Well-posedness of a Newtonian two-phase flow with Boussinesq-Scriven surface fluid

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Dedicated to Katarina
and my parents
Simone & Michael

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Introduction

The motion of two immiscible fluids like oil and water can be modeled as a moving boundary problem for the two-phase Navier-Stokes equations. In a sharp-interface model, the interface between both fluid phases is considered as a geometric hypersurface. Physicists expect that interfacial properties such as surface tension play a prominent role when the interfacial area is large compared to the fluid volume. In this regard, Boussinesq [Bou13] proposed to consider certain surface viscosities that are related to intrinsic frictional forces within the interface. Several decades later, Scriven [Scr60] generalized Boussinesq's approach and obtained a model for arbitrary coordinate systems. This model is nowadays called the two-phase Navier-Stokes equations with Boussinesq-Scriven surface fluid and is denoted by (N) in this thesis. From a mathematician's point of view, it is fundamental to investigate whether this problem is well-posed; that is, whether it admits a uniquely determined solution that depends continuously on the initial state. Such a theory also has practical advantages. In particular, it can clarify admissible ranges of relevant parameters and indicate general limitations of the model that might be difficult to explore with experiments or numerical simulations alone. In this spirit, Bothe and Prüss [BP10] formally analyzed a related linear model problem and proved that its well-posedness depends on a condition for the interfacial velocity. The purpose of the present thesis is to extend their work and to investigate whether the original nonlinear problem is well-posed.

Let us formulate the model (N). We assume that the adjacent fluid phases occupy time-dependent disjoint open subsets $\Omega_+(t)$ and $\Omega_-(t)$ in $\mathbb{R}^n$ ($n \geq 2$), which are separated by the sharp interface $\Gamma(t) = \partial\Omega_+(t) \cap \partial\Omega_-(t)$. Both bulk phases $\Omega_\pm(t)$ and the interface $\Gamma(t)$ fill a rigid container $\Omega = \Omega_+(t) \cup \Gamma(t) \cup \Omega_-(t)$, which is a stationary domain. We employ the mass densities $\rho_\pm$, the velocity fields $u_\pm$, and the stress tensor $T_\pm = S_\pm - \pi_\pm I$ with viscous stress tensor $S_\pm$ and pressure $\pi_\pm$. With the characteristic function $\chi_\pm$ of $\Omega_\pm$, we put $\rho = \rho_+\chi_+ + \rho_-\chi_-$ and analogously for the other quantities. The principles of conservation of mass and momentum in $\Omega_\pm$ yield the continuity equation and the Navier-Stokes equation

$$\partial_t \rho + \operatorname{div}(\rho u) = 0, \quad \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u - T) = \rho f.$$

We restrict our considerations to incompressible Newtonian flows for which $\rho_\pm$ are positive constants and the viscous stress tensor $S_\pm = 2\mu_\pm D_\pm$ depends linearly on the rate-of-strain tensor $D_\pm = (\nabla u_\pm + [\nabla u_\pm]^\top)/2$ with constant positive shear viscosities $\mu_\pm$. By putting also $f_\pm = 0$, we neglect external forces like gravity.

Additional conditions must be imposed on the fluid-solid boundary $\partial\Omega$ and the interface Γ. For simplicity, the latter should not touch the boundary $\partial\Omega$ and hence one of the bulk phases, say Ω_-, should have its boundary $\partial\Omega_- = \Gamma$ in Ω. Furthermore, we let the flow satisfy the no-slip conditions $u_+ = 0$ on $\partial\Omega$ and $[\![u]\!] = 0$ on Γ, where $[\![u]\!] := u_+|_\Gamma - u_-|_\Gamma$ denotes the jump of u across Γ. We exclude phase transitions and assume that the interface is material in the sense that the normal velocity V_Γ of Γ is given by $V_\Gamma = u_\pm|_\Gamma \cdot \nu_\Gamma$, where ν_Γ denotes the unit normal directed into Ω_+. Thus, Γ is advected with the flow of the bulk phases. Conservation of momentum also yields the interfacial momentum balance

$$-[\![T]\!]\nu_\Gamma = \operatorname{div}_\Gamma T_\Gamma,$$

where T_Γ is the surface stress tensor and $\operatorname{div}_\Gamma T_\Gamma$ denotes its surface divergence. When surface viscosities are negligible, we can put $T_\Gamma = \sigma P_\Gamma$ with the surface tension coefficient σ and the tangential projection $P_\Gamma = I - \nu_\Gamma \otimes \nu_\Gamma$. With the $(n-1)$-fold mean curvature $H_\Gamma = -\operatorname{div}_\Gamma \nu_\Gamma$, this yields the well-known Laplace-Young law $-[\![T]\!]\nu_\Gamma = \sigma H_\Gamma \nu_\Gamma$ if σ is constant, and otherwise $-[\![T]\!]\nu_\Gamma = \sigma H_\Gamma \nu_\Gamma + \nabla_\Gamma \sigma$ with Marangoni force $\nabla_\Gamma \sigma$, where ∇_Γ denotes the surface gradient. In order to incorporate surface viscosities, we assume that T_Γ is given by the Boussinesq-Scriven constitutive law [cf. SSO07]

$$T_\Gamma = \sigma P_\Gamma + (\lambda_s - \mu_s) \operatorname{div}_\Gamma u \, P_\Gamma + 2\mu_s D_\Gamma,$$

where λ_s and μ_s are the surface viscosities and D_Γ is the interfacial rate-of-strain tensor

$$D_\Gamma = 2^{-1} P_\Gamma (\nabla_\Gamma u + [\nabla_\Gamma u]^\top) P_\Gamma.$$

We can decompose T_Γ into

$$T_\Gamma = \{\sigma + (\lambda_s + (3-n)\mu_s/(n-1)) \operatorname{div}_\Gamma u\} P_\Gamma + 2\mu_s [D_\Gamma - (\operatorname{tr} D_\Gamma/(n-1)) P_\Gamma],$$

where the first summand is an isotropic tensor field and the second one has vanishing trace. Thus, we call μ_s the surface shear viscosity and

$$\kappa_s = \lambda_s + (3-n)\mu_s/(n-1)$$

the surface dilational viscosity. The latter equals λ_s in the case $n = 3$.

Bothe and Prüss [BP10] already noticed that the tangential part of the interfacial force $\operatorname{div}_\Gamma T_\Gamma$ is of second order in v but only of first order in w. Accordingly, when we reformulate problem (N), we should handle the tangential and normal components separately; a complication that is not present in the situation without surface viscosities that was investigated by Köhne, Prüss, and Wilke [KPW13], where simply $\operatorname{div}_\Gamma T_\Gamma = \sigma H_\Gamma \nu_\Gamma$ with $(n-1)$-fold mean curvature H_Γ. In our situation, we decompose the velocity field u near Γ into $u = v + w\nu_\Gamma$ with tangential component $v := P_\Gamma u$ and normal component $w := \nu_\Gamma \cdot u$ and decompose the vector field $\operatorname{div}_\Gamma T_\Gamma$ accordingly. Then it can be shown that $\operatorname{div}_\Gamma T_\Gamma$ has the following structure.

$$\begin{aligned}\operatorname{div}_\Gamma T_\Gamma = {} & \mu_s \widetilde{\Delta}_\Gamma v + \lambda_s \nabla_\Gamma \operatorname{div}_\Gamma v + \mu_s H_\Gamma [\nabla_\Gamma v] \nu_\Gamma - \mu_s L_\Gamma^2 v \\ & - 2\mu_s L_\Gamma \nabla_\Gamma w + (\mu_s - \lambda_s) \nabla_\Gamma w \, H_\Gamma - (\mu_s + \lambda_s) w \nabla_\Gamma H_\Gamma \\ & + [(\lambda_s - \mu_s) H_\Gamma \operatorname{div}_\Gamma v + 2\mu_s \operatorname{tr}(L_\Gamma D_\Gamma(v))] \nu_\Gamma \\ & + [\sigma H_\Gamma - (\lambda_s - \mu_s) H_\Gamma^2 w - 2\mu_s \operatorname{tr}(L_\Gamma^2) w] \nu_\Gamma.\end{aligned}$$

Here we employ a Laplace-Beltrami operator $\widetilde{\Delta}_\Gamma$ that acts on tangential vector fields, the scalar Laplace-Beltrami operator $\Delta_\Gamma = \operatorname{div}_\Gamma \nabla_\Gamma$, and the Weingarten tensor L_Γ.

We summarize these considerations in the aforementioned free boundary problem

$$\text{(N)} \qquad \left\{ \begin{aligned} \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u - T) &= 0 && \text{in } \Omega \setminus \Gamma(t), \\ \operatorname{div} u &= 0 && \text{in } \Omega \setminus \Gamma(t), \\ [\![u]\!] &= 0 && \text{on } \Gamma(t), \\ -[\![T]\!]\nu_\Gamma - \operatorname{div}_\Gamma T_\Gamma &= 0 && \text{on } \Gamma(t), \\ V_\Gamma - u \cdot \nu_\Gamma &= 0 && \text{on } \Gamma(t), \\ u|_{\partial\Omega} &= 0 && \text{on } \partial\Omega, \\ \Gamma(0) &= \Gamma_0, \\ u|_{t=0} &= u_0 && \text{in } \Omega \setminus \Gamma_0. \end{aligned} \right.$$

This model is considered as an initial value problem for a given initial velocity $u_0 \colon \Omega \to \mathbb{R}^n$ and a given initial interface $\Gamma_0 \subset \Omega$ and we ask for short-time existence and uniqueness of the unknown solution (u, π, Γ) and its continuous dependence with respect to (u_0, Γ_0). More information related to this model is given in the monographs of Aris [Ari89]; Edwards, Brenner, and

Wasan [EBW91]; and Slattery, Sagis, and Oh [SSO07]. A more recent survey on related models is given by Sagis [Sag11]. These authors mainly deal with theoretical properties in special situations and with experimental results. Furthermore, Barrett, Garcke, and Nürnberg [BGN14] analyzed a semi-discretized version of (N), where surface tension and surface viscosities depend on the concentration of a surfactant, whose distribution is governed by a convection-diffusion equation on the interface. For the simplified situation of a spherical droplet Ω_- in a Stokes-Poiseuille flow, Reusken and Zhang [RZ13] carried out numerical experiments and studied the migration velocity of that droplet.

On the other hand, the theoretical understanding of problem (N) in general bounded configurations is still limited. Bothe and Prüss [BP10] have shown that the energy functional

$$\frac{1}{2}\int_\Omega \rho|u(t,x)|^2\,dx + \sigma|\Gamma(t)|$$

is always a strict Ljapunov functional for sufficiently smooth solutions and that its critical points for constant phase volumes $|\Omega_\pm|$ are precisely the stationary states of (N). However, the well-posedness of problem (N) has not been proved by rigorous mathematical analysis. Even worse, they found an additional condition that determines the well-posedness of a linear model problem in the whole space $\Omega = \mathbb{R}^n$ with flat reference interface $\Sigma = \mathbb{R}^{n-1}\times\{0\}$. In terms of some reference velocity u_* related to u_0, this condition is given by

$$d_0^{\mathrm{BP}} := \sigma + (\lambda_s - \mu_s)\operatorname{div}_\Sigma(P_\Sigma u_*) + 2\mu_s \min_{\zeta\in\mathbb{R}^n,\,|\zeta|=1} \zeta\cdot[\nabla_\Sigma(P_\Sigma u_*)]\zeta > 0.$$

In case $d_0^{\mathrm{BP}} < 0$, the interface symbol is not invertible. Hence it is not clear whether problem (N) is well-posed for arbitrary velocities u_0, not even for short times.

This thesis attempts to fill this gap. We will reformulate problem (N) as an equivalent transformed problem (T) where the unkown interface $\Gamma(t)$ is replaced by a stationary interface Σ and a height fucntion $h(t,\cdot)\colon \Sigma\to\mathbb{R}$. As our main result, we prove that problem (T) is locally well-posed for initial velocities subject to the following well-posedness condition:

$$\text{(WPC)}\qquad \inf_\Sigma\left(\sigma + (\lambda_s-\mu_s)\operatorname{div}_\Sigma u_0 + 2\mu_s \min_{\zeta\in\mathbb{R}^n,\,|\zeta|=1}\zeta\cdot[\nabla_\Sigma u_0]\zeta\right) > 0.$$

Thus, compared to the linear model problem of Bothe and Prüss, not only the tangential velocity $P_\Sigma u_0|_\Sigma$, but the full velocity is important for the well-posedness of the nonlinear problem. We further show that the corresponding condition is not only sufficient, but also necessary for the invertibility of the interface symbol of a corresponding linear model problem.

We mainly follow the strategy of Köhne, Prüss, and Wilke [KPW13] and employ a time-dependent diffeomorphism $\Theta(t,\cdot)$ of the underlying domain Ω, which maps a fixed hypersurface $\Sigma\subset\Omega$ onto $\Gamma(t) = \Theta(\{t\}\times\Sigma)$. One such diffeomorphism is the well-known *Hanzawa map* Θ_h^{Han} [Han81, (2.1)], which was first used by Hanzawa for transforming the one-phase Stefan problem. It is an extension to Ω of the parametrization

$$\theta_h(t,x) = x + h(t,x)\nu_\Sigma(x) \in \Gamma(t) \quad \text{for } t\in J,\ x\in\Sigma.$$

The Hanzawa map was also applied by Escher, Prüss, and Simonett [EPS02] for transforming a two-phase Stefan problem and by Köhne, Prüss, and Wilke [KPW13] for transforming the two-phase Navier-Stokes equations with surface tension. For the latter, the authors considered the transformed functions

$$\overline{u}(t,x) = u(t,\Theta_h(t,x)), \quad \overline{\pi}(t,x) = \pi(t,\Theta_h(t,x)),$$

and reformulated their original problem for (u,π,Γ) as a transformed problem for $(\overline{u},\overline{\pi},h)$.

However, this velocity transformation does not seem appropriate for transforming our problem (N) with additional surface viscosities, since, on the one hand, both $v = P_\Gamma u$ and

$w = \nu_\Gamma \cdot u$ would depend on both $\overline{v} = P_\Sigma \overline{u}$ and $\overline{w} = \nu_\Sigma \cdot \overline{u}$, but on the other hand, the interface momentum balance requires different orders of differentiability for v and w. We therefore employ both a different diffeomorphism and a different velocity transformation for ensuring that these velocity components are transformed separately. We consider a class of maps $\Theta\colon J \times \overline{\Omega} \to \overline{\Omega}$ that we call the normal-preserving admissible maps. For such a map, the normal derivative $\partial_{\nu_\Sigma}\Theta(t,x)$ is a multiple of the original normal vector field $\nu_\Gamma(t,\Theta(t,x))$, whereas the Hanzawa map satisfies $\partial_{\nu_\Sigma}\Theta_h^{\mathrm{Han}} = \nu_\Sigma$. Moreover, the Jacobian $\partial_x\Theta(t,x)$ maps the normal space $\mathbb{R}\nu_\Sigma(x)$ onto the normal space $\mathbb{R}\nu_\Gamma(t,\Theta(t,x))$ and the tangent space $T_x\Sigma$ onto $T_{\Theta(t,x)}\Gamma(t)$. We will construct such a map Θ_h in terms of a height function h by using a similar method as Abels and Terasawa [AT09], who transformed a Stokes problem with variable viscosity in a bent half-space. Our map Θ_h has several advantages when we consider the velocity transformation

$$u(t,\Theta_h(t,x)) = [\partial_x\Theta_h(t,x)]\overline{u}(t,x).$$

First, we have

$$v(t,\Theta_h(t,x)) = [\partial_x\Theta_h(t,x)]\overline{v}(t,x), \quad w(t,\Theta_h(t,x)) = \nu_\Gamma(t,\Theta_h(t,x)) \cdot \nu_\Sigma(x)\overline{w}(t,x),$$

and thus the velocity components are transformed separately. Second, the advected moving interface condition $V_\Gamma = \nu_\Gamma \cdot u$ is transformed to the simple identity

$$\partial_t h = \overline{w}.$$

Thus, compared to Prüss and Simonett [PS11], we can avoid perturbations in this equation.

In this way, problem (N) can be reformulated as a transformed problem

$$\text{(T)} \qquad \begin{cases} \rho\partial_t\overline{u} - \mu\Delta\overline{u} + \nabla\overline{\pi} = F_u(\overline{u},\overline{\pi},h) & \text{in } J \times \Omega \setminus \Sigma, \\ \operatorname{div}\overline{u} = F_d(\overline{u},h) & \text{in } J \times \Omega \setminus \Sigma, \\ [\![\overline{u}]\!] = 0 & \text{on } J \times \Sigma, \\ L_u(\overline{u},\overline{\pi},h;\overline{u}_*) = G_u(\overline{u},\overline{\pi},h;\overline{u}_*,\overline{\pi}_*,h_*) & \text{on } J \times \Sigma, \\ \partial_t h - \overline{u}\cdot\nu_\Sigma = 0 & \text{on } J \times \Sigma, \\ \overline{u}|_{\partial\Omega} = 0 & \text{on } J \times \partial\Omega, \\ h|_{t=0} = h_0 & \text{on } J \times \Sigma, \\ \overline{u}|_{t=0} = \overline{u}_0 & \text{in } \Omega \setminus \Sigma. \end{cases}$$

Here the left-hand sides are linear with respect to $(\overline{u},\overline{\pi},h)$, the functions $\overline{u}_*$, $\overline{\pi}_*$, and h_* are chosen according to the initial data, and F_u, F_d, and G_u are nonlinear perturbations that have to be controlled in a suitable way. In the following, we omit the bars over $\overline{u}$, $\overline{u}_*$, $\overline{\pi}$, and $\overline{\pi}_*$. For solving problem (T), we also employ its principal linearization

$$\text{(PL)} \qquad \begin{cases} \rho\partial_t u - \mu\Delta u + \nabla\pi = f_u & \text{in } J \times \Omega \setminus \Sigma, \\ \operatorname{div} u = f_d & \text{in } J \times \Omega \setminus \Sigma, \\ [\![u]\!] = 0 & \text{on } J \times \Sigma, \\ L_u(u,\pi,h;u_*) = g_u & \text{on } J \times \Sigma, \\ \partial_t h - u\cdot\nu_\Sigma = 0 & \text{on } J \times \Sigma, \\ u|_{\partial\Omega} = 0 & \text{on } J \times \partial\Omega, \\ h|_{t=0} = 0 & \text{on } \Sigma, \\ u|_{t=0} = 0 & \text{in } \Omega \setminus \Sigma. \end{cases}$$

In order to define the operator L_u, we decompose $u_* = v_* + w_*\nu_\Sigma$ and $g_u = g_v + g_w\nu_\Sigma$ as well as $L_u(u, \pi, h; u_*) = L_v(u, h; u_*) + L_w(u, \pi, h; u_*)\nu_\Sigma$. Then we have

$$\begin{aligned} L_v(u, h; u_*) &= -\mu_s \widetilde{\Delta}_\Sigma v - \lambda_s \nabla_\Sigma \operatorname{div}_\Sigma v - [\![\mu \nabla_\Sigma w]\!] - [\![\mu \partial_\nu v]\!] + (\lambda_s + \mu_s) w_* \nabla_\Sigma \Delta_\Sigma h, \\ L_w(u, \pi, h; u_*) &= -\operatorname{tr}\left([(\lambda_s - \mu_s) H_\Sigma + 2\mu_s L_\Sigma] \nabla_\Sigma v\right) - 2[\![\mu \partial_\nu w]\!] + [\![\pi]\!] \\ &\qquad - \operatorname{tr}\left([\sigma + (\lambda_s - \mu_s)(\operatorname{div}_\Sigma v_* - 2H_\Sigma w_*) + 2\mu_s D_\Sigma(v_*) - 4\mu_s w_* L_\Sigma] \nabla_\Sigma^2 h\right). \end{aligned}$$

A crucial task is to verify that problem (PL) has optimal regularity, which means that the solution-to-data map $(u, \pi, h) \mapsto (f_u, f_d, g_u)$ is a topological linear isomorphism between suitable function spaces. Hence the regularity conditions on the data must be both necessary and sufficient for the existence and regularity of the solution. In this case, the well-posedness of the nonlinear problem (T) can be proved simply by Banach's fixed-point theorem. We are interested in spaces for which the velocity $u(t,x)$ and pressure $\pi(t,x)$ satisfy the regularity conditions

$$u \in H_p^1(J; L_p(\Omega)^n) \cap L_p(J; H_p^2(\Omega \setminus \Sigma)^n), \quad \pi \in L_p(J; \dot{H}_p^1(\Omega \setminus \Sigma)),$$

where the Lebesgue exponent $p \in (1, \infty)$ will be chosen sufficiently large for controlling the nonlinear perturbations in problem (T). In order to construct such spaces, we solve a linear model problem for (PL) in the whole space $\Omega = \mathbb{R}^{n+1}$ with a flat interface $\Sigma = \mathbb{R}^n \times \{0\}$ under the restriction $(f_u, f_d, u_0, h_0) = 0$. The generic element of $\mathbb{R}^{n+1}$ is denoted by (x, y) with $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$. Let $\vartheta_w \in \mathbb{R}$, $\vartheta_L \in \mathbb{R}^{n\times n}$, and $\vartheta_{Dv} \in \mathbb{R}^{n\times n}$ denote the values of w_*, L_Σ, and $D_\Sigma(v_*)$ at some fixed position and define the parameters

$$\begin{aligned} &c_1 := (\lambda_s + \mu_s)\vartheta_w, && c_2 := (\lambda_s - \mu_s) \operatorname{tr} \vartheta_L, \\ &C_3 := \mu_s \vartheta_L, && C_4 := 2\mu_s(\vartheta_{Dv} - 2\vartheta_w \vartheta_L), \\ &c_{5,6} \in \{0, 1\}, && c_\sigma := \sigma + (\lambda_s - \mu_s) \operatorname{tr}(\vartheta_{Dv} - 2\vartheta_w\vartheta_L). \end{aligned}$$

Then the aforementioned model problem is given by

$$\text{(MP)} \quad \left\{ \begin{aligned} \rho(\tau + \partial_t)u - \mu\Delta u + \nabla\pi &= 0 && \text{in } \mathbb{R}_+ \times \dot{\mathbb{R}}^{n+1}, \\ \operatorname{div} u &= 0 && \text{in } \mathbb{R}_+ \times \dot{\mathbb{R}}^{n+1}, \\ [\![u]\!] &= 0 && \text{on } \mathbb{R}_+ \times \mathbb{R}^n, \\ -\mu_s \Delta_x v - \lambda_s \nabla_x \operatorname{div}_x v - c_5 [\![\mu \nabla_x w]\!] - c_6 [\![\mu \partial_y v]\!] + c_1 \nabla_x \Delta_x h &= g_v && \text{on } \mathbb{R}_+ \times \mathbb{R}^n, \\ -\operatorname{tr}((c_2 + 2C_3)\nabla_x v) - 2[\![\mu \partial_y w]\!] + [\![\pi]\!] - \operatorname{tr}((c_\sigma + C_4)\nabla_x^2 h) &= g_w && \text{on } \mathbb{R}_+ \times \mathbb{R}^n, \\ (\tau + \partial_t)h - w &= g_h && \text{on } \mathbb{R}_+ \times \mathbb{R}^n, \\ h|_{t=0} &= 0 && \text{on } \mathbb{R}^n, \\ u|_{t=0} &= 0 && \text{in } \mathbb{R}^{n+1}. \end{aligned} \right.$$

Here, $\tau > 0$ will be a sufficiently large number and we allow for $g_h \neq 0$. The term $c_5[\![\mu\nabla_x w]\!]$ is of lower order in our functional analytic setting and therefore negligible, in contrast to the situation without surface viscosities. Moreover, we will choose $c_6 = 1$ for proving well-posedness, but also allow for $c_6 \in \{0, 1\}$ in the symbolic calculations.

A basic version of problem (MP) without surface viscosities was solved in an L_p-setting by Prüss and Simonett [PS10] for the parameters $(c_1, c_2, C_3, C_4) = 0$, $c_{5,6} = 1$, and $c_\sigma = \sigma$. They also included gravity acting in the negative x_{n+1}-direction and studied the modified equation $\partial_t h - w + b \cdot \nabla h = g_h$ in [PS11]. Here the additional term $b \cdot \nabla h$ with $b \in \mathbb{R}^n$ arises when the free-interface problem is transformed by means of the Hanzawa diffeomorphism Θ_h and the velocity transformation $u(t, \Theta_h(t, x)) = \overline{u}(t, x)$ and when the transformed problem is linearized at a non-trivial reference velocity. In this thesis we can neglect the term $b \cdot \nabla h$.

A linear problem including surface viscosities λ_s, $\mu_s > 0$ was derived and analyzed by Bothe and Prüss [BP10]. Roughly speaking, their model corresponds to (MP) with $c_1 = 0$, $c_2 = 0$, $C_3 = 0$, $c_\sigma = \sigma + (\lambda_s - \mu_s)\vartheta_{dv}$, $C_4 = 2\mu_s\vartheta_{Dv}$, $c_{5,6} = 1$. In particular, all terms arising for non-trivial ϑ_w are not present and their g_v-equation is only of second order in h. As mentioned above, we require the well-posedness condition

$$d_0(\vartheta_{Du}) := \sigma + (\lambda_s - \mu_s)\operatorname{tr}\vartheta_{Du} + 2\mu_s \min_{\xi\in\mathbb{R}^n\setminus\{0\}} \xi\cdot[\vartheta_{Du}]\xi|\xi|^{-2} > 0.$$

This condition is also necessary for the invertibility of the associated interface symbol.

Denk and Kaip [DK13, Section 4.7] solved a variant of (MP) for vanishing c_1, c_2, C_3, C_4 which combines the models of [BP10] and [PS11] for surface viscosities and gravity. They derived function spaces for the interface quantities for which the corresponding interface operator is an isomorphism and their results cover both cases λ_s, $\mu_s = 0$ and λ_s, $\mu_s > 0$. Fortunately, we can adapt their method to the present situation, but we shall employ somewhat different function spaces, due to the additional leading order term $c_1\nabla_x\Delta_x h$. We will compute the interface symbol and prove that it is invertible for all cases $c_{5,6} \in \{0,1\}$. Moreover, we will see that the order structure of the system and hence also the spaces for optimal regularity depend on c_6 but not on c_5. Suitable function spaces for solving problem (MP) are only constructed in the case $c_6 = 1$ since these spaces allow for better time regularity than the case $c_6 = 0$. Unfortunately, the g_v-equation is not invariant under the parabolic scaling $v(t,x) = v_\zeta(\zeta t, \sqrt{\zeta}x)$ and in this situation the author does not know how to perform perturbation theory on $J = (0,\infty)$ for arbitrary initial states. Therefore we deal with short time intervals $J = (0,T)$ and use a small end time T instead of a large number τ as a perturbation parameter.

To transfer optimal regularity of the model problem to the principal linearization (PL), we adapt the localization procedures of Köhne, Prüss, and Wilke [KPW13]; Abels and Terasawa [AT09]; Denk, Hieber, and Prüss [DHP03]; Amann, Hieber, and Simonett [AHS94]; and Ladyzhenskaya, Solonnikov, and Ural'tseva [LSU68]. With also provide a theory on an elliptic transmission problem

$$\text{(TP)}\qquad \begin{cases} \operatorname{div}(\mu\nabla\psi) = \operatorname{div} u & \text{in } \Omega\setminus\Sigma,\\ \partial_\nu\psi = \nu\cdot u & \text{on } \partial\Omega,\\ [\![\mu\partial_\nu\psi]\!] = [\![\mu\nu\cdot u]\!] & \text{on } \Sigma,\\ [\![\psi]\!] = 0 & \text{on } \Sigma, \end{cases}$$

and its weak version

$$\int_\Omega \mu\nabla\psi\cdot\nabla\varphi\,dx = \int_\Omega u\cdot\nabla\varphi\,dx \qquad \text{for } \varphi\in C_c^\infty(\mathbb{R}^n), \quad [\![\psi]\!] = 0 \text{ on } \Sigma.$$

This theory suffices to determine the bulk pressure π and to handle the inhomogeneity f_d in the divergence equation. For these problems we prove the optimal a priori estimates

$$\|\nabla\psi\|_{H_p^k(\Omega\setminus\Sigma)} \le C\|u\|_{H_p^k(\Omega\setminus\Sigma)} \qquad \text{for } k\in\{0,1,2\},$$

by means of a localization procedure based on the methods of Simader and Sohr [SS92] and Köhne, Prüss, and Wilke [KPW13].

In this way we can conclude that (PL) induces a topological isomorphism and that the linear solution operator corresponding to (PL) is uniformly bounded with respect to the length of the time interval $T \to 0+$ and certain reference velocities u_* which satisfy (WPC). By means of Banach's fixed point theorem we show that problem (T) is well-posed for small T, for small h_0 and for possibly large u_0 that satisfies (WPC).

This thesis is organized as follows. In Chapter 1, we derive problem (N) in a mathematically rigorous way and study some properties of this model. Chapter 2 provides an optimal regularity theory for the transmission problem (TP), which is employed later on. Optimal regularity for the principal linearization (PL) is proved in Chapter 3. Finally, we establish well-posedness for the transformed problem (T) in Chapter 4. For keeping this thesis self-contained, we provide relevant results on differential geometry and functional analysis in Appendices A and B.

CHAPTER 1

Modeling of moving interface flows

In this chapter we derive the model (N) in a rigorous way from basic principles and constitutive assumptions. To this end, we also study the concepts of moving domains and moving hypersurfaces and recall important divergence theorems and transport theorems.

Basic notation. Throughout this thesis, the symbols $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of the positive integers, the integers, the real numbers, and the complex numbers, and we let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$. We also put $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ := [0, \infty)$, $\mathbb{R}_- := (-\infty, 0]$, and $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Re} z \geq 0\}$. The imaginary unit is denoted by i. For a real number x we let $\lfloor x \rfloor := [x] := \max\{k \in \mathbb{Z} : k \leq x\}$, $\lceil x \rceil := \min\{k \in \mathbb{Z} : k \geq x\}$, and $\{x\} := x - \lfloor x \rfloor$.

The n-dimensional Euclidean space $\mathbb{R}^n$ ($n \in \mathbb{N}$) is equipped with the scalar product $v \cdot w = (v|w) = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n$ and the norm $|v| = \sqrt{v \cdot v}$ for $v = (v_1, v_2, \ldots, v_n)$ and $w = (w_1, w_2, \ldots, w_n)$. The vector space $\mathbb{C}^n$ is equipped with the scalar product $(v|w) = v_1 \bar{w}_1 + v_2 \bar{w}_2 + \cdots + v_n \bar{w}_n$, where the bar denotes complex conjugation. We let $\langle v, w \rangle = v \cdot w = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n$ denote the bilinear product of two vectors v, $w \in \mathbb{C}^n$. The canonical basis of $\mathbb{K}^n$ as a $\mathbb{K}$-vector space consists of the unit vectors $e^j = e_j = (\delta_{ij})_i$, where δ_{ij}, δ_i^j, and δ^{ij} denote the Kronecker delta.

Matrices are denoted by $A = [a_{ij}]_{ij} \in \mathbb{K}^{n \times m}$ for $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$. The transposed matrix of A is given by $A^\top = [a_{ji}]_{ij}$. The symmetric part of a quadratic matrix $A \in \mathbb{K}^{n \times n}$ is $\operatorname{sym} A := 2^{-1}(A + A^\top)$ and the Kronecker product of two vectors $v \in \mathbb{K}^n$ and $w \in \mathbb{K}^m$ is defined by $v \otimes w := [v_i w_j]_{ij}$. The symbol $|A|$ denotes the induced matrix norm of the Euclidean vector norm; that is, $|A| = \max\{|Av| : v \in \mathbb{K}^n \text{ with } |v| = 1\}$. The trace of $A \in \mathbb{K}^{n \times n}$ is $\operatorname{tr} A = a_{11} + \cdots + a_{nn}$. Using Einstein's summation convention, we write $\operatorname{tr} A = e_j \cdot A e^j$. For two matrices A, $B \in \mathbb{R}^{n \times n}$ we put $A : B := \operatorname{tr}(A^\top B) = a_{ij} b_{ij}$.

For two sets U and V, we write $U \subset V$, if U is a subset of V. We also write $U \dot\cup V$ for the union $U \cup V$ of disjoint sets U and V. The power set of U, which consists of all subsets of U, is denoted by 2^U. We write $U \subset\subset V$ if U and V are subsets of some metric space such that U is bounded and its closure $\overline{U}$ is contained in V.

The notation $f \colon X \supset U \to V \subset Y$ or $f \colon U \subset X \to V \subset Y$ indicates that f is a mapping from the subset U of the set X into the subset V of the set Y. The set $\operatorname{gr} f = \{(x, f(x)) : x \in U\}$ is the graph of f. For a set-valued map $F \colon U \to 2^Y$ we put $\operatorname{gr} F := \cup_{x \in U} (\{x\} \times F(x)) \subset U \times Y$. If U and V are subspaces of topological spaces X and Y, then the vector space $C(U; V)$ contains all maps $f \colon U \to V$ that are continuous with respect to the topologies induced by X and Y. We will abbreviate $C(U; \mathbb{K}) =: C(U)$.

The partial derivatives of a C^1-map f defined in $U \subset \mathbb{R}^n$ are denoted by $\partial_i f = \partial f / \partial x_i$ and the (Fréchet) derivative ∂f of f at $x_* \in U$ is the linear map $v \mapsto [\partial f(x_*)] v = (d/dh) f(x_* + hv)|_{h=0}$. The nabla operator $\nabla = (\partial_1, \partial_2, \ldots, \partial_n)^\top$ is defined by $\nabla f = (\partial_1 f, \ldots, \partial_n f)^\top$ for a scalar field f and $\nabla v = e^j \otimes \partial_j v = [\partial_i v_j]_{ij}$ for a vector field v. The divergence is defined by $\operatorname{div} v = \partial_i v_i$ for a vector field v and $\operatorname{div} S = (\partial_j S_{ij})_i$ for a symmetric matrix field S. Thus, $\operatorname{div}(Sv) = \operatorname{div} S \cdot v + S : \nabla v$.

Let Σ be a C^1-hypersurface in $\mathbb{R}^n$ with local parametrization $U \subset \mathbb{R}^{n-1} \to \Sigma$, $u \mapsto x = \phi(u)$. We employ the tangent vectors $\tau_j(x) = \partial_j \phi(u)$, which span the tangent space $T_x \Sigma$, and the cotangent vectors $\tau^k(x)$, which are uniquely determined by the relations $\tau_j(x) \cdot \tau^k(x) = \delta_j^k$. The partial derivatives of a C^1-map $f\colon \Sigma \to X$ are denoted by $\partial_i f(x) := \partial_i(f \circ \phi)(u)$. We define the surface gradient $\nabla_\Sigma f = \tau^j \partial_j f$ for a scalar field f and $\nabla_\Sigma u = \tau^j \otimes \partial_j u$ for a (not necessarily tangential) vector field u. Moreover, we define the surface divergence $\operatorname{div}_\Sigma u = \partial_j u \cdot \tau^j$ and $\operatorname{div}_\Sigma S = (\partial_j S)\tau^j$ for a symmetric matrix field S. Thus, $\operatorname{div}_\Sigma(Su) = \operatorname{div}_\Sigma S \cdot u + S : \nabla_\Sigma u$. Moreover, $\nu_\Sigma\colon \Sigma \to \mathbb{R}^n$ is a unit normal field of Σ and, if Σ is of class C^2, $L_\Sigma = -\nabla_\Sigma \nu_\Sigma$ denotes the Weingarten tensor and $H_\Sigma = -\operatorname{div}_\Sigma \nu_\Sigma = \operatorname{tr} L_\Sigma$ denotes the ($(n-1)$-fold) mean curvature.

For a metric space (X, d_X), the symbols $B_R^X(x)$ or $\mathbb{B}_R^X(x)$ denote the open ball $\{y \in X : d_X(x,y) < R\}$ of radius R and center $x \in X$. If $(X, \|\cdot\|_X)$ is a normed vector space, we abbreviate $B_R^X := B_R^X(0) := \{x \in X : \|x\|_X < R\}$. We will write $B_R(x)$ instead of $B_R^X(x)$ if X is known from the context. For a subset M of X we define $B_R(M) = \{y \in X : \operatorname{dist}(x, M) < R\}$, where $\operatorname{dist}(x, M) := \inf\{\operatorname{dist}(x,y) : y \in M\}$. Two normed vector spaces X and $Y$are equal if they coincide as sets and have equivalent norms. We write $Y \hookrightarrow X$, if Y is continuously embedded into X and we write $Y \hookrightarrow^d X$, if the embedding is also dense. The *complexification* of a real vector space $X = X_\mathbb{R}$ is denoted by $X_\mathbb{C} = \{x_1 + ix_2 : x_1, x_2 \in X\}$.

The vector space of all linear operators $A\colon X \to Y$ between vector spaces X and Y is denoted by $\mathcal{L}(X;Y)$ and we abbreviate $\mathcal{L}(X) := \mathcal{L}(X;X)$. We let $N(A) = \{x \in X : Ax = 0\}$ and $R(A) = \{Ax : x \in X\}$ denote the null space an range of a linear operator $A\colon X \to Y$. The *complexification* of an $\mathbb{R}$-linear operator $A\colon X \to Y$ is given by $A_\mathbb{C}\colon X_\mathbb{C} \to Y_\mathbb{C}$, $(x_1 + ix_2) \mapsto Ax_1 + iAx_2$. The space of bounded linear operators $A\colon X \to Y$ between normed vector spaces X and Y is denoted by $\mathcal{B}(X;Y)$, and it is equipped with the operator norm $\|A\|_{\mathcal{B}(X;Y)} = \|A\|_{X \to Y}$. The space of bounded k-linear maps $A\colon X^k \to Y$ for $k \in \mathbb{N}$ is denoted by $\mathcal{B}^k(X^k;Y)$, and its norm is denoted by

$$\|A\|_{\mathcal{B}^k(X^k;Y)} = \sup\{\|A(x_1, \dots, x_k)\|_Y : x_1, \dots, x_k \in X \text{ with } \|x_1\| = \cdots = \|x_k\| = 1\}.$$

The space of bounded linear isomorphisms from X to Y is denoted by $\mathcal{B}_{\text{isom}}(X;Y)$ and that of linear isomorphisms by $\mathcal{L}_{\text{isom}}(X;Y)$. We let $I_X\colon x \mapsto x$ denote the identity on X and $\langle x^*, x\rangle_{X^* \times X} = \langle x^*, x\rangle = x^*(x)$ denote the duality pairing for $x^* \in X^*$ and $x \in X$.

We employ the theory of moving hypersurfaces and Riemannian manifolds as given by do Carmo [Car92], Kimura [Kim08], and Prüss and Simonett [PS13]. More background information on differential geometry and the theory of function spaces is given in Appendices A and B.1.

1.1. Moving hypersurfaces and integral theorems

In order to define moving domains and hypersurfaces, we consider the initial-value problem

$$\dot{x}(t) = u(t, x(t)) \text{ for } t \in J, \quad x(t_0) = x_0, \tag{1.1}$$

where J is an open interval and $u\colon G \to \mathbb{R}^n$ ($n \in \mathbb{N}$) is a given vector field on an open subset G of $J \times \mathbb{R}^n$. It is custom to understand the map $t \mapsto x(t)$ as the trajectory of a moving particle that starts at position x_0 at time t_0 and moves with velocity $u(t, x(t))$. We say that x_0 is the *convected coordinate* of the moving particle [cf. Old50; Scr60].

A local *solution* of (1.1) is a C^1-map $x\colon J(t_0, x_0) \to \mathbb{R}^n$ on some interval $J(t_0, x_0) \subset \mathbb{R}$ that contains t_0, such that $(t, x(t))$ belongs to G for all $t \in J(t_0, x_0)$ and such that (1.1) is satisfied on $J(t_0, x_0)$. If $(t, x) \mapsto u(t, x)$ is continuous on G and locally Lipschitz with respect to x, then the Picard-Lindelöf theorem implies that for every $(t_0, x_0) \in G$, there exists a unique local solution on some interval $(t_0 - \delta, t_0 + \delta)$. Moreover, the solution has a unique extension to a maximal interval of existence, which is again denoted by $J(t_0, x_0)$. This interval is open and for any finite $t_* \in \partial J(t_0, x_0)$, the function $(t, x(t))$ tends to ∂G or it blows up as $t \to t_*$, in the sense that $\operatorname{dist}((t, x(t)), \partial G) \to 0$ or $|x(t)| \to \infty$.

1.1. Proposition. *Let $J \subset \mathbb{R}$ be an open interval, let $G \subset J \times \mathbb{R}^n$ be open, let $u \in C(G)^n$ be locally Lipschitz with respect to x, and let $t \mapsto x_{(t_0,x_0)}(t)$ denote the unique solution to* (1.1) *for $(t_0, x_0) \in G$.*

Then the map

$$\Phi\colon (t, t_0, x_0) \mapsto x_{(t_0,x_0)}(t), \quad \{(t, t_0, x_0) \in J \times G : t \in J(t_0, x_0)\} \to \mathbb{R}^n$$

has the following properties.

(i) $(t, \Phi(t, s, x))$ *belongs to* G *for all* $t \in J(s, x)$ *and* $(s, x) \in G$.
(ii) $\Phi(t, t, x) = x$ *for all* $(t, x) \in G$.
(iii) $\Phi(t, s, \Phi(s, r, x)) = \Phi(t, r, x)$ *for all* $(r, x) \in G$ *and* $t, s \in J(r, x)$.
(iv) $\Phi(\cdot, s, x)$ *is continuously differentiable in* $J(s, x)$ *for all* $(s, x) \in G$.
(v) $\Phi(t, \cdot, \cdot)$ *is locally Lipschitz in* $\{(s, x) \in G : t \in J(s, x)\}$ *for all* $t \in J$.

Proof. Since $\Phi(\cdot, t, x)$ is a solution, it satisfies (i) and (ii). Next, the functions $\Phi(\cdot, s, \Phi(s, r, x))$ and $\Phi(\cdot, r, x)$ are solutions to (1.1) and coincide at $t = s$, since $\Phi(s, r, x) = \Phi(s, s, \Phi(s, r, x))$. By uniqueness, we have $\Phi(\cdot, s, \Phi(s, r, x)) = \Phi(\cdot, r, x)$ on $J(r, x)$. The C^1-regularity of $\Phi(\cdot, s, x)$ follows from $\partial_t\Phi(t, s, x) = u(t, \Phi(t, s, x))$, and the local Lipschitz condition is a consequence of Gronwall's Lemma [see PW10, Satz 4.1.2]. □

1.2. Remark. We can guarantee that every solution exists for all $t \in J$, when we also assume that $G = J \times \mathbb{R}^n$ and that u is *linearly bounded* with respect to x; that is, there are $a, b \in C(J; \mathbb{R}_+)$ such that $|u(t, x)| \le a(t) + b(t)|x|$ for all $t \in J$, $x \in \mathbb{R}^n$ [see PW10, Korollar 2.5.1].

The map Φ induces a *local flow* in G in the following sense.

1.3. Definition. Let G be a topological space and U be an open subset of $\mathbb{R} \times G$ that contains $\{0\} \times G$. A continuous map $\tilde{\Phi}\colon U \subset \mathbb{R} \times G \to G$ is called a *local flow* in G, if

(i) $\tilde{\Phi}(0, z) = z$ for all $z \in G$.
(ii) $\tilde{\Phi}(t + s, z) = \tilde{\Phi}(t, \tilde{\Phi}(s, z))$ for all $(s, z) \in U$ and $t \in \mathbb{R}$ with $(t, \tilde{\Phi}(s, z)) \in U$.

If $\mathcal{U} = \mathbb{R} \times G$, in addition, then we call $\tilde{\Phi}$ a *(global) flow* in G.

1.4. Corollary. *In the situation of Proposition 1.1, the mapping*

$$\begin{aligned} \tilde{\Phi}\colon (s, (t_0, x_0)) &\mapsto (t_0 + s, \Phi(t_0 + s, t_0, x_0)), \\ \{(s, t_0, x_0) &\in \mathbb{R} \times \operatorname{gr}\Omega : t_0 + s \in J(t_0, x_0)\} \to \operatorname{gr}\Omega \end{aligned} \tag{1.2}$$

is a local flow in $\operatorname{gr}\Omega$. *We also call* Φ *the flow in* $\operatorname{gr}\Omega$ *induced by the velocity field* u.

1.5. Definition. Let $J \subset \mathbb{R}$ be an open interval, $n \in \mathbb{N}$, and $\Omega\colon J \ni t \mapsto \Omega(t)$ be a set-valued map such that each $\Omega(t)$ is a domain in $\mathbb{R}^n$. We call Ω a *moving domain*, if there is a flow $\Phi\colon J \times \operatorname{gr}\Omega \to \operatorname{gr}\Omega$ induced by some velocity field $u\colon \operatorname{gr}\Omega \to \mathbb{R}^n$ such that

$$\Omega(t) = \Phi(t, t_0, \Omega(t_0)) := \{\Phi(t, t_0, x) : x \in \Omega(t_0)\} \quad \text{for all } t,\ t_0 \in J.$$

This definition allows to describe fluid volumes, since $\Omega(t)$ can be obtained by following the trajectories $\Phi(\cdot, t_0, x_0)$ of the particles with initial position $x_0 \in \Omega(t_0)$. We note that there may be different velocity fields that describe the same moving domain; for instance $\Omega(t) := (-t, t)$ moves according to the velocity field $u(t, x) = x/t$, but also according to $u(t, x) = x^3/t^3$ for $|x| \le t$. However, the normal component $u(t, x) \cdot \nu_{\partial\Omega(t)}(x)$ at $\partial\Omega(t)$ does not depend on the choice of such a velocity field; a property that holds true for general moving hypersurfaces, which are defined as follows [cf. Kim08, Definition 5.1].

1.6. Definition. Let $J \subset \mathbb{R}$ be an open interval, $n \ge 2$, and $k, l \in \mathbb{N}_0$.

(i) A set-valued map $\Gamma\colon J \ni t \mapsto \Gamma(t)$ is called a *moving hypersurface* (of class C^1), if each $\Gamma(t)$ is an oriented C^1-hypersurface in $\mathbb{R}^n$ and its graph $\operatorname{gr}\Gamma$ is a C^1-hypersurface in $\mathbb{R}^{1+n}$.
(ii) A moving hypersurface Γ is of class C^k [$C^{(k,l)}$], if all its local height functions $h\colon J' \times U \subset J \times \nu_0^\perp \to \operatorname{gr}\Gamma$ in the sense of Definition A.1 on page 129 are of class $C^k(J' \times U)$ [$C^{(k,l)}(J' \times U)$].
(iii) A moving hypersurface Γ is compact, if each $\Gamma(t)$ is compact.

(iv) A moving hypersurface Γ is induced by a moving domain $\Omega\colon J \in t \mapsto \Omega(t)$ with flow $\Phi\colon J \times \operatorname{gr}\Omega \to \operatorname{gr}\Omega$, if $\operatorname{gr}\Gamma \subset \operatorname{gr}\Omega$ and $\Gamma(t) = \Phi(t, t_0, \Gamma(t_0))$ for all $t, t_0 \in J$.

Proposition A.5 implies that for every moving C^{k+1}-hypersurface Γ, the $(n+1)$-dimensional normal $\nu_{\operatorname{gr}\Gamma}$ of $\operatorname{gr}\Gamma$ belongs to the class $C^k(\operatorname{gr}\Gamma)^{1+n}$ and the n-dimensional normal $\nu_{\Gamma(t)}$ of $\Gamma(t)$ belongs to $C^k(\Gamma(t))^n$. Later on, we will show that every compact moving C^2-hypersurface is induced by some flow.

1.7. Proposition ([cf. Kim08, Definition 5.4]). *Suppose that Γ is a moving C^1-hypersurface in $\mathbb{R}^n$. Then there exists a unique function $V_\Gamma\colon \operatorname{gr}\Gamma \to \mathbb{R}$, called the* normal velocity *of Γ, which satisfies the identity*

$$V_\Gamma(t,x) = \gamma'(t) \cdot \nu_{\Gamma(t)}(x) \quad \textit{for all } t \in J,\ x \in \Gamma(t), \tag{1.3}$$

for every C^1-path $(t-\delta, t+\delta) \ni s \mapsto \gamma(s) \in \Gamma(s)$ with $\gamma(t) = x$.

Moreover, the unit normal $\nu_{\operatorname{gr}\Gamma} = (\nu_{\operatorname{gr}\Gamma,t}, \nu_{\operatorname{gr}\Gamma,x}) \in \mathbb{R}^{1+n}$ of $\operatorname{gr}\Gamma$ is given by

$$\nu_{\operatorname{gr}\Gamma} = (1+V_\Gamma^2)^{-1/2}(-V_\Gamma, \nu_\Gamma) \quad \textit{with } V_\Gamma = -\nu_{\operatorname{gr}\Gamma,t}(1-\nu_{\operatorname{gr}\Gamma,t}^2)^{-1/2}. \tag{1.4}$$

In particular, if Γ is induced by a flow with velocity u, then

$$V_\Gamma(t,x) = u(t,x) \cdot \nu_{\Gamma(t)}(x) \quad \textit{for all } t \in J,\ x \in \Gamma(t).$$

Proof. Since $\operatorname{gr}\Gamma$ has dimension n and each $\Gamma(t)$ has dimension $n-1$, we conclude that every tangent space $T_{(t,x)}\operatorname{gr}\Gamma$ must have the form $T_{(t,x)}\operatorname{gr}\Gamma = \{\nu_{\operatorname{gr}\Gamma}(t,x)\}^\perp = \mathbb{R} \times \nu_{\operatorname{gr}\Gamma,x}(t,x)^\perp$ with $\nu_{\operatorname{gr}\Gamma,x}(t,x) \neq 0$, and hence $|\nu_{\operatorname{gr}\Gamma,t}(t,x)| < 1$. Moreover, for every $(t,x) \in \operatorname{gr}\Gamma$ and $\tau \in T_x\Gamma(t)$, the vector $(0,\tau)$ belongs to $T_{(t,x)}\operatorname{gr}\Gamma$, and hence $\nu_{\operatorname{gr}\Gamma,x}(t,x)$ must be parallel to $\nu_{\Gamma(t)}(x)$. Therefore the identity $|\nu_{\operatorname{gr}\Gamma}|^2 = \nu_{\operatorname{gr}\Gamma,t}^2 + |\nu_{\operatorname{gr}\Gamma,x}|^2 = 1$ yields $\nu_{\operatorname{gr}\Gamma,x} = |\nu_{\operatorname{gr}\Gamma,x}|\nu_{\Gamma(t)} = (1-\nu_{\operatorname{gr}\Gamma,t}^2)^{1/2}\,\nu_{\Gamma(t)}$.

Uniqueness of V_Γ. Let V_Γ satisfy (1.3) and consider a C^1-path $s \mapsto (s, \gamma(s))$ in $\operatorname{gr}\Gamma$ with $\gamma(t) = x$. Then its derivative $(1, \gamma'(t))$ belongs to $T_{(t,x)}\operatorname{gr}\Gamma$ and we have

$$0 = (1, \gamma'(t)) \cdot \nu_{\operatorname{gr}\Gamma}(t,x) = \nu_{\operatorname{gr}\Gamma,t}(t,x) + (1-\nu_{\operatorname{gr}\Gamma,t}(t,x)^2)^{1/2}\,\gamma'(t) \cdot \nu_{\Gamma(t)}(x). \tag{1.5}$$

Since $|\nu_{\operatorname{gr}\Gamma,t}|$ is smaller than 1, we obtain $V_\Gamma = \gamma'(t) \cdot \nu_{\Gamma(t)}(x) = -\nu_{\operatorname{gr}\Gamma,t}(1-\nu_{\operatorname{gr}\Gamma,t}^2)^{-1/2}$. Therefore $V_\Gamma(t,x)$ is uniquely determined.

Existence of V_Γ. The function $V_\Gamma = -\nu_{\operatorname{gr}\Gamma,t}(1-\nu_{\operatorname{gr}\Gamma,t}^2)^{-1/2}$ is well-defined and thus (1.5) implies (1.3). Finally, the identity (1.4) follows from those of $\nu_{\operatorname{gr}\Gamma,x}$ and V_Γ in terms of $\nu_{\operatorname{gr}\Gamma,t}$. □

1.8. Proposition. *Every compact moving C^2-hypersurface is induced by some flow.*

Proof. Let $\Gamma\colon J \ni t \mapsto \Gamma(t)$ be a moving C^2-hypersurface in $\mathbb{R}^n$ with normal velocity V_Γ. From $\nu_{\operatorname{gr}\Gamma} \in C^1(\operatorname{gr}\Gamma)$ and (1.4) we infer that $u := V_\Gamma \nu_\Gamma$ is of class $C^1(\operatorname{gr}\Gamma)^n$. By compactness of $\Gamma(t)$ and Proposition A.12, the vector field $u(t,\cdot)$ is $L(t)$-Lipschitz on $\Gamma(t)$; that is, we have $|u(t,x) - u(t,y)| \le L(t)|x-y|$ for all $x, y \in \Gamma(t)$, $t \in J$. Then the McShane-Whitney extension [cf. Hei05, p. 5]

$$\tilde{u}^j(t,x) := \inf_{y \in \Gamma(t)} (u^j(t,y) + L(t)|x-y|) \quad \text{for } j \in \{1,\dots,n\},\ x \in \mathbb{R}^n,\ t \in J$$

of $u = (u^j)_j$ is $\sqrt{n}L(t)$-Lipschitz on $\mathbb{R}^n$ and linearly bounded. According to Proposition 1.1 and Remark 1.2, there is a flow $\Phi\colon J \times J \times \mathbb{R}^n \to J \times \mathbb{R}^n$ with velocity $\tilde{u}$, which induces Γ. □

We will frequently employ the following version of the divergence theorem.

1.9. Theorem (Divergence theorem). *Let $\Omega \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) be a bounded open set with C^1-boundary $\partial\Omega$ or a bent half-space $\{(x', x_n) \in \mathbb{R}^n : x_n > \omega(x')\}$ with $\omega \in C_c^1(\mathbb{R}^{n-1})$. Then*

$$\int_\Omega \operatorname{div} u\, dx = \int_{\partial\Omega} u \cdot \nu_{\partial\Omega}\, d(\partial\Omega) \quad \textit{for all } u \in H_1^1(\Omega; \mathbb{R}^n). \tag{1.6}$$

Here $\int_{\partial\Omega} \dots \, d(\partial\Omega)$ denotes integration with respect to the $(n-1)$-dimensional Hausdorff measure on $\partial\Omega$, and $H^k_p = W^k_p$ denotes the Sobolev space of order $k \in \mathbb{N}_0$ and power $p \in [1, \infty)$.

Proof of Theorem 1.9. The assertion for the case $\partial\Omega \in C^2$ and $u \in C^1_c(\mathbb{R}^n; \mathbb{R}^n)$ is well-known. In the general case, it follows from an approximation argument. □

Next, we state the surface divergence theorem for tangential vector fields of class. The Sobolev space $H^k_p(\Gamma; T\Gamma)$ of tangential vector fields is defined by means of trivializing coordinate systems for the tangent bundle $T\Gamma$ (see page 163).

1.10. Theorem (Surface divergence theorem [cf. BPS05, Theorem A])**.** *Let $\Gamma \subset \mathbb{R}^n$ be a compact C^2-hypersurface with boundary $\partial\Gamma$ of class C^2, whose normal within $T\Gamma$ is denoted by $n_{\partial\Gamma}$. Then*

$$\int_\Gamma \operatorname{div}_\Gamma v \, d\Gamma = \int_{\partial\Gamma} v \cdot n_{\partial\Gamma} \, d(\partial\Gamma) \quad \textit{for all } v \in H^1_1(\Gamma; T\Gamma).$$

Proof. By [BPS05, Theorem A], the surface divergence theorem applies to tangential vector fields v of class C^1, and hence also to $v \in H^1_1(\Gamma; T\Gamma)$ by approximation. □

The next theorem allows to differentiate integrals $\int_{\Omega(t)} \psi(t, x) \, dx$ with respect to time. Assume that the velocity of a moving domain Ω belongs to the Banach space $BUC^{(0,1)}(\operatorname{gr}\Omega)^n$ of all bounded, uniformly continuous vector fields $u\colon \operatorname{gr}\Omega \to \mathbb{R}^n$ whose first-order spatial derivatives are bounded and uniformly continuous on $\operatorname{gr}\Omega$. Then the induced flow $\Phi\colon J \times \operatorname{gr}\Omega \to \operatorname{gr}\Omega$ is continuously differentiable [cf. PW10, Satz 4.3.1], its Jacobian $\partial_x\Phi$ with respect to the spatial variables is invertible, and we have $\det \partial_x \Phi > 0$ on $J \times \operatorname{gr}\Omega$. Hence $\Phi(t, t_0, \cdot)\colon \Omega(t_0) \to \Omega(t)$ is a C^1-*diffeomorphism*; that is, a bijective C^1-map whose Jacobian is invertible everywhere in $\Omega(t_0)$. For a C^1-function ψ on $\operatorname{gr}\Omega$, the *material derivative $D\psi/Dt$* with respect to the flow Φ with velocity u is defined by

$$\frac{D\psi(t,x)}{Dt} := \frac{d}{ds}\psi(t+s, \Phi(t+s,t,x))\bigg|_{s=0} = \partial_t\psi(t,x) + [\partial_x\psi(t,x)]u(t,x). \tag{1.7}$$

1.11. Theorem (Reynolds transport theorem)**.** *Let $\Omega\colon J \ni t \mapsto \Omega(t)$ be a moving domain in $\mathbb{R}^n$ with velocity $u \in BUC^{(0,1)}(\operatorname{gr}\Omega)^n$. Then, given a function $\psi \in H^1_1(\operatorname{gr}\Omega)$, we have*

$$\frac{d}{dt}\int_\Omega \psi \, dx = \int_\Omega \left(\frac{D\psi}{Dt} + \psi \operatorname{div} u\right) dx \quad \textit{a. e. in } J. \tag{1.8}$$

Proof. Let Φ denote the flow induced by u. For fixed t and x, the matrices $Y(s) := \partial_x\Phi(t+s, t, x)$ and $A(s) := \partial_x u(t+s, \Phi(t+s,t,x))$ satisfy $Y'(s) = A(s)Y(s)$ by the chain rule. A well-known identity [see e. g. PW10, Lemma 3.1.2] yields $\det Y'(s) = \operatorname{tr} A(s) \det Y(s)$. Thus,

$$(d/ds) \det \partial_x\Phi(t+s,t,x) = \operatorname{div} u(t+s, \Phi(t+s,t,x)) \det \partial_x\Phi(t+s,t,x).$$

Having in mind that $\partial_x\Phi(t,t,x) = 1$ and that $\Phi(t+s,t,\cdot)\colon \Omega(t) \to \Omega(t+s)$ is bijective, we conclude that $\Phi(t+s,t,\cdot)$ is a diffeomorphism. Therefore the change of variables formula gives

$$\int_{\Omega(t+s)} \psi(t+s, y) \, dy = \int_{\Omega(t)} \psi(t+s, \Phi(t+s,t,x)) \det \partial_x\Phi(t+s,t,x) \, dx. \tag{1.9}$$

By differentiating (1.9) with respect to s at $s = 0$, we obtain (1.8). □

In Theorem 1.11, the Sobolev space $H^1_1(\operatorname{gr}\Omega)$ has the usual meaning, since $\operatorname{gr}\Omega$ is an open subset of $\mathbb{R}^{1+n}$; a fact that does not hold true for $\operatorname{gr}\Gamma$ and can not be used for defining anisotropic spaces. Therefore, we employ the diffeomorphism

$$\tilde{\Phi}_{t_0}\colon J \times \Omega(t_0) \to \operatorname{gr}\Omega, \quad \tilde{\Phi}_{t_0}(t,x) = (t, \Phi(t,t_0,x))$$

and the pull-back $(\tilde\Phi_{t_0}^*\psi)(t,x) := (\psi\circ\tilde\Phi_{t_0})(t,x) = \psi(t,\Phi(t,t_0,x))$, and we assume that J is bounded. Having in mind that $\partial_x\Phi$ is bounded and $\det\partial_x\Phi$ is strictly positive on $\operatorname{gr}\Omega$, we conclude that $\tilde\Phi_{t_0}^*\colon H_1^1(\operatorname{gr}\Omega)\to H_1^1(J\times\Omega(t_0))$ is a topological linear isomorphism. This motivates the definitions

$$H_p^k(\operatorname{gr}\Omega) := \tilde\Phi_{t_0}^* H_p^k(J\times\Omega(t_0)),\quad H_p^{(k,l)}(\operatorname{gr}\Omega) := \tilde\Phi_{t_0}^* H_p^{(k,l)}(J\times\Omega(t_0)),$$
$$H_p^k(\operatorname{gr}\Gamma) := \tilde\Phi_{t_0}^* H_p^k(J\times\Gamma(t_0)),\quad H_p^{(k,l)}(\operatorname{gr}\Gamma) := \tilde\Phi_{t_0}^* H_p^{(k,l)}(J\times\Gamma(t_0)),$$

where $H_p^{(k,l)}(J\times X) = H_p^k(J;L_p(X))\cap L_p(J;H_p^l(X))$ for $k,\, l\in\mathbb{N}_0$, $p\in[1,\infty)$. Their vector-valued versions are defined as on page 163.

The following theorem allows to differentiate integrals over moving hypersurfaces.

1.12. Theorem (Surface transport theorem [cf. BPS05, Theorem B])**.** *Let $\Gamma\colon J\ni t\mapsto\Gamma(t)$ be a compact moving C^2-hypersurface with velocity $u\in BUC^{(0,1)}(\operatorname{gr}\Gamma)^n$ and let $\psi\in H_1^1(\operatorname{gr}\Gamma)$. Then*

$$\frac{d}{dt}\int_\Gamma \psi\, d\Gamma = \int_\Gamma\left(\frac{D\psi}{Dt}+\psi\operatorname{div}_\Gamma u\right)d\Gamma\quad \textit{a. e. in } J.$$

Proof. The special case $u\in BUC^1(\operatorname{gr}\Gamma)^n$ and $\psi\in C^1(\operatorname{gr}\Gamma)$ is treated in [BPS05, Theorem B] and therefore our assertion follows from a straightforward approximation argument. □

1.2. Derivation of the model

In this section we derive problem (N) from integral balance equations and constitutive assumptions. More information on the mathematical modeling of fluid dynamics can be found for instance in [Ari89; And+07; BP10; BPS05; Den94; DS95; Old50; Scr60; SS82; SSO07; Tan93; Tan95].

We consider a bounded domain $\Omega\subset\mathbb{R}^n$ that contains a compact moving C^2-hypersurface $\Gamma(t)$ on a bounded open interval $J\subset\mathbb{R}$. Then we can decompose $\Omega=\Omega_+(t)\,\dot\cup\,\Gamma(t)\,\dot\cup\,\Omega_-(t)$ with moving domains $\Omega_\pm(t)$ (see Corollary A.19 on page 138). In particular, each $\Gamma(t)$ is a compact subset of Ω and therefore the interface does not touch the boundary. We may assume that $\partial\Omega_-(t)=\Gamma(t)$, and hence $\partial\Omega\subset\partial\Omega_+$. Let $u_\pm\in BUC(\operatorname{gr}\Omega_\pm)^n$ be corresponding velocity fields and define

$$u(t,\cdot)\colon\Omega\setminus\Gamma(t)\to\mathbb{R}^n,\quad u(t,x):=u_\pm(t,x)\quad\text{for } x\in\Omega_\pm(t),\ t\in J.$$

For the sake of brevity, we omit the argument t if no confusion seems likely; that is, we write $\Omega\setminus\Gamma$ and Γ instead of $\Omega\setminus\Gamma(t)$ and $\Gamma(t)$ when we consider some fixed t, and we understand that

$$u|_\Gamma(t,x) = u(t,\cdot)|_{\Gamma(t)}(x) = u|_{\operatorname{gr}\Gamma}(t,x)\quad\text{for } x\in\Gamma(t),\ t\in J.$$

Let $\nu_\pm$ denote the outward normal on $\partial\Omega_\pm$ and let $\nu_\Gamma=\nu_-=-\nu_+$ denote the normal at Γ. With the Sobolev space $H_p^k=W_p^k$ of order $k\in\mathbb{N}_0$ and exponent $p\in[1,\infty)$, we write

$$u\in H_p^k(\Omega\setminus\Gamma;\mathbb{R}^n)\quad\text{if and only if } u_+\in H_p^k(\Omega_+;\mathbb{R}^n)\text{ and } u_-\in H_p^k(\Omega_-;\mathbb{R}^n).$$

Other function spaces on $\Omega\setminus\Gamma$ are defined analogously. The *jump* of $u\in H_p^1(\Omega\setminus\Gamma;\mathbb{R}^n)$ on Γ,

$$[\![u]\!] := u_+|_\Gamma - u_-|_\Gamma,$$

is well-defined in the sense of traces. Then the following divergence theorem applies.

1.13. Theorem (Divergence theorem with interface)**.** *Let $\Omega\subset\mathbb{R}^n$ be an open set with C^1-boundary such that the divergence theorem (1.6) is valid and let $\Gamma\subset\Omega$ be a C^1-hypersurface. Then*

$$\int_{\Omega\setminus\Gamma}\operatorname{div}u\,dx = \int_{\partial\Omega}\nu_{\partial\Omega}\cdot u\,d(\partial\Omega) - \int_{\Omega\cap\Gamma}\nu_\Gamma\cdot[\![u]\!]\,d\Gamma\quad \textit{for all } u\in H_1^1(\Omega\setminus\Gamma;\mathbb{R}^n).$$

Proof. This follows by separating the integral over $\Omega\setminus\Gamma$ into integrals over Ω_+ and Ω_-, and by applying the divergence theorem to the separate integrals. □

1.2.1. Balance equations. Our next goal is to derive differential balance equations for a scalar quantity $\psi\colon J\times\Omega\to\mathbb{R}$ that satisfies certain integral balance equations. In order to apply the previous integral theorems, we assume that ψ is of class $H_1^1(J\times\Omega)$ and that

$$u_\pm \in BUC^{(0,1)}(\operatorname{gr}\Omega_\pm)^n;$$

that is, the vector fields u_+ and u_- and their first-order spatial derivatives are bounded and uniformly continuous. We further assume that u is continuous across Γ and that Γ is advected with the flow induced by u; that is,

$$[\![u]\!] = 0 \text{ on } \Gamma, \quad V_\Gamma = u|_\Gamma\cdot\nu_\Gamma.$$

We also assume that $\nu_{\partial\Omega}\cdot u|_{\partial\Omega}=0$, so that Ω is a trivial moving domain with velocity u.

We consider the density $\psi(t,x)$ of an extensive scalar quantity like the mass density ρ or the kinetic energy density $\rho|u|^2$. Let V be a *control volume* in Ω; that is, a moving domain $V\colon J\ni t\mapsto V(t)\subset\Omega$ with the same velocity u. Suppose that $\psi\in H_1^1(J\times\Omega)$ satisfies an *integral balance equation*

$$\frac{d}{dt}\int_V \psi\,dx = \int_V g\,dx + \int_{V\cap\Gamma} g_{\Gamma\to\Omega}\,d\Gamma - \int_{\partial V} j\cdot\nu_{\partial V}\,d(\partial V) \quad \text{a. e. in } J \tag{1.10}$$

for every control volume V with appropriate quantities g, $g_{\Gamma\to\Omega}$, and j. Here

(i) $\int_V g\,dx$ are the sources of ψ in V with volume density g,
(ii) $\int_{\Gamma\cap V} g_{\Gamma\to\Omega}\,d\sigma_\Gamma$ are the sources of ψ on $\Gamma\cap V$ with surface density $g_{\Gamma\to\Omega}$, and
(iii) $\int_{\partial V} j\cdot\nu_{\partial V}\,d(\partial V)$ is the molecular flow of ψ through ∂V with *flux* j.

It is sufficient to impose the regularity assumptions

$$\psi\in H_1^1(J\times\Omega),\ j\in H_1^{(0,1)}(J\times\Omega;\mathbb{R}^n),\ g\in L_1(J\times\Omega),\ g_{\Gamma\to\Omega}\in L_1(\operatorname{gr}\Gamma). \tag{1.11}$$

We wish to derive a differential balance from (1.10). First, Theorem 1.13 yields

$$\frac{d}{dt}\int_V\psi\,dx = \int_V (g-\operatorname{div} j)\,dx + \int_{V\cap\Gamma}(g_{\Gamma\to\Omega} - [\![j]\!]\cdot\nu_\Gamma)\,d\Gamma.$$

With the transport theorem (1.8) we obtain the identity

$$\int_V\left(\frac{\partial\psi}{\partial t} + \operatorname{div}(\psi u + j) - g\right)dx + \int_{V\cap\Gamma}([\![j]\!]\cdot\nu_\Gamma - g_{\Gamma\to\Omega})\,d\Gamma = 0. \tag{1.12}$$

For fixed t, equation (1.12) is valid for every bounded smooth subset $V(t)$ of $\Omega\setminus\Gamma(t)$. From the Lebesgue's integration theory we infer that the first integrand must vanish almost everywhere in $\Omega\setminus\Gamma(t)$. Therefore the following *differential balance equation* is valid a. e. in $J\times\Omega$.

$$\partial_t\psi + \operatorname{div}(\psi u + j) = g \quad \text{in } \Omega\setminus\Gamma.$$

Hence the surface integral in (1.12) vanishes for every time t and every control volume V in Ω. It is not difficult to show that every domain in $\Gamma(t)$ with C^2-boundary can be represented as $V(t)\cap\Gamma$ with some control volume V. Therefore the following *jump condition* is satisfied.

$$[\![j]\!]\cdot\nu_\Gamma = g_{\Gamma\to\Omega} \quad \text{on } \Gamma.$$

Next, assume that there are a scalar surface density $\psi_\Gamma(t,x)$ for $x\in\Gamma(t)$ and quantities g_Γ and j_Γ such that the following *surface integral balance equation* is valid for every control volume.

$$\frac{d}{dt}\int_{V\cap\Gamma}\psi_\Gamma\,d\Gamma = \int_{V\cap\Gamma}(g_\Gamma - g_{\Gamma\to\Omega})\,d\Gamma - \int_{C=\partial V\cap\Gamma} j_\Gamma\cdot n_C\,dC. \tag{1.13}$$

Here g_Γ is the interface source density, the integral $\int_{C=\partial V\cap\Gamma} j_\Gamma\cdot n_C\,dC$ is the molecular flow through the $(n-2)$-dimensional surface $C=\Gamma\cap\partial V$ with outward normal $n_C(t,x)\in T_x\Gamma(t)$ and the *interface flux* j_Γ is tangential vector field on Γ. Sufficient regularity conditions are

$$\psi_\Gamma\in H_1^1(\operatorname{gr}\Gamma),\ j_\Gamma\in H_1^{(0,1)}(\operatorname{gr}\Gamma;T\Gamma),\ g_\Gamma\in L_1(\operatorname{gr}\Gamma), \tag{1.14}$$

We combine (1.10) and (1.13) to

$$\begin{aligned}\frac{d}{dt}\left(\int_V \psi\,dx + \int_{V\cap\Gamma}\psi_\Gamma\,d\Gamma\right) &= \int_V g\,dx + \int_{V\cap\Gamma} g_\Gamma\,d\Gamma \\ &\quad - \int_{\partial V} j\cdot\nu_{\partial V}\,d(\partial V) - \int_{\partial V\cap\Gamma} j_\Gamma\cdot n_C\,dC.\end{aligned} \tag{1.15}$$

Again, we wish to derive the differential balance equation that corresponds to (1.13). First, the surface divergence theorem yields

$$\frac{d}{dt}\int_{V\cap\Gamma}\psi_\Gamma\,d\Gamma = \int_{V\cap\Gamma}(-\operatorname{div}_\Gamma j_\Gamma + g_\Gamma - g_{\Gamma\to\Omega})\,d\Gamma,$$

and the surface transport theorem (Theorem 1.12) implies

$$\int_{V\cap\Gamma}\left(\frac{D\psi_\Gamma}{Dt} + \psi_\Gamma\operatorname{div}_\Gamma u + \operatorname{div}_\Gamma j_\Gamma - g_\Gamma + g_{\Gamma\to\Omega}\right)d\Gamma = 0.$$

Since V is arbitary, we obtain the *surface differential balance equation*

$$D\psi_\Gamma/Dt + \psi_\Gamma\operatorname{div}_\Gamma u + \operatorname{div}_\Gamma j_\Gamma = g_\Gamma - g_{\Gamma\to\Omega} \quad \text{on } \Gamma.$$

Consequently, we have shown that if the quantities ψ, ψ_Γ, j, j_Γ, g, and g_Γ satisfy the integral balance equations (1.10) and (1.13) and the regularity conditions (1.11) and (1.14), then these quantities also satisfy the *differential balance equations*

$$\partial_t\psi + \operatorname{div}(\psi u + j) = g \qquad \text{in } \Omega\setminus\Gamma, \tag{1.16a}$$

$$[\![j]\!]\cdot\nu_\Gamma = g_{\Gamma\to\Omega} \qquad \text{on } \Gamma, \tag{1.16b}$$

$$D\psi_\Gamma/Dt + \psi_\Gamma\operatorname{div}_\Gamma u + \operatorname{div}_\Gamma j_\Gamma = g_\Gamma - g_{\Gamma\to\Omega} \qquad \text{on } \Gamma. \tag{1.16c}$$

1.2.2. Balance of mass. In order to derive the balance equations for the mass from the differential balances (1.16), we let $\psi = \rho$ and $\psi_\Gamma = \rho_\Gamma$ and obtain the *continuity equation*

$$\partial_t\rho + \operatorname{div}(\rho u + j) = g \quad \text{in } \Omega\setminus\Gamma.$$

In this thesis we study the *incompressible* case $\rho =$ constant and $j(\rho) = 0$. We also neglect interface mass and therefore let $\rho_\Gamma = 0$ and $j_\Gamma(\rho) = 0$. Assuming that Ω represents a closed system, we further neglect sources of mass; that is, $g(\rho) = 0$ and $g_\Gamma(\rho) = 0$. Hence

$$\operatorname{div} u = 0 \quad \text{in } \Omega\setminus\Gamma.$$

1.2.3. Balance of momentum. The momentum density $\psi = \rho u$ is not scalar and thus we can not apply (1.16) directly. Instead, we consider the scalar densities $\psi(e) := \psi\cdot e = \rho e\cdot u$ for suitable vector fields e. This well-known approach was modified by Scriven [Scr60] for deriving the Boussinesq-Scriven law. For every constant vector e we have $\operatorname{div}(\psi(e)u) = \partial_i((\rho e_j u_j)u_i) = e_j\partial_i(\rho u_i u_j) = e\cdot\operatorname{div}(\rho u\otimes u)$. We neglect external forces such as gravity and therefore let $g(e) = 0$. It will suffice to assume that

$$u_\pm \in BUC^{(0,1)}(\operatorname{gr}\Omega_\pm)^n \cap H_1^{(1,0)}(J\times\Omega_\pm)^n \cap H_1^{(0,2)}(\operatorname{gr}\Omega_\pm)^n. \tag{1.17}$$

Then $\psi(e)$ belongs to $H_1^1(J\times\Omega)$ and (1.16a) implies

$$e\cdot\partial_t(\rho u) + e\cdot\operatorname{div}(\rho u\otimes u) = \operatorname{div} j(e) \quad \text{in } \Omega\setminus\Gamma,$$

for every constant vector e. We shall prescribe a flux of the form $j(e) = j_\pi(e) + j_S(e) = e\cdot T$ that consists of a pressure part $j_\pi(e)$ and a viscous part $j_S(e)$. The quantity T is the *stress tensor*.

Consider a control volume V in Ω with $V(t) \subset \Omega\setminus\Gamma(t)$. Then either $\overline{V}(t) \subset \Omega_+(t)$ or $\overline{V}(t) \subset \Omega_-(t)$. One force acting on V is the *pressure force* $f_\pi = -\int_{\partial V}\pi\nu_{\partial V}\,d(\partial V)$ with *pressure*

$$\pi_\pm \in H_1^{(0,1)}(\operatorname{gr}\Omega_\pm). \tag{1.18}$$

Here $\pi\nu_{\partial V}\, d(\partial V)$ can be understood as the pressure force which acts on a surface element perpendicular to $\nu_{\partial V}$. For every constant vector e, the divergence theorem yields

$$e \cdot f_\pi = -\int_{\partial V} \pi e \cdot \nu_{\partial V}\, d(\partial V) = -\int_V \operatorname{div}(\pi e)\, dx.$$

In view of the second identity, we let $j_\pi(e) := -\pi e$. The tensor $J_\pi := -\pi I$ yields the desired linear relations $j_\pi(e) = e \cdot J_\pi$ and $\operatorname{div} j_\pi(e) = e \cdot \operatorname{div} J_\pi$ with respect to e.

Due to friction on ∂V, there is another force acting on V, the stress $f_S = -\int_{\partial V} S\nu_{\partial V}\, d(\partial V)$ with the *viscous stress tensor* S. We assume that both $\Omega_\pm$ consist of *Newtonian fluids*, which means that the viscous stress tensor depends linearly on the *rate-of-strain tensor*

$$D := D(u) := \operatorname{sym}[\nabla u] = 2^{-1}(\nabla u + [\nabla u]^\top).$$

Therefore we define the viscous stress tensor

$$S := S(u) := 2\mu D(u) = \mu(\nabla u + [\nabla u]^\top),$$

where the number $\mu_\pm$ is the *shear viscosity* of the fluid $\Omega_\pm$. If e is constant, then

$$e \cdot f_S = \int_{\partial V} e \cdot S\nu_{\partial V}\, d(\partial V) = \int_V \operatorname{div}(Se)\, dx.$$

Thus we let $j_S(e) := Se$ and $J_S := S$ and hence $\operatorname{div} j_S(e) = \operatorname{div}(e \cdot J_S) = e \cdot \operatorname{div} S$. We call

$$T := T(u, \pi) := J_S + J_\pi = 2\mu D(u) - \pi I \tag{1.19}$$

the *(total) stress tensor*. Since the vector e is arbitrary, we obtain the *differential momentum balance*

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u - T) = 0 \quad \text{in } \Omega \setminus \Gamma. \tag{1.20}$$

1.2.4. Interface momentum balance. We recall that the momentum density $\psi = \rho u$ induces scalar densities $\psi(e) = e \cdot \psi$, whose bulk fluxes are $j(e) = e \cdot T$ for a given vector field e. The latter is allowed to have a possibly non-tangential restriction $e|_\Gamma = e^\alpha \tau_\alpha + e_\nu \nu_\Gamma$ on Γ. Here we differ from Scriven [Scr60, p. 101] and Aris [Ari89, p. 238], who only considered vector fields with vanishing covariant derivatives, which do not cover constant vectors unless Γ is flat. Since the interface has vanishing mass density, we have $\psi_\Gamma := \rho_\Gamma u|_\Gamma = 0$ and then equations (1.16b) and (1.16c) yield

$$-e \cdot [\![T]\!]\nu_\Gamma = \operatorname{div}_\Gamma(j_\Gamma(e)) \quad \text{on } \Gamma \tag{1.21}$$

for every vector field e. Here the interface flux $j_\Gamma(e) = j_{\Gamma,\sigma}(e) + j_{\Gamma,S}(e)$ will consist of a surface tension part $j_{\Gamma,\sigma}(e)$ and a viscous part $j_{\Gamma,S}(e)$. We first let $j_{\Gamma,\sigma}(e) := e \cdot (\sigma P_\Gamma)$, where σ is the constant surface tension coefficient. If $e = e_0 \in \mathbb{R}^n$ is constant, then $\operatorname{div}_\Gamma(j_{\Gamma,\sigma}(e)) = e \cdot \operatorname{div}_\Gamma(\sigma P_\Gamma)$.

We define the viscous flux $j_{\Gamma,S}(e) := e \cdot S_\Gamma$ with viscous surface stress tensor S_Γ. Following Scriven [Scr60], we regard Γ as an $(n-1)$-dimensional fluid with *rate-of-strain* tensor

$$D_\Gamma := D_\Gamma(u) := 2^{-1} Dg_{\alpha\beta}/Dt\, \tau^\alpha \otimes \tau^\beta.$$

Similar to Sekomb and Skalak [SS82], we can derive the usual expression of D_Γ in Euclidean coordinates. For every parametrization $y \mapsto \varphi(y)$ of $\Gamma(t)$, the map $y \mapsto \Phi(t+s, t, \varphi(y))$ is a parametrization of $\Gamma(t+s)$. Thus the tangent vectors of $\Gamma(t+s)$ are related to those of $\Gamma(t)$ by

$$\tau_i(t+s, \Phi(t+s,t,x)) = \partial_{x_i}\Phi(t+s,t,x) = \partial_x\Phi(t+s,t,x)\tau_i(t,x).$$

Having in mind that $\nabla_\Gamma u := \tau_\Gamma^j \otimes \partial_j u$, we obtain

$$(D/Dt)\tau_i(t,x) = (d/ds)[\partial_x\Phi(t+s,t,x)]|_{s=0}\tau_i(t,x) = [\nabla_\Gamma u(t,x)]^\top \tau_i(t,x).$$

Then the relations $g_{ij} = \tau_i \cdot \tau_j$ and $P_\Gamma = \tau_i \otimes \tau^i = I - \nu_\Gamma \otimes \nu_\Gamma$ yield

$$D_\Gamma = \operatorname{sym}(P_\Gamma[\nabla_\Gamma u]P_\Gamma) = 2^{-1}P_\Gamma(\nabla_\Gamma u + [\nabla_\Gamma u]^\top)P_\Gamma.$$

Scriven [Scr60] proposed to consider *Newtonian surface fluids* for which S_Γ depends linearly on D_Γ. Hence we define the *viscous surface stress tensor*

$$S_\Gamma := S_\Gamma(u) := (\lambda_s - \mu_s)(\operatorname{div}_\Gamma u)P_\Gamma + 2\mu_s D_\Gamma,$$

where λ_s and μ_s are constant real numbers. The *(total) surface stress tensor* is defined by

$$T_\Gamma := T_\Gamma(u) := \sigma P_\Gamma + S_\Gamma(u) = \sigma P_\Gamma + (\lambda_s - \mu_s)(\operatorname{div}_\Gamma u)P_\Gamma + 2\mu_s D_\Gamma. \tag{1.22}$$

Then the flux $j_\Gamma(e) = e \cdot T_\Gamma$ satisfies $\operatorname{div}_\Gamma(j_\Gamma(e)) = e \cdot \operatorname{div}_\Gamma T_\Gamma + T_\Gamma : \nabla_\Gamma e$. In Section 1.3 we will see that $j_\Gamma(e)$ belongs to the class $H_1^{(0,1)}(\operatorname{gr}\Gamma; T\Gamma)$, provided that

$$v = P_\Gamma u|_\Gamma \in H_1^{(0,2)}(\operatorname{gr}\Gamma; T\Gamma), \quad w = \nu_\Gamma \cdot u|_\Gamma \in H_1^{(0,1)}(\operatorname{gr}\Gamma), \quad \Gamma(t) \in C^3. \tag{1.23}$$

By choosing the constant vectors $e = e_i$ in (1.21), we obtain the *interface momentum balance*

$$-[\![T]\!]\nu_\Gamma = \operatorname{div}_\Gamma T_\Gamma \quad \text{on } \Gamma. \tag{1.24}$$

By imposing the no-slip condition $u|_{\partial\Omega} = 0$, the derivation of the model (N) is complete.

1.3. Properties of the model

Similar to [BP10], we will decompose the interface momentum balance (1.24) into tangential and normal parts and derive an energy identity in arbitrary control volumes; but, in contrast to [BP10], we employ covariant derivatives.

Let each $\Gamma(t)$ be of class C^3. According to Einstein's summation convention, we always sum over repeated greek indices $\alpha, \beta, \ldots \in \{1, \ldots, n-1\}$, whereas latin indices $i, j, \ldots \in \{1, \ldots, n-1\}$ denote free indices. We will use the Weingarten tensor $L = l_{\alpha\beta}\tau^\alpha \otimes \tau^\beta = l^{\alpha\beta}\tau_\alpha \otimes \tau_\beta$, the mean curvature $H = g^{\alpha\beta}l_{\alpha\beta}$ and the Cristoffel symbols $\Lambda_{ij,k} = \partial_i\tau_j \cdot \tau_k$ and $\Lambda_{ij}^k = \partial_i\tau_j \cdot \tau^k = g^{kl}\Lambda_{ij,l}$. Then we define *covariant derivatives* as follows: For a tangential vector field $v \in C^1(\Gamma; T\Gamma)$ and a co-vector field $\omega \in C^1(\Gamma; T^*\Gamma)$, we let

$$v_{;k} = \widetilde{\nabla}_k v = P_\Gamma \partial_k v = v^\alpha{}_{;k}\,\tau_\alpha = (\partial_k v^\alpha + \Lambda_{k\beta}^\alpha v^\beta)\tau_\alpha,$$
$$\omega_{;k} = \widetilde{\nabla}_k \omega = P_\Gamma \partial_k \omega = \omega_{\alpha;k}\,\tau^\alpha = (\partial_k \omega_\alpha - \Lambda_{k\alpha}^\beta \omega_\beta)\tau^\alpha;$$

for a possibly non-tangential vector field $u = v + w\nu_\Gamma \in C^1(\Gamma; \mathbb{R}^n)$, we let

$$u_{;k} = \widetilde{\nabla}_k u = P_\Gamma \partial_k u = v^\alpha{}_{;k}\,\tau_\alpha + w\partial_k\nu = (\partial_k v^\alpha + \Lambda_{k\beta}^\alpha v^\beta - w l_{k\beta} g^{\beta\alpha})\tau_\alpha;$$

and for second-order tensor fields $T \in C^1(\Gamma; T\Gamma \otimes T\Gamma)$ and $D \in C^1(\Gamma; T^*\Gamma \otimes T^*\Gamma)$, we let

$$T_{;k} = \widetilde{\nabla}_k T = T^{\alpha\beta}{}_{;k}\,\tau_\alpha \otimes \tau_\beta = (\partial_k T^{\alpha\beta} + \Lambda_{k\gamma}^\alpha T^{\gamma\beta} + \Lambda_{k\gamma}^\beta T^{\alpha\gamma})\tau_\alpha \otimes \tau_\beta,$$
$$D_{;k} = \widetilde{\nabla}_k D = D_{\alpha\beta;k}\,\tau^\alpha \otimes \tau^\beta = (\partial_k D_{\alpha\beta} - \Lambda_{k\alpha}^\gamma D_{\gamma\beta} - \Lambda_{k\beta}^\gamma D_{\alpha\gamma})\tau^\alpha \otimes \tau^\beta.$$

The usage of covariant derivatives (i) ensures that the derivative of a section of some bundle is again a section of that bundle, (ii) provides the simple relations

$$g_{ij;k} = 0, \quad g^{ij}{}_{;k} = 0, \tag{1.25}$$

and (iii) provides the general product rule

$$(T^{i_1\cdots}{}_{j_1\ldots}\,S^{k_1\cdots}{}_{l_1\ldots})_{;m} = T^{i_1\cdots}{}_{j_1\ldots;m}\,S^{k_1\cdots}{}_{l_1\ldots} + T^{i_1\cdots}{}_{j_1\ldots}\,S^{k_1\cdots}{}_{l_1\ldots;m}. \tag{1.26}$$

Some relations to surface differential operators are given by

$$\operatorname{div}_\Gamma(v^\alpha\tau_\alpha + w\nu_\Gamma) = v^\alpha{}_{;\alpha} - wH, \tag{1.27a}$$

$$D_\Gamma(v^\alpha\tau_\alpha + w\nu_\Gamma) = 2^{-1}\tau^\alpha \otimes \tau^\beta(v_{\alpha;\beta} + v_{\beta;\alpha}) - wL, \tag{1.27b}$$

$$\operatorname{div}_\Gamma(T^{\alpha\beta}\tau_\alpha \otimes \tau_\beta) = T^{\alpha\beta}{}_{;\alpha}\tau_\beta + T^{\alpha\beta}l_{\alpha\beta}\nu_\Gamma \quad (\text{if } T^{\alpha\beta} = T^{\beta\alpha}), \tag{1.27c}$$

$$l_{ij;k} = l_{ik;j} = l_{jk;i}. \tag{1.27d}$$

Second-order covariant derivatives are denoted by $\widetilde{\nabla}_k\widetilde{\nabla}_l = (\cdot)_{;lk}$. The covariant derivatives of tangential vector fields do not necessarily commute, but satisfy the relations

$$v^i{}_{;jk} - v^i{}_{;kj} = R^i{}_{\alpha jk}v^\alpha, \quad v_{i;jk} - v_{i;kj} = -v_\alpha R^\alpha{}_{ijk}, \tag{1.28a}$$

$$R^i{}_{jkl} = g^{i\alpha}R_{\alpha jkl}, \qquad R_{ijkl} = l_{ik}l_{jl} - l_{il}l_{jk}. \tag{1.28b}$$

The Laplace-Beltrami operators for $\psi \in C^2(\Gamma)$ and $u = v^\alpha\tau_\alpha + w\nu_\Gamma \in C^2(\Gamma)^n$ are given by

$$\begin{aligned}\Delta_\Gamma\psi &= \operatorname{div}_\Gamma \nabla_\Gamma\psi \quad = g^{\alpha\beta}(\partial_\alpha\partial_\beta\psi - \Lambda^\gamma_{\alpha\beta}\partial_\gamma\psi),\\ \widetilde{\Delta}_\Gamma u &= g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta u = (g^{\alpha\beta}v^\gamma{}_{;\alpha\beta} - \partial_\alpha w l^{\alpha\gamma} - wH_{;\delta}g^{\delta\gamma})\tau_\gamma.\end{aligned}$$

We refer to Appendix A.4 for more information on these identities.

1.3.1. Decomposition of the interface balance. We decompose (1.24) into its tangential and normal parts. From (1.22), (1.27a), and (1.27b), we infer that $T_\Gamma = T_\Gamma^{\alpha\beta}\tau_\alpha \otimes \tau_\beta$ has the components

$$T_\Gamma^{\alpha\beta} = \sigma g^{\alpha\beta} + (\lambda_s - \mu_s)(v^\gamma{}_{;\gamma} - Hw)g^{\alpha\beta} + \mu_s g^{\alpha\gamma}g^{\beta\delta}(v_{\gamma;\delta} + v_{\delta;\gamma} - 2wl_{\gamma\delta}). \tag{1.29}$$

With equations (1.25), (1.26), and (1.27c), we decompose $\operatorname{div}_\Gamma T_\Gamma = T_\Gamma^{\alpha\beta}{}_{;\alpha}\tau_\beta + T_\Gamma^{\alpha\beta}l_{\alpha\beta}\nu_\Gamma$ as

$$\begin{aligned}&\Big[\mu_s g^{\alpha\gamma}g^{\beta\delta}(v_{\delta;\gamma\alpha} + v_{\gamma;\delta\alpha} - 2wl_{\gamma\delta;\alpha} - 2w_{;\alpha}l_{\gamma\delta}) + (\lambda_s - \mu_s)(v^\gamma{}_{;\gamma\alpha} - H_{;\alpha}w - Hw_{;\alpha})g^{\alpha\beta}\Big]\,\tau_\beta\\ &\quad + \Big[\sigma H + (\lambda_s - \mu_s)(v^\gamma{}_{;\gamma} - Hw)H + \mu_s l_{\alpha\beta}g^{\alpha\gamma}g^{\beta\delta}(v_{\gamma;\delta} + v_{\delta;\gamma} - 2wl_{\gamma\delta})\Big]\,\nu_\Gamma.\end{aligned}$$

Let us rewrite this equation in vector notation. We have $g^{\alpha\gamma}g^{\beta\delta}v_{\delta;\gamma\alpha}\tau_\beta = \widetilde{\Delta}_\Gamma v$, and with (1.28a) and (1.28b), we obtain $g^{\alpha\gamma}g^{\beta\delta}v_{\gamma;\delta\alpha} = \nabla_\Gamma \operatorname{div}_\Gamma v$. Identity (1.27d) yields $g^{\alpha\gamma}g^{\beta\delta}l_{\gamma\delta;\alpha}\tau_\beta = \nabla_\Gamma H$. We proceed in a similar way with the remaining terms and obtain

$$\begin{aligned}\operatorname{div}_\Gamma T_\Gamma = {}&\mu_s\widetilde{\Delta}_\Gamma v + \lambda_s\nabla_\Gamma \operatorname{div}_\Gamma v\\ &- (\lambda_s + \mu_s)\,w\nabla_\Gamma H + [(\mu_s - \lambda_s)H - 2\mu_s L]\nabla_\Gamma w\\ &+ [(\lambda_s - \mu_s)\operatorname{div}_\Gamma v\, H + 2\mu_s\, L : D_\Gamma(v)]\nu_\Gamma\\ &+ [\sigma H - (\lambda_s - \mu_s)\,w\,H^2 - 2\mu_s\,w\operatorname{tr}(L^2)]\nu_\Gamma.\end{aligned} \tag{1.30}$$

We conclude that the interface momentum balance (1.24) has the tangential part

$$\begin{aligned}-P_\Gamma[\![T]\!]\nu_\Gamma &= -[\![\mu]\!][\nabla_\Gamma v]\nu_\Gamma - [\![\mu]\!]\nabla_\Gamma w - [\![\mu\partial_\nu v]\!]\\ &= \mu_s\widetilde{\Delta}_\Gamma v + \lambda_s\nabla_\Gamma \operatorname{div}_\Gamma v - (\lambda_s + \mu_s)w\nabla_\Gamma H + [(\mu_s - \lambda_s)H - 2\mu_s L]\nabla_\Gamma w,\end{aligned} \tag{1.31a}$$

and the normal part

$$\begin{aligned}-\nu_\Gamma \cdot [\![T]\!]\nu_\Gamma &= -2[\![\mu\partial_\nu w]\!] + [\![\pi]\!]\\ &= \sigma H + (\lambda_s - \mu_s)\operatorname{div}_\Gamma u\, H + 2\mu_s\, D_\Gamma : L.\end{aligned} \tag{1.31b}$$

1.3.2. Energy identity. We consider the kinetic energy $\int_V 2^{-1}\rho|u|^2\,dx$ of a control volume V in Ω. By applying the transport theorem, the divergence theorem, the identity $\operatorname{div} u = 0$, and the differential momentum balance (1.20), we obtain the *kinetic energy balance*

$$\frac{d}{dt}\int_V \frac{\rho}{2}|u|^2\,dx = -\int_V 2\mu D : D\,dx + \int_{\partial V} Tu \cdot \nu_{\partial V}\,d(\partial V) - \int_{V\cap\Gamma}[\![Tu]\!]\cdot\nu_\Gamma\,d\Gamma. \tag{1.32}$$

In view of the integral balance (1.10), we see that the scalar quantity $\psi = 2^{-1}\rho|u|^2$ has the bulk source density $g = -2\mu D : D$, the bulk flux $j = -Tu$, and the interface source density $g_{\Gamma\to\Omega} = -[\![Tu]\!]\cdot\nu_\Gamma$. The interface momentum balance (1.24) and identity (A.19) imply

$$\begin{aligned}-[\![Tu]\!]\cdot\nu_\Gamma &= u\cdot\operatorname{div}_\Gamma T_\Gamma = \operatorname{div}_\Gamma(T_\Gamma u) - T_\Gamma : D_\Gamma\\ &= \operatorname{div}_\Gamma(T_\Gamma u) - \sigma\operatorname{div}_\Gamma u - (\lambda_s - \mu_s)(\operatorname{div}_\Gamma u)^2 - 2\mu_s D_\Gamma : D_\Gamma.\end{aligned}$$

Thus, the surface transport theorem and the surface divergence theorem yield the *energy identity*

$$
\begin{aligned}
&\frac{d}{dt}\left(\int_V \frac{\rho}{2}|u|^2\,dx + \int_{V\cap\Gamma} \sigma\,d\Gamma\right) \\
&\quad = -\int_V 2\mu D : D\,dx - \int_{V\cap\Gamma} \big((\lambda_s-\mu_s)(\operatorname{div}_\Gamma u)^2 + 2\mu_s D_\Gamma : D_\Gamma\big)\,d\Gamma \\
&\qquad + \int_{\partial V} Tu\cdot\nu_{\partial V}\,d(\partial V) + \int_{C=\partial V\cap\Gamma} T_\Gamma u\cdot n_C\,dC.
\end{aligned}
\tag{1.33}
$$

In the special case $V=\Omega$ and imposing the no-slip boundary condition $u|_{\partial\Omega}=0$, we recover the energy identity from [BP10, Theorem 3.1],

$$
\begin{aligned}
&\frac{d}{dt}\left(\int_\Omega \frac{\rho}{2}|u|^2\,dx + \int_\Gamma \sigma\,d\Gamma\right) \\
&\quad = -\int_\Omega 2\mu D : D\,dx - \int_\Gamma \big((\lambda_s-\mu_s)(\operatorname{div}_\Gamma u)^2 + 2\mu_s D_\Gamma : D_\Gamma\big)\,d\Gamma.
\end{aligned}
\tag{1.34}
$$

By comparing (1.33) with the general integral balance (1.15), we see that the energy has the bulk density $\psi = 2^{-1}\rho|u|^2$, the bulk flux $j=-Tu$, the interface density $\psi_\Gamma=\sigma$ and the interface flux $j_\Gamma = -T_\Gamma u = -T_\Gamma v$. Moreover, if $\lambda_s\ge\mu_s\ge 0$, then the bulk source density $g=-2\mu D : D$ and the interface source density $g_\Gamma = -(\lambda_s-\mu_s)(\operatorname{div}_\Gamma u)^2 - 2\mu_s\operatorname{tr}(D_\Gamma^2)$ are non-positive and thus responsible for dissipation.

CHAPTER 2

Linear elliptic transmission problems

In this chapter we investigate the elliptic transmission problem (TP) in both a strong and a weak sense. We restate problem (TP) as the *strong transmission problem*

$$\tag{2.1} \begin{cases} -\operatorname{div}(\mu\nabla u) = f & \text{in } \Omega\setminus\Sigma, \\ \mu\partial_\nu u = g & \text{on } \partial\Omega, \\ [\![\mu\partial_\nu u]\!] = h_1 & \text{on } \Sigma, \\ [\![u]\!] = h_2 & \text{on } \Sigma, \end{cases}$$

considered in a domain Ω that contains a C^1-hypersurface Σ. Here $u\colon \Omega\setminus\Sigma \to \mathbb{K}$ is an unknown scalar field, (f, g, h_1, h_2) are given data, $\mu : \Omega\setminus\Sigma \to (0,\infty)$ is a variable coefficient, and the jump $[\![\cdot]\!]$ was defined on page 16. We also study the *weak transmission problem*

$$\tag{2.2} \begin{cases} \displaystyle\int_\Omega \mu\nabla u\cdot\nabla\phi\,dx = \langle F|\phi\rangle & \text{for all } \phi\in\mathcal{D}(\mathbb{R}^n), \\ [\![u]\!] = h_2 & \text{on } \Sigma, \end{cases}$$

for given data (F, h_2). We will see that (2.2) can be obtained from (2.1) by multiplying the first equation with ϕ and integrating by parts. In the case $\Sigma = \emptyset$ and $\mu_\pm = 1$, problem (2.2) is called the *weak Neumann problem*. Both problems (2.1) and (2.2) can be used to eliminate the pressure and divergence in the more complex linear problem (PL); we adopt this strategy from Köhne, Prüss, and Wilke [KPW13; Wil13]. Both problems were solved in [KPW13] for constant coefficients $\mu_\pm = 1/\rho_\pm$ in a bounded domain Ω and the authors established optimal H^2_p-regularity for (2.1), optimal $\dot H^1_p$-regularity for (2.2), and optimal W^{2+s}_p-regularity for (2.1) under the restriction $(g, h_1, h_2) = 0$. Similar transmission problems are investigated in the forthcoming monograph [PS15].

Our goal is to prove that both (2.1) and (2.2) have *optimal regularity* in the sense that the solution-to-data maps $u\mapsto(f,g,h_1,h_2)$ and $u\mapsto(F,h_2)$ are topological linear isomorphisms between suitable Banach spaces. We impose the following basic assumption on Ω and Σ.

2.1. Assumption. $\Omega\subset\mathbb{R}^n$ $(n\geq 2)$ is a domain with C^1-boundary $\partial\Omega$ and $\Sigma\subset\Omega$ is a closed C^1-hypersurface such that one of the following conditions is satisfied.

- (i) Ω is the whole space $\mathbb{R}^n$ and Σ is empty.
- (ii) Ω is a *bent half-space* $\mathbb{R}^n_\omega = \{(x', x_n)\in\mathbb{R}^n : x_n > \omega(x')\}$ with $\omega\in C^1_c(\mathbb{R}^{n-1})$ and Σ is empty.
- (iii) Ω is the whole space $\mathbb{R}^n$, Σ is a *bent hyperplane* $\Sigma_\omega = \{(x', \omega(x')) : x'\in\mathbb{R}^{n-1}\}$ with $\omega\in C^1_c(\mathbb{R}^{n-1})$, and $\Omega\setminus\Sigma$ consists of the bent half-spaces $\Omega_\pm = \{(x', x_n)\in\mathbb{R}^n : x_n \gtrless \omega(x')\}$.
- (iv) Ω is a bounded domain with C^1-boundary, Σ is compact and possibly empty, and $\Omega\setminus\Sigma$ consists of disjoint open sets $\Omega_\pm$ with $\partial\Omega\subset\partial\Omega_+$ and $\Sigma = \partial\Omega_-$.

We let $\nu_{\partial\Omega}$, $\nu_{\partial\Omega_\pm}$, and ν_Σ denote the exterior unit normal fields on $\partial\Omega$, $\partial\Omega_\pm$, and Σ, and we choose the orientation of Σ such that $\nu_\Sigma = -\nu_{\partial\Omega_+} = \nu_{\partial\Omega_-}$ on Σ.

In order to define suitable solution spaces, we recall that u belongs to $H^1_p(\Omega\setminus\Sigma)$ if and only if its restrictions $u_\pm = u|_{\Omega_\pm}$ belong to $H^1_p(\Omega_\pm)$. Other function spaces on $\Omega\setminus\Sigma$ are defined analogously. For an open subset $G\subset\mathbb{R}^n$ we consider the vector space

$$\dot{\mathcal{H}}^k_p(G) := \{u\in H^k_{1,\mathrm{loc}}(\overline{G}) : \nabla^k u\in L_p(G)\},\qquad \text{for } k\in\mathbb{N}_0,\ p\in[1,\infty).$$

This space is semi-normed with respect to $\|\nabla^k\cdot\|_p$. We call a function $u\colon \Omega\setminus\Sigma \to \mathbb{K}$ a *strong solution* to (2.1), if u belongs to the space $(\dot{\mathcal{H}}^2_p \cap \dot{\mathcal{H}}^1_p)(\Omega\setminus\Sigma) := \dot{\mathcal{H}}^2_p(\Omega\setminus\Sigma)\cap\dot{\mathcal{H}}^1_p(\Omega\setminus\Sigma)$ and if (2.1) is satisfied in the sense of distributions. In particular, the first equation is understood in $\mathcal{D}'(\Omega\setminus\Sigma)$; that is,

$$-\int_\Omega \operatorname{div}(\mu\nabla u)\phi\,dx = \int_\Omega f\phi\,dx \quad \text{ for all } \phi\in\mathcal{D}(\Omega\setminus\Sigma),$$

where $\mathcal{D}(\Omega\setminus\Sigma)$ denotes the space of smooth functions in Ω that vanish near $\partial\Omega\cup\Sigma$. Obviously, every constant function is a strong solution of (2.1) for vanishing data; hence, we shall choose a semi-norm on $(\dot{\mathcal{H}}^2_p \cap \dot{\mathcal{H}}^1_p)(\Omega\setminus\Sigma)$ whose null-space consists of all constant functions. Such a semi-norm is given by

$$\|u\|_{\mathbb{E}^0} := \|\nabla^2 u\|_{L_p(\Omega)} + \|\nabla u\|_{L_p(\Omega)} + \|[\![u]\!]\|_{L_p(\Sigma')},$$

where $\Sigma'\subset\Sigma$ is a bounded open subset with C^1-boundary that has positive measure, provided that $\Sigma\neq\emptyset$. We will prove that strong solutions are uniquely determined within the space

$$\mathbb{E}^0 := \Big((\dot{\mathcal{H}}^2_p \cap \dot{\mathcal{H}}^1_p)(\Omega\setminus\Sigma), \|\cdot\|_{\mathbb{E}^0}\Big)\Big/\mathbb{K}.$$

We will also study strong solutions within spaces of lower or higher regularity

$$\mathbb{E}^k := \left(\bigcap_{j=1}^{k+2}\dot{\mathcal{H}}^j_p(\Omega\setminus\Sigma), \|\cdot\|_{\mathbb{E}^k}\right)\Big/\mathbb{K}, \quad \|u\|_{\mathbb{E}^k} := \sum_{j=1}^{k+2}\|\nabla^j u\|_p + \|[\![u]\!]\|_{L_p(\Sigma')}, \quad k\in\mathbb{N}_0\cup\{-1\}.$$

Next, we derive suitable conditions on the data (f,g,h_1,h_2) that are necessary for the existence of a strong solution $u\in\mathbb{E}^0$ of (2.1). We assume in addition that $\partial\Omega$ and Σ are of class C^{2-} and that μ belongs to $W^1_\infty(\Omega\setminus\Sigma)$; that is, $\mu_\pm$ are weakly differentiable in $\Omega_\pm$ and both $\mu_\pm$ and $\nabla\mu_\pm$ belong to $L_\infty(\Omega_\pm)$; thus, $\mu_\pm$ are Lipschitz functions. Given a strong solution $u\in\mathbb{E}^0$ of problem (2.1), the corresponding data (f,g,h_1,h_2) satisfy the regularity conditions

$$(f,g,h_1,h_2)\in L_p(\Omega)\times W^{1-1/p}_p(\partial\Omega)\times W^{1-1/p}_p(\Sigma)\times\Big(\dot{\mathcal{W}}^{2-1/p}_p(\Sigma)\cap\dot{\mathcal{W}}^{1-1/p}_p(\Sigma)\cap L_p(\Sigma')\Big).$$

Here the semi-normed Sobolev-Slobodeckiĭ spaces $(\dot{\mathcal{W}}^{k+s}_p(\Sigma), [\![\nabla^k\cdot]\!]_{W^s_p(\Sigma)})$ are defined by

$$\dot{\mathcal{W}}^{k+s}_p(\Sigma) := \{u\in H^k_{p,\mathrm{loc}}(\Sigma) : [\![\nabla^k u]\!]_{W^s_p(\Sigma)}<\infty\} \quad \text{ for } k\in\mathbb{N}_0,\ s\in(0,1),\ p\in[1,\infty),$$

and the semi-norm $[\![\cdot]\!]_{W^s_p(\Sigma)}$ is defined intrinsically by

$$[\![v]\!]_{W^s_p(\Sigma)} := \left(\int_\Sigma\int_\Sigma \frac{|v(x)-v(y)|^p}{\operatorname{dist}_\Sigma(x,y)^{n+sp}}\,d\Sigma(x)\,d\Sigma(y)\right)^{1/p}.$$

Bothe and Prüss [BP07] noticed that another joint regularity condition for (f,g,h_1) is necessary. Indeed, let $\phi\in\mathcal{D}(\mathbb{R}^n)$ be a test function. Then an integration by parts yields

$$\int_\Omega \mu\nabla u\cdot\nabla\phi\,dx = -\int_\Omega\operatorname{div}(\mu\nabla u)\phi\,dx + \int_{\partial\Omega}\mu\partial_\nu u\,\phi\,d(\partial\Omega) - \int_\Sigma [\![\mu\partial_\nu u]\!]\phi\,d\Sigma.$$

The right-hand side can be expressed in terms of the data as a functional

$$\big\langle F_{(f,g,h_1)}\big|\phi\big\rangle := \int_\Omega f\phi\,dx + \int_{\partial\Omega} g\phi\,d(\partial\Omega) - \int_\Sigma h_1\phi\,d\Sigma = \int_\Omega\mu\nabla u\cdot\nabla\phi\,dx. \tag{2.3}$$

Thus the triple (f,g,h_1) induces a continuous linear functional $\phi\mapsto\big\langle F_{(f,g,h_1)}\big|\phi\big\rangle$ on the normed vector space $(\mathcal{D}(\mathbb{R}^n), \|\nabla\cdot\|_{L_{p'}(\Omega)})$, where $1/p+1/p'=1$. We can also define such a functional by

$$\langle F_{\mu\nabla u}|\phi\rangle := \int_\Omega \mu\nabla u\cdot\nabla\phi\,dx \quad \text{ for } \phi\in\mathcal{D}(\mathbb{R}^n).$$

The completion of $(\mathcal{D}(\mathbb{R}^n), \|\nabla\cdot\|_{L_{p'}(\Omega)})$ is the homogeneous Sobolev space

$$\dot{H}^1_{p'}(\Omega) := \dot{\mathcal{H}}^1_{p'}(\Omega)/\mathbb{K}, \quad \|\phi\|_{\dot{H}^1_{p'}(\Omega)} := \|\nabla\phi\|_{L_{p'}(\Omega)},$$

considered modulo constant functions [Gal11; Sob63]. Its topological dual space is denoted by

$$\hat{H}^{-1}_p(\Omega) := \dot{H}^1_{p'}(\Omega)^*, \quad \|F\|_{\hat{H}^{-1}_p(\Omega)} = \sup_{0\neq\phi\in\dot{H}^1_{p'}(\Omega)} \frac{|\langle F|\phi\rangle|}{\|\nabla\phi\|_{L_{p'}(\Omega)}}.$$

The data (f, g, h_1, h_2) must therefore satisfy the joint regularity condition

$$F_{(f,g,h_1)} \in \hat{H}^{-1}_p(\Omega). \tag{2.4}$$

If Ω is bounded, then this regularity condition reduces to the compatibility condtion

$$\int_\Omega f\,dx + \int_{\partial\Omega} g\,d(\partial\Omega) - \int_\Sigma h_1\,d\Sigma = 0. \tag{2.5}$$

Indeed, by choosing $\phi = 1$, we see that (2.4) implies (2.5). For proving the converse implication, we consider a test function $\phi \in \mathcal{D}(\mathbb{R}^n)$, we let $\langle\phi\rangle_\Omega := |\Omega|^{-1}\int_\Omega \phi\,dx$ denote the mean value of ϕ, and we recall the *Poincaré-Wirtinger inequality*

$$\|\phi - \langle\phi\rangle_\Omega\|_{p'} \le C(\Omega, p')\|\nabla\phi\|_{p'} \quad \text{for } \phi \in H^1_{p'}(\Omega). \tag{2.6}$$

Then, for a given tuple $(f, g, h_1) \in L_p(\Omega) \times L_p(\partial\Omega) \times L_p(\Sigma)$ satisfying (2.5), inequality (2.6) yields

$$|\langle F_{(f,g,h_1)}|\phi\rangle| = |\langle F_{(f,g,h_1)}|\phi - \langle\phi\rangle_\Omega\rangle| \le C\|\nabla\phi\|_{p'};$$

that is, $F_{(f,g,h_1)}$ belongs to $\hat{H}^{-1}_p(\Omega)$. In this sense, (2.4) and (2.5) are equivalent, if Ω is bounded.

For a strong solution of class $\mathbb{E}^k$ ($k \in \mathbb{N}_0$), the corresponding data belong to the spaces

$$\mathbb{F}^k_{\mathrm{cc}} := \left\{(f, g, h_1, h_2) \in \mathbb{F}^k : F_{(f,g,h_1)} \in \hat{H}^{-1}_p(\Omega)\right\},$$
$$\mathbb{F}^k := H^k_p(\Omega\setminus\Sigma) \times W^{k+1-1/p}_p(\partial\Omega) \times W^{k+1-1/p}_p(\Sigma) \times \left(\bigcap_{j=0}^{k+1} \dot{\mathcal{W}}^{j+1-1/p}_p(\Sigma) \cap L_p(\Sigma')\right).$$

Now we are ready to state the main result for the strong transmission problem (2.1).

2.2. Theorem (Optimal H^{k+2}_p-regularity for (2.1))**.** *Let Ω and Σ satisfy Assumption 2.1, let $k \in \mathbb{N}_0$, suppose that $\partial\Omega$ and Σ are of class C^{k+2-}, and let $p \in (1,\infty)$.*

If Ω is bounded, then for given $\mu \in W^{k+1}_\infty(\Omega\setminus\Sigma)$ with $\mu_0 \le \mu \le \mu_0^{-1}$, the solution-to-data map

$$u \mapsto (-\operatorname{div}(\mu\nabla u), \mu\partial_\nu u, [\![\mu\partial_\nu u]\!], [\![u]\!]), \quad \mathbb{E}^k \to \mathbb{F}^k_{cc} \tag{2.7}$$

is a topological linear isomorphism.

If Ω is unbounded, then for given $\mu_0 \in (0,1]$ there exists $\eta > 0$ such that if

(i) $\omega \in C^{k+2-}_c(\mathbb{R}^{n-1})$ with $\|\nabla\omega\|_\infty \le \eta$ in case $\Omega = \mathbb{R}^n_\omega$ or $\Sigma = \Sigma_\omega$,

(ii) $\mu \in W^{k+1}_\infty(\Omega\setminus\Sigma)$ with $\mu_0 \le \mu \le \mu_0^{-1}$ and $\|\mu_\pm - \mu^*_\pm\|_\infty \le \eta$ for some $\mu^*_\pm \in [\mu_0, \mu_0^{-1}]$,

then the map (2.7) is a topological linear isomorphism.

In order to prove Theorem 2.2, we first establish a corresponding result for the regularized operator $\lambda - \operatorname{div}(\mu\nabla\cdot)$ with some sufficiently large $\lambda > 0$ (see Theorem 2.18) by means of a localization procedure as in [LSU68; AHS94; DHP03; KPW13]. For the case $\lambda = 0$ we employ a spectral theoretic argument as in [KPW13; Wil13] and the localization procedure of Simader and Sohr [SS92]. Our main result on the weak transmission problem (2.2) is the following.

2.3. Theorem (Optimal H^1_p-regularity for (2.2))**.** *Let Ω and Σ satisfy Assumption 2.1 and let $p \in (1,\infty)$.*

If Ω is bounded, then for given $\mu_\pm \in C(\overline{\Omega}_\pm)$ with $\inf \mu_\pm > 0$, *the solution-to-data map*

$$u \mapsto (F_{\mu\nabla u}, [\![u]\!])\,, \quad \mathbb{E}^{-1} \to \mathbb{F}^{-1}_{cc} = \hat{H}^{-1}_p(\Omega) \times \left(\dot{\mathcal{W}}^{1-1/p}_p(\Sigma) \cap L_p(\Sigma')\right), \tag{2.8}$$

is a topological linear isomorphism.

If Ω is unbounded, then for given $\mu_0 \in (0,1]$ there exists $\eta > 0$ such that if

(i) $\omega \in C^1_c(\mathbb{R}^{n-1})$ with $\|\nabla\omega\|_\infty \le \eta$ in case $\Omega = \mathbb{R}^n_\omega$ or $\Sigma = \Sigma_\omega$,

(ii) $\mu_\pm \in L_\infty(\Omega_\pm)$ with $\mu_0 \le \mu_\pm \le \mu_0^{-1}$ and $\|\mu_\pm - \mu^*_\pm\|_\infty \le \eta$ for some $\mu^*_\pm \in [\mu_0, \mu_0^{-1}]$,

then the map (2.8) *is a topological linear isomorphism.*

2.1. The strong transmission problem for $\lambda - \operatorname{div}(\mu\nabla\cdot)$

We consider the linear operator

$$A_\lambda \colon u \mapsto (\lambda u - \operatorname{div}(\mu\nabla u), \mu\partial_\nu u, [\![\mu\partial_\nu u]\!], [\![u]\!]) \qquad \text{for } \lambda \in \mathbb{C} \setminus \mathbb{R}_-,$$

which is induced by the strong transmission problem

$$\begin{cases} \lambda u - \operatorname{div}(\mu\nabla u) = f & \text{in } \Omega \setminus \Sigma, \\ \mu\partial_\nu u = g & \text{on } \partial\Omega, \\ [\![\mu\partial_\nu u]\!] = h_1 & \text{on } \Sigma, \\ [\![u]\!] = h_2 & \text{on } \Sigma. \end{cases} \tag{2.9}$$

Our goal is to prove that A_λ is a topological linear isomorphism from the solution space

$$\mathbb{E}^k = \mathbb{E}^k(\Omega \setminus \Sigma) := H^{k+2}_p(\Omega \setminus \Sigma) \qquad \text{for } k \in \mathbb{N}_0,$$

onto the space of data

$$\mathbb{F}^k = \mathbb{F}^k(\Omega \setminus \Sigma) := H^k_p(\Omega \setminus \Sigma) \times W^{k+1-1/p}_p(\partial\Omega) \times W^{k+1-1/p}_p(\Sigma) \times W^{k+2-1/p}_p(\Sigma),$$

provided that $|\lambda|$ is sufficiently large and $\partial\Omega$, Σ, and μ are sufficiently regular. We identify

$$\mathbb{F}^k(\Omega \setminus \Sigma) \cong \begin{cases} H^k_p(\mathbb{R}^n) & \text{if } \Omega = \mathbb{R}^n,\ \Sigma = \emptyset, \\ H^k_p(\Omega) \times W^{k+1-1/p}_p(\partial\Omega) & \text{if } \Omega \ne \mathbb{R}^n,\ \Sigma = \emptyset, \\ H^k_p(\mathbb{R}^n \setminus \Sigma) \times W^{k+1-1/p}_p(\Sigma) \times W^{k+2-1/p}_p(\Sigma) & \text{if } \Omega = \mathbb{R}^n,\ \Sigma \ne \emptyset. \end{cases}$$

Our strategy to solve problem (2.9) is based on solving basic model problems, perturbed model problems, and on localization. In a basic model problem, we assume that μ is constant, Ω is the whole space $\mathbb{R}^n$ or a half-space $\mathbb{R}^n_+$, and Σ is a hyperplane $\mathbb{R}^{n-1} \times \{0\}$ or empty. In a perturbed model problem, we also allow for bent half-spaces $\Omega = \mathbb{R}^n_\omega = \{(x', x_n) \in \mathbb{R}^n : x_n > \omega(x')\}$, bent hyperplanes $\Sigma = \Sigma_\omega = \{(x', \omega(x')) : x' \in \mathbb{R}^{n-1}\}$, and variable coefficients with small oscillations. In a small region of $\overline{\Omega}$, problem (2.9) looks like a perturbed model problem, after an appropriate rotation and translation. Hence, if these perturbed model problems have appropriate "local" solution operators, then we can construct a "global" solution operator for problem (2.9) in terms of the local solution operators. Such a localization technique is provided in Section 2.1.1.

During the localization procedure, we have to control leading-order and lower-order perturbations, and this can be achieved by using a smallness parameter η and λ-dependent norms for $\mathbb{E}^k_\lambda$ and $\mathbb{F}^k_\lambda$, as defined in Section 2.1.2. These norms have useful scaling properties and allow to reduce the operator A_λ to A_1 for the basic model problems. Hence, if A_1 is invertible, then A_λ is uniformly invertible with respect to λ. The basic model problems for $\Sigma = \emptyset$ are well-known and we therefore turn our attention to the flat-interface model problem in Section 2.1.3. It is solved by means of the Fourier transform and with the joint $\mathcal{H}^\infty$ functional calculus. In Section 2.1.4, we investigate the perturbed model problem for $\Sigma = \Sigma_\omega$ with variable coefficient and derive the corresponding results for the remaining model problems. Here the parameter η bounds the oscillations of the coefficient μ and the gradient of ω and allows to control leading-order perturbations, whereas the parameter λ is used to control lower-order perturbations. Finally, we prove optimal regularity for problem (2.9) in a bounded domain in Section 2.1.5.

2.1.1. Localization technique. We provide a localization technique that allows to invert a "global" operator $A_\lambda\colon E \to F$ having invertible "local" versions $A_{\lambda,j}\colon E_j \to F_j$. This technique is similar to the corresponding procedures in [LSU68; AHS94; DHP03].

2.4. Definition. Let E and E_j ($j \in J \subset \mathbb{N}$) be Banach spaces, let $q \in [1,\infty)$, and define

$$\mathbf{E} := \prod_j E_j, \quad l_q(\mathbf{E}) := \left\{(x_j)_{j\in J} \in \mathbf{E} : \|x\|_{l_q(\mathbf{E})} < \infty\right\}, \quad \|x\|_{l_q(\mathbf{E})} := \left(\sum_{j\in J} \|x_j\|_{E_j}^q\right)^{1/q}.$$

Let further $\Phi_{E,j} \in \mathcal{B}(E; E_j)$ and $\Psi_{E,j} \in \mathcal{B}(E_j; E)$ be bounded linear operators such that

$$\sum_{j\in J} \Psi_{E,j}\Phi_{E,j}x = x \qquad \text{for all } x \in E,$$

where the series converges in E; and suppose that the maps

$$\begin{aligned} r_E\colon l_q(\mathbf{E}) &\to E, & (x_j)_{j\in J} &\mapsto \sum_{j\in J} \Psi_{E,j}x_j, \\ r_E^c\colon \quad E &\to l_q(\mathbf{E}), & x &\mapsto (\Phi_{E,j}x)_{j\in J}, \end{aligned}$$

are linear and bounded (hence, r_E is a retraction with co-retraction r_E^c). Then we say that the triple $(\mathbf{E}, (\Phi_{E,j})_{j\in J}, (\Psi_{E,j})_{j\in J})$ is an *l_q-approximation system* for E.

The spaces E and E_j are related to linear operators A_λ and $A_{\lambda,j}$ as follows.

2.5. Assumption. (i) E and F are Banach spaces over the same scalar field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, which have l_q-approximation systems $(\mathbf{E}, (\Phi_{E,j})_{j\in J}, (\Psi_{E,j})_{j\in J})$ and $(\mathbf{F}, (\Phi_{F,j})_{j\in J}, (\Psi_{F,j})_{j\in J})$ for some $q \in [1,\infty)$.

(ii) For some unbounded set $\Lambda \subset \mathbb{K}$, the families $\{\|\cdot\|_{X,\lambda} : \lambda \in \Lambda\}$ consist of equivalent norms on $X \in \{E, F, E_j, F_j : j \in J\}$ and we have

$$\sup_{\lambda\in\Lambda} \|r_E\|_{\mathcal{B}(l_q(\mathbf{E});E)),\lambda} < \infty, \quad \sup_{\lambda\in\Lambda} \|r_F^c\|_{\mathcal{B}(F;l_q(\mathbf{F})),\lambda} < \infty.$$

(iii) $A_\lambda\colon E \to F$ ($\lambda \in \Lambda$) are bounded linear operators such that the maps $A_\lambda\colon (E, \|\cdot\|_{E,\lambda}) \to (F, \|\cdot\|_{F,\lambda})$ are uniformly bounded with respect to $\lambda \in \Lambda$.

(iv) There exist invertible operators $A_{\lambda,j} \in \mathcal{B}_{\mathrm{isom}}(E_j; F_j)$ ($j \in J$, $\lambda \in \Lambda$) such that

$$\sup_{\lambda\in\Lambda} \|(f_j)_j \mapsto (A_{\lambda,j}^{-1}f_j)_j\|_{\mathcal{B}(l_q(\mathbf{F});l_q(\mathbf{E})),\lambda} < \infty.$$

(v) The operators $B_{\lambda,j} := \Phi_{F,j}A_\lambda - A_{\lambda,j}\Phi_{E,j} \in \mathcal{B}(E; F_j)$ satisfy

$$\lim_{|\lambda|\to\infty} \|u \mapsto (B_{\lambda,j}u)_j\|_{\mathcal{B}(E;l_q(\mathbf{F})),\lambda} = 0.$$

(vi) The operators $C_{\lambda,j} := A_\lambda\Psi_{E,j} - \Psi_{F,j}A_{\lambda,j} \in \mathcal{B}(E_j; F)$ satisfy

$$\lim_{|\lambda|\to\infty} \left\|(u_j) \mapsto \sum_j C_{\lambda,j}u_j\right\|_{\mathcal{B}(l_q(\mathbf{E});F),\lambda} = 0.$$

For later applications, it is important to establish uniform bounds for data-to-solution maps. A parameter-dependent operator $A_\lambda \in \mathcal{B}_{\mathrm{isom}}(E; F)$ is called *uniformly invertible* with respect to λ, if there is a number C such that $\|A_\lambda^{-1}\|_{F\to E} \le C$ for all λ.

2.6. Proposition (cf. [AHS94, Proposition 3.2])**.** *If Assumption 2.5 is satisfied, then there is $\lambda_0 > 0$ such that $A_\lambda\colon (E, \|\cdot\|_{E,\lambda}) \to (F, \|\cdot\|_{F,\lambda})$ is uniformly invertible with respect to $\lambda \in \Lambda$ with $|\lambda| \ge \lambda_0$.*

Proof. We consider the approximate inverse

$$R_\lambda \colon F \to E, \quad R_\lambda f := \sum\nolimits_j \Psi_{E,j} A_{\lambda,j}^{-1} \Phi_{F,j} f \quad \text{for } f \in F.$$

Let us write

$$R_\lambda A_\lambda - I_E = \sum\nolimits_j \Psi_{E,j} A_{\lambda,j}^{-1} (\Phi_{F,j} A_\lambda - A_{\lambda,j} \Phi_{E,j}) = \sum\nolimits_j \Psi_{E,j} A_{\lambda,j}^{-1} B_{\lambda,j},$$

$$A_\lambda R_\lambda - I_F = \sum\nolimits_j (A_\lambda \Psi_{E,j} - \Psi_{F,j} A_{\lambda,j}) A_{\lambda,j}^{-1} \Phi_{F,j} = \sum\nolimits_j C_{\lambda,j} A_{\lambda,j}^{-1} \Phi_{F,j}.$$

With Assumption 2.5 we can choose an upper bound $M > 0$ for the numbers

$$\sup_{\lambda \in \Lambda} \|r_E\|_{\mathcal{B}(l_q(\mathbf{E});E),\lambda}, \quad \sup_{\lambda \in \Lambda} \left\| (f_j)_j \mapsto (A_{\lambda,j}^{-1} f_j)_j \right\|_{\mathcal{B}(l_q(\mathbf{F});l_q(\mathbf{E})),\lambda}, \quad \sup_{\lambda \in \Lambda} \|r_F^c\|_{\mathcal{B}(F;l_q(\mathbf{F})),\lambda}.$$

Then R_λ is bounded by M^3 and we obtain the following estimates for $f \in F$ and $u \in E$:

$$\|R_\lambda A_\lambda u - u\|_{E,\lambda} = \left\| \sum\nolimits_j \Psi_{E,j} A_{\lambda,j}^{-1} B_{\lambda,j} u \right\|_{E,\lambda} \le M^2 \, \|u \mapsto (B_{\lambda,j} u)_j\|_{\mathcal{B}(E;l_q(\mathbf{F})),\lambda} \, \|u\|_{E,\lambda},$$

$$\|A_\lambda R_\lambda f - f\|_{F,\lambda} = \left\| \sum\nolimits_j C_{\lambda,j} A_{\lambda,j}^{-1} \Phi_{F,j} f \right\|_{F,\lambda} \le \left(\left\| (u_j) \mapsto \sum\nolimits_j C_{\lambda,j} u_j \right\|_{\mathcal{B}(l_q(\mathbf{E});F),\lambda} \right) M^2 \|f\|_{F,\lambda}.$$

Therefore we can find some $\lambda_0 \ge 0$ such that

$$\|A_\lambda R_\lambda - I_F\|_{\mathcal{B}(F),\lambda} \le 2^{-1}, \quad \|R_\lambda A_\lambda - I_E\|_{\mathcal{B}(E),\lambda} \le 2^{-1} \quad \text{for } \lambda \in \Lambda, \ |\lambda| \ge \lambda_0.$$

Hence the operators $A_\lambda R_\lambda = I_F - (I_F - A_\lambda R_\lambda) \in \mathcal{B}(F)$ and $R_\lambda A_\lambda = I_E - (I_E - R_\lambda A_\lambda) \in \mathcal{B}(E)$ are invertible. Consequently, $R_\lambda (A_\lambda R_\lambda)^{-1} \in \mathcal{B}(F;E)$ is a right-inverse and $(A_\lambda R_\lambda)^{-1} R_\lambda \in \mathcal{B}(F;E)$ is a left-inverse for A_λ. Thus, A_λ is invertible for all $\lambda \in \Lambda$ with $|\lambda| \ge \lambda_0$ and its inverse $A_\lambda^{-1} = R_\lambda (A_\lambda R_\lambda)^{-1} = (A_\lambda R_\lambda)^{-1} R_\lambda$ is bounded by $2M^3$. □

Next, we provide a localization set-up that can be used to construct approximation systems.

