On Robust Optimization

A Unified Approach to Robustness Using a Nonlinear Scalarizing Functional and Relations to Set Optimization

Dissertation

zur Erlangung des Doktorgrades der Naturwissenschaften (Dr. rer. nat.)

der

Naturwissenschaftlichen Fakultät II

der Martin-Luther-Universität Halle-Wittenberg,

vorgelegt

von Frau Elisabeth Anna Sophia Köbis

geb. am 31.07.1986 in Merseburg

Gutachter:
Frau Prof. Dr. Christiane Tammer (Martin-Luther-Universität Halle-Wittenberg)
Herr Prof. Dr. Akhtar Khan (Rochester Institute of Technology)

Tag der Verteidigung: 19.06.2014

Acknowledgements

I wish to thank my advisor Prof. Dr. Christiane Tammer for her guidance and all the support she has given. I am truly grateful for our inspiring discussions which led to new ideas and challenging projects that contributed to this thesis. In addition, I would like to thank Akhtar Khan, Basca Jadamba and Kathrin Klamroth for welcoming me so warmly to their institutions and providing support. Furthermore, I wish to express my sincere thanks to Christiane Tammer, Kathrin Klamroth, Anita Schöbel, Jonas Ide and Daishi Kuroiwa, for our inspiring collaborations have significantly influenced this thesis. I enjoyed very much the nice working atmosphere at our institute, especially in our working group, which made it a great pleasure to do research here. I am grateful to the "Stiftung Theoretische Physik/Mathematik" for supporting this work by providing a generous travel grant which enabled me to present our projects at conferences and to collaborate with researchers outside of Halle. It is my great pleasure to offer warm thanks to my parents who have always supported and encouraged me. Finally, I wish to thank my husband Markus for his constant support. I feel most fortunate to have you by my side!

Contents

Chapter 1

Introduction

Uncertain data contaminate most optimization problems in various applications ranging from science and engineering to industry and thus represent an essential component in optimization. From a mathematical point of view, many problems can be modeled as an optimization problem and be solved, but in real life, having exact data is very rare and seems almost impossible. Due to a lack of complete information, uncertain data can highly affect solutions and thus influence the decision making process. Hence, it is crucial to address this important issue in optimization theory.

The goal of this work is to provide and study concepts for treating uncertain data in optimization problems and hence to facilitate a decision maker's choice when aiming for a solution that performs well in some sense.

Reasons for uncertain data in optimization problems are measurement and numerical errors, incomplete information and various future scenarios that are not known prior to solving a problem. Goerigk [37] distinguishes between *microscopic* and *macroscopic* errors. Microscopic errors comprise the following numerical errors that result from

- limited precision of computations on a computer system;
- approximate solutions obtained when numerical models simplify an equation;
- rounding.

Macroscopic errors consist of a broader variety of errors such as

- forecast errors: As a prominent example, consider weather forecasting. When not predicted accurately, forecast errors can result in expensive implications, like redirecting planes, evacuation or flooded areas that could have been avoided if the weather had been forecasted precisely.
- changing environments: If a solution has been computed or is in the process of being computed and some of the data change – this is typical in a long-term context – the computed solution may be no longer feasible or optimal. One example is computing train schedules: If one train is delayed (for instance due to some weather condition),

then this affects other trains as well, and the timetable would not be feasible or optimal anymore.

Potential applications of uncertain optimization include supply and inventory management, since demand and tools needed for the production process can easily be exposed to uncertain changes. Further examples for uncertain data in optimization problems can be found in the field of market analysis, share prices, transportation science, timetabling and location theory.

Two ways of dealing with uncertainties in optimization problems are described in the literature: Firstly, *robust optimization* assumes the uncertain parameter to belong to a given uncertainty set. This approach is very practical in many applications, especially since one is not troubled to deal with probabilities. One simply puts the values of the uncertain parameter in consideration that seem likely enough to be attained. Robust optimization has first been studied by Soyster [92] for linear programming problems and was later intensely investigated by Ben-Tal, El Ghaoui and Nemirovski, see [8] for an extensive collection of results.

Secondly, *stochastic programming* is another conspicuous concept that deals with uncertain data in optimization problems. We refer to Birge and Louveaux [15] for an introduction to this field of science. Contrary to robust optimization, stochastic programming assumes some knowledge about the probability distribution of the uncertain parameter. Usually, the problem consists of optimizing the expected value of a cost function subject to some constraints that have to remain feasible for the solution within a certain probability. Of course, one particular challenge using this approach is to find such a probability distribution.

The above mentioned approaches to modeling uncertain data in optimization problems have thus far been considered fundamentally different. One of our goals in this work is, however, to present both concepts in a unifying framework, allowing to establish connections between them.

In addition to discussing robust and stochastic approaches to scalar optimization, i.e., where only one objective function is considered, we will present concepts for obtaining robust solutions of uncertain multi-objective optimization problems.

Optimizing conflicting goals at the same time has been of great interest in the optimization community since the fundamental work by Pareto [82] and Edgeworth [23] and resulted in the field of *multicriteria optimization*. The first robust concepts for uncertain *multicriteria* optimization problems was introduced by Deb and Gupta [21]. Using an idea by Branke [16], the authors define robustness as some sensitivity against disturbances in the decision space. They call a solution to a problem robust if small perturbations in the decision space result in only small disturbances in the objective space. Additional research on robust multicriteria optimization was done in [4, 41, 28]. Kuroiwa and Lee [68] presented the first scenario-based approach by directly transferring the main idea of robust scalar optimization to multicriteria optimization. This concept was recently generalized by Ehrgott et al. [25] who implicitly used a set-order relation to define robust solutions for uncertain multicriteria optimization problems. One of the objectives of this thesis is to reveal close relations between robust multicriteria optimization problems, as defined in [25], and set optimization. Furthermore, using different set order relations, new concepts for deriving robustness concepts for uncertain vector-valued optimization will be introduced in Chapter 5.

1.1 Uncertain Scalar Optimization Problems

A deterministic optimization problem is given by

$$(Q) \qquad \begin{aligned} \min\ & f(x) \\ \text{s.t. } & F_i(x) \leq 0,\ i = 1, \dots, m \\ & x \in \mathbb{R}^n, \end{aligned} \tag{1.1}$$

with objective function $f : \mathbb{R}^n \to \mathbb{R}$ and m constraints $F_i : \mathbb{R}^n \to \mathbb{R},\ i = 1, \dots, m$. The goal is to obtain a solution x^0 that minimizes the objective function subject to the given constraints. Throughout this work the notions optimization and minimization are used equivalently.

We will now formulate an optimization problem with uncertainties. We denote the uncertainty set by $\mathcal{U} \subset \mathbb{R}^N$, which is the set of all uncertain parameters. Now let $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R},\ F_i : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R},\ i = 1, \dots, m$. Then an *uncertain* optimization problem is defined as a parametrized optimization problem

$$(Q(\xi), \xi \in \mathcal{U}), \tag{1.2}$$

where for a given $\xi \in \mathcal{U}$ the optimization problem $(Q(\xi))$ is given by

$$(Q(\xi)) \qquad \begin{aligned} \min\ & f(x, \xi) \\ \text{s.t. } & F_i(x, \xi) \leq 0,\ i = 1, \dots, m, \\ & x \in \mathbb{R}^n. \end{aligned} \tag{1.3}$$

When solving for a solution of the uncertain minimization problem (1.2), it is not known which value $\xi \in \mathcal{U}$ is going to be realized. Now the straightforward question that arises is: How can we deal with such a family of parametrized optimization problems? Clearly, since we have not specified the structure of the uncertainty set $\mathcal{U}$ yet, there may be infinitely many optimization problems. The goal of robust optimization as well as stochastic programming is to convert the family of parametrized optimization problems (1.2) into a single problem which is then solved in order to obtain a solution that is optimal in some sense.

We call $\hat{\xi} \in \mathcal{U}$ the *nominal value*, i.e., the value of ξ that we believe is true today. This may be the value of ξ that we consider the most likely to be attained. The corresponding *nominal problem* is denoted by $(Q(\hat{\xi}))$. The nominal value will be of importance in the definition of the reliably robust optimization problem in Section 3.1.3.

1.2 Approaches to Uncertain Optimization in the Literature

In this section, we give an overview of stochastic and robust optimization. As has been outlined before, these two concepts have been treated in a conceptually different manner in the literature. In Chapter 3 we will present a unifying methodology for specifying optimality in this uncertain problem structure.

1.2.1 Stochastic Programming

When using stochastic minimization concepts for dealing with uncertain data, one needs to assume some knowledge about the probability distribution of the uncertain parameter. The most common approach is to optimize the expected value of the objective function (or some cost function) subject to constraints that are required to be satisfied within a certain probability. If the set of feasible solutions is fixed, the problem consists of minimizing the expected value of a function $f : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$,

$$\min_{x \in \mathcal{X}} \mathbb{E}[f(x,\xi)], \tag{1.4}$$

where $\mathbb{R}^n \supseteq \mathcal{X} = \{x \in \mathbb{R}^n | \ \forall\ \xi \in \mathcal{U} : \ F_i(x,\xi) \leq 0, \ i = 1, \dots, m\}$ is the set of feasible solutions. Here ξ belongs to the probability space $(\mathcal{U}, \mathfrak{F}, P)$ with a given probability measure P. If $\mathcal{U}$ is a finite set, then the expected value of the cost function reads

$$\mathbb{E}[f(x,\xi)] = \sum_{k=1}^{q} p_k f(x, \xi_k),$$

where for each $\xi_k \in \mathcal{U}$ the probability $p_k \geq 0, \ k = 1, \dots, q, \ \sum_{k=1}^{q} p_k = 1$ is known.

