# Scalarization Functionals in Mathematical Finance and Vector Optimization

## - A New View on Acceptance Sets and Risk Measures -

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Herrn Marcel Marohn

geboren am 18. Juli 1990 in Leipzig

**Gutachter:**
Frau Prof. Dr. Christiane Tammer (Martin-Luther-Universität Halle-Wittenberg)
Frau Prof. Dr. Gemayqzel Bouza Allende (Universidad de la Habana)

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*To my mother.*

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# Contents

# List of Symbols and Abbreviations

In the following, we present a list of symbols and abbreviations used in this thesis. Most of these are explained in more detail when they appear in the text for the first time. The reader might look up here if a symbol is used later without explanation once again.

### Specific sets and spaces

| | |
|---|---|
| $\emptyset$ | empty set |
| $\mathbb{N}$ | set of positive natural numbers |
| $\mathbb{R}^n$ | space of $n$-dimensional vectors of real numbers |
| $\mathbb{R}^n_+$ | non-negative orthant of $\mathbb{R}^n$ |
| $\mathbb{R}_+$ | set of non-negative real numbers |
| $\mathbb{R}_>$ | set of positive real numbers |
| $\mathbb{R}_-$ | set of non-positive real numbers |
| $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ | set of extended real numbers |
| $\mathcal{B}(\mathbb{R})$ | Borel-$\sigma$-Algebra on $\mathbb{R}$ |
| $\mathcal{L}^0 = \mathcal{L}^0(\Omega, \mathcal{F})$ | space of measurable real-valued functions on a measurable space $(\Omega, \mathcal{F})$ |
| $\mathcal{L}^p = \mathcal{L}^p(\Omega, \mathcal{F}, \mathbb{P})$ | space of $p$-integrable real-valued functions on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ |
| $\mathcal{L}^\infty = \mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$ | space of essential bounded measurable real-valued functions on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ |
| $\mathcal{B}(\Omega, \mathcal{F})$ | space of bounded measurable real-valued functions on a measurable space $(\Omega, \mathcal{F})$ |

### Topological and vector spaces

| | |
|---|---|
| $\tau$ | topology on a set $\mathcal{X}$ |
| $\mathcal{T}$ | basis for a topology $\tau$ |
| $(\mathcal{X}, \tau)$ | topological (vector) space with topology $\tau \subseteq \mathcal{P}(\mathcal{X})$ |
| $\mathcal{N}_X$ | set of all neighborhoods of $X \in \mathcal{X}$ |
| $\mathcal{R}, \leq$ | binary relation on $\mathcal{X}$ |
| $\leq_\mathcal{C}$ | binary relation on $\mathcal{X}$ given by a cone $\mathcal{C}$ |

| | |
|---|---|
| $\mathcal{X}_+$ | positive cone of $\mathcal{X}$ |
| $\mathcal{X}^*$ | topological dual space of $\mathcal{X}$ |
| $\mathcal{X}^n$ | ($n$-times) product space of $\mathcal{X}$ with itself |
| $d$ | metric on a set $\mathcal{X}$ |
| $(\mathcal{X}, d)$ | metric space with metric $d\colon \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ |
| $\|\cdot\|$ | norm on a vector space $\mathcal{X}$ |
| $(\mathcal{X}, \|\cdot\|)$ | normed space with norm $\|\cdot\| : \mathcal{X} \times \mathbb{R}_+$ |
| $\dim \mathcal{X}$ | dimension of the real vector space $\mathcal{X}$ |
| $\Omega$ | sample space |
| $\mathcal{F}$ | event space, being a $\sigma$-Algebra on $\Omega$ |
| $\mathbb{P}$ | probability measure on $\mathcal{F}$ |
| $(\Omega, \mathcal{F})$ | measurable space |
| $(\Omega, \mathcal{F}, \mathbb{P})$ | probability space |

## Functions and operators

| | |
|---|---|
| $\operatorname{Im} f$ | image of a functional $f$ |
| $f^{-1}(\mathcal{D})$ | preimage of a set $\mathcal{D}$ under the functional $f$ |
| $\operatorname{dom} f$ | domain of a functional $f$ |
| $\operatorname{epi} f$ | epigraph of a functional $f$ |
| $\ker f$ | kernel or nullity of a linear functional $f$ |
| $\operatorname{lev}_{f,\leq}(t)$ | sublevel set of $f$ to the level $t$ |
| $\operatorname{lev}_{f,<}(t)$ | strict sublevel set of $f$ to the level $t$ |
| $\operatorname{lev}_{f,=}(t)$ | level line of $f$ to the level $t$ |
| $\mathcal{P}(\mathcal{D})$ | power set, i.e., set of all subsets of $\mathcal{D}$ |
| $\operatorname{span} \mathcal{D}$ | span of the set $\mathcal{D}$ (the subspace generated by $\mathcal{D}$) |
| $\operatorname{conv} \mathcal{D}$ | convex hull of the set $\mathcal{D}$ |
| $\operatorname{rec} \mathcal{D}$ | recession cone of the set $\mathcal{D}$ |
| $\mathcal{B}_r(X)$ | open ball with radius $r > 0$ and center $X$ |
| $\operatorname{int} \mathcal{D}$ | interior of the set $\mathcal{D}$ |
| $\operatorname{cl} \mathcal{D}$ | closure of the set $\mathcal{D}$ |
| $\operatorname{bd} \mathcal{D}$ | boundary of the set $\mathcal{D}$ |
| $\operatorname{int}_{-U}(\mathcal{D})$ | $(-U)$-directional interior of the set $\mathcal{D}$ |
| $\operatorname{cl}_{-U}(\mathcal{D})$ | $(-U)$-directional closure of the set $\mathcal{D}$ |
| $\operatorname{bd}_{-U}(\mathcal{D})$ | $(-U)$-directional boundary of the set $\mathcal{D}$ |
| $\varphi_{\mathcal{D},K}$ | Gerstewitz functional on a vector space $\mathcal{X}$ with $\emptyset \neq \mathcal{D} \subseteq \mathcal{X}$, $K \in \mathcal{X} \backslash \{\mathbb{0}\}$, and $\mathcal{D} - \mathbb{R}_+ K \subseteq \mathcal{D}$ |

### Financial markets and risk measures

| | |
|---|---|
| $\mathcal{X}$ | space of capital positions (in general, a real vector space) |
| $S^i := (S_0^i, S_1^i)^T$ | $i$-th eligible asset with price $S_0^i \in \mathbb{R}$ and payoff $S_1^i \in \mathcal{X}$ |
| $\mathcal{S} := (S^i)_{i=0}^n$ | collection of the eligible assets |
| $S_0 := (S_0^i)_{i=0}^n$ | vector of prices of the eligible assets |
| $S_1 := (S_1^i)_{i=0}^n$ | vector of eligible payoffs |
| $U$ | positive eligible payoff in $\mathcal{M} \cap \mathcal{X}_+$ (by Ass. 2: $\pi(U) = 1$) |
| $\mathcal{M} := \operatorname{span}\{S_1^0, S_1^1, \dots, S_1^n\}$ | space of eligible payoffs |
| $\pi$ | pricing functional on $\mathcal{M}$ |
| $\mathcal{A}$ | acceptance set in $\mathcal{X}$ |
| $\rho$ | monetary risk measure on $\mathcal{X}$ |
| $\rho_{\mathcal{A},\mathcal{M},\pi}$ | risk measure on $\mathcal{X}$ associated with $\mathcal{A}$, $\mathcal{M}$, and $\pi$ |
| $\mathcal{A}_\rho$ | acceptance set associated with the risk measure $\rho$ |
| (FM) | financial market model |
| $q_{(\alpha)}$ | lower $\alpha$-quantile |
| $q^{(\alpha)}$ | upper $\alpha$-quantile |
| $\mathbb{1}_\Omega$ | random variable that equals 1 for each state $\omega \in \Omega$ |
| $\mathbb{P}$-a.s. | $\mathbb{P}$-almost surely, i.e., with probability 1 |
| $\mathrm{VaR}_\alpha$ | Value-at-Risk to the level $\alpha \in (0, 1)$ |
| $\mathrm{CVaR}_\alpha$ | Conditional Value-at-Risk to the level $\alpha \in (0, 1)$ |
| $\mathrm{AVaR}_\alpha$ | Average Value-at-Risk to the level $\alpha \in (0, 1)$ |
| $\mathrm{ES}_\alpha$ | Expected Shortfall to the level $\alpha \in (0, 1)$ |

### Optimization

| | |
|---|---|
| $(P_\pi(X))$ | optimization problem with given $X \in \mathcal{X}$ in (FM): $\pi(Z) \to \min\limits_{X+Z\in\mathcal{A}, Z\in\mathcal{M}}!$ |
| $(V_\mathcal{A})$ | vector optimization problem with domination set $\mathcal{D}$ in (FM): $X^0 \to \mathcal{D}\text{-}\min\limits_{X^0\in\mathcal{A}}$ |
| $\mathrm{Eff}(\mathcal{A}, \mathcal{D})$ | set of efficient points of $\mathcal{A}$ with respect to $\mathcal{D}$ |
| $\mathrm{Eff}_w(\mathcal{A}, \mathcal{D})$ | set of weakly efficient points of $\mathcal{A}$ with respect to $\mathcal{D}$ |
| $\mathcal{C}_\pi$ | price cone |
| $\mathcal{C}_{\ker}$ | kernel cone |

# Notations

Besides the named symbols and abbreviations listed before, there are certain general conventions that will be followed throughout this thesis.

- Sets (including vector spaces and topological spaces) are denoted by calligraphic upper case latin letters (e.g., $\mathcal{A}$ and $\mathcal{X}$). There are some less exceptions if there is a more common notion in the literature (e.g., $\tau$ for a topology, which is a set itself).
- Elements of sets (and, therefore, vectors as well) that are not explicitly scalars of a real vector space are denoted by upper case latin letters (e.g., $X$).
- Scalars are denoted by lower case greek letters (e.g., $\lambda$).
- Constants and elements of sequences of real numbers are denoted by lower case latin letters (e.g., $c$ and $(t_n)_{n\in\mathbb{N}}$). There are some less exceptions if there is a more common notion in the literature (e.g., $\epsilon$ and $\delta$ to describe continuity, and $\alpha$ for significance levels).
- Sequences in vector spaces (except $\mathbb{R}$) are denoted by upper case latin letters with superscript index in round brackets, followed by the index set written as subscript (e.g., $(X^n)_{n\in\mathbb{N}}$). For sequences in $\mathbb{R}$, lower case latin letters with subscript index are used (e.g., $(t_n)_{n\in\mathbb{N}}$).
- Maps do not follow any of the listed notations before. For example, we write $\pi$ for a pricing functional, $\mathcal{E}$ for a set-valued optimal payoff map, and $d$ for a distance function. We do so to preserve literature references.
- Vectors $X \in \mathbb{R}^n$ are written as column vectors with entries $X_i$ for $i = 1, \dots, n$, i.e., $X = (X_1, \dots, X_n)^T$ by use of the transpose operator $^T$. The only exception is given by portfolios $x \in \mathbb{R}^{n+1}$, which have an index set $\{0, 1, \dots, n\}$ to highlight the reference of each component $x_i$ to the corresponding asset with number $i \in \{0, 1, \dots, n\}$.
- The null vector of a real vector space is denoted by $\mathbb{0}$.
- If nothing else is stated, $\mathbb{R}^n$ is understood to be the Euclidean space, i.e., $\mathbb{R}^n$ equipped with the natural scalar product $\langle X, Y\rangle = X^T Y$, the Euclidean norm $\|\cdot\| = \sqrt{\langle X, X\rangle}$, and, if needed, the standard topology induced by $\|\cdot\|$.

# Introduction

In 2008, the financial crisis shook the international financial markets and heralded a new age for the agents in them. The crisis revealed a massive misbehavior of banks in investing and risk managing, leading to big failure ratios and a wave of insolvencies, compare Hull [112] for an overview of the crisis, and Federico, Vazquez [193] for some empirical studies about weak liquidity and higher leverage before the crisis resulting in failures during the crisis. With the insolvency of the investment bank Lehman Brothers, institutions realized that they can not rely on the theory of "too big to fail" because this "big player" was not rescued by the government despite its systemic relevance. Afterwards, financial institutions and their activities, especially, in investment and risk management, got much more into focus to protect small investors and guard the stability of the economic system. Expanded regulatories, e.g., by Basel III (see [21], [22]) mean much more restrictions for the possible actions of financial institutions. Hence, it is from special interest to find profitable investment strategies and most important to make efficient decisions, especially, in the current challenging period of structural low interests. For example, mortgage loans in Germany belong to the basic products in bank portfolios, which were very attractive for the institutions in the past, especially, with the real estate as a security. The contract involves a legally defined 10-year special right of cancellation (see section 489 (1) no. 2 of the German Civil Code (Bürgerliches Gesetzbuch – BGB)). If a rational loan taker took a mortgage loan before 2008, he will exercise that special right to secure the, nowadays, lower interest rates for the remaining time of the lending period by taking a new mortgage loan at current low interest rates. The consequences for the bank are sharp losses through write-downs in the books. Compensation can be reached by cost reducing, e.g., by closing local affiliates (which is limited by the number of affiliates), or additional risk taking, generating hopefully higher returns. Since there are more restrictions in risk-taking by the intense regulatory influence, there are also restrictions in generating returns with investments by the risk-return-trade-off, see, e.g., Bali and Peng in [18] for explanations and empirical evidence of this trade-off.

In this thesis, we study portfolio optimization problems and risk measures associated with acceptance sets. Acceptance sets are suitable to model regulatory restrictions of financial institutions and their capital positions. Portfolio optimization problems, i.e., the division of an amount of capital into assets with respect to some criteria like expected return maximization or risk minimization (see Markowitz [138], Elbannan [62], Gupta et al. [93], and references therein), represent one popular problem class in financial mathematics (which is not to confuse with capital allocation problems, see, e.g., Canna et al. [38], [39], Denault [51], and Kalkbrener [121]). Portfolio optimization problems have been widely studied, having its origin in the landmark paper "Portfolio Selection" of Harry M. Markowitz in 1952, see [138]. In the so called *Markowitz model*, we suppose a rational, utility-maximizing and risk averse investor, who has to decide about the portions $x_i \in \mathbb{R}, i \in \{1, \ldots, n\}$ of his capital that shall be invested in the $i$-th of $n \in \mathbb{N}$ given assets for a certain period of time. Each unit of an asset $i$ has a return $R_i$ after the period of time, which is a random variable. For the chosen portions $x = (x_1, \ldots, x_n)^T$, the corresponding portfolio return is given by $R(x) := \sum_{i=1}^n x_i R_i$. In the classical Markowitz model, the risk of the investment is measured by the variance $\mathrm{Var}(\cdot)$ of the portfolio return and the gain is quantified by the expected value $\mathbb{E}(\cdot)$ of the portfolio return. The investor wants to determine the portions $x$ such that the gain is as large and the risk as small as possible. That leads to the following multiobjective Markowitz model with two, in general, contradicting objectives:

$$\begin{pmatrix} -\mathbb{E}(R(x)) \\ \mathrm{Var}(R(x)) \end{pmatrix} \longrightarrow \underset{x \geq 0,\ \sum_{i=1}^{n} x_i = 1}{\mathcal{D}\text{-Min}}\ , \qquad (P_{\text{Markowitz}})$$

where $\mathcal{D}$-Min stands for minimization in the sense of vector optimization with a nonempty domination set $\mathcal{D} \subseteq \mathbb{R}^2$ (see Section 1.2), $\mathrm{Var}(R(x))$ is the variance of the portfolio return, and $\mathbb{E}(R(x))$ the expected total portfolio return.

The variance was one of the first choices for measuring risk, but there are various options for modeling and measuring risk, see [159]. While risk could be understood as the extend of the deviation of an outcome from an expected value, it could also be measured as the valuation of the potential of a possible probability-based loss. It turned out that the variance does not fulfill desirable properties being from interest for practical purposes (see, e.g., Emmer et al. [65] for practically preferred measures). Artzner et al. initiated an axiomatic approach for suitable risk measures in [14] to face that deficit, known as coherent risk measures, which was generalized to monetary risk measures (see Föllmer, Schied [78]) and convex risk measures (see Föllmer, Schied [76]). Examples for widely used risk measures are the Value-at-Risk (see Christoffersen [42], Duffie [54], and Pritsker [161]) and the Conditional-Value-at-Risk (or Expected Shortfall, see Acerbi, Tasche [3], [4], Pflug [158], and Rockafellar, Uryasev [166], [167]). Spectral risk measures, as special coherent risk measures, provide a connection with deviation measures like the (often used for risk measuring in financial industry) standard deviation (see Acerbi [1] and Rockafellar et al. [168], [169]). Hence, modern publications concentrate on other measures than the variance in general. Following that, this thesis is devoted to monetary risk measures being more suitable for many applications because of their important properties, especially, translation invariance (the risk is reduced by adding a secure capital amount) and monotonicity (capital positions with larger payoffs in each scenario have less risk). We consider the monetary risk measures $\rho_{\mathcal{A},\mathcal{M},\pi} \colon \mathcal{X} \to \overline{\mathbb{R}}$ given by

$$\mathcal{X} \ni X \mapsto \rho_{\mathcal{A},\mathcal{M},\pi}(X) := \inf\{\pi(Z) \mid Z \in \mathcal{M}, X + Z \in \mathcal{A}\} \qquad (1)$$

following Farkas et al. [71] and Baes et al. [17], where $\pi \colon \mathcal{M} \to \mathbb{R}$ is a linear pricing functional defined on a subspace $\mathcal{M} \subseteq \mathcal{X}$ of the real vector space of capital positions $\mathcal{X}$, and $\mathcal{A} \subseteq \mathcal{X}$ is an acceptance set. $\mathcal{M}$ represents the payoffs of given eligible assets in the space of capital positions. The functional $\rho_{\mathcal{A},\mathcal{M},\pi}$ is a monetary risk measure that quantifies the minimal costs for a financial institution to satisfy regulatory preconditions with its initial capital position $X \in \mathcal{X}$. This is also called reaching acceptability to highlight the relationship to the acceptance set $\mathcal{A}$. In this thesis, we aim to show new important properties of the functional $\rho_{\mathcal{A},\mathcal{M},\pi}$ and applications. Especially, we characterize finiteness, properness, convexity, subadditivity, (strict) sublevel sets, and level lines of $\rho_{\mathcal{A},\mathcal{M},\pi}$. These properties are important, for example, for including $\rho_{\mathcal{A},\mathcal{M},\pi}$ in optimization problems and determining the solution set of the problem ($P_\pi(X)$) below (with optimal value $\rho_{\mathcal{A},\mathcal{M},\pi}(X)$ for each $X \in \mathcal{X}$), but also for developing algorithms for finding solutions numerically.

The risk measure $\rho_{\mathcal{A},\mathcal{M},\pi}$ given by (1) is a generalization of the nonlinear functional $\varphi_{\mathcal{A},K} : \mathcal{X} \to \overline{\mathbb{R}}$ known as Gerstewitz-functional in the literature, which is given by

$$\varphi_{\mathcal{D},K}(X) := \inf\{t \in \mathbb{R} \mid X \in tK + \mathcal{D}\} \tag{2}$$

with $\mathcal{D}$ being a nonempty subset of the vector space $\mathcal{X}$, $K \in \mathcal{X}\backslash\{\mathbb{0}\}$, and $\mathcal{D} - \mathbb{R}_+K \subseteq \mathcal{D}$. Indeed, in $\rho_{\mathcal{A},\mathcal{M},\pi}$, a set of directions is simultaneously allowed instead of one fixed direction $K \in \mathcal{X}\backslash\{\mathbb{0}\}$ by using the subspace $\mathcal{M}$ of $\mathcal{X}$ . Moreover, the movements $Z \in \mathcal{M}$ corresponding to the directions are more generally valuated by a linear functional $\pi$. Functionals given by (2) are used by Gerstewitz in [88] for deriving separation theorems for not necessarily convex sets where $\varphi_{\mathcal{A},K}$ is employed as separation functional, and for scalarization of vector optimization problems. Scalarization and separation of sets are important for many fields of research, e.g., functional analysis, optimization, production theory, and financial mathematics, especially, with respect to risk theory, such that functionals defined by (2) are applicable in many settings. Although the risk measure $\rho_{\mathcal{A},\mathcal{M},\pi}$ given by (1) includes generalized components as mentioned before, the so called Reduction Lemma (see Farkas et al. [71]) shows that $\rho_{\mathcal{A},\mathcal{M},\pi}$ can be nevertheless reduced to a functional from type (2) if we assume the existence of some positive payoff $U \in \mathcal{M} \cap \mathcal{X}_+$ with $\pi(U) > 0$, where $\mathcal{X}_+$ denotes the positive cone in $\mathcal{X}$. Our results provide new relationships between properties of $\rho_{\mathcal{A},\mathcal{M},\pi}$ and the assumptions on the financial market model, especially, on the acceptance set $\mathcal{A}$ and the subspace $\mathcal{M}$ of $\mathcal{X}$. To do so, we exploit the relationship between $\rho_{\mathcal{A},\mathcal{M},\pi}$ and a functional $\varphi_{\mathcal{D},K}$ from type (2) with a suitable set $\mathcal{D}$ and direction $K$. Nevertheless, we are not just referring to results for Gerstewitz functionals because this is not adequate for our purposes even if it would be possible sometimes. As noticed before, we want to gain deeper insights in the role of the given financial market by deriving characterizations for the properties of $\rho_{\mathcal{A},\mathcal{M},\pi}$ that are not just primarily given by properties of $\mathcal{D}$ and $K$. For example, we have to consider $\mathcal{D} := \mathcal{A} + \ker \pi$ in the related Gerstewitz functional $\varphi_{\mathcal{D},K}$, but there is no direct relationship between properties of $\mathcal{A} + \ker \pi$ and the acceptance set $\mathcal{A}$ itself in general.

Motivated by [17], we aim to apply the gained results to the optimization problem

$$\pi(Z) \to \min_{X+Z\in\mathcal{A},\ Z\in\mathcal{M}} \tag{$P_\pi(X)$}$$

for given $X \in \mathcal{X}$, where the optimal value of $(P_\pi(X))$ is given by $\rho_{\mathcal{A},\mathcal{M},\pi}(X)$. In [17], $\mathcal{X}$ is assumed to be a first-countable locally convex Hausdorff space, and $\mathcal{A}$ is assumed to be closed. Here, we derive new results for solutions of $(P_\pi(X))$ under weaker assumptions, especially, in a vector space $\mathcal{X}$ without any topology requirements on $\mathcal{A}$. The solution set of $(P_\pi(X))$ is given by the set-valued optimal (eligible) payoff map $\mathcal{E}\colon \mathcal{X} \rightrightarrows \mathcal{M}$ (introduced in [17] under the listed stronger assumptions) as

$$\mathcal{E}(X) := \{Z \in \mathcal{M} \mid X + Z \in \mathcal{A}, \pi(Z) = \rho_{\mathcal{A},\mathcal{M},\pi}(X)\}. \tag{3}$$

We derive a characterization of $\mathcal{E}(X)$ that highlights the algebraic properties of $\mathcal{A}$, especially, the directional boundary of $\mathcal{A}$. It turns out that our description is also suitable for the setting in [17] and more precisely. Moreover, we use this characterization to derive generalized uniqueness and existence results for solutions of $(P_\pi(X))$.

As seen so far, multiobjective portfolio optimization problems, risk measures, and acceptance sets are from special interest for researchers and practitioners, providing an enormous wide of research topics. This thesis is devoted to vector optimization problems and risk measures with respect to capital positions and financial institutions that have to fulfill regulatory restrictions. Thus, our results directly fit into the current development of research in financial mathematics and vector optimization. We assume a real vector space $\mathcal{X}$ of capital positions instead of a topological vector space (or, especially, an $\mathcal{L}^p$-space). Although it is quite natural in mathematics to make minimal assumptions, there are important practical reasons to do so. In mathematical finance and, especially, in arbitrage theory (see, e.g., the financial market models in [46, Section 2] and Riedel [165]), considering general vector spaces instead of topological vector spaces $\mathcal{X}$ is from interest for addressing practitioners and improving the practical applicability. Industrial users or economic researchers often work with financial data or samples to deal with random variables and, thus, real vector spaces. Readers like them might not know, which topology is suitable to choose in the given situation. In the literature about monetary risk measures, capital positions are supposed to be elements of a vector space $\mathcal{X}$ of bounded functions containing the constants (see Föllmer, Schied [78]). Sometimes, it is not convenient to consider $\mathcal{X}$ endowed with the supremum norm (inducing the topology). Hence, it is preferable to consider a vector space $\mathcal{X}$ for directly and easily application of our outcomes, and we do not pose the danger of generating a lack of interest for non-mathematicians by unnecessary mathematical or non-economical assumptions.

The portfolio optimization problem $(P_{\text{Markowitz}})$ in the Markowitz model is a vector optimization problem. Vector optimization is an important branch of optimization theory (see Ehrgott [57], Göpfert et al. [91], Jahn [116], Khan et al. [124], and references therein). It deals with the optimization of vector-valued mappings defined on a vector space, where the image space is a vector space equipped with a preorder. Vector optimization problems have received special attention due to their applications in different fields of sciences like financial mathematics (see Feinstein, Rudloff [73] and Hamel et al. [101], [104]), social welfare economics (see Bao, Mordukhovich [20], and Islam, Craven [115]), energetic and medical engineering (see Bischoff, Jahn, Köbis [30], Eichfelder [59], and Küfer et al. [130]), game theory (see Corley [44], Nieuwenhuis [155], and Hamel, Löhne [103]), location theory (see Günther, Tammer [92], and Nickel, Puerto [154]), and robust multiobjective decision making (see Klamroth et al. [125], [126], Köbis [127], and Wiecek, Dranichak [197]). There are different solution concepts for vector optimization problems, see, e.g., Ansari et al. [11], Eichfelder, Jahn [60], Khan et al. [124], Heyde, Löhne [108], and references therein. Fundamental tools for solving vector optimization problems theoretically and numerically, as well, are scalarization techniques, compare, e.g., Bouza et al. [36], [37], Eichfelder [58], Gerstewitz [88], Göpfert et al. [91], Gutiérrez et al. [94], [95], [96], Luc [136], Miglierina, Molho [148], Tammer, Zălinescu [186], Tammer, Weidner [185], and references therein. Well known scalarization functionals for vector optimization problems are special cases of the Gerstewitz functional given by (2) (see Tammer, Weidner [185, Ch. 6]). For example, in the literature, one studies surrogate problems for solving ($P_{\text{Markowitz}}$). Applying the $\epsilon$-constraint method by Haimes et al. (see [98]) as a special case of (2), the decision maker can choose priory a lower bound $r_{\min} > 0$ for the expected total return and minimizes the risk. He determines a portfolio with expected total return greater or equal than $r_{\min}$ and minimal risk. The corresponding scalar $\epsilon$-constraint optimization problem is called *Min–risk problem*:

$$\operatorname{Var}(R(x)) \longrightarrow \min_{\mathbb{E}(R(x)) \geq r_{\min},\; x \geq \mathbb{0},\; \sum_{i=1}^{n} x_i = 1} . \qquad (P_{\text{Min-risk}})$$

It is well known (compare Hamel [99], Tammer [184], and references therein) that the scalar $\epsilon$-constraint problem ($P_{\text{Min-risk}}$) of the original vector optimization problem ($P_{\text{Markowitz}}$) can be described as special case of a scalarization by the Gerstewitz functional $\varphi_{\mathcal{D},K}$ given by (2) with $K := (1,0)^T$ and $\mathcal{D} := (0, r_{\min})^T + \mathcal{C}$ by use of the convex cone $\mathcal{C} := \{X \in \mathbb{R}^2 : \; X_1 \leq 0, \; X_2 \geq 0\}$, and setting $\inf \emptyset = +\infty$. Indeed, the Min–risk problem ($P_{\text{Min-risk}}$) is equivalent to

$$\varphi_{\mathcal{A},K}((\operatorname{Var}(R(x)), \mathbb{E}(R(x)))^T) \longrightarrow \min_{x \geq \mathbb{0},\; \sum_{i=1}^{n} x_i = 1}$$

with $\varphi_{\mathcal{A},K} : \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\}$ given by

$$\varphi_{\mathcal{A},K}(X) := \inf\{t \in \mathbb{R} \mid \; X \in t(1,0)^T + (0, r_{\min})^T + \mathcal{C}\}.$$

Now, we return to the optimization problem ($P_\pi(X)$). Here, the capital positions $X \in \mathcal{X}$ are viewed as payoffs of corresponding portfolios, which can be changed by investments into eligible assets with eligible payoffs modeled by the subspace $\mathcal{M}$ of $\mathcal{X}$. Hence, it represents a (scalar) portfolio optimization problem, too. With $\rho_{\mathcal{A},\mathcal{M},\pi}$ given by (1) being the optimal value of ($P_\pi(X)$) for $X \in \mathcal{X}$, $-\rho_{\mathcal{A},\mathcal{M},\pi}$ can be used as scalarization functional for a vector optimization problem

$$X^0 \to \mathcal{D}\text{-}\min_{X^0 \in \mathcal{A}} \qquad (V_{\mathcal{A}})$$

in the space of capital positions under a suitable domination set $\mathcal{D}$ to distinguish between the (for the given $X \in \mathcal{X}$ possibly interesting) acceptable capital positions $X^0 \in \mathcal{A}$. In this thesis, we will derive new results for the solution set of the vector optimization problem ($V_{\mathcal{A}}$) by showing characterizations of the sets of efficient and weakly efficient points of the acceptance set $\mathcal{A}$, and outlining their relationship to the set of (cost-optimal) acceptable capital positions denoted by $\mathcal{A}'$. The set $\mathcal{A}'$ is from special interest because, for directionally closed acceptance sets $\mathcal{A}$, it holds that

$$\mathcal{A}' = \bigcup_{\substack{X \in \mathcal{X}, \\ \mathcal{E}(X) \neq \emptyset}} (\{X\} + \mathcal{E}(X)) \subseteq \mathcal{A}, \tag{4}$$

i.e., $\mathcal{A}'$ is the set of all acceptable positions resulting by the solution set $\mathcal{E}(X)$ given by (3) through any $X \in \mathcal{X}$. For gaining a deep understanding of the limits of our chosen financial model, we consider two domination sets motivated by economical interpretation of the interests of institutional decision makers. These are non-pointed, convex cones, and our results provide new extensive insights in the role of the domination set for efficient points of acceptance sets in (not necessary topological) vector spaces, and also for weakly efficient points in topological vector spaces. We will see that properties of the subspace $\mathcal{M}$ of $\mathcal{X}$ are crucial for our observations with respect to the set $\mathcal{A}'$ (which is by (4) directly connected with the solution set $\mathcal{E}(X)$), since it does not have to be possible to reach any acceptable position by another capital position in $\mathcal{A}$. Hence, we also provide an intensive study of properties of $\mathcal{A}'$ and its relationship to the assumptions on the financial market.

As noticed before, the Markowitz model was the origin of modern portfolio theory, and there have been arisen many extensions of it (see, e.g., Ehrgott, Klamroth, Schwehm [57]). A very important and common extension is the Capital Asset Pricing Model (CAPM), which is one of the most popular and widely used equilibrium asset pricing models in theory and practice, see Sharpe [180], Lintner [135], and Mossin [151]. The popularity arose because it was a first model proposing a relationship between expected return and risk of assets. Moreover, general risk measures as well as deviation measures have been involved in portfolio analysis as objective functions or risk constraints (see, e.g., Akume et al. [7], Gabih et al.[83], Gaivoronski, Pflug [86], Krokhmal et al. [129], Rockafellar et al. [170], and Sun et. al. [183]). In this thesis, risk is considered from an institutional point of view. Nevertheless, as seen by the financial crisis in 2008, measuring systemic risk is from economical interest, too, see, e.g., Acharya et al. [6] and Feinstein et al. [73]. One approach using multidimensional acceptance sets is studied by Biagini et al. in [28]. Financial institutions face many different risks like credit risk (or default risk), market risk, and liquidity risk, which are issue of many research outcomes, see, e.g., Ghabri et al. [89] for a study about the influence of Bitcoin on liquidity risk, or Redeker, Wunderlich [164] for a study of credit risk with asymmetric information. It is shown and well-known that the class of functionals with the property (2) coincides with the class of translation invariant functions and, thus, functionals given by (2) are employed as coherent risk measures in risk theory (see Artzner et al. [14], Jaschke, Küchler [118], and also [82], [101], [107], [177], and references therein), since translation invariance is a basic property of risk measures. Hence, functionals given by (2) are applicable in many settings and for different problems in mathematics and mathematical economics. In financial mathematics, risk measures with respect to acceptance sets have been studied for many years. Artzner et al. proved in [14] an one-to-one-correspondence between coherent risk measures and convex, closed acceptance sets, which was generalized by Föllmer and Schied in [78] to monetary risk measures and general acceptance sets. Thus, acceptance sets naturally occur when we are working with risk measures. Obviously, it is from interest for practical purposes to find effective algorithms to solve the corresponding portfolio problems, see, e.g., Feng et al. [75] for portfolio optimization with Value-at-Risk. For developing algorithms, it is important to study the mathematical properties of the solution set of the optimization problem, first, as we will do it in this thesis for the optimization problem ($P_\pi(X)$).

Summing the previous up, we make new contributions to the current research on practical motivated optimization problems in financial markets concerning risk measures associated with acceptance sets:

- We show important properties of the risk measure $\rho_{\mathcal{A},\mathcal{M},\pi}$ given by (1), which quantifies the minimal costs of reaching acceptability with respect to some acceptance set $\mathcal{A}$, a pricing functional $\pi$, and a subspace of allowed actions $\mathcal{M} \subseteq \mathcal{X}$ being generated by the payoffs of eligible assets. These properties are crucial for interpreting $\rho_{\mathcal{A},\mathcal{M},\pi}$ as a monetary risk measure, e.g., if it supports diversification in portfolio management.

- By use of these properties of $\rho_{\mathcal{A},\mathcal{M},\pi}$, we give a new, most precisely description of the solution set $\mathcal{E}(X)$ with $X \in \mathcal{X}$ given by (3) of the optimization problem ($P_\pi(X)$) in a generalized setting and outline important relationships between $\mathcal{E}(X)$ and the acceptance set $\mathcal{A}$.

- With respect to the vector optimization problem (V), we derive useful results for characterizing (weakly) efficient points of the acceptance set $\mathcal{A}$ and provide new relationships of these points with solutions of ($P_\pi(X)$).

- With these new results, we are able to get an intense and fully new view on the interplay of the mathematical objects in our financial market, namely, the space of capital positions $\mathcal{X}$, acceptance set $\mathcal{A}$, the space of eligible payoffs $\mathcal{M}$, and the pricing functional $\pi$. By outlining the minimal requirements for the proved results (excepting weakly efficient points, without any topology), we provide widely applicability in practice and research, and unite current research outcomes in our general setting.

In the following, we summarize the research issues and outcomes of this thesis: For providing the necessary mathematical background with the aim of making the thesis as self-contained as possible to improve the readability, we collect the basic preliminaries from functional analysis, optimization and financial mathematics in Chapter 1. We also unite different definitions for Value-at-Risk, and give an overview about varying assumptions on general risk measures and unnecessary distinction of Expected Shortfall, Conditional Value-at-Risk, and Average Value-at-Risk in the literature. In Chapter 2, we present our considered one-period model of a financial market (FM) within a vector space $\mathcal{X}$ of capital positions, by use of the terminology from Chapter 1. Moreover, we introduce the mathematical economical optimization problem ($P_\pi(X)$) in more detail and show new important properties of the nonlinear risk measure $\rho_{\mathcal{A},\mathcal{M},\pi}\colon \mathcal{X} \to \overline{\mathbb{R}}$ given by (1) (see [17], [71], [82], and [177]), where $\pi\colon \mathcal{M} \to \mathbb{R}$ is a pricing functional, $\mathcal{M} \subseteq \mathcal{X}$ is a subspace of the vector space $\mathcal{X}$, and $\mathcal{A} \subseteq \mathcal{X}$ is an acceptance set. The functional $\rho_{\mathcal{A},\mathcal{M},\pi}$ is the optimal value of ($P_\pi(X)$) for $X \in \mathcal{X}$. For instance, we characterize the finiteness, (strict) sublevel sets, and level lines of $\rho_{\mathcal{A},\mathcal{M},\pi}$. In Chapter 3, we focus on the solution set $\mathcal{E}(X)$ (see (3)) of the optimization problem ($P_\pi(X)$) for $X \in \mathcal{X}$ with a directionally closed acceptance set $\mathcal{A}$. We obtain a characterization of $\mathcal{E}(X)$ that is more precisely than the one in [17]. Also, we show existence and uniqueness results for solutions of ($P_\pi(X)$). Moreover, we highlight how to determine the set of optimal acceptable capital positions $\mathcal{A}'$, which is given by (4) for directionally closed acceptance sets and is directly connected with $\mathcal{E}(X)$.We focus on the vector optimization problem ($V_\mathcal{A}$) in Chapter 4, and introduce two domination sets given by non-pointed, convex cones $\mathcal{C}_\pi$ and $\mathcal{C}_{\text{ker}}$, which we derive by the view of institutional decision makers. With these cones, we gain a new view on efficient and weakly efficient points of acceptance sets. Especially, we provide deep insights in efficient points of $\mathcal{A}$ for vector spaces without use of any topology. For example, we show that the set of efficient points coincides with $\mathcal{A}'$ in many cases. Finally, in the conclusions, we summarize the main results and give an outlook on some further interesting research, and summarize the author's contributions to the chapters of this thesis afterwards.

# Chapter 1

# Mathematical Preliminaries

In this chapter, we present an overview of the basic terminology and results from functional analysis, optimization, and financial mathematics that is relevant for this thesis. It aims to improve the readability by providing a glossary about the mathematical background such that this thesis is as self-contained as possible. Although its exhaustiveness, we assume that the reader is familiar with basic operations of set theory. The chapter is organized as follows:

- In Section 1.1, we begin by recalling algebraic and topological properties of sets and spaces. Especially, we will deal with directional properties of subsets of real vector spaces, since directionally closed acceptance sets will be subject of our studies. Moreover, we sketch some results and examples for normed vector spaces because many considered vector spaces can be naturally equipped with a norm. We will pay particular attention to extended real-valued functionals, which are important for working with risk measures and scalarization functionals.

- In Section 1.2, basic concepts of vector optimization are collected. This includes efficient and weakly efficient points. We present properties of the sets of (weakly) efficient points and sketch the solution concept of using scalarization functionals.

- At last, since this thesis is about optimization problems related to financial risk management, we summarize the financial mathematical background in Section 1.3. Capital positions and the payoff of financial assets are assumed to be random, in general. Hence, we shortly recall the basics from probability theory first. Afterwards, we present one-period models of financial markets, including financial assets and pricing functionals. Finally, we give a short overview about monetary risk measures, and present practical and theoretical important examples like the Value-at-Risk or the Expected Shortfall with their corresponding main properties. Here, we prove some results in the given setting, since, e.g., these risk measures are not defined identically in the literature and, especially, are not always considered as being extended real-valued.

## 1.1 Fundamentals of functional analysis

Throughout this thesis, the set of natural numbers is denoted by $\mathbb{N}$, the set of real numbers is denoted by $\mathbb{R}$, the extended set of real numbers is denoted by $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$, the set of non-negative real numbers by $\mathbb{R}_+$, the set of positive real numbers by $\mathbb{R}_>$, and the set of non-positive real numbers by $\mathbb{R}_-$. Let $\mathcal{A}, \mathcal{B}$ be two arbitrary sets. As usual, the union, intersection, difference, and Cartesian product of these sets are denoted by $\mathcal{A} \cup \mathcal{B}, \mathcal{A} \cap \mathcal{B}, \mathcal{A} \backslash \mathcal{B}$, and $\mathcal{A} \times \mathcal{B}$, respectively. The cardinality of $\mathcal{A}$ is denoted by $|\mathcal{A}|$. We write $A \subsetneq B$ for $\mathcal{A} \subseteq \mathcal{B}$ if we want to emphasize that $\mathcal{A} \neq \mathcal{B}$ holds. $\mathcal{P}(\mathcal{A})$ denotes the power set of $\mathcal{A}$, i.e., the set of all subsets of $\mathcal{A}$. The complement of a subset $\mathcal{A} \subseteq \mathcal{X}$ of a vector (or topological) space $\mathcal{X}$ is $\mathcal{X} \backslash \mathcal{A}$. We assume that the reader is familiar with these terms and refer to standard text books in linear algebra, e.g., [182].

### 1.1.1 Topological and vector spaces

First, we recall some terminology of topological and vector spaces, which are from interest in our work.

**Definition 1.1.1** (see [52, Def. 1.1.1])**.** Let $\tau$ be nonempty subset of a given set $\mathcal{X}$. $\tau$ is said to be a *topology* for $\mathcal{X}$ if the following conditions are satisfied:

(i) $\emptyset, \mathcal{X} \in \tau$,

(ii) For any family of sets $(\mathcal{U}_i)_{i \in I} \subseteq \tau$ with index set $I$, it holds that $\bigcup_{i \in I} \mathcal{U}_i \in \tau$,

(iii) For finitely many sets $\mathcal{U}_j \in \tau$, $j = 1, \ldots, n$ with $n \in \mathbb{N}$, it holds that $\bigcap_{j=1}^{n} \mathcal{U}_j \in \tau$.

Given a topology $\tau$, each set $\mathcal{U} \in \tau$ is called *open (with respect to $\tau$) in $\mathcal{X}$*, and $\mathcal{V} \subseteq \mathcal{X}$ with $\mathcal{X} \backslash \mathcal{V} \in \tau$ is called *closed (with respect to $\tau$) in $\mathcal{X}$*. The tupel $(\mathcal{X}, \tau)$ is called *topological space*. A set of subsets $\mathcal{B} \subseteq \mathcal{P}(\mathcal{X})$ of the topological space $(\mathcal{X}, \tau)$ is called *basis of the topology $\tau$* if every open set $\mathcal{U} \in \tau$ is an union of elements in $\mathcal{B}$.

We just write $\mathcal{X}$ for the topological space $(\mathcal{X}, \tau)$ if the topology is clear or not from interest. Topologies can be described by sets that generate them in the following sense:

**Definition 1.1.2** (see [153, § 13])**.** Let $\mathcal{X}$ be a nonempty set. $\mathcal{T} \subseteq \mathcal{X}$ is called *basis for a topology on $\mathcal{X}$* if the following conditions hold:

(i) For each $X \in \mathcal{X}$, there is some $\mathcal{U} \in \mathcal{T}$ with $X \in \mathcal{U}$,

(ii) For all $X \in \mathcal{X}$ and $\mathcal{U}_1, \mathcal{U}_2 \in \mathcal{T}$ with $X \in \mathcal{U}_1 \cap \mathcal{U}_2$, there is some $\mathcal{U}_3 \in \mathcal{T}$ with $X \in \mathcal{U}_3$ and $\mathcal{U}_3 \subseteq \mathcal{U}_1 \cap \mathcal{U}_2$.

If $\mathcal{T} \subseteq \mathcal{X}$ is a basis for a topology on $\mathcal{X}$, then the *topology $\tau$ generated by $\mathcal{T}$* is given by

$$\tau = \{\mathcal{U} \subseteq \mathcal{X} \mid \forall X \in \mathcal{U} \; \exists \mathcal{V} \in \mathcal{T} : \; X \in \mathcal{V}, \mathcal{V} \subseteq \mathcal{U}\}.$$

A set of subsets $\mathcal{B} \subseteq \mathcal{P}(\mathcal{X})$ of the topological space $(\mathcal{X}, \tau)$ is basis of the topology $\tau$ if every open set $\mathcal{U} \in \tau$ is an union of elements in $\mathcal{B}$. The relationship between closed and open sets as being complements to each other leads to the following rules for closed sets:

**Lemma 1.1.3** (see [153, Th. 17.1])**.** *Let $(\mathcal{X}, \tau)$ be a topological space. Then, the following conditions hold:*

(i) *$\mathcal{X}$ and $\emptyset$ are closed,*

(ii) *For any family of closed sets $(\mathcal{D}_i)_{i \in I} \subseteq \mathcal{X}$ with index set $I$, the intersection $\bigcap_{i \in I} \mathcal{D}_i$ is closed,*

(iii) *For finitely many closed sets $\mathcal{D}_j \subseteq \mathcal{X}$, $j = 1, \ldots, n$ with $n \in \mathbb{N}$, the union $\bigcup_{j=1}^{n} D_j$ is closed.*

The following terminology is standard in topology:

**Definition 1.1.4** (see [153, § 17])**.** Let $(\mathcal{X}, \tau)$ be a topological space and $\mathcal{A} \subseteq \mathcal{X}$ be an arbitrary subset of $\mathcal{X}$. We call

$$\operatorname{int} \mathcal{A} := \bigcup_{\mathcal{U} \subseteq \mathcal{A},\ \mathcal{U} \text{ open}} \mathcal{U}$$

the *interior of* $\mathcal{A}$ and

$$\operatorname{cl} \mathcal{A} := \bigcap_{\mathcal{A} \subseteq \mathcal{D},\ \mathcal{D} \text{ closed}} \mathcal{D}$$

the *closure of* $\mathcal{A}$. The *boundary of* $\mathcal{A}$ is given by $\operatorname{bd} \mathcal{A} := \operatorname{cl} \mathcal{A} \backslash \operatorname{int} \mathcal{A}$.

**Remark 1.1.5.** *Let $(\mathcal{X}, \tau)$ be a topological space and $\mathcal{A} \subseteq \mathcal{X}$ arbitrary. By Definition 1.1.1 and Lemma 1.1.3, $\operatorname{int} \mathcal{A}$ is open and $\operatorname{cl} \mathcal{A}$ is closed. By definition of the interior and closure of $\mathcal{A}$, the relationship*

$$\operatorname{int} A \subseteq \mathcal{A} \subseteq \operatorname{cl} A$$

*holds. Obviously,*

$$\operatorname{int} \mathcal{A} = \{X \in \mathcal{X} \mid \exists \mathcal{U} \in \tau : \ X \in \mathcal{U}, \mathcal{U} \subseteq \mathcal{A}\}$$

*is fulfilled and, furthermore, it holds that (see [153, Th. 17.5])*

$$\operatorname{cl} \mathcal{A} = \{X \in \mathcal{X} \mid \forall \mathcal{U} \in \tau \text{ with } X \in \mathcal{U} : \ \mathcal{U} \cap \mathcal{A} \neq \emptyset\}.$$

*Moreover, we obtain by definition of the boundary of $\mathcal{A}$*

$$\operatorname{bd} \mathcal{A} = \{X \in \mathcal{X} \mid \forall \mathcal{U} \in \tau \text{ with } X \in \mathcal{U} : \mathcal{U} \cap \mathcal{A} \neq \emptyset, \mathcal{U} \not\subseteq \mathcal{A}\}.$$

The following relationship for closed and open sets is well known:

**Lemma 1.1.6** (see [185, Lemma 2.1.11])**.** *Let $(\mathcal{X},\tau)$ be a topological space and $\mathcal{A} \subseteq \mathcal{X}$ arbitrary. Then, the following holds:*

(i) *$\mathcal{A}$ is open if and only if* $\operatorname{int} A = \mathcal{A}$ *holds,*

(ii) *$\mathcal{A}$ is closed if and only if* $\operatorname{cl} A = \mathcal{A}$ *holds.*

Considering open sets containing some specific element $X \in \mathcal{X}$ (as, e.g., in the equivalent characterization of the closure and interior of a set in Remark 1.1.5) leads to the following terminology:

**Definition 1.1.7** (see [198, Def. 1-6-2 and Rem. 1-6-9])**.** Let $(\mathcal{X},\tau)$ be a topological space and $X \in \mathcal{X}$. Every $\mathcal{U} \subseteq \tau$ with $X \in \mathcal{U}$ is called *neighborhood of* $X$. The set of all neighborhoods of a given point $X$ will be written as $\mathcal{N}_X$. We call $\mathcal{B}_X \subseteq \mathcal{N}_X$ a *local base (of neighborhoods) at* $X$ if for every $\mathcal{U} \in \mathcal{N}_X$ there is some $\mathcal{B} \in \mathcal{B}_X$ with $\mathcal{B} \subseteq \mathcal{U}$.

**Remark 1.1.8.** *Given a topological space $(\mathcal{X},\tau)$, we do not highlight in the symbol $\mathcal{N}_X$ the related topology $\tau$ for $X \in \mathcal{X}$. Even if we deal with more than one topological space, the element $X$ (and, thus, the space $\mathcal{X}$ it belongs to) in the symbol $\mathcal{N}_X$ implies, which topology is meant. Only if $X$ could belong to another considered topological space, as well, we will explain the meant topology.*

**Example 1.1.9** (see [35, Def. 4.1.1], [146, § 2], and [153, Th. 19.1])**.** Let $\mathcal{I}$ be an index set and $(\mathcal{X}_i,\tau_i)$ be topological spaces for $i \in \mathcal{I}$. Consider the Cartesian product (see [160, Def. 2.32])

$$\mathcal{X} := \prod_{i\in\mathcal{I}} \mathcal{X}_i = \left\{ f\colon \mathcal{I} \to \bigcup_{i\in\mathcal{I}} \mathcal{X}_i \mid \forall i \in \mathcal{I}:\ f(i) \in \mathcal{X}_i \right\}$$

and the canonical projections $p_i\colon \mathcal{X} \to \mathcal{X}_i$ defined by $p_i(f) := f(i)$ for $i \in \mathcal{I}$ and $f \in \mathcal{X}$ arbitrary. Note that we obtain in the (in this thesis considered) case of $\mathcal{I} = \mathbb{N}$

$$\mathcal{X} = \prod_{i\in\mathbb{N}} \mathcal{X}_i = \{(X_i)_{i\in\mathbb{N}} \mid \forall i \in \mathcal{I}:\ X_i \in \mathcal{X}_i\}$$

and $p_i(X) = X_i$ for all $i \in \mathbb{N}$ with $X = (X_i)_{i\in\mathbb{N}} \in \mathcal{X}$. The *product topology on* $\mathcal{X}$ is the coarsest topology $\tau$ on $\mathcal{X}$ (i.e., for each topology $\nu$ on $\mathcal{X}$ it holds that $\tau \subseteq \nu$) such that every $p_i$ with $i \in \mathcal{I}$ arbitrary is continuous with respect to $\tau_i$ (see Definition 1.1.16). $(\mathcal{X},\tau)$ with $\mathcal{X} = \prod_{i\in\mathcal{I}} X_i$ and $\tau$ being the product topology on $\mathcal{X}$ is called *product space*. It holds that

$$\tau = \left\{ \bigcup_{j\in\mathcal{J}} \mathcal{T}_j \mid \mathcal{J} \subseteq \mathcal{I}, \mathcal{T}_j \in \mathcal{T} \right\},$$

where

$$\mathcal{T} := \left\{ \prod_{i\in\mathcal{I}} \mathcal{U}_i \mid \mathcal{U}_i \in \tau_i \textit{ for all } i \in \mathcal{I}, \mathcal{U}_j \neq \mathcal{X}_j \text{ only for finitely many } j \in \mathcal{I} \right\},$$

i.e., $\mathcal{T}$ is a basis of $\tau$ called *natural basis* of the product space $\mathcal{X}$. Furthermore, it holds that

$$\mathcal{T} = \left\{ \bigcap_{j\in\mathcal{J}} \mathcal{D}_j \mid \mathcal{J} \subseteq \mathcal{I}, \mathcal{D}_j \subseteq \mathcal{S} \text{ and } |\mathcal{D}_j| < +\infty \text{ for all } j \in \mathcal{J} \right\},$$

where $\mathcal{S} := \{p_i^{-1}(\mathcal{U}) \mid i \in \mathcal{I}, \mathcal{U} \in \tau_i\}$. Note that in the case of finitely many topological spaces, i.e., $\mathcal{I} = \{1, \ldots, n\}$, we obtain that the topology $\tau$ is given by the basis

$$\mathcal{T} = \left\{ \prod_{i=1}^{n} \mathcal{U}_i \mid \forall i \in \{1, \ldots, n\} : \ \mathcal{U}_i \in \tau_i \right\}.$$

Moreover, $\mathcal{T} \neq \tau$ holds in general. For example, if $\mathcal{X}_1 = \mathcal{X}_2 = \{A, B\}$ with arbitrary elements $A, B$, and $\tau_1 = \tau_2 = \{\emptyset, \{A, B\}, \{B\}\}$, we obtain the product topology $\tau$ for $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2$ as

$$\tau = \{\emptyset, \mathcal{X}_1 \times \mathcal{X}_2, (B, B), \{(B, B), (A, B)\}, \{(B, B), (B, A)\}, \{(B, B), (A, B), (B, A)\}\}$$

and, thus, $\tau \neq \tau_1 \times \tau_2$, since $\{(B, B), (A, B), (B, A)\} \notin \tau_1 \times \tau_2$. ◊

Note that we write $\mathcal{X}^n$ for the $n$-times product of the same topological space $\mathcal{X}$

$$\mathcal{X}^n = \prod_{i=1}^{n} \mathcal{X}.$$

We return to Definition 1.1.4 and want to recall an even more practical characterization for the closure than in Remark 1.1.5 for topological spaces with the following property:

**Definition 1.1.10** (see [153, § 21])**.** A topological space $(\mathcal{X}, \tau)$ is call *first-countable* if it fulfills the following condition, known as *first axiom of countability*:

$$\forall X \in \mathcal{X},\ \exists (\mathcal{N}_i)_{i\in\mathbb{N}} \subseteq \mathcal{N}_X,\ \forall \mathcal{N} \in \mathcal{N}_X, \exists i \in \mathbb{N} : \quad \mathcal{N}_i \subseteq \mathcal{N}.$$

For the noticed characterization, we need limits of sequences in topological spaces. Hence, we shortly recall the definition:

**Definition 1.1.11** (see [122, Def. 1.2.6])**.** Let $(\mathcal{X}, \tau)$ be a topological space and $(X_n)_{n\in\mathbb{N}} \subseteq \mathcal{X}$ be some sequence of elements in $\mathcal{X}$. We say that $(X_n)_{n\in\mathbb{N}}$ *converges to a point* $X \in \mathcal{X}$ and call $X$ *limit of the sequence* $(X_n)_{n\in\mathbb{N}}$ if

$$\forall \mathcal{U} \in \mathcal{N}_X, \exists N \in \mathbb{N} : \quad X_n \in \mathcal{U} \text{ for all } n \geq N.$$

The sequence is called *convergent* then. We write $X_n \to X$ or $\lim_{n\to+\infty} X_n = X$ for a sequence $(X_n)_{n\in\mathbb{N}}$ that converges to $X$.

Now, we can formulate the announced result for the closure of an arbitrary subset in topological spaces:

**Lemma 1.1.12** (see [153, Th. 30.1])**.** *Let $(\mathcal{X}, \tau)$ be a first-countable topological space and $\mathcal{A} \subseteq \mathcal{X}$ arbitrary. Then,*

$$\operatorname{cl} \mathcal{A} = \{X \in \mathcal{X} \mid \exists (X_n)_{n\in\mathbb{N}} \subseteq \mathcal{A} : \lim_{n\to+\infty} X_n = X\}.$$

In general, the limit of a convergent sequence does not have to be unique. Since the uniqueness is very helpful or, in general, needed, there is often some additional assumption on the topological space:

**Definition 1.1.13** (see [52, Def. 1.1.14])**.** A topological space $(\mathcal{X}, \tau)$ is called *Hausdorff* if two arbitrary distinct points $X, Y \in \mathcal{X}$ have disjoint neighborhoods, i.e., there are $\mathcal{U} \in \mathcal{N}_X$ and $\mathcal{V} \in \mathcal{N}_Y$ with $\mathcal{U} \cap \mathcal{V} = \emptyset$.

Some important consequence of working in Hausdorff topological spaces $\mathcal{X}$ is that singletons $\{X\}$ with $X \in \mathcal{X}$ arbitrary are closed (see [52, Remark 1.1.15]). Furthermore, as announced before, Hausdorff spaces secure that the limit of a convergent sequence is unique:

**Lemma 1.1.14** (see [153, Th. 17.10])**.** *Let $(\mathcal{X}, \tau)$ be a Hausdorff topological space and $(X_n)_{n\in\mathbb{N}} \subseteq \mathcal{X}$ be a sequence in $\mathcal{X}$. Then, $(X_n)_{n\in\mathbb{N}}$ converges to at most one point in $\mathcal{X}$.*

Now, we recall some standard terminology concerning maps:

**Definition 1.1.15** (see [173, Def. 1.16])**.** Let $\mathcal{X}, \mathcal{Y}$ be sets, $f\colon \mathcal{X} \to \mathcal{Y}$ be a mapping (or simply a map) from $\mathcal{X}$ into $\mathcal{Y}$, $\mathcal{A} \subseteq \mathcal{X}$, and $\mathcal{D} \subseteq \mathcal{Y}$. The *image of $\mathcal{A}$ under $f$* and the *preimage of $\mathcal{D}$ under $f$* are given by

$$f(\mathcal{A}) := \{f(X) \mid X \in \mathcal{A}\} \subseteq \mathcal{Y}$$

and

$$f^{-1}(\mathcal{D}) := \{X \in \mathcal{X} \mid f(X) \in \mathcal{D}\} \subseteq \mathcal{X},$$

respectively. The *Image of $f$* is

$$\operatorname{Im} f := \{f(X) \mid X \in \mathcal{X}\} \subseteq \mathcal{Y}.$$

Continuity is an important property of maps in many situations, which we recall next.

**Definition 1.1.16** (see [52, Def. 1.1.23])**.** Let $(\mathcal{X}, \tau_{\mathcal{X}})$ and $(\mathcal{Y}, \tau_{\mathcal{Y}})$ be topological spaces, $f\colon \mathcal{X} \to \mathcal{Y}$ be a map, and $X \in \mathcal{X}$. We say that $f$ is *continuous at $X$* if

$$\forall \mathcal{V} \in \mathcal{N}_{f(X)}, \exists \mathcal{U} \in \mathcal{N}_X : \quad f(\mathcal{U}) \subseteq \mathcal{V}.$$

We call $f$ *continuous on $\mathcal{X}$* if it holds that

$$\forall \mathcal{V} \in \tau_{\mathcal{Y}} : \quad f^{-1}(\mathcal{V}) \in \tau_{\mathcal{X}}.$$

Continuity depends on the function $f\colon \mathcal{X} \to \mathcal{Y}$, but also on the topologies specified for $\mathcal{X}$ and $\mathcal{Y}$. If we want to emphasize that, we will speak of continuity *with respect to the topologies $\tau_{\mathcal{X}}$ and $\tau_{\mathcal{Y}}$*.

Next, we recall fundamentals of vector space theory, since these spaces are of main interest for this thesis. Although we define vector spaces generally for fields $\mathbb{F}$ (i.e., informally spoken, sets with addition- and multiplication-operator similar to those in $\mathbb{R}$, and existing additive and multiplicative inverse, see [178]), we will only consider real vector spaces, i.e., the case $\mathbb{F} = \mathbb{R}$ (see Remark 1.1.18).

**Definition 1.1.17** (see [122, Def. 1.1.1])**.** Let $\mathcal{X}$ be a nonempty set and $\mathbb{F}$ be a field (which elements are called *scalars*). Furthermore, let $+\colon \mathcal{X}\times\mathcal{X}\to\mathcal{X}$ and $\cdot\colon \mathbb{F}\times\mathcal{X}\to\mathcal{X}$ be mappings representing addition and scalar multiplication on $\mathcal{X}$, respectively. Then, $\mathcal{X}$ is called *vector space over* $\mathbb{F}$ and the elements $X\in\mathcal{X}$ are called *vectors* if the following conditions are satisfied:

(i) $\forall X,Y\in\mathcal{X}\colon\ X+Y=Y+X$,

(ii) $\forall X,Y,Z\in\mathcal{X}\colon\ X+(Y+Z)=(X+Y)+Z$,

(iii) There is an unique *null vector* $\mathbb{0}\in\mathcal{X}$ such that $X+\mathbb{0}=X$ for all $X\in\mathcal{X}$,

(iv) For every $X\in\mathcal{X}$, there is an unique *inverse vector* $-X\in\mathcal{X}$ with $X+(-X)=\mathbb{0}$,

(v) $\forall X\in\mathcal{X}\colon\ 1\cdot X=X$ with $1\in\mathbb{F}$ being the *multiplicative identity*,

(vi) $\forall a,b\in\mathbb{F},\forall X\in\mathcal{X}\colon\ a\cdot(b\cdot X)=(a\cdot b)\cdot X$,

(vii) $\forall a,b\in\mathbb{F},\forall X\in\mathcal{X}\colon\ (a+b)\cdot X=a\cdot X+b\cdot X$,

(viii) $\forall a\in\mathbb{F},\forall X,Y\in\mathcal{X}\colon\ a\cdot(X+Y)=a\cdot X+a\cdot Y$.

**Remark 1.1.18.** *In Definition 1.1.17*(vi) *and* (vii)*, we did not distinguish between addition in $\mathcal{X}$ and addition in $\mathbb{F}$, also not between scalar multiplication and multiplication of elements in $\mathcal{F}$. Hence, we used the same symbol $+$ and $\cdot$ , respectively, because it is obvious, which operation is meant. In the literature, vector spaces are also known as* linear spaces*. We will only consider the field $\mathbb{F}=\mathbb{R}$ in this thesis. Thus, we will always assume vector spaces over $\mathbb{R}$ and only speak of* (real) vector spaces *in the following. For details about general vector spaces over arbitrary fields, see, e.g., [178, Sec. 2]. Given a vector space $\mathcal{X}$, we will distinguish between the null vector $\mathbb{0}\in\mathcal{X}$ and vector of ones $\mathbb{1}\in\mathcal{X}$ written double-struck, and the scalars $0\in\mathbb{R}$ and $1\in\mathbb{R}$ in general to emphasize if we mean a vector or a scalar. Note that we will left out the symbol $\cdot$ for the scalar multiplication and just write $aX:=a\cdot X$ for scalar multiplication of $X\in\mathcal{X}$ with some $a\in\mathbb{R}$ to avoid confusion with, e.g., scalar products in the literature.*

**Example 1.1.19.** We denote by $\mathbb{R}^{m\times n}$ the vector space of matrices $A=(A_{ij})_{\substack{i=1,\dots,m\\ j=1,\dots,n}}$ with $m\in\mathbb{N}$ rows, $n\in\mathbb{N}$ columns, and real entries, i.e., $A_{ij}\in\mathbb{R}$ for all $i=1,\dots,m$ and $j=1,\dots n$. Furthermore, we denote by $\mathbb{R}^n$ the $n$-dimensional real vector space, which consists of all ordered $n$-tupels with real entries, i.e.,

$$\mathbb{R}^n:=\{X=(X_1,\dots,X_n)^T \mid X_i\in\mathbb{R}\text{ for all }1\le i\le n\}.$$

The operator $^T$ denotes the *transposed* vector and switches the rows and columns indices of a matrix, i.e., the element $A_{ij}$ becomes $A_{ji}$ for each $i$ and $j$. If the matrix consists of one column or one row, respectively, like the vectors $X\in\mathbb{R}^n$, we leave out the corresponding index to that single column or row. Thus, since elements of $\mathbb{R}^n$ are understood to be column vectors here, $\mathbb{R}^n$ coincides with $\mathbb{R}^{n\times 1}$. As usual, we consider the componentwise addition and scalar multiplication on $\mathbb{R}^{m\times n}$ and $\mathbb{R}^n$. ♢

Now, we recall some useful terminology from set theory for vector spaces.

**Definition 1.1.20** (see [143])**.** Let $\mathcal{X}$ be a vector space and $\mathcal{A}, \mathcal{B} \subseteq \mathcal{X}$. Then,

$$\mathcal{A} + \mathcal{B} := \{X + Y \mid X \in \mathcal{A}, Y \in \mathcal{B}\}$$

is the *Minkowski sum* of $\mathcal{A}$ and $\mathcal{B}$. The *Minkowski product* of $\mathcal{A}$ and $\mathcal{B}$ is defined by

$$\mathcal{A}\mathcal{B} := \{XY \mid X \in \mathcal{A}, Y \in \mathcal{B}\}.$$

**Remark 1.1.21.** *Let $\mathcal{X}$ be a vector space and $\mathcal{A} \subseteq \mathcal{X}$. For simplicity, we replace $\{X\}$ by the shortcut $X$ in a Minkowski sum with a set consisting of only one arbitrary element $X \in \mathcal{X}$, i.e.,*

$$X + \mathcal{A} = \mathcal{A} + X := \mathcal{A} + \{X\} \quad \text{and} \quad \mathcal{A}X := \mathcal{A}\{X\}.$$

*Especially, we write for the scalar multiplication of a subset $\mathcal{A} \subseteq \mathcal{X}$ with $\lambda \in \mathbb{R}$*

$$\lambda\mathcal{A} := \{\lambda X \mid X \in \mathcal{A}\}.$$

*Furthermore, we use the shortcut $-\mathcal{B} := (-1)\mathcal{B}$ for scalar multiplication of $-1 \in \mathbb{R}$ with a given subset $\mathcal{B} \subseteq \mathcal{X}$ and define the Minkowski subtraction as the Minkowski sum of $\mathcal{A}$ and $-\mathcal{B}$, i.e.,*

$$\mathcal{A} - \mathcal{B} := \mathcal{A} + (-\mathcal{B}) = \{X - Y \mid X \in \mathcal{A}, Y \in \mathcal{B}\}.$$

**Definition 1.1.22** (see [185, Def. 2.1.6 and 2.2.2] and [17])**.** Let $\mathcal{X}$ be a vector space and $\mathcal{A} \subseteq \mathcal{X}$.

(i) $\mathcal{A}$ is called *star-shaped (around $\mathbb{0} \in \mathcal{X}$)* if

$$\forall X \in \mathcal{A}, \forall \lambda \in [0,1]: \quad \lambda X \in \mathcal{A},$$

(ii) $\mathcal{A}$ is called *convex* if

$$\forall X, Y \in \mathcal{A}, \forall \lambda \in [0,1]: \quad X + (1-\lambda)Y \in \mathcal{A},$$

(iii) $\mathcal{A}$ is called a *cone* if $\mathcal{A}$ is nonempty and

$$\forall\ X \in \mathcal{A}, \forall \lambda \geq 0: \ \lambda X \in \mathcal{A},$$

(iv) $\mathcal{A}$ is called *polyhedral* if

$$\mathcal{A} = \bigcap_{i=1}^{n} \{X \in \mathcal{X} \mid \varphi_i(X) \geq \alpha_i\}$$

with linear functionals $\varphi_i \colon \mathcal{X} \to \mathbb{R}$ and scalars $\alpha_i \in \mathbb{R}$ for $i = 1, \ldots, n$.

**Remark 1.1.23.** *Let $\mathcal{X}$ be a vector space and $\mathcal{A} \subseteq \mathcal{X}$. If $\mathcal{A} \subseteq \mathcal{X}$ is convex with $\mathbb{0} \in \mathcal{A}$, then $\mathcal{A}$ is obviously star-shaped. Moreover, if $\mathcal{A}$ is polyhedral, $\mathcal{A}$ is also convex and (in topological vector spaces) closed. Also, every cone $\mathcal{C} \subseteq \mathcal{X}$ is star-shaped.*

**Definition 1.1.24** (see [91, Def. 2.1.11]). Let $\mathcal{X}$ be a vector space and $\mathcal{C} \subseteq \mathcal{X}$ be a cone.

(i) $\mathcal{C}$ is said to be *nontrivial* or *proper* if $\{0\} \subsetneq \mathcal{C} \neq \mathcal{X}$,

(ii) $\mathcal{C}$ is said to be *pointed* if $\mathcal{C} \cap (-\mathcal{C}) = \{0\}$,

(iii) $\mathcal{C}$ is said to be *generating* or *reproducing* if $\mathcal{C} - \mathcal{C} = \mathcal{X}$.

Obviously, if $\mathcal{C} \subseteq \mathcal{X}$ is a generating cone in a vector space $\mathcal{X}$, then $\operatorname{span}(C) = \mathcal{X}$ holds (compare also Lemma 1.1.41) with $\operatorname{span}(\mathcal{C})$ being the subspace of $\mathcal{X}$ generated by $\mathcal{C}$, known as *linear span of* $\mathcal{C}$, see Definition 1.1.32.

**Lemma 1.1.25** (see [185, Lemma 2.2.6]). *Let $\mathcal{X}$ be a vector space and $\emptyset \neq \mathcal{A} \subseteq \mathcal{X}$. Then, the following conditions are equivalent:*

(i) *$\mathcal{A}$ is a convex cone,*

(ii) *$\mathcal{A}$ is a cone fulfilling $\mathcal{A} + \mathcal{A} \subseteq \mathcal{A}$,*

(iii) *$\mathcal{A}$ is convex with $\mathcal{A} + \mathcal{A} \subseteq \mathcal{A}$ and $0 \in \mathcal{A}$.*

A cone from special interest is the *recession cone* of a subset $\mathcal{A} \subset \mathcal{X}$, i.e.,

$$\operatorname{rec} \mathcal{A} := \{X \in \mathcal{X} \mid Y + \lambda X \in \mathcal{A} \text{ for all } Y \in \mathcal{A}, \lambda \in [0, +\infty)\}.$$

Now, we introduce a algebraic (directional) concept, which can be seen as a fitting equivalent in vector spaces for the closure, interior, and boundary in topological spaces.

**Definition 1.1.26** (see [97] and [185]). Let $\mathcal{X}$ be a vector space, $\mathcal{A} \subseteq \mathcal{X}$ and $K \in \mathcal{X} \backslash \{0\}$. The *$K$-directional closure of* $\mathcal{A}$ is given by

$$\operatorname{cl}_K(\mathcal{A}) := \{X \in \mathcal{X} \mid \forall \lambda \in \mathbb{R}_> \ \exists t \in \mathbb{R}_+ \text{ with } t < \lambda \text{ and } X - tK \in \mathcal{A}\}.$$

$\mathcal{A}$ is said to be *$K$-directionally closed* if $\mathcal{A} = \operatorname{cl}_K(\mathcal{A})$. The *$K$-directional interior of* $\mathcal{A}$ is given by

$$\operatorname{int}_K(\mathcal{A}) := \{X \in \mathcal{X} \mid \exists \lambda \in \mathbb{R}_> \ \forall t \in [0, \lambda] : X + tK \in \mathcal{A}\},$$

and the *$K$-directional boundary of* $\mathcal{A}$ is given by

$$\operatorname{bd}_K(\mathcal{A}) := \operatorname{cl}_K(\mathcal{A}) \backslash \operatorname{int}_K(\mathcal{A}).$$

The origin for Definition 1.1.26 is [97]. Our definition of the directional closure is from [195]. A similar definition of the directional interior and directional boundary under the assumption $-K \in \operatorname{rec} \mathcal{A}$ can be found in [102]. We recall some useful results with respect to Definition 1.1.26, which are collected from [185]:

**Lemma 1.1.27** (see [185, Lemma 2.3.24]). *Let $\mathcal{X}$ be a vector space, $\mathcal{A} \subseteq \mathcal{X}$, and $K \in \mathcal{X} \backslash \{0\}$. Then,*

$$\operatorname{cl}_K(\mathcal{A}) = \{X \in \mathcal{X} \mid \exists (t_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}_+ : t_n \downarrow 0 \ \forall n \in \mathbb{N} : X - t_n K \in \mathcal{A}\}.$$

The property in Lemma 1.1.27 is used in [194] to define the $K$-directional closure. A similar definition of the $K$-directional closure can be found in [162] as so called *$K$-vector closure*.

**Lemma 1.1.28** (see [185, Lemma 2.3.26 and 2.3.42, Prop. 2.3.48 and 2.3.49]). *Let $\mathcal{X}$ be a vector space, $\mathcal{A} \subseteq \mathcal{X}$, and $K \in \mathcal{X}\backslash\{\mathbb{0}\}$. Then, the following holds:*

(i) $\mathcal{A} \subseteq \operatorname{cl}_K(\mathcal{A}) \subseteq \mathcal{A} + \mathbb{R}_+ K$,

(ii) $\operatorname{cl}_K(\operatorname{cl}_K(\mathcal{A})) = \operatorname{cl}_K(\mathcal{A})$,

(iii) $\operatorname{cl}_K(\mathcal{A}) - \mathbb{R}_+ K = \operatorname{cl}_K(\mathcal{A} - \mathbb{R}_+ K) = \operatorname{cl}_K(\mathcal{A} - \mathbb{R}_> K)$,

(iv) $\mathcal{A} - \mathbb{R}_> K = \operatorname{cl}_K(\mathcal{A}) - \mathbb{R}_> K = \operatorname{int}_K(\mathcal{A} - \mathbb{R}_> K) = \operatorname{int}_K(\mathcal{A} - \mathbb{R}_+ K)$.

**Lemma 1.1.29** (see [185, Prop. 2.3.29 and 2.3.53]). *Let $\mathcal{X}$ be a vector space, $\mathcal{A} \subseteq \mathcal{X}$, and $K \in \mathcal{X}\backslash\{\mathbb{0}\}$. Suppose $K \in -\operatorname{rec}\mathcal{A}$. Then, the following holds:*

(i) $\operatorname{cl}_K(\mathcal{A}) = \{X \in \mathcal{X} \mid X - \mathbb{R}_> K \subseteq \mathcal{A}\}$,

(ii) $\operatorname{int}_K(\mathcal{A}) = \{X \in \mathcal{A} \mid \exists t \in \mathbb{R}_> : X + tK \in \mathcal{A}\} = \mathcal{A} - \mathbb{R}_> K$,

(iii) $\operatorname{bd}_K(\mathcal{A}) = \{X \in \mathcal{X} \mid \forall t \in \mathbb{R}_> : X + tK \notin \mathcal{A} \text{ and } X - tK \in \mathcal{A}\}$.

The following lemma delivers relationships between topological and directional properties of a subset of a topological vector space:

**Lemma 1.1.30** (see [185, Prop. 2.3.54 and 2.3.55]). *Let $\mathcal{X}$ be a topological vector space, $\mathcal{A} \subseteq \mathcal{X}$, and $K \in \mathcal{X}\backslash\{\mathbb{0}\}$. Then, the following holds:*

(i) $\operatorname{cl}_K(\mathcal{A}) \subseteq \operatorname{cl}\mathcal{A}$,

(ii) $\operatorname{int}\mathcal{A} \subseteq \operatorname{int}_K(\mathcal{A}) \subseteq \mathcal{A}$,

(iii) $\operatorname{bd}_K(\mathcal{A}) \subseteq \operatorname{bd}\mathcal{A}$.

In the following, we recall the terminology concerning the basis and dimension of a vector space.

**Definition 1.1.31** (see [122, Def. 1.1.3 and 1.1.4]). Let $\mathcal{X}$ be a real vector space and $\mathcal{B} := \{X_1, \dots, X_n\} \subseteq \mathcal{X}$ with $n \in \mathbb{N}$. A vector of the form

$$\sum_{i=1}^{n} \lambda_i X_i \text{ with } \lambda_i \in \mathbb{R}, i = 1, \dots, n \tag{1.1}$$

is called *linear combination of* $X_1, \dots, X_n$. We say that $X_1, \dots, X_n$ are *linear independent* if it holds that

$$\sum_{i=1}^{n} \lambda_i X_i = 0 \text{ for } \lambda_i \in \mathbb{R}, i = 1, \dots, n \quad \Longrightarrow \quad \forall i \in \{1, \dots, n\}: \ \lambda_i = 0.$$

Otherwise, $X_1, \dots, X_n$ are called *linear dependent*. An infinite set $\mathcal{D}$ of vectors in $\mathcal{X}$ is called *linear independent* (and, otherwise, *linear dependent*) if every finite subset of $\mathcal{D}$ fulfills (1.1).

**Definition 1.1.32** (see [122, Def. 1.1.5]). Let $\mathcal{X}$ be a real vector space and $\mathcal{B} \subseteq \mathcal{X}$. The set of all finite linear combinations of vectors in $\mathcal{B}$ is given by

$$\operatorname{span} \mathcal{B} := \left\{ \sum_{i=1}^{n} \lambda_i X_i, \lambda_i \in \mathbb{R}, X_i \in \mathcal{B} \text{ for all } i = 1, \ldots, n \text{ with } n \in \mathbb{N} \right\}$$

and called *linear span of* $\mathcal{B}$. We call $\mathcal{B}$ a *basis of* $\mathcal{X}$ if it is maximal linear independent, i.e., $\mathcal{B}$ is linear independent and $\mathcal{B} \cup \{X\}$ is linear dependent for every $X \in \mathcal{X}$.

The set $\operatorname{span} \mathcal{B}$ in Definition 1.1.32 is the smallest subspace of $\mathcal{X}$ containing $\mathcal{B}$ and also called the *subspace generated by* $\mathcal{B}$ (see [122]). If $\mathcal{B}$ is a basis of $\mathcal{X}$, then every vector $X \in \mathcal{X}$ can be written as a (finite) linear combination of vectors from $\mathcal{B}$. Note that every vector space has a basis and two arbitrary bases of $\mathcal{X}$ have the same cardinality (see [122, Prop. 1.1.1]). Thus, the following definition is justified:

**Definition 1.1.33** (see [122, Def. 1.1.6]). Let $\mathcal{X}$ be a real vector space. $\mathcal{X}$ is said to be *finite dimensional* if it admits a basis with finite cardinality and *infinite dimensional* otherwise. The *dimension of* $\mathcal{X}$ is denoted $\dim \mathcal{X}$, which is the number of elements in a basis of $\mathcal{X}$ if $\mathcal{X}$ is finite dimensional and, otherwise, infinity.

