Vector Optimization Problems with Variable Ordering Structures

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To my parents, **Parviz** *and* **Shahla**

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Chapter 1

Introduction

Many real world optimization problems require the minimization of multiple conflicting objectives, e.g. in finance, the maximization of the expected return versus the minimization of risk in portfolio optimization; in production theory, the minimization of production time versus the minimization of the cost of manufacturing equipment; or the maximization of tumor control versus the minimization of normal tissue complication in radiotherapy treatment design. Such problems can be formulated as vector optimization problems.

Recently, vector optimization problems with variable ordering structures are studied intensively in the literature because they have important applications in economics, engineering design, management science and many other fields (see Bao, Mordukhovich [5], Eichfelder [24], Engau [29] and Huang, Yang, Chan [40]).

Approximate solutions of vector optimization problems with variable ordering structures play an important role from the theoretical as well as computational point of view. It is well known that one needs compactness assumptions in order to show existence results for optimization problems. Such compactness assumptions are not fulfilled for many optimization problems. Also, we know that under weak assumptions and without compactness conditions, we have to deal with approximate solutions and we can show several assertions without any compactness assumptions for these solutions. Also, if we apply numerical algorithms for solving optimization problems, then these algorithms usually generate approximate solutions which are close to the exact solutions. Here, we will introduce approximate solutions of vector optimization problems with variable ordering structures. Many papers deal with different concepts for approximate solutions with respect to a fixed ordering structure; see [39, 54, 56, 60, 68–74] for different definitions, concepts and properties of these elements. Gutiérrez , Jiménez and Novo in [36] introduced a new concept of approximate solutions of vector optimization problems and they unified some different concepts of approximate solutions with respect to fixed ordering structures. In this thesis, we deal with approximate minimizers, approximately nondominated and approximately minimal solutions with respect to variable ordering structures.

In the second chapter, we will present some necessary mathematical backgrounds and concepts which will be used in the next chapters. Vector optimization problems have close relationships with partial orderings in objective spaces. Also, it is known that convex cone helps us to define an ordering in vector spaces. Therefore, we will introduce binary relations and some of their properties. Another important topic in the second chapter is cones and their properties. At the end, we will show some relationships between cone properties and binary ordering.

In the third chapter, we will consider two different orderings such partially orderings with fixed cones and variable orderings with variable structures. In the first section, we deal with the vector optimization problem with a fixed cone and later we introduce concepts for approximately minimal, approximately nondominated solutions and approximate minimizers of vector optimization problems with respect to variable ordering structures. In order to describe solution concepts, we use a set-valued map and this map is not a (pointed, convex) cone-valued map necessarily. We illustrate the different concepts for approximate solutions by several examples. Important properties of these three different kinds of approximate solutions of vector optimization problems with respect to variable ordering structures will be shown. Eichfelder [24] studied relationships between exact nondominated and minimal solutions of vector optimization problems with variable ordering structures. In the third chapter, we will show relationships between different kinds of approximately optimal elements (εk^0-nondominated, εk^0-minimal and εk^0-minimizers) of vector optimization problems with respect to variable ordering structures and relationships between sets of approximate solutions choosing different parameters will be discussed. At the end of the third chapter, it will be obvious to see that concepts of approximately nondominated, approximately minimal elements and approximate minimizers coincide in the case of vector optimization problems with fixed ordering structures.

In scalarization methods for vector optimization problems, we replace a vector optimization problem by a suitable scalar optimization problem to characterize optimal elements. In the fourth chapter, we characterize εk^0-optimal elements by scalarization via nonlinear functionals. By this scalarization, we show that an approximate solution of the original vector optimization problem is also a solution for the scalar problem and vice versa. We present a scalarization method with the help of nonlinear functionals. This scalarization method for vector optimization problems was introduced by Gerstewitz (1983) in [31] (see also [32], [33, Theorem 2.3.1], [35, Theorem 3.38]) and one year later by Pascoletti and Serafini (1984) in [61]. Some generalizations of this scalarization method for vector optimization problems with a variable ordering structure where the ordering map is pointed, closed, convex and cone-valued can be found in [12, 15, 16, 22]. Here, we have a generalization of the Tammer-Weidner functional without any cone or convexity assumptions and we use it for the characterization of all of the three different kinds of approximate solutions. In fact, our ordering map is just a set-valued map with certain properties. For sure, our scalarization also works when the ordering map is a convex and cone-valued map.

Ekeland's variational principle is a very deep assertion concerning the existence of an exact solution of a slightly perturbed optimization problem in a neighborhood of an approximate solution of the original optimization problem. Applications of Ekeland's variational principle can be seen in economics, control theory, game theory, nonsmooth analysis and many other fields. Several generalizations of Ekeland's variational principle [28] for vector optimization problems with a fixed ordering structure are given in [3, 4, 8, 9, 13, 14, 37, 38, 41, 44, 55, 70]. In the fifth chapter, we will use results from third and fourth chapters in order to derive variational principles for vector optimization problems with variable ordering structures and an extension of Ekeland's variational principle for vector optimization problems with a variable ordering structure will be given.

In the last chapter, we present optimality conditions for approximate solutions of vector optimization problems with variable ordering structures. We will use the variational principles presented in the fifth chapter in order to derive necessary conditions for approximate solutions of vector optimization problems with variable ordering structures. Bao and Mordukhovich [5] have shown necessary conditions for nondominated points of sets and nondominated solutions of vector optimization problems with variable ordering structures and general geometric constraints, applying methods of variational analysis and generalized differentiation (see Mordukhovich [58] and Mordukhovich, Shao [59]). In our result we use both Mordukhovich and generic approach to subdifferentials (compare [21]). We prove the necessary condition for approximate solutions using a vector-valued variant of Ekeland's variational principle (see [34, Corollary 9]). After that we will give second-order optimality conditions by concept of tangential derivatives of second-order for set-valued optimization problems with variable ordering structures.

Chapter 2

Preliminaries

In this chapter, we will present some necessary mathematical backgrounds and concepts which will be used in the next chapters. Vector optimization problems have close relation with partial orderings in objective spaces. Also, it is known that convex cones help us to define orderings in vector spaces. Therefore, we will introduce binary relations and some of their properties. The second important topic is cones and some of their properties. At the end, we will show some cone properties and their relationships with order properties.

2.1 Order Relations

In this section, we introduce binary and partial relations and some of their properties. Partial orders are the most important classes of relations in vector optimization problems. Partial ordering is given in many real linear spaces because of its important role for introducing solution concepts and practical interests. In the following we suppose that X and Y are real linear spaces.

Definition 2.1.1. Let $A \neq \emptyset$ be a nonempty set, the set of ordered pairs of elements of A is defined as following: $A \times A := \{(x_1, x_2) \mid x_1, x_2 \in A\}$.

A binary relation on X is a subset $\mathscr{R}$ of $X \times X$. We write $x\mathscr{R}y$ for $(x,y) \in \mathscr{R}$.

Definition 2.1.2. A binary relation $\mathscr{R}$ on X is called

- reflexive iff $(x,x) \in \mathscr{R}$ for all $x \in X$,
- symmetric iff $(x,y) \in \mathscr{R} \implies (y,x) \in \mathscr{R}$ for all $x,y \in X$,
- antisymmetric iff $(x,y) \in \mathscr{R}$ and $(y,x) \in \mathscr{R} \implies x = y$ for all $x,y \in \mathscr{R}$,
- transitive iff $(x,y) \in \mathscr{R}$ and $(y,z) \in \mathscr{R} \implies (x,z) \in \mathscr{R}$ for all $x,y,z \in \mathscr{R}$.

A binary relation $\mathscr{R}$ is called preorder if it is transitive.

Definition 2.1.3. Every binary relation $\leq$ on X is called a partial ordering if it is reflexive, transitive and if for arbitrary $a,b,x,y \in X$, the following properties hold:

1. $x \leq y$ and $a \leq b \implies x+a \leq y+b$,

2. $x \leq y$ and $\lambda \in \mathbb{R}_+ \implies \lambda x \leq \lambda y$.

Definition 2.1.4. A partially ordered linear space is a real linear space equipped with a partial order.

It is important to note that we can not compare two arbitrary elements in a partially ordered space in general. But there are some binary relations which any two arbitrary elements are comparable.

Definition 2.1.5. A binary relation $\mathscr{R}$ is called a total order if $\mathscr{R}$ is a partial order and if every two elements are comparable, i.e., for all $x,y \in X$, either $x\mathscr{R}y$ or $y\mathscr{R}x$.

Definition 2.1.6. If each subset $A \subseteq X$ has a first element $\bar{x}$, i.e., $\bar{x}\mathscr{R}x$ for all $x \in A$, then we say that X is well ordered relative to $\mathscr{R}$.

Minimal (maximal) elements of a set A relative to relation $\mathscr{R}$ are defined as following:

Definition 2.1.7. Suppose that $\mathscr{R}$ is an order relation on the nonempty set S and A is a subset of S. We say $\bar{x}$ is a minimal (maximal) element of A relative to $\mathscr{R}$ if

$$\forall x \in A,\ x\mathscr{R}\bar{x} \implies \bar{x}\mathscr{R}x \qquad (\forall x \in A \quad \bar{x}\mathscr{R}x \implies x\mathscr{R}\bar{x}).$$

The set of minimal (maximal) element of A relative to $\mathscr{R}$ are denoted by $\mathrm{Min}(A,\mathscr{R})$ $(\mathrm{Max}(A,\mathscr{R}))$. Note that if the order relation $\mathscr{R}$ is antisymmetric, then $\bar{x}$ is a minimal (maximal) element of A if and only if

$$\forall x \in A, \quad x\mathscr{R}\bar{x} \implies x = \bar{x}, \qquad (\forall x \in A, \quad \bar{x}\mathscr{R}x \implies x = \bar{x}).$$

Remark 2.1.8. If $\mathscr{R}$ is a partial order on X, then a subset $A \subseteq X$ can have no, one or several minimal (maximal) elements. But if $\mathscr{R}$ is a total order, then every subset A of X has at most one minimal (maximal) element.

Definition 2.1.9. If $\mathscr{R}$ is an order relation on X and $A \subseteq X$, then $\mathscr{R}_A = \mathscr{R} \cap (A \times A)$ is an order relation of A and has the following properties:

1. If $\mathscr{R}$ is a partial order (preorder, total order) on X, then $\mathscr{R}_A$ is also a partial order (preorder, total order) on A.

2. $\bar{x}$ is a minimal (maximal) element of A relative to $\mathscr{R}$ if and only if $\bar{x}$ is a minimal (maximal) element of A relative to $\mathscr{R}_A$.

Example 2.1.10. Suppose that S is a nonempty set and let $\mathscr{P}$=$P(S)$ be the power set of S, i.e., the set of all subsets of S, then binary relation

$$\mathscr{R} := \{(A,B) \in \mathscr{P} \times \mathscr{P} : A \subset B\}$$

is a partial order. If X has more than two element then $\mathscr{R}$ is not a total order.

2.2 Cone Properties

Definition 2.2.1. Let $C \subset Y$ be a nonempty subset of a real linear space Y. The set C is called a cone iff $\lambda c \in C$ for all $c \in C$ and for all $\lambda \in \mathbb{R}_+ := \{t \in \mathbb{R} \mid t \geqq 0\}$.

We define multiplication of a set with a scalar by

$$\alpha S := \{\alpha s : s \in S\},$$

specially $-S = \{-s : s \in S\}$. Furthermore, algebraic sum of two sets S and T is as following

$$S + T := \{s + t : s \in S, t \in T\}.$$

We write $s + T$ instead of $\{s\} + T$, if $S = \{s\}$ is a singleton.

Some further notation used in this thesis are as following:

- $\operatorname{int} S$ is the interior of S,
- $\operatorname{bd} S$ is the boundary of S,
- $\operatorname{cl} S := \operatorname{int} S \cup \operatorname{bd} S$,
- $\operatorname{rint} S$ is the relative interior of S,
- $\operatorname{conv} S$ is the convex hull of S.
- $\operatorname{cone} S$ is the conic hull of S.

Definition 2.2.2. Suppose Y is a real linear space and $C \subseteq Y$.

1. A cone/set C is called proper or nontrivial iff $C \neq Y$ and $C \neq \{\mathbf{0}_Y\}$.
2. A cone/set C is called solid iff $\operatorname{int} C \neq \emptyset$.
3. A cone/set C is called pointed iff $C \cap (-C) = \{\mathbf{0}_Y\}$, i.e., if $c \in C$ and $c \neq \mathbf{0}_Y$, then $c \notin -C$.
4. A cone/set C is called convex iff $\lambda c_1 + (1-\lambda)c_2 \in C$ for all $0 \leq \lambda \leq 1$ for all $c_1, c_2 \in C$.
5. We say that cone C generates Y or C is called reproducing iff $C - C = Y$.

6. Let nonempty convex set B be a subset of convex cone $C \neq \{\mathbf{0}_Y\}$. The set B is called a base for convex cone C iff each $c \in C \backslash \{\mathbf{0}_Y\}$ has a unique representation of the form $c = \beta b$ for some $\beta > 0$ and for some $b \in B$.

Convexity of B and uniqueness of β imply $\mathbf{0}_Y \notin B$. Note that a cone C is convex if and only if it is closed under addition and this means closedness under addition of a cone C is sufficient for convexity.

Theorem 2.2.3. Suppose that C is a convex cone with base B in real linear space Y, then C is pointed.

Proof. Suppose both c and $-c$ belong to C. By definition of base, we get $c = \lambda b_1$ and $-c = \mu b_2$ and this means $c = \lambda b_1 = -\mu b_2$. Therefore,

$$\lambda b_1 + \mu b_2 = \mathbf{0} \implies \frac{\lambda}{\lambda + \mu} b_1 + \frac{\mu}{\lambda + \mu} b_2 = \mathbf{0}.$$

By convexity of B, we get $\mathbf{0} \in B$. But this is a contradiction to $\mathbf{0} \notin B$. $\square$

Proposition 2.2.4. Let C be a convex cone in a real linear space Y with a nonempty interior, then $\operatorname{int} C = C + \operatorname{int} C$.

Proof. We know for $c_1 \in \operatorname{int} C$, $c_1 = c_1 + \mathbf{0}_Y$. Therefore, obviously

$$\operatorname{int} C = \operatorname{int} C + \{\mathbf{0}_Y\} \subseteq \operatorname{int} C + C.$$

Now suppose that $c_1 \in \operatorname{int} C, c_2 \in C$ and $y \in Y$. By $c_1 \in \operatorname{int} C$, there exists $\overline{\varepsilon} > 0$ such that $c_1 + \varepsilon y \in C$ for every $\varepsilon \in [0, \overline{\varepsilon}]$. Since C is a convex cone, $c_1 + \varepsilon y + c_2 \in C$. Because c_1, c_2, y were arbitrary and $c_1 + c_2 + \varepsilon y \in C$, then $c_1 + c_2$ is an interior point of C and $C + \operatorname{int} C \subseteq \operatorname{int} C$ and proof is complete. $\square$

Definition 2.2.5. Let Y be a vector space over a field $\mathbb{F}$. The continuous dual space of Y, denoted Y^*, is the set of all linear maps from Y to $\mathbb{F}$.

Definition 2.2.6. The dual cone C^* of a set C is the following set

$$C^* = \{y^* \in Y^* : y^*(y) \geq 0 \ \ \forall y \in C\}.$$

Note that dual cone is always a convex cone, even if C is neither convex nor a cone. Furthermore, the set

$$C^\sharp = \{y^* \in Y^* : y^*(y) > 0 \ \ \forall y \in C\}$$

is called quasi interior of cone C. Note that if $C^\# \neq \emptyset$, then C is pointed. This implication is an equivalence if Y is a finite dimensional space; see page 2 of [33] for the proof and more details. For example, if $C = \{\mathbf{0}_Y\}$, then $C^* = Y^*$ and if $C = Y$, then $C^* = \{\mathbf{0}\}$. Suppose that C_1 and C_2 are two convex cone with dual cones C_1^* and C_2^* respectively and $C_1 \subseteq C_2$, then $C_2^* \subseteq C_1^*$.

Proposition 2.2.7. Let C be a convex cone in Y with $\operatorname{int} C \neq \emptyset$, then

$$\operatorname{int} C \subset \{y \in Y : y^*(y) > 0 \quad \forall y^* \in C^* \setminus \{\mathbf{0}\}\ \}.$$

Proof. Suppose that $y \in \operatorname{int} C$ and $y^* \in C^* \setminus \{\mathbf{0}\}$, then there exists $z \in Y$ such that $y^*(z) < 0$. Now, since $y \in \operatorname{int} C$, there exists $\overline{\varepsilon} > 0$ such that $y + \varepsilon z \in C$ for all $\varepsilon \in [0, \overline{\varepsilon}]$. Since $y + \varepsilon z \in C$ and y^* is linear, we have

$$y^*(y + \varepsilon z) \geq 0 \implies y^*(y) \geq -\varepsilon y^*(z) > 0 \qquad \text{and}$$

$$\operatorname{int} C \subset \{y \in Y : y^*(y) > 0 \quad \forall y^* \in C^* \setminus \{\mathbf{0}\}\}.$$

This completes the proof. □

If C characterizes a partial ordering in Y, we say C is an ordering cone. Now, we are ready to show relationships between order properties and cone properties.

Theorem 2.2.8. Suppose that C is a pointed convex cone in Y, then the binary relation

$$\leq_C = \{(x, y) \in Y \times Y : y - x \in C\} \tag{2.1}$$

is a partial ordering in Y.

Proof. Proof is easy and we just prove transitivity. Suppose $x \leq_C y$ and $y \leq_C z$, so

$$y - x, z - y \in C \implies \frac{1}{2}(y - x) + \frac{1}{2}(z - y) \in C.$$

This means $z - x \in C$ and $x \leq_C z$. □

Proposition 2.2.9. Suppose that order relation $\mathscr{R}$ is compatible with scalar multiplication, i.e., $(\alpha x, \alpha y) \in \mathscr{R}$ for all $(x, y) \in \mathscr{R}$ and $\alpha \in \mathbb{R}_+$, then $C_{\mathscr{R}} := \{y - x : (x, y) \in \mathscr{R}\}$ is a cone.

Proof. Suppose that $a \in C_{\mathscr{R}}$, then there exists $(x, y) \in \mathscr{R}$ such that $a = y - x$ and by compatibility with scalar multiplication, we get $(\alpha x, \alpha y) \in \mathscr{R}$. Therefore, $\alpha a = \alpha y - \alpha x \in C_{\mathscr{R}}$ for all $\alpha > 0$ and this means $C_{\mathscr{R}}$ is a cone. □

Similarly, we define strict partial ordering by

$$<_C = \{(x, y) \in Y \times Y : y - x \in \operatorname{int} C\}.$$

For example, when $C = \mathbb{R}_+$, then partial ordering $\leq_C$ is usual ordering $\leq$ and the strict partial ordering $<_C$ is usual strict ordering $<$ on $\mathbb{R}$.

We say a relation $\mathscr{R}$ is compatible with addition if $(x + z, y + z) \in \mathscr{R}$ holds for all $(x, y) \in \mathscr{R}$ and $z \in Y$.

Lemma 2.2.10. If $\mathscr{R}$ have addition compatibility, then for all $a \in C_{\mathscr{R}}$ we have $\mathbf{0}\mathscr{R}a$.

Proof. Suppose that $a \in C_{\mathscr{R}}$, then there exists $(x,y) \in \mathscr{R}$ such that $a = y - x$. By addition compatibility, we get $(x+z, y+z) \in \mathscr{R}$ for $z \in \mathbb{R}^m$. Now, set $z = -x$, then $(\mathbf{0}, y-x) \in \mathscr{R}$ and $\mathbf{0}\mathscr{R}a$. □

The following theorem [33, Theorem 2.1.13] shows relationships between some geometrical properties of cones and order relation properties.

Proposition 2.2.11. Let Y be a linear space and C be a cone in Y. Then $\leq_C$ defined by (2.1) is compatible with addition and scalar multiplication. Moreover the following properties hold:

1. $\leq_C$ is reflexive if and only if $\mathbf{0} \in C$.

2. $\leq_C$ is antisymmetric if and only if C is pointed.

3. $\leq_C$ is transitive if and only if C is convex.

Proof.

1. Suppose that $\leq_C$ is reflexive, then for any $x \in Y$, $x - x = \mathbf{0} \in C$. Now, if $\mathbf{0} \in C$ and $x \in Y$, then $x - x = \mathbf{0} \in C$ and $x \leq_c x$ for any $x \in Y$.

2. First suppose that $\leq_C$ is antisymmetric and $a, -a \in C$. Since $a, -a \in C$, we can say that $\mathbf{0} \leq_C a$ and $\mathbf{0} \leq_c -a$. By addition compatibility, we get $a \leq_C \mathbf{0}$. Since $\leq_C$ is antisymmetric, $a = \mathbf{0}$. Now, suppose that C is pointed and there exist some $x, y \in Y$ such that $x \leq_C y$ and $y \leq_C x$. Therefore, $y - x = a \in C$ and $x - y = -a \in C$, but since C is pointed, $a = -a = \mathbf{0}$ and this means $x = y$.

3. First suppose that $x, y, z \in Y$, $x \leq_C y$ and $y \leq_C z$, then we have $y - x, z - y \in C$. By convexity of cone C, we get $y - x + z - y \in C$ and this means $x \leq_C z$. Now, suppose that $a, b \in C$ and $\leq_C$ is transitive. By $b \in C$, $\mathbf{0} \leq_C b$ and addition compatibility, we get $a \leq_C a + b$. By transitivity of $\leq_C$, we get $\mathbf{0} \leq_C a + b$ and this means $a + b \in C$ and C is a convex cone. □

Chapter 3

Solution Concepts of Vector Optimization Problems

Many real world optimization problems require the minimization of multiple conflicting objectives, e.g. the maximization of the expected return versus the minimization of risk in portfolio optimization, the minimization of production time versus the minimization of the cost of manufacturing equipment, or the maximization of tumor control versus the minimization of normal tissue complication in radiotherapy treatment design. Such problems can be formulated as vector optimization problems. In this chapter, we will consider two different orderings such partially orderings with fixed cones and variable orderings with variable structures and we will define optimal points with respect to these orderings. When we are talking about optimal points of a set in a partially ordered space, we are mostly interested in minimal or maximal elements. But also sometimes for us is important to find weakly (strongly) minimal or maximal points and locally optimal solutions in the case of nonconvex optimization. Let Y be a real linear space and $C \subset Y$ be an ordering cone. Since every maximal point of a set $\Omega \subset Y$ with respect to an ordering cone $C \subset Y$ is also a minimal point with respect to the ordering cone $-C$, we just will deal with minimal points here. We study the following vector optimization problem

$$\min\, y \quad \text{subject to} \;\; y \in \Omega. \tag{VOP}$$

Here, we will introduce approximate solutions of vector optimization problems with fixed and variable ordering structures. Many papers deal with different concepts for approximate solutions of vector optimization problems with respect to fixed ordering structures; see [39, 54, 56, 60, 68–74] for different definitions, concepts and properties of these elements. Gutiérrez , Jiménez and Novo in [36] introduced a new concept of approximate solutions of vector optimization problems and they unified some different concepts of approximate solutions with respect to fixed ordering structures. First, we define several notions of approximate elements of vector optimization problems with fixed and variable ordering structures and later, relationships between sets of approximate solutions choosing different parameters ε will be discussed. Last section is devoted to the presentation of the relationships between different concepts of approximate solutions of vector optimization problems with variable ordering structures. Obviously, an exact solution of a vector optimization problem is the special case of approximate solution and all our results can be used for exact solutions.

3.1 Approximate Solutions of Vector Optimization Problems w.r.t. Fixed Ordering Structures

It is well known that one needs compactness assumptions in order to show existence results for solutions of optimization problems. Such compactness assumptions are not fulfilled for many optimization problems. Also, we know that under weak assumptions and without compactness conditions, we have to deal with approximate solutions and we can show several assertions without any compactness assumptions for these solutions. Also, if we apply numerical algorithms for solving optimization problems, then these algorithms usually generate approximate solutions that are close to exact solutions. Therefore, first in this section, we will define approximate solutions of vector optimization problems with respect to fixed ordering structures and in the next section, we will deal with approximate solutions of vector optimization problems with respect to variable ordering structures. Let that Y be a real linear space, $\Omega \subset Y$ be a closed subset of Y and $C \subset Y$ be an ordering, proper, closed, convex and pointed cone. Consider the vector optimization problem (VOP).

Definition 3.1.1.

- An element $\bar{y} \in \Omega$ is called a minimal point of the set Ω with respect to the ordering cone C iff

$$\Omega \cap (\bar{y} - C \backslash \{\mathbf{0}\}) = \emptyset. \tag{3.1}$$

- Let $\operatorname{int} C \neq \emptyset$. An element $\bar{y} \in \Omega$ is called a weakly minimal point of the set Ω with respect to cone C iff

$$(\bar{y} - \operatorname{int} C) \cap \Omega = \emptyset.$$

We can define similar definitions for local and local weakly minimal point of the set Ω. For more details about local minimal points see [11, 19, 49, 75].

Definition 3.1.2. Consider the problem (VOP).

- An element $\bar{y} \in \Omega$ is called a local minimal point of the set Ω with respect to the ordering cone C iff there exists a neighborhood U of $\bar{y}$ such that

$$(\bar{y} - C \backslash \{\mathbf{0}\}) \cap (\Omega \cap U) = \emptyset.$$

- An element $\bar{y} \in \Omega$ is called a local weakly minimal point of the set Ω with respect to the ordering cone C iff there exists a neighborhood U of $\bar{y}$ such that
$$(\bar{y} - \operatorname{int} C) \cap (\Omega \cap U) = \emptyset.$$

If we apply numerical and iterative algorithms for solving optimization problems, then these algorithms usually generate approximate solutions. Also for showing existence results for solutions of optimization problems, one need to apply compactness assumptions but these compactness assumptions are not fulfilled for many optimization problems and without these assumptions, we have to deal with approximate solutions. In the following, we bring definition of approximate solutions of vector optimization problems with fixed ordering structures.

Let Y be a linear topological space and $k^0 \in Y \backslash \{\mathbf{0}\}$. For any $\varepsilon \geq 0$, we define approximately minimal elements of the set Ω with respect to a fixed ordering cone $C \subset Y$ as following.

Definition 3.1.3. Let $y_\varepsilon \in \Omega$, $\varepsilon \geq 0$ and consider the problem (VOP).

1. y_ε is said to be an εk^0-minimal element of Ω with respect to C iff
$$(y_\varepsilon - \varepsilon k^0 - C \backslash \{\mathbf{0}\}) \cap \Omega = \emptyset.$$

2. Let $\operatorname{int} C \neq \emptyset$. y_ε is said to be a weakly εk^0-minimal element of Ω with respect to C iff
$$(y_\varepsilon - \varepsilon k^0 - \operatorname{int} C) \cap \Omega = \emptyset.$$

3.2 Approximate Solutions of Vector Optimization Problems w.r.t. Variable Ordering Structures

Recently, vector optimization problems w.r.t. variable ordering structures (VVOP) are studied intensively in the literature because they have important applications in economics, engineering design, management science and many other fields (see Bao, Mordukhovich [5], Bao, Mordukhovich, Soubeyran [6], Eichfelder [24], Engau [29] and Huang, Yang, Chan [40]). In these papers, solution concepts are introduced using cone-valued map whereas in this section, set-valued maps are considered. In this chapter we impose the following assumption.

Assumption (A). Let Y be a linear topological space, $k^0 \in Y \backslash \{\mathbf{0}\}$, $\varepsilon \geq 0$ and Ω be a closed set in Y. Suppose that $C : Y \rightrightarrows Y$ is a set-valued map where $C(y)$ is a proper and closed set with $C(y) + [0, +\infty)k^0 \subseteq C(y)$ for all $y \in \Omega$.

The natural way to extend solution concepts of vector optimization problems with a fixed ordering cone given by (3.1) to the case of variable ordering structures is as follows.

We are looking for elements $\bar{y} \in \Omega$ such that

$$(\bar{y} - C(\bar{y}) \setminus \{\mathbf{0}\}) \cap \Omega = \emptyset \tag{3.2}$$

or

$$\forall y \in \Omega : \qquad (\bar{y} - C(y) \setminus \{\mathbf{0}\}) \cap \Omega = \emptyset. \tag{3.3}$$

Solutions $\bar{y} \in \Omega$ with the property (3.2) are called *minimal elements* of Ω with respect to $C(\cdot)$. Minimal points of Ω with respect to $C(\cdot)$ are introduced by Chen, Huang and Yang [10–12]. Solutions with the property given by (3.3) are called *minimizers* of Ω with respect to $C(\cdot)$, (see [12, Definition 1.11] under a different name).

In the case of fixed ordering structure, we can write (3.1) equivalently in the following form

$$\forall y \in \Omega : \qquad \bar{y} \notin y + C \setminus \{\mathbf{0}\}. \tag{3.4}$$

But if we want to generalize this definition to vector optimization problems with variable ordering structures, then the concept in (3.4) leads us to *minimal* (see (3.6)) as well as to so called *nondominated points* (see (3.5)) of Ω with respect to $C(\cdot)$. A minimal point $\bar{y}$ of a set Ω is a candidate element which is not dominated by another point y of Ω with respect to the associated set $C(\bar{y})$ at this candidate point $\bar{y}$. In the definition of minimal elements, an ordering set is a set associated to the minimal point but for nondominated elements, an ordering set is a set associated to another point. Important properties of these points can be found in [10–12, 15, 23, 24, 76–78].

If we want to define a solution concept for vector optimization problems with respect to ordering $C : Y \rightrightarrows Y$ in a natural way from (3.4) we get

$$\forall y \in \Omega : \qquad \bar{y} \notin y + C(y) \setminus \{\mathbf{0}\} \tag{3.5}$$

or

$$\forall y \in \Omega : \qquad \bar{y} \notin y + C(\bar{y}) \setminus \{\mathbf{0}\}. \tag{3.6}$$

The concept in (3.5) was introduced by Yu [78] in 1974, the so called *nondominated points*. Furthermore, (3.6) leads to the definition of *minimal points* considered by Chen, Huang and Yang [10–12].

However, it is not possible to derive the concept of *minimizers* from (3.4) because we change in the definition of minimizers the set $C(y)$ independently from the elements belonging to Ω.

For sure, sets of all these points coincide in vector optimization problems with fixed ordering structures.

Suppose that $C : Y \rightrightarrows Y$ is a set-valued map where $C(y)$ is a closed and pointed set for every $y \in Y$. We define three different domination relations. For $y^1, y^2, y^3 \in Y$

$$y^1 \leq_1 y^2 \text{ if } y^2 \in y^1 + C(y^1) \backslash \{\mathbf{0}\}, \tag{3.7}$$

$$y^1 \leq_2 y^2 \text{ if } y^2 \in y^1 + C(y^2) \backslash \{\mathbf{0}\}, \tag{3.8}$$

$$y^1 \leq_3 y^2 \text{ if for all } \mathrm{y}^3 \in \mathrm{Y}: \; y^2 \in y^1 + C(y^3) \backslash \{\mathbf{0}\}. \tag{3.9}$$

If $C(y^1) = C(y^2) = C(y^3)$ for all $y^1, y^2, y^3 \in Y$, then these three domination relations are the same and a vector optimization problem with a variable ordering structure reduces to a vector optimization problem with a standard fixed domination structure.

3.2.1 Approximate Minimizers

We introduce the concept of approximate minimizers based on the domination relation (3.9). More details and properties of these points are given in [66, 67].

Definition 3.2.1. Let assumption (A) be fulfilled and $y_\varepsilon \in \Omega$.

1. y_ε is said to be an εk^0-minimizer of Ω with respect to the ordering map $C : Y \rightrightarrows Y$ iff $y^1 \not\leq_3 y_\varepsilon - \varepsilon k^0$ for all $y^1 \in \Omega$, i.e.,

$$\forall y, y^1 \in \Omega : \qquad (y_\varepsilon - \varepsilon k^0 - C(y) \backslash \{\mathbf{0}\}) \cap \{y^1\} = \emptyset.$$

2. Let $\operatorname{int} C(y) \neq \emptyset$ for all $y \in \Omega$. y_ε is said to be a weak εk^0- minimizer of Ω with respect to the ordering map $C : Y \rightrightarrows Y$ iff

$$\forall y, y^1 \in \Omega : \qquad (y_\varepsilon - \varepsilon k^0 - \operatorname{int} C(y)) \cap \{y^1\} = \emptyset.$$

3. y_ε is said to be a strong εk^0-minimizer of Ω with respect to the map $C : Y \rightrightarrows Y$ iff

$$\forall y^1, y^2 \in \Omega : \qquad y_\varepsilon - \varepsilon k^0 \in y^1 - C(y^2).$$

Remark 3.2.2.

- We denote the set of all εk^0-minimizers of Ω with respect to the ordering map $C : Y \rightrightarrows Y$ by $\varepsilon k^0\text{-MZ}(\Omega, C)$.
- We denote the set of all weak εk^0-minimizers of Ω with respect to the ordering map C by $\varepsilon k^0\text{-WMZ}(\Omega, C)$.
- We denote the set of all strong εk^0-minimizers of Ω with respect to C by $\varepsilon k^0\text{-SMZ}(\Omega, C)$.

If $\varepsilon = 0$, then these definitions are definitions of minimizers, weak minimizers and strong minimizers (see [12]) and we denote them by $\text{MZ}(\Omega, C)$, $\text{WMZ}(\Omega, C)$ and $\text{SMZ}(\Omega, C)$ respectively.

Lemma 3.2.3. Let $C : Y \rightrightarrows Y$ be a set-valued map and $k^0 \in Y \setminus \{\mathbf{0}\}$, then $C(y) + [0,+\infty)k^0 \subseteq C(y)$ implies $\operatorname{int} C(y) + [0,+\infty)k^0 \subseteq \operatorname{int} C(y)$.

Proof. Suppose that there exists $c \in \operatorname{int} C(y)$ and $\varepsilon > 0$ such that $c + \varepsilon k^0 \notin \operatorname{int} C(y)$. Since $C(y) + [0,+\infty)k^0 \subseteq C(y)$ and $c + \varepsilon k^0 \notin \operatorname{int} C(y)$, we have $c + \varepsilon k^0 \in \operatorname{bd} C(y)$. By $c \in \operatorname{int} C(y)$ and $c + \varepsilon k^0 \in \operatorname{bd} C(y)$, we get the following implication for any $\gamma > 0$:

$$c + \varepsilon k^0 + \gamma k^0 \notin C(y) \implies c + (\varepsilon + \gamma)k^0 \notin C(y).$$

But this is a contradiction to the assumption $C(y) + [0,+\infty)k^0 \subseteq C(y)$. Therefore, we can conclude that $\operatorname{int} C(y) + [0,+\infty)k^0 \subseteq \operatorname{int} C(y)$. □

The following theorem shows several properties of approximate minimizers, weak approximate minimizers and strong approximate minimizers and their relationships to each other. Later, this theorem will help us to see relationships between sets of exact minimizers and εk^0-minimizers.

Theorem 3.2.4. Let assumption (A) be fulfilled, $C(y)$ be a pointed set for all $y \in \Omega$ and additionally $\varepsilon, \varepsilon_1, \varepsilon_2 \geq 0$. The following properties hold:

1. $\mathrm{MZ}(\Omega, C) \subseteq \varepsilon_1 k^0\text{-}\mathrm{MZ}(\Omega, C) \subseteq \varepsilon_2 k^0\text{-}\mathrm{MZ}(\Omega, C) \quad \text{if } 0 \leq \varepsilon_1 \leq \varepsilon_2.$
2. Suppose that $\operatorname{int} C(y) \neq \emptyset$ for all $y \in \Omega$, then the following holds: $\mathrm{WMZ}(\Omega, C) \subseteq \varepsilon_1 k^0\text{-}\mathrm{WMZ}(\Omega, C) \subseteq \varepsilon_2 k^0\text{-}\mathrm{WMZ}(\Omega, C) \quad \text{if } 0 \leq \varepsilon_1 \leq \varepsilon_2.$
3. $\mathrm{SMZ}(\Omega, C) \subseteq \varepsilon_1 k^0\text{-}\mathrm{SMZ}(\Omega, C) \subseteq \varepsilon_2 k^0\text{-}\mathrm{SMZ}(\Omega, C) \quad \text{if } 0 \leq \varepsilon_1 \leq \varepsilon_2.$
4. Suppose that $\operatorname{int} C(y) \neq \emptyset$ for all $y \in \Omega$, then the following holds: $\varepsilon k^0\text{-}\mathrm{SMZ}(\Omega, C) \subseteq \varepsilon k^0\text{-}\mathrm{MZ}(\Omega, C) \subseteq \varepsilon k^0\text{-}\mathrm{WMZ}(\Omega, C) \quad \text{if } \varepsilon \geq 0.$

Proof. 1. First, we prove if $\varepsilon_1 \geq 0$ then $\mathrm{MZ}(\Omega, C) \subseteq \varepsilon_1 k^0\text{-}MZ(\Omega, C)$. The proof is obvious for $\varepsilon_1 = 0$ and we suppose $\varepsilon_1 > 0$. Suppose that $y_\varepsilon \in \mathrm{MZ}(\Omega, C)$ but $y_\varepsilon \notin \varepsilon_1 k^0\text{-}\mathrm{MZ}(\Omega, C)$. This means that there exist $y^1, y \in \Omega$ and $c_1 \in C(y) \setminus \{\mathbf{0}\}$ such that

$$y^1 \in (y_\varepsilon - \varepsilon_1 k^0 - C(y) \setminus \{\mathbf{0}\}) \cap \Omega \quad \text{and} \quad y_\varepsilon - \varepsilon_1 k^0 - c_1 = y^1. \tag{3.10}$$

Since $C(y) + [0,+\infty)k^0 \subseteq C(y)$, there exists $c_2 \in C(y)$ such that

$$c_1 + \varepsilon_1 k^0 = c_2. \tag{3.11}$$

We prove that $c_2 \neq \mathbf{0}$. Suppose that $c_2 = \mathbf{0}$, then by (3.11), we get

$$c_1 + \varepsilon_1 k^0 = \mathbf{0} \implies c_1 = -\varepsilon_1 k^0. \tag{3.12}$$

By $c_1 \in C(y)$, we get $-\varepsilon_1 k^0 \in C(y)$. By pointedness of $C(y)$ and $C(y) + [0,+\infty)k^0 \subseteq C(y)$ for all $y \in \Omega$, we get

$$\mathbf{0} + \varepsilon_1 k^0 = \varepsilon_1 k^0 \in C(y). \tag{3.13}$$

Since $\varepsilon_1 \neq 0$ and $k^0 \neq \mathbf{0}$, we have $\varepsilon_1 k^0 \neq 0$ and by (3.12) and (3.13), we get

$$\{\varepsilon_1 k^0, -\varepsilon_1 k^0\} \in C(y)\backslash\{\mathbf{0}\} \cap (-C(y)),$$

but this is contradiction to the pointedness of $C(y)$. This means that $c_2 \in C(y)\backslash\{\mathbf{0}\}$ and $c_2 \neq \mathbf{0}$. Moreover, by (3.10) and (3.11), we get

$$y^1 = y_\varepsilon - c_2 \implies (y_\varepsilon - C(y)\backslash\{\mathbf{0}\}) \cap \{y^1\} \neq \emptyset.$$

But this is a contradiction to $y_\varepsilon \in \mathrm{MZ}(\Omega, C)$.

Now suppose that $0 < \varepsilon_1 < \varepsilon_2$, then there exists $\gamma > 0$ such that $\varepsilon_2 = \varepsilon_1 + \gamma$. Suppose that $y_\varepsilon \in \varepsilon_1 k^0$-$\mathrm{MZ}(\Omega, C)$ but $y_\varepsilon \notin \varepsilon_2 k^0$-$\mathrm{MZ}(\Omega, C)$. This means that there exist $y^1, y \in \Omega$

$$y^1 \in (y_\varepsilon - (\varepsilon_1 + \gamma)k^0 - C(y)\backslash\{\mathbf{0}\}) \implies y^1 \in (y_\varepsilon - \varepsilon_1 k^0 - (\gamma k^0 + C(y)\backslash\{\mathbf{0}\}).$$

Since $C(y) + [0, +\infty)k^0 \subseteq C(y)$, then $\gamma k^0 + C(y)\backslash\{\mathbf{0}\} \subseteq C(y)$ and there exists $c_1 \in C(y)$ such that $y_\varepsilon - \varepsilon_1 k^0 - c_1 = y^1$. Also by $C(y) + [0, +\infty)k^0 \subseteq C(y)$, $\varepsilon_1 \neq 0, k^0 \neq 0$ and pointedness of $C(y)$ for all $y \in \Omega$, we get $(C(y)\backslash\{\mathbf{0}\}) + [0, +\infty)k^0 \subseteq (C(y)\backslash\{\mathbf{0}\})$. This means that $c_1 \neq \mathbf{0}$ and $c_1 \in C(y)\backslash\{\mathbf{0}\}$. Since $c_1 \neq \mathbf{0}$, we have $y^1 \in (y_\varepsilon - \varepsilon_1 k^0 - C(y)\backslash\{\mathbf{0}\})$. But this is a contradiction to $y_\varepsilon \in \varepsilon_1 k^0$-$\mathrm{MZ}(\Omega, C)$.

2. Suppose that $y_\varepsilon \in \mathrm{WMZ}(\Omega, C)$, i.e., y_ε is a weak minimizer with respect to the ordering map C and $(y_\varepsilon - \operatorname{int} C(y)) \cap \{y^1\} = \emptyset$ for all $y, y^1 \in \Omega$. Since $\operatorname{int} C(y) + [0, +\infty)k^0 \subseteq \operatorname{int} C(y)$, for any $\varepsilon > 0$ we can write

$$(y_\varepsilon - \varepsilon k^0 - \operatorname{int} C(y)) \cap \{y^1\} \subseteq (y_\varepsilon - \operatorname{int} C(y)) \cap \{y^1\} = \emptyset \qquad \forall y, y^1 \in \Omega,$$

$y_\varepsilon \in \varepsilon k^0$-$\mathrm{WMZ}(\Omega, C)$ for all $\varepsilon > 0$ and $\mathrm{WMZ}(\Omega, C) \subseteq \varepsilon_1 k^0$-$\mathrm{WMZ}(\Omega, C)$.

Now suppose that $\varepsilon_1 < \varepsilon_2$, then there exists $\gamma > 0$ such that $\varepsilon_2 = \varepsilon_1 + \gamma$. Suppose there exists $y_\varepsilon \in \varepsilon_1 k^0$-$\mathrm{WMZ}(\Omega, C)$ such that $y_\varepsilon \notin \varepsilon_2 k^0$-$\mathrm{WMZ}(\Omega, C)$. This means there exist elements $y, y^1 \in \Omega$ such that $y^1 \in (y_\varepsilon - \varepsilon_2 k^0 - \operatorname{int} C(y))$ and $y_\varepsilon - \varepsilon_2 k^0 - c_1 = y^1 \in \Omega$ for some $c_1 \in \operatorname{int} C(y)$. Therefore, we can write

$$y_\varepsilon - (\varepsilon_1 + \gamma)k^0 - c_1 \in \{y^1\} \implies y_\varepsilon - \varepsilon_1 k^0 - (\gamma k^0 + c_1) \in \{y^1\}. \tag{3.14}$$

By $\operatorname{int} C(y) + [0, +\infty)k^0 \subseteq \operatorname{int} C(y)$ and (3.14), we get $c_1 + \gamma k^0 = c_2 \in \operatorname{int} C(y)$ and

$$y_\varepsilon - \varepsilon_1 k^0 - c_2 \in \{y^1\} \implies (y_\varepsilon - \varepsilon_1 k^0 - \operatorname{int} C(y)) \cap \{y^1\} \neq \emptyset.$$

But this is a contradiction to $y_\varepsilon \in \varepsilon_1 k^0$-$\mathrm{WMZ}(\Omega, C)$.

3. Now we prove that $\mathrm{SMZ}(\Omega,C) \subseteq \varepsilon_1 k^0$-$\mathrm{SMZ}(\Omega,C)$ for $\varepsilon_1 \geq 0$. We just need to consider $\varepsilon_1 > 0$, otherwise $\varepsilon_1 = 0$ and obviously $\mathrm{SMZ}(\Omega,C) = \varepsilon_1 k^0$-$\mathrm{SMZ}(\Omega,C)$. Suppose that $y_\varepsilon \in \mathrm{SMZ}(\Omega,C)$, then for all $\varepsilon_1 > 0$,

$$y_\varepsilon \in y_1 - C(y_2)\backslash\{\mathbf{0}\} \implies y_\varepsilon - \varepsilon_1 k^0 \in y_1 - (C(y_2)\backslash\{\mathbf{0}\} + \varepsilon_1 k^0).$$

By $C(y) + [0,+\infty)k^0 \subseteq C(y)$, $\varepsilon_1 \neq 0, k^0 \neq \mathbf{0}$ and the pointedness of $C(y)$ for all $y \in \Omega$, we get $(C(y_2)\backslash\{\mathbf{0}\}) + [0,+\infty)k^0 \subseteq C(y_2)\backslash\{\mathbf{0}\}$. Therefore

$$y_\varepsilon - \varepsilon_1 k^0 = y_1 - C(y_2)\backslash\{\mathbf{0}\} \quad \text{and} \quad y_\varepsilon \in \varepsilon_1 k^0 - \mathrm{SMZ}(\Omega,C).$$

Now suppose that $y_\varepsilon \in \varepsilon_1 k^0$-$\mathrm{SMZ}(\Omega,C)$ and $\varepsilon_1 < \varepsilon_2$. Therefore, there exists $\gamma > 0$ such that $\varepsilon_1 + \gamma = \varepsilon_2$. By $y_\varepsilon \in \varepsilon_1 k^0$-$\mathrm{SMZ}(\Omega,C)$, we get

$$y_\varepsilon - \varepsilon_2 k^0 = y_\varepsilon - \varepsilon_1 k^0 - \gamma k^0 \in y_1 - C(y_2)\backslash\{\mathbf{0}\} - \gamma k^0. \tag{3.15}$$

Similar to the above $(C(y)\backslash\{\mathbf{0}\}) + [0,+\infty)k^0 \subseteq C(y)\backslash\{\mathbf{0}\}$ for all $y \in \Omega$. By this and (3.15), we can write

$$y_\varepsilon - \varepsilon_2 k^0 \in y_1 - (C(y_2)\backslash\{\mathbf{0}\} + \gamma k^0) \subseteq y_1 - C(y_2)\backslash\{\mathbf{0}\}$$

and this completes the proof.

4. Suppose that y_ε is a strong εk^0-minimizer of Ω with respect to the ordering map C, then for all $y \in \Omega$:

$$y_\varepsilon - \varepsilon k^0 \in y_1 - C(y_2) \implies y_\varepsilon - \varepsilon k^0 - y_1 \in -C(y_2).$$

Now suppose that y_ε is not an εk^0-minimizer and there exist $y_1, y_2 \in \Omega$ and $c_1 \in C(y_2)\backslash\{\mathbf{0}\}$ such that

$$y_\varepsilon - \varepsilon k^0 - y_1 \in C(y_2)\backslash\{\mathbf{0}\} \quad \text{and} \quad y_\varepsilon = y_1 + \varepsilon k^0 + c_1.$$

Therefore $y_\varepsilon - \varepsilon k^0 - y_1 \in C(y_2)\backslash\{\mathbf{0}\} \cap -C(y_2)$. But this is a contradiction to the pointedness of $C(y)$. This means that each strongly εk^0-minimizer is an εk^0-minimizer and also εk^0-$\mathrm{SMZ}(\Omega,C) \subseteq \varepsilon k^0$-$\mathrm{MZ}(\Omega,C)$.

Now, we show that each εk^0-minimizer is a weak εk^0-minimizer. By pointedness of $C(y)$, we get $\mathbf{0} \in \mathrm{bd}\, C(y)$, $\mathrm{int}\, C(y) \subseteq C(y)\backslash\{\mathbf{0}\}$ and

$$(y_\varepsilon - \varepsilon k^0 - \mathrm{int}\, C(y_2)) \cap \Omega \subseteq (y_\varepsilon - \varepsilon k^0 - C(y_2)\backslash\{\mathbf{0}\}) \cap \Omega = \emptyset.$$

This means that if $y_\varepsilon \in \varepsilon k^0$-$\mathrm{MZ}(\Omega,C)$, then $y_\varepsilon \in \varepsilon k^0$-$\mathrm{WMZ}(\Omega,C)$. □

The following Theorem helps us to see the relationships between sets of (weak) εk^0-minimizers and exact minimizers of the set Ω with respect to $C : Y \rightrightarrows Y$.

Theorem 3.2.5. Let assumption (A) be fulfilled.

1. Let $\operatorname{int} C(y) \neq \emptyset$ for all $y \in \Omega$, then $\bigcap_{\varepsilon>0} \varepsilon k^0\text{-MZ}(\Omega, C) \subseteq \text{WMZ}(\Omega, C)$.

2. $\text{MZ}(\Omega, C) \subseteq \bigcap_{\varepsilon>0} \varepsilon k^0\text{-MZ}(\Omega, C)$

3. Let $\operatorname{int} C(y) \neq \emptyset$ for all $y \in \Omega$, then $\text{WMZ}(\Omega, C) = \bigcap_{\varepsilon>0} \varepsilon k^0\text{-WMZ}(\Omega, C)$.

Proof. 1. Suppose that y_ε belongs to the set $\bigcap_{\varepsilon>0} \varepsilon k^0\text{-MZ}(\Omega, C)$ but y_ε is not a weak minimizer of the set Ω.

$$\exists y, y^1 \in \Omega : \ (y_\varepsilon - \operatorname{int} C(y)) \cap \{y^1\} \neq \emptyset \implies \exists c_1 \in \operatorname{int} C(y) : \ y_\varepsilon - c_1 = y^1.$$

By $c_1 \in \operatorname{int} C(y)$, there exists $\varepsilon_1 > 0$ such that ball $B(c_1, \varepsilon_1) \subseteq C(y)$ and $c_1 - \varepsilon_1 k^0$ belongs to $C(y) \backslash \{\mathbf{0}\}$. This means that there exists $c_2 \in C(y) \backslash \{\mathbf{0}\}$ such that $c_1 = c_2 + \varepsilon_1 k^0$ and

$$(y_\varepsilon - c_1) \in \Omega \implies (y_\varepsilon - c_2 - \varepsilon_1 k^0) \in \Omega \implies (y_\varepsilon - \varepsilon_1 k^0 - C(y) \backslash \{\mathbf{0}\}) \cap \Omega \neq \emptyset.$$

This means that $y_\varepsilon \notin \bigcap_{\varepsilon>0} \varepsilon k^0\text{-MZ}(\Omega, C)$. But this is a contradiction because we supposed that y_ε belongs to the set $\bigcap_{\varepsilon>0} \varepsilon k^0\text{-MZ}(\Omega, C)$.

2. By the first part of Theorem 3.2.4, we get $\text{MZ}(\Omega, C) \subseteq \varepsilon_1 k^0\text{-MZ}(\Omega, C)$ for all $\varepsilon_1 > 0$. Therefore, $\text{MZ}(\Omega, C) \subseteq \bigcap_{\varepsilon>0} \varepsilon k^0\text{-MZ}(\Omega, C)$.

3. By the second part of Theorem 3.2.4, $\text{WMZ}(\Omega, C) \subseteq \varepsilon_1 k^0\text{-WMZ}(\Omega, C)$ holds for every arbitrary $\varepsilon_1 > 0$ and this means $\text{WMZ}(\Omega, C) \subseteq \bigcap_{\varepsilon>0} \varepsilon k^0\text{-WMZ}(\Omega, C)$. Now, suppose that $y_\varepsilon \in \bigcap_{\varepsilon>0} \varepsilon k^0\text{-WMZ}(\Omega, C)$, then for any $\varepsilon > 0$

$$(y_\varepsilon - \varepsilon k^0 - \operatorname{int} C(y)) \cap \Omega = \emptyset \implies \bigcup_{\varepsilon>0} \big((y_\varepsilon - \varepsilon k^0 - \operatorname{int} C(y)) \cap \Omega\big) = \emptyset.$$

By $\operatorname{int} C(y) \subseteq \bigcup_{\varepsilon>0} (\operatorname{int} C(y) + \varepsilon k^0)$ we can write,

$$(y_\varepsilon - \operatorname{int} C(y)) \cap \Omega \subseteq \bigcup_{\varepsilon>0} \big((y_\varepsilon - \varepsilon k^0 - \operatorname{int} C(y)) \cap \Omega\big) = \emptyset.$$

Therefore y_ε is a weak minimizer of Ω with respect to the ordering map $C : Y \rightrightarrows Y$. □

3.2.2 Approximately Nondominated Elements

We define the concept of approximately nondominated solutions of vector optimization problems with respect to a variable ordering structure based on the domination relation (3.7). More details and properties of these points are given in [67].

Definition 3.2.6. Let assumption (A) be fulfilled and $y_\varepsilon \in \Omega$.

1. y_ε is said to be an εk^0-nondominated element of Ω with respect to the ordering map $C : Y \rightrightarrows Y$ iff $y \nleq_1 y_\varepsilon - \varepsilon k^0$ for all $y \in \Omega$, i.e.,

$$\forall y \in \Omega : \qquad (y_\varepsilon - \varepsilon k^0 - C(y) \backslash \{\mathbf{0}\}) \cap \{y\} = \emptyset.$$

2. Let $\operatorname{int} C(y) \neq \emptyset$ for all $y \in \Omega$. y_ε is said to be a weakly εk^0-nondominated element of Ω with respect to the ordering map $C : Y \rightrightarrows Y$ iff

$$\forall y \in \Omega : \qquad (y_\varepsilon - \varepsilon k^0 - \operatorname{int} C(y)) \cap \{y\} = \emptyset.$$

3. y_ε is said to be a strongly εk^0-nondominated element of Ω with respect to the ordering map $C : Y \rightrightarrows Y$ iff

$$\forall y \in \Omega : \qquad y_\varepsilon - \varepsilon k^0 \in y - C(y).$$

Remark 3.2.7.

- We denote the set of all εk^0-nondominated elements of Ω with respect to the ordering map $C : Y \rightrightarrows Y$ by εk^0-$\mathrm{N}(\Omega, C)$.
- We denote the set of all weakly εk^0-nondominated elements of Ω with respect to the ordering map C by εk^0-$\mathrm{WN}(\Omega, C)$.
- We denote the set of all strongly εk^0-nondominated elements of Ω with respect to C by εk^0-$\mathrm{SN}(\Omega, C)$.

If $\varepsilon = 0$, then these definitions coincide with standard definitions of nondominated points (see [24, 78]). The sets of nondominated, weakly nondominated and strongly nondominated elements of the set Ω with respect to the ordering map C will be denoted by $\mathrm{N}(\Omega, C)$, $\mathrm{WN}(\Omega, C)$ and $\mathrm{SN}(\Omega, C)$ respectively.

The following theorem shows several properties of the approximately nondominated, weakly approximately nondominated and strongly approximately nondominated solutions. This theorem will help us later to show the relationships between the sets of exact nondominated elements and approximately nondominated elements.

Theorem 3.2.8. Let assumption (A) be fulfilled and $C(y)$ be a pointed set for all $y \in \Omega$.

1. $\mathrm{N}(\Omega, C) \subseteq \varepsilon_1 k^0\text{-}\mathrm{N}(\Omega, C) \subseteq \varepsilon_2 k^0\text{-}\mathrm{N}(\Omega, C) \quad \text{if } 0 \leq \varepsilon_1 \leq \varepsilon_2.$

2. Suppose that $\operatorname{int} C(y) \neq \emptyset$ for all $y \in \Omega$, then the following holds:

   $\mathrm{WN}(\Omega, C) \subseteq \varepsilon_1 k^0\text{-}\mathrm{WN}(\Omega, C) \subseteq \varepsilon_2 k^0\text{-}\mathrm{WN}(\Omega, C) \quad \text{if } 0 \leq \varepsilon_1 \leq \varepsilon_2.$

3. $\mathrm{SN}(\Omega, C) \subseteq \varepsilon_1 k^0\text{-}\mathrm{SN}(\Omega, C) \subseteq \varepsilon_2 k^0\text{-}\mathrm{SN}(\Omega, C) \quad \text{if } 0 \leq \varepsilon_1 \leq \varepsilon_2.$

4. Suppose that $\operatorname{int} C(y) \neq \emptyset$ for all $y \in \Omega$, then the following holds:

   εk^0-SN$(\Omega, C) \subseteq \varepsilon k^0$-N$(\Omega, C) \subseteq \varepsilon k^0$-WN$(\Omega, C)$ if $\varepsilon \geq 0$.

