Algorithms and Decomposition Methods for Multiobjective Location and Approximation Problems

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Abstract

Considering the importance of multiobjective optimization in combination with locational analysis, we study in this thesis a class of extended multiobjective location and approximation problems, where the objective function includes distances as well as cost functions. This class is very important and has a lot of applications in economy, engineering and physics. In particular, we give an example for an application in radiotherapy treatment.

The principle aims of our work are the following: First we prove duality assertions for extended multiobjective location and approximation problems using Lagrange duality. Furthermore, the extended multiobjective location problem is decomposed, such that a multiobjective location problem is obtained as a subproblem. We study this multiobjective location problem comprehensively and get through Pareto reducibility a new characterization of the set of weakly minimal solutions using its well-known duality-based geometrical structure. An implementable partition algorithm for the set of minimal solutions of the multiobjective location problem is also derived. This algorithm is the base for developing decomposition algorithms, that provide minimal solutions of the extended multiobjective location problem. In the final part of the thesis, we study a multi-facility location problem by transforming it into a single-facility approximation problem in a higher dimension. We derive an algorithm for solving it applying the method of partial inverse and the proximal point algorithm. Finally, an interactive procedure for a multiobjective multi-facility problem is developed using this algorithm.

Zusammenfassung

Uns ist heutzutage sehr bewusst, welche wichtige Rolle die mehrkriterielle Optimierung kombiniert mit Standorttheorie spielt. In dieser Arbeit beschäftigen wir uns mit einer Klasse von erweiterten mehrkriteriellen Standort- und Approximationsproblemen mit einer vektorwertigen Zielfunktion, die sowohl die Abstände als auch die Kostenfunktionen enthält. Diese Klasse ist sehr wichtig und hat viele Anwendungen in der Wirtschaft, den Ingenieurwissenschaften und in der Physik. Es wird insbesondere ein Beispiel in der Radiotherapie präsentiert.

Die Hauptziele dieser Arbeit sind: Die Dualitätsaussagen für die erweiterten mehrkriteriellen Standort- und Approximationsprobleme mit Hilfe der Lagrange-Dualität zu beweisen. Danach werden erweiterte mehrkriterielle Standortprobleme zerlegt, wobei das mehrkriterielle Standortproblem ein Teilproblem ist. Das mehrkriterielle Standortproblem wird ausführlich untersucht. Die geometrische Struktur der Menge der Minimallösungen dieses Problems wird danach benutzt, um eine neue Charakterisierung der Menge der schwachen Minimallösungen mit Hilfe von Pareto-Reduzierbarkeit zu erhalten. Weiterhin wird ein implementabler Zerlegungsalgorithmus entwickelt, um die Menge der Minimallösungen des mehrkriteriellen Standortproblems zu endlich vielen Rechtecken zu zerlegen. Dieser Algorithmus ist die Basis der Entwicklung weiterer Dekompositionsalgorithmen zur Lösung von erweiterten mehrkriteriellen Standortproblemen.

Im letzten Teil dieser Arbeit untersuchen wir N-Standortprobleme. Das N-Standortproblem wird zu einem Approximationsproblem in einer höheren Dimension transformiert. Zur Lösung des entstandenen Approximationsproblems wird ein Proximal-Point-Algorithmus hergeleitet. Unter Verwendung des Algorithmus wird eine interaktive Prozedur zur Lösung von mehrkriteriellen N-Standortproblemen entwickelt.

Arabic Abstract

نبذة عن الرسالة

تختص هذه الرسالة بالحديث عن الأَمْثَلَةِ المتجهية ونظرية تحديد المواقع (تسمى أيضاً نظرية المؤْضَعَة). إن نظرية الأَمْثَلَة المتجهية (متعددة المعايير) والتي نشأت من مفكرين اقتصاديين في نهاية القرن التاسع عشر ذات تطبيقات واسعة في بحوث العمليات والأنظمة الصناعية ونظرية التحكم ونظرية اتخاذ القرار والاتصالات. نحاول في هذه الرسالة تطبيق أدوات هذه النظرية ونتائجها وخاصة نظرية الثنوية على مسائل تحديد المواقع وتعميماتها (مسائل التقريب).

مسائل تحديد المواقع تتلخص في البحث عن موقع أَمْثَل لبناء مُرْفَق جديد بحيث تكون المسافات بالنسبة لمجموعة مواقع أخرى أصغرية. كما تسمى مسألة تحديد المواقع متعددة المرافق عندما يكون البحث عن مواقع لعدة مرافق جديدة بدلاً من واحد. لهذه النظرية تطبيقات كثيرة غير المرافق الاجتماعية، منها الهندسة التقنية وتخطيط الإنتاج والاقتصاد والطب. تعمم مسائل تحديد المواقع إلى مسائل تقريب عندما يكون من اللازم إجراء تحويلات (مثل تدوير الإحداثيات) على المواقع الموجودة قبل حساب المسافات.

في هذه الرسالة اخترنا نموذجاً لمسائل تحديد المواقع التقريبية ندرس فيه إيجاد القيمة الصغرى لدالة متجهية ملحقة بدالة أخرى تعبر عن التكاليف. هذا النموذج الرياضي له تطبيقات كثيرة أعطينا له مثالاً في المعالجة الشعاعية وتحديداً مسائل حساب القيمة المثلى للكثافة الشعاعية المطَبَّقة على كل من الأنسجة المريضة أو السليمة.

من أجل حل هذا النموذج تم تطبيق نظرية الثنوية بتقنية مضروبات لاغرانج حيث تم إثبات مسائل الثنوية للمسألة المدروسة، وهي مبرهنات الثنوية الضعيفة والثنوية القوية المباشرة والثنوية القوية العكسية. كما نحاول تطبيق نتائج التجزئة والتخفيض على مسألتنا وتطوير خوارزميات قابلة للبرمجة تختص بحل هذه المسألة عن طريق التجزئة. بتجزئة هذا النموذج نحصل على مسألة تحديد المواقع المتجهية المعروفة كمسألة جزئية. لقد تم استخدام البنية الهندسية المعروفة لمجموعة الحلول الصغرى لهذه المسألة الجزئية في إيجاد نتائج جديدة تتلخص في وصف بنية مجموعة الحلول الصغرى الضعيفة والتي لم تكن معروفة سابقاً، وكذلك نتائج جديدة تتعلق بتجزئة مجموعة الحلول الصغرى إلى مجموعات محدبة. هذه التجزئة مكنتنا من إيجاد خوارزميات عددية هامة ذات تطبيقات واسعة، مثلاً في مجال بناء المرافق الغير مرغوب قربها كمحطات معالجة المياه. هذه النتائج المهمة والتي تمت برمجتها (بجهود المبرمج كريستيان غنتر) تم نشرها في أكثر من مجلة علمية.

أما مسائل تحديد المواقع متعددة المرافق فتمّت دراستها في شكلها السُلَّمي وتحويلها إلى مسألة تحديد موقع واحد ولكن في فضاء أعلى درجة. من أجل الحل تم تطبيق خوارزمية النقطة الأقرب والمسألة العكسية لسبينغرام. بعد ذلك قمنا بإيجاد طريقة تفاعلية لحل مسألة تحديد المواقع المتجهية عن طريق الخوارزمية السابقة.

Contents

List of Symbols and Notations

$x^j \in X$	Vectors in X
x_i	The component i of the vector x
$A + B$	The Minkowski sum of the sets A and B
$\mathrm{int}\, A$	The interior of a set A
$\mathrm{cl}\, A$	The closure of a set A
$\mathrm{bd}\ A$	The boundary of a set A
$\mathrm{conv}\ A$	The convex hull of a set A
$\mathrm{dom}\ f$	The domain of a function f
$\mathrm{epi}\ f$	The epigraph of a function f
$\|\cdot\|$	A norm $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$
$\langle \cdot, \cdot \rangle$	A scalar product $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$
X', X^*	The algebraic and the dual spaces of the space X respectively
$x^*(\cdot)$	A linear continuous functional
K^*	The dual cone of K
$\mathcal{X}_S(x)$	The indicator function of the set S at x
$N_M(x^0)$	The normal cone at x^0 with respect to the set M
$\mathbb{R}^p_+$	The standard ordering cone in $\mathbb{R}^p$
$A^\perp$	The orthogonal of the set A

$\partial f(x^0)$	The subdifferential of a function f at x^0
I_p	The set of the indices $1, \cdots, p$
I	A subset of indices of I_p
$\lvert I \rvert$	The cardinality of set of indices I
Eff	The set of efficient elements in the image space
$\mathrm{Eff}_{\mathrm{w}}$	The set of weakly efficient elements in the image space
Min	The set of minimal solutions in the pre-image space
$\mathrm{Min}_{\mathrm{w}}$	The set of weakly minimal solutions in the pre-image space
E_a	The set of the existing facilities
New	The set of the new facilities
$\mathbb{N}$	The set of the natural numbers
$\mathcal{N}_1(A)$	The Manhattan rectangular hull of a set A
$\mathcal{N}_\infty(A)$	The maximum rectangular hull of a set A
PPA	Proximal Point Algorithm

(P)	A general multiobjective optimization problem
(P_1)	A scalar approximation problem
(P_2)	A multiobjective approximation problem
$(\mathscr{P})$	The extended multiobjective approximation problem
$(\mathscr{P}_{\mathscr{L}})$	The linearized multiobjective approximation problem
$(\mathscr{P}_1)$	The multiobjective location problem
$(\mathscr{P}_2)$	The extended multiobjective location problem
$(\mathscr{P}_C)$	The multiobjective linear subproblem
(MFP)	The scalar multi-facility location problem

$(\mathsf{SFP}_{\mathsf{HD}})$	The single-facility approximation problem in higher dimensions
(MMFP)	The multiobjective multi-facility location problem

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CHAPTER 1

Introduction

Location problems appear in many variants and with different constraints depending on the application in practice, for instance in urban development, regional planning, social applications, engineering, production planning, economics or medicine. Such problems and corresponding algorithms are well-studied in the literature [30, 49, 52, 73, 81] and an overview is provided in [92].

In many problems of locational analysis the decision maker looks for new facilities such that the distances between the new facilities and existing facilities are minimal in a certain sense. An important question in modeling is how to choose the distances and the composition of the distances corresponding to the application. One possibility for the composition of the distances is the weighted sum (see for example [30]), which is known as the median problem. On the other hand, sometimes it is more convenient to minimize the largest distance to the existing facilities, e.g., in locating emergency service facilities. This kind of problems is called a center problem.

If someone uses the weighted sum for the composition of the distances, then the importance of each particular existing location is represented by a weight, which is usually chosen by the decision maker. However, it is often difficult to choose these weights. It is also possible that the solution of a scalar problem with some selected weights is not practicable. This leads to multiobjective location problems. By formulating a multiobjective location problem with the distances in the components of the vector-valued objective function, the decision maker gets an overview over the whole solution set, even on special solutions of the scalar problems.

The decision maker has to describe the distance functions in order to formulate the multiobjective location problem. The description may be done by norms, which in fact may be chosen in different ways. For instance, in the case of an air-way distance we choose the Euclidean norm. In many applications in locational analysis the road system is related to the Manhattan norm or to the maximum norm.

An extension of location theory is approximation theory, where some transformations of the variables

in the problem are included. Such problems occur, for example, if one is looking for approximate solutions for partial differential equations, which appear in many and different applications (see [49]). Furthermore, we study multi-facility location problems, where finitely many new facilities are to be located. We reformulate the multi-facility location problem as an approximation problem in higher dimensions.
On the other hand, one of the advantages of multiobjective models in locational analysis is that we can add further criteria to the mathematical model, where accruing costs could be considered as a specific example.
The aim of this work is to combine these extensions and to study a general class of multiobjective location and approximation problems. We support the decision makers in finding solutions corresponding to their preferences by developing new algorithms and solution methods concerning such problems.

We use tools from convex analysis in order to prove duality assertions for multiobjective location and approximation problems in the scope of this work. The application of duality results gives sometimes a perception of the original problem and provides frequently a simpler solution process.
Moreover, we apply methods, which reduce the complexity of the given multiobjective optimization problem. The complexity rises essentially by increasing the number of the objectives. A simpler optimization problem can be obtained by eliminating the non-consequential or non-crucial criteria and possibly including it in the restrictions. Other approaches suggest a decomposition of the multiobjective optimization problem to a family of subproblems and then to study the relationships between the solutions of the original problem and the solutions of the subproblems. Through such decompositions, we may obtain some minimal solutions or all minimal solutions of the original problem with a simpler and faster process.
Many authors have worked on these decomposition methods during the last decades and several interesting results concerning minimal solutions of multiobjective optimization problems have been obtained. A main research point is how the sets of minimal solutions of the original and reduced (or extended) problems act together, see for example [38, 78, 80, 95, 96, 99, 113] from the point of view of reducing methods, and [42, 85] from the perspective of adding new criteria to a given multiobjective optimization problem.

In location problems the number of the objectives is often greater than the dimension of the pre-image space. Other examples where the number of the objectives is greater than the dimension of the pre-image space can occur in decision making theory. Therefore, it is very convenient to derive algorithms computing the set of efficient solutions in the pre-image space in difference to the image approach in Benson's algorithm (see [10]). The dimension of the pre-image space and the number of the criteria play a crucial role in finding relationships between minimal and weakly minimal solutions of a given multiobjective optimization problem, see [113] and related results in [36, 78, 86, 96, 99].

The overarching goal of this thesis is to study extended models of multiobjective location and approximation problems. This includes several specific goals: The first goal is to formulate a generalized Lagrange dual problem for the extended multiobjective location and approximation problem and to prove duality assertions.
Second, we develop a convenient decomposition method for this class of problems and derive new duality-based implementable algorithms. These algorithms generate minimal and weakly minimal solutions of extended multiobjective location problems.
Furthermore, the third goal is to derive algorithms for solving scalar and multiobjective multi-facility location problems by using the proximal point algorithm.
Our main purpose is to generate impressive results which support the decision making process in many fields of applications, such as medicine, industry, landscape and urban planning.

This work is structured as follows:

- We begin with an introduction to location theory in Chapter 2. Different models and a literature review are given. Then we introduce the model of an extended multiobjective location and approximation problem.

- In Chapter 3, we give the mathematical and analytical background of the notions and tools used in the thesis.

- We study different solution concepts of multiobjective optimization problems in Chapter 4. Furthermore, we introduce decomposition and reducibility methods for multiobjective optimization problems and contribute with some new results, that are important for a characterization of minimal solutions of extended multiobjective location problems.

- In Chapter 5, we apply Lagrange-duality techniques to derive duality assertions for the extended model given in Chapter 2. In the second part of this chapter, we use a characterization of the set of minimal solutions of a multiobjective location problem, as well as Pareto reducibility introduced by Popovici [96, 99], in order to prove that the set of weakly minimal solutions of the multiobjective location problem coincides with the rectangular Manhattan hull of the existing facilities.

- In Chapter 6, we derive a partition algorithm for the set of minimal solutions of a multiobjective location problem that we apply for developing implementable decomposition algorithms for solving extended multiobjective location problems.

- In Chapter 7, we apply the well-known proximal point algorithm in order to solve a scalar multi-facility location problem, which is converted to a scalar approximation problem in higher dimensions. We also give an interactive procedure based on the proximal point algorithm for the scalar multi-facility location problem in order to solve multiobjective location and approximation problems.

CHAPTER 2

Modeling in Location Theory

In location theory, we essentially differentiate between two branches. The first one is discrete location theory, like the problem of distributing the departments and the belonging facilities of a hospital and health center, which can be a question of saving lives. Such models are also called network location problems, which are discrete location problems. To know more about discrete location problems see for example [24, 91]. The second one, which is studied in this work, is continuous location theory.

We consider the location problem here as follows: We are looking for one new location x in the plane, with respect to p existing locations $a^1, \cdots, a^p$. Our aim is to minimize the distances from the new location x to each of the existing locations a^i, for $i = 1, \ldots, p$. Minimizing the distances is not the only criterion which can be studied, but it is the most used by formulating location problems, because minimizing the distances represents minimizing the length of the route, the transport cost, fuel consumption and environment pollution.

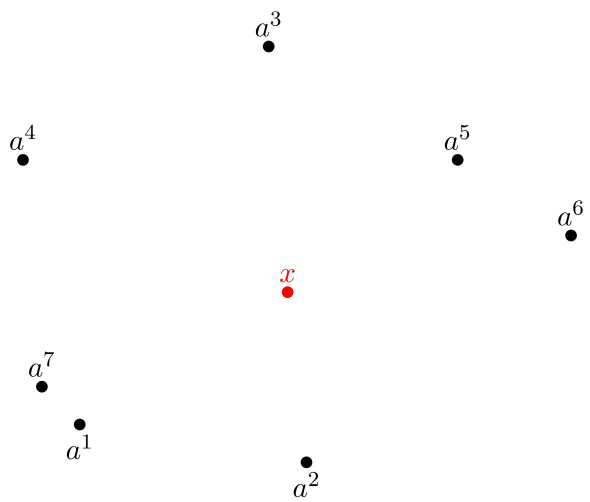


Figure 2.1: Finding a new facility x with respect to the given facilities $a^1, \ldots, a^7$.

Two questions arise now: How do we define these distances, and how do we associate these distances together in the objective function. Concerning the first question, in this work we define the distances by norms (see Definition 3.10). The examples in Section 3.1.2 show the different motivations in order to decide what is the most suitable norm for setting the right model, which depends essentially on the nature of the facilities and its environment. We can also choose different norms in the same model, as we see going along this chapter. Apart from the norm, more general distance functions can be used such as gauges, especially when the distances between the facilities are not symmetric.
Answering the second question, i.e., the kind of composing the distances, is very crucial for solving the location problem. Examples of objective functions in location theory are: Minimizing the weighted sum of the distances, minimizing the largest distance or considering every distance as a component of a vector-valued objective function.
For a more comprehensive overview, we refer to Hamacher [52] and Hamacher and Nickel [53]. They introduced five criteria for the classification of location problems represented by means of five positions, which can be shortly explained as the following:

- The number of the new facilities: Single-facility or multi-facility problem.
- The type of the problem: Planar (in $\mathbb{R}^2$), network (a discrete problem) or $\mathbb{R}^n$.
- Specialities: Restrictions such as barriers, forbidden regions or capacity limitation.
- The type of the distance function: Norms (e.g. the Euclidean norm or the maximum norm), or using other general distance functions like gauges.
- The type of the objective function: Minimizing the weighted sum of the distances, minimizing the largest distance, or considering a vector-valued objective function.

More information about classifications and modeling of location problems as well as solution algorithms can be found in [27, 44, 52, 53, 81, 91].

We introduce different models representing scalar and multiobjective locations problems in the following sections. To this end and throughout this work, we use the following notation for any index set $\{1, \ldots, n\}$ with $n \in \mathbb{N}$:

$$I_n := \{1, \ldots, n\}.$$

2.1 Scalar Single-Facility Location Problems

As we mentioned above, the **single-facility location problem** is represented in the search for a new facility $x \in \mathbb{R}^2$ with respect to the existing facilities $a^1, \ldots, a^p \in \mathbb{R}^2$. Some facilities can be represented

by a line segment or closed regions (see [113]). The existing facilities are always represented by a single point a^i from the considered space in this study.
However, after determining the suitable norm we still have to associate these distances together in order to formulate a location problem. Thus, the choice of the type of the objective function plays an important and crucial role in location theory.
Formulating a planar SFLP as a **median location problem** implies using the sum to associate these distances:

$$\begin{cases} \text{Minimize} & \sum_{i=1}^{p} w_i \|x - a^i\| \\ \text{subject to} & x \in \mathbb{R}^2, \end{cases} \tag{2.1}$$

where $\|\cdot\|$ denotes a norm on $\mathbb{R}^2$ and $w_i \geq 0, i \in I_p$ are weights being determined by the decision maker. Weights represent the importance of the particular facility. For example, when the facility a^i represents some ward in a city, then the weight w_i may represent the population density or the tourist attraction of a^i.
As for the case, where the new location x must possibly be centered to all existing facilities, i.e., must still be optimal for the largest distance (also called the worst case scenario), then the maximum of the distances $\|x - a^i\|$ is to be minimized:

$$\begin{cases} \text{Minimize} & \max_{1 \leq i \leq p} w_i \|x - a^i\| \\ \text{subject to} & x \in \mathbb{R}^2. \end{cases} \tag{2.2}$$

The problem (2.2) is called a **center location problem**.
Many algorithms and solution methods are derived to solve the problems (2.1) and (2.2) with different norms, see for example [27, 52, 55, 81, 91].

However, these problems can also be generalized or extended. Some facilities do not fit in the wanted model, unless we do some transformations $A_i, i \in I_p$ such as rotations of the street nets or movement in a specific direction.
Including such transformations is very interesting, like using it in finding approximate solutions for partial differential equations (see for example [49, Section 4.1.4]). Section 2.5 introduces a corresponding application in the radiotherapy treatment.
For understanding how such transformations work, the following two cases can be introduced as an example:

- If the facility $\tilde{a}_i$ has some deformation, then for $x, \tilde{a}_i \in \mathbb{R}^2$ (or $\mathbb{R}^n$) we apply the transformation $A_i \in L(\mathbb{R}^2, \mathbb{R}^2)$ (where $L(X, Y)$ is the set of all linear and continuous operators from X into Y):

$$\|A_i(x - \tilde{a}^i)\| = \|A_i(x) - a^i\|,$$

for $i \in I_p$. If the original facility has no deformation, then A_i is equal to the unit matrix and $\tilde{a}^i = a^i$.

- More generally, if $x \in \mathbb{R}^n$ and $a^i \in \mathbb{R}^k$, then $A_i \in L(\mathbb{R}^n, \mathbb{R}^k), i \in I_p$.

In this thesis we study extensions of the problem (2.1). Some of our results are derived for models in general spaces (see Section 5.2), such that we formulate the problem in normed spaces (see Definition 3.10).

Let $(X, \|\cdot\|)$, $(V, \|\cdot\|)$ be normed spaces and $c : X \to \mathbb{R}$. We consider now the additional cost function c, which is not included or not related to the distances, for instance, additional operational costs, utility costs and overheads.

The consideration of the cost function c, together with the transformations A_i allow us to extend the median problem defined in (2.1) to the following **scalar approximation problem**:

$$(P_1) \qquad \begin{cases} \text{Minimize} & c(x) + \sum\limits_{i=1}^{p} \alpha_i \|A_i(x) - a^i\|_{(i)}^{\beta_i} \\ \text{subject to} & x \in D, \end{cases} \tag{2.3}$$

where $x \in X$, $a^i \in V$, $A_i \in L(X, V)$, $\alpha_i \geq 0$ and $\beta_i \geq 1$ for all $i \in I_p$, with $D \subset X$ is some restriction set. Furthermore, $\|\cdot\|_{(i)}$ $(i \in I_p)$ denote different norms in V.

Note the use of the powers β_i and the mixed norms $\|\cdot\|_{(i)}$ in the term $\|A_i(x) - a^i\|_{(i)}^{\beta_i}$. The motivation to work with such powers is to cover many applications appearing e.g. in the control theory. On the other hand, investigating models with mixed norms, or generally, mixed distance functions, has been studied in the literature, see [17, 18, 57, 64, 65, 94]. The ability of assigning a convenient distance function for every particular facility gives an advantage to the problem (2.3), especially in decision making by modeling real world problems.

A proposed solution method to solve (P_1) is the proximal point algorithm (PPA) based on the method of the partial inverse introduced by Spingarn in [105]. This method is presented in Chapter 3.

2.2 Multi-Facility Location Problems

The problem turns to be a **multi-facility location problem**, if the decision maker is going to locate more than one facility. It is represented in looking for a set of N new facilities $x^1, \dots, x^N \in \mathbb{R}^2$ with respect to p existing facilities $a^i \in \mathbb{R}^2$ such that the distances, between the new and the existing facilities as well as between the new facilities among themselves are minimal. As an example consider the problem of locating N new containers for recyclable waste with respect to p blocks of flats in a ward of a city, or the problem of locating N helicopter emergency stations in skiing area.

This kind of problems are also called N-Location problems, multi-Weber problems or multi-facility

location-allocation problems and were formulated first by Cooper in 1963 [22]. Note that a single-facility location problem with $N = 1$ is a special case of a multi-facility location problem.
In Chapter 7, we apply a version of proximal point algorithm in order to solve scalar as well as multiobjective multi-facility location problems, motivated by an example in [107] and [103].

2.3 Multiobjective Location Problems

The importance and the wide application spectrum of multiobjective optimization is well-known nowadays. Therefore, working with the tools of multiobjective optimization in the field of location theory is of a big interest to the researchers and decision makers, since the weights in the previous scalar problems are mostly difficult to be determined precisely.
Many authors worked on multiobjective location modelings, see [11, 16, 17, 20, 21, 46, 49, 53, 81, 86, 90, 113, 114] and the references therein. Our participation in this field focuses on working with decompositions and reducing methods for multiobjective location problems, see Chapter 4, Chapter 5, Chapter 6 and [1, 2, 3, 4].
Multiobjective location problems have a vector-valued objective function with the components $f_i(x) = \|x - a^i\|$, $i \in I_p$. Solution algorithms for characterizing minimal and weakly minimal solutions of multiobjective location problems are discussed in Section 5.3.
The consideration of a specific linear cost function is very practical and useful in many applications, as mentioned above. Multiobjective approximation problems including a vector-valued linear cost function $C : X \to \mathbb{R}^p$ are given in the form:

$$(P_2) \qquad \begin{cases} \text{Minimize} \quad f(x) = C(x) + \begin{pmatrix} \alpha_1 \|A_1(x) - a^1\|_1^{\beta_1} \\ \cdots \\ \alpha_p \|A_p(x) - a^p\|_p^{\beta_p} \end{pmatrix} \\ \text{subject to} \quad x \in D \end{cases} \tag{2.4}$$

are studied in [49], where $x \in X$ and X, D, a^i, α_i, A_i, β_i are defined as in (P_1) for all $i \in I_p$.
However, we deal with another new model in the frame of this study, which is introduced in the next section.

2.4 Extended Multiobjective Approximation Problems

We introduce a vector-valued objective function different to the model (2.4), which includes the locational components as well as the linear cost components $C_{p+j} : X \to \mathbb{R}$ $(j = 1, \cdots, m)$:

$$(\mathscr{P}) \qquad \begin{cases} \text{Minimize} \quad f(x) = \begin{pmatrix} \alpha_1 \|A_1(x) - a^1\|_1^{\beta_1} \\ \cdots \\ \alpha_p \|A_p(x) - a^p\|_p^{\beta_p} \\ C_{p+1}(x) \\ \cdots \\ C_{p+m}(x) \end{pmatrix} \\ \text{subject to} \quad x \in \mathcal{A}, \end{cases} \tag{2.5}$$

concerning some feasible set $\mathcal{A} \subset X$, where $x \in X$ and a^i, α_i, A_i, β_i are defined as in (P_1) for all $i \in I_p$. This model is very useful for the decision maker, since we include the cost functions in the vector-valued objective function. The advantage appears by applying scalarization methods. By using one vector-valued objective function the distance components and the cost components get more different components of the scalarizing functional, which is not the case by formulating models with two vectors such as (2.4). In addition to that, by using this structure it is possible to apply reducing methods and derive decomposition algorithms (see Section 4.3 and 6.2).
A special case of (2.4) was first introduced in [1].

There are many procedures to generate solutions of such multiobjective location and approximation problems. In this thesis we discuss the following approaches:

- Lagrange-duality techniques in Chapter 5.
- Decomposition and reduction methods in Chapter 6.
- A proximal point algorithm and the methods of partial inverse in Chapter 7.

Furthermore, the problem $(\mathscr{P})$ is able to be linearized for special norms and solved through multiobjective linear programming as shown also in Chapter 5.

2.5 Application in the Radiotherapy Treatment

There are many useful applications for multiobjective location and approximation problems given by $(\mathscr{P})$. We formulate a model in the radiotherapy treatment as multiobjective approximation problem.
In the radiotherapy, a tumor is planned to be treated through available beams $k = 1, \ldots, p$. Known research points are, for example, the selection of beam angles or computation of an intensity map for

each selected beam angle (intensity problem). These problems and other models were studied by many authors, for example see [31, 32, 104]. For intensity problems in particular see [32, 37, 52, 104].

We suppose that each beam consists of bixels $j = 1, ..., n$, and voxels, which are indexed by $i = 1, ..., m$. Also let (a_{ijk}) denote the dose deposited in voxel i at unit intensity for bixel j of beam k (or the rate at which radiation along sub-beam j in beam k is deposited into dose-point i), where (a_{ijk}) is positive for each (i, j, k).

These rates are patient-specific constants, and hence, the mapping between intensity (or fluence) and dose is linear.

Furthermore, we denote

- Dose deposition matrix A (defined by the values (a_{ijk})) by indexing rows by i and columns by (j, k).
- Beam intensity: $x \in \mathbb{R}^{np}$, x_{jk} represents the intensity of bixel j, $j = 1, ..., n$ of beam k, $k = 1, ..., p$.
- T represents the tumor, C represents critical organs (K critical organs or organs at risk (OARs) are represented by $C_1, ..., C_K$), N represents normal tissue,
- m: total number of voxels, $m = m_T + m_C + m_N$, where $m_C = m_{C_1} + \cdots + m_{C_K}$.
- A_T, A_C, A_N : A can be partitioned and reordered into sub-matrices $A_T \in \mathbb{R}^{m_T \times np}$, $A_C \in \mathbb{R}^{m_C \times np}$ and $A_N \in \mathbb{R}^{m_C \times np}$ (according to the rows corresponding to tumor, critical organ and normal tissue voxels (A_i: row i of A).

Now for the treatment planning suppose the following:

$TG \in \mathbb{R}^{m_T}$: desired dose to tumor voxels,

$TLB \in \mathbb{R}^{m_T}$: lower bounds on the dose to tumor voxels,

$TUB \in \mathbb{R}^{m_T}$: upper bounds on the dose to tumor voxels,

$CUB \in \mathbb{R}^{m_C}$: upper bounds on the dose to critical organ voxels,

$NUB \in \mathbb{R}^{m_N}$: upper bounds on dose to normal tissue voxels.

Desired dose distribution can not always be obtained due to physical limitations and trade-offs between the various conflicting treatment goals. Therefore, we choose a multiobjective characteristic of inverse planning. Hence, we consider the problem of finding the optimal intensity x of the dose:

$$\begin{cases} \text{Minimize} & f_R(x) \\ \text{subject to} & x \in \mathbb{R}^{np}_+, \end{cases} \tag{2.6}$$

where

$$f_R(x) := \begin{pmatrix} \|A_T x - TG\|_1 \\ \|A_N x\|_2 \\ \|(A_C x - CUB)_+\|_3 \end{pmatrix}, \tag{2.7}$$

$x \in \mathbb{R}^{np}_+$, $\|\cdot\|_i : \mathbb{R}^{np} \to \mathbb{R}$, $i = 1, 2, 3$ are norms.

The first criterion $\|A_T x - TG\|_1$ can be interpreted as the deviation from the prescribed dose to the tumor, $\|A_N x\|_2$ is the dose to the normal tissue, and $\|(A_C x - CUB)_+\|_3$ represents the overdose to the critical organ.

We observe that the model defined in (2.6) is a special case of the extended multiobjective location and approximation problem $(\mathscr{P})$. For solving the problem (2.6), we can apply a proximal point algorithm (see Section 3.4) and an interactive procedure like proposed in Algorithm 7.6.

Remark 2.1.

1. The following additional restrictions (aside from $x \in \mathbb{R}^{np}_+$) can be considered for the model defined in (2.6):

$$\begin{aligned} TLB \leq A_T x &\leq TUB \\ A_C x &\leq CUB. \end{aligned}$$

2. Some other models use the squared Euclidean norm $\|x\|_2^2 = \sqrt{\sum_{i=1}^{np} x_i^2}$ for describing the average deviation for doses in the above described criteria. This model is also a special case of $(\mathscr{P})$, but we point to the fact that the proximal point algorithm studied in this work is appropriate for the special case $\beta_i = 1$. For other versions of the proximal point algorithm appropriate for the case $\beta_i > 1$ (see [49, Section 4.2.1] or [107]).

3. It is also possible to extend the model (2.6) to a multiobjective approximation problem with certain additional criteria. Algorithm 7.6 can be also applied for solving such an extended model.

CHAPTER 3

Convex Analysis

3.1 Basic Concepts

In this section, we mainly recall the mathematical concepts which are used in this work.
We set that the concept of a linear space is well known, and refer that we are dealing throughout with real linear spaces only.

3.1.1 Convex Sets and Convex Functions

Convex sets and convex functions have a great importance in optimization theory. The useful properties of convex sets and the differential ability of convex functions make the search for a minimum much easier. Of course, not all the models in the applications deal with convexity, but when that is the case, it is much easier to guarantee the existence of solutions and to set algorithms which deliver optimal solutions for the problem.

Definition 3.1. *Let S be a subset of a linear space X. S is called* ***convex*** *if $\alpha x + (1-\alpha)y \in S$ whenever $x, y \in S$ and $\alpha \in [0, 1]$.*

We call the set $[x, y] := \{\alpha x + (1-\alpha)y \in S\}$ the **line segment** connecting the points x and y. So geometrically, a set S is convex if and only if the line segment of each two points of S is completely included in S.

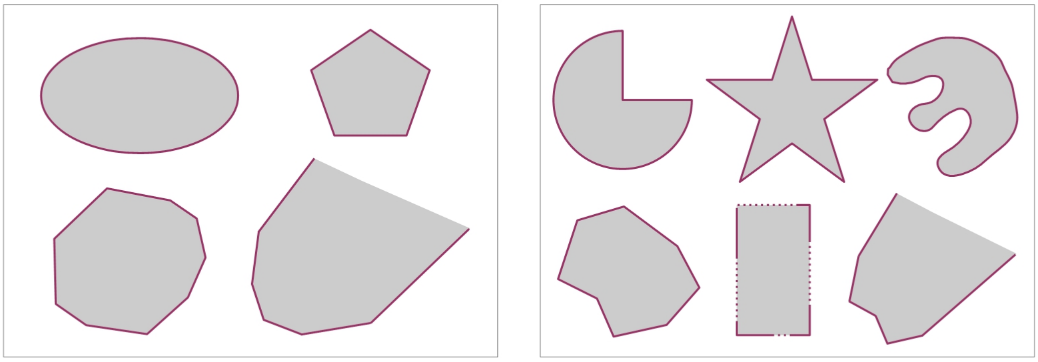


Figure 3.1: Examples for convex sets (group left) and nonconvex sets (group right).

Examples 3.2.

(1) The whole space X, the empty set and any singleton set are all convex sets.

(2) Hyperplanes and halfspaces in $\mathbb{R}^n$ are convex sets.

(3) The intersection of any collection of convex sets is convex.

(4) If A and B are convex sets, then their **sum** (also called the **Minkowski sum**)

$$A + B := \{a + b \mid a \in A,\ b \in B\}$$

is convex. Note that if A={a} then the sum $\{a\} + B$ is usually written $a + B$. The set $a + B$ is a convex set whenever B is convex. Also the set $-A$ is convex, whenever A is convex.

(5) Figure 3.1 visualizes some examples for convex as well as non convex sets.

Let X be a linear space, a **convex combination** of the points $x^i \in X$ for $i = 1, \cdots, k$, is defined by $x := \sum_{i=1}^k \alpha_i x^i$ with $\sum_{i=1}^k \alpha_i = 1$ and $\alpha_i \geq 0$ for $i = 1, \cdots, k$. The set of all convex combinations of the points x^i for $i = 1, \cdots, k$ is called the **convex hull** of the points $x^1, \cdots, x^k$. We write

$$\operatorname{conv}(\{x^1, \cdots, x^k\}) := \{x \in X \mid x = \sum_{i=1}^k \alpha_i x^i \text{ with } \alpha_i \geq 0\ (i = 1, \cdots, k) \text{ and} \sum_{i=1}^k \alpha_i = 1\}. \quad (3.1)$$

Accordingly, the convex hull of a set A is the set

$$\operatorname{conv} A := \bigcap\{C \subset X \mid A \subset C, C \text{ is convex}\}. \quad (3.2)$$

Certainly, the set A is generally not convex and $\operatorname{conv} A$ is the smallest convex set containing A.
An important class of convex sets are convex cones. Thus, we introduce next the concept of a cone and its corresponding properties.

Definition 3.3 (Cones)**.** *Let K be a nonempty subset of a linear space X, K is called a* **cone** *if $\alpha x \in K$ whenever $x \in K$ and $\alpha \geq 0$, i.e., if $\alpha\, K \subset K$ holds.*

If K is a cone, then we have $0 \in K$. A cone $K \subset X$ is called **convex**, if for all $x^1, x^2 \in K$, the relation $x^1 + x^2 \in K$ holds. In other words, a cone K is convex if $K + K \subset K$.
A cone K is said to be **nontrivial** or **proper**, if $K \neq \{0\}$ and $K \neq X$. Furthermore, a cone K is **pointed**, if $K \cap (-K) = \{0\}$.

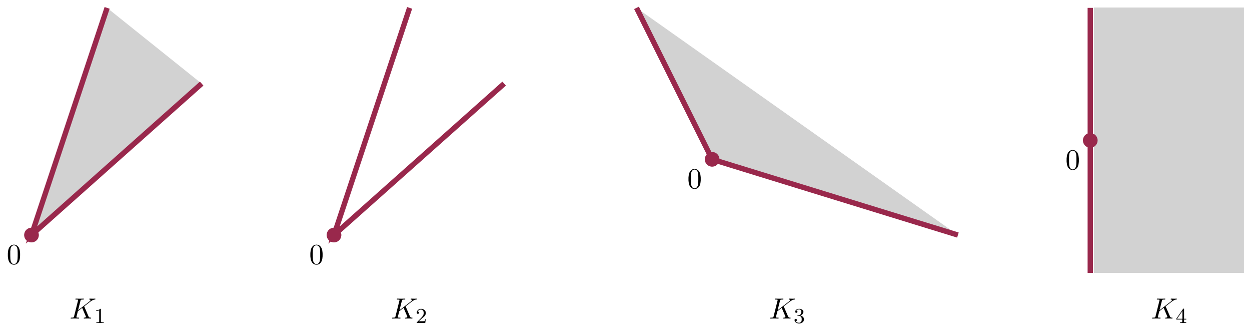


Figure 3.2: Examples of cones in $\mathbb{R}^2$.

Figure 3.2 presents some examples of cones, where K_1, K_3 are convex, proper and pointed cones, K_2 is a proper and pointed cone but not convex, K_4 is a proper and convex cone but not pointed.
Cones with these properties can generate order relations, for instance an order relation in $\mathbb{R}^n$. This makes cones of great interest in the theory of multiobjective optimization.

Before we call up the definition of convex functions, it is important to be clear about the idea of **extended real-valued functions**. In this work location and approximation problems are studied considering some restrictions. Using such extended real-valued functions is very important and useful, since it can change a restricted optimization problem into a free (not restricted) optimization problem. Also generally by working with some other operators like directional derivative, some functions come out with values in the set of extended real numbers.

An extended real-valued functions can be described as $f : X \to \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$, where X is real linear space. That means we allow the function in some cases to take the values $-\infty$ and $+\infty$, in this work rather $+\infty$.

Remark 3.4. In order to deal with these values we arrange for $\alpha \in \mathbb{R}$ that: $\alpha < +\infty,\ \alpha \pm \infty = \pm\infty$ and for $\alpha > 0$: $\alpha \cdot (+\infty) = +\infty$. We set also $\infty\ + (-\infty) = +\infty,\ 0 \cdot (+\infty) = -\infty$. To learn more about the nature of those operations in $\mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ see [56].

As an example for that we take the indicator function, which will be used later in our study.

Example 3.5. Consider the set $S \subset X$, the indicator function concerning the set S is a mapping $\mathcal{X}_S : X \to \mathbb{R} \cup \{+\infty\}$ with

$$\mathcal{X}_S(x) := \begin{cases} 0 & \text{for} \quad x \in S, \\ +\infty & \text{for} \quad x \in X \setminus S. \end{cases} \tag{3.3}$$

To illustrate the use of the indicator function in converting a restricted optimization problem into a free one we consider the following general optimization problem concerning the objective function $f : X \to \mathbb{R}$, which is restricted by a feasible set $S \subset X$:

$$\min_{x \in S} f(x). \tag{3.4}$$

This problem becomes a free optimization problem by extending the objective function f to a new function $\breve{f} : X \to \mathbb{R} \cup \{+\infty\}$ with $\breve{f}(x) := f(x) + \mathcal{X}_S(x)$. We get the free problem

$$\min_{x \in X} \breve{f}(x). \tag{3.5}$$

Moreover, we remind for the function $f : X \to \mathbb{R} \cup \{+\infty\}$ that the set

$$\text{dom } f := \{x \in X \mid f(x) < +\infty\} \tag{3.6}$$

is called the **domain** of f and the effective domain when f is extended real-valued.

Definition 3.6. *Let X be a linear space and let $f : X \to \mathbb{R} \cup \{+\infty\}$. The function f is called **convex**, if for all $x, y \in X$ and for all $\alpha \in [0, 1]$:*

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y). \tag{3.7}$$

*Furthermore, the function f is called **concave** if the function $-f$ is convex.*

Definition 3.7. *Let X be a linear space and let $f : X \to \mathbb{R} \cup \{+\infty\}$. The function f is called **quasi convex**, if the level set $L_{\leqslant}(r) := \{x \in X \mid f(x) \leq r\}$ is convex for all $r \in \mathbb{R}$.*

Since a big part of this work is dealing with multiobjective functions, we speak about a generalized type of convex vector-valued functions by means of an ordering cone in Section 4.2.