In this thesis, finite dimensional vector spaces like $\mathbb{R}^n$ will be considered, but also infinite dimensional vector spaces like $\mathcal{L}^p$ (see Example 1.3.13) will be from special interest with respect to applications in finance. Now, we recall some terminology for maps defined on vector spaces:

**Definition 1.1.34** (see [122, Def. 1.1.8 and 1.1.9] and [52, Def. 3.1.13]). Let $\mathcal{X}, \mathcal{Y}$ be real vector spaces. A map $f\colon \mathcal{X} \to \mathcal{Y}$ is called *linear operator* or just *linear* if it holds that

$$\forall X, Y \in \mathcal{X}, \forall \lambda, \mu \in \mathbb{R}: \qquad f(\lambda X + \mu Y) = \lambda f(X) + \mu f(Y).$$

In the case $\mathcal{Y} = \overline{\mathbb{R}}$, a linear operator $f\colon \mathcal{X} \to \overline{\mathbb{R}}$ is called a *linear functional on* $\mathcal{X}$. For a linear operator $f\colon \mathcal{X} \to \mathcal{Y}$, the image $\operatorname{Im} f$ is also called *range of* $f$ and its dimension is called *rank of* $f$. Furthermore, the *kernel* or *null space* of a linear operator $f\colon \mathcal{X} \to \mathcal{Y}$ is given by

$$\ker f := \{X \in \mathcal{X} \mid f(X) = \mathbb{0}\}.$$

**Remark 1.1.35** (see [122, Def. 1.1.9]). *Given a linear operator $f\colon \mathcal{X} \to \mathcal{Y}$ between real vector spaces $\mathcal{X}$ and $\mathcal{Y}$, the range of $f$ and the kernel of $f$ are subspaces of $\mathcal{Y}$ and $\mathcal{X}$, respectively.*

The following lemma provides an important relationship between the dimension of a vector space and linear maps defined on this space. Lemma 1.1.36 is taken from [16] and is also known as *rank-nullity-theorem*, see, e.g., [123].

**Lemma 1.1.36** (Rank-Nullity-Theorem, see [16]). *Let $\mathcal{X}, \mathcal{Y}$ be vector spaces and $f\colon \mathcal{X} \to \mathcal{Y}$ be linear. Then,*

$$\dim \mathcal{X} = \dim (\ker f) + \dim (\operatorname{Im} f).$$

We will make use of topological properties (e.g., with respect to continuity of maps) in vector spaces, as well, which leads to the following definition:

**Definition 1.1.37.** (see [198, Def. 4-1-1]) Let $\mathcal{X}$ be a real vector space. Then, $(\mathcal{X}, \tau)$ is called *(real) topological vector space* if $\mathcal{X}$ is endowed with a topology $\tau \subseteq \mathcal{X}$ such that

$$\mathcal{X} \times \mathcal{X} \ni (X, Y) \mapsto X + Y \in \mathcal{X} \quad \text{and} \quad \mathbb{R} \times \mathcal{X} \ni (\alpha, X) \mapsto \alpha \cdot X \in \mathcal{X}$$

are continuous for $\mathcal{X} \times \mathcal{X}$ and $\mathbb{R} \times \mathcal{X}$ being endowed with the corresponding product topologies. In this case, we say that $\tau$ is *compatible with the vector space (or linear) structure of* $\mathcal{X}$.

We write $\mathcal{X}$ for the (real) topological vector space $(\mathcal{X}, \tau)$ (and call $\mathcal{X}$ so) if the topology $\tau$ is obvious or not from interest. Moreover, we only speak of topological vector spaces, since we always consider real ones. Since many spaces from interest in functional analysis are Hausdorff spaces, some authors additional require for $(\mathcal{X}, \tau)$ to be a Hausdorff topological vector space in Definition 1.1.37 (see, e.g., [52, Def. 3.1.8]). This additional Hausdorff-property can also be replaced by requiring that every point of $\mathcal{X}$ is a closed set, since a topological vector space $(\mathcal{X}, \tau)$ in the sense of Definition 1.1.37 is a Hausdorff topological vector space then (see, e.g., [173, Th. 1.12]). That requirement can also be relaxed to $\{\mathbb{0}\}$ being closed, since a topological vector space $(\mathcal{X}, \tau)$ is Hausdorff if $\{\mathbb{0}\}$ is closed, i.e., $\{\mathbb{0}\} \notin \tau$ with $\mathbb{0} \in \mathcal{X}$ being the null vector (see [117, Cor. 2.1.6]).

**Remark 1.1.38.** *Topological vector spaces unite the topological framework of the beginning of this Section with a linear structure of elements in this space in a suitable way. In this thesis, we will always consider topological vector spaces (instead of general topological spaces) if a topology is needed, since we will work as far as possible with a vector space only. Since we will work with linear pricing functionals being defined on a finite dimensional vector space, we will make use of the following fact: Each linear real-valued function being defined on a finite dimensional Hausdorff topological vector space* $\mathcal{X}$ *is continuous (see, e.g., [173, Item 1.19 and 1.20]).*

For topological vector spaces, the topology can be localized at $\mathbb{0}$:

**Lemma 1.1.39** (see [198, Theorem 4-1-4])**.** *Let* $(\mathcal{X}, \tau)$ *be a real topological vector space,* $X \in \mathcal{X}$*, and* $\mathcal{U} \subseteq \mathcal{X}$*. Then,* $\mathcal{U}$ *is a neighborhood of* $X$*, i.e.,* $\mathcal{U} \in \mathcal{N}_X$*, if and only if* $\mathcal{U} - \{X\} \in \mathcal{N}_{\mathbb{0}}$*.*

We will also make use of the following result:

**Lemma 1.1.40** (see [198, Th. 4-2-3 and Th. 4-2-4])**.** *Let* $(\mathcal{X}, \tau)$ *be a real topological vector space and* $\mathcal{U} \in \mathcal{N}_{\mathbb{0}}$*. Then,* $\mathcal{U}$ *is absorbing, i.e., it holds that*

$$\forall X \in \mathcal{X}, \exists \epsilon > 0 : \quad tX \in \mathcal{U} \ \text{ for all } |t| < \epsilon.$$

*Furthermore,* $t\mathcal{U} \in \mathcal{N}_{\mathbb{0}}$ *holds for each* $t \in \mathbb{R} \backslash \{0\}$*.*

As a consequence of Lemma 1.1.40, it holds for the closure of an arbitrary subset $\mathcal{A} \subseteq \mathcal{X}$ of a topological vector spaces $(\mathcal{X}, \tau)$ (see [198, Th. 4-2-6]) that

$$\operatorname{cl} \mathcal{A} = \bigcap \{\mathcal{A} + \mathcal{U} \mid \mathcal{U} \in \mathcal{N}_{\mathbb{0}}\}.$$

The following is a interesting and useful consequence of Lemma 1.1.40 for cones. We could not find a reference in the literature, so we give a proof here.

**Lemma 1.1.41.** *Let $(\mathcal{X}, \tau)$ be a topological vector space and $\mathcal{C} \subseteq \mathcal{X}$ be a cone. If* $\operatorname{int} \mathcal{C} \neq \emptyset$*, then $\mathcal{C}$ is generating with* $\operatorname{span} \mathcal{C} = \mathcal{X}$*.*

**Proof.** Let $X \in \mathcal{C}$ and $\mathcal{U} \in \tau$ with $X \in \mathcal{U} \subseteq \mathcal{C}$. By Lemma 1.1.39, $\mathcal{V} := \mathcal{U} - \{X\} \in \tau$ with

$$\mathcal{V} \subseteq \mathcal{U} - \mathcal{U} \subseteq \mathcal{C} - \mathcal{C} \tag{1.2}$$

and $\mathbb{0} \in \mathcal{V}$, i.e., $\mathcal{V} \in \mathcal{N}_{\mathbb{0}}$. Take $Y \in \mathcal{X}$ arbitrary. Then, there is some $t \in \mathbb{R}_{>}$ with $tY \in \mathcal{V}$ by Lemma 1.1.40 because $\mathcal{V} \in \mathcal{N}_{\mathbb{0}}$ and, thus, $tY \in \mathcal{C} - \mathcal{C}$ by (1.2). Since $\mathcal{C}$ is a cone, it is easy to see that $\mathcal{C} - \mathcal{C}$ is a cone, too. Hence, $Y = \frac{1}{t}(tY) \in \mathcal{C} - \mathcal{C} \subseteq \operatorname{span}(\mathcal{C})$. Hence, $\mathcal{X} \subseteq \mathcal{C} - \mathcal{C} \subseteq \operatorname{span}(\mathcal{C})$ and, thus, $\mathcal{X} = \operatorname{span}(\mathcal{C})$ by $\operatorname{span}(\mathcal{C}) \subseteq \mathcal{X}$ being a subspace of $\mathcal{X}$. □

We will see that the interior of the positive cone $\mathcal{X}_+$ of a finite-dimensional topological vector space $\mathcal{X}$ is always nonempty (see Remark 4.3.1). Sometimes we refer to spaces being known as locally convex spaces, when these are used in literature references:

**Definition 1.1.42** (see [198, Sec. 7.1])**.** A topological vector space $(\mathcal{X}, \tau)$ is called *locally convex* if there is a local base $\{\mathcal{U}_i\}_{i \in I} \subseteq \tau$ at $\mathbb{0}$ with $\mathcal{U}_i$ being convex sets.

For example, every normed space is locally convex (see [198, Sec. 7.1]). Many spaces that are used in practice and we consider can be naturally equipped with a norm or, at least, a metric. Hence, we shortly recall some main definitions and facts.

**Definition 1.1.43** (see [122, Def. 2.1.2])**.** Let $\mathcal{X}$ be a real vector space. We call $\|\cdot\| : \mathcal{X} \to \mathbb{R}_+$ a *norm on* $\mathcal{X}$ if the following conditions hold:

(i) $\|X\| = 0$ for $X \in \mathcal{X}$ if and only if $X = \mathbb{0}$,

(ii) $\|\alpha X\| = |\alpha| \, \|X\|$ for each $\alpha \in \mathbb{R}, X \in \mathcal{X}$,

(iii) The *triangle inequality* is satisfied, i.e.,

$$\forall X, Y \in \mathcal{X}: \qquad \|X + Y\| \leq \|X\| + \|Y\|.$$

A tupel $(\mathcal{X}, \|\cdot\|)$ with a real vector space $\mathcal{X}$ and a norm $\|\cdot\| : \mathcal{X} \to \mathbb{R}_+$ on $\mathcal{X}$ is called *normed (vector) space*.

**Example 1.1.44.** The *euclidean norm* on $\mathbb{R}^n$ is given by

$$\forall X = (X_1, X_2, \ldots, X_n)^T \in \mathbb{R}^n: \quad \|X\|_2 = \sqrt{\sum_{i=1}^{n} X_i^2}.$$

The vector space $\mathbb{R}^n$ equipped with the usual inner product or scalar product $\langle X, Y \rangle := X^T Y$ for $X, Y \in \mathbb{R}^n$ and the euclidean norm $\|\cdot\|_2$ is called *($n$-dimensional) Euclidean space*, and it holds that $\|X\|_2 = \sqrt{\langle X, X\rangle}$ for each $X \in \mathbb{R}^n$. If we just write $\|\cdot\|$ in the context of $\mathbb{R}^n$, we mean the euclidean norm. It is well known that all norms on $\mathbb{R}^n$ (as for every finite dimensional vector space) are equivalent, see [196, Satz I.2.5]. The euclidean norm is an example for a *p-norm* on $\mathbb{R}^n$ (namely, $p = 2$). For $1 \leq p < +\infty$, a $p$-norm is defined by

$$\forall X = (X_1, X_2, \dots, X_n)^T \in \mathbb{R}^n : \quad \|X\|_p := \left(\sum_{i=1}^{n} |X_i|^p\right)^{\frac{1}{p}}.$$

◊

We write just $\mathcal{X}$ for $(\mathcal{X}, \|\cdot\|)$ if the norm is obvious or not from interest. The norm of a vector $X \in \mathcal{X}$ can be understood as a generalization of the geometrical length of a vector in Euclidean spaces $\mathbb{R}^n$. Analogously, there is a generalization for the distance between two vectors:

**Definition 1.1.45** (see [174, Def. 2.15])**.** Let $\mathcal{X}$ be a nonempty set. We call $d\colon \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ a *metric* or *distance function* if the following conditions hold:

(i) $d(X, Y) = 0$ for $X, Y \in \mathcal{X}$ if and only if $X = Y$,

(ii) $d(X, Y) = d(Y, X)$ for all $X, Y \in \mathcal{X}$,

(iii) $d(X, Y) \leq d(X, Z) + d(Z, Y)$ for all $X, Y, Z \in \mathcal{X}$.

A tupel $(\mathcal{X}, d)$ with $d$ being a metric on the nonempty set $\mathcal{X}$ is called *metric space*. For each $X, Y \in \mathcal{X}$, we call $d(X, Y) \in \mathbb{R}_+$ the *distance between $X$ and $Y$*.

**Example 1.1.46.** Note that a normed space always defines a metric space by equipping $\mathcal{X}$ with the metric $d(X, Y) := \|X - Y\|$ for all $X, Y \in \mathcal{X}$. Hence, for the euclidean space $\mathbb{R}^n$ with $n \in \mathbb{N}$, equipped with the euclidean norm $\|\cdot\|_2 : \mathbb{R}^n \to \mathbb{R}$ (see Example 1.1.44), the distance between two arbitrary points can be defined by the euclidean metric

$$\forall X, Y \in \mathbb{R}^n : \qquad d(X, Y) := \|X - Y\|_2 = \sqrt{\sum_{i=1}^{n} (X_i - Y_i)^2}.$$

Especially, we obtain for $n = 1$ the well known case $d(X, Y) = |X - Y|$ for all $X, Y \in \mathbb{R}$. Thus, the norm of a vector $X \in \mathcal{X}$ is its distance from the origin. ◊

**Remark 1.1.47** (see [153, § 20])**.** *Let $(\mathcal{X}, d)$ be a metric space. An* open ball *centered at $X \in \mathcal{X}$* with radius $r > 0$ *is defined by*

$$\mathcal{B}_r(X) := \{Y \in \mathcal{X} \mid d(X, Y) < r\}.$$

*In the literature, there is sometimes an index $d$ to emphasize that the open ball also depends on the metric $d$, i.e., some authors write $\mathcal{B}_{r,d}(X)$. It is easy to see that $\mathcal{B}_r(X)$ is a convex set for all $X \in \mathcal{X}, r \in \mathbb{R}_>$. Open balls play an important role with respect to the topology induced by the metric $d$ (see [153, § 20]): In every metric space $(\mathcal{X}, d)$ (and, thus, in every normed space, see Example 1.1.46), a basis of the topology induced by $d$ is given by the collection*

$\{\mathcal{B}_r(X) \mid X \in \mathcal{X}, r > 0\}$ *of all open balls. The corresponding topology is called* metric topology*. That explains why we call* $\mathcal{B}_r(X)$ open *ball. Note that* $\mathcal{U} \subseteq \mathcal{X}$ *is open with respect to the metric topology induced by* $d$ *if and only if*

$$\forall X \in \mathcal{U}, \exists r > 0 : \quad \mathcal{B}_r(X) \subseteq \mathcal{U}.$$

*Consequently, in metric or normed spaces, we only need to check topological properties including open sets (like continuity of maps, see Definition 1.1.16) by use of open balls* $\mathcal{B}_r(X)$*. Furthermore, note that metric spaces (and, thus, normed spaces) fulfill the axiom of first countability in the sense of Definition 1.1.10 (see [8, Sec. 2.2, p. 27]) and, thus, we can apply Lemma 1.1.12 for characterizing the closure of sets. Furthermore, a metric space equipped with the metric topology is Hausdorff (see [122, Sec. 1.2]).*

We recall the useful characterization of convergent sequences in metric spaces:

**Lemma 1.1.48** (see [122, Prop.1.2.1])**.** *Let* $(\mathcal{X}, d)$ *be a metric space and* $(X_k)_{k\in\mathbb{N}} \subseteq \mathcal{X}$ *be a sequence. Then,* $(X_k)_{k\in\mathbb{N}}$ *converges to the limit* $X \in \mathcal{X}$ *if*

$$\forall \epsilon > 0, \exists N \in \mathbb{N} : \quad d(X_k, X) < \epsilon \text{ for all } k \geq N.$$

Lemma 1.1.48 implies for normed vector spaces $(\mathcal{X}, \|.\|)$ that $(X_k)_{k\in\mathbb{N}} \subseteq \mathcal{X}$ converges to the limit $X \in \mathcal{X}$ if

$$\lim_{k\to+\infty} \|X_k - X\| = 0.$$

In metric spaces, we have the following more practical characterization of continuity that can be derived by Lemma 1.1.48:

**Lemma 1.1.49** (see [153, Th. 21.1 and 21.3])**.** *Let* $(\mathcal{X}, d_{\mathcal{X}}), (\mathcal{Y}, d_{\mathcal{Y}})$ *be metric spaces and* $f : \mathcal{X} \to \mathcal{Y}$ *be a map. Then,* $f$ *is continuous at* $X \in \mathcal{X}$ *if and only if the following holds:*

$$\forall \epsilon > 0, \exists \delta = \delta(\epsilon, X) : \qquad d_{\mathcal{X}}(X, Y) < \delta \quad \Longrightarrow \quad d_{\mathcal{Y}}(f(X), f(Y)) < \epsilon.$$

*Moreover,* $f$ *is continuous at* $X \in \mathcal{X}$ *if and only if* $f(X_n) \to f(X)$ *for* $n \to +\infty$ *holds for every sequence* $(X_n)_{n\in\mathbb{N}} \subseteq \mathcal{X}$ *with* $X_n \to X$ *for* $n \to +\infty$*.*

We say that a metric space $(\mathcal{X}, d)$ is *complete* if each Cauchy sequence (see [122, Def. 1.2.8]) converges in $\mathcal{X}$, see [122, Def. 1.2.9]. A complete normed vector space is called *Banach space.*

### 1.1.2 Binary relations and order cones

In vector optimization and other applications, it is necessary to compare elements $X, Y \in \mathcal{X}$ of a vector space $\mathcal{X}$. Hence, let $\mathcal{R} \subseteq \mathcal{X} \times \mathcal{X}$ be a binary relation on $\mathcal{X}$. In the literature, $X\mathcal{R}Y$ is equivalently used for $(X, Y) \in \mathcal{R}$, compare [91] for standard terminology and examples. In the following, we write $\leq$ for $\mathcal{R}$.

**Definition 1.1.50** (see [185, Def. 2.1.40])**.** Let $\leq$ be a binary relation on a vector space $\mathcal{X}$. We say that $\leq$ is

(i) *reflexive* if $X \leq X$ for all $X \in \mathcal{X}$,

(ii) *symmetric* if $X \leq Y$ implies $Y \leq X$ for all $X, Y \in \mathcal{X}$,

(iii) *antisymmetric* if $(X \leq Y) \wedge (Y \leq X)$ implies $X = Y$ for all $X, Y \in \mathcal{X}$,

(iv) *transitive* if $(X \leq Y) \wedge (Y \leq Z)$ implies $X \leq Z$ for all $X, Y, Z \in \mathcal{X}$,

(v) *total* if $X \leq Y$ or $Y \leq X$ for all $X, Y \in \mathcal{X}$,

(vi) *compatible with the linear structure of* $\mathcal{X}$ if for all $X, Y \in \mathcal{X}$ with $X \leq Y$, it holds that

$$\forall \lambda \in \mathbb{R}_+ : \quad \lambda X \leq \lambda Y \qquad \text{and} \qquad \forall Z \in \mathcal{X} : \quad X + Z \leq Y + Z.$$

**Definition 1.1.51** (see [185, Def. 2.1.40])**.** Let $\leq$ be a binary relation on a vector space $\mathcal{X}$. We call $\leq$ a *preorder on* $\mathcal{X}$ if $\leq$ is reflexive and transitive. Moreover, $\leq$ is called a *partial order on* $\mathcal{X}$ if it is a preorder and antisymmetric. $\mathcal{X}$ is said to be *partially ordered by* $\leq$ if the binary relation $\leq$ is a partial order on $\mathcal{X}$. We say that $\leq$ is a *total order on* $\mathcal{X}$ if $\leq$ is a partial order and total.

**Example 1.1.52** (see [185, Expl. 2.2.13])**.** Let $\mathcal{X} = \mathbb{R}^n$. We give some examples of ordering structures on $\mathbb{R}^n$:

(a) The strict product order (or strict componentwise order) $<$ and product order (or componentwise order) $\leq$ are partial orders on $\mathbb{R}^n$, which are given for $X, Y \in \mathbb{R}^n$ by

$$X < Y \quad :\Longleftrightarrow \quad \forall i \in \{1, \dots, n\} : \ X_i < Y_i$$

and

$$X \leq Y \quad :\Longleftrightarrow \quad \forall i \in \{1, \dots, n\} : \ X_i \leq Y_i.$$

(b) The lexicographical strict order $<_{\text{lex}}$ and lexicographical order $\leq_{\text{lex}}$ are total orders on $\mathbb{R}^n$, which are given for $X, Y \in \mathbb{R}^n$ by

$$X <_{\text{lex}} Y \quad :\Longleftrightarrow \quad \exists k \in \{2, \dots, n\} : X_k < Y_k, \ X_i = Y_i \text{ for } i = 1, \dots, k-1$$

and

$$X \leq_{\text{lex}} Y \quad :\Longleftrightarrow \quad X <_{\text{lex}} Y \text{ or } X = Y.$$

$\Diamond$

Cones are suitable for describing binary relations on a vector space $\mathcal{X}$, especially, partial orders, see the following theorem:

**Theorem 1.1.53** (see [91, Th. 2.1.13])**.** *Let $\mathcal{X}$ be a vector space.*

(i) *Let $\mathcal{C} \subseteq \mathcal{X}$ be a cone. Consider*

$$\leq_{\mathcal{C}} := \{(X,Y) \in \mathcal{X} \times \mathcal{X} \mid Y - X \in \mathcal{C}\}. \tag{1.3}$$

*Then, $\leq_{\mathcal{C}}$ is a binary relation on $\mathcal{X}$ and fulfills the following properties:*

(a) *$\leq_{\mathcal{C}}$ is reflexive and compatible with the linear structure of $\mathcal{X}$,*

(b) *$\leq_{\mathcal{C}}$ is transitive if and only if $\mathcal{C}$ is convex,*

(c) *$\leq_{\mathcal{C}}$ is antisymmetric if and only if $\mathcal{C}$ is pointed.*

(ii) *Let $\leq$ a reflexive binary relation on $\mathcal{X}$ that is compatible with the linear structure of $\mathcal{X}$. Consider*

$$\mathcal{C}_{\leq} := \{X \in \mathcal{X} \mid \mathbb{0} \leq X\}.$$

*Then, $\mathcal{C}_{\leq}$ is a cone fulfilling*

$$\forall X, Y \in \mathcal{X}: \quad X \leq Y \quad \Longleftrightarrow \quad X \leq_{(\mathcal{C}_{\leq})} Y.$$

Let $\mathcal{X}$ be a vector space. By Theorem 1.1.53, $\leq_{\mathcal{C}}$ given by (1.3) is a partial order if and only if $\mathcal{C} \subseteq \mathcal{X}$ is a convex, pointed cone. Thus, we call a convex, pointed cone $\mathcal{C}$ an *ordering cone* in $\mathcal{X}$. The corresponding partial order $\leq_{\mathcal{C}}$ is given by

$$\forall X, Y \in \mathcal{X}: \quad X \leq_{\mathcal{C}} Y \quad :\Longleftrightarrow \quad Y - X \in \mathcal{C}. \tag{1.4}$$

For example, if $\mathcal{X}$ is partially ordered by $\leq$, the natural ordering cone in $\mathcal{X}$ is the *positive cone*

$$\mathcal{X}_+ = \{X \in \mathcal{X} \mid \mathbb{0} \leq X\}. \tag{1.5}$$

The corresponding partial order $\leq_{\mathcal{X}_+}$ given by (1.3) coincides with $\leq$ by Theorem 1.1.53(ii). Every element $X \in \mathcal{X}_+$ is called *positive*. An example for $\leq_{\mathcal{X}_+}$ is the componentwise ordering on $\mathcal{X} = \mathbb{R}^n$ with $\mathcal{X}_+ = \mathbb{R}^n_+$ (see Example 1.1.52). In this thesis, we say that a vector space $\mathcal{X}$ is *partially ordered by* $\mathcal{C} \subseteq \mathcal{X}$ if $\mathcal{C}$ is an ordering cone in $\mathcal{X}$ and, then, we consider the partial order $\leq_{\mathcal{C}}$ on $\mathcal{X}$ given by (1.4).

### 1.1.3 Extended real-valued functions

In this thesis, extended real-valued functions will be substantial, for example, when we deal with risk measures (see Section 1.3.3). Hence, we collect some well-known notions and results for mappings $f\colon \mathcal{X} \to \overline{\mathbb{R}}$ with $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ and $\mathcal{X}$ being a vector space. Note that we make use of the so called *inf-addition rule* in this thesis (see [150]):

$$(+\infty) + (-\infty) = (-\infty) + (+\infty) = +\infty.$$

**Definition 1.1.54** (see [185, Def. 3.2.2])**.** Let $\mathcal{X}$ be a vector space and $f\colon \mathcal{X} \to \overline{\mathbb{R}}$. We call

$$\operatorname{dom} f := \{X \in \mathcal{X} \mid f(X) < +\infty\}$$

the *(effective) domain* of $f$ and

$$\operatorname{epi} f := \{(X,t) \in \mathcal{X} \times \mathbb{R} \mid f(X) \leq t\}$$

the *epigraph* of $f$. We say that $f$ is *finite-valued (on $\mathcal{X}$)* if $f(X) \in \mathbb{R}$ for each $X \in \mathcal{X}$. Moreover, $f$ is said to be *proper* if $\operatorname{dom} f \neq \emptyset$ and $f(X) > -\infty$ for all $X \in \mathcal{X}$.

It is equivalent to say that $f\colon \mathcal{X} \to \overline{\mathbb{R}}$ is proper if and only if $\operatorname{dom} f \neq \emptyset$ and $f$ is finite-valued on $\operatorname{dom} f$. We recall the following terminology for basic properties of extended real-valued functions that sometimes may slightly differ from the ordinary definitions for real-valued functions:

**Definition 1.1.55** (see [185])**.** Let $\mathcal{X}$ be a vector space and $f\colon \mathcal{X} \to \overline{\mathbb{R}}$.

(i) $f$ is called *convex* (on $\mathcal{X}$), if $\operatorname{epi} f$ is a convex set,

(ii) $f$ is called *affine* (on $\mathcal{X}$), if $f$ is convex (on $\mathcal{X}$) and $-f$ is convex (on $\mathcal{X}$),

(iii) $f$ is called *linear* (on $\mathcal{X}$), if $f$ is affine (on $\mathcal{X}$) with $f(\mathbb{0}) = 0$,

(iv) $f$ is called *positively homogeneous* (on $\mathcal{X}$) if $\operatorname{epi} f$ is a nonempty cone,

(v) $f$ is called *homogeneous* (on $\mathcal{X}$) if $f$ is positively homogeneous and odd, i.e., $f(-X) = -f(X)$ for each $X \in \mathcal{X}$,

(vi) $f$ is called *subadditive* (on $\mathcal{X}$) if $\operatorname{epi} f + \operatorname{epi} f \subseteq \operatorname{epi} f$ holds,

(vii) $f$ is called *additive* (on $\mathcal{X}$) if $f$ is subadditive and superadditive, i.e., $-f$ is subadditive.

(viii) $f$ is called *sublinear* (on $\mathcal{X}$) if $\operatorname{epi} f$ is a nonempty, convex cone.

In Definition 1.1.55(i), the characterization of convex functions by use of the epigraph was introduced by Fenchel in [74, Sec. 3.1] for real-valued functions on a finite-dimensional vector space. The following lemma shows that some of the given definitions in Definition 1.1.55 coincide with the well-known usual definitions for real-valued functions:

**Lemma 1.1.56** (see [185, Th. 3.4.4. and 3.5.11])**.** *Let $\mathcal{X}$ be a vector space and $f\colon \mathcal{X} \to \overline{\mathbb{R}}$. Then, the following holds:*

(i) *$f$ is convex if and only if*

$$\forall X, Y \in \mathcal{X}, \forall \lambda \in [0,1]: \quad f(\lambda X + (1-\lambda)Y) \leq \lambda f(X) + (1-\lambda) f(Y).$$

(ii) *$f$ is positively homogeneous with $f(\mathbb{0}) = 0$ if and only if*

$$\forall \lambda \in \mathbb{R}_+, \forall X \in \mathcal{X}: \quad f(\lambda X) = \lambda f(X),$$

(iii) *f is homogeneous if and only if*

$$\forall \lambda \in \mathbb{R}, \forall X \in \mathcal{X}: \quad f(\lambda X) = \lambda f(X),$$

(iv) *f is subadditive if and only if*

$$\forall X, Y \in \mathcal{X}: \quad f(X+Y) \leq f(X) + f(Y),$$

(v) *Suppose that f is proper or* $\operatorname{dom} f \in \{\emptyset, \mathcal{X}\}$. *Then, f is additive if and only if*

$$\forall X, Y \in \mathcal{X}: \quad f(X+Y) = f(X) + f(Y).$$

Lemma 1.1.56(i) and (iv) need the assumption of the inf-addition rule as noticed at the beginning of this section. Without this rule, we have to assume that $f$ only attains the value $+\infty$ or $-\infty$, and have to make the same assumption in (iv) as in (v). It will be useful to recall the following result that outlines the relationship between some properties in Definition 1.1.55 for a vector space $\mathcal{X}$:

**Lemma 1.1.57** (see [185, Lemma 3.5.12])**.** *Let $\mathcal{X}$ be a vector space and $f\colon \mathcal{X} \to \overline{\mathbb{R}}$. Then, f is sublinear if and only if one of the following equivalent conditions is fulfilled:*

(i) *f is convex and positively homogeneous,*

(ii) *f is convex and subadditive with $f(\mathbb{0}) \leq 0$,*

(iii) *f is subadditive and positively homogeneous.*

**Remark 1.1.58.** *Let $\mathcal{X}$ be a vector space. Each linear functional fulfills Lemma 1.1.57*(ii) *by Definition 1.1.55*(iii)*. Thus, linear functionals are sublinear and, especially, positively homogeneous. More exactly, it holds for a functional $f\colon \mathcal{X} \to \overline{\mathbb{R}}$ that (see [185, Th. 3.5.18])*

$$f \text{ is linear} \quad \Longleftrightarrow \quad f \text{ is additive and homogeneous.}$$

*Continuous linear functionals are finite-valued (see [185, Cor. 3.5.7]). Moreover, a linear functional $f\colon \mathcal{X} \to \overline{\mathbb{R}}$ is finite-valued if f is proper or* $\operatorname{dom} f = \mathcal{X}$ *(see [185, Cor. 3.5.20]). Note that a real-valued functional f is linear if and only if it holds that (see [185, Cor. 3.5.19])*

$$\forall X^1, X^2 \in \mathcal{X}, \forall \lambda_1, \lambda_2 \in \mathbb{R}: \quad f(\lambda_1 X^1 + \lambda_2 X^2) = \lambda_1 f(X^1) + \lambda_2 f(X^2).$$

**Definition 1.1.59** (see [185, Def. 3.3.5])**.** Let $(\mathcal{X}, \tau)$ be a topological vector space and $f\colon \mathcal{X} \to \overline{\mathbb{R}}$. We call $f$ *lower semicontinuous at* $X^0 \in \mathcal{X}$ if $f(X^0) = -\infty$ or

$$\forall h \in \mathbb{R} \text{ with } h < f(X^0), \exists \mathcal{U} \in \mathcal{N}_{X^0}: \quad f(X) > h \text{ for each } X \in \mathcal{U}.$$

$f$ is said to be *lower semicontinuous (on $\mathcal{X}$)* if $f$ is lower semicontinuous at each $X \in \mathcal{X}$.

We will make use of the following terminology:

**Definition 1.1.60** (see [185, Def. 3.3.10])**.** Let $X$ be a vector space, $f\colon \mathcal{X} \to \overline{\mathbb{R}}$, and $t \in \mathbb{R}$ arbitrary. We call

$$\operatorname{lev}_{f,\leq}(t) := \{X \in \mathcal{X} \mid f(X) \leq t\}$$

*sublevel set* of $f$ to the level $t$,

$$\operatorname{lev}_{f,<}(t) := \{X \in \mathcal{X} \mid f(X) < t\}$$

*strict sublevel set* of $f$ to the level $t$, and

$$\operatorname{lev}_{f,=}(t) := \{X \in \mathcal{X} \mid f(X) = t\}$$

*level line* of $f$ to the level $t$.

The following lemma provides a useful characterization of lower semicontinuity:

**Lemma 1.1.61** (see [185, Th. 3.3.12])**.** *Let $\mathcal{X}$ be a topological space and $f\colon \mathcal{X} \to \overline{\mathbb{R}}$ be a map. Then, the following conditions are equivalent:*

(i) *$f$ is lower semicontinuous,*

(ii) *For all $t \in \mathbb{R}$, the sublevel sets $\operatorname{lev}_{f,\leq}(t)$ are closed,*

(iii) *$\operatorname{epi} f$ is closed in $\mathcal{X} \times \mathbb{R}$ with respect to the product topology on $\mathcal{X} \times \mathbb{R}$.*

**Remark 1.1.62.** *Let $\mathcal{X}$ be a topological vector space. In the literature, the set*

$$\mathcal{X}^* := \{\varphi\colon \mathcal{X} \to \overline{\mathbb{R}} \mid \varphi \text{ linear and continuous}\}$$

*usually denotes the topological dual space. For working with linear functionals, it is useful to have the following consequence of the Hahn-Banach-Theorem in mind (see [91, Ch. 2.2]): There is always some non-trivial linear continuous functional if $\mathcal{X}$ is a locally convex Hausdorff space with $\mathcal{X} \neq \{\mathbb{0}\}$, i.e., $\mathcal{X}^*$ does not only consist of the null functional. That is equivalent to*

$$\forall X \in \mathcal{X}\backslash\{\mathbb{0}\},\ \exists \varphi \in \mathcal{X}^* : \qquad \varphi(X) > 0.$$

*Consequently, for given $X, Y \in \mathcal{X}$ with $X \neq Y$, there is some $\varphi \in \mathcal{X}^*$ separating the elements $X$ and $Y$, i.e., it holds that $\varphi(X) \neq \varphi(Y)$. Moreover, if $f\colon \mathcal{X} \to \overline{\mathbb{R}}$ is a continuous, positively homogeneous function, then $f$ is finite-valued or constant with value $-\infty$ (see [185, Prop. 4.14.11]). Note that linear functionals are positively homogeneous (see Remark 1.1.58). Moreover, if $\mathcal{X}$ is a finite-dimensional Hausdorff topological vector space and $\mathcal{Y}$ a topological vector space like $\overline{\mathbb{R}}$, then each linear functional $f\colon \mathcal{X} \to \overline{\mathcal{Y}}$ is continuous (see [173, Lemma 1.20 and Th. 1.21].*

## 1.2 Vector optimization

*Vector optimization* is an important branch of optimization theory and deals with the problem of optimizing a vector-valued function, which is defined on a vector space and has a vector space equipped with a preorder as an image space. There is a bride range of literature concerning vector optimization, see, e.g., Göpfert et al. [91], Ehrgott [56], Sawaragi et al. [176], Jahn [116], and Eichfelder, Jahn [60]. In the literature, the image space $\mathbb{R}^m$ with $m \in \mathbb{N}$ is often considered, which is known as *multiobjective optimization* and focuses on the optimization of multiple, conflicting objectives simultaneously. An origin can be seen in the works of Edgeworth [55], and Pareto [156] (see also [157]), and, in the branch of mathematical optimization, the work of Kuhn, Tucker [131]. For more details about the historical references, see Eichfelder [60], and Ansari, Yao [12] for some overview about recent developments in vector optimization. Vector optimization problems are from special interest due to a wield field of applications, for example, in financial mathematics, location theory, social welfare economics, robust multiobjective decision making, and energetic and medical engineering (see, e.g., Bao, Mordukhovich [20], Bischoff et al. [30], Eichfelder [59], Feinstein, Rudloff [73], Günther, Tammer [92], Hamel et al. [101], [104], Klamroth et al. [125], [126], Köbis [127], Küfer et al. [130]), and Nickel, Puerto [154]).

In vector optimization, the decision maker is interested in finding non-dominated solutions of a function $f\colon \mathcal{X} \to \mathcal{Y}$ with $\mathcal{X}, \mathcal{Y}$ being vector spaces, $\mathcal{D} \subseteq \mathcal{Y}$ being a nonempty subset of $\mathcal{Y}$ representing a preference relation (in general, a preorder) $\mathcal{R} \subseteq \mathcal{Y} \times \mathcal{Y}$, and $\mathcal{B} \subseteq \mathcal{X}$ being the feasible set. We denote this optimization problem by

$$f(X) \longrightarrow \mathcal{D}\text{-}\underset{X \in \mathcal{B}}{\mathrm{Min}}. \tag{V}$$

It is not obvious what $\mathcal{D}\text{-}\underset{X \in \mathcal{B}}{\mathrm{Min}}$ as minimization in the sense of vector optimization should mean. There are various solution concepts for vector optimization problems, compare Ansari et al. [11], Eichfelder [60], Heyde, Löhne [108], Khan et al. [124], and references therein. We present two main characterizations of non-dominated points of an arbitrary subset $\mathcal{A} \subseteq \mathcal{X}$. For use as a solution concept of (V), the set $\mathcal{A}$ has just to be replaced by the image $f(\mathcal{B})$, see Definition 1.2.4. All concepts have some preference relation of the decision maker defined on the image space in common. For example, with respect to multiobjective optimization problems, we need preference information from the decision maker to resolve the conflicts, resulting by considering multiple objectives simultaneously, to find satisfying solutions. The preference relation is induced by a nonempty subset $\mathcal{D} \subseteq \mathcal{Y}$ called *domination set*. In general, this preference relation does not have to be a partial order on $\mathcal{Y}$ (see Section 1.1.2), and, thus, $\mathcal{D}$ is not a cone in general. Mostly, $\mathcal{D}$ is assumed to induce a preorder on the image space $\mathcal{Y}$.

**Definition 1.2.1** (see [185, Sec. 6.2.1])**.** Let $\mathcal{X}$ be a vector space. Consider nonempty subsets $\mathcal{A}, \mathcal{D} \subseteq \mathcal{X}$. An element $X^0 \in \mathcal{A}$ is called an *efficient point of $\mathcal{A}$ with respect to $\mathcal{D}$* if

$$\mathcal{A} \cap (\{X^0\} - \mathcal{D}) \subseteq \{X^0\}$$

holds. The set of the efficient points of $\mathcal{A}$ with respect to $\mathcal{D}$ is denoted by $\mathrm{Eff}\,(\mathcal{A}, \mathcal{D})$.

Obviously, for any vector space $\mathcal{X}$ and nonempty subsets $\mathcal{A}, \mathcal{B}, \mathcal{D} \subseteq \mathcal{X}$, it holds that

$$\mathcal{D} \subseteq \mathcal{B} \quad \Longrightarrow \quad \operatorname{Eff}(\mathcal{A}, \mathcal{B}) \subseteq \operatorname{Eff}(\mathcal{A}, \mathcal{D}).$$

If the vector space $\mathcal{X}$ is partially ordered by the pointed convex cone $\mathcal{C} \subseteq \mathcal{X}$ (see (1.4)), efficient points with respect to Definition 1.2.1 are optimal elements of $\mathcal{A} \subseteq \mathcal{X}$ with respect to the preference relation $\leq_{\mathcal{C}}$ given by (1.4) by choosing $\mathcal{D} = \mathcal{C}$. Consequently, Definition 1.2.1 of efficient points $X^0 \in \mathcal{A}$ with respect to a convex pointed cone $\mathcal{C}$ is equivalent to

$$\nexists X \in \mathcal{A} \backslash \{X^0\} : \; X \leq_{\mathcal{C}} X^0.$$

It turns out that, in general, it is easier to consider open sets as domination sets for determining solutions of vector optimization problems (see, e.g., [185, Sec. 6.2]). That leads to the following definition:

**Definition 1.2.2** ([185, Sec. 6.2.2])**.** Let $\mathcal{X}$ be a topological vector space. Consider nonempty subsets $\mathcal{A} \subseteq \mathcal{X}$ and $\mathcal{D} \subseteq \mathcal{X}$ with $\operatorname{int} \mathcal{D} \neq \emptyset$. An element $X^0 \in \mathcal{A}$ is called a *weakly efficient point of $\mathcal{A}$ with respect to $\mathcal{D}$* if $X^0 \in \operatorname{Eff}(\mathcal{A}, \operatorname{int} \mathcal{D})$ holds. The set of the weakly efficient points of $\mathcal{A}$ with respect to $\mathcal{D}$ is denoted by $\operatorname{Eff}_w(\mathcal{A}, \mathcal{D})$.

For linear spaces, there is also a definition for weakly efficient points by use of the algebraic interior, see [116]. We recall the main relationships between the weakly efficient points and efficient points of a subset $\mathcal{A} \subseteq \mathcal{X}$ in a topological vector space $\mathcal{X}$:

**Lemma 1.2.3** (see [185, Sec. 6.2])**.** *Let $\mathcal{X}$ be a topological vector space and $\mathcal{A}, \mathcal{D} \subseteq \mathcal{X}$ be nonempty subsets of $\mathcal{X}$. Then, the following conditions hold:*

(i) *If* $\operatorname{int} \mathcal{D} \neq \emptyset$*, then* $\operatorname{Eff}(\mathcal{A}, \mathcal{D}) \subseteq \operatorname{Eff}_w(\mathcal{A}, \mathcal{D})$*,*

(ii) *If* $\mathbb{0} \in \operatorname{bd}(\mathcal{D} \backslash \{\mathbb{0}\})$*, then* $\operatorname{Eff}(\mathcal{A}, \mathcal{D}) \subseteq \operatorname{bd} \mathcal{A}$*,*

(iii) *If* $\operatorname{int} \mathcal{D} \neq \emptyset$ *and* $\mathbb{0} \in \operatorname{bd}(\operatorname{int} \mathcal{D} \backslash \{\mathbb{0}\})$*, then* $\operatorname{Eff}_w(\mathcal{A}, \mathcal{D}) \subseteq \operatorname{bd} \mathcal{A}$*.*

With respect to the introduced concepts of efficiency and weakly efficiency, we can define the corresponding solution concepts for (V):

**Definition 1.2.4.** Let $\mathcal{X}, \mathcal{Y}$ be vector spaces and $f \colon \mathcal{X} \to \mathcal{Y}$. Consider the optimization problem (V) with the feasible set $\mathcal{B} \subseteq \mathcal{X}$ and domination set $\emptyset \neq \mathcal{D} \subseteq \mathcal{Y}$. We call $X \in \mathcal{X}$ *efficient solution of* (V) *with respect to $\mathcal{D}$* if $f(X) \in \mathcal{Y}$ is an efficient point of the image $f(\mathcal{B})$, i.e., $f(X) \in \operatorname{Eff}(f(\mathcal{B}), \mathcal{D})$. If $\mathcal{Y}$ is a topological vector space and $\operatorname{int} \mathcal{D} \neq \emptyset$, we call $X \in \mathcal{X}$ *weakly efficient solution of* (V) *with respect to $\mathcal{D}$* if $f(X) \in \mathcal{Y}$ is a weakly efficient point of the image $f(\mathcal{B})$, i.e., $\operatorname{Eff}_w(f(\mathcal{B}), \mathcal{D})$.

Note that there are more solution concepts for vector optimization problems then the presented ones. For example, one can define solutions as properly efficient points of the image set, see, e.g., [26] and [87]. As usual for optimization problems, the solutions of (V) can not be determined explicitly in general. Hence, there are several algorithms for solving vector optimization problems numerically, e.g., Benson-type algorithms (see [104] and references therein) or algorithms using adaptive parameter control (see [58] and references therein). Scalarization techniques provide a fundamental tool for solving vector optimization problems given by (V) theoretically and numerically, see Bouza et al. [36], [37], Eichfelder [58], Gerstewitz [88], Göpfert et al. [91], Gutiérrez et al. [94], [95], [96], Luc [136], Miglierina, Molho [148], Tammer, Zălinescu [186], Tammer, Weidner [185], and references therein. A unified characterization of scalarization functionals can be found in [36]. Well established scalarization functionals for vector optimization problems are special cases of the Gerstewitz functional (see [185, Ch. 6]):

**Definition 1.2.5.** Let $\mathcal{X}$ be a vector space, $\emptyset \neq \mathcal{A} \subseteq \mathcal{X}$, and $K \in \mathcal{X}\backslash\{\mathbb{0}\}$ such that $\mathcal{A} - \mathbb{R}_+ K \subseteq \mathcal{A}$. The nonlinear functional $\varphi_{\mathcal{A},K} : \mathcal{X} \to \overline{\mathbb{R}}$ given by

$$\varphi_{\mathcal{A},K}(X) := \inf\{t \in \mathbb{R} \mid X \in tK + \mathcal{A}\} \tag{1.6}$$

is called *Gerstewitz-functional.*

The class of functionals given by (1.6) coincides with the class of translation invariant functionals (see [185, Th. 4.2.3]), i.e., with $\mathcal{X}$ being a vector space, the class

$$\{\varphi_{\mathcal{A},K} \colon \mathcal{X} \to \overline{\mathbb{R}} \mid \emptyset \neq \mathcal{A} \subsetneq \mathcal{X}\}$$

of functionals from type (1.6) coincides for each $K \in \mathcal{X}\backslash\{\mathbb{0}\}$ with the class

$$\{\varphi \colon \mathcal{X} \to \overline{\mathbb{R}} \mid \forall X \in \mathcal{X}, \forall t \in \mathbb{R} : \; \varphi(X + tK) = \varphi(X) + t\}.$$

A detailed study of these functionals can be found in [185]. Translation invariant functionals given by (1.6) are used by Gerstewitz in [88] in order to show separation theorems by use of $\varphi_{\mathcal{A},K}$ as separation functional for sets that are not necessarily convex, and, as already noticed, as scalarization functional in vector optimization. We want to collect some of the most important properties of the functional $\varphi_{\mathcal{A},K}$ given by (1.6):

**Lemma 1.2.6** (see [185, Sec. 4.2])**.** *Let $\mathcal{X}$ be a vector space, $\emptyset \neq \mathcal{A} \subsetneq \mathcal{X}$, $K \in \mathcal{X}\backslash\{\mathbb{0}\}$. Consider the nonlinear functional $\varphi_{\mathcal{A},K} : \mathcal{X} \to \overline{\mathbb{R}}$ given by* (1.6)*. Then, the following conditions hold:*

(i) $\operatorname{dom}(\varphi_{\mathcal{A},K}) = \mathcal{A} + \mathbb{R}K$,

(ii) $\varphi_{\mathcal{A},K}$ *is translation invariant along $K$, i.e.,*

$$\forall X \in \mathcal{X}, \forall t \in \mathbb{R} : \quad \varphi_{\mathcal{A},K}(X + tK) = \varphi_{\mathcal{A},K}(X) + t,$$

(iii) $\operatorname{lev}_{\varphi_{\mathcal{A},K},\leq}(t) = \operatorname{cl}_K(\mathcal{A} - \mathbb{R}_+K) + tK = \operatorname{cl}_K(\mathcal{A}) - \mathbb{R}_+K + tK$ *for each* $t \in \mathbb{R}$,

(iv) $\operatorname{lev}_{\varphi_{\mathcal{A},K},<}(t) = \operatorname{int}_K(\mathcal{A} - \mathbb{R}_+K) + tK = \mathcal{A} - \mathbb{R}_>K + tK$ *for each* $t \in \mathbb{R}$,

(v) $\operatorname{lev}_{\varphi_{\mathcal{A},K},=}(t) = \operatorname{bd}_K(\mathcal{A} - \mathbb{R}_+K) + tK$ *for each* $t \in \mathbb{R}$,

(vi) $\varphi_{\mathcal{A},K}$ *is convex if and only if* $\operatorname{cl}_K(\mathcal{A} - \mathbb{R}_+ K)$ *is convex,*

(vii) $\varphi_{\mathcal{A},K}$ *is positively homogeneous if and only if* $\operatorname{cl}_K(\mathcal{A} - \mathbb{R}_+ K)$ *is a cone,*

(viii) $\varphi_{\mathcal{A},K}$ *is subadditive if and only if* $\operatorname{cl}_K(\mathcal{A} - \mathbb{R}_+ K) + \operatorname{cl}_K(\mathcal{A} - \mathbb{R}_+ K) \subseteq \operatorname{cl}_K(\mathcal{A} - \mathbb{R}_+ K)$*,*

(ix) $\varphi_{\mathcal{A},K}$ *is sublinear if and only if* $\operatorname{cl}_K(\mathcal{A} - \mathbb{R}_+ K)$ *is a convex cone,*

(x) $\varphi_{\mathcal{A},K}$ *is* $\mathcal{B}$*-monotone with* $\mathcal{B} \subseteq \mathcal{X}$ *if and only if* $\mathcal{A} - \mathcal{B} \subseteq \operatorname{cl}_K(\mathcal{A}) - \mathbb{R}_+ K$*,*

(xi) *Suppose* $\mathcal{A} - \mathbb{R}_+ K \subseteq \mathcal{A}$*. Then,* $\varphi_{\mathcal{A},K}$ *is proper if and only if* $\mathcal{A}$ *does not contain lines parallel to* $K$*, i.e.,*

$$\forall X \in \mathcal{A}: \quad X + \mathbb{R}K \not\subseteq \mathcal{A}. \tag{1.7}$$

(xii) $\varphi_{\mathcal{A},K}$ *is finite if and only if* $\operatorname{bd}_K(\mathcal{A} - \mathbb{R}_+ K) + \mathbb{R}K = \mathcal{X}$*.*

*Moreover, the conditions for* $\operatorname{cl}_K(\mathcal{A} - \mathbb{R}_+ K)$ *on the right-hand sight of* (vi), (vii), (viii) *and* (ix) *are fulfilled if they hold for* $\operatorname{cl}_K(\mathcal{A})$ *or* $\mathcal{A}$*. Now, suppose that* $(\mathcal{X}, \tau)$ *is a topological vector space. Then, the following holds in addition:*

(xiii) $\varphi_{\mathcal{A},K}$ *is lower semicontinuous if* $\operatorname{cl}\mathcal{A} - \mathbb{R}_> K \subseteq \mathcal{A}$*,*

(xiv) $\varphi_{\mathcal{A},K}$ *is continuous if* $\operatorname{cl}\mathcal{A} - \mathbb{R}_> K \subseteq \operatorname{int}\mathcal{A}$*.*

In this thesis, we study extensions of the functional given by (1.6) (see Definition 2.3.1), and show corresponding properties to those noticed in Lemma 1.2.6. Although all translation invariant functionals (like risk measures, see Section 1.3.3) can be replaced by a corresponding functional of type (1.6), we will not just refer to the properties in Lemma 1.2.6 and give direct proofs for our results. Note that it is not trivial to choose the corresponding functional of type (1.6) and, thus, to link to the properties above. In general, even if it would be possible, we do not just refer to Lemma 1.2.6 to gain a better understanding of the specific situation and assumptions with respect to the financial market, which allows alternative proofs. Indeed, it turns out (see Remark 2.3.13) that the risk measure considered in this thesis can be reduced to some functional from type (1.6), but is described by an augmented set and not directly by the original subset $\mathcal{A} \subseteq \mathcal{X}$ of the financial market anymore. Hence, we have to provide a more intensive study to derive properties with respect to the original objects of interest.

## 1.3 Financial mathematics

### 1.3.1 Basics from probability theory

Throughout this thesis, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, i.e., a set of all possible outcomes $\Omega$ called *sample space*, a $\sigma$-algebra $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ called *event space*, and a probability measure $\mathbb{P}\colon \mathcal{F} \to [0,1]$. We recall the definitions of these objects for convenience of the reader:

**Definition 1.3.1** (see [145, Def. 1.7])**.** Let $\mathcal{X}$ be an arbitrary set. We call a subset $\mathcal{F} \subseteq \mathcal{P}(\mathcal{X})$ of the power set of $\mathcal{X}$ a *$\sigma$-algebra on $\mathcal{X}$* if the following holds:

(i) $\mathcal{X} \in \mathcal{F}$,

(ii) $\mathcal{X} \backslash \mathcal{A} \in \mathcal{F}$ for all $\mathcal{A} \in \mathcal{F}$,

(iii) For each $(\mathcal{A}_n)_{n\in\mathbb{N}} \subseteq \mathcal{F}$, it holds that

$$\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{F}.$$

If $\mathcal{F}$ is a $\sigma$-algebra on $\mathcal{X}$, we call $(\mathcal{X}, \mathcal{F})$ a *measurable space* or *Borel space*. Moreover, let $\mathcal{D} \subseteq \mathcal{P}(\mathcal{X})$ be an arbitrary family of subsets of $\mathcal{X}$. We call the smallest $\sigma$-algebra on $\mathcal{X}$ containing $\mathcal{D}$ the *$\sigma$-algebra generated by* $\mathcal{D}$, which is denoted by $\sigma(\mathcal{D})$. Especially, if $(\mathcal{X}, \tau)$ is a topological space, we call $\mathcal{B}(\mathcal{X}) := \sigma(\tau)$ the *Borel-$\sigma$-Algebra on $\mathcal{X}$*.

**Definition 1.3.2** (see [145, Def. 1.32 and 3.2])**.** Let $(\Omega, \mathcal{F})$ be a measurable space. We call $\mathbb{P}\colon \mathcal{F} \to [0,1]$ a *probability measure on $\mathcal{F}$* if the following holds:

(i) $\mathbb{P}(\Omega) = 1$,

(ii) For each $(\mathcal{A}_n)_{n\in\mathbb{N}} \subseteq \mathcal{F}$ of pairwise disjoint sets (i.e., $\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$ for all $i \neq j$), it holds that

$$\mathbb{P}\left(\bigcup_{n\in\mathbb{N}} \mathcal{A}_n\right) = \sum_{n\in\mathbb{N}} \mathbb{P}(\mathcal{A}_n).$$

If $\mathbb{P}$ is a probability measure on $\mathcal{F}$, we call $(\Omega, \mathcal{F}, \mathbb{P})$ a *probability space*.

By definition of probability measures as mappings $\mathbb{P}\colon \mathcal{F} \to [0,1]$, $\mathbb{P}(\mathcal{A}) \geq 0$ holds automatically for each $\mathcal{A} \in \mathcal{F}$ (which is sometimes required in the literature by choosing the image space $\mathbb{R}$). We also recall some well-known important properties of probability measures:

**Lemma 1.3.3** (see [145, Satz 3.4])**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Then,*

(i) $\mathbb{P}(\Omega \backslash \mathcal{A}) = 1 - \mathbb{P}(\mathcal{A})$ *for all* $\mathcal{A} \in \mathcal{F}$,

(ii) $\mathbb{P}(\mathcal{A}) \leq \mathbb{P}(\mathcal{B})$ *for all* $\mathcal{A}, \mathcal{B} \in \mathcal{F}$ *with* $\mathcal{A} \subseteq \mathcal{B}$,

(iii) *Let* $(\mathcal{A}_i)_{i\in\mathbb{N}} \subseteq \mathcal{F}$. *It holds that* $\mathbb{P}\left(\bigcup_{i\in\mathbb{N}} \mathcal{A}_i\right) \leq \sum_{i\in\mathbb{N}} \mathbb{P}(\mathcal{A}_i)$.

**Definition 1.3.4** (see [145, Def. 1.23])**.** Let $(\mathcal{X}, \mathcal{F})$ and $(\mathcal{Y}, \mathcal{G})$ be measurable spaces, and $f : \mathcal{X} \to \mathcal{Y}$. We call $f$ *measurable (with respect to* $\mathcal{F}$ *and* $\mathcal{G}$*)* if it holds that

$$\forall \mathcal{A} \in \mathcal{G} : \quad f^{-1}(\mathcal{A}) \subseteq \mathcal{F}.$$

**Definition 1.3.5** (see [145, Def. 1.42, 3.12, and 4.1])**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We call $X \colon \Omega \to \mathbb{R}$ a *(real) random variable* if $X$ is measurable with respect to $\mathcal{F}$ and the Borel-$\sigma$-Algebra $\mathcal{B}(\mathbb{R})$. $\mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$ denotes the vector space of all (real) random variables. We call $\mathbb{P}_X \colon \mathcal{B}(\mathbb{R}) \to [0, 1]$ given by $\mathbb{P}_X(\mathcal{D}) := \mathbb{P}(X^{-1}(\mathcal{D}))$ *(probability) distribution of* $X$ and write $X \sim \mathbb{P}_X$.

Note that it is enough to consider a measurable space $(\Omega, \mathcal{F})$ in the first part of Definition 1.3.5 because random variables are measurable functions, i.e., $\mathcal{L}^0(\Omega, \mathcal{F})$ denotes the space of measurable functions (see Example 1.3.13). Nevertheless, we speak of random variables for measurable functions on $\Omega$ with respect to $\mathcal{F}$ and $\mathcal{B}(\mathbb{R})$ if we consider a probability space since random variables provide a probability distribution naturally and we focus on the probabilistic aspects then. By definition, the distribution of a random variable $X$ is the image measure (or push-forward measure) of $X$ under the probability measure $\mathbb{P}$. Since we only consider real-valued random variables in this thesis, we just speak of random variables. We only write $\mathcal{L}^0$ instead of $\mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$ for the space of random variables if the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is clear or does not matter. Recall that the distribution function $F_X \colon \mathbb{R} \to [0, 1]$ of a real random variable $X \in \mathcal{L}^0$ given by $F_X(x) := \mathbb{P}(X \leq x)$ is monotonically increasing and right-continuous with

$$\lim_{x \to -\infty} F_X(x) = 0 \quad \text{and} \quad \lim_{x \to +\infty} F_X(x) = 1,$$

see, e.g., [145].

**Remark 1.3.6.** *Consider some probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *and a random variable* $X \colon \Omega \to \mathbb{R}$. *Let* $\mathcal{A} \in \mathcal{F}$ *be some arbitrary event. We use the shortcuts*

$$\{X \in \mathcal{A}\} \quad \textit{for} \quad \{\omega \in \Omega \mid X(\omega) \in \mathcal{A}\}$$

*and*

$$\mathbb{P}(X \in \mathcal{A}) \quad \textit{for} \quad \mathbb{P}(\{X \in \mathcal{A}\}) = \mathbb{P}(\{\omega \in \Omega \mid X(\omega) \in \mathcal{A}\}),$$

*respectively. Moreover, we set*

$$X \in \mathcal{A} \quad := \quad \mathbb{P}(X \in \mathcal{A}) = 1. \tag{1.8}$$

*Hence, we understand relations between random variables* $\mathbb{P}$*-almost surely (with the shortcut "*$\mathbb{P}$*-a.s.") and write for random variables* $X, Y \in \mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$

$$\begin{aligned} X = Y &\quad :\Longleftrightarrow \quad \mathbb{P}(X = Y) = 1, \\ X \neq Y &\quad :\Longleftrightarrow \quad \mathbb{P}(X = Y) = 0, \\ X \leq Y &\quad :\Longleftrightarrow \quad \mathbb{P}(X \leq Y) = 1, \\ X < Y &\quad :\Longleftrightarrow \quad \mathbb{P}(X < Y) = 1. \end{aligned}$$

*Note that we leave out the term $\mathbb{P}$-a.s. in general. For constant random variables $X = c\mathbb{1}_\Omega$ with $c \in \mathbb{R}$, we often use the shortcut $X = c$.*

Distributions of random variables can be often described by densities, see the following definition.

**Definition 1.3.7** (see [145, Def. 2.35]). Let $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathbb{P})$ a probability space with the Borel-$\sigma$-Algebra $\mathcal{B}(\mathbb{R})$ on $\mathbb{R}$. We call $f\colon \mathbb{R} \to \mathbb{R}_+$ fulfilling

$$\forall \mathcal{A} \in \mathcal{B}(\mathbb{R}): \quad \mathbb{P}(\mathcal{A}) = \int_{\mathcal{A}} f \; d\mathbb{P}$$

*density function of* $\mathbb{P}$, where $\int_{\mathcal{A}} f \; d\mathbb{P}$ is the Lebesgue-integral of $f$ over $\mathcal{A}$ with respect to $\mathbb{P}$.

**Example 1.3.8** (see [145, Chapter 3.3]). Let $X$ be a real random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Properties of the distribution of $X$ are said to be properties of $X$, e.g., $X$ is called finite if $\mathbb{P}_X$ is finite. Analogously, $X$ is said to be *normally distributed with the parameters* $\mu \in \mathbb{R}$ *and* $\sigma^2 \in \mathbb{R}_>$ and write $X \sim \mathcal{N}(\mu, \sigma^2)$ if $\mathbb{P}_X$ is the normal distribution $\mathcal{N}(\mu, \sigma^2)$, i.e., $\mathbb{P}_X$ has the density function $f\colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}},$$

where $e$ denotes the exponential function. $\mathcal{N}(0,1)$ is called *standard normal distribution* and $X \sim \mathcal{N}(0,1)$ is called *standard normally distributed*. To highlight this special case, the density function of the standard normal distribution is denoted by $\varphi$ and the related distribution function is denoted by $\Phi$. With $f$ being the density function and $F$ being the distribution function of $\mathcal{N}(\mu, \sigma^2)$, it holds for each $x \in \mathbb{R}$ that

$$f(x) = \frac{1}{\sigma} \varphi\left(\frac{x-\mu}{\sigma}\right) \quad \text{and} \quad F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right).$$

Hence, $X \sim \mathcal{N}(\mu, \sigma^2)$ if and only if $\frac{X-\mu}{\sigma} \sim \mathcal{N}(0,1)$. $\frac{X-\mu}{\sigma}$ is known as *standardization* of $X$. $\Diamond$

As seen in Section 1.3.3, many important risk measures are given by quantiles:

**Definition 1.3.9** (see [49, Def. 6.1], [3, Def. 2.1]). Let $X$ be a real random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\alpha \in (0,1)$. We call $q \in \mathbb{R}$ *$\alpha$-quantile of $X$* if

$$\mathbb{P}(X < q) \leq \alpha \leq \mathbb{P}(X \leq q).$$

Furthermore,

$$q_{(\alpha)}(X) := \inf\{m \in \mathbb{R} \mid \mathbb{P}(X \leq m) \geq \alpha\}$$

denotes the *lower $\alpha$-quantile of $X$* and

$$q^{(\alpha)}(X) := \inf\{m \in \mathbb{R} \mid \mathbb{P}(X \leq m) > \alpha\}$$

denotes the *upper $\alpha$-quantile of $X$*.

Obviously, we have $q_{(\alpha)}(X) \leq q^{(\alpha)}(X)$. The $\alpha$-quantiles of $X$ are contained in the interval $[q_{(\alpha)}(X), q^{(\alpha)}(X)]$, which motivates the terminology from Definition 1.3.9 of *lower* and *upper* $\alpha$-quantile for these interval endpoints. The following properties will be useful, which are given without a proof in [24, Sec. 11.2]:

**Lemma 1.3.10** (see [24])**.** *Let $X$ be a real random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\alpha \in (0,1)$. Then, the following holds:*

(i) $q^{(\alpha)}(X) = \sup\{m \in \mathbb{R} \mid \mathbb{P}(X \leq m) \leq \alpha\}$,

(ii) $q_{(\alpha)}(X) = \sup\{m \in \mathbb{R} \mid \mathbb{P}(X < m) < \alpha\}$.

**Proof.** First, we show (i). Let $q := \sup\{m \in \mathbb{R} \mid \mathbb{P}(X \leq m) \leq \alpha\}$. By definition of the upper $\alpha$-quantile $q^{(\alpha)}(X)$, it holds that

$$\forall m \in \mathbb{R} \text{ with } m < q^{(\alpha)}(X): \quad \mathbb{P}(X \leq m) \leq \alpha$$

and, thus, $q \geq q^{(\alpha)}(X)$. Since the distribution function $F_X(x) := \mathbb{P}(X \leq x)$ is monotonically increasing, we obtain by definition of the upper $\alpha$-quantile $q^{(\alpha)}(X)$ furthermore

$$\forall t \in \mathbb{R}_> : \quad \mathbb{P}\left(X \leq q^{(\alpha)}(X) + t\right) > \alpha$$

and $\mathbb{P}\left(X \leq q^{(\alpha)}(X)\right) \geq \alpha$, which implies $q \leq q^{(\alpha)}(X)$. Thus, we have $q = q^{(\alpha)}(X)$, i.e.,

$$q^{(\alpha)}(X) = \sup\{m \in \mathbb{R} \mid \mathbb{P}(X \leq m) \leq \alpha\},$$

showing (i). Now, we prove (ii). Let $q := \sup\{m \in \mathbb{R} \mid \mathbb{P}(X < m) < \alpha\}$. Then,

$$\forall t \in \mathbb{R}_> : \quad \alpha \leq \mathbb{P}(X < q + t) \leq \mathbb{P}(X \leq q + t).$$

Hence, $q_{(\alpha)}(X) \leq q$ by definition of the lower $\alpha$-quantile $q_{(\alpha)}(X)$. On the other hand, $q_{(\alpha)}(X) \geq q$ holds. Indeed, if $q_{(\alpha)}(X) < q$, then there is by definition of $q$ some $t \in \mathbb{R}_>$ with

$$\mathbb{P}\left(X \leq q_{(\alpha)}(X)\right) \leq \mathbb{P}\left(X < q_{(\alpha)} + t\right) < \alpha$$

in contradiction to the definition of $q_{(\alpha)}(X)$, since $\mathbb{P}(X \leq q_{(\alpha)}(X)) \geq \alpha$ by $F_X(x) := \mathbb{P}(X \leq x)$ being right-continuous. Hence, $q_{(\alpha)}(X) \geq q$ holds and, thus, $q_{(\alpha)}(X) = q$, which shows (ii). □

There is many research concerning estimation of quantiles. For example, Embrechts et al. studied in [63] the estimation of quantiles for extreme value distributions. In [187], Taylor estimated time-varying quantiles by exponentially weighted quantile regression to estimate the Value-at-Risk and Expected Shortfall, which are very popular risk measures in the sense of Definition 1.3.23 (see Section 1.3.3).

### 1.3.2 One-period model of financial markets

In this section, we present the basic financial mathematical framework that we will consider here. In this thesis, a financial market is represented by an one-period model with times $t = 0$ and $t = 1$ through eligible assets in the sense of the following definition, which is motivated by the description in [78, Sec. 1.1]:

**Definition 1.3.11.** Let $\mathcal{X}$ be a vector space. An *one-period model of a financial market with times* $t = 0$ *and* $t = 1$ is given by a set $\mathcal{S} \subseteq \mathbb{R} \times \mathcal{X}$ of *(eligible) assets*, where each eligible asset is given by a tupel $S = (S_0, S_1)^T \in \mathcal{S}$ with *price* $S_0 \in \mathbb{R}$ in $t = 0$ and *(eligible) payoff* $S_1 \in \mathcal{X}$ for one share of the asset $S$.

Recall that the superscript $^T$ always denotes transposed vectors. In [78], it is required for each asset $S = (S_0, S_1) \in \mathcal{S}$ that $S_0 \in \mathbb{R}_+$ and $S_1$ is a non-negative measurable function on a measurable space $(\Omega, \mathcal{F})$. Within an one-period model of a financial market, the investors trade in $t = 0$ and obtain a payoff in $t = 1$ with respect to their investment. Note that our definition of an one-period model of a financial market is quite general and, also, minimal because a financial market is often naturally equipped with additional financial objects for meaningful studies, e.g., a pricing functional in the sense of Definition 1.3.20 below. Since we only consider one-period models in this thesis, we often speak simply of *financial markets* or *financial market models*.

**Remark 1.3.12.** *Let $\mathcal{X}$ be a vector space. In this thesis, we always consider finitely many assets in the financial market, i.e.,*

$$\mathcal{S} := \{(S_0^i, S_1^i)^T \in \mathbb{R} \times \mathcal{X} \mid i = 0, 1, \ldots, n\} \quad \textit{with } n \in \mathbb{N}.$$

*For shortcut, we denote by*

$$S^i := (S_0^i, S_1^i)^T \in \mathbb{R} \times \mathcal{X} \tag{1.9}$$

*the $i$-th eligible asset ($i = 0, 1, \ldots, n$). Moreover, we denote by*

$$S_0 := (S_0^i)_{i=0}^n \in \mathbb{R}^{n+1} \quad \textit{and} \quad S_1 := (S_1^i)_{i=0}^n \in \mathcal{X}^{n+1} \tag{1.10}$$

*the vector of prices and the vector of eligible payoffs of the eligible assets $S^i$ ($i = 0, 1, \ldots, n$), respectively. We call*

$$\mathcal{M} := \operatorname{span}\left\{S_1^i \mid i = 0, 1, \ldots, n\right\} \tag{1.11}$$

*the space of eligible payoffs, which is a subspace of $\mathcal{X}$. By assuming finitely many assets in the market, it holds that $\dim \mathcal{M} < +\infty$.*

Eligible assets (which are often just called *assets*) are traded elements in a financial market and, thus, the investment opportunities. Typical examples are shares, bonds, commodities, and currencies. The payoff $S_1^i \in \mathcal{X}$ of an asset $i \in \{0, 1, \ldots, n\}$ is random in general. Hence, we may think for the vector space $\mathcal{X}$ as a space like $\mathcal{X} = \mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$ such that $S_1^i$ is a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Thus, we consider $\mathcal{X}$ together with a probability space to highlight the general randomness of payoffs. Then, each $\omega \in \Omega$ represents a particular scenario of the evolution of the financial market. Note that the payoff $S_1^i$ is not always uncertain. For example, it is typical that there is a secure investment opportunity in the market (e.g., a treasury bond of the government) paying a secure amount of money at $t = 1$. An asset with a secure payoff is often called *riskless or riskfree*, while assets with non-secure payoffs are called *risky*. We will assume in our financial market model (FM) in this thesis (see Section 2.2) that $S^0$ is a riskless asset, i.e., the index $i = 0$ will distinguish the riskless asset from the risky assets $i = 1, \ldots, n$.

**Example 1.3.13.** In a financial market model as given by Definition 1.3.11, there are many possible choices for the vector space $\mathcal{X}$ of capital positions. There are some widely used and suitable spaces, which we want to outline (see, e.g., [69, Expl. 2.1] and [133] for an overview, and also Section 2.1). Obviously, we will deal with random variables describing payoffs of assets and portfolios, see the remarks before this example, and, hence, consider vector spaces $\mathcal{X}$ of random variables with a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}$ is a probability measure on a sigma-algebra $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ with sample space $\Omega$ (see Section 1.3.1). In the following, let $\mathcal{L}^0 = \mathcal{L}^0(\Omega, \mathcal{F})$ be the space of measurable functions on a measurable space $(\Omega, \mathcal{F})$ given by

$$\mathcal{L}^0(\Omega, \mathcal{F}) := \{X : \Omega \to \mathbb{R} \mid X \text{ is } \mathcal{F} - \text{measurable}\}.$$

If we have some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$ represents the space of all real random variables (see Definition 1.3.5). Some authors (see, e.g., [15]) just consider the general space $\mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$ for modeling payoffs (or even just a set of real mappings, see [14]). Another well-known example for the vector space $\mathcal{X}$ of capital positions in the literature is the *space of bounded measurable functions* (see [70])

$$\mathcal{B}(\Omega, \mathcal{F}) := \{X \in \mathcal{L}^0(\Omega, \mathcal{F}) \mid \|X\|_\infty < +\infty\}$$

with $\|X\|_\infty := \sup_{\omega \in \Omega} |X(\omega)|$. The space $(\mathcal{B}(\Omega, \mathcal{F}), \|\cdot\|_\infty)$ is a Banach space. Moreover, many authors consider so called $\mathcal{L}^p$-spaces (see [122, Sec. 6.1]): If we identify $X \equiv Y$ if and only if $\mathbb{P}(X \neq Y) = 0$, then the space $\mathcal{L}^p = \mathcal{L}^p(\Omega, \mathcal{F}, \mathbb{P})$ for $p \in [1, +\infty)$ is the *space of p-integrable random variables on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ given by

$$\mathcal{L}^p(\Omega, \mathcal{F}, \mathbb{P}) := \{X \in \mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P}) | \, \|X\|_{\mathcal{L}^p} < +\infty\}$$

with $\|X\|_{\mathcal{L}^p} = (\mathbb{E}(|X|^p))^{\frac{1}{p}} = \left(\int_\Omega |X|^p \ d\mathbb{P}\right)^{\frac{1}{p}}$. To extend the definition to $p = +\infty$, we set

$$\mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}) := \{X \in \mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P}) | \, \|X\|_{\mathcal{L}^\infty} < +\infty\}$$

with $\|X\|_{\mathcal{L}^\infty} = \inf\{c \geq 0 | \, |X(\omega)| \leq c \ \mathbb{P}\text{-a.s.}\}$ and call $\mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$ the *space of essential bounded random variables*. If the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is clear or not from interest, we just write $\mathcal{L}^p$ for the corresponding $\mathcal{L}^p$-space with $1 \leq p \leq +\infty$. The spaces $(\mathcal{L}^p, \|\cdot\|_{\mathcal{L}^p})$ and $(\mathcal{L}^\infty, \|\cdot\|_{\mathcal{L}^\infty})$ are Banach spaces (and, thus, normed spaces), since we identify almost-sure identical random variables (see [122, Sec. 6.1]). Furthermore, $\mathcal{L}^q \subseteq \mathcal{L}^p$ for $1 \leq p \leq q \leq +\infty$ holds because of $\mathbb{P}(\Omega) < +\infty$ (see [122, Prop. 6.1.3]). $\Diamond$

In the literature, the case of one eligible asset (which coincides with $\dim \mathcal{M} = 1$) was studied extensively first, also the question of how to choose the eligible asset for discounting, see, e.g., Farkas et al. for a defaultable bond within the space $\mathcal{X} = \mathcal{B}(\Omega, \mathcal{F})$ in [69] and [70] (see Example 1.3.13). Other spaces were studied in the literature, as well, see, e.g., $\mathcal{X} = \mathcal{L}^p$ in [120] and Orlicz spaces in [41]. We refer to Section 2.1 for a short literature review. In practice, the case of multiple eligible assets is more from interest and, hence, we will assume $\dim \mathcal{M} > 1$ in our model (FM) for $\mathcal{M}$ being the subspace of $\mathcal{X}$ given by (1.11), see (2.11) and also Remark 2.2.17. Nevertheless, most of our results can be easily transferred to the case of one eligible asset, as well. To present our results in a general setting, we suppose that the payoffs of the assets are elements of an arbitrary vector space $\mathcal{X}$ instead of a specific space of random variables. From a mathematical point of view, the way of modeling an asset is crucial. For example, payoffs of defaultable bonds and also of shares are often modeled by random variables having the property of not being essentially bounded away from zero (e.g., with a log-normal distribution), see [69].

**Remark 1.3.14.** *In financial mathematics, research also deals with multi-period models of financial markets. These are subdivided in discrete market models with time periods $\{0, 1, \dots, T\} \subseteq \mathbb{N} \cup \{0\}$ and continuous market models with non-countable time periods (in general, $[0, T] \subseteq \mathbb{R}_+$). We consider an one-period model here for several reasons: First, as described in the introduction of this thesis, our research focuses on an specific economical research question that justifies the assumption of trading only once with respect to a current valid regulatory situation (see Section 2.2). A transfer into a multi-period setting leads to another assumption on the economical background (for example, regulatory restrictions changing over time or depending on time-varying parameters). In general, an institution has to fulfill these restrictions anytime, but it is only validated by a regulatory audit at specific dates (mostly every few years). Hence, the problem of fulfilling regulatory preconditions can be reduced for our purposes and sake of convenience to one trading time point that is addressed to pass the acceptability test at the time of audit.*

**Definition 1.3.15** (see [152])**.** Consider a vector space $\mathcal{X}$ and an one-period model of a financial market with assets $S^i = (S_0^i, S_1^i) \in \mathbb{R} \times \mathcal{X}$ for $i \in \{0, 1, \dots, n\}$. A *portfolio (of the assets $S^i$)* is a vector $x = (x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}$ with $x_i \in \mathbb{R}$ representing the number of shares of the asset $i \in \{0, 1, \dots, n\}$ hold by an investor. The *value of the portfolio $x$ in time $t \in \{0, 1\}$* is given by

$$V_t(x) := \sum_{i=0}^{n} x_i S_t^i = S_t^T x$$

with $S_t^i$ defined as in (1.10). $V_0(x)$ is also called *price of the portfolio $x$* and $V_1(x)$ is also called *payoff of the portfolio $x$.*

**Remark 1.3.16.** *Consider a portfolio $x = (x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}$ according to Definition 1.3.15 within a one-period model of a financial market given by a vector space $\mathcal{X}$ and a set $(S^i)_{i=0}^n \subseteq \mathbb{R} \times \mathcal{X}$ of eligible assets. For simplicity, we assume that buying and selling does not generate extra costs (known as transaction costs). That might be realistic for large investors (like financial institutions as it is considered here), but it is not suitable for small (private) investors* *(see [78]). If $x_i < 0$ holds with $S^i$ being a risky asset, then the investor is selling the amount of assets from his properties or he is* short-selling *the asset $S^i$, i.e., he sells $x_i$ shares of the asset (which he lend from the broker) without actually owning it in $t = 0$ and earns $x_i S_0^i$, but he has to repurchase the same volume in the future (if in $t = 1$ for the price $x_i S_1^i$) to give it back to the broker. In this thesis, we abstract from the part of re-buying the asset later. If $S^i$ is a secure asset with $x_i < 0$, it corresponds to taking a loan with receiving $x_i S_0^i$ in $t = 0$ and paying back $x_i S_1^i$ in $t = 1$. In any of these cases of short selling or lending, the amount of generated capital can be used to finance the other buys of shares in the portfolio. As with transaction costs, the situation of $x_i < 0$ for some $i \in \{0, 1, \dots, n\}$ is only realistic in practice for large investors in general.*

A basic concept in financial mathematics is known as *market efficiency*, which is subject of the arbitrage theory. Market efficiency refers to the situation that there is no trading opportunity providing a profit without any downside risk, see the following definition and the remarks afterwards.