Proof. 1. We prove that if $\varepsilon_1 \geq 0$, then $\mathrm{N}(\Omega, C) \subseteq \varepsilon_1 k^0$-N$(\Omega, C)$. If $\varepsilon_1 = 0$, then we have $\mathrm{N}(\Omega, C) = \varepsilon_1 k^0$-N$(\Omega, C)$. Therefore let $\varepsilon_1 > 0$ and suppose there exists $y_\varepsilon \in \mathrm{N}(\Omega, C)$ such that $y_\varepsilon \notin \varepsilon_1 k^0$-N$(\Omega, C)$. This means that there exist $y \in \Omega$ and $c_1 \in C(y) \setminus \{\mathbf{0}\}$ such that

$$y \in (y_\varepsilon - \varepsilon_1 k^0 - C(y) \setminus \{\mathbf{0}\}) \quad \text{and} \quad y_\varepsilon - \varepsilon_1 k^0 - c_1 = y. \tag{3.16}$$

Since $C(y) + [0, +\infty)k^0 \subseteq C(y)$, there exists $c_2 \in C(y)$ such that

$$c_1 + \varepsilon_1 k^0 = c_2. \tag{3.17}$$

Similar to the first part of Theorem 3.2.4, $c_2 \neq \mathbf{0}$. By (3.16) and (3.17), we get

$$y = y_\varepsilon - c_2 \implies (y_\varepsilon - C(y) \setminus \{\mathbf{0}\}) \cap \{y\} \neq \emptyset.$$

But this is a contradiction to $y_\varepsilon \in \mathrm{N}(\Omega, C)$. Now, suppose that $0 < \varepsilon_1 < \varepsilon_2$, then there exists $\gamma > 0$ such that $\varepsilon_2 = \varepsilon_1 + \gamma$. Suppose that $y_\varepsilon \in \varepsilon_1 k^0$-N$(\Omega, C)$ but $y_\varepsilon \notin \varepsilon_2 k^0$-N$(\Omega, C)$, then there exist $y \in \Omega$ and $c_1 \in C(y) \setminus \{\mathbf{0}\}$ such that

$$y_\varepsilon - \varepsilon_2 k^0 - c_1 = y. \tag{3.18}$$

By (3.18) and $\varepsilon_2 = \varepsilon_1 + \gamma$, we can write

$$y \in (y_\varepsilon - (\varepsilon_1 + \gamma)k^0 - C(y) \setminus \{\mathbf{0}\}) \implies y \in (y_\varepsilon - \varepsilon_1 k^0 - (\gamma k^0 + C(y) \setminus \{\mathbf{0}\}).$$

This implies that there exists $c_2 \in \gamma k^0 + C(y) \setminus \{\mathbf{0}\}$ such that $y = y_\varepsilon - \varepsilon k^0 - c_2$. By assumption $C(y) + [0, +\infty)k^0 \subseteq C(y)$, we get $c_2 \in \gamma k^0 + C(y) \setminus \{\mathbf{0}\} \subset C(y)$ and $c_2 \in C(y)$. Similar to the above, we have $c_2 \neq \mathbf{0}$ and $c_2 \in C(y) \setminus \{\mathbf{0}\}$. Therefore $y_\varepsilon - \varepsilon_1 k^0 - c_2 = y$ and since $c_2 \neq \mathbf{0}$, we can write $y \in (y_\varepsilon - \varepsilon_1 k^0 - C(y) \setminus \{\mathbf{0}\})$. But this is a contradiction to our assumption $y_\varepsilon \in \varepsilon k^0$-N$(\Omega, C)$. Therefore $\varepsilon_1 k^0$-N$(\Omega, C) \subseteq \varepsilon_2 k^0$-N$(\Omega, C)$.

2. Suppose that $y_\varepsilon \in \mathrm{WN}(\Omega, C)$, i.e., y_ε is a weakly nondominated point with respect to the ordering map C and $(y_\varepsilon - \operatorname{int} C(y)) \cap \{y\} = \emptyset$ for all $y \in \Omega$. By Lemma 3.2.3, we get $\operatorname{int} C(y) + [0, +\infty)k^0 \subseteq \operatorname{int} C(y)$ for all $y \in \Omega$ and therefore for any $\varepsilon > 0$, we can write

$$(y_\varepsilon - \varepsilon k^0 - \operatorname{int} C(y)) \cap \{y\} \subseteq (y_\varepsilon - \operatorname{int} C(y)) \cap \{y\} = \emptyset.$$

This means that $y_\varepsilon \in \varepsilon k^0$-WN$(\Omega, C)$ for all $\varepsilon > 0$ and $\mathrm{WN}(\Omega, C) \subseteq \varepsilon_1 k^0$-WN$(\Omega, C)$. Let $0 < \varepsilon_1 < \varepsilon_2$, then there exists $\gamma > 0$ such that $\varepsilon_2 = \varepsilon_1 + \gamma$. Suppose that there exists an element $y_\varepsilon \in \varepsilon_1 k^0$-WN$(\Omega, C)$ such that $y_\varepsilon \notin \varepsilon_2 k^0$-WN$(\Omega, C)$. This means there exist

$y \in \Omega$ and $c_1 \in \mathrm{int}C(y)$ such that $(y_\varepsilon - \varepsilon_2 k^0 - \mathrm{int}C(y)) \cap \{y\} \neq \emptyset$ and $y_\varepsilon - \varepsilon_2 k^0 - c_1 = y$. Therefore, we can write

$$y_\varepsilon - (\varepsilon_1 + \gamma)k^0 - c_1 = y \implies y_\varepsilon - \varepsilon_1 k^0 - (\gamma k^0 + c_1) = y. \tag{3.19}$$

By $\mathrm{int}C(y) + [0,+\infty)k^0 \subseteq \mathrm{int}C(y)$, we get $c_2 := c_1 + \gamma k^0 \in \mathrm{int}C(y)$ and by (3.19),

$$y_\varepsilon - \varepsilon_1 k^0 - c_2 = y \implies (y_\varepsilon - \varepsilon_1 k^0 - \mathrm{int}C(y)) \cap \{y\} \neq \emptyset.$$

But this is a contradiction to $y_\varepsilon \in \varepsilon_1 k^0$-WN$(\Omega, C)$.

3. Now we prove that SN$(\Omega, C) \subseteq \varepsilon_1 k^0$-SN$(\Omega, C)$ for $\varepsilon_1 \geq 0$. We just need to consider $\varepsilon_1 > 0$, otherwise $\varepsilon_1 = 0$ and obviously SN$(\Omega, C) = \varepsilon_1 k^0$-SN$(\Omega, C)$. Suppose that $y_\varepsilon \in$ SN(Ω, C), then for all $\varepsilon_1 > 0$,

$$y_\varepsilon \in y - C(y) \backslash \{\mathbf{0}\} \implies y_\varepsilon - \varepsilon_1 k^0 \in y - (C(y) \backslash \{\mathbf{0}\} + \varepsilon_1 k^0).$$

By $C(y) + [0,+\infty)k^0 \subseteq C(y)$, $\varepsilon_1 \neq 0, k^0 \neq \mathbf{0}$ and the pointedness of $C(y)$ for all $y \in \Omega$, we get $(C(y) \backslash \{\mathbf{0}\}) + [0,+\infty)k^0 \subseteq C(y) \backslash \{\mathbf{0}\}$. Therefore

$$y_\varepsilon - \varepsilon_1 k^0 = y - C(y) \backslash \{\mathbf{0}\} \quad \text{and} \quad y_\varepsilon \in \varepsilon_1 k^0\text{-SN}(\Omega, C).$$

Now, suppose that $y_\varepsilon \in \varepsilon_1 k^0$-SN$(\Omega, C)$ and $\varepsilon_1 < \varepsilon_2$. Therefore, there exists $\gamma > 0$ such that $\varepsilon_1 + \gamma = \varepsilon_2$. Since $y_\varepsilon \in \varepsilon_1 k^0$-SN$(\Omega, C)$, for all $y \in \Omega$

$$y_\varepsilon - \varepsilon_2 k^0 = y_\varepsilon - \varepsilon_1 k^0 - \gamma k^0 \in y - C(y_\varepsilon) \backslash \{\mathbf{0}\} - \gamma k^0. \tag{3.20}$$

Similar to the above $(C(y) \backslash \{\mathbf{0}\}) + [0,+\infty)k^0 \subseteq C(y) \backslash \{\mathbf{0}\}$ for all $y \in \Omega$. By this and (3.20), we get

$$y_\varepsilon - \varepsilon_2 k^0 \in y - \left(C(y) \backslash \{\mathbf{0}\} + \gamma k^0\right) \subseteq y - C(y) \backslash \{\mathbf{0}\}$$

and this completes the proof.

4. Suppose that y_ε is a strongly εk^0- nondominated element of Ω with respect to the ordering map C, then for all $y \in \Omega$ we have:

$$y_\varepsilon - \varepsilon k^0 \in y - C(y) \implies y_\varepsilon - \varepsilon k^0 - y \in -C(y).$$

Now suppose that y_ε is not an εk^0-nondominated element and there exist an element $y \in \Omega$ and $c_1 \in C(y) \backslash \{\mathbf{0}\}$ such that

$$y_\varepsilon = y + \varepsilon k^0 + c_1 \quad \text{and} \quad y_\varepsilon - \varepsilon k^0 - y \in C(y) \backslash \{\mathbf{0}\}.$$

Therefore $y_\varepsilon - \varepsilon k^0 - y \in C(y)\setminus\{\mathbf{0}\}) \cap -C(y)$. But this is a contradiction to the pointedness of $C(y)$. This means each strongly εk^0-nondominated element is an εk^0-nondominated element and εk^0-SN$(\Omega, C) \subseteq \varepsilon k^0$-N$(\Omega, C)$.

Now, we show that each εk^0-nondominated element is a weakly εk^0-nondominated element. By pointedness of $C(y)$, we get $\mathbf{0} \in \operatorname{bd} C(y)$, $\operatorname{int} C(y) \subseteq C(y)\setminus\{\mathbf{0}\}$ and

$$(y_\varepsilon - \varepsilon k^0 - \operatorname{int} C(y)) \cap \{y\} \subseteq (y_\varepsilon - \varepsilon k^0 - C(y)\setminus\{\mathbf{0}\}) \cap \{y\} = \emptyset.$$

This means if $y_\varepsilon \in \varepsilon k^0$-N$(\Omega, C)$, then $y_\varepsilon \in \varepsilon k^0$-WN$(\Omega, C)$ and the proof is complete. □

Now, we show relationships between (weakly) εk^0-nondominated elements and exact nondominated elements of the set Ω with respect to the ordering map $C : Y \rightrightarrows Y$

Theorem 3.2.9. Let assumption (A) be fulfilled, then the following holds:

1. Let $\operatorname{int} C(y) \neq \emptyset$ for all $y \in \Omega$, then $\bigcap_{\varepsilon>0} \varepsilon k^0$-N$(\Omega, C) \subseteq$ WN(Ω, C).

2. N$(\Omega, C) \subseteq \bigcap_{\varepsilon>0} \varepsilon k^0$-N$(\Omega, C)$.

3. Let $\operatorname{int} C(y) \neq \emptyset$ for all $y \in \Omega$, then WN$(\Omega, C) = \bigcap_{\varepsilon>0} \varepsilon k^0$-WN$(\Omega, C)$.

Proof. 1. The proof is similar to that of part 1 of Theorem 3.2.5.

2. By the first part of Theorem 3.2.8, we get N$(\Omega, C) \subseteq \varepsilon_1 k^0$-N$(\Omega, C)$ for all $\varepsilon_1 > 0$. Therefore, N$(\Omega, C) \subseteq \bigcap_{\varepsilon>0} \varepsilon k^0$-N$(\Omega, C)$.

3. By the second part of Theorem 3.2.8, we get WN$(\Omega, C) \subseteq \varepsilon_1 k^0$-WN$(\Omega, C)$ for all $\varepsilon_1 > 0$. Therefore, WN$(\Omega, C) \subseteq \bigcap_{\varepsilon>0} \varepsilon k^0$-WN$(\Omega, C)$. The proof of reverse implication is similar to that of part 3 of Theorem 3.2.5. □

3.2.3 Approximately Minimal Elements

In this subsection we deal with approximately minimal solutions of vector optimization problems with variable ordering structures and their properties.

Definition 3.2.10. Let assumption (A) be fulfilled and $y_\varepsilon \in \Omega$.

1. y_ε is said to be an εk^0-minimal element of Ω with respect to the ordering map $C : Y \rightrightarrows Y$ iff $y \not\leq_2 y_\varepsilon - \varepsilon k^0$ for all $y \in \Omega$, i.e.,

$$(y_\varepsilon - \varepsilon k^0 - C(y_\varepsilon)\setminus\{\mathbf{0}\}) \cap \Omega = \emptyset.$$

2. Let $\operatorname{int} C(y_\varepsilon) \neq \emptyset$. y_ε is said to be a weakly εk^0-minimal element of Ω with respect to the ordering map $C : Y \rightrightarrows Y$ iff

$$(y_\varepsilon - \varepsilon k^0 - \operatorname{int} C(y_\varepsilon)) \cap \Omega = \emptyset.$$

3. y_ε is said to be a strongly εk^0-minimal element of Ω with respect to the ordering map $C : Y \rightrightarrows Y$ iff

$$\forall y \in \Omega : \qquad y_\varepsilon - \varepsilon k^0 \in y - C(y_\varepsilon).$$

Remark 3.2.11.
- We denote the set of all εk^0-minimal elements of Ω with respect to the ordering map $C : Y \rightrightarrows Y$ by εk^0-$\mathrm{M}(\Omega, C)$.
- We denote the set of all weakly εk^0-minimal elements of Ω with respect to the ordering map C by εk^0-$\mathrm{WM}(\Omega, C)$.
- We denote the set of all strongly εk^0-minimal elements of Ω with respect to the ordering map C by εk^0-$\mathrm{SM}(\Omega, C)$.

If $\varepsilon = 0$, then these definitions coincide with standard definitions of minimal points (see [24, 40]). The sets of minimal, weakly minimal and strongly minimal elements of the set Ω with respect to the ordering map C will be denoted by $\mathrm{M}(\Omega, C)$, $\mathrm{WM}(\Omega, C)$ and $\mathrm{SM}(\Omega, C)$, respectively.

Theorem 3.2.12. Let assumption (A) be fulfilled and $C(y)$ be a pointed set for all $y \in \Omega$.

1. $\mathrm{M}(\Omega, C) \subseteq \varepsilon_1 k^0\text{-}\mathrm{M}(\Omega, C) \subseteq \varepsilon_2 k^0\text{-}\mathrm{M}(\Omega, C) \quad \text{if } 0 \le \varepsilon_1 \le \varepsilon_2.$

2. Suppose that $\operatorname{int} C(y) \neq \emptyset$ for all $y \in \Omega$, then the following holds:
   $\mathrm{WM}(\Omega, C) \subseteq \varepsilon_1 k^0\text{-}\mathrm{WM}(\Omega, C) \subseteq \varepsilon_2 k^0\text{-}\mathrm{WM}(\Omega, C) \quad \text{if } 0 \le \varepsilon_1 \le \varepsilon_2.$

3. $\mathrm{SM}(\Omega, C) \subseteq \varepsilon_1 k^0\text{-}\mathrm{SM}(\Omega, C) \subseteq \varepsilon_2 k^0\text{-}\mathrm{SM}(\Omega, C) \quad \text{if } 0 \le \varepsilon_1 \le \varepsilon_2.$

4. Suppose that $\operatorname{int} C(y) \neq \emptyset$ for all $y \in \Omega$, then the following holds:
   $\varepsilon k^0\text{-}\mathrm{SM}(\Omega, C) \subseteq \varepsilon k^0\text{-}\mathrm{M}(\Omega, C) \subseteq \varepsilon k^0\text{-}\mathrm{WM}(\Omega, C) \quad \text{if } \varepsilon \ge 0.$

Proof. The proof is similar to that of Theorem 3.2.8. □

In the following theorem, we show relationships between (weakly) εk^0-minimal elements of the set Ω with respect to the ordering map $C : Y \rightrightarrows Y$ and minimal elements of Ω with respect to C.

Theorem 3.2.13. Let assumption (A) be fulfilled.

1. Let $\operatorname{int} C(y) \neq \emptyset$ for all $y \in \Omega$, then $\bigcap_{\varepsilon>0} \varepsilon k^0\text{-}\mathrm{M}(\Omega, C) \subseteq \mathrm{WM}(\Omega, C)$.

2. $\mathrm{M}(\Omega, C) \subseteq \bigcap_{\varepsilon>0} \varepsilon k^0\text{-}\mathrm{M}(\Omega, C)$.

3. Let $\operatorname{int} C(y) \neq \emptyset$ for all $y \in \Omega$, then $\mathrm{WM}(\Omega, C) = \bigcap_{\varepsilon>0} \varepsilon k^0\text{-}\mathrm{WM}(\Omega, C)$.

Proof.
1. The proof is similar to that of part 1 of Theorem 3.2.5.

2. By the first part of Theorem 3.2.12, we get $\mathrm{M}(\Omega, C) \subseteq \varepsilon_1 k^0\text{-}\mathrm{M}(\Omega, C)$ for all $\varepsilon_1 > 0$. Therefore, $\mathrm{M}(\Omega, C) \subseteq \bigcap_{\varepsilon>0} \varepsilon k^0\text{-}\mathrm{M}(\Omega, C)$.

3. By the second part of Theorem 3.2.12, we get $WM(\Omega,C) \subseteq \varepsilon_1 k^0$-WM$(\Omega,C)$ for all $\varepsilon_1 > 0$. Therefore, WM$(\Omega,C) \subseteq \bigcap_{\varepsilon>0} \varepsilon k^0$-WM$(\Omega,C)$. The proof of reverse implication is similar to that of part 3 of Theorem 3.2.5. □

In general, the set of weakly nondominated points is not a subset of $\bigcap_{\varepsilon>0} \varepsilon k^0$-N$(\Omega,C)$. This statement is also true for weakly minimal and weak minimizers. Consider Figure 3.1 where for all $y \in \Omega$, $C(y) = \mathbb{R}^2_+$ and $k^0 = (1,0)^T$. It is not difficult to see that $\{\{(1,y_2)\} \cup \{(y_1,1)\}\} \cap \Omega$ is the set of weakly nondominated points and $\{(1,y_2)\} \cap \Omega$ is the set $\bigcap_{\varepsilon>0} \varepsilon k^0$-N$(\Omega,C)$. Therefore, we can see that the set of weakly nondominated points is not a subset of $\bigcap_{\varepsilon>0} \varepsilon k^0$-N$(\Omega,C)$. This example also shows that the set of weakly minimal and weak minimizers are not subsets of $\bigcap_{\varepsilon>0} \varepsilon k^0$-M$(\Omega,C)$ and $\bigcap_{\varepsilon>0} \varepsilon k^0$-MZ$(\Omega,C)$, respectively.

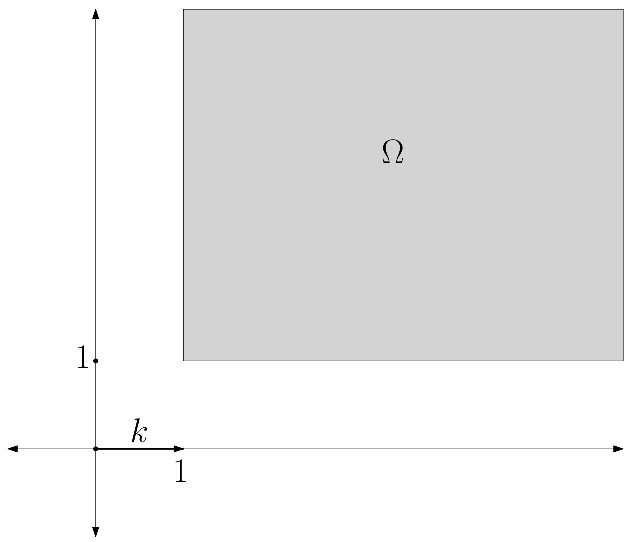

FIGURE 3.1: Set Ω, $C(y) = \mathbb{R}^2_+$ for all $y \in \Omega$ and $k^0 = (1,0)$.

Definitions of local εk^0-MZ(Ω,C), εk^0-N(Ω,C) and εk^0-M(Ω,C) are similar. We just need to substitute Ω with $\Omega \cap U$ in Definitions 3.2.1, 3.2.6 and 3.2.10 where U is a neighborhood of a candidate point. If Ω is a convex set, then each locally εk^0- optimal element is also a globally εk^0-optimal element. This is also true for weakly (strongly) εk^0-optimal elements.

With some examples we show that sets of approximately optimal elements of vector optimization problems with variable ordering structures do not coincide.

Example 3.2.14. Let $\varepsilon = \frac{1}{100}$ and $k^0 = (1,0)^T$. Also, suppose that

$$\Omega = \{(y_1,y_2) \in \mathbb{R}^2 \mid \; y_1 + y_2 \geq 2, \; y_1 \geq 0, \;\; 0 \leq y_2 \leq 2\}$$

and

$$C(y_1,y_2) = \begin{cases} \mathbb{R}^2_+, & \text{if } y_1 = 0 \text{ or } y_2 = 0 \\ \text{cone conv } \{(2,0)^T, (y_1,y_2)\}, & \text{otherwise.} \end{cases}$$

It is easy to see that $C(y)+[0,+\infty)k^0 \subseteq C(y)$ for all $y \in \Omega$ and elements of the set

$$\{(y_1,y_2) \in \Omega \mid y_1+y_2 \leq 2+\frac{1}{100}\}$$

are εk^0-nondominated, εk^0-minimizer and also εk^0-minimal elements and the sets of all these points coincide. Furthermore,

$$\left\{(y_1,y_2) \in \Omega \mid y_1+y_2 \leq 2+\frac{1}{100}\right\} \cup \{(y_1,0)\}$$

describes the set of weakly εk^0-nondominated, weak εk^0-minimizer and weakly εk^0-minimal elements (see Fig. 3.2).

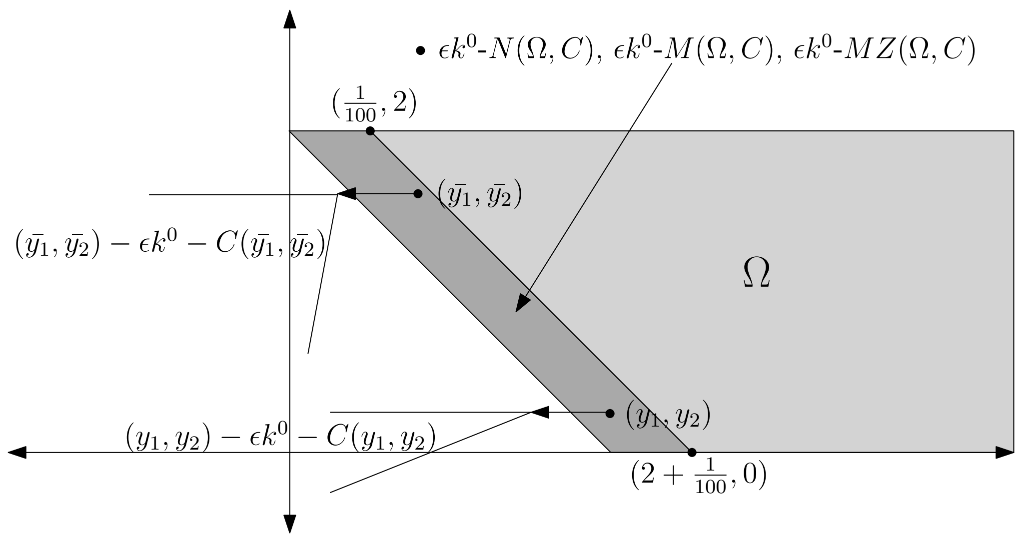

FIGURE 3.2: Example 3.2.14 where sets of εk^0-N(Ω,C), εk^0-MZ(Ω,C) and εk^0-M(Ω,C) of Ω coincide.

For vector optimization problems with respect to fixed ordering structures, (weakly, strongly) εk^0-nondominated elements and (weakly, strongly) εk^0-minimal elements coincide, but the following examples show that this is not true when we are dealing with vector optimization problems with variable ordering structures.

Example 3.2.15. Let $\varepsilon = \frac{1}{100}$ and $k^0 = (1,0)^T$. Also suppose that

$$\Omega = \left\{(y_1,y_2) \in \mathbb{R}^2 \mid y_1+y_2 \geq -1, \quad y_1 \leq 0, \quad y_2 \leq 0\right\}$$

and

$$C(y_1,y_2) = \begin{cases} \{(d_1,d_2) \in \mathbb{R}^2 \mid d_1 \geq 0,\ d_2 \leq 0\}, & \text{for } (-1,0)^T \\ \mathbb{R}^2_+, & \text{otherwise.} \end{cases}$$

It is easy to see that $C(y)+[0,+\infty)k^0 \subseteq C(y)$ for all $y \in \Omega$. Then

$$\{(y_1,y_2) \in \Omega \mid y_1+y_2 \leq -1+\varepsilon\}$$

is the set of εk^0-minimal elements but just the elements of the set

$$\{(y_1,y_2) \in \Omega \mid y_1 < -1+\varepsilon\} \cup \{(-1+\varepsilon,0)\}$$

are εk^0-nondominated and εk^0-minimizers (see Fig. 3.3).

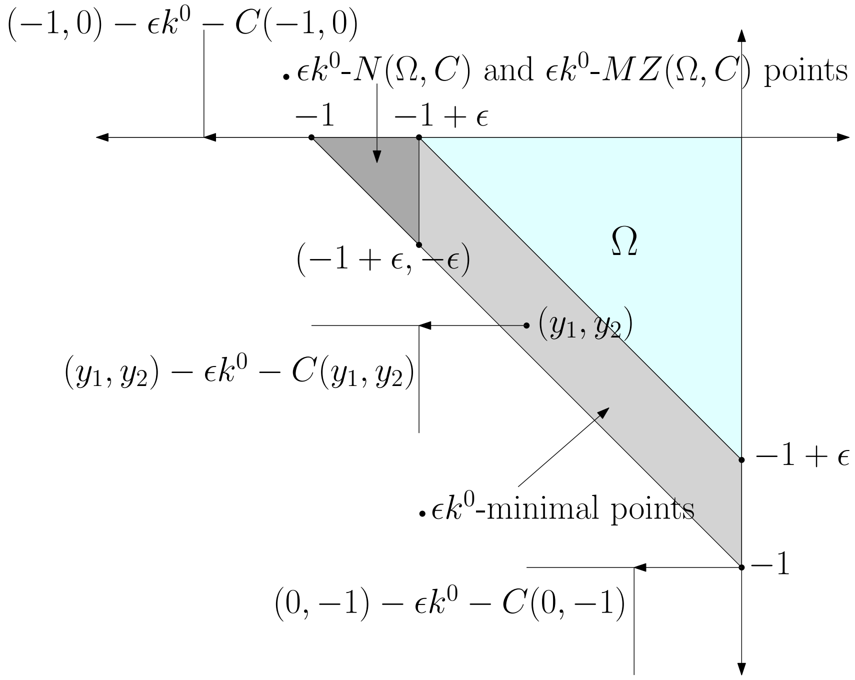

FIGURE 3.3: Example 3.2.15 where there exists an εk^0-minimal element of Ω which is neither εk^0-nondominated element nor εk^0-minimizer.

Example 3.2.16. Let $\varepsilon = \frac{1}{100}$ and $k^0=(1,1)^T$. Also suppose that

$$\Omega = \left\{(y_1,y_2) \in \mathbb{R}^2 \mid \; y_1+y_2 \geq -1, \quad y_1 \leq 0, \quad y_2 \leq 0\right\}$$

and

$$C(y_1,y_2) = \begin{cases} \{(d_1,d_2) \in \mathbb{R}^2 \mid d_2 \geq 0,\ d_1+d_2 \geq -1\}, & \text{for } (y_1,y_2)=(-1,0)^T \\ \mathbb{R}^2_+, & \text{otherwise.} \end{cases}$$

It is easy to see that $C(y)+[0,+\infty)k^0 \subseteq C(y)$ for all $y \in \Omega$. Then

$$\left\{(y_1,y_2) \in \Omega \mid y_1+y_2 \leq -\frac{98}{100},\ y_1 \neq -1\right\}$$

is the set of εk^0-minimal elements, but $(-1,0)^T$ is not a minimal element. However, $(-1,0)^T$ is an εk^0-nondominated element and $\left\{(y_1,y_2) \in \Omega \mid \; y_1+y_2 \leq -\frac{98}{100}\right\}$ is the set of εk^0-nondominated elements. Obviously, $(-1,0)^T$ is not also an εk^0-minimizer and $\{(y_1,y_2) \in \Omega \mid -1 < y_2 < -1+\varepsilon\}$ is the set of εk^0-minimizers (see Fig. 3.4).

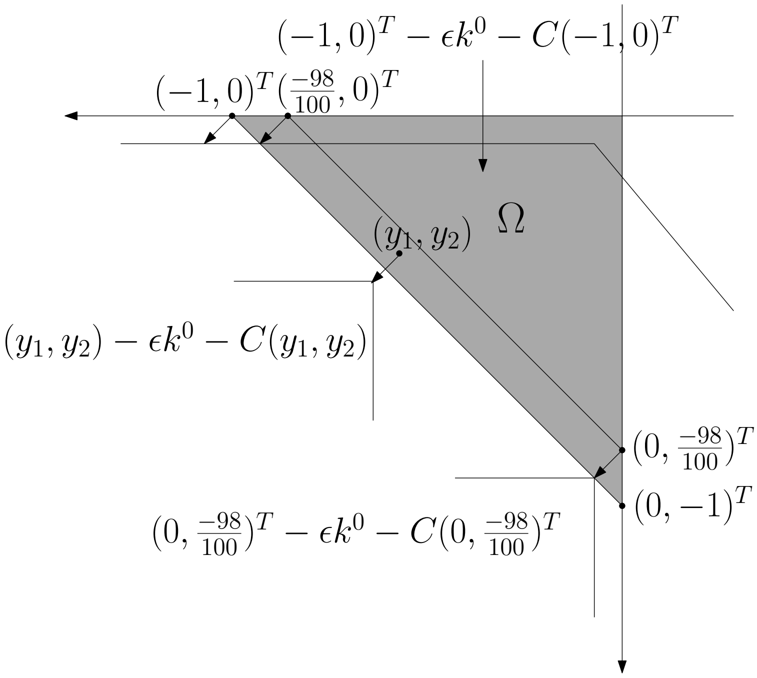

FIGURE 3.4: Example 3.2.16 where $(-1,0)^T$ is an εk^0-nondominated element of the set Ω, but it is neither εk^0-minimizer nor εk^0-minimal element.

In the following example, we show that there are some elements which belong to the set of εk^0-nondominated elements and also εk^0-minimal elements but they do not belong to the set of εk^0-minimizers.

Example 3.2.17. Let $\varepsilon = \frac{1}{100}$ and $k^0 = (1,0)^T$. Also suppose that

$$\Omega = \left\{(y_1,y_2) \in \mathbb{R}^2 \mid y_1+y_2 \geq -1, \quad y_1 \leq 0, \quad y_2 \leq 0\right\}$$

and

$$C(y_1,y_2) = \begin{cases} \{(d_1,d_2) \in \mathbb{R}^2 \mid d_2 \geq 0,\; d_1+d_2 \geq 0\}, & \text{for } (y_1,y_2) = (0,0) \\ \mathbb{R}^2_+, & \text{otherwise.} \end{cases}$$

It is easy to see that $C(y) + [0,+\infty)k^0 \subseteq C(y)$ for all $y \in \Omega$. Then

$$\left\{(y_1,y_2) \in \Omega \mid y_1+y_2 \leq -\frac{99}{100}\right\}$$

is the set of εk^0-minimal and εk^0-nondominated points. But only points of the set $\left\{(y_1, y_2) \in \Omega |\ y_1 + y_2 < -\frac{99}{100}\right\}$ are εk^0-minimizers and points of

$$\left\{(y_1, y_2) \in \Omega |\ y_1 + y_2 = -\frac{99}{100}\right\}$$

are not εk^0-minimizers. This shows that there exist elements of Ω which are both εk^0-nondominated and εk^0-minimal but not εk^0-minimizer (see Fig. 3.5).

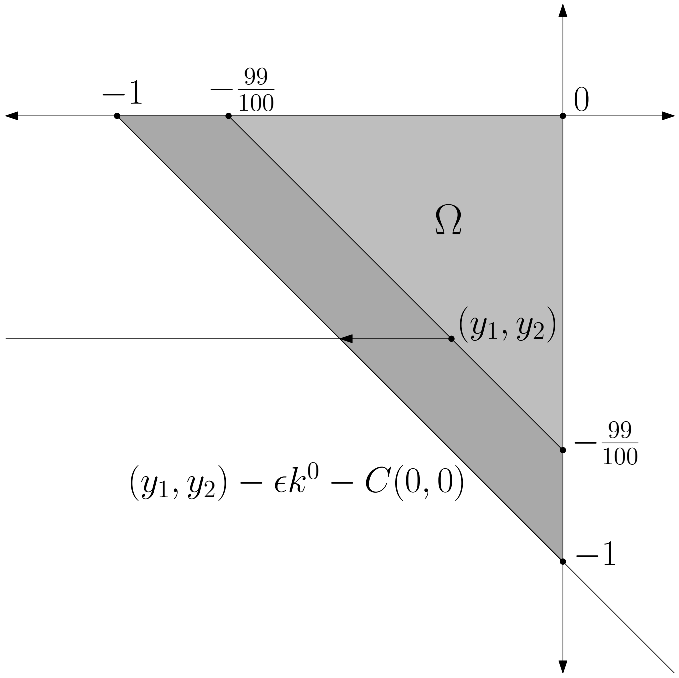

FIGURE 3.5: Example 3.2.17 where there exists an element which is both εk^0-nondominated and εk^0-minimal element but not εk^0-minimizer.

3.2.4 Relationships between Different Concepts of Approximately Optimal Elements

Eichfelder [24] studied relationships between exact nondominated and minimal solutions of vector optimization problems with variable ordering structures. In this section, we will show relationships between different kinds of approximately optimal elements (εk^0-nondominated, εk^0-minimal and εk^0-minimizers) of vector optimization problems with respect to variable ordering structures. At the end of this section, it will be obvious to see that concepts of approximately nondominated, approximately minimal and approximate minimizers coincide in the case of vector optimization with fixed ordering structures. First, we will show relationships between (weak, strong) εk^0-minimizers and (weakly, strongly) εk^0-nondominated elements of Ω with respect to $C : Y \rightrightarrows Y$. These theorems shows us that in vector optimization problems with fixed ordering structures, there is no difference between the sets of approximately nondominated and approximate minimizers and they do coincide.

Theorem 3.2.18. Let assumption (A) be fulfilled.

1. Every εk^0-minimizer of Ω with respect to C is also an εk^0-nondominated element.

2. Every εk^0-nondominated element of Ω with respect to C is also an εk^0-minimizer if $C(y) = C(y')$ for all $y, y' \in \Omega$. This means that each approximately nondominated element is approximate minimizer if the ordering set is fixed for all $y \in \Omega$.

3. Suppose that $\operatorname{int} C(y) \neq \emptyset$ for all $y \in \Omega$, then every weak εk^0-minimizer of Ω with respect to C is also a weakly εk^0-nondominated element.

4. Suppose that $\operatorname{int} C(y) \neq \emptyset$ for all $y \in \Omega$, then every weakly εk^0-nondominated element of Ω with respect to C is also a weak εk^0-minimizer if $\operatorname{int} C(y) = \operatorname{int} C(y')$ for all $y, y' \in \Omega$.

5. Every strong εk^0-minimizer of Ω with respect to C is also a strongly εk^0-nondominated element.

6. Every strongly εk^0-nondominated element of Ω with respect to the ordering map C is also a strong εk^0-minimizer if $C(y) = C(y')$ for all $y, y' \in \Omega$.

Proof.

1. This is obvious from the first parts of Definition 3.2.6 and Definition 3.2.10.

2. Let y_ε be an εk^0-nondominated element. Then $(y_\varepsilon - \varepsilon k^0 - C(y)\backslash\{\mathbf{0}\}) \cap \{y\} = \emptyset$ for all $y \in \Omega$. Since $C(y) = C(y^1)$ for all $y, y^1 \in \Omega$, we have
$$(y_\varepsilon - \varepsilon k^0 - C(y^1)\backslash\{\mathbf{0}\}) \cap \{y\} = (y_\varepsilon - \varepsilon k^0 - C(y)\backslash\{\mathbf{0}\}) \cap \{y\} = \emptyset \ \ \forall y, y^1 \in \Omega.$$
This means that $(y_\varepsilon - \varepsilon k^0 - C(y)\backslash\{\mathbf{0}\}) \cap y^1 = \emptyset$ for all $y, y^1 \in \Omega$ and therefore y_ε is an εk^0-minimizer.

3. This is obvious from the second parts of Definition 3.2.1 and Definition 3.2.6.

4. If $\operatorname{int} C(y) = \operatorname{int} C(y^1)$ for all $y, y^1 \in \Omega$ and y_ε is a weakly εk^0-nondominated element, then
$$(y_\varepsilon - \varepsilon k^0 - \operatorname{int} C(y)) \cap \{y^1\} = (y_\varepsilon - \varepsilon k^0 - \operatorname{int} C(y^1)) \cap \{y^1\} = \emptyset$$
for all $y, y^1 \in \Omega$. This means that $(y_\varepsilon - \varepsilon k^0 - \operatorname{int} C(y)) \cap \{y^1\} = \emptyset$ for all $y, y^1 \in \Omega$ and y_ε is a weak εk^0-minimizer.

5. This part is obvious from the third parts of Definition 3.2.1 and Definition 3.2.6.

6. Suppose that y_ε is a strongly εk^0-nondominated element, then $y_\varepsilon - \varepsilon k^0 \in y - C(y)\backslash\{\mathbf{0}\}$ for all $y \in \Omega$. Since $C(y) = C(y^1)$ for all $y, y^1 \in \Omega$, we have
$$y_\varepsilon - \varepsilon k^0 \in y - C(y)\backslash\{\mathbf{0}\} \implies y_\varepsilon - \varepsilon k^0 \in y - C(y^1)\backslash\{\mathbf{0}\} \ \ \forall y, y^1 \in \Omega$$
and this means that y_ε is a strong εk^0-minimizer. □

The following examples show that the condition $C(y) = C(y')$ for all $y, y' \in \Omega$ in the second part of Theorem 3.2.18 is not a necessary condition and this condition is just sufficient condition.

Example 3.2.19. Consider Example 3.2.15. Obviously $(-1,0)^T$ is an εk^0-nondominated element and also it is an εk^0-minimizer but $C(y') = C(y)$ does not hold for all $y, y' \in \Omega$.

Example 3.2.20. Consider Example 3.2.16. It is easy to see that $(-1,0)^T$ is an εk^0-nondominated element but not εk^0-minimizer. it is obvious that $\{(d_1,d_2) \in \mathbb{R}^2 | d_2 \geq 0,\ d_1 + d_2 \geq -1\}$ is not a subset of $\mathbb{R}^2_+$.

Now, we discuss the relationships between sets of (weak, strong) εk^0-minimizer and (weakly, strongly) εk^0-minimal elements with respect to the ordering map C. We know that in the case of vector optimization problems with fixed ordering structures, these sets coincide but in vector optimization problems with a variable ordering structure, Examples 3.2.15 and 3.2.17 show that there are some approximately minimal elements which are not approximate minimizers.

Theorem 3.2.21. Let assumption (A) be fulfilled.

1. Every εk^0-minimizer of Ω with respect to C is also an εk^0-minimal element.

2. Suppose that y_ε is an εk^0-minimal element of Ω with respect to C, then y_ε is also an εk^0-minimizer if $C(y) \subseteq C(y_\varepsilon)$ for all $y \in \Omega$.

3. Suppose that $\operatorname{int} C(y) \neq \emptyset$ for all $y \in \Omega$, then every weak εk^0-minimizer of Ω with respect to C is also a weakly εk^0-minimal element.

4. Suppose that $\operatorname{int} C(y) \neq \emptyset$ for all $y \in \Omega$ and y_ε is a weakly εk^0-minimal element of Ω with respect to C, then y_ε is also a weak εk^0-minimizer if $\operatorname{int} C(y) \subseteq \operatorname{int} C(y_\varepsilon)$ for all $y \in \Omega$.

5. Every strong εk^0-minimizer of Ω with respect to C is also a strongly εk^0-minimal element.

6. Suppose that y_ε is a strongly εk^0-minimal element of Ω with respect to C, then y_ε is also a strong εk^0-minimizer if $C(y_\varepsilon) \subseteq C(y)$ for all $y \in \Omega$.

Proof. 1. This is obvious from the first parts of Definition 3.2.10 and Definition 3.2.1.

2. Suppose that y_ε is an εk^0-minimal element, then $(y_\varepsilon - \varepsilon k^0 - C(y_\varepsilon)\backslash\{\mathbf{0}\}) \cap \Omega = \emptyset$. Since $C(y) \subseteq C(y_\varepsilon)$ for all $y \in \Omega$, we have

$$(y_\varepsilon - \varepsilon k^0 - C(y)\backslash\{\mathbf{0}\}) \cap \{y^1\} \subseteq (y_\varepsilon - \varepsilon k^0 - C(y_\varepsilon)\backslash\{\mathbf{0}\}) \cap \{y^1\} = \emptyset \ \ \forall y, y^1 \in \Omega.$$

This means that y_ε is a εk^0-minimizer.

3. This is obvious from the second parts of Definition 3.2.10 and Definition 3.2.1.

4. If $\operatorname{int} C(y) \subseteq \operatorname{int} C(y_\varepsilon)$ and y_ε is a weakly εk^0-minimal element, then for all $y, y^1 \in \Omega$

$$(y_\varepsilon - \varepsilon k^0 - \operatorname{int} C(y)) \cap \{y^1\} \subseteq (y_\varepsilon - \varepsilon k^0 - \operatorname{int} C(y_\varepsilon)) \cap \{y^1\} = \emptyset.$$

This means that $(y_\varepsilon - \varepsilon k^0 - \operatorname{int} C(y)) \cap \{y^1\} = \emptyset$ for all element $y, y^1 \in \Omega$ and y_ε is a weak εk^0-minimizer.

5. This part is obvious from the third parts of Definition 3.2.10 and Definition 3.2.1.

6. Suppose that y_ε is a strongly εk^0-minimal element, then $y_\varepsilon - \varepsilon k^0 \in y - C(y_\varepsilon) \setminus \{\mathbf{0}\}$ for all $y \in \Omega$. Since $C(y_\varepsilon) \subseteq C(y^1)$ for all $y^1 \in \Omega$, we have

$$y_\varepsilon - \varepsilon k^0 \in y - C(y_\varepsilon) \setminus \{\mathbf{0}\} \implies y_\varepsilon - \varepsilon k^0 \in y - C(y^1) \setminus \{\mathbf{0}\} \;\; \forall y, y^1 \in \Omega.$$

This means that y_ε is a strong εk^0-minimizer. $\square$

The following examples show that the condition $C(y) \subseteq C(y_\varepsilon)$ for all $y \in \Omega$ in the second part of Theorem 3.2.21 is not a necessary condition.

Example 3.2.22. Consider Example 3.2.15. Obviously $(-1,0)^T$ is an εk^0-minimal element and εk^0-minimizer but $C(y) \subseteq C(y_\varepsilon)$ does not hold for all $y \in \Omega$.

Example 3.2.23. Consider Example 3.2.15. It is easy to see that $(0,-1)^T$ is an εk^0-minimal element but not εk^0-minimizer. It is obvious that $\{(d_1, d_2) \in \mathbb{R}^2 | d_1 \geq 0, \; d_2 \leq 0\}$ is not a subset of $\mathbb{R}^2_+$.

In the following theorem, we show the relationships between (weakly, strongly) εk^0-minimal and (weakly, strongly) εk^0-nondominated elements of Ω with respect to the map $C : Y \rightrightarrows Y$.

Theorem 3.2.24. Let assumption (A) be fulfilled.

1. Suppose that y_ε is an εk^0- minimal element of Ω with respect to the map $C : Y \rightrightarrows Y$, then y_ε is also an εk^0-nondominated element of Ω if $C(y) \subseteq C(y_\varepsilon)$ for all $y \in \Omega$.

2. Suppose that y_ε is an εk^0- nondominated element of Ω with respect to the ordering map $C : Y \rightrightarrows Y$, then y_ε is also an εk^0-minimal element of Ω if $C(y_\varepsilon) \subseteq C(y)$ for all $y \in \Omega$.

3. Suppose that $\operatorname{int} C(y) \neq \emptyset$ for all $y \in \Omega$ and y_ε is a weakly εk^0- minimal element of Ω with respect to the ordering map $C : Y \rightrightarrows Y$, then y_ε is also a weakly εk^0-nondominated element of Ω if $\operatorname{int} C(y) \subseteq \operatorname{int} C(y_\varepsilon)$ for all $y \in \Omega$.

4. Suppose that $\operatorname{int} C(y) \neq \emptyset$ for all $y \in \Omega$ and y_ε is a weakly εk^0- nondominated element of Ω with respect to the ordering map $C : Y \rightrightarrows Y$, then y_ε is also a weakly εk^0-minimal element of Ω if $\operatorname{int} C(y_\varepsilon) \subseteq \operatorname{int} C(y)$ for all $y \in \Omega$.

5. Suppose that y_ε is a strongly εk^0- minimal element of Ω with respect to the ordering map $C : Y \rightrightarrows Y$, then y_ε is also a strongly εk^0-nondominated element of Ω if $C(y_\varepsilon) \subseteq C(y)$ for all $y \in \Omega$.

6. Suppose that y_ε is a strongly εk^0- nondominated element of Ω with respect to the ordering map $C : Y \rightrightarrows Y$, then y_ε is also an εk^0-minimal element of Ω if $C(y) \subseteq C(y_\varepsilon)$ for all $y \in \Omega$.

Proof. 1. By $C(y) \subseteq C(y_\varepsilon)$ and the second part of Theorem 3.2.21, we know that y_ε is an εk^0-minimizer. Now, from the first part of Theorem 3.2.18, it is obvious that y_ε is a εk^0-nondominated element.

2. Since y_ε is a εk^0-nondominated element and $C(y_\varepsilon) \subseteq C(y)$ for all $y \in \Omega$, we can write

$$(y_\varepsilon - \varepsilon k^0 - C(y)\backslash\{\mathbf{0}\}) \cap \{y\} = \emptyset \implies (y_\varepsilon - \varepsilon k^0 - C(y_\varepsilon)\backslash\{\mathbf{0}\}) \cap \{y\} = \emptyset \;\; \forall y \in \Omega.$$

This means that $(y_\varepsilon - \varepsilon k^0 - C(y_\varepsilon)\backslash\{\mathbf{0}\}) \cap \Omega = \emptyset$ and y_ε is εk^0-minimal element.

3. By $C(y) \subseteq C(y_\varepsilon)$ and the part 4 of Theorem 3.2.21, y_ε is a weak εk^0-minimizer. Now, by the third part of Theorem 3.2.18, it is obvious that y_ε is a weakly εk^0-nondominated element.

4. The proof is similar to that of part 2 by considering $\operatorname{int} C(y_\varepsilon) \subseteq \operatorname{int} C(y)$ for all $y \in \Omega$.

5. By $C(y_\varepsilon) \subseteq C(y)$ and the part 6 of Theorem 3.2.21, we know y_ε is a strong εk^0-minimizer. Now, by the part 5 of Theorem 3.2.18, it is obvious that y_ε is a strongly εk^0-nondominated element.

6. Suppose that y_ε is a strongly εk^0-nondominated element of Ω. By $C(y) \subseteq C(y_\varepsilon)$ for all $y \in \Omega$, we get

$$y_\varepsilon - \varepsilon k^0 \in y - C(y)\backslash\{\mathbf{0}\} \implies y_\varepsilon - \varepsilon k^0 \in y - C(y_\varepsilon)\backslash\{\mathbf{0}\} \;\; \forall y \in \Omega.$$

This means that y_ε is a strongly εk^0-minimal element of Ω. □

The following examples show that the condition $C(y) \subseteq C(y_\varepsilon)$ for all $y \in \Omega$ in the first part of Theorem 3.2.24 is a sufficient condition but it is not a necessary condition.

Example 3.2.25. Consider Example 3.2.15. Obviously $(-1,0)^T$ is an εk^0-minimal element and also it is an εk^0-nondominated element but $C(y) \subseteq C(y_\varepsilon)$ does not hold for all $y \in \Omega$.

Example 3.2.26. Consider Example 3.2.15 where $(0,-1)^T$ is an εk^0-minimal element but not εk^0-nondominated element. Obviously $\mathbb{R}^2_+$ is not a subset of $\{(d_1,d_2) \in \mathbb{R}^2 | d_1 \geq 0,\ d_2 \leq 0\}$.

The following examples show that the condition $C(y_\varepsilon) \subseteq C(y)$ for all $y \in \Omega$ in the second part of Theorem 3.2.24 is a sufficient condition but it is not a necessary condition.

Example 3.2.27. Consider Example 3.2.15. It is easy to see that $(-1,0)^T$ is an εk^0-nondominated element and also it is an εk^0-minimal element but $C(y_\varepsilon) \subseteq C(y)$ does not hold for all $y \in \Omega$.

Example 3.2.28. Consider Example 3.2.16 where $(-1,0)^T$ is an εk^0-nondominated element but not εk^0-minimal element. It is obvious that $\{(d_1,d_2) \in \mathbb{R}^2 | d_2 \geq 0,\ d_1 + d_2 \geq -1\}$ is not a subset of $\mathbb{R}^2_+$.

Chapter 4

Characterization of Solutions of Vector Optimization Problems

In this chapter, we study nonlinear scalarization method by means of a nonlinear separating functional $\varphi_{C,k^0} : Y \to \overline{\mathbb{R}}$ introduced by Gerstewitz (Tammer) [31]. Indeed, solutions of vector optimization problems can be found through scalarization procedures and we use properties of scalar optimization problems to characterize solutions of original vector optimization problems. First, we bring the definition of Tammer-Weidner scalarization method for vector optimization problems with fixed ordering structures and then we generalize this scalarization for vector optimization problems with variable ordering structures. We study characterization and properties of solutions of both cases. In the first section, we deal with vector optimization problems with fixed ordering structures. We prove that if $\bar{y}$ is a minimal element of the original vector optimization problem, then $\bar{y}$ is a minimal solution of the scalar optimization problem and functional φ_{C,k^0}. With some conditions on ordering cone C, we will show that the optimal solution of the original vector optimization problem with fixed ordering structure can be characterized by Tammer-Weidner nonlinear functional. In the second section, we study vector optimization problems with respect to variable structures and variable ordering cones. We prove that we can characterize all approximately optimal elements of the set of feasible solutions with respect to a variable ordering structure by scalarization functionals.

4.1 Characterization of Approximate Solutions of Vector Optimization Problems with Fixed Ordering Structures

Scalarization of a given vector optimization problem (VOP) means the replacement of (VOP) by a suitable scalar optimization problem with a real-valued objective function. Since the scalar optimization theory is widely developed, it is important to transform vector optimization problems to scalar optimization problems and in this case, we can use methods of usual scalar optimization problems. In fact, solutions of the scalarized problem are also solutions of the given vector optimization problem (VOP). There are many scalarization approaches for determining of the solutions of (VOP), for instance weighted sum method, $\varepsilon-$constraint method, Pascoletti-Serafini (PS) scalarization and method introduced by Gerstewitz (Tammer) and Weidner (TSP). In this section, we concentrate on the (TSP) scalarization method. We know that we have wide variety of scalarization approaches for determining minimal solution of (VOP) but not all are suitable for nonconvex problems or arbitrary partial orderings. For example, we know that weighted sum method has disadvantage in nonconvex case and we can not determine all the efficient points. (TSP) method also is important for us because other scalarization methods such weighted sum, ε-constraint method, Pascoletti-Serafini (PS), etc are special case of (TSP) scalarization approach.

Let Y be a real linear space, $\Omega \subset Y$, $k^0 \in Y\backslash\{\mathbf{0}\}$ and C be a closed and proper subset of Y such that $C+[0,\infty)k^0 \subset C$. Nonlinear separating functional $\varphi_{C,k^0} : \Omega \to \overline{\mathbb{R}}$ defined by Gerstewitz (Tammer) and Weidner is as the following

$$\varphi_{C,k^0}(y) := \inf\{t \in \mathbb{R} \mid y \in tk^0 - C\}. \tag{4.1}$$

With this scalarization, we can have all important properties of a scalarization approach of vector optimization problems should have. If $\bar{y}$ is a minimal solution of (4.1), then depending on monotonicity properties of φ_{C,k^0}, one can show we get kind of solution (minimal, weakly minimal) for the original problem (VOP) and by variation of parameters, we can characterize all optimal elements of the original vector optimization problem (VOP). We say $\bar{y}$ is a minimal solution of φ_{C,k^0} if there is no other feasible solution y for φ_{C,k^0} such that $\varphi_{C,k^0}(y) < \varphi_{C,k^0}(\bar{y})$. In the following theorem, we bring characterization of (weakly) minimal elements of the original problem (VOP); see [32] for more properties of solutions of functional defined by (4.1) and see [23] for similar results using Pascoletti-Serafini scalarization functional.

Theorem 4.1.1. Let Y be a real linear space, $\Omega \subset Y$, C be a closed, proper, solid and pointed subset of Y, $k^0 \in Y\backslash\{\mathbf{0}\}$, $\operatorname{int} C \neq \emptyset$, $C+(0,\infty)k^0 \subset \operatorname{int} C$ and $\bar{y} \in \Omega$.

1. Let $\bar{y}$ be a weakly minimal element of Ω with respect to cone C in the sense of Definition 3.1.1, then $\varphi_{C,k^0,\bar{y}}(\bar{y}) \leq \varphi_{C,k^0,\bar{y}}(y)$ for all $y \in \Omega$ where

$$\varphi_{C,k^0,\bar{y}}(y) := \inf\{t \in \mathbb{R} \mid tk^0 + \bar{y} - y \in C\}.$$

2. Suppose that $\bar{y}$ is a minimal element of Ω with respect to C in the sense of Definition 3.1.1, then $\varphi_{C,k^0,\bar{y}}(\bar{y}) \leq \varphi_{C,k^0,\bar{y}}(y)$ for all $y \in \Omega$.

3. Suppose that $C + \operatorname{int} C \subset \operatorname{int} C$ and $\bar{y}$ is a minimal solution of (TSP) scalarization functional φ_{C,k^0} in (4.1), then $\bar{y}$ is a weakly minimal element of Ω in the sense of Definition 3.1.1.

4. Suppose that $C+C \subset C$ and $\bar{y}$ is an unique minimal solution of (TSP) scalarization functional in (4.1), i.e., $\varphi_{C,k^0}(y) > \varphi_{C,k^0}(\bar{y})$ for all $y \in \Omega$, then $\bar{y}$ is a minimal element of Ω in the sense of Definition 3.1.1.

Proof. 1. By $k^0 \in \operatorname{int} C$, we get

$$0k^0 + \bar{y} - \bar{y} = \mathbf{0} + \bar{y} - \bar{y} = \mathbf{0} \in C \quad \text{and} \quad \varphi_{C,k^0,\bar{y}}(\bar{y}) \leq 0.$$

Suppose that there exists another feasible solution $y \in \Omega$ such that

$$t = \varphi_{C,k^0,\bar{y}}(y) < \varphi_{C,k^0,\bar{y}}(\bar{y}) \leq 0 \quad \text{and} \quad tk^0 + \bar{y} - y = c_2 \in C.$$

By $t < 0$, there exists $\varepsilon > 0$ such that $t + \varepsilon = 0$. By $0k^0 + \bar{y} - \bar{y} = \mathbf{0}$ and $tk^0 + \bar{y} - y = c_2$, we get

$$0k^0 + \bar{y} - \bar{y} = tk^0 + \varepsilon k^0 + \bar{y} + y - y - \bar{y} = c_2 + \varepsilon k^0 + y - \bar{y} = \mathbf{0}.$$

By $C + (0, +\infty)k^0 \subset \operatorname{int} C$, we get $c_2 + \varepsilon k^0 \in \operatorname{int} C$ and $\bar{y} \in y + \operatorname{int} C$. But this is a contradiction to weak minimality of $\bar{y}$.

2. By applying the first part and because every minimal element is also a weakly minimal element, we get $\varphi_{C,k^0,\bar{y}}(\bar{y}) \leq \varphi_{C,k^0,\bar{y}}(y)$ for all $y \in \Omega$.

3. Suppose that $\varphi_{C,k^0}(\bar{y}) = \bar{t}$ and $\bar{y}$ is not weakly minimal solution of the original vector optimization problem (VOP), then there exists $c_1 \in \operatorname{int} C$ and $y \in \Omega$ such that $\bar{y} = y + c_1$. Since $\bar{y}$ is a minimal solution of (4.1), there exists $c_2 \in C$ such that $\bar{t}k^0 - \bar{y} = c_2$. By $c_1 \in \operatorname{int} C$ and $C + \operatorname{int} C \subseteq \operatorname{int} C$, we get $c_1 + c_2 \in \operatorname{int} C$. This means there exists an $\varepsilon > 0$ such that $c_1 + c_2 - \varepsilon k^0 \in \operatorname{int} C$. By $\bar{t}k^0 - \bar{y} = c_2$ and $\bar{y} = y + c_1$, we get

$$\bar{t}k^0 - \bar{y} = \bar{t}k^0 - (y + c_1) = c_2 \implies \bar{t}k^0 - y = c_1 + c_2.$$

Let $t = \bar{t} - \varepsilon$, then we get

$$tk^0 - y = (\bar{t} - \varepsilon)k^0 - y = \bar{t}k^0 - \varepsilon k^0 - y = c_1 + c_2 - \varepsilon k^0 \in C.$$

This means y is a feasible solution and $\varphi_{C,k^0}(y) < \varphi_{C,k^0}(\bar{y})$. But this is a contradiction because we supposed that $\bar{y}$ is a minimal solution of the scalar problem (4.1).

4. Suppose that $\bar{y}$ is an unique minimal solution of (4.1) but $\bar{y}$ is not a minimal element of (VOP), then there exists $c_1 \in C$ and $y \in \Omega$ such that $\bar{y} = y + c_1$. Since $\bar{y}$ is a minimal solution of (4.1), there exist $c_2 \in C$ and $\bar{t} \in \mathbb{R}$ such that $\bar{t}k^0 - \bar{y} = c_2$. By this and $\bar{y} = y + c_1$, we get

$$\bar{t}k^0 - \bar{y} = \bar{t}k^0 - (y + c_1) = c_2 \implies \bar{t}k^0 - y = c_1 + c_2.$$

By $C + C \subset C$, we have $c_1 + c_2 \in C$, $\bar{t}k^0 - y \in C$ and $\varphi_{C,k^0}(y) = \bar{t}$. This means that there exists another feasible solution y of (4.1) such that $\varphi_{C,k^0}(y) = \varphi_{C,k^0}(\bar{y})$. But this is a contradiction because we supposed $\varphi_{C,k^0}(y) > \varphi_{C,k^0}(\bar{y})$ for all other feasible solutions y. Therefore, $\bar{y}$ is a minimal element of (VOP) with respect to the ordering cone C. $\square$

Definition 4.1.2. Let Y be a real linear space, $C \subset Y$ and $y_1, y_2 \in Y$. We say y_1 dominates y_2 with respect to C iff there exists an element $c \in C$ such that $y_2 = y_1 + c$.