Numerous extensions of (1.4) are possible and have been proposed in the literature: For instance, the set of feasible solutions $\mathcal{X}$ may be given as a set of constraints of expected value functions. Another extension of (1.4) is two stage stochastic programming, see Beale [5], Dantzig [20] and Tintner [97] for early references. Such an approach takes into account that some knowledge about the uncertainty may be revealed after a decision has been made on the variable at stage one. Thus, at a second stage, when the realization of some of the uncertainty is known, the decision maker uses this knowledge to take a recourse action on the remaining variables. If we again assume that the uncertainty set $\mathcal{U}$ is finite, each scenario $\xi_k \in \mathcal{U}$ is associated to a probability $p_k \geq 0, \ k = 1, \dots, q,$ $\sum_{k=1}^{q} p_k = 1$. In this situation, a *two-stage stochastic counterpart* can be formulated as

$$\min_{x \in \mathcal{X}} \rho_{SP}(x) \tag{1.5}$$

where $\rho_{SP}(x) := \mathbb{E}[Q(x,\xi)] = \sum_{k=1}^{q} p_k Q(x, \xi_k)$. Here, $\mathcal{X}$ denotes the feasible set of the first-stage problem which could, for example, be defined based on the nominal scenario as $\mathcal{X} = \{x \in \mathbb{R}^n | F_i(x, \hat{\xi}) \leq 0, \, i = 1, \dots, m\}$, or as the set of solutions which satisfy the constraints for every possible realization of the uncertain parameter, $\mathcal{X} = \{x \in \mathbb{R}^n | \forall\ \xi \in \mathcal{U} : \ F_i(x,\xi) \leq 0, \, i = 1, \dots, m\}$. The objective is to minimize the expectation of the overall cost $Q(x,\xi)$ that involves, for given $x \in \mathcal{X}$ and known $\xi \in \mathcal{U}$, an optimal recourse action u, i.e., an optimal solution of the second-stage problem

$$\begin{aligned} Q(x,\xi) = &\min f(x,u,\xi) \\ &\text{s.t. } u \in \mathcal{G}(x,\xi). \end{aligned} \tag{1.6}$$

The second-stage objective function $f(x,u,\xi)$ and the feasible set $\mathcal{G}(x,\xi)$ of the second-stage problem are both parametrized with respect to the stage one solution $x \in \mathcal{X}$ and the scenario $\xi \in \mathcal{U}$. In terms of the uncertain optimization problem (1.2), we assume here that the objective function f in (1.2) depends both on the first-stage and the second-stage variables, i.e., on the nominal cost and the cost of the recourse action. We hence consider the following specification of problem (1.5) with objective function $\rho_{SP}(x,u) := \sum_{k=1}^{q} p_k f(x,u_k,\xi_k)$:

$$(SP) \qquad \begin{aligned} &\min \rho_{SP}(x,u) \\ &\text{s.t. } \forall\ \xi_k \in \mathcal{U}:\ F_i(x,\xi_k) - \delta_k(u_k) \le 0,\ i = 1,\dots,m, \\ &\qquad x \in \mathbb{R}^n, \\ &\qquad u_k \in \mathcal{G}(x,\xi_k),\ k = 1,\dots,q, \end{aligned} \tag{1.7}$$

with compensations $\delta_k : \mathbb{R}^{\overline{n}} \to \mathbb{R}$ that depend on the second-stage decisions $u_k \in \mathbb{R}^{\overline{n}}$, $k = 1,\dots,q$. If we set $\mathcal{G}(x,\xi) = \emptyset$ in the two-stage stochastic programming formulation (1.7), we obtain the static model (1.4) as a specification in which the second-stage variables $u \in \mathbb{R}^{\overline{n}\cdot q}$ are omitted.

Further possible objective functions in stochastic programming include a utility function

$$-\mathbb{E}(u(f(x,\xi))),$$

and a Markowitz model

$$-\mathbb{E}(f(x,\xi)) + \lambda \operatorname{Var} f(x,\xi)$$

with $\lambda > 0$, where f represents payments of an investment and Var denotes the variance.

For an introduction to stochastic programming and other concepts incorporating stochastic effects, we refer to [57, 85, 15, 90, 89].

1.2.2 Robust Optimization

If no knowledge about the probability distribution of an uncertain parameter is present, there is another concept for dealing with uncertain optimization problems called *robust optimization*. Robust optimization is an active and relatively new field of science. The first researcher who studied what is now referred to as robust optimization problems was Soyster [92] in 1973. He considered robust *linear* optimization problems with uncertain constraints by assuming the column vector of the constraint matrix to belong to compact and convex uncertainty sets. In 1976, Falk [29] contributed by investigating linear programs whose parameters in the objective function are assumed to belong to a convex uncertainty set. He proposed to use a maxmin-approach for the objective function and presented optimality criteria that strengthened Soyster's results. In 1982, Singh [91] followed this work line.

Neither of them, however, explicitly used the term *robustness*. The first time the expression robustness was used in optimization was more than 20 years later by Ben-Tal and Nemirovski [10, 11, 12] and El-Ghaoui et al. [70]. They propose to treat uncertain scalar optimization problems by minimizing the worst-case objective function over all possible realizations of the uncertain parameter.

The traditional scope on robust optimization is built on three assumptions [8]:

A1: The decision variables represent "here and now" decisions, meaning that the uncertain data is only revealed *after* an optimal decision has been reached.

A2: The decision maker is only responsible for the resulting decision if the uncertain data belong to the uncertainty set.

A3: Violations of constraints are not tolerated for any $\xi \in \mathcal{U}$, thus the constraints are hard.

Ben-Tal et al. [8] call the resulting problem the *robust counterpart* to an uncertain optimization problem (1.2):

$$(RC) \qquad \min \sup_{\xi \in \mathcal{U}} f(x,\xi) \quad \text{s.t. } \forall\, \xi \in \mathcal{U}: \; F_i(x,\xi) \leq 0, \; i = 1,\dots,m, \; x \in \mathbb{R}^n. \tag{1.8}$$

The crucial assumption of this robust approach consists of supposing that the uncertain parameter belongs to a set that is given prior to solving the robust counterpart. Most studies are concerned with finding tractable representations of the robust counterpart, i.e., simplifying the robust counterpart so it can be solved using algorithms that are already known.

Consider, for instance, the robust counterpart of an uncertain linear program

$$\begin{aligned} \min c^T x \quad \text{s.t. } \forall\, \xi \in \mathcal{U}: \; & F_i(x,\xi) \leq 0, \; i = 1,\dots,m, \\ & x \in \mathbb{R}^n, \end{aligned} \tag{1.9}$$

where $c \in \mathbb{R}^n$ is given. The research interest here lies in investigating which cases of robust counterparts of an uncertain conic problem are computationally tractable, i.e., adapt an equivalent formulation that can be processed computationally. Ben-Tal et al. [8] show that if $\{x \in \mathbb{R}^n | \forall\, \xi \in \mathcal{U}: \; F_i(x,\xi) \leq 0, \; i = 1,\dots,m\}$ is a computationally tractable convex cone and if $\mathcal{U}$ is given as a convex hull of a finite set of scenarios ξ, then the robust counterpart (1.9) is computationally tractable.

Deviations from the above assumptions result in different robust counterparts. For instance, revising Assumption A2 leads to a *globalized* robust counterpart: This approach relies on the more realistic possibility that some of the uncertain data may run outside of the uncertainty set. This approach is based on the assumption that a decision maker prefers not to obtain

- solutions that are too pessimistic or

- an infeasible robust counterpart

when using the traditional robust optimization approach. Thus, Ben-Tal et al. [8] propose to incorporate an uncertainty set for the "typical" data and to control the deterioration of a solution in case the uncertain data deviate from the uncertainty set. The resulting model proposed in [8] now reads as follows:

$$(gRC) \qquad \begin{array}{ll} \min t & \text{s.t. } f(x,\xi) \le t + \alpha_0 \operatorname{dist}(\xi,\mathcal{U}), \\ & \forall\, \xi \in \mathcal{U}: \; F_i(x,\xi) \le \alpha_i \operatorname{dist}(\xi,\mathcal{U}),\; i = 1,\dots,m, \\ & x \in \mathbb{R}^n, \end{array} \tag{1.10}$$

where $\operatorname{dist}(\xi,\mathcal{U}) := \inf_{\xi' \in \mathcal{U}} ||\xi - \xi'||$ and $\alpha_i \in \mathbb{R}$, $i = 1,\dots,m$, given. Note that for $\xi \in \mathcal{U}$ we have $\operatorname{dist}(\xi,\mathcal{U}) = 0$, resulting in the traditional hard constraints. (gRC) is referred to as *globalized robust counterpart* [8].

Revising Assumption A1, on the other hand, results in the field of *adjustable robust optimization*, introduced in Ben-Tal et al. [9]. This approach proposes that some of the decision variables are "here and now" variables, while part of the decision variables may be adjusted at a later stage. Applications of adjustable robust optimization include uncertain network flow and design problems [1] and circuit design [73]. For an overview to robust multi-stage decision making, we refer to [8, Chapters 13,15].

Combining the globalized robust counterpart with adjustable robust optimization leads to the so called *comprehensive robust counterpart*, as proposed by Ben-Tal et al. [7].

Robust Integer Programming has been intensely studied by Kouvelis and Yu [64]. The authors in [64] also give numerous examples of applications for robust optimization.

Of course, finding an uncertainty set remains a difficult task when modeling such a problem and ensuring that the solution set be nonempty. In terms of obtaining the uncertainty set, Brown [17] derives procedures to construct the uncertainty set based on a decision maker's attitude toward risk.

More recently, Beck and Ben-Tal [6] studied duality results between the robust counterpart of an uncertain scalar minimization problem and the corresponding optimistic counterpart, which consists of the problem minimizing the best-case objective function.