**Definition 1.3.17** (see Irle [114, Def. 1.10])**.** Consider an one-period model of a financial market given by a vector space $\mathcal{X}$ and eligible assets $S^i = (S_0^i, S_1^i)^T \in \mathbb{R} \times \mathcal{X}$. A portfolio $x \in \mathbb{R}^{n+1}$ is called *arbitrage (opportunity)* if it holds that

$$V_0(x) \leq 0 \ \wedge \ \mathbb{P}(V_1(x) \geq \mathbb{0}) = 1 \quad \text{and additionally} \quad V_0(x) < 0 \ \vee \ \mathbb{P}(V_1(x) > \mathbb{0}) > 0,$$

where $V_0(x)$ and $V_1(x)$ are the price and the payoff of the portfolio $x$, respectively (see Definition 1.3.15). We call the market *arbitrage-free or efficient* if there are no arbitrage opportunities in the market, and say that the *no-arbitrage principle* is fulfilled.

**Remark 1.3.18.** *Let $\mathcal{X}$ be a vector space. Suppose that the market with eligible assets $S^i = (S_0^i, S_1^i)^T \in \mathbb{R} \times \mathcal{X}$ is arbitrage-free. Then, it holds for all $x \in \mathbb{R}^{n+1}$ that*

$$(V_0(x) \leq 0 \ \wedge \ V_1(x) \geq \mathbb{0}) \quad \Longrightarrow \quad V_0(x) = 0 = \mathbb{P}(V_1(x) > \mathbb{0}), \tag{1.12}$$

*where $V_0(x)$ and $V_1(x)$ are the price and the payoff of the portfolio $x$, respectively (see Definition 1.3.15). Note that relations $V_1(x) \geq \mathbb{0}$ have to be understood as $\mathbb{P}$ - a.s. in this thesis, since $V_1(x)$ is a random variable. Moreover,*

$$(V_1(x) \geq \mathbb{0} \ \wedge \ \mathbb{P}(V_1(x) > \mathbb{0}) = 0) \quad \Longrightarrow \quad V_1(x) = \mathbb{0} \tag{1.13}$$

*holds in (1.12) (see Remark 1.3.6). Proving that the market is arbitrage-free can often be simplified as noticed in [114, Anmerkung 1.11]: An one-period-model is arbitrage-free if there is no arbitrage opportunity $x \in \mathbb{R}^{n+1}$ fulfilling $V_0(x) = 0$. We want to mention that arbitrage is not defined uniformly in the literature. For example, Föllmer and Schied do not include the case $V_0(x) < 0$ in their definition of arbitrage opportunities in [78, Def. 1.2]. Nevertheless, we chose our definition from Irle [114] to distinguish between two types of arbitrage here, namely arbitrage opportunities $x \in \mathbb{R}^{n+1}$ fulfilling $V_0(x) < 0$ in Definition 1.3.17 (called* free lunch*) and* *arbitrage opportunities $x$ fulfilling $\mathbb{P}(V_1(x) > \mathbb{0}) > 0$ in Definition 1.3.17 (called* free lottery *or* money machine*), see also Bamberg [19]. Interested readers can find an illustrative introduction to the arbitrage principle in [192], where also derivative assets are considered. For a more general mathematical introduction, we refer to [50]. The economical background with a distinction between arbitrage, hedging, and speculation can be found in the famous economical literature [32] and [111], where also the role of arbitrage for capital market theory is highlighted. In the real world, examples for arbitrage are hard to find and disappear very fast due to increased digital trading and fast information gathering. For an explicit real-world example of arbitrage in Japan from a society orientated point of view, see [149].*

In a financial market, the Law of One Price is often assumed to hold, i.e., two portfolios generating the same payoff in $t = 1$ have the same price in $t = 0$, see the following definition:

**Definition 1.3.19.** Let $\mathcal{X}$ be a vector space. Consider an one-period model of a financial market with assets $\mathcal{S} = \{(S_0^i, S_1^i)^T \in \mathbb{R} \times \mathcal{X} \mid i = 0, 1, \dots, n\}$ and the space of eligible payoffs $\mathcal{M}$ given by (1.11). We say that the *Law on One Price holds* if for all $Z \in \mathcal{M}$ there is some $c \in \mathbb{R}$ such that

$$\forall x \in \mathbb{R}^{n+1} \text{ with } V_1(x) = Z: \quad V_0(x) = c, \tag{1.14}$$

where $S_j$, $j \in \{0, 1\}$, is the vector given by (1.10).

If the Law of One Price holds, it is possible to define a pricing functional in the following sense:

**Definition 1.3.20.** Let $\mathcal{X}$ be a vector space. Consider an one-period model of a financial market with assets $\mathcal{S} = \{(S_0^i, S_1^i)^T \in \mathbb{R} \times \mathcal{X} \mid i = 0, 1, \dots, n\}$, and space of eligible payoffs $\mathcal{M}$ given by (1.11). Suppose that the Law on One Price holds (see Definition 1.3.19). We call a functional $\pi\colon \mathcal{M} \to \mathbb{R}$ with $\pi(S_1^i) = S_0^i$ for each $i \in \{0, 1, \dots, n\}$ a *pricing functional* or *price functional*.

Although it might be possible to define arbitrary pricing functionals, it makes sense with respect to (1.14) to define a pricing functional $\pi\colon \mathcal{M} \to \mathbb{R}$ as in Definition 1.3.20 by

$$\forall Z \in \mathcal{M}: \quad \pi(Z) := V_0(x) = S_0^T x \quad \text{with } x \in \mathbb{R}^{n+1} \text{ s.t. } Z = V_1(x). \tag{1.15}$$

Obviously, $\pi$ given by (1.15) is linear. Note that the functional defined by (1.15) is the only *linear* pricing functional on $\mathcal{M}$ because it holds for $\pi$ defined by (1.15) and any linear pricing functional $\tilde{\pi}$ on $\mathcal{M}$ according to Definition 1.3.20

$$\tilde{\pi}(Z) = \tilde{\pi}(S_1^T x) = \tilde{\pi}\left(\sum_{j=0}^{n} x_j S_1^j\right) = \sum_{j=0}^{n} x_j \tilde{\pi}(S_1^j) = \sum_{j=0}^{n} x_j S_0^j = S_0^T x = \pi(Z).$$

In the following, we only consider $\pi$ given by (1.15). For that $\pi\colon \mathcal{M} \to \mathbb{R}$, the definition of an arbitrage-free market (see Definition 1.3.17) can be equivalently formulated by use of the subspace $\mathcal{M}$ of $\mathcal{X}$ as follows: Consider an one-period model of a financial market with vector space $\mathcal{X}$, eligible assets $\mathcal{S} = \{(S_0^i, S_1^i)^T \in \mathbb{R} \times \mathcal{X} \mid i = 0, 1, \dots, n\}$, the space of eligible payoffs $\mathcal{M}$ given by (1.11), and the pricing functional $\pi \colon \mathcal{M} \to \mathbb{R}$ given by (1.15) according to Definition 1.3.20. Then, the market is arbitrage-free if it holds for all $Z \in \mathcal{M}$ that

$$(\pi(Z) \leq 0 \;\wedge\; Z \geq \mathbb{0} \; \mathbb{P} - a.s.) \quad \Longrightarrow \quad \pi(Z) = 0 = \mathbb{P}(Z > \mathbb{0}). \tag{1.16}$$

The following lemma shows that the linear pricing functional $\pi \colon \mathcal{M} \to \mathbb{R}$ given by (1.15) is monotonically increasing on $\mathcal{M}$, i.e., for the real vector space $\mathcal{X}$ being partially ordered by the positive cone $\mathcal{X}_+$, it holds that

$$\forall Z^1, Z^2 \in \mathcal{M} : \quad Z^2 - Z^1 \in \mathcal{X}_+ \quad \Longrightarrow \quad \pi(Z^1) \leq \pi(Z^2). \tag{1.17}$$

Indeed, $\pi$ is *strictly* monotonically increasing, see Remark 1.3.22 below.

**Lemma 1.3.21** (see Marohn, Tammer [140, Lemma 3.1])**.** *Let $\mathcal{X}$ be a vector space partially ordered by the positive cone $\mathcal{X}_+$. Consider an one-period model of a financial market with assets $\mathcal{S} = \{(S_0^i, S_1^i)^T \in \mathbb{R} \times \mathcal{X} \mid i = 0, 1, \dots, n\}$, the space of eligible payoffs $\mathcal{M}$ given by* (1.11)*, and a pricing functional $\pi \colon \mathcal{M} \to \mathbb{R}$ according to Definition 1.3.20. Suppose that $\pi$ is linear and, thus, given by* (1.15)*. Moreover, assume that the no-arbitrage principle is fulfilled. Then, $\pi$ is monotonically increasing on $\mathcal{M}$, i.e.,* (1.17) *holds.*

**Proof.** Let $Z^1, Z^2 \in \mathcal{M}$ fulfill $Z^2 - Z^1 \in \mathcal{X}_+$ and $Z^1 \neq Z^2$ $\mathbb{P}$-a.s. Then,

$$\mathbb{P}(Z^2 - Z^1 > \mathbb{0}) = 1$$

holds. If $\pi(Z^2) < \pi(Z^1)$ holds, we obtain $\pi(Z^2 - Z^1) < 0$ by linearity of $\pi$. Thus, $Z^2 - Z^1 \in \mathcal{M}$ is a free lunch - arbitrage (see Remark 1.3.18), in contradiction to the no-arbitrage principle (see (1.16)). Consequently, $\pi(Z^2) \geq \pi(Z^1)$ holds. □

**Remark 1.3.22.** *Let $\mathcal{X}$ be a vector space partially ordered by the positive cone $\mathcal{X}_+$. Consider an one-period model of a financial market with assets $\mathcal{S} = \{(S_0^i, S_1^i)^T \in \mathbb{R} \times \mathcal{X} \mid i = 0, 1, \dots, n\}$, the space of eligible payoffs $\mathcal{M}$ given by* (1.11)*, and the linear pricing functional $\pi \colon \mathcal{M} \to \mathbb{R}$ according to Definition 1.3.20, i.e., $\pi$ is given by* (1.15)*. It holds that*

$$\ker \pi \cap \mathcal{X}_+ = \{\mathbb{0}\}$$

*if there are no arbitrage opportunities in the market (see* (1.13) *and* (1.16)*). Note that we identify a random variable $Z \in \mathcal{M}$ with the random variable $\mathbb{0} \in \mathcal{M}$ if and only if $\mathbb{P}(Z = \mathbb{0}) = 1$, see Remark 1.3.6. Thus, under the no-arbitrage principle, we obtain*

$$\forall Z \in (\mathcal{M} \cap \mathcal{X}_+) \backslash \{\mathbb{0}\} : \qquad \pi(Z) > 0$$

*by monotonicity of $\pi$. Hence, to be more precisely, in a arbitrage-free financial market as assumed in Lemma 1.3.21, the pricing functional $\pi$ is* strictly *monotonically increasing, i.e.,*

$$\forall Z^1, Z^2 \in \mathcal{M} : \quad Z^2 - Z^1 \in \mathcal{X}_+ \backslash \{\mathbb{0}\} \quad \Longrightarrow \quad \pi(Z^1) < \pi(Z^2).$$

*Indeed, if $\pi(Z^1) = \pi(Z^2)$ holds in the end of the proof of Lemma 1.3.21, we obtain that $Z^2 - Z^1$ is a free-lottery arbitrage opportunity, in contradiction to the assumption that the market is arbitrage-free. It is important to mention that the converse in Lemma 1.3.21 is not true: Monotonicity of $\pi$ does not imply that the no-arbitrage principle is fulfilled. For example, consider $\mathcal{X} = \mathbb{R}^2 = \mathcal{M}$ and $\pi(Z) = \pi(Z_1, Z_2) := Z_2$. Then, $\ker \pi \cap \mathcal{X}_+ = \mathbb{R}_+ \times \{0\} \neq \{\mathbb{0}\}$ holds, i.e., the no-arbitrage principle is not fulfilled.*

### 1.3.3 Monetary risk measures

There are many options how to measure and manage risk. Originally, in the famous work [138] of portfolio selection by Markowitz, risk is quantified by the variance or standard deviation of portfolio returns. In modern financial mathematics, the following terminology and axiomatic approach of risk measures has been established, which was introduced by Artzner et al. in [14]. We will give an overview about properties of general risk measures and some of the most common risk measures in theory and practice (also used for acceptance sets, see Example 2.2.16), which are not always defined identically in the literature. Moreover, we did not find a publication that presents an overview about the different definitions used for them in theory and practice, and additionally emphasizes the relationships between these practical risk measures.

**Definition 1.3.23** (see [78, Def. 4.1], also [14],[77], [81])**.** Let $\mathcal{X}$ be a vector space partially ordered by the positive cone $\mathcal{X}_+$. We call $\rho \colon \mathcal{X} \to \overline{\mathbb{R}}$ a *(monetary) risk measure* if the following conditions are satisfied:

(i) $\rho$ is monotonically decreasing on $\mathcal{X}$, i.e., $\forall X, Y \in \mathcal{X}: \; Y - X \in \mathcal{X}_+ \Longrightarrow \rho(Y) \leq \rho(X)$,

(ii) $\rho$ is translation invariant, i.e., $\forall m \in \mathbb{R}, \forall X \in \mathcal{X}: \; \rho(X + m) = \rho(X) - m$.

**Remark 1.3.24.** *Föllmer and Schied already introduced the terminology as above in their book [78] in an edition from 2004. Nevertheless, the authors assume a space of bounded functions and real-valued risk measures. The properties required in Definition 1.3.23 provide natural economical interpretations. By property* (i)*, for a given capital position $X \in \mathcal{X}$, a capital position $Y \in \mathcal{X}$ with larger (or equal) payoffs in each scenario has no larger risk than $X$. Property* (ii) *in Definition 1.3.23 is also known as* cash invariance *in the standard literature of financial mathematics and means that the risk of a given capital position $X \in \mathcal{X}$ can be reduced in the amount of $m \in \mathbb{R}$ by adding the same capital amount $m$ to the position $X$. That highlights why we speak of* monetary *risk measures: The risk of a capital position represents a capital amount and* (ii) *can be interpreted that this risk capital is reduced if we externally provide or rise a extra amount of secure money in each scenario that is not prior part of the capital position $X$.*

*Note that we consider* extended *real-valued functionals on $\mathcal{X}$ in Definition 1.3.23, i.e., the image space $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$. In the literature, there are various definitions for risk measures. Some authors define a risk measure as any real-valued (and, thus, finite) functional $\rho \colon \mathcal{X} \to \mathbb{R}$ (see Artzner et al. [14, Def. 2.1]), while others consider $\rho \colon \mathcal{X} \to \mathbb{R}$ with the properties in Definition 1.3.23 (see, as noticed before, Föllmer, Schied [78, Def. 4.1]). Moreover, some* *authors consider* $\rho\colon \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ *with the properties in Definition 1.3.23, but additionally require* $\rho(\mathbb{0}) \in \mathbb{R}$ *(see Föllmer, Schied [77, Def. 2.1]).*

Risk measures are usually defined for a space of capital positions $\mathcal{X}$ like a $\mathcal{L}^p$-space, especially, with $1 \le p \le +\infty$, see, e.g., [76] and [81]. If $(\mathcal{X}, \|\cdot\|_\infty)$ is a normed vector space with supremum norm $\|\cdot\|_\infty$, then any real-valued risk measure $\rho\colon \mathcal{X} \to \mathbb{R}$ is Lipschitz continuous with respect to $\|\cdot\|_\infty$ (see Föllmer, Schied [78, Lemma 4.3]). Many risk measures are defined by quantiles. The Value-at-Risk is one of the most famous risk measure. It is widely used in practice and studied in research, and provides an alternative to the classic expected utility approach, see, e.g., Bouchaud, Potters [34], Embrechts et al. [64], and Fabozzi et al. [67]. The importance of this measure is highlighted by its central role in the Basel Accords and Solvency II, which are the main regulatory preconditions for financial institutions in the European Union. Hence, we want to present an overview about important properties of the Value-at-Risk in the following.

**Definition 1.3.25** (see Delbaen [49, Def. 6.2])**.** Let $X$ be a real random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\alpha \in (0,1)$. The *Value-at-Risk of* $X$ *at the level* $\alpha$ is given by

$$\mathrm{VaR}_\alpha(X) := -q^{(\alpha)}(X),$$

where $q^{(\alpha)}(X)$ denotes the upper-$\alpha$-quantile (see Definition 1.3.9).

An practical overview about Value-at-Risk can be found in [27]. The Value-at-Risk is well-defined for $X \in \mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$ as given in Definition 1.3.5. The Value-at-Risk is illustrated for the density function $f(x)$ of a normally distributed random variable $X$ in Figure 1.1. By Definition 1.3.25, we suppose gain distributions for a random variable, but it is easy to convert our definition for loss distributions.

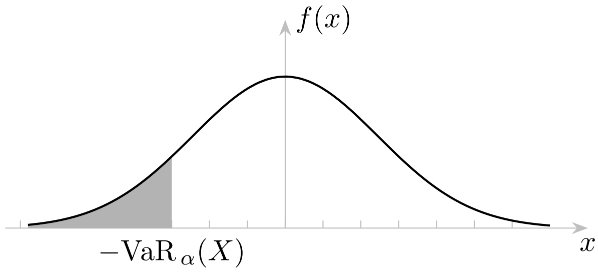


Figure 1.1: Value-at-Risk

**Remark 1.3.26.** *In general, the level* $\alpha \in (0,1)$ *is chosen very small for working with* $\mathrm{VaR}_\alpha$*, e.g,* $\alpha = 0.05$ *or* $\alpha = 0.01$*. In the literature, there can be found some slightly different definitions of the Value-at-Risk. For example, Gaivoronski and Pflug defined the Value-at-Risk in [86] by*

$$\mathrm{V@R} := \mathbb{E}(X) - q^{(\alpha)}(X).$$

*We used the expression* V@R *from [86] here in order to avoid confusing with our Definition 1.3.25. In general, these differences do not influence the results in this thesis, especially, for the* *case of short observation periods of a few days, where $\mathbb{E}(X) \approx 0$ holds for the payoff of the most reasonable portfolios.*

The following alternative expressions for the Value-at-Risk are taken from [3] and [14]. Since there is no direct proof of the equivalence to Definition 1.3.25, we give a proof here to show that the choice of definition does not have any effect on the results:

**Lemma 1.3.27** (see [4, Def. 2.2] and [14])**.** *Let $X$ be a real random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\alpha \in (0,1)$. Then, it holds that:*

$$\mathrm{VaR}_{\alpha}(X) = q_{(1-\alpha)}(-X) = \inf\{m \in \mathbb{R} \mid \mathbb{P}(X + m < \mathbb{0}) \leq \alpha\}.$$

**Proof.** First, we show

$$\mathrm{VaR}_{\alpha}(X) = q_{(1-\alpha)}(-X). \tag{1.18}$$

We obtain from Definition 1.3.25 and Lemma 1.3.10(i)

$$\mathrm{VaR}_{\alpha}(X) = -q^{(\alpha)}(X) = -\sup\{m \in \mathbb{R} \mid \mathbb{P}(X \leq m) \leq \alpha\}.$$

Furthermore, it holds that

$$\begin{aligned}
-\sup\{m \in \mathbb{R} \mid \mathbb{P}(X \leq m) \leq \alpha\} &= \inf\{m \in \mathbb{R} \mid \mathbb{P}(X \leq -m) \leq \alpha\} \\
&= \inf\{m \in \mathbb{R} \mid 1 - \mathbb{P}(X > -m) \leq \alpha\} \\
&= \inf\{m \in \mathbb{R} \mid \mathbb{P}(X > -m) \geq 1 - \alpha\} \\
&= \inf\{m \in \mathbb{R} \mid \mathbb{P}(-X < m) \geq 1 - \alpha\}.
\end{aligned}$$

Now, we need to show

$$\inf\{m \in \mathbb{R} \mid \mathbb{P}(-X < m) \geq 1 - \alpha\} = \inf\{m \in \mathbb{R} \mid \mathbb{P}(-X \leq m) \geq 1 - \alpha\} \tag{1.19}$$

because the latter equals $q_{(1-\alpha)}(-X)$ (see Definition 1.3.9). Let

$$m_l := \inf\{m \in \mathbb{R} \mid \mathbb{P}(-X < m) \geq 1 - \alpha\} \quad \text{and} \quad m_{le} := \inf\{m \in \mathbb{R} \mid \mathbb{P}(-X \leq m) \geq 1 - \alpha\}.$$

We need to show $m_l = m_{le}$. By $\mathbb{P}$ being a probability measure (see Lemma 1.3.3), we obtain

$$\forall m \in \mathbb{R}: \quad \mathbb{P}(-X < m) \leq \mathbb{P}(-X \leq m)$$

and, thus, $m_{le} \leq m_l$. Suppose that $m_l = m_{le} + \epsilon$ for some $\epsilon > 0$ and $\tilde{m} := \frac{m_{le}+m_l}{2}$. Then, it holds that $m_{le} < \tilde{m} < m_l$ and, thus,

$$1 - \alpha \leq \mathbb{P}(-X \leq m_{le}) \leq \mathbb{P}(-X < \tilde{m}) \leq \mathbb{P}(-X < m_l),$$

i.e., $1 - \alpha < \mathbb{P}(-X \leq \tilde{m})$ with $\tilde{m} < l_l$, in contradiction to definition of $m_l$ as an infimum. Consequently, $m_l = m_{le}$, i.e., (1.19) holds. That completes the proof of (1.18).

Now, we prove

$$\mathrm{VaR}_{\alpha}(X) = \inf\{m \in \mathbb{R} \mid \mathbb{P}(X + m < \mathbb{0}) \leq \alpha\}. \tag{1.20}$$

We obtain by (1.18)

$$\begin{aligned}\mathrm{VaR}_{\alpha}(X) = q_{(1-\alpha)}(-X) &= \inf\{m \in \mathbb{R} \mid \mathbb{P}(-X \leq m) \geq 1-\alpha\} \\ &= \inf\{m \in \mathbb{R} \mid \mathbb{P}(X \geq -m) \geq 1-\alpha\} \\ &= \inf\{m \in \mathbb{R} \mid 1 - \mathbb{P}(X \geq -m) \leq \alpha\} \\ &= \inf\{m \in \mathbb{R} \mid \mathbb{P}(X < -m) \leq \alpha\} \\ &= \inf\{m \in \mathbb{R} \mid \mathbb{P}(X + m < \mathbb{0}) \leq \alpha\}.\end{aligned}$$

That shows (1.20). □

**Remark 1.3.28.** *By Lemma 1.3.27, we can interpret the Value-at-Risk as follows: Given* $X \in \mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$ *(see Definition 1.3.5),* $\mathrm{VaR}_{\alpha}(X)$ *is the smallest amount of capital that has to be added to the financial position* $X$ *to reach a probability of a loss that is not higher than* $\alpha$*.*

**Example 1.3.29** (see [78, Equ. 4.10])**.** Let $X \in \mathcal{N}(\mu, \sigma^2)$, i.e., $X$ be a normally distributed real random variable with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 \in \mathbb{R}_+$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (see Example 1.3.8). Furthermore, let $\Phi\colon \mathbb{R} \to [0,1]$ denote the cumulative distribution function of the standard normal distribution $\mathcal{N}(0,1)$. Take $\alpha \in (0,1)$ arbitrary. Then,

$$\mathrm{VaR}_{\alpha}(X) = -\mu + \sigma\Phi^{-1}(1-\alpha). \tag{1.21}$$

Indeed, by standardizing $X$ as $Z := \frac{X-\mu}{\sigma}$ (see Example 1.3.8), we obtain by $\Phi$ (the distribution function of $\mathcal{N}(\mu, \sigma^2)$) being continuous, strictly monotonically increasing, and symmetric around 0 (see, e.g., [145])

$$\begin{aligned}\mathrm{VaR}_{\alpha}(X) = m \in \mathbb{R} \quad &\Longleftrightarrow \quad \mathbb{P}(X \leq m) = \alpha \\ &\Longleftrightarrow \quad \mathbb{P}\left(Z \leq \frac{-m-\mu}{\sigma}\right) = \alpha \\ &\Longleftrightarrow \quad \Phi\left(\frac{-m-\mu}{\sigma}\right) = \alpha \\ &\Longleftrightarrow \quad m = -\mu - \sigma\Phi^{-1}(\alpha) \\ &\Longleftrightarrow \quad m = -\mu + \sigma\Phi^{-1}(1-\alpha).\end{aligned}$$

As a result, (1.21) holds. ◊

We collect some properties of the Value-at-Risk:

**Lemma 1.3.30** (see [24, Th. 11.1])**.** *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space,* $\alpha \in (0,1)$ *and* $\mathrm{VaR}_{\alpha}\colon \mathcal{L}^0 \to \overline{\mathbb{R}}$ *the Value-at-Risk from Definition 1.3.25. Then, the following conditions hold:*

(i) $\mathrm{VaR}_{\alpha}$ *is a monetary risk measure,*

(ii) $\mathrm{VaR}_\alpha$ *is positively homogeneous, i.e.,*

$$\forall X \in \mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P}), \forall \lambda \in \mathbb{R}_+ : \quad \mathrm{VaR}_\alpha(\lambda X) = \lambda \mathrm{VaR}_\alpha(X),$$

(iii) $\mathrm{VaR}_\alpha$ *is distribution invariant, i.e.,* $\mathrm{VaR}_\alpha(X) = \mathrm{VaR}_\alpha(Y)$ *for all random variables* $X, Y \in \mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$ *having the same distribution,*

(iv) $\mathrm{VaR}_\alpha(X) \leq \mathrm{VaR}_\beta(X)$ *for all* $X \in \mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$ *and* $\alpha, \beta \in (0,1)$ *with* $\alpha \leq \beta$.

It is a well-known and widely used practical method to invest capital among different assets to reduce the total risk of the investment, called *diversification*. We refer to Lhabitant [132] for a detailed overview about diversification and methods to measure it. Diversification means that the decision maker invests a portion $\lambda \in [0,1]$ into a possible strategy or investment opportunity with output $X \in \mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$ and the remaining part into another one with output $Y \in \mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$. Of course, this idea is generalized to splitting up the capital into multiple assets and strategies in practice. Convex risk measures (as introduced in Definition 1.3.31) are useful because these risk measures take diversification into account, which is an important point of view for investment decisions. Convexity assumes that diversification should not increase the risk, which will be highlighted by (1.25) in Lemma 1.3.33 below.

The Value-at-Risk does not support diversification in general, i.e., the Value-at-Risk of a combination of sub-portfolios might be larger then the sum of the single Value-at-Risks of the sub-portfolios, see, e.g., [47] for an easy example that Value-at-Risk is not generally subadditive. Risk professionals realized that there is a gap between theoretical modeling and market practice. Moreover, from a regulatory point of view, Value-at-Risk does not face the so-called *tail risk*, which is generated beyond the corresponding quantile of the distribution. Hence, there was searched for an alternative for the widely used Value-at-Risk, fulfilling properties that can be united with portfolio practice (see Acerbi [4]). The answer was delivered by Artzner, Delbaen, Eber and Heath in [13], and, afterwards, in more detail in their landmark paper [14], where that gap was closed by a axiomatic description of the concept of *coherent risk measures*. This can be seen as the beginning of risk management as an own science with an own specific framework, resulting in other explicit risk measures with more suitable properties than the Value-at-Risk (see Definition 1.3.35). Later, a generalized concept of *convex risk measures* was extensively studied in the works from Föllmer, Schied [76] and Frittelli, Rosazza Gianin [81]. Under the terminology *weakly coherent risk measures*, convex risk measures were introduced by Heath in [105] and Heath, Ku in [106] (the paper is cited in [76] as a version being also published in 2002).

**Definition 1.3.31** (see [14], [77], [81])**.** Let $\mathcal{X}$ be a real vector space and $\rho \colon \mathcal{X} \to \overline{\mathbb{R}}$ a monetary risk measure. We call $\rho$ a *convex risk measure* if it holds that

$$\forall X, Y \in \mathcal{X}, \forall \lambda \in [0,1] : \quad \rho(\lambda X + (1-\lambda)Y) \leq \lambda \rho(X) + (1-\lambda)\rho(Y). \tag{1.22}$$

A convex risk measure $\rho$ is called *coherent risk measure* if it is positively homogeneous, i.e., it holds that

$$\forall X \in \mathcal{X}, \forall \lambda \in \mathbb{R}_+ : \quad \rho(\lambda X) = \lambda \rho(X). \tag{1.23}$$

**Remark 1.3.32.** *Let $\mathcal{X}$ be a real vector space and $\rho\colon \mathcal{X} \to \overline{\mathbb{R}}$ be a monetary risk measure. If $\rho$ is positively homogeneous (see (1.23)), it holds that $\rho(\mathbb{0}) = 0$ because of $\mathbb{0} \in \mathcal{X}$ and*

$$\forall \lambda \in \mathbb{R}_+ : \quad \rho(\mathbb{0}) = \rho(\lambda \mathbb{0}) = \lambda \rho(\mathbb{0}).$$

*A risk measure $\rho$ fulfilling $\rho(\mathbb{0}) = 0$ is called normalized. Furthermore, if $\rho$ is a coherent risk measure, then $\rho$ is also subadditive, i.e.,*

$$\forall X, Y \in \mathcal{X} : \quad \rho(X + Y) \leq \rho(X) + \rho(Y). \tag{1.24}$$

*Indeed, by convexity of $\rho$ (see (1.22)), it holds that*

$$\frac{1}{2}\rho(X + Y) = \rho\left(\frac{1}{2}X + \frac{1}{2}Y\right) \leq \frac{1}{2}\rho(X) + \frac{1}{2}\rho(Y),$$

*which leads to (1.24). Note that Artzner et al. originally defined coherent risk measures in [14, Def. 2.4] by subadditivity instead of convexity. It is well known that subadditive, positively homogeneous mappings are convex (see also Lemma 1.1.57). Thus, the definition in [14] is equivalent to Definition 1.3.31. Moreover, the chosen space of capital positions $\mathcal{X}$ has much influence on the possible risk measures. For example, as proved in [29], there are no finite convex risk measures for $\mathcal{X} = \mathcal{L}^p$ with $0 \leq p < 1$, which are not constant.*

For monetary risk measures as introduced in Definition 1.3.23, convexity is equivalent to *quasi-convexity*. Since this property is formulated and proved for real-valued monetary risk measures $\rho\colon \mathcal{X} \to \mathbb{R}$ in [72], we reformulate it for general translation invariant maps (not necessary being risk measures) $\rho\colon \mathcal{X} \to \overline{\mathbb{R}}$ to emphasize the minimal necessary preconditions here and insert a similar proof:

**Lemma 1.3.33** (see [72, Sec. 2.2.3])**.** *Let $\mathcal{X}$ be a real vector space and $\rho\colon \mathcal{X} \to \overline{\mathbb{R}}$ be a map fulfilling Definition 1.3.23*(ii) *(translation invariance). Then, $\rho$ is convex (see (1.22)) if and only if $\rho$ is quasi-convex, i.e.,*

$$\forall X, Y \in \mathcal{X}, \forall \lambda \in [0,1] : \quad \rho(\lambda X + (1-\lambda)Y) \leq \max\{\rho(X), \rho(Y)\}. \tag{1.25}$$

**Proof.** Let $X, Y \in \mathcal{X}$. Assume without loss of generality that $\rho(X) \leq \rho(Y)$ holds. By translation invariance of $\rho$, there is some $m \in \mathbb{R}_+$ with $\rho(X - m) = \rho(X) + m = \rho(Y)$. Thus,

$$\forall \lambda \in [0,1] : \quad \rho(Y) = \lambda\rho(X) + (1-\lambda)\rho(Y) + m. \tag{1.26}$$

First, we assume that $\rho$ is convex and show (1.25). We obtain by (1.26) that

$$\begin{aligned}
\forall \lambda \in [0,1] : \quad \rho(\lambda X + (1-\lambda)Y) &\leq \rho(\lambda X + (1-\lambda)Y) + m \\
&\leq \lambda\rho(X) + (1-\lambda)\rho(Y) + m \\
&= \rho(Y) = \max\{\rho(X), \rho(Y)\}
\end{aligned}$$

holds, since $\rho(X) \leq \rho(Y)$ is fulfilled, i.e., $\rho$ is quasi-convex. Conversely, assume that $\rho$ is quasi-convex, and show that $\rho$ is convex. We obtain by $\rho(X) \leq \rho(Y)$ and (1.25)

$$\forall \lambda \in [0,1] : \quad \rho(\lambda X + (1-\lambda)Y) + m \leq \max\{\rho(X), \rho(Y)\} = \rho(Y) = \lambda\rho(X) + (1-\lambda)\rho(Y) + m.$$

Consequently,

$$\forall \lambda \in [0,1]: \quad \rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y),$$

i.e., $\rho$ is convex. □

Although non-coherent risk measures like Value-at-Risk do not provide the subadditivity property, it is controversial if this property is always desirable, see, e.g., [53] and [163]. Moreover, Danielsson et al. observed in [47] that, for many practicable situations (especially, with sufficiently low probability levels), the Value-at-Risk is subadditive, indeed. From this point of view, risk measures like VaR are nevertheless from interest for practical purposes. For example, normally distributed random variables $X$ are often considered. It can be shown that the Value-at-Risk is convex (and, thus, by Lemma 1.3.30, a coherent risk measure) then if $\alpha$ is sufficiently small:

**Lemma 1.3.34** (see [24, Th. 11.2])**.** *Let* $\mathcal{N}(\Omega, \mathcal{F}, \mathbb{P})$ *denote the space of all normally distributed random variables on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, and* $\alpha \in \left(0, \frac{1}{2}\right]$*. Then,* $\mathrm{VaR}_\alpha \colon \mathcal{N}(\Omega, \mathcal{F}, \mathbb{P}) \to \overline{\mathbb{R}}$ *is a coherent risk measure.*

Although considered in many situations, gain distributions fail in general to be normally distributed and exhibit fat tails, which leads to the problem that the Value-at-Risk (since it does not respect the distribution of the random variable beyond the upper $\alpha$-quantile) ignores the so called *tail risk*, see [167]. These circumstances together with the missing general coherence of the Value-at-Risk (and, thus, disregarding diversification in general) motivated the study of other risk measures in financial mathematics and economics, for example those given in Definition 1.3.35 below. Note that there were also practical deliberations beside the mathematical properties for modeling risk by other measures then Value-at-Risk, namely for gaining some more suitable interpretation of risk with respect to the practical situation. Recall for $X \colon \Omega \to \mathbb{R}$ with $\Omega \neq \emptyset$ being a nonempty set that

$$X^+(\omega) := \begin{cases} X(\omega) & , \ X(\omega) > 0, \\ 0 & , \ \text{else} \end{cases} \quad \text{and} \quad X^-(\omega) := (-X)^+(\omega) = \begin{cases} -X(\omega) & , \ X(\omega) < 0, \\ 0 & , \ \text{else} \end{cases}$$

define the *positive part* $X^+ \colon \Omega \to \mathbb{R}_+$ and the *negative part* $X^- \colon \Omega \to \mathbb{R}_+$ of $X$, respectively.

**Definition 1.3.35** (see [3, Def. 2.5 and 2.6])**.** Let $X$ be a real-valued random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\alpha \in (0,1)$. Suppose $\mathbb{E}(X^-) < +\infty$ with $X^-$ being the negative part of $\mathcal{X}$.

a) The *Average Value-at-Risk of* $X$ *at the level* $\alpha$ is given by

$$\mathrm{AVaR}_\alpha(X) := -\frac{1}{\alpha}\int_0^\alpha q_{(\beta)}(X)\, d\beta.$$

b) The *Conditional Value-at-Risk of $X$ at the level $\alpha$* is given by

$$\mathrm{CVaR}_{\alpha}(X) := \inf_{s\in\mathbb{R}} \left\{ \frac{\mathbb{E}((X-s)^-)}{\alpha} - s \right\}$$

c) The *Expected Shortfall of $X$ at the level $\alpha$* is given by

$$\mathrm{ES}_{\alpha}(X) := -\frac{1}{\alpha}\left(\mathbb{E}\left(X\mathbb{1}_{\{X\leq q_{(\alpha)}\}}\right) + q_{(\alpha)}\left(\alpha - \mathbb{P}\left(X \leq q_{(\alpha)}\right)\right)\right).$$

The Conditional Value-at-Risk was introduced in [158] and [190], the Expected Shortfall in [2], and the Average Value-at-Risk in [167]. Acerbie and Tasche studied in [3] the Conditional Value-at-Risk, the Expected Shortfall, and relationships between them and other risk measures in detail, compare also [158] for a study of relationships between Value-at-Risk and Conditional-Value-at-Risk. In [4], the authors showed how the Expected Shortfall arises in a natural way as a coherent alternative to Value-at-Risk.

**Remark 1.3.36.** *To see that term "Average Value-at-Risk" is justified, we remember that $q_{(\alpha)}(X) \neq q^{(\alpha)}(X)$ can only hold for countable many $\alpha \in (0,1)$ and, thus, the integral value does not change by replacing one by the other (see [24, Def. 11.6]), i.e.,*

$$\mathrm{AVaR}_{\alpha}(X) = -\frac{1}{\alpha}\int_0^{\alpha} q_{(\beta)}\ d\beta = -\frac{1}{\alpha}\int_0^{\alpha} q^{(\beta)}\ d\beta = \frac{1}{\alpha}\int_0^{\alpha} \mathrm{VaR}_{\beta}(X)\ d\beta$$

*holds by Definition 1.3.25. Furthermore, the Conditional Value-at-Risk can be calculated by use of an arbitrary $\alpha$-quantile $q$ of $X$ because, for every real integrable random variable $X$ and $\alpha \in (0,1)$ fixed, it holds that (see [24, Def. 11.8 and Bemerkung 11.3])*

$$\mathrm{CVaR}_{\alpha}(X) = \frac{1}{\alpha}\mathbb{E}\left((X-q)^-\right) - q \quad \text{for each } q \in [q_{(\alpha)}, q^{(\alpha)}]. \tag{1.27}$$

*Moreover, it holds that*

$$\mathrm{ES}_{\alpha}(X) = -\frac{1}{\alpha}\left(\mathbb{E}\left(X\mathbb{1}_{\{X<q\}}\right) + q\left(\alpha - \mathbb{P}(X<q)\right)\right) \quad \text{for each } q \in [q_{(\alpha)}, q^{(\alpha)}] \tag{1.28}$$

*by [3, Equ. (4.12)]. If the cumulative distribution function $F_X$ of $X$ is continuous, we obtain for each $\alpha$-quantile $q$*

$$\mathrm{ES}_{\alpha}(X) = -\frac{1}{\alpha}\mathbb{E}\left(X\mathbb{1}_{\{X<q\}}\right) = -\mathbb{E}(X \mid X < q).$$

*That means (and explains the terminology) that the expected shortfall is the conditional expectation of $X$ under the condition of a realization below the $\alpha$-quantile $q$.*

Although there are three different terms that are used in the literature, it can be shown that this is not necessary because they are all defining the same:

**Theorem 1.3.37** (see [24, Th. 11.3 and 11.4])**.** *Let $X$ be a real random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\alpha \in (0,1)$. Suppose $\mathbb{E}(X^-) < +\infty$ with $X^-$ being the negative part of $\mathcal{X}$. Then, it holds that*

$$\mathrm{AVaR}_{\alpha}(X) = \mathrm{CVaR}_{\alpha}(X) = \mathrm{ES}_{\alpha}(X).$$

*Furthermore, the corresponding risk measures $\mathrm{AVaR}_{\alpha}$, $\mathrm{CVaR}_{\alpha}$ and $\mathrm{ES}_{\alpha}$ with $\alpha \in (0,1)$ are coherent risk measures in the sense of Definition 1.3.31.*

**Remark 1.3.38.** *Let* $X \in \mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$ *(see Definition 1.3.5). Suppose* $\mathbb{E}(X^-) < +\infty$ *with* $X^-$ *being the negative part of* $\mathcal{X}$*. Then, it holds that*

$$\forall \alpha \in (0,1) : \quad \mathrm{AVaR}_\alpha(X) \le \mathrm{VaR}_\alpha(X)$$

*because we have*

$$\mathrm{AVaR}_\alpha(X) = \frac{1}{\alpha} \int_0^\alpha \mathrm{VaR}_\beta(X)\ d\beta \le \mathrm{VaR}_\alpha(X) \frac{1}{\alpha} \int_0^\alpha\ d\beta = \mathrm{VaR}_\alpha(X)$$

*by Lemma 1.3.30*(iv)*. By Theorem 1.3.37, the other coherent risk measures* $\mathrm{ES}_\alpha$ *and* $\mathrm{CVaR}_\alpha$ *do not extend* $\mathrm{VaR}_\alpha$ *for fixed* $X \in \mathcal{X}$ *and* $\alpha \in (0,1)$*, as well.*

There are many more risk measures that are considered in research and practice, like Tail Conditional Expectation or Worst Conditional Expectation, see [14]. The considered risk measures do not provide closed-form expressions, in general. Nevertheless, it is obviously from special interest for practical financial risk management to calculate the Value-at-Risk or Conditional Value-at-Risk numerically as efficient as possible. We refer to [110] for a review on Monte-Carlo-methods for calculating $\mathrm{VaR}_\alpha$ and $\mathrm{CVaR}_\alpha$. There are also other methods to calculate or estimate these measures, see, e.g., [43] for an estimation of $\mathrm{VaR}_\alpha$ by single index quantile regression, and [10] for a comparison of different methods in calculating $\mathrm{VaR}_\alpha$, including historical simulation.

# Chapter 2

# Risk Measurement Regimes

In this chapter, we describe our specific financial market model and study a practical motivated monetary risk measure. The chapter is organized as follows:

- In Section 2.1, we give a short literature overview about portfolio optimization and risk measures, especially, those associated with acceptance sets. We start with the origin of modern portfolio optimization, namely the mean-variance portfolio optimization problem introduced by Harry M. Markowitz. We sketch the historical development from the classical coherent risk measure - framework from Artzner et al. in 1999 to the latest publications concerning risk measures associated to acceptance sets under multiple eligible assets, which are the main focus of our studies.

- Afterwards, we specify the financial model we are working with in Section 2.2. This includes the space of capital positions $\mathcal{X}$, the subspace of eligible payoffs $\mathcal{M} \subset \mathcal{X}$, the pricing functional $\pi\colon \mathcal{M} \to \mathbb{R}$, and the acceptance set $\mathcal{A} \subseteq \mathcal{X}$. Especially, acceptance sets $\mathcal{A}$ are used to model a system of regulatory preconditions in the literature and describe capital positions that are allowed to attend by financial institutions.

- In Section 2.3, we introduce the nonlinear risk measure $\rho_{\mathcal{A},\mathcal{M},\pi}\colon \mathcal{X} \to \overline{\mathbb{R}}$ that is from interest in this thesis. Given a capital position $X \in \mathcal{X}$ and regulatory restrictions by an acceptance set $\mathcal{A}$, the risk measure $\rho_{\mathcal{A},\mathcal{M},\pi}$ determines the minimal costs to transfer $\mathcal{X}$ into an acceptable capital position in $\mathcal{A}$ if the investor is only allowed to invest into the eligible assets (i.e., only by use of the eligible payoffs in $\mathcal{M}$). The corresponding cost-minimization problem for $X \in \mathcal{X}$ with optimal value $\rho_{\mathcal{A},\mathcal{M},\pi}$ is denoted by ($P_\pi(X)$). We prove important results concerning sublevel sets, strict sublevel sets, and level lines of $\rho_{\mathcal{A},\mathcal{M},\pi}$, but also finiteness, convexity, and their relationship with properties of $\mathcal{A}$. These results will be useful to characterize solutions of the optimization problem ($P_\pi(X)$) in Chapter 3.

The main results of this chapter are published in [140].

## 2.1 Literature review

### Modern portfolio theory: The Markowitz model, CAPM, and their extensions

Portfolio optimization problems trace back a long time of research. The process of portfolio selection can be viewed as a two part working process. At first, observations are collected and used to generate beliefs about the performance (or, more generally, the development of factors influencing the advantageousness) of assets in the future. Afterwards, these beliefs are used to derive a portfolio choice. In this thesis, we focus on the last one and assume that the investor (or decision maker) has already future beliefs.

Portfolio theory explains rational investors how to invest their capital, e.g., for utility maximization, but also how prices of risky assets are determined. One of the first popular portfolio optimization models (compare our remarks in the introduction of this thesis) was developed 1952 by Harry M. Markowitz in [138]. It is viewed as the origin of the so called modern portfolio theory. This so called Markowitz model is sometimes referred to as the problem of "mean-variance (portfolio) optimization" and, thus, the modern portfolio theory is also called "mean-variance analysis". Since its publication in 1952, many attempts were made for improving the Markowitz model, especially, with respect to more realistic assumptions. Extensions of the Markowitz model belong to the so called *post-modern portfolio theory*. One of the most famous models is the *capital asset pricing model* (CAPM), which was independently introduced in the 1960's by Treynor in [188] and [189], Sharpe in [179] and [180], Lintner in [134] and [135], and Mossin in [151]. It is an equilibrium model that allows predictions concerning the relationship between risk and expected return of an asset under certain assumptions (see, e.g., [32]). For example, it supposes that all investors engage mean-variance-optimization and have the same risk attitude. Furthermore, investors are allowed to borrow or lend at a common risk-free return rate. The CAPM postulates that the return on a risky asset in equilibrium is determined by an unique risk factor of the asset and the market price of risk (which all risky assets have in common), namely

$$\mathbb{E}(R_i) = R_f + \beta_i(\mathbb{E}(R_M) - R_f),$$

where $M$ is the market portfolio (a theoretical efficient portfolio consisting of all financial assets), $i$ is the index of a risky asset, $R_i$ and $R_M$ are random variables describing the return of a risky asset $i$ and the market portfolio, respectively, $R_f$ is a constant random variable describing the risk-free return rate, and $\beta_i$ is the ratio

$$\beta_i = \frac{\mathrm{Cov}(R_i, R_M)}{\sigma_M^2}$$

with $\sigma_M^2$ being the variance of the market portfolio. $\beta_i(\mathbb{E}(R_M) - R_f)$ is called *risk premium*. It is the product of the relative risk of the particular asset measured by $\beta_i$ and the benchmark risk premium. Thus, the risk premium does not depend on the total volatility of the investment itself. By that, one important result is that risk premium is only rewarded for systemic risk:

firm-specific risk is not priced and can be eliminated by diversification. That seems to be verified regarding the recent Corona-crisis, as seen for example in March 2020, when all global stock markets fell around 30 % - 40 % in regard to the beginning of the year due to the pandemic, see, e.g., Lyócsa et al. [137], and Belhassine, Karamti [25], and references therein for an impact analysis of COVID-19 pandemic on interconnectedness of financial markets.

CAPM was the first model that tries to explain expected returns by economic factors. From a practical point of view, CAPM is one of the most famous models for calculating asset prices, see, e.g., [80]. Although its great popularity, there was also a lot of criticism about it. Roll [171] complains for example about the testability of the CAPM. It can not be tested in the case we do not know the exact composition of the true market portfolio, i.e., *all* individual assets have to be included in the sample. Thus, it is necessary to use proxies, e.g., an index like the S&P 500, for the market portfolio, which is problematic since the proxy might or might not be mean-variance efficient, when the market portfolio is (or, maybe, is not). Additionally, using different proxies might lead to different conclusions, which is also known as benchmark error.

After introduction of the CAPM, many generalizations have been developed, which face different assumptions or problems of the model. Some of these extensions shall be listed here. Black developed a slightly different version of CAPM in [31], which is known as Black CAPM or Zero-Beta CAPM. The model does not assume a riskless asset, and, furthermore, includes restrictions on borrowing. A less restrictive model than the CAPM is given by the *arbitrage price theory*, which was purposed by Ross [172]. Here, the expected return is a linear function of various factors (e.g., market indices). The assumption of the CAPM that all assets are tradable was early criticized because private businesses, as an important part of the economy, and human capital, as the earning power of individuals, do not trade, see [32, Chap. 9.2]. The investment demands differ, for example, by the personal income: by prudence, one might not invest into a company, he or she is employed by. Mayers derived in [144] an equilibrium model for investors, which have varying labor incomes relative to their non-labor capital. Merton relaxed the assumption of a one-period- to a multi-period-model of CAPM, the so called *intertemporal capital asset pricing model* ICAPM , including lifetime consumption as one additional aim, since not only mean and variance of investments matter to investors (see Merton [147], and also Fama [68] for a first multi-period consumption-investment model). The ICAPM was also extended to allow different sources of extra-market risk. Finally, transaction costs and the role of liquidity for risk premium were studied, e.g., by Amihud [9], and Acharya, Pedersen [5].

Nevertheless this short insight in portfolio models, there are much more approaches for portfolio theories with different focus. For example, Shefrin and Statman introduced in [181] some approach, where value maximization does not have to be the ultimate motivation for investing, and founded the so called *behavioral portfolio theory*. It suggests that there are more goals an optimal portfolio might have to satisfy. Moreover, in later years, other issues are added to the classical return-risk-optimization problem, as well, see, e.g., Utz et al. [191] for a tri-criterion portfolio optimization problem taking social responsibility into account.

### Modeling risk with acceptance sets

Other extensions of the Markowitz model arose by the critics about the choice of variance as a measure of portfolio risk. Variance or standard deviation do not have mathematical properties that are suitable for many economical applications. For example, the measure variance is not subadditive and, thus, can not take diversification into account (see Section 1.3.3. Moreover, variance and standard deviation are both not translation invariant and, thus, an additional secure capital amount does not reduce the risk. Thus, for quantifying risk in a monetary sense, risk measures as in Definition 1.3.23 like Value-at-Risk or Expected Shortfall (also known as Conditional-Value-at-Risk) have been considered. Especially, coherent risk measures, as introduced by Artzner et al. 1999 in [14], are from special interest as noticed in Section 1.3.3). Nevertheless, standard deviation is also used today in research in practice for quantifying risk with a different point of view, namely for measuring uncertainty of payoffs in the sense of deviation measures (see Rockafellar et al. [168]).

In [14], risk measures associated with acceptance sets were also introduced. The authors consider an one-period model with times 0 and $T$, a function space $\mathcal{X}$ on a finite sample space $\Omega$, an acceptance set $\mathcal{A} \subseteq \mathcal{X}$ (see Definition 2.2.9), and a risk-free reference asset $S = (1, r\mathbb{1}_\Omega)$ with price 1 today and a total rate of return $r \in \mathbb{R}_>$ in each possible state in future time $T$. In this framework, Artzner et al. defined a cash-additive risk measure associated with an acceptance set $\mathcal{A}$ as a real-valued function $\rho_{\mathcal{A},r} \colon \mathcal{X} \to \mathbb{R}$ with

$$\rho_{\mathcal{A},r}(X) := \inf\{m \in \mathbb{R} \mid X + mr\mathbb{1}_\Omega \in \mathcal{A}\} \tag{2.1}$$

where $X \colon \Omega \to \mathbb{R}$ is a random variable representing the final net worth of a position for each outcome $\omega \in \Omega$ in time $T$, which are assumed to be finitely many. Cash-additive means that the considered eligible asset is a risk-free bond or other risk-free asset, see Farkas et al. [69]. Moreover, Artzner et al. introduced in [14] the terminology of coherent risk measures and coherent acceptance sets for finite sample spaces. Delbaen extended the terminology of coherent risk measures and coherent acceptance sets to general probability spaces in [49]. He studied cash-additive risk measures in the space of bounded measurable functions $\mathcal{X} = \mathcal{L}^\infty$ under the assumption that the risk-free return rate is zero, namely

$$\forall X \in \mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}) : \quad \rho_{\mathcal{A}}(X) := \inf\{m \in \mathbb{R} \mid X + m\mathbb{1}_\Omega \in \mathcal{A}\}. \tag{2.2}$$

The author mentions in [49] that the case of no interest rate can be simply reduced to the case in [14] by discounting.

Many research focused on cash-additive risk measures afterwards. Föllmer, Schied [76], and Frittelli, Rosazza Gianin [81] studied *convex risk measures* as convex, real-valued maps $\rho \colon \mathcal{X} \to \mathbb{R}$ fulfilling the monotonicity and translation invariance property in Definition 1.3.23. Hence, they generalized the concept of coherent risk measures in their works from 2002. Föllmer, Schied mention in [76] that Heath and Ku already stated a representation theorem for convex risk measures (see [105], [106, Prop. 2.7]). Although this paper can only be found as a version of

2004, Heath and Ku have introduced convex risk measures first and called them *weakly coherent risk measures.*

### Random return rates and non-cash-additive risk measures

With respect to the considered deterministic return rates of the risk-free asset, further research extended that approach by considering random return rates $r\colon \Omega \to \mathbb{R}_>$ (see, e.g., [15] and [128]). In the paper from Farkas, Koch-Medina and Munari [69, Def. 2.4] from 2014, the risk measure in (2.1) is formulated more generally for a topological vector space $\mathcal{X}$ partially ordered by the positive cone $\mathcal{X}_+$ by a reference asset $S := (S_0, S_T)$ with initial price $S_0 > 0$ and terminal payoff $S_T \in \mathcal{X}_+$, namely

$$\forall X \in \mathcal{X}: \quad \rho_{\mathcal{A},S}(X) := \inf\left\{m \in \mathbb{R} \mid X + \frac{m}{S_0} S_T \in \mathcal{A}\right\}, \tag{2.3}$$

where $\mathcal{A} \subseteq \mathcal{X}$ is assumed to be a nonempty, proper, monotone subset of $\mathcal{X}$. Here, $\frac{m}{S_0} S_T$ can be interpreted as the payoff of $\frac{m}{S_0}$ units of the asset $S$. Farkas et al. also mention in [69] that $\rho_{\mathcal{A},S}$ can only be reduced to some monetary risk measure $\rho_{\mathcal{A}}$ (see (2.2)) by discounting if the chosen numeraire fulfills that $S_T$ is a non-zero, positive payoff, which is *(essentially) bounded away from zero*, i.e.,

$$\exists \epsilon > 0: \quad \mathbb{P}(|S_T| \geq \epsilon) = 1.$$

This is fulfilled for a constant, positive return of the eligible asset $S$. If $S_T$ is bounded away from zero, the discounted capital position $\tilde{X} := \frac{X}{S_T}$ with respect to this numeraire $S$ provides

$$\forall X \in \mathcal{L}^p: \quad \rho_{\mathcal{A},S}(X) = S_0 \rho_{\tilde{\mathcal{A}}}(\tilde{X})$$

with $\tilde{\mathcal{A}} := \frac{1}{S_T}\mathcal{A}$ (see Definition 1.1.20). Thus, the risk measure $\rho_{\mathcal{A},S}$ given by (2.3) can be expressed in terms of $\rho_{\tilde{\mathcal{A}}}$ given by (2.2), which is a cash-additive risk measure. Note that $S_T$ being bounded away from zero is crucial for discounting with the numeraire $S$, since, otherwise, discounting is impossible if $\mathbb{P}(S_T = \mathbb{0}) > 0$ or the discounted positions $X_T/S_T$ do not have to belong to $\mathcal{X}$ (e.g., the space of bounded real random variables) if $\mathbb{P}(S_T = \mathbb{0}) = 0$ with $S_T$ not being bounded away from zero (see [69]).

In the literature, the case of one eligible asset was studied extensively first, also the question of how to choose it. As mentioned above, Farkas et al. considered general eligible assets $r\colon \Omega \to \mathbb{R}_+$, including defaultable bonds, and, thus, risk measures that are not cash-additive for an arbitrary Hausdorff topological vector space $\mathcal{X}$ in [69] and for $\mathcal{X} = \mathcal{L}^\infty$ in [70]. Other spaces were studied, as well, see, e.g., Kaina, Rüschendorf [120] for $\mathcal{X} = \mathcal{L}^p$ with $1 \leq p < +\infty$, and Cheridito, Li [41] for Orlicz spaces. Considering general eligible assets is from special interest because it is more realistic to assume that the future outcome of the eligible asset does not have to be positive, e.g., even government bonds can default (as it could be observed during the financial crisis). Thus, considering *S-additive risk measures* as introduced in [69] is useful, i.e., $\rho_{\mathcal{A},S}$ given by (2.3) fulfilling

$$\forall X \in \mathcal{X}, \lambda \in \mathbb{R}: \quad \rho_{\mathcal{A},S}(X + \lambda S_T) = \rho_{\mathcal{A},S}(X) - \lambda S_0$$

with $\mathcal{X}$ being a Hausdorff topological vector space and $\mathcal{A}$ being an acceptance set. Given a space $\mathcal{X}$ of functions $X\colon \Omega \to \mathbb{R}$, an eligible asset $S = (S_0, S_T) \in \mathbb{R}_> \times \mathcal{A}$ with an acceptance set $\mathcal{A} \subseteq \mathcal{X}$, Farkas et al. introduced in [72] *intrinsic risk measures* $R_{\mathcal{A},S}\colon \mathbb{R}_> \times \mathcal{X} \to \mathbb{R}$ that take the initial price of a financial position $X = (X_0, X_T) \in \mathbb{R}_> \times \mathcal{X}$ into account. The intrinsic risk measure is given by

$$R_{\mathcal{A},S} := \inf\left\{\lambda \in [0,1] \mid (1-\lambda)X_T + \lambda\frac{X_0}{S_0}S_T \in \mathcal{A}\right\}.$$

Here, the investor is interested in the smallest portion $\lambda$ such that selling this portion of the initial position and investing $\lambda X_0$ into the eligible asset $S$ makes the resulting position acceptable.

### Multiple eligible assets

Already in the early 2000's, the case of multiple eligible assets was considered. Scandolo [177], and Frittelli, Scandolo [82] introduced for an one-period model of a financial market the risk measure $\rho_{\mathcal{A}}\colon \mathcal{L} \to \mathbb{R}$ with $\mathcal{L} \subseteq \mathcal{X}$, $\mathcal{A} \subseteq \mathcal{X}$ being arbitrary subsets of a partially ordered real vector space $\mathcal{X}$ as

$$\rho_{\mathcal{A}}(X) := \inf\{\pi(Y) \mid Y \in \mathcal{L}, X + Y \in \mathcal{A}\} \tag{2.4}$$

and called $\rho_{\mathcal{A}}$ *capital requirement*. In [82], a multi-period market model is considered as well. Artzner, Delbaen, Koch-Medina investigated $\rho_{\mathcal{A}}$ in [15] in more detail, and defined the following *multi-eligible-asset risk measure* $\rho_{\mathcal{A},\mathcal{M},\pi}\colon \mathcal{X} \to \overline{\mathbb{R}}$

$$\rho_{\mathcal{A},\mathcal{M},\pi}(X) := \inf\{\pi(Z) \mid Z \in \mathcal{M}, X + Z \in \mathcal{A}\}, \tag{2.5}$$

where $\mathcal{X}$ is a topological real vector space, $\mathcal{M} \subseteq \mathcal{X}$ is a subspace spanned by the eligible assets, $\mathcal{A} \subseteq \mathcal{X}$ is an acceptance set and $\pi\colon \mathcal{M} \to \mathbb{R}$ is a pricing functional. Obviously, (2.4) is more general than (2.5) with respect to the components, since in (2.4), the set $\mathcal{C}$ does not have to be a subspace of $\mathcal{X}$ and $\mathcal{A}$ does not have to be an acceptance set. Nevertheless, (2.4) is assumed to be real-valued, while (2.5) is extended real-valued. The authors Farkas, Koch-Medina and Munari studied in [71] finiteness and continuity properties of the risk measure $\rho_{\mathcal{A},\mathcal{M},\pi}$. In this thesis, we focus on (2.5) under more general assumptions.

Farkas et al. considered in [71] applications like set-valued risk measures and optimal risk sharing, too. The functional given by (2.5) was studied by Artzner et al. in [15] for coherent acceptance sets and the space of measurable functions $\mathcal{X} = \mathcal{L}^0(\Omega, \mathcal{F})$ with a measure space $(\Omega, \mathcal{F})$. For our best knowledge, Artzner et al. named the numeraire an *eligible asset* in the case of one reference asset. Furthermore, Artzner et al. considered also a multi-currency setting in [15]. The research concerning the functional (2.5) is extended by Baes et al. in [17] for $\mathcal{X}$ being a locally convex Hausdorff space and $\mathcal{M}$ being a finite dimensional subspace of $\mathcal{X}$. Baes et al. focused on the set-valued *optimal payoff map* $\mathcal{E}\colon \mathcal{X} \rightrightarrows \mathcal{M}$ given by

$$\mathcal{E}(X) := \{Z \in \mathcal{M} \mid X + Z \in \mathcal{A}, \pi(Z) = \rho_{\mathcal{A},\mathcal{M},\pi}(X)\}. \tag{2.6}$$

The authors studied conditions for the existence of optimal payoffs $Z \in \mathcal{E}(X)$ for a given financial position $X \in \mathcal{X}$ and for $\mathcal{E}(X)$ being a singleton. Moreover, they analyzed the stability of optimal payoffs, i.e., their behavior under perturbation or approximation of $\mathcal{X}$.

Although we work in a financial institutional setting, we want to mention another important risk measure class known as *systemic risk measures*. Here, the univariate framework is extended by taking into account a complete system instead of a single financial institute. For example, system risk can be interpreted as the minimal amount of money that is necessary for the system security after aggregating individual risks by some aggregation rule, see, e.g., Chen et al. [40], and Hoffmann et al. [109]. There is a huge literature about systemic risk measures, see, e.g., Fouque, Langsam [79], Hurd [113], Biagini et al. [28], and references therein for an overview. Many different main focuses and models are considered in the literature, for example, the classical contagion model (see, e.g., Eisenberg, Noe [61], and Gai, Kapadia [85]), and the default model (see Gai, Kapadia [84]). Empirical studies on banking networks with respect to system risk can be found in [33] and [45]. Biagini et al. present in [28] some approach for a general methodological framework for system risk measures using multidimensional acceptance sets and aggregation functions.

## 2.2 The financial market model (FM)

The short research overview shows that risk measures, acceptance sets and portfolio optimization are intensively studied for many decades and are still from interest nowadays. In this section, we introduce the basic financial market model we are working with in this thesis, and use the basic terminology from Section 1.3. The model will be summarized in (FM) after introducing acceptance sets with respect to risk measures at the end of this section. The idea is derived by the research results we have sketched in Section 2.1, especially, [17] and [71].

Throughout this thesis, $\mathcal{X}$ is a real vector space partially ordered by the positive cone $\mathcal{X}_+$, which is a convex, pointed cone and provides the partial order relation $\leq$, see (1.5). $\mathcal{X}$ is the *space of capital positions* and $X \in \mathcal{X}$ is called *capital position* or also *financial position*. If nothing else is stated, $\mathcal{X}$ is assumed to be a space of random variables. For modeling probabilities and coincidence, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, i.e., $\Omega$ is the set of all possible future states in $t = 1$, $\mathcal{F}$ is a $\sigma$-Algebra on $\Omega$, and $\mathbb{P}\colon \mathcal{F} \to [0, 1]$ is a probability measure on $\mathcal{F}$.

**Remark 2.2.1.** *Baes et al. assume in [17] a locally convex Hausdorff space fulfilling the first axiom of countability. In [71], Farkas et al. suppose a topological vector space, instead. Despite this, we only equip our real vector space $\mathcal{X}$ with a topology and further properties where it is really necessary. Moreover, many authors suppose specific spaces that are typical for applications like $\mathcal{L}^p$ with $1 \leq p \leq +\infty$ (see Example 1.3.13). The reader who is more familiar with working in the specific financial spaces named before might think of the more familiar space instead of our general real vector space in this thesis. As emphasized by these publications and argued in the introduction of this thesis, assuming a general real vector space $\mathcal{X}$ is not standard for financial market modeling, but justified by various reasons. One the one hand, we do so to* *derive generalized results (for example, with respect to [17] and [139]) by weaker assumptions on $\mathcal{X}$. On the other hand, it is often of interest to consider general real vector spaces $\mathcal{X}$ instead of topological vector spaces in mathematical finance, especially, in arbitrage theory, to improve the applicability for practical purposes, see, e.g., the class of financial market models in [46, Section 2] and Riedel [165]. Especially, an industrial user or economic researcher might not know which topology to choose that is suitable for the corresponding situation, while he is working with data samples for random variables such that a linear space is sufficient. An economical user can easily apply our outcomes to practical problems, which does not pose the danger of generating an lack of interest by unnecessary mathematical and non-economical assumptions.*

We assume an one-period model of a financial market with times $t \in \{0, 1\}$ that is specified in the following. In $t = 0$ ("*today*"), the investor choices his portfolio. Starting with an initial portfolio, he can reallocate the money invested into this portfolio, e.g., sell some shares and buy some other with the sales proceeds, or invest additional money in market assets. This results into a capital position that delivers some (in general) random payoff at the future time $t = 1$ ("*tomorrow*"). As mentioned above, $X \in \mathcal{X}$ represents the capital of an investor in the future time $t = 1$. It is the residual value of assets and liabilities. Thus, gains are positive outcomes ($X(\omega) > 0$) and losses are negative outcomes ($X(\omega) < 0$). Hence, $X$ provides a gain distribution if it is a random variable. Sometimes, we consider in examples a finite set $\Omega$ for convenience, i.e., $\Omega = (\omega_1, \ldots, \omega_n)$, and, thus, vectors $X \in \mathbb{R}^n$ with $X_k = X(\omega_k)$ $(k = 1, \ldots, n)$. Nevertheless, remember that $\mathcal{X}$ could especially, be any normed vector space, but if we consider a space of random variables like $\mathcal{X} = \mathcal{L}^p(\Omega, \mathcal{F}, \mathbb{P})$, relations like $\mathbb{0} \leq X$, which means $X \in \mathcal{X}_+$, are understood in the sense of $\mathbb{P}$-a.s. (see (1.8)).

Recall that the superscript $^T$ always denotes transposed vectors. As noticed in Remark 1.3.12, we consider a finitely number $n \in \mathbb{N}$ of liquid eligible assets in the market, which are represented by the finite set $\mathcal{S} = (S^i)_{i=0}^n \subseteq \mathbb{R} \times \mathcal{X}$ with $S^i$ given by (1.9), i.e.,

$$S^i := (S_0^i, S_1^i)^T \in \mathbb{R} \times \mathcal{X}, \quad i \in \{0, 1, \ldots, n\}.$$

Since the index $i \in \{0, 1, \ldots, n\}$ denotes the number of the eligible asset, we also speak from $S^i$ as the $i$-th eligible asset. $S_0^i \in \mathbb{R}$ denotes the price in $t = 0$ for one unit of the asset $i$. Furthermore, $S_1^i \in \mathcal{X}$ denotes the eligible payoff in $t = 1$ for each unit, which is random in general. For convenience, we collect the prices and payoffs in vectors $S_0 \in \mathbb{R}^{n+1}$ and $S_1 \in \mathcal{X}^{n+1}$, respectively, as given by (1.10), see Remark 1.3.12. As noticed in Section 1.3.2, the asset $i = 0$ denotes a risk-free (or secure) investment opportunity with risk-free rate of return $r \in \mathbb{R}_+$, which is here given by

$$S^0 := (1, (1 + r)\mathbb{1}_\Omega)^T = (1, 1 + r)^T. \tag{2.7}$$

The terminology "risk-free" highlights that the payoff is a constant. We shortly refer to $S^0$ as *the secure asset* and to $S^i$ with $i \in \{1, \ldots, n\}$ as *a risky asset*. Moreover, we call $S_1^0$ *secure eligible payoff*, and $S_1^i$ for $i = 1, \ldots, n$ *risky eligible payoff*. We suppose that $S_1^0$ and $S_1^i$ are linear independent for each $i = 1, \dots, n$. Hence, there is no other risk-free investment opportunity, and, by $n \in \mathbb{N}$, there is at least one risky asset in the market. Remember that we shortly write $X = c$ for a random variable that is constantly $c \in \mathbb{R}$, i.e., $X = c\mathbb{1}_\Omega$ with $\mathbb{1}_\Omega \in \mathcal{X}$ being the random variable that equals 1 in each scenario $\omega \in \Omega$ (see Section 1.3.1).

**Remark 2.2.2.** *The assumption of a secure investment opportunity have many economic research models like the CAPM in common. The CAPM (capital asset pricing model) is a famous and well-used model in economics (see Section 2.1). Nevertheless, the secure asset $S^0$ is not directly assumed by Baes et al. in [17] or Farkas et al. in [71]. Examples of a secure investment opportunity are U.S. Treasury Bonds or German Government Bonds. Furthermore, we could imagine a central bank account as a secure investment opportunity. Note that the expression "secure" represents only a theoretical secureness of the asset: Of course, the opportunity is not riskless at all, since, e.g., interest risks or market risks even exist if we assume that treasury bonds do not default (which also is not true for every emitting government in practice). For convenience, we assume no interest payments, i.e., $r = 0$ and, thus, (2.7) is simplified to*

$$S^0 = (1, \mathbb{1}_\Omega)^T = (1, 1)^T. \tag{2.8}$$

*Of course, $r = 0$ is a major simplification as well as that there is only one secure alternative, which is especially, independent from the time horizon. The latter is no problem at all, since we only consider an one-period-model such that the secure interest rate corresponds to this time horizon, but it is important to have that in mind if the model is generalized to a multi-period setting. Moreover, it is well known (especially, since the negative interest rate policy after the financial crisis 2008) that interest rates do not have to be non-negative, i.e., $r \geq 0$ is not always true in practice. For instance, European banks are penalized since March 2016 by an interest rate of $r = -0.4\%$, the so-called deposit facility, for parking money at the European Central Bank (ECB) instead of investing or lending it [66]. An interest rate being negative means that the bank pays money for lending (or, in the example of the ECB account, saving) it. This phenomena is also challenging for assurances: These have to invest parts of their capital for security of their customers into theoretical secure assets like government bonds by law. The problem is that these bonds partly have negative effective interest rates, too, see for example German government bonds, which yields were negative for every maturity on the 2nd August 2019.*

The possible actions of a decision maker in $t = 0$ can be described by portfolios

$$x = (x_0, x_1, \dots, x_n)^T \in \mathbb{R}^{n+1}$$

of the eligible assets $S^i$, $i = 0, 1, \dots, n$, given by (1.9) (see Definition 1.3.15 for the portfolio terminology). Without repeating it, we assume that each portfolio vector $x$ has the index set $\{0, 1, \dots, n\}$ in the following, to highlight the referenced eligible asset of each component. The resulting payoff for a portfolio $x$ (which corresponds to the change of the origin capital position of the financial institute) is its value at time 1, i.e.,

$$V_1(x) := \sum_{i=0}^{n} x_i S_1^i = S_1^T x. \tag{2.9}$$

Thus, each payoff $V_1(x)$ of a portfolio consisting of the secure eligible asset $S^0$, and the other (random) risky eligible assets $S^i$ with $i = 1, \ldots, n$ is an element of the space of eligible payoffs $\mathcal{M} \subseteq \mathcal{X}$ given by (1.11). By (2.8), we obtain

$$\mathcal{M} = \operatorname{span}\{1, S_1^1, \ldots, S_1^n\} \tag{2.10}$$

in our model. The space of eligible payoffs $\mathcal{M}$ is a subspace of $\mathcal{X}$. Since we assume that the secure eligible payoff $S_1^0$ is linear independent from the other $n \in \mathbb{N}$ risky eligible payoffs, we obtain

$$1 < \dim \mathcal{M} \leq n + 1 < +\infty, \tag{2.11}$$

and each $Z \in \mathcal{M}$ corresponds to some portfolio $x \in \mathbb{R}^{n+1}$ fulfilling $Z = V_1(x)$ with $V_1(x)$ given by (2.9). Moreover, the portfolio $x$ fulfilling $Z = V_1(x)$ is uniquely determined if all risky eligible payoffs $S_1^i$, $i = 1, \ldots, n$ (which are assumed to be linear independent from $S_1^0$) are linear independent, too. If $\mathcal{X}$ is a topological vector space, we equip $\mathcal{M}$ with the relative topology induced by $\mathcal{X}$.

**Remark 2.2.3.** *Let $\mathcal{X}$ be a vector space and $\mathcal{M}$ be the subspace of $\mathcal{X}$ given by (2.10). Then,*

$$\forall m \in \mathbb{R}: \qquad m = mS_1^0 \in \mathcal{M} \tag{2.12}$$

*because of $S_1^0 = 1\!\!1_\Omega = 1$ by (2.8) and, thus, $\operatorname{span}\{S_1^0\} = \mathbb{R} \subseteq \mathcal{M}$. Note that $S_1^0 = 1\!\!1_\Omega \in \mathcal{M} \subseteq \mathcal{X}$ is secured for spaces like $\mathcal{X} = \mathcal{L}^p$, $p \in [1, +\infty)$, by considering random variables and, thus, a probability space with probability measure $\mathbb{P}$, i.e., $\mathbb{P}(\Omega) = 1$ for arbitrary sample set $\Omega$. Hence, all constant random variables $c = c1\!\!1_\Omega$ are eligible payoffs, i.e., elements of $\mathcal{M}$. By (2.12), we will see in (2.14) that the price of a secure payoff $m \in \mathbb{R}$ given by the pricing functional in (2.13) equals $mS_0^0 = m$ in an arbitrage-free market.*

As noticed in Section 1.3.2, we make the following typical assumptions about our market:

**Assumption 1.** *Consider an one-period model of a financial market with vector space $\mathcal{X}$ and eligible assets $S^i \in \mathbb{R} \times \mathcal{X}$ with $i = 0, 1, \ldots, n$ given by (1.9). Let $\mathcal{M}$ be the subspace of $\mathcal{X}$ given by (2.10). The Law of One Price holds (see Definition 1.3.19) and the no-arbitrage principle is fulfilled (see Definition 1.3.17).*

For some notes about different definitions and types of arbitrage opportunities, see Remark 1.3.18. With respect to Definition 1.3.20, the Law of One Price in Assumption 1 allows us to define a pricing functional on $\mathcal{M}$ given by (2.10), since every portfolio $x \in \mathbb{R}^{n+1}$ with the same payoff $Z \in \mathcal{M}$ has the same initial price, which, especially, leads to a unique price for every eligible payoff $Z \in \mathcal{M}$. Note that the Law of One Price is automatically fulfilled for linear independent eligible payoffs $S_1^i$, $i = 0, 1, \ldots, n$, since every payoff $Z \in \mathcal{M}$ delivers an unique portfolio $x \in \mathbb{R}^{n+1}$ with $Z = V_1(x)$ then, and, thus, an unique price. Following Baes et al. [17], we define a pricing functional $\pi\colon \mathcal{M} \to \mathbb{R}$ with respect to Definition 1.3.20 by use of the eligible assets $S^i \in \mathbb{R} \times \mathcal{X}$, $i = 0, 1, \ldots, n$, given by (1.9) as follows:

$$\pi(Z) := S_0^T x \;\; \text{for all } x \in \mathbb{R}^{n+1} : \; Z = S_1^T x \tag{2.13}$$

with $S_j$, $j \in \{0, 1\}$ given by (1.10). Obviously, $\pi$ is a linear operator fulfilling

$$\forall i \in \{0, 1, \ldots, n\} : \quad \pi(S_1^i) = S_0^i.$$

Especially, we obtain

$$\forall m \in \mathbb{R} : \quad \pi(m) = m\pi(S_1^0) = mS_0^0 = m, \tag{2.14}$$

since $S^0 = (1, 1)^T$ by (2.8) and $m \in \mathbb{R} \subseteq \mathcal{M}$ by (2.12) (see Remark 2.2.3). Moreover, $\pi$ is continuous on $\mathcal{M}$ if $\mathcal{X}$ is a Hausdorff topological vector space, since $\dim \mathcal{M} < +\infty$ by (2.11), see Remark 1.1.38 and 1.1.58. By Lemma 1.3.21, $\pi$ given by (2.13) is monotonically increasing on $\mathcal{M}$ in the sense of (1.17) (to be precisely, strictly monotonically increasing, see Remark 1.3.22) if the no-arbitrage principle is fulfilled.

**Remark 2.2.4.** *Assumption 1 is crucial for working with the pricing functional* $\pi\colon \mathcal{M} \to \mathbb{R}$ *given by* (2.13)*: The Law of One Price (see Definition 1.3.19) secures that* $\pi$ *is well-defined and, as seen in the proof of Lemma 1.3.21, the no-arbitrage principle (see Definition 1.3.17) implies the monotonicity of* $\pi$*. As noticed in Remark 1.3.22, it holds under the no-arbitrage principle that*

$$\ker \pi \cap \mathcal{X}_+ = \{\mathbb{0}\}$$

*and, thus,*

$$\forall Z \in (\mathcal{M} \cap \mathcal{X}_+) \backslash \{\mathbb{0}\} : \qquad \pi(Z) > 0 \tag{2.15}$$

*by monotonicity of* $\pi$ *(see Lemma 1.3.21). Note that random variables* $Z \in \mathcal{M}$ *are identified with* $\mathbb{0} \in \mathcal{M}$ *if and only if* $\mathbb{P}(Z = \mathbb{0}) = 1$*, as noticed in Remark 1.3.6.*

We will make use of the following comfortable notion: all eligible payoffs of portfolios with the same price $m \in \mathbb{R}$ are summarized by

$$\pi_m := \pi^{-1}(m) = \{Z \in \mathcal{M} \mid \pi(Z) = m\} \subseteq \mathcal{X}. \tag{2.16}$$

Note that $\mathcal{M} \cap \mathcal{X}_+ \neq \{\mathbb{0}\}$ holds by existence of a secure investment opportunity with payoff $S_1^0 = 1 \in \mathcal{M}$, see Remark 2.2.3. Nevertheless, as in [17], we always state the general assumption of the existence of any payoff $U \in \mathcal{M} \cap \mathcal{X}_+$ with strict positive price, i.e.,

$$\mathcal{X}_+ \cap \bigcup_{m>0} \pi_m \neq \emptyset,$$

and only where it is really necessary, to allow that the reader might easier apply our results to a setting without a secure investment opportunity, i.e., $1 \notin \mathcal{M}$, or markets with arbitrage opportunities such that $\pi$ is not necessarily monotonically increasing, i.e., (2.15) does not hold.