Theorem 4.1.3. Let C be a convex cone, $\bar{y}$ be a minimal solution of $\varphi_{C,k^0}(.)$ in (4.1) and $y \in \Omega$ dominates $\bar{y}$, then y is also a minimal solution for (4.1) with respect to ordering cone C.

Proof. Since $\bar{y}$ is a minimal solution, there exists $\bar{t} \in \mathbb{R}$ such that $\bar{t}k^0 - \bar{y} = c_1 \in C$. Since y dominates $\bar{y}$, there exists $c_2 \in C$ such that $\bar{y} = y + c_2$. First we prove $c_2 \in \mathrm{bd}C$. By contrary, suppose that $c_2 \in \mathrm{int}C$, then $\bar{y}$ is not even weakly minimal solution of (VOP) and the third part of Theorem 4.1.1 implies that $\bar{y}$ can not be a minimal solution for (4.1). By $\bar{t}k^0 - \bar{y} = c_1$ and $\bar{y} = y + c_2$, we get

$$\bar{t}k^0 - y = \bar{t}k^0 - \bar{y} + c_2 = c_1 + c_2.$$

By convexity of the cone C, we get $c_1 + c_2 \in C$ and y is a feasible solution for (4.1). Since $\bar{y}$ is a minimal solution, $\bar{t}$ is smallest t with respect to k^0 for objective function in (4.1) and therefore, y is a minimal solution for (4.1). $\square$

By this theorem, we can say that if $\bar{y}$ is a unique minimal solution of scalar problem (4.1), then $\bar{y}$ is a minimal solution of vector optimization problem (VOP) with respect to cone C.

Theorem 4.1.4. Let C be a solid and convex cone. A point $\bar{y}$ is a minimal solution for vector optimization problem (VOP) with respect to cone C in the sense of Definition 3.1.1 if

(i) there exists $\bar{t} \in \mathbb{R}$ such that $\varphi(\bar{y}) = \bar{t}$ and $\bar{y}$ is a minimal solution for scalar problem (TSP) and $\varphi_{C,k^0}(.)$ where $k^0 \in \mathrm{int}C$.

(ii) And also the following equation holds

$$\Omega \cap (\bar{t}k^0 - \partial C) \cap (\bar{y} - \partial C) = \{\bar{y}\}.$$

Proof. Suppose that conditions (i) and (ii) hold and $\bar{y}$ is not a minimal solution for the vector optimization problem (VOP), then there exists $c_2 \in C$ and $y \in \Omega$ such that $\bar{y} = y + c_2$. Similar to the proof of Theorem 4.1.3, we know that c_2 belongs to the boundary of C. Since y dominates $\bar{y}$, by Theorem 4.1.3, y is a minimal solution of (TSP) with respect to cone C and $\varphi_{C,k^0}(y) \leq \bar{t}$. By the first condition, we get $\bar{t}k^0 - \bar{y} = c_1$ and

$$\bar{t}k^0 - y = \bar{t}k^0 - y + \bar{y} - \bar{y} = \bar{t}k^0 - \bar{y} + \bar{y} - y = c_1 + c_2.$$

Since y is a minimal solution, we have $c_1 + c_2 \in C$. Now, we need to prove that $c_1 + c_2$ belongs to the boundary of cone C. By contrary, suppose $c_1 + c_2 \in \mathrm{int}C$, then there exists $\varepsilon > 0$ such that $c_1 + c_2 - \varepsilon k^0 \in C$ and $\bar{t}k^0 - y - \varepsilon k^0 = (\bar{t} - \varepsilon)k^0 - y = (c_1 + c_2 - \varepsilon k^0)$. This means y is a feasible solution with $\varphi_{C,k^0}(y) \leq \bar{t} - \varepsilon < \bar{t}$. But this is a contradiction because we suppose that $\bar{y}$

is minimal solution of (4.1). Therefore, $c_1 + c_2 \in \operatorname{bd} C$ and $y = \bar{t}k^0 - (c_1 + c_2) \in \bar{t}k^0 - \partial C$. Also, we know that $y \in \Omega$ and from dominating property $y = \bar{y} - c_2 \in \bar{y} - \partial C$. Finally we can write,

$$\Omega \cap (\bar{t}k^0 - \partial C) \cap (\bar{y} - \partial C) = \{\bar{y}, y\}.$$

This leads us to a contradiction and completes the proof. □

Similar results to this theorem for Pascolleti-Serafini scalarization method were given in [61, Theorem 3.7].

4.2 Characterization of Approximate Solutions of Vector Optimization Problems with Variable Ordering Structure

In this section, we present a scalarization method with the help of nonlinear functionals (see 4.1). This scalarization method for vector optimization problems with resect to a fixed ordering structure was introduced by Gerstewitz (1983) in [31] (see also [32], [33, Theorem 2.3.1], [35, Theorem 3.38]) and one year later by Pascoletti and Serafini (1984) in [61]. Some generalizations of this scalarization method for vector optimization problems with a variable ordering structure where the ordering map is pointed, closed, convex and cone-valued can be found in [12, 15, 16, 22]. Here, we give a generalization of the Tammer-Weidner functional without any cone or convexity assumptions and we use it for the characterization of all of the three different kinds of approximate solutions. In fact, our ordering map is just a set-valued map with certain properties. For sure, our scalarization also works when the ordering map is convex and cone-valued.

4.2.1 Scalarizing Functionals and Their Properties

First, we give generalizations of the nonlinear separating functional defined by Gerstewitz (Tammer) and Weidner and some of its properties. With the help of this functional, we will characterize approximate solutions of vector optimization problems with variable ordering structures. In this section we impose the following assumption.

Assumption (A1). Let Ω be a subset of Y and $k^0 \in Y \backslash \{\mathbf{0}\}$. Furthermore, let $C : Y \rightrightarrows Y$ be a set-valued map where $C(\omega)$ is a proper, pointed, closed and solid set with $C(\omega) + [0, +\infty)k^0 \subseteq C(\omega)$ for any $\omega \in Y$.

For $\omega \in Y$, we define a functional $\theta_\omega : Y \to \overline{\mathbb{R}}$ in the following way

$$\theta_\omega(y) := \inf\{t \in \mathbb{R} \mid y \in tk^0 - C(\omega)\}. \tag{4.2}$$

Remark 4.2.1. For fixed $\omega \in Y$, the functional defined by (4.2) coincide with (4.1).

Eichfelder gave a generalization of nonlinear separating functional (4.1) as following. Eichfelder [22] used $\varphi : Y \to \overline{\mathbb{R}}$ where

$$\varphi(y) := \inf\{t \in \mathbb{R} \mid y \in tk^0 - C(y)\} \tag{4.3}$$

for a characterization of exact nondominated and minimal solutions where $C(y)$ is a pointed, closed and convex cone for all $y \in Y$. Note that in (4.3), $C(y)$ is a pointed, closed and convex cone associated to the same y but for the functional in (4.2), $C(\omega)$ is independent from y.

Another generalization of nonlinear separating functional defined by Gerstewitz (Tammer) and Weidner is given by Chen, Yang and Yu. Chen et al. [16] considered $\operatorname{int} C(y) \neq \emptyset$ for all $y \in y$ and $k : Y \to Y$ as a vector-valued map and used the following functional $\zeta(y,z) : Y \times Y \to \overline{\mathbb{R}}$ where

$$\zeta(y,z) := \inf\{t \in \mathbb{R} \mid z \in tk(y) - C(y)\} \tag{4.4}$$

for characterizations of exact nondominated and minimal solutions where $C(y)$ is a pointed, closed and convex cone and $k(y) \in \operatorname{int} C(y)$ for all $y \in Y$.

Remark 4.2.2. Note that for fixed $y \in Y$ and fixed $k(y) = k^0$, the functional defined by (4.4) coincides with (4.1).

Chen and Yang [15] proved a similar result to Theorem 4.2.7. Chen, Yang and Yu [16] proved that the functional (4.4) is subadditive in the second variable if $C : Y \rightrightarrows Y$ is a linear set-valued map and if $C(y)$ is a pointed, closed and convex cone for all $y \in Y$. Chen, Huang and Yang [12] proved that the functional ξ defined by (4.4) is lower semicontinuous if $C : Y \rightrightarrows Y$ is a continuous set-valued map and they also proved that ξ is positively homogenous in the second variable if $C(y)$ is a pointed, closed and convex cone for all $y \in Y$.

Lemma 4.2.3. Let assumption (A1) be fulfilled, $\omega, y \in Y$ and $\theta_\omega(y) = t_1$. Then for any $t_2 \geq t_1$,

$$y \in t_2 k^0 - C(\omega).$$

Proof. By $C(\omega) + [0,+\infty)k^0 \subseteq C(\omega)$, $y \in t_1 k^0 - C(\omega)$ and $t_2 - t_1 \geq 0$, we can write

$$y \in t_1 k^0 - C(\omega) = t_2 k^0 - [(t_2 - t_1)k^0 + C(\omega)] \subseteq t_2 k^0 - C(\omega). \qquad \square$$

It is important to show that the scalarizing functional (4.2) is proper. In the following theorem, this property will be shown. Compare the following theorem with results given by Göpfert et. al. in Theorem 2.3.1 of [33].

Theorem 4.2.4. Let assumption (A1) be fulfilled. The functional θ_ω defined in (4.2) is proper for all $\omega \in Y$ if one of the following properties holds,

1. For all $\omega \in Y$, $C(\omega)$ does not contain any line parallel to k^0, i.e.,

$$\forall y \in Y,\ \exists t \in \mathbb{R}\ :\ y \notin tk^0 - C(\omega).$$

2. There exists a cone $D \subseteq Y$ such that $k^0 \in \operatorname{int} D$ and $C(\omega) + \operatorname{int} D \subseteq C(\omega)$ for all $\omega \in Y$.

Proof. 1. Suppose that there exists $\omega \in \Omega$ such that $\theta_\omega(y) = -\infty$. By Lemma 4.2.3 and for any $t > -\infty$, we get $y \in tk^0 - C(\omega)$ and $\{tk^0 - y \mid t \in \mathbb{R}\} \subset C(\omega)$. This means that there exists $y \in Y$ such that $C(\omega)$ contains a line parallel to k^0 and this leads to a contradiction.

2. By Proposition 2.3.4 of [33], we get $C(\omega)$ does not contain any parallel line to k^0. The rest of proof is similar to that of part 1. □

In the special case, when $C(\omega)$ in assumption (A1) is a convex cone-valued map, we have the following corollary (see Remark 2.1 and Proposition 2.2 of [15]).

Corollary 4.2.5. Let assumption (A1) be fulfilled. Additionally let $C(\omega)$ be a convex cone-valued map and $k^0 \in \bigcap_{\omega \in Y} \operatorname{int} C(\omega)$, then θ_ω is proper for all $\omega \in Y$.

Proof. Obviously, $C(\omega) + \operatorname{int} C(\omega) \subseteq \operatorname{int} C(\omega)$ when $C(\omega)$ is a pointed and convex cone and by the second part of Theorem 4.2.4, θ_ω is proper. □

Remark 4.2.6. Note that the assumption $k^0 \in \bigcap_{\omega \in Y} \operatorname{int} C(\omega)$ in Corollary 4.2.5 for convex cones coincide with the assumption $C(\omega) + (0, +\infty)k^0 \subseteq \operatorname{int} C(\omega)$ for sets.

The following theorem shows some important properties of the functional θ_ω in (4.2) and it will help us to prove subadditivity, monotonicity and other theorems in the next sections.

Theorem 4.2.7. Let assumption (A1) be fulfilled and additionally $C(y) + (0, +\infty)k^0 \subseteq \operatorname{int} C(y)$ for all $y \in Y$. For any $y, \omega \in Y$, we have the following properties:

1. $\theta_\omega(y) < t \Longleftrightarrow y \in tk^0 - \operatorname{int} C(\omega)$.

2. $\theta_\omega(y) \leq t \Longleftrightarrow y \in tk^0 - C(\omega)$.

3. $\theta_\omega(y) = t \Longleftrightarrow y \in tk^0 - \operatorname{bd} C(\omega)$.

4. $\theta_\omega(y) \geq t \Longleftrightarrow y \notin tk^0 - \operatorname{int} C(\omega)$.

5. $\theta_\omega(y) > t \Longleftrightarrow y \notin tk^0 - C(\omega)$.

Proof. 1. First, we prove that the following implication holds:

$$\forall \lambda \in \mathbb{R},\ y \in \lambda k^0 - C(\omega) \implies \forall \mu > \lambda : \ y \in \mu k^0 - \operatorname{int} C(\omega). \tag{4.5}$$

Indeed, for $y \in \lambda k^0 - C(\omega)$ and $\mu > \lambda$ holds

$$y - \mu k^0 = y - \lambda k^0 + (\lambda - \mu)k^0 \in -C(\omega) + (\lambda - \mu)k^0 \subseteq -\operatorname{int} C(\omega).$$

Since $C(\omega)+(0,+\infty)k^0 \subseteq \operatorname{int} C(\omega)$. If we suppose $\theta_\omega(y) < t$, then there exists at least one $\lambda < t$ such that $y \in \lambda k^0 - C(\omega)$. Now by (4.5), we have $y \in tk^0 - \operatorname{int} C(\omega)$.

Now suppose that $y \in tk^0 - \operatorname{int} C(\omega)$, therefore there exists $c^1 \in \operatorname{int} C(\omega)$ such that

$$y = tk^0 - c^1. \tag{4.6}$$

Since $c^1 \in \operatorname{int} C(\omega)$, there exists $\gamma > 0$ such that $c^1 - \gamma k^0 \in C(\omega)$. By this and (4.6), we get

$$y = (t-\gamma)k^0 - (c^1 - \gamma k^0) \implies y \in (t-\gamma)k^0 - C(\omega).$$

Hence $\theta_\omega(y) \le (t-\gamma) < t$.

2. Suppose that $\theta_\omega(y) \le t$, then $\theta_\omega(y) = t$ or $\theta_\omega(y) < t$. If $\theta_\omega(y) < t$, by the previous part, we get

$$y \in tk^0 - \operatorname{int} C(\omega) \implies y \in tk^0 - C(\omega).$$

Now suppose $\theta_\omega(y) = t$ and there exists a sequence $t_n \to t$ such that $t < t_n$ and $\theta_\omega(y) < t_n$. By the first part, we get $y \in t_n k^0 - \operatorname{int} C(\omega)$, i.e., $t_n k^0 - y \in C(\omega)$. By $t_n k^0 - y \to tk^0 - y$ and since $C(\omega)$ is a closed set, $tk^0 - y \in C(\omega)$ and $y \in tk^0 - C(\omega)$.

Now suppose $y \in tk^0 - C(\omega)$, then obviously from the definition of θ_ω, we get $\theta_\omega(y) \le t$.

3. Suppose that $\theta_\omega(y) = t$, then by the second part, we get $y \in tk^0 - C(\omega)$ and this means either $y \in tk^0 - \operatorname{bd} C(\omega)$ or $y \in tk^0 - \operatorname{int} C(\omega)$. If $y \in tk^0 - \operatorname{bd} C(\omega)$, the result holds. But suppose that $y \in tk^0 - \operatorname{int} C(\omega)$, then by first part, $\theta_\omega(y) < t$ and this is a contradiction to our assumption.

Now suppose that $y \in tk^0 - \operatorname{bd} C(\omega)$, then $y \in tk^0 - C(\omega)$ and $\theta_\omega(y) \le t$. If $\theta_\omega(y) \ne t$, then $\theta_\omega(y) < t$ and by the first part, we get $y \in tk^0 - \operatorname{int} C(\omega)$ which is a contradiction to our assumption.

4. This follows from the first part.

5. This follows from the second part. □

In the following, we will prove that our scalarizing functional is lower semicontinuous (l. s. c), subadditive, positively homogenous, monotone and continuous in the case that some assumptions hold. These properties are important for us and they will be used in the next chapters in order to show a generalization of Ekeland's variational principle for vector optimization problems with variable ordering structures. Furthermore, this properties will be used in the last chapter for deriving necessary conditions for different kinds of approximate solutions of vector optimization problems with variable ordering structures. First we remember definition of lower and upper semicontinuity and then we will prove that our scalarization functional is lower semicontinuous.

Definition 4.2.8. Let $\theta : Y \to \mathbb{R}$ and $y \in Y$.

1. (a) θ is upper semicontinuous at y iff for every sequence $\{y^n\}$ in Y with $y^n \to y$, the following holds: $\limsup \theta(y^n) \leq \theta(y)$.

   (b) f is upper semi-continuous iff it is upper semicontinuous at every $y \in Y$.

2. (a) θ is lower semicontinuous at y iff for every sequence $\{y^n\}$ in Y with $y^n \to y$, the following holds: $\liminf \theta(y^n) \geq \theta(y)$.

   (b) f is lower semicontinuous iff it is lower semi-continuous at every $y \in Y$.

Theorem 4.2.9. Let Y be a topological space and let $\theta : Y \to \mathbb{R}$. The followings are equivalent.

1. The functional θ is lower semicontinuous on Y.

2. For any $t \in \mathbb{R}$, the set $\{y \in Y \mid \theta_\omega(y) > t\}$ is an open set in Y.

3. For any $t \in \mathbb{R}$, the set $\{y \in Y \mid \theta_\omega(y) \leq t\}$ is a closed set in Y.

Proof. See [53, Theorem 7.1.1]. □

Using the above theorem, we show that θ_ω defined in (4.2) is lower semicontinuous. For the case of fix ordering case see Theorem 2.3.1 of [33].

Theorem 4.2.10. Let assumption (A1) be fulfilled and additionally $C(y) + (0, +\infty)k^0 \subseteq \operatorname{int} C(y)$ for all $y \in Y$, then θ_ω in (4.2) is lower semicontinuous for any $\omega \in Y$.

Proof. We have to show that for any $t \in \mathbb{R}$, the set

$$S_t := \{y \in Y \mid \theta_\omega(y) \leq t\}$$

is a closed set. For this, we show that for any sequence $\{y^n\} \in S_t$ with $y^n \to y^0$, the limit point of the sequence belongs to the set S_t and this proves that S_t is a closed set. Since $y^n \in S_t$, we have $\theta_\omega(y^n) \leq t$. Now by the second part of Theorem 4.2.7, we get

$$y^n \in tk^0 - C(\omega) \implies tk^0 - y^n \in C(\omega).$$

Since $C(\omega)$ is a closed set, the limit point of the sequence $tk^0 - y^n \to tk^0 - y^0$ also belongs to $C(\omega)$ and $y^0 \in tk^0 - C(\omega)$ and by the second part of Theorem 4.2.7, we get $\theta_\omega(y^0) \leq t$. This means that S_t is a closed set for any $t \in \mathbb{R}$. By Theorem 4.2.9, θ_ω is lower semicontinuous for any $\omega \in Y$. □

Lemma 4.2.11. Suppose that assumption (A1) holds. If $f : X \to Y$ is a continuous function, then $(\theta_\omega \circ f)(\cdot) = \theta_\omega(f(\cdot)) : X \to \mathbb{R}$ is a lower semicontinuous functional for all $\omega \in Y$.

Proof. We prove that $S := \{x \mid \theta_\omega(f(x)) \le t\}$ is a closed set for all $t \in \mathbb{R}$. Choose $t \in \mathbb{R}$ arbitrarily and suppose that $\{x^n\}$ is a sequence in S such that $x^n \to x^0$. We prove that $x^0 \in S$ and this means that S is a closed set. By $x^n \in S$, we get $\theta_\omega(f(x^n)) \le t$ and

$$tk^0 - f(x^n) \in C(\omega).$$

This means that $f(x^n) \in tk^0 - C(\omega)$. By continuity of f, we get $f(x^n) \to f(x^0)$ and therefore $f(x^0) \in tk^0 - C(\omega)$ and this means that $\theta_\omega(f(x)) \le t$ and this completes the proof. $\square$

Now we prove that our functional is positively homogenous and for this first we remember definition of positively homogenous function.

Definition 4.2.12. Let $\theta : Y \to \mathbb{R}$ and $y \in Y$. θ is positively homogenous iff $\lambda\theta(y) = \theta(\lambda y)$ for all $y \in Y$ and $\lambda \ge 0$.

The following theorem shows that θ_ω is positively homogenous under some assumptions. This theorem was proven by Göpfert et. al. for the case of fixed ordering; see [33, Theorem 2.3.1].

Theorem 4.2.13. Let assumption (A1) be fulfilled and additionally $C(y) + (0,+\infty)k^0 \subseteq \operatorname{int} C(y)$ for all $y \in Y$. For each $\omega \in Y$, θ_ω in (4.2) is positively homogeneous if and only if $C(\omega)$ is a cone.

Proof. First assume that $\lambda > 0$, then for any $y, \omega \in Y$, we have

$$\theta_\omega(\lambda y) = \inf\{t \in \mathbb{R} \mid \lambda y \in tk^0 - C(\omega)\}.$$

Since $C(\omega)$ is a cone, we have $C(\omega) = \lambda C(\omega)$ and

$$\theta_\omega(\lambda y) = \inf\{t \in \mathbb{R} \mid \lambda y \in tk^0 - \lambda C(\omega)\} = \lambda \inf\{\frac{t}{\lambda} \in \mathbb{R} \mid y \in \frac{t}{\lambda}k^0 - C(\omega)\},$$

so by $t' = \frac{t}{\lambda}$, we get

$$\theta_\omega(\lambda y) = \lambda \inf\{t' \in \mathbb{R} \mid y \in t'k^0 - C(\omega)\} = \lambda\theta_\omega(y).$$

Suppose now $\lambda = 0$, then obviously $0\theta_\omega(y) = 0$ and we just need to prove that $\theta_\omega(\mathbf{0}) = 0$. By pointedness $C(\omega)$, we have $\mathbf{0} \in \operatorname{bd} C(\omega)$ for all $\omega \in Y$ and by Theorem 4.2.7, we get $\theta_\omega(\mathbf{0}) = 0$.

Now assume that θ_ω is positively homogenous and take $y \in C(\omega)$. By the second part of Theorem 4.2.7, we get $\theta_\omega(-y) \le 0$. Since θ_ω is positively homogeneous, we obtain

$$\theta_\omega(-\lambda y) \le \lambda\theta_\omega(-y) \le 0.$$

Again by the second part of Theorem 4.2.7, we get $\lambda y \in C(\omega)$ and $\lambda C(\omega) \subseteq C(\omega)$. Now suppose that $y \in C(\omega)$, then by the second part of Theorem 4.2.7, we get

$$\theta_\omega(-y) \leq 0 \implies \lambda \theta_\omega(-\frac{y}{\lambda}) \leq 0.$$

By $\lambda > 0$, we get $\frac{y}{\lambda} \in C(\omega)$ and $y \in \lambda C(\omega)$ and this implies $C(\omega) \subseteq \lambda C(\omega)$. Therefore it holds $C(\omega) = \lambda C(\omega)$ for any $\lambda > 0$ and $\omega \in Y$. This means $C(\omega)$ is a cone. □

Subadditivity of the scalarizing functional is important for us and we need this property in the next section for the characterization of approximately nondominated, minimal and minimizers. Furthermore, subadditivity is an important property for the deriving a variational principle for vector optimization problems with a variable ordering structure. Therefore, first we bring definition of subadditivity and then we prove that our scalarizing functional is subadditive functional.

Definition 4.2.14. Suppose that $f : X \to \mathbb{R}$. f is said to be subadditive function if and only if $f(x+y) \leq f(x) + f(y)$ for all $x, y \in X$.

Theorem 4.2.15. Let assumption (A1) be fulfilled and additionally $C(y) + (0,+\infty)k^0 \subseteq \operatorname{int} C(y)$ for all $y \in Y$, then for each $\omega \in Y$, θ_ω in (4.2) is subadditive if and only if $C(\omega) + C(\omega) \subseteq C(\omega)$.

Proof. Assume that $C(\omega) + C(\omega) \subseteq C(\omega)$ for all $\omega \in Y$. Let $y^1, y^2 \in Y$ and $t_1, t_2 \in \mathbb{R}$ such that $\theta_\omega(y^1) = t_1$ and $\theta_\omega(y^2) = t_2$. By the second part of Theorem 4.2.7, we get

$$\theta_\omega(y^1) = t_1 \implies y^1 \in t_1 k^0 - C(\omega). \tag{4.7}$$

$$\theta_\omega(y^2) = t_2 \implies y^2 \in t_2 k^0 - C(\omega). \tag{4.8}$$

By (4.7), (4.8) and $C(\omega) + C(\omega) \subseteq C(\omega)$, we get

$$y^1 + y^2 \in (t_1 + t_2)k^0 - (C(\omega) + C(\omega)) \subseteq (t_1 + t_2)k^0 - C(\omega).$$

Again, by the second part of Theorem 4.2.7, we get $\theta_\omega(y^1 + y^2) \leq t_1 + t_2 = \theta_\omega(y^1) + \theta_\omega(y^2)$. Now assume that θ_ω is subadditive. We show that $C(\omega) + C(\omega) \subseteq C(\omega)$. Take $y^1, y^2 \in C(\omega)$. By the second part of Theorem 4.2.7 and $y^1, y^2 \in C(\omega)$, we get $\theta_\omega(-y^1) \leq 0$ and $\theta_\omega(-y^2) \leq 0$. Since θ_ω is subadditive,

$$\theta_\omega(-y^1 - y^2) \leq \theta_\omega(-y^1) + \theta_\omega(-y^2) \leq 0.$$

By the second part of Theorem 4.2.7, we get $y^1 + y^2 \in C(\omega)$ and this completes our proof. □

For sure, there are a lot of new things about vector optimization problems with a variable ordering structure and one of the important things is how to define the convexity of a functional with respect to an ordering map. In vector optimization problems with a fixed ordering structure, convexity of a functional is equal to the convexity of its epigraph, i.e., the scalarizing functional is convex if and only if its epigraph is convex. But unfortunately, for definitions of convexity in vector optimization problems with a variable ordering structure in literature, this is not true and convexity of a epigraph does not imply convexity of its functional; see [12] for more details about convexity in vector optimization problems with a variable ordering structure. Still there exist no unified definition of convexity in vector optimization problems with a variable ordering structure and relationships between the convexity of epigraph and convexity of its functional is not known yet. For all $\omega \in Y$, we say that our functional $\theta_\omega : \Omega \to \overline{\mathbb{R}}$ is convex if for all $y^1, y^2 \in Y$ and $0 \leq \lambda \leq 1$ the following inequality holds,

$$\theta_\omega(\lambda y^1 + (1-\lambda)y^2)) \leq \lambda \theta_\omega(y^1) + (1-\lambda)\theta_\omega(y^2).$$

Theorem 4.2.16. Let assumption (A1) be fulfilled and additionally $C(y) + (0, +\infty)k^0 \subseteq \operatorname{int} C(y)$ for all $y \in Y$. For all $\omega \in Y$, θ_ω is convex if and only if $C(\omega)$ be a convex set.

Proof. Assume that $\lambda \in [0,1]$, $y^1, y^2 \in Y$ such that $\theta_\omega(y^1) = t_1$ and $\theta_\omega(y^2) = t_2$. By the second part of Theorem 4.2.7, we get $y^1 \in t_1 k^0 - C(\omega)$ and $y^2 \in t_2 k^0 - C(\omega)$. By convexity of $C(\omega)$, we have

$$\begin{aligned} \lambda y^1 + (1-\lambda)y^2 &\in \lambda t_1 k^0 + (1-\lambda)t_2 k^0 - \lambda C(\omega) + (1-\lambda)C(\omega) \\ &\subseteq (\lambda t_1 + (1-\lambda)t_2)k^0 - C(\omega). \end{aligned}$$

Therefore

$$\theta_\omega(\lambda y^1 + (1-\lambda)y^2) \leq \lambda \theta_\omega(y) + (1-\lambda)\theta_\omega(y^2),$$

and this means that θ_ω is convex.

Now suppose that θ_ω is convex, $y^1, y^2 \in C(\omega)$ and $\lambda \in [0,1]$. By $y^1, y^2 \in C(\omega)$ and the second part of Theorem 4.2.7, we get $\theta_\omega(-y^1) \leq 0$ and $\theta_\omega(-y^2) \leq 0$. By convexity of θ_ω, we get

$$\theta_\omega(-(\lambda y^1 + (1-\lambda)y^2)) \leq \lambda \theta_\omega(-y^1) + (1-\lambda)\theta_\omega(-y^2) \leq 0.$$

By the second part of Theorem 4.2.7, we get $\lambda y^1 + (1-\lambda)y^2 \in C(\omega)$ and $C(\omega)$ is convex. $\square$

In the last theorem of this section, we prove some monotonicity properties of our scalarization functional and these properties will be used in the next section in order to characterize approximately optimal solutions of vector optimization problems with variable ordering structures and later for the proof of variational principle of vector optimization problems with variable ordering structures; see Theorem 2.3.1 of [33] for the case of fixed ordering case. First we recall definition of monotonicity.

Definition 4.2.17. Let $D \subset Y$ be a set. A function $\theta : Y \to \mathbb{R}$ is called D-monotone function if for every $d \in D$ and $y^1, y^2 \in Y$, the following holds,

$$y^2 \in y^1 + D \implies \theta(y^1) \leq \theta(y^2).$$

Theorem 4.2.18. Let assumption (A1) be fulfilled. Additionally let $C(\omega) + (0, +\infty)k^0 \subseteq \operatorname{int} C(\omega)$ for all $\omega \in Y$ and $D \subseteq Y$. Then for each $\omega \in Y$, the following properties hold for θ_ω in (4.2).

1. θ_ω is D-monotone $\Longleftrightarrow C(\omega) + D \subseteq C(\omega)$.

2. $\forall\, y \in Y,\ t_1 \in \mathbb{R}: \qquad \theta_\omega(y + t_1 k^0) = \theta_\omega(y) + t_1 \quad$ (translation property).

3. θ_ω is continuous for all $\omega \in Y$.

4. If θ_ω is proper, then θ_ω is D-monotone $\Leftrightarrow C(\omega) + D \subseteq C(\omega) \Leftrightarrow \operatorname{bd} C(\omega) + D \subseteq C(\omega)$. Moreover, if θ_ω is finite-valued, then θ_ω is strictly D-monotone iff

$$C(\omega) + D \backslash \{\mathbf{0}\} \subseteq \operatorname{int} C(\omega) \quad \Longleftrightarrow \quad \operatorname{bd} C(\omega) + D \backslash \{\mathbf{0}\} \subseteq \operatorname{int} C(\omega).$$

Proof. 1. Assume that for all $\omega \in Y$, $C(\omega) + D \subseteq C(\omega)$. Consider $y^1, y^2 \in Y$ with $y^1 \leq_D y^2$. We prove that $\theta_\omega(y^1) \leq \theta_\omega(y^2)$ for any $\omega \in Y$. Suppose that $\theta_\omega(y^2) = t$. By the second part of Theorem 4.2.7, we get

$$y^2 \in tk^0 - C(\omega). \tag{4.9}$$

Since $y^1 \leq_D y^2$, there exists $d \in D$ such that $y^1 + d = y^2$. By (4.9), we get

$$y^2 = y^1 + d \in tk^0 - C(\omega) \implies y^1 \in tk^0 - (C(\omega) + d) \subseteq tk^0 - C(\omega).$$

Again by the second part of Theorem 4.2.7, we get $\theta_\omega(y^1) \leq t = \theta_\omega(y^2)$.

Now let θ_ω be D-monotone and choose $d \in D$ and $y^1 \in C(\omega)$ arbitrarily. By $y^1 \in C(\omega)$ and the second part of Theorem 4.2.7, we get $\theta_\omega(-y^1) \leq 0$. Since θ_ω is D-monotone, $\theta_\omega(-y^1 - d) \leq 0$ and again by the first part of Theorem 4.2.7, we get

$$-y^1 - d \in -C(\omega) \implies y^1 + d \in C(\omega) \qquad \forall y^1 \in C(\omega),\ \forall d \in D.$$

Since y^1, d were chosen arbitrarily, we get $C(\omega) + D \subseteq C(\omega)$.

2. Suppose that $\theta_\omega(y) = t$. By the third part of Theorem 4.2.7, for $t_1 \in \mathbb{R}$, we get

$$y \in tk^0 - \operatorname{bd} C(\omega) \implies y + t_1 k^0 \in (t + t_1)k^0 - \operatorname{bd} C(\omega) \implies \theta_\omega(y + t_1 k^0) = t + t_1$$

and this means that $\theta_\omega(y + t_1 k^0) = \theta_\omega(y) + t_1$.

3. By Theorem 4.2.10, we know that θ_ω is lower semicontinuous and we just need to prove that it is also upper semicontinuous. Therefore we need to show that for any $t \in \mathbb{R}$, the set

$$\overline{S}_t := \{y \in Y |\ \theta_\omega(y) \geq t\}$$

is a closed set. For this, we suppose that $y^n \to y^0$ is a sequence and $y^n \in \overline{S}_t$. We show that the limit point of this sequence belongs to the set $\overline{S}_t$ and this proves that $\overline{S}_t$ is a closed set. Since $y^n \in \overline{S}_t$, $\theta_\omega(y^n) \geq t$. Now by the part 4 of Theorem 4.2.7, we get

$$y^n \notin tk^0 - \operatorname{int} C(\omega) \implies tk^0 - y^n \notin \operatorname{int} C(\omega) \implies tk^0 - y^n \in (\operatorname{int} C(\omega))^c.$$

Since $\operatorname{int} C(\omega)$ is an open set, its complement $(\operatorname{int} C(\omega))^c$ is a closed set and includes all the limit points. Therefore $tk^0 - y^0 \in (\operatorname{int} C(\omega))^c$ and this means

$$tk^0 - y^0 \notin \operatorname{int} C(\omega) \implies y^0 \notin tk^0 - \operatorname{int} C(\omega).$$

Again by the part 4 of Theorem 4.2.7, we get $\theta_\omega(y^0) \geq t$ and this implies that $\overline{S}_t$ is a closed set and θ_ω is upper semicontinuous. Since θ_ω is also lower semicontinuous, θ_ω is continuous.

4. We just prove the second part and the proof for the first part is similar to that. Assume that θ_ω is strictly D-monotone and take $y^1 \in \operatorname{bd} C(\omega)$ and $d \in D \backslash \{\mathbf{0}\}$. Since $y^1 \in \operatorname{bd} C(\omega)$, by the third part of Theorem 4.2.7, we get $\theta_\omega(-y^1) = 0$ and $\theta_\omega(-y^1 - d) < 0$. By the first part of Theorem 4.2.7, we get

$$-y^1 - d \in -\operatorname{int} C(\omega) \implies y^1 + d \in \operatorname{int} C(\omega) \qquad \forall y^1 \in C(\omega),\ \forall d \in D.$$

Now, suppose that $\operatorname{bd} C(\omega) + (D \backslash \{\mathbf{0}\}) \subseteq \operatorname{int} C(\omega)$ and $y^1, y^2 \in Y$ with $y^2 - y^1 \in D \backslash \{\mathbf{0}\}$. This means that there is an element $d \in D \backslash \{\mathbf{0}\}$ with $y^2 = y^1 + d$. By the second part of Theorem 4.2.7, we get $y^2 \in \theta_\omega(y^2)k^0 - \operatorname{bd} C(\omega)$ and

$$y^2 = y^1 + d \in \theta_\omega(y^2)k^0 - \operatorname{bd} C(\omega)$$

$$\implies y^1 \in \theta_\omega(y^2)k^0 - (\operatorname{bd} C(\omega) + (D \backslash \{\mathbf{0}\})) \subseteq \theta_\omega(y^2)k^0 - \operatorname{int} C(\omega).$$

By the first part of Theorem 4.2.7, we get $\theta_\omega(y^1) < \theta_\omega(y^2)$. The remaining implication is obvious. □

Definition 4.2.19. Function θ_ω is a sublinear function iff it is positively homogenous and subadditive.

Theorem 4.2.20. Assume that $C(\omega)$ is a proper, closed, convex and solid cone for all $\omega \in Y$ and $k^0 \in \operatorname{int} C(\omega)$, then θ_ω is a finite-valued continuous sublinear function for all $\omega \in Y$.

Proof. See Corollary 2.3.5 of [33]. □

First we bring definition of Lipschitz continuous functions and later we show that our scalarization functional is Lipschitz continuous under special assumptions. Suppose that (X,d) and (Y,ρ) are metric spaces with metrics d on X and ρ on Y.

Definition 4.2.21. A function $f : X \to Y$ is called Lipschitz continuous if there exists a real constant $K \geq 0$ such that for all $x_1, x_2 \in X$, $\rho(f(x_1), f(x_2)) \leq Kd(x_1, x_2)$.

Theorem 4.2.22. Let all the assumptions of Theorem 4.2.20 be fulfilled. Then θ_ω is Lipschitz continuous for all $\omega \in Y$

Proof. By [20, Proposition 2.1], every continuous sublinear function is Lipschtiz continuous. Now by Theorem 4.2.20, θ_ω is Lipschitz continuous. □

We already introduce monotone function in Definition 4.2.17 with respect to a fixed set. Now we introduce monotone function with respect to a set-valued map as following.

Definition 4.2.23. Suppose that Y is a linear topological space, $B : Y \rightrightarrows Y$ is a set-valued map, $\omega \in Y$ and consider functionals $\theta_\omega : Y \to \mathbb{R}$. We say that θ_ω is a monotone functional with respect to a set-valued map $B : Y \rightrightarrows Y$ if the following holds for all $\omega, y^1, y^2 \in Y$,

$$y^1 \in y^2 + B(\omega) \backslash \{\mathbf{0}\} \;\text{ implies }\; \theta_\omega(y^1) \geqq \theta_\omega(y^2).$$

Also, we say θ_ω is strictly B-monotone, if for all $\omega, y^1, y^2 \in Y$

$$y^1 \in y^2 + B(\omega) \backslash \{\mathbf{0}\} \;\text{ implies }\; \theta_\omega(y^1) > \theta_\omega(y^2).$$

Remark 4.2.24. If $B = B(y^1) = B(y^2)$ and $\theta_{y^1}(y) = \theta_{y^2}(y)$ for all $y, y^1, y^2 \in Y$, then the above definition coincide with the usual definition on monotone function with respect to the set B,

$$y^1 \in y^2 + B \backslash \{\mathbf{0}\} \;\text{ implies }\; \theta(y^1) \geqq \theta(y^2).$$

The following nonconvex separation theorem will be used in the next section for our proofs; see [33, Theorem 2.3.6] for vector optimization problems with fixed ordering structures.

Theorem 4.2.25. Let assumption (A1) be fulfilled. Additionally let $S \subseteq Y$ be a nonempty set, $C(\omega) + (0, +\infty)k^0 \subseteq \operatorname{int} C(\omega)$ and $S \cap (-\operatorname{int} C(\omega)) = \emptyset$ for all $\omega \in Y$. Let $B : Y \rightrightarrows Y$ be a cone-valued map such that $k^0 \in \operatorname{int} B(\omega)$ and $C(\omega) + \operatorname{int} B(\omega) \subseteq C(\omega)$ for all $\omega \in Y$. Then for all $\omega \in Y$, θ_ω is finite-valued continuous functional and

$$\theta_\omega(-z) < 0 \leq \theta_\omega(s) \qquad \forall \omega \in Y, \forall z \in \operatorname{int} C(\omega),\ \forall s \in S.$$

Proof. By Theorem 4.2.4 and [33, Proposition 2.3.4], we get θ_ω is finite-valued and by the third part of Theorem 4.2.18, θ_ω is continuous. By the first part of Theorem 4.2.7, we get $-\operatorname{int} C(\omega) = \{z \in Y \mid \theta_\omega(z) < 0\}$. Now since $S \cap (-\operatorname{int} C(\omega)) = \emptyset$ for all $\omega \in Y$, we can write:

$$\theta_\omega(-z) < 0 \leq \theta_\omega(s) \qquad \forall \omega \in Y, \forall z \in \operatorname{int} C(\omega),\ \forall s \in S$$

and this completes the proof. □

Definition 4.2.26. Let X, Y be Banach spaces. We say that $f : X \to Y$ is bounded from below over $\mathfrak{S} \subset X$ with respect to $y^0 \in Y$ and $\Theta \subset Y$ iff $f(\mathfrak{S}) \subseteq y^0 + \Theta$.

Lemma 4.2.27. Let assumption (A1) be fulfilled, $C(\omega) + (0, +\infty)k^0 \subseteq \operatorname{int} C(\omega)$ for all $\omega \in Y$, X be a Banach space and $\mathfrak{S} \subset X$. For $\bar{x} \in \mathfrak{S}$, set $\bar{y} = f(\bar{x})$ and consider the functional $\theta_{\bar{y}}$. Additionally suppose that there exists a cone $D \subset Y$ such that $k^0 \in D$ and $C(\bar{y}) + \operatorname{int} D \subset C(\bar{y})$ and $f : X \to Y$ is bounded from below with respect to $y^0 \in Y$ and the set $\Theta = C(\bar{y})$ in the sense of Definition 4.2.26, then $\theta_{\bar{y}} \circ f$ is bounded below.

Proof. By Definition 4.2.26, we have $f(\mathfrak{S}) \subset y^0 + C(\bar{y})$. By the first part of [33, Proposition 2.3.4], there exists $\hat{t}$ such

$$\hat{t}k^0 - y^0 \notin C(\bar{y}) \tag{4.10}$$

Assume that $\theta_{\bar{y}} \circ f$ is not bounded from below and there exists $x \in \mathfrak{S}$ such that $\theta_{\bar{y}}(f(x)) < \hat{t}$. Since f is bounded from below, there exists $c_1 \in C(\bar{y})$ such that $f(x) = y^0 + c_1$. By $\theta_{\bar{y}}(f(x)) < \hat{t}$, Lemma 4.2.3 and Theorem 4.2.7, we have

$$f(x) \in \hat{t}k^0 - C(f(x)) \implies y^0 + c_1 \in \hat{t}k^0 - C(\bar{y}) \implies y^0 \in \hat{t}k^0 - (C(\bar{y}) + c_1).$$

By $C(\bar{y}) + c_1 \subseteq C(\bar{y})$, we get $y^0 \in \hat{t}k^0 - C(\bar{y})$ which is a contradiction to (4.10). This completes the proof and $\theta_{\bar{y}}(f(.))$ is bounded below. □

4.2.2 Characterization of Approximate Minimizers by Scalarizing Functionals

In this section, we characterize εk^0-minimizers of vector optimization problems with respect to variable ordering structures by scalarization via nonlinear functionals. First in the following theorem, we show that each εk^0-minimizer element of the set Ω is a solution of the scalar optimization problem.

Theorem 4.2.28. Let assumption (A1) be fulfilled. Additionally let $C(\omega) + (0, +\infty)k^0 \subseteq \operatorname{int} C(\omega)$ for all $\omega \in Y$ and $\varepsilon \geq 0$.

1. If $y_\varepsilon \in \Omega$ is an εk^0-minimizer of the set $\Omega \subseteq Y$, then $\theta_\omega(\mathbf{0}) \leq \theta_\omega(y - y_\varepsilon) + \varepsilon$ for all $y, \omega \in \Omega$, where $\theta_\omega(y) = \inf\{t \in \mathbb{R} \mid tk^0 - y \in C(\omega)\}$.

2. If $y_\varepsilon \in \Omega$ is a weak εk^0-minimizer of Ω, then $\theta_\omega(\mathbf{0}) \leq \theta_\omega(y - y_\varepsilon) + \varepsilon$ for all $y, \omega \in \Omega$.

3. If $y_\varepsilon \in \Omega$ is a strong εk^0-minimizer of Ω, then $\theta_\omega(\mathbf{0}) < \theta_\omega(y - y_\varepsilon) + \varepsilon$ for all $y, \omega \in \Omega$.

Proof. 1. Suppose that y_ε is an εk^0-minimizer of the set Ω and there exist $y, \omega \in \Omega$ such that $\theta_\omega(y - y_\varepsilon) + \varepsilon < \theta_\omega(\mathbf{0}) := t$. First, we prove that $t = 0$. By the second part of Theorem 4.2.7, we get

$$\theta_\omega(\mathbf{0}) = t \implies tk^0 - \mathbf{0} \in C(\omega) \implies tk^0 \in C(\omega).$$

By pointedness of $C(\omega)$, we get $\mathbf{0} \in C(\omega)$ and $t \leq 0$. Again by pointedness of $C(\omega)$, $tk^0 \in C(\omega)$ and $C(\omega) + [0, +\infty)k^0 \subseteq C(\omega)$, we get $t \geq 0$ and therefore we can write $t = 0$. By $\theta_\omega(\mathbf{0}) = 0$ and $\theta_\omega(y - y_\varepsilon) + \varepsilon < \theta_\omega(\mathbf{0})$, there exists $\gamma > 0$ such that $\theta_\omega(y - y_\varepsilon) = -\gamma - \varepsilon$ and by the second part of Theorem 4.2.7, we get

$$(-\gamma - \varepsilon)k^0 + y_\varepsilon - y = c^1 \in C(\omega) \implies y_\varepsilon - y - \varepsilon k^0 \in C(\omega) + \gamma k^0.$$

By $\gamma > 0$ and $C(\omega) + [0, +\infty)k^0 \subseteq \operatorname{int} C(\omega)$, we get $y_\varepsilon - \varepsilon k^0 - y \in C(\omega) \backslash \{\mathbf{0}\}$ and

$$(y_\varepsilon - \varepsilon k^0 - C(\omega) \backslash \{\mathbf{0}\}) \cap \Omega \neq \emptyset$$

which is a contradiction to our assumption.

2. The proof is similar to that of the previous part.

3. From the first part, we know that $\theta_\omega(\mathbf{0}) \leq \theta_\omega(y - y_\varepsilon) + \varepsilon$ for all $y, \omega \in \Omega$. We just need to show that $\theta_\omega(\mathbf{0}) \neq \theta_\omega(y - y_\varepsilon) + \varepsilon$ for all $y, \omega \in \Omega$ and this means that we need to show $\theta_\omega(y - y_\varepsilon) + \varepsilon \neq 0$ for all $y, \omega \in \Omega$. If $y_\varepsilon = y$ and $\varepsilon > 0$, then $\theta_\omega(y - y_\varepsilon) = 0$ and obviously $\theta_\omega(y - y_\varepsilon) + \varepsilon \neq 0$.

   Again, if $y_\varepsilon = y$ and $\varepsilon = 0$, then our assumption $(\theta_\omega(\mathbf{0}) < \theta_\omega(y - y_\varepsilon) + \varepsilon)$ can not be fulfilled. Therefore suppose $y_\varepsilon \neq y$ and there exist $y, \omega \in \Omega$ such that $\theta_\omega(y - y_\varepsilon) + \varepsilon = 0$, then by the second part of Theorem 4.2.7, we get

$$y_\varepsilon - \varepsilon k^0 - y \in C(\omega). \tag{4.11}$$

   Also, by definition of strong εk^0-minimizers, for all $\omega \in \Omega, y \in \Omega \backslash \{y_\varepsilon\}$, we get

$$y_\varepsilon - \varepsilon k^0 \in y - C(\omega) \backslash \{\mathbf{0}\} \implies y_\varepsilon - \varepsilon k^0 - y \in -C(\omega) \backslash \{\mathbf{0}\}. \tag{4.12}$$

   By (4.11) and (4.12), we get $(y_\varepsilon - \varepsilon k^0 - y) \in C(\omega) \cap -C(\omega) \backslash \{\mathbf{0}\}$. But this is a contradiction to the pointedness of $C(\omega)$. Therefore $\theta_\omega(\mathbf{0}) \neq \theta_\omega(y - y_\varepsilon) + \varepsilon$ holds and $\theta_\omega(\mathbf{0}) < \theta_\omega(y - y_\varepsilon) + \varepsilon$ for all $y, \omega \in \Omega$. □

In the special case that $\varepsilon = 0$, we have:

Corollary 4.2.29. Let assumption (A1) be fulfilled and additionally $C(\omega)+(0,+\infty)k^0 \subseteq \operatorname{int} C(\omega)$ for all $\omega \in Y$.

1. If $\bar{y} \in \Omega$ is a minimizer of the set $\Omega \subseteq Y$, then $\theta_\omega(\mathbf{0}) \leq \theta_\omega(y-\bar{y})$ for all $y, \omega \in \Omega$.

2. If $\bar{y} \in \Omega$ is a weak minimizer of Ω, then $\theta_\omega(\mathbf{0}) \leq \theta_\omega(y-\bar{y})$ for all $y, \omega \in \Omega$.

3. If $\bar{y} \in \Omega$ is a strong minimizer of Ω, then $\theta_\omega(\mathbf{0}) < \theta_\omega(y-\bar{y})$ for all $y, \omega \in \Omega$.

In Theorem 4.2.28, we showed that each εk^0-minimizer of the set Ω is a solution for the scalar optimization problem. The following theorem completes characterizations of εk^0-minimizers of vector optimization problems with respect to variable ordering structures.

Theorem 4.2.30. Let assumption (A1) be fulfilled. Additionally let $C(\omega)+(0,+\infty)k^0 \subseteq \operatorname{int} C(\omega)$ for all $\omega \in Y$ and $\varepsilon \geq 0$.

1. If $y_\varepsilon \in \Omega$ and $\theta_\omega(\mathbf{0}) < \theta_\omega(y-y_\varepsilon)+\varepsilon$ for all $y, \omega \in \Omega$, then y_ε is an εk^0-minimizer of the set Ω.

2. If $y_\varepsilon \in \Omega$ and $\theta_\omega(\mathbf{0}) \leq \theta_\omega(y-y_\varepsilon)+\varepsilon$ for all $y, \omega \in \Omega$, then y_ε is a weak εk^0-minimizer of the set Ω.

Proof. 1. Similar to the proof of the first part of Theorem 4.2.28, $\theta_{y^2}(\mathbf{0}) = 0$. Now suppose that $\theta_\omega(\mathbf{0}) < \theta_\omega(y-y_\varepsilon)+\varepsilon$ for all $y, \omega \in \Omega$ but y_ε is not an εk^0-minimizer and there exist $y^1, y^2 \in \Omega$ such that $y^1 - y_\varepsilon \in -\varepsilon k^0 - C(y^2)\backslash\{\mathbf{0}\}$ and

$$-\varepsilon k^0 + y_\varepsilon - y^1 \in C(y^2)\backslash\{\mathbf{0}\}.$$

By the second part of Theorem 4.2.7, we get $\theta_{y^2}(y^1-y_\varepsilon)+\varepsilon \leq 0 = \theta_{y^2}(\mathbf{0})$ which is a contradiction to our assumption.

2. Suppose that $\theta_\omega(\mathbf{0}) \leq \theta_\omega(y-y_\varepsilon)+\varepsilon$ for all $y, \omega \in \Omega$ but y_ε is not a weak εk^0-minimizer and there exist $y^1, y^2 \in \Omega$ such that $y^1 \in y_\varepsilon - \varepsilon k^0 - \operatorname{int} C(y^2)$ and

$$-\varepsilon k^0 + y_\varepsilon - y^1 \in \operatorname{int} C(y^2).$$

Similar to the first part of Theorem 4.2.28, $\theta_{y^2}(\mathbf{0}) = 0$ and by the first part of Theorem 4.2.7, we get $\theta_{y^2}(y^1-y_\varepsilon)+\varepsilon < 0 = \theta_{y^2}(\mathbf{0})$ which is a contradiction to our assumption. □

Corollary 4.2.31. Let assumption (A1) be fulfilled and additionally $C(\omega)+(0,+\infty)k^0 \subseteq \operatorname{int} C(\omega)$ for all $\omega \in Y$

1. If $\bar{y} \in \Omega$ and $\theta_\omega(\mathbf{0}) < \theta_\omega(y-\bar{y})$ for all $y, \omega \in \Omega$, then $\bar{y}$ is a minimizer of the set Ω.

2. If $\bar{y} \in \Omega$ and $\theta_\omega(\mathbf{0}) \leq \theta_\omega(y-\bar{y})+$ for all $y, \omega \in \Omega$, then $\bar{y}$ is a weak minimizer of Ω.

4.2.3 Characterization of Approximately Nondominated Elements by Scalarizing Functionals

In the last section, we characterized approximate minimizers by the scalarization via nonlinear functional methods. We can also use this method for characterizing the approximately nondominated elements. First in the following theorem, we show that each εk^0-nondominated element of the set Ω is a solution of the scalar optimization problem.

Theorem 4.2.32. Let assumption (A1) be fulfilled. Additionally let $C(\omega)+(0,+\infty)k^0 \subseteq \operatorname{int} C(\omega)$ for all $\omega \in Y$ and $\varepsilon \geq 0$.

1. If $y_\varepsilon \in \Omega$ is an εk^0-nondominated element of the set $\Omega \subseteq Y$, then $\theta_\omega(\mathbf{0}) \leq \theta_\omega(\omega - y_\varepsilon) + \varepsilon$ for all $\omega \in \Omega$ where $\theta_\omega(y) = \inf\{t \in \mathbb{R} \mid tk^0 - y \in C(\omega)\}$.

2. If $y_\varepsilon \in \Omega$ is a weakly εk^0-nondominated element of Ω, then $\theta_\omega(\mathbf{0}) \leq \theta_\omega(\omega - y_\varepsilon) + \varepsilon$ for all $\omega \in \Omega$.

3. If $y_\varepsilon \in \Omega$ is a strongly εk^0-nondominated element of Ω, then $\theta_\omega(\mathbf{0}) < \theta_\omega(\omega - y_\varepsilon) + \varepsilon$ for all $\omega \in \Omega$.

Proof. 1. Suppose that y_ε is an εk^0-nondominated element of the set Ω and there exists $\omega \in \Omega$ such that $\theta_\omega(\omega - y_\varepsilon) + \varepsilon < \theta_\omega(\mathbf{0}) = t$. Similar to the proof of the first part of Theorem 4.2.28, $t = 0$. By $\theta_\omega(\omega - y_\varepsilon) + \varepsilon < \theta_\omega(\mathbf{0})$ and $\theta_\omega(\mathbf{0}) = 0$, there exists $\gamma > 0$ such that $\theta_\omega(\omega - y_\varepsilon) = -\gamma - \varepsilon$ and by the second part of Theorem 4.2.7, we get

$$(-\varepsilon - \gamma)k^0 + y_\varepsilon - \omega = c^1 \in C(\omega) \implies y_\varepsilon - \omega - \varepsilon k^0 = c^1 + \gamma k^0 \in C(\omega) + \gamma k^0.$$

By $\gamma > 0$ and $C(\omega) + [0,+\infty)k^0 \subseteq \operatorname{int} C(\omega)$, we get $y_\varepsilon - \varepsilon k^0 - \omega \in C(\omega) \backslash \{\mathbf{0}\}$ and

$$(y_\varepsilon - \varepsilon k^0 - C(\omega) \backslash \{\mathbf{0}\}) \cap \{\omega\} \neq \emptyset$$

which is a contradiction to our assumption.

2. The proof is similar to that of part 1.

3. By the first part, the proof is similar to that of part 3 of Theorem 4.2.28. □

In the special case $\varepsilon = 0$, we have the following corollary.

Corollary 4.2.33. Let assumption (A1) be fulfilled and additionally $C(\omega)+(0,+\infty)k^0 \subseteq \operatorname{int} C(\omega)$ for all $\omega \in Y$.

1. If $\bar{y} \in \Omega$ is a nondominated element of $\Omega \subseteq Y$, then for all $\omega \in \Omega$, $\theta_\omega(\mathbf{0}) \leq \theta_\omega(\omega - \bar{y})$.

2. If $y_\varepsilon \in \Omega$ is a weakly nondominated element of Ω, then for all $\omega \in \Omega$, $\theta_\omega(\mathbf{0}) \leq \theta_\omega(\omega - \bar{y})$.

3. If $y_\varepsilon \in \Omega$ is a strongly nondominated element of Ω, then for all $\omega \in \Omega$, $\theta_\omega(\mathbf{0}) < \theta_\omega(\omega - \bar{y})$.

In Theorem 4.2.32, we showed that each εk^0-nondominated element of the set Ω is a solution for the scalar optimization problem. The following theorem completes characterizations of approximately nondominated elements of Ω with respect to ordering map $C : Y \rightrightarrows Y$.

Theorem 4.2.34. Let assumption $(A1)$ be fulfilled. Additionally let $C(\omega) + (0, +\infty)k^0 \subseteq \operatorname{int} C(\omega)$ for all $\omega \in Y$ and $\varepsilon \geq 0$.

1. Let $y_\varepsilon \in \Omega$ and $\theta_\omega(\mathbf{0}) < \theta_\omega(\omega - y_\varepsilon) + \varepsilon$ for all $\omega \in \Omega$, then y_ε is an εk^0-nondominated element of the set Ω.

2. Let $y_\varepsilon \in \Omega$ and $\theta_\omega(\mathbf{0}) \leq \theta_\omega(\omega - y_\varepsilon) + \varepsilon$ for all $\omega \in \Omega$, then y_ε is a weakly εk^0-nondominated element of the set Ω.

Proof. 1. Similar to the proof of the first part of Theorem 4.2.28, $\theta_\omega(\mathbf{0}) = 0$. Suppose that $\theta_\omega(\mathbf{0}) < \theta_\omega(\omega - y_\varepsilon) + \varepsilon$ for all $\omega \in \Omega$ but y_ε is not an εk^0-nondominated element of the set Ω and there exists $\omega \in \Omega$ such that $y_\varepsilon - \varepsilon k^0 \in \omega + C(\omega) \backslash \{\mathbf{0}\}$ and

$$-\varepsilon k^0 + y_\varepsilon - \omega \in C(\omega) \backslash \{\mathbf{0}\}.$$

By the second part of Theorem 4.2.7, we get $\theta_\omega(\omega - y_\varepsilon) + \varepsilon \leq 0 = \theta_\omega(\mathbf{0})$ which is a contradiction to our assumption.