2.7. Remark. Let Ω be a bounded domain in $\mathbb{R}^n$ ($n \ge 2$) with C^1-boundary $\partial\Omega$ and let $\Sigma \subset \Omega$ be a compact C^1-hypersurface. We say that a family $(U_j)_{j \in J}$ of open subsets of $\mathbb{R}^n$ is a *finite open covering* for $\overline{\Omega}$ in $\mathbb{R}^n$, if J is finite and $\overline{\Omega}$ is contained in $\bigcup_{j \in J} U_j$. Since $\overline{\Omega}$, $\partial\Omega$, and Σ are compact, there exists $r_0 > 0$ such that for every $r \in (0, r_0]$ we can choose

(i) a finite open covering of balls $U_j = B_r(p_j)$ with $p_j \in \overline{\Omega}$ such that the index set can be decomposed as $J = J_1 \cup J_2 \cup J_3$ with

$$\begin{aligned} &p_j \in \Omega \setminus \Sigma \text{ and } \overline{U}_j \subset \Omega \setminus \Sigma && \text{if } j \in J_1, \\ &p_j \in \partial\Omega \text{ and } U_j \cap \Sigma = \emptyset && \text{if } j \in J_2, \\ &p_j \in \Sigma \text{ and } U_j \subset \Omega && \text{if } j \in J_3, \end{aligned}$$

(ii) a family $(\Theta_j)_{j \in J}$ of rigid transformations

$$\Theta_j \colon x \mapsto p_j + Q_j x, \quad B_r(0) \to U_j = B_r(p_j),$$

with an orthogonal matrix $Q_j = \partial_x \Theta_j \in \mathbb{R}^{n \times n}$ such that

$$\begin{aligned} Q_j &= I && \text{if } j \in J_1, \\ -Q_j e_n &= \nu_{\partial\Omega}(p_j) && \text{if } j \in J_2, \\ Q_j e_n &= \nu_\Sigma(p_j) && \text{if } j \in J_3. \end{aligned}$$

2.8. Definition (Localization set-up)**.** Let Ω be a bounded domain in $\mathbb{R}^n$ ($n \ge 2$) with C^1-boundary $\partial\Omega$ and let $\Sigma \subset \Omega$ be a compact C^1-hypersurface. For $\omega \colon \mathbb{R}^{n-1} \to \mathbb{R}$ we put

$$\mathbb{R}^n_\omega := \{(x', x_n) \in \mathbb{R}^n : x_n > \omega(x')\}, \quad \Sigma_\omega := \{(x', x_n) \in \mathbb{R}^n : x_n = \omega(x')\}.$$

Let $r > 0$ and $\eta > 0$ be given and suppose that

(i) $(U_j)_{j \in J}$ is a finite open covering for $\overline{\Omega}$ in $\mathbb{R}^n$ with $U_j = B_r(p_j)$ as in Remark 2.7.(i),

(ii) $(\Theta_j)_{j\in J}$ is a family of rigid transformations as in Remark 2.7.(ii),
(iii) $(\omega_j)_{j\in J}$ is a family of functions of class $C_c^1(\mathbb{R}^{n-1})$ which satisfy

$$\omega_j(0) = |\nabla\omega_j(0)| = 0, \quad \|\nabla\omega_j\|_{L_\infty(\mathbb{R}^{n-1})} \le \eta \quad \text{for all } j \in J,$$

and suppose that

$$\begin{aligned} \omega_j &= 0 && \text{if } j \in J_1,\\ \Theta_j(B_r(0)\cap\mathbb{R}^n_{\omega_j}) &= U_j \cap \Omega && \text{if } j \in J_2,\\ \Theta_j(B_r(0)\cap\mathbb{R}^n\setminus\Sigma_{\omega_j}) &= U_j\cap\Omega\setminus\Sigma && \text{if } j \in J_3. \end{aligned}$$

Then we call $(U_j, \Theta_j, \omega_j)_{j\in J}$ an *(η, r)-localization set-up* for (Ω, Σ).

2.9. Lemma. *Let $\Omega \subset \mathbb{R}^n$ ($n \ge 2$) be a bounded domain with C^1-boundary $\partial\Omega$ and $\Sigma \subset \Omega$ be a compact C^1-hypersurface.*

(i) If $\eta > 0$ is given, then there exists $r_0 > 0$ such that for every $r \in (0, r_0]$ we can find an (η, r)-localization set-up $(U_j, \Theta_j, \omega_j)_{j\in J}$ for (Ω, Σ).

(ii) If, additionally, $\partial\Omega$ and Σ are of class C^{k-} for some $k \ge 2$, then the ω_j belong to $C_c^{k-}(\mathbb{R}^{n-1})$ and there exists $C = C(n, p, \partial\Omega, \Sigma) > 0$ such that

$$\|\omega_j\|_{H_p^2(\mathbb{R}^{n-1})} \le C r^{(n-1)/p} \|\nabla^2\omega_j\|_{L_\infty(\mathbb{R}^{n-1})} \quad \text{for all } j \in J_2 \cup J_3.$$

Proof. As in Remark 2.7, we let $U_j = B_r(p_j)$ form a finite open covering for $\overline{\Omega}$ and consider the rigid transformations $\Theta_j\colon x \mapsto p_j + Q_j x$. The case $j \in J_1$ is trivial and since the cases $j \in J_2$ and $j \in J_3$ are analogous, we concentrate on $j \in J_3$.

We first construct the functions ω_j and prove that $\|\nabla\omega_j\|_\infty$ is small. For every $p \in \Sigma$ we can find a number $r_1(p) > 0$ and a unique height function ω_p on $B_{r_1(p)} \subset \mathbb{R}^{n-1}$, such that for $\Sigma_{\omega_p} := \{(x', \omega_s(x')) : x' \in B_{r_1(p)}\}$ we have $\Theta(\Sigma_{\omega_p}) \subset \Sigma$ for some rigid transformation $\Theta\colon x \mapsto p + Qx$ with $Qe_n = \nu_\Sigma(p)$. The function $\nabla\omega_p$ is related to ν_Σ by (see also (A.3) on page 130)

$$\nabla\omega_p = -\frac{QP'Q^\top(\nu_\Sigma\circ\Theta)}{Qe_n\cdot\nu_\Sigma\circ\Theta} \quad \text{on } B_p, \quad \text{with } P' = I_n - e_n\otimes e_n.$$

Moreover, it satisfies $\omega_p(0) = |\nabla\omega_p(0)| = 0$. Since ν_Σ is uniformly continuous on Σ, we obtain $\|\nabla\omega_p|_{B_t}\|_\infty \to 0$ as $t \to 0$, uniformly with respect to $p \in \Sigma$. By compactness of Σ, we may choose the number r_1 uniform in $p \in \Sigma$.

Let $\chi \in \mathcal{B}(\mathbb{R}^{n-1})$ with $0 \le \chi \le 1$, $\chi(x') = 1$ for $|x'| \le 1$ and $\chi(x') = 0$ for $|x'| \ge 2$. For $r \in (0, r_1/2]$ we define a function $\tilde\omega_{p,r}\colon \mathbb{R}^{n-1} \to \mathbb{R}$ with support in B_{2r} by

$$\tilde\omega_{p,r}(x') := \begin{cases} \chi(x'/r)\omega_p(x') & \text{for } |x'| < 2r,\\ 0 & \text{for } |x'| \ge 2r. \end{cases}$$

Then $\tilde\omega_{p,r}(x') = \omega_p(x')$ for all $x' \in B_r$. From $\omega_p(0) = 0$ and the fundamental theorem of calculus, we obtain the inequality $\|\omega_p|_{B_r}\|_\infty \le r\|\nabla'\omega_p|_{B_r}\|_\infty$. The uniform continuity of ν_Σ further implies that $\|\nabla'\omega_p|_{B_r}\|_\infty \to 0$ as $r \to 0$, uniformly in $p \in \Sigma$. Therefore

$$\begin{aligned} \|\nabla\tilde\omega_{p,r}\|_\infty &\le r^{-1}\|\nabla\chi\|_\infty\|\omega_p|_{B_{2r}}\|_\infty + \|\nabla\omega_p|_{B_{2r}}\|_\infty\\ &\le (\|\nabla\chi\|_\infty + 1)\|\nabla\omega_p|_{B_{2r}}\|_\infty \to 0 \quad \text{as } r \to 0, \end{aligned}$$

uniformly in $p \in \Sigma$. Thus for given $\eta > 0$ we can choose a number $r_0 \in (0, r_1/2]$ such that $\|\nabla\tilde\omega_{p,r}\|_\infty \le \eta$ for all $p \in \Sigma$, $r \in (0, r_0]$. We finally put $\omega_j := \tilde\omega_{p_j,r}$ for $j \in J$ with a suitable finite index set $J(r)$. Hence assertion (i) is valid.

Having in mind that every $\omega \in W^2_\infty(\mathbb{R}^{n-1})$ with $\omega(0) = |\nabla\omega(0)| = 0$ satisfies the estimates $|\nabla^k\omega(x)| \le |x|^{2-k}\|\nabla^k\omega\|_\infty$ for $k \in \{0, 1, 2\}$, and using the substitution $x = ry$, we obtain

$$\|\nabla^k\tilde\omega_r\|_p = \|\nabla^k(\chi(\cdot/r)\omega)\|_p \le C r^{(n-1)/p}\|\chi\|_{H_p^k}\|\nabla^k\omega\|_\infty \quad \text{for } k \in \{0, 1, 2\}.$$

This proves assertion (ii). □

2.1.2. λ-dependent norms. Let Ω and Σ satisfy Assumption 2.1. For $p \in (1,\infty)$, $k \in \mathbb{N}_0$, and $\lambda \in \mathbb{C} \setminus \{0\}$, we define the Banach spaces

$$\mathbb{E}^k_\lambda = (\mathbb{E}^k, \|\cdot\|_{\mathbb{E}^k_\lambda}), \quad \mathbb{E}^k = H^{k+2}_p(\Omega \setminus \Sigma),$$
$$\mathbb{F}^k_\lambda = (\mathbb{F}^k, \|\cdot\|_{\mathbb{F}^k_\lambda}), \quad \mathbb{F}^k = H^k_p(\Omega \setminus \Sigma) \times W^{k+1-1/p}_p(\partial\Omega) \times W^{k+1-1/p}_p(\Sigma) \cap W^{k+2-1/p}_p(\Sigma),$$

which are equipped with the equivalent λ-dependent norms

$$\|u\|_{\mathbb{E}^k_\lambda} := \|u\|_{H^{k+2}_p(\Omega\setminus\Sigma),\lambda} := \sum_{j=0}^{k+2} \|\lambda^{(k+2-j)/2}\nabla^j u\|_{L_p(\Omega)},$$
$$\|(f,g,h_1,h_2)\|_{\mathbb{F}^k_\lambda} := \|f\|_{H^k_p(\Omega\setminus\Sigma),\lambda} + \|g\|_{W^{k+1-1/p}_p(\partial\Omega),\lambda} + \|h_1\|_{W^{k+1-1/p}_p(\Sigma),\lambda} + \|h_2\|_{W^{k+2-1/p}_p(\Sigma),\lambda},$$

where

$$\|f\|_{H^k_p(\Omega\setminus\Sigma),\lambda} := \sum_{j=0}^{k} \|\lambda^{(k-j)/2}\nabla^j f\|_{L_p(\Omega)},$$
$$\|g\|_{W^{k+1-1/p}_p(\partial\Omega),\lambda} := [\![\nabla^k_{\partial\Omega} g]\!]_{W^{1-1/p}_p(\partial\Omega)} + \sum_{j=0}^{k} |\lambda|^{1/2-1/2p} \|\lambda^{(k-j)/2}\nabla^j_{\partial\Omega} g\|_{L_p(\partial\Omega)},$$
$$\|h_1\|_{W^{k+1-1/p}_p(\Sigma),\lambda} := [\![\nabla^k_{\Sigma} h_1]\!]_{W^{1-1/p}_p(\Sigma)} + \sum_{j=0}^{k} |\lambda|^{1/2-1/2p} \|\lambda^{(k-j)/2}\nabla^j_{\Sigma} h_1\|_{L_p(\Sigma)},$$
$$\|h_2\|_{W^{k+2-1/p}_p(\Sigma),\lambda} := [\![\nabla^{k+1}_{\Sigma} h_2]\!]_{W^{1-1/p}_p(\Sigma)} + \sum_{j=0}^{k+1} |\lambda|^{1/2-1/2p} \|\lambda^{(k+1-j)/2}\nabla^j_{\Sigma} h_2\|_{L_p(\Sigma)}.$$

Let us first derive these norms and have a look at its advantages. We consider the scaling

$$u_\lambda(x) := \lambda^\alpha u(\lambda^{-\beta}x) \quad \text{for } x \in \Omega \setminus \Sigma \text{ with some } \alpha,\ \beta \in \mathbb{R}.$$

We only consider the cases $\Omega \in \{\mathbb{R}^n, \mathbb{R}^n_+\}$ and $\Sigma \in \{\mathbb{R}^{n-1} \times \{0\}, \emptyset\}$ since these are invariant under the transformation $x \mapsto \lambda^{-\beta}x$. Then

$$\lambda u(x) - \Delta u(x) = \lambda^{-\alpha}\left(\lambda u_\lambda(\lambda^\beta x) - \lambda^{2\beta}\Delta u_\lambda(\lambda^\beta x)\right).$$

Since the local operators $A_{\lambda,j}$ should be uniformly invertible in λ, we want to achieve that the equations for u_λ do not depend on λ and therefore must choose $\beta = 1/2$. Next, the norm of the transformation $u \mapsto u_\lambda$ should satisfy $\|u\|_{\mathbb{E}^k_\lambda} = \|u_\lambda\|_{\mathbb{E}^k_1}$ and hence we require that

$$\|u\|_{\mathbb{E}^k_\lambda} = \sum_{j=0}^{k+2} \|\nabla^j u_\lambda\|_{L_p(\Omega)} = \sum_{j=0}^{k+2} \|\lambda^{\alpha-\beta j+\beta n/p}\nabla^j u\|_{L_p(\Omega)} = \sum_{j=0}^{k+2} \|\lambda^{\alpha-j/2+n/2p}\nabla^j u\|_{L_p(\Omega)}.$$

Finally, we choose $\alpha = (k+2-n/p)/2$ so that the highest order term in this norm does not depend on λ. This yields precisely the aforementioned $\mathbb{E}^k_\lambda$-norm. We keep in mind that

$$u_\lambda := \lambda^{(k+2)/2-n/2p} u(\lambda^{-1/2}\cdot).$$

Similarly, we define the rescaled data

$$f_\lambda := \lambda^{k/2-n/2p} f(\lambda^{-1/2}\cdot),$$
$$g_\lambda := \lambda^{(k+1)/2-n/2p} g(\lambda^{-1/2}\cdot),$$
$$h_{1\lambda} := \lambda^{(k+1)/2-n/2p} h_1(\lambda^{-1/2}\cdot),$$
$$h_{2\lambda} := \lambda^{(k+2)/2-n/2p} h_2(\lambda^{-1/2}\cdot).$$

When we replace the functions (u,f,g,h_1,h_2) by $(u_\lambda, f_\lambda, g_\lambda, h_{1\lambda}, h_{2\lambda})$ in (2.9), we see that the system $A_\lambda u = (f,g,h_1,h_2)$ is equivalent to $A_1 u_\lambda = (f_\lambda, g_\lambda, h_{1\lambda}, h_{2\lambda})$ in case $\Omega \in \{\mathbb{R}^n, \mathbb{R}^n_+\}$ and $\Sigma \in \{\mathbb{R}^{n-1} \times \{0\}, \emptyset\}$. Moreover,

$$\|u\|_{\mathbb{E}^k_\lambda} = \|u_\lambda\|_{\mathbb{E}^k_1}, \quad \|(f,g,h_1,h_2)\|_{\mathbb{F}^k_\lambda} = \|(f_\lambda, g_\lambda, h_{1\lambda}, h_{2\lambda})\|_{\mathbb{F}^k_1}.$$

2.1.3. Basic model problems. We first consider the system $A_\lambda u = (f, h_1, h_2)$ in the situation of a whole space $\Omega := \mathbb{R}^n$ ($n \geq 2$) with flat interface $\Sigma := \mathbb{R}^{n-1} \times \{0\} \cong \mathbb{R}^{n-1}$ and constant coefficients $\mu_\pm \in (0,\infty)$; that is,

$$(2.10)\qquad \begin{cases} \lambda u - \mu\Delta u = f & \text{in } \dot{\mathbb{R}}^n, \\ [\![\mu\partial_n u]\!] = h_1 & \text{on } \mathbb{R}^{n-1}, \\ [\![u]\!] = h_2 & \text{on } \mathbb{R}^{n-1}. \end{cases}$$

Here we have put $\dot\Omega := \Omega \setminus \Sigma = \dot{\mathbb{R}}^n$ and $\Omega_\pm := \mathbb{R}^{n-1} \times \pm(0,\infty)$. The elements of Ω are denoted by $x = (x', x_n)$ or (x', y) with $x' \in \mathbb{R}^{n-1}$ and $x_n = y \in \mathbb{R}$, and we let $\Delta = \partial_1^2 + \cdots + \partial_n^2$, $\Delta' = \partial_1^2 + \cdots + \partial_{n-1}^2$, $\nabla = (\partial_1, \ldots, \partial_n)$, and $\nabla' = (\partial_1, \ldots, \partial_{n-1})$. The parameter λ belongs to the open sector $\Sigma_\phi := \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg\lambda| < \phi\}$ for $\phi \in (0,\pi)$.

We shall prove that problem (2.10) has optimal H_p^{k+2}-regularity in the following sense.

2.10. Lemma. *Let $\mu_0 \in (0,1]$, $k \in \mathbb{N}_0$, $\phi \in (0,\pi)$, and $p \in (1,\infty)$. Then the operator*

$$A_\lambda \colon \mathbb{E}_\lambda^k(\dot{\mathbb{R}}^n) \to \mathbb{F}_\lambda^k(\dot{\mathbb{R}}^n), \quad u \mapsto (\lambda u - \operatorname{div}(\mu\nabla u), [\![\mu\partial_n u]\!], [\![u]\!])$$

is uniformly invertible with respect to $\mu_\pm \in [\mu_0, \mu_0^{-1}]$ and $\lambda \in \Sigma_\phi$.

Proof. (i) In order to prove *uniqueness*, it is sufficient to consider a solution $u \in H_p^2(\dot{\mathbb{R}}^n)$ to (2.10) for trivial data $A_\lambda u = (f, h_1, h_2) = 0$. When we consider u as a function $y \mapsto u(\cdot, y)$ that belongs to the space $H_p^2(\dot{\mathbb{R}}; L_p(\mathbb{R}^{n-1})) \cap L_p(\mathbb{R}; H_p^2(\mathbb{R}^{n-1}))$, we see that both functions $y \mapsto u_\pm(\cdot, y)$, $\pm[0,\infty) \to \mathcal{S}'(\mathbb{R}^{n-1})$ are continuous. The functions $\omega_\pm(\xi) := (\lambda\mu_\pm^{-1} + |\xi|^2)^{1/2}$ satisfy $\operatorname{Re}\omega_\pm(\xi) > 0$ for all $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ and $\xi \in \mathbb{R}^{n-1}$. Then the partially Fourier transformed equations with respect to $x \in \mathbb{R}^{n-1}$ with covariable $\xi \in \mathbb{R}^{n-1}$ are given by

$$(2.11)\qquad \begin{cases} \omega^2\tilde u - \partial_y^2\tilde u = 0 & \text{in } \mathcal{D}'(\dot{\mathbb{R}}; \mathcal{S}'(\mathbb{R}^{n-1})), \\ \mu_+\partial_y\tilde u_+(\cdot,0) - \mu_-\partial_y\tilde u_-(\cdot,0) = 0 & \text{in } \mathcal{S}'(\mathbb{R}^{n-1}), \\ \tilde u_+(\cdot,0) - \tilde u_-(\cdot,0) = 0 & \text{in } \mathcal{S}'(\mathbb{R}^{n-1}). \end{cases}$$

The first equation in (2.11) must be understood in the following sense:

$$(2.12)\qquad \int_{-\infty}^{\infty} \tilde u(\cdot, y)\left(\omega^2\varphi(y) - \partial_y^2\varphi(y)\right) dy = 0 \text{ in } \mathcal{S}'(\mathbb{R}^{n-1}) \quad \text{for } \varphi \in \mathcal{D}(\dot{\mathbb{R}}).$$

We claim that (2.12) implies $\tilde u(\cdot, \pm y) = (\xi \mapsto e^{-\omega_\pm(\xi)y})c_\pm$ for all $y \geq 0$ and some $c_\pm \in \mathcal{S}'(\mathbb{R}^{n-1})$. Indeed, in order to check this for $\tilde u_+(\cdot, y)$, we write an arbitrary $\varphi \in \mathcal{D}(\mathbb{R}_+)$ as

$$(2.13)\qquad \varphi(y) = (\omega_+^2 - \partial_y^2)\psi_\varphi(y) + h_+(y)\left\langle e^{\omega_+\cdot}|\varphi\right\rangle + h_-(y)\left\langle e^{-\omega_+\cdot}\middle|\varphi\right\rangle.$$

Here $\langle\cdot|\cdot\rangle$ denotes bilinear integration over $\mathbb{R}_+$, the functions $h_\pm \in \mathcal{D}(\mathbb{R}_+)$ with $\langle e^{\pm\omega_+\cdot}|h_\pm\rangle = 1$ and $\langle e^{\pm\omega_+\cdot}|h_\mp\rangle = 0$ are fixed (independent of φ), and we can calculate the solution $\psi_\varphi \in \mathcal{D}(\mathbb{R}_+)$ of (2.13) by using Green's functions (see Lemma 3.3 on page 56). Then it can be readily checked that

$$\langle\tilde u_+|\varphi\rangle = \langle\tilde u_+|h_+\rangle\left\langle e^{\omega_+\cdot}|\varphi\right\rangle + \langle\tilde u_+|h_-\rangle\left\langle e^{-\omega_+\cdot}\middle|\varphi\right\rangle \quad \text{for } \varphi \in \mathcal{D}(\mathbb{R}_+),$$

with the constant distributions $c_{+,\pm} := \langle\tilde u_+|h_\pm\rangle$. Since $y \mapsto u(\cdot, y)$ belongs to $H_p^2(\mathbb{R}_+; L_p(\mathbb{R}^{n-1}))$, we must have $c_{+,+} = 0$ and hence $\tilde u_+(\cdot, y) = (\xi \mapsto e^{-\omega_+(\xi)y})c_+$ for $y \geq 0$ with $c_+ := c_{+,-}$. Analogously, we have $\tilde u(\cdot, -y) = (\xi \mapsto e^{-\omega_-(\xi)y})c_-$ for $y \geq 0$ with some $c_- \in \mathcal{S}'(\mathbb{R}^{n-1})$. The remaining equations yield $c_+ = c_-$ and $-\mu_+c_+\omega_+ - \mu_-c_-\omega_- = 0$, and thus $c_\pm = 0$. Therefore (2.10) has at most one solution in $H_p^2(\dot{\mathbb{R}}^n)$.

(ii) *Existence for $k = 0$ and $f = 0$.* We construct a solution u of (2.10) for given $(0, h_1, h_2) \in \mathbb{F}^0_\lambda$. The partially Fourier transformed function $y \mapsto \tilde{u}(\cdot, y)$, $\dot{\mathbb{R}} \to \mathcal{S}'(\mathbb{R}^{n-1})$ must satisfy the system

$$\left\{ \begin{aligned} \omega^2 \tilde{u} - \partial_y^2 \tilde{u} &= 0 && \text{in } \mathcal{D}'(\dot{\mathbb{R}}; \mathcal{S}'(\mathbb{R}^{n-1})), \\ \mu_+ \partial_y \tilde{u}_+(\cdot, 0) - \mu_- \partial_y \tilde{u}_-(\cdot, 0) &= \tilde{h}_1 && \text{in } \mathcal{S}'(\mathbb{R}^{n-1}), \\ \tilde{u}_+(\cdot, 0) - \tilde{u}_-(\cdot, 0) &= \tilde{h}_2 && \text{in } \mathcal{S}'(\mathbb{R}^{n-1}). \end{aligned} \right. \tag{2.14}$$

Problem (2.14) has the following $\mathcal{D}'(\dot{\mathbb{R}}; \mathcal{S}'(\mathbb{R}^{n-1}))$-solution.

$$\begin{bmatrix} \tilde{u}_+(\cdot, y) \\ \tilde{u}_-(\cdot, -y) \end{bmatrix} = \frac{1}{\mu_+ + \mu_-} \begin{bmatrix} -\frac{e^{-\omega_+ y}}{\omega_+} & \mu_- e^{-\omega_+ y} \\ -\frac{e^{-\omega_- y}}{\omega_-} & -\mu_+ e^{-\omega_- y} \end{bmatrix} \begin{bmatrix} \tilde{h}_1 \\ \tilde{h}_2 \end{bmatrix}.$$

In order to invert the partial Fourier transform $u \mapsto \tilde{u}$, we employ the joint functional calculus for ∇' from Theorem B.69 on page 166. Here we consider $\nabla' = (\partial_1 \ldots, \partial_{n-1})$ as an operator tuple $\mathbf{T} = (T_1, \ldots, T_{n-1})$ in $X = L_p(\mathbb{R}^{n-1})$ in the sense of Remark B.65. For the symbols $\omega_{\pm,\lambda}(z) = (\lambda/\mu_\pm - z \cdot z)^{1/2}$, we define $\omega_\pm(\nabla') = (\lambda/\mu_\pm - \Delta')^{1/2} =: L_{\pm,\lambda} \colon H^1_p(\mathbb{R}^{n-1}) \to L_p(\mathbb{R}^{n-1})$. With Theorem B.25 on page 155 we define the extensions $(x, y) \mapsto (e^{-L_{\pm,\lambda} y} h_2)(x) \in H^2_p(\mathbb{R}^n_+)$ and $(x, y) \mapsto (e^{-L_{\pm,\lambda} y} h_1)(x) \in H^1_p(\mathbb{R}^n_+)$. Then a solution to (2.10) is given by

$$\begin{bmatrix} u_+(\cdot, y) \\ u_-(\cdot, -y) \end{bmatrix} = \frac{1}{\mu_+ + \mu_-} \begin{bmatrix} -L_{+,\lambda}^{-1} e^{-y L_{+,\lambda}} & \mu_- e^{-y L_{+,\lambda}} \\ -L_{-,\lambda}^{-1} e^{-y L_{-,\lambda}} & -\mu_+ e^{-y L_{-,\lambda}} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}. \tag{2.15}$$

(iii) *A uniform bound with respect to λ.* We employ the dilations $\sigma_t \in \mathcal{B}_{\text{isom}}(L_p(\mathbb{R}^n))$ and $\sigma'_t \in \mathcal{B}_{\text{isom}}(L_p(\mathbb{R}^{n-1}))$ with $t \in (0, \infty)$ defined by $\sigma_t u := u(t\cdot)$ for $u \in L_p(\mathbb{R}^n)$ and $\sigma'_t h := h(t\cdot)$ for $h \in L_p(\mathbb{R}^{n-1})$. Then $\Delta' \sigma'_{\lambda^{1/2}} = \lambda \sigma'_{\lambda^{1/2}} \Delta'$ and, with $L_\pm := L_{\pm,1}$, we obtain

$$L^2_{\pm,\lambda} = \lambda/\mu_\pm - \Delta' = \lambda \sigma'_{\lambda^{1/2}} (1/\mu_\pm - \Delta') \sigma'_{\lambda^{-1/2}} = \left(\lambda^{1/2} \sigma'_{\lambda^{1/2}} L_\pm \sigma'_{\lambda^{-1/2}} \right)^2.$$

Hence $L_{\pm,\lambda} = \lambda^{1/2} \sigma'_{\lambda^{1/2}} L_\pm \sigma'_{\lambda^{-1/2}}$ on $D(L_\pm) = H^1_p(\mathbb{R}^{n-1})$. For $h \in L_p(\mathbb{R}^{n-1})$, we have

$$\sigma_{\lambda^{-1/2}} \left((x, y) \mapsto e^{-y L_{\pm,\lambda}} h \right) = \exp\left(-\lambda^{-1/2} y \sigma'_{\lambda^{-1/2}} L_{\pm,\lambda} \sigma'_{\lambda^{1/2}} \right) \sigma'_{\lambda^{-1/2}} h = \exp\left(-y L_\pm \right) \sigma'_{\lambda^{-1/2}} h.$$

Then the rescaled functions $u_{\pm,\lambda} := \lambda^{1-n/2p} \sigma_{\lambda^{-1/2}} u_\pm$ and $h_{j,\lambda} := \lambda^{j/2-n/2p} \sigma'_{\lambda^{-1/2}} h_j$ satisfy

$$\begin{bmatrix} u_{+,\lambda}(\cdot, y) \\ u_{-,\lambda}(\cdot, -y) \end{bmatrix} = \frac{1}{\mu_+ + \mu_-} \begin{bmatrix} -L_+^{-1} e^{-y L_+} & \mu_- e^{-y L_+} \\ -L_-^{-1} e^{-y L_-} & -\mu_+ e^{-y L_-} \end{bmatrix} \begin{bmatrix} h_{1,\lambda} \\ h_{2,\lambda} \end{bmatrix} \quad \text{for } y > 0.$$

For given $\mu_0 \in (0, 1)$ and $\vartheta \in (0, \pi)$, there exists $M > 0$ such that $L^2_\pm$ are operators of positive type $\mathcal{P}_1(H^2_p(\mathbb{R}^{n-1}), L_p(\mathbb{R}^{n-1}), M, \vartheta)$ for all $\mu_\pm \in [\mu_0, \mu_0^{-1}]$ (see page 154) and therefore Theorem B.25 yields the assertion for $f = 0$.

(iv) If $f \in L_p(\mathbb{R}^n)$ is arbitrary, then a solution to (2.10) is given by $u + v + w$, where u is defined by (2.15), $v_\pm := (\lambda - \mu_\pm \Delta)^{-1} f_\pm \in H^2_p(\mathbb{R}^n_\pm)$ with $v_\pm|_{y=0} = 0$ are the half-space solutions from [DHP03, Theorem 7.3], and w is the solution to $\lambda w - \mu \Delta w = 0$, $[\![\rho w]\!] = -[\![\rho v]\!]$, $[\![\partial_y w]\!] = -[\![\partial_y v]\!]$, which is defined analogously as u in (2.15). Therefore the assertion for $k = 0$ is proved.

(v) *Existence for $k \geq 0$.* Let $(f, h_1, h_2) \in \mathbb{F}^k_\lambda$ be given. We shall construct a solution to (2.10) of the form $u = v + w$, where v, w are defined as follows. Let $E_\pm \in \mathcal{B}(H^k_p(\mathbb{R}^n_\pm); H^k_p(\mathbb{R}^n))$ and $E_{\pm,\alpha} \in \mathcal{B}(H^{k-|\alpha|}_p(\mathbb{R}^n_\pm); H^{k-|\alpha|}_p(\mathbb{R}^n))$ denote the extension operators from Theorem B.6 on page 147 with the property $\partial_x^\alpha E_\pm g = E_{\pm,\alpha}\partial_x^\alpha g$ for $g \in H_p^k(\mathbb{R}_+^n)$ and $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le k$. Then the functions $v_\pm := (\lambda - \mu_\pm\Delta)^{-1}E_\pm(f|_{\mathbb{R}_\pm^n})$ belong to $H_p^{k+2}(\mathbb{R}^n)$ and satisfy

$$\|v_\pm\|_{H_p^2(\mathbb{R}^n),\lambda} \le C(n,p,\mu_0)\|f\|_{L_p(\mathbb{R}^n)} \quad \text{for } f \in H_p^k(\dot{\mathbb{R}}^n),\ \lambda \in \Sigma_\phi,\ \mu_\pm \in [\mu_0, \mu_0^{-1}].$$

Differentiating the equation $(\lambda - \mu_\pm\Delta)v_\pm = E_\pm(f|_{\mathbb{R}_\pm^n})$ shows that $(\lambda - \mu_\pm\Delta)\partial_x^\alpha v = E_{\pm,\alpha}\partial_x^\alpha f \in L_p(\mathbb{R}^n)$ for $|\alpha| \le k$. Hence the function $v := \chi_{\mathbb{R}_+^n} v_+ + \chi_{\mathbb{R}_-^n} v_-$ belongs to $H_p^{k+2}(\dot{\mathbb{R}}^n)$, solves the equation $(\lambda - \mu\Delta)v = f$ in $\dot{\mathbb{R}}^n$, and satisfies

$$\|v\|_{\mathbb{E}_\lambda^k(\dot{\mathbb{R}}^n)} \le C(n,p,\mu_0)\|f\|_{H_p^k(\dot{\mathbb{R}}^n),\lambda} \quad \text{for } f \in H_p^k(\dot{\mathbb{R}}^n),\ \lambda \in \Sigma_\phi,\ \mu_\pm \in [\mu_0, \mu_0^{-1}].$$

The function $w \in H_p^2(\dot{\mathbb{R}}^n)$ is defined as the solution to $(\lambda - \mu\Delta)w = 0$, $[\![\mu\partial_n w]\!] = h_1 - [\![\mu\partial_n v]\!]$, $[\![w]\!] = h_2 - [\![v]\!]$. From uniqueness and (2.15) we derive the representation

$$\begin{bmatrix} w_+(\cdot, y) \\ w_-(\cdot, -y) \end{bmatrix} = \frac{1}{\mu_+ + \mu_-}\begin{bmatrix} -L_{+,\lambda}^{-1}e^{-yL_{+,\lambda}} & \mu_- e^{-yL_{+,\lambda}} \\ -L_{-,\lambda}^{-1}e^{-yL_{-,\lambda}} & -\mu_+ e^{-yL_{-,\lambda}} \end{bmatrix}\begin{bmatrix} h_1 - [\![\mu\partial_n v]\!] \\ h_2 - [\![v]\!] \end{bmatrix}.$$

In order to verify that w belongs to $H_p^{k+2}(\dot{\mathbb{R}}^n)$, we let $\alpha \in \mathbb{N}_0^{n-1}$ with $|\alpha| \le k$. By Theorem B.69, the operators $\nabla'^{|\alpha|}L_{\pm,\lambda}^{-|\alpha|}$ are isomorphisms in $W_p^s(\mathbb{R}^{n-1})$ for every $s \ge 0$. Hence, by using the commutativity $L_{\pm,\lambda}e^{-yL_{\pm,\lambda}} = e^{-yL_{\pm,\lambda}}L_{\pm,\lambda}$ and by applying Theorem B.25, we see that

$$\|\partial_{x'}^\alpha w_\pm\|_{H_p^2(\mathbb{R}_\pm^n)} \lesssim \|L_{\pm,\lambda}^{|\alpha|}w_\pm\|_{H_p^2(\mathbb{R}_\pm^n)} \lesssim \|\nabla'^{|\alpha|}g_1\|_{W_p^{1-1/p}(\mathbb{R}^{n-1})} + \|\nabla'^{|\alpha|}g_2\|_{W_p^{2-1/p}(\mathbb{R}^{n-1})}.$$

The normal derivatives can be estimated similarly by means of $\partial_y^j e^{-yL_{\pm,\lambda}} = e^{-yL_{\pm,\lambda}}(-L_{\pm,\lambda})^j$. This shows that w belongs to $H_p^{k+2}(\dot{\mathbb{R}}^n)$ and satisfies $(\lambda - \mu\Delta)w = 0$. Hence $u = v + w$ belongs to $\mathbb{E}_\lambda^k(\dot{\mathbb{R}}^n)$ and solves $A_\lambda u = (f, h_1, h_2)$. Uniform bounds for $\|A_\lambda^{-1}\|$ with respect to $|\arg\lambda| < \phi$ can be shown again by a scaling argument. □

Lemma 2.10 includes optimal H_p^{k+2}-regularity of the whole space model problem without interface, since we can choose $\mu_+ = \mu_-$ and restrict the operator A_λ to the case $h_1 = h_2 = 0$.

2.11. Corollary. *Let $\mu_0 \in (0,1]$, $k \in \mathbb{N}_0$, $\phi \in (0,\pi)$, and $p \in (1,\infty)$. Then the operator*

$$A_\lambda \colon \mathbb{E}_\lambda^k(\mathbb{R}^n) \to \mathbb{F}_\lambda^k(\mathbb{R}^n), \quad u \mapsto \lambda u - \mathrm{div}(\mu\nabla u)$$

is uniformly invertible with respect to $\mu \in [\mu_0, \mu_0^{-1}]$ and $\lambda \in \Sigma_\phi$.

Finally, we consider the remaining model problem for $\Omega = \mathbb{R}_+^n$ and $\Sigma = \emptyset$.

2.12. Lemma. *Let $\mu_0 \in (0,1]$, $k \in \mathbb{N}_0$, $\phi \in (0,\pi)$, and $p \in (1,\infty)$. Then the operator*

$$A_\lambda \colon \mathbb{E}_\lambda^k(\mathbb{R}_+^n) \to \mathbb{F}_\lambda^k(\mathbb{R}_+^n), \quad u \mapsto (\lambda u - \mathrm{div}(\mu\nabla u), -\mu\partial_n u)$$

is uniformly invertible with respect to $\mu \in [\mu_0, \mu_0^{-1}]$ and $\lambda \in \Sigma_\phi$.

Proof. We obtain the assertion by following the lines of the proof of Lemma 2.10, except for the elimination of the boundary condition. Here a solution u to $A_\lambda u = (0, g)$ is given by

$$u(\cdot, y) = \frac{1}{\mu}L_\lambda^{-1}e^{-yL_\lambda}g, \quad L_\lambda = \sqrt{\lambda - \mu\Delta'}. \qquad \square$$

2.1.4. Perturbed model problems. We next consider the model problem $A_\lambda u = (f, h_2, h_2)$ for $\Omega = \mathbb{R}^n$, for a bent hyperplane $\Sigma_\omega := \theta_\omega(\mathbb{R}^{n-1})$ with $\theta_\omega(x') := (x', \omega(x'))$ for $\omega \in C_c^{2-}(\mathbb{R}^{n-1})$ and for constants parameters $\mu_\pm > 0$. This model problem reads as follows.

$$\begin{cases} \lambda u - \mu\Delta u = f & \text{in } \mathbb{R}^n \setminus \Sigma_\omega, \\ [\![\mu\partial_\nu u]\!] = h_1 & \text{on } \Sigma_\omega, \\ [\![u]\!] = h_2 & \text{on } \Sigma_\omega. \end{cases} \tag{2.16}$$

2.13. Remark. Problem (2.16) can be reduced to a flat interface problem with the following transformation. We only assume that ω is of class $C^1(\mathbb{R}^{n-1})$ and we consider the map

$$\Theta_\omega\colon \mathbb{R}^n \to \mathbb{R}^n, \quad (x', x_n) \mapsto (x', x_n + \omega(x')).$$

(i) It is easy to check that Θ_ω^{-1} is given by $(x', x_n) \mapsto (x', x_n - \omega(x'))$ and that

$$\partial\Theta_\omega = \begin{bmatrix} I & 0 \\ \partial\omega & 1 \end{bmatrix}, \quad [\partial\Theta_\omega]^{-1} = \begin{bmatrix} I & 0 \\ -\partial\omega & 1 \end{bmatrix}, \quad \det \partial\Theta_\omega = 1.$$

Hence both $\Theta_\omega\colon \mathbb{R}^n \to \mathbb{R}^n$ and $\Theta_\omega|_{\Sigma_0}\colon \Sigma_0 \to \Sigma_\omega$ are C^1-diffeomorphisms.

(ii) The hypersurface Σ_ω has

(a) the tangent vectors $\tau_j \circ \Theta_\omega = e_j + \partial_j\omega\, e_n$ for $j < n$,

(b) the unit normal vector $\nu \circ \Theta_\omega = \beta(e_n - \nabla\omega)$ with $\beta := (1 + |\nabla\omega|^2)^{-1/2}$,

(c) the cotangent vectors $\tau^j \circ \Theta_\omega = e_j + \beta^2\partial_j\omega(e_n - \nabla\omega)$.

For $p \in (1,\infty)$, $k \in \mathbb{N}_0$, $\omega \in C_c^{k+2-}(\mathbb{R}^{n-1})$, and $\lambda \in \mathbb{C} \setminus \{0\}$, we employ the function spaces

$$\begin{aligned} \mathbb{E}_\lambda^k &= \mathbb{E}_\lambda^k(\mathbb{R}^n \setminus \Sigma_\omega) = H_p^{k+2}(\mathbb{R}^n \setminus \Sigma_\omega), \\ \mathbb{F}_\lambda^k &= \mathbb{F}_\lambda^k(\mathbb{R}^n \setminus \Sigma_\omega) = H_p^{k}(\mathbb{R}^n \setminus \Sigma_\omega) \times W_p^{k+1-1/p}(\Sigma_\omega) \times W_p^{k+2-1/p}(\Sigma_\omega), \end{aligned}$$

equipped with the λ-dependent norms from Section 2.1.2.

2.14. Lemma. *Let $\mu_0 \in (0,1]$, $k \in \mathbb{N}_0$, $\phi \in (0,\pi)$, and $p \in (1,\infty)$. Then there exists $\eta > 0$ such that for every $M > 0$ we can find some $\lambda_0 \geq 1$ such that the operator*

$$A_\lambda\colon \mathbb{E}_\lambda^k(\mathbb{R}^n \setminus \Sigma_\omega) \to \mathbb{F}_\lambda^k(\mathbb{R}^n \setminus \Sigma_\omega), \quad u \mapsto (\lambda u - \operatorname{div}(\mu\nabla u), [\![\mu\partial_\nu u]\!], [\![u]\!]),$$

is uniformly invertible with respect to

$$\omega \in C_c^{k+2-}(\mathbb{R}^{n-1}),\ \|\nabla\omega\|_{W_\infty^{k+1}} \leq M,\ \|\nabla\omega\|_\infty \leq \eta, \quad \mu_\pm \in [\mu_0, \mu_0^{-1}], \quad \lambda \in \Sigma_\phi,\ |\lambda| \geq \lambda_0.$$

Proof. (i) We study a transformation of the functions $u \in \mathbb{E}_\lambda^k$ and $(f, h_1, h_2) \in \mathbb{F}_\lambda^k$ to a flat interface situation. The map $\Theta = \Theta_\omega$ from Remark 2.13 is a C^{k+2-}-diffeomorphism from $\mathbb{R}_\pm^n := \mathbb{R}^{n-1} \times \pm(0,\infty)$ onto $\Omega_\pm := \{(x', x_n) \in \mathbb{R}^n : x_n \gtrless \omega(x')\}$ and from Σ_0 onto Σ_ω. Both $\partial\Theta$ and $\partial\Theta^{-1}$ belong to $W_\infty^{k+1}(\mathbb{R}^n)$. We consider the pull-backs

$$\overline{u} = u \circ \Theta, \quad \overline{f} = f \circ \Theta, \quad \overline{h}_j = h_j \circ \Theta.$$

By means of the chain rule (B.19) and the substitution formula (A.12), it follows that

$$\begin{aligned} \overline{u} &\in \overline{\mathbb{E}}_\lambda^k := H_p^{k+2}(\dot{\mathbb{R}}^n), \\ (\overline{f}, \overline{h}_1, \overline{h}_2) &\in \overline{\mathbb{F}}_\lambda^k := H_p^{k}(\dot{\mathbb{R}}^n) \times W_p^{k+1-1/p}(\mathbb{R}^{n-1}) \times W_p^{k+2-1/p}(\mathbb{R}^{n-1}), \end{aligned}$$

and that $u \mapsto \overline{u}$, $\mathbb{E}_\lambda^k \to \overline{\mathbb{E}}_\lambda^k$ and $(f, h_1, h_2) \mapsto (\overline{f}, \overline{h}_1, \overline{h}_2)$, $\mathbb{F}_\lambda^k \to \overline{\mathbb{F}}_\lambda^k$ are topological linear isomorphisms. To be more precise, let $1 \leq j \leq k+2$. Then

$$\|\lambda^{(k+2-j)/2}\nabla^j\overline{u}\|_p \leq \sum_{i=1}^{j}\sum_{\beta,\sigma} \frac{|\lambda|^{-(j-i)/2}}{i!\beta!}\|\lambda^{(k+2-i)/2}\nabla^i u \circ \Theta\|_p \|\partial^{\beta_1}\Theta\|_\infty \cdots \|\partial^{\beta_i}\Theta\|_\infty,$$

where the sum is taken over multi-indices $\beta \in \mathbb{N}^i$ such that $|\beta| = j$ and all $j!$ permutations σ of $\{1, \ldots, j\}$. From $\det \partial\Theta = 1$ we infer that $\|\nabla^i u \circ \Theta\|_p = \|\nabla^i u\|_p$. This shows that

$$C(n,k,M)^{-1}\|u\|_{\mathbb{E}_\lambda^k} \leq \|\overline{u}\|_{\overline{\mathbb{E}}_\lambda^k} \leq C(n,k,M)\|u\|_{\mathbb{E}_\lambda^k}, \tag{2.17}$$

where $C(n,k,M)$ is uniform with respect to those ω and λ that satisfy $\|\nabla\omega\|_{W_\infty^{k+1}} \leq M$ and $|\lambda| \geq 1$. The relevant estimates for $\overline{f}$ in $H_p^k(\dot{\mathbb{R}}^n)$ follow analogously and those of $\overline{h}_j$ in $H_p^{k-1+j}(\mathbb{R}^{n-1})$

follow from (A.12). Finally, since $|\partial\Theta| \le (1+|\nabla\omega|^2)^{1/2}$ and $|\partial\Theta^{-1}| \le (1+|\nabla\omega|^2)^{1/2}$, we infer again from (A.12) that the Slobodeckiĭ semi-norm for $s \in (0,1)$ satisfies

$$[\![g\circ\Theta]\!]_{W_p^s(\mathbb{R}^{n-1})} \le (1+\|\nabla\omega\|_\infty^2)^{s/2+(n-1)/2p}\,[\![g]\!]_{W_p^s(\Sigma_\omega)}\,,$$
$$[\![\overline{g}\circ\Theta^{-1}]\!]_{W_p^s(\Sigma_\omega)} \le (1+\|\nabla\omega\|_\infty^2)^{s/2+(n+1)/2p}\,[\![\overline{g}]\!]_{W_p^s(\mathbb{R}^{n-1})}\,.$$

We conclude that

$$C(n,k,p,M)^{-1}\|(f,h_1,h_2)\|_{\mathbb{F}_\lambda^k} \le \|(\overline{f},\overline{h}_1,\overline{h}_2)\|_{\overline{\mathbb{F}}_\lambda^k} \le C(n,k,p,M)\|(f,h_1,h_2)\|_{\mathbb{F}_\lambda^k}. \tag{2.18}$$

(ii) We derive the transformed problem. From $(\nabla u)\circ\Theta = [\partial\Theta]^{-\top}\nabla\overline{u}$ we infer that

$$\partial_\nu u\circ\Theta = (\nu_{\Sigma_\omega}\cdot\nabla u)\circ\Theta = -\beta\nabla'\omega\cdot\nabla'\overline{u} + \beta^{-1}\partial_n\overline{u}, \quad \beta = (1+|\nabla'\omega|^2)^{-1}.$$

With $\Theta_m^{-1} := (\Theta^{-1})_m$ and $\partial_j\Theta_m^{-1} = \delta_{jm} - \delta_{mn}\partial_j\omega$ and $\Delta\Theta_m^{-1} = -\delta_{mn}\Delta\omega$, we obtain

$$\begin{aligned}(\Delta u)\circ\Theta &= \Delta\overline{u} + \sum\nolimits_{jlm}\partial_l\partial_m\overline{u}\left(\partial_j\Theta_l^{-1}\partial_j\Theta_m^{-1} - \delta_{jl}\delta_{jm}\right) + \sum\nolimits_l\partial_l\overline{u}\,\Delta\Theta_l^{-1},\\ &= \Delta\overline{u} - 2\nabla\partial_n\overline{u}\cdot\nabla\omega + \partial_n^2\overline{u}\,|\nabla\omega|^2 + \partial_n\overline{u}\,\Delta\omega.\end{aligned}$$

Therefore problem (2.16) is transformed to

$$\begin{aligned}\lambda\overline{u} - \mu\Delta\overline{u} &= \overline{f} + F_2\overline{u} + F_1\overline{u} && \text{in } \dot{\mathbb{R}}^n,\\ [\![\mu\partial_n\overline{u}]\!] &= \overline{h}_1 + H\overline{u} && \text{on } \mathbb{R}^{n-1},\\ [\![\overline{u}]\!] &= \overline{h}_2 && \text{on } \mathbb{R}^{n-1},\end{aligned}$$

where the perturbations $F_l = F_l(\mu,\omega)$ and $H = H(\mu,\omega)$ are given by

$$\begin{aligned}F_1\overline{u} &= -\mu\Delta'\omega\,\partial_n\overline{u},\\ F_2\overline{u} &= \mu|\nabla'\omega|^2\partial_n^2\overline{u} - 2\mu\partial_n\nabla'\overline{u}\cdot\nabla'\omega,\\ H\overline{u} &= \beta\nabla'\omega\cdot[\![\mu\nabla'\overline{u}]\!] + (1-\beta^{-1})[\![\mu\partial_n\overline{u}]\!].\end{aligned}$$

(iii) Let us derive suitable estimates for F_l and H. Our goal is to show that

$$\|F_l(\mu,\omega)\|_{\mathcal{B}(\overline{E}_\lambda^k;H_p^k(\dot{\mathbb{R}}^n)),\lambda} + \|H(\mu,\omega)\|_{\mathcal{B}(\overline{\mathbb{E}}_\lambda^k;W_p^{k+1-1/p}(\mathbb{R}^{n-1})),\lambda} \to 0 \quad \text{as } \eta\to 0 \text{ and } \lambda_0\to\infty.$$

To be precise, we shall show that for given $\varepsilon > 0$, there exist $\eta = \eta(n,\mu_0,k,\phi,p,\varepsilon) \in (0,1]$ and $\lambda_0 = \lambda_0(n,\mu_0,k,\phi,p,M,\varepsilon) \ge M^{-1}$ such that the estimate

$$\|F_l(\mu,\omega)\overline{u}\|_{H_p^k(\dot{\mathbb{R}}^n),\lambda} + \|H(\mu,\omega)\overline{u}\|_{W_p^{k+1-1/p}(\mathbb{R}^{n-1}),\lambda} \le \varepsilon\|\overline{u}\|_{\overline{\mathbb{E}}_\lambda^k}$$

is valid for all $\overline{u} \in \overline{\mathbb{E}}_\lambda^k$, all $l \in \{1,2\}$, all $\omega \in C_c^{k+2-}(\mathbb{R}^{n-1})$ with $\|\nabla\omega\|_{W_\infty^{k+1}} \le M$ and $\|\nabla\omega\|_\infty \le \eta$, all $\mu_\pm \in [\mu_0,\mu_0^{-1}]$ and all $\lambda \in \Sigma_\phi$ with $|\lambda| \ge \lambda_0$.

For an estimation of F_l we let $0 \le j \le k$. The product rule and Hölder's inequality yield

$$\begin{aligned}\|\lambda^{(k-j)/2}\nabla^j(F_1(\mu,\omega)\overline{u})\|_p &\le C(n,\mu_0,k)\sum_{i=0}^{j}|\lambda|^{-1/2-(j-i)/2}\|\nabla^{j-i+2}\omega\|_\infty\|\lambda^{(k+2-i)/2}\nabla^i\overline{u}\|_p\\ &\le C(n,\mu_0,k,M)|\lambda|^{-1/2}\|\overline{u}\|_{\overline{\mathbb{E}}_\lambda^k}.\end{aligned}$$

In the norm of $F_2\overline{u}$ we control the leading order terms with a factor η as follows. For $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = k$ and for $0 \le j \le k$ we have

$$\begin{aligned}\|\lambda^{(k-|\alpha|)/2}F_2(\mu,\omega)(\partial_x^\alpha\overline{u})\|_p &\le \eta\,C(n,\mu_0,k)\|\overline{u}\|_{\overline{\mathbb{E}}_\lambda^k},\\ \|\lambda^{(k-j)/2}\nabla^j(F_2(\mu,\omega)\overline{u})\|_p &\le \eta\,C(n,\mu_0,k)\|\overline{u}\|_{\overline{\mathbb{E}}_\lambda^k} + |\lambda|^{-1/2}C(n,\mu_0,k,M)\|\overline{u}\|_{\overline{\mathbb{E}}_\lambda^k}.\end{aligned}$$

We emphasize that the coefficient $C(n,\mu_0,k)$ near η does not depend on the bound M for the derivatives of ω. For the estimations of H, we use the property $\|1-\beta(\omega)\|_\infty \to 0$ as $\|\nabla\omega\|_\infty \to 0$ and the pointwise multiplication estimate (B.7). Furthermore, a scaling argument yields

$$\|v|_{x_n=0}\|_{W_p^{m-1/p}(\mathbb{R}^{n-1}),\lambda} \le C(n,p,m)\|v\|_{H_p^m(\mathbb{R}^n_+),\lambda} \quad \text{for } v\in H_p^m(\mathbb{R}^n_+),\ m\in\mathbb{N},\ \lambda\in\mathbb{C}\setminus\{0\}.$$

Then the leading order terms in the norm of $H\overline{u}$ are estimated by means of

$$[\![H\partial^\alpha\overline{u}]\!]_{W_p^{1-1/p}} \le \eta\, C(n,\mu_0,p)\,[\![\partial^\alpha\overline{u}]\!]_{W_p^{1-1/p}} + C(n,\mu_0,p,M)\|\partial^\alpha\overline{u}\|_p, \quad \text{for } |\alpha|=k,$$

where we have used $|\nabla\omega|\le\eta\le 1$ and $\beta\le 1$. We therefore obtain the estimate

$$\|H(\eta,\omega)\overline{u}\|_{W_p^{k+1-1/p}(\mathbb{R}^{n-1}),\lambda} \le \eta\, C(n,\mu_0,k,p)\|\overline{u}\|_{\overline{\mathbb{E}}^k_\lambda} + |\lambda|^{-1/2+1/2p}C(n,\mu_0,k,p,M)\|\overline{u}\|_{\overline{\mathbb{E}}^k_\lambda}.$$

(iv) We finally consider the operators

$$\overline{A}_\lambda\colon \overline{u}\mapsto ((\lambda-\mu\Delta)\overline{u}, [\![\mu\partial_n\overline{u}]\!], [\![\overline{u}]\!]),$$
$$P(\mu,\omega)\colon \overline{u}\mapsto (F_2\overline{u}+F_1\overline{u}, H_2\overline{u}+H_1\overline{u}, 0).$$

In Lemma 2.10 we have proved that $\overline{A}_\lambda\colon \overline{\mathbb{E}}^k_\lambda \to \overline{\mathbb{F}}^k_\lambda$ is invertible and that

$$\|(\overline{A}_\lambda)^{-1}(\overline{f},\overline{h}_1,\overline{h}_2)\|_{\overline{\mathbb{E}}^k_\lambda} \le C(n,\mu_0,p,\phi,k)\|(\overline{f},\overline{h}_1,\overline{h}_2)\|_{\overline{\mathbb{F}}^k_\lambda}.$$

With step (iii) we can choose numbers $\eta(n,\mu_0,k,\phi,p)\in(0,1]$ and $\lambda_0(n,\mu_0,k,\phi,p,M)\ge 1$ such that $\|(\overline{A}_\lambda)^{-1}P(\mu,\omega)\overline{u}\| \le 2^{-1}\|\overline{u}\|_{\overline{\mathbb{E}}^k_\lambda}$. Then a Neumann series argument and the pull-back estimates (2.17) and (2.18) imply that the desired solution $u\in\mathbb{E}^k_\lambda$ to (2.16) is given by

$$u = A_\lambda^{-1}(f,h_1,h_2) = \Big((I-(\overline{A}_\lambda^{-1})P(\mu,\omega))^{-1}(\overline{A}_\lambda)^{-1}(f\circ\Theta^{-1}, h_1\circ\Theta^{-1}, h_2\circ\Theta^{-1})\Big)\circ\Theta.$$

This representation also shows the uniform bounds for A_λ^{-1}. □

Next, we consider the perturbed model problem

$$\begin{cases} \lambda u - \operatorname{div}(\mu\nabla u) = f & \text{in } \mathbb{R}^n\setminus\Sigma_\omega, \\ [\![\mu\partial_\nu u]\!] = h_1 & \text{on } \Sigma_\omega, \\ [\![u]\!] = h_2 & \text{on } \Sigma_\omega, \end{cases} \tag{2.19}$$

with variable coefficients

$$\mu_\pm\colon \Omega_\pm \to (0,\infty), \quad \Omega_\pm := \{(x',x_n)\in\mathbb{R}^n : x_n \gtrless \omega(x')\}.$$

2.15. Lemma. *Let $\mu_0\in(0,1]$, $k\in\mathbb{N}_0$, $\phi\in(0,\pi)$, and $p\in(1,\infty)$. Then there exists $\eta>0$ such that for every $M>0$ we can find some $\lambda_0\ge 1$ such that the operator*

$$A_\lambda\colon \mathbb{E}^k_\lambda(\mathbb{R}^n\setminus\Sigma_\omega)\to\mathbb{F}^k_\lambda(\mathbb{R}^n\setminus\Sigma_\omega), \quad u\mapsto(\lambda u-\operatorname{div}(\mu\nabla u), [\![\mu\partial_\nu u]\!], [\![u]\!])$$

is uniformly invertible with respect to
(i) $\omega\in C_c^{k+2-}(\mathbb{R}^{n-1})$ with $\|\nabla\omega\|_{W^{k+1}_\infty}\le M$ and $\|\nabla\omega\|_\infty\le\eta$,
(ii) $\mu_\pm\in W^{k+1}_\infty(\Omega_\pm)$ with $\mu_0\le\mu_\pm\le\mu_0^{-1}$, $\|\mu_\pm\|_{W^{k+1}_\infty}\le M$, $\sup\{|\mu_\pm(x)-\mu_\pm(y)| : x,y\in\Omega_\pm\}\le 2\eta$,
(iii) $\lambda\in\Sigma_\phi$ with $|\lambda|\ge\lambda_0$.

Proof. We may choose constants $\mu^*_\pm\in[\mu_0,\mu_0^{-1}]$ such that $\sup\{|\mu_\pm(x)-\mu^*_\pm| : x\in\Omega_\pm\}\le\eta$, for instance $\mu^*_\pm := (\sup\mu_\pm+\inf\mu_\pm)/2$. Then problem (2.19) can be written as

$$\begin{aligned} \lambda u-\mu^*\Delta u &= f+\operatorname{div}((\mu-\mu^*)\nabla u) && \text{in } \mathbb{R}^n\setminus\Sigma_\omega, \\ [\![\mu^*\partial_\nu u]\!] &= h_1+[\![(\mu^*-\mu)\partial_\nu u]\!] && \text{on } \Sigma_\omega, \\ [\![u]\!] &= h_2 && \text{on } \Sigma_\omega. \end{aligned}$$

Let $A^*_\lambda \colon u \mapsto ((\lambda - \mu^*\Delta)u, [\![\mu^*\partial_\nu u]\!], [\![u]\!])$ and $P\colon u \mapsto (\mathrm{div}((\mu - \mu^*)\nabla u), [\![(\mu^* - \mu)\partial_\nu u]\!], 0)$. With $\|\mu_\pm - \mu^*_\pm\|_\infty \le \eta$ and similar estimates as for the perturbations in Lemma 2.14, we obtain

$$\|\mathrm{div}((\mu - \mu^*)\nabla u)\|_{H^k_p(\mathbb{R}^n\setminus\Sigma_\omega),\lambda} \le \Big(\eta\, C(n,\mu_0,k) + |\lambda|^{-1/2}C(n,\mu_0,k,M)\Big)\,\|u\|_{\mathbb{E}^k_\lambda},$$
$$\|[\![(\mu^* - \mu)\partial_\nu u]\!]\|_{W^{k+1-1/p}_p(\Sigma_\omega),\lambda} \le \Big(\eta\, C(n,\mu_0,k,p) + |\lambda|^{-1/2+1/2p}C(n,\mu_0,k,p,M)\Big)\,\|u\|_{\mathbb{E}^k_\lambda}.$$

The operators A^*_λ are uniformly invertible by Lemma 2.14 and a Neumann series argument implies that for some $\eta > 0$ and $\lambda_0 \ge 1$, the operator $A_\lambda = A^*_\lambda - P$ is uniformly invertible. □

Lemma 2.15 includes the case $\Omega \setminus \Sigma = \mathbb{R}^n$, since μ is allowed to be continuous across Σ_0.

2.16. Corollary. *Let $\mu_0 \in (0,1]$, $k \in \mathbb{N}_0$, $\phi \in (0,\pi)$, and $p \in (1,\infty)$. Then there exists $\eta > 0$ such that for every $M > 0$ we can find some $\lambda_0 \ge 1$ such that the operator*

$$A_\lambda \colon \mathbb{E}^k_\lambda(\mathbb{R}^n) \to \mathbb{F}^k_\lambda(\mathbb{R}^n), \quad u \mapsto \lambda u - \mathrm{div}(\mu\nabla u),$$

is uniformly invertible with respect to

(i) $\mu \in W^{k+1}_\infty(\mathbb{R}^n)$ with $\mu_0 \le \mu \le \mu_0^{-1}$, $\|\mu\|_{W^{k+1}_\infty} \le M$ and $\sup\{|\mu(x) - \mu(y)| : x, y \in \mathbb{R}^n\} \le 2\eta$,
(ii) $\lambda \in \Sigma_\phi$ with $|\lambda| \ge \lambda_0$.

The bent half-space problem can be solved analogously as the bent interface problem, by using the half-space result Lemma 2.12 instead of the flat interface result Lemma 2.10.

2.17. Corollary. *Let $\mu_0 \in (0,1]$, $k \in \mathbb{N}_0$, $\phi \in (0,\pi)$, and $p \in (1,\infty)$. Then there exists $\eta > 0$ such that for every $M > 0$ we can find some $\lambda_0 \ge 1$ such that the operator*

$$A_\lambda \colon \mathbb{E}^k_\lambda(\mathbb{R}^n_\omega) \to \mathbb{F}^k_\lambda(\mathbb{R}^n_\omega), \quad u \mapsto (\lambda u - \mathrm{div}(\mu\nabla u), \mu\partial_\nu u)\,,$$

is uniformly invertible with respect to

(i) $\omega \in C^{k+2-}_c(\mathbb{R}^{n-1})$ with $\|\nabla\omega\|_{W^{k+1}_\infty} \le M$ and $\|\nabla\omega\|_\infty \le \eta$,
(ii) $\mu \in W^{k+1}_\infty(\mathbb{R}^n_\omega)$ with $\mu_0 \le \mu \le \mu_0^{-1}$, $\|\mu\|_{W^{k+1}_\infty} \le M$ and $\sup\{|\mu(x) - \mu(y)| : x, y \in \Omega_\pm\} \le 2\eta$,
(iii) $\lambda \in \Sigma_\phi$ with $|\lambda| \ge \lambda_0$.

2.1.5. Bounded domains. We solve the strong transmission problem (2.9) in a bounded domain $\Omega \subset \mathbb{R}^n$ ($n \ge 2$) with boundary $\partial\Omega \in C^{k+2-}$ ($k \in \mathbb{N}_0$) and compact interface $\Sigma \subset \Omega$ of class C^{k+2-} for variable coefficients $\mu_\pm \colon \Omega_\pm \to (0,\infty)$.

2.18. Theorem. *Let $\mu_0 \in (0,1]$, $\phi \in (0,\pi)$, and $p \in (1,\infty)$. Then there exists $\eta > 0$ such that for all $M > 0$ we can find some $\lambda_0 \ge 1$ such that the operator*

$$A_\lambda \colon \mathbb{E}^k_\lambda(\Omega \setminus \Sigma) \to \mathbb{F}^k_\lambda(\Omega \setminus \Sigma), \quad u \mapsto (\lambda u - \mathrm{div}(\mu\nabla u), \mu\partial_\nu u, [\![\mu\partial_\nu u]\!], [\![u]\!])\,,$$

is uniformly invertible with respect to

(i) $\mu \in W^{k+1}_\infty(\Omega \setminus \Sigma)$ with $\mu_0 \le \mu_\pm \le \mu_0^{-1}$ and $\|\mu_\pm\|_{W^{k+1}_\infty} \le M$,
(ii) $\lambda \in \Sigma_\phi$ with $|\lambda| \ge \lambda_0$.

Proof. We apply our localization technique from Section 2.1.1.

(i) We define the *global spaces*

$$E := E_\lambda := \mathbb{E}^k_\lambda(\Omega \setminus \Sigma), \quad F := F_\lambda := \mathbb{F}^k_\lambda(\Omega \setminus \Sigma),$$

equipped with the λ-dependent norms from Section 2.1.2.

For definining local spaces we employ Lemma 2.9, which implies that for every $\eta > 0$ there exist a number $r_0 = r_0(\eta) > 0$ and an $(\eta, r_0(\eta))$-localization set-up. In particular, we can find a finite set $I = I(\eta, r_0(\eta))$, an open covering for $\overline{\Omega}$ of balls $U_j = B_{r_0}(p_j)$ ($j \in I$), rigid transformations

$$\Theta_j \colon x \mapsto p_j + Q_j x, \quad B_{r_0}(0) \to U_j,$$

and height functions $\omega_j \in C_c^{k+2-}(\mathbb{R}^{n-1})$ with $\|\nabla\omega_j\|_\infty \le \eta$ and $\|\nabla\omega\|_{W_\infty^{k+1}} \le M(r)$. Furthermore, the index set can be decomposed into $I = I_1 \cup I_2 \cup I_3$, where $j \in I_1$ corresponds to the whole space case $\Omega \cap U_j = \Theta_j(\mathbb{R}^n \cap B_{r_0})$, $j \in I_2$ corresponds to the bent half-space case $\Omega \cap U_j = \Theta_j(\mathbb{R}^n_{\omega_j} \cap B_{r_0})$, and $j \in I_3$ corresponds to the bent hyperplane case $\Sigma \cap U_j = \Theta_j(\Sigma_{\omega_j} \cap B_{r_0})$. Then we define

$$\begin{aligned} \Omega_j &:= \mathbb{R}^n, & \Sigma_j &:= \emptyset && \text{for } j \in I_1, \\ \Omega_j &:= \mathbb{R}^n_{\omega_j}, & \Sigma_j &:= \emptyset && \text{for } j \in I_2, \\ \Omega_j &:= \mathbb{R}^n, & \Sigma_j &:= \Sigma_{\omega_j} && \text{for } j \in I_3. \end{aligned}$$

Now we define the *local spaces*

$$E_j := E_{j,\lambda} := \mathbb{E}^k_\lambda(\Omega_j \setminus \Sigma_j), \quad F_j := F_{j,\lambda} := \mathbb{F}^k_\lambda(\Omega_j \setminus \Sigma_j) \quad \text{for } j \in I_1 \cup I_2 \cup I_3.$$

We keep in mind that these definitions depend on the localization set-up and this will be fixed in step (iv) during the definition of the local operators.

(ii) We next define *approximation systems* for E and F. Choose a smooth partition of unity $(\varphi_j)_{j\in I}$ for $\overline{\Omega}$ in $\mathbb{R}^n$ subordinate to $(U_j)_{j\in I}$ and choose smooth cut-off functions $(\psi_j)_{j\in I}$ with $\operatorname{supp}\psi_j \subset B_{r_0}$ and $\psi_j \circ \Theta_j^{-1} = 1$ on $\operatorname{supp}\varphi_j$. Then we have $\sum_j \psi_j \circ \Theta_j^{-1} \varphi_j = 1$ in $\overline{\Omega}$. Define

$$\begin{aligned} \Phi_{E,j} u &:= (\varphi_j u) \circ \Theta_j, & \Phi_{F,j}(f,g,h_1,h_2) &:= (\varphi_j f, \varphi_j g, \varphi_j h_1, \varphi_j h_2) \circ \Theta_j, \\ \Psi_{E,j} u_j &:= (\psi_j u_j) \circ \Theta_j^{-1}, & \Psi_{F,j}(f_j,g_j,h_{1j},h_{2j}) &:= (\psi_j f_j, \psi_j g_j, \psi_j h_{1j}, \psi_j h_{2j}) \circ \Theta_j^{-1}. \end{aligned}$$

The triples $(\mathbf{E}, (\Phi_{E,j}), (\Psi_{E,j}))$ and $(\mathbf{F}, (\Phi_{F,j}), (\Psi_{F,j}))$ are indeed l_q-approximation systems for E and F, as can be checked by means of pointwise multiplication $W_p^{1-1/p} \times W_\infty^1 \to W_p^{1-1/p}$ (B.7), the chain rule, the substitution formula (A.12), and the regularity condition $\omega_j \in C_c^{k+2-}(\mathbb{R}^{n-1})$. Moreover, we may choose any $q \in [1,\infty)$, since the set I is finite. Furthermore, the retractions r_E and r_F and the co-retractions r_E^c and r_F^c are defined by

$$\begin{aligned} r_E^c u &:= (\Phi_{E,j} u)_{j\in J}, & r_F^c(f,g,h_1,h_2) &:= (\Phi_{F,j}(f,g,h_1,h_2))_{j\in J}, \\ r_E(u_j)_{j\in J} &:= \sum\nolimits_{j\in J} \Psi_{E,j} u_j, & r_F(f_j,g_j,h_{1j},h_{2j})_{j\in J} &:= \sum\nolimits_{j\in J} \Psi_{F,j}(f_j,g_j,h_{1j},h_{2j}). \end{aligned}$$

These operators satisfy the estimate

$$\|r_X\|_{\mathcal{B}(l_q(\mathbf{X});X),\lambda} + \|r_X^c\|_{\mathcal{B}(X;l_q(\mathbf{X})),\lambda} \le C(n,p,k,I(\eta,r),M(r),q) \quad \text{for } X \in \{E,F\}.$$

The numbers η and r will be fixed below for proving optimal regularity of the relevant model problems. Then the remaining perturbations will be controlled only by the largeness of λ_0.

(iii) In order to define the local operators $A_{\lambda,j}$, we first have to define *local coefficients*. As for the construction of ω_j in Lemma 2.9, we fix a smooth cut-off function $\chi \in \mathcal{B}(\mathbb{R}^n)$ with $0 \le \chi \le 1$, $\chi(x) = 1$ for $|x| \le 1$ and $\chi(x) = 0$ for $|x| \ge 2$. For $j \in I_3$, we consider the transformed coefficients $\overline{\mu}_j := \mu \circ \Theta_j$ that are defined on $B_{r_0} \cap \Omega_j \setminus \Sigma_j$. Given a radius $r \in (0, r_0/2]$, we define

$$\tilde{\mu}_{j,r,\pm}(x) := \overline{\mu}_{j,\pm}(0) + \begin{cases} \chi(x/r)(\overline{\mu}_{j,\pm}(x) - \overline{\mu}_{j,\pm}(0)) & \text{for } x \in \Omega_{j,\pm},\ |x| < 2r, \\ 0 & \text{for } x \in \Omega_{j,\pm},\ |x| \ge 2r. \end{cases}$$

Then $\tilde{\mu}_{j,r,\pm}(x) = \overline{\mu}_{j,\pm}(x)$ for all $x \in B_r \cap \Omega_{j,\pm}$ and $\|\tilde{\mu}_{j,r,\pm} - \overline{\mu}_{j,\pm}(0)\|_\infty \to 0$ as $r \to 0$ by uniform continuity of $\overline{\mu}_{j,\pm}$. Hence, for given $\eta > 0$ we can fix a number $r = r(\eta) \in (0, r_0/2]$ to ensure that the local coefficients $\mu_j := \tilde{\mu}_{j,r}$ satisfy $|\mu_{j,\pm}(x) - \mu_{j,\pm}(y)| \le 2\eta$ for all x, $y \in \Omega_{j,\pm}$, and all $j \in I_3$. In the case $j \in I_1 \cup I_2$, we define μ_j analogously.

(iv) Now we define *local operators* $A_{\lambda,j}$ and fix the chart radius r such that these operators are invertible and satisfy Assumption 2.5.(iv). Given a function $u_j \in E_j$, we let

$$A_{\lambda,j}u_j := \begin{cases} \lambda u_j - \operatorname{div}(\mu_j \nabla u_j) & \text{if } j \in J_1, \\ (\lambda u_j - \operatorname{div}(\mu_j \nabla u_j), \mu_j \partial_\nu u_j) & \text{if } j \in J_2, \\ (\lambda u_j - \operatorname{div}(\mu_j \nabla u_j), [\![\mu_j \partial_\nu u_j]\!], [\![u_j]\!]) & \text{if } j \in J_3. \end{cases}$$

By Corollaries 2.16 and 2.17 and Lemma 2.15, we can find a number $\eta(n, \mu_0, k, \phi, p) > 0$ such that for all $M \geq 1$, there exists $\lambda_0(n, \mu_0, k, \phi, p, M) \geq 1$ such that the operators $A_{\lambda,j}$ ($j \in I$) are uniformly invertible with respect to $\omega_j \in C_c^{k+2-}(\mathbb{R}^{n-1})$ with $\|\nabla \omega_j\|_{W_\infty^{k+1}} \leq M$, $\|\nabla \omega_j\|_\infty \leq \eta$; and $\mu_j \in W_\infty^{k+1}(\Omega_j \setminus \Sigma_j)$ with $\mu_0 \leq \mu_j \leq \mu_0^{-1}$, $\|\mu_{j,\pm}\|_{W_\infty^{k+1}} \leq M$, and $\sup\{|\mu_{j,\pm}(x) - \mu_{j,\pm}(y)| : x, y \in \Omega_{j,\pm}\} \leq 2\eta$; and $\lambda \in \Sigma_\phi$ with $|\lambda| \geq \lambda_0$. In order to fulfill these conditions, we now fix a number $r \in (0, r_0/2]$ and an (η, r)-localization set-up $(U_j, \Theta_j, \omega_j)_{j \in I(\eta,r)}$ such that $\|\nabla \omega_j\|_\infty \leq \eta$ and $|\mu_\pm(x) - \mu_\pm(p_j)| \leq \eta$ for all $x \in U_j \cap \Omega_\pm$ and all $j \in I(\eta, r)$. By compactness of $\partial\Omega$ and Σ and since I is finite, there exists $M = M(\Omega, \Sigma, r) > 0$ such that $\|\nabla \omega_j\|_{W_\infty^{k+1}} \leq M$ and $\|\mu_j\|_{W_\infty^{k+1}} \leq M$ for all $j \in I(\eta, r)$. Now the aforementioned results yield suitable numbers λ_0 and C such that

$$\|A_{\lambda,j}^{-1}\|_{\mathcal{B}(F_j;E_j),\lambda} \leq C \quad \text{for } j \in I,\ \lambda \in \Sigma_\phi,\ |\lambda| \geq \lambda_0.$$

(v) Finally, we consider the perturbations $B_{\lambda,j}$ and $C_{\lambda,j}$. Since the mappings Θ_j are affine and since $\mu_j(x) = \mu(\Theta_j(x))$ for $x \in B_r \cap \Omega_j$, we obtain

$$\begin{aligned} B_{\lambda,j}u &= \Phi_{F,j}A_\lambda u - A_{\lambda,j}\Phi_{E,j}u = (\varphi_j A_\lambda u) \circ \Theta_j - A_{\lambda,j}((\varphi u) \circ \Theta_j) \\ &= (\mu \nabla \varphi_j \cdot \nabla u + \operatorname{div}(\mu \nabla \varphi_j\, u), -\mu u \partial_\nu \varphi_j,\ -[\![\mu u]\!]\partial_\nu \varphi_j, 0) \circ \Theta_j. \end{aligned}$$

This commutator is of lower order and therefore

$$\|B_{\lambda,j}u\|_{F_j,\lambda} \leq |\lambda|^{-1/2+1/2p} C(n, \mu_0, k, \phi, p, M)\|u\|_{E,\lambda} \quad \text{for } u \in E,\ \lambda \in \mathbb{C} \setminus \{0\},\ |\lambda| \geq 1,\ j \in I.$$

Since I is finite and $q \in [1, \infty)$, it follows that

$$\sup_{0 \neq u \in E} \frac{\|(B_{\lambda,j}u)_{j \in I}\|_{l_q(\mathbf{F}),\lambda}}{\|u\|_{E,\lambda}} \leq |\lambda|^{-1/2+1/2p} C(n, \mu_0, k, \phi, p, M, |I|, q).$$

For the perturbations $C_{\lambda,j} = A_\lambda \Psi_{E,j} - \Psi_{F,j} A_{\lambda,j}$ we obtain

$$\sup_{0 \neq (u_j)_{j \in I} \in l_q(\mathbf{E})} \frac{\|\sum_j C_{\lambda,j} u_j\|_{F,\lambda}}{\|(u_j)_{j \in I}\|_{L_q(\mathbf{E}),\lambda}} \leq |\lambda|^{-1/2+1/2p} C(n, \mu_0, k, \phi, p, M, |I|, q).$$

Therefore Assumption 2.5 is satisfied and Proposition 2.6 yields the assertion. □

2.2. Transmission problems for $\operatorname{div}(\mu\nabla\cdot)$

We prove optimal $\dot{H}_p^{k+2}$-regularity for the strong transmission problem (2.1) and optimal $\dot{H}_p^1$-regularity for the weak transmission problem (2.2). In Section 2.2.1 we define the solution space $\mathbb{E}^k$ and the data space $\mathbb{F}^k$ that are equipped with equivalent λ-dependent norms. For the basic model problems in $\dot{\mathbb{R}}^n$, $\mathbb{R}_+^n$, and $\mathbb{R}^n$, we prove optimal $\mathbb{E}^k$-regularity in Section 2.2.2, uniformly with respect to λ and ω. Perturbed model problems are solved in Section 2.2.3 for sufficiently large λ. For a bounded domain $\Omega \subset \mathbb{R}^n$ with compact hypersurface $\Sigma \subset \Omega$, we solve the weak transmission problem in Section 2.2.4 and the strong transmission problem in Section 2.2.5.

2.2.1. λ-dependent norms for $\operatorname{div}(\mu\nabla\cdot)$. For $p \in (1,\infty)$, $k \in \mathbb{N}_0$, and an open set $G \subset \mathbb{R}^n$, we consider the semi-normed vector space $\dot{\mathcal{H}}_p^k(G)$ from page 23. The semi-norm $\|\phi\|_{\dot{\mathcal{H}}_p^k(G)} = \|\nabla^k\phi\|_{L_p(G)}$ vanishes if and only if ϕ belongs to the vector space $\mathcal{P}_{k-1}$ of all polynomials of degree not larger than $k-1$ [cf. Gru09, Theorem 4.19]. Therefore its quotient space

$$\dot{H}_p^k(G) := \dot{\mathcal{H}}_p^k(G)/\mathcal{P}_{k-1}, \quad \|\phi\|_{\dot{H}_p^k(G)} := \|\nabla^k\phi\|_{L_p(G)},$$

the *homogeneous Sobolev space*, is a Banach space [cf. Gal11, Exercise III.1.2]. Alternatively, the vector space $\dot{\mathcal{H}}_p^k(G)$ becomes a Banach space when it is endowed with the norm

$$\|\phi\|_{\dot{\mathcal{H}}_p^k(G)\cap L_p(G')} := \|\nabla^k\phi\|_{L_p(G)} + \|\phi\|_{L_p(G')},$$

with some non-empty bounded smooth subdomain $G' \subset G$. The corresponding norms for different subdomains G' are equivalent [cf. Gal11, Section III.1].

Let Ω and Σ satisfy Assumption 2.1. We consider the semi-normed vector space

$$\dot{\mathcal{H}}_p^k(\dot{\Omega}) = \dot{\mathcal{H}}_p^k(\Omega\setminus\Sigma) := \left\{u \in H^k_{p,\mathrm{loc}}(\Omega\setminus\Sigma) : u_\pm \in H^k_{p,\mathrm{loc}}(\overline{\Omega_\pm}), \nabla^k u_\pm \in L_p(\Omega_\pm)\right\},$$

whose semi-norm $\|u\|_{\dot{\mathcal{H}}_p^k(\Omega\setminus\Sigma)} := \|\nabla^k u\|_{L_p(\Omega)}$ vanishes if and only if $u_\pm \in \mathcal{P}_{k-1}$. Then, given $k \in \mathbb{N}_0\cup\{-1\}$, $\lambda \in (0,\infty)$, and $p \in (1,\infty)$, we define the *solution space*

$$\mathbb{E}^k_\lambda := \left(\mathbb{E}^k, \|\cdot\|_{\mathbb{E}^k_\lambda}\right), \quad \mathbb{E}^k := \left(\bigcap\nolimits_{j=1}^{k+2}\dot{\mathcal{H}}_p^j(\Omega\setminus\Sigma)\right)\Big/\mathbb{K}, \tag{2.20}$$

which is a Banach space with respect to the equivalent λ-dependent norm

$$\|u\|_{\mathbb{E}^k_\lambda} := \sum\nolimits_{j=1}^{k+2}\|\lambda^{(k+2-j)/2}\nabla^j u\|_{L_p(\Omega)} + \|\lambda^{(k+2)/2-1/2p}[\![u]\!]\|_{L_p(\Sigma'_\lambda)} + \|\lambda^{(k+2)/2}(u-\langle u\rangle_{\Omega'_\lambda})\|_{L_p(\Omega'_\lambda)}.$$

Here Ω'_λ bounded subdomain of $\Omega\setminus\Sigma$ with C^1-boundary and, in the case $\Sigma\neq\emptyset$, we let $\Sigma'_\lambda\neq\emptyset$ be a bounded subdomain of Σ with C^1-boundary. If $\Sigma=\emptyset$, then we let $\Sigma'_\lambda=\emptyset$. If the semi-norm $\|u\|_{\mathbb{E}^k_\lambda}$ vanishes, then both $u_\pm$ are constant in $\Omega_\pm$ and these constants coincide because of $[\![u]\!]=0$ on Σ'_λ. Hence the null space of the semi-norm $\|\cdot\|_{\mathbb{E}^k_\lambda}$ consists of all constant functions and we will see that this is precisely the space of solutions with trivial data. The parameter λ will again be useful for controlling lower-order perturbations in perturbed model problems.

In order to define the space of data, we recall from page 24 that the functionals $F_{(f,g,h_1)}$ and $F_{\mu\nabla u}$ are considered as elements of the dual space

$$\hat{H}_p^{-1}(\Omega) := \dot{H}^1_{p'}(\Omega)^*, \quad \|F\|_{\hat{H}_p^{-1}(\Omega)} := \sup_{0\neq\phi\in\dot{H}^1_{p'}(\Omega)}\frac{|\langle F|\phi\rangle|}{\|\nabla\phi\|_{L_{p'}(\Omega)}}, \quad \frac{1}{p}+\frac{1}{p'}=1.$$

Then the *space of data* for $k=-1$ is defined by

$$\mathbb{F}^{-1}_{\mathrm{cc},\lambda} := \left(\mathbb{F}^{-1}_{\mathrm{cc}}, \|\cdot\|_{\mathbb{F}^{-1}_{\mathrm{cc},\lambda}}\right), \quad \mathbb{F}^{-1}_{\mathrm{cc}} := \hat{H}_p^{-1}(\Omega)\times\dot{\mathcal{W}}_p^{1-1/p}(\Sigma),$$

which is a Banach space with respect to the equivalent λ-dependent norm

$$\|(F,h_2)\|_{\mathbb{F}^{-1}_{\mathrm{cc},\lambda}} := \|F\|_{\hat{H}_p^{-1}(\Omega)} + [\![h_2]\!]_{W_p^{1-1/p}(\Sigma)} + \|\lambda^{1/2-1/2p}h_2\|_{L_p(\Sigma'_\lambda)}.$$

For $k\geq 0$, it is defined by

$$\mathbb{F}^k_{\mathrm{cc},\lambda} := \left(\mathbb{F}^k_{\mathrm{cc}}, \|\cdot\|_{\mathbb{F}^k_{\mathrm{cc},\lambda}}\right), \quad \mathbb{F}^k_{\mathrm{cc}} := \left\{(f,g,h_1,h_2)\in\mathbb{F}^k : F_{(f,g,h_1)}\in\hat{H}_p^{-1}(\Omega)\right\},$$

where

$$\mathbb{F}^k := H_p^k(\Omega\setminus\Sigma)\times W_p^{k+1-1/p}(\partial\Omega)\times W_p^{k+1-1/p}(\Sigma)\times\bigcap\nolimits_{j=1}^{k+2}\dot{\mathcal{W}}_p^{j-1/p}(\Sigma),$$

which is a Banach space with respect to the equivalent λ-dependent norm

$$\|(f,g,h_1,h_2)\|_{\mathbb{F}^k_{\mathrm{cc},\lambda}} := \|(f,g,h_1,h_2)\|_{\mathbb{F}^k_\lambda} + \|\lambda^{(k+1)/2}F_{(f,g,h_1)}\|_{\hat{H}^{-1}_p(\Omega)},$$
$$\|(f,g,h_1,h_2)\|_{\mathbb{F}^k_\lambda} := \|f\|_{H^k_p(\Omega\setminus\Sigma),\lambda} + \|g\|_{W^{k+1-1/p}_p(\partial\Omega),\lambda} + \|h_1\|_{W^{k+1-1/p}_p(\Sigma),\lambda} + \|h_2\|_{\bigcap_{j=1}^{k+2}\mathcal{W}^{j-1/p}_p(\Sigma)\cap L_p(\Sigma'_\lambda),\lambda},$$

where

$$\|f\|_{H^k_p(\Omega\setminus\Sigma),\lambda} := \sum\nolimits_{j=0}^k \|\lambda^{(k-j)/2}\nabla^j f\|_{L_p(\Omega)},$$
$$\|g\|_{W^{k+1-1/p}_p(\partial\Omega),\lambda} := [\![\nabla^k_\Sigma g]\!]_{W^{1-1/p}_p(\partial\Omega)} + \sum\nolimits_{j=0}^k |\lambda|^{1/2-1/2p}\|\lambda^{(k-j)/2}\nabla^j_{\partial\Omega} g\|_{L_p(\partial\Omega)},$$
$$\|h_1\|_{W^{k+1-1/p}_p(\Sigma),\lambda} := [\![\nabla^k_\Sigma h_1]\!]_{W^{1-1/p}_p(\Sigma)} + \sum\nolimits_{j=0}^k |\lambda|^{1/2-1/2p}\|\lambda^{(k-j)/2}\nabla^j_\Sigma h_1\|_{L_p(\Sigma)},$$
$$\|h_2\|_{\bigcap_{j=1}^{k+2}\mathcal{W}^{j-1/p}_p(\Sigma)\cap L_p(\Sigma'_\lambda),\lambda} := [\![\nabla^{k+1}_\Sigma h_2]\!]_{W^{1-1/p}_p(\Sigma)} + \sum\nolimits_{j=1}^{k+1} |\lambda|^{1/2-1/2p}\|\lambda^{(k+1-j)/2}\nabla^j_\Sigma h_2\|_{L_p(\Sigma)} + \|\lambda^{(k+2)/2-1/2p}h_2\|_{L_p(\Sigma'_\lambda)}.$$

In the basic situations $\Omega\setminus\Sigma \in \{\mathbb{R}^n, \mathbb{R}^n_+, \dot{\mathbb{R}}^n\}$, $\Omega'_\lambda = \lambda^{-1/2}\Omega'_1$, and $\Sigma'_\lambda = \lambda^{-1/2}\Sigma'_1$, we obtain

$$\|u\|_{\mathbb{E}^k_\lambda} = \|u_\lambda\|_{\mathbb{E}^k_1}, \quad \|(f,g,h_1,h_2)\|_{\mathbb{F}^k_\lambda} = \|(f_\lambda,g_\lambda,h_{1\lambda},h_{2\lambda})\|_{\mathbb{F}^k_1},$$

where the rescaled functions u_λ, f_λ, g_λ, $h_{1\lambda}$, and $h_{2\lambda}$ were defined in Section 2.1.2.

2.2.2. Basic model problems. In order to solve problem (2.1), we have to determine the null space and range of the operator $L := \mathrm{div}(\mu\nabla\cdot)$, considered as an unbounded operator in $L_p(\Omega)$. It is clear that all constant functions belong to $N(L)$ and the converse inclusion follows for $p \geq 2n/(n+2)$ from an integration by parts. For the remaining case $p \in (1, 2n/(n+2))$, which is more involved, Simader and Sohr [SS92] obtained the following weak a priori estimate.

2.19. Theorem ([cf. SS92]). *Let $n \geq 2$, $p \in (1,\infty)$, and let $\Omega \subset \mathbb{R}^n$ be either*
- *(i) the whole space $\mathbb{R}^n$,*
- *(ii) the half space $\mathbb{R}^n_+ = \mathbb{R}^{n-1}\times(0,\infty)$,*
- *(iii) a bent half space $\mathbb{R}^n_\omega = \{(x',x_n)\in\mathbb{R}^n : x_n > \omega(x')\}$ whose defining function $\omega \in C^1_c(\mathbb{R}^{n-1})$ satisfies $\|\nabla'\omega\|_\infty \leq \eta$ for some $\eta = \eta(n,p) > 0$,*
- *(iv) a bounded domain in $\mathbb{R}^n$ with C^1-boundary,*
- *(v) or an exterior domain in $\mathbb{R}^n$ with C^1-boundary; that is, $\mathbb{R}^n\setminus\Omega$ is a bounded domain.*

Then there exists a constant $C = C(n,p,\eta,\Omega) > 0$ such that

$$\|\nabla u\|_{L_p(\Omega)} \leq C \sup\left\{\frac{|\int_\Omega \nabla u\cdot\nabla\phi\,dx|}{\|\nabla\phi\|_{L_{p'}(\Omega)}} : \phi \in \dot{H}^1_{p'}(\Omega)\setminus\{0\}\right\} \quad \text{for } u \in \dot{\mathcal{H}}^1_p(\Omega). \tag{2.21}$$

We start with the analog of Lemma 2.10 for the case $\Omega = \mathbb{R}^n$, $\Sigma = \mathbb{R}^{n-1}\times\{0\}$ and $\lambda = 0$; that is, we consider the strong and the weak transmission problem

$$\left\{\begin{aligned} -\mu\Delta u &= f && \text{in } \dot{\mathbb{R}}^n, \\ [\![\mu\partial_n u]\!] &= h_1 && \text{on } \mathbb{R}^{n-1}, \\ [\![u]\!] &= h_2 && \text{on } \mathbb{R}^{n-1}. \end{aligned}\right\}, \quad \left\{\begin{aligned} \int_{\mathbb{R}^n}\mu\nabla u\cdot\nabla\phi\,dx &= \langle F|\phi\rangle && \text{for all } \phi \in \dot{H}^1_{p'}(\mathbb{R}^n), \\ [\![u]\!] &= h_2 && \text{on } \mathbb{R}^{n-1}, \end{aligned}\right\}, \tag{2.22}$$

with constant coefficients $\mu_\pm > 0$. Here the functionals $F_{\mu\nabla u}$ and $F_{(f,h_1)}$ on $\dot{H}^1_{p'}(\mathbb{R}^n)$ are given

$$\langle F_{\mu\nabla u}|\phi\rangle := \int_{\mathbb{R}^n}\mu\nabla u\cdot\nabla\phi\,dx, \quad \langle F_{(f,h_1)}|\phi\rangle := \int_{\mathbb{R}^n} f\phi\,dx - \int_{\mathbb{R}^{n-1}} h_1\phi\,dx', \quad \text{for } \phi \in \dot{H}^1_{p'}(\mathbb{R}^n).$$

Our goal is to prove that the induced operator

$$A\colon \mathbb{E}^k_\lambda \to \mathbb{F}^k_\lambda, \quad Au = \begin{cases} (-\operatorname{div}(\mu\nabla u), [\![\mu\partial_n u]\!], [\![u]\!]) & \text{if } k \geq 0, \\ (F_{\mu\nabla u}, [\![u]\!]) & \text{if } k = -1, \end{cases} \tag{2.23}$$

is an isomorphism. In order to deal with $k = -1$, we modify the strategy of [SS92, Lemma 3.3]; thus, we first derive a variant of the Calderòn-Zygmund estimate $\|\nabla^2\phi\|_p \lesssim \|\Delta\phi\|_p$ for $\phi \in \mathcal{D}(\mathbb{R}^n)$.

2.20. Lemma. *Let $n \geq 2$, $\Sigma := \mathbb{R}^{n-1} \times \{0\}$, $\mu_\pm > 0$, and $p \in (1,\infty)$, and define the vector spaces*

$$Y := Y_\mu := \left\{(x', x_n) \mapsto a' \cdot x' + b\mu(x_n)^{-1}x_n + c : a' \in \mathbb{K}^{n-1},\, b, c \in \mathbb{K}\right\},$$
$$X := X_{p,\mu} := \left\{u \in \dot{\mathcal{H}}^2_p(\dot{\mathbb{R}}^n) : [\![\mu\partial_n u]\!] = [\![u]\!] = 0 \text{ on } \Sigma\right\}, \quad \|u\|_X = \|\nabla^2 u\|_{L_p(\mathbb{R}^n)}.$$

Then X/Y is a Banach space and the map

$$-\mu\Delta\colon X/Y \to L_p(\mathbb{R}^n)$$

is a topological linear isomorphism. In particular, there exists $C(n, p, \mu_\pm) > 0$ such that

$$C^{-1}\|\nabla^2 u\|_{L_p(\mathbb{R}^n)} \leq \|\mu\Delta u\|_{L_p(\mathbb{R}^n)} \leq C\|\nabla^2 u\|_{L_p(\mathbb{R}^n)} \quad \textit{for all } u \in X. \tag{2.24}$$

Furthermore, the map

$$A\colon u \mapsto (-\mu\Delta u, [\![\mu\partial_n u]\!], [\![u]\!]), \quad \dot{\mathcal{H}}^2_p(\dot{\mathbb{R}}^n)/Y \to L_p(\mathbb{R}^n) \times \dot{W}^{1-1/p}_p(\mathbb{R}^{n-1}) \times \dot{W}^{2-1/p}_p(\mathbb{R}^{n-1})$$

is uniformly invertible with respect to $\mu_\pm \in [\mu_0, \mu_0^{-1}]$, for every $\mu_0 \in (0, 1]$.

Proof. (i) For the *injectivity* of $-\mu\Delta$ modulo Y we adapt an argument of Wilke [Wil13, p. 104–105]. Suppose that $u \in X$ satisfies $-\mu\Delta u = 0$ in the sense of $\mathcal{D}'(\dot{\mathbb{R}}^n)$. Then we even have $\Delta u = 0$ in $\mathcal{D}'(\dot{\mathbb{R}}^n)$, but not necessarily in $\mathcal{D}'(\mathbb{R}^n)$. We put $v_+ := u_+ - Ru_-$ on $\mathbb{R}^n_+$ and $v_- := -Rv_+$ on $\mathbb{R}^n_-$ where $(R\phi)(x', x_n) := \phi(x', -x_n)$ denotes even reflection. From $[\![u]\!] = 0$ we infer that $v = 0$ on Σ and hence also $[\![\nabla v]\!] = e_n[\![\partial_n v]\!]$ on Σ. But since $\partial_n v_+ = -\partial_n(Rv_-) = \partial_n v_-$, we have $[\![\partial_n v]\!] = 0$, which yields $v \in \dot{\mathcal{H}}^2_p(\mathbb{R}^n)$ and integrating by parts yields $-\Delta v = 0$ in $\mathcal{D}'(\mathbb{R}^n)$. Here the negative Laplacian represents the Riesz potential $\dot{J}_2 = -\Delta\colon \dot{H}^2_p(\mathbb{R}^n) \to L_p(\mathbb{R}^n)$, which is a topological isomorphism (Theorem B.15). Hence v must be a linear map.

In an analogous way we can check that $w_+ := \mu_+u_+ + \mu_-Ru_-$ on $\mathbb{R}^n_+$ and $w_- := Rw_+$ on $\mathbb{R}^n_-$ yield a function $w \in \dot{\mathcal{H}}^2_p(\mathbb{R}^n)$ with $[\![w]\!] = 0$, $\partial_n w = 0$ on $\mathbb{R}^{n-1}$, and $-\Delta w = 0$. Hence also w is a linear map. By using that $u_\pm$ are linear combinations of $v_\pm$ and $w_\pm$, we easily see that $u \in Y$.

(ii) For the *surjectivity* of $-\mu\Delta$, we construct $u = v + w \in X$ where $v \in \dot{\mathcal{H}}^2_p(\mathbb{R}^n)$ is a representative of $(-\Delta)^{-1}(\mu^{-1}f) \in \dot{H}^2_p(\mathbb{R}^n)$ and $w \in \dot{\mathcal{H}}^2_p(\dot{\mathbb{R}}^n)$ satisfies

$$-\Delta w = 0 \text{ in } \mathcal{D}'(\dot{\mathbb{R}}^n), \quad [\![w]\!] = 0 \text{ on } \Sigma, \quad [\![\mu\partial_n w]\!] = -[\![\mu\partial_n v]\!] \text{ on } \Sigma.$$

By applying the partial Fourier transform and solving the resulting system, we obtain

$$\tilde{w}(\xi', x_n) = ((\mu_+ + \mu_-)|\xi'|)^{-1}e^{-|\xi' x_n|}[\![\mu\partial_n\tilde{v}]\!](\xi') \quad \text{for } \xi' \in \mathbb{R}^{n-1},\, x_n \in \dot{\mathbb{R}}.$$

Therefore w has the following representation, which can be seen by using Jawerth's trace theorem $\dot{H}^2_p(\mathbb{R}^n)|_{x_n=0} = \dot{W}^{2-1/p}_p(\mathbb{R}^{n-1})$ from Theorem B.31, the Riesz potential $\dot{J}'_{-1} = (-\Delta')^{-1/2}$, and the Poisson semigroup $P(x_n) = e^{-x_n(-\Delta')^{1/2}}$:

$$w(\cdot, x_n) = (\mu_+ + \mu_-)^{-1}e^{-|x_n|(-\Delta')^{1/2}}((-\Delta')^{-1/2}[\![\mu\partial_n v]\!]).$$

Hence, w belongs to $\dot{\mathcal{H}}^2_p(\dot{\mathbb{R}}^n)$ and satisfies the asserted a priori estimate. Therefore the operator $-\mu\Delta\colon X \to Y$ is surjective and has a bounded right-inverse.

(iii) Finally, we consider the map $A\colon u \mapsto (-\mu\Delta u, [\![\mu\partial_n u]\!], [\![u]\!])$, which is injective by step (i). For proving surjectivity, we let $(f, h_1, h_2) \in L_p(\mathbb{R}^n) \times \dot{W}_p^{1-1/p}(\mathbb{R}^{n-1}) \times \dot{W}_p^{2-1/p}(\mathbb{R}^{n-1})$ be given. We construct $u = v + w$ with $v = (-\Delta)^{-1}(\mu^{-1}f) \in \dot{\mathcal{H}}_p^2(\mathbb{R}^n)$ and $-\mu\Delta w = 0$, $[\![\mu\partial_n w]\!] = h_1 - [\![\mu\partial_n v]\!]$, $[\![w]\!] = h_2$. The function w can be constructed as in step (ii) and is given by

$$w(\cdot, \pm x_n) = (\mu_+ + \mu_-)^{-1} e^{-|x_n|(-\Delta')^{1/2}} (-(-\Delta')^{-1/2}(h_1 - [\![\mu\partial_n v]\!]) \pm \mu_\mp h_2).$$

Therefore A is uniformly invertible with respect to $\mu_\pm \in [\mu_0, \mu_0^{-1}]$. □

2.21. Remark. The space $X_{p,\mu} = \{u \in \dot{\mathcal{H}}_p^2(\dot{\mathbb{R}}^n) : [\![\mu\partial_n u]\!] = [\![u]\!] = 0 \text{ on } \Sigma\}$ can be identified with the standard space $\dot{\mathcal{H}}_p^2(\mathbb{R}^n)$ by means of the bijection

$$T_\mu := \dot{\mathcal{H}}_p^2(\mathbb{R}^n) \to X_{p,\mu}, \quad (T_\mu u)(x', x_n) = u(x', \mu(x_n)^{-1} x_n).$$

The semi-norms $\|\nabla^k T_\mu \cdot\|_{L_p(\mathbb{R}^n)}$ and $\|\nabla^k \cdot\|_{L_p(\mathbb{R}^n)}$ are equivalent on $\dot{\mathcal{H}}_p^k(\mathbb{R}^n)$ for $k \in \mathbb{N}_0$.

In order to deal with the case $k = -1$, we provide some density results.

2.22. Lemma. *Let $n \geq 2$, $\Sigma := \mathbb{R}^{n-1} \times \{0\}$, $\mu_\pm > 0$, and $p \in (1, \infty)$.*

(i) For $u \in \dot{\mathcal{H}}_p^1(\mathbb{R}^n)$ and $\varepsilon > 0$ there exists $u_\varepsilon \in X_{p,\mu} \cap \dot{\mathcal{H}}_p^1(\mathbb{R}^n)$ such that $\|\nabla(u_\varepsilon - u)\|_{L_p(\mathbb{R}^n)} \leq \varepsilon$.

(ii) For $u \in X_{p,\mu}$ and $\varepsilon > 0$ there exists $u_\varepsilon \in X_{p,\mu} \cap \dot{\mathcal{H}}_p^1(\mathbb{R}^n)$ such that $\|\nabla^2(u_\varepsilon - u)\|_{L_p(\mathbb{R}^n)} \leq \varepsilon$.

Proof. (i) We shall construct u_ε by an *anisotropic mollification*. Let φ_r denote the *Friedrichs mollifier* with support $B_r(0) \subset \mathbb{R}^n$; that is, $\varphi_r(x) = r^{-n}\varphi(x/r)$ with some $\varphi \in \mathcal{D}(\mathbb{R}^n)$ such that $\varphi \geq 0$, $\int_{\mathbb{R}^n} \varphi \, dx = 1$, and $\operatorname{supp} \varphi = B_1(0)$. Then we consider the function

$$u_r := T_\mu(\varphi_r * (T_\mu^{-1} u)) \quad \text{for } r > 0.$$

Then $\varphi_r * (T_\mu^{-1} u)$ belongs to $C^\infty(\mathbb{R}^n)$ and hence $[\![u_\varepsilon]\!] = [\![\mu\partial_n u_\varepsilon]\!] = 0$. Moreover,

$$\nabla(T_\mu^{-1} u_r) = \varphi_r * \nabla(T_\mu^{-1} u) \to \nabla(T_\mu^{-1} u) \text{ in } L_p(\mathbb{R}^n) \text{ as } r \to 0,$$

and hence also $\nabla u_r \to \nabla u$ in $L_p(\mathbb{R}^n)$. Finally, from $T_\mu^{-1} u \in \dot{\mathcal{H}}_p^1(\mathbb{R}^n)$ and $\varphi_r \in \mathcal{D}(\mathbb{R}^n)$ we infer that $\varphi_r * (T_\mu^{-1} u)$ belongs to $\dot{\mathcal{H}}_p^2(\mathbb{R}^n)$. Hence, for some sufficiently small $r = r(\varepsilon) > 0$, there exists some $u_\varepsilon := u_{r(\varepsilon)} \in X_{p,\mu} \cap \dot{\mathcal{H}}_p^1(\mathbb{R}^n)$ with the desired properties.

(ii) By Remark 2.21, the function $T_\mu^{-1} u$ belongs to the usual homogeneous space $\dot{\mathcal{H}}_p^2(\mathbb{R}^n)$ and thanks to Remark B.12, there is a linear function $v_0\colon \mathbb{R}^n \to \mathbb{K}$ such that $\varphi_r * (\chi_R \cdot (T_\mu^{-1} u - v_0))$ converges to $T_\mu^{-1} u - v_0$ in $\dot{\mathcal{H}}_p^2(\mathbb{R}^n)$ as $r \to 0$ and $R \to \infty$. Here $\chi_R \in \mathcal{D}(\mathbb{R}^n)$ denotes the radial Sobolev cut-off function with support $B_R(0)$. Since $\varphi_r * (\chi_R \cdot (T_\mu^{-1} u - v_0))$ belongs to $\mathcal{D}(\mathbb{R}^n)$, the function $u_\varepsilon = T_\mu(\varphi_r * (\chi_R \cdot (T_\mu^{-1} u - v_0)))$ belongs to $X_{p,\mu} \cap \dot{\mathcal{H}}_p^1(\mathbb{R}^n)$ and satisfies the assertion for some small $r = r(\varepsilon) > 0$ and some large $R = R(\varepsilon) > 0$. □

We are ready to prove optimal $\mathbb{E}^{-1}$-regularity in the case $\Sigma \cong \mathbb{R}^{n-1}$ and $[\![u]\!] = 0$.

2.23. Lemma. *Let $n \geq 2$, $\Sigma := \mathbb{R}^{n-1} \times \{0\}$, $\dot{\mathbb{R}}^n := \mathbb{R}^n \setminus \Sigma$, $\mu_0 \in (0, 1]$, and $p \in (1, \infty)$. Then the map*

$$u \mapsto F_{\mu\nabla u}, \quad \dot{H}_p^1(\mathbb{R}^n) \to \hat{H}_p^{-1}(\mathbb{R}^n)$$

is uniformly invertible with respect to $\mu_\pm \in [\mu_0, \mu_0^{-1}]$.

Proof. (i) Similar as in [SS92, Lemma 3.3], we prove that $u \mapsto F_{\mu\nabla u}$ is injective and bounded from below. For $u \in (\dot{\mathcal{H}}_p^2 \cap \dot{\mathcal{H}}_p^1)(\dot{\mathbb{R}}^n) := \dot{\mathcal{H}}_p^2(\dot{\mathbb{R}}^n) \cap \dot{\mathcal{H}}_p^1(\dot{\mathbb{R}}^n)$ and $\phi \in (\dot{\mathcal{H}}_{p'}^2 \cap \dot{\mathcal{H}}_{p'}^1)(\dot{\mathbb{R}}^n)$ we obtain

$$\int_{\mathbb{R}^n} \partial_j u \, \Delta\phi \, dx = \int_{\mathbb{R}^n} \mu\nabla u \cdot \nabla(\partial_j(\mu^{-1}\phi)) \, dx + \int_\Sigma [\![\delta_{jn} \nabla u \cdot \nabla\phi - \partial_j u \, \partial_n \phi]\!] \, dx'.$$

(i.a) Let $j = n$ and assume that $[\![u]\!] = 0$ and $[\![\phi]\!] = 0$. Then $[\![\nabla' u]\!] = \nabla'[\![u]\!] = 0$ and $[\![\nabla'\phi]\!] = 0$. Thus, the integrand in the interface integral vanishes; that is,

$$[\![\delta_{jn}\nabla u\cdot\nabla\phi - \partial_j u\,\partial_n\phi]\!] = [\![\nabla' u\cdot\nabla'\phi]\!] = 0.$$

Let $Z_n := \{\phi\in(\dot{\mathcal{H}}^2_{p'}\cap\dot{\mathcal{H}}^1_{p'})(\dot{\mathbb{R}}^n) : [\![\mu^{-1}\partial_n\phi]\!] = [\![\phi]\!] = 0\} = X_{p',\mu^{-1}}\cap\dot{\mathcal{H}}^1_{p'}(\mathbb{R}^n)$. Then Lemma 2.22.(ii) implies that Z_n is dense in $X_{p',\mu^{-1}}$. By Lemma 2.20, the map $\mu\Delta\colon X_{p',\mu^{-1}}\to L_{p'}(\mathbb{R}^n) = \mu L_{p'}(\mathbb{R}^n)$ is bounded and surjective and the estimate $\|\nabla^2\phi\|_{L_{p'}(\mathbb{R}^n)}\le C(n,p,\mu)\|\Delta\phi\|_{L_{p'}(\mathbb{R}^n)}$ applies to all $\phi\in X_{p',\mu^{-1}}$. Therefore the space ΔZ_n is dense in $L_{p'}(\mathbb{R}^n)$. Furthermore, $\mu^{-1}\partial_n Z_n$ is a subspace of $\dot{\mathcal{H}}^1_{p'}(\mathbb{R}^n)$. Hence for every $u\in\dot{\mathcal{H}}^2_p(\dot{\mathbb{R}}^n)\cap\dot{\mathcal{H}}^1_p(\mathbb{R}^n)$, we obtain

$$\begin{aligned}\|\partial_n u\|_{L_p(\mathbb{R}^n)} &= \sup_{\phi\in Z_n,\,\Delta\phi\neq 0}\frac{|\int_{\mathbb{R}^n}\partial_n u\,\Delta\phi\,dx|}{\|\Delta\phi\|_{L_{p'}(\mathbb{R}^n)}} = \sup_{\phi\in Z_n,\,\Delta\phi\neq 0}\frac{|\int_{\mathbb{R}^n}\mu\nabla u\cdot\nabla(\mu^{-1}\partial_n\phi)\,dx|}{\|\Delta\phi\|_{L_{p'}(\mathbb{R}^n)}}\\ &\le C(n,p,\mu)\sup_{\phi\in Z_n,\,\Delta\phi\neq 0}\frac{|\int_{\mathbb{R}^n}\mu\nabla u\cdot\nabla(\mu^{-1}\partial_n\phi)\,dx|}{\|\nabla^2\phi\|_{L_{p'}(\mathbb{R}^n)}}\le C'(n,p,\mu)\|F_{\mu\nabla u}\|_{\hat{H}^{-1}_p(\mathbb{R}^n)}.\end{aligned}$$

By Lemma 2.22.(i), the inequality also applies to all $u\in\dot{\mathcal{H}}^1_p(\mathbb{R}^n)$.

(i.b) Let $j<n$, $[\![u]\!]=0$, and $[\![\mu^{-1}\phi]\!] = [\![\partial_n\phi]\!] = 0$. Then the interface integral vanishes, since

$$-[\![\delta_{jn}\nabla u\cdot\nabla\phi - \partial_j u\,\partial_n\phi]\!] = [\![\partial_j u\,\partial_n\phi]\!] = 0.$$

We now let $Z_j := \{\phi\in\dot{\mathcal{H}}^2_{p'}(\dot{\mathbb{R}}^n)\cap\dot{\mathcal{H}}^1_{p'}(\dot{\mathbb{R}}^n) : [\![\partial_n\phi]\!] = [\![\mu^{-1}\phi]\!] = 0\}$. Then it is easy to check that $\mu^{-1}Z_j = X_{p',\mu}\cap\dot{\mathcal{H}}^1_{p'}(\mathbb{R}^n)$ and $\mu^{-1}\partial_j Z_j\subset\dot{\mathcal{H}}^1_{p'}(\mathbb{R}^n)$ and that ΔZ_j is dense in $L_{p'}(\mathbb{R}^n)$. Therefore the desired inequality follows in the same way as before.

(ii) It remains to show that $u\mapsto F_{\mu\nabla u}$ is surjective. Let $F\in\hat{H}^{-1}_p(\mathbb{R}^n) = \dot{H}^1_{p'}(\mathbb{R}^n)^*$. Since we may identify $\dot{H}^1_{p'}(\mathbb{R}^n)$ with the closed subspace $\nabla\dot{H}^1_{p'}(\mathbb{R}^n)$ of $L_{p'}(\mathbb{R}^n)^n$, there exists some $f\in L_p(\mathbb{R}^n)^n$ with $\|f\|_{L_p(\mathbb{R}^n)^n} = \|F\|_{\hat{H}^{-1}_p(\mathbb{R}^n)}$ such that $\langle F|\phi\rangle = \int_{\mathbb{R}^n} f\cdot\nabla\phi\,dx$ for all $\phi\in\dot{H}^1_{p'}(\mathbb{R}^n)$ [cf. AF03, Theorem 3.9]. Let $(f_k)\subset H^1_p(\dot{\mathbb{R}}^n)^n$ be a sequence such that $f_k\to f$ in $L_p(\mathbb{R}^n)^n$ as $k\to\infty$ and define

$$\langle F_k|\phi\rangle := \int_{\mathbb{R}^n} f_k\cdot\nabla\phi\,dx = -\int_{\mathbb{R}^n}\operatorname{div} f_k\,\phi\,dx - \int_\Sigma[\![e_n\cdot f_k]\!]\phi\,dx'.$$

Hence $F_k\to F$ in $\hat{H}^{-1}_p(\mathbb{R}^n)$. With Lemma 2.20, we let $u_k\in\dot{\mathcal{H}}^2_p(\dot{\mathbb{R}}^n)\cap\dot{\mathcal{H}}^1_p(\mathbb{R}^n)$ solve the system

$$-\mu\Delta u_k = -\operatorname{div} f_k,\quad [\![\mu\partial_n u_k]\!] = [\![e_n\cdot f_k]\!],\quad [\![u_k]\!] = 0.$$

Then we have $F_{\mu\nabla u_k} = F_k$ and $\|\nabla u_k - \nabla u_{k'}\|_{L_p(\mathbb{R}^n)}\le C\|F_k - F_{k'}\|_{\hat{H}^{-1}_p(\mathbb{R}^n)}$. Therefore $\nabla u_k\to\nabla u$ in $L_p(\mathbb{R}^n)$ for some $u\in\dot{\mathcal{H}}^1_p(\mathbb{R}^n)$ and this limit satisfies $F_{\mu\nabla u} = F$. □

We now prove optimal $\mathbb{E}^k$-regularity for the transmission problems (2.22) in the flat interface case $\Omega = \mathbb{R}^n$ and $\Sigma = \mathbb{R}^{n-1}\times\{0\}$. In the definition of the norms, we let $\Omega'_\lambda = \lambda^{-1/2}\Omega'$ and $\Sigma'_\lambda = \lambda^{-1/2}\Sigma'$, where $\Omega'\neq\emptyset$ and $\Sigma'\neq\emptyset$ are bounded open subsets of $\mathbb{R}^n$ and Σ with C^1-boundaries.

2.24. Lemma. *Let $n\ge 2$, $\mu_0\in(0,1]$, $k\in\mathbb{N}_0\cup\{-1\}$, and $p\in(1,\infty)$. Then the map $A\colon\mathbb{E}^k_\lambda(\dot{\mathbb{R}}^n)\to\mathbb{F}^k_{cc,\lambda}(\dot{\mathbb{R}}^n)$ in (2.23) is uniformly invertible with respect to $\mu_\pm\in[\mu_0,\mu_0^{-1}]$ and $\lambda\in(0,\infty)$.*

Proof. (i) *Uniqueness.* Let $k=-1$ and let $u\in\dot{\mathcal{H}}^1_p(\dot{\mathbb{R}}^n)$ satisfy $F_{\mu\nabla u}=0$ and $[\![u]\!]=0$. Then u belongs to $\dot{\mathcal{H}}^1_p(\mathbb{R}^n)$ and Lemma 2.23 implies that u is constant. For $k\ge 0$ we consider a function $u\in\mathbb{E}^k_\lambda\subset(\dot{\mathcal{H}}^2_p\cap\dot{\mathcal{H}}^1_p)(\dot{\mathbb{R}}^n)$ such that $\mu\nabla u = 0$ and $[\![\mu\partial_n u]\!] = [\![u]\!] = 0$. By Lemma 2.20, u is constant.

(ii) *Existence for* $k = -1$. Given $F \in \hat{H}_p^{-1}(\mathbb{R}^n)$ and $h_2 \in \dot{\mathcal{W}}_p^{1-1/p}(\mathbb{R}^{n-1})$, we construct $u = v + w \in \dot{\mathcal{H}}_p^1(\dot{\mathbb{R}}^n)/\mathbb{K}$ as follows. Let $\mathcal{E}_+ \in \mathcal{B}(\dot{W}_p^{1-1/p}(\mathbb{R}^{n-1}); \dot{H}_p^1(\mathbb{R}_+^n))$ be the extension operator from Theorem B.31. Then the equivalence class $\mathcal{E}_+(h_2 + \mathbb{K}) \in \dot{H}_p^1(\mathbb{R}_+^n)$ has a representative $v_+ \in \dot{\mathcal{H}}_p^1(\mathbb{R}_+^n)$ with $v_+|_{x_n=0} = h_2$. By choosing $v_- := 0$, the function v belongs to $\dot{\mathcal{H}}_p^1(\dot{\mathbb{R}}^n)$ and satisfies $[\![v]\!] = h_2$. Next, we determine $w \in \dot{\mathcal{H}}_p^1(\mathbb{R}^n)$ as a solution to $\int_{\mathbb{R}^n} \mu\nabla w \cdot \nabla\phi\, dx = \langle F - F_{\mu\nabla v}|\phi\rangle$ for $\phi \in \dot{H}_{p'}^1(\mathbb{R}^n)$ by means of Lemma 2.23. Then $u = v + w$ belongs to $\mathbb{E}_\lambda^{-1}$ and solves $F_{\mu\nabla u} = F$ and $[\![u]\!] = h_2$.

(iii) *Existence for* $k \geq 0$. We construct a solution $u = u^1 + u^2 + u^3$ with

$$\begin{aligned} u_\pm^1 &:= \mu_\pm^{-1} \left((-\Delta)^{-1}(E_\pm f_\pm)\right)\big|_{\mathbb{R}_\pm^n}, \\ u_\pm^2(\cdot, \pm x_n) &:= -(\mu_+ + \mu_-)^{-1} e^{-x_n(-\Delta')^{1/2}}(-\Delta')^{-1/2}(h_1 - [\![\mu\partial_n u^1]\!]), \\ u_\pm^3(\cdot, \pm x_n) &:= \pm\mu_\mp(\mu_+ + \mu_-)^{-1} e^{-x_n(-\Delta')^{1/2}}(h_2 - [\![u^1]\!]), \end{aligned}$$

where $x_n > 0$ and $E_\pm \colon H_p^k(\mathbb{R}_\pm^n) \to H_p^k(\mathbb{R}^n)$ is an extension operator. Indeed, the function u^1 belongs to $\bigcap_{j=2}^{k+2} \dot{\mathcal{H}}_p^2(\dot{\mathbb{R}}^n)$ by Theorem B.15 and satisfies $-\mu\Delta u^1 = f$. Hence $h_1 - [\![\mu\partial_n u^1]\!]$ belongs to $\bigcap_{j=1}^{k+1} \dot{\mathcal{W}}_p^{j-1/p}(\mathbb{R}^{n-1})$ and $h_2 - [\![u^1]\!]$ belongs to $\bigcap_{j=2}^{k+2} \dot{\mathcal{W}}_p^{j-1/p}(\mathbb{R}^{n-1})$. Then Theorems B.15 and B.28 imply $u \in \bigcap_{j=2}^{k+2} \dot{\mathcal{H}}_p^j(\dot{\mathbb{R}}^n)$ and we have $[\![\mu\partial_n u]\!] = h_1$ and $[\![u]\!] = h_2$. Finally, Lemma 2.23 yields the estimate $\|\nabla u\|_p \lesssim \|F_{(f,h_1)}\|_{\hat{H}_p^{-1}(\mathbb{R}^n)}$ and therefore u belongs to $\mathbb{E}_\lambda^k$.

(iv) *Uniform estimates with respect to* λ. We employ the rescaled functions u_λ, f_λ, $h_{1\lambda}$, and $h_{2\lambda}$ from page 30. Then the identity $Au = (f, h_1, h_2)$ is equivalent to $Au_\lambda = (f_\lambda, h_{1\lambda}, h_{2\lambda})$ and we have

$$\|u\|_{\mathbb{E}_\lambda^k} = \|u_\lambda\|_{\mathbb{E}_1^k}, \quad \|(f, h_1, h_2)\|_{\mathbb{F}_\lambda^k} = \|(f_\lambda, h_{1\lambda}, h_{2\lambda})\|_{\mathbb{F}_1^k}.$$

Therefore A^{-1} is uniformly bounded with respect to $\lambda \in (0, \infty)$ and $\mu_\pm \in [\mu_0, \mu_0^{-1}]$. □

It remains to study the *half-space problems*

$$\left\{\begin{array}{ll} -\mu\Delta u = f & \text{in } \mathbb{R}_+^n, \\ -\mu\partial_n u = g & \text{on } \mathbb{R}^{n-1}. \end{array}\right\}, \quad \left\{\int_{\mathbb{R}^n} \mu\nabla u \cdot \nabla\phi\, dx = \langle F|\phi\rangle \quad \text{for all } \phi \in \dot{H}_{p'}^1(\mathbb{R}^n).\right\},$$

with a constant coefficient $\mu > 0$. The right one is (except for $\mu \neq 1$) the weak Neumann problem, which is covered by Theorem 2.19. However, we still have to verify the mapping properties with respect to the higher regularity conditions.

2.25. Lemma. *Let* $n \geq 2$, $\Omega = \mathbb{R}_+^n$, $\mu_0 \in (0, 1]$, $k \in \mathbb{N}_0 \cup \{-1\}$, *and* $p \in (1, \infty)$. *Then the operator*

$$A \colon \mathbb{E}_\lambda^k(\mathbb{R}_+^n) \to \mathbb{F}_{cc,\lambda}^k(\mathbb{R}_+^n), \quad u \mapsto Au = \begin{cases} (-\operatorname{div}(\mu\nabla u), -\mu\partial_n u) & \text{if } k \geq 0, \\ F_{\mu\nabla u} & \text{if } k = -1, \end{cases}$$

is uniformly invertible with respect to $\mu \in [\mu_0, \mu_0^{-1}]$ *and* $\lambda \in (0, \infty)$.

Proof. (i) *Uniqueness* follows from Theorem 2.19.

(ii) *Existence for* $k \geq 0$. We construct a solution $u = u^1 + u^2$ by

$$\begin{aligned} u^1 &:= \mu^{-1} \left((-\Delta)^{-1}(E_+ f)\right)\big|_{\mathbb{R}_+^n}, \\ u^2(\cdot, x_n) &:= -\mu^{-1} e^{-x_n(-\Delta')^{1/2}}(-\Delta')^{-1/2}\left(g + \mu\partial_n u^1(\cdot, 0)\right). \end{aligned} \tag{2.25}$$

Then u^1 belongs to $\bigcap_{j=2}^{k+2} \dot{\mathcal{H}}_p^2(\mathbb{R}_+^n)$ by Theorem B.15 and satisfies $-\mu\Delta u^1 = f$. Hence $g + \mu\partial_n u^1(\cdot, 0)$ belongs to $\bigcap_{j=1}^{k+1} \dot{\mathcal{W}}_p^{j-1/p}(\mathbb{R}^{n-1})$ and Theorems B.15 and B.28 imply $u \in \bigcap_{j=2}^{k+2} \dot{H}_p^j(\mathbb{R}_+^n)$ and we have $-\mu\Delta u = f$ and $-\mu\partial_n u(\cdot, 0) = g$. Finally, the weak a priori estimate implies $\|\nabla u\|_p \lesssim \|F_{(f,g)}\|_{\hat{H}_p^{-1}(\mathbb{R}_+^n)}$ and therefore u belongs to $\mathbb{E}_\lambda^k$.

(iii) *Existence for* $k=-1$. Let $F \in \hat{H}_p^{-1}(\mathbb{R}^n_+) = \dot{H}^1_{p'}(\mathbb{R}^n_+)^*$. Since we may identify $\dot{H}^1_{p'}(\mathbb{R}^n_+)$ isometrically with the closed subspace $\nabla\dot{H}^1_{p'}(\mathbb{R}^n_+)$ of $L_{p'}(\mathbb{R}^n_+)^n$, there exists $f \in L_p(\mathbb{R}^n_+)^n$ with $\|f\|_p = \|F\|_{\hat{H}_p^{-1}(\mathbb{R}^n_+)}$ such that $\langle F|\phi\rangle = \int_{\mathbb{R}^n_+} f\cdot\nabla\phi\,dx$ for all $\phi \in \dot{H}^1_{p'}(\mathbb{R}^n_+)$ [cf. AF03, Theorem 3.9]. Let $(f_k) \subset H^1_p(\dot{\mathbb{R}}^n_+)^n$ be a sequence such that $f_k \to f$ in $L_p(\mathbb{R}^n_+)^n$ as $k\to\infty$ and let

$$\langle F_k|\phi\rangle := \int_{\mathbb{R}^n_+} f_k\cdot\nabla\phi\,dx = -\int_{\mathbb{R}^n_+}\operatorname{div} f_k\,\phi\,dx + \int_{\mathbb{R}^{n-1}} e_n\cdot f_k\,\phi\,dx'.$$

Then $F_k \to F$ in $\hat{H}_p^{-1}(\mathbb{R}^n_+)$.

Next, we construct solutions $u_k = u_k^1 + u_k^2 \in \dot{\mathcal{H}}^2_p(\mathbb{R}^n_+)\cap\dot{\mathcal{H}}^1_p(\mathbb{R}^n_+)$ of the systems

$$\mu\Delta u_k = \operatorname{div} f_k \text{ in } \mathbb{R}^n_+,\quad \mu\partial_n u_k = e_n\cdot f_k \text{ on } \mathbb{R}^{n-1},$$

by using (2.25) with f replaced by $-\operatorname{div} f_k$. Then the identity $F_{\mu\nabla u_k} = F_k$ is valid and we have $u_k \in \dot{\mathcal{H}}^1_p(\mathbb{R}^n_+)$ and $\|\nabla u_k - \nabla u_{k'}\|_p \le C\|F_k - F_{k'}\|_{\hat{H}_p^{-1}(\mathbb{R}^n_+)}$. Therefore $\nabla u_k \to \nabla u$ in $L_p(\mathbb{R}^n_+)$ for some $u \in \dot{\mathcal{H}}^1_p(\mathbb{R}^n_+)$ and this limit satisfies $F_{\mu\nabla u} = F$.

(iv) The *uniform estimates* again follow from a scaling argument as on page 45. □

2.2.3. Perturbed model problems. We next solve the transmission problems

$$(2.26)\quad \left\{\begin{aligned} -\operatorname{div}(\mu\nabla u) &= f && \text{in } \Omega\setminus\Sigma,\\ \mu\partial_\nu u &= g && \text{on } \partial\Omega,\\ [\![\mu\partial_\nu u]\!] &= h_1 && \text{on } \Sigma,\\ [\![u]\!] &= h_2 && \text{on } \Sigma. \end{aligned}\right\},\quad \left\{\begin{aligned} \int_\Omega \mu\nabla u\cdot\nabla\phi\,dx &= \langle F|\phi\rangle && \text{for all } \phi\in\dot{H}^1_{p'}(\Omega),\\ [\![u]\!] &= h_2 && \text{on } \Sigma. \end{aligned}\right\}.$$

for the bent interface case $\Omega = \mathbb{R}^n$ and $\Sigma = \Sigma_\omega = \{(x',\omega(x')) : x' \in \mathbb{R}^{n-1}\}$ with variable coefficients $\mu_\pm\colon \Omega_\pm \to (0,\infty)$, where $\Omega_\pm = \{(x',x_n)\in\mathbb{R}^n : x_n \gtrless \omega(x')\}$. In the definitions of the norms of $\mathbb{E}^k_\lambda$ and $\mathbb{F}^k_{\mathrm{cc},\lambda}$ from page 40, we let $\Omega'_\lambda := \Theta_\omega(\lambda^{-1/2}\Omega')$ and $\Sigma'_\lambda := \Theta_\omega(\lambda^{-1/2}\Sigma')$, where $\Omega' \ne \emptyset$ and $\Sigma' \ne \emptyset$ are bounded open subsets of $\mathbb{R}^n$ and $\mathbb{R}^{n-1}\times\{0\}$ with C^1-boundaries. The C^1-diffeomorphism $\Theta_\omega\colon (x',x_n)\mapsto(x',x_n+\omega(x'))$ was studied on page 34.

2.26. Lemma. *Let $n \ge 2$, $\mu_0 \in (0,1]$, $k \in \mathbb{N}_0\cup\{-1\}$, and $p\in(1,\infty)$. Then there exists $\eta > 0$ such that for every $M > 0$ we can find some $\lambda_0 \ge 1$ such that the operator*

$$A\colon \mathbb{E}^k_\lambda(\mathbb{R}^n\setminus\Sigma_\omega) \to \mathbb{F}^k_{cc,\lambda}(\mathbb{R}^n\setminus\Sigma_\omega),\quad u\mapsto Au = \begin{cases} (-\operatorname{div}(\mu\nabla u), [\![\mu\partial_\nu u]\!], [\![u]\!]) & \text{if } k\ge 0,\\ (F_{\mu\nabla u}, [\![u]\!]) & \text{if } k=-1,\end{cases}$$

is uniformly invertible with respect to

(i) $\omega \in C^1_c(\mathbb{R}^{n-1})\cap C^{k+2-}_c(\mathbb{R}^{n-1})$ *with* $\|\nabla\omega\|_{W^{k+1}_\infty} \le M$ *and* $\|\nabla\omega\|_\infty \le \eta$,

(ii) $\mu_\pm \in C(\overline{\Omega_\pm})\cap W^{k+1}_\infty(\Omega_\pm)$ *with* $\mu_0 \le \mu \le \mu_0^{-1}$, $\|\mu\|_{W^{k+1}_\infty} \le M$, *and* $\sup\{|\mu_\pm(x)-\mu_\pm(y)| : x,y\in\Omega_\pm\} \le 2\eta$,

(iii) $\lambda \in [\lambda_0,\infty)$.

Proof. (i) We first consider the case of constant coefficients $\mu_\pm \in [\mu_0,\mu_0^{-1}]$.