**Assumption 2.** *Consider an one-period model of a financial market with a vector space $\mathcal{X}$ being partially ordered by the positive cone $\mathcal{X}_+$ and eligible assets $S^i \in \mathbb{R} \times \mathcal{X}$ with $i = 0, 1, \ldots, n$ given by (1.9). For $\mathcal{M}$ being the subspace of $\mathcal{X}$ given by (2.10) and $\pi \colon \mathcal{M} \to \mathbb{R}$ being the linear pricing functional in (2.13), there is some positive payoff $U \in \mathcal{M} \cap \mathcal{X}_+$ with $\pi(U) = 1$, where $X_+$ is the positive cone in $\mathcal{X}$.*

Note that the Law of One Price from Assumption 1 is automatically fulfilled by Assumption 2 because the existence of $\pi$ is required in Assumption 2.

**Remark 2.2.5.** *It is no restriction to assume $\pi(U) = 1$ instead of $\pi(U) \in \mathbb{R}_>$ for $U \in \mathcal{M} \cap \mathcal{X}_+$ in Assumption 2 because $\pi$ given by (2.13) is a linear functional. As mentioned before, we can set $U = S_1^0 = 1$ for $\mathcal{M}$ given by (2.10). Assumption 2 is quite natural, since it can be easily satisfied by choosing the payoff of a bond, which is, in general, an element of $\mathcal{X}_+$ and, thus, has a positive price if we assume no arbitrage opportunities in the market, i.e., Assumption 1 is fulfilled. Since bonds are less risky than other securities or derivatives in the market and bonds (at least by U.S. and German government) can be assumed to provide high liquidity and quality, there should be a bond that is also an eligible asset, e.g., a treasury bond of the U.S. federal government.*

**Remark 2.2.6.** *For our results, the subspace $\ker \pi$ with $\pi \colon \mathcal{M} \to \mathbb{R}$ defined as in (2.13) will be very important. The kernel describes portfolios with their corresponding payoffs that can be realized by zero costs. The Rank-Nullity Theorem (see Lemma 1.1.36) implies*

$$\dim(\ker \pi) = \dim \mathcal{M} - 1 \qquad \text{for } \pi \not\equiv 0$$

*because of $\operatorname{Im}(\pi) = \mathbb{R}$ then. Note that $\pi \equiv 0$ and, thus, $\ker \pi = \mathcal{M}$ is excluded under Assumption 2 through existence of $U \notin \ker \pi$. Hence, $\dim(\ker \pi) \geq 1$ holds because we suppose $\dim \mathcal{M} > 1$ by (2.11), and, thus, $\{\mathbb{0}\} \subsetneq \ker \pi$.*

The following result was observed in [71] and proved in [139] for a topological vector space $\mathcal{X}$. Since there were no topological properties required in the proof, we can rewrite (2.16) by Assumption 2 for arbitrary real vector spaces $\mathcal{X}$ as stated in [140]:

**Lemma 2.2.7** (see Marohn, Tammer [140, Lemma 3.2])**.** *Let $\mathcal{X}$ be a real vector space partially ordered by the positive cone $\mathcal{X}_+$. Take $U \in \mathcal{M} \cap \mathcal{X}_+$ arbitrary with $\pi(U) = 1$ according to Assumption 2 and $m \in \mathbb{R}$. Then, for $\pi_m$ given by (2.16), it holds that*

$$\pi_m = mU + \ker \pi.$$

**Proof.** For $m \in \mathbb{R}$ arbitrary, it holds that

$$\forall V \in \ker \pi : \qquad \pi(mU + V) = m\pi(U) + \pi(V) = m$$

by linearity of $\pi$ and $\pi(U) = 1$ by Assumption 2. Hence, $mU + V \in \pi_m$ holds, which shows $\pi_m \supseteq mU + \ker \pi$.

Now, take $Z \in \pi_m$ arbitrary and let $V := Z - mU$. Then, $V \in \mathcal{M}$ holds because $\mathcal{M}$ is a subspace of $\mathcal{X}$. Hence, we obtain

$$\pi(V) = \pi(Z) - m\pi(U) = 0$$

by linearity of $\pi$, since it holds that $\pi(Z) = m$ by $Z \in \pi_m$ and $\pi(U) = 1$ by Assumption 2. That shows $\pi_m \subseteq mU + \ker \pi$ and completes the proof. □

**Remark 2.2.8.** *Lemma 2.2.7 can be reformulated if we assume the existence of $U \in \mathcal{M} \cap \mathcal{X}_+$ with $\pi(U) \in \mathbb{R}_+ \backslash \{0\}$ arbitrary in Assumption 2. Then, it holds that (see [71])*

$$\forall m \in \mathbb{R}: \qquad \pi_m = \frac{m}{\pi(U)} U + \ker \pi.$$

Now, we introduce a suitable set that specifies those capital positions, which are allowed to be occupied by regulatory preconditions. These positions are called *acceptable*. If the current capital position of the financial institution is not acceptable, the decision maker has to decide, which actions can be undertaken such that the resulting new capital position is acceptable. On the other hand, if the current position is already acceptable, the decision maker could set money free for other uses without loosing acceptability. As mentioned earlier, the possible actions for modifying the current capital position are investing into and selling eligible assets, and, thus, are modeled by the space of eligible payoffs $\mathcal{M}$ given by (2.10). The set of all capital positions being acceptable capitalized with respect to regulatory constraints is modeled by an acceptance set according to the following definition (see also [17] and [14]):

**Definition 2.2.9.** Let $\mathcal{X}$ be a real vector space partially ordered by the positive cone $\mathcal{X}_+$. We call $\mathcal{A} \subseteq \mathcal{X}$ an *acceptance set* if the following conditions hold:

(i) $\mathbb{0} \in \mathcal{A}$,

(ii) $\mathcal{A}$ is proper: $\mathcal{A} \subsetneq \mathcal{X}$,

(iii) $\mathcal{A}$ is monotone: $\mathcal{A} + \mathcal{X}_+ \subseteq \mathcal{A}$, i.e., $\mathcal{X}_+ \subseteq \operatorname{rec} \mathcal{A}$.

**Remark 2.2.10.** *By Definition 2.2.9*(i) *and* (iii)*, acceptance sets $\mathcal{A} \subseteq \mathcal{X}$ provide $\mathcal{X}_+ \subseteq \mathcal{A}$. Especially, acceptance sets are nonempty and*

$$\forall m \in \mathbb{R}: \qquad m = m\mathbb{1}_\Omega \in \mathcal{A}.$$

*Moreover, Definition 2.2.9*(iii) *directly implies*

$$\mathcal{A} + \mathcal{X}_+ = \mathcal{A} \tag{2.17}$$

*because of $\mathbb{0} \in \mathcal{A} \cap \mathcal{X}_+$. Property* (2.17) *is known under the terminology* free disposal assumption *with respect to the positive cone $\mathcal{X}_+$. It is a typical assumption in production theory, see [119] and references therein for this condition. Thus, acceptance sets $\mathcal{A}$ are also called* free disposal sets*, since they fulfill* (2.17)*. In optimization and mathematical economics, free disposal sets are* *widely used, see, e.g., [48]. For the case* $\mathcal{X}_+ = \mathbb{R}^k_+$ *with* $k \in \mathbb{N}$*, sets with the property* (2.17) *are studied as so called* downward sets*, see, e.g., [142].*

*If we consider* $U \in \mathcal{M} \cap \mathcal{X}_+$ *arbitrary (e.g., by Assumption 2), we obtain* $\mathbb{R}_+ U \subseteq \mathcal{X}_+$ *and, thus, by Definition 2.2.9*(iii)

$$\mathcal{A} + \mathbb{R}_+ U \subseteq \mathcal{A},$$

*i.e.,* $U \in \operatorname{rec} \mathcal{A}$*. Obviously, we have more exactly*

$$\mathcal{A} + \mathbb{R}_+ U = \mathcal{A}. \tag{2.18}$$

*Moreover, we get* $U \in \mathcal{A}$ *by* (i) *and* (2.18)*.*

**Remark 2.2.11.** *Definition 2.2.9 is motivated by [17], but, in difference to the authors, we do not assume an acceptance set* $\mathcal{A} \subseteq \mathcal{X}$ *to be closed, in general. We will see that, for example, the properties of the risk measure that we study in Section 2.3 do not need any topological properties that need to be required by* $\mathcal{A}$*.*

*In the literature, there are more other definitions of acceptance sets than ours. In [71], Farkas et al. do only require* $\emptyset \neq \mathcal{A} \subsetneq \mathcal{X}$ *and* $\mathcal{A} + \mathcal{X}_+ \subseteq \mathcal{A}$ *for an acceptance set, and call* $\mathcal{A}$ *a capital adequacy test. The authors argue that these two properties can be united with expectations from nontrivial capital adequacy tests, which we can agree with by the natural interpretation of the properties. Especially, positions dominating any other acceptable positions should be automatically acceptable themselves (being required by the monotonicity of* $\mathcal{A}$*), and not every capital position should be acceptable to model sensible regulatory frameworks (being required by the properness of* $\mathcal{A}$*).* $\mathbb{0} \in \mathcal{A}$ *is from technical interest, since many acceptance sets are given by monetary risk measures implying this property, see Definition 2.2.12, and the requirement can be easily fulfilled by translation without endangering the other properties.*

*In [17], Baes et al. observe that requiring* $\mathbb{0} \in \mathcal{A}$*, closedness of* $\mathcal{A}$ *and the properties in [71] are widely assumed in practice. Many authors additionally require convexity of* $\mathcal{A}$*, but some essential acceptance sets do not fulfill this (see Example 2.2.16). In dependence of specific properties, there is further terminology introduced by several authors, e.g., convex acceptance sets by Föllmer, Schied in [76], and Frittelli, Rosazza Gianin in [81] for convex* $\mathcal{A}$*, conic acceptance set by Farkas, Koch-Medina, Munari in [71] for a cone* $\mathcal{A}$*, and coherent acceptance sets by Artzner et al. in [14] for a convex cone* $\mathcal{A}$*.*

In practice, acceptance sets are mostly given by risk measures (see Section 1.3.3), which leads to the following definition (see [14, Def. 2.3]):

**Definition 2.2.12.** Let $\mathcal{X}$ be a real vector space and $\rho \colon \mathcal{X} \to \overline{\mathbb{R}}$ be a (monetary) risk measure in the sense of Definition 1.3.23 with $-\infty < \rho(\mathbb{0}) \leq 0$. We call

$$\mathcal{A}_\rho := \{X \in \mathcal{X} \mid \rho(X) \leq 0\}$$

*acceptance set associated with the risk measure* $\rho$*.*

Since many practicable acceptance sets are given by sets of the type $\mathcal{A}_\rho$, we study this set in more detail. The following lemma provides a justification for calling $\mathcal{A}_\rho$ in Definition 2.2.12 an acceptance set in the sense of Definition 2.2.9. Lemma 2.2.13 is derived from Föllmer, Schied [78, Prop. 4.6], and Artzner et al. [14, Prop. 2.2.], where a real-valued risk measure $\rho\colon \mathcal{X} \to \mathbb{R}$ is considered. In the literature, the set $\mathcal{A}_\rho$ is often considered under many different assumptions on the acceptance set $\mathcal{A}$, the risk measure $\rho$ or the space of capital positions $\mathcal{X}$ itself. Moreover, in [78], there are not considered extended-real-valued risk measures and not all properties that we require by an acceptance set are covered. Hence, we give a proof of the lemma here (see also [99, Prop. 3, 5, and 6] for some of the observations in Lemma 2.2.13, partially without a proof).

**Lemma 2.2.13.** *Let $\mathcal{X}$ be a real vector space and $\rho\colon \mathcal{X} \to \overline{\mathbb{R}}$ be a (monetary) risk measure in the sense of Definition 1.3.23 with $-\infty < \rho(\mathbb{0}) \le 0$. Then,*

$$\mathcal{A}_\rho := \operatorname{lev}_{\rho,\le}(0) = \{X \in \mathcal{X} \mid \rho(X) \le 0\}$$

*is an acceptance set in the sense of Definition 2.2.9, where $\operatorname{lev}_{\rho,\le}(0)$ denotes the sublevel set of $\rho$ to the level 0 (see Definition 1.1.60). Furthermore, the following properties hold:*

(i) $\rho(X) = \inf\{m \in \mathbb{R} \mid X + m \in \mathcal{A}_\rho\}$ *for all $X \in \mathcal{X}$,*

(ii) $\mathcal{A}_\rho$ *is convex if and only if $\rho$ is convex,*

(iii) $\mathcal{A}_\rho$ *is a cone if and only if $\rho$ is positively homogeneous with $\rho(\mathbb{0}) = 0$.*

**Proof.** First, we show that $\mathcal{A}_\rho$ fulfills the properties (i)-(iii) in Definition 2.2.9 and, thus, is an acceptance set. By assuming $\rho(\mathbb{0}) \le 0$, it holds that $\mathbb{0} \in \mathcal{A}_\rho$. Now, we prove that $\mathcal{A}_\rho$ is proper, i.e., $\mathcal{A}_\rho \subsetneq \mathcal{X}$. By $\rho(\mathbb{0}) \in (-\infty, 0] \subseteq \mathbb{R}$,

$$\forall m \in \mathbb{R}\backslash\{0\}: \quad \rho(m) = \rho(\mathbb{0}) - m \neq \rho(\mathbb{0})$$

because of the translation invariance of $\rho$ (see Definition 1.3.23(ii)). Consequently, $\rho$ is not constant. Thus,

$$\forall m \in \mathbb{R} \text{ with } m < \rho(\mathbb{0}): \quad \rho(m) = \rho(\mathbb{0}) - m > 0,$$

i.e., $m \in \mathcal{X}\backslash\mathcal{A}_\rho$, by translation invariance of the risk measure $\rho$ (see Definition 1.3.23(ii)), showing $\mathcal{A}_\rho \subsetneq \mathcal{X}$. Finally, take $X, Y \in \mathcal{X}$ with $X \in \mathcal{A}_\rho$ and $Y - X \in \mathcal{X}_+$. Then, $\rho(Y) \le \rho(X) \le 0$ by monotonicity of $\rho$ (see Def. 1.3.23(i)), i.e., $Y \in \mathcal{A}_\rho$. That completes the proof of $\mathcal{A}_\rho$ being an acceptance set.

Now, we prove the remaining properties:

(i) For $X \in \mathcal{X}$ arbitrary, it holds that

$$\begin{aligned}\rho(X) &= \inf\{m \in \mathbb{R} \mid \rho(X) \le m\}\\ &= \inf\{m \in \mathbb{R} \mid \rho(X + m) \le 0\}\\ &= \inf\{m \in \mathbb{R} \mid X + m \in \mathcal{A}_\rho\}\end{aligned}$$

by translation invariance of $\rho$, showing (i). Note that the translation invariance also covers $\rho(X) \in \{-\infty, +\infty\}$ by

$$\forall X \in \mathcal{X} \text{ with } \rho(X) \in \{-\infty, +\infty\}, \forall m \in \mathbb{R}: \quad \rho(X+m) = \rho(X).$$

(ii) Suppose $\rho$ is convex and let $X, Y \in \mathcal{A}_\rho$, $\lambda \in [0,1]$ arbitrary. Then,

$$\rho(\lambda X + (1-\lambda)Y) \leq \lambda \underbrace{\rho(X)}_{\leq 0} + (1-\lambda) \underbrace{\rho(Y)}_{\leq 0} \leq 0,$$

i.e., $\mathcal{A}_\rho$ is convex.

Conversely, suppose that $\mathcal{A}_\rho$ is convex. First, for $X \in \mathcal{X}$ with $\rho(X) \in \mathbb{R}$ arbitrary, we obtain $X + \rho(X) \in \mathcal{A}_\rho$ because of

$$\rho(X + \rho(X)) = \rho(X) - \rho(X) = 0 \tag{2.19}$$

by translation invariance of $\rho$. Thus, for $X, Y \in \mathcal{X}$ with $\rho(X), \rho(Y) \in \mathbb{R}$ and $\lambda \in [0,1]$ arbitrary, we obtain

$$\rho(\lambda(\underbrace{X + \rho(X)}_{\in \mathcal{A}_\rho}) + (1-\lambda)(\underbrace{Y + \rho(Y)}_{\in \mathcal{A}_\rho})) \leq 0$$

by convexity of $\mathcal{A}_\rho$. Thus, by translation invariance of $\rho$,

$$\begin{aligned}\rho(\lambda(X + \rho(X)) + (1-\lambda)(Y + \rho(Y))) &= \rho(\lambda X + (1-\lambda)Y + \underbrace{\lambda\rho(X) + (1-\lambda)\rho(Y)}_{\in \mathbb{R}}) \\ &= \rho(\lambda X + (1-\lambda)Y) - \lambda\rho(X) - (1-\lambda)\rho(Y) \\ &\leq 0,\end{aligned}$$

i.e.,

$$\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y). \tag{2.20}$$

On the other hand, (2.20) holds if $\rho(X) = +\infty$ or $\rho(Y) = +\infty$ by the inf-addition rule (see Section 1.1.3 and Lemma 1.1.56). It is left to show (2.20) for $X, Y \in \mathcal{X}$ with $\rho(X) = -\infty$ and $\rho(Y) < +\infty$ (without loss of generality). We have to prove

$$\forall \lambda \in [0,1]: \quad \lambda\rho(X) + (1-\lambda)\rho(Y) = -\infty.$$

By translation invariance of $\rho$, we obtain

$$\forall m \in \mathbb{R}: \quad \rho(X+m) = \rho(X) - m = -\infty. \tag{2.21}$$

Suppose there is some $\lambda \in [0,1)$ such that $Z := \lambda X + (1-\lambda)Y$ fulfills

$$\rho(Z) = \rho(\lambda X + (1-\lambda)Y) > -\infty. \tag{2.22}$$

Then, $Z \notin \{X\} + \mathbb{R}$ by (2.21). Let $\tilde{Z} := Z + \rho(Z) - \epsilon$ with $\epsilon > 0$ arbitrary. Then, we have by translation invariance of $\rho$

$$\rho(\tilde{Z}) = \rho(Z) - \rho(Z) + \epsilon = \epsilon > 0. \tag{2.23}$$

Now, we show that $\rho(\tilde{Z}) > 0$ contradicts the convexity of $\mathcal{A}_\rho$ (see Figure 2.1): We take some $\tilde{m} \in \mathbb{R}$ such that $\tilde{Y} := Y + \tilde{m}$ fulfills

$$\rho(\tilde{Y}) = \rho(Y) - \tilde{m} \leq 0. \tag{2.24}$$

Furthermore, we consider $\tilde{X} \in (X + \mathbb{R})$ such that $\tilde{Z}$ is an element of the line segment between $\tilde{X}$ and $\tilde{Y}$, i.e.,

$$\exists \tilde{\lambda} \in [0,1]: \quad \tilde{Z} = \tilde{\lambda}\tilde{X} + (1 - \tilde{\lambda})\tilde{Y}. \tag{2.25}$$

Note that $\tilde{X}$ exists because $Z$ is on the line segment of $X$ and $Y$, and $\tilde{Y}$ and $\tilde{Z}$ are just $Y$ and $Z$ shifted along $\mathbb{R}$. Then, $\tilde{X} \in \{X\} + \mathbb{R}$ implies $\rho(\tilde{X}) = -\infty$ by (2.21). Moreover, $\rho(\tilde{Y}) \leq 0$ by (2.24). Hence, we obtain $\tilde{X}, \tilde{Y} \in \mathcal{A}_\rho$ and, thus, $\tilde{Z} \in \mathcal{A}_\rho$ because of (2.25) by convexity of $\mathcal{A}_\rho$, i.e., $\rho(\tilde{Z}) \leq 0$, which contradicts (2.23). As a result, (2.21) does not hold, which implies

$$\forall \lambda \in [0,1]: \quad \rho(\lambda X + (1-\lambda)Y) = -\infty,$$

i.e., (2.20) is shown for $\rho(X) = -\infty$ and $\rho(Y) < +\infty$. That completes the proof that $\rho$ is convex if $\mathcal{A}_\rho$ is convex.

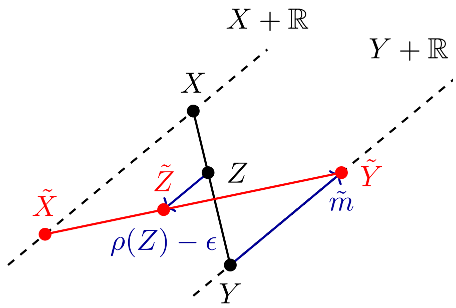


Figure 2.1: Illustration of the proof of Lemma 2.2.13(ii)

(iii) Assume that $\rho$ is positively homogeneous and let $X \in \mathcal{A}_\rho$, $\lambda \in \mathbb{R}_+$ be arbitrary. Then,

$$\rho(\lambda X) = \lambda \underbrace{\rho(X)}_{\leq 0} \leq 0,$$

i.e., $\mathcal{A}_\rho$ is a cone.

Conversely, assume that $\mathcal{A}_\rho$ is a cone. First, we show $\rho(\mathbb{0}) = 0$. By assumption, it holds that $\rho(\mathbb{0}) \in (-\infty, 0]$. Suppose $\rho(\mathbb{0}) < 0$. Then, $-\rho(\mathbb{0}) > 0$ and $\rho(\rho(\mathbb{0})) = \rho(\mathbb{0}) - \rho(\mathbb{0}) = 0$ by translation invariance of $\rho$. Furthermore,

$$\forall m \leq \rho(\mathbb{0}) : \quad \rho(m) = 0 \tag{2.26}$$

holds. Indeed, for each $m \in \mathbb{R}$ with $m \leq \rho(\mathbb{0}) < 0$,

$$\rho(m) \geq \rho(\rho(\mathbb{0})) = 0$$

holds by monotonicity of $\rho$, and $\rho(m) \leq 0$ is fulfilled because

$$m = \underbrace{\left(\frac{m}{\rho(\mathbb{0})}\right)}_{\in \mathbb{R}_>} \underbrace{\rho(\mathbb{0})}_{\in \mathcal{A}_\rho} \in \mathcal{A}_\rho$$

by $\mathcal{A}_\rho$ being a cone. Hence, we obtain by (2.26)

$$\rho(\mathbb{0}) = \inf\{m \in \mathbb{R} \mid \rho(m) \leq 0\} = -\infty$$

because of (i), in contradiction to $\rho(\mathbb{0}) > -\infty$. Hence, $\rho(\mathbb{0}) = 0$ holds.

Now, let $X \in \mathcal{X}$ arbitrary with $\rho(X) \in \mathbb{R}$ and $\lambda \in \mathbb{R}_+$ arbitrary. Then, by (2.19), $\rho(X + \rho(X)) = 0$, i.e., $X + \rho(X) \in \mathcal{A}_\rho$. Since $\mathcal{A}_\rho$ is a cone,

$$\lambda(X + \rho(X)) \in \mathcal{A}_\rho,$$

i.e., $\rho(\lambda(X + \rho(X))) \leq 0$. Since $\rho(X) \in \mathbb{R}$, we have

$$\rho(\lambda(X + \rho(X))) = \rho(\lambda X) - \lambda\rho(X) \leq 0$$

by translation invariance of $\rho$. Thus, we have shown

$$\forall \lambda \in \mathbb{R}_+ : \quad \rho(\lambda X) \leq \lambda\rho(X). \tag{2.27}$$

Now, we need to show the converse relation in (2.27). It holds that

$$\forall m < \rho(X) : \quad X + m \notin \mathcal{A}_\rho$$

by (i) and, thus,

$$\forall m < \rho(X), \forall \lambda \in \mathbb{R}_> : \quad \lambda(X + m) \notin \mathcal{A}_\rho$$

by $\mathcal{A}_\rho$ being a cone. As a result,

$$\forall m < \rho(X), \forall \lambda \in \mathbb{R}_> : \quad \rho(\lambda(X + m)) = \rho(\lambda X) - \lambda m > 0$$

holds by translation invariance of $\rho$. Hence,

$$\forall m < \rho(X), \forall \lambda \in \mathbb{R}_> : \quad \lambda m < \rho(\lambda X),$$

and $\rho(\lambda X) \leq \lambda\rho(X)$ by (2.27), which implies for $m \to \rho(X)$

$$\forall \lambda \in \mathbb{R}_> : \quad \rho(\lambda X) = \lambda\rho(X) \tag{2.28}$$

for $X \in \mathcal{X}$ with $\rho(X) \in \mathbb{R}$ arbitrary. Note that we already showed $\rho(\mathbb{0}) = 0$ and, thus, (2.28) holds for $\lambda = 0$, too.

It is left to show (2.28) for $X \in \mathcal{X}$ with $\rho(X) \in \{-\infty, +\infty\}$. Suppose $\rho(X) = -\infty$. Then, $X + m \in \mathcal{A}_\rho$ for all $m \in \mathbb{R}$ by (i), i.e.,

$$\forall m \in \mathbb{R} : \quad \rho(X + m) \leq 0.$$

Since $\mathcal{A}_\rho$ is a cone, we obtain $\lambda(X + m) \in \mathcal{A}_\rho$ for all $\lambda \in \mathbb{R}_+$, i.e.,

$$\forall m \in \mathbb{R}, \forall \lambda \in \mathbb{R}_+ : \quad \rho(\lambda X + \lambda m) = \rho(\lambda X) - \lambda m \leq 0$$

by translation invariance of $\rho$. Thus, $\rho(\lambda X) \leq \lambda m$ for all $m \in \mathbb{R}, \lambda \in \mathbb{R}_+$, i.e., $\rho(\lambda X) = -\infty$ for every $\lambda \in \mathbb{R}_+$. On the other hand, if $\rho(X) = +\infty$, we obtain by (i)

$$\nexists m \in \mathbb{R} : \quad \rho(X + m) \leq 0.$$

Since $\mathcal{A}_\rho$ is a cone, it must also hold

$$\forall \lambda \in \mathbb{R}_+, \nexists m \in \mathbb{R} : \quad \rho(\lambda X + \lambda m) \leq 0$$

Thus, for every $\lambda \in \mathbb{R}_+$ holds $\rho(\lambda X) > \lambda m$ for all $m \in \mathbb{R}$ by translation invariance of $\rho$, i.e., $\rho(\lambda X) = +\infty$ for every $\lambda \in \mathbb{R}_+$. That completes the proof of $\rho$ being positively homogeneous.

□

**Remark 2.2.14.** *Consider a vector space $\mathcal{X}$ and $\mathcal{A}_\rho \subseteq \mathcal{X}$ given by Definition 2.2.12 with $\rho\colon \mathcal{X} \to \overline{\mathbb{R}}$ being a risk measure (see Definition 1.3.23). In the proof of $\mathcal{A}_\rho$ being an acceptance set in Lemma 2.2.13, $\rho(\mathbb{0}) \leq 0$ is necessary to show that $\mathbb{0} \in \mathcal{A}_\rho$. Moreover, $\rho$ being equally a constant $\rho \equiv c$ with $c \in [-\infty, 0]$ implies $\mathcal{A}_\rho = \mathcal{X}$, but as seen in the proof and by use of $\rho(\mathbb{0}) \in \mathbb{R}$, $\rho$ is never constant and simultaneously real-valued by Definition 1.3.23*(ii)*. Hence, we used $\rho(\mathbb{0}) > -\infty$ indirectly to conclude $\rho \not\equiv -\infty$ such that Lemma 2.2.13 can be relaxed to sets $\mathcal{A} \subseteq \mathcal{X}$ that fulfill every requirements of an acceptance set in Definition 2.2.9 except for $\mathbb{0} \in \mathcal{A}$.*

**Remark 2.2.15.** *Consider a vector space $\mathcal{X}$ and $\mathcal{A}_\rho$ given by Definition 2.2.12. Lemma 2.2.13 provides a characterization of coherent risk measures by the acceptance set $\mathcal{A}_\rho$: a risk measure $\rho\colon \mathcal{X} \to \overline{\mathbb{R}}$ in the sense of Definition 1.3.23 is a coherent risk measure in the sense of Definition 1.3.31 if and only if $\mathcal{A}_\rho$ is a convex cone. Moreover, in a topological vector space $\mathcal{X}$, $\mathcal{A}_\rho$ is not closed in general. If the risk measure $\rho\colon \mathcal{X} \to \overline{\mathbb{R}}$ is continuous, then $\mathcal{A}_\rho$ is obviously closed as the preimage of the closed set $[-\infty, 0]$ of $\rho$. Moreover, $\mathcal{A}_\rho = \operatorname{lev}_{\rho,\leq}(0)$ is closed as a sublevel set of $\rho$ if the risk measure $\rho$ is lower semicontinuous (see Lemma 1.1.61). For example, if*

*$(\mathcal{X}, \|\cdot\|_\infty)$ is a Banach space of random variables with supremum norm, then each real-valued coherent risk measure $\rho\colon \mathcal{X} \to \mathbb{R}$ is Lipschitz continuous (see [78, Lemma 4.3]) and, thus, lower semicontinuous (more exactly, $\rho$ is continuous, see, e.g., [185, Prop. 4.4.7]). Another example is given by convex, real-valued risk measures $\rho\colon \mathcal{X} \to \mathbb{R}$ on a Banach lattice $(\mathcal{X}, \|\cdot\|)$ (e.g., $\mathcal{L}^p$ with $\|\cdot\|_{\mathcal{L}^p}$ and $1 \leq p \leq +\infty$), which follows from [175, Prop. 1] (see also [133, Remark 2.4]).*

**Example 2.2.16** (see Föllmer, Schied [78] and Baes et al. [17])**.** Let $\mathcal{X}$ be a real vector space of random variables with some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Furthermore, let $\rho\colon \mathcal{X} \to \overline{\mathbb{R}}$ be a (monetary) risk measure in the sense of Definition 1.3.23 with $\rho(\mathbb{0}) \leq 0$. In the following, let $X \in \mathcal{X}$ represent a random payoff and, thus, be equipped with a gain distribution. Furthermore, let $\alpha \in (0, 1)$ be an arbitrary confidence level. Popular examples of acceptance sets $\mathcal{A}_\rho$ defined by risk measures $\rho$ are those defined by the Value-at-Risk $\mathrm{VaR}_\alpha$ (see Definition 1.3.25 and Lemma 1.3.27), i.e.,

$$\mathcal{A}_{\mathrm{VaR}_\alpha} = \{X \in \mathcal{X} \mid \mathrm{VaR}_\alpha(X) \leq 0\} = \{X \in \mathcal{X} \mid \inf\{m \in \mathbb{R} \mid \mathbb{P}(X < -m) \leq \alpha\} \leq 0\},$$

and defined by the Average-Value-at-Risk $\mathrm{AVaR}_\alpha$, i.e.,

$$\mathcal{A}_{\mathrm{AVaR}_\alpha} = \{X \in \mathcal{X} \mid \mathrm{AVaR}_\alpha(X) \leq 0\} = \left\{X \in \mathcal{X} \mid \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_\beta(X)\, d\beta \leq 0\right\}.$$

Equivalently, Conditional-Value-at-Risk $\mathrm{CVaR}_\alpha$ or Expected Shortfall $\mathrm{ES}_\alpha$ can be considered instead of $\mathrm{AVaR}_\alpha$, see Definition 1.3.35 and Theorem 1.3.37 for the equivalence of these risk measures. For the Value-at-Risk, the corresponding acceptance set $\mathcal{A}_{\mathrm{VaR}_\alpha}$ is a cone, since $\mathrm{VaR}_\alpha$ is a positively homogeneous (monetary) risk measure (see Lemma 1.3.30). Hence, $\mathcal{A}_{\mathrm{VaR}_\alpha}$ is a conic acceptance set, which is not necessarily convex, since $\mathrm{VaR}_\alpha$ is not convex (see Section 1.3.3). For the Average-Value-at-Risk, the corresponding acceptance set $\mathcal{A}_{\mathrm{AVaR}_\alpha}$ is a convex cone, since $\mathrm{AVaR}_\alpha$ is a coherent risk measure (see Lemma 1.3.37), and, thus, $\mathcal{A}_{\mathrm{AVaR}_\alpha}$ is a coherent acceptance set. Note that $\mathcal{A}_{\mathrm{AVaR}_\alpha}$ is more restrictive then $\mathcal{A}_{\mathrm{VaR}_\alpha}$ because of

$$\mathcal{A}_{\mathrm{AVaR}_\alpha} \subseteq \mathcal{A}_{\mathrm{VaR}_\alpha}$$

by Remark 1.3.38. Of course, there is in general more than one regulatory precondition an financial institution has to fulfill. If finitely many acceptance sets $\mathcal{A}_j$ with $j = 1, \ldots, m$ describe the single preconditions, then, obviously,

$$\mathcal{A}_{reg} := \bigcap_{j=1}^{m} \mathcal{A}_j$$

is an acceptance set, too. ◊

Now, we have completed our financial market setting, which we want to summarize in the following. We will consider this financial model (with respect to our remarks at the beginning of Section 2.2) throughout this thesis, and refer to it by the shortcut (FM):

$$\begin{aligned}
&\mathcal{X} \text{ is the real vector space of capital positions, partially ordered by } \mathcal{X}_+,\\
&(\Omega, \mathcal{F}, \mathbb{P}) \text{ is a probability space,}\\
&S^i \in \mathbb{R} \times \mathcal{X} \ (i = 0, 1, \ldots, n) \text{ is the } i\text{-th eligible asset from (1.9)}\\
&\text{with } S^0 \text{ being the secure investment opportunity given by (2.7),}\\
&\mathcal{M} \subseteq \mathcal{X} \text{ is the space of eligible payoffs given by (2.10) with } 1 < \dim \mathcal{M} < +\infty,\\
&\pi \colon \mathcal{M} \to \mathbb{R} \text{ is the linear pricing functional given by (2.13),}\\
&\mathcal{A} \subseteq \mathcal{X} \text{ is an acceptance set according to Definition 2.2.9,}\\
&\text{and Assumption 1 is fulfilled.}
\end{aligned} \tag{FM}$$

**Remark 2.2.17.** *Note that we consider multiple eligible assets by* (FM)*, i.e.,* $\dim \mathcal{M} > 1$ *and, thus,* $\ker \pi \neq \{\mathbb{0}\}$ *by applying the rank-nullity-theorem for* $\pi \colon \mathcal{M} \to \mathbb{R}$ *given by* (2.13) *(see Remark 2.2.6). The case* $\dim \mathcal{M} = 1$ *coincides with the case of one eligible asset that is well-studied, see, e.g., [14] and [70]. We do not consider only one eligible asset because of practical purposes, where the multi-dimensional case is standard. Nevertheless, the results in this thesis can also be proven for one eligible asset, since most of them do not require* $\ker \pi \neq \{\mathbb{0}\}$*.*

While we always require Assumption 1 to be fulfilled in (FM) as a minimal (for financial markets often typical) requirement, we only make use of the other assumptions where it is really necessary. For example, another often made assumption in financial mathematics is that there are no *good deals* in the market:

**Definition 2.2.18** (see [17])**.** Consider (FM). We call an eligible payoff $Z \in (\mathcal{A} \cap \mathcal{M}) \backslash \{\mathbb{0}\}$ with $\pi(Z) \leq 0$ *good deal*.

**Assumption 3** (absence of good deals)**.** *Consider* (FM)*. It holds that*

$$\forall Z \in (\mathcal{A} \cap \mathcal{M}) \backslash \{\mathbb{0}\} : \qquad \pi(Z) > 0.$$

A characterization for the absence of good deals, i.e., a validation of Assumption 3, is given in the following lemma, which was derived in [17, Prop. 2.6 (iii)] for closed acceptance sets in a locally convex Hausdorff space above $\mathbb{R}$ fulfilling the first axiom of countability. However, it is also possible to prove a corresponding result for (FM) with $\mathcal{X}$ being a real vector space, which is more precisely then the one shown in [140].

**Lemma 2.2.19** (see [140, Lemma 4.1])**.** *Consider* (FM)*. Let* $U \in \mathcal{M} \cap \mathcal{X}_+$ *be a payoff fulfilling Assumption 2. If Assumption 3 is fulfilled, then* $\mathcal{A} \cap \ker \pi = \{\mathbb{0}\}$ *holds. The converse is true if*

$$\mathcal{A} \cap (-\mathbb{R}_> U) = \emptyset \tag{2.29}$$

*is fulfilled.*

**Proof.** If Assumption 3 is fulfilled, $\mathcal{A} \cap \ker \pi = \{\mathbb{0}\}$ holds, obviously, since $\ker \pi \subseteq \mathcal{M}$ and $\{\mathbb{0}\} \subsetneq \ker \pi$ by Remark 2.2.6.

Now, conversely, let $\mathcal{A} \cap \ker \pi = \{\mathbb{0}\}$. Suppose Assumption 3 does not hold, i.e., let $Z \in (\mathcal{A} \cap \mathcal{M}) \backslash \{\mathbb{0}\}$ with $\pi(Z) \leq 0$. First, $\pi(Z) < 0$ holds, since $Z \in \ker \pi$ and $\mathcal{A} \cap \ker \pi \neq \{\mathbb{0}\}$, otherwise. Then, there is some $V \in \ker \pi$ with

$$Z = \pi(Z)U + V$$

by Lemma 2.2.7. Because of $\pi(Z) < 0$ and monotonicity of $\mathcal{A}$ (see Definition 2.2.9(iii)), we obtain by $Z \in \mathcal{A}$

$$V = Z - \pi(Z)U \in \mathcal{A}$$

and, thus, $V = \mathbb{0}$ by $\mathcal{A} \cap \ker \pi = \{\mathbb{0}\}$. As a result, $Z = \pi(Z)U \in -\mathbb{R}_{>}U$, which contradicts (2.29) by $Z \in \mathcal{A}$. Hence, Assumption 3 holds and the proof is complete. □

An example that (2.29) is necessary for the characterization in Lemma 2.2.19 can be found in [17, Example 2.9].

**Remark 2.2.20.** *Instead of (2.29), there are other further sufficient conditions for the converse direction in Lemma 2.2.19 that can be found in [17, Prop. 2.6], namely $\mathcal{A} \cap (-\mathcal{A}) = \{\mathbb{0}\}$ or $\mathcal{A} \cap (-\mathcal{X}_+) = \{\mathbb{0}\}$. Although [17, Prop. 2.6] requires $\mathcal{A}$ to be closed, this property is not used in the proof and, thus, the result also holds in our setting.*

## 2.3 Risk associated with acceptance sets

Consider the financial market (FM). A financial institution is obviously interested in passing an acceptability test (given by an acceptance set $\mathcal{A} \subseteq \mathcal{X}$, see Definition 2.2.9) with minimal costs. The corresponding optimization problem for given $X \in \mathcal{X}$ is given by

$$\pi(Z) \to \min_{X+Z \in \mathcal{A},\, Z \in \mathcal{M}}. \tag{$P_\pi(X)$}$$

In this section, we focus on the functional quantifying the optimal value of $\pi(Z)$ in ($P_\pi(X)$). In Chapter 3, we study the solution set of ($P_\pi(X)$). Hence, the following questions are from interest:

- What is the minimal capital amount that the decision maker has to raise and invest into the eligible assets $S^i$ $(i = 0, 1, \dots, n)$ given by (1.9) to reach a (in general) new capital position $X^0 \in \mathcal{A}$?
- If it is even possible, how can an acceptable capital position corresponding to the minimal costs be reached?

As noticed before, the second will be studied in Chapter 3, while, in this section, we study the following functional that answers the first question:

**Definition 2.3.1** (see Frittelli, Scandolo [82], and also Artzner et al. [15], Baes et al. [17], and Farkas et al. [71])**.** Let $\mathcal{X}$ be a real vector space, $\mathcal{M}$ be a subset of $\mathcal{X}$, $\pi\colon \mathcal{M} \to \mathbb{R}$ be functional on $\mathcal{M}$, and $\mathcal{A} \subseteq \mathcal{M}$ be an arbitrary subset of $\mathcal{X}$. The functional $\rho_{\mathcal{A},\mathcal{M},\pi}\colon \mathcal{X} \to \overline{\mathbb{R}}$ defined by

$$\rho_{\mathcal{A},\mathcal{M},\pi}(X) := \inf\{\pi(Z) \mid Z \in \mathcal{M}, X + Z \in \mathcal{A}\} \tag{2.30}$$

is called *risk measure on $\mathcal{X}$ associated with $\mathcal{A}$, $\mathcal{M}$ and $\pi$.*

As mentioned in Section 2.1, (2.30) is introduced in [82] and [177]. Note that we have generally defined $\rho_{\mathcal{A},\mathcal{M},\pi}$ without any probability measure, $\mathcal{A}$ necessarily being an acceptance set, $\mathcal{M}$ being a subspace of $\mathcal{X}$ represented by any eligible payoffs, and $\pi$ being a pricing functional. Nevertheless, we will be interested in considering the economical background modeled by (FM) in this thesis. In Lemma 2.3.5, we will see that $\rho_{\mathcal{A},\mathcal{M},\pi}$ is a risk measure, indeed, if we consider (FM). In the literature, the triple $(\mathcal{A}, \mathcal{M}, \pi)$ in (FM) is also known as *risk measurement regime* (see Farkas et al. [71, Def. 2]) and describes a setting for interpreting $\rho_{\mathcal{A},\mathcal{M},\pi}$ as a risk measure and allowing economical interpretations of it.

**Remark 2.3.2.** *In the following, we will always consider the functional $\rho_{\mathcal{A},\mathcal{M},\pi}\colon \mathcal{X} \to \overline{\mathbb{R}}$ from Definition 2.3.1 always together with a risk measurement regime $(\mathcal{A}, \mathcal{M}, \pi)$ by assuming the financial market model (FM). Hence, $\rho_{\mathcal{A},\mathcal{M},\pi}(X)$ can be interpreted as the capital requirement for making a current capital position acceptable, as mentioned before Definition 2.3.1. If $\rho_{\mathcal{A},\mathcal{M},\pi}(X) < 0$, the decision maker can set money free to change the current capital position $X$ to reach acceptability. Note that a negative value of $\rho_{\mathcal{A},\mathcal{M},\pi}(X)$, even $\rho_{\mathcal{A},\mathcal{M},\pi}(X) = 0$, does not mean that $X$ is already acceptable, i.e.,*

$$\rho_{\mathcal{A},\mathcal{M},\pi}(X) \leq 0 \quad \not\Rightarrow \quad X \in \mathcal{A}, \tag{2.31}$$

*see [139, Remark 4.4]. Conversely, if the position $X$ is already acceptable, $\rho_{\mathcal{A},\mathcal{M},\pi}(X)$ has to be non-positive, since $\mathbb{0} \in \mathcal{M}$ and $\pi(\mathbb{0}) = 0$, i.e.,*

$$X \in \mathcal{A} \quad \Longrightarrow \quad \rho_{\mathcal{A},\mathcal{M},\pi}(X) \leq 0. \tag{2.32}$$

We illustrate the described relations (2.32) and (2.31) in the following example (see also Figure 2.2):

**Example 2.3.3.** Consider $\mathcal{X} = \mathcal{M} = \mathbb{R}^2$ and $\pi\colon \mathcal{M} \to \mathbb{R}$ given by $\pi(Z) = \frac{Z_1+Z_2}{2}$. Then,

$$\ker \pi = \{(Z_1, Z_2)^T \in \mathbb{R}^2 \mid Z_2 = -Z_1\}$$

and $U = (1, 1)^T$ fulfills Assumption 2, i.e., $\pi(U) = 1$. Consider $X^1 = (2, 1.5)^T \in \mathcal{X}$ and $X^2 = (4, -1.5)^T \in \mathcal{X}$ as illustrated in Figure 2.2. Then, $X^1$ is an example for (2.32): $X^1$ fulfills $X^1 \in \mathcal{A}$ and, thus, $\rho_{\mathcal{A},\mathcal{M},\pi}(X^1) \leq 0$ because $\mathbb{0} \in \mathcal{M}$. Moreover, $\rho_{\mathcal{A},\mathcal{M},\pi}(X^1) < 0$ holds because the dashed arrow represents a vector

$$Z^1 := m_1 U \in \mathcal{M} \text{ with some } m_1 \in \mathbb{R}_< \text{ such that } X^1 + Z^1 \in \mathcal{A}$$

and, thus,

$$\rho_{\mathcal{A},\mathcal{M},\pi}(X^1) \leq \pi(Z^1) = m_1 < 0$$

holds. More precisely, in Figure 2.2, $Z^1 = m_1 U$ is given by $m_1 = -1.75$, i.e.,

$$X^1 + Z^1 = (0.25, -0.25)^T \in \mathcal{A}$$

with $\rho_{\mathcal{A},\mathcal{M},\pi}(X^1) \leq -1.75$ .

On the other hand, $X^2 = (4, -1.5)^T$ fulfills $X^2 \notin \mathcal{A}$ and is an example for (2.31): the dashed arrow starting at $X^2$ represents the vector $Z^2 := (-3, 0.5)^T$ and, thus, fulfills $\pi(Z^2) = -1.25 < 0$. Consequently, $\rho_{\mathcal{A},\mathcal{M},\pi}(X^2) < 0$ holds by $X^2 + Z^2 = (1, -1)^T \in \mathcal{A}$. $\diamond$

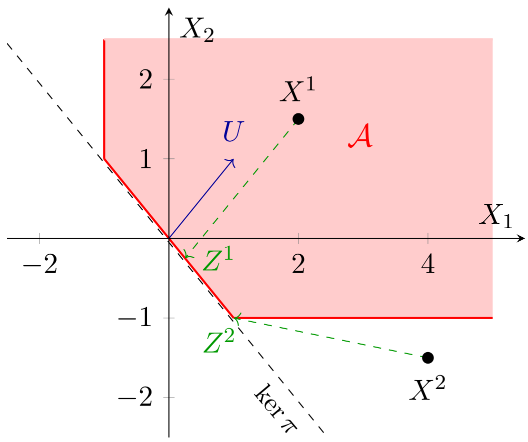


Figure 2.2: Examples for $\rho_{\mathcal{A},\mathcal{M},\pi} \leq 0$

**Remark 2.3.4.** *We want to highlight some observations in Example 2.3.3 with respect to* $\rho_{\mathcal{A},\mathcal{M},\pi}\colon \mathcal{X} \to \overline{\mathbb{R}}$ *given by* (2.30).

- *The values* $\rho_{\mathcal{A},\mathcal{M},\pi}(X^1)$ *and* $\rho_{\mathcal{A},\mathcal{M},\pi}(X^2)$ *can be determined by considering* $\mathcal{A} + \ker\pi$*: It holds* $Z^2 \in m_2 U + \ker\pi$ *with* $m_2 = \pi(Z^1) = -1.75 \in \mathbb{R}_<$*. Since we can rewrite every* $Z \in \mathcal{M}$ *by some multiple* $m \in \mathbb{R}$ *of* $U$ *and some vector in* $\ker\pi$ *(see Lemma 2.2.7), it seems natural that we can reduce the problem of determining* $\rho_{\mathcal{A},\mathcal{M},\pi}(X)$ *for arbitrary given* $X \in \mathcal{X}$ *to the approach*

$$m \longrightarrow \min_{X+mU\in\mathcal{A}+\ker\pi}! \; .$$

*We will see that this is true in Lemma 2.3.11.*

- *The vectors* $Z^1$ *and* $Z^2$ *in Example 2.3.3 provide both minimal costs for reaching acceptability for the given position* $X^1$ *and* $X^2$*, respectively, i.e.,*

$$\rho_{\mathcal{A},\mathcal{M},\pi}(X^1) = \pi(Z^1) \text{ and } \rho_{\mathcal{A},\mathcal{M},\pi}(X^2) = \pi(Z^2). \tag{2.33}$$

*It is conspicuous that* $Z^1$ *and* $Z^2$ *change the financial positions* $X^1$ *and* $X^2$ *into positions in* $\operatorname{bd}_{-U}(\mathcal{A}) \subseteq \operatorname{bd}\mathcal{A} \subseteq \mathcal{A}$*. We will prove a mathematical precisely result in Chapter 3 in Theorem 3.1.6 (see also the result* (3.7) *in [17, Th. 3.2]) that validates our impression in this example.*

- *Although* $Z^2$ *fulfills* $\rho_{\mathcal{A},\mathcal{M},\pi}(X^2) = \pi(Z^2)$*, the vector* $Z^2$ *does not also deliver the shortest distance from* $X^2$ *to* $\mathcal{A}$ *if we consider the Euclidean metric on* $\mathbb{R}^2$*. Hence, there is no direct relationship between the shortest distance from* $X^2$ *to* $\mathcal{A}$ *and* $\rho_{\mathcal{A},\mathcal{M},\pi}(X^2)$*, and we can not focus on this geometric problem.*

- *Moreover,* $Z^1$ *and* $Z^2$ *are not unique vectors fulfilling* (2.33)*. For example, we can add* $Z = (-1,1)^T \in \ker\pi$ *to any of the vectors* $Z^1$ *and* $Z^2$ *and obtain*

$$X^j + Z^j + Z \in \mathcal{A} \quad \text{with } \pi(Z^j + Z) = \pi(Z^j) = \rho_{\mathcal{A},\mathcal{M},\pi}(X^j) \text{ for } j \in \{1,2\}. \tag{2.34}$$

*All positions along the line segment between* $(-1,1)^T$ *and* $(1,-1)^T$ *can be reached along* $\ker\pi$*, i.e., with price zero, and still belong to* $\mathcal{A}$*. So, there are infinitely many*

$$Z \in \ker\pi \cap \operatorname{conv}\{(-1,1)^T, (1,-1)^T\}$$

*(depending on the considered vector* $Z^1$ *or* $Z^2$*) such that* (2.34) *holds for* $j = 1$ *and* $j = 2$*, respectively. We consider the set of optimal eligible payoffs*

$$\mathcal{E}(X) := \{Z \in \mathcal{M} \mid X + Z \in \mathcal{A}, \pi(Z) = \rho_{\mathcal{A},\mathcal{M},\pi}(X)\} \tag{2.35}$$

*later in Chapter 3, and obtain characterizations for uniqueness of* $\mathcal{E}(X)$*, i.e.,* $|\mathcal{E}(X)| = 1$ *(see Theorem 3.2.14), but also for* $\mathcal{E}(X) \neq \emptyset$ *(see Theorem 3.1.10).*

Now, we want to study properties like monotonicity, finiteness, and sublevel sets of the risk measure $\rho_{\mathcal{A},\mathcal{M},\pi}\colon \mathcal{X} \to \overline{\mathbb{R}}$ given by (2.30) with respect to (FM) in more detail. Our first result justifies the terminology "risk measure" for $\rho_{\mathcal{A},\mathcal{M},\pi}$ in the sense of Definition 1.3.23 and is derived by a result from Farkas, Koch-Medina, Munari [71], who worked in topological vector spaces. We gave a proof in [139] that did not use any topological properties and, thus, the result also works for real vector spaces.

**Lemma 2.3.5** (see [71, Lemma 2] and [139, Lemma 3.14])**.** *Consider* (FM) *and the functional* $\rho_{\mathcal{A},\mathcal{M},\pi}\colon \mathcal{X} \to \overline{\mathbb{R}}$ *given by* (2.30)*. Let Assumption 2 be fulfilled. Then,* $\rho_{\mathcal{A},\mathcal{M},\pi}$ *has the following properties:*

(i) $\rho_{\mathcal{A},\mathcal{M},\pi}(X) \geq \rho_{\mathcal{A},\mathcal{M},\pi}(Y)$ *for all* $X, Y \in \mathcal{X}$ *with* $Y \in X + \mathcal{X}_+$*,*

(ii) $\rho_{\mathcal{A},\mathcal{M},\pi}(X + Z) = \rho_{\mathcal{A},\mathcal{M},\pi}(X) - \pi(Z)$ *for all* $X \in \mathcal{X}, Z \in \mathcal{M}$*.*

**Proof.** Let $X \in \mathcal{X}$ arbitrary and $U \in \mathcal{M} \cap \mathcal{X}_+$ according to Assumption 2.

(i) Take $Y \in X + \mathcal{X}_+$ and $Z \in \mathcal{M}$ with $X + Z \in \mathcal{A}$ and $\pi(Z) = \rho_{\mathcal{A},\mathcal{M},\pi}(X)$ arbitrary. Then, $Y + Z \in \mathcal{A}$, too, because of

$$Y + Z = (X + Z) + (Y - X) \in \mathcal{A} + \mathcal{X}_+ \subseteq \mathcal{A}$$

by $\mathcal{A}$ being a monotone set (see Definition 2.2.9(iii)). Hence, $\rho_{\mathcal{A},\mathcal{M},\pi}(Y) \leq \rho_{\mathcal{A},\mathcal{M},\pi}(X)$ holds.

(ii) Take $Z \in \mathcal{M}$ arbitrary and fixed. Then, we obtain for each $W \in \mathcal{M}$ arbitrary

$$(X + Z) + W \in \mathcal{A} \quad \Longleftrightarrow \quad X + \tilde{Z} \in \mathcal{A} \text{ with } \tilde{Z} := Z + W.$$

Thus,

$$\begin{aligned}
\rho_{\mathcal{A},\mathcal{M},\pi}(X + Z) &= \inf\{\pi(W) \mid W \in \mathcal{M}, X + Z + W \in \mathcal{A}\} \\
&= \inf\{\pi(\tilde{Z} - Z) \mid \tilde{Z} \in \mathcal{M}, X + \tilde{Z} \in \mathcal{A}\} \\
&= \inf\left\{\pi(\tilde{Z}) - \pi(Z) \mid \tilde{Z} \in \mathcal{M}, X + \tilde{Z} \in \mathcal{A}\right\} \\
&= \rho_{\mathcal{A},\mathcal{M},\pi}(X) - \pi(Z)
\end{aligned}$$

by linearity of $\pi$.

□

**Remark 2.3.6.** *Lemma 2.3.5 implies that $\rho_{\mathcal{A},\mathcal{M},\pi} \colon \mathcal{X} \to \overline{\mathbb{R}}$ given by (2.30) is a risk measure, indeed: By property* (i), *$\rho_{\mathcal{A},\mathcal{M},\pi}$ is monotone, while* (ii) *implies*

$$\forall m \in \mathbb{R}, \forall X \in \mathcal{X} : \qquad \rho_{\mathcal{A},\mathcal{M},\pi}(X + m) = \rho_{\mathcal{A},\mathcal{M},\pi}(X) - m\pi(\mathbb{1}_\Omega). \tag{2.36}$$

*As we mentioned earlier, we include the secure asset $S^0 = (S^0_0, S^0_1)^T = (1, \mathbb{1}_\Omega)^T$ with price $\pi(S^0_1) = \pi(\mathbb{1}_\Omega) = S^0_0 = 1$ in* (FM) *and, thus, all constants are eligible payoffs, i.e., $\mathbb{R} \subseteq \mathcal{M}$, see Remark 2.2.2 and Remark 2.2.3. (As mentioned there, we can choose $U = S^0_1 = \mathbb{1}_\Omega \in \mathcal{M} \cap \mathcal{X}_+$ to fulfill Assumption 2 in* (FM)*.) Hence,* (2.36) *delivers*

$$\forall m \in \mathbb{R}, \forall X \in \mathcal{X} : \qquad \rho_{\mathcal{A},\mathcal{M},\pi}(X + m) = \rho_{\mathcal{A},\mathcal{M},\pi}(X) - m,$$

*which is known as* cash additivity*. Thus, $\rho_{\mathcal{A},\mathcal{M},\pi}$ is a risk measure in the sense of Definition 1.3.23. In models without constants being eligible payoffs, i.e., $\mathbb{1}_\Omega \notin \mathcal{M}$, Lemma 2.3.5(ii) does not hold. Our definition coincides with $S$-additive risk measures for $S = \mathbb{1}_\Omega$ or, more generally, $\mathcal{M}$-additive risk measures in the literature, compare [69].*

*As mentioned in Remark 1.3.24 and in contrast to some other definitions in the literature, we do not assume $\rho_{\mathcal{A},\mathcal{M},\pi}(\mathbb{0}) \in \mathbb{R}$ for a risk measure. $\rho_{\mathcal{A},\mathcal{M},\pi}(\mathbb{0}) \leq 0$ is sensible for many practical situations, see also Lemma 2.2.13, and is fulfilled for each acceptance set $\mathcal{A}$ according to Definition 2.2.9 by $\mathbb{0} \in \mathcal{A}$ (compare also Remark 2.3.2). Especially, since $\rho_{\mathcal{A},\mathcal{M},\pi}(\mathbb{0}) \neq 0$ in general, $\rho_{\mathcal{A},\mathcal{M},\pi}$ is no coherent risk measure in the sense of Definition 1.3.23, in general, i.e., $\rho_{\mathcal{A},\mathcal{M},\pi}$ is not always positively homogeneous or convex, see Remark 1.3.24. Nevertheless, we will see in Lemma 2.3.44 that $\rho_{\mathcal{A},\mathcal{M},\pi}$ is a coherent risk measure if $\mathcal{A}$ is a convex cone.*

**Remark 2.3.7.** *Property* (ii) *in Lemma 2.3.5 is also known as $\mathcal{M}$-additivity or translation invariance and has interesting implications. Let $X \in \mathcal{X}$ arbitrary, $U \in \mathcal{M} \cap \mathcal{X}_+$ given as in Assumption 2 and $\rho_{\mathcal{A},\mathcal{M},\pi} \colon \mathcal{X} \to \overline{\mathbb{R}}$ be defined as in* (2.30)*. Then, it holds that*

$$\forall m \in \mathbb{R}: \quad \rho_{\mathcal{A},\mathcal{M},\pi}(X + mU) = \rho_{\mathcal{A},\mathcal{M},\pi}(X) - m$$

*and, thus, for $\rho_{\mathcal{A},\mathcal{M},\pi}(X) \in \mathbb{R}$,*

$$\rho_{\mathcal{A},\mathcal{M},\pi}(X + \rho_{\mathcal{A},\mathcal{M},\pi}(X)U) = 0.$$

*If $\rho_{\mathcal{A},\mathcal{M},\pi}$ is normalized, i.e., $\rho_{\mathcal{A},\mathcal{M},\pi}(\mathbb{0}) = 0$, then*

$$\forall Z \in \mathcal{M}: \ \rho_{\mathcal{A},\mathcal{M},\pi}(Z) = \rho_{\mathcal{A},\mathcal{M},\pi}(\mathbb{0}) - \pi(Z) = -\pi(Z).$$

*holds and, thus, Lemma 2.3.5*(ii) *is equivalent to*

$$\forall X \in \mathcal{X}, \ \forall Z \in \mathcal{M}: \quad \rho_{\mathcal{A},\mathcal{M},\pi}(X + Z) = \rho_{\mathcal{A},\mathcal{M},\pi}(X) + \rho_{\mathcal{A},\mathcal{M},\pi}(Z). \tag{2.37}$$

According to Assumption 3 (see also Definition 2.2.18), we observe in the following lemma that the absense of good deals is sufficient for $\rho_{\mathcal{A},\mathcal{M},\pi}$ being normalized, which we already mentioned as a remark in [140], but with an unnecessary additional assumption and without a proof:

**Lemma 2.3.8** (see Marohn, Tammer [140, Remark 5.2])**.** *Consider* (FM) *and the functional $\rho_{\mathcal{A},\mathcal{M},\pi} \colon \mathcal{X} \to \overline{\mathbb{R}}$ given by* (2.30)*. Then, $\rho_{\mathcal{A},\mathcal{M},\pi}$ is normalized, i.e., $\rho_{\mathcal{A},\mathcal{M},\pi}(\mathbb{0}) = 0$ holds if Assumption 3 is fulfilled. Moreover, $\mathbb{0} \in \operatorname{bd}_{-U}(\mathcal{A})$ holds for every $U \in \mathcal{M} \cap \mathcal{X}_+$ fulfilling Assumption 2 if $\rho_{\mathcal{A},\mathcal{M},\pi}$ is normalized.*

**Proof.** Suppose that $\rho_{\mathcal{A},\mathcal{M},\pi}(\mathbb{0}) \neq 0$. Then,

$$\rho_{\mathcal{A},\mathcal{M},\pi}(\mathbb{0}) < 0 \text{ holds because of } \mathbb{0} \in \mathcal{A}$$

by Definition 2.2.9(i) and $\mathbb{0} \in \mathcal{M}$. As a result,

$$\exists Z \in \mathcal{M}: \quad Z = \mathbb{0} + Z \in \mathcal{A} \text{ and } \pi(Z) < 0,$$

which contradicts Assumption 3 by Definition 2.2.18. Consequently, $\rho_{\mathcal{A},\mathcal{M},\pi}(\mathbb{0}) \neq 0$ holds.

Now, let $U \in \mathcal{M} \cap \mathcal{X}_+$ be arbitrary according to Assumption 2. By $\mathbb{0} \in \mathcal{A}$ and Definition 2.2.9(iii), we obtain $\mathbb{0} + mU \in \mathcal{A}$ for all $m \in \mathbb{R}_+$. If $\mathbb{0} \notin \operatorname{bd}_{-U}(\mathcal{A})$ holds, then there is some $\tilde{m} \in \mathbb{R}_>$ such that $\mathbb{0} - mU \in \mathcal{A}$ for all $m \in (0, \tilde{m}]$, which contradicts $\rho_{\mathcal{A},\mathcal{M},\pi}(\mathbb{0}) = 0$, since $-mU \in \mathcal{M}$ with $\pi(-mU) = -m < 0$. Hence, $\mathbb{0} \in \operatorname{bd}_{-U}(\mathcal{A})$ holds, which completes the proof. □

**Remark 2.3.9.** *In Lemma 2.3.8, $\mathbb{0} \in \operatorname{bd}_{-U}(\mathcal{A})$ for $U \in \mathcal{M} \cap \mathcal{X}_+$ arbitrary was proved without use of Assumption 3. Only $\rho_{\mathcal{A},\mathcal{M},\pi}$ being normalized was necessary to show that property. Thus, we have shown*

$$\textit{Assumption 3 is fulfilled} \quad \Longrightarrow \quad \rho_{\mathcal{A},\mathcal{M},\pi}(\mathbb{0}) = 0 \quad \Longrightarrow \quad \mathbb{0} \in \operatorname{bd}_{-U}(\mathcal{A}).$$

*For topological vector spaces $\mathcal{X}$, $\rho_{\mathcal{A},\mathcal{M},\pi} \colon \mathcal{X} \to \overline{\mathbb{R}}$ given by* (2.30) *being normalized also implies $\mathbb{0} \in \operatorname{bd} \mathcal{A}$ by $\operatorname{bd}_{-U}(\mathcal{A}) \subseteq \operatorname{bd} \mathcal{A}$ for all $U \in \mathcal{M} \cap \mathcal{X}_+$.*

While absence of good deals, i.e., Assumption 3, secures that $\rho_{\mathcal{A},\mathcal{M},\pi}\colon \mathcal{X} \to \overline{\mathbb{R}}$ given by (2.30) is normalized (as seen in Lemma 2.3.8), the reverse does not hold, but we can give an additional condition to make that true:

**Corollary 2.3.10.** *Consider (FM) and the functional $\rho_{\mathcal{A},\mathcal{M},\pi}\colon \mathcal{X} \to \overline{\mathbb{R}}$ given by (2.30). Let $\rho_{\mathcal{A},\mathcal{M},\pi}$ be normalized, i.e., $\rho_{\mathcal{A},\mathcal{M},\pi}(\mathbb{0}) = 0$ is fulfilled. Then, the following conditions hold:*

(i) *If $Z \in (\mathcal{M} \cap \mathcal{A})\backslash\{\mathbb{0}\}$ is a good deal in the sense of Definition 2.2.18, then $\pi(Z) = 0$ holds.*

(ii) *Let $\mathcal{A} \cap \ker \pi = \{\mathbb{0}\}$ hold. Then, there are no good deals, i.e., Assumption 3 is fulfilled.*

**Proof.**

(i) Suppose there is some $Z \in (\mathcal{M} \cap \mathcal{A})\backslash\{\mathbb{0}\}$ with $\pi(Z) < 0$. Then, $\mathbb{0} + Z \in \mathcal{A}$ and, thus, $\rho_{\mathcal{A},\mathcal{M},\pi}(\mathbb{0}) \le \pi(Z) < 0$ in contradiction to $\rho_{\mathcal{A},\mathcal{M},\pi}$ being normalized. As a result, since every good deal fulfills $\pi(Z) \le 0$, we obtain $Z \in \ker \pi$ for every good deal $Z \in (\mathcal{M} \cap \mathcal{A})\backslash\{\mathbb{0}\}$.

(ii) By Lemma 2.3.8, $\rho_{\mathcal{A},\mathcal{M},\pi}(\mathbb{0}) = 0$ implies $\mathbb{0} \in \mathrm{bd}_{-U}(\mathcal{A})$ for every $U \in \mathcal{M} \cap \mathcal{X}_+$, and, thus, (2.29) holds, i.e.,

$$\mathcal{A} \cap (-\mathbb{R}_> U) = \emptyset.$$

Because we have $\mathcal{A} \cap \ker \pi = \{\mathbb{0}\}$ by precondition and (2.29) holds, Assumption 3 is fulfilled by Lemma 2.2.19.

□

The following so called *Reduction Lemma* shows that it is sufficient in (FM) to consider the direction given by $U$ according to Assumption 2 for determining the values of $\rho_{\mathcal{A},\mathcal{M},\pi}$ given by (2.30). With this approach, the augmented set $\mathcal{A} + \ker \pi$ has to be considered. We already motivated that relationship between $\rho_{\mathcal{A},\mathcal{M},\pi}$ and $U$ in Remark 2.3.4. The lemma is from [71] under assumption of topological vector spaces, but the proof does not use any topological properties. Thus, we can formulate it here for vector spaces and refer for the proof to the original source.

**Lemma 2.3.11** (Reduction Lemma, see Farkas et al. [71, Lemma 3])**.** *Consider (FM). Let Assumption 2 be fulfilled by $U \in \mathcal{M} \cap \mathcal{X}_+$ and $\rho_{\mathcal{A},\mathcal{M},\pi}\colon \mathcal{X} \to \overline{\mathbb{R}}$ be the functional given by (2.30). Then,*

$$\forall X \in \mathcal{X}: \quad \rho_{\mathcal{A},\mathcal{M},\pi}(X) = \inf\{m \in \mathbb{R} \mid X + mU \in \mathcal{A} + \ker \pi\}.$$

**Remark 2.3.12.** *The idea of the Reduction Lemma 2.3.11 is illustrated in Figure 2.3: For the given $X \in \mathcal{X}$, the vector $Z \in \mathcal{M}$ fulfills*

$$X + Z \in \mathcal{A} \quad \textit{and} \quad \pi(Z) = \rho_{\mathcal{A},\mathcal{M},\pi}(X), \tag{2.38}$$

*i.e., $Z \in \mathcal{E}(X)$ with $\mathcal{E}(X)$ being the set of optimal eligible payoffs given by (2.35). On the other hand, the vector $mU$ with $U \in \mathcal{M} \cap \mathcal{X}_+$ being the eligible payoff according to Assumption 2 differs from $Z$ only by an element of $\ker \pi$, i.e.,*

$$Z - mU \in \ker \pi.$$

*Hence, $mU$ and $Z$ have the same price, i.e.,*

$$\rho_{\mathcal{A},\mathcal{M},\pi}(X) = \pi(Z) = \pi(mU) = m$$

*holds. Moreover, $mU = \rho_{\mathcal{A},\mathcal{M},\pi}(X)U$ fulfills*

$$X + mU = X + \rho_{\mathcal{A},\mathcal{M},\pi}(X)U \in \mathcal{A} + \ker \pi.$$

*By definition of $\rho_{\mathcal{A},\mathcal{M},\pi}$, $mU$ with $m := \rho_{\mathcal{A},\mathcal{M},\pi}(X)$ is a minimal element of $\mathcal{A} + \ker \pi$ in the sense that*

$$\forall \tilde{m} < m : \quad X + \tilde{m}U \notin \mathcal{A} + \ker \pi.$$

*Summing these observations up, we obtain that each optimal eligible payoff $Z \in \mathcal{M}$ (in the sense that $Z$ fulfills (2.38)) can be represented by $\rho_{\mathcal{A},\mathcal{M},\pi}(X)U$ and an element $Z^0 \in \ker \pi$ as*

$$Z = \rho_{\mathcal{A},\mathcal{M},\pi}(X)U + Z^0.$$

*Recall that this can be united with our observation $\pi_m = mU + \ker \pi$ for each $m \in \mathbb{R}$ with $\pi_m \subseteq \mathcal{M}$ given by (2.16), compare also Lemma 2.2.7.*

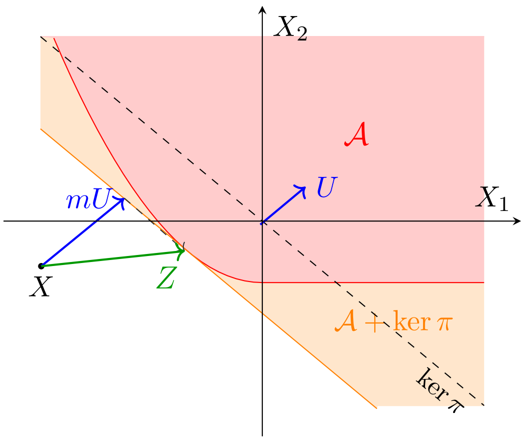


Figure 2.3: Illustration of the Reduction Lemma 2.3.11

**Remark 2.3.13.** *Consider (FM). The Reduction Lemma provides a direct relationship between the functional $\rho_{\mathcal{A},\mathcal{M},\pi} \colon \mathcal{X} \to \overline{\mathbb{R}}$ defined by (2.30) and the nonlinear Gerstewitz-functional $\varphi_{\mathcal{D},K} \colon \mathcal{X} \to \overline{\mathbb{R}}$ given by (see Definition 1.2.5)*

$$\varphi_{\mathcal{D},K}(X) := \inf\{t \in \mathbb{R} \mid X \in tK + \mathcal{D}\} \tag{2.39}$$

*under a given nonempty subset* $\mathcal{D} \subseteq \mathcal{X}$ *and an element* $\mathbb{0} \neq K \in \mathcal{X}$ *such that* $\mathcal{D} - \mathbb{R}_+ K \subseteq \mathcal{D}$*. As mentioned in Remark 2.3.7,* $\rho_{\mathcal{A},\mathcal{M},\pi} \colon \mathcal{X} \to \overline{\mathbb{R}}$ *given by* (2.30) *is a translation invariant functional by Lemma 2.3.5*(ii)*.* $\varphi_{\mathcal{D},K}$ *also has monotonicity and translation invariance properties. Thus, the functional* $\rho_{\mathcal{A},\mathcal{M},\pi}$ *can be seen as a generalization of the nonlinear functional (2.39) with a subspace* $\mathcal{M}$ *of directions* $K$*, and evaluating the shift of the set* $\mathcal{D}$ *by a linear pricing functional* $\pi$*. The Reduction Lemma 2.3.11 shows that, although it is a generalization,* $\rho_{\mathcal{A},\mathcal{M},\pi}$ *can be reduced to a functional of the type* (2.39) *through the payoff* $U \in \mathcal{M} \cap \mathcal{X}_+$ *according to Assumption 2: It holds that*

$$\rho_{\mathcal{A},\mathcal{M},\pi} = \varphi_{\mathcal{A}+\ker \pi, -U}$$

*with* $U \in \mathcal{M} \cap \mathcal{X}_+$ *being a vector according to Assumption 2. Note that* $\mathcal{D} - \mathbb{R}_+ K \subseteq \mathcal{D}$ *is fulfilled with* $\mathcal{D} := \mathcal{A} + \ker \pi$ *and* $K := -U$*, since* $\mathcal{A} + \ker \pi$ *fulfills the monotonicity property in the sense of Definition 2.2.9*(iii)*, see Lemma 2.3.14, and, thus,*

$$\mathcal{A} + \ker \pi + \mathbb{R}_> U \subseteq \mathcal{A} + \ker \pi$$

*holds. Note that* $\mathcal{A} + \ker \pi$ *is not always an acceptance set, see Remark 2.3.18.*

Now, we give a closer look to the set $\mathcal{A} + \ker \pi$, which can be interpreted as the set of capital positions that can be made acceptable by non-positive costs. As already motivated by the Reduction Lemma 2.3.11, it will turn out that the set $\mathcal{A} + \ker \pi$ is most important for properties of the functional $\rho_{\mathcal{A},\mathcal{M},\pi} \colon \mathcal{X} \to \overline{\mathbb{R}}$ given by (2.30) and also the set of optimal eligible payoffs (see Chapter 3). Especially, properties of $\mathcal{A} + \ker \pi$ with respect to the direction $U \in \mathcal{M} \cap \mathcal{X}_+$ from Assumption 2 will be crucial. First, we study which properties of the acceptance set $\mathcal{A} \subseteq \mathcal{X}$ (see Definition 2.2.9) are transferred to the set $\mathcal{A} + \ker \pi$. Obviously, $\mathcal{A} + \ker \pi \neq \emptyset$ always holds, since $\mathbb{0} \in \mathcal{A} + \ker \pi$ by $\mathbb{0} \in \mathcal{A}$ and $\mathbb{0} \in \ker \pi$. The following lemma shows that $\mathcal{A} + \ker \pi$ also fulfills the monotonicity property:

**Lemma 2.3.14** (see Marohn, Tammer [140, Lemma 5.3])**.** *Consider* (FM)*. Then, it holds that*

$$\forall X \in \mathcal{A} + \ker \pi : \quad Y \in \mathcal{A} + \ker \pi \textit{ for each } Y \in \mathcal{X} \textit{ with } Y - X \in \mathcal{X}_+.$$

**Proof.** Take $X, Y \in \mathcal{X}$ arbitrary such that $X \in \mathcal{A} + \ker \pi$ and $Y - X \in \mathcal{X}_+$. Then,

$$\exists X^0 \in \mathcal{A}, \exists Z^0 \in \ker \pi : \quad X^0 = X - Z^0.$$

Since

$$(Y - Z^0) - X^0 = (Y - Z^0) - (X - Z^0) = Y - X \in \mathcal{X}_+,$$

we obtain $Y - Z^0 \in \mathcal{A}$ by $X^0 \in \mathcal{A}$ and monotonicity of $\mathcal{A}$, see Definition 2.2.9(iii). As a result,

$$Y = (Y - Z^0) + Z^0 \in \mathcal{A} + \ker \pi$$

holds, which completes the proof. □

**Remark 2.3.15.** *Because the given $X \in \mathcal{A} + \ker \pi$ does not necessary have to be an element of the acceptance set $\mathcal{A}$, the proof of Lemma 2.3.14 does not trivially follow by monotonicity of $\mathcal{A}$.*

Since we often need it in later proofs, the following corollary applies Lemma 2.3.14 for elements that can be reached in direction of an element $U \in \mathcal{M} \cap \mathcal{X}_+$:

**Corollary 2.3.16** (see Marohn, Tammer [140, Cor. 5.1])**.** *Consider (FM) and $X \in \mathcal{A} + \ker \pi$ arbitrary. Then, it holds that*

$$\forall m \in \mathbb{R}_+, \forall U \in \mathcal{M} \cap \mathcal{X}_+ : \quad X + mU \in \mathcal{A} + \ker \pi. \tag{2.40}$$

*More precisely, we have*

$$\mathcal{A} + \ker \pi + \mathbb{R}_+ U = \mathcal{A} + \ker \pi. \tag{2.41}$$

**Proof.** For $U \in \mathcal{M} \cap \mathcal{X}_+$ and $m \in \mathbb{R}_+$ arbitrary, assertion (2.40) follow directly from Lemma 2.3.14 by

$$(X + mU) - X = mU \in \mathcal{X}_+,$$

and (2.41) follows from $\mathcal{A} + \mathbb{R}_+ U = \mathcal{A}$, see (2.18). □

Corollary 2.3.16 allows us to give the following easy characterization for the risk measure $\rho_{\mathcal{A},\mathcal{M},\pi} \colon \mathcal{X} \to \overline{\mathbb{R}}$ given by (2.30) to attain a value larger than $-\infty$:

**Corollary 2.3.17.** *Consider (FM) and the functional $\rho_{\mathcal{A},\mathcal{M},\pi} \colon \mathcal{X} \to \overline{\mathbb{R}}$ defined by (2.30). Take $X \in \mathcal{X}$ arbitrary. Then, it holds that*

$$\rho_{\mathcal{A},\mathcal{M},\pi}(X) > -\infty \quad \iff \quad \exists m \in \mathbb{R} : \; X + mU \notin \mathcal{A} + \ker \pi.$$

**Proof.** By the Reduction Lemma 2.3.11, it holds for $X \in \mathcal{X}$ arbitrary that

$$\rho_{\mathcal{A},\mathcal{M},\pi}(X) > -\infty \quad \iff \quad \exists \tilde{m} \in \mathbb{R} \cup +\infty : \; X + mU \notin \mathcal{A} + \ker \pi \text{ for all } m < \tilde{m}.$$

Thus, we only need to show ($\Leftarrow$). Let $m \in \mathbb{R}$ with $X + mU \notin \mathcal{A} + \ker \pi$. Then, the following implication holds by Corollary 2.3.16:

$$\exists \tilde{m} \in \mathbb{R}_> : \; X + (m - \tilde{m})U \in \mathcal{A} + \ker \pi \quad \Longrightarrow \quad X + mU \in \mathcal{A} + \ker \pi.$$

Since $X + mU \notin \mathcal{A} + \ker \pi$ holds, we obtain

$$\forall \tilde{m} \in \mathbb{R}_+ : \quad X + (m - \tilde{m})U \notin \mathcal{A} + \ker \pi.$$

By the Reduction Lemma 2.3.11, we obtain $\rho_{\mathcal{A},\mathcal{M},\pi}(X) > -\infty$, showing ($\Leftarrow$). That completes the proof. □

**Remark 2.3.18.** *Consider* (FM). *Because of* $\mathbb{0} \in \mathcal{A} + \ker \pi$ *and Lemma 2.3.14,* $\mathcal{A} + \ker \pi$ *is an acceptance set in the sense of Definition 2.2.9, too, if it is also proper. The following easy Example 2.3.19 shows that* $\mathcal{A} + \ker \pi$ *is not generally proper although* $\mathcal{A}$ *is proper. Nevertheless, Corollary 2.3.17 implies that this case is not from interest because it holds that*

$$\mathcal{A} + \ker \pi \text{ is proper, i.e., an acceptance set} \quad \Longleftrightarrow \quad \exists X \in \mathcal{X} : \ \rho_{\mathcal{A},\mathcal{M},\pi}(X) > -\infty \tag{2.42}$$

*with* $\rho_{\mathcal{A},\mathcal{M},\pi} \colon \mathcal{X} \to \overline{\mathbb{R}}$ *given by* (2.30). *This is easy to see because, for each* $X \in \mathcal{X}$,

$$X \notin \mathcal{A} + \ker \pi \quad \Longrightarrow \quad \rho_{\mathcal{A},\mathcal{M},\pi}(X) > -\infty$$

*holds by Corollary 2.3.17.*

**Example 2.3.19.** Let $\mathcal{X} = \mathcal{M} = \mathbb{R}^2$ and $\pi \colon \mathcal{M} \to \mathbb{R}$ with $\pi(Z) := Z_1$ for $Z = (Z_1, Z_2) \in \mathcal{M}$. Then, we obtain $\ker \pi = \{\mathbb{0}\} \times \mathbb{R}$. Consider $\mathcal{A} := \mathbb{R} \times \mathbb{R}_+ \subseteq \mathcal{X}$, which is obviously an acceptance set in the sense of Definition 2.2.9 and, especially, a proper subset of $\mathcal{X}$. Clearly, $\mathcal{A} + \ker \pi = \mathbb{R}^2$ fulfills $\mathbb{0} \in \mathcal{A} + \ker \pi$ and the monotonicity property (see Definition 2.2.9(iii)), but $\mathcal{A} + \ker \pi$ is no proper subset of $\mathbb{R}^2$ and, thus, no acceptance set. Obviously, $\rho_{\mathcal{A},\mathcal{M},\pi} \equiv -\infty$ holds for $\rho_{\mathcal{A},\mathcal{M},\pi} \colon \mathcal{X} \to \overline{\mathbb{R}}$ given by (2.30). ◊

The following useful result for (FM) shows that properties of the acceptance set $\mathcal{A} \subseteq \mathcal{X}$ like convexity transfer to the augmented set $\mathcal{A} + \ker \pi$. We give a direct proof although the assertion is clear, since the sum of two convex sets, cones and sets that are closed under addition, respectively, has the same property (and $\ker \pi$ as a subspace of $\mathcal{X}$ has all of these properties):

**Lemma 2.3.20.** *Consider* (FM). *Then, the following properties hold:*

(i) $\mathcal{A}$ *is convex* $\Longrightarrow$ $\mathcal{A} + \ker \pi$ *is convex,*

(ii) $\mathcal{A}$ *is a cone* $\Longrightarrow$ $\mathcal{A} + \ker \pi$ *is a cone,*

(iii) $\mathcal{A}$ *is closed under addition* $\Longrightarrow$ $\mathcal{A} + \ker \pi$ *is closed under addition.*

**Proof.** Let $X, Y \in \mathcal{A} + \ker \pi$ arbitrary. Then, there are $X^0, Y^0 \in \mathcal{A}$ and $Z^X, Z^Y \in \ker \pi$ with $X = X^0 + Z^X$ and $Y = Y^0 + Z^Y$.