2. The proof is similar to that of part 2 of Theorem 4.2.30. □

In the special case that $\varepsilon = 0$, we have:

Corollary 4.2.35. Let assumption $(A1)$ be fulfilled and additionally $C(\omega) + (0, +\infty)k^0 \subseteq \operatorname{int} C(\omega)$ for all $\omega \in Y$

1. Let $\bar{y} \in \Omega$ and $\theta_\omega(\mathbf{0}) < \theta_\omega(\omega - \bar{y})$ for all $\omega \in \Omega$, then $\bar{y}$ is a nondominated element of the set Ω.

2. Let $\bar{y} \in \Omega$ and $\theta_\omega(\mathbf{0}) \leq \theta_\omega(\omega - \bar{y})$ for all $\omega \in \Omega$, then $\bar{y}$ is a weakly nondominated element of the set Ω.

4.2.4 Characterization of Approximately Minimal Elements by Scalarizing Functionals

In the special case, when $\varepsilon = 0$ and C is a cone-valued map and each $C(y)$ is a pointed and convex cone, Eichfelder [22] gave characterization of exact solutions of vector optimization problems with variable ordering structures for nondominated and minimal solutions. In the following theorem, we characterize approximately minimal elements of the set Ω with respect to the set-valued map C by scalarization via nonlinear functionals.

Theorem 4.2.36. Let assumption (A1) be fulfilled. Additionally let $C(\omega)+(0,+\infty)k^0 \subseteq \operatorname{int} C(\omega)$ for all $\omega \in Y$ and $\varepsilon \geq 0$.

1. If $y_\varepsilon \in \Omega$ is an εk^0-minimal element of the set $\Omega \subseteq Y$, then $\theta_{y_\varepsilon}(\mathbf{0}) \leq \theta_{y_\varepsilon}(y-y_\varepsilon)+\varepsilon$ for all $y \in \Omega$ where $\theta_{y_\varepsilon}(y) = \inf\{t \in \mathbb{R} \mid tk^0 - y \in C(y_\varepsilon)\}$.

2. If $y_\varepsilon \in \Omega$ be a weakly εk^0-minimal element of Ω, then $\theta_{y_\varepsilon}(\mathbf{0}) \leq \theta_{y_\varepsilon}(y-y_\varepsilon)+\varepsilon$ for all $y \in \Omega$.

3. If $y_\varepsilon \in \Omega$ is a strongly εk^0-minimal element of Ω, then $\theta_{y_\varepsilon}(\mathbf{0}) < \theta_{y_\varepsilon}(y-y_\varepsilon)+\varepsilon$ for all $y \in \Omega$.

Proof. 1. Suppose that y_ε is an εk^0-minimal element of the set Ω and there exists $y \in \Omega$ such that $\theta_{y_\varepsilon}(y-y_\varepsilon)+\varepsilon < \theta_{y_\varepsilon}(\mathbf{0}) = t$. Similar to the proof of Theorem 4.2.28, $t = 0$. Since $\theta_{y_\varepsilon}(y-y_\varepsilon)+\varepsilon < 0$, there exists $\gamma > 0$ such that

$$\theta_{y_\varepsilon}(y-y_\varepsilon) = -\gamma - \varepsilon.$$

By the second part of Theorem 4.2.7, we get

$$(-\gamma-\varepsilon)k^0 + y_\varepsilon - y = c^1 \in C(y_\varepsilon) \implies y_\varepsilon - \varepsilon k^0 - y = c^1 + \gamma k^0 \in C(y_\varepsilon) + \gamma k^0.$$

By $\gamma > 0$ and $C(y_\varepsilon)+[0,+\infty)k^0 \subseteq \operatorname{int} C(y_\varepsilon)$, we get $y_\varepsilon - \varepsilon k^0 - y \in C(y_\varepsilon)\backslash\{\mathbf{0}\}$ and

$$(y_\varepsilon - \varepsilon k^0 - C(y_\varepsilon)\backslash\{\mathbf{0}\}) \cap \Omega \neq \emptyset$$

which is a contradiction to our assumption.

2. The proof is similar to that of part 1.

3. By the first part, the proof is similar to that of part 3 of Theorem 4.2.28. □

In the special case that $\varepsilon = 0$, we have

Corollary 4.2.37. Let assumption (A1) be fulfilled and additionally $C(\omega)+(0,+\infty)k^0 \subseteq \operatorname{int} C(\omega)$ for all $\omega \in Y$.

1. If $\bar{y} \in \Omega$ is a minimal element of the set $\Omega \subseteq Y$, then for all $y \in \Omega$, $\theta_{\bar{y}}(\mathbf{0}) \leq \theta_{\bar{y}}(y-\bar{y})$.

2. If $\bar{y} \in \Omega$ be a weakly minimal element of Ω, then $\theta_{\bar{y}}(\mathbf{0}) \leq \theta_{\bar{y}}(y-\bar{y})$ for all $y \in \Omega$.

3. If $\bar{y} \in \Omega$ is a strongly minimal element of Ω, then $\theta_{\bar{y}}(\mathbf{0}) < \theta_{\bar{y}}(y-\bar{y})$ for all $y \in \Omega$.

Theorem 4.2.36 proves that each εk^0-minimal element of the set Ω is a solution for the scalar optimization problem. The following theorem completes characterizations of εk^0-minimal elements of the set Ω with respect to the ordering map $C : Y \rightrightarrows Y$.

Theorem 4.2.38. Let assumption (A1) be fulfilled. Additionally let $C(\omega)+(0,+\infty)k^0 \subseteq \operatorname{int} C(\omega)$ for all $\omega \in Y$ and $\varepsilon \geq 0$.

1. Let $y_\varepsilon \in \Omega$ and $\theta_{y_\varepsilon}(\mathbf{0}) < \theta_{y_\varepsilon}(y - y_\varepsilon) + \varepsilon$ for all $y \in \Omega$, then y_ε is an εk^0-minimal element of the set Ω.

2. Let $y_\varepsilon \in \Omega$ such that $\theta_{y_\varepsilon}(\mathbf{0}) \leq \theta_{y_\varepsilon}(y - y_\varepsilon) + \varepsilon$ for all $y \in \Omega$, then y_ε is a weakly εk^0-minimal element of the set Ω.

Proof. 1. Again similar to the proof of Theorem 4.2.28, $\theta_{y_\varepsilon}(\mathbf{0}) = 0$. Now suppose that $\theta_{y_\varepsilon}(\mathbf{0}) < \theta_{y_\varepsilon}(y - y_\varepsilon) + \varepsilon$ for all $y \in \Omega$ but y_ε is not an εk^0-minimal element. this means there exists $y \in \Omega$ such that $y_\varepsilon - \varepsilon k^0 \in y + C(y_\varepsilon) \setminus \{\mathbf{0}\}$ and

$$-\varepsilon k^0 + y_\varepsilon - y \in C(y_\varepsilon) \setminus \{\mathbf{0}\}.$$

By the second part of Theorem 4.2.7, we get $\theta_{y_\varepsilon}(y - y_\varepsilon) + \varepsilon \leq 0 = \theta_{y_\varepsilon}(\mathbf{0})$ which is a contradiction to our assumption.

2. The proof is similar to that of part 2 of Theorem 4.2.30. □

In the special case that $\varepsilon = 0$, we have:

Corollary 4.2.39. Let assumption (A1) be fulfilled and additionally $C(\omega) + (0, +\infty)k^0 \subseteq \operatorname{int} C(\omega)$ for all $\omega \in Y$.

1. Let $\bar{y} \in \Omega$ and $\theta_{\bar{y}}(\mathbf{0}) < \theta_{\bar{y}}(y - \bar{y})$ for all $y \in \Omega$, then $\bar{y}$ is a minimal element of the set Ω.

2. Let $\bar{y} \in \Omega$ such that $\theta_{\bar{y}}(\mathbf{0}) \leq \theta_{\bar{y}}(y - \bar{y})$ for all $y \in \Omega$, then $\bar{y}$ is a weakly minimal element of the set Ω.

Chapter 5

Variational Principles in Vector Optimization with Variable Order Structures

We know that for vector optimization problems with variable ordering structures (VVOP), we have three different solution types (see section 2 of Chapter 3) and it is of interest to formulate variational principles for these solutions. Ekeland (1974) formulated in [27] a variational principle, which has applications in many domains of mathematics. Ekeland's variational principle (EVP) is a deep assertion concerning the existence of an exact solution of a slightly perturbed optimization problem in a neighborhood of an approximate solution of the original optimization problem under the assumption that the objective function of the original problem is bounded from below and lower semicontinuous (l.s.c). Several generalization of Ekeland's variational principle for vector optimization problems with fixed ordering structures are given in [3, 4, 8, 9, 13, 14, 28, 37, 38, 41, 44, 55, 70]. The aim of this chapter is to establish new variational principles of Ekeland's type for three different kinds of solutions of vector optimization problems with variable ordering structures by using a nonlinear scalarization technique (see Chapter 4) and derive from them necessary conditions for approximate solutions of vector optimization problems with variable ordering structures in the next chapter. Applications of Ekeland's variational principle can be seen in economics, control theory, game theory, nonsmooth analysis and many other fields.

Theorem 5.0.1. [28] Let $(X, \|\cdot\|)$ be a real Banach space, $\varepsilon > 0$ and $g : X \to \mathbb{R}$ be a real-valued lower semicontinuous function which is bounded below on the closed subset $\mathfrak{S}$ of X. Let x' be an element in $\mathfrak{S}$ such that $g(x') \leqq \inf\{g(x) \mid x \in \mathfrak{S}\} + \varepsilon$, then there exists a point $x_\varepsilon \in \operatorname{dom} g \cap \mathfrak{S}$ such that

(a) $g(x_\varepsilon) \leqq g(x') \leqq \inf\{g(x)|x \in \mathfrak{S}\} + \varepsilon$.

(b) $\|x_\varepsilon - x'\| \leqq \sqrt{\varepsilon}$.

(c) $g(x_\varepsilon) < g(x) + \sqrt{\varepsilon}\,\|x - x_\varepsilon\|$ for all $x \in \mathfrak{S}$.

Remark 5.0.2. **(Strong form of Ekeland's variational principle).** Theorem 5.0.1 is known as the weak version of Ekeland's variational principle since we can find an element $x_\varepsilon \in \operatorname{dom} g \cap \mathfrak{S}$ which additionally satisfies the following condition (see [28])

(c′) $g(x_\varepsilon) + \sqrt{\varepsilon}\,\|x' - x_\varepsilon\| \leq g(x')$.

In this chapter we impose the following standing assumptions.

Assumption (A2). X is a real Banach space, $\mathfrak{S}$ is a subset of X, Y is a topological linear space, $\varepsilon \geq 0$, $k^0 \in Y\backslash\{\mathbf{0}\}$ and $f : X \to Y$ is a vector-valued function. Let $C : Y \rightrightarrows Y$ be a set-valued map where $C(y)$ is a proper, pointed and closed set which satisfies $C(y) + [0,+\infty)k^0 \subseteq C(y)$ for all $y \in Y$.

Furthermore, in some cases, we consider the following assumptions.

(A3) The nonzero vector $k^0 \in Y\backslash\{\mathbf{0}\}$ satisfies $C(y) + (0,+\infty)k^0 \subseteq \operatorname{int} C(y)$ for all $y \in Y$.

(A4) For all $y \in f(\mathfrak{S})$, $\quad C(y) + C(y) \subseteq C(y)$.

(A5) $B : Y \rightrightarrows Y$ be a cone-valued map such that for each $y \in f(\mathfrak{S})$, $k^0 \in \operatorname{int} B(y)$.

(A6) For all $y \in f(\mathfrak{S})$, $\quad C(y) + (B(y)\backslash\{\mathbf{0}\}) \subseteq C(y)$.

By (A5), (A6) and the second part of Theorem 4.2.4, we get the functional θ_ω in (4.2) is proper. Also, by Theorem 4.2.15, assumption (A4) is necessary and sufficient for subadditivity of θ_ω in (4.2). By the first part of Theorem 4.2.18, we know that (A6) is necessary and sufficient for θ_ω in (4.2) to be a B-monotone function. Also by the third part of Theorem 4.2.18 and (A3), our functional θ_ω in (4.2) is continuous.

Remark 5.0.3. Figure 5.1 and 5.2 give examples for sets C where assumptions (A2), (A3) and (A4) are fulfilled (for $C(y) \equiv C$).

In the following we will consider the following vector optimization problem with respect to a variable ordering structure.

$$\varepsilon k^0\text{–Min} \;\; f(x) \text{ subject to } \; x \in \mathfrak{S} \text{ with respect to } \; C. \tag{VVOP}$$

In the third chapter, we defined approximate minimizers, approximate nondominated and approximate minimal elements of the set $\Omega \subset Y$ with respect to variable ordering structures in the image space Y; see (Definition 3.2.1), (Definition 3.2.6) and (Definition 3.2.10). For reader convenience, we define these solution concepts of vector optimization problems with respect to

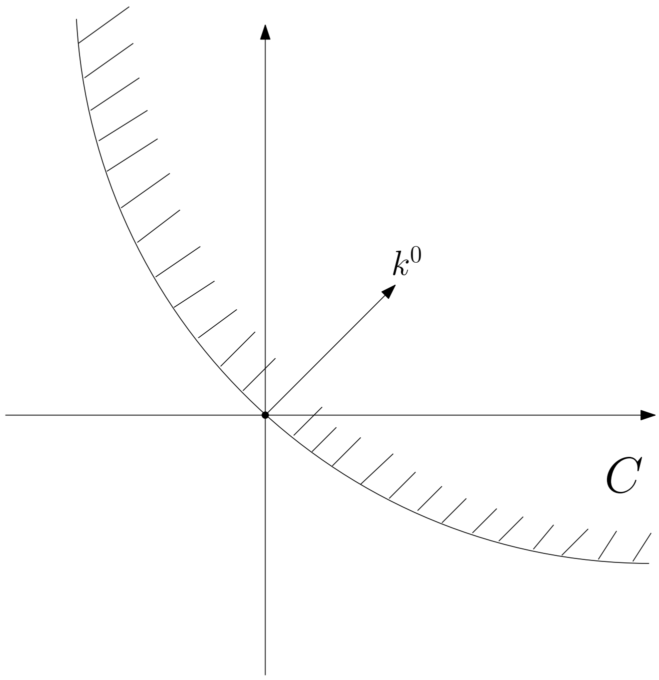

FIGURE 5.1: A convex set C satisfies assumptions (A2), (A3) and (A4).

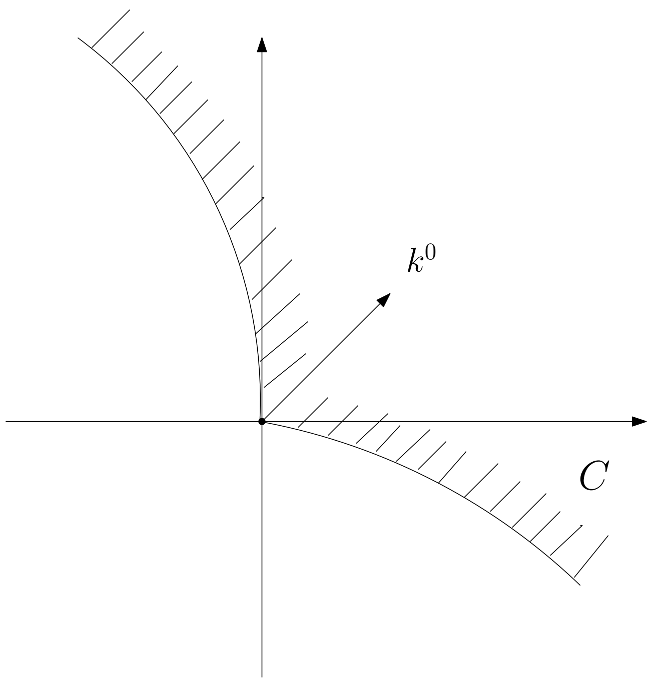

FIGURE 5.2: A nonconvex set C satisfies assumptions (A2), (A3) and (A4).

variable ordering structures (VVOP) for $\mathfrak{S} \subset X$ and the vector-valued function $f : X \to Y$ in the original space X.

Definition 5.0.4. Let assumption (A2) be fulfilled and $x_\varepsilon \in \mathfrak{S}$. Consider problem (VVOP).

1. x_ε is said to be an εk^0-minimizer of (VVOP) with respect to the map $C : Y \rightrightarrows Y$ iff

$$\forall x, x^1 \in \mathfrak{S} : \qquad (f(x_\varepsilon) - \varepsilon k^0 - C(f(x)) \backslash \{\mathbf{0}\}) \cap \{f(x^1)\} = \emptyset.$$

2. Let $\operatorname{int} C(y) \neq \emptyset$ for all $y \in Y$. y_ε is said to be a weak εk^0-minimizer of (VVOP) with respect to the ordering map $C : Y \rightrightarrows Y$ iff

$$\forall x, x^1 \in \mathfrak{S} : \qquad (f(x_\varepsilon) - \varepsilon k^0 - (\operatorname{int} C(f(x))) \cap \{f(x^1)\} = \emptyset.$$

Remark 5.0.5.

- We denote the set of all εk^0-minimizers of (VVOP) with respect to the ordering map $C : Y \rightrightarrows Y$ by εk^0-$\mathrm{MZ}(\mathfrak{S}, f, C)$.
- We denote the set of all weak εk^0-minimizers of (VVOP) with respect to the ordering map C by εk^0-$\mathrm{WMZ}(\mathfrak{S}, f, C)$.

When $\varepsilon = 0$, it coincide with the usual definition of (weak) minimizers; see e.g. [12]. We denote the set of minimizers and weak minimizers by $\mathrm{MZ}(\mathfrak{S}, f, C)$ and $\mathrm{WMZ}(\mathfrak{S}, f, C)$, respectively.

Definition 5.0.6. Let assumption (A2) be fulfilled and $x_\varepsilon \in \mathfrak{S}$. Consider problem (VVOP).

1. x_ε is said to be an εk^0-nondominated solution of (VVOP) with respect to the ordering map $C : Y \rightrightarrows Y$ iff

$$\forall x \in \mathfrak{S} : \qquad (f(x_\varepsilon) - \varepsilon k^0 - C(f(x)) \backslash \{\mathbf{0}\}) \cap \{f(x)\} = \emptyset.$$

2. Let $\operatorname{int} C(y) \neq \emptyset$ for all $y \in Y$. y_ε is said to be a weak εk^0-nondominated solution of (VVOP) with respect to the ordering map $C : Y \rightrightarrows Y$ iff

$$\forall x \in \mathfrak{S} : \qquad (f(x_\varepsilon) - \varepsilon k^0 - (\operatorname{int} C(f(x))) \cap \{f(x)\} = \emptyset.$$

Remark 5.0.7.

- We denote the set of all εk^0-nondominated solutions of (VVOP) with respect to the ordering map $C : Y \rightrightarrows Y$ by εk^0-$\mathrm{N}(\mathfrak{S}, f, C)$.
- We denote the set of all weak εk^0-nondominated solutions of (VVOP) with respect to the ordering map C by εk^0-$\mathrm{WN}(\mathfrak{S}, f, C)$.

When $\varepsilon = 0$, it coincide with the usual definition of (weakly) nondominated solutions; see e.g. [24, 78]. We denote the set of nondominated solutions and weakly nondominated solutions by $\mathrm{N}(\mathfrak{S}, f, C)$ and $\mathrm{WN}(\mathfrak{S}, f, C)$, respectively.

Definition 5.0.8. Let assumption (A2) be fulfilled and $x_\varepsilon \in \mathfrak{S}$. Consider problem (VVOP).

1. x_ε is said to be an εk^0-minimal solution of (VVOP) with respect to the ordering map $C : Y \rightrightarrows Y$ iff
$$(f(x_\varepsilon) - \varepsilon k^0 - C(f(x_\varepsilon))) \cap \{f(\mathfrak{S})\} = \emptyset.$$

2. Let $\operatorname{int} C(y) \neq \emptyset$ for all $y \in Y$. y_ε is said to be a weak εk^0-minimal solution of (VVOP) with respect to the ordering map $C : Y \rightrightarrows Y$ iff
$$(f(x_\varepsilon) - \varepsilon k^0 - (\operatorname{int} C(f(x_\varepsilon))) \cap \{f(\mathfrak{S})\} = \emptyset.$$

Remark 5.0.9.
- We denote the set of all εk^0-minimal solutions of (VVOP) with respect to the ordering map $C : Y \rightrightarrows Y$ by εk^0-$\mathrm{M}(\mathfrak{S}, f, C)$.
- We denote the set of all weak εk^0-minimal solutions of (VVOP) with respect to the ordering map C by εk^0-$\mathrm{WM}(\mathfrak{S}, f, C)$.

When $\varepsilon = 0$, it coincide with the usual definition of (weakly) minimal solutions; see e.g. [24, 40]. We denote the set of minimal solutions and weakly minimal solutions by $\mathrm{M}(\mathfrak{S}, f, C)$ and $\mathrm{WM}(\mathfrak{S}, f, C)$, respectively.

5.1 Variational Principle for Approximately Minimal Solutions

Note that in many Ekeland-type results in the literature; see, e.g. [4–6] and the references therein, the function f is assumed to be C-level-closed, known also as C-lower semicontinuous and C-semicontinuous [17, Definition 2.4], where C is a fixed ordering cone of the ordered image space. Therefore we begin this section with the following definitions of lower semicontinuity in fixed and variable ordering structures.

Definition 5.1.1. Consider problem (VVOP), $\bar{x} \in \mathfrak{S} \cap \operatorname{dom} f$, $\bar{y} := f(\bar{x})$ and $\overline{C} := C(\bar{y})$ is fixed. The function f is said to be $\overline{C}$-lower semicontinuous over $\mathfrak{S}$ iff the sets
$$\operatorname{lev}(y; f) := \left\{x \in \mathfrak{S} |\ f(x) \in y - \overline{C}\right\}$$
are closed in X for all $y \in Y$.

Definition 5.1.2. Consider problem (VVOP), $\bar{x} \in \mathfrak{S} \cap \operatorname{dom} f$, $\bar{y} := f(\bar{x})$ and $\overline{C} := C(\bar{y})$ is fixed. The function f is $(k^0, \overline{C})$-lower semicontinuous over $\mathfrak{S}$ iff the sets
$$M(t) := \left\{x \in \mathfrak{S} |\ f(x) \in tk^0 - \overline{C}\right\}$$
are closed in X for all $t \in \mathbb{R}$.

Definition 5.1.3. We say that $f : X \to Y$ is lower semicontinuous with respect to the ordering map $C : Y \rightrightarrows Y$, $k^0 \in Y\backslash\{\mathbf{0}\}$ and $\mathfrak{S} \subseteq X$ (for short $(k^0, C, \mathfrak{S})$-lsc), if

$$M^X_{(\omega,t)} := \{x \in \mathfrak{S} \mid f(x) \in tk^0 - \mathrm{cl}\, C(\omega)\}$$

is a closed set for all $\omega \in f(\mathfrak{S})$ and each $t \in \mathbb{R}$.

If $C = C(\omega_1) = C(\omega_2)$ is a fixed set, then Definition 5.1.3 coincides with Tammer's definition on page 133 in [70]. Moreover, if $Y = \mathbb{R}$, then our definition coincide with the standard definition of lower semicontinuity. In order to prove the main theorem of this section, first we have to prove the following lemmas.

Lemma 5.1.4. Let $C : Y \rightrightarrows Y$ be a set-valued map and assumptions $(\mathrm{A2})$ and $(\mathrm{A3})$ be fulfilled. For each $\omega \in f(\mathfrak{S})$, consider the functional θ_ω defined by (4.2). If the function $f : X \to Y$ in (VVOP) is $(k^0, C, \mathfrak{S})$-lsc, then $(\theta_\omega \circ f)(\cdot) = \theta_\omega(f(\cdot))$ is a lower semicontinuous functional for each $\omega \in f(\mathfrak{S})$.

Proof. Since the function $f : X \to Y$ is $(k^0, C, \mathfrak{S})$-lsc, the set

$$M^X_{(\omega,t)} = \{x \in \mathfrak{S} \mid f(x) \in tk^0 - C(\omega)\}$$

is closed for all $\omega \in f(\mathfrak{S})$ and $t \in \mathbb{R}$.

Now consider that $M^Y_{(\omega,t)} = tk^0 - C(\omega) \subseteq Y$. By (A3) and the third part of Theorem 4.2.18, we know that $\theta_\omega : Y \to (-\infty, \infty)$ is a continuous functional for each $\omega \in f(\mathfrak{S})$ and by Theorem 4.2.7, we get

$$M^Y_{(\omega,t)} = tk^0 - C(\omega) = \{y \in Y \mid y \in tk^0 - C(\omega)\}$$

$$= \{y \in Y \mid \theta_\omega(y) \leqq \theta_\omega(tk^0)\} = \{y \in Y \mid \theta_\omega(y) \leqq t\} := M^Y_{(\omega,\theta_\omega,t)}$$

for each $\omega \in f(\mathfrak{S})$ and $t \in \mathbb{R}$. This means for all $\omega \in f(\mathfrak{S})$ and $t \in \mathbb{R}$,

$$M^X_{(\omega,\theta_\omega,t)} = \{x \in \mathfrak{S} \mid \theta_\omega(f(x)) \leqq t\} = \{x \in \mathfrak{S} \mid f(x) \in M^Y_{(\omega,\theta_\omega,t)}\} = \{x \in \mathfrak{S} \mid f(x) \in M^Y_{(\omega,t)}\} = M^X_{(\omega,t)}$$

is a closed set and $\theta_\omega \circ f$ is lower semicontinuous for all $\omega \in f(\mathfrak{S})$. $\square$

Lemma 5.1.5. Suppose that assumptions $(\mathrm{A2})$–$(\mathrm{A3})$ hold and let $B : Y \rightrightarrows Y$ be a cone-valued map satisfying assumptions $(\mathrm{A5})$–$(\mathrm{A6})$. Consider the problem (VVOP). If $x_\varepsilon \in \varepsilon k^0$-M$(\mathfrak{S}, f, C)$, then there exists a continuous functional $\theta_{f(x_\varepsilon)} : Y \to \mathbb{R}$ which is strictly $B(f(x_\varepsilon))$-monotone in the sense of Definition 4.2.17 and

$$\forall x \in \mathfrak{S}, \qquad \theta_{f(x_\varepsilon)}(f(x_\varepsilon)) \leq \theta_{f(x_\varepsilon)}(f(x) + \varepsilon k^0).$$

Moreover, if $C(f(x_\varepsilon))+C(f(x_\varepsilon))\subseteq C(f(x_\varepsilon))$ holds, then $\theta_{f(x_\varepsilon)}$ is subadditive on Y and

$$\forall x\in\mathfrak{S},\qquad \theta_{f(x_\varepsilon)}(f(x_\varepsilon))\le\theta_{f(x_\varepsilon)}(f(x))+\theta_{f(x_\varepsilon)}(\varepsilon k^0).$$

Proof. Suppose that $k^0\in Y\backslash\{\mathbf{0}\}$, $\varepsilon>0$ and $x_\varepsilon\in\varepsilon k^0$-M$(\mathfrak{S},f,C)$. This means that,

$$(f(x_\varepsilon)-\varepsilon k^0-C(f(x_\varepsilon))\backslash\{\mathbf{0}\})\cap f(\mathfrak{S})=\emptyset.$$

This implies $(f(x_\varepsilon)-C(f(x_\varepsilon))\backslash\{\mathbf{0}\})\cap(f(\mathfrak{S})+\varepsilon k^0)=\emptyset$.

We consider $V(x_\varepsilon):=(f(x_\varepsilon)-C(f(x_\varepsilon))\backslash\{\mathbf{0}\})$ and $f(\mathfrak{S})+\varepsilon k^0=U$. Taking assumptions on maps C and B and by Theorem 2.3.6 of [33], we get desired functional. Therefore, there exists a continuous functional $\theta_{f(x_\varepsilon)}:Y\to\mathbb{R}$ such that $\theta_{f(x_\varepsilon)}(f(x_\varepsilon))\le\ \theta_{f(x_\varepsilon)}(f(\mathfrak{S})+\varepsilon k^0)$. By $C(f(x_\varepsilon))+C(f(x_\varepsilon))\subseteq C(f(x_\varepsilon))$ and Theorem 4.2.15, $\theta_{f(x_\varepsilon)}$ is a subadditive functional and

$$\theta_{f(x_\varepsilon)}(f(x_\varepsilon))\le\theta_{f(x_\varepsilon)}(f(\mathfrak{S}))+\theta_{f(x_\varepsilon)}(\varepsilon k^0).$$

This completes the proof. □

The following lemma gives some properties of the functional in Lemma 5.1.5 and these properties will be used later in the proof of other lemmas and our main theorem about the vectorial Ekeland's variational principle for minimal solutions of (VVOP).

Lemma 5.1.6. Let assumptions (A2)–(A3) and (A5)–(A6) be fulfilled, then we can choose the functional $\theta_{f(x_\varepsilon)}:Y\to\mathbb{R}$ in Lemma 5.1.5 in a way such that:

1. $\theta_{f(x_\varepsilon)}(k^0)=1$.
2. $\theta_{f(x_\varepsilon)}(\mathbf{0})=0$.
3. $\theta_{f(x_\varepsilon)}(\varepsilon k^0)=\varepsilon$ and $\theta_{f(x_\varepsilon)}(-\varepsilon k^0)=-\theta_{f(x_\varepsilon)}(\varepsilon k^0)=-\varepsilon$.

Proof.

1. By definition of separating functional $\theta_{f(x_\varepsilon)}:Y\to\mathbb{R}$ in (4.2), we have
$$\theta_{f(x_\varepsilon)}(y):=\inf\{t\in\mathbb{R}\mid y\in tk^0-C(f(x_\varepsilon))\}.$$
By pointedness of $C(f(x_\varepsilon))$ and (A3), we get $\mathbf{0}\in\operatorname{bd}C(f(x_\varepsilon))$ and $k^0\in k^0-\operatorname{bd}C(f(x_\varepsilon))$. Taking into account the third part of Theorem 4.2.7, we get $\theta_{f(x_\varepsilon)}(k^0)=1$.
2. By $\mathbf{0}\in\operatorname{bd}C(f(x_\varepsilon))$ and the third part of Theorem 4.2.7, we get $\theta_{f(x_\varepsilon)}(\mathbf{0})=0$.
3. We prove that $\theta_{f(x_\varepsilon)}(\varepsilon k^0)=\varepsilon$. The proof of the rest is similar. By the second part of Theorem 4.2.18, the following holds for all $y\in Y$ and $t\in\mathbb{R}$,
$$\theta_{f(x_\varepsilon)}(y+tk^0)=\theta_{f(x_\varepsilon)}(y)+t.$$
Therefore $\theta_{f(x_\varepsilon)}(\mathbf{0}+\varepsilon k^0)=\theta_{f(x_\varepsilon)}(\mathbf{0})+\varepsilon$ and $\theta_{f(x_\varepsilon)}(\varepsilon k^0)=\varepsilon$. □

Lemma 5.1.7. Let X be a real Banach space, $\mathfrak{S} \subset X$, $x_\varepsilon \in \mathfrak{S}$, Y be a topological linear space, $\varepsilon \geq 0$, $k^0 \in Y \backslash \{\mathbf{0}\}$, $f : X \to Y$ is a vector-valued function with $\operatorname{dom} f \neq \emptyset$ and $B : Y \rightrightarrows Y$ be a cone-valued map satisfying (A5).

(i) Furthermore, suppose that for the strictly B-monotone (in the sense of Definition 4.2.23), continuous and subadditive functional $\theta_{f(x_\varepsilon)} : Y \to \mathbb{R}$, the following inequality holds

$$\forall x \in \mathfrak{S}, \qquad \theta_{f(x_\varepsilon)}(f(x_\varepsilon)) \leq \theta_{f(x_\varepsilon)}(f(x)) - \theta_{f(x_\varepsilon)}(-\varepsilon k^0),$$

then $x_\varepsilon \in \varepsilon k^0$-WM$(\mathfrak{S}, f, C)$ for some set-valued map $C : Y \rightrightarrows Y$ such that $\mathbf{0} \in \operatorname{cl} C(f(x_\varepsilon)) \backslash C(f(x_\varepsilon))$, $B(f(x_\varepsilon)) \backslash \{\mathbf{0}\} \subseteq C(f(x_\varepsilon))$ and $\operatorname{cl} C(f(x_\varepsilon)) + (B(f(x_\varepsilon)) \backslash \{\mathbf{0}\}) \subseteq C(f(x_\varepsilon))$.

Proof. We define $C(f(x_\varepsilon))$ as

$$C(f(x_\varepsilon)) := \{y \in Y \mid \theta_{f(x_\varepsilon)}(-y + f(x_\varepsilon) - \varepsilon k^0) < \theta_{f(x_\varepsilon)}(f(x_\varepsilon) - \varepsilon k^0)\}, \tag{5.1}$$

and a functional $\hat{\theta}_{f(x_\varepsilon)}(y) : Y \to \mathbb{R}$ as

$$\hat{\theta}_{f(x_\varepsilon)}(y) := \theta_{f(x_\varepsilon)}(y + f(x_\varepsilon) - \varepsilon k^0). \tag{5.2}$$

By (5.2) and (i) and since $\theta_{f(x_\varepsilon)}$ is subadditive, we get

$$\begin{aligned} &\hat{\theta}_{f(x_\varepsilon)}(f(\mathfrak{S}) + \varepsilon k^0 - f(x_\varepsilon)) = \theta_{f(x_\varepsilon)}(f(\mathfrak{S})) \geq \\ &\theta_{f(x_\varepsilon)}(f(x_\varepsilon)) + \theta_{f(x_\varepsilon)}(-\varepsilon k^0) \geq \\ &\theta_{f(x_\varepsilon)}(f(x_\varepsilon) - \varepsilon k^0) = \hat{\theta}_{f(x_\varepsilon)}(\mathbf{0}). \end{aligned}$$

Now by (5.1) and (5.2), we get

$$\hat{\theta}_{f(x_\varepsilon)}(-C(f(x_\varepsilon))) = \theta_{f(x_\varepsilon)}(-C(f(x_\varepsilon)) + f(x_\varepsilon) - \varepsilon k^0) < \theta_{f(x_\varepsilon)}(f(x_\varepsilon) - \varepsilon k^0) = \hat{\theta}_{f(x_\varepsilon)}(\mathbf{0}),$$

therefore

$$(-\operatorname{int} C(f(x_\varepsilon))) \cap (f(\mathfrak{S}) + \varepsilon k^0 - f(x_\varepsilon)) = \emptyset \implies (f(x_\varepsilon) - \varepsilon k^0 - \operatorname{int} C(f(x_\varepsilon))) \cap f(\mathfrak{S}) = \emptyset.$$

Since $\theta_{f(x_\varepsilon)}$ is a strictly B-monotone functional, then $B(f(x_\varepsilon)) \backslash \{\mathbf{0}\} \subseteq C(f(x_\varepsilon))$. Now we show that $\operatorname{cl} C(f(x_\varepsilon)) + (B(f(x_\varepsilon)) \backslash \{\mathbf{0}\}) \subseteq C(f(x_\varepsilon))$. Choose $y \in \operatorname{cl} C(f(x_\varepsilon))$ and $b \in y + B(f(x_\varepsilon)) \backslash \{\mathbf{0}\}$. Since $\hat{\theta}_{f(x_\varepsilon)}$ is strictly B-monotone and $y \in \operatorname{cl} C(f(x_\varepsilon)) \subseteq \{y \mid \hat{\theta}_{f(x_\varepsilon)}(-y) \leq \hat{\theta}_{f(x_\varepsilon)}(\mathbf{0})\}$,

$$\hat{\theta}_{f(x_\varepsilon)}(-b) < \hat{\theta}_{f(x_\varepsilon)}(-y) \leq \hat{\theta}_{f(x_\varepsilon)}(\mathbf{0}).$$

Therefore $b \in \operatorname{cl} C(f(x_\varepsilon)) + (B(f(x_\varepsilon)) \backslash \{\mathbf{0}\})$ implies $b \in C(f(x_\varepsilon))$. Assumption $k^0 \in \operatorname{int} B(f(x_\varepsilon))$ and $\operatorname{int} C(f(x_\varepsilon)) + (B(f(x_\varepsilon)) \backslash \{\mathbf{0}\}) \subseteq C(f(x_\varepsilon))$ implies $C(f(x_\varepsilon)) + \varepsilon k^0 \subseteq C(f(x_\varepsilon))$.

Moreover, by $\mathbf{0} \in \mathrm{cl}(B(f(x_\varepsilon))\backslash\{\mathbf{0}\})$, $B(f(x_\varepsilon))\backslash\{\mathbf{0}\} \subseteq C(f(x_\varepsilon))$ and $\mathbf{0} \notin C(f(x_\varepsilon))$, we get easily $\mathbf{0} \in \mathrm{cl}\,C(f(x_\varepsilon))\backslash C(f(x_\varepsilon))$. $\square$

The following theorem gives the first generalization of the Ekeland's variational principle for εk^0-minimal solutions of (VVOP) provided that $f : X \to Y$ is $(k^0, C, \mathfrak{S})$-lower semicontinuous and bounded from below.

Theorem 5.1.8. Consider the problem (VVOP) and let $\bar{x} \in \varepsilon k^0\text{-M}(\mathfrak{S}, f, C)$. Impose in addition to (A2)–(A6) the following assumptions:

$(A7)$ $C(y) \subseteq C(f(\bar{x}))$ for all $y \in f(\mathfrak{S})$.

$(A8)$ f is $(k^0, C, \mathfrak{S})$-lower semicontinuous over $\mathfrak{S}$ in the sense of Definition 5.1.3.

$(A9)$ f is bounded from below over closed subset $\mathfrak{S}$ of X with respect to $f(\bar{x})$ and $C(f(\bar{x}))$ in the sense of Definition 4.2.26.

Then there exists an element $x_\varepsilon \in \mathrm{dom}\, f \cap \mathfrak{S}$ such that

1. $x_\varepsilon \in \varepsilon k^0\text{-WM}(\mathfrak{S}, f, B)$,
2. $\|\bar{x} - x_\varepsilon\| \leq \sqrt{\varepsilon}$,
3. $x_\varepsilon \in \mathrm{WM}(\mathfrak{S}, f_{\varepsilon k^0}, B)$ with $f_{\varepsilon k^0}(x) = f(x) + \sqrt{\varepsilon}\,\|x - x_\varepsilon\|\, k^0$. (5.3)

Proof. Suppose that $\bar{x} \in \varepsilon k^0\text{-M}(\mathfrak{S}, f, C)$, then by the definition of approximately minimal solutions (Definition 5.0.8), we get

$$(f(\bar{x}) - \varepsilon k^0 - C(f(\bar{x}))\backslash\{\mathbf{0}\}) \cap f(\mathfrak{S}) = \emptyset.$$

Now suppose that $\overline{f} := f - f(\bar{x})$, then we have

$$(\overline{f}(\bar{x}) - \varepsilon k^0 - C(f(\bar{x}))\backslash\{\mathbf{0}\}) \cap \overline{f}(\mathfrak{S}) = \emptyset.$$

By Lemma 5.1.5, the inclusion $C(f(\bar{x})) + C(f(\bar{x})) \subseteq C(f(\bar{x}))$ by (A4) and Lemma 5.1.6, the functional $\theta_{f(\bar{x})} : Y \to \mathbb{R}$ defined by (4.2) is a strictly B-monotone, continuous and subadditive functional. Furthermore,

$$\forall x \in \mathfrak{S}, \qquad \theta_{f(\bar{x})}(\overline{f}(\bar{x})) \leq \theta_{f(\bar{x})}(\overline{f}(x)) + \theta_{f(\bar{x})}(\varepsilon k^0) = \theta_{f(\bar{x})}(\overline{f}(x)) + \varepsilon.$$

This means that

$$\theta_{f(\bar{x})}(\overline{f}(\bar{x})) \leq \inf_{x \in \mathfrak{S}} \theta_{f(\bar{x})}(\overline{f}(x)) + \varepsilon, \qquad \varepsilon > 0.$$

Observe that the validity of (A8)–(A9) ensures $(k^0, C, \mathfrak{S})$-lower semicontinuity and the boundedness from below of f and $\overline{f}$. By Lemma 4.2.27, Lemma 5.1.4, Theorem 5.0.1 and Remark 5.0.2, there exists an element $x_\varepsilon \in \mathfrak{S}$ such that

1. $\theta_{f(\bar{x})}(\overline{f}(x_\varepsilon)) \leq \theta_{f(\bar{x})}(\overline{f}(\bar{x})) \leq \inf_{x\in\mathfrak{S}} \theta_{f(\bar{x})}(\overline{f}(x)) + \varepsilon,$ (5.4)

2. $\|x_\varepsilon - \bar{x}\| \leq \sqrt{\varepsilon},$

3. for all $x \in \mathfrak{S}$, $\theta_{f(\bar{x})}(\overline{f}(x_\varepsilon)) \leq \theta_{f(\bar{x})}(\overline{f}(x)) + \sqrt{\varepsilon}\,\|x - x_\varepsilon\|,$ (5.5)

4. $\theta_{f(\bar{x})}(\overline{f}(x_\varepsilon)) + \sqrt{\varepsilon}\,\|\bar{x} - x_\varepsilon\| \leq \theta_{f(\bar{x})}(\overline{f}(\bar{x})).$ (5.6)

By Lemma 5.1.6 and (5.4), for all $x \in \mathfrak{S}$ we get

$$\theta_{f(\bar{x})}(\overline{f}(x_\varepsilon)) \leq \inf_{x\in\mathfrak{S}} \theta_{f(\bar{x})}(\overline{f}(x)) + \varepsilon \leq \theta_{f(\bar{x})}(\overline{f}(x)) + \theta_{f(\bar{x})}(\varepsilon k^0) = \theta_{f(\bar{x})}(\overline{f}(x)) - \theta_{f(\bar{x})}(-\varepsilon k^0).$$

By Lemma 5.1.7, the inclusion $B(f(x_\varepsilon)) \subseteq C(f(x_\varepsilon)) \subseteq C(f(\bar{x}))$ by (A6)–(A7) and $\overline{f} = f - f(\bar{x})$,

$$(\overline{f}(x_\varepsilon) - \varepsilon k^0 - \operatorname{int} B(f(x_\varepsilon))) \cap \overline{f}(\mathfrak{S}) = \emptyset.$$

This implies that $x_\varepsilon \in \varepsilon k^0\text{-WM}(\mathfrak{S}, f, B)$. Now we prove (5.3) and for this, suppose that there exists an element $x \in \mathfrak{S}$ such that

$$\begin{aligned} & f(x) \in f(x_\varepsilon) - \sqrt{\varepsilon}\,\|x - x_\varepsilon\|\,k^0 - \operatorname{int} B(f(x_\varepsilon)) \\ \Longrightarrow \quad & \overline{f}(x) \in \overline{f}(x_\varepsilon) - \sqrt{\varepsilon}\,\|x - x_\varepsilon\|\,k^0 - \operatorname{int} B(f(x_\varepsilon)). \end{aligned}$$

Since $\theta_{f(\bar{x})}$ is a strictly B-monotone, continuous and subadditive functional,

$$\theta_{f(\bar{x})}(\overline{f}(x)) < \theta_{f(\bar{x})}(\overline{f}(x_\varepsilon) - \sqrt{\varepsilon}\,\|x - x_\varepsilon\|\,k^0) \leq \theta_{f(\bar{x})}(\overline{f}(x_\varepsilon)) + \theta_{f(\bar{x})}(-\sqrt{\varepsilon}\,\|x - x_\varepsilon\|\,k^0).$$

Now by Lemma 5.1.6, we get

$$\theta_{f(\bar{x})}(-\sqrt{\varepsilon}\,\|x - x_\varepsilon\|\,k^0) = -\sqrt{\varepsilon}\,\|x - x_\varepsilon\| \implies \theta_{f(\bar{x})}(\overline{f}(x_\varepsilon)) > \theta_{f(\bar{x})}(\overline{f}(x)) + \sqrt{\varepsilon}\,\|x - x_\varepsilon\|.$$

But this yields a contradiction because of (5.5). □

In the special case that $C : Y \rightrightarrows Y$ is a solid, closed, pointed and convex cone-valued map, we have the following corollary.

Corollary 5.1.9. Let $C : Y \rightrightarrows Y$ be a cone-valued map where $C(y)$ is a solid and convex cone for all $y \in f(\mathfrak{S})$, $k^0 \in \bigcap_{y\in f(\mathfrak{S})} \operatorname{int} C(y)$ and $\varepsilon > 0$. Consider the problem (VVOP) and furthermore, let $\bar{x} \in \varepsilon k^0\text{-M}(\mathfrak{S}, f, C)$. Impose the following assumptions:

(A7) $C(y) \subseteq C(f(\bar{x}))$ for all $y \in f(\mathfrak{S})$.

(A8) f is $(k^0, C, \mathfrak{S})$-lower semicontinuous over $\mathfrak{S}$ in the sense of Definition 5.1.3.

(A9) f is bounded from below over closed subset $\mathfrak{S}$ of X with respect to $f(\bar{x})$ and $C(f(\bar{x}))$ in the sense of Definition 4.2.26.

Then there exists an element $x_\varepsilon \in \operatorname{dom} f \cap \mathfrak{S}$ such that

1. $x_\varepsilon \in \varepsilon k^0\text{-WM}(\mathfrak{S}, f, C)$,
2. $\|\bar{x} - x_\varepsilon\| \leq \sqrt{\varepsilon}$,
3. $x_\varepsilon \in \text{WM}(\mathfrak{S}, f_{\varepsilon k^0}, C)$ with $f_{\varepsilon k^0}(x) = f(x) + \sqrt{\varepsilon}\,\|x - x_\varepsilon\|\,k^0$.

In the special case, if $C(y_1) = C(y_2) = C$ is a fixed, solid and convex cone, Corollary 5.1.9 covers Corollary 1 [3], Theorem 5.1 [8], Theorem 2 [9], Theorem 3.1 for a vector-valued map [13], Theorem 2.1 [14], Theorem 3.1 [38], Theorem 10 [44] and Theorem 4.1 [70]. For sure in the case $Y = \mathbb{R}$, we have the classical Ekeland's variational principle [28].

In Theorem 5.1.8 and Corollary 5.1.9, the existence of an element belonging to the set of εk^0-minimal solutions of the original problem that is a weakly minimal solution of a perturbed optimization problem is shown. We show a sharper result, namely that there exists an εk^0-minimal solution of the original problem that is a minimal solution of a perturbed vector optimization problem with variable ordering structure; see also [2].

Theorem 5.1.10. **(Variational principle for εk^0-minimal solutions, solid case).** Consider problem (VVOP), let $\bar{x} \in \varepsilon k^0\text{-M}(\mathfrak{S}, f, C)$ and set $\bar{y} = f(\bar{x})$. Assume that in addition to (A2) the following conditions hold:

$(A3')$ The image $\overline{C} := C(\bar{y})$ is a proper, closed, pointed, and solid set satisfying $\mathbb{R}k^0 - \overline{C} = Y$.

$(A5')$ There exists a cone-valued mapping $B : Y \rightrightarrows Y$ such that $k^0 \in \operatorname{int} \overline{B}$ with $\overline{B} := B(\bar{y})$.

$(A6')$ $\overline{C} + \overline{B} \backslash \{\mathbf{0}\} \subset \operatorname{int} \overline{C}$, and $B(f(x)) \subset \overline{B}$ for all $x \in \mathfrak{S}$ with $\|x - \bar{x}\| \leq \sqrt{\varepsilon}$.

$(A8')$ f is $(k^0, \overline{C})$-lower semicontinuous over $\mathfrak{S}$ in the sense of Definition 5.1.2.

$(A9')$ f is bounded from below over $\mathfrak{S}$ with respect to an element $\underline{y}$ and the cone $\overline{C}$ in the sense of Definition 4.2.26.

Then, there exists some $x_\varepsilon \in \mathfrak{S} \cap \operatorname{dom} f$ such that

(i$'$) $x_\varepsilon \in \varepsilon k^0\text{-M}(\mathfrak{S}, f, B)$, i.e., $\big(f(x_\varepsilon) - \varepsilon k^0 - B(f(x_\varepsilon)) \setminus \{\mathbf{0}\}\big) \cap f(\mathfrak{S}) = \emptyset$.

(ii) $\|x_\varepsilon - \bar{x}\| \leq \sqrt{\varepsilon}$.

(iii$'$) $x_\varepsilon \in \text{M}(\mathfrak{S}, f_{\varepsilon k^0}, B)$, where $f_{\varepsilon k^0}(x) := f(x) + \sqrt{\varepsilon}\|x - x_\varepsilon\| k^0$.

Proof. Consider the nonlinear scalarization function $\theta_{f(\bar{x})}$ defined by 4.2,

$$\theta_{f(\bar{x})}(z) = \inf\big\{t \in \mathbb{R} \mid z \in tk^0 - \overline{C}\big\}.$$

By Theorem 4.2.18, $\theta_{f(\bar{x})}$ has the following properties under the assumption made in the theorem:

- $\theta_{f(\bar{x})}(y+\lambda k^0) = \theta_{f(\bar{x})}(y)+\lambda$ for all $y \in Y$ and for all $\lambda \in \mathbb{R}$.
- $\theta_{f(\bar{x})}$ is continuous.
- $\theta_{f(\bar{x})}$ is strictly $\overline{B}$-monotone in the sense that

$$y_2 - y_1 \in \overline{B} \setminus \{\mathbf{0}\} \implies \theta_{f(\bar{x})}(y_1) < \theta_{f(\bar{x})}(y_2).$$

We prove that $\bar{x}$ is an ε-minimal solution of some scalar optimization problem. To proceed, set $g(x) := f(x) - f(\bar{x})$ with $\operatorname{dom} g = \operatorname{dom} f$. Obviously, $g(\bar{x}) = \mathbf{0}$. We get from the εk^0-minimality of $\bar{x}$ to the function f with respect to the order structure C in Definition 5.0.8 that

$$\begin{aligned}
& (f(x) - f(\bar{x}) + \varepsilon k^0) \notin -C(f(\bar{x})),\ \forall\, x \in \mathfrak{S} \cap \operatorname{dom} f \text{ with } f(x) \neq f(\bar{x}) \\
\iff\ & (g(x) + \varepsilon k^0) \notin -\overline{C},\ \forall\, x \in \mathfrak{S} \cap \operatorname{dom} g \text{ with } g(x) \neq 0 \\
\implies\ & \theta_{f(\bar{x})}(g(x) + \varepsilon k^0) = \theta_{f(\bar{x})}(g(x)) + \varepsilon \geq 0,\ \forall\, x \in \mathfrak{S} \cap \operatorname{dom} g,
\end{aligned}$$

where the implication holds due to the strict $\overline{B}$-monotonicity of $\theta_{f(\bar{x})}$ and $\theta_{f(\bar{x})}(\mathbf{0}) = 0$ which only holds because of (A2) and the pointedness of $\overline{C}$ in assumption (A3′). This together with $\theta_{f(\bar{x})}(g(\bar{x})) = \theta_{f(\bar{x})}(\mathbf{0}) = 0$ yields

$$\inf_{x \in \mathfrak{S} \cap \operatorname{dom} g} \theta_{f(\bar{x})}(g(x)) + \varepsilon \geq \theta_{f(\bar{x})}(g(\bar{x})), \tag{5.7}$$

i.e., $\bar{x}$ is an ε-minimal solution of the composition $\theta_{f(\bar{x})} \circ g : X \to \mathbb{R} \cup \{+\infty\}$ over $\mathfrak{S}$.

Observe that the validity of $(A8')$ and $(A9')$ ensures the boundedness from below and the lower semicontinuity of the composition $\theta_{f(\bar{x})} \circ g$, respectively. Employing now the classical Ekeland's variational principle in Theorem 5.0.1 to the function $\theta_{f(\bar{x})} \circ g$ and its ε-minimal solution $\bar{x}$, we can find some $x_\varepsilon \in \mathfrak{S} \cap \operatorname{dom} g = \mathfrak{S} \cap \operatorname{dom} f$ such that

(a) $\theta_{f(\bar{x})}(g(x_\varepsilon)) \leq \theta_{f(\bar{x})}(g(\bar{x})) = 0$;

(b) $\|x_\varepsilon - \bar{x}\| \leq \sqrt{\varepsilon}$;

(c) $\theta_{f(\bar{x})}(g(x)) + \sqrt{\varepsilon}\|x - \bar{x}\| > \theta_{f(\bar{x})}(g(x_\varepsilon))$ for all $x \in \operatorname{dom} f \cap \mathfrak{S}$ and $x \neq x_\varepsilon$.

Next, we will show that x_ε satisfies also two major relations (i′) and (iii′) in the theorem. Arguing by contradiction, we assume that (i′) does not hold, i.e., x_ε is not a εk^0-minimal solution of (VVOP) with respect to the order structure B. By Definition 5.0.8 and $x_\varepsilon \notin \varepsilon k^0{-}\mathrm{M}(\mathfrak{S}, f, B)$, we get the existence of $x \in \mathfrak{S}$ such that

$$\begin{aligned}
& f(x) \in f(x_\varepsilon) - \varepsilon k^0 - B(f(x_\varepsilon)) \setminus \{\mathbf{0}\} \\
\iff\ & f(x) - f(\bar{x}) + \varepsilon k^0 \in (f(x_\varepsilon) - f(\bar{x})) - B(f(x_\varepsilon)) \setminus \{\mathbf{0}\} \\
\iff\ & g(x) + \varepsilon k^0 \in g(x_\varepsilon) - B(f(x_\varepsilon)) \setminus \{\mathbf{0}\} \overset{(\mathrm{A6}')}{\subset} g(x_\varepsilon) - \overline{B} \setminus \{\mathbf{0}\}.
\end{aligned}$$

By the strict $\overline{B}$-monotonicity of $\theta_{f(\bar{x})}$ we get from the last inclusion that

$$\theta_{f(\bar{x})}(g(x_\varepsilon)) > \theta_{f(\bar{x})}(g(x)+\varepsilon k^0) = \theta_{f(\bar{x})}(g(x))+\varepsilon \geq \inf_{u\in\mathfrak{S}\cap\operatorname{dom} f} \theta_{f(\bar{x})}(g(u))+\varepsilon \geq \theta_{f(\bar{x})}(g(\bar{x}))$$

where the last estimate (≥ 0) holds due to (5.7). The latter contradicts (a). This contradiction ensures that the validity of (i′) in the theorem.

It remains to show the fulfillment of condition (iii′). Arguing by contradiction, assume that x_ε is not a minimal solution of the perturbed function $f_{\varepsilon k^0} = f + \sqrt{\varepsilon}\|\cdot - x_\varepsilon\|k^0$ with respect to $B(\cdot)$, i.e., there is some $x \in \mathfrak{S}\cap\operatorname{dom} f = \mathfrak{S}\cap\operatorname{dom} g$ and $x \neq x_\varepsilon$ such that

$$\begin{aligned}
& f(x)+\sqrt{\varepsilon}\|x-x_\varepsilon\|k^0 \in f(x_\varepsilon)-B(f(x_\varepsilon)) \\
\iff\quad & f(x)-f(\bar{x})+\sqrt{\varepsilon}\|x-x_\varepsilon\|k^0 \in f(x_\varepsilon)-f(\bar{x})-B(f(x_\varepsilon)) \\
\iff\quad & g(x)+\sqrt{\varepsilon}\|x-x_\varepsilon\|k^0 \in g(x_\varepsilon)-B(f(x_\varepsilon))\setminus\{\mathbf{0}\} \overset{(A6')}{\subset} g(x_\varepsilon)-\overline{B}\setminus\{\mathbf{0}\}. \\
\implies\quad & \theta_{f(\bar{x})}\left(g(x)+\sqrt{\varepsilon}\|x-x_\varepsilon\|k^0\right) = \theta_{f(\bar{x})}(g(x))+\sqrt{\varepsilon}\|x-x_\varepsilon\| \leq \theta_{f(\bar{x})}(g(x_\varepsilon))
\end{aligned}$$

where the implication holds due to the $\overline{B}$-monotonicity of $\theta_{f(\bar{x})}$. The latter inequality contradicts (c). The contradiction justifies (iii′) and thus completes the proof of the theorem. □

The next result is another improved version of [63, Theorem 5.1] for the nonsolid case assuming that $(X, ||\cdot||)$ is a Banach space. In the proof of the next variational principle we will use Theorem 3.4 by Bao and Mordukhovich [4] such that we adapt our assumptions concerning boundedness as well as lower semicontinuity to this theorem. Furthermore, we suppose in the next results that X and Y are Banach spaces.

Definition 5.1.11. Consider the problem (VVOP). We say that $f : X \to Y$ is bounded from below over $\mathfrak{S}$ with respect to $\Theta \subset Y$ if there is a bounded set $M \subset Y$ such that $f(\mathfrak{S}) \subseteq M+\Theta$.

Remark 5.1.12. Of course, in the case of Banach spaces X and Y, the boundedness in the sense of Definition 5.1.11 is weaker than the boundedness in the sense of Definition 4.2.26. However, in Definition 5.1.11 the boundedness of the set M is supposed and we are dealing with Banach spaces. The boundedness in the sense of Definition 5.1.11 is used in [4] and called quasiboundedness there.

The following theorem gives a variational principle for approximate minimal solution of vector optimization problems with variable ordering structures; see [2].

Theorem 5.1.13. **(Variational principle for εk^0-minimal solutions, nonsolid case).** Suppose that X and Y are Banach spaces and consider (VVOP). Let $\bar{x} \in \varepsilon k^0$-M$(\mathfrak{S}, f, C)$. Set $\bar{y} := f(\bar{x})$ and $\overline{C} := C(\bar{y})$. Assume in addition to the standing assumption (A2) the following one holds:

(A3″) $\overline{C}$ is a proper, closed, convex and pointed cone.