(i.a) *Transformation to the flat interface.* As for Lemma 2.14, we consider the pull-backs

$$\overline{u} = u\circ\Theta,\quad \overline{h}_2 = h_2\circ\Theta\quad (\text{for } k\ge -1),$$

where the C^1-diffeomorphism $\Theta = \Theta_\omega : (x',x_n)\mapsto(x',x_n+\omega(x'))$ satisfies $\partial\Theta$, $\partial\Theta^{-1} \in W^{k+1}_\infty(\mathbb{R}^n)$ (cf. p. 34). We further define

$$\overline{f} = f\circ\Theta,\quad \overline{h}_1 = (1+|\nabla\omega|^2)^{1/2}h_1\circ\Theta\quad (\text{for } k\ge 0).$$

For $\phi \in \dot{\mathcal{H}}^1_{p'}(\mathbb{R}^n)$, $(f, h_1, h_2) \in \mathbb{F}^0_{\mathrm{cc},\lambda}$, $\overline{\phi} = \phi \circ \Theta$, and $u \in \mathbb{E}^{-1}_\lambda$, we obtain the transformed functionals

$$\begin{aligned}
\langle F_{(f,h_1)} | \phi \rangle &= \int_{\mathbb{R}^n} f\phi \, dx - \int_{\Sigma_\omega} h_1 \phi \, d\sigma \\
&= \int_{\mathbb{R}^n} \overline{f}\,\overline{\phi} |\det \partial\Theta| \, dx - \int_{\mathbb{R}^{n-1}} \sqrt{1 + |\nabla\omega|^2} (h_1 \circ \Theta) \, \overline{\phi} \, dx' = \left\langle F_{(\overline{f},\overline{h}_1)} \middle| \overline{\phi} \right\rangle, \\
\langle F_{\mu\nabla u} | \phi \rangle &= \int_{\mathbb{R}^n} \mu\nabla u \cdot \nabla\phi \, dx = \int_{\mathbb{R}^n} \overline{\mu} [\partial\Theta]^{-\top} \nabla\overline{u} \cdot [\partial\Theta]^{-\top} \nabla\overline{\phi} |\det \partial\Theta| \, dx \\
&= \langle F_{\overline{\mu}\nabla\overline{u}} | \overline{\phi} \rangle + \int_{\mathbb{R}^n} \overline{\mu}\nabla\overline{u} \cdot \left([\partial\Theta]^{-1} [\partial\Theta]^{-\top} - I \right) \nabla\overline{\phi} \, dx.
\end{aligned}$$

Let $\overline{\mathbb{E}}^k_\lambda$ and $\overline{\mathbb{F}}^k_{\mathrm{cc},\lambda}$ denote the corresponding spaces on $\dot{\mathbb{R}}^n$. Then the maps $u \mapsto \overline{u}$, $\mathbb{E}^k_\lambda \to \overline{\mathbb{E}}^k_\lambda$ $(k \geq -1)$ and $(f, h_1, h_2) \mapsto (\overline{f}, \overline{h}_1, \overline{h}_2)$, $\mathbb{F}^k_{\mathrm{cc},\lambda} \to \overline{\mathbb{F}}^k_{\mathrm{cc},\lambda}$ $(k \geq 0)$ are linear bijections and we obtain the estimates

$$\begin{aligned}
C(n,k,M)^{-1} \|u\|_{\mathbb{E}^k_\lambda} &\leq \|\overline{u}\|_{\overline{\mathbb{E}}^k_\lambda} \leq C(n,k,M) \|u\|_{\mathbb{E}^k_\lambda}, \\
C(n,k,p,M)^{-1} \|(f,h_1,h_2)\|_{\mathbb{F}^k_{\mathrm{cc},\lambda}} &\leq \|(\overline{f},\overline{h}_1,\overline{h}_2)\|_{\overline{\mathbb{F}}^k_{\mathrm{cc},\lambda}} \leq C(n,k,p,M) \|(f,h_1,h_2)\|_{\mathbb{F}^k_{\mathrm{cc},\lambda}}.
\end{aligned}$$

Here the numbers $C(n,k,M)$ and $C(n,k,p,M)$ are uniform with respect to $\lambda \in [1,\infty)$ and with respect to those $\omega \in C_c^{k+2-}(\mathbb{R}^{n-1})$ which satisfy $\|\nabla\omega\|_{W^{k+1}_\infty} \leq M$. Since $2^{-1}\|\nabla\phi\|_{p'} \leq \|\nabla\overline{\phi}\|_{p'} \leq 2\|\nabla\phi\|_{p'}$ for $\phi \in \dot{\mathcal{H}}^1_{p'}(\mathbb{R}^n)$, the map $F \mapsto \overline{F}$, defined by $\langle \overline{F} | \overline{\phi} \rangle := \langle F | \phi \rangle$ for $\overline{\phi} \in \dot{H}^1_{p'}(\mathbb{R}^n)$, is an isomorphism of $\hat{H}^{-1}_p(\mathbb{R}^n)$, and we have

$$2^{-1}\|F\|_{\hat{H}^{-1}_p(\mathbb{R}^n)} \leq \|\overline{F}\|_{\hat{H}^{-1}_p(\mathbb{R}^n)} \leq 2\|F\|_{\hat{H}^{-1}_p(\mathbb{R}^n)} \quad \text{for } F \in \hat{H}^{-1}_p(\mathbb{R}^n).$$

(i.b) *The transformed problems* are given by (cf. p. 35)

$$\left\{ \begin{aligned}
\lambda\overline{u} - \mu\Delta\overline{u} &= \overline{f} + P_2\overline{u} + P_1\overline{u} && \text{in } \dot{\mathbb{R}}^n, \\
[\![\mu\partial_n\overline{u}]\!] &= \overline{h}_1 + H\overline{u} && \text{on } \mathbb{R}^{n-1}, \\
[\![\overline{u}]\!] &= \overline{h}_2 && \text{on } \mathbb{R}^{n-1}.
\end{aligned} \right\}, \quad
\left\{ \begin{aligned}
F_{\overline{\mu}\nabla\overline{u}} &= \overline{F} + P_3\overline{u} && \text{in } \hat{H}^{-1}_p(\mathbb{R}^n), \\
[\![\overline{u}]\!] &= \overline{h}_2 && \text{on } \mathbb{R}^{n-1}.
\end{aligned} \right\}.$$

Here the perturbations $P_l = P_l(\mu,\omega)$ and $H = H(\mu,\omega)$ are given by

$$\begin{aligned}
P_1\overline{u} &= -\mu\Delta'\omega\,\partial_n\overline{u}, \\
P_2\overline{u} &= \mu|\nabla'\omega|^2\partial_n^2\overline{u} - 2\mu\partial_n\nabla'\overline{u} \cdot \nabla'\omega, \\
\langle P_3\overline{u} | \overline{\phi} \rangle &= \int_{\mathbb{R}^n} \overline{\mu}\nabla\overline{u} \cdot \left([\partial\Theta]^{-1}[\partial\Theta]^{-\top} - I \right) \nabla\overline{\phi} \, dx, \\
H\overline{u} &= \nabla'\omega \cdot [\![\mu\nabla'\overline{u}]\!] - |\nabla\omega|^2 [\![\mu\partial_n\overline{u}]\!].
\end{aligned}$$

For $\overline{u} \in \overline{\mathbb{E}}^k_\lambda$ and $\lambda \in [\lambda_0, \infty)$ we obtain the following estimates (cf. p. 35).

$$\begin{aligned}
&\|\lambda^{(k-j)/2}\nabla^j(P_1\overline{u})\|_p \leq \lambda^{-1/2} C(n,\mu_0,k,M) \|\overline{u}\|_{\overline{\mathbb{E}}^k_\lambda} && \text{for } 0 \leq j \leq k, \\
&\|\lambda^{(k-j)/2}\nabla^j(P_2\overline{u})\|_p \leq \left(\eta\, C(n,\mu_0,k) + \lambda^{-1/2} C(n,\mu_0,k,M) \right) \|\overline{u}\|_{\overline{\mathbb{E}}^k_\lambda} && \text{for } 0 \leq j \leq k, \\
&\|\lambda^{(k+1)/2} P_3\overline{u}\|_{\hat{H}^{-1}_p(\mathbb{R}^n)} \leq \eta\, C(n,\mu_0) \|\overline{u}\|_{\overline{\mathbb{E}}^k_\lambda}, \\
&\|H\overline{u}\|_{W^{k+1-1/p}_p(\mathbb{R}^{n-1}),\lambda} \leq (\eta + \lambda^{-1/2+1/2p})\, C(n,\mu_0,k,p,M,\lambda_0) \|\overline{u}\|_{\overline{\mathbb{E}}^k_\lambda} && \text{for } k \geq 0.
\end{aligned}$$

Therefore a Neumann series argument as on page 36 yields the invertibility of A and the uniform bounds in the case of constant coefficients $\mu_\pm \in [\mu_0, \mu_0^{-1}]$.

(ii) For variable coefficients $\mu_\pm$, we proceed as in the proof of Lemma 2.15. We study the perturbed problems (cf. p. 36)

$$\left\{\begin{aligned} \lambda u - \mu^*\Delta u &= f + P_4 u && \text{in } \mathbb{R}^n \setminus \Sigma_\omega, \\ [\![\mu^*\partial_\nu u]\!] &= h_1 + H_2 u && \text{on } \Sigma_\omega, \\ [\![u]\!] &= \overline{h}_2 && \text{on } \Sigma_\omega. \end{aligned}\right\}, \quad \left\{\begin{aligned} F_{\mu^*\nabla u} &= F + P_5 u && \text{in } \hat{H}_p^{-1}(\mathbb{R}^n), \\ [\![\overline{u}]\!] &= \overline{h}_2 && \text{on } \Sigma_\omega. \end{aligned}\right\},$$

where the perturbations $P_l = P_l(\mu, \omega)$ and $H_2 = H_2(\mu, \omega)$ are given by

$$\begin{aligned} P_4 u &= \operatorname{div}((\mu - \mu^*)\nabla u), \\ \langle P_5 u | \phi \rangle &= \int_{\mathbb{R}^n} (\mu - \mu^*)\nabla u \cdot \nabla \phi \, dx, \\ H_2 u &= [\![(\mu^* - \mu)\partial_\nu u]\!]. \end{aligned}$$

These perturbations can be estimated as follows.

$$\begin{aligned} \|P_4 u\|_{H_p^k(\mathbb{R}^n\setminus\Sigma_\omega),\lambda} &\le \Big(\eta\, C(n,\mu_0,k) + \lambda^{-1/2} C(n,\mu_0,k,M)\Big) \|u\|_{\mathbb{E}_\lambda^k} && \text{if } k \ge 0, \\ \|\lambda^{(k+1)/2} P_5 u\|_{\hat{H}_p^{-1}(\mathbb{R}^n)} &\le \eta\, \|u\|_{\mathbb{E}_\lambda^k}, \\ \|H_2 u\|_{W_p^{k+1-1/p}(\Sigma_\omega),\lambda} &\le \Big(\eta\, C(n,\mu_0,k,p) + \lambda^{-1/2+1/2p} C(n,\mu_0,k,p,M)\Big) \|u\|_{\mathbb{E}_\lambda^k} && \text{if } k \ge 0. \end{aligned}$$

Again, a Neumann series argument yields the uniform invertibility of $A\colon \mathbb{E}_\lambda^k \to \mathbb{F}_{\mathrm{cc},\lambda}^k$. □

The solvability of the perturbed model problem (2.26) in case $\Omega = \mathbb{R}^n$ and $\Omega'_\lambda = \lambda^{-1/2}\Omega'$ for $\Omega' \subset \mathbb{R}^n$ and $\Sigma = \Sigma' = \emptyset$ follows again by considering continuous coefficient functions.

2.27. Corollary. *Let $n \ge 2$, $\mu_0 \in (0,1]$, $k \in \mathbb{N}_0 \cup \{-1\}$, and $p \in (1,\infty)$. Then there exists $\eta > 0$ such that for every $M > 0$ we can find some $\lambda_0 \ge 1$ such that the operator*

$$A\colon \mathbb{E}_\lambda^k(\mathbb{R}^n) \to \mathbb{F}_{cc,\lambda}^k(\mathbb{R}^n), \quad u \mapsto Au = \begin{cases} -\operatorname{div}(\mu\nabla u) & \text{if } k \ge 0, \\ F_{\mu\nabla u} & \text{if } k = -1, \end{cases}$$

is uniformly invertible with respect to $\mu \in C(\mathbb{R}^n) \cap W_\infty^{k+1}(\mathbb{R}^n)$ with $\mu_0 \le \mu \le \mu_0^{-1}$, $\|\mu\|_{W_\infty^{k+1}} \le M$, and $\sup\{|\mu(x) - \mu(y)| : x, y \in \mathbb{R}^n\} \le 2\eta$, and $\lambda \in [\lambda_0, \infty)$.

The bent half-space problems (2.26) for $\Omega = \mathbb{R}_\omega^n$ and $\Omega'_\lambda = \Theta_\omega(\lambda^{-1/2}\Omega')$ with $\Omega' \subset \mathbb{R}_+^n$ and $\Sigma = \Sigma' = \emptyset$ can be solved analogously as the bent interface problem, by following the lines of the proof of Lemma 2.26 and by using the half-space result Lemma 2.25.

2.28. Corollary. *Let $n \ge 2$, $\mu_0 \in (0,1]$, $k \in \mathbb{N}_0 \cup \{-1\}$, and $p \in (1,\infty)$. Then there exists $\eta > 0$ such that for every $M > 0$ we can find some $\lambda_0 \ge 1$ such that the operator*

$$A\colon \mathbb{E}_\lambda^k(\mathbb{R}_\omega^n) \to \mathbb{F}_{cc,\lambda}^k(\mathbb{R}_\omega^n), \quad u \mapsto Au = \begin{cases} (-\operatorname{div}(\mu\nabla u), \mu\partial_\nu u) & \text{if } k \ge 0, \\ F_{\mu\nabla u} & \text{if } k = -1, \end{cases}$$

is uniformly invertible with respect to

(i) $\omega \in C_c^1(\mathbb{R}^{n-1}) \cap C_c^{k+2-}(\mathbb{R}^{n-1})$ with $\|\nabla\omega\|_{W_\infty^{k+1}} \le M$ and $\|\nabla\omega\|_\infty \le \eta$,

(ii) $\mu \in C(\overline{\mathbb{R}_\omega^n}) \cap W_\infty^{k+1}(\mathbb{R}_\omega^n)$ with $\mu_0 \le \mu \le \mu_0^{-1}$, $\|\mu\|_{W_\infty^{k+1}} \le M$, and $\sup\{|\mu(x) - \mu(y)| : x, y \in \mathbb{R}_\omega^n\} \le 2\eta$,

(iii) $\lambda \in [\lambda_0, \infty)$.

2.2.4. The weak transmission problem in bounded domains. We next consider the problems (2.26) in bounded domain $\Omega \subset \mathbb{R}^n$ with C^1-boundary $\partial\Omega$ and C^1-interface $\Sigma \subset \Omega$ and variable coefficients $\mu_\pm \in C(\overline{\Omega}_\pm)$. We first study uniqueness of weak solutions.

2.29. Lemma. *Let Ω and Σ be bounded, let $\mu_\pm \in C(\overline{\Omega_\pm})$ with $\inf \mu_\pm > 0$ and $p \in (1,\infty)$. Then every solution $u \in H^1_p(\Omega)$ of the problem*

$$\int_\Omega \mu\nabla u \cdot \nabla\phi \, dx = 0 \quad \textit{for all } \phi \in \mathcal{D}(\mathbb{R}^n)$$

is a constant function.

Proof. The proof of this lemma is easy for $p \in [2,\infty)$, since we can choose $\phi = \overline{u}$, the complex conjugate of u. For $p \in (1,2)$ we employ the localization procedure of [SS92, Lemma 3.9].

(i) First, we assume that ∇u belongs to $L_2(\Omega)$. If $p \geq 2$, then u belongs to $H^1_2(\Omega)$ by the Poincaré-Wirtinger inequality. By choosing $\phi = \overline{u} \in H^1_2(\Omega)$ we obtain $\int_\Omega \mu\nabla u \cdot \nabla\overline{u}\, dx = 0$ and hence $\nabla u = 0$ in Ω, which yields the assertion. In the case $p \in (1,2)$, we let

$$1/q_j := 1/p - j/n, \quad j \in \{0,1,\dots,k\}, \tag{2.27}$$

where $k \in \mathbb{N}_0$ is chosen such that $1/q_k \leq 1/2 < 1/q_{k-1}$. From the Sobolev embedding theorem we obtain the embedding $H^1_{q_j}(\Omega) \hookrightarrow L_{q_{j+1}}(\Omega)$. For $j \leq k$ we have $\nabla u \in L_2(\Omega) \hookrightarrow L_{q_j}(\Omega)$ and therefore induction yields $u \in H^1_{q_k}(\Omega) \hookrightarrow L_{q_{k+1}}(\Omega) \hookrightarrow L_2(\Omega)$. Hence u belongs to $H^1_2(\Omega)$ and we again obtain $\nabla u = 0$ in Ω. It remains to prove that $\nabla u \in L_2(\Omega)$ for all $u \in H^1_p(\Omega)$ with $F_{\mu\nabla u} = 0$.

(ii) *Localization set-up.* Lemma 2.9 implies that for every given $\eta > 0$ there exists $r_0(\eta) > 0$ such that for all $r \in (0, r_0(\eta)]$ we can find an (η,r)-localization set-up $(U_j,\Theta_j,\omega_j)_{j\in I(\eta,r)}$ for $\Omega\setminus\Sigma$. Let $I = I_1 \cup I_2 \cup I_3$, $\Theta_j\colon x \mapsto p_j + Q_j x$, Ω_j, and Σ_j have the same meaning as in the proof of Theorem 2.18 on page 37. We may also assume that the sets $(\Theta_j(B_{r2^{-k-1}}))_{j\in I}$ cover $\overline{\Omega}$ with k from step (i).

There exists $\mu_0 \in (0,1]$ such that $\mu_0 \leq \mu_\pm \leq \mu_0^{-1}$ in $\Omega\setminus\Sigma$. We now choose the number $\eta(n,\mu_0,p) > 0$ such that Lemma 2.26 and Corollaries 2.27 and 2.28 are applicable. Then there exists $r_\mu(\eta) > 0$ such that $|\mu_\pm(x) - \mu_\pm(y)| \leq 2\eta$ for all $x,y \in \Omega_\pm$ with $|x-y| \leq 2r_\mu(\eta)$. We define local coefficient functions μ_j as on page 38 and obtain $\|\mu_{j,\pm} - \mu^*_{j,\pm}\|_\infty \leq \eta$ for some constants $\mu^*_{j,\pm}$ and all $j \in I(\eta,r)$, provided that $r \in (0, r_0(\eta)/2] \cap (0, r_\mu(\eta)/2]$. Now the aforementioned results are applicable and yield suitable numbers $\lambda_0 \geq 1$ and $C \geq 1$ such that

$$\|\nabla(A_j^{-1}F_j)\|_{L_p(\Omega_j)} \leq C\|F_j\|_{\hat{H}^{-1}_p(\Omega_j)} \quad \text{for } F_j \in \hat{H}^{-1}_p(\Omega_j),\ j \in I,\ \lambda \in [\lambda_0,\infty),$$

(iii) We now show that ∇u belongs to $L_2(\Omega)$ by refining the argument in step (i). Let $j \in I_3$ be fixed. We define the numbers q_l ($l \in \{0,1,\dots,k\}$) by (2.27) and let $r_l := r2^{-l}$ ($l \in \{0,1,\dots,k+1\}$). We further choose $\psi_l \in \mathcal{D}(B_{r_l})$ such that $0 \leq \psi_l \leq 1$ and $\psi_l = 1$ on $B_{r_{l+1}} \subset \mathbb{R}^n$. For every $v \in \dot{\mathcal{H}}^1_{p'}(\mathbb{R}^n)$, we let $v_l := v|_{B_{r_l}} - \langle v\rangle_{B_{r_l}}$. Since $(\psi_l v_l)\circ\Theta_l^{-1}$ belongs to $\dot{\mathcal{H}}^1_{p'}(\Omega)$ and since $\partial_x\Theta_j$ is orthogonal, we obtain

$$\int_{\mathbb{R}^n} \mu_j \nabla(u\circ\Theta_j)\cdot\nabla(\psi_l v_l)\, dx = \int_\Omega \mu\nabla u \cdot \nabla((\psi_l v_l)\circ\Theta_j^{-1})\, dx = 0.$$

With $\overline{u} := u\circ\Theta_j$, this yields

$$\int_{\mathbb{R}^n} \mu_j\nabla(\psi_l\overline{u})\cdot\nabla v_l\, dx = \int_{B_{r_l}} \mu_j\overline{u}\nabla\psi_l\cdot\nabla v_l\, dx - \int_{B_{r_l}} \mu_j\nabla\overline{u}\cdot(v_l\nabla\psi_l)\, dx. \tag{2.28}$$

From (2.28) we shall deduce that $\psi_l\overline{u} \in H^1_{q_l}(\mathbb{R}^n)$ for $l \in \{0,1,\dots,k\}$ by induction. For $l = 0$ the assertion is valid since $q_0 = p$. Next, suppose that $\psi_{l-1}\overline{u}$ belongs to $H^1_{q_{l-1}}(\mathbb{R}^n)$ for some $l \in \{1,\dots,k\}$. With $\psi_{l-1} = 1$ on $B_{r_l} = B_{r_{l-1}/2}$, this implies $\overline{u} \in H^1_{q_{l-1}}(B_{r_l}) \hookrightarrow L_{q_l}(B_{r_l})$ by the Sobolev embedding theorem. With the dual exponents q_l', defined by $1/q_l' := 1 - 1/q_l =$

$1/p' + l/n$ ($l \in \{0, 1, \dots, k\}$), we obtain $v \in \dot{\mathcal{H}}^1_{p'}(\mathbb{R}^n) \subset \dot{\mathcal{H}}^1_{q'_l,\mathrm{loc}}(\mathbb{R}^n) \subset L_{q'_{l-1},\mathrm{loc}}(\mathbb{R}^n)$ from the Poincaré-Wirtinger inequality and the Sobolev embedding theorem. Hence

$$\left|\int_{B_{r_l}} \mu_j \overline{u} \nabla\psi_l \cdot \nabla v_l \, dx\right| \le \|\mu_j \overline{u}\|_{L_{q_l}(B_{r_l})} \|\nabla\psi_l\|_\infty \|\nabla v_l\|_{L_{q'_l}(B_{r_l})} \le C_1(\mu, u, \psi_l) \|\nabla v_l\|_{L_{p'}(\mathbb{R}^n)},$$

$$\left|\int_{B_{r_l}} \mu_j \nabla\overline{u} \cdot (v_l \nabla\psi_l) \, dx\right| \le \|\mu_j \nabla\overline{u}\|_{L_{q_{l-1}}(B_{r_l})} \|v\|_{L_{q'_{l-1}}(B_{r_l})} \|\nabla\psi_l\|_\infty \le C_2(\mu, u, \psi_l) \|\nabla v_l\|_{L_{p'}(\mathbb{R}^n)}.$$

Since the map $v \mapsto v_l + \mathbb{K}$, $\dot{\mathcal{H}}^1_{p'}(\mathbb{R}^n) \to \dot{H}^1_{p'}(B_{r_l})$ is surjective, the identity (2.28) and Lemma 2.26 imply $\nabla(\psi_l \overline{u}) \in L_{q_l}(\mathbb{R}^n)$ and hence $\psi_l \overline{u} \in H^1_{q_l}(\mathbb{R}^n)$. Induction therefore yields $\psi_k \overline{u} \in H^1_{q_k}(B_{r_k}) \hookrightarrow H^1_2(B_{r_k})$ and hence $\overline{u}|_{B_{r2^{-k-1}}} \in H^1_2(B_{r_j 2^{-k-1}})$. In the case $j \in I_1 \cup I_2$ we proceed analogously, by using Corollaries 2.27 and 2.28 instead of Lemma 2.26. Since the open sets $\Theta_j(B_{r_j 2^{-k-1}})$ cover $\overline{\Omega}$, we obtain $u \in H^1_2(\Omega \setminus \Sigma)$. Then step (i) yields the assertion. □

We are ready to prove that the weak transmission problem (2.2) has optimal $\dot{H}^1_p$-regularity.

Proof of Theorem 2.3. The cases $\Omega \setminus \Sigma \in \{\mathbb{R}^n, \mathbb{R}^n_\omega, \mathbb{R}^n \setminus \Sigma_\omega\}$ were solved in Lemma 2.26 and Corollaries 2.27 and 2.28. For the remaining case we follow the proof of [SS92, Theorem 1.3].

(i) We prove the *weak a priori estimate*

$$\|\mu\nabla u\|_{L_p(\Omega)} \le C\|F_{\mu\nabla u}\|_{\hat{H}^{-1}_p(\Omega)} \quad \text{for } u \in \dot{\mathcal{H}}^1_p(\Omega). \tag{2.29}$$

Assume that it is not true. Then we find a sequence $(u_k) \subset \dot{\mathcal{H}}^1_p(\Omega)$ such that

$$1 = \|\mu\nabla u_k\|_{L_p(\Omega)} \ge k\|F_{\mu\nabla u_k}\|_{\hat{H}^{-1}_p(\Omega)} \quad \text{for all } k \in \mathbb{N}.$$

We may assume that $\int_\Omega u_k \, dx = 0$, so that the sequence $\|u_k\|_{L_p(\Omega)}$ is bounded by the Poincaré-Wirtinger inequality. Since $H^1_p(\Omega)$ is compactly embedded into $L_p(\Omega)$, we may also assume that the sequence (u_k) converges in $L_p(\Omega)$ to some limit $u \in L_p(\Omega)$. Furthermore, the space $Z := \{v \in \dot{\mathcal{H}}^1_p(\Omega) : \int_\Omega v \, dx = 0\}$ with norm $\|\mu\nabla\cdot\|_{L_p(\Omega)}$ is isomorphic to the closed subspace $\mu\nabla Z$ of $L_p(\Omega)^n$ and therefore Z is reflexive. Hence we may even assume that u belongs to Z and that (u_k) converges weakly to u; that is, on the one hand $F_{\mu\nabla u_k} \to F_{\mu\nabla u}$ in $\hat{H}^{-1}_p(\Omega)$, but also

$$\lim_{k\to\infty} \int_\Omega \mu\nabla u_k \cdot f \, dx = \int_\Omega \mu\nabla u \cdot f \, dx \quad \text{for all } f \in L_{p'}(\Omega)^n. \tag{2.30}$$

Thus, $F_{\mu\nabla u} = 0$ and hence Lemma 2.29 implies that $\nabla u = 0$.

Next, as in step (ii) in the proof of Lemma 2.29, we consider an open covering (U_j) for $\overline{\Omega}$ with $j \in I = I_1 \cup I_2 \cup I_3$, and rigid transformations $\Theta_j \colon B_r(0) \subset \mathbb{R}^n \to U_j \subset \mathbb{R}^n$. We assume that the smaller sets $\Theta_j(B_{r_j/2})$ form an open covering for $\overline{\Omega}$ and choose functions $\psi_j \in \mathcal{D}(B_r)$ with $0 \le \psi_j \le 1$ and $\psi_j = 1$ on $B_{r_j/2}$. The weak a priori estimates for the model problems in Lemma 2.26 and Corollaries 2.27 and 2.28 imply that there is a number $C(n, \mu, p, \eta) > 0$ such that

$$\|\nabla(\psi_j \, u_k \circ \Theta_j)\|_{L_p(\Omega_j)} \le C\|F_{\mu_j\nabla(\psi_j \, u_k \circ \Theta_j)}\|_{\hat{H}^{-1}_p(\Omega_j)} \quad \text{for } j \in I,\ k \in \mathbb{N},$$

where the height functions $\omega_j \in C^1_c(\mathbb{R}^{n-1})$, $j \in I_2 \cup I_3$, satisfy $\|\nabla'\omega_j\|_\infty \le \eta$ and the local coefficients μ_j satisfy $\|\mu_j - \mu^*_j\|_\infty \le \eta$ for some locally constant functions μ^*_j.

Let j be fixed and put $B := B_r$, $\overline{u}_k := u_k \circ \Theta_j$. We choose a sequence $(v_k) \subset \dot{\mathcal{H}}^1_{p'}(B)$ with $\int_B v_k \, dx = 0$ and $\|\nabla v_k\|_{L_{p'}(B)} = 1$ which converges strongly to some v in $L_{p'}(B)$ and satisfies

$$\left|\int_B \mu_j \nabla(\psi_j \overline{u}_k) \cdot \nabla v_k \, dx\right| \ge d_k - \frac{1}{k}, \quad \text{with } d_k := \|F_{\mu_j\nabla(\psi_j\overline{u}_k)}\|_{\hat{H}^{-1}_p(\Omega_j)}.$$

In order to show that $d_k \to 0$, we compute

$$\int_B \mu_j \nabla(\psi_j \overline{u}_k) \cdot \nabla v_k \, dx$$
$$= \int_B \mu_j \nabla \overline{u}_k \cdot \nabla(\psi_j v_k) \, dx + \int_B \mu_j \overline{u}_k \nabla \psi_j \cdot \nabla v_k \, dx - \int_B \mu_j \nabla \overline{u}_k \cdot (v_k \nabla \psi_j) \, dx.$$

Here the first summand on the right-hand side vanishes for $k \to \infty$, which can be seen by transforming the integral from B to $\Theta_j(B)$ with the orthogonality of $\partial_x \Theta_j$ and by using that $F_{\mu\nabla u_k} \to 0$ in $\hat{H}_p^{-1}(\Omega)$. The second integral vanishes, since $u_k \to 0$ in $L_p(\Omega)$ and since $\|\mu_j\|_\infty$, $\|\nabla\psi_j\|_\infty$, $\|\nabla v_k\|_{p'}$, $\|\nabla' \omega_j\|_\infty$ are bounded. Finally, since $\|v_k\|_{p'}$ is bounded, we may use (2.30) and $\nabla u = 0$ to conclude that also the third integral vanishes.

We have shown that $\lim_{k\to\infty} F_{\mu_j \nabla(\psi_j \, u_k \circ \Theta_j)} = 0$ in $\hat{H}_p^{-1}(\Omega_j)$ for each j. The weak a priori estimates for the model problems therefore imply that $\lim_{k\to\infty} \nabla(\psi_j \, u_k \circ \Theta_j) = 0$ in $L_p(B_r)^n$ for every j. With $\psi_j = 1$ on $B_{r_j/2}$ and since the sets $\Theta_j(B_{r_j/2})$ cover $\overline{\Omega}$, we conclude that $\nabla u_k \to 0$ in $L_p(\Omega)$. This is a contradiction to $\|\mu\nabla u_k\|_{L_p(\Omega)} = 1$. Therefore estimate (2.29) is valid.

(ii) *Existence for given $F \in \hat{H}_p^{-1}(\Omega)$ and $h_2 = 0$.* We employ the strategy from [SS92, Lemma 3.1]. Since the space $\nabla\dot{\mathcal{H}}_p^1(\Omega)$ is closed in $L_p(\Omega)^n$, it follows from step (i) that $X := \{F_{\mu\nabla u} : u \in \dot{\mathcal{H}}_p^1(\Omega)\}$ is a closed subspace of $\hat{H}_p^{-1}(\Omega)$. We assume that $X \neq \hat{H}_p^{-1}(\Omega)$ and seek a contradiction. The Hahn-Banach theorem yields a non-trivial functional $J \in (\hat{H}_p^{-1}(\Omega))^* \setminus \{0\}$ such that $J|_X = 0$. Since closed subspaces and quotient spaces of reflexive spaces are again reflexive, we may identify $(\hat{H}_p^{-1}(\Omega))^* = (\dot{H}_{p'}^1(\Omega))^{**} \cong \dot{H}_{p'}^1(\Omega)$. Hence there exists a unique $\phi \in \dot{H}_{p'}^1(\Omega)$ with $\|\nabla\phi\|_{L_{p'}(\Omega)^n} = \|J\|_{\hat{H}_p^{-1}(\Omega)^*} \neq 0$ such that $\langle J|F\rangle = \langle F|\phi\rangle$ for every $F \in \hat{H}_p^{-1}(\Omega)$. Using assertion (i) for p' instead of p and considering only the functionals $F_{\mu\nabla u}$ for $u \in \dot{\mathcal{H}}_p^1(\Omega)$, we see that

$$\|\nabla\phi\|_{L_{p'}(\Omega)} \lesssim \sup_{0\neq u \in \dot{H}_p^1(\Omega)} \frac{|\int_\Omega \mu\nabla\phi \cdot \nabla u \, dx|}{\|\nabla u\|_{L_p(\Omega)}} = \sup_{0\neq u \in \dot{H}_p^1(\Omega)} \frac{|\langle J|_X | F_{\mu\nabla u}\rangle|}{\|\nabla u\|_{L_p(\Omega)}} = 0.$$

This is a contradiction to $\nabla\phi \neq 0$. Therefore the map $u \mapsto F_{\mu\nabla u}$, $\dot{\mathcal{H}}_p^1(\Omega) \to \hat{H}_p^{-1}(\Omega)$ is surjective.

(iii) *Existence for given $F \in \hat{H}_p^{-1}(\Omega)$ and $h_2 \in W_p^{1-1/p}(\Sigma)$.* We construct $u = v + w$ as follows. Let $\mathcal{E}_+ \in \mathcal{B}(W_p^{1-1/p}(\Sigma); H_p^1(\Omega_+))$ denote a co-retraction for the trace operator $H_p^1(\Omega_+) \to W_p^{1-1/p}(\Sigma)$. Then we define $v_+ := \mathcal{E}_+ h_2$ and $v_- := 0$, so that $v \in H_p^1(\Omega \setminus \Sigma)$ with $[\![v]\!] = h_2$. Finally, we determine $w \in H_p^1(\Omega)$ as a solution to $\int_\Omega \mu\nabla w \cdot \nabla\phi \, dx = \langle F - F_{\mu\nabla v}|\phi\rangle$ for $\phi \in H_{p'}^1(\Omega)$. Then $u = v + w$ solves (2.2) and hence $u \mapsto (F_{\mu\nabla u}, [\![u]\!])$ is surjective. We conclude that the map $u \mapsto (F_{\mu\nabla u}, [\![u]\!])$, $\mathbb{E}^{-1} \to \mathbb{F}^{-1}$ induced by the weak transmission problem (2.2) is invertible. ☐

2.2.5. The strong transmission problem in bounded domains. In order to solve the strong transmission problem (2.1) in the case $\lambda = 0$, we employ the following fact.

2.30. Proposition (cf. [EN00, Corollary IV.1.19] and [Lun95, Remark A.2.4])**.** *Let $A\colon D(A) \subset X \to X$ be a densely defined linear operator in a Banach space X with compact resolvent. Then $\sigma(A)$ consists only of poles of $\lambda \mapsto (\lambda - A)^{-1}$ with finite algebraic multiplicity. If $\lambda \in \sigma(A)$ satisfies $N(\lambda_0 - A) = N((\lambda_0 - A)^2)$, then $X = N(\lambda - A) \oplus R(\lambda - A)$ as a topological direct sum.*

Proof of Theorem 2.2. The result for the cases $\Omega\setminus\Sigma \in \{\mathbb{R}^n\setminus\Sigma_\omega, \mathbb{R}^n_\omega, \mathbb{R}^n\}$ was proved in Lemma 2.26 and Corollaries 2.27 and 2.28 and it remains to consider a bounded domain.

(i) *Homogenous boundary conditions.* We define $L_{p,0}(\Omega) := \{f \in L_p(\Omega) : \int_\Omega f \, dx = 0\}$ and, for $k \geq 0$, we consider the operator

$$L = -\operatorname{div}(\mu\nabla\cdot), \quad D(L) = \left\{u \in H_p^{k+2}(\Omega \setminus \Sigma) : \mu\partial_\nu u = 0 \text{ on } \partial\Omega,\ [\![\mu\partial_\nu u]\!] = [\![u]\!] = 0 \text{ on } \Sigma\right\}.$$

Theorem 2.18 implies that $\lambda - L\colon D(L) \to H_p^k(\Omega\setminus\Sigma)$ is invertible for $|\lambda| \geq \lambda_0$ and hence the resolvent set of L is not empty. The resolvent is also compact. From Lemma 2.29 we infer that $N(L) = \mathbb{K}$ and an integration by parts shows that $R(L) \subset H_p^k(\Omega\setminus\Sigma) \cap L_{p,0}(\Omega)$. We also have the topological direct sum $L_p(\Omega) = L_{p,0}(\Omega) \oplus \mathbb{K}$ where the projection onto $L_{p,0}(\Omega)$ is given by $u \mapsto u - \langle u\rangle_\Omega$ where $\langle u\rangle_\Omega := |\Omega|^{-1}\int_\Omega u\,dx$ denotes the mean value of u in Ω. Hence also

$$H_p^k(\Omega\setminus\Sigma) = \Big(H_p^k(\Omega\setminus\Sigma) \cap L_{p,0}(\Omega)\Big) \oplus \mathbb{K}.$$

In order to apply Proposition 2.30, we let $u \in N(L^2)$. Then $Lu \in R(L) \cap N(L) \subset L_{p,0}(\Omega) \cap \mathbb{K}$ which yields $Lu = 0$ and hence $u \in N(L)$. Therefore we also have $H_p^k(\Omega\setminus\Sigma) = R(L) \oplus \mathbb{K}$ which yields $R(L) = H_p^k(\Omega\setminus\Sigma)\cap L_{p,0}(\Omega)$. Thus, the operator $L\colon D(L)\cap L_{p,0}(\Omega) \to H_p^k(\Omega\setminus\Sigma)\cap L_{p,0}(\Omega)$ is therefore bijective and bounded and therefore invertible by the closed graph theorem. As a consequence, the strong transmission problem admits at most one solution within $H_p^{k+2}(\Omega\setminus\Sigma) \cap L_{p,0}(\Omega)$.

(ii) *Existence.* For given data (f, g, h_1, h_2), we construct a solution $u = u^1 + u^2$ to (2.1), by solving the subproblems

$$\left\{\begin{aligned} \lambda u_1 - \operatorname{div}(\mu\nabla u_1) &= \langle f\rangle_\Omega && \text{in } \Omega, \\ \mu\partial_\nu u_1 &= g && \text{on } \partial\Omega, \\ [\![\mu\partial_\nu u_1]\!] &= h_1 && \text{on } \Sigma, \\ [\![u_1]\!] &= h_2 && \text{on } \Sigma. \end{aligned}\right\}, \qquad \left\{\begin{aligned} -\operatorname{div}(\mu\nabla u_2) &= \lambda u_1 + f - \langle f\rangle_\Omega && \text{in } \Omega, \\ \mu\partial_\nu u_2 &= 0 && \text{on } \partial\Omega, \\ [\![\mu\partial_\nu u_2]\!] &= 0 && \text{on } \Sigma, \\ [\![u_2]\!] &= 0 && \text{on } \Sigma. \end{aligned}\right\}.$$

The first problem is solvable for some sufficiently large $\lambda \in [1,\infty)$ by Theorem 2.18. Then the compatibility condition on (f, g, h_1) implies $\langle \lambda u_1\rangle_\Omega = 0$ and therefore $\lambda u_1 + f - \langle f\rangle_\Omega$ belongs to $H_p^k(\Omega\setminus\Sigma)\cap L_{p,0}(\Omega)$. Hence the problem for u^2 is solvable by step (i). The proof of Theorem 2.2 is complete. ☐

CHAPTER 3

The linearized problem

We investigate the linear problem (PL), which we restate as

$$
\left\{
\begin{aligned}
\rho\partial_t u - \mu\Delta u + \nabla\pi &= f_u && \text{in } J\times\Omega\setminus\Sigma,\\
\operatorname{div} u &= f_d && \text{in } J\times\Omega\setminus\Sigma,\\
[\![u]\!] &= 0 && \text{on } J\times\Sigma,\\
L_v(u,h;u_*) &= g_v && \text{on } J\times\Sigma,\\
L_w(u,\pi,h;u_*) &= g_w && \text{on } J\times\Sigma,\\
\partial_t h - u\cdot\nu_\Sigma &= g_h && \text{on } J\times\Sigma,\\
u|_{\partial\Omega} &= 0 && \text{on } J\times\partial\Omega,\\
h|_{t=0} &= 0 && \text{on } \Sigma,\\
u|_{t=0} &= 0 && \text{in } \Omega\setminus\Sigma.
\end{aligned}
\right.
\tag{3.1}
$$

Here we consider a bounded domain $\Omega\subset\mathbb{R}^n$ ($n\geq 2$) with smooth boundary $\partial\Omega$ and compact smooth interface $\Sigma\subset\Omega$ such that $\Omega\setminus\Sigma$ consists of disjoint open sets Ω_+ and Ω_- with $\partial\Omega_+\cap\partial\Omega_-=\Sigma$. We choose the unit normal vector field $\nu_\Sigma=\nu_{\partial\Omega_-}=-\nu_{\partial\Omega_+}$ that points into Ω_+. Given two functions $\psi_\pm$ on $\Omega_\pm$, we put $\psi:=\psi_+\chi_++\psi_-\chi_-$ with the characteristic functions $\chi_\pm$ of $\Omega_\pm$, and we define the jump $[\![\psi]\!]:=\psi_+|_\Sigma-\psi_-|_\Sigma$. In this way we define the density $\rho=\rho_+\chi_++\rho_-\chi_-$ and viscosity $\mu=\mu_+\chi_++\mu_-\chi_-$ with positive constants $\rho_\pm$ and $\mu_\pm$. Moreover, $J=(0,T)$ is a bounded interval with $T\in(0,\infty)$ and $u_*\colon J\times\Sigma\to\mathbb{R}^n$ is a possibly non-tangential vector field. In a tubular neighborhood $B_r(\Sigma)\subset\Omega$ of Σ, there exists a nonlinear projection $\Pi: B_r(\Sigma)\to\Sigma$ and we decompose the velocity field u into

$$u=v+w\,\nu_\Sigma\circ\Pi,\quad v:=[P_\Sigma\circ\Pi]u,\quad w:=(\nu_\Sigma\circ\Pi|u).$$

Analogously, we let $u_*=v_*+w_*\nu_\Sigma$ on Σ. Then the operators L_v and L_w are defined by

$$
\begin{aligned}
L_v(u,h;u_*) &:= -\mu_s\widetilde{\Delta}_\Sigma v-\lambda_s\nabla_\Sigma\operatorname{div}_\Sigma v-[\![\mu\partial_\nu v]\!]-[\![\mu]\!]\nabla_\Sigma w+(\lambda_s+\mu_s)w_*\nabla_\Sigma\Delta_\Sigma h,\\
L_w(u,\pi,h;u_*) &:= -\operatorname{tr}\left([(\lambda_s-\mu_s)H_\Sigma+2\mu_s L_\Sigma]\nabla_\Sigma v\right)-2[\![\mu\partial_\nu w]\!]+[\![\pi]\!]\\
&\qquad -\operatorname{tr}\left([\sigma+(\lambda_s-\mu_s)(\operatorname{div}_\Sigma v_*-2H_\Sigma w_*)+2\mu_s(D_\Sigma(v_*)-2w_*L_\Sigma)]\nabla_\Sigma^2 h\right).
\end{aligned}
$$

Here the surface shear viscosity μ_s is a positive constant and λ_s (the surface dilational visocosity if $n=3$) is a real number. Moreover, we employ the surface gradient $\nabla_\Sigma w=\tau^j\partial_j w$, the surface divergence $\operatorname{div}_\Sigma u=\tau^j\cdot\partial_j u$, the scalar Laplace-Beltrami operator $\Delta_\Sigma h=\operatorname{div}_\Sigma\nabla_\Sigma h$, the tangential Laplace-Beltrami operator $\widetilde{\Delta}_\Sigma v=g^{jk}\widetilde{\nabla}_j\widetilde{\nabla}_k v$, the Weingarten tensor $L_\Sigma=-\nabla_\Sigma\nu_\Sigma$, and the $(n-1)$-fold mean curvature $H_\Sigma=\operatorname{tr}L_\Sigma$. More information on the differential geometric quantities is given in Appendix A.

In this chapter we prove that problem (3.1) has optimal regularity in the sense that, for suitable Banach spaces ${}_0\mathbb{E}$ and ${}_0\mathbb{F}$, the solution-to-data map of problem (3.1), given by $(u,\pi,h)\mapsto(f_u,f_d,g_v,g_w,g_h)$, ${}_0\mathbb{E}\to{}_0\mathbb{F}$, is a topological linear isomorphism. To this end, it is crucial to understand the situation of a flat interface $\Sigma=\mathbb{R}^n\times\{0\}\cong\mathbb{R}^n$ in the whole space $\Omega=\mathbb{R}^{n+1}$. For the corresponding model problem (MP) we prove optimal regularity in Section 3.1 (see Theorems 3.1 and 3.14). Next, in Section 3.2, we prove optimal regularity for a perturbed model problem with a bent hyperplane and variable coefficients (see Theorem 3.16). Finally, Section 3.3 contains the main result on optimal regularity for problem (3.1) in a bounded configuration (see Theorem 3.21).

3.1. The interface conditions

In this section we prove optimal regularity for the model problem (MP) that corresponds to problem (3.1) in the situation of a flat interface $\Sigma = \mathbb{R}^n \times \{0\} \cong \mathbb{R}^n$ in the whole space $\Omega = \mathbb{R}^{n+1}$ ($n \in \mathbb{N}$) with $(f_u, f_d) = 0$. We restate this problem as

$$
\left\{
\begin{aligned}
\rho(\tau + \partial_t)u - \mu\Delta u + \nabla\pi &= 0 && \text{in } J \times \dot{\mathbb{R}}^{n+1},\\
\operatorname{div} u &= 0 && \text{in } J \times \dot{\mathbb{R}}^{n+1},\\
[\![u]\!] &= 0 && \text{on } J \times \mathbb{R}^n,\\
-\mu_s\Delta_x v - \lambda_s\nabla_x \operatorname{div}_x v - c_5[\![\mu\nabla_x w]\!] - c_6[\![\mu\partial_y v]\!] + c_1\nabla_x\Delta_x h &= g_v && \text{on } J \times \mathbb{R}^n,\\
-\operatorname{tr}((c_2 + 2C_3)\nabla_x v) - 2[\![\mu\partial_y w]\!] + [\![\pi]\!] - \operatorname{tr}((c_\sigma + C_4)\nabla_x^2 h) &= g_w && \text{on } J \times \mathbb{R}^n,\\
(\tau + \partial_t)h - w &= g_h && \text{on } J \times \mathbb{R}^n,\\
h|_{t=0} &= 0 && \text{on } \mathbb{R}^n,\\
u|_{t=0} &= 0 && \text{in } \mathbb{R}^{n+1}.
\end{aligned}
\right. \tag{3.2}
$$

In this section we let $\rho_\pm$, $\mu_\pm$, σ, and μ_s be positive constants, λ_s be a real number, $\tau \in [0, \infty)$ be a constant, $J = (0, T)$ or $J = (0, \infty)$, and $\dot{\mathbb{R}}^{n+1} = \mathbb{R}^n \times (\mathbb{R}\backslash\{0\})$. The elements of $\mathbb{R}^{n+1}$ are denoted by (x, y) with $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$. The parameters c_1, c_2, C_3, C_4, c_5, c_6, and c_σ are defined by

$$
\left\{
\begin{aligned}
c_1 &:= (\lambda_s + \mu_s)\vartheta_w, & c_2 &:= (\lambda_s - \mu_s)\operatorname{tr}\vartheta_L,\\
C_3 &:= \mu_s\vartheta_L, & C_4 &:= 2\mu_s(\vartheta_{Dv} - 2\vartheta_w\vartheta_L),\\
c_5,\, c_6 &\in \{0, 1\}, & c_\sigma &:= \sigma + (\lambda_s - \mu_s)\operatorname{tr}(\vartheta_{Dv} - 2\vartheta_w\vartheta_L),
\end{aligned}
\right. \tag{3.3}
$$

and depend on

$$\vartheta = (\vartheta_w, \vartheta_L, \vartheta_{Dv}) \quad \text{for } \vartheta_w \in \mathbb{R},\ \vartheta_L \in \mathbb{R}^{n\times n},\ \vartheta_{Dv} \in \mathbb{R}^{n\times n}.$$

In Section 3.3 we will relate these parameters to the normal reference velocity w_*, the Weingarten map L_Σ, and the tangential rate-of-strain tensor $D_\Sigma(v_*)$ in problem (3.1). We further abbreviate

$$\vartheta_H := \operatorname{tr}(\vartheta_L), \quad \vartheta_{dv} := \operatorname{tr}(\vartheta_{Dv}), \quad \vartheta_{Du} := \vartheta_{Dv} - \vartheta_w\vartheta_L, \quad \vartheta_{du} := \operatorname{tr}(\vartheta_{Du}) = \vartheta_{dv} - \vartheta_w\vartheta_H,$$

and we define

$$d_0(\vartheta_{Du}) := \sigma + (\lambda_s - \mu_s)\operatorname{tr}\vartheta_{Du} + 2\mu_s \min_{\xi\in\mathbb{R}^n\backslash\{0\}} |\xi|^{-2}\xi^\top[\vartheta_{Du}]\xi \quad \text{for } \vartheta_{Du} \in \mathbb{R}^{n\times n}.$$

Then we define the following parameter set for problem (3.2) and a given number $M > 0$:

$$\mathcal{P}_M := \left\{\vartheta = (\vartheta_w, \vartheta_L, \vartheta_{Dv}) \in \mathbb{R} \times \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} : |\vartheta| \le M,\ d_0(\vartheta_{Du}) \ge 1/M\right\}. \tag{3.4}$$

Our main result on problem (3.2) reads as follows, where the solution space is denoted by

$$
\begin{aligned}
{}_0\mathbb{E} = {}_0\mathbb{E}(J, \tau) := \{(u, \pi, h) \in {}_0\mathbb{E}_{u,v,w}(J) \times {}_0\mathbb{E}_{\pi,[\![\pi]\!]}(J) \times {}_0\mathbb{E}_h(J) :\\
\rho(\tau + \partial_t)u - \mu\Delta u + \nabla\pi = 0,\ \operatorname{div} u = 0\}.
\end{aligned} \tag{3.5}
$$

Here the relevant function spaces are defined in Figure 3.1 on the next page.

3.1. Theorem. *Let $\lambda_s + \mu_s > 0$, $c_5 \in \{0, 1\}$, $c_6 = 1$, $J = (0, \infty)$, $p \in (1, \infty)$, and $M > 0$.*

Then there exists $\tau \in (0, \infty)$ such that the solution-to-data map $(u, \pi, h) \mapsto (g_v, g_w, g_h)$, ${}_0\mathbb{E} \to {}_0\mathbb{G}_v \times {}_0\mathbb{G}_w \times {}_0\mathbb{G}_h$ of problem (3.2) is uniformly invertible with respect to $\vartheta \in \mathcal{P}_M$.

$$
\begin{aligned}
{}_0\mathbb{E}_u &:= \{u \in {}_0H_p^1(J; L_p(\mathbb{R}^{n+1})^{n+1}) \cap L_p(J; H_p^2(\dot{\mathbb{R}}^{n+1})^{n+1}) : [\![u]\!] = 0\}, \\
{}_0\mathbb{E}_v &:= {}_0W_p^{1-1/2p}(J; L_p(\mathbb{R}^n)^n) \cap {}_0W_p^{1/2-1/2p}(J; H_p^2(\mathbb{R}^n)^n) \cap L_p(J; W_p^{3-1/p}(\mathbb{R}^n)^n), \\
{}_0\mathbb{E}_w &:= {}_0W_p^{1-1/2p}(J; H_p^1(\mathbb{R}^n)) \cap L_p(J; W_p^{3-1/p}(\mathbb{R}^n)), \\
{}_0\mathbb{E}_{u,v,w} &:= \{u = (v, w) \in {}_0\mathbb{E}_u : v|_{y=0} \in {}_0\mathbb{E}_v,\, w|_{y=0} \in {}_0\mathbb{E}_w\}, \\
\mathbb{E}_\pi &:= L_p(J; \dot{H}_p^1(\dot{\mathbb{R}}^{n+1})), \\
{}_0\mathbb{E}_{\pi,[\![\pi]\!]} &:= \{\pi \in \mathbb{E}_\pi : [\![\pi]\!] \in {}_0\mathbb{G}_w\}, \\
{}_0\mathbb{E}_h &:= {}_0W_p^{2-1/2p}(J; H_p^1(\mathbb{R}^n)) \cap {}_0H_p^1(J; W_p^{3-1/p}(\mathbb{R}^n)) \cap L_p(J; W_p^{4-1/p}(\mathbb{R}^n)), \\
{}_0\mathbb{G}_v &:= {}_0W_p^{1/2-1/2p}(J; L_p(\mathbb{R}^n)^n) \cap L_p(J; W_p^{1-1/p}(\mathbb{R}^n)^n), \\
{}_0\mathbb{G}_w &:= {}_0W_p^{1/2-1/2p}(J; H_p^1(\mathbb{R}^n)) \cap L_p(J; W_p^{2-1/p}(\mathbb{R}^n)), \\
{}_0\mathbb{G}_h &:= {}_0W_p^{1-1/2p}(J; H_p^1(\mathbb{R}^n)) \cap L_p(J; W_p^{3-1/p}(\mathbb{R}^n)).
\end{aligned}
$$

FIGURE 3.1. Function spaces ${}_0\mathbb{E}_{\dots}$ and ${}_0\mathbb{G}_{\dots}$ for problem (MP).

This theorem is a central result of this thesis, as it provides the basic functional analytic framework for proving that the linear problem (3.1) has optimal regularity, and these function spaces are also appropriate for proving that problem (T) is locally well-posed. We can easily conclude the following result on bounded time intervals.

3.2. Corollary. *Let $\lambda_s + \mu_s > 0$, $c_5 \in \{0, 1\}$, $c_6 = 1$, $\tau = 0$, $p \in (1, \infty)$, $T_0 \in (0, \infty)$, and $M > 0$. Then the solution-to-data map $(u, \pi, h) \mapsto (g_v, g_w, g_h)$, ${}_0\mathbb{E}(J, 0) \to {}_0\mathbb{G}_v(J) \times {}_0\mathbb{G}_w(J) \times {}_0\mathbb{G}_h(J)$ of problem (3.2) is uniformly invertible with respect to $\vartheta \in \mathcal{P}_M$ and $J = (0, T)$ with $T \in (0, T_0]$.*

Proof. As in [PSS07, p. 720] and [DK13, Remark 1.70], we consider the multiplication operator

$$(\mathcal{M}_\tau u)(t) := e^{\tau t} u(t) \qquad \text{for } u \in L_{1,\mathrm{loc}}(\mathbb{R}_+; X),\ \tau \in \mathbb{R},$$

with *exponential weight* $t \mapsto e^{\tau t}$. Then it is easy to verify the operator identities

$$\mathcal{M}_\tau^{-1} = \mathcal{M}_{-\tau}, \quad \partial_t \mathcal{M}_\tau = \mathcal{M}_\tau(\tau + \partial_t).$$

Hence we have $\partial_t = \mathcal{M}_\tau(\tau + \partial_t)\mathcal{M}_{-\tau}$.

Theorem 3.1 yields a number $\tau \geq 0$ such that the solution-to-data map

$$S_{\infty,\tau} \colon (u, \pi, h) \mapsto (g_v, g_w, g_h), \quad {}_0\mathbb{E}(\mathbb{R}_+, \tau) \to {}_0\mathbb{G}_v(\mathbb{R}_+) \times {}_0\mathbb{G}_w(\mathbb{R}_+) \times {}_0\mathbb{G}_h(\mathbb{R}_+)$$

of problem (3.2) is uniformly invertible with respect to $\vartheta \in \mathcal{P}_M$. By Lemma B.9 on page 148, there exist linear extension operators $\mathcal{E}_{J,j} \colon {}_0\mathbb{G}_j(J) \to {}_0\mathbb{G}_j(\mathbb{R}_+)$ ($j \in \{v, w, h\}$) that are uniformly bounded with respect to $T \in (0, \infty)$. The data-to-solution map for (3.2) on J is therefore given by

$$(g_v, g_w, g_h) \mapsto (u, \pi, h) = \left(\mathcal{M}_\tau S_{\infty,\tau}^{-1} \mathcal{M}_{-\tau} \left(\mathcal{E}_{J,v} g_v, \mathcal{E}_{J,w} g_w, \mathcal{E}_{J,h} g_h\right)\right)\Big|_{[0,T]},$$

and its asserted mapping properties can be easily checked. □

The proof of Theorem 3.1 is prepared in the following subsections and given on page 68. In Section 3.1.1, we apply the Fourier-Laplace transformation to problem (3.2) and express the transformed solution $(\hat{u}, [\![\hat{\pi}]\!], \hat{h})$ by means of Green's functions and the values of $(\hat{u}, \partial_y \hat{u}_\pm, \hat{\pi}, \hat{h})$ at $y = 0$. The latter satisfy a linear system (3.12) for given unknowns $(\hat{g}_v, \hat{g}_w, \hat{g}_h)$ whose determinant (3.13), which we call the *interface symbol* of problem (3.2), does not vanish. Moreover, the system (3.12) fits into the theory of Denk and Kaip [DK13] on N-parabolic mixed order systems, as will be shown in Section 3.1.2. By applying their theory in Section 3.1.3, we obtain suitable function spaces on $J \times \Sigma$ such that the map $(u|_\Sigma, \partial_y u_\pm|_\Sigma, [\![\pi]\!], h) \mapsto (g_v, g_w, g_h)$ is uniformly invertible with respect to the parameter $\vartheta \in \mathcal{P}_M$. In Section 3.1.4, we employ an appropriate extension technique as in [DHP03; DHP07; DK13] for proving that (u, π) satisfy the desired interior regularity conditions. Finally, inhomogeneous bulk data (f_u, f_d) are resolved in Section 3.1.5 by using optimal regularity of elliptic transmission problems from Chapter 2 and of the Stokes problem in a half space from [DHP01].

3.1.1. The interface symbol. We first adapt the computations of Denk and Kaip [DK13, Section 4.7] and derive the linear system (3.12) for the transformed interface values of (u, π, h). Assume that (u, π, h) is a solution of problem (3.2), which can be transformed with the Fourier transformation $x \rightsquigarrow \xi$, $\nabla_x \rightsquigarrow i\xi$ and the Laplace transformation $t \rightsquigarrow \lambda$. The transformed functions are denoted by $\hat{u}(\lambda, \xi, y)$, $\hat{\pi}(\lambda, \xi, y)$, and $\hat{h}(\lambda, \xi)$. For $j \in \{1, 2\}$, we define

$$\hat{u}_j(\lambda, \xi, y) := \hat{u}(\lambda, \xi, (-1)^j y), \quad \hat{\pi}_j(\lambda, \xi, y) := \hat{\pi}(\lambda, \xi, (-1)^j y) \quad \text{for } y > 0.$$

The transformed tangential and normal velocities $\hat{v}_j$ and $\hat{w}_j$ are defined analogously and we let $\rho_2 := \rho_+$, $\rho_1 := \rho_-$, and so on. We consider the parabolic case $\lambda \in \Sigma_\phi = \{\lambda \in \mathbb{C} : |\arg \lambda| < \phi\}$ with $\phi \in (\pi/2, \pi)$. Since $\tau + \Sigma_\phi$ is a subset of Σ_ϕ for $\tau \geq 0$, we may replace $\tau + \lambda \in \tau + \Sigma_\phi$ by $\lambda \in \Sigma_\phi$ in the following computations.

The Fourier-Laplace transformed equation of $[\![u]\!] = 0$ is $[\![\hat{u}]\!] = 0$, and hence $\hat{u}_2 = \hat{u}_1$, $\hat{v}_2 = \hat{v}_1 =: \hat{v}$, and $\hat{w}_2 = \hat{w}_1 =: \hat{w}$ at $y = 0$. For $j \in \{1, 2\}$, $\lambda \in \Sigma_\phi$, $\xi \in \mathbb{R}^n$, and $k \in \{3, 4\}$, we define

$$\omega_j(\lambda, \xi) := (\rho_j \mu_j^{-1} \lambda + |\xi|^2)^{1/2}, \quad c_k(\xi) := |\xi|^{-2} \xi^\top C_k \xi.$$

Then (3.2) is transformed to the following system.

$$\mu_j \omega_j^2 \hat{u}_j - \mu_j \partial_y^2 \hat{u}_j + (i\xi, (-1)^j \partial_y)^\top \hat{\pi}_j = 0, \tag{3.6a}$$

$$i\xi \cdot \hat{v}_j + (-1)^j \partial_y \hat{w}_j = 0, \tag{3.6b}$$

$$[\![\hat{u}]\!] = 0, \tag{3.6c}$$

$$\left(\mu_s |\xi|^2 + \lambda_s \xi \otimes \xi\right) \hat{v} - c_6 \left(\mu_2 \partial_y \hat{v}_2 + \mu_1 \partial_y \hat{v}_1\right) - c_5 [\![\mu]\!] i\xi \hat{w} - c_1 i\xi |\xi|^2 \hat{h} = \hat{g}_v, \tag{3.6d}$$

$$-\left(c_2 i\xi + 2C_3 i\xi\right) \cdot \hat{v} - 2\left(\mu_2 \partial_y \hat{w}_2 + \mu_1 \partial_y \hat{w}_1\right) + [\![\hat{\pi}]\!] + (c_\sigma + c_4(\xi))|\xi|^2 \hat{h} = \hat{g}_w, \tag{3.6e}$$

$$\lambda \hat{h} - \hat{w} = \hat{g}_h. \tag{3.6f}$$

Equation (3.6a) can be eliminated with the following result on Green's functions

$$k_\pm(y, s) := k_\pm(y, s; \tau) := (e^{-\tau|y-s|} \pm e^{-\tau(y+s)})/2\tau \quad \text{for } y, s \geq 0,\ \tau \in \mathbb{C} \setminus \{0\}.$$

3.3. Lemma. *For $\mu \in \mathbb{C}$ and $f \in C([0, \infty))$ with $(s \mapsto e^{-\tau s} f(s)) \in L_1(0, \infty)$, the functions*

$$v_\pm(y) := \mu e^{-\tau y} - \int_0^\infty k_\pm(y, s) f(s)\, ds \quad \textit{for } y \geq 0,$$

solve the initial value problems

$$\partial_y^2 v_+ - \tau^2 v_+ = f, \quad v_+(0) = \mu - \frac{1}{\tau} \int_0^\infty e^{-\tau s} f(s)\, ds, \quad \partial_y v_+(0) = -\tau\mu,$$

$$\partial_y^2 v_- - \tau^2 v_- = f, \quad v_-(0) = \mu, \qquad \partial_y v_-(0) = -\tau\mu - \int_0^\infty e^{-\tau s} f(s)\, ds.$$

Proof. These assertions can be verified easily. □

Consequently, equation (3.6a) can be eliminated when we represent the transformed functions $(\hat{v}, \hat{w}, \hat{\pi})$ in terms of unknown transformed functions $\hat{p}_j(\lambda, \xi)$, $\hat{\Phi}_v^j(\lambda, \xi)$, and $\hat{\Phi}_w^j(\lambda, \xi)$ as follows:

$$\hat{\pi}_j(\lambda,\xi,y) := \hat{p}_j(\lambda,\xi)e^{-|\xi|y}, \tag{3.7a}$$

$$\hat{v}_j(\lambda,\xi,y) := \hat{\Phi}_v^j(\lambda,\xi)e^{-\omega_j(\lambda,\xi)y} - \int_0^\infty k_-(y,s;\omega_j(\lambda,\xi))\frac{i\xi\hat{\pi}_j(\lambda,\xi,s)}{\mu_j}\,ds, \tag{3.7b}$$

$$\hat{w}_j(\lambda,\xi,y) := \hat{\Phi}_w^j(\lambda,\xi)e^{-\omega_j(\lambda,\xi)y} - \int_0^\infty k_+(y,s;\omega_j(\lambda,\xi))\frac{(-1)^j\partial_s\hat{\pi}_j(\lambda,\xi,s)}{\mu_j}\,ds. \tag{3.7c}$$

These functions satisfy the following interface conditions.

$$\hat{\pi}_j|_{y=0} = \hat{p}_j, \qquad [\![\hat{\pi}]\!] = \hat{p}_2 - \hat{p}_1, \tag{3.8a}$$

$$\hat{v}_j|_{y=0} = \hat{\Phi}_v^1 = \hat{\Phi}_v^2 =: \hat{\Phi}_v, \qquad \partial_y\hat{v}_j|_{y=0} = -\omega_j\hat{\Phi}_v - \frac{i\xi\hat{p}_j}{\mu_j(\omega_j+|\xi|)}, \tag{3.8b}$$

$$\hat{w}_j|_{y=0} = \hat{\Phi}_w^j + \frac{(-1)^j|\xi|\hat{p}_j}{\mu_j\omega_j(\omega_j+|\xi|)}, \quad \partial_y\hat{w}_j|_{y=0} = -\omega_j\hat{\Phi}_w^j. \tag{3.8c}$$

We employ the abbreviations $\alpha_j := \mu_j\omega_j(\omega_j+|\xi|)$ and $\Omega_+ := \alpha_1+\alpha_2$. Then, with $\hat{p}_1 = \hat{p}_2 - [\![\hat{\pi}]\!]$, $\hat{w}_2|_{y=0} = \hat{w}_1|_{y=0}$, and (3.8c), we represent $\hat{p}_1$ and $\hat{p}_2$ as

$$\hat{p}_j = \frac{\alpha_1\alpha_2}{|\xi|\Omega_+}\left(\hat{\Phi}_w^1 - \hat{\Phi}_w^2\right) + \frac{(-1)^j\alpha_j}{\Omega_+}[\![\hat{\pi}]\!]. \tag{3.9}$$

Hence the transformed functions $\hat{w}_j$ and $\partial_y\hat{v}_j$ are given by

$$\hat{w}_1 = \hat{w}_2 = \frac{\alpha_2}{\Omega_+}\hat{\Phi}_w^2 + \frac{\alpha_1}{\Omega_+}\hat{\Phi}_w^1 + \frac{|\xi|}{\Omega_+}[\![\hat{\pi}]\!], \tag{3.10a}$$

$$\partial_y\hat{v}_1 = \frac{\alpha_2\omega_1 i\xi}{\Omega_+|\xi|}\hat{\Phi}_w^2 - \frac{\alpha_2\omega_1 i\xi}{\Omega_+|\xi|}\hat{\Phi}_w^1 - \omega_1\hat{\Phi}_v + \frac{\omega_1 i\xi[\![\hat{\pi}]\!]}{\Omega_+}, \tag{3.10b}$$

$$\partial_y\hat{v}_2 = \frac{\alpha_1\omega_2 i\xi}{\Omega_+|\xi|}\hat{\Phi}_w^2 - \frac{\alpha_1\omega_2 i\xi}{\Omega_+|\xi|}\hat{\Phi}_w^1 - \omega_2\hat{\Phi}_v - \frac{\omega_2 i\xi[\![\hat{\pi}]\!]}{\Omega_+}. \tag{3.10c}$$

It remains to formulate a linear system for the unknowns $\hat{\Phi}_v$, $\hat{\Phi}_w^2$, $\hat{\Phi}_w^1$, $\hat{h}$, and $[\![\hat{\pi}]\!]$. We abbreviate

$$\begin{aligned}
\Omega' &:= c_6\mu_1\omega_1 + c_6\mu_2\omega_2 + \mu_s|\xi|^2,\\
L_w^1 &:= c_6\mu_1\omega_1\alpha_2 + c_6\mu_2\omega_2\alpha_1 - c_5[\![\mu]\!]|\xi|\alpha_1,\\
L_w^2 &:= -c_6\mu_1\omega_1\alpha_2 - c_6\mu_2\omega_2\alpha_1 - c_5[\![\mu]\!]|\xi|\alpha_2,\\
L_q &:= c_6\mu_2\omega_2 - c_6\mu_1\omega_1 - c_5[\![\mu]\!]|\xi|.
\end{aligned}$$

Then equations (3.6d) to (3.6f), (3.9) and (3.10) yield

$$\left(\Omega'\,\mathrm{id}_n + \lambda_s\xi\otimes\xi\right)\hat{\Phi}_v + \frac{i\xi L_w^2}{|\xi|\Omega_+}\hat{\Phi}_w^2 + \frac{i\xi L_w^1}{|\xi|\Omega_+}\hat{\Phi}_w^1 + \frac{i\xi L_q}{\Omega_+}[\![\hat{\pi}]\!] - c_1 i\xi|\xi|^2\hat{h} = \hat{g}_v, \tag{3.11a}$$

$$-\left(c_2 i\xi + 2C_3 i\xi\right)\cdot\hat{\Phi}_v + 2\mu_2\omega_2\hat{\Phi}_w^2 + 2\mu_1\omega_1\hat{\Phi}_w^1 + [\![\hat{\pi}]\!] + (c_\sigma + c_4)|\xi|^2\hat{h} = \hat{g}_w, \tag{3.11b}$$

$$\lambda\hat{h} - \frac{\alpha_2}{\Omega_+}\hat{\Phi}_w^2 - \frac{\alpha_1}{\Omega_+}\hat{\Phi}_w^1 - \frac{|\xi|}{\Omega_+}[\![\hat{\pi}]\!] = \hat{g}_h. \tag{3.11c}$$

Consequently, system (3.6) becomes

$$\underbrace{\begin{bmatrix} \Omega' I_n - \lambda_s i\xi \otimes i\xi & \frac{L_w^2 i\xi}{\Omega_+|\xi|} & \frac{L_w^1 i\xi}{\Omega_+|\xi|} & -c_1 i\xi|\xi|^2 & L_q \frac{i\xi}{\Omega_+} \\ i\xi^\top & -\omega_2 & 0 & 0 & 0 \\ i\xi^\top & 0 & \omega_1 & 0 & 0 \\ 0 & -\frac{\alpha_2}{\Omega_+} & -\frac{\alpha_1}{\Omega_+} & \lambda & -\frac{|\xi|}{\Omega_+} \\ -c_2 i\xi^\top - 2i\xi^\top C_3 & 2\mu_2\omega_2 & 2\mu_1\omega_1 & (c_\sigma + c_4)|\xi|^2 & 1 \end{bmatrix}}_{\hat{\mathcal{L}}(\lambda,\xi)} \begin{bmatrix} \hat{\Phi}_v \\ \hat{\Phi}_w^2 \\ \hat{\Phi}_w^1 \\ \hat{h} \\ [\![\hat{\pi}]\!] \end{bmatrix} = \begin{bmatrix} \hat{g}_v \\ 0 \\ 0 \\ \hat{g}_h \\ \hat{g}_w \end{bmatrix}. \tag{3.12}$$

In order to compute the interface symbol $\det \mathcal{L}(\lambda, \xi)$, we
(i) subtract row $n+1$ from row $n+2$,
(ii) add $c_2 \cdot (\text{row } n+1)$ to row $n+4$,
(iii) add $\lambda_s i\xi \otimes (\text{row } n+1)$ to rows $1, \dots, n$,
(iv) add $-\Omega'^{-1} i\xi^\top \cdot (\text{rows } 1, \dots, n)$ to row $n+1$, and
(v) add $2\Omega'^{-1} i\xi^\top C_3 \cdot (\text{rows } 1, \dots, n)$ to row $n+4$.
In this way we calculcate

$$\begin{aligned} &\det \hat{\mathcal{L}}(\lambda, \xi) \\ &= \det \begin{bmatrix} \Omega' I_n & * & * & * & * \\ 0 & -\omega_2 + \frac{|\xi|^2}{\Omega'}\left(\frac{L_w^2}{\Omega_+|\xi|} - \lambda_s\omega_2\right) & \frac{|\xi|^2}{\Omega'}\frac{L_w^1}{\Omega_+|\xi|} & -c_1\frac{|\xi|^4}{\Omega'} & \frac{L_q}{\Omega'\Omega_+}|\xi|^2 \\ 0 & \omega_2 & \omega_1 & 0 & 0 \\ 0 & -\frac{\alpha_2}{\Omega_+} & -\frac{\alpha_1}{\Omega_+} & \lambda & -\frac{|\xi|}{\Omega_+} \\ 0 & \begin{matrix}(2\mu_2 - c_2)\omega_2 \\ -\frac{2|\xi|^2 c_3}{\Omega'}\left(\frac{L_w^2}{\Omega_+|\xi|} - \lambda_s\omega_2\right)\end{matrix} & \begin{matrix}2\mu_1\omega_1 \\ -\frac{2|\xi|^2 c_3}{\Omega'}\frac{L_w^1}{\Omega_+|\xi|}\end{matrix} & \begin{matrix}(c_\sigma + c_4)|\xi|^2 \\ -\frac{2c_1c_3|\xi|^4}{\Omega'}\end{matrix} & 1 - \frac{2c_3L_q|\xi|^2}{\Omega'\Omega_+} \end{bmatrix} \\ &= \frac{\Omega'^{n-2}}{\Omega_+^2} \det \begin{bmatrix} -\omega_2\Omega'\Omega_+ + L_w^2|\xi| - \lambda_s\omega_2\Omega_+|\xi|^2 & L_w^1|\xi| & -c_1|\xi|^4 & L_q|\xi|^2 \\ \omega_2 & \omega_1 & 0 & 0 \\ -\alpha_2 & -\alpha_1 & \lambda & -|\xi| \\ \begin{matrix}(2\mu_2 - c_2)\omega_2\Omega'\Omega_+ \\ -2c_3L_w^2|\xi| + 2\lambda_s c_3\omega_2\Omega_+|\xi|^2\end{matrix} & \begin{matrix}2\mu_1\omega_1\Omega'\Omega_+ \\ -2c_3L_w^1|\xi|\end{matrix} & \begin{matrix}(c_\sigma + c_4)|\xi|^2\Omega' \\ -2c_1c_3|\xi|^4\end{matrix} & \Omega'\Omega_+ - 2c_3L_q|\xi|^2 \end{bmatrix}. \end{aligned}$$

Here an asterisk $*$ denotes a non-specified entry. The remaining (4×4)-determinant can be calculated with the software Maxima [Max] and we refer to page 173 in Appendix B.5 for the source code. Therefore the interface symbol can be written as

$$\det \hat{\mathcal{L}}(\lambda, \xi) = -\omega_1(\lambda, \xi)\omega_2(\lambda, \xi)\Omega_+(\lambda, \xi)^{-1}\Omega'(\lambda, \xi)^{n-1}P(\lambda, \xi), \tag{3.13}$$

where the symbol $P(\lambda, \xi)$ is defined as follows. Define the mean value $\langle\!\langle \psi \rangle\!\rangle := (\psi_1 + \psi_2)/2$, the jump $[\![\psi]\!] := \psi_2 - \psi_1$, and let

$$\begin{aligned} d(\xi) &:= c_\sigma + 2\vartheta_w c_3(\xi) + c_4(\xi) + \vartheta_w c_2 \\ &= \sigma + (\lambda_s - \mu_s)(\vartheta_{dv} - 2\vartheta_H\vartheta_w) \\ &\quad - 2\vartheta_w|\xi|^{-2} i\xi^\top[\mu_s\vartheta_L]i\xi - |\xi|^{-2} i\xi^\top[2\mu_s(\vartheta_{Dv} - 2\vartheta_w\vartheta_L)]i\xi + (\lambda_s - \mu_s)\vartheta_w\vartheta_H \\ &= \sigma + (\lambda_s - \mu_s)(\vartheta_{dv} - \vartheta_H\vartheta_w) + |\xi|^{-2}\xi^\top[\vartheta_w\vartheta_L + 2\vartheta_w 2\mu_s(\vartheta_{Dv} - \vartheta_w\vartheta_L)]\xi \\ &= \sigma + (\lambda_s - \mu_s)\vartheta_{du} + 2\mu_s|\xi|^{-2}\xi^\top[\vartheta_{Du}]\xi. \end{aligned}$$

Then, with $\beta_s := \lambda_s + \mu_s$, the symbol $P(\lambda, \xi)$ in (3.13) is given by

$$
\begin{aligned}
P(\lambda,\xi) = {} & \underline{\beta_s d(\xi)|\xi|^5} + 2\langle\!\langle \mu \rangle\!\rangle (c_6 c_\sigma + c_4 c_6)|\xi|^4 \\
& + \underline{c_1 \left([\![\mu\omega]\!] - [\![\mu]\!]|\xi| \right) |\xi|^4} \\
& \quad + \left(2c_6 c_\sigma \langle\!\langle \mu\omega \rangle\!\rangle + (2\vartheta_3 + c_2)c_5 [\![\mu]\!]\lambda - [\![\mu]\!]^2 c_5 \lambda + 2c_4 c_6 \langle\!\langle \mu\omega \rangle\!\rangle \right) |\xi|^3 \\
& \quad + \left(4c_6 \langle\!\langle \mu\omega \rangle\!\rangle^2 + 2c_6 \mu_1\mu_2 \langle\!\langle \omega^2 \rangle\!\rangle + 4c_6 \langle\!\langle \mu^2\omega \rangle\!\rangle |\xi| + (c_5 [\![\mu]\!] - c_2 c_6 - 2c_3 c_6)[\![\mu\omega]\!]|\xi| \right) \lambda |\xi| \\
& + \underline{2\beta_s \left(\langle\!\langle \mu\omega^2 \rangle\!\rangle + \langle\!\langle \mu\omega \rangle\!\rangle |\xi| \right) \lambda |\xi|^2} \\
& + \underline{4c_6 \langle\!\langle \mu\omega^2 \rangle\!\rangle \langle\!\langle \mu\omega \rangle\!\rangle \lambda}.
\end{aligned}
$$

Here the underlined terms are the principal parts as we will see in the next section.

3.1.2. Invertibility of the interface symbol. Our goal is to show that $\hat{\mathcal{L}}$ is an N-parabolic mixed-order system in the sense of Definition B.77 on page 168. Moreover, since our localization procedure will require uniform invertibility with respect to the reference velocity $u_* = v_* + w_* \nu_\Sigma$ and L_Σ, we will also study the dependence on the related parameters $\vartheta = (\vartheta_w, \vartheta_L, \vartheta_{Dv}) \in \mathcal{P}_M$. First, we show that the interface symbol $\det \hat{\mathcal{L}}$ of problem (3.2) is an N-parabolic symbol. To this end, we replace

$$
i\xi \rightsquigarrow z \in \overline{B\Sigma_\delta^n}, \quad |\xi| \rightsquigarrow |z|_- = \sqrt{-z \cdot z},
$$

and we define the complex $(n+4) \times (n+4)$-matrix

$$
\hat{\mathcal{L}}(\lambda, z; \vartheta) := \begin{bmatrix}
\Omega' I_n - \lambda_s z \otimes z & \frac{L_w^2 z}{\Omega_+ |z|_-} & \frac{L_w^1 z}{\Omega_+ |z|_-} & -c_1(\vartheta) z |z|_-^2 & L_q \frac{z}{\Omega_+} \\
z^\top & -\omega_2 & 0 & 0 & 0 \\
z^\top & 0 & \omega_1 & 0 & 0 \\
0 & -\frac{\alpha_2}{\Omega_+} & -\frac{\alpha_1}{\Omega_+} & \lambda & -\frac{|z|_-}{\Omega_+} \\
-c_2(\vartheta) z^\top - 2z^\top C_3(\vartheta) & 2\mu_2\omega_2 & 2\mu_1\omega_1 & c_\sigma(\vartheta)|z|_-^2 - z^\top C_4(\vartheta) z & 1
\end{bmatrix}. \tag{3.14}
$$

We replace the functions $d(\xi)$ and $P(\lambda, \xi)$ as

$$
\begin{aligned}
d(z;\vartheta) &:= \sigma + (\lambda_s - \mu_s)\operatorname{tr}\vartheta_{Du} + 2\mu_s |z|_-^{-2} i z^\top [\vartheta_{Du}] i z \quad \text{ with } c_j(z;\vartheta) := -|z|_-^{-2} z^\top C_j(\vartheta) z, \\
P(\lambda, z;\vartheta) &:= \beta_s d(z;\vartheta)|z|_-^5 + 2\langle\!\langle \mu \rangle\!\rangle (c_6 c_\sigma(\vartheta) + c_4(z;\vartheta)c_6)|z|_-^4 \\
& \quad + c_1(\vartheta) \left([\![\mu\omega]\!] - [\![\mu]\!]|z|_- \right) |z|_-^4 \\
& \qquad + \left(2c_6 c_\sigma(\vartheta) \langle\!\langle \mu\omega \rangle\!\rangle + (2c_3(z;\vartheta) + c_2(\vartheta))c_5 [\![\mu]\!]\lambda - [\![\mu]\!]^2 c_5 \lambda + 2c_4(z;\vartheta) c_6 \langle\!\langle \mu\omega \rangle\!\rangle \right) |z|_-^3 \\
& \qquad + \left(4c_6 \langle\!\langle \mu\omega \rangle\!\rangle^2 + 2c_6 \mu_1\mu_2 \langle\!\langle \omega^2 \rangle\!\rangle + 4c_6 \langle\!\langle \mu^2\omega \rangle\!\rangle |z|_- \right) \lambda |z|_- \\
& \qquad + (c_5 [\![\mu]\!] - c_2(\vartheta) c_6 - 2c_3(z;\vartheta) c_6)[\![\mu\omega]\!] \lambda |z|_-^2 \\
& \quad + 2\beta_s \left(\langle\!\langle \mu\omega^2 \rangle\!\rangle + \langle\!\langle \mu\omega \rangle\!\rangle |z|_- \right) \lambda |z|_-^2 \\
& \quad + 4c_6 \langle\!\langle \mu\omega^2 \rangle\!\rangle \langle\!\langle \mu\omega \rangle\!\rangle \lambda.
\end{aligned}
$$

It is straightforward to check that P belongs to the symbol class in Definition B.72 on page 166.

Next, we employ the γ-orders and γ-principal parts of the symbols ω_j, Ω', Ω_+, and P, which are defined in Definition B.73 and given in Figure 3.2 on the following page. Due to Theorem B.75, it is sufficient to show that the principal parts of ω_j, Ω_+, Ω', and P do not vanish, and therefore the function $d(z;\vartheta)$ should not vanish. Let us derive a condition on the parameter tuple $\vartheta \in \mathcal{P}_M$ which ensures that

$$
\text{there is } \delta > 0 \text{ such that } \operatorname{Re} d(z;\vartheta) > 0 \text{ for all } z \in B\Sigma_\delta^n. \tag{3.15}
$$

$$d_\gamma(\omega_j) = \begin{cases} 1 & : \gamma \in (0,2], \\ \gamma/2 & : \gamma \in [2,\infty]. \end{cases} \qquad d_\gamma(\Omega_+) = \begin{cases} 2 & : \gamma \in (0,2], \\ \gamma & : \gamma \in [2,\infty]. \end{cases}$$

$$d_\gamma(\Omega') = \begin{cases} 2 & : \gamma \in (0,\infty],\ c_6 = 0, \\ 2 & : \gamma \in (0,4],\ c_6 = 1, \\ \gamma/2 & : \gamma \in [4,\infty],\ c_6 = 1. \end{cases} \qquad d_\gamma(P) = \begin{cases} 5 & : \gamma \in (0,1], \\ 4+\gamma & : \gamma \in [0,2], \\ 2+2\gamma & : \gamma \in [2,\infty],\ c_6 = 0, \\ 2+2\gamma & : \gamma \in [2,4],\ c_6 = 1, \\ 5\gamma/2 & : \gamma \in [4,\infty],\ c_6 = 1. \end{cases}$$

$$\pi_\gamma \omega_j(\lambda, z) = \begin{cases} |z|_- & : \gamma \in (0,2), \\ (\rho_j \mu_j^{-1} \lambda + |z|_-^2)^{1/2} & : \gamma = 2, \\ (\rho_j \mu_j^{-1})^{1/2} \lambda^{1/2} & : \gamma \in (2,\infty]. \end{cases}$$

$$\pi_\gamma \Omega_+(\lambda, z) = \begin{cases} 4\langle\!\langle \mu \rangle\!\rangle |z|_-^2 & : \gamma \in (0,2), \\ \Omega_+(\lambda, z) & : \gamma = 2, \\ 2\langle\!\langle \rho \rangle\!\rangle \lambda & : \gamma \in (2,\infty]. \end{cases}$$

$$\pi_\gamma \Omega'(\lambda, z) = \begin{cases} \mu_s |z|_-^2 & : \gamma \in (0,\infty],\ c_6 = 0, \\ \mu_s |z|_-^2 & : \gamma \in (0,4),\ c_6 = 1, \\ 2\langle\!\langle \sqrt{\rho\mu} \rangle\!\rangle \lambda^{1/2} + \mu_s |z|_-^2 & : \gamma = 4,\ c_6 = 1, \\ 2\langle\!\langle \sqrt{\rho\mu} \rangle\!\rangle \lambda^{1/2} & : \gamma \in (4,\infty],\ c_6 = 1. \end{cases}$$

$$\pi_\gamma P(\lambda, z; \vartheta) = \begin{cases} \beta_s d(z;\vartheta) |z|_-^5 & : \gamma \in (0,1), \\ \beta_s d(z;\vartheta) |z|_-^5 + 4\beta_s \langle\!\langle \mu \rangle\!\rangle \lambda |z|_-^4 & : \gamma = 1, \\ 4\beta_s \langle\!\langle \mu \rangle\!\rangle \lambda |z|_-^4 & : \gamma \in (1,2), \\ 2\beta_s \left(\langle\!\langle \mu\omega^2 \rangle\!\rangle + \langle\!\langle \mu\omega \rangle\!\rangle |z|_- \right) \lambda |z|_-^2 & : \gamma = 2, \\ 2\beta_s \langle\!\langle \rho \rangle\!\rangle \lambda^2 |z|_-^2 & : \gamma \in (2,4), \\ 2\beta_s \langle\!\langle \rho \rangle\!\rangle \lambda^2 |z|_-^2 + 4c_6 \langle\!\langle \rho \rangle\!\rangle \langle\!\langle \sqrt{\rho\mu} \rangle\!\rangle \lambda^{5/2} & : \gamma = 4, \\ 2\beta_s \langle\!\langle \rho \rangle\!\rangle \lambda^2 |z|_-^2 & : \gamma \in (4,\infty], c_6 = 0, \\ 4\langle\!\langle \rho \rangle\!\rangle \langle\!\langle \sqrt{\rho\mu} \rangle\!\rangle \lambda^{5/2} & : \gamma \in (4,\infty],\ c_6 = 1. \end{cases}$$

FIGURE 3.2. The γ-orders and γ-principal parts of the symbols ω_j, Ω_+, Ω', and P.

It is shown in Lemma B.55 that an estimate $C^{-1}|z| \le ||z|_-| \le C|z|$ applies for $z \in \overline{B\Sigma_\delta^n} \setminus \{0\}$ and $\delta \in (0, \pi/4)$. Moreover, Lemma B.55 yields the following estimates for $j \in \{3,4\}$.

$$|c_j(z;\vartheta)| \le n^{1/2} |C_j(\vartheta)|, \quad |\mathrm{Im}\, c_j(z;\vartheta)| \le \sin(4\delta) |c_j(z;\vartheta)| \quad \text{for } z \in \overline{B\Sigma_\delta^n} \setminus \{0\},\ \delta \in (0, \pi/8].$$

Hence a sufficient condition for (3.15) is

$$d_0(\vartheta_{Du}) = \sigma + (\lambda_s - \mu_s)\, \mathrm{tr}\, \vartheta_{Du} + 2\mu_s \min_{\xi \in \mathbb{R}^n \setminus \{0\}} |\xi|^{-2} \xi^\top [\vartheta_{Du}] \xi > 0. \tag{3.16}$$

Indeed, suppose that $d_0(\vartheta_{Du}) \ge 1/M$ and $|\vartheta_{Du}| \le M$ for some $M > 0$. Then

$$\mathrm{Re}\, d(z;\vartheta) \ge d_0(\vartheta_{Du}) + 2\mu_s \left(\min_{z \in \overline{B\Sigma_\delta^n} \setminus \{0\}} \mathrm{Re}\, \frac{iz^\top [\vartheta_{Du}] iz}{|z|_-^2} - \min_{\xi \in \mathbb{R}^n \setminus \{0\}} \frac{\xi^\top [\vartheta_{Du}] \xi}{|\xi|^2} \right) \to d_0(\vartheta_{Du})$$

as $\delta \to 0$. Hence there exists $\delta = \delta(R) \in (0, \pi/8)$ such that $\operatorname{Re} d(z;\vartheta) \geq 1/(2M)$ for all $z \in B\Sigma_\delta^n$. In view of the inclusion $i\mathbb{R}^n \subset \overline{B\Sigma_\delta^n}$, we see that condition (3.16) is also necessary for (3.15).

3.4. Lemma. *Let ρ_j, μ_j, σ, and μ_s be positive constants and let λ_s be a real number.*

(i) If $\beta_s = \lambda_s + \mu_s > 0$, then for given $M > 0$ and $\phi \in (\pi/2, \pi)$ there exists $\delta \in (0, \pi/8]$ such that $P\colon \overline{\Sigma}_\phi \times \overline{B\Sigma_\delta^n} \times \mathcal{P}_M \to \mathbb{C}$ is N-parabolic.

(ii) Conversely, if $P(\cdot, \cdot; \vartheta)\colon \overline{\Sigma}_\phi \times \overline{B\Sigma_\delta^n} \to \mathbb{C}$ is N-parabolic for some $\phi \in (\pi/2, \pi)$, $\delta \in (0, \pi/8]$, and $\vartheta \in \mathbb{R} \times \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$, then $\lambda_s + \mu_s > 0$ and ϑ satisfies (3.16).

Proof. (i) In view of Theorem B.75, it is sufficient to show that the principal parts of the symbol P do not vanish, in the sense that

$$\pi_\gamma P(\lambda, z; \vartheta) \neq 0 \quad \text{for all } \gamma \in (0,\infty],\ \lambda \in \overline{\Sigma}_\phi \setminus \{0\},\ z \in \overline{B\Sigma_\delta^n} \setminus \{0\},\ \vartheta \in \mathcal{P}_M.$$

First, we choose $\delta(M) \in (0, \pi/8]$ such that $\operatorname{Re} d(z;\vartheta) \geq 1/(2M)$ for $z \in \overline{B\Sigma_\delta^n}$ with $\delta \in (0, \delta(M)]$. Then Lemma B.55 implies that $\pi_\gamma P(\lambda, z; \vartheta)$ does not vanish for all $\gamma \in (0,1)$. Next, let $\gamma \in [1, 2)$. Since $\arg(d(z;\vartheta)|z|_-) \leq 5\delta$, there exists $\delta_1(M) \in (0, \delta(M)]$ such that $|d(z;\vartheta)|z|_-| \geq n^{-1/4}|z|/(2M)$ for all $\delta \in (0, \delta_1(M)]$ and $z \in \overline{B\Sigma_\delta^n}$. Hence for some $\delta = \delta(M, \phi) \leq \delta_1(M)$ with $5\delta + \phi < \pi$ the number $\beta_s d(z;\vartheta)|z|_- + 4\beta_s \langle\!\langle \mu \rangle\!\rangle\, \lambda$ belongs to $\overline{\Sigma}_\phi \setminus \{0\}$, which implies that $\pi_\gamma P(\lambda, z;\vartheta)$ does not vanish for $\gamma \in [1, 2)$. In the case $\gamma = 2$, we write

$$\pi_2 P(\lambda, z; \vartheta) = 2\beta_s \langle\!\langle \mu\omega \rangle\!\rangle \left(\frac{\langle\!\langle \mu\omega^2 \rangle\!\rangle / \lambda^{1/2}}{\langle\!\langle \mu\omega \rangle\!\rangle / \lambda^{1/2}} + |z|_- \right) \lambda |z|_-^2. \tag{3.17}$$

Recall that $\omega_j(\lambda, z)^2 = \rho_j \mu_j^{-1}\lambda + |z|_-^2$ belongs to $\overline{\Sigma}_\phi \setminus \{0\}$. Hence we obtain $|\arg(\langle\!\langle \mu\omega^2 \rangle\!\rangle / \lambda^{1/2})| \leq \phi/2 + 2\delta$ and $|\arg(\langle\!\langle \mu\omega \rangle\!\rangle / \lambda^{1/2})| \leq \phi/2 + \delta$ and therefore $|\arg(\langle\!\langle \mu\omega^2 \rangle\!\rangle / \langle\!\langle \mu\omega \rangle\!\rangle)| \leq \phi + 3\delta$. By choosing $\delta < (\pi - \phi)/4$, it follows that $\langle\!\langle \mu\omega^2 \rangle\!\rangle / \langle\!\langle \mu\omega \rangle\!\rangle + |z|_- \neq 0$ and then $\pi_2 P(\lambda, z;\vartheta) \neq 0$. The remaining cases $\gamma \in (2, \infty]$ can be treated similary. Therefore P has non-vanishing principal parts and hence, by Theorem B.75, it is N-parabolic.

(ii) To prove the converse assertion, let $P(\cdot, \cdot; \vartheta)$ be N-parabolic. Then $\pi_\gamma P(\cdot, \cdot; \vartheta)$ does not vanish for all $\gamma \in (0, 1)$ and we conclude that $\beta_s \neq 0$ and $d(i\xi; \vartheta) \neq 0$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$. Next,

$$\pi_4 P(\lambda, i\xi; \vartheta) = (2\beta_s \langle\!\langle \rho \rangle\!\rangle |\xi|^2 + 4\langle\!\langle \rho \rangle\!\rangle \langle\!\langle \rho\mu \rangle\!\rangle \lambda^{1/2})\lambda^2 \neq 0 \quad \text{for all } \lambda > 0,\ \xi \in \mathbb{R}^n \setminus \{0\}.$$

This yields $\beta_s > 0$ by using $\langle\!\langle \rho \rangle\!\rangle > 0$ and $\langle\!\langle \rho\mu \rangle\!\rangle > 0$. Finally,

$$\pi_1 P(\lambda, i\xi; \vartheta) = ((\lambda_s + \mu_s) d(i\xi;\vartheta)|\xi| + 2\beta_s \langle\!\langle \mu \rangle\!\rangle \lambda)\, |\xi|^4 \neq 0 \quad \text{for all } \lambda > 0,\ \xi \in \mathbb{R}^n \setminus \{0\},$$

and this implies $d(i\xi;\vartheta) > 0$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$, since $\beta_s > 0$ and $\langle\!\langle \mu \rangle\!\rangle > 0$. This in turn yields $d_0(\vartheta_{Du}) = \min\{d(i\xi;\vartheta) : \xi \in \mathbb{R}^n,\ |\xi| = 1\} > 0$. □

3.5. Corollary. *Let ρ_j, μ_j, σ, and μ_s be positive constants and let λ_s be a real number.*

(i) If $\lambda_s + \mu_s > 0$, then for given $M > 0$ and $\phi \in (\pi/2, \pi)$ there exists $\delta \in (0, \pi/8]$ such that $\det \hat{\mathcal{L}}\colon \overline{\Sigma}_\phi \times \overline{B\Sigma_\delta^n} \times \mathcal{P}_M \to \mathbb{C}$ is N-parabolic.

(ii) Conversely, if $\det \hat{\mathcal{L}}(\cdot, \cdot; \vartheta)\colon \overline{\Sigma}_\phi \times \overline{B\Sigma_\delta^n} \to \mathbb{C}$ is N-parabolic for some $\phi \in (\pi/2, \pi)$, $\delta \in (0, \pi/8]$, and $\vartheta \in \mathbb{R} \times \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$, then $\lambda_s + \mu_s > 0$ and ϑ satisfies (3.16).

Proof. (i) It is easy to check that $\omega_1\omega_2\Omega_+^{-1}$ satisfies the homogeneity property

$$(\omega_1\omega_2\Omega_+^{-1})(\eta^2\lambda, \eta z) = (\omega_1\omega_2\Omega_+^{-1})(\lambda, z) \quad \text{for all } \eta > 0,$$

and therefore belongs to the symbol class $S_N(\overline{\Sigma}_\phi \times \overline{B\Sigma_\delta^n})$ with $\phi \in (\pi/2, \pi)$ and $\delta \in (0, \pi/8)$. We further have $\Omega' = c_6\mu_1\omega_1 + c_6\mu_2\omega_2 + \mu_s|z|_-^2 \in S_N(\overline{\Sigma}_\phi \times \overline{B\Sigma_\delta^n})$ and $P \in S_N(\overline{\Sigma}_\phi \times \overline{B\Sigma_\delta^n} \times \mathcal{P}_M)$ if $\phi \in (\pi/2, \pi)$ and if $\delta = \delta(M, \phi)$ is chosen as in Lemma 3.4. These symbols have strictly positive order functions and therefore [DK13, Lemma 3.33] yields

$$\det \hat{\mathcal{L}} = -\omega_1\omega_2\Omega_+^{-1} \cdot \Omega'^{\,n-1} \cdot P \in S_N(\overline{\Sigma}_\phi \times \overline{B\Sigma_\delta^n} \times \mathcal{P}_M).$$

(ii) Let $\det \hat{\mathcal{L}}(\cdot,\cdot;\vartheta)$ be N-parabolic for some $\vartheta \in \mathcal{P}_M$ and recall that ω_j, Ω_+^{-1}, and Ω' are N-parabolic. Hence their principal parts do not vanish and the above representation of $\det \hat{\mathcal{L}}$ and the identities in Figure 3.2 show that $\pi_\gamma P(\lambda, i\xi; \vartheta) \neq 0$ for all $\lambda > 0$ and $\xi \in \mathbb{R}^n \setminus \{0\}$. Therefore Lemma 3.4 implies that $\lambda_s + \mu_s > 0$ and ϑ satisfies (3.16). □

3.1.3. Function spaces. Next, we construct spaces $\mathbb{H}$ and $\mathbb{F}$ such that the *interface operator*

$$\hat{\mathcal{L}}(\tau + \mathcal{D}_t, \mathcal{D}_x; \vartheta) \colon \mathbb{H} \to \mathbb{F}, \quad (\Phi_v, \Phi_w^2, \Phi_w^1, h, [\![\pi]\!]) \mapsto (g_v, 0, 0, g_h, g_w)$$

is uniformly invertible with respect to $\vartheta \in \mathcal{P}_M$, for every $M > 0$. Here the operator $\hat{\mathcal{L}}(\tau + \mathcal{D}_t, \mathcal{D}_x; \vartheta)$ is defined by the joint functional calculus of $(\mathcal{D}_t, \mathcal{D}_x)$ from Theorem B.70 on page 166. We note that every component of $\hat{\mathcal{L}}(\lambda, z; \vartheta)$ belongs to the symbol class $S(\overline{\Sigma}_\phi \times \overline{B\Sigma}_\delta^n \times \mathcal{P}_M)$ from page 166 for $\phi \in (\pi/2, \pi)$ and some $\delta \in (0, \pi/8]$. From now on we restict our considerations to the case

$$c_5 \in \{0, 1\}, \quad c_6 = 1.$$

In order to apply Theorem B.79, we first estimate the γ-orders of the components of $\hat{\mathcal{L}}$.

$$d_\gamma(\hat{\mathcal{L}}) \leq \begin{bmatrix} \max\{2,\gamma/2\} & \max\{1,\gamma/2\} & \max\{1,\gamma/2\} & 3 & 1 - \max\{1,\gamma/2\} \\ 1 & \max\{1,\gamma/2\} & -\infty & -\infty & -\infty \\ 1 & -\infty & \max\{1,\gamma/2\} & -\infty & -\infty \\ -\infty & 0 & 0 & \gamma & 1 - \max\{2,\gamma\} \\ 1 & \max\{1,\gamma/2\} & \max\{1,\gamma/2\} & 2 & 0 \end{bmatrix}.$$

Here the relation $\leq$ is considered component-wise and an entry $-\infty$ corresponds to a vanishing component of $\hat{\mathcal{L}}$.

We define the row-wise order functions s_j and the column-wise order functions t_i by

$$\begin{aligned} s_1(\gamma) = \cdots = s_n(\gamma) &:= 1, & t_1(\gamma) = \cdots = t_n(\gamma) &:= \max\{2,\gamma/2\} - 1, \\ s_{n+1}(\gamma) = s_{n+2}(\gamma) &:= 0, & t_{n+1}(\gamma) = t_{n+2}(\gamma) &:= \max\{1,\gamma/2\}, \\ s_{n+3}(\gamma) &:= -1, & t_{n+3}(\gamma) &:= \max\{1,\gamma\} + 1, \\ s_{n+4}(\gamma) &:= 0, & t_{n+4}(\gamma) &:= 0. \end{aligned}$$

Then it follows that

$$\begin{aligned} \sum\nolimits_j s_j(\gamma) + \sum\nolimits_i t_i(\gamma) &= n \max\{2,\gamma/2\} + 2\max\{1,\gamma/2\} + \max\{1,\gamma\} \\ &= \max\{2n+3, 2n+2+\gamma, 2n+2\gamma, (n+4)\gamma/2\} = d_\gamma(\det \hat{\mathcal{L}}). \end{aligned}$$

Moreover, for all $i, j \in \{1, \dots, n+4\}$, the function $s_j + t_i$ is an upper order function for $\hat{\mathcal{L}}_{ji}$.

3.6. Corollary. *Let ρ_j, μ_j, σ, and μ_s be positive constants and let λ_s be a real number such that $\lambda_s + \mu_s > 0$. Then for given numbers $M > 0$ and $\phi \in (\pi/2, \pi)$, there exists $\delta \in (0, \pi/8]$ such that the symbol $\hat{\mathcal{L}} \colon \overline{\Sigma}_\phi \times \overline{B\Sigma}_\delta^n \times \mathcal{P}_M \to \mathbb{C}^{(n+4)\times(n+4)}$ is an N-parabolic mixed-order system.*

3.7. Remark. The preceding choice of the order functions differs from [DK13, p. 223]. In particular, we take care of the additional entries $\hat{\mathcal{L}}_{i,n+3}$ ($i \leq n$) with γ-order lesser or equal to 3 and we avoid the difference $\max\{2,\gamma/2\} - \max\{1,\gamma/2\}$, which is neither convex nor concave and hence not an order function.

	$l=0, \gamma \in (0,1]$	$l=1, \gamma \in (1,2]$	$l=2, \gamma \in (2,4]$	$l=3, \gamma \in (4,\infty)$
t_i	$b_0(t_i)+m_0(t_i)\gamma$	$b_1(t_i)+m_1(t_i)\gamma$	$b_2(t_i)+m_2(t_i)\gamma$	$b_3(t_i)+m_3(t_i)\gamma$
$t_1=\cdots=t_n$	$1+0\gamma$	$1+0\gamma$	$1+0\gamma$	$-1+\frac{1}{2}\gamma$
$t_{n+1}=t_{n+2}$	$1+0\gamma$	$1+0\gamma$	$0+\frac{1}{2}\gamma$	$0+\frac{1}{2}\gamma$
t_{n+3}	$2+0\gamma$	$1+1\gamma$	$1+1\gamma$	$1+1\gamma$
t_{n+4}	$0+0\gamma$	$0+0\gamma$	$0+0\gamma$	$0+0\gamma$
s_j	$b_0(s_j)+m_0(s_j)\gamma$	$b_1(s_j)+m_1(s_j)\gamma$	$b_2(s_j)+m_2(s_j)\gamma$	$b_3(s_j)+m_3(s_j)\gamma$
$s_1=\cdots=s_n$	$1+0\gamma$	$1+0\gamma$	$1+0\gamma$	$1+0\gamma$
$s_{n+1}=s_{n+2}$	$0+0\gamma$	$0+0\gamma$	$0+0\gamma$	$0+0\gamma$
s_{n+3}	$-1+0\gamma$	$-1+0\gamma$	$-1+0\gamma$	$-1+0\gamma$
s_{n+4}	$0+0\gamma$	$0+0\gamma$	$0+0\gamma$	$0+0\gamma$

FIGURE 3.3. Upper order functions for the symbol $\hat{\mathcal{L}}$.

$$
\begin{aligned}
\mathbb{H}_1=\cdots=\mathbb{H}_n &:= L_p(W_p^{3-\frac{1}{p}})\cap L_p(W_p^{3-\frac{1}{p}}) \cap {}_0W_p^{\frac{1}{2}-\frac{1}{2p}}(H_p^2)\cap {}_0W_p^{1-\frac{1}{2p}}(L_p) && (\text{for } \Phi_v),\\
\mathbb{H}_{n+1}=\mathbb{H}_{n+2} &:= L_p(W_p^{3-\frac{1}{p}})\cap L_p(W_p^{3-\frac{1}{p}}) \cap {}_0W_p^{1-\frac{1}{2p}}(H_p^1)\cap {}_0W_p^{1-\frac{1}{2p}}(H_p^1) && (\text{for } \Phi_w^j),\\
\mathbb{H}_{n+3} &:= L_p(W_p^{4-\frac{1}{p}})\cap {}_0H_p^1(W_p^{3-\frac{1}{p}}) \cap {}_0W_p^{\frac{3}{2}-\frac{1}{2p}}(H_p^2)\cap {}_0W_p^{\frac{3}{2}-\frac{1}{2p}}(H_p^2) && (\text{for } h),\\
\mathbb{H}_{n+4} &:= L_p(W_p^{2-\frac{1}{p}})\cap L_p(W_p^{2-\frac{1}{p}}) \cap {}_0W_p^{\frac{1}{2}-\frac{1}{2p}}(H_p^1)\cap {}_0W_p^{\frac{1}{2}-\frac{1}{2p}}(H_p^1) && (\text{for } [\![\pi]\!]),\\
\mathbb{F}_1=\cdots=\mathbb{F}_n &:= L_p(W_p^{1-\frac{1}{p}})\cap L_p(W_p^{1-\frac{1}{p}}) \cap {}_0W_p^{\frac{1}{2}-\frac{1}{2p}}(L_p)\cap {}_0W_p^{\frac{1}{2}-\frac{1}{2p}}(L_p) && (\text{for } g_v),\\
\mathbb{F}_{n+1}=\mathbb{F}_{n+2} &:= L_p(W_p^{2-\frac{1}{p}})\cap L_p(W_p^{2-\frac{1}{p}}) \cap {}_0W_p^{\frac{1}{2}-\frac{1}{2p}}(H_p^1)\cap {}_0W_p^{\frac{1}{2}-\frac{1}{2p}}(H_p^1), &&\\
\mathbb{F}_{n+3} &:= L_p(W_p^{3-\frac{1}{p}})\cap L_p(W_p^{3-\frac{1}{p}}) \cap {}_0W_p^{\frac{1}{2}-\frac{1}{2p}}(H_p^2)\cap {}_0W_p^{\frac{1}{2}-\frac{1}{2p}}(H_p^2) && (\text{for } g_h),\\
\mathbb{F}_{n+4} &:= L_p(W_p^{2-\frac{1}{p}})\cap L_p(W_p^{2-\frac{1}{p}}) \cap {}_0W_p^{\frac{1}{2}-\frac{1}{2p}}(H_p^1)\cap {}_0W_p^{\frac{1}{2}-\frac{1}{2p}}(H_p^1) && (\text{for } g_w).
\end{aligned}
$$

Here we abbreviate $H_p^\alpha(W_p^\beta) := H_p^\alpha(\mathbb{R}_+; W_p^\beta(\mathbb{R}^n))$ and $W_p^\alpha(H_p^\beta) := W_p^\alpha(\mathbb{R}_+; H_p^\beta(\mathbb{R}^n))$.

FIGURE 3.4. Function spaces for the operator $\hat{\mathcal{L}}(\tau + \mathcal{D}_t, \mathcal{D}_x; \vartheta)$.

We choose the following parameters and scales for the construction of the spaces $\mathbb{H}_i$ and $\mathbb{F}_j$.

$$
\begin{array}{llll}
(\gamma_0,\gamma_1]=(0,1], & (\gamma_1,\gamma_2]=(1,2], & (\gamma_2,\gamma_3]=(2,4], & (\gamma_3,\gamma_4]=(4,\infty],\\
s_0'=0, & s_1'=0, & s_2'=1/2-1/2p, & s_3'=1/2-1/2p,\\
r_0'=2-1/p, & r_1'=2-1/p, & r_2'=1, & r_3'=1,\\
\mathcal{F}_0(\mathcal{K}_0)=H_p(B_{p,p}), & \mathcal{F}_1(\mathcal{K}_1)=H_p(B_{p,p}), & \mathcal{F}_2(\mathcal{K}_2)=B_{p,p}(H_p), & \mathcal{F}_3(\mathcal{K}_3)=B_{p,p}(H_p).
\end{array}
$$

By using the definitions of t_i and s_j, we obtain the representations in Figure 3.3 on this page. Then we define the spaces $\mathbb{H}_i$ and $\mathbb{F}_j$ by

$$
\mathbb{H}_i := \bigcap_{l=0}^{3} {}_0\mathcal{F}_l^{s_l'+m_l(t_i)}\left(\mathcal{K}_l^{r_l'+b_l(t_i)}\right), \quad \mathbb{F}_j := \bigcap_{l=0}^{3} {}_0\mathcal{F}_l^{s_l'-m_l(s_j)}\left(\mathcal{K}_l^{r_l'-b_l(s_j)}\right).
$$

In our situation this yields the spaces in Figure 3.4 on the current page. These spaces are admissible in the sense of Definition B.78 and therefore Theorem B.79 yields the following result.

3.8. Lemma. *Let ρ_j, μ_j, σ, and μ_s be positive constants and let λ_s be a real number such that $\lambda_s+\mu_s>0$. Then, given $p\in(1,\infty)$, $M>0$, and $\phi\in(\pi/2,\pi)$, there exist $\delta\in(0,\pi/8)$ and $\tau\in[0,\infty)$ such that*

$$\hat{\mathcal{L}}(\tau+\mathcal{D}_t,\mathcal{D}_x;\vartheta)\colon \prod_{i=1}^{n+4}\mathbb{H}_i\to\prod_{j=1}^{n+4}\mathbb{F}_j$$

is uniformly invertible with respect to $\vartheta\in\mathcal{P}_M$.

By restricting the inverse of $\hat{\mathcal{L}}(\tau+\mathcal{D}_t,\mathcal{D}_x;\vartheta)$ to tuples of the form $(g_v,0,0,g_h,g_w)$, and by using that $\partial_t h=g_h+w$ belongs to the subspace ${}_0\mathbb{G}_h$ of $\mathbb{F}_{n+3}$ for $g_h\in{}_0\mathbb{G}_h$ and $w\in{}_0\mathbb{E}_w$, we obtain the following result for the spaces from Figure 3.1.

3.9. Corollary. *In the situation of Lemma 3.8, the map*

$$(g_v,g_w,g_h)\mapsto(\Phi_v,\Phi_w^2,\Phi_w^1,h,[\![\pi]\!])^\top=[\hat{\mathcal{L}}(\tau+\mathcal{D}_t,\mathcal{D}_x;\vartheta)]^{-1}(g_v,0,0,g_h,g_w)^\top,$$
$${}_0\mathbb{G}_v\times{}_0\mathbb{G}_w\times{}_0\mathbb{G}_h\to{}_0\mathbb{E}_v\times{}_0\mathbb{E}_w\times{}_0\mathbb{E}_w\times{}_0\mathbb{E}_h\times{}_0\mathbb{G}_w$$

is uniformly bounded with respect to $\vartheta\in\mathcal{P}_M$.

3.1.4. Interior regularity. Our next goal is to verify the interior regularity conditions

$$u\in H_p^{1,2}(\mathbb{R}_+\times\dot{\mathbb{R}}^{n+1}):=H_p^1(\mathbb{R}_+;L_p(\mathbb{R}^{n+1}))\cap L_p(\mathbb{R}_+;H_p^2(\dot{\mathbb{R}}^{n+1})),\quad \pi\in L_p(\mathbb{R}_+;\dot{H}_p^1(\dot{\mathbb{R}}^{n+1}))$$

for the functions u and π from (3.7) and Corollary 3.9. We can easily obtain the pressure regularity from the properties of the Poisson semigroup, but for the velocity we need to study more involved extension operators. The Fourier-Laplace transformed functions $\hat{\pi}_j$, $\hat{v}_j$, and $\hat{w}_j$ of π and $u|_\Sigma=(v,w)$ were given in (3.7), where we employed Green's functions $k_\pm$ from Lemma 3.3, the Poisson extension symbol $e^{-|\xi|y}$ and the parabolic extension symbol $e^{-\omega_j(\lambda,\xi)y}$ with

$$\omega_j(\lambda,\xi)=(\rho_j\mu_j^{-1}(\tau+\lambda)+|\xi|^2)^{1/2}\qquad\text{for }\lambda\in\overline{\Sigma}_\phi,\ \xi\in\mathbb{R}^n.$$

Here $\tau>0$ is chosen as in Lemma 3.8. For computing the integrals in $\hat{v}$ and $\hat{w}$, we let $y>0$, $\omega\in\mathbb{C}$, and $\eta:=|z|_-\in\mathbb{C}$ with $\operatorname{Re}\omega$, $\operatorname{Re}\eta$, and $\operatorname{Re}(\omega-\eta)>0$. Then

$$\begin{aligned}\int_0^\infty k_\pm(y,s;\omega)e^{-\eta s}\,ds&=\int_0^\infty\frac{e^{-\omega|y-s|}\pm e^{-\omega(y+s)}}{2\omega}e^{-\eta s}\,ds\\&=\int_0^y\frac{e^{-\omega y}}{2\omega}\left(e^{\omega s}\pm e^{-\omega s}\right)e^{-\eta s}\,ds+\int_y^\infty\frac{e^{\omega y}\pm e^{-\omega y}}{2\omega}e^{-\omega s-\eta s}\,ds\\&=\frac{e^{-\omega y}}{2\omega}\left(\frac{e^{(\omega-\eta)y}-1}{\omega-\eta}\mp\frac{e^{-(\omega+\eta)y}-1}{\omega+\eta}\right)+\frac{e^{\omega y}\pm e^{-\omega y}}{2\omega}\frac{e^{-(\omega+\eta)y}}{\omega+\eta}\\&=\frac{e^{-\eta y}-e^{-\omega y}}{2\omega(\omega-\eta)}+\frac{e^{-\eta y}\pm e^{-\omega y}}{2\omega(\omega+\eta)}.\end{aligned}$$

From the identites $g_h=(\tau+\partial_t)h-w$ and (3.8c) we infer that

$$\hat{p}_j(\lambda,\xi)=\frac{(-1)^j\mu_j\omega_j(\omega_j+|\xi|)}{|\xi|}\left((\tau+\lambda)\hat{h}-\hat{g}_h-\hat{\Phi}_w^j\right).$$

Plugging in these identities into (3.7) yields the representations in Figure 3.5 on the facing page.

In order to prove the interior regularity of the velocity, we employ general extension operators $E[s]$ induced by an extension symbol $s(\lambda,\xi,y)$ like the parabolic extension symbol $e^{-\omega(\lambda,\xi)y}$ or the Stokes extension symbol $\omega\frac{e^{-\omega y}-e^{-|\xi|y}}{\omega-|\xi|}$ and apply these to the extension symbols $V_{w,j}$, $V_{h,j}$, $W_{w,j}$, and $W_{h,j}$. To this purpose, we first prove the boundedness of certain integral operators by comparing their kernels with the Hilbert transform. Similar results were established by Denk, Hieber and Prüss [DHP03, Section 6.4], [DHP07, Section 4], and by Denk and Kaip [DK13, Section 3.5].

Let $\omega_j(\lambda,\xi) = (\rho_j\mu_j^{-1}(\tau+\lambda)+|\xi|^2)^{1/2}$ and define the extension symbols

$$V_{w,j}(\lambda,\xi,y) := \frac{(-1)^{j+1}i\xi\omega_j}{|\xi|}\frac{e^{-\omega_j y}-e^{-|\xi|y}}{\omega_j-|\xi|}, \qquad V_{h,j}(\lambda,\xi,y) := \frac{(-1)^{j}i\xi\omega_j}{|\xi|}\frac{e^{-\omega_j y}-e^{-|\xi|y}}{\omega_j-|\xi|},$$

$$W_{w,j}(\lambda,\xi,y) := \omega_j\frac{e^{-\omega_j y}-e^{-|\xi|y}}{\omega_j-|\xi|}, \qquad W_{h,j}(\lambda,\xi,y) := -\frac{|\xi|e^{-\omega_j y}-\omega_j e^{-|\xi|y}}{\omega_j-|\xi|}.$$

The Fourier-Laplace transforms $\hat{u}_j(\lambda,\xi,y) := \hat{u}(\lambda,\xi,(-1)^j y)$ and $\hat{\pi}_j(\lambda,\xi,y) := \hat{\pi}(\lambda,\xi,(-1)^j y)$ of $u_j=(v_j,w_j)$ and π_j are given by

$$\hat{v}_j(\lambda,\xi,y) = e^{-\omega_j y}\hat{\Phi}_v(\lambda,\xi) + V_{w,j}(\lambda,\xi,y)\hat{\Phi}_w^j(\lambda,\xi) + V_{h,j}(\lambda,\xi,y)((\tau+\lambda)\hat{h}(\lambda,\xi)-\hat{g}_h(\lambda,\xi)),$$

$$\hat{w}_j(\lambda,\xi,y) = W_{w,j}(\lambda,\xi,y)\hat{\Phi}_w^j(\lambda,\xi) + W_{h,j}(\lambda,\xi,y)((\tau+\lambda)\hat{h}(\lambda,\xi)-\hat{g}_h(\lambda,\xi)),$$

$$\hat{\pi}_j(\lambda,\xi,y) = e^{-|\xi|y}(-1)^j\mu_j\omega_j(\omega_j+|\xi|)|\xi|^{-1}((\tau+\lambda)\hat{h}(\lambda,\xi)-\hat{g}_h(\lambda,\xi)-\hat{\Phi}_w^j(\lambda,\xi)).$$

FIGURE 3.5. The interior Fourier-Laplace transformed velocity and pressure.

3.10. Lemma. *Let E be a Banach space of class $\mathcal{HT}$ with property (α) and let $\phi\in(\pi/2,\pi)$ and $\delta\in(0,\pi/2)$ such that $\phi+2\delta<\pi$. Suppose that the mapping*

$$k\colon (\lambda,z,y,\bar{y})\mapsto k(\lambda,z,y,\bar{y}), \quad \Sigma_\phi\times B\Sigma_\delta^n\times\mathbb{R}_+\times\mathbb{R}_+\to\mathcal{B}(E)$$

is holomorphic with respect to (λ,z) for every $(y,\bar{y})$ and that

$$M(k) := \sup\left\{|(y+\bar{y})k(\lambda,z,y,\bar{y})|_{\mathcal{B}(E)} : \lambda\in\Sigma_\phi,\ z\in B\Sigma_\delta^n,\ y,\bar{y}\in\mathbb{R}_+\right\}<\infty.$$

With the joint functional calculus for $(\mathcal{D}_t,\mathcal{D}_x)$ from Theorem B.70 we define

$$k(\mathcal{D}_t,\mathcal{D}_x,y,\bar{y})\in\mathcal{B}\left(L_p(\mathbb{R}_+\times\mathbb{R}^n;E)\right) \quad \text{for } y,\ \bar{y}\in\mathbb{R}_+,\ p\in(1,\infty).$$

Define an integral operator $G[k]$ by

$$(G[k]u)(y) = \int_0^\infty k(\mathcal{D}_t,\mathcal{D}_x,y,\bar{y})u(\cdot,\cdot,\bar{y})\,d\bar{y}, \quad \text{for } y\in(0,\infty),\ u\in C_c^\infty(\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}_+;E).$$

Then $G[k]$ can be extended uniquely to a bounded operator in $L_p(\mathbb{R}_+;L_p(\mathbb{R}_+^{n+1};E))$ such that

$$\|G[k]\|_{\mathcal{B}(L_p(\mathbb{R}_+;L_p(\mathbb{R}_+^{n+1};E)))}\le CM(k)\|\mathcal{HT}\|_{\mathcal{B}(L_p(\mathbb{R}_+))}.$$

Here $\mathcal{HT}$ is the one-sided Hilbert transform on $L_p(\mathbb{R}_+)$ and the number C satisfies

$$\|f(\mathcal{D}_t,\mathcal{D}_x)\|_{\mathcal{B}(L_p(\mathbb{R}_+\times\mathbb{R}^n;E))}\le C\|f\|_\infty \quad \text{for all } f\in\mathcal{H}^\infty(\Sigma_\phi\times B\Sigma_\delta^n).$$

Proof. The one-sided scalar Hilbert transform

$$(\mathcal{HT}u)(y) = \int_0^\infty \frac{u(\bar{y})}{y+\bar{y}}\,d\bar{y} \quad \text{for } y\in\mathbb{R}_+,\ u\in L_p(\mathbb{R}_+)$$

is bounded in $L_p(\mathbb{R}_+)$ with norm $\tan(\pi/2p)$ if $p\in(1,2]$ and $\cot(\pi/2p)$ if $p\in[2,\infty)$ [see ME88]. By applying Theorem B.70 to the family of bounded holomorphic functions $\{(y+\bar{y})k(\cdot,\cdot,y,\bar{y}) : y,\bar{y}\in\mathbb{R}_+\}\subset\mathcal{H}^\infty(\Sigma_\phi\times B\Sigma_\delta^n)$, we obtain

$$\begin{aligned}\|(G[k]u)(t)\|_{L_p(\mathbb{R}_+^{n+1};E)} &\le CM(k)\left\|y\mapsto\int_0^\infty\frac{1}{y+\bar{y}}|u(t,\cdot,\bar{y})|_E\,d\bar{y}\right\|_{L_p(\mathbb{R}_+^{n+1})}\\ &\le CM(k)\|\mathcal{HT}\|_{L_p(\mathbb{R}_+)}\|u(t,\cdot,\cdot)\|_{L_p(\mathbb{R}_+^{n+1};E)}.\end{aligned}$$

Next, we may rearrange the t- and y- variables using Fubini's theorem:

$$\|t\mapsto u(t,\cdot,\cdot)\|_{L_p(\mathbb{R}_+;L_p(\mathbb{R}_+^{n+1};E))} = \|y\mapsto u(\cdot,\cdot,y)\|_{L_p(\mathbb{R}_+;L_p(\mathbb{R}_+^{n+1};E))}.$$

Hence the assertion follows. □

Next we define the aforementioned extension operators with the *Volevich trick* (3.18). We employ the anisotropic Sobolev-Slobodeckiĭ spaces

$$ {}_0W_p^{t,s}(J\times\Omega;E) := {}_0W_p^t(J;L_p(\Omega;E))\cap L_p(J;W_p^s(\Omega;E)). $$

3.11. Lemma (Extension operators from $y=0$ to $y\in\mathbb{R}_+$). *Let E be a Banach space of class $\mathcal{HT}$ with property (α) and let $\phi\in(\pi/2,\pi)$ and $\delta\in(0,\pi/8]$ such that $\phi+2\delta<\pi$. Suppose that the mapping*

$$ s\colon(\lambda,z,y)\mapsto s(\lambda,z,y),\quad \Sigma_\phi\times B\Sigma_\delta^n\times\mathbb{R}_+\to\mathcal{B}(E) $$

is holomorphic and bounded with respect to (λ,z) for every y. Then we define $S(y):=s(\mathcal{D}_t,\mathcal{D}_x,y)\in\mathcal{B}(L_p(\mathbb{R}_+\times\mathbb{R}^n;E))$ by Theorem B.70 and we consider the operator

$$ (3.18)\qquad (E[s]f)(y)=S(y)(f|_{y=0})=-\int_0^\infty\left(\partial_{\bar y}S(y+\bar y)f(\bar y)+S(y+\bar y)\partial_{\bar y}f(\bar y)\right)d\bar y,\quad y>0. $$

acting on appropriate functions $f:\mathbb{R}_+\times\mathbb{R}_+^{n+1}\to E$, which are specified below. Since $E[s]f$ only depends on $f|_{y=0}$, we may also consider $E[s]$ as an extension operator which maps functions $f:\mathbb{R}_+\times\mathbb{R}^n\to E$ to functions $E[s]f\colon\mathbb{R}_+\times\mathbb{R}_+^{n+1}\to E$.

(i) Let $\omega(\lambda,z):=(\tau+\lambda-z\cdot z)^{1/2}$ with some $\tau\geq 0$ and suppose that s satisfies

$$ (3.19)\qquad \sup\left\{|y\omega(\lambda,z)^{1-j}\partial_y^j s(\lambda,z,y)| : y>0,\ \lambda\in\Sigma_\phi,\ z\in B\Sigma_\delta^n,\ j\in\{0,1,2,3\}\right\}<\infty. $$

Then the operator $E[s]$ is bounded in ${}_0H_p^{1,2}(\mathbb{R}_+\times\mathbb{R}_+^{n+1};E)$. Hence, considered as an extension operator, $E[s]$ is bounded as

$$ E[s]\colon {}_0W_p^{1-1/2p,2-1/p}(\mathbb{R}_+\times\mathbb{R}^n;E)\to {}_0H_p^{1,2}(\mathbb{R}_+\times\mathbb{R}_+^{n+1};E). $$

(ii) Suppose that s satisfies the weaker condition

$$ \sup\left\{|yzs(\lambda,z,y)|, |y\omega(\lambda,z)^{1-j}\partial_y^j s(\lambda,z,y)| : y>0,\ \lambda\in\Sigma_\phi,\ z\in B\Sigma_\delta^n,\ j\in\{1,2,3\}\right\}<\infty. $$

Let $P(y)=e^{-\sqrt{-\Delta_x}y}$ denote the Poisson semigroup. Then $E[s]$ is a bounded operator as a map

$$ \left\{f\in{}_0H_p^{1,2}(\mathbb{R}_+\times\mathbb{R}_+^{n+1};E) : f(\cdot,\cdot,y)=P(y)f(\cdot,\cdot,0)\text{ for }y>0\right\}\to{}_0H_p^{1,2}(\mathbb{R}_+\times\mathbb{R}_+^{n+1};E). $$

Hence, considered as an extension operator, $E[s]$ is bounded as [cf. PS10, Proposition 3.3]

$$ E[s]\colon {}_0H_p^1(\mathbb{R}_+;\dot W_p^{-1/p}(\mathbb{R}^n;E))\cap W_p^{1-1/2p,2-1/p}(\mathbb{R}_+\times\mathbb{R}^n;E)\to{}_0H_p^{1,2}(\mathbb{R}_+\times\mathbb{R}_+^{n+1};E). $$

Proof. In order to apply Lemma 3.10 we consider the kernels

$$ (\lambda,z,y,\bar y)\mapsto s(\lambda,z,y+\bar y),\quad (\lambda,z,y,\bar y)\mapsto\partial_{\bar y}s(\lambda,z,y+\bar y), $$

which are again denoted by s end $\partial_{\bar y}s$, respectively. Then

$$ E[s]=-G[\partial_{\bar y}s]-G[s]\partial_{\bar y}. $$

(i) By means of Theorem B.70 and Remarks B.65 we define the operator $L:=\omega(\mathcal{D}_t,\mathcal{D}_x)$. By the Kalton-Weis-Theorem B.47 and by using Theorem B.34 and Corollary B.37, the operators

$$ L^2\colon\ {}_0H_p^{1,2}(\mathbb{R}_+\times\mathbb{R}^n;E)\to L_p(\mathbb{R}_+\times\mathbb{R}^n;E), $$
$$ L=(L^2)^{1/2}\colon {}_0H_p^{1/2,1}(\mathbb{R}_+\times\mathbb{R}^n;E)\to L_p(\mathbb{R}_+\times\mathbb{R}^n;E) $$

are invertible with $\mathcal{R}$-bounded $\mathcal{H}^\infty$-calculi of angle not larger than $\pi/2$ and $\pi/4$, respectively.

For given numbers $\phi\in(\pi/2,\pi)$ and $\delta\in(0,\pi/8]$ with $\phi+2\delta<\pi$, the functions $z_j\omega(\lambda,z)^{-1}$, $z_jz_k\omega(\lambda,z)^{-2}$, and $\lambda\omega(\lambda,z)^{-2}$ are bounded with respect to $\lambda\in\overline{\Sigma}_\phi$ and $z\in\overline{B\Sigma_\delta^n}$ by Example B.56. Therefore Theorem B.70 implies that the operators $\partial_{x_j}L^{-1}$, $\partial_{x_j}\partial_{x_k}L^{-2}$, and ∂_tL^{-2} are bounded in $L_p(\mathbb{R}_+\times\mathbb{R}_+^{n+1};E)$. In order to prove that $E[s]$ maps ${}_0H_p^{1,2}(\mathbb{R}_+\times\mathbb{R}_+^{n+1};E)$ into itself, we use the multiplicative property $(\omega s)(\mathcal{D}_t,\mathcal{D}_x,y)=Ls(\mathcal{D}_t,\mathcal{D}_x,y)$ of the joint functional calculus and deduce the following identity for $f \in {}_0H_p^{1,2}(\mathbb{R}_+ \times \mathbb{R}_+^{n+1}; E)$ and l, $m \in \{0,1,2\}$ with $0 \le l+m \le 2$:

$$\begin{aligned} L^l\partial_y^m E[s]f &= L^l\partial_y^m \left(-G[\partial_y s]f - G[\omega s]L^{-1}\partial_y f\right) \\ &= -G[\omega^{-m}\partial_y^{1+m}s]L^{l+m}f - G[\omega^{1-m}\partial_y^m s]L^{l+m-1}\partial_y f. \end{aligned}$$

Since s satisfies (3.19), it follows from Lemma 3.10 that the operators $G[\omega^{1-j}\partial_y^j s]$ ($j \in \{0,1,2,3\}$) are bounded in $L_p(\mathbb{R}_+ \times \mathbb{R}_+^{n+1}; E)$. Moreover, the functions $L^{l+m}f$ and $L^{l+m-1}\partial_y f$ belong to $L_p(\mathbb{R}_+ \times \mathbb{R}_+^{n+1}; E)$ and depend continuously on $f \in {}_0H_p^{1,2}(\mathbb{R}_+ \times \mathbb{R}_+^{n+1}; E)$. Therefore assertion (i) is valid.

(ii) We represent the relevant derivatives of $E[s]f$ as

$$\begin{aligned} \nabla_x^j E[s]f &= -G[\partial_y s]\nabla_x^j f - G[|z|_- s]\nabla_x^j \sqrt{-\Delta_x}^{-1}\partial_y f \quad &&\text{for } j \in \{0,1,2\}, \\ \partial_t E[s]f &= -G[\partial_y s]\partial_t f - G[|z|_- s]\partial_t \sqrt{-\Delta_x}^{-1}\partial_y f, \\ \nabla_x^j\partial_y E[s]f &= -G[\omega^{-1}\partial_y^2 s]\nabla_x Lf - G[\partial_y s]\nabla_x^j\partial_y f &&\text{for } j \in \{0,1\}, \\ \partial_y^2 E[s]f &= -G[\omega^{-2}\partial_y^3 s]L^2 f - G[\omega^{-1}\partial_y^2 s]L\partial_y f. \end{aligned}$$

The operators $G[|z|_- s]$ and $G[\omega^{1-j}\partial_y^j s]$ ($j \in \{1,2,3\}$) are bounded in $L_p(\mathbb{R}_+ \times \mathbb{R}_+^{n+1}; E)$ by Lemma 3.10. For functions $f \in {}_0H_p^{1,2}(\mathbb{R}_+ \times \mathbb{R}_+^{n+1}; E)$ of the form $f(y) = P(y)f(0)$, we have $\partial_y f = -\sqrt{-\Delta_x} f$ and hence $\sqrt{-\Delta_x}^{-1}\partial_y f = -f$. For a given function

$$\begin{aligned} g &\in {}_0H_p^1(\mathbb{R}_+; \dot{W}_p^{-1/p}(\mathbb{R}^n; E)) \cap W_p^{1-1/2p,2-1/p}(\mathbb{R}_+ \times \mathbb{R}^n; E) \\ &\hookrightarrow {}_0H_p^1(\mathbb{R}_+; \dot{W}_p^{-1/p}(\mathbb{R}^n; E)) \cap L_p(\mathbb{R}_+; (\dot{W}_p^{-1/p} \cap \dot{W}_p^{2-1/p})(\mathbb{R}^n; E)), \end{aligned}$$

the Poisson extension $f(\cdot,\cdot,y) = P(y)g$ belongs to ${}_0H_p^{1,2}(\mathbb{R}_+ \times \mathbb{R}_+^{n+1}; E)$ by Theorem B.28. Therefore $E[s]$ satisfies assertion (ii). ☐

In order to deal with the symbols $V_{w,j}$ and $W_{w,j}$, we study the *Stokes extension symbol*

$$s(\lambda,z,y) := \omega(\lambda,z)\frac{e^{-\omega(\lambda,z)y} - e^{-\eta(z)y}}{\omega(\lambda,z) - \eta(z)}, \quad \omega(\lambda,z) := (\tau + \lambda + |z|_-^2)^{1/2},\ \eta(z) := |z|_-, \tag{3.20}$$

for $\lambda \in \Sigma_\phi$, $z \in B\Sigma_\delta^n$, and $y \in (0,\infty)$.