Examples 3.8.

(1) There are functions, which are neither convex nor concave. The only functions, which are convex and concave at the same time are linear functions.

(2) More generally, affine and polyhedral functions are convex [62].

(3) Norms and distance functions are convex (See Section 3.1.2).

(4) The indicator function $\mathcal{X}_S$ is convex, whenever S is convex.

There is a strong relationship between convex sets and convex functions, which can be realized through the next definition.

Definition 3.9. *Let X be a linear space and let $f : X \to \mathbb{R} \cup \{+\infty\}$. The set*

$$epi f := \{(x, r) \in X \times \mathbb{R} \mid f(x) \leq r\} \tag{3.8}$$

*is called the **epigraph** of f.*

We can easily prove that f is a convex function if and only if $epi f$ is a convex set, (for the proof see for example [50]).

3.1.2 Distance Functions

This work is dealing with location theory, therefore it will be convenient to take a big view at distance functions. The main purpose in location problems is to minimize the distances between the new facilities and the existing facilities, and the used tool to formulate the corresponding objective function is an appropriate distance function. Since the main distance functions in this work are norms, we start with the definition of norms and Banach spaces and get to know the open sets in this environment.

Normed linear Spaces

The algebraic structure of linear spaces is not enough to present some analytic notions as open sets and convergence, this brings us to the concept of linear normed spaces, which provide such additional structures. Next, we define norms and introduce some important examples and properties.

Definition 3.10. *Let X be a linear space. A **norm** on X is a function $\|\cdot\| : X \to \mathbb{R}$ which assigns to each element x in the space X a real number $\|x\|$ in such a manner that for all $x, y \in X$ and for all $\alpha \in \mathbb{R}$:*

(1) $\|x\| = 0 \Leftrightarrow x = 0$ *(definiteness),*

(2) $\|\alpha x\| = |\alpha| \, \|x\|$ *(positive homogeneity),*

(3) $\|x + y\| \leq \|x\| + \|y\|$ *(the triangle inequality).*

*We call $(X, \|\cdot\|)$ a **normed space**.*

From the properties (2) and (3), we conclude that $\|x\| \geq 0$ for all $x \in X$ and that every norm is a convex function.
We notice that a normed space is a metric space with respect to the induced metric defined by

$$\forall x, y \in X : \quad d(x, y) := \|x - y\| \,. \tag{3.9}$$

A complete normed space is called a **Banach space**, i.e., if it is complete as a metric space with a metric defined in (3.9). For instance, the linear space $\mathbb{R}^n$ with a norm $\|\cdot\|$ is a Banach space, also the set $C(X, \mathbb{R})$ of all continuous functions defined on a metric space X is a real Banach space.
However, a Banach space is a linear space with a topological structure. But we still have to keep in mind that a linear space equipped with a topology is generally not a Banach space, not even a normed space.
A special important class of normed linear spaces are inner product spaces.

Definition 3.11. *Let X be real linear space. An inner product is a mapping $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$, which satisfy the next conditions, for all x, y and z in X and scalars $\alpha \in \mathbb{R}$:*

(1) $\langle x, y \rangle = \langle y, x \rangle$,

(2) $\langle x, x \rangle \geq 0$, *and* $\langle x, x \rangle = 0 \Leftrightarrow x = 0$,

(3) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$,

(4) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$.

The space $(X, \langle \cdot, \cdot \rangle)$ is called an inner product space (or pre-Hilbert space).

Note that inner products over complex spaces is slightly different, particularly in (1) and (4).
The standard example is the inner product on $\mathbb{R}^n$ defined for $x, y \in \mathbb{R}^n$ by

$$\langle x, y \rangle = \sum_{j=1}^{n} x_j y_j. \tag{3.10}$$

If $\langle \cdot, \cdot \rangle$ is an inner product on a linear space X, then for all x, y in X it holds the CauchySchwarz inequality:

$$|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle.$$

Any inner product on a linear space X defines the norm

$$\|x\| := \sqrt{\langle x, x \rangle}. \tag{3.11}$$

Thus, an inner product space is also a normed linear space.

A **Hilbert Space** is a complete linear space with an inner product. So a Banach space is a Hilbert space when the norm is defined from an inner product. It holds also the parallelogram equality for any vectors x and y in a Hilbert space X:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \tag{3.12}$$

Conversely, a normed space is an inner product space if and only if the parallelogram equality is fulfilled.
In a normed linear space $(X, \|\cdot\|)$ the **open ball** and the **closed ball** with the center $x \in X$ and the radius $r > 0$ are defined as the sets

$$B(x,r) := \{y \in X \mid \|x-y\| < r\}, \tag{3.13}$$

$$B[x,r] := \{y \in X \mid \|x-y\| \leq r\}, \tag{3.14}$$

respectively. In particular, $B[0,1]$ is the unit ball. It is clear that the open ball and the closed ball are convex sets.
Consider a subset S of $(X, \|\cdot\|)$. An element $x^0 \in S$ is called an interior point of S if there exists a positive real number r such that $B(x^0, r) \subset S$. The set

$$\operatorname{int} S := \{x \in S \mid \exists\, r > 0 : B(x^0, r) \subset S\} \tag{3.15}$$

is called the interior of S. The set S is said to be open if $\operatorname{int} S = S$. We define also the closure of a set $S \subset X$ by

$$\operatorname{cl} S := \{x \in X \mid \exists\, \{x_n\}_{n=1}^{\infty} \text{ with } x_n \in S, n \in \mathbb{N}, \text{and } \lim_{n\to+\infty} x_n = x\}. \tag{3.16}$$

S is said to be closed if $\operatorname{cl} S = S$. It is obvious that $\operatorname{int} S$ is an open set and $\operatorname{cl} S$ is a closed set.
Furthermore, the set $\operatorname{cl} A \setminus \operatorname{int} A =: \operatorname{bd} A$ is called the boundary of A.
In a liner space X, a point $x^0 \in A \subset X$ is called an algebraic interior point of A, if for every $y \in X$ there is an $\alpha > 0$ with $[x^0 - \alpha y, x^0 + \alpha y] \subset A$. The set of all algebraic interior points of A is called the core of A and is denoted by $\operatorname{core} A$.

We introduce now some examples of norms in different spaces concentrating on norms which are important for location theory.

Example 3.12 (Euclidean norm)**.** The Euclidean norm in $\mathbb{R}^n$ represents the length of the vector $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ in the form

$$\|x\|_2 := \sqrt{x_1^2 + \cdots + x_n^2}. \tag{3.17}$$

The Euclidean norm is the most known norm in $\mathbb{R}^n$. It allows the movement in all directions, which is not the most case that we face in the application, but rather in measuring bee-line distances which are also known as the crow flies distances.

Example 3.13 (Manhattan norm, city block norm or rectangular norm)**.** It is defined for all $x \in \mathbb{R}^n$ by

$$\|x\|_1 := \sum_{i=1}^{n} |x_i|. \tag{3.18}$$

In location theory the Manhattan norm plays a crucial important role, in this work as well. Apart from its use in the city networks, the Manhattan norm is also used in machine engineering and the branch of robotics.

Example 3.14 (Maximum norm)**.** The Maximum norm is defined for all $x \in \mathbb{R}^n$ by

$$\|x\|_\infty := \max\{|x_1|, \cdots, |x_n|\}. \tag{3.19}$$

The norms in Examples 3.12 - 3.14 belong to a family of norms called p-norms.

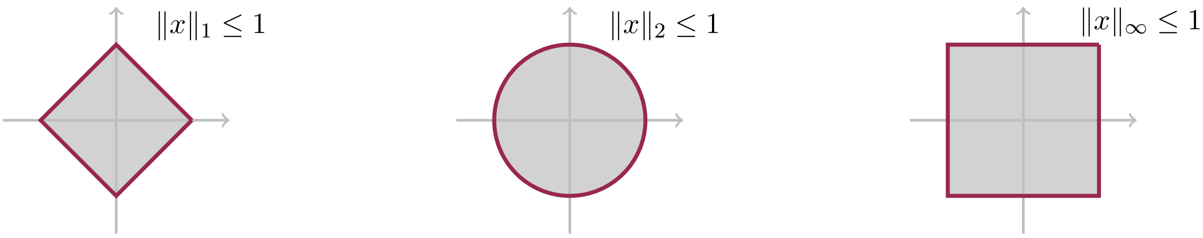


Figure 3.3: The unit balls for the Manhattan norm, the Euclidean norm and the maximum norm in $\mathbb{R}^2$.

Let $1 \leq p \leq \infty$, then the p-norm is defined for all $x \in \mathbb{R}^n$ through

$$\|x\|_p := \begin{cases} \left(\sum\limits_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}, & \text{for } 1 \leq p < \infty \\ \max\{|x_1|, \cdots, |x_n|\}, & \text{for } p = \infty. \end{cases} \tag{3.20}$$

Obviously, for $p = 1$ we get the Manhattan norm and for $p = 2$ we obtain the Euclidean norm. The limit $p \to \infty$ in (3.20) implies the maximum norm (Example 3.14). Examples for unit balls of special p-norms are given in Figure 3.3.

The previous examples presented norms in finite-dimensional spaces. Example 3.15 is about a norm in an infinite-dimensional space.

Example 3.15 (l^p norm)**.** Choose a value of $p \geq 1$, and let $l^p = l^p(\mathbb{N})$ denote the set of all sequences $x := \{a^n\}_{n=1}^{\infty}$ of complex numbers (indexed by the positive integers $\mathbb{N}$) for which $\sum_{n=1}^{\infty} |a^n|^p < \infty$. In l^p space a norm is define for $x = \{a^n\} \in l^p$ by

$$\|x\|_{(p)} := \left(\sum_{n=1}^{\infty} |a^n|^p\right)^{\frac{1}{p}}. \tag{3.21}$$

We can include the choice $p = \infty$ by modifying this definition in the expected way:

$$l^\infty = \left\{ x = \{a^n\}_{n=1}^\infty : \sup_{n\in\mathbb{N}} |a^n| < \infty \right\} \tag{3.22}$$

and for $x = \{a^n\}_{n=1}^\infty \in l^\infty$:

$$\|x\|_{(\infty)} := \sup_{n\in\mathbb{N}} |a^n| \,. \tag{3.23}$$

3.1.3 Dual Spaces

For a linear space X the algebraic dual space of X is given by

$$X' := \left\{ \acute{x} : X \to \mathbb{R} \mid x' \text{ is linear} \right\}, \tag{3.24}$$

and for a linear topological space X the topological dual space of X is given by

$$X^* := \{ x^* : X \to \mathbb{R} \mid x^* \text{ is linear and continuous} \} \,. \tag{3.25}$$

A linear functional $x' : X \to \mathbb{R}$ is said to be bounded, if there exists a number $M \geq 0$ with $|\acute{x}(x)| \leq M \|x\|$ for all $x \in X$. There is an equivalence between linear bounded and linear continuous functionals.
The norm of $x^* \in X^*$ is defined as

$$\|x^*\|_* := \sup_{x\neq 0} \frac{|x^*(x)|}{\|x\|}. \tag{3.26}$$

It is easy to check that $\|\cdot\|_*$ is a norm on the space X^*. We say that $\|\cdot\|_*$ is the dual norm of $\|\cdot\|$. Furthermore, it holds

$$\|x^*\|_* = \sup_{x\neq 0} \frac{|x^*(x)|}{\|x\|} = \sup_{\|x\|=1} |x^*(x)| = \sup_{\|x\|\leq 1} |x^*(x)| = \sup_{\|x\|\leq 1, x\neq 0} \frac{|x^*(x)|}{\|x\|}. \tag{3.27}$$

The nature of the set X^* of all linear continuous functionals on X will be described in the following theorem:

Theorem 3.16 ([50, Theorem 3.1])**.** *Let $(X, \|\cdot\|)$ be a normed space and let X^* be the set of all linear bounded functionals on X. The set X^* itself is a linear space and a normed space with the norm $\|x^*\|_*$. The generalized Schwarz's inequality holds:*

$$|x^*(x)| \leq \|x^*\|_* \, \|x\| \quad (\textit{for all } x \in X,\ x^* \in X^*). \tag{3.28}$$

It can be proved that the dual space $(X^*, \|\cdot\|_*)$ is always complete, i.e., it is always a Banach space. The dual space X^* has also a dual space, which is called the double dual space and denoted by X^{**}. The double dual space is also a Banach space with the norm $\|\cdot\|_{**}$. The normed space $(X, \|\cdot\|)$ is called reflexive when $X = X^{**}$ noticing that the inclusion $X \subset X^{**}$ always holds (see [50, Theorem 3.5]). The spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ and all Hilbert spaces are reflexive (where $\mathbb{C}$) is the set of complex numbers.

The Dual Cone

Now we present some additional properties concerning cones. Let X be a normed space and let $K \subset X$ be a cone.
The set

$$K^* = \{x^* \in X^* \mid \forall x \in K \;:\; x^*(x) \geq 0\} \tag{3.29}$$

is called the **dual cone** of K.

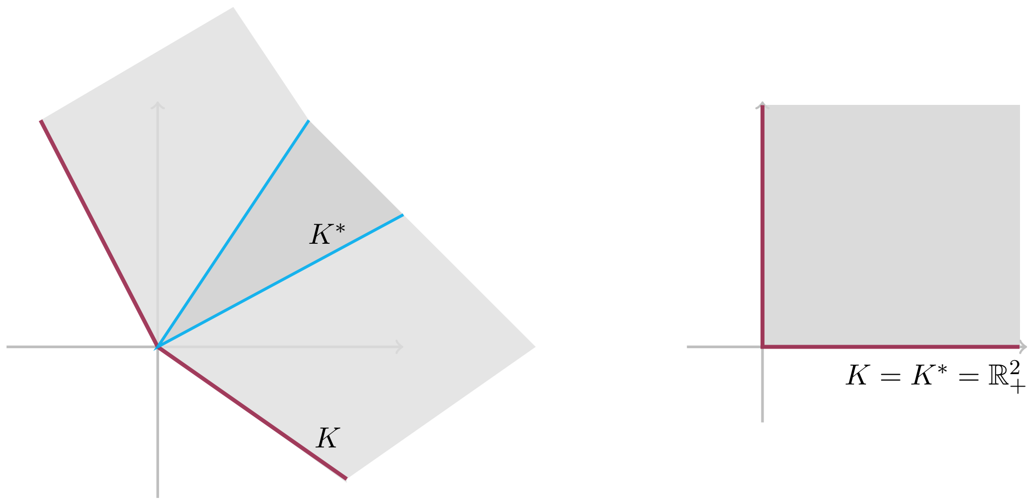


Figure 3.4: Examples for dual cones in $\mathbb{R}^2$.

Examples 3.17.

1. We consider the standard ordering cone in $\mathbb{R}^p$:

$$\mathbb{R}^p_+ = \{y \in \mathbb{R}^p \mid \forall i = 1, \cdots, p :\; y_i \geq 0\}, \tag{3.30}$$

   then

$$(\mathbb{R}^p_+)^* = \{z^* \in (\mathbb{R}^p)^* = \mathbb{R}^p \mid \forall y \in \mathbb{R}^p_+ :\; (z^*)^T y \geq 0\} = \mathbb{R}^p_+.$$

   In particular, $(\mathbb{R}^2_+)^* = \mathbb{R}^2_+$ as we see in Figure 3.4.

2. If $K = \mathbb{R}^p$, then $K^* = \{z^* \in (\mathbb{R}^p)^* = \mathbb{R}^p \mid \forall y \in \mathbb{R}^p :\; (z^*)^T y \geq 0\} = \{0\}$. Conversely, if $K = \{0\}$, then $K^* = \{z^* \in (\mathbb{R}^p)^* = \mathbb{R}^p \mid (z^*)^T 0 \geq 0\} = \mathbb{R}^p$.

3. An additional example is given in Figure 3.4.

From the previous examples we observe that $(K^*)^* = K$.
Furthermore, the **normal cone** N_M of the set M at the point x^0 is the set:

$$N_M(x^0) := \begin{cases} \{x^* \in X^* \mid \forall x \in M \;:\; x^*(x - x^0) \le 0\} & \text{if} \quad x^0 \in M, \\ \emptyset & \text{otherwise.} \end{cases} \tag{3.31}$$

We can make the observation that, if x^0 is an interior point of the set M, then the normal cone of M at x^0 is $N_M(x^0) = \{0\}$.

Orthogonality

Geometrical properties, like orthogonality and projection, can well be described in Hilbert spaces through the structure of the inner product. The property of orthogonality is defined as follows.

Definition 3.18. *Let X be a Hilbert space. For $x, y \in X$ we say that x is **orthogonal** to y, denoted by $x \perp y$, if $\langle x, y\rangle = 0$. For the sets $A, B \subset X$, we say that $A \perp B$ if $\langle x, y\rangle = 0$ for all $x \in A$ and all $y \in B$. Furthermore, we define for a set $A \subset X$ the **orthogonal complement** through*

$$A^\perp := \{x \in X \mid \forall y \in A : \; x \perp y\}. \tag{3.32}$$

We can show for any subset $A \subset X$ that $A^\perp$ is a closed linear subspace of the Hilbert space X, also it is clear that $A \cap A^\perp = \{0\}$.

Theorem 3.19 (Complementary subspaces [60])**.** *If A is a complete subspace of the inner product space X, then every $x \in X$ can be uniquely represented as:*

$$x = u + v \;:\quad u \in A, v \in A^\perp;$$

in other words it holds the orthogonal decomposition $A \oplus A^\perp = X$. The subspaces A and $A^\perp$ are called complementary subspaces.

Theorem 3.19 also holds in the special case, if X is a Hilbert space and A is a closed subspace. For the existence and the uniqueness of optimal solutions of a general optimization problem in inner product spaces we introduce these two theorems.

Theorem 3.20 ([60, Theorem 21.1])**.** *Let X be an inner product space. If $S \neq \emptyset$ is a convex and complete subset of X (e.g. a complete subspace of X), then for $x \in X$ the problem $\|x - y\| \to \min_{y\in S}$ has a unique solution in S, i.e., there exist a unique element $y^0 \in S$ with $\|x - y^0\| \le \|x - y\|$ for all $y \in S$.*

In particular, the assumptions of Theorem 3.20 are also fulfilled, when X is a complete space and S is a nonempty, convex and closed subset of X. For the proof of Theorem 3.20, the completeness of S and the parallelogram equality play the main role.

Theorem 3.21 ([60, Theorem 21.2]). *Let S be a linear subspace of an inner product space $(X, \langle\cdot,\cdot\rangle)$. If the problem $\|x-y\| \to \min_{y\in S}$ for some element $x \in X$ has ever a solution $y^0 \in S$, then y^0 is the only solution in S and $x - y^0 \perp S$.*

3.1.4 Separation Theorems for Convex Sets

There are many famous fundamental theorems, which play a role in the background of our results, such as Zorn's Lemma and Hahn-Banach-Theorem.
One of the important theorems from Functional Analysis, which is based on the Hahn-Banach-Theorem, is the following separation theorem. Such a separation theorem is an important tool for deriving characterizations of solutions of vector optimization problems (see Theorem 4.12) and for the proofs of duality assertions (see Theorem 3.37, Theorem 5.12).

Theorem 3.22 (Separation Theorem: [50, Theorem 5.11]). *Let X be a real normed space, $A, B \subset X$ be nonempty convex sets with $\operatorname{int} A \neq \emptyset$ and $\operatorname{int} A \cap B = \emptyset$. Then the two sets A and B can be separated through a non-trivial continuous linear functional $x^* \in X^*$. If A and B are open, then the separation is strict and made by a continuous linear functional $x^* \in X^* \setminus \{0\}$ and a real number α*

$$\forall s \in A,\ \forall t \in B : x^*(s) < \alpha < x^*(t). \tag{3.33}$$

Theorem 3.23 (Separation Theorem: [73, Theorem 3.18]). *Let X be a real locally convex space[i], A be a nonempty closed convex subset of X. Then $x \in X \setminus A$ if and only if there exist a continuous linear functional $x^* \in X^*$ and a real number α with*

$$\forall s \in A: \quad x^*(x) < \alpha \leq x^*(s). \tag{3.34}$$

3.2 Differentiability Properties of Functions

In this section, we recall the definitions of directional derivatives in order to define the subdifferential of convex function, especially of the norm. The subdifferential is our main tool to formulate the optimality conditions in Section 7.1.

3.2.1 Directional Derivative

The directional derivative is an extension of the well-known derivative in the real space. As an example of the literature see [50, Definition 3.24],[73, Definition 2.12.] or the famous book [100].

Definition 3.24 (Gâteaux Derivative). *Let X be a linear space, S a nonempty subset of X, Y a normed space, and let $f : S \to Y$ be a mapping. For $x^0 \in S$, $h \in X$, the mapping f is called Gâteaux*

[i] X is a locally convex space, if the origin $0 \in X$ has a neighborhood base formed by convex sets, see [73, Definition 1.33]

differentiable at x^0 in the direction h if there exists an $\varepsilon > 0$ with $[x^0 - \varepsilon h, x^0 + \varepsilon h] \subset S$ and if the limit

$$f'(x^0, h) := \lim_{t \to 0} \frac{f(x^0 + th) - f(x^0)}{t} \tag{3.35}$$

*exist. $f'(x^0, h)$ is called the **Gâteaux derivative** of f at x^0 in the direction h. If this limit exists for all $h \in X$, then f is called Gâteaux differentiable at x^0, and $f'(x^0, \cdot)$ is called the Gâteaux derivative of f at x^0.*

The next definition does not consider the whole interval $[x^0 - \varepsilon h, x^0 + \varepsilon h]$. We see later that this case could be sometimes enough for our study.

Definition 3.25 (Right-Hand side Gâteaux Derivative)**.** *Consider the assumptions in Definition 3.24. For $x^0 \in S,\ h \in X$, if $\varepsilon > 0$ exists with only $[x^0, x^0 + \varepsilon h] \subset S$ and the if the limit*

$$f'_+(x^0, h) := \lim_{t \to +0} \frac{f(x^0 + th) - f(x^0)}{t} \tag{3.36}$$

*exist, then f is called directionally differentiable at x^0 in the direction h and $f'_+(x^0, h)$ is called the **right-hand side direction derivative** (or direction derivative) of f at x^0 in the direction h.*
In the same way, if $[x^0 - \varepsilon h, x^0] \subset S$, and $t \to -0$ in the limit, then we talk about left-hand side direction derivative denoted by $f'_-(x^0, h)$.

We illustrate some properties of Gâteaux derivative and the direction derivative (in Definitions 3.24 and 3.25) and relationships between them. It is easy to show that the following statements are true:

- $f'(x^0, \cdot)$ is positively homogeneous but not necessarily linear. (A mapping $A : X \to Y$ is called positively homogeneous if $A(\alpha x) = \alpha A(x)$ for all $x \in X, \alpha \in \mathbb{R}_+$).
- Consider the assumptions in Definition 3.24. The function f is Gâteaux differentiable at $x^0 \in S$ in direction h, if and only if f is right-hand side and left-hand side directionally differentiable at x^0 in direction h and $f'_+(x^0, h) = f'_-(x^0, h)$. It holds $f'(x^0, h) = f'_+(x^0, h) = f'_-(x^0, h)$.
- The function f is left-hand side differentiable at x^0 in direction h, if and only if f is right-hand side differentiable at x^0 in direction $-h$. And the equality $f'_-(x^0, h) = f'_+(x^0, -h)$ always holds.

In the literature, there are some other generalized definitions of derivatives. Fréchet derivative, for instance, is a generalization of the directional derivative in Banach spaces and is definitely a stronger condition than Gâteaux derivative.

Now, we present some differential properties of function under convexity assumptions. Let X be a linear space, $S \subset X$ is convex. If $f : S \to \mathbb{R} \cup \{+\infty\}$ is convex, then f is right hand-side and left-hand side Gâteaux differentiable at every algebraic interior point x^0 of S with $f(x^0) \in \mathbb{R}$, in every direction $h \in \mathbb{R}$, and the Gâteaux derivative mapping $f'(x^0, \cdot) : X \to \mathbb{R}$ is linear (see [50, Theorem 3.32]). Moreover, it holds the monotonicity of the of the difference quotient. The next theorem presents an important inequality for deriving optimality conditions under convexity assumptions.

Theorem 3.26 ([50, Theorem 3.33]). *Let X be a linear space, $S \subset X$ a convex set with $S = core(S)$ (i.e., S consists only of algebraic interior points). Moreover, let the function $f : X \to \mathbb{R}$ be Gâteaux differentiable in every point in S , then the following statements are equivalent:*

1. *f is convex.*
2. *$f'(x, \cdot)$ is linear for all $x \in S$, and the following subgradient inequality holds for all $x, x^0 \in S$:*

$$f'(x^0, x - x^0) \le f(x) - f(x^0). \tag{3.37}$$

In the following example, we compute the Gâteaux derivative of certain convex functions, namely the norm. The norm as a distance function is the main tool for studying our locations problems in Chapters 5,6,7. In particular, the differentiability properties of the norm appear by formulating the optimality conditions in Chapter 7.

Example 3.27 ([50]). Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space with the norm $\|x\| := \sqrt{\langle x, x \rangle}$. Consider the function $f(x) := \|x - x^0\|^2$ for a fixed point $x^0 \in X$.
We compute the Gâteaux derivative $f'(x, h)$ for the function f for $x, h \in X$. To this end, we compute the following quotient for $t \in \mathbb{R}$:

$$\begin{aligned}
\frac{f(x+th) - f(x)}{t} &= \frac{\langle x + th - x^0, x + th - x^0 \rangle - \langle x - x^0, x - x^0 \rangle}{t} \\
&= \frac{\langle x - x^0, x + th - x^0 \rangle + \langle th, x + th - x^0 \rangle - \langle x - x^0, x - x^0 \rangle}{t} \\
&= \frac{\langle x - x^0, x - x^0 \rangle + \langle x - x^0, th \rangle + \langle th, x + thx^0 \rangle - \langle x - x^0, x - x^0 \rangle}{t} \\
&= \frac{2\langle x - x^0, th \rangle + \langle th, th \rangle}{t} = \frac{2t\langle x - x^0, h \rangle + t^2\langle h, h \rangle}{t} \\
&= 2\langle x - x^0, h \rangle + t\langle h, h \rangle.
\end{aligned}$$

By computing the limit we get

$$f'(x, h) = \lim_{t \to 0} \frac{f(x^0 + th) - f(x^0)}{t} = 2\langle x - x^0, h \rangle.$$

3.2.2 The Subdifferential of a Convex Function

The subdifferential of convex functions play a crucial role in nonlinear optimization. Generally, distance functions are not differentiable. The search for more general notion for differentiability leads to the subdifferential of a function, which turn to be an important tool to formulate the optimality conditions, especially in location theory (see Chapter 7).

Definition 3.28 (The Subdifferential [73]). *Let X be a real Banach space, $f : X \to \mathbb{R} \cup \{+\infty\}$, $x, x^0 \in X$ at x^0 in direction x. The set*

$$\partial_G f(x^0) := \{x^* \in X^* \mid \forall x \in X : \; x^*(x) \leq f'_+(x^0, x)\} \tag{3.38}$$

is called the ***subdifferential*** *of f at x^0, the elements of $\partial_G f(x^0)$ are called* ***subgradients****.*
If $f : X \to \mathbb{R} \cup \{+\infty\}$ is ***convex****, then the subdifferential of f at $x^0 \in \operatorname{dom} f$ is the set*

$$\partial f(x^0) := \{x^* \in X^* \mid \forall x \in X : \; x^*(x - x^0) \leq f(x) - f(x^0)\}. \tag{3.39}$$

Remark 3.29.

1. We note again that the elements of the subdifferential are functionals from the dual space which we call the subgradients, that means the subdifferential $\partial(\cdot)$ is a set-valued operator.

2. If f is convex, then $\partial_G f(x^0) = \partial f(x^0)$ for $x^0 \in X$

The next theorem shows, under which assumptions the subdifferential of a convex function exists.

Theorem 3.30 ([50, Theorem 5.12]). *Let X be a Banach space, $x^0 \in X$ and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a convex functional. If $f(x^0) < +\infty$ and f is continuous at x^0, then*

$$\partial f(x^0) \neq \emptyset.$$

In order to prove this theorem one can apply the separation theorem (e.g. Theorem 3.22).

3.2.3 Subdifferential Calculus

To apply the optimality conditions in some applications and algorithms, we have to compute the subdifferential of the sum of functions. Therefore, we recall the next theorem concerning the sum rule of convex functions.

Theorem 3.31 (The Subdifferential Sum Rule [50, Theorem 5.13]). *For $n \geq 2$ let $f_1, \cdots, f_n : X \to \mathbb{R} \cup \{+\infty\}$ be convex functionals on a Banach space X. If for an element x^0 the values $f_1(x^0), \cdots, f_n(x^0) < +\infty$ exist and if $f_1, \cdots, f_{n-1}$ are continuous at x^0, then it holds for all $x \in X$ that*

$$\partial\left(\sum_{i=1}^{n} f_i(x)\right) = \sum_{i=1}^{n} \partial f_i(x). \tag{3.40}$$

Note that the right side of the equation (3.40) is understood as the Minkowski sum of the sets of the subdifferentials $\partial f_i(x)$ of the functions f_i.
Now, we compute the subdifferential of some special functions, which we use later in our algorithms. We start with the subdifferential of the norm. Note that norms are convex functions.

Lemma 3.32 ([50]). *Let X be a Banach space. The norm is subdifferentiable and*

$$\begin{aligned} \forall x \in X \setminus \{0\} \ &: \ \partial \left\|\cdot\right\|(x) = \{x^* \in X^* \mid \langle x^*, x\rangle = \|x\| \textit{ and } \|x^*\|_* = 1\}, \\ \textit{at } x = 0 \ &: \ \partial \left\|\cdot\right\|(x) = \{x^* \in X^* \mid \|x^*\|_* \leq 1\}. \end{aligned}$$

Furthermore, for proving the duality statements for extended multiobjective approximation problems in Chapter 5, we introduce the following Lemma for computing the subdifferential of a norm with an exponent.

Lemma 3.33 ([5]). *Let X be a Banach space. If $\beta > 1$ and $x \neq 0$, then*

$$\forall x \in X : \quad \partial \left(\frac{1}{\beta} \left\|\cdot\right\|^{\beta}\right)(x) = \left\{x^* \in X^* \mid \langle x^*, x\rangle = \|x\|^{\beta} \textit{ and } \|x^*\|_* = \|x\|^{\beta-1}\right\}.$$

Moreover, we need to compute the subdifferential of the indicator function (defined in Example 3.5) for some convex set $S \subset X$.
Since S is convex the indicator function $\mathcal{X}_S$ is also convex, then according to (3.39) it holds for the subgradients $x^* \in X^*$ of $\mathcal{X}_S$ at a point $x^0 \in S$ the following:

$$\forall x \in X : \ x^*(x - x^0) \leq \mathcal{X}_S(x) - \mathcal{X}_S(x^0). \tag{3.41}$$

If $x \in S$ then $\mathcal{X}_S(x) = 0$, also $\mathcal{X}_S(x^0) = 0$ for $x^0 \in S$. We conclude that $x^*(x - x^0) \leq 0$ for all $x \in S$. For $x \in X \setminus S$ the inequality in (3.41) is also fulfilled. If $x^0 \notin S$ then there is no $x^* \in X^*$, for which the inequality in (3.41) holds for all $x \in X$, i.e., $\partial \mathcal{X}_S(x^0) = \emptyset$. We get

$$\partial \mathcal{X}_S(x^0) = \begin{cases} \{x^* \in X^* \mid \forall x \in S : x^*(x - x^0) \leq 0\} & \text{if } x^0 \in S \\ \emptyset & \text{if } x^0 \notin S. \end{cases} \tag{3.42}$$

Comparing with the definition of the normal cone in (3.31), we observe that the subdifferential of the indicator function $\partial \mathcal{X}_S(x^0)$ of a convex set S at a point $x^0 \in S$ coincides with the normal cone N_S of the set S at x^0:

$$\partial \mathcal{X}_S(x^0) = N_S(x^0).$$

3.2.4 Optimality Conditions

The search for the optimal solution and answering the question of its existence is one of the main focuses in the optimization theory. In this section, the right-hand side direction derivative (see

Definition 3.25), Gâteaux-derivative and the subdifferential of convex functions are used in order to introduce different formulations of necessary and sufficient optimality conditions. As mentioned above, convexity assumptions yield the existence of Gâteaux-derivative (see also [50, Theorem 3.32]).
The following necessary and sufficient optimality condition is given by a variational inequality.

Theorem 3.34 ([50]). *Let X be a linear space, S a convex set and $f : S \to \mathbb{R}$ a convex function. Then for $x, x^0 \in S$ it holds:*

1. *x^0 is a minimal solution of the nonlinear optimization problem $\min\limits_{x\in S} f(x)$, if and only if for all $x \in S$:*
$$f'_+(x^0, x - x^0) \geq 0.$$
2. *If S is a linear subspace and f is Gâteaux-differentiable, then x^0 is a minimal solution of the nonlinear optimization problem $\min\limits_{x\in S} f(x)$, if and only if for all $x \in S$:*
$$f'(x^0, x) = 0.$$

In the following theorem, a necessary and sufficient optimality condition for minimal solutions of $\min_{x\in S} f(x)$ are given. We use this kind of optimality condition later in our results in Chapter 7.

Theorem 3.35 ([50, Theorem 5.14]). *Let X be a Banach space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be convex with $f(x) < +\infty$. Then for $x^0 \in S$ it holds:*
$x^0 \in X$ is a minimal solution of the nonlinear optimization problem $\min\limits_{x\in X} f(x)$, if and only if
$$0 \in \partial f(x^0). \tag{3.43}$$

The proof follows easily from (3.39).

3.3 Duality for Convex Optimization Problems

In the linear and convex optimization, it can be easier and shorter to solve the dual problem for a given original problem. When the dual problem is found and a solution of it exists, then we get more information and perhaps a solution of the original problem. For instance, if the primal problem is an approximation problem, we can get a dual problem with a linear objective function, which is definitely easier to compute.
Let A and B be nonempty sets, and let $f : A \to \mathbb{R}$, $g : B \to \mathbb{R}$. In general, consider the following problems

$$(P): \quad \inf_{x\in A} f(x) =: \alpha, \tag{3.44}$$
$$(D): \quad \sup_{u\in B} g(u) =: \beta. \tag{3.45}$$

We say that a pair of problems are dual [48], if

$$\forall x \in A, \forall u \in B: \; g(u) \le f(x). \tag{3.46}$$

The Property (3.46) is called **weak duality** of the pair (P) and (D). If

$$\alpha = \inf_{x\in A} f(x) < \beta = \sup_{u\in B} g(u),$$

then we state that there exists a duality gap between (P) and (D). If α, β are finite with $\alpha = \beta$ and one of the two values $\alpha = f(x^0)$ or $\beta = g(u^0)$ can be obtained feasibly (for some $x^0 \in A$ or $u^0 \in B$), then we speak of a **strong duality** between (P) and (D).
This means, for all $x^0 \in A$, $u^0 \in B$ with $f(x^0) = g(u^0)$ we get the next properties for the solution of the pair (P) and (D):

$$\begin{aligned} &\alpha = \beta \\ &x^0 \text{ is the solution of } (P) \\ &u^0 \text{ is the solution of } (D). \end{aligned}$$

It is important to examine the relationships between α and β and to look for sufficient conditions for $\beta = \alpha$.
There are several possibilities to construct a dual problem, such as Fenchel's duality or Lagrange duality.
Here we introduce a general approach to Lagrange-duality, for example see [50].

Definition 3.36. *Let A and B be non-empty sets, and let $L : A \times B \to \mathbb{R}$. Then $(x^0, u^0) \in A \times B$ is said to be* ***saddle point*** *of L with respect to $A \times B$, if*

$$\max_{u\in B} L(x^0, u) = L(x^0, u^0) = \min_{x\in A} L(x, u^0).$$

For example, consider $L : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with $L(x, u) = x^2 - u^2$, then it s easy to see that $(0, 0)$ is saddle point of L with respect to $\mathbb{R} \times \mathbb{R}$.

The function $L : A \times B \to \mathbb{R}$ is called the **Lagrange function** and will be used in order to form the next optimization problem.

Now for the primal problem

$$(P) \quad \inf_{x\in A} f(x) = \alpha; \qquad -\infty \le \alpha \le +\infty \tag{3.47}$$

we assume that: For all $x \in A$ it holds $f(x) = \sup_{u\in B} L(x, u)$.
Analogously for the appropriated dual problem

$$(D) \quad \sup_{u\in B} g(u) = \beta; \qquad -\infty \le \beta \le +\infty \tag{3.48}$$

and for $g(u)$ we assume that: For all $u \in B$ holds $g(u) = \inf_{x \in A} L(x, u)$.
Both problems (P) and (D) can now be formulated as the following:

$$(P) \quad \inf_{x \in A} \sup_{u \in B} L(x, u) = \alpha, \tag{3.49}$$

$$(D) \quad \sup_{u \in B} \inf_{x \in A} L(x, u) = \beta. \tag{3.50}$$

We now present the main theorem of Lagrange duality (see [50, Theorem 5.19] and [125, Theorem 49.B]) including the existence of saddle points of the Lagrange function given in (3.49) and (3.50).

Theorem 3.37 (Main Theorem: Duality between (P) and (D)). *Let A and B be non-empty sets, $L : A \times B \to \mathbb{R}$. Then:*

I. ***Double Duality:*** *(D) and (P) are respectively equivalent to:*

$$(D) \quad \inf_{u \in B} \sup_{x \in A} -L(x, u) = -\beta,$$

$$(P) \quad \sup_{x \in A} \inf_{u \in B} -L(x, u) = -\alpha$$

and in this sense (P) is the dual problem of (D).

II. ***Weak Duality:*** *$\beta \leq \alpha$ always holds and for $x^0 \in A$ and $u^0 \in B$ with $f(x^0) = g(u^0)$ we get that $\alpha = \beta$ and x^0 is a solution of (P) and u^0 is a solution of (D).*

III. ***Duality:*** *(x^0, u^0) is saddle point of the Lagrange function L with respect to $A \times B$ if and only if x^0 is a solution of (P) and u^0 is a solution of (D) and $\alpha = \beta$ holds. In addition to that $\alpha = f(x^0) = L(x^0, u^0) = g(u^0) = \beta$ is fulfilled.*

IV. ***Existence Statements:*** *L has a saddle point with respect to the set $A \times B$, if the next six conditions hold:*

(C_1) *The set $A \subset X$ is closed and convex, where X is reflexive Banach space.*

(C_2) *The set $B \subset Y$ is closed and convex, where Y is reflexive Banach space.*

(C_3) *$x \to L(x, u)$ is convex and lower semi continuous over the set A for every $u \in B$.*

(C_4) *$u \to -L(x, u)$ is convex and lower semi continuous over the set B for every $x \in A$.*

(C_5) *A is bounded or*
$\exists u_0 \in B$ with $L(x, u_0) \longrightarrow +\infty$ for $\|x\| \longrightarrow +\infty$, $x \in A$.

(C_6) *B is bounded or*
$\exists x_0 \in A$ with $-L(x_0, u) \longrightarrow +\infty$ for $\|u\| \longrightarrow +\infty$, $u \in B$.

V. ***Strong Duality:***

a) *If the conditions* (C_1) – (C_5) *are fulfilled and* $\alpha < +\infty$ *holds, then* (P) *has a solution* $x^0 \in A$ *and* $\alpha = \beta$ *holds.*

b) *If the conditions* (C_1) – (C_4) *and* (C_6) *are fulfilled and* $\beta > -\infty$ *holds, then* (D) *has a solution* $u^0 \in B$ *and* $\alpha = \beta$ *holds.*

3.4 The Method of Partial Inverse (Spingarn)

In this section the method of partial inverse of Spingarn [105] is introduced, see also [88, 101]. This method is an instrument for developing different versions of the proximal point algorithm, which is an important tool for solving several problems in the approximation theory. Usually a **Proximal Point Algorithm** (shortly called PPA) uses this method to solve the optimality conditions of the corresponding problem.

The PPA is used to find the zero element of a set-valued maximal monotone operator $T : E \rightrightarrows E$, i.e. to solve the problem:

$$\text{Find} \quad v \in E \quad \text{s.t.} \quad 0 \in T(v), \tag{3.51}$$

where E is a Hilbert space.
The next definition illustrates the maximal monotone property for set-valued operators.