(A7′) $C(f(x)) \subset \overline{C}$ for all $y \in \mathfrak{S}$ with $\|x-\bar{x}\| \leq \varepsilon$.

(A8″) f is $\overline{C}$-lower semicontinuous over $\mathfrak{S}$ in the sense of Definition 5.1.1.

(A9″) f is bounded from below over $\mathfrak{S}$ with respect to the cone $\overline{C}$ in the sense of Definition 5.1.11.

Then, there exists $x_\varepsilon \in \operatorname{dom} f \cap \mathfrak{S}$ such that

(i) $f(x_\varepsilon) \in f(\bar{x}) - C(f(\bar{x}))$, and thus $x_\varepsilon \in \varepsilon k^0$-M$(\mathfrak{S}, f, C)$.

(ii) $||\bar{x} - x_\varepsilon|| \leq \sqrt{\varepsilon}$.

(iii) $x_\varepsilon \in \operatorname{Min}(\mathfrak{S}, f_{\varepsilon k^0}, C)$, where $f_{\varepsilon k^0}(x) := f(x) + \sqrt{\varepsilon}||x_\varepsilon - x||k^0$.

Proof. By the $\overline{C}$-lower semicontinuity of f over $\mathfrak{S}$ in (A8″) and the continuity of the norm, the function $f(\cdot) + \sqrt{\varepsilon}\|\bar{x} - \cdot\|k^0$ is $\overline{C}$-lower semicontinuous over Ω and thus the $\bar{y}$-level-set of $f(\cdot) + \sqrt{\varepsilon}\|\bar{x} - \cdot\|k^0$ with respect to the ordering cone $\overline{C}$ denoted by

$$\Xi := \operatorname{lev}(\bar{y}; f) = \left\{x \in \mathfrak{S} \mid f(x) + \sqrt{\varepsilon}\|\bar{x} - x\|k^0 \in \bar{y} - \overline{C}\right\}$$

is a closed set in X. Observe that the restriction f_Ξ of f on Ξ with $\operatorname{dom} f_\Xi = \Xi$ satisfies all the assumptions of the vector version of Ekeland's variational principle in vector optimization with ordering cone/ fixed ordering structure; see, e.g., [4, Theorem 3.4]. Observe also that $\bar{x}$ is an εk^0-minimal solution of f_Ξ with respect to the closed, convex and pointed cone $\overline{C}$, i.e.,

$$f_\Xi(x) \notin f_\Xi(\bar{x}) - \varepsilon k^0 - \overline{C} \backslash \{\mathbf{0}\}, \ \forall x \in \Xi.$$

Employing [4, Theorem 3.4] to the function f_Ξ, its εk^0-minimal solution $\bar{x}$, $\overline{C} = C(f(\bar{x}))$, k^0, ε, and $\lambda = \sqrt{\varepsilon}$, we can find some $x_\varepsilon \in \Xi$ with $\|\bar{x} - x_\varepsilon\| \leq \sqrt{\varepsilon}$ such that

$$f(x) + \sqrt{\varepsilon}\|x_\varepsilon - x\|k^0 \notin f(x_\varepsilon) - C(f(\bar{x})), \ \forall\, x \in \Xi \setminus \{x_\varepsilon\}. \tag{5.8}$$

Obviously, (ii) is satisfied. (i) follows directly from $x_\varepsilon \in \Xi$ as follows:

$$\begin{aligned} x_\varepsilon \in \Xi \quad &\Longleftrightarrow \quad f(x_\varepsilon) + \sqrt{\varepsilon}\|\bar{x} - x_\varepsilon\|k^0 \in \bar{y} - \overline{C} && (5.9)\\ &\Longleftrightarrow \quad f(x_\varepsilon) \in f(\bar{x}) - \left(\sqrt{\varepsilon}\|\bar{x} - x_\varepsilon\|k^0 + \overline{C}\right) \\ &\overset{(A2)}{\Longrightarrow} \quad f(x_\varepsilon) \in f(\bar{x}) - \overline{C} = f(\bar{x}) - C(f(\bar{x})). && (5.10) \end{aligned}$$

Obviously, (5.10) verifies the first part of (i). To justify the second part of (i), we use (5.10), the εk^0-minimality of $\bar{x}$, the inclusion $C(f(x_\varepsilon)) \subset \overline{C}$ by assumption (A7′), and the convexity of the cone $\overline{C}$ in (A3$''$) ensuring that $\overline{C}+\overline{C}\setminus\{\mathbf{0}\}\subset\overline{C}\setminus\{\mathbf{0}\}$. Details below.

$$\begin{aligned}
\bar{x}\in\varepsilon k^0\text{-M}(\mathfrak{S},f,C) &\iff \big(f(\bar{x})-\varepsilon k^0-(\overline{C}\setminus\{\mathbf{0}\})\big)\cap f(\mathfrak{S})=\emptyset\\
&\iff \big(f(\bar{x})-\overline{C}-\varepsilon k^0-\overline{C}\setminus\{\mathbf{0}\}\big)\cap f(\mathfrak{S})=\emptyset\\
&\overset{(5.10)}{\Longrightarrow} \big(f(x_\varepsilon)-\varepsilon k^0-C(f(x_\varepsilon))\setminus\{\mathbf{0}\}\big)\cap f(\mathfrak{S})=\emptyset\\
&\iff x_\varepsilon\in\varepsilon k^0\text{-M}(\mathfrak{S},f,C).
\end{aligned}$$

Finally, we will justify (iii) by contradiction. Assume that it does not hold, and then find some $x\in\mathfrak{S}\cap\operatorname{dom}f$ with $x\neq x_\varepsilon$ such that $f(x)+\sqrt{\varepsilon}\|x_\varepsilon-x\|k^0\in f(x_\varepsilon)-C(f(x_\varepsilon))$. By (A7$'$), we get

$$f(x)+\sqrt{\varepsilon}\|x_\varepsilon-x\|k^0\in f(x_\varepsilon)-C(f(\bar{x}))=f(x_\varepsilon)-\overline{C}. \tag{5.11}$$

Using (5.8) this implies $x\notin\Xi$. Summing up this inclusion (5.11) and the one in (5.9) gives

$$f(x)+\sqrt{\varepsilon}\big(\|x_\varepsilon-x\|+\|\bar{x}-x_\varepsilon\|\big)k^0\in f(\bar{x})-\overline{C}-\overline{C}=f(\bar{x})-\overline{C}, \tag{5.12}$$

where $\overline{C}+\overline{C}=\overline{C}$ holds due to the convexity of the cone $\overline{C}$ in (A3$''$).

Since $\|x_\varepsilon-x\|+\|\bar{x}-x_\varepsilon\|-\|\bar{x}-x\|\geq 0$ by the triangle inequality of the norm, we will further manipulate (5.12) as follows:

$$\begin{aligned}
& f(x)+\sqrt{\varepsilon}\|\bar{x}-x\|k^0\\
\in\;& f(\bar{x})-\sqrt{\varepsilon}(\|x_\varepsilon-x\|+\|\bar{x}-x_\varepsilon\|-\|\bar{x}-x\|)k^0-\overline{C}\\
\subset\;& f(\bar{x})-\overline{C},
\end{aligned}$$

where the inclusion holds due to (A2). By the construction of Ξ, we have $x\in\Xi$ and arrive at a contradition. This contradiction verifies the validity of (iii) and thus completes the proof of the theorem. □

5.2 Variational Principle for Approximately Nondominated Solutions

In this section, we give an extension of Ekeland's theorem for εk^0-nondominated solutions of vector optimization problems with variable ordering structures, where the εk^0-nondominatedness for solutions of (VVOP) is defined in Definition 5.0.6. It is important to emphasize that there is no difference between εk^0-nondominated and εk^0-minimal solutions in the case of fixed ordering structure. The reader can find many examples illustrating that this statement is in general not true in the case of variable ordering structure in [22, 24, 67].

Lemma 5.2.1. Suppose that assumptions (A2)–(A3) hold and let $B : Y \rightrightarrows Y$ be a cone-valued map satisfying assumptions (A5)–(A6). Consider the problem (VVOP). If $x_\varepsilon \in \varepsilon k^0$-N$(\mathfrak{S}, f, C)$, then for every element $x \in \mathfrak{S}$, there exists a continuous functional $\theta_{f(x)} : Y \to \mathbb{R}$ which is strictly B-monotone in the sense of Definition 4.2.23 and

$$\forall x \in \mathfrak{S}, \qquad \theta_{f(x)}(f(x_\varepsilon)) \leqq \theta_{f(x)}(f(x) + \varepsilon k^0).$$

Moreover, if (A4) holds, then each $\theta_{f(x)}$ is subadditive on Y and

$$\forall x \in \mathfrak{S}, \qquad \theta_{f(x)}(f(x_\varepsilon)) \leqq \theta_{f(x)}(f(x)) + \theta_{f(x)}(\varepsilon k^0).$$

Proof. Suppose that $k^0 \in Y \backslash \{\mathbf{0}\}$, $\varepsilon > 0$ and $x_\varepsilon \in \varepsilon k^0$-N$(\mathfrak{S}, f, C)$. This means that for all $x \in \mathfrak{S}$,

$$(f(x_\varepsilon) - \varepsilon k^0 - C(f(x)) \backslash \{\mathbf{0}\}) \cap f(x) = \emptyset,$$

and

$$(f(x_\varepsilon) - C(f(x)) \backslash \{\mathbf{0}\}) \cap (f(x) + \varepsilon k^0) = \emptyset.$$

We consider $D(f(x)) := (f(x_\varepsilon) - C(f(x)) \backslash \{\mathbf{0}\})$ and $f(x) + \varepsilon k^0 = U$. Taking into account (A5), (A6) and Theorem 4.2.25, we get desired functionals and for all $x \in \mathfrak{S}$, the continuous functional $\theta_{f(x)} : Y \to \mathbb{R}$ satisfies $\theta_{f(x)}(f(x_\varepsilon)) \leqq \theta_{f(x)}(f(x) + \varepsilon k^0)$. Now if (A4) holds, then for all $x \in \mathfrak{S}$, $\theta_{f(x)}$ is subadditive and

$$\theta_{f(x)}(f(x_\varepsilon)) \leqq \theta_{f(x)}(f(x)) + \theta_{f(x)}(\varepsilon k^0)$$

and proof is complete. □

The following lemma gives some properties of the functional in Lemma 5.2.1 and these properties will be used later in the proof of other lemmas and our main theorem about extension of Ekeland's theorem for εk^0-nondominated solutions of vector optimization problems with variable ordering structures.

Lemma 5.2.2. Let assumptions (A2)–(A3) and (A5)–(A6) be fulfilled, then for all $x \in \mathfrak{S}$, we can choose the functional $\theta_{f(x)} : Y \to \mathbb{R}$ in Lemma 5.2.1 in a way such that:

1. $\theta_{f(x)}(k^0) = 1$.
2. $\theta_{f(x)}(\mathbf{0}) = 0$.
3. $\theta_{f(x)}(\varepsilon k^0) = \varepsilon$ and $\theta_{f(x)}(-\varepsilon k^0) = -\theta_{f(x)}(\varepsilon k^0) = -\varepsilon$.

Proof. The proof is similar to that of Lemma 5.1.6. □

Lemma 5.2.3. Let X be a real Banach space, $\mathfrak{S} \subset X$, $x_\varepsilon \in \mathfrak{S}$, Y be a topological linear space, $\varepsilon \geq 0$, $k^0 \in Y \backslash \{\mathbf{0}\}$, $f : X \to Y$ is a vector-valued function with $\operatorname{dom} f \neq \emptyset$ and $B : Y \rightrightarrows Y$ be a cone-valued map satisfying (A5).

(j) Furthermore, suppose that for any $x \in \mathfrak{S}$ and for all strictly B-monotone (in the sense of Definition 4.2.23), continuous, subadditive functionals $\theta_{f(x)} : Y \to \mathbb{R}$ the following inequality holds

$$\forall x \in \mathfrak{S}, \qquad \theta_{f(x)}(f(x_\varepsilon)) \leqq \theta_{f(x)}(f(x)) - \theta_{f(x)}(-\varepsilon k^0),$$

then $x_\varepsilon \in \varepsilon k^0$-WN$(\mathfrak{S}, f, C)$ for some set-valued map $C : Y \rightrightarrows Y$ such that $B(f(x))\backslash\{\mathbf{0}\} \subseteq C(f(x))$, $\mathbf{0} \in \operatorname{cl} C(f(x))\backslash C(f(x))$, $\operatorname{cl} C(f(x)) + (B(f(x))\backslash\{\mathbf{0}\}) \subseteq C(f(x))$ for all $x \in \mathfrak{S}$.

Proof. For each $x \in \mathfrak{S}$, we define $C(f(x))$ as following,

$$C(f(x)) = \{y \in Y \mid \theta_{f(x)}(-y + f(x_\varepsilon) - \varepsilon k^0) < \theta_{f(x)}(f(x_\varepsilon) - \varepsilon k^0)\}, \tag{5.13}$$

and a functional $\hat{\theta}_{f(x)}(y) : Y \to \mathbb{R}$ with

$$\hat{\theta}_{f(x)}(y) := \theta_{f(x)}(y + f(x_\varepsilon) - \varepsilon k^0). \tag{5.14}$$

By (5.14) and (j) and since $\theta_{f(x)}$ is subadditive for all $x \in \mathfrak{S}$, we get

$$\begin{aligned}
&\hat{\theta}_{f(x)}(f(x) + \varepsilon k^0 - f(x_\varepsilon)) = \theta_{f(x)}(f(x)) \geqq \\
&\theta_{f(x)}(f(x_\varepsilon)) + \theta_{f(x)}(-\varepsilon k^0) \geqq \\
&\theta_{f(x)}(f(x_\varepsilon) - \varepsilon k^0) = \hat{\theta}_{f(x)}(\mathbf{0}).
\end{aligned}$$

Now by (5.13) and (5.14), we can write

$$\hat{\theta}_{f(x)}(-C(f(x))) = \theta_{f(x)}(-C(f(x)) + f(x_\varepsilon) - \varepsilon k^0) < \theta_{f(x)}(f(x_\varepsilon) - \varepsilon k^0) = \hat{\theta}_{f(x)}(\mathbf{0}),$$

and therefore for each $x \in \mathfrak{S}$,

$$(-\operatorname{int} C(f(x))) \cap (f(x) + \varepsilon k^0 - f(x_\varepsilon)) = \emptyset \implies (f(x_\varepsilon) - \varepsilon k^0 - \operatorname{int} C(f(x))) \cap f(x) = \emptyset.$$

Since $\theta_{f(x)}$ is a strictly B-monotone functional for any $x \in \mathfrak{S}$, then $B(f(x))\backslash\{\mathbf{0}\} \subseteq C(f(x))$ for all $x \in \mathfrak{S}$. Now we show that $\operatorname{cl} C(f(x)) + (B(f(x))\backslash\{\mathbf{0}\}) \subseteq C(f(x))$. Choose $y \in \operatorname{cl} C(f(x))$ and $b \in y + B(f(x))\backslash\{\mathbf{0}\}$. Since $y \in \operatorname{cl} C(f(x)) \subseteq \{y \mid \hat{\theta}_{f(x)}(-y) \leqq \hat{\theta}_{f(x)}(0)\}$ and $\hat{\theta}_{f(x)}$ is strictly B-monotone, we have

$$\hat{\theta}_{f(x)}(-b) < \hat{\theta}_{f(x)}(-y) \leqq \hat{\theta}_{f(x)}(\mathbf{0}).$$

Therefore $b \in \operatorname{cl} C(f(x)) + (B(f(x))\backslash\{\mathbf{0}\})$ implies $b \in C(f(x))$. Now by the assumption (A5) and $\operatorname{cl} C(f(x)) + (B(f(x))\backslash\{\mathbf{0}\}) \subseteq C(f(x))$, we get $C(f(x)) + \varepsilon k^0 \subseteq C(f(x))$. By $\mathbf{0} \in \operatorname{cl}(B(f(x))\backslash\{\mathbf{0}\})$, $B(f(x))\backslash\{\mathbf{0}\} \subseteq C(f(x))$ and $\mathbf{0} \notin C(f(x))$, we get $\mathbf{0} \in \operatorname{cl} C(f(x))\backslash C(f(x))$. □

In Definition 4.2.26, we defined bounded from below function over $\mathfrak{S}$ with respect to a set. Now we generalize this definition to bounded from below function over $\mathfrak{S}$ with respect to a set-valued map.

Definition 5.2.4. Let X,Y be Banach spaces and $C : Y \rightrightarrows Y$ be a set-valued map. We say that $f : X \to Y$ is bounded from below over $\mathfrak{S}$ with respect to the set-valued map C if for any $y \in f(\mathfrak{S})$ there exists y^0 such that $f(\mathfrak{S}) \subseteq y^0 + C(y)$.

Lemma 5.2.5. Let assumptions (A2) and (A3) be fulfilled and $B : Y \rightrightarrows Y$ be a cone-valued map satisfying (A5) and (A6). Suppose $f : X \to Y$ is bounded from below over $\mathfrak{S}$ with respect to C in the sense of Definition 5.2.4, then $\theta_y \circ f$ is bounded below for all $y \in \mathfrak{S}$.

Proof. Proof is similar to that of Lemma 4.2.27. □

The following theorem gives the first generalization of the Ekeland's variational principle for εk^0-nondominated solutions of (VVOP) provided that $f : X \to Y$ is bounded from below and $(k^0, C, \mathfrak{S})$-lower semicontinuous (see [64]).

Theorem 5.2.6. Consider the problem (VVOP) and let $\bar{x} \in \varepsilon k^0\text{-N}(\mathfrak{S}, f, C)$. Impose in addition to (A2)–(A6) the following assumptions:

$(A8)$ f is $(k^0, C, \mathfrak{S})$-lower semicontinuous over $\mathfrak{S}$ in the sense of Definition 5.1.3.

$(A9''')$ f is bounded from below over $\mathfrak{S}$ with respect to C in the sense of Definition 5.2.4.

Then there exists an element $x_\varepsilon \in \operatorname{dom} f \cap \mathfrak{S}$ such that

1. $x_\varepsilon \in \varepsilon k^0\text{-WN}(\mathfrak{S}, f, B)$,
2. $\|\bar{x} - x_\varepsilon\| \leqq \sqrt{\varepsilon}$,
3. $x_\varepsilon \in \mathrm{WN}(\mathfrak{S}, f_{\varepsilon k^0}, B)$ with $f_{\varepsilon k^0}(x) = f(x) + \sqrt{\varepsilon}\,\|x - x_\varepsilon\|\, k^0.$ (5.15)

Proof. Suppose that $\bar{x} \in \varepsilon k^0\text{-N}(\mathfrak{S}, f, C)$, then by the definition of approximately nondominated solutions (Definition 5.0.6), we have $(f(\bar{x}) - \varepsilon k^0 - C(f(x)) \backslash \{\mathbf{0}\}) \cap f(x) = \emptyset$. Now suppose that $\overline{f} := f - f(\bar{x})$, then we have

$$(\overline{f}(\bar{x}) - \varepsilon k^0 - C(f(x)) \backslash \{\mathbf{0}\}) \cap \overline{f}(x) = \emptyset.$$

By (A4), Lemma 5.2.1 and Lemma 5.2.2, the functional $\theta_{f(x)} : Y \to \mathbb{R}$ defined by (4.2) is strictly B-monotone, continuous and subadditive for all $x \in \mathfrak{S}$. Furthermore,

$$\forall\, x \in \mathfrak{S}, \quad \theta_{f(x)}(\overline{f}(\bar{x})) \leqq \theta_{f(x)}(\overline{f}(x)) + \theta_{f(x)}(\varepsilon k^0) = \theta_{f(x)}(\overline{f}(x)) + \varepsilon.$$

This means that for all $x \in \mathfrak{S}$,

$$\theta_{f(x)}(\overline{f}(\bar{x})) \leqq \inf_{x \in \mathfrak{S}} \theta_{f(x)}(\overline{f}(x)) + \varepsilon, \qquad \varepsilon > 0.$$

Observe that the validity of (A8)–(A9$'''$) ensures the boundedness from below and $(k^0, C, \mathfrak{S})$-lower semicontinuity of f and $\overline{f}$. By Lemma 5.1.4, Lemma 5.2.5, Theorem 5.0.1 and Remark 5.0.2, there exists $x_\varepsilon \in \mathfrak{S}$ such that for all $x \in \mathfrak{S}$,

1. $\theta_{f(x)}(\overline{f}(x_\varepsilon)) \leqq \theta_{f(x)}(\overline{f}(\overline{x})) \leqq \inf_{x\in\mathfrak{S}} \theta_{f(x)}(\overline{f}(x)) + \varepsilon,$ (5.16)

2. $\|x_\varepsilon - \overline{x}\| \leqq \sqrt{\varepsilon},$

3. for all $x \in \mathfrak{S}$, $\theta_{f(x)}(\overline{f}(x_\varepsilon)) \leqq \theta_{f(x)}(\overline{f}(x)) + \sqrt{\varepsilon}\,\|x - x_\varepsilon\|.$ (5.17)

By Lemma 5.2.2 and (5.16), for all $x \in \mathfrak{S}$, we get

$$\theta_{f(x)}(\overline{f}(x_\varepsilon)) \leqq \inf_{x\in\mathfrak{S}} \theta_{f(x)}(\overline{f}(x)) + \varepsilon \leqq \theta_{f(x)}(\overline{f}(x)) + \theta_{f(x)}(\varepsilon k^0) = \theta_{f(x)}(\overline{f}(x)) - \theta_{f(x)}(-\varepsilon k^0).$$

Now by Lemma 5.2.3, the inclusion $B(f(x)) \subseteq C(f(x))$ by assumption (A6) and $\overline{f} = f - f(\overline{x})$,

$$\forall x \in \mathfrak{S}, \qquad (\overline{f}(x_\varepsilon) - \varepsilon k^0 - \operatorname{int} B(f(x))) \cap \{\overline{f}(x)\} = \emptyset.$$

This implies that $x_\varepsilon \in \varepsilon k^0$-WN$(\mathfrak{S}, f, B)$. Now we prove (5.15) and for this, suppose that there exists an element $x \in \mathfrak{S}$ such that

$$\begin{aligned} & f(x) \in f(x_\varepsilon) - \sqrt{\varepsilon}\,\|x - x_\varepsilon\|\,k^0 - \operatorname{int} B(f(x)) \\ \Longrightarrow \quad & \overline{f}(x) \in \overline{f}(x_\varepsilon) - \sqrt{\varepsilon}\,\|x - x_\varepsilon\|\,k^0 - \operatorname{int} B(f(x)). \end{aligned}$$

Since for all $x \in \mathfrak{S}$, $\theta_{f(x)}$ is a strictly B-monotone continuous subadditive functional, for all $x \in \mathfrak{S}$, we can write

$$\theta_{f(x)}(\overline{f}(x)) < \theta_{f(x)}(\overline{f}(x_\varepsilon) - \sqrt{\varepsilon}\,\|x - x_\varepsilon\|\,k^0) \leqq \theta_{f(x)}(\overline{f}(x_\varepsilon)) + \theta_{f(x)}(-\sqrt{\varepsilon}\,\|x - x_\varepsilon\|\,k^0).$$

Now by Lemma 5.2.2, we get

$$\theta_{f(x)}(-\sqrt{\varepsilon}\,\|x - x_\varepsilon\|\,k^0) = -\sqrt{\varepsilon}\,\|x - x_\varepsilon\| \implies \theta_{f(x)}(\overline{f}(x_\varepsilon)) > \theta_{f(x)}(\overline{f}(x)) + \sqrt{\varepsilon}\,\|x - x_\varepsilon\|,$$

but this yields a contradiction because of (5.17). □

In the special case that $C : Y \rightrightarrows Y$ is a solid, closed, pointed and convex cone-valued map, we have the following corollary.

Corollary 5.2.7. Let $C : Y \rightrightarrows Y$ be a cone-valued map where $C(f(x))$ is a solid convex cone for all $x \in \mathfrak{S}$, $k^0 \in \bigcap_{x\in\mathfrak{S}} \operatorname{int} C(f(x))$ and $\varepsilon > 0$. Consider the problem (VVOP) and furthermore, let $\overline{x} \in \varepsilon k^0$-N$(\mathfrak{S}, f, C)$. Impose the following assumptions:

(A8) f is $(k^0, C, \mathfrak{S})$-lower semicontinuous over $\mathfrak{S}$ in the sense of Definition 5.1.3.

(A9‴) f is bounded from below over $\mathfrak{S}$ with respect to C in the sense of Definition 5.2.4.

Then there exists an element $x_\varepsilon \in \operatorname{dom} f \cap \mathfrak{S}$ such that

1. $x_\varepsilon \in \varepsilon k^0$-WN$(\mathfrak{S}, f, C)$,

2. $\|\bar{x} - x_\varepsilon\| \leqq \sqrt{\varepsilon}$,

3. $x_\varepsilon \in \mathrm{WN}(\mathfrak{S}, f_{\varepsilon k^0}, C)$ with $f_{\varepsilon k^0}(x) = f(x) + \sqrt{\varepsilon}\,\|x - x_\varepsilon\|\, k^0$.

In Theorem 5.2.6, we gave an extension of Ekeland's variational principle for εk^0-nondominated solutions of (VVOP) where for each $y \in f(\mathfrak{S})$, we have a different functional. Now, using the functional $\varphi_{\bar{y},C,k^0}$ defined by (5.18), we give another extension of Ekeland's theorem for approximately nondominated solutions of vector optimization problems with variable ordering structures. In order to prove the next main theorem, we use the functional $\varphi_{\bar{y},C,k^0} \colon Y \to \overline{\mathbb{R}}$ to some $\bar{y} \in Y$ and some $k^0 \in Y$ defined by

$$\varphi_{\bar{y},C,k^0}(y) = \inf\{t \in \mathbb{R} \mid \bar{y} + tk^0 - y \in C(y)\} \text{ for all } y \in Y \tag{5.18}$$

which is a slightly modification of the functional φ defined by (4.3). This functional was studied in [25]; see also [22] for characterizing nondominated elements with respect to a variable ordering structure which is defined by a cone-valued map. A generalization of this scalarization was studied already by Chen and Yang [15] and later also by Chen and colleagues [12, 16]. The following lemma is proven in [24, Theorem 5.11].

Lemma 5.2.8. Let (A2) and (A3) hold and let $C : Y \rightrightarrows Y$ be a set-valued map where $C(y)$ is closed for each $y \in Y$ satisfying the following condition for some $k^0 \in Y \setminus \{\mathbf{0}\}$:

(C1) $(-\infty, 0)k^0 \cap C(y) = \emptyset$ and $\mathbf{0} \in \mathrm{bd}\, C(y)$ for all $y \in f(\mathfrak{S})$.

We consider the functional $\varphi_{\bar{y},C,k^0} \colon Y \to \mathbb{R}$ defined in (5.18) for some $\bar{y} \in f(\mathfrak{S})$. Then the following hold:

(a) Under condition (C1), one has
$$\varphi_{\bar{y},C,k^0}(\bar{y}) = 0$$

(b) Let $\bar{x} \in \mathfrak{S}$ and $\bar{y} = f(\bar{x})$. Then $\bar{x} \in \mathrm{WN}(\mathfrak{S}, f, C)$ if and only if
$$\inf_{y \in f(\mathfrak{S})} \varphi_{\bar{y},C,k^0}(y) = 0.$$

(c) Let $\varepsilon \geq 0$, $\bar{x} \in \mathfrak{S}$, and $\bar{y} = f(\bar{x})$. Then $\bar{x} \in \varepsilon k^0$-WN$(\mathfrak{S}, f, C)$ if and only if
$$\inf_{y \in f(\mathfrak{S})} \varphi_{\bar{y},C,k^0}(y) \geq -\varepsilon.$$

Proof. We set $\varphi(y) := \varphi_{\bar{y},C,k^0}(y)$ for all $y \in f(\mathfrak{S})$. As (b) follows from (c) for $\varepsilon = 0$, we prove only (a) and (c).

(a) We have $\varphi(\bar{y}) = \inf\{t \in \mathbb{R} \mid tk^0 \in C(\bar{y})\}$. By $\mathbf{0} \in \mathrm{bd}\, C(y)$ for all $y \in f(\mathfrak{S})$ and assumption $(-\infty, 0)k^0 \cap C(y) = \emptyset$ we get $\varphi(\bar{y}) = 0$.

(c) Assume $\varphi(y) \geq -\varepsilon$ for all $y \in f(\mathfrak{S})$ but $\bar{x} \notin \varepsilon k^0$-WN$(\mathfrak{S}, f, C)$. Then there exists $y \in f(\mathfrak{S})$ with $\bar{y} - \varepsilon k^0 - y \in \operatorname{int} C(y)$. Thus there is a scalar $t < 0$ such that

$$(\bar{y} - y) + (t - \varepsilon) k^0 \in C(y),$$

i. e. $\bar{y} + (t - \varepsilon) k^0 - y \in C(y)$ and hence $\varphi(y) \leq t - \varepsilon < -\varepsilon$, which is a contradiction.

Next, let $\bar{x} \in \varepsilon k^0$-WN$(\mathfrak{S}, f, C)$ but assume the existence of $t \in \mathbb{R}$, $t < -\varepsilon$ and $y \in f(\mathfrak{S})$ such that

$$\bar{y} + t k^0 - y \in C(y).$$

As $C(y) + (-t - \varepsilon) k^0 \subseteq \operatorname{int} C(y)$ by (A3), we have

$$\bar{y} - \varepsilon k^0 \in y + C(y) + (-t - \varepsilon) k^0 \subseteq y + \operatorname{int} C(y)$$

in contradiction to the weakly εk^0-nondominatedness of $\bar{x}$ to the problem (VVOP). □

Lemma 5.2.9. Consider problem (VVOP), $\bar{x} \in \mathfrak{S}$, $\bar{y} = f(\bar{x})$, and the scalarization functional $\varphi_{\bar{y},C,k^0}$ defined by (5.18). Assume that the ordering structure $C : Y \rightrightarrows Y$ satisfies condition (C2):

(C2) C has a closed graph over $f(\mathfrak{S})$ in the sense that for every sequence of pairs $\{(y_n, v_n)\}$, if $y_n \in f(\mathfrak{S})$ and $v_n \in C(y_n)$ for all $n \in \mathbb{N}$ and $(y_n, v_n) \to (y_*, v_*)$ as $n \to +\infty$, then $y_* \in f(\mathfrak{S})$ and $v_* \in C(y_*)$.

Then if f is a continuous function over $\mathfrak{S}$, the composition $(\varphi_{\bar{y},C,k^0} \circ f)(\cdot) = \varphi_{\bar{y},C,k^0}(f(\cdot))$ is a lower semicontinuous functional over $\mathfrak{S}$.

Proof. Assume that f is a continuous function over $\mathfrak{S}$. To prove the lowersemicontinuity of $\varphi_{\bar{y},C,k^0} \circ f$ over $\mathfrak{S}$, it is sufficient to show that the set

$$A := \operatorname{lev}(t; \varphi_{\bar{y},C,k^0} \circ f) = \left\{x \in \mathfrak{S} \mid \varphi_{\bar{y},C,k^0}(f(x)) \leq t\right\}$$

is closed in X for all $t \in \mathbb{R}$. Fix $t \in \mathbb{R}$ arbitrarily and take any sequence $\{x_n\}$ in A such that $x_n \to x_*$ as $n \to +\infty$. By the description of A, we have $\varphi_{\bar{y},C,k^0}(f(x_n)) \leq t$ and thus

$$\bar{y} + t k^0 - f(x_n) \in C(f(x_n)).$$

Since f is continuous over the set $\mathfrak{S}$, the sequence of pairs $(y_n, v_n) \in \operatorname{gph} C$ with $y_n := f(x_n)$ and $v_n := \bar{y} + t k^0 - f(x_n)$ converges to $(f(x_*), \bar{y} + t k^0 - f(x_*))$. By (C2), we have

$$\bar{y} + t k^0 - f(x_*) \in C(f(x_*))$$

and thus $\varphi_{\bar{y},C,k^0}(f(x_*)) \leq t$ by the definition of $\varphi_{\bar{y},C,k^0}$ in (5.18). The last inequality justifies $x_* \in A$ and thus the closedness of the set A. The proof is complete. □

Lemma 5.2.10. Suppose (A2), (A3). Consider problem (VVOP), $\bar{y} \in Y$, and the scalarization functional $\varphi_{\bar{y},C,k^0}$ defined by (5.18). Assume that the ordering structure $C : Y \rightrightarrows Y$ satisfies conditions (C1) and $(\widehat{A9})$:

$(\widehat{A9})$ f is bounded from below over $\mathfrak{S}$ with respect to $\underline{y} \in Y$ and $\Theta = C(\underline{y})$ in the sense of Definition 4.2.26.

Furthermore, let the ordering structure $C : Y \rightrightarrows Y$ satisfies for $\underline{y}$ from assumption $(\widehat{A9})$:

(C3) $C(y) + C(\underline{y}) \subset C(\underline{y})$ for all $y \in f(\mathfrak{S})$. Furthermore, suppose that there exists a cone D with $k^0 \in \operatorname{int} D$ and $C(\underline{y}) + \operatorname{int} D \subset C(\underline{y})$.

Then the functional $\varphi_{\bar{y},C,k^0} \circ f$ is bounded from below over $\mathfrak{S}$.

Proof. Consider the element $\underline{y}$ given by assumption $(\widehat{A9})$. Taking into account assumption (C3) and [33, Theorem 2.3.1 and Theorem 2.3.4], there exists $\underline{t} \in \mathbb{R}$ such that

$$\bar{y} + \underline{t}k^0 - \underline{y} \notin C(\underline{y}). \tag{5.19}$$

Assume now that f is bounded from below over $\mathfrak{S}$ by $\underline{y}$ with respect to $C(\underline{y})$, but $\varphi_{\bar{y},C,k^0} \circ f$ is not bounded from below over $\mathfrak{S}$. The former ensures that $-\underline{y} \in -f(x) + C(\underline{y})$. The latter allows us to find some $x \in \mathfrak{S}$ such that $\varphi_{\bar{y},C,k^0}(f(x)) < \underline{t}$. By (5.18) and (A3), we have

$$\bar{y} + \underline{t}k^0 - f(x) \in C(f(x)).$$

Combining the last two inclusions while taking into account (C3), we have

$$\bar{y} + \underline{t}k^0 - \underline{y} \in C(f(x)) + C(\underline{y}) \subset C(\underline{y})$$

which contradicts (5.19). The contradiction clearly verifies the lower boundedness from below of $\varphi_{\bar{y},C,k^0}$ over $\mathfrak{S}$ and completes the proof. □

We are now ready to present an extension of Ekeland's theorem for εk^0-nondominated solutions of vector optimization problems with variable ordering structures.

Theorem 5.2.11. Consider (VVOP) and let $\bar{x} \in \varepsilon k^0$-N$(\mathfrak{S}, f, C)$, $\underline{y} \in Y$, and set $\bar{y} := f(\bar{x})$. Impose, in addition to (A2)–(A3), (C1)–(C3) and $(\widehat{A9})$ in Lemmata 5.2.8, 5.2.9 and 5.2.10, the following condition hold:

(A10) f is continuous over $\mathfrak{S}$.

Then, there exists a point $x_\varepsilon \in \operatorname{dom} f \cap \mathfrak{S}$ such that

(i) $\varphi_{\bar{y},C,k^0}(f(x_\varepsilon)) \le \inf_{x \in \mathfrak{S}} \varphi_{\bar{y},C,k^0}(f(x)) + \varepsilon$,

(ii) $\|\bar{x}-x_\varepsilon\| \leq \sqrt{\varepsilon}$,

(iii) $x_\varepsilon \in \mathfrak{S}$ is an exact solution of the scalar problem

$$\min_{x\in\mathfrak{S}} \ \varphi_{\bar{y},C,k^0}(f(x)) + \sqrt{\varepsilon}||x-x_\varepsilon||,$$

(iv) $\varphi_{\bar{y},C,k^0}(f(x_\varepsilon)) + \sqrt{\varepsilon}\,\|\bar{x}-x_\varepsilon\| \leq \varphi_{\bar{y},C,k^0}(f(\bar{x}))$.

Proof. Consider $\bar{x} \in \varepsilon k^0$-N$(\mathfrak{S}, f, C)$, $\bar{y} := f(\bar{x})$, and the functional $\varphi_{\bar{y},C,k^0}$ defined in (5.18). By (A3), we have $\operatorname{int} C(f(\bar{x})) \neq \emptyset$, and thus $\bar{x} \in \varepsilon k^0$-WN$(\mathfrak{S}, f, C)$. By Lemma 5.2.8, we get under the imposed conditions (C1) and (A3) that

$$\varphi_{\bar{y},C,k^0}(\bar{y}) \leq \varphi_{\bar{y},C,k^0}(\bar{y}) + \varepsilon = \inf_{y\in f(\mathfrak{S})} \varphi_{\bar{y},C,k^0}(y) + \varepsilon,$$

i.e., $\bar{y}$ is an ε-minimal solution of $\varphi_{\bar{y},C,k^0} \circ f$ over $\mathfrak{S}$. Under the assumptions made in the theorem, the functional $\varphi_{\bar{y},C,k^0} \circ f$ is lower semicontinuous and bounded from below on $\mathfrak{S}$ because of Lemmas 5.2.9 and 5.2.10. This means that all the assumptions of Theorem 5.0.1 are fulfilled. Therefore, we get from Theorem 5.0.1 and Remark 5.0.2, the existence of $x_\varepsilon \in \mathfrak{S}$ such that

(i) $\varphi_{\bar{y},C,k^0}(f(x_\varepsilon)) \leq \varphi_{\bar{y},C,k^0}(f(\bar{x})) \leq \inf_{x\in\mathfrak{S}} \varphi_{\bar{y},C,k^0}(f(x)) + \varepsilon$.

(ii) $\|x_\varepsilon - \bar{x}\| \leq \sqrt{\varepsilon}$.

(iii) $\varphi_{\bar{y},C,k^0}(f(x)) + \sqrt{\varepsilon}\,\|x-x_\varepsilon\| > \varphi_{\bar{y},C,k^0}(f(x_\varepsilon))$ for all $x \in \mathfrak{S}$ and $x \neq x_\varepsilon$.

(iv) $\varphi_{\bar{y},C,k^0}(f(x_\varepsilon)) + \sqrt{\varepsilon}\,\|\bar{x}-x_\varepsilon\| \leq \varphi_{\bar{y},C,k^0}(f(\bar{x}))$.

The proof is complete. □

Remark 5.2.12. In Theorem 5.2.11 the assumptions imposed on the set-valued mapping C are weaker than the assumptions in the variational principles for εk^0-minimal elements in the previous section, however, the assertions in Theorem 5.2.6 are weaker too.

5.3 Variational Principle for Approximate Minimizers

In this section, we give an extension of Ekeland's theorem for εk^0-minimizers of vector optimization problems with variable ordering structures. It is important to emphasize that there is no difference between εk^0-minimizers, εk^0-nondominated and εk^0-minimal solutions in the case of fixed ordering structures. The reader can find many examples illustrating that this statement is in general not true in the case of variable ordering structure in [22, 24, 67]. In order to prove the main theorem of this section, first we prove the following lemmas.

Lemma 5.3.1. Suppose that assumptions (A2)–(A3) hold and let $B : Y \rightrightarrows Y$ be a cone-valued map satisfying assumptions (A5)–(A6). Consider the problem (VVOP). If $x_\varepsilon \in \varepsilon k^0$-MZ$(\mathfrak{S}, f, C)$, then for each $\omega \in f(\mathfrak{S})$, there exists a continuous functional $\theta_\omega : Y \to \mathbb{R}$ which is strictly B-monotone in the sense of Definition 4.2.23 and

$$\forall x \in \mathfrak{S}, \omega \in f(\mathfrak{S}) \qquad \theta_\omega(f(x_\varepsilon)) \leqq \theta_\omega(f(x) + \varepsilon k^0).$$

Moreover if (A4) holds, then for each $\omega \in f(\mathfrak{S})$, θ_ω is subadditive on Y and

$$\forall x \in \mathfrak{S}, \omega \in f(\mathfrak{S}) \qquad \theta_\omega(f(x_\varepsilon)) \leqq \theta_\omega(f(x)) + \theta_\omega(\varepsilon k^0).$$

Proof. Suppose that $k^0 \in Y \backslash \{\mathbf{0}\}$, $\varepsilon > 0$ and $x_\varepsilon \in \varepsilon k^0$-MZ$(\mathfrak{S}, f, C)$. This means for all $\omega \in f(\mathfrak{S})$, $(f(x_\varepsilon) - \varepsilon k^0 - C(\omega) \backslash \{\mathbf{0}\}) \cap f(\mathfrak{S}) = \emptyset$ and therefore

$$\forall \omega \in f(\mathfrak{S}), \qquad (f(x_\varepsilon) - C(\omega) \backslash \{\mathbf{0}\}) \cap (f(\mathfrak{S}) + \varepsilon k^0) = \emptyset.$$

We consider $D(\omega) := (f(x_\varepsilon) - C(\omega) \backslash \{\mathbf{0}\})$ and $f(\mathfrak{S}) + \varepsilon k^0 = U$. Taking into account (A5) and (A6) and applying Theorem 4.2.25, we get desired functionals. Therefore for any $\omega \in f(\mathfrak{S})$, there exist a continuous functional $\theta_\omega : Y \to \mathbb{R}$ such that $\theta_\omega(f(x_\varepsilon)) \leqq \ \theta_\omega(f(\mathfrak{S}) + \varepsilon k^0)$. Now if (A4) holds, then θ_ω is subadditive for all $\omega \in f(\mathfrak{S})$ and

$$\theta_\omega(f(x_\varepsilon)) \leqq \theta_\omega(f(\mathfrak{S})) + \theta_\omega(\varepsilon k^0)$$

and the proof is complete. □

The following lemma gives some properties of functionals in Lemma 5.3.1 and these properties will be used later in the proof of other lemmas and our main theorem about extension of Ekeland's theorem for εk^0-minimizers of vector optimization problems with variable ordering structures.

Lemma 5.3.2. Let assumptions (A2)–(A3) and (A5)–(A6) be fulfilled, then for each $\omega \in f(\mathfrak{S})$, we can choose the functional $\theta_\omega : Y \to \mathbb{R}$ in Lemma 5.3.1 in a way such that the followings hold.

1. $\theta_\omega(k^0) = 1$.

2. $\theta_\omega(\mathbf{0}) = 0$.

3. $\theta_\omega(\varepsilon k^0) = \varepsilon$ and $\theta_\omega(-\varepsilon k^0) = -\theta_\omega(\varepsilon k^0) = -\varepsilon$.

Proof. 1. By definition of separating functional θ_ω in (4.2), for each $\omega \in f(\mathfrak{S})$, we get

$$\theta_\omega(y) = \inf\{t \mid y \in tk^0 - C(\omega)\}.$$

By pointedness of $C(\omega)$ and (A3), we get $\mathbf{0} \in \mathrm{bd}\, C(\omega)$ and $k^0 \in k^0 - \mathrm{bd}\, C(\omega)$ for all $\omega \in f(\mathfrak{S})$. Therefore by the third part of Theorem 4.2.7, we get $\theta_\omega(k^0) = 1$.

2. By $\mathbf{0} \in \mathrm{bd}\, C(\omega)$ for all $\omega \in f(\mathfrak{S})$ and the third part of Theorem 4.2.7, we get $\theta_\omega(\mathbf{0}) = 0$ for all $\omega \in f(\mathfrak{S})$.

3. By the second part of Theorem 4.2.18, we know that for all $y \in Y$, $t \in \mathbb{R}$, $\omega \in f(\mathfrak{S})$ the following equation holds:
$$\theta_\omega(y + tk^0) = \theta_\omega(y) + t,$$
therefore $\theta_\omega(\mathbf{0} + \varepsilon k^0) = \theta_\omega(\mathbf{0}) + \varepsilon$ and $\theta_\omega(\varepsilon k^0) = \varepsilon$. Proofs of other parts are similar. □

Lemma 5.3.3. Let X be a real Banach space, $\mathfrak{S} \subset X$, $x_\varepsilon \in \mathfrak{S}$, Y be a topological linear space, $\varepsilon \geq 0$, $k^0 \in Y \backslash \{\mathbf{0}\}$, $f : X \to Y$ is a vector-valued function with $\mathrm{dom}\, f \neq \emptyset$ and $B : Y \rightrightarrows Y$ be a cone-valued map satisfying (A5).

(k) Furthermore, suppose that for any $\omega \in f(\mathfrak{S})$ and strictly B-monotone (in the sense of Definition 4.2.23), continuous, subadditive functional $\theta_\omega : Y \to \mathbb{R}$ the following inequality holds
$$\forall x \in \mathfrak{S}, \omega \in f(\mathfrak{S}) \qquad \theta_\omega(f(x_\varepsilon)) \leqq \theta_\omega(f(x)) - \theta_\omega(-\varepsilon k^0),$$
then $x_\varepsilon \in \varepsilon k^0\text{-WMZ}(\mathfrak{S}, f, C)$ for some set-valued map $C : Y \rightrightarrows Y$ such that $B(\omega)\backslash\{\mathbf{0}\} \subseteq C(\omega)$, $\mathbf{0} \in \mathrm{cl}\, C(\omega) \backslash C(\omega)$, $\mathrm{cl}\, C(\omega) + (B(\omega)\backslash\{\mathbf{0}\}) \subseteq C(\omega)$ for all $\omega \in f(\mathfrak{S})$.

Proof. For each $\omega \in f(\mathfrak{S})$, we define $C(\omega)$ and functional $\hat{\theta}_\omega : Y \to \mathbb{R}$ as following,
$$C(\omega) = \{y \in Y \mid \theta_\omega(-y + f(x_\varepsilon) - \varepsilon k^0) < \theta_\omega(f(x_\varepsilon) - \varepsilon k^0)\}, \tag{5.20}$$
$$\hat{\theta}_\omega(y) := \theta_\omega(y + f(x_\varepsilon) - \varepsilon k^0). \tag{5.21}$$

By (5.21) and (k) and since θ_ω is subadditive for all $\omega \in f(\mathfrak{S})$, we get
$$\begin{aligned}&\hat{\theta}_\omega(f(\mathfrak{S}) + \varepsilon k^0 - f(x_\varepsilon)) = \theta_\omega(f(\mathfrak{S})) \geqq\\ &\theta_\omega(f(x_\varepsilon)) + \theta_\omega(-\varepsilon k^0) \geqq\\ &\theta_\omega(f(x_\varepsilon) - \varepsilon k^0) = \hat{\theta}_\omega(\mathbf{0}).\end{aligned}$$

Now by (5.20) and (5.21), we get
$$\hat{\theta}_\omega(-C(\omega)) = \theta_\omega(-C(\omega) + f(x_\varepsilon) - \varepsilon k^0) < \theta_\omega(f(x_\varepsilon) - \varepsilon k^0) = \hat{\theta}_\omega(\mathbf{0}),$$
therefore for each $\omega \in f(\mathfrak{S})$,
$$(-\mathrm{int}\, C(\omega)) \cap (f(\mathfrak{S}) + \varepsilon k^0 - f(x_\varepsilon)) = \emptyset \implies (f(x_\varepsilon) - \varepsilon k^0 - \mathrm{int}\, C(\omega)) \cap f(\mathfrak{S}) = \emptyset.$$

Since θ_ω is strictly B-monotone functional in the sense of Definition 4.2.23, for all $\omega \in f(\mathfrak{S})$, $B(\omega)\backslash\{\mathbf{0}\} \subseteq C(\omega)$. Now we show that $\mathrm{cl}\, C(\omega) + (B(\omega)\backslash\{\mathbf{0}\}) \subseteq C(\omega)$. Choose $y \in \mathrm{cl}\, C(\omega)$ and

$b \in y + B(\omega)\backslash\{\mathbf{0}\}$. Since $\hat{\theta}_\omega$ is strictly B-monotone and $y \in \mathrm{cl}\, C(\omega) \subseteq \{y \mid \hat{\theta}_\omega(-y) \leqq \hat{\theta}_\omega(\mathbf{0})\}$, we have

$$\hat{\theta}_\omega(-b) < \hat{\theta}_\omega(-y) \leqq \hat{\theta}_\omega(\mathbf{0}).$$

Therefore $b \in \mathrm{cl}\, C(\omega) + (B(\omega)\backslash\{\mathbf{0}\})$ implies $b \in C(\omega)$. By the assumption (A5) and the inclusion $\mathrm{cl}\, C(\omega) + (B(\omega)\backslash\{\mathbf{0}\}) \subseteq C(\omega)$, we get $C(\omega) + \varepsilon k^0 \subseteq C(\omega)$. Also since $\mathbf{0} \in \mathrm{cl}(B(\omega)\backslash\{\mathbf{0}\})$, $B(\omega)\backslash\{\mathbf{0}\} \subseteq C(\omega)$ and $\mathbf{0} \notin C(\omega)$, therefore $\mathbf{0} \in \mathrm{cl}\, C(\omega)\backslash C(\omega)$. □

We are now ready to present an extension of Ekeland's theorem for εk^0-minimizers of vector optimization problem (VVOP) with a variable ordering structure.

Theorem 5.3.4. Consider the problem (VVOP) and let $\bar{x} \in \varepsilon k^0$-MZ$(\mathfrak{S}, f, C)$. Impose in addition to (A2)–(A6) the following assumptions:

$(A8)$ f is $(k^0, C, \mathfrak{S})$-lower semicontinuous over $\mathfrak{S}$ in the sense of Definition 5.1.3.

$(A9''')$ f is bounded from below over $\mathfrak{S}$ with respect to C in the sense of Definition 5.2.4.

Then there exists an element $x_\varepsilon \in \mathrm{dom}\, f \cap \mathfrak{S}$ such that

1. $x_\varepsilon \in \varepsilon k^0$-WMZ$(\mathfrak{S}, f, B)$,
2. $\|\bar{x} - x_\varepsilon\| \leqq \sqrt{\varepsilon}$,
3. $x_\varepsilon \in$ WMZ$(\mathfrak{S}, f_{\varepsilon k^0}, B)$ with $f_{\varepsilon k^0}(x) = f(x) + \sqrt{\varepsilon}\, \|x - x_\varepsilon\|\, k^0$. (5.22)

Proof. Let $\bar{x} \in \varepsilon k^0$-MZ$(\mathfrak{S}, f, C)$. By the definition of εk^0-minimizer (Definition 5.0.4), we get

$$\forall \omega \in f(\mathfrak{S}), \qquad (f(\bar{x}) - \varepsilon k^0 - C(\omega)\backslash\{\mathbf{0}\}) \cap f(\mathfrak{S}) = \emptyset.$$

Now suppose that $\bar{f} := f - f(\bar{x})$, then we have

$$\forall \omega \in f(\mathfrak{S}), \qquad (\bar{f}(\bar{x}) - \varepsilon k^0 - C(\omega)\backslash\{\mathbf{0}\}) \cap \bar{f}(\mathfrak{S}) = \emptyset.$$

By (A4), Lemma 5.3.1 and 5.3.2, for all $\omega \in f(\mathfrak{S})$, the functional $\theta_\omega : Y \to \mathbb{R}$ in (4.2) is a strictly B-monotone, continuous and subadditive functional such that

$$\forall x \in \mathfrak{S}, \qquad \theta_\omega(\bar{f}(\bar{x})) \leqq \theta_\omega(\bar{f}(x)) + \theta_\omega(\varepsilon k^0) = \theta_\omega(\bar{f}(x)) + \varepsilon.$$

This means that for all $\omega \in f(\mathfrak{S})$,

$$\theta_\omega(\bar{f}(\bar{x})) \leqq \inf_{x \in \mathfrak{S}} \theta_\omega(\bar{f}(x)) + \varepsilon, \qquad \varepsilon > 0.$$

Observe that the validity of (A8)–(A9''') ensures $(k^0, C, \mathfrak{S})$-lower semicontinuity and the boundedness from below of f and $\bar{f}$. By Lemma 5.1.4, Lemma 5.2.5, Theorem 5.0.1 and Remark 5.0.2, there exists $x_\varepsilon \in \mathfrak{S}$ such that for all $\omega \in f(\mathfrak{S})$,

1. $\theta_\omega(\overline{f}(x_\varepsilon)) \leqq \theta_\omega(\overline{f}(\bar{x})) \leqq \inf_{x\in\mathfrak{S}} \theta_\omega(\overline{f}(x)) + \varepsilon,$ (5.23)

2. $\|x_\varepsilon - \bar{x}\| \leqq \sqrt{\varepsilon},$

3. for all $x, \omega \in \mathfrak{S},\ \theta_\omega(\overline{f}(x_\varepsilon)) \leqq \theta_\omega(\overline{f}(x)) + \sqrt{\varepsilon}\,\|x - x_\varepsilon\|,$ (5.24)

4. $\theta_\omega(\overline{f}(x_\varepsilon)) + \sqrt{\varepsilon}\,\|\bar{x} - x_\varepsilon\| \leq \theta_\omega(\overline{f}(\bar{x})).$ (5.25)

By Lemma 5.3.2 and (5.23), for all $x \in \mathfrak{S}, \omega \in f(\mathfrak{S})$, we get

$$\theta_\omega(\overline{f}(x_\varepsilon)) \leqq \inf_{x\in\mathfrak{S}} \theta_\omega(\overline{f}(x)) + \varepsilon \leqq \theta_\omega(\overline{f}(x)) + \theta_\omega(\varepsilon k^0) = \theta_\omega(\overline{f}(x)) - \theta_\omega(-\varepsilon k^0).$$

By Lemma 5.3.3, the inclusion $B(\omega) \subseteq C(\omega)$ by assumption (A6) and $\overline{f} = f - f(\bar{x})$, we get

$$(\overline{f}(x_\varepsilon) - \varepsilon k^0 - \operatorname{int} B(\omega)\backslash\{\mathbf{0}\}) \cap \overline{f}(\mathfrak{S}) = \emptyset.$$

This implies that $x_\varepsilon \in \varepsilon k^0$-WMZ$(\mathfrak{S}, f, B)$. Now we prove (5.22) and for this, suppose that there exist elements $x, \omega \in \mathfrak{S}$ such that

$$\begin{aligned} & f(x) \in f(x_\varepsilon) - \sqrt{\varepsilon}\,\|x - x_\varepsilon\|\, k^0 - \operatorname{int} B(\omega) \\ \Longrightarrow \quad & \overline{f}(x) \in \overline{f}(x_\varepsilon) - \sqrt{\varepsilon}\,\|x - x_\varepsilon\|\, k^0 - \operatorname{int} B(\omega). \end{aligned}$$

Since for all $\omega \in f(\mathfrak{S})$, θ_ω is a strictly B-monotone continuous subadditive functional, then

$$\theta_\omega(\overline{f}(x)) < \theta_\omega(\overline{f}(x_\varepsilon) - \sqrt{\varepsilon}\,\|x - x_\varepsilon\|\, k^0) \leqq \theta_\omega(\overline{f}(x_\varepsilon)) + \theta_\omega(-\sqrt{\varepsilon}\,\|x - x_\varepsilon\|\, k^0).$$

Now by Lemma 5.3.2, we get

$$\theta_\omega(-\sqrt{\varepsilon}\,\|x - x_\varepsilon\|\, k^0) = -\sqrt{\varepsilon}\,\|x - x_\varepsilon\| \implies \theta_\omega(\overline{f}(x_\varepsilon)) > \theta_\omega(\overline{f}(x)) + \sqrt{\varepsilon}\,\|x - x_\varepsilon\|,$$

but this yields a contradiction because of (5.24). □

In the special case that $C : Y \rightrightarrows Y$ is a solid, closed, pointed and convex cone-valued map, we have the following corollary.

Corollary 5.3.5. Suppose that $C : Y \rightrightarrows Y$ is a cone-valued map where $C(\omega)$ is a solid convex cone for all $\omega \in f(\mathfrak{S})$, $k^0 \in \bigcap_{\omega\in f(\mathfrak{S})} \operatorname{int} C(\omega)$ and $\varepsilon > 0$. Consider the problem (VVOP) and let $\bar{x} \in \varepsilon k^0$-MZ$(\mathfrak{S}, f, C)$. Impose the following assumptions:

(A8) f is $(k^0, C, \mathfrak{S})$-lower semicontinuous over $\mathfrak{S}$ in the sense of Definition 5.1.3.

(A9‴) f is bounded from below over $\mathfrak{S}$ with respect to C in the sense of Definition 5.2.4.

Then there exists an element $x_\varepsilon \in \mathfrak{S}$ such that

1. $x_\varepsilon \in \varepsilon k^0$-WMZ$(\mathfrak{S}, f, C)$,

2. $\|\bar{x} - x_\varepsilon\| \leqq \sqrt{\varepsilon}$,

3. $x_\varepsilon \in WMZ(\mathfrak{S}, f_{\varepsilon k^0}, C)$ with $f_{\varepsilon k^0}(x) = f(x) + \sqrt{\varepsilon}\,\|x - x_\varepsilon\|\,k^0$.

Chapter 6

Optimality Conditions

The aim of this chapter is to derive new optimality conditions for approximate solutions of vector optimization problems with variable ordering structures. Bao and Mordukhovich [4, 5] have shown necessary conditions for nondominated points of sets and nondominated solutions of vector optimization problems with variable ordering structures and general geometric constraints, applying methods of variational analysis and generalized differentiation (see Mordukhovich [58] and Mordukhovich, Shao [59]). Furthermore, Bao, Eichfelder, Soleimani and Tammer [2] have shown necessary conditions for approximately nondominated solutions of vector optimization problems with variable ordering structures in Asplund spaces using a vector-valued variant of Ekeland's variational principle. New necessary conditions for approximate minimizers and approximately minimal solution of vector optimization problems with variable ordering structures is shown by Soleimani and Tammer in [65].

First, we bring some necessary definitions that will be used later in this chapter.