3.12. Lemma. *Let $n \in \mathbb{N}$, $\phi \in (\pi/2, \pi)$, $\delta \in (0, \pi/8]$ with $\phi + 2\delta < \pi$, and $\tau > 0$ and define s by (3.20). Then for every $j \in \mathbb{N}$, there exists $C > 0$ such that*

$$\sup\left\{|yzs(\lambda,z,y)|, |y\omega(\lambda,z)^{1-j}\partial_y^j s(\lambda,z,y)| : \lambda \in \Sigma_\phi, z \in B\Sigma_\delta^n, y \in (0,\infty)\right\} \le C.$$

Proof. It is useful [cf. SS08, p. 186] to represent the difference quotient as

$$\frac{e^{-\omega y} - e^{-\eta y}}{\omega - \eta} = \int_0^1 \frac{d}{ds}\frac{e^{-g(s)y}}{\omega - \eta}\,ds = -y\int_0^1 e^{-g(s)y}\,ds, \quad \text{where } g(s) := (1-s)\eta + s\omega.$$

Lemma B.55 and Example B.56 imply $|\eta| \sim \operatorname{Re}\eta$ and $|\omega| \sim \operatorname{Re}\omega$ and hence $\operatorname{Re} g(s) \gtrsim |z|$. For $\alpha \in (0,\infty)$ and $x \in (0,\infty)$, the inequality $e^{-x} \le (\alpha/ex)^\alpha$ is valid. Therefore

$$|yzs(\lambda,z,y)| \le y^2|z\omega|\int_0^1 e^{-\operatorname{Re} g(s)y}\,ds \le y^2|z\omega|\int_0^1 \frac{4\,ds}{e^2y^2(\operatorname{Re} g(s))^2} = \frac{4|z\omega|}{e^2\operatorname{Re}\eta\operatorname{Re}\omega} \lesssim 1.$$

Let us show that $|y\omega^{1-j}\partial_y^j s| \lesssim 1$ for $j \in \mathbb{N}$. Since $\phi + 2\delta < \pi$, the inequality (B.14) yields $|\eta| \lesssim |\omega|$. In the case $|\eta| \le 2^{-1}|\omega|$, we have $|\omega - \eta| \ge 2^{-1}|\omega|$ and hence

$$|y\omega^{1-j}\partial_y^j s| \le y\left|\omega^{2-j}\frac{\omega^j e^{-\omega y} - \eta^j e^{-\eta y}}{\omega - \eta}\right| \le y|\omega|^{2-j}\frac{|\omega|^j e^{-\operatorname{Re}\omega y} + |\eta|^j e^{-\operatorname{Re}\eta y}}{|\omega - \eta|} \lesssim 1.$$

Next, the Leibniz rule yields

$$y\partial_y^j s = -y\partial_y^j \left(y\omega \int_0^1 e^{-g(s)y}\, ds \right)$$
$$= (-1)^{j+1} y^2 \omega \int_0^1 g(s)^j e^{-g(s)y}\, ds + j(-1)^j y\omega \int_0^1 g(s)^{j-1} e^{-g(s)y}\, ds.$$

Hence, in the remaining case $2^{-1}|\omega| \le |\eta| \lesssim |\omega|$, we have $\operatorname{Re} g(s) \sim |\omega|$ and thus

$$|y\omega^{1-j}\partial_y^j s| \le y^2|\omega|^{2-j} \int_0^1 |\omega|^j e^{-\operatorname{Re} g(s) y}\, ds + jy|\omega|^{2-j} \int_0^1 |\omega|^{j-1} e^{-\operatorname{Re} g(s)y}\, ds$$
$$\le |\omega|^2 \int_0^1 \frac{4}{e^2(\operatorname{Re} g(s))^2}\, ds + j|\omega| \int_0^1 \frac{1}{e(\operatorname{Re} g(s))^2}\, ds \lesssim 1. \qquad \square$$

We are ready to prove Theorem 3.1.

Proof of Theorem 3.1. The boundedness of the solution-to-data map $(u, \pi, h) \mapsto (g_v, g_w, g_h)$ follows from the mixed derivative embeddings on page 159 and the spatial trace theorem on page 156. Moreover, the functions u, π, and h are Fourier-Laplace transformable in the sense of distributions and their transforms have the representations in (3.7). Hence the uniqueness in Lemma 3.8 and Corollary 3.9 imply that problem (3.2) has at most one solution.

In order to construct a solution, we let $(g_v, g_w, g_h) \in {}_0\mathbb{G}_v \times {}_0\mathbb{G}_w \times {}_0\mathbb{G}_h$ be given and define the functions u, π, and h as in Figure 3.5, Lemma 3.8, and Corollary 3.9. Then (u, π, h) solves problem (3.2), which follows from the injectivity of the Fourier-Laplace transformation. It remains to prove that the data-to-solution map $(g_v, g_w, g_h) \mapsto (u, \pi, h)$ is uniformly bounded with respect to $\vartheta \in \mathcal{P}_M$.

(i) Corollary 3.9 implies that $(g_v, g_w, g_h) \mapsto h$ is uniformly bounded with respect to $\vartheta \in \mathcal{P}_M$.

(ii) The pressure π has the symbol $\hat{\pi}_j(\lambda, \xi, y) = e^{-|\xi|y}\hat{p}_j(\lambda, \xi)$ where $e^{-|\xi|y}$ is the symbol of the Poisson semigroup $P(y)$. Therefore Theorem B.28.(iv) and $\partial_y P(y) = \sqrt{-\Delta_x} P(y)$ yield

$$\|\nabla_{(x,y)} P(y) p_j\|_{L_p(\mathbb{R}_+ \times \mathbb{R}_+^{n+1})} \lesssim \|p_j\|_{L_p(\mathbb{R}_+; \dot{W}_p^{1-1/p}(\mathbb{R}^n))}.$$

In view of the divergence conditions (3.6b) and the identity (3.9) for $\hat{p}_j$, we obtain

$$\hat{p}_j = -\frac{\alpha_1\alpha_2}{\Omega_+(\omega_1 + \omega_2)} \frac{i\xi}{|\xi|} \cdot \hat{\Phi}_v + \frac{(-1)^j \alpha_j}{\Omega_+} [\![\hat{\pi}]\!].$$

Hence, by Corollary 3.9, the interface pressures $p_j = \pi_j|_{y=0}$ satisfy

$$p_j \in {}_0W_p^{1/2-1/2p}(\mathbb{R}_+; L_p(\mathbb{R}^n)) \cap L_p(\mathbb{R}_+; W_p^{2-2/p}(\mathbb{R}^n)) \hookrightarrow L_p(\mathbb{R}_+; \dot{W}_p^{1-1/p}(\mathbb{R}^n)),$$

and therefore π_j belongs to $\mathbb{E}_\pi = L_p(\mathbb{R}_+; \dot{H}_p^1(\mathbb{R}_+^{n+1}))$ and satisfies $[\![\pi]\!] \in {}_0\mathbb{G}_w$; thus, $\pi \in {}_0\mathbb{E}_{\pi,[\![\pi]\!]}$. Moreover, the map $(g_v, g_w, g_h) \mapsto \pi$ is uniformly bounded with respect to $\vartheta \in \mathcal{P}_M$.

(iii) Corollary 3.9 yields $p_j \in {}_0\mathbb{G}_w$, $\Phi_v \in {}_0\mathbb{E}_v$, and $\Phi_w^j \in {}_0\mathbb{E}_w$. Therefore the identities (3.8b) and (3.8c) yield $v|_{y=0} = \Phi_v \in {}_0\mathbb{E}_v$ and $w|_{y=0} \in {}_0\mathbb{E}_w$. Since Φ_v belongs to the Dirichlet trace space ${}_0W_p^{1-1/2p,2-1/p}(\mathbb{R}_+ \times \mathbb{R}^n)$ of ${}_0H_p^{1,2}(\mathbb{R}_+ \times \mathbb{R}_+^{n+1})$, we conclude from Theorem B.25 that the parabolic extension $[y \mapsto e^{-L_j y}]\Phi_v$ with $L_j := \omega_j(\mathcal{D}_t, \mathcal{D}_x)^{-1}$ belongs to ${}_0H_p^{1,2}(\mathbb{R}_+ \times \mathbb{R}_+^{n+1})$. Next, from (3.6b) we infer that $\hat{\Phi}_w^j = (-1)^j i\xi\omega_j^{-1}\hat{\Phi}_v^j$, and with ${}_0W_p^{1-1/2p,2-1/p}(\mathbb{R}_+ \times \mathbb{R}^n) \hookrightarrow {}_0H_p^{1/2}(\mathbb{R}_+; W_p^{1-1/p}(\mathbb{R}^n))$ (see Proposition B.44) we obtain

$$\Phi_w^j = (-1)^j \dot{J}_1 L_j^{-1} \Phi_v^j \in \dot{J}_1 L_j^{-1} {}_0H_p^{1/2}(\mathbb{R}_+; W_p^{1-1/p}(\mathbb{R}^n)) \hookrightarrow {}_0H_p^1(\mathbb{R}_+; \dot{W}_p^{-1/p}(\mathbb{R}^n)).$$

Hence the Poisson extension $(t, x, y) \mapsto (P(y)\Phi_w^j)(t, x)$ belongs to ${}_0H_p^1(\mathbb{R}_+; L_p(\mathbb{R}_+^{n+1}))$. By Lemma 3.12 and Example B.56, the Stokes extension symbols $W_{w,j}$ and $V_{w,j}$ satisfy the assumption of Lemma 3.11.(ii). Since $V_{h,j} = -V_{w,j}$ and since $(\tau + \partial_t)h - g_h$ belongs to the space

${}_0H^1_p(\mathbb{R}_+; \dot{W}_p^{-1/p}(\mathbb{R}^n)) \cap {}_0W_p^{1-1/2p,2-1/p}(\mathbb{R}_+ \times \mathbb{R}^n)$, we conclude that v belongs to ${}_0H^{1,2}_p(\mathbb{R}_+ \times \dot{\mathbb{R}}^{n+1})$ and that $(g_v, g_w, g_h) \mapsto v$ is uniformly bounded with respect to $\vartheta \in \mathcal{P}_M$. Finally, the symbol $W_{h,j}$ also satisfies the assumption of Lemma 3.11.(ii) and therefore w belongs to ${}_0H^{1,2}_p(\mathbb{R}_+ \times \dot{\mathbb{R}}^{n+1})^n$. We conclude that $u = (v, w)$ belongs to ${}_0\mathbb{E}_{u,v,w}$ and that $(g_v, g_w, g_h) \mapsto u$ is uniformly bounded with respect to $\vartheta \in \mathcal{P}_M$. The proof of Theorem 3.1 is complete. □

3.1.5. Inhomogeneous bulk equations. The next step towards optimal regularity of problem (3.1) is to allow for additional data (f_u, f_d); that is, we consider the problem

$$(3.21)\quad \begin{cases} \rho\partial_t u - \mu\Delta u + \nabla\pi = f_u & \text{in } J \times \dot{\mathbb{R}}^{n+1}, \\ \operatorname{div} u = f_d & \text{in } J \times \dot{\mathbb{R}}^{n+1}, \\ [\![u]\!] = 0 & \text{on } J \times \mathbb{R}^n, \\ -\mu_s\Delta_x v - \lambda_s \nabla_x \operatorname{div}_x v - [\![\mu\partial_y v]\!] - c_5[\![\mu\nabla_x w]\!] + c_1\nabla_x\Delta_x h = g_v & \text{on } J \times \mathbb{R}^n, \\ -\operatorname{tr}((c_2 + 2C_3)\nabla_x v) - 2[\![\mu\partial_y w]\!] + [\![\pi]\!] - \operatorname{tr}((c_\sigma + C_4)\nabla_x^2 h) = g_w & \text{on } J \times \mathbb{R}^n, \\ (\tau + \partial_t)h - w = g_h & \text{on } J \times \mathbb{R}^n, \\ h|_{t=0} = 0 & \text{on } \mathbb{R}^n, \\ u|_{t=0} = 0 & \text{in } \mathbb{R}^{n+1}. \end{cases}$$

Here we still consider a flat interface $\Sigma = \mathbb{R}^n \times \{0\} \cong \mathbb{R}^n$ in the whole space $\Omega = \mathbb{R}^{n+1}$, but restrict our investigation to a bounded time interval $J = (0, T)$ with $T \in (0, \infty)$ and $\tau = 0$. The physical parameters ρ_1, ρ_2, μ_1, μ_2, σ, λ_s, and μ_s, and the abbreviations c_1, c_2, C_3, C_4, and c_σ are the same as on page 54. For the additional data (f_u, f_d) we consider the conditions

$$\begin{aligned} &f_u \in \mathbb{F}_u := L_p(J; L_p(\mathbb{R}^{n+1})^{n+1}), \\ &f_d \in {}_0\mathbb{F}_d := {}_0H^1_p(J; \dot{H}_p^{-1}(\mathbb{R}^{n+1})) \cap L_p(J; H^1_p(\dot{\mathbb{R}}^{n+1})). \end{aligned}$$

In the previously considered case $\operatorname{div} u = f_d = 0$, the term $\partial_y w$ was of class ${}_0\mathbb{G}_w$, since

$$\partial_y w = \operatorname{div} u - \operatorname{div}_x v = -\operatorname{div}_x v \in {}_0\mathbb{G}_w \quad \text{for } u \in {}_0\mathbb{E}_{u,v,w} \text{ with } \operatorname{div} u = 0.$$

In order to maintain this property for $f_d \neq 0$, we consider the additional conditions

$$\partial_y w_\pm|_\Sigma \in {}_0\mathbb{G}_w, \quad f_{d\pm}|_\Sigma \in {}_0\mathbb{G}_w.$$

Then the space of suitable divergence data can be characterized as follows.

3.13. Lemma. *The divergence operator*

$$(3.22)\quad \begin{aligned} \operatorname{div}\colon {}_0\mathbb{E}_{u,v,w,\partial_y w} &:= \{u \in {}_0\mathbb{E}_{u,v,w} : \partial_y w_\pm|_{y=0} \in {}_0\mathbb{G}_w\} \\ &\to {}_0\mathbb{F}_{d,\Sigma} := \{f_d \in {}_0\mathbb{F}_d : f_{d\pm}|_{y=0} \in {}_0\mathbb{G}_w\} \end{aligned}$$

is a retraction.

Proof. We have to show that $\operatorname{div}\colon {}_0\mathbb{E}_{u,v,w,\partial_y w} \to {}_0\mathbb{F}_{d,\Sigma}$ is bounded and surjective and has a bounded right-inverse. The divergence theorem with interface implies that the map (3.22) is bounded. In order to construct a bulk velocity field $u \in \mathbb{E}_{u,v,w,\partial_\nu w}$ for given divergence data $f_d \in {}_0\mathbb{F}_{d,\Sigma}$, we employ the data-to-solution operators $S_\pm\colon f_{d\pm} \mapsto (u_\pm, \pi_\pm)$ for the one-phase Stokes problems

$$(\partial_t - \Delta)u_\pm + \nabla\pi_\pm = 0 \text{ in } J \times \Omega_\pm,\ \operatorname{div} u_\pm = f_{d\pm} \text{ in } J \times \Omega_\pm,\ u_\pm|_{y=0} = 0 \text{ on } J \times \mathbb{R}^n$$

for $\Omega_\pm = \pm\mathbb{R}^{n+1}_+$ from [BP07, Theorem 6.1]. The function u from $(u_\pm, \pi_\pm) = S_\pm(f_{d\pm})$ belongs to ${}_0\mathbb{E}_u$, the traces $v|_{y=0}$ and $w|_{y=0}$ vanish, and hence belong to ${}_0\mathbb{E}_v$ and ${}_0\mathbb{E}_w$, respectively. Moreover, $\partial_y w_\pm|_{y=0} = f_{d\pm}|_{y=0}$ belong to ${}_0\mathbb{G}_w$. Therefore $f_d \mapsto u$ is a bounded right-inverse for (3.22). □

$$
\begin{aligned}
{}_0\mathbb{E}_u &:= \{u \in {}_0H^1_p(J; L_p(\mathbb{R}^{n+1})^{n+1}) \cap L_p(J; H^2_p(\dot{\mathbb{R}}^{n+1})^{n+1}) : [\![u]\!] = 0\},\\
{}_0\mathbb{E}_v &:= {}_0W^{1-1/2p}_p(J; L_p(\mathbb{R}^n)^n) \cap {}_0W^{1/2-1/2p}_p(J; H^2_p(\mathbb{R}^n)^n) \cap L_p(J; W^{3-1/p}_p(\mathbb{R}^n)^n),\\
{}_0\mathbb{E}_w &:= {}_0W^{1-1/2p}_p(J; H^1_p(\mathbb{R}^n)) \cap L_p(J; W^{3-1/p}_p(\mathbb{R}^n)),\\
{}_0\mathbb{E}_{u,v,w,\partial_y w} &:= \{u = (v,w) \in {}_0\mathbb{E}_u : v|_{y=0} \in {}_0\mathbb{E}_v,\ w|_{y=0} \in {}_0\mathbb{E}_w,\ \partial_y w_\pm|_{y=0} \in {}_0\mathbb{G}_w\},\\
\mathbb{E}_\pi &:= L_p(J; \dot{H}^1_p(\dot{\mathbb{R}}^{n+1})),\\
{}_0\mathbb{E}_{\pi,[\![\pi]\!]} &:= \{\pi \in \mathbb{E}_\pi : [\![\pi]\!] \in {}_0\mathbb{G}_w\},\\
{}_0\mathbb{E}_h &:= {}_0W^{2-1/2p}_p(J; H^1_p(\mathbb{R}^n)) \cap {}_0H^1_p(J; W^{3-1/p}_p(\mathbb{R}^n)) \cap L_p(J; W^{4-1/p}_p(\mathbb{R}^n)),\\
\mathbb{F}_u &:= L_p(J; L_p(\mathbb{R}^{n+1})^{n+1}),\\
{}_0\mathbb{F}_d &:= {}_0H^1_p(J; \dot{H}^{-1}_p(\mathbb{R}^{n+1})) \cap L_p(J; H^1_p(\dot{\mathbb{R}}^{n+1})),\\
{}_0\mathbb{F}_{d,\Sigma} &:= \{f_d \in {}_0\mathbb{F}_d : f_{d\pm}|_{y=0} \in {}_0\mathbb{G}_w\},\\
{}_0\mathbb{G}_v &:= {}_0W^{1/2-1/2p}_p(J; L_p(\mathbb{R}^n)^n) \cap L_p(J; W^{1-1/p}_p(\mathbb{R}^n)^n),\\
{}_0\mathbb{G}_w &:= {}_0W^{1/2-1/2p}_p(J; H^1_p(\mathbb{R}^n)) \cap L_p(J; W^{2-1/p}_p(\mathbb{R}^n)),\\
{}_0\mathbb{G}_h &:= {}_0W^{1-1/2p}_p(J, H^1_p(\mathbb{R}^n)) \cap L_p(J, W^{3-1/p}_p(\mathbb{R}^n)).
\end{aligned}
$$

FIGURE 3.6. Function spaces ${}_0\mathbb{E}_{\dots}$, ${}_0\mathbb{F}_{\dots}$, and ${}_0\mathbb{G}_{\dots}$ on $(J, \mathbb{R}^{n+1}, \mathbb{R}^n)$.

We are ready to prove optimal regularity for problem (3.21). The relevant function spaces are summarized in Figure 3.6 on this page.

3.14. Theorem. *Let* $\lambda_s + \mu_s > 0$, $c_5 \in \{0,1\}$, $c_6 = 1$, $p \in (1,\infty)$, $T_0 \in (0,\infty)$, *and* $M > 0$. *Then the solution-to-data map*

$$
\begin{gathered}
(u,\pi,h) \mapsto (f_u, f_d, g_v, g_w, g_h),\\
{}_0\mathbb{E}_{u,v,w,\partial_y w} \times {}_0\mathbb{E}_{\pi,[\![\pi]\!]} \times {}_0\mathbb{E}_h \to \mathbb{F}_u \times {}_0\mathbb{F}_{d,\Sigma} \times {}_0\mathbb{G}_v \times {}_0\mathbb{G}_w \times {}_0\mathbb{G}_h
\end{gathered}
$$

of problem (3.21) *is uniformly invertible with respect to* $T \in (0, T_0]$ *and* $\vartheta \in \mathcal{P}_M$.

Proof. Boundedness of the solution-to-data map follows from the mixed derivative embeddings on page 159, the divergence theorem with interface, and the spatial trace theorem on page 156. Injectivity follows from Corollary 3.2. For proving surjectivity, we construct a solution

$$(u,\pi,h) = (u_1,0,0) + (u_2,\pi_2,0) + (u_3,\pi_3,h_3).$$

First, with the co-retraction $\operatorname{div}^c\colon {}_0\mathbb{F}_{d,\Sigma} \to {}_0\mathbb{E}_{u,v,w,\partial_y w}$ from Lemma 3.13, we choose $u_1 = \operatorname{div}^c f_d$. Second, let $f_{u,2} := f_u - (\rho\partial_t - \mu\Delta)u_1$ and let $P = I - \nabla\Delta^{-1}\operatorname{div}$ denote the Helmholtz projection in $L_p(\mathbb{R}^{n+1})^{n+1}$. Then $Pf_{u,2}$ belongs to $\mathbb{F}_u$ and we seek a solution $u_2 \in {}_0\mathbb{E}_u$ of the Stokes problem

$$\rho\partial_t u_2 - \mu\Delta u_2 = Pf_{u,2}, \quad \operatorname{div} u_2 = 0, \quad u_2|_\Sigma = 0. \tag{3.23}$$

Since (3.23) consists of two separated one-phase Stokes problems in $J \times \mathbb{R}^{n+1}_\pm$, we obtain the desired solution map $Pf_{u,2} \mapsto u_2$ from [DHP01, Theorem 7.6]. We trivially have $0 = v_2|_\Sigma \in {}_0\mathbb{E}_v$, $0 = w_2|_\Sigma \in {}_0\mathbb{E}_w$, and $0 = \partial_y w_2|_\Sigma = \operatorname{div} u_2|_\Sigma - \operatorname{div}_x v_2|_\Sigma \in {}_0\mathbb{G}_w$. Therefore u_2 belongs to ${}_0\mathbb{E}_{u,v,w,\partial_y w}$. With Lemma 2.23, we define π_2 as the solution to the weak Neumann transmission problem

$$\langle\nabla\pi_2(t,\cdot), \nabla\varphi\rangle_{\mathbb{R}^{n+1}} = \langle(I-P)f_{u,2}(t,\cdot), \nabla\varphi\rangle_{\mathbb{R}^{n+1}} \quad \text{for all } \varphi \in \dot{H}^1_{p'}(\mathbb{R}^{n+1}), \quad [\![\pi_2]\!] = 0.$$

Hence π_2 belongs to ${}_0\mathbb{E}_{\pi,[\![\pi]\!]}$ and satisfies $\nabla\pi_2 = (I-P)f_{u,2}$ in $\mathbb{F}_u$. The uniform boundedness of $(f_u, f_d) \mapsto (u_1+u_2, \pi_2, 0)$ with respect to $T \in (0, T_0]$ follows by extension of the data from $J = (0,T)$ to $(0,\infty)$ with Lemma B.9 and by restriction to $(0, T_0)$.

Finally, Corollary 3.2 yields a unique solution $(u_3, \pi_3, h_3) \in {}_0\mathbb{E}$ of

$$\begin{aligned} \rho\partial_t u_3 - \mu\Delta u_3 + \nabla\pi_3 &= 0 && \text{in } J\times\dot{\mathbb{R}}^{n+1},\\ \operatorname{div} u_3 &= 0 && \text{in } J\times\dot{\mathbb{R}}^{n+1},\\ L_u(u_3,\pi_3,h_3;\vartheta) &= (g_v,g_w) - L_u(u_1+u_2,\pi_2,0;\vartheta) && \text{on } J\times\mathbb{R}^n,\\ \partial_t h_3 - w_3 &= g_h + w_1 + w_2 && \text{on } J\times\mathbb{R}^n, \end{aligned}$$

and the map $(f_u, f_d, g_v, g_w, g_h) \mapsto (u_3, \pi_3, h_3)$ is uniformly bounded with respect to $T \in (0, T_0]$ and $\vartheta \in \mathcal{P}_M$. Hence the proof of Theorem 3.14 is complete. ☐

3.2. Bent hyperplanes and variable coefficients

We generalize Theorem 3.14 to a situation where the interface is a bent hyperplane

$$\Sigma = \Sigma_\omega := \{(x', \omega(x')) : x' \in \mathbb{R}^{n-1}\} \quad \text{with } \omega \in BC^4(\mathbb{R}^{n-1}) \tag{3.24}$$

in $\Omega = \mathbb{R}^n$ ($n \geq 2$) and the coefficients on the interface may depend on (t, x'). In a tubular neighborhood $B_r(\Sigma)$ with projection $\Pi_\Sigma\colon B_r(\Sigma) \to \Sigma$, we decompose $u = v + w\ \nu_\Sigma \circ \Pi$ with $v := [P_\Sigma \circ \Pi_\Sigma]u$ and $w := (\nu_\Sigma \circ \Pi_\Sigma|u)$. We consider the perturbed model problem

$$\left\{\begin{aligned} \rho\partial_t u - \mu\Delta u + \nabla\pi &= f_u && \text{in } J\times(\mathbb{R}^n\setminus\Sigma_\omega),\\ \operatorname{div} u &= f_d && \text{in } J\times(\mathbb{R}^n\setminus\Sigma_\omega),\\ [\![u]\!] &= 0 && \text{on } J\times\Sigma_\omega,\\ L_v(u,h;\vartheta,\omega) &= g_v && \text{on } J\times\Sigma_\omega,\\ L_w(u,\pi,h;\vartheta,\omega) &= g_w && \text{on } J\times\Sigma_\omega,\\ \partial_t h - w &= g_h && \text{on } J\times\Sigma_\omega,\\ h|_{t=0} &= 0 && \text{on } \Sigma_\omega,\\ u|_{t=0} &= 0 && \text{in } \mathbb{R}^n. \end{aligned}\right. \tag{3.25}$$

Here $J = (0,T)$ is bounded, the parameter triple $\vartheta = (\vartheta_L, \vartheta_w, \vartheta_{Dv})$ consists of fixed functions

$$\vartheta_L\colon \Sigma_\omega \to \mathbb{R}^{n\times n}, \quad (\vartheta_w, \vartheta_{Dv})\colon J\times\Sigma_\omega \to \mathbb{K}\times\mathbb{K}^{n\times n},$$

and, similar to (3.3) on page 54, we define further parameters

$$\left\{\begin{array}{ll} \vartheta_1 := (\lambda_s+\mu_s)\vartheta_w, & \vartheta_2 := (\lambda_s-\mu_s)\operatorname{tr}\vartheta_L,\\ \vartheta_3 := \mu_s\vartheta_L, & \vartheta_4 := 2\mu_s[\vartheta_{Dv} - 2\vartheta_w\vartheta_L],\\ c_5 \in \{0,1\}, & \vartheta_\sigma := \sigma + (\lambda_s-\mu_s)\operatorname{tr}[\vartheta_{Dv} - 2\vartheta_w\vartheta_L]. \end{array}\right. \tag{3.26}$$

Then the operators L_v and L_w are given by

$$\begin{aligned} L_v(u,h;\omega,\vartheta) &= -\mu_s\widetilde{\Delta}_{\Sigma_\omega} v - \lambda_s\nabla_{\Sigma_\omega}\operatorname{div}_{\Sigma_\omega} v - [\![\mu\partial_{\nu_{\Sigma_\omega}} v]\!] - c_5[\![\mu\nabla_{\Sigma_\omega} w]\!] + \vartheta_1\nabla_{\Sigma_\omega}\Delta_{\Sigma_\omega} h,\\ L_w(u,\pi,h;\omega,\vartheta) &= -\operatorname{tr}\left([\vartheta_2+2\vartheta_3]\nabla_{\Sigma_\omega} v\right) - 2[\![\mu\partial_{\nu_{\Sigma_\omega}} w]\!] + [\![\pi]\!] - \operatorname{tr}\left([\vartheta_\sigma+\vartheta_4]\nabla^2_{\Sigma_\omega} h\right). \end{aligned}$$

More details on these operators will be given in Figure 3.8 on page 73 and Section 4.3.

We will prove optimal regularity for problem (3.25) for the following class of parameters and by using the function spaces from Figure 3.7 on the next page.

3.15. Definition. Given $M, T, \eta, R \in (0,\infty)$, the set $\mathcal{P}_{M,T,\eta,R}$ consists of all $(\vartheta^*, \omega, \vartheta)$ such that

(i) the constant tuple $\vartheta^* = (\vartheta^*_L, \vartheta^*_w, \vartheta^*_{Dv})$ belongs to the parameter set $\mathcal{P}_M$ from page 54,

(ii) the map $\omega \in BC^4(\mathbb{R}^{n-1})$ satisfies $\|\omega\|_{BC^1\cap H^2_p} \leq \eta$, $\|\omega\|_{BC^4} \leq R$, and $\omega(0) = |\nabla\omega(0)| = 0$,

$$
\begin{aligned}
{}_0\mathbb{E}_u &:= \{u \in {}_0H_p^1(J; L_p(\Omega)^n) \cap L_p(J; H_p^2(\Omega\setminus\Sigma)^n) : u|_{\partial\Omega} = 0\},\\
{}_0\mathbb{E}_v &:= {}_0W_p^{1-1/2p}(J; L_p(\Sigma; T\Sigma))\\
&\qquad \cap {}_0W_p^{1/2-1/2p}(J; H_p^2(\Sigma; T\Sigma)) \cap L_p(J; W_p^{3-1/p}(\Sigma; T\Sigma)),\\
{}_0\mathbb{E}_w &:= {}_0W_p^{1-1/2p}(J; H_p^1(\Sigma)) \cap L_p(J; W_p^{3-1/p}(\Sigma)),\\
{}_0\mathbb{E}_{u,v,w,\partial_\nu w} &:= \{u \in {}_0\mathbb{E}_u : [\![u]\!] = 0,\ v|_\Sigma \in {}_0\mathbb{E}_v,\ w|_\Sigma \in {}_0\mathbb{E}_w,\ \partial_\nu w_\pm \in {}_0\mathbb{G}_w\},\\
\mathbb{E}_\pi &:= L_p(J; \dot{H}_p^1(\Omega\setminus\Sigma)),\\
{}_0\mathbb{E}_{\pi,[\![\pi]\!]} &:= \{\pi \in \mathbb{E}_\pi : [\![\pi]\!] \in {}_0\mathbb{G}_w\},\\
{}_0\mathbb{E}_h &:= {}_0W_p^{2-1/2p}(J; H_p^1(\Sigma)) \cap {}_0H_p^1(J; W_p^{3-1/p}(\Sigma)) \cap L_p(J; W_p^{4-1/p}(\Sigma)),\\
\mathbb{F}_u &:= L_p(J\times\Omega)^n,\\
{}_0\mathbb{F}_d &:= {}_0H_p^1(J; \hat{H}_p^{-1}(\Omega)) \cap L_p(J; H_p^1(\Omega\setminus\Sigma)),\\
{}_0\mathbb{F}_{d,\Sigma} &:= \{f_d \in {}_0\mathbb{F}_d : f_{d\pm}|_\Sigma \in {}_0\mathbb{G}_w\},\\
{}_0\mathbb{G}_v &:= {}_0W_p^{1/2-1/2p}(J; L_p(\Sigma; T\Sigma)) \cap L_p(J; W_p^{1-1/p}(\Sigma; T\Sigma)),\\
{}_0\mathbb{G}_w &:= {}_0W_p^{1/2-1/2p}(J; H_p^1(\Sigma)) \cap L_p(J; W_p^{2-1/p}(\Sigma)),\\
{}_0\mathbb{G}_h &:= {}_0W_p^{1-1/2p}(J; H_p^1(\Sigma)) \cap L_p(J; W_p^{3-1/p}(\Sigma)).
\end{aligned}
$$

The spaces L_p, H_p^k, and W_p^s ($p \in (1,\infty)$, $k \in \mathbb{N}_0$, $s \in [0,\infty)$) are endowed with the intrinsic norm (B.5). The corresponding spaces $\mathbb{E}_{\dots}$, $\mathbb{F}_{\dots}$, and $\mathbb{G}_{\dots}$ are defined by replacing ${}_0W_p^s$ by W_p^s and ${}_0H_p^k$ by H_p^k. The scalar-valued versions of ${}_0\mathbb{E}_v$ and ${}_0\mathbb{G}_v$ are denoted by the same symbol.

FIGURE 3.7. Function spaces ${}_0\mathbb{E}_{\dots}$, ${}_0\mathbb{F}_{\dots}$, and ${}_0\mathbb{G}_{\dots}$ on (J,Ω,Σ).

(iii) the triple $\vartheta = (\vartheta_L, \vartheta_w, \vartheta_{Dv})$ consists of functions $\vartheta_L : \Sigma_\omega \to \mathbb{R}^{n\times n}$ and $(\vartheta_w, \vartheta_{Dv}) \colon (0,T)\times\Sigma_\omega \to \mathbb{R}\times\mathbb{R}^{n\times n}$ that satisfy the inequalities

$$
\begin{aligned}
\|\vartheta_L - \vartheta_L^*\|_{BC(\Sigma_\omega)\cap H_p^1(\Sigma_\omega)} &\le \eta, & \|\vartheta_L - \vartheta_L^*\|_{BC^2(\Sigma_\omega)} &\le R,\\
\|\vartheta_w - \vartheta_w^*\|_{C([0,T];BC(\Sigma_\omega)\cap H_p^1(\Sigma_\omega))} &\le \eta, & \|\vartheta_w - \vartheta_w^*\|_{\mathbb{G}_w(T)} &\le R,\\
\|\vartheta_{Dv} - \vartheta_{Dv}^*\|_{C([0,T];BC(\Sigma_\omega)\cap H_p^1(\Sigma_\omega))} &\le \eta, & \|\vartheta_{Dv} - \vartheta_{Dv}^*\|_{\mathbb{G}_w(T)} &\le R.
\end{aligned}
$$

3.16. Theorem. *Let $\rho_\pm$, $\mu_\pm$, σ, μ_s, $\lambda_s + \mu_s > 0$ and let $p \in (n+2,\infty)$, $M > 0$, and $T_1 > 0$ be fixed. Then there exists $\eta > 0$ such that for given $R > 0$ we can find a number $T_0 \in (0, T_1]$ such that the solution-to-data map*

$$
\begin{gathered}
(u,\pi,h) \mapsto (f_u, f_d, g_v, g_w, g_h),\\
{}_0\mathbb{E} := {}_0\mathbb{E}_{u,v,w,\partial_\nu w} \times {}_0\mathbb{E}_{\pi,[\![\pi]\!]} \times {}_0\mathbb{E}_h \to {}_0\mathbb{F} := \mathbb{F}_u \times {}_0\mathbb{F}_{d,\Sigma} \times {}_0\mathbb{G}_v \times {}_0\mathbb{G}_w \times {}_0\mathbb{G}_h
\end{gathered}
$$

of problem (3.25) is uniformly invertible with respect to $T \in (0,T_0]$ and $(\vartheta^*,\omega,\vartheta) \in \mathcal{P}_{M,T_1,\eta,R}$.

We point out that the number η depends on the bound M for ϑ^* but not on the bound R for ω and ϑ. This will be important for the localization procedure for a bounded domain. The proof of Theorem 3.16 will be reduced to an application of Theorem 3.14 for the flat interface problem (3.21). To this end, we consider the usual defining diffeomorphism

$$
\Theta_\omega(x', x_n) = (x', x_n + \omega(x')) \quad \text{for } x' \in \mathbb{R}^{n-1},\ x_n \in \mathbb{R}. \tag{3.27}
$$

Let $\partial = (\partial', \partial_n)$, $\partial' = (\partial_1, \dots, \partial_{n-1})$, $\nabla = \partial^\top$, $\nabla' = \partial'^\top$ and $\Sigma_\omega = \{(x', \omega(x')) : x' \in \mathbb{R}^{n-1}\}$. The defining diffeomorphism $\Theta_\omega(x', x_n) = (x', x_n + \omega(x'))$ for Σ_ω satisfies

$$\partial\Theta_\omega = I + e_n \otimes \nabla\omega = \begin{bmatrix} I' & 0 \\ \partial'\omega & 1 \end{bmatrix}, \quad [\partial\Theta_\omega]^{-1} = I - e_n \otimes \nabla\omega = \begin{bmatrix} I' & 0 \\ -\partial'\omega & 1 \end{bmatrix}.$$

The tangent vectors τ_j, cotangent vectors τ^j, unit normal vector ν, metric components g_{jk}, g^{jk} and Christoffel symbols $\Lambda_{jk,l}$, Λ^l_{kj} of Σ_ω are given by

$$\begin{aligned} \nu \circ \Theta_\omega &= \beta(e_n - \nabla\omega), & \beta &= (1 + |\nabla\omega|^2)^{-1/2}, \\ \tau_j \circ \Theta_\omega &= e_j + \partial_j\omega\, e_n, & \tau^j \circ \Theta_\omega &= e_j + \beta^2\partial_j\omega(e_n - \nabla\omega), \\ g_{jk} \circ \Theta_\omega &= \delta_{jk} + \partial_j\omega\partial_k\omega, & g^{jk} \circ \Theta_\omega &= \delta_{jk} - \beta^2\partial_j\omega\partial_k\omega, \\ \Lambda_{jk,l} \circ \Theta_\omega &= \partial_j\partial_k\omega\partial_l\omega, & \Lambda^l_{jk} \circ \Theta_\omega &= \beta^2\partial_j\partial_k\omega\partial_l\omega. \end{aligned}$$

The projections $P' = I - e_n \otimes e_n$ and $P_{\Sigma_\omega} = I - \nu_{\Sigma_\omega} \otimes \nu_{\Sigma_\omega}$ satisfy

$$P' = \begin{bmatrix} I' & 0 \\ 0 & 0 \end{bmatrix}, \quad P_{\Sigma_\omega} \circ \Theta_\omega = \begin{bmatrix} I' - \beta^2\nabla'\omega \otimes \nabla'\omega & \beta^2\nabla'\omega \\ \beta^2\partial'\omega & 1 - \beta^2 \end{bmatrix}.$$

For a scalar field φ, a tangential vector field v and a not necessarily tangential vector field u, the gradient $\nabla_{\Sigma_\omega}\varphi$, $\nabla_{\Sigma_\omega}u$, divergence $\operatorname{div}_{\Sigma_\omega} u$, scalar Laplace-Beltrami operator $\Delta_{\Sigma_\omega}\varphi = \operatorname{div}_{\Sigma_\omega} \nabla_{\Sigma_\omega}\varphi$ and tangential Laplace-Beltrami operator $\widetilde{\Delta}_{\Sigma_\omega} v = g^{jk}\widetilde{\nabla}_j\widetilde{\nabla}_k v$ are given by

$$\begin{aligned} \nabla_{\Sigma_\omega}\varphi &= \tau^j\partial_j\varphi, & \nabla_{\Sigma_\omega}u &= \tau^j \otimes \partial_j u, \\ \Delta_{\Sigma_\omega}\varphi &= g^{jk}(\partial_j\partial_k\varphi - \Lambda^l_{jk}\partial_l\varphi), & \operatorname{div}_{\Sigma_\omega} u &= \tau^j \cdot \partial_j u, \\ & & \widetilde{\Delta}_{\Sigma_\omega} v &= g^{jk}P_{\Sigma_\omega}\partial_j(P_{\Sigma_\omega}\partial_k v). \end{aligned}$$

FIGURE 3.8. Differential geometric identities for bent hyperplanes.

Since $x' \mapsto \Theta_\omega(x', 0)$ is a global parametrization for Σ_ω, we can compute the relevant differential geometric quantities of Σ_ω by a straightforward application of Appendix A. The relevant identities are collected in Figure 3.8 on this page.

In Lemma 3.17 we will prove that the induced transformations for solutions and data induce isomorphisms between the function spaces in $\mathbb{R}^n \setminus \Sigma_\omega$ and the corresponding spaces in $\dot{\mathbb{R}}^n = \mathbb{R}^n \setminus \Sigma_0$, which are uniformly bounded and uniformly invertible with respect to ω. We also derive transformation identities for the velocity components v and w which are collected in Figure 3.9 on the next page. These will be employed for deriving the transformed version (3.48) of problem (3.25). This transformed problem corresponds to the basic model problem (3.21) with additional perturbations. We will control those pertubations by means of appropriate interval-dependent estimates and estimates for pointwise multiplication and continuous embeddings (see Lemmas 3.18 and 3.19). For proving Theorem 3.16, we require smallness of η in order to control the leading-order perturbations and smallness of T for controlling the lower-order perturbations.

3.17. Lemma. *Let $\omega \in BC^4(\mathbb{R}^{n-1})$ and $J = (0, T)$ with $T \in (0, \infty]$. Consider the transformations*

$$\begin{aligned} (u, \pi, h) \circ \Theta_\omega &= ([\partial\Theta_\omega]\overline{u}, \overline{\pi}, \overline{h}), \\ (f_u, f_d, g_u, g_h) \circ \Theta_\omega &= ([\partial\Theta_\omega]\overline{f}_u, \overline{f}_d, [\partial\Theta_\omega]\overline{g}_u, \overline{g}_h), \end{aligned}$$

and the decompositions $u = v + w\,\nu_{\Sigma_\omega}$, $\overline{u} = \overline{v} + \overline{w}\,e_n$, $g_u = g_v + g_w\,\nu_{\Sigma_\omega}$, and $\overline{g}_u = \overline{g}_v + \overline{g}_w\,e_n$.

Define Θ_ω as in Figure 3.8 and let

$$u\circ\Theta_\omega=[\partial\Theta_\omega]\overline{u},\quad u=v+w\,\nu_{\Sigma_\omega},\quad \overline{u}=\overline{v}+\overline{w}\,e_n.$$

Then the following identities are valid.

(3.29a) $\overline{v}=P'(v\circ\Theta_\omega)+Q_v(\omega)w\circ\Theta_\omega,\qquad Q_v(\omega)=-\beta\nabla\omega,$

(3.29b) $\overline{w}=w\circ\Theta_\omega+Q_w(\omega)w\circ\Theta_w,\qquad Q_w(\omega)=\beta^{-1}-1,$

(3.29c) $\partial_n\overline{w}=\beta^2(\partial_\nu w)\circ\Theta_\omega+\beta^3\partial_j\omega(\partial_j w)\circ\Theta_\omega,$

(3.29d) $v\circ\Theta_\omega=[I+e_n\otimes\nabla\omega]\overline{v}+\{(1-\beta^2)e_n-\beta^2\nabla\omega\}\overline{w},$

(3.29e) $w\circ\Theta_\omega=\overline{w}+(\beta-1)\overline{w},$

(3.29f) $(\partial_\nu w)\circ\Theta_\omega=\partial_n\overline{w}-\beta\nabla'\omega\cdot\nabla'(\beta\overline{w}).$

Their derivations are given in the proof of Lemma 3.17.

FIGURE 3.9. Identities for the transformed velocity field.

Then, given $R>0$, the operators

(3.28a)	$\overline{u}\mapsto u,$	$\mathbb{E}_u(\mathbb{R}^n\setminus\Sigma_0)\to\mathbb{E}_u(\mathbb{R}^n\setminus\Sigma_\omega),$
(3.28b)	$(\overline{v},\overline{w})\mapsto(v,w),$	$\mathbb{E}_v(\Sigma_0)\times\mathbb{E}_w(\Sigma_0)\to\mathbb{E}_v(\Sigma_\omega)\times\mathbb{E}_w(\Sigma_\omega),$
(3.28c)	$\overline{u}\mapsto u,$	$\mathbb{E}_{u,v,w,\partial_n w}(\mathbb{R}^n\setminus\Sigma_0)\to\mathbb{E}_{u,v,w,\partial_\nu w}(\mathbb{R}^n\setminus\Sigma_\omega),$
(3.28d)	$\overline{\pi}\mapsto\pi,$	$\mathbb{E}_\pi(\mathbb{R}^n\setminus\Sigma_0)\to\mathbb{E}_\pi(\mathbb{R}^n\setminus\Sigma_\omega),$
(3.28e)	$\overline{\pi}\mapsto\pi,$	$\mathbb{E}_{\pi,[\![\pi]\!]}(\mathbb{R}^n\setminus\Sigma_0)\to\mathbb{E}_{\pi,[\![\pi]\!]}(\mathbb{R}^n\setminus\Sigma_\omega),$
(3.28f)	$\overline{h}\mapsto h,$	$\mathbb{E}_h(\Sigma_0)\to\mathbb{E}_h(\Sigma_\omega),$
(3.28g)	$\overline{f}_u\mapsto f_u,$	$\mathbb{F}_u(\mathbb{R}^n\setminus\Sigma_0)\to\mathbb{F}_u(\mathbb{R}^n\setminus\Sigma_\omega),$
(3.28h)	$\overline{f}_d\mapsto f_d,$	$\mathbb{F}_d(\mathbb{R}^n\setminus\Sigma_0)\to\mathbb{F}_d(\mathbb{R}^n\setminus\Sigma_\omega),$
(3.28i)	$\overline{f}_d\mapsto f_d,$	$\mathbb{F}_{d,\Sigma}(\mathbb{R}^n\setminus\Sigma_0)\to\mathbb{F}_{d,\Sigma}(\mathbb{R}^n\setminus\Sigma_\omega),$
(3.28j)	$(\overline{g}_v,\overline{g}_w)\mapsto(g_v,g_w),$	$\mathbb{G}_v(\Sigma_0)\times\mathbb{G}_w(\Sigma_0)\to\mathbb{G}_v(\Sigma_\omega)\times\mathbb{G}_w(\Sigma_\omega),$
(3.28k)	$\overline{g}_h\mapsto g_h,$	$\mathbb{G}_h(\Sigma_0)\to\mathbb{G}_h(\Sigma_\omega)$

are uniformly bounded and uniformly invertible with respect to $\|\nabla\omega\|_{BC^3}\le R$ and $T\in(0,\infty]$.

Proof. (i) In order to estimate the norms of transformed functions, we employ the chain rule from Remark B.85 on page 171 for representing their derivatives. Define $\Theta=\Theta_\omega$ as in (3.27).

(i.a) For $\overline{x}\in\mathbb{R}^n$ and $\overline{\alpha}\in\mathbb{R}^n$ we put $x=\Theta(\overline{x})$ and $\alpha=[\partial\Theta(\overline{x})]\overline{\alpha}$. Then the derivatives of Θ and Θ^{-1} read as follows.

$$[\partial\Theta(\overline{x})]\overline{\alpha}=\overline{\alpha}+e_n(\nabla\omega(\overline{x}')|\overline{\alpha}),\qquad [\partial^j\Theta(\overline{x})](\overline{\alpha}_1,\dots,\overline{\alpha}_j)=e_n[\partial^j\omega(\overline{x}')](\overline{\alpha}_1,\dots,\overline{\alpha}_j)\text{ for } j\ge 2,$$
$$[\partial\Theta^{-1}(x)]\alpha=\alpha-e_n(\nabla\omega(\overline{x}')|\overline{\alpha}),\quad [\partial^j\Theta^{-1}(x)](\alpha_1,\dots,\alpha_j)=-e_n[\partial^j\omega(\overline{x}')](\overline{\alpha}_1,\dots,\overline{\alpha}_j)\text{ for } j\ge 2.$$

(i.b) For a sufficiently smooth function $\varphi:\mathbb{R}^n\to\mathbb{R}$ we put $\overline{\varphi}:=\varphi\circ\Theta$ and obtain

$$[\partial\varphi(x)]\alpha=[\partial\overline{\varphi}(\overline{x})]\overline{\alpha},$$
$$[\partial^2\varphi(x)](\alpha_1,\alpha_2)=[\partial^2\overline{\varphi}(\overline{x})](\overline{\alpha}_1,\overline{\alpha}_2)-[\partial\overline{\varphi}(\overline{x})][\partial\Theta(\overline{x})]^{-1}[\partial^2\Theta(\overline{x})](\overline{\alpha}_1,\overline{\alpha}_2).$$

Omitting the arguments x and $\overline{x}$ and the square brackets around derivatives, we have

$$\begin{aligned}
&\partial^3\varphi(\alpha_1,\alpha_2,\alpha_3)\\
&= \partial^3\overline{\varphi}(\overline{\alpha}_1,\overline{\alpha}_2,\overline{\alpha}_3)\\
&\quad - \partial^2\overline{\varphi}\{(\partial\Theta^{-1}\partial^2\Theta(\overline{\alpha}_1,\overline{\alpha}_2),\overline{\alpha}_3) + (\partial\Theta^{-1}\partial^2\Theta(\overline{\alpha}_1,\overline{\alpha}_3),\overline{\alpha}_2) + (\partial\Theta^{-1}\partial^2\Theta(\overline{\alpha}_2,\overline{\alpha}_3),\overline{\alpha}_1)\}\\
&\quad + \partial\overline{\varphi}\,\partial\Theta^{-1}\partial^2\Theta\{(\partial\Theta^{-1}\partial^2\Theta(\overline{\alpha}_1,\overline{\alpha}_2),\overline{\alpha}_3) + (\partial\Theta^{-1}\partial^2\Theta(\overline{\alpha}_1,\overline{\alpha}_3),\overline{\alpha}_2) + (\partial\Theta^{-1}\partial^2\Theta(\overline{\alpha}_2,\overline{\alpha}_3),\overline{\alpha}_1)\}\\
&\quad - \partial\overline{\varphi}\,\partial\Theta^{-1}\partial^3\Theta(\overline{\alpha}_1,\overline{\alpha}_2,\overline{\alpha}_3).
\end{aligned}$$

(i.c) For $\overline{u} : \mathbb{R}^n \to \mathbb{R}^n$ we let $u(x) := [\partial\Theta(\overline{x})]\overline{u}(\overline{x})$ and write $u = \partial\Theta\,\overline{u}$, to be short. Then

$$\begin{aligned}
\partial u\,\alpha_1 &= \partial\Theta\,\partial\overline{u}\,\overline{\alpha} + \partial^2\Theta(\overline{u},\overline{\alpha}),\\
\partial^2 u(\alpha_1,\alpha_2) &= \partial\Theta\,\partial^2\overline{u}(\overline{\alpha}_1,\overline{\alpha}_2) + \partial^2\Theta\{(\partial\overline{u}\,\overline{\alpha}_1,\overline{\alpha}_2) + (\overline{\alpha}_1,\partial\overline{u}\,\overline{\alpha}_2)\}\\
&\quad - \partial\Theta\,\partial\overline{u}\,\partial\Theta^{-1}\partial^2\Theta(\overline{\alpha}_1,\overline{\alpha}_2) - \partial^2\Theta(\overline{u},\partial\Theta^{-1}\partial^2\Theta(\overline{\alpha}_1,\overline{\alpha}_2)) + \partial^3\Theta(\overline{u},\overline{\alpha}_1,\overline{\alpha}_2).
\end{aligned}$$

(ii) Lemma B.10 on page 148 yields the pointwise multiplication estimate

$$[\![uv]\!]_{W_p^\sigma(\Sigma_\omega)} \le \|u\|_{L_\infty(\Sigma_\omega)}\,[\![v]\!]_{W_p^\sigma(\Sigma_\omega)} + C(n,s,p,\|\nabla\omega\|_{BC^1})\|u\|_{W^1_\infty(\Sigma_\omega)}\|v\|_{L_p(\Sigma)} \tag{3.30}$$

for $u \in W^1_\infty(\Sigma_\omega)$, $v \in W^\sigma_p(\Sigma_\omega)$, $\sigma \in (0,1)$, $p \in [1,\infty)$.

(iii) Now we are prepared for proving that the operators (3.28) are uniformly invertible. We will frequently employ the differential geometric identities in Figure 3.8.

(iii.a) The invertibility of $\overline{u} \mapsto u$ in (3.28a), $\overline{\pi} \mapsto \pi$ in (3.28d) and $\overline{f}_u \mapsto f_u$ in (3.28g) easily follows from (i). These operators are uniformly invertible with respect to $\|\nabla\omega\|_{BC^2} \le M$.

(iii.b) We recall that $W^s_p(\Sigma_\omega)$ is equipped with the intrinsic Sobolev-Slobodeckiĭ norm (B.5). Hence, from $\nabla^{k+1}_{\Sigma_\omega} h = (\nabla_{\Sigma_\omega}\otimes)^k\nabla_{\Sigma_\omega} h$ ($k \le 2$) and (3.30) we infer that the operator $\overline{h} \mapsto h$ in (3.28f) is invertible, uniformly with respect to $\|\nabla\omega\|_{BC^3} \le M$. In a similar way we see that $\overline{g}_h \mapsto g_h$ in (3.28k) is uniformly invertible with respect to $\|\nabla\omega\|_{BC^2} \le M$.

(iii.c) The transformed normal velocity satisfies

$$\begin{aligned}
\overline{w} = e_n\cdot\overline{u} &= e_n\cdot[\partial\Theta]^{-1}(v + \nu_{\Sigma_\omega} w)\circ\Theta_\omega\\
&= (1+|\nabla\omega|^2)^{1/2} w\circ\Theta_\omega\\
&= w\circ\Theta_\omega + Q_w(\omega) w\circ\Theta_\omega \qquad \text{with } Q_w(\omega) := (1+|\nabla\omega|^2)^{1/2} - 1 = \beta^{-1} - 1.
\end{aligned}$$

Hence the identities (3.29b) and (3.29e) are valid and the operator $w \mapsto \overline{w}$, $\mathbb{E}_w(\Sigma_\omega) \to \mathbb{E}_w(\Sigma_0)$ is uniformly invertible with respect to $\|\nabla\omega\|_{BC^3} \le M$.

(iii.d) The projection $P' = I - e_n\otimes e_n$ satisfies

$$P'[\partial\Theta_\omega]^{-1}[P_{\Sigma_\omega}\circ\Theta_\omega] = P' + \beta\nabla\omega\otimes(\nu_{\Sigma_\omega}\circ\Theta_\omega).$$

Therefore v is related to $(\overline{v},\overline{w})$ by

$$\begin{aligned}
\overline{v} = P'\overline{u} &= P'[\partial\Theta_\omega]^{-1}(P_{\Sigma_\omega} v + w\,\nu_{\Sigma_\omega})\circ\Theta_\omega\\
&= P'(v\circ\Theta_\omega) + Q_v(\omega) w\circ\Theta_\omega \qquad \text{with } Q_v(\omega) := -\nabla\omega(1+|\nabla\omega|^2)^{-1/2} = -\beta\nabla\omega.
\end{aligned}$$

This yields (3.29a). With $\mathbb{E}_w \hookrightarrow \mathbb{E}_v$, it follows that $(v,w) \mapsto \overline{v}$, $\mathbb{E}_v(\Sigma_\omega)\times {}_0\mathbb{E}_w(\Sigma_\omega) \to {}_0\mathbb{E}_v(\Sigma_0)$ is uniformly bounded with respect to $\|\nabla\omega\|_{BC^3} \le M$. The inverse representation is given by

$$\begin{aligned}
v\circ\Theta_\omega &= [P_{\Sigma_\omega}\circ\Theta_\omega][I + e_n\otimes\nabla\omega](\overline{v} + \overline{w}\,e_n)\\
&= [I + e_n\otimes\nabla\omega]\overline{v} + \{(1-\beta^2)e_n + \beta^2\nabla\omega\}\overline{w}.
\end{aligned}$$

Therefore identity (3.29d) is true and the operator $(\overline{v},\overline{w}) \mapsto (v,w)$ in (3.28b) is uniformly invertible with respect to $\|\nabla\omega\|_{BC^3} \le M$.

(iii.e) With $w \circ \Theta_\omega = \beta\overline{w}$ we obtain

$$
\begin{aligned}
(\partial_\nu w) \circ \Theta_\omega &= \beta(e_n - \nabla\omega) \cdot \nabla((\beta\overline{w}) \circ \Theta_\omega^{-1}) \circ \Theta_\omega \\
&= \beta[I - e_n \otimes \nabla\omega](e_n - \nabla\omega) \cdot \nabla(\beta\overline{w}) \\
&= \beta(e_n - \nabla\omega + e_n|\nabla\omega|^2) \cdot \nabla(\beta\overline{w}) \\
&= \partial_n\overline{w} - \beta\nabla'\omega \cdot \nabla'(\beta\overline{w}), \\
\partial_n\overline{w} &= \beta e_n \cdot \nabla(w \circ \Theta_\omega) \\
&= \beta[I + e_n \otimes \nabla\omega]e_n \cdot (\nu\partial_\nu w + \tau^j\partial_j w) \circ \Theta_\omega \\
&= \beta^2(\partial_\nu w) \circ \Theta_\omega + \beta^3\partial_j\omega(\partial_j w) \circ \Theta_\omega.
\end{aligned}
$$

Thus, equations (3.29c) and (3.29f) are valid and the operator $u \mapsto \overline{u}$ in (3.28c) is uniformly invertible with respect to $\|\nabla\omega\|_{BC^3} \le M$.

(iii.f) The relation between $g_u \circ \Theta_\omega$ and $\overline{g}_u$ is analogous to that of $u|_\Sigma \circ \Theta_\omega$ and $\overline{u}|_{\Sigma_0}$. Hence equations (3.29a), (3.29b), (3.29d) and (3.29e) yield

$$
\begin{aligned}
\overline{g}_v &= P'(g_v \circ \Theta_\omega) + Q_v(\omega)g_w \circ \Theta_\omega, \\
g_v \circ \Theta_\omega &= [I + e_n \otimes \nabla\omega]\overline{g}_v + \beta^2(\nabla\omega - e_n|\nabla\omega|^2)\overline{g}_w, \qquad g_w \circ \Theta_\omega = \beta\overline{g}_w.
\end{aligned} \tag{3.31}
$$

Therefore (3.28j) and (3.28e) are uniformly invertible with respect to $\|\nabla\omega\|_{BC^2} \le M$.

(iii.g) For given $\overline{f}_d \in \mathbb{F}_d(\mathbb{R}^n \setminus \Sigma_0)$ and $\varphi \in \dot{H}^1_{p'}(\mathbb{R}^n)$, we obtain

$$
\int_{\mathbb{R}^n} f_d\,\varphi\,dx = \int_{\mathbb{R}^n} \overline{f}_d\,\varphi \circ \Theta_\omega\,dx, \qquad \text{with } \det\partial\Theta_\omega = 1.
$$

The map $\varphi \mapsto \varphi \circ \Theta_\omega$, $\dot{H}^1_{p'}(\mathbb{R}^n) \to \dot{H}^1_{p'}(\mathbb{R}^n)$ is uniformly invertible and therefore

$$
\overline{f}_d \mapsto f_d, \quad H^1_p(J; \hat{H}^{-1}_p(\mathbb{R}^n)) \to H^1_p(J; \hat{H}^{-1}_p(\mathbb{R}^n))
$$

is uniformly invertible with respect to $\|\nabla\omega\|_\infty \le M$. The estimates for the remaining norms follow similarly as above and therefore (3.28h) and (3.28i) are uniformly invertible with respect to $\|\nabla\omega\|_{BC^2} \le M$. The proof of Lemma 3.17 is complete. □

In order to take advantage of short time intervals we will frequently employ the following *interval-dependent estimates*, where we study the time-dependence of certain embedding constants.

3.18. Lemma. *Let X be a Banach space, $J = (0,T)$ with $T \in (0,\infty)$, and $p \in [1,\infty)$. Then*

$$\|u\|_{L_p(J)} \le T^{1/p-1/q}\|u\|_{L_q(J)}, \qquad \text{for } u \in L_q(J;X),\ q \in [p,\infty], \tag{3.32a}$$

$$\|v\|_{L_p(J)} \le T\tfrac{1}{1-1/p}\|\partial_t v\|_{L_p(J)}, \qquad \text{for } v \in {}_0W^1_p(J;X),\ p > 1, \tag{3.32b}$$

$$\|v\|_{L_p(J)} \le T^\alpha \tfrac{1}{2^{1/p}}\tfrac{1+\alpha-1/p}{\alpha-1/p}\,[\![v]\!]_{W^\alpha_p(J)}, \qquad \text{for } v \in {}_0W^\alpha_p(J;X),\ \alpha \in (1/p,1), \tag{3.32c}$$

$$[\![u]\!]_{W^\alpha_p(J)} \le T^{\beta-\alpha}\,[\![u]\!]_{W^\beta_p(J)}, \qquad \text{for } u \in W^\beta_p(J;X),\ \alpha \in (0,1),\ \beta \in (\alpha,1), \tag{3.32d}$$

$$[\![u]\!]_{W^\alpha_p(J)} \le T^{1-\alpha}\tfrac{2^{1/p}}{\alpha(p-\alpha p)^{1/p}}\,\|\partial_t u\|_{L_p(J)}, \qquad \text{for } u \in W^1_p(J;X),\ \alpha \in (0,1). \tag{3.32e}$$

Proof. (3.32a) follows from Hölder's inequality. To prove (3.32b), we apply Hardy's inequality (B.4) to $\partial_t v$. (3.32c) follows from Lemma B.5. (3.32d) can be verified directly:

$$
[\![u]\!]_{W^\alpha_p(0,T)} = \left(\int_0^T\int_0^T |t-s|^{(\beta-\alpha)p}\frac{|u(t)-u(s)|^p_X}{|t-s|^{1+\beta p}}\,ds\,dt\right)^{1/p} \le T^{\beta-\alpha}\,[\![u]\!]_{W^\beta_p(0,T)}.
$$

Estimate (3.32e) follows from Hardy's inequality as in [PS11, Proposition 5.1]. □

Next we provide appropriate estimates for controlling perturbations in ${}_0\mathbb{G}_v$, ${}_0\mathbb{G}_w$, and ${}_0\mathbb{G}_h$. Basically, such estimates were used in [PSS07] and [PS10].

3.19. Lemma. *Let Ω be a domain in $\mathbb{R}^n$ ($n \geq 2$) containing a smooth (possibly empty) hypersurface Σ such that Assumption 2.1 on page 23 is satisfied and let $p \in (1,\infty)$. Then the following assertions are valid.*

(i) If $p > 2$, then for all $\delta \in (0,1/2)$ and $T_0 \in (0,\infty)$ there exists $C(\delta,T_0) > 0$ such that

$$\|(T^{-1}u, T^{-\delta}\nabla u)\|_{\mathbb{F}_u(T)} \leq C(\delta,T_0)\|u\|_{{}_0\mathbb{E}_u(T)} \tag{3.33}$$

for all $u \in {}_0\mathbb{E}_u(T)$ and $T \in (0,T_0]$.

(ii) If $p > 2$, then for all $T_0 \in (0,\infty)$ there exists $C(T_0) > 0$ such that

$$\|(T^{-1/2}v, T^{-1/4}\nabla' v)\|_{{}_0\mathbb{G}_v(T)} + \|T^{-1/4}v\|_{{}_0\mathbb{G}_w(T)} \leq C(T_0)\|v\|_{{}_0\mathbb{E}_v(T)} \tag{3.34}$$

for all $v \in {}_0\mathbb{E}_v(T)$ and $T \in (0,T_0]$.

(iii) If $p > 2$, then for all $\delta \in (0,1/2)$ and $T_0 \in (0,\infty)$ there exists $C(\delta,T_0) > 0$ such that

$$\|(T^{-1/2}w, T^{-\delta}\nabla' w)\|_{{}_0\mathbb{G}_v(T)} + \|T^{-\delta}w\|_{{}_0\mathbb{G}_w(T)} \leq C(\delta,T_0)\|w\|_{{}_0\mathbb{E}_w(T)} \tag{3.35}$$

for all $w \in {}_0\mathbb{E}_w(T)$ and $T \in (0,T_0]$.

(iv) If $p > 3$, then for all $\delta \in (0,3/2)$ and $T_0 \in (0,\infty)$ there exists $C(\delta,T_0) > 0$ such that

$$\|(T^{-3/2}h, T^{-\delta}\nabla' h, T^{-1}\nabla'^2 h)\|_{{}_0\mathbb{G}_v(T)} + \|(T^{-3/2}h, T^{-1}\nabla' h)\|_{{}_0\mathbb{G}_w(T)} \leq C(\delta,T_0)\|h\|_{{}_0\mathbb{E}_h(T)} \tag{3.36}$$

for all $h \in {}_0\mathbb{E}_h(T)$ and $T \in (0,T_0]$.

(v) There exists $C > 0$ such that for all $T \in (0,\infty)$, $\varphi \in BC^1(\Sigma)$, and $g_v \in \mathbb{G}_v(T)$ we have

$$\|\varphi g_v\|_{\mathbb{G}_v(T)} \leq C\|\varphi\|_\infty \|g_v\|_{\mathbb{G}_v(T)} + C\|\varphi\|_\infty^{1/p}\|\nabla'\varphi\|_\infty^{1-1/p}\|g_v\|_{L_p(0,T;L_p(\Sigma))}. \tag{3.37}$$

and if $p > 3$, then for all $\delta \in (0,1/2-1/2p)$ and $T_0 \in (0,\infty)$ there exists $C(\delta,T_0) > 0$ such that

$$\|g_v\|_{L_p(0,T;L_p(\Sigma))} \leq T^\delta C(\delta,T_0)\|g_v\|_{{}_0\mathbb{G}_v(T)} \quad \text{for all } g_v \in {}_0\mathbb{G}_v(T),\ T \in (0,T_0]. \tag{3.38}$$

There exists $C > 0$ such that for all $T \in (0,\infty)$ we have

$$\|\vartheta g_v\|_{\mathbb{G}_v(T)} \leq C\|\vartheta\|_\infty\|g_v\|_{\mathbb{G}_v(T)} + C\|\vartheta\|_{\mathbb{G}_v(T)}\|g_v\|_\infty \quad \text{for all } \vartheta, g_v \in \mathbb{G}_v(T) \cap L_\infty(J\times\Sigma), \tag{3.39}$$

and if $p > n+2$, then we have the continuous embedding

$$\mathbb{G}_v(T) \hookrightarrow C([0,T];BC(\Sigma)),$$

and for all $\delta \in (0,1/2-(n+2)/2p)$ and $T_0 \in (0,\infty)$ there exists $C(\delta,T_0) > 0$ such that

$$\|g_v\|_{C([0,T];BC(\Sigma))} \leq T^\delta C(\delta,T_0)\|g_v\|_{{}_0\mathbb{G}_v(T)} \quad \text{for all } g_v \in {}_0\mathbb{G}_v(T),\ T \in (0,\infty). \tag{3.40}$$

(vi) There exists $C > 0$ such that for all $T \in (0,\infty)$, $\varphi \in BC^2(\mathbb{R}^{n-1})$, and $g_w \in \mathbb{G}_w(T)$ we have

$$\|\varphi g_w\|_{\mathbb{G}_w(T)} \leq C\|\varphi\|_{L_\infty\cap H^1_p}\|g_w\|_{\mathbb{G}_w(T)} + C\|\varphi\|_{BC^2}\|g_w\|_{L_p(0,T;H^1_p(\Sigma))}, \tag{3.41}$$

and if $p > 3$, then for all $\delta \in (0,1/2-1/2p)$ and $T_0 \in (0,\infty)$ there exists $C(\delta,T_0) > 0$ such that

$$\|g_w\|_{L_p(0,T;H^1_p(\Sigma))} \leq T^\delta C(\delta,T_0)\|g_w\|_{{}_0\mathbb{G}_w(T)} \quad \text{for } g_w \in {}_0\mathbb{G}_w(T),\ T \in (0,T_0]. \tag{3.42}$$

If $p \in (n+2,\infty)$, then we have the continuous embedding

$$\mathbb{G}_w(T) \hookrightarrow \tilde{\mathbb{G}}_w(T) := C([0,T];H^1_p(\Sigma)) \cap L_p(0,T;BC^1(\Sigma)),$$

there exists $C > 0$ such that for all $T \in (0,\infty)$, $\vartheta \in \mathbb{G}_w(T)$, and $g_w \in \mathbb{G}_w(T)$ we have

$$\|\vartheta g_w\|_{\mathbb{G}_w(T)} \leq C\|\vartheta\|_{C([0,T];H^1_p(\Sigma))}\|g_w\|_{\mathbb{G}_w(T)} + C\|\vartheta\|_{\mathbb{G}_w(T)}\|g_w\|_{\tilde{\mathbb{G}}_w(T)}, \tag{3.43}$$

and for all $\delta \in (0,1/2-3/2p)$ and $T_0 \in (0,\infty)$ there exists $C(\delta,T_0) > 0$ such that

$$\|g_w\|_{{}_0\tilde{\mathbb{G}}_w(T)} \leq T^\delta C(\delta,T_0)\|g_w\|_{{}_0\mathbb{G}_w(T)} \quad \text{for all } g_w \in {}_0\mathbb{G}_w(T),\ T \in (0,\infty). \tag{3.44}$$

(vii) There is $C > 0$ such that for all $T \in (0, \infty)$, $\varphi \in BC^3(\mathbb{R}^{n-1})$, and $g_h \in \mathbb{G}_h(T)$ we have

$$\|\varphi g_h\|_{\mathbb{G}_h(T)} \le C\|\varphi\|_{L_\infty \cap H^1_p}\|g_h\|_{\mathbb{G}_h(T)} + C\|\varphi\|_{BC^3}\|g_h\|_{L_p(0,T;H^2_p(\Sigma))}, \tag{3.45}$$

and if $p > 3$, then for all $T_0 \in (0, \infty)$ there exists $C(T_0) > 0$ such that

$$\|g_h\|_{L_p(0,T;H^2_p(\Sigma))} \le T^{1/2-1/2p}C(T_0)\|g_h\|_{{}_0\mathbb{G}_h(T)} \quad \textit{for all } g_h \in {}_0\mathbb{G}_h(T),\ T \in (0, T_0]. \tag{3.46}$$

Proof. We will frequently employ the embeddings (B.1), (B.2), (B.3) page 145, and the mixed derivative embeddings from Proposition B.44 on page 159. From Lemma B.9 on page 148 we infer that the embedding constants for the relevant subspaces with vanishing initial values are uniformly bounded with respect to $T \in (0, T_0]$, by extension to $(0, \infty)$ and restriction to $(0, T_0)$. Moreover, Lemma 3.18 on page 76 yields a factor $T^\delta C(\delta, T_0)$ for the norm bound of an embedding into a space with lower temporal regularity.

For τ, $\sigma \in (0, 1)$, $p \in [1, \infty)$, and $q \in [1, \infty]$, we abbreviate

$$[\![\cdot]\!]_{\tau,p;q} := [\![\cdot]\!]_{W^\tau_p(0,T;L_q(\Sigma))}, \quad [\![\cdot]\!]_{q;\sigma,p} := [\![\cdot]\!]_{L_q(0,T;W^\sigma_p(\Sigma))}, \quad \|\cdot\|_{p;q} := \|\cdot\|_{L_p(0,T;L_q(\Sigma))}.$$

We may assume that the norms of $\mathbb{G}_v$, $\mathbb{G}_w$, and $\mathbb{G}_h$ are given by

$$\begin{aligned}
\|v\|_{\mathbb{G}_v} &= [\![v]\!]_{1/2-1/2p,p;p} + \|v\|_{p;p} + [\![v]\!]_{p;1-1/p,p}, \\
\|w\|_{\mathbb{G}_w} &= [\![(w, \nabla w)]\!]_{1/2-1/2p,p;p} + \|(w, \nabla w)\|_{p;p} + [\![\nabla w]\!]_{p;1-1/p,p}, \\
\|h\|_{\mathbb{G}_h} &= [\![(h, \nabla h)]\!]_{1-1/2p,p;p} + \|(h, \nabla h, \nabla^2 h)\|_{p;p} + [\![\nabla^2 h]\!]_{p;1-1/p,p},
\end{aligned}$$

since these norms are equivalent to the usual ones and the corresponding constants only depend on p and n but not on T. Lemma B.10 yields the estimate

$$[\![uv]\!]_{W^\sigma_p} \le \|u\|_\infty [\![v]\!]_{W^\sigma_p} + C(n, p, \sigma)\|u\|^{1-\sigma}_\infty \|\nabla u\|^\sigma_\infty \|v\|_p \tag{3.47}$$

for $u \in W^1_\infty(\Omega)$ and $v \in W^\sigma_p(\Omega)$.

The inequality (3.32c) and the mixed derivative embeddings yield

$$\|\nabla u\|_{L_p(0,T;L_p(\mathbb{R}^n))} \le T^\delta C(\delta, T_0)\|u\|_{{}_0W^\delta_p(0,T;H^1_p(\mathbb{R}^n))} \le T^\delta C(\delta, T_0)\|u\|_{{}_0\mathbb{E}_u},$$

for $\delta \in (0, 1/2)$, provided $1/2 > 1/p$ which is true for $p > 2$. Together with (3.32b), this proves assertion (i). With (3.32c) and (3.32d) we obtain

$$\begin{aligned}
\|v\|_{{}_0\mathbb{G}_v(T)} &= [\![v]\!]_{1/2-1/2p,p;p} + \|v\|_{p;p} + [\![v]\!]_{p;1-1/p,p} \\
&\le C\{T^{1/2} + T^{1-1/2p}\}\,[\![v]\!]_{1-1/2p,p;p} + CT^\delta\|v\|_{{}_0W^\delta_p(0,T;W^{1-1/p}_p(\Sigma))}
\end{aligned}$$

for $v \in {}_0\mathbb{E}_v(T)$ and $\delta \in (1/p, 1)$. Moreover, for $\rho \in (0, 1/2]$ with $1/p < \delta < 1/2 - 1/2p + \rho \le 1 - 1/2p$ and $2 - 4\rho > 1 - 1/p$ the mixed derivative embeddings yield

$$\|v\|_{W^\delta_p(0,T;W^{1-1/p}_p(\Sigma))} \le C(T_0)\|v\|_{H^{1/2-1/2p+\rho}_p(0,T;W^{2-4\rho}_p(\Sigma))} \le C(T_0)\|v\|_{{}_0\mathbb{E}_v(T)}.$$

The number ρ must belong to $(0, 1/2] \cap (\delta - 1/2 + 1/2p, 1/4 + 1/4p)$ and this interval is nonempty if $\delta < 3/4 - 1/4p$, which is true for $\delta \le 1/2$. The embedding estimates (3.34), (3.35), (3.36), (3.38), (3.40), (3.42), (3.44), and (3.46) follow similarly and hence assertions (ii), (iii), and (iv) are valid.

The bilinear estimates (3.37), (3.41), and (3.45) can be verified by means of the spatial pointwise multiplication inequality (3.47), since the factor φ does not depend on time. Hence (vii) is valid.

Finally, the bilinear estimates (3.39) and (3.43) follow from (3.47), Sobolev's embedding, and the pointwise multiplication estimate in Lemma B.81. Therefore (v) and (vi) are also true. □

Proof of Theorem 3.16. For a given parameter tuple $(\vartheta^*, \omega, \vartheta) \in \mathcal{P}_{M,T_1,\eta,R}$, we define ϑ_1^*, ϑ_2^*, ϑ_3^*, ϑ_4^*, and ϑ_σ^* according to (3.26). Let $z = (u, \pi, h)$, $\overline{z} = (\overline{u}, \overline{\pi}, \overline{h})$, $f = (f_u, f_d, g_v, g_w, g_h)$, and $\overline{f} = (\overline{f}_u, \overline{f}_d, \overline{g}_v, \overline{g}_w, \overline{g}_h)$ be related as in Lemma 3.17. We introduce the transformed operators

$$\begin{aligned}
\overline{L}_v(\overline{u}, \overline{h}; \vartheta^*) &:= -\mu_s \Delta' \overline{v} - \lambda_s \nabla' \operatorname{div}' \overline{v} - [\![\mu \partial_n \overline{v}]\!] - c_5 [\![\mu \nabla' \overline{w}]\!] + \vartheta_1^* \nabla' \Delta' \overline{h}, \\
\overline{L}_w(\overline{u}, \overline{\pi}, \overline{h}; \vartheta^*) &:= -\operatorname{tr}([\vartheta_2^* + 2\vartheta_3^*] \nabla' \overline{v}) - 2[\![\mu \partial_n \overline{w}]\!] + [\![\overline{\pi}]\!] - \operatorname{tr}([\vartheta_\sigma^* + \vartheta_4^*] \nabla'^2 \overline{h}), \\
F_u(\overline{u}, \overline{\pi}; \omega) &:= \overline{\mu}([\partial \Theta_\omega]^{-1} (\Delta u) \circ \Theta_\omega - \Delta \overline{u}) + (I - [\partial \Theta_\omega]^{-1} [\partial \Theta_\omega]^{-\top}) \nabla \overline{\pi}, \\
G_v(\overline{u}, \overline{\pi}, \overline{h}; \vartheta^*, \omega, \vartheta) &:= \overline{L}_v(\overline{u}, \overline{h}; \vartheta^*) - P' L_v(u, h; \omega, \vartheta) \circ \Theta_\omega - Q_v(\omega) L_w(u, \pi, h; \omega, \vartheta) \circ \Theta_\omega, \\
G_w(\overline{u}, \overline{\pi}, \overline{h}; \vartheta^*, \omega, \vartheta) &:= \overline{L}_w(\overline{u}, \overline{\pi}, \overline{h}; \vartheta^*) - L_w(u, \pi, h; \omega, \vartheta) \circ \Theta_\omega - Q_w(\omega) L_w(u, \pi, h; \omega, \vartheta) \circ \Theta_\omega, \\
G_h(\overline{w}; \omega) &:= ((1 + |\nabla \omega|^2)^{-1/2} - 1) \overline{w}.
\end{aligned}$$

Here actually G_v and G_w do not depend on $\overline{\pi}$ and we will therefore write $G_j(\overline{u}, \overline{h}; \vartheta^*, \omega, \vartheta)$ for $j \in \{v, w\}$. More details on these operators will be given below and we will show that problem (3.25) is equivalent to the following problem for $\Sigma_0 = \mathbb{R}^{n-1} \times \{0\}$.

$$(3.48) \qquad \left\{ \begin{aligned}
\overline{\rho} \partial_t \overline{u} - \overline{\mu} \Delta \overline{u} + \nabla \overline{\pi} &= \overline{f}_u + F_u(\overline{u}, \overline{\pi}; \omega) && \text{in } J \times \dot{\mathbb{R}}^n, \\
\operatorname{div} \overline{u} &= \overline{f}_d && \text{in } J \times \dot{\mathbb{R}}^n, \\
[\![\overline{u}]\!] &= 0 && \text{on } J \times \Sigma_0, \\
\overline{L}_v(\overline{u}, \overline{h}; \vartheta^*) &= \overline{g}_v + G_v(\overline{u}, \overline{h}; \vartheta^*, \omega, \vartheta) && \text{on } J \times \Sigma_0, \\
\overline{L}_w(\overline{u}, \overline{\pi}, \overline{h}; \vartheta^*) &= \overline{g}_w + G_w(\overline{u}, \overline{h}; \vartheta^*, \omega, \vartheta) && \text{on } J \times \Sigma_0, \\
\partial_t \overline{h} - \overline{w} &= \overline{g}_h + G_h(\overline{w}; \omega) && \text{on } J \times \Sigma_0, \\
\overline{h}|_{t=0} &= 0 && \text{on } \Sigma_0, \\
\overline{u}|_{t=0} &= 0 && \text{in } \mathbb{R}^n.
\end{aligned} \right.$$

Let us abbreviate

$$S(\vartheta^*) \overline{z} := \begin{bmatrix} \overline{\rho} \partial_t \overline{u} - \overline{\mu} \Delta \overline{u} + \nabla \overline{\pi} \\ \operatorname{div} \overline{u} \\ \overline{L}_v(\overline{u}, \overline{h}; \vartheta^*) \\ \overline{L}_w(\overline{u}, \overline{\pi}, \overline{h}; \vartheta^*) \\ \partial_t \overline{h} - \overline{w} \end{bmatrix}, \quad F(\vartheta^*, \omega, \vartheta) \overline{z} := \begin{bmatrix} F_u(\overline{u}, \overline{\pi}; \omega) \\ 0 \\ G_v(\overline{u}, \overline{h}; \vartheta^*, \omega, \vartheta) \\ G_w(\overline{u}, \overline{h}; \vartheta^*, \omega, \vartheta) \\ G_h(\overline{w}; \omega) \end{bmatrix}.$$

Analogously as in Figure 3.7 on page 72 we let

$${}_0\overline{\mathbb{E}} := {}_0\overline{\mathbb{E}}_{u,v,w,\partial_\nu w} \times {}_0\overline{\mathbb{E}}_{\pi, [\![\pi]\!]} \times {}_0\overline{\mathbb{E}}_h, \quad {}_0\overline{\mathbb{F}} := \overline{\mathbb{F}}_u \times {}_0\overline{\mathbb{F}}_{d,\Sigma} \times {}_0\overline{\mathbb{G}}_v \times {}_0\overline{\mathbb{G}}_w \times {}_0\overline{\mathbb{G}}_h$$

denote the corresponding spaces defined with Σ_0 instead of Σ_ω. Our goal is to prove that

$$(3.49) \qquad \overline{z} \mapsto \overline{f} = [S(\vartheta^*) - F(\vartheta^*, \omega, \vartheta)] \overline{z}, \quad {}_0\overline{\mathbb{E}}(T) \to {}_0\overline{\mathbb{F}}(T)$$

is uniformly invertible with respect to $T \in (0, T_1]$ and $(\vartheta^*, \omega, \vartheta) \in \mathcal{P}_{M,T_1,\eta,R}$.

Theorem 3.14 shows that $S(\vartheta^*)\colon {}_0\overline{\mathbb{E}}(T) \to {}_0\overline{\mathbb{F}}(T)$ is uniformly invertible with respect to $T \in (0, T_1]$ and $\vartheta^* \in \mathcal{P}_M$, for every $T_1 \in (0, \infty)$ and $M \in (0, \infty)$, where $\mathcal{P}_M$ is defined in equation (3.4) on page 54. In order to apply a Neumann series argument, it remains to ensure that

$$\|[S(\vartheta^*)]^{-1} F(\vartheta^*, \omega, \vartheta)\|_{\mathcal{B}({}_0\overline{\mathbb{E}}(T))} < 1.$$

(i) We first compute the perturbations in more detail and we abbreviate

$$X(\overline{x}) := \Theta_\omega(\overline{x}), \quad \overline{X}(x) := \Theta_\omega^{-1}(x), \quad \text{for } x, \overline{x} \in \mathbb{R}^n.$$

By using the summation convention, we have

$$\partial_l X_k = \delta_{lk} + \delta_{kn}\partial_l\omega, \quad \partial_j \overline{X}_m = \delta_{jm} - \delta_{mn}\partial_j\omega.$$

Then the following identity is valid, where the values of u and $\overline{X}$ are taken at $(t,x) \in J \times (\mathbb{R}^n \setminus \Sigma_\omega)$ and those of $\overline{u}$ and X at $(t,\overline{x}) \in J \times (\mathbb{R}^n \setminus \Sigma_0)$ with $x = X(\overline{x})$.

$$\begin{aligned}\Delta u_k = \Delta \overline{u}_k &+ (\partial_l X_k\, \partial_j \overline{X}_m\, \partial_j \overline{X}_p - \delta_{kl}\delta_{jm}\delta_{jp})\, \partial_m\partial_p \overline{u}_l \\ &+ \big(\partial_l\partial_p X_k\, \partial_j \overline{X}_m\, \partial_j \overline{X}_p - \partial_p \overline{X}_m\, \partial_j\partial_r X_p\, \partial_j \overline{X}_r\, \partial_l X_k\big)\, \partial_m \overline{u}_l \\ &+ \big(\partial_l\partial_m\partial_p X_k\, \partial_j \overline{X}_m\, \partial_j \overline{X}_p - \partial_p \overline{X}_m\, \partial_j\partial_r X_p\, \partial_j \overline{X}_r\, \partial_l\partial_m X_k\big)\, \overline{u}_l.\end{aligned} \tag{3.50}$$

Moreover, a straightforward computation shows that the divergence satisfies

$$\operatorname{div} \overline{u} = \partial_k \left([\partial\Theta_\omega]^{-1}_{kl} u_l \circ \Theta_\omega\right) = \operatorname{div} u \circ \Theta_\omega,$$

and therefore no perturbations of the divergence equation appear.

The representations of G_v and G_w and the corresponding equations in (3.48) follow from (3.31).

(ii) In order to control the perturbation $F_u(\overline{u},\overline{\pi};\omega)$ in $\overline{\mathbb{F}}_u(T) = L_p(0,T;L_p(\mathbb{R}^n))$ we note that $\|I - \partial\Theta_\omega\|_\infty \to 0$ as $\|\nabla\omega\|_\infty \to 0$. Hence there exists $\eta > 0$ such that

$$\|(I - [\partial\Theta_\omega]^{-1}[\partial\Theta_\omega]^{-\top})\nabla\overline{\pi}\|_{L_p(0,T;L_p(\mathbb{R}^n))} \le \varepsilon \|\nabla\overline{\pi}\|_{L_p(0,T;L_p(\mathbb{R}^n))} \quad \text{for } T \in (0,\infty),\ \|\nabla\omega\|_\infty \le \eta.$$

Next, we rewrite the transformation formula (3.50) as

$$(\Delta u)\circ\Theta_\omega - \Delta\overline{u} = a^{jk}(\omega)\partial_j\partial_k\overline{u} + b^j(\omega)\partial_j\overline{u} + c(\omega)\overline{u},$$

where the coefficients $a^{jk}(\omega)$, $b^j(\omega)$, and $c(\omega)$ are functions on $\mathbb{R}^n$ which satisfy the following estimate. For given $\varepsilon > 0$ and $R > 0$ there exist $\eta > 0$, $R_b \ge 0$, and $R_c \ge 0$ such that

$$\|a^{jk}(\omega)\|_\infty \le \varepsilon, \quad \|b^j(\omega)\|_\infty \le R_b, \quad \|c(\omega)\|_\infty \le R_c,$$

for all $\omega \in BC^3(\mathbb{R}^{n-1})$ with $\|\nabla\omega\|_\infty \le \eta$ and $\|\nabla\omega\|_{BC^2} \le R$. By controlling the lower-order terms $\overline{u}$ and $\nabla\overline{u}$ with estimate (3.33), we conclude that for given $\varepsilon > 0$ there exists $\eta > 0$ such that

$$\begin{aligned}\|(\Delta u)\circ\Theta_\omega - \Delta\overline{u}\|_{L_p(0,T;L_p(\mathbb{R}^n))} &\le \varepsilon\|u\|_{{}_0\overline{\mathbb{E}}_u(T)} + C(R)\|\overline{u}\|_{L_p(0,T;H^1_p(\mathbb{R}^n))} \\ &\le \varepsilon\|u\|_{{}_0\overline{\mathbb{E}}_u(T)} + C(R)T^\delta C(\delta,T_1)\|\overline{u}\|_{{}_0\overline{\mathbb{E}}_u(T)},\end{aligned}$$

provided that $u \in {}_0\overline{\mathbb{E}}_u(T)$, $T \in (0,T_1]$, $\delta \in (1/p,1/2)$, $T_1 \in (0,\infty)$, $\|\nabla\omega\|_\infty \le \eta$, and $\|\nabla\omega\|_{BC^2} \le R$. We conclude that for M, T_1, ε, and $R > 0$ there are $\eta(M,T_1,\varepsilon) > 0$ and $T_0(M,T_1,\varepsilon,R) \in (0,T_1]$ such that

$$\|\overline{z} \mapsto F_u(\overline{u},\overline{\pi};\omega)\|_{{}_0\overline{\mathbb{E}}(T)\to\overline{\mathbb{F}}_u(T))} \le \varepsilon \quad \text{for all } (\vartheta^*,\omega,\vartheta) \in \mathcal{P}_{M,T_1,\eta,R}.$$

(iii) We next control the perturbation G_v in ${}_0\overline{\mathbb{G}}_v(T)$. First, the estimates (3.37) and (3.38) and Lemma 3.17 yield an estimate

$$\begin{aligned}\|Q_v(\omega)L_w(z;\omega,\vartheta)\circ\Theta_\omega\|_{{}_0\overline{\mathbb{G}}_v(T)} &\le C\eta\|\overline{z}\|_{{}_0\overline{\mathbb{E}}(T)} + C(R)\|L_w(z;\omega,\vartheta)\circ\Theta_\omega\|_{L_p(0,T;L_p(\Sigma))} \\ &\le C\eta\|\overline{z}\|_{{}_0\overline{\mathbb{E}}(T)} + C(R)T^\delta C(\delta,T_1)\|\overline{z}\|_{{}_0\overline{\mathbb{E}}(T)},\end{aligned}$$

uniformly with respect to $\overline{z} \in {}_0\overline{\mathbb{E}}(T)$, $T \in (0,T_1]$, $\|\nabla\omega\|_\infty \le \eta$, and $\|\nabla\omega\|_{BC^3} \le R$.

It remains to estimate the difference $\overline{L}_v(\overline{u},\overline{h};\vartheta^*) - P'L_v(u,h;\omega,\vartheta)\circ\Theta_\omega$ in ${}_0\overline{\mathbb{G}}_v(T)$. In view of

$$\|P' - P_{\Sigma_\omega}\circ\Theta_\omega\|_\infty \to 0 \quad \text{as } \|\nabla\omega\|_\infty \to 0,$$

and estimate (3.38), we may omit the projection P' in the above difference and therefore it remains to estimate the following differences in ${}_0\overline{\mathbb{G}}_v(T)$.

(3.51a) $$\Delta'\overline{v} - (\widetilde{\Delta}_{\Sigma_\omega} v) \circ \Theta_\omega,$$

(3.51b) $$\nabla' \operatorname{div}' \overline{v} - (\nabla_{\Sigma_\omega} \operatorname{div}_{\Sigma_\omega} v) \circ \Theta_\omega,$$

(3.51c) $$[\![\overline{\mu}\partial_n \overline{v}]\!] - [\![\mu \partial_{\nu_{\Sigma_\omega}} v]\!] \circ \Theta_\omega,$$

(3.51d) $$[\![\overline{\mu}\nabla'\overline{w}]\!] - [\![\mu \nabla_{\Sigma_\omega} w]\!] \circ \Theta_\omega,$$

(3.51e) $$\vartheta_w^* \nabla' \Delta' \overline{h} - (\vartheta_w \nabla_{\Sigma_\omega} \Delta_{\Sigma_\omega} h) \circ \Theta_\omega.$$

The differences (3.51a), (3.51b), and (3.51d) can be controlled by applying the identities in Figures 3.8 and 3.9, Lemma 3.17, and the estimates (3.34), (3.35), (3.37) and (3.38) and we obtain

$$\begin{aligned}&\|(\Delta'\overline{v} - (\widetilde{\Delta}_{\Sigma_\omega} v) \circ \Theta_\omega, \nabla' \operatorname{div}' \overline{v} - (\nabla_{\Sigma_\omega} \operatorname{div}_{\Sigma_\omega} v) \circ \Theta_\omega, [\![\overline{\mu}\nabla'\overline{w}]\!] - [\![\mu \nabla_{\Sigma_\omega} w]\!] \circ \Theta_\omega)\|_{{}_0\overline{\mathbb{G}}_v(T)} \\ &\leq \{C\eta + C(R)T^{1/4}C(T_1)\}\|(\overline{v}, \overline{w})\|_{{}_0\overline{\mathbb{E}}_v(T)\times {}_0\overline{\mathbb{E}}_w(T)},\end{aligned}$$

uniformly with respect to $\overline{v} \in {}_0\overline{\mathbb{E}}_v(T)$, $\overline{w} \in {}_0\overline{\mathbb{E}}_w(T)$, $T \in (0, T_1]$, $T_1 \in (0, \infty)$, $\|\nabla\omega\|_\infty \leq \eta$, and $\|\nabla\omega\|_{BC^3} \leq R$. For (3.51e) we employ the estimates (3.36), (3.39) and (3.40) and obtain

$$\|\vartheta_w^* \nabla' \Delta' \overline{h} - (\vartheta_w \nabla_{\Sigma_\omega} \Delta_{\Sigma_\omega} h) \circ \Theta_\omega\|_{{}_0\overline{\mathbb{G}}_v(T)} \leq \{C\eta + C(R)T^\delta C(\delta, T_1)\}\|\overline{h}\|_{{}_0\overline{\mathbb{E}}_h(T)}.$$

In order to deal with $\partial_\nu v$, we note that $v = [P_{\Sigma_\omega} \circ \Pi_{\Sigma_\omega}]u$ near Σ_ω with the nonlinear projection Π_{Σ_ω} onto Σ_ω from on page 138. Hence we obtain

$$\begin{aligned}\|[\![\overline{\mu}\partial_n \overline{v}]\!] - [\![\mu \partial_{\nu_{\Sigma_\omega}} v]\!] \circ \Theta_\omega\|_{{}_0\overline{\mathbb{G}}_v(T)} &\leq C\eta\|\overline{u}\|_{{}_0\overline{\mathbb{E}}_u(T)} + C(R)\|(\overline{u}, \nabla'\overline{u})\|_{{}_0\overline{\mathbb{G}}_v(T)} \\ &\leq \{C\eta + C(R)T^{-1/4}C(T_1)\}\|\overline{u}\|_{{}_0\overline{\mathbb{E}}_{u,v,w,\partial_\nu w}(T)}.\end{aligned}$$

We conclude that for $\varepsilon > 0$ there is $\eta > 0$ such that for $R, T_1 \in (0, \infty)$ there is $T_0 \in (0, T_1]$ such that

$$\|\overline{z} \mapsto G_v(\overline{u}, \overline{h}; \vartheta^*, \omega, \vartheta)\|_{{}_0\overline{\mathbb{E}}(T)\to {}_0\overline{\mathbb{G}}_v(T)} \leq \varepsilon \quad \text{for all } (\vartheta^*, \omega, \vartheta) \in \mathcal{P}_{M,T_1,\eta,R},\ T \in (0, T_0].$$

(iv) We next control G_w in ${}_0\overline{\mathbb{G}}_w(T)$. The estimates (3.41) and (3.42) and Lemma 3.17 yield

$$\begin{aligned}\|Q_w(\omega)L_w(z; \omega, \vartheta) \circ \Theta_\omega\|_{{}_0\overline{\mathbb{G}}_w(T)} &\leq C\eta\|\overline{z}\|_{{}_0\overline{\mathbb{E}}(T)} + C(R)\|L_w(z; \omega, \vartheta) \circ \Theta_\omega\|_{L_p(0,T;H_p^1(\Sigma))} \\ &\leq C\eta\|\overline{z}\|_{{}_0\overline{\mathbb{E}}(T)} + C(R)T^\delta C(\delta, T_1)\|\overline{z}\|_{{}_0\overline{\mathbb{E}}(T)},\end{aligned}$$

uniformly with respect to $\overline{z} \in {}_0\overline{\mathbb{E}}(T)$, $T \in (0, T_1]$, $\|\nabla\omega\|_\infty \leq \eta$, and $\|\nabla\omega\|_{BC^3} \leq R$. It remains to estimate the difference $\overline{L}_w(\overline{u}, \overline{\pi}, \overline{h}; \vartheta^*) - L_w(u, \pi, h; \omega, \vartheta) \circ \Theta_\omega$ which consists of

(3.52a) $$\vartheta_L^* \nabla' \overline{v} - (\vartheta_L \nabla_{\Sigma_\omega} v) \circ \Theta_\omega,$$

(3.52b) $$[\![\overline{\mu}\partial_n \overline{w}]\!] - [\![\mu \partial_{\nu_{\Sigma_\omega}} w]\!] \circ \Theta_\omega,$$

(3.52c) $$\operatorname{tr}(\nabla'^2 \overline{h}) - \operatorname{tr}(\nabla_{\Sigma_\omega}^2 h) \circ \Theta_\omega,$$

(3.52d) $$\operatorname{tr}(\operatorname{tr} \vartheta_{Dv}^* \nabla'^2 \overline{h}) - \operatorname{tr}(\operatorname{tr} \vartheta_{Dv} \nabla_{\Sigma_\omega}^2 h) \circ \Theta_\omega,$$

(3.52e) $$\operatorname{tr}(\vartheta_w^* \operatorname{tr} \vartheta_L^* \nabla'^2 \overline{h}) - \operatorname{tr}(\vartheta_w \operatorname{tr} \vartheta_L \nabla_{\Sigma_\omega}^2 h) \circ \Theta_\omega,$$

(3.52f) $$\operatorname{tr}(\vartheta_{Dv}^* \nabla'^2 \overline{h}) - \operatorname{tr}(\vartheta_{Dv} \nabla_{\Sigma_\omega}^2 h) \circ \Theta_\omega,$$

(3.52g) $$\operatorname{tr}(\vartheta_w^* \vartheta_L^* \nabla'^2 \overline{h}) - \operatorname{tr}(\vartheta_w \vartheta_L \nabla_{\Sigma_\omega}^2 h) \circ \Theta_\omega.$$

Again we control lower order terms by using (3.34), (3.35), and (3.36). The differences (3.52a) to (3.52c) can be controlled by means of the estimates (3.41) and (3.42). For the terms (3.52d) to (3.52g) we employ the estimates (3.43) and (3.44). We conclude that for $\varepsilon > 0$ there is $\eta > 0$ such that for $R, T_1 \in (0, \infty)$ there is $T_0 \in (0, T_1]$ such that

$$\|\overline{z} \mapsto G_w(\overline{u}, \overline{h}; \vartheta^*, \omega, \vartheta)\|_{{}_0\overline{\mathbb{E}}_u(T)\to {}_0\overline{\mathbb{G}}_w(T)} \leq \varepsilon \quad \text{for all } (\vartheta^*, \omega, \vartheta) \in \mathcal{P}_{M,T_1,\eta,R},\ T \in (0, T_0].$$

(v) The relevant estimates of G_h in ${}_0\overline{\mathbb{G}}_h(T)$ follow from estimates (3.45) and (3.46) and we conclude that for $T_1 > 0$ and $\varepsilon > 0$ there is $\eta > 0$ such that for $R > 0$ there is $T_0 \in (0, T_1]$ such that

$$\|\overline{z} \mapsto G_h(\overline{w}; \omega)\|_{{}_0\overline{\mathbb{E}}(T) \to {}_0\overline{\mathbb{G}}_h(T)} \leq \varepsilon \quad \text{for all } (\vartheta^*, \omega, \vartheta) \in \mathcal{P}_{M,T_1,\eta,R},\ T \in (0, T_0].$$

(vi) The preceding steps show that for given M, T_1, ε, $R > 0$ there are $\eta = \eta(M, T_1, \varepsilon) > 0$ and $T_0 = T_0(M, T_1, \varepsilon, R) \in (0, T_1]$ such that

$$\|F(\vartheta^*, \omega, \vartheta)\|_{{}_0\overline{\mathbb{E}}(T) \to {}_0\overline{\mathbb{F}}(T)} \leq \varepsilon \quad \text{for all } (\vartheta^*, \omega, \vartheta) \in \mathcal{P}_{M,T_1,\eta,R}. \tag{3.53}$$

For fixed M and T_1 we apply Theorem 3.14 and obtain a finite number

$$C := \sup\left\{\|[S(\vartheta^*)]^{-1}\|_{{}_0\overline{\mathbb{F}}(T) \to {}_0\overline{\mathbb{E}}(T)} : \vartheta^* \in \mathcal{P}_M,\ T \in (0, T_1]\right\}.$$

Next we fix $\varepsilon \in (0, C^{-1})$ and for given $R > 0$ we choose $\eta(M, T_1, \varepsilon) > 0$ and $T_0(M, T_1, \varepsilon, R) > 0$ such that (3.53) is valid. Then the operator (3.49) is uniformly invertible with respect to $(\vartheta^*, \omega, \vartheta) \in \mathcal{P}_{M,T_1,\eta,R}$ and $T \in (0, T_0]$ and the proof of Theorem 3.16 is complete. □

3.3. Bounded domains

We consider problem (3.1) = (PL) in a bounded domain $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) with smooth boundary $\partial\Omega$ and with a compact smooth hypersurface $\Sigma \subset \Omega$ such that $\Omega \setminus \Sigma$ consists of disjoint open sets Ω_+ and Ω_- with $\partial\Omega_+ \cap \partial\Omega_- = \Sigma$. We will establish optimal regularity for this problem on some short time interval $(0, T)$ for the following class of reference velocities u_*. The involved function spaces are collected in Figure 3.7 on page 72.

3.20. Definition. Let $p \in (\max\{3, (n+2)/2\}, \infty)$, $M > 0$, and $T > 0$. The parameter set $\mathcal{P}_{M,T}$ consists of all vector fields $u_* = v_* + w_*\nu_\Sigma \in \mathbb{E}_v(T) + \mathbb{E}_w(T) \cdot \nu_\Sigma$ such that

$$\|w_*\|_{\mathbb{G}_w(T)} \leq M, \quad \|D_\Sigma(v_*)\|_{\mathbb{G}_w(T)} \leq M,$$

and

$$\inf_{(0,T)\times\Sigma} d_0(D_\Sigma(u_*)) = \inf_{(0,T)\times\Sigma} \left(\sigma + (\lambda_s - \mu_s)\operatorname{div}_\Sigma u_* + 2\mu_s \min_{\zeta\in\mathbb{R}^n,\ |\zeta|=1} \zeta^\top D_\Sigma(u_*)\zeta\right) \geq M^{-1}. \tag{3.54}$$

Note that condition (3.54) makes sense due to the embeddings $\mathbb{E}_v \hookrightarrow C([0,T]; C^1(\Sigma; T\Sigma))$ and $\mathbb{E}_w \hookrightarrow C([0,T]; C^1(\Sigma))$, which are valid for $p > \max\{3, (n+2)/2\}$.

3.21. Theorem. *Let $\rho_\pm$, $\mu_\pm$, σ, μ_s, $\lambda_s + \mu_s > 0$, and let $p \in (\max\{5, n+2\}, \infty)$ and M, $T_1 > 0$. Then there exists $T_0 \in (0, T_1]$ such that the solution-to-data map*

$$(u, \pi, h) \mapsto (f_u, f_d, g_v, g_w, g_h),$$
$${}_0\mathbb{E} = {}_0\mathbb{E}_{u,v,w,\partial_\nu w} \times {}_0\mathbb{E}_{\pi,[\![\pi]\!]} \times {}_0\mathbb{E}_h \to {}_0\mathbb{F} = {}_0\mathbb{F}_u \times {}_0\mathbb{F}_{d,\Sigma} \times {}_0\mathbb{G}_v \times {}_0\mathbb{G}_w \times {}_0\mathbb{G}_h$$

of problem (3.1) is uniformly invertible with respect to $T \in (0, T_0]$ and $u_* \in \mathcal{P}_{M,T_1}$.

For the proof we apply a modified version of the elliptic localization technique from Section 2.1.1, which will be presented in Section 3.3.1. As in [KPW13; Wil13], we localize problem (3.1) in both time and space and we construct both a left- and a right-inverse for the solution-to-data map. A different approach was used in [Gei+12] for a stationary Stokes problem, where the authors localize in space, establish $\mathcal{R}$-bounds for the data-to-solution map, and apply Weis' characterization of maximal L_p-regularity [Wei01].

As in Section 3.2, we employ T-dependent estimates for controlling lower-order perturbations; however, in order to control the commutator $[\nabla, \varphi_j]\pi = \nabla\varphi_j\,\pi$, the usual elliptic localization does not suffice. In addition, we employ certain projections on subspaces with vanishing momentum and divergence data. These projections are constructed similarly as in [Gei+12; KPW13; Wil13], by resolving non-trivial momentum and divergence data by means of weak

Neumann transmission problems and one-phase Stokes problems. Then, similar to [Köh13; KPW13; Wil13], we prove in Section 3.3.2 that the pressure has additional temporal regularity, if it belongs to the range of the aforementioned projection. Section 3.3.3 contains the local spaces, approximation systems, and local operators and in Section 3.3.4 we prove the relevant commutator estimates.

3.3.1. Localization technique. For $T \in (0, T_1]$ with some fixed number $T_1 \in (0, \infty)$ we will consider Banach spaces $E = E(T)$ and $F = F(T)$ of functions on $(0, T)$ and linear operators $A_T : E(T) \to F(T)$. For the sake of brevity, we wish to omit the T-dependence occasionally. To justify this we always assume that the spaces and operators are compatible in the sense that for $0 < T \leq T' \leq T_1$ their realizations over $(0, T)$ coincide with the restrictions to $(0, T)$ of their realizations over $(0, T')$.

We fix a number $q \in [1, \infty)$ and an index set $I \subset \mathbb{N}_0$ and consider l_q-approximation systems $(\mathbf{E}, (\Phi_{E,j})_{j\in I}, (\Psi_{E,j})_{j\in I})$ and $(\mathbf{F}, (\Phi_{F,j})_{j\in I}, (\Psi_{F,j})_{j\in I})$ for E and F in the sense of Definition 2.4 on page 27. Our goal is to show that a given linear operator $A \in \mathcal{B}(E; F)$ is uniformly invertible with respect to $T \in (0, T_0]$ for some $T_0 \in (0, T_1]$. Let us therefore assume that

(i) there are invertible linear operators $A_j \in \mathcal{B}_{\text{isom}}(E_j; F_j)$, the *local operators*, such that

$$\sup_{T\in(0,T_1]} \|(f_j)_{j\in I} \mapsto (A_j^{-1} f_j)_{j\in I}\|_{l_q(\mathbf{F}(T))\to l_q(\mathbf{E}(T))} < \infty.$$

Indeed, these operators A_j will correspond to certain model problems and the uniform invertibility of A_j will follow from the boundedness of the relevant coefficients related to Σ and u_*. We further assume that

(ii) we can find a projection $P_F \in \mathcal{B}(F)$ and an operator $R_0 \in \mathcal{B}(F; E)$ such that

$$(I_F - P_F)AR_0(I_F - P_F) = I_F - P_F.$$

We wish to choose the projection

$$P_F : (f_u, f_d, g_v, g_w, g_h) \mapsto (0, 0, g_v, g_w, g_h)$$

and therefore the operator $R_0 : (f_u, f_d, 0, 0, 0) \mapsto (u, \pi, 0)$ should produce functions $(u, \pi, 0) \in E$ with $(\rho\partial_t - \mu\Delta)u + \nabla\pi = f_u$ and $\operatorname{div} u = f_d$. Moreover, the operator

$$P_E := I_E - R_0(I_F - P_F)A$$

is a projection in $\mathcal{B}(E)$ and we obtain

$$\begin{aligned} P_F A P_E &= P_F A(I_E - R_0(I_F - P_F)A) \\ &= A - (I_F - P_F)A + (I_F - P_F)AR_0(I_F - P_F)A - AR_0(I_F - P_F)A \\ &= A - AR_0(I_F - P_F)A = AP_E. \end{aligned}$$

In particular, for given $z = (u, \pi, h) \in P_E E$ and $Az = (f_u, f_d, g_v, g_w, g_h)$ we have $(f_u, f_d) = 0$.

We also consider the *local projections*

$$P_{F,j} : (f_{uj}, f_{dj}, g_{vj}, g_{wj}, g_{hj}) \mapsto (0, 0, g_{vj}, g_{wj}, g_{hj}), \quad P_{E,j} := A_j^{-1} P_{F,j} A_j,$$

and we assume that

(iii) the projections $P_{F,j}$ satisfy

$$\Phi_{F,j} P_F = P_{F,j} \Phi_{F,j}.$$

This property will be trivial in our situation, since it means that for given $(0, 0, g_v, g_w, g_h) \in F$, the tuple $\Phi_{F,j}(0, 0, g_v, g_w, g_h)$ has the form $(0, 0, g_{vj}, g_{wj}, g_{hj})$.

Now we define an *approximate inverse* for A by

$$R : F \to E, \quad R := \sum\nolimits_j \Psi_{E,j} A_j^{-1} \Phi_{F,j} P_F (I_F - AR_0(I_F - P_F)) + R_0(I_F - P_F).$$

Note that $\sum_j \Psi_{E,j} A_j^{-1} \Phi_{F,j}$ is the usual approximate inverse in the elliptic and parabolic theory and that the operator $R_0(I_F - P_F)$ takes care of the momentum and divergence data (f_u, f_d). The latter is constructed in Lemma 3.27 on page 91. From our assumptions (i) to (iii) we infer that

$$AR - I_F = \sum_j (A\Psi_{E,j} - \Psi_{F,j} A_j) A_j^{-1} \Phi_{F,j} P_F (I_F - AR_0(I_F - P_F)), \tag{3.55a}$$

$$RA - I_E = \sum_j \Psi_{E,j} A_j^{-1} (\Phi_{F,j} A - A_j \Phi_{E,j}) P_E. \tag{3.55b}$$

In order to apply a Neumann series argument we wish to guarantee that

$$\|AR - I_F\|_{F \to F} \leq 2^{-1}, \quad \|RA - I_E\|_{E \to E} \leq 2^{-1} \quad \text{for } T \in (0, T_0].$$

If this is true then the operators $AR = I_F - (I_F - AR) \in \mathcal{B}(F(T))$ and $RA = I_E - (I_E - RA) \in \mathcal{B}(E(T))$ are invertible for all $T \in (0, T_0]$ and A has the inverse $R(AR)^{-1} = (RA)^{-1}R$. If further $M > 0$ is a bound for R, then $2M$ is a bound for A^{-1}.

Hence, in view of (3.55), it remains to guarantee the *commutator estimates*

$$\|A\Psi_{E,j} - \Psi_{F,j} A_j\|_{P_{E,j} E_j \to F} \leq \varepsilon, \tag{3.56a}$$

$$\|\Phi_{F,j} A - A_j \Phi_{E,j}\|_{P_E E \to F_j} \leq \varepsilon, \tag{3.56b}$$

for every given ε by choosing A_j, Φ_j, and Ψ_j suitably and T_0 sufficiently small, whereas the other operators should remain uniformly bounded.

3.3.2. Time regularity of the pressure. We consider the equations

$$\begin{cases} \rho \partial_t u - \mu \Delta u + \nabla \pi = f_u & \text{in } J \times \Omega \setminus \Sigma, \\ \operatorname{div} u = 0 & \text{in } J \times \Omega \setminus \Sigma, \\ u|_{\partial\Omega} \cdot \nu = 0 & \text{on } J \times \partial\Omega, \\ [\![u]\!] \cdot \nu = 0 & \text{on } J \times \Sigma, \end{cases} \tag{3.57}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is a domain with (possibly empty) C^{2-}-boundary and $\Sigma \subset \Omega$ is a (possibly empty) closed C^{2-}-hypersurface. Let further $p \in (1, \infty)$ and assume that Ω and Σ satisfy Assumption 2.1 where the bound $\eta = \eta(n, p, \rho^{-1}) > 0$ for $\|\nabla\omega\|_\infty$ is chosen such that Theorem 2.2 is applicable. In this case we obtain a bounded solution operator

$$g_0 \mapsto \psi, \quad (L_p \cap \hat{H}_p^{-1})(\Omega) \to (\dot{\mathcal{H}}_p^2 \cap \dot{\mathcal{H}}_p^1)(\Omega \setminus \Sigma)/(\rho^{-1}\mathbb{K})$$

for the elliptic transmission problem

$$-\Delta\psi = g_0 \text{ in } \Omega, \quad \partial_\nu \psi = 0 \text{ on } \partial\Omega, \quad [\![\partial_\nu \psi]\!] = 0 \text{ on } \Sigma, \quad [\![\rho\psi]\!] = 0 \text{ on } \Sigma.$$

With methods from [Köh13, Proposition 7.14], [KPW13, Corollary 1], and [Wil13, Lemma 2.1.1], we will prove the following temporal regularity result for the pressure π, where we let $\langle \phi \rangle_K := |K|^{-1} \int_K \phi \, dx$ denote the mean value of $\phi \in L_1(K)$ for a bounded domain K.

3.22. Lemma. *Let ρ_1, ρ_2, μ_1, $\mu_2 \in (0, \infty)$, $J = (0, T)$ with $T \in (0, \infty]$, $p \in (1, \infty)$, and $\alpha \in (0, 1/2 - 1/2p]$. Let K be a bounded C^1-subdomain of Ω and suppose that (u, π, f_u) satisfies (3.57) and*

$$\begin{cases} u \in \mathbb{E}_u = H_p^1(J; L_p(\Omega)^n) \cap L_p(J; H_p^2(\Omega \setminus \Sigma)^n), \\ \pi \in \mathbb{E}_\pi = L_p(J; \dot{H}_p^1(\Omega \setminus \Sigma)), \\ [\![\pi]\!] \in W_p^\alpha(J; L_p(\Sigma)), \\ f_u = f_{u,\alpha} + \rho f_{u,\sigma} \in W_p^\alpha(J; L_p(\Omega)^n) + \rho L_p(J; L_{p,\sigma}(\Omega)). \end{cases} \tag{3.58}$$

Then the following estimate is valid with some $C = C(n, p, K, T) > 0$.

$$\|\pi - \langle \pi \rangle_K\|_{W_p^\alpha(J; L_p(K))} \leq C \left(\|u\|_{\mathbb{E}_u} + \|f_{u,\alpha}\|_{W_p^\alpha(J; L_p(\Omega))} + \|[\![\pi]\!]\|_{W_p^\alpha(J; L_p(\Sigma))} \right). \tag{3.59}$$

Moreover, the number C is uniform with respect to $T \in (0,\infty]$ under the restrictions

$$u \in {}_0\mathbb{E}_u, \quad [\![\pi]\!] \in {}_0W_p^\alpha(J; L_p(\Sigma)), \quad f_{u,\alpha} \in {}_0W_p^\alpha(J; L_p(\Omega)^n), \quad \alpha \neq 1/p.$$

Proof. For $g \in L_{p'}(K)$ we define a function $g_0 \in L_{p'}(\Omega)$ by $g_0(x) := g(x) - \langle g \rangle_K$ for $x \in K$ and $g_0(x) = 0$ for $x \in \Omega \setminus K$. The Poincaré-Wirtinger inequality for $H_p^1(K)$ implies

$$\|g_0\|_{\hat{H}_{p'}^{-1}(\Omega)} = \sup_{\phi \in \mathcal{D}(\mathbb{R}^n)\setminus\{0\}} \frac{|\int_\Omega g_0 \phi \, dx|}{\|\nabla \phi\|_{L_p(\Omega)}} = \sup_{\phi \in \mathcal{D}(\mathbb{R}^n)\setminus\{0\}} \frac{|\int_K g_0(\phi - \langle \phi \rangle_K) \, dx|}{\|\nabla \phi\|_{L_p(\Omega)}} \leq C(K)\|g\|_{L_{p'}(K)}.$$

By Theorem 2.2, we can find some $\rho\psi \in \dot{\mathcal{H}}_{p'}^2(\Omega \setminus \Sigma) \cap \dot{\mathcal{H}}_{p'}^1(\Omega)$ such that

$$-\Delta\psi = g_0 \text{ in } \Omega, \quad \partial_\nu \psi = 0 \text{ on } \partial\Omega, \quad [\![\partial_\nu \psi]\!] = 0 \text{ on } \Sigma, \quad [\![\rho\psi]\!] = 0 \text{ on } \Sigma,$$

which satisfies the estimate $\|\nabla \psi\|_{H_{p'}^1(\Omega\setminus\Sigma)} \leq C\|g\|_{L_{p'}(K)}$.

For $\pi_0 := \pi - \langle \pi \rangle_K$ and g, g_0, and ψ as above, an integration by parts yields

$$-\int_K \pi_0 g \, dx = -\int_\Omega \pi_0 g_0 \, dx = \int_\Omega \pi_0 \Delta\psi \, dx = -\int_\Omega \nabla\pi \cdot \nabla\psi \, dx - \int_\Sigma [\![\pi]\!] \partial_\nu \psi \, d\sigma.$$

By using the equations in (3.57) and integrating by parts, we obtain

$$\begin{aligned} -\int_K \pi_0 g \, dx = &\int_\Omega \mu \nabla u : \nabla^2 \psi \, dx - \int_{\partial\Omega} \mu \partial_\nu u \cdot \nabla \psi \, d\sigma + \int_\Sigma [\![\mu \partial_\nu u \cdot \nabla\psi]\!] \, d\sigma \\ &- \int_\Omega f_{u,\alpha} \cdot \nabla \psi \, dx - \int_\Sigma [\![\pi]\!] \partial_\nu \psi \, d\sigma =: \langle F_{u, f_{u,\alpha}, [\![\pi]\!]}, g \rangle. \end{aligned}$$

The duality $L_p(K)^* \cong L_{p'}(K)$ yields the estimate

$$\begin{aligned} \|\pi_0(t)\|_{L_p(K)} = \|F_{u,f_{u,\alpha},[\![\pi]\!]}(t)\|_{L_{p'}(K)^*} \lesssim &\|u(t)\|_{H_p^1(\Omega)} + \|\partial_\nu u(t)\|_{L_p(\partial\Omega)} + \|\mu_\pm \partial_\nu u_\pm(t)\|_{L_p(\Sigma)} \\ &+ \|f_{u,\alpha}(t)\|_{L_p(\Omega)} + \|[\![\pi(t)]\!]\|_{L_p(\Sigma)}. \end{aligned}$$

In order to apply the $W_p^\alpha(0,T)$-seminorm, we observe that

$$\|\pi_0(t) - \pi_0(s)\|_{L_p(K)} = \|F_{u(t)-u(s), f_{u,\alpha}(t) - f_{u,\alpha}(s), [\![\pi(t)]\!] - [\![\pi(s)]\!]}\|_{L_{p'}(K)^*}.$$

Hence, for some number $C = C(n,p,K)$, which does not depend on $T \in (0,\infty]$, we have

$$\begin{aligned} \|\pi_0\|_{W_p^\alpha(J;L_p(K))} \leq\ &C \left(\|u\|_{W_p^\alpha(J;H_p^1(\Omega))} + \|\partial_\nu u\|_{W_p^\alpha(J;L_p(\partial\Omega)} + \|\partial_\nu u_\pm\|_{W_p^\alpha(J;L_p(\Sigma))} \right) \\ &+ C \left(\|f_{u,\alpha}\|_{W_p^\alpha(J;L_p(\Omega))} + \|[\![\pi]\!]\|_{W_p^\alpha(J;L_p(\Sigma))} \right). \end{aligned}$$

Since $\alpha \leq 1/2 - 1/2p$, the trace theorem (Theorem B.32) and the mixed derivative embeddings (Proposition B.44) yield a constant $C = C(n,p,K,T)$ such that

$$\|\pi_0\|_{W_p^\alpha(J;L_p(K))} \leq C \left(\|u\|_{\mathbb{E}_u(T)} + \|f_{u,\alpha}\|_{W_p^\alpha(J;L_p(\Omega))} + \|[\![\pi]\!]\|_{W_p^\alpha(J;L_p(\Sigma))} \right).$$

Therefore the asserted estimate (3.59) is valid. Uniform estimates with respect to T follow by extension and restriction (Lemma B.9). □

3.3.3. Local operators. With the spaces from page 72, we define the space of solutions

$$E(T) := {}_0\mathbb{E}(J,\Omega,\Sigma) := {}_0\mathbb{E}_{u,v,w,\partial_\nu w}(J,\Omega,\Sigma) \times {}_0\mathbb{E}_{\pi,[\![\pi]\!]}(J,\Omega,\Sigma) \times {}_0\mathbb{E}_h(J,\Sigma),$$

and the space of data

$$F(T) := {}_0\mathbb{F}(J,\Omega,\Sigma) := \mathbb{F}_u(J,\Omega) \times {}_0\mathbb{F}_{d,\Sigma}(J,\Omega,\Sigma) \times {}_0\mathbb{G}_v(J,\Sigma) \times {}_0\mathbb{G}_w(J,\Sigma) \times {}_0\mathbb{G}_h(J,\Sigma).$$

In order to define the local spaces E_j and F_j we employ Lemma 2.9 on page 29, which implies that for every $\eta > 0$ there is $r_0(\eta) > 0$ such that for every $r \in (0, r_0(\eta)]$ we can find an (η, r)-localization set-up for (Ω, Σ) in the sense of Definition 2.8. Hence for some finite set

$I = I(\eta, r)$ there exist an open covering for $\overline{\Omega}$ of balls $U_j = B_r(p_j)$ ($j \in I$) and there are rigid transformations

$$\Theta_j \colon x \mapsto p_j + Q_j x, \quad B_r(0) \to U_j,$$

and height functions $\omega_j \in C_c^\infty(\mathbb{R}^{n-1})$ with $\|\omega_j\|_{BC^1 \cap H_p^2} \le \eta$. Furthermore, the index set can be decomposed into $I = I_1 \cup I_2 \cup I_3$, where $j \in I_1$ corresponds to the whole space case $\Omega \cap U_j = \Theta_j(\mathbb{R}^n \cap B_r)$, $j \in I_2$ corresponds to the bent half-space case $\Omega \cap U_j = \Theta_j(\mathbb{R}^n_{\omega_j} \cap B_r)$, and $j \in I_3$ corresponds to the bent hyperplane case $\Sigma \cap U_j = \Theta_j(\Sigma_{\omega_j} \cap B_r)$. We define

$$\begin{aligned}
&\Omega_j := \mathbb{R}^n, \quad && \Sigma_j := \emptyset \quad && \text{for } j \in I_1,\\
&\Omega_j := \mathbb{R}^n_{\omega_j}, \quad && \Sigma_j := \emptyset \quad && \text{for } j \in I_2,\\
&\Omega_j := \mathbb{R}^n, \quad && \Sigma_j := \Sigma_{\omega_j} \quad && \text{for } j \in I_3.
\end{aligned}$$

Then we define the local spaces

$$E_j(T) = {}_0\mathbb{E}(J, \Omega_j, \Sigma_j), \qquad F_j(T) = {}_0\mathbb{F}(J, \Omega_j, \Sigma_j) \qquad \text{for } j \in I_1 \cup I_2 \cup I_3,$$

where in the case $j \in I_1 \cup I_2$ we identify

$${}_0\mathbb{E}(J, \Omega_j, \emptyset) \cong \{u \in {}_0\mathbb{E}_u(J, \Omega_j, \emptyset) : u|_{\partial\Omega_j} = 0\} \times \mathbb{E}_\pi, \qquad {}_0\mathbb{F}(J, \Omega_j, \emptyset) \cong \mathbb{F}_u(J, \Omega_j) \times {}_0\mathbb{F}_d(J, \Omega_j).$$

We choose a partition of unity $(\varphi_j)_{j \in I}$ for $\overline{\Omega}$ in $\mathbb{R}^n$ subordinate to $(U_j)_{j \in I}$ and choose another family of cut-off functions $(\psi_j)_{j \in I}$ with $\operatorname{supp} \psi_j \subset U_j$ and $\psi_j = 1$ on $\operatorname{supp} \varphi_j$. Then we have $\sum_j \psi_j \varphi_j = 1$ in $\overline{\Omega}$ and we define approximation systems for E and F by

$$\begin{aligned}
\Phi_{E,j}(u, \pi, h) &:= (Q_j^\top(\varphi_j u), (\varphi_j \pi), (\varphi_j h)) \circ \Theta_j,\\
\Psi_{E,j}(u_j, \pi_j, h_j) &:= (Q_j(\psi_j u_j), (\psi_j \pi_j), (\psi_j h_j)) \circ \Theta_j^{-1},\\
\Phi_{F,j}(f_u, f_d, g_v, g_w, g_h) &:= (Q_j^\top(\varphi_j f_u), (\varphi_j f_d), Q_j^\top(\varphi_j g_v), (\varphi_j g_w), (\varphi_j g_h)) \circ \Theta_j,\\
\Psi_{F,j}(f_{uj}, f_{dj}, g_{vj}, g_{wj}, g_{hj}) &:= (Q_j(\psi_j f_{uj}), (\psi_j f_{dj}), Q_j(\psi_j g_{vj}), (\psi_j g_{wj}), (\psi_j g_{hj})) \circ \Theta_j^{-1}.
\end{aligned}$$

The relevant mapping properties of these maps follow as in Lemma 3.17.

Problem (3.1) induces a bounded linear operator $A \colon E \to F$ by

$$A(u, \pi, h) := \begin{bmatrix} \rho\partial_t u - \mu\Delta u + \nabla\pi \\ \operatorname{div} u \\ L_v(u, h; u_*) \\ L_w(u, \pi, h; u_*) \\ \partial_t h - u \cdot \nu_\Sigma \end{bmatrix} \quad \text{for } (u, \pi, h) \in E. \tag{3.60}$$

For $j \in I_1 \cup I_2$ we define the local operators $A_j \colon E_j \to F_j$ by

$$A_j(u, \pi) := \begin{bmatrix} \rho\partial_t u - \mu\Delta u + \nabla\pi \\ \operatorname{div} u \end{bmatrix} \quad \text{for } (u, \pi) \in E_j,\ j \in I_1 \cup I_2. \tag{3.61}$$

The results of Bothe and Prüss ([BP07, Theorem 5.1, Theorem 6.1]) imply that $A_j \colon E_j \to F_j$ is invertible for $j \in I_1 \cup I_2$ and $\omega_j = 0$. For $j \in I_2$ and $\omega_j \neq 0$ we employ the following result.

3.23. Lemma. *Let $n \ge 2$, ρ, $\mu > 0$, and $p \in (n+2, \infty)$.*

Then there exists $\eta > 0$ such that for given $R > 0$ we can find a number $T_0(R) > 0$ such that the solution-to-data map $(u, \pi) \mapsto (f_u, f_d)$, ${}_0\mathbb{E}_u \times \mathbb{E}_\pi \to \mathbb{F}_u \times {}_0\mathbb{F}_d$ of problem

$$\left\{ \begin{aligned} \rho\partial_t u - \mu\Delta u + \nabla\pi &= f_u \quad \text{in } J \times \mathbb{R}^n_\omega, \\ \operatorname{div} u &= f_d \quad \text{in } J \times \mathbb{R}^n_\omega, \end{aligned} \right. \tag{3.62}$$

is uniformly invertible with respect to $T \in (0, T_0]$ *and*

$$\omega \in BC^3(\mathbb{R}^{n-1}),\ \|\omega\|_{BC^1} \le \eta,\ \|\omega\|_{BC^3} \le R. \tag{3.63}$$

Proof. We employ the transformation $\Theta_{\omega_j} : \mathbb{R}^n_+ \to \mathbb{R}^n_{\omega_j}$, $(x', x_n) \mapsto (x', x_n + \omega_j(x'))$ from equation (3.27) on page 72. As in Lemma 3.17, we define the transformed functions

$$\overline{u} = [\partial\Theta_{\omega_j}]^{-1} u \circ \Theta_{\omega_j}, \quad \overline{\pi} = \pi \circ \Theta_{\omega_j}, \quad \overline{f}_u := [\partial\Theta_{\omega_j}]^{-1} f_u \circ \Theta_{\omega_j} \quad \overline{f}_d := f_d \circ \Theta_{\omega_j}.$$

Then (3.62) is equivalent to

$$\begin{cases} (\rho\partial_t - \mu\Delta)\overline{u} + \nabla\overline{\pi} = \overline{f}_u + F_u(\overline{u}, \overline{\pi}; \omega_j) & \text{in } J \times \mathbb{R}^n_+, \\ \operatorname{div} \overline{u} = \overline{f}_d & \text{in } J \times \mathbb{R}^n_+, \end{cases} \tag{3.64}$$

where the perturbation F_u is defined by

$$F_u(\overline{u}, \overline{\pi}; \omega_j) := \overline{\mu}([\partial\Theta_{\omega_j}]^{-1}(\Delta u) \circ \Theta_{\omega_j} - \Delta\overline{u}) + (I - [\partial\Theta_{\omega_j}]^{-1}[\partial\Theta_{\omega_j}]^{-\top})\nabla\overline{\pi},$$

and the difference $(\Delta u)\circ\Theta_{\omega_j} - \Delta\overline{u}$ can be expressed by (3.50). As for Theorem 3.16 it follows that the map $(\overline{u}, \overline{\pi}) \to (\overline{f}_u, \overline{f}_d)$ induced by (3.64) is uniformly invertible with respect to $T \in (0, T_0]$ (3.63) for some $\eta > 0$ and $T_0(\eta) > 0$. The proof of Lemma 3.17 shows that the transformation $(\overline{u}, \overline{\pi}, \overline{f}_u, \overline{f}_d) \to (u, \pi, f_u, f_d)$ is uniformly invertible and this yields the assertion. ☐

For the case $j \in I_3$ we first define the local coefficients of A_j. These depend on the functions

$$L_\Sigma = \tau^k_\Sigma \otimes \partial_k \nu_\Sigma, \quad w_* = \nu_\Sigma \cdot u_*, \quad D_\Sigma(v_*) = \operatorname{sym}([\tau^k_\Sigma \otimes \partial_k v_*]P_\Sigma),$$

with $v_* = P_\Sigma u_*$. Their transforms under the rigid map $\Theta_j : x \mapsto Q_j x + p_j$ are given by

$$L_{\Sigma_j} = Q_j^\top [L_\Sigma \circ \Theta_{\omega_j}] Q_j, \quad \overline{w}_* = w_* \circ \Theta_{\omega_j}, \quad D_{\Sigma_j}(\overline{v}_*) = Q_j^\top [(D_\Sigma(v_*)) \circ \Theta_{\omega_j}] Q_j,$$

where $\overline{v}_* = Q_j^\top (v_* \circ \Theta_{\omega_j})$.

As for the construction of ω_j in Lemma 2.9, we fix a cut-off function $\chi \in \mathcal{B}(\mathbb{R}^n)$ with $0 \le \chi \le 1$, $\chi(x) = 1$ for $|x| \le 1$, and $\chi(x) = 0$ for $|x| \ge 2$. For a given function ψ on $J \times (\Sigma_j \cap B_{r_0})$ and $r \in (0, r_0/2]$ we define another function $\tilde{\psi}_r$ on $J \times \Sigma_j$ by

$$\tilde{\psi}_r(t, x) := (S_r\psi)(t, x) := \psi(0, 0) + \begin{cases} \chi(x/r)(\psi(t, x) - \psi(0, 0)) & \text{for } |x| < 2r \le r_0, \\ 0 & \text{for } |x| \ge 2r. \end{cases}$$

Then $\tilde{\psi}_r(t, x) = \psi(t, x)$ for all $(t, x) \in J \times (\Sigma_j \cap B_r)$.

3.24. Proposition (Properties of $S_r : \psi \mapsto \tilde{\psi}_r$)**.** *Let* $\Sigma = \Sigma_\omega$ *be a bent* C^2*-hyperplane in* $\mathbb{R}^n$*.*

(i) For all $r_0 > 0$ *there exists* $C > 0$ *such that*

$$\|\tilde{\psi}_r - \psi(0)\|_{BC(\Sigma)\cap H^1_p(\Sigma)} \le C r^{\max\{1,(n-1)/p\}} \|\psi\|_{BC^1(\Sigma\cap B_{2r})} \tag{3.65}$$

for all $r \in (0, r_0/2]$ *and* $\psi \in BC^1(\Sigma \cap B_{r_0})$*.*

(ii) For all $r_0 > 0$ *there exists* $C > 0$ *such that*

$$\|\tilde{\psi}_r - \psi(0)\|_{BC^1(\Sigma)\cap H^2_p(\Sigma)} \le C r^{\max\{1,(n-1)/p\}} \|\psi\|_{BC^2(\Sigma\cap B_{2r})}. \tag{3.66}$$

for all $r \in (0, r_0/2]$*,* $\psi \in BC^2(\Sigma \cap B_{r_0})$ *with* $\nabla\psi(0) = 0$*.*

(iii) For all $T_1 > 0$*,* $r_0 > 0$*, and* $\gamma \in (0, 1)$ *there exists* $C > 0$ *such that*

$$\|\tilde{\psi}_r - \psi(0, 0)\|_{C([0,T];BC(\Sigma)\cap H^1_p(\Sigma))} \le C r^{\max\{1,(n-1)/p\}} (1 + T^\gamma/r) \|\psi\|_{C^\gamma([0,T];BC^1(\Sigma\cap B_{2r}))} \tag{3.67}$$

for all $r \in (0, r_0/2]$ *and* $\psi \in C^\gamma([0, T_1]; BC^1(\Sigma \cap B_{r_0}))$*.*

Given $\eta > 0$*, the number* C *in (i) to (iii) is uniform with respect to* $\|\nabla\omega\|_\infty \le \eta$*.*

Proof. With the substitution $x = ry$ we obtain the identities $\|\chi(\cdot/r)\|_p = r^{(n-1)/p}\|\chi\|_p$ and $\|(\nabla\chi)(\cdot/r)\|_p = r^{(n-1)/p}\|\nabla\chi\|_p$. We will also use the inequalities

$$|\psi(x)-\psi(0)| \le (1+\eta^2)^{1/2}|x|\|\nabla_\Sigma\psi\|_\infty,$$
$$|\psi(x)-\psi(0)| \le (1+\eta^2)^{1/2}|x|^2\|\nabla_\Sigma^2\psi\|_\infty \quad \text{if } \nabla_\Sigma\psi(0)=0,$$

which follow from Proposition A.12 on page 133. Then (i) and (ii) are readily checked.