(i) For $\lambda \in [0, 1]$ arbitrary, we obtain

$$\lambda X + (1-\lambda)Y = \underbrace{\lambda X^0 + (1-\lambda)Y^0}_{\in \mathcal{A}} + \underbrace{\lambda Z^X + (1-\lambda)Z^Y}_{\in \ker \pi} \in \mathcal{A} + \ker \pi$$

by convexity of $\mathcal{A}$ and convexity of $\ker \pi$ (note that $\lambda Z^X + (1-\lambda)Z^Y \in \mathcal{M}$, since $\mathcal{M}$ is a vector space, and $\pi(\lambda Z^X + (1-\lambda)Z^Y) = 0$ by linearity of $\pi$). Hence, $\mathcal{A} + \ker \pi$ is convex.

(ii) For $\lambda \in \mathbb{R}_+$ arbitrary, it holds that

$$\lambda X = \underbrace{\lambda X^0}_{\in \mathcal{A}} + \underbrace{\lambda Z^X}_{\in \ker \pi} \in \mathcal{A} + \ker \pi$$

because $\mathcal{A}$ is a cone and, obviously, since $\mathcal{M}$ is a vector space and $\pi$ linear, $\lambda Z^X \in \mathcal{M}$ with $\pi(\lambda Z^X) = 0$.

(iii) It holds that

$$X + Y = \underbrace{X^0 + Y^0}_{\in \mathcal{A}} + \underbrace{Z^X + Z^Y}_{\in \ker \pi} \in \mathcal{A} + \ker \pi$$

because $\mathcal{A}$ is closed under addition and, obviously, $Z^X + Z^Y \in \mathcal{M}$ with $\pi(Z^X + Z^Y) = 0$ by linearity of $\pi$.

□

Now, we study the recession cone of $\mathcal{A}$ and $\mathcal{A} + \ker \pi$, respectively.

**Lemma 2.3.21** (see Marohn, Tammer [140, Lemma 5.4])**.** *Consider* (FM)*. For all* $V \in \mathcal{X}$*, the following holds:*

$$V \in \operatorname{rec} \mathcal{A} \qquad \Longrightarrow \qquad V \in \operatorname{rec}(\mathcal{A} + \ker \pi).$$

**Proof.** Take $V \in \operatorname{rec} \mathcal{A}$ and $X \in \mathcal{A}$ arbitrary. Then, there are $X^0 \in \mathcal{A}$ and $Z^0 \in \ker \pi$ with $X = X^0 + Z^0$, which implies

$$\forall \lambda \in \mathbb{R}_+ : \quad X + \lambda V = (\underbrace{X^0 + \lambda V}_{\in \mathcal{A}}) + Z^0 \in \mathcal{A} + \ker \pi$$

because of $V \in \operatorname{rec} \mathcal{A}$. That shows $V \in \operatorname{rec}(\mathcal{A} + \ker \pi)$. □

In Remark 2.2.10, we noticed $U \in \operatorname{rec} \mathcal{A}$ for every $U \in \mathcal{M} \cap \mathcal{X}_+$. Some properties of the risk measure $\rho_{\mathcal{A},\mathcal{M},\pi} \colon \mathcal{X} \to \overline{\mathbb{R}}$ given by (2.30) will depend on if $-U \in \operatorname{rec} \mathcal{A}$ and $-U \in \operatorname{rec}(\mathcal{A} + \ker \pi)$ holds, respectively, for $U \in \mathcal{M} \cap \mathcal{X}_+$ according to Assumption 2. We collect the previous results with respect to $U$ for convenient use in the proofs of our main results in Theorem 2.3.25 in the following corollary:

**Corollary 2.3.22** (see Marohn, Tammer [140, Cor. 5.2])**.** *Consider* (FM)*. It holds that*

$$\forall U \in \mathcal{M} \cap \mathcal{X}_+ : \qquad U \in \operatorname{rec} \mathcal{A} \quad \text{and} \quad U \in \operatorname{rec}(\mathcal{A} + \ker \pi).$$

*Furthermore, it holds that*

$$\forall U \in \mathcal{M} \cap \mathcal{X}_+ : \qquad -U \in \operatorname{rec} \mathcal{A} \quad \Longrightarrow \quad -U \in \operatorname{rec}(\mathcal{A} + \ker \pi).$$

**Proof.** We have shown $U \in \operatorname{rec} \mathcal{A}$ for $U \in \mathcal{M} \cap \mathcal{X}_+$ arbitrary in Remark 2.2.10. The remaining assertions follow from Lemma 2.3.21. □

**Remark 2.3.23.** *In Corollary 2.3.22, the converse direction does not generally hold, i.e., for* $U \in \mathcal{M} \cap \mathcal{X}_+$*,*

$$-U \in \operatorname{rec}(\mathcal{A} + \ker \pi) \quad \not\Longrightarrow \quad -U \in \operatorname{rec} \mathcal{A}$$

*although* $U$ *and* $-U$ *are no elements of* $\ker \pi$*. We refer for illustration to the following example.*

**Example 2.3.24** (see [140, Expl. 5.1]). Let $\mathcal{X} = \mathcal{M} = \mathbb{R}^2$ and consider $\pi\colon \mathcal{M} \to \mathbb{R}$ defined by

$$\pi(Z) = \pi(Z_1, Z_2) := \frac{1}{2}(Z_1 + Z_2),$$

i.e., $\ker \pi = \{Z \in \mathbb{R}^2 \mid Z_2 = -Z_1\}$. Let $U = (1,1)^T$ and $\mathcal{A} \subseteq \mathbb{R}^2$ with

$$\mathcal{A} = \{X \in \mathbb{R}^2 \mid X_1 \geq 0\}.$$

Obviously, $\mathcal{A}$ is an acceptance set with $\mathcal{A} + \ker \pi = \mathbb{R}^2$ and, thus, $-U \in \operatorname{rec}(\mathcal{A} + \ker \pi)$, while $-U \notin \operatorname{rec} \mathcal{A}$, since $-U = \mathbb{0} - U \notin \mathcal{A}$ with $\mathbb{0} \in \mathcal{A}$. ◊

Now, we are able to study the risk measure $\rho_{\mathcal{A},\mathcal{M},\pi}\colon \mathcal{X} \to \overline{\mathbb{R}}$ given by (2.30) in more detail. Corresponding results for general translation invariant functionals $\varphi_{\mathcal{D},K}\colon \mathcal{X} \to \overline{\mathbb{R}}$ given by (2.39) can be found in [185, Ch. 4]. We provide new assertions by focusing on the specific financial market (FM), which leads to new proofs and insights in the needed requirements. First, we want to present one of our main results that gives detailed information about the epigraph, (strict) sublevel sets, and level lines of $\rho_{\mathcal{A},\mathcal{M},\pi}$, which we will use to derive more results, e.g., concerning the domain of $\rho_{\mathcal{A},\mathcal{M},\pi}$. The following theorem (compare also [185, Prop. 4.2.1]) generalizes results from Baes et al. in [17, Lemma 2.12] for closed acceptance sets $\mathcal{A} \subseteq \mathcal{X}$ in locally convex Hausdorff spaces (compare also [71])).

**Theorem 2.3.25** (see Marohn, Tammer [140, Theorem 5.1]). *Consider* (FM). *Let Assumption 2 be fulfilled by the eligible payoff $U \in \mathcal{M} \cap \mathcal{X}_+$ and $\rho_{\mathcal{A},\mathcal{M},\pi}\colon \mathcal{X} \to \overline{\mathbb{R}}$ be the functional introduced in* (2.30). *Then, for $m \in \mathbb{R}$ arbitrary, the following conditions hold:*

(i) $\operatorname{lev}_{\rho_{\mathcal{A},\mathcal{M},\pi},<}(m) = \operatorname{int}_{-U}(\mathcal{A} + \ker \pi) - mU = \mathcal{A} + \ker \pi + \mathbb{R}_{>}U - mU \subseteq \mathcal{A} + \ker \pi - mU,$

(ii) $\operatorname{lev}_{\rho_{\mathcal{A},\mathcal{M},\pi},\leq}(m) = \operatorname{cl}_{-U}(\mathcal{A} + \ker \pi) - mU,$

(iii) $\operatorname{lev}_{\rho_{\mathcal{A},\mathcal{M},\pi},=}(m) = \operatorname{bd}_{-U}(\mathcal{A} + \ker \pi) - mU.$

*Furthermore, it holds that*

$$\operatorname{epi} \rho_{\mathcal{A},\mathcal{M},\pi} = \{(X, m) \in \mathcal{X} \times \mathbb{R} \mid X \in \operatorname{cl}_{-U}(\mathcal{A} + \ker \pi) - mU\}.$$

**Proof.**

(i) First, we note that $U \in \operatorname{rec} \mathcal{A}$ by Corollary 2.3.22. Thus, we obtain by Lemma 1.1.29 (ii)

$$\operatorname{int}_{-U}(\mathcal{A} + \ker \pi) = \mathcal{A} + \ker \pi + \mathbb{R}_{>}U,$$

which implies the second equation in (i), i.e.,

$$\operatorname{int}_{-U}(\mathcal{A} + \ker \pi) - mU = \mathcal{A} + \ker \pi + \mathbb{R}_{>}U - mU. \tag{2.43}$$

Moreover,

$$\mathcal{A} + \ker \pi + \mathbb{R}_{>}U - mU \subseteq \mathcal{A} + \ker \pi - mU$$

holds because of $\mathcal{A} + \mathbb{R}_{>}U \subseteq \mathcal{A}$ by $U \in \operatorname{rec}\mathcal{A}$, showing the last relation in (i). Applying the Reduction Lemma 2.3.11 delivers

$$\begin{aligned}
\operatorname{lev}_{\rho_{\mathcal{A},\mathcal{M},\pi},<}(m) &= \{X \in \mathcal{X} \mid \rho_{\mathcal{A},\mathcal{M},\pi}(X) < m\} \\
&= \{X \in \mathcal{X} \mid \exists t \in \mathbb{R}_{>} : X + (m-t)U \in \mathcal{A} + \ker\pi\} \\
&= \{X \in \mathcal{X} \mid \exists t \in \mathbb{R}_{>} : X \in \mathcal{A} + \ker\pi - mU + tU\} \\
&= \mathcal{A} + \ker\pi + \mathbb{R}_{>}U - mU.
\end{aligned}$$

By (2.43), that shows

$$\operatorname{lev}_{\rho_{\mathcal{A},\mathcal{M},\pi},<}(m) = \operatorname{int}_{-U}(\mathcal{A} + \ker\pi) - mU,$$

completing the proof of (i).

(ii) First, we show

$$\operatorname{cl}_{-U}(\mathcal{A} + \ker\pi) + \mathbb{R}_{+}U - mU = \operatorname{cl}_{-U}(\mathcal{A} + \ker\pi) - mU. \tag{2.44}$$

Lemma 1.1.28(iii) delivers

$$\operatorname{cl}_{-U}(\mathcal{A} + \ker\pi) + \mathbb{R}_{+}U = \operatorname{cl}_{-U}(\mathcal{A} + \ker\pi + \mathbb{R}_{+}U).$$

We have $\mathcal{A} + \ker\pi + \mathbb{R}_{+}U = \mathcal{A} + \ker\pi$ by Corollary 2.3.16. Hence, we have

$$\operatorname{cl}_{-U}(\mathcal{A} + \ker\pi) + \mathbb{R}_{+}U = \operatorname{cl}_{-U}(\mathcal{A} + \ker\pi),$$

which implies (2.44). It is left to prove

$$\operatorname{lev}_{\rho_{\mathcal{A},\mathcal{M},\pi},\leq}(m) = \operatorname{cl}_{-U}(\mathcal{A} + \ker\pi) - mU. \tag{2.45}$$

We show ($\subseteq$) in (2.45), first: consider $X \in \mathcal{X}$ with $\rho_{\mathcal{A},\mathcal{M},\pi}(X) = m$. Because $\rho_{\mathcal{A},\mathcal{M},\pi}$ is defined in (2.30) as an infimum, we obtain by the Reduction Lemma 2.3.11 that there is a sequence $(m_n) \subseteq \mathbb{R}$ with $m_n \downarrow m$ for $n \to +\infty$ such that

$$X \in \mathcal{A} + \ker\pi - m_nU = \mathcal{A} + \ker\pi - (m_n - m)U - mU$$

holds, i.e.,

$$X + mU - (m_n - m)(-U) \in \mathcal{A} + \ker\pi.$$

That implies $X + mU \in \operatorname{cl}_{-U}(\mathcal{A} + \ker\pi)$ because of $(m_n - m) \downarrow 0$, see Lemma 1.1.27, which shows

$$\operatorname{lev}_{\rho_{\mathcal{A},\mathcal{M},\pi},=}(m) \subseteq \operatorname{cl}_{-U}(\mathcal{A} + \ker\pi) - mU. \tag{2.46}$$

On the other hand, we obtain by (i)

$$\operatorname{lev}_{\rho_{\mathcal{A},\mathcal{M},\pi},<}(m) = \mathcal{A} + \ker\pi + \mathbb{R}_{>}U - mU.$$

Hence, by $\mathcal{A} + \ker \pi \subseteq \operatorname{cl}_{-U}(\mathcal{A} + \ker \pi)$ because of Lemma 1.1.28(i) and $\mathbb{R}_{>}U \subseteq \mathbb{R}_{+}U$,

$$\operatorname{lev}_{\rho_{\mathcal{A},\mathcal{M},\pi},<}(m) \subseteq \operatorname{cl}_{-U}(\mathcal{A} + \ker \pi) + \mathbb{R}_{+}U - mU,$$

which is by (2.44) equivalent to

$$\operatorname{lev}_{\rho_{\mathcal{A},\mathcal{M},\pi},<}(m) \subseteq \operatorname{cl}_{-U}(\mathcal{A} + \ker \pi) - mU. \tag{2.47}$$

As a result, (2.46) and (2.47) imply together

$$\operatorname{lev}_{\rho_{\mathcal{A},\mathcal{M},\pi},\leq}(m) \subseteq \operatorname{cl}_{-U}(\mathcal{A} + \ker \pi) - mU,$$

showing ($\subseteq$) in (2.45).

It is left to prove ($\supseteq$) in (2.45): We obtain by Lemma 1.1.28(iv)

$$\operatorname{cl}_{-U}(\mathcal{A} + \ker \pi) + \mathbb{R}_{>}U = \operatorname{int}_{-U}(\mathcal{A} + \ker \pi + \mathbb{R}_{+}U),$$

which is by $\mathcal{A} + \mathbb{R}_{+}U = \mathcal{A}$ given by (2.18) equivalent to

$$\operatorname{cl}_{-U}(\mathcal{A} + \ker \pi) + \mathbb{R}_{>}U = \operatorname{int}_{-U}(\mathcal{A} + \ker \pi).$$

Therefore, we have by (i)

$$\operatorname{cl}_{-U}(\mathcal{A} + \ker \pi) + \mathbb{R}_{>}U - mU = \operatorname{lev}_{\rho_{\mathcal{A},\mathcal{M},\pi},<}(m) \subseteq \operatorname{lev}_{\rho_{\mathcal{A},\mathcal{M},\pi},\leq}(m),$$

showing ($\supseteq$) in (2.45) and completing the proof of (2.45).

(iii) By use of the proved results (i) and (ii), we obtain

$$\begin{aligned}\operatorname{lev}_{\rho_{\mathcal{A},\mathcal{M},\pi},=}(m) &= \operatorname{lev}_{\rho_{\mathcal{A},\mathcal{M},\pi},\leq}(m) \setminus \operatorname{lev}_{\rho_{\mathcal{A},\mathcal{M},\pi},<}(m)\\ &= (\operatorname{cl}_{-U}(\mathcal{A} + \ker \pi) - mU) \setminus (\operatorname{int}_{-U}(\mathcal{A} + \ker \pi) - mU)\\ &= (\operatorname{cl}_{-U}(\mathcal{A} + \ker \pi) \setminus \operatorname{int}_{-U}(\mathcal{A} + \ker \pi)) - mU\\ &= \operatorname{bd}_{-U}(\mathcal{A} + \ker \pi) - mU,\end{aligned}$$

showing (iii).

Theorem 2.3.25(ii) also directly provides the description of epi $\rho_{\mathcal{A},\mathcal{M},\pi}$. □

The following corollary is derived by [185, Prop. 4.2.1].

**Corollary 2.3.26** (see Marohn, Tammer [140, Cor. 5.3])**.** *Consider* (FM)*. Let Assumption 2 be fulfilled by the eligible payoff $U \in \mathcal{M} \cap \mathcal{X}_{+}$ and $\rho_{\mathcal{A},\mathcal{M},\pi} \colon \mathcal{X} \to \overline{\mathbb{R}}$ be the functional given by* (2.30)*. Then, for $m \in \mathbb{R}$ arbitrary, the following holds:*

(i) $\operatorname{lev}_{\rho_{\mathcal{A},\mathcal{M},\pi},\leq}(m) \supseteq \mathcal{A} + \ker \pi - mU$*,*

(ii) $\operatorname{lev}_{\rho_{\mathcal{A},\mathcal{M},\pi},\leq}(m) = \mathcal{A} + \ker \pi - mU \iff \mathcal{A} + \ker \pi$ *is* $(-U)$*-directionally closed.*

**Proof.**

(i) By Lemma 1.1.28(i),

$$\mathcal{A} + \ker \pi - mU \subseteq \operatorname{cl}_{-U}(\mathcal{A} + \ker \pi) - mU$$

holds. Hence, Theorem 2.3.25(ii) implies the assertion.

(ii) The assertion is a consequence of [185, Prop. 4.2.1(b)].

□

Baes et al. studied in [17, Lemma 2.12] (compare also Farkas et al. [71]) the sets $\operatorname{lev}_{\rho_{\mathcal{A},\mathcal{M},\pi},<}(m)$, $\operatorname{lev}_{\rho_{\mathcal{A},\mathcal{M},\pi},\leq}(m)$ and $\operatorname{lev}_{\rho_{\mathcal{A},\mathcal{M},\pi},=}(m)$ for the special case $m = 0$ under supposing $\rho_{\mathcal{A},\mathcal{M},\pi} \colon \mathcal{X} \to \overline{\mathbb{R}}$ given by (2.30) being continuous and finite on $\mathcal{X}$, and $\mathcal{A}$ being a closed acceptance set. They proved the following result:

**Lemma 2.3.27** (see Baes et al. [17, Lemma 2.12])**.** *Consider* (FM). *Let $\mathcal{X}$ be a locally convex Hausdorff space fulfilling the first axiom of countability and $\mathcal{A} \subseteq \mathcal{X}$ be a closed acceptance set. Moreover, let Assumption 2 be fulfilled and $\rho_{\mathcal{A},\mathcal{M},\pi} \colon \mathcal{X} \to \overline{\mathbb{R}}$ be the functional given by* (2.30)*. Assume that $\rho_{\mathcal{A},\mathcal{M},\pi}$ is continuous and finite on $\mathcal{X}$. Then, the following hold:*

(i) $\operatorname{lev}_{\rho_{\mathcal{A},\mathcal{M},\pi},<}(0) = \operatorname{int}(\mathcal{A} + \ker \pi)$,

(ii) $\operatorname{lev}_{\rho_{\mathcal{A},\mathcal{M},\pi},\leq}(0) = \operatorname{cl}(\mathcal{A} + \ker \pi)$,

(iii) $\operatorname{lev}_{\rho_{\mathcal{A},\mathcal{M},\pi},=}(0) = \operatorname{bd}(\mathcal{A} + \ker \pi)$.

Since the properties in Lemma 2.3.27 look very similar to our results for $m = 0$ in Theorem 2.3.25, where we derived $\operatorname{int}_{-U}(\mathcal{A}+\ker \pi)$, $\operatorname{cl}_{-U}(\mathcal{A}+\ker \pi)$ and $\operatorname{bd}_{-U}(\mathcal{A}+\ker \pi)$ for the (strict) sublevel set and level line, it is clear to ask how these results can be united. The following theorem gives an answer that is, for convenient use of sequences, formulated for normed vector spaces instead of the weaker assumption of locally convex Hausdorff spaces:

**Theorem 2.3.28** (see Marohn, Tammer [140, Theorem 5.2])**.** *Consider* (FM)*. Let $(\mathcal{X}, \|\cdot\|)$ be a normed real vector space, Assumption 2 be fulfilled by the eligible payoff $U \in \mathcal{M} \cap \mathcal{X}_+$ and $\rho_{\mathcal{A},\mathcal{M},\pi} \colon \mathcal{X} \to \overline{\mathbb{R}}$ be the functional given by* (2.30)*. Furthermore, we assume that one of the following conditions hold:*

(a) *The functional $\rho_{\mathcal{A},\mathcal{M},\pi}$ is continuous on $\mathcal{X}$,*

(b) *The set $\mathcal{A} + \ker \pi$ fulfills*

$$\mathcal{A} + \ker \pi + \mathbb{R}_{>} U \subseteq \operatorname{int}(\mathcal{A} + \ker \pi) \tag{2.48}$$

*and*

$$\operatorname{cl}(\mathcal{A} + \ker \pi) + \mathbb{R}_{>} U \subseteq \mathcal{A} + \ker \pi. \tag{2.49}$$

*Then, the following properties are satisfied:*

(i) $\operatorname{int}_{-U}(\mathcal{A} + \ker \pi) = \operatorname{int}(\mathcal{A} + \ker \pi)$,

(ii) $\operatorname{cl}_{-U}(\mathcal{A} + \ker \pi) = \operatorname{cl}(\mathcal{A} + \ker \pi)$,

(iii) $\operatorname{bd}_{-U}(\mathcal{A} + \ker \pi) = \operatorname{bd}(\mathcal{A} + \ker \pi)$.

**Proof.** First, we assume that condition (a) is fulfilled.

(i) By Lemma 1.1.30(ii), it holds that

$$\operatorname{int}_{-U}(\mathcal{A} + \ker \pi) \supseteq \operatorname{int}(\mathcal{A} + \ker \pi).$$

Hence, we only need to show

$$\operatorname{int}_{-U}(\mathcal{A} + \ker \pi) \subseteq \operatorname{int}(\mathcal{A} + \ker \pi). \tag{2.50}$$

Assume that (2.50) does not hold and take $X \in \operatorname{int}_{-U}(\mathcal{A}+\ker \pi)$ with $X \notin \operatorname{int}(\mathcal{A} + \ker \pi)$. Then, by definition of $\operatorname{int}_{-U}(\mathcal{A} + \ker \pi)$, it holds that

$$(X - \mathbb{R}_{>}U) \cap (\mathcal{A} + \ker \pi) \neq \emptyset.$$

Hence, we obtain $\rho_{\mathcal{A},\mathcal{M},\pi}(X) < 0$ by the Reduction Lemma 2.3.11. Since we assume $X \notin \operatorname{int}(\mathcal{A} + \ker \pi)$, we can find a sequence $(X_n)_{n\in\mathbb{N}} \subseteq \mathcal{X}$ with

$$\forall n \in \mathbb{N}: \quad X_n \in \mathcal{B}_{\frac{1}{n}}(X) \text{ and } X_n \notin \mathcal{A} + \ker \pi \tag{2.51}$$

where $\mathcal{B}_{\frac{1}{n}}(X) = \left\{Y \in \mathcal{X} \mid \|Y - X\| < \frac{1}{n}\right\}$ is the open ball with center $X$ and radius $\frac{1}{n}$, see Remark 1.1.47. The sequence fulfills $X_n \to X$ for $n \to +\infty$, since $\|X_n - X\| \to 0$ for $n \to +\infty$, and, by continuity of $\rho_{\mathcal{A},\mathcal{M},\pi}$ from (a),

$$\rho_{\mathcal{A},\mathcal{M},\pi}(X_n) \to \rho_{\mathcal{A},\mathcal{M},\pi}(X) < 0 \text{ for } n \to +\infty.$$

Because $(X_n)_{n\in\mathbb{N}}$ converges to $X$, it holds that

$$\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}: \quad |\rho_{\mathcal{A},\mathcal{M},\pi}(X_n) - \rho_{\mathcal{A},\mathcal{M},\pi}(X)| < \epsilon \text{ for all } n > N(\epsilon).$$

We can choose $\epsilon = \left|\frac{\rho_{\mathcal{A},\mathcal{M},\pi}(X)}{2}\right|$ and obtain $\rho_{\mathcal{A},\mathcal{M},\pi}(X_n) < 0$ for each $n > N(\epsilon)$. Hence,

$$X_n \in \operatorname{int}_{-U}(\mathcal{A} + \ker \pi) \subseteq \mathcal{A} + \ker \pi \text{ for all } n > N(\epsilon)$$

by Theorem 2.3.25(i) and Lemma 1.1.30(ii), which contradicts the definition of $(X_n)_{n\in\mathbb{N}}$ in (2.51). Consequently, there is no such sequence and, thus,

$$\exists n_0 \in \mathbb{N}: \quad \mathcal{B}_{\frac{1}{n_0}}(X) \subseteq \mathcal{A} + \ker \pi,$$

i.e., $X \in \operatorname{int}(\mathcal{A} + \ker \pi)$, which is a contradiction. Hence, (2.50) must hold and, therefore, the proof of (i) is completed.

(ii) We only need to show

$$\operatorname{cl}_{-U}(\mathcal{A}+\ker\pi) \supseteq \operatorname{cl}(\mathcal{A}+\ker\pi) \tag{2.52}$$

because the relation ($\subseteq$) follows by Lemma 1.1.30(i). Take $X \in \operatorname{cl}(\mathcal{A}+\ker\pi)$ arbitrary. Then, by fundamental properties of the closure,

$$\exists (X_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}+\ker\pi : \quad X_n \to X \text{ for } n \to +\infty.$$

Since $X_n \in \mathcal{A}+\ker\pi$ for all $n \in \mathbb{N}$, we obtain $\rho_{\mathcal{A},\mathcal{M},\pi}(X_n) \leq 0$ for each $n \in \mathbb{N}$ by the Reduction Lemma 2.3.11, which implies $\rho_{\mathcal{A},\mathcal{M},\pi}(X) \leq 0$ by continuity of $\rho_{\mathcal{A},\mathcal{M},\pi}$ from (a). As a result, $X \in \operatorname{cl}_{-U}(\mathcal{A}+\ker\pi)$ must hold by Theorem 2.3.25(ii), showing (2.52) and completing the proof of (ii).

(iii) From the proved properties (i) and (ii), we obtain the assertion by

$$\begin{aligned}\operatorname{bd}_{-U}(\mathcal{A}+\ker\pi) &= \operatorname{cl}_{-U}(\mathcal{A}+\ker\pi)\backslash\operatorname{int}_{-U}(\mathcal{A}+\ker\pi)\\ &= \operatorname{cl}(\mathcal{A}+\ker\pi)\backslash\operatorname{int}(\mathcal{A}+\ker\pi)\\ &= \operatorname{bd}(\mathcal{A}+\ker\pi).\end{aligned}$$

Now, we prove the properties (i), (ii), and (iii) under the assumption that (b) holds.

(i) As under assumption of (a), we only have to prove (2.50). Take $X \in \operatorname{int}_{-U}(\mathcal{A}+\ker\pi)$ arbitrary. By definition of $\operatorname{int}_{-U}(\mathcal{A}+\ker\pi)$,

$$\exists m \in \mathbb{R}_{>} : \quad X - mU \in \mathcal{A}+\ker\pi.$$

Consequently,

$$X \in \mathcal{A}+\ker\pi+\mathbb{R}_{>}U \subseteq \operatorname{int}(\mathcal{A}+\ker\pi)$$

holds by (2.48), showing property (i).

(ii) Also, as under assumption of (a), we only need to prove (2.52), but that is a direct implication of (2.49) through Lemma 1.1.29(i).

(iii) The proof is identical to the one under assumption of (a).

□

**Remark 2.3.29.** *Theorem 2.3.28 shows that the results in [17, Lemma 2.12] occur by Theorem 2.3.25 if we assume that $\rho_{\mathcal{A},\mathcal{M},\pi}\colon \mathcal{X} \to \overline{\mathbb{R}}$ given by (2.30) is continuous in $\mathcal{X}$. Note that the assumption of $\mathcal{A} \subseteq \mathcal{X}$ being a closed acceptance set as in [17] is not necessary. For the conditions in Theorem 2.3.28*(b)*, it holds that*

$$\begin{aligned}(2.48) \quad &\Longleftrightarrow \quad \operatorname{int}_{-U}(\mathcal{A}+\ker\pi) = \operatorname{int}(\mathcal{A}+\ker\pi),\\ (2.49) \quad &\Longleftrightarrow \quad \operatorname{cl}_{-U}(\mathcal{A}+\ker\pi) = \operatorname{cl}(\mathcal{A}+\ker\pi)\end{aligned}$$

*by [185, Prop. 2.3.54 and 2.3.55] with $U \in \mathcal{M} \cap \mathcal{X}_+$ according to Assumption 2, since $U \in \operatorname{rec}(\mathcal{A} + \ker \pi)$ by Corollary 2.3.22.*

*Under assumption of a closed acceptance set $\mathcal{A}$, non of the properties in Theorem 2.3.28(b) has to be fulfilled. Moreover, although the subspace $\ker \pi$ (as $\pi \colon \mathcal{M} \to \mathbb{R}$ is linear and continuous) is closed even if $\dim \mathcal{X} = +\infty$ and, thus, $\mathcal{A} + \ker \pi$ is a sum of two closed sets, the augmented set $\mathcal{A} + \ker \pi$ is not necessarily closed or $(-U)$-directionally closed.*

**Remark 2.3.30.** *As noticed in Remark 2.3.6, $\rho_{\mathcal{A},\mathcal{M},\pi} \colon \mathcal{X} \to \overline{\mathbb{R}}$ defined as in (2.30) is a monetary risk measure with $\rho_{\mathcal{A},\mathcal{M},\pi}(\mathbb{0}) \le 0$ by $\mathbb{0} \in \mathcal{A}$, see Remark 2.3.2. Indeed, $\rho_{\mathcal{A},\mathcal{M},\pi}(\mathbb{0}) = 0$ if $\rho_{\mathcal{A},\mathcal{M},\pi}$ is normalized, which is given if Assumption 3 is satisfied, see Lemma 2.3.8. Hence, the sublevel set*

$$\mathcal{A}_{\rho_{\mathcal{A},\mathcal{M},\pi}} := \operatorname{lev}_{\rho_{\mathcal{A},\mathcal{M},\pi},\le}(0) = \{X \in \mathcal{X} \mid \rho_{\mathcal{A},\mathcal{M},\pi}(X) \le 0\}$$

*from Lemma 2.3.27(ii) is an acceptance set itself by Lemma 2.2.13 if $\rho_{\mathcal{A},\mathcal{M},\pi}(\mathbb{0}) \neq -\infty$, e.g., if $\rho_{\mathcal{A},\mathcal{M},\pi}$ is normalized. More generally, as mentioned in Remark 2.2.14, it is sufficient to require that there is some arbitrary $X \in \mathcal{X}$ with $\rho_{\mathcal{A},\mathcal{M},\pi}(X) > -\infty$ to secure that $\mathcal{A}_{\rho_{\mathcal{A},\mathcal{M},\pi}}$ is proper. Moreover, $U \in \mathcal{M} \cap \mathcal{X}_+$ from Assumption 2 fulfills $U \in \mathcal{A}_{\rho_{\mathcal{A},\mathcal{M},\pi}}$ and $U \in \operatorname{rec} \mathcal{A}_{\rho_{\mathcal{A},\mathcal{M},\pi}}$, as well, see Remark 2.2.10.*

The relationship between the risk measure $\rho_{\mathcal{A},\mathcal{M},\pi} \colon \mathcal{X} \to \overline{\mathbb{R}}$ introduced in (2.30) and the acceptance set $\mathcal{A}_{\rho_{\mathcal{A},\mathcal{M},\pi}} := \operatorname{lev}_{\rho_{\mathcal{A},\mathcal{M},\pi}}(0) \subseteq \mathcal{X}$ can be described as follows (compare also [185, Th. 4.2.4]):

**Theorem 2.3.31** (see Marohn, Tammer [140, Theorem 5.3])**.** *Consider (FM). Let Assumption 2 be fulfilled by the eligible payoff $U \in \mathcal{M} \cap \mathcal{X}_+$, $\rho_{\mathcal{A},\mathcal{M},\pi} \colon \mathcal{X} \to \overline{\mathbb{R}}$ be the functional given by (2.30) and $\mathcal{A}_{\rho_{\mathcal{A},\mathcal{M},\pi}} := \operatorname{lev}_{\rho_{\mathcal{A},\mathcal{M},\pi},\le}(0) \subseteq \mathcal{X}$. Then, the following holds:*

$$\forall m \in \mathbb{R}: \quad \operatorname{lev}_{\rho_{\mathcal{A},\mathcal{M},\pi},\le}(m) = \mathcal{A}_{\rho_{\mathcal{A},\mathcal{M},\pi}} - mU. \tag{2.53}$$

*Moreover, the set $\mathcal{A}_{\rho_{\mathcal{A},\mathcal{M},\pi}}$ satisfies the following properties:*

(i) *$\mathcal{A}_{\rho_{\mathcal{A},\mathcal{M},\pi}}$ is $(-U)$-directionally closed,*

(ii) *$\forall X \in \mathcal{X}: \ \rho_{\mathcal{A}_{\rho_{\mathcal{A},\mathcal{M},\pi}},\mathcal{M},\pi}(X) = \rho_{\mathcal{A},\mathcal{M},\pi}(X)$.*

**Proof.** Theorem 2.3.25(ii) implies for $m = 0$

$$\mathcal{A}_{\rho_{\mathcal{A},\mathcal{M},\pi}} = \operatorname{cl}_{-U}(\mathcal{A} + \ker \pi) + \mathbb{R}_+ U,$$

which yields, also by Theorem 2.3.25(ii),

$$\forall m \in \mathbb{R}: \quad \mathcal{A}_{\rho_{\mathcal{A},\mathcal{M},\pi}} - mU = \operatorname{cl}_{-U}(\mathcal{A} + \ker \pi) + \mathbb{R}_+ U - mU = \operatorname{lev}_{\rho_{\mathcal{A},\mathcal{M},\pi},\le}(m),$$

i.e., (2.53) is proved. As a result of (2.53), we obtain

$$\mathcal{A}_{\rho_{\mathcal{A},\mathcal{M},\pi}} = \operatorname{lev}_{\rho_{\mathcal{A},\mathcal{M},\pi},\le}(0) = \operatorname{cl}_{-U}(\mathcal{A} + \ker \pi) \tag{2.54}$$

by Theorem 2.3.25(ii) for $m = 0$, which is obviously $(-U)$-directionally closed, showing (i). For proving (ii), we show

$$\mathcal{A}_{\rho_{\mathcal{A},\mathcal{M},\pi}} + \ker \pi = \mathcal{A}_{\rho_{\mathcal{A},\mathcal{M},\pi}}. \tag{2.55}$$

Since $\mathbb{0} \in \ker \pi$, we only need to prove

$$\mathcal{A}_{\rho_{\mathcal{A},\mathcal{M},\pi}} + \ker \pi \subseteq \mathcal{A}_{\rho_{\mathcal{A},\mathcal{M},\pi}}.$$

Take $X \in \mathcal{A}_{\rho_{\mathcal{A},\mathcal{M},\pi}} + \ker \pi$ arbitrary. Then,

$$\exists Z^0 \in \ker \pi : \quad X^0 := X + Z^0 \in \mathcal{A}_{\rho_{\mathcal{A},\mathcal{M},\pi}} = \mathrm{cl}_{-U}(\mathcal{A} + \ker \pi)$$

by (2.54). As a result of $X^0 \in \mathrm{cl}_{-U}(\mathcal{A} + \ker \pi)$ with $U \in \mathrm{rec}\,(\mathcal{A} + \ker \pi)$ (see Corollary 2.3.22),

$$\forall m \in \mathbb{R}_> : \quad X^0 + mU \in \mathcal{A} + \ker \pi$$

holds by Lemma 1.1.29. We obtain

$$\forall m \in \mathbb{R}_> : \quad X^0 + mU - Z^0 \in \mathcal{A} + \ker \pi$$

by $-Z^0 \in \ker \pi$, especially,

$$\forall n \in \mathbb{N} : \quad X^0 + \frac{1}{n}U - Z^0 = X + \frac{1}{n}U \in \mathcal{A} + \ker \pi,$$

which yields $X \in \mathrm{cl}_{-U}(\mathcal{A} + \ker \pi)$ by Lemma 1.1.27, showing $X \in \mathcal{A}_{\rho_{\mathcal{A},\mathcal{M},\pi}}$ and, thus, (2.55) holds. Now, the proved formulas (2.53) and (2.55) imply for all $X \in \mathcal{X}$

$$\begin{aligned}
\rho_{\mathcal{A},\mathcal{M},\pi}(X) &= \inf\{m \in \mathbb{R} \mid X \in \mathrm{lev}_{\rho_{\mathcal{A},\mathcal{M},\pi},\leq}(m)\} \\
&= \inf\{m \in \mathbb{R} \mid X \in \mathcal{A}_{\rho_{\mathcal{A},\mathcal{M},\pi}} - mU\} \\
&= \inf\{m \in \mathbb{R} \mid X + mU \in \mathcal{A}_{\rho_{\mathcal{A},\mathcal{M},\pi}}\} \\
&= \inf\{m \in \mathbb{R} \mid X + mU \in \mathcal{A}_{\rho_{\mathcal{A},\mathcal{M},\pi}} + \ker \pi\} \\
&= \rho_{\mathcal{A}_{\rho_{\mathcal{A},\mathcal{M},\pi}},\mathcal{M},\pi}(X),
\end{aligned}$$

where the last equation occurs by the Reduction Lemma 2.3.11. That shows (ii) and completes the proof of Theorem 2.3.31.

□

By Theorem 2.3.31, we can replace in (FM) any acceptance set $\mathcal{A}$ by the $(-U)$-directionally closed acceptance set $\mathcal{A}_{\rho_{\mathcal{A},\mathcal{M},\pi}} := \mathrm{lev}_{\rho_{\mathcal{A},\mathcal{M},\pi},\leq}(0) = \mathrm{cl}_{-U}(\mathcal{A} + \ker \pi)$ (compare Theorem 2.3.25) without changing the values of the risk measure $\rho_{\mathcal{A},\mathcal{M},\pi}$ given by (2.30). Now, we want to study in which range we can vary the acceptance set $\mathcal{A} \subseteq \mathcal{X}$ to some (not necessary acceptance) set $\mathcal{D} \subseteq \mathcal{X}$ without changing the values of the risk measure $\rho_{\mathcal{A},\mathcal{M},\pi} \colon \mathcal{X} \to \overline{\mathbb{R}}$ given by (2.30). The following lemma gives some condition to characterize these sets $\mathcal{D}$ (compare also [185, Prop. 4.2.1]):

**Lemma 2.3.32** (see Marohn, Tammer [140, Lemma 5.6])**.** *Consider* (FM)*. Let Assumption 2 be fulfilled by the eligible payoff* $U \in \mathcal{M} \cap \mathcal{X}_+$ *and* $\rho_{\mathcal{A},\mathcal{M},\pi} \colon \mathcal{X} \to \overline{\mathbb{R}}$ *the functional given by* (2.30)*. Then,*

$$\rho_{\mathcal{A},\mathcal{M},\pi} = \rho_{\mathcal{D},\mathcal{M},\pi}$$

*holds for every subset* $\mathcal{D} \subseteq \mathcal{X}$ *that fulfills the following condition:*

$$\mathcal{A} + \ker \pi \subseteq \mathcal{D} + \ker \pi \subseteq \operatorname{cl}_{-U}(\mathcal{A} + \ker \pi). \tag{2.56}$$

**Proof.** Consider $\mathcal{D} \subseteq \mathcal{X}$ fulfilling (2.56). Then,

$$\operatorname{cl}_{-U}(\mathcal{A} + \ker \pi) \subseteq \operatorname{cl}_{-U}(\mathcal{D} + \ker \pi).$$

Furthermore, we get

$$\operatorname{cl}_{-U}(\mathcal{D} + \ker \pi) \subseteq \operatorname{cl}_{-U}(\operatorname{cl}_{-U}(\mathcal{A} + \ker \pi)) = \operatorname{cl}_{-U}(\mathcal{A} + \ker \pi)$$

by (2.56) and Lemma 1.1.28(ii). Thus,

$$\operatorname{cl}_{-U}(\mathcal{A} + \ker \pi) = \operatorname{cl}_{-U}(\mathcal{D} + \ker \pi).$$

This yields

$$\operatorname{cl}_{-U}(\mathcal{A} + \ker \pi) - mU = \operatorname{cl}_{-U}(\mathcal{D} + \ker \pi) - mU$$

for all $m \in \mathbb{R}$, i.e., $\operatorname{lev}_{\rho_{\mathcal{A},\mathcal{M},\pi},\leq}(m) = \operatorname{lev}_{\rho_{\mathcal{D},\mathcal{M},\pi},\leq}(m)$ by Theorem 2.3.25(ii). As a result, we obtain $\rho_{\mathcal{A},\mathcal{M},\pi} = \rho_{\mathcal{D},\mathcal{M},\pi}$. □

**Remark 2.3.33.** *Lemma 2.3.32 allows to change the acceptance set* $\mathcal{A} \subseteq \mathcal{X}$ *into another set (for example, acceptance set)* $\mathcal{D} \subseteq \mathcal{X}$ *that might be able to handle, describe or calculate than the original set* $\mathcal{A}$*. On the first sight, it might seem that the range in Lemma 2.3.32 given by* (2.56) *is very small. Although* $\mathcal{D} \subseteq \mathcal{X}$ *in* (2.56) *does not even have to be an acceptance set, the following example, nevertheless, shows that there can be strictly smaller (and more easy) acceptance sets* $\mathcal{D} \subsetneq \mathcal{A} \subseteq \mathcal{X}$ *fulfilling* (2.56)*. Note that* $\mathcal{D} := \operatorname{cl}_{-U}(\mathcal{A} + \ker \pi) = \mathcal{A}_{\rho_{\mathcal{A},\mathcal{M},\pi}}$ *in Lemma 2.3.31 fulfills the condition* (2.56)*.*

**Example 2.3.34.** Let $\mathcal{X} = \mathcal{M} = \mathbb{R}^2$, $\pi \colon \mathcal{M} \to \mathbb{R}$ with $\pi(Z) = \pi(Z_1, Z_2) = \frac{Z_1+Z_2}{2}$ and $U = (1,0)^T$ according to Assumption 2. Consider the acceptance set

$$\mathcal{A} := \left[\left((\mathbb{N} \cup \{0,-1\})(-1,1)^T\right) + \mathbb{R}^2_+\right] \setminus \left[\left((\mathbb{N}(-1,1)^T\right) + ([0,1)U)\right) \cup (1,-1)^T\right],$$

which is not closed (or $(-U)$-directional closed) and illustrated in Figure 2.4. In the definition of $\mathcal{A}$, recall that the product of a subset $\mathcal{B} \subseteq \mathbb{R}$ (like $\mathcal{B} = \mathbb{N}$ or $\mathcal{B} = [0,1)$) with a vector $X \in \mathcal{X}$ is defined by

$$\mathcal{B}X := \{\lambda X \mid \lambda \in \mathcal{B}\},$$

see Definition 1.1.20. Moreover, Assumption 3 is fulfilled, which is equivalent to $\mathcal{A} \cap \ker \pi = \{\mathbb{0}\}$ because of $\mathcal{A} \cap (-\mathbb{R}_{>}U) = \emptyset$ (see Lemma 2.2.19). Obviously,

$$\mathcal{A} + \ker \pi = \operatorname{cl}_{-U}(\mathcal{A} + \ker \pi) = \{X = (X_1, X_2)^T \in \mathbb{R}^2 \mid X_2 \geq -X_1\}$$

holds. Then, $\mathcal{D} := \mathbb{R}^2_+$ fulfills (2.56) and is an (even closed) acceptance set itself. Hence, we can work with $\rho_{\mathcal{D},\mathcal{M},\pi} \colon \mathcal{X} \to \overline{\mathbb{R}}$ defined as in (2.30) instead of considering $\rho_{\mathcal{A},\mathcal{M},\pi}$. Interestingly, even the solution sets coincide for both acceptance sets $\mathcal{A}$ and $\mathcal{D}$, which does not have to be true in general. Of course, if we transform the acceptance set $\mathcal{A}$ into a $(-U)$-directional closed acceptance set by $\tilde{\mathcal{A}}_1 := \mathcal{A} \cup \{(1, -1)^T\}$, or, also, into a closed acceptance set by

$$\tilde{\mathcal{A}}_2 := \left[\left((\mathbb{N} \cup \{0, -1\})(-1, 1)^T\right) + \mathbb{R}^2_+\right],$$

the acceptance set $\mathcal{D}$ fulfills (2.56) for $\tilde{\mathcal{A}}_1$ and $\tilde{\mathcal{A}}_2$ furthermore. Hence, these properties do not effect the statement in Remark 2.3.33. ♢

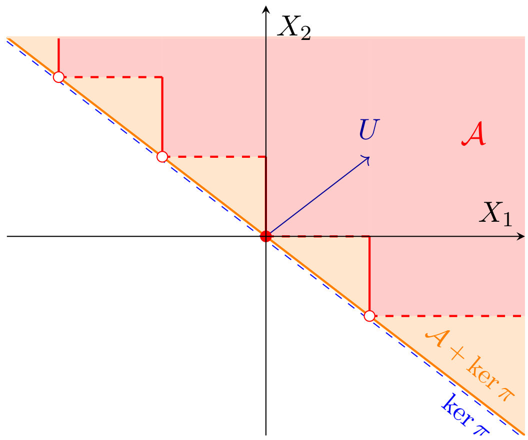


Figure 2.4: Illustration of Example 2.3.34

Characterizations of the finiteness of a functional are a main issue of study, which follows next. By Theorem 2.3.25(iii), we already know that the following holds for $X \in \mathcal{X}$ arbitrary:

$$\rho_{\mathcal{A},\mathcal{M},\pi}(X) \in \mathbb{R} \qquad \Longleftrightarrow \qquad X \in \operatorname{bd}_{-U}(\mathcal{A} + \ker \pi) + \mathbb{R}U. \tag{2.57}$$

**Theorem 2.3.35** (see Marohn, Tammer [140, Theorem 5.4])**.** *Consider* (FM)*. Let Assumption 2 be fulfilled by the eligible payoff $U \in \mathcal{M} \cap \mathcal{X}_+$ and $\rho_{\mathcal{A},\mathcal{M},\pi} \colon \mathcal{X} \to \overline{\mathbb{R}}$ be the functional given by* (2.30)*. Then, the domain of the risk measure $\rho_{\mathcal{A},\mathcal{M},\pi}$ is given by*

$$\operatorname{dom} \rho_{\mathcal{A},\mathcal{M},\pi} = \mathcal{A} + \ker \pi + \mathbb{R}U = \mathcal{A} + \mathcal{M}. \tag{2.58}$$

**Proof.** For $X \in \mathcal{X}$ arbitrary, it holds that

$$\begin{aligned} X \in \operatorname{dom} \rho_{\mathcal{A},\mathcal{M},\pi} \quad &\Longleftrightarrow \quad \rho_{\mathcal{A},\mathcal{M},\pi}(X) < +\infty \\ &\Longleftrightarrow \quad \exists m \in \mathbb{R} : \; X + mU \in \mathcal{A} + \ker \pi \\ &\Longleftrightarrow \quad \exists m \in \mathbb{R} : \; X \in \mathcal{A} + \ker \pi - mU \\ &\Longleftrightarrow \quad X \in \mathcal{A} + \ker \pi + \mathbb{R}U \end{aligned}$$

by the Reduction Lemma 2.3.11, showing

$$\operatorname{dom} \rho_{\mathcal{A},\mathcal{M},\pi} = \mathcal{A} + \ker \pi + \mathbb{R}U.$$

Now, we need to show that

$$\mathcal{A} + \ker \pi + \mathbb{R}U = \mathcal{A} + \mathcal{M}. \tag{2.59}$$

Note that

$$\mathcal{M} \supseteq \ker \pi + \mathbb{R}U$$

is obviously fulfilled, since $U \in \mathcal{M}$ and $\mathcal{M}$ being a subspace of $\mathcal{X}$. Hence, for the proof of (2.59), it is left to show

$$\mathcal{M} \subseteq \ker \pi + \mathbb{R}U \tag{2.60}$$

Take $Z \in \mathcal{M}$ arbitrary. By linearity of $\pi$, it holds that $Z - \pi(Z)U \in \mathcal{M}$ with

$$\pi(Z - \pi(Z)U) = \pi(Z) - \pi(Z)\pi(U) = 0$$

because of $\pi(U) = 1$ by Assumption 2, i.e., $Z - \pi(Z)U \in \ker \pi$. As a result,

$$Z = (Z - \pi(Z)U) + \pi(Z)U \in \ker \pi + \mathbb{R}U$$

holds, which shows (2.60). Hence, (2.59) is proved, which completes the proof of (2.58). □

**Remark 2.3.36.** *The direct proof of $\mathcal{M} = \ker \pi + \mathbb{R}U$ in Theorem 2.3.35 can also be replaced by the following shorter proof: Because of Lemma 2.2.7, it holds that $\pi_m = mU + \ker \pi$ for $\pi_m \subseteq \mathcal{M}$ defined as in (2.16) and, thus,*

$$\mathcal{M} = \bigcup_{m \in \mathbb{R}} \pi_m = \bigcup_{m \in \mathbb{R}} (mU + \ker \pi) = \mathbb{R}U + \ker \pi,$$

*where the last equation is obvious.*

The following Lemma is observed by Baes et al. in [17] for locally convex Hausdorff topological real vector spaces $\mathcal{X}$. Since there is no proof of it, we gave one without any topological properties in our paper [139]. Hence, we can reformulate it here for real vector spaces.

**Lemma 2.3.37** (see Baes, Koch-Medina, Munari [17, Remark 2.11], and Marohn, Tammer [140, Lemma 5.7])**.** *Consider (FM). Let Assumption 2 be fulfilled by the eligible payoff $U \in \mathcal{M} \cap \mathcal{X}_+$ and $\rho_{\mathcal{A},\mathcal{M},\pi} \colon \mathcal{X} \to \overline{\mathbb{R}}$ be the functional given by (2.30). Then, the following holds:*

$$\mathcal{A} + \ker \pi = \mathcal{X} \quad \Longleftrightarrow \quad \rho_{\mathcal{A},\mathcal{M},\pi} \equiv -\infty$$

**Proof.** The assertion follows directly from Corollary 2.3.17. □

In addition to Lemma 2.3.37, we can give the following characterization for capital positions in the domain of the risk measure $\rho_{\mathcal{A},\mathcal{M},\pi} \colon \mathcal{X} \to \overline{\mathbb{R}}$ given by (2.30):

**Lemma 2.3.38** (see [140, Lemma 5.8])**.** *Consider* (FM)*. Let Assumption 2 be fulfilled by the eligible payoff* $U \in \mathcal{M} \cap \mathcal{X}_+$ *and* $\rho_{\mathcal{A},\mathcal{M},\pi} \colon \mathcal{X} \to \overline{\mathbb{R}}$ *be the functional given by* (2.30)*. Then, the following holds:*

$$-U \in \operatorname{rec} \mathcal{A} \quad \Longrightarrow \quad \forall X \in \operatorname{dom} \rho_{\mathcal{A},\mathcal{M},\pi} : \ \rho_{\mathcal{A},\mathcal{M},\pi}(X) = -\infty.$$

**Proof.** By Corollary 2.3.22, $-U \in \operatorname{rec} \mathcal{A}$ implies $-U \in \operatorname{rec}(\mathcal{A} + \ker \pi)$. Hence, for every $X \in \operatorname{dom} \rho_{\mathcal{A},\mathcal{M},\pi}$, it holds that

$$\exists m \in \mathbb{R} : \quad X + mU \in \mathcal{A} + \ker \pi$$

by the Reduction Lemma 2.3.11 which yields

$$\forall t \in \mathbb{R}_+ : \quad X + mU - tU = X + (m - t)U \in \mathcal{A} + \ker \pi,$$

because of $-U \in \operatorname{rec}(\mathcal{A} + \ker \pi)$, i.e., $\rho_{\mathcal{A},\mathcal{M},\pi}(X) \leq m - t$ for every $t \in \mathbb{R}_+$ by the Reduction Lemma 2.3.11. Consequently, $\rho_{\mathcal{A},\mathcal{M},\pi}(X) = -\infty$ must hold. □

**Remark 2.3.39.** *In general, we can secure the finiteness of* $\rho_{\mathcal{A},\mathcal{M},\pi} \colon \mathcal{X} \to \overline{\mathbb{R}}$ *given by* (2.30) *only for capital positions* $X \in \operatorname{dom} \rho_{\mathcal{A},\mathcal{M},\pi}$ *by Lemma 2.3.38 and not for the whole space of capital positions* $\mathcal{X}$*. Furthermore, note that under* (FM) *and Assumption 2,* $U \in \operatorname{rec} \mathcal{A}$ *is fulfilled (see Remark 2.2.10) and, thus,* $U \in \operatorname{rec} \mathcal{A}$ *and* $-U \in \operatorname{rec} \mathcal{A}$ *are supposed in Lemma 2.3.38.*

*As noticed in [140, Remark 5.7], the condition* $-U \in \operatorname{rec} \mathcal{A}$ *is neither necessary nor sufficient for* $\mathcal{A} + \ker \pi = \mathcal{X}$*: On the one hand, Example 2.3.24 shows that*

$$\mathcal{A} + \ker \pi = \mathcal{X} \quad \not\Longrightarrow \quad -U \in \operatorname{rec} \mathcal{A}$$

*is true although it holds that* $U \in \operatorname{rec} \mathcal{A}$ *for* $U \in \mathcal{M} \cap \mathcal{X}_+$ *arbitrary. Hence, by*

$$\mathcal{A} + \ker \pi = \mathcal{X} \quad \Longrightarrow \quad \rho_{\mathcal{A},\mathcal{M},\pi} \equiv -\infty \quad \Longrightarrow \quad \operatorname{dom} \rho_{\mathcal{A},\mathcal{M},\pi} = \mathcal{X}$$

*because of Lemma 2.3.37, the example highlights that the converse direction in Lemma 2.3.38 does not hold.*

*On the other hand, the following Example 2.3.40 shows that*

$$-U \in \operatorname{rec} \mathcal{A} \quad \not\Longrightarrow \quad \mathcal{A} + \ker \pi = \mathcal{X}$$

*also holds in general. However, if* $\mathcal{M} = \mathcal{X}$ *holds (and, thus,* $\dim \mathcal{X} < +\infty$*, too), then* $-U \in \operatorname{rec} \mathcal{A}$ *is sufficient for* $\mathcal{A} + \ker \pi = \mathcal{X}$*, indeed, because the direct sum of the subspaces* $\mathbb{R}U$ *and* $\ker \pi$ *of* $\mathcal{M}$ *fulfills*

$$\mathcal{M} = \mathbb{R}U + \ker \pi,$$

*see Remark 2.3.36. Hence,* $-U \in \operatorname{rec} \mathcal{A}$ *implies by* $\mathbb{0} \in \mathcal{A}$ *that* $\mathbb{R}U \subseteq \mathcal{A}$ *and, thus,*

$$\mathcal{X} = \mathcal{M} = \mathbb{R}U + \ker \pi \subseteq \mathcal{A} + \ker \pi \subseteq \mathcal{X}$$

*hold, i.e.,* $\mathcal{A} + \ker \pi = \mathcal{X}$*.*

**Example 2.3.40** (see Marohn, Tammer [140, Remark 5.7])**.** Consider $\mathcal{X} = \mathbb{R}^3$ and suppose $\mathcal{M} = \{0\} \times \mathbb{R} \times \mathbb{R}$. Furthermore, let $\pi\colon \mathcal{M} \to \mathbb{R}$ with $\pi(Z) = \pi(Z_1, Z_2, Z_3) := Z_3$. We choose $U = (0,0,1)^T \in \mathcal{M} \cap \mathcal{X}_+$ as the eligible payoff fulfilling Assumption 2 and

$$\mathcal{A} := \mathbb{R}U + \mathbb{R}^3_+$$

as the acceptance set. Then, $-U \in \operatorname{rec} \mathcal{A}$ and $U \in \operatorname{rec} \mathcal{A}$ are fulfilled. Moreover, we obtain

$$\mathcal{X} = \mathbb{R}^3 \neq \mathcal{A} + \ker \pi = \mathcal{A} + \mathbb{R}(0,1,0)^T = \{(X_1, X_2, X_3)^T \in \mathbb{R}^3 \mid X_1 \geq 0\}.$$

$\Diamond$

**Remark 2.3.41.** *If $\mathcal{A} + \ker \pi \neq \mathcal{X}$ is fulfilled under consideration of* (FM)*, it is impossible to reach acceptability for every capital position without any costs. Artzner et al. call this phenomena absence of acceptability arbitrage in [15]. For topological vector spaces $\mathcal{X}$, the authors of [17] and [71] observe different sufficient conditions for $\rho_{\mathcal{A},\mathcal{M},\pi}\colon \mathcal{X} \to \overline{\mathbb{R}}$ given by* (2.30) *being finite and continuous if $\mathcal{A} + \ker \pi \neq \mathcal{X}$ holds, for example, $\operatorname{int} \mathcal{X}_+ \cap \mathcal{M} \neq \emptyset$. We refer to [71, Section 3] for characterizations of the finiteness of $\rho_{\mathcal{A},\mathcal{M},\pi}$ when the acceptance set $\mathcal{A}$ is equipped with special properties like convexity.*

The observations in Lemma 2.3.37 and Lemma 2.3.38 lead to the following equivalence, which gives more details about the situation that $\rho_{\mathcal{A},\mathcal{M},\pi}\colon \mathcal{X} \to \overline{\mathbb{R}}$ given by (2.30) does not take the value $-\infty$ on the whole space of capital positions $\mathcal{X}$:

**Theorem 2.3.42** (see Marohn, Tammer [140, Theorem 5.5])**.** *Consider* (FM)*. Let Assumption 2 be fulfilled by the eligible payoff $U \in \mathcal{M} \cap \mathcal{X}_+$ and $\rho_{\mathcal{A},\mathcal{M},\pi}\colon \mathcal{X} \to \overline{\mathbb{R}}$ be the functional given by* (2.30)*. Then, the following conditions are equivalent:*

(i) *$\rho_{\mathcal{A},\mathcal{M},\pi}$ is proper,*

(ii) *$\mathcal{A} + \ker \pi$ does not contain lines parallel to $U$, i.e.,*

$$\forall X \in \mathcal{A} + \ker \pi : \quad X + \mathbb{R}U \not\subseteq \mathcal{A} + \ker \pi. \tag{2.61}$$

**Proof.** Note that

$$(2.61) \quad \Longleftrightarrow \quad \forall X \in \mathcal{A} + \ker \pi \ \exists m \in \mathbb{R} : \ X + mU \notin \mathcal{A} + \ker \pi.$$

holds. Suppose that (i) is fulfilled. Then,

$$\forall X \in \mathcal{X} : \quad \rho_{\mathcal{A},\mathcal{M},\pi}(X) > -\infty,$$

i.e., by the Reduction Lemma 2.3.11, there is some $m \in \mathbb{R}$ such that

$$\forall t < m : \quad X + tU \notin \mathcal{A} + \ker \pi,$$

which implies that (ii) holds.

Conversely, we assume that (ii) is fulfilled and show that $\rho_{\mathcal{A},\mathcal{M},\pi}$ is proper. First, it holds that $\operatorname{dom}\rho_{\mathcal{A},\mathcal{M},\pi} \neq \emptyset$ because of $\mathbb{0} \in \mathcal{A}$ and, thus, $\rho_{\mathcal{A},\mathcal{M},\pi}(\mathbb{0}) < +\infty$. Now, take $X \in \mathcal{X}$ arbitrary with $\rho_{\mathcal{A},\mathcal{M},\pi}(X) < +\infty$. Then, there is some $m \in \mathbb{R}$ with $X + mU \in \mathcal{A} + \ker\pi$, which yields

$$\exists t \in \mathbb{R}_> : \quad X + (m-t)U \notin \mathcal{A} + \ker\pi,$$

i.e., $\rho_{\mathcal{A},\mathcal{M},\pi}(X) > -\infty$ holds. As a result, (i) is fulfilled. □

**Remark 2.3.43.** *As seen in the proof of Theorem 2.3.42, we can replace $X \in \mathcal{A} + \ker\pi$ by $X \in \mathcal{X}$ in (2.61). Note that*

$$(2.61) \quad \Longrightarrow \quad -U \notin \operatorname{rec}\mathcal{A}$$

*holds for $U \in \mathcal{M} \cap \mathcal{X}_+$ fulfilling Assumption 2 because of*

$$-U \in \operatorname{rec}\mathcal{A} \quad \Longrightarrow \quad \forall X \in \mathcal{A}, \forall m \in \mathbb{R}_+ : \; X - mU \in \mathcal{A} \subseteq \mathcal{A} + \ker\pi,$$

*which contradicts (2.61) by $X + mU \in \mathcal{A} + \ker\pi$ for all $m \in \mathbb{R}_+$ (see Corollary 2.3.22). On the other hand, Example 2.3.24 highlights*

$$-U \notin \operatorname{rec}\mathcal{A} \quad \not\Longrightarrow \quad (2.61).$$

We already mentioned in Remark 2.3.6 that the risk measure $\rho_{\mathcal{A},\mathcal{M},\pi} \colon \mathcal{X} \to \overline{\mathbb{R}}$ given by (2.30) is not always a coherent risk measure, but the following lemma from [71] gives sufficient conditions for $\rho_{\mathcal{A},\mathcal{M},\pi}$ to be a convex or coherent risk measure. Since there was no proof, we give one that is more accurate than that we presented in [139, Lemma 3.20] and that does not use any topological properties (compare also [91, Theorem 2.3.1]).

**Lemma 2.3.44** (see Farkas et al. [71, Lemma 2])**.** *Consider (FM). Let Assumption 2 be fulfilled and $\rho_{\mathcal{A},\mathcal{M},\pi} \colon \mathcal{X} \to \overline{\mathbb{R}}$ be the functional given by (2.30). Then, the following properties hold:*

(i) *$\rho_{\mathcal{A},\mathcal{M},\pi}$ is convex if and only if $\mathcal{A}$ is convex,*

(ii) *$\rho_{\mathcal{A},\mathcal{M},\pi}$ is subadditive if and only if $\mathcal{A}$ is closed under addition, i.e., $X + Y \in \mathcal{A}$ for all $X, Y \in \mathcal{A}$,*

(iii) *$\rho_{\mathcal{A},\mathcal{M},\pi}$ is positively homogeneous if and only if $\mathcal{A}$ is a cone.*

*Especially, $\rho_{\mathcal{A},\mathcal{M},\pi}$ is sublinear. Moreover, if $\rho_{\mathcal{A},\mathcal{M},\pi}$ fulfills two of the properties of being convex, subadditive or positively homogeneous, then the third property is automatically fulfilled.*

**Proof.** It holds that $\rho_{\mathcal{A},\mathcal{M},\pi}(\mathbb{0}) \leq 0$ because of $\mathbb{0} \in \mathcal{A}$ by Definition 2.2.9(i) and the Reduction Lemma 2.3.11. Hence, two arbitrary of the properties (i), (ii) and (iii) are equivalent to $\rho_{\mathcal{A},\mathcal{M},\pi}$ being sublinear and, thus, imply the third remaining property, see Lemma 1.1.57. The assertions (i), (ii), and (iii) follow from $\rho_{\mathcal{A},\mathcal{M},\pi} \equiv \rho_{\mathcal{A}_{\rho_{\mathcal{A},\mathcal{M},\pi}},\mathcal{M},\pi}$ by Theorem 2.3.31 and Lemma 2.2.13. Nevertheless, we want to give direct proofs for the influence of given properties of $\mathcal{A}$ on the risk measure $\rho_{\mathcal{A},\mathcal{M},\pi}$ in the following. Let $U \in \mathcal{M} \cap \mathcal{X}_+$ be the eligible payoff according to Assumption 2. First, we note that $\mathcal{A}$ being convex, a cone or closed under addition implies $\mathcal{A} + \ker\pi$ being convex, a cone or closed under addition, respectively, by Lemma 2.3.20.

(i) Take $X, Y \in \mathcal{X}$ and $\lambda \in [0,1]$ arbitrary. Let

$$m := \lambda \rho_{\mathcal{A},\mathcal{M},\pi}(X) + (1-\lambda)\rho_{\mathcal{A},\mathcal{M},\pi}(Y).$$

Then,

$$W := \lambda X + (1-\lambda)Y + mU = \lambda(X + \rho_{\mathcal{A},\mathcal{M},\pi}(X)U) + (1-\lambda)(Y + \rho_{\mathcal{A},\mathcal{M},\pi}(Y)U)$$

by convexity of $\mathcal{A}$ and, thus, convexity of $\mathcal{A}+\ker\pi$. Because $\rho_{\mathcal{A},\mathcal{M},\pi}(X+\rho_{\mathcal{A},\mathcal{M},\pi}(X)U) = 0$ and $\rho_{\mathcal{A},\mathcal{M},\pi}(Y+\rho_{\mathcal{A},\mathcal{M},\pi}(Y)U) = 0$ hold by translation invariance of $\rho_{\mathcal{A},\mathcal{M},\pi}$ with $\pi(U) = 1$ by Assumption 2 (see Lemma 2.3.5(ii)), we obtain

$$X + \rho_{\mathcal{A},\mathcal{M},\pi}(X)U \in \operatorname{bd}_{-U}(\mathcal{A}+\ker\pi), \quad Y + \rho_{\mathcal{A},\mathcal{M},\pi}(Y)U \in \operatorname{bd}_{-U}(\mathcal{A}+\ker\pi)$$

by Theorem 2.3.25(iii). Suppose that $W \notin \operatorname{cl}_{-U}(\mathcal{A}+\ker\pi)$ holds. Then, for $t > 0$ sufficiently small, $W + tU \notin \mathcal{A}+\ker\pi$ holds, but

$$W + tU = \lambda\underbrace{((X + \rho_{\mathcal{A},\mathcal{M},\pi}(X)U) + tU)}_{\in\mathcal{A}+\ker\pi} + (1-\lambda)\underbrace{((Y + \rho_{\mathcal{A},\mathcal{M},\pi}(Y)U) + tU)}_{\in\mathcal{A}+\ker\pi} \in \mathcal{A}+\ker\pi$$

by definition of $\operatorname{bd}_{-U}(\mathcal{A}+\ker\pi)$ and convexity of $\mathcal{A}+\ker\pi$, which is a contradiction. As a result,

$$W = \lambda(X + \rho_{\mathcal{A},\mathcal{M},\pi}(X)U) + (1-\lambda)(Y + \rho_{\mathcal{A},\mathcal{M},\pi}(Y)U) \in \operatorname{cl}_{-U}(\mathcal{A}+\ker\pi).$$

holds, i.e., $\rho_{\mathcal{A},\mathcal{M},\pi}(W) \leq 0$ by Theorem 2.3.25. Hence,

$$\rho_{\mathcal{A},\mathcal{M},\pi}(\lambda X + (1-\lambda)Y) = \rho_{\mathcal{A},\mathcal{M},\pi}(W - mU) = \rho_{\mathcal{A},\mathcal{M},\pi}(W) + m\pi(U) \leq m$$

holds by translation invariance of $\rho_{\mathcal{A},\mathcal{M},\pi}$ from Lemma 2.3.5(ii) and $\pi(U) = 1$ by Assumption 2, showing

$$\rho_{\mathcal{A},\mathcal{M},\pi}(\lambda X + (1-\lambda)Y) \leq \lambda\rho_{\mathcal{A},\mathcal{M},\pi}(X) + (1-\lambda)\rho_{\mathcal{A},\mathcal{M},\pi}(Y).$$

Thus, $\rho_{\mathcal{A},\mathcal{M},\pi}$ is convex.

(ii) We obtain

$$X + Y + (\rho_{\mathcal{A},\mathcal{M},\pi}(X) + \rho_{\mathcal{A},\mathcal{M},\pi}(Y))U \in \operatorname{cl}_{-U}(\mathcal{A}+\ker\pi)$$

because

$$\begin{aligned}\forall t > 0: \quad & X + Y + (\rho_{\mathcal{A},\mathcal{M},\pi}(X) + \rho_{\mathcal{A},\mathcal{M},\pi}(Y) + t)U \\ &= \underbrace{X + \left(\rho_{\mathcal{A},\mathcal{M},\pi}(X) + \frac{t}{2}\right)U}_{\in\mathcal{A}+\ker\pi} + \underbrace{Y + \left(\rho_{\mathcal{A},\mathcal{M},\pi}(Y) + \frac{t}{2}\right)U}_{\in\mathcal{A}+\ker\pi} \in \mathcal{A}+\ker\pi\end{aligned}$$

by the Reduction Lemma 2.3.11 and $\mathcal{A}+\ker\pi$ being closed under addition. Hence, by Theorem 2.3.25,

$$\rho_{\mathcal{A},\mathcal{M},\pi}(X+Y) \leq \rho_{\mathcal{A},\mathcal{M},\pi}(X) + \rho_{\mathcal{A},\mathcal{M},\pi}(Y)$$

holds.

(iii) First, we note $\rho_{\mathcal{A},\mathcal{M},\pi}(\mathbb{0}) \in \{-\infty, 0\}$ because $\rho_{\mathcal{A},\mathcal{M},\pi}(\mathbb{0}) \leq 0$ by $\mathbb{0} \in \mathcal{A}$ (see Definition 2.2.9(i)) and, if $\rho_{\mathcal{A},\mathcal{M},\pi}(\mathbb{0}) < 0$ holds, then

$$\exists m \in \mathbb{R}_> : \quad \mathbb{0} - mU = -mU \in \mathcal{A} + \ker \pi$$

by the Reduction Lemma 2.3.11 which implies

$$\forall \lambda \in \mathbb{R}_+ : \quad -\lambda m U \in \mathcal{A} + \ker \pi$$

by $\mathcal{A} + \ker \pi$ being a cone, i.e., $\rho_{\mathcal{A},\mathcal{M},\pi}(\mathbb{0}) = -\infty$.

Now, we consider $X \in (\mathcal{A} + \ker \pi) \backslash \{\mathbb{0}\}$ and $\lambda \in \mathbb{R}_>$ arbitrary, first. Suppose that $\rho_{\mathcal{A},\mathcal{M},\pi}(X) = -\infty$. Then,

$$\forall m \in \mathbb{R} : \quad X + mU \in \mathcal{A} + \ker \pi$$

by the Reduction Lemma 2.3.11 and, thus,

$$\forall m \in \mathbb{R} : \quad \lambda X + \lambda m U = \lambda(X + mU) \in \mathcal{A} + \ker \pi$$

because $\mathcal{A} + \ker \pi$ is a cone. Hence, $\rho_{\mathcal{A},\mathcal{M},\pi}(\lambda X) = -\infty$ holds, too. Analogously, if $\rho_{\mathcal{A},\mathcal{M},\pi}(X) = +\infty$, then

$$\forall m \in \mathbb{R} : \quad X + mU \notin \mathcal{A} + \ker \pi$$

holds by the Reduction Lemma 2.3.11 and, thus, $\rho_{\mathcal{A},\mathcal{M},\pi}(\lambda X) = +\infty$, too, because otherwise

$$\exists m \in \mathbb{R} : \ \lambda X + mU \in \mathcal{A} + \ker \pi \quad \Longrightarrow \quad \frac{1}{\lambda}(\lambda X + mU) = X + \frac{m}{\lambda} U \in \mathcal{A} + \ker \pi$$

by $\mathcal{A} + \ker \pi$ being a cone, which would be a contradiction.

Now, we suppose $\rho_{\mathcal{A},\mathcal{M},\pi}(X) \in \mathbb{R}$. Then,

$$W := \lambda X + \lambda \rho_{\mathcal{A},\mathcal{M},\pi}(X) U = \lambda(X + \rho_{\mathcal{A},\mathcal{M},\pi}(X) U)$$

with $X + \rho_{\mathcal{A},\mathcal{M},\pi}(X)U \in \operatorname{bd}_{-U}(\mathcal{A} + \ker \pi)$. Suppose that $W \notin \operatorname{bd}_{-U}(\mathcal{A} + \ker \pi)$ holds. Consider the case $W \notin \mathcal{A} + \ker \pi$ first. Then, there is some $t > 0$ sufficiently small with $W + tU \notin \mathcal{A} + \ker \pi$, but

$$W + tU = \lambda \left( X + \rho_{\mathcal{A},\mathcal{M},\pi}(X)U + \frac{t}{\lambda} U \right) \in \mathcal{A} + \ker \pi$$

by $\mathcal{A} + \ker \pi$ being a cone, which is a contradiction. Hence, $W \in \mathcal{A} + \ker \pi$ must hold, but then, by $W \notin \operatorname{bd}_{-U}(\mathcal{A} + \ker \pi)$, we obtain analogously that there is some $t > 0$ sufficiently small with $W - tU \in \mathcal{A} + \ker \pi$ and, thus,

$$\frac{1}{\lambda}(W - tU) = \left( X + \rho_{\mathcal{A},\mathcal{M},\pi}(X)U - \frac{t}{\lambda} U \right) \in \mathcal{A} + \ker \pi,$$

since $\mathcal{A} + \ker \pi$ is a cone and $\frac{1}{\lambda} > 0$. Hence,

$$\rho_{\mathcal{A},\mathcal{M},\pi}(X) \leq \rho_{\mathcal{A},\mathcal{M},\pi}(X) - \frac{t}{\lambda} < \rho_{\mathcal{A},\mathcal{M},\pi}(X),$$

which is a contradiction. As a result, $W \in \operatorname{bd}_{-U}(\mathcal{A}+\ker \pi)$ must hold, i.e., $\rho_{\mathcal{A},\mathcal{M},\pi}(W) = 0$ by Theorem 2.3.25(iii) and, thus,

$$\rho_{\mathcal{A},\mathcal{M},\pi}(\lambda X) = \rho_{\mathcal{A},\mathcal{M},\pi}(W - \lambda\rho_{\mathcal{A},\mathcal{M},\pi}(X)U) = \rho_{\mathcal{A},\mathcal{M},\pi}(W) + \lambda\rho_{\mathcal{A},\mathcal{M},\pi}(X)\pi(U) = \lambda\rho_{\mathcal{A},\mathcal{M},\pi}(X)$$

by translation invariance of $\rho_{\mathcal{A},\mathcal{M},\pi}$ from Lemma 2.3.5(ii) and $\pi(U) = 1$ by Assumption 2, showing

$$\rho_{\mathcal{A},\mathcal{M},\pi}(\lambda X) = \lambda\rho_{\mathcal{A},\mathcal{M},\pi}(X).$$

That completes the proof that $\rho_{\mathcal{A},\mathcal{M},\pi}$ is positively homogeneous.