Definition 6.0.1. Consider a convex functional $f : X \to \mathbb{R}$. For a given point $\bar{x} \in \operatorname{dom} f$, Fenchel subdifferential is defined as following.

$$\partial f(\bar{x}) := \{x^* \in X^* \mid f(y) - f(\bar{x}) \geq \langle y - \bar{x}, x^* \rangle \quad \forall y \in X\}.$$

Definition 6.0.2. Let X and Y be Banach spaces, and $U \subset X$ be an open subset of X. A function $f : U \to Y$ is called Frèchet differentiable at $x \in U$ if there exists a bounded linear operator $A_x : X \to Y$ such that

$$\lim_{h \to 0} \frac{\|f(x+h) - f(x) - A_x(h)\|_Y}{\|h\|_X} = 0.$$

Definition 6.0.3. A Banach space is Asplund if every convex continuous function $\varphi : U \to \mathbb{R}$ defined on an open convex subset U of X is Fréchet differentiable on a dense subset of U.

The class of Asplund spaces is quite broad including every reflexive Banach space and every Banach space with a separable dual; in particular, l^p and $L^p[0,1]$ for $1 < p < \infty$ are Asplund spaces, but ℓ_1 and ℓ_∞ are not Asplund spaces.

Definition 6.0.4. Let $\mathfrak{S}$ be a subset of Banach space X and $\bar{x} \in \mathfrak{S}$.

(a) The *Fréchet normal cone* of $\mathfrak{S}$ at $\bar{x} \in \mathfrak{S}$ is defined by

$$\hat{N}(\bar{x};\mathfrak{S}) := \left\{x^* \in X^* \;\middle|\; \limsup_{x \xrightarrow{\mathfrak{S}} \bar{x}} \frac{\langle x^*, x-\bar{x}\rangle}{\|x-\bar{x}\|} \leq 0\right\}, \tag{6.1}$$

where $x \xrightarrow{\mathfrak{S}} \bar{x}$ means $x \to \bar{x}$ with $x \in \mathfrak{S}$.

(b) Assume that X is an Asplund space and $\mathfrak{S}$ is locally closed around $\bar{x} \in \mathfrak{S}$, i.e., there is a neighborhood U of $\bar{x}$ such that $\mathfrak{S} \cap \operatorname{cl} U$ is a closed set. The *(basic, limiting, Mordukhovich) normal cone* of $\mathfrak{S}$ at $\bar{x}$ is defined by

$$\begin{aligned} N(\bar{x};\mathfrak{S}) &:= \operatorname*{Lim\,sup}_{x\to\bar{x}} \widehat{N}(x;\mathfrak{S}) \\ &= \left\{x^* \in X^* \;\middle|\; \exists\, x_k \to \bar{x},\ x_k^* \xrightarrow{w^*} x^* \text{ with } x_k^* \in \widehat{N}(x_k;\mathfrak{S})\right\}, \end{aligned} \tag{6.2}$$

where Lim sup stands for the sequential Painlevé-Kuratowski outer limit of the Fréchet normal cone to $\mathfrak{S}$ at x as x tends to $\bar{x}$.

Note that, in contrast to (6.1), the basic normal cone (6.2) is often nonconvex enjoying nevertheless *full calculus*, and that both the cones (6.1) and (6.2) reduce to the normal cone of convex analysis when $\mathfrak{S}$ is convex.

Definition 6.0.5. F is called upper semicontinuous at $\bar{x} \in X$ if for any neighborhood $\mathcal{N}(F(\bar{x}))$ of $F(\bar{x})$, there exists a neighborhood $\mathcal{N}(\bar{x})$ of $\bar{x}$ such that

$$\forall x \in \mathcal{N}(\bar{x}) \qquad F(x) \subseteq \mathcal{N}(F(\bar{x})).$$

F is called upper semicontinuous on X if F is upper semi-continuous at every $x \in X$.

Definition 6.0.6. F is called lower semicontinuous at $\bar{x} \in X$ if for any $y \in F(\bar{x})$ and any neighborhood $\mathcal{N}(y)$ of y , there exists a neighborhood $\mathcal{N}(\bar{x})$ of x such that

$$\forall x \in \mathcal{N}(\bar{x}) \qquad F(x) \cap \mathcal{N}(y) \neq \emptyset.$$

F is called lower semicontinuous on X if F is lower semi-continuous at every $x \in X$ and F is called continuous at X if F is both upper semi-continuous and lower semi-continuous at every $x \in X$.

6.1 Optimality Conditions for Approximately Minimal Solutions

In this section, we present optimality conditions for approximately minimal solutions of vector optimization problems with variable ordering structures with different approaches namely Mordukhovich subdifferential approach and general generic approach. Also second order optimality condition using second order contingent derivatives and epiderivatives will be shown for weakly minimal solutions of set-valued optimization in the last subsection of this section. Our results in previous chapters like variational principle and characterization of approximately minimal solutions of vector optimization problems with variable ordering structures will be used here in order to derive optimality conditions for approximately minimal solutions of vector optimization problems with variable ordering structures.

6.1.1 Mordukhovich Subdifferential Approach

In the following, we bring definition of Mordukhovich (basic, limiting) subdifferential which will be used for deriving optimality condition without convexity assumption for solutions of (VVOP) in the following and sections 6.2 and 6.3.

Definition 6.1.1. Consider a functional $f : X \to \mathbb{R}$ and a point $\bar{x} \in \operatorname{dom} f$.

(a) The set

$$\partial_M f(\bar{x}) := \left\{x^* \in X^* \mid (x^*,-1) \in N((\bar{x}, f(\bar{x})); \operatorname{epi} f)\right\}$$

is the *(basic, limiting) subdifferential* of f at $\bar{x}$, and its elements are basic subgradients of φ at this point.

(b) The set

$$\partial^\infty f(\bar{x}) := \left\{x^* \in X^* \mid (x^*,0) \in N((\bar{x}, f(\bar{x})); \operatorname{epi} f)\right\}$$

is the singular subdifferential of f at $\bar{x}$, and its elements are singular subgradients of f at this point.

If f is locally Lipschitz at $\bar{x}$, then $\partial^\infty f(\bar{x}) = \{\mathbf{0}\}$. If f is strictly Lipschitz continuous at $\bar{x}$; in particular, it is $C^{1,1}$, then $\partial_M f(\bar{x}) = \{\nabla f(\bar{x})\}$.

Lemma 6.1.2. ([58, Theorem 3.36 and Corollary 3.43]) Assume that X is Asplund.

(a) Suppose that $\varphi_1, \varphi_2 : X \to \mathbb{R}$ are proper functionals and there exists a neighborhood U of $\bar{x} \in \operatorname{dom} \varphi_1 \cap \operatorname{dom} \varphi_2$ such that φ_1 is Lipschitz and φ_2 is lower semicontinuous on U, then

$$\partial_M(\varphi_1 + \varphi_2)(\bar{x}) \subset \partial_M \varphi_1(\bar{x}) + \partial_M \varphi_2(\bar{x}).$$

(b) If $f : X \to Y$ is strictly Lipschitz at $\bar{x}$ and $\varphi : Y \to \mathbb{R}$ is finite and lower semicontinuous on some neighborhood of $f(\bar{x})$, then

$$\partial_M(\varphi \circ f)(\bar{x}) \subset \bigcup_{y^* \in \partial_M \varphi(f(\bar{x}))} \partial_M(y^* \circ f)(\bar{x})$$

provided that the pair of functions (φ, f) satisfies the qualification condition

$$\partial^\infty \varphi(\bar{y}) \cap \ker \partial_M \langle \cdot, f\rangle(\bar{x}) = \{\mathbf{0}\}, \tag{6.3}$$

where $\ker \partial_M \langle \cdot, f\rangle(\bar{x}) = \{y^* \in Y^* \mid \mathbf{0} \in \partial_M \langle y^*, f\rangle(\bar{x})\}$.

Assumption (A11). Let X and Y be Asplund spaces, $k^0 \in Y \backslash \{\mathbf{0}\}$, $\varepsilon \geq 0$, $f : X \to Y$ be a continuous and strictly Lipschitz function, $\mathfrak{S}$ be a closed subset of X and $C : Y \rightrightarrows Y$ be a set-valued map such that $C(y)$ is a pointed closed set with $C(y) + (0, +\infty)k^0 \subseteq \operatorname{int} C(y)$ for all $y \in Y$.

Theorem 6.1.3. Consider problem (VVOP), let $\bar{x} \in \varepsilon k^0$-M$(\mathfrak{S}, f, C)$ and set $\bar{y} := f(\bar{x})$. Assume that in addition to (A11) the following conditions hold:

$(A5)$ $B : Y \rightrightarrows Y$ be a cone-valued map such that for all $y \in f(\mathfrak{S})$, $k^0 \in \operatorname{int} B(y)$.

$(A6'')$ $C(y) + B(y) \backslash \{\mathbf{0}\} \subseteq \operatorname{int} C(y)$ and $B(f(x)) \subset C(f(\bar{x}))$ for all $\|x - \bar{x}\| \leq \sqrt{\varepsilon}$.

$(A7)$ $C(y) \subseteq C(\bar{y})$ for all $y \in f(\mathfrak{S})$.

$(A9)$ f is bounded from below over $\mathfrak{S}$ with respect to $\bar{y} \in Y$ and $\Theta = C(\bar{y})$ in the sense of Definition 4.2.26.

Let $\theta_{\bar{y}} \circ f$ satisfies the qualification condition (6.3) for all $x \in \mathfrak{S}$ such that $\|x - \bar{x}\| \leq \sqrt{\varepsilon}$. Then, there exist $x_\varepsilon \in \mathfrak{S}$ and $v^* \in \partial_M(\theta_{\bar{y}}(f(x_\varepsilon))$ such that

$$\mathbf{0} \in \partial_M(v^* \circ f)(x_\varepsilon) + N(x_\varepsilon; \mathfrak{S}) + \sqrt{\varepsilon} B_{X^*}.$$

Proof. By Theorem 5.1.8 and (5.5), there exists $x_\varepsilon \in \operatorname{dom} f \cap \mathfrak{S}$ such that it is an exact solution of minimizing a functional $h : X \to \mathbb{R} \cup \{+\infty\}$ over $\mathfrak{S}$ with

$$h(x) := (\theta_{\bar{y}} \circ f)(x) + \sqrt{\varepsilon}\,\|x - x_\varepsilon\| \quad \text{for all } x \in X.$$

By [58, Proposition 5.1], we get

$$\mathbf{0} \in \partial_M h(x_\varepsilon) + N(x_\varepsilon; \mathfrak{S}).$$

By Lemma 4.2.11, the composition $\theta_{\bar{y}} \circ f$ is lower-semicontinuous on a neighborhood of x_ε. Employing Lemma 6.1.2 (a) to the lower semicontinuous functional $\theta_{\bar{y}} \circ f$ and the Lipschitz continuous function $\|.\|$, we have

$$\partial_M h(x_\varepsilon) \quad \subset \quad \partial_M\big(\theta_{\bar{y}} \circ f\big)(x_\varepsilon) + \partial_M(\sqrt{\varepsilon}\,\|\cdot - x_\varepsilon\|)(x_\varepsilon).$$

By Lemma 6.1.2 (b), we get

$$\partial_M(\theta_{\bar{y}} \circ f)(x_\varepsilon) \subset \bigcup \{\partial_M(v^* \circ f)(x_\varepsilon) \mid v^* \in \partial_M \theta_{\bar{y}}(f(x_\varepsilon))\}.$$

Combining three inclusions together while taking into account the subdifferential of the norm $\partial_M \|\cdot - x_\varepsilon\|(x_\varepsilon) = B_{X^*}$, we can find $v^* \in \partial_M \theta_{\bar{y}}(f(x_\varepsilon))$ satisfying

$$\mathbf{0} \in \partial_M(v^* \circ f)(x_\varepsilon) + N(x_\varepsilon; \mathfrak{S}) + \sqrt{\varepsilon} B_{X^*}$$

and proof is complete. □

Corollary 6.1.4. Consider problem (VVOP). Let $\bar{x} \in \mathrm{M}(\mathfrak{S}, f, C)$ be an exact minimal solution of problem (VVOP), set $\bar{y} := f(\bar{x})$ and let all assumptions of Theorem 6.1.3 be fulfilled. Assume that $\theta_{\bar{y}} \circ f$ satisfies the qualification condition (6.3) at $\bar{x}$. Then, for any $\lambda > 0$, there exists $v^* \in \partial_M(\theta_{\bar{y}}(f(\bar{x}))$ such that

$$\mathbf{0} \in \partial_M(v^* \circ f)(\bar{x}) + N(\bar{x}; \mathfrak{S}) + \lambda B_{X^*}. \tag{6.4}$$

Proof. Since $\bar{x} \in \mathrm{M}(\mathfrak{S}, f, C)$, i.e., it is a $0k^0$-minimal solution of (VVOP), it is also εk^0-minimal of (VVOP) with $\varepsilon = \lambda^2 > 0$ for all $\lambda > 0$ and a weakly minimal solution. By Theorem 5.1.8 and Theorem 6.1.3, the only point which satisfies (5.6) is $\bar{x}$ and we can find $v^* \in \partial_M \theta_{\bar{y}}(f(\bar{x}))$ such that

$$\mathbf{0} \in \partial_M(v^* \circ f)(\bar{x}) + N(\bar{x}; \mathfrak{S}) + \sqrt{\varepsilon} B_{X^*}$$

clearly verifying (6.4). The proof is complete. □

6.1.2 Generic Approach

It is possible to derive optimality conditions using general generic approach. For doing this we use an abstract subdifferentials. We introduce a generic approach to subdifferentials that includes many well-known subdifferentials.

Let $\mathscr{X}$ be a class of Banach spaces which contains the class of finite dimensional normed vector spaces. By an abstract subdifferential ∂ we mean a map which associates to every lsc function $h : X \in \mathscr{X} \to \overline{\mathbb{R}}$ and to every $x \in X$ a (possible empty) subset $\partial h(x) \subset X^*$. Let $X, Y \in \mathscr{X}$ and denote by $\mathscr{F}(X,Y)$ a class of functions acting between X and Y having the property that by composition at left with a lsc function from Y to $\overline{\mathbb{R}}$ the resulting function is still lsc.

In the following we work with the next properties of the abstract subdifferential ∂:

(H1) If h is convex, then $\partial h(x)$ coincides with the Fenchel subdifferential.

(H2) If x is a local minimum point for h, then $0 \in \partial h(x)$; $\partial h(u) = \emptyset$ if $u \notin \operatorname{dom} h$.

Note that (H1) and (H2) are very natural requirements for any subdifferential.

(H3) If $\varphi : Y \to \mathbb{R} \cup \{+\infty\}$ is convex and $\psi \in \mathscr{F}(X,Y)$, then for every x,

$$\partial(\varphi \circ \psi)(x) \subseteq \bigcup_{y^* \in \partial\varphi(\psi(x))} \partial(y^* \circ \psi)(x).$$

(H4) If $\varphi : Y \to \mathbb{R} \cup \{+\infty\}$ is convex, $\psi \in \mathscr{F}(X,Y)$, and $S \subset X$ is a closed set containing x, then

$$\partial(\varphi \circ \psi + I_S)(x) \subseteq \partial(\varphi \circ \psi)(x) + \partial I_S(x).$$

(H5) If h is convex and $g : X \to \mathbb{R} \cup \{+\infty\}$ is locally Lipschitz, then for every $x \in \operatorname{Dom} h \cap \operatorname{Dom} g$,

$$\partial(h+g)(x) \subseteq \partial h(x) + \partial g(x).$$

(H6) If $X \in \mathscr{X}$, $\varphi : X \to \mathbb{R}$ is a locally Lipschitz functions and $x \in \operatorname{dom} h$, then

$$\partial(h+\varphi)(x) \subset \|\cdot\|^* - \limsup_{y \xrightarrow{h} x, z \to x} (\partial h(y) + \partial\varphi(z)).$$

(H7) If $\varphi : Y \to \mathbb{R}$ is locally Lipschitz and $\psi \in \mathscr{F}(X,Y)$, then for every x,

$$\partial(\varphi \circ \psi)(x) \subset \|\cdot\|^* - \limsup_{u \xrightarrow{\psi} x, v \to \psi(x)} \bigcup_{u^* \in \partial\varphi(v)} \partial(u^* \circ \psi)(u),$$

where the following notations are used:

1. $u \xrightarrow{h} x$ means that $u \to x$ and $h(u) \to h(x)$; note that if h is continuous, then $u \xrightarrow{h} x$ is equivalent with $u \to x$.

2. $x^* \in \|\cdot\|^* - \limsup_{u \to x} \partial h(u)$ means that for every $\varepsilon > 0$ there exist x_ε and x_ε^* such that $x_\varepsilon^* \in \partial h(x_\varepsilon)$ and $\|x_\varepsilon - x\| < \varepsilon$, $\|x_\varepsilon^* - x^*\| < \varepsilon$.
The notation $x^* \in \|\cdot\|^* - \limsup_{u \xrightarrow{h} x} \partial h(u)$ has a similar interpretation and it is equivalent with $x^* \in \|\cdot\|^* - \limsup_{u \to x} \partial h(u)$ provided that h is continuous.

6.1.2.1 Exact Optimality Conditions

In this section, we give exact necessary conditions for approximately minimal solutions of vector optimization problems with variable ordering structures using generic subdifferential approach with the help of nonlinear separating functionals defined by Tammer and Weidner and its properties [32].

Assumption (A12). X, Y are Banach spaces, $\mathfrak{S} \subset X$ is a closed set in X, $f \in \mathscr{F}(X,Y)$ is a function with $\operatorname{dom} f \neq \emptyset$ and $\varepsilon \geq 0$.

Assumption (A13). $C : Y \rightrightarrows Y$ is a set-valued map such that $C(y)$ is a closed, solid and pointed set for all $y \in Y$. The nonzero vector $k^0 \in Y \setminus \{\mathbf{0}\}$ satisfies $C(y) + (0, +\infty)k^0 \subseteq \operatorname{int} C(y)$ for all $y \in Y$.

In order to drive necessary optimality condition for approximately minimal solutions of (VVOP), for $\bar{x} \in \mathfrak{S}$, we use the scalarization functional $\theta_{\bar{x}} : Y \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ defined by

$$\theta_{\bar{x}}(y) := \inf\{t \in \mathbb{R} \mid y \in tk^0 + f(\bar{x}) - C(f(\bar{x}))\} \tag{6.5}$$

which is slightly modification of the functional defined by (4.2). Note that all the properties and theorems given in section 4.2 hold also for this functional and additionally we have the following theorem.

Theorem 6.1.5. [33, Theorem 2.3.1] Let assumptions (A12) and (A13) be fulfilled and $\bar{x} \in X$. The functional $\theta_{\bar{x}} : Y \to \overline{\mathbb{R}}$ defined by (6.5) has the following properties.

(a) The functional $\theta_{\bar{x}}$ is finite-valued if and only if $C(f(\bar{x}))$ does not contain lines parallel to k^0 and $\mathbb{R}k^0 - C(f(\bar{x})) = Y$.

(b) The domain of $\theta_{\bar{x}}$ is $\mathbb{R}k^0 - C(f(\bar{x}))$ and

$$\theta_{\bar{x}}(y + \lambda k^0) = \theta_{\bar{x}}(y) + \lambda \quad \forall y \in Y,\ \forall \lambda \in \mathbb{R}.$$

(c) $\theta_{\bar{x}}$ is convex if and only if $C(f(\bar{x}))$ is convex.

If the functional $\theta_{\bar{x}}$ is proper and convex, we get the following result concerning the classical (Fenchel) subdifferential ∂ of $\theta_{\bar{x}}$.

Theorem 6.1.6. [21, Theorem 2.2] Let $\bar{x} \in X$, $C(f(\bar{x})) \subset Y$ be a closed convex proper set, $k^0 \in Y \setminus \{0\}$ such that $C(f(\bar{x})) + [0, +\infty)k^0 \subset C(f(\bar{x}))$ holds and for every $y \in Y$, there exists $t \in \mathbb{R}$ such that $y + tk^0 \notin C(f(\bar{x})) - f(\bar{x})$. Consider the function $\theta_{\bar{x}}$ given by (6.5) and let $\hat{y} \in \operatorname{dom} \theta_{\bar{x}}$. Then

$$\partial \theta_{\bar{x}}(\hat{y}) = \{v^* \in Y^* \mid \forall d \in D :\ v^*(k^0) = 1, v^*(d) + v^*(\hat{y}) - \theta_{\bar{x}}(\hat{y}) \geq 0\}, \tag{6.6}$$

where $D := C(f(\bar{x})) - f(\bar{x})$.

The following theorem gives a characterization of approximately minimal solutions of (VVOP) by using a scalarization by means of the functional $\theta_{\bar{x}} : Y \to \overline{\mathbb{R}}$ defined by (6.5).

Theorem 6.1.7. Suppose that assumptions (A12) and (A13) hold. Let $\bar{x} \in \mathfrak{S}$ be an εk^0-minimal solution of (VVOP). Consider the function $\theta_{\bar{x}}$ given by (6.5). Then $\theta_{\bar{x}}(f(\bar{x})) \leq \inf\limits_{x \in \mathfrak{S}} \theta_{\bar{x}}(f(x)) + \varepsilon$ for all $x \in \mathfrak{S}$.

Proof. Set $\bar{y} = f(\bar{x})$ and suppose that $\theta_{\bar{x}}(\bar{y}) = \bar{t}$. First, we prove that $\bar{t} = 0$. By $\theta_{\bar{x}}(\bar{y}) = \bar{t}$ and Theorem 4.2.7, we get

$$\bar{t}k^0 + \bar{y} - \bar{y} \in C(\bar{y}) \implies \bar{t}k^0 \in C(\bar{y}).$$

By pointedness of $C(\bar{y})$, we get $\mathbf{0} \in \operatorname{bd} C(\bar{y})$ and $\bar{t} \leq 0$. We claim that $\bar{t} = 0$. Suppose that $\bar{t} < 0$, then again by pointedness of $C(\bar{y})$, $-\bar{t} > 0$, $k^0 \neq \mathbf{0}$ and $C(f(\bar{x})) + [0, +\infty)k^0 \subset C(f(\bar{x}))$, we get $-\bar{t}k^0 \in C(\bar{y}) \backslash \{\mathbf{0}\}$ and therefore $\bar{t}k^0 \in C(\bar{y}) \backslash \{\mathbf{0}\} \cap (-C(\bar{y}))$. But this is a contradiction to pointedness of $C(\bar{y})$ in assumption (A13) and therefore $\bar{t} = 0$. Now by contrary, suppose that there exists an element $x \in \mathfrak{S}$ such that $\theta_{\bar{x}}(f(x)) + \varepsilon < \theta_{\bar{x}}(\bar{y}) = 0$. This means that there exists $\gamma > 0$ such that $\theta_{\bar{x}}(f(x)) + \varepsilon + \gamma = 0$ and $\theta_{\bar{x}}(f(x)) = -\varepsilon - \gamma$. Again by Theorem 4.2.7, we get

$$(-\varepsilon - \gamma)k^0 + \bar{y} - f(x) \in C(\bar{y}) \implies \bar{y} - \varepsilon k^0 - y \in C(\bar{y}) + \gamma k^0 \subset C(\bar{y}) \setminus \{\mathbf{0}\}.$$

This means that $(\bar{y} - \varepsilon k^0 - C(\bar{y}) \setminus \{\mathbf{0}\}) \cap f(\mathfrak{S}) \neq \emptyset$. But this is a contradiction to approximate minimality of $\bar{x}$ and therefore $\theta_{\bar{x}}(f(\bar{x})) \leq \inf_{x \in \Omega} \theta_{\bar{x}}(f(x)) + \varepsilon$ for all $x \in \mathfrak{S}$. □

Lemma 6.1.8. Consider problem (VVOP) and let $\bar{x} \in \mathfrak{S}$ and the functional $\theta_{\bar{x}}$ given by (6.5) and set $\bar{y} := f(\bar{x})$. Impose in addition to (A12)–(A13) the following assumptions:

(*C*5) $C(\bar{y}) + C(\bar{y}) \subseteq C(\bar{y})$.

(*C*6) f is bounded from below over $\mathfrak{S}$ with respect to an element $y \in Y$ with $\theta_{\bar{x}}(y) > -\infty$ and $\Theta := C(\bar{y})$ in the sense of Definition 4.2.26.

Then the functional $\theta_{\bar{x}} \circ f$ is bounded from below.

Proof. Under the assumption $C(\bar{y}) + C(\bar{y}) \subseteq C(\bar{y})$ by (C5), the functional $\theta_{\bar{x}}$ is $C(\bar{y})$-monotone taking into account Theorem 4.2.18. The $C(\bar{y})$-monotonicity of $\theta_{\bar{x}}$ and $f(\mathfrak{S}) \subseteq y + C(\bar{y})$ implies

$$\forall x \in \mathfrak{S}: \qquad \theta_{\bar{x}}(f(x)) \geq \theta_{\bar{x}}(y),$$

i.e., $\theta_{\bar{x}} \circ f$ is bounded from below. □

In the next theorem we show necessary conditions for approximately minimal solutions of vector optimization problems with variable ordering structures.

Theorem 6.1.9. Consider problem (VVOP) and let $\bar{x} \in \varepsilon k^0\text{-M}(\mathfrak{S}, f, C)$ and the functional $\theta_{\bar{x}}$ given by (6.5). Set $\bar{y} := f(\bar{x})$ and let $C(\bar{y})$ be a convex set. Impose in addition to (A12)–(A13) the following assumptions:

(*C*5) $C(\bar{y}) + C(\bar{y}) \subseteq C(\bar{y})$.

(*C*6) f is bounded from below over $\mathfrak{S}$ with respect to an element $y \in Y$ with $\theta_{\bar{x}}(y) > -\infty$ and $\Theta := C(\bar{y})$ in the sense of Definition 4.2.26.

(*C*7) $f \in \mathscr{F}(X, Y)$ is locally Lipschitz.

Consider an abstract subdifferential ∂ for that (H1) – (H5) are satisfied. Then, there exists $x_\varepsilon \in \operatorname{dom} f \cap \mathfrak{S}$ and $v^* \in \partial(\theta_{\bar{x}}(f(x_\varepsilon))$ such that

$$\mathbf{0} \in \partial(v^* \circ f)(x_\varepsilon) + N(x_\varepsilon; \mathfrak{S}) + \sqrt{\varepsilon} B_{X^*}.$$

Proof. Let $\bar{x} \in \varepsilon k^0$-M($\mathfrak{S}$,f,C). By Theorem 6.1.7, we get $\theta_{\bar{x}}(f(\bar{x})) \leq \inf_{x \in \mathfrak{S}} \theta_{\bar{x}}(f(x)) + \varepsilon$. Therefore $\bar{x}$ is an approximate solution of the scalar problem with the objective functional $\theta_{\bar{x}} \circ f$. By $f \in \mathscr{F}(X,Y)$ and Theorem 4.2.10, $(\theta_{\bar{x}} \circ f)$ is lower semicontinuous. Furthermore, $(\theta_{\bar{x}} \circ f)$ is bounded from below because of Lemma 6.1.8. This yields that the assumptions of the scalar Ekeland's variational principle (Theorem 5.0.1) and strong form of Ekeland's variational principle (Remark 5.0.2) are fulfilled.

By Theorem 5.0.1 and Remark 5.0.2, there exists an element $x_\varepsilon \in \operatorname{dom} f \cap \mathfrak{S}$ such that it satisfies parts (a), (b) and (c) of Theorem 5.0.1 and it is an exact solution of minimizing a functional $h : X \to \mathbb{R} \cup \{+\infty\}$ over $\mathfrak{S}$ with

$$h(x) := (\theta_{\bar{x}} \circ f)(x) + \sqrt{\varepsilon}\, \|x - x_\varepsilon\| \quad \text{for all } x \in X.$$

Taking into account (H2) and (H4), we get

$$\mathbf{0} \in \partial h(x_\varepsilon) + N(x_\varepsilon; \mathfrak{S}).$$

Under the given assumptions the functional $\theta_{\bar{x}}$ is convex and continuous taking into account Theorem 6.1.5 (c) and third part of Theorem 4.2.18. Since f is locally Lipschitz and $\theta_{\bar{x}}$ is convex and continuous (and hence locally Lipschitz; see [62, Proposition 1.6]), it is clear that $\theta_{\bar{x}} \circ f$ is also locally Lipschitz. This implies together with the convexity of $\|\cdot\|$ and (H5) that

$$\partial h(x_\varepsilon) \quad \subseteq \quad \partial\big(\theta_{\bar{x}} \circ f\big)(x_\varepsilon) + \partial(\sqrt{\varepsilon}\, \|\cdot - x_\varepsilon\|)(x_\varepsilon).$$

By (H3), we get

$$\partial(\theta_{\bar{x}} \circ f)(x_\varepsilon) \subseteq \bigcup \big\{ \partial(v^* \circ f)(x_\varepsilon) \mid v^* \in \partial \theta_{\bar{x}}(f(x_\varepsilon)) \big\}.$$

Because of the convexity of the norm and (H1), we get $\partial \|\cdot - x_\varepsilon\|(x_\varepsilon) = B_{X^*}$ and by the last three inclusions, we can find $v^* \in \partial \theta_{\bar{x}}(f(x_\varepsilon))$ satisfying

$$\mathbf{0} \in \partial(v^* \circ f)(x_\varepsilon) + N(x_\varepsilon; \mathfrak{S}) + \sqrt{\varepsilon} B_{X^*}$$

and proof is complete. □

Remark 6.1.10. Taking into account Theorem 6.1.6 we get in Theorem 6.1.9, the existence of $v^* \in Y^*$ such that (6.6) holds.

6.1.2.2 Fuzzy Optimality Conditions

In the proof of the next result we use the functional $\theta_{\bar{y}} : Y \to \mathbb{R}$ defined by (4.2)

$$\theta_{\bar{y}}(y) = \inf\{t \in \mathbb{R} \mid y \in tk^0 - C(\bar{y})\}. \tag{6.7}$$

where $C(\bar{y}) \subset Y$ is a proper, closed and convex cone with nonempty interior and $k^0 \in \operatorname{int} C(\bar{y})$.

Under the given assumptions this functional is continuous and convex and its subdifferential is given by

$$\partial \theta_{\bar{y}}(u) = \{v^* \in C(\bar{y})^* \mid v^*(k^0) = 1, v^*(u) = \theta_{\bar{y}}(u)\} \tag{6.8}$$

(see [21, Lemma 2.1]).

In the next theorem we show necessary conditions for approximately minimal elements of a vector optimization problem with a variable ordering structure following the line of the proof of [21, Theorem 5.3].

Theorem 6.1.11. Let assumptions (A12) and (A13) be fulfilled, $X, Y \in \mathscr{X}$, $f \in \mathscr{F}(X,Y)$ be a L-Lipschitz function and $\mathfrak{S}$ be a closed subset of the Banach space X. Let $x_\varepsilon \in \varepsilon k^0$-M$(\mathfrak{S}, f, C)$ and $C(f(x_\varepsilon))$ is a closed convex cone with nonempty interior. Then for every $k^0 \in \operatorname{int} C(f(x_\varepsilon))$ and $\mu > 0$, there exist elements $u \in B(x_\varepsilon, \sqrt{\varepsilon} + \mu), z \in B(x_\varepsilon, \sqrt{\varepsilon} + \mu/2) \cap \mathfrak{S}$, $u^* \in (C(f(x_\varepsilon)))^*$, $u^*(k^0) = 1$, $x^* \in X^*$, $\|x^*\| \leq 1$ such that

$$\mathbf{0} \in \partial(u^* \circ f)(u) + \sqrt{\varepsilon} u^*(k^0) x^* + N_\partial(\mathfrak{S}, z) + B(0, \mu),$$

provided that ∂ satisfies (H1), (H2), (H6), (H7).
Moreover, for some elements $x \in B(x_\varepsilon, \sqrt{\varepsilon} + \mu/2)$ and $v \in B(f(x) - f(x_\varepsilon), L\sqrt{\varepsilon} + \mu)$ it holds $u^*(v) = \theta_{f(x_\varepsilon)}(v)$.

Proof. We consider $x_\varepsilon \in \varepsilon k^0$-M$(\mathfrak{S}, f, C)$. Taking into account Definition 5.0.8 we have

$$(f(x_\varepsilon) - \varepsilon k^0 - C(f(x_\varepsilon)) \setminus \{\mathbf{0}\}) \cap f(\mathfrak{S}) = \emptyset.$$

The function f is supposed to be Lipschitz, so it is continuous as well and since $\mathfrak{S}$ is a closed set in a Banach space it is a complete metric space endowed with the distance induced by the norm. Thus, the assumptions of the vector-valued variant of Ekeland's variational principle given in [34, Corollary 9] are fulfilled. Applying this variational principle we get the existence of an element $\bar{x} \in \mathfrak{S}$ with $\|\bar{x} - x_\varepsilon\| < \sqrt{\varepsilon}$ and having the property that

$$h(\mathfrak{S}) \cap (h(\bar{x}) - C(f(x_\varepsilon)) \setminus \{\mathbf{0}\}) = \emptyset,$$

where

$$h(x) := f(x) + \sqrt{\varepsilon}\, \|x - \bar{x}\|\, k^0.$$

Let $\mu > 0$. Applying now [21, Theorem 4.2] for a positive number δ with properties that $2\delta < \mu$ and $\sqrt{\varepsilon}\left\|k^0\right\|\delta/2+\delta/2 < \mu$ and using the functional $\theta_{f(x_\varepsilon)}$ defined by (4.2), we can find elements $\overline{u} \in B(\overline{x},\delta) \subset B(x_\varepsilon,\sqrt{\varepsilon}+\delta)$, $x \in B(\overline{x},\delta/2) \subset B(x_\varepsilon,\sqrt{\varepsilon}+\delta/2)$, $v \in B(h(x)-h(\overline{x}),\delta/2)$, $z \in B(\overline{x},\delta/2)\cap\mathfrak{S} \subset B(x_\varepsilon,\sqrt{\varepsilon}+\delta/2)\cap\mathfrak{S}$, $u^* \in \partial\theta_{f(x_\varepsilon)}(v)$, such that

$$\mathbf{0} \in \partial(u^*\circ h)(\overline{u})+N_\partial(\mathfrak{S},z)+B(0,\delta). \tag{6.9}$$

We get the properties $u^* \in (C(f(x_\varepsilon)))^*, u^*(k^0) = 1$ from (6.8).

Consider the element $\overline{x}^* \in \partial(u^*\circ h)(\overline{u})$ involved in (6.9). Because of

$$\partial(u^*\circ h)(\overline{u}) = \partial(u^*\circ(f(\cdot)+\sqrt{\varepsilon}\,\|\cdot-\overline{x}\|\,k^0))(\overline{u}),$$

taking into account (H1) and (H6), there exist $u \in B(\overline{u},\delta) \subset B(x_\varepsilon,\sqrt{\varepsilon}+2\delta)$ and $u' \in B(\overline{u},\delta)$ such that

$$\overline{x}^* \in \partial(u^*\circ f)(u)+\sqrt{\varepsilon}u^*(k^0)\partial(\|\cdot-\overline{x}\|)(u')+B(0,\delta). \tag{6.10}$$

Taking into account the well-known structure of the subdifferential of the norm and combining relations (6.9) and (6.10) it follows that there exists $x^* \in X^*$ with $||x^*|| = 1$ such that

$$\mathbf{0} \in \partial(u^*\circ f)(u)+\sqrt{\varepsilon}u^*(k^0)x^*+N_\partial(\mathfrak{S},z)+B(0,2\delta).$$

Because $2\delta < \mu$, it remains only to prove the estimation about the ball which contains v. Then,

$$\begin{aligned}
\|v-(f(x)-f(x_\varepsilon))\| &\le \|v-(h(x)-h(\overline{x}))\|+\|(h(x)-h(\overline{x}))-(f(x)-f(x_\varepsilon))\|\\
&\le \delta/2+\left\|\sqrt{\varepsilon}k^0\,\|x-\overline{x}\|-f(\overline{x})+f(x_\varepsilon)\right\|\\
&\le \delta/2+\sqrt{\varepsilon}\left\|k^0\right\|\delta/2+L\sqrt{\varepsilon}\\
&< L\sqrt{\varepsilon}+\mu,
\end{aligned}$$

where the last inequality follows because of the assumptions made on δ. Moreover, we get $u^*(v) = \theta_{\overline{y}}(v)$ from (6.8). This completes the proof. □

6.1.3 Second Order Optimality Conditions

It is known that duality principles in vector optimization, fuzzy optimization, inverse problems, etc can be studied using approaches from set-valued optimization. There are two important approaches to optimality conditions for set-valued optimization. One is using derivatives of the involved set-valued maps and another one is using the alternative type theorems. Corley (1988) used contingent derivatives in [18] in order to give optimality conditions in set-valued optimization and later many papers come out giving optimality conditions for set-valued optimization;

see [7, 30, 42, 45–48, 50–52, 57]. Jahn [47] introduced second order contingent epiderivatives and using this, he gave second order optimality conditions for set-valued optimization with fixed ordering structures. In the paper by Isac and Khan [42], second order optimality conditions are given when the underlying second order contingent sets are empty. Here we generalize optimality conditions given by Isac and Khan [42] for set-valued optimization problems with variable ordering structures. In the beginning of this chapter, we recalled first and second order contingent derivatives, contingent epiderivatives, second order tangential derivatives and epiderivatives. Now we used them in order to give second order optimality conditions for set-valued optimization problems with variable ordering structures. Let X and Y be two separated topological vector space and $F : X \rightrightarrows Y$ be a set-valued map. The domain and the graph of F are given by $\operatorname{dom} F := \{x \in X \mid F(x) \neq \emptyset\}$ and $\operatorname{gph} F = \{(x,y) \in X \times Y \mid y \in F(x),\ x \in \operatorname{dom} F\}$, respectively.

In order to give second order optimality conditions for approximate solutions of set-valued optimization problems with variable ordering structures, we bring some definitions which will be used later.

Definition 6.1.12. Let $K : X \rightrightarrows Y$ be a cone-valued map where $K(x)$ is a pointed, solid, closed and convex cone for all $x \in X$.

$$\operatorname{epi} F := \{(x,y) \in X \times Y | \ y \in F(x) + K(x)\}.$$

The profile F_+ with respect to the map $K : X \rightrightarrows Y$ is defined as following:

$$F_+(x) = F(x) + K(x).$$

It is easy to see that $\operatorname{epi}(F) = \operatorname{gph}(F_+)$.

We bring following definitions of second order tangent derivative and second order epiderivative of set-valued maps in order to give second order optimality condition for weak minimality in set-valued optimization with variable ordering structures. For this, first we bring definitions of contingent cones, second order contingent sets and contingent derivatives.

Definition 6.1.13. Let X be a real normed space and $\mathfrak{S}$ be a nonempty subset of X.

1. The contingent cone $T(\mathfrak{S},\bar{x})$ of $\mathfrak{S}$ at $\bar{x} \in \operatorname{cl}\mathfrak{S}$ is the set of all $x \in X$ such that there are the sequences $(t_n) \subset \mathbb{R}_+$ with $t_n \downarrow 0$ and $(x_n) \subset X$ with $x_n \to x$ such that $\bar{x} + t_n x_n \in \mathfrak{S}$.

2. The interiorly contingent cone $IT(\mathfrak{S},\bar{x})$ of $\mathfrak{S}$ at $\bar{x} \in \operatorname{cl}\mathfrak{S}$ is the set of all $x \in X$ such that for any sequences $(t_n) \subset \mathbb{R}_+$ with $t_n \downarrow 0$ and $(x_n) \subset X$ with $x_n \to x$, there exists $n_1 \in \mathbb{N}$ such that $\bar{x} + t_n x_n \in \mathfrak{S}$ for all $n \geq n_1$.

3. The second order contingent set $T^2(\mathfrak{S},\bar{x},d)$ of the set $\mathfrak{S}$ at $\bar{x} \in \operatorname{cl}\mathfrak{S}$ in the direction of $d \in X$ is the set of all $x \in X$ such that there are the sequences $t_n \subset \mathbb{R}_+$ with $(t_n) \downarrow 0$ and $(x_n) \subset X$ with $x_n \to x$ such that $\bar{x} + t_n d + (\frac{t_n^2}{2}) x_n \in \mathfrak{S}$.

Remark 6.1.14. The following properties hold for (interiorly) contingent cones and second order contingent sets.

1. $T(\mathfrak{S},\bar{x})$ is nonempty closed cone.
2. $T^2(\mathfrak{S},\bar{x},d)$ is closed set, possible empty and nonconnected.
3. If $d \in T(\mathfrak{S},\bar{x})$, then $T^2(\mathfrak{S},\bar{x},d)$ may be nonempty.
4. $IT(\mathfrak{S},\bar{x})$ is an open cone and $IT(\mathfrak{S},\bar{x}) = X \setminus T(X \setminus \mathfrak{S},\bar{x})$.
5. $T(\mathfrak{S},\bar{x}) = T(\mathrm{cl}\,\mathfrak{S},\bar{x})$, $\mathrm{int}(T(\mathfrak{S},\bar{x})) = IT(\mathfrak{S},\bar{x})$, $IT(\mathfrak{S},\bar{x}) = IT(\mathrm{int}\,\mathfrak{S},\bar{x})$ and $\mathrm{cl}(IT(\mathfrak{S},\bar{x})) = T(\mathfrak{S},\bar{x})$.

Definition 6.1.15. Let X,Y be normed spaces, $F : X \rightrightarrows Y$ be a set-valued map and $K : X \rightrightarrows Y$ be a cone-valued map where $K(x)$ is a proper, convex, and pointed cone for all $x \in X$.

1. The contingent derivative of F at $(\bar{x},\bar{y})$ is the set-valued map $D_cF(\bar{x},\bar{y}) : X \rightrightarrows Y$ defined by
$$\mathrm{gph}\,(D_cF(\bar{x},\bar{y})) := T(\mathrm{gph}\,(F),(\bar{x},\bar{y})).$$
2. Let $(\bar{x},\bar{y}) \in \mathrm{gph}\,(F)$ be given. A single-valued map $DF(\bar{x},\bar{y}) : X \to Y$ whose epigraph equals contingent cone to the epigraph of F at $(\bar{x},\bar{y})$, i.e.,
$$\mathrm{epi}(DF(\bar{x},\bar{y})) = T(\mathrm{epi}(F),(\bar{x},\bar{y})),$$
is called contingent epiderivative of F at $(\bar{x},\bar{y})$.

For more details about contingent epiderivative see [48]. Now we are ready to bring definitions of second order contingent derivatives and second order tangential derivative/epiderivatives. These definitions will be used later in order to give second order optimality conditions for local weakly minimal solutions of set-valued optimization problems with variable ordering structures.

Definition 6.1.16. The second order contingent derivative of $F : X \rightrightarrows Y$ at $(\bar{x},\bar{y}) \in \mathrm{gph}\,(F)$ in the direction $(\bar{u},\bar{v})$ is a set-valued map $D_c^2F(\bar{x},\bar{y},\bar{u},\bar{v}) : X \rightrightarrows Y$ defined by

$$D_c^2F(\bar{x},\bar{y},\bar{u},\bar{v})(x) = \{y \in Y |\ (x,y) \in T^2(\mathrm{gph}\,(F),(\bar{x},\bar{y}),(\bar{u},\bar{v}))\}.$$

Remark 6.1.17. If $(\bar{u},\bar{v}) = (\mathbf{0}_X,\mathbf{0}_Y)$, then from $D_c^2F(\bar{x},\bar{y},\bar{u},\bar{v})$, we recover the contingent derivative $D_cF(\bar{x},\bar{y})$ of F at $(\bar{x},\bar{y})$.

Definition 6.1.18. Let X,Y be real normed spaces, $F,K : X \rightrightarrows Y$ be set-valued maps where $K(x)$ is a pointed, closed and convex cone for all $x \in X$. Let $(\bar{x},\bar{y}) \in \mathrm{gph}\,(F)$ and $(\bar{u},\bar{v}) \in X \times Y$.

1. A set-valued map $D^2F(\bar{x},\bar{y},\bar{u},\bar{v}) : X \rightrightarrows Y$ given by

$$D^2F(\bar{x},\bar{y},\bar{u},\bar{v}) = \{y \in Y \mid (x,y) \in T(T(\mathrm{gph}(F),(\bar{x},\bar{y})),(\bar{u},\bar{v}))\}$$

is called second order tangential derivative of F at $(\bar{x},\bar{y})$ in the direction of $(\bar{u},\bar{v})$.

2. A single-valued map $D_e^2F(\bar{x},\bar{y},\bar{u},\bar{v}) : X \to Y$ given by

$$\mathrm{epi}(D_e^2F(\bar{x},\bar{y},\bar{u},\bar{v})) = T(T(\mathrm{epi}(F),(\bar{x},\bar{y})),(\bar{u},\bar{v}))$$

is called second-order tangential epiderivative of F at $(\bar{x},\bar{y})$ at the direction $(\bar{u},\bar{v})$.

Assumption (A14). Let X,Y be normed spaces, $\mathfrak{S} \subset X$ and $K : X \rightrightarrows Y$ be a set-valued map where $K(x)$ is a nontrivial, pointed, solid, closed and convex cone for all $x \in X$. Let $F : \mathfrak{S} \rightrightarrows Y$ be a set-valued map.

Under assumption (A14) we consider the following optimization problem with respect to a variable ordering structure:

$$\text{Minimize } F(x) \quad \text{subject to } x \in \mathfrak{S}. \tag{P_1}$$

$(\bar{x},\bar{y}) \in \mathrm{gph}(F)$ is called a weakly minimal solution of (P_1) with respect to K if and only if $(\bar{y} - \mathrm{int}K(\bar{x})) \cap F(\mathfrak{S}) = \emptyset$. Also $(\bar{x},\bar{y}) \in \mathrm{gph}(F)$ is called a local weakly minimal solution of (P_1) iff there exists a neighborhood U of $\bar{x}$ such that

$$(\bar{y} - \mathrm{int}K(\bar{x}) \cap F(\mathfrak{S} \cap U) = \emptyset.$$

Remark 6.1.19. Observe that in the definition of optimal solutions in the third chapter, our ordering map $C : Y \rightrightarrows Y$ has the same origin and image space Y while here our ordering map $K : X \rightrightarrows Y$ is defined from origin space X to the image space Y. This will be important in the following and subsections 6.2.3 and 6.3.3 because multifunction $F : X \rightrightarrows Y$ will be used instead of $f : X \to Y$ and output of $F(x)$ is a set for each $x \in X$.

In the following theorem, we give necessary optimality conditions for local weakly minimal solutions of (P_1); see [43] for the case of fixed ordering structure. Let $(\bar{x},\bar{y})$ be a weakly minimal solution of (P_1), we define $(F + K(\bar{x})) : X \rightrightarrows Y$ as $(F + K(\bar{x}))(x) := F(x) + K(\bar{x})$. Please note that $K(\bar{x})$ is fixed here and is different from profile $F_+(x) = F(x) + K(x)$.

Theorem 6.1.20. Let assumption (A14) be fulfilled and $(\bar{x},\bar{y}) \in \mathrm{gph}(F)$ be a local weakly minimal solution of problem (P_1), then for every $\bar{u} \in \mathrm{dom}(D(F + K(\bar{x}))(\bar{x},\bar{y}))$ and for every $\bar{v} \in D(F + K(\bar{x}))(\bar{x},\bar{y})(\bar{u}) \cap (-K(\bar{x}))$, the following holds:

$$D^2(F + K(\bar{x}))(\bar{x},\bar{y},\bar{u},\bar{v})(x) \cap IT(-\mathrm{int}K(\bar{x}),\bar{v}) = \emptyset \tag{6.11}$$

for all $x \in \operatorname{dom}(D^2(F+K(\bar{x}))(\bar{x},\bar{y},\bar{u},\bar{v}))$.

Proof. Suppose that (6.11) does not hold and there exists $x \in \operatorname{dom}(D^2(F+K(\bar{x}))(\bar{x},\bar{y},\bar{u},\bar{v}))$ such that

$$y \in D^2(F+K(\bar{x}))(\bar{x},\bar{y},\bar{u},\bar{v})(x) \cap IT(-\operatorname{int} K(\bar{x}),\bar{v})),$$

then $(x,y) \in T(T(\operatorname{gph}(F)+K(\bar{x}),(\bar{x},\bar{y})),(\bar{u},\bar{v}))$. Therefore there exist sequences $(t_n) \in \mathbb{R}_+$ with $t_n \downarrow 0$ and $(x_n,y_n) \subset X \times Y$ with $(x_n,y_n) \to (x,y)$ such that

$$(\bar{u}+t_nx_n,\bar{v}+t_ny_n) \in T(\operatorname{gph}(F)+K(\bar{x}),(\bar{x},\bar{y})) \qquad \forall n \in \mathbb{N}.$$

By $t_n \downarrow 0$, $y_n \to y$ and $y \in IT(-\operatorname{int} K(\bar{x}),\bar{v})$, there exists $n_1 \in \mathbb{N}$ such that

$$\bar{v}+t_ny_n \in -\operatorname{int} K(\bar{x}), \qquad \forall n > n_1.$$

For any $n > n_1$, we fix an element $(u_n,v_n) = (\bar{u}+t_nx_n,\bar{v}+t_ny_n)$ and notice that

$$(u_n,v_n) \in T(\operatorname{gph}(F)+K(\bar{x}),(\bar{x},\bar{y})).$$

By the definition of contingent cones, for (u_n,v_n), there exist sequences $(x_m,y_m) \subset X \times Y$ with $(x_m,y_m) \to (u_n,v_n)$ and $(t_m) \subset \mathbb{R}_+$ with $t_m \downarrow 0$ such that

$$\bar{y}+t_my_m \in F(\bar{x}+t_mx_m)+K(\bar{x}).$$

Furthermore, by $v_n \in -\operatorname{int} K(\bar{x})$ and $y_m \to v_n$, there exists $m_1 \in \mathbb{N}$ such that $y_m \in -\operatorname{int} K(\bar{x})$ for all $m > m_1$. Since $K(\bar{x})$ is a cone, this implies that $t_my_m \in -\operatorname{int} K(\bar{x})$. Now assume that $\omega_m \in F(\bar{x}+t_mx_m)$ such that $\bar{y}+t_my_m \in \omega_m+K(\bar{x})$. Then

$$\omega_m \in \bar{y}-\operatorname{int} K(\bar{x}).$$

Since $b_m := (\bar{x}+t_mx_m) \to \bar{x}$, there exists $m_2 > 0$ such that $b_m \in \mathscr{N}(\bar{x})$ where $\mathscr{N}(\bar{x})$ is a suitable neighborhood of $\bar{x}$. Therefore, we showed that there exists a sequence $\{\omega_m\}$ such that

$$\omega_m \in F(b_m) \cap (\bar{y}-\operatorname{int} K(\bar{x})) \qquad \text{for all } m > \{m_1,m_2\}.$$

This is a contradiction to weak minimality of $(\bar{x},\bar{y})$ and proof is complete. $\square$

If we set $(\bar{x},\bar{y}) = (\mathbf{0}_X,\mathbf{0}_Y)$, we have the following corollary.

Corollary 6.1.21. Let assumption (A14) be fulfilled and $(\bar{x},\bar{y}) \in \operatorname{gph}(F)$ be a local weakly minimal solution of problem (P_1), then

$$D(F+K(\bar{x}))(\bar{x},\bar{y}) \cap -\operatorname{int} K(\bar{x}) = \emptyset \quad \text{for all} \quad x \in \operatorname{dom}(D(F+K(\bar{x}))(\bar{x},\bar{y})).$$

Example 6.1.22. Suppose that $X = Y = \mathbb{R}$, $\mathfrak{S} = [1,2] \subset X$, $f(x) = 2x$, $g(x) = 3x$ and $F : X \rightrightarrows Y$ be a set-valued map such that

$$F(x) = \{y \in Y |\ f(x) \leq y \leq g(x)\}.$$

We also define $K : X \rightrightarrows Y$ as following

$$K(x) = \begin{cases} [0,+\infty) & \text{if } x = 1,2 \\ (-\infty, 0] & \text{otherwise.} \end{cases}$$

Then $F(\mathfrak{S}) = \bigcup_{x\in\mathfrak{S}} F(x) = [2,6]$ and $(1,2)$ is a weakly minimal solution because

$$F(\mathfrak{S}) \cap \big(\bar{y} - \operatorname{int} K(\bar{x})\big) = [2,6] \cap (2 - (0,+\infty)) = [2,6] \cap (-\infty, 2) = \emptyset.$$

Also $D(F+K(1))(1,2) \subset [0,+\infty)$ and $\operatorname{gph}(D(F+K(1))(1,2)) = \{(x,y) \in \mathbb{R}^2_+ |\ y \geq 2x\}$. Therefore, for any $(x,y) \in T(\operatorname{gph}(F+K(1)),(1,2))$, we have $y \geq \mathbf{0}$. This means that

$$D(F+K(1))(1,2) \cap -\operatorname{int} K(1) = \emptyset.$$

The following theorem shows that (6.11) is in fact a sufficient optimality condition under a certain convexity assumption; see [43] for the case of fixed ordering structure.

Theorem 6.1.23. Let assumption (A14) be fulfilled, $\bar{x} \in \mathfrak{S}$ and $\operatorname{gph}(F+K(\bar{x}))$ be convex. Let for every $\bar{u} \in \operatorname{dom}(D(F+K(\bar{x}))(\bar{x},\bar{y}))$, every $\bar{v} \in D(F+K(\bar{x}))(\bar{x},\bar{y})(\bar{u}) \cap \big(-\operatorname{bd} K(\bar{x})\big)$ and for all $x \in \operatorname{dom}(D^2(F+K(\bar{x}))(\bar{x},\bar{y},\bar{u},\bar{v}))$, the following holds:

$$D^2(F+K(\bar{x}))(\bar{x},\bar{y},\bar{u},\bar{v})(x) \cap IT(-\operatorname{int} K(\bar{x}),\bar{v}) = \emptyset \tag{6.12}$$

Then $(\bar{x},\bar{y}) \in \operatorname{gph}(F)$ is a weakly minimal solution of (P_1).

Proof. By setting $(\bar{u},\bar{v}) = (\mathbf{0}_X, \mathbf{0}_Y)$ and by (6.12), we obtain

$$D(F+K(\bar{x}))(\bar{x},\bar{y}) \cap -\operatorname{int} K(\bar{x}) = \emptyset.$$

Notice that $(x-\bar{x}) \in \operatorname{dom}(D(F+K(\bar{x}))(\bar{x},\bar{y}))$ for $x \in \mathfrak{S}$ and under the convexity assumption, we have $y - \bar{y} \subset D(F)(\bar{x},\bar{y})(x-\bar{x})$, therefore

$$(y-\bar{y}) \cap -\operatorname{int} K(\bar{x}) = \emptyset.$$

Therefore $F(\mathfrak{S}) \cap (\bar{y} - \operatorname{int} K(\bar{x})) = \emptyset$ and this completes the proof. □

Assumption (A15). Let X,Y,Z be normed spaces and $K : X \rightrightarrows Y$ be a set-valued map where $K(x)$ is a nontrivial, pointed, closed, solid and convex cone for all $x \in X$. Let $F : \mathfrak{S} \rightrightarrows Y$ and $G : \mathfrak{S} \rightrightarrows Z$ be set-valued maps and $D \subset Z$ is a nontrivial, solid, pointed, closed and convex cone.

Under assumption (A15) we consider the following optimization problem with respect to a variable ordering structure:

$$\text{Minimize } F(x) \quad \text{subject to } x \in \mathfrak{S}_1 := \{x \in \mathfrak{S} \mid G(x) \cap -D \neq \emptyset\}. \tag{P_2}$$

The following theorem gives a necessary optimality condition for local weakly minimal solutions of (P_2); see [43] for the case of fixed ordering structure.

Theorem 6.1.24. Let assumption (A15) be fulfilled. Suppose that $(\bar{x},\bar{y}) \in \mathrm{gph}\,(F)$ is a local weakly minimal solution of (P_2) and $\bar{z} \in G(\bar{x})$. Then for every $\bar{u} \in \mathrm{dom}\,(D(F,G)(\bar{x},\bar{y},\bar{z}))$, every $(\bar{v},\overline{\omega}) \in D(F,G)(\bar{x},\bar{y},\bar{z}) \cap \{-K(\bar{x}) \times -D\}$ and every $x \in D_1 := \mathrm{dom}\,(D^2(F,G)(\bar{x},\bar{y},\bar{z},\bar{u},\bar{v},\overline{\omega}))$, we have

$$D^2(F,G)(\bar{x},\bar{y},\bar{z},\bar{u},\bar{v},\overline{\omega}))(x) \cap IT(-K(\bar{x}),\bar{v}) \times IT(IT(-D,\bar{z}),\overline{\omega}) = \emptyset \tag{6.13}$$

where the notation $(F,G)(x)$ represents $(F(x)+K(\bar{x})) \times (G(x)+D)$.