Apart from the *robust counterpart* approach (compare (1.8)), there exist other definitions of robustness for uncertain scalar optimization problems. These will be discussed in detail and presented in a unifying framework in Chapter 3. In the following chapter we will introduce a nonlinear scalarizing functional that will be used to characterize the objective functions in each discussed robustness concept. Chapter 4 is concerned with exposing relationships that exist between certain kinds of robust problems and an particularly chosen unconstrained vector optimization problem. In Chapter 5, we present new approaches to uncertain vector optimization using set order relations. We derive scalarization and vectorization results to obtain solution procedures for computing solutions of uncertain multi-objective problems. Finally, Chapter 6 is devoted to deriving optimality conditions for one of the presented robustness concepts based on abstract subdifferentials by means of a nonlinear scalarizing functional.

Chapter 2

Vector Optimization and Scalarizing Functionals

One main part of this thesis is devoted to applying a scalarization technique to multicriteria optimization, which allows for robustness concepts as well as stochastic programming to be presented within a unifying framework. Although robustness approaches and stochastic programming have, for the most part, been considered fundamentally different, we will demonstrate how these concepts may be obtained by a variation of parameters involved in a prominent scalarization method. In Chapter 3, we will show how robust and stochastic scalar optimization problems can be characterized using a nonlinear scalarizing functional. This functional will be discussed in this chapter and important properties which will be beneficial for our later analysis will be mentioned here.

Throughout this chapter, let Y be a linear topological space, $k \in Y \setminus \{0\}$ and let $\mathcal{F}, B$ be proper subsets of Y. We suppose that

$$B + [0, +\infty) \cdot k \subset B. \tag{2.1}$$

Under these assumptions, we are now able to introduce the nonlinear scalarizing functional $z^{B,k} : Y \to \mathbb{R} \cup \{+\infty\} \cup \{-\infty\} =: \overline{\mathbb{R}}$

$$z^{B,k}(y) := \inf\{t \in \mathbb{R} | y \in tk - B\}. \tag{2.2}$$

The nonlinear scalarizing approach can now be formulated in the following way:

$$(P_{k,B,\mathcal{F}}) \qquad \inf_{y \in \mathcal{F}} z^{B,k}(y). \tag{2.3}$$

Note that the functional $z^{B,k}$ operates in the objective space Y of some (vector-valued) function $f : X \to Y$, where X is a linear space. When searching for minimal solutions $x \in \mathcal{X} \subseteq X$ of f, the functional $z^{B,k}$ can be used to scalarize f. Since the functional's well-studied properties, mainly monotonicity properties, allow for connections to vector-valued optimization problems, $z^{B,k}$ may be used to gain minimal solutions of f.

Many well-known scalarization concepts are indeed special cases of this nonlinear scalarization method which was first introduced by Gerstewitz (Tammer) [33], see also

Gerth (Tammer), Weidner [34], Pascoletti, Serafini [83], Göpfert, Tammer, Zălinescu [40] and Göpfert, Riahi, Tammer, Zălinescu [39]. Specifically, this scalarization concept includes, for example, weighted-sums, Tschebyscheff and ϵ-constraint scalarization. In order to show that this scalarization method also comprises a large number of different models from robust optimization and stochastic programming as specifications, we present some important notations and preliminaries in this chapter that will be useful later.

The above problem $(P_{k,B,\mathcal{F}})$ has been intensely studied in various works, see, for instance, [99]. Initially, the scalarizing functional $z^{B,k}$ was used in [34] to prove separation theorems for not necessarily convex sets.

Monotonicity and continuity properties of $z^{B,k}$ were studied by Gerth (Tammer) and Weidner in [34], and later in [99, 39]. Further important properties of the functional $z^{B,k}$, for example the translation property and sublinearity, were shown in [39]. Applications of $z^{B,k}$ include coherent risk measures in financial mathematics (see, e.g., Heyde [44]).

In the following chapter it is shown that one may obtain a unifying concept for a variety of robustness concepts and stochastic programming. Specifically, in Chapter 3 we show that different concepts of uncertain scalar optimization problems can be described by means of the functional $z^{B,k}$ and problem $(P_{k,B,\mathcal{F}})$ by choosing the parameters $B,\ k$ and $\mathcal{F}$ accordingly. Later on, we will observe that the well-studied properties of this scalarizing functional allow for connections to multi-objective optimization. Based on the interpretation of uncertain scalar optimization problems by means of the nonlinear scalarizing functional $z^{B,k}$, we will formulate multiple objective counterparts and observe that their minimal sets comprise optimal solutions of the considered uncertain scalar optimization problems.

Before considering numerous properties of the nonlinear scalarizing functional, we recall some further notations.

Definition 1. *Let Y be a linear topological space, $D \subset Y,\ D \neq \emptyset$. A functional $z : Y \to \overline{\mathbb{R}}$ is D-**monotone**, if for*

$$y_1,\ y_2 \in Y :\ y_1 \in y_2 - D \Rightarrow z(y_1) \leq z(y_2).$$

*Moreover, z is said to be **strictly** D-**monotone**, if for*

$$y_1, y_2 \in Y :\ y_1 \in y_2 - D \setminus \{0\} \Rightarrow z(y_1) < z(y_2).$$

*Its **domain** and **epigraph** are denoted by*

$$\text{dom } z := \{y \in Y | z(y) < +\infty\}, \quad \text{epi } z := \{(y,t) \in Y \times \mathbb{R} | z(y) \leq t\}.$$

*The functional z is said to be **proper** if* dom $z \neq \emptyset$ *and z does not take the value $-\infty$. z is **lower semi-continuous** if* epi z *is closed. $A \subset Y$ is a **convex** set if $\forall\ \lambda \in [0,1],\ \forall\ a_1, a_2 \in A :\ \lambda a_1 + (1-\lambda)a_2 \in A$. z is **convex** on the convex set A if $\forall\ \lambda \in [0,1],\ \forall\ a_1, a_2 \in A :\ z(\lambda a_1 + (1-\lambda)a_2) \leq \lambda z(a_1) + (1-\lambda)z(a_2)$. z is **quasiconvex** if*

$$\forall\ y_1, \ldots, y_p \in Y,\ \forall\ \lambda_i \in [0,1],\ i = 1, \ldots, p,\ \sum_{i=1}^{p} \lambda_i = 1 :\ z(\sum_{i=1}^{p} \lambda_i y_i) \leq \max\{z(y_1), \ldots, z(y_p)\}.$$

z is called ***subadditive*** *if* $\forall\ y_1, y_2 \in Y:\ \ z(y_1 + y_2) \leq z(y_1) + z(y_2)$. *$z$ is* ***positively homogeneous*** *if* $\forall\ y \in Y,\ \forall\ \lambda \in \mathbb{R},\ \lambda \geq 0:\ \ z(\lambda y) = \lambda z(y)$. *If z is subadditive and positively homogeneous, then z is* ***sublinear***. *A set $C \subset Y$ is called a* ***cone*** *if* $\forall\ \lambda \in \mathbb{R},\ \lambda \geq 0,\ \forall\ y \in C:\ \ \lambda y \in C$. *The* ***dual cone*** *to C is denoted by* $C^* := \{y^* \in Y^* |\ \forall\ y \in C:\ \ y^*(y) \geq 0\}$ *and the* ***quasi-interior*** *of C^* is defined by* $C^{\#} := \{y^* \in C^* |\ \forall\ y \in C \setminus \{0\}:\ \ y^*(y) > 0\}$. *A cone C is* ***pointed*** *if* $C \cap (-C) = \{0\}$. *A cone C is* ***convex*** *if $y_1 \in C$ and $y_2 \in C$ implies that $y_1 + y_2 \in C$. Finally, the cone C is* ***proper*** *if* $C \neq \{0\}$ *and* $C \neq Y$.

Now we recall the definition of minimal solutions that is used in multi-objective optimization and then present important properties of the nonlinear scalarizing functional used in this thesis.

Definition 2. *Let Y be a linear topological space, $\mathcal{F} \subset Y$, $\mathcal{F} \neq \emptyset$ and $C \subset Y$ a proper pointed closed convex cone. We call an element $y \in \mathcal{F}$ C-**minimal** in $\mathcal{F}$, if*

$$\mathcal{F} \cap (y - (C \setminus \{0\})) = \emptyset. \tag{2.4}$$

Moreover, if additionally $\operatorname{int} \mathrm{C} \neq \emptyset$, *$y \in \mathcal{F}$ is **weakly** C-**minimal** in $\mathcal{F}$, if*

$$\mathcal{F} \cap (y - \operatorname{int} C) = \emptyset. \tag{2.5}$$

*Furthermore, we call an element $y \in \mathcal{F}$ **strictly** C-**minimal** in $\mathcal{F}$, if*

$$(\mathcal{F} \setminus \{y\}) \cap (y - C) = \emptyset. \tag{2.6}$$

We denote the set of all C-minimal elements in $\mathcal{F}$ by $\mathrm{Min}(\mathcal{F}, C \setminus \{0\})$, *the set of all weakly C-minimal elements in $\mathcal{F}$ is denoted by* $\mathrm{Min}(\mathcal{F}, \operatorname{int} C)$, *and the set of all strictly C-minimal elements is defined as* $\mathrm{Min}(\mathcal{F}, C)$.

In Chapter 5 we will study maximal points of sets, which are defined as $\mathrm{Max}(\mathcal{F}, Q) := \mathrm{Min}(\mathcal{F}, -Q)$ for $Q = C \setminus \{0\},\ Q = \operatorname{int} C,\ Q = C$, respectively.