Next, for $t\in[0,T]$ we have

$$\begin{aligned}\|\nabla\tilde\psi_r(t,\cdot)\|_p &\le Cr^{(n-1)/p}\|\chi\|_{H^1_p}\|\nabla\psi(t,\cdot)\|_{BC(\Sigma\cap B_{2r})} + \|r^{-1}\nabla\chi(\cdot/r)\|_p|\psi(t,0)-\psi(0,0)|\\ &\le Cr^{(n-1)/p}\|\chi\|_{H^1_p}(1+T^\gamma/r)\|\psi\|_{C^\gamma([0,T];BC(\Sigma\cap B_{2r}))}.\end{aligned}$$

The estimate of $\|\tilde\psi_r(t,\cdot)\|_p$ is similar and hence (iii) is valid. □

For given $u_*\in\mathcal{P}_{M,T_1}$ and $r>0$ we define

$$\text{(3.68)}\qquad \begin{cases} \vartheta_j := (\vartheta_{L,j},\vartheta_{w,j},\vartheta_{Dv,j}), & \vartheta_j^* := (\vartheta^*_{L,j},\vartheta^*_{w,j},\vartheta^*_{Dv,j}),\\ \vartheta_{L,j} := S_r(Q_j^\top[L_\Sigma\circ\Theta_{\omega_j}]Q_j), & \vartheta^*_{L,j} := Q_j^\top[L_\Sigma(p_j)]Q_j,\\ \vartheta_{w,j} := S_r(w_*\circ\Theta_{\omega_j}), & \vartheta^*_{w,j} := w_*(0,p_j),\\ \vartheta_{Dv,j} := S_r(Q_j^\top[(D_\Sigma v_*)\circ\Theta_{\omega_j}]Q_j), & \vartheta^*_{Dv,j} := Q_j^\top[(D_\Sigma v_*)(0,p_j)]Q_j.\end{cases}$$

Then the local operators $A_j\colon E_j\to F_j$ for $j\in I_3$ are defined by

$$\text{(3.69)}\qquad A_j(u,\pi,h) := \begin{bmatrix}\rho\partial_t u-\mu\Delta u+\nabla\pi\\ \operatorname{div} u\\ L_v(u,h;\omega_j,\vartheta_j)\\ L_w(u,\pi,h;\omega_j,\vartheta_j)\\ \partial_t h - u\cdot\nu_{\Sigma_j}\end{bmatrix}\qquad \text{for } (u,\pi,h)\in E_j,\ j\in I_3,$$

where L_v and L_w are defined on page 71.

3.25. Corollary. *Let $p\in(n+2,\infty)$ and M, $T_1>0$. Then there are positive functions $r(\cdot)$ and $T_0(\cdot)$ such that for some $\eta_0>0$ and every $\eta\in(0,\eta_0]$, the pair (Ω,Σ) has an $(\eta,r(\eta))$-localization set-up and the local operators $A_j\colon E_j\to F_j$ ($j\in I_1\cup I_2\cup I_3$) defined by equations (3.61) and (3.69) are uniformly invertible with respect to $T\in(0,T_0(\eta)]$, $j\in I$, and $u_*\in\mathcal{P}_{M,T_1}$.*

Proof. For given $M>0$ there exists $M_1 = M_1(M,\Sigma)\ge M$ such that

$$\sup_{x\in\Sigma}|(L_\Sigma(x),w_*(0,x),D_\Sigma v_*(0,x))|\le M_1,\quad \inf_\Sigma d_0(D_\Sigma(u_*|_{t=0}))\ge M_1^{-1}$$

for all $u_*\in\mathcal{P}_{M,T_1}$ and $T_1\in(0,\infty)$. For given $\eta>0$, Lemma 2.9 yields a positive number $r_0(\eta)$ such that for every $r\in(0,r_0(\eta)]$ the pair (Ω,Σ) has an (η,r)-localization set-up such that $\|\omega_j\|_{BC^1\cap H^2_p}\le\eta$ for all $j\in I_2\cup I_3$ and there exists $R(r)>0$ such that $\|\omega_j\|_{BC^4}\le R(r)$ for $j\in I_2\cup I_3$. Sobolev's embedding and the mixed derivative embeddings yield an estimate

$$\|(w_*,D_\Sigma v_*)\|_{C^\gamma([0,T];BC^1(\Sigma))}\le CM\qquad\text{for all } u_*\in\mathcal{P}_{M,T_1}$$

for some $\gamma>0$ and $C\ge1$. By Proposition 3.24 we can find a positive number $r_1(\eta)\le r_0(\eta)$ and a positive function $r\mapsto R(r)$ such that the parameters $(\vartheta_j^*,\vartheta_j)$ from (3.68) satisfy

$$\begin{aligned}\|\vartheta_{L,j}-\vartheta^*_{L,j}\|_{BC(\Sigma_\omega)\cap H^1_p(\Sigma_\omega)}&\le\eta, & \|\vartheta_{L,j}-\vartheta^*_{L,j}\|_{BC^2(\Sigma_\omega)}&\le R(r),\\ \|\vartheta_{w,j}-\vartheta^*_{w,j}\|_{C([0,T];BC(\Sigma_\omega)\cap H^1_p(\Sigma_\omega))}&\le\eta, & \|\vartheta_{w,j}-\vartheta^*_{w,j}\|_{\mathbb{G}_w(T)}&\le R(r),\\ \|\vartheta_{Dv,j}-\vartheta^*_{Dv,j}\|_{C([0,T];BC(\Sigma_\omega)\cap H^1_p(\Sigma_\omega))}&\le\eta, & \|\vartheta_{Dv,j}-\vartheta^*_{Dv,j}\|_{\mathbb{G}_w(T)}&\le R(r)\end{aligned}$$

for all $r\in(0,r_1(\eta)]$, $j\in I_3$, and $u_*\in\mathcal{P}_{M,T_1}$. Hence $(\vartheta_j^*,\omega_j,\vartheta_j)$ belongs to the set $\mathcal{P}_{CM_1,T_1,\eta,R(r)}$ from page 71 for all $j\in I_3$. By Theorem 3.16 and Lemma 3.23, there exist a positive number η_0

and a function $R \mapsto T_0(R)$ such that if $\eta \le \eta_0$ and $r \in (0, r_1(\eta)]$, then the operators $A_j \colon E_j(T) \to F_j(T)$ are uniformly invertible with respect to $T \in (0, T_0(R(r))]$, $j \in I$, and $u_* \in \mathcal{P}_{M,T_1}$. □

3.3.4. Commutator estimates. For proving Theorem 3.21 it remains to verify the commutator estimates (3.56) and to construct the operator R_0.

3.26. Lemma. *Let $p \in (\max\{5, n+2\}, \infty)$ and let M, T_1, $(U_j, \Theta_j, \omega_j)$, A_j, and T_0 be as in Corollary 3.25. Then for all $\varepsilon > 0$ there exists $T_0' \in (0, T_0]$ such that*

$$\|A\Psi_{E,j} - \Psi_{F,j}A_j\|_{P_{E,j}E_j(T)\to F(T)} \le \varepsilon, \tag{3.70a}$$

$$\|\Phi_{F,j}A - A_j\Phi_{E,j}\|_{P_E E(T)\to F_j(T)} \le \varepsilon, \tag{3.70b}$$

for all $T \in (0, T_0']$, $j \in I$, and $u_* \in \mathcal{P}_{M,T_1}$.

Proof. It is sufficient to prove estimate (3.70a), since (3.70b) can be proved analogously.

For given $z_j = (u_j, \pi_j, h_j) \in P_{E,j}E_j$, the pair (u_j, π_j) satisfies the assumptions of Lemma 3.22 in $\Omega_j \setminus \Sigma_j$ and we conclude that $\pi_{j0} := \pi_j - \langle \pi_j \rangle_{K_j}$ belongs to ${}_0W_p^{1/2-1/2p}(J; L_p(K_j))$ for every bounded smooth domain $K_j \subset \Omega_j$ which contains the support of $\nabla\psi_j$ and we have

$$\|\pi_j - \langle \pi_j \rangle_{K_j}\|_{{}_0W_p^{1/2-1/2p}(J;L_p(K_j))} \le C(K_j)\left(\|u_j\|_{{}_0\mathbb{E}_u(J,\Omega_j,\Sigma_j)} + \|[\![\pi_j]\!]\|_{{}_0\mathbb{G}_w(J,\Sigma_j)}\right) \tag{3.71}$$

for all $z_j \in P_{E,j}E_j(T)$ and $T \in (0, T_0]$. We next deal with the cases $j \in I_1$, I_2, and I_3 separately.

(1) *Perturbed whole-space problem.* Let $j \in I_1$ be fixed and let $z = (u, \pi_0) \in P_{E,j}E_j$. Then

$$(A\Psi_{E,j} - \Psi_{F,j}A_j)z = \begin{bmatrix} \pi_0 Q_j \nabla\psi_j - \mu Q_j[\Delta, \psi_j]u \\ \nabla\psi_j \cdot u \end{bmatrix} \circ \Theta_j^{-1} =: \begin{bmatrix} Q_j F_{uj}(u, \pi_0) \\ F_{dj}(u) \end{bmatrix} \circ \Theta_j^{-1}. \tag{3.72}$$

Here we let $[S, T] = ST - TS$ denote the *commutator* of linear operators S and T.

We show that the perturbations F_{uj} and F_{dj} satisfy the estimate

$$\|F_{uj}(u, \pi_0)\|_{\mathbb{F}_u(T)} + \|F_{dj}(u)\|_{{}_0\mathbb{F}_d(T)} \le CT^{1/2-1/2p}\|u\|_{{}_0\mathbb{E}_u(T)}, \tag{3.73}$$

where $\mathbb{F}_u = \mathbb{F}_u(J, \mathbb{R}^n, \emptyset)$ and ${}_0\mathbb{F}_d := {}_0\mathbb{F}_d(J, \mathbb{R}^n, \emptyset) = {}_0\mathbb{F}_{d,\Sigma}(J, \mathbb{R}^n, \emptyset)$. From estimate (3.71), the mixed derivative embeddings and the interval dependent estimates in Lemma 3.18 we obtain the following estimates. For all $\delta \in (1/p, 1/2)$ and $T_0 > 0$ we have

$$\|[\Delta, \psi_j]u\|_{\mathbb{F}_u(T)} \le C(T_0)\|u\|_{L_p(0,T;H_p^1)} \le C(\delta, T_0)T^\delta\|u\|_{{}_0W_p^\delta(0,T;H_p^1)} \le C(\delta, T_0)T^\delta\|u\|_{{}_0\mathbb{E}_u(T)},$$

$$\|\pi_0\nabla\psi_j\|_{\mathbb{F}_u(T)} \le C(T_0)T^{1/2-1/2p}\|\pi_0\|_{{}_0W_p^{1/2-1/2p}(0,T;L_p)} \le C(T_0)T^{1/2-1/2p}\|u\|_{{}_0\mathbb{E}_u(T)}.$$

The estimates in the divergence space ${}_0\mathbb{F}_d(T) = H_p^1(0, T; \dot{H}_p^{-1}(\mathbb{R}^n)) \cap L_p(0, T; H_p^1(\mathbb{R}^n))$ are obtained in two steps. First, for $\delta \in (1/p, 1/2)$ and $T_0 > 0$ we have

$$\|\nabla\psi_j \cdot u\|_{L_p(0,T;H_p^1)} \le C(T_0)\|u\|_{L_p(0,T;H_p^1)} \le T^\delta C(\delta, T_0)\|u\|_{{}_0\mathbb{E}_u(T)}.$$

Second, the term $\nabla\psi_j \cdot u$ acts as a functional on $\phi \in \dot{H}_{p'}^1(\mathbb{R}^n)$ in virtue of $\phi \mapsto \int_\Omega \nabla\psi_j \cdot u\, \phi\, dx$. The condition $\operatorname{div} u = 0$ yields $\int_{\mathbb{R}^n} \nabla\psi_j \cdot u\, dx = 0$. Hence $\int_{\mathbb{R}^n} \nabla\psi_j \cdot u\, \phi\, dx = \int_{\mathbb{R}^n} \nabla\psi_j \cdot u\, \phi_0\, dx$, where $\phi_0 = \phi - \langle \phi \rangle_{K_j}$ and thus

$$\begin{aligned} \partial_t \int_{\mathbb{R}^n} \nabla\psi_j \cdot u\, \phi\, dx &= \int_{\mathbb{R}^n} \nabla\psi_j \cdot \left(\frac{\mu}{\rho}\Delta u - \frac{1}{\rho}\nabla\pi\right)\phi_0\, dx \\ &= -\int_{\mathbb{R}^n} \frac{\mu}{\rho}(\nabla u)^\top : \nabla(\nabla\psi_j\phi_0)\, dx + \int_{\mathbb{R}^n} \frac{1}{\rho}\pi_0 \operatorname{div}(\nabla\psi_j\phi_0)\, dx. \end{aligned} \tag{3.74}$$

Applying the Poincaré-Wirtinger inequality to $\phi_0 \in H_{p'}^1(K_j)$ and using (3.71), we obtain

$$\|\nabla\psi_j \cdot u\|_{{}_0H_p^1(0,T;\dot{H}_p^{-1})} \le C(T_0)\|u\|_{L_p(0,T;H_p^1)} + C(T_0)\|\pi_0\|_{L_p(0,T;L_p)} \le C(T_0)T^{1/2-1/2p}\|u\|_{{}_0\mathbb{E}_u(T)}.$$

Therefore estimate (3.73) is valid.

(2) *Perturbed half-space problem.* In the case $j \in I_2$, $\Omega_j = \mathbb{R}^n_{\omega_j}$, and $\Sigma_j = \emptyset$, the commutator is also given by (3.72) and F_{uj} in $\mathbb{F}_u(J, \Omega_j, \Sigma_j)$ and F_{dj} in $L_p(0,T; H^1_p(\Omega_j))$ can be estimated in the same way as above. In view of $\operatorname{div} u = 0$ and $u|_{\partial\Omega_j} = 0$ it remains to estimate the functional

$$F_{dj}\colon \phi \mapsto \langle \nabla\psi_j \cdot u, \phi\rangle = -\int_{\Omega_j} \psi_j u \cdot \nabla\phi \, dx$$

in $H^1_p(0,T; \hat{H}^{-1}_p(\Omega_j))$. As for (3.74) we obtain

$$\partial_t \int_{\Omega_j} \nabla\psi_j \cdot u\, \phi \, dx = -\int_{\Omega_j} \frac{\mu}{\rho}(\nabla u)^\top : \nabla(\nabla\psi_j\phi_0)\, dx + \int_{\Omega_j} \frac{1}{\rho}\pi_0 \operatorname{div}(\nabla\psi_j\phi_0)\, dx \\ + \int_{\partial\Omega_j} \left(\frac{\mu}{\rho}\nabla\psi \cdot \partial_\nu u\, \phi - \frac{\mu}{\rho}\nabla^2\psi : \nabla u\, \phi - \frac{\partial_\nu\psi}{\rho}\pi\phi\right) d(\partial\Omega_j).$$

For every $\delta \in (1/p, 1/2 - 3/2p)$ the trace operator

$$\pi_0 \mapsto \pi_0|_{\partial\Omega_j}\colon W^{1/2-1/2p}_p(0,T; L_p(\Omega_j)) \cap L_p(0,T; H^1_p(\Omega_j)) \hookrightarrow W^\delta_p(0,T; L_p(\partial\Omega_j))$$

is bounded since $p > 5$. Therefore the Poincaré-Wirtinger inequality and Lemma 3.22 yield

$$\|\partial_t\langle\nabla\psi_j \cdot u, \cdot\rangle\|_{L_p(0,T;\hat{H}^{-1}_p(\Omega_j))} \le C\|(\nabla u, \pi_0)\|_{L_p(0,T;L_p(\Omega_j))} + C\|(\partial_\nu u, \pi)\|_{L_p(0,T;L_p(\partial\Omega_j))} \\ \le C(\delta, T_0)T^\delta \|(u,\pi)\|_{{}_0\mathbb{E}_u(T,\Omega_j,\emptyset)\times\mathbb{E}_\pi(J,\Omega_j,\emptyset)}.$$

(3) *Perturbed interface problem.* Let $j \in I_3$ be fixed. For $z = (u, \pi_0, h) \in P_{E,j}E_j$ we have

$$(A\Psi_{E,j} - \Psi_{F,j}A_j)z = \begin{bmatrix} Q_jF_{uj}(u,\pi_0) \\ F_{dj}(u) \\ Q_jG_{vj}(u,h) \\ G_{wj}(u) \\ 0 \end{bmatrix} \circ \Theta_j^{-1}, \tag{3.75}$$

where F_{uj}, F_{dj}, G_{vj}, and G_{wj} are the commutators

$$\begin{aligned} F_{uj}(u,\pi_0) &= \pi_0\nabla\psi_j - \Delta\psi_j\, u - 2[\nabla u]^\top\nabla\psi_j, \\ F_{dj}(u) &= \nabla\psi_j \cdot u, \\ G_{vj}(u,h) &= -[\mu_s\widetilde{\Delta}_{\Sigma_j}, \psi_j]v - \lambda_s[\nabla_{\Sigma_j}\operatorname{div}_{\Sigma_j}, \psi_j]v - [\![\mu]\!]\partial_\nu\psi_j\, v \\ &\quad - [\![\mu]\!]\nabla_{\Sigma_j}\psi_j\, w - (\lambda_s + \mu_s)\vartheta_{w,j}[\nabla_{\Sigma_j}\Delta_{\Sigma_j}, \psi_j]h, \\ G_{wj}(u,h) &= -\operatorname{tr}([(\lambda_s - \mu_s)\operatorname{tr}\vartheta_{L,j} + 2\mu_s\vartheta_{L,j}]\nabla_{\Sigma_j}\psi_j)v - 2[\![\mu]\!]\partial_\nu\psi_j\, w \\ &\quad - \operatorname{tr}([\sigma + (\lambda_s - \mu_s)(\operatorname{tr}\vartheta_{Dv,j} - 2\operatorname{tr}\vartheta_{L,j}\vartheta_{w,j}) + 2\mu_s(\vartheta_{Dv,j} - 2\vartheta_{w,j}\vartheta_{L,j})][\nabla^2_{\Sigma_j}, \psi_j]h). \end{aligned}$$

Clearly, F_{uj} in $L_p(J \times \Omega_j)$ and F_{dj} in $L_p(0,T; H^1_p(\Omega_j \setminus \Sigma_j))$ can be estimated as in the case $j \in I_1$. Due to $\operatorname{div} u = 0$ and $[\![\nu_\Sigma \cdot u]\!] = 0$, the functional $F_{dj} \in H^1_p(0,T; \hat{H}^{-1}_p(\Omega_j))$ is given by

$$\phi \mapsto \langle\nabla\psi_j \cdot u, \phi\rangle = \int_{\Omega_j} \nabla\psi_j \cdot u\, \phi\, dx = -\int_{\Omega_j} \psi_j u \cdot \nabla\phi\, dx.$$

Let $\delta \in (1/p, 1/2 - 3/2p)$. With

$$\partial_t \int_{\Omega_j} \nabla\psi_j \cdot u\, \phi\, dx = -\int_{\Omega_j} \rho^{-1}\mu(\nabla u)^\top : \nabla(\nabla\psi_j\phi_0)\, dx + \int_{\Omega_j} \rho^{-1}\pi_0 \operatorname{div}(\nabla\psi_j\phi_0)\, dx \\ - \int_{\Sigma_j} [\![\rho^{-1}\mu\nabla\psi \cdot \partial_\nu u\, \phi - \rho^{-1}\mu\nabla^2\psi : \nabla u\, \phi - \rho^{-1}\partial_\nu\psi\pi\phi]\!]\, d\Sigma_j$$

and the pressure estimate (3.71), we obtain

$$\begin{aligned}\|\partial_t\langle\nabla\psi_j\cdot u,\cdot\rangle\|_{L_p(0,T;\hat{H}_p^{-1}(\Omega_j))} &\le C\|(\nabla u,\pi_0)\|_{L_p(0,T;L_p(\Omega_j))}+C\|(\partial_\nu u,\pi_\pm)\|_{L_p(0,T;L_p(\Sigma_j))}\\ &\le C(\delta,T_0)T^\delta\|(u,\pi)\|_{{}_0\mathbb{E}_{u,v,w,\partial_\nu w}(J,\Omega_j,\Sigma_j)\times{}_0\mathbb{E}_{\pi,[\![\pi]\!]}(J,\Omega_j,\Sigma_j)}.\end{aligned}$$

The remaining terms G_{vj} in ${}_0\mathbb{G}_v$ and $F_{dj\pm}|_\Sigma$ and G_{wj} in ${}_0\mathbb{G}_w$ are lower order differential operators in (u,h) and therefore the assertions (ii) to (iv) in Lemma 3.19 yield the estimate

$$\|G_{vj}(u,h)\|_{{}_0\mathbb{G}_v(T)}+\|(F_{dj\pm}(u)|_\Sigma,G_{wj}(u,h))\|_{{}_0\mathbb{G}_w(T)}\le T^{1/4}C(\delta,T_0)\|z\|_{E_j(T)}.$$

Hence, given $\varepsilon>0$, there exists $T_0'(\varepsilon)\in(0,T_0]$ such that (3.70a) is valid. Estimate (3.70b) follows analogously. □

3.27. Lemma (Construction of R_0)**.** *Let Ω be a bounded domain in $\mathbb{R}^n$ ($n\ge 2$) with smooth boundary $\partial\Omega$, let $\Sigma\subset\Omega$ be a compact smooth hypersurface, and let T_1, $\rho_\pm$, and $\mu_\pm>0$ be fixed. Then the operator*

(3.76) $\quad (u,\pi)\mapsto((\rho\partial_t-\mu\Delta)u+\nabla\pi,\operatorname{div}u),\quad {}_0\mathbb{E}_{u,v,w,\partial_\nu w}(T)\times{}_0\mathbb{E}_{\pi,[\![\pi]\!]}(T)\to\mathbb{F}_u(T)\times{}_0\mathbb{F}_{d,\Sigma}(T)$

is a retraction and it has a uniformly bounded co-retraction with respect to $T\in(0,T_1]$.

Proof. The spatial trace theorem, the divergence theorem and the identity $\operatorname{div}u=\operatorname{div}_\Sigma v-H_\Sigma w-\partial_\nu w$ near Σ imply that $\operatorname{div}\colon{}_0\mathbb{E}_{u,v,w,\partial_\nu w}\to{}_0\mathbb{F}_{d,\Sigma}$ is bounded. Hence (3.76) is bounded.

From Theorem 2.3 we obtain the Helmholtz decomposition $f_u=\nabla F_u+f_{u\sigma}$ where $f_{u\sigma}:=f_u-\nabla F_u$ belongs to $L_p(0,T;L_{p,\sigma}(\Omega))$ and $F_u\in L_p(0,T;\dot{H}_p^1(\Omega))$ is defined as the solution to the weak Neumann problem $\langle\nabla F_u,\nabla\phi\rangle_\Omega=\langle f_u,\nabla\phi\rangle_\Omega$ for all $\phi\in\dot{H}^1_{p'}(\Omega)$.

Next, we define $u^1:=\nabla U$, where the functions U solves the transmission problem

$$\Delta U=f_d\text{ in }J\times\Omega\setminus\Sigma,\quad \partial_\nu U_+|_{\partial\Omega}=0,\quad [\![\rho U]\!]=0,\quad [\![\partial_\nu U]\!]=0.$$

By Theorem 2.2 and Theorem 2.3, the operator

$$f_d\mapsto u^1=\nabla U,\quad {}_0\mathbb{F}_{d,\Sigma}\to{}_0H_p^{(1,2)}(J\times(\Omega\setminus\Sigma))={}_0H^1_p(J;L_p(\Omega))\cap L_p(J;H^2_p(\Omega\setminus\Sigma))$$

is bounded. Since the traces $u^1|_{\partial\Omega}$ and $u^1_\pm|_\Sigma$ do not necessarily vanish, we construct another function $u^2\in{}_0H_p^{(1,2)}(J\times(\Omega\setminus\Sigma))$ by solving the problem

$$\begin{aligned}(\rho\partial_t-\mu\Delta)u^2&=f_{u\sigma} &&\text{in } J\times\Omega\setminus\Sigma,\\ \operatorname{div}u^2&=0 &&\text{in } J\times\Omega\setminus\Sigma,\\ u^2_+|_{\partial\Omega}&=-u^1|_{\partial\Omega} &&\text{on } J\times\partial\Omega,\\ u^2_\pm|_\Sigma&=-P_\Sigma u^1_\pm|_\Sigma &&\text{on } J\times\Sigma.\end{aligned}$$

This problem can be decoupled into one-phase Stokes problems in the components of $\Omega\setminus\Sigma$ which can be solved by means of [BP07, Theorem 4.1] and the Helmholtz projection. Hence there is a bounded solution operator $(f_{u\sigma},u^1)\mapsto u^2$, the function $u:=u^1+u^2$ satisfies $u|_{\partial\Omega}=0$, $P_\Sigma u|_\Sigma=0$, and $\operatorname{div}u=f_d$, and therefore u belongs to ${}_0\mathbb{E}_{u,v,w,\partial_\nu w}$.

The pressure π is defined as the solution to the weak transmission problem

$$\langle\nabla\pi,\nabla\phi\rangle=\langle\nabla(F_u-(\rho\partial_t-\mu\Delta)U),\nabla\phi\rangle_\Omega\text{ for all }\phi\in\dot{H}^1_{p'}(\Omega),\quad [\![\pi]\!]=-[\![(\rho\partial_t-\mu\Delta)U]\!]=[\![\mu]\!]f_d,$$

and hence belongs to ${}_0\mathbb{E}_{\pi,[\![\pi]\!]}$. It is now straightforward to check that the operator $R_0\colon(f_u,f_d)\mapsto(u^1+u^2,\pi)$, $\mathbb{F}_u\times{}_0\mathbb{F}_{d,\Sigma}\to{}_0\mathbb{E}_{u,v,w,\partial_\nu w}\times{}_0\mathbb{E}_{\pi,[\![\pi]\!]}$ is a uniformly bounded co-retraction for (3.76). □

Proof of Theorem 3.21. The assertions of the theorem follow from the strategy in Section 3.3.1, by applying Lemmas 3.26 and 3.27. □

CHAPTER 4

The nonlinear problem

Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a bounded smooth domain. In this chapter we transform problem (N) with compact moving interface $\Gamma(t) \subset \Omega$ to problem (T) over a fixed interface $\Sigma \subset \Omega$ and prove that problem (T) is well-posed on a sufficiently short interval $J = (0, T)$. The notion of well-posedness is based on the function spaces in Figure 4.1 on the next page, and our basically follows the strategy of Köhne, Prüß, and Wilke [KPW13]. However, we restrict our considerations to the case where the initial interface $\Gamma_0 = \theta_{h_0}(\Sigma)$ is already parametrized over Σ.

In order to transform problem (N), we need a time-dependent diffeomorphism $\Theta(t, \cdot)$ of the underlying domain Ω, which maps a fixed hypersurface $\Sigma \subset \Omega$ onto $\Gamma(t) = \Theta(\{t\} \times \Sigma)$. Such maps are studied in Section 4.1, where we construct a normal-preserving admissible map $\Theta_h \colon J \times \overline{\Omega} \to \overline{\Omega}$ induced by a height function $h(t, \cdot) \colon \Sigma \to \mathbb{R}$ and extending the parametrization

$$\theta_h(t, x) = x + h(t, x)\nu_\Sigma(x) \in \Gamma(t) \quad \text{for } t \in J,\ x \in \Sigma \tag{4.1}$$

to $J \times \overline{\Omega}$. The map Θ_h yields useful identities for the velocity transformation

$$u(t, \Theta_h(t, x)) = [\partial_x \Theta_h(t, x)]\overline{u}(t, x). \tag{4.2}$$

These identities are used to derive problem (T) in Section 4.2 and Section 4.3.

For proving well-posedness of (T), we will apply the following fixed point theorem.

4.1. Theorem (Banach's fixed point theorem, [DM07]). *Let (M, d) be a complete metric space, A be a topological space, and $F \colon M \times A \to M$ be a map with the following properties:*

(i) There exists $q \in (0, 1)$ such that

$$d(F(x, a), F(y, a)) \leq q d(x, y) \quad \text{for all } x, y \in M \text{ and all } a \in A.$$

(ii) For every $x \in M$, the mapping $a \mapsto F(x, a)$ is continuous on A.

Then for every $a \in A$ there is a unique $\varphi(a) \in M$ such that $F(\varphi(a), a) = \varphi(a)$. Moreover, the map $\varphi \colon A \to M$ is continuous.

This tool is applied in Section 4.4, where we prove our main result Theorem 4.33 with the following technique. First, in order to eliminate the initial condition $(u, h)|_{t=0} = (u_0, h_0) =: z_0$, we will construct a triple $z_* = (u_*, \pi_*, h_*)$ with $(u_*, h_*)|_{t=0} = z_0$ by means of semigroup theory and Chapter 2. Then the desired solution is given by $z = z_\bullet + z_*$, where $z_\bullet = (u_\bullet, \pi_\bullet, h_\bullet)$ should satisfy the identity $L(z_\bullet; u_*) = N(z_\bullet; z_*)$, the operator $L(\cdot; u_*)$ is the solution-to-data map of problem (PL), and N contains the nonlinear perturbations that arise during the transformation. Hence, with Theorem 3.21, we can define the map $F(z_\bullet; z_0) := [L(\cdot; u_*)]^{-1} N(z_\bullet; z_*)$. Thus, in view of the desired identity $z_\bullet = F(z_\bullet; z_0)$, it remains to show that F satisfies the assumptions of Theorem 4.1. To this end, we will show that $F(z_\bullet; z_0)$ and $\partial_{z_\bullet} F(z_\bullet; z_0)$ become as small as we wish, when we choose $z_\bullet$, T, and h_0 sufficiently small. Since $L(\cdot; u_*)$ is uniformly invertible, it remains to control the perturbation $N(z_\bullet; z_0)$ and its derivative $\partial_{z_\bullet} N(z_\bullet; z_0)$.

We control these perturbations in the context of their derivations. In Section 4.2, we deal with the transformed momentum balance and the transformed divergence equation, where we do not yet employ an explicit representation of Θ. In Section 4.3, we control the perturbations for the transformed interface momentum balance when the moving interface is represented as $\Gamma(t) = \Theta_h(\{t\} \times \Sigma)$. Here we also specialize the results from Section 4.2 to the case of a normal-preserving admissible map.

For $n \geq 2$, $p \in (3,\infty)$, and $J = (0,T)$, we let

$$
\begin{aligned}
\mathbb{E}_u &= \{u \in H^1_p(J; L_p(\Omega)^n) \cap L_p(J; H^2_p(\Omega \setminus \Sigma)^n) : u|_{\partial\Omega} = 0, [\![u]\!] = 0 \text{ on } \Sigma\},\\
\mathbb{E}_v &= W^{1-1/2p}_p(J; L_p(\Sigma; T\Sigma)) \cap W^{1/2-1/2p}_p(J; H^2_p(\Sigma; T\Sigma)) \cap L_p(J; W^{3-1/p}_p(\Sigma; T\Sigma)),\\
\mathbb{E}_w &= W^{1-1/2p}_p(J; H^1_p(\Sigma)) \cap L_p(J; W^{3-1/p}_p(\Sigma)),\\
\mathbb{E}_{u,v,w} &= \{u \in \mathbb{E}_u : v|_\Sigma \in \mathbb{E}_v,\ w|_\Sigma \in \mathbb{E}_w\},\\
\mathbb{E}_{u,v,w,\partial_\nu w} &= \{u \in \mathbb{E}_{u,v,w} : \partial_\nu w_\pm|_\Sigma \in \mathbb{G}_w\},\\
\mathbb{E}_\pi &= L_p(J; \dot{H}^1_p(\Omega \setminus \Sigma)),\\
\mathbb{E}_{\pi,[\![\pi]\!]} &= \{\pi \in \mathbb{E}_\pi : [\![\pi]\!] \in \mathbb{G}_w\},\\
\mathbb{E}_h &= W^{2-1/2p}_p(J; H^1_p(\Sigma)) \cap \tilde{\mathbb{E}}_h,\\
\tilde{\mathbb{E}}_h &= H^1_p(J; W^{3-1/p}_p(\Sigma)) \cap L_p(J; W^{4-1/p}_p(\Sigma)),\\
\mathbb{E}_\Theta &= H^{3/2}_p(J; H^2_p(\mathbb{R}^n)) \cap H^1_p(J; H^3_p(\mathbb{R}^n)) \cap L_p(J; H^4_p(\mathbb{R}^n)),\\
\mathbb{E}_{\partial\Theta} &= W^{2-1/2p}_p(J; L_p(\Sigma)) \cap \tilde{\mathbb{E}}_{\partial\Theta},\\
\tilde{\mathbb{E}}_{\partial\Theta} &= H^1_p(J; W^{2-1/p}_p(\Sigma)) \cap L_p(J; W^{3-1/p}_p(\Sigma)),\\
\mathbb{E} &= \mathbb{E}_{u,v,w,\partial_\nu w} \times \mathbb{E}_{\pi,[\![\pi]\!]} \times \mathbb{E}_h,\\
\tilde{\mathbb{E}} &= \mathbb{E}_{u,v,w,\partial_\nu w} \times \mathbb{E}_{\pi,[\![\pi]\!]} \times \tilde{\mathbb{E}}_h,\\
\mathbb{F}_u &= L_p(J; L_p(\Omega)^n),\\
\mathbb{F}_d &= H^1_p(J; \hat{H}^{-1}_p(\Omega)) \cap L_p(J; H^1_p(\Omega \setminus \Sigma)),\\
\mathbb{F}_{d,\Sigma} &= \{f_d \in \mathbb{F}_d : f_{d\pm}|_\Sigma \in \mathbb{G}_w\},\\
\mathbb{G}_v &= W^{1/2-1/2p}_p(J; L_p(\Sigma; T\Sigma)) \cap L_p(J; W^{1-1/p}_p(\Sigma; T\Sigma)),\\
\mathbb{G}_w &= W^{1/2-1/2p}_p(J; H^1_p(\Sigma)) \cap L_p(J; W^{2-1/p}_p(\Sigma)),\\
\mathbb{G}_h &= W^{1-1/2p}_p(J; H^1_p(\Sigma)) \cap L_p(J; W^{3-1/p}_p(\Sigma)),\\
\tilde{\mathbb{F}} &= \mathbb{F}_u \times \mathbb{F}_{d,\Sigma} \times \mathbb{G}_v \times \mathbb{G}_w \times \mathbb{G}_h \times \mathbb{E}_{u,v,w,\partial_\nu w}|_{t=0} \times \mathbb{E}_h|_{t=0},\\
\mathbb{F} &= \{(f_u, f_d, g_v, g_w, u_0, h_0) \in \tilde{\mathbb{F}} : f_d|_{t=0} = \operatorname{div} u_0,\ L_v(u_0, h_0; u_*|_{t=0}) = g_v|_{t=0}\}.
\end{aligned}
$$

Here we decompose $u = v + w\nu_\Sigma$ near Σ with $v = P_\Sigma u$ and $w = \nu_\Sigma \cdot u$. We will also write $\mathbb{E}(T)$ or $\mathbb{E}(J,\Omega,\Sigma)$ instead of $\mathbb{E}$ for indicating the dependence on T or (J,Ω,Σ) (analogously for the other spaces).

FIGURE 4.1. Function spaces $\mathbb{E}_{\dots}$, $\mathbb{F}_{\dots}$, and $\mathbb{G}_{\dots}$ on (J,Ω,Σ).

4.1. Diffeomorphism and transformation

We study time-dependent diffeomorphisms $\Theta(t,\cdot)$ in a domain Ω that map a fixed hypersurface $\Sigma \subset \Omega$ onto a moving hypersurface $\Gamma(t)$. In Section 4.1.1 we define and study admissible maps, admissible moving hypersurfaces, and normal-preserving admissible maps and derive useful identities for the velocity transformation $u(t,\Theta(t,x)) = [\partial_x\Theta(t,x)]\overline{u}(t,x)$. In Section 4.1.2 we revisit the Hanzawa map Θ_h and prove that it is admissible but not normal-preserving. In Section 4.1.3 we construct a normal-preserving admissible map $\Theta_h\colon J \times \overline{\Omega} \to \overline{\Omega}$, which depends analytically on its inducing height function h.

4.1.1. General admissible maps. First, we consider general admissible maps, admissible moving hypersurfaces, and normal-preserving admissible maps.

4.2. Definition. Let $J \subset \mathbb{R}$ be an interval and $\Omega \subset \mathbb{R}^n$ be a domain.
(i) A map

$$\Theta\colon (t,x) \mapsto x = \Theta(t,x), \quad J \times \overline{\Omega} \to \overline{\Omega}$$

of class $C^1(J \times \overline{\Omega})^n$ is called an *admissible map*, if (a) the Jacobian $\partial_x \Theta(t,x)$ is invertible for all $t \in J$ and all $x \in \overline{\Omega}$, (b) the map

$$\tilde{\Theta}\colon (t,x) \mapsto (t, \Theta(t,x)), \quad J \times \Omega \to J \times \Omega$$

is a diffeomorphism, and (c) we have $\Theta(t,x) = x$ for all $t \in J$ and all $x \in \partial\Omega$.
(ii) A map $\Theta\colon \overline{\Omega} \to \overline{\Omega}$ is called admissible, if $(t,x) \mapsto \Theta(x)$, $\mathbb{R} \times \overline{\Omega} \to \overline{\Omega}$ is admissible.
(iii) A moving hypersurface $\Gamma\colon J \to 2^{\Omega}$, $t \mapsto \Gamma(t)$ is called *admissible*, if there exist a C^1-hypersurface $\Sigma \subset \Omega$ and an admissible map $\Theta\colon J \times \overline{\Omega} \to \overline{\Omega}$ such that

$$\Gamma(t) = \Theta(\{t\} \times \Sigma) \quad \text{for all } t \in J.$$

We easily obtain the following properties, which are useful for transforming problem (N).

4.3. Proposition. *Let $J \subset \mathbb{R}$ be a compact interval, $\Omega \subset \mathbb{R}^n$ be a bounded domain, and $\Theta : J \times \overline{\Omega} \to \overline{\Omega}$ be admissible. Then also $(t,x) \mapsto \Theta(t,\cdot)^{-1}(x)$, $J \times \overline{\Omega} \to \overline{\Omega}$ is admissible.*

4.4. Proposition. *Let $t \mapsto \Gamma(t) = \Theta(\{t\} \times \Sigma)$ be an admissible moving hypersurface.*
(i) The tangent vectors of $\Gamma(t)$ are given by

$$\tau_j^{\Gamma}(t,x) = [\partial_{\overline{x}} \Theta(t,\overline{x})]\tau_j^{\Sigma}(\overline{x}) \quad \text{for all } x = \Theta(t,\overline{x}),\ \overline{x} \in \Sigma, \tag{4.3}$$

and a continuous unit normal field on $\Gamma(t)$ is given by

$$\nu_{\Gamma}(t,x) = \frac{[\partial_{\overline{x}} \Theta(t,\overline{x})]^{-\top} \nu_{\Sigma}(\overline{x})}{|[\partial_{\overline{x}} \Theta(t,\overline{x})]^{-\top} \nu_{\Sigma}(\overline{x})|} \quad \text{for all } x = \Theta(t,\overline{x}),\ \overline{x} \in \Sigma. \tag{4.4}$$

(ii) The normal velocity of $\Gamma(t)$ is given by

$$V_{\Gamma}(t,x) = \nu_{\Gamma}(t,x) \cdot \partial_t \Theta(t,\overline{x}) \quad \text{for all } x = \Theta(t,\overline{x}),\ \overline{x} \in \Sigma.$$

Proof. (i) Since $\Gamma(t)$ is oriented and $\det \partial_x \Theta$ is either positive or negative in all of $\overline{\Omega}$, the hypersurface Σ must be orientable. Let ν_Σ denote a unit normal field on Σ and let $\varphi\colon \mathbb{R}^{n-1} \supset U \to \Sigma$ be a parametrization for Σ. Since the restriction $\Theta(t,\cdot)|_\Sigma\colon \Sigma \to \Gamma(t)$ is a diffeomorphism, the map $y \mapsto \Theta(t,\varphi(y))\colon \mathbb{R}^{n-1} \supset U \to \Gamma(t)$ is a parametrization for $\Gamma(t)$ and the vectors

$$\tau_j^{\Gamma}(t,x) := \partial_j(\Theta(t,\cdot) \circ \varphi)(u) = \partial_{\overline{x}} \Theta(t,\overline{x})\tau_j^{\Sigma}(\overline{x}) \quad \text{for } x = \Theta(t,\overline{x}),\ \overline{x} = \varphi(u) \in \Sigma$$

form a basis of the tangent space $T_x\Gamma(t)$. Hence (4.3) is valid. Since we have $\tau_j^{\Gamma} \cdot \nu_\Gamma = 0$ for all j, the normal $\nu_\Gamma(t,x)$ must be parallel to $[\partial_{\overline{x}} \Theta(t,\overline{x})]^{-\top} \nu_\Sigma(\overline{x})$. From the identity $|\nu_\Gamma| = 1$, it follows that either $\nu_\Gamma(t,x)$ or $-\nu_\Gamma(t,x)$ satisfy (4.4).
(ii) The second assertion follows from Proposition 1.7, by using the trajectories $\gamma = \Theta(\cdot,\overline{x})$. □

Next, we introduce normal-preserving admissible maps and study their geometric properties as well as their kinematic properties associated to the velocity transformation (4.2).

4.5. Definition. Given an admissible map $\Theta\colon J \times \overline{\Omega} \to \overline{\Omega}$ and a C^1-hypersurface $\Sigma \subset \Omega$, we put $\Gamma(t) = \Theta(\{t\} \times \Sigma)$ and we say that Θ is *normal-preserving* for Σ, if the vectors $\partial_{\nu_\Sigma} \Theta(t,x)$ and $\nu_{\Gamma(t)}(\Theta(t,x))$ are parallel for every $t \in J$ and every $x \in \Sigma$; that is, there exists $\beta\colon J \times \Sigma \to \mathbb{R} \setminus \{0\}$ such that $\partial_{\nu_\Sigma} \Theta(t,x) = \beta(t,x)\nu_{\Gamma(t)}(\Theta(t,x))$.

4.6. Proposition. *Let $\Theta\colon J\times\overline{\Omega}\to\Omega$ be a normal-preserving admissible map for a C^1-hypersurface $\Sigma\subset\Omega$ and put $\Gamma(\cdot)=\Theta(\{\cdot\}\times\Sigma)$. Then the following identities are valid on $J\times\Sigma$:*

$$\nu_\Gamma\circ\tilde{\Theta}=\frac{[\partial_x\Theta]^{-\top}\nu_\Sigma}{|[\partial_x\Theta]^{-\top}\nu_\Sigma|}=\frac{\partial_{\nu_\Sigma}\Theta}{|\partial_{\nu_\Sigma}\Theta|},\quad |\partial_{\nu_\Sigma}\Theta|=|[\partial_x\Theta^{-\top}]\nu_\Sigma|^{-1},\tag{4.5a}$$

$$\tau_j^\Gamma\circ\tilde{\Theta}=[\partial_x\Theta]\tau_j^\Sigma,\tag{4.5b}$$

$$\tau_\Gamma^j\circ\tilde{\Theta}=[\partial_x\Theta]^{-\top}\tau_\Sigma^j,\tag{4.5c}$$

$$P_\Gamma\circ\tilde{\Theta}=[\partial_x\Theta]P_\Sigma[\partial_x\Theta]^{-1}=[\partial_x\Theta]^{-\top}P_\Sigma[\partial_x\Theta]^\top.\tag{4.5d}$$

Let two vector fields $u\colon J\times\Omega\to\mathbb{R}^n$ and $\overline{u}\colon J\times\Omega\to\mathbb{R}^n$ be related by $u\circ\tilde{\Theta}=[\partial_x\Theta]\overline{u}$, and decompose $u|_\Gamma=v+w\,\nu_\Gamma$ with $v=P_\Gamma u|_\Gamma$ and $\overline{u}|_\Sigma=\overline{v}+\overline{w}\,\nu_\Sigma$ with $\overline{v}=P_\Sigma\overline{u}|_\Sigma$. Then we have

$$v\circ\tilde{\Theta}=[\partial_x\Theta]\overline{v},\tag{4.6a}$$

$$w\circ\tilde{\Theta}=|\partial_{\nu_\Sigma}\Theta|\,\overline{w}.\tag{4.6b}$$

Proof. Since Θ is normal-preserving, the general identity (4.4) in Proposition 4.4 yields

$$\nu_\Gamma\circ\tilde{\Theta}=\frac{\partial_{\nu_\Sigma}\Theta}{(\nu_\Gamma\circ\tilde{\Theta}|\partial_{\nu_\Sigma}\Theta)}=|[\partial_x\Theta]^{-\top}\nu_\Sigma|\partial_{\nu_\Sigma}\Theta,$$

and therefore (4.5a) is valid. Identity (4.5b) is a repetition of (4.3). From the relations $\tau_j^\Gamma\cdot\tau_\Gamma^k=\delta_j^k$ and $\tau_j^\Gamma\cdot\nu_\Gamma=\tau_\Gamma^j\cdot\nu_\Gamma=0$ we obtain (4.5c), and then (4.5d) is readily checked.

The remaining identities can be verified as follows.

$$v\circ\tilde{\Theta}=[P_\Gamma\circ\tilde{\Theta}][\partial_x\Theta]\overline{u}=[\partial_x\Theta]P_\Sigma\overline{u}=[\partial_x\Theta]\overline{v},$$
$$w\circ\tilde{\Theta}=(u\circ\tilde{\Theta}|\nu_\Gamma\circ\tilde{\Theta})=(\overline{u}|[\partial_x\Theta]^\top(|\partial_{\nu_\Sigma}\Theta|[\partial_x\Theta]^{-\top}\nu_\Sigma))=|\partial_{\nu_\Sigma}\Theta|\,\overline{w}.\qquad\square$$

4.1.2. The Hanzawa map. Next, we revisit the Hanzawa map Θ_h and prove that it is admissible but not normal-preserving. In order to construct it, we recall that

$$x\mapsto\theta_h(x)=x+h(x)\nu_\Sigma(x),\quad \Sigma\to\Gamma_h$$

is a parametrization for $\Gamma_h=\theta_h(\Sigma)$ over Σ. If Σ is of class C^2, then for $|h(x)|<|L_\Sigma(x)|$, the matrix

$$M_h(x):=[I_x-h(x)L_\Sigma(x)]^{-1}$$

from page 138 is invertible, maps $T_x\Sigma$ onto itself, and satisfies $M_h\nu_\Sigma=\nu_\Sigma$. Moreover,

$$\nu_{\Gamma_h}\circ\theta_h=\beta_h(\nu_\Sigma-M_h\nabla_\Sigma h),\quad\text{with }\beta_h:=(\nu_\Sigma|\nu_{\Gamma_h}\circ\theta_h)=(1+|M_h\nabla_\Sigma h|^2)^{-1/2}.\tag{4.7}$$

A hypersurface $\Sigma\subset\mathbb{R}^n$ is said to have a *tubular neighborhood* of radius $r>0$, if the map

$$X\colon(p,t)\mapsto p+t\nu_\Sigma(p),\quad \Sigma\times(-r,r)\to B_r(\Sigma):=\{x\in\mathbb{R}^n:\operatorname{dist}(x,\Sigma)<r\}$$

is a *homeomorphism*; that is, X is bijective and continuous and has a continuous inverse (see Definition A.16). The inverse of X is denoted by

$$X^{-1}(x)=(\Pi(x),d(x))=(p,t)\quad\text{for }x=p+t\nu_\Sigma(p)\in B_r(\Sigma).$$

Proposition A.17 implies that every compact C^2-hypersurface has a tubular neighborhood.

4.7. Definition (cf. [Han81, p. 309])**.** Let $\Omega\subset\mathbb{R}^n$ ($n\geq 2$) be a domain and let $\Sigma\subset\Omega$ be a closed C^2-hypersurface with tubular neighborhood $B_r(\Sigma)\subset\Omega$ of radius $r>0$. Choose a function $\chi\in C^\infty(\mathbb{R};[0,1])$ such that $\chi(s)=1$ if $|s|\leq r/3$ and $\chi(s)=0$ if $|s|\geq 2r/3$ and $\|\chi'\|_\infty<6/r$.

Then, for a given height function $h\colon\Sigma\to\mathbb{R}$, we define the *stationary Hanzawa map*

$$\Theta_h(x):=\begin{cases}x+\chi(d(x))\,h(\Pi(x))\,\nu_\Sigma(\Pi(x)) & \text{for }x\in B_r(\Sigma),\\ x & \text{for }x\in\overline{\Omega}\setminus B_r(\Sigma).\end{cases}$$

For $J \subset \mathbb{R}$ and a height function $h\colon J \times \Sigma \to \mathbb{R}$, we define the *time-dependent Hanzawa map*

$$\Theta_h(t,x) := \Theta_{h(t,\cdot)}(x) \quad \text{for } (t,x) \in J \times \overline{\Omega}.$$

4.8. Theorem. *Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a domain, $k \geq 1$, and let $\Sigma \subset \Omega$ be a compact C^{k+1}-hypersurface with tubular neighborhood $B_r(\Sigma) \subset \Omega$ of radius $r < \|L_\Sigma\|_\infty^{-1}$. Then, for a given height function*

$$h \in C^k(\Sigma) \quad \text{with } \|h\|_\infty < 1/\|\chi'\|_\infty,$$

the stationary Hanzawa map $\Theta_h\colon \Omega \to \Omega$ is a C^k-diffeomorphism, and in $B_r(\Sigma)$ we have

$$\begin{aligned} \partial_x\Theta_h &= P_\Sigma\circ\Pi - \chi\circ d\,[hL_\Sigma]\circ\Pi\,[P_\Sigma\circ\Pi - d\,L_\Sigma\circ\Pi]^{-1} && \textit{(purely tangential part)} \\ &\quad + (1+\chi'\circ d\ h\circ\Pi)[\nu_\Sigma\otimes\nu_\Sigma]\circ\Pi && \textit{(purely normal part)} \\ &\quad + \chi\circ d\,[\nu_\Sigma\otimes\nabla_\Sigma h]\circ\Pi\,[P_\Sigma\circ\Pi - d\,L_\Sigma\circ\Pi]^{-1} && \textit{(tangential-to-normal part)} \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} [\partial_x\Theta_h]^{-1} &= [P_\Sigma\circ\Pi - (d+\chi\circ d\ h\circ\Pi)L_\Sigma\circ\Pi]^{-1}[P_\Sigma\circ\Pi - d\,L_\Sigma\circ\Pi] \\ &\quad + (1+\chi'\circ d\ h\circ\Pi)^{-1}[\nu_\Sigma\otimes\nu_\Sigma]\circ\Pi \\ &\quad - \chi\circ d(1+\chi'\circ d\ h\circ\Pi)^{-1}[\nu_\Sigma\otimes\nabla_\Sigma h]\circ\Pi[P_\Sigma\circ\Pi - (d+\chi\circ d\ h\circ\Pi)L_\Sigma\circ\Pi]^{-1}. \end{aligned} \tag{4.9}$$

In particular, the following identities are valid on Σ:

$$\partial_x\Theta_h|_\Sigma = P_\Sigma - hL_\Sigma + \nu_\Sigma\otimes\nu_\Sigma + \nu_\Sigma\otimes\nabla_\Sigma h, \tag{4.10a}$$

$$[\partial_x\Theta_h|_\Sigma]^{-1} = [P_\Sigma - hL_\Sigma]^{-1}P_\Sigma + \nu_\Sigma\otimes\nu_\Sigma - \nu_\Sigma\otimes\nabla_\Sigma h[P_\Sigma - hL_\Sigma]^{-1}. \tag{4.10b}$$

Proof. *Local invertibility.* We check that the inverse $[\partial_x\Theta_h]^{-1}$ exists everywhere in Ω. Clearly, it suffices to consider the case $x \in B_r(\Sigma)$. Proposition A.20 and a straightforward calculation show that (4.8) is valid in $B_r(\Sigma)$. The purely tangential part of $\partial_x\Theta_h$ can be written as

$$[P_\Sigma\circ\Pi - (d+\chi\circ d\ h\circ\Pi)L_\Sigma\circ\Pi]\,[I - d\,L_\Sigma\circ\Pi]^{-1}.$$

The conditions $\|h\|_\infty < r/3$ and $\chi(s) = 0$ for $|s| \geq 2r/3$ yield $|d+\chi\circ d\ h\circ\Pi|_\infty \leq r < \|L_\Sigma\|_\infty^{-1}$ in $B_r(\Sigma)$. Hence the purely tangential part is a linear isomorphism of $T_{\Pi(x)}\Sigma$. The purely normal part is a linear isomorphism of $\mathbb{R}\nu_\Sigma(x)$, since $\|h\|_\infty < \|\chi'\|_\infty^{-1}$. Therefore $\partial_x\Theta_h(x)$ is an isomorphism of $\mathbb{R}^n$. For every invertible $A \in \mathbb{C}^{n\times n}$ and $a, b \in \mathbb{C}^n$ we have

$$[A + b\otimes a]^{-1} = A^{-1} - \frac{A^{-1}b\otimes A^{-\top}a}{1 + A^{-1}b\cdot a}. \tag{4.11}$$

Then also (4.9) follows by straightforward calculations. Hence $[\partial_x\Theta_h(\cdot)]^{-1}$ is bounded in Ω and, by the implicit function theorem, $\Theta_h\colon \Omega \to \Omega$ is a local C^k-diffeomorphism.

Surjectivity. Since the map $\Theta_h\colon \Omega \to \Omega$ is a local homeomorphism, the set $\Theta_h(\Omega)$ is an open subset of Ω. We now show that it is closed as a subset of Ω. Let $(y_n)_n \subset \Theta_h(\Omega)$ converge to $y \in \Omega$. Since $\overline{B}_r(\Sigma)$ is compact and $\Theta_h(x) = x$ in $\Omega \setminus B_r(\Sigma)$, the preimages $x_n = \Theta_h^{-1}(y_n)$ have a convergent subsequence $x_{n_k} \to x \in \Omega$. Therefore $y = \lim_k y_{n_k} = \lim_k \Theta_h(x_{n_k}) = \Theta_h(x)$ also belongs to $\Theta_h(\Omega)$. Consequently, $\Theta_h(\Omega)$ is open, closed, and nonempty in Ω, thus $\Theta_h(\Omega) = \Omega$, which implies that $\Theta_h\colon \Omega \to \Omega$ is surjective.

Injectivity. It suffices to show that the restriction of Θ_h to $B_r(\Sigma)$ is injective. If $\Theta_h(x) = \Theta_h(y)$, then the tubular neighborhood property of Σ implies

$$\Pi(x) + (d(x) + \chi(d(x))h(\Pi(x)))\,\nu_\Sigma(\Pi(x)) = \Pi(y) + (d(y) + \chi(d(y))h(\Pi(y)))\,\nu_\Sigma(\Pi(y))$$

and hence $\Pi(x) = \Pi(y)$ and $d(x) + \chi(d(x))h(\Pi(x)) = d(y) + \chi(d(y))h(\Pi(y))$. Since $s \mapsto s + \chi(s)h(\Pi(x))$ is injective by $|h\chi'| < 1$, we obtain $d(x) = d(y)$, and thus $x = y$.

We conclude that the stationary Hanzawa map Θ_h is a C^k-diffeomorphism of Ω. $\square$

4.9. Corollary. *Let $\Sigma \subset \Omega$ be a compact C^2-hypersurface with tubular neighborhood $B_r(\Sigma) \subset \Omega$ of radius $r < \|L_\Sigma\|_\infty^{-1}$, and let Θ_h denote the time-dependent Hanzawa map induced by*

$$h \in C^1([0,T] \times \Sigma) \quad \text{with } \|h\|_\infty < \|\chi'\|_\infty^{-1}. \tag{4.12}$$

Then the map $\Theta_h : [0,T] \times \overline{\Omega} \to \overline{\Omega}$ and the moving hypersurface $t \mapsto \Gamma_h(t) = \Theta_h(\{t\} \times \Sigma)$ are admissible, but the map Θ_h is not normal-preserving unless $\nabla_\Sigma h = 0$.

Proof. Admissibility follows from Theorem 4.8 and identity (4.10) yields

$$\partial_{\nu_\Sigma} \Theta_h = \nu_\Sigma, \quad \nu_\Gamma \circ \tilde{\Theta} = \frac{\nu_\Sigma - [I_x - hL_\Sigma]^{-1} \nabla_\Sigma h}{(1 + |[I_x - hL_\Sigma]^{-1} \nabla_\Sigma h|^2)^{1/2}}.$$

Thus, $\partial_{\nu_\Sigma} \Theta$ and $\nu_\Gamma \circ \tilde{\Theta}_h$ are not parallel for $\nabla_\Sigma h \neq 0$. □

4.1.3. A normal-preserving admissible map. We will construct a normal-preserving admissible map $\Theta_h \colon J \times \overline{\Omega} \to \overline{\Omega}$, which, considered as an element of some Banach space $\mathbb{E}_\Theta$, depends analytically on its inducing height function $h \in \mathbb{E}_h$. We first construct a diffeomorphism $\Theta_h \colon \mathbb{R}^n \to \mathbb{R}^n$ which maps a compact smooth hypersurface Σ onto $\Gamma := \Theta_h(\Sigma)$ such that $\partial_{\nu_\Sigma} \Theta_h(x)$ is parallel to $\nu_\Gamma(\Theta_h(x))$. For this construction, we employ a co-retraction $\mathfrak{S}$ for the trace operator

$$u \mapsto (u|_\Sigma, \partial_\nu u|_\Sigma, \dots, \partial_\nu^k u|_\Sigma), \quad W_p^s(\mathbb{R}^n) \to \prod_{j=0}^k W_p^{s-j-1/p}(\Sigma).$$

4.10. Lemma. *Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a domain, $\Sigma \subset \Omega$ be a compact smooth hypersurface, and let $p \in (1,\infty)$, $k \in \mathbb{N}_0$, and $s \in (k + 1/p, \infty)$. Then there exists a bounded linear operator*

$$\mathfrak{S} \colon \prod_{j=0}^k W_p^{s-j-1/p}(\Sigma) \to W_p^s(\mathbb{R}^n)$$

with the properties

$$\mathfrak{S}(f_0, \dots, f_k)|_{\mathbb{R}^n \setminus \Omega} = 0, \quad \partial_{\nu_\Sigma}^j (\mathfrak{S}(f_0, \dots, f_k))|_\Sigma = f_j \quad \text{for } j \in \{0, 1, \dots, k\},$$

for all

$$(f_0, \dots, f_k) \in \prod_{j=0}^k W_p^{s-j-1/p}(\Sigma).$$

The operator $\mathfrak{S}$ only depends on Σ and k but not on s or p.

Proof. With Corollary A.19 we decompose $\Omega \setminus \Sigma = \Omega_+ \dot{\cup} \Omega_-$ such that $\Sigma = \partial\Omega_-$ and $\nu_\Sigma = \nu_{\partial\Omega_-} = -\nu_{\partial\Omega_+}$. Let $\Omega' \subset \Omega$ be a bounded smooth domain which still contains Σ and let $\Omega'_\pm := \Omega' \cap \Omega_\pm$. Triebel [Tri10, p. 3.3.3] has shown that there exist bounded linear operators

$$S_\pm \colon \prod_{j=0}^k W_p^{s-j-1/p}(\partial\Omega'_\pm) \to W_p^s(\Omega')$$

with the property

$$\partial_j S_\pm(g_0, \dots, g_k) = g_j \text{ on } \partial\Omega'_\pm \quad \text{for all } j \in \{0, \dots, k\},\ (g_0, \dots, g_k) \in \prod_{j=0}^k W_p^{s-j-1/p}(\partial\Omega'_\pm).$$

For a tuple $(f_0, \dots, f_k) \in \prod_{j=0}^k W_p^{s-j-1/p}(\Sigma)$ we define a linear operator $\mathfrak{S}$ by

$$\mathfrak{S}(f_0, \dots, f_k) := \begin{cases} 0 & \text{in } \mathbb{R}^n \setminus \Omega', \\ S_+(\chi_\Sigma \cdot ((-1)^j f_j)_{j=0}^k + \chi_{\partial\Omega'} \cdot (0, \dots, 0)) & \text{in } \Omega'_+, \\ S_-(f_0, \dots, f_k) & \text{in } \Omega'_-. \end{cases}$$

This operator has the asserted properties. □

We are ready to construct a normal-preserving map.

4.11. Definition. Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a domain and $\Sigma \subset \Omega$ be a compact smooth hypersurface.

(i) Let $p \in (1, \infty)$ and $s \in (1 + n/p)$, and let $h \in W_p^{s-1/p}(\Sigma)$ be a height function satisfying

$$\|h\|_\infty < \|L_\Sigma\|_\infty^{-1}. \tag{4.13}$$

Then we define the *stationary normal-preserving map*

$$\Theta_h(x) := x + \mathfrak{S}(h\nu_\Sigma, g_h)(x) \quad \text{for } x \in \mathbb{R}^n, \tag{4.14}$$

where $\mathfrak{S}$ denotes the linear operator from Lemma 4.10 with $k = 1$, and

$$g_h := [(\nu_{\Gamma_h} \otimes \nu_{\Gamma_h}) \circ \theta_h - I]\nu_\Sigma = \beta_h \, \nu_{\Gamma_h} \circ \theta_h - \nu_\Sigma. \tag{4.15}$$

(ii) Let $J = (0, T)$ with $T \in (0, \infty)$, and let $h\colon J \times \Sigma \to \mathbb{R}$ be a height function such that $h(t, \cdot)$ satisfies the assumptions of (i) for almost all $t \in J$. Then we define the *time-dependent normal-preserving map*

$$\Theta_h(t, x) := \Theta_{h(t,\cdot)}(x) = x + \mathfrak{S}(h(t, \cdot)\nu_\Sigma, g_{h(t,\cdot)})(x) \quad \text{for } t \in J,\, x \in \mathbb{R}^n. \tag{4.16}$$

4.12. Proposition. *Let Θ_h denote the stationary normal-preserving map.*

(i) Θ_h maps Σ onto $\Gamma_h := \theta_h(\Sigma)$.

(ii) $\partial_{\nu_\Sigma}\Theta_h = [\partial_x\Theta_h]\nu_\Sigma = [(\nu_{\Gamma_h} \otimes \nu_{\Gamma_h}) \circ \theta_h]\nu_\Sigma = \beta_h \, \nu_{\Gamma_h} \circ \theta_h$ on Σ.

(iii) $\Theta_h = \mathrm{Id}_x$ in $\mathbb{R}^n \setminus \Omega$.

(iv) For $M_h := (I - hL_\Sigma)^{-1}$ as on page 138, we have

$$\partial_x\Theta_h|_\Sigma = P_\Sigma - hL_\Sigma - \beta_h^2 M_h \nabla_\Sigma h \otimes \nu_\Sigma + \nu_\Sigma \otimes \nabla_\Sigma h + \beta_h^2 \nu_\Sigma \otimes \nu_\Sigma, \tag{4.17}$$

$$[\partial_x\Theta_h|_\Sigma]^{-1} = M_h - \beta_h^2 M_h^2 \nabla_\Sigma h \otimes M_h \nabla_\Sigma h + \beta_h^2 M_h^2 \nabla_\Sigma h \otimes \nu_\Sigma - \nu_\Sigma \otimes M_h \nabla_\Sigma h. \tag{4.18}$$

Proof. All assertions are obvious, except for (4.18), which can be derived from (4.11). □

It remains to prove that the stationary normal-preserving map is a diffeomorphism of $\mathbb{R}^n$ and that its time-dependent version is an admissible map. Compared to the Hanzawa map whose Jacobian has an explicit inverse (4.9) in all of Ω, the normal-preserving map Θ_h as in (4.14) and (4.16) lacks such a representation. We therefore want to show that

$$\sup\nolimits_{t,x} |I_x - [\partial_x\Theta_h(t, x)]^{-1}| < 1,$$

and this can be shown for height functions which are sufficiently small in an appropriate norm. As in Chapter 3 we consider height functions in the class

$$\mathbb{E}_h = W_p^{2-1/2p}(J; H_p^1(\Sigma)) \cap H_p^1(J; W_p^{3-1/p}(\Sigma)) \cap L_p(J; W_p^{4-1/p}(\Sigma)).$$

Then the Jacobian $\partial_x\Theta_h|_\Sigma$ belongs to the space

$$\mathbb{E}_{\partial\Theta} := W_p^{2-1/2p}(J; L_p(\Sigma)) \cap H_p^1(J; W_p^{2-1/p}(\Sigma)) \cap L_p(J; W_p^{3-1/p}(\Sigma)).$$

The space $\mathbb{E}_{\partial\Theta}$ is considered as the target space of the nonlinear map $h \mapsto \partial_x\Theta_h|_\Sigma$, and for proving analyticity of the latter, we employ the following properties of $\mathbb{E}_{\partial\Theta}$.

4.13. Lemma. *Let $\Sigma \subset \mathbb{R}^n$ ($n \geq 2$) be a compact smooth hypersurface, $J = (0, T)$ be bounded, $p \in ((n+2)/2, \infty)$, and $m \in \mathbb{N}$. Then the following assertions are valid:*

(i) The space $\mathbb{E}_{\partial\Theta}$ is continuously embedded into $C^1(\overline{J} \times \Sigma)$.

(ii) $\mathbb{E}_{\partial\Theta}$ is a multiplication algebra and there exists $C(T) \geq 1$ such that

$$\|fg\|_{\mathbb{E}_{\partial\Theta}} \leq C(T) \Big(\|f\|_{\mathbb{E}_{\partial\Theta}} \|g\|_{C^1(\overline{J}\times\Sigma)} + \|f\|_{C^1(\overline{J}\times\Sigma)} \|g\|_{\mathbb{E}_{\partial\Theta}} \Big) \quad \textit{for all } f, g \in \mathbb{E}_{\partial\Theta}. \tag{4.19}$$

(iii) The operator $A \mapsto A^{-1}$, $\{A \in \mathbb{E}_{\partial\Theta}^{m\times m} : \sup_{J\times\Sigma} |A^{-1}(\cdot)| < \infty\} \to \mathbb{E}_{\partial\Theta}$ is analytic.

(iv) The operator $f \mapsto f^{1/2}$, $\{f \in \mathbb{E}_{\partial\Theta} : \inf_{J\times\Sigma} \mathrm{dist}(f(\cdot), \mathbb{R}_-) > 0\} \to \mathbb{E}_{\partial\Theta}$ is analytic.

Proof. (i) We abbreviate $W_p^t(W_p^s) := W_p^t(J; W_p^s(\Sigma))$, $C^k(C^l) := C^k(\overline{J}; C^l(\Sigma))$, and similarly for the other spaces. The mixed derivative embeddings and Sobolev's embedding (B.1) imply

$$W_p^{2-1/2p}(L_p) \cap H_p^1(W_p^{2-1/p}) \hookrightarrow W_p^{1+\theta}(H_p^{2-1/p-2\theta)}) \hookrightarrow W_p^{1+1/p+\varepsilon_t}(W_p^{(n-1)/p+\varepsilon_s}) \hookrightarrow C^1(C),$$

for sufficiently small ε_t, $\varepsilon_s > 0$, provided that $\theta \in (0, 1 - 1/2p)$ satisfies $1 + \theta > 1 + 1/p$ and $2 - 1/p - 2\theta > (n-1)/p$. Such a number θ exists if $1/p < 1 - n/2p$, and this is true for $p > (n+2)/2$. Analogously, we obtain

$$H_p^1(W_p^{2-1/p}) \cap L_p(W_p^{3-1/p}) \hookrightarrow W_p^\theta(H_p^{3-1/p-\theta}) \hookrightarrow C(C^1),$$

provided that $\theta \in (0,1)$ satisfies $\theta > 1/p$ and $3 - 1/p - \theta > 1 + (n-1)/p$. Such a number θ exists if $1/p < 2 - n/p$, and this is true if $p > (n+1)/2$. Hence we have $\mathbb{E}_{\partial\Theta} \hookrightarrow C^1(\overline{J} \times \Sigma)$.

(ii) The norm of $\mathbb{E}_{\partial\Theta}$ consists of the semi-norms

$$[\![\partial_t \cdot]\!]_{1-1/2p,p;p}, \quad [\![(\partial_t\partial_x, \partial_x^2, \partial_x)\cdot]\!]_{p;1-1/p,p}, \quad \|(1, \partial_t, \partial_t\partial_x, \partial_x, \partial_x^2)\cdot\|_p,$$

where we recall the following abbreviations from page 78:

$$[\![\cdot]\!]_{t,p;p} := [\![\cdot]\!]_{W_p^t(L_p)}, \quad [\![\cdot]\!]_{p;s,p} := [\![\cdot]\!]_{L_p(W_p^s)}, \quad \|\cdot\|_p := \|\cdot\|_{L_p(L_p)}.$$

With Lemma B.81 and Lemma B.10 we control some of the leading-order terms of $\|fg\|_{\mathbb{E}_{\partial\Theta}}$ by

$$\begin{aligned}
[\![\partial_t f\, g]\!]_{1-1/2p,p;p} &\lesssim [\![\partial_t f]\!]_{1-1/2p,p;p}\|g\|_\infty + \|\partial_t f\|_\infty\, [\![g]\!]_{1-1/2p,p;p}\,, \\
[\![\partial_t\partial_x f\, g]\!]_{p;1-1/p,p} &\lesssim [\![\partial_t\partial_x f]\!]_{p;1-1/p,p}\|g\|_\infty + \|\partial_t\partial_x f\|_p\|(g, \partial_x g)\|_\infty, \\
[\![\partial_x^2 f\, g]\!]_{p;1-1/p,p} &\lesssim [\![\partial_x^2 f]\!]_{p;1-1/p,p}\|g\|_\infty + \|\partial_x^2 f\|_p\|(g, \partial_x g)\|_\infty.
\end{aligned}$$

These terms and the remaining ones can be estimated by the right-hand side of (4.19). Therefore the pointwise multiplication estimate (4.19) is valid and $\mathbb{E}_{\partial\Theta}$ is a multiplication algebra.

(iii) Let us check that A^{-1} belongs to $\mathbb{E}_{\partial\Theta}^{m\times m}$ for every $A \in \mathbb{E}_{\partial\Theta}^{m\times m}$ with $A^{-1} \in L_\infty(J \times \Sigma)$. Abbreviating $\tau = 1 - 1/2p$ and using Lemma B.81, we obtain

$$[\![\partial_t A^{-1}]\!]_{\tau,p;p} = [\![A^{-1}[\partial_t A]A^{-1}]\!]_{\tau,p;p} \lesssim [\![\partial_t A]\!]_{\tau,p;p}\|A^{-1}\|_\infty^2 + [\![A^{-1}]\!]_{\tau,p;p}\|A^{-1}\|_\infty\|\partial_t A\|_\infty.$$

Next, from the inequality

$$|A(t,x)^{-1} - A(t',x)^{-1}| \le \|A^{-1}\|_\infty^2 |A(t,x) - A(t',x)|,$$

we infer that $[\![A^{-1}]\!]_{\tau,p;p} \lesssim \|A^{-1}\|_\infty^2 [\![A]\!]_{\tau,p;p}$ is finite and therefore $[\![\partial_t A^{-1}]\!]_{\tau,p;p}$ is finite. Analogously,

$$[\![A^{-1}]\!]_{p;\sigma,p} \lesssim \|A^{-1}\|_\infty^2 [\![A]\!]_{p;\sigma,p} < \infty,$$

with $\sigma = 1 - 1/p$. Hence, for $j \in \{1, \dots, n-1\}$, we obtain

$$\begin{aligned}
[\![\partial_t\partial_j A^{-1}]\!]_{p;\sigma,p} &= [\![A^{-1}[\partial_j A]A^{-1}[\partial_t A]A^{-1} + A^{-1}[\partial_t A]A^{-1}[\partial_j A]A^{-1} - A^{-1}[\partial_t\partial_j A]A^{-1}]\!]_{p;\sigma,p} \\
&\lesssim \|A^{-1}\|_\infty^4 \|A\|_{p;\sigma,p}\|\partial_j A\|_\infty\|\partial_t A\|_\infty + \|A^{-1}\|_\infty^3 [\![\partial_j A]\!]_{p;\sigma,p}\|\partial_t A\|_\infty + \|A^{-1}\|_\infty^3 [\![\partial_t A]\!]_{p;\sigma,p}\|\partial_j A\|_\infty \\
&\quad + \|A^{-1}\|_\infty^2 [\![\partial_t\partial_j A]\!]_{p;\sigma,p} + \|(A^{-1}, \nabla_\Sigma A^{-1})\|_\infty \|\partial_t\partial_j A\|_p \|A^{-1}\|_\infty < \infty.
\end{aligned}$$

The semi-norm $[\![\partial_j\partial_k A^{-1}]\!]_{p;\sigma,p}$ can be estimated analogously. We further have

$$\|A^{-1}\|_p \le T^{1/p}|\Sigma|^{1/p}\|A^{-1}\|_\infty < \infty,$$

and the remaining terms in $\|A^{-1}\|_{\mathbb{E}_{\partial\Theta}}$ are also finite. Therefore A^{-1} belongs to $\mathbb{E}_{\partial\Theta}^{m\times m}$, and Proposition B.88 yields analyticity of the inversion operator $A \mapsto A^{-1}$.

(iv) Every bounded function f with $\inf \operatorname{dist}(f(\cdot), \mathbb{R}_-) > 0$ satisfies both $\inf|f| > 0$ and $\sup|\arg f| < \pi$. Hence $f^{1/2}$ and f^{-1} are bounded. The Cauchy-Schwarz inequality yields

$$\|f^{1/2}\|_p = \left(\int_0^T \int_\Sigma 1 \cdot |f|^{p/2}\, d\Sigma\, dt\right)^{1/p} \le T^{1/2p}|\Sigma|^{1/2p}\|f\|_p^{1/2} < \infty.$$

Since $\sup|\arg f^{1/2}| < \pi/2$, there exists $c > 0$ such that

$$|f(t,x)^{1/2} + f(t',x')^{1/2}| \geq c(|f(t,x)|^{1/2} + |f(t',x')|^{1/2})$$

by Lemma B.54. Then the estimate

$$|f(t,x)^{1/2} - f(t',x)^{1/2}| = \frac{|f(t,x) - f(t',x)|}{|f(t,x)^{1/2} + f(t',x)^{1/2}|} \lesssim \frac{|f(t,x) - f(t',x)|}{\inf|f|^{1/2}},$$

yields

$$[\![f^{1/2}]\!]_{\tau,p;p} \lesssim (\inf|f|)^{-1/2}[\![f]\!]_{\tau,p;p} < \infty$$

for $\tau = 1 - 1/2p$, and therefore

$$\begin{aligned}[\![\partial_t f^{1/2}]\!]_{\tau,p;p} &= [\![2^{-1}f^{-1}f^{1/2}\partial_t f]\!]_{\tau,p;p} \\ &\lesssim [\![f^{-1}]\!]_{\tau,p;p}\|f\|_\infty^{1/2}\|\partial_t f\|_\infty + \|f^{-1}\|_\infty \left(\|f^{1/2}\|_{\tau,p;p}\|\partial_t f\|_\infty + \|f^{1/2}\|_\infty[\![\partial_t f]\!]_{\tau,p;p}\right)\end{aligned}$$

is finite. For $\sigma = 1 - 1/p$ we similarly obtain

$$[\![f^{1/2}]\!]_{p;\sigma,p} \lesssim (\inf|f|)^{-1/2}[\![f]\!]_{p;\sigma,p} < \infty.$$

Next,

$$\begin{aligned}[\![\nabla_\Sigma\partial_t f^{1/2}]\!]_{p;\sigma,p} &= 2^{-1}[\![2^{-1}f^{-3/2}\nabla_\Sigma f\partial_t f + f^{-1/2}\nabla_\Sigma\partial_t f]\!]_{p;\sigma,p} \\ &\lesssim \|f^{-1}\|_{\mathbb{E}_{\partial\Theta}}^2(\|f^{1/2}\|_\infty + \|f^{1/2}\|_{p;\sigma,p})\|f\|_{\mathbb{E}_{\partial\Theta}}^2 \\ &\quad + \|(1,\nabla_\Sigma)f^{-1/2}\|_\infty([\![\nabla_\Sigma\partial_t f]\!]_{p;\sigma,p} + \|\nabla_\Sigma\partial_t f\|_p) < \infty.\end{aligned}$$

The remaining terms in $\|f^{1/2}\|_{\mathbb{E}_{\partial\Theta}}$ can be estimated similarly. Hence $f^{1/2}$ belongs to $\mathbb{E}_{\partial\Theta}$ for every $f \in \mathbb{E}_{\partial\Theta}$ with $\inf_{J\times\Sigma}\operatorname{dist}(f(\cdot),\mathbb{R}_-) > 0$, and Proposition B.89 yields analyticity of $f \mapsto f^{1/2}$. ☐

The next step towards analyticity of $h \mapsto \Theta_h$ is to show that $h \mapsto \nu_{\Gamma_h} \circ \theta_h$ is analytic.

4.14. Lemma. *Let $\Sigma \subset \mathbb{R}^n$ ($n \geq 2$) be a compact smooth hypersurface.*

(i) Let $p \in (1,\infty)$, $s \in (1+n/p,\infty)$, and $\tau \in (1+n/p, s]$. Then there exists $\delta_{h_0} = \delta_{h_0}(\Sigma,p,\tau) > 0$ such that all height functions

$$h \in W_p^{s-1/p}(\Sigma) \cap \mathcal{U}_{h_0}, \quad \text{with } \mathcal{U}_{h_0} := \{h \in W_p^{\tau-1/p}(\Sigma) : \|h\|_{W_p^{\tau-1/p}(\Sigma)} < \delta_{h_0}\},$$

satisfy (4.13). In this case the map

$$h \mapsto \nu_{\Gamma_h} \circ \theta_h, \quad W_p^{s-1/p}(\Sigma) \cap \mathcal{U}_{h_0} \to W_p^{s-1-1/p}(\Sigma)^n$$

is analytic.

(ii) Let $p \in ((n+2)/2,\infty)$, $\tau \in (1+n/p, 4-1/p]$, and let $J = (0,T)$ be bounded. Then there exists $\delta_h = \delta_h(\Sigma,p,\tau) > 0$ such that all height functions

$$h \in \mathbb{E}_h \cap \mathcal{U}_h, \quad \text{with } \mathcal{U}_h := \{h \in L_\infty(0,T;W_p^{\tau-1/p}(\Sigma)) : \|h\|_{L_\infty(J;W_p^{\tau-1/p}(\Sigma))} < \delta_h\}, \tag{4.20}$$

satisfy (4.13). In this case the map

$$h \mapsto \nu_{\Gamma_h} \circ \tilde{\Theta}_h, \quad \mathbb{E}_h \cap \mathcal{U}_h \to \mathbb{E}_{\partial\Theta}^n$$

is analytic.

Proof. (i) The identity (4.15) shows that the values of $h\nu_\Sigma$ and g_h depend analytically on those of $(h,\nabla_\Sigma h) \in \mathbb{R}\times\mathbb{R}^n$ such that $|h| < \|L_\Sigma\|_\infty^{-1}$ and $|\nabla_\Sigma h| < (1 - |h|\|L_\Sigma\|_\infty^{-1})^{-1}$. From Sobolev's embedding $W_p^{s-1-1/p}(\Sigma) \hookrightarrow BC(\Sigma)$ we infer that there exists δ_{h_0} such that (4.13) is satisfied if $\|h\|_{W_p^{\tau-1/p}(\Sigma)} < \delta_{h_0}$. By Remark B.80, the space $W_p^{s-1-1/p}(\Sigma)$ is a multiplication algebra, and since ν_Σ and L_Σ are smooth, we infer from Lemma B.10 that

$$h \mapsto (h\nu_\Sigma, I - hL_\Sigma), \quad W_p^{s-1/p}(\Sigma) \to W_p^{s-1/p}(\Sigma)^n \times W_p^{s-1-1/p}(\Sigma)^{n\times n}$$

is affine and continuous. The inversion operator

$$A \mapsto A^{-1}, \quad \{A \in W_p^{s-1-1/p}(\Sigma)^{n\times n} : \|A^{-1}\|_\infty < \infty\} \to W_p^{s-1-1/p}(\Sigma)^{n\times n}$$

is analytic by Lemma B.90. Hence, by Corollary B.86, the map

$$h \mapsto (I - hL_\Sigma)^{-1}, \quad W_p^{s-1/p}(\Sigma) \cap \mathcal{U}_{h_0} \to W_p^{s-1-1/p}(\Sigma)^{n\times n}$$

is analytic, and therefore also

$$h \mapsto |(I - hL_\Sigma)^{-1}\nabla_\Sigma h|^2, \quad W_p^{s-1/p}(\Sigma) \cap \mathcal{U}_{h_0} \to W_p^{s-1-1/p}(\Sigma)^{n\times n}$$

is analytic. Again by Lemma B.90 and Sobolev's embedding, the square root operator and the inversion operator are analytic operators from $\{u \in W_p^{s-1-1/p}(\Sigma) : \inf_\Sigma \operatorname{dist}(u(\cdot), \mathbb{R}_-) > 0\}$ to $W_p^{s-1-1/p}(\Sigma)$. Thus, in view of (4.15), we conclude that

$$h \mapsto \nu_{\Gamma_h} \circ \theta_h, \quad W_p^{s-1/p}(\Sigma) \cap \mathcal{U}_{h_0} \to W_p^{s-1-1/p}(\Sigma)$$

is analytic.

(ii) The temporal trace theorem yields the embedding

$$\mathbb{E}_h(T) \hookrightarrow C([0,T]; W_p^{4-2/p}(\Sigma)) \hookrightarrow C([0,T]; W_p^{\tau-1/p}(\Sigma)).$$

By employing Lemma 4.13 instead of Lemma B.90, assertion (ii) follows analogously. □

Now we can prove that the normal-preserving map Θ_h is a diffeomorphism and that it depends analytically on the height function h. We consider Θ_h as an element of $\mathrm{Id}_x + \mathbb{E}_\Theta{}^n$, where

$$\mathbb{E}_\Theta := H_p^{3/2}(J; H_p^2(\mathbb{R}^n)) \cap H_p^1(J; H_p^3(\mathbb{R}^n)) \cap L_p(J; H_p^4(\mathbb{R}^n)).$$

4.15. Theorem. *Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a domain, $\Sigma \subset \Omega$ be a compact smooth hypersurface, and $p \in (1,\infty)$.*

(i) Let $s \in (1 + n/p, \infty)$ and $\tau \in (1 + n/p, s]$. Then for some $\delta_{h_0} > 0$ and all height functions

$$h \in W_p^{s-1/p}(\Sigma) \cap \mathcal{U}_{h_0} \quad \textit{with } \mathcal{U}_{h_0} := \{h \in W_p^{\tau-1/p}(\Sigma) : \|h\|_{W_p^{\tau-1/p}(\Sigma)} < \delta_{h_0}\},$$

the inequality (4.13) is satisfied, the stationary normal-preserving map $\Theta_h \colon \mathbb{R}^n \to \mathbb{R}^n$ from (4.14) is an admissible map, and the map

$$h \mapsto \Theta_h - \mathrm{Id}_x, \quad W_p^{s-1/p}(\Sigma) \cap \mathcal{U}_{h_0} \to W_p^s(\mathbb{R}^n)^n$$

is analytic.

(ii) Let $p \in ((n+2)/2, \infty)$ and $\tau \in (1 + n/p, 4 - 1/p]$. Then there exists $\delta_h > 0$ such that for all $T \in (0,\infty)$ and all height functions

$$h \in \mathbb{E}_h(T) \cap \mathcal{U}_h \quad \textit{with } \mathcal{U}_h := \{h \in L_\infty(0,T; W_p^{\tau-1/p}(\Sigma)) : \|h\|_{L_\infty(0,T;W_p^{\tau-1/p}(\Sigma))} < \delta_h\}, \tag{4.21}$$

the following assertions are true:

(ii.a) The inequality (4.13) is satisfied by $h(t,\cdot)$ for all $t \in [0,T]$.

(ii.b) $\Theta_h \colon [0,T] \times \overline{\Omega} \to \overline{\Omega}$ is a normal-preserving admissible map for Σ.

(ii.c) The following maps are analytic:

$$h \mapsto \Theta_h - \mathrm{Id}_x, \qquad \mathbb{E}_h(T) \cap \mathcal{U}_h \to \mathbb{E}_\Theta(T), \tag{4.22}$$

$$h \mapsto [\partial_x \Theta_h]|_\Sigma = I_x + \partial_j(h\nu_\Sigma) \otimes \tau_\Sigma^j + g_h \otimes \nu_\Sigma, \quad \mathbb{E}_h(T) \cap \mathcal{U}_h \to \mathbb{E}_{\partial\Theta}^{n\times n}(T). \tag{4.23}$$

Proof. (i) From Lemma 4.14.(i) we infer that the map

$$h \mapsto g_h, \quad W_p^{\tau-1/p}(\Sigma) \cap \mathcal{U}_{h_0} \to W_p^{\tau-1-1/p}(\Sigma)$$

is analytic for every $\tau \in (1+n/p, s]$. Moreover,

$$\|h\nu_\Sigma\|_{W_p^{\tau-1/p}(\Sigma)} + \|g_h\|_{W_p^{\tau-1-1/p}(\Sigma)} \to 0 \quad \text{as } \|h\|_{W_p^{\tau-1/p}(\Sigma)} \to 0.$$

Therefore the map $\Theta_h - \mathrm{Id}_x$ belongs to $W_p^s(\mathbb{R}^n)^n \cap C^1(\mathbb{R}^n)^n$, satisfies $\Theta_h|_\Sigma = \theta_h$ and $\partial_{\nu_\Sigma}\Theta_h|_\Sigma = [(\nu_{\Gamma_h} \otimes \nu_{\Gamma_h}) \circ \theta_h]\nu_\Sigma$, and depends analytically on $h \in W_p^{s-1/p}(\Sigma) \cap \mathcal{U}_{h_0}$. From Lemma 4.10 we infer that $\mathfrak{S}(h\nu_\Sigma, g_h)$ has compact support in $\overline{\Omega}$, and hence $\Theta_h = \mathrm{Id}_x$ in $\mathbb{R}^n \setminus \Omega$. In order to guarantee that Θ_h is a diffeomorphism, we observe that

$$\|\partial_x\Theta_h - I_x\|_{L_\infty(\mathbb{R}^n)} \lesssim \|\mathfrak{S}(h\nu_\Sigma, g_h)\|_{W_p^\tau(\mathbb{R}^n)} \lesssim \|h\nu_\Sigma\|_{W_p^{\tau-1/p}(\Sigma)} + \|g_h\|_{W_p^{\tau-1-1/p}(\Sigma)}.$$

Hence, if $\|h\|_{W_p^{\tau-1/p}(\Sigma)}$ is sufficiently small, then $\|\partial_x\Theta_h - I\|_\infty < 1$, and thus Θ_h is a global diffeomorphism of $\mathbb{R}^n$.

(ii) It is shown in Lemma 4.13 that $\mathbb{E}_{\partial\Theta}$ is a multiplication algebra and the subset $\{u \in \mathbb{E}_{\partial\Theta} : \inf_\Sigma \operatorname{dist}(u(\cdot), \mathbb{R}_+) > 0\}$ is invariant under pointwise inversion and square root. Let $J = (0,T)$. From Lemma 4.14.(ii) we infer that $h\nu_\Sigma \in \mathbb{E}_h$ and $g_h \in \mathbb{E}_{\partial\Theta}$ defined by (4.15) depend analytically on $h \in \mathbb{E}_h(T) \cap \mathcal{U}_h$. The mixed derivative embeddings yield

$$h\nu_\Sigma \in W_p^{2-1/2p-\rho}(J; H_p^{1+2\rho}(\Sigma))^n, \quad g_h \in W_p^{2-1/2p-\rho}(J; H_p^{2\rho}(\Sigma))^n \quad \text{for all } \rho \in [0, 1-1/2p).$$

We choose $\rho := 1/2 - 1/2p$. By Lemma 4.10, the map (4.22) is well-defined and analytic. Analyticity of (4.23) follows from analyticity of $h \mapsto (h\nu_\Sigma, g_h)$ and Lemma B.10. The diffeomorphism property follows from assertion (i).

(iii) From Sobolev's embedding (B.1) we deduce

$$W_p^\theta(J; W_p^{4-\theta}(\mathbb{R}^n)) \hookrightarrow C(\overline{J}; BC^2(\mathbb{R}^n)),$$

provided that $\theta > 1/p$ and $4 - \theta - n/p > 2$. Since $p > (n+2)/2$, we have

$$H_p^{3/2}(J; H_p^2(\mathbb{R}^n)) \cap H_p^1(J; H_p^3(\mathbb{R}^n)) \hookrightarrow C^1(\overline{J}; BC^1(\mathbb{R}^n)).$$

Hence the map Θ_h is admissible. □

We complete this section with a collection of useful transformation identities for the normal-preserving admissible map Θ_h and the velocity transformation $u \circ \tilde{\Theta}_h = [\partial_x\Theta_h]\overline{u}$.

4.16. Lemma. *The relations* (4.24) *and* (4.25) *on the next page are valid.*

Proof. Most identities follow from Propositions 4.6 and 4.12 and equation (4.15). The remaining identity (4.25c) can be verified as follows.

$$(\partial_{\nu_\Gamma} w) \circ \tilde{\Theta}_h = (\nabla((\beta_h\overline{w}) \circ \tilde{\Theta}_h^{-1})|\nu_\Gamma) \circ \Theta_h = \nabla(\beta_h\overline{w}) \cdot [\partial_x\Theta_h]^{-1}\beta_h^{-1}[\partial_x\Theta_h]\nu_\Sigma = \partial_{\nu_\Sigma}\overline{w}. \quad \square$$

4.2. The transformed bulk equations

In this section we transform the momentum balance and the divergence equation

$$\rho\partial_t u + \rho(u \cdot \nabla)u - \mu\Delta u + \nabla\pi = 0 \quad \text{for } t \in J,\ x \in \Omega \setminus \Gamma(t), \tag{4.26}$$

$$\operatorname{div} u = 0 \quad \text{for } t \in J,\ x \in \Omega \setminus \Gamma(t). \tag{4.27}$$

Here $J = (0,T)$ is a bounded interval, $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is a domain, and Γ is an admissible moving hypersurface in Ω, which is induced by an admissible map $\Theta\colon J \times \overline{\Omega} \to \overline{\Omega}$ and a compact smooth hypersurface $\Sigma \subset \Omega$. We do not yet employ an explicit representation of Θ.

Let $\Theta_h \colon J \times \overline{\Omega} \to \overline{\Omega}$ denote the normal-preserving map for $\Sigma \subset \Omega$ as defined in Theorem 4.15, $\Gamma(t) = \Theta_{h(t)}(\Sigma)$, and $M_h = (I_x - hL_\Sigma)^{-1}$. Then the following identities are valid on $J \times \Sigma$:

(4.24a) $$\beta_h = (\nu_\Gamma \circ \tilde{\Theta}_h | \nu_\Sigma) = |[\partial_x \Theta_h]\nu_\Sigma| = |[\partial_x \Theta_h]^{-\top} \nu_\Sigma|^{-1},$$

(4.24b) $$\nu_\Gamma \circ \tilde{\Theta}_h = \beta_h^{-1} [\partial_x \Theta_h] \nu_\Sigma = \beta_h [\partial_x \Theta_h]^{-\top} \nu_\Sigma,$$

(4.24c) $$\tau_j^{\Gamma_h} \circ \tilde{\Theta}_h = [\partial_x \Theta_h] \tau_j^\Sigma,$$

(4.24d) $$\tau_{\Gamma_h}^j \circ \tilde{\Theta}_h = [\partial_x \Theta_h]^{-\top} \tau_\Sigma^j,$$

(4.24e) $$P_\Gamma \circ \tilde{\Theta}_h = [\partial_x \Theta_h] P_\Sigma [\partial_x \Theta_h]^{-1},$$

(4.24f) $$\partial_x \Theta_h = P_\Sigma - hL_\Sigma - \beta_h^2 M_h \nabla_\Sigma h \otimes \nu_\Sigma + \nu_\Sigma \otimes \nabla_\Sigma h + \beta_h^2 \nu_\Sigma \otimes \nu_\Sigma,$$

(4.24g) $$[\partial_x \Theta_h]^{-1} = M_h - \beta_h^2 M_h^2 \nabla_\Sigma h \otimes M_h \nabla_\Sigma h + \beta_h^2 M_h^2 \nabla_\Sigma h \otimes \nu_\Sigma - \nu_\Sigma \otimes M_h \nabla_\Sigma h.$$

Let $u = v + w\nu_\Gamma$ and $\overline{u} = \overline{v} + \overline{w}\nu_\Sigma$ be related by $u \circ \tilde{\Theta}_h = [\partial_x \Theta_h]\overline{u}$. Then we also have

(4.25a) $$v \circ \tilde{\Theta}_h = [\partial_x \Theta_h] \overline{v},$$

(4.25b) $$w \circ \tilde{\Theta}_h = \beta_h \, \overline{w},$$

(4.25c) $$\partial_{\nu_\Gamma} w \circ \tilde{\Theta}_h = \partial_{\nu_\Sigma} \overline{w},$$

(4.25d) $$V_{\Gamma_h} \circ \tilde{\Theta}_h = \beta_h \partial_t h.$$

FIGURE 4.2. Transformation identities for the normal-preserving map Θ_h.

Our first task is to derive the transformed equations

(4.28) $$\overline{\rho}\partial_t \overline{u} - \overline{\mu}\Delta \overline{u} + \nabla \overline{\pi} = F_u(\overline{u}, \overline{\pi}, \Theta) \quad \text{in } J \times (\Omega \setminus \Sigma),$$

(4.29) $$\operatorname{div} \overline{u} = F_d(\overline{u}, \Theta) \quad \text{in } J \times (\Omega \setminus \Sigma),$$

for the transformed velocity

(4.30) $$\overline{u}(t, \overline{x}) = [\partial_{\overline{x}} \Theta(t, \overline{x})]^{-1} u(t, \Theta(t, \overline{x})),$$

and the transformed pressure

(4.31) $$\overline{\pi}(t, \overline{x}) := \pi(t, \Theta(t, \overline{x})).$$

The nonlinear perturbations F_u and F_d are derived in Lemmas 4.17 and 4.19.

Second, for proving well-posedness of the transformed problem (T) with Banach's fixed point theorem, we have to *control the perturbations* F_u and F_d. To be precise, we will show that their values and their first order Fréchet derivatives can be deemed as small as we wish, by choosing T sufficiently small and by requiring that $\Theta|_{t=0}$ is sufficiently close to the identity (Lemmas 4.21 and 4.23). These perturbations are polynomial Nemytskiĭ operators with respect to the functions $(\overline{u}, \overline{\pi}, \Theta, [\partial_{\overline{x}} \Theta]^{-1})$ and some of their derivatives. In order to prove their analyticity, we employ their polynomial structure and certain T-dependent embedding estimates.

4.17. Lemma. *Assume that Θ is of class $C^1(J; C^1(\overline{\Omega})) \cap C(J; C^3(\overline{\Omega}))$ and put*

$$X(t, \overline{x}) := \Theta(t, \overline{x}), \quad \overline{X}(t, x) := \Theta(t, \cdot)^{-1}(x).$$

For given $\pi \in L_{1,loc}(J; H^1_{1,loc}(\Omega \setminus \Gamma))$ and $u \in H^1_{1,loc}(J; L_{1,loc}(\Omega; \mathbb{R}^n)) \cap L_{1,loc}(J; H^2_{1,loc}(\Omega \setminus \Gamma; \mathbb{R}^n))$ we define $\overline{u}$ and $\overline{\pi}$ as in (4.30) and (4.31). Then the identities in Figure 4.3 on the facing page are valid.

For an admissible map $\Theta\colon J \times \overline{\Omega} \to \overline{\Omega}$ and for $u(t,x) = [\partial_{\overline{x}}\Theta(t,\overline{x})]\overline{u}(t,\overline{x})$ and $\pi(t,x) = \overline{\pi}(t,\overline{x})$ with $x = X(t,\overline{x})$ and $\overline{x} = \overline{X}(t,x)$ as in Lemma 4.17, we have

$$\begin{aligned}
\partial_t \overline{X}_m &= -\partial_i \overline{X}_m\, \partial_t X_i,\\
\partial_j u_k &= \partial_j \overline{u}_k + (\partial_l X_k\, \partial_j \overline{X}_m - \delta_{kl}\delta_{jm})\partial_m \overline{u}_l + \partial_l\partial_m X_k\, \partial_j \overline{X}_m\ \overline{u}_l,\\
\partial_t u_k &= \partial_t \overline{u}_k + (\partial_l X_k - \delta_{kl})\partial_t \overline{u}_l - \partial_t X_i\, \partial_l X_k\, \partial_i \overline{X}_m\ \partial_m \overline{u}_l\\
&\quad + (\partial_t\partial_l X_k - \partial_t X_i\, \partial_l\partial_m X_k\, \partial_i \overline{X}_m)\, \overline{u}_l,\\
\operatorname{div} u &= \operatorname{div} \overline{u} + \partial_l\partial_m X_j\, \partial_j \overline{X}_m\ \overline{u}_l,\\
\Delta u_k &= \Delta \overline{u}_k + (\partial_l X_k\, \partial_j \overline{X}_m\, \partial_j \overline{X}_i - \delta_{kl}\delta_{jm}\delta_{ji})\, \partial_m\partial_i \overline{u}_l\\
&\quad + \big(\partial_l\partial_i X_k\, \partial_j \overline{X}_m\, \partial_j \overline{X}_i - \partial_i \overline{X}_m\, \partial_j\partial_r X_i\, \partial_j \overline{X}_r\, \partial_l X_k\big)\, \partial_m \overline{u}_l\\
&\quad + \big(\partial_l\partial_m\partial_i X_k\, \partial_j \overline{X}_m\, \partial_j \overline{X}_i - \partial_i \overline{X}_m\, \partial_j\partial_r X_i\, \partial_j \overline{X}_r\, \partial_l\partial_m X_k\big)\, \overline{u}_l,\\
u_j\partial_j u_k &= \partial_l X_k\, \partial_i X_j\, \partial_j \overline{X}_m\ \overline{u}_i \partial_m \overline{u}_l + \partial_l\partial_m X_k\, \partial_i X_j\, \partial_j \overline{X}_m\ \overline{u}_l \overline{u}_i,\\
\partial_j \pi &= \partial_j \overline{\pi} + (\partial_j \overline{X}_m - \delta_{jm})\partial_m \overline{\pi}.
\end{aligned}$$

Here the values of u, π, and $\overline{X}$ are taken at (t,x), and those of $\overline{u}$, $\overline{\pi}$, and X at $(t,\overline{x})$.

FIGURE 4.3. Transformed differential operators.

Proof. By the inverse function theorem we have

$$\begin{aligned}
\partial_x \overline{X}(t,x) &= \partial_x(\Theta(t,\cdot))^{-1}(x) = [\partial_{\overline{x}}\Theta(t,\overline{x})]^{-1},\\
\partial_t \overline{X}(t,x) &= -[\partial_{\overline{x}}\Theta(t,x)]^{-1}\partial_t\Theta(t,\overline{x}) = -\partial_x \overline{X}(t,x)\partial_t X(t,\overline{x}), \qquad \partial_t \overline{X}_m = -\partial_n \overline{X}_m \partial_t X_n.
\end{aligned}$$

In order to transform $\partial_j u_k$, we apply the chain rule for weak derivatives ([Hun13, Proposition 3.21]). Neglecting the dependence on t, we obtain

$$\begin{aligned}
\partial_{x_j} u_k(x) &= \partial_{x_j}\left(\partial_{\overline{x}_l} X_k(\overline{X}(x))\, \overline{u}_l(\overline{X}(x))\right)\\
&= \partial_{\overline{x}_j} \overline{u}_k(\overline{x}) + (\partial_{\overline{x}_l} X_k(\overline{x})\, \partial_{x_j} \overline{X}_m(x) - \delta_{kl}\delta_{jm})\partial_{\overline{x}_m} \overline{u}_l(\overline{x})\\
&\quad + \partial_{\overline{x}_l}\partial_{\overline{x}_m} X_k(x)\, \partial_{x_j} \overline{X}_m(x)\, \overline{u}_l(\overline{x}).
\end{aligned}$$

The remaining equations follow by straightforward computations. □

In every transformation formula in Figure 4.3, the first summand on the right-hand side is the principal part and the remaining summands are treated as perturbations. In order to abstract their polynomial structure, we employ the following convention.

4.18. Convention. For a map $f\colon E_1 \times \cdots \times E_k \to Y$ between Banach spaces $E_1, \ldots, E_k$, and Y over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and the induced Nemytskiĭ operator

$$F\colon u \mapsto f \circ u, \quad F(u_1,\ldots,u_k)(x) = f(u_1(x),\ldots,u_k(x)),$$

acting on $E_1 \times \cdots \times E_k$-valued maps $u = (u_1,\ldots,u_k)$, we write

$$\begin{aligned}
F(u) &= L(u) && \text{if } f \text{ is linear and bounded,}\\
F(u_1,\ldots,u_k) &= M(u_1^{\alpha_1},\ldots,u_k^{\alpha_k}) && \text{if } f \text{ is a monomial of degree } \alpha \in \mathbb{N}_0^k,\\
F(u_1,\ldots,u_k) &= P(u_1,\ldots,u_k) && \text{if } f \text{ is a polynomial,}\\
F(u_1,\ldots,u_k) &= P_\alpha(u_1,\ldots,u_k) && \text{if } f \text{ is a polynomial of degree at most } \alpha \in \mathbb{N}_0^k,\\
F(u_1,\ldots,u_k) &= P_{\alpha,0}(u_1,\ldots,u_k) && \text{if } f(u_1,\ldots,u_k) = P_\alpha(u_1,\ldots,u_k) \text{ and } f(0) = 0.
\end{aligned}$$

The symbols L, M, P, P_α, and $P_{\alpha,0}$ may denote a different mapping at every occurrence.

For $x = X(t,\overline{x}) = \Theta(t,\overline{x})$, $\partial_x \overline{X}(t,x) = [\partial\Theta(t,\overline{x})]^{-1}$, *and* $\tilde{\Theta}(t,\overline{x}) = (t, \Theta(t,\overline{x}))$ *we have*

$$\begin{aligned} F_u(\overline{u},\overline{\pi},\Theta) := &-((\rho\partial_t u)\circ\tilde{\Theta} - \overline{\rho}\partial_t\overline{u}) \\ &- ((\rho u\cdot\nabla)u)\circ\tilde{\Theta} \\ &- ((\nabla\pi)\circ\tilde{\Theta} - \nabla\overline{\pi}) \\ &+ ((\mu\Delta u)\circ\tilde{\Theta} - \overline{\mu}\Delta\overline{u}), \end{aligned}$$

where

$$\begin{aligned} (\partial_t u_k)\circ\tilde{\Theta} - \partial_t\overline{u}_k &= (\partial_l X_k - \delta_{lk})\partial_t\overline{u}_l - \partial_l X_k \partial_t X_n \partial_n \overline{X}_m\, \partial_m \overline{u}_l \\ &\quad + \left(\partial_t\partial_l X_k - \partial_l\partial_m X_k \partial_t X_n \partial_n \overline{X}_m\right)\overline{u}_l \\ &= M\left(\partial_{\overline{x}}\Theta - I, \partial_t\overline{u}\right) + M\left(\partial_t X, \partial_{\overline{x}} X, \partial_x\overline{X}, \partial_{\overline{x}}\overline{u}\right) \\ &\quad + M\left((\partial_t\partial_{\overline{x}} X, \partial_t X), P\left(\partial_{\overline{x}}^2 X, \partial_x\overline{X}\right), \overline{u}\right), \\ (u_j\partial_j u_k)\circ\tilde{\Theta} &= \partial_l X_k \partial_n X_j \partial_j \overline{X}_m\, \overline{u}_n \partial_m \overline{u}_l + \partial_l\partial_m X_k \partial_n X_j \partial_j \overline{X}_m\, \overline{u}_l \overline{u}_n \\ &= M\left(P\left(\partial_{\overline{x}}^2 X, \partial_{\overline{x}} X, \partial_x \overline{X}\right), \overline{u}, (\overline{u}, \partial_{\overline{x}}\overline{u})\right), \\ \Delta u\circ\tilde{\Theta} - \Delta\overline{u} &= \left(\partial_l X_k\, \partial_j\overline{X}_m \partial_j\overline{X}_n - \delta_{lk}\delta_{jm}\delta_{jn}\right)\partial_m\partial_n\overline{u}_l \\ &\quad + \left(2\partial_l\partial_n X_k \partial_j\overline{X}_m\, \partial_j\overline{X}_n + \partial_l X_k\, \Delta\overline{X}_m\right)\partial_m\overline{u}_l \\ &\quad + \left(\partial_l\partial_m\partial_n X_k\, \partial_j\overline{X}_m \partial_j\overline{X}_n - \partial_j\partial_l X_k \partial_l\partial_m X_k \partial_k\overline{X}_m \partial_j\overline{X}_l\right)\overline{u}_l \\ &= M\left(P_{(1,2),0}\left(\partial_{\overline{x}} X - I, \partial_x\overline{X} - I\right), \partial_{\overline{x}}^2\overline{u}\right) \\ &\quad + M\left(P\left(\partial_{\overline{x}}^2 X, \partial_{\overline{x}} X, \partial_x\overline{X}\right), (\overline{u}, \partial_{\overline{x}}\overline{u})\right) + M\left(\partial_{\overline{x}}^3 X, (\partial_x\overline{X})^2, \overline{u}\right), \\ (\partial_j\pi)\circ\tilde{\Theta} - \partial_j\overline{\pi} &= -(\partial_j\overline{X}_m - \delta_{jm})\partial_m\overline{\pi} = M\left(\partial_{\overline{x}} X - I, \partial_{\overline{x}}\overline{\pi}\right). \end{aligned}$$

Here the values of u, π, *and* $\overline{X}$ *are taken at* $(t,x) \in J\times(\Omega\setminus\Gamma(t))$, *and those of* $\overline{u}$, $\overline{\pi}$, *and* X *at* $(t,\overline{x}) \in J\times(\Omega\setminus\Sigma)$ *with* $x = X(t,\overline{x})$.

FIGURE 4.4. The perturbations in the transformed momentum equation.

4.2.1. The transformed momentum equation. In the next Lemma 4.19, we derive the transformed momentum equation for admissible diffeomorphisms $\Theta(t,\cdot)$. The map $F_u(u,\pi,\Theta)$ is a polynomial operator in $(u,\pi,\Theta,[\partial_x\Theta]^{-1})$. For suitable $\Theta\in\mathcal{U}_\Theta$ we obtain analyticity of F_u and smallness of a certain Fréchet derivative of F_u in Lemma 4.21. Sufficient T-dependent embeddings are given in Lemma 4.20. Later on we will specialize this result to normal-preserving diffeomorphisms $\Theta_h(t,\cdot)$ with $h\in\mathcal{U}_h$ (Corollary 4.27).

4.19. Lemma. *The momentum equation* (4.26) *corresponds to the transformed momentum equation* (4.28), *where the vector-field* $F_u(\overline{u},\overline{\pi},\Theta)\colon J\times(\Omega\setminus\Sigma)\to\mathbb{R}^n$ *is given in Figure 4.4 on the current page. Therefore* F_u *is a polynomial Nemytskiĭ operator with respect to* $(\overline{u},\overline{\pi},\Theta,[\partial_{\overline{x}}\Theta]^{-1})$ *of the form*

$$\begin{aligned} F_u(\overline{u},\overline{\pi},\Theta) = M&\left(P_{(1,2),0}\left(\partial_{\overline{x}}\Theta - I, [\partial_{\overline{x}}\Theta]^{-1} - I\right), \left(\partial_t\overline{u}, \partial_{\overline{x}}^2\overline{u}\right)\right) + M\left(\partial_{\overline{x}}\Theta - I, \partial_{\overline{x}}\overline{\pi}\right) \\ &+ M\left(P\left(\partial_{\overline{x}}^2\Theta, \partial_{\overline{x}}\Theta, [\partial_{\overline{x}}\Theta]^{-1}\right), \overline{u}, (\overline{u}, \partial_{\overline{x}}\overline{u})\right) + M\left(P\left(\partial_{\overline{x}}^2\Theta, \partial_{\overline{x}}\Theta, [\partial_{\overline{x}}\Theta]^{-1}\right), \partial_{\overline{x}}\overline{u}\right) \\ &+ M\left(\left(\partial_t\partial_{\overline{x}}\Theta, \partial_t\Theta, \partial_{\overline{x}}^3\Theta\right), P\left(\partial_{\overline{x}}^2\Theta, [\partial_{\overline{x}}\Theta]^{-1}\right), \overline{u}\right). \end{aligned}$$

Proof. This follows from Lemma 4.17 by straightforward calculations. □

In the remainder of this section we omit the bars on $\overline{u}$ and $\overline{\pi}$; that is, u and π denote the transformed velocity and pressure. We recall from Figure 4.1 that

$$\mathbb{E}_u = \{u \in H_p^1(J; L_p(\Omega)^n) \cap L_p(J; H_p^2(\Omega \setminus \Sigma)^n) : u|_{\partial\Omega} = 0, [\![u]\!] = 0 \text{ on } \Sigma\},$$
$$\mathbb{E}_\Theta = H_p^{3/2}(J; H_p^2(\mathbb{R}^n)) \cap H_p^1(J; H_p^3(\mathbb{R}^n)) \cap L_p(J; H_p^4(\mathbb{R}^n)).$$

We consider the map $(u, \pi, \Theta) \mapsto F_u(u, \pi, \Theta)$ with target space $\mathbb{F}_u$, defined for $u \in \mathbb{E}_u$, $\pi \in \mathbb{E}_\pi$, and $\Theta \in \mathbb{E}_\Theta$, provided that $[\partial_x\Theta]^{-1}$ is bounded on $J \times \Omega$. Therefore we let

$$\mathcal{U}_\Theta := \{\Theta \in \mathbb{E}_\Theta^n : \Theta|_{J\times\overline{\Omega}} \colon J \times \overline{\Omega} \to \overline{\Omega} \text{ is an admissible map}\}. \tag{4.32}$$

From Proposition 4.3 we infer that for $\Theta \in \mathbb{E}_\Theta \cap \mathcal{U}_\Theta$, the map $[\partial_x\Theta]^{-1}$ is bounded on $J \times \Omega$. In order to control the nonlinearities on small time intervals, we consider the closed subspaces

$$_0\mathbb{E}_u := \{u_\bullet \in \mathbb{E}_u : u_\bullet|_{t=0} = 0\}, \quad {_0\mathbb{E}_\Theta} := \{\eta_\bullet \in \mathbb{E}_\Theta : \eta_\bullet|_{t=0} = 0,\ \partial_t\eta_\bullet|_{t=0} = 0\}.$$

4.20. Lemma. *Let $p \in (1,\infty) \setminus \{3/2, 3\}$ and $T \in (0,\infty)$. Then the continuous embeddings*

$$\mathbb{E}_u(T) \hookrightarrow C([0,T] \times \overline{\Omega})^n \qquad \text{if } p > (n+2)/2, \tag{4.33}$$
$$\mathbb{E}_\Theta(T) \hookrightarrow H_p^1(0,T; H_p^1(\Omega)^n) \cap C([0,T]; C^2(\overline{\Omega})^n) \cap L_p(0,T; H_p^3(\Omega)^n) \qquad \text{if } p > (n+1)/2, \tag{4.34}$$
$$\mathbb{E}_\Theta \hookrightarrow C^1([0,T]; C^1(\overline{\Omega})^n) \cap C([0,T]; C^3(\overline{\Omega})^n) \qquad \text{if } p > n+2, \tag{4.35}$$

are valid, and for some $\delta_0 > 0$ and all $\delta \in (0, \delta_0]$, $T_0 > 0$, and $T \in (0, T_0]$ we have

$$\|u_\bullet\|_{L_p(0,T;H_p^1(\Omega\setminus\Sigma))} \leq T^\delta C(\delta, T_0)\|u_\bullet\|_{_0\mathbb{E}_u(T)} \qquad \text{if } p > 2, \tag{4.36}$$
$$\|u_\bullet\|_{_0C([0,T]\times\overline{\Omega})} \leq T^\delta C(\delta, T_0)\|u_\bullet\|_{_0\mathbb{E}_u(T)} \qquad \text{if } p > (n+2)/2, \tag{4.37}$$
$$\|\eta_\bullet\|_{_0H_p^1(0,T;H_p^1(\Omega))\cap_0C([0,T];C^2(\overline{\Omega}))\cap L_p(0,T;H_p^3(\Omega))} \leq T^\delta C(\delta, T_0)\|\eta_\bullet\|_{_0\mathbb{E}_\Theta(T)} \qquad \text{if } p > (n+1)/2. \tag{4.38}$$

Proof. We proceed as in the proof of Lemma 3.19.

Assertions (4.33) and (4.37) follow from Proposition B.44, (B.2), (3.32c), and (3.32d), since

$$\begin{aligned}\mathbb{E}_u &\hookrightarrow W_p^{\rho-\tau}(0,T; W_p^{2(1-\rho)}(\Omega\backslash\Sigma))^n \\ &\hookrightarrow W_p^{1/p+\varepsilon_t}(0,T; W_p^{n/p+\varepsilon_s}(\Omega\backslash\Sigma))^n \hookrightarrow C([0,T]; BUC(\Omega\backslash\Sigma))^n\end{aligned}$$

for some τ, ε_t, $\varepsilon_s > 0$, and $\rho \in (0,1)$, if $\rho > 1/p$ and $2(1-\rho) > n/p$, and this is possible if $p > (n+2)/2$. Since $u \in \mathbb{E}_u$ satisfies $[\![u]\!] = 0$, we obtain $u \in C([0,T] \times \overline{\Omega})^n$. Estimate (4.36) follows from Lemma 3.19.(i). Again by Proposition B.44 and Sobolev's embedding (B.1) we have

$$\mathbb{E}_\Theta \hookrightarrow W_p^\theta(J; W_p^{4-\theta}(\mathbb{R}^n)) \hookrightarrow C(\overline{J}; BC^2(\mathbb{R}^n)),$$

provided that $\theta > 1/p$ and $4 - \theta - n/p > 2$, and this is possible if $p > (n+1)/2$. Therefore (4.34) is valid, and estimate (4.38) also follows from (B.2), (3.32c), and (3.32d). The embedding (4.35) can be verified similarly. □

We are ready to control the perturbation F_u.

4.21. Lemma. *Let $p \in (n+2,\infty)$ and $T \in (0,\infty)$. Then the map*

$$F_u \colon \{(u,\pi,\Theta) \in \mathbb{E}_u(T) \times \mathbb{E}_\pi(T) \times \mathbb{E}_\Theta(T) : \Theta \in \mathcal{U}_\Theta\} \to \mathbb{F}_u(T)$$

is analytic and has the following properties:

(i) For given $T_0 \in (0,\infty)$, $R \in [1,\infty)$, $u \in \mathbb{E}_u(T_0)$, $\pi \in \mathbb{E}_\pi(T_0)$, and $\Theta \in \mathbb{E}_\Theta(T_0) \cap \mathcal{U}_\Theta$, we have

$$\|F_u(u,\pi,\Theta)\|_{\mathbb{F}_u(T)} \to 0 \quad \text{as } T \to 0,\ \|\partial_x\Theta - I_x\|_{C([0,T]\times\overline{\Omega})} \to 0,$$

and this convergence is uniform with respect to

$$\|u\|_{\mathbb{E}_u(T_0)} + \|\pi\|_{\mathbb{E}_\pi(T_0)} + \|\Theta\|_{\mathbb{E}_\Theta(T_0)} + \|[\partial_x\Theta]^{-1}\|_{C([0,T_0]\times\overline{\Omega})} \leq R. \tag{4.39}$$

(ii) For given $T_0 \in (0,\infty)$, $R \in [1,\infty)$, $u_* \in \mathbb{E}_u(T_0)$, and $\Theta_* \in \mathbb{E}_\Theta(T_0)$, the map

$$(u_\bullet, \pi, \eta_\bullet) \mapsto F_u(u_* + u_\bullet, \pi, \Theta_* + \eta_\bullet),$$
$$\{(u_\bullet, \pi, \eta_\bullet) \in {}_0\mathbb{E}_u(T) \times \mathbb{E}_\pi(T) \times {}_0\mathbb{E}_\Theta(T) : \Theta_* + \eta_\bullet \in \mathcal{U}_\Theta\} \to \mathbb{F}_u(T),$$

satisfies

$$\|\partial_{(u_\bullet,\pi,\eta_\bullet)} F_u(u_* + u_\bullet, \pi, \Theta_* + \eta_\bullet)\|_{\mathcal{B}({}_0\mathbb{E}_u(T)\times\mathbb{E}_\pi(T)\times{}_0\mathbb{E}_\Theta(T);\mathbb{F}_u(T))} \to 0$$
$$\text{as } T \to 0,\ \|\partial_x\Theta - I_x\|_{C([0,T]\times\overline{\Omega})} \to 0,$$

and this convergence is uniform with respect to

$$\|(u_*, u_\bullet)\|_{\mathbb{E}_u(T_0)} + \|\pi\|_{\mathbb{E}_\pi(T_0)} + \|(\Theta_*, \eta_\bullet)\|_{\mathbb{E}_\Theta(T_0)} + \|[\partial_x(\Theta_* + \eta_\bullet)]^{-1}\|_{C([0,T_0]\times\overline{\Omega})} \le R. \tag{4.40}$$

Proof. Lemma 4.19 shows that $F_u(u,\pi,\Theta)$ depends polynomially on $(u,\pi,\Theta,[\partial_x\Theta]^{-1})$. With Theorem 4.15 and the embeddings (4.33) and (4.34), it is straightforward to check that F_u is analytic. It is also bounded and uniformly continuous with respect to (4.39).

(i) Assume that $T \le 1$. First, Hardy's inequality (B.4) yields the estimate

$$\begin{aligned}\|u\|_{\mathbb{F}_u(T)} &\le \left(\int_0^T \|u_0\|_{L_p(\Omega)}^p\,dt\right)^{1/p} + \left(\int_0^T \left\|\frac{T}{t}\int_0^t \partial_s u(s)\,ds\right\|_{L_p(\Omega)}^p dt\right)^{1/p} \\ &\le CT^{1/p}\left(\|u_0\|_{L_p(\Omega)} + \|\partial_t u\|_{\mathbb{F}_u(T_0)}\right) \le CT^{1/p}R.\end{aligned} \tag{4.41}$$

Here the latter inequality follows from $\mathbb{E}_u(T_0) \hookrightarrow C([0,T_0]\times\overline{\Omega}) \hookrightarrow C([0,T_0];L_p(\Omega))$ (see also (4.33)). Second, with inequality (3.32c) and the mixed derivative embeddings we obtain

$$\|\partial_x u\|_{\mathbb{F}_u(T)} \le CT^{1/p}\|u_0\|_{H_p^1(\Omega)} + CT^\alpha [\![u - u_0]\!]_{{}_0W_p^\alpha(0,T;H_p^1(\Omega))} \le CT^{1/p}R \tag{4.42}$$

for some $\alpha \in (1/p, 1/2)$. Hence, by using embedding (4.35) and choosing $T > 0$ sufficiently small, we can control those terms in $F_u(u,\pi,\Theta)$ which contain a lower-order factor u or $\partial_x u$. The leading-order terms $\partial_t u$, $\partial_x^2 u$, and $\partial_x\pi$ only appear in products with a factor $\partial_x\Theta - I_x$ or $[\partial_x\Theta]^{-1} - I_x = [\partial_x\Theta]^{-1}[I_x - \partial_x\Theta]$, and can therefore be controlled with the smallness of $\|\partial_x\Theta - I_x\|_{C([0,T]\times\overline{\Omega})}$.

(ii) For given $u_* \in \mathbb{E}_u(T_0)$, $u_\bullet \in {}_0\mathbb{E}_u(T_0)$, $\Theta_* \in \mathbb{E}_\Theta(T_0)$, and $\eta_\bullet \in {}_0\mathbb{E}_\Theta(T_0)$, we let

$$u = u_* + u_\bullet \in \mathbb{E}_u(T_0), \quad \Theta = \Theta_* + \eta_\bullet \in \mathbb{E}_\Theta(T_0).$$

With Convention 4.18 we can express the derivative of $u_\bullet \mapsto F_u(u_* + u_\bullet, \pi, \Theta)$ applied to $\tilde{u}_\bullet \in {}_0\mathbb{E}_u(T)$ as

$$\begin{aligned}[\partial_{u_\bullet} F_u(u,\pi,\Theta)]\tilde{u}_\bullet = {}& M\left(P_{(1,2),0}\left(\partial_x\Theta - I, [\partial_x\Theta]^{-1} - I\right), \left(\partial_t\tilde{u}_\bullet, \partial_x^2\tilde{u}_\bullet\right)\right) \\ &+ M\left(P\left(\partial_x^2\Theta, \partial_x\Theta, [\partial_x\Theta]^{-1}\right), \tilde{u}_\bullet, (u, \partial_x u)\right) \\ &+ M\left(P\left(\partial_x^2\Theta, \partial_x\Theta, [\partial_x\Theta]^{-1}\right), u, (\tilde{u}_\bullet, \partial_x\tilde{u}_\bullet)\right) \\ &+ M\left(P\left(\partial_x^2\Theta, \partial_x\Theta, [\partial_x\Theta]^{-1}\right), \partial_x\tilde{u}_\bullet\right) \\ &+ M\left(P\left(\partial_t\partial_x\Theta, \partial_t\Theta, \partial_x^3\Theta, \partial_x^2\Theta, [\partial_x\Theta]^{-1}\right), \tilde{u}_\bullet\right).\end{aligned}$$

Together with (4.36) and (4.37), a straightforward estimation yields

$$\|\partial_{u_\bullet} F_u(u,\pi,\Theta)\|_{\mathcal{B}({}_0\mathbb{E}_u(T);\mathbb{F}_u(T))} \to 0 \quad \text{as } T \to 0,\ \|\partial_x\Theta - I_x\|_{C([0,T]\times\overline{\Omega})} \to 0,$$

uniformly with respect to (4.40).

Next, the derivative of $\mathbb{E}_\pi(T) \ni \pi \mapsto F_u(u,\pi,\Theta)$ applied to $\tilde{\pi} \in \mathbb{E}_\pi(T)$ is given by

$$[\partial_\pi F_u(u,\pi,\Theta)]\tilde{\pi} = M\left([\partial_x\Theta]^{-1} - I, \partial_x\tilde{\pi}\right),$$

and therefore

$$\|\partial_\pi F_u(u,\pi,\Theta)\|_{\mathcal{B}(\mathbb{E}_\pi(T);\mathbb{F}_u(T))} \to 0 \quad \text{as } \|\partial_x\Theta - I_x\|_{C([0,T]\times\overline{\Omega})} \to 0.$$

Finally, we study the derivative of ${}_0\mathbb{E}_\Theta(T) \ni \eta_\bullet \mapsto F_u(u, \pi, \Theta_* + \eta_\bullet)$. With

$$[\partial_{\eta_\bullet}([\partial_x\Theta]^{-1})]\tilde{\eta}_\bullet = -[\partial_x\Theta]^{-1}[\partial_x\tilde{\eta}_\bullet][\partial_x\Theta]^{-1} = M\left(\partial_x\tilde{\eta}_\bullet, ([\partial_x\Theta]^{-1})^2\right), \tag{4.43}$$

we obtain

$$[\partial_{\eta_\bullet}P_{(1,2),0}(\partial_x\Theta - I, [\partial_x\Theta]^{-1} - I)]\tilde{\eta}_\bullet = M\left(\partial_x\tilde{\eta}, P(\partial_x\Theta, [\partial_x\Theta]^{-1})\right).$$

Hence

$$\begin{aligned}[\partial_{\eta_\bullet}F_u(u,\pi,\Theta)]\tilde{\eta}_\bullet = {} & M\left(\partial_x\tilde{\eta}_\bullet, P\left(\partial_x\Theta, [\partial_x\Theta]^{-1}\right), (\partial_t u, \partial_x^2 u, \partial_x\pi)\right) \\ & + M\left((\partial_x^2\tilde{\eta}_\bullet, \partial_x\tilde{\eta}_\bullet), P\left(\partial_x\Theta, \partial_x\Theta, [\partial_x\Theta]^{-1}\right), P_1(u), (u, \partial_x u)\right) \\ & + M\left((\partial_t\partial_x\tilde{\eta}_\bullet, \partial_t\tilde{\eta}_\bullet, \partial_x^3\tilde{\eta}_\bullet), P\left(\partial_x^2\Theta, [\partial_x\Theta]^{-1}\right), u\right) \\ & + M\left((\partial_t\partial_x\Theta, \partial_t\Theta, \partial_x^3\Theta), (\partial_x^2\tilde{\eta}_\bullet, \partial_x\tilde{\eta}_\bullet), P\left(\partial_x^2\Theta, [\partial_x\Theta]^{-1}\right), u\right).\end{aligned}$$

By a straightforward estimation and by using (4.33), (4.34), and (4.38) we conclude that

$$\|\partial_{\eta_\bullet}F_u(u,\pi,\Theta)\|_{\mathcal{B}({}_0\mathbb{E}_\Theta(T);\mathbb{F}_u(T))} \to 0 \quad \text{as } T \to 0.$$ □

4.2.2. The transformed divergence equation. We have transformed the equation

$$\operatorname{div} u(t,x) = 0 \quad \text{for } t \in J,\ x \in \Omega \setminus \Gamma(t),$$

to the following equation for $(\overline{u}, \Theta)$ with $u \circ \tilde{\Theta} = [\partial_x\Theta]\overline{u}$:

$$\operatorname{div} \overline{u} = F_d(\overline{u}, \Theta) \quad \text{in } J \times (\Omega \setminus \Sigma).$$

Here the perturbation $F_d(\overline{u}, \Theta)\colon J \times \Omega \to \mathbb{R}$ is given by

$$F_d(\overline{u},\Theta)(t,\overline{x}) = -\partial_l\partial_m\Theta_j(t,\overline{x})[\partial_x\Theta(t,\overline{x})]^{-1}_{mj}\,\overline{u}_l(t,\overline{x}) = M\left(\partial_x^2\Theta, [\partial_x\Theta]^{-1}, \overline{u}\right). \tag{4.44a}$$

Again we replace $\overline{u}$ by u. Abels and Wilke [AW13] noticed that the identities $\operatorname{div} u = F_d(u, \Theta)$ and $\int_\Omega \operatorname{div} u\, dx = 0$ imply that the integral $\int_\Omega F_d(u,\Theta)dx$ vanishes, but this might be false for arbitrary $u \in \mathbb{E}_u$. Therefore we replace $F_d(u, \Theta)$ by its part

$$\tilde{F}_d(u,\Theta) := F_d(u,\Theta) - \frac{1}{|\Omega|}\int_\Omega F_d(u,\Theta)\,dx \tag{4.44b}$$

with vanishing mean value $|\Omega|^{-1}\int_\Omega \tilde{F}_d(u,\Theta)\,dx = 0$. We will exploit the fact that $F_d(u,\Theta)$ is trilinear in $(u, \partial_x^2\Theta, [\partial_x\Theta]^{-1})$, and with the embeddings in Lemma 4.22 we will show that $\tilde{F}_d$ is analytic and can be controlled in a similar way as F_u (Lemma 4.23).