Proof. Suppose that the assertion is not true and there exists $x \in \mathrm{dom}\,(D^2(F,G)(\bar{x},\bar{y},\bar{z},\bar{u},\bar{v},\overline{\omega}))$ such that $(y,z) \in D^2(F,G)(\bar{x},\bar{y},\bar{z},\bar{u},\bar{v},\overline{\omega})(x) \cap IT(-K(\bar{x}),\bar{v}) \times IT(IT(-\mathrm{int}\,D,\bar{z}),\overline{\omega})$ which implies that $(x,y,z) \in T(T(\mathrm{gph}\,(F,G),(\bar{x},\bar{y},\bar{z})),(\bar{u},\bar{v},\overline{\omega}))$. This means there exist sequences (t_n) in $\mathbb{R}_+$ with $t_n \downarrow 0$ and (x_n,y_n,z_n) in the product space $X \times Y \times Z$ with $(x_n,y_n,z_n) \to (x,y,z)$ such that

$$(\bar{u}+t_n x_n, \bar{v}+t_n y_n, \overline{\omega}+t_n z_n) \in T(\mathrm{gph}\,(F,G),(\bar{x},\bar{y},\bar{z})).$$

By $y \in IT(-\mathrm{int}\,K(\bar{x}),\bar{v})$, there exists $n_1 \in \mathbb{N}$ such that

$$\bar{v}+t_n y_n \in -\mathrm{int}\,K(\bar{x}) \qquad n \geq n_1.$$

Analogously by $z \in IT(IT(-\mathrm{int}\,D,\bar{z}),\overline{\omega})$, there exists $n_2 \in \mathbb{N}$ such that $\overline{\omega}+t_n z_n \in IT(-\mathrm{int}\,D,\bar{z})$ for all $n > n_2$. For $n \geq \max\{n_1,n_2\}$, we fix $u_n := \bar{u}+t_n x_n$, $v_n := \bar{v}+t_n y_n$ and $\omega_n := \overline{\omega}+t_n z_n$. Then

$$(u_n,v_n,\omega_n) \in T(\mathrm{gph}\,(F,G),(\bar{x},\bar{y},\bar{z}))$$

$$(v_n,\omega_n) \in -\mathrm{int}\,K(\bar{x}) \times IT(-\mathrm{int}\,D,\bar{z}).$$

By definition of contingent cones, there exist sequences $(t_m) \subset \mathbb{R}_+$ with $t_m \downarrow 0$ and (x_m,y_m,z_m) in $X \times Y \times Z$ with $(x_m,y_m,z_m) \to (u_n,v_n,\omega_n)$ such that $(\bar{y}+t_m y_m, \bar{z}+t_m z_m) \in (F,G)(\bar{x}+t_m x_m)$ which means

$$\bar{y}+t_m y_m \in F(\bar{x}+t_m x_m)+K(\bar{x}),$$

$$\bar{z}+t_m z_m \in G(\bar{x}+t_m x_m)+D.$$

By $y_m \to v_n$ and $v_n \in -\mathrm{int}\,K(\bar{x})$, there exists $m_1 > 0$ such that

$$t_m y_m \in -\mathrm{int}\,K(\bar{x}) \qquad \forall m > m_1.$$

Let $a_m \in F(\bar{x}+t_m x_m)$ be such that $\bar{y}-\operatorname{int}K(\bar{x}) \in a_m + K(\bar{x})$ and consequently we have:

$$a_m \in \bar{y}-\operatorname{int}K(\bar{x}).$$

By $\omega_n \in IT(-\operatorname{int}D, \bar{z})$, there exists $m_2 > 0$ such that $\bar{z}+t_m z_m \in -\operatorname{int}D$ for every $m > m_2$. Let $b_m \in G(\bar{x}+t_m x_m)$ be such that $\bar{z}+t_m z_m \in b_m + D$ and therefore we have $b_m \in -\operatorname{int}D$. Therefore, for sufficiently large m, we have $c_m := \bar{x}+t_m x_m \in \mathscr{N}(\bar{x})$, $G(c_m)\cap -D \neq \emptyset$ and

$$F(c_m)\cap(\bar{y}-\operatorname{int}K(\bar{x})) \neq \emptyset.$$

But this is a contradiction to the local weak minimality of $(\bar{x},\bar{y})$ and proof is complete. □

In assumption (A15), we suppose that $D \subset Z$ is a fixed, nontrivial, solid, pointed and convex cone for all $x \in X$. In the following assumption, we suppose that D is not fixed and it is defined by a convex cone-valued map $D : X \rightrightarrows Z$.

Assumption (A16). Let X,Y,Z be normed spaces, $K : X \rightrightarrows Y$ be a set-valued map such that $K(x) \subset Y$ is a nontrivial, pointed, solid, closed, and convex cone for all $x \in X$, $F : \mathfrak{S} \rightrightarrows Y$ and $G : \mathfrak{S} \rightrightarrows Z$ be set-valued maps and $D : X \rightrightarrows Z$ be a cone-valued map where $D(x) \subset Z$ is a nontrivial, solid, pointed, closed and convex cone for all $x \in X$.

Under assumption (A16) we consider the following optimization problem with respect to a variable ordering structure:

$$\text{Minimize } F(x) \quad \text{subject to } x \in \mathfrak{S}_2 := \{x \in \mathfrak{S} \mid G(x)\cap -D(x) \neq \emptyset.\} \tag{P_3}$$

The following theorem gives a necessary optimality condition for local weakly minimal solution of (P_3).

Theorem 6.1.25. Let assumption (A16) be fulfilled. Suppose that $(\bar{x},\bar{y}) \in \operatorname{gph}(F)$ is a local weakly minimal solution of problem (P_3), $\bar{z} \in G(\bar{x})$ and $D(\bar{x}) \subseteq D(x)$ for all $x \in \mathscr{N}(\bar{x})$. Then for every $\bar{u} \in \operatorname{dom}(D(\widehat{F,G})(\bar{x},\bar{y},\bar{z}))$, for every $(\bar{v},\bar{\omega}) \in D(\widehat{F,G})(\bar{x},\bar{y},\bar{z}) \cap \{-K(\bar{x}) \times -D(\bar{x})\}$ and for every $x \in D_1 := \operatorname{dom}(D^2(\widehat{F,G})(\bar{x},\bar{y},\bar{z},\bar{u},\bar{v},\bar{\omega}))$, we have

$$D^2(\widehat{F,G})(\bar{x},\bar{y},\bar{z},\bar{u},\bar{v},\bar{\omega})(x) \cap IT(-K(\bar{x}),\bar{v}) \times IT(IT(-D(\bar{x}),\bar{z}),\bar{\omega}) = \emptyset$$

where the notation $(\widehat{F,G})(x)$ represents $(F(x)+K(\bar{x}))\times(G(x)+D(x))$.

Proof. Suppose that assertion is not true and there exists $x \in \operatorname{dom}(D^2(\widehat{F,G})(\bar{x},\bar{y},\bar{z},\bar{u},\bar{v},\bar{\omega}))$ such that $(y,z) \in D^2(\widehat{F,G})(\bar{x},\bar{y},\bar{z},\bar{u},\bar{v},\bar{\omega})(x) \cap IT(-K(\bar{x}),\bar{v}) \times IT(IT(-\operatorname{int}D(\bar{x}),\bar{z}),\bar{\omega})$ which implies that $(x,y,z) \in T(T(\operatorname{gph}(\widehat{F,G}),(\bar{x},\bar{y},\bar{z})),(\bar{u},\bar{v},\bar{\omega}))$. Therefore, there exist sequences $(t_n) \subset \mathbb{R}_+$ with $t_n \downarrow 0$ and $(x_n,y_n,z_n) \subset X\times Y\times Z$ with $(x_n,y_n,z_n) \to (x,y,z)$ such that

$$(\bar{u}+t_n x_n, \bar{v}+t_n y_n, \bar{\omega}+t_n z_n) \in T(\operatorname{gph}(\widehat{F,G}),(\bar{x},\bar{y},\bar{z})).$$

By $y \in IT(-\operatorname{int} K(\bar{x}), \bar{v})$, there exists $n_1 \in \mathbb{N}$ such that

$$\bar{v} + t_n y_n \in -\operatorname{int} K(\bar{x}) \qquad n \geq n_1.$$

By $z \in IT(IT(-\operatorname{int} D(\bar{x}), \bar{z}), \overline{\omega})$, there exists $n_2 \in \mathbb{N}$ such that $\overline{\omega} + t_n z_n \in IT(-\operatorname{int} D(\bar{x}), \bar{z})$ for all $n > n_2$. For $n \geq \max\{n_1, n_2\}$, we fix $u_n := \bar{u} + t_n x_n$, $v_n := \bar{v} + t_n y_n$ and $\omega_n := \overline{\omega} + t_n z_n$. Then

$$(u_n, v_n, \omega_n) \in T(\operatorname{gph}(\widehat{F,G}), (\bar{x}, \bar{y}, \bar{z})),$$
$$(v_n, \omega_n) \in -\operatorname{int} K(\bar{x}) \times IT(-\operatorname{int} D(\bar{x}), \bar{z}).$$

By definition of contingent cones, there exist sequences $(t_m) \subset \mathbb{R}_+$ with $t_m \downarrow 0$ and (x_m, y_m, z_m) in $X \times Y \times Z$ with $(x_m, y_m, z_m) \to (u_n, v_n, \omega_n)$ such that $(\bar{y} + t_m y_m, \bar{z} + t_m z_m) \in (\widehat{F,G})(\bar{x} + t_m x_m)$ and

$$\bar{y} + t_m y_m \in F(\bar{x} + t_m x_m) + K(\bar{x}),$$
$$\bar{z} + t_m z_m \in G(\bar{x} + t_m x_m) + D(\bar{x} + t_m x_m).$$

By $y_m \to v_n$ and $v_n \in -\operatorname{int} K(\bar{x})$, there exists $m_1 > 0$ such that

$$t_m y_m \in -\operatorname{int} K(\bar{x}) \qquad \forall m > m_1.$$

Let $a_m \in F(\bar{x} + t_m x_m)$ be such that $\bar{y} - \operatorname{int} K(\bar{x}) \in a_m + K(\bar{x})$ and consequently we have:

$$a_m \in \bar{y} - \operatorname{int} K(\bar{x}).$$

By $\omega_n \in IT(-\operatorname{int} D(\bar{x}), \bar{z})$, there exists $m_2 > 0$ such that $\bar{z} + t_m z_m \in -\operatorname{int} D(\bar{x})$ for every $m > m_2$. Let $b_m \in G(\bar{x} + t_m x_m)$ be such that $\bar{z} + t_m z_m \in b_m + D(\bar{x} + t_m x_m)$. By this assumption we get $b_m \in -\operatorname{int} D(\bar{x}) - D(\bar{x} + t_m x_m)$. By $D(\bar{x}) \subseteq D(x)$ for all $x \in \mathscr{N}(\bar{x})$ and since $D(x)$ is a convex cone for all x, we get

$$b_m \in -\operatorname{int} D(\bar{x}) - D(\bar{x} + t_m x_m) \subseteq -\operatorname{int} D(\bar{x} + t_m x_m) - D(\bar{x} + t_m x_m) \subseteq -\operatorname{int} D(\bar{x} + t_m x_m).$$

Therefore, for sufficiently large m, $c_m := \bar{x} + t_m x_m \in \mathscr{N}(\bar{x})$, $G(c_m) \cap -D(c_m) \neq \emptyset$ and

$$F(c_m) \cap (\bar{y} - \operatorname{int} K(\bar{x})) \neq \emptyset.$$

This is a contradiction to the local weakly minimality of $(\bar{x}, \bar{y})$. □

6.2 Optimality Conditions for Approximately Nondominated Solutions

In this section, we present optimality conditions for approximately nondominated solutions of vector optimization problems with variable ordering structures with different approaches namely

Mordukhovich subdifferential approach and general generic approach. In the last subsection, contingent derivatives and epiderivatives will be used in order to derive second order optimality condition for weakly nondominated solutions of set-valued optimization. Given variational principle in the previous chapter for approximately nondominated solutions and characterization of approximately nondominated solutions of vector optimization problems with variable ordering structures in the fourth chapter will be used here in order to derive optimality conditions for approximately nondominated solutions of vector optimization problems with variable ordering structures.

6.2.1 Mordukhovich Subdifferential Approach

In this subsection, we give some optimality conditions for approximately nondominated solutions of vector optimization problems with variable ordering structures using the functional $\varphi_{\bar{y},C,k^0}$ defined in (5.18) with Mordukhovich subdifferential approach. The following optimality condition is given by Bao, Eichfelder, Soleimani and Tammer in [2] and the same result for the functional defined in (4.2) can be proven in similar ways. It worth to remember that all solution concepts of vector optimization problems with variable ordering structures coincide in the case of vector optimization problems with fixed ordering structures.

Theorem 6.2.1. Consider problem (VVOP), let $\bar{x} \in \varepsilon k^0\text{-N}(\mathfrak{S}, f, C)$ and set $\bar{y} := f(\bar{x})$. Assume that in addition to (A11) the following conditions hold:

$(\widehat{A9})$ f is bounded from below over $\mathfrak{S}$ with respect to $\underline{y} \in Y$ and $\Theta = C(\underline{y})$ in the sense of Definition 4.2.26.

(C2) C has a closed graph over $f(\mathfrak{S})$ in the sense that for every sequence of pairs $\{(y_n, v_n)\}$, if $y_n \in f(\mathfrak{S})$ and $v_n \in C(y_n)$ for all $n \in \mathbb{N}$ and $(y_n, v_n) \to (y_*, v_*)$ as $n \to +\infty$, then $y_* \in f(\mathfrak{S})$ and $v_* \in C(y_*)$.

$(C5')$ $C(y) + C(\underline{y}) \subset C(\underline{y})$ for all $y \in f(\mathfrak{S})$.

$(C8)$ There exists a cone D with $k^0 \in \operatorname{int} D$ and $C(\underline{y}) + \operatorname{int} D \subset C(\underline{y})$.

Let $\varphi_{\bar{y},C,k^0} \circ f$ satisfies the qualification condition (6.3) for all $x \in \mathfrak{S}$ such that $\|x - \bar{x}\| \leq \sqrt{\varepsilon}$. Then, there exist $x_\varepsilon \in \operatorname{dom} f \cap \mathfrak{S}$ and $v^* \in \partial_M(\varphi_{\bar{y},C,k^0}(f(x_\varepsilon))$ such that

$$\mathbf{0} \in \partial_M(v^* \circ f)(x_\varepsilon) + N(x_\varepsilon; \mathfrak{S}) + \sqrt{\varepsilon} B_{X^*}.$$

Proof. By Theorem 5.2.11, there exists $x_\varepsilon \in \operatorname{dom} f \cap \mathfrak{S}$ such that it is an exact solution of minimizing a functional $h : X \to \mathbb{R} \cup \{+\infty\}$ over $\mathfrak{S}$ with

$$h(x) := (\varphi_{\bar{y},C,k^0} \circ f)(x) + \sqrt{\varepsilon}\,\|x - x_\varepsilon\| \quad \text{for all } x \in X$$

By [58, Proposition 5.1], we get

$$\mathbf{0} \in \partial_M h(x_\varepsilon) + N(x_\varepsilon; \mathfrak{S}).$$

By Lemma 5.2.9, the composition $\varphi_{\bar{y},C,k^0} \circ f$ is lower-semicontinuous on a neighborhood of x_ε. Employing Lemma 6.1.2 (a) to the lower semicontinuous functional $\varphi_{\bar{y},C,k^0} \circ f$ and the Lipschitz continuous function $\|.\|$, we have

$$\partial_M h(x_\varepsilon) \quad \subset \quad \partial_M \big(\varphi_{\bar{y},C,k^0} \circ f\big)(x_\varepsilon) + \partial_M (\sqrt{\varepsilon}\, \|\cdot - x_\varepsilon\|)(x_\varepsilon).$$

By Lemma 6.1.2 (b), we get

$$\partial_M (\varphi_{\bar{y},C,k^0} \circ f)(x_\varepsilon) \subset \bigcup \big\{ \partial_M (v^* \circ f)(x_\varepsilon) \mid v^* \in \partial_M \varphi_{\bar{y},C,k^0}(f(x_\varepsilon)) \big\}.$$

Combining three inclusions together while taking into account the subdifferential of the norm $\partial_M \|\cdot - x_\varepsilon\| (x_\varepsilon) = B_{X^*}$, we can find $v^* \in \partial_M \varphi_{\bar{y},C,k^0}(f(x_\varepsilon))$ satisfying

$$\mathbf{0} \in \partial_M (v^* \circ f)(x_\varepsilon) + N(x_\varepsilon; \mathfrak{S}) + \sqrt{\varepsilon} B_{X^*}.$$

The proof is complete. □

Corollary 6.2.2. Consider problem (VVOP) and let all assumptions of Theorem 6.2.1 be fulfilled. Let $\bar{x} \in \mathrm{N}(\mathfrak{S}, f, C)$ be an exact nondominated solution of problem (VVOP) and set $\bar{y} := f(\bar{x})$. Assume that $\varphi_{\bar{y},C,k^0} \circ f$ satisfies the qualification condition (6.3) at $\bar{x}$. Then, for any $\lambda > 0$, there exists $v^* \in \partial_M (\varphi_{\bar{y},C,k^0}(f(\bar{x}))$ such that

$$\mathbf{0} \in \partial_M (v^* \circ f)(\bar{x}) + N(\bar{x}; \mathfrak{S}) + \lambda B_{X^*}. \tag{6.14}$$

Proof. Since $\bar{x} \in \mathrm{N}(\mathfrak{S}, f, C)$, i.e., it is a $0k^0$-nondominated solution of problem (VVOP), it is also εk^0-nondominated solution of (VVOP) with $\varepsilon = \lambda^2$ for all $\lambda > 0$ and a weakly nondominated solution of (VVOP). By Theorem 5.2.11 and Theorem 6.2.1, the only point which satisfies the part (iv) of Theorem 5.2.11 is $\bar{x}$ and we can find $v^* \in \partial_M \varphi_{\bar{y},C,k^0}(f(\bar{x}))$ such that

$$\mathbf{0} \in \partial_M (v^* \circ f)(\bar{x}) + N(\bar{x}; \mathfrak{S}) + \sqrt{\varepsilon} B_{X^*}$$

clearly verifying (6.14). The proof is complete. □

Note that the necessary conditions for nondominated solutions of problem (VVOP) obtained in this section are different from those in [5, 26].

6.2.2 Generic Approach

Similar to approximately minimal solutions of vector optimization problems with variable ordering structures, it is possible to derive optimality conditions for approximately nondominated solutions of vector optimization problems with variable ordering structures using general generic approach; see section 6.1.2 for the symbols and assumptions.

6.2.2.1 Exact Optimality Conditions

Similar to the previous section we use the functional $\varphi_{\bar{y},C,k^0}$ defined in (5.18). In order to prove the main result of this section, we need to prove that this functional is convex functional. For this, first we bring the following definition.

Definition 6.2.3. A set-valued map C is convex if and only if

$$\lambda C(x_1)+(1-\lambda)C(x_2)\subseteq C(\lambda x_1+(1-\lambda)x_2)$$

for all $\lambda\in[0,1]$ and all $x_1,x_2\in\operatorname{dom}C$.

Lemma 6.2.4. Let assumption (A13) be fulfilled and additionally C be a convex set-valued map, then $\varphi_{\bar{y},C,k^0}$ defined by (5.18) is convex.

Proof. Let $\lambda\in[0,1]$ and $y^1,y^2\in Y$ such that $\varphi_{\bar{y},C,k^0}(y^1)=t_1$ and $\varphi_{\bar{y},C,k^0}(y^2)=t_2$. By (5.18) we have:

$$\begin{aligned}\bar{y}+t_1k^0-y^1\in C(y^1) &\implies \lambda\bar{y}+\lambda t_1k^0-\lambda y^1\in\lambda C(y^1)\\ \bar{y}+t_2k^0-y^2\in C(y^2) &\implies (1-\lambda)\bar{y}+(1-\lambda)t_2k^0-(1-\lambda)y^2\in(1-\lambda)C(y^2)\end{aligned}$$

and thus

$$\bar{y}+\big(\lambda t_1+(1-\lambda)t_2\big)k^0-\big(\lambda y^1+(1-\lambda)y^2\big)\in\lambda C(y^1)+(1-\lambda)C(y^2).$$

By convexity of the set-valued map C, we get

$$\bar{y}+\big(\lambda t_1+(1-\lambda)t_2\big)k^0-\big(\lambda y^1+(1-\lambda)y^2\big)\in C(\lambda y^1+(1-\lambda)(y^2)).$$

Now again by (5.18), we get

$$\varphi_{\bar{y},C,k^0}(\lambda y^1+(1-\lambda)y^2)\le\lambda t_1+(1-\lambda)t_2=\lambda\varphi_{\bar{y},C,k^0}(y^1)+(1-\lambda)\varphi_{\bar{y},C,k^0}(y^2)$$

and this completes the proof. □

Another important property that we need to prove about the functional defined by (5.18) is continuity. In following lemmata, we prove this functional is lower semicontinuous and upper semicontinuous.

Lemma 6.2.5. Let assumption (A13) be fulfilled and $C:Y\rightrightarrows Y$ satisfies condition (C2) in Theorem 6.2.1. Then the functional $\varphi_{\bar{y},C,k^0}$ defined by (5.18) is a lower semicontinuous functional.

Proof. To prove the lower semicontinuity of $\varphi_{\bar{y},C,k^0}$, it is sufficient to prove that

$$\underline{S}:=\operatorname{lev}(t,\varphi_{\bar{y},C,k^0})=\{y\in Y\mid\varphi_{\bar{y},C,k^0}(y)\le t\}$$

is a closed set in Y for all $t \in \mathbb{R}$. Fix $t \in \mathbb{R}$ arbitrarily and take any sequence $\{y^n\}$ in $\underline{S}$ such that $y^n \to y^0$ as $n \to +\infty$. By the description of $\underline{S}$, we have $\varphi_{\underline{y},C,k^0}(y^n) \le t$ and thus

$$\underline{y} + tk^0 - y^n \in C(y^n).$$

By (C2), we have $\bar{y} + tk^0 - y^0 \in C(y^0)$ and $\varphi_{\bar{y},C,k^0}(y^0) \le t$. The last inequality justifies $y^0 \in \underline{S}$ and thus closedness of $\underline{S}$. The proof is complete. □

Now we prove that under some conditions the functional $\varphi_{\bar{y},C,k^0}$ defined by (5.18) is upper semicontinuous. For this, we define set-valued map $\underline{C} : Y \rightrightarrows Y$ as following;

$$\forall y \in Y \qquad \underline{C}(y) := (\operatorname{int} C(y))^c,$$

i.e, for each $y \in Y$, $\underline{C}(y)$ is the set of complement of $\operatorname{int} C(y)$. Since interior of $C(y)$ is an open set, then $\underline{C}(y)$ is closed for all $y \in Y$.

Lemma 6.2.6. Let assumptions (A13) be fulfilled and additionally $\underline{C}(y) : Y \rightrightarrows Y$ satisfies condition (C2) in Theorem 6.2.1. Then the functional $\varphi_{\bar{y},C,k^0}$ defined by (5.18) is a upper semicontinuous functional.

Proof. To prove the upper semicontinuity of $\varphi_{\bar{y},C,k^0}$, it is sufficient to prove that

$$\overline{S} := \{y \in Y \mid \varphi_{\bar{y},C,k^0}(y) \ge t\}$$

is a closed set in Y for all $t \in \mathbb{R}$. Fix $t \in \mathbb{R}$ arbitrarily and take any sequence $\{y^n\}$ in $\overline{S}$ such that $y^n \to y^0$ as $n \to +\infty$. By the description of $\overline{S}$, we have $\varphi_{\underline{y},C,k^0}(y^n) \ge t$ and thus

$$\underline{y} + tk^0 - y^n \notin \operatorname{int} C(y^n) \implies \underline{y} + tk^0 - y^n \in \underline{C}(y^n).$$

By (C2), we have $\bar{y} + tk^0 - y^0 \in \underline{C}(y^0)$ and $\varphi_{\bar{y},C,k^0}(y^0) \ge t$. The last inequality justifies $y^0 \in \overline{S}$ and thus closedness of $\overline{S}$. The proof is complete. □

Corollary 6.2.7. Let assumptions (A13) be fulfilled and set-valued maps C and $\underline{C}$ satisfy condition (C2) in Theorem 6.2.1. Then the functional $\varphi_{\bar{y},C,k^0}$ defined by (5.18) is a continuous functional.

Theorem 6.2.8. Consider problem (VVOP) and let $\bar{x} \in \varepsilon k^0\text{-N}(\mathfrak{S}, f, C)$, the functional $\varphi_{\bar{y},C,k^0}$ given by (5.18) and set $\bar{y} := f(\bar{x})$. Let C be a convex set-valued map and set-valued maps C and $\underline{C}$ satisfy condition (C2) in Theorem 6.2.1. Impose in addition to (A12)–(A13), (C7) in Theorem 6.1.9, (C5′) and (C8) in Lemma 6.2.1 the following assumptions:

($\widehat{A9}$) f is bounded from below over $\mathfrak{S}$ with respect to $\underline{y} \in Y$ and $\Theta = C(\underline{y})$ in the sense of Definition 4.2.26.

Consider an abstract subdifferential ∂ for that (H1) – (H5) are satisfied. Then, there exists $x_\varepsilon \in \operatorname{dom} f \cap \mathfrak{S}$ and $v^* \in \partial(\varphi_{\bar{y},C,k^0}(f(x_\varepsilon))$ such that

$$\mathbf{0} \in \partial(v^* \circ f)(x_\varepsilon) + N(x_\varepsilon; \mathfrak{S}) + \sqrt{\varepsilon} B_{X^*}.$$

Proof. Let $\bar{x} \in \varepsilon k^0\text{-N}(\mathfrak{S}, \mathrm{f}, \mathrm{C})$. Applying Lemma 5.2.8, we get

$$\varphi_{\bar{y},C,k^0}(f(\bar{x})) \leq \inf_{x \in \mathfrak{S}} \varphi_{\bar{y},C,k^0}(f(x)) + \varepsilon.$$

Therefore $\bar{x}$ is an approximate solution of the scalar problem with the objective functional $\varphi_{\bar{y},C,k^0} \circ f$. By Corollary 6.2.7, we get that $\varphi_{\bar{y},C,k^0} \circ f$ is lower semicontinuous because of $f \in \mathscr{F}(X,Y)$. Furthermore, $\varphi_{\bar{y},C,k^0} \circ f$ is bounded from below because of Lemma 5.2.10. This yields that the assumptions of the scalar Ekeland's variational principle (Theorem 5.0.1) and strong form of Ekeland's variational principle (Remark 5.0.2) are fulfilled.

By Theorem 5.0.1 and Remark 5.0.2, there exists an element $x_\varepsilon \in \operatorname{dom} f \cap \mathfrak{S}$ such that it satisfies parts (a), (b) and (c) of Theorem 5.0.1 and it is an exact solution of minimizing a functional $h : X \to \mathbb{R} \cup \{+\infty\}$ over $\mathfrak{S}$ with

$$h(x) := (\varphi_{\bar{y},C,k^0} \circ f)(x) + \sqrt{\varepsilon}\,\|x - x_\varepsilon\| \quad \text{for all } x \in X.$$

Taking into account (H2) and (H4), we get

$$\mathbf{0} \in \partial h(x_\varepsilon) + N(x_\varepsilon; \mathfrak{S}).$$

Under the given assumptions the functional $\varphi_{\bar{y},C,k^0}$ is convex and continuous taking into account Lemma 6.2.4 and Corollary 6.2.7. Since f is locally Lipschitz and $\varphi_{\bar{y},C,k^0}$ is convex and continuous (and hence locally Lipschitz, see [62, Proposition 1.6]), it is clear that $\varphi_{\bar{y},C,k^0} \circ f$ is also locally Lipschitz. This implies together with the convexity of $\|\cdot\|$ and (H5) that

$$\partial h(x_\varepsilon) \quad \subseteq \quad \partial\big(\varphi_{\bar{y},C,k^0} \circ f\big)(x_\varepsilon) + \partial(\sqrt{\varepsilon}\,\|\cdot - x_\varepsilon\|)(x_\varepsilon).$$

By (H3), we get

$$\partial(\varphi_{\bar{y},C,k^0} \circ f)(x_\varepsilon) \subseteq \bigcup \big\{\partial(v^* \circ f)(x_\varepsilon) \mid v^* \in \partial\theta_{\bar{x}}(f(x_\varepsilon))\big\}.$$

Because of the convexity of the norm and (H1), we get $\partial\,\|\cdot - x_\varepsilon\|\,(x_\varepsilon) = B_{X^*}$ and by the last three inclusions, we can find $v^* \in \partial\varphi_{\bar{y},C,k^0}(f(x_\varepsilon))$ satisfying

$$\mathbf{0} \in \partial(v^* \circ f)(x_\varepsilon) + N(x_\varepsilon; \mathfrak{S}) + \sqrt{\varepsilon} B_{X^*}$$

and proof is complete. □

6.2.2.2 Fuzzy Optimality Conditions

Now we give fuzzy optimality conditions for approximately nondominated solutions of vector optimization problems with variable ordering structures. We use the functional $\theta_{\bar{y}} : Y \to \mathbb{R}$ defined by (6.7)

$$\theta_{\bar{y}}(y) = \inf\{t \in \mathbb{R} \mid y \in tk^0 - C(\bar{y})\}.$$

where $C(\bar{y}) \subset Y$ is a proper closed convex cone with nonempty interior and $k^0 \in \operatorname{int} C(\bar{y})$.

Theorem 6.2.9. Let assumptions (A12) and (A13) be fulfilled, $X,Y \in \mathscr{X}$, $f \in \mathscr{F}(X,Y)$ be a L-Lipschitz function and $\mathfrak{S}$ be a closed subset of the Banach space X. Let $x_\varepsilon \in \varepsilon k^0$-N$(\mathfrak{S}, f, C)$ and $C(f(x_\varepsilon))$ is a closed convex cone with nonempty interior and $C(f(x_\varepsilon)) \subseteq C(f(x))$ for all $x \in \mathfrak{S}$.

Then for every $k^0 \in \operatorname{int} C(f(x_\varepsilon))$ and $\mu > 0$, there exist elements $z \in B(x_\varepsilon, \sqrt{\varepsilon} + \mu/2) \cap \mathfrak{S}$, $u \in B(x_\varepsilon, \sqrt{\varepsilon} + \mu), u^* \in (C(f(x_\varepsilon)))^*, u^*(k^0) = 1, x^* \in X^*, \|x^*\| \leq 1$ such that

$$\mathbf{0} \in \partial(u^* \circ f)(u) + \sqrt{\varepsilon} u^*(k^0) x^* + N_\partial(\mathfrak{S}, z) + B(0, \mu),$$

provided that ∂ satisfies (H1), (H2), (H6), (H7).
Moreover, for some elements $x \in B(x_\varepsilon, \sqrt{\varepsilon} + \mu/2)$ and $v \in B(f(x) - f(x_\varepsilon), L\sqrt{\varepsilon} + \mu)$ it holds $u^*(v) = \theta_{f(x_\varepsilon)}(v)$.

Proof. Consider $x_\varepsilon \in \varepsilon k^0$-N$(\mathfrak{S}, f, C)$. By $C(f(x_\varepsilon)) \subseteq C(f(x))$ for all $x \in \mathfrak{S}$ and the second part of Theorem 3.2.24, x_ε is an approximately minimal solution of (VVOP) the proof is completed by applying Theorem 6.1.11. □

6.2.3 Second Order Optimality Conditions

In this section, we give second order optimality conditions for nondominated solutions of set-valued optimization problems with variable ordering structures. Consider the optimization problem (P_1). We say $(\bar{x}, \bar{y}) \in \operatorname{gph}(F)$ is a weakly nondominated solution of (P_1) if and only if

$$(\bar{y} - \operatorname{int} K(x)) \cap F(x) = \emptyset \qquad \forall x \in \mathfrak{S}$$

and $(\bar{x}, \bar{y}) \in \operatorname{gph}(F)$ is called a local weakly nondominated solution if and only if there exists a neighborhood U of $\bar{x}$ such that

$$(\bar{y} - \operatorname{int} K(x)) \cap F(x) = \emptyset \qquad \forall x \in \mathfrak{S} \cap U.$$

By second part of Theorem 3.2.24, each weakly nondominated element $\bar{x}$ is also a weakly minimal point if $K(\bar{x}) \subset K(x)$ for all $x \in \mathfrak{S}$. Therefore by Theorem 3.2.24 and if $K(\bar{x}) \subset K(x)$, then all the results in section 6.1.3 work for weakly nondominated solution $\bar{x}$. But again it is worth to remember that the set of nondominated solutions and minimal solutions do not coincide in the case of variable ordering structure and there exists a nondominated solution which is not a minimal solution and for this reason, we consider the general case. First we bring definition of convex process; see [1] for more details.

Definition 6.2.10. A set-valued map K is a process if and only if for all $x \in X$ and $\lambda > 0$, the followings hold:

$$\lambda K(x) = K(\lambda x) \quad \text{and} \quad \mathbf{0} \in K(\mathbf{0}).$$

Definition 6.2.11. A set-valued map K is a convex process if and only if it is a process satisfying

$$K(x_1) + K(x_2) \subseteq K(x_1 + x_2).$$

In the following theorem, we give a necessary optimality condition for local weakly nondominated solutions of (P_1); see [43] for the case of fixed ordering structure. Remember the definition of profile $F_+(x) : X \rightrightarrows Y$ as $F_+(x) = F(x) + K(x)$.

Theorem 6.2.12. Let assumption (A14) be fulfilled. Additionally suppose that $(\bar{x}, \bar{y}) \in \operatorname{gph}(F)$ is a local weakly nondominated solution of (P_1) and $K : X \rightrightarrows Y$ is a convex process, then for every $\bar{u} \in \operatorname{dom}(D(F_+)(\bar{x}, \bar{y}))$ and every $\bar{v} \in D(F_+)(\bar{x}, \bar{y})(\bar{u}) \cap (-K(\bar{x}))$, the following holds:

$$D^2(F_+)(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) \cap IT(-\operatorname{int} K(\bar{x}), \bar{v}) = \emptyset \tag{6.15}$$

for all $x \in \operatorname{dom}(D^2(F_+)(\bar{x}, \bar{y}, \bar{u}, \bar{v}))$.

Proof. Suppose that (6.15) does not hold and there exists $x \in \operatorname{dom}(D^2(F_+)(\bar{x}, \bar{y}, \bar{u}, \bar{v}))$ such that

$$y \in D^2(F_+)(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) \cap IT(-\operatorname{int} K(\bar{x}), \bar{v}),$$

then $(x, y) \in T(T(\operatorname{gph}(F_+), (\bar{x}, \bar{y})), (\bar{u}, \bar{v}))$. Therefore there exist sequences $(t_n) \in \mathbb{R}_+$ with $t_n \downarrow 0$ and $(x_n, y_n) \subset X \times Y$ with $(x_n, y_n) \to (x, y)$ such that

$$(\bar{u} + t_n x_n, \bar{v} + t_n y_n) \in T(\operatorname{gph}(F_+), (\bar{x}, \bar{y})) \qquad \forall n \in \mathbb{N}.$$

By $t_n \downarrow 0$, $y_n \to y$ and $y \in IT(-\operatorname{int} K(\bar{x}), \bar{v})$, there exists $n_1 \in \mathbb{N}$ such that

$$\bar{v} + t_n y_n \in -\operatorname{int} K(\bar{x}), \qquad \forall n > n_1.$$

For any $n > n_1$, we fix an element $(u_n, v_n) = (\bar{u} + t_n x_n, \bar{v} + t_n y_n)$ and notice that

$$(u_n, v_n) \in T(\operatorname{gph}(F_+), (\bar{x}, \bar{y})).$$

By the definition of contingent cones, for (u_n, v_n), there exist sequences $(x_m, y_m) \subset X \times Y$ with $(x_m, y_m) \to (u_n, v_n)$ and $(t_m) \subset \mathbb{R}_+$ with $t_m \downarrow 0$ such that

$$\bar{y} + t_m y_m \in F_+(\bar{x} + t_m x_m) = F(\bar{x} + t_m x_m) + K(\bar{x} + t_m x_m).$$

Furthermore, by $v_n \in -\operatorname{int} K(\bar{x})$ and $y_m \to v_n$, there exists $m_1 \in \mathbb{N}$ such that $y_m \in -\operatorname{int} K(\bar{x})$ for all $m > m_1$. Since $K(\bar{x})$ is a cone, this implies that $t_m y_m \in -\operatorname{int} K(\bar{x})$. Since K is a convex process, $K(x)$ is a convex cone and $\mathbf{0} \in K(x)$ for all $x \in X$, then we have

$$K(\bar{x}) \subseteq K(\bar{x}) + \mathbf{0} \subseteq K(\bar{x}) + K(t_m x_m) \subseteq K(\bar{x} + t_m x_m). \tag{6.16}$$

By (6.16) and $t_m y_m \in -\operatorname{int} K(\bar{x})$, we get $t_m y_m \in -\operatorname{int} K(\bar{x} + t_m x_m)$. Furthemore let elements $\omega_m \in F(\bar{x} + t_m x_m)$ be such that $\bar{y} + t_m y_m \in \omega_m + K(\bar{x} + t_m x_m)$. Then by convexity of $K(\bar{x} + t_m x_m)$ and $t_m y_m \in -\operatorname{int} K(\bar{x} + t_m x_m)$, we get $\omega_m \in \bar{y} - \operatorname{int} K(\bar{x} + t_m x_m)$.
By $b_m := (\bar{x} + t_m x_m) \to \bar{x}$, there exists $m_2 > 0$ such that $b_m \in \mathscr{N}(\bar{x})$ where $\mathscr{N}(\bar{x})$ is a suitable neighborhood of $\bar{x}$. Therefore, we have shown that there exists a sequence $\{\omega_m\}$ such that

$$\omega_m \in F(b_m) \cap (\bar{y} - \operatorname{int} K(b_m)) \qquad \text{for all } m > \{m_1, m_2\}.$$

This is a contradiction to local weakly nondominatedness of $(\bar{x}, \bar{y})$. □

If we set $(\bar{x}, \bar{y}) = (\mathbf{0}_X, \mathbf{0}_Y)$, we have the following corollary.

Corollary 6.2.13. Let all the assumptions of Theorem 6.2.12 be fulfilled and $(\bar{x}, \bar{y}) \in \operatorname{gph}(F)$ be a local weakly nondominated solution of problem (P_1), then

$$D(F_+)(\bar{x}, \bar{y}) \cap -\operatorname{int} K(\bar{x}) = \emptyset \quad \text{for all} \quad x \in \operatorname{dom}(D(F_+)(\bar{x}, \bar{y})).$$

In the following theorem, we give a necessary optimality condition for local weakly nondominated solution of problem (P_2).

Theorem 6.2.14. Let assumption (A15) be fulfilled, $(\bar{x}, \bar{y}) \in \operatorname{gph}(F)$ be a local weakly nondominated solution of (P_2), $\bar{z} \in G(\bar{x})$ and additionally $K : X \rightrightarrows Y$ be a convex process.

Then for every $\bar{u} \in \operatorname{dom}(D(F_+, G)(\bar{x}, \bar{y}, \bar{z}))$, every $(\bar{v}, \bar{\varpi}) \in D(F_+, G)(\bar{x}, \bar{y}, \bar{z}) \cap \{-K(\bar{x}) \times -D\}$ and for every $x \in D_1 := \operatorname{dom}(D^2(F_+, G)(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{\varpi}))$, we have

$$D^2(F_+, G)(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{\varpi}))(x) \cap IT(-K(\bar{x}), \bar{v}) \times IT(IT(-D, \bar{z}), \bar{\varpi}) = \emptyset$$

where the notation $(F_+, G)(x)$ represents $(F(x) + K(x)) \times (G(x) + D)$.

Proof. Suppose that assertion is not true and there exists $x \in \operatorname{dom}(D^2(F_+, G)(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{\varpi}))$ such that $(y, z) \in D^2(F_+, G)(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{\varpi})(x) \cap IT(-K(\bar{x}), \bar{v}) \times IT(IT(-\operatorname{int} D, \bar{z}), \bar{\varpi})$ which implies that $(x, y, z) \in T(T(\operatorname{gph}(F_+, G), (\bar{x}, \bar{y}, \bar{z})), (\bar{u}, \bar{v}, \bar{\varpi}))$. This means that there exist sequences

$(t_n) \subset \mathbb{R}_+$ with $t_n \downarrow 0$ and $(x_n, y_n, z_n) \subset X \times Y \times Z$ with $(x_n, y_n, z_n) \to (x, y, z)$ such that

$$(\overline{u} + t_n x_n, \overline{v} + t_n y_n, \overline{\omega} + t_n z_n) \in T(\operatorname{gph}(F_+, G), (\overline{x}, \overline{y}, \overline{z})).$$

By $y \in IT(-\operatorname{int} K(\overline{x}), \overline{v})$, there exists $n_1 \in \mathbb{N}$ such that

$$\overline{v} + t_n y_n \in -\operatorname{int} K(\overline{x}) \qquad n \geq n_1.$$

Analogously by $z \in IT(IT(-\operatorname{int} D, \overline{z}), \overline{\omega})$, there exists $n_2 \in \mathbb{N}$ such that $\overline{\omega} + t_n z_n \in IT(-\operatorname{int} D, \overline{z})$ for all $n > n_2$. For $n \geq \max\{n_1, n_2\}$, we fix $u_n := \overline{u} + t_n x_n$, $v_n := \overline{v} + t_n y_n$ and $\omega_n := \overline{\omega} + t_n z_n$. Then

$$(u_n, v_n, \omega_n) \in T(\operatorname{gph}(F_+, G), (\overline{x}, \overline{y}, \overline{z}))$$

$$(v_n, \omega_n) \in -\operatorname{int} K(\overline{x}) \times IT(-\operatorname{int} D, \overline{z}).$$

By definition of contingent cones, there exist sequences $(t_m) \subset \mathbb{R}_+$ with $t_m \downarrow 0$ and (x_m, y_m, z_m) in $X \times Y \times Z$ with $(x_m, y_m, z_m) \to (u_n, v_n, \omega_n)$ such that $(\overline{y} + t_m y_m, \overline{z} + t_m z_m) \in (F_+, G)(\overline{x} + t_m x_m)$ which means

$$\overline{y} + t_m y_m \in F_+(\overline{x} + t_m x_m) = F(\overline{x} + t_m x_m) + K(\overline{x} + t_m x_m),$$

$$\overline{z} + t_m z_m \in G(\overline{x} + t_m x_m) + D.$$

By $y_m \to v_n$ and $v_n \in -\operatorname{int} K(\overline{x})$, there exists $m_1 > 0$ such that

$$t_m y_m \in -\operatorname{int} K(\overline{x}) \qquad \forall m > m_1.$$

Let $a_m \in F(\overline{x} + t_m x_m)$ be such that $\overline{y} - \operatorname{int} K(\overline{x}) \in a_m + K(\overline{x} + t_m x_m)$. Since K is a convex process and for all $x \in X$, $K(x)$ is a convex cone and $\mathbf{0} \in K(x)$, we have

$$K(\overline{x}) \subseteq K(\overline{x}) + \mathbf{0} \subseteq K(\overline{x}) + K(t_m x_m) \subseteq K(\overline{x} + t_m x_m).$$

Consequently we have:

$$a_m \in \overline{y} - \operatorname{int} K(\overline{x}) - K(\overline{x} + t_m x_m) \subseteq \overline{y} - \operatorname{int} K(\overline{x} + t_m x_m).$$

By $\omega_n \in IT(-\operatorname{int} D, \overline{z})$, there exists $m_2 > 0$ such that $\overline{z} + t_m z_m \in -\operatorname{int} D$ for every $m > m_2$. Let $b_m \in G(\overline{x} + t_m x_m)$ be such that $\overline{z} + t_m z_m \in b_m + D$ and therefore we have $b_m \in -\operatorname{int} D$. Therefore, for sufficiently large m, we have $c_m := \overline{x} + t_m x_m \in \mathcal{N}(\overline{x})$, $G(c_m) \cap -D \neq \emptyset$ and

$$F(c_m) \cap (\overline{y} - \operatorname{int} K(c_m)) \neq \emptyset.$$

This is a contradiction because we supposed that $(\overline{x}, \overline{y})$ is a local weakly nondominated solution of the problem (P_2). $\square$

The following theorem gives a necessary condition for local weakly nondominated solution of (P_3).

Theorem 6.2.15. Let assumption (A16) be fulfilled, $(\bar{x},\bar{y}) \in \operatorname{gph}(F)$ be a local weakly nondominated solution of problem (P_3), $\bar{z} \in G(\bar{x})$ and $K : X \rightrightarrows Y$ be a convex process. Additionally, suppose that $D : X \rightrightarrows Z$ satisfies one of the following conditions:

(i) D is a convex process.

(ii) For all $x \in \mathscr{N}(\bar{x})$, $D(\bar{x}) \subseteq D(x)$.

Then for every $\bar{u} \in \operatorname{dom}(D(\widehat{F_+,G})(\bar{x},\bar{y},\bar{z}))$, every $(\bar{v},\overline{\omega}) \in D(\widehat{F_+,G})(\bar{x},\bar{y},\bar{z}) \cap \{-K(\bar{x}) \times -D(\bar{x})\}$ and for every $x \in D_1 := \operatorname{dom}(D^2(\widehat{F_+,G})(\bar{x},\bar{y},\bar{z},\bar{u},\bar{v},\overline{\omega}))$, we have

$$D^2(\widehat{F_+,G})(\bar{x},\bar{y},\bar{z},\bar{u},\bar{v},\overline{\omega})(x) \cap IT(-K(\bar{x}),\bar{v}) \times IT(IT(-D(\bar{x}),\bar{z}),\overline{\omega}) = \emptyset$$

where the notation $(\widehat{F_+,G})(x)$ represents $(F(x)+K(x)) \times (G(x)+D(x))$.

Proof. Suppose that assertion is not true and there exists $x \in \operatorname{dom}(D^2(\widehat{F_+,G})(\bar{x},\bar{y},\bar{z},\bar{u},\bar{v},\overline{\omega}))$ such that $(y,z) \in D^2(\widehat{F_+,G})(\bar{x},\bar{y},\bar{z},\bar{u},\bar{v},\overline{\omega})(x) \cap IT(-K(\bar{x}),\bar{v}) \times IT(IT(-\operatorname{int} D(\bar{x}),\bar{z}),\overline{\omega})$ which implies that $(x,y,z) \in T(T(\operatorname{gph}(\widehat{F_+,G}),(\bar{x},\bar{y},\bar{z})),(\bar{u},\bar{v},\overline{\omega}))$. This means that there exist sequences $(t_n) \subset \mathbb{R}_+$ with $t_n \downarrow 0$ and $(x_n,y_n,z_n) \subset X \times Y \times Z$ with $(x_n,y_n,z_n) \to (x,y,z)$ such that

$$(\bar{u}+t_nx_n,\bar{v}+t_ny_n,\overline{\omega}+t_nz_n) \in T(\operatorname{gph}(\widehat{F_+,G}),(\bar{x},\bar{y},\bar{z})).$$

By $y \in IT(-\operatorname{int} K(\bar{x}),\bar{v})$, there exists $n_1 \in \mathbb{N}$ such that

$$\bar{v}+t_ny_n \in -\operatorname{int} K(\bar{x}) \qquad n \geq n_1.$$

By $z \in IT(IT(-\operatorname{int} D(\bar{x}),\bar{z}),\overline{\omega})$, there exists $n_2 \in \mathbb{N}$ such that $\overline{\omega}+t_nz_n \in IT(-\operatorname{int} D(\bar{x}),\bar{z})$ for all $n > n_2$. For $n \geq \max\{n_1,n_2\}$, we fix $u_n := \bar{u}+t_nx_n$, $v_n := \bar{v}+t_ny_n$ and $\omega_n := \overline{\omega}+t_nz_n$. Then

$$(u_n,v_n,\omega_n) \in T(\operatorname{gph}(\widehat{F_+,G}),(\bar{x},\bar{y},\bar{z}))$$

$$(v_n,\omega_n) \in -\operatorname{int} K(\bar{x}) \times IT(-\operatorname{int} D(\bar{x}),\bar{z}).$$

By definition of contingent cones, there exist sequences $(t_m) \subset \mathbb{R}_+$ with $t_m \downarrow 0$ and (x_m,y_m,z_m) in $X \times Y \times Z$ with $(x_m,y_m,z_m) \to (u_n,v_n,\omega_n)$ such that $(\bar{y}+t_my_m,\bar{z}+t_mz_m) \in (\widehat{F_+,G})(\bar{x}+t_mx_m)$ which means

$$\bar{y}+t_my_m \in F_+(\bar{x}+t_mx_m) = F(\bar{x}+t_mx_m)+K(\bar{x}+t_mx_m),$$

$$\bar{z}+t_mz_m \in G(\bar{x}+t_mx_m)+D(\bar{x}+t_mx_m).$$

By $y_m \to v_n$ and $v_n \in -\operatorname{int} K(\bar{x})$, there exists $m_1 > 0$ such that

$$t_m y_m \in -\operatorname{int} K(\bar{x}) \qquad \forall m > m_1.$$

Let $a_m \in F(\bar{x}+t_m x_m)$ be such that $\bar{y} - \operatorname{int} K(\bar{x}) \in a_m + K(\bar{x}+t_m x_m)$. Since K is a convex process and for all $x \in X$, $K(x)$ is a convex cone and $\mathbf{0} \in K(x)$, we have

$$K(\bar{x}) \subseteq K(\bar{x}) + \mathbf{0} \subseteq K(\bar{x}) + K(t_m x_m) \subseteq K(\bar{x}+t_m x_m).$$

Consequently we have:

$$a_m \in \bar{y} - \operatorname{int} K(\bar{x}) - K(\bar{x}+t_m x_m) \subseteq \bar{y} - \operatorname{int} K(\bar{x}+t_m x_m).$$

By $\omega_n \in IT(-\operatorname{int} D(\bar{x}), \bar{z})$, there exists $m_2 > 0$ such that $\bar{z} + t_m z_m \in -\operatorname{int} D(\bar{x})$ for every $m > m_2$. Let $b_m \in G(\bar{x}+t_m x_m)$ be such that $\bar{z} + t_m z_m \in b_m + D(\bar{x}+t_m x_m)$ and

$$b_m \in -\operatorname{int} D(\bar{x}) - \operatorname{int} D(\bar{x}+t_m x_m). \tag{6.17}$$

Now if condition (i) on D holds we have,

$$D(\bar{x}) \subseteq D(\bar{x}) + \mathbf{0} \subseteq D(\bar{x}) + D(t_m x_m) \subseteq D(\bar{x}+t_m x_m)$$

or if condition (ii) on D holds, again we have directly $D(\bar{x}) \subseteq D(\bar{x}+t_m x_m)$. Therefore by (6.17) and one of the conditions (i) or (ii), we have $b_m \in -\operatorname{int} D(\bar{x}+t_m x_m)$. This means that for sufficiently large m, we have $c_m := \bar{x}+t_m x_m \in \mathscr{N}(\bar{x})$ such that $G(c_m) \cap -D(c_m) \neq \emptyset$ and

$$F(c_m) \cap (\bar{y} - \operatorname{int} K(c_m)) \neq \emptyset.$$

But this is a contradiction because we supposed that $(\bar{x}, \bar{y})$ is a local weakly nondominated solution of (P_3). □

6.3 Optimality Conditions for Approximate Minimizers

In this section, we present optimality conditions for approximate minimizers of vector optimization problems with variable ordering structures with different approaches namely Mordukhovich subdifferential approach and general generic approach. Our results about variational principles for approximate minimizers and characterization of approximate minimizers of vector optimization problems with variable ordering structures in previous chapters will be used here in order to derive optimality conditions for approximate minimizers of vector optimization problems with variable ordering structures. Finally, second order optimality condition for weak minimizers of set-valued optimization with variable ordering structures will be presented using second order contingent derivatives and epiderivatives.

Remark 6.3.1. By Theorem 3.2.18 and Theorem 3.2.21, each (approximate) minimizer of vector optimization problems with variable ordering structures is both (approximately) minimal and nondominated solution of vector optimization problems with variable ordering structures (VVOP). This means that all optimality conditions for (approximately) minimal and nondominated solutions of (VVOP) can be used for (approximate) minimizers of vector optimization problems with variable ordering structures. It worth to remember that all these solution concepts coincide in the case of vector optimization problems with fixed ordering structures.

Let assumptions (A12) and (A13) be fulfilled and $\bar{x} \in X$. In order to drive necessary conditions for approximate minimizers of vector optimization problems with variable ordering structures, we use the following functional which is a slight modification of the functional defined by Chen and Yang [15], especially concerning the assumptions for the set-valued map C. We define $\xi_{\bar{x}}(z,y) : Y \times Y \to \overline{\mathbb{R}}$ as following:

$$\xi_{\bar{x}}(z,y) := \inf\{t \in \mathbb{R} \mid z \in tk^0 + f(\bar{x}) - C(y)\}. \tag{6.18}$$

Lemma 6.3.2. Let assumptions (A12) and (A13) be fulfilled and $\bar{x} \in X$. For each $t \in \mathbb{R}$ and $y, z \in Y$, followings hold.

$$\xi_{\bar{x}}(z,y) > t \Longleftrightarrow z \notin tk^0 + f(\bar{x}) - C(y),$$

$$\xi_{\bar{x}}(z,y) \geqq t \Longleftrightarrow z \notin tk^0 + f(\bar{x}) - \operatorname{int} C(y),$$

$$\xi_{\bar{x}}(z,y) = t \Longleftrightarrow z \in tk^0 + f(\bar{x}) - \operatorname{bd} C(y),$$

$$\xi_{\bar{x}}(z,y) \leqq t \Longleftrightarrow z \in tk^0 + f(\bar{x}) - C(y),$$

$$\xi_{\bar{x}}(z,y) < t \Longleftrightarrow z \in tk^0 + f(\bar{x}) - \operatorname{int} C(y).$$

Proof. Proof is similar to that of Theorem 4.2.7. □

Theorem 6.3.3. Suppose assumptions (A12) and (A13) hold, then for each arbitrary fixed $y \in Y$, $\xi_{\bar{x}}(\cdot, y)$ is continuous.

Proof. Proof is similar to that of third part of Theorem 4.2.18. □

Now we prove that $\xi_{\bar{x}}(\cdot, y)$ is a convex functional for each $y \in Y$. Chen and Yang [15] proved that ξ is a convex functional under strong assumptions and for convex cone-valued map $C : Y \rightrightarrows Y$.

Theorem 6.3.4. Suppose that assumptions (A12) and (A13) hold, $\bar{x} \in X$ and additionally $C(y)$ is a convex set for all $y \in Y$. Then $\xi_{\bar{x}}(\cdot, y)$ is convex for all $y \in Y$.

Proof. Let $y \in Y$ be an arbitrary but fixed element. Assume that $\lambda \in [0,1]$ and $z^1, z^2 \in Y$ such that $\xi_{\bar{x}}(z^1, y) = t_1$ and $\xi_{\bar{x}}(z^2, y) = t_2$. By Lemma 6.3.2, we have the followings

$$\xi_{\bar{x}}(z^1, y) = t_1 \Longrightarrow y^1 \in t_1 + f(\bar{x}) - C(y),$$

$$\xi_{\bar{x}}(z^1,y) = t_2 \Longrightarrow y^2 \in t_2 + f(\bar{x}) - C(y).$$

This means there exist $c,d \in C(y)$ such that $z^1 = t_1k^0 + f(\bar{x}) - c$ and $z^2 = t_2k^0 + f(\bar{x}) - d$ and

$$\begin{aligned} & \lambda z^1 + (1-\lambda)z^2 \\ = \ & \lambda t_1 k^0 + \lambda f(\bar{x}) - \lambda c + (1-\lambda)t_2k^0 + (1-\lambda)f(\bar{x}) - (1-\lambda)d \\ = \ & \big(\lambda t_1 + (1-\lambda)t_2\big)k^0 + f(\bar{x}) - \big(\lambda c + (1-\lambda)d\big), \end{aligned}$$

by $c,d \in C(y)$ and convexity of $C(y)$, we get $\lambda c + (1-\lambda)d \in C(y)$ and therefore

$$\lambda z^1 + (1-\lambda)z^2 \in (\lambda t_1 + (1-\lambda)t_2)k^0 + f(\bar{x}) - C(y).$$

Again by Lemma 6.3.2, $\xi_{\bar{x}}(\lambda z^1 + (1-\lambda)z^2, y) \leq \lambda t_1 + (1-\lambda)t_2$ and

$$\xi_{\bar{x}}(\lambda z^1 + (1-\lambda)z^2, y) \leq \lambda \xi_{\bar{x}}(z^1,y) + (1-\lambda)\xi_{\bar{x}}(z^2,y).$$

This means that $\xi_{\bar{x}}(\cdot,y)$ is convex and the proof is complete. □

Definition 6.3.5. Consider $\bar{x} \in X$ and the functional $\xi_{\bar{x}} : Y \times Y \to \overline{\mathbb{R}}$ given by (6.18). $f : X \to Y$ is called bounded from below over $\mathfrak{S}$ with respect to $\xi_{\bar{x}}$ if and only if for all $\mathfrak{s} \in \mathfrak{S}$, there exists a real number $\alpha > -\infty$ such that

$$\inf_{x \in \mathfrak{S}} \xi_{\bar{x}}(f(x), f(\mathfrak{s})) > \alpha.$$

The following theorem gives a characterization of approximate minimizers of (VVOP) by using a scalarization by means of the functional $\xi_{\bar{x}} : Y \times Y \to \overline{\mathbb{R}}$ defined by (6.18).