Notice that (2.4) is equivalent to

$$\nexists\ \overline{y} \in \mathcal{F}:\ \ \overline{y} \in y - C \setminus \{0\}.$$

Furthermore,

$$\begin{aligned} (2.5) &\iff \nexists \overline{y}\ \in \mathcal{F}:\ \overline{y} \in y - \operatorname{int} C, \\ (2.6) &\iff \nexists \overline{y}\ \in \mathcal{F} \setminus \{y\}:\ \overline{y} \in y - C. \end{aligned}$$

If it is clear which cone C is used, we will replace ($\cdot$, weak, strict) C-minimality by ($\cdot$, weak, strict) minimality.

Note that if we set $Y = \mathbb{R}^k$ in the above definition and if the ordering cone C is given by the nonnegative orthant $C = \mathbb{R}^k_{\geqq}$ (see the definition below), we obtain the so called concept of Pareto optimality. In that case, we call elements in the decision space ($\cdot$, weakly, strictly) Pareto optimal. It is worth mentioning that this notation is historically not correct, since both Pareto [82] as well as Edgeworth [23] introduced this approach, compare the brief historic remark in [27]. Nevertheless, we will follow the notation of Pareto optimality, as it is widely accepted in the literature. We refer to the books of Ehrgott [24] and Jahn [52, 53] for a detailed introduction to multiple objective optimization.

For the special case of $Y = \mathbb{R}^k$, we define for $y_1, y_2 \in \mathbb{R}^k$

$$\begin{aligned} y_1 \leqq y_2 &:\Leftrightarrow y_2^i \in [y_1^i, +\infty) \ \forall\, i = 1, \ldots, k, \\ y_1 \leq y_2 &:\Leftrightarrow y_1 \leqq y_2 \text{ and } y_1 \neq y_2, \\ y_1 < y_2 &:\Leftrightarrow y_2^i \in (y_1^i, +\infty) \ \forall\, i = 1, \ldots, k. \end{aligned}$$

Additionally, we define the sets $\mathbb{R}^k_{\geqq}$, $\mathbb{R}^k_{\geq}$, $\mathbb{R}^k_{>}$ as follows:

$$\mathbb{R}^k_{[\geqq/\geq/>]} := \{x \in \mathbb{R}^k : \ x[\geqq / \geq / >]0\}.$$

Furthermore, for the special case $\mathcal{F} \subseteq \mathbb{R}^k$, we call the set of minimal solutions $\mathrm{Min}(\mathcal{F}, \mathbb{R}^k_{\geq})$ **externally stable** (see [24, Def. 2.20.]) if for all $y \in Y \setminus (\mathrm{Min}(\mathcal{F}, \mathbb{R}^k_{\geq}))$ there exists $y^0 \in \mathrm{Min}(\mathcal{F}, \mathbb{R}^k_{\geq})$ with $y^0 \leq y$.

Some of the above properties for the linear topological space Y are now used to describe connections of monotone scalarizing functionals to multi-objective optimization.

Theorem 1 ([51, Theorem 2.2],[34, Theorem 3.3])**.** *Let Y be a linear topological space, $C \subset Y$ a proper pointed closed convex cone, and $\mathcal{F}$ a nonempty subset of Y. Then it holds:*

(i) If there exists a strictly C-monotone functional $z : Y \to \mathbb{R}$, where $\forall\, y \in \mathcal{F} : \ z(y^0) \leq z(y)$ holds, then $y^0 \in \mathrm{Min}(\mathcal{F}, C \setminus \{0\})$.

(ii) If there exists a C-monotone functional $z : Y \to \mathbb{R}$, where $\forall\, y \in \mathcal{F} \setminus \{y^0\} : \ z(y^0) < z(y)$, then $y^0 \in \mathrm{Min}(\mathcal{F}, C \setminus \{0\})$.

Additionally, if $\operatorname{int} C \neq \emptyset$ and if there exists a strictly $(\operatorname{int} C)$-monotone functional $z : Y \to \mathbb{R}$ where $\forall\, y \in \mathcal{F} : \ z(y^0) \leq z(y)$, then $y^0 \in \mathrm{Min}(\mathcal{F}, \operatorname{int} C)$.

Part (i) in Theorem 1 can be found in [34, Theorem 3.3]. A proof of the theorem is presented in [51, Theorem 2.2].

Theorem 2 below shows that the nonlinear scalarizing functional $z = z^{B,k}$ introduced in (2.2) satisfies, under quite general assumptions, the properties given in Theorem 1 and thus immediately connects to minimal solutions in multiple objective optimization.

Theorem 2 ([34, 39])**.** *Let Y be a linear topological space, $B \subset Y$ a closed proper set and $D \subset Y$. Furthermore, let $k \in Y \setminus \{0\}$ such that* (2.1) *is satisfied. Then the following properties hold for $z = z^{B,k}$:*

(a) z is lower semi-continuous.

(b) *z is convex* $\Longleftrightarrow$ *B is convex,*
$[\forall\, y \in Y,\ \forall\, r > 0 :\ z(ry) = rz(y)] \Longleftrightarrow$ *B is a cone.*

(c) *z is proper* $\Longleftrightarrow$ *B does not contain lines parallel to k, i.e.,* $\forall\, y \in Y\ \exists\, r \in \mathbb{R} :\ y + rk \notin B$.

(d) *z is D-monotone* $\Longleftrightarrow B + D \subset B$.

(e) *z is subadditive* $\Longleftrightarrow B + B \subset B$.

(f) $\forall\, y \in Y,\ \forall\, r \in \mathbb{R} :\ z(y) \le r \Longleftrightarrow y \in rk - B$.

(g) $\forall\, y \in Y,\ \forall\, r \in \mathbb{R} :\ z(y + rk) = z(y) + r$ *(translation property).*

(h) *z is finite-valued* $\Longleftrightarrow$ *B does not contain lines parallel to k and* $\mathbb{R}k - B = Y$.

Let furthermore $B + (0, +\infty) \cdot k \subset \operatorname{int} B$. *Then*

(i) *z is continuous.*

(j) $\forall\, y \in Y,\ \forall\, r \in \mathbb{R} :\ z(y) = r \Longleftrightarrow y \in rk - \operatorname{bd} B$,
$\forall\, y \in Y,\ \forall\, r \in \mathbb{R} :\ z(y) < r \Longleftrightarrow y \in rk - \operatorname{int} B$.

(k) *If z is proper, then z is D-monotone* $\Longleftrightarrow B + D \subset B \Longleftrightarrow \operatorname{bd} B + D \subset B$.

(l) *If z is finite-valued, then z is strictly D-monotone* $\Longleftrightarrow B + (D \setminus \{0\}) \subset \operatorname{int} B \Longleftrightarrow \operatorname{bd} B + (D \setminus \{0\}) \subset \operatorname{int} B$.

(m) *Suppose z is proper. Then z is subadditive* $\Longleftrightarrow B + B \subset B \Longleftrightarrow \operatorname{bd} B + \operatorname{bd} B \subset B$.

A proof of the above theorem can be found in [39, Theorem 2.3.1].

The following corollary summarizes the above results for the special case of C being a proper closed convex cone and k belonging to $\operatorname{int} C$.

Corollary 1 ([39, Corollary 2.3.5.]). *Let C be a proper closed convex cone and $k \in \operatorname{int} C$. Then $z = z^{C,k}$, defined by (2.2), is a finite-valued continuous sublinear and strictly $(\operatorname{int} C)$-monotone functional such that*

$$\forall\, y \in Y,\ \forall\, r \in \mathbb{R} :\ z(y) \le r \Longleftrightarrow y \in rk - C,$$

$$\forall\, y \in Y,\ \forall\, r \in \mathbb{R} :\ z(y) < r \Longleftrightarrow y \in rk - \operatorname{int} C.$$

In the following chapter, we will formulate various robustness concepts by means of the functional $z^{B,k}$ (compare (2.2)) and study the functionals properties based on Theorem 2.

Chapter 3

A Unified Approach to Robust Optimization and Stochastic Programming

As indicated in Chapter 1, uncertainties in optimization lead to a family of parametrized optimization problems. The goal of robust optimization is to transfer this family of optimization problems to one optimization problem that produces *robust* solutions, i.e., solutions that perform well in several scenarios, depending on the considered definition of robustness. Contrary to *stochastic optimization*, a robust approach does not depend on a probabilistic structure of the uncertain parameter but relies on an uncertainty set. Thus, it is assumed that the uncertain parameter ξ belongs to a given uncertainty set $\mathcal{U}$. Some works are devoted to finding an uncertainty set, e.g., in [17], the author investigates how one may compute an uncertainty set that represents the decision maker's attitude toward risk (see also [13]). Although the issue of finding such a set $\mathcal{U}$ is itself a difficult task, we will presume that the uncertainty set is given.

In this chapter, while focusing on scalar robust problems, we present a concept that allows for a unifying approach to various definitions of robustness using a nonlinear scalarizing functional as discussed in Chapter 2. It will further be shown that a stochastic programming approach also fits into the unifying concept. The results of the following sections provide new insights into the nature of scalar robust optimization problems, ranging from continuity properties that are revealed to connections to multi-objective optimization problems. Furthermore, we will illustrate that new robustness concepts may be achieved by using a nonlinear scalarizing functional.

Throughout this chapter, we suppose that the minimum of each described optimization problem exists.

3.1 Discrete Uncertainty Set

Throughout Section 3.1, we consider the case where the uncertainty set consists of finitely many elements, i.e., $\mathcal{U} = \{\xi_1, \dots, \xi_q\} \subset \mathbb{R}^N$. This means that we consider q possible objective functions $f(x, \xi)$. In the following, we will present several concepts for scalar robust optimization problems that are known from the literature. It will be shown how these concepts can be formulated using the nonlinear scalarizing functional discussed in Chapter 2 by varying the parameters in the objective function and in the set of feasible solutions. This scalarizing functional possesses interesting properties which will be revealed to hold as well for the objective functions describing the robustness concepts. The results in this section are applicable to a wide range of problems, for instance if $\mathcal{U}$ is given by the convex hull of finitely many scenarios ξ, see Section 3.2.3. The following results, with the exception of the weighted robustness concept, are based on Klamroth, Köbis, Schöbel and Tammer [59].