Definition 3.38. *Let* E *be a real Hilbert space with inner product* $\langle \cdot, \cdot \rangle$*. A set-valued operator* $T : E \rightrightarrows E$ *is said to be **monotone** when,*

$$\forall y \in T(x), \grave{y} \in T(\grave{x}),\ x, \grave{x} \in E : \quad \langle x - \grave{x}, y - \grave{y} \rangle \geq 0.$$

The operator T *is **maximal monotone**, if its graph is not strictly contained in the graph of any other monotone operator.*

For a maximal monotone operator T we consider the mapping

$$P := (I + cT)^{-1} : E \to E, \quad c > 0, \tag{3.52}$$

this mapping is called the proximal-mapping and has a single-valued image. This result can be found in [88], where the result says that maximal monotone operators have the property: For every $x \in E,\ c > 0$ there is a unique $P(x) \in E$ such that $x - P(x) \in cT(P(x))$, which is equal to the fact that the mapping $P = (I + cT)^{-1}$ is single-valued. The proximal-mapping is the basic of the iteration of the PPA, which takes for solving (3.51) a sequence $(c_n),\ c_n \in \mathbb{R}$ with $c_i > k > 0$ for all $i = 1, \dots, n$ and an arbitrary starting point $x^1 \in E$. Then the iteration

$$x^{n+1} := (I + c_n T)^{-1}(x^n) \tag{3.53}$$

either converges to a solution x^0 with $0 \in T(x^0)$ or $\|x^n\| \to \infty$.

The method of Spingarn [105] propose to choose complementary subspaces $\mathbb{A}$ and $\mathbb{B}$ of E, which means $E = \mathbb{A} \oplus \mathbb{B}$, with $\mathbb{A} = \mathbb{B}^\perp$ (see Definition 3.18 and Theorem 3.19), then to consider the problem of finding

$$y^0 \in \mathbb{A}, \; p^0 \in \mathbb{B} \quad \text{s.t.} \quad p^0 \in T(y^0). \tag{3.54}$$

In order to solve this problem Spingarn [105] has introduced the inverse $T_{\mathbb{A}}$ of T through the definition:

Definition 3.39 ([105])**.** *Let $\mathbb{A}$ and $\mathbb{B}$ be complementary subspaces of the Hilbert space E. Let T be a monotone set-valued operator defined on E. The partial inverse $T_{\mathbb{A}}$ of T with respect to the subspace $\mathbb{A}$ is a set-valued operator defined on the space E by: $u \in T_{\mathbb{A}}(v)$ if and only if there exist $y, p \in E$ with $p \in T(y)$, $v = y_{\mathbb{A}} + p_{\mathbb{B}}$ and $u = p_{\mathbb{A}} + y_{\mathbb{B}}$. This implies that $T_{\mathbb{A}}$ has the graph:*

$$graph(T_{\mathbb{A}}) = \{(y_{\mathbb{A}} + p_{\mathbb{B}},\; p_{\mathbb{A}} + y_{\mathbb{B}}) : p \in T(y)\}. \tag{3.55}$$

If we compare (3.55) with the graph of T:

$$graph(T) = \{(y_{\mathbb{A}} + y_{\mathbb{B}},\; p_{\mathbb{A}} + p_{\mathbb{B}}) : p \in T(y)\}, \tag{3.56}$$

we see that $y_{\mathbb{B}}$ and $p_{\mathbb{B}}$ are exchanged. One can also observe that if $\mathbb{B} = \{0\}$ and $\mathbb{A} = E$, then $T_{\mathbb{A}} = T$; and if $\mathbb{B} = E$ and $\mathbb{A} = \{0\}$, then $T_{\mathbb{A}}$ is equal to the inverse of T.

Furthermore, we mention one more important result of Spingarn, which describes the maximal monotonicity relation between the operator T and its partial inverse $T_{\mathbb{A}}$.

Theorem 3.40 ([105])**.** *The operator $T_{\mathbb{A}}$ is maximal monotone if and only if T is maximal monotone.*

The following theorem shows the idea of using the concept of the partial inverse by Spingarn.

Theorem 3.41 ([105])**.** *Let v be an element from E with $v = v_{\mathbb{A}} + v_{\mathbb{B}}$, where $v_{\mathbb{A}} \in \mathbb{A}$ and $v_{\mathbb{B}} \in \mathbb{B}$. Furthermore, let the operator $T : E \rightrightarrows E$ be monotone. Then*

$$0 \in T_{\mathbb{A}}(v) \Leftrightarrow v_{\mathbb{B}} \in T(v_{\mathbb{A}}). \tag{3.57}$$

The equation $0 \in T_{\mathbb{A}}(v)$ is equivalent to $v_{\mathbb{B}} \in T(v_{\mathbb{A}})$, where $v = v_{\mathbb{A}} + v_{\mathbb{B}}$ with $v_{\mathbb{A}} \in \mathbb{A}$ and $v_{\mathbb{B}} \in \mathbb{B}$. Applying of this equivalence on the iteration (3.53) leads to

$$\begin{aligned}
v^{k+1} &:= (I + T_{\mathbb{A}})^{-1} v^k \\
v^k &\in v^{k+1} + T_{\mathbb{A}}(v^{k+1}), \qquad v^k = y^k + p^k,\;\; y^k \in \mathbb{A},\; p^k \in \mathbb{B} \\
(y^k - y^{k+1}) + (p^k - p^{k+1}) &\in T_{\mathbb{A}}(y^{k+1} + p^{k+1}) \\
\underbrace{(y^k - y^{k+1}) + p^{k+1}}_{=:\tilde{p}^k} &\in T(\underbrace{y^{k+1} + (p^k - p^{k+1})}_{=:\tilde{y}^k})
\end{aligned}$$

Putting $\tilde{y}^k := y^{k+1} + (p^k - p^{k+1})$ and $\tilde{p}^k := (y^k - y^{k+1}) + p^{k+1}$ we get the equations

$$\tilde{p}^k \in T(\tilde{y}^k) \quad \text{with} \quad \tilde{y}^k + \tilde{p}^k = y^k + p^k \tag{3.58}$$

$$y^{k+1} := \tilde{y}^k_{\mathbb{A}} \quad p^{k+1} := \tilde{p}^k_{\mathbb{B}}. \tag{3.59}$$

Solving of (3.58) is called the **proximal step**, and solving of (3.59) is called the **projection step**.

CHAPTER 4

Multiobjective Optimization Problems

Multiobjective optimization (also known as vector optimization or multiple objective programming) is a very good approach for solving many real world problems occurring in operation research, industrial systems, networks, control theory, management sciences, decision making, and politics.

A ”good” solution for a multiobjective optimization problem based on a vector-valued objective function is originally proposed by the Irish economist Francis Ysidro Edgeworth (1881) and the Swiss economist Vilfredo Pareto (1896) [29], [93]. It is based on the idea of finding good compromises, instead of finding a single solution in the scalar case. In this work, we apply such Pareto optimal notions rather in the field of location and approximation theory.

Many books and works have introduced the foundations of multiobjective optimization such as [30, 47, 48, 73]; for a limited example of its different applications see [31, 34, 54, 61, 77]

In this Chapter, we introduce some concepts of the solutions in multiobjective optimization. Then a view of scalarizations for multiobjective optimization problems is given. Furthermore, we describe the decomposition and the reduction of multiobjective optimization problems and introduce some corresponding properties and results.

Now, let X be a linear space and let $f_i(x) : X \to \mathbb{R}, i \in \{1, \cdots, p\},\ p \in \mathbb{N}, p \geq 2$. We consider the vector-valued objective function $f : X \to \mathbb{R}^p$ with

$$f(x) = \begin{pmatrix} f_1(x) \\ \cdots \\ f_p(x) \end{pmatrix}. \tag{4.1}$$

Usually a general multiobjective optimization problem is described as

$$(P) \quad \begin{cases} \text{Minimize} & f(x) \\ \text{subject to} & x \in \mathcal{A}. \end{cases} \tag{4.2}$$

for a given nonempty feasible set $\mathcal{A} \subset X$. Given the emphasis of locational analysis and approximation theory within this work, X is considered to be simply $\mathbb{R}^n$.

In the following section we study the solutions concept concerning the problem (P), in order to understand the minimization in the image space $\mathbb{R}^p$.

4.1 Solution Concepts in Multiobjective Optimization

We introduce a number of well-known solution concepts for multiobjective optimization problems, see [30, 47, 48, 73].
In multiobjective optimization we use cones with certain properties as a generalization of the order relation in $\mathbb{R}$. Let $K \subset \mathbb{R}^p$ be a proper, convex, closed and pointed cone, K therefore characterizes a partial ordering relation in $\mathbb{R}^p$ and is called an *ordering cone* (or a positive cone).
The order relation generated by the cone K can be described as the following:

$$y^1 \leq_K y^2 \;\Leftrightarrow\; y^2 - y^1 \in K, \quad \text{for } y^1, y^2 \in \mathbb{R}^p.$$

The definition of the cone itself includes the reflexivity of the relation defined above, as $0 \in K$. The convexity of the cone K implies the transitivity, and K being pointed leads to the antisymmetry. The antisymmetry is omitted in some literature by setting the cone not to be necessarily pointed, but in this work the ordering cone will be always pointed. Often, the ordering cone used in $\mathbb{R}^p$ is the standard ordering cone, which is $\mathbb{R}^p_+$.

Now we present the first concept of a solution, the so called Pareto optimality.

Definition 4.1. *Let $F \subset \mathbb{R}^p$ and let $K \subset \mathbb{R}^p$ be a proper, convex, pointed and closed cone. An element $y^0 \in F$ is said to be **efficient** with respect to the cone K, if*

$$F \cap \big(y^0 - (K \setminus \{0\})\big) = \emptyset.$$

The set of all efficient elements in F with respect to the cone K is denoted by $\mathrm{Eff}(F, K)$.

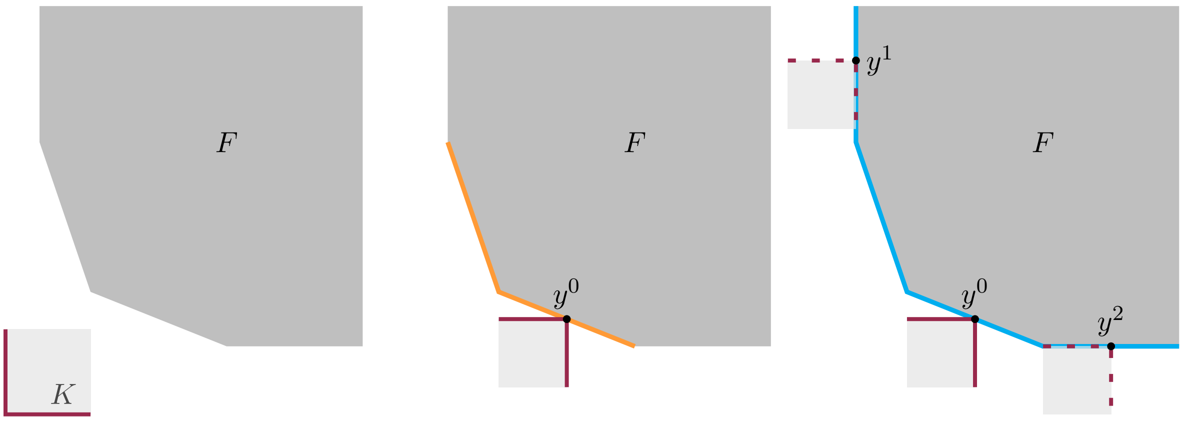
Remark 4.2. In the above definition, we did not use any index when denoting the set of efficient elements in F with respect to the cone K concerning the minimization problem by $\mathrm{Eff}(F, K)$. Later we use some indices in order to distinguish the minimization problem $\mathrm{Eff}_{\mathrm{Min}}$ from the maximization problem by using the symbol $\mathrm{Eff}_{\mathrm{Max}}$.

In the next definition we introduce weakly efficient elements under the assumption that the ordering cone has a nonempty interior; this is important from the mathematical point of view. Furthermore, numerical algorithms usually generate weakly efficient elements.

Definition 4.3 (Weak Efficiency)**.** *Let $F \subset \mathbb{R}^p$ and let $K \subset \mathbb{R}^p$ be a proper, convex, pointed and closed cone. Further let* $\operatorname{int} K \neq \emptyset$*. $y^0 \in F$ is said to be a **weakly efficient** element of F with respect to K, if*

$$F \cap \big(y^0 - \operatorname{int} K\big) = \emptyset.$$

The set of all weakly efficient elements in F with respect to the cone K is denoted by $\mathrm{Eff}_{\mathrm{w}}(F, K)$.



$F \subset \mathbb{R}^2,\ K = \mathbb{R}^2_+$ $\qquad$ $y^0 \in \mathrm{Eff}(F, K)$ $\qquad$ $y^0, y^1, y^2 \in \mathrm{Eff}_{\mathrm{w}}(F, K)$

Figure 4.1: The set $\mathrm{Eff}(F, K)$ and the set $\mathrm{Eff}_{\mathrm{w}}(F, K)$ of a set F w.r.t. $K = \mathbb{R}^2_+$.

Figure 4.1 gives a visualized example of the definitions above.
Let X be linear space. Now consider $f : \mathcal{A} \to \mathbb{R}^p$ for $\mathcal{A} \subset X$, which is partially ordered by the cone $K \subset \mathbb{R}^p$, and consider the multiobjective optimization problem (P) defined in (4.2). We take $F := f[\mathcal{A}] := \{f(x) \mid x \in \mathcal{A}\}$ in the Definitions 4.1 and 4.3.
An element $x^0 \in \mathcal{A}$, for which $f(x^0) \in \mathrm{Eff}(f[\mathcal{A}], K)$, is called a **minimal solution**. We denote the set of minimal solutions of $\mathcal{A}$ with respect to the objective function f and the cone K by $\mathrm{Min}(\mathcal{A}, f)$. This implies that

$$\mathrm{Min}(\mathcal{A}, f) := \{x \in \mathcal{A} \mid f(x) \in \mathrm{Eff}(f[\mathcal{A}], K)\}\,. \tag{4.3}$$

In the same way, we denote the set of **weakly minimal solutions** of $\mathcal{A}$ with respect to the objective function f and K by $\mathrm{Min}_{\mathrm{w}}(\mathcal{A}, f)$. This implies that

$$\mathrm{Min}_{\mathrm{w}}(\mathcal{A}, f) := \{x \in \mathcal{A} \mid f(x) \in \mathrm{Eff}_{\mathrm{w}}(f[\mathcal{A}], K)\}\,. \tag{4.4}$$

Using the previous solution concepts and their notations we formulate the problem (4.2) in the following way:

$$(P) \qquad \mathrm{Eff}(f[\mathcal{A}], K). \tag{4.5}$$

The formulations (4.5) and (4.2) are both used in the literature and in the following chapters.

Next we introduce further efficiency definitions, which are going to help us in the duality assertions.

Definition 4.4 (Proper Efficiency)**.** *Let $F \subset \mathbb{R}^p$ and let $K \subset \mathbb{R}^p$ be a proper, convex, pointed and closed cone. An element $y^0 \in F$ is said to be a* ***properly efficient*** *element of F with respect to K, if* *there exists a proper convex cone* $\tilde{K} \subset \mathbb{R}^p$ *with* $K \setminus \{0\} \subset \operatorname{int} \tilde{K}$ *such that* y^0 *is an efficient element with respect to* $\tilde{K}$*, i.e.,*

$$F \cap \left(y^0 - \left(\tilde{K} \setminus \{0\}\right)\right) = \emptyset.$$

The set of all properly efficient elements in F *with respect to the cone* K *is denoted by* $\mathrm{Eff}_{\mathrm{p}}(F, K)$.

Under the assumption $\operatorname{int} K \neq \emptyset$, it can be easily shown that every properly efficient element of a set F is an efficient element of F, and that every efficient element of F is a weakly efficient element of it with respect to a certain cone K.

$$\mathrm{Eff}_{\mathrm{p}}(F, K) \subset \mathrm{Eff}(F, K) \subset \mathrm{Eff}_{\mathrm{w}}(F, K). \tag{4.6}$$

We can describe proper efficiency differently under convexity assumptions through scalarization by linear continuous functionals λ^* belonging to the interior of the dual cone $\operatorname{int} K^*$, as defined below.

Definition 4.5 (Schönfeld [102])**.** *Let* $F \subset \mathbb{R}^p$ *be a convex set and let* $K \subset \mathbb{R}^p$ *be a proper, convex, pointed and closed cone. We call* $y^0 \in F$ *a properly efficient element of* F *with respect to the cone* K *(in the sense of Schönfeld), if an element* $\lambda^* \in \operatorname{int} K^*$ *exists with*

$$\forall\, y \in F: \quad \lambda^*(y^0) \leq \lambda^*(y).$$

The set of all properly efficient elements in F *with respect to the cone* K *in the sense of Schönfeld is denoted by* $\mathrm{Eff}_{\mathrm{pSch}}(F, K)$.

4.2 Scalarization

The idea of the scalarization is that we replace a multiobjective optimization problem by a surrogate scalar problem, i.e., an optimization problem with a real-valued objective function. The solutions of the multiobjective optimization problem can, under some assumptions, be characterized through the solutions of the scalar problem. This is very interesting, as the study of a scalar optimization problem is greatly developed. There are many possibilities to formulate the surrogate scalar problem (see e.g. [30, 37, 48]). One of the well-known methods is the weighted sum scalarization, where a multiobjective optimization problem like (4.2) can be transformed into a scalar problem like (2.1) by choosing some suitable weights. We also use this method in this work, namely in Algorithm 6.8.
In this section, we give some properties and conditions for characterizing properly efficient elements (cf. Definition 4.5) through scalarization (e.g. by using a scalarizing functional belonging to the dual cone). To this end we introduce monotonicity properties of a scalarization functional $z : \mathbb{R}^p \to \mathbb{R}$:

Definition 4.6. *Let* $K \subset \mathbb{R}^p$ *be a proper, convex, pointed and closed cone and let* $y^1, y^2 \in \mathbb{R}^p$*. The functional* $z : \mathbb{R}^p \to \mathbb{R}$ *is said to be:*

- ***K-monotone*** *increasing, if* $y^2 \in y^1 + K$ *implies* $z(y^1) \leq z(y^2)$.
- ***strictly K-monotone*** *increasing, if* $y^2 \in y^1 + (K \setminus \{0\})$ *implies* $z(y^1) < z(y^2)$.

The next example shows the monotonicity properties of elements belonging to the dual cone of K.

Example 4.7. We consider again the standard ordering cone in $\mathbb{R}^p$:

$$\mathbb{R}^p_+ = \{y \in \mathbb{R}^p \mid y_i \geq 0, \forall i = 1, \cdots, p\}.$$

According to Example 3.17 $(\mathbb{R}^p_+)^* = \mathbb{R}^p_+$. The element $z^* \in \mathbb{R}^p_+ \setminus \{0\}$ is a $\mathbb{R}^p_+$-monotone increasing functional. Actually by taking $y^2 \in y^1 + \mathbb{R}^p_+$ we infer that $z^*(y^2 - y^1) \geq 0$, having the definition of the dual cone in mind. It is therefore clear that $z^*(y^1) \leq z^*(y^2)$. On the other hand, $z^* \in \operatorname{int} \mathbb{R}^p_+$ is strictly $\mathbb{R}^p_+$-monotone increasing, because from $y^2 \in y^1 + (\mathbb{R}^p_+ \setminus \{0\})$ we get $z^*(y^2 - y^1) > 0$, i.e., $z^*(y^1) < z^*(y^2)$.

Theorem 4.8 ([48]). *Let $F \subset \mathbb{R}^p$ be a nonempty set, and $K \subset \mathbb{R}^p$ be a proper, convex, pointed and closed cone. For $y^0 \in F$ and $z : \mathbb{R}^p \to \mathbb{R}$ let $z(y^0) \leq z(y)$ for all $y \in F$. If z is strictly K-monotone, then $y^0 \in \operatorname{Eff}(F, K)$.*

For the other direction, we still need some convexity assumptions which are represented in the next definition and theorem.

Definition 4.9. *Let $K \subset \mathbb{R}^p$ be a proper, convex, pointed and closed cone and let $X \subset \mathbb{R}^n$ be a convex set. We call the function $f : X \longrightarrow \mathbb{R}^p$* ***K-convex****, if*

$$f(\alpha x^1 + (1 - \alpha)x^2) \in \alpha f(x^1) + (1 - \alpha)f(x^2) \; - K \tag{4.7}$$

holds for all $x^1, x^2 \in X$ and all $\alpha \in [0, 1]$.

Remark 4.10. For the special case $p = 1$ we get

$$\forall x^1, x^2 \in X, \forall \alpha \in [0, 1] : \quad f(\alpha x^1 + (1 - \alpha)x^2) \leq \alpha f(x^1) + (1 - \alpha)f(x^2),$$

which is the classic definition of the convexity of a function $f : X \longrightarrow \mathbb{R}$.

Now we investigate the convexity of the set $f[X]$ in the image space under the convexity assumptions of X and f.

Theorem 4.11 ([48, Theorem 2.11]). *Let $K \subset \mathbb{R}^p$ be a proper, convex, pointed and closed cone, let $X \subset \mathbb{R}^n$ be convex and let the function $f : X \longrightarrow \mathbb{R}^p$ be K-convex. Then the set $f[X] + K$ is convex.*

We introduce the following scalarization result, which states that for every minimal solution x^0 (with $f(x^0) \in \operatorname{Eff}(f[X], K)$), there exits a functional $\lambda^* \in K^* \setminus \{0\}$ such that x^0 is a minimal solution of the scalarized problem. See also [73, Theorem 5.4].

Theorem 4.12. *Let $K \subset \mathbb{R}^p$ be a proper, convex, pointed and closed cone with* $\operatorname{int} K \neq \emptyset$*, the set* $X \subset \mathbb{R}^n$ *is convex and let the function* $f : X \longrightarrow \mathbb{R}^p$ *be K-convex. Then*

$$f(x^0) \in \operatorname{Eff}(f[X], K) \Longrightarrow \Big(\exists \lambda^* \in K^* \setminus \{0\}, \quad \forall x \in X : \lambda^*(f(x^0)) \leq \lambda^*(f(x)) \Big).$$

Proof. Let $f(x^0) \in \operatorname{Eff}(f[X], K)$, then $f[X] \cap \big(f(x^0) - (K \setminus \{0\})\big) = \emptyset$. This implies

$$(f[X] + K) \cap \big(f(x^0) - (K \setminus \{0\})\big) = \emptyset.$$

The last statement holds, because supposing the contrary, i.e., supposing that

$$(f[X] + K) \cap \big(f(x^0) - (K \setminus \{0\})\big) \neq \emptyset$$

implies the existence of $x^1 \in X, k^1 \in K$ with $(x^1) + k^1 \in f(x^0) - (K \setminus \{0\})$

$$f(x^1) \in f(x^0) - (K \setminus \{0\}) - k^1.$$

It follows $f(x^1) \in f(x^0) - (K \setminus \{0\})$, since K is a convex and pointed cone. This contradicts the fact that $f(x^0) \in \operatorname{Eff}(f[X], K)$. Now we apply the separation Theorem 3.22 on the two sets

$$A := (f(x^0) - K) \quad \text{and} \quad B := f[X] + K,$$

where A is nonempty and convex with $\operatorname{int} A \neq \emptyset$, B is nonempty and convex, and $\operatorname{int} A \cap B = \emptyset$ (see Theorem 4.11).

Taking into account Theorem 3.22 there exists $\lambda^* \in \mathbb{R}^p \setminus \{0\}, \alpha \in \mathbb{R}$ such that for all $x \in X$ and all $k^1, k^2 \in K$:

$$\lambda^* \left(f(x^0) - k^1\right) \leq \alpha \leq \lambda^* \left(f(x) + k^2\right). \tag{4.8}$$

Since K is a cone we get $\lambda^* \in K^* \setminus \{0\}$: Otherwise, suppose that $\lambda^* \notin K^*$, then $\lambda^*(k) < 0$ for some $k \in K$. With some $n \in \mathbb{N}$ we make $\lambda^*(n \cdot k)$ small enough such that $\lambda^*(n \cdot k) < \alpha - \lambda^*(f(x))$ for some fixed $x \in X$, which contradicts (4.8). Consequently we obtain

$$\forall x \in X : \quad \lambda^*(f(x^0)) \leq \lambda^*(f(x)).$$

□

4.3 Decomposition of Multiobjective Optimization Problems

Often it is easier to solve complex problems using certain subproblems. We mentioned above that the complexity of a multiobjective optimization problem rises essentially by increasing the number of the objectives. We obtain a simpler optimization problem, for instance by eliminating some criteria and including it in the restrictions. However, in many papers decomposition methods for multiobjective

optimization problems are derived. In these papers the authors decompose a multiobjective optimization problem to a family of subproblems and study the relationships between the solutions of the original problem and the solutions of the subproblems, see [38, 78, 80, 95, 96, 99, 113]. Furthermore, in [42, 85] the authors added new criteria to a given multiobjective optimization problem.
Building on the aforementioned works, the aim of this section is to derive new results in order to obtain efficient elements by decomposing a multiobjective optimization problem. These results are used in Sections 6.2 in order to generate minimal solutions of extended multiobjective location problems through the decomposition algorithms derived in Sections 6.1 and 6.2.

Consider again the general multiobjective optimization problems (P) in (4.2) and the index set $I_p = \{1, \cdots, p\}$. For every nonempty selection of indices, $I \subset I_p$, the notation f_I will represent the function

$$f_I = (f_{i_1}, \cdots, f_{i_k}) : X \to \mathbb{R}^k,$$

where $I := \{i_1 < \cdots < i_k\}$.
Given any nonempty feasible domain $\mathcal{A} \subset X$, we consider the optimization problem:

$$(P_I) \quad \begin{cases} \text{Minimize} & f_I(x) \\ \text{subject to} & x \in \mathcal{A}. \end{cases} \tag{4.9}$$

We observe that (P_I) represents the multiobjective optimization problems (P) itself when $I = I_p$, and when $I = \{i\}$ then the corresponding (P_I) is a scalar problem with the objective function f_i. If $|I|$ is the cardinality of the set I, then for $0 < |I| < p$ the optimization problem (P_I) can be considered as a subproblem, i.e., a reduced problem obtained from $\left(P_{I_p}\right)$ by eliminating certain criteria.
As defined in Section 4.1 and for the ordering cone $K_I := \mathbb{R}^{|I|}_+$, the set of the minimal solutions of (P_I) is given by

$$\begin{aligned} \mathrm{Min}(\mathcal{A}, f_I) &= \left\{x \in \mathcal{A} \mid f_I(x) \in \mathrm{Eff}(f_I[\mathcal{A}], \mathbb{R}^{|I|}_+)\right\} && (4.10) \\ &= \left\{x \in \mathcal{A} \mid f_I[\mathcal{A}] \cap \left(f_I(x) - (\mathbb{R}^{|I|}_+ \setminus \{0\})\right) = \emptyset\right\}. && (4.11) \end{aligned}$$

One can easily check that for every subset I of I_p it holds that

$$\mathrm{Min}(\mathcal{A}, f_I) \subset \mathrm{Min}_{\mathrm{w}}(\mathcal{A}, f_I) \subset \mathrm{Min}_{\mathrm{w}}(\mathcal{A}, f_{I_p}). \tag{4.12}$$

In particular, for every $i \in I_p$, letting $I := \{i\}$ and identifying f_I with f_i we have

$$\mathrm{Min}(\mathcal{A}, f_i) = \mathrm{Min}_{\mathrm{w}}(\mathcal{A}, f_i) = \operatorname*{argmin}_{x \in \mathcal{A}} f_i(x). \tag{4.13}$$

Theorem 4.13 ([3]). *For every proper subset I of I_p we have*

$$\mathrm{Min}\left(\mathrm{Min}(\mathcal{A}, f_I), f_{I_p \setminus I}\right) \subset \mathrm{Min}(\mathcal{A}, f_{I_p}).$$

Proof. Let $I \subset I_p$ with $0 < |I| < p$ and let $x^0 \in \mathrm{Min}\left(\mathrm{Min}(\mathcal{A}, f_I), f_{I_p \setminus I}\right)$. Then

$$x^0 \in \mathrm{Min}(\mathcal{A}, f_I); \tag{4.14}$$

$$\nexists\, x \in \mathrm{Min}(\mathcal{A}, f_I) \;\text{ such that }\; f_{I_p \setminus I}(x) \in f_{I_p \setminus I}(x^0) - \mathbb{R}_+^{p-|I|} \setminus \{0\}. \tag{4.15}$$

Suppose to the contrary that $x^0 \notin \mathrm{Min}(\mathcal{A}, f_{I_p})$. Since $x^0 \in \mathcal{A}$ by (4.14), we infer the existence of an element $\tilde{x} \in \mathcal{A}$ satisfying $f_{I_p}(\tilde{x}) \in f_{I_p}(x^0) - \mathbb{R}_+^p \setminus \{0\}$, i.e.,

$$f_I(\tilde{x}) \in f_I(x^0) - \mathbb{R}_+^{|I|}; \tag{4.16}$$

$$f_{I_p \setminus I}(\tilde{x}) \in f_{I_p \setminus I}(x^0) - \mathbb{R}_+^{p-|I|}; \tag{4.17}$$

$$f_{I_p}(\tilde{x}) \neq f_{I_p}(x^0). \tag{4.18}$$

By (4.14) and (4.16) it follows that

$$f_I(\tilde{x}) = f_I(x^0) \tag{4.19}$$

and $\tilde{x} \in \mathrm{Min}(\mathcal{A}, f_I)$. But the relations (4.16), (4.17), (4.18) together with (4.19) imply $f_{I_p \setminus I}(\tilde{x}) \in f_{I_p \setminus I}(x^0) - \mathbb{R}_+^{p-|I|} \setminus \{0\}$, contradicting (4.15). $\square$

Furthermore, we can show the following result.

Corollary 4.14 ([3]). *Let I be a proper subset of I_p. For every weight vector $\lambda \in \operatorname{int} \mathbb{R}_+^{p-|I|}$ we have*

$$\underset{x \in \mathrm{Min}(\mathcal{A}, f_I)}{\operatorname{argmin}} \;\lambda(f_{I_p \setminus I}(x)) \subset \mathrm{Min}(\mathcal{A}, f_{I_p}).$$

Proof. Let $\lambda \in \operatorname{int} \mathbb{R}_+^{p-|I|}$. According to the weighted sum method in multiobjective optimization, we have

$$\underset{x \in C}{\operatorname{argmin}}\, \lambda(f_{I_p \setminus I}(x)) \subset \mathrm{Min}(C, f_{I_p \setminus I}),$$

for any nonempty set $C \subset \mathcal{A}$. In particular, for $C := \mathrm{Min}(\mathcal{A}, f_I)$, we obtain

$$\underset{x \in \mathrm{Min}(\mathcal{A}, f_I)}{\operatorname{argmin}} \;\lambda(f_{I_p \setminus I}(x)) \subset \mathrm{Min}\left(\mathrm{Min}(\mathcal{A}, f_I), f_{I_p \setminus I}\right).$$

Finally, by Theorem 4.13 we get the desired inclusion. $\square$

In particular, when $I := I_p \setminus \{i\}$ and $\lambda = 1$, then we have for each $i \in I_p$

$$\underset{x \in \mathrm{Min}(\mathcal{A}, f_{I_p \setminus \{i\}})}{\operatorname{argmin}} \; f_i(x) \subset \mathrm{Min}(\mathcal{A}, f_{I_p}).$$

In the following definition, we learn about the concept of Pareto reducibility for multiobjective optimization problems as introduced by Popovici [96]. Pareto reducibility is an important tool in our results which aim to find a characterization of the set of weakly minimal solutions of a multiobjective location problem with the maximum norm or Manhattan norm (see Theorem 5.18).

Definition 4.15 ([96]). *A problem $\left(P_{I_p}\right)$ is said to be* ***Pareto reducible*** *if its weakly efficient solutions are efficient solutions either for the problem $\left(P_{I_p}\right)$ itself or for some subproblem (P_I) of it. We can formulate Pareto reducibility as follows*

$$\text{A problem } \left(P_{I_p}\right) \text{ is Pareto reducible} \quad \Leftrightarrow \quad \mathrm{Min_w}(\mathcal{A}, f) = \bigcup_{\emptyset \neq I \subset I_n} \mathrm{Min}(\mathcal{A}, f_I) \tag{4.20}$$

Theorem 4.16 ([3]). *If the multiobjective optimization problem $\left(P_{I_p}\right)$ is Pareto reducible, then the following assertions are equivalent*:

1. $\mathrm{Min_w}(\mathcal{A}, f_{I_p}) = \mathrm{Min}(\mathcal{A}, f_{I_p})$.
2. $\mathrm{Min}(\mathcal{A}, f_I) \subset \mathrm{Min}(\mathcal{A}, f_{I_p})$ *for each proper subset I of I_p.*

Proof. The conclusion directly follows from (4.20). □

Corollary 4.17 ([3]). *Let $p = 2$. If the bicriteria optimization problem (P_{I_2}) is Pareto reducible, then the following assertions are equivalent*:

1. $\mathrm{Min_w}(\mathcal{A}, f_{I_2}) = \mathrm{Min}(\mathcal{A}, f_{I_2})$.
2. $\mathrm{argmin}_{x\in\mathcal{A}} f_1(x) \bigcup \mathrm{argmin}_{x\in\mathcal{A}} f_2(x) \subset \mathrm{Min}(\mathcal{A}, f_{I_2})$.
3. *The outcome sets $f_{I_2}\left(\mathrm{argmin}_{x\in\mathcal{A}} f_1(x)\right)$ and $f_{I_2}\left(\mathrm{argmin}_{x\in\mathcal{A}} f_2(x)\right)$ are either empty or singletons.*

Proof. In view of (4.13), the equivalence $1 \Leftrightarrow 2$ is a straightforward consequence of Theorem 4.16. In order to prove the equivalence $2 \Leftrightarrow 3$ let $i \in I_2$. It suffices to show that $\mathrm{argmin}_{x\in\mathcal{A}} f_i(x) \subset \mathrm{Min}(\mathcal{A}, f_{I_2})$ if and only if $\mathrm{card}\left(f_{I_2}\left(\mathrm{argmin}_{x\in\mathcal{A}} f_i(x)\right)\right) \leqslant 1$.
Assume that $\mathrm{argmin}_{x\in\mathcal{A}} f_i(x) \subset \mathrm{Min}(\mathcal{A}, f_{I_2})$ and suppose to the contrary that there exist $x', x'' \in \mathrm{argmin}_{x\in\mathcal{A}} f_i(x)$ with $f_{I_2}(x') \neq f_{I_2}(x'')$. Then $f_i(x') = f_i(x'') = \min f_i(D)$ and $f_{3-i}(x') \neq f_{3-i}(x'')$. It follows that $f_{I_2}(x'') \in f_{I_2}(x') - \mathbb{R}^2_+ \setminus \{0\}$ or $f_{I_2}(x') \in f_{I_2}(x'') - \mathbb{R}^2_+ \setminus \{0\}$, hence $x' \notin \mathrm{Min}(\mathcal{A}, f_{I_2})$ or $x'' \notin \mathrm{Min}(\mathcal{A}, f_{I_2})$, which contradicts our assumption.
If $card\left(f_{I_2}\left(\mathrm{argmin}_{x\in\mathcal{A}} f_i(x)\right)\right) = 0$, then $\mathrm{argmin}_{x\in\mathcal{A}} f_i(x) = \emptyset \subset \mathrm{Min}(\mathcal{A}, f_{I_2})$. Assume now that $\mathrm{card}\left(f_{I_2}\left(\mathrm{argmin}_{x\in\mathcal{A}} f_i(x)\right)\right) = 1$ and suppose to the contrary that there exists $x^0 \in \mathrm{argmin}_{x\in\mathcal{A}} f_i(x) \setminus \mathrm{Min}(\mathcal{A}, f_{I_2})$. Then there exists $\tilde{x} \in \mathcal{A}$ such that $f_{I_2}(\tilde{x}) \in f_{I_2}(x^0) - \mathbb{R}^2_+ \setminus \{0\}$, i.e., $f_i(\tilde{x}) \leqslant f_i(x^0)$ and $f_{3-i}(\tilde{x}) \leqslant f_{3-i}(x^0)$, with one of those inequalities being strict. More precisely, since $f_i(x^0) = \min f_i(D) \leqslant f_i(\tilde{x})$ we should have $f_i(\tilde{x}) = f_i(x^0)$ and $f_{3-i}(\tilde{x}) < f_{3-i}(x^0)$. It follows that $\tilde{x} \neq x_0$ and $x^0, \tilde{x} \in \mathrm{argmin}_{x\in\mathcal{A}} f_i(x)$, which is a contradiction. □

Corollary 4.18 ([3]). *If $\mathcal{A}$ is a nonempty convex subset of a linear space and all criteria $f_1, \cdots, f_p$ of the optimization problem $\left(P_{I_p}\right)$ are convex, each of them attaining its minimum on $\mathcal{A}$ in a unique point, then*

$$\operatorname{Min}(\mathcal{A}, f_I) = \operatorname{Min}_{\mathrm{w}}(\mathcal{A}, f_I) \ \textit{for all} \ \ I \subset I_p \ \textit{with} \ |I| = 2.$$

The next corollary is derived simply from other interesting results which give sufficient conditions for Pareto reducibility. The next result was given by [80] (1984), it gives sufficient conditions for what later in [96] (2005) called Pareto reducibility.

Corollary 4.19. *Let X be a convex subset of a real linear space and let $f = (f_1, \cdots, f_p) : X \to \mathbb{R}^p$. If $f_1, \cdots, f_p$ are convex, then the problem $\left(P_{I_p}\right)$ is Pareto reducible.*

CHAPTER 5

Duality Assertions for Extended Approximation Problems

Many authors along several decades have studied multiobjective approximation problems, and used duality as an effective tool not only in the theory of scalar but also for multiobjective approximation problems, see [12, 49, 59, 71, 73, 106, 108, 109, 110, 111, 112]. In particular, we focus on Lagrange duality approaches like in [49, 71, 73, 106, 108]. The idea of Lagrange duality is already introduced in Section 3.3. Duality assertions for approximation problems based on conjugation are derived in [109, 110, 112].

In this chapter, we describe the extended multiobjective location and approximation problem $(\mathscr{P})$ which is introduced in Section 2.4, where we discussed also the difference to the known formulations in the literature (such as [49] or [112]). We construct the dual problem of the problem $(\mathscr{P})$ with this specific structure and prove the duality assertions according to Section 3.3. As such, we work with scalarization methods in order to do that. In the first part of this chapter we therefore introduce the dual problem of a scalar approximation and control problem, which is already known in the literature mentioned above. Then we use Lagrange duality similar to [49] and [106] in order to prove weak, strong direct, and converse duality assertions. The weak duality statement is easy to prove, while proving strong duality statements requires some additional assumptions as we see in Section 5.2. In the third part of this chapter, we comprehensively study a special case of $(\mathscr{P})$, namely a multiobjective location problem. We use geometric characterizations of the set of minimal solution of the multiobjective location problem, especially that is given in [46], in order to derive new results, which characterize the set of weakly minimal solutions of the multiobjective location problem by using Pareto reducibility (see [96, 99]). These results are used to derive implementable decomposition algorithms in Chapter 6.

In order to describe the extended multiobjective location and approximation problem $(\mathscr{P})$, we consider the following assumptions:

(A_1) Let $(X, \|\cdot\|_X)$, $(U, \|\cdot\|_U)$ be reflexive Banach spaces, which are partially ordered by the nontrivial, convex, pointed and closed cones $K_X \subset X$, $K_U \subset U$. The corresponding dual cones are K_X^*, K_U^*.

(A_2) Define $I_p := \{1, \cdots, p\}$. Let $(V, \|\cdot\|_{(i)})$ be a normed space and let $a^i \in V$, $\alpha_i \in \mathbb{R}_+$ and $\beta_i \geq 1$ for each $i \in I_p$. Furthermore, let $b \in U$.

(A_3) Consider the operators $A_i \in L(X, V)$ for $i \in I_p$, $B \in L(X, U)$ and $c \in L(X, \mathbb{R})$.

The reflexivity of the spaces X and U is needed in order to prove strong duality statements for extended multiobjective approximation problems by applying the main duality theorem (see Theorem 3.37).

In the following section, we introduce a scalar approximation problem under the previous assumptions. Furthermore, we formulate a corresponding dual problem and take a look on a very well-known special case of it, which are both given in the literature mentioned above.

5.1 Duality Assertions for Scalar Approximation Problems

Under the assumptions (A_1), (A_2) and (A_3) we consider the following scalar approximation problem (P_1) (compare with (2.3))

$$(P_1) \qquad \begin{cases} \text{Minimize} & c(x) + \sum_{i=1}^{p} \alpha_i \|A_i(x) - a^i\|_{(i)}^{\beta_i} \\ \text{subject to} & x \in D. \end{cases} \tag{5.1}$$

The feasible set D is defined corresponding to the assumptions (A_1), (A_2) and (A_3) as follows

$$D := \{x \in X \mid x \in K_X, B(x) - b \in K_U\}.$$

Consider a special case of (P_1) by taking $\beta_i = 1$ in (5.1) and the same norm $\|\cdot\|_{(i)} = \|\cdot\|$ in (5.1) for all $i \in I_p$. For this special case of (P_1) the dual problem is given in [49] as

$$\begin{cases} \text{Maximize} & \sum_{i=1}^{p} \alpha_i y_i(a^i) + z(b) \\ \text{subject to} & (y, z) \in D^*, \end{cases} \tag{5.2}$$

with

$$\begin{aligned} D^* := \{(y, z) \mid y = (y_1, \cdots, y_p), y_i \in L(V, \mathbb{R}), \alpha_i \|y_i\|_* \leq \alpha_i \\ z \in K_U^*,\ c - \sum_{i=1}^{p} \alpha_i A_i^* y_i - B^* z \in K_X^*\}. \end{aligned} \tag{5.3}$$

We get from Theorem 3.37 that for the primal problem (5.1) and the dual problem (5.2) weak, and under certain additional assumptions, strong duality assertions hold (see [49, 50]).