4.22. Lemma. *The embedding*

$$\mathbb{E}_\Theta \hookrightarrow H^1_p(0,T;C^2(\overline{\Omega}))^n \cap C([0,T];H^3_p(\Omega))^n \quad \text{if } p > n, \tag{4.45}$$

is continuous, and for some $\delta_0 > 0$ and all $\delta \in (0,\delta_0]$, $T_0 > 0$, and $T \in (0, T_0]$, we have

$$\|\eta_\bullet\|_{{}_0H^1_p(0,T;C^2(\overline{\Omega}))\cap{}_0C([0,T];H^3_p(\overline{\Omega}))} \le T^\delta C(\delta, T_0)\|\eta_\bullet\|_{{}_0\mathbb{E}_\Theta(T)} \quad \text{if } p > n+2. \tag{4.46}$$

Proof. Sobolev's embedding (B.1) yields (4.45). Next, Hölder's inequality yields

$$\|\eta_\bullet\|_{{}_0C([0,T];H^3_p(\Omega))} \le T^{1-1/p}\|\partial_t\eta_\bullet\|_{L_p(0,T;H^3_p(\Omega))} \le T^{1-1/p}\|\eta_\bullet\|_{{}_0\mathbb{E}_\Theta(T)}.$$

With the embeddings (B.3) and (B.1), the estimates (3.32b), (3.32c), and (3.32d), and the mixed derivative embeddings (Proposition B.44) we obtain

$$\begin{aligned}\|\eta_\bullet\|_{{}_0H^1_p(0,T;C^2(\overline{\Omega}))} &\lesssim T^{1/p+\varepsilon}\|\eta_\bullet\|_{{}_0W^{1+1/p+\varepsilon}_p(0,T;C^2(\overline{\Omega}))} \\ &\lesssim T^{1/p+\varepsilon}\|\eta_\bullet\|_{{}_0H^{3/2-\rho}_p(0,T;H^{2+2\rho}_p(\Omega))} \lesssim T^{1/p+\varepsilon}\|\eta_\bullet\|_{{}_0\mathbb{E}_\Theta(T)},\end{aligned}$$

with suitable numbers $\varepsilon > 0$ and $\rho \in (0, 1/2)$ which exist if $p > n+2$. Thus (4.46) is valid. □

4.23. Lemma. *For $p \in (n, \infty)$ and $T \in (0, \infty)$, the map*

$$\tilde{F}_d \colon \{(u, \Theta) \in \mathbb{E}_u(T) \times \mathbb{E}_\Theta(T) : \Theta \in \mathcal{U}_\Theta\} \to \mathbb{F}_d(T)$$

is analytic. Assume in addition that $p > n + 2$. Then the following assertions are valid:

(i) For given $T_0 \in (0, \infty)$, $R \in [1, \infty)$, $u \in \mathbb{E}_u(T_0)$, and $\Theta \in \mathbb{E}_\Theta(T_0) \cap \mathcal{U}_\Theta$, we have

$$\|\tilde{F}_d(u, \Theta)\|_{\mathbb{F}_d(T)} \to 0 \quad \text{as } T \to 0,\ \|\partial_x \Theta - I_x\|_{C([0,T];C^1(\overline{\Omega}))} \to 0,$$

and this convergence is uniform with respect to

$$\|u\|_{\mathbb{E}_u(T_0)} + \|\Theta\|_{\mathbb{E}_\Theta(T_0)} + \|[\partial_x \Theta]^{-1}\|_{C([0,T_0]\times\overline{\Omega})} \le R.$$

(ii) For given $T_0 \in (0, \infty)$, $R \in [1, \infty)$, $u_* \in \mathbb{E}_u(T_0)$, $u_\bullet \in {}_0\mathbb{E}_u(T_0)$, $\Theta = \Theta_* + \eta_\bullet \in \mathbb{E}_\Theta(T_0) \cap \mathcal{U}_\Theta$ with $\Theta_* \in \mathbb{E}_\Theta(T_0)$, and $\eta_\bullet \in {}_0\mathbb{E}_\Theta(T_0)$, we have

$$\|\partial_{(u_\bullet,\eta_\bullet)} \tilde{F}_d(u_* + u_\bullet, \Theta_* + \eta_\bullet)\|_{\mathcal{B}({}_0\mathbb{E}_u(T)\times{}_0\mathbb{E}_\Theta(T);\mathbb{F}_d(T))} \to 0 \quad \text{as } T \to 0,\ \|\partial_x^2 \Theta\|_{C([0,T];C^1(\overline{\Omega}))} \to 0,$$

and this convergence is uniform with respect to

$$\|(u_*, u_\bullet)\|_{\mathbb{E}_u(T_0)} + \|(\Theta_*, \eta_\bullet)\|_{\mathbb{E}_\Theta(T_0)} + \|[\partial_x(\Theta_* + \eta_\bullet)]^{-1}\|_{C([0,T_0]\times\overline{\Omega})} \le R.$$

Proof. The divergence theorem implies

$$\int_\Omega \operatorname{div} f \, \varphi \, dx = -\int_\Omega f \cdot \nabla\varphi \, dx, \quad \text{for } \varphi \in \dot{H}^1_{p'}(\Omega),\ f \in H^1_{p,0}(\Omega).$$

Therefore we can extend the divergence operator to a bounded operator on $L_p(\Omega)$ such that

$$\|\operatorname{div} f\|_{\dot{H}^1_{p'}(\Omega)^*} \le \|f\|_{L_p(\Omega)} \quad \text{for } f \in L_p(\Omega). \tag{4.47}$$

For $f \in L_p(\Omega)$ and $\varphi \in C_c^\infty(\mathbb{R}^n)$, we have

$$\int_\Omega (f - \langle f\rangle_\Omega)\varphi \, dx = \int_\Omega (f - \langle f\rangle_\Omega)(\varphi - \langle \varphi\rangle_\Omega) \, dx.$$

Hence the Poincaré-Wirtinger inequality $\|\varphi - \langle\varphi\rangle_\Omega\|_{p'} \le C_{PW}\|\nabla\varphi\|_{p'}$ for $\varphi \in \dot{H}^1_{p'}(\Omega)$ implies

$$\|f - \langle f\rangle_\Omega\|_{\dot{H}^1_{p'}(\Omega)^*} \le C_{PW}\|f - \langle f\rangle_\Omega\|_p \le C_{PW}\left(1 + |\Omega|^{-1/p}\right)\|f\|_p \quad \text{for } f \in L_p(\Omega). \tag{4.48}$$

The inequality (4.48) implies

$$\|f - \langle f\rangle_\Omega\|_{H^1_p(0,T;\dot{H}^1_{p'}(\Omega)^*)} \le C_{PW}\left(1 + |\Omega|^{-1/p}\right)\|f\|_{H^1_p(0,T;L_p(\Omega))}. \tag{4.49}$$

From the embedding $H^1_p(0,T;X) \hookrightarrow C([0,T];X)$ we infer that pointwise multiplication

$$\bullet \colon H^1_p(0,T;L_\infty(\Omega)) \times H^1_p(0,T;L_p(\Omega)) \to H^1_p(0,T;L_p(\Omega))$$

is continuous and that $H^1_p(0,T;L_\infty(\Omega))$ is a multiplication algebra. Moreover,

$$\bullet \colon L_\infty(0,T;H^1_p(\Omega)) \times L_p(0,T;H^1_p(\Omega)) \to L_p(0,T;H^1_p(\Omega))$$

is continuous and $L_\infty(0,T;H^1_p(\Omega))$ is a multiplication algebra for $p > n$. Thus

$$Y := H^1_p(0,T;L_\infty(\Omega)) \cap L_\infty(0,T;H^1_p(\Omega))$$

also is a multiplication algebra, and from (4.49) we infer that

$$\|gf - \langle gf\rangle_\Omega\|_{\mathbb{F}_d} \lesssim \|gf\|_{H^1_p(J\times\Omega)} \lesssim \|g\|_Y\|f\|_{H^1_p(J\times\Omega)}. \tag{4.50}$$

Estimate (4.50) and multiplication in Y imply that the trilinear map

$$(A, B, u) \mapsto a_{lmj}b_{mj}u_l - \langle a_{lmj}b_{mj}u_l\rangle_\Omega, \quad Y^{n\times n\times n} \times Y^{n\times n} \times H^1_p((0,T)\times\Omega)^n \to \mathbb{F}_d(T)$$

is continuous. The map $\Theta \mapsto A := \partial_x^2\Theta$, $\mathbb{E}_\Theta \to Y^{n\times n\times n}$ is linear and bounded by (4.45), and the map $\Theta \mapsto B := [\partial_x\Theta]^{-1}, \mathcal{U}_\Theta \to Y^{n\times n}$ is analytic by Proposition B.88. Therefore $\tilde{F}_d$ is analytic.

(i) With estimate (4.42) we obtain

$$\|a_{lmj}b_{mj}u_l\|_{L_p(J;H^1_p(\Omega))} \le \|A\|_Y\|B\|_Y \cdot T^{1/p}C(T_0)\|u\|_{\mathbb{E}_u(T_0)}, \tag{4.51a}$$

$$\|\partial_t(a_{lmj}b_{mj})u_l\|_{L_p(J\times\Omega)} \le (\|A_t\|_p\|B\|_\infty + \|A\|_\infty\|B_t\|_p) \cdot T^{1/p}C(T_0)\|u\|_{\mathbb{E}_u(T_0)}, \tag{4.51b}$$

$$\|a_{lmj}b_{mj}\partial_t u_l\|_{L_p(J\times\Omega)} \le \|A\|_\infty\|B\|_\infty\|u\|_{\mathbb{E}_u(T)}. \tag{4.51c}$$

Hence the first estimate in (4.50) yields the assertion.

(ii) With (4.44a) and (4.43) we obtain the partial Fréchet derivatives

$$[\partial_{u_\bullet}F_d(u_*+u_\bullet,\Theta_*+\eta_\bullet)]\tilde u_\bullet = F_d(\tilde u_\bullet,\Theta) = M\left(\partial_x^2\Theta,[\partial_x\Theta]^{-1},\tilde u_\bullet\right),$$

$$[\partial_{\eta_\bullet}F_d(u_*+u_\bullet,\Theta_*+\eta_\bullet)]\tilde\eta_\bullet = M\left(\partial_x^2\tilde\eta_\bullet,[\partial_x\Theta]^{-1},u\right) + M\left(\partial_x^2\Theta,\left([\partial_x\Theta]^{-1}\right)^2,\partial_x\tilde\eta_\bullet,u\right).$$

From (4.46) with $p>n+2$ we infer that

$$\|\partial_x^j\eta_\bullet\|_Y \le T^\delta C(\delta,T_0)\|\eta_\bullet\|_{_0\mathbb{E}_\Theta(T)} \text{ for } j\in\{1,2\}.$$

This estimate and (4.51) yield the assertion. □

4.3. The transformed interface equations

In this section we transform the interface momentum balance

$$-[\![T(u,\pi)]\!]\nu_\Gamma - \operatorname{div}_\Gamma T_\Gamma(u) = 0 \quad \text{on } \Gamma(t),\ t\in J, \tag{4.52}$$

which was derived on page 19. We assume that the unknown moving interface is represented as $\Gamma(t)=\Gamma_h(t)=\Theta_h(\{t\}\times\Sigma)$ in terms of the unknown height function h and the normal-preserving map Θ_h from page 99 and Theorem 4.15.(ii). Our goal is to decompose (4.52) into a principal linear part and a remaining nonlinear part, and to handle the latter as a perturbation with respect to the function spaces on page 94. An explicit description of these perturbations is given on the following page. As the main result of this section we prove that the nonlinear perturbations can be deemed as small as we wish provided that the time interval $J=(0,T)$ and the initial height function $h_0=h|_{t=0}$ are sufficiently small (Lemma 4.26). In Corollaries 4.27 and 4.28 we prove the corresponding results for the transformed bulk equations by specializing Lemmas 4.21 and 4.23.

Let us take a closer look at (4.52). From the identities (1.19) and (1.22) we recall that

$$\begin{cases} T(u,\pi) = 2\mu D(u) - \pi I, \\ D(u) = 2^{-1}(\nabla u + [\nabla u]^\top), \\ T_\Gamma(u) = \sigma P_\Gamma + (\lambda_s-\mu_s)(\operatorname{div}_\Gamma u)P_\Gamma + 2\mu_s D_\Gamma(u), \\ D_\Gamma(u) = 2^{-1}P_\Gamma(\nabla_\Gamma u + [\nabla_\Gamma u]^\top)P_\Gamma. \end{cases} \tag{4.53}$$

Define a tangential vector field $N_v(u,\Gamma)$ and a scalar field $N_w(u,\pi,\Gamma)$ by

$$\begin{aligned} N_v(u,\Gamma) &:= -P_\Gamma[\![T(u,0)]\!]\nu_\Gamma - P_\Gamma\operatorname{div}_\Gamma T_\Gamma(u) \\ &= -([\![\mu\underline{\partial_\nu v}]\!] + [\![\mu]\!][\nabla_\Gamma v]\nu_\Gamma + [\![\mu]\!]\nabla_\Gamma w) \\ &\quad - (\mu_s\underline{\widetilde\Delta_\Gamma v} + \lambda_s\underline{\nabla_\Gamma\operatorname{div}_\Gamma v} + (\mu_s+\lambda_s)w\underline{\nabla_\Gamma H_\Gamma} + [(\mu_s-\lambda_s)H_\Gamma - 2\mu_s L_\Gamma]\nabla_\Gamma w), \\ N_w(u,\pi,\Gamma) &:= -\nu_\Gamma\cdot[\![T(u,\pi)]\!]\nu_\Gamma - \nu_\Gamma\cdot\operatorname{div}_\Gamma T_\Gamma(u) \\ &= -(2[\![\mu\underline{\partial_\nu w}]\!] - \underline{[\![\pi]\!]}) \\ &\quad - (\sigma\underline{H}_\Gamma + (\lambda_s-\mu_s)\,\underline{\operatorname{div}_\Gamma u\,H_\Gamma} + 2\mu_s\,\underline{D_\Gamma(u):L_\Gamma}). \end{aligned}$$

Here we have again used the decomposition $u=v+w\nu_\Gamma$ near Γ, and the underlined terms are considered as the principal part with respect to the chosen function spaces. Thus, the interface momentum balance (4.52) can be written as

$$N_v(u,\Gamma) + N_w(u,\pi,\Gamma)\nu_\Gamma = 0 \quad \text{on } \Gamma(t),\ t\in J.$$

For given

$$z_* = (\overline{u}_*, \overline{\pi}_*, h_*) \in \tilde{\mathbb{E}}, \quad z_\bullet = (\overline{u}_\bullet, \overline{\pi}_\bullet, h_\bullet) \in {}_0\tilde{\mathbb{E}}, \quad \text{with } h = h_* + h_\bullet \in \mathcal{U}_h,$$

we define Θ_h as in Theorem 4.15, $\tilde{\Theta}_h(t,x) := (t, \Theta_h(t,x))$, $\Gamma_h(t) := \Theta_h(t,\Sigma)$, and

$$\overline{u} := \overline{u}_* + \overline{u}_\bullet, \quad u_h := ([\partial_x \Theta_h]\overline{u}) \circ \tilde{\Theta}_h^{-1},$$

and decompose $u_h = v_h + w_h \nu_{\Gamma_h}$ near Γ_h, and $\overline{u}_* = \overline{v}_* + \overline{w}_* \nu_\Sigma$ and $\overline{u}_\bullet = \overline{v}_\bullet + \overline{w}_\bullet \nu_\Sigma$ near Σ. Then the maps G_v and G_w from (4.57) have the representations

$$\begin{aligned}
&G_v(z_\bullet; z_*) = G_v(\overline{u}_\bullet, h_\bullet; \overline{u}_*, h_*) \\
&= [\![\mu\{[\partial_x\Theta_h]^{-1}(\partial_{\nu_{\Gamma_h}} v_h) \circ \tilde{\Theta}_h - \partial_\nu \overline{v}_\bullet\}]\!] && \text{(4.55a)}\\
&\quad + \mu_s\{[\partial_x\Theta_h]^{-1}(\widetilde{\Delta}_{\Gamma_h} v_h) \circ \tilde{\Theta}_h - \widetilde{\Delta}_\Sigma \overline{v}_\bullet\} && \text{(4.55b)}\\
&\quad + \lambda_s\{[\partial_x\Theta_h]^{-1}(\nabla_{\Gamma_h} \operatorname{div}_{\Gamma_h} v_h) \circ \tilde{\Theta}_h - \nabla_\Sigma \operatorname{div}_\Sigma \overline{v}_\bullet\} && \text{(4.55c)}\\
&\quad + (\lambda_s + \mu_s)\{[\partial_x\Theta_h]^{-1}(w_h \nabla_{\Gamma_h} H_{\Gamma_h}) \circ \tilde{\Theta}_h - \overline{w}_* \nabla_\Sigma \Delta_\Sigma h_\bullet\} && \text{(4.55d)}\\
&\quad + [\partial_x\Theta_h]^{-1}([(\mu_s - \lambda_s) H_{\Gamma_h} - 2\mu_s L_{\Gamma_h}] \nabla_{\Gamma_h}(w_h)) \circ \tilde{\Theta}_h && \text{(4.55e)}\\
&\quad + [\![\mu]\!][\partial_x\Theta_h]^{-1}([\nabla_{\Gamma_h} v_h]\nu_{\Gamma_h}) \circ \tilde{\Theta}_h && \text{(4.55f)}\\
&\quad + [\![\mu]\!]\{[\partial_x\Theta_h]^{-1}(\nabla_{\Gamma_h} w_h) \circ \tilde{\Theta}_h - \nabla_\Sigma \overline{w}_\bullet\}, && \text{(4.55g)}
\end{aligned}$$

$$\begin{aligned}
&G_w(z_\bullet; z_*) = G_w(\overline{u}_\bullet, h_\bullet; \overline{u}_*, \overline{\pi}_*, h_*) \\
&= 2[\![\mu \partial_{\nu_\Sigma} \overline{w}_*]\!] + [\![\overline{\pi}_*]\!] && \text{(4.56a)}\\
&\quad + \sigma\{H_{\Gamma_h} \circ \tilde{\Theta}_h - \Delta_\Sigma h_\bullet\} && \text{(4.56b)}\\
&\quad + (\lambda_s - \mu_s)\{(\operatorname{div}_{\Gamma_h} u_h\, H_{\Gamma_h}) \circ \tilde{\Theta}_h - H_\Sigma \operatorname{div}_\Sigma \overline{v}_\bullet - (\operatorname{div}_\Sigma \overline{v}_* - 2H_\Sigma \overline{w}_*)\Delta_\Sigma h_\bullet\} && \text{(4.56c)}\\
&\quad + 2\mu_s\{(D_{\Gamma_h}(u_h) : L_{\Gamma_h}) \circ \tilde{\Theta}_h - D_\Sigma(\overline{v}) : L_\Sigma - [D_\Sigma(\overline{v}_*) - 2\overline{w}_* L_\Sigma] : \nabla_\Sigma^2 h_\bullet\}. && \text{(4.56d)}
\end{aligned}$$

FIGURE 4.5. The perturbations G_v and G_w.

Next, we derive the transformed version of (4.52). For given transformed functions $\overline{u} \in \mathbb{E}_u$, $\overline{\pi} \in \mathbb{E}_\pi$, and $h \in \mathbb{E}_h \cap \mathcal{U}_h$, and with $\tilde{\Theta}_h(t,x) = (t, \Theta_h(t,x))$, we define

$$\begin{aligned}
u_h &:= ([\partial_x\Theta_h]\overline{u}) \circ \tilde{\Theta}_h^{-1}, & \overline{N}_v(\overline{u}, h) &:= [\partial_x\Theta_h]^{-1} N_v(u_h, \Gamma_h) \circ \tilde{\Theta}_h,\\
\pi_h &:= \overline{\pi} \circ \tilde{\Theta}_h^{-1}, & \overline{N}_w(\overline{u}, \overline{\pi}, h) &:= N_w(u_h, \pi_h, \Gamma_h) \circ \tilde{\Theta}_h.
\end{aligned}$$

Then the transformed interface momentum balance is given by

$$\overline{N}_v(\overline{u}, h) + \overline{N}_w(\overline{u}, \overline{\pi}, h)\nu_\Sigma = 0 \quad \text{on } J \times \Sigma. \tag{4.54}$$

In order to resolve this interface condition, we decompose both $\overline{N}_j$ ($j \in \{v, w\}$) into a principal linear part L_j and a nonlinear perturbation G_j. By means of Lemma 4.16, it is straightforward to compute more explicit representations of G_v and G_w, and we will employ the identities (4.55) and (4.56) in Figure 4.5 on the current page.

For controlling the perturbations G_v and G_w and for proving their analyticity, we first provide some estimates for the lower-order terms in Lemma 4.24. Then we study pointwise multiplication, inversion, and square root in the function spaces $\mathbb{G}_v$ and $\mathbb{G}_w$ in Lemma 4.25. It is sufficient to consider the larger class of height functions

$$\tilde{\mathbb{E}}_h := H^1_p(J; W^{3-1/p}_p(\Sigma)) \cap L_p(J; W^{4-1/p}_p(\Sigma)),$$

which contains $\mathbb{E}_h$. Then $\partial_x\Theta_h|_\Sigma$ belongs to the space

$$\tilde{\mathbb{E}}_{\partial\Theta} := H^1_p(J; W^{2-1/p}_p(\Sigma)) \cap L_p(J; W^{3-1/p}_p(\Sigma)),$$

which satisfies $\mathbb{E}_{\partial\Theta} \hookrightarrow \tilde{\mathbb{E}}_{\partial\Theta} \hookrightarrow \mathbb{G}_v \cap \mathbb{G}_w$.

We will consider triples $z = z_* + z_\bullet$ of the form

$$z_* = (\overline{u}_*, \overline{\pi}_*, h_*) \in \tilde{\mathbb{E}}(T_0) := \mathbb{E}_{u,v,w,\partial_\nu w}(T_0) \times \mathbb{E}_{\pi,[\![\pi]\!]}(T_0) \times \tilde{\mathbb{E}}_h(T_0),$$
$$z_\bullet = (\overline{u}_\bullet, \overline{\pi}_\bullet, h_\bullet) \in {}_0\tilde{\mathbb{E}}(T) := {}_0\mathbb{E}_{u,v,w,\partial_\nu w}(T) \times {}_0\mathbb{E}_{\pi,[\![\pi]\!]}(T) \times {}_0\tilde{\mathbb{E}}_h(T).$$

The operators L_j are chosen as follows:

$$L_v(\overline{u}_\bullet, h_\bullet; \overline{u}_*) = -\mu_s\widetilde{\Delta}_\Sigma\overline{v}_\bullet - \lambda_s\nabla_\Sigma \operatorname{div}_\Sigma \overline{v}_\bullet - [\![\mu\partial_\nu\overline{v}_\bullet]\!] - [\![\mu]\!]\nabla_\Sigma\overline{w}_\bullet - (\lambda_s+\mu_s)\overline{w}_*\nabla_\Sigma\Delta_\Sigma h_\bullet,$$
$$\begin{aligned}L_w(\overline{u}_\bullet, \overline{\pi}_\bullet, h_\bullet; \overline{u}_*) = &-\operatorname{tr}\left([(\lambda_s-\mu_s)H_\Sigma + 2\mu_s L_\Sigma]\nabla_\Sigma\overline{v}_\bullet\right) - 2[\![\mu\partial_\nu\overline{w}_\bullet]\!] + [\![\overline{\pi}_\bullet]\!] \\ &-\operatorname{tr}\left([\sigma + (\lambda_s-\mu_s)(\operatorname{div}_\Sigma \overline{v}_* - 2\overline{w}_*H_\Sigma) + 2\mu_s(D_\Sigma(\overline{v}_*) - 2\overline{w}_*L_\Sigma)]\nabla^2_\Sigma h_\bullet\right).\end{aligned}$$

These operators are linear with respect to $z_\bullet$. The nonlinear perturbations G_j are given by

$$G_v(\overline{u}_\bullet, h_\bullet; \overline{u}_*, h_*) := L_v(\overline{u}_\bullet, h_\bullet; \overline{u}_*) - \overline{N}_v(\overline{u}_* + \overline{u}_\bullet, h_* + h_\bullet), \tag{4.57a}$$

$$G_w(\overline{u}_\bullet, h_\bullet; \overline{u}_*, \overline{\pi}_*, h_*) := L_w(\overline{u}_\bullet, 0, h_\bullet; \overline{u}_*) - \overline{N}_w(\overline{u}_* + \overline{u}_\bullet, \overline{\pi}_*, h_* + h_\bullet). \tag{4.57b}$$

Note that the right-hand side of (4.57b) satisfies

$$\begin{aligned}&L_w(\overline{u}_\bullet, \overline{\pi}_\bullet, h_\bullet; \overline{u}_*) - \overline{N}_w(\overline{u}_* + \overline{u}_\bullet, \overline{\pi}_* + \overline{\pi}_\bullet, h_* + h_\bullet)\\ &= L_w(\overline{u}_\bullet, 0, h_\bullet; \overline{u}_*) - \overline{N}_w(\overline{u}_* + \overline{u}_\bullet, \overline{\pi}_*, h_* + h_\bullet)\end{aligned}$$

and hence does not depend on $\overline{\pi}_\bullet$.

The lower-order terms of G_v and G_w will be controlled with the following estimates.

4.24. Lemma. *Let $p \in (n+2, \infty)$.*

(i) There exists $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0]$, $T_0 > 0$, and $T \in (0, T_0]$, we have

$$\|(h_\bullet, \nabla_\Sigma h_\bullet, \nabla^2_\Sigma h_\bullet)\|_{{}_0\mathbb{G}_v(T)} \le C(\delta, T_0)T^\delta\|h_\bullet\|_{{}_0\tilde{\mathbb{E}}_h(T)}, \tag{4.58a}$$

$$\|(h_\bullet, \nabla_\Sigma h_\bullet)\|_{{}_0\mathbb{G}_w(T)} \le C(\delta, T_0)T^\delta\|h_\bullet\|_{{}_0\tilde{\mathbb{E}}_h(T)}, \tag{4.58b}$$

$$\|(u_\bullet, \nabla_\Sigma u_\bullet)\|_{{}_0\mathbb{G}_v(T)} \le C(\delta, T_0)T^\delta\|u_\bullet\|_{{}_0\mathbb{E}_v(T)}, \tag{4.58c}$$

$$\|u_\bullet\|_{{}_0\mathbb{G}_w(T)} \le C(\delta, T_0)T^\delta\|u_\bullet\|_{{}_0\mathbb{E}_v(T)}, \tag{4.58d}$$

for all $h_\bullet \in {}_0\tilde{\mathbb{E}}_h(T)$ and all not necessarily tangential vector fields $u_\bullet \in {}_0\mathbb{E}_v(T)$.

(ii) Given $\varepsilon > 0$, there exists $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0]$, $T_0 > 0$, and $T \in (0, T_0]$, we have

$$\|h_\bullet\|_{{}_0C([0,T];C^3(\Sigma))} \le C(\delta, T_0)T^\delta\|h_\bullet\|_{{}_0\tilde{\mathbb{E}}_h(T)}, \tag{4.59a}$$

$$\|u_\bullet\|_{C([0,T];C^1(\Sigma))} \le C(\delta, T_0)T^\delta\|u_\bullet\|_{{}_0\mathbb{E}_v(T)}, \tag{4.59b}$$

for all $h_\bullet \in {}_0\tilde{\mathbb{E}}_h(T)$ and $u_\bullet \in {}_0\mathbb{E}_v(T)$, and

$$\|(h, \nabla_\Sigma h)\|_{\mathbb{G}_v(T)} \le C(\delta, T_0)\left(T^\delta\|h\|_{\tilde{\mathbb{E}}_h(T)} + \|h_0\|_{W^{2-2/p}_p(\Sigma)}\right), \tag{4.60a}$$

$$\|h\|_{C([0,T];C^3(\Sigma))} \le C(\delta, T_0)\left(T^\delta\|h\|_{\tilde{\mathbb{E}}_h(T)} + \|h_0\|_{W^{3+(n-1)/p+\varepsilon}_p(\Sigma)}\right), \tag{4.60b}$$

$$\|(h, \nabla_\Sigma h)\|_{\mathbb{G}_w(T)} \le C(\delta, T_0)\left(T^\delta\|h\|_{\tilde{\mathbb{E}}_h(T)} + \|h_0\|_{W^{3-2/p}_p(\Sigma)}\right), \tag{4.60c}$$

for all $T \in (0, T_0]$ and $h \in \tilde{\mathbb{E}}_h(T)$ with $h|_{t=0} = h_0$.

Proof. We proceed as in the proofs of Lemmas 3.19 and 4.20 and we also employ temporal extension operators of initial values from Corollaries B.26, B.58 and B.59 on pages 155, 163 and 164.

(i) For proving (4.58), we first observe that (with all spaces considered over $(0,T)\times\Sigma$))

$$(h_\bullet, \nabla_\Sigma h_\bullet, \nabla_\Sigma^2 h_\bullet) \in {}_0H_p^1(W_p^{1-1/p}) \hookrightarrow {}_0W_p^{1/2-1/2p}(L_p)\cap L_p(W_p^{1-1/p}) = {}_0\mathbb{G}_v,$$

and estimate (4.58a) follows by using the inequalities (3.32b) and (3.32e). Similarly,

$$(h_\bullet, \nabla_\Sigma h_\bullet) \in {}_0H_p^1(W_p^{2-1/p}) \hookrightarrow {}_0W_p^{3/4-1/2p}(L_p)\cap {}_0W_p^{1/2-1/2p}(H_p^1)\cap L_p(W_p^{2-1/p}) = {}_0\mathbb{G}_w,$$

and hence (4.58b) is valid. Next,

$$(u_\bullet, \nabla_\Sigma u_\bullet) \in {}_0W_p^{3/4-1/2p}(L_p)\cap {}_0W_p^{1/2-1/p}(H_p^1) \hookrightarrow {}_0W_p^{1/2-1/2p}(L_p)\cap L_p(W_p^{1-1/p}) = {}_0\mathbb{G}_v,$$

and therefore estimate (4.58c) follows from Lemma 3.18. Similarly, (4.58d) follows from

$$u_\bullet \in {}_0\mathbb{E}_v \hookrightarrow {}_0W_p^{1-1/2p}(L_p)\cap {}_0W_p^{3/4-1/2p}(H_p^1)\cap {}_0H_p^{1/2-1/4p}(W_p^{2-1/p}) \hookrightarrow {}_0\mathbb{G}_w.$$

(ii) Estimates (4.59) follow similarly, by using Sobolev's embedding (B.1).

In order to prove the estimates (4.60) for $h\in\tilde{\mathbb{E}}_h(T)$ we employ the decomposition

$$h = \mathcal{E}_T h_0 + (h-\mathcal{E}_T h_0), \quad (\mathcal{E}_T h_0)(t) := e^{-t\sqrt{\mu-\Delta_\Sigma}}h_0, \quad h_0 := h|_{t=0}.$$

From Corollaries B.26 and B.58 we infer that the realizations

$$\begin{aligned}&\mathcal{E}_T\colon W_p^{2-2/p}(\Sigma)\to W_p^{1-1/p}(0,T;H_p^1(\Sigma))\cap L_p(0,T;W_p^{2-1/p}(\Sigma)),\\ &\mathcal{E}_T\colon W_p^{4-2/p}(\Sigma)\to \tilde{\mathbb{E}}_h(T)\end{aligned}$$

are bounded, uniformly with respect to $T\in(0,T_0]$. With estimate (4.58a) we obtain

$$\begin{aligned}\|\nabla_\Sigma h\|_{\mathbb{G}_v(T)} &\le \|\nabla_\Sigma(h-\mathcal{E}_T h_0)\|_{{}_0\mathbb{G}_v(T)} + \|\nabla_\Sigma \mathcal{E}_T h_0\|_{\mathbb{G}_v(T)}\\ &\le C(\delta,T_0)\left(T^\delta\|h\|_{\tilde{\mathbb{E}}_h(T)} + \|h_0\|_{W_p^{2-2/p}(\Sigma)}\right).\end{aligned}$$

Therefore (4.60a) is valid. Next, the realization

$$\mathcal{E}_T\colon W_p^{3+(n-1)/p+\varepsilon}(\Sigma)\to H_p^1(0,T;W_p^{2+n/p+\varepsilon}(\Sigma))\cap L_p(0,T;W_p^{3+n/p+\varepsilon}(\Sigma))$$

is also bounded and its target space is embedded into $C([0,T];C^3(\Sigma))$. This yields an estimate

$$\begin{aligned}\|h\|_{C([0,T];C^3(\Sigma))} &\le \|h-\mathcal{E}_T h_0\|_{{}_0C([0,T];C^3(\Sigma))} + \|\mathcal{E}_T h_0\|_{C([0,T];C^3(\Sigma))}\\ &\le C(\delta,T_0)\left(T^\delta\|h\|_{\tilde{\mathbb{E}}_h(T)} + \|h_0\|_{W_p^{3+(n-1)/p+\varepsilon}(\Sigma)}\right),\end{aligned}$$

which proves (4.60b). With the boundedness of

$$\mathcal{E}_T\colon W_p^{3-2/p}(\Sigma)\to H_p^1(0,T;W_p^{2-1/p}(\Sigma))\cap L_p(0,T;W_p^{3-1/p}(\Sigma)),$$

and with estimate (4.58b) we obtain

$$\begin{aligned}\|(h,\nabla_\Sigma h)\|_{\mathbb{G}_w(T)} &\le \|(1,\nabla_\Sigma)(h-\mathcal{E}_T h_0)\|_{{}_0\mathbb{G}_w(T)} + \|(1,\nabla_\Sigma)\mathcal{E}_T h_0\|_{\mathbb{G}_w(T)}\\ &\le C(\delta,T_0)\left(T^\delta\|h\|_{\tilde{\mathbb{E}}_h(T)} + \|h_0\|_{W_p^{3-2/p}(\Sigma)}\right),\end{aligned}$$

and thus (4.60c) is valid. □

Next, we provide estimates for controlling products with leading-order terms. Let $X(T)$ denote the scalar version of one of the spaces $\tilde{\mathbb{E}}_{\partial\Theta}(T)$, $\mathbb{G}_v(T)$, or $\mathbb{G}_w(T)$ from Figure 4.1 on page 94. Analogously as for $\mathbb{E}_{\partial\Theta}(T)$ in Lemma 4.13, we will show that $X(T)$ is multiplication algebra, and that pointwise inversion and square root are analytic operators in suitable subsets of $X(T)$. We also consider certain larger spaces $Y(T) = C([0,T];C^k(\Sigma)) \supset X(T)$ with the property

$$\|f\|_{{}_0Y(T)} \le T^\delta C(\delta,T_0)\|f\|_{{}_0X(T)} \quad \text{for } f\in{}_0X(T),\ T\in(0,T_0],$$

where ${}_0X(T) := \{f \in X(T) : f|_{t=0} = 0\}$ and ${}_0Y(T) := \{f \in Y(T) : f|_{t=0} = 0\}$. Moreover, the temporal trace space $\gamma_0 X$ of $X(T)$ is embedded into a larger space Z for which we obtain a T-dependent estimate

$$\|f\|_{X(T)} \le T^\delta C(\delta, T_0)(\|f\|_{X(T)} + \|f|_{t=0}\|_{\gamma_0 X}) + C(T_0)\|f|_{t=0}\|_Z \quad \text{for } f \in X(T),\ T \in (0, T_0].$$

Hence, together with a bilinear estimate

$$\|fg\|_{X(T)} \le C(T)(\|f\|_{X(T)}\|g\|_{Y(T)} + \|f\|_{Y(T)}\|g\|_{X(T)}),$$

we can control $\|fg\|_{X(T)}$ by choosing T, $\|f|_{t=0}\|_Z$, and $\|g|_{t=0}\|_Z$ sufficiently small.

4.25. Lemma. *Let $\Sigma \subset \mathbb{R}^n$ ($n \ge 2$) be a compact smooth hypersurface and let*

(4.61a) $$X(T) = \tilde{\mathbb{E}}_{\partial\Theta}(T), \quad Y(T) = C([0,T]; C^2(\Sigma)), \quad Z = W_p^{2+(n-1)/p+\varepsilon}(\Sigma),$$

(4.61b) *or* $$X(T) = \mathbb{G}_v(T), \quad Y(T) = C([0,T]; C(\Sigma)), \quad Z = W_p^{(n-1)/p+\varepsilon}(\Sigma),$$

(4.61c) *or* $$X(T) = \mathbb{G}_w(T), \quad Y(T) = C([0,T]; C^1(\Sigma)), \quad Z = W_p^{1+(n-1)/p+\varepsilon}(\Sigma),$$

where $p \in (n+2, \infty)$, $T \in (0,\infty)$, and $\varepsilon \in (0, 1-(n+2)/p]$. Then the following assertions are valid:

(i) We have $X(T) \hookrightarrow Y(T)$, and for some $\delta_0 > 0$ and all $\delta \in (0, \delta_0]$, $T_0 > 0$, and $T \in (0, T_0]$ we have

(4.62) $$\|f\|_{Y(T)} \le T^\delta C(\delta, T_0)\|f\|_{{}_0X(T)} \quad \text{for } f \in {}_0X(T) = \{f \in X(T) : f|_{t=0} = 0\}.$$

(ii) For $\varepsilon \in (0, 1-(n+2)/p]$ there is $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0]$, $T_0 > 0$, and $T \in (0, T_0]$ we have

(4.63) $$\|f\|_{Y(T)} \le T^\delta C(\delta, T_0)\left(\|f\|_{X(T)} + \|f|_{t=0}\|_{\gamma_0 X}\right) + C(T_0)\|f|_{t=0}\|_Z \quad \text{for } f \in X(T).$$

(iii) $X(T)$ is a multiplication algebra, and there exists $C(T) \ge 1$ such that

(4.64) $$\|fg\|_{X(T)} \le C(T)\left(\|f\|_{X(T)}\|g\|_{Y(T)} + \|f\|_{Y(T)}\|g\|_{X(T)}\right) \quad \text{for } f, g \in X(T),$$

and for given $T_0 \in (0,\infty)$ there exists $C(T_0)$ such that for all $T \in (0, T_0]$ we have

(4.65)
$$\|fg\|_{{}_0X(T)} \le C(T_0)\left(\|f\|_{{}_0X(T)}\|g\|_{Y(T_0)} + \|f\|_{{}_0Y(T)}\|g\|_{X(T_0)}\right) \quad \text{for } f \in {}_0X(T),\ g \in X(T_0).$$

(iv) The inversion operator $A \mapsto A^{-1}$, $\{A \in X^{m\times m} : \sup_{J\times\Sigma}|A^{-1}| < \infty\} \to X$ is analytic.

(v) The square root operator $f \mapsto f(\cdot)^{1/2}$, $\{f \in X : \inf_{J\times\Sigma} \operatorname{dist}(f(\cdot), \mathbb{R}_-) > 0\} \to X$ is analytic.

Proof. We only deal with (4.61a) since the remaining assertions can be proved analogously.

(i) We abbreviate $W_p^t(W_p^s) := W_p^t(J; W_p^s(\Sigma))$, $C^k(C^l) := C^k(\overline{J}; C^l(\Sigma))$, and similarly for the other spaces. The mixed derivative embeddings and Sobolev's embedding (B.1) imply

$$H_p^1(W_p^{2-1/p}) \cap L_p(W_p^{3-1/p}) \hookrightarrow W_p^\theta(H_p^{3-1/p-\theta}) \hookrightarrow C(C^2),$$

provided that $\theta \in (0,1)$ satisfies $\theta > 1/p$ and $3 - 1/p - \theta > 2 + (n-1)/p$. Such a number θ exists if $1/p < 1 - n/p$, and this is true if $p > n+1$. Moreover, Lemma 3.18 yields the estimate (4.62).

(ii) By Corollaries B.26 and B.58 on pages 155 and 163, the extension operator

$$R_A \colon x \mapsto [t \mapsto e^{-At}x], \quad W_p^{s+2-2/p}(\Sigma) \to H_p^1(0,T; W_p^s(\Sigma)) \cap L_p(0,T; W_p^{s+2}(\Sigma))$$

for $A = 1 - \Delta_\Sigma$ and $s \in [0,\infty)$ is uniformly bounded with respect to $T \in (0, T_0]$. By decomposing $f(t) = (f(t) - e^{-tA}f(0)) + e^{-tA}f(0)$ and applying (4.62), we obtain (4.63).

(iii) The norm of $X(T)$ consists of the semi-norms

$$[\![(\nabla_\Sigma, \partial_t\nabla_\Sigma, \nabla_\Sigma^2)\cdot]\!]_{p;1-1/p,p}, \quad \|(1, \partial_t, \partial_t\nabla_\Sigma, \nabla_\Sigma, \nabla_\Sigma^2)\cdot\|_p,$$

where $[\![\cdot]\!]_{p;\sigma,p} := [\![\cdot]\!]_{L_p(W_p^\sigma)}$ and $\|\cdot\|_p := \|\cdot\|_{L_p(L_p)}$. With Lemma B.10 we control the leading-order terms of $\|fg\|_X$ by

$$\begin{aligned}
[\![\partial_t \nabla_\Sigma f\, g]\!]_{p;1-1/p,p} &\lesssim [\![\partial_t \nabla_\Sigma f]\!]_{p;1-1/p,p}\|g\|_\infty + \|\partial_t \nabla_\Sigma f\|_p \|(g, \nabla_\Sigma g)\|_\infty,\\
[\![\partial_t f\, \nabla_\Sigma g]\!]_{p;1-1/p,p} &\lesssim [\![\partial_t f]\!]_{p;1-1/p,p}\|\nabla_\Sigma g\|_\infty + \|\partial_t f\|_p \|(\nabla_\Sigma g, \nabla_\Sigma^2 g)\|_\infty,\\
[\![\nabla_\Sigma^2 f\, g]\!]_{p;1-1/p,p} &\lesssim [\![\nabla_\Sigma^2 f]\!]_{p;1-1/p,p}\|g\|_\infty + \|\nabla_\Sigma^2 f\|_p \|(g, \nabla_\Sigma g)\|_\infty.
\end{aligned}$$

These terms and the remaining ones can be estimated by the right-hand side of (4.64). Therefore (4.64) is valid and the uniform estimate (4.65) follows by means of extension ${}_0X(T) \to {}_0X(\infty)$ and restriction ${}_0X(\infty) \to {}_0X(T_0)$, where the temporal extension operator ${}_0X(T) \to {}_0X(\infty)$ from Lemma B.9 on page 148 is uniformly bounded with respect to $T \in (0, T_0]$.

(iv) Let us check that A^{-1} belongs to $X^{m\times m}$ for every $A \in X^{m\times m}$ with $A^{-1} \in C(\overline{J} \times \Sigma)$. From the inequality $|A(t,x)^{-1} - A(t,x')^{-1}| \le \|A^{-1}\|_\infty^2 |A(t,x) - A(t,x')|$ we infer that $[\![A^{-1}]\!]_{p;\sigma,p} \lesssim \|A^{-1}\|_\infty^2 [\![A]\!]_{p;\sigma,p} < \infty$ for $\sigma = 1 - 1/p$. For given $j \in \{1, \ldots, n-1\}$, Lemma B.10 yields

$$\begin{aligned}
[\![\partial_t\partial_j A^{-1}]\!]_{p;\sigma,p} &= \big[\!\big[A^{-1}[\partial_j A]A^{-1}[\partial_t A]A^{-1} + A^{-1}[\partial_t A]A^{-1}[\partial_j A]A^{-1} - A^{-1}[\partial_t\partial_j A]A^{-1}\big]\!\big]_{p;\sigma,p}\\
&\lesssim \|A^{-1}\|_\infty^3 \|\partial_j A\|_\infty [\![\partial_t A]\!]_{p;\sigma,p} + \big(\|A^{-1}\|_\infty^4 \|(A, \nabla_\Sigma A)\|_\infty^2 + \|A^{-1}\|_\infty^3 [\![(\nabla_\Sigma A, \nabla_\Sigma^2 A)]\!]_\infty\big) \|\partial_t A\|_p\\
&\quad + \|A^{-1}\|_\infty^2 [\![\partial_t\partial_j A]\!]_{p;\sigma,p} + \|(A^{-1}, \nabla_\Sigma A^{-1})\|_\infty \|A^{-1}\|_\infty \|\partial_t\partial_j A\|_p < \infty.
\end{aligned}$$

The semi-norm $[\![\partial_j\partial_k A^{-1}]\!]_{p;\sigma,p}$ can be estimated analogously. We further have

$$\|A^{-1}\|_p \le T^{1/p}|\Sigma|^{1/p}\|A^{-1}\|_\infty < \infty,$$

and the remaining terms in $\|A^{-1}\|_X$ are also finite. Therefore A^{-1} belongs to $X^{m\times m}$, and then Proposition B.88 on page 172 yields analyticity of the inversion operator $A \mapsto A^{-1}$.

(v) Assertion (v) follows by a similar proof as on page 100. □

We are ready to control the perturbations G_v and G_w. The triple $z_\bullet = (\overline{u}_\bullet, \overline{\pi}_\bullet, h_\bullet) \in {}_0\tilde{\mathbb{E}}(T)$ has vanishing initial values, and $z_* = (\overline{u}_*, \overline{\pi}_*, h_*) \in \tilde{\mathbb{E}}(T_0)$ should satisfy the compatibility conditions

$$G_j(0; z_*)|_{t=0} = 0 \quad \text{for } j \in \{v, w\}. \tag{4.66}$$

Then we can control $G_j(z_\bullet; z_*)$ in ${}_0\mathbb{G}_j(T)$ by choosing $T \in (0, T_0]$ and $h_*|_{t=0}$ sufficiently small. Even without requiring (4.66) we can control the partial Fréchet derivative

$$\partial_{z_\bullet} G_j(z_\bullet; z_*) \in \mathcal{B}({}_0\tilde{\mathbb{E}}(T); {}_0\mathbb{G}_j(T)).$$

4.26. Lemma. *Let $p \in (n+2, \infty)$, $\tau \in (1 + n/p, 4 - 1/p]$, $T_0 \in (0, \infty)$, $T \in (0, T_0]$, and*

$$\mathcal{U}_h = \{h \in L_\infty(0, T; W_p^{\tau - 1/p}(\Sigma)) : \|h\|_{L_\infty(0,T;W_p^{\tau-1/p}(\Sigma))} < \delta_h\}$$

with $\delta_h(\Omega, \Sigma, p, \tau) > 0$ as in Theorem 4.15. Then the maps

$$\begin{aligned}
&(z_\bullet, z_*) \mapsto L_v(\overline{u}_\bullet, h_\bullet; u_*), && {}_0\tilde{\mathbb{E}}(T) \times \tilde{\mathbb{E}}(T_0) \to {}_0\mathbb{G}_v(T),\\
&(z_\bullet, z_*) \mapsto L_w(\overline{u}_\bullet, \overline{\pi}_\bullet, h_\bullet; u_*), && {}_0\tilde{\mathbb{E}}(T) \times \tilde{\mathbb{E}}(T_0) \to {}_0\mathbb{G}_w(T),\\
&(z_\bullet, z_*) \mapsto G_v(\overline{u}_\bullet, h_\bullet; \overline{u}_*, h_*), && \{(z_\bullet; z_*) \in {}_0\tilde{\mathbb{E}}(T) \times \tilde{\mathbb{E}}(T_0) : h_\bullet + h_* \in \mathcal{U}_h\} \to \mathbb{G}_v(T),\\
&(z_\bullet, z_*) \mapsto G_w(\overline{u}_\bullet, h_\bullet; \overline{u}_*, \overline{\pi}_*, h_*), && \{(z_\bullet; z_*) \in {}_0\tilde{\mathbb{E}}(T) \times \tilde{\mathbb{E}}(T_0) : h_\bullet + h_* \in \mathcal{U}_h\} \to \mathbb{G}_w(T)
\end{aligned}$$

are analytic and depend polynomially on $z_\bullet$, z_*, $\partial_x\Theta_{h_*+h_\bullet}$, $[\partial_x\Theta_{h_*+h_\bullet}]^{-1}$, $\beta_{h_*+h_\bullet}$, and $\beta_{h_*+h_\bullet}^{-1}$.

In addition, let $\tau \in (3 + n/p, 4 - 1/p)$. Then G_v and G_w have the following properties:

(i) For $z_\bullet \in {}_0\tilde{\mathbb{E}}(T_0)$, $z_* \in \tilde{\mathbb{E}}(T_0)$ with $h = h_* + h_\bullet \in \mathcal{U}_h(T_0)$ and (4.66), we have

$$\|G_v(z_\bullet; z_*)\|_{\mathbb{G}_v(T)} + \|G_w(z_\bullet; z_*)\|_{\mathbb{G}_w(T)} \to 0 \quad \text{as } T \to 0,\ h_0 = h_*|_{t=0} \to 0 \text{ in } W_p^{\tau-1/p}(\Sigma).$$

Given $R \in [1, \infty)$, this convergence is uniform with respect to

$$\|(z_\bullet, z_*)\|_{\tilde{\mathbb{E}}(T_0)} + \|(\partial_x\Theta_{h_*+h_\bullet}, [\partial_x\Theta_{h_*+h_\bullet}]^{-1})\|_{\tilde{\mathbb{E}}_{\partial\Theta}(T_0)} + \|(\beta_{h_*+h_\bullet}, \beta^{-1}_{h_*+h_\bullet})\|_{\tilde{\mathbb{E}}_{\partial\Theta}(T_0)} \le R. \tag{4.67}$$

(ii) For given $T_0 \in (0, \infty)$, $z_\bullet \in {}_0\tilde{\mathbb{E}}(T_0)$, and $z_* \in \tilde{\mathbb{E}}(T_0)$ with $h = h_* + h_\bullet \in \mathcal{U}_h(T_0)$, we have

$$\|\partial_{z_\bullet} G_v(z_\bullet; z_*)\|_{{}_0\tilde{\mathbb{E}}(T) \to {}_0\mathbb{G}_v(T)} + \|\partial_{z_\bullet} G_w(z_\bullet; z_*)\|_{{}_0\tilde{\mathbb{E}}(T) \to {}_0\mathbb{G}_w(T)} \to 0, \tag{4.68}$$

as $T \to 0$, $h_0 \to 0$ in $W_p^{\tau-1/p}(\Sigma)$. Given $R \ge 1$, this convergence is uniform with respect to (4.67).

Proof. *Analyticity.* We first note that the scalar-valued versions of the spaces $\mathbb{G}_v$ and $\mathbb{G}_w$ are multiplication algebras by Lemma 4.25. The maps L_v and L_w consist of linear and bilinear differential operators, and hence their analyticity follows from the mixed derivative embeddings and the spatial trace theorem. In order to prove the analyticity of G_v and G_w, it is sufficient to prove that the map

$$(\overline{u}, \overline{\pi}, h) \mapsto \left(\overline{N}_v(\overline{u}, h), \overline{N}_w(\overline{u}, \overline{\pi}, h)\right) : \mathbb{E}_{u,v,w,\partial_\nu w} \times \mathbb{E}_{\pi,[\![\pi]\!]} \times \tilde{\mathbb{E}}_h \cap \mathcal{U}_h \to \mathbb{G}_v \times \mathbb{G}_w$$

is analytic. Theorem 4.15, the identities (4.24) and Lemma 4.25 imply that the quantities

$$\beta_h,\ \beta_h^{-1},\ \nu_{\Gamma_h} \circ \tilde{\Theta}_h,\ [\partial_x\Theta_h]|_\Sigma,\ [\partial_x\Theta_h]^{-1}|_\Sigma,\ \tau_j^{\Gamma_h} \circ \tilde{\Theta}_h,\ \tau^j_{\Gamma_h} \circ \tilde{\Theta}_h$$

considered in $\tilde{\mathbb{E}}_{\partial\Theta}$ depend analytically on $h \in \tilde{\mathbb{E}}_h \cap \mathcal{U}_h$. Next, the Weingarten tensor L_Γ and the mean curvature H_Γ are given by

$$L_\Gamma = -\nabla_\Gamma \nu_\Gamma = -\tau^j_\Gamma \otimes \partial_j \nu_\Gamma = l^\Gamma_{jk} \tau^j_\Gamma \otimes \tau^k_\Gamma = l^{jk}_\Gamma \tau_j^\Gamma \otimes \tau_k^\Gamma, \quad H_\Gamma = \operatorname{tr} L_\Gamma = -\operatorname{div}_\Gamma \nu_\Gamma. \tag{4.69}$$

Therefore the maps $h \mapsto L_{\Gamma_h} \circ \tilde{\Theta}_h$, $\tilde{\mathbb{E}}_h \cap \mathcal{U}_h \to (\mathbb{G}_v \cap \mathbb{G}_w)^{n\times n}$ and $h \mapsto \nabla_{\Gamma_h} H_{\Gamma_h} \circ \tilde{\Theta}_h$, $\tilde{\mathbb{E}}_h \cap \mathcal{U}_h \mapsto \mathbb{G}_v^n$ are analytic, and their values depend polynomially on $(\partial_x\Theta_h, [\partial_x\Theta_h]^{-1}, \beta_h, \beta_h^{-1})$. Lemma 4.16 yields

$$v_h \circ \tilde{\Theta}_h = [\partial_x\Theta_h]\overline{v}, \quad w_h \circ \tilde{\Theta}_h = \beta_h \overline{w}, \quad (\partial_{\nu_{\Gamma_h}} w_h) \circ \tilde{\Theta}_h = \partial_{\nu_\Sigma} \overline{w}, \tag{4.70}$$

and we conclude that, given $h \in \tilde{\mathbb{E}}_h \cap \mathcal{U}_h$, the map $\overline{u} \mapsto \overline{N}_v(\overline{u}, h)$, $\mathbb{E}_{u,v,w} \to \mathbb{G}_v$ is linear and continuous, and $(\overline{u}, \overline{\pi}) \mapsto \overline{N}_w(\overline{u}, \overline{\pi}, h)$, $\mathbb{E}_{u,v,w,\partial_\nu w} \times \mathbb{E}_{\pi,[\![\pi]\!]} \to \mathbb{G}_w$ is affine and continuous. Therefore $\overline{N}_v$ and $\overline{N}_w$ are analytic and depend polynomially on $(\overline{u}, \overline{\pi}, h, \partial_x\Theta_h, [\partial_x\Theta_h]^{-1}, \beta_h, \beta_h^{-1})$.

(i) *Smallness of $G_j(z_\bullet; z_*)$.* With $G_j(0; z_*)|_{t=0} = 0$ we rewrite

$$G_j(z_\bullet; z_*) = G_j(z_\bullet; z_*) - R_j(G_j(0; z_*)|_{t=0}), \tag{4.71}$$

where we employ the temporal extension operators

$$R_v : g_{v0} \mapsto [t \mapsto e^{-t(1-\widetilde{\Delta}_\Sigma)} g_{v0}], \quad R_w : g_{w0} \mapsto [t \mapsto e^{-(1-\Delta_\Sigma)} g_{w0}]$$

from Corollaries B.26, B.58 and B.59. Then we can rewrite the representations (4.55) and (4.56) for G_j in such a way that every difference has a vanishing initial value. For instance, with $h_0 = h_*|_{t=0}$, the first difference (4.55a) in $G_v(z_\bullet; z_*)$ becomes

$$\left[\!\!\left[\mu \left\{ [\partial_x\Theta_h]^{-1} (\partial_{\nu_{\Gamma_h}} v_h) \circ \tilde{\Theta}_h - \partial_{\nu_\Sigma} \overline{v}_\bullet - R_v \left([\partial_x\Theta_{h_0}]^{-1} (\partial_{\nu_{\Gamma_{h_0}}} v_{h_0}) \circ \Theta_{h_0} \right) \right\} \right]\!\!\right].$$

With (4.24) and (4.25) we rewrite and decompose the difference in the curled brackets as

$$\begin{aligned}
&[\partial_x\Theta_h]^{-1}\partial_x\left([\partial_x\Theta_h](\overline{v}_* + \overline{v}_\bullet)\right)[\partial_x\Theta_h]^{-1}\left(\beta_h^{-1}[\partial_x\Theta_h]\nu_\Sigma\right)\\
&\quad - \partial_{\nu_\Sigma}\overline{v}_\bullet - R_v\left([\partial_x\Theta_{h_0}]^{-1}(\partial_{\nu_{\Gamma_{h_0}}} v_{h_0})\circ\Theta_{h_0}\right)\\
&= \partial_{\nu_\Sigma}\overline{v}_\bullet(\beta_h^{-1} - 1)\\
&\quad + \beta_h^{-1}\partial_{\nu_\Sigma}\overline{v}_* - R_v\left(\beta_{h_0}^{-1}\partial_{\nu_\Sigma}\overline{v}_0\right)\\
&\quad + \beta_h^{-1}[\partial_x\Theta_h]^{-1}[\partial_x[\partial_x\Theta_h](\overline{v}_* + \overline{v}_\bullet)][\partial_x\Theta_h]\nu_\Sigma] - R_v\left(\beta_{h_0}[\partial_x\Theta_{h_0}]^{-1}[\partial_x[\partial_x\Theta_{h_0}]\overline{v}_0][\partial_x\Theta_{h_0}]\nu_\Sigma\right).
\end{aligned}$$

These differences belong to ${}_0\mathbb{G}_v(T)$, and from the estimates (4.63) and (4.65) we infer that they tend to zero in ${}_0\mathbb{G}_v(T)$ as $T \to 0$ and $[\partial_x\Theta_{h_0}]^{-1} \to I_x$ in $W_p^{2+(n+1)/p+\varepsilon}(\Sigma)$ for some $\varepsilon \in (0, 1 - (n+1)/p]$. The latter follows from $h_0 \to 0$ in $W_p^{\tau-1/p}(\Sigma)$ since $\tau - 1/p \geq 3 + (n+1)/p + \varepsilon$ for some $\varepsilon > 0$. The remaining differences in (4.71) can be estimated similarly, and therefore assertion (i) is valid.

(ii) *Smallness of* $\partial_{\overline{u}_\bullet} G_v$. For proving estimate (4.68) we first investigate the directional derivative $\partial_{\overline{u}_\bullet} G_v(\overline{u}_\bullet, h_\bullet; \overline{u}_*, h_*)$ applied to $\tilde{\overline{u}}_\bullet \in {}_0\mathbb{E}_{u,v,w,\partial_\nu w}(T)$. The map $\tilde{\overline{u}}_\bullet \mapsto G_v(\overline{u}_\bullet + \tilde{\overline{u}}_\bullet, h_\bullet; \overline{u}_*, h_*)$ is affine, and therefore G_v satisfies

$$[\partial_{\overline{u}_\bullet} G_v(\overline{u}_\bullet, h_\bullet; \overline{u}_*, h_*)]\tilde{\overline{u}}_\bullet = G_v(\tilde{\overline{u}}_\bullet, h_\bullet; 0, h_*),$$

and has the lower-order terms (4.55d) to (4.55g) with respect to $\overline{u} = \overline{u}_* + \overline{u}_\bullet$. Their directional derivatives with respect to $\overline{u}_\bullet$ applied to $\tilde{\overline{u}}_\bullet$ only depend on the values of $\tilde{\overline{u}}_\bullet$ and $\nabla_\Sigma\tilde{\overline{u}}_\bullet|_\Sigma$, and with estimate (4.58c) we can control these terms by choosing T small. Applying the identities in Figure 4.2, the leading-order terms in the $\overline{u}_\bullet$-derivatives of (4.55a) to (4.55c) are given by

$$[\partial_x\tilde{\overline{v}}_\bullet]\{\beta_h^{-1}[\partial_x\Theta_h]\nu_\Sigma - \nu_\Sigma\},\ \{(g^{ij}_{\Gamma_h}\circ\tilde{\Theta}_h) - g^{ij}_\Sigma\}\partial_i\partial_j\tilde{\overline{v}}_\bullet,\ \{[\tau^i_{\Gamma_h}\otimes\tau^j_{\Gamma_h}]\circ\tilde{\Theta}_h - [\tau^i_\Sigma\otimes\tau^j_\Sigma]\}\partial_i\partial_j\tilde{\overline{v}}_\bullet.$$

By means of estimate (4.60a), we can further control $(h, \nabla_\Sigma h)$ in the $\mathbb{G}_v(T)$-norm and obtain

$$\|(\beta_h - 1, \beta_h^{-1} - 1)\|_{\mathbb{G}_v(T)} + \|([\partial_x\Theta_h] - I_x, [\partial_x\Theta_h]^{-1} - I_x)\|_{\mathbb{G}_v(T)} \to 0$$

as $T \to 0$, $h_0 \to 0$ in $W_p^{2-2/p}(\Sigma)$, and therefore

$$\|\partial_{\overline{u}_\bullet} G_v(\overline{u}_\bullet, h_\bullet; \overline{u}_*, h_*)\|_{{}_0\mathbb{E}_{u,v,w,\partial_\nu w}(T)\to{}_0\mathbb{G}_v(T)} \to 0 \quad \text{as } T \to 0,\ h_0 \to 0 \text{ in } W_p^{2-2/p}(\Sigma),$$

uniformly with respect to (4.67).

Smallness of $\partial_{\overline{u}_\bullet} G_w$. For the computation of $\partial_{\overline{u}_\bullet} G_w(\overline{u}_\bullet, h_\bullet; \overline{u}_*, \overline{\pi}_*, h_*)]\tilde{\overline{u}}_\bullet$ we only have to consider the differences (4.56c) and (4.56d) where $\nabla_\Sigma\tilde{\overline{u}}_\bullet|_\Sigma$ and $\nabla^2_\Sigma h$ are of leading order. The lower-order terms can be controlled with estimate (4.58d). Concerning the leading-order terms we note that by using the identities (4.7), (4.53), and (4.69), it remains to control the products $[\nabla_\Sigma\tilde{\overline{v}}_\bullet][\nabla^2_\Sigma h]$ and $[\nabla_\Sigma\tilde{\overline{v}}_\bullet]\nabla_\Sigma h$ in the ${}_0\mathbb{G}_w(T)$-norm. From Lemma 4.25 we infer that

$$\|[\nabla_\Sigma\overline{v}]([\nabla^2_\Sigma h], \nabla_\Sigma h)\|_{{}_0\mathbb{G}_w(T)} \lesssim \|\nabla_\Sigma\overline{v}\|_{{}_0C(\overline{J};C^1(\Sigma))}\|h\|_{\mathbb{E}_h(T_0)} + \|\overline{v}\|_{{}_0\mathbb{E}_v(T)}\|h\|_{C(\overline{J};C^3(\Sigma))},$$

and therefore, by using the estimates (4.59a) and (4.60b), we obtain

$$\|\partial_{\overline{u}} G_w(\overline{u}_\bullet, h_\bullet; \overline{u}_*, \overline{\pi}_*, h_*)\|_{{}_0\mathbb{E}_{u,v,w}(T)\to{}_0\mathbb{G}_v(T)} \to 0 \quad \text{as } T \to 0,\ h_0 \to 0 \text{ in } W_p^{\tau-1/p}(\Sigma),$$

for some $\tau > 3 + n/p$, and this convergence is uniform with respect to (4.67).

Smallness of $\partial_{h_\bullet} G_v$. We control $[\partial_{h_\bullet} G_v(\overline{u}_\bullet, h_\bullet; \overline{u}_*, h_*)]\tilde{h}_\bullet$ in the ${}_0\mathbb{G}_v(T)$-norm for $\tilde{h}_\bullet \in {}_0\tilde{\mathbb{E}}_h(T)$. Estimate (4.58a) allows to control all the terms in $[\partial_{h_\bullet} G_v(\overline{u}_\bullet, h_\bullet; \overline{u}_*, h_*)]\tilde{h}_\bullet$ by T, except for

$$\left[\partial_{h_\bullet}([\partial_x\Theta_h]^{-1}(w_h\nabla_{\Gamma_h} H_{\Gamma_h})\circ\tilde{\Theta}_h)\right]\tilde{h}_\bullet - \overline{w}\nabla_\Sigma\Delta_\Sigma\tilde{h}_\bullet, \quad h := h_* + h_\bullet,$$

which contains the leading-order term

$$[\partial_x\Theta_h]^{-1}\beta_h\overline{w}[\partial_x\Theta_h]^{-\top}\nabla_\Sigma\left([\partial_x\Theta_h]^{-\top}\tau^j_\Sigma\middle|\partial_j(\beta_h[I - hL_\Sigma]^{-1}\nabla_\Sigma\tilde{h}_\bullet)\right) - \overline{w}\nabla_\Sigma \operatorname{div}_\Sigma \nabla_\Sigma\tilde{h}_\bullet.$$

In order to control this term in ${}_0\mathbb{G}_v(T)$, it suffices to control $\partial_x\Theta_h - I_x$ and thus $(h, \nabla_\Sigma h)$ in $\mathbb{G}_v(T)$, and that was already done in (4.60a). We conclude that

$$\|\partial_{h_\bullet} G_v(\overline{u}_\bullet, h_\bullet; \overline{u}_*, h_*)\|_{{}_0\tilde{\mathbb{E}}_h(T)\to{}_0\mathbb{G}_v(T)} \to 0 \quad \text{as } T \to 0,\ h_0 \to 0 \text{ in } W_p^{1+(n-1)/p+\varepsilon}(\Sigma),$$

for some $\varepsilon > 0$, uniformly with respect to (4.67).

Smallness of $\partial_{h_\bullet} G_w$. It remains to control $[\partial_{h_\bullet} G_w(\overline{u}_\bullet, h_\bullet; \overline{u}_*, \overline{\pi}_*, h_*)]\tilde{h}_\bullet$ in ${}_0\mathbb{G}_w(T)$. All its summands which only contain $(\tilde{h}_\bullet, \nabla_\Sigma \tilde{h}_\bullet)$ but not $\nabla^2_\Sigma \tilde{h}_\bullet$ can be controlled by T with estimate (4.58b). With estimate (4.60c) we can also control all terms which contain $(h, \nabla_\Sigma h)$ but not $\nabla^2_\Sigma h$. Furthermore, with the estimates (4.60b) and (4.64), we can also control the bilinear leading-order term

$$\begin{aligned}\|[\nabla^2_\Sigma h][\nabla^2_\Sigma \tilde{h}_\bullet]\|_{{}_0\mathbb{G}_w(T)} &\lesssim \|\nabla^2_\Sigma h\|_{\mathbb{G}_w(T_0)}\|\nabla^2_\Sigma \tilde{h}_\bullet\|_{{}_0C([0,T];C^1)} + \|\nabla^2_\Sigma h\|_{C([0,T_0];C^1)}\|\nabla^2_\Sigma \tilde{h}_\bullet\|_{{}_0\mathbb{G}_w(T)} \\ &\lesssim \|h\|_{\tilde{\mathbb{E}}_h(T_0)} \cdot T^\delta \|\tilde{h}_\bullet\|_{{}_0\tilde{\mathbb{E}}_h(T)} + \|h_0\|_{W_p^{3+(n-1)/p+\varepsilon}} \|\tilde{h}_\bullet\|_{{}_0\tilde{\mathbb{E}}_h(T)}.\end{aligned}$$

Among the leading-order terms, we consider the directional derivative

$$[\partial_{h_\bullet}(H_{\Gamma_h} \circ \tilde{\Theta}_h)]\tilde{h}_\bullet = -[\partial_{h_\bullet}((\tau^j_{\Gamma_h}|\partial_j \nu_{\Gamma_h}) \circ \tilde{\Theta}_h)]\tilde{h}_\bullet, \quad h := h_* + h_\bullet.$$

Its leading-order part containing $\nabla^2_\Sigma \tilde{h}_\bullet$ is given by

$$[\partial_x\Theta_h]^{-\top}\tau^j_\Sigma \cdot \left(\beta_h[I - hL_\Sigma]^{-1}\partial_j\nabla_\Sigma \tilde{h}_\bullet + \partial_j([\partial_{h_\bullet}\beta_h]\tilde{h}_\bullet)\,[I - hL_\Sigma]^{-1}\nabla_\Sigma h\right).$$

With estimate (4.60c) we can estimate the first summand by

$$\left\|\tilde{h}_\bullet \mapsto \left([\partial_x\Theta_h]^{-\top}\tau^j_\Sigma \cdot ([I - hL_\Sigma]^{-1}\partial_j\nabla_\Sigma \tilde{h}_\bullet) - \Delta_\Sigma \tilde{h}_\bullet\right)\right\|_{{}_0\tilde{\mathbb{E}}_h(T)\to{}_0\mathbb{G}_w(T)} \to 0,$$

as $T \to 0$ and $h_0 \to 0$ in $W_p^{3-2/p}(\Sigma)$. For the second summand we use

$$[\partial_{h_\bullet}\beta_h]\tilde{h}_\bullet = -\beta_h^2\left(\nabla_\Sigma h\middle|(I - hL_\Sigma)^{-2}\left(\nabla_\Sigma \tilde{h}_\bullet - \tilde{h}_\bullet L_\Sigma(I - hL_\Sigma)^{-1}\nabla_\Sigma h\right)\right),$$

and therefore the estimates (4.60b) and (4.60c) yield $\partial_j([\partial_{h_\bullet}\beta_h]\tilde{h}_\bullet) \to 0$ in ${}_0\mathbb{G}_w(T)$ as $T \to 0$ and $h_0 \to 0$ in $W_p^{3+(n-1)/p+\varepsilon}(\Sigma)$. Therefore

$$\|[\partial_{h_\bullet}(L_{\Gamma_h} \circ \tilde{\Theta}_h)]\tilde{h}_\bullet - \nabla^2_\Sigma \tilde{h}_\bullet\|_{{}_0\mathbb{G}_w(T)} \to 0, \tag{4.72}$$

as $T \to 0$ and $h_0 \to 0$ in $W_p^{3+(n-1)/p+\varepsilon}(\Sigma)$. This allows to control the directional derivative of (4.56b). Concerning the remaining terms (4.56c) and (4.56d), we note that (see (A.17) on page 140)

$$\operatorname{div}_\Gamma u = (\tau^j_\Gamma|\partial_j v) - wH_\Gamma, \quad D_\Gamma(u) = D_\Gamma(v) - wL_\Gamma = \operatorname{sym}(\tau^j_\Gamma \otimes P_\Gamma\partial_j v) - wL_\Gamma.$$

Therefore it is sufficient to consider the differences

$$[\partial_{h_\bullet}((\operatorname{div}_{\Gamma_h}(u_h)) \circ \tilde{\Theta}_h)]\tilde{h}_\bullet\, H_{\Gamma_h} \circ \tilde{\Theta}_h + \overline{w}\,\Delta_\Sigma \tilde{h}_\bullet\, H_\Sigma, \tag{4.73a}$$

$$\operatorname{div}_{\Gamma_h}(u_h) \circ \tilde{\Theta}_h\,[\partial_{h_\bullet}(H_{\Gamma_h} \circ \tilde{\Theta}_h)]\tilde{h}_\bullet - \operatorname{div}_\Sigma(\overline{u})\,\Delta_\Sigma \tilde{h}_\bullet, \tag{4.73b}$$

$$[\partial_{h_\bullet}((D_{\Gamma_h}(u_h) \circ \tilde{\Theta}_h)]\tilde{h}_\bullet : [L_{\Gamma_h} \circ \tilde{\Theta}_h] + \overline{w}\nabla^2_\Sigma \tilde{h}_\bullet : L_\Sigma, \tag{4.73c}$$

$$[D_{\Gamma_h}(u_h) \circ \tilde{\Theta}_h] : [\partial_{h_\bullet}(L_{\Gamma_h} \circ \tilde{\Theta}_h)]\tilde{h}_\bullet - D_\Sigma(\overline{u}) : \nabla^2_\Sigma \tilde{h}_\bullet. \tag{4.73d}$$

With the estimates (4.60b), (4.60c), (4.64), and (4.72), we can control the directional derivatives of (4.73) in ${}_0\mathbb{G}_w(T)$ with $T \to 0$ and $h_0 \to 0$ in $W_p^{3+(n-1)/p+\varepsilon}(\Sigma)$. The proof of the lemma is complete. □

We also have to specialize the corresponding results for F_u and F_d to the case of a normal-preserving admissible map. Theorem 4.15 implies that the map

$$h \mapsto \Theta_h - \mathrm{Id}_x = \mathfrak{S}(h\nu_\Sigma, g_h), \quad \mathbb{E}_h \cap \mathcal{U}_h \to \mathbb{E}_\Theta \cap \mathcal{U}_\Theta$$

is analytic for $p \in ((n+2)/2, \infty)$ and $\tau \in (1+n/p, 4-1/p]$, where the subsets $\mathcal{U}_h$ and $\mathcal{U}_\Theta$ were defined on pages 102 and 107. For $h = h_* + h_\bullet$ with $h_* \in \mathbb{E}(T_0)$ and $h_\bullet \in {}_0\mathbb{E}(T_0)$, the Fréchet derivative of $\Theta_h - \mathrm{Id}_x$ is given by

$$[\partial_{h_\bullet}(\Theta_h - \mathrm{Id}_x)]\tilde{h}_\bullet = \mathfrak{S}(\tilde{h}_\bullet \nu_\Sigma, [\partial_{h_\bullet}(g_h)]\tilde{h}_\bullet),$$

and becomes $\mathfrak{S}(\tilde{h}_\bullet \nu_\Sigma, -\nabla_\Sigma \tilde{h}_\bullet)$ at $h = 0$.

4.27. Corollary. *Let $p \in (n+2, \infty) \setminus \{3\}$, $\tau \in (1+n/p, 4-1/p)$, $T \in (0,\infty)$, and*

$$F_u(u,\pi,h) := F_u(u,\pi,\Theta_h) \quad \text{for } u \in \mathbb{E}_u,\ \pi \in \mathbb{E}_\pi,\ h \in \mathbb{E}_h \cap \mathcal{U}_h.$$

Then $F_u \colon \{(u,\pi,h) \in \mathbb{E}_u \times \mathbb{E}_\pi \times \mathbb{E}_h : h \in \mathcal{U}_h\} \to \mathbb{F}_u$ is analytic and has the following properties:

(i) Given $T_0 \in (0,\infty)$, $R \in (\delta_h^{-1}, \infty)$, $u \in \mathbb{E}_u(T_0)$, $\pi \in \mathbb{E}_\pi(T_0)$, and $h \in \mathbb{E}_h(T_0) \cap \mathcal{U}_h$, we have

$$\|F_u(u,\pi,h)\|_{\mathbb{F}_u(T)} \to 0 \quad \text{as } T \to 0,\ h_0 := h|_{t=0} \to 0 \text{ in } W_p^{\tau-1/p}(\Sigma),$$

uniformly with respect to

$$\|u\|_{\mathbb{E}_u(T_0)} + \|\pi\|_{\mathbb{E}_\pi(T_0)} + \|h\|_{\mathbb{E}_h(T_0)} \le R, \quad \|h\|_{L_\infty(0,T_0;W_p^{\tau-1/p}(\Sigma))} \le \delta_h - R^{-1}.$$

(ii) Given $T_0 \in (0,\infty)$, $R \in (\delta_h^{-1},\infty)$, $u_* \in \mathbb{E}_u(T_0)$, $u_\bullet \in {}_0\mathbb{E}_u(T_0)$, $\pi \in \mathbb{E}_\pi(T_0)$, $h_* \in \mathbb{E}_h(T_0)$, and $h_\bullet \in {}_0\mathbb{E}_h(T_0)$ with $h = h_* + h_\bullet \in \mathcal{U}_h$, we have

$$\|\partial_{(u_\bullet,\pi,h_\bullet)} F_u(u_* + u_\bullet, \pi, h_* + h_\bullet)\|_{{}_0\mathbb{E}_u(T) \times \mathbb{E}_\pi(T) \times {}_0\mathbb{E}_h(T) \to \mathbb{F}_u(T)} \to 0,$$

as $T \to 0$ and $h_0 \to 0$ in $W_p^{\tau-1/p}(\Sigma)$. This convergence is uniform with respect to

$$\|(u_*, u_\bullet)\|_{\mathbb{E}_u(T_0)} + \|\pi\|_{\mathbb{E}_\pi(T_0)} + \|(h_*, h_\bullet)\|_{\mathbb{E}_h(T_0)} \le R,$$
$$\|h_* + h_\bullet\|_{L_\infty(0,T_0;W_p^{\tau-1/p}(\Sigma))} \le \delta_h - R^{-1}.$$

Proof. In order to apply Theorem 4.15 and Lemma 4.21, it remains to show that $\|\partial_x \Theta_h - I_x\|_\infty = \|\mathfrak{S}(h\nu\Sigma, g_h)\|_\infty \to 0$ as $T \to 0$ and $h_0 \to 0$ in $W_p^{\tau-1/p}(\Sigma)$. Given $\tau > 1 + n/p$ we have

$$\begin{aligned} \|\partial_x \Theta_h - I_x\|_{L_\infty((0,T)\times\Omega)} &= \|\mathfrak{S}(h\nu_\Sigma, g_h)\|_{L_\infty((0,T)\times\Omega)} \\ &\lesssim \|h\nu_\Sigma\|_{L_\infty(0,T;W_p^{\tau-1/p}(\Sigma))} + \|g_h\|_{L_\infty(0,T;W_p^{\tau-1-1/p}(\Sigma))} \to 0, \end{aligned}$$

as $\|h\|_{L_\infty(0,T;W_p^{\tau-1/p}(\Sigma))} \to 0$. As on page 114, we decompose $h = (h - \mathcal{E}_T h_0) + \mathcal{E}_T h_0$. The embedding

$$C(W_p^{\tau-1/p}) \supset H_p^{1/p+\varepsilon}(W_p^{\tau-1/p}) \supset H_p^1(W_p^{\tau-1+\varepsilon}) \cap L_p(W_p^{\tau+\varepsilon}) \supset \mathbb{E}_h,$$

and Lemma 3.18 yield an estimate

$$\|h\|_{C([0,T],W_p^{\tau-1/p}(\Sigma))} \le C(\delta, T_0)(T^\delta \|h\|_{\mathbb{E}_h(T_0)} + \|h_0\|_{W_p^{\tau-1/p}(\Sigma)}), \tag{4.74}$$

for some $\delta_0 > 0$ and all $\delta \in (0,\delta_0]$ and $T \in (0,T_0]$, provided that $\tau < 4 - 1/p$. This yields the required convergence $\|h\|_{L_\infty(0,T;W_p^{\tau-1/p}(\Sigma))} \to 0$ as $T \to 0$ and $h_0 \to 0$ in $W_p^{\tau-1/p}(\Sigma)$. □

For an estimation of the divergence perturbation $\tilde{F}_d$ we also use the compatibility condition

$$\operatorname{div} u_0 = \tilde{F}_d(u_0, h_0) \quad \text{in } \Omega. \tag{4.75}$$

4.28. Corollary. *Let $p \in (n+2,\infty)$, $\tau \in (3+n/p, 4-1/p)$, $T \in (0,\infty)$, and*

$$\tilde{F}_d(u,h) := \tilde{F}_d(u,\Theta_h) \quad \text{for } u \in \mathbb{E}_{u,v,w,\partial_\nu w},\ h \in \mathbb{E}_h \cap \mathcal{U}_h.$$

Then the map $\tilde{F}_d \colon \{(u,h) \in \mathbb{E}_{u,v,w,\partial_\nu w}(T) \times \mathbb{E}_h(T) : h \in \mathcal{U}_h\} \to \mathbb{F}_{d,\Sigma}(T)$ is analytic and has the following properties:

(i) For given $T_0 \in (0,\infty)$, $R \in (\delta_h^{-1},\infty)$, $u \in \mathbb{E}_{u,v,w,\partial_\nu w}(T_0)$, $h \in \mathbb{E}(T_0) \cap \mathcal{U}_h$, and $h_0 = h|_{t=0}$, we have

$$\|\tilde{F}_d(u,h) - \operatorname{div} u\|_{\mathbb{F}_{d,\Sigma}(T)} \to 0 \quad \text{as } T \to 0,\ \|h_0\|_{W_p^{\tau-1/p}(\Sigma)} \to 0,$$

and this convergence is uniform with respect to

$$\|u\|_{\mathbb{E}_{u,v,w,\partial_\nu w}(T_0)} + \|h\|_{\mathbb{E}_h(T_0)} \le R, \quad \|h\|_{L_\infty(0,T_0;W_p^{\tau-1/p}(\Sigma))} \le \delta_h - R^{-1}.$$

(ii) For given $T_0 \in (0,\infty)$, $R \in (\delta_h^{-1},\infty)$, $u_* \in \mathbb{E}_{u,v,w,\partial_\nu w}(T_0)$, $u_\bullet \in {}_0\mathbb{E}_{u,v,w,\partial_\nu w}(T_0)$, $h_* \in \mathbb{E}(T_0)$, $h_\bullet \in {}_0\mathbb{E}_h(T_0)$ with $h = h_* + h_\bullet \in \mathcal{U}_h$, and $h_0 = h_*|_{t=0}$, we have

$$\|\partial_{(u_\bullet,h_\bullet)}\tilde{F}_d(u_* + u_\bullet, h_* + h_\bullet)\|_{{}_0\mathbb{E}_{u,v,w,\partial_\nu w}(T)\times {}_0\mathbb{E}_h(T)\to \mathbb{F}_{d,\Sigma}(T)} \to 0,$$

as $T \to 0$ and $\|h_0\|_{W_p^{\tau-1/p}(\Sigma)} \to 0$, and this convergence is uniform with respect to

$$\|(u_*,u_\bullet)\|_{\mathbb{E}_{u,v,w,\partial_\nu w}(T_0)} + \|(h_*,h_\bullet)\|_{\mathbb{E}_h(T_0)} \le R, \tag{4.76}$$

$$\|h_* + h_\bullet\|_{L_\infty(0,T_0;W_p^{\tau-1/p}(\Sigma))} \le \delta_h - R^{-1}. \tag{4.77}$$

Proof. We recall from pages 94 and 109 that $\tilde{F}_d(u,h)$ is trilinear in $(\partial_x^2\Theta_h, [\partial_x\Theta_h]^{-1}, u)$, and that

$$\|f_d\|_{\mathbb{F}_{d,\Sigma}} = \|f_d\|_{\mathbb{F}_d} + \|(f_{d+}|_\Sigma, f_{d-}|_\Sigma)\|_{\mathbb{G}_w},$$

where $J = (0,T)$. The assertions for the $\mathbb{F}_d$-norm follow from Theorem 4.15 and Lemma 4.23, as soon as we have ensured that the difference $\partial_x\Theta_h - I_x = \partial_x\mathfrak{S}(h\nu_\Sigma, g_h)$ tends to zero in the $C([0,T];C^1(\overline{\Omega}))$-norm. Since $\tau > 2 + n/p$, we have

$$\|\partial_x\Theta_h - I_x\|_{L_\infty((0,T)\times\Omega)} \lesssim \|h\nu_\Sigma\|_{L_\infty(0,T;W_p^{\tau-1/p}(\Sigma))} + \|g_h\|_{L_\infty(0,T;W_p^{\tau-1-1/p}(\Sigma))} \to 0,$$

as $\|h\|_{L_\infty(0,T;W_p^{\tau-1/p}(\Sigma))} \to 0$. By using (4.74) and $\tau < 4 - 1/p$, the assertions for the $\mathbb{F}_d$-norm follow.

The space $\mathbb{G}_w$ is a multiplication algebra by Lemma 4.25, and from the mixed derivative embeddings and the T-dependent estimates in Lemma 3.18, we obtain the estimate

$$\|u_\bullet|_\Sigma\|_{{}_0\mathbb{G}_w(T)} \le T^\delta C(\delta,T_0)\|u_\bullet\|_{{}_0\mathbb{E}_{u,v,w,\partial_\nu w}(T)}. \tag{4.78}$$

With $\partial_{\nu_\Sigma}^2\mathfrak{S}(h\nu_\Sigma,g_h)|_\Sigma = 0$, it follows that the values of $\partial_x^2\mathfrak{S}(h\nu_\Sigma,g_h)|_\Sigma$ depend linearly on those of $(h, \nabla_\Sigma h, \nabla_\Sigma^2 h, g_h, \nabla_\Sigma g_h)$. Therefore $h \mapsto \partial_x^2\mathfrak{S}(h\nu_\Sigma,g_h)$, $\tilde{\mathbb{E}}_h(T) \cap \mathcal{U}_h \to \mathbb{G}_w(T)$ is analytic. Moreover, with the estimates (4.60b), (4.65), and (4.78), we can control $\tilde{F}_d(u,h) - \operatorname{div} u$ in ${}_0\mathbb{G}_w(T)$ and $\partial_{(u_\bullet,h_\bullet)}\tilde{F}_d(u,h)$ in $\mathcal{B}({}_0\mathbb{E}_{u,v,w,\partial_\nu w}(T) \times {}_0\tilde{\mathbb{E}}_h(T); {}_0\mathbb{G}_w(T))$ by choosing T and h_0 in $W_p^{3+(n-1)/p+\varepsilon}(\Sigma)$ sufficiently small. Therefore both assertions of Corollary 4.28 are true. □

4.4. Local well-posedness of the transformed problem

Finally, we prove well-posedness for the transformed problem (T), which we restate as

$$\rho\partial_t u - \mu\Delta u + \nabla\pi = F_u(u,\pi,h) \quad \text{in } J\times\Omega\setminus\Sigma, \tag{4.79a}$$
$$\operatorname{div} u = F_d(u,h) \quad \text{in } J\times\Omega\setminus\Sigma, \tag{4.79b}$$
$$[\![u]\!] = 0 \quad \text{on } J\times\Sigma, \tag{4.79c}$$
$$\overline{N}_v(u,h) + \overline{N}_w(u,\pi,h)\nu_\Sigma = 0 \quad \text{on } J\times\Sigma, \tag{4.79d}$$
$$\partial_t h - u\cdot\nu_\Sigma = 0 \quad \text{on } J\times\Sigma, \tag{4.79e}$$
$$u|_{\partial\Omega} = 0 \quad \text{on } J\times\partial\Omega, \tag{4.79f}$$
$$h|_{t=0} = h_0 \quad \text{on } \Sigma, \tag{4.79g}$$
$$u|_{t=0} = u_0 \quad \text{in } \Omega\setminus\Sigma. \tag{4.79h}$$

Here $J=(0,T)$ is a bounded interval and Ω is a bounded smooth domain in $\mathbb{R}^n$ ($n\geq 2$) that contains a compact smooth hypersurface Σ. We employ the operators F_u from page 106, F_d from page 109, and $\overline{N}_v$ and $\overline{N}_w$ from page 112. We decompose $u = v + w\nu_\Sigma\circ\Pi_\Sigma$ near Σ. Both u and π denote transformed quantities; that is, we omit the bars over u and π.

An $\mathbb{E}$-*solution* of problem (4.79) = (T) on $J=(0,T)$ is a triple

$$(u,\pi,h)\in\mathbb{E}(T) := \mathbb{E}_{u,v,w,\partial_\nu w}(T)\times\mathbb{E}_{\pi,[\![\pi]\!]}(T)\times(\mathbb{E}_h(T)\cap\mathcal{U}_h),$$

which satisfies (4.79) pointwise almost everywhere. The relevant function spaces are collected in Figure 4.1. The nonlinearities are well-defined if the height function satisfies the smallness condition $h\in\mathcal{U}_h$ from Theorem 4.15.(ii) on page 102.

We will consider $\mathbb{E}$-solutions of the form

$$(u,\pi,h) = (u_*+u_\bullet, \pi_*+\pi_\bullet, h_*+h_\bullet) \quad \text{with } (u_\bullet,\pi_\bullet,h_\bullet)\in{}_0\mathbb{E},\ (u_*,\pi_*,h_*)\in\mathbb{E},$$

where $(u_\bullet,\pi_\bullet,h_\bullet)$ has vanishing initial values and (u_*,π_*,h_*) satisfies the initial conditions. In Definition 4.29 we define the state space X_p of initial data (u_0,h_0), which is a subset of some Banach space $\mathbb{X}_p$. It is shown in Lemma 4.31 that for every $(u_0,h_0)\in X_p$, there exists a tuple $(u_*,\pi_*,h_*)\in\mathbb{E}(T)$ which satisfies $(u_*,h_*)|_{t=0}=(u_0,h_0)$ and depends linearly and continuously on (u_0,h_0). Then it remains to solve a variant of problem (4.79) with vanishing initial values. In Theorem 4.33 we finally prove that (4.79) is locally well-posed in X_p with respect to $\mathbb{E}$ in the sense of Definition 4.32 on page 125, and that the trajectories $t\mapsto(u(t),h(t))$ remain in X_p.

First, we deal with the non-homogeneous initial conditions in problem (4.79). For $p>3$ and a given tuple $(u,\pi,h)\in\mathbb{E}(T)$, the temporal trace theorem yields

$$\begin{aligned} u &\in C(\overline{J};W_p^{2-2/p}(\Omega\setminus\Sigma)^n),\ u|_{\partial\Omega}=0,\ [\![u]\!]=0,\\ u|_\Sigma &\in C(\overline{J};W_p^{3-3/p}(\Sigma)^n),\\ \partial_\nu w|_\Sigma,\ [\![\pi]\!] &\in C(\overline{J};W_p^{2-3/p}(\Sigma)),\\ h &\in C(\overline{J};W_p^{4-2/p}(\Sigma)). \end{aligned}$$

This observation motivates the following definition of initial states.

4.29. Definition (State space $X_p(\tau,M)\subset\mathbb{X}_p$)**.** (i) Given $p\in(3,\infty)$, we let $\mathbb{X}_p$ denote the Banach space of all pairs (u_0,h_0), which satisfy the conditions

$$\begin{aligned} u_0 &\in W_p^{2-2/p}(\Omega\setminus\Sigma;\mathbb{R}^n),\ u_0|_{\partial\Omega}=0,\ [\![u_0]\!]=0,\\ u_0|_\Sigma &\in W_p^{3-3/p}(\Sigma;\mathbb{R}^n),\\ \partial_\nu w_0|_\Sigma &\in W_p^{2-3/p}(\Sigma;\mathbb{R}),\\ h_0 &\in W_p^{4-2/p}(\Sigma;\mathbb{R}); \end{aligned}$$

and $\mathbb{X}_p$ is equipped with the norm

$$\|(u_0,h_0)\|_{\mathbb{X}_p} := \|u_0\|_{W_p^{2-2/p}(\Omega\setminus\Sigma)} + \|u_0|_\Sigma\|_{W_p^{3-3/p}(\Sigma)} + \|\partial_\nu w_0|_\Sigma\|_{W_p^{2-3/p}(\Sigma)} + \|h_0\|_{W_p^{4-2/p}(\Sigma)}.$$

(ii) Given $p \in (\max\{3,(n+2)/2\},\infty)$ and $\tau \in (1+n/p, 4-1/p]$, we choose the number $\delta_h = \delta_h(\Omega,\Sigma,p,\tau) > 0$ such that both assertions of Theorem 4.15 are valid. For given $M \in (\delta_h^{-1},\infty]$, the (nonlinear) *state space* $X_p(\tau,M)$ consists of all pairs $(u_0,h_0) \in \mathbb{X}_p$ with

$$\|(u_0,h_0)\|_{\mathbb{X}_p} < M,$$

which satisfy the compatibility conditions

$$\operatorname{div} u_0 = F_d(u_0,h_0) = -\sum\nolimits_{j,l,m} (u_0)_l \partial_l \partial_m (\Theta_{h_0})_j \partial_j (\Theta_{h_0}^{-1})_m, \tag{4.80a}$$

$$G_v(0,0;u_0,h_0) = 0, \tag{4.80b}$$

the smallness condition

$$\|h_0\|_{W_p^{\tau-1/p}(\Sigma)} < \delta_h - M^{-1}, \tag{4.81}$$

and the well-posedness condition

(4.82)

$$\inf_{x\in\Sigma} d_0(D_\Sigma(u_0)(x)) = \inf_\Sigma \Big(\sigma + (\lambda_s - \mu_s)\operatorname{tr} D_\Sigma(u_0) + 2\mu_s \min_{\zeta\in\mathbb{R}^n,\,|\zeta|=1} \zeta^\top [D_\Sigma(u_0)]\zeta\Big) > M^{-1}.$$

For given $\eta \in (0,\infty)$, we further let

$$X_p(\tau,M,\eta) := \{(u_0,h_0) \in X_p(\tau,M) : \|h_0\|_{W_p^{\tau-1/p}(\Sigma)} < \eta\}.$$

4.30. Remark. The compatibility conditions (4.80) arise since both spaces $\mathbb{F}_d$ and $\mathbb{G}_v$ have well-defined initial traces. There is no compatibility condition for L_w and G_w since the initial value $[\![\pi]\!]|_{t=0}$ is not prescribed. Condition (4.81) and $p > (n+2)/3$ allow to define the diffeomorphism $\Theta_{h_0}\colon \Omega \to \Omega$ with $\Theta_{h_0}|_\Sigma\colon \Sigma \to \Gamma_{h_0}$ by means of Theorem 4.15. With $p > \max\{3,(n+2)/2\}$ we obtain $u_0|_\Sigma \in C^1(\Sigma)$, and condition (4.82) will be used to employ the linear solution operator from Theorem 3.21. Equipped with the induced metric of $\mathbb{X}_p$, the space $X_p(\tau,M)$ is a metric space. If $M_0 \le M$ and $\eta_0 \le \eta$, then $X_p(\tau,M_0,\eta_0) \subset X_p(\tau,M,\eta)$.

Next, we construct functions $(u,\pi,h) \in \mathbb{E}(T_0)$ satisfying the initial condition $(u,h)|_{t=0} = (u_0,h_0)$ for given $(u_0,h_0) \in X_p \in \{X_p(\tau,M), X_p(\tau,M,\eta)\}$, together with corresponding interior data $(f_u,f_d) \in \mathbb{F}_u \times \mathbb{F}_{d,\Sigma}$. We also show that the trajectories $t \mapsto (u(t,\cdot),h(t,\cdot))$ remain in X_p.

4.31. Lemma. *Let $p \in (\max\{3,(n+2)/2\},\infty)$ and $\tau \in (1+n/p, 4-1/p]$.*

(i) For every $T_0 \in (0,\infty)$, there exists a bounded linear operator

$$(u_0,h_0) \mapsto (u,\pi,h,f_u,f_d),$$
$$X_p(\tau,\infty) \to \mathbb{E}_{u,v,w,\partial_\nu w}(T_0) \times \mathbb{E}_{\pi,[\![\pi]\!]}(T_0) \times \mathbb{E}_h(T_0) \times \mathbb{F}_u(T_0) \times \mathbb{F}_{d,\Sigma}(T_0)$$

whose values satisfy

$$(\rho\partial_t - \mu\Delta)u + \nabla\pi = f_u \quad \textit{in } J\times\Omega, \tag{4.83a}$$

$$\operatorname{div} u = f_d \quad \textit{in } J\times\Omega, \tag{4.83b}$$

$$\partial_t h - \nu_\Sigma\cdot u|_\Sigma = 0 \quad \textit{on } J\times\Sigma, \tag{4.83c}$$

$$u|_{t=0} = u_0 \quad \textit{in } \Omega, \tag{4.83d}$$

$$h|_{t=0} = h_0 \quad \textit{on } \Sigma. \tag{4.83e}$$

(ii) Moreover, if $\tau < 4-1/p$ and $M_0 < M$, then there exists $T \in (0,T_0]$ such that

$$(u,h) \in C([0,T];X_p(\tau,M)) \quad \textit{for all } (u_0,h_0) \in X_p(\tau,M_0), \tag{4.84}$$

and for $M_0 < M$ and $\eta_0 < \eta$, there exists $T \in (0, T_0]$ such that

$$(u, h) \in C([0, T]; X_p(\tau, M, \eta)) \quad \text{for all } (u_0, h_0) \in X_p(\tau, M_0, \eta_0). \tag{4.85}$$

Proof. (i) Let $(u_0, h_0) \in X_p(\tau, \infty)$ be given. With $w_0 := \nu_\Sigma \cdot u_0|_\Sigma$, we define

$$h(t) := h_A(t) + h_B(t) := \left(2e^{-tA} - e^{-2tA}\right) h_0 + \left(e^{-tB} - e^{-2tB}\right) B^{-1} w_0, \quad w(t) := \partial_t h(t),$$

where the operators $A = \sqrt{1 - \Delta_\Sigma}$ and $B = 1 - \Delta_\Sigma$ are realized in $L_p(\Sigma)$. With $h_0 \in D_A(4 - 2/p, p)$ and Corollaries B.26 and B.58 on pages 155 and 163, we see that h_A belongs to $\mathbb{E}_h$. Similarly, with $B^{-1}w_0 \in W_p^{4-3/p}(\Sigma) = D_B(2 - 3/2p, p)$ and $w_0 \in W_p^{3-3/p}(\Sigma) = D_B(3/2 - 3/2p, p)$, we obtain $h_B \in W_p^{4-1/p}(J \times \Sigma)$ and $\partial_t h_B \in W_p^{3/2-1/2p}(J; L_p(\Sigma)) \cap L_p(J; W_p^{3-1/p}(\Sigma))$, and thus h_B belongs to $\mathbb{E}_h$. We conclude that h is well-defined in $\mathbb{E}_h$, the function $w = \partial_t h$ belongs to $\mathbb{E}_w$, both functions depend linearly and continuously on $w_0 \in W_p^{3-3/p}(\Sigma)$ and $h_0 \in W_p^{4-2/p}(\Sigma)$, and (4.83c) is satisfied.

Next, with the operator $\widetilde{\Delta}_\Sigma = g^{\alpha\beta}\widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta \colon H_p^2(\Sigma; T\Sigma) \to L_p(\Sigma; T\Sigma)$, we define

$$v(t, \cdot) := e^{-t(1-\widetilde{\Delta}_\Sigma)}(P_\Sigma u_0|_\Sigma).$$

The complexification of $1 - \widetilde{\Delta}_\Sigma$ belongs to the class $\mathcal{RS}(L_p(\Sigma; (T\Sigma)_\mathbb{C}))$ with $\mathcal{R}$-angle zero by Corollary B.59, and to $\mathcal{H}^\infty(W_p^{1-1/p}(\Sigma; (T\Sigma)_\mathbb{C}))$ by Theorem B.27. Hence the semigroup $e^{t\widetilde{\Delta}_\Sigma}$ is analytic in $W_p^{1-1/p}(\Sigma; (T\Sigma)_\mathbb{C})$, and from Theorem B.25 we infer that v belongs to

$$W_p^{3/2-1/2p}(J; L_p(\Sigma; T\Sigma)) \cap L_p(J; W_p^{3-1/p}(\Sigma; T\Sigma)) \hookrightarrow \mathbb{E}_v.$$

Let us construct the divergence data f_d on $J \times (\Omega \setminus \Sigma)$. From $p > (n+2)/2$, Sobolev's embedding (B.1), and Lemma B.81, we deduce that $W_p^{2-2/p}(\Omega \setminus \Sigma)$ and $W_p^{2-2/p}(\Sigma)$ are multiplication algebras. The compatibility condition (4.80a) implies $\operatorname{div} u_0 \in W_p^{2-2/p}(\Omega \setminus \Sigma)$ and $\operatorname{div} u_0|_\Sigma \in W_p^{2-2/p}(\Sigma)$. By Corollary B.58 and Theorem B.25, the function $f_{d\Sigma}(t) := e^{-t(1-\Delta_\Sigma)}(\operatorname{div} u_0|_\Sigma)$ belongs to $H_p^1(J; L_p(\Sigma)) \cap L_p(J; H_p^2(\Sigma)) \hookrightarrow \mathbb{G}_w$. Let $\tilde{f}_d$ solve the heat problem $(\partial_t - \Delta)\tilde{f}_d = 0$ in $J \times \Omega_\pm$, $\tilde{f}_d|_{\partial\Omega} = 0$, $\tilde{f}_d|_\Sigma = f_{d\Sigma}$, and $\tilde{f}_d|_{t=0} = \operatorname{div} u_0$. Then we let $f_d|_{\Omega_\pm} := \tilde{f}_d|_{\Omega_\pm} - \langle \tilde{f}_d \rangle_{\Omega_\pm} \mp |\Sigma| \langle w \rangle_\Sigma$, and this function f_d belongs to $H_p^1(J; L_p(\Omega)) \cap L_p(J; H_p^2(\Omega \setminus \Sigma))$. In view of $\int_\Omega f_d \, dx = 0$ and the Poincaré-Wirtinger inequality, it also belongs to $H_p^1(J; \hat{H}_p^{-1}(\Omega))$, and hence $f_d \in \mathbb{F}_{d,\Sigma}$.

Next, we obtain the bulk velocity field $u \in \mathbb{E}_{u,v,w,\partial_\nu w}$ from the solution (u, q) of the one-phase Stokes problems $(\rho_\pm \partial_t - \mu_\pm \Delta)u_\pm + \nabla q_\pm = 0$ in $\Omega_\pm$, $\operatorname{div} u = f_d$ in Ω, $u_+|_{\partial\Omega} = 0$, and $u_\pm|_\Sigma = v + w\nu_\Sigma$ with [BP07, Theorem 4.1]. For the construction of π, we first define $g_0 = [\![q_0]\!]$ by eliminating $[\![q_0]\!]$ from the equation $G_w(0, 0; u_0, q_0, h_0) = 0$. Then $t \mapsto g(t) := e^{-t(1-\Delta_\Sigma)} g_0$ belongs to $\mathbb{G}_w$. The function $\pi \in \mathbb{E}_{\pi,[\![\pi]\!]}$ is defined with Theorem 2.3 as the unique solution of the weak transmission problem $\langle \nabla\pi, \nabla\varphi \rangle_\Omega = -\langle (\rho\partial_t - \mu\Delta)u, \nabla\varphi \rangle_\Omega$ for all $\varphi \in \dot{H}_{p'}^1(\Omega)$ and $[\![\pi]\!] = g$. Finally, the function $f_u := (\rho\partial_t - \mu\Delta)u + \nabla\pi$ belongs to $\mathbb{F}_u$. Therefore assertion (i) is valid.

(ii) We employ the following estimates, which follow from Sobolev's embedding, Lemma 3.18, the mixed derivative embeddings, and Theorem B.25. For $\delta \in (0, 1 - 1/p]$, we have

$$\begin{aligned} \|h_A\|_{H_p^{1/p+\delta}(0,T_0; W_p^{s-1/p}(\Sigma))} &\le C(T_0) \|h_A\|_{H_p^1(0,T_0; W_p^{s-1+\delta}(\Sigma)) \cap L_p(0,T_0; W_p^{s+\delta}(\Sigma))} \\ &\le C(T_0) \|h_0\|_{W_p^{s-1/p+\delta}(\Sigma)} \le C(T_0) M_0, \end{aligned}$$

provided that $1 - \delta \le s < 4 - 1/p$ and $\delta \le 4 - 1/p - s$; and

$$\begin{aligned} \|h_B\|_{H_p^{1/p+\delta}(0,T_0; W_p^{s-1/p}(\Sigma))} &\le C(T_0) \|h_B\|_{H_p^1(0,T_0; W_p^{s-2+2\delta}(\Sigma)) \cap L_p(0,T_0; W_p^{s+1/p+2\delta}(\Sigma))} \\ &\le C(T_0) \|B^{-1} w_0\|_{W_p^{s-1/p+2\delta}(\Sigma)} \le C(T_0) M_0, \end{aligned}$$

provided that $2 - 2\delta \le s < 5 - 2/p$ and $\delta \le 5/2 - 1/p - s/2$. Thus, for $s = \tau$, some $\delta_0 > 0$, and all $\delta \in (0, \delta_0]$, $T \in (0, T_0]$, and $(u_0, h_0) \in X_p(\tau, M_0)$, we have

$$\|h(\cdot) - h_0\|_{{}_0C([0,T];W_p^{\tau-1/p}(\Sigma))} \le T^{\delta/2} C(\delta, T_0) \|h(\cdot) - h_0\|_{{}_0H_p^{1/p+\delta}(0,T;W_p^{\tau-1/p})} \le T^{\delta/2} C(\delta, T_0) M_0.$$

Next, recall that $w = \partial_t h_A + \partial_t h_B$. For $\delta \in (0, 1 - 1/p]$, we have

$$\begin{aligned}\|\partial_t h_A\|_{H_p^{1/p+\delta}(0,T_0;W_p^{s-1/p}(\Sigma))} &\le C(T_0) \|A h_A\|_{H_p^1(0,T_0;W_p^{s-1+\delta}(\Sigma)) \cap L_p(0,T_0;W_p^{s+\delta}(\Sigma))} \\ &\le C(T_0) \|A h_0\|_{W_p^{s-1/p+\delta}(\Sigma)} \le C(T_0) M_0,\end{aligned}$$

provided that $1 - \delta \le s < 3 - 1/p$ and $\delta \le 3 - 1/p - s$;

$$\begin{aligned}\|\partial_t h_B\|_{H_p^{1/p+\delta}(0,T_0;W_p^{s-1/p}(\Sigma))} &\le C(T_0) \|B h_B\|_{H_p^1(0,T_0;W_p^{s-2+2\delta}(\Sigma)) \cap L_p(0,T_0;W_p^{s+1/p+2\delta}(\Sigma))} \\ &\le C(T_0) \|w_0\|_{W_p^{s-1/p+2\delta}(\Sigma)} \le C(T_0) M_0,\end{aligned}$$

provided that $2 - 2\delta \le s < 3 - 2/p$ and $\delta \le 3/2 - 1/p - s/2$; and

$$\begin{aligned}\|v\|_{H_p^{1/p+\delta}(0,T_0;W_p^{s-1/p}(\Sigma))} &\le C(T_0) \|v\|_{H_p^1(0,T_0;W_p^{s-2+2\delta}(\Sigma)) \cap L_p(0,T_0;W_p^{s+1/p+2\delta}(\Sigma))} \\ &\le C(T_0) \|v_0\|_{W_p^{s-1/p+2\delta}(\Sigma)} \le C(T_0) M_0,\end{aligned}$$

provided that $2 - 2\delta \le s < 3 - 2/p$ and $\delta \le 3/2 - 1/p - s/2$. Thus, for given $s \in (1 + n/p, 3 - 2/p)$ there exists $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0]$, $T \in (0, T_0]$, and $(u_0, h_0) \in X_p(\tau, M_0)$, we have

$$\begin{aligned}\|u(\cdot) - u_0\|_{{}_0C([0,T];C^1(\Sigma))} &\le T^{\delta/2} C(\delta, T_0) \|v(\cdot) - v_0 + (w(\cdot) - w_0)\nu_\Sigma\|_{{}_0W_p^{1/p+\delta}(0,T;W_p^{s-2/p})(\Sigma)} \\ &\le T^{\delta/2} C(\delta, T_0) M_0.\end{aligned}$$

We conclude that there exists $T = T(M_0, M) > 0$ such that

$$\sup_{t \le T} \|h(t, \cdot)\|_{W_p^{\tau-1/p}(\Sigma)} < \delta_h - M^{-1}, \quad \inf_{t \le T} \inf_{\Sigma} d_0(u(t, \cdot)) > M^{-1},$$

for all $(u_0, h_0) \in X_p(\tau, M_0)$. The assertion for $X_p(\tau, M, \eta)$ follows similarly. □

In order to formulate our main result, we first specify our notion of well-posedness.

4.32. Definition. Let $X_p = X_p(\tau, M, \eta)$ and $\mathbb{E}$ have the same meaning as on pages 94 and 122. Problem (4.79) = (T) is called *locally well-posed* in X_p with respect to $\mathbb{E}$, if

(i) for every $z_{0*} \in X_p$ there exist $T > 0$ and $\delta > 0$ such for all $z_0 = (u_0, h_0) \in X_p \cap B_\delta^{\mathbb{X}_p}(z_{0*})$, problem (4.79) has a unique $\mathbb{E}$-solution (u, π, h) on $(0, T)$,

(ii) the map $z_0 \mapsto (u, \pi, h)$, $X_p \cap B_\delta^{\mathbb{X}_p}(z_{0*}) \to \mathbb{E}(T)$ is continuous,

(iii) the trajectory $t \mapsto (u(t), h(t))$ belongs to $C([0, T]; X_p)$, and the map $z_0 \mapsto (u, h)$, $X_p \cap B_\delta^{\mathbb{X}_p}(z_{0*}) \to C([0, T]; X_p)$ is continuous.

Our main result for the transformed problem (4.79) = (T) is the following.

4.33. Theorem (Main result). *Let $p > n + 2$, $\tau \in (3 + n/p, 4 - 1/p)$, and $M < \infty$. Then there exists $\eta > 0$ such that problem (4.79) is locally well-posed in $X_p(\tau, M, \eta)$ with respect to $\mathbb{E}$.*

Proof. For given $z_0 = (u_0, h_0) \in X_p(\tau, \infty)$ we seek a solution of the form

$$z = (u, \pi, h) = z_* + z_\bullet \in \mathbb{E}(T) \quad \text{with } z_\bullet = (u_\bullet, \pi_\bullet, h_\bullet) \in {}_0\mathbb{E}(T),\ z_* = (u_*, \pi_*, h_*) \in \mathbb{E}(T)$$

on some small time interval $J = (0, T)$ such that

$$z_\bullet|_{t=0} = (u_\bullet, h_\bullet)|_{t=0} = (0, 0), \quad z_*|_{t=0} = (u_*, h_*)|_{t=0} = (u_0, h_0) = z_0.$$

We abbreviate the transformed problem (4.79) as

$$L(z_\bullet; z_*) = N(z_\bullet; z_*), \quad z_\bullet|_{t=0} = 0, \quad z_*|_{t=0} = z_0, \tag{4.86}$$

where $L(z_\bullet; z_*) = L(u_\bullet, \pi_\bullet, h_\bullet; u_*)$ and $N(z_\bullet; z_*) = N(u_\bullet, \pi_\bullet, h_\bullet; u_*, \pi_*, h_*)$ are given by

$$L(z_\bullet; z_*) = \begin{bmatrix} (\rho\partial_t - \mu\Delta)u_\bullet + \nabla\pi_\bullet \\ \operatorname{div} u_\bullet \\ L_v(u_\bullet, h_\bullet; u_*) \\ L_w(u_\bullet, \pi_\bullet, h_\bullet; u_*) \\ \partial_t h_\bullet - w_\bullet \end{bmatrix}, \quad N(z_\bullet; z_*) = \begin{bmatrix} F_u(u_* + u_\bullet, \pi_* + \pi_\bullet, h_* + h_\bullet) \\ F_d(u_* + u_\bullet, h_* + h_\bullet) - \operatorname{div} u_* \\ G_v(u_\bullet, h_\bullet; u_*, h_*) \\ G_w(u_\bullet, h_\bullet; u_*, \pi_*, h_*) \\ 0 \end{bmatrix}.$$

(i) *Construction of* z_*. Let $M \in (\delta_h^{-1}, \infty]$, let $(u_0, h_0) \mapsto (u_*, \pi_*, h_*, f_u, f_d)$ denote the bounded linear operator from Lemma 4.31, put $E_T(u_0, h_0) := (u_*, \pi_*, h_*)$, and let $\mathcal{P}_{R,T_0}$ denote the set of admissible parameters u_* from page 82. Then for given $M_0 < M$ and $\eta_0 < \eta$, there are $T_0 > 0$ and $R \geq 1$ such that the realizations

$$\begin{aligned} E_T \colon X_p(\tau, M_0) &\to \{(u_*, \pi_*, h_*) \in \mathbb{E}(T) : u_* \in \mathcal{P}_{R,T_0},\ (u_*, h_*) \in C([0,T]; X_p(\tau, M))\}, \\ E_T \colon X_p(\tau, M_0, \eta_0) &\to \{(u_*, \pi_*, h_*) \in \mathbb{E}(T) : u_* \in \mathcal{P}_{R,T_0},\ (u_*, h_*) \in C([0,T]; X_p(\tau, M, \eta))\} \end{aligned}$$

are linear and bounded for every $T \in (0, T_0]$.

(ii) *Strategy to determine* $z_\bullet$. It remains to determine the solution $z_\bullet \in {}_0\mathbb{E}(T)$ of the equation $L(z_\bullet; E_T(z_0)) = N(z_\bullet; E_T(z_0))$. In Theorem 3.21, we have shown that $L(\cdot; z_*) \colon {}_0\mathbb{E}(T) \to {}_0\mathbb{F}(T)$ is uniformly invertible with respect to $T \in (0, T_0]$ and $u_* \in \mathcal{P}_{R,T_0}$, for given $T_0 \in (0, \infty)$ and $R \in [1, \infty)$. Therefore we intend to apply Banach's fixed point theorem to the map

$$\begin{aligned} F \colon (z_\bullet, z_0) &\mapsto [L(\cdot; E_T(z_0))]^{-1} N(z_\bullet; E_T(z_0)), \\ \{(z_\bullet, z_0) &\in {}_0\mathbb{E}(T) \times X_p(\tau, M, \eta)\} \to {}_0\mathbb{E}(T), \end{aligned}$$

with suitable $\eta > 0$ and $T > 0$, depending on $M \in (\delta_h^{-1}, \infty)$. To this end we will show that

$$\|F(z_\bullet; z_0)\|_{{}_0\mathbb{E}(T)} + \|\partial_{z_\bullet} F(z_\bullet; z_0)\|_{\mathcal{B}({}_0\mathbb{E}(T))} \to 0 \quad \text{as } T \to 0,\ \|h_0\|_{W_p^{\tau - 1/p}(\Sigma)} \to 0, \tag{4.87}$$

uniformly with respect to $z_0 \in X_p(\tau, M)$ and $z_\bullet \in \overline{\mathbb{B}}_r(T)$, where

$$\overline{\mathbb{B}}_r(T) := \{z_\bullet = (u_\bullet, \pi_\bullet, h_\bullet) \in {}_0\mathbb{E}(T) : \|z_\bullet\|_{{}_0\mathbb{E}(T)} \leq r\}.$$

(iii) *Properties of* F. Let $T_0 \in (0, \infty)$, $T \in (0, T_0]$, and $R \in [1, \infty)$. The map $u_* \mapsto L(\cdot; u_*)$, $\mathbb{E}_{u,v,w}(T_0) \to \mathcal{B}({}_0\mathbb{E}(T); {}_0\mathbb{F}(T))$ is affine and therefore uniformly continuous on $B_R^{\mathbb{E}_{u,v,w}(T_0)}$. Theorem 3.21 implies that $u_* \mapsto [L(\cdot; u_*)]^{-1}$, $\mathcal{P}_{R,T_0} \to \mathcal{B}({}_0\mathbb{F}(T); {}_0\mathbb{E}(T))$ is uniformly bounded. Since E_T is linear and bounded, the map $z_0 \mapsto [L(\cdot; E_T(z_0))]^{-1}$, $X_p(\tau, M) \to \mathcal{B}({}_0\mathbb{F}(T); {}_0\mathbb{E}(T))$ is uniformly continuous. The function $N(z_\bullet; E_T(z_0))$ and its derivative with respect to $z_\bullet$ depend polynomially on the functions $z_\bullet$, $z_* = E_T(z_0)$, $\partial_x \Theta_h$, $[\partial_x \Theta_h]^{-1}$, β_h, and β_h^{-1}, where $h = h_* + h_\bullet$. From estimate (4.74) we infer that there exists $\delta > 0$ such that

$$\|h_\bullet\|_{{}_0C([0,T]; W_p^{\tau - 1/p}(\Sigma))} \leq T^\delta C(\delta, T_0) \|h_\bullet\|_{{}_0\mathbb{E}_h(T)} \quad \text{for } h_\bullet \in {}_0\mathbb{E}_h(T),\ T \in (0, T_0].$$

Hence for given $M \in (\delta_h^{-1}, \infty)$ there exist T_0, $r \in (0, \infty)$ such that the maps

$$\begin{aligned} (z_\bullet; z_0) &\mapsto F(z_\bullet; z_0), & \{(z_\bullet, z_0) \in \overline{\mathbb{B}}_r(T) \times X_p(\tau, M)\} &\to {}_0\mathbb{E}(T), \\ (z_\bullet; z_0) &\mapsto \partial_{z_\bullet} F(z_\bullet; z_0), & \{(z_\bullet, z_0) \in \overline{\mathbb{B}}_r(T) \times X_p(\tau, M)\} &\to \mathcal{B}({}_0\mathbb{E}(T)) \end{aligned}$$

are well-defined and uniformly continuous for every $T \in (0, T_0]$.

Corollaries 4.27 and 4.28 and Lemma 4.26 yield

$$\|N(z_\bullet; E_T(z_0))\|_{{}_0\mathbb{F}(T)} + \|\partial_{z_\bullet} N(z_\bullet; E_T(z_0))\|_{{}_0\mathbb{E}(T) \to {}_0\mathbb{F}(T)} \to 0 \quad \text{as } T \to 0,\ \|h_0\|_{W_p^{\tau - 1/p}(\Sigma)} \to 0,$$

uniformly with respect to $z_0 \in X_p(\tau, M)$ and $z_\bullet \in \overline{\mathbb{B}}_r(T)$. Therefore (4.87) is valid.

(iv) *Strict contraction.* Let us prove that $F(\cdot; z_0)$ is a strict contraction within the closed set $\overline{\mathbb{B}}_r(T)$ for some r, $T > 0$. From estimate (4.87) we infer that for given $q \in (0,1)$ and $M \in (\delta_h^{-1}, \infty)$ there exist positive numbers η, T_0, and r such that

$$\|\partial_{z_\bullet} F(z_\bullet; z_0)\|_{\mathcal{B}({}_0\mathbb{E}(T))} \le q \quad \text{for all } z_\bullet \in \overline{\mathbb{B}}_r(T),\ T \in (0, T_0],\ z_0 \in X_p(\tau, M, \eta).$$

Estimate (4.87) and the differentiability of $F(\cdot, z_0)$ imply that there are positive numbers η, T_0, and r such that

$$\|F(z_\bullet; z_0)\|_{{}_0\mathbb{E}(T)} \le q\, \|z_\bullet\|_{{}_0\mathbb{E}(T)} + \|F(0; z_0)\|_{{}_0\mathbb{E}(T)} \le r$$

for all $z_\bullet \in \overline{\mathbb{B}}_r(T)$, $T \in (0, T_0]$, and $z_0 \in X_p(\tau, M, \eta)$. Therefore $F(\cdot; z_0)$ maps $\overline{\mathbb{B}}_r(T)$ into itself as a q-contraction.

(v) Banach's fixed point theorem implies that $F(\cdot; z_0)$ has a unique fixed point $z_\bullet$ within $\overline{\mathbb{B}}_r(T) \subset {}_0\mathbb{E}(T)$ and $z_\bullet$ depends continuously on $z_0 \in X_p(\tau, M, \eta)$. Moreover, $z_\bullet + E_T(z_0)$ is an $\mathbb{E}$-solution of problem (4.79) on $(0, T)$ and $(u, h) = (u_* + u_\bullet, h_* + h_\bullet)$ belongs to $C([0, T]; X_p(\tau, \infty))$. Let us show that its trajectory remains in $X_p(\tau, M, \eta)$. For given $z_{0*} \in X_p(\tau, M, \eta)$ there are $M_0 < M$ and $\eta_0 < \eta$ such that z_{0*} belongs to $X_p(\tau, M_0, \eta_0)$. Therefore Lemma 4.31 yields some numbers $T_1 \in (0, T_0]$ and $r_1 \le r$ such that, given $z_0 \in X_p(\tau, M, \eta) \cap B_\delta^{\mathbb{X}_p}(z_{0*})$, $T \in (0, T_1]$, and $z_\bullet \in \overline{\mathbb{B}}_{r_1}(T)$, the solution $z = z_\bullet + E_T(z_0)$ satisfies $(u, h) \in C([0, T]; X_p(\tau, M, \eta))$.

(vi) In order to prove uniqueness within the larger space $\mathbb{E}(T) = E_T(z_0) + {}_0\mathbb{E}(T)$ for $T < T_1$, we assume that there is a different $\mathbb{E}$-solution $z^1 = z_* + z_\bullet^1$ on $(0, T)$. Since $F(\cdot, z_0)$ has at most one fixed point within $\overline{\mathbb{B}}_r(T)$, the triple $z_\bullet^1 \in {}_0\mathbb{E}(T)$ does not belong to $\overline{\mathbb{B}}_r(T)$. But since the norm of $\mathbb{E}(T)$ consists of integrals over $(0, T)$, we have $\|z_\bullet^1\|_{\mathbb{E}(T')} \to 0$ as $T' \to 0$, and hence $z_\bullet$ and $z_\bullet^1$ coincide on some interval $(0, T')$. We may assume that this interval is maximal in the sense that for every $\varepsilon > 0$, the triples $z_\bullet$ and $z_\bullet^1$ do not coincide on $(T', T' + \varepsilon)$. Since $z^1|_{t=T'} = z|_{t=T'}$ belongs to $X_p(\tau, M, \eta)$, we can repeat the fixed point argument and obtain a contradiction. Hence problem (4.79) has at most one $\mathbb{E}$-solution. The proof of Theorem 4.33 is complete. □

APPENDIX A

Differential geometry of hypersurfaces in $\mathbb{R}^n$

We provide results on hypersurfaces in the n-dimensional Euclidean space $\mathbb{R}^n$ that are used in the main part of this thesis. Kimura [Kim08] and Prüss and Simonett [PS13] developed such a theory of hypersurfaces that is applicable for moving boundary problems.

A.1. Classes of hypersurfaces in $\mathbb{R}^n$

We will define hypersurfaces in terms of parametrizations over hyperplanes, where the hypersurface is locally represented as a translated and rotated graph of a scalar height function. The regularity of that surface is defined by the regularity of its height functions. We next introduce tangent vectors, normal vectors, and differential operators and characterize the regularity of a hypersurface by the regularity of its normal. According to *Einstein's summation convention*, we always sum over repeated greek indices $\alpha, \beta, \ldots \in \{1, \ldots, n-1\}$, whereas latin indices $i, j, \ldots \in \{1, \ldots, n-1\}$ denote free indices.

A.1. Definition. We say that $\Sigma \subset \mathbb{R}^n$ is a *Lipschitz hypersurface* or *hypersurface of class* C^{1-}, if every point $p \in \Sigma$ has a neighborhood U of p in Σ, such that there are a hyperplane

$$\nu_0^\perp := \{x \in \mathbb{R}^n : x \cdot \nu_0 = 0\} \quad \text{with } \nu_0 \in \mathbb{R}^n,\ |\nu_0| = 1,$$

a point $p_0 \in \mathbb{R}^n$, a number $r > 0$, and a Lipschitz function

$$h\colon \nu_0^\perp \cap \overline{B}_r(0) := \{u \in \nu_0^\perp : |u| \leq r\} \to \mathbb{R},$$

such that U is parametrized by

$$\varphi\colon \nu_0^\perp \cap B_r(0) \subset \nu_0^\perp \to \Sigma, \quad u \mapsto p_0 + u + h(u)\nu_0.$$

We call φ a *parametrization of* Σ *over the hyperplane* $\nu_0^\perp$ with *height function* h.

(i) Σ is called C^k*-hypersurface* ($k \geq 1$) or hypersurface of class C^k, if the height function in every parametrization satisfies $h \in C^k(\overline{B}_r(0))$.

(ii) The notions of C^{k-}*-hypersurfaces*, *analytic* or C^ω*-hypersurfaces*, and W^s_p*-hypersurfaces* are defined accordingly, where $k \geq 1$, $s \geq 0$, and $p \in [1, \infty]$.

(iii) Σ is called *closed* resp. *compact* if it is closed resp. compact as a subset of $\mathbb{R}^n$.

A.2. Remark. Our definition of hypersurfaces exhibits the following topological properties.

(i) Clearly, every hypersurface is a C^{1-}-submanifold of $\mathbb{R}^n$ of dimension $n-1$. Therefore it may have a boundary and even an infinite number of connected components, but no self-intersections. A closed hypersurface has no boundary and a compact hypersurface has a finite number of connected components.

(ii) If Σ is a connected compact hypersurface, then Jordan's theorem asserts that Σ separates $\mathbb{R}^n$ in a bounded and an exterior domain, both having the same boundary Σ [Bro11]. Any connected closed hypersurface Σ separates $\mathbb{R}^n$ in precisely two domains [cf. Sam69].

A.3. Definition (Tangents, normal, differential operators)**.** Let Σ be a C^1-hypersurface in $\mathbb{R}^n$ and $\varphi\colon B_r(0) \subset \nu_0^\perp \to \Sigma$, $u \mapsto p_0 + u + h(u)\nu_0$ be a local parametrization.

(i) Let $f\colon \Sigma \to X$ be a map with values in a Banach space X. For a given parametrization φ and a basis $\{e_i\}_i$ of the hyperplane $\nu_0^\perp$, we define the *partial derivatives* of f by

$$\partial_i^\Sigma f(p) = \partial_i f(p) := \partial u_i (f \circ \varphi)(u) \quad \text{for } p = \varphi(u).$$

(ii) The *tangent space* $T_p\Sigma$ is the vector space $\{\varphi'(0)u : u \in \nu_0^\perp\}$, its elements are the *tangent vectors*. In particular, $\partial_i \varphi(u) =: \tau_i(p)$ with $p = \varphi(u)$ are tangent vectors and the set $\{\tau_i(p)\}_i$ is a basis for $T_p\Sigma$. The dual basis $\{\tau^i(p)\}_i$ of the *cotangent vectors* $\tau^i(p) = \tau^i_\Sigma(p)$ is defined by the relation $\tau_i(p) \cdot \tau^j(p) = \delta_i^j$. We will also use the notation $\tau_j^\Sigma(p)$ to indicate the dependence on the hypersurface Σ. In terms of the parametrization φ, we can choose the tangent vectors

$$\tau_j(\varphi(u)) = \tau_j^\Sigma(\varphi(u)) = e_j + \partial_j h(u)\nu_0.$$

(iii) For a closed connected C^1-hypersurface $\Sigma \subset \mathbb{R}^n$, there exists a *continuous unit normal field* $\nu_\Sigma\colon \Sigma \to \mathbb{R}^n$, also called *Gauss map* [cf. Sam69]. Locally, the unit normal can be chosen as

$$\nu_\Sigma(\varphi(u)) = \frac{\nu_0 - \nabla_u h(u)}{\sqrt{1 + |\nabla_u h(u)|^2}}, \tag{A.1}$$

where the $(n-1)$-dimensional *gradient* $\nabla_u h := e_\alpha \partial_{u_\alpha} h$ is considered as an element of $\mathbb{R}^n$.

(iv) The tangential projection $P_\Sigma(p)\colon \mathbb{R}^n \to T_p\Sigma$ onto $T_p\Sigma$ is given by

$$P_\Sigma = \tau^\alpha \otimes \tau_\alpha = \tau_\alpha \otimes \tau^\alpha = I - \nu_\Sigma \otimes \nu_\Sigma,$$

where $I \in \mathbb{R}^{n\times n}$ denotes the identity matrix.

(v) For a scalar function $f\colon \Sigma \to \mathbb{K}$, a possibly non-tangential vector field $u\colon \Sigma \to \mathbb{R}^n$, and a matrix field $S\colon \Sigma \to \mathbb{K}^{n\times n}$, we define the surface gradient

$$\nabla_\Gamma f := \tau^\alpha \partial_\alpha f, \quad \nabla_\Gamma u := \tau^\alpha \otimes \partial_\alpha u,$$

and the surface divergence

$$\operatorname{div}_\Gamma u := (\partial_\alpha u | \tau^\alpha), \quad \operatorname{div}_\Gamma S := (\partial_\alpha S)\tau^\alpha.$$

If Σ is of class C^{2-}, then we define the scalar Laplace-Beltrami operator

$$\Delta_\Gamma f := \operatorname{div}_\Gamma \nabla_\Gamma f = g^{\alpha\beta}(\partial_\alpha \partial_\beta f - \Lambda^\gamma_{\alpha\beta}\partial_\gamma f),$$

where $g_{ij} = \tau_i \cdot \tau_j$ are the compontents of the Riemannian metric tensor, the components g^{ij} are defined via $g^{i\alpha}g_{\alpha j} = \delta^i_j$, and $\Lambda_{ij,k} = \partial_i \tau_j \cdot \tau_k$ and $\Lambda^k_{ij} = g^{k\alpha}\Lambda_{ij,\alpha}$ are the *Christoffel symbols*.

A.4. Remark. We shall use further relations between height function h and normal ν_Σ of a C^1-hypersurface Σ in order to extend a given parametrization $\varphi\colon B_r(0) \subset \nu_0^\perp \to \Sigma$, $u \mapsto p_0 + u + h(u)\nu_0$. With the projection $P_0 := I - \nu_0 \otimes \nu_0$ of $\mathbb{R}^n$ onto the hyperplane $\nu_0^\perp$, we obtain $|P_0 \nu_\Sigma(\varphi(u))|^2 = 1 - (\nu_0|\nu_\Sigma(\varphi(u)))^2$ and therefore (A.1) implies

$$|\nabla h(u)|^2 = \frac{|P_0\nu_\Sigma(p)|^2}{1 - |P_0\nu_\Sigma(p)|^2} = \frac{1 - (\nu_0|\nu_\Sigma(p))^2}{(\nu_0|\nu_\Sigma(p))^2} \quad \text{for } p = \varphi(u),\ u \in B_r(0) \subset \nu_0^\perp. \tag{A.2}$$

This shows that, if we want to extend the domain $B_r(0)$, we have to ensure that ∇h remains bounded, which is equivalent to $\nu_\Sigma(\varphi(u)) \cdot \nu_0 \geq \eta$ for some $\eta \in (0, 1]$ and all u, where the optimal η and the Lipschitz constant $\|\nabla h\|_\infty$ are related by

$$\eta = (1 + \|\nabla h\|_\infty^2)^{-1/2}, \quad \|\nabla h\|_\infty = (\eta^{-2} - 1)^{1/2}.$$

For the height function h we obtain

$$\nabla h(u) = -\frac{P_0 \nu_\Sigma(p)}{\nu_0 \cdot \nu_\Sigma(p)} \quad \text{for } p = \varphi(u). \tag{A.3}$$

For a fixed basis $\{e_i\}_i$ of $\nu_0^\perp$, the parametrization φ induces the tangent vectors

$$\tau_j^\Sigma(p) = \partial_{u_j}\varphi(u) = e_j + \partial_j h(u)\nu_0 = e_j - \frac{e_j \cdot \nu_\Sigma(p)}{\nu_0 \cdot \nu_\Sigma(p)}\nu_0 \quad \text{for } p = \varphi(u).$$

Further properties of φ in terms of the intrinsic distance are given in Proposition A.12.

In the spirit of Prüss and Simonett [PS13], we can also characterize the regularity of a hypersurface by the regularity of its normal.

A.5. Proposition. *For a compact C^{1-}-hypersurface $\Sigma \subset \mathbb{R}^n$ with normal $\nu_\Sigma \in L_\infty(\Sigma; \mathbb{R}^n)$ and $k \in \mathbb{N}$, the following characterizations are valid.*

(i) Σ is a C^{k+1}-hypersurface if and only if $\nu_\Sigma \in C^k(\Sigma; \mathbb{R}^n)$.

(ii) Σ is a C^{k+1-}-hypersurface if and only if $\nu_\Sigma \in C^{k-}(\Sigma; \mathbb{R}^n)$.

(iii) Σ is an analytic hypersurface if and only if $\nu_\Sigma \in C^\omega(\Sigma; \mathbb{R}^n)$.

Proof. We employ local coordinates $u \in U \subset \mathbb{R}^{n-1}$ and, for simplicity, we neglect the rotation and translation; that is, we assume $Q = I$ and $p_0 = 0$. Then we can express the normal ν_Σ as

$$\nu(u, h(u)) = \beta(u)(-\nabla h(u), 1), \quad \beta(u) = (1 + |\nabla h(u)|^2)^{-1/2} \quad \text{for } u \in U \subset \mathbb{R}^{n-1}. \tag{A.4}$$

(i) If $\Sigma \in C^{k+1}$, then we have $h \in C^{k+1}$. With identity (A.1), this implies that $u \mapsto \nu(u, h(u))$ is C^k in local coordinates, which means that $\nu_\Sigma \in C^k(\Sigma; \mathbb{R}^n)$. Conversely, let $\nu_\Sigma \in C^k(\Sigma; \mathbb{R}^n)$. Then $\beta = \nu \cdot e_n$ is C^k and therefore also $\nabla h = -\beta^{-1} P_{1,\dots,n-1}\nu$ belongs to C^k by (A.3). Together with $h \in C^{1-}$ this gives $h \in C^{k+1}$.

(ii) This equivalence follows analogously.