3.1.1 Weighted Robustness

The first considered concept for obtaining robust solutions of an uncertain optimization problem will be referred to as *weighted robustness.*

Let weights $w_k > 0,\ k = 1, \dots, q$, be given and consider for the functional $\rho_{wRC}(x) := \max_{k=1,\dots,q} w_k f(x, \xi_k)$ the **weighted robust counterpart**

$$(wRC) \qquad \begin{aligned} &\min \rho_{wRC}(x) \\ &\text{s.t. } \forall\, \xi \in \mathcal{U} : \ F_i(x, \xi) \leq 0, \ i = 1, \dots, m, \\ &\quad x \in \mathbb{R}^n. \end{aligned} \tag{3.1}$$

A feasible solution of problem (wRC) will be called **weighted robust**. The set of all feasible weighted robust solutions is denoted by

$$\mathfrak{A} := \{x \in \mathbb{R}^n |\ \forall\, \xi \in \mathcal{U} : \ F_i(x, \xi) \leq 0, \ i = 1, \dots, m\}. \tag{3.2}$$

Such a weighted robust approach to an uncertain optimization problem was proposed by Sayin and Kouvelis [87, 63] to compute solutions of a vector-valued optimization problem.

For $w_k = 1, \ k = 1, \dots, q$, this concept coincides with an approach called *strict robustness.* It was first studied by Soyster [92] who considered linear optimization problems, i.e., a linear objective function that is minimized over constraints which are described by a set of linear inequalities. The term *robustness*, however, was introduced by Ben-Tal, El Ghaoui, and Nemirovski in [10] who studied robust optimization in numerous publications, see e.g. [35] for an early contribution and [8] for an extensive collection of results. A classification of strict robustness within a unifying framework by means of the nonlinear scalarizing functional $z^{B,k}$ (see (2.2)) is presented in [59].

To obtain a solution that is *weighted robust* for the uncertain optimization problem (1.2), the aim is to minimize the weighted worst possible objective function value in order to yield a solution that performs well even in the weighted worst case scenario. In terms of the uncertain constraints given in the problem (1.3), a *weighted robust* solution is required to satisfy these constraints in every possible future scenario $\xi \in \mathcal{U}$.

In the following theorem it is shown that the nonlinear scalarizing functional $z^{B,k}$ (see (2.2)) can be used to express the weighted robust optimization problem (wRC) when embedding the problem in $Y = \mathbb{R}^q$ and choosing the involved parameters B, k and $\mathcal{F}$ accordingly.

Theorem 3. *Consider for* $Y = \mathbb{R}^q$

$$W := \begin{pmatrix} w_1 & 0 & \dots & 0 \\ 0 & w_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & w_q \end{pmatrix}, \tag{3.3}$$

$$\mathfrak{A}_1 := \mathfrak{A}, \tag{3.4}$$

$$B_1 := \{y \in Y |\ Wy \geqq 0\}, \tag{3.5}$$

$$k_1 := (w_1^{-1}, \dots, w_q^{-1})^T, \tag{3.6}$$

$$\mathcal{F}_1 := \{(f(x, \xi_1), \dots, f(x, \xi_q))^T |\ x \in \mathfrak{A}_1\}. \tag{3.7}$$

For $k = k_1,\ B = B_1$*, condition* (2.1) *is satisfied and with* $\mathcal{F} = \mathcal{F}_1$*, problem* $(P_{k,B,\mathcal{F}})$ *(see* (2.3)*) is equivalent to problem* (wRC) *(see* (3.1)*) in the following sense:*

$$\begin{aligned} \min\{z^{B_1,k_1}(y) |\ y \in \mathcal{F}_1\} &= z^{B_1,k_1}(y^0) \\ &= \min\{\rho_{wRC}(x) |\ x \in \mathfrak{A}_1\} \\ &= \rho_{wRC}(x^0), \end{aligned}$$

with $y^0 = (f(x^0, \xi_1), \dots, f(x^0, \xi_q))^T$.

Proof. Since $w_k > 0$ for each $k = 1, \dots, q$, we obtain $B_1 + [0, +\infty) \cdot k_1 \subset B_1$, and condition (2.1) is fulfilled. Since $k_1 \in \operatorname{int} B_1$ and B_1 is closed, the infimum in the definition of z^{B_1,k_1} is finite and attained and the infimum can be replaced by a minimum:

$$\begin{aligned} \min_{y \in \mathcal{F}_1} z^{B_1,k_1}(y) &= \min_{y \in \mathcal{F}_1} \min\{t \in \mathbb{R} |\ y \in tk_1 - B_1\} \\ &= \min_{y \in \mathcal{F}_1} \min\{t \in \mathbb{R} |\ y - tk_1 \in -B_1\} \\ &= \min_{x \in \mathfrak{A}_1} \min\{t \in \mathbb{R} |\ (f(x, \xi_1), \dots, f(x, \xi_q))^T - t \cdot (w_1^{-1}, \dots, w_q^{-1})^T \in -B_1\} \\ &= \min_{x \in \mathfrak{A}_1} \min\{t \in \mathbb{R} |\ (w_1 f(x, \xi_1), \dots, w_q f(x, \xi_q))^T \leqq t \cdot (1, \dots, 1)^T\} \\ &= \min\{\max_{k=1,\dots,q} w_k f(x, \xi_k) |\ x \in \mathfrak{A}_1\} \\ &= \min\{\rho_{RC}(x) |\ x \in \mathfrak{A}_1\}. \end{aligned}$$

$\square$

Note that B_1 above is equal to $\mathbb{R}^q_{\geqq}$. The matrix W has only been introduced in order to simplify the representation of the proof of Theorem 3. The vector $k_1 = (w_1^{-1}, \dots, w_q^{-1})$ depends on the selection of the weights w_i, and thus represents the decision maker's preferences toward the different scenarios. For the special case of *strict robustness*, i.e, the weighted robust problem (wRC) (see (3.1)) with $w_k = 1,\ k = 1, \dots, q$, the selection of k_1 symbolizes that all possible objective functions are regarded in parallel and no objective function is preferred to another one.

Remark 1. *Since B_1 is a proper closed convex cone and $k_1 \in \operatorname{int} B_1$, the functional z^{B_1,k_1} is continuous, finite-valued, $\mathbb{R}^q_{\geqq}$-monotone, strictly $\mathbb{R}^q_{>}$-monotone and sublinear, taking into account Corollary 1.*

Remark 2. *Note that the weighted robustness approach coincides with the weighted Tschebyscheff scalarization with the origin as reference point. It is well known that the Tschebyscheff scalarization is a special case of functional $z^{B,k}$ (compare (2.2)), see [99]. Furthermore, Theorem 3 shows that (wRC) (see (3.1)) can be interpreted as a weighted max-ordering problem as defined in multiple objective optimization, see [24]. This relationship was also observed by Kouvelis and Sayin [63, 87] where it was used to determine the nondominated set of discrete bicriteria optimization problems.*

3.1.2 Deviation Robustness

The following robustness approach to uncertain optimization will be called *deviation robustness*, sometimes it is referred to as *minmax regret robustness*. This approach takes into account the best possible objective values for each future scenario, while minimizing the worst possible objective function value at the same time. The function to be minimized is $\max_{\xi\in\mathcal{U}}(f(x,\xi) - f^0(\xi))$, where $f^0(\xi) \in \mathbb{R}$ is the optimal value of problem $(Q(\xi))$ (see (1.3)) for each parameter $\xi \in \mathcal{U}$. Analogous to the concept of weighted robustness, a deviation robust solution should fulfill the constraints for every future scenario $\xi \in \mathcal{U}$. This robustness approach has proven to be very useful in many applications such as scheduling or location theory, mostly if no uncertainty in the constraints is present. We refer to [64] for a collection of many applications. The **deviation robust counterpart** of (1.2) can now be introduced as

$$(dRC) \qquad \begin{aligned} &\min \rho_{dRC}(x) \\ &\text{s.t. } \forall\, \xi \in \mathcal{U}:\ F_i(x,\xi) \le 0,\ i = 1, \dots, m, \\ &\qquad x \in \mathbb{R}^n, \end{aligned} \tag{3.8}$$

where $\rho_{dRC}(x) := \max_{\xi\in\mathcal{U}}(f(x,\xi) - f^0(\xi))$. Feasible solutions of (dRC) will be called **deviation robust**. We denote by

$$f^0 := (f^0(\xi_1), \dots, f^0(\xi_q))^T \tag{3.9}$$

the vector consisting of the individual minimizers for the respective scenarios which can be interpreted as an *ideal solution vector*. Now we are able to formulate the following theorem, which shows that *deviation robust* solutions may be computed by minimizing the nonlinear scalarizing functional $z^{B,k}$ for specific parameter values B, k and $\mathcal{F}$:

Theorem 4. *Consider for* $Y = \mathbb{R}^q$

$$\mathfrak{A}_2 := \mathfrak{A} \quad (\textit{compare } (3.2)), \tag{3.10}$$

$$B_2 := \mathbb{R}^q_{\geqq} - f^0, \tag{3.11}$$

$$k_2 := 1_q = (1, \dots, 1)^T, \tag{3.12}$$

$$\mathcal{F}_2 := \{(f(x,\xi_1), \dots, f(x,\xi_q))^T |\ x \in \mathfrak{A}_2\}. \tag{3.13}$$

For $k = k_2,\ B = B_2$*, condition* (2.1) *is fulfilled and with* $\mathcal{F} = \mathcal{F}_2$*, problem* $(P_{k,B,\mathcal{F}})$ *(see* (2.3)*) is equivalent to problem* (dRC) *(see* (3.8)*) in the following sense:*

$$\begin{aligned}\min\{z^{B_2,k_2}(y) |\ y \in \mathcal{F}_2\} &= z^{B_2,k_2}(y^0)\\ &= \min\{\rho_{dRC}(x) |\ x \in \mathfrak{A}_2\}\\ &= \rho_{dRC}(x^0),\end{aligned}$$

with $y^0 = (f(x^0,\xi_1), \dots, f(x^0,\xi_q))^T$.