The Fermat-Weber-Problem

If we additionally consider the special case $D = V = X$, $c = 0$ and $A_i = \mathbb{I}$ for $i \in I_p$, we obtain the Fermat-Weber-Problem introduced in (2.1)

$$\begin{cases} \text{Minimize} & \sum\limits_{i=1}^{p} \alpha_i \|x - a^i\| \\ \text{subject to} & x \in X, \end{cases}$$

with the corresponding dual problem

$$\begin{cases} \text{Maximize} & \sum\limits_{i=1}^{p} \alpha_i y_i(a^i) \\ \text{subject to} & y_i \in X^*,\ \|y_i\|_* \leq 1, i \in I_p,\ \sum\limits_{i=1}^{p} \alpha_i y_i = 0. \end{cases}$$

We now focus on multiobjective problems and introduce the class of extended multiobjective location and approximation problems.

5.2 Duality Assertions for Extended Multiobjective Approximation Problems

In this Section, we introduce in detail the class of multiobjective location and approximation problems given in (2.5). We minimize a vector-valued objective function, where the first p components are given by distances to the existing facilities. Furthermore, the vector-valued objective function also contains m additional components representing occurring cost functions. We formulate its dual problem and prove the corresponding duality assertions. To achieve this, we use the general Lagrange duality principle introduced in Section 3.3.

The Primal Problem

In addition to the assumptions (A_1), (A_2) and (A_3) we suppose

(A_4) $C \in L(X, \mathbb{R}^m)$ is a vector-valued cost function with $C_{p+j} \in L(X, \mathbb{R})$ for $j \in \{1, \cdots, m\} =: I_m$:

$$C = \begin{pmatrix} C_{p+1} \\ \cdots \\ C_{p+m} \end{pmatrix}.$$

(A_5) We assume that $K \subset \mathbb{R}^{p+m}$ is a nontrivial, convex, pointed and closed cone with $\mathbb{R}^{p+m}_{+} \subset K$ (for example see Figure 3.4) and let K^* be the corresponding dual cone of K with $\operatorname{int} K^* \neq \emptyset$.

Taking into account the assumptions (A_1) – (A_5) we introduce the primal vector-valued objective function $f : X \to \mathbb{R}^{p+m}$ with

$$f(x) := \begin{pmatrix} \alpha_1 \|A_1(x) - a^1\|_1^{\beta_1} \\ \cdots \\ \alpha_p \|A_p(x) - a^p\|_p^{\beta_p} \\ C_{p+1}(x) \\ \cdots \\ C_{p+m}(x) \end{pmatrix} \tag{5.4}$$

and the primal feasible set

$$\mathcal{A} = \{x \in X \mid x \in K_X, B(x) - b \in K_U\}\,. \tag{5.5}$$

The extended multiobjective location and approximation problem we study in this section is given by

$$(\mathscr{P}) \qquad \mathrm{Eff}_{\mathrm{Min}}\,(f[\mathcal{A}], K). \tag{5.6}$$

In order to applying the duality principle described in Section 3.3, we introduce a Lagrange function in a generalized form.
For $x \in X$ and $Y = (Y^1, \cdots, Y^p)$ with $Y^i \in L(V, \mathbb{R})$, $i \in I_p$, $\lambda^* \in \operatorname{int} K^*$, $u^* \in L(U, \mathbb{R})$ we define

$$L_{\lambda^*}(x, Y, u^*) := \lambda^* \begin{pmatrix} \frac{\alpha_1}{\beta_1}\, Y^1(a^1 - A_1(x)) \\ \cdots \\ \frac{\alpha_p}{\beta_p}\, Y^p(a^p - A_p(x)) \\ C_{p+1}(x) \\ \cdots \\ C_{p+m}(x) \end{pmatrix} + u^*(b - B(x)). \tag{5.7}$$

Remark 5.1.

1. In difference to the classical Lagrange function (see for example [13]), in (5.7) the objective function of the primal problem is not involved.

2. We observe from (5.7) that the Lagrange function is linear in x whenever $Y,\ u^*$ are fixed, linear in Y whenever $x,\ u^*$ are fixed and linear in u^* whenever $x,\ Y$ are fixed.

Construction of the Dual Problem to $(\mathscr{P})$

We now formulate the dual problem to the problem $(\mathscr{P})$. We begin with the dual objective function $f^* : \mathcal{B} \to \mathbb{R}^{p+m}$ defined for $(Y, Z) \in \mathcal{B}$ by

$$f^*(Y,Z) := \begin{pmatrix} \frac{\alpha_1}{\beta_1} Y^1(a^1) \\ \cdots \\ \frac{\alpha_p}{\beta_p} Y^p(a^p) \\ 0 \\ \cdots \\ 0 \end{pmatrix} + Z(b), \tag{5.8}$$

where $Z \in L(U, \mathbb{R}^{p+m})$. The dual feasible set is given by

$$\begin{aligned} \mathcal{B} = \{(Y,Z) \mid Y = (Y^1, \dots, Y^p) : Y^i \in L(V, \mathbb{R}), Z \in L(U, \mathbb{R}^{p+m}),& \\ \alpha_i \|Y^i\|_{(i)^*} = \beta_i \alpha_i \|a^i - A_i(x)\|_{(i)}^{\beta_i - 1} \quad \forall i = 1, \cdots, p,& \\ \exists \lambda^* \in \operatorname{int} K^* : \quad \sum_{j=1}^{m} \lambda^*_{p+j} C_{p+j} - \sum_{i=1}^{p} \lambda^*_i \, \frac{\alpha_i}{\beta_i} \, A_i^* Y^i - (Z(B))^* \lambda^* \in K_X^*& \\ \text{where } Z^* \lambda^* \in K_U^* \},& \end{aligned} \tag{5.9}$$

where $\|\cdot\|_{(i^*)}$ is the dual norm of the norm $\|\cdot\|_{(i)}$, and $(ZA)^*$ is, in general, an adjoint operator. The dual problem of the multiobjective approximation problem $(\mathscr{P})$ is given by

$$(\mathscr{D}) \qquad \operatorname{Eff}_{\mathrm{Max}}(f^*[\mathcal{B}], K). \tag{5.10}$$

Special Cases of $(\mathscr{P})$

The following special cases can be taken from the extended multiobjective location and approximation problem $(\mathscr{P})$:

1. If $\alpha_1 = \ldots = \alpha_p = 0$, then the problem $(\mathscr{P})$ is a multiobjective linear optimization problem (a multiobjective linear program). We additionally consider $m = 1$, then we have a scalar linear optimization problem.

2. If $C_{p+j} = 0$ for $j \in I_m$, then by eliminating the trivial criteria we have a multiobjective approximation problem.

3. If $C_{p+j} = 0$ (and the trivial criteria are eliminated), $\beta_i = 1$ and $A_i = \mathbb{I}$ (certainly when $X = V$) for each $i \in I_p$, we get the well-known multiobjective location problem, which is studied comprehensively in Section 5.3.

4. If $A_i = \mathbb{I}$ and $\beta_i = 1$ for each $i \in I_p$, we have an extended multiobjective location problem. Decomposition algorithms for this special case are derived in Section 6.2.

In order to prove the duality statements for $(\mathscr{P})$ and $(\mathscr{D})$, we use a scalarization by means of linear continuous functionals and Theorem 3.37 taking into account the Lagrange function (5.7).
Since we use a scalarization in the proof of the duality statements, it is convenient to define a scalarized form of the dual feasible set and of the elements of the image space by introducing the following sets. For $\lambda^* \in \operatorname{int} K^*$:

$$\begin{aligned}\mathcal{B}_{\lambda^*} := \{(Y, u^*) \mid Y = (Y^1, \ldots, Y^p) : Y^i \in L(V, \mathbb{R}),\ u^* \in L(U, \mathbb{R}),\\ \alpha_i \|Y^i\|_{(i)^*} = \beta_i \alpha_i \|a^i - A_i(x)\|_{(i)}^{\beta_i - 1}, \forall\, i = 1, \cdots, p,\\ \sum_{j=1}^{m} \lambda^*_{p+j} C_{p+j} - \sum_{i=1}^{p} \lambda^*_i \frac{\alpha_i}{\beta_i} A_i^* Y^i - B^*(u^*) \in K_X^* \text{ with } u^* \in K_U^*\}.\end{aligned} \tag{5.11}$$

$$D_1 := \{d \in \mathbb{R}^{p+m} \mid \exists\, \lambda^* \in \operatorname{int} K^*, \exists\, (Y, u^*) \in \mathcal{B}_{\lambda^*} : \lambda^* d = \sum_{i=1}^{p} \lambda^*_i \frac{\alpha_i}{\beta_i} Y^i(a^i) + u^*(b)\} \tag{5.12}$$

$$D_2 := \{d \in \mathbb{R}^{p+m} \mid \exists\, (Y, Z) \in \mathcal{B} : \quad d = \begin{pmatrix} \frac{\alpha_1}{\beta_1} Y^1(a^1) \\ \cdots \\ \frac{\alpha_p}{\beta_p} Y^p(a^p) \\ 0 \\ \cdots \\ 0 \end{pmatrix} + Z(b)\ \}. \tag{5.13}$$

Furthermore, for more abbreviation we denote

$$\mathcal{Y} := \{Y = (Y^1, \ldots, Y^p) \mid Y^i \in L(V, \mathbb{R}) \text{ with } \alpha_i \|Y^i\|_{(i)^*} = \beta_i \alpha_i \|a^i - A_i(x)\|_{(i)}^{\beta_i - 1}\}. \tag{5.14}$$

The next step is to compute both the supremum and the infimum of the Lagrange function in (5.7), which helps by proving the duality assertions.
Note the use of Lemma 3.33 for computing the supremum of the Lagrange function.

Lemma 5.2. *For every $\lambda^* \in \operatorname{int} K^*$ we have*

$$\sup_{\substack{u^* \in K_U^* \\ Y \in \mathcal{Y}}} L_{\lambda^*}(x, Y, u^*) =$$
$$\begin{cases} \sum\limits_{i=1}^{p} \lambda^*_i \alpha_i \|a^i - A_i(x)\|_{(i)}^{\beta_i} + \sum\limits_{j=1}^{m} \lambda^*_{p+j} C_{p+j}(x) & \textit{if} \quad B(x) - b \in K_U \\ +\infty & \textit{otherwise.} \end{cases}$$

Proof. We have

$$\sup_{u^*\in K_U^*} u^*(b-B(x)) = \begin{cases} 0 & \text{if} \quad B(x)-b\in K_U \\ +\infty & \text{otherwise.} \end{cases}$$

Furthermore, we have according to the Hahn-Banach-Theorem (see for example [50, Theorem 5.1]) and considering the subdifferential of the norm in Lemma 3.33 for $i=1,\cdots,p$

$$\sup_{Y\in\mathcal{Y}} Y^i(a^i-A_i(x)) = \beta_i \, \|a^i-A_i(x)\|_{(i)}^{\beta_i}.$$

This leads to

$$\begin{aligned}
&\sup_{\substack{u^*\in K_U^*\\ Y\in\mathcal{Y}}} L_{\lambda^*}(x,Y,u^*) = \\
&= \sup_{\substack{u^*\in K_U^*\\ Y\in\mathcal{Y}}} \left[\sum_{i=1}^{p} \lambda_i^* \frac{\alpha_i}{\beta_i} \, Y^i(a^i-A_i(x)) + \sum_{j=1}^{m} \lambda_{p+j}^* C_{p+j}(x) + u^*(b-B(x))\right] \\
&= \sum_{i=1}^{p} \lambda_i^* \frac{\alpha_i}{\beta_i} \sup_{Y\in\mathcal{Y}} Y^i(a^i-A_i(x)) + \sum_{j=1}^{m} \lambda_{p+j}^* C_{p+j}(x) + \sup_{u^*\in K_U^*} u^*(b-B(x)) \\
&= \begin{cases} \sum\limits_{i=1}^{p} \lambda_i^*\alpha_i \|a^i-A_i(x)\|_{(i)}^{\beta_i} + \sum\limits_{j=1}^{m} \lambda_{p+j}^* C_{p+j}(x) & \text{if} \quad B(x)-b\in K_U \\ +\infty & \text{otherwise.} \end{cases}
\end{aligned}$$

□

Now we compute the infimum of the Lagrange function, which we use for proving the weak duality statement.

Lemma 5.3. *For every $\lambda^*\in \operatorname{int} K^*$ we have*

$$\begin{aligned}
&\inf_{x\in K_X} L_{\lambda^*}(x,Y,u^*) = \\
&= \begin{cases} \sum\limits_{i=1}^{p} \lambda_i^* \frac{\alpha_i}{\beta_i} Y^i(a^i) + u^*(b) & \textit{if} \quad \sum\limits_{j=1}^{m} \lambda_{p+j}^* C_{p+j} - \sum\limits_{i=1}^{p} \lambda_i^* \frac{\alpha_i}{\beta_i} A_i^* Y^i - B^*(u^*) \in K_X^* \\ -\infty & \textit{otherwise.} \end{cases}
\end{aligned}$$

Proof.

$$
\begin{aligned}
&\inf_{x\in K_X} L_{\lambda^*}(x,Y,u^*) = \\
&= \inf_{x\in K_X}\left[\sum_{i=1}^{p}\lambda_i^*\frac{\alpha_i}{\beta_i}\,Y^i(a^i - A_i(x)) + \sum_{j=1}^{m}\lambda_{p+j}^* C_{p+j}(x) + u^*(b - B(x))\right] \\
&= \inf_{x\in K_X}\left[\sum_{i=1}^{p}\lambda_i^*\frac{\alpha_i}{\beta_i}Y^i(a^i) + u^*(b) + \sum_{j=1}^{m}\lambda_{p+j}^* C_{p+j}(x) - \sum_{i=1}^{p}\lambda_i^*\frac{\alpha_i}{\beta_i}[A_i^*Y^i](x) - [B^*(u^*)](x)\right] \\
&= \sum_{i=1}^{p}\lambda_i^*\frac{\alpha_i}{\beta_i}Y^i(a^i) + u^*(b) + \inf_{x\in K_X}\left[\sum_{j=1}^{m}\lambda_{p+j}^* C_{p+j} - \sum_{i=1}^{p}\lambda_i^*\frac{\alpha_i}{\beta_i}A_i^*Y^i - B^*(u^*)\right](x) \\
&= \begin{cases} \sum\limits_{i=1}^{p}\lambda_i^*\frac{\alpha_i}{\beta_i}Y^i(a^i) + u^*(b) & \text{if } \sum\limits_{j=1}^{m}\lambda_{p+j}^* C_{p+j} - \sum\limits_{i=1}^{p}\lambda_i^*\frac{\alpha_i}{\beta_i}A_i^*Y^i - B^*(u^*) \in K_X^* \\ -\infty & \text{otherwise.} \end{cases}
\end{aligned}
$$

□

By computing the supremum and the infimum of the Lagrange function, we observe that they coincide with the scalarized form of the primal and the dual objective functions respectively.
Now we show the relationship between D_1 and D_2, defined in (5.12) and (5.13), respectively.

Theorem 5.4. *It holds $D_2 \subset D_1$.*

Proof. Let $d_0 \in D_2 : d_0 \in \mathbb{R}^{p+m}, \exists (Y,Z) \in \mathcal{B}$ with

$$
d_0 - \begin{pmatrix} \frac{\alpha_1}{\beta_1}Y^1(a^1) \\ \cdots \\ \frac{\alpha_p}{\beta_p}Y^p(a^p) \\ 0 \\ \cdots \\ 0 \end{pmatrix} = Z(b). \tag{5.15}
$$

According to (5.9) we see that $(Y,Z) \in \mathcal{B}$ means

$$
\exists \lambda^* \in \operatorname{int} K^* : \quad \sum_{j=1}^{m}\lambda_{p+j}^* C_{p+j} - \sum_{i=1}^{p}\lambda_i^*\frac{\alpha_i}{\beta_i}\,A_i^*Y^i - (Z(B))^*\lambda^* \in K_X^*, \tag{5.16}
$$

$\alpha_i\|Y^i\|_{(i)^*} = \beta_i\alpha_i\|a^i - A_i(x)\|_{(i)}^{\beta_i-1}$ and $Z^*\lambda^* \in K_U^*$. We see that $Z^*\lambda^* \in L(U,\mathbb{R})$. Let us now set

$u^* := (Z^*\lambda^*) \in K_U^*$, so we get $\lambda^*(Zb) = u^*(b)$, with (5.15) we get

$$\lambda^*(d_0 - \begin{pmatrix} \frac{\alpha_1}{\beta_1} Y^1(a^1) \\ \cdots \\ \frac{\alpha_p}{\beta_p} Y^p(a^p) \\ 0 \\ \cdots \\ 0 \end{pmatrix}) = u^*(b)$$

$$\lambda^* d_0 = \sum_{i=1}^{p} \lambda_i^* \frac{\alpha_i}{\beta_i} Y^i(a^i) + u^*(b). \tag{5.17}$$

Furthermore, from (5.16) with $u^* := (Z^*\lambda^*)$ we get

$$\sum_{j=1}^{m} \lambda_{p+j}^* C_{p+j} - \sum_{i=1}^{p} \lambda_i^* \frac{\alpha_i}{\beta_i} A_i^* Y^i - B^*(u^*) \in K_X^* \text{ with } u^* \in K_U^* \tag{5.18}$$

$$\Rightarrow (Y, u^*) \in \mathcal{B}_{\lambda^*}$$

so from (5.17) and (5.18) we find that $d_0 \in D_1$. Thus $D_2 \subset D_1$. □

For proving the other direction, namely $D_1 \subset D_2$, we assume that $b \neq 0$ in $(\mathscr{P})$. This assumption depends on the next theorem.

Theorem 5.5 ([73, Theorem 2.3])**.** *Let X and Y be real separated locally convex linear spaces, and let the elements $x \in X, x^* \in X^*, y \in Y$ and $y^* \in Y^*$ be given.*

(a) If there is a linear map $T : X \to Y$ with $y = T(x)$ and $x^* = T^*(y^*)$, then $y^*(y) = x^*(x)$.

(b) If $x \neq 0_X, y^* \neq 0_{Y^*}$ and $y^*(y) = x^*(x)$, then there is a continuous linear map $T : X \to Y$ with $y = T(x)$ and $x^* = T^*(y^*)$.

Now under the assumption $b \neq 0$ we prove the second inclusion between D_1, D_2.

Theorem 5.6. *If $b \neq 0$ then $D_1 \subset D_2$.*

Proof. Let $d \in D_1$. This means $d \in \mathbb{R}^{p+m}$ and there exist $\lambda^* \in \operatorname{int} K^*$, and $(Y, u^*) \in \mathcal{B}_{\lambda^*}$ such that

$$\lambda^* d = \sum_{i=1}^{p} \lambda_i^* \frac{\alpha_i}{\beta_i} Y^i(a^i) + u^*(b) \quad \Rightarrow \quad \lambda^*(d - \begin{pmatrix} \frac{\alpha_1}{\beta_1} Y^1(a^1) \\ \cdots \\ \frac{\alpha_p}{\beta_p} Y^p(a^p) \\ 0 \\ \cdots \\ 0 \end{pmatrix}) = u^*(b).$$

We set

$$y = d - \begin{pmatrix} \frac{\alpha_1}{\beta_1} Y^1(a^1) \\ \cdots \\ \frac{\alpha_p}{\beta_p} Y^p(a^p) \\ 0 \\ \cdots \\ 0 \end{pmatrix} \quad \Rightarrow \quad \lambda^*(y) = u^*(b).$$

Since $b \neq 0$ and according to Theorem 5.5 (b):

$$\exists Z \in (U, \mathbb{R}^{p+m}) \quad \text{with} \quad y = Z(b), u^* = Z^* \lambda^*. \tag{5.19}$$

This implies that

$$d - \begin{pmatrix} \frac{\alpha_1}{\beta_1} Y^1(a^1) \\ \cdots \\ \frac{\alpha_p}{\beta_p} Y^p(a^p) \\ 0 \\ \cdots \\ 0 \end{pmatrix} = Zb \quad \Rightarrow \quad d = \begin{pmatrix} \frac{\alpha_1}{\beta_1} Y^1(a^1) \\ \cdots \\ \frac{\alpha_p}{\beta_p} Y^p(a^p) \\ 0 \\ \cdots \\ 0 \end{pmatrix} + Zb.$$

Now we prove that $(Y, Z) \in \mathcal{B}$. From the definition of D_1 we get that there exists some $(Y, u^*) \in \mathcal{B}_{\lambda^*}$. This means that there exists $u^* \in L(U, \mathbb{R})$ with $u^* \in K_U^*$, $\alpha_i \|Y^i\|_{(i)^*} = \beta_i \alpha_i \|a^i - A_i(x)\|_{(i)}^{\beta_i - 1}$ and there exists $\lambda^* \in \operatorname{int} K^*$ with

$$\sum_{j=1}^{m} \lambda_{p+j}^* C_{p+j} - \sum_{i=1}^{p} \lambda_i^* \frac{\alpha_i}{\beta_i} A_i^* Y^i - B^*(u^*) \in K_X^*.$$

Since $u^* = Z^* \lambda^*$ from (5.19) we get

$$\sum_{j=1}^{m} \lambda_{p+j}^* C_{p+j} - \sum_{i=1}^{p} \lambda_i^* \frac{\alpha_i}{\beta_i} A_i^* Y^i - (Z(B))^* \lambda^* \in K_X^*.$$

Thus $(Y, Z) \in \mathcal{B}$ and this leads to $d \in D_2$. $\square$

Theorems 5.4 and 5.6 yield that $D_1 = D_2$, which confirm the representation of the elements of the image space and its scalarized forms.

In the following we show the duality assertions for $(\mathscr{P})$ and $(\mathscr{D})$ starting with the weak duality. Proving the duality assertions is similar to [49, 106], where we generalize and modify these results for our model.

Weak Duality

First, we prove the weak duality theorem for $(\mathscr{P})$ and $(\mathscr{D})$. It can be easily proved without any convexity assumptions.

Theorem 5.7 (Weak Duality for $(\mathscr{P})$ and $(\mathscr{D})$).

$$f[\mathcal{A}] \cap (f^*[\mathcal{B}] - (K \setminus \{0\})) = \emptyset. \tag{5.20}$$

Proof. Let $\lambda^* \in \operatorname{int} K^*$, and let x be an arbitrary fixed element from $\mathcal{A}$.

$$\begin{aligned}
\lambda^*(f(x)) &\geq \inf_{x\in\mathcal{A}} \lambda^*(f(x)) \\
&= \inf_{x\in\mathcal{A}} \left(\sum_{i=1}^{p} \lambda_i^* \alpha_i \|a^i - A_i(x)\|_{(i)}^{\beta_i} + \sum_{j=p+1}^{p+m} \lambda_j^* C_j(x) \right) \\
&= \inf_{x\in K_X} \sup_{\substack{u^*\in K_U^* \\ Y\in\mathcal{Y}}} L_{\lambda^*}(x,Y,u^*) \qquad \text{(see Lemma 5.2)} \\
&= \inf_{x\in K_X} \sup_{\substack{u^*\in K_U^* \\ Y\in\mathcal{Y}}} \left[\sum_{i=1}^{p} \lambda_i^* \frac{\alpha_i}{\beta_i} Y^i(a^i - A_i(x)) + \sum_{j=1}^{m} \lambda_{p+j}^* C_{p+j}(x) + u^*(b - B(x)) \right] \\
&\geq \inf_{x\in K_X} \left[\sum_{i=1}^{p} \lambda_i^* \frac{\alpha_i}{\beta_i} Y^i(a^i - A_i(x)) + \sum_{j=1}^{m} \lambda_{p+j}^* C_{p+j}(x) + u^*(b - B(x)) \right] \quad (\forall u^* \in K_U^*,\ \forall Y \in \mathcal{Y}) \\
&= \sum_{i=1}^{p} \lambda_i^* \frac{\alpha_i}{\beta_i} Y^i(a^i) + u^*(b) + \inf_{x\in K_X} \Big[\underbrace{\Big(\sum_{j=1}^{m} \lambda_{p+j}^* C_{p+j} - \sum_{i=1}^{p} \lambda_i^* \frac{\alpha_i}{\beta_i} A_i^* Y^i - B^*(u^*) \Big)}_{\in K_X^*}(x) \Big] \\
&= \begin{cases} \sum\limits_{i=1}^{p} \lambda_i^* \frac{\alpha_i}{\beta_i} Y^i(a^i) + u^*(b) & \text{if} \quad (Y,u^*) \in \mathcal{B}_{\lambda^*} \\ -\infty & \text{otherwise.} \end{cases}
\end{aligned}$$

Taking into account the definition of D_1 we see that:

$$\forall d \in D_1, \forall x \in \mathcal{A} : \lambda^* f(x) \geq \lambda^* d. \tag{5.21}$$

Assuming that $\exists\, x \in \mathcal{A}, \exists\, d \in D_1 : \quad f(x) \in d - (K \setminus \{0\})$

$$\Rightarrow \lambda^* f(x) < \lambda^* d \quad \text{because} \quad \lambda^* \in \operatorname{int} K^*$$

which contradicts (5.21).

$$\begin{aligned}
\text{So} \qquad & f[\mathcal{A}] \cap (D_1 - (K \setminus \{0\})) = \emptyset \\
\Rightarrow \qquad & f[\mathcal{A}] \cap (D_2 - (K \setminus \{0\})) = \emptyset \qquad (D_2 \subset D_1) \\
\Rightarrow \qquad & f[\mathcal{A}] \cap (f^*[\mathcal{B}] - (K \setminus \{0\})) = \emptyset \qquad \text{(definition of } D_2\text{).}
\end{aligned}$$

$\square$

Strong Duality

In order to prove the strong duality between $(\mathscr{P})$ and $(\mathscr{D})$ applying Theorem 3.37 we need additional assumptions. Some of these assumptions (especially (C_5) and (C_6) in Theorem 3.37) are hard to prove, therefore we show in the next two propositions sufficient conditions (generalized Slater conditions) for (C_5) and (C_6) in Theorem 3.37.

Proposition 5.8. *Let $\lambda^* \in \operatorname{int} K^*$ and assume that there exists $(\bar{Y}, \bar{Z}) \in \mathcal{B}$ with*

$$\sum_{j=1}^{m} \lambda^*_{p+j} C_{p+j} - \sum_{i=1}^{p} \frac{\alpha_i}{\beta_i} \lambda^*_i A^*_i \bar{Y}_i - (\bar{Z}(B))^* \lambda^* \in \operatorname{int} K^*_X.$$

Put $\bar{u}^* := \bar{Z}^* \lambda^*$. Then $L_{\lambda^*}(x, \bar{Y}, \bar{u}^*) \longrightarrow +\infty$ for $\|x\|_X \longrightarrow +\infty$, where $x \in K_X$, i.e., the condition (C_5) in Theorem 3.37 is fulfilled.

Proof. We have $\sum\limits_{j=1}^{m} \lambda^*_{p+j} C_{p+j} - \sum\limits_{i=1}^{p} \frac{\alpha_i}{\beta_i} \lambda^*_i A^*_i \bar{Y}_i - (\bar{Z}(B))^* \lambda^* \in \operatorname{int} K^*_X$. For $\bar{u}^* = \bar{Z}^* \lambda^*$, we can find $\gamma > 0$ with

$$(\sum_{j=1}^{m} \lambda^*_{p+j} C_{p+j} - \sum_{i=1}^{p} \frac{\alpha_i}{\beta_i} \lambda^*_i A^*_i \bar{Y}_i - B^*(\bar{u}^*))(x) \geq \gamma, \qquad \forall x \in K_X \quad \text{with} \quad \|x\|_X = 1.$$

We consider the sequence $\{x^i\} : \|x^i\|_X \longrightarrow +\infty \quad (i \longrightarrow +\infty)$ and then we set

$$\tilde{x}^i := \frac{1}{\|x^i\|_X} x^i \quad \Rightarrow \quad \|\tilde{x}^i\|_X = 1.$$

$$\begin{aligned} L_{\lambda^*}(x^i, \bar{Y}, \bar{u}^*) &= (\sum_{j=p+1}^{p+m} \lambda^*_j C_j - \sum_{i=1}^{p} \lambda^*_i \frac{\alpha_i}{\beta_i} A^*_i \bar{Y}^i - B^*(\bar{u}^*))(x^i) + \sum_{i=1}^{p} \lambda^*_i \frac{\alpha_i}{\beta_i} \bar{Y}^i(a^i) + \bar{u}^*(b) \\ &= \|x^i\|_X (\sum_{j=p+1}^{p+m} \lambda^*_j C_j - \sum_{i=1}^{p} \lambda^*_i \frac{\alpha_i}{\beta_i} A^*_i \bar{Y}^i - B^*(\bar{u}^*))(\tilde{x}^i) + \sum_{i=1}^{p} \lambda^*_i \frac{\alpha_i}{\beta_i} \bar{Y}^i(a^i) + \bar{u}^*(b) \\ &\geq \|x^i\| \gamma + \sum_{i=1}^{p} \lambda^*_i \frac{\alpha_i}{\beta_i} \bar{Y}^i(a^i) + \bar{u}^*(b) \quad \longrightarrow +\infty \quad \text{for} \quad \|x^i\| \longrightarrow +\infty \end{aligned}$$

We infer that (C_5) in Theorem 3.37 is fulfilled. □

The following proposition for the condition (C_6) in Theorem 3.37.

Proposition 5.9. *Let $\lambda^* \in \operatorname{int} K^*$ and assume that there exists $\bar{x} \in K_X$ with $B(\bar{x}) - b \in \operatorname{int} K_U$. Then it holds that $-L_{\lambda^*}(\bar{x}, Y, u^*) \longrightarrow +\infty$ for $\|(Y, u^*)\| \longrightarrow +\infty$, where $(Y, u^*) \in \mathcal{B}_{\lambda^*}$, i.e., the condition (C_6) in Theorem 3.37 is fulfilled.*

Proof. According to the assumption there exists $\bar{x} : B(\bar{x}) - b \in \operatorname{int} K_U$, then we can find $\delta > 0$: $u^*(b - B(\bar{x})) \leq -\delta$ for all $u^* \in K_U^*$ and $\|u^*\|_* = 1$.
We take an arbitrary sequence $\{(Y^k, u_k^*)\} \in \mathcal{B}_{\lambda^*}$ with $\|(Y^k, u_k^*)\|_* \longrightarrow +\infty$.
Considering $\alpha_i \|Y^i\|_{(i)^*} = \beta_i \alpha_i \|a^i - A_i(x)\|_{(i)}^{\beta_i - 1}$ the next two statements are fulfilled:

$$\{\|u_k^*\|\}_* \longrightarrow +\infty \quad \text{and} \quad \sum_{i=1}^{p} \lambda_i^* \frac{\alpha_i}{\beta_i} Y_i^k(a^i - A_i \bar{x}) \quad \text{is bounded.}$$

We set also $\tilde{u}_k^* = \frac{1}{\|u_k^*\|_*} u_k^* \quad \Rightarrow \quad \|\tilde{u}_k^*\|_* = 1,$ which leads to

$$u_k^*(b - B(\bar{x})) = \|u_k^*\|_* \tilde{u}_k^*(b - B(\bar{x})) \leq -\|u_k^*\|_* \delta.$$

$$\begin{aligned} L_{\lambda^*}(\bar{x}, Y^k, u_k^*) &= \sum_{i=1}^{p} \lambda_i^* \frac{\alpha_i}{\beta_i} Y_i^k(a^i - A_i \bar{x}) + \sum_{j=1}^{m} \lambda_{j+p}^* C_{j+p}(\bar{x}) + u_k^*(b - B(\bar{x})) \\ &\leq \sum_{i=1}^{p} \lambda_i^* \frac{\alpha_i}{\beta_i} Y_i^k(a^i - A_i \bar{x}) + \sum_{j=1}^{m} \lambda_{j+p}^* C_{j+p}(\bar{x}) - \delta \|u_k^*\|_* \end{aligned}$$

Hence, $L_{\lambda^*}(\bar{x}, Y^k, u_k^*) \longrightarrow -\infty$ when $\|(Y^k, u_k^*)\|_* \longrightarrow +\infty$, i.e., the condition (C_6) in Theorem 3.37 is fulfilled. $\square$

Under certain assumptions supposed in Propositions 5.8 and 5.9, the conditions (C_5) and (C_6) in Theorem 3.37 are fulfilled. We can therefore apply Theorem 3.37 in order to prove the strong direct duality assertion for $(\mathscr{P})$ and $(\mathscr{D})$, where we consider properly efficient elements $f(x^0)$ of $(\mathscr{P})$ in the sense of Definition 4.5, i.e., $f(x^0) \in \operatorname{Eff}_{\text{pSch}}(f[\mathcal{A}], K)$.

Theorem 5.10 (Strong Direct Duality for $(\mathscr{P})$ and $(\mathscr{D})$)**.** *Assume that*

- $b \neq 0$.
- *For every* $\lambda^* \in \operatorname{int} K^*$, *there exists* $(Y^0, Z^0) \in \mathcal{B}$ *with*

$$\sum_{j=1}^{m} \lambda_{p+j}^* C_{p+j} - \sum_{i=1}^{p} \frac{\alpha_i}{\beta_i} \lambda_i^* A_i^* Y_i^0 - (Z^0(B))^* \lambda^* \in \operatorname{int} K_X^*.$$

Then for every $f(x^0) \in \operatorname{Eff}_{\text{pSch}}(f[\mathcal{A}], K)$ *there exists an element* $f^*(Y^0, Z^0) \in \operatorname{Eff}_{\text{Max}}(f^*[\mathcal{B}], K)$ *with*

$$f(x^0) = f^*(Y^0, Z^0). \tag{5.22}$$

Proof. $f(x^0)$ is a properly efficient solution of $(\mathscr{P})$ in the sense of Definition 4.5. $f(x^0) \in \operatorname{Eff}_{\text{pSch}}(f[\mathcal{A}], K)$, this means that

$$\begin{aligned} \exists\, \lambda^* \in \operatorname{int} K^* : \quad \lambda^*(f(x^0)) &= \inf_{x \in \mathcal{A}} \lambda^*(f(x)) \\ &= \inf_{x \in \mathcal{A}} \sup_{\substack{u^* \in K_U^* \\ Y \in \mathcal{Y}}} L_{\lambda^*}(x, Y, u^*) \end{aligned}$$

by taking into account Lemma 5.2. Since (C_1) – (C_4) and (C_6) are fulfilled (see Proposition 5.9), then according to the strong duality assertion in Theorem 3.37 there exists $(Y^0, u_0^*) \in \mathcal{B}_{\lambda^*}$ such that:

$$\lambda^*(f(x^0)) = \sum_{i=1}^{p} \lambda_i^* \frac{\alpha_i}{\beta_i} Y_i^0(a^i) + u_0^*(b),$$

this means that $f(x^0) \in D_1$ and since $b \neq 0 \;\Rightarrow f(x^0) \in D_2$. So there exists $(Y^0, Z^0) \in \mathcal{B}$ with

$$f(x^0) = \begin{pmatrix} \frac{\alpha_1}{\beta_1} Y_1^0(a^1) \\ \cdots \\ \frac{\alpha_p}{\beta_p} Y_p^0(a^p) \\ 0 \\ \vdots \\ 0 \end{pmatrix} + Z^0 b = f^*(Y^0, Z^0).$$

According to weak duality (Theorem 5.7) we have $f[\mathcal{A}] \cap (f^*[\mathcal{B}] - K \backslash \{0\}) = \emptyset$, such that we get for $f(x^0) = f^*(Y^0, Z^0)$

$$\big(f^*(Y^0, Z^0) + (K \backslash \{0\})\big) \cap f^*[\mathcal{B}] = \emptyset.$$

This means

$$f(x^0) = f^*(Y^0, Z^0) \in \operatorname{Eff}_{\mathrm{Max}}(f^*[\mathcal{B}], K).$$

$\square$

For proving the converse duality, we first give the next helping result.

Lemma 5.11. *$d^0 \in \operatorname{Eff}_{\mathrm{Max}}(D_1, K)$ if and only if $\lambda_d^*(d^0) \geq \lambda_d^*(d)$ for all $d \in D_1$, where λ_d^* corresponds to d according to the definition of $\mathcal{B}_{\lambda_d^*}$.*

Proof. (a) Suppose that $d^0 \in \operatorname{Eff}_{\mathrm{Max}}(D_1, K)$. If $d' \in d^0 + K \setminus \{0\}$ (leading to $\lambda^*(d') > \lambda^*(d^0) = \sum_{i=1}^{p} \lambda^* \frac{\alpha_i}{\beta_i} Y^i(a^i) + u^*(b)$, $d^0 \in D_1$), then $d' \notin D_1$. So, for every $d \in D_1$ and all $\lambda_d^* \in \operatorname{int} K^*$ we have

$$\lambda_d^*(d^0) \geq \lambda_d^*(d) = \sum_{i=1}^{p} \lambda_{d,i}^* \frac{\alpha_i}{\beta_i} Y^i(a^i) + u^*(b),$$

for every $(Y, u^*) \in \mathcal{B}_{\lambda_d^*}$.

(b) Now let $d^0 \in D_1$, but $d^0 \in \operatorname{Eff}_{\mathrm{Max}}(D_1, K)$. Then there exists $d \in D_1$ with $d \in d^0 + K \setminus \{0\}$. Then

$$\forall \lambda^* \in \operatorname{int} K^* : \;\lambda^*(d^0) < \lambda^*(d),$$

which contradicts our assumption.

$\square$

Theorem 5.12 (Strong Converse Duality for $(\mathscr{P})$ and $(\mathscr{D})$)**.** *Assume that*

1. $b \neq 0$ *and* $\operatorname{int} K \neq \emptyset$.

2. $f[\mathcal{A}] + K$ *is closed.*

3. *There exists* $x^0 \in \mathcal{A}$ *with* $B(x^0) - b \in \operatorname{int} K_{\mathcal{U}}$.

4. *For every* $\lambda^* \in \operatorname{int} K^*$ *with* $\inf\{\lambda^*(f(x)) \mid x \in \mathcal{A}\} > -\infty$, *there exists* $(Y^0, Z^0) \in \mathcal{B}$ *with*
$$\sum_{j=1}^{m} \lambda^*_{p+j} C_{p+j} - \sum_{i=1}^{p} \frac{\alpha_i}{\beta_i} \lambda^*_i A^*_i Y^0_i - (Z^0(B))^* \lambda^* \in \operatorname{int} K^*_X.$$

5. *Furthermore, for every* $\lambda^* \in \operatorname{int} K^*$ *with* $\inf\{\lambda^*(f(x)) \mid x \in \mathcal{A}\} > -\infty$, *there exists* $x^\lambda \in \mathcal{A}$ *with* $\inf\{\lambda^*(f(x)) \mid x \in \mathcal{A}\} = \lambda^*(f(x^\lambda))$.

Then for every $f^*(Y^0, Z^0) \in \operatorname{Eff}_{\mathrm{Max}}(f^*[\mathcal{B}], K)$ *there is an element* $f(x^0) \in \operatorname{Eff}_{\mathrm{pSch}}(f[\mathcal{A}], K)$ *with*
$$f^*(Y^0, Z^0) = f(x^0). \tag{5.23}$$

Proof. Let $d^0 := f^*(Y^0, Z^0) \in \operatorname{Eff}_{\mathrm{Max}}(f^*[\mathcal{B}], K)$. That means d^0 is a (maximal) efficient element of $f^*[\mathcal{B}]$. Then it is also a (maximal) efficient element of D_2, and since $b \neq 0$ then d^0 is a (maximal) efficient element of D_1 too. This yields according to Lemma 5.11 $\lambda^*_d(d^0) \geq \lambda^*_d(d)$ for all $d \in D_1$ and all $\lambda^*_d \in \operatorname{int} K^*$. Then we can state that $d^0 := f^*(Y^0, Z^0) \in f[\mathcal{A}] + K$, otherwise

if $d^0 \notin f[\mathcal{A}] + K$, since $f[\mathcal{A}] + K$ is closed and convex we get according to Theorem 3.23 that there exists $\lambda^*_1 \in \mathbb{R}^{p+m} \setminus \{0\}$ and a real number γ with
$$\forall w \in f[\mathcal{A}] + K: \quad \lambda^*_1(d^0) < \gamma \leq \lambda^*_1(w). \tag{5.24}$$
We can conclude that $\lambda^*_1 \in K^* \setminus \{0\}$, because of (5.24) and the cone properties of K. Furthermore, from (5.21) we get for $d^0 \in D_1$ and for all $\lambda^*_2 \in \operatorname{int} K^*$ that
$$\forall w \in f[\mathcal{A}] + K: \quad \lambda^*_2(d^0) \leq \lambda^*_2(w). \tag{5.25}$$
Taking $\alpha \in (0,1)$, we consider $\lambda^*_\alpha := \alpha\lambda^*_2 + (1-\alpha)\lambda^*_1$. Then it holds that $\lambda^*_\alpha \in \operatorname{int} K^*$. By applying λ^*_α to d^0 we get
$$\begin{aligned} \lambda^*_\alpha(d^0) &= \alpha\lambda^*_2(d^0) + (1-\alpha)\lambda^*_1(d^0) \\ &< \alpha\lambda^*_2(f(x)) + (1-\alpha)\lambda^*_1(f(x)) = \lambda^*_\alpha(f(x)) \,, \forall x \in \mathcal{A} \end{aligned} \tag{5.26}$$
The inequality (5.26) is directly concluded from (5.24), (5.25) and from the fact that $0 \in K$. This implies that $\inf\{\lambda^*_\alpha(f(x)) \mid x \in \mathcal{A}\} > -\infty$ for $\lambda^*_\alpha \in \operatorname{int} K^*$. Then according to Assumption 5. there exists an element $x^\lambda \in \mathcal{A}$ with
$$\inf\{\lambda^*_\alpha(f(x)) \mid x \in \mathcal{A}\} = \lambda^*_\alpha(f(x^\lambda)). \tag{5.27}$$

From (5.26) and (5.27) we get that

$$\lambda_\alpha^*(d^0) < \lambda_\alpha^*(f(x^\lambda)). \tag{5.28}$$

Furthermore, from the strong duality we get that

$$f(x^\lambda) \in D_1$$

with the corresponding λ_α^*. Then by Lemma 5.11

$$\lambda_\alpha^*\left(f(x^\lambda)\right) \leq \lambda_\alpha^*(d^0),$$

which contradicts (5.28).