(iii) It is sufficient to note that $h \in C^{1-}$ and $\nabla h \in C^\omega$ imply $h \in C^\omega$, since

$$h(u) - h(u_0) = \int_0^1 \nabla h(u_0 + s(u - u_0)) \cdot (u - u_0) ds = \sum_{k=0}^\infty \frac{\nabla h^{(k)}(u_0)}{(k+1)!}(u - u_0)^k \cdot (u - u_0). \quad \square$$

A.6. Remark. For a C^{2-}-hypersurface Σ in $\mathbb{R}^n$, we define the *Weingarten tensor*

$$L := L_\Sigma := -\nabla_\Sigma \nu_\Sigma = -\tau^\alpha \otimes \partial_\alpha \nu_\Sigma \colon \Sigma \to \mathbb{R}^{n\times n}.$$

For every $p \in \Sigma$, the matrix $L(p)$ is symmetric and vanishes on $\mathbb{R}\nu_\Sigma(p)$ so that $L(p)T_p\Sigma \subset T_p\Sigma$. The Weingarten tensor induces the *second fundamental form* $II_p(v, w) = L_\Sigma(p)v \cdot w = l_{\alpha\beta}(p)v^\alpha w^\beta$ for v, $w \in T_p\Sigma$. The eigenvalues $\kappa_j(p)$ of $L(p)$ are the *principal curvatures* of Σ at p and the corresponding eigenvectors are the principal directions [Kim08, Theorem 2.10]. For every C^{2-}-path $\gamma\colon [a, b] \to \Sigma$ with $|\gamma'(t)| = 1$ for all $t \in [a, b]$, the curvature of γ at $\gamma(t)$ is $\gamma''(t)$ and we have $|\gamma''(t)| \le |L(\gamma(t))|$. The $(n-1)$-fold *mean curvature* is given by

$$H_\Sigma := \kappa_1 + \cdots + \kappa_{n-1} = \operatorname{tr} L_\Sigma = -\operatorname{div}_\Sigma \nu_\Sigma.$$

The Christoffel symbols satisfy the following relations [PS13, (12), (14)],

$$\partial_i \tau_j = \Lambda_{ij}^\alpha \tau_\alpha + l_{ij}\nu, \quad \partial_i \tau^j = -\Lambda_{i\alpha}^j \tau^\alpha + l_i^j \nu.$$

We recall a well-known fact from differential geometry.

A.7. Theorem (see e. g. [Ale62]). *Let Σ be a compact connected C^2-hypersurface in $\mathbb{R}^n$ $(n \ge 2)$ with constant mean curvature. Then Σ is a sphere.*

A.2. The intrinsic distance of a hypersurface

In this thesis, a C^{1-}-hypersurface Σ is equipped with the Riemannian metric induced by the scalar product of $\mathbb{R}^n$; that is, for $p \in \Sigma$, a scalar product in $T_p\Sigma$ is defined by $(\tau|\tilde\tau)_p := \tau \cdot \tilde\tau$ and therefore $T_p\Sigma$ has the induced norm $|\tau| = (\sum_{j=1}^n (e_j \cdot \tau)^2)^{1/2}$. The *intrinsic distance* $\operatorname{dist}_\Sigma(p, q)$ for p, $q \in \Sigma$ is defined as the infimum of the lengths of all C^{1-}-curves in Σ joining p to q. In

Proposition A.12, we prove that the intrinsic distance and the induced norm of $\mathbb{R}^n$ are equivalent for compact connected C^{1-}-hypersurfaces. If Σ is of class C^{3-}, then we can find a curve γ from p to q with minimal length $l(\gamma) = \mathrm{dist}_\Sigma(p,q)$, which is a *minimizing geodesic* [see e. g. Car92, Chapter 3].

A.8. Remark. Geodesics can be defined for every C^{3-}-hypersurface $\Sigma \subset \mathbb{R}^n$ $(n \geq 2)$. Let $\gamma\colon [a,b] \to \Sigma$ be a C^1-curve and let $v\colon [a,b] \to T\Sigma$ be a *tangential vector field* along γ; that is, $v(t)$ belongs to the $T_{\gamma(t)}\Sigma$ and therefore has a representation $v(t) = v^\alpha(t)\tau_\alpha(\gamma(t))$. The *covariant derivative* of v along γ is defined by

$$\frac{Dv(t)}{Dt} := P_\Sigma(\gamma(t))\frac{dv(t)}{dt} = \partial_t v^\alpha(t)\tau_\alpha(\gamma(t)) + v^\alpha(t)\partial_t\gamma^\beta(t)P_\Sigma(\gamma(t))\partial_\beta\tau_\alpha(\gamma(t)).$$

We call a C^2-curve $\gamma\colon [a,b] \to \Sigma$ a *geodesic*, if it satisfies the *geodesic equation*

$$\frac{D}{Dt}\frac{d\gamma}{dt} = (P_\Sigma \circ \gamma)\gamma'' = 0 \quad \text{in } (a,b).$$

In local coordinates $x^i(t) = e^i \cdot \varphi^{-1}(\gamma(t))$, the geodesic equation becomes a system of ordinary differential equations

$$\frac{d^2x^i}{dt^2} + (\Lambda^i_{\alpha\beta} \circ x)\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} = 0. \tag{A.5}$$

Here the Christoffel symbols $\Lambda^k_{ij} = g^{k\alpha}(\partial_i\tau_j|\tau_\alpha)$ are locally Lipschitz continuous. From the theory of ordinary differential equations, we infer that (A.5) has a unique local C^{3-}-solution $t \mapsto x(t) = x(t;x_0,x_1)$ that satisfies prescribed initial conditions $x(t_0) = x_0$ and $x'(t_0) = x_1$. Moreover, $x(t;x_0,x_1)$ depends continuously on (x_0,x_1). Consequently, for given $p \in \Sigma$ and $v \in T_p\Sigma$, there exists a unique geodesic $t \mapsto \gamma(t;p,v)$ such that $\gamma(0;p,v) = p$ and $\gamma'(0;p,v) = v$.

The geodesics are homogeneous in the sense that for a geodesic $\gamma(\cdot;p,v)$ on $(-\delta,\delta)$ and every $\alpha > 0$, the map $t \mapsto \gamma(\alpha t;p,v)$ is also a geodesic on $(-\delta/\alpha,\delta/\alpha)$ and the identity $\gamma(\alpha t;p,v) = \gamma(t;p,\alpha v)$ applies to all $t \in (-\delta,\delta)$ [Car92, Lemma 3.2.6]. Moreover, the map

$$(t,(p,v)) \mapsto (\gamma(t;p,v),\gamma'(t;p,v))$$

is a local flow on $T\Sigma$, called the *geodesic flow*. We say that Σ is *(geodesically) complete*, if every geodesic $\gamma\colon [a,b] \to \Sigma$ can be extended to a geodesic $\tilde\gamma\colon \mathbb{R} \to \Sigma$. In this case, the geodesic flow is global with respect to $t \in \mathbb{R}$, $p \in \Sigma$, and $v \in T_p\Sigma$.

A geodesic locally minimizes the distance between two points in the sense that for every $t \in [a,b]$ there is $\varepsilon > 0$ such that $\mathrm{dist}_\Sigma(\gamma(t_1),\gamma(t_2)) = \int_{t_1}^{t_2}|\gamma'(s)|ds$ for all $t_1, t_2 \in [a,b]\cap(t-\varepsilon,t+\varepsilon)$ with $t_1 < t_2$ [Car92, Remark 3.3.8]. Conversely, if p, $q \in \Sigma$ are given, then every piecewise differentiable curve joining p to q with minimal length is a geodesic [Car92, Corollary 3.3.9].

A.9. Remark. For a C^{3-}-hypersurface Σ, we define the *exponential map*

$$\exp_p(v) := \gamma(1;p,v) = \gamma(|v|;p,v/|v|) \quad \text{for } p \in \Sigma,\ v \in B_r(0) \subset T_p\Sigma,$$

for some $r > 0$ [see Car92, Chapters 3, 13]. In view of

$$\left(\frac{d\exp_p}{dv}(0)\right)v = \frac{d}{dt}\exp_p(tv)\big|_{t=0} = \frac{d}{dt}\gamma(t;p,v)|_{t=0} = \gamma'(0;p,v) = v,$$

we see that $(d/dv)\exp_p(0) = P_\Sigma(p)$. Therefore, by the inverse function theorem, $\exp_p$ is a local C^1-diffeomorphism at $0 \in T_p\Sigma$ into Σ. The number

$$i(\Sigma,p) = \sup\{r > 0 : \exp_p\colon B_r(0) \subset T_p\Sigma \to \exp_p(B_r(0)) \subset \Sigma \text{ is a diffeomorphism}\}$$

is called the *injectivity radius* of Σ at p and $i(\Sigma) := \inf\{i(\Sigma,p) : p \in \Sigma\}$ is the injectivity radius of Σ. Clearly, if Σ is compact, then $i(\Sigma) > 0$. If $q \in \exp_p(B_{i(\Sigma)}(0))$, then there exists a unique geodesic joining p and q that minimizes $\mathrm{dist}_\Sigma(p,q)$ [Car92, Corollary 13.2.8].

The Hopf-Rinow theorem [HR31] characterizes geodesic completeness of general C^{3-}-class Riemannian manifolds.

A.10. Theorem (Hopf-Rinow [cf. Car92, Theorem 7.2.8])**.** *Let M be a Riemannian C^{3-}-manifold. Then the following assertions are equivalent.*

(i) The exponential map $\exp_p$ is defined on all of T_pM for every $p \in M$.
(ii) Every closed and bounded subset of M is compact.
(iii) The metric space $(M, \operatorname{dist}_M(\cdot,\cdot))$ is complete.
(iv) M is geodesically complete.
(v) There exists a sequence of compact subsets $K_j \subset M$ with $K_j \subset K_{j+1}$ and $\bigcup_j K_j = M$ such that if $q_j \notin K_j$, then $\operatorname{dist}_M(p, q_j) \to \infty$ for every $p \in M$.

If, in addition, M is connected, then any of the statements above implies that

(vi) For any $p, q \in M$, there exists a geodesic γ joining p and q with $l(\gamma) = \operatorname{dist}_M(p,q)$.

A.11. Corollary ([Car92, Corollaries 7.2.9, 7.2.10])**.** *Every compact Riemannian C^{3-}-manifold is complete and every closed C^{3-}-submanifold of a complete Riemannian C^{3-}-manifold is complete in the induced metric. In particular, every closed C^{3-}-hypersurface in $\mathbb{R}^n$ ($n \geq 2$) is complete.*

The following relations between intrinsic distance and Euclidean distance will be used later on for dealing with the intrinsic Slobodeckiĭ semi-norm.

A.12. Proposition. *Let $\Sigma \subset \mathbb{R}^n$ be a C^{1-}-hypersurface.*

(i) Let $p \in \Sigma$ be fixed, let $\varphi\colon T_p\Sigma \supset U \to \Sigma$, $u \mapsto p + u + h(u)\nu_\Sigma(p)$ be a parametrization over the tangent space where U is convex, and let $h \in C^{1-}(\overline{U})$ so that $\|\nabla h\|_\infty < \infty$. Then

$$|u - v| \leq |\varphi(u) - \varphi(v)| \leq \operatorname{dist}_\Sigma(\varphi(u), \varphi(v)) \leq (1 + \|\nabla h\|_\infty^2)^{1/2}|u - v| \quad \textit{for all } u,\ v \in U.$$

(ii) Let $\Sigma \in C^{1-}$ be compact and connected. Then $\operatorname{dist}_\Sigma(\cdot,\cdot)$ is bounded and for some $C \geq 1$ we have

$$|p - q| \leq \operatorname{dist}_\Sigma(p,q) \leq C|p - q| \quad \textit{for all } p, q \in \Sigma.$$

(iii) Let $\Sigma \in C^{2-}$ and $\|L_\Sigma\|_\infty < \infty$. If $p, q \in \Sigma$ satisfy $\operatorname{dist}_\Sigma(p,q) < \sqrt{2}\,\|L_\Sigma\|_\infty^{-1}$, then

$$|\nu_\Sigma(p) - \nu_\Sigma(q)| \leq \operatorname{dist}_\Sigma(p,q)\|L_\Sigma\|_\infty, \quad \nu_\Sigma(p)\cdot\nu_\Sigma(q) > 0.$$

(iv) Let $\Sigma \in C^{3-}$ and $\|L_\Sigma\|_\infty < \infty$. If $p, q \in \Sigma$ satisfy $\operatorname{dist}_\Sigma(p,q) < 2\,\|L_\Sigma\|_\infty^{-1}$, then

$$|p - q| \leq \operatorname{dist}_\Sigma(p,q) \leq \frac{|p - q|}{1 - \frac{1}{2}\operatorname{dist}_\Sigma(p,q)\,\|L_\Sigma\|_\infty}.$$

Proof. *(i)* For almost all $u \in U$, we have $\varphi'(u) = P_\Sigma(p) + \nu_\Sigma(p) \otimes \nabla h(u)$ and $|\varphi'(u)|^2 = 1 + |\nabla h(u)|^2$. The map $[0,1] \mapsto \varphi(u + (v-u)t)$ is a curve from $\varphi(u)$ to $\varphi(v)$ in Σ and therefore

$$\begin{aligned}
&\operatorname{dist}_\Sigma(\varphi(u), \varphi(v)) \leq \|\varphi'\|_\infty|u - v| \leq (1 + \|\nabla h\|_\infty^2)^{1/2}|u - v|,\\
&\operatorname{dist}_\Sigma(\varphi(u), \varphi(v)) \geq |\varphi(u) - \varphi(v)| = (|u - v|^2 + |(h(u) - h(v))\nu_\Sigma(p)|^2)^{1/2} \geq |u - v|.
\end{aligned}$$

(ii) For every $p \in \Sigma$, there exists $r(p) > 0$ such that $\Sigma \cap B_{r(p)}(p)$ can be parametrized over $T_p\Sigma$ via $\varphi_p(u) = p + u + h_p(u)\nu_\Sigma(p)$, where h satisfies $\|\nabla h_p\|_\infty \leq 1$. From (i) we obtain that

$$|q - \tilde{q}| \leq \operatorname{dist}_\Sigma(q, \tilde{q}) \leq \sqrt{2}|q - \tilde{q}| \quad \text{for } q, \tilde{q} \in \Sigma \cap B_{r(p)}(p).$$

In particular, we have $\operatorname{dist}_\Sigma(p,q) < \sqrt{2}r(p)$ for every $q \in \Sigma \cap B_{r(p)}(p)$. By compactness, there exist finitely many sets $\Sigma \cap B_{r_j}(p)$ with the above properties and $r_j = r(p_j)$ such that Σ is the union of these sets. Since Σ is connected, the numbers $\operatorname{dist}_\Sigma(p_j, p_k)$ have a finite maximum R. Therefore $\operatorname{dist}_\Sigma(\cdot,\cdot)$ is bounded by $R + 2\sqrt{2}\max r_j$.

Assume that the assertion is false. Then there exist $p_n, q_n \in \Sigma$ with $\operatorname{dist}_\Sigma(p_n, q_n) > n|p_n - q_n|$. Since Σ is compact we may assume that $p_n \to p \in \Sigma$ and $q_n \to q \in \Sigma$ and since $\operatorname{dist}_\Sigma$ is bounded, the p_n and q_n converge to the same limit $p = q$. But then almost all p_n, q_n are contained in some

$\Sigma \cap B_{r_0}(p_0)$ and hence $n|p_n - q_n| < \operatorname{dist}_\Sigma(p_n, q_n) \le \sqrt{2}|p_n - q_n|$, a contradiction. Therefore (ii) is valid.

(iii) For a curve γ joining q to p in Σ, we have

$$|\nu_\Sigma(p) - \nu_\Sigma(q)| = \left| \int_0^{l(\gamma)} \frac{d}{dt} \nu_\Sigma(\gamma(t))\, dt \right| = \left| \int_0^{l(\gamma)} L_\Sigma(\gamma(t))\gamma'(t)\, dt \right| \le l(\gamma)\, \|L_\Sigma\|_\infty .$$

The inequality $\nu_\Sigma(p) \cdot \nu_\Sigma(q) > 0$ is valid if and only if $|\nu_\Sigma(p) - \nu_\Sigma(q)|^2 = 2 - 2\nu_\Sigma(p) \cdot \nu_\Sigma(q) < 2$ and this is true if $\operatorname{dist}_\Sigma(p, q) < \sqrt{2}\|L_\Sigma\|_\infty^{-1}$.

(iv) Let γ be a geodesic from q to p with minimal length $l(\gamma) = \operatorname{dist}_\Sigma(p, q)$ and $|\gamma'| = 1$. Since $P_\Sigma(\gamma(t))\gamma''(t) = 0$ and $\nu_\Sigma(\gamma(t)) \cdot \gamma'(t) = 0$, we obtain $\gamma''(t) = \nu_\Sigma(\gamma(t))\,(L_\Sigma(\gamma(t))\gamma'(t) \cdot \gamma'(t))$. Hence

$$p - q = \int_0^{l(\gamma)} \gamma'(t)\, dt = l(\gamma)\, \gamma'(0) + \int_0^{l(\gamma)} \int_0^t \gamma''(s)\, ds\, dt,$$

$$|p - q| \ge l(\gamma) - \int_0^{l(\gamma)} \int_0^t |L_\Sigma(\gamma(s))|\, ds\, dt \ge l(\gamma) - \frac{l(\gamma)^2}{2}\, \|L_\Sigma\|_\infty .$$

This yields the assertion. □

The next result provides parametrizations that are defined on balls with uniform radius.

A.13. Proposition. *Let $\Sigma \subset \mathbb{R}^n$ $(n \ge 2)$ be a C^{2-}-hypersurface such that L_Σ is bounded and put $R_* := \sqrt{2}\|L_\Sigma\|_\infty^{-1} \in (0, \infty]$, $\delta(R) := 1 - R^2\|L_\Sigma\|_\infty^2/2 \in (0, 1]$, and $r(R) := R\delta(R)$ for $R \in (0, R_*)$. Then for every $x \in \Sigma$, there exists a parametrization*

$$\varphi_x \colon B_{r(R)}(0) \subset T_x\Sigma \to B_R^\Sigma(x), \quad u \mapsto x + u + h_x(u)\nu_\Sigma(x)$$

with height function $h_x \in C^{2-}(\overline{B_{r(R)}(0)})$ such that $h_x(0) = |\nabla h_x(u)| = 0$.

Proof. Given $x \in \Sigma$, there exists a parametrization $\varphi_x \colon V_x \to \Sigma$ on some small neighborhood V_x of the origin such that $\varphi_x(u) = x + u + h_x(u)\nu_\Sigma(x)$ for some $h_x \in C^{2-}(\overline{V_x})$ with $h_x(0) = |\nabla h_x(0)| = 0$. Our goal is to show that φ_x can be extended to map with the asserted properties. Such an extension is uniquely determined by the representation (A.3) of ∇h_x in terms of ν_Σ. The identity (A.2) shows that

$$|\nabla h_x(u)|^2 = (\nu_\Sigma(x)|\nu_\Sigma(\varphi_x(u)))^{-2} - 1 \quad \text{for } u \in \overline{V_x}.$$

With Remark A.4 we obtain

$$(\nu_\Sigma(x)|\nu_\Sigma(\varphi_x(u))) \ge \delta, \quad |\nabla h_x(u)|^2 \le \frac{1 - \delta^2}{\delta^2} \quad \text{for all } u \in \overline{V_x},\ x \in \Sigma.$$

Therefore we can extend h_x and φ_x uniquely onto $\overline{B_{r(R)}(0)}$. □

A.3. Neighborhoods of hypersurfaces

We show that every compact hypersurface of class C^{2-} satisfies a uniform ball condition and has a tubular neighborhood with uniform thickness. Within such a neighborhood, we study further hypersurfaces that are induced by height functions. For the corresponding diffeomorphism between the original and the new hypersurface, we derive an integral transformation formula that does not use local coordinates (see (A.12)). We also provide a level function for a possibly disconnected compact hypersurface.

A.14. Definition. A hypersurface $\Sigma \subset \mathbb{R}^n$ satisfies the *ball condition* of radius $r > 0$ at the point $p \in \Sigma$, if the open balls $B_r(p - r\nu_\Sigma(p))$ and $B_r(p + r\nu_\Sigma(p))$ do not intersect Σ. We say that Σ satisfies the *uniform ball condition* of radius $r > 0$, if it satisfies the ball condition of the same radius r at every $p \in \Sigma$.

A.15. Remark. Let S be a closed subset of $\mathbb{R}^n$ of the form $S = \mathbb{R}^n \setminus (\Omega_+ \cup \Omega_-)$ with disjoint open subsets Ω_+ and Ω_- of $\mathbb{R}^n$. As in [Kim08, Section 3.1], we define the *signed distance*

$$d(x) = \begin{cases} \operatorname{dist}(x, S) & \text{for } x \in \Omega_+, \\ 0 & \text{for } x \in S, \\ -\operatorname{dist}(x, S) & \text{for } x \in \Omega_-. \end{cases}$$

By [Kim08, Theorem 3.2], both maps $\operatorname{dist}(\cdot, S)$ and $d(\cdot)$ are Lipschitz continuous.

A.16. Definition ([cf. PS13]). A hypersurface $\Sigma \subset \mathbb{R}^n$ has a *tubular neighborhood* of radius $r > 0$, if the map

$$X\colon (p,t) \mapsto p + t\nu_\Sigma(p), \quad \Sigma \times (-r, r) \to B_r(\Sigma) := \{x \in \mathbb{R}^n : \operatorname{dist}(x, \Sigma) < r\} \tag{A.6}$$

is a homeomorphism; that is, X is continuous and bijective and therefore has a continuous inverse. We say that the tubular neighborhood is of class C^k ($k \geq 1$), if X is a C^k-diffeomorphism; that is, X is of class C^k and $\partial X(p,t)\colon T_p\Sigma \times \mathbb{R} \to \mathbb{R}^n$ is invertible for all (p,t). We decompose

$$X^{-1}(x) = (\Pi(x), d(x)) \quad \text{with } \Pi(x) = p \in \Sigma,\ d(x) = t \in (-r, r),\ x = \Pi(x) + d(x)\nu_\Sigma(\Pi(x)).$$

A.17. Proposition. *The following assertions are valid.*

(i) A closed hypersurface $\Sigma \subset \mathbb{R}^n$ has a tubular neighborhood of radius r if and only if it satisfies the uniform ball condition of radius r. In this case, it also has a tubular neighborhood of radius

$$r_\Sigma := \sup\{r > 0 : \Sigma \text{ has a tubular neighborhood of radius } r\}.$$

(ii) If Σ is a compact C^{2-}-hypersurface in $\mathbb{R}^n$, then it has a tubular neighborhood of radius $r_\Sigma > 0$, the homeomorphism X in (A.6) has a locally Lipschitz continuous inverse and the principal curvatures of Σ and the Weingarten map L_Σ are bounded by $1/r_\Sigma$ almost everywhere.

(iii) If Σ is a compact C^{k+1}-hypersurface ($k \geq 1$), then X is a C^k-diffeomorphism with derivative

$$\partial_{(p,t)}X(p,t)(v,s) = v + s\nu_\Sigma(p) - tL_\Sigma(p)v, \quad (p,t) \in \Sigma \times (-r,r), \quad (v,s) \in T_p\Sigma \times \mathbb{R}. \tag{A.7}$$

Proof. (i) *Ball condition $\Rightarrow$ tubular neighborhood.* Suppose that Σ satisfies the uniform ball condition of radius r. It remains to show that the continuous map $X\colon (p,t) \mapsto p + t\nu_\Sigma(p)$, $\Sigma \times (-r,r) \to B_r(\Sigma) := \{x \in \mathbb{R}^n : \operatorname{dist}(x,\Sigma) < r\}$ is bijective.

Surjectivity. Given $x \in B_r(\Sigma)$, we put $\delta := \operatorname{dist}(x,\Sigma) < r$. Then $B_\delta(x) \cap \Sigma$ is empty and, since Σ is closed, there exists $p \in \Sigma$ with $|x - p| = \delta$, so that $p \in \partial B_\delta(x)$. Since none of the balls $B_{\delta'}(x)$ ($\delta' < \delta$) intersects Σ, the vector $p - x$ is normal to $T_p\Sigma$ and hence x belongs to the image of X.

Injectivity. Suppose that for some $x \in B_r(\Sigma)$ there are two different points $p, q \in \Sigma$ such that $X(p,s) = X(q,t) = x$ for some $s, t \in (-r,r)$. Then we must have $|s| = |p - x| = |q - x| = |t|$. Therefore the balls $B_\sigma(p + \sigma\nu_\Sigma(p))$ with $\sigma \in (s,r)$ if $s > 0$ and $\sigma \in (-r,s)$ if $s < 0$ are tangent to Σ at p and contain the point $q \in \Sigma$. But this means $\Sigma \cap B_r(p + (\operatorname{sign} s) r\nu_\Sigma(p)) \neq \emptyset$, a contradiction. Hence $q = p$. The map X is therefore bijective and continuous and therefore Σ has a tubular neighborhood of radius r.

Tubular neighborhood $\Rightarrow$ ball condition. Suppose that Σ has a tubular neighborhood of radius r. We show that none of the balls $B_r(p \pm r\nu_\Sigma(p))$ ($p \in \Sigma$) intersects Σ. Assuming the contrary, there exist two different $p, q \in \Sigma$ such that $q \in B_r(x_0)$ with $x_0 := p + r\nu_\Sigma(p)$ (the case $p - r\nu_\Sigma(p)$ can be handled analogously). Then for some $\delta < r$, the ball $B_\delta(x_0)$ touches Σ; that is, there exists $q_0 \in \Sigma \cap \partial B_\delta(x_0)$ with $\delta := |q_0 - x_0| = \operatorname{dist}(x_0, \Sigma) < r$ and thus $T_{q_0}\Sigma$ coincides with $T_{q_0}\partial B_\delta(x_0)$ and hence $x_0 = q_0 + \delta\nu_\Sigma(q_0)$. But since $X\colon \Sigma \times (-r,r) \to B_r(\Sigma)$ is bijective, we have $q_0 = \Pi(x_0) = p$ which implies $p \in \overline{B_\delta}(p + r\nu_\Sigma(p))$, a contradiction to $p \notin B_r(p + \nu_\Sigma(p))$. Hence Σ satisfies the uniform ball condition of radius r.

The number r_Σ. Suppose that $B_r(p + sr\nu_\Sigma(p)) \cap \Sigma = \emptyset$ for all $p \in \Sigma$, $s \in \{-1, 1\}$, and $r \in (0, r_\Sigma)$. For fixed $s \in \{-1,1\}$ and $p \in \Sigma$, the ball $B_{r_\Sigma}(p + sr_\Sigma\nu_\Sigma(p))$ is the union of the balls $B_r(p + sr\nu_\Sigma(p))$ $(r \in (0, r_\Sigma))$ and hence we obtain $B_{r_\Sigma}(p + sr_\Sigma\nu_\Sigma(p)) \cap \Sigma = \emptyset$. Therefore Σ satisfies the uniform ball condition of radius r_Σ and has a tubular neighborhood of radius r_Σ.

(ii) We show that every compact C^{2-}-hypersurface Σ satisfies a uniform ball condition.

(ii.a) For given $\varepsilon > 0$, we first prove that there is a number $\delta > 0$ such that every surface piece $\Sigma \cap B_\delta(p_0)$ $(p \in \Omega)$ satisfies a ball condition at p_0. It is sufficient to consider the case $p_0 = 0$ and $\nu_\Sigma(p_0) = e_n = (0, \dots, 0, 1)$, since the assertion for general $p_0 \in \Sigma$ and $\nu_\Sigma(p_0)$ follows by using an appropriate rotation and translation. We choose a local coordinate system $\Sigma \cap B_\delta(0) = \{(u, h(u)) : u \in U_\delta\}$ for some open subset $U_\delta \subset \mathbb{R}^{n-1}$ and a $C^{1,1}$-function $h\colon U_\delta \to \mathbb{R}$. The lower half sphere of $\partial B_r(re_n)$ and the upper half sphere of $\partial B_r(-re_n)$ are parametrized by $(u, k_r(u))$ and $(u, -k_r(u))$, respectively, where k_r is defined by

$$k_r(u) := r - \sqrt{r^2 - |u|^2} \quad \text{for } u \in B_r(0) \subset \mathbb{R}^{n-1}.$$

The ball condition requires that the following inequality is satisfied.

$$-k_r(u) \le h(u) \le k_r(u) \quad \text{for } u \in B_r(0) \cap U_\delta. \tag{A.8}$$

In order to guarantee this condition, we seek a sufficient upper bound for the radius r. Fix $v \in \mathbb{R}^{n-1}$ with $|v| = 1$ and consider the rescaled functions $\tilde{h}(s) := h(rsv)$ and $\tilde{k}(s) := k_r(rsv) = r(1 - (1 - s^2)^{1/2})$ for $s \in (-1, 1)$. Then $\tilde{h}(0) = \tilde{h}'(0) = \tilde{k}(0) = \tilde{k}'(0) = 0$ and $\tilde{h}''(s) = r^2 v \cdot (\nabla^2 h(srv))v$ and $\tilde{k}''(s) = r(1 - s^2)^{-3/2}$. For $t \in (0, 1)$ this yields

$$\begin{aligned}
\tilde{k}(t) - \tilde{h}(t) &= \int_0^t \int_0^s (\tilde{k}''(s') - \tilde{h}''(s'))ds'ds \\
&= \int_0^t (t - s)\left(r(1 - s^2)^{-3/2} - r^2 v \cdot (\nabla^2 h(srv))v\right) ds \\
&\ge \int_0^t (t - s)\left(r - r^2|\nabla^2 h(srv)|\right) ds.
\end{aligned}$$

The integrand is non-negative if we choose $r \le \sup\{|\nabla^2 h(u)| : u \in U_\delta\}^{-1}$ and $\delta > 0$ such that the supremum is finite. In this case both $\tilde{k} - \tilde{h}$ and $\tilde{k} + \tilde{h}$ are non-negative and hence the local ball condition (A.8) is satisfied. This means that $\Sigma \cap B_\delta$ satisfies the ball condition at $p_0 = 0$.

(ii.b) Next we prove an estimate of $\sup\{|\nabla^2 h(u)| : u \in U_\delta\}$ in terms of the global quantity $\|L_\Sigma\|_\infty$. We may again assume that $p_0 = 0$ and $\nu_\Sigma(p_0) = e_n$. Letting where $p = (u, h(u))$, we want to express the local map $U_\delta \ni u \mapsto \nabla^2 h(u)$ in terms of the global map $\Sigma \ni p \mapsto L_\Sigma(p) = -\nabla_\Sigma \nu_\Sigma(p)$. Let $\nu(u) := \nu_\Sigma(u, h(u))$ and $L(u) := L_\Sigma(u, h(u))$. From (A.4) we obtain $\nu(u) = (\nu(u)|e_n)(e_n - \nabla h(u))$ and therefore $P\nu(u) = -(\nu(u)|e_n)\nabla h(u)$ with $P := I - e_n \otimes e_n$. The identities $h(0) = 0$ and $\nabla h(0) = 0$ yield $\nabla_\Sigma \nu_\Sigma(p_0) = \nabla \nu(0) = \nabla^2 h(0)$. Moreover,

$$-L(u) = \nabla_\Sigma \nu_\Sigma(p) = \tau_\Sigma^j(p) \otimes \partial_{u_j} \nu_\Sigma(u, h(u)) = \nabla \nu(u) + (\tau_\Sigma^j(p) - e_j) \otimes \partial_j \nu(u),$$

where $\tau_\Sigma^j(p)$ tends to e^j as $p \to p_0$. A straightforward computation gives

$$\nabla^2 h(u) = \nabla\left(-\frac{P\nu(u)}{\nu(u) \cdot e_n}\right) = -\frac{P\nabla\nu(u)P}{\nu(u) \cdot e_n} + \frac{P\nu(u) \otimes ((\nabla\nu(u))e_n)}{(\nu(u)|e_n)^2}. \tag{A.9}$$

Since Σ is compact, the quantities ν_Σ and τ_Σ^j are uniformly continuous and thus $\nu(u) \to \nu(0)$ and $\tau_\Sigma^j(p) \to e_j$ as p tends to p_0, uniformly with respect to $p_0 \in \Sigma$. Hence for some $\delta(\varepsilon) > 0$ and all p_0, we have

$$\|\nabla^2 h\|_{L_\infty(U_\delta)} \le \|\nabla_\Sigma \nu_\Sigma\|_{L_\infty(\Sigma)} + \varepsilon.$$

(ii.c) The previous steps imply that for all $r < \|\nabla_\Sigma \nu_\Sigma\|_\infty^{-1} = \|L_\Sigma\|_\infty^{-1}$, there is a number $\delta = \delta(r) > 0$ such that every part $\Sigma \cap B_\delta(p_0)$ $(p_0 \in \Sigma)$ satisfies the uniform ball condition of radius r at the point p_0. Since Σ is compact, there exists $r \in (0, \|L_\Sigma\|^{-1})$ such that Σ satisfies the uniform ball condition of radius r.

Next, we prove the local Lipschitz continuity of $X^{-1} = (\Pi, d)\colon x = p + s\nu_\Sigma(p) \mapsto (p, s)$. The signed distance d is Lipschitz with constant 1 and by Proposition A.12, the map X^{-1} is continuous. For every $x_0 \in B_r(\Sigma)$ there exists $\varepsilon > 0$ such that $\overline{B_\varepsilon}(x_0) \subset B_r(\Sigma)$ and $\operatorname{dist}_\Sigma(\Pi(x_2), \Pi(x_1)) < 2\|L_\Sigma\|_\infty$ for x_1, $x_2 \in B_\varepsilon(x_0)$. For $x_j = p_j + d(x_j)\nu_\Sigma(p_j) \in \overline{B_\varepsilon}(x_0)$ ($j \in \{1, 2\}$), we obtain

$$\begin{aligned}|x_2 - x_1| &\geq |p_2 - p_1| - |d(x_2)||\nu_\Sigma(p_2) - \nu_\Sigma(p_1)| - |d(x_2) - d(x_1)||\nu_\Sigma(p_2)| \\ &\geq |p_2 - p_1| - |d(x_2)|\frac{\|L_\Sigma\|_\infty}{\sqrt{2}} \operatorname{dist}_\Sigma(p_2, p_1) - |x_2 - x_1| \\ &\geq |p_2 - p_1| - |d(x_2)|\frac{\|L_\Sigma\|_\infty}{\sqrt{2}} \frac{|p_2 - p_1|}{2 - \|L_\Sigma\|_\infty \operatorname{dist}_\Sigma(p_2, p_1)} - |x_2 - x_1|.\end{aligned}$$

Therefore Proposition A.12 implies that X^{-1} is locally Lipschitz.

It remains to prove the estimate for the principal curvatures. Fix an arbitrary point $p \in \Sigma$ and a principal curvature direction $v \in T_p\Sigma$ so that $L_\Sigma v = \kappa v$ and $|\kappa| \leq \|L_\Sigma\|_\infty$. By means of a parametrization over $T_p\Sigma$ and by reduction to the case $p = 0$ and $\nu_\Sigma(p) = e_n$, the ball condition yields the inequality $|h(tv)| \leq k_r(t) := r - \sqrt{r^2 - t^2}$ for some $\delta \in (0, r]$ and all $|t| \leq \delta$. Using $k_r(0) = k_r'(0) = 0$ and $k_r''(0) = 1/r$, we obtain $|d^2h(tv)/dt^2|_{t=0}| \leq 1/r$ and therefore $|\kappa| = |\kappa v \cdot v| = |\nabla^2 h(0)(v, v)| \leq 1/r$. Taking sequences $(p_n)_n$, $(v_n)_n$, and $(\kappa_n)_n$ with $v_n \in T_{p_n}\Sigma$, $L_\Sigma v_n = \kappa_n v_n$, and $|\kappa_n| \to \|L_\Sigma\|_\infty$, we obtain the desired inequality $\|L_\Sigma\|_\infty \leq 1/r$.

(iii) Let Σ be of class C^{k+1} ($k \geq 1$). For $\tau \in T_p\Sigma$ and $s \in \mathbb{R}$, we obtain $X'(p, t)(\tau, s) = \tau + t(\nabla_\Sigma \nu_\Sigma)(p)\tau + s\nu_\Sigma(p)$. If $X'(p, t)(\tau, s) = 0$, then $(P_\Sigma(p) + t(\nabla_\Sigma \nu_\Sigma)(p))\tau = 0$ and $s\nu_\Sigma(p) = 0$. Using $|P_\Sigma(p) + t(\nabla_\Sigma \nu_\Sigma)(p)| \geq 1 - |t||\nabla_\Sigma \nu_\Sigma| > 0$ and $|\nu_\Sigma(p)| = 1$, this implies $(\tau, s) = 0$ and therefore $X'(p, t)\colon T_p\Sigma \times \mathbb{R} \to \mathbb{R}^n$ is bijective. By the inverse function theorem, the map X is a C^k-diffeomorphism. □

A.18. Lemma (A level function [cf. PS13]). *Let $\Sigma \subset \mathbb{R}^n$ be a closed (possibly unbounded and possibly disconnected) hypersurface with tubular neighborhood of radius $r > 0$. Then Σ is a level set $\Sigma = \varphi^{-1}(\{0\})$ with a function $\varphi \in C^1(\mathbb{R}^n)$, which has the following properties:*

(i) $\nabla\varphi|_\Sigma$ is a continuous unit normal field on Σ,

(ii) $\varphi(x) \in \{-1, 1\}$ for $x \in \mathbb{R}^n \setminus B_r(\Sigma)$.

Proof. We extend the construction of [PS13, Section 4.2], which is valid for compact connected hypersurfaces. Let Σ_j and Ω_k denote the at most countably many connected components of Σ and $\mathbb{R}^n \setminus \Sigma$, respectively. For every j, the component Σ_j is a closed connected hypersurface and hence there are precisely two domains Ω_k such that $\Sigma_j \subset \partial\Omega_k$ [cf. Sam69].

In the terminology of graph theory, the vertices $\mathcal{V} := \{\Omega_k\}$ and the edges $\mathcal{E} = \{\Sigma_j\}$ form a connected graph $(\mathcal{V}, \mathcal{E})$ with two vertices Ω_k, Ω_l being adjacent if and only if $\overline{\Omega}_k \cap \overline{\Omega}_l \neq \emptyset$. Since there exists precisely one edge $\Sigma_j \subset \partial\Omega_k \cap \partial\Omega_l$ that joins Ω_k to Ω_l, the graph is undirected and simple. Suppose that $(\Omega_{k_1}, \Omega_{k_2}, \ldots, \Omega_{k_m})$ is a cycle with distinct vertices Ω_{k_l} ($l \in \{1, 2, \ldots, m\}$) and corresponding edges $(\Sigma_{j_1}, \Sigma_{j_2}, \ldots, \Sigma_{j_m})$. This means $\Sigma_{j_l} \subset \partial\Omega_{k_l} \cap \partial\Omega_{k_{l+1}}$ for $l < m$ and $\Sigma_{j_m} \subset \partial\Omega_{k_m} \cap \partial\Omega_{k_1}$. Then there exists a closed curve $\gamma\colon [1/2, m+1/2] \to \mathbb{R}^n$ such that $\gamma(l) \in \Sigma_{j_l}$ for all l, $\gamma(l - t) \in \Omega_{k_l}$ for all l, $t \in (0, 1)$ and $\gamma(m + t) \in \Omega_{k_1}$ for all $t \in (0, 1/2)$. The component Σ_{j_1} separates $\mathbb{R}^n$ in two components U_1 and U_2 such that $\Omega_{k_1} \subset U_1$. But then $\gamma(t)$ belongs to U_2 for all $t \in (1, m + 1/2]$, which is a contradiction. Hence the graph contains no cycles and is therefore bipartite. Consequently, there exists a function $\chi\colon \mathbb{R}^n \to \{-1, 0, 1\}$ such that

(i) $\chi(x) = 0$ if and only if $x \in \Sigma$,

(ii) χ is constant in every connected component Ω_k of $\mathbb{R}^n \setminus \Sigma$,

(iii) if $\overline{\Omega}_k \cap \overline{\Omega}_l \neq \emptyset$, then $(\chi(\Omega_k), \chi(\Omega_l)) = (-1, 1)$ or $(\chi(\Omega_k), \chi(\Omega_l)) = (1, -1)$.

On the connected components Σ_j of Σ, we can therefore choose the orientation in such a way that the normal $\nu|_{\Sigma_j}$ points into $\Omega_+ := \{x \in \mathbb{R}^n : \chi(x) = 1\}$. We also put $\Omega_- := \{x \in \mathbb{R}^n : \chi(x) = -1\}$. Then $\Omega_+ \cup \Omega_- = \mathbb{R}^n \setminus \Sigma$ and the signed distance satisfies $d(x) > 0$ for

$x \in \Omega_+ \cap B_r(\Sigma)$ and $d(x) < 0$ for $x \in \Omega_- \cap B_r(\Sigma)$. We fix some $\psi \in \mathcal{D}(\mathbb{R})$ such that $\psi(t) = 1$ for $|t| \leq 1/3$, $\psi(t) = 0$ for $|t| \geq 2/3$. Then a possible choice for φ is

$$\varphi(x) = \begin{cases} d(x)\psi(d(x)/r) + (1 - \psi(d(x)/r))\,\mathrm{sign}(d(x)) & \text{for } x \in B_r(\Sigma), \\ \chi_{\Omega_+}(x) - \chi_{\Omega_-}(x) & \text{for } x \notin B_r(\Sigma), \end{cases} \tag{A.10}$$

In particular, we obtain $\nabla\varphi(x) = \nabla d(x) = \nu_\Sigma(\Pi(x))$ for $x \in B_{r/3}(\Sigma)$. □

A.19. Corollary. *Every domain $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) that contains a compact C^2-hypersurface Σ can be decomposed into*

$$\Omega = \Omega_+ \cup \Sigma \cup \Omega_- \quad \textit{with } \Omega_\pm = \{x \in \Omega : \varphi_\Sigma(x) \gtrless 0\},$$

where φ_Σ denotes a level function as in Lemma A.18.

The next results deals with important geometric quantities of C^{2-}-hypersurfaces.

A.20. Proposition. *Let $\Sigma \subset \mathbb{R}^n$ be a compact C^{2-}-hypersurface with tubular neighborhood $B_r(\Sigma)$ and $X\colon (p,s) \mapsto x = p + s\nu_\Sigma(p)$ be the corresponding diffeomorphism with inverse $X^{-1} = (\Pi, d)$. Then the following assertions are valid.*

(i) The map

$$M\colon x \mapsto [I - d(x)L_\Sigma(\Pi(x))]^{-1}, \quad B_r(\Sigma) \to \mathbb{R}^{n\times n}$$

is essentially bounded and, for almost all $x \in B_r(\Sigma)$ and $p = \Pi(x) \in \Sigma$, the linear map $M(x)$ satisfies $M(x)\nu_\Sigma(p) = \nu_\Sigma(p)$ and maps $T_p\Sigma$ onto itself.

(ii) The map d is of class C^{2-} and satisfies $\nabla d(x) = M(x)\nu_\Sigma(p) = \nu_\Sigma(p)$.

(iii) The map Π is of class C^{1-} and satisfies

$$\Pi'(x) = P_\Sigma(\Pi(x))M(x) = M(x) - \nu_\Sigma(p) \otimes \nu_\Sigma(p) = \Pi'(x)^\top.$$

Proof. Let $x \in B_r(\Sigma)$ and $p = \Pi(x)$. In view of $|d(x)L_\Sigma(p)| < r \cdot r^{-1} = 1$, the matrix $I - d(x)L_\Sigma(p)$ has maximal rank. It is also symmetric and in view of $L_\Sigma\nu_\Sigma = 0$, it satisfies $[I - d(x)L_\Sigma(p)]\nu_\Sigma(p) = \nu_\Sigma(p)$ for all $p \in \Sigma$. Therefore its inverse $M(x) = [I - d(x)L_\Sigma(p)]^{-1}$ satisfies $M(x)\nu_\Sigma(p) = \nu_\Sigma(p)$ and maps $T_p\Sigma$ onto itself.

Let $\phi\colon U \subset \mathbb{R}^{n-1} \to \Sigma$, $u \mapsto p = \phi(u)$ be a chart for Σ. For each $\lambda \in (-r, r)$, the identities $\Pi(X(\phi(u),\lambda)) = \phi(u)$ and $x = X(\phi(u),\lambda)$ imply

$$\tau_i(p) = \partial_{u_i}\phi(u) = \Pi'(x)(\tau_i(p) + \lambda\partial_i\nu_\Sigma(p)) = \Pi'(x)(I - \lambda L_\Sigma(p))\tau_i(p),$$

$$\Pi'(x)(I - \lambda L_\Sigma(p))\nu_\Sigma(p) = \Pi'(x)\nu_\Sigma(p) = \lim_{s\to 0}\frac{1}{s}(\Pi(x + s\nu_\Sigma(p)) - \Pi(x)) = 0.$$

Hence $\Pi'[I - dL_\Sigma \circ \Pi] = P_\Sigma$ and this yields $\Pi' = P_\Sigma[I - dL_\Sigma \circ \Pi]^{-1} = P_\Sigma M = M - \nu_\Sigma \otimes \nu_\Sigma = MP_\Sigma = \Pi'^\top$. Using the relations $d(x) = (x - p)\cdot\nu_\Sigma(p)$ and $(\partial_i\nu_\Sigma(p)|\nu_\Sigma(p)) = 0$, we obtain

$$\nabla d(x) = \nu_\Sigma(p) - \nu_\Sigma(p)\Pi'(x) + \Pi'(x)\nabla_\Sigma\nu_\Sigma(p)d(x)\nu_\Sigma(p) = \nu_\Sigma(p).$$ □

As in [PS13], we consider hypersurfaces that are defined in tubular neighborhoods of a given hypersurface.

A.21. Proposition. *Let $\Sigma \subset \mathbb{R}^n$ be a closed hypersurface of class C^{k+1} [resp. C^{k+1-}] ($k \geq 1$) with tubular neighborhood of radius $r > 0$ and let $h \in C^k(\Sigma)$ [resp. $h \in C^{k-}(\Sigma)$] satisfy $\|h\|_\infty < r$. Then the following assertions are valid.*

(i) The image $\Sigma_h = \theta_h(\Sigma)$ of $\theta_h\colon p \mapsto p + h(p)\nu_\Sigma(p)$ is a closed hypersurface of class C^k [resp. C^{k-}] and the map $\theta_h\colon \Sigma \to \Sigma_h$ is a C^k-diffeomorphism [resp. C^{k-}-diffeomorphism if $k \geq 2$ and a homeomorphism with locally Lipschitz continuous inverse if $k = 1$].

(ii) The normal ν_{Σ_h} of Σ_h is [in the case $k = 1$ almost everywhere] given by

$$\nu_{\Sigma_h}(p + h(p)\nu_\Sigma(p)) = \frac{\nu_\Sigma(p) - M_h(p)\nabla_\Sigma h(p)}{\sqrt{1 + |M_h(p)\nabla_\Sigma h(p)|^2}} \quad \textit{with } M_h(p) := (I - h(p)L_\Sigma(p))^{-1}. \tag{A.11}$$

(iii) Suppose in addition that $\|\nabla_\Sigma h\|_\infty < \infty$. Then the normal of Σ_h satisfies the inequality

$$\nu_{\Sigma_h}(p + h(p)\nu_\Sigma(p)) \cdot \nu_\Sigma(p) \geq (1 + \|M_h \nabla_\Sigma h\|_\infty^2)^{-1/2} > 0 \quad \textit{for } p \in \Sigma.$$

(iv) The following integral transformation formula is valid for $f \in L_1(\Sigma_h)$.

$$\int_{\Sigma_h} f \, d\Sigma_h = \int_\Sigma f \circ \theta_h \det(P_\Sigma - hL_\Sigma) \sqrt{1 + |M_h \nabla_\Sigma h|^2} \, d\Sigma. \tag{A.12}$$

Proof. *(i)* Let $\phi \colon \mathbb{R}^{n-1} \supset U \in u \mapsto p \in \Sigma$ be a parametrization for Σ and let $\theta_h \colon \Sigma \to \Sigma_h$, $p \mapsto p + h(p)\nu_\Sigma(p)$. Then the derivative $\theta_h' \colon T_p\Sigma \to T_{\theta_h(p)}\Sigma_h$ is given by

$$\theta_h' = P_\Sigma + \nu_\Sigma \otimes \nabla_\Sigma h + h\nabla_\Sigma \nu_\Sigma = P_\Sigma + \nu_\Sigma \otimes \nabla_\Sigma h - hL_\Sigma. \tag{A.13}$$

If $\theta_h'(p)u = 0$ for some $u \in T_p\Sigma$, then $u - h(p)L_\Sigma(p)u = 0$. The assumption $\|h\|_\infty < r$ implies $|h(p)(L_\Sigma)(p)| < 1$ and this yields $u = 0$. Therefore θ_h' is bijective for all $p \in \Sigma$ and thereby $\theta_h \colon \Sigma \to \Sigma_h$ is a local diffeomorphism. The map θ_h is also surjective and coincides with a restriction of the map $X \colon B_r(\Sigma) \to \Sigma \times (-r, r)$ from Proposition A.17 to $\{(p,t) \in \Sigma \times (-r,r) : t = h(p)\}$. Since X is bijective, the map θ_h is a global diffeomorphism and $\theta_h \circ \phi$ is a parametrization for Σ_h, which shows that Σ_h is of class C^k [C^{k-}].

(ii) A derivation of (A.11) can be found in [PS13, Section 3.2]. The inverse $M_h(p)$ of $I - h(p)L_\Sigma(p)$ is well-defined because of $\|h\|_\infty < r$.

(iii) This estimate is a direct consequence of (A.11).

(iv) By means of the parametrization $p = \varphi(u)$ we obtain $d\Sigma(p) = \sqrt{g(u)}\, du$ where $g = \det G$, $G = \varphi'^\top \varphi'$, as well as $d\Sigma_h(\theta_h(p)) = \sqrt{g_h(u)}\, du$, $g_h = \det G_h$. Since $\theta_h \circ \varphi$ is a parametrization for Σ_h, we have $G_h = [\theta_h' \circ \varphi \;\, \varphi']^\top \theta_h' \circ \varphi \;\, \varphi' = \varphi'^\top [\theta_h' \circ \varphi]^\top [\theta_h' \circ \varphi] \varphi'$. Hence identity (A.13) yields

$$[\theta_h' \circ \varphi]^\top [\theta_h' \circ \varphi] = (P_\Sigma - hL_\Sigma)^2 + \nabla_\Sigma h \otimes \nabla_\Sigma h = M_h^{-2}(P_\Sigma + M_h^2 \nabla_\Sigma h \otimes \nabla_\Sigma h).$$

For computing $\det G_h$, we recall two facts from linear algebra. First, for any two isomorphisms $A \colon X \to Y$ and $B \colon Y \to X$ between n-dimensional vector spaces X and Y we have $\det_X(BA) = \det_Y(AB)$, since the determinant $\det_X$ in X is given by the identity $\det_X(C) = V(Cx_1, \ldots, Cx_n)/V(x_1, \ldots, x_n)$ for any $C \in \mathcal{L}(X)$, any volume form V in X and any basis (x_j) of X. Second, we have $\det(I + a \otimes b) = 1 + a \cdot b$ for $a, b \in \mathbb{R}^n$. These facts yield

$$\begin{aligned} g_h &= \det\nolimits_{\mathbb{R}^{n-1}}(\varphi'^\top \varphi') \det\nolimits_{\mathbb{R}^{n-1}}(\varphi'^{-1}[\theta_h' \circ \varphi]^\top [\theta_h' \circ \varphi]\varphi') \\ &= g \det\nolimits_{\mathbb{R}^{n-1}}(\varphi'^{-1} M_h^{-2} \varphi') \det\nolimits_{\mathbb{R}^{n-1}}(\varphi'^{-1} P_\Sigma \varphi' + \varphi'^{-1} M_h^2 \nabla_\Sigma h \otimes \varphi'^\top \nabla_\Sigma h) \\ &= g \det\nolimits_{T_p\Sigma}(P_\Sigma - hL_\Sigma)^2 (1 + |M_h \nabla_\Sigma h|^2). \end{aligned}$$

Therefore the asserted equation (A.12) follows. □

A.4. Covariant differentiation

Let Γ be a C^3-hypersurface of $\mathbb{R}^n$, equipped with the induced Euclidean metric $(v|w)_{g(p)} = v \cdot w$ for v, $w \in T_p\Gamma \subset \mathbb{R}^n$ and $p \in \Gamma$. We let $\tau_1, \ldots, \tau_{n-1}$ be a basis of tangent vectors on $T_p\Gamma$ with dual basis $\tau^1, \ldots, \tau^{n-1}$ so that $\tau_j \cdot \tau^k = \delta_j^k$, we let ν denote the unit normal on Γ, and we let $P = I - \nu \otimes \nu$ denote the projection onto the tangent space. Moreover, we let $C^k(\Gamma; T\Gamma)$ ($k \in \mathbb{N}_0$) denote the Banach space of all tangential vector fields $v = v^\alpha \tau_\alpha$ of class C^k on Γ.

A.4.1. First order covariant derivatives. We define the *(partial) covariant derivative* $\widetilde{\nabla}_j v$ with respect to the coordinate x_j by

$$v_{;j} := \widetilde{\nabla}_j v := P\partial_j(v^\alpha \tau_\alpha) = (\partial_j v^\alpha + \Lambda^\alpha_{j\beta} v^\beta)\tau_\alpha =: v^\alpha{}_{;j} \tau_\alpha.$$

Here Λ^k_{ij} is the Christoffel symbol of the second kind. Moreover, we let

$$\widetilde{\nabla} v := \widetilde{\nabla}_\Gamma v := v^\alpha{}_{;\beta}\, \tau_\alpha \otimes \tau^\beta = (\partial_\beta v^\alpha + \Lambda^\alpha_{\beta\gamma} v^\gamma)\tau_\alpha \otimes \tau^\beta \quad \text{for } v \in C^1(\Gamma; T\Gamma),$$

so that

$$\widetilde{\nabla}_u v := [\widetilde{\nabla} v]u = (\partial_\beta v^\alpha u^\beta + \Lambda^\alpha_{\beta\gamma} v^\gamma u^\beta)\tau_\alpha \quad \text{for } u \in C(\Gamma; T\Gamma).$$

This definition of $\widetilde{\nabla}_u v$ coincides with the Levi-Civita connection on Γ and ensures that $\widetilde{\nabla}_u v$ is again a tangential vector field.

For a possibly non-tangential vector field $u = v + w\nu = v^\alpha \tau_\alpha + w\nu \in C^1(\Gamma; \mathbb{R}^n)$ we define

$$u_{;k} := \widetilde{\nabla}_k u := P\partial_k(v^\alpha \tau_\alpha + w\nu) = \widetilde{\nabla}_k(v^\alpha \tau_\alpha) - w l_{k\alpha}\tau^\alpha, \tag{A.14}$$

where l_{jk} denote the components of the Weingarten tensor $L = l_{\alpha\beta}\tau^\alpha \otimes \tau^\beta = -\nabla_\Gamma \nu$. Then

$$\partial_k u = \widetilde{\nabla}_k u + [\nu \otimes \nu]\partial_k u = \left(v^\alpha{}_{;k} - w g^{\alpha\beta} l_{k\beta}\right)\tau_\alpha + (v^\alpha l_{k\alpha} + \partial_k w)\,\nu.$$

By abbreviating $v_\alpha := g_{\alpha\gamma} v^\gamma$, we rewrite the surface gradient $\nabla_\Gamma u = \tau^\alpha \otimes \partial_\alpha u$ as

$$\begin{aligned} \nabla_\Gamma u &= \tau^\alpha \otimes \tau^\beta\,(v_{\beta;\alpha} - w l_{\alpha\beta}) + \tau^\alpha \otimes \nu (v^\beta l_{\alpha\beta} + \partial_\alpha w) \\ &= [\widetilde{\nabla} v]^\top - wL + (Lv + \nabla_\Gamma w) \otimes \nu. \end{aligned} \tag{A.15}$$

With the mean curvature $H := \operatorname{tr} L = -\operatorname{div}_\Gamma \nu$, the surface divergence $\operatorname{div}_\Gamma u$ satisfies

$$\operatorname{div}_\Gamma u := \tau^\alpha \cdot \partial_\alpha u = v^\alpha{}_{;\alpha} - wH = \operatorname{div}_\Gamma v - wH. \tag{A.16}$$

The symmetric part $D_\Gamma(u)$ of $P[\nabla_\Gamma u]P$ is given by

$$D_\Gamma(u) := \operatorname{sym}(P[\nabla_\Gamma u]P) = 2^{-1}\tau^\alpha \otimes \tau^\beta (v_{\alpha;\beta} + v_{\beta;\alpha}) - wL = 2^{-1}(\widetilde{\nabla} v + [\widetilde{\nabla} v]^\top) - wL. \tag{A.17}$$

We note that $\operatorname{tr} D_\Gamma(u) = \operatorname{div}_\Gamma u$.

Second order tensors have the form $S^{\alpha\beta}\tau_\alpha \otimes \tau_\beta$, $S_{\alpha\beta}\tau^\alpha \otimes \tau^\beta$, $S^\alpha{}_\beta \tau_\alpha \otimes \tau^\beta$, or $S_\alpha{}^\beta \tau^\alpha \otimes \tau_\beta$, and their first order covariant derivatives are given by

$$\begin{aligned} S^{ij}{}_{;k} &= \partial_k S^{ij} + \Lambda^i_{\alpha k} S^{\alpha j} + \Lambda^j_{\alpha k} S^{i\alpha}, &\qquad S^i{}_{j;k} &= \partial_k S^i{}_j + \Lambda^i_{\alpha k} S^\alpha{}_j - \Lambda^\alpha_{jk} S^i{}_\alpha, \\ S_{ij;k} &= \partial_k S_{ij} - \Lambda^\alpha_{ik} S_{\alpha j} - \Lambda^\alpha_{jk} S_{i\alpha}, &\qquad S_i{}^j{}_{;k} &= \partial_k S_i{}^j - \Lambda^\alpha_{ik} S_\alpha{}^j + \Lambda^j_{\alpha k} S_i{}^\alpha. \end{aligned}$$

Then the surface divergence of a second order symmetric tensor $S^{\alpha\beta}\tau_\alpha \otimes \tau_\beta$ is given by

$$\operatorname{div}_\Gamma S = \operatorname{div}_\Gamma(S^{\alpha\beta}\tau_\alpha \otimes \tau_\beta) := [\partial_\gamma(S^{\alpha\beta}\tau_\alpha \otimes \tau_\beta)]\tau^\gamma = S^{\alpha\beta}{}_{;\alpha}\tau_\beta + S^{\alpha\beta} l_{\alpha\beta}\nu_\Gamma. \tag{A.18}$$

For symmetric $S = S^{\alpha\beta}\tau_\alpha \otimes \tau_\beta$ and $u = v^\alpha \tau_\alpha + w\nu$ we have

$$\operatorname{div}_\Gamma(Su) = \operatorname{div}_\Gamma S \cdot u + S : \nabla_\Gamma u. \tag{A.19}$$

By using the identity $\partial_k \tau_i \cdot \tau_j = \Lambda^\alpha_{ik} g_{\alpha j}$, we can easily deduce the useful identities

$$g_{ij;k} = 0, \quad g^{ij}{}_{;k} = 0. \tag{A.20}$$

Thus, the metric tensor $P = g^{ij}\tau_i \otimes \tau_j = g_{ij}\tau^i \otimes \tau^j$ satisfies

$$\operatorname{div}_\Gamma P = H\nu. \tag{A.21}$$

The components $l_{ij} = -\tau_i \cdot \partial_j \nu$ of the Weingarten tensor satisfy the relations

$$l_{ij;k} = l_{ik;j} = l_{jk;i}. \tag{A.22}$$

More generally, let T be a tensor with the components $T^{i_1\cdots i_a}_{j_1\cdots j_b}$, where we agree on not raising or lowering indices when the order between the contravariant and the covariant indices is not indicated. Then the covariant derivative of T with respect to x_k is given by

$$T^{i_1\cdots i_a}_{j_1\cdots j_b;m} = \partial_m T^{i_1\cdots i_a}_{j_1\cdots j_b} + \sum_{p=1}^{a} \Lambda^{i_p}_{\alpha m} T^{i_1\cdots i_{p-1}\alpha i_{p+1}\cdots i_a}_{j_1\cdots j_b} - \sum_{p=1}^{b} \Lambda^{\alpha}_{j_i m} T^{i_1\cdots i_a}_{j_1\cdots j_{p-1}\alpha j_{p+1}\cdots j_b}. \tag{A.23}$$

For two tensors S and T the following product rule is valid.

$$\left(S^{i_1\cdots i_a}_{j_1\cdots j_b} T^{k_1\cdots k_c}_{l_1\cdots l_d}\right)_{;m} = S^{i_1\cdots i_a}_{j_1\cdots j_b;m} T^{k_1\cdots k_c}_{l_1\cdots l_d} + S^{i_1\cdots i_a}_{j_1\cdots j_b} T^{k_1\cdots k_c}_{l_1\cdots l_d;m}. \tag{A.24}$$

A.4.2. Relation to bulk differential operators. Let $\Omega \subset \mathbb{R}^n$ be open and let $\Gamma \subset \Omega$ be a C^3-hypersurface which admits a C^1-class tubular neighborhood map $(x,s) \mapsto X(x,s) = x + s\nu(x)$ from an open subset $U \subset \Gamma \times \mathbb{R}$ with $U \supset \Gamma \times \{0\}$ onto $V \subset \Omega$. Let $(\Pi, d) = X^{-1}$ so that $\Pi(x + s\nu(x)) = x$ and $d(x + s\nu(x)) = s$. For a vector field $u\colon V \to \mathbb{R}^n$ we let

$$u = v + w\nu \circ \Pi, \quad v := [P \circ \Pi]u, \quad w := (\nu \circ \Pi | u).$$

Then we easily find the following identities on Γ.

$$\nabla u = \nabla_\Gamma u + \nu \otimes \partial_\nu u, \tag{A.25a}$$

$$\nabla_\Gamma u = \nabla_\Gamma v - wL + \nabla_\Gamma w \otimes \nu, \tag{A.25b}$$

$$\operatorname{div} u = \operatorname{div}_\Gamma u - (\nu | \partial_\nu u). \tag{A.25c}$$

A.4.3. Second order covariant derivatives. For a tangential vector field $v = v^\alpha \tau_\alpha$, we consider the second order covariant derivatives

$$\widetilde{\nabla}_j \widetilde{\nabla}_k (v^\alpha \tau_\alpha) = \widetilde{\nabla}_j (v^\alpha{}_{;k} \tau_\alpha) = v^\alpha{}_{;kj} \tau_\alpha.$$

The operators $\widetilde{\nabla}_j$ and $\widetilde{\nabla}_k$ do not necessarily commute but satisfy the relations

$$v^i{}_{;jk} - v^i{}_{;kj} = R^i{}_{\alpha jk} v^\alpha, \quad v_{i;jk} - v_{i;kj} = -v_\alpha R^\alpha{}_{ijk}, \tag{A.26}$$

where $R^i{}_{ljk} = \tau^i \cdot R(\tau_j, \tau_k)\tau_l$ are the components of the Riemann curvature tensor R, given by

$$R^i{}_{ljk} = \partial_j \Lambda^i_{kl} - \partial_k \Lambda^i_{jl} + \Lambda^i_{j\alpha}\Lambda^\alpha_{kl} - \Lambda^i_{k\alpha}\Lambda^\alpha_{jl}, \quad g_{im} R^i{}_{ljk} =: R_{mljk}.$$

This tensor has the symmetries

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0, \quad R_{ijkl} = -R_{ijlk} = -R_{jikl} = R_{klij}.$$

For a hypersurface Γ in $\mathbb{R}^n$ we have

$$R_{ijkl} = l_{ik} l_{jl} - l_{il} l_{jk}. \tag{A.27}$$

A.4.4. The tangential Laplace-Beltrami operator. We define

$$\widetilde{\Delta} v := \widetilde{\Delta}_\Gamma v := g^{\alpha\beta} \widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta v = g^{\alpha\beta} v^\gamma{}_{;\alpha\beta}\, \tau^\gamma \quad \text{for } v \in C^2(\Gamma; T\Gamma).$$

This definition is consistent with $\widetilde{\Delta} v = -\widetilde{\nabla}^* \widetilde{\nabla} v$, where $\widetilde{\nabla}^*$ is the formal $L_2(\Gamma)$-adjoint of $\widetilde{\nabla}$, which means that $\widetilde{\nabla}^* W$ ($W \in C^1(\Gamma; T\Gamma \otimes T^*\Gamma)$) is defined by the relation

$$(\widetilde{\nabla}^* W | v)_{L_2(\Gamma; T\Gamma)} = (W | \widetilde{\nabla} v)_{L_2(\Gamma; T\Gamma \otimes T^*\Gamma)} \quad \text{for all } v \in C^1_c(\Gamma; T\Gamma).$$

To check this, we write $W = W^\alpha{}_\beta\, \tau_\alpha \otimes \tau^\beta$ and $\widetilde{\nabla} v = v^\alpha{}_{;\beta}\, \tau_\alpha \otimes \tau^\beta$ and obtain

$$\begin{aligned}(W | \widetilde{\nabla} v)_{L_2(\Gamma; T\Gamma \otimes T^*\Gamma)} &= \int_\Gamma W\tau^\alpha \cdot \widetilde{\nabla}_\alpha v \, d\Gamma \\ &= \int_\Gamma g^{\alpha\beta} W^\gamma{}_\beta \tau_\gamma \cdot v^\delta{}_{;\alpha} \tau_\delta \, d\Gamma = \int_\Gamma g^{\alpha\beta} W^\gamma{}_\beta g_{\gamma\delta} v^\delta{}_{;\alpha} \, d\Gamma.\end{aligned}$$

From $\operatorname{div}_\Gamma v = v^\alpha{}_{;\alpha}$, identities (A.20), and the surface divergence theorem, we infer that

$$(W|\widetilde{\nabla} v)_{L_2(\Gamma;T\Gamma\otimes T^*\Gamma)} = -\int_\Gamma g^{\alpha\beta} W^\gamma{}_{\beta;\alpha} g_{\gamma\delta} v^\delta \, d\Gamma = -(g^{\alpha\beta} W^\gamma{}_{\beta;\alpha} \tau_\gamma | v)_{L_2(\Gamma;T\Gamma)},$$

and thus $\widetilde{\nabla}^* W = -g^{\alpha\beta} w^\gamma{}_{\beta;\alpha} \tau_\gamma$. This yields $\widetilde{\Delta} v = -\widetilde{\nabla}^* \widetilde{\nabla} v$.

Finally, for $u \in C^1(\Gamma; T\Gamma)$ and $v \in C^2(\Gamma; T\Gamma)$, we calculate

$$\begin{aligned}(\widetilde{\Delta} v|u)_{L_2(\Gamma;T\Gamma)} &= \int_\Gamma g^{\alpha\beta} v^\gamma{}_{;\alpha\beta} \tau_\gamma \cdot \tau_\delta u^\delta \, d\Gamma \\ &= -\int_\Gamma v^\gamma{}_{;\alpha} u^\delta{}_{;\beta} g^{\alpha\beta} g_{\gamma\delta} \, d\Gamma = -\int_\Gamma \widetilde{\nabla} v : \widetilde{\nabla} u \, d\Gamma,\end{aligned} \tag{A.28}$$

where $S : T = \operatorname{tr}(S^\top T) = (S\tau_\alpha | T\tau^\alpha)$. Therefore $\widetilde{\Delta}$ is symmetric and negative semi-definite.

For a non-tangential vector field $u = v + w\nu$, equation (A.14) yields

$$\widetilde{\Delta}(v + w\nu) = \widetilde{\Delta} v - L\nabla_\Gamma w - w\nabla_\Gamma H. \tag{A.29}$$

APPENDIX B

Functional analytic methods

B.1. Function spaces

B.1.1. Classical function spaces. Let Ω be an open subset of $\mathbb{R}^n$ ($n \in \mathbb{N}$), let X be a Banach space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and let $k \in \mathbb{N}_0$. The vector space $C^k(\Omega; X)$ consists of the k times continuously Fréchet-differentiable functions from Ω to X. We abbreviate $C^k(\Omega) := C^k(\Omega; \mathbb{K})$ and $C^k(\Omega)^n := C^k(\Omega; \mathbb{K}^n)$, analogously for all subsequent spaces. The subspace $C^k(\overline{\Omega}; X)$ consists of those $u \in C^k(\Omega; X)$ that have a continuous extension onto the closure $\overline{\Omega}$ of Ω, together will all derivatives up to order k. The Banach space $BC^k(\Omega; X)$ consists of all bounded functions $u \in C^k(\Omega; X)$ with bounded derivatives up to order k, equipped with the norm

$$\|u\|_{BC^k(\Omega;X)} := \sup\{\|\partial_x^\beta u(x)\|_X : \beta \in \mathbb{N}_0^n,\ |\beta| \le k,\ x \in \Omega\}.$$

The space $BUC^k(\Omega; X)$ consists of all bounded, uniformly continuous functions $u \in C^k(\Omega; X)$ with bounded, uniformly continuous derivatives up to order k. For an interval $J \subset \mathbb{R}$, we let $C_0(J; X) = \{u \in C(J; X) : \|u(t)\|_X \to 0 \text{ as } t \to \infty\}$. For given Banach spaces X and Y and an open subset $U \subset X$, the spaces $C^k(U; Y)$, $BC^k(U; Y)$, and $BUC^k(U; Y)$ are defined analogously. For $k \in \mathbb{N}$, $\alpha \in (0, 1]$, and $f\colon \mathbb{R}^n \to X$, we define the seminorm

$$[\![f]\!]_{C^{k,\alpha}} := \sup_{x \neq y} \frac{\|f^{(k)}(x) - f^{(k)}(y)\|_X}{|x - y|^\alpha}.$$

The space $C^{k,\alpha}(\mathbb{R}^n; X) := \{f \in C^k(\mathbb{R}^n; X) : [\![f]\!]_{C^{k,\alpha}} < \infty\}$ is called *Hölder space* if $\alpha < 1$ and *Lipschitz space* if $\alpha = 1$. We also write $C^{k+\alpha} := C^{k,\alpha}$ if $\alpha \in (0, 1)$ and $C^{k-} := \{f \in C^{k-1} : f^{(k-1)} \text{ is locally Lipschitz}\}$. Rademacher's theorem implies that a function $u \in C(\mathbb{R}^n)$ belongs to $C^{0,1}(\mathbb{R}^n)$ if and only if it is almost everywhere differentiable and its derivative is bounded.

The support $\operatorname{supp} u$ of a function $u \in C(\Omega; X)$ is the closure of the set $\{x \in \Omega : \varphi(x) \neq 0\}$ in $\mathbb{R}^n$. The space $C_c^k(\Omega; X)$ consists of all $u \in C^k(\Omega; X)$ such that $\operatorname{supp} u$ is compact and a subset of Ω. We let $\mathcal{D}(\Omega; X) = C_c^\infty(\Omega; X)$ denote the Fréchet space of test functions. The *space of distributions* $\mathcal{D}'(\Omega; X)$ consists of all continuous linear maps $\mathcal{D}(\Omega) \to X$. A function $u \in C^\infty(\mathbb{R}^n; X)$ is called rapidly decreasing, if $x \mapsto |x|^{|\alpha|}\partial_x^\beta u(x)$ is bounded on $\mathbb{R}^n$ for every pair of multi-indices α, $\beta \in \mathbb{N}_0^n$. Here we let $|\alpha| = |\alpha_1| + \cdots + |\alpha_n|$ and $\partial_x^\beta = \partial_1^{\beta_1} \cdots \partial_n^{\beta_n}$. The *Schwartz space* of rapidly decreasing functions is denoted by $\mathcal{S}(\mathbb{R}^n; X)$ and the space of *tempered distributions* $\mathcal{S}'(\mathbb{R}^n; X)$ consists of all continuous linear maps $\mathcal{S}(\mathbb{R}^n) \to X$.

Given a measure space $(\Omega, \mathcal{A}, \mu)$, we let $L_0(\Omega; X) := L_0(\Omega, \mathcal{A}, \mu; X)$ denote the vector space of all equivalence classes of strongly μ-measurable functions $\Omega \to X$. Given $p \in [1, \infty]$ and $m \in \mathbb{N}_0$, we employ the usual *Bochner-Lebesgue space* $L_p(\Omega; X) = L_p(\Omega, \mathcal{A}, \mu; X)$ and the *Sobolev space* $W_p^m(\Omega; X)$. The *Bessel potential space* of order $s \in \mathbb{R}$ is defined by

$$H_p^s(\mathbb{R}^n; X) := \left\{u \in \mathcal{S}'(\mathbb{R}^n; X) : \|f\|_{H_p^s(\mathbb{R}^n;X)} = \|\mathcal{F}^{-1}(\xi \mapsto (1 + |\xi|^2)^{s/2}(\mathcal{F}f)(\xi))\|_{L_p(\mathbb{R}^n)} < \infty\right\}.$$

The operator $J^\sigma\colon f \mapsto \mathcal{F}^{-1}(\xi \mapsto (1 + |\xi|^2)^{\sigma/2}(\mathcal{F}f)(\xi))$ is called *Bessel potential* of order σ and its realization $J^\sigma\colon H_p^{s+\sigma}(\mathbb{R}^n; X) \to H_p^s(\mathbb{R}^n; X)$ is an isomorphism; that is, a bijective, bounded linear map with bounded inverse.

For $s = m \in \mathbb{N}_0$ and $p \in (1, \infty)$, the spaces $H_p^m(\mathbb{R}^n; X)$ and $W_p^m(\mathbb{R}^n; X)$ coincide with equivalent norms if and only if X is of class $\mathcal{HT}$ [McC84; Zim89]. For $p \in [1, \infty)$, we have $(H_p^s(\mathbb{R}^n))' = H_{p'}^{-s}(\mathbb{R}^n)$ [BL76]. The *Sobolev-Slobodeckiĭ space* $W_p^s(\Omega; X)$ of order $s \in (0, \infty) \setminus \mathbb{N}_0$ with $s = [s] + \{s\}$, $[s] \in \mathbb{N}_0$, $\{s\} \in (0, 1)$, and $p \in [1, \infty)$ is defined by

$$W_p^s(\Omega; X) := \left\{ u \in \mathcal{D}'(\Omega; X) : \|u\|_{W_p^s(\Omega;X)} := \|u\|_{W_p^{[s]}(\Omega;X)} + \sum_{|\alpha|=[s]} [\![\partial^\alpha u]\!]_{W_p^{\{s\}}(\Omega;X)} < \infty \right\},$$

as in [Ama97, p. 10] and [Tri10, Theorem 2.5.7], where the seminorm $[\![\cdot]\!]_{W_p^\theta(\Omega;X)}$ is defined by

$$[\![u]\!]_{W_p^\theta(\Omega;X)} := \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|_X^p}{|x - y|^{n+\theta p}} \, dx \, dy \right)^{1/p} \qquad \text{for } \theta \in (0, 1).$$

We also refer to [Tri95, Theorem 4.2.4] and [Lud14] for some properties of this norm and [KPW13, Section 3.2] for an equivalent norm. Following [Joh95; RS96; Tri10; SSS12], we introduce Besov spaces and Triebel-Lizorkin spaces over $\mathbb{R}^n$ in terms of the Fourier transform $\mathcal{F}$ on $\mathcal{S}'(\mathbb{R}^n)$ and a partition of unity. Let $\{\varphi_j\}_{j=0}^\infty \subset \mathcal{S}(\mathbb{R}^n)$ satisfy the following properties.

(i) There exist $A, B, C \in (0, \infty)$ such that $\operatorname{supp} \varphi_0 \subset \overline{\mathbb{B}}_A$, $\operatorname{supp} \varphi_j \subset \overline{\mathbb{B}}_{C2^{j+1}} \setminus \mathbb{B}_{B2^{j-1}}$ for $j \in \mathbb{N}$.
(ii) For every $\alpha \in \mathbb{N}_0^n$ there is $c_a \in (0, \infty)$ such that $2^{j|\alpha|} |D^\alpha \varphi_j(x)| \le c_\alpha$ for all $x \in \mathbb{R}^n$, $j \in \mathbb{N}_0$.
(iii) $\sum_{j=0}^\infty \varphi_j(x) = 1$ for all $x \in \mathbb{R}^n$.

The *Besov space* $B_{pq}^s(\mathbb{R}^n; X)$ and the *Triebel-Lizorkin space* $F_{pq}^s(\mathbb{R}^n; X)$ of order $s \in \mathbb{R}$, integral-exponent $p \in [1, \infty]$, and sum-exponent $q \in [1, \infty]$ are defined by

$$B_{pq}^s(\mathbb{R}^n; X) := \{u \in \mathcal{S}'(\mathbb{R}^n; X) : \|u\|_{B_{pq}^s} := \big\| \{2^{sj} \mathcal{F}^{-1}[\varphi_j \mathcal{F} u]\}_j \big\|_{l_q(L_p)} < \infty\},$$
$$F_{pq}^s(\mathbb{R}^n; X) := \{u \in \mathcal{S}'(\mathbb{R}^n; X) : \|u\|_{F_{pq}^s} := \big\| \{2^{sj} \mathcal{F}^{-1}[\varphi_j \mathcal{F} u]\}_j \big\|_{L_p(l_q)} < \infty\}.$$

We recall that the identity

$$W_p^s(\mathbb{R}^n; X) = B_{pp}^s(\mathbb{R}^n; X) \qquad \text{for } s \in (0, \infty) \setminus \mathbb{N},\ p \in (1, \infty),$$

is valid for every Banach space X [see Ama97, (5.8), (5.9)], whereas

$$H_p^m(\mathbb{R}^n; X) = W_p^m(\mathbb{R}^n; X) \qquad \text{for } m \in \mathbb{N}_0,\ p \in (1, \infty),$$

is valid if and only if X is a Banach space of class $\mathcal{HT}$ [McC84; SSS12]. Moreover,

$$H_p^s(\mathbb{R}^n; X) = F_{p2}^s(\mathbb{R}^n; X) \qquad \text{for } s \in \mathbb{R},\ p \in (1, \infty),$$

is valid if and only if X can be renormed as a Hilbert space [SSS12, Section 2.2].

We collect some properties of interpolation spaces from the monographs [BL76; Lun09; Tri95]. Let X_0 and X_1 be Banach spaces with dense embeding $X_1 \hookrightarrow^d X_0$ and let also X, Y, Y_0, and Y_1 be Banach spaces. The space Y is called *interpolation space* for the couple (X_0, X_1), if $X_1 \hookrightarrow Y \hookrightarrow X_0$ and if every operator $T \in \mathcal{B}(X_0)$ with $T|_{X_1} \in \mathcal{B}(X_1)$ satisfies $T|_Y \in \mathcal{B}(Y)$. For $q \in [1, \infty)$ and $\theta \in (0, 1)$, we let $X_{\theta,q} = (X_0, X_1)_{\theta,q}$ denote the *real interpolation space* and $X_\theta = [X_0, X_1]_\theta$ denote the *complex interpolation space*. The following inequalities are valid.

$$\|\cdot\|_{(X_0,X_1)_{\theta,q}} \le C(\theta, q) \|\cdot\|_0^{1-\theta} \|\cdot\|_1^\theta \text{ on } X_1, \qquad \|\cdot\|_{[X_0,X_1]_\theta} \le \|\cdot\|_0^{1-\theta} \|\cdot\|_1^\theta \text{ on } X_1.$$

If $X_1 \hookrightarrow^d Y \hookrightarrow^d X_0$, then

$$(X_0, X_1)_{\theta,q} \hookrightarrow^d (X_0, Y)_{\theta,q}, \qquad [X_0, X_1]_\theta \hookrightarrow^d [X_0, Y]_\theta.$$

If $\theta \in (0, 1)$, $q \in (1, \infty)$ and $r_j \colon X_j \to Y_j$ are isomorphisms with $r_1 = r_0|_{X_1}$, then

$$(r_0 X_0, r_1 X_1)_{\theta,q} = r_0 (X_0, X_1)_{\theta,q}, \qquad [r_0 X_0, r_1 X_1]_\theta = r_0 [X_0, X_1]_\theta.$$

For $p \in [1,\infty)$ and a σ-finite measure space $(\Omega,\mathcal{A},\mu)$, we have

$$(L_p(\Omega;X_0),L_p(\Omega;X_1))_{\theta,q} = L_p(\Omega;(X_0,X_1)_{\theta,q}),$$
$$[L_p(\Omega;X_0),L_p(\Omega;X_1)]_\theta = L_p(\Omega;[X_0,X_1]_\theta).$$

For two sectorial operators A and B in X with commuting resolvents, we have

$$(X,D(A)\cap D(B))_{\theta,q} = D_A(\theta,q)\cap D_B(\theta,q) \quad \text{for } \theta\in(0,1),\ q\in[1,\infty].$$

Let us abbreviate $\mathfrak{F} := \mathfrak{F}(\mathbb{R}^n;X)$ for $\mathfrak{F} \in \{L_p, W_p^m, W_p^s\}$. In terms of real interpolation, [Ama97, (5.8), (5.9)] yields the representation

$$W_p^s = B_{pp}^s = \big(L_p, W_p^m\big)_{s/m,p} \qquad \text{for } p\in[1,\infty),\ s\in(0,\infty)\setminus\mathbb{N},\ m\in\mathbb{N},\ s<m.$$

From [Ama97, (5.2)-(5.6), (5.8), (5.15)] we derive the embedding

$$W_p^s \hookrightarrow W_q^t \quad \text{if } s,t\in[0,\infty),\ s-n/p>t-n/q, \quad 1\ge 1/p>1/q>0.$$

Moreover, from [Ama97, (5.2)-(5.6), (5.8), (5.16)] we obtain

$$W_p^s \hookrightarrow BUC^t \quad \text{if } s-n/p>t,\ p\in[1,\infty). \tag{B.1}$$

By [Ama97, (5.2), (5.15)], we further have

$$B_{pp}^{m+\varepsilon} \hookrightarrow W_p^m \hookrightarrow B_{pp}^{m-\delta} \quad \text{for } \varepsilon>0,\ \delta\in(0,m),\ m\in\mathbb{N}_0,\ p\in[1,\infty). \tag{B.2}$$

By [SSS12, Proposition 2.13, Theorem 2.20] and [Ama97, (5.2)] we have

$$B_{pp}^{s+\varepsilon} \hookrightarrow H_p^s \hookrightarrow B_{pp}^{s-\varepsilon} \quad \text{for } \varepsilon>0,\ s\in\mathbb{R},\ p\in[1,\infty). \tag{B.3}$$

B.1.2. Regularity of domains, embeddings, and extensions.

B.1. Definition (Cone condition)**.** Given $x\in\mathbb{R}^n$, $r>0$, $\theta>0$, $v\in\mathbb{R}^n\setminus\{0\}$, the set

$$x+C_{r,\theta,v} = x+\{y\in\mathbb{R}^n : y=0 \text{ or } |y|\in(0,r], \angle(y,v)\le\theta/2\}$$

is called finite cone with vertex x, height r, direction $v(x)$ and opening angle θ. The angle $\alpha=\angle(y,v)\in[0,\pi]$ between $y,v\in\mathbb{R}^n\setminus\{0\}$ is defined by $y\cdot v=|y|\,|v|\cos\alpha$.

A domain $\Omega\subset\mathbb{R}^n$, $(n\in\mathbb{N})$, satisfies the *cone condition* if there exist $r>0$, $\theta>0$ such that each $x\in\Omega$ is the vertex of a finite cone $x+C_{r,\theta,v(x)}\subset\Omega$, for some $v(x)\in\mathbb{R}^n\setminus\{0\}$.

B.2. Definition (Local Lipschitz Condition)**.** A bounded domain $\Omega\subset\mathbb{R}^n$ satisfies the *local Lipschitz condition*, if each $x\in\partial\Omega$ has a neighborhood $U_x\subset\mathbb{R}^n$ such that $U_x\cap\partial\Omega$ is the graph of a Lipschitz continuous function; that is, there is $V\subset\mathbb{R}^{n-1}$, $f\in C^{0,1}(V;\mathbb{R})$ and an orthogonal transformation Q such that $U_x\cap\partial\Omega = x+Q\,\mathrm{graph}\,f=\{x+Q(v,f(v)) : v\in V\}$.

B.3. Theorem. *Let $\Omega\subset\mathbb{R}^n$ be a domain, X be a Banach space, $p\in[1,\infty)$, $q\in[1,\infty]$, $s\in\mathbb{R}$, $k\in\mathbb{N}_0$. If Ω satisfies the cone condition, then*

$$B_{pq}^{s+k}(\Omega;X)\hookrightarrow BC^k(\Omega;X), \qquad \textit{if } s-n/p>0.$$

If Ω satisfies the strong local Lipschitz condition, then

$$B_{pq}^{s+k}(\Omega;X)\hookrightarrow BUC^k(\Omega;X), \qquad \textit{if } s-n/p>0.$$

Proof. The assertions for the scalar-valued case $X=\mathbb{K}$ are known [AF03, Theorem 7.34, Theorem 7.37]. To obtain the vector-valued result we consider $u\in W_p^{s+k}(\Omega;X)$ and $x'\in X'$. Then $x'\circ u$ belongs to $W_p^{s+k}(\Omega)$ with $\|x'\circ u\|_{W_p^{s+k}}\le\|x'\|_{X'}\|u\|_{W_p^{s+k}(X)}$. The scalar-valued embedding implies $x'\circ u\in BC^k(\Omega)$ with $\|x'\circ u\|_{BC^k}\le C\|x'\circ u\|_{W_p^{s+k}}$, where C denotes the embedding constant for $W_p^{s+k}(\Omega)\hookrightarrow BC^k(\Omega)$. Assume in addition that $u\in\mathcal{S}(\mathbb{R}^n;X)$. Then

$$\|u\|_{BC^k(X)} = \sup_{\|x'\|\le1}\|x'(u)\|_{BC^k}\le\sup_{\|x'\|\le1}C\|x'(u)\|_{W_p^{s+k}}\le C\|u\|_{W_p^{s+k}(X)}.$$

Hence the identity is bounded from a dense subset of $W_p^{s+k}(\Omega;X)$ into $BC^k(\Omega;X)$. Approximation yields $W_p^{s+k}(\Omega;X) \hookrightarrow BC^k(\Omega;X)$. The second embedding can be shown analogously. □

B.4. Lemma (Hardy's inequality [cf. Dur70, Appendix B])**.** *Let $p \in [1,\infty)$, $r \in (-1+1/p,\infty)$, $T \in (0,\infty]$ and let X be a Banach space. If $[t \mapsto t^{-r} g(t)] \in L_p(0,T;X)$, then $[x \mapsto x^{-r-1} \int_0^x g(t)\,dt] \in L_p(0,T;X)$ and the following inequality is valid.*

$$\left(\int_0^T \left\| \frac{1}{x^{1+r}} \int_0^x g(t)\,dt \right\|^p dx\right)^{1/p} \le \frac{1}{1+r-1/p} \left(\int_0^T \frac{1}{t^{rp}} \|g(t)\|^p \,dt\right)^{1/p}. \tag{B.4}$$

Proof. The result can be proved similarly as in [Dur70, Appendix B], where the case $r=0$ is considered. First let $T<\infty$. We employ the substitution $t = xs/T$ and the continuous version

$$\left\| \int_0^T f(t,\cdot)\,dt \right\|_{L_p(\mu)} \le \int_0^T \|f(t,\cdot)\|_{L_p(\mu)}\,dt$$

of Minkowski's inequality with respect to the measure $d\mu(x) = dx/x^{rp}$. Then

$$\begin{aligned}\left(\int_0^T \left(\frac{1}{x^{1+r}} \int_0^x \|g(t)\|_X\,dt\right)^p dx\right)^{1/p} &= \left(\int_0^T \left(\frac{1}{Tx^r} \int_0^T \left\|g\left(\frac{xs}{T}\right)\right\|_X ds\right)^p dx\right)^{1/p} \\ &\le \frac{1}{T}\int_0^T \left(\int_0^T \frac{1}{x^{rp}} \left\|g\left(\frac{xs}{T}\right)\right\|_X^p dx\right)^{1/p} ds,\end{aligned}$$

provided that the right-hand side is finite. But this follows with the substitution $xs/T = u$,

$$\begin{aligned}&\frac{1}{T}\int_0^T \left(\int_0^T \frac{1}{x^{rp}} \left\|g\left(\frac{xs}{T}\right)\right\|_X^p dx\right)^{1/p} ds \\ &\le \frac{1}{T^{1+r-1/p}} \int_0^T t^{r-1/p} \left(\int_0^t \frac{1}{u^{rp}} \|g(u)\|_X^p\,du\right)^{1/p} dt \\ &\le \frac{1}{1+r-1/p} \left(\int_0^T \frac{1}{u^{rp}} \|g(u)\|_X^p\,du\right)^{1/p}.\end{aligned}$$

By Fubini's theorem and the finiteness of the right-hand side, the left-hand side is also finite and this proves Hardy's inequality for the case $T<\infty$. The assertion for $T=\infty$ follows by taking limits as $T\to\infty$. □

B.5. Lemma. *Let X be a Banach space, $p \in [1,\infty)$, $T \in (0,\infty)$, $\alpha \in (1/p,\infty)$. Then the following inequality is valid for every $u \in L_0(0,T;X)$ with $[t \mapsto t^{-\alpha} u(t)] \in L_p(0,T;X)$.*

$$\left(\int_0^T \frac{1}{t^{\alpha p}} \|u(t)\|_X^p\,dt\right)^{1/p} \le \frac{1}{2^p}\frac{1+\alpha-1/p}{\alpha-1/p} \left(\int_0^T\int_0^T \frac{\|u(t)-u(s)\|_X^p}{|t-s|^{1+\alpha p}}\,ds\,dt\right)^{1/p}.$$

Proof. This inequality can be checked by an inspection of the proof of [PSS07, (6.8)]. □

The spaces $L_p(\Omega;X)$, $W_p^m(\Omega;X)$, $W_p^s(\Omega;X)$ were defined intrinsically; that is, by using only the values of functions at points in Ω. Alternatively, we consider the corresponding spaces of restrictions to Ω of functions on $\mathbb{R}^n$, defined by

$$\begin{aligned}\mathfrak{F}(\Omega;X) &:= \mathfrak{F}(\mathbb{R}^n;X)|_\Omega := \{u|_\Omega : u \in \mathfrak{F}(\mathbb{R}^n;X)\}, \\ \|u\|_{\mathfrak{F}(\mathbb{R}^n;X)|_\Omega} &:= \inf\{\|v\|_{\mathfrak{F}(\mathbb{R}^n;X)} : v \in \mathfrak{F}(\mathbb{R}^n;X),\ v|_\Omega = u\},\end{aligned}$$

where $\mathfrak{F} \in \{L_p, W_p^m, W_p^s\}$. Then we obtain the embeddings

$$L_p(\mathbb{R}^n;X)|_\Omega \hookrightarrow L_p(\Omega;X), \quad W_p^m(\mathbb{R}^n;X)|_\Omega \hookrightarrow W_p^m(\Omega;X), \quad W_p^s(\mathbb{R}^n;X)|_\Omega \hookrightarrow W_p^s(\Omega;X).$$

If $\Omega \subset \mathbb{R}^n$ is a bounded domain of cone-type (see [Tri95, Definition 4.2.3]), then

$$F^m_{p,2}(\mathbb{R}^n)|_\Omega = H^m_p(\mathbb{R}^n)|_\Omega = W^m_p(\Omega)$$

for $m \in \mathbb{N}$, $p \in (1,\infty)$ [Tri95, Theorem 4.2.4]. The Besov space $B^s_{pq}(\Omega)$ is also given as the real interpolation space

$$B^s_{pq}(\Omega) = \big(L_p(\Omega), H^m_p(\Omega)\big)_{s/m,q}, \quad s \in (0,\infty),\, p \in [1,\infty),\, q \in [1,\infty],$$

where m is the smallest integer larger than s [AF03, p. 7.32]. If $\Omega = \mathbb{R}^n$, we can choose any $m \in \mathbb{N}$ with $m > s$ [Tri95, 2.4.2 Remark 2].

The trivial extension by zero is bounded from $L_p(\Omega; X)$ to $L_p(\mathbb{R}^n)$, thus the spaces $L_p(\Omega; X)$ and $L_p(\mathbb{R}^n)|_\Omega$ coincide with equal norms. However, this operator does not map continuous functions on Ω to continuous functions on $\mathbb{R}^n$ and is hence not necessarily bounded from $W^m_p(\Omega; X)$ to $W^m_p(\mathbb{R}^n)$. In fact, function spaces on domains defined via restriction maybe smaller then those defined intrinsically, by nonexistence of extension operators [AF03, Paragraphs 3.20, 6.47.1, 7.32].

Extension theorems guarantee the existence of bounded extension operators, if the boundary of Ω is sufficiently regular. Then it follows immediately, that the space of restrictions coincides with the intrinsically defined space and the corresponding norms are equivalent, see Corollary B.8 for an example. This is very useful to transfer properties of function spaces on $\mathbb{R}^n$ to those on domains.

We will employ the following extension operators from $\Omega = \mathbb{R}^n_+$ to $\mathbb{R}^n$ ($n \in \mathbb{N}$), which are defined in [AF03, Theorem 5.19] by higher order reflections.

B.6. Theorem. *Let $k \in \mathbb{N}_0$. We define extension operators E^k and E^k_α ($\alpha \in \mathbb{N}^n_0$, $|\alpha| \le k$) from $\mathbb{R}^n_+$ to $\mathbb{R}^n$ by (the sum over $1 \le j \le 0$ is considered as zero)*

$$E^k u(x', -x_n) := \sum\nolimits_{j=1}^k \lambda_{j,k} u(x', j x_n),$$

$$E^k_\alpha u(x', -x_n) := \sum\nolimits_{j=1}^k (-j)^{\alpha_n} \lambda_{j,k} u(x', j x_n),$$

where $u \in L_{1,loc}(\overline{\mathbb{R}^n_+})$, $x' \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}_+$, and the numbers $\lambda_{j,k}$ solve the linear system

$$\sum\nolimits_{j=1}^k (-j)^l \lambda_{j,k} = 1 \quad \text{for all } l \in \{0,1,\dots,k-1\}.$$

Then

$$E^k \in \mathcal{B}(H^l_p(\mathbb{R}^n_+); H^l_p(\mathbb{R}^n)), \quad E^k_\alpha \in \mathcal{B}(H^{l-|\alpha|}_p(\mathbb{R}^n_+); H^{l-|\alpha|}_p(\mathbb{R}^n)), \quad \partial^\alpha_x E = E_\alpha \partial^\alpha_x,$$

for all $p \in [1,\infty)$, $l \in \{0,1,\dots,k\}$, $\alpha \in \mathbb{N}^n_0$ with $|\alpha| \le l$.

B.7. Theorem (Stein's extension theorem [Ste70], [AF03, Theorem 5.24]). *If Ω is a domain in $\mathbb{R}^n$ that satisfies the strong local Lipschitz condition, then there exists a linear extension operator, which is bounded from $W^m_p(\Omega)$ to $W^m_p(\mathbb{R}^n)$ for all $m \in \mathbb{N}_0$ and all $p \in [1,\infty)$.*

B.8. Corollary. *Let $p \in [1,\infty)$ and suppose that the domain $\Omega \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) satisfies the strong local Lipschitz condition. Then the following norms on $W^m_p(\Omega; X)$ are equivalent:*

$$\|u\|_{1,\Omega} = \left(\sum\nolimits_{|\alpha|\le m} \|\partial^\alpha u\|^p_{L_p(\Omega)}\right)^{1/p}, \quad \|u\|_{2,\Omega} = \inf\{\|v\|_{1,\mathbb{R}^n} : v \in W^m_p(\mathbb{R}^n; X), v|_\Omega = u\}.$$

For a bounded interval $(0,T)$ and a fixed order of differentiability $k \in \mathbb{N}_0$ or $s \in [0,\infty)$, it is possible to construct an extension operator with a uniform norm bound with respect to $T \in (0,\infty)$ and power $p \in [1,\infty)$. We also refer to [PSS07, Proposition 6.1].

For $s \in [0,\infty)$ with $s - 1/p \notin \mathbb{N}_0$ and a Banach space X, we define the space

$${}_0W^s_p(0,T;X) := \overline{C^\infty_c((0,T];X)}^{\|\cdot\|_{W^s_p}} = \left\{u \in W^s_p(0,T;X) : \partial^j_t u|_{t=0} = 0 \text{ for } j \le [s-1/p]\right\}.$$

Here $[s-1/p] := \min\{k \in \mathbb{Z} : k \le s-1/p\}$ denotes the integer part of $s-1/p \in \mathbb{R} \setminus \mathbb{N}_0$ and the above characterization of ${}_0W_p^s(0,T;X)$ follows from [Ama09, Theorem 4.6.2].

B.9. Lemma ([MS12, Lemma 2.5]). *Let $J=(0,T)$ be finite, $p \in (1,\infty)$, $\mu \in (1/p,1]$ and X be a Banach space of class $\mathcal{HT}$. Given $k \in \mathbb{N}$, there is an extension operator $\mathcal{E}_J$ from J to $\mathbb{R}_+$ with*

$$\mathcal{E}_J \in \mathcal{B}\left(W_{p,\mu}^s(J;X); W_{p,\mu}^s(\mathbb{R}_+;X)\right) \cap \mathcal{B}\left(H_{p,\mu}^s(J;X); H_{p,\mu}^s(\mathbb{R}_+;X)\right), \quad \text{for all } s \in [0,k].$$

Here we can replace W by ${}_0W$ and H by ${}_0H$. There is further an extension operator

$$\mathcal{E}_J^0 \in \mathcal{B}\left({}_0W_{p,\mu}^s(J;X); {}_0W_{p,\mu}^s(\mathbb{R}_+;X)\right) \cap \mathcal{B}\left({}_0H_{p,\mu}^s(J;X); {}_0H_{p,\mu}^s(\mathbb{R}_+;X)\right), \quad \text{for all } s \in [0,2],$$

which is independent of the space X and whose operator norm has a uniform bound with respect to $T \in (0,\infty)$. Moreover,

$$\mathcal{E}_J,\, \mathcal{E}_J^0 \in \mathcal{B}(L_\infty(J;X); L_\infty(\mathbb{R}_+;X)),$$

where the operator norms have a uniform bound with respect to $T \in (0,\infty)$.

B.1.3. Intrinsic spaces on hypersurfaces. Let $\Sigma \subset \mathbb{R}^{n+1}$ ($n \in \mathbb{N}$) be a compact smooth hypersurface (without boundary) and let $p \in [1,\infty]$ and $s \in [0,\infty)$. There are two approaches to define the vector-valued Sobolev-Slobodeckiĭ spaces $W_p^s(\Sigma;X)$ for a Banach space X over $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$. For the *intrinsic* approach we define it as the closure of $C^\infty(\Sigma;X)$ in the norm

$$\text{(B.5)} \quad \|u\|_{W_p^s} := \|u\|_{W_p^{\lfloor s\rfloor}} + [\![\partial^{\lfloor s\rfloor}u]\!]_{W_p^{\{s\}}}, \quad [\![v]\!]_{W_p^{\{s\}}} := \left(\int_{\Sigma\times\Sigma} \frac{|v(x)-v(y)|^p}{\operatorname{dist}_\Sigma(x,y)^{n+\{s\}p}}\, d\Sigma^2(x,y)\right)^{1/p},$$

where we require that $p<\infty$ if $s \notin \mathbb{N}_0$ and in the case $s \in \mathbb{N}_0$, the seminorm $[\![\partial^{\lfloor s\rfloor}u]\!]_{W_p^{\{s\}}}$ is omitted. The intrinsic distance $\operatorname{dist}_\Sigma$ is studied on page 133. Note that for a function $u\colon \Sigma \to X$, the derivative $\partial_\Sigma^j u(p)$ of order j belongs to $\mathcal{B}^j((T_p\Sigma)^j;X)$ and can be identified with some element of $\mathcal{B}^j((\mathbb{R}^{n+1})^j;X)$.

It is useful to relate the space $W_p^s(\Sigma;X)$ to the corresponding spaces on the whole space $\mathbb{R}^n$, for the purpose of using the known embedding and interpolation properties of the latter spaces. To this end we consider the *extrinsic* definition of $W_p^s(\Sigma;X)$ as a retract of $W_p^s(\mathbb{R}^n;X)^N$ with some $N \in \mathbb{N}$. A bounded linear operator $r\colon X \to Y$ between normed vector spaces X and Y is called *retraction* if there exists a bounded linear operator $r^c\colon Y \to X$ such that $rr^c = I_Y$ In this case we say that r^c is a *co-retraction* for r and Y is a retract of X. As in [Tri10, Definition 3.2.2/2], we shall show that the map r defined by $r(u) = ((\chi_j u)\circ\varphi_j^{-1})_j$ is a retraction, where $(\varphi_j,U_j)_{j=1}^N$ is an atlas and $(\chi_j)_{j=1}^N$ is a finite partition of unity for Σ, subordinate to $(U_j)_{j=1}^N$.

B.10. Lemma. *Let $n \in \mathbb{N}$, $p \in [1,\infty)$, $s \in (0,1)$ and let X, Y, Z be Banach spaces with continuous multiplication $X \times Y \to Z$, $(x,y) \mapsto xy$. Let ω_n denote the $(n-1)$-dimensional area of $\{x \in \mathbb{R}^n : |x| = 1\}$.*

(i) For $u \in W_\infty^1(\mathbb{R}^n;X)$ and $v \in W_p^s(\mathbb{R}^n;Y)$ we have

$$\text{(B.6)} \qquad [\![uv]\!]_{W_p^s} \le \|u\|_\infty [\![v]\!]_{W_p^s} + 2^{1-s}\left(\frac{\omega_n}{s(1-s)p}\right)^{1/p} \|u\|_\infty^{1-s}\, \|\nabla u\|_\infty^s\, \|v\|_p\,.$$

(ii) For $n=1$, $T \in (0,\infty)$, $u \in W_\infty^1(0,T;X)$, and $v \in W_p^s(0,T;Y)$, we also have (B.6).

(iii) Let $\Sigma \subset \mathbb{R}^{n+1}$ be a C^{1-}-hypersurface such that the numbers

$$C_1(R) = \sup_{x\in\Sigma}\left(\int_{B_R^\Sigma(x)} \frac{\operatorname{dist}_\Sigma(x,y)^p\, d\Sigma(y)}{\operatorname{dist}_\Sigma(x,y)^{n+sp}}\right)^{1/p}, \quad C_2(R) = 2\sup_{x\in\Sigma}\left(\int_{\Sigma\setminus B_R^\Sigma(x)} \frac{d\Sigma(y)}{\operatorname{dist}_\Sigma(x,y)^{n+sp}}\right)^{1/p}$$

are finite for some $R>0$. Then for all $u \in W_\infty^1(\Sigma;X)$ and $v \in W_p^s(\Sigma;Y)$ we have

$$[\![uv]\!]_{W_p^s(\Sigma)} \le \|u\|_{L_\infty(\Sigma)} [\![v]\!]_{W_p^s(\Sigma)} + \left(C_1(R)^p \|\nabla_\Sigma u\|_{L_\infty(\Sigma)}^p + C_2(R)^p \|u\|_{L_\infty(\Sigma)}^p\right)^{1/p} \|v\|_{L_p(\Sigma)}.$$

(iv) Let $\Sigma \subset \mathbb{R}^{n+1}$ be a C^{3-}-hypersurface. If $L_\Sigma = -\nabla_\Sigma \nu_\Sigma$ is bounded, then for $R_* := \sqrt{2}\|L_\Sigma\|_\infty^{-1}$, $R \in (0, R_*)$, and $\delta = \delta(R) := 1 - R^2\|L_\Sigma\|_\infty^2/2 \in (0,1]$, we have

$$C_1(R) \le \frac{\omega_n^{1/p} R^{1-s}}{\delta^{1-s-1/p}((1-s)p)^{1/p}} < \infty.$$

If Σ is compact, then

$$C_2(R) \le \frac{2|\Sigma|^{1/p}}{R^{s+n/p}} < \infty.$$

If Σ is a perturbed hyperplane $\{(x, h(x)) : x \in \mathbb{R}^n\}$ with $h \in C_c^{3-}(\mathbb{R}^n)$, then

$$C_2(R) \le \frac{2\omega_n^{1/p}(1 + \|\nabla h\|_\infty^2)^{1/2p}}{(sp)^{1/p} R^s} < \infty.$$

Proof. (i) First, Minkowski's inequality yields

$$\begin{aligned}
[\![uv]\!]_{W_p^s(\mathbb{R}^n)} &= \left(\iint_{\mathbb{R}^{2n}} \frac{|u(x+y)\,(v(x+y)-v(x)) + (u(x+y)-u(x))\,v(x)|^p}{|y|^{n+sp}}\,d(x,y)\right)^{\frac{1}{p}} \\
&\le \|u\|_{L_\infty(\mathbb{R}^n)}\,[\![v]\!]_{W_p^s(\mathbb{R}^n)} + \left(\iint_{\mathbb{R}^{2n}} \frac{|u(x+y)-u(x)|^p|v(x)|^p}{|y|^{n+sp}}\,d(x,y)\right)^{\frac{1}{p}}.
\end{aligned}$$

Next, we may consider the case $\nabla u \ne 0$ and let $R := 2\,\|u\|_\infty / \|\nabla u\|_\infty$. Then

$$\begin{aligned}
&\iint_{\mathbb{R}^{2n}} \frac{|u(x+y)-u(x)|^p|v(x)|^p}{|y|^{n+sp}}\,d(x,y) \\
&\quad\le \int_{\mathbb{R}^n}\left(\int_{|y|\le R} \frac{\|\nabla u\|_{L_\infty(\mathbb{R}^n)}^p}{|y|^{n-(1-s)p}}\,dx + \int_{|y|>R} \frac{2^p\,\|u\|_{L_\infty(\mathbb{R}^n)}^p}{|y|^{n+sp}}\,dx\right)|v(x)|^p\,dx \\
&\quad= \frac{2^{(1-s)p}\omega_n}{s(1-s)p}\,\|u\|_{L_\infty(\mathbb{R}^n)}^{(1-s)p}\,\|\nabla u\|_{L_\infty(\mathbb{R}^n)}^{sp}\,\|v\|_{L_p(\mathbb{R}^n)}^p.
\end{aligned}$$

Combining these estimates, we obtain inequality (B.6).

(ii) For $R = 2\|u\|_\infty/\|u'\|_\infty$ we obtain

$$\begin{aligned}
&\iint_{(0,T)^2} \frac{|u(x)-u(y)|^p|v(x)|^p}{|x-y|^{1+sp}}\,d(x,y) \\
&\quad\le \int_0^T \left(\int_{x-R}^{x+R} \frac{\|u'\|_\infty^p}{|x-y|^{sp}}\,dy + 2^p\|u\|_\infty^p 2\int_R^\infty y^{-1-sp}\,dy\right)|v(x)|^p\,dx.
\end{aligned}$$

Hence, with $2 = \omega_1$, the assertion follows analogously as above.

(iii) From Minkowski's inequality and Fubini's theorem we infer that

$$\begin{aligned}
[\![uv]\!]_{W_p^s} &\le \|u\|_\infty\,[\![v]\!]_{W_p^s} + \left(\iint_{\Sigma^2} \frac{|u(x+y)-u(x)|^p|v(x)|^p}{\mathrm{dist}_\Sigma(x,y)^{n+sp}}\,d\Sigma^2(x,y)\right)^{1/p} \\
&\le \|u\|_\infty\,[\![v]\!]_{W_p^s} + \sup_{x\in\Sigma}\left(\int_\Sigma \frac{|u(x+y)-u(x)|^p}{\mathrm{dist}_\Sigma(x,y)^{n+sp}}\,d\Sigma(y)\right)^{1/p}\|v\|_p.
\end{aligned}$$

Clearly,

$$|u(x+y)-u(x)| \le \min\left\{\|\nabla_\Sigma u\|_\infty\,\mathrm{dist}_\Sigma(x,y), 2\|u\|_\infty\right\},$$

and therefore

$$\int_\Sigma \frac{|u(x+y)-u(x)|^p}{\mathrm{dist}_\Sigma(x,y)^{n+sp}}\,d\Sigma(y) \le C_1(R)^p\|\nabla_\Sigma u\|_\infty^p + C_2(R)^p\|u\|_\infty^p,$$

which yields the asserted estimate.

(iv) With Proposition A.13 we can parametrize every geodesic ball $B_R^\Sigma(x)$ by

$$\varphi_x\colon U_x \to B_R^\Sigma(x), \quad \varphi_x(u) = x + Q_x u + h_x(u)\nu_\Sigma(x),$$

where $U_x \subset \mathbb{R}^n$ is a neighborhood of the origin, $Q_x \in \mathbb{R}^{(n+1)\times(n+1)}$ is an orthogonal matrix with $Q_x e_{n+1} = \nu_\Sigma(x)$, and $h_x\colon \overline{U_x} \to \mathbb{R}$ is a C^{3-}-function with $h_x(0) = |\nabla h_x(0)| = 0$. We further have $B_{R\delta(R)} \subset U_x \subset B_R$ and

$$(\nu_\Sigma(x)|\nu_\Sigma(\varphi_x(u))) \geq \delta, \quad |\nabla h_x(u)|^2 \leq \frac{1-\delta^2}{\delta^2} \qquad \text{for all } u \in \overline{U_x},\ x \in \Sigma.$$

Proposition A.12 yields $|u| \leq \operatorname{dist}_\Sigma(x, \varphi_x(u)) \leq (1 + |\nabla h_x(u)|^2)^{1/2}|u| \leq \delta^{-1}|u|$ and thus

$$\begin{aligned}\int_{B_R^\Sigma(x)} \operatorname{dist}_\Sigma(x,y)^{(1-s)p-n}\, d\Sigma(y) &= \int_{U_x} \operatorname{dist}_\Sigma(x,\varphi_x(u))^{(1-s)p-n}\sqrt{1+|\nabla h_x(u)|^2}\, du \\ &\leq \frac{1}{\delta}\int_{\partial B_1}\int_0^R \chi_{U_x}(t\zeta)|t\zeta|^{(1-s)p}|t\zeta|^{-n}t^{n-1}\, dt\, d\zeta \\ &\leq \frac{\omega_n R^{(1-s)p}}{\delta^{(1-s)p+1}(1-s)p}.\end{aligned}$$

This yields the estimate for $C_1(R)$. The other estimates follows easily. □

The previous estimates allow pointwise multiplication with test functions; for instance,

$$\text{(B.7)} \quad \|uv\|_{W_p^s(\mathbb{R}^n)} \leq C(n,p,s)\left(\|u\|_\infty [\![v]\!]_{W_p^s} + \|u\|_{W_\infty^1}\|v\|_p\right) \qquad \text{for } u \in W_\infty^1(\mathbb{R}^n),\ v \in W_p^s(\mathbb{R}^n).$$

This in turn yields the equivalence of intrinsic and extrinsic spaces.

B.11. Lemma. *Let $\Sigma \subset \mathbb{R}^{n+1}$ ($n \in \mathbb{N}$) be a smooth bounded hypersurface with smooth compact boundary $\partial\Sigma$, let $s \in [0,\infty)$, $p \in [1,\infty)$, and let X be a Banach space. Then the space $W_p^s(\Sigma;X)$ endowed with the intrinsic norm* (B.5) *is a retract of $W_p^s(\mathbb{R}^n;X)^N$ for some $N \in \mathbb{N}$.*

Proof. By Lemma B.10 we can find a smooth atlas $(\varphi_j(U_j), \varphi_j^{-1})_{j=1}^N$ for Σ and a smooth partition of unity $(\chi_j)_{j=1}^N$ subordinate to $(U_j)_{j=1}^N$ where $B_{R\delta(R)} \subset U_j \subset B_R \subset \mathbb{R}^n$ and $\varphi_j(u) = x_j + u + h_j(u)\nu_\Sigma(x_j)$ with $x_j \in \Sigma$ and $h_j \in C^\infty(\overline{U_j})$. Then Lemma B.10, the chain rule (B.19), the transformation formula (A.12), and Proposition A.12 imply that

$$r\colon W_p^s(\Sigma;X) \to W_p^s(\mathbb{R}^n;X)^N, \quad u \mapsto ((\chi_j u)\circ\varphi_j)_{j=1}^N$$

is well-defined and bounded.

Let further $(\psi_j)_{j=1}^N$ be a collection of smooth functions $\psi_j \in \mathcal{D}(\varphi_j(U_j))$ with $\psi_j = 1$ on $\operatorname{supp}\varphi_j(U_j)$. Then it is also straightforward to check that

$$r^c\colon W_p^s(\mathbb{R}^n;X)^N \to W_p^s(\Sigma;X), \quad (v_j)_{j=1}^N \mapsto \sum\nolimits_{j=1}^N \psi_j(v_j|_{U_j}\circ\varphi_j^{-1})$$

is a co-retraction for r. □

B.1.4. Homogeneous function spaces. We define the homogeneous spaces $\dot H_p^s(\Omega)$, $\dot B_{pq}^s(\Omega)$, and $\dot F_{pq}^s(\Omega)$ and collect some of their properties. Further information on homogeneous spaces is given by Bergh and Löfström [BL76], Kozono and Sohr [KS91], Simader and Sohr [SS96], Maz'ya [Maz11], and Triebel [Tri10].

There are two main approaches to define the homogeneous spaces on a domain $\Omega \subset \mathbb{R}^n$ or on a possibly disconnected open subset $\Omega \subset \mathbb{R}^n$. In the case $\Omega = \mathbb{R}^n$, these spaces can be defined as subspaces of $\mathcal{S}_0'(\mathbb{R}^n)$, see Definition B.13. These spaces consist of distributions modulo polynomials. Then the spaces on domains can be defined extrinsically as spaces of restrictions of functions over $\mathbb{R}^n$. For instance we can define $\dot H_p^s(\Omega) := \dot H_p^s(\mathbb{R}^n)|_\Omega := \{u\colon \Omega \to X : \exists v \in \dot H_p^s(\mathbb{R}^n) : v|_\Omega = u\}$, equipped with the norm $\|u\|_{\dot H_p^s(\Omega)} := \inf\{\|v\|_{\dot H_p^s(\mathbb{R}^n)} : v|_\Omega = u\}$. The extrinsic approach allows to transfer known results on embeddings, lifting properties (Theorem B.15) and traces (Theorem B.31, Theorem B.28) to the spaces over domains.

The homogeneous spaces can also be defined intrinsically as equivalence classes of functions on Ω modulo polynomials whose degree does not exceed some $k \in \mathbb{N}_0$. The norm only depends on the chosen subset $\Omega \subset \mathbb{R}^n$ and is independent of the representative. In the whole space case $\Omega = \mathbb{R}^n$, the intrinsic and extrinsic norms are equivalent (Remark B.14). The same is true for a subset $\Omega \subset \mathbb{R}^n$ that admits a bounded extension operator from Ω to $\mathbb{R}^n$ for the intrinsic norm.

B.12. Remark (Approximation of $\dot{H}^k_p$-functions by C^∞_c-functions)**.** As in [Gal11, Theorem II.7.1] we consider *Sobolev's cut-off function* (see Sobolev [Sob63])

$$\chi_R(x) = \chi\left(\frac{\log\log|x|}{\log\log R}\right) \text{ with } R > e \text{ and } \chi \in C^\infty([0,\infty);[0,1]),\ \chi = 1 \text{ on } [0,\tfrac{1}{2}],\ \chi = 0 \text{ on } [1,\infty).$$

Thus $\chi_R(x) = 1$ if $|x| \le e^{\sqrt{\log R}}$, $\chi_R(x) = 0$ if $|x| \ge R$ and the support of $\nabla\chi_R$ is contained in $\{e^{\sqrt{\log R}} \le |x| \le R\}$. For every $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \ge 1$ and $R_0 > e$ we have the estimate

$$|\partial^\alpha \chi_R(x)| \le \frac{c_{\alpha,R_0}}{\log\log R}\frac{1}{|x|^{|\alpha|}\log|x|}, \qquad \text{for } e^{\sqrt{\log R}} \le |x| \le R,\ R \ge R_0 > e.$$

Furthermore, let $\rho_r(x) = r^{-n}\rho(x/r)$ denote *Friedrichs' mollifiers* with some $\rho \in C^\infty_c(\mathbb{R}^n)$ such that $\operatorname{supp}\rho = \overline{B_1(0)}$, $\rho \ge 0$, $\rho(x) > 0$ for $|x| < 1$, and $\int_{\mathbb{R}^n}\rho(x)\,dx = 1$.

Let $n \ge 2$, $p \in [1,\infty)$, $k \in \mathbb{N}_0$ and let $u \in \dot{\mathcal{H}}^k_p(\mathbb{R}^n)$. Then we can find a polynomial u_0 in $\mathbb{R}^n$ of degree $\le k-1$ such that $u - u_0$ can be approximated in the norm $\|\nabla^k\cdot\|_{L_p(\mathbb{R}^n)}$ by test functions

$$u_k = \rho_{r_k} * (\chi_k \cdot (u - u_0)), \quad k \in \mathbb{N},$$

with some sequence $(r_k)_k$ such that $r_k \to 0$ as $k \to \infty$.

B.13. Definition (Extrinsic definition of homogeneous spaces)**.** Let $\mathcal{S}_0(\mathbb{R}^n) := \{\varphi \in \mathcal{S}(\mathbb{R}^n) : (\partial^\alpha\mathcal{F}\varphi)(0) = 0 \text{ for all } \alpha \in \mathbb{N}_0^n\}$ The dual space $\mathcal{S}_0'(\mathbb{R}^n)$ can be identified with $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}$, where $\mathcal{P} = \cup_{k\ge 0}\mathcal{P}_k$ and $\mathcal{P}_k$ is the linear space of all polynomials of degree not larger than k. Then the *homogeneous Besov space* and the *homogeneous Triebel-Lizorkin space* are defined by

$$\dot{B}^s_{pq}(\mathbb{R}^n) := \left\{u \in \mathcal{S}_0'(\mathbb{R}^n) : \|u\|_{\dot{B}^s_{pq}(\mathbb{R}^n)} := \left(\sum\nolimits_{j\in\mathbb{Z}}\left(2^{js}\left\|\mathcal{F}^{-1}\varphi_j\mathcal{F}u\right\|_{L_p(\mathbb{R}^n)}\right)^q\right)^{1/p} < \infty\right\},$$
$$\dot{F}^s_{pq}(\mathbb{R}^n) := \left\{u \in \mathcal{S}_0'(\mathbb{R}^n) : \|u\|_{\dot{F}^s_{pq}(\mathbb{R}^n)} := \left\|\left(\sum\nolimits_{j\in\mathbb{Z}}|2^{js}\mathcal{F}^{-1}\varphi_j\mathcal{F}u|^q\right)^{1/q}\right\|_{L_p(\mathbb{R}^n)} < \infty\right\}$$

Furthermore, we define the *homogeneous Bessel potential space* $H^s_p(\mathbb{R}^n) := F^s_{p2}(\mathbb{R}^n)$. We refer to [Tri10, Chapter 5], [BL76, Chapter 6] and [RS96, Section 2.6] for further information.

B.14. Remarks (Properties of homogeneous spaces)**.** (i) For $p \in (1,\infty)$, $s \in \mathbb{R}$, the following embeddings are continuous and dense (see [Tri10, Theorem 5.1.5].

$$\mathcal{S}_0(\mathbb{R}^n) \hookrightarrow \dot{H}^s_p(\mathbb{R}^n) \hookrightarrow \mathcal{S}_0'(\mathbb{R}^n), \quad \mathcal{S}_0(\mathbb{R}^n) \hookrightarrow \dot{B}^s_{pp}(\mathbb{R}^n) \hookrightarrow \mathcal{S}_0'(\mathbb{R}^n).$$

(ii) We have $\|f\|_{\dot{H}^s_p} = 0$ if and only if f is a polynomial [see BL76, Section 6.3].

(iii) If $s \in \mathbb{R}$, $p, q \in [1,\infty]$, and $\theta \in (0,1)$, then [see BL76, Theorem 6.3.1],

$$(\dot{H}^{s_0}_p(\mathbb{R}^n), \dot{H}^{s_1}_p(\mathbb{R}^n))_{\theta,q} = \dot{B}^s_{pq}(\mathbb{R}^n), \quad \text{if } s = (1-\theta)s_0 + \theta s_1,\ \theta \in (0,1).$$

(iv) If $s \in (0,\infty)$ and $p, q \in [1,\infty]$, then [see BL76, Theorem 6.3.2]

$$B^s_{pq}(\mathbb{R}^n) = L_p(\mathbb{R}^n) \cap \dot{B}^s_{pq}(\mathbb{R}^n), \quad H^s_p(\mathbb{R}^n) = L_p(\mathbb{R}^n) \cap \dot{H}^s_p(\mathbb{R}^n).$$

(v) If $m \in \mathbb{N}_0$ and $p \in (1,\infty)$, then $u \mapsto \sum_{|\alpha|=m}\|\partial^\alpha u\|_{L_p(\mathbb{R}^n)}$ the space $\dot{H}^m_p(\mathbb{R}^n)$ can be identified with the space $\dot{W}^m_p(\mathbb{R}^n)$ (see [Tri10, Theorem 5.2.3/1]).

(vi) If $s \in (0,1)$ and $p, q \in [1,\infty)$, then the space $\dot{B}^s_{pp}(\mathbb{R}^n)$ can be identified with the space $\dot{W}^s_p(\mathbb{R}^n)$ (see [Tri10, Theorem 5.2.3/2]).

B.15. Theorem ([Tri10, Theorem 5.2.3/1], [Ste70, Section V.1])**.** *Let*

$$\dot{J}_\sigma u := (-\Delta)^{\sigma/2} u = \mathcal{F}^{-1}(\xi \mapsto |\xi|^\sigma \mathcal{F}u(\xi)) \quad \textit{for } \sigma \in \mathbb{R},\ u \in \mathcal{S}_0'(\mathbb{R}^n)$$

denote the Riesz potential.

(i) *If* $s \in \mathbb{R}$, $\sigma \in \mathbb{R}$, $q \in [1,\infty]$, $p \in [1,\infty]$, *then* $\dot{J}_\sigma \colon \dot{B}^s_{pq}(\mathbb{R}^n) \to \dot{B}^{s-\sigma}_{pq}(\mathbb{R}^n)$ *is an isomorphism.*

(ii) *If in addition* $p \in [1,\infty)$, *then* $\dot{J}_\sigma \colon \dot{F}^s_{pq}(\mathbb{R}^n) \to \dot{F}^{s-\sigma}_{pq}(\mathbb{R}^n)$ *is an isomorphism.*

B.2. Sectorial operators and maximal regularity

B.16. Definition ([cf. DHP03, Definition 3.1])**.** A family of operators $\mathcal{T} \subset \mathcal{B}(X)$ is called $\mathcal{R}$-*bounded*, if there are numbers $C > 0$ and $p \in [1,\infty)$ such that the inequality

$$\left\| \sum\nolimits_{j=1}^N \varepsilon_j T_j x_j \right\|_{L_p(\Omega;X)} \le C \left\| \sum\nolimits_{j=1}^N \varepsilon_j x_j \right\|_{L_p(\Omega;X)}$$

is valid for all $N \in \mathbb{N}$, $T_j \in \mathcal{T}$, $x_j \in X$ and for all independent, symmetric $\{-1,1\}$-valued random variables ε_j on a probability space $(\Omega, \mathcal{M}, \mu)$. The smallest such number C is called the $\mathcal{R}$-bound of $\mathcal{T}$, denoted by $\mathcal{R}(\mathcal{T})$.

B.17. Definition ([cf. AHS94; DHP03])**.** Let X be a complex Banach space and let Σ_θ denote the open sector

$$\Sigma_\theta := \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| < \theta\} = \{re^{i\varphi} : r \in (0,\infty),\ \varphi \in (-\theta,\theta)\}, \quad \theta \in (0,\pi].$$

We write $f \in \mathcal{H}_0^\infty(\Sigma_\theta)$ if $f \colon \Sigma_\theta \to \mathbb{C}$ is a bounded holomorphic function such that there exists $s > 0$ such that $|f|(\lambda) \le \frac{c|\lambda|^s}{1+|\lambda|^{2s}}$ in Σ_θ for some $c \ge 0$.

(i) A linear operator $A \colon D(A) \to X$ is called *sectorial* (of type (K,ϑ) with $K \ge 1$, $\vartheta \in (0,\pi)$) if both $D(A)$ and $R(A)$ are dense in X and

$$\Sigma_\vartheta \subset \rho(-A) \quad \text{and} \quad \|\lambda(\lambda + A)^{-1}\|_{\mathcal{B}(X)} \le K \quad \text{for all } \lambda \in \Sigma_\vartheta.$$

We call $\phi_A := \inf\{\pi - \vartheta : \exists K \ge 1 : A \text{ is of type } (K,\vartheta)\}$ the *spectral angle* of A.

(ii) A sectorial operator $A \colon D(A) \to X$ is called $\mathcal{R}$-*sectorial* (of type (K,ϑ)) if

$$\Sigma_\vartheta \subset \rho(-A) \quad \text{and} \quad \mathcal{R}\left(\{\lambda(\lambda + A)^{-1} : \lambda \in \Sigma_\vartheta\}\right) \le K.$$

We call $\phi_A^{\mathcal{R}} := \inf\{\pi - \vartheta : \{\lambda(\lambda + A)^{-1} : \lambda \in \Sigma_\vartheta\} \text{ is } \mathcal{R}\text{-bounded}\}$ the $\mathcal{R}$-angle of A.

(iii) A sectorial operator A is said to be *of type* (K,ϑ), if

$$\overline{\Sigma}_\vartheta := \overline{\Sigma_\vartheta} \subset \rho(-A) \quad \text{and} \quad (1 + |\lambda|)\|(\lambda + A)^{-1}\|_{\mathcal{B}(X)} \le K \quad \text{for all } \lambda \in \overline{\Sigma}_\vartheta.$$

(iv) A sectorial operator A is said to have *bounded imaginary powers* (of type (C,θ)), if $A^{it} \in \mathcal{B}(X)$ for all $t \in \mathbb{R}$ and

$$\|A^{it}\|_{\mathcal{B}(X)} \le Ce^{\theta|t|} \quad \text{for } t \in \mathbb{R}.$$

(v) We say that a A has a *bounded* $\mathcal{H}^\infty$*-calculus* (of type (M,ϑ)), if A is sectorial of type (K,ϑ) with some $K \ge 1$ and

$$\|f(A)\|_{\mathcal{B}(X)} \le M\|f\|_\infty, \quad \text{for } f \in \mathcal{H}^\infty(\Sigma_{\pi-\vartheta}),$$

where $f(A)$ is defined by the extended functional calculus, see Remark B.33.

(vi) We say that A has an $\mathcal{R}$*-bounded* $\mathcal{H}^\infty$*-calculus* (of type (M,ϑ)), if

$$M := \mathcal{R}\left(\{f(A) \in \mathcal{B}(X) : f \in \mathcal{H}^\infty(\Sigma_{\pi-\vartheta}),\ \|f\|_\infty \le 1\}\right) < \infty.$$

We introduce the abbreviations

$$\begin{aligned}
\mathcal{S}(X;K,\vartheta) &:= \{A : A \text{ is sectorial in } X \text{ of type } (K,\vartheta)\}\,,\\
\mathcal{RS}(X;K,\vartheta) &:= \{A : A \text{ is } \mathcal{R}\text{-sectorial in } X \text{ of type } (K,\vartheta)\}\,,\\
\mathcal{P}(X;K,\vartheta) &:= \{A : A \text{ is of type } (K;\vartheta) \text{ in } X\}\,,\\
\mathcal{BIP}(X;C,\theta) &:= \{A : A \text{ has bounded imaginary powers of type } (C,\theta) \text{ in } X\}\,,\\
\mathcal{H}^\infty(X;M,\vartheta) &:= \{A : A \text{ has a bounded } \mathcal{H}^\infty\text{-calculus in } X \text{ of type } (M,\vartheta) \text{ in } X\}\,,\\
\mathcal{RH}^\infty(X;M,\vartheta) &:= \{A : A \text{ has an } \mathcal{R}\text{-bounded } \mathcal{H}^\infty\text{-calculus in } X \text{ of type } (M,\vartheta) \text{ in } X\}\,.
\end{aligned}$$

Furthermore, we define $\mathcal{S}(X;\vartheta) := \cup_K \mathcal{S}(X;K,\vartheta)$ and $\mathcal{S}(X) := \cup_\vartheta \mathcal{S}(X;\vartheta)$ and we will write $\mathcal{S}(K,\vartheta) := \mathcal{S}(X;K,\vartheta)$ and $\mathcal{S}(\vartheta) := \mathcal{S}(X;\vartheta)$ if no confusion seems likely, analogously for the other classes. Then we define the angles

$$\begin{aligned}
\phi_A &:= \inf\{\phi \geq 0 : A \in \mathcal{S}(\pi-\phi)\},\\
\phi_A^{\mathcal{R}} &:= \inf\{\phi \geq 0 : A \in \mathcal{RS}(\pi-\phi)\},\\
\theta_A &:= \limsup_{|t|\to\infty} \frac{\log\|A^{it}\|}{|t|} = \inf\{\theta \geq 0 : A \in \mathcal{BIP}(\theta)\},\\
\phi_A^\infty &:= \inf\{\phi \geq 0 : A \in \mathcal{H}^\infty(\pi-\phi)\},\\
\phi_A^{\mathcal{R}\infty} &:= \inf\{\phi \geq 0 : A \in \mathcal{RH}^\infty(\pi-\phi)\}.
\end{aligned}$$

B.2.1. Maximal L_p-regularity. We collect some material on analytic semigroups and maximal L_p-regularity from [Dor93], [Ama95], [Lun95], [Wei01], [Prü02], and [DHP03]. We assume that $A\colon D(A) \to X$ is a closed linear operator in a complex Banach space X and that $D(A)$ is equipped with the graph norm $\|\cdot\|_X + \|A\cdot\|_X$.

B.18. Definition (Analytic semigroup)**.** Let $\theta \in (0,\pi/2]$ and $\Sigma_\theta = \{\lambda \in \mathbb{C}\setminus\{0\} : |\arg\lambda| < \theta\}$. A family $T := \{T(t) : t \in \Sigma_\theta \cup \{0\}\} \subset \mathcal{B}(X)$ is called *(strongly continuous) analytic semigroup*, if

(i) the map $t \mapsto T(t)\colon \Sigma_\theta \to \mathcal{B}(X)$ is analytic,
(ii) $T(0) = I$ and $T(t)T(s) = T(t+s)$ for all t, $s \in \Sigma_\theta \cup \{0\}$ (semigroup property),
(iii) $T(t_n)x \to x$ in X as $\Sigma_{\theta'} \ni t_n \to 0$ for all $x \in X$, $\theta' \in (0,\theta)$ (strong continuity).