Proof. Since $B_2 + [0,+\infty) \cdot k_2 = (\mathbb{R}^q_{\geqq} - f^0) + [0,+\infty) \cdot 1_q \subset \mathbb{R}^q_{\geqq} - f^0 = B_2$, condition (2.1) is satisfied. Moreover,

$$\begin{aligned}\min_{y\in\mathcal{F}_2} z^{B_2,k_2}(y) &= \min_{y\in\mathcal{F}_2} \min\{t \in \mathbb{R} |\ y \in tk_2 - B_2\}\\ &= \min_{x\in\mathfrak{A}_2} \min\{t \in \mathbb{R} |\ (f(x,\xi_1), \dots, f(x,\xi_q))^T - (f^0(\xi_1), \dots, f^0(\xi_q))^T - t \cdot 1_q \leqq 0_q\}\\ &= \min_{x\in\mathfrak{A}_2} \min\{t \in \mathbb{R} |\ (f(x,\xi_1), \dots, f(x,\xi_q))^T - (f^0(\xi_1), \dots, f^0(\xi_q))^T \leqq t \cdot 1_q\}\\ &= \min\{\max_{\xi\in\mathcal{U}}(f(x,\xi) - f^0(\xi)) |\ x \in \mathfrak{A}_2\}\\ &= \min\{\rho_{dRC}(x) |\ x \in \mathfrak{A}_2\}.\end{aligned}$$

□

Note that the same result would have been achieved if we had chosen $\widetilde{B_2} := \mathbb{R}^q_{\geqq}$ to minimize $z^{\widetilde{B_2},k_2}$ on the set of feasible elements $\widetilde{\mathcal{F}_2} := \{(f(x,\xi_1), \dots, f(x,\xi_q))^T - (f^0(\xi_1), \dots, f^0(\xi_q))^T |\ x \in \mathfrak{A}_2\}$. This means that (dRC) (see (3.8)) is a shifted version of (wRC) (see (3.1)) for $w_k = 1,\ k = 1, \dots, q$. Thus, under the assumption of a finite uncertainty set $\mathcal{U}$ and if the ideal solution vector f^0 is known, one can conclude that (dRC) and (wRC) can be solved in the same complexity range. Notice that it would be entirely possible to formulate a *weighted* deviation robust optimization problem, i.e., a deviation robust problem with objective function $\rho_{dRC}(x) := \max_{k=1,\dots,q}(w_k f(x,\xi_k) - f^0(\xi_k))$.

Some properties of the functional z^{B_2,k_2} that are gained from Theorem 2 are presented in the following remark.

Remark 3. *Using Theorem 2 and the fact that* $B_2 + (0, +\infty) \cdot k_2 \subset \operatorname{int} B_2$, *we can conclude that the functional* z^{B_2,k_2} *is continuous, finite-valued, convex,* $\mathbb{R}^q_{\geqq}$*-monotone and strictly* $\mathbb{R}^q_{>}$*-monotone. Note that for our proposed approach to formulate* (dRC) *(see* (3.8)*), Corollary 1 cannot be applied, since* B_2 *is not a cone in general.*

Remark 4. *Similar to the case of weighted robustness, the concept of deviation robustness can be described by the Tschebyscheff scalarization, however, not with the origin as reference point but with the ideal point* f^0 *defined in* (3.9) *as reference point. This shows once again the close relationship between these two robustness concepts, see also Kouvelis and Sayin [63, 87].*

3.1.3 Reliable Robustness

The following concept will be called *reliable robustness* and describes the possibility of a robust solution to satisfy slightly adapted constraints. Since it sometimes may not seem realistic for a solution to fulfill all the constraints at the same time or at the cost of optimality of the objective function, it is proposed here to replace the original hard constraints $F_i(x, \xi) \leq 0$ for each $\xi \in \mathcal{U},\ i = 1, \ldots, m$, by soft constraints $F_i(x, \xi) \leq \delta_i$ for each $\xi \in \mathcal{U}$, where $\delta_i \in \mathbb{R},\ i = 1, \ldots, m$. Note that Assumption A3 (see Chapter 1) is not required to be satisfied here. The infeasibility tolerances δ_i are assumed to be given by the decision maker. However, the original constraints for the nominal value $\hat{\xi}$ should be fulfilled, i.e., $F_i(x, \hat{\xi}) \leq 0,\ i = 1, \ldots, m$. Then the **reliably robust counterpart** of (1.2) introduced by Ben-Tal and Nemirovski in [12], is proposed in the following way.

$$(rRC) \qquad \begin{aligned} &\min \rho_{rRC}(x) \\ &\text{s.t. } F_i(x, \hat{\xi}) \leq 0,\ i = 1, \ldots, m, \\ &\qquad \forall\, \xi \in \mathcal{U} :\ F_i(x, \xi) \leq \delta_i,\ i = 1, \ldots, m, \\ &\qquad x \in \mathbb{R}^n, \end{aligned} \tag{3.14}$$

with $\rho_{rRC}(x) := \max_{\xi \in \mathcal{U}} f(x, \xi)$. A feasible solution of (rRC) is called **reliably robust**. Note that strict robustness (i.e., the weighted robust problem (3.1) with $w_k = 1,\ k = 1, \ldots, q$) is a special case of reliable robustness, since both problems are equivalent for $\delta_i = 0$ for all $i = 1, \ldots, m$.

The following theorem describes how the reliably robust problem (rRC) can be expressed using the nonlinear scalarizing functional $z^{B,k}$. Since the proof is mostly similar to that of Theorem 3 with the only exception being the set $\mathcal{F}_3$ of feasible solutions and setting $w_k = 1,\ k = 1, \ldots, q$, the proof is omitted.

Theorem 5. *Consider for* $Y = \mathbb{R}^q$

$$\mathfrak{A}_3 := \{x \in \mathbb{R}^n |\ F_i(x, \hat{\xi}) \leq 0,\ \forall\, \xi \in \mathcal{U} :\ F_i(x, \xi) \leq \delta_i,\ i = 1, \ldots, m\}, \tag{3.15}$$

$$B_3 := \mathbb{R}^q_{\geqq}, \tag{3.16}$$

$$k_3 := 1_q, \tag{3.17}$$

$$\mathcal{F}_3 := \{(f(x, \xi_1), \ldots, f(x, \xi_q))^T |\ x \in \mathfrak{A}_3\}. \tag{3.18}$$

For $k = k_3,\ B = B_3$, condition (2.1) is satisfied and with $\mathcal{F} = \mathcal{F}_3$, problem $(P_{k,B,\mathcal{F}})$ (see (2.3)) is equivalent to problem (rRC) (see (3.14)) in the following sense:

$$\begin{aligned}&\min\{z^{B_3,k_3}(y)|\ y \in \mathcal{F}_3\}\\ &\quad = z^{B_3,k_3}(y^0)\\ &\quad = \min\{\rho_{rRC}(x)|\ x \in \mathfrak{A}_3\}\\ &\quad = \rho_{rRC}(x^0),\end{aligned}$$

where $y^0 = (f(x^0, \xi_1), \dots, f(x^0, \xi_q))^T$.

Remark 5. *For the special case of strict robustness (i.e., the weighted robust problem (wRC), (3.1), with $w_k = 1,\ k = 1, \dots, q$), it holds $z^{B_3,k_3} = z^{B_1,k_1}$ and the functional z^{B_3,k_3} is again continuous, finite-valued, $\mathbb{R}^q_{\geqq}$-monotone, strictly $\mathbb{R}^q_{>}$-monotone and sublinear, taking into account Corollary 1, compare Remark 1.*

Remark 6. *The concept of reliable robustness is described by the Tschebyscheff scalarization with the origin as reference point and on the basis of a relaxed feasible set, as a special case of functional $z^{B,k}$ (see (2.2)).*

3.1.4 Light Robustness

By considering a variation of the constraints $F_i(x, \xi) \leq \delta_i$, where $F_i,\ \delta_i,\ i = 1, \dots, m$, are defined as in the definition of the reliably robust optimization problem (rRC) (see (3.14)), one may wish to minimize these tolerances, which describes the key essence of the present robustness concept, called light robustness. This approach was first mentioned in 2008 by Fischetti and Monaci in [30] for linear programs with the Γ-uncertainty set introduced by Bertsimas and Sim [14] and generalized to a broader class of uncertain robust optimization problems by Schöbel [88]. Applications of the concept of light robustness include timetabling [88, 31] and timetable information [38].