Thus, $d^0 = f^*(Y^0, Z^0) \in f[\mathcal{A}] + K$, that means there exists $x^0 \in \mathcal{A}$ with $f^*(Y^0, Z^0) \in f(x^0) + K$, i.e.,

$$f(x^0) \in f^*(Y^0, Z^0) - K.$$

On the other hand, weak duality (Theorem 5.7) yields that

$$f(x^0) \notin f^*(Y^0, Z^0) - (K \setminus \{0\}). \tag{5.29}$$

Hence,

$$f^*(Y^0, Z^0) = f(x^0). \tag{5.30}$$

Furthermore, from (5.21) and (5.30) we obtain for $\bar{\lambda}^* \in \operatorname{int} K^*$ (which corresponds to $(Y^0, Z^0) \in \mathcal{B}$):

$$\forall x \in \mathcal{A}: \quad \bar{\lambda}^*(f(x^0)) = \bar{\lambda}^*(f^*(Y^0, Z^0)) \leq \bar{\lambda}^*(f(x)).$$

Then according to Definition 4.5 we get $f(x^0) \in \mathrm{Eff}_{\mathrm{pSch}}(f[\mathcal{A}], K)$. □

Linearization of the Multiobjective Approximation Problem ($\mathscr{P}$)

It will be very interesting if the extended multiobjective approximation problem ($\mathscr{P}$) can also be solved as a multiobjective linear problem. This would enable us to make use of the well-known methods of the widely developed theory of multiobjective linear optimization.
In the following, we show that this is possible for some special cases of ($\mathscr{P}$), where block norms are used in the objective function, see also [72, 89].
Block norms are norms with polyhedral unit balls, for example the Manhattan norm and the maximum norm are block norms. It is also convenient to mention that every norm can be described through its unit ball and conversely. Therefore we describe the block norm through the structure of its polyhedral

unit ball in this approach. A polyhedral set can be characterized as the intersection of closed half spaces or as the convex hull of its extreme points. For a better overview see for example [100].

We take the following special case: $\beta_i = 1$ and $\alpha_i = 1$ for $i \in I_p = \{1, \cdots, p\}$, $x \in \mathbb{R}^{n_1}, a^i \in \mathbb{R}^{n_2}, A_i \in L(\mathbb{R}^{n_1}, \mathbb{R}^{n_2})$.

$$f(x) = \begin{pmatrix} \|A_1(x) - a^1\|_1 \\ \cdots \\ \|A_p(x) - a^p\|_p \\ C_{p+1}(x) \\ \cdots \\ C_{p+m}(x) \end{pmatrix} \tag{5.31}$$

Furthermore, for each $i \in I_p$ let the norms $\|\cdot\|_i : \mathbb{R}^{n_2} \to \mathbb{R}$ in (5.31) be block norms described through their unit balls $\mathbf{B}_i$. These unit balls $\mathbf{B}_i$ are described through their extremal points $b^{i1}, \cdots, b^{iq_i} \in \mathbb{R}^{n_2}$ by

$$\mathbf{B}_i = \{y \in \mathbb{R}^{n_2} \mid \langle b^{ij}, y\rangle \leq \gamma^{ij},\ \gamma^{ij} \in \mathbb{R},\ j = 1, \cdots, q_i\}. \tag{5.32}$$

The corresponding dual norms $\|\cdot\|_{i*}$ are also block norms described through their unit balls $\mathbf{B}_i^*$, which are again described through their extremal points $\tilde{b}^{i1}, \cdots, \tilde{b}^{i\tilde{q}_i} \in \mathbb{R}n_2$:

$$\mathbf{B}_i^* = \{y \in \mathbb{R}^{n_2} \mid \langle \tilde{b}^{ij}, y\rangle \leq \tilde{\gamma}^{ij},\ \tilde{\gamma}^{ij} \in \mathbb{R},\ j = 1, \cdots, \tilde{q}_i\} \tag{5.33}$$

The multiobjective location and approximation problem with the objective function described in (5.31) can be transformed into a **multiobjective linear program** described as follows:

$$(\mathscr{P}_{\mathscr{L}}) \quad \begin{cases} \text{Minimize} \quad F_{\mathscr{L}}(z) := (t^1, \cdots, t^p, t^{p+1}, \cdots, t^{p+m}) : \mathbb{R}^{n_3} \to \mathbb{R}^{p+m} \\ \text{subject to} \quad z \in \mathscr{A}, \end{cases} \tag{5.34}$$

for some feasible set $\mathscr{A}$, which will be described next together with n_3.
The variables $t^1, \cdots, t^p$ are real numbers, for which the following inequalities hold

$$\begin{gathered} \|A_p(x) - a^1\|_1 \leq t^1 \\ \cdots \\ \|A_p(x) - a^p\|_p \leq t^p. \end{gathered} \tag{5.35}$$

For $t^{p+1}, \cdots, t^{p+m}$ it hold that $t^{p+j} = C_{p+j}(x)$, for each $j \in I_m$.
The inequalities (5.35) are restrictions for the problem $(\mathscr{P}_{\mathscr{L}})$. However, these restrictions are not linear. Our next aim, is to reformulate the restrictions (5.35) in the linear form using the dual norms.
We know from the definition of the dual norm that

$$\|x^*\|_* = \sup_{\|x\| \leq 1} x^*(x) \quad \text{or} \quad \|x\| = \sup_{\|x^*\|_* \leq 1} x(x^*)$$

Applying this on (5.35) we get the following p inequalities:

$$\sup_{\|Y^{i*}\|_{i*}\leq 1} Y^{i*}(A_i(x)-a^i)\leq t^i, \quad i\in I_p, \tag{5.36}$$

where $Y^{i*}\in\mathbb{R}^{n_2}$. In order to get rid of the supremum in (5.36), the number of the inequalities is doubled to $2\cdot p$ as follows

$$Y^{i*}(A_i(x)-a^i)\leq t^i \tag{5.37a}$$

$$\|Y^{i*}\|_{i*}\leq 1 \tag{5.37b}$$

The inequalities (5.37a) are now linear. The inequalities (5.37b) mean that $Y^{i*}\in\mathbf{B}_i^*$ for $i\in I_p$. Thus, from (5.33) the variables Y^{i*} fulfill the inequality system:

$$Y^{i*}(\tilde{b}^{ij})\leq\tilde{\gamma}^{ij}, \tag{5.38}$$

where $\tilde{\gamma}^{ij}\in\mathbb{R}$ for $i=1,\cdots,p$ and $j=1,\cdots,\tilde{q}_i$.
Now all the restrictions $Y^{i*}(A_i(x)-a^i)\leq t^i$, $Y^{i*}(\tilde{b}^{ij})\leq\tilde{\gamma}^{ij}$ for $i=1,\cdots,p$ and $j=1,\cdots,\tilde{q}_i$ are linear.

Let $n_3:=n_1+p\cdot(n_2+1)$. Then the set $\mathscr{A}$ can be described as

$$\begin{aligned}\mathscr{A}=\{z\in\mathbb{R}^{n_3}\mid z=(x,t^1,\cdots,t^p,Y^{1*},\cdots,Y^{p*}),\ \text{with } x\in\mathbb{R}^{n_1},\ t^1,\cdots,t^p\in\mathbb{R},\\ Y^{1*},\cdots,Y^{p*}\in\mathbb{R}^{n_2},\forall\, i=1,\cdots,p,\ \forall\, j=1,\cdots,\tilde{q}_i:\\ Y^{i*}(A_i(x)-a^i)\leq t^i,\ \ Y^{i*}(\tilde{b}^{ij})\leq\tilde{\gamma}^{ij}\}.\end{aligned} \tag{5.39}$$

We conclude that the problem $(\mathscr{P})$ is able to be linearized in the special case described above (with $\beta_i=1$ and block norms), i.e., it is possible to transform the problem $(\mathscr{P})$ to a multiobjective linear problem $(\mathscr{P}_{\mathscr{L}})$ (given in (5.34)) with $n_1+p\cdot(n_2+1)=n_3$ variables, and $p+\sum_{i=1}^{p}\tilde{q}_i$ linear restrictions.
Among the well known methods for solving multiobjective linear problems, we suggest the online solver BENSOLVE. BENSOLVE is an open solver project based on Benson's outer approximation algorithm:

http://www.bensolve.org/

Benson's algorithms and its extensions can be found in [83].

5.3 Duality-based Characterizations of Solutions of Multiobjective Location Problems

The purpose of this section is to discuss the characterizations of solutions of Multiobjective Location Problems. If we observe the multiobjective location problems described below, we note that they are also a special case of the problem $(\mathscr{P})$ introduced in (5.6). We focus on the solutions of multiobjective location problems in the original space, since it is more convenient, due to the nature of location problems, to derive algorithms for computing the set of optimal solutions in the original space in difference to the image approaches, for example Benson's algorithm. Thus we study the sets of minimal and weakly minimal solutions of such multiobjective location problems.

In this section, first using duality assertions we give the characterizations obtained in [46] for minimal solutions of multiobjective location problems. Then using reducibility results we give a new characterization of weakly minimal solutions of multiobjective location problems [3]. These results are used in Chapter 6 in order to derive implementable algorithms for solving the extended multiobjective location problems through decomposition.

A multiobjective location problem, which is a special case of $(\mathscr{P})$ from Section 5.2, can be described as the following: Let $p \geq 2$, we consider the existing facilities

$$a^1 = (a_1^1, a_2^1), \cdots, a^p = (a_1^p, a_2^p) \in \mathbb{R}^2,$$

and denote $E_a := \{a^1, \cdots, a^p\}$.

The multiobjective single-facility location problem is to find a new location $x \in \mathbb{R}^2$ by considering the functions $f_i(x) := \|x - a^i\|, i \in I_p$, where $\|\cdot\|$ is a norm in $\mathbb{R}^2$.

The primal objective function of a multiobjective location problem can be described by

$$f_{I_p}(x) := \begin{pmatrix} \|x - a^1\| \\ \cdots \\ \|x - a^P\| \end{pmatrix}. \tag{5.40}$$

We observe that the function given by (5.40) is a special case of (5.4) by taking $m = 0$, $\beta_i = 1$, $\|\cdot\|_{(i)} = \|\cdot\|$ and $A_i = I$ for $i \in I_p$.

We consider the following multiobjective location problem

$$(\mathscr{P}_1) \qquad \operatorname{Eff}(f_{I_p}[\mathbb{R}^2], \mathbb{R}^p_+). \tag{5.41}$$

As mentioned above, it is appreciable to find the set of minimal solutions of the planar multiobjective location problem in the original space, i.e., the set $\operatorname{Min}(\mathbb{R}^2, f_{I_p})$ (see(4.3)). Many characterization of the set of minimal solutions of a planar multiobjective location problem $(\mathscr{P}_1)$ can be found in the literature, for instance in [21, 46, 80, 120] and the references cited in these papers. We introduce the characterization of the set of minimal solutions $\mathrm{Min}(\mathbb{R}^2, f_{I_p})$ of the multiobjective location problem $(\mathscr{P}_1)$ used in Gerth and Pöhler [46], and then we give a new result for characterizing the set of weakly minimal solutions $\mathrm{Min}_{\mathrm{w}}(\mathbb{R}^2, f_{I_p})$ of $(\mathscr{P}_1)$.

This characterization (see [46]) is based on the duality assertions shown in Section 5.2. Its geometrical structure is related to the **the maximum norm ($\|\cdot\|_\infty$) or the Manhattan norm ($\|\cdot\|_1$)**. Thus, from now on we set the norms used in (5.40) to be either the Manhattan norm or the maximum norm. This does not weaken our problem or limit it when we consider the importance of these two norms. The Manhattan norm is very useful for representing the street nets and the electronic semiconductor, which match many applications, while the maximum norm is an approximation of the p-norms.

It is meaningful to study the dual problem to the primal multiobjective location problem $(\mathscr{P}_1)$. Taking into account the construction of the dual problem in Section 5.2, we get the following dual objective function concerning the dual problem to the primal problem $(\mathscr{P}_1)$ with the maximum norm or the Manhattan norm

$$f^*_{I_p}(Y) = \begin{pmatrix} Y_1(a^1) \\ \cdots \\ Y_p(a^p) \end{pmatrix},$$

with the dual feasible set

$$\mathcal{B}_{I_p} = \{Y := (Y^1, \cdots, Y^p), Y^i \in L(\mathbb{R}^2, \mathbb{R}^1) : \exists \lambda^* \in \operatorname{int} \mathbb{R}^p_+ \text{ with } \sum_{i=1}^{p} \lambda^* Y^i = 0, \|Y^i\|_* \leq 1\};$$

where $\|\cdot\|_*$ is the dual norm of the given norm. Note that the maximum norm and the Manhattan norm are dual to each other. This means if we use the maximum norm in (5.40), then the norm used in $\mathcal{B}_{I_p}$ is the Manhattan norm and conversely.
The dual problem of $(\mathscr{P}_1)$ is

$$(\mathscr{D}_1) \quad \mathrm{Eff}_{\mathrm{Max}}\,(f^*[\mathcal{B}_{I_p}], \mathbb{R}^p_+)$$

Before going through the geometrical algorithm for characterizing the set of minimal solutions of $(\mathscr{P}_1)$, we introduce the next basic theorem for characterizing a minimal solution $x \in \mathrm{Min}(\mathbb{R}^2, f_{I_p})$. For p existing facilities $a^1, \cdots, a^p$, we say that a point $x = (x_1, x_2) \in \mathbb{R}^2$ satisfies the **condition** $\mathfrak{C}(x)$ if and only if for every existing facility a^i there exists a^j such that

$$\|x - a^i\| + \|x - a^j\| = \|a^i - a^j\|. \tag{5.42}$$

Theorem 5.13 ([21]). *$x = (x_1, x_2)$ is a minimal solution of $(\mathscr{P}_1)$ with the maximum norm or Manhattan norm, if and only if x satisfies the condition $\mathfrak{C}(x)$.*

Proof. Let x be a given point such that condition $\mathfrak{C}(x)$ holds. We suppose that x is not a minimal solution, which means that there exists a point y such that $\|y-a^i\| \le \|x-a^i\|$ for all $i = 1, \cdots, p$ and $\|y-a^s\| < \|x-a^s\|$ for some s. Since $\mathfrak{C}(x)$ holds, there exists a facility a^t with $\|x-a^s\|+\|x-a^t\| = \|a^s-a^t\|$. But this leads to

$$\|y-a^s\| + \|y-a^t\| < \|x-a^s\| + \|x-a^t\| = \|a^s-a^t\|$$

which contradicts the triangle inequality. So x is a minimal solution.
Let $x^0 = (x_1^0, x_2^0)$ be a given point, for which the condition $\mathfrak{C}(x)$ does not hold. We want to prove that there is some point, which dominates x^0. Define Q_1, Q_2, Q_3 and Q_4 as the four ordinary quadrants of $\mathbb{R}^2$ with the point x^0 as the origin.
$\mathfrak{C}(x)$ does not hold, that means there is a facility $a^i = (a_1^i, a_2^i)$ such that for all $a^j \in \{1, \cdots, p\}$:

$$\|x^0 - a^i\| + \|x^0 - a^j\| > \|a^i - a^j\|$$

Without loss of generality, let $a^i \in Q_1$. We consider the following three cases:

- Case 1: $a_1^i = x_1^0,\ a_2^i > x_2^0$
- Case 2: $a_1^i > x_1^0,\ a_2^i = x_2^0$
- Case 3: $a_1^i > x_1^0,\ a_2^i > x_2^0$

Since Case 2 is a complete analog of Case 1, we shall only consider Case 1 and Case 3.

Case 1: For all facilities a^j in Q_3 and Q_4 we have

$$\|x^0 - a^i\|_1 + \|x^0 - a^j\|_1 = \|a^i - a^j\|_1$$

Therefore, there are no facilities in Q_3 and Q_4. Since not all facilities are on a line, there is a facility in $\operatorname{int} Q_1$ or $\operatorname{int} Q_2$. Without less of generality, we assume that $\operatorname{int} Q_1$ contains a facility. Let

$$c := \min_{a_1^i > x_1^0} (a_1^i - x_1^0), \quad d := \min_{a_2^i > x_2^0} (a_2^i - x_2^0).$$

Define the point $y^0 = (y_1^0, y_2^0)$ as the following. If $d \le c$, then y^0 is intersection of the lines $x - y = x_1^0 - x_2^0$ and $y = x_2^0 + d$. If $d > c$, then y^0 is intersection of the lines $x - y = x_1^0 - x_2^0$ and $x = x_1^0 + c$. Note that $\|x^0 - y^0\| > 0$. We prove now that y^0 dominates x^0. For all facilities a^j in Q_2 we have $a_1^j \le x_1^0$ and $a_2^j \ge x_2^0 + d$. Therefore,

$$\|y^0 - a^j\| = (y_1^0 - a_1^j) + (a_2^j - y_2^0) \quad \text{and} \quad \|x^0 - a^j\| = (x_1^0 - a_1^j) + (a_2^j - x_2^0).$$

Since $x_1^0 - x_2^0 = y_1^0 - y_2^0$, we have $\|x^0 - a^j\| = \|y^0 - a^j\|$. For all a^j in $\operatorname{int} Q_1$ we have $a_1^j \geq x_1^0 + c$ and $a_2^j \geq x_2^0 + d$, therefore

$$\|x^0 - a^j\| = \|y^0 - a^j\| + \|x^0 - y^0\| > \|y^0 - a^j\|.$$

Since $\operatorname{int} Q_1$ contains at least one facility we have proven that y^0 dominates x^0.

Case 3: Again, we define y^0 as in Case 1 and prove that y^0 dominates x^0. There can be no facilities in Q_3 since for all facilities a^j in Q_3 we have

$$\|x^0 - a^i\| + \|x^0 - a^j\| = \|a^i - a^j\|.$$

For all facilities a^j in Q_2 we have $\|x^0 - a^j\| = \|y^0 - a^j\|$ using the same argument as in Case 1. For all facilities a^j in Q_4 we have $a_1^j \geq x_1^0 + c$ and $a_2^j \leq x_2^0$: therefore

$$\|y^0 - a^j\| = (a_1^j - y_1^0) + (y_2^0 - a_2^j) \quad \text{and} \quad \|x^0 - a^j\| = (a_1^j - x_1^0) + (x_2^0 - a_2^j).$$

Since $x_1^0 - x_2^0 = y_1^0 - y_2^0$ again we have $\|x^0 - a^j\| = \|y^0 - a^j\|$. For all facilities a^j in $\operatorname{int} Q_1$ we have $a_1^j \geq x_1^0 + c$ and $a_2^j \geq x_2^0 + d$. Therefore

$$\|x^0 - a^j\| = \|y^0 - a^j\| + \|x^0 - y^0\| > \|y^0 - a^j\|.$$

Since $\operatorname{int} Q_1$ contains at least one facility we have proven that y^0 dominates x^0.

□

Theorem 5.13 leads to a significant result, that is the whole set of minimal solutions can not lie outside a rectangle, whose lines correspond to the level lines of the dual norm of the given norm, which contains the existing facilities $a^1, \cdots, a^p$. This rectangular set will be often used by our algorithms and results in order to characterize the set of minimal and weakly minimal solution of the multiobjective location problem $(\mathscr{P}_1)$. Therefore, we describe it geometrically and analytically as follows:
Geometrically, the level lines mentioned above are the lines going through the existing points $a^1, \cdots, a^p$, which are either:

- The lines ($x_1 = a_1^i$ and $x_2 = a_2^i$ for $i \in I_p$) which are parallel to the level lines of the maximum norm, when we use the Manhattan norm in the objective function described in (5.41). In this case, the smallest rectangular constructed by these lines containing $a^1, \cdots, a^p$ is denoted by $\mathcal{N}_\infty$ and called **the maximum rectangular hull** of the existing facilities $a^1, \cdots, a^p$.

- Or the lines ($x_1 + x_2 = a_1^i + a_2^i$ and $x_1 - x_2 = a_1^i - a_2^i$ for $i \in I_p$) which are parallel to the level lines of the Manhattan norm, when we use the maximum norm in (5.41). In this case, the smallest rectangular constructed by these lines containing $a^1, \cdots, a^p$ is denoted by $\mathcal{N}_1$ and called **the Manhattan rectangular hull** of the existing facilities $a^1, \cdots, a^p$.

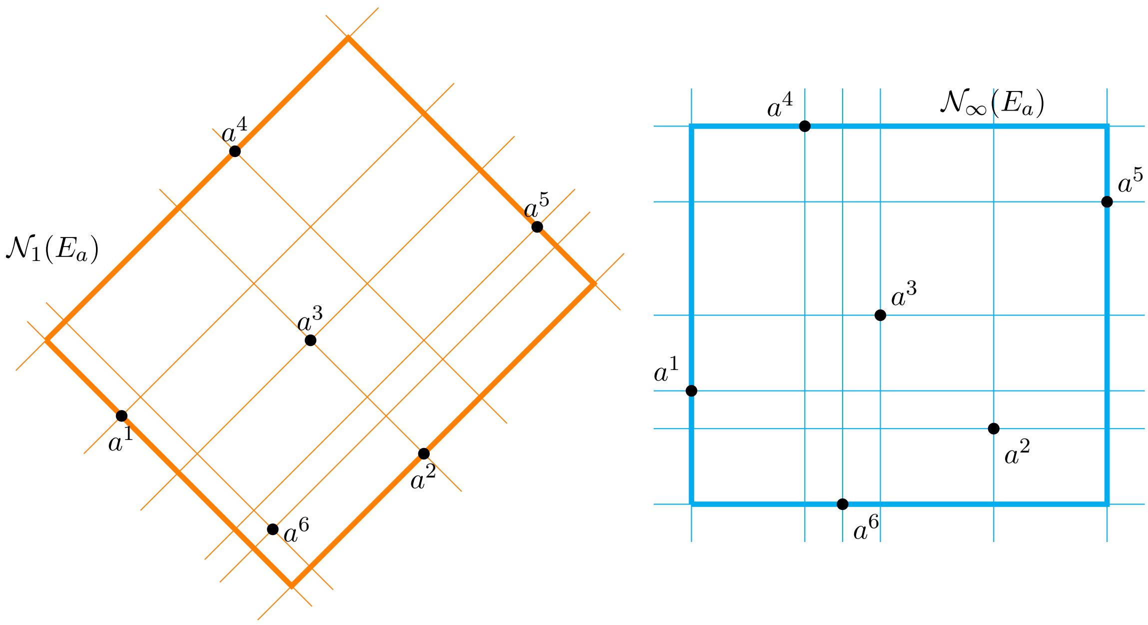


Figure 5.1: The Manhattan rectangular hull and the maximum rectangular hull of the points $a^1, \cdots, a^5$.

Figure 5.1 illustrates the above described Manhattan rectangular hull (on the left side) and the maximum rectangular hull (on the right side) with the constructing lines. Moreover, see Figure 5.2 in order to compare the Manhattan rectangular hull and the maximum rectangular hull with the well-known convex hull.

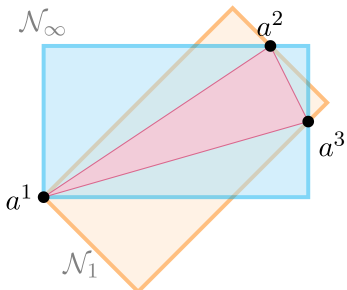


Figure 5.2: The convex hull, the Manhattan rectangular hull and the maximum rectangular hull of the points a^1, a^2, a^3.

From now on we set the maximum norm $\|\cdot\|_\infty$ to be used in (5.40). Now we define the Manhattan rectangular hull $\mathcal{N}_1$ mathematically (the maximum rectangular hull $\mathcal{N}_\infty$ can be analogously described in a similar even easier way). In the following we drop the index in $\mathcal{N}_1$ taking into account that $\mathcal{N}$ without any index is understood as the Manhattan rectangular hull.

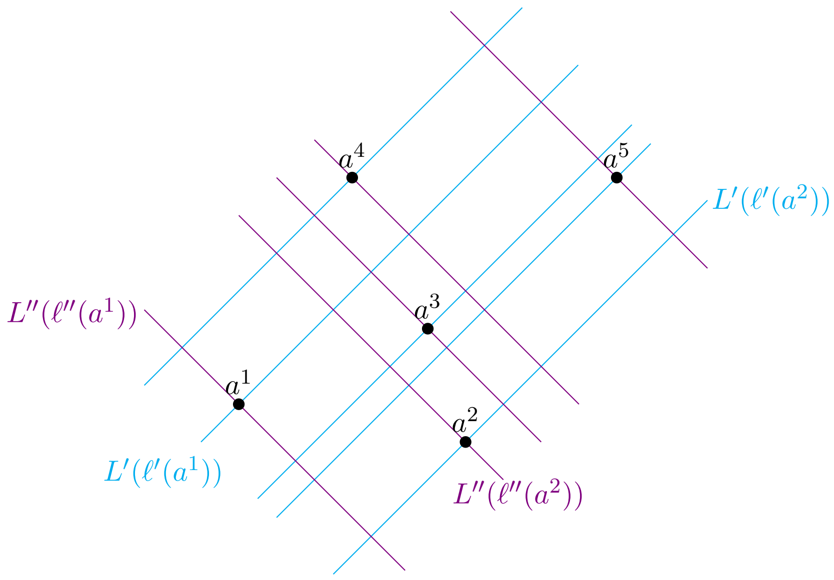


Figure 5.3: The lines $L'(a^i), L''(a^i)$

The **Manhattan rectangular hull** of any nonempty bounded subset A of $\mathbb{R}^2$ is defined as the intersection of all Manhattan closed balls containing A, i.e.,

$$\mathcal{N}(A) := \bigcap \left\{ B(x,r) \;\middle|\; x \in \mathbb{R}^2, r > 0, A \subset B(x,r) \right\},$$

where $B(x,r) := \{y \in \mathbb{R}^2 \mid \|y - x\|_1 \leqslant r\}$. Now, we describe the Manhattan hull analytically by introducing the linear functions $\ell' \colon \mathbb{R}^2 \to \mathbb{R}$ and $\ell'' \colon \mathbb{R}^2 \to \mathbb{R}$, given by

$$\ell'(x) := -x_1 + x_2 \quad \text{and} \quad \ell''(x) := x_1 + x_2$$

for all $x = (x_1, x_2) \in \mathbb{R}^2$. It can be easily seen that

$$N(A) = \{x \in \mathbb{R}^2 | \inf \ell'(A) \leqslant \ell'(x) \leqslant \sup \ell'(A), \inf \ell''(A) \leqslant \ell''(x) \leqslant \sup \ell''(A)\}. \tag{5.43}$$

Geometrically, the level sets of the linear functions ℓ' and ℓ'' play an important role for describing the rectangular Manhattan hull and for deriving later results. For any real number α we denote

$$\begin{aligned} L'_{\leqslant}(\alpha) &:= \{x \in \mathbb{R}^2 \mid \ell'(x) \leqslant \alpha\}, \quad L''_{\leqslant}(\alpha) := \{x \in \mathbb{R}^2 \mid \ell''(x) \leqslant \alpha\}, \\ L'_{\geqslant}(\alpha) &:= \{x \in \mathbb{R}^2 \mid \ell'(x) \geqslant \alpha\}, \quad L''_{\geqslant}(\alpha) := \{x \in \mathbb{R}^2 \mid \ell''(x) \geqslant \alpha\}. \end{aligned} \tag{5.44}$$

We observe that these level sets are closed half-planes, which are bounded by straight-lines parallel to the first bisector and the second bisector of the plane, respectively:

$$L'(\alpha) \;:= \{x \in \mathbb{R}^2 \mid \ell'(x) = \alpha\}, \quad L''(\alpha) := \{x \in \mathbb{R}^2 \mid \ell''(x) = \alpha\}. \tag{5.45}$$

In terms of level sets, $\mathcal{N}(A)$ can be expressed (see [3])

$$\mathcal{N}(A) = L'_{\geqslant}\left(\inf \ell'(A)\right) \cap L'_{\leqslant}\left(\sup \ell'(A)\right) \cap L''_{\geqslant}\left(\inf \ell''(A)\right) \cap L''_{\leqslant}\left(\sup \ell''(A)\right), \tag{5.46}$$

which shows that $\mathcal{N}(A)$ is a polyhedral set, as being the intersection of four closed half-planes. More precisely it is a rectangle, a line segment or a singleton.

In [46] a characterization of the set $\mathrm{Min}(\mathbb{R}^2, f_{I_p})$ is given. We use the Manhattan hull of the existing facilities, denoted for simplicity by

$$\mathcal{N} := \mathcal{N}(\{a^1, \cdots, a^p\})$$

and certain sets related to the structure of the subdifferential of the maximum norm. More precisely, for each $i \in I_p$ one defines four open sets bounded by the straight lines $L'(\ell'(a^i))$ and $L''(\ell''(a^i))$ passing through a^i and being parallel to the first and the second bisectors of the plane, namely

$$\begin{aligned}
S_1(a^i) &:= \{x \in \mathbb{R}^2 \mid a_2^i - x_2 > |a_1^i - x_1|\} = \operatorname{int} L'_{\leqslant}(\ell'(a^i)) \cap \operatorname{int} L''_{\leqslant}(\ell''(a^i)) \\
S_2(a^i) &:= \{x \in \mathbb{R}^2 \mid x_2 - a_2^i > |a_1^i - x_1|\} = \operatorname{int} L'_{\geqslant}(\ell'(a^i)) \cap \operatorname{int} L''_{\geqslant}(\ell''(a^i)) \\
S_3(a^i) &:= \{x \in \mathbb{R}^2 \mid a_1^i - x_1 > |a_2^i - x_2|\} = \operatorname{int} L'_{\geqslant}(\ell'(a^i)) \cap \operatorname{int} L''_{\leqslant}(\ell''(a^i)) \\
S_4(a^i) &:= \{x \in \mathbb{R}^2 \mid x_1 - a_1^i > |a_2^i - x_2|\} = \operatorname{int} L'_{\leqslant}(\ell'(a^i)) \cap \operatorname{int} L''_{\geqslant}(\ell''(a^i))
\end{aligned}$$

and then, for every $r \in \{1, 2, 3, 4\}$, one constructs the set

$$S_r := \{x \in \mathcal{N} \mid \exists\, i \in I_p : x \in S_r(a^i)\} = \mathcal{N} \cap \left(\cup_{i=1}^p S_r(a^i)\right).$$

The following preliminary result was established in [46].

Lemma 5.14 ([46]). *The set of minimal solutions of the multiobjective location problem* $(\mathscr{P}_1)$ *with the maximum norm has the following representation:*

$$\begin{aligned}
\mathrm{Min}(\mathbb{R}^2, f_{I_p}) = &\left[(\operatorname{cl} S_1 \cap \operatorname{cl} S_2) \cup \left((\mathcal{N} \setminus S_1) \cap (\mathcal{N} \setminus S_2)\right)\right] \\
&\cap \left[(\operatorname{cl} S_3 \cap \operatorname{cl} S_4) \cup \left((\mathcal{N} \setminus S_3) \cap (\mathcal{N} \setminus S_4)\right)\right].
\end{aligned}$$

As we see, Lemma 5.14 generates the set of minimal solutions of the multiobjective location problem $(\mathscr{P}_1)$, noticing that the set $\mathcal{N} := \mathcal{N}(\{a^1, \cdots, a^p\})$ determined in (5.46) is a closed rectangle, possibly degenerated into a line segment or a singleton.

Remark 5.15. We can formulate the characterization of solutions of $(\mathscr{P}_1)$ given in Lemma 5.14 analogously also for the multiobjective location problem $(\mathscr{P}_1)$ with Manhattan norm. We observe only that the structure of the sets $S_1(a^i)$ – $S_1(a^i)$ is different.

In order to learn more about the structure of the set of minimal solutions generated by Lemma 5.14, we introduce the following characterizing result.

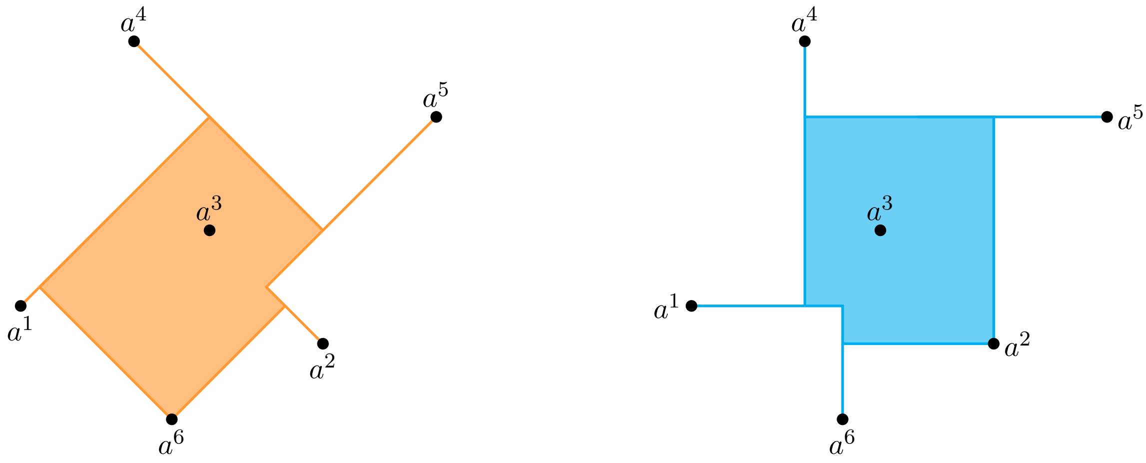


Figure 5.4: The set of minimal solutions $\mathrm{Min}(\mathbb{R}^2, f_{I_p})$ of $(\mathscr{P}_1)$ with the maximum norm and the Manhattan norm.

Lemma 5.16 ([4, Lemma 3.2])**.** *The set of minimal solutions* $\mathrm{Min}(\mathbb{R}^2, f_{I_p})$ *of the multiobjective location problem* $(\mathscr{P}_1)$ *with the maximum norm has the following properties:*

1. $E_a \subset \mathrm{Min}(\mathbb{R}^2, f_{I_p})$.
2. $\mathrm{Min}(\mathbb{R}^2, f_{I_p})$ *can be represented as a finite union of (possibly degenerated) rectangles in the plane.*
3. $\mathrm{Min}(\mathbb{R}^2, f_{I_p}) = \mathrm{Min_p}(\mathbb{R}^2, f_{I_p})$.

The previous lemma creates the basis of the ability to partition the set $\mathrm{Min}(\mathbb{R}^2, f_{I_p})$ into convex sets, which is our result in Section 6.1. This partition plays the key role for the decomposition methods for solving extended multiobjective location problems in Section 6.2.

Our aim now, is to prove that the Manhattan rectangular hull $\mathcal{N}$ is actually the set of weakly minimal solutions of the multiobjective location problem $(\mathscr{P}_1)$. To this end, we introduce a result on the structure of Manhattan hulls [3], which can be also used for not necessarily finite sets.

Lemma 5.17 ([3])**.** *If A is a nonempty compact subset of* $\mathbb{R}^2$*, then*

$$\mathcal{N}(A) = \bigcup_{x,y\in A} \mathcal{N}(\{x,y\}). \tag{5.47}$$

Proof. Assume that $A \subset \mathbb{R}^2$ is a nonempty compact set. According to its definition, the hull operator $\mathcal{N}(\cdot)$ is isotonic, i.e., $\mathcal{N}(S) \subset \mathcal{N}(A)$ for every nonempty bounded set $S \subset A$. Hence for all $x, y \in A$ we have $\mathcal{N}(\{x,y\}) \subset \mathcal{N}(A)$, which entails the inclusion ”$\supset$” in (5.47). Notice that this inclusion is still true even if A is not closed.

Now let us prove the inclusion "$\subset$" in (5.47). By compactness of A and continuity of ℓ' and ℓ'', we can choose four (not necessarily distinct) points :

$$\underline{a}' \in \operatorname*{argmin}_{a\in A} \ell'(a),\ \overline{a}' \in \operatorname*{argmax}_{a\in A} \ell'(a),\ \underline{a}'' \in \operatorname*{argmin}_{a\in A} \ell''(a),\ \text{and } \overline{a}'' \in \operatorname*{argmax}_{a\in A} \ell''(a).$$

On the one hand, by (5.43) we can deduce that

$$\begin{aligned} \mathcal{N}(A) &= \{x \in \mathbb{R}^2 \mid \ell'(\underline{a}') \leqslant \ell'(x) \leqslant \ell'(\overline{a}'),\, \ell''(\underline{a}'') \leqslant \ell''(x) \leqslant \ell''(\overline{a}'')\} \\ &= \mathcal{N}(\{\underline{a}', \overline{a}', \underline{a}'', \overline{a}''\}). \end{aligned}$$

On the other hand, it is a simple exercise of planar geometry to check that

$$\mathcal{N}(\{\underline{a}', \overline{a}', \underline{a}'', \overline{a}''\}) = \bigcup_{x,y\in\{\underline{a}', \overline{a}', \underline{a}'', \overline{a}''\}} \mathcal{N}(\{x, y\}).$$

Thus the inclusion "$\subset$" in (5.47) holds true. □

Furthermore, we state that the multiobjective location problem $(\mathscr{P}_1)$ is Pareto reducible, since all distance functions $f_i(x) = \|x - a^i\|$ given in (5.40) are convex for $i \in I_p$, (cf. Corollary 4.19). This implies due to (4.20) that the equality

$$\operatorname{Min}_{\mathrm{w}}(\mathbb{R}^2, f_{I_p}) = \bigcup_{\emptyset\neq I\subset I_p} \operatorname{Min}(\mathbb{R}^2, f_I) \tag{5.48}$$

holds for the problem $(\mathscr{P}_1)$.

Now, with the help of the Pareto reducibility of the problem $(\mathscr{P}_1)$, Lemma 5.17 and the characterization of the set of minimal solutions of $(\mathscr{P}_1)$ in Lemma 5.14, we introduce the following characterization of the set of weakly minimal solutions of the multiobjective location problem $(\mathscr{P}_1)$.

Theorem 5.18 ([3]). *The following properties hold for the multiobjective location problem $(\mathscr{P}_1)$ with the maximum norm*:

1. *For every index set $I \subset I_p$ with cardinality $|I| \in \{1, 2\}$,*

$$\operatorname{Min}_{\mathrm{w}}(\mathbb{R}^2, f_I) = \operatorname{Min}(\mathbb{R}^2, f_I) = \mathcal{N}(\{a^i \mid i \in I\}).$$

2. *The set of weakly minimal solutions is given by*

$$\operatorname{Min}_{\mathrm{w}}(\mathbb{R}^2, f_{I_p}) = \bigcup_{I\subset I_p, |I|=2} \operatorname{Min}(\mathbb{R}^2, f_I) = \mathcal{N}(\{a^1, \cdots, a^p\}) =: \mathcal{N}.$$

Proof. In order to prove 1, consider a set $I \subset I_p$ with $|I| \in \{1, 2\}$.