B.19. Definition. The *generator* $A\colon D(A) \to X$ of an analytic semigroup T is defined by

$$Ax := \lim_{t\to 0+} \frac{T(t)x - x}{t}, \quad D(A) := \left\{x \in X : \lim_{t\to 0+} \frac{T(t)x - x}{t} \text{ exists in } X\right\}.$$

From now on we let $-A$ be the negative generator of the analytic semigroup $e^{-tA} := T(t)$. For a given function $f \in L_{1,\mathrm{loc}}([0,\infty);X)$ we consider the *abstract Cauchy problem*

$$\partial_t u(t) + Au(t) = f(t), \quad t \in (0,\infty), \quad u(0) = 0. \tag{B.8}$$

It is known [Ama95, Remarks II.2.1.2] that the unique *mild solution* $u \in C([0,\infty);X)$ of (B.8) is given by the *variation of parameters formula*

$$u(t) = \int_0^t e^{-(t-s)A} f(s)\,ds, \quad t \in [0,\infty).$$

We study the solvability of problem (B.8) with respect to the function spaces

$${}_0\mathbb{E}(T) := {}_0H_p^1(0,T;X) \cap L_p(0,T;D(A)), \quad \mathbb{F}(T) = L_p(0,T;X),$$

where $T \in (0,\infty]$ and $p \in (1,\infty)$.

B.20. Definition. We say that A has *maximal $L_p(0,T;X)$-regularity* or *maximal L_p-regularity* on $(0,T)$ in X if for every $f \in \mathbb{F}(T)$, the mild solution of problem (B.8) belongs to ${}_0\mathbb{E}(T)$. We let $\mathcal{MR}_p(J;X)$ denote the class of all operators with maximal $L_p(0,T;X)$-regularity.

B.21. Remarks. The following facts are shown in [Dor93], [Ama95], and [Prü02]. (i) If $A \in \mathcal{MR}_p(J;X)$ is valid for some $p \in (1,\infty)$, then it is valid for all $p \in (1,\infty)$. We will therefore simply write $\mathcal{MR}$ instead of $\mathcal{MR}_p$ in the following. (ii) If $A \in \mathcal{MR}((0,T_0);X)$ for some $T_0 \in (0,\infty]$, then $A \in \mathcal{MR}((0,T);X)$ for all $T \in (0,\infty)$. (iii) If $A \in \mathcal{MR}((0,1);X)$, then there is $\mu > 0$ such that $\mu + A \in \mathcal{MR}(\mathbb{R}_+;X)$.

The translations $\mu + A$ with large $\mu > 0$ can be avoided if the spectrum $\sigma(A)$ of A is contained in the positive right half-plane.

B.22. Theorem (Kato [see Dor93, Theorem 2.4])**.** *If $\mu + A \in \mathcal{MR}(\mathbb{R}_+;X)$ for some $\mu \in \mathbb{C}$ and if $\sigma(A) \subset \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}$, then $A \in \mathcal{MR}(\mathbb{R}_+;X)$.*

It is useful to define the larger class ${}_0\mathcal{MR}(\mathbb{R}_+;X) \supset \mathcal{MR}(\mathbb{R}_+;X)$ of all $A \in \mathcal{MR}((0,1);X)$ for which the mild solution u to (B.8) satisfies the weaker a priori estimate

$$\|\partial_t u\|_{L_p(\mathbb{R}_+;X)} + \|Au\|_{L_p(\mathbb{R}_+;X)} \le C\|f\|_{L_p(\mathbb{R}_+;X)} \quad \text{for all } f \in L_p(\mathbb{R}_+;X).$$

We note that $A \in \mathcal{MR}(\mathbb{R}_+;X)$ if and only if $A \in {}_0\mathcal{MR}(\mathbb{R}_+;X)$ and $0 \in \rho(A)$ [see PS15]. The following characterization of maximal L_p-regularity is very important and useful.

B.23. Theorem (Weis, [Wei01, Theorem 4.2], [cf. DHP03, Theorem 4.4])**.** *Let X be a Banach space of class $\mathcal{HT}$ and let A generate a bounded analytic semigroup in X. Then A belongs to ${}_0\mathcal{MR}(\mathbb{R}_+;X)$ if and only if $\{\lambda(\lambda + A)^{-1} : \lambda \in \Sigma_\theta\}$ is $\mathcal{R}$-bounded for some $\theta > \pi/2$.*

Next, we study exponentially decaying solutions of the abstract initial value problem

$$\partial_t u + Au = f \text{ on } J, \quad u(0) = x.$$

Let E, $\mathbb{X}(J)$ be Banach spaces such that $\mathbb{X}(J) \hookrightarrow L_{1,\text{loc}}(J;E)$ where $J = (0,T)$ for $T \in (0,\infty]$ and let $\omega \in \mathbb{R}$. We employ the *exponentially weighted space*

$$e^{-\omega}\mathbb{X}(J) := \{u \in L_{1,\text{loc}}(J;E) : [t \mapsto e^{\omega t}u(t)] \in \mathbb{X}(J)\},$$

equipped with the norm $\|u\|_{e^{-\omega}\mathbb{X}(J)} := \|[t \mapsto e^{\omega t}u(t)]\|_{\mathbb{X}(J)}$.

B.24. Proposition ([cf. Ama95, Proposition III.1.5.3])**.** *Suppose that $\omega + A \colon D(A) \to X$ has maximal $L_p(\mathbb{R}_+;X)$-regularity for some $\omega \in \mathbb{R}$. Then*

$$(\partial_t + A, \gamma_0) : e^{\omega\cdot}\left[H^1_p(\mathbb{R}_+;X) \cap L_p(\mathbb{R}_+;D(A))\right] \to e^{\omega\cdot}L_p(\mathbb{R}_+;X) \times D_A(1-1/p,p)$$

is an isomorphism.

B.2.2. Fractional domains and abstract trace spaces. For two Banach spaces X_0 and X_1 with dense embedding $X_1 \hookrightarrow X_0$ and for $M \ge 1$ and $\vartheta \in (\pi/2,\pi)$, we define the class

$$\mathcal{P}_1(X_1,X_0;M,\vartheta) := \Big\{A \in \mathcal{P}(X_0;M,\vartheta) \cap \mathcal{B}_{\text{isom}}(X_1;X_0) : \|A\|_{\mathcal{B}(X_1;X_0)} \le M,$$
$$(1+|\lambda|)^{1-j}\|(\lambda + A)^{-1}\|_{\mathcal{B}(X_0;X_j)} \le M \text{ for } j \in \{0,1\},\ \lambda \in \overline{\Sigma}_\vartheta\Big\}.$$

If A belongs to $\mathcal{P}_1(X_1,X_0;M,\vartheta)$, then $-A$ generates an exponentially stable analytic semigroup $t \mapsto e^{-tA}$. Arguing as in [AHS94, Section 1], it can be shown that there are $\omega_0 = \omega_0(X_1,X_0,M,\vartheta) > 0$ and $M' \ge 1$ such that for all $\omega \in (0,\omega_0)$ we have $A - \omega \in \mathcal{P}_1(X_1,X_0;M',\vartheta)$.

For $\alpha \in (0,1)$ and $p \in (1,\infty)$ we define the seminorms

$$[x]_{D_A(\alpha,p)} := \left(\int_0^\infty |t^{1-\alpha}Ae^{-tA}x|^p_{X_0}\frac{dt}{t}\right)^{1/p}, \quad [\![x]\!]_{D_A(\alpha,p)} := \left(\int_0^\infty |t^{-\alpha}(e^{-tA} - I)x|^p_{X_0}\frac{dt}{t}\right)^{1/p}.$$

It is shown in [Lun95, Proposition 2.2.4] that these seminorms are equivalent. The *fractional domains* of A for $\alpha \in (0,1)$ and $p \in (1,\infty)$ are defined by

$$D_A(\alpha,p) := \left\{x \in X_0 : [x]_{D_A(\alpha,p)} < \infty\right\}, \quad |x|_{D_A(\alpha,p)} := |x|_{X_0} + [x]_{D_A(\alpha,p)}.$$

We also put $D_A(1,p) := D(A)$ with $[x]_{D_A(1,p)} := |Ax|_{X_0}$ and we let $(R_A x)(t) = e^{-tA}x$.

B.25. Theorem (cf. [Lun95, Section 2.2.1], [Ama95, Proposition III.4.10.3])**.** *Let X_0, X_1 be Banach spaces with dense embedding $X_1 \hookrightarrow X_0$ and let $M \geq 1$, $\vartheta \in (\pi/2, \pi)$, $p \in (1,\infty)$, $\alpha \in (1/p, 1]$ be fixed. Then the following norms are equivalent in $x \in D(A)$ with uniform constants with respect to $A \in \mathcal{P}_1(X_1, X_0; M, \vartheta)$ and $T \in (0,\infty]$.*

$$|x|_{D_A(\alpha-1/p,p)}, \quad |x|_{(X_0,D(A))_{\alpha-1/p,p}}, \quad \|R_A x\|_{L_p(0,T;D_A(\alpha,p))}, \quad \|R_A x\|_{W_p^\alpha(0,T;X_0)}.$$

In particular, the operator

$$R_A\colon x \mapsto \left(t \mapsto e^{-tA}x\right), \quad D_A(\alpha - 1/p, p) \to W_p^\alpha(0,T;X_0) \cap L_p(0,T;D_A(\alpha,p))$$

is uniformly bounded with respect to $A \in \mathcal{P}_1(X_1, X_0; M, \vartheta)$ and $T \in (0,\infty]$.

For the spaces $D_A(k+\alpha, p) := (D(A^k), D(A^{k+1}))_{\alpha,p}$ we obtain the following result.

B.26. Corollary. *Let $A\colon D(A) \to X$ be the negative generator of a bounded analytic semigroup in X such that A is invertible and let $k \in \mathbb{N}_0$, $p \in (1,\infty)$, $\alpha \in (1/p, 1]$. Then the operator*

$$R_A\colon u \mapsto \left(t \mapsto e^{-tA}u\right), \quad D_A(k+\alpha-1/p,p) \to W_p^{k+\alpha}(\mathbb{R}_+;X) \cap L_p(\mathbb{R}_+;D_A(k+\alpha,p))$$

is a bounded right-inverse for trace operator $\cdot|_{t=0}$.

Proof. This follows from Theorem B.25 and the identity $\partial_t e^{-tA} = -Ae^{-tA} = e^{-tA}A$. □

B.27. Theorem ([Dor99])**.** *Let A be invertible and sectorial in X with spectral angle ϕ_A. Then A has a bounded $\mathcal{H}^\infty$ functional calculus in $D_A(\alpha,p)$ ($\alpha \in (0,1)$, $p \in (1,\infty)$) with $\phi_A^\infty \leq \phi_A$.*

B.2.3. Some concrete trace spaces.

B.28. Theorem (Poisson semigroup [Tri10, Remark 5.2.3/4], [Tri95, p. 2.5.3])**.** *Let $n \in \mathbb{N}$, let*

$$p(x) = \frac{c_n}{(1+|x|^2)^{(n+1)/2}}, \quad \text{with } c_n > 0 \text{ such that } \int_{\mathbb{R}^n} p(x)\,dx = 1,$$

denote the Poisson kernel and put $p_t(x) = t^{-n}p(x/t)$. Then the following assertions are valid.

(i) The Poisson semigroup

$$(P(t)u)(x) := (p_t * u)(x) = \int_{\mathbb{R}^n} \frac{c_n t\, u(y)\,dy}{(|x-y|^2+t^2)^{(n+1)/2}}, \quad u \in L_p(\mathbb{R}^n),\ t > 0,$$

is a bounded analytic C_0-semigroup in $L_p(\mathbb{R}^n)$, $p \in (1,\infty)$.

(ii) The identity $P(t)u = \mathcal{F}^{-1}(\xi \mapsto e^{-|\xi|t}\mathcal{F}u(\xi))$ is valid for every $u \in \mathcal{S}(\mathbb{R}^n)$.

(iii) Let Λ denote the generator of P. Then $\Lambda^{2m} = (-1)^m\Delta^m$, $D(\Lambda^{2m}) = H_p^{2m}(\mathbb{R}^n)$ for $m \in \mathbb{N}$.

(iv) For $s \in (0,\infty)$, $q \in [1,\infty]$, $m \in \mathbb{N}$, $m > s$, the following norms are equivalent.

$$\|u\|_{B_{pq}^s(\mathbb{R}^n)} \sim \|u\|_{L_p(\mathbb{R}^n)} + \left(\int_0^\infty t^{(m-s)q} \left\|\frac{\partial^m P(t)u}{\partial t^m}\right\|_{L_p(\mathbb{R}^n)}^q \frac{dt}{t}\right)^{1/q}, \quad u \in B_{pq}^s(\mathbb{R}^n),$$

$$\|u\|_{\dot{B}_{pq}^s(\mathbb{R}^n)} \sim \left(\int_0^\infty t^{(m-s)q} \left\|\frac{\partial^m P(t)u}{\partial t^m}\right\|_{L_p(\mathbb{R}^n)}^q \frac{dt}{t}\right)^{1/q}, \quad u \in \dot{B}_{pq}^s(\mathbb{R}^n).$$

B.29. Theorem ([Zac03, Theorem 3.2.1])**.** *Let X be a Banach space of class $\mathcal{HT}$, $p \in (1,\infty)$, $\gamma \in [0,1/p)$ and $s+\gamma > n+1/p$ with $n \in \mathbb{N}_0$. Let further $J = [0,T]$ or $\mathbb{R}_+$, and A be an $\mathcal{R}$-sectorial operator in X with $\mathcal{R}$-angle $\phi_A^{\mathcal{R}} < \pi/s$. Then for all $0 \leq k \leq n$,*

$$H_p^{s+\gamma}(J;X) \cap H_p^\gamma(J;D_{A^s}) \hookrightarrow BUC^k(J;D_A(s+\gamma-k-1/p,p))$$

and

$$B_{pp}^{s+\gamma}(J;X) \cap H_p^\gamma(J;D_A(s,p)) \hookrightarrow BUC^k(J;D_A(s+\gamma-k-1/p,p)).$$

B.30. Theorem ([SSS12, Theorem 4.19])**.** *Let X be a Banach space and $p \in (1,\infty)$, $m \in \mathbb{N}$, $s \in (1/p,\infty)$. For the restriction operator $\varphi \mapsto \varphi|_{\mathbb{R}^{n-1}}$, $C(\mathbb{R}^n;X) \to C(\mathbb{R}^{n-1};X)$, the following assertions are valid.*

(i) The restriction operator can be extended uniquely to a continuous surjective mapping

$$\operatorname{tr}\colon W_p^m(\mathbb{R}^n;X) \to B_{pp}^{m-1/p}(\mathbb{R}^{n-1};X)$$

and tr *has a continuous right-inverse* $\operatorname{ext}\colon B_{pp}^{m-1/p}(\mathbb{R}^{n-1};X) \to W_p^m(\mathbb{R}^n;X)$.

(ii) The restriction operator can be extended uniquely to a continuous surjective mapping

$$\operatorname{tr}\colon H_p^s(\mathbb{R}^n;X) \to B_{pp}^{s-1/p}(\mathbb{R}^{n-1};X)$$

and tr *has a continuous right-inverse* $\operatorname{ext}\colon B_{pp}^{s-1/p}(\mathbb{R}^{n-1};X) \to H_p^s(\mathbb{R}^n;X))$.

B.31. Theorem ([Jaw77, Theorem 2.1], [Jaw78, Theorem 5.1]). *Let* $p \in [1,\infty)$, $q \in [1,\infty]$, $s \in (1/p,\infty)$, $n \in \mathbb{N}$, $n \geq 2$. *For the restriction operator* $\varphi \mapsto \varphi|_{\mathbb{R}^{n-1}}$, $\mathcal{S}_0(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^{n-1})$, *the following assertions are valid.*

(i) *The restriction operator can be extended to a continuous surjective mapping*

$$\operatorname{tr}\colon \dot{B}_{pq}^s(\mathbb{R}^n) \to \dot{B}_{pq}^{s-1/p}(\mathbb{R}^{n-1})$$

and there exists a linear operator $\operatorname{ext}\colon \mathcal{S}_0'(\mathbb{R}^{n-1}) \to \mathcal{S}_0'(\mathbb{R})$ *(independent of* p, q, s*) such that the realization* $\operatorname{ext}\colon \dot{B}_{pq}^{s-1/p}(\mathbb{R}^{n-1}) \to \dot{B}_{pq}^s(\mathbb{R}^n)$ *is a continuous right-inverse of* tr.

(ii) *The restriction operator can be extended to a continuous surjective mapping*

$$\operatorname{tr}\colon \dot{F}_{pq}^s(\mathbb{R}^n) \to \dot{B}_{pp}^{s-1/p}(\mathbb{R}^{n-1})$$

and there exists a linear operator $\operatorname{ext}\colon \mathcal{S}_0'(\mathbb{R}^{n-1}) \to \mathcal{S}_0'(\mathbb{R})$ *(independent of* p, q, s*) such that the realization* $\operatorname{ext}\colon \dot{B}_{pp}^{s-1/p}(\mathbb{R}^{n-1}) \to \dot{F}_{pq}^s(\mathbb{R}^n)$ *is a continuous right-inverse of* tr.

B.32. Theorem (Spatial trace theorem [cf. MS12, Theorem 4.5]). *Let* E *be a Banach space of class* $\mathcal{HT}$, $J = (0,T)$ *be finite or infinite,* $p \in (1,\infty)$, $m \in \mathbb{N}$, $s \in (0,1]$, *such that* $2ms \in \mathbb{N}$. *Assume that* $\Omega \subset \mathbb{R}^n$ *is a domain with compact smooth boundary, or* $\Omega \in \{\mathbb{R}^n, \mathbb{R}^n_+\}$. *Then the trace*

$$\begin{aligned} u \mapsto u|_{\partial\Omega}\colon & H_p^s(J;L_p(\Omega;E)) \cap L_p(J;H_p^{2ms}(\Omega;E)) \\ & \to W_p^{s-1/2mp}(J;L_p(\partial\Omega;E)) \cap L_p(J;W_p^{2ms-1/p}(\partial\Omega;E)) \end{aligned}$$

is continuous and surjective and has a continuous right-inverse. The restriction of the trace to

$${}_0H_p^s(J;L_p(\Omega;E)) \cap L_p(J;H_p^{2ms}(\Omega;E)),$$

is uniformly bounded with respect to the length of J.

B.2.4. Functional calculus for sectorial operators.

B.33. Remark (Functional calculus). Let $A \in \mathcal{S}(X;\vartheta)$.

(i) The *(primary)* $\mathcal{H}^\infty$*-functional calculus* $\Phi_A\colon \mathcal{H}_0^\infty(\Sigma_{\pi-\vartheta}) \to \mathcal{B}(X)$ is defined by

$$(\Phi_A(f))(x) := f(A)x := \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda + A)^{-1}\,d\lambda, \qquad \text{for } x \in X,$$

where the curve $\Gamma = e^{-i\psi}[0,\infty) \cup e^{i\psi}[0,\infty) \subset \rho(-A)$ surrounds $\sigma(-A)$ counterclockwise.

(ii) Extended functional calculus. …

B.34. Theorem (Spectral mapping theorem, [Haa06, Theorem 2.7.8]). *Let* $A \in \mathcal{S}(X)$, $\phi \in (\phi_A, \pi)$ *and let* $f \in \mathcal{H}_P(\Sigma_\phi)$ *have polynomial limits at* $\{0,\infty\}$. *Then*

$$f(\tilde{\sigma}(A)) = \tilde{\sigma}(f(A)),$$

where $\tilde{\sigma}(A) := \sigma(A)$ *if* A *is bounded and* $\tilde{\sigma}(A) := \sigma(A) \cup \{\infty\}$ *otherwise.*

B.2.5. Fractional powers. Set $\rho(\lambda) = \lambda(1+\lambda)^{-2}$. This function belongs to $\mathcal{H}_0^\infty(\Sigma_\sigma)$ for each $\sigma \in (0,\pi)$ with norm $|\rho|_{L_\infty(\Sigma_\sigma)} = (2(1+\cos\sigma))^{-1}$. For given $\sigma \in (0,\pi)$ the function $\lambda \mapsto \lambda^\alpha$ is holomorphic and bijective from Σ_σ to $\Sigma_{|\alpha|\sigma}$ for all $\alpha \in \mathbb{R}$ with $|\alpha| < \pi/\sigma$.

Let $A \in \mathcal{S}(X, \phi_A)$ and let

$$\Phi_A = (f \mapsto f(A))\colon \mathcal{H}_0^\infty(\Sigma_\sigma) \to \mathcal{B}(X),$$
$$\bar\Phi_A = (f \mapsto f(A))\colon \mathcal{H}_p(\Sigma_\sigma) \to \mathcal{B}(X)$$

denote the primary and the extended $\mathcal{H}^\infty$-calculi of A, respectively. Then the fractional powers A^α, $\alpha \in \mathbb{C}$, are defined by

$$A^\alpha = \bar\Phi_A(\lambda \mapsto \lambda^\alpha) = \Phi_A(\rho)^{-k}\Phi_A(\lambda \mapsto \rho(\lambda)^k\lambda^\alpha),$$

for $k \in \mathbb{N}$, $k > \alpha$. Their natural domains are given by

$$D(A^\alpha) = \left\{x \in X : \Phi_A(\lambda \mapsto \rho(\lambda)^k\lambda^\alpha)x \in D(A^k) \cap R(A^k)\right\}.$$

B.35. Theorem ([DHP03, Theorem 2.3])**.** *Let A be sectorial in X with spectral angle ϕ_A and let $\alpha \in (-\pi/\phi_A, \pi, \phi/A)$. Then A^α is also sectorial in X with $\phi_{A^\alpha} \le |\alpha|\phi_A$.*

A sufficient condition for $A^\alpha \in \mathcal{H}^\infty(X)$ can be derived with the following *composition rule.*

B.36. Theorem ([cf. Haa06, Theorem 2.4.2])**.** *Let $A \in \mathcal{S}(\omega)$ ($\omega \in [0,\pi)$) be injective and, for some $\phi \in (\omega,\pi)$ and $\omega' \in [0,\pi)$, let $g \in \mathcal{H}_P(\Sigma_\phi)$ be a function such that $g(A) \in \mathcal{S}(\omega')$ and such that for every $\phi' \in (\omega',\pi)$ there exists $\phi \in (\omega,\pi)$ such that $g \in \mathcal{H}_P(\Sigma_\phi)$ and $g(\Sigma_\phi) \subset \overline{\Sigma_{\phi'}}$. Then*

$$(f \circ g)(A) = f(g(A)) \quad \text{for all } \phi' \in (\omega',\pi),\ f \in \mathcal{H}_P(\Sigma_{\phi'}).$$

B.37. Corollary. *The following implications are valid.*

$$A \in \mathcal{H}^\infty(X),\ -\tfrac{\pi}{|\phi_A^\infty|} < \alpha < \tfrac{\pi}{|\phi_A^\infty|} \quad \Rightarrow \quad A^\alpha \in \mathcal{H}^\infty(X),\ \phi_{A^\alpha}^\infty \le |\alpha|\phi_A^\infty,$$
$$A \in \mathcal{RH}^\infty(X),\ -\tfrac{\pi}{|\phi_A^{\mathcal{R}\infty}|} < \alpha < \tfrac{\pi}{|\phi_A^{\mathcal{R}\infty}|} \quad \Rightarrow \quad A^\alpha \in \mathcal{RH}^\infty(X),\ \phi_{A^\alpha}^{\mathcal{R}\infty} \le |\alpha|\phi_A^{\mathcal{R}\infty}.$$

B.38. Corollary. *For $A \in \mathcal{H}^\infty(X; M, \vartheta)$, the following assertions are valid.*

(i) $A \in \mathcal{BIP}(M, \pi - \vartheta)$.

(ii) *For $s \in [0, \pi/(\pi-\vartheta))$, we have $A^s \in \mathcal{H}^\infty(M, \vartheta + (1-s)(\pi-\vartheta))$.*

(iii) *For $\varepsilon > 0$, we have $\varepsilon + A \in \mathcal{P}(M_1, \vartheta_1)$ for every $\vartheta_1 \in (0,\vartheta)$, where $M_1 = 2Mc(1+\varepsilon^{-2})^{1/2}$ and $c = 1/\min\{1, 1+\cos(\pi-(\vartheta-\vartheta_1))\}$.*

Proof. *(i)* The assertion follows from $|z^{it}| = |e^{it(\ln|z|+i\arg z)}| = e^{-t\arg z} \le e^{|t|\arg z}$.

(ii) The function $g_s\colon z \mapsto z^s$ maps $\Sigma_{\pi-\vartheta}$ onto $\Sigma_{s(\pi-\vartheta)}$. Hence for $f \in \mathcal{H}^\infty(\Sigma_{s(\pi-\vartheta)})$ we have $f \circ g_s \in \mathcal{H}^\infty(\Sigma_{\pi-\vartheta})$ with the same L_∞-norm. Moreover, the composition rule implies $f(A^s) = (f \circ g_s)(A)$ and this yields the assertion.

(iii) Using $(\lambda + \varepsilon + A)^{-1} = (\lambda + \varepsilon + \cdot)^{-1}(A)$ and (B.14), we obtain

$$\begin{aligned}(1+|\lambda|)\|(\lambda+\varepsilon+A)^{-1}\| &\le (1+|\lambda|)M\|(\lambda+\varepsilon+\cdot)^{-1}\|_{L_\infty(\Sigma_{\pi-\vartheta})} \\ &\le \frac{\sqrt{2}M}{\min\{1, 1+\cos(\pi-(\vartheta-\vartheta_1))\}^{1/2}} \frac{1+|\lambda|}{|\lambda+\varepsilon|} \\ &\le \frac{2M}{\min\{1, 1+\cos(\pi-(\vartheta-\vartheta_1))\}}\sqrt{1+\frac{1}{\varepsilon^2}}.\end{aligned}$$ □

B.39. Theorem (cf. [Ama95, (I.2.9.9)] and [Tri95, Theorem 1.15.3])**.** *If $A \in \mathcal{BIP}(X)$, then*

$$D(A^\alpha) \cong [X, D(A)]_\alpha, \quad \text{for } \alpha \in [0,1],$$

where $D(A^\alpha)$ is equipped with the norm $x \mapsto \|x\|_X + \|A^\alpha x\|_X$. Moreover, given $\theta \ge 0$, $\vartheta \in (0,\pi)$, $M \ge 1$, $\alpha \in [0,1]$, there exists $C \ge 1$ such that for all $A \in \mathcal{BIP}(X; M, \theta) \cap \mathcal{P}(X; M, \vartheta)$, we have

$$C^{-1}\|x\|_{D(A^\alpha)} \le \|x\|_{[X,D(A)]_\alpha} \le C\|x\|_{D(A^\alpha)} \quad \text{for } x \in D(A^\alpha).$$

B.40. Corollary. *Let $M \geq 1$, $\vartheta \in (0, \pi)$, $s \in (0,1)$, $\varepsilon > 0$, let X_0, X_1 be Banach spaces with dense embedding $X_1 \hookrightarrow X_0$ and let $0 < \phi < \vartheta_s := s\vartheta + (1-s)\pi$. Then there exists $N \geq 1$ such that if*

$$A \in \mathcal{P}_1(X_1, X_0; M, \vartheta) \cap \mathcal{H}^\infty(X_0; M, \vartheta),$$

then

$$\varepsilon + A^s \in \mathcal{P}_1([X_0, X_1]_s, X_0; N, \phi) \cap \mathcal{H}^\infty(X_0; N, \phi).$$

Proof. First, the norms $\|x\|_{D(A)} = \|x\|_{X_0} + \|Ax\|_{X_0}$ and $\|x\|_{X_1}$ are equivalent, uniformly with respect to $A \in \mathcal{P}_1(X_1, X_0; M, \vartheta)$, since both $\|A\|_{\mathcal{B}(X_1;X_0)}$ and $\|A^{-1}\|_{\mathcal{B}(X_0;X_1)}$ are bounded by M. Hence, by Theorem B.39 and Corollary B.38, the norms of $[X_0, X_1]_s$, $[X_0, D(A)]_s$ and $D(A^s)$ are equivalent, uniformly with respect to $A \in \mathcal{P}_1(X_1, X_0; M, \vartheta) \cap \mathcal{H}^\infty(X_0; M, \vartheta)$, and this implies

$$\|A^s\|_{\mathcal{B}([X_0,X_1]_s;X_0)} \sim \|A^s\|_{\mathcal{B}(D(A^s);X_0)} \leq 1.$$

Therefore $\|\varepsilon + A^s\|_{\mathcal{B}([X_0,X_1]_s;X_0)}$ is uniformly bounded. Again by Corollary B.38 and basic resolvent identities like $A^s(\lambda + A^s)^{-1} = I - \lambda(\lambda + A^s)^{-1}$ we obtain $\varepsilon + A^s \in \mathcal{P}_1(D(A^s), X_0; M_1, \phi)$ for all $A \in \mathcal{P}_1(X_1, X_0; M, \vartheta) \cap \mathcal{H}^\infty(X_0; M, \vartheta)$. By the uniform equivalence of the norms of $D(A^s)$ and $[X_0, X_1]_s$, there exists $N \geq 1$ with $\varepsilon + A^s \in \mathcal{P}_1([X_0, X_1]_s, X_0; N, \phi)$, uniformly in A. □

B.2.6. Sums and products of sectorial operators. Let $(A, D(A))$ and $(B, D(B))$ be densely defined closed linear operators in a Banach space X. We collect several results for the sum $A + B$ and the product AB under the condition that A and B are resolvent commuting. We define the sum $A+B$, the product AB and the commutator $[A, B]$ by

$$\begin{aligned} (A+B)x &= Ax + Bx, & D(A+B) &:= D(A) \cap D(B), \\ (AB)x &= A(Bx), & D(AB) &:= \{x \in D(B) : Bx \in D(A)\}, \\ [A,B]x &= ABx - BAx, & D([A,B]) &:= D(AB) \cap D(BA). \end{aligned}$$

B.41. Remark (Commuting operators). *(i)* A bounded operator $T \in \mathcal{B}(X)$ is said to commute with a closed operator $A\colon D(A) \subset X \to X$, if $TA = AT$ on $D(A)$. If $\rho(A) \neq 0$, then this is equivalent to $[T, R(\lambda, A)] = 0$ for some (and hence all) $\lambda \in \rho(A)$ [Ama95; Haa06].

(ii) Suppose that $\rho(A) \neq \emptyset$, $\rho(B) \neq \emptyset$. We say that A and B are *resolvent commuting*, if $[(\lambda - A)^{-1}, (\mu - B)^{-1}] = 0$ for some (and hence all) $\lambda \in \rho(A)$, $\mu \in \rho(B)$. It can be shown that if A and B are resolvent-commuting, then $ABx = BAx$ for all $x \in D(AB) \cap D(BA)$.

(iii) If $A, B \in \mathcal{S}(X)$ are resolvent commuting, then also $f(A)$, $g(B)$ are resolvent commuting for all $f \in \mathcal{H}_A(\Sigma_\psi)$, $g \in \mathcal{H}_B(\Sigma_\rho)$, $\psi \in (\phi_A, \pi)$, $\rho \in (\phi_B, \pi)$.

Next we state a version of the mixed derivative theorem of Sobolevskiĭ [Sob75, Theorem 6]. A linear operator $A\colon D(A) \subset X \to X$ is called *positive* if it has the properties of a sectorial operator (Definition B.17) except that $R(A)$ does not need to be dense in X and the resolvent estimate is valid in a set $\{\lambda = re^{i\varphi} : |\varphi| \in [\theta, \pi],\ r \in [r_0, \infty)\}$ with some $r_0 \geq 0$, which may be smaller than the sector $-\Sigma_\theta$. In this case $k + A$ for $k \geq r_0$ is sectorial and invertible and $D(A)$ is a Banach space for the norm $\|(k + A)\cdot\|_X$. We say that two linear operators A and B in X form a *coercive pair* if for some numbers $M \geq 0$ and $k \in \mathbb{N}_0$ we have the estimate

$$\|(k + A)x\| + \|(k + B)x\| \leq M\|(k + A)x + (k + B)x\| \quad \text{for all } x \in D(A) \cap D(B).$$

B.42. Theorem (Mixed derivatives [cf. Sob75, Theorem 6]). *Let X be a Banach space and let A and B form a coercive pair of positive operators with commuting resolvents such that their spectral angles satisfy $\phi_A + \phi_B < \pi$. Then $A + B$ is positive, and for sufficiently large k and arbitrary $0 \leq \alpha \leq 1$ we have the continuous embeddings*

$$D(A + B) \hookrightarrow D((k + A)^\alpha (k + B)^{1-\alpha}) \cap D((k + B)^{1-\alpha}(k + A)^\alpha).$$

B.43. Corollary ([MS12, Proposition 1.1])**.** *Let X be a Banach space of class $\mathcal{HT}$ and suppose that the operators A, $B \in \mathcal{BIP}(X)$ are resolvent commuting and satisfy $\theta_A + \theta_B < \pi$. If A or B is invertible, then $A+B$ is invertible, $A+B \in \mathcal{BIP}(X)$ with $\theta_{A+B} \le \theta_A + \theta_B$ and $A^\alpha B^{1-\alpha}(A+B)^{-1}$ is bounded in X for every $\alpha \in [0,1]$.*

B.44. Proposition (Mixed derivative embeddings [cf. MS12, Proposition 3.2])**.** *Let $J = (0,T)$ be finite or infinite, $p \in (1,\infty)$, let X be a Banach space of class $\mathcal{HT}$ and let $\Omega \subset \mathbb{R}^n$ be a domain with compact smooth boundary, or $\Omega \in \{\mathbb{R}^n, \mathbb{R}^n_+\}$. Let further t, $s \ge 0$, $\alpha \in (0,2)$, $\beta > 0$, $\rho \in [0,1]$, and set $H^t_p(H^s_p) := H^t_p(J; H^s_p(\Omega; X))$, and analogously for the other anisotropic spaces. Then*

$$H^{t+\alpha}_p(H^s_p) \cap H^t_p(H^{s+\beta}_p) \hookrightarrow H^{t+\rho\alpha}_p(H^{s+(1-\rho)\beta}_p),$$

and moreover each of the spaces

$$H^{t+\alpha}_p(W^s_p) \cap H^t_p(W^{s+\beta}_p),\ W^{t+\alpha}_p(H^s_p) \cap W^t_p(H^{s+\beta}_p),\ W^{t+\alpha}_p(H^s_p) \cap H^t_p(W^{s+\beta}_p)$$

is continuously embedded into

$$W^{t+\rho\alpha}_p(H^{s+(1-\rho)\beta}_p) \cap H^{t+\rho\alpha}_p(W^{s+(1-\rho)\beta}_p),$$

provided that all the occurring W_p-spaces have a non-integer order of differentiability. Finally, assuming all orders of differentiability to be non-integer, we have

$$W^{t+\alpha}_p(W^s_p) \cap W^t_p(W^{s+\beta}_p) \hookrightarrow W^{t+\rho\alpha}_p(W^{s+(1-\rho)\beta}_p).$$

These embeddings remain true if Ω is replaced by its boundary. They are also valid if all H_p-, W_p-spaces with respect to time are replaced by ${}_0H_p$-, ${}_0W_p$-spaces, respectively. Restricting in the latter case to $t+\alpha \le 2$, the embedding constants have a uniform bound with respect to to the length of J.

B.45. Remark. The following mixed derivative embeddings are valid.

$$H^{s+\alpha}_p(\mathbb{R}_+; \dot H^r_p(\mathbb{R}^n)) \cap H^s_p(\mathbb{R}_+; \dot H^{r+\beta}_p(\mathbb{R}^n)) \hookrightarrow H^{s+\theta\alpha}_p(\mathbb{R}_+; \dot H^{r+(1-\theta)\beta}_p(\mathbb{R}^n)), \tag{B.9}$$

if s, $\beta \in [0,\infty)$, $\alpha \in [0,2]$, $r \in \mathbb{R}$, $\theta \in [0,1]$ and

$$W^{s+\alpha}_p(\mathbb{R}_+; \dot H^r_p(\mathbb{R}^n)) \cap W^s_p(\mathbb{R}_+; \dot H^{r+\beta}_p(\mathbb{R}^n)) \hookrightarrow H^{s+\theta\alpha}_p(\mathbb{R}_+; \dot W^{r+(1-\theta)\beta}_p(\mathbb{R}^n)), \tag{B.10}$$

if s, $\beta \in (0,\infty)$, $\alpha \in (0,2)$, $r \in \mathbb{R}$, $\theta \in (0,1)$ and s, $s+\alpha$, $r+(1-\theta)\beta \notin \mathbb{Z}$ and

$$H^{s+\alpha}_p(\mathbb{R}_+; \dot W^r_p(\mathbb{R}^n)) \cap H^s_p(\mathbb{R}_+; \dot W^{r+\beta}_p(\mathbb{R}^n)) \hookrightarrow W^{s+\theta\alpha}_p(\mathbb{R}_+; \dot H^{r+(1-\theta)\beta}_p(\mathbb{R}^n)), \tag{B.11}$$

if $s \in [0,\infty)$, $\alpha \in (0,2)$, $\beta \in (0,\infty)$, $r \in \mathbb{R}$, $\theta \in (0,1)$ and r, $r+\beta$, $s+\theta\alpha \notin \mathbb{Z}$.

Proof. The strategy is the same as for the non-homogeneous spaces ([MS12, Proposition 3.2]). We abbreviate $\mathcal{F}(\mathcal{K}) := \mathcal{F}(\mathbb{R}_+; \mathcal{K}(\mathbb{R}^n))$ for $\mathcal{F} \in \{\dot H^s_p, \dot W^s_p\}$, $\mathcal{K} \in \{\dot H^r_p, \dot W^r_p\}$. By applying the mixed derivative theorem B.42 to the $\mathcal{BIP}$-operators $(1-\partial_t)^\alpha \colon H^{s+\alpha}_p(\dot H^r_p) \to H^s_p(\dot H^r_p)$ and $(-\Delta)^{\beta/2} \colon H^s_p(\dot H^r_p \cap \dot H^{r+\beta}_p) \to H^s_p(\dot H^r_p)$ (see theorems B.68, B.15 and Section B.2.5), we obtain

$$\|(1-\partial_t)^{\theta\alpha}(-\Delta)^{(1-\theta)\beta/2} w\|_{H^s_p(\dot H^r_p)} \lesssim \|w\|_{H^{s+\alpha}_p(\dot H^r_p) \cap H^s_p(\dot H^{r+\beta}_p)}, \qquad \text{for all } \theta \in [0,1].$$

By using the invertibility of the operators

$$(1-\partial_t)^{\theta\alpha} \colon H^{s+\theta\alpha}_p(\dot H^r_p) \to H^s_p(\dot H^r_p),$$
$$(-\Delta)^{(1-\theta)\beta/2} \colon H^{s+\theta\alpha}_p(\dot H^{r+(1-\theta)\beta}_p) \to H^{s+\theta\alpha}_p(\dot H^r_p),$$

we further have

$$\|(1-\partial_t)^{\theta\alpha}(-\Delta)^{(1-\theta)\beta/2} w\|_{H^s_p(\dot H^r_p)} \sim \|(-\Delta)^{(1-\theta)\beta/2} w\|_{H^{s+\theta\alpha}_p(\dot H^r_p)} \sim \|w\|_{H^{s+\theta\alpha}_p(\dot H^{r+(1-\theta)\beta}_p)}.$$

Hence (B.9) is shown.

Next, for proving the embedding (B.10), we choose some sufficiently small $\varepsilon > 0$ and put $s_\pm := s \pm \alpha\varepsilon$, $\theta_\pm := \theta \mp \varepsilon$. Then $s + \alpha \pm \alpha\varepsilon = s_\pm + \alpha$, $s \pm \alpha\varepsilon = s_\pm$, $s + \theta\alpha = s_\pm + \theta_\pm\alpha$ and $r + (1-\theta)\beta \pm \varepsilon\beta = r + (1-\theta_\pm)\beta$. We now apply (B.9) with $s_\pm$, $\theta_\pm$ instead of s, θ and obtain

$$H_p^{s+\alpha\pm\alpha\varepsilon}(\dot H_p^r) \cap H_p^{s\pm\alpha\varepsilon}(\dot H_p^{r+\beta}) \hookrightarrow H_p^{s+\theta\alpha}(\dot H_p^{r+(1-\theta)\beta\mp\varepsilon\beta}).$$

Applying the real interpolation functor $(\cdot,\cdot)_{1/2,p}$ and the identity $(\dot H_p^{t_0}, \dot H_p^{t_1})_{\theta,p} = \dot W_p^t$ for $t = (1-\theta)t_0 + \theta t_1 \in \mathbb{R}\setminus\mathbb{Z}$, we obtain (B.10). The embedding (B.11) follows similarly by choosing $r_\pm := r \mp \beta\varepsilon$, $\theta_\pm := \theta \mp \varepsilon$ so that $s + \theta\alpha = s + \theta_\pm\alpha \pm \alpha\varepsilon$ and $r + (1-\theta)\beta = r_\pm + (1-\theta_\pm)\beta$. ☐

B.46. Theorem ([PS90, Theorem 5]). *Let X be a Banach space of class $\mathcal{HT}$, let $A, B \in \mathcal{BIP}(X)$ with $\theta_A + \theta_B < \pi$ be resolvent commuting, and let $\theta = \max(\theta_A, \theta_B)$, $\theta_A \neq \theta_B$. Then $A + B \in \mathcal{BIP}(X)$ with $\theta_{A+B} \le \theta$.*

B.47. Theorem (Kalton-Weis [KW01, Theorem 6.3]). *Suppose $A \in \mathcal{H}^\infty(X)$ and $B \in \mathcal{RS}(X)$ are resolvent commuting and $\phi_A^\infty + \phi_B^{\mathcal{R}} < \pi$. Then $A + B$ with domain $D(A) \cap D(B)$ is a closed operator and there is a constant C such that*

$$|Ax| + |Bx| \le C|Ax + Bx|, \quad \text{for all } x \in D(A)\cap D(B).$$

Thus, $A + B$ is invertible if either A or B is invertible. Furthermore, if X has property (α), then $A + B \in \mathcal{RS}(X)$ with $\phi_{A+B}^{\mathcal{R}} \le \max(\phi_A^\infty, \phi_B^{\mathcal{R}})$.

B.48. Corollary ([PS07]). *Suppose $A \in \mathcal{H}^\infty(X)$ and $B \in \mathcal{RH}^\infty(X)$ are commuting such that $\phi_A^\infty + \phi_B^{\mathcal{R}\infty} < \pi$. Then $A + B \in \mathcal{H}^\infty(X)$.*

Next, we consider the product AB of two sectorial operators A, B.

B.49. Theorem ([PS90, Corollary 3]). *Suppose X is of class $\mathcal{HT}$, let $A, B \in \mathcal{BIP}(X)$ with $0 \le \theta_A + \theta_B < \pi$ be resolvent commuting. Then AB is closable and $\overline{AB} \in \mathcal{BIP}(X)$ with $\theta_{AB} \le \theta_A + \theta_B$. If in addition A is invertible, then AB is closed.*

B.50. Corollary ([HDH05, Corollary 2.2]). *Let X be a Banach space and assume that $A \in \mathcal{H}^\infty(X)$ and $B \in \mathcal{RS}(X)$ are resolvent commuting such that $0 \in \rho(A)$ and $\phi_A^\infty + \phi_B^{\mathcal{R}} < \pi$. Then $AB \in \mathcal{S}(X)$ and $\phi_{AB} \le \phi_A^\infty + \phi_B^{\mathcal{R}}$. If in addition $B \in \mathcal{RH}^\infty(X)$ with $\phi_A^\infty + \phi_B^{\mathcal{R}\infty} < \pi$, then $AB \in \mathcal{H}^\infty(X)$ with $\phi_{AB}^\infty \le \phi_A^\infty + \phi_B^{\mathcal{R}\infty}$.*

B.2.7. Estimates for Fourier-Laplace symbols. In order to obtain the mapping properties of linear pseudo-differential operators, we will establish estimates of their Fourier-Laplace-symbols with respect to the temporal and spatial covariables λ, z for ∂_t and $\sqrt{-\Delta}$, respectively.

B.51. Remark (Laplace transform). Let $\mathbb{R}_+ = [0,\infty)$ and $\mathbb{C}_{>0} = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. Let $f \in L_{1,loc}(\mathbb{R}_+; X)$ be of exponential growth; that is, the integral $\int_0^\infty e^{-\omega t}|f(t)|_X\,dt$ is finite for some $\omega \in \mathbb{R}$. Then we define the Laplace transform $\mathcal{L}f\colon \omega + \mathbb{C}_{>0} \to X$ of f by

$$\widehat{f}(\lambda) := (\mathcal{L}f)(\lambda) := \int_0^\infty e^{-\lambda t} f(t)\,dt, \quad (\operatorname{Re}\lambda \ge \omega).$$

Then $\mathcal{L}f \in BUC(\omega + \overline{\mathbb{C}_{>0}}; X) \cap \mathcal{H}(\omega + \mathbb{C}_{>0}; X)$. For given $\theta_0 \in (0,\pi]$, let $\mathcal{H}_{bc}(\Sigma_{\theta_0})$ denote the vector space of all functions on Σ_{θ_0} that are holomorphic in Σ_{θ_0} and bounded and continuous on each closed sector $\overline{\Sigma_\theta}$, $\theta \in [0,\theta_0)$. Then Cauchy's theorem leads to

$$\{\mathcal{L}f : f \in \mathcal{H}_{bc}(\Sigma_{\theta_0})\} = \left\{g \in \mathcal{H}(\Sigma_{\pi/2+\theta_0}) : \lambda \mapsto \lambda g(\lambda) \in \mathcal{H}_{bc}(\Sigma_{\pi/2+\theta_0})\right\}.$$

See [Prü93, Theorem 0.1]. Uniqueness of the Laplace transform: The complex inversion formula

$$f(t) = \lim_{N\to\infty} \frac{1}{2\pi} \int_{-N}^{N} \left(1 - \frac{|\rho|}{N}\right) e^{(\sigma+i\rho)t}\mathcal{L}f(\sigma+i\rho)\,d\rho$$

applies for almost all $t \in \mathbb{R}_+$ and each $\sigma > \omega$. The real inversion formula

$$f(t) = \lim_{\sigma\to\infty} \sum_{n=0}^{\infty} \frac{(-1)^n(\sigma^2 t)^{n+1}}{n!(n+1)!} \mathcal{L}f^{(n)}(\sigma)$$

applies for almost all $t > 0$. The Laplace transform has the following relation to the Fourier transform $\mathcal{F}$. For each $f \in L_1(\mathbb{R}; X)$ such that $f(t) = 0$, $(t < 0)$, it is $\mathcal{F}f(\rho) = \mathcal{L}f(i\rho)$, $(\rho \in \mathbb{R})$.

B.52. Definition ($\alpha \lesssim \beta$, $\alpha \sim \beta$)**.** Let D be a set, $(X, |\cdot|)$ be a normed vector space and consider two functions $\alpha, \beta\colon D \to X$. We say that α is dominated by β (in D) and write

$$\alpha \lesssim \beta \text{ (in } D) \quad \text{if and only if} \quad \exists C > 0 \forall x \in D : |\alpha(x)| \le C|\beta(x)|.$$

The functions α, β are said to be equivalent, and we write $\alpha \sim \beta$, if $\alpha \lesssim \beta$ and $\beta \lesssim \alpha$.

B.53. Example. By the concavity of the logarithm and the monotonicity of the exponential function, we obtain the following estimate.

$$\forall a, b \in (0,\infty),\ \theta \in [0,1] : a^\theta b^{1-\theta} \le \theta a + (1-\theta) b \le \max\{\theta, 1-\theta\}(a+b).$$

We will use it frequently. In particular, it implies that $a + b \le a + a^\theta b^{1-\theta} + b \le 2(a+b)$; that is,

$$a^\theta b^{1-\theta} \lesssim a + b \text{ in } (0,\infty)^2, \quad a + b \sim a + a^\theta b^{1-\theta} + b \text{ in } (0,\infty)^2 \tag{B.12}$$

for every $\theta \in [0,1]$.

Let us generalize these estimates to complex numbers.

B.54. Lemma. *For $\phi \in (0,\pi)$, let $\Sigma_\phi := \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| < \phi\}$ denote the open sector centered at zero with opening angle 2ϕ. Then*

$$\lambda_1 + \lambda_2 \sim |\lambda_1| + |\lambda_2| \quad \textit{in } \Sigma_{\phi_1} \times \Sigma_{\phi_2} \textit{ if } \phi_1 + \phi_2 < \pi, \tag{B.13}$$

$$|\lambda_1 + \lambda_2| \ge 2^{-1/2}\sqrt{1 + \min\{0, \cos(|\arg \lambda_1| + |\arg \lambda_2|)\}}\,(|\lambda_1| + |\lambda_2|) \quad \textit{in } \mathbb{C} \times \mathbb{C}, \tag{B.14}$$

$$\lambda_1 + \lambda_2 \sim \lambda_1 + \lambda_1^{1-\theta}\lambda_2^\theta + \lambda_2 \quad \textit{in } \Sigma_{\phi_1} \times \Sigma_{\phi_2} \textit{ if } \theta \in [0,1],\ \phi_j \in [0,\pi/2),\ \phi_1 + \phi_2 < \pi/2. \tag{B.15}$$

Proof. *(i)* Clearly, the estimate $\lambda_1 + \lambda_2 \lesssim |\lambda_1| + |\lambda_2|$ applies in $\mathbb{C}^2$ by the triangle inequality. Let $\phi_1 \in [0,\pi)$, $\phi_2 \in [0, \pi - \phi_1)$ and $\lambda_1 \in \Sigma_{\phi_1}$, $\lambda_2 \in \Sigma_{\phi_2}$. Then $\operatorname{Re}(\lambda_1 \lambda_2) \ge |\lambda_1\lambda_2| \cos(\phi_1 + \phi_2)$ where $\cos(\phi_1 + \phi_2) \in (-1, 1]$. For $a, b, c, s \in \mathbb{R}$, $s \le 1$ we obtain

$$a^2 + 2abc + b^2 = s(a+b)^2 + (1-s)(a+b)^2 + 2ab(c-1) \ge s(a+b)^2 + 4ab(1-s) + 2ab(c-1).$$

Choosing $s = \frac{1}{2}(1+c)$ yields $a^2 + 2abc + b^2 \ge \frac{1}{2}(1+c)(a+b)^2$. Taking $c = \cos(\phi_1 + \phi_2)$ yields

$$|\lambda_1 + \lambda_2|^2 = |\lambda_1|^2 + 2\operatorname{Re}(\lambda_1 \bar{\lambda}_2) + |\lambda_2|^2 \ge \tfrac{1}{2}(1 + \cos(\phi_1 + \phi_2))(|\lambda_1| + |\lambda_2|)^2, \quad \text{for } \lambda_j \in \Sigma_{\phi_j}.$$

This inequality and the triangle inequality yield the asserted inequalities.

(ii) Let $\lambda_j \in \Sigma_{\phi_j}$ for $\phi_j \in [0,\pi)$, $\phi_1 + \phi_2 < \pi$ and $\theta \in [0,1]$. Then (B.12) implies

$$\lambda_1 + \lambda_1^{1-\theta}\lambda_2^\theta + \lambda_2 \lesssim |\lambda_1| + |\lambda_1|^{1-\theta}|\lambda_2|^\theta + |\lambda_2| \lesssim |\lambda_1| + |\lambda_2| \lesssim \lambda_1 + \lambda_2 \text{ in } \Sigma_{\phi_1} \times \Sigma_{\phi_2}.$$

To prove the converse, we estimate as follows.

$$|\lambda_1 + \lambda_1^{1-\theta}\lambda_2^\theta + \lambda_2|^2 = |\lambda_1|^2 + 2\operatorname{Re}\left(\lambda_1\bar{\lambda}_2\right) + |\lambda_2|^2 + |\lambda_1^{1-\theta}\lambda_2^\theta|^2 + 2\operatorname{Re}\left((\lambda_1 + \lambda_2)\bar{\lambda}_1^{1-\theta}\bar{\lambda}_2^\theta\right),$$

$$2\operatorname{Re}\left((\lambda_1 + \lambda_2)\bar{\lambda}_1^{1-\theta}\bar{\lambda}_2^\theta\right) \ge 2\cos(\theta(\phi_1 + \phi_2))|\lambda_1|^{2-\theta}|\lambda_2|^\theta + 2\cos((1-\theta)(\phi_1 + \phi_2))|\lambda_1|^{1-\theta}|\lambda_2|^{1+\theta}.$$

Let us abbreviate $c_s := \cos(s(\phi_1 + \phi_2))$ for $s \in \mathbb{R}$. Since both c_θ and $c_{1-\theta}$ are non-negative, we have $2\operatorname{Re}((\lambda_1 + \lambda_2)\bar{\lambda}_1^{1-\theta}\bar{\lambda}_2^\theta) \ge 0$ and (B.15) is established. □

Let us derive some inequalities for the elements $z = (z_j)_{j=1}^n \in \overline{B\Sigma}_\delta^n$ of the closed bisector

$$\overline{B\Sigma}_\delta^n := \left(\overline{B\Sigma}_\delta\right)^n, \quad B\Sigma_\delta := \left\{z = re^{i\varphi} \in \mathbb{C} : r \in \dot{\mathbb{R}},\ |\arg\varphi - \pi/2| < \delta\right\} = i\Sigma_\delta \cup -i\Sigma_\delta.$$

B.55. Lemma. *Let $n \in \mathbb{N}$, $\delta \in [0,\pi)$ and define $|z|_- := \sqrt{-z\cdot z} = \sqrt{-\sum_j z_j^2}$ for $z \in \overline{B\Sigma_\delta^n} \setminus \{0\}$.*

(i) If $\delta \in [0,\pi/4)$, then $|z|_- \in \overline{\Sigma}_\delta$ for every $z \in \overline{B\Sigma_\delta^n} \setminus \{0\}$.

(ii) If $\delta \in [0,\pi/4)$ satisfies $1+(n-1)\cos(4\delta) > 0$, then

$$||z|_-| \geq \left(\frac{1+(n-1)\cos(4\delta)}{n}\right)^{1/4}|z| \quad \textit{for all } z \in \overline{B\Sigma_\delta^n} \setminus \{0\}.$$

(iii) If $\delta \in [0,\pi/8]$, then every $z \in \overline{B\Sigma_\delta^n} \setminus \{0\}$ satisfies

$$n^{-1/4}|z| \leq ||z|_-| \leq |z|, \quad \mathrm{Re}|z|_- \geq n^{-1/4}\cos(\delta)|z|, \quad \mathrm{Im}|z|_- \leq \sin(\delta)|z|. \tag{B.16}$$

Proof. (i) For every $z = (z_j) \in \overline{B\Sigma_\delta^n} \setminus \{0\}$ we have $-z_j^2 \in \overline{\Sigma}_{2\delta}$ and hence $-z\cdot z = -\sum_j z_j^2 \in \overline{\Sigma}_{2\delta}$ by (B.13) and therefore $|z|_- = \sqrt{-\sum_j z_j^2}$ belongs to $\overline{\Sigma}_\delta$.

(ii) Hölder's inequality yields

$$|z|^2 = |z|_2^2 = \sum\nolimits_j |z_j|^2 \leq n^{1/2}\left(\sum\nolimits_j |z_j|^4\right)^{1/2} = n^{1/2}|z|_4^2 \quad \text{for } z \in \mathbb{C}^n.$$

Then the assertion follows from the following estimate.

$$\begin{aligned}
||z|_-|^4 &= |-\sum\nolimits_j z_j^2|^2 = \sum\nolimits_{j,k} z_j^2\overline{z_k^2} = \sum\nolimits_j |z_j^2|^2 + 2\sum\nolimits_{j<k}\mathrm{Re}\left(z_j^2\overline{z_k^2}\right)\\
&\geq \sum\nolimits_j |z_j^2|^2 + 2\cos(4\delta)\sum\nolimits_{j<k}|z_j^2||z_k^2|\\
&= (1-\cos(4\delta))\sum\nolimits_j |z_j^2|^2 + \cos(4\delta)\left(\sum\nolimits_j |z_j^2|^2 + 2\sum\nolimits_{j<k}|z_j^2||z_k^2|\right)\\
&\geq \left(\frac{1-\cos(4\delta)}{n} + \cos(4\delta)\right)|z|_2^4.
\end{aligned}$$

(iii) This is a simple consequence of (i) and (ii). □

B.56. Example. We consider the parabolic symbol

$$\omega(\lambda,z) = \sqrt{\rho(\tau+\lambda)-\mu z^2}, \quad \lambda \in \Sigma_\phi,\ z \in B\Sigma_\delta^n,$$

where $\rho > 0$, $\tau > 0$ and $\mu > 0$ are constants and $\phi \in (\pi/2,\pi)$ and $\delta \in (0,\pi/8]$ satisfy $\phi + 2\delta < \pi$. Clearly, $|\omega| \lesssim 1 + |\lambda|^{1/2} + |z|$. Then from inequality (B.14) and Hölder's inequality $a+b+c \leq \sqrt{3}\sqrt{a^2+b^2+c^2}$ for $a, b, c \geq 0$ we obtain

$$\begin{aligned}
|\omega(\lambda,z)| &\geq \sqrt{\sqrt{\frac{c_1}{2}}\rho\tau + \sqrt{\frac{c_1}{2}}\sqrt{\frac{c_2}{2}}\left(\rho|\lambda| + \mu|z^2|\right)}\\
&\geq \sqrt{\frac{1}{3}\sqrt{\frac{c_1}{2}}\rho\tau} + \sqrt{\frac{\sqrt{c_1c_2}}{6}\rho|\lambda|} + \sqrt{\frac{\sqrt{c_1c_2}}{6}\mu|z^2|} \quad \text{for } \lambda \in \Sigma_\phi,\ z \in B\Sigma_\delta^n,
\end{aligned}$$

where $c_1^2 = 1 + \cos\phi > 0$ and $c_2^2 = 1 + \cos(\phi + 2n\delta) > 0$. From (B.16) we conclude that

$$\omega(\lambda,z) = \sqrt{\rho(\tau+\lambda)-\mu z^2} \sim 1 + \lambda^{1/2} + \sqrt{-z^2} \quad \text{for } \lambda \in \Sigma_\phi,\ z \in B\Sigma_\delta^n.$$

B.2.8. Elliptic differential operators on manifolds. The subsequent theorem of Amann, Hieber and Simonett [AHS94] guarantees that certain elliptic operators on compact manifolds are $\mathcal{R}$-sectorial and have a bounded $\mathcal{H}^\infty$ functional calculus in $L_p(M;G)$. Here (M,g) is a compact n-dimensional Riemannian C^m-manifold without boundary ($m \in \mathbb{N}$) and $G := (G,\pi,M)$ is a C^m-class vector bundle over M whose fibers $\pi^{-1}(\{x\})$ ($x \in M$) are isomorphic to a Banach space $E \cong \mathbb{C}^N$ of finite dimension N. A trivializing coordinate system (κ,χ_κ) for G consists of a chart $\kappa\colon U_\kappa \to E$ and a trivializing map $\pi^{-1}(U_\kappa) \to U_\kappa \times E$, $g \mapsto (\pi(g), \chi_\kappa(g))$. The local representation of a section u of G with respect to (κ, χ_κ) is given by $u_\kappa := \chi_\kappa \circ u \circ \kappa^{-1}$, $\kappa(U_\kappa) \to E$.

For given $p \in (1,\infty)$ and $k \in \{0,1,\dots,m\}$, we define the Sobolev spaces $H_p^k(M;G)$ of all sections u of G such that φu_κ belongs to $H_p^k(\mathbb{R}^n;E)$ for every $\varphi \in C_c^k(\kappa(U_\kappa))$ and for every trivializing coordinate system (κ,χ_κ). Moreover, $H_p^k(M;G)$ is a Banach space with respect to the norm $\|u\|_{H_p^k(M;G)} := \sum_\kappa \|(\varphi_\kappa \circ \kappa^{-1})u_\kappa\|_{H_p^k(\mathbb{R}^n;E)}$ where the sum is taken over a finite partition of unity $(\varphi_\kappa)_\kappa$ for M, subordinate to $(U_\kappa)_\kappa$. Hence for $K = |\{\kappa\}|$, the map $r\colon H_p^k(\mathbb{R}^n;\mathbb{C}^N)^K \to H_p^k(M;G)$, $(u_\kappa)_\kappa \mapsto u$ is a retraction and thus $H_p^k(M;G)$ inherits many embedding and interpolation properties from the space $H_p^k(\mathbb{R}^n)$. The Lebesgue spaces $L_p(M;G)$ and the Sobolev-Slobodeckiĭ spaces $W_p^s(M;G)$ for $s \in [0,m]$ are defined analogously.

Let $\mathcal{A}\colon H_p^m(M;G) \to L_p(M;G)$ be differential operator with representation $\mathcal{A}_\kappa(y,D) = \sum_{|\alpha|\le m} a_{\kappa,\alpha}(y)D^\alpha$ for $y \in \kappa(U_\kappa)$, where $a_{\kappa,\alpha} \in C(\kappa(U_\kappa);\mathcal{L}(E))$ and $D_j := -i\partial_j$ for $1 \le j \le n$. The operator $\mathcal{A}$ is called θ-elliptic ($\theta \in [0,\pi)$), if its principal symbol $\mathcal{A}_\pi$ satisfies

$$\sigma(\mathcal{A}_\pi(\xi_x^*)) \subset \{\lambda \in \mathbb{C}\setminus\{0\} : |\arg\lambda| \le \theta\}, \qquad \text{for all } x \in M,\ \xi_x^* \in T_x^*M \setminus \{0\}.$$

Note that $\mathcal{A}_\pi(\xi_x^*) \in \mathcal{L}(\pi^{-1}(\{x\}))$ is a homogeneous polynomial in ξ_x^*.

B.57. Theorem ([AHS94, Theorem 10.1, Theorem 10.3])**.** *Let M be a compact n-dimensional C^m-manifold ($m \in \mathbb{N}$) without boundary, let G be a complex $C^{\min\{m,2\}}$-vector bundle over M and let $p \in (1,\infty)$. Let $\mathcal{A}\colon H_p^m(M;G) \to L_p(M;G)$ be a linear differential operator with continuous coefficients such that $\mathcal{A}$ is θ_0-elliptic for some $\theta_0 \in [0,\pi)$.*

(i) For every $\theta \in (\theta_0,\pi)$ there exists $\mu_\theta > 0$ such that $\mu_\theta + \mathcal{A}\colon H_p^m(M;G) \to L_p(M;G)$ is an isomorphism and $\mathcal{R}$-sectorial with $\mathcal{R}$-angle θ.

(ii) If $\mathcal{A}_\pi$ has C^ε-coefficients for some $\varepsilon \in (0,1)$, then for every $\theta \in (0,\theta_0)$ there exists $\mu_\theta > 0$ such that $\mu_\theta + \mathcal{A}$ has a bounded $\mathcal{H}^\infty$ functional calculus in $L_p(M;G)$ with $\mathcal{H}^\infty$-angle θ.

B.58. Corollary. *Let $\Sigma \subset \mathbb{R}^n$ ($n \ge 2$) be a compact C^2-hypersurface, let $\Delta_\Sigma = \operatorname{div}_\Sigma \nabla_\Sigma = g^{ij}(\partial_i\partial_j - \Lambda_{ij}^k\partial_k)$ denote the scalar Laplace-Beltrami operator and let $p \in (1,\infty)$, $\mathbb{K} = \mathbb{C}$.*

(i) For every $\mu \in (0,\infty)$, the operator $\mu - \Delta_\Sigma\colon H_p^2(\Sigma) \to L_p(\Sigma)$ is invertible and $\mathcal{R}$-sectorial with $\mathcal{R}$-angle zero.

(ii) If $\Sigma \in C^{2+\varepsilon}$ for some $\varepsilon \in (0,1)$, then for every $\theta \in (0,\pi)$ there exists $\mu_\theta \in (0,\infty)$ such that $\mu_\theta - \Delta_\Sigma$ has a bounded $\mathcal{H}^\infty$ functional calculus in $L_p(\Sigma)$ with $\mathcal{H}^\infty$-angle θ.

(iii) Let $\lambda_1 > 0$ denote the smallest non-zero eigenvalue of $-\Delta_\Sigma$. Then for every $\mu \in (-\lambda_1,\infty)$, the operator $\mu - \Delta_\Sigma\colon H_p^2(\Sigma) \cap L_{p,0}(\Sigma) \to L_{p,0}(\Sigma)$ is invertible and $\mathcal{R}$-sectorial with $\mathcal{R}$-angle zero.

(iv) Let $s \in [0,\infty)$ and assume that Σ be smooth and let $s \in [0,\infty)$. Then for every $\mu \in (0,\infty)$, the operator $\mu - \Delta_\Sigma\colon W_p^{s+2}(\Sigma) \to W_p^s(\Sigma)$ is invertible and $\mathcal{R}$-sectorial with $\mathcal{R}$-angle zero.

Proof. (i) The domain $H_p^2(\Sigma)$ of Δ_Σ is compactly embedded into the ground space $L_p(\Sigma)$ and therefore the spectrum of Δ_Σ consists solely of eigenvalues with finite multiplicity. The surface divergence theorem implies that all eigenvalues are non-positive and that zero is an eigenvalue with multiplicity one (the corresponding eigenfunctions are the constant functions). Hence, by considering the operators $-e^{i\psi}\Delta_\Sigma$ ($\psi \in (-\pi/2,\pi/2)$) and using Theorem B.57.(i), Theorem B.22 and Theorem B.23, we obtain the assertion.

(ii) Since $g^{ij} \in C^{1+\varepsilon}$ and $\Lambda_{ij}^k \in C^\varepsilon$, the assertion follows from Theorem B.57.(ii).

(iii) We have the direct decomposition $L_p(\Sigma) = L_{p,0}(\Sigma) \oplus \mathbb{K}$, where we identify $\mathbb{K}$ with the constant functions, which form the eigenspace for the eigenvalue zero. Hence the spectrum of the realization of $-\Delta_\Sigma$ in $L_{p,0}(\Sigma)$ is contained in $[\lambda_1,\infty)$ and the assertion follows as in (i).

(iv) By means of a localization procedure as in Section 2.1 it can be shown that the operator $\mu - \Delta_\Sigma\colon H_p^{k+2}(\Sigma) \to H_p^k(\Sigma)$ is invertible for large μ. This assertion holds true for all $\mu > 0$ by Theorem B.22. By means of retractions $r\colon W_p^s(\mathbb{R}^{n-1})^K \to W_p^s(\Sigma)$ and real interpolation, it follows that $\mu - \Delta_\Sigma \colon W^{s+2}_p(\Sigma) \to W^s_p(\Sigma)$ is invertible and $\mathcal{R}$-sectorial with $\mathcal{R}$-angle zero for every $s \in [0,\infty)$. □

B.59. Corollary. *Let $\Sigma \subset \mathbb{R}^n$ ($n \geq 2$) be a compact C^2-hypersurface, let $p \in (1,\infty)$, let $\widetilde{\nabla}_\Sigma$ denote the covariant derivative on Σ and let $\widetilde{\Delta}_\Sigma \colon H^2_p(\Sigma; T\Sigma) \to L_p(\Sigma; T\Sigma)$, $\widetilde{\Delta}_\Sigma v = g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta v$ denote the Laplace-Beltrami operator for tangential vector fields on Σ (see page 141).*

(i) For every $\mu \in (0,\infty)$, the operator $\mu - \widetilde{\Delta}_\Sigma \colon H^2_p(\Sigma; T\Sigma) \to L_p(\Sigma; T\Sigma)$ is invertible.
(ii) Let $(T\Sigma)_\mathbb{C}$ denote the complexification of $T\Sigma$. Then for every $\mu \in (0,\infty)$, the operator $\mu - \widetilde{\Delta}_\Sigma \colon H^2_p(\Sigma; (T\Sigma)_\mathbb{C}) \to L_p(\Sigma; (T\Sigma)_\mathbb{C})$ is an isomorphism and $\mathcal{R}$-sectorial with $\mathcal{R}$-angle zero.
(iii) If $\Sigma \in C^{2+\varepsilon}$ for some $\varepsilon \in (0,1)$, then for every $\theta \in (0,\pi)$ there exists $\mu_\theta \in (0,\infty)$ such that $\mu_\theta - \widetilde{\Delta}_\Sigma$ has a bounded $\mathcal{H}^\infty$ functional calculus in $L_p(\Sigma; (T\Sigma)_\mathbb{C})$ with $\mathcal{H}^\infty$-angle θ.

Proof. The proof is similar as for Corollary B.58. □

B.3. Joint functional calculus and mixed-order systems

B.3.1. The joint $\mathcal{H}^\infty$-functional calculus for (∂_t, ∇). We collect some results of Denk and Kaip [DK13] on the joint functional calculus operator tuples like $\nabla = (\partial_1, \ldots, \partial_n)$ and (∂_t, ∇).

B.60. Definition ([KW04, p. 4.9]). We say that a Banach space X has *property* (α), if there exists $C > 0$ such that

$$\int_0^1\int_0^1 \Big\|\sum\nolimits_{i,j=1}^n r_i(u)r_j(v)\alpha_{ij}x_{ij}\Big\|_X \,du\,dv \leq C\int_0^1\int_0^1 \Big\|\sum\nolimits_{i,j=1}^n r_i(u)r_j(v)x_{ij}\Big\|_X \,du\,dv$$

for all $n \in \mathbb{N}$, $\alpha_{ij} \in \mathbb{C}$ with $|\alpha_{ij}| \leq 1$, $x_{ij} \in X$.

B.61. Remarks ([KW04, p. 4.10], [DK13, Remark 1.15]). Let X have property (α).

(i) If Y is a closed subspace of X, then Y has property (α).
(ii) If Y is isomorphic to X, then Y has property (α).
(iii) If (Ω, μ) is a σ-finite measure space and $p \in [1,\infty)$, then $L_p(\Omega, \mu; X)$ has property (α).
(iv) Every Hilbert space has property (α).

B.62. Definition (Ground space $\mathcal{W}$, [DK13, Definition 1.71], [Kai12, Definition 2.25]). Let $n \in \mathbb{N}$, p_0, p_1, q_0, $q_1 \in (1,\infty)$, s, $\omega \in [0,\infty)$, $r \in \mathbb{R}$ and X be a Banach space of class $\mathcal{HT}$ with property (α). Then we let

$$\mathcal{W} := e^{\omega\cdot}{}_0\mathcal{F}^s(\mathbb{R}_+; \mathcal{K}^r(\mathbb{R}^n; X)), \quad \mathcal{K} \in \{B_{p_1,q_1}, H_{p_1}\}, \quad \mathcal{F} \in \begin{cases} \{B_{p_0,q_0}, H_{p_0}\}, & \text{if } s > 0, \\ \{H_{p_0}\}, & \text{if } s = 0. \end{cases}$$

Here the space $e^{\omega\cdot}{}_0\mathcal{F}^s(\mathbb{R}_+; Y)$ consists of all functions $t \mapsto e^{\omega t}u(t)$ such that $u \in {}_0\mathcal{F}^s(\mathbb{R}_+; Y)$, equipped with the norm $\|t \mapsto e^{\omega t}u(t)\|_{{}_0\mathcal{F}^s(\mathbb{R}_+;Y)}$.

B.63. Definition (Sectors, bisectors, curves, cf. [DK13, Definition 1.1]). For $\phi \in (0,\pi)$, let $\Sigma_\phi \subset \mathbb{C}$ denote the open sector

$$\Sigma_\phi = \{z = re^{i\varphi} : r \in (0,\infty),\ |\varphi| < \phi\} = \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \phi\}.$$

For $\delta \in (0,\pi/2)$, let $B\Sigma_\delta \subset \mathbb{C}$ denote the open bisector

$$B\Sigma_\delta = \{z = re^{i\varphi} : r \in \mathbb{R} \setminus \{0\},\ |\varphi - \pi/2| < \delta\} = i\Sigma_\delta \cup -i\Sigma_\delta.$$

For $\psi \in (0,\pi)$ we define the curve $\Gamma_\psi \subset \mathbb{C}$ by means of the parametrization $\mathbb{R} \ni r \mapsto |r|e^{-i\psi\operatorname{sign} r}$. Hence $\Gamma_\psi = \partial\Sigma_\psi$ is oriented counter-clockwise around Σ_ψ.

B.64. Definition ([cf. DK13, Definition 1.17]). Let $n \in \mathbb{N}$, let $\Omega \subset \mathbb{C}^n$ be open and let Y be a Banach space. We define

(i) $\mathcal{H}(\Omega; Y)$, the vector space of all holomorphic Y-valued functions on Ω,
(ii) $\mathcal{H}^\infty(\Omega; Y)$, the vector space of all bounded holomorphic Y-valued functions on Ω,
(iii) $\mathcal{H}^\infty_0(\Omega; Y) = \{f \in \mathcal{H}^\infty(\Omega; Y) : \exists C,\ s > 0\, \forall z \in \Omega : |f(z)|_Y \leq C\prod_{j=1}^n(\min\{|z_j|, |z_j|^{-1}\})^s\}$,

(iv) $\mathcal{H}_P(\Omega;Y) = \{f \in \mathcal{H}(\Omega;Y) : \exists C,\, s > 0\, \forall z \in \Omega : |f(z)|_Y \le C\prod_{j=1}^n(\max\{|z_j|, |z_j|^{-1}\})^s\}$.

Thus the spaces $\mathcal{H}_0^\infty(\Omega;Y)$ and $\mathcal{H}_P(\Omega;Y)$ consists of functions that have a polynomial decay or a polynomial growth at zero and at infinity.

B.65. Remarks ([DK13, Definition 1.20])**.** Let T_j ($j \in \{1,\dots,n\}$) be closed linear operators in a complex Banach space X such that each T_j is sectorial or bi-sectorial and such that all resolvents the $(\lambda - T_j)^{-1}$, $(\mu - T_l)^{-1}$ commute ($\lambda \in \rho(T_j)$, $\mu \in \rho(T_l)$, $j, l \in \{1,\dots,n\}$).

For sectorial T_j we choose $\Omega_j := \Sigma_{\theta_j}$ with some $\theta_j \in (\varphi_{T_j}, \pi)$ and $\Gamma_j := \Gamma_{\varphi_j}\partial\Sigma \dots$. For bi-sectorial T_j we choose $\Omega_j := B\Sigma_{\delta_j}$ with some $\delta_j \in (\varphi^{\mathrm{bi}}_{T_j}, \pi/2)$ and $\Gamma_j := \partial\Sigma_{\phi_j}$ with some $\phi_j \in ()$. We put $\Omega := \Omega_1 \times \dots \times \Omega_n$ and $\Gamma := \Gamma_1 \times \dots \times \Gamma_n$ and we let $\mathcal{B}_T \subset \mathcal{B}(X)$ denote the commutator algebra of $\{(\lambda - T_j)^{-1} : \lambda \in \rho(T_j),\, j \in \{1,\dots,n\}\}$ in $\mathcal{B}(X)$.

(i) *Joint $\mathcal{H}^\infty$-functional calculus.* For $f \in \mathcal{H}_0^\infty(\Omega;\mathcal{B}_T)$ we define

$$f(T) := \frac{1}{(2\pi i)^n}\int_\Gamma f(z)\prod_{j=1}^n (z_j - T_j)^{-1}\,dz \in \mathcal{B}(X).$$

(ii) *Joint $\mathcal{H}_P$-functional calculus.* For $k, n \in \mathbb{N}$, the functions

$$\psi_{k,n}(z) := \frac{k^2 z_1}{(1+kz_1)(k+z_1)}\cdots\frac{k^2 z_n}{(1+kz_n)(k+z_n)}$$

belong to $\mathcal{H}_0^\infty(\Omega)$ and for every $f \in \mathcal{H}_P(\Omega;Y)$ there exists $m \in \mathbb{N}_0$ such that $\psi_{k,n}^m f \in \mathcal{H}_0^\infty(\Omega;\mathcal{B}_T)$ for all $k \in \mathbb{N}$. Hence, for $f \in \mathcal{H}_P(\Omega;\mathcal{B}_T)$ we may choose $m \in \mathbb{N}_0$ with $\psi_{1,n}^m f \in \mathcal{H}_0^\infty(\Omega;\mathcal{B}_T)$ and define

$$f(T)\colon D(f(T)) \to X, \quad x \mapsto \psi(T)^{-m}(\psi^m f)(T)x$$

with domain $D(f(T)) := \{x \in X : (\psi^m f)(T)x \in R(\psi(T)^m)\}$.

B.66. Definition (Joint $\mathcal{H}^\infty$-functional calculus for $(\mu + \partial_t, \nabla_x)$, [Kai12, Definition 1.10])**.** Put

$$\Omega := \Sigma_\phi \times B\Sigma_{\delta_1} \times \cdots B\Sigma_{\delta_n}, \qquad \text{where } \phi \in (\pi/2, \pi),\ \delta_j \in (0, \pi/2).$$

Let $\mu \in [0,\infty)$ and consider the operators ∂_t, ∂_{x_1}, …, ∂_{x_n} as closed operators in $\mathcal{W}$. For the tuple $(\mu + \partial_t, \nabla)$ we define the joint $\mathcal{H}^\infty(\Omega)$-functional calculus

$$f \mapsto f(\mu + \partial_t, \nabla) := \frac{1}{(2\pi i)^{1+n}}\int_{\Gamma_{\phi'}}\int_{\prod_j(\Gamma_{\delta_j'}\cup(-\Gamma_{\delta_j'}))} f(\tau,z)(\tau - \mu - \partial_t)^{-1}\prod\nolimits_j (z_j - \partial_{x_j})^{-1} d(\tau, z),$$

where $f \in \mathcal{H}_0^\infty(\Omega)$ and $\phi' \in (\pi/2, \phi)$, $\delta_j' \in (0, \delta_j)$ (the integrals do not depend on ϕ', δ_j'). The resolvents $(\tau - \mu - \partial_t)^{-1}$ and $(z_j - \partial_{x_j})^{-1}$ are considered as bounded linear operators in the same ground space $\mathcal{W}$ according to Definition B.62.

B.67. Theorem (Time derivative, [DK13, Theorems 1.83, 1.84])**.** *Let r, s, $\omega \in [0,\infty)$ and let $\mathcal{F}$, $\mathcal{K}$, X be as in Definition B.62. Let*

$$\mathcal{D}_t\colon u \mapsto \partial_t u, \quad e^{\omega\cdot}{}_0\mathcal{F}^{s+1}(\mathbb{R}_+;X) \to e^{\omega\cdot}{}_0\mathcal{F}^{s}(\mathbb{R}_+;X)$$

denote the realization in $e^{\omega\cdot}{}_0\mathcal{F}^{s}(\mathbb{R}_+;X)$ of the time derivative. Then the following assertions are valid.

(i) $\mathcal{D}_t$ has an $\mathcal{R}$-bounded $\mathcal{H}^\infty$-calculus with $\phi^{\mathcal{R}\infty}_{\mathcal{D}_t} = \pi/2$.

(ii) The operator $\mathcal{D}_t$ has

(a) the resolvent set $\rho(\mathcal{D}_t) = \{z \in \mathbb{C} : \operatorname{Re} z < \omega\}$,

(b) the residual spectrum $\sigma_r(\mathcal{D}_t) = \{z \in \mathbb{C} : \operatorname{Re} z > \omega\}$,

(c) the point spectrum $\sigma_p(\mathcal{D}_t) = \emptyset$,

(d) the continuous spectrum $\sigma_c(\mathcal{D}_t) = i\mathbb{R} + \omega$.

B.68. Theorem ([MS12, Proposition 2.9]). *Let* $p \in (1,\infty)$, $X \in \mathcal{HT}$, $s \in [0,\infty)$, $\alpha \in (0,2)$, $\omega \in (0,\infty)$. *Then the operators*

$$(\omega - \partial_t)^\alpha \quad \text{in } H_p^s(\mathbb{R}_+;X) \text{ with domain } H_p^{s+\alpha}(\mathbb{R}_+;X),$$
$$(\omega - \partial_t)^\alpha \quad \text{in } W_p^s(\mathbb{R}_+;X) \text{ with domain } W_p^{s+\alpha}(\mathbb{R}_+;X),\ s,\ s+\alpha \notin \mathbb{N}_0$$

are invertible and have bounded $\mathcal{H}^\infty$ *functional calculi with angle not larger than* $\alpha\pi/2$.

B.69. Theorem (Space derivatives, [DK13, Theorem 1.81]). *Let*

$$\mathcal{D}_{x_j}\colon u \mapsto \partial_{x_j} u, \quad e^{\omega\cdot}{}_0\mathcal{F}^s(\mathbb{R}_+;\mathcal{K}^{r+1}(\mathbb{R}^n;X)) \to e^{\omega\cdot}{}_0\mathcal{F}^s(\mathbb{R}_+;\mathcal{K}^r(\mathbb{R}^n;X))$$

denote the realizations of the partial derivatives in $e^{\omega\cdot}{}_0\mathcal{F}^s(\mathbb{R}_+;\mathcal{K}^r(\mathbb{R}^n;X))$. *Let* $\delta_j \in (0,\pi/2)$ *and* $\Omega_x = \prod_{j=1}^n B\Sigma_{\delta_j}$. *Then the tuple* $\mathcal{D}_x = (\mathcal{D}_{x_1},\dots,\mathcal{D}_{x_n})$ *has a bounded joint* $\mathcal{H}^\infty(\Omega_x)$*-calculus.*

B.70. Theorem ([Kai12, Definition 1.15, Theorem 2.47]). *Let* $\mathcal{W} = e^{\omega\cdot}{}_0\mathcal{F}^s(\mathbb{R}_+;\mathcal{K}^r(\mathbb{R}^n;X))$ *be as in Definition B.62, let* $\Omega = \Sigma_\phi \times B\Sigma_{\delta_1} \times \cdots \times B\Sigma_{\delta_n}$ *be as in Definition B.66 and let* $\sigma \geq 0$. *Then the tuple* $(\sigma + \mathcal{D}_t, \mathcal{D}_x)$ *has a bounded joint* $\mathcal{H}^\infty(\Omega)$*-functional calculus in* $\mathcal{W}$.

B.3.2. Parabolic mixed-order systems. We define order functions and Newton polygons. An example is given in Figure B.1 on the facing page. Then we consider a class of parameter-dependent symbols $S(\overline{\Sigma}_\phi \times \overline{B\Sigma}_\delta^n \times K)$, which are used in Section 3.1 for solving the Fouier-Laplace transformed differential equations.

B.71. Remarks ([DK13]). (i) A continuous and piecewise linear function $\mu\colon [0,\infty) \to \mathbb{R}$ is called an *order function* if μ is convex or concave. In this case there exist $M \in \mathbb{N}$ and $\gamma_l > 0$, $m_l(\mu), b_l(\mu) \in \mathbb{R}$ for $l \in \{0,\dots,M\}$ with $0 =: \gamma_0 < \gamma_1 < \cdots < \gamma_M < \gamma_{M+1} := \infty$ such that

$$\mu(\gamma) = b_l(\mu) + m_l(\mu)\gamma \quad \text{for } \gamma \in (\gamma_l, \gamma_{l+1}),$$

and we have

$$m_{l-1}(\mu) \leq m_l(\mu), \quad b_{l-1}(\mu) \geq b_l(\mu) \quad \text{for } l \in \{1,\dots,M\}$$

(that is, μ is convex) or

$$m_{l-1}(\mu) \geq m_l(\mu), \quad b_{l-1}(\mu) \leq b_l(\mu) \quad \text{for } l \in \{1,\dots,M\}$$

(that is, μ is concave). If μ is convex, then we have

$$\mu(\gamma) = \max\{b_l(\mu) + m_l(\mu)\gamma : l \in \{0,\dots,M\}\} \quad \text{for } \gamma \in [0,\infty).$$

(ii) A convex [concave] order function μ is *increasing* [*decreasing*] if $m_l(\mu) \geq 0$ [$m_l(\mu) \leq 0$] for all $l \in \{0,\dots,M\}$. A convex [concave] order function μ is *increasing* [*decreasing*] if $m_l(\mu) \geq 0$ [$m_l(\mu) \leq 0$] for all $l \in \{0,\dots,M\}$.

(iii) An order function μ is called *strictly positive* if μ is convex and $m_l(\mu) \geq 0$ and $b_l(\mu) \geq 0$ for all $l \in \{0,\dots,M\}$. An order function μ is called *strictly negative* if $-\mu$ is strictly positive.

(iv) For a given finite set $\nu = (r_j, s_j)_{j=0}^{J+1} \subset [0,\infty)^2$, $J \in \mathbb{N}_0$, the associated *Newton polygon* $N(\nu)$ is defined as the convex hull in $\mathbb{R}^2$ of the set of vertices $(0,0)$, $(0,s_j)$, $(r_j,0)$, (r_j,s_j), $j \in \{0,\dots,J+1\}$.

B.72. Definition (Symbol class $S(\overline{\Sigma}_\phi \times \overline{B\Sigma}_\delta^n \times K)$, cf. [DK13]). Let $K \subset \mathbb{C}^m$ be compact, $\phi \in (\pi/2,\pi)$, $\delta \in (0,\pi/2)$. Then we let $S(\overline{\Sigma}_\phi \times \overline{B\Sigma}_\delta^n \times K)$ be the set of all functions

$$P\colon \overline{\Sigma}_\phi \times \overline{B\Sigma}_\delta^n \times K \to \mathbb{C}, \quad (\lambda,z,\vartheta) \mapsto P(\lambda,z;\vartheta) = \sum_{j\in J} \chi_j(\vartheta)\omega_j(\lambda,z)\varphi_j(\lambda)\psi_j(z), \tag{B.17}$$

where J is a finite index set and for all $j \in J$,

(i) $\chi_j\colon K \to \mathbb{C}$ is continuous and nontrivial,

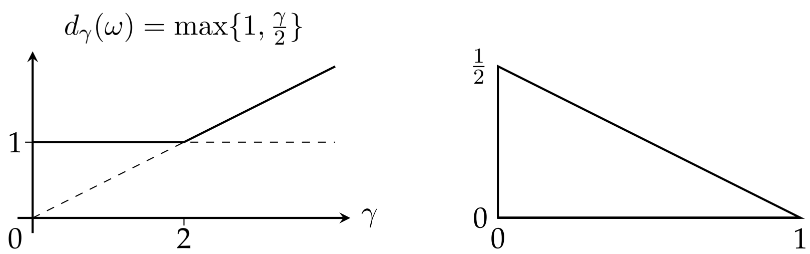

FIGURE B.1. γ-order and Newton polygon of the symbol $\omega(\lambda, z) = (\lambda + |z|_-^2)^{1/2}$.

(ii) ω_j is holomorphic in $\Sigma_\phi \times B\Sigma_\delta^n$, continuous in $\overline{\Sigma}_\phi \times \overline{B\Sigma_\delta^n}$ and satisfies

$$\omega_j(\eta^2\lambda, \eta z) = \eta^{N_j}\omega_j(\lambda, z) \neq 0 \quad \text{for } \eta > 0,\ (\lambda, z) \in \overline{\Sigma}_\phi \times \overline{B\Sigma_\delta^n} \setminus \{(0,0)\}$$

with some $N_j \in [0, \infty)$,

(iii) φ_j is holomorphic in Σ_ϕ, continuous in $\overline{\Sigma}_\phi$ and satisfies

$$\varphi_j(\eta\lambda) = \eta^{M_j}\varphi_j(\lambda) \neq 0 \quad \text{for } \eta > 0,\ \lambda \in \overline{\Sigma}_\phi \setminus \{0\}$$

with some $M_j \in [0, \infty)$,

(iv) ψ_j is holomorphic in $B\Sigma_\delta^n$, continuous in $\overline{B\Sigma_\delta^n}$ and satisfies

$$\psi_j(\eta z) = \eta^{L_j}\psi_j(z) \neq 0 \quad \text{for } \eta > 0,\ z \in \overline{B\Sigma_\delta^n} \setminus \{0\}$$

with some $L_j \in [0, \infty)$,

(v) for every $\gamma \in (0, \infty]$, the γ-principal part $\pi_\gamma P$ (see below) is not identical zero.

B.73. Definition (γ-order and γ-principal part, cf. [DK13, cf. Definition 2.10])**.** Let $P \in S(\overline{\Sigma}_\phi \times \overline{B\Sigma_\delta^n} \times K)$ with representation (B.17). We put $P[\vartheta] := P(\cdot, \cdot; \vartheta)$ and $J[\vartheta] := \{j \in J : \chi_j(\vartheta) \neq 0\}$. For $\vartheta \in K$ with $J[\vartheta] \neq \emptyset$ we define the γ-order

$$d_\gamma(P[\vartheta]) := \begin{cases} \max\limits_{j \in J[\vartheta]} \{\gamma M_j + N_j \max\{\gamma/2, 1\} + L_j\} & \text{for } \gamma \in (0, \infty), \\ \max\limits_{j \in J[\vartheta]} \{M_j + N_j/2\} & \text{for } \gamma = \infty. \end{cases}$$

Let

$$J_\gamma[\vartheta] := \begin{cases} \{j \in J : \gamma M_j + N_j \max\{\gamma/2, 1\} + L_j = d_\gamma(P[\vartheta])\} & \text{for } \gamma \in (0, \infty), \\ \{j \in J : M_j + N_j/2 = d_\infty(P[\vartheta])\} & \text{for } \gamma = \infty. \end{cases}$$

We define the γ-principal part

$$\pi_\gamma P(\lambda, z; \vartheta) := \begin{cases} \lim\limits_{\eta \to \infty} \dfrac{P(\eta^\gamma \lambda, \eta z; \vartheta)}{\eta^{d_\gamma(P[\vartheta])}} = \sum\limits_{j \in J_\gamma[\vartheta]} \chi_j(\vartheta)\pi_\gamma\omega_j(\lambda, z)\varphi_j(\lambda)\varphi_j(z) & \text{for } \gamma \in (0, \infty), \\ \lim\limits_{\eta \to \infty} \dfrac{P(\eta\lambda, z; \vartheta)}{\eta^{d_\infty(P[\vartheta])}} = \sum\limits_{j \in J_\infty[\vartheta]} \chi_j(\vartheta)\omega_j(\lambda, 0)\varphi_j(\lambda)\psi_j(z) & \text{for } \gamma = \infty, \end{cases}$$

where

$$\pi_\gamma\omega_j(\lambda, z) := \begin{cases} \omega_j(0, z) & \text{for } \gamma \in (0, 2), \\ \omega_j(\lambda, z) & \text{for } \gamma = 2, \\ \omega_j(\lambda, 0) & \text{for } \gamma \in (2, \infty]. \end{cases}$$

B.74. Definition (N-parabolic symbol, cf. [Kai12],[DK13])**.** The class $S_N(\overline{\Sigma}_\phi \times \overline{B\Sigma_\delta^n} \times K)$ of *N-parabolic symbols* consists of all functions $P \in S(\overline{\Sigma}_\phi \times \overline{B\Sigma_\delta^n} \times K)$ such that

(i) $\pi_\gamma P(\cdot,\cdot,\vartheta)$ is nontrivial for all $\gamma \in (0,\infty]$, $\vartheta \in K$ and all $\vartheta \mapsto d_\gamma(P(\cdot,\cdot,\vartheta))$ are constant,
(ii) P satisfies a two-sided estimate

$$|P(\lambda,z,\vartheta)| \sim \sum\nolimits_{(r,s)\in N_V} |\lambda|^s|z|^r \quad \text{uniformly with respect to } (\lambda,z,\vartheta) \in \overline{\Sigma}_\phi \times \overline{B\Sigma_\delta^n} \times K,$$

where N_V denotes the set of the vertices of the Newton polygon associated to $d_\gamma(P)$.

B.75. Theorem (cf. [DK13, Theorem 2.56, Corollary 2.57])**.** *The symbol $P \in S(\overline{\Sigma}_\phi \times \overline{B\Sigma_\delta^n} \times K)$ is N-parabolic if and only if*

$$\pi_\gamma P(\lambda,z,\vartheta) \neq 0 \quad \textit{for all } \gamma \in (0,\infty],\ \lambda \in \overline{\Sigma}_\phi \setminus \{0\},\ z \in \overline{B\Sigma_\delta^n} \setminus \{0\},\ \vartheta \in K.$$

The next result implies that every N-parabolic symbol induces a topological linear isomorphism with uniform bounds with respect to a compact parameter set K. We let $\mathcal{F}^s(\mathcal{K}^r)$ be as in Definition B.62 and apply the joint functional calculus for $(\mu+\partial_t,\partial_{x_1},\dots,\partial_{x_n})$ from Remark B.65.

B.76. Theorem (cf. [Kai12; DK13])**.** *Let $P \in S_N(\overline{\Sigma}_\phi \times \overline{B\Sigma_\delta^n} \times K)$, $\phi \in (\pi/2,\pi)$, $\delta > 0$. Then there exists $\mu_0 \geq 0$ such that $\vartheta \mapsto P(\mu_0+\cdot,\cdot,\vartheta)^{-1}$, $K \to H^\infty(\Sigma_\phi \times B\Sigma_\delta^n)$ is bounded. Moreover, for every such μ_0, there exists $C > 0$ such that for all $\mu \in [\mu_0,\infty)$ and $\vartheta \in K$, the realization*

$$P(\mu+\partial_t,\partial_{x_1},\dots,\partial_{x_n},\vartheta)\colon \bigcap\nolimits_{(r,s)\in N_V(P)} \left({}_0\mathcal{F}^{s'+s}(\mathcal{K}^{r'+r})\right) \to {}_0\mathcal{F}^{s'}(\mathcal{K}^{r'})$$

is an isomorphism and both $P(\mu+\partial_t,\partial_{x_1},\dots,\partial_{x_n},\vartheta)$, $[P(\mu+\partial_t,\partial_{x_1},\dots,\partial_{x_n},\vartheta)]^{-1}$ are bounded by C.

B.77. Definition (cf. [DK13])**.** Let $\phi \in (\pi/2,\pi)$, $\delta \in (0,\pi/2)$ and K be a compact topological space. A function $\mathcal{L}\colon \overline{\Sigma}_\phi \times \overline{B\Sigma_\delta^n} \times K \to \mathbb{C}^{m\times m}$ is called an *N-parabolic mixed-order system* if
(i) $\mathcal{L}(\cdot,\cdot,\vartheta)$ is holomorphic and polynomially bounded, uniformly in $\vartheta \in K$,
(ii) $\det \mathcal{L}$ is N-parabolic in the sense of Definition B.74,
(iii) there are order functions s_j and t_i such that $s_j + t_i$ is an upper order function for $\mathcal{L}_{j,i}$ for all $j, i \in \{1,\dots,m\}$,
(iv) $d_\gamma(\det \mathcal{L}) = \sum_{j=1}^m (s_j(\gamma)+t_i(\gamma))$ for all $\gamma \in (0,\infty]$.

B.78. Definition (cf. [DK13, Definition 2.78])**.** Let μ_1 and μ_2 be convex increasing order functions such that $\mu_1 - \mu_2$ is an order function. Let $p \in (1,\infty)$. The scale

$$(\mathcal{F}_l,\mathcal{K}_l) \in \{(B_{pp},H_p),(H_p,B_{pp})\}, \quad l \in \{0,\dots,M\},$$

is called (μ_1,μ_2)-admissible, if that there exists $k \in \{0,\dots,M-1\}$ such that

$$(\mathcal{F}_0,\mathcal{K}_0) = \dots = (\mathcal{F}_k,\mathcal{K}_k) = (H_p,B_{pp}),$$
$$(\mathcal{F}_{k+1},\mathcal{K}_{k+1}) = \dots = (\mathcal{F}_M,\mathcal{K}_M) = (B_{pp},H_p),$$

and

$$(b_k(\mu_2),m_k(\mu_2)) \neq (b_{k+1}(\mu_2),m_{k+1}(\mu_2)) \quad \text{if } \mu_1-\mu_2 \text{ is convex},$$
$$(b_k(\mu_1),m_k(\mu_1)) \neq (b_{k+1}(\mu_1),m_{k+1}(\mu_1)) \quad \text{if } \mu_1-\mu_2 \text{ is concave}.$$

B.79. Theorem (cf. [DK13, Theorem 2.69, Corollary 2.80])**.** *Let X be a Banach space of class $\mathcal{HT}$ with property (α). Let $\mathcal{L}\colon \overline{\Sigma}_\phi \times \overline{B\Sigma_\delta^n} \times K \to \mathbb{C}^{m\times m}$ be an N-parabolic mixed order system such that for each $i, j \in \{1,\dots,m\}$, the order function s_j+t_i is convex and increasing or concave and decreasing. Let $\rho \geq 0$, $s_l' \geq 0$, $r_l' \in \mathbb{R}$, $l \in \{0,\dots,M\}$, such that*

$$\mu_{\mathbb{H}_i}(\gamma) := \max_l\{[s_l'+m_l(t_i)]\gamma + r_l' + b_l(t_i)\}, \quad \gamma \geq 0,$$
$$\mu_{\mathbb{F}_j}(\gamma) := \max_l\{[s_l'-m_l(s_j)]\gamma + r_l' - b_l(s_j)\}, \quad \gamma \geq 0, \quad i,j \in \{1,\dots,m\},$$

are convex increasing order functions. Furthermore, let $p \in (1,\infty)$ and let the scale

$$(\mathcal{F}_l,\mathcal{K}_l) \in \{(B_{pp},H_p),(H_p,B_{pp})\}, \quad l \in \{0,\dots,M\},$$

be $(\mu_{\mathbb{H}_i}, \mu_{\mathbb{F}_j})$-admissible for all $i, j \in \{1, \dots, m\}$ and let

$$s'_l > \max\{\max\{-m_l(t_i), m_l(s_j)\} : i, j \in \{1, \dots, m\}\} \quad \text{for all } l \in \{0, \dots, k\},$$

with k from Definition B.78. With ${}_0\mathcal{F}^s_l(\mathcal{K}^r_l) := {}_0\mathcal{F}^s_l(\mathbb{R}_+; \mathcal{K}^r_l(\mathbb{R}^n; X))$ we define the spaces

$$\mathbb{H}_i := \bigcap\nolimits_{l=0}^{M} {}_0\mathcal{F}_l^{s'_l + m_l(t_i)}\left(\mathcal{K}_l^{r'_l + b_l(t_i)}\right), \qquad \mathbb{F}_j := \bigcap\nolimits_{l=0}^{M} {}_0\mathcal{F}_l^{s'_l - m_l(s_j)}\left(\mathcal{K}_l^{r'_l - b_l(s_j)}\right).$$

Then there exists $\tau_0 > 0$ such that for all $\tau \geq \tau_0$,

$$\mathcal{L}(\tau + \mathcal{D}_t, \mathcal{D}_x, \vartheta) \colon \prod\nolimits_{i=1}^{m} \mathbb{H}_i \to \prod\nolimits_{j=1}^{m} \mathbb{F}_j$$

is a topological linear isomorphism and its inverse

$$(\mathcal{L}(\tau + \mathcal{D}_t, \mathcal{D}_x, \vartheta))^{-1} = \mathcal{L}^{-1}(\tau + \mathcal{D}_t, \mathcal{D}_x, \vartheta)$$

is uniformly bounded with respect to $\vartheta \in K$.

B.4. Analytic Nemytskiĭ operators

The nonlinear problem (T) contains nonlinear operators $(u, \pi, h, t, x) \mapsto F(u, \pi, h)(t, x)$ where $F(u, \pi, h)(t, x)$ only depends on the values of (u, π, h) and its derivatives at (t, x). These so-called Nemytskiĭ operators are studied in Section B.4. In order to prove the analyticity of a Nemytskiĭ operator, we define it in an open subset of a Banach space X such that
(i) $X \hookrightarrow BUC(M; \mathbb{K})$ for some metric space M,
(ii) X is a Banach algebra with respect to pointwise multiplication,
(iii) we have $u^{-1} \in X$ for every $u \in X$ with $\inf\{|u(x)| : x \in M\} > 0$.

B.80. Remark. Let $\Sigma \subset \mathbb{R}^n$ be a compact smooth hypersurface and let $\theta \in (0, 1)$, $p \in (1, \infty)$. Then we have

$$[\![uv]\!]_{\theta,p} \leq \|u\|_\infty [\![v]\!]_{\theta,p} + [\![u]\!]_{\theta,p} \|v\|_\infty \quad \text{for } u, v \in W^\theta_p(\Sigma) \cap L_\infty(\Sigma).$$

Therefore the spaces $W^{k+\theta}_p(\Sigma) \cap W^k_\infty(\Sigma)$ ($k \in \mathbb{N}_0$, $\theta \in [0, 1]$, $p \in (1, \infty)$) are multiplication algebras.

One more general result is given in

B.81. Lemma (cf. [Mey10, Lemma 1.3.19])**.** *Let $\Omega \subset \mathbb{R}^n$ be a domain with compact smooth boundary, or $\Omega \in \{\mathbb{R}^n, \mathbb{R}^n_+\}$, or let Ω be the boundary of such a domain. Let further X be a Banach space of class $\mathcal{HT}$, let $s \in (0, \infty)$ and $p \in (1, \infty)$. Then there exists $C > 0$ such that*

$$\|fg\|_{W^s_p(\Omega;X)} \leq C\|f\|_{L_\infty(\Omega;\mathcal{B}(X))}\|g\|_{W^s_p(\Omega;X)} + C\|f\|_{W^s_p(\Omega;\mathcal{B}(X))}\|g\|_{L_\infty(\Omega;X)}$$

for all $f \in W^s_p(\Omega; \mathcal{B}(X)) \cap L_\infty(\Omega; \mathcal{B}(X))$ and $g \in W^s_p(\Omega; X) \cap L_\infty(\Omega; X)$.

Hence $W^s_p(\Sigma)$ is a multiplication algebra for $s \in (0, \infty)$, $p \in (1, \infty)$ with $s - (n-1)/p > 0$.

B.82. Definition. Let M be a measure space, let X, Y be Banach spaces, $U \subset X$ be open and $f \colon M \times U \to Y$ be a *Carathéodory function*; that is,
(i) $u \mapsto f(x, u)$ is continuous for almost all $x \in M$,
(ii) $x \mapsto f(x, u)$ is measurable for all $u \in U$.
Then the map

$$F \colon U^M \to Y^M, \quad u \mapsto [x \mapsto F(u)(x) := f(x, u(x))]$$

is called the *Nemytksiĭ operator* (of order zero) induced by f.

B.83. Definition. Let M be a set and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. A Banach space $X \subset \mathbb{K}^M$ is called a *multiplication algebra* if pointwise multiplication

$$X \times X \to X, \quad (u, v) \mapsto uv = [x \mapsto u(x)v(x)]$$

is continuous. In this case there exists $C_X > 0$ such that $\|uv\|_X \leq C_X \|u\|_X \|v\|_X$ for all $u, v \in X$.

We collect some information on analytic operators between open subsets of Banach spaces from Appell and Zabrejko [AZ90], Deimling [Dei10], and Zeidler [Zei86].

Let $X_1, \ldots, X_k$, Y be Banach spaces over the same scalar field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We say that a k-linear operator $T\colon X_1 \times \cdots \times X_k \to Y$ is *bounded*, if there exists a number $C \geq 0$ such that $\|T(x_1, \ldots, x_k)\|_Y \leq C\|x_1\|_{X_1} \cdots \|x_k\|_{X_1}$ for all tuples $(x_1, \ldots, x_k)$. The infimum of such numbers C is the norm of T, denoted by $\|T\|$ or $\|T\|_{\mathcal{B}^k(X_1\times\cdots\times X_k;Y)}$. We put

$$\mathcal{B}^k(X_1 \times \cdots \times X_k; Y) := \{T : X_1 \times \cdots \times X_k \to Y : T \text{ is } k\text{-linear and bounded}\}.$$

For $k = 0$, we put $X_1 \times \cdots \times X_k := \{0\}$ and $\mathcal{B}^0(\{0\}; Y) := Y$. For a multi-index $\alpha \in \mathbb{N}_0^k$ we let $K = |\{j \in \{1, \ldots, k\} : \alpha_j \neq 0\}|$ and $\phi\colon \{1, \ldots, K\} \to \{1, \ldots, k\}$ be strictly increasing such that $\alpha_{\phi(j)} \neq 0$ for all j. Then we identify $X_1^{\alpha_1} \times \cdots \times X_k^{\alpha_k}$ with $X_{\varphi(1)}^{\alpha_{\varphi(1)}} \times \cdots \times X_{\varphi(K)}^{\alpha_{\varphi(K)}}$ and define

$$\mathcal{B}^\alpha(X_1^{\alpha_1} \times \cdots \times X_k^{\alpha_k}; Y) := \mathcal{B}^{|\alpha|}(X_1^{\alpha_1} \times \cdots \times X_k^{\alpha_k}; Y).$$

A map $T\colon X^k = X \times \cdots \times X \to Y$ is called *symmetric* if $T(x_1, \ldots, x_k) = T(x_{\sigma(1)}, \ldots, x_{\sigma(k)})$ for all tuples $(x_1, \ldots, x_k)$ and all permutations σ of $\{1, 2, \ldots, k\}$. The map $T\colon X_1^{\alpha_1} \times \cdots \times X_k^{\alpha_k}$ ($\alpha \in \mathbb{N}_0^k$) is called *partially symmetric*, if it is symmetric with respect to every tuple $(x_{j,1}, \ldots, x_{j,\alpha_j}) \in X_j^{\alpha_j}$ when the other variables are fixed. We define

$$\mathcal{B}^k_{\text{sym}}(X^k; Y) := \{T \in \mathcal{B}^k(X^k; Y) : T \text{ is symmetric}\},$$
$$\mathcal{B}^\alpha_{\text{sym}}(X_1^{\alpha_1} \times \cdots \times X_k^{\alpha_k}; Y) := \{T \in \mathcal{B}^{|\alpha|}(X_1^{\alpha_1} \times \cdots \times X_k^{\alpha_k}; Y) : T \text{ is partially symmetric}\}.$$

A map $M\colon X_1 \times \cdots \times X_k \to Y$ is called *monomial (operator)* of degree $\alpha \in \mathbb{N}_0^k$ induced by the multilinear map $T \in \mathcal{B}^\alpha_{\text{sym}}(X_1^{\alpha_1} \times \cdots \times X_k^{\alpha_k}; Y)$ if

$$M(x_1, \ldots, x_k) = T(x_1^{\alpha_1}, \ldots, x_k^{\alpha_k}) \quad \text{for all } (x_1, \ldots, x_k) \in X_1 \times \cdots \times X_k,$$

where $x_j^{\alpha_j}$ denotes the tuple $(x_j, \ldots, x_j) \in X_j^{\alpha_j}$.

A map $P\colon X_1 \times \cdots \times X_k \to Y$ is called *polynomial (operator)* of degree lesser than or equal to $\alpha \in \mathbb{N}_0^k$, if there exist finitely many monomials $M^{(i)}\colon X_1 \times \cdots \times X_k \to Y$ of degree $\alpha^{(i)} \in \mathbb{N}_0^k$ with $\alpha^{(i)} \leq \alpha$ such that $P = \sum_i M^{(i)}$.

B.84. Definition (Analytic operator)**.** Let X, Y be Banach spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $U \subset X$ be open. We say that $F\colon U \subset X \to Y$ is ($\mathbb{K}$-) *analytic* at $u \in U$, if there exists $r > 0$ and symmetric operators $F_k \in \mathcal{B}^k_{\text{sym}}(X^k; Y)$, $k \geq 0$, such that

$$\text{(B.18)} \qquad \sum\nolimits_{k=0}^\infty \|F_k\|_{\mathcal{B}^k(X^k;Y)} \|h\|_X^k < \infty \text{ and } F(u + h) = \sum\nolimits_{k=0}^\infty F_k h^k \quad \text{for all } h \in B_r^X.$$

A function is called *analytic* in U, if it is analytic at every point $u_0 \in U$.

If F is analytic at u, then F is C^∞ near u and we have $F_k = F^{(k)}(u)/k!$. We next define

$$r_F(u) := \min\left\{\operatorname{dist}_X(u, \partial U), C_F(u)^{-1}\right\}, \quad C_F(u) := \limsup\nolimits_{k\to\infty} \|F^{(k)}(u)/k!\|^{1/k}_{\mathcal{B}^k(X^k;Y)}.$$

Then the *Taylor series* $\sum_{k=0}^\infty F^{(k)}(u)h^k/k!$ converges in Y and equals $F(u+h)$ for $\|h\|_X < r_F(u)$. If $\mathbb{K} = \mathbb{C}$, then a function is analytic in U if and only if it is holomorphic in U. We then have

$$F^{(k)}(u)h^k = \frac{k!}{2\pi i} \int_{|\zeta|=\rho} \frac{F(u + \zeta h)}{\zeta^{k+1}}\, d\zeta \quad \text{for } 0 < \rho\|h\|_X < r_F(u),\ k \in \mathbb{N}_0,$$

and *Cauchy's estimates* are valid:

$$\|F^{(k)}(u)\|_{\mathcal{B}^k(X^k;Y)} \leq \frac{k!}{\delta^k} \|F\|_{L_\infty(B_\delta(u);Y)} \quad \text{for } 0 < \delta < r_F(u),\ k \in \mathbb{N}_0.$$

B.85. Remark (Chain rule). Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^k$ be open subsets. The following chain rule is valid for sufficiently smooth maps $f\colon V \to \mathbb{R}$ and $u\colon U \to V$ at $x \in U$ (see [RS96, (5.2.1/6)]).

$$\partial^\alpha (f \circ u)(x) = \sum\nolimits_{j=1}^{|\alpha|} \sum c_{\alpha,j,\beta^{(1)},\ldots,\beta^{(j)}} f^{(j)}(u(x)) \partial^{\beta^{(1)}} u(x) \cdots \partial^{\beta^{(j)}} u(x).$$

Here the second sum is taken over all multi-indices $\beta^{(1)}, \ldots, \beta^{(j)} \in \mathbb{N}_0^n \setminus \{0\}$ such that $\beta^{(1)} + \cdots + \beta^{(j)} = \alpha$ and $c_{\alpha,j,\beta^{(1)},\ldots,\beta^{(j)}}$ are some constants that do not depend on f and u.

We next state Fraenkel's chain rule [Fra78, Formula A]. Let X, Y, Z be Banach spaces, $U \subset X$, $V \subset Y$ be open sets and let $f \in C^N(V; Z)$, $u \in C^N(U; V)$. Then $f \circ u \in C^N(U; Z)$ and for $n \in \{1, \ldots, N\}$, $x \in U$ and $(v_1, \ldots, v_n) \in X^n$, we have

$$(f \circ u)^{(n)} v_1 \cdots v_n = \sum_{j=1}^{n} \sum_{\beta,\sigma} \frac{f^{(j)} \circ u}{j!\beta!} (u^{(\beta_1)} v_{\sigma(1)} \cdots v_{\sigma(\beta_1)}) \cdots (u^{(\beta_j)} v_{\sigma(n-\beta_j+1)} \cdots v_{\sigma(n)}), \tag{B.19}$$

where the sum $\sum_{\beta,\sigma}$ is taken over multi-indices $\beta \in \mathbb{N}^j$ such that $|\beta| = n$ and all $n!$ permutations σ of $\{1, \ldots, n\}$. It can be shown that

$$\sum\nolimits_{\beta \in \mathbb{N}^j, |\beta|=1} 1 = |\{\beta \in \mathbb{N}^j : |\beta| = n\}| = \binom{n-1}{j-1} \quad \text{for } 1 \le j \le n. \tag{B.20}$$

B.86. Corollary. *If $F\colon U \subset X \to V \subset Y$ and $G\colon V \subset Y \to Z$ are analytic, then $G \circ F\colon U \subset X \to Z$ is analytic.*

Proof. Let $M_F := \sup_{j\in\mathbb{N}} \|F^{(j)}(x)/j!\|^{1/j}$, $M_G := \sup_{j\in\mathbb{N}} \|G^{(j)}(F(x))/j!\|^{1/j}$ for $x \in U$. The chain rule (B.19) yields

$$\begin{aligned} \frac{\|(G \circ F)^{(n)}(x)\|}{n!} &\le M_F^n \sum_{j=1}^{n} M_G^j |\{\beta \in \mathbb{N}^j : |\beta| = n\}| \\ &\le M_F^n \sum_{j=1}^{n} M_G^j \binom{n-1}{j-1} = M_F^n M_G (1 + M_G)^{n-1}. \end{aligned}$$

Therefore $\sum_{n\ge0} (G \circ F)^{(n)}(x) h^n/n!$ converges for $\|h\| < [M_F(1 + M_G)]^{-1}$. The representation

$$G(F(x+h)) = \sum_{j\ge0} \frac{G^{(j)}(F(x))}{j!} (F(x+h) - F(x))^j = \sum_{j\ge0} \frac{G^{(j)}(F(x))}{j!} \left(\sum_{l\ge1} \frac{F^{(l)}(x) h^l}{l!} \right)^j,$$

is valid for small h. As in the proof of [Fra78, Formula A] we rewrite the right-hand side as

$$G(F(x)) + \sum_{n\ge1} \sum_{j=1}^{n} \sum_{\beta \in \mathbb{N}^j, |\beta|=n} \sum_{\sigma} \frac{G^{(j)}(F(x))}{j!\beta!} (F^{(\beta_1)}(x) h^{\beta_1}) \cdots (F^{(\beta_j)}(x) h^{\beta_j}).$$

By the chain rule, $G(F(x+h))$ coincides with its Taylor series for small h. Therefore $G \circ F$ is analytic at x. □

B.87. Proposition. *Let M be a metric space, $X \hookrightarrow BUC(M; \mathbb{K})$ be a multiplication algebra, $U \subset \mathbb{K}^m$ ($m \in \mathbb{N}$) be open and $f\colon U \subset \mathbb{K}^m \to \mathbb{K}$ be analytic. Define*

$$\mathcal{U} := \{u \in X^m : u(M) \subset U, \inf r_f(u(M)) > 0, f \circ u \in X, C_F(u) < \infty\},$$
$$C_F(u) = \limsup\nolimits_{j\to\infty} \|\partial^j f \circ u/j!\|_{\mathcal{B}^j((X^m)^j;X)}^{1/j} \text{ for } u \in X^m \text{ with } u(M) \subset U.$$

Then $F\colon u \mapsto f \circ u, \mathcal{U} \subset X^m \to X$ is analytic.

Proof. For all $u \in X^m$ with $u(M) \subset U$ and all $h \in X^m$ with $\|h\|_{X^m} < C(u)^{-1}$, the Taylor series $\sum_{j\geq 0} F^{(j)}(u)h^j/j!$ converges in X. Since f is analytic on $u(M) \subset U$, we obtain the representation

$$F(u+h)(x) = f(u(x)+h(x)) = \sum\nolimits_{j\geq 0} \partial^j f(u(x))h(x)^j/j! \quad \text{for } x \in M,$$

provided $|h(x)| < r_f(u(x))$. From $X \hookrightarrow BUC(M;\mathbb{K})$ we infer that

$$|h(x)| \leq \|h\|_{BUC(M)^m} \leq \|I\|_{X\to BUC(M)}\|h\|_{X^m} < \inf r_f(u(M)) \leq r_f(u(x)) \quad \text{for all } x \in M,$$

for $h \in X^m$ with $\|h\|_{X^m} < \|I\|^{-1}_{X\to BUC(M)} \inf r_f(u(M))$. Therefore F is analytic at u with

$$r_F(u) \geq \min\{\|I\|^{-1}_{X\to BUC(M)} \inf r_f(u(M)), C_F(u)^{-1}\} > 0. \qquad \square$$

B.88. Proposition. *Let M be a metric space, $X \hookrightarrow BUC(M;\mathbb{K})$ be a multiplication algebra and $m \in \mathbb{N}$. Then the map $A \mapsto [A(\cdot)]^{-1}$, $\{A(\cdot) \in X^{m\times m} : [A(\cdot)]^{-1} \in X^{m\times m}\} \to X^{m\times m}$ is analytic.*

Proof. Let $U := \{A \in \mathbb{K}^{m\times m} : A \text{ is invertible}\}$ and $f\colon A \mapsto A^{-1}$, $U \subset \mathbb{K}^{m\times m} \to \mathbb{K}^{m\times m}$. Then

$$\text{(B.21)} \quad \partial^j f(A)(B_1,\dots,B_k) = (-1)^j \sum\nolimits_\sigma \left(\prod\nolimits_{i=1}^j \left(A^{-1}B_{\sigma(i)}\right)\right) A^{-1} \quad \text{for } j \in \mathbb{N}_0,\ B \in \mathbb{K}^{m\times m},$$

where the sum is taken over all $j!$ permutations σ of $\{1,\dots,j\}$. Hence $C_f(A) = |A^{-1}|$ for $A \in U$. For $A \in U$ and $B \in \mathbb{K}^{m\times m}$ with $|B| < |A^{-1}|^{-1}$ we have $A + B = A(I + A^{-1}B) \in U$ and thus $\operatorname{dist}_{\mathbb{K}^{m\times m}}(A, \partial U) \geq |A^{-1}|^{-1}$. Therefore f is analytic with $r_f(A) = |A^{-1}|^{-1}$.

The space $X^{m\times m}$ with norm $\|\cdot\|_X := \|\cdot\|_{X^{m\times m}}$ is a Banach algebra with respect to pointwise matrix multiplication and there exists $C_X > 0$ such that $\|AB\|_X \leq C_X\|A\|_X\|B\|_X$. Hence

$$\|\partial^j f(A(\cdot))\|_X \leq C_X^{2j}\|A^{-1}\|_X^{j+1} j! \quad \text{for } j \in \mathbb{N}_0,\ A \in X^{m\times m} \text{ with } A^{-1} \in X^{m\times m}.$$

Proposition B.87 with $C_F(A) = C_X^2\|A^{-1}\|_X < \infty$ yields the assertion. $\square$

B.89. Proposition. *Let M be a metric space and $X \hookrightarrow BUC(M;\mathbb{K})$ be a multiplication algebra. Then the map $F\colon u \mapsto u(\cdot)^{1/2}$, $\{u \in X : \inf_M \operatorname{dist}(u(\cdot),\mathbb{R}_-) > 0,\ u^{1/2}, u^{-1} \in X\} \to X$ is analytic.*

Proof. The map $f\colon z \mapsto z^{1/2}$, $\mathbb{K}\setminus\mathbb{R}_- \to \mathbb{K}$ is analytic with $C_f(z) = |z|$ and $r_f(z) = \operatorname{dist}(z,\mathbb{R}_-)$ and

$$\left\|\frac{\partial^k f\circ u}{k!}\right\|_X^{1/k} = \left|\frac{1}{2\cdot 1}\cdot\left(-\frac{1}{2\cdot 2}\right)\cdots\left(-\frac{2k-3}{2\cdot k}\right)\right|^{1/k} \|u^{-k+1/2}\|_X^{1/k} \leq c_k C_X \|u^{1/2}\|_X^{1/k}\|u^{-1}\|_X,$$

with $\lim_{k\to\infty} c_k = 1$. Hence $C_F(u) \leq C_X\|u^{-1}\|_X$ and Proposition B.87 yields analyticity. $\square$

B.90. Lemma. *Let $\Sigma \subset \mathbb{R}^n$ ($n \geq 2$) be a compact smooth hypersurface and let $s \in [0,\infty)$, $p \in (1,\infty)$, and $m \in \mathbb{N}$. Then the pointwise matrix inversion operator*

$$A(\cdot) \mapsto A(\cdot)^{-1}, \quad \{A \in (W_p^s \cap C)(\Sigma;\mathbb{K}^{m\times m}) : \|A(\cdot)^{-1}\|_\infty < \infty\} \to (W_p^s \cap C)(\Sigma;\mathbb{K}^{m\times m})$$

and the pointwise square root operator

$$u(\cdot) \mapsto \sqrt{u(\cdot)}, \quad \{u \in (W_p^s \cap C)(\Sigma) : \inf\nolimits_\Sigma \operatorname{dist}(u(\cdot),\mathbb{R}_-) > 0\} \to (W_p^s \cap C)(\Sigma;\mathbb{K}^{m\times m})$$

are analytic.

Proof. The matrix inversion operator is well-defined, since we can control A^{-1} in the W_p^s-norm by means of the identity (B.21), Lemma B.81 and the inequalities

$$\|A^{-1}\|_p \leq |\Sigma|^{1/p}\|A^{-1}\|_\infty, \quad [\![A^{-1}]\!]_{\theta,p} \leq \|A^{-1}\|_\infty^2 [\![A]\!]_{\theta,p}.$$

Then Proposition B.88 yields analyticity. The second assertion follows from the estimates

$$\|\sqrt{u}\|_p \leq |\Sigma|^{1/p}\|u\|_\infty^{1/2}, \quad [\![\sqrt{u}]\!]_{\theta,p} \leq \left(2\inf\nolimits_\Sigma |u|^{1/2}\right)^{-1} [\![u]\!]_{\theta,p}. \qquad \square$$

B.5. Computation of the boundary symbol

For the derivation of the boundary symbol in Section 3.1.1 we have employed the identity (3.13) on page 58. This identity can be checked with the software Maxima [Max] with the following source code.

```
gam1: %omega[1]+z;
gam2: %omega[2]+z;
%alpha[1]: %mu[1]*%omega[1]*gam1;
%alpha[2]: %mu[2]*%omega[2]*gam2;
%Omega[p]: %mu[1]*%omega[1]*gam1+%mu[2]*%omega[2]*gam2;
%Omega[s]: %mu[s]*z^2
c[6]*%mu[1]*%omega[1]+c[6]*%mu[2]*%omega[2]+;
Lw1: c[5]*z*%alpha[1]*(%mu[1]-%mu[2])
  + c[6]*%mu[1]*%omega[1]*%alpha[2]
  + c[6]*%mu[2]*%omega[2]*%alpha[1];
Lw2: c[5]*z*%alpha[2]*(%mu[1]-%mu[2])
  - c[6]*%mu[1]*%omega[1]*%alpha[2]
  - c[6]*%mu[2]*%omega[2]*%alpha[1];
Lq: c[5]*z*(%mu[1]-%mu[2])
  + c[6]*%mu[2]*%omega[2] - c[6]*%mu[1]*%omega[1];
ratvars(z,%mu[1],%mu[2],%omega[1],%omega[2],%lambda);
B: matrix([-%omega[2]*%Omega[s]*%Omega[p]
      +z*Lw2-z^2*%Omega[p]*%lambda[s]*%omega[2],
    Lw1*z, -c[1]*z^4, Lq*z^2],
  [%omega[2], %omega[1], 0, 0],
  [-%alpha[2], -%alpha[1], %lambda, -z],
  [(2*%mu[2]-c[2])*%omega[2]*%Omega[p]*%Omega[s]
      -2*%theta[3]*z*Lw2
      +2*%theta[3]*%lambda[s]*%omega[2]*z^2*%Omega[p],
    2*%mu[1]*%omega[1]*%Omega[p]*%Omega[s]
      -2*%theta[3]*Lw1*z,
    (c[%sigma]+%theta[4])*%Omega[s]*z^2
      +2*c[1]*%theta[3]*z^4,
    %Omega[s]*%Omega[p]
      -2*%theta[3]*z^2*Lq]);
detB: expand(determinant(B));
factor(detB);
P: expand(divide(detB,%omega[1]*%omega[2]
  *(%mu[2]*%omega[2]*z+%mu[1]*%omega[1]*z
  +%mu[2]*%omega[2]^2+%mu[1]*%omega[1]^2)
  *(%mu[s]*z^2+c[6]*%mu[2]*%omega[2]
  +c[6]*%mu[1]*%omega[1]))[1]);
Q: -scsimp(P,
  (%theta[4]+c[%sigma])*(%mu[s]+%lambda[s])
  +c[1]*c[2]+2*c[1]*%theta[3]=%beta[s]*d,
  %mu[s]+%lambda[s]=%beta[s]);
```

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List of symbols

Numbers

$\mathbb{C}$ the complex numbers. 11

$\mathbb{C}_+$ $\{z \in \mathbb{C} : \operatorname{Re} z \geq 0\}$, the complex numbers with non-negative real part. 11

$\mathbb{K}$ either $\mathbb{R}$ or $\mathbb{C}$. 11

$\mathbb{N}$ $\{1, 2, \ldots\}$, the positive integers. 11

$\mathbb{N}_0$ $\{0, 1, 2, \ldots\}$, the non-negative integers. 11

$\mathbb{R}$ the real numbers. 11

$\mathbb{R}_+$ $[0, \infty)$, the non-negative real numbers. 11

$\mathbb{R}_-$ $(-\infty, 0]$, the non-positive real numbers. 11

$\mathbb{Z}$ $\{\ldots, -1, 0, 1, \ldots\}$, the integers. 11

Surface differential operators

$\widetilde{\Delta}_\Gamma$ $g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta$, Laplace-Beltrami operator for tangential vector fields. 141, 164

$D_\Gamma(u)$ $\operatorname{sym}(P_\Gamma[\nabla_\Gamma u]P_\Gamma)$, surface rate-of-strain tensor for $u \colon \Gamma \to \mathbb{R}^n$. 19, 140

$\operatorname{div}_\Gamma S$ $[\partial_\alpha S]\tau^\alpha$, surface divergence of a symmetric tensor $S^{\alpha\beta}\tau_\alpha \otimes \tau_\beta$. 140

$\operatorname{div}_\Gamma u$ $\tau^\alpha \cdot \partial_\alpha u$, surface divergence of a vector field $u = v^\alpha \tau_\alpha + w\nu$. 140

∇_Γ $\tau_\Gamma^j \otimes \partial_j$, surface gradient of Γ. 140

$\widetilde{\nabla}_\Gamma$ covariant derivative of Γ. 139, 140

Symbols related to basic function spaces

B^s_{pq} Besov space of order s with exponents p and q. 144

BUC^k space of all bounded and uniformly continuous functions with bounded and uniformly continuous derivatives up to order k. 143

$C(U; V)$ space of all continuous functions $f \colon U \subset X \to V \subset Y$. 11

$C_0(J; X)$ Banach space of all $u \in C(J; X)$ such that $\|u(t)\|_X \to 0$ as $t \to \infty$. 143

C^{k-} space of all $u \in C^{k-1}$ such that ∇^{k-1} is locally Lipschitz. 143

$C^{k,\alpha}$ Hölder space of all $u \in C^k$ such that $\nabla^k u \in C^{0,\alpha}$. 143

$\mathcal{D}(\Omega; X)$ space of test functions in Ω. 143

$\mathcal{D}'(\Omega; X)$ space of distributions in Ω. 143

F^s_{pq} Triebel-Lizorkin space of order s with exponents p, q. 144

$\mathcal{H}(\Omega; Y)$ space of all holomorphic Y-valued functions on Ω. 164

$\mathcal{H}^\infty(\Omega; Y)$ space of all bounded holomorphic Y-valued functions on Ω. 164

$\mathcal{H}^\infty_0(\Omega; Y)$ subspace of $\mathcal{H}^\infty(\Omega; Y)$ with polynomial decay near 0 and ∞. 164

$\mathcal{H}_P(\Omega; Y)$ subspace of $\mathcal{H}(\Omega; Y)$ with polynomial growth near 0 and ∞. 165

$\dot{H}^k_p$ homogeneous Sobolev space of order k with exponent p. 40

$\dot{\mathcal{H}}^k_p$ semi-normed version of $\dot{H}^k_p$. 23, 40

$\hat{H}^{-1}_p$ dual space of $\dot{H}^1_{p'}$. 25, 40

H^s_p Bessel-potential space of order s with exponent p. 143

L_p Lebesgue space with exponent p. 143

$\mathcal{P}_k$ space of all polynomials. 151

$\mathcal{P}_k$ space of all polynomials of degree lesser or equal than k. 151

$\mathcal{S}$ space of rapidly decreasing functions. 143

$\mathcal{S}'$ space of tempered distributions. 143

$\mathcal{S}_0$ subspace of all $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with $(\partial^\alpha \mathcal{F}\varphi)(0) = 0$ for all $\alpha \in \mathbb{N}^n_0$. 151

$\mathcal{S}'_0$ dual space of $\mathcal{S}_0$, equivalence classes of $\mathcal{S}'$ modulo polynomials. 151

Σ_θ $\{re^{i\varphi} : r \in (0, \infty), \varphi \in (-\theta, \theta)\}$, open sector. 152, 161, 164

W^s_p Sobolev-Slobodeckiĭ space of order s with exponent p. 143, 144

Special function spaces

$\mathbb{E}_{\partial\Theta}$ space of interface regularity for the Jacobian of the normal-preserving map. 99

$\tilde{\mathbb{E}}_{\partial\Theta}$ a larger space than $\mathbb{E}_{\partial\Theta}$. 113, 115

$\mathbb{E}_h$ space of time-dependent height functions h. 99

$\tilde{\mathbb{E}}_h$ a larger space than $\mathbb{E}_h$. 112

$\mathbb{E}_\Theta$ space of interior regularity for the normal-preserving map. 102

$\mathbb{E}_{u,v,w,[\![\mu\partial_\nu w]\!]}$ space of all $u \in \mathbb{E}_u$ with $P_\Sigma u|_\Sigma \in \mathbb{E}_v$, $w := \nu_\Sigma \cdot u|_\Sigma \in \mathbb{E}_w$, $\partial_\nu w_\pm|_\Sigma \in \mathbb{G}_w$. 69

$\mathbb{F}_{d,\Sigma}$ space of all $f_d \in \mathbb{F}_d$ with $f_{d,\pm}|_\Sigma \in \mathbb{G}_w$. 69, 123

$\mathbb{G}_v$ space for the tangential interface stress balance. 115

$\mathbb{G}_w$ space for the normal interface stress balance. 115

$\mathcal{P}_{M,T}$ parameter set for perturbed version of problem (MP). 82

$\mathcal{P}_{M,T,\eta,R}$ parameter set for problem (PL). 71

$\mathcal{U}_h$ set of certain $h\colon J \times \Sigma \to \mathbb{R}$ for which Θ_h is a diffeomorphism. 101, 102

$\mathcal{U}_{h_0}$ set of certain $h_0\colon \Sigma \to \mathbb{R}$ for which Θ_{h_0} is a diffeomorphism. 101, 102

Linear operators

$D_A(\alpha, p)$ fractional domain of A. 154

$\mathcal{H}(X)$ class of linear operators $A\colon D(A) \subset X \to X$ with bounded $\mathcal{H}^\infty$-calculus. 153

$\mathcal{P}_1(X_1, X_0; M, \vartheta)$ class of invertible linear operators $A\colon X_1 \to X_0$ satisfying the estimate $(1+|\lambda|)\|(\lambda+A)^{-1}\| \le K$ for $|\arg\lambda| \le \vartheta$. 154

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Eidesstattliche Erklärung

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Arbeit selbständig und ohne fremde Hilfe verfasst habe, keine anderen als die von mir angegebenen Quellen und Hilfsmittel benutzt habe und die den benutzen Werken wörtlich oder inhaltlich entnommenen Stellen als solche kenntlich gemacht habe.

Halle (Saale), 28. November 2015

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