We denote by z^0 the optimal value of the nominal problem $(Q(\hat{\xi}))$, and suppose that z^0 be positive, i.e., $z^0 > 0$. One of our aims consists of providing an upper bound for the nominal value $f(x, \hat{\xi})$. Thus, $f(x, \hat{\xi}) \leq (1 + \gamma)z^0$, with a given $\gamma \geq 0$. Then the **lightly robust counterpart** of (1.2) is defined by

$$(lRC) \qquad \begin{aligned}&\min \rho_{lRC}(\delta)\\ &\text{s.t. } F_i(x, \hat{\xi}) \leq 0,\ i = 1, \dots, m,\\ &\qquad f(x, \hat{\xi}) \leq (1 + \gamma)z^0,\\ &\qquad \forall\, \xi \in \mathcal{U}:\ F_i(x, \xi) \leq \delta_i,\ i = 1, \dots, m,\\ &\qquad x \in \mathbb{R}^n,\\ &\qquad \delta_i \in \mathbb{R},\ i = 1, \dots, m,\end{aligned} \qquad (3.19)$$

where $\rho_{lRC}(\delta) := \sum_{i=1}^m w_i\delta_i$, with given weights $w_i \geq 0,\ i = 1, \dots, m,\ \sum_{i=1}^m w_i = 1$. A feasible solution of (lRC) will be called **lightly robust**.

The essential observation in the next theorem is the representation of (lRC) (see (3.19)) by the nonlinear scalarizing functional $z^{B,k}$ (see (2.2)) for a specific choice of the parameters $B, \mathcal{F}$ and k.

Theorem 6. *Consider for $Y = \mathbb{R}^m$*

$$B_4 := \{(\delta_1, \dots, \delta_m)^T | \sum_{i=1}^m w_i\delta_i \geq 0,\ \delta_i \in \mathbb{R},\ i = 1, \dots, m\}, \tag{3.20}$$

$$k_4 := 1_m, \tag{3.21}$$

$$\begin{aligned} \mathcal{F}_4 := \{(\delta_1, \dots, \delta_m)^T | \ \exists\ x \in \mathbb{R}^n :\ & F_i(x, \hat{\xi}) \leq 0,\ f(x, \hat{\xi}) \leq (1 + \gamma)z^0, \\ \forall\ \xi \in \mathcal{U} :\ & F_i(x, \xi) \leq \delta_i,\ \delta_i \in \mathbb{R},\ i = 1, \dots, m\}. \end{aligned} \tag{3.22}$$

For $k = k_4,\ B = B_4$, condition (2.1) *is satisfied and with $\mathcal{F} = \mathcal{F}_4$, problem $(P_{k,B,\mathcal{F}})$ (see* (2.3)*) is equivalent to problem (lRC) (see* (3.19)*) in the following sense:*

$$\begin{aligned} &\min\{z^{B_4,k_4}(y)|\ y \in \mathcal{F}_4\} \\ &\quad = z^{B_4,k_4}(y^0) \\ &\quad = \min\{\rho_{lRC}(\delta)|\ \delta \in \mathcal{F}_4\} \\ &\quad = \rho_{lRC}(\delta^0), \end{aligned}$$

where $y^0 = \delta^0 = (\delta_1^0, \dots, \delta_m^0)^T$.

Proof. In this case, $B_4 + [0, +\infty) \cdot k_4 = \{(\delta_1, \dots, \delta_m)^T \in \mathbb{R}^m |\ \sum_{i=1}^m w_i\delta_i \geq 0\} + [0, +\infty) \cdot 1_m \subset B_4$, and (2.1) is satisfied in $\mathbb{R}^m$. Moreover,

$$\begin{aligned} \min_{y \in \mathcal{F}_4} z^{B_4,k_4}(y) &= \min_{y \in \mathcal{F}_4} \min\{t \in \mathbb{R}|\ y \in tk_4 - B_4\} \\ &= \min_{y \in \mathcal{F}_4} \min\{t \in \mathbb{R}|\ y - tk_4 \in -B_4\} \\ &= \min_{\delta \in \mathcal{F}_4} \min\{t \in \mathbb{R}|\ \sum_{i=1}^m w_i(\delta_i - t) \leq 0\} \\ &= \min_{\delta \in \mathcal{F}_4} \min\{t \in \mathbb{R}|\ \sum_{i=1}^m w_i\delta_i \leq t \cdot \underbrace{\sum_{i=1}^m w_i}_{=1}\} \\ &= \min\{\sum_{i=1}^m w_i\delta_i|\ \delta \in \mathcal{F}_4\} \\ &= \min\{\rho_{lRC}(\delta)|\ \delta \in \mathcal{F}_4\}. \end{aligned}$$

□

Remark 7. *Note that B_4 is a proper closed convex cone with $k_4 \in \operatorname{int} B_4$ and Corollary 1 implies that the functional z^{B_4,k_4} is continuous, finite-valued, $\mathbb{R}^m_{\geqq}$-monotone, strictly $\mathbb{R}^m_{>}$-monotone and sublinear.*

Remark 8. *As the concept of light robustness sums the weighted upper bounds δ_i, $i = 1, \ldots, m$, it can be regarded as a weighted sum approach with constraints in the weighted objective function. Note that the nonlinear scalarizing functional now operates in $\mathbb{R}^m$, the space of dimension of number of constraints.*

3.1.5 Stochastic Programming

Stochastic programming models differ fundamentally from robust optimization models as they assume some knowledge about the probability distribution of the uncertain data. For an introduction to stochastic programming we refer to Birge and Louveaux [15] and Shapiro et al. [89]. Note that since we assume that the uncertainty set $\mathcal{U}$ is finite, each scenario $\xi_k \in \mathcal{U}$ now is associated to a probability $p_k \geq 0$, $k = 1, \ldots, q$, $\sum_{k=1}^q p_k = 1$. In this situation, a **two stage stochastic counterpart** can be formulated as

$$(SP) \qquad \begin{aligned} &\min \rho_{SP}(x,u) \\ &\text{s.t. } \forall\, k \in \{1, \ldots, q\}: \; F_i(x, \xi_k) - \delta_k(u_k) \leq 0,\; i = 1, \ldots, m, \\ &\qquad x \in \mathbb{R}^n, \\ &\qquad u_k \in \mathcal{G}(x, \xi_k),\; k = 1, \ldots, q, \end{aligned} \tag{3.23}$$

with $\rho_{SP}(x,u) := \sum_{k=1}^q p_k f(x, u_k, \xi_k)$ and compensations $\delta_k : \mathbb{R}^{\overline{n}} \to \mathbb{R}$, $k = 1, \ldots, q$, that depend on the second stage decision $u_k \in \mathbb{R}^{\overline{n}}$, $k = 1, \ldots, q$, and $\mathcal{G}(x, \xi_k) \subset \mathbb{R}^{\overline{n}}$.

The following reformulation holds and verifies that the above stochastic programming problem is a special case of minimizing the nonlinear scalarizing functional $z^{B,k}$ (see (2.2)) as well.

Theorem 7. *Let in $Y = \mathbb{R}^q$*

$$\begin{aligned} \mathfrak{A}_5 := \{(x,u) := (x, u_1, \ldots, u_q) \in \mathbb{R}^{n \times \overline{n} \cdot q} |\; \forall\, \xi_k \in \mathcal{U}: \\ F_i(x, \xi_k) - \delta_k(u_k) \leq 0,\; i = 1, \ldots, m,\; u_k \in \mathcal{G}(x, \xi_k),\; k = 1, \ldots, q\}, \end{aligned} \tag{3.24}$$

$$B_5 := \{(y_1, \ldots, y_q)^T |\; \sum_{k=1}^q p_k y_k \geq 0,\; y_k \in \mathbb{R},\; k = 1, \ldots, q\}, \tag{3.25}$$

$$k_5 := 1_q, \tag{3.26}$$

$$\mathcal{F}_5 := \{(f(x, u_1, \xi_1), \ldots, f(x, u_q, \xi_q))^T |\; (x,u) \in \mathfrak{A}_5\}. \tag{3.27}$$

For $k = k_5$, $B = B_5$, condition (2.1) is satisfied and with $\mathcal{F} = \mathcal{F}_5$, problem $(P_{k,B,\mathcal{F}})$ (see (2.3)) is equivalent to problem (SP) (see (3.23)) in the following sense:

$$\begin{aligned} &\min\{z^{B_5,k_5}(y) |\; y \in \mathcal{F}_5\} \\ &= z^{B_5,k_5}(y^0) \\ &= \min\{\rho_{SP}(x,u) |\; (x,u) \in \mathfrak{A}_5\} \\ &= \rho_{SP}(x^0, u^0), \end{aligned}$$

where $y^0 = (f(x^0, u_1^0, \xi_1), \ldots, f(x^0, u_q^0, \xi_q))^T$.

Proof. We have $B_5+[0,+\infty)\cdot k_5 = \{(y_1,\dots,y_q)^T \in \mathbb{R}^q \mid \sum_{k=1}^q p_k y_k \geq 0\} + [0,+\infty)\cdot 1_m \subset B_5$, thus (2.1) is satisfied. Moreover,

$$\begin{aligned}
\min_{y\in\mathcal{F}_5} z^{B_5,k_5}(y) &= \min_{y\in\mathcal{F}_5} \min\{t\in\mathbb{R} \mid y \in tk_5 - B_5\} \\
&= \min_{y\in\mathcal{F}_5} \min\{t\in\mathbb{R} \mid y - tk_5 \in -B_5\} \\
&= \min_{y\in\mathcal{F}_5} \min\{t\in\mathbb{R} \mid \sum_{k=1}^{q} p_k(y_k - t) \leq 0\} \\
&= \min_{y\in\mathcal{F}_5} \min\{t\in\mathbb{R} \mid \sum_{k=1}^{q} p_k y_k \leq t\cdot \underbrace{\sum_{k=1}^{q} p_k}_{=1}\} \\
&= \min\{\sum_{k=1}^{q} p_k y_k \mid y \in \mathcal{F}_5\} \\
&= \min\{\rho_{SP}(x,u) \mid (x,u) \in \mathfrak{A}_5\}.
\end{aligned}$$

□

Remark 9. *B_5 is a proper closed convex cone with $k_5 \in \operatorname{int} B_5$ and Corollary 1 implies that the functional z^{B_5,k_5} is continuous, finite-valued, $\mathbb{R}^q_{\geqq}$-monotone, strictly $\mathbb{R}^q_{>}$-monotone and sublinear.*

Remark 10. *Similar to the case of light robustness, the above formulated two-stage stochastic programming problem can be interpreted as a weighted sums approach, however, in this case with a relaxed feasible set. This relation was also observed by Gast [32] in the multiple objective context. Note that in the special case of the static model (sSP) (see* (3.33) *in Section 3.1.8), the feasible set is in fact identical to the set of strictly robust solutions $\mathfrak{A}$ (and not relaxed), see* (3.2).