If $|I| = 1$, then $I = \{i\}$ for some $i \in I_p$ and, in view of (4.12) and taking into account that $\|v\|_\infty = 0$ if and only if $v = (0,0)$, it is easily seen that

$$\mathrm{Min_w}(\mathbb{R}^2, f_I) = \mathrm{Min}(\mathbb{R}^2, f_I) = \underset{x\in\mathbb{R}^2}{\mathrm{argmin}}\, f_i(x) = \{a^i\} = \mathcal{N}(\{a^i\}).$$

If $|I| = 2$, then we have $\mathrm{Min_w}(\mathbb{R}^2, f_I) = \mathrm{Min}(\mathbb{R}^2, f_I)$ according to Corollary 4.18, since for every $i \in I$ the distance function $f_i(\cdot) = \|\cdot - a^i\|_\infty$ is convex and attains its minimal value on $\mathbb{R}^2$ at a unique point, namely a^i. On the other hand, by applying Lemma 5.14 for the bicriteria location problem (P_I), one can check geometrically that $\mathrm{Min}(\mathbb{R}^2, f_I) = \mathcal{N}(\{a^i \mid i \in I\})$. We are now going to prove 2. Observe that given any $I \subset I_p$ with $|I| = 2$ it follows from 1 that $\mathrm{Min}(\mathbb{R}^2, f_i) \subset \mathrm{Min}(\mathbb{R}^2, f_I) = \mathrm{Min}(\mathbb{R}^2, f_I) = \mathcal{N}(\{a^i \mid i \in I\})$ for each $i \in I$. Hence

$$\bigcup_{\substack{I\subset I_p\\|I|=2}} \mathrm{Min}(\mathbb{R}^2, f_I) = \bigcup_{\substack{I\subset I_p\\1\leqslant|I|\leqslant 2}} \mathrm{Min}(\mathbb{R}^2, f_I) = \bigcup_{\substack{I\subset I_p\\1\leqslant|I|\leqslant 2}} \mathcal{N}(\{a^i \mid i \in I\}).$$

On the other hand, as a direct consequence of Lemma 5.17 we infer that

$$\bigcup_{I\subset I_p, 1\leqslant|I|\leqslant 2} \mathcal{N}(\{a^i \mid i \in I\}) = \mathcal{N}(\{a^1, \cdots, a^p\}).$$

Thus, one of the claimed equalities holds true:

$$\bigcup_{I\subset I_p, |I|=2} \mathrm{Min}(\mathbb{R}^2, f_I) = \mathcal{N}(\{a^1, \cdots, a^p\}). \tag{5.49}$$

Further we observe that, in view of (4.12), we have $\mathrm{Min}(\mathbb{R}^2, f_I) \subset \mathrm{Min_w}(\mathbb{R}^2, f_{I_p})$ for any nonempty $I \subset I_p$. Thus (5.49) entails

$$\mathcal{N}(\{a^1, \cdots, a^p\}) \subset \mathrm{Min_w}(\mathbb{R}^2, f_{I_p}). \tag{5.50}$$

By applying Lemma 5.14 to any subproblem (P_I) with $\emptyset \neq I \subset I_p$, we deduce that

$$\mathrm{Min}(\mathbb{R}^2, f_I) \subset \mathcal{N}(\{a^i \mid i \in I\}). \tag{5.51}$$

Since the problem $(\mathscr{P}_1)$ is Pareto reducible and the Manhattan hull operator $\mathcal{N}(\cdot)$ is isotone, it follows from (5.51) and the equality (5.48) that

$$\mathrm{Min_w}(\mathbb{R}^2, f_{I_p}) \subset \bigcup_{\emptyset\neq I\subset I_p} \mathcal{N}(\{a^i \mid i \in I\}) \subset \mathcal{N}(\{a^1, \cdots, a^p\}). \tag{5.52}$$

Finally, from (5.50) and (5.52) we conclude that

$$\mathrm{Min_w}(\mathbb{R}^2, f_{I_p}) = \mathcal{N}(\{a^1, \cdots, a^p\})$$

as claimed. Thus the assertion 2 is true. □

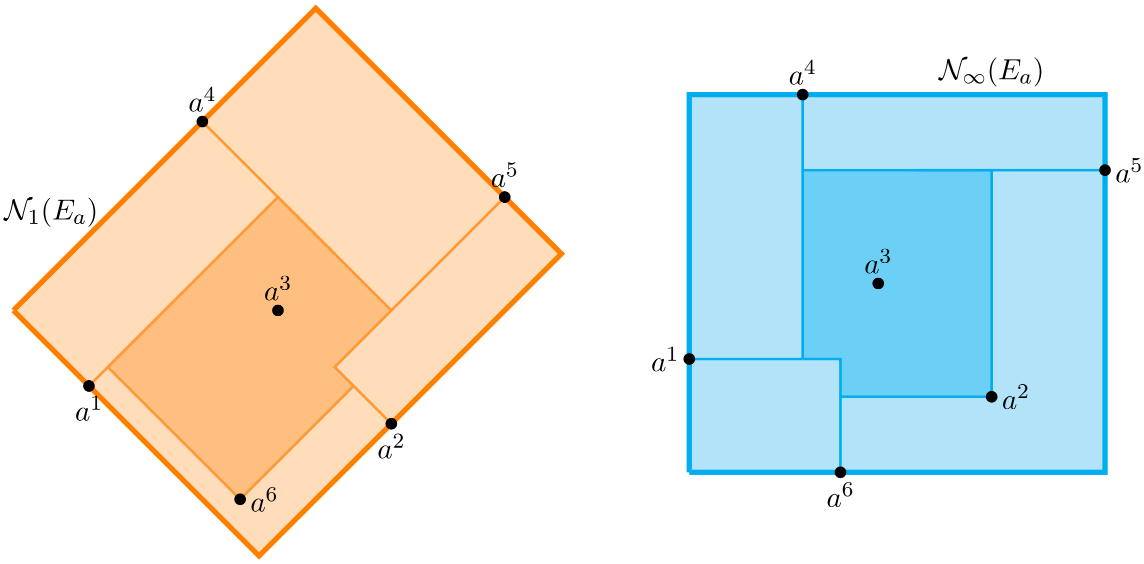


Figure 5.5: The set of (weakly) minimal solutions of $(\mathscr{P}_1)$ with the maximum norm and the Manhattan norm.

The results Lemma 5.17 – Theorem 5.18 can be analogously developed for the multiobjective location problem $(\mathscr{P}_1)$ with the Manhattan norm using the maximum rectangular hull $\mathcal{N}_\infty$.

Example 5.19. For the points $a^1, \cdots, a^6$ in the plane (see Figure 5.5), we can compute the set of minimal solutions of the multiobjective location problem $(\mathscr{P}_1)$ concerning $a^1, \cdots, a^6$ with both the maximum norm and the Manhattan norm by applying Lemma 5.14. Furthermore, if we compute the Manhattan rectangular hull $\mathcal{N}_1$ and the maximum rectangular hull $\mathcal{N}_\infty$, we obtain by Theorem 5.18 the set of weakly minimal solutions of the multiobjective location problem $(\mathscr{P}_1)$ concerning $a^1, \cdots, a^6$. Figure 5.5 on the left, shows the set of the minimal solutions with respect to the maximum norm in dark orange, as well as the set of weakly minimal solutions in light orange, which coincides with the the Manhattan rectangular hull $\mathcal{N}_1$ of the points $a^1, \cdots, a^6$. On the right, we see the set of the minimal solutions with respect to the Manhattan norm in dark cyan, as well as the set of weakly minimal solutions in light cyan, which coincides with the maximum rectangular hull $\mathcal{N}_\infty$ of the points $a^1, \cdots, a^6$.

CHAPTER 6

Decomposition Methods for Multiobjective Single-Facility Location Problems

In this chapter, we develop decomposition algorithms for solving extended multiobjective location problems (a special case of $(\mathscr{P})$ with $\alpha_i = 1,\ \beta_i = 1,\ A_i = \mathbb{I}$ and the norms $\|\cdot\|_{(i)}$ are either $\|\cdot\|_1$ for $i = 1, \dots, p$ or $\|\cdot\|_\infty$ for $i = 1, \dots, p$) by dividing it into simpler subproblems. One of these subproblems is the multiobjective location problem, which we introduced as $(\mathscr{P}_1)$ with the objective function defined in (5.40).

In Section 5.3 we derived a duality-based characterization of the solution set $\mathrm{Min}(\mathbb{R}^2, f_{I_p})$ of $(\mathscr{P}_1)$. In order to develop a decomposition method for the extended multiobjective location problem, a partition of the solution set $\mathrm{Min}(\mathbb{R}^2, f_{I_p})$ of the subproblem $(\mathscr{P}_1)$ is important. Implementable algorithms for this partition are given in Section 6.1. Furthermore, we develop decomposition algorithms (with different applications) for solving the extended multiobjective location problem (see [1, 2, 3, 4, 51]) using this partition in Section 6.2.

6.1 Partition Algorithm for Multiobjective Location Problems

Now we consider the multiobjective location problem $(\mathscr{P}_1)$ with the maximum norm or the Manhattan norm. As mentioned above the problem $(\mathscr{P}_1)$ is intensively studied in Section 5.3. The duality-based characterization of the solutions set $\mathrm{Min}(\mathbb{R}^2, f_{I_p})$ of $(\mathscr{P}_1)$ is given in Lemma 5.14.

Our goal is to find an algorithm, to partition the set $\mathrm{Min}(\mathbb{R}^2, f_{I_p})$, which is generally not convex, and divide it into a finite number of convex subsets. These subsets are (possibly degenerated) rectangles, which can be constructed using the structure of the geometrical characterization of $\mathrm{Min}(\mathbb{R}^2, f_{I_p})$, see Lemma 5.14 and Lemma 5.16.

In order to formulate the partition algorithm (Algorithm 6.3), we take a look on a method, which serves to be an efficient reduction step for our partition algorithm. This method is an algorithm for computing all efficient elements of the discrete set $E_a = \{a^1, \cdots, a^p\}$ with respect to a certain cone in $\mathbb{R}^2$. That is the **Jahn-Graef-Younes method**, (see [73, 74] and [4]). The Jahn-Graef-Younes method can be formulated for some proper, pointed, closed and convex cone. Other versions, which are developed in [4], formulate the algorithm after sorting the set of the facilities $a^1, \cdots, a^p$ within the set E_a. The version we introduce next, also developed in [4], uses contrarily to the other versions the special structure of the cones $K_1, \cdots, K_4$ defined below.

$$K_1 := \mathbb{R}_+ \times \mathbb{R}_+, \qquad K_2 := \mathbb{R}_- \times \mathbb{R}_-,$$
$$K_3 := \mathbb{R}_- \times \mathbb{R}_+, \qquad K_4 := \mathbb{R}_+ \times \mathbb{R}_-.$$

For the notations used in the algorithm, we define $E_{a_j} = \{a_j^1, \cdots, a_j^p\}$ for $j = 1, 2$, where $a^i = (a_1^i, a_2^i) \in E_a$, $i \in I_p$. We can now easily order the components within the sets E_{a_1} and E_{a_2}.

Algorithm 6.1 ([4]). Input: $E_a = \{a^1, \cdots, a^p\}$ and $K \in \{K_1, K_2, K_3, K_4\}$.

1. Choosing the cone K from $\{K_1, K_2, K_3, K_4\}$:
   If $K = K_1$ or $K = K_3$, then reorder the existing facilities such that $a_2^1 \leq \cdots \leq a_2^p$.
   If $K = K_2$ or $K = K_4$, then reorder the existing facilities such that $a_2^1 \geq \cdots \geq a_2^p$.

2. Let $T := \{a^1\}$ and $l = 2$.

3. For all $i := 2, \cdots, p$:

   - If $K = K_1$, then check:
     If $a_2^l = a_2^i$ and $a_1^l > a_1^i$, then $T := (T \setminus \{a^l\}) \cup \{a^i\}$ and $l := i$.
     If $a_2^l < a_2^i$ and $a_1^l > a_1^i$, then $T := T \cup \{a^i\}$ and $l = i$.
   - If $K = K_2$, then check:
     If $a_2^l = a_2^i$ and $a_1^l < a_1^i$, then $T := (T \setminus \{a^l\}) \cup \{a^i\}$ and $l := i$.
     If $a_2^l > a_2^i$ and $a_1^l < a_1^i$, then $T := T \cup \{a^i\}$ and $l = i$.
   - If $K = K_3$, then check:
     If $a_2^l = a_2^i$ and $a_1^l < a_1^i$, then $T := (T \setminus \{a^l\}) \cup \{a^i\}$ and $l := i$.
     If $a_2^l < a_2^i$ and $a_1^l < a_1^i$, then $T := T \cup \{a^i\}$ and $l = i$.
   - If $K = K_4$, then check:
     If $a_2^l = a_2^i$ and $a_1^l > a_1^i$, then $T := (T \setminus \{a^l\}) \cup \{a^i\}$ and $l := i$.
     If $a_2^l > a_2^i$ and $a_1^l > a_1^i$, then $T := T \cup \{a^i\}$ and $l = i$.

Output: $T = \text{Eff}(E_a, K)$, i.e., the set of efficient elements of the discrete set E_a in the image space.

The problem $(\mathscr{P}_1)$ is reduced by considering the objectives which correspond only to the facilities a^i from E_a which belong to $T = \text{Eff}(E_a, K)$ (the output of Algorithm 6.1).
The following theorem which is proved in [4] shows that the set of minimal solutions of $(\mathscr{P}_1)$ does not change by applying the reduction introduced in Algorithm 6.1.

Theorem 6.2 ([4, Theorem 11]). *Let $i^0 \in I_p$. Then it holds that*

$$a^{i^0} \in E_a \setminus \left(\bigcup_{r=1}^{4} \text{Eff}(E_a, K_r) \right) \Leftrightarrow \text{Min}(\mathbb{R}^2, f_{I_p}) = \text{Min}(\mathbb{R}^2, f_{I_p \setminus \{i^0\}}).$$

In the following, we formulate the partition algorithm (Algorithm 6.3) of the set of minimal solutions of $(\mathscr{P}_1)$. We first apply Algorithm 6.1 for reducing the number of the facilities, then we proceed with the other steps, which provide the partition of the set $\text{Min}(\mathbb{R}^2, f_{I_p})$ into (possibly degenerated) rectangles.

The advantage of applying Algorithm 6.1, as the first step in the partition algorithm (Algorithm 6.3), comparing with no reduction for the problem $(\mathscr{P}_1)$; or even the advantage of applying this version comparing with the first versions of the Jahn-Graef-Younes method is the remarkable improvement in the computational order by the implementation[i] of our partition algorithm (see [4]).

The Partition Algorithm for $\text{Min}(\mathbb{R}^2, f_{I_p})$ with Manhattan Norm

Now we formulate our partition algorithm (also called the Geometric Rectangular Decomposition Algorithm [4]) for computing the set of minimal solutions $\text{Min}(\mathbb{R}^2, f_{I_p})$ of the problem $(\mathscr{P}_1)$ with the Manhattan norm $\|\cdot\|_1$ and then dividing it into a finite union of (possibly degenerated) rectangles.

Algorithm 6.3 ([4]). Input: $(\mathscr{P}_1)$ concerning the existing facilities $E_a = \{a^1, \cdots, a^p\}$ and the norm $\|\cdot\|_1$.

1. Compute for all $r = 1, 2, 3, 4$ the set

$$T_r := \text{Eff}(E_a, K_r)$$

through Algorithm 6.1.

2. Define the new set of existing facilities through

$$\tilde{E}_a := \bigcup_{r=1}^{4} T_r \subset E_a.$$

Let $k := |\tilde{E}_a|$ and $I_k := \{1, \cdots, k\}$, where $k \leq p$. Then there exist $\tilde{a}^i := (\tilde{a}^i, \tilde{a}^i)$ belonging to E_a, such that

$$\tilde{E}_a = \{\tilde{a}^1, \cdots, \tilde{a}^k\} \subset E_a \subset \mathbb{R}^2.$$

[i] See Section 6.3.

3 Sort the components $\tilde{a}_i^1, \cdots, \tilde{a}_i^k, (i = 1, 2)$ of the existing facilities from the new set $\tilde{E}_a$ and eliminate the duplicated values. The new ordered values of the components $\tilde{a}_1^j$ are denoted by $x_1 < \cdots < x_{s_1}$, and the new ordered values of the components $\tilde{a}_2^j$ are denoted by $y_1 < \cdots < y_{s_2}$ for $s_1, s_2 \leq k$ and $j \in I_k$.

4 Define the four elements e^1, e^2, e^3, e^4 as the following:

$$e^1 := (x_1, y_1),\ e^2 := (x_{s_1}, y_{s_2}),\ e^3 := (x_{s_1}, y_1),\ e^4 := (x_1, y_{s_2}).$$

Compute the sets:

$$\tilde{T}_1 := \{b \in T_2 \mid e^1 \in b + \operatorname{int} K_2\},$$
$$\tilde{T}_2 := \{b \in T_1 \mid e^2 \in b + \operatorname{int} K_1\},$$
$$\tilde{T}_3 := \{b \in T_4 \mid e^3 \in b + \operatorname{int} K_4\},$$
$$\tilde{T}_4 := \{b \in T_3 \mid e^4 \in b + \operatorname{int} K_3\}.$$

Introduce the sets $\mathcal{C}_i := \emptyset$, $i = 1, \ldots, s_1$.
Now, for all $i = 1, \ldots, s_1$, for all $j = 1, \ldots, s_2$:

(a) Define
$$D_r := \left\{(x_i - b_1, y_j - b_2) \mid b := (b_1, b_2) \in \tilde{T}_r\right\}$$
for all $r = 1, 2, 3, 4$.
If $(0, 0) \in D_r$, for some $r \in \{1, 2, 3, 4\}$, then go to (f).

(b) Define $bool_1 := 0$ and $bool_2 := 0$.
If there exists $\beta \in D_1$ such that $\beta \in K_2$ holds, then define $bool_1 := 1$.
If $bool_1 = 1$, then check:
If there exists $\beta \in D_2$ such that $\beta \in K_1$ holds, then define $bool_2 := 1$.

If $bool_2 = 1$, then go to (d), else go to (c).

(c) If $bool_1 = 0$, then $bool_1 := 1$, else check:
If there exists $\beta \in D_1$ such that $\beta \in \operatorname{int} K_2$ holds, then define $bool_1 := 0$.

Define $bool_2 := 0$. If $bool_1 = 1$, then $bool_2 := 1$ and check:
If there exists $\beta \in D_2$ such that $\beta \in \operatorname{int} K_1$ holds,
then define $bool_2 := 0$.

If $bool_2 = 1$, then go to (d), else choose the next point.

(d) Define $bool_1 := 0$ and $bool_2 := 0$.
If there exists $\beta \in D_3$ such that $\beta \in K_4$ holds, then define $bool_1 := 1$.
If $bool_1 = 1$, then check:
If there exists $\beta \in D_4$ such that $\beta \in K_3$ holds, then define $bool_2 := 1$.

If $bool_2 = 1$, then go to (f), else go to (e).

(e) If $bool_1 = 0$, then $bool_1 := 1$, else check:
If there exists $\beta \in D_3$ such that $\beta \in \operatorname{int} K_4$ holds, then define $bool_1 := 0$.

Define $bool_2 := 0$. If $bool_1 = 1$, then $bool_2 := 1$ and check:
If there exists $\beta \in D_4$ such that $\beta \in \operatorname{int} K_3$ holds, then define $bool_2 := 0$.

If $bool_2 = 1$, then go to (f), else choose the next point.

(f) Define $\mathcal{C}_i := \mathcal{C}_i \cup \{y_j\}$ and choose the next point.

5 Define

$$\mathcal{C}_i^{\min} := \min \mathcal{C}_i \ \text{ and } \ \mathcal{C}_i^{\max} := \max \mathcal{C}_i$$

for all $i \in \{1, \dots, s_1\}$. Now, consider two cases:

Case 1: Let $s_1 = 1$. Then define

$$\mathcal{R}_1^* := \operatorname{conv}\{(x_1, \mathcal{C}_1^{\min}), (x_1, \mathcal{C}_1^{\max})\} \ \text{ and } \ \mathcal{R}_2^* := \emptyset.$$

Case 2: Let $s_1 \geq 2$. Define

$$\overline{\mathcal{C}_i} := \max\left\{\mathcal{C}_i^{\min}, \mathcal{C}_{i+1}^{\min}\right\} \ \text{ and } \ \underline{\mathcal{C}_i} := \min\left\{\mathcal{C}_i^{\max}, \mathcal{C}_{i+1}^{\max}\right\}$$

for all $i \in \{1, \dots, s_1 - 1\}$. Moreover, define

$$\mathcal{R}_2^* := \bigcup_{i=1}^{s_1-1} \operatorname{conv}\{(x_i, \underline{\mathcal{C}_i}),\ (x_i, \overline{\mathcal{C}_i}),\ (x_{i+1}, \overline{\mathcal{C}_i}),\ (x_{i+1}, \underline{\mathcal{C}_i})\} \ \text{ and } \ \mathcal{R}_1^* := \emptyset.$$

Now check:

If $\mathcal{C}_1^{\min} < \underline{\mathcal{C}_1}$, then $\mathcal{R}_1^* := \mathcal{R}_1^* \cup \operatorname{conv}\{(x_1, \underline{\mathcal{C}_1}), (x_1, \mathcal{C}_1^{\min})\}$.

If $\mathcal{C}_1^{\max} > \overline{\mathcal{C}_1}$, then $\mathcal{R}_1^* := \mathcal{R}_1^* \cup \operatorname{conv}\{(x_1, \overline{\mathcal{C}_1}), (x_1, \mathcal{C}_1^{\max})\}$.

If $\mathcal{C}_{s_1}^{\min} < \underline{\mathcal{C}_{s_1-1}}$, then $\mathcal{R}_1^* := \mathcal{R}_1^* \cup \operatorname{conv}\{(x_{s_1}, \underline{\mathcal{C}_{s_1-1}}), (x_{s_1}, \mathcal{C}_{s_1}^{\min})\}$.

If $\mathcal{C}_{s_1}^{\max} > \overline{\mathcal{C}_{s_1-1}}$, then $\mathcal{R}_1^* := \mathcal{R}_1^* \cup \operatorname{conv}\{(x_{s_1}, \overline{\mathcal{C}_{s_1-1}}), (x_{s_1}, \mathcal{C}_{s_1}^{\max})\}$.

Suppose that $s_1 \geq 3$ holds. In addition, check for all $i \in \{2, \dots, s_1 - 1\}$:

If $\mathcal{C}_i^{\min} < \mathcal{C}^* := \min\{\underline{\mathcal{C}_{i-1}}, \underline{\mathcal{C}_i}\}$, then $\mathcal{R}_1^* := \mathcal{R}_1^* \cup \operatorname{conv}\{(x_i, \mathcal{C}^*), (x_i, \mathcal{C}_i^{\min})\}$.

If $\mathcal{C}_i^{\max} > \mathcal{C}^* := \max\{\overline{\mathcal{C}_{i-1}}, \overline{\mathcal{C}_i}\}$, then $\mathcal{R}_1^* := \tilde{\mathcal{R}}_1^* \cup \operatorname{conv}\{(x_i, \mathcal{C}^*), (x_i, \mathcal{C}_i^{\max})\}$.

If $\underline{\mathcal{C}_{i-1}} > \overline{\mathcal{C}_i}$, then $\mathcal{R}_1^* := \mathcal{R}_1^* \cup \operatorname{conv}\{(x_i, \overline{\mathcal{C}_i}), (x_i, \underline{\mathcal{C}_{i-1}})\}$.

If $\overline{\mathcal{C}_{i-1}} < \underline{\mathcal{C}_i}$, then $\mathcal{R}_1^* := \mathcal{R}_1^* \cup \operatorname{conv}\{(x_i, \overline{\mathcal{C}_{i-1}}), (x_i, \underline{\mathcal{C}_i})\}$.

Output: The whole set of minimal solutions $\operatorname{Min}(\mathbb{R}^2, f_{I_p}) = \mathcal{R}_1^* \cup \mathcal{R}_2^*$ of the problem $(\mathscr{P}_1)$ as a union of rectangles, possibly degenerated.

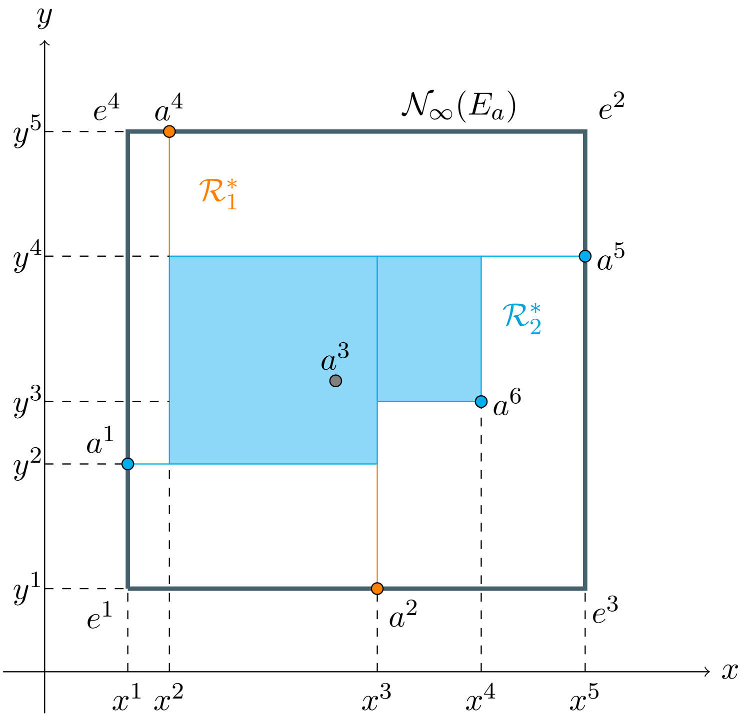


Figure 6.1: Output of the Algorithm 6.3

For an applied example of the output of Algorithm 6.3 see Figure 6.1.

The correctness of the partition Algorithm (Algorithm 6.3)

We observe that Algorithm 6.3 divide the set of minimal solutions $\mathrm{Min}(\mathbb{R}^2, f_{I_p})$ (of the multiobjective location problem $(\mathscr{P}_1)$) into a finite number of rectangles, represented in the set $\mathcal{R}_1^*$ of all vertical line segments and the set $\mathcal{R}_2^*$ of all rectangles and all horizontal line segments. The correctness and the analysis of Algorithm 6.3 is extensively studied in [4]. But it is important to mention the following result, which state that Algorithm 6.3 indeed generate the whole set of minimal solutions, which is given through Lemma 5.14.

Taking into account the reduction result, i.e. Theorem 6.2, let

$$\tilde{I} := \left\{ i \in I_p \mid a^i \in \bigcup_{r=1}^{4} \mathrm{Eff}(E_a, K_r) \right\},$$

then the following result holds:

Theorem 6.4 ([4, Theorem 18]). *Let $\mathcal{R}_1^*$ and $\mathcal{R}_2^*$ be generated by Algorithm 6.3. Then we have:*

$$\mathrm{Min}(\mathbb{R}^2, f_{I_p}) = \mathrm{Min}(\mathbb{R}^2, f_{\tilde{I}}) = \mathcal{R}_1^* \cup \mathcal{R}_2^*.$$

The Partition Algorithm for $\mathrm{Min}(\mathbb{R}^2, f_{I_p})$ with Maximum Norm

Analogously to Algorithm 6.3 we can derive an algorithm to partition the set $\mathrm{Min}(\mathbb{R}^2, f_{I_p})$ into a finite union of closed rectangles, possibly degenerated, while using the maximum norm in f_{I_p}. There is a possibility to do this by means of the grid composed by the two families of the straight lines $L'(\ell'(a^i))$ and $L''(\ell''(a^i))$, passing through a^i for all $i \in I_p$, which are defined in (5.45). However, here we benefit simply from the implementable form of Algorithm 6.3 and profit from the relation between Manhattan and maximum norms described as follows:

If T is a linear transformation with,

$$T := \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \tag{6.1}$$

then according to [52] and [76] we get for some $x^1, x^2 \in \mathbb{R}^2$ the relation

$$\|x^1 - x^2\|_\infty = \|Tx^1 - Tx^2\|_1.$$

With the help of the linear transformation T defined in (6.1), we can simply apply Algorithm 6.3 also for the maximum norm.

Figure 6.2 shows an example of the set of the minimal solutions with the maximum norm decomposed through the partition algorithm into (possibly degenerated) rectangles.

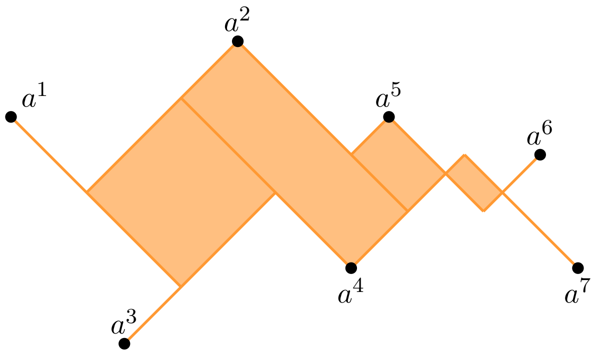


Figure 6.2: Example for the partition algorithm with the maximum norm.

6.2 Decomposition Methods for the Extended Multiobjective Location Problems

Consider a special case of the extended multiobjective approximation and location problem $(\mathscr{P})$ with the objective function defined in (5.4) (namely by taking $\alpha_i = 1,\ \beta_i = 1,\ A_i = \mathbb{I}$ and the norms $\|\cdot\|_{(i)}$ are either $\|\cdot\|_1$ for $i = 1, \dots, p$ or $\|\cdot\|_\infty$ for $i = 1, \dots, p$). Our next aim is to use the results derived in Section 5.3 and Section 6.1 to develop a decomposition method for solving a special case of the problem $(\mathscr{P})$, namely an extended multiobjective location problem. We present several possibilities to derive decomposition algorithms for solving this problem.

Furthermore, through a numerical example we show, that our decomposition method is also able to be applied by considering attraction and repulsion facilities in a location model. This field of the location theory appears in many applications, for example, when the distances to pollution stations or nuclear plants are desired to be maximal (see e.g. [75]).

We consider a special case of $(\mathscr{P})$ by taking $\alpha_i = 1,\ \beta_i = 1,\ A_i = \mathbb{I}$ and the norms $\|\cdot\|_{(i)}$ are either $\|\cdot\|_1$ for $i = 1, \dots, p$ or $\|\cdot\|_\infty$ for $i = 1, \dots, p$ with a bounded feasible set $X \subset \mathbb{R}^2$:

$$(\mathscr{P}_2) \qquad \mathrm{Eff}(f_{I_{p+m}}[X], \mathbb{R}^{p+m}_+), \tag{6.2}$$

where the objective function is given by

$$f_{I_{p+m}}(x) = \begin{pmatrix} \|x - a^1\| \\ \cdots \\ \|x - a^p\| \\ C_{p+1}(x) \\ \cdots \\ C_{p+m}(x) \end{pmatrix}. \tag{6.3}$$

What we do now, as insinuated in the beginning of this chapter, is decomposing the problem $(\mathscr{P}_2)$ by dividing it into a multiobjective location subproblem (by selecting the criteria $1, \cdots, p$ in (6.3)) and a multiobjective linear subproblem (by selecting the criteria $p+1, \ldots, p+m$). The first problem, which we denoted above as $(\mathscr{P}_1)$, is extensively studied in Section 5.3. The set of its minimal solution is successfully decomposed into closed and convex subsets in Section 6.1 through Algorithm 6.3.
The next algorithm for solving the problem $(\mathscr{P}_2)$ introduces in detail the above described decomposing method. For the feasibility of this algorithm we suppose that $\mathcal{N} \subset X$.

Algorithm 6.5. Input: The existing facilities $E_a = \{a^1, \ldots, a^p\}$ and the extended multiobjective location problem $(\mathscr{P}_2)$ with the maximum norm or the Manhattan norm.

1. Decompose $(\mathscr{P}_2)$ into two problems:
The first problem is a multiobjective location problem $(\mathscr{P}_1)$: $\operatorname{Eff}(f_{I_p}[X], \mathbb{R}^p_+)$ with the multiobjective function:
$$f_{I_p}(x) = \begin{pmatrix} \|x - a^1\| \\ \cdots \\ \|x - a^p\| \end{pmatrix}.$$
The second problem is a multiobjective linear problem $(\mathscr{P}_C)$: $\operatorname{Eff}(C[X], \mathbb{R}^m_+)$, where
$$C(x) = \begin{pmatrix} C_{p+1}(x) \\ \cdots \\ C_{p+m}(x) \end{pmatrix}.$$

2. Generate the Manhattan rectangular hull $\mathcal{N}_1(E_a)$ by using the analytical characterization (5.43) (when the maximum norm is used in (6.3)), or alternatively the Maximum rectangular hull $\mathcal{N}_\infty(E_a)$ (when the Manhattan norm is used in (6.3)).

3. Compute the solution set of minimal solutions of $(\mathscr{P}_1)$ represented in the set $\operatorname{Min}(X, f_{I_p})$ through Lemma 5.14 (according to this characterization, the set $\operatorname{Min}(X, f_{I_p})$ is always contained in $\mathcal{N}$).

4. Divide $\operatorname{Min}(X, f_{I_p})$ through Algorithm 6.3 into nonempty, possibly degenerated rectangles $\mathcal{R}_1, \ldots, \mathcal{R}_l$ belonging to $\mathcal{R}^*_1$, $\mathcal{R}^*_2$, so we get
$$\operatorname{Min}(\mathbb{R}^2, f_{I_p}) = \mathcal{R}^*_1 \cup \mathcal{R}^*_2 = \bigcup_{i=1}^{l} \mathcal{R}_i.$$

5. Minimize the multiobjective function $C(x)$ of the problem $(\mathscr{P}_C)$ over the sets $\mathcal{R}_i \subset \mathcal{R}^*_1 \cup \mathcal{R}^*_2$ for $i = 1, \ldots, l$. In other words, compute l solution sets by solving each of the l subproblems $\operatorname{Eff}(C[\mathcal{R}_i], \mathbb{R}^m_+)$.

6 Compute the following set:

$$\mathrm{Eff}\left(\bigcup_{i=1}^{l} \mathrm{Eff}(C[\mathcal{R}_i], \mathbb{R}^m_+), \mathbb{R}^m_+\right) =: \mathcal{E}. \tag{6.4}$$

Output: x^0 with $f(x^0) \in \mathcal{E}$ is a minimal solution of the extended problem $(\mathscr{P}_2)$, i.e., $x^0 \in \mathrm{Min}(X, f_{I_{p+m}})$.

In order to prove that a solution x^0 with $f(x^0) \in \mathcal{E}$ is a minimal solution of the extended problem $(\mathscr{P}_2)$ we introduce a corresponding result given in [51]. This result is proved for a general multiobjective function $f_{I_q} : Y \to \mathbb{R}^q$ defined on a nonempty set Y with $f_{I_q} := (f_1, \dots, f_p, f_{p+1}, \dots, f_{p+m})$ (where $p, m, p+m = q \in \mathbb{N}$) and the restriction set $\emptyset \neq X \subset Y$. Initially, we say that the multiobjective function f_{I_q} fulfils the domination property over the set X, if

$$\forall x \in X\ \exists \tilde{x} \in \mathrm{Min}(X, f_{I_q}):\ f(x) \in f(\tilde{x}) + \mathbb{R}^q_+. \tag{6.5}$$

A multiobjective function f_{I_q} fulfils according to [58] the domination property over the set X, if X is compact and f_{I_q} is continuous on X.
Now the following result from [51, Theorem 2.9, Corollary 2.10] can be introduced.

Theorem 6.6 ([51]). *Let $l \in \mathbb{N}$ and $Z_1, \dots, Z_l$ nonempty subsets of X such that $\mathrm{Min}(X, f_{I_p}) = \bigcup_{i=1}^{l} Z_i$ holds. If $f_{I_{q\setminus p}}$ fulfils the dominance property (6.5) over Z_i for all $i = 1, \dots, l$, then for all $x^0 \in \mathrm{Min}(X, f_{I_p})$ with*

$$f_{I_{q\setminus p}}(x^0) \in \mathrm{Eff}\left(\bigcup_{i=1}^{l} \mathrm{Eff}(f_{I_{q\setminus p}}[Z_i], \mathbb{R}^m_+), \mathbb{R}^m_+\right)$$

it holds that

$$x^0 \in \mathrm{Min}(X, f_{I_q}).$$

According to Theorem 6.6 and by replacing the sets Z_i by the rectangles $\mathcal{R}_i \subset \mathcal{R}_1^* \cup \mathcal{R}_2^*$ for $i = 1, \dots, l$ which are generated by Algorithm 6.3, we conclude that x^0 with $f(x^0) \in \mathcal{E}$ (given by (6.4)) is a minimal solution of the extended problem $(\mathscr{P}_2)$, i.e., $x^0 \in \mathrm{Min}(X, f_{I_{p+m}})$. Note that the function $C \in L(X, \mathbb{R}^m)$ in the extended problem $(\mathscr{P}_2)$ is continuous and the rectangles $\mathcal{R}_1, \dots, \mathcal{R}_l$ are compact.

Remark 6.7.

- Step 2 in Algorithm 6.5 computes the Manhattan rectangular hull $\mathcal{N}_1(E_a)$, which serves not only characterizing the set of minimal solution in the coming Step 3, but also gives the advantage in Algorithm 6.5 of generating the set of weakly minimal solutions $\mathrm{Min}_w(X, f_{I_p})$ by

Theorem 5.18. In addition, according to (4.12) the elements of $\mathcal{N}_1(E_a)$ are also weakly minimal solutions of the extended problem $(\mathscr{P}_2)$. That means it holds that

$$\mathcal{N}_1(E_a) \subset \mathrm{Min_w}(X, f_{I_{p+m}})$$

- For the implementation of the Steps (2) – (4) see Section 6.3.

Furthermore, we have the possibility to derive additional algorithms for generating minimal solutions of $(\mathscr{P}_2)$ starting with the first steps of Algorithm 6.5. For example, if we consider the results presented in Theorem 4.13 and Corollary 4.14, then instead of doing Step (5) in Algorithm 6.5 we are able to minimize the scalarized version of the multiobjective problem $(\mathscr{P}_C)$ over the rectangles $\mathcal{R}_i \subset \mathcal{R}_1^* \cup \mathcal{R}_2^*$, $i \in \{1, \dots, l\} =: I_l$. The following algorithm describes this method, see also [3].

Algorithm 6.8 ([3])**.** Input: The existing facilities $E_a = \{a^1, \cdots, a^p\}$ and the problem $(\mathscr{P}_2)$ with the maximum norm or the Manhattan norm.

(1) Do the Steps (1), (2), (3) and (4) from Algorithm 6.5.

(2) Choose some positive numbers $\alpha_1, \cdots, \alpha_m$ according to the importance of the cost functions $C_{p+1}, \cdots, C_{p+m}$.

(3) For every $i \in I_l$ find an optimal solution of the scalar optimization problem, which is the weighted sum of the cost functions $C_{p+1}, \cdots, C_{p+m}$, as the following:

$$x^i \in \operatorname*{argmin}_{x \in \mathcal{R}_i} \sum_{j=1}^{m} \alpha_j C_{p+j}(x).$$

(4) Find $i_0 \in I_l$ such that

$$\sum_{j=1}^{m} \alpha_j C_{p+j}(x^{i_0}) = \min_{i \in I_l} \left\{ \min_{x \in \mathcal{R}_i} \sum_{j=1}^{m} \alpha_j C_{p+j}(x) \right\}.$$

Output: x^{i_0} as a minimal solution of the extend problem $(\mathscr{P}_2)$, i.e., $x^{i_0} \in \mathrm{Min}(X, f_{I_{p+m}})$.

Remark 6.9. The solution x^{i_0} obtained from Step (4) in Algorithm 6.8 is the solution of the scalarized multiobjective problem $(\mathscr{P}_C)$ obtained through the decomposition in Algorithm 6.5, i.e.,

$$x^{i_0} \in \operatorname*{argmin}_{x \in \mathcal{R}_1^* \cup \mathcal{R}_2^*} \sum_{j=1}^{m} \alpha_j C_{p+j}(x) = \operatorname*{argmin}_{x \in \mathrm{Min}(\mathbb{R}^2, f_{I_p})} \sum_{j=1}^{m} \alpha_j C_{p+j}(x). \tag{6.6}$$

From (6.6), Theorem 4.13 and Corollary 4.14 we state that x^{i_0} is a minimal solution of the original problem $(\mathscr{P}_2)$, that means $x^{i_0} \in \mathrm{Min}(X, f_{I_{p+m}})$. Hence, we can conclude that Algorithm 6.8 generates minimal solutions of the original problem $(\mathscr{P}_2)$.

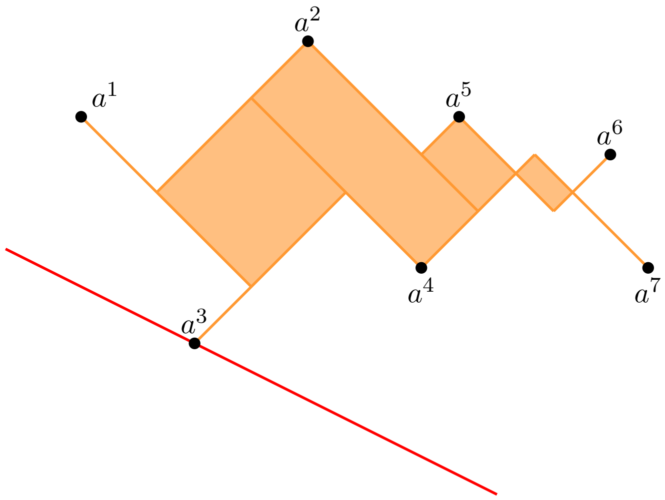


Figure 6.3: Illustration of Example 6.10.

In this section we present, in addition, two examples. The following example (Example 6.10) serves rather introducing an implemented special case of Algorithm 6.5. The second example (Example 6.11) is an example about applying Algorithm 6.5 and Algorithm 6.8 by considering attraction and repulsion facilities.