□

# Chapter 3

# Optimal Eligible Payoffs

In this chapter, we consider (FM) and focus on the solution set of the optimization problem ($P_\pi(X)$) for given $X \in \mathcal{X}$ and $(-U)$-directional closed acceptance set $\mathcal{A}$ with $U \in \mathcal{M} \cap \mathcal{X}_+$ according to Assumption 2, i.e.,

$$\pi(Z) \to \min_{X+Z \in \mathcal{A},\, Z \in \mathcal{M}}. \tag{$P_\pi(X)$}$$

The optimization problem ($P_\pi(X)$) formalizes the economical problem of making a given capital position $X$ of an financial institution acceptable by minimal costs. Hence, for $\rho_{\mathcal{A},\mathcal{M},\pi} \colon \mathcal{X} \to \overline{\mathbb{R}}$ being the risk measure given by (2.30), the solution set of ($P_\pi(X)$) is given by

$$\mathcal{E}(X) := \{Z \in \mathcal{M} \mid X + Z \in \mathcal{A}, \pi(Z) = \rho_{\mathcal{A},\mathcal{M},\pi}(X)\}$$

and $\rho_{\mathcal{A},\mathcal{M},\pi}(X)$ is the optimal value of ($P_\pi(X)$). The set-valued map $\mathcal{E} \colon \mathcal{X} \rightrightarrows \mathcal{M}$ is introduced as *optimal payoff map* in [17] for closed acceptance sets. The chapter is organized as follows:

- In Section 3.1, we analyze the consequences for $\mathcal{E}(X)$ with $X \in \mathcal{X}$ arbitrary if we assume directionally closed acceptance sets (instead of closed acceptance sets as in [17]). As we will see, we will be able to derive a generalized characterization of $\mathcal{E}(X)$ that can also be applied for the setting in [17]. To argue the generalized character of the characterization, we outline the relationship between the boundary and $(-U)$-directional boundary of $\mathcal{A}$ and $\mathcal{A} + \ker \pi$, respectively. Moreover, we give some existence and uniqueness results for solutions of ($P_\pi(X)$), which generalize results in [17].

- Afterwards, we study the set of cost-optimal acceptable capital positions

  $$\mathcal{A}' := \operatorname{bd}_{-U}(\mathcal{A}) \cap \operatorname{bd}_{-U}(\mathcal{A} + \ker \pi).$$

  in Section 3.2. By our characterization for $\mathcal{E}(X)$ derived in Section 3.1, it is obvious that $\mathcal{A}'$ consists of capital positions resulting from solutions of ($P_\pi(X)$) for any $X \in \mathcal{X}$. We derive interesting properties of $\mathcal{A}'$ that will be useful for determining (weakly) efficient points of the acceptance set $\mathcal{A}$, and highlight the role of the choice of $X \in \mathcal{X}$ for determining $\mathcal{A}'$.

The main results in this chapter are published in [141].

## 3.1 Optimal payoff map

In [17] and [139], the following set-valued mapping is considered for a closed acceptance set $\mathcal{A}$, which we want to study here in a more general setting:

**Definition 3.1.1** (see Baes et al. [17]). Consider (FM) and let $\rho_{\mathcal{A},\mathcal{M},\pi}\colon \mathcal{X} \to \overline{\mathbb{R}}$ be the risk measure given by (2.30). We call $\mathcal{E}\colon \mathcal{X} \rightrightarrows \mathcal{M}$ given by

$$\mathcal{E}(X) := \{Z \in \mathcal{M} \mid X + Z \in \mathcal{A}, \pi(Z) = \rho_{\mathcal{A},\mathcal{M},\pi}(X)\} \tag{3.1}$$

*optimal (eligible) payoff map* and every $Z \in \mathcal{E}(X)$ *optimal (eligible) payoff for* $X \in \mathcal{X}$. Moreover, for $X \in \mathcal{X}$ arbitrary, each $X^0 \in \mathcal{X}$ fulfilling

$$X^0 = X + Z \in \mathcal{A} \quad \text{with some} \quad Z \in \mathcal{E}(X)$$

is called a *(cost-)optimal acceptable capital position* for $X$.

For (FM) and given $X \in \mathcal{X}$, the set $\mathcal{E}(X)$ from Definition 3.1.1 is the solution set of the optimization problem

$$\pi(Z) \to \min_{X+Z\in\mathcal{A},\, Z\in\mathcal{M}}. \tag{$P_\pi(X)$}$$

Our aim is to generalize the results in [17] and [139] for locally convex Hausdorff spaces $\mathcal{X}$ and closed acceptance sets $\mathcal{A}$. In our setting (FM), $\mathcal{X}$ is a vector space partially ordered by the positive cone $\mathcal{X}_+$, which is not necessarily equipped with a topology. Moreover, we make weaker assumption on the acceptance set $\mathcal{A}$:

**Assumption 4.** *Consider* (FM) *and let* $U \in \mathcal{M} \cap \mathcal{X}_+$ *be an eligible payoff fulfilling Assumption 2. The acceptance set* $\mathcal{A}$ *is* $(-U)$*-directionally closed.*

Note that directionally closedness is an algebraic property and not a topological, see Definition 1.1.26. Acceptance sets are defined in various ways in the literature (see Remark 2.2.11). Directionally closed acceptance sets are considered, e.g., in [102] for studies of set-valued risk measures, where the acceptance set is assumed to be directionally closed with respect to a set of directions, and in the survey [100] about monetary risk measures.

**Remark 3.1.2.** *It is useful to consider directionally closed sets in different situations. For example, Artzner et al. studied in [14] the risk measure* $\rho_{\mathcal{A},r_0}\colon \mathcal{X} \to \overline{\mathbb{R}}$ *given by*

$$\rho_{\mathcal{A},r_0}(X) := \inf\{m \in \mathbb{R} \mid X + m r_0 1\!\!1_\Omega \in \mathcal{A}\}$$

*for a space of random variables* $\mathcal{X}$ *on a finite probability space with* $r_0 \in \mathbb{R}_>$ *being the return of a risk-free reference instrument,* $1\!\!1_\Omega \in \mathcal{X}$ *being the random variable that equals 1 in each scenario, and an acceptance set* $\mathcal{A} \subseteq \mathcal{X}$*. In our setting, the risk-free reference instrument is given by* $S^0$ *and, thus, the return by* $r_0 = S^0_1 = 1 + r$*, which leads to* $r_0 = 1$ *by our assumption of* $r = 0$ *for* $S^0$ *(see Remark 2.2.2). If we consider* (FM) *with only one eligible asset with payoff* $U = r_0 1\!\!1_\Omega \in \mathcal{M} \cap \mathcal{X}_+$ *according to Assumption 2, we obtain*

$$\rho_{\mathcal{A},r_0}(X) = \rho_{\mathcal{A},\mathcal{M},\pi}(X)$$

*with* $\rho_{\mathcal{A},\mathcal{M},\pi}\colon \mathcal{X} \to \overline{\mathbb{R}}$ *given by* (2.30) *by the Reduction Lemma 2.3.11. Hence, if* $X \in \mathcal{X}$ *fulfills* $\rho_{\mathcal{A},r_0}(X) \in \mathbb{R}$*, we obtain*

$$X + \rho_{\mathcal{A},r_0}(X) r_0 \mathbb{1}_\Omega \in \operatorname{bd}_{-r_0\mathbb{1}_\Omega}(\mathcal{A})$$

*by Theorem 2.3.25*(iii)*. Consequently, to secure that every* $X \in \mathcal{X}$ *with* $\rho_{\mathcal{A},r_0}(X) \in \mathbb{R}$ *can be made acceptable by transformation into an optimal acceptable capital position* $X^0 \in \mathcal{A}$ *by* $Z \in \mathcal{E}(X)$ *(see Definition 3.1.1), we have to require*

$$\operatorname{bd}_{-r_0\mathbb{1}_\Omega}(\mathcal{A}) \subseteq \mathcal{A}.$$

*Because of* $\operatorname{bd}_{-r_0\mathbb{1}_\Omega}(\mathcal{A}) \subseteq \operatorname{cl}_{-r_0\mathbb{1}_\Omega}(\mathcal{A})$*, that requirement is fulfilled by assuming* $\mathcal{A}$ *to be a* $(-U)-$*directionally closed acceptance set according to Assumption 4 with* $U = r_0\mathbb{1}_\Omega$*.*

Directionally closed acceptance sets can also directly occur in economical situations, e.g., by considering scenario-based acceptance sets. Regulatory preconditions for financial institutions are sometimes formulated with respect to pre-defined scenarios as, e.g., in the Basel framework. One scenario might be $\omega_1$: "*The market development is good*" (or another kind of best case), while a second scenario might be $\omega_2$: "*The market development is bad*" (or another kind of worst case). Of course, these scenarios will be (mathematically or economically) specified with much more details like assumptions about the development of the interest rates that clarify what "*good*" or "*bad*" might mean, for example, an interest rate shock simulated by an increase or a decline of 200 basis points for a market interest rate in a month. Furthermore, since two scenarios are too coarse in general, there are more scenarios in real situations, see, e.g., the standardized interest rate shock scenarios for banking books in Basel III [23, Sub-item 31.90]. Nevertheless, regulatory institutions like the Federal Institute for Financial Services Supervision (BaFin) in Germany could formulate restrictions with respect to scenarios as in the following illustrating, easy example:

**Example 3.1.3.** Consider (FM). According to the regulatory restrictions of a given regulator, the financial net worth of a bank's capital position $X \in \mathcal{X} = \mathbb{R}^2$ has to be non-negative in scenario $\omega_1$ ("good" market situation), i.e., $X_1 := X(\omega_1) \geq 0$, while it has to fulfill $X_2 := X(\omega_2) > -c$ for a given $c \in \mathbb{R}_>$ in scenario $\omega_2$ ("bad" market situation). The resulting acceptance set is then given by

$$\mathcal{A} = \left\{ X = (X_1, X_2)^T \in \mathbb{R}^2 \mid X_1 \geq 0, X_2 > -c \right\}.$$

While the former solvency condition $X_1 \geq 0$ is clear from a regulatory point of view, the latter one $X_2 > -1$ can be justified by the circumstance that a financial institution can easily take debts from the central bank (in Europe the European Central Bank) for a short time in general. Thus, short-time refinancing of a (not unlimited tall) amount of money $c$ with low costs is no problem at all, but the bank has to achieve a capital position taller than $-c$ to be able to pay the (even small) interests. Since these interest rates differ for different lending periods and may depend on the time point (while the regulatory preconditions and, thus, $\mathcal{A}$

will not be adjusted at each time point), there can not be fixed a specific boundary such that an institution with a financial position in this boundary is always acceptable. Now, consider a typical defaultable eligible asset $U \in \mathcal{M} \cap \mathcal{X}_+$ with price $\pi(U) = 1$ and (scenario based) financial net worth $U = (U(\omega_1), U(\omega_2))^T := (2, 0)^T$. Then, we see that $\mathcal{A}$ is obviously not closed, but $(-U)$-directional closed (see Figure 3.1). $\Diamond$

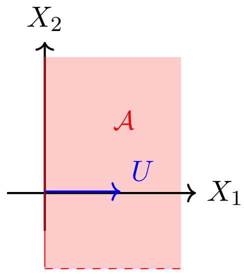


Figure 3.1: Illustration of the scenario based acceptance set $\mathcal{A}$ in Example 3.1.3

As we will see later in Theorem 3.1.6, Assumption 4 does not automatically lead to the existence of an optimal acceptable capital position $X^0 \in \mathcal{A}$ for $X \in \mathcal{X}$ in (FM) in general. Nevertheless, we will prove that $X^0 \in \operatorname{bd}_{-U}(\mathcal{A})$ holds for each optimal acceptable capital position with $U \in \mathcal{M} \cap \mathcal{X}_+$ arbitrary according to Assumption 2. The reason is that $\mathcal{E}(X)$ with $\mathcal{E} \colon \mathcal{X} \rightrightarrows \mathcal{M}$ given by (2.6) does not have to be non-empty for given $X \in \mathcal{X}$. We will show in Theorem 3.1.6 that $\mathcal{E}(X) \neq \emptyset$ is equivalent to $X$ being able to be changed into a position $X^0 \in \operatorname{bd}_{-U}(\mathcal{A}) \cap \operatorname{bd}_{-U}(\mathcal{A} + \ker \pi)$ if $\mathcal{A}$ is a $(-U)$-directionally closed acceptance set $\mathcal{A}$. Note that the case of one eligible asset as in [14] implies $\ker \pi = \{\mathbb{0}\}$ such that it is sufficient that $X$ can be transformed into a position $X \in \operatorname{bd}_{-U}(\mathcal{A})$ then, but $\ker \pi = \{\mathbb{0}\}$ is excluded here by (2.11) (see Remark 2.2.17).

Baes et al. [17] studied the optimal payoff map $\mathcal{E} \colon \mathcal{X} \rightrightarrows \mathcal{M}$ from Definition 3.1.1 in detail, but additionally assumed a closed acceptance set in a locally convex Hausdorff space. Since closed subsets of topological vector spaces $\mathcal{X}$ are also $K$-directional closed for any $K \in \mathcal{X} \backslash \{\mathbb{0}\}$ (see, e.g., [185, Prop. 2.3.54]), the results in this chapter can be applied for closed acceptance sets in topological vector spaces like in [17], as well. Nevertheless, in (FM), Assumption 4 leads automatically to a closed acceptance set if a certain additional assumption concerning $\mathcal{A}$ is fulfilled, see the following lemma and also [185, Prop. 2.3.54]:

**Lemma 3.1.4** (see [185, Prop. 2.3.54]). *Consider (FM). Let $(\mathcal{X}, \tau)$ be a topological vector space and $\mathcal{A} \subseteq \mathcal{X}$ be an acceptance set fulfilling Assumption 4 with some $U \in \mathcal{M} \cap \mathcal{X}_+$. Suppose that*

$$\operatorname{cl} \mathcal{A} + \mathbb{R}_> U \subseteq \mathcal{A} \tag{3.2}$$

*holds. Then, $\mathcal{A}$ is closed.*

**Proof.** $\mathcal{A}$ is closed if and only if $\mathcal{A} = \operatorname{cl} \mathcal{A}$ holds. Since $\mathcal{A}$ is $(-U)$-directionally closed by Assumption 4, we have $\operatorname{cl}_{-U}(\mathcal{A}) = \mathcal{A}$. Hence, $\mathcal{A}$ is closed if and only if

$$\operatorname{cl}_{-U}(\mathcal{A}) = \operatorname{cl} \mathcal{A} \tag{3.3}$$

holds. First, we proof

$$\operatorname{cl}_{-U}(\mathcal{A}) \subseteq \operatorname{cl}\mathcal{A}. \tag{3.4}$$

We obtain from $\mathcal{A} \subseteq \operatorname{cl}\mathcal{A}$

$$\operatorname{cl}_{-U}(\mathcal{A}) \subseteq \operatorname{cl}_{-U}(\operatorname{cl}\mathcal{A}).$$

Consequently, for (3.4), we need to show $\operatorname{cl}_{-U}(\operatorname{cl}\mathcal{A}) = \operatorname{cl}\mathcal{A}$, i.e., $\operatorname{cl}\mathcal{A}$ is $(-U)$-directionally closed. Take $X \in \operatorname{cl}_{-U}(\operatorname{cl}\mathcal{A})$ arbitrary. Then, it holds that

$$\exists (m_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}_+ \text{ with } m_n \downarrow 0: \quad X + m_n U = X - m_n(-U) \in \operatorname{cl}\mathcal{A}. \tag{3.5}$$

Let $\mathcal{U} \in \mathcal{N}_X$ be an arbitrary neighborhood of $X$. By Lemma 1.1.39, $\mathcal{U} - X \in \mathcal{N}_0$ is a neighborhood of $0 \in \mathcal{X}$. Because of Lemma 1.1.40, $\mathcal{U} - X$ is absorbing which implies by $m_n \downarrow 0$

$$\exists k \in \mathbb{N}: \quad m_k U \in \mathcal{U} - X$$

because of $0 < m_k < \epsilon$ for an arbitrary given $\epsilon > 0$ if $k \in \mathbb{N}$ is sufficiently tall. Hence,

$$X + m_k U \in \mathcal{U} \cap \operatorname{cl}\mathcal{A}$$

holds by (3.5) which leads to $\mathcal{U} \cap \operatorname{cl}\mathcal{A} \neq \emptyset$ for any neighborhood $\mathcal{U}$ of $X$. That implies $X \in \operatorname{cl}(\operatorname{cl}\mathcal{A})$ by Remark 1.1.5. Since $\operatorname{cl}(\operatorname{cl}\mathcal{A}) = \mathcal{A}$ holds, we obtain $X \in \mathcal{A}$ for any $X \in \operatorname{cl}_{-U}(\operatorname{cl}\mathcal{A})$, i.e.,

$$\operatorname{cl}_{-U}(\operatorname{cl}\mathcal{A}) \subseteq \operatorname{cl}\mathcal{A}.$$

Because the inverse relation holds by definition of $\operatorname{cl}_{-U}(\cdot)$, we obtain $\operatorname{cl}_{-U}(\operatorname{cl}\mathcal{A}) = \operatorname{cl}\mathcal{A}$, showing $\operatorname{cl}\mathcal{A}$ is $(-U)$-directionally closed which completes the proof of (3.4). For (3.3), it remains to show

$$\operatorname{cl}_{-U}(\mathcal{A}) \supseteq \operatorname{cl}\mathcal{A}. \tag{3.6}$$

As noticed in Remark 2.2.10, $U \in \operatorname{rec}\mathcal{A}$ and, thus, $-U \in -\operatorname{rec}\mathcal{A}$ hold. Therefore,

$$\operatorname{cl}_{-U}(\mathcal{A}) = \{X \in \mathcal{X} \mid X + \mathbb{R}_{>}U \subseteq \mathcal{A}\}$$

holds by Lemma 1.1.29(i). By the precondition (3.2), the relationship (3.6) follows. Taking into account (3.4), we obtain (3.3), i.e, $\operatorname{cl}_{-U}(\mathcal{A}) = \operatorname{cl}\mathcal{A}$. As mentioned at the beginning of the proof, this implies that $\mathcal{A}$ is closed by Assumption 4. □

Of course, as in Example 3.1.3, (3.2) is not always fulfilled and, thus, not each acceptance set according to Assumption 4 is closed.

**Example 3.1.5.** Consider (FM) as in Example 3.1.3 again, i.e., let $\mathcal{X} = \mathcal{M} = \mathbb{R}^2$, $U = (2,0)^T$ and $\mathcal{A} \subseteq \mathbb{R}^2$ be given by

$$\mathcal{A} = \left\{X = (X_1, X_2)^T \in \mathbb{R}^2 \mid X_1 \geq 0, X_2 > -c\right\}.$$

Then, $\mathcal{A}$ is an $(-U)$-directionally closed acceptance set with respect to Definition 2.2.9, but $\mathcal{A}$ is obviously not closed, since

$$(0, -c)^T + \mathbb{R}_+ U \not\subseteq \mathcal{A}$$

holds. Indeed, (3.2) in Lemma 3.1.4 is not fulfilled: It is $(0, -c)^T + \mathbb{R}_+ U \subseteq \operatorname{cl} \mathcal{A}$, but for $X = (0, -c)^T \in \operatorname{cl} \mathcal{A}$, it holds that $X + \mathbb{R}_> U \notin \mathcal{A}$, which contradicts (3.2). $\Diamond$

Now, as announced at the beginning of this chapter, we focus on the optimal payoff map $\mathcal{E} \colon \mathcal{X} \rightrightarrows \mathcal{M}$ given by (3.1) with respect to (FM). In [17, Th. 3.2], the authors assume a locally convex Hausdorff space $\mathcal{X}$ and derived the following description of $\mathcal{E}(X)$ for given $X \in \mathcal{X}$ and a closed acceptance set $\mathcal{A} \subseteq \mathcal{X}$ with $\rho_{\mathcal{A},\mathcal{M},\pi} \colon \mathcal{X} \to \overline{\mathbb{R}}$ given by (2.30) being finite and continuous:

$$\forall X \in \mathcal{X}: \quad \mathcal{E}(X) = \{Z \in \mathcal{M} \mid X + Z \in \operatorname{bd} \mathcal{A} \cap \operatorname{bd} (\mathcal{A} + \ker \pi)\}. \tag{3.7}$$

In this thesis, we consider mostly real vector spaces $\mathcal{X}$ without any topology. Especially, the acceptance set $\mathcal{A} \subseteq \mathcal{X}$ is not assumed to be a closed subset of a topological vector space in (FM). Nevertheless, we could derive in [141] the following, more general characterization of the set of optimal eligible payoffs $\mathcal{E}(X)$ for $(-U)$-directionally closed acceptance sets with $U \in \mathcal{M} \cap \mathcal{X}_+$ arbitrary according to Assumption 2.

**Theorem 3.1.6** (see Marohn, Tammer [141, Theorem 4.5])**.** *Consider* (FM)*. Let Assumption 2 be fulfilled by some $U \in \mathcal{M} \cap \mathcal{X}_+$. Furthermore, let $\mathcal{E} \colon \mathcal{X} \rightrightarrows \mathcal{M}$ be the set-valued optimal payoff map introduced in* (3.1)*. Take $X \in \mathcal{X}$ arbitrary. Then,*

$$\mathcal{E}(X) \subseteq \{Z \in \mathcal{M} \mid X + Z \in \operatorname{bd}_{-U}(\mathcal{A}) \cap \operatorname{bd}_{-U}(\mathcal{A} + \ker \pi)\}. \tag{3.8}$$

*Furthermore,*

$$\mathcal{E}(X) = \{Z \in \mathcal{M} \mid X + Z \in \operatorname{bd}_{-U}(\mathcal{A}) \cap \operatorname{bd}_{-U}(\mathcal{A} + \ker \pi)\} \tag{3.9}$$

*holds if one of the following conditions is fulfilled:*

(i) $\rho_{\mathcal{A},\mathcal{M},\pi}(X) \in \{-\infty, +\infty\}$,

(ii) *$\mathcal{A}$ is a $(-U)$-directionally closed acceptance set according to Assumption 4.*

**Proof.** We set

$$\mathcal{Z}(X) := \{Z \in \mathcal{M} \mid X + Z \in \operatorname{bd}_{-U}(\mathcal{A}) \cap \operatorname{bd}_{-U}(\mathcal{A} + \ker \pi)\} \tag{3.10}$$

for convenience and improved readability. First, let $\mathcal{A} \subseteq \mathcal{X}$ be an arbitrary acceptance set and $X \in \mathcal{X}$. We show (3.8) and, since the case $\mathcal{E}(X) = \emptyset$ is trivial, we suppose $\mathcal{E}(X) \neq \emptyset$ which implies $\rho_{\mathcal{A},\mathcal{M},\pi}(X) \in \mathbb{R}$ by definition of $\mathcal{E}(X)$ (see (2.6)). Take $Z \in \mathcal{E}(X)$ arbitrary and show $Z \in \mathcal{Z}(X)$ according to (3.10): By $Z \in \mathcal{E}(X)$, it holds that

$$X + Z \in \mathcal{A} \quad \text{and} \quad \pi(Z) = \rho_{\mathcal{A},\mathcal{M},\pi}(X).$$

Let $(t_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}_>$ arbitrary with $t_n \downarrow 0$ for $n \to +\infty$. Then, by monotonicity of $\mathcal{A}$ (see Definition 2.2.9(iii)), we obtain

$$\forall n \in \mathbb{N}: \quad X + Z + t_n U \in \mathcal{A}$$

because of $t_n U \in \mathcal{X}_+$ for each $n \in \mathbb{N}$. Moreover, it holds that

$$\forall n \in \mathbb{N}: \quad X + Z - t_n U \notin \mathcal{A}$$

because $X + Z - t_n U \in \mathcal{A}$ contradicts $\rho_{\mathcal{A},\mathcal{M},\pi}(X) = \pi(Z)$ by

$$\pi(Z - t_n U) = \pi(Z) - t_n < \pi(Z),$$

since $\pi$ is linear, $t_n \in \mathbb{R}_>$ for all $n \in \mathbb{N}$ and $\pi(U) = 1$ by Assumption 2. As a result, we obtain

$$X + Z \in \operatorname{bd}_{-U}(\mathcal{A}). \tag{3.11}$$

Now, it is left to prove $X + Z \in \operatorname{bd}_{-U}(\mathcal{A} + \ker \pi)$. Let $m := \rho_{\mathcal{A},\mathcal{M},\pi}(X)$. Since $Z \in \mathcal{E}(X)$, we have $X + Z \in \mathcal{A}$ and, thus,

$$X + Z \in \mathcal{A} + \ker \pi.$$

Let $m := \rho_{\mathcal{A},\mathcal{M},\pi}(X)$. Then, there is some $Z^0 \in \ker \pi$ with $Z = mU + Z^0$ by Lemma 2.2.7, i.e.,

$$X + mU + Z^0 \in \mathcal{A} + \ker \pi.$$

By monotonicity of $\mathcal{A} + \ker \pi$ (see Lemma 2.3.14), it holds that

$$\forall t \in \mathbb{R}_>: \quad X + (m + t)U + Z^0 \in \mathcal{A} + \ker \pi. \tag{3.12}$$

Suppose

$$\exists \tilde{t} \in \mathbb{R}_>: \quad X + (m - \tilde{t})U + Z^0 \in \mathcal{A} + \ker \pi. \tag{3.13}$$

Then, there are $X' \in \mathcal{A}$ and $Z' \in \ker \pi$ such that

$$X' = X + (m - \tilde{t})U + Z^0 + Z' \in \mathcal{A}$$

holds. Because of

$$\pi((m - \tilde{t})U + Z^0 + Z') = m - \tilde{t} < m = \pi(Z),$$

we obtain a contradiction to $\pi(Z) = \rho_{\mathcal{A},\mathcal{M},\pi}(X)$. Hence, (3.13) can not be fulfilled, i.e., it holds that

$$\forall t \in \mathbb{R}_>: \quad X + (m - t)U + Z^0 \notin \mathcal{A} + \ker \pi.$$

Hence, by (3.12), it holds that $X + Z \in \operatorname{bd}_{-U}(\mathcal{A} + \ker \pi)$. Taking (3.11) into account, $Z \in \mathcal{Z}(X)$ according to (3.10) holds which completes the proof of $\mathcal{E}(X) \subseteq \mathcal{Z}(X)$, i.e., (3.8) is shown.

In the following, we prove (3.9) and need to show that, for $X \in \mathcal{X}$ arbitrary,

$$\mathcal{E}(X) \supseteq \mathcal{Z}(X) \tag{3.14}$$

holds with $\mathcal{Z}(X)$ defined as in (3.10) if (i) or (ii) is fulfilled. First, we suppose that (i) is fulfilled by taking $X \in \mathcal{X}$ with $\rho_{\mathcal{A},\mathcal{M},\pi}(X) \notin \mathbb{R}$. Then, $\mathcal{E}(X) = \emptyset$ holds by definition of $\mathcal{E}(X)$ (see (3.1)), since $\pi(Z) \in \mathbb{R}$ holds for all $Z \in \mathcal{M}$. Now, we show $\mathcal{Z}(X) = \emptyset$. Assume that

$$\exists Z \in \mathcal{M}: \quad X + Z \in \operatorname{bd}_{-U}(\mathcal{A} + \ker \pi)$$

holds. Then, by Theorem 2.3.25(iii), $\rho_{\mathcal{A},\mathcal{M},\pi}(X+Z) = 0$ holds, implying $\rho_{\mathcal{A},\mathcal{M},\pi}(X) = \pi(Z) \in \mathbb{R}$ by translation invariance of $\rho_{\mathcal{A},\mathcal{M},\pi}$ (see Lemma 2.3.5(ii)) in contradiction to $\rho_{\mathcal{A},\mathcal{M},\pi}(X) \notin \mathbb{R}$. As a result,

$$\forall Z \in \mathcal{M}: \quad X + Z \notin \operatorname{bd}_{-U}(\mathcal{A} + \ker \pi)$$

and, thus,

$$\mathcal{Z}(X) = \emptyset = \mathcal{E}(X)$$

hold by definition of $\mathcal{Z}(X)$ in (3.10). Hence, we have proved (3.14) for condition (i) being fulfilled and, since we also proved (3.8), the proof of (3.9) is complete, i.e., $\mathcal{Z}(X) = \mathcal{E}(X)$ for $X \in \mathcal{X}$ fulfilling $\rho_{\mathcal{A},\mathcal{M},\pi}(X) \in \{-\infty, +\infty\}$.

Now, it is left to prove (3.9) for the case of (ii) being fulfilled. Hence, let $\mathcal{A}$ be a $(-U)$-directionally closed acceptance set. Because we already have proved (3.9) for (i) being fulfilled, we consider $X \in \mathcal{X}$ arbitrary with $\rho_{\mathcal{A},\mathcal{M},\pi}(X) \in \mathbb{R}$. At first, we assume $\mathcal{E}(X) = \emptyset$. Then, it holds that

$$\forall Z \in \mathcal{M} \text{ with } \pi(Z) = \rho_{\mathcal{A},\mathcal{M},\pi}(X): \quad X + Z \notin \mathcal{A}. \tag{3.15}$$

We need to show $\mathcal{Z}(X) = \emptyset$ for $\mathcal{Z}(X)$ as in (3.10). By Theorem 2.3.25(iii), we obtain

$$\forall Z \in \mathcal{M}: \quad X + Z \in \operatorname{bd}_{-U}(\mathcal{A} + \ker \pi) \quad \Longleftrightarrow \quad \rho_{\mathcal{A},\mathcal{M},\pi}(X + Z) = 0$$

which is equivalent to $\rho_{\mathcal{A},\mathcal{M},\pi}(X) = \pi(Z)$ by $\rho_{\mathcal{A},\mathcal{M},\pi}$ being translation invariant (see Lemma 2.3.5(ii)). Thus, we get

$$\forall Z \in \mathcal{M}: \quad X + Z \in \operatorname{bd}_{-U}(\mathcal{A} + \ker \pi) \quad \Longleftrightarrow \quad \pi(Z) = \rho_{\mathcal{A},\mathcal{M},\pi}(X). \tag{3.16}$$

As a result, (3.15) leads to

$$\forall Z \in \mathcal{M}: \qquad X + Z \in \operatorname{bd}_{-U}(\mathcal{A} + \ker \pi) \quad \Longrightarrow \quad X + Z \notin \mathcal{A}.$$

Because of Assumption 4, we obtain $\operatorname{bd}_{-U}(\mathcal{A}) \subseteq \operatorname{cl}_{-U}(\mathcal{A}) = \mathcal{A}$ and, thus,

$$\forall Z \in \mathcal{M}: \qquad X + Z \in \operatorname{bd}_{-U}(\mathcal{A} + \ker \pi) \quad \Longrightarrow \quad X + Z \notin \operatorname{bd}_{-U}(\mathcal{A})$$

which leads by definition of $\mathcal{Z}(X)$ as in (3.10) to

$$\mathcal{Z}(X) = \emptyset = \mathcal{E}(X),$$

showing (3.9). Finally, we show (3.9) for $X \in \mathcal{X}$ with $\mathcal{E}(X) \neq \emptyset$ if (ii) is fulfilled. Then, $\rho_{\mathcal{A},\mathcal{M},\pi}(X) \in \mathbb{R}$ holds. Again by (3.8), it is left to show $\mathcal{E}(X) \supseteq \mathcal{Z}(X)$: Take $Z \in \mathcal{Z}(X)$ arbitrary. Then, by definition of $\mathcal{Z}(X)$ as in (3.10), $Z \in \mathcal{M}$ with

$$X + Z \in \operatorname{bd}_{-U}(\mathcal{A}) \quad \text{and} \quad X + Z \in \operatorname{bd}_{-U}(\mathcal{A} + \ker \pi) \tag{3.17}$$

holds. From the first equation in (3.17), we get $X + Z \in \mathcal{A}$ because of $\operatorname{bd}_{-U}(\mathcal{A}) \subseteq \operatorname{cl}_{-U}(\mathcal{A}) = \mathcal{A}$ by Assumption 4. The second equation in (3.17) delivers (as in the proofs before)

$$\rho_{\mathcal{A},\mathcal{M},\pi}(X) = \pi(Z), \tag{3.18}$$

since $\rho_{\mathcal{A},\mathcal{M},\pi}(X + Z) = \rho_{\mathcal{A},\mathcal{M},\pi}(X) - \pi(Z) = 0$ holds by Theorem 2.3.25(iii) and translation invariance of $\rho_{\mathcal{A},\mathcal{M},\pi}$ (see Lemma 2.3.5(ii)). Hence, by definition of $\mathcal{E}(X)$ (see (3.1)), $Z \in \mathcal{E}(X)$ holds, showing $\mathcal{Z}(X) \subseteq \mathcal{E}(X)$. As a result, the proof of (3.9) is complete. □

**Remark 3.1.7.** *As mentioned before and illustrated in the following Example 3.1.8, Theorem 3.1.6 (and, especially, the characterization (3.9)) is really a generalization of the result (3.7) in [17, Theorem 3.2] for $\mathcal{A} \subseteq \mathcal{X}$ being a closed acceptance set in a locally convex Hausdorff space $\mathcal{X}$. Indeed, we do not make any assumptions on the finiteness and continuity of $\rho_{\mathcal{A},\mathcal{M},\pi}$ or use any topological properties of $\mathcal{A}$ in Theorem 3.1.6. Of course, if $\mathcal{A} \subseteq \mathcal{X}$ is closed for a topological vector space $\mathcal{X}$, $\mathcal{A}$ is $(-U)$-directionally closed, too, such that Theorem 3.1.6 may also be applied. Furthermore, Assumption 4, i.e., $\mathcal{A}$ is $(-U)$-directionally closed with $U \in \mathcal{M} \cap \mathcal{X}_+$ according to Assumption 2, was only necessary in Theorem 3.1.6 for the proof of*

$$\mathcal{E}(X) \supseteq \{Z \in \mathcal{M} \mid X + Z \in \operatorname{bd}_{-U}(\mathcal{A}) \cap \operatorname{bd}_{-U}(\mathcal{A} + \ker \pi)\} =: \mathcal{Z}(X)$$

*to conclude $X + Z \in \mathcal{A}$ for each $X \in \mathcal{X}, Z \in \mathcal{M}$ fulfilling $X + Z \in \operatorname{bd}_{-U}(\mathcal{A})$. Hence, we obtain $\mathcal{E}(X) \subseteq \mathcal{Z}(X)$ for each $X \in \mathcal{X}$ and arbitrary acceptance set $\mathcal{A} \subseteq \mathcal{X}$. Furthermore, it holds that*

$$\operatorname{bd}_{-U}(\mathcal{A}) \cap \operatorname{bd}_{-U}(\mathcal{A} + \ker \pi) \subseteq \operatorname{bd} \mathcal{A} \cap \operatorname{bd}(\mathcal{A} + \ker \pi)$$

*for arbitrary acceptance sets $\mathcal{A}$ in a topological vector space $\mathcal{X}$. Consequently, Theorem 3.1.6 shows that $\operatorname{bd} \mathcal{A} \setminus \operatorname{bd}_{-U}(\mathcal{A})$ and $\operatorname{bd}(\mathcal{A} + \ker \pi) \setminus \operatorname{bd}_{-U}(\mathcal{A} + \ker \pi)$ do not have to be taken into account for determining $\mathcal{E}(X)$, even if $\mathcal{A}$ is a closed acceptance set. Moreover, it holds that $\operatorname{bd}_{-U}(\mathcal{A}) \subseteq \operatorname{bd} \mathcal{A} \subseteq \mathcal{A}$, but $\operatorname{bd} \mathcal{A} = \operatorname{bd}_{-U}(\mathcal{A})$ does not hold in general, see also Example 3.1.8. Nevertheless, we showed in Theorem 2.3.28 that $\rho_{\mathcal{A},\mathcal{M},\pi} \colon \mathcal{X} \to \overline{\mathbb{R}}$ given by (2.30) being continuous is sufficient for $\operatorname{bd}_{-U}(\mathcal{A} + \ker \pi) = \operatorname{bd}(\mathcal{A} + \ker \pi)$, which was additionally assumed in [17].*

**Example 3.1.8** (see [141, Expl. 4.7]). Consider (FM) with $\mathcal{X} = \mathbb{R}^3$, $\mathcal{M} = \{Z \in \mathbb{R}^3 \mid Z_3 = 0\}$ and $\pi(Z) := Z_1 + Z_2$. Then,

$$\ker \pi = \{Z \in \mathcal{M} \mid Z_2 = -Z_1\}$$

holds. Choose $U := (0,1,0)^T$ according to Assumption 2 and consider $\mathcal{A} := \mathcal{X}_+ = \mathbb{R}^3_+$ which is a (-$U$)-directionally closed acceptance set and also closed with respect to the natural topology on $\mathbb{R}^3$. Obviously,

$$\operatorname{bd}_{-U}(\mathcal{A}) = \{X \in \mathbb{R}^3_+ \mid X_2 = 0\} \subsetneq \operatorname{bd}\mathcal{A}$$

is fulfilled and, furthermore,

$$\mathcal{A} + \ker \pi = \{X \in \mathbb{R}^3 \mid X_2 \geq -X_1, X_3 \geq 0\}$$

holds. Consequently, we obtain

$$\operatorname{bd}_{-U}(\mathcal{A} + \ker \pi) = \{X \in \mathbb{R}^3 \mid X_2 = -X_1, X_3 \geq 0\}$$

and

$$\operatorname{bd}(\mathcal{A} + \ker \pi) = \operatorname{bd}_{-U}(\mathcal{A} + \ker \pi) \cup \{X \in \mathbb{R}^3 \mid X_2 \geq -X_1, X_3 = 0\}$$

which shows $\operatorname{bd}_{-U}(\mathcal{A} + \ker \pi) \subsetneq \operatorname{bd}(\mathcal{A} + \ker \pi)$. As a result, we get

$$\operatorname{bd}_{-U}(\mathcal{A}) \cap \operatorname{bd}_{-U}(\mathcal{A} + \ker \pi) = \mathbb{R}_+(0,0,1)^T,$$

while

$$\operatorname{bd}\mathcal{A} \cap \operatorname{bd}(\mathcal{A} + \ker \pi) = \mathbb{R}_+(0,0,1)^T \cup \{X \in \mathbb{R}^3 \mid X_1, X_2 \geq 0, X_3 = 0\}$$

holds. Thus, by Theorem 3.1.6, the solution set of ($P_\pi(X)$) for $X \in \mathcal{X}$ with $X_3 \geq 0$ is

$$\mathcal{E}(X) = \{Z \in \mathbb{R}^3 \mid X + Z \in \mathbb{R}_+(0,0,1)^T, Z_3 = 0\} = \begin{cases} \{(-X_1, -X_2, 0)^T\} & \text{, if } X_3 \geq 0, \\ \emptyset & \text{, else} \end{cases}$$

which implies

$$\rho_{\mathcal{A},\mathcal{M},\pi}(X) = \begin{cases} -X_1 - X_2 & \text{, if } X_3 \geq 0, \\ +\infty & \text{, else} \end{cases}$$

for $\rho_{\mathcal{A},\mathcal{M},\pi} \colon \mathcal{X} \to \overline{\mathbb{R}}$ given by (2.30). Especially, $\rho_{\mathcal{A},\mathcal{M},\pi}$ is not continuous and not finite on $\mathcal{X}$.

◊

**Remark 3.1.9.** *Consider (FM) with a $(-U)$-directional acceptance set $\mathcal{A}$, where $U \in \mathcal{M} \cap \mathcal{X}_+$ is an eligible payoff according to Assumption 2. A decision maker with initial capital position $X \in \mathcal{X}$ can find optimal capital positions $X^0 \in \mathcal{A}$ according to solutions of ($P_\pi(X)$) by the following three geometric steps, which illustrate Theorem 3.1.6 and can be useful for algorithmic purposes (see also Figure 3.2):*

*1.) Find a movement $Z \in \mathcal{M}$ that transforms $X$ into a position*

$$X + Z \in \operatorname{bd}_{-U}(\mathcal{A} + \ker \pi). \tag{3.19}$$

*An eligible payoff $Z \in \mathcal{M}$ fulfilling (3.19) exists if and only if $\rho_{\mathcal{A},\mathcal{M},\pi}(X) \in \mathbb{R}$ with $\rho_{\mathcal{A},\mathcal{M},\pi} \colon \mathcal{X} \to \overline{\mathbb{R}}$ being the functional given by (2.30). Indeed,*

$$\forall Z \in \mathcal{M}: \quad X + Z \in \operatorname{bd}_{-U}(\mathcal{A} + \ker \pi) \quad \Longrightarrow \quad \rho_{\mathcal{A},\mathcal{M},\pi}(X + Z) = \rho_{\mathcal{A},\mathcal{M},\pi}(X) - \pi(Z) = 0$$

*by Theorem 2.3.25(iii) and translation invariance of $\rho_{\mathcal{A},\mathcal{M},\pi}$ by Lemma 2.3.5. Hence, step 1 determines the minimal costs for reaching acceptability with respect to the initial position $X$ because $\rho_{\mathcal{A},\mathcal{M},\pi}(X) = \pi(Z)$ with $Z \in \mathcal{M}$ fulfilling (3.19). As seen in the Reduction Lemma 2.3.11, this step can be simplified because it is sufficient to consider $Z = U \in \mathcal{M} \cap \mathcal{X}_+$ and finding $m \in \mathbb{R}$ with $X + mU \in \operatorname{bd}_{-U}(\mathcal{A} + \ker \pi)$. Indeed, it holds that*

$$\rho_{\mathcal{A},\mathcal{M},\pi}(X) = m \quad \Longleftrightarrow \quad X + mU \in \operatorname{bd}_{-U}(\mathcal{A} + \ker \pi)$$

*by Theorem 2.3.25(iii).*

*2.) If step 1 was successful (i.e., there is a $Z \in \mathcal{M}$ fulfilling (3.19) and, thus, $\rho_{\mathcal{A},\mathcal{M},\pi}(X) \in \mathbb{R}$), find a costless movement $Z^0 \in \ker \pi$ that transforms the in step 1 resulting position $X + Z$ into a capital position $X^0 \in \operatorname{bd}_{-U}(\mathcal{A})$, which is acceptable (i.e., $X^0 \in \mathcal{A}$) by Assumption 4 and, thus, $\operatorname{bd}_{-U}(\mathcal{A}) \subseteq \mathcal{A}$. If there is no such $Z^0 \in \ker \pi$, then $\mathcal{E}(X) = \emptyset$. Note that*

$$\forall X \in \mathcal{X}: \quad \mathcal{E}(X) = \emptyset \quad \not\Rightarrow \quad \rho_{\mathcal{A},\mathcal{M},\pi}(X) \notin \mathbb{R}$$

*holds in general, since step 1 can be successful while step 2 is not, see Example 3.1.14.*

*3.) If both movements $Z, Z^0$ from step 1 and step 2, respectively, exist, then $Z + Z^0 \in \mathcal{E}(X)$ is an optimal eligible payoff with minimal costs*

$$\rho_{\mathcal{A},\mathcal{M},\pi}(X) = \pi(Z - Z^0) = \pi(Z) \in \mathbb{R}$$

*for reaching acceptability, i.e., $X^0 = X + Z + Z^0 \in \mathcal{A}$ is an (cost-)optimal acceptable capital position (see Definition 3.1.1).*

*The procedure is illustrated in Figure 3.2. Moreover, the illustrated $\tilde{X} \in \mathcal{X} \backslash \mathcal{A}$ in the figure shows that*

$$\rho_{\mathcal{A},\mathcal{M},\pi}(\tilde{X}) < 0 \quad \not\Rightarrow \quad \tilde{X} \in \mathcal{A}$$

*holds in general. Indeed, for $\tilde{X} \in \mathcal{X} \backslash \mathcal{A}$ in Figure 3.2, step 1 delivers $\tilde{Z} := \tilde{m}U$ with $\pi(\tilde{Z}) = \tilde{m} < 0$ fulfilling (3.19), i.e.,*

$$\tilde{X} + \tilde{Z} \in \operatorname{bd}_{-U}(\mathcal{A} + \ker \pi),$$

*which leads in step 2 to $\tilde{Z}^0 \in \ker \pi$ with*

$$\tilde{X} + \tilde{Z} + \tilde{Z}^0 \in \operatorname{bd}_{-U}(\mathcal{A}).$$

*Thus,*

$$\rho_{\mathcal{A},\mathcal{M},\pi}(\tilde{X}) = \pi(\tilde{Z} + \tilde{Z}^0) = \tilde{m} < 0$$

*holds. Consequently, if $\tilde{X}$ is the capital position of an institute, the decision maker can set money free to reach acceptability although the origin position $\tilde{X}$ was not acceptable. As mentioned in Remark 2.3.2, the figure also highlights (for example, by considering $X = \mathbb{0}$)*

$$X \in \mathcal{A} \quad \not\Rightarrow \quad \rho_{\mathcal{A},\mathcal{M},\pi}(X) = 0$$

*in general because the risk of $X \in \mathcal{A}$ could also be negative. Indeed, by definition of $\rho_{\mathcal{A},\mathcal{M},\pi}$ in* (2.30) *and $\mathbb{0} \in \mathcal{M}$ with $\pi(\mathbb{0}) = 0$, it holds that*

$$X \in \mathcal{A} \quad \Longrightarrow \quad \rho_{\mathcal{A},\mathcal{M},\pi}(X) \leq 0.$$

*A negative value of $\rho_{\mathcal{A},\mathcal{M},\pi}$ can be interpreted as that the decision maker is to gorgeous in holding an acceptable capital position $X$. That means there is an amount of money bounded by $X$ that could be used otherwise without loosing acceptability. Furthermore, although different positions in Figure 3.2 lead to the same (unique) acceptable capital position $X^0 \in \mathcal{A}$, it is not true in general that there is always an unique acceptable capital position $X^0$ which all initial positions result in by following the steps presented above (see Example 3.1.13 and Theorem 3.2.14).*

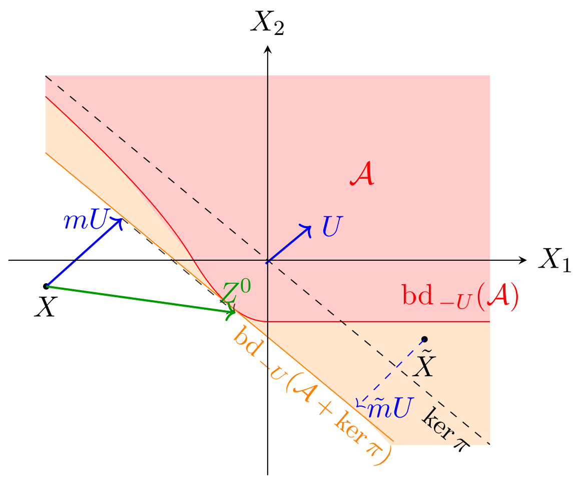


Figure 3.2: Procedure for determining $Z^0 \in \mathcal{E}(X)$ for $\mathcal{X} = \mathcal{M} = \mathbb{R}^2$

As noticed in step 2,

$$\forall X \in \mathcal{X}: \quad \rho_{\mathcal{A},\mathcal{M},\pi}(X) \in \mathbb{R} \quad \not\Rightarrow \quad \mathcal{E}(X) \neq \emptyset \tag{3.20}$$

holds for $\rho_{\mathcal{A},\mathcal{M},\pi} \colon \mathcal{X} \to \overline{\mathbb{R}}$ given by (2.30), but it obviously holds that

$$\forall X \in \mathcal{X}: \quad \rho_{\mathcal{A},\mathcal{M},\pi}(X) \notin \mathbb{R} \quad \Longrightarrow \quad \mathcal{E}(X) = \emptyset \tag{3.21}$$

and

$$\forall X \in \mathcal{X}: \quad \mathcal{E}(X) \neq \emptyset \quad \Longrightarrow \quad \rho_{\mathcal{A},\mathcal{M},\pi}(X) \in \mathbb{R}. \tag{3.22}$$

Indeed, the following theorem shows that the existence of an optimal acceptable capital position according to ($P_\pi(X)$) for any $X \in \mathcal{X}$ depends on the $(-U)$-directional closedness of the augmented set $\mathcal{A} + \ker \pi$ with $U \in \mathcal{M} \cap \mathcal{X}_+$ according to Assumption 2. By (3.22), it is necessary for $\mathcal{E}(X) \neq \emptyset$ with $X \in \mathcal{X}$ that $\rho_{\mathcal{A},\mathcal{M},\pi}(X)$ is finite. The theorem is a generalization of a corresponding result in [17, Prop. 4.1] for closed acceptance sets in a locally convex Hausdorff space $\mathcal{X}$, and $\rho_{\mathcal{A},\mathcal{M},\pi}$ being finite and continuous on $\mathcal{X}$.

**Theorem 3.1.10.** *Consider (FM). Let $\rho_{\mathcal{A},\mathcal{M},\pi}\colon \mathcal{X} \to \overline{\mathbb{R}}$ be the functional given by (2.30) and $U \in \mathcal{M} \cap \mathcal{X}_+$ an eligible payoff according to Assumption 2. Then, the following conditions are equivalent:*

(i) *$\mathcal{E}(X) \neq \emptyset$ for all $X \in \mathcal{X}$ with $\rho_{\mathcal{A},\mathcal{M},\pi}(X) \in \mathbb{R}$,*

(ii) *$\mathcal{E}(X) \neq \emptyset$ for all $X \in \operatorname{bd}_{-U}(\mathcal{A} + \ker \pi)$,*

(iii) *$\mathcal{A} + \ker \pi$ is $(-U)$-directionally closed.*

**Proof.** Let $\rho_{\mathcal{A},\mathcal{M},\pi}$ be finite on $\mathcal{X}$. Obviously, (i) $\Rightarrow$ (ii) holds by Theorem 2.3.25. Suppose that (ii) is fulfilled and prove that (iii) holds. Then, for $X \in \operatorname{bd}_{-U}(\mathcal{A} + \ker \pi)$ arbitrary, we obtain

$$\rho_{\mathcal{A},\mathcal{M},\pi}(X) = 0$$

by Theorem 2.3.25(iii). Because of $\mathcal{E}(X) \neq \emptyset$ by (ii) and definition of $\mathcal{E}(X)$ (see (2.6)), there is some $Z \in \mathcal{M}$ with $X + Z \in \mathcal{A}$ and $Z \in \ker \pi$ by $\pi(Z) = \rho_{\mathcal{A},\mathcal{M},\pi}(X) = 0$. Hence, $X \in \mathcal{A} + \ker \pi$ holds, showing

$$\operatorname{bd}_{-U}(\mathcal{A} + \ker \pi) \subseteq \mathcal{A} + \ker \pi,$$

i.e., (iii) holds. It is left to show (iii) $\Rightarrow$ (i). Suppose $\mathcal{A} + \ker \pi$ is $(-U)$-directionally closed and take $X \in \mathcal{X}$ arbitrary with $\rho_{\mathcal{A},\mathcal{M},\pi}(X) \in \mathbb{R}$. Then,

$$\rho_{\mathcal{A},\mathcal{M},\pi}(X + \rho_{\mathcal{A},\mathcal{M},\pi}(X)U) = 0$$

by translation invariance of $\rho_{\mathcal{A},\mathcal{M},\pi}$ (see Lemma 2.3.5(ii)) and $\pi(U) = 1$ from Assumption 2. Hence,

$$X + \rho_{\mathcal{A},\mathcal{M},\pi}(X)U \in \operatorname{bd}_{-U}(\mathcal{A} + \ker \pi)$$

holds by Theorem 2.3.25(iii). Because $\operatorname{bd}_{-U}(\mathcal{A} + \ker \pi) \subseteq \mathcal{A} + \ker \pi$ holds by (iii), there is some $Z^0 \in \ker \pi$ with

$$X + \rho_{\mathcal{A},\mathcal{M},\pi}(X)U + Z^0 \in \mathcal{A}.$$

Thus, $Z := \rho_{\mathcal{A},\mathcal{M},\pi}(X)U + Z^0 \in \mathcal{M}$ fulfills $Z \in \mathcal{E}(X)$ because of $\pi(Z) = \rho_{\mathcal{A},\mathcal{M},\pi}(X)$ by linearity of $\pi$, $\pi(U) = 1$ by Assumption 2 and $\pi(Z^0) = 0$. Consequently, $\mathcal{E}(X) \neq \emptyset$ holds, which completes the proof of (iii) $\Rightarrow$ (i). □

**Remark 3.1.11.** *Consider (FM), $U \in \mathcal{M} \cap \mathcal{X}_+$ according to Assumption 2, the risk measure $\rho_{\mathcal{A},\mathcal{M},\pi}\colon \mathcal{X} \to \overline{\mathbb{R}}$ given by (2.30) and the optimal payoff map $\mathcal{E}\colon \mathcal{X} \rightrightarrows \mathcal{M}$ given by (2.6). As seen in Example 3.1.8, $(-U)$-directional closedness of $\mathcal{A}+\ker\pi$ alone does not secure $\rho_{\mathcal{A},\mathcal{M},\pi}(X) \in \mathbb{R}$ and, thus, $\mathcal{E}(X) \neq \emptyset$ for all $X \in \mathcal{X}$. Thus, we have to assume finiteness of $\rho_{\mathcal{A},\mathcal{M},\pi}(X)$ in Theorem 3.1.10*(i) *and the theorem delivers for the case of $\rho_{\mathcal{A},\mathcal{M},\pi}$ being finite on the whole space $\mathcal{X}$ a characterization for*

$$\forall X \in \mathcal{X}: \quad \mathcal{E}(X) \neq \emptyset$$

*by the directionally closedness of $\mathcal{A}+\ker\pi$. Note that*

$$\mathcal{A}+\ker\pi \text{ is } (-U)\text{-directionally closed} \quad \not\Rightarrow \quad \mathcal{A} \text{ is } (-U)\text{-directionally closed}, \tag{3.23}$$

*see Example 3.1.12, and*

$$\mathcal{A} \text{ is } (-U)\text{-directionally closed} \quad \not\Rightarrow \quad \mathcal{A}+\ker\pi \text{ is } (-U)\text{-directionally closed}, \tag{3.24}$$

*see Example 3.1.14. Thus, the same holds for closed acceptance sets in topological vector spaces. Hence, Theorem 3.1.10 can also be applied if Assumption 4 is not fulfilled and, thus, for arbitrary acceptance sets $\mathcal{A}$.*

**Example 3.1.12.** Consider (FM) with $\mathcal{X} = \mathcal{M} = \mathbb{R}^2$, $\mathcal{A} = (-1,+\infty) \times \mathbb{R}_+$, $U = (0,1)^T$, and $\pi(Z) = \pi(Z_1,Z_2) = Z_2$. Then, $\mathcal{A}$ is not $(-U)$-directionally closed, but $\mathcal{A}+\ker\pi = \mathbb{R} \times \mathbb{R}_+$ is $(-U)$-directionally closed. Moreover, $(-X_1,-X_2) \in \mathcal{E}(X)$ for each $X \in \mathcal{X}$ and $\rho_{\mathcal{A},\mathcal{M},\pi}(X) = -X_2$ for each $X = (X_1,X_2)^T \in \mathbb{R}^2$ with $\mathcal{E}(X)$ given by (3.1) and $\rho_{\mathcal{A},\mathcal{M},\pi}\colon \mathcal{X} \to \overline{\mathbb{R}}$ given by (2.30). That confirms Theorem 3.1.10 and that Assumption 4 is not necessary. ♢

The following examples for (FM) are motivated by those from Baes et al. in [17] and outline with respect to Theorem 3.1.10 that even desirable properties of the acceptance set $\mathcal{A} \subseteq \mathcal{X}$ like convexity do not necessarily lead to an unique solution of ($P_\pi(X)$) or, even, a nonempty solution set $\mathcal{E}(X) \neq \emptyset$ for $X \in \mathcal{X}$ arbitrary and $\mathcal{E}\colon \mathcal{X} \rightrightarrows \mathcal{M}$ given by (3.1). In the figures belonging to these examples, we have already illustrated the set of optimal acceptable capital positions $\mathcal{A}' := \operatorname{bd}_{-U}(\mathcal{A}) \cap \operatorname{bd}_{-U}(\mathcal{A}+\ker\pi)$ (see Definition 3.2.1), which we will study in the next section. In these examples, $\mathcal{A}'$ represents all acceptable capital positions, which are suitable positions for the given $X \in \mathcal{X}$ to change into by $Z \in \mathcal{E}(X)$ for making the position $X$ acceptable.

Example 3.1.13 shows that $\mathcal{E}(X)$ can consist of infinitely many elements even if $\mathcal{A}$ is convex and, therefore, star-shaped.

**Example 3.1.13** (see Marohn, Tammer [139])**.** Consider (FM) with $\mathcal{X} = \mathbb{R}^2 = \mathcal{M}$ and $\pi(Z) := \frac{Z_1+Z_2}{2}$ for $Z = (Z_1,Z_2)^T \in \mathcal{X}$. Furthermore, take $U = (1,1)^T \in \mathcal{M} \cap \mathcal{X}_+$ which fulfills $\pi(U) = 1$ and, therefore, Assumption 2. Consider

$$\mathcal{A} := \left\{X \in \mathbb{R}^2 \mid X_1 \geq -1, X_2 \geq \max\{-1,-X_1\}\right\},$$

which is a convex acceptance set $\mathcal{A} \subseteq \mathcal{X}$ according to Definition 2.2.9 and, thus, star-shaped, see also Figure 3.3. Moreover, $\mathcal{A}$ is closed for the Euclidean topology on $\mathbb{R}^2$ and, thus, $(-U)$-directionally closed. Then,

$$\ker \pi = \{Z \in \mathbb{R}^2 \mid Z_2 = -Z_1\}$$

delivers

$$\mathcal{A} + \ker \pi = \{X \in \mathbb{R}^2 \mid X_2 \geq -X_1\},$$

which is a closed half-space of $\mathcal{X}$ with respect to the Euclidean topology on $\mathbb{R}^2$. Then, we obtain

$$\operatorname{bd}_{-U}(\mathcal{A} + \ker \pi) = \{X \in \mathbb{R}^2 \mid X_2 = -X_1\}$$

and

$$\operatorname{bd}_{-U}(\mathcal{A}) \cap \operatorname{bd}_{-U}(\mathcal{A} + \ker \pi) = \left\{X \in \mathbb{R}^2 \mid -1 \leq X_1 \leq 1, X_2 = -X_1\right\}.$$

Consequently, for arbitrary $X \in \mathbb{R}^2$, we have

$$X - mU \in \operatorname{bd}_{-U}(\mathcal{A} + \ker \pi)$$

with $m := \frac{X_1+X_2}{2}$, and $\pi(-mU) = -\frac{X_1+X_2}{2}$ by linearity of $\pi$ and $\pi(U) = 1$. Since $mU - X \in \mathcal{M}$ holds, we obtain for

$$Z^0 := -mU + (mU - X) = X$$

that

$$\forall X \in \mathbb{R}^2: \qquad X + Z^0 = \mathbb{0} \in \operatorname{bd}_{-U}(\mathcal{A}) \cap \operatorname{bd}_{-U}(\mathcal{A} + \ker \pi)$$

holds. Because $\mathcal{A}$ is $(-U)$-directionally closed, it holds that $Z^0 \in \mathcal{E}(X)$ and, especially, $\mathcal{E}(X) \neq \emptyset$ for each $X \in \mathbb{R}^2$ by Theorem 3.1.6. More precisely,

$$\mathcal{E}(X) = \{Z \in \mathbb{R}^2 \mid -1 - X_1 \leq Z_1 \leq 1 - X_1, Z_2 = -(Z_1 + X_1) - X_2\}.$$

and, thus, $|\mathcal{E}(X)| = +\infty$ hold. By $Z^0 \in \mathcal{E}(X)$, we obtain

$$\forall X \in \mathcal{X}: \quad \rho_{\mathcal{A},\mathcal{M},\pi}(X) = \pi(Z^0) = -\frac{X_1 + X_2}{2} \in \mathbb{R}$$

with $\rho_{\mathcal{A},\mathcal{M},\pi}\colon \mathcal{X} \to \overline{\mathbb{R}}$ being the functional given by (2.30).

◊

Next, we give an example for (FM) with $\mathcal{A} \subseteq \mathcal{X}$ being a convex acceptance set, but $\mathcal{E}(X) = \emptyset$ holds for each $X \in \mathcal{X}$ and $\mathcal{E}\colon \mathcal{X} \rightrightarrows \mathcal{M}$ defined as in (2.6).

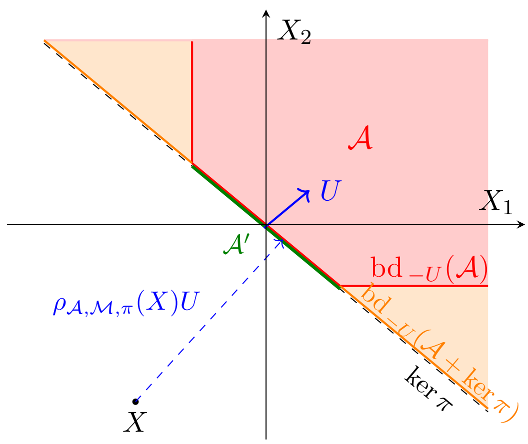


Figure 3.3: Starshaped $\mathcal{A}$ with $|\mathcal{E}(X)| = +\infty$ for all $X \in \mathcal{X}$

**Example 3.1.14** (see Marohn, Tammer [139]). Consider (FM) with $\mathcal{X}, \mathcal{M}, \pi$ and $U$ as in Example 3.1.13. Let

$$\mathcal{A} := \left\{X \in \mathbb{R}^2 \mid X_1 > -1, X_2 \geq \frac{-(X_1+1)^2 - 1}{X_1 + 1}\right\}.$$

Then, $\mathcal{A} \subseteq \mathcal{X}$ is a convex (and, thus, star-shaped) acceptance set according to Definition 2.2.9. $\mathcal{A}$ is closed with respect to the Euclidean topology and, thus, also $(-U)$-directionally closed. Furthermore,

$$\mathcal{A} + \ker\pi = \{X \in \mathbb{R}^2 \mid X_2 > -X_1 - 1\},$$

which is an open half-space with respect to the Euclidean topology. It holds that

$$\operatorname{bd}_{-U}(\mathcal{A} + \ker\pi) = \{X \in \mathbb{R}^2 \mid X_2 = -X_1 - 1\} \not\subseteq \mathcal{A} + \ker\pi,$$

implying

$$\operatorname{bd}_{-U}(\mathcal{A}) \cap \operatorname{bd}_{-U}(\mathcal{A} + \ker\pi) = \emptyset.$$

Hence, $\mathcal{E}(X) = \emptyset$ for all $X \in \mathbb{R}^2$ by (3.8) in Theorem 3.1.6 as illustrated in Figure 3.4. On the other hand, it holds that

$$\forall X \in \mathbb{R}^2: \quad \rho_{\mathcal{A},\mathcal{M},\pi}(X) = -\frac{X_1 + X_2}{2} - \frac{1}{2} \in \mathbb{R}$$

with $\rho_{\mathcal{A},\mathcal{M},\pi}\colon \mathcal{X} \to \overline{\mathbb{R}}$ given by (2.30). That highlights our remark according to (3.20): the finiteness of $\rho_{\mathcal{A},\mathcal{M},\pi}(X)$ for $X \in \mathcal{X}$ does not imply the existence of solutions of ($P_\pi(X)$), i.e., $\mathcal{E}(X) \neq \emptyset$.

$\Diamond$

Finally, we give an example for a non-star-shaped acceptance set $\mathcal{A}$ star-shaped with unique solution of ($P_\pi(X)$) for each $X \in \mathcal{X}$.

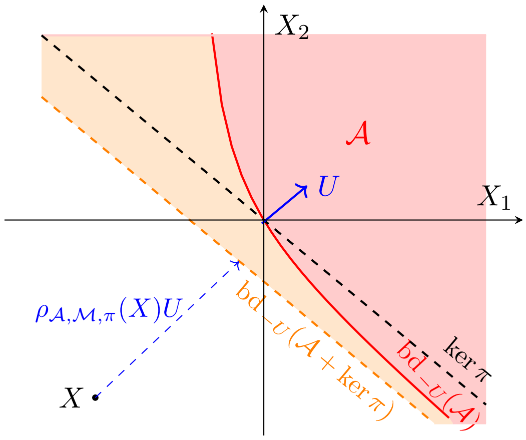


Figure 3.4: Starshaped $\mathcal{A}$ with $\mathcal{E}(X) = \emptyset$ for all $X \in \mathcal{X}$

**Example 3.1.15** (see Marohn, Tammer [139]). Consider (FM) with $\mathcal{X}, \mathcal{M}, \pi$ and $U$ as in Example 3.1.13. Let

$$\mathcal{A} := \left\{ X \in \mathbb{R}^2 \middle| X_1 \leq 0, X_2 \geq \frac{-X_1^2 + 2X_1}{X - 1} \right\} \cup \mathcal{X}_+,$$

which is a non-star-shaped (especially, non-convex) acceptance set as required in Definition 2.2.9. Moreover, $\mathcal{A}$ is closed with respect to the Euclidean topology and, thus, $(-U)$-directionally closed. Then,

$$\mathcal{A} + \ker \pi = \{X \in \mathbb{R}^2 \mid X_2 \geq -X_1\}$$

is a closed half-space with respect to the Euclidean topology, see also Figure 3.5. Furthermore, it holds that

$$\operatorname{bd}_{-U}(\mathcal{A}) \cap \operatorname{bd}_{-U}(\mathcal{A} + \ker \pi) = \{\mathbb{0}\}$$

and, hence, $\mathcal{E}(X) = \{-X\} \neq \emptyset$ for each $X \in \mathbb{R}^2$ by Theorem 3.1.6 and $\mathcal{M} = \mathbb{R}^2$.

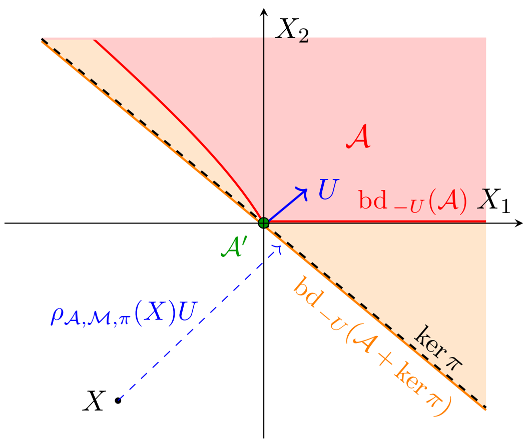


Figure 3.5: Non-starshaped $\mathcal{A}$ with $|\mathcal{E}(X)| = 1$ for all $X \in \mathcal{X}$

◊

Further existence and uniqueness results for closed acceptance sets in locally convex Hausdorff topological vector spaces can be found in [17].

## 3.2 Cost-optimal acceptable capital positions

Consider (FM). In this section, we follow [139] and [141], respectively, and study the following set, which is motivated by Theorem 3.1.6:

**Definition 3.2.1.** Consider (FM) and an eligible payoff $U \in \mathcal{M} \cap \mathcal{X}_+$ according to Assumption 2. The *set of (cost-)optimal acceptable capital positions* is defined by

$$\mathcal{A}' := \operatorname{bd}_{-U}(\mathcal{A}) \cap \operatorname{bd}_{-U}(\mathcal{A} + \ker \pi). \tag{3.25}$$

As seen in Theorem 3.1.6 and reformulated in Theorem 3.2.2 for directionally closed acceptance sets $\mathcal{A}$, $\mathcal{A}'$ given by (3.25) is directly connected with the optimal payoff map $\mathcal{E}\colon \mathcal{X} \rightrightarrows \mathcal{M}$ given by (2.6) and, thus, with our economical problem ($P_\pi(X)$) for $X \in \mathcal{X}$. Moreover, $\mathcal{A}'$ is from special interest with respect to our studies of efficient points of the acceptance set $\mathcal{A}$ in Chapter 4. As noticed in Lemma 1.2.3 for topological vector spaces, efficient points of $\mathcal{A}$ are elements of the boundary under some assumption on $\mathcal{A}$. In Theorem 4.2.4, we will see that the efficient points of $\mathcal{A}$ are elements of $\operatorname{bd}_{-U}(\mathcal{A})$ with $U \in \mathcal{M} \cap \mathcal{X}_+$ according to Assumption 2 if we consider a vector space $\mathcal{X}$ without any topology. Indeed, we will prove in Theorem 4.2.6 the interesting result that the set of efficient points of $\mathcal{A}$ is a subset of $\mathcal{A}'$ Hence, there are extraordinary points in $\operatorname{bd} \mathcal{A}$ (or, more exactly, $\operatorname{bd}_{-U}(\mathcal{A})$) that are from interest for being candidates for efficient points, namely those which also belong to the $(-U)$-directional boundary of the augmented set $\mathcal{A} + \ker \pi$. Considering $\mathcal{A} + \ker \pi$ emphasizes that we are relaxing the set of acceptable points being from interest for the investor (see also Remark 2.3.18 about $\mathcal{A} + \ker \pi$ being an acceptance set). $\mathcal{A} + \ker \pi$ is the set of elements in $\mathcal{X}$ which can be made acceptable by zero costs. This set fits better any preference relation of the investor because an acceptable capital position $X$ and a capital position that can be transformed into that position $X$ without any additional costs should be equally valued for the investor if he is focused on the acceptance set. We already used the approach of searching a position in $\operatorname{bd}_{-U}(\mathcal{A} + \ker \pi)$ first in our algorithmic notes in Remark 3.1.9 in step 1. By considering $\mathcal{A}'$ given by (3.25), the part $\operatorname{bd}_{-U}(\mathcal{A})$ in $\mathcal{A}'$ will secure for $(-U)$-directionally closed acceptance sets that the transformation of a position in $\operatorname{bd}_{-U}(\mathcal{A} + \ker \pi)$ into some position in $\mathcal{A}$ is really possible, since $\operatorname{bd}_{-U}(\mathcal{A}) \subseteq \mathcal{A}$ then.

Note that we assumed $\mathcal{A}$ to be a closed acceptance set in a locally convex Hausdorff space and $\rho_{\mathcal{A},\mathcal{M},\pi}\colon \mathcal{X} \to \overline{\mathbb{R}}$ given by (2.30) to be continuous and finite in [139]. Moreover, $\mathcal{A}'$ is defined in [139] by the boundary of $\mathcal{A}$ and $\mathcal{A} + \ker \pi$, respectively, instead of the $(-U)$-directional boundary. As noticed in [141] and showed in the following, these assumptions can be relaxed. The first theorem concludes by Theorem 3.1.6 that the terminology for $\mathcal{A}'$ defined as in (3.25) is obviously justified, i.e., $\mathcal{A}'$ really coincides with the set of all optimal acceptable capital positions (see Definition 3.1.1).

**Theorem 3.2.2** (see [141, Theorem 4.8])**.** *Consider* (FM)*. Let $\mathcal{A} \subseteq \mathcal{X}$ be an acceptance set such that Assumption 4 is fulfilled with $U \in \mathcal{M} \cap \mathcal{X}_+$ being an eligible payoff according to Assumption 2. Moreover, let $\mathcal{E}\colon \mathcal{X} \rightrightarrows \mathcal{M}$ be the optimal payoff map as in Definition 3.1.1 and $\mathcal{A}' \subseteq \mathcal{X}$ given by* (3.25)*. Then, the following holds:*

$$\mathcal{A}' = \{X + Z \in \mathcal{X} \mid X \in \mathcal{X}, Z \in \mathcal{E}(X) \neq \emptyset\}.$$

**Proof.** The relationship $(\supseteq)$ is a direct consequence of Theorem 3.1.6. Analogously, we obtain $\mathbb{0} \in \mathcal{E}(X)$ for each $X \in \mathcal{A}'$ by Theorem 3.1.6 because $\mathcal{A}$ is assumed to be $(-U)$-directionally closed, showing $(\subseteq)$. □

**Remark 3.2.3.** *As seen in the proof, Theorem 3.2.2 can be seen as a reformulation of Theorem 3.1.6 and highlights the importance of Assumption 4: For* (FM)*, $\mathcal{A}' \subseteq \mathcal{X}$ defined as in* (3.25)*, $\mathcal{E}\colon \mathcal{X} \rightrightarrows \mathcal{M}$ the optimal payoff map given by* (3.1) *and $\mathcal{A} \subseteq \mathcal{X}$ being an arbitrary acceptance set, it holds that*

$$\mathcal{A}' \supseteq \{X + Z \in \mathcal{X} \mid X \in \mathcal{X}, Z \in \mathcal{E}(X) \neq \emptyset\}. \tag{3.26}$$

*If $\mathcal{A}$ fulfills Assumption 4 with $U \in \mathcal{M} \cap \mathcal{X}_+$ being a eligible payoff according to Assumption 2, we obtain the equality in* (3.26) *by Theorem 3.2.2 and it holds that*

$$\forall X \in \mathcal{X}: \quad \mathcal{E}(X) = \{Z \in \mathcal{M} \mid X + Z \in \mathcal{A}'\} = \mathcal{M} \cap (\mathcal{A}' - \{X\}). \tag{3.27}$$

*Hence, we obtain for the case $\mathcal{M} = \mathcal{X}$ (which is only possible for $\dim \mathcal{X} < +\infty$ by* (2.11)*)*

$$\mathcal{A}' \neq \emptyset \quad \Longrightarrow \quad \forall X \in \mathcal{X}: \; \mathcal{E}(X) \neq \emptyset.$$

The following characterization of $\mathcal{A} \backslash \mathcal{A}'$ with $\mathcal{A}'$ given by (3.25) for arbitrary acceptance sets $\mathcal{A} \subseteq \mathcal{X}$ will be very useful for proving further properties of $\mathcal{A}'$, deriving an uniqueness result for ($P_\pi(X)$) in Theorem 3.2.14, and our studies of (weakly) efficient points in Chapter 4:

**Lemma 3.2.4.** *Consider* (FM)*. Let Assumption 2 be fulfilled by some eligible payoff $U \in \mathcal{M} \cap \mathcal{X}_+$. Furthermore, let $\mathcal{A}' \subseteq \mathcal{X}$ be the set optimal acceptable capital positions given by* (3.25)*. Then,*

$$\mathcal{A} \backslash \mathcal{A}' \subseteq \operatorname{int}_{-U}(\mathcal{A} + \ker \pi). \tag{3.28}$$

**Proof.** Take $X \in \mathcal{A}$ with $X \notin \mathcal{A}' = \operatorname{bd}_{-U}(\mathcal{A}) \cap \operatorname{bd}_{-U}(\mathcal{A} + \ker \pi)$ arbitrary. If $X \in \operatorname{bd}_{-U}(\mathcal{A})$, then $X \notin \operatorname{bd}_{-U}(\mathcal{A} + \ker \pi)$ by definition of $\mathcal{A}'$ (see (3.25)) and, thus, $X \in \operatorname{int}_{-U}(\mathcal{A} + \ker \pi)$. Otherwise, if $X \in \operatorname{int}_{-U}(\mathcal{A})$, then,

$$\exists \epsilon > 0, \; \forall t \in [0, \epsilon]: \quad X - tU \in \mathcal{A} \subseteq \mathcal{A} + \ker \pi.$$

Thus, $X \in \operatorname{int}_{-U}(\mathcal{A} + \ker \pi)$, too. That completes the proof of (3.28). □

Especially, with respect to Theorem 3.2.2, it will be helpful to characterize acceptable positions that can be reached by elements of $\mathcal{A}'$ by non-positive costs with $\mathcal{A}'$ given by (3.25). The following lemma shows that there are no acceptable positions that can be reached by negative costs. Moreover, these elements belong to $\mathcal{A}'$ again, and for directionally closed acceptance sets, $\mathcal{A}'$ coincides with the set of these acceptable positions. Lemma 3.2.5 is mentioned in [141, Equation (4.13)] for directionally closed acceptance sets without a proof:

**Lemma 3.2.5** (see [141])**.** *Consider* (FM)*. Let $U \in \mathcal{M} \cap \mathcal{X}_+$ be an eligible payoff according to Assumption 2 and $\mathcal{A}' \subseteq \mathcal{X}$ given by* (3.25)*. Then, it holds that*

$$\left(\mathcal{A}' + \bigcup_{m\leq 0} \pi_m\right) \cap \mathcal{A} = (\mathcal{A}' + \ker \pi) \cap \mathcal{A} \subseteq \mathcal{A}' \tag{3.29}$$

*with $\pi_m \subseteq \mathcal{M}$ given by* (2.16) *being the set of all eligible payoffs with price $m \in \mathbb{R}$. If $\mathcal{A} \subseteq \mathcal{X}$ is an acceptance set such that Assumption 4 is fulfilled with respect to $U$, then*

$$\left(\mathcal{A}' + \bigcup_{m\leq 0} \pi_m\right) \cap \mathcal{A} = (\mathcal{A}' + \ker \pi) \cap \mathcal{A} = \mathcal{A}'. \tag{3.30}$$

**Proof.** First, let $\mathcal{A}$ be an arbitrary acceptance set. We show

$$\left(\mathcal{A}' + \bigcup_{m\leq 0} \pi_m\right) \cap \mathcal{A} = (\mathcal{A}' + \ker \pi) \cap \mathcal{A}. \tag{3.31}$$

Because of $\pi_0 = \ker \pi$, we only need to show

$$\left(\mathcal{A}' + \bigcup_{m< 0} \pi_m\right) \cap \mathcal{A} = \emptyset. \tag{3.32}$$

Suppose there are $Y \in \mathcal{A}$ and $X \in \mathcal{A}'$ with

$$Y = X + Z \quad \text{ for some } Z \in \mathcal{M} \text{ with } \pi(Z) < 0.$$

Then,

$$\exists m \in \mathbb{R}_>, \exists Z^0 \in \ker \pi : \quad Y = X - mU + Z^0$$

by the Reduction Lemma 2.3.11. Thus, $X - mU \in \mathcal{A} + \ker \pi$ by $Y \in \mathcal{A}$. Because of the monotonicity of $\mathcal{A} + \ker \pi$ (see Lemma 2.3.14) and $U \in \mathcal{X}_+$, we obtain

$$\forall t \in \mathbb{R}_> : \quad X - (m - t)U \in \mathcal{A} + \ker \pi,$$

which implies $X \in \operatorname{int}_{-U}(\mathcal{A} + \ker \pi)$, in contradiction to $X \in \mathcal{A}' \subseteq \operatorname{bd}_{-U}(\mathcal{A} + \ker \pi)$. As a result, (3.32) and, also, (3.31) must hold. Now, we prove

$$(\mathcal{A}' + \ker \pi) \cap \mathcal{A} \subseteq \mathcal{A}'. \tag{3.33}$$

Suppose

$$\exists X \in \mathcal{A}', \exists Z^0 \in \ker \pi : \quad X + Z^0 \in \mathcal{A} \backslash \mathcal{A}'.$$

By Lemma 3.2.4, $X + Z^0 \in \operatorname{int}_{-U}(\mathcal{A} + \ker \pi)$. Hence,

$$\exists \epsilon \in \mathbb{R}_>, \forall m \in [0, \epsilon] : \quad X + Z^0 - mU \in \mathcal{A} + \ker \pi.$$

By $Z^0 \in \ker \pi$, this is implies

$$\exists \epsilon \in \mathbb{R}_>, \forall m \in [0, \epsilon] : \quad X - mU \in \mathcal{A} + \ker \pi,$$

i.e., $X \in \operatorname{int}_{-U}(\mathcal{A} + \ker \pi)$, in contradiction to $X \in \mathcal{A}' \subseteq \operatorname{bd}_{-U}(\mathcal{A} + \ker \pi)$. Thus, (3.33) is shown and the proof of (3.29) is complete.