Theorem 6.3.6. Suppose that assumptions (A12) and (A13) hold. Let $\bar{x} \in \Omega$ be an εk^0-minimizer of (VVOP), then for all $\mathfrak{s} \in \mathfrak{S}$,

$$\xi_{\bar{x}}(f(\bar{x}), f(\mathfrak{s})) \leq \inf_{x \in \mathfrak{S}} \xi_{\bar{x}}(f(x), f(\mathfrak{s})) + \varepsilon. \tag{6.19}$$

Proof. Let $\mathfrak{s}$ be an arbitrary but fixed element of $\mathfrak{S}$ and set $\bar{y} = f(\bar{x})$. First we prove that $\xi_{\bar{x}}(f(\bar{x}), f(\mathfrak{s})) = 0$. Suppose $\xi_{\bar{x}}(f(\bar{x}), f(\mathfrak{s})) = \bar{t}$. By Theorem 6.3.2, we get

$$tk^0 + \bar{y} - \bar{y} \in C(f(\mathfrak{s})) \implies \bar{t}k^0 \in C(f(\mathfrak{s})).$$

By pointedness of $C(f(\mathfrak{s}))$, we get $\mathbf{0} \in \mathrm{bd}\, C(f(\mathfrak{s}))$ and $\bar{t} \leq 0$. If $\xi_{\bar{x}}(f(\bar{x}), f(\mathfrak{s})) < 0$, then $\bar{t} < 0$ and $-\bar{t} > 0$. By pointedness of $C(f(\mathfrak{s}))$, $-\bar{t} > 0$, $k^0 \neq \mathbf{0}$ and $C(f(\mathfrak{s})) + [0,+\infty)k^0 \subset C(f(\mathfrak{s}))$, we get $-\bar{t}k^0 \in C(f(\mathfrak{s}))\backslash\{\mathbf{0}\}$ and therefore $\bar{t}k^0 \in C(f(\mathfrak{s}))\backslash\{\mathbf{0}\} \cap (-C(f(\mathfrak{s})))$ and we arrive at a contradiction because we supposed $C(y)$ is a pointed set for all $y \in Y$. This means that $\bar{t} = 0$. Now we prove that $\xi_{\bar{x}}(f(\bar{x}), f(\mathfrak{s})) \leq \inf_{x \in \mathfrak{S}} \xi_{\bar{x}}(f(x), f(\mathfrak{s})) + \varepsilon$. Suppose by contrary (6.19) does not hold and there exists an element $x \in \mathfrak{S}$ such that $\xi_{\bar{x}}(f(x), f(\mathfrak{s})) + \varepsilon < \xi_{\bar{x}}(f(\bar{x}), f(\mathfrak{s})) = 0$. This means that there exists $\beta > 0$ such that $\xi_{\bar{x}}(f(x), f(\mathfrak{s})) = -\varepsilon - \beta$. By Theorem 6.3.2, we get

$$\begin{aligned} & (-\varepsilon-\beta)k^0 + \bar{y} - f(x) \in C(f(\mathfrak{s})) \\ \Longrightarrow \quad & \bar{y} - \varepsilon k^0 - f(x) \in C(f(\mathfrak{s})) + \beta k^0 \subset C(f(\mathfrak{s})) \setminus \{\mathbf{0}\}. \end{aligned}$$

This means that there exists $\mathfrak{s} \in \mathfrak{S}$ such that

$$(\bar{y} - \varepsilon k^0 - C(f(\mathfrak{s})) \setminus \{\mathbf{0}\}) \cap f(\mathfrak{S}) \neq \emptyset$$

and therefore $\bar{x} \notin \varepsilon k^0$-MZ$(\mathfrak{S}, f, C)$. But this is a contradiction because we supposed that $\bar{x}$ is an εk^0-minimizer of (VVOP). Therefore

$$\xi_{\bar{x}}(f(\bar{x}), f(\mathfrak{s})) \leq \inf_{x \in \mathfrak{S}} \xi_{\bar{x}}(f(x), f(\mathfrak{s})) + \varepsilon.$$

for all $x, \mathfrak{s} \in \mathfrak{S}$. □

6.3.1 Mordukhovich Subdifferential Approach

In this section, we give some optimality condition for approximate minimizers of vector optimization problems with variable ordering structures using the functional $\xi_{\bar{x}}$ defined in (6.18).

Theorem 6.3.7. Consider problem (VVOP), let $\bar{x} \in \varepsilon k^0$-MZ$(\mathfrak{S}, f, C)$ and $\bar{y} := f(\bar{x})$. Assume that in addition to (A11) the following conditions hold:

$(A5)$ $B : Y \rightrightarrows Y$ be a cone-valued map such that for all $y \in f(\mathfrak{S})$, $k^0 \in \operatorname{int} B(y)$.

$(A6'')$ $C(y) + B(y) \setminus \{\mathbf{0}\} \subseteq \operatorname{int} C(y)$ and $B(f(x)) \subset C(f(\bar{x}))$ for all $\|x - \bar{x}\| \leq \sqrt{\varepsilon}$.

$(C9)$ f is bounded from below in the sense of Definition 6.3.5 over $\mathfrak{S}$ with respect to $\xi_{\bar{x}}$.

Let $\xi_{\bar{x}}(\cdot, f(\mathfrak{s})) \circ f$ satisfies the condition (6.3) for all $x, \mathfrak{s} \in \mathfrak{S}$ such that $\|x - \bar{x}\| \leq \sqrt{\varepsilon}$. Then for all $\mathfrak{s} \in \mathfrak{S}$, there exist $x_{\mathfrak{s}} \in \operatorname{dom} f \cap \mathfrak{S}$ and $v_{\mathfrak{s}}^* \in \partial_M(\xi_{\bar{x}}(f(x_{\mathfrak{s}}), f(\mathfrak{s}))$ such that

$$\mathbf{0} \in \partial_M(v_{\mathfrak{s}}^* \circ f)(x_{\mathfrak{s}}) + N(x_{\mathfrak{s}}; \mathfrak{S}) + \sqrt{\varepsilon} B_{X^*}.$$

Proof. Let $\bar{x} \in \varepsilon k^0$-MZ$(\mathfrak{S}, \mathrm{f}, \mathrm{C})$. Applying Theorem 6.3.6, we get for all $\mathfrak{s} \in \mathfrak{S}$

$$\xi_{\bar{x}}(f(\bar{x}), f(\mathfrak{s})) \leq \inf_{x \in \mathfrak{S}} \xi_{\bar{x}}(f(x), f(\mathfrak{s})) + \varepsilon.$$

Therefore $\bar{x}$ is an approximate minimizer of the scalar problem with the objective functionals $\xi_{\bar{x}}(\cdot, f(\mathfrak{s}))$ for all $\mathfrak{s} \in \mathfrak{S}$. Taking into account Theorem 6.3.3 and since f is strictly Lipschitz by (A11), we get that $\xi_{\bar{x}}(\cdot, f(\mathfrak{s}))$ is lower semicontinuous for all $\mathfrak{s} \in \mathfrak{S}$. Furthermore, $\xi_{\bar{x}}(\cdot, f(\mathfrak{s}))$ is bounded from below for all $\mathfrak{s} \in \mathfrak{S}$. By the scalar Ekeland's variational principle (Theorem 5.0.1), for all $\mathfrak{s} \in \mathfrak{S}$ there exists an element $x_{\mathfrak{s}} \in \operatorname{dom} f \cap \mathfrak{S}$ such that it satisfies parts (a), (b) and (c) of

Theorem 5.0.1 and it is an exact solution of an optimization problem with the objective function $h_{\mathfrak{s}} : X \to \mathbb{R} \cup \{+\infty\}$ over $\mathfrak{S}$ with

$$h_{\mathfrak{s}}(x) := \xi_{\bar{x}}(f(x), f(\mathfrak{s})) + \sqrt{\varepsilon}\,\|x - x_{\mathfrak{s}}\| \quad \text{for all } x \in X$$

By [58, Proposition 5.1], we get

$$\mathbf{0} \in \partial_M h_{\mathfrak{s}}(x_{\mathfrak{s}}) + N(x_{\mathfrak{s}}; \mathfrak{S}).$$

By Theorem 6.3.3 and (A11), the composition $\xi_{\bar{x}}(\cdot, f(\mathfrak{s})) \circ f$ is lower-semicontinuous on a neighborhood of $x_{\mathfrak{s}}$ for all $\mathfrak{s} \in \mathfrak{S}$. Employing Lemma 6.1.2 (a) to the lower semicontinuous functional $\xi_{\bar{x}}(\cdot, f(\mathfrak{s})) \circ f$ and the Lipschitz continuous function $\|.\|$, we have

$$\partial_M h_{\mathfrak{s}}(x_{\mathfrak{s}}) \quad \subset \quad \partial_M\big(\xi_{\bar{x}}(\cdot, f(\mathfrak{s})) \circ f\big)(x_{\mathfrak{s}}) + \partial_M(\sqrt{\varepsilon}\,\|\cdot - x_{\mathfrak{s}}\|)(x_{\mathfrak{s}}).$$

By Lemma 6.1.2 (b), we get

$$\partial_M(\xi_{\bar{x}}(\cdot, f(\mathfrak{s})) \circ f)(x_{\mathfrak{s}}) \subset \bigcup \big\{\partial_M(v_{\mathfrak{s}}^* \circ f)(x_{\mathfrak{s}}) \mid v_{\mathfrak{s}}^* \in \partial_M \xi_{\bar{x}}(f(x_{\mathfrak{s}}), f(\mathfrak{s}))\big\}.$$

Combining three inclusions together while taking into account the subdifferential of the norm $\partial_M \|\cdot - x_{\mathfrak{s}}\|(x_{\mathfrak{s}}) = B_{X^*}$, we can find $v_{\mathfrak{s}}^* \in \partial_M \xi_{\bar{x}}(f(x_{\mathfrak{s}}), f(\mathfrak{s}))$ satisfying

$$\mathbf{0} \in \partial_M(v_{\mathfrak{s}}^* \circ f)(x_{\mathfrak{s}}) + N(x_{\mathfrak{s}}; \mathfrak{S}) + \sqrt{\varepsilon} B_{X^*}.$$

The proof is complete. □

Corollary 6.3.8. Consider problem (VVOP), $\bar{x} \in \mathrm{MZ}(\mathfrak{S}, f, C)$ be an exact minimizer of problem (VVOP) and $\bar{y} := f(\bar{x})$. Let all the assumptions of Theorem 6.1.3 be fulfilled. Also assume that $\xi_{\bar{x}}(\cdot, f(\mathfrak{s})) \circ f$ satisfies the qualification condition (6.3) at $\bar{x}$ for all $\mathfrak{s} \in \mathfrak{S}$. Then, for any $\lambda > 0$ and $\mathfrak{s} \in \mathfrak{S}$, there exists $v_{\mathfrak{s}}^* \in \partial_M(\xi_{\bar{x}}(f(\bar{x}), f(\mathfrak{s}))$ such that

$$\mathbf{0} \in \partial_M(v_{\mathfrak{s}}^* \circ f)(\bar{x}) + N(\bar{x}; \mathfrak{S}) + \lambda B_{X^*}. \tag{6.20}$$

such that $\|x_{\varepsilon}^*\| \leq \varepsilon$.

Proof. Since $\bar{x} \in \mathrm{MZ}(\mathfrak{S}, f, C)$, i.e., it is a $0k^0$-minimizer of (VVOP), it is also εk^0-minimizer of (VVOP) with $\varepsilon = \lambda^2$ for all $\lambda > 0$. By Theorem 5.3.4 and Theorem 6.3.7, the only point which satisfies (5.25) is $\bar{x}$ and we can find $v_{\mathfrak{s}}^* \in \partial_M \xi_{\bar{x}}(f(\bar{x}), f(\mathfrak{s}))$ such that

$$\mathbf{0} \in \partial_M(v_{\mathfrak{s}}^* \circ f)(\bar{x}) + N(\bar{x}; \mathfrak{S}) + \sqrt{\varepsilon} B_{X^*}$$

clearly verifying (6.20). The proof is complete. □

6.3.2 Generic Approach

Similar to approximately minimal and nondominated solutions of vector optimization problems with variable ordering structures, it is also possible to derive optimality conditions for approximate minimizers of vector optimization problems with variable ordering structures using general generic approach; see section 6.1.2 for symbols and assumptions.

6.3.2.1 Exact Optimality Conditions

In this section, we give necessary conditions for approximate minimizers of vector optimization problems with variable ordering structures. Theorem 6.3.6 and Ekeland's variational principle (Theorem 5.0.1) will be used in order to drive necessary conditions for minimizers of vector optimization problems with variable ordering structures.

Theorem 6.3.9. Consider problem (VVOP) and the functional $\xi_{\bar{x}}$ given by (6.18). Let $C(y)$ be a convex set for all $y \in Y$, $\bar{x} \in \varepsilon k^0\text{-MZ}(\mathfrak{S},\mathrm{f},\mathrm{C})$ and set $\bar{y} := f(\bar{x})$. Impose in addition to (A12)–(A13) the following assumptions:

$(C7)$ $f \in \mathscr{F}(X,Y)$ is locally Lipschitz.

$(C9)$ f is bounded from below in the sense of Definition 6.3.5 over $\mathfrak{S}$ with respect to $\xi_{\bar{x}}$.

Consider an abstract subdifferential ∂ for that (H1) – (H5) are satisfied. Then for all $\mathfrak{s} \in \mathfrak{S}$, there exist $x_{\mathfrak{s}} \in \operatorname{dom} f \cap \mathfrak{S}$ and $v_{\mathfrak{s}}^* \in \partial(\xi_{\bar{x}}(f(x_{\mathfrak{S}}), f(\mathfrak{s}))$ such that

$$\mathbf{0} \in \partial(v_{\mathfrak{s}}^* \circ f)(x_{\mathfrak{s}}) + N(x_{\mathfrak{s}};\mathfrak{S}) + \sqrt{\varepsilon} B_{X^*}.$$

Proof. Let $\bar{x} \in \varepsilon k^0\text{-MZ}(\mathfrak{S},\mathrm{f},\mathrm{C})$. Applying Theorem 6.3.6, we get for all $\mathfrak{s} \in \mathfrak{S}$

$$\xi_{\bar{x}}(f(\bar{x}), f(\mathfrak{s})) \leq \inf_{x \in \mathfrak{S}} \xi_{\bar{x}}(f(x), f(\mathfrak{s})) + \varepsilon.$$

Therefore $\bar{x}$ is an approximate minimizer of the scalar problem with the objective functionals $\xi_{\bar{x}}(\cdot, f(\mathfrak{s}))$ for all $\mathfrak{s} \in \mathfrak{S}$. Taking into account Theorem 6.3.3 and $f \in \mathscr{F}(X,Y)$ we get that $\xi_{\bar{x}}(\cdot, f(\mathfrak{s}))$ is lower semicontinuous for all $\mathfrak{s} \in \mathfrak{S}$. Furthermore, $\xi_{\bar{x}}(\cdot, f(\mathfrak{s}))$ is bounded from below for all $\mathfrak{s} \in \mathfrak{s}$. By the scalar Ekeland's variational principle (Theorem 5.0.1), for all $\mathfrak{s} \in \mathfrak{S}$ there exists an element $x_{\mathfrak{s}} \in \operatorname{dom} f \cap \mathfrak{S}$ such that it satisfies parts (a), (b) and (c) of Theorem 5.0.1 and it is an exact solution of an optimization problem with the objective function $h_{\mathfrak{s}} : X \to \mathbb{R} \cup \{+\infty\}$ over $\mathfrak{S}$ with

$$h_{\mathfrak{s}}(x) := \xi_{\bar{x}}(f(x), f(\mathfrak{s})) + \sqrt{\varepsilon}\, \|x - x_{\mathfrak{s}}\| \quad \text{for all } x \in X.$$

By (H2) and (H4), we get

$$\mathbf{0} \in \partial h_{\mathfrak{s}}(x_{\mathfrak{s}}) + N(x_{\mathfrak{s}};\mathfrak{S}).$$

Under the given assumptions and taking into account Theorem 6.3.3 and Theorem 6.3.4, the functional $\xi_{\bar{x}}(\cdot, f(\mathfrak{s}))$ is convex and continuous. Since f is locally Lipschitz and $\xi_{\bar{x}}(\cdot, f(\mathfrak{s}))$ is convex and continuous and hence locally Lipschitz, then the composition $\xi_{\bar{x}}(f(\cdot), f(\mathfrak{s}))$ is also locally Lipschitz. This implies together with the convexity of the norm $\|\cdot\|$ and (H5) that

$$\partial h_{\mathfrak{s}}(x_{\mathfrak{s}}) \quad \subseteq \quad \partial\big(\xi_{\bar{x}}(f(\cdot), f(\mathfrak{s}))\big)(x_{\mathfrak{s}}) + \partial(\sqrt{\varepsilon}\,\|\cdot - x_{\mathfrak{s}}\|)(x_{\mathfrak{s}}).$$

By (H3), we get

$$\partial(\xi_{\bar{x}}(f(\cdot), f(\mathfrak{s})))(x_{\mathfrak{s}}) \subseteq \bigcup \big\{\partial(v_{\mathfrak{s}}^* \circ f)(x_{\mathfrak{s}}) \mid v_{\mathfrak{s}}^* \in \partial\xi_{\bar{x}}(f(x_{\mathfrak{s}}), f(\mathfrak{s}))\big\}.$$

Because of the convexity of the norm and (H1), we get $\partial\,\|\cdot - x_{\mathfrak{s}}\|\,(x_{\mathfrak{s}}) = B_{X^*}$ and by the last three inclusions, we can find $v_{\mathfrak{s}}^* \in \partial\xi_{\bar{x}}(f(x_{\mathfrak{s}}), f(\mathfrak{s}))$ satisfying

$$\mathbf{0} \in \partial(v_{\mathfrak{s}}^* \circ f)(x_{\mathfrak{s}}) + N(x_{\mathfrak{s}}; \mathfrak{S}) + \sqrt{\varepsilon} B_{X^*}$$

and the proof is complete. □

6.3.2.2 Fuzzy Optimality Conditions

Now we are ready to give fuzzy optimality conditions for approximate minimizers of vector optimization problems with variable ordering structures. In the proof of the main result of this section, we use the functional $\xi : Y \times Y \to \mathbb{R}$ defined by (6.18)

$$\xi(z, y) := \inf\{t \in \mathbb{R} \mid z \in tk^0 - C(y)\} \tag{6.21}$$

where $C(y) \subset Y$ is a proper, closed and convex cone with nonempty interior and $k^0 \in \operatorname{int} C(y)$ for all $y \in Y$. By Theorem 6.3.3 and Theorem 6.3.4, we already proved that $\xi_{\bar{x}}(\cdot, y)$ is continuous and convex for all $y \in Y$ if $C(y) \subset Y$ is a proper, closed and convex cone with nonempty interior and $k^0 \in \operatorname{int} C(y)$ for all $y \in Y$. Therefore, for all fixed arbitrary $y \in Y$ and under the given assumptions, the functional $\xi_{\bar{x}}(\cdot, y)$ is continuous and convex and its subdifferential is given by

$$\partial\xi(u, y) = \{v^* \in C(y)^* \mid v^*(k^0) = 1, v^*(u) = \xi(u, y)\} \tag{6.22}$$

(see [21, Lemma 2.1]).

In the next theorem we show necessary conditions for approximate minimizers of a vector optimization problem with a variable ordering structure following the line of the proof of [21, Theorem 5.3].

Theorem 6.3.10. Suppose that assumptions (A12) and (A13) are fulfilled and additionally $C(f(\mathfrak{s}))$ is a closed convex cone with nonempty interior for all $\mathfrak{s} \in \mathfrak{S}$. Let $X, Y \in \mathscr{X}$, $f \in \mathscr{F}(X, Y)$ be a L-Lipschitz function, $\mathfrak{S}$ be a closed subset of the Banach space X and $x_\varepsilon \in \varepsilon k^0$-MZ$(\mathfrak{S}, f, C)$. Then for every $\mathfrak{s} \in \mathfrak{S}$ and $k^0 \in \operatorname{int} C(f(\mathfrak{s}))$ and $\mu > 0$, there exist elements $u_{\mathfrak{s}} \in B(x_\varepsilon, \sqrt{\varepsilon} + \mu)$,

$z_{\mathfrak{s}} \in B(x_\varepsilon, \sqrt{\varepsilon}+\mu/2) \cap \mathfrak{S}, u^*_{\mathfrak{s}} \in (C(f(\mathfrak{s})))^*, u^*_{\mathfrak{s}}(k^0) = 1, x^*_{\mathfrak{s}} \in X^*, \|x^*_{\mathfrak{s}}\| \le 1$ such that

$$\mathbf{0} \in \partial(u^*_{\mathfrak{s}} \circ f)(u_{\mathfrak{s}}) + \sqrt{\varepsilon} u^*_{\mathfrak{s}}(k^0) x^*_{\mathfrak{s}} + N_\partial(\mathfrak{S}, z_{\mathfrak{s}}) + B(0,\mu),$$

provided that ∂ satisfies (H1), (H2), (H6), (H7).
Moreover, for some elements $x \in B(x_\varepsilon, \sqrt{\varepsilon}+\mu/2)$ and $v_{\mathfrak{s}} \in B(f(x)-f(x_\varepsilon), L\sqrt{\varepsilon}+\mu)$ it holds $u^*_{\mathfrak{s}}(v_{\mathfrak{s}}) = \xi(v_{\mathfrak{s}}, \mathfrak{s})$.

Proof. Let $x_\varepsilon \in \varepsilon k^0$-MZ$(\mathfrak{S}, f, C)$. Taking into account Definition 5.0.4, for all $\mathfrak{s} \in \mathfrak{S}$ we have

$$(f(x_\varepsilon) - \varepsilon k^0 - C(f(\mathfrak{s}))\backslash\{\mathbf{0}\}) \cap f(\mathfrak{S}) = \emptyset.$$

The function f is supposed to be Lipschitz, so it is continuous as well and since $\mathfrak{S}$ is a closed set in a Banach space it is a complete metric space endowed with the distance induced by the norm. Thus, the assumptions of the vector-valued variant of Ekeland's variational principle given in [34, Corollary 9] are fulfilled for all $\mathfrak{s} \in \mathfrak{S}$. Applying this variational principle for all $\mathfrak{s} \in \mathfrak{S}$, we get the existence of an element $x_{\mathfrak{s}} \in \mathfrak{S}$ with $\|x_{\mathfrak{s}} - x_\varepsilon\| < \sqrt{\varepsilon}$ and having the property that

$$h_{\mathfrak{s}}(\mathfrak{S}) \cap (h_{\mathfrak{s}}(x_{\mathfrak{s}}) - C(f(\mathfrak{s})) \setminus \{\mathbf{0}\}) = \emptyset,$$

for all $\mathfrak{s} \in \mathfrak{S}$ where

$$h_{\mathfrak{s}}(x) := f(x) + \sqrt{\varepsilon}\,\|x - x_{\mathfrak{s}}\|\, k^0.$$

Let $\mu > 0$. Applying now [21, Theorem 4.2] for a positive number δ with properties that $2\delta < \mu$ and $\sqrt{\varepsilon}\,\|k^0\|\,\delta/2 + \delta/2 < \mu$ and using the functional $\xi(\cdot, f(\mathfrak{s}))$ defined by (6.21), we can find $\overline{u}_{\mathfrak{s}} \in B(x_{\mathfrak{s}}, \delta) \subset B(x_\varepsilon, \sqrt{\varepsilon}+\delta)$, $x \in B(x_{\mathfrak{s}}, \delta/2) \subset B(x_\varepsilon, \sqrt{\varepsilon}+\delta/2)$, $v_{\mathfrak{s}} \in B(h_{\mathfrak{s}}(x) - h_{\mathfrak{s}}(x_{\mathfrak{s}}), \delta/2)$, $z_{\mathfrak{s}} \in B(x_{\mathfrak{s}}, \delta/2) \cap \mathfrak{S} \subset B(x_\varepsilon, \sqrt{\varepsilon}+\delta/2) \cap \mathfrak{S}$, $u^*_{\mathfrak{s}} \in \partial\xi(v_{\mathfrak{s}}, f(\mathfrak{s}))$, such that

$$\mathbf{0} \in \partial(u^*_{\mathfrak{s}} \circ h_{\mathfrak{s}})(\overline{u}_{\mathfrak{s}}) + N_\partial(\mathfrak{S}, z_{\mathfrak{s}}) + B(0,\delta). \tag{6.23}$$

We get the properties $u^*_{\mathfrak{s}} \in (C(f(\mathfrak{s})))^*, u^*_{\mathfrak{s}}(k^0) = 1$ from (6.22).
Consider the element $\overline{x}^*_{\mathfrak{s}} \in \partial(u^*_{\mathfrak{s}} \circ h_{\mathfrak{s}})(\overline{u}_{\mathfrak{s}})$ involved in (6.23). Because of

$$\partial(u^*_{\mathfrak{s}} \circ h_{\mathfrak{s}})(\overline{u}_{\mathfrak{s}}) = \partial(u^*_{\mathfrak{s}} \circ (f(\cdot) + \sqrt{\varepsilon}\,\|\cdot - x_{\mathfrak{s}}\|\, k^0))(\overline{u}_{\mathfrak{s}}),$$

taking into account (H1) and (H6), there exist $u_{\mathfrak{s}} \in B(\overline{u}_{\mathfrak{s}}, \delta) \subset B(x_\varepsilon, \sqrt{\varepsilon}+2\delta)$ and $u'_{\mathfrak{s}} \in B(\overline{u}_{\mathfrak{s}}, \delta)$ such that

$$\overline{x}^*_{\mathfrak{s}} \in \partial(u^*_{\mathfrak{s}} \circ f)(u_{\mathfrak{s}}) + \sqrt{\varepsilon} u^*_{\mathfrak{s}}(k^0)\partial(\|\cdot - x_{\mathfrak{s}}\|)(u'_{\mathfrak{s}}) + B(0,\delta). \tag{6.24}$$

Taking into account the well-known structure of the subdifferential of the norm and combining relations (6.23) and (6.24) it follows that there exists $x^*_{\mathfrak{s}} \in X^*$ with $||x^*_{\mathfrak{s}}|| = 1$ such that

$$\mathbf{0} \in \partial(u^*_{\mathfrak{s}} \circ f)(u_{\mathfrak{s}}) + \sqrt{\varepsilon} u^*_{\mathfrak{s}}(k^0) x^*_{\mathfrak{s}} + N_{\partial}(\mathfrak{S}, z_{\mathfrak{s}}) + B(0, 2\delta).$$

Because $2\delta < \mu$, it remains only to prove the estimation about the ball which contains $v_{\mathfrak{s}}$. Then,

$$\begin{aligned}\|v_{\mathfrak{s}} - (f(x) - f(x_\varepsilon))\| &\leq \|v_{\mathfrak{s}} - (h_{\mathfrak{s}}(x) - h_{\mathfrak{s}}(x_{\mathfrak{s}}))\| + \|(h_{\mathfrak{s}}(x) - h_{\mathfrak{s}}(x_{\mathfrak{s}})) - (f(x) - f(x_\varepsilon))\| \\ &\leq \delta/2 + \left\|\sqrt{\varepsilon} k^0 \|x - x_{\mathfrak{s}}\| - f(x_{\mathfrak{s}}) + f(x_\varepsilon)\right\| \\ &\leq \delta/2 + \sqrt{\varepsilon}\left\|k^0\right\| \delta/2 + L\sqrt{\varepsilon} \\ &< L\sqrt{\varepsilon} + \mu,\end{aligned}$$

where the last inequality follows because of the assumptions made on δ. Moreover, we get $u^*_{\mathfrak{s}}(v_{\mathfrak{s}}) = \xi(v_{\mathfrak{s}}, \mathfrak{s})$ from (6.22). This completes the proof. □

6.3.3 Second Order Optimality Conditions

In this section, we give second order optimality conditions for minimizers of set-valued optimization problems with variable ordering structures. Consider the optimization problem (P_1). We say $(\bar{x}, \bar{y}) \in \operatorname{gph}(F)$ is a weak minimizer of (P_1) if and only if

$$(\bar{y} - \operatorname{int} K(x)) \cap F(\mathfrak{S}) = \emptyset \qquad \forall x \in \mathfrak{S}$$

and $(\bar{x}, \bar{y}) \in \operatorname{gph}(F)$ is called a local weak minimizer if and only if there exists a neighborhood U of $\bar{x}$ such that

$$(\bar{y} - \operatorname{int} K(x)) \cap F(\mathfrak{S}) = \emptyset \qquad \forall x \in \mathfrak{S} \cap U.$$

By first parts of Theorem 3.2.18 and Theorem 3.2.21, each weak minimizer is both weakly minimal and weakly nondominated solution and all the results in sections 6.1.3 and 6.2.3 hold also for weak minimizers.

In the following theorem, we give necessary optimality conditions for local weak minimizers of (P_1); see [43] for the case of fixed ordering structure. For $\mathfrak{s} \in \mathfrak{S}$, $(F + K(\mathfrak{s})) : X \rightrightarrows Y$ is defined as $(F + K(\mathfrak{s}))(x) = F(x) + K(\mathfrak{s})$.

Theorem 6.3.11. Let assumption (A14) be fulfilled and $(\bar{x}, \bar{y}) \in \operatorname{gph}(F)$ be a local weak minimizer of problem (P_1), then for every $\mathfrak{s} \in \mathfrak{S}$, $\bar{u} \in \operatorname{dom}(D(F + K(\mathfrak{s}))(\bar{x}, \bar{y}))$ and for every $\bar{v} \in D(F + K(\mathfrak{s}))(\bar{x}, \bar{y})(\bar{u}) \cap (-K(\mathfrak{s}))$, the following holds:

$$D^2(F + K(\mathfrak{s}))(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) \cap IT(-\operatorname{int} K(\mathfrak{s}), \bar{v}) = \emptyset \tag{6.25}$$

for all $x \in \operatorname{dom}(D^2(F + K(\mathfrak{s}))(\bar{x}, \bar{y}, \bar{u}, \bar{v}))$.

Proof. Suppose that (6.25) does not hold and for some arbitrary $\mathfrak{s} \in \mathfrak{S}$, there exists an element $x \in \operatorname{dom}(D^2(F+K(\mathfrak{s})))(\bar{x},\bar{y},\bar{u},\bar{v})$ such that

$$y \in D^2(F+K(\mathfrak{s}))(\bar{x},\bar{y},\bar{u},\bar{v})(x) \cap IT(-\operatorname{int} K(\mathfrak{s}),\bar{v})).$$

Then $(x,y) \in T(T(\operatorname{gph}(F)+K(\mathfrak{s}),(\bar{x},\bar{y})),(\bar{u},\bar{v}))$. Therefore there exist sequences $(t_n) \in \mathbb{R}_+$ with $t_n \downarrow 0$ and $(x_n,y_n) \subset X \times Y$ with $(x_n,y_n) \to (x,y)$ such that

$$(\bar{u}+t_n x_n, \bar{v}+t_n y_n) \in T(\operatorname{gph}(F)+K(\mathfrak{s}),(\bar{x},\bar{y})) \qquad \forall n \in \mathbb{N}.$$

By $t_n \downarrow 0$, $y_n \to y$ and $y \in IT(-\operatorname{int} K(\mathfrak{s}),\bar{v})$, there exists $n_1 \in \mathbb{N}$ such that

$$\bar{v}+t_n y_n \in -\operatorname{int} K(\mathfrak{s}), \qquad \forall n > n_1.$$

For any $n > n_1$, we fix an element $(u_n,v_n) = (\bar{u}+t_n x_n, \bar{v}+t_n y_n)$ and notice that

$$(u_n,v_n) \in T(\operatorname{gph}(F)+K(\mathfrak{s}),(\bar{x},\bar{y})).$$

By the definition of contingent cones, for (u_n,v_n), there exist sequences $(x_m,y_m) \subset X \times Y$ with $(x_m,y_m) \to (u_n,v_n)$ and $(t_m) \subset \mathbb{R}_+$ with $t_m \downarrow 0$ such that

$$\bar{y}+t_m y_m \in F(\bar{x}+t_m x_m)+K(\mathfrak{s}).$$

Furthermore, by $v_n \in -\operatorname{int} K(\mathfrak{s})$ and $y_m \to v_n$, there exists $m_1 \in \mathbb{N}$ such that $y_m \in -\operatorname{int} K(\mathfrak{s})$ for all $m > m_1$. Since $K(\mathfrak{s})$ is a cone, this implies that $t_m y_m \in -\operatorname{int} K(\mathfrak{s})$. Now assume that $a_m \in F(\bar{x}+t_m x_m)$ such that $\bar{y}+t_m y_m \in a_m + K(\mathfrak{s})$. Then

$$a_m \in \bar{y} - \operatorname{int} K(\mathfrak{s}).$$

By $b_m := (\bar{x}+t_m x_m) \to \bar{x}$, there exists $m_2 > 0$ such that $b_m \in \mathscr{N}(\bar{x})$ where $\mathscr{N}(\bar{x})$ is a suitable neighborhood of $\bar{x}$. Therefore, we have shown that there exists an arbitrary element $\mathfrak{s} \in \mathfrak{S}$ and a sequence $\{a_m\}$ such that

$$a_m \in F(b_m) \cap (\bar{y} - \operatorname{int} K(\mathfrak{s})) \qquad \text{for all } m > \{m_1, m_2\}.$$

This is a contradiction because we supposed that $(\bar{x},\bar{y})$ is a local weak minimizer. $\square$

If we set $(\bar{x},\bar{y}) = (\mathbf{0}_X, \mathbf{0}_Y)$, we have the following corollary.

Corollary 6.3.12. Let assumption (A14) be fulfilled and $(\bar{x},\bar{y}) \in \operatorname{gph}(F)$ be a local weak minimizer of problem (P_1), then for all $\mathfrak{s} \in \mathfrak{S}$

$$D(F+K(\mathfrak{s}))(\bar{x},\bar{y}) \cap -\operatorname{int} K(\mathfrak{s}) = \emptyset \quad \text{for all} \quad x \in \operatorname{dom}(D(F+K(\mathfrak{s}))(\bar{x},\bar{y})).$$

The following theorem gives a necessary optimality condition for local weak minimizer of (P_2); see [43] for the case of fixed ordering structure. For each $\mathfrak{s} \in \mathfrak{S}_1$ in (P2), $F_{\mathfrak{s}}(x)$ is defined as $F(x) + K(\mathfrak{s})$.

Theorem 6.3.13. Let assumption (A15) be fulfilled, $(\bar{x}, \bar{y}) \in \operatorname{gph}(F)$ be a local weak minimizer of (P_2) and $\bar{z} \in G(\bar{x})$. Then for every $\mathfrak{s} \in \mathfrak{S}$, for every $\bar{u} \in \operatorname{dom}(D(F_{\mathfrak{s}}, G)(\bar{x}, \bar{y}, \bar{z}))$, for every $(\bar{v}, \overline{\omega}) \in D(F_{\mathfrak{s}}, G)(\bar{x}, \bar{y}, \bar{z}) \cap \{-K(\mathfrak{s}) \times -D\}$ and for every $x \in D_1 := \operatorname{dom}(D^2(F_{\mathfrak{s}}, G)(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \overline{\omega}))$, we have

$$D^2(F_{\mathfrak{s}}, G)(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \overline{\omega}))(x) \cap IT(-K(\mathfrak{s}), \bar{v}) \times IT(IT(-D, \bar{z}), \overline{\omega}) = \emptyset \tag{6.26}$$

where the notation $(F_{\mathfrak{s}}, G)(x)$ represents $(F(x) + K(\mathfrak{s})) \times (G(x) + D)$ for each $\mathfrak{s} \in \mathfrak{S}_1$.

Proof. Suppose that assertion (6.26) is not true and there exists an arbitrary element $\mathfrak{s} \in \mathfrak{S}_1$ and $x \in \operatorname{dom}(D^2(F_{\mathfrak{s}}, G)(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \overline{\omega}))$ such that

$$(y, z) \in D^2(F_{\mathfrak{s}}, G)(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \overline{\omega})(x) \cap IT(-K(\mathfrak{s}), \bar{v}) \times IT(IT(-\operatorname{int} D, \bar{z}), \overline{\omega})$$

which implies that $(x, y, z) \in T(T(\operatorname{gph}(F_{\mathfrak{s}}, G), (\bar{x}, \bar{y}, \bar{z})), (\bar{u}, \bar{v}, \overline{\omega}))$. Therefore, there exist sequences $(t_n) \subset \mathbb{R}_+$ with $t_n \downarrow 0$ and $(x_n, y_n, z_n) \subset X \times Y \times Z$ with $(x_n, y_n, z_n) \to (x, y, z)$ such that

$$(\bar{u} + t_n x_n, \bar{v} + t_n y_n, \overline{\omega} + t_n z_n) \in T(\operatorname{gph}(F_{\mathfrak{s}}, G), (\bar{x}, \bar{y}, \bar{z})).$$

By $y \in IT(-\operatorname{int} K(\mathfrak{s}), \bar{v})$, there exists $n_1 \in \mathbb{N}$ such that

$$\bar{v} + t_n y_n \in -\operatorname{int} K(\mathfrak{s}) \qquad n \geq n_1.$$

By $z \in IT(IT(-\operatorname{int} D, \bar{z}), \overline{\omega})$, there exists $n_2 \in \mathbb{N}$ such that $\overline{\omega} + t_n z_n \in IT(-\operatorname{int} D, \bar{z})$ for all $n > n_2$. For $n \geq \max\{n_1, n_2\}$, we fix $u_n := \bar{u} + t_n x_n$, $v_n := \bar{v} + t_n y_n$ and $\omega_n := \overline{\omega} + t_n z_n$. Then

$$(u_n, v_n, \omega_n) \in T(\operatorname{gph}(F_{\mathfrak{s}}, G), (\bar{x}, \bar{y}, \bar{z})),$$
$$(v_n, \omega_n) \in -\operatorname{int} K(\mathfrak{s}) \times IT(-\operatorname{int} D, \bar{z}).$$

By definition of contingent cones, there exist sequences $(t_m) \subset \mathbb{R}_+$ with $t_m \downarrow 0$ and (x_m, y_m, z_m) in $X \times Y \times Z$ with $(x_m, y_m, z_m) \to (u_n, v_n, \omega_n)$ such that $(\bar{y} + t_m y_m, \bar{z} + t_m z_m) \in (F_{\mathfrak{s}}, G)(\bar{x} + t_m x_m)$ which means

$$\bar{y} + t_m y_m \in F(\bar{x} + t_m x_m) + K(\mathfrak{s}),$$
$$\bar{z} + t_m z_m \in G(\bar{x} + t_m x_m) + D.$$

By $y_m \to v_n$ and $v_n \in -\operatorname{int} K(\mathfrak{s})$, there exists $m_1 > 0$ such that

$$t_m y_m \in -\operatorname{int} K(\mathfrak{s}) \qquad \forall m > m_1.$$

Let $d_m \in F(\bar{x} + t_m x_m)$ be such that $\bar{y} - \operatorname{int} K(\mathfrak{s}) \in d_m + K(\mathfrak{s})$ and consequently we have:

$$d_m \in \bar{y} - \operatorname{int} K(\mathfrak{s}).$$

By $\omega_n \in IT(-\operatorname{int} D, \bar{z})$, there exists $m_2 > 0$ such that $\bar{z} + t_m z_m \in -\operatorname{int} D$ for every $m > m_2$. Let $b_m \in G(\bar{x} + t_m x_m)$ be such that $\bar{z} + t_m z_m \in b_m + D$ and therefore we have $b_m \in -\operatorname{int} D$. Therefore, for sufficiently large m, we have $c_m := \bar{x} + t_m x_m \in \mathscr{N}(\bar{x})$, $G(c_m) \cap -D \neq \emptyset$ and

$$F(c_m) \cap (\bar{y} - \operatorname{int} K(\mathfrak{s})) \neq \emptyset.$$

But this is a contradiction because we supposed that $(\bar{x}, \bar{y})$ is a local weak minimizer of (P_2). $\square$

The following theorem gives a necessary optimality condition for local weak minimizers of (P_3). For each $\mathfrak{s} \in \mathfrak{S}_2$, $F_{\mathfrak{s}}(x)$ is defined as $F(x) + K(\mathfrak{s})$

Theorem 6.3.14. Let assumption (A16) be fulfilled. Suppose that $(\bar{x}, \bar{y}) \in \operatorname{gph}(F)$ is a local weak minimizer of problem (P_3), $\bar{z} \in G(\bar{x})$ and $D : X \rightrightarrows Z$ satisfies one of the following conditions:

(i^*) D is a convex process.

(ii^*) For all $x \in \mathscr{N}(\bar{x})$, $D(\bar{x}) \subseteq D(x)$.

Then for every element $\mathfrak{s} \in \mathfrak{S}_2$, for every $(\bar{v}, \overline{\omega}) \in D(\widehat{F_{\mathfrak{s}}, G})(\bar{x}, \bar{y}, \bar{z}) \cap \{-K(\bar{x}) \times -D(\bar{x})\}$, for every $\bar{u} \in \operatorname{dom}(D(\widehat{F_{\mathfrak{s}}, G})(\bar{x}, \bar{y}, \bar{z}))$ and for every $x \in D_1 := \operatorname{dom}(D^2(\widehat{F_{\mathfrak{s}}, G})(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \overline{\omega}))$, we have

$$D^2(\widehat{F_{\mathfrak{s}}, G})(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \overline{\omega})(x) \cap IT(-K(\mathfrak{s}), \bar{v}) \times IT(IT(-D(\bar{x}), \bar{z}), \overline{\omega}) = \emptyset$$

where the notation $(\widehat{F_{\mathfrak{s}}, G})(x)$ represents $(F(x) + K(\mathfrak{s})) \times (G(x) + D(x))$.

Proof. Suppose that assertion is not true and there exists an arbitrary element $\mathfrak{s} \in \mathfrak{S}_2$ and element $x \in \operatorname{dom}(D^2(\widehat{F_{\mathfrak{s}}, G})(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \overline{\omega}))$ such that

$$(y, z) \in D^2(\widehat{F_{\mathfrak{s}}, G})(\bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \overline{\omega})(x) \cap IT(-K(\mathfrak{s}), \bar{v}) \times IT(IT(-\operatorname{int} D(\bar{x}), \bar{z}), \overline{\omega})$$

which implies that $(x, y, z) \in T(T(\operatorname{gph}(\widehat{F_{\mathfrak{s}}, G}), (\bar{x}, \bar{y}, \bar{z})), (\bar{u}, \bar{v}, \overline{\omega}))$. Therefore, there exist sequences $(t_n) \subset \mathbb{R}_+$ with $t_n \downarrow 0$ and $(x_n, y_n, z_n) \subset X \times Y \times Z$ with $(x_n, y_n, z_n) \to (x, y, z)$ such that

$$(\bar{u} + t_n x_n, \bar{v} + t_n y_n, \overline{\omega} + t_n z_n) \in T(\operatorname{gph}(\widehat{F_{\mathfrak{s}}, G}), (\bar{x}, \bar{y}, \bar{z})).$$

By $y \in IT(-\operatorname{int} K(\mathfrak{s}), \bar{v})$, there exists $n_1 \in \mathbb{N}$ such that

$$\bar{v} + t_n y_n \in -\operatorname{int} K(\mathfrak{s}) \qquad n \geq n_1.$$

By $z \in IT(IT(-\operatorname{int} D(\bar{x}), \bar{z}), \overline{\omega})$, there exists $n_2 \in \mathbb{N}$ such that $\overline{\omega} + t_n z_n \in IT(-\operatorname{int} D(\bar{x}), \bar{z})$ for all $n > n_2$. For $n \geq \max\{n_1, n_2\}$, we fix $u_n := \bar{u} + t_n x_n$, $v_n := \bar{v} + t_n y_n$ and $\omega_n := \overline{\omega} + t_n z_n$. Then

$$(u_n, v_n, \omega_n) \in T(\operatorname{gph}(\widehat{F_{\mathfrak{s}}, G}), (\bar{x}, \bar{y}, \bar{z}))$$

$$(v_n, \omega_n) \in -\operatorname{int} K(\mathfrak{s}) \times IT(-\operatorname{int} D(\bar{x}), \bar{z}).$$

By definition of contingent cones, there exist sequences $(t_m) \subset \mathbb{R}_+$ with $t_m \downarrow 0$ and (x_m, y_m, z_m) in $X \times Y \times Z$ with $(x_m, y_m, z_m) \to (u_n, v_n, \omega_n)$ such that $(\bar{y} + t_m y_m, \bar{z} + t_m z_m) \in (\widehat{F_{\mathfrak{s}}, G})(\bar{x} + t_m x_m)$ which means

$$\bar{y} + t_m y_m \in F(\bar{x} + t_m x_m) + K(\mathfrak{s}),$$

$$\bar{z} + t_m z_m \in G(\bar{x} + t_m x_m) + D(\bar{x} + t_m x_m).$$

By $y_m \to v_n$ and $v_n \in -\operatorname{int} K(\mathfrak{s})$, there exists $m_1 > 0$ such that

$$t_m y_m \in -\operatorname{int} K(\mathfrak{s}) \qquad \forall m > m_1.$$

Let $d_m \in F(\bar{x} + t_m x_m)$ be such that $\bar{y} - \operatorname{int} K(\mathfrak{s}) \in d_m + K(\mathfrak{s})$ and consequently we have:

$$d_m \in \bar{y} - \operatorname{int} K(\mathfrak{s}).$$

By $\omega_n \in IT(-\operatorname{int} D(\bar{x}), \bar{z})$, there exists $m_2 > 0$ such that $\bar{z} + t_m z_m \in -\operatorname{int} D(\bar{x})$ for every $m > m_2$. Let $b_m \in G(\bar{x} + t_m x_m)$ be such that $\bar{z} + t_m z_m \in b_m + D(\bar{x} + t_m x_m)$ and $b_m \in -\operatorname{int} D(\bar{x}) - D(\bar{x} + t_m x_m)$. Now if condition (i*) on D holds we have,

$$D(\bar{x}) \subseteq D(\bar{x}) + \mathbf{0} \subseteq D(\bar{x}) + D(t_m x_m) \subseteq D(\bar{x} + t_m x_m)$$

or if condition (ii*) on D holds, again we have directly $D(\bar{x}) \subseteq D(\bar{x} + t_m x_m)$ and since $D(x)$ is a convex cone for all x, we get

$$b_m \in -\operatorname{int} D(\bar{x} + t_m x_m) - D(\bar{x} + t_m x_m) \subseteq -\operatorname{int} D(\bar{x} + t_m x_m).$$

Therefore, for sufficiently large m, $c_m := \bar{x} + t_m x_m \in \mathscr{N}(\bar{x})$, $G(c_m) \cap -D(c_m) \neq \emptyset$ and

$$F(c_m) \cap (\bar{y} - \operatorname{int} K(\mathfrak{s})) \neq \emptyset.$$

But this is a contradiction because we supposed that $(\bar{x}, \bar{y})$ is a local weak minimizer of (P_3) and proof is complete. $\square$

Chapter 7

Conclusions

In this thesis, we studied approximate solutions of vector optimization problems with variable ordering structures and their properties. After introducing concepts of approximate solutions of vector optimization problems with variable ordering structures, we characterized them using generalizations of nonlinear scalarizing functionals defined by Tammer and Weidner. We derived variational principles for vector optimization problems with variable ordering structures and these variational principles were used in the last chapter in order to show optimality conditions for approximate solutions of vector optimization problems with variable ordering structures.

Approximate solutions of vector optimization problems play an important role from the theoretical as well as computational point of view. It is well known that one needs compactness assumptions in order to show existence results for solutions of optimization problems. Such compactness assumptions are not fulfilled for many optimization problems. Also we know that under weak assumptions and without compactness conditions, we have to deal with approximate solutions and we can show several assertions without any compactness assumptions for these solutions. Furthermore, if we apply numerical algorithms for solving optimization problems, then these algorithms usually generate approximate solutions that are close to the exact solutions. In the third chapter, we introduced several notions of approximate elements of vector optimization problems with fixed and variable ordering structures and later, relationships between sets of approximate solutions choosing different parameters were discussed. The last section of third chapter was devoted to the presentation of relationships between different concepts of approximate solutions of vector optimization problems with variable ordering structures. Obviously, exact solution of vector optimization problems is the special case of approximate solution and all our results in this thesis can be used for exact solutions.

Scalarization of a given vector optimization problem means the replacement of it by a suitable scalar optimization problem with a real-valued objective function. Indeed, solutions of vector optimization problems can be found through scalarization procedures and we use properties of scalar optimization problems to characterize solutions of original vector optimization problems. Characterization of approximate solutions of vector optimization problems with respect to variable ordering structures by means of suitable nonlinear functionals were discussed in the fourth chapter. We characterized approximate minimizer, approximate nondominated and approximate minimal solutions of vector optimization problems with respect to a variable ordering structure by generalization of nonlinear separating functional $\theta(y) : Y \to \overline{\mathbb{R}}$ defined by Tammer and Weidner as following:

$$\theta(y) = \inf\{t \in \mathbb{R} \mid y \in tk - C\}.$$

In the fifth chapter, we used the concepts for approximate solutions of vector optimization problems with variable ordering structures in order to derive variational principles for problems with variable ordering structure. Ekeland's variational principle is a deep assertion concerning the existence of an exact solution of a slightly perturbed optimization problem in a neighborhood of an approximate solution of the original optimization problem under the assumption that the objective function of the original problem is bounded from below and lower semicontinuous. Firstly, we supposed that our ordering set-valued map $C : Y \rightrightarrows Y$ associates a set $C(y)$ for any $y \in Y$ with nonempty interior and we gave an extension of Ekeland's variational principle for approximate solutions of vector optimization problems with respect to this ordering map. In fact, we show the existence of an element belonging to the set of approximately minimal solutions of the original problem that is a weakly minimal solution of a perturbed optimization problem. Later, we showed a sharper result, namely that there exists not just weakly minimal but minimal solution of the original problem that is a minimal solution of a perturbed vector optimization problem with variable ordering structure. After proving results for solid case, we supposed that our ordering set-valued map C associates a set $C(y)$ for any $y \in Y$ with empty interior and we gave an extension of Ekelenad's variational principle for approximate minimal solutions of vector optimization problems with variable ordering structures.

In the last chapter, we presented necessary optimality conditions for approximate minimizers and approximately nondominated and approximately minimal solutions of vector optimization problems with variable ordering structures. We used our results in fourth and fifth chapters in order to show optimality condition for approximate solutions. After that we gave second-order optimality conditions by concept of tangential derivatives of second-order for set-valued optimization problems with variable ordering structures.

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List of Symbols and Abbreviations

$:=$	equal by definition	6
$\leq$	partial ordering on a linear space	5
$\leq_C$	partial ordering induced by a convex cone C	8
$<_C$	strict partial ordering by cone C	8
$x^n \to x^0$	x^n converges x^0	42
$x_k^* \xrightarrow{w^*} x^*$	x_k^* converges x^* (in weak* topology)	84
$\liminf$	lower limit for real numbers	41
$\limsup$	upper limit for real numbers	41
$C^{1,1}$	continuously differentiable functions with Lipschitzian derivatives	85
$\mathbb{R}$	real line $(-\infty, +\infty)$	41
$\mathbb{R}_+$	positive real line $[0, +\infty)$	6
$\overline{\mathbb{R}}$	extended real line $[-\infty, +\infty]$	33
$\mathbb{R}^2_+$	nonnegative orthant of $\mathbb{R}^2$	24
L^p	standard Lebesgue spaces where $1 \leq p \leq \infty$	84
l^p	sequences of real numbers where $1 \leq p \leq \infty$	84
(X, d)	metric space with metric d	47
Y^*	continuous dual space of Y	7
$\emptyset$	empty set	4
$\mathbf{0}_Y$	zero vector of space Y	6
$[0, \overline{\varepsilon}]$	closed interval between 0 and $\overline{\varepsilon}$	7
$(0, +\infty)$	open interval between 0 and $+\infty$	98
$[0, +\infty)$	half open interval between 0 and $+\infty$	98
B_{X^*}	unit ball in X^*	87
$\operatorname{bd} S$	boundary of S	6
C^*	dual cone of the cone C	7
$C^\sharp$	quasi interior of the cone C^+	7

$\operatorname{cone} S$	conic hull of S	6
$\operatorname{conv} S$	convex hull of S	6
$\operatorname{cl} S$	closure of S	6
$\operatorname{int} S$	interior of S	6
$\operatorname{rint} S$	relative interior of S	6
$(\operatorname{int} C(\omega))^c$	complement of $\operatorname{int} C(\omega)$ i.e., $Y \setminus \operatorname{int} C(\omega)$	46
εk^0-$\mathrm{M}(\Omega, C)$	approximately minimal elements of Ω w.r.t. C	23
εk^0-$\mathrm{MZ}(\Omega, C)$	approximate minimizers of Ω w.r.t. C	14
εk^0-$\mathrm{N}(\Omega, C)$	approximately nondominated elements of Ω w.r.t. C	19
εk^0-$\mathrm{SM}(\Omega, C)$	approximately strong minimal elements of Ω w.r.t. C	23
εk^0-$\mathrm{SMZ}(\Omega, C)$	strong approximate minimizers of Ω w.r.t. C	14
εk^0-$\mathrm{SN}(\Omega, C)$	approximately strong nondominated of Ω w.r.t. C	19
εk^0-$\mathrm{WM}(\Omega, C)$	approximately weak minimal elements of Ω w.r.t. C	23
εk^0-$\mathrm{WMZ}(\Omega, C)$	weak approximate minimizers of Ω w.r.t. C	14
εk^0-$\mathrm{WN}(\Omega, C)$	approximately weak nondominated of Ω w.r.t. C	19
εk^0-$\mathrm{M}(\mathfrak{S}, f, C)$	approximately minimal solutions of $VVOP$	59
εk^0-$\mathrm{MZ}(\mathfrak{S}, f, C)$	approximate minimizer solutions of $VVOP$	58
εk^0-$\mathrm{N}(\mathfrak{S}, f, C)$	approximately nondominated solutions of $VVOP$	58
εk^0-$\mathrm{WM}(\mathfrak{S}, f, C)$	approximately weak minimal solutions of $VVOP$	59
εk^0-$\mathrm{WMZ}(\mathfrak{S}, f, C)$	weak approximate minimizer solutions of $VVOP$	58
εk^0-$\mathrm{WN}(\mathfrak{S}, f, C)$	approximately weak nondominated solutions of $VVOP$	58
$\mathrm{Max}(A, \mathscr{R})$	the class of maximal elements of Ω with respect to $\mathscr{R}$	5
$\mathrm{Min}(A, \mathscr{R})$	the class of minimal elements of A with respect to $\mathscr{R}$	5
$\mathrm{M}(\Omega, C)$	minimal elements of Ω w.r.t. C	23
$\mathrm{MZ}(\Omega, C)$	minimizers of Ω w.r.t. C	14
$\mathrm{N}(\Omega, C)$	nondominated elements of Ω w.r.t. C	19
$\mathrm{SM}(\Omega, C)$	strongly minimal elements of Ω w.r.t. C	23
$\mathrm{SMZ}(\Omega, C)$	strong minimizers of Ω w.r.t. C	14
$\mathrm{SN}(\Omega, C)$	strongly nondominated elements of Ω w.r.t. C	19
$\mathrm{WM}(\Omega, C)$	weakly minimal elements of Ω w.r.t. C	23
$\mathrm{WMZ}(\Omega, C)$	weak minimizers of Ω w.r.t. C	14
$\mathrm{WN}(\Omega, C)$	weakly nondominated elements of Ω w.r.t. C	19
$\mathrm{M}(\mathfrak{S}, f, C)$	minimal solutions of $VVOP$	59
$\mathrm{MZ}(\mathfrak{S}, f, C)$	minimizer solutions of $VVOP$	58
$\mathrm{N}(\mathfrak{S}, f, C)$	nondominated solutions of $VVOP$	58
$\mathrm{WM}(\mathfrak{S}, f, C)$	weakly minimal solutions of $VVOP$	59
$\mathrm{WMZ}(\mathfrak{S}, f, C)$	weak minimizer solutions of $VVOP$	58
$\mathrm{WN}(\mathfrak{S}, f, C)$	weakly nondominated solutions of $VVOP$	58
$\mathscr{N}(\bar{x})$	neighborhood of x	84

$\mathbb{N}$	set of natural numbers	97
$N(\bar{x};\mathfrak{S})$	basic/limiting normal cone to $\mathfrak{S}$ at $\bar{x}$	84
$\widehat{N}(\bar{x};\mathfrak{S})$	Fréchet normal cone to $\mathfrak{S}$ at $\bar{x}$	84
$\mathscr{P}$	power set	6
$S+T$	algebraic sum of two sets S and T	6
$T(\mathfrak{S},\bar{x})$	contingent cone to $\mathfrak{S}$ at $\bar{x}$	94
$IT(\mathfrak{S},\bar{x})$	interiorly contingent cone to $\mathfrak{S}$ at $\bar{x}$	94
$T^2(\mathfrak{S},\bar{x},d)$	second order contingent set to $\mathfrak{S}$ at $\bar{x}$ in direction d	94
$f : X \to Y$	single-valued mapping from X to Y	41
$F : X \rightrightarrows Y$	set-valued mapping from X to Y	94
$\operatorname{dom} f$	domain of function f	62
$\operatorname{epi} f$	epigraph of function f	85
$\operatorname{gph} F$	graph of function F	94
$\partial f(\bar{x})$	Fenchel subdifferential of f at $\bar{x}$	83
$\partial_M f(\bar{x})$	basic/limiting subdifferential of f at $\bar{x}$	85
$\partial^\infty f(\bar{x})$	singular subdifferential of f at $\bar{x}$	85
$DF(\bar{x},\bar{y})$	contingent epiderivative of F at $(\bar{x},\bar{y}) \in \operatorname{gph} F$	95
$D^2F(\bar{x},\bar{y},\bar{u},\bar{v})$	second order tangential derivative of F at $(\bar{x},\bar{y}) \in \operatorname{gph} F$ in the direction of $(\bar{u},\bar{v})$	96
$D_cF(\bar{x},\bar{y})$	contingent derivative of F at $(\bar{x},\bar{y}) \in \operatorname{gph} F$	95
$D_c^2F(\bar{x},\bar{y},\bar{u},\bar{v})$	second order contingent derivative of F at $(\bar{x},\bar{y}) \in \operatorname{gph} F$ in the direction of $(\bar{u},\bar{v})$	95
$D_e^2F(\bar{x},\bar{y},\bar{u},\bar{v})$	second order tangential epiderivative of F at $(\bar{x},\bar{y}) \in \operatorname{gph} F$ in the direction of $(\bar{u},\bar{v})$	96
EVP	Ekeland's variational principle	55
$\operatorname{lev}(y;f)$	sublevel set	59
l.s.c	lower semicontinuous	40
PS	Pascoletti-Serafini scalarization method	34
TSP	Tammer-Weidner scalarization method	34
VOP	vector optimization problem	10
VVOP	vector optimization problem with a variable ordering structure	12, 56
w.r.t.	with respect to	12

Index

Selbständigkeitserklärung

Hiermit erkläre ich Behnam Soleimani an Eides statt, dass ich die vorliegende Dissertation selbständig und ohne fremde Hilfe angefertigt habe. Ich habe keine anderen als die angegebenen Quellen und Hilfsmittel benutzt und die den benutzten Werken wörtlich oder inhaltlich entnommenen Stellen als solche kenntlich gemacht.

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