Example 6.10. Given the points $a^1, \ldots, a^7$, which are located in the plane as the following (see Figure 6.3):

$$a^1 = (1,5), \quad a^2 = (4,6), \quad a^3 = (2.5,2),$$
$$a^4 = (5.5,3), \quad a^5 = (6,5), \quad a^6 = (8,4.5), \quad a^7 = (8.5,3).$$

We consider additionally the cost function $C(x) = x_1 + 2x_2$ for $x = (x_1, x_2) \in \mathbb{R}^2$. Then the extended multiobjective location problem with the maximum norm of the type $(\mathscr{P}_2)$ is to minimize the vector-valued function $f_{I_{p+m}} = (f_1, \ldots, f_8) : \mathbb{R}^2 \to \mathbb{R}^{10}$ over the whole plane $\mathbb{R}^2$, where $f_i(x) := \|x - a^i\|_\infty$ for all $i \in I_p := \{1, \ldots, 7\}$ and $f_8(x) := C(x)$.
By solving this problem with Algorithm 6.5 (for $m = 1$), we obtain the set of minimal solutions $\mathrm{Min}(\mathbb{R}^2, f_{I_p})$ divided into eight (possibly degenerated) rectangles $\mathcal{R}_1, \cdots, \mathcal{R}_8$ through Step 3 and

Step 4 of Algorithm 6.5, where

$$\begin{aligned}
\mathcal{R}_1 &:= \operatorname{conv}\{(2.5, 2), (3.25, 2.75)\},\\
\mathcal{R}_2 &:= \operatorname{conv}\{(2, 4), (1, 5)\},\\
\mathcal{R}_3 &:= \operatorname{conv}\{(3.25, 2.75), (2, 4), (3.25, 5.25), (4.5, 4)\},\\
\mathcal{R}_4 &:= \operatorname{conv}\{(3.25, 5.25), (4, 6), (6.25, 3.75), (5.5, 3)\},\\
\mathcal{R}_5 &:= \operatorname{conv}\{(5.5, 4.5), (6.25, 3.75), (6.75, 4.25), (6, 5)\},\\
\mathcal{R}_6 &:= \operatorname{conv}\{(6.75, 4.25), (7.25, 3.75), (7.5, 4), (7, 4.5)\},\\
\mathcal{R}_7 &:= \operatorname{conv}\{(7.5, 4), (8, 4.5)\},\\
\mathcal{R}_8 &:= \operatorname{conv}\{(7.5, 4), (8.5, 3.5)\}.
\end{aligned}$$

Figure 6.3 shows the output of these two steps. By minimizing the cost function $C(x)$ over each of the sets $\mathcal{R}_j$, $j \in \{1, \ldots, 8\} := I_l$ we get the eight minimal values

$$x^j \in \operatorname*{argmin}_{x \in \mathcal{R}_j} C(x)$$

for $j \in I_l$. This implies the values:

$$\begin{array}{llll}
x^1 = (2.5, 2), & x^2 = (2, 4), & x^3 = (3.25, 2.75), & x^4 = (5.5, 3),\\
x^5 = (6.25, 3.75), & x^6 = (7.25, 3.75), & x^7 = (7.5, 4), & x^8 = (8, 4.5).
\end{array}$$

Then we apply Step 6 of Algorithm 6.5 in the scalar form (since $m = 1$):

$$C(x^0) = \min_{j \in I_l} \left\{ C(x^j) \right\}. \tag{6.7}$$

We compute the minimum from (6.7) and we obtain the point x^1, which is the facility a^3. Then through Theorem 4.13 the following relation holds

$$x^0 = a^3 \in \operatorname*{argmin}_{x \subset \operatorname{Min}(\mathbb{R}^2, f_{I_p})} C(x) \in \operatorname{Min}(\mathbb{R}^2, f_{I_{p+m}}).$$

If we generalize the cost functions $C_{p+j},\ (j \in I_m)$ in (6.3) to be not necessarily linear, for example, concave functions, then as mentioned above we are able to apply Algorithm 6.8 by considering repulsion facilities in addition to the criteria $1, \ldots, p$ in (6.3) (see also [3, Remark 5.1]).
It is also important to mention that this problem is a nonconvex multiobjective optimization problem

Example 6.11 gives a numerical application of the above mentioned problems. It is also a direct application of Lemma 5.14, Theorem 5.18, Algorithm 6.3 and Algorithm 6.8.

Example 6.11 ([3]). We want to locate a new facility $x \in \mathbb{R}^2$ with minimal distances to six existing facilities ($p = 6$ attraction points)

$$a^1 = (0,3), \qquad a^2 = (5,0.5), \qquad a^3 = (2.5,1.5),$$
$$a^4 = (3,3.5), \qquad a^5 = (7,3.5), \qquad a^6 = (5.5,3.5).$$

Concerning these points $a^1,\dots,a^6$, we consider the following distances using the maximum norm

$$f_i(x) := \|x - a^i\|_\infty, \ \forall\, x \in \mathbb{R}^2, \ \forall\, i \in I_p := \{1,\dots,6\}.$$

In addition to the attraction points $a^1,\dots,a^6$, we consider two repulsion points (i.e., undesirable facilities)

$$b^1 := (5,5) \quad \text{and} \quad b^2 := (1,1),$$

and we want to maximize the distance from the new location point x to each of the points b^1 and b^2. By considering the Euclidean distance, which is appropriate to model the propagation of waves, we obtain two ($m = 2$) concave criteria

$$f_{p+i}(x) := -\|x - b^i\|_2, \ \forall\, x \in \mathbb{R}^2, \ \forall\, i \in I_m := \{1,2\}.$$

This problem, involving the attraction points $a^1,\dots,a^6$ as well as the repulsion points b^1, b^2, leads us to a multiobjective optimization problem, where the vector-valued objective function $f_{I_{p+m}} := (f_1,\dots,f_8) : \mathbb{R}^2 \to \mathbb{R}^8$ over the set $\mathbb{R}^2$ is given by

$$f_{I_{p+m}}(x) = \begin{pmatrix} \|x - a^1\|_\infty \\ \cdots \\ \|x - a^6\|_\infty \\ -\|x - b^1\|_2 \\ -\|x - b^2\|_2 \end{pmatrix}.$$

The resulting problem is the following extended multiobjective location problem:

$$(\mathscr{P}_2) \qquad \begin{cases} \text{Minimize} & f_{I_{p+m}}(x) \\ \text{subject to} & x \in \mathbb{R}^2. \end{cases}$$

Since in practice the repulsion points are not equally undesirable, we assign to b^1 and b^2 certain weights and apply rather Algorithm 6.8.

According to the first steps of the decomposition Algorithm 6.8, we minimize the vector-valued function $f_{I_p} = (f_1,\dots,f_6) : \mathbb{R}^2 \to \mathbb{R}^6$ over $\mathbb{R}^2$ (i.e., we solve the multiobjective location problem of the type $(\mathscr{P}_1)$ obtained from decomposing $(\mathscr{P}_2)$).

The set of weakly minimal solutions is computed by generating the Manhattan hull:

$$\mathrm{Min_w}(\mathbb{R}^2, f_{I_p}) = \mathcal{N} := \mathrm{conv}\{(0,3), (3.75,6.75), (7.5,3), (3.75,-0.75)\}.$$

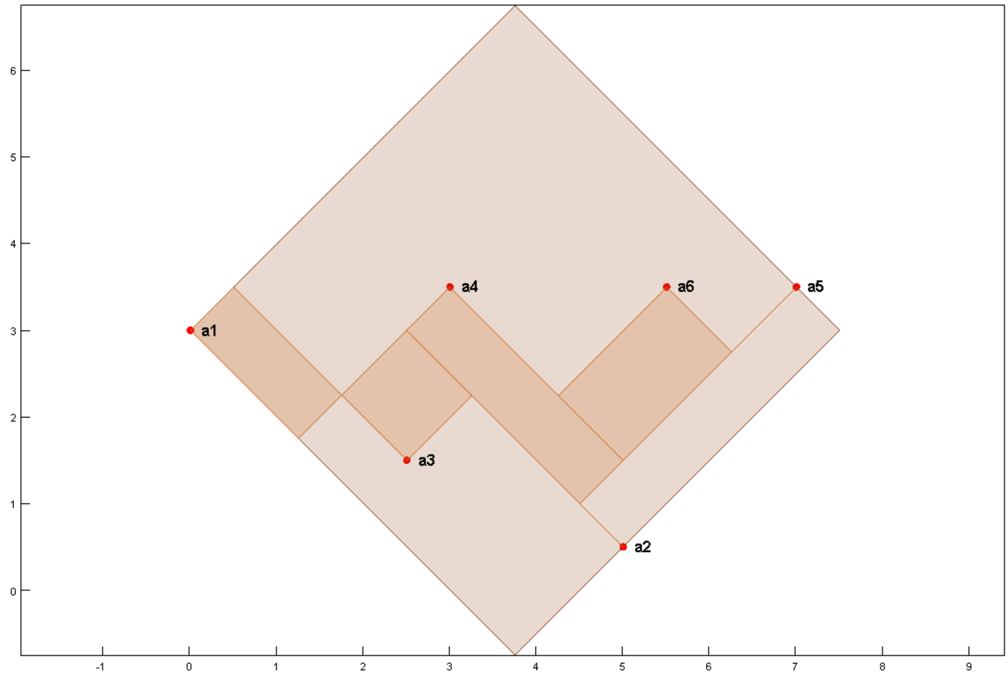


Figure 6.4: The sets of minimal and weakly minimal solutions of the location problem $(\mathscr{P}_1)$.

Then applying Algorithm 6.3 we obtain the set of minimial solutions (also using Lemma 5.14) divided into closed (possibly degenerated) rectangles.
More precisely, we obtain the set of minimal solutions $\mathrm{Min}(\mathbb{R}^2, f_{I_p})$ divided into six closed rectangles, two of them being degenerated into line segments:

$$\begin{aligned}
\mathcal{R}_1 &:= \mathrm{conv}\{(0,3),(0.5,3.5),(1.75,2.25),(1.25,1.75)\},\\
\mathcal{R}_2 &:= \mathrm{conv}\{(1.75,2.25),(2.5,3),(3.25,2.25),(2.5,1.5)\},\\
\mathcal{R}_3 &:= \mathrm{conv}\{(2.5,3),(3,3.5),(5,1.5),(4.5,1)\},\\
\mathcal{R}_4 &:= \mathrm{conv}\{(4.25,2.25),(5.5,3.5),(6.25,2.75),(5,1.5)\},\\
\mathcal{R}_5 &:= \mathrm{conv}\{(4.5,1),(5,0.5)\},\\
\mathcal{R}_6 &:= \mathrm{conv}\{(6.25,2.75),(7,3.5)\}.
\end{aligned}$$

We have $\mathrm{Min}(\mathbb{R}^2, f_{I_p}) = \bigcup_{j=1}^{6} \mathcal{R}_j$. This decomposition of the set of minimal solutions and the Manhattan hull $\mathcal{N} = \mathrm{Min}_{\mathrm{w}}(\mathbb{R}^2, f_{I_p})$ are shown in Figure 6.4.
Now, according to Step 2 in Algorithm 6.8, we assign to b^1 and b^2 certain weights, as for instance

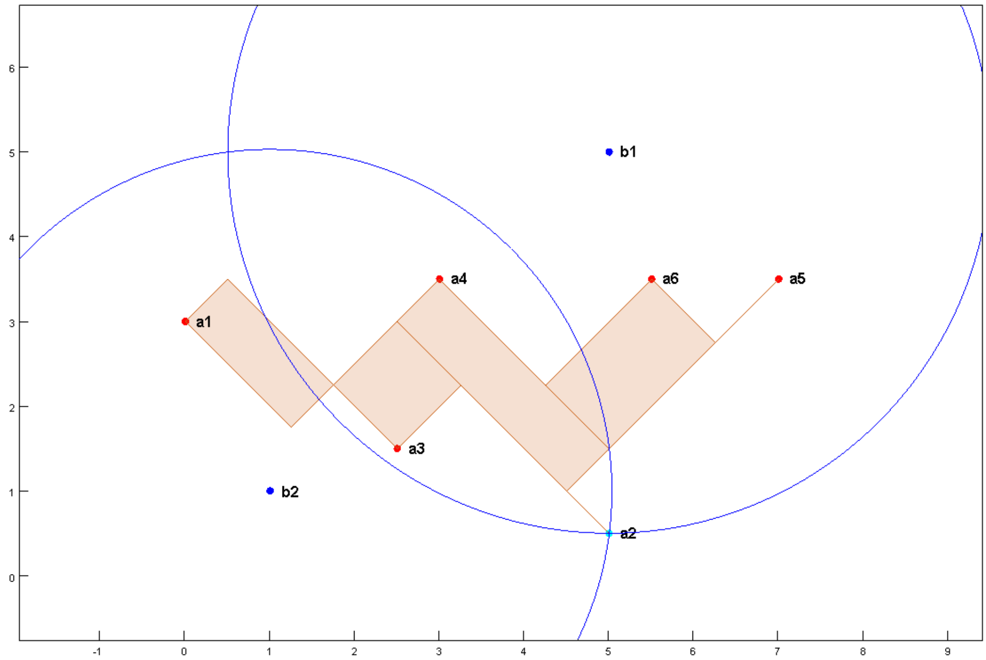


Figure 6.5: The minimal solution a^2 of the extended location problem.

$\alpha_1 = 2$ and $\alpha_2 = 1$ and minimize the weighted sum function, $h : \mathbb{R}^2 \to \mathbb{R}$, defined for all $x \in \mathbb{R}^2$ by

$$h(x) := \sum_{i=1}^{m} \alpha_i f_{p+i}(x) = -2\|x - b^1\| - \|x - b^2\|$$

over the set $\mathrm{Min}(\mathbb{R}^2, f_{I_p})$.

It is quite easy to minimize the concave function h over $\mathrm{Min}(\mathbb{R}^2, f_{I_p}) = \bigcup_{j=1}^{6} \mathcal{R}_j$, since for every $j \in \{1, \ldots, 6\}$ a minimum point x^j of h over the closed (possibly degenerated) rectangle $\mathcal{R}_j$ can be chosen among the extreme points of $\mathcal{R}_j$. The extreme points x^j generated by Algorithm 6.3 are listed in the following table:

	$\mathcal{R}_1$	$\mathcal{R}_2$	$\mathcal{R}_3$	$\mathcal{R}_4$	$\mathcal{R}_5$	$\mathcal{R}_6$
x^j	(0, 3)	(2.5, 1.5)	(4.5, 1)	(5, 1.5)	(5, 0.5)	(7, 3.5)
$h(x^j)$	-13.01	-10.18	-11.56	-11.03	-13.03	-11.50

Finally, we get a minimal solution of the extended multiobjective location problem $(\mathscr{P}_2)$, namely $x^{j_0} = x^5 = (5, 0.5) = a^2 \in \mathrm{Min}(\mathbb{R}^2, f_{I_{p+m}})$. Figure 6.5 shows the level curves of the objectives f_{p+1} and f_{p+2} at $x^{j_0} = a^2$.

6.3 Implementations with MATLAB

It is now clear that finding new results and algorithms is not enough to convince the decision makers and the investors. The implementation of these algorithms encourage not only them but also the researchers themselves. Developing such programs performs many functions, they help to make sure that the algorithms are accurate and faultless, they check the generality and they accelerate the work. Furthermore, the visualization made in many programs gives a graphical variant for the solutions, which is very useful also for the researchers to develop new results out of that. This holds in all applied mathematical branches, but since location theory becomes more important in the modern industry, many programmers have improved a lot of solving algorithms and visualized them.
The algorithms presented in the previous sections are implemented in a new software with the name FLO - Facility Location Optimizer developed by Christian Günther. Furthermore, in this software FLO many other well-known algorithms are included. In the following we describe the software FLO, but first we give additionally a short overview of a well-known software for solving location problems.

LoLA and LoLoLA

LoLA (Library of Location Algorithms) is one of the older software libraries, which are designed to solve location problems by suggesting an optimal location for the wanted facility (or facilities) for a specified location problem. Actually, it is a long list of algorithms, which are available in LOLA. Briefly we can say, LoLA can solve median and center SFLPs with or without restrictions using different distances, multi-facility center problems, discrete problems and network problems. The software was developed in the university of Kaiserslautern (Working group: Optimization).
During the ongoing project ”StanLay” the working group optimization is developing the new software package LoLoLA - Library of Location and Layout Algorithms as a replacement. The new software is supposed to include the algorithms given in LOLA with a new interface and better features. It is based on Python language and supposed to be released this year.

FLO - Facility Location Optimizer

FLO is a practical MATLAB user interface to compute and plot the solutions of many types of location optimization problems. The software was developed by Christian Günther from Martin Luther University (see [51]).
By running the software we get two windows. The main window shows the locations and the solution sets and the outputs of the algorithms. The second window is the module window, where the types of the problems and the parameter are determined. We can see a screenshot of both windows of the software FLO in Figure 6.6.

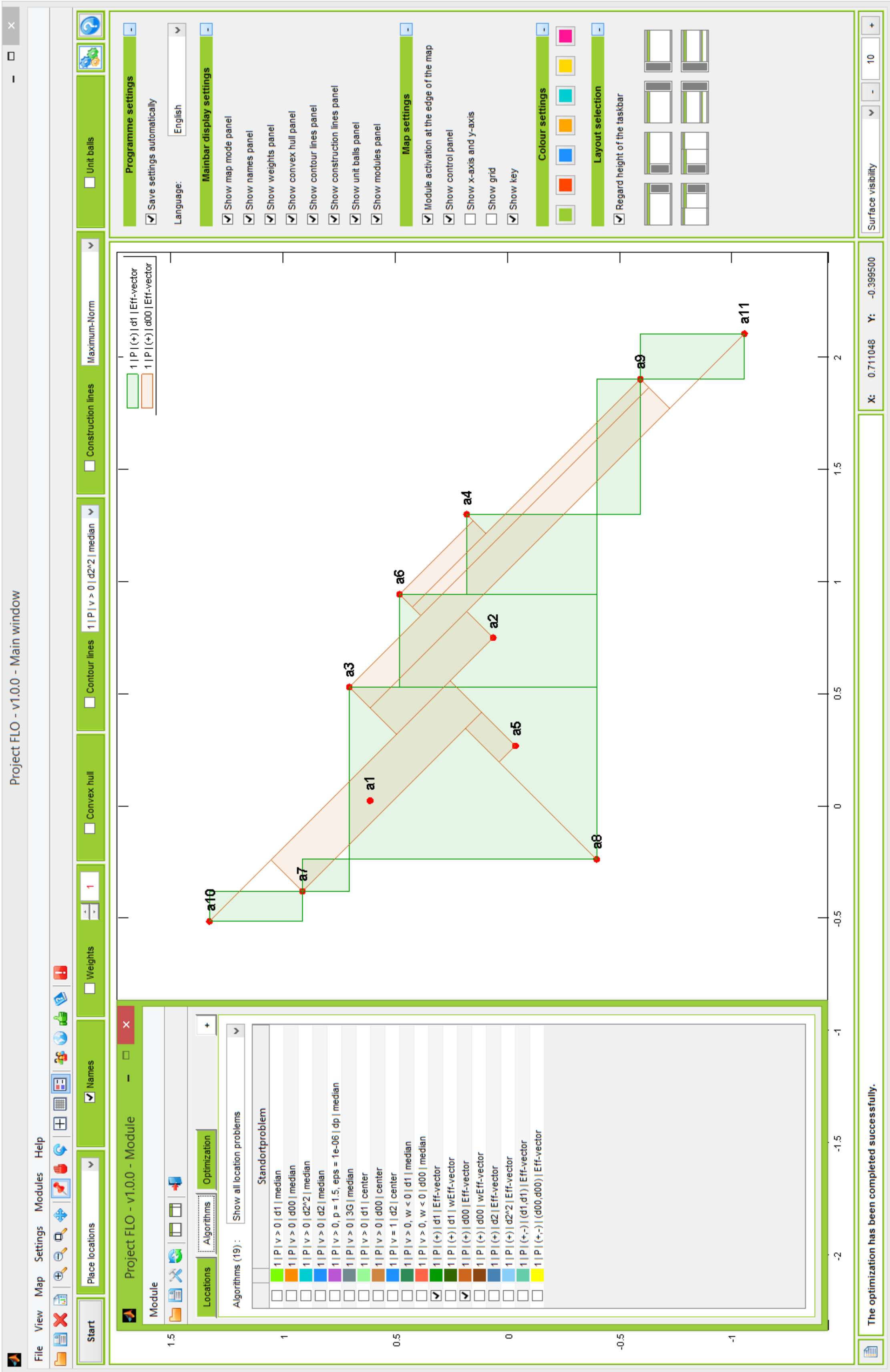


Figure 6.6: The main window of the Software FLO.

According to the current version of the software we can locate the existing facilities directly on the coordinate system in the main window, or we can give its coordinates manually. This can also be made directly on a loaded real map by choosing the corresponding option.
Some of the type of the location problems, which can be solved by FLO, are listed below.

- Median and center single-facility location problem with different distance functions such as norms and polyhedral gauges.
- Median and center single-facility location problem with restrictions (variable facilities, barriers).
- Multiobjective location problems with Manhattan and the maximum distances.
- Extended multiobjective location problems with an additional linear cost function.
- Multiobjective location problems involving attraction and repulsion points.

Concerning this thesis, the following results are implemented in FLO: Lemma 5.14, Theorem 5.18, Algorithm 6.1, Algorithm 6.3, Algorithm 6.5 in the case $m = 1$ (Example 6.10), Algorithm 6.8 and Example 6.11.
By choosing the desired type of the problem in the module-window, for example multiobjective location problem with the maximum norm, the software can generate the set of minimal and weakly minimal solutions and partition the set of minimal solutions into possibly degenerated rectangles. This was shown in Figure 6.2.
An access to FLO and more details about this software can be found under the following link:

http://www.project-flo.de/

Furthermore, some additional features and advantages are emphasized in the following remark.

Remark 6.12.

- As mentioned above, the user can deal directly with a real map, which can be loaded by the software. The location of the facilities and the solution sets of the solutions can be visualized with different colors on the real map.
- The software can show the set of (weakly) minimal solutions with different distance functions in the same coordinate system (or directly on a map). Moreover, FLO can plot the convex hull and the level lines of the objective function. The intersections of these sets of the solutions and the convex hull can be of huge interest for the decision maker, when there are difficulties choosing the norm. (Note that the convex hull is sufficient for considering the set of minimal solutions for the p-norms).

- The ability of changing the algorithm settings with detailed information of the outcome of the algorithms.
- Options of customizing the interface and the language.

CHAPTER 7

Duality-based Algorithms for Multi-Facility Location Problems

Multi-facility location problems are an important class of location problems, since most of the applications are not limited to locate only one new facility, but generally more. The problem is summarized by finding the location of a set of N new facilities $x^1, \ldots, x^N$ with respect to p existing facilities such that the distances are minimal. This kind of problems are also called N-Location problems, multi-Weber problems or multi-facility location-allocation problems and were formulated first by Cooper in 1963 [22]. Two examples of such problems are mentioned in Section 2.2.

Aspiring to accuracy this study is not going to ignore the distances between the new facilities to each other, like some previous works did. We set that the distances from the new to the existing facilities as well as the distances under the new facilities have to be minimized simultaneously.

There are many approaches for solving multi-facility location problems, for instance, the heuristics methods (integer programming) as in [23, 79], methods based on a decomposition into smaller multi-facility location problems as in [15] or the generalizations of Weiszfeld's methods as in [70].

In this chapter, we study scalar as well as multiobjective multi-facility location problems. We convert the scalar multi-facility location problem into a single-facility approximation problem in a higher dimension motivated by [103] and [107]. We use the method of the partial inverse by Spingarn [105], which we introduced in Section 3.4, in order to develop a convenient proximal point algorithm for solving the scalar multi-facility location problem converted into a single-facility approximation problem in a higher dimension.

Proximal point algorithms for solving single-facility location problem or the generalizations of the Fermat-Weber-Problem (such as (P_1) introduced in (2.3)) were studied for example by [7, 8, 9, 107] as extensions of the results in [64, 66, 87]. The idea of the solutions is formulating the optimality condition for the problem using the subdifferential calculus, then the application of a suitable Spingarn's partial inverse method for solving the optimality conditions approximately.

Starting from [107, Section 4] and [50, Section 5.10] we formulate a proximal point algorithm for a multi-facility location problem by converting it into a scalar approximation problem. We derive the

optimality conditions and define a suitable operator in order to apply Spingarn's method.
Thereafter we discuss multiobjective multi-facility location problem and develop an interactive procedure for solving such problems using the results for the scalar multi-facility location problems.

7.1 A Proximal Point Algorithm for Scalar Multi-Facility Location Problems

Let $X = \mathbb{R}^n$ and $N \geq 2$. In a scalar multi-facility location problem we locate a set of N new facilities $New := \{x^1, \cdots, x^N\} \subset X$ with respect to a set $E_a = \{a^1, \cdots, a^p\}$ of p existing facilities given in X. We point out that the results of this section can be derived also for Hilbert spaces.
The search for the new facilities is with the consideration of the minimization of both: the distances between the new and the existing facilities and the distances between the new facilities among each other. Also we suppose that the costs of building the N new facilities are $c^i \in X$ for $i \in \{1, \cdots, N\} =: I_N$.

We assign to each distance the weights $h_{ij} > 0$ and $w_{kl} > 0$. These weights can represent, for instance, accruing costs between the locations i, j and k, l respectively. These costs can also be proportional to the distances between these locations. Then we can define the objective function of the scalar multi-facility location problem as

$$f(New, E_a) := \sum_{i=1}^{N} {c^i}^T x^i + \sum_{i=1}^{N} \sum_{j=1}^{p} h_{ij} \|x^i - a^j\| + \sum_{k=1}^{N-1} \sum_{l=k+1}^{N} w_{kl} \|x^k - x^l\|. \tag{7.1}$$

Furthermore, we suppose that $x^i \in D_i \subset X$, where D_i is a convex and closed set with non-empty interior, for each $i \in I_N$.

A feasible set D of the the scalar multi-facility location problem can be obtained from the sets $D_1, \ldots, D_N$ as

$$D_1 \times \cdots \times D_N =: D \subset X^N, \tag{7.2}$$

where $X^N := \overbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}^{N \text{ times}}$.
Then the scalar multi-facility location problem can be described as

$$(\mathsf{MFP}) \qquad \begin{cases} \text{Minimize} & f(New, E_a) \\ \text{subject to} & New \in D. \end{cases} \tag{7.3}$$

Remark 7.1. Taking the feasible set D with the specific structure given in (7.2) is new and different from the above mentioned literature. They chose the feasible set to be the intersection of the sets D_i. The advantage of building the Cartesian product of the sets D_i is considering the case when D_i are, for instance, disjoint or they have no convenient intersection.

We can see through the next proposition that the set D is convex, closed and has nonempty interior, which are very important assumptions in order to solve the problem with the proposed method of Spingarn (for applying the optimality conditions and being able to use the sum rule).

Proposition 7.2. *Let D_i for $i = 1, \cdots, m$ be convex and closed subsets of $\mathbb{R}^k$ with nonempty interiors and let D be the Cartesian product of the sets D_i, i.e.,*

$$D := \{x = (x^1, \dots, x^m) \mid x^i \in D_i,\ (i = 1, \cdots, m)\}.$$

Then the set D is convex, closed and has nonempty interior in $\mathbb{R}^{k \cdot m}$.

Proof. The convexity and the closedness of D can be easily shown (see also [123]). Now we prove that D has a nonempty interior. Since the sets D_i have nonempty interiors, then in every set D_i there is an interior point x^i with a neighborhood of it U_i contained in D_i. Then for the point $x = (x^1, \dots, x^m) \in D$ there exists a neighborhood, which is the open set $U := U_1 \times \cdots \times U_m$ contained in D since $U_i \subset D_i$. Then we conclude that x is an interior point of D. □

Converting to a Single-Facility Approximation Problem with Higher Dimensions

We can exchange (MFP) with an equivalent single-facility median problem by defining a new variable $z := (x^1, \dots, x^N) \in X^N$, i.e.,

$$z = (x_1^1, \dots, x_n^1, \cdots, x_1^N, \dots, x_n^N) \tag{7.4}$$

The distances between the new and the existing facilities can be converted through the matrices $H^i \in L(X^N, X)$ for $i \in I_N$, which consists of the zero matrices $O \in L(X, X)$ and the identity matrix $\mathbb{I} \in L(X, X)$ next to each other with $\mathbb{I}$ in the digit i, i.e., for each $i \in I_N$:

$$H^i := (O, \cdots, O, \underbrace{\mathbb{I}}_{\text{digit } i}, O, \cdots, O). \tag{7.5}$$

Doing the same for the distances between the new facilities among each other we get the matrices $W^{kl} \in L(X^N, X)$, which consist of the zero matrices with the identity matrix $\mathbb{I}$ in the digit k and $-\mathbb{I}$ in the digit l:

$$W^{kl} := (O, \cdots, O, \underbrace{\mathbb{I}}_{\text{digit } k}, O, \cdots, O, \underbrace{-\mathbb{I}}_{\text{digit } l}, O, \cdots, O), \tag{7.6}$$

for $k = 1, \cdots, N-1$ and $l = k+1, \cdots, N$.

Furthermore, let

$$c := (c^1, \dots, c^N) \in X^N \tag{7.7}$$

The new objective function with the new variable $z = (x^1, \cdots, x^N) \in X^N$, which is equivalent to (7.1), can be formulated as follows

$$f_1(z, E_a) := c^T z + \sum_{i=1}^{N} \sum_{j=1}^{p} h_{ij} \|H^i z - a^j\| + \sum_{k=1}^{N-1} \sum_{l=k+1}^{N} w_{kl} \|W^{kl} z\|, \tag{7.8}$$

with $z, c \in X^N$, $a^i \in X$ and $h_{ij} \geq 0$, $w_{kl} \geq 0$.
Using the objective function defined in (7.8), the problem (MFP) is equivalent to the following single-facility approximation problem in a higher dimension, namely in X^N:

$$\begin{cases} \text{Minimize} & f_1(z, E_a) \\ \text{subject to} & z \in D. \end{cases} \tag{7.9}$$

We observe that this problem is a special case of (P_1) defined in (2.3).
In order to solve this problem using the proximal point algorithm illustrated in Section 3.4, we use an indicator function (defined in (3.3)) in order to obtain an unrestricted problem. The objective function of the unrestricted problem is

$$f_2(z, E_a) := c^T z + \sum_{i=1}^{N} \sum_{j=1}^{p} h_{ij} \|H^i z - a^j\| + \sum_{k=1}^{N-1} \sum_{l=k+1}^{N} w_{kl} \|W^{kl} z\| + \mathcal{X}_D(z). \tag{7.10}$$

This leads us to a single-facility approximation problem in a higher dimension given by

$$(\mathsf{SFP_{HD}}) \quad \begin{cases} \text{Minimize} & f_2(z, E_a) \\ \text{subject to} & z \in X^N. \end{cases} \tag{7.11}$$

The problem $(\mathsf{SFP_{HD}})$ in (7.11) is equivalent to the problem (MFP) in (7.3).

The Optimality Conditions

Under the given assumptions we observe that the objective function f_2 described in (7.10) is convex, where the norm terms and the linear term are convex. The indicator function is also convex, since the set D is convex (see Proposition 7.2). Thus, the sufficient and necessary optimality condition for an optimal point $z^0 \in X^N$ with respect to the problem $(\mathsf{SFP_{HD}})$ is the following subdifferential condition:

$$0 \in \partial f_2(z^0, E_a). \tag{7.12}$$

For computing the subdifferential of $f_2(z^0, E_a)$ given in (7.10) we have to apply the Subdifferential Sum Rule (see (3.40)). The assumptions of Theorem 3.31 are fulfilled, where D has nonempty interior according to Proposition 7.2. Therefore, we compute the subdifferential of the norm parts and of the indicator function.

The subdifferential of the indicator function for the set D coincides with the normal cone N_D with respect to the set D (cf. (3.42)). This means

$$\partial \mathcal{X}_D(z^0) = N_D(z^0) = \begin{cases} \{z^* \in X^N \mid \langle z^*, z - z^0 \rangle \leq 0 \quad \forall z \in D\} & \text{if} \quad z^0 \in D, \\ \emptyset & \text{otherwise.} \end{cases} \tag{7.13}$$

Hence, the condition (7.12) can be described equivalently by

$$\alpha_{ij} \in \partial(h_{ij}\|H^i z^0 - a^j\|) \qquad i = 1, \ldots, N;\ j = 1, \ldots, p, \tag{7.14}$$

$$\beta_{kl} \in \partial(w_{kl}\|W^{kl} z^0\|) \qquad k = 1, \ldots, N-1;\ l = 1, \ldots, N, \tag{7.15}$$

$$\gamma \in N_D(z^0) \tag{7.16}$$

$$c + \sum_{i=1}^{N}\sum_{j=1}^{p} \alpha_{ij} + \sum_{k=1}^{N-1}\sum_{l=k+1}^{N} \beta_{kl} + \gamma \ = \ 0 \tag{7.17}$$

Spingarn's Problem

In order to apply the method of the partial inverse, we introduce a space $\mathbb{E}$ with the corresponding complementary subspaces $\mathbb{A}$ and $\mathbb{B}$ (see Section 3.4). These spaces with the help of an operator T are our tool for reformulating the optimality conditions given in (7.14) - (7.17) in a more practical way. In the above mentioned literature there are many possibilities for defining the spaces $\mathbb{E}$, $\mathbb{A}$ and $\mathbb{B}$. We follow the method introduced in [107] and apply it for our problem $(\mathsf{SFP_{HD}})$.
For defining the space $\mathbb{E}$, we consider again the objective function described in (7.10):

$$f_2(z, E_a) = c^T z + \sum_{i=1}^{N}\sum_{j=1}^{p} h_{ij}\|H^i z - a^j\| + \sum_{k=1}^{N-1}\sum_{l=k+1}^{N} w_{kl}\|W^{kl} z\| + \mathcal{X}_D(z).$$

First, we compute the number of the addends in the objective function in (7.10). In the first term $c^T z$ and in the last term $\mathcal{X}_D(z)$ there is only one addend. In the second term: $\sum\limits_{i=1}^{N}\sum\limits_{j=1}^{p} h_{ij}\|H^i z - a^j\|$ we have $N \cdot p$ addends. The third term $\sum\limits_{k=1}^{N-1}\sum\limits_{l=k+1}^{N} w_{kl}\|W^{kl} z\|$ is assorted lexicographically and has $\frac{N(N-1)}{2}$ addend, since $k = 1, \cdots, N-1$ and $l = 1, \cdots, N$. Thus, we can give the matrices W^{kl} a new index μ from 1 till $\frac{N(N-1)}{2} =: t$, i.e., W^μ $(\mu = 1, \cdots, t)$ are now the matrices

$$W^1 := W^{12},\ W^2 := W^{13}, \cdots, W^t := W^{N(N-1)}.$$

We do the same to the weights w_{kl} for $k = 1, \cdots, N-1$ and $l = 1, \cdots, N$ and give them the new index μ from 1 till $\frac{N(N-1)}{2} = t$ and get

$$w_1 := w_{12},\ w_2 := w_{13}, \cdots, w_t := w_{N(N-1)}.$$

If we put
$$q := Np + t + 2 \tag{7.18}$$
then we can introduce the space $\mathbb{E}$ as follows
$$\mathbb{E} := \{e = (e_1, \cdots, e_q) \mid e_i \in X \ (i = 1, \cdots, q)\} \ \subset X^q, \tag{7.19}$$
where $X = \mathbb{R}^n$. The space $\mathbb{E}$ is a Hilbert space equipped with the usual scalar product.
Furthermore, we define an operator $S : \mathbb{E} \to X^N$ in order to describe the complementary subspaces $\mathbb{A}$ and $\mathbb{B}$ of $\mathbb{E}$:
$$S(e) := \sum_{i=1}^{N}\sum_{j=1}^{p}(H^i)^T \, e_{(i-1)p+j} \ + \ \sum_{\mu=1}^{t}(W^\mu)^T \, e_{Np+\mu} \ + \ e_{Np+t+1} \ + \ e_{Np+t+2}, \tag{7.20}$$
where $e \in \mathbb{E}$. Now we define the subspaces $\mathbb{A}$ and $\mathbb{B}$ of $\mathbb{E}$ using the following abbreviations
$$\begin{aligned}
&x_1^i(z) := (H^i z, \cdots, H^i z) && \in X^p, && i = 1, \ldots, N,\\
&x_2(z) := (W^1 z, \cdots, W^t z) && \in X^t, && t = \frac{N(N-1)}{2},\\
&x_3(z) := (z, z) && \in X^2.
\end{aligned}$$
Considering the previous indications we now define $\mathbb{A}$ and $\mathbb{B}$ by
$$\mathbb{A} := \{y = (y_1, \cdots, y_q) \in \mathbb{E} \mid y = (x_1^1(z), \cdots, x_1^N(z), x_2(z), x_3(z)), z \in X^N\}, \tag{7.21}$$
$$\mathbb{B} := \{p = (p_1, \cdots, p_q) \in \mathbb{E} \mid S(p) = 0\}. \tag{7.22}$$
Our aim now is to prove that $\mathbb{A}^\perp = \mathbb{B}$ and to conclude that $\mathbb{A} \oplus \mathbb{B} = \mathbb{E}$.
Take $y = (y_1, \cdots, y_q) \in \mathbb{A}$ and $v = (v_1, \cdots, v_q) \in \mathbb{E}$, then
$$\langle y, v\rangle = \sum_{i=1}^{N}\sum_{j=1}^{p}\langle H^i z, v_{(i-1)p+j}\rangle + \sum_{\mu=1}^{t}\langle W^\mu z, v_{Np+\mu}\rangle + \langle z, v_{Np+t+1}\rangle + \langle z, v_{Np+t+2}\rangle \tag{7.23}$$
$$= \sum_{i=1}^{N}\sum_{j=1}^{p}\langle z, (H^i)^T v_{(i-1)p+j}\rangle + \sum_{\mu=1}^{t}\langle z, (W^\mu)^T v_{Np+\mu}\rangle + \langle z, v_{Np+t+1}\rangle + \langle z, v_{Np+t+2}\rangle$$
$$= \langle z, S(v)\rangle. \tag{7.24}$$
If $v \in \mathbb{B}$ then $\langle y, v\rangle = 0$, then $v \in \mathbb{A}^\perp$, which means that $\mathbb{B} \subset \mathbb{A}^\perp$. If $v \in \mathbb{A}^\perp$, then $\langle y, v\rangle = 0$ for all $y \in \mathbb{A}$ and consequently for all $z \in X^N$ from (7.24), i.e., $\langle z, S(v)\rangle = 0$ for all $z \in X^N$. Hence $S(v) = 0$, which implies that $v \in \mathbb{B}$ and $\mathbb{A}^\perp \subset \mathbb{B}$. Hence, $\mathbb{A}^\perp = \mathbb{B}$. From the closedness of the subspaces $\mathbb{A}$ and $\mathbb{B}$ we get $\mathbb{A} \oplus \mathbb{B} = \mathbb{E}$.
We proceed to the proximal step in the PPA and define the set-valued operator T on the space $\mathbb{E}$, whose components are related to the subdifferentials of the particular terms in the objective function.

For $y_{(i-1)p+j} := H^i z$, $(i = 1, \dots, N;\ j = 1, \dots, p)$, and $y_{Np+\mu} := W^{kl} z$, $(\mu = 1, \dots, t)$ and $y_{Np+t+s} := z$, $(s = 1, 2)$ we define:

$$
\begin{aligned}
T :& \mathbb{E} \rightrightarrows \mathbb{E} && (7.25)\\
y &\to T(y) = (T_1(y_1), \cdots, T_q(y_q)), && (7.26)
\end{aligned}
$$

where

$$
\begin{aligned}
T_{(i-1)p+j}(y_{(i-1)p+j}) &:= \partial(h_{ij}\|y_{(i-1)p+j} - a^j\|), \quad && i = 1, \dots, N;\ j = 1, \dots, p && (7.27)\\
T_{Np+\mu}(y_{Np+\mu}) &:= \partial(w_\mu \|y_{Np+\mu}\|), && \mu = 1, \dots, t && (7.28)\\
T_{Np+t+1}(y_{Np+t+1}) &:= N_D(y_{Np+t+1}), && && (7.29)\\
T_{Np+t+2}(y_{Np+t+2}) &:= c. && && (7.30)
\end{aligned}
$$

If we put

$$
\begin{aligned}
\alpha_{ij} &:= (H^i)^T p^0_{(i-1)p+j} && \in T_{(i-1)p+j}(y_{(i-1)p+j}), && (7.31)\\
\beta_\mu &:= W^\mu p^0_{Np+\mu} && \in T_{Np+\mu}(y_{Np+\mu}), && (7.32)\\
\gamma &:= p^0_{Np+t+1} && \in T_{Np+t+1}(y_{Np+t+1}), && (7.33)\\
c &:= p^0_{Np+t+2} && \in T_{Np+t+1}(y_{Np+t+1}), && (7.34)\\
y^0 &= (x_1^1(z^0), \cdots, x_1^N(z^0), x_2(z^0), x_3(z^0)), && && (7.35)
\end{aligned}
$$

then the optimality conditions in (7.14) - (7.17) can be reformulated as:

$$
\text{Find} \quad (y^0, p^0) \in \mathbb{A} \times \mathbb{B} \quad \text{s.t.} \quad p^0 \in T(y^0). \tag{7.36}
$$

So the problem (7.12) (or equivalently the conditions (7.14) - (7.17)) is equivalent to (7.36). If one of them has a solution, then the other one has a solution as well, i.e., the solution can be obtained from one of them.