Now, suppose that $\mathcal{A}$ is $(-U)$-directionally closed. Because of $\{\mathbb{0}\} \subseteq \ker \pi = \pi_0$ and $\mathcal{A}' \subseteq \operatorname{bd}_{-U}(\mathcal{A}) \subseteq \mathcal{A}$ for $\mathcal{A}$ being $(-U)$-directionally closed, it holds that

$$\mathcal{A}' = \left( \mathcal{A}' + \bigcup_{m \le 0} \pi_m \right) \cap \mathcal{A}' \subseteq \left( \mathcal{A}' + \bigcup_{m \le 0} \pi_m \right) \cap \mathcal{A} = (\mathcal{A}' + \ker \pi) \cap \mathcal{A} \subseteq \mathcal{A}'$$

by (3.29), i.e., all sets coincide. That shows (3.30) and everything is proved. □

Lemma 3.2.5 explains that if we are able to move from $X \in \mathcal{A}' \subseteq \mathcal{A}$ along $\ker \pi$ with preserving acceptability (i.e., staying in $\mathcal{A}$), we obtain again a capital position belonging to $\mathcal{A}'$. Moreover, we can not reach another position in $\mathcal{A}$ (and, thus, $\mathcal{A}'$, too) by considering movements with price $m < 0$, i.e.,

$$\forall X \in \mathcal{A}', \forall m < 0, \forall Z \in \pi_m : \quad X + Z \notin \mathcal{A}.$$

As a result, the difference $X - Y$ of two arbitrary capital positions $X, Y \in \mathcal{A}'$ belongs to $\ker \pi$ if it represents an eligible payoff, i.e., $X - Y \in \mathcal{M}$. Note that, in general, it holds that $\mathcal{A}' - \mathcal{A}' \nsubseteq \mathcal{M}$. This is stated in the following lemma, which will be crucial for our studies:

**Corollary 3.2.6** (see [141, Lemma 4.9]). *Consider (FM). Let $U \in \mathcal{M} \cap \mathcal{X}_+$ be an eligible payoff according to Assumption 2 and $\mathcal{A}' \subseteq \mathcal{X}$ given by (3.25). Moreover, take $X^0, Y^0 \in \mathcal{A}'$ with $X^0 - Y^0 \in \mathcal{M}$ arbitrary. Then, $X^0 - Y^0 \in \ker \pi$ holds.*

**Proof.** Suppose that there are $X^0, Y^0 \in \mathcal{A}'$ with $X^0 - Y^0 \in \mathcal{M}$ such that $\pi(X^0 - Y^0) \neq 0$. Without loss of generality, assume $\pi(X^0 - Y^0) < 0$. Let $m := -\pi(X^0 - Y^0)$. Then, there is some $Z^0 \in \ker \pi$ with $X^0 - Y^0 = -mU + Z^0$ by the Reduction Lemma 2.3.11, i.e.,

$$X = Y^0 + (X^0 - Y^0) = Y^0 - mU + Z^0. \tag{3.34}$$

Since $X \in \mathcal{A}' \subseteq \operatorname{bd}_{-U}(\mathcal{A})$, it holds that

$$\forall t > 0 : \quad X + tU \in \mathcal{A}$$

and, thus,

$$\forall t > 0: \quad Y + (t - m)U = X + tU - Z^0 \in \mathcal{A} + \ker \pi$$

by (3.34) and $Z^0 \in \ker \pi$. As a result, $Y \in \operatorname{int}_{-U}(\mathcal{A} + \ker \pi)$ in contradiction to $Y \in \mathcal{A}'$, since $\mathcal{A}' \subseteq \operatorname{bd}_{-U}(\mathcal{A} + \ker \pi)$. Hence, $\pi(X^0 - Y^0) \geq 0$ holds. Suppose $\pi(X^0 - Y^0) > 0$. Then, $\pi(Y^0 - X^0) < 0$, which also delivers a contradiction as above, since we can switch the roles of $X^0$ and $Y^0$ such that the former proof can be analogously applied. Thus, $X^0 - Y^0 \in \ker \pi$ holds by $X^0 - Y^0 \in \mathcal{M}$ and $\pi(X^0 - Y^0) \in \mathbb{R}$, and the proof of Corollary 3.2.6 is complete. □

Corollary 3.2.6 implies

$$\mathcal{A}' - \mathcal{A}' \subseteq \mathcal{M} \quad \Longrightarrow \quad \mathcal{A}' - \mathcal{A}' \subseteq \ker \pi,$$

which leads to the following more precise characterization of $\mathcal{A}'$ given by (3.25): if we are able to move one capital position in $\mathcal{A}'$ to another via $\mathcal{M}$, i.e., $\mathcal{A}' - \mathcal{A}' \in \mathcal{M}$ holds, then the set $\mathcal{A}'$ coincides with the optimal acceptable capital positions for a *single* arbitrary given $X \in \mathcal{X}$ fulfilling $\mathcal{E}(X) \neq \emptyset$. Also, the theorem gives more details about (3.27) for this situation.

**Theorem 3.2.7** (see [141, Theorem 4.10])**.** *Consider (FM). Let $\mathcal{A} \subseteq \mathcal{X}$ be an acceptance set such that Assumption 4 is fulfilled with $U \in \mathcal{M} \cap \mathcal{X}_+$ being a eligible payoff according to Assumption 2 and $\mathcal{A}' \subseteq \mathcal{X}$ given by (3.25). Furthermore, let $\mathcal{E} \colon \mathcal{X} \rightrightarrows \mathcal{M}$ be the optimal payoff map as in (3.1). Take $X \in \mathcal{X}$ with $\mathcal{E}(X) \neq \emptyset$ arbitrary and suppose $\mathcal{A}' - \mathcal{A}' \subseteq \mathcal{M}$, i.e., it holds that*

$$\forall X^0, Y^0 \in \mathcal{A}': \quad X^0 - Y^0 \in \mathcal{M}. \tag{3.35}$$

*Then, the following conditions are fulfilled:*

(i) *It holds that $\mathcal{A}' = \{X + Z \in \mathcal{X} \mid Z \in \mathcal{E}(X)\}$,*

(ii) *For all $Y \in \mathcal{X}$ with $\mathcal{E}(Y) \neq \emptyset$, it holds that*

$$\mathcal{E}(Y) = \{Y^0 - Y \mid Y^0 \in \mathcal{A}'\} = \{X + Z - Y \mid Z \in \mathcal{E}(X)\}.$$

**Proof.** Let $X \in \mathcal{X}$ be arbitrary and fixed. First, we show (i). By Theorem 3.2.2, we only need to prove ($\subseteq$). Let $X^0 := X + Z^0$ for some $Z^0 \in \mathcal{E}(X)$. Take $Y^0 \in \mathcal{A}'$ arbitrary. Because of Theorem 3.2.2, it holds that

$$\exists Y \in \mathcal{X}, \exists Z \in \mathcal{E}(Y): \quad Y^0 = Y + Z$$

and, hence, $Y^0 - X^0 \in \ker \pi$ by (3.35) and Corollary 3.2.6. Thus, we obtain

$$Y^0 = X^0 + (Y^0 - X^0) = X + \underbrace{Z^0 + Y^0 - X^0}_{\in \mathcal{M}} \in \mathcal{A}'.$$

Consequently, $Z^0 + Y^0 - X^0 \in \mathcal{E}(X)$ holds by (3.27), i.e.,

$$Y^0 \in \{X + Z \in \mathcal{X} \mid Z \in \mathcal{E}(X)\}.$$

That shows ($\subseteq$) in (i) and completes the proof of

$$\mathcal{A}' = \{X + Z \mid Z \in \mathcal{E}(X)\}.$$

For the proof of Theorem 3.2.7(ii), we only need to show the first equation by (i) and, according to (3.27), we only have to prove

$$\forall Y \in \mathcal{X} \text{ with } \mathcal{E}(Y) \neq \emptyset : \quad \{Y^0 - Y \mid Y^0 \in \mathcal{A}'\} \subseteq \mathcal{M}.$$

Take $Y \in \mathcal{X}$ with $\mathcal{E}(Y) \neq \emptyset$ and $Y^0 \in \mathcal{A}'$ arbitrary. Then,

$$\exists Z \in \mathcal{E}(Y) : \quad Y + Z \in \mathcal{A}'$$

by Theorem 3.2.2. Because of the assumption of (3.35), it holds that

$$Y^0 - Y = \underbrace{Y^0 - (Y + Z)}_{\subseteq \mathcal{M}} + Z \in \mathcal{M}$$

by $Y^0 \in \mathcal{A}'$ and $Y + Z \in \mathcal{A}'$. Hence, we obtain $Y^0 - Y \in \mathcal{E}(Y)$ by (3.27), showing (ii). □

**Remark 3.2.8.** *Theorem 3.2.7 shows that we only need one arbitrary capital position $X \in \mathcal{X}$ with nonempty solution set $\mathcal{E}(X)$ of ($P_\pi(X)$) to determine all optimal acceptable capital positions $X^0 \in \mathcal{A}'$ that are suitable for any other capital positions $Y \in \mathcal{X}$ with nonempty solution set $\mathcal{E}(Y)$ of ($P_\pi(Y)$), too. In other words, if we have determined the optimal acceptable capital positions according to Definition 3.1.1 for one $X$, we do not have to solve the problem ($P_\pi(Y)$) for other positions $Y$, too, because the solutions $\mathcal{E}(Y)$ can be derived by $\mathcal{E}(X)$ as noticed in Theorem 3.2.7*(ii)*. The reason for the possible transfer of one problem into the other is that it holds under the preconditions of Theorem 3.2.7 that*

$$\forall X, Y \in \mathcal{X} : \quad \mathcal{E}(X) \neq \emptyset \wedge \mathcal{E}(Y) \neq \emptyset \quad \Longrightarrow \quad Y - X \in \mathcal{M}$$

*because of*

$$\exists Z \in \mathcal{E}(X), \exists Z^0 \in \mathcal{E}(Y) : \quad X^0 = X + Z \in \mathcal{A}', \; Y^0 = Y + Z^0 \in \mathcal{A}'$$

*by Theorem 3.2.7 and, thus,*

$$Y - X = Z + (Y^0 - X^0) - Z^0 \in \mathcal{M}$$

*by $\mathcal{M}$ being a vector space with $Y^0 - X^0 \in \ker \pi \subseteq \mathcal{M}$ because of Corollary 3.2.6.*

We give an example that (3.35) is really necessary in Theorem 3.2.7:

**Example 3.2.9** (see [139, Expl. 4.13])**.** Consider (FM) with $\mathcal{X} = \mathbb{R}^3$, $\mathcal{M} = \{Z \in \mathbb{R}^3 \mid Z_3 = 0\}$ and $\pi(Z) := Z_2$. Then,

$$\ker \pi = \{Z \in \mathbb{R}^3 \mid Z_2 = Z_3 = 0\}$$

holds. Consider $U = (0,1,0)^T \in \mathcal{M} \cap \mathbb{R}^3_+$ according to Assumption 2 and

$$\mathcal{A} = \{X = (X_1, X_2, X_3)^T \in \mathbb{R}^3 \mid X_1, X_2 \geq 0\}$$

Then, $\mathcal{A}$ is an $(-U)$-directional closed acceptance set with

$$\operatorname{bd}_{-U}(\mathcal{A}) = \{X \in \mathbb{R}^3 \mid X_1 \geq 0,\ X_2 = 0\}$$

and

$$\mathcal{A} + \ker \pi = \{X \in \mathbb{R}^3 \mid X_2 \geq 0\},$$

i.e.,

$$\operatorname{bd}_{-U}(\mathcal{A} + \ker \pi) = \{X \in \mathbb{R}^3 \mid X_2 = 0\}.$$

Thus,

$$\mathcal{A}' = \operatorname{bd}_{-U}(\mathcal{A}) \cap \operatorname{bd}_{-U}(\mathcal{A} + \ker \pi) = \{X \in \mathbb{R}^3 \mid X_1 \geq 0, X_2 = 0\}.$$

Hence, it holds that

$$\forall X^0, Y^0 \in \mathcal{A}' : \quad X^0 - Y^0 \in \mathcal{M} \quad \Longleftrightarrow \quad X_3^0 = Y_0^3.$$

Take $X^0 \in \mathcal{A}'$ arbitrary. If $Y \in \mathbb{R}^3$ fulfills $Y_3 \neq X_3^0$, we get $X^0 - Y \notin \mathcal{M}$ which, therefore, violates (3.35) in Theorem 3.2.7 such that $X^0 - Y \notin \mathcal{E}(Y)$ although $\mathcal{A}' \neq \emptyset$ and

$$\mathcal{E}(Y) = \{Z \in \mathbb{R}^3 \mid Y_1 + Z_1 \geq 0, Z_2 = -Y_2, Z_3 = 0\} \neq \emptyset$$

hold. $\Diamond$

The preconditions of Theorem 3.2.7 can be relaxed if the financial market is complete in the following sense:

**Definition 3.2.10** (see [141, Def. 4.12])**.** Consider (FM). If $\mathcal{M} = \mathcal{X}$, the financial market (FM) is called *complete*. Otherwise, the market is said to be *incomplete*.

**Remark 3.2.11.** *In* (FM)*, we assume finitely many assets. Hence,* $\dim \mathcal{M} < +\infty$ *(see* (2.10)*) is fulfilled. Thus, for infinite dimensional vector spaces* $\mathcal{X}$*, the market* (FM) *is always incomplete.*

In Theorem 3.2.7, the precondition (3.35) is obviously fulfilled if (FM) is complete in the sense of Definition 3.2.10. In that case, the following corollary shows that it can be considered any arbitrary capital position $X \in \mathcal{X}$ in Theorem 3.2.7, nevertheless if $\mathcal{E}(X) \neq \emptyset$ or not (e.g., $X = 0$ such that the problem reduces to determine the set $\mathcal{A}' \subseteq \mathcal{X}$, see also (3.27)). The corollary is stated in [141] without a proof.

**Corollary 3.2.12** (see [141, Corollary 4.13]). *Consider* (FM). *Let $\mathcal{A} \subseteq \mathcal{X}$ be an acceptance set such that Assumption 4 is fulfilled with $U \in \mathcal{M} \cap \mathcal{X}_+$ being a eligible payoff according to Assumption 2 and $\mathcal{A}' \subseteq \mathcal{X}$ given by* (3.25)*. Furthermore, let $\mathcal{E}\colon \mathcal{X} \rightrightarrows \mathcal{M}$ be the optimal payoff map as in* (3.1)*. Suppose that the market* (FM) *is complete (see Definition 3.2.10) and take $X \in \mathcal{X}$ arbitrary. Then, the following conditions are fulfilled:*

(i) *It holds that $\mathcal{A}' = \{X + Z \in \mathcal{X} \mid Z \in \mathcal{E}(X)\}$,*

(ii) *For all $Y \in \mathcal{X}$, it holds that*

$$\mathcal{E}(Y) = \{Y^0 - Y \mid Y^0 \in \mathcal{A}'\}.$$

**Proof.** First, we suppose $X \in \mathcal{X}$ with $\mathcal{E}(X) \neq \emptyset$. Then, (i) follows directly from Theorem 3.2.7(i) and we only need to prove (ii). Take $Y \in \mathcal{X}$ arbitrary. With respect to Theorem 3.2.7(ii), we need to show $\mathcal{E}(Y) \neq \emptyset$. By Theorem 3.2.2, it holds that

$$\forall Z^0 \in \mathcal{E}(X): \quad X^0 := X + Z^0 \in \mathcal{A}'.$$

Since the market is complete, $\mathcal{M} = \mathcal{X}$ and, thus, $X^0 - Y \in \mathcal{M}$ hold. Hence, we obtain $X^0 - Y \in \mathcal{E}(Y)$ by (3.27) because of

$$Y + (X^0 - Y) = X^0 \in \mathcal{A}',$$

showing Corollary 3.2.12(ii) for the case $\mathcal{E}(X) \neq \emptyset$.

Now, we assume $\mathcal{E}(X) = \emptyset$. Suppose there is some $Y \in \mathcal{X}$ with $\mathcal{E}(Y) \neq \emptyset$. By Theorem 3.1.6,

$$\exists Z^0 \in \mathcal{E}(Y): \quad Y^0 := Y + Z^0 \in \mathcal{A}'.$$

Since the market is complete, it holds that $Y^0 - X \in \mathcal{M}$. As a result,

$$X + (Y^0 - X) = Y^0 \in \mathcal{A}'$$

which implies $Y^0 - X \in \mathcal{E}(X)$ by (3.27) and, thus, contradicts $\mathcal{E}(X) = \emptyset$. As a result, $\mathcal{E}(Y) = \emptyset$ holds for every $Y \in \mathcal{X}$ and, thus, we obtain $\mathcal{A}' = \emptyset = \mathcal{E}(X)$ by Theorem 3.2.2, which shows (i) and (ii) for the case $\mathcal{E}(X) = \emptyset$. That completes the proof of the corollary. □

**Remark 3.2.13.** *As stated before Corollary 3.2.12, it holds for complete markets* (FM) *that $\mathcal{A}' = \mathcal{E}(\mathbb{0})$ with $\mathcal{E}\colon \mathcal{X} \rightrightarrows \mathcal{M}$ being the optimal payoff map given by* (2.6) *and $\mathcal{A}' \subseteq \mathcal{X}$ being the set from* (3.25)*, since $X \in \mathcal{X}$ can be chosen in Corollary 3.2.12 arbitrarily. Hence, $\mathcal{E}(\mathbb{0}) = \emptyset$ implies $\mathcal{E}(X) = \emptyset$ for all $X \in \mathcal{X}$ in complete markets* (FM).

For conclusion of this section, we present some conditions for unique solutions of ($P_\pi(X)$) with $X \in \mathcal{X}$ arbitrary (i.e., $|\mathcal{E}(X)| = 1$). The following result generalizes a corresponding result in [17, Prop. 4.11] for closed acceptance sets in locally convex Hausdorff spaces $\mathcal{X}$.

**Theorem 3.2.14.** *Consider* (FM)*. Let Assumption 4 be fulfilled by some eligible payoff* $U \in \mathcal{M} \cap \mathcal{X}_+$ *and an acceptance set* $\mathcal{A} \subseteq \mathcal{X}$*. Furthermore, let* $\mathcal{A}' \subseteq \mathcal{X}$ *be the set optimal acceptable capital positions given by* (3.25) *and* $\mathcal{E}\colon \mathcal{X} \rightrightarrows \mathcal{M}$ *be the optimal payoff map given by Definition 3.1.1. Then, the following conditions are equivalent:*

(i) $|\mathcal{E}(X)| = 1$ *for all* $X \in \mathcal{X}$,

(ii) $|\mathcal{E}(X)| = 1$ *for all* $X \in \mathcal{A}'$,

(iii) *It holds that* $\mathcal{A}' \cap (\operatorname{bd}_{-U}(\mathcal{A}) + (\ker \pi \backslash \{\mathbb{0}\})) = \emptyset$,

(iv) *It holds that* $\operatorname{bd}_{-U}(\mathcal{A}) \cap (\operatorname{bd}_{-U}(\mathcal{A}) + (\ker \pi \backslash \{\mathbb{0}\})) \subseteq \operatorname{int}_{-U}(\mathcal{A} + \ker \pi)$.

**Proof.** Obviously, (i) $\Rightarrow$ (ii) holds. Thus, we show (ii) $\Rightarrow$ (iii) now. Suppose there is some $X \in \mathcal{A}'$ with $X \in \operatorname{bd}_{-U}(\mathcal{A}) + (\ker \pi \backslash \{\mathbb{0}\})$. Then, $\mathbb{0} \in \mathcal{E}(X)$ by (3.27). Moreover,

$$\exists Z \in \ker \pi \backslash \{\mathbb{0}\} : \quad X + Z \in \operatorname{bd}_{-U}(\mathcal{A}).$$

For $\rho_{\mathcal{A},\mathcal{M},\pi}\colon \mathcal{X} \to \overline{\mathbb{R}}$ given by (2.30), it holds that $\rho_{\mathcal{A},\mathcal{M},\pi}(X) = 0$, since $\mathbb{0} \in \mathcal{E}(X)$ with $\pi(\mathbb{0}) = 0$ by linearity of $\pi$ given by (2.13). Because $\pi(Z) = 0$, too, and $X + Z \in \operatorname{bd}_{-U}(\mathcal{A}) \subseteq \mathcal{A}$ by $\mathcal{A}$ being $(-U)$-directionally closed, we obtain $Z \in \mathcal{E}(X)$ by definition of $\mathcal{E}(X)$. Hence, $\{\mathbb{0}, Z\} \subseteq \mathcal{E}(X)$ holds, which contradicts $|\mathcal{E}(X)| = 1$. Consequently, (iii) holds, which shows (ii) $\Rightarrow$ (iii).

Now, we prove (iii) $\Rightarrow$ (iv). Take $X \in \operatorname{bd}_{-U}(\mathcal{A}) \cap (\operatorname{bd}_{-U}(\mathcal{A}) + (\ker \pi \backslash \{\mathbb{0}\}))$ arbitrary. Then, $X \notin \mathcal{A}'$ by (iii). Because $X \in \operatorname{bd}_{-U}(\mathcal{A}) \subseteq \mathcal{A}$ by $\mathcal{A}$ being $(-U)$-directionally closed, we obtain $X \in \mathcal{A} \backslash \mathcal{A}' \subseteq \operatorname{int}_{-U}(\mathcal{A} + \ker \pi)$ by Lemma 3.2.4, showing (iv).

Finally, we prove (iv) $\Rightarrow$ (i). Suppose there is some $X \in \mathcal{X}$ with $Z^1, Z^2 \in \mathcal{E}(X)$ and $Z^1 \neq Z^2$. Then, $Z^2 - Z^1 \in \ker \pi \backslash \{\mathbb{0}\}$, since $\pi(Z^1) = \pi(Z^2)$ by definition of $\mathcal{E}(X)$. Moreover, $X + Z^1 \in \mathcal{A}'$ and $X + Z^2 \in \mathcal{A}'$ by (3.27), which implies $X + Z^1 \in \operatorname{bd}_{-U}(\mathcal{A})$ and $X + Z^2 \in \operatorname{bd}_{-U}(\mathcal{A})$ by definition of $\mathcal{A}'$. Hence,

$$X + Z^2 = X + Z^1 + (Z^2 - Z^1) \in \operatorname{bd}_{-U}(\mathcal{A}) + (\ker \pi \backslash \{\mathbb{0}\}),$$

which contradicts (iv) by $X + Z^2 \in \mathcal{A}' = \operatorname{bd}_{-U}(\mathcal{A}) \cap \operatorname{bd}_{-U}(\mathcal{A} + \ker \pi)$. Consequently, (i) holds and the proof is complete. □

Example 3.1.15 illustrates $|\mathcal{E}(X)| = 1$ for each $X \in \mathcal{X}$ with Theorem 3.2.14 (iii) and (iv) being obviously fulfilled, see also Figure 3.5. An example with $|\mathcal{E}(X)| > 1$ for each $X \in \mathcal{X}$ is illustrated in Example 3.1.13, see also Figure 3.3.

# Chapter 4

# Efficient and Weakly Efficient Points of Acceptance Sets

In this chapter, we consider (FM) and analyze efficient and weakly efficient points of acceptance sets. In our paper [139], we investigated efficiency for closed acceptance sets and, moreover, by a less common definition of efficient points. Here, we consider a $(-U)$-directional closed acceptance set $\mathcal{A} \subseteq \mathcal{X}$ with $U \in \mathcal{M} \cap \mathcal{X}_+$ according to Assumption 2 in a vector space $\mathcal{X}$, and use the more common definition of efficient points from Section 1.2. The chapter is organized as follows:

- In Section 4.1, we introduce the *price cone* $\mathcal{C}_\pi$ and the *kernel cone* $\mathcal{C}_{\text{ker}}$. Note that the kernel cone was introduced in our paper [139], but it was denoted by $\mathcal{C}_\pi$ instead of $\mathcal{C}_{\text{ker}}$ there, which we mention here to avoid misunderstandings in the following. We present a motivation for determining efficient points with respect to these cones and study basic properties of $\mathcal{C}_\pi$ and $\mathcal{C}_{\text{ker}}$.

- Afterwards, we study the sets of efficient points $\text{Eff}\,(\mathcal{A}, \mathcal{C}_\pi)$ and $\text{Eff}\,(\mathcal{A}, \mathcal{C}_{\text{ker}})$ of a $(-U)$-directionally closed acceptance set $\mathcal{A}$ in Section 4.2. We will focus on the relationship of these sets with the set of optimal acceptable capital positions $\mathcal{A}'$ given by (3.25), which consists of capital positions resulting from solutions of $(P_\pi(X))$ for any $X \in \mathcal{X}$ as seen in Section 3.2. We will show that $\text{Eff}\,(\mathcal{A}, \mathcal{C}_\pi)$ and $\text{Eff}\,(\mathcal{A}, \mathcal{C}_{\text{ker}})$ are subsets of $\mathcal{A}'$. Moreover, we prove that $\text{Eff}\,(\mathcal{A}, \mathcal{C}_\pi)$ coincides with $\mathcal{A}' \neq \emptyset$ if $\text{Eff}\,(\mathcal{A}, \mathcal{C}_\pi)$ is non-empty, too, while $\text{Eff}\,(\mathcal{A}, \mathcal{C}_{\text{ker}}) = \mathcal{A}'$ only holds if $\text{Eff}\,(\mathcal{A}, \mathcal{C}_{\text{ker}})$ is non-empty and $\mathcal{A}' - \mathcal{A}' \in \mathcal{M}$ holds.

- The results from the previous section will be directly used in Section 4.3 to derive a characterization of weakly efficient points of an acceptance set $\mathcal{A} \subseteq \mathcal{X}$ in a topological vector space $\mathcal{X}$ with respect to the price cone $\mathcal{C}_\pi$ and the kernel cone $\mathcal{C}_{\text{ker}}$, respectively. To do so, we will study the interior of these cones, and we will observe that the set of weakly efficient points $\text{Eff}_w(\mathcal{A}, \mathcal{C}_\pi)$ is only well-defined in complete markets. Nevertheless, we will also obtain a relationship of the sets of weakly efficient points and $\mathcal{A}'$, as well.

The main results of this sections are published as an overview without any proofs in [141].

## 4.1 Kernel and price cones

Consider (FM) and the optimization problem ($P_\pi(X)$) for some $X \in \mathcal{X}$, where $\rho_{\mathcal{A},\mathcal{M},\pi}(X)$ given by (2.30) is the optimal value of ($P_\pi(X)$) and $\mathcal{E}(X)$ given by (2.6) is the solution set of ($P_\pi(X)$). In this chapter, we present a new view on the efficient and weakly efficient points of the acceptance set $\mathcal{A}$. The motivation for studying efficient and weakly efficient points of $\mathcal{A}$ is that $-\rho_{\mathcal{A},\mathcal{M},\pi}$ can be used as scalarization functional for a vector optimization problem

$$X^0 \to \underset{X^0 \in \mathcal{A}}{\mathcal{D}\text{-Min}} \tag{$V_\mathcal{A}$}$$

in the space of capital positions $\mathcal{X}$, where $\mathcal{D} \subseteq \mathcal{X}$ is a suitable domination set (compare Section 1.2) to distinguish between the acceptable capital positions $X^0 \in \mathcal{A}$. For $\mathcal{A}$ being $(-U)$-directionally closed with $U \in \mathcal{M} \cap \mathcal{X}_+$ according to Assumption 2, the set of cost-optimal acceptable capital positions $\mathcal{A}'$ given by (3.25) is a subset of $\mathcal{A}$ that is directly connected with the solution set $\mathcal{E}(X)$ of ($P_\pi(X)$) by (see Theorem 3.2.2)

$$\mathcal{A}' = \bigcup_{\substack{X \in \mathcal{X}, \\ \mathcal{E}(X) \neq \emptyset}} (\{X\} + \mathcal{E}(X)) \subseteq \mathcal{A},$$

i.e., $\mathcal{A}'$ is the set of all acceptable positions resulting by the solution set $\mathcal{E}(X)$ given by (3) through any $X \in \mathcal{X}$. Thus, we aim to show important relationships between $\mathcal{A}'$ and the sets of efficient and weakly efficient points of $\mathcal{A}$, respectively (and, thus, relationships between solutions of ($P_\pi(X)$) and ($V_\mathcal{A}$)).

In this Section, we introduce two domination sets that we will consider for determining (weakly) efficient points of $\mathcal{A}$ to gain a deep understanding of the limits of our chosen model (FM) and to outline important assumptions (especially, on the subspace $\mathcal{M}$ given by (2.10) and $\mathcal{A}'$ given by (3.25), respectively) for our results. That provides new extensive insights in the role of the domination set for efficient points of $\mathcal{A}$ in (not necessary topological) vector spaces.

Consider a $(-U)$-directionally closed acceptance set $\mathcal{A}$ with $U \in \mathcal{M} \cap \mathcal{X}_+$ according to Assumption 2. We start with some geometrical motivation, which we illustrated in Figure 3.2: cost-optimal acceptable capital positions $X^0 \in \mathcal{A}'$ with $\mathcal{A}' \subseteq \operatorname{bd}_{-U}(\mathcal{A})$ defined as in (3.25) (and, thus, optimal eligible payoffs) can be determined by a shift of $\ker \pi$ along the acceptance set $\mathcal{A}$ in direction $-U$ as far as possible with $U \in \mathcal{M} \cap \mathcal{X}_+$ according to Assumption 2. If this procedure leads to a non-empty intersection with $\operatorname{bd}_{-U}(\mathcal{A})$, we obtain $\mathcal{A}' \neq \emptyset$. Hence, as shown in Lemma 3.2.5, it holds that

$$\left(\mathcal{A}' - \bigcup_{m<0} \pi_m\right) \cap \mathcal{A} = \emptyset, \tag{4.1}$$

and

$$\big(\mathcal{A}' - \pi_0\big) \cap \mathcal{A} = \big(\mathcal{A}' - \ker \pi\big) \cap \mathcal{A} \subseteq \mathcal{A}', \tag{4.2}$$

where $\pi_m \subseteq \mathcal{M}$ is given by (2.16) and consists of all eligible payoffs with price $m \in \mathbb{R}$. Note that $\mathcal{A}' \subseteq \mathcal{A}$ for $\mathcal{A}$ being $(-U)$-directionally closed. Thus, (4.1) and (4.2) imply for $X^0 \in \mathcal{A}'$

arbitrary

$$\forall X \in \mathcal{A} \text{ with } X - X^0 \in - \bigcup_{m \in \mathbb{R}_+} \pi_m : \qquad X - X^0 \in \ker \pi \text{ and } X \in \mathcal{A}'. \tag{4.3}$$

Because $\mathcal{A}'$ represents all acceptable capital positions from interest for making a given capital position acceptable with minimal costs (see Section 3.2), it is naturally interesting to find conditions that lead for given $X^0 \in \mathcal{A}'$ to

$$\left( \{X^0\} - \bigcup_{m \leq 0} \pi_m \right) \cap \mathcal{A} \subseteq \{X^0\},$$

i.e., if we changed a position into an acceptable $X^0 \in \mathcal{A}'$ with minimal costs, there is no other acceptable capital position of interest that can be reached by zero (or negative) costs. Since we do not focus on one single $X^0$, we want to study conditions for

$$\forall X^0 \in \mathcal{A}',\ \nexists X \in \mathcal{A} \backslash \{X^0\} : \qquad X - X^0 \in - \bigcup_{m \geq 0} \pi_m = \bigcup_{m \leq 0} \pi_m, \tag{4.4}$$

which is equivalent by (4.1) and (4.3) to

$$\forall X^0 \in \mathcal{A}',\ \nexists X \in \mathcal{A} \backslash \{X^0\} : \qquad X - X^0 \in \ker \pi.$$

Note that this situation does not imply $\operatorname{card} \mathcal{A}' = 1$: $X^0, Y^0 \in \mathcal{A}'$ implies $X^0 - Y^0 \in \ker \pi$ or $X^0 - Y^0 \notin \mathcal{M}$ by Lemma 3.2.5. With respect to (4.4), we want to determine conditions implying that it is impossible to switch from one optimal acceptable capital position to another without positive costs, which can be useful to derive uniqueness results for solutions of $(P_\pi(X))$ for any $X \in \mathcal{X}$. On the other hand, we can study for given $X^0 \in \mathcal{A}'$ if there is no acceptable capital position that can be reached costless that provides a larger payoff in each scenario than any other (maybe unreachable) acceptable capital position, i.e.,

$$\left( \{X^0\} + \ker \pi - \mathcal{X}_+ \right) \cap \mathcal{A} \subseteq \{X^0\},$$

Hence, we are also interested in study conditions such that

$$\forall X^0 \in \mathcal{A}',\ \nexists X \in \mathcal{A} \backslash \{X^0\} : \qquad X - X^0 \in \ker \pi - \mathcal{X}_+. \tag{4.5}$$

This is from interest for portfolio management, especially, for requirements on the choice of eligible assets and modeling sensible acceptance sets. We studied (4.5) in [139] for closed acceptance sets and under more restrictive preconditions than here. Note that (4.5) does not necessary describe *all* positions that fulfill (4.4). The reason is that considering $\ker \pi - \mathcal{X}_+$ in (4.5) is more restrictive than considering $\bigcup_{m \leq 0} \pi_m$ in (4.4), in general, see Lemma 4.1.5.

The condition (4.4) is more interesting from an economical or practical point of view with respect to our model (FM). We will see that $\mathcal{A}' - \mathcal{A}' \subseteq \mathcal{M}$ is sufficient for (4.4) and (4.5). Moreover, it is obviously that $|\mathcal{E}(X)| \leq 1$ for every $X \in \mathcal{X}$ is necessary for both conditions, but it is not sufficient, see Remark 4.2.9 later. Indeed, if $|\mathcal{E}(X)| > 1$ for some $X \in \mathcal{X}$, two optimal acceptable capital positions $X^0, Y^0 \in \mathcal{A}'$ corresponding to $X$ in the sense of Definition 3.1.1 fulfill $Y^0 - X^0 \in \ker \pi$ by Corollary 3.2.6 and, thus, $Y^0 - X^0 \in -\bigcup_{m \leq 0} \pi_m \subseteq \ker \pi - \mathcal{X}_+$. Some results for necessary and sufficient conditions for uniqueness of optimal payoffs can be found in Theorem 3.2.14 and some more for closed acceptance sets in [17, Section 4.2].

If we look at (4.4) and (4.5), respectively, we directly discover similarities to efficiency properties of $\mathcal{A}$ and, thus, solutions of ($V_{\mathcal{A}}$). Hence, we want to study the relationship between $\mathcal{A}'$ given by (3.25) and (weakly) efficient points of the (-$U$)-directionally closed acceptance set $\mathcal{A}$ in this chapter. Note that $\mathcal{A}$ is $(-U)$-directionally closed with $U \in \mathcal{M} \cap \mathcal{X}_+$ according to Assumption 2 if $\mathcal{A}$ is a closed subset of a topological vector space as considered in [17] and [139]. As an additional difference to [139], we also consider a new cone (the price cone), and also seize some of the facts for the in [139] introduced kernel cone, showing that these even hold by use of a more common definition for efficient points. Furthermore, we formulate and prove some more precisely theorems as those in [139] such that it is more clear where the additional assumption $\mathcal{A}' - \mathcal{A}' \subseteq \mathcal{M}$ is really necessary.

In this section, we want to introduce and study the mentioned cones of interest themselves.

**Definition 4.1.1.** Consider (FM). Let Assumption 2 be fulfilled by the eligible payoff $U \in \mathcal{M} \cap \mathcal{X}_+$. The *price cone* is defined by

$$\mathcal{C}_\pi := \bigcup_{m \geq 0} \pi_m = \pi^{-1}(\mathbb{R}_+) \subseteq \mathcal{M} \subseteq \mathcal{X}$$

with $\pi_m \subseteq \mathcal{M}$ given by (2.16).

**Remark 4.1.2.** *Consider* (FM). *The price cone $\mathcal{C}_\pi \subseteq \mathcal{X}$ is a cone in the sense of Definition 1.1.22. Indeed, it holds that*

$$\forall Z \in \mathcal{C}_\pi, \forall \lambda \in \mathbb{R}_+ : \quad \lambda Z \in \mathcal{C}_\pi$$

*because $\pi(Z) \geq 0$ for each $Z \in \mathcal{C}_\pi$ by Definition 4.1.1 and, thus, $\pi(\lambda Z) = \lambda \pi(Z) \geq 0$ for each $\lambda \in \mathbb{R}_+$ by linearity of $\pi$. Hence, $\mathcal{C}_\pi$ is a cone.*

For $\mathcal{C}_\pi$ given as in Definition (4.1.1), it holds that

$$\mathcal{C}_\pi = \ker \pi + \mathbb{R}_+ U \tag{4.6}$$

by Lemma 2.2.7. Indeed,

$$\mathcal{C}_\pi = \bigcup_{m \geq 0} \pi_m = \bigcup_{m \geq 0} (\ker \pi + mU) = \ker \pi + \mathbb{R}_+ U.$$

For $X \in \mathcal{X}$, the set $X - \mathcal{C}_\pi$ can be interpreted as those capital positions that can be reached by $X$ with price less or equal than zero.

**Definition 4.1.3** (see [139])**.** Consider (FM). The *kernel cone* is given by

$$\mathcal{C}_{\ker} := \ker \pi + \mathcal{X}_+ \subseteq \mathcal{X}.$$

**Remark 4.1.4.** *Consider (FM). The kernel cone $\mathcal{C}_{ker} \subseteq \mathcal{X}$ is also a cone in the sense of Definition 1.1.22. Indeed, for $X \in \mathcal{C}_{\ker}$ arbitrary,*

$$\exists Z^0 \in \ker \pi, \exists X^0 \in \mathcal{X}_+ : \quad X = X^0 + Z^0.$$

*Thus, we obtain for each $\lambda \in \mathbb{R}_+$*

$$\lambda X = \underbrace{\lambda Z^0}_{\in \ker \pi} + \underbrace{\lambda X^0}_{\in \mathcal{X}_+} \in \ker \pi + \mathcal{X}_+ = \mathcal{C}_{\ker},$$

*i.e., $\mathcal{C}_{\ker}$ is a cone.*

The kernel cone is introduced in our paper [139]. It is important to mention that the kernel cone was denoted $\mathcal{C}_\pi$ instead of $\mathcal{C}_{\ker}$ there, which we want to highlight here for avoiding misunderstandings. Note that $\mathcal{C}_{\ker} \not\subseteq \mathcal{M}$ holds in general for the kernel cone $\mathcal{C}_{\ker}$ from Definition 4.1.3, while $\mathcal{C}_\pi \subseteq \mathcal{M}$ for $\mathcal{C}_\pi$ from Definition 4.1.1. For $X \in \mathcal{X}$, the set $X - \mathcal{C}_{\ker}$ can be interpreted as the set of all capital positions smaller (in the sense of the natural partial order given by $\mathcal{X}_+$) or equal then any capital position that can be reached by $X$ with zero costs.

As seen in Remark 4.1.2 and 4.1.4, $\mathcal{C}_\pi$ and $\mathcal{C}_{\ker}$ are cones, indeed, and, thus, naming them cones is justified. First, we observe the following relationship between the price cone $\mathcal{C}_\pi$ from Definition 4.1.1 and the kernel cone $\mathcal{C}_{\ker}$ from Definition 4.1.3 that we mentioned in the motivation at the beginning of this section:

**Lemma 4.1.5.** *Consider (FM). Let Assumption 2 be fulfilled by some eligible payoff $U \in \mathcal{M} \cap \mathcal{X}_+$. Furthermore, let $\mathcal{C}_\pi \subseteq \mathcal{M}$ be the price cone given by Definition 4.1.1 and $\mathcal{C}_{\ker} \subseteq \mathcal{X}$ be the kernel cone given by Definition 4.1.3. Then,*

$$\mathcal{C}_\pi \subseteq \mathcal{C}_{\ker}. \tag{4.7}$$

*If $\mathcal{X}_+ \subseteq \mathcal{M}$ holds, then $\mathcal{C}_\pi = \mathcal{C}_{\ker} \subseteq \mathcal{M}$.*

**Proof.** Because of (4.6), we obtain by $U \in \mathcal{X}_+$

$$\mathcal{C}_\pi = \bigcup_{m \geq 0} \pi_m = \mathbb{R}_+ U + \ker \pi \subseteq \mathcal{X}_+ + \ker \pi = \mathcal{C}_{\ker},$$

i.e., (4.7) holds. Assume $\mathcal{X}_+ \subseteq \mathcal{M}$ now. Then,

$$\mathcal{X}_+ \subseteq \mathcal{M} = \bigcup_{m \in \mathbb{R}} \pi_m.$$

Because of Lemma 1.3.21 and $\mathbb{0} \in \mathcal{M} \cap \mathcal{X}_+$, we obtain $\pi(Z) \geq 0$ for all $Z \in \mathcal{X}_+$, i.e.,

$$\mathcal{X}_+ \subseteq \bigcup_{m \geq 0} \pi_m.$$

That implies

$$\mathcal{C}_{\ker} = \ker \pi + \mathcal{X}_+ \subseteq \underbrace{\ker \pi}_{= \pi_0} + \bigcup_{m \geq 0} \pi_m = \bigcup_{m \geq 0} \pi_m = \mathcal{C}_\pi,$$

i.e., $\mathcal{C}_{\ker} \subseteq \mathcal{C}_\pi \subseteq \mathcal{M}$ holds. By (4.7), the cones $\mathcal{C}_{\ker}$ and $\mathcal{C}_\pi$ coincide. □

As highlighted in Lemma 4.1.5, it holds that $\mathcal{C}_{\ker} \not\subseteq \mathcal{M}$, in general. In Figure 4.1, the cones $\mathcal{C}_\pi$ and $\mathcal{C}_{\ker}$ are illustrated for $\mathcal{X} = \mathcal{M} = \mathbb{R}^2$. Since $\mathcal{X}_+ = \mathbb{R}^2_+ \subseteq \mathcal{M}$ holds, both sets coincide (see Lemma 4.1.5). Note that $\mathcal{C}_\pi$ and $\mathcal{C}_{\ker}$ are no half spaces in general. For the necessity of $\mathcal{X}_+ \subseteq \mathcal{M}$ in Lemma 4.1.5, see Example 4.2.1 or [139, Expl. 5.6 and 5.7]. Furthermore,

$$\mathcal{C}_{\ker} = \ker \pi + \mathcal{X}_+ \subseteq \ker \pi + \mathcal{A}$$

holds because of $\mathcal{X}_+ \subseteq \mathcal{A}$ by Definition 2.2.9(iii) and $\mathbb{0} \in \mathcal{A}$. Thus, if $\rho_{\mathcal{A},\mathcal{M},\pi}(X) > -\infty$ holds for some $X \in \mathcal{X}$ with $\rho_{\mathcal{A},\mathcal{M},\pi} \colon \mathcal{X} \to \overline{\mathbb{R}}$ being the risk measure in (2.30), $\mathcal{A} + \ker \pi$ is an acceptance set and, especially, proper (see Remark 2.3.18), i.e.,

$$\mathcal{C}_{\ker} \subseteq \mathcal{A} + \ker \pi \neq \mathcal{X}. \tag{4.8}$$

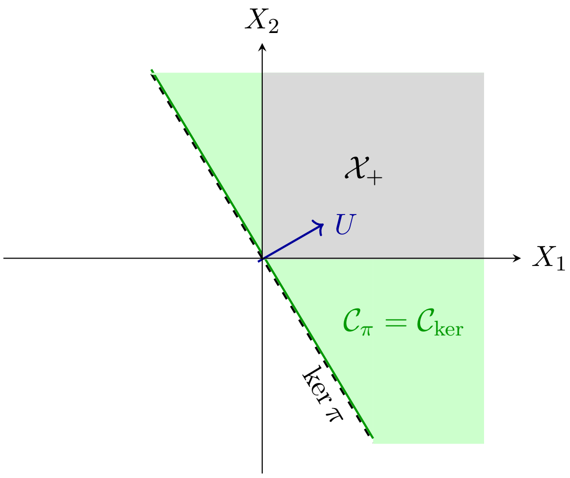


Figure 4.1: Illustration of $\mathcal{C}_{\ker}$

The following lemma includes properties of the kernel cone $\mathcal{C}_{\ker}$ from [139, Lemma 5.1] and shows that $\mathcal{C}_\pi$ has similar properties like $\mathcal{C}_{\ker}$, but is additionally directionally closed.

**Lemma 4.1.6.** *Consider (FM). Let Assumption 2 be fulfilled by some eligible payoff $U \in \mathcal{M} \cap \mathcal{X}_+$, $\mathcal{C}_\pi \subseteq \mathcal{M}$ be the price cone given by Definition 4.1.1, and $\mathcal{C}_{\ker} \subseteq \mathcal{X}$ be the kernel cone given by Definition 4.1.3. Then, the following holds:*

(i) *$\mathcal{C}_\pi$ is a nontrivial, convex, $(-U)$-directionally closed cone with $\mathbb{0} \in \operatorname{bd}_{-U}(\mathcal{C}_\pi)$,*

(ii) *$\mathcal{C}_{\ker}$ is a convex cone with $\mathbb{0} \in \operatorname{bd}_{-U}(\mathcal{C}_{\ker})$, which is nontrivial if $\mathcal{A} + \ker \pi \neq \mathcal{X}$ holds.*

**Proof.** In (FM), $\ker \pi \supsetneq \{\mathbb{0}\}$ holds by (2.11). Consider $\mathcal{C}_\pi$ first. $\mathcal{C}_\pi$ is a cone by Remark 4.1.2. Take $Z^1, Z^2 \in \mathcal{C}_\pi$ arbitrary. Then, $\pi(Z^1) \geq 0$ and $\pi(Z^2) \geq 0$ hold. Then, we have

$$\pi(Z^1 + Z^2) = \pi(Z^1) + \pi(Z^2) \geq 0$$

by linearity of $\pi$ and, thus, $Z^1 + Z^2 \in \mathcal{C}_\pi$. Hence, because $\mathcal{C}_\pi$ is a cone, Lemma 1.1.25 implies that $\mathcal{C}_\pi$ is convex. Obviously, $\ker \pi \subseteq \mathcal{C}_\pi$ holds. Furthermore, $\mathbb{0} \in \mathcal{C}_\pi \neq \mathcal{X}$ holds because of

$\mathbb{0} \in \ker \pi$ and $-U \notin \mathcal{C}_\pi$ by $\pi(-U) = -\pi(U) < 0$. As a result, $\mathcal{C}_\pi$ is nontrivial (see Definition 1.1.24). Since

$$\forall \lambda \in \mathbb{R}_> : \quad \pi(\lambda U) > 0 \quad \text{and} \quad \pi(-\lambda U) < 0,$$

we obtain $\mathbb{0} \in \operatorname{bd}_{-U}(\mathcal{C}_\pi)$. Now, take $Z \in \mathcal{X}$ and $(t_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}_>$ with $t_n \downarrow 0$ arbitrary such that $Z + t_n U \in \mathcal{C}_\pi$ holds for all $n \in \mathbb{N}$. Then, $Z \in \mathcal{M}$ holds because of

$$Z = \underbrace{(Z - t_n U)}_{\in \mathcal{C}_\pi \subseteq \mathcal{M}} + \underbrace{t_n U}_{\in \mathcal{M}} \in \mathcal{M}$$

for $n \in \mathbb{N}$ arbitrary by $\mathcal{M}$ being a subspace of $\mathcal{X}$. As a result,

$$0 \leq \pi(Z + t_n U) = \pi(Z) + t_n$$

by $Z + t_n U \in \mathcal{C}_\pi$ for each $n \in \mathbb{N}$. Consequently, $\pi(Z) \geq -t_n$ holds for all $n \in \mathbb{N}$, which implies $\pi(Z) \geq 0$ by $t_n \downarrow 0$ for $n \to +\infty$. As a result, $Z \in \mathcal{C}_\pi$ holds, i.e., $\mathcal{C}_\pi$ is $(-U)$-directionally closed. That completes the proof of (i).

Now, we consider $\mathcal{C}_{\text{ker}}$: since $\mathbb{0} \in \mathcal{X}_+$, we have $\ker \pi \subseteq \mathcal{C}_{\text{ker}}$. By (4.8), $\mathcal{C}_{\text{ker}}$ is nontrivial if $\mathcal{A} + \ker \pi$ is proper, i.e., $\mathcal{A} + \ker \pi \neq \mathcal{X}$. Because $\mathcal{C}_{\text{ker}}$ is a (Minkowski) sum of two convex sets, $\mathcal{C}_{\text{ker}}$ is convex, too. $\mathcal{C}_{\text{ker}}$ is a cone by Remark 4.1.4. Finally,

$$\forall \lambda \in \mathbb{R}_> : \quad -\lambda U \in \{\mathbb{0}\} - (\mathcal{X}_+ \backslash \{\mathbb{0}\}) \not\subseteq \mathcal{C}_{\text{ker}} \quad \text{and} \quad \lambda U \in \mathcal{X}_+ \subseteq \mathcal{C}_{\text{ker}}.$$

Hence, $\mathbb{0} \in \operatorname{bd}_{-U}(\mathcal{C}_{\text{ker}})$ holds and (ii) is proved. □

**Remark 4.1.7.** *Let $\mathcal{C}_\pi$ be the price cone given by Definition 4.1.1 and $\mathcal{C}_{\text{ker}}$ be the kernel cone given by Definition 4.1.3. It holds that $\ker \pi \subseteq \mathcal{C}_\pi$, $\ker \pi \subseteq \mathcal{C}_{\text{ker}}$, and*

$$\mathcal{C}_{\text{ker}} \text{ and } \mathcal{C}_\pi \text{ are pointed} \quad \Longleftrightarrow \quad \ker \pi = \{\mathbb{0}\}.$$

*For $\mathcal{M} \subseteq \mathcal{X}$ given by (2.10) fulfilling (2.11), we obtain $\ker \pi \neq \{\mathbb{0}\}$ by $\dim \mathcal{M} > 1$ (see Remark 2.2.6). The situation $\dim \mathcal{M} = 1$ and, thus, $\ker \pi = \{\mathbb{0}\}$ coincides with the case of one eligible asset which we excluded in our setting (FM) as noticed in Remark 2.2.17. Thus, the cones $\mathcal{C}_\pi$ and $\mathcal{C}_{\text{ker}}$ are never pointed in (FM). Indeed,*

$$\forall X \in \ker \pi \backslash \{\mathbb{0}\} : \quad \{X, -X\} \subseteq \ker \pi \tag{4.9}$$

*is fulfilled, since $\ker \pi$ is a subspace of $\mathcal{M}$. Thus, $\mathcal{C}_\pi$ and $\mathcal{C}_{\text{ker}}$ are not pointed in (FM). Furthermore, it is sufficient for $\mathcal{A} + \ker \pi \neq \mathcal{X}$ in Lemma 4.1.6(ii) that $\rho_{\mathcal{A},\mathcal{M},\pi} : \mathcal{X} \to \overline{\mathbb{R}}$ given by (2.30) is proper (see (2.42) and Lemma 2.3.37).*

**Remark 4.1.8.** *Let $\mathcal{X}$ be a Hausdorff topological vector space partially ordered by the positive cone $\mathcal{X}_+$, $\mathcal{C}_\pi$ be the price cone given by Definition 4.1.1 and $\mathcal{C}_{\text{ker}}$ be the kernel cone given by Definition 4.1.3. Then, $\mathcal{C}_\pi = \pi^{-1}(\mathbb{R}_+)$ is always closed by $\pi$ being continuous because of $\dim \mathcal{M} < +\infty$. If $\dim \mathcal{X} < +\infty$, $\mathcal{X}_+$ is polyhedral, since $\mathcal{X}_+$ is a finite intersection of closed half spaces, and $\mathcal{C}_{\text{ker}}$ is polyhedral if $\mathcal{C}_{\text{ker}} = \mathcal{X}_+ + \ker \pi \neq \mathcal{X}$ (follows by [17, Lemma 4.7] with $\mathcal{B} = \mathcal{X}_+$ and $\dim(\ker \pi) < +\infty$). Hence, $\mathcal{C}_{\text{ker}}$ is closed for $\dim \mathcal{X} < +\infty$, too. Nevertheless, $\mathcal{C}_{\text{ker}}$ does not have to be closed in infinite dimensional topological vector spaces $\mathcal{X}$.*

## 4.2 Efficient points of the acceptance set

In this section, we consider (FM) and study efficient points of the acceptance set $\mathcal{A} \subseteq \mathcal{X}$ with respect to the price cone $\mathcal{C}_\pi$ in Definition 4.1.1 and the kernel cone $\mathcal{C}_{\ker}$ in Definition 4.1.3. In contrast to [139], we work with a more common definition of efficient points here: The *set of efficient points of $\mathcal{A}$ with respect to the convex cone $\mathcal{C}_\pi$* are defined by (see Definition 1.2.1 with $\mathcal{D} = \mathcal{C}_\pi$)

$$\mathrm{Eff}\,(\mathcal{A}, \mathcal{C}_\pi) = \{X \in \mathcal{A} \mid \mathcal{A} \cap (\{X\} - \mathcal{C}_\pi) \subseteq \{X\}\}. \tag{4.10}$$

Analogously, we define $\mathrm{Eff}\,(\mathcal{A}, \mathcal{C}_{\ker})$ as the set of efficient points of $\mathcal{A}$ with respect to $\mathcal{C}_{\ker}$. In topological vector spaces $\mathcal{X}$, efficient points are elements of $\mathrm{bd}\,\mathcal{A}$ by Lemma 1.2.3 if $\mathbb{0} \in \mathrm{bd}\,(\mathcal{C}_\pi \backslash \{\mathbb{0}\})$ and $\mathbb{0} \in \mathrm{bd}\,(\mathcal{C}_{\ker} \backslash \{\mathbb{0}\})$, respectively. Since $\mathcal{C}_\pi \subsetneq \mathcal{C}_{\ker}$ holds by (4.7), we obtain

$$\mathrm{Eff}\,(\mathcal{A}, \mathcal{C}_{\ker}) \subseteq \mathrm{Eff}\,(\mathcal{A}, \mathcal{C}_\pi). \tag{4.11}$$

The following example illustrates that $\mathrm{Eff}\,(\mathcal{A}, \mathcal{C}_{\ker}) \subseteq \mathrm{Eff}\,(\mathcal{A}, \mathcal{C}_\pi)$ and $\mathcal{C}_\pi \subseteq \mathcal{C}_{\ker}$ hold, in general.

**Example 4.2.1.** Consider (FM) with $\mathcal{X} = \mathbb{R}^3$, $\mathcal{M} = \{Z \in \mathbb{R}^3 \mid Z_2 = 0\}$, $U = (1, 0, 0)^T$ and $\pi(Z) := Z_1 + Z_3$. Then,

$$\ker \pi = \{Z \in \mathbb{R}^3 \mid Z_3 = -Z_1, Z_2 = 0\}$$

and

$$\mathcal{C}_\pi = \{Z \in \mathbb{R}^3 \mid Z_3 \geq -Z_1, Z_2 = 0\}.$$

Moreover,

$$\mathcal{C}_{\ker} = \{Z \in \mathbb{R}^3 \mid Z_3 \geq -Z_1, Z_2 \geq 0\}.$$

Consider the acceptance set $\mathcal{A} = \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$, which is closed with respect to the euclidean topology on $\mathbb{R}^3$. Then,

$$\mathrm{Eff}\,(\mathcal{A}, \mathcal{C}_{\ker}) = \emptyset \quad \text{and} \quad \mathrm{Eff}\,(\mathcal{A}, \mathcal{C}_\pi) = \{\mathbb{0}\} \times \mathbb{R} \times \{\mathbb{0}\} = \mathcal{A}',$$

where $\mathcal{A}' \subseteq \mathcal{X}$ is the set of optimal acceptable capital positions given by (3.25). $\Diamond$

With respect to our remarks at the beginning of Section 4.1 (see also our remarks after Definition 4.1.1), $\mathrm{Eff}\,(\mathcal{A}, \mathcal{C}_\pi)$ consists of acceptable capital positions $X \in \mathcal{A}$ that can not be transferred by some $Z \in \mathcal{M}$ with price $\pi(Z) \leq 0$ to another acceptable capital position $X^0 \in \mathcal{A}$, which is similar to the condition (4.4) for $\mathcal{A}' \subseteq \mathcal{X}$ defined in (3.25). With respect to the remarks after Definition 4.1.3, we can interpret the set $\mathrm{Eff}\,(\mathcal{A}, \mathcal{C}_{\ker})$ by (4.11) as the acceptable capital positions in $\mathrm{Eff}\,(\mathcal{A}, \mathcal{C}_\pi)$ for which there is no position along $\ker \pi$ that is greater (in the sense of the order relation given by $\mathcal{X}_+$) than any other acceptable capital position.

As observed in Example 4.2.1, we conjecture $\mathrm{Eff}\,(\mathcal{A}, \mathcal{C}_\pi) \subseteq \mathcal{A}'$ and, thus, $\mathrm{Eff}\,(\mathcal{A}, \mathcal{C}_{\ker}) \subseteq \mathcal{A}'$ by (4.11) for closed acceptance sets $\mathcal{A} \subseteq \mathcal{X}$, i.e., the sets of efficient points of $\mathcal{A}$ are subsets of $\mathrm{bd}_{-U}(\mathcal{A})$ with $U \in \mathcal{M} \cap \mathcal{X}_+$ according to Assumption 2. With respect to Example 4.2.1, the following example strengthens our conjecture that $\mathrm{Eff}(\mathcal{A}, \mathcal{C}_{\mathrm{ker}}) \subseteq \mathrm{Eff}(\mathcal{A}, \mathcal{C}_\pi) \subseteq \mathcal{A}'$ even holds for just $(-U)$-directional closed acceptance sets not being closed, as well, with respect to Assumption 4 and non-empty sets of efficient points of $\mathcal{A}$.

**Example 4.2.2.** Consider (FM) with $\mathcal{X} = \mathbb{R}^3$, $\mathcal{M} = \{Z = (Z_1, Z_2, Z_3)^T \in \mathbb{R}^3 \mid Z_2 = -Z_1\}, \pi(Z) := Z_3$ and $U = (0,0,1)^T$. Let $R = (0,0,-1)^T, S = (1,0,-2)^T$ and

$$\mathcal{A} = \{\mathbb{0}\} \cup ((\mathrm{conv}\{R,S\} + \mathcal{X}_+)\backslash \mathrm{conv}\{R,\mathbb{0}\}) \cup \left(U + \mathrm{conv}\left\{\mathrm{cone}\left\{(-1,0,0)^T, (0,1,1)^T, (0,0,1)^T\right\}\right\}\right),$$

where conv$\{.\}$ denotes the convex hull and cone$\{.\}$ denotes the cone hull of a set. $\mathcal{A}$ is illustrated in Figure 4.2 and an acceptance set with respect to Definition 2.2.9 which is not closed because of $\mathrm{conv}\{R,\mathbb{0}\}\backslash\{\mathbb{0}\} \not\subseteq \mathcal{A}$. Nevertheless, $\mathcal{A}$ fulfills Assumption 4, i.e., $\mathcal{A}$ is $(-U)$-directionally closed because $\mathbb{0} \in \mathcal{A}$ and $\mathrm{bd}_{-U}(\mathcal{A}) = \mathrm{bd}(\mathcal{A})\backslash \mathcal{D} \subseteq \mathcal{A}$ with

$$D := (\mathrm{conv}\{R,\mathbb{0}\}\backslash\{\mathbb{0}\}) \cup \left(R + \mathbb{R}_>(0,1,0)^T + \mathbb{R}_>(0,0,1)^T\right) \cup \left(U + \mathbb{R}_>(-1,0,0)^T + \mathbb{R}_>(0,0,1)^T\right).$$

Furthermore,

$$\ker \pi = \{Z \in \mathbb{R}^3 \mid Z_2 = -Z_1, Z_3 = 0\}$$

and $\rho_{\mathcal{A},\mathcal{M},\pi}(X) \in \mathbb{R}$ for every $X \in \mathcal{X}$ with $\rho_{\mathcal{A},\mathcal{M},\pi} \colon \mathcal{X} \to \overline{\mathbb{R}}$ given by (2.30). Moreover, $\mathcal{C}_\pi \subsetneq \mathcal{C}_{\mathrm{ker}}$ with

$$\mathcal{C}_\pi = \{X \in \mathbb{R}^3 \mid X_2 = -X_1, X_3 \geq 0\}$$

being the price cone (see Definition 4.1.1) and

$$\mathcal{C}_{\mathrm{ker}} = \{X \in \mathbb{R}^3 \mid X_2 \geq -X_1, X_3 \geq 0\}$$

being the kernel cone (see Definition 4.1.3). We have

$$\mathcal{A} \cap \{X \in \mathbb{R}^3 \mid X_3 = 0\} = \{X \in \mathbb{R}^3 \mid X_1, X_2 \geq 0, X_3 = 0\}$$

and, thus,

$$\mathcal{A} \cap \ker \pi = \mathcal{A} \cap \{Z \in \mathbb{R}^3 \mid Z_2 = -Z_1, Z_3 = 0\} = \{\mathbb{0}\}.$$

Furthermore,

$$\mathrm{Eff}(\mathcal{A}, \mathcal{C}_{\mathrm{ker}}) = \{\mathbb{0}\} \cup (\mathrm{conv}\{R,S\}\backslash R))$$

and

$$\mathrm{Eff}(\mathcal{A}, \mathcal{C}_\pi) = \mathrm{Eff}(\mathcal{A}, \mathcal{C}_{\mathrm{ker}}) \cup (U + \mathbb{R}_>(-1,0,0)^T).$$

Thus, $\mathrm{Eff}(\mathcal{A}, \mathcal{C}_{\mathrm{ker}}) \subsetneq \mathrm{Eff}(\mathcal{A}, \mathcal{C}_\pi)$. Moreover, it is easy to see (and will be verified in Theorem 4.2.6) that $\mathrm{Eff}(\mathcal{A}, \mathcal{C}_\pi) \subseteq \mathcal{A}'$ holds with $\mathcal{A}' \subseteq \mathcal{X}$ given by (3.25), but

$$R + \mathbb{R}_>(0,1,0)^T \subseteq \mathcal{A}',$$

while $R+\mathbb{R}_>(0,1,0)^T \not\subseteq \text{Eff}\,(\mathcal{A}, \mathcal{C}_\pi)$. Indeed, take $X \in R+\mathbb{R}_>(0,1,0)^T$ arbitrary and $(m,-m,0)^T \in \ker\pi \subseteq \mathcal{C}_\pi$ with $m \in \mathbb{R}_>$ sufficiently small, namely, $0 < m < X_2$. Then, $X + (m,-m,0)^T \in \mathcal{A}$ and, thus, $X \notin \text{Eff}\,(\mathcal{A}, \mathcal{C}_\pi)$. Hence, we obtain

$$\emptyset \neq \text{Eff}\,(\mathcal{A}, \mathcal{C}_{\ker}) \subsetneq \text{Eff}\,(\mathcal{A}, \mathcal{C}_\pi) \subsetneq \mathcal{A}'.$$

$\diamond$

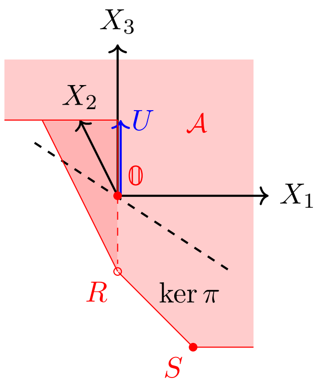


Figure 4.2: Illustration of Example 4.2.2 with $\emptyset \neq \text{Eff}\,(\mathcal{A}, \mathcal{C}_{\ker\pi}) \subsetneq \text{Eff}\,(\mathcal{A}, \mathcal{C}_\pi) \subsetneq \mathcal{A}'$

**Remark 4.2.3.** *Consider (FM). We will focus on $(-U)$-directionally closed acceptance sets $\mathcal{A}$ with $U \in \mathcal{M} \cap \mathcal{X}_+$ according to Assumption 2. Like in Example 4.2.2 and characterized in Lemma 3.1.4, the acceptance set $\mathcal{A} \subseteq \mathcal{X}$ does not have to be closed in topological vector spaces $\mathcal{X}$. As already mentioned earlier, $\mathcal{A}$ is $(-U)$-directionally closed, too, if $\mathcal{A}$ is closed. Hence, the results that we derive in this chapter also work for closed acceptance sets. As noticed before, we conjecture that the sets of efficient points of $\mathcal{A}$ are subsets of $\text{bd}_{-U}(\mathcal{A})$. For arbitrary acceptance sets $\mathcal{A}$ in topological vector spaces, we obtain $\text{bd}_{-U}(\mathcal{A}) \subseteq \text{bd}\,\mathcal{A}$, in general, and both sets do not have to coincide even if $\mathcal{A}$ is closed. The proof of Theorem 2.3.28(b) also works if we replace $\mathcal{A} + \ker\pi$ by the acceptance set $\mathcal{A}$. Hence,*

$$\mathcal{A} + \mathbb{R}_> U \subseteq \text{int}\,\mathcal{A}$$

*and*

$$\text{cl}\,\mathcal{A} + \mathbb{R}_> U \subseteq \mathcal{A}.$$

*are sufficient for $\text{bd}_{-U}(\mathcal{A}) = \text{bd}\,\mathcal{A}$. In contrast, Theorem 2.3.28(a) implies $\text{bd}_{-U}(\mathcal{A} + \ker\pi) = \text{bd}\,(\mathcal{A} + \ker\pi)$ if $\rho_{\mathcal{A},\mathcal{M},\pi} \colon \mathcal{X} \to \overline{\mathbb{R}}$ defined in (2.30) is continuous on $\mathcal{X}$, but it does not secure $\text{bd}_{-U}(\mathcal{A}) = \text{bd}\,\mathcal{A}$, i.e., Theorem 2.3.28(a) is not sufficient for $\text{bd}_{-U}(\mathcal{A}) = \text{bd}\,\mathcal{A}$. An easy example is given by (FM) with $\mathcal{X} = \mathcal{M} = \mathbb{R}^2$, $\pi(Z) := Z_2$, $U = (0,1)^T$ and $\mathcal{A} = \mathbb{R}^2_+$: It holds that $\rho_{\mathcal{A},\mathcal{M},\pi}(X) = -X_2$ for all $X \in \mathbb{R}^2$ and, thus, $\rho_{\mathcal{A},\mathcal{M},\pi}$ is obviously finite and continuous, but we have*

$$\text{bd}_{-U}(\mathcal{A}) = \mathbb{R}_+ \times \{\mathbb{0}\} \subsetneq (\mathbb{R}_+ \times \{0\}) \cup (\{0\} \times \mathbb{R}_+) = \text{bd}\,(\mathbb{R}^2_+)$$

*while, nevertheless,* $\mathrm{bd}_{-U}(\mathcal{A} + \ker \pi) = \mathrm{bd}(\mathcal{A} + \ker \pi) = \mathbb{R} \times \{\mathbb{0}\}$ *holds (as stated in Theorem 2.3.28).*

Our aim is to study the relationship between the sets of efficient points $\mathrm{Eff}(\mathcal{A}, \mathcal{C}_{\ker})$ and $\mathrm{Eff}(\mathcal{A}, \mathcal{C}_\pi)$, respectively, and the set $\mathcal{A}' \subseteq \mathcal{X}$ given by (3.25) in detail. Especially, we will outline assumptions which imply that $\mathrm{Eff}(\mathcal{A}, \mathcal{C}_{\ker})$, $\mathrm{Eff}(\mathcal{A}, \mathcal{C}_\pi)$ and $\mathcal{A}'$ coincide. We suppose $(-U)$-directionally closed acceptance sets $\mathcal{A}$, i.e., Assumption 4 is fulfilled. First, we observe the following result, which strengthens our conjecture that there is a relationship between the sets of efficient points of $\mathcal{A}$ and $\mathcal{A}' \subseteq \mathrm{bd}_{-U}(\mathcal{A})$:

**Theorem 4.2.4.** *Consider (FM). Let Assumption 2 be fulfilled by some eligible payoff* $U \in \mathcal{M} \cap \mathcal{X}_+$. *Furthermore, let* $\mathcal{C}_\pi \subseteq \mathcal{M}$ *be the price cone given by Definition 4.1.1 and* $\mathcal{C}_{\ker} \subseteq \mathcal{X}$ *be the kernel cone given by Definition 4.1.3. Then,*

$$\mathrm{Eff}(\mathcal{A}, \mathcal{C}_{\ker}) \subseteq \mathrm{Eff}(\mathcal{A}, \mathcal{C}_\pi) \subseteq \mathrm{bd}_{-U}(\mathcal{A}).$$

**Proof.** Since (4.7) holds, we only need to show $\mathrm{Eff}(\mathcal{A}, \mathcal{C}_\pi) \subseteq \mathrm{bd}_{-U}(\mathcal{A})$. Let $X \in \mathrm{Eff}(\mathcal{A}, \mathcal{C}_\pi)$. Then, $X \in \mathcal{A}$ by definition of $\mathrm{Eff}(\mathcal{A}, \mathcal{C}_\pi)$. Suppose $X \notin \mathrm{bd}_{-U}(\mathcal{A}) = \mathrm{cl}_{-U}(\mathcal{A}) \backslash \mathrm{int}_{-U}(\mathcal{A})$. Since $\mathrm{int}_{-U}(\mathcal{A}) \subseteq \mathcal{A} \subseteq \mathrm{cl}_{-U}(\mathcal{A})$ and $X \in \mathcal{A}$, we obtain $X \notin \mathcal{A} \backslash \mathrm{int}_{-U}(\mathcal{A})$ and, thus, $X \in \mathrm{int}_{-U}(\mathcal{A})$. Then,

$$\exists t \in \mathbb{R}_> : \quad X - tU \in \mathcal{A}.$$

Hence,

$$\mathcal{A} \cap \left( \{X\} - \bigcup_{m>0} \pi_m \right) \not\subseteq \{X\}$$

with $\pi_m \subseteq \mathcal{M}$ given by (2.16), since $\pi(tU) = t > 0$ by linearity of $\pi$ and $\pi(U) = 1$ by Assumption 2. Consequently,

$$\mathcal{A} \cap (\{X\} - \mathcal{C}_\pi \backslash \{\mathbb{0}\}) \not\subseteq \{X\},$$

in contradiction to $X \in \mathrm{Eff}(\mathcal{A}, \mathcal{C}_\pi)$. As a result, $X \in \mathrm{bd}_{-U}(\mathcal{A})$ holds. □

**Remark 4.2.5.** *Theorem 4.2.4 is, especially, interesting with respect to Lemma 1.2.3: In topological vector spaces* $\mathcal{X}$, *Lemma 1.2.3 shows that the efficient points of* $\mathcal{A} \subseteq \mathcal{X}$ *are elements of* $\mathrm{bd}\,\mathcal{A}$ *under some certain additional assumption. Hence, we proved in Theorem 4.2.4 a corresponding result for the directional boundary of* $\mathcal{A}$ *without any topological properties and with* $\mathcal{A}$ *being a subset of a vector space. Moreover, we did not make any additional assumptions on* $\mathcal{A}$. *More precisely, although we assume an acceptance set in (FM), we did not even use that* $\mathcal{A}$ *is an acceptance set in the proof, indeed.*

Since the set of optimal acceptable capital positions $\mathcal{A}' \subseteq \mathcal{X}$ given by (3.25) fulfills $\mathcal{A}' \subseteq \mathrm{bd}_{-U}(\mathcal{A})$, we can use Theorem 4.2.4 to prove explicit relationships between $\mathcal{A}'$ and the sets of efficient points of $\mathcal{A}$, see Theorem 4.2.6. Note that we can rewrite $\mathcal{A}'$ by $\mathcal{C}_\pi$ and $\mathcal{C}_{\ker}$ (see Definition 4.1.1 and Definition 4.1.3) as

$$\mathcal{A}' = \operatorname{bd}_{-U}(\mathcal{A}) \cap \operatorname{bd}_{-U}(\mathcal{A}+\mathcal{C}_\pi) = \operatorname{bd}_{-U}(\mathcal{A}) \cap \operatorname{bd}_{-U}(\mathcal{A}+\mathcal{C}_{\ker}) \tag{4.12}$$

because

$$\mathcal{A}+\mathcal{C}_\pi \overset{(4.6)}{=} \mathcal{A}+\mathbb{R}_+U+\ker\pi \overset{(2.18)}{=} \mathcal{A}+\ker\pi \overset{(2.17)}{=} \mathcal{A}+\mathcal{X}_+ +\ker\pi = \mathcal{A}+\mathcal{C}_{\ker} \tag{4.13}$$

with $U \in \mathcal{M}\cap\mathcal{X}_+$ according to Assumption 2, where the last equation follows from definition of $\mathcal{C}_{\ker}$. Now, we present the first main result of this chapter:

**Theorem 4.2.6.** *Consider (FM). Let Assumption 2 be fulfilled by some eligible payoff $U \in \mathcal{M}\cap\mathcal{X}_+$. Furthermore, let $\mathcal{C}_\pi \subseteq \mathcal{M}$ be the price cone given by Definition 4.1.1, $\mathcal{C}_{\ker} \subseteq \mathcal{X}$ be the kernel cone given by Definition 4.1.3 and $\mathcal{A}' \subseteq \mathcal{X}$ the set in (3.25). Then, it holds that*

$$\operatorname{Eff}(\mathcal{A},\mathcal{C}_{\ker}) \subseteq \operatorname{Eff}(\mathcal{A},\mathcal{C}_\pi) \subseteq \mathcal{A}'. \tag{4.14}$$

*Moreover, if $\mathcal{M}$ fulfills (3.35), i.e., $\forall X^0, Y^0 \in \mathcal{A}' :\ X^0 - Y^0 \in \mathcal{M}$, then the following conditions hold:*

(i) *If $\operatorname{Eff}(\mathcal{A},\mathcal{C}_\pi) \neq \emptyset$, then $\operatorname{Eff}(\mathcal{A},\mathcal{C}_\pi) = \mathcal{A}'$.*

(ii) *If $\operatorname{Eff}(\mathcal{A},\mathcal{C}_{\ker}) \neq \emptyset$, then*

$$\operatorname{Eff}(\mathcal{A},\mathcal{C}_{\ker}) = \operatorname{Eff}(\mathcal{A},\mathcal{C}_\pi) = \mathcal{A}'.$$

**Proof.** First, we prove (4.14) for an arbitrary acceptance set $\mathcal{A}$. By (4.11), we only need to show $\operatorname{Eff}(\mathcal{A},\mathcal{C}_\pi) \subseteq \mathcal{A}'$. Take $X \in \operatorname{Eff}(\mathcal{A},\mathcal{C}_\pi)$ arbitrary. Then, $X \in \operatorname{bd}_{-U}(\mathcal{A})$ by Theorem 4.2.4. Suppose that $X \notin \operatorname{bd}_{-U}(\mathcal{A}+\ker\pi)$. Then, $X \notin \operatorname{bd}_{-U}(\mathcal{A}+\mathcal{C}_\pi)$ by (4.13), and $X \in \mathcal{A}+\mathcal{C}_\pi$ by $\mathbb{0} \in \mathcal{C}_\pi$ and $X \in \mathcal{A}$ by definition of $\operatorname{Eff}(\mathcal{A},\mathcal{C}_\pi)$. Thus, $X \in \operatorname{int}_{-U}(\mathcal{A}+\mathcal{C}_\pi)$, i.e.,

$$\exists \epsilon > 0,\ \forall t \in (0,\epsilon) : \quad X - tU \in \mathcal{A}+\mathcal{C}_\pi.$$

Take $t \in (0,\epsilon)$ arbitrary. Then, $X - tU \notin \mathcal{A}$ by $X \in \operatorname{bd}_{-U}(\mathcal{A})$, and, thus,

$$X - tU \in \mathcal{A}+\mathcal{C}_\pi\backslash\{\mathbb{0}\}.$$

Indeed, if $X - tU \in \mathcal{A}$ for some $t \in \mathbb{R}_+$, we obtain $X - mU \in \mathcal{A}$ for each $m < t$ by monotonicity of $\mathcal{A}$ (see Definition 2.2.9(iii)) in contradiction to $X \in \operatorname{bd}_{-U}(\mathcal{A})$. Hence, there is some $X^0 \in \mathcal{A}\backslash\{X\}$ with

$$X - tU \in \{X^0\} + \mathcal{C}_\pi\backslash\{\mathbb{0}\}$$

or, equivalently,

$$X^0 \in \{X\} - (\mathcal{C}_\pi\backslash\{\mathbb{0}\} + \{tU\}). \tag{4.15}$$

It holds that $tU \in \pi_t$ for $\pi_t \subseteq \mathcal{M}$ given by (2.16) with $t > 0$ by linearity of $\pi$ and $\pi(U) = 1$ for $U \in \mathcal{X}_+ \cap \mathcal{M}$ according to Assumption 2. Consequently, $tU \in \mathcal{C}_\pi \backslash \ker \pi$ and, thus,

$$\mathcal{C}_\pi \backslash \{\mathbb{0}\} + \{tU\} \subseteq \mathcal{C}_\pi \backslash \ker \pi, \tag{4.16}$$

since

$$\forall Z \in \mathcal{C}_\pi \backslash \{\mathbb{0}\} : \quad \pi(Z + tU) \geq \pi(tU) = t > 0$$

by linearity of $\pi$. Especially, $Z + tU \in \mathcal{C}_\pi \backslash \{\mathbb{0}\}$ for each $Z \in \mathcal{C}_\pi$. Consequently, (4.15) implies

$$X^0 \in \{X\} - (\mathcal{C}_\pi \backslash \{\mathbb{0}\} + \{tU\}) \overset{(4.16)}{\subseteq} \{X\} - (\mathcal{C}_\pi \backslash \ker \pi) \subseteq \{X\} - \mathcal{C}_\pi \backslash \{\mathbb{0}\},$$

showing

$$X^0 \in \mathcal{A} \cap (\{X\} - \mathcal{C}_\pi \backslash \{\mathbb{0}\}) \not\subseteq \{X\}$$

because of $X^0 \in \mathcal{A}$. Hence, we obtain a contradiction to $X \in \operatorname{Eff}(\mathcal{A}, \mathcal{C}_\pi)$. As a result, $X \in \operatorname{bd}_{-U}(\mathcal{A} + \ker \pi)$ holds and the proof of (4.14) is complete.