The Proximal Step

Observing (7.27) - (7.29), the set-valued operator T is composed by subdifferentials, therefore it is maximal monotone. But the PPA solves the equations of the kind $0 \in T(v)$. In order to bring the problem (7.36) to this form, we use the method of Spingarn illustrated in Section 3.4.
Applying the equivalence (3.57) in Theorem 3.41, the problem in (7.36) is equivalent to

$$
\text{Find} \quad (y^0, p^0) \in \mathbb{A} \times \mathbb{B} \quad \text{s.t.} \quad 0 \in T_{\mathbb{A}}(y^0 + p^0), \tag{7.37}
$$

where $T_{\mathbb{A}}$ is the partial inverse of T w.r.t $\mathbb{A}$. By considering the iterations in Section 3.4, we state that the problem we have to solve now is solving

$$
\tilde{p}^k \in T(\tilde{y}^k) \quad \text{with} \quad \tilde{y}^k + \tilde{p}^k = y^k + p^k.
$$

Which is denoted above as the proximal step. For that it holds that

$$\tilde{p} \in T(\tilde{y}) \;\Leftrightarrow\; \tilde{p}_i \in T_i(\tilde{y}_i) \quad \text{for } i = 1, \cdots, q.$$

Considering the last formulation of the optimality conditions, we define the iteration for the PPA:

$$\tilde{p}^k_{(i-1)p+j} \in \partial(h_{ij}\|\tilde{y}^k_{(i-1)p+j} - a^j\|) \tag{7.38a}$$

$$\text{with} \quad \tilde{p}^k_{(i-1)p+j} + \tilde{y}^k_{(i-1)p+j} = p^k_{(i-1)p+j} + y^k_{(i-1)p+j}, \tag{7.38b}$$

$$\tilde{p}^k_{Np+\mu} \in \partial(w_\mu\|\tilde{y}^k_{Np+\mu}\|) \tag{7.39a}$$

$$\text{with} \quad \tilde{p}^k_{Np+\mu} + \tilde{y}^k_{Np+\mu} = p^k_{Np+\mu} + y^k_{Np+\mu}, \tag{7.39b}$$

$$\tilde{p}^k_{Np+t+1} \in N_D(\tilde{y}^k_{Np+t+1}) \tag{7.40a}$$

$$\text{with} \quad \tilde{p}^k_{Np+t+1} + \tilde{y}^k_{Np+t+1} = p^k_{Np+t+1} + y^k_{Np+t+1}, \tag{7.40b}$$

$$\tilde{p}^k_{Np+t+2} \;=\; c \tag{7.41a}$$

$$\text{with} \quad \tilde{y}^k_{Np+t+2} = p^k_{Np+t+2} + y^k_{Np+t+2} \; - c \tag{7.41b}$$

for $i = 1, \ldots, N;\; j = 1, \ldots, p;\; \mu = 1, \ldots, t.$

In the following, we have to compute the subdifferentials in (7.38) – (7.40).

The Subdifferential for the Norm Parts and the Indicator Function

(a) *Computing the subdifferential of the norm parts in* (7.38) *and* (7.39)

For computing the subdifferential in (7.38), we discuss two cases:

I) $\tilde{y}^k_{(i-1)p+j} - a^j \neq 0$

In this case the structure of the subdifferential of the norm (see Lemma 3.32) provides the next two statements

$$\|\tilde{p}^k_{(i-1)p+j}\|_* = h_{ij}, \tag{7.42}$$

$$\left\langle \tilde{p}^k_{(i-1)p+j} \,,\, \tilde{y}^k_{(i-1)p+j} - a^j \right\rangle = h_{ij}\|\tilde{y}^k_{(i-1)p+j} - a^j\| \tag{7.43}$$

From (7.43) and the consideration of (7.38b) and the multiplication with $\frac{1}{h_{ij}}$ we get

$$\langle \frac{\tilde{p}^k_{(i-1)p+j}}{h_{ij}}\ ,\ \underbrace{p^k_{(i-1)p+j} + y^k_{(i-1)p+j} - a^j}_{:=b_{(i-1)p+j}} - \tilde{p}^k_{(i-1)p+j}\rangle$$

$$= \|\overbrace{p^k_{(i-1)p+j} + y^k_{(i-1)p+j} - a^j} - \tilde{p}^k_{(i-1)p+j}\| \tag{7.44}$$

We observe that (cf. (3.14))

$$B^*[0,1] = \{z^* \in X^N : \|z^*\|_* \leq 1\} \quad \text{(later only } B^*) \tag{7.45}$$

and apply the Cauchy-Schwartz-inequality, (i.e., $\langle x, y\rangle \leq \|x\| \cdot \|y\|$) in an inner-product space:

$$\langle z^*, z\rangle \leq \|z^*\|_* \cdot \|z\| \quad \Rightarrow$$

$$\langle z^*, z\rangle \leq \|z\| \tag{7.46}$$

now from (7.44) with the consideration of the previous inequality (7.46) leads to

$$\langle \frac{\tilde{p}^k_{(i-1)p+j}}{h_{ij}}\ ,\ b_{(i-1)p+j} - \tilde{p}^k_{(i-1)p+j}\rangle \ = \ \|b_{(i-1)p+j} - \tilde{p}^k_{(i-1)p+j}\| \ \geq \ \langle z^*\ ,\ b_{(i-1)p+j} - \tilde{p}^k_{(i-1)p+j}\rangle.$$

This means

$$\begin{aligned}
&\langle \frac{\tilde{p}^k_{(i-1)p+j}}{h_{ij}} - z^*\ ,\ b_{(i-1)p+j} - \tilde{p}^k_{(i-1)p+j}\rangle \ \geq\ 0, \qquad \forall z^* \in B^*\\
&\langle \frac{\tilde{p}^k_{(i-1)p+j}}{h_{ij}} - z^*\ ,\ \frac{b_{(i-1)p+j}}{h_{ij}} - \frac{\tilde{p}^k_{(i-1)p+j}}{h_{ij}}\rangle \ \geq\ 0, \qquad \forall z^* \in B^*\\
&\langle \frac{\tilde{p}^k_{(i-1)p+j}}{h_{ij}} - \frac{b_{(i-1)p+j}}{h_{ij}}\ ,\ z^* - \frac{\tilde{p}^k_{(i-1)p+j}}{h_{ij}}\rangle \ \geq\ 0, \qquad \forall z^* \in B^*
\end{aligned}$$

and concerning the best approximation of $\frac{b_{(i-1)p+j}}{h_{ij}}$ with respect to B^* in Pre-Hilbert spaces the last inequality is equivalent to

$$\tilde{p}^k_{(i-1)p+j} \ = \ h_{ij}\ P_{B^*}(\frac{b_{(i-1)p+j}}{h_{ij}}). \tag{7.47}$$

II) $\tilde{y}^k - a^j = 0$

For this case, we get the structure of the subdifferential of the norm also from Lemma 3.32 as

$$\tilde{p}^k_{(i-1)+j} \in h_{ij}\ \partial\|0\| = \{z^* \in X^N \mid \|z^*\|_* \leq h_{ij}\}, \tag{7.48}$$

which leads to

$$\|\tilde{p}^k_{(i-1)+j}\|_* \leq h_{ij}. \tag{7.49}$$

Now from (7.38b) and the assumption that is $\tilde{y}^k = a^j$ (of the case II)) we get

$$\tilde{p}^k_{(i-1)+j} = p^k_{(i-1)+j} \; + \; y^k_{(i-1)+j} \; - \; a^j \tag{7.50}$$

from (7.49) and (7.50):

$$\|b_{(i-1)p+j}\|_* := \|p^k_{(i-1)+j} + y^k_{(i-1)+j} - a^j\|_* = \|\tilde{p}^k_{(i-1)+j}\|_* \leq h_{ij}. \tag{7.51}$$

By summarizing both cases I) and II) for the norm parts in (7.38) we find:

$$\begin{aligned} b_{(i-1)p+j} &:= p^k_{(i-1)+j} + y^k_{(i-1)+j} - a^j \\ \tilde{p}^k_{(i-1)+j} &:= \begin{cases} b_{(i-1)p+j} & : \|\frac{b_{(i-1)p+j}}{h_{ij}}\|_* \leq 1 \\ h_{ij}\; P_{B^*}(\frac{b_{(i-1)p+j}}{h_{ij}}) & : \|\frac{b_{(i-1)p+j}}{h_{ij}}\|_* > 1. \end{cases} \end{aligned}$$

Analogously to the result of computing the norm parts in (7.38), the subdifferential in (7.39) is computed as follows

$$\begin{aligned} b_{Np+\mu} &:= p^k_{Np+\mu} + y^k_{Np+\mu} \\ \tilde{p}^k_{Np+\mu} &:= \begin{cases} b_{Np+\mu} & : \|\frac{b_{Np+\mu}}{w_\mu}\|_* \leq 1 \\ w_\mu\; P_{B^0}(\frac{b_{Np+\mu}}{w_\mu}) & : \|\frac{b_{Np+\mu}}{w_\mu}\|_* > 1. \end{cases} \end{aligned}$$

(b) *Computing the subdifferential of the indicator function in* (7.40)

That is to solve

$$\tilde{p}^k_{Np+t+1} \in N_D(\tilde{y}^k_{Np+t+1}),$$

with

$$\tilde{p}^k_{Np+t+1} + \tilde{y}^k_{Np+t+1} = p^k_{Np+t+1} + y^k_{Np+t+1}.$$

By reformulating the last equation:

$$\tilde{p}^k_{Np+t+1} = p^k_{Np+t+1} + y^k_{Np+t+1} - \tilde{y}^k_{Np+t+1}$$

and then

$$\begin{aligned} p^k_{Np+t+1} + y^k_{Np+t+1} - \tilde{y}^k_{Np+t+1} &\in N_D(\tilde{y}^k_{Np+t+1}) \\ p^k_{Np+t+1} + y^k_{Np+t+1} &\in \tilde{y}^k_{Np+t+1} + N_D(\tilde{y}^k_{Np+t+1}) \\ p^k_{Np+t+1} + y^k_{Np+t+1} &\in (I + N_D)(\tilde{y}^k_{Np+t+1}). \end{aligned}$$

Since the inverse of $(I + N_D)$ is the projection onto the set D, we can write

$$\tilde{y}^k_{Np+t+1} := P_D(p^k_{Np+t+1} + y^k_{Np+t+1}) \tag{7.52}$$

and $\tilde{p}^k_{Np+t+1}$ can be obtained from $\tilde{y}^k_{Np+t+1}$ as follows

$$\tilde{p}^k_{Np+t+1} = p^k_{Np+t+1} + y^k_{Np+t+1} - \tilde{y}^k_{Np+t+1}. \tag{7.53}$$

The Projection Step (The Projection on the Subspaces)

For an arbitrary element $v \in \mathbb{E}$ with $v = v_{\mathbb{A}} + v_{\mathbb{B}}$, $v_{\mathbb{A}} \in \mathbb{A}$ and $v_{\mathbb{B}} \in \mathbb{B}$, we have

$$v_{\mathbb{A}} = (x_1^1(z), \cdots, x_1^N(z), x_2(z), x_3(z)) \in \mathbb{A} \tag{7.54}$$

with

$$\begin{aligned} x_1^i(z) &:= (H^i z, \cdots, H^i z) && \in X^N && i = 1, \ldots, N \\ x_2(z) &:= (W^1 z, \cdots, W^t z) && \in X^t && t = \frac{N(N-1)}{2} \\ x_3(z) &:= (z, z) && \in X^2. \end{aligned}$$

We apply the definition of the operator S given in (7.20) on $v \in \mathbb{E}$ with $v = v_{\mathbb{A}} + v_{\mathbb{B}}$, where $v_{\mathbb{A}} \in \mathbb{A}$ and $v_{\mathbb{B}} \in \mathbb{B}$:

$$S(v) = S(v_{\mathbb{A}}) \qquad (S(v_{\mathbb{B}}) = 0) \tag{7.55}$$

$$S(v) = \sum_{i=1}^{N} \sum_{j=1}^{p} (H^i)^T \, v_{(i-1)p+j} \; + \; \sum_{\mu=1}^{t} (W^\mu)^T \, v_{Np+\mu} \; + \; v_{Np+t+1} \; + \; v_{Np+t+2} \tag{7.56}$$

$$S(v_{\mathbb{A}}) = \sum_{i=1}^{N} \sum_{j=1}^{p} (H^i)^T \, v_{\mathbb{A}(i-1)p+j} \; + \; \sum_{\mu=1}^{t} (W^\mu)^T \, v_{\mathbb{A}Np+\mu} \; + \; v_{\mathbb{A}Np+t+1} \; + \; v_{\mathbb{A}Np+t+2} \tag{7.57}$$

Considering (7.54), then (7.57) leads to

$$\begin{aligned} S(v_{\mathbb{A}}) &= \sum_{i=1}^{N} \sum_{j=1}^{p} (H^i)^T H^i z \; + \; \sum_{\mu=1}^{t} (W^\mu)^T \, W^\mu z \; + \; z \; + \; z \\ &= \left[p \left(\sum_{i=1}^{N} (H^i)^T H^i \right) \; + \; \sum_{\mu=1}^{t} (W^\mu)^T \, W^\mu \; + \; 2I \right] z \\ &=: \Phi z \end{aligned}$$

with

$$\Phi := p \left(\sum_{i=1}^{N} (H^i)^T H^i \right) \; + \; \sum_{\mu=1}^{t} (W^\mu)^T \, W^\mu \; + \; 2I \tag{7.58}$$

and by putting

$$\begin{aligned} S(v) = S(v_{\mathbb{A}}) &= \sum_{i=1}^{N} \sum_{j=1}^{p} (H^i)^T \, v_{(i-1)p+j} \; + \; \sum_{\mu=1}^{t} (W^\mu)^T \, v_{Np+\mu} \; + \; v_{Np+t+1} \; + \; v_{Np+t+2} \\ &=: u \end{aligned} \tag{7.59}$$

then we state that z can be obtained from solving the following equation being obtained from (7.58) and (7.59):

$$\Phi z = u. \tag{7.60}$$

In order to solve this equation we want to study the regularity of the operator Φ, this is to study weather $\det(\Phi) \neq 0$.
Because of the special structure of the matrices H^i and $W^\mu = W^{kl}$ we have

$$\sum_{j=1}^{p}\sum_{i=1}^{N}(H^i)^T H^i = p \begin{pmatrix} E & O & \cdots & O \\ O & E & \cdots & O \\ \vdots & & \ddots & \vdots \\ O & \cdots & O & E \end{pmatrix} =: pI_{(nN,nN)},$$

$$\sum_{\mu=1}^{t}(W^\mu)^T\, W^\mu = \begin{pmatrix} (N-1)E & -E & \cdots & -E \\ -E & (N-1)E & \cdots & -E \\ \vdots & & \ddots & \vdots \\ -E & \cdots & -E & (N-1)E \end{pmatrix}_{(nN,nN)}.$$

Then Φ can be obtained as

$$\begin{pmatrix} rE & -E & \cdots & -E \\ -E & rE & \cdots & -E \\ \vdots & & \ddots & \vdots \\ -E & \cdots & -E & rE \end{pmatrix}_{(nN,nN)} \quad \text{with} \quad \boxed{r = p + (N-1) + 2 > 0}$$

for a multi-facility problem is $p \geq 1$ and $N \geq 2$ and so

$$r > N - 1, \quad \text{and} \quad r > 4. \tag{7.61}$$

Proposition 7.3. *For $N \geq 2$ it holds*

$$\det \Phi = \det \begin{pmatrix} rE & -E & \cdots & -E \\ -E & rE & \cdots & -E \\ \vdots & & \ddots & \vdots \\ -E & \cdots & -E & rE \end{pmatrix} \neq 0.$$

Proof. In order to compute the determinant of the matrix Φ we begin with the switch of $N-1$ rows, in the way that the odd rows go up and the even rows go down. In the same way we switch $N-1$ columns in the way that the odd columns go to the left and the even columns go to the right. In this way the matrix Φ has the new form:

$$\Phi = \begin{pmatrix} M_N^r & 0 \\ 0 & M_N^r \end{pmatrix}_{(nN,nN)} \quad \text{with} \quad M_N^r = \begin{pmatrix} r & -1 & \cdots & -1 \\ -1 & r & \cdots & -1 \\ \vdots & & \ddots & \vdots \\ -1 & -1 & \cdots & r \end{pmatrix}_{(N,N)}.$$

If we leave the first row and multiplicate the rest with r and add to the first row, then the determinant of M_N^r can be obtained as the following

$$\begin{aligned}\det(M_N^r) &= \frac{1}{r^{N-1}} \det \begin{pmatrix} r & -1 & \cdots & -1 \\ 0 & r^2-1 & \cdots & -(r+1) \\ \vdots & & \ddots & \vdots \\ 0 & -(r+1) & \cdots & r^2-1 \end{pmatrix} \\ &= \frac{1}{r^{N-1}}\, r \det\left((r+1)M_{N-1}^{r-1}\right) \\ &= \frac{1}{r^{N-2}}\,(r+1)^{N-1} \det\left(M_{N-1}^{r-1}\right).\end{aligned}$$

$r = -1$ makes this determinant equal to zero, and since $r > 3$ we have to prove that $\det(M_N^r) \neq 0$ by induction:

1. For $N = 2$ and $r > 3$ we have:

$$\begin{aligned}\det(M_2^r) &= (r+1)\det(M_1^{r-1}) \\ &= (r+1)(r-1) \\ &= r^2 - 1 \\ \Rightarrow \det(M_2^r) &> 0\end{aligned}$$

2. It holds that $\det(M_N^r) \neq 0$ for $N = k$ and $r > k-1$:

$$\det(M_k^r) = \frac{1}{r^{k-2}}\,(r+1)^{k-1}\det\left(M_{k-1}^{r-1}\right) \neq 0$$

3. We want now to prove that $\det(M_{k+1}^r) \neq 0$ for $N = k+1$ and $r > k$:

$$\begin{aligned}\det(M_N^r) &= \frac{1}{r^{N-2}}\,(r+1)^{N-1}\det\left(M_{N-1}^{r-1}\right) \\ \det(M_{k+1}^r) &= \frac{1}{r^{k-1}}\,(r+1)^k \det M_k^{r-1}\end{aligned}$$

Let us put $s := r-1$ then $s > k-1$:

$$\det(M_{k+1}^r) = \frac{1}{(s+1)^{k-1}}\,((s+1)+1)^k \det(M_k^s)$$

We see that $\frac{1}{(s+1)^{k-1}}\,((s+1)+1)^k \neq 0$ and also $\det(M_k^s) \neq 0$ for $s > k-1$ thus we conclude that $\det(M_N^r) \neq 0$.

Then $\det(\Phi) = (\det(M_N^r))^2 \neq 0$. □

Now the operator Φ is invertible, since $\det(\Phi) \neq 0$. Hence, we have a solution of (7.60), which represents the projection of $v \in \mathbb{E}$ onto the subspace $\mathbb{A}$:

$$
\begin{aligned}
z :&= \Phi^{-1}\, u \\
&= \Phi^{-1}\left(\sum_{i=1}^{N}\sum_{j=1}^{p}(H^i)^T\, v_{(i-1)p+j} \;+\; \sum_{\mu=1}^{t}(W^\mu)^T\, v_{Np+\mu} \;+\; v_{Np+t+1}v_{Np+t+2}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
v_{\mathbb{A}} &= (y_1, \cdots, y_q) \\
&:= (x_1^1(z), \cdots, x_1^N(z), x_2(z), x_3(z)) \in \mathbb{A}
\end{aligned}
$$

with

$$
\begin{aligned}
&x_1^i(z) := (H^i z, \cdots, H^i z) && \in X^N && i = 1, \dots, N \\
&x_2(z) := (W^1 z, \cdots, W^t z) && \in X^t && t = \frac{N(N-1)}{2} \\
&x_3(z) := (z, z) && \in X^2.
\end{aligned}
$$

Since $\mathbb{A} \oplus \mathbb{B}$, the projection of $v \in \mathbb{E}$ onto the subspace $\mathbb{B}$ can be computed as

$$
v_{\mathbb{B}} = v - v_{\mathbb{A}} \tag{7.62}
$$

The points $\tilde{p}^k$ and $\tilde{y}^k$ have the relation (see (3.58)):

$$
\tilde{y}^k + \tilde{p}^k = y^k + p^k, \quad (y^k \in \mathbb{A}, p^k \in \mathbb{B})
$$

and through the additivity of the projection on a linear subspace we obtain

$$
\begin{aligned}
\tilde{y}^k &= y^k + p^k - \tilde{p}^k \\
P_{\mathbb{A}}(\tilde{y}^k) &= P_{\mathbb{A}}(y^k) + P_{\mathbb{A}}(p^k) - P_{\mathbb{A}}(\tilde{p}^k) \\
y^{k+1} - y^k &= -P_{\mathbb{A}}(\tilde{p}^k).
\end{aligned}
$$

and a PPA (with the illustration of the iterations) for solving the problem $(\mathsf{SFP_{HD}})$ or (MFP) is ready to be introduced.

A PPA for solving the problem (MFP)

The Algorithm 7.4 is a version of the PPA for computing an approximate solution of a multi-facility location problem (MFP) in (7.3) converted into the approximation problem $(\mathsf{SFP_{HD}})$ in (7.11). By considering the sum $y^k + p^k$ as a variable our PPA is described as follows.

Algorithm 7.4. Input: The problem (MFP), i.e., the sets $New = \{x^1, \cdots, x^N\}$, $E_a = \{a^1, \cdots, a^p\}$, c^i, h_{ij} and w_{kl} for $i \in I_N$, $j \in I_p$, $k = 1, \cdots, N-1$, $l = k+1, \cdots, N$.

① The initialization

- Convert the problem (MFP) into the form ($\mathsf{SFP_{HD}}$), i.e., compute H^i, W^{kl} for $i \in I_N$, $k = 1, \cdots, N-1$, $l = k+1, \cdots, N$.
- Give the matrices W^{kl} for $k = 1, \cdots, N-1$ and $l = 1, \cdots, N$ a new numbering from 1 till $\frac{N(N-1)}{2} =: t$.
- Take $z^1 \in X^N,\ p^1 \in \mathbb{B}$ and set:

$$\begin{aligned}(p^1 + y^1)_{(i-1)p+j} &= p^1_{(i-1)p+j} \ + \ H^i z^1 \\ (p^1 + y^1)_{Np+\mu} &= p^1_{Np+\mu} \ + \ W^\mu z^1 \\ (p^1 + y^1)_{Np+t+s} &= p^1_{Np+t+s} \ + \ z^1\end{aligned}$$

  for $i = 1, \ldots, N;\ j = 1, \ldots, p;\ \mu = 1, \ldots, t;\ s = 1, 2$

- Compute Φ^{-1}

② The Proximal Step

- For $i = 1, \ldots, N;\ j = 1, \ldots, p;\ \mu = 1, \ldots, t$ set

$$\begin{aligned}b_{(i-1)p+j} &:= p^k_{(i-1)+j} + y^k_{(i-1)+j} - a^j \\ \tilde{p}^k_{(i-1)+j} &:= \begin{cases} b_{(i-1)p+j} & : \|\frac{b_{(i-1)p+j}}{h_{ij}}\|_* \le 1 \\ h_{ij}\ P_{B^0}(\frac{b_{(i-1)p+j}}{h_{ij}}) & : \|\frac{b_{(i-1)p+j}}{h_{ij}}\|_* > 1. \end{cases}\end{aligned}$$

$$\begin{aligned}b_{Np+\mu} &:= p^k_{Np+\mu} + y^k_{Np+\mu} \\ \tilde{p}^k_{Np+\mu} &:= \begin{cases} b_{Np+\mu} & : \|\frac{b_{Np+\mu}}{w_\mu}\|_* \le 1 \\ w_\mu\ P_{B^0}(\frac{b_{Np+\mu}}{w_\mu}) & : \|\frac{b_{Np+\mu}}{w_\mu}\|_* > 1. \end{cases}\end{aligned}$$

- For $s = 1$ set

$$\tilde{p}^k_{Np+t+1} = p^k_{Np+t+1} + y^k_{Np+t+1} - P_D(p^k_{Np+t+1} + y^k_{Np+t+1})$$

- For $s = 2$ set

$$\tilde{p}^k_{Np+t+2} = c$$

③ The Projection Step

Compute

- $\tilde{p}^k := \Phi^{-1}(\underbrace{\sum_{i=1}^{N}\sum_{j=1}^{p}(H^i)^T\, p^k_{(i-1)p+j} \;+\; \sum_{\mu=1}^{t}(W^\mu)^T\, p^k_{Np+\mu} \;+\; p^k_{Np+t+1} + \; p^k_{Np+t+2}}_{:=\sigma_1})$.
- $z^{k+1} := z^k - \tilde{p}^k$
- $p^{k+1} + y^{k+1} := \tilde{p}^k + \big(x_1^1(z^k - 2\tilde{p}^k), \cdots, x_1^N(z^k - 2\tilde{p}^k),\; x_2(z^k - 2\tilde{p}^k),\; x_3(z^k - 2\tilde{p}^k)\big)$

4 A Stop Criterion:

Stop, if for a given value $\varepsilon > 0$:

$$\|\sigma_1\| + \|\tilde{p}^k\| = \|S(\tilde{p}^k)\| + \|\tilde{p}^k\| < \varepsilon$$

and

$$\|(p^{k+1} + y^{k+1}) - (p^k + y^k)\| < \varepsilon.$$

Otherwise, set $k = k + 1$ and go back to Step 2.

Output: z^k the approximate solution of the condition (7.12), which solves the problem $(\mathsf{SFP}_{\mathsf{HD}})$.

Duality for Scalar Multi-Facility Location Problem

After finding a solution procedure for solving scalar multi-facility location problems through the proximal point algorithm, we take a look on duality for this problem (see Section 5.1).
We consider the primal approximation problem in (7.9) with the objective function

$$f_1(z, E_a) := c^T z + \sum_{i=1}^{N}\sum_{j=1}^{p} h_{ij}\|H^i z - a^j\| + \sum_{k=1}^{N-1}\sum_{l=k+1}^{N} w_{kl}\|W^{kl} z\|. \tag{7.63}$$

This function is of the form of the objective function in (5.1) such that it is possible to apply the duality results given in Section 5.1.
For $K_{X^N} \subset X^N$, $B \in L(X^N, X^N)$, $b \in X^N$, we suppose that the feasible set D has the following structure:

$$D := \{z \in X^N \mid z \in K_{X^N}, B(z) - b \in K_{X^N}\}.$$

According to the assumptions of our problem in (7.9) the spaces X and V from (5.1) are equal to X^N in this case.
For the primal problem given in

$$\begin{cases} \text{Minimize} & f_1(z, E_a) \\ \text{subject to} & x \in D, \end{cases} \tag{7.64}$$

we introduce the following dual problem

$$\begin{cases} \text{Maximize} \quad f_1^*(y,u,v) \\ \text{subject to} \quad x \in D^*, \end{cases} \tag{7.65}$$

where

$$f^*(y,u,v) := \sum_{i=1}^{N}\sum_{j=1}^{p} h_{ij} y_{ij}(a^j) + v(b)$$

and

$$\begin{aligned} D^* := \{(y,u,v) \mid y = (y_{11},\cdots,y_{Np}), y_{ij} \in L(X,\mathbb{R}), u = (u_{11},\cdots,u_t) = (u_{kl}) \in L(X,\mathbb{R}), \\ h_{ij}\|y_j\|_* \le h_{ij},\ w_{kl}\|u_{kl}\|_* \le w_{kl},\ v \in K^*_{X^N}, \\ c - \sum_{i=1}^{N}\sum_{j=1}^{p} h_{ij}(H^i)^T y_{ij} + \sum_{k=1}^{N-1}\sum_{l=k+1}^{N} w_{kl}(W^{kl})^T u_{kl} - B^T v \in K^*_{X^N}\}. \end{aligned}$$

Remark 7.5. The duality statements given in Chapter 5 hold also for the primal problem (7.64) and the dual problem (7.65). Taking into account that the scalar multi-facility location problem (MFP) given in (7.3) is equivalent to the approximation problem (7.9) we get weak (and under additional assumptions) strong duality statements for the problems (7.3) and (7.65). These duality statements can be used for deriving a slopping criteria in the proximal point algorithm (Algorithm 7.4) much the same as in [49].

7.2 An Interactive Procedure for Solving Multiobjective Approximation Problems

This section is devoted to taking a look at solving multiobjective location problems by using a suitable scalarization and the proximal point algorithm (Algorithm 7.4).

An interactive procedure for solving (P_2), given in (2.4), through the PPA is introduced in [49, 107] for two types of assumptions. We want to find such an approach for a multiobjective multi-facility location problem.

The scalar multi-facility location problem (MFP) described in (7.3) depends on the choice of the weights h_{ij}, w_{kl}. If the decision maker is not able to give concrete values for these weights, then it is very convenient to study a multi-facility location problem with a vector-valued objective function.

A multiobjective multi-facility location problem, concerning the new facilities $New = \{x^1,\cdots,x^N\}$ from $X = \mathbb{R}^n$ and the exiting facilities $E_a = \{a^1,\cdots,a^p\}$ and $c^i \in X \ \ (i \in I_N)$, can be formulated for $s := Np + t + N$ as follows (cf. (7.1)):

$$F(New, E_a) := (F_{11},\ldots,F_{1p},\ldots,F_{N1},\ldots,F_{Np},F'_{12},\ldots,F'_{N,N-1},F''_1,\ldots,F''_N) : X \to \mathbb{R}^s \tag{7.66}$$

with

$$\begin{aligned}
F_{ij}(New, E_a) &:= \|x^i - a^j\|, \\
F'_{kl}(New, E_a) &:= \|x^k - x^l\|, \\
F''_i(New, E_a) &:= c^{i^T} x^i
\end{aligned}$$

for all $i = 1, \dots, N$, $j = 1, \dots, p$, $k = 1, \dots, N-1$, $l = k+1, \dots, N$ with $t = \frac{N(N-1)}{2}$ and $s = Np + t + N$.

In order to formulate the multiobjective multi-facility location problem we introduce a proper, convex, pointed and closed cone $K \subset \mathbb{R}^s$ with $\operatorname{int} K \neq \emptyset$. The multiobjective multi-facility location problem is given by

$$(\mathsf{MMFP}) \quad \begin{cases} \text{Minimize} & F(New, E_a) \\ \text{subject to} & x \in D, \end{cases} \tag{7.67}$$

where $D \subset X^N$ is defined as in (7.2). Note that the minimum in (7.67) is to be understood with respect to the cone K.

According to Theorem 4.12 with the fact that $F(New, E_a)$ is $\mathbb{R}^s_+$-convex (which can be easily observed), we know that there exists $\lambda^* \in K^* \setminus \{0\}$ such that we get the scalarized problem:

$$\begin{cases} \text{Minimize} \quad F(New, \lambda^*) := \sum\limits_{i=1}^{N} \sum\limits_{j=1}^{p} h_{ij} \|x^i - a^j\| + \sum\limits_{k=1}^{N-1} \sum\limits_{l=k+1}^{N} w_{kl} \|x^k - x^l\| + \sum\limits_{i=1}^{N} \lambda_i^* c^{i^T} x^i \\ \text{subject to} \quad x \in D, \end{cases} \tag{7.68}$$

where

$$\lambda^* := (h_{11}, \dots, h_{Np}, w_{12}, \dots, w_{N,N-1}, \lambda_1^*, \dots, \lambda_N^*) \in K^* \setminus \{0\},$$

where h_{ij}, w_{kl}, $\lambda_i^* \in \mathbb{R}$ for all $i = 1, \dots, N$, $j = 1, \dots, p$, $k = 1, \dots, N-1$, $l = k+1, \dots, N$. For the problems (7.67) and (7.68) it holds that for every minimal solution of the multiobjective problem (MMFP) given in (7.67) there exits a functional $\lambda^* \in K^* \setminus \{0\}$ such that it is also a minimal solution of the scalarized problem (7.67). It is important to mention that h_{ij}, w_{kl} in (7.68) can be selected according to the choice of λ^* from the dual cone in contrast to the weights h_{ij}, w_{kl} in (7.1) which are determined by the decision maker.

We observe that the problem obtained is a scalar multi-facility location problem of the type (7.1). Therefore, we convert this problem to an approximation problem as in Section 7.1. This means we set $z = (x^1, \dots, x^N) \in X^N$ and $c = (c^1, \dots, c^N) \in X^N$ as shown in (7.4), (7.7) and define the matrices H^i and W^{kl} for all $i = 1, \dots, N$, $j = 1, \dots, p$, $k = 1, \dots, N-1$, $l = k+1, \dots, N$ as in (7.5) and (7.6) respectively. Furthermore, we define the set $D \subset X^N$ as in (7.2). This implies the problem

$$\begin{cases} \text{Minimize} \quad F(z,\lambda^*) := \sum\limits_{i=1}^{N} \sum\limits_{j=1}^{p} h_{ij} \|H^i z - a^j\| + \sum\limits_{k=1}^{N-1} \sum\limits_{l=k+1}^{N} w_{kl} \|W^{kl} z\| + \sum\limits_{i=1}^{N} \lambda_i^* c^T z \\ \text{subject to} \quad z \in D. \end{cases} \tag{7.69}$$

For deriving an interactive procedure for solving the multiobjective multi-facility location problem (MMFP) we define the optimal-value function

$$\phi(\lambda^*) := \inf\{F(z,\lambda^*) \mid z \in D\}, \tag{7.70}$$

and the optimal set mapping $\psi: \ D \rightrightarrows \mathbb{R}$ with

$$\psi(\lambda^*) := \{z \in D \mid F(z,\lambda^*) = \phi(\lambda^*)\}. \tag{7.71}$$

In [49, Section 4.2.3] continuity properties of the mappings ϕ and ψ are shown, namely the continuity of ϕ and the upper continuity of ψ (see [49, Definition 4.2.7]). These stability properties are fulfilled for the problem (7.69) such that it is possible to derive an effective algorithm for solving (MMFP). Taking into account these continuity properties we formulate the following interactive solution procedure for the multiobjective approximation problem (MMFP) given in (7.67). This interactive procedure uses a scalarization by means of linear continuous functionals and the PPA (Algorithm 7.4) in order to solve (MMFP) involving the decision maker by accepting solutions according to his preferences.

Algorithm 7.6 (An Interactive Procedure for Solving (MMFP))**.**

1
- Choose $\bar{\lambda}^* \in \operatorname{int} K^*$.
- Compute the approximate solution (y^0, p^0) of the scalarized problem (7.69) with Algorithm 7.4.
- If the solution (y^0, p^0) is accepted by the decision maker, then STOP.

2 Set $k = 0$ and $t_0 = 0$. Choose $\bar{\bar{\lambda}}^* \in \operatorname{int} K^*, \bar{\bar{\lambda}}^* \neq \bar{\lambda}^*$.

3 Choose t_{k+1} with $t_k < t_{k+1} < 1$, set $\lambda_k^* := \bar{\lambda}^* + t_{k+1}(\bar{\bar{\lambda}}^* - \bar{\lambda}^*)$ and compute an approximate solution (y^{k+1}, p^{k+1}) with Algorithm 7.4 and (y^k, p^k) as a starting point. If an approximate solution is not found for $t > t_k$, then go to Step 1.

4 If the point (y^{k+1}, p^{k+1}) is accepted by the decision maker, then STOP.

5
- If $t_{k+1} = 1$, then go to Step 1.
- Otherwise, set $k = k + 1$ and go to Step 3.

Remark 7.7. A similar approach can be developed for solving the multiobjective location and approximation problem $(\mathscr{P})$ defined in (5.6).
Consider a special case of $(\mathscr{P})$ with $x, a^1, \cdots, a^p \in X = \mathbb{R}^n$, $A_i \in L(X, X)$, $\beta_i = 1$ and $\|\cdot\|_{(i)} = \|\cdot\|$ for $i \in I_p$. The assumptions (A_4) and (A_5) are still holding. We refer also that concerning the case $\beta_i > 1$ another version of the PPA can be derived for more general assumptions (cf. [49]).
We minimize the vector-valued objective function $f(x)$ from (5.4) over a convex and closed set $D \subset X$. From Section 5.2 we know that there exists $\lambda^* \in \operatorname{int} K^*$ such that we get the scalarized problem

$$\begin{cases} \text{Minimize} \quad f(x, \lambda) := \sum\limits_{i=1}^{p} \lambda_i^* \|A_i(x) - a^i\| + \sum\limits_{j=1}^{m} \lambda_{p+j}^* C_{p+j}(x) \\ \text{subject to} \quad x \in D. \end{cases} \tag{7.72}$$

We observe that the problem (7.72) is a special case of (7.68). Thus we can apply Algorithm 7.6 for solving the special case of the multiobjective location and approximation problem $(\mathscr{P})$ mentioned above.

CHAPTER 8

Conclusions and Outlook

Conclusions

The results of this thesis can be summarized as follows:

- In this thesis we studied a class of extended multiobjective location and approximation problems $(\mathscr{P})$, where the objective function includes distances as well as cost functions. This class is very important and has a lot of applications, for instance, in economy, engineering and physics. In particular, we gave an example for an application in radiotherapy treatment.

  The structure of the objective function in $(\mathscr{P})$ is more useful for the decision maker because the cost functions are included in the vector-valued objective function as additional criteria, which is different to the known literature. This model is more natural than to study the sum of a cost function and a vector of approximation terms in the objective function. Furthermore, by using this structure of the objective function it is possible to apply reducing methods (studied in Section 4.3) and to derive decomposition algorithms, that provide minimal solutions of the extended multiobjective location problem.

- We proved duality assertions for the problem $(\mathscr{P})$ using generalized Lagrange duality in Section 5.2. For proving strong duality statements we have shown that the assumptions (C_5) and (C_6) in Theorem 3.37 are fulfilled under generalized Slater conditions.

- A brief overview of the linearization ability of the problem $(\mathscr{P})$ using block norms is given. Then the well-know algorithms for solving multiobjective linear optimization problems can by applied, for example the open solver BENSOLVE (which is based on Benson's outer approximation algorithm).

- In Section 5.3 we studied the multiobjective location problem $(\mathscr{P}_1)$ (given in (5.41)). We studied the duality-based geometrical structure of the set of minimal solutions of $(\mathscr{P}_1)$, and we derived the following new results:

  - We characterized the set of weakly minimal solutions of $(\mathscr{P}_1)$ with help of the Pareto reducibility of $(\mathscr{P}_1)$, and we proved that this set coincides with the Manhattan (or the maximum) rectangular hull of the exiting facilities.
  - We derived an implementable partition algorithm (Algorithm 6.3) for decomposing the set of minimal solutions of $(\mathscr{P}_1)$ (which is generally not convex).

- The partition algorithm is the base for developing decomposition algorithms of the extended multiobjective location problems $(\mathscr{P}_2)$ (given in (6.2)). These decomposition algorithms generate minimal solutions of the extended problem $(\mathscr{P}_2)$ taking into account the decomposition results in Section 4.3.

- We gave numerical implementable examples of the application of the previous decomposition algorithms, especially for solving location problems where attraction and repulsion points are involved.

- Scalar multi-facility location problems are also studied in this thesis. First, we transformed the multi location problem into a single-facility approximation problem in higher dimensions. Second, we used the method of the partial inverse by Spingarn and a corresponding proximal point algorithm for solving scalar multi-facility location problems.

- Furthermore, we showed that multiobjective multi-facility location problems can be solved through an interactive procedure using the PPA developed for the scalar multi-facility location problem.

Outlook

During this study we have discovered several interesting topics that can be investigated in the future. Some of these interesting research points are listed below.

- Deriving duality assertions for extended multiobjective location and approximation problems involving gauges instead of norms.

- It is also interesting to derive duality assertions for extended multiobjective location and approximation problems formulated by considering some uncertainties (duality for robust multiobjective optimization problems).

- The extended multiobjective location and approximation problems can be studied with not necessarily linear additional cost functions, but with more general nonlinear additional functions.

- Developing proximal point algorithms for scalar and multiobjective multi-facility location problems involving mixed gauges.

- The linearization approach that we introduced for $(\mathscr{P})$ can be extended by applying different methods of multiobjective linear programming. Furthermore, we can derive decomposition algorithms (and apply the results of Section 6.2) for the resulting multiobjective linear problem.

- The possibility of applying our partition and decomposition algorithms on extended multiobjective location problem where forbidden regions are considered.

- Some of the results of this thesis are shown in general spaces. It would be interesting to use these results for deriving algorithms for solving approximation problems in general spaces.

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Selbständigkeitserklärung

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Dissertation selbständig und ohne fremde Hilfe angefertigt habe. Ich habe keine anderen als die angegebenen Quellen und Hilfsmittel benutzt und die den benutzten Werken wörtlich oder inhaltlich entnommenen Stellen als solche kenntlich gemacht.

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2003 - 2006	Übung und Tutorium, Institute für Informatik, Institut für Ingenieurswissenschaften, Universität Damaskus
12/2005 - 6/2006	Mathematiklehrerin, Realschule in Damaskus
2003 - 2004	Ersatzmathematiklehrerin, verschiedene Schulen in Damaskus

Publikationsliste

S. Alzorba. Duality Statements for Multicriteria Location Problems and Applications. In: Tagungsband / 19. Workshop für (Diskrete) Optimierung Holzhau, pp. 1 – 6, 2010.

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