In the following, we assume that $\mathcal{A}$ is a $(-U)$-directionally closed acceptance set and $\mathcal{M}$ fulfills (3.35). We prove (i) and assume that $\operatorname{Eff}(\mathcal{A}, \mathcal{C}_\pi) \neq \emptyset$. By (4.14), we obtain $\mathcal{A}' \neq \emptyset$. Take $X \in \mathcal{A}'$ arbitrary and suppose $X \notin \operatorname{Eff}(\mathcal{A}, \mathcal{C}_\pi)$. Since $\operatorname{Eff}(\mathcal{A}, \mathcal{C}_\pi) \neq \emptyset$, there is some $X^0 \in \operatorname{Eff}(\mathcal{A}, \mathcal{C}_\pi)$ . Thus, $X^0 \in \mathcal{A}'$ by (4.14) with $X^0 \neq X$. By (3.35), it holds that $X^0 - X \in \mathcal{M}$. Hence, we obtain

$$X^0 - X \in \ker \pi \backslash \{\mathbb{0}\} \subseteq \mathcal{C}_\pi \backslash \{\mathbb{0}\}$$

by Corollary 3.2.6 and, thus,

$$X \in \{X^0\} - \mathcal{C}_\pi \backslash \{\mathbb{0}\},$$

in contradiction to $X^0 \in \operatorname{Eff}(\mathcal{A}, \mathcal{C}_\pi)$. As a result, $X \in \operatorname{Eff}(\mathcal{A}, \mathcal{C}_\pi)$, which shows $\operatorname{Eff}(\mathcal{A}, \mathcal{C}_\pi) = \mathcal{A}'$ and the proof of (i) is complete.

Finally, we show (ii) and suppose $\operatorname{Eff}(\mathcal{A}, \mathcal{C}_{\ker}) \neq \emptyset$. By (4.11), it is sufficient to prove $\operatorname{Eff}(\mathcal{A}, \mathcal{C}_{\ker}) = \mathcal{A}'$. Because of $\ker \pi \subseteq \mathcal{C}_{\ker}$, the proof of (i) works for (ii), as well: For $X \in \mathcal{A}' \backslash \operatorname{Eff}(\mathcal{A}, \mathcal{C}_{\ker})$ and $X^0 \in \operatorname{Eff}(\mathcal{A}, \mathcal{C}_{\ker}) \subseteq \mathcal{A}'$, we conclude $X^0 - X \in \ker \pi \backslash \{\mathbb{0}\} \subseteq \mathcal{C}_{\ker} \backslash \{\mathbb{0}\}$ and, thus, $X \in \{X^0\} - \mathcal{C}_{\ker} \backslash \{\mathbb{0}\}$, in contradiction to $X^0 \in \operatorname{Eff}(\mathcal{A}, \mathcal{C}_{\ker})$. Hence, $\mathcal{A}' = \operatorname{Eff}(A, \mathcal{C}_{\ker})$ holds and (ii) is shown. □

**Remark 4.2.7.** *Theorem 4.2.6 shows that the sets of efficient points with respect to the cones $\mathcal{C}_\pi$ and $\mathcal{C}_{\ker}$, respectively, are subsets of $\mathcal{A}'$ given by (3.25) for any arbitrary acceptance set $\mathcal{A}$ in (FM), i.e., no directional closedness assumption is necessary. Moreover, the theorem highlights the observations in our examples:*

- *In Example 4.2.1, we obtained $\emptyset = \operatorname{Eff}(\mathcal{A}, \mathcal{C}_{\ker}) \subsetneq \operatorname{Eff}(\mathcal{A}, \mathcal{C}_\pi) = \mathcal{A}'$ with $\mathcal{A}' - \mathcal{A}' \not\subseteq \mathcal{M}$,*
- *In Example 4.2.2, we obtained $\emptyset \neq \operatorname{Eff}(\mathcal{A}, \mathcal{C}_{\ker}) \subsetneq \operatorname{Eff}(\mathcal{A}, \mathcal{C}_\pi) \subsetneq \mathcal{A}'$ with $\mathcal{A}' - \mathcal{A}' \not\subseteq \mathcal{M}$.*

*The last example shows the necessity of* (3.35) *in Theorem 4.2.6, which does not hold in Example 4.2.2, e.g., because of* $X^0 - Y^0 \notin \mathcal{M}$ *for each* $X^0, Y^0 \in R + \mathbb{R}_>(0,1,0)^T \subseteq \mathcal{A}'$.

Lemma 3.2.4 provides a useful characterization of $\mathcal{A}\backslash\mathcal{A}'$ with $\mathcal{A}'$ given by (3.25). Note that $\mathcal{A}\backslash\mathcal{A}'$, especially, consists of no efficient points of $\mathcal{A}$ by Theorem 4.2.6. We will use the result

$$\mathcal{A}'\backslash\mathcal{A} \subseteq \operatorname{int}_{-U}(\mathcal{A} + \ker\pi)$$

from Lemma 3.2.4 with $U \in \mathcal{M} \cap \mathcal{X}_+$ according to Assumption 2 in the proof of the following corollary with respect to special cases of Theorem 4.2.6 and also in some proofs of results for weakly efficient points later.

**Corollary 4.2.8.** *Consider* (FM). *Let Assumption 2 be fulfilled by some eligible payoff* $U \in \mathcal{M} \cap \mathcal{X}_+$. *Furthermore, let* $\mathcal{C}_\pi \subseteq \mathcal{M}$ *be the price cone in Definition 4.1.1,* $\mathcal{C}_{\ker} \subseteq \mathcal{X}$ *be the kernel cone in Definition 4.1.3 and* $\mathcal{A}' \subseteq \mathcal{X}$ *be the set of optimal acceptable capital positions given by* (3.25). *Suppose that one of the following conditions is fulfilled:*

(i) $\mathcal{A}' = \emptyset$,

(ii) $|\mathcal{A}'| = 1$.

*Then,* $\operatorname{Eff}(\mathcal{A}, \mathcal{C}_{\ker}) = \operatorname{Eff}(\mathcal{A}, \mathcal{C}_\pi) = \mathcal{A}'$.

**Proof.** If (i) holds, (4.14) in Theorem 4.2.6 implies $\operatorname{Eff}(\mathcal{A}, \mathcal{C}_{\ker}) = \emptyset = \operatorname{Eff}(\mathcal{A}, \mathcal{C}_\pi)$, showing $\operatorname{Eff}(\mathcal{A}, \mathcal{C}_{\ker}) = \operatorname{Eff}(\mathcal{A}, \mathcal{C}_\pi) = \mathcal{A}' = \emptyset$. Now, we assume (ii) and let $\mathcal{A}' = \{X^0\}$ for some $X^0 \in \mathcal{A}$. It is sufficient to show $\operatorname{Eff}(\mathcal{A}, \mathcal{C}_{\ker}) = \mathcal{A}'$ by (4.11). Suppose $X^0 \notin \operatorname{Eff}(\mathcal{A}, \mathcal{C}_{\ker})$. Then, there is some $X \in \mathcal{A}$ with $X \neq X^0$ fulfilling

$$X \in \{X^0\} - \mathcal{C}_{\ker}\backslash\{\mathbb{0}\}. \tag{4.17}$$

Since $\mathcal{A}' = \{X^0\}$, we obtain $X \notin \mathcal{A}'$ and, thus,

$$X \in \operatorname{int}_{-U}(\mathcal{A} + \ker\pi) = \operatorname{int}_{-U}(\mathcal{A} + \mathcal{C}_{\ker})$$

by Lemma 3.2.4. Hence,

$$\exists \epsilon > 0: \quad \{X\} - [0,\epsilon]U \subseteq \mathcal{A} + \ker\pi,$$

where $[0,\epsilon]U := \{tU \mid 0 \leq t \leq \epsilon\}$. Consider the direction $K := X^0 - X$. Then, $K \in \mathcal{C}_{\ker}\backslash\{\mathbb{0}\}$ by (4.17). Moreover,

$$\mathcal{A} + \ker\pi + \mathcal{C}_{\ker} = \mathcal{A} + \ker\pi + \ker\pi + \mathcal{X}_+ \subseteq \mathcal{A} + \ker\pi$$

by monotonicity of $\mathcal{A} + \ker\pi$ (see Lemma 2.3.14) and $\ker\pi + \ker\pi \subseteq \ker\pi$, since $\ker\pi$ is a subspace of $\mathcal{M}$. Consequently,

$$\mathcal{C}_{\ker} \subseteq \operatorname{rec}(\mathcal{A} + \ker\pi)$$

and, thus,

$$\{X^0\} - [0,\epsilon]U = \{X\} - [0,\epsilon]U + \{K\} \subseteq \mathcal{A} + \ker\pi$$

because of $\{X\} - [0,\epsilon]U \in \mathcal{A} + \ker\pi$ and $K \in \mathcal{C}_{\ker}\backslash\{\mathbb{0}\}$. As a result, $X^0 \in \operatorname{int}_{-U}(\mathcal{A} + \ker\pi)$, which contradicts $X^0 \in \mathcal{A}'$ by $\mathcal{A}' \subseteq \operatorname{bd}_{-U}(\mathcal{A} + \ker\pi)$. Consequently, $X \in \operatorname{Eff}(\mathcal{A}, \mathcal{C}_{\ker})$ holds and the proof is complete. □

**Remark 4.2.9.** *Consider* (FM). Corollary 4.2.8 really provides special cases with respect to the assertion in Theorem 4.2.6. In Corollary 4.2.8(ii), the condition (3.35) is automatically fulfilled. Hence, the proof just shows $\operatorname{Eff}(\mathcal{A}, \mathcal{C}_{\ker}) \neq \emptyset$. Nevertheless, we can not replace the assumption $|\mathcal{A}'| = 1$ by $|\mathcal{E}(X)| = 1$ for every $X \in \mathcal{X}$, where $\mathcal{E}(X)$ is the set of optimal payoffs given by (3.1), see [139, Expl. 5.6 and 5.7]. Although there is a direct relationship between $\mathcal{A}'$ and the optimal payoff map $\mathcal{E}\colon \mathcal{X} \rightrightarrows \mathcal{M}$ as seen in Section 3.2 (especially, Theorem 3.2.2 and Theorem 3.2.7), it holds that

$$|\mathcal{A}'| = 1 \quad \not\Leftrightarrow \quad \forall X \in \mathcal{X}: \quad |\mathcal{E}(X)| = 1. \tag{4.18}$$

In Example 4.2.1, it holds that $\mathcal{E}(X) = \{(-X_1, 0, -X_3)^T\}$ for $X \in \mathbb{R}^3$ arbitrary and $\operatorname{Eff}(\mathcal{A}, \mathcal{C}_{\ker}) = \emptyset \subsetneq \operatorname{Eff}(\mathcal{A}, \mathcal{C}_\pi) \subsetneq \mathcal{A}'$, which shows that the condition on the right side in (4.18) is not sufficient for the assertion $\operatorname{Eff}(\mathcal{A}, \mathcal{C}_{\ker}) = \operatorname{Eff}(\mathcal{A}, \mathcal{C}_\pi) = \mathcal{A}'$.

## 4.3 Weakly efficient points and the role of complete markets

Now, we consider (FM) and study weakly efficient points of the acceptance set $\mathcal{A} \subseteq \mathcal{X}$ with respect to $\mathcal{C}_\pi$ given by Definition 4.1.1 and $\mathcal{C}_{\ker}$ given by Definition 4.1.3. Hence, we will consider topological vector spaces, which we assume to be equipped with the standard topology if there is a norm on $\mathcal{X}$. For our purposes, we remember that a market is said to be complete if $\mathcal{X} = \mathcal{M}$ (see Definition 3.2.10). As noticed in Remark 3.2.11, $\mathcal{X}$ has to be finite dimensional if the market is complete, since $\mathcal{M}$ is a finite dimensional subspace by (2.10). Moreover, the market is complete if each positive element (i.e., $X \in \mathcal{X}_+$) is an eligible payoff, i.e.,

$$\mathcal{X}_+ \subseteq \mathcal{M} \quad \overset{\dim \mathcal{M} < +\infty}{\Longrightarrow} \quad \mathcal{X} = \mathcal{M} \text{ (complete market) and } \dim \mathcal{X} < +\infty. \tag{4.19}$$

**Remark 4.3.1.** *Note that the positive cone $\mathcal{X}_+$ does not have to be generating in a vector space $\mathcal{X}$ in general, i.e., $\operatorname{span} \mathcal{X}_+ \subsetneq \mathcal{X}$. It is sufficient for $\mathcal{X}_+$ being generating in a partial ordered topological vector space $\mathcal{X}$ that $\operatorname{int} \mathcal{X}_+ \neq \emptyset$, since $\mathcal{X}_+$ is a cone (see Lemma 1.1.41). For example, $\operatorname{int} C[a,b]_+ \neq \emptyset$ holds for the space of continuous functions $C[a,b]$ on an interval $[a,b] \subseteq \mathbb{R}$ with the supremum norm (see [124, Expl. 2.2.13]), while $\operatorname{int} l^p_+ = \emptyset$ holds for the space of bounded real sequences with respect to the $p-$norm for $p \in [1,+\infty)$ (see [90, Expl. 2.1]). Nevertheless, it holds that $\operatorname{int} \mathcal{X}_+ \neq \emptyset$ if $\mathcal{X}$ is a finite dimensional vector space equipped with the standard topology induced by some norm. Indeed, let $\mathcal{B} := \{X^1, \ldots, X^n\}$ be a basis on a vector space $\mathcal{X}$*

*with* $\dim \mathcal{X} = n < +\infty$. *The basis* $\mathcal{B}$ *can be chosen such that* $\mathcal{B} \subseteq \mathcal{X}_+$. *Then, it is well-known that*

$$\|X\|_1 := \sum_{i=1}^{n} |\alpha_i| \quad \textit{for each } X \in \mathcal{X} \textit{ with } X = \sum_{i=1}^{n} \alpha_i X^i$$

*is a norm on each finite dimensional vector space* $\mathcal{X}$. *Hence, each finite dimensional vector space can be equipped with the standard topology, i.e., a topology induced by a norm. Moreover, all norms on a finite dimensional space are equivalent, i.e., each norm on* $\mathcal{X}$ *induces the same topology (see [196, Satz I.2.5]). Note that the representation of* $X \in \mathcal{X}$ *as a linear combination of the vectors in* $\mathcal{B}$ *is unique. More exactly,* $\varphi\colon \mathbb{R}^n \to \mathcal{X}$ *with*

$$\varphi(\alpha) := \sum_{i=1}^{n} \alpha_i X^i$$

*is bijective and continuous. Take* $\alpha \in \mathbb{R}^n_+$ *with* $\min_i \alpha_i$ *sufficiently tall. Then, there is some* $r > 0$ *sufficiently small such that the open ball* $\mathcal{B}_r(\alpha)$ *(defined by the Euclidean metric in* $\mathbb{R}^n$*) with radius* $r$ *and center* $\alpha$ *fulfills* $\mathcal{B}_r(\alpha) \subseteq \mathbb{R}^n_+$. *The ball* $\mathcal{B}_r(\alpha)$ *is open with respect to the Euclidean topology in* $\mathbb{R}^n$ *(see Remark 1.1.47). Moreover,* $\varphi(\mathcal{B}_r(\alpha)) \subseteq \mathcal{X}_+$ *holds. Since* $\varphi$ *is bijective and continuous,* $\varphi(\mathcal{B}_r(\alpha))$ *is open with respect to the topology in* $\mathcal{X}$, *which does not depend on the considered norm* $\|\cdot\|_1$ *by the equivalence of each norm on* $\mathcal{X}$ *as noticed above. Thus,* $\operatorname{int} \mathcal{X}_+ \neq \emptyset$.

As for efficient points of the acceptance set $\mathcal{A}$, we want to outline a relationship between the weakly efficient points of $\mathcal{A}$ and the set of optimal acceptable capital positions $\mathcal{A}' \subseteq \operatorname{bd}_{-U}(\mathcal{A})$ given by (3.25). Consequently, we need to determine the interior of the cones $\mathcal{C}_\pi$ and $\mathcal{C}_{\ker}$ in topological vector spaces $\mathcal{X}$, respectively.

**Remark 4.3.2.** *Consider* (FM), $\mathcal{C}_\pi$ *given by Definition 4.1.1 and* $\mathcal{C}_{\ker}$ *given by Definition 4.1.3. Note that Assumption 2 is not sufficient for* $\operatorname{int} \mathcal{C}_{\ker} \neq \emptyset$ *and* $\operatorname{int} \mathcal{C}_\pi \neq \emptyset$. *Indeed, let* $U \in \mathcal{M} \cap \mathcal{X}_+$ *according to Assumption 2. Then,* $U$ *fulfills* $U \in \mathcal{C}_{\ker} \cap \mathcal{C}_\pi$, *but* $U \notin \operatorname{int} \mathcal{C}_{\ker}$ *and* $U \notin \operatorname{int} \mathcal{C}_\pi$, *in general: In Example 4.2.1, it holds that*

$$\operatorname{int} \mathcal{C}_\pi = \emptyset \quad \textit{and} \quad \operatorname{int} \mathcal{C}_{\ker} = \{Z \in \mathbb{R}^3 \mid Z_3 > -Z_2, Z_2 > 0\}.$$

*Thus,*

$$U = (1,0,0)^T \notin \operatorname{int} \mathcal{C}_\pi \cup \operatorname{int} \mathcal{C}_{\ker}.$$

*Especially,* $U \notin \operatorname{int} \mathcal{X}_+$ *holds in general. Of course,* $U$ *is an element of relative interior and the algebraic interior (also known as core) of* $\mathcal{C}_\pi$ *with respect to the subspace* $\mathcal{M}$ *given by* (2.10). *Moreover, we recall that* $\mathcal{C}_{\ker}$ *does not have to be closed, while* $\mathcal{C}_\pi$ *is always closed. Nevertheless,* $\mathcal{C}_{\ker}$ *is at least closed in complete markets, see Remark 4.1.8.*

As a first step to determine the interior of the cones $\mathcal{C}_{\ker}$ and $\mathcal{C}_\pi$ in topological vector spaces $\mathcal{X}$, we observe the following:

**Lemma 4.3.3.** *Consider (FM). Let Assumption 2 be fulfilled by some eligible payoff* $U \in \mathcal{M} \cap \mathcal{X}_+$*. Furthermore, let* $\mathcal{C}_\pi \subseteq \mathcal{M}$ *be the price cone given by Definition 4.1.1 and* $\mathcal{C}_{\ker} \subseteq \mathcal{X}$ *be the kernel cone given by Definition 4.1.3. Then, it holds that*

$$\ker \pi \subseteq \mathrm{bd}_{-U}(\mathcal{C}_{\ker}) \quad \text{and} \quad \ker \pi = \mathrm{bd}_{-U}(\mathcal{C}_\pi). \tag{4.20}$$

*If* $\mathcal{X}_+ \subseteq \mathcal{M}$*, then* $\ker \pi = \mathrm{bd}_{-U}(\mathcal{C}_{\ker})$*.*

**Proof.** First, we prove $\ker \pi \subseteq \mathrm{bd}_{-U}(\mathcal{C}_{\ker})$. Take $Z \in \ker \pi$ arbitrary. Then,

$$\forall t \in \mathbb{R}_> : \quad Z + tU \in \ker \pi + \mathcal{X}_+ = \mathcal{C}_{\ker},$$

taking into account the definition of $\mathcal{C}_{\ker}$. It holds that

$$\forall Z^0 \in (\ker \pi + \mathcal{X}_+) \cap \mathcal{M} : \quad \pi(Z^0) \geq 0. \tag{4.21}$$

Indeed, for $\tilde{Z} \in \ker \pi$ arbitrary, and $Z^0 \in (\{\tilde{Z}\} + \mathcal{X}_+) \cap \mathcal{M}$, we obtain $Z^0 - \tilde{Z} \in \mathcal{X}_+$ and, thus,

$$\pi(Z^0) \geq \pi(\tilde{Z}) = 0,$$

since $\pi$ is monotonically increasing on $\mathcal{M}$ by Lemma 1.3.21. Furthermore,

$$\forall t \in \mathbb{R}_> : \quad \pi(Z - tU) = -t < 0 \tag{4.22}$$

by linearity of $\pi$ and $\pi(U) = 1$ by Assumption 2, since $Z - tU \in \mathcal{M}$ by $\mathcal{M}$ being a subspace of $\mathcal{M}$ and $Z \in \ker \pi$. Because of $Z - tU \in \mathcal{M}$ for each $t \in \mathbb{R}_>$ (since $\mathcal{M}$ is a vector space),

$$\forall t \in \mathbb{R}_> : \quad Z - tU \notin \ker \pi + \mathcal{X}_+ = \mathcal{C}_{\ker}$$

by (4.21), taking into account (4.22) and the definition of $\mathcal{C}_{\ker}$. As a result, $\ker \pi \subseteq \mathrm{bd}_{-U}(\mathcal{C}_{\ker})$ holds.

Now, we show

$$\ker \pi = \mathrm{bd}_{-U}(\mathcal{C}_\pi). \tag{4.23}$$

First, we prove $(\subseteq)$ in (4.23). Take $Z \in \ker \pi$ arbitrary. It holds that

$$\forall t \in \mathbb{R}_> : \quad \pi(Z + tU) = t > 0 \quad \text{and } \pi(Z - tU) = -t < 0,$$

since $\pi$ is linear and $\pi(U) = 1$ by Assumption 2. Hence,

$$\forall t \in \mathbb{R}_> : \quad Z + tU \in \pi_t \subseteq \mathcal{C}_\pi \quad \text{and } Z - tU \in \pi_{-t} \not\subseteq \mathcal{C}_\pi,$$

with $\pi_t \subseteq \mathcal{M}$ given by (2.16), showing $\ker \pi \subseteq \mathrm{bd}_{-U}(\mathcal{C}_\pi)$ in (4.23). Now, we prove $\mathrm{bd}_{-U}(\mathcal{C}_\pi) \subseteq \ker \pi$. Take $Z \in \mathrm{bd}_{-U}(\mathcal{C}_\pi)$ arbitrary. Then, $Z \in \mathcal{C}_\pi \subseteq \mathcal{M}$ because $\mathcal{C}_\pi$ is $(-U)$-directionally closed by Lemma 4.1.6. Because of $Z \in \mathrm{bd}_{-U}(\mathcal{C}_\pi)$ and monotonicity of $\pi$ (see Lemma 1.3.21),

$$\forall t \in \mathbb{R}_> : \quad \pi(Z + tU) \in \mathcal{C}_\pi \quad \text{and } \pi(Z - tU) \notin \mathcal{C}_\pi$$

and, thus,

$$\forall t \in \mathbb{R}_> : \quad \pi(Z + tU) = \pi(Z) + t \geq 0 \quad \text{and } \pi(Z - tU) = \pi(Z) - t < 0.$$

Consequently,

$$\forall t \in \mathbb{R}_> : \quad -t < \pi(Z) \leq t,$$

which implies $Z \in \ker \pi$. As a result, $\mathrm{bd}_{-U}(\mathcal{C}_\pi) \subseteq \ker \pi$ holds, which completes the proof of (4.23) and (4.20). Finally, we assume $\mathcal{X}_+ \subseteq \mathcal{M}$ holds. Then, we obtain $\mathcal{C}_{\ker} = \mathcal{C}_\pi$ by Lemma 4.1.5 and, thus, $\mathrm{bd}_{-U}(\mathcal{C}_{\ker}) = \ker \pi$ by (4.20). □

For the necessity of $\mathcal{X}_+ \subseteq \mathcal{M}$ in Lemma 4.3.3, see Example 4.2.1 or [139, Expl. 5.6 and 5.7]: While $\ker \pi = \mathrm{bd}_{-U}(\mathcal{C}_\pi)$ always holds, we obtain $\ker \pi \subsetneq \mathrm{bd}\, \mathcal{C}_\pi = \mathcal{C}_\pi$ in Example 4.2.1. Thus, in topological vector spaces, we can not replace $\mathrm{bd}_{-U}(\mathcal{C}_\pi)$ by $\mathrm{bd}\, \mathcal{C}_\pi$ in Lemma 4.3.3, but, with respect to (4.19), we obtain the following corresponding result for $\mathcal{X}_+ \subseteq \mathcal{M}$ and, thus, complete markets (see Definition 3.2.10 and (4.19)):

**Corollary 4.3.4.** *Consider (FM). Let $(\mathcal{X}, \tau)$ be a topological vector space and Assumption 2 be fulfilled by some eligible payoff $U \in \mathcal{M} \cap \mathcal{X}_+$. Furthermore, let $\mathcal{C}_\pi \subseteq \mathcal{M}$ be the price cone given by Definition 4.1.1 and $\mathcal{C}_{\ker} \subseteq \mathcal{X}$ be the kernel cone given by Definition 4.1.3. Suppose $\mathcal{X}_+ \subseteq \mathcal{M}$. Then,*

$$\ker \pi = \mathrm{bd}\, \mathcal{C}_\pi = \mathrm{bd}\, \mathcal{C}_{\ker}.$$

**Proof.** By Lemma 4.1.5 and $\mathcal{X}_+ \subseteq \mathcal{M}$, we obtain $\mathcal{C}_{\ker} = \mathcal{C}_\pi$ and, thus,

$$\ker \pi = \mathrm{bd}_{-U}(\mathcal{C}_\pi) = \mathrm{bd}_{-U}(\mathcal{C}_{\ker}) \tag{4.24}$$

by Lemma 4.3.3. Because of $\mathcal{C}_{\ker} = \mathcal{C}_\pi$, we only need to prove

$$\ker \pi = \mathrm{bd}\, \mathcal{C}_\pi.$$

By 1.1.30(iii), it is sufficient to show

$$\mathrm{bd}\, \mathcal{C}_\pi \subseteq \mathrm{bd}_{-U}(\mathcal{C}_\pi). \tag{4.25}$$

Let $Z \in \mathrm{bd}\, \mathcal{C}_\pi$ arbitrary. As noticed in Remark 4.1.8, $\mathcal{C}_\pi$ is closed and, thus, $Z \in \mathcal{C}_\pi$. Consequently, $\pi(Z) \geq 0$ holds. Suppose $Z \notin \mathrm{bd}_{-U}(\mathcal{C}_\pi)$. Then, $\pi(Z) > 0$ because $\mathrm{bd}_{-U}(\mathcal{C}_\pi) = \ker \pi$ by (4.24). By $\mathcal{X}_+ \subseteq \mathcal{M}$, we have $\mathcal{X} = \mathcal{M}$ (see (4.19)). Thus, $\mathcal{M}$ can be equipped with the same topology $\tau$ as $\mathcal{X}$. Take $\epsilon > 0$ sufficiently small with $(\pi(Z) - \epsilon, \pi(Z) + \epsilon) \subseteq \mathbb{R}_>$. Because $\pi$ is continuous by $\dim \mathcal{M} < +\infty$ (see Remark 1.1.38) and $(\pi(Z) - \epsilon, \pi(Z) + \epsilon)$ is open in $\mathbb{R}$ with respect to the Euclidean topology, we obtain

$$\exists \mathcal{U} \in \mathcal{N}_Z \subseteq \tau : \quad \pi^{-1}((\pi(Z) - \epsilon, \pi(Z) + \epsilon)) \subseteq \mathcal{U},$$

where $\tau$ denotes the topology on $\mathcal{X}$ and $\mathcal{N}_Z$ the set of open neighborhoods of $Z$ (see Definition 1.1.7). Thus, $\mathcal{U} \subseteq \mathcal{C}_\pi \setminus \ker \pi$ holds, since each $W \in \mathcal{U}$ fulfills $\pi(W) > 0$ by choice of $\epsilon > 0$. As a result, $Z \in \mathrm{int}\, \mathcal{C}_\pi$, in contradiction to $Z \in \mathrm{bd}\, \mathcal{C}_\pi$. Consequently, $\pi(Z) = 0$ holds, i.e., $Z \in \ker \pi = \mathrm{bd}_{-U}(\mathcal{C}_\pi)$ by (4.24), showing (4.25) and completing the proof. □

With Lemma 4.3.3 and Corollary 4.3.4, we are able to study weakly efficient points of the acceptance set $\mathcal{A}$ with respect to $\mathcal{C}_\pi$ in Definition 4.1.1 and $\mathcal{C}_{\text{ker}}$ in Definition 4.1.3 for (FM) in topological vector spaces,respectively. Especially, we identify the weakly efficient points for complete markets (FM). Suppose $\operatorname{int}\mathcal{C}_\pi \neq \emptyset$ and $\operatorname{int}\mathcal{C}_{\text{ker}} \neq \emptyset$. Then, the sets of *weakly efficient points of $\mathcal{A}$ with respect to $\mathcal{C}_\pi$ and $\mathcal{C}_{\text{ker}}$* are given by

$$\operatorname{Eff}_w(\mathcal{A},\mathcal{C}_\pi) := \operatorname{Eff}(\mathcal{A},\operatorname{int}\mathcal{C}_\pi) \quad \text{and} \quad \operatorname{Eff}_w(\mathcal{A},\mathcal{C}_{\text{ker}}) = \operatorname{Eff}(\mathcal{A},\operatorname{int}\mathcal{C}_{\text{ker}}). \tag{4.26}$$

Hence, we need to use our previous results to describe the interior of the cones $\mathcal{C}_\pi$ and $\mathcal{C}_{\text{ker}}$. It holds

$$\ker\pi = \operatorname{bd}_{-U}(\mathcal{C}_\pi) \subseteq \operatorname{bd}\mathcal{C}_\pi \quad \text{and} \quad \ker\pi \subseteq \operatorname{bd}_{-U}(\mathcal{C}_{\text{ker}}) \subseteq \operatorname{bd}\mathcal{C}_{\text{ker}}$$

by Lemma 4.3.3 and Lemma 1.1.30(iii). Thus,

$$\operatorname{int}(\mathcal{C}_\pi) \subseteq \mathcal{C}_\pi\setminus\ker\pi \quad \text{and} \quad \operatorname{int}(\mathcal{C}_{\text{ker}}) \subseteq \mathcal{C}_{\text{ker}}\setminus\ker\pi$$

and, therefore,

$$\operatorname{Eff}(\mathcal{A},\mathcal{C}_\pi\setminus\ker\pi) \subseteq \operatorname{Eff}_w(\mathcal{A},\mathcal{C}_\pi) \tag{4.27}$$

and

$$\operatorname{Eff}(\mathcal{A},\mathcal{C}_{\text{ker}}\setminus\ker\pi) \subseteq \operatorname{Eff}_w(\mathcal{A},\mathcal{C}_{\text{ker}}), \tag{4.28}$$

respectively. Note that $\mathbb{0} \notin \mathcal{C}_\pi\setminus\ker\pi$ and $\mathbb{0} \notin \mathcal{C}_{\text{ker}}\setminus\ker\pi$ hold. Therefore, $\mathcal{C}_\pi\setminus\ker\pi$ and $\mathcal{C}_{\text{ker}}\setminus\ker\pi$ are no cones, but it holds that

$$\mathbb{0} \in \operatorname{bd}(\mathcal{C}_\pi\setminus\ker\pi) \quad \text{and} \quad \mathbb{0} \in \operatorname{bd}(\mathcal{C}_{\text{ker}}\setminus\ker\pi), \tag{4.29}$$

respectively. For complete markets, we obtain $\mathcal{C}_\pi = \mathcal{C}_{\text{ker}}$ by Lemma 4.1.5 and, thus,

$$\operatorname{int}(\mathcal{C}_{\text{ker}}) = \operatorname{int}(\mathcal{C}_\pi) = \mathcal{C}_\pi\setminus\ker\pi \quad \text{if } \mathcal{X}_+ \subseteq \mathcal{M}$$

by Corollary 4.3.4 and $\mathcal{C}_\pi$ being closed, see Remark 4.1.8. Hence,

$$\mathbb{0} \in \operatorname{bd}(\operatorname{int}\mathcal{C}_\pi) = \operatorname{bd}(\operatorname{int}\mathcal{C}_{\text{ker}}) \quad \text{if } \mathcal{X}_+ \subseteq \mathcal{M}$$

by (4.29) and, thus,

$$\operatorname{Eff}_w(\mathcal{A},\mathcal{C}_{\text{ker}}) = \operatorname{Eff}_w(\mathcal{A},\mathcal{C}_\pi) = \operatorname{Eff}(\mathcal{A},\mathcal{C}_\pi\setminus\ker\pi) \subseteq \operatorname{bd}\mathcal{A} \quad \text{if } \mathcal{X}_+ \subseteq \mathcal{M} \tag{4.30}$$

because efficient points of $\mathcal{A}$ are elements of $\operatorname{bd}\mathcal{A}$ by $\mathbb{0} \in \operatorname{bd}(\operatorname{int}\mathcal{C}_\pi) = \operatorname{bd}(\operatorname{int}\mathcal{C}_\pi\setminus\{\mathbb{0}\})$ and Lemma 1.2.3.

**Remark 4.3.5.** *Consider* (FM)*. Let $\mathcal{C}_\pi$ be the price cone given by Definition 4.1.1. Since we require* $\dim\mathcal{M} < +\infty$ *by* (2.11) *with $\mathcal{M}$ being the subspace of the vector space $\mathcal{X}$ given by* (2.10)*, we obtain in topological vector spaces $\mathcal{X}$ (compare Remark 4.3.1)*

$$\operatorname{int}(\mathcal{C}_\pi) \neq \emptyset \overset{\mathcal{C}_\pi\subseteq\mathcal{M}}{\Longrightarrow} \operatorname{int}\mathcal{M} \neq \emptyset \overset{(2.10)}{\Longrightarrow} \mathcal{M} = \mathcal{X} \overset{(2.11)}{\Longrightarrow} \dim\mathcal{X} = \dim\mathcal{M} < +\infty$$

*by Lemma 1.1.41 because each subspace $\mathcal{M}$ of $\mathcal{X}$ is a (convex) cone. Consequently, in our setting (FM), it is only possible to study* $\text{Eff}_w(\mathcal{A}, \mathcal{C}_\pi)$ *(which requires* $\text{int}\, \mathcal{C}_\pi \neq \emptyset$*) in a complete market, i.e.,* $\mathcal{X} = \mathcal{M}$*, and, thus, with a finite dimensional topological vector space* $\mathcal{X}$*. Nevertheless,* $\text{Eff}_w(\mathcal{A}, \mathcal{C}_{\text{ker}})$ *can be studied in non-complete markets.*

We will use the derived properties (4.27), (4.28) and (4.30) to gain some better understanding of weakly efficient points of acceptance sets. The following theorem will be crucial for that and highlights with respect to (4.27) and (4.28) that there is a relationship to $\mathcal{A}' \subseteq \mathcal{X}$ given by (3.25), too:

**Theorem 4.3.6.** *Consider (FM). Let* $(\mathcal{X}, \tau)$ *be a topological vector space and Assumption 2 be fulfilled by some eligible payoff* $U \in \mathcal{M} \cap \mathcal{X}_+$*. Furthermore, let* $\mathcal{C}_\pi \subseteq \mathcal{M}$ *be the price cone given by Definition 4.1.1,* $\mathcal{C}_{\text{ker}} \subseteq \mathcal{X}$ *be the kernel cone given by Definition 4.1.3 and* $\mathcal{A}' \subseteq \mathcal{X}$ *be the set of optimal acceptable capital positions in (3.25). Then,*

$$\mathcal{A}' \subseteq \text{Eff}\,(\mathcal{A}, \mathcal{C}_{\text{ker}} \setminus \ker \pi) \subseteq \text{Eff}\,(\mathcal{A}, \mathcal{C}_\pi \setminus \ker \pi).$$

**Proof.** Take $X^0 \in \mathcal{A}'$ arbitrary. Obviously,

$$\mathcal{C}_\pi \setminus \ker \pi \subseteq \mathcal{C}_{\text{ker}} \setminus \ker \pi$$

by (4.7) and, thus, $\text{Eff}\,(\mathcal{A}, \mathcal{C}_{\text{ker}} \setminus \ker \pi) \subseteq \text{Eff}\,(\mathcal{A}, \mathcal{C}_\pi \setminus \ker \pi)$. Hence, it is sufficient to show $X^0 \in \text{Eff}\,(\mathcal{A}, \mathcal{C}_{\text{ker}} \setminus \ker \pi)$. Because of (4.14) in Theorem 4.2.6 and $\mathcal{C}_{\text{ker}} \setminus \ker \pi \subseteq \mathcal{C}_{\text{ker}}$, we obtain

$$\text{Eff}\,(\mathcal{A}, \mathcal{C}_{\text{ker}}) \subseteq \mathcal{A}' \quad \text{and} \quad \text{Eff}\,(\mathcal{A}, \mathcal{C}_{\text{ker}}) \subseteq \text{Eff}\,(\mathcal{A}, \mathcal{C}_{\text{ker}} \setminus \ker \pi).$$

Hence, we can assume $X^0 \notin \text{Eff}\,(\mathcal{A}, \mathcal{C}_{\text{ker}})$. Suppose there is some $X \in \mathcal{A}$ with $X^0 \neq X$ and $X \in \{X^0\} - (\mathcal{C}_{\text{ker}} \setminus \ker \pi)$, i.e.,

$$X^0 - X \in \mathcal{C}_{\text{ker}} \setminus \ker \pi. \tag{4.31}$$

Then, $X \notin \mathcal{A}'$ holds. Indeed, if $X \in \mathcal{A}'$, then $X^0 \in \mathcal{A}'$ implies $X^0 - X \in \ker \pi$ by Corollary 3.2.6, in contradiction to (4.31). Hence,

$$X \in \text{int}_{-U}(\mathcal{A} + \ker \pi)$$

by Lemma 3.2.4 with $U \in \mathcal{M} \cap \mathcal{X}_+$ being the eligible payoff according to Assumption 2. Thus,

$$\exists \epsilon > 0, \forall t \in [0, \epsilon] : \quad X - tU \in \mathcal{A} + \ker \pi.$$

Take $t \in [0, \epsilon]$ arbitrary with $X - tU \in \mathcal{A} + \ker \pi = \mathcal{A} + \mathcal{C}_{\text{ker}}$ by (4.12). Since $\mathcal{C}_{\text{ker}}$ is convex by Lemma 4.1.6, we obtain

$$\mathcal{C}_{\text{ker}} + (\mathcal{C}_{\text{ker}} \setminus \ker \pi) \subseteq \mathcal{C}_{\text{ker}} + \mathcal{C}_{\text{ker}} \subseteq \mathcal{C}_{\text{ker}}$$

by Lemma 1.1.25. Thus,

$$X^0 - tU = X - tU + (X^0 - X) \in \mathcal{A} + \mathcal{C}_{\text{ker}} = \mathcal{A} + \ker \pi$$

because of $X - tU \in \mathcal{A} + \mathcal{C}_{\ker}$ and $X^0 - X \in \mathcal{C}_{\ker} \setminus \ker \pi$ by (4.31). Consequently,

$$X^0 \in \operatorname{int}_{-U}(\mathcal{A} + \ker \pi)$$

in contradiction to $X^0 \in \mathcal{A}' \subseteq \operatorname{bd}_{-U}(\mathcal{A} + \ker \pi)$. As a result, $X^0 \in \operatorname{Eff}(\mathcal{A}, \mathcal{C}_{\ker} \setminus \ker \pi)$ holds and the proof is complete. □

**Remark 4.3.7.** *Consider (FM). Let $\mathcal{C}_\pi \subseteq \mathcal{M}$ be the price cone given by Definition 4.1.1, $\mathcal{C}_{\ker} \subseteq \mathcal{X}$ be the kernel cone given by Definition 4.1.3 and $\mathcal{A}' \subseteq \mathcal{X}$ the set of optimal acceptable capital positions in (3.25). By Theorem 4.2.6, the sets $\operatorname{Eff}(\mathcal{A}, \mathcal{C}_{\ker})$ and $\operatorname{Eff}(\mathcal{A}, \mathcal{C}_\pi)$ are subsets of $\mathcal{A}'$. Theorem 4.3.6, especially, characterizes the elements of $\mathcal{A}' \setminus \operatorname{Eff}(\mathcal{A}, \mathcal{C}_{\ker})$ and $\mathcal{A}' \setminus \operatorname{Eff}(\mathcal{A}, \mathcal{C}_\pi)$, too: Even if points of $\mathcal{A}'$ are not efficient points of $\mathcal{A}$ with respect to $\mathcal{C}_{\ker}$ or $\mathcal{C}_\pi$, these are efficient points of the acceptance set $\mathcal{A}$ with respect to $\mathcal{C}_{\ker} \setminus \ker \pi$ and $\mathcal{C}_\pi \setminus \ker \pi$. Hence, Theorem 4.3.6 shows that if $X \in \mathcal{A}'$ is not an efficient point of the acceptance set $\mathcal{A}$ with respect to one of the cones $\mathcal{C}_\pi$ or $\mathcal{C}_{\ker}$, respectively, then there is some other point $X^0 \in \mathcal{A}'$ along $\ker \pi$, i.e.,*

$$(\{X\} + \ker \pi) \cap \mathcal{A}' \neq \emptyset,$$

*because $X$ is an efficient point of $\mathcal{A}$ if $\ker \pi$ is removed from the cone by Theorem 4.3.6. Note that*

$$(\{X\} + \ker \pi) \cap \mathcal{A} \subseteq \mathcal{A}'$$

*holds because of Lemma 3.2.5. Hence, no other element in $\mathcal{A} \setminus \mathcal{A}'$ contradicts the efficiency property (4.10) for $X$ with respect to $\mathcal{C}_\pi$ or $\mathcal{C}_{\ker}$.*

Next, we show the following relationship with respect to (4.27) and (4.28):

**Lemma 4.3.8.** *Consider (FM). Let $(\mathcal{X}, \tau)$ be a topological vector space and Assumption 2 be fulfilled by some eligible payoff $U \in \mathcal{M} \cap \mathcal{X}_+$. Furthermore, let $\mathcal{C}_\pi \subseteq \mathcal{M}$ be the price cone given by Definition 4.1.1 and $\mathcal{C}_{\ker} \subseteq \mathcal{X}$ be the kernel cone given by Definition 4.1.3. Suppose that $\operatorname{int} \mathcal{C}_\pi \neq \emptyset$ and $\operatorname{int} \mathcal{C}_{\ker} \neq \emptyset$ hold. Then,*

$$\operatorname{Eff}_w(\mathcal{A}, \mathcal{C}_{\ker}) \subseteq \operatorname{Eff}_w(\mathcal{A}, \mathcal{C}_\pi).$$

**Proof.** By (4.26), we have to prove

$$\operatorname{int} \mathcal{C}_\pi \subseteq \operatorname{int} \mathcal{C}_{\ker}. \tag{4.32}$$

Let $X \in \operatorname{int} \mathcal{C}_\pi$ arbitrary and $\tau$ be the considered topology on $\mathcal{X}$. We need to show $X \in \operatorname{int} \mathcal{C}_{\ker}$. By $X \in \operatorname{int} \mathcal{C}_\pi$, there is some $\mathcal{U} \in \tau$ with $X \in \mathcal{U}$ and $\mathcal{U} \subseteq \mathcal{C}_\pi$. Thus,

$$\forall Z \in \mathcal{U}: \quad \pi(Z) \geq 0.$$

Take $Z \in \mathcal{U}$ arbitrary. If $\pi(Z) = 0$, then $Z \in \ker \pi \subseteq \mathcal{C}_{\ker}$. Suppose $\pi(Z) > 0$ now. By Lemma 2.2.7, it holds that

$$Z - \pi(Z)U \in \ker \pi.$$

Hence,

$$Z = (Z - \pi(Z)U) + \pi(Z)U \in \ker \pi + \mathcal{X}_+ = \mathcal{C}_{\ker}$$

because of $\pi(Z)U \in \mathcal{X}_+$ by $\pi(Z) > 0$ and $U \in \mathcal{X}_+$ by Assumption 2. Thus, $Z \in \mathcal{C}_{\ker}$. Since $Z \in \mathcal{U} \subseteq \mathcal{C}_\pi$ was chosen arbitrarily, we obtain $\mathcal{U} \subseteq \mathcal{C}_{\ker}$. As a result, (4.32) holds and the proof is complete. □

**Remark 4.3.9.** *We gave a direct proof of Lemma 4.3.8, although it was not necessary. Indeed, with respect to Remark 4.3.5, the assumption* $\operatorname{int} \mathcal{C}_\pi \neq \emptyset$ *implies* $\mathcal{X} = \mathcal{M}$ *for* (FM) *and, thus,* $\mathcal{C}_\pi = \mathcal{C}_{\ker}$ *by Lemma 4.1.5. Hence, we obtain* $\operatorname{Eff}_w(\mathcal{A}, \mathcal{C}_{\ker}) = \operatorname{Eff}_w(\mathcal{A}, \mathcal{C}_\pi)$ *trivially. Nevertheless, proving Lemma 4.3.8 directly allows the reader to transfer the result to a setting with* $\dim \mathcal{M} = +\infty$, *because* $\dim \mathcal{M} < +\infty$ *was not necessary for the proof.*

Now, we present the main result concerning our studies of weakly efficient points:

**Theorem 4.3.10.** *Consider* (FM). *Let* $(\mathcal{X}, \tau)$ *be a topological vector space and Assumption 2 be fulfilled by some eligible payoff* $U \in \mathcal{M} \cap \mathcal{X}_+$. *Furthermore, let* $\mathcal{C}_\pi \subseteq \mathcal{M}$ *be the price cone given by Definition 4.1.1 and* $\mathcal{C}_{\ker} \subseteq \mathcal{X}$ *be the kernel cone given by Definition 4.1.3. Let* $\mathcal{A}' \subseteq \mathcal{X}$ *be the set of optimal acceptable capital positions in* (3.25). *Then, the following conditions hold:*

(i) *Suppose* $\operatorname{int} \mathcal{C}_{\ker} \neq \emptyset$. *Then,* $\mathcal{A}' \subseteq \operatorname{Eff}_w(\mathcal{A}, \mathcal{C}_{\ker})$.

(ii) *Suppose* $\operatorname{int} \mathcal{C}_\pi \neq \emptyset$. *Then,*

$$\mathcal{A}' = \operatorname{Eff}_w(\mathcal{A}, \mathcal{C}_{\ker}) = \operatorname{Eff}_w(\mathcal{A}, \mathcal{C}_\pi). \tag{4.33}$$

**Proof.** First, we prove (i) and assume $\operatorname{int} \mathcal{C}_{\ker} \neq \emptyset$. By Theorem 4.3.6 and (4.27), we obtain

$$\mathcal{A}' \subseteq \operatorname{Eff}(\mathcal{A}, \mathcal{C}_{\ker} \setminus \ker \pi) \subseteq \operatorname{Eff}_w(\mathcal{A}, \mathcal{C}_{\ker}),$$

which shows (i). Now, we prove (ii) and suppose $\operatorname{int} \mathcal{C}_\pi \neq \emptyset$. Because of $\mathcal{C}_\pi \subseteq \mathcal{C}_{\ker}$ by Lemma 4.1.5, we obtain $\operatorname{int} \mathcal{C}_{\ker} \neq \emptyset$ by $\operatorname{int} \mathcal{C}_\pi \neq \emptyset$. Consequently,

$$\mathcal{A}' \subseteq \operatorname{Eff}_w(\mathcal{A}, \mathcal{C}_{\ker}) \subseteq \operatorname{Eff}_w(\mathcal{A}, \mathcal{C}_\pi) \tag{4.34}$$

by (i) and Lemma 4.3.8.

Finally, for the proof of (4.33), it is sufficient to show

$$\operatorname{Eff}_w(\mathcal{A}, \mathcal{C}_\pi) \subseteq \mathcal{A}'$$

by (4.34) and Lemma 4.3.8. By Remark 4.3.5, we obtain $\mathcal{X}_+ \subseteq \mathcal{M}$ because of $\mathcal{X} = \mathcal{M}$ by $\operatorname{int} \mathcal{C}_\pi \neq \emptyset$. Thus,

$$\operatorname{Eff}_w(\mathcal{A}, \mathcal{C}_\pi) = \operatorname{Eff}(\mathcal{A}, \mathcal{C}_\pi \setminus \ker \pi)$$

by (4.30). Suppose there is some $X \in \mathrm{Eff}_w(\mathcal{A}, \mathcal{C}_\pi)$ with $X \notin \mathcal{A}'$. Since $X \in \mathcal{A}$ by definition of $\mathrm{Eff}_w(\mathcal{A}, \mathcal{C}_\pi)$ in (4.26), we obtain

$$X \in \mathrm{int}_{-U}(\mathcal{A} + \ker \pi)$$

by Lemma 3.2.4. Consequently,

$$(\{X\} - \mathbb{R}_> U) \cap (\mathcal{A} + \ker \pi) \neq \emptyset, \tag{4.35}$$

since $U \in \mathcal{M} \cap \mathcal{X}_+$ by Assumption 2 (which is fulfilled by Assumption 4). Because of

$$\{X\} - \mathbb{R}_> U \subseteq \{X\} - (\mathcal{C}_\pi \setminus \ker \pi)$$

and (4.35), there is some $X^0 \in \mathcal{A} + \ker \pi$ with

$$X^0 \in \{X\} - (\mathcal{C}_\pi \setminus \ker \pi). \tag{4.36}$$

Since $X^0 \in \mathcal{A} + \ker \pi$,

$$\mathcal{A} \cap (\{X^0\} + \ker \pi) \neq \emptyset$$

with $X \notin \mathcal{A} \cap (\{X^0\} + \ker \pi)$ by (4.36). Consequently,

$$\mathcal{A} \cap (\{X\} - (\mathcal{C}_\pi \setminus \ker \pi)) \nsubseteq \{X\},$$

which contradicts $X \in \mathrm{Eff}(\mathcal{A}, \mathcal{C}_\pi \setminus \ker \pi) = \mathrm{Eff}_w(\mathcal{A}, \mathcal{C}_\pi)$. As a result, $\mathrm{Eff}_w(\mathcal{A}, \mathcal{C}_\pi) \subseteq \mathcal{A}'$ holds and the proof of (4.33) is complete. □

**Remark 4.3.11.** *Note that the proof of Theorem 4.3.10 did not use* $\dim \mathcal{M} < +\infty$*. Although* $\mathcal{X} = \mathcal{M}$ *and* $\mathcal{C}_\pi = \mathcal{C}_{\ker}$ *hold as consequences of* $\mathrm{int}\, \mathcal{C}_\pi \neq \emptyset$ *in Theorem 4.3.10*(ii) *(see Remark 4.3.5), we only used* $\mathcal{C}_\pi \subseteq \mathcal{C}_{\ker}$ *to show*

$$\mathcal{A}' \subseteq \mathrm{Eff}_w(\mathcal{A}, \mathcal{C}_{\ker}) \subseteq \mathrm{Eff}_w(\mathcal{A}, \mathcal{C}_\pi)$$

*by use of* (i)*. Moreover, we only needed* $\mathcal{X}_+ \subseteq \mathcal{M}$ *in the proof of Theorem 4.3.10*(ii)

$$\mathrm{Eff}_w(\mathcal{A}, \mathcal{C}_{\ker}) = \mathrm{Eff}(\mathcal{A}, \mathcal{C}_\pi \setminus \ker \pi)$$

*to conclude can be applied for settings with* $\dim \mathcal{M} = +\infty$*, as well. In the proof of* (4.33) *in Theorem 4.3.10*(ii)*, we used* $\mathcal{X}_+ \subseteq \mathcal{M}$ *as a consequence of* $\mathrm{int}\, \mathcal{C}_\pi \neq \emptyset$ *(which implies* $\mathcal{X} = \mathcal{M}$*, see Remark 4.3.5, i.e.,* (FM) *is a complete market, see Definition 3.2.10). In our setting,* $\mathcal{X} = \mathcal{M}$ *implies that* $\mathcal{X}$ *is a finite dimensional topological space by* $\dim \mathcal{M} < +\infty$*, but*

$$\mathcal{A}' = \mathrm{Eff}_w(\mathcal{A}, \mathcal{C}_{\ker}) = \mathrm{Eff}_w(\mathcal{A}, \mathcal{C}_\pi)$$

*also holds for settings with* $\dim \mathcal{M} = +\infty = \dim \mathcal{X}$ *under assumption of* $\mathrm{int}\, \mathcal{C}_\pi \neq \emptyset$*, because we did not use* $\dim \mathcal{M} < +\infty$ *in the proof.*

For conclusion, we give some examples for our assertions in Theorem 4.3.10:

**Example 4.3.12.** Let $\mathcal{X} = \mathbb{R}^2 = \mathcal{M}$, $U = (0,1)^T$ and $\pi(Z) = Z_2$. Consider

$$\mathcal{A} = ((0,-1)^T + \mathbb{R}^2_+)\backslash((0,1](0,-1)^T).$$

Then, $\mathcal{A}$ is a $(-U)$-directionally closed acceptance set, which is not closed with respect to the Euclidean topology on $\mathbb{R}^2$. Furthermore, $\ker\pi = \mathbb{R}\times\{0\}$ and

$$\mathcal{A}' = (0,-1)^T + \mathbb{R}_>(1,0)^T$$

hold. Since $\mathcal{X}_+ \subseteq \mathcal{M}$ is obviously fulfilled, it holds that $\mathcal{C}_\pi = \mathcal{C}_{\ker}$ by Lemma 4.1.5 and

$$\operatorname{int}\mathcal{C}_\pi = \mathcal{C}_\pi\backslash\ker\pi = \operatorname{int}\mathcal{C}_{\ker}$$

by Corollary 4.3.4. Because of $\mathcal{C}_\pi\backslash\ker\pi = \mathbb{R}\times\mathbb{R}_>$, we obtain

$$\operatorname{Eff}_w(\mathcal{A},\mathcal{C}_\pi) = (0,-1)^T + \mathbb{R}_>(1,0)^T = \operatorname{Eff}_w(\mathcal{A},\mathcal{C}_{\ker}),$$

and, thus,

$$\operatorname{Eff}_w(\mathcal{A},\mathcal{C}_\pi) = \operatorname{Eff}_w(\mathcal{A},\mathcal{C}_{\ker}) = \mathcal{A}'.$$

Note that

$$\operatorname{cl}\mathcal{A}\cap\big(\{(0,-1)^T\} - (\mathcal{C}_\pi\backslash\ker\pi)\big) = \emptyset$$

with $(0,-1)^T \in \operatorname{bd}\mathcal{A}\backslash\operatorname{bd}_{-U}(\mathcal{A})$, i.e., $(0,-1)^T \in \operatorname{Eff}_w(\operatorname{cl}\mathcal{A},\mathcal{C}_\pi)$, but $(0,-1)^T \notin \operatorname{Eff}_w(\mathcal{A},\mathcal{C}_\pi)$ because $(0,-1)^T \notin \mathcal{A}$. Thus, there may be points $X \in \operatorname{bd}\mathcal{A}\backslash\operatorname{bd}_{-U}(\mathcal{A})$ fulfilling the weak efficiency condition $\mathcal{A}\cap(\{X\} - \operatorname{int}\mathcal{C}_\pi) \subseteq \{X\}$, but not satisfying $X \in \mathcal{A}$ (which is required for being a weakly efficient point of $\mathcal{A}$) because $\mathcal{A}$ is not closed. ◊

Next, we refer to Example 4.2.1, where $\operatorname{int}\mathcal{C}_\pi = \emptyset$ holds.

**Example 4.3.13.** Consider the setting (FM) in Example 4.2.1, i.e., the vector space $\mathcal{X} = \mathbb{R}^3$ (here, being equipped with the Euclidean topology $\tau$), $\mathcal{M} = \{Z \in \mathbb{R}^3 \mid Z_2 = 0\}$, $\mathcal{A} = \mathbb{R}_+\times\mathbb{R}\times\mathbb{R}_+$, $U = (1,0,0)^T$ and $\pi(Z) := Z_1 + Z_3$, i.e.,

$$\ker\pi = \{Z \in \mathbb{R}^3 \mid Z_3 = -Z_1, Z_2 = 0\}.$$

The cones $\mathcal{C}_\pi$ and $\mathcal{C}_{\ker}$ in Definition 4.1.1 and Definition 4.1.3 are given by

$$\mathcal{C}_\pi = \{Z \in \mathbb{R}^3 \mid Z_3 \geq -Z_1, Z_2 = 0\}$$

and

$$\mathcal{C}_{\ker} = \{Z \in \mathbb{R}^3 \mid Z_3 \geq -Z_1, Z_2 \geq 0\},$$

respectively. We obtained

$$\operatorname{Eff}(\mathcal{A},\mathcal{C}_{\ker}) = \emptyset \quad \text{and} \quad \operatorname{Eff}(\mathcal{A},\mathcal{C}_\pi) = \{0\}\times\mathbb{R}\times\{0\} = \mathcal{A}'$$

in Example 4.2.1 with $\mathcal{A}' \subseteq \mathcal{X}$ given by (3.25). Moreover,

$$\operatorname{int} C_\pi = \emptyset \quad \text{and} \quad \operatorname{int} \mathcal{C}_{\ker} = \{Z \in \mathbb{R}^3 \mid Z_3 > -Z_1, Z_2 > 0\}$$

hold. Thus, $\operatorname{Eff}_w(\mathcal{A}, \mathcal{C}_\pi)$ is not defined and the observations in Remark 4.3.5 can not be applied (especially, $\mathcal{M} \subsetneq \mathcal{X}$ and $\mathcal{C}_\pi \neq \mathcal{C}_{\ker}$ hold). Nevertheless, since $\operatorname{int} \mathcal{C}_{\ker} \neq \emptyset$, we obtain

$$\operatorname{Eff}_w(\mathcal{A}, \mathcal{C}_{\ker}) = \mathcal{A}'.$$

◊

The following finale example shows that $\mathcal{A}' \subsetneq \operatorname{Eff}_w(\mathcal{A}, \mathcal{C}_{\ker})$ holds in general, even if $\mathcal{X}$ is a finite dimensional topological vector space. Moreover, Example 4.3.14 shows that the strict relations $\operatorname{Eff}(\mathcal{A}, \mathcal{C}_{\ker}) \subsetneq \mathcal{A}'$ and $\mathcal{A}' \subsetneq \operatorname{Eff}_w(\mathcal{A}, \mathcal{C}_{\ker})$ hold in general, even if $\operatorname{Eff}(\mathcal{A}, \mathcal{C}_{\ker}) \neq \emptyset$.

**Example 4.3.14.** Consider (FM) with $\mathcal{X} = \mathbb{R}^3$ being equipped with the Euclidean topology $\tau$, $\mathcal{M} = \mathbb{R} \times \{0\} \times \mathbb{R}$, $\pi(Z) = \frac{Z_1 + Z_3}{2}$ and, thus,

$$\ker \pi = \{Z \in \mathbb{R}^3 \mid Z_3 = -Z_1, Z_2 = 0\}.$$

Moreover, let $\mathcal{A} = \mathcal{X}_+$ and $U = (1, 0, 1)^T$. $\mathcal{A}$ fulfills Assumption 4 with respect to $\mathcal{U}$. It holds that

$$\mathcal{A}' = \{0\} \times \mathbb{R}_+ \times \{0\}$$

with $\mathcal{A}' \subseteq \mathcal{X}$ given by (3.25). Obviously, $\operatorname{int} \mathcal{C}_\pi = \emptyset$, since $\mathcal{C}_\pi \subseteq \mathcal{M}$ and $\mathcal{M}$ is a proper subspace of $\mathcal{X}$ (see Remark 4.3.5). Moreover,

$$\mathcal{C}_{\ker} = \{X \in \mathbb{R}^3 \mid X_3 \geq X_1, X_2 \geq 0\}$$

and, thus,

$$\operatorname{int} \mathcal{C}_{\ker} = \{X \in \mathbb{R}^3 \mid X_3 > X_1, X_2 > 0\}.$$

Hence, we obtain $\operatorname{Eff}(\mathcal{A}, \mathcal{C}_{\ker}) = \{\mathbb{0}\}$ and, thus,

$$\emptyset \neq \operatorname{Eff}(\mathcal{A}, \mathcal{C}_{\ker}) \subsetneq \mathcal{A}' \subsetneq \operatorname{Eff}_w(\mathcal{A}, \mathcal{C}_{\ker}) = (\mathbb{R}_+ \times \mathbb{R}_+ \times \{0\}) \cup (\mathbb{R}_+ \times \{0\} \times \mathbb{R}_+).$$

◊

# Chapter 5

# Conclusions and Outlook

In this thesis, we studied multiobjective optimization problems related to risk measures and acceptance sets. The problems are economical motivated by the current developments in risk and portfolio management of financial institutes, especially, with respect to regulatory restrictions. These restrictions can be modeled by an acceptance set $\mathcal{A}$ as a subset of a real vector space $\mathcal{X}$ of capital positions. The allowed actions of the decision maker for changing a capital position can be modeled as a subspace $\mathcal{M}$ of $\mathcal{X}$ representing payoffs of eligible assets. By use of these objects, we focused on the optimization problem ($P_\pi(X)$) of reaching acceptability with minimal costs by means of the pricing functional $\pi\colon \mathcal{M} \to \mathbb{R}$, i.e.,

$$\pi(Z) \to \min_{X+Z\in\mathcal{A},\, Z\in\mathcal{M}}.$$

The resulting optimal value is the risk of the capital position, given by the risk measure $\rho_{\mathcal{A},\mathcal{M},\pi}\colon \mathcal{X} \to \overline{\mathbb{R}}$. The optimal eligible payoffs are summarized by the optimal payoff map $\mathcal{E}\colon \mathcal{X} \rightrightarrows \mathcal{M}$ as

$$\mathcal{E}(X) := \{Z \in \mathcal{M} \mid X + Z \in \mathcal{A}, \pi(Z) = \rho_{\mathcal{A},\mathcal{M},\pi}(X)\}.$$

With respect to our financial market model (FM), our outcomes can be summarized as follows:

- We introduced our basic financial market model (FM) in Chapter 2, especially, the risk measurement regime $(\mathcal{A}, \mathcal{M}, \pi)$, acceptance set $\mathcal{A}$, and the risk measure $\rho_{\mathcal{A},\mathcal{M},\pi}$. Although our model and studied objects are motivated by the works [17] and [71], we started with a new approach: We worked in a general real vector space $\mathcal{X}$ without any topology in general and assumed a secure investment opportunity in the market (as it is usual for many economic practical models). Furthermore, we considered general acceptance sets and extended real-valued risk measures. By our minimal assumptions, our results can be comfortably applied to other settings and in practice.

- In Section 2.3, we studied the risk measure $\rho_{\mathcal{A},\mathcal{M},\pi}$ in detail. Our main results are given in Theorem 2.3.25, where we characterize the level line, sublevel sets, and strict sublevel sets of $\rho_{\mathcal{A},\mathcal{M},\pi}$ by means of the augmented set $\mathcal{A} + \ker \pi$. We used these important results to characterize $\operatorname{dom}(\rho_{\mathcal{A},\mathcal{M},\pi})$ in Theorem 2.3.35, and outline conditions for the finiteness of $\rho_{\mathcal{A},\mathcal{M},\pi}$ in Lemma 2.3.37 and Lemma 2.3.38. Moreover, we characterize the properness of $\rho_{\mathcal{A},\mathcal{M},\pi}$ in Theorem 2.3.42, and properties like convexity, subadditivity and positively homogeneity of the risk measure by means of properties of $\mathcal{A}$ in Lemma 2.3.44. These properties are important for interpreting risk measures in mathematical finance, especially, for characterizing $\rho_{\mathcal{A},\mathcal{M},\pi}$ as a coherent (or convex) risk measure that supports the idea of diversification. Moreover, these properties are crucial for our results concerning the characterization of $\mathcal{E}(X)$, and for studying optimization problems including $\rho_{\mathcal{A},\mathcal{M},\pi}$, e.g., with respect to deriving optimality conditions or algorithms for finding solutions.

- In Chapter 3, we used the properties of $\rho_{\mathcal{A},\mathcal{M},\pi}$ derived in Chapter 2 to prove a new description of the solution set $\mathcal{E}(X)$ of $(P_\pi(X))$ with $X \in \mathcal{X}$ . The main contributions of Chapter 3 are given in Theorem 3.1.6 for directionally closed acceptance sets $\mathcal{A}$, which generalize a corresponding result for closed acceptance sets in [17]. We also highlighted relationships between $\mathcal{A}$ and $\mathcal{E}(X)$ if the acceptance set is not directionally closed. Moreover, we proved new characterizations for existence and uniqueness of optimal eligible payoffs in Theorem 3.1.10 and Theorem 3.2.14, respectively. The results of Chapter 3 are important for deriving useful properties of (weakly) efficient elements of the acceptance set $\mathcal{A}$ in Chapter 4. Furthermore, the results are crucial for applications and developing algorithms for solving $(P_\pi(X))$ in real and more complex situations.

- In Chapter 4, we provide a better geometric understanding of the relationship between solutions of $(P_\pi(X))$ and the $(-U)$-directionally closed acceptance set $\mathcal{A}$. For that, we characterized efficient and weakly efficient points of $\mathcal{A}$ with respect to the price cone $\mathcal{C}_\pi$ and the kernel cone $\mathcal{C}_{\text{ker}}$ as solutions of the vector optimization problem $(V_\mathcal{A})$. Our main contributions of Chapter 4 are presented in Theorem 4.2.6 and Theorem 4.3.10, which show the important relationship between the (weakly) efficient sets and $\mathcal{A}' \subseteq \operatorname{bd}_{-U}(\mathcal{A})$ with $\mathcal{A}'$ given by (3.25). For our results concerning efficient points of $\mathcal{A}$, we do not even need a topology and provide a most intense insight in the crucial requirements on the financial market. Moreover, we highlight the role of the complete markets for weakly efficient points in topological vector spaces. Our results are useful for deriving solution concepts that unite the optimization problems $(P_\pi(X))$ and $(V_\mathcal{A})$. Especially, it motivates new approaches for deriving existence and uniqueness results of $(P_\pi(X))$.

From the observations in this thesis, there arise many interesting ideas for further research topics. Especially, it would be interesting:

(i) to consider extensions of our financial market model (FM):

  - We do not directly integrate transaction costs or taxes with respect to buying or selling assets. Nevertheless, it is possible that the transaction costs are limited by a maximum value or that there is a minimal amount of transaction costs that always has to be paid without taking into account the invested capital or the bought volume of assets. Thus, the transaction costs would be nonlinear in that case. This more general case should be studied.

- We do not suppose capital restrictions for investing or leveraging in our model, which is not realistic in general. For further research, it is of interest to consider models where more realistic restrictions are involved.
- Although our definitions of acceptance sets, eligible assets, and the space of eligible payoffs $\mathcal{M}$ are economical reasonable, it is possible that institutions are not allowed to invest unrestricted into some of these assets, for example, to restrict default risk. As a consequence, the set $\mathcal{M}$ of possible movements is no subspace of $\mathcal{X}$ anymore. Moreover, one could consider variable order structures with respect to the acceptance set and the set of eligible payoffs, i.e., $\mathcal{A}\colon \mathcal{X} \rightrightarrows \mathcal{X}$ and $\mathcal{M}\colon \mathcal{X} \rightrightarrows \mathcal{X}$. Hence, it would be interesting to study properties of the functional or efficient points of the acceptance set with variable dominance structures (see, e.g., [37]).
- Instead of an one-period model, the consequences of considering a multi-period model (see, e.g., [15] and [82]) on our results may be from interest.

(ii) to study multiobjective portfolio optimization problems where the risk measure $\rho_{\mathcal{A},\mathcal{M},\pi}$ is involved. Also, it would be interesting to see how the choice of the eligible assets and institutional restrictions affect the resulting optimal portfolios.

(iii) to execute a stability analysis of the optimal payoff map $\mathcal{E}\colon \mathcal{X} \rightrightarrows \mathcal{M}$ for (FM).

In [17], the stability with respect to the semicontinuity of $\mathcal{E}$ is studied for more specific framework, especially, with a locally convex Hausdorff space $\mathcal{X}$ and a closed acceptance set $\mathcal{A}$. It would be interesting, which of these results can be generalized.

(iv) to characterize weakly efficient points for incomplete markets.

In Theorem 4.3.10, we could only determine $\operatorname{Eff}_w(\mathcal{A}, \mathcal{C}_\pi) = \emptyset$ and $\mathcal{A}' \subseteq \operatorname{Eff}_w(\mathcal{A}, \mathcal{C}_{\text{ker}})$ for incomplete markets. The reason is that $\operatorname{int} \mathcal{C}_\pi = \emptyset$ holds for incomplete markets, and that we could not fully describe $\operatorname{int} \mathcal{C}_{\text{ker}}$. Hence, $\operatorname{int} \mathcal{C}_{\text{ker}}$ has to be studied in more detail. Moreover, points with other types of efficiency (like proper efficient points) of $\mathcal{A}$ would be interesting to characterize, especially, with respect to $\mathcal{C}_\pi$ for incomplete markets.

(v) to extend the studies of the nonlinear risk measure $\rho_{\mathcal{A},\mathcal{M},\pi}$.

Since we focused on vector spaces without a topology, it would be useful to study properties of $\rho_{\mathcal{A},\mathcal{M},\pi}$ with respect to some topology (e.g., continuity) for (FM). Moreover, it would be interesting to apply well-established and useful results from variational analysis on $\rho_{\mathcal{A},\mathcal{M},\pi}$.

# Summary of Contributions

Several results presented in this thesis are published in the following three publications in international peer-reviewed journals:

[141] Marohn, M., Tammer, C. (2022): *Optimal payoffs for directional closed acceptance sets.* Journal of Nonlinear and Variational Analysis, Vol. 6 (6), 641 - 659.

[140] Marohn, M., Tammer, C. (2021): *A new view on risk measures associated with acceptance sets.* Applied Set-Valued Analysis and Optimization, Vol. 3 (3), 355 - 380.

[139] Marohn, M., Tammer, C. (2020): *Characterization of efficient points of acceptance sets.* Journal Applied Analysis and Optimization, Vol. 4 (1), 79 - 114.

In the following, we outline which parts of the outcomes, listed in the conclusion before, are published in these articles, and summarize the author's new contributions to the chapters of this thesis:

- Motivated by the short preliminaries used in our published articles above, we gave an extended overview about basic terminology of set theory, topology, functional analysis, probability theory, and vector optimization for stand-alone readability of this thesis in Chapter 1, using standard literature, e.g., [52], [78], [122], [153], [173], [198], but also newer publications like [97] and [185]. The resulting preliminaries in financial mathematics for one-period models in Section 1.3 are particularly new (see Definition 1.3.11), since there is no similar systematic standard with these minimal and abstract setting for an one-period model in the literature. Also, we provide a detailed overview about some practical interesting risk measures like (Conditional) Value-at-Risk and the differently used definitions in the literature.

- The financial market model (FM) is used in each of our articles above and first published under this shortcut in [140]. The main results of Chapter 2 are presented in Section 2.3 and published in [140], although we assume in this thesis that not all of the eligible payoffs $S_1^i$ have to be linear independent, such that the Law of One Price is not automatically fulfilled. The literature review in Section 2.1 and Lemma 2.2.13 (as well as the remarks afterwards) are new. In Lemma 2.2.19, we formulated and proved a more detailed version of the corresponding published lemma in [140, Lemma 4.1]. Also, Example 2.3.3, Corollary 2.3.10, Corollary 2.3.17, our studies of $\mathcal{A} + \ker \pi$ in Example 2.3.19 and Lemma 2.3.20, as well as Example 2.3.34 are new. In Lemma 2.3.44, we proved a more precise characterization of properties of $\rho_{\mathcal{A},\mathcal{M},\pi}$ than in [139, Lemma 3.20].

- Chapter 3 is based on our article [141]. We newly included Example 3.1.3 and Example 3.1.5 for illustrating the situation for directionally closed acceptance sets and economical motivations for considering them. Theorem 3.1.10 and Theorem 3.2.14 about existence and uniqueness of optimal eligible payoffs are new, and generalize results in [17]. Also, Lemma 3.2.4 is new and we inserted the missing proofs of Lemma 3.2.5, Corollary 3.2.6, and Corollary 3.2.12.

- We studied efficient and weakly efficient points in [139] and [141]. In Chapter 4 of this thesis, we derived the results in a more general context. Especially, we consider a more common definition of efficiency and not necessary closed cones. The main results were published without proofs in [141]. Hence, the whole chapter contains new results, including the proofs, examples, and most of the remarks.

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# List of Figures

# Selbstständigkeitserklärung

Ich erkläre an Eides statt, dass ich die vorliegende Arbeit

**Scalarization Functionals in Mathematical Finance and Vector Optimization**
**- A New View on Acceptance Sets and Risk Measures -**

selbstständig und ohne fremde Hilfe verfasst, keine anderen als die von mir angegebenen Quellen und Hilfsmittel benutzt und die den benutzten Werken wörtlich oder inhaltlich entnommenen Stellen als solche kenntlich gemacht habe.

Leipzig, den 20. Mai 2022

Marcel Marohn

# Curriculum Vitae

Marcel Marohn, born on July 18, 1990 in Leipzig

| | |
|---|---|
| **Oct. 2017 - May 2022** | Doctoral Studies in Applied Mathematics<br>Martin-Luther-Universität Halle-Wittenberg |
| **Aug. 2021 - Today** | Research Fellow at<br>Brandenburg University of Technology Cottbus-Senftenberg,<br>Faculty 5 - Business, Law and Social Sciences<br>Chair of Finance |
| **Aug. 2018 - Sep. 2021** | Research Fellow at University of Leipzig,<br>Faculty of Economics and Management Science,<br>Chair of Investment and Finance |
| **Jan. 2016 - Sept. 2017** | Consultant in Finance at ICnova AG Karlsruhe |
| **Aug. 2015** | Diploma in Economic Mathematics |
| **Oct. 2010 - Aug. 2015** | Diploma Studies in Economic Mathematics<br>University of Leipzig |
| **Jul. 2009 - Mar. 2010** | Civilian Service at Ev. Diakonissenkrankenhaus Leipzig |
| **Jun. 2009** | Abitur |
| **Aug. 2007 - Jun. 2009** | Max-Klinger-Gymnasium, Leipzig |

Leipzig, May 20, 2022

Marcel Marohn

# Publications

- M. Marohn and C. Tammer (2020): *Efficient points of acceptance sets.* Journal Applied Analysis and Optimization 4(1), 79–114.
- M. Marohn and C. Tammer (2021): *A new view on risk measures associated with acceptance sets.* Applied Set-Valued Analysis and Optimization 3(3), 355 – 380.
- M. Marohn and C. Tammer (2022): *Optimal payoffs for directional closed acceptance sets.* Journal of Nonlinear and Variational Analysis 6(6), 641 – 659 .