3.1.6 New Concepts for Robustness

This section is devoted to showing that the nonlinear scalarizing functional $z^{B,k}$ (see (2.2)) is beneficial to obtain new robustness concepts when dealing with uncertain scalar optimization problems that do not rely on a probabilistic nature. It is well known that the functional $z^{B,k}$ contains many scalar problems as specifications (see [95]), for instance the weighted Tschebyscheff scalarization, weighted sum scalarization, or ϵ-constraint scalarization, see [99] for details. Since these observations are well studied, they serve as a motivation to investigate whether it is suitable to gather new robustness concepts from the nonlinear scalarizing functional $z^{B,k}$ as well. This goal will be attained by a variation of the parameters B and k, as well as a suitable choice of the set of feasible elements $\mathcal{F}$. As an example, we will introduce a new approach toward robustness, which we will call ϵ-constraint robustness. In the following we analyze which type of robust counterpart is defined by this scalarization. To this end, let some $j \in \{1,\dots,q\}$ and real values $\epsilon_l \in \mathbb{R}$,

$l = 1, \ldots, q,\ l \neq j$ be given. Then we use the following components for the ϵ-constraint scalarization:

$$k_6 = (k_6^1, \ldots, k_6^q)^T \text{ where } k_6^l = \begin{cases} 1 & \text{for } l = j, \\ 0 & \text{for } l \neq j, \end{cases} \tag{3.28}$$

$$B_6 := \mathbb{R}^q_{\geqq} - \bar{b}, \text{ with } \bar{b} = (\bar{b}^1, \ldots, \bar{b}^q)^T,\ \bar{b}^l = \begin{cases} 0 & \text{for } l = j, \\ \epsilon_l & \text{for } l \neq j, \end{cases} \tag{3.29}$$

$$\mathcal{F}_6 = \{(f(x, \xi_1), \ldots, f(x, \xi_q))^T |\ x \in \mathfrak{A}\}. \tag{3.30}$$

Note that the set of feasible elements $\mathcal{F}_6$ coincides with the set of weighted robust feasible points $\mathcal{F}_1$ (see (3.7)). With these parameters the functional z^{B_6,k_6} describes the ϵ-constraint-method (cf. Eichfelder [26] and Haimes, Lasdon, D. A. Wismer [42]). Now the following reformulation holds.

Theorem 8. *Let $\epsilon := (\epsilon_1, \ldots, \epsilon_q)^T \in \mathbb{R}^q$ and $j \in \{1, \ldots, q\}$. Then for $k = k_6,\ B = B_6$,* (2.1) *holds and with $\mathcal{F} = \mathcal{F}_6$, problem $(P_{k,B,\mathcal{F}})$ (see* (2.3)*) is equivalent to*

$$(\epsilon RC) \qquad \begin{aligned} &\min \rho_{\epsilon RC}(x) \\ &s.t.\ \forall\ \xi \in \mathcal{U}:\ F_i(x, \xi) \leq 0,\ i = 1, \ldots, m, \\ &\qquad x \in \mathbb{R}^n, \\ &\qquad f(x, \xi_l) \leq \epsilon_l,\ l \in \{1, \ldots, q\},\ l \neq j, \end{aligned} \tag{3.31}$$

where $\rho_{\epsilon RC}(x) := f(x, \xi_j)$.

Proof. Since $B_6 + [0, +\infty) \cdot k_6 \subset B_6$, condition (2.1) is satisfied. Moreover,

$$\begin{aligned} \min_{y \in \mathcal{F}_6} z^{B_6,k_6}(y) &= \min_{y \in \mathcal{F}_6} \min\{t \in \mathbb{R}|\ y \in tk_6 - B_6\} \\ &= \min_{y \in \mathcal{F}_6} \min\{t \in \mathbb{R}|\ y - tk_6 \in -B_6\} \\ &= \min_{x \in \mathfrak{A}} \min\{t \in \mathbb{R}|\ f(x, \xi_j) \leq t,\ f(x, \xi_l) \leq \epsilon_l,\ l \in \{1, \ldots, q\},\ l \neq j\} \\ &= \min\{\rho_{\epsilon RC}(x)|\ x \in \mathbb{R}^n,\ \forall\ \xi \in \mathcal{U}:\ F_i(x, \xi) \leq 0,\ i = 1, \ldots, m, \\ &\qquad f(x, \xi_l) \leq \epsilon_l,\ l \in \{1, \ldots, q\},\ l \neq j\}. \end{aligned}$$

□

Note that the above suggested analysis can be performed for any possible variation of the parameters B, k and $\mathcal{F}$ in order to obtain new concepts for robustness. Such an approach may be beneficial for a decision maker whose attitude has not yet been represented by a given robustness concept. Thus, a new concept may be developed that fits the specific needs of the decision maker, taking his preferences in terms of risk and uncertainty into account.

Theorem 8 shows that the problem of minimizing the nonlinear scalarizing functional $z^{B,k}$ can be formulated as (ϵRC) (see (3.31)). We call (ϵRC) the **ϵ-constraint robust counterpart** of an uncertain optimization problem $(Q(\xi))$ (see (1.3)). In a next step, we analyze its meaning for robust optimization. Contrary to the other robustness concepts, the parameter k_6 symbolizes that only a single objective function is minimized. In particular, the decision maker chooses one specific objective function that he wishes to minimize subject to the constraints that are known from weighted and deviation robustness (although other constraints are entirely possible and the above concept may be adapted to a different set of feasible solutions $\mathcal{F}$). Furthermore, the former objective functions $f(x, \xi_l),\ l \in \{1, \dots, q\},\ l \neq j$, are shifted to and treated as constraints. This approach is useful if a solution is required with a given nominal quality for every scenario $\xi_l, l \in \{1, \dots, q\}, l \neq j$, while finding the best possible objective value for the remaining scenario j. When applying this concept, one difficulty is immediately revealed, namely, how to pick the upper bounds ϵ_l for the constraints. If they are chosen too small, the set of feasible solutions of (ϵRC) (see (3.31)) may be empty, or the objective function value $f(x, \xi_j)$ may not perform well enough. On the other hand, if the bounds ϵ_l are chosen too large, the optimality, meaning the value $f(x, \xi_l),\ l \neq j$, for the other scenarios decreases. Such a concept may be beneficial for a decision maker whose preferences have not yet been represented by any other robustness approach or to provide him with a wider choice of options. In addition, the values ϵ could, for instance, represent a company's regulations or safety standards which have to be satisfied.

Remark 11. *Note that we could have included the constraints $f(x, \xi_l) \leq \epsilon_l$, $l \in \{1, \dots, q\}$, $l \neq j$ in the set of feasible points $\widetilde{\mathcal{F}_6}$, and we would have obtained*

$$\begin{aligned}\widetilde{\mathcal{F}_6} = \{(f(x, \xi_1), \dots, f(x, \xi_q))^T |\ x \in \mathbb{R}^n :\ & f(x, \xi_l) \leq \epsilon_l,\ l \in \{1, \dots, q\},\ l \neq j, \\ & \forall\ \xi \in \mathcal{U} :\ F_i(x, \xi) \leq 0,\ i = 1, \dots, m\}.\end{aligned}$$

Then we would have obtained $\widetilde{B_6} = \mathbb{R}^q_{\geqq}$ instead of B_6 and we would have gained the same ϵ-constraint robust problem as above. The set of feasible points $\widetilde{\mathcal{F}_6}$, however, then could have been smaller and that could possibly introduce some difficulties in implementation.

Finally, some properties of the nonlinear scalarizing functional z^{B_6,k_6} are presented in the following corollary.

Corollary 2. *The functional z^{B_6,k_6} is lower semi-continuous, convex, $\mathbb{R}^q_{\geqq}$-monotone, strictly $\mathbb{R}^q_{>}$-monotone, proper, and the properties (f) and (g) from Theorem 2 hold.*

Proof. Since condition (2.1) is satisfied, Theorem 2 implies that z^{B_6,k_6} is lower semi-continuous, convex, proper, $\mathbb{R}^q_{\geqq}$-monotone and the properties (f) and (g) hold true. However, in the case of ϵ-constraint robustness we have $B_6 + (0, +\infty) \cdot k_6 \not\subset \operatorname{int} B_6$ for B_6 given by (3.29). Therefore, we show directly that z^{B_6,k_6} is strictly $\mathbb{R}^q_{>}$-monotone: Consider $t \in \mathbb{R},\ y \in tk_6 - \operatorname{int} B_6$. Then $tk_6 - y \in \operatorname{int} B_6$. Consequently, there exists an $s > 0$ such that $tk_6 - y - sk_6 \in \operatorname{int} B_6 \subset B_6$. Using (f) from Theorem 2, we deduce $z^{B_6,k_6}(y) \leq t - s < t$, and thus

$$tk_6 - \operatorname{int} B_6 \subset \{y \in \mathbb{R}^q |\ z^{B_6,k_6}(y) < t\}. \tag{3.32}$$

Furthermore, for $y_1 \in y_2 - \mathbb{R}^q_>$, it holds

$$\begin{aligned} y_1 \in y_2 - \mathbb{R}^q_> &\overset{\text{Thm. 2 } (f)}{\subset} z^{B_6,k_6}(y_2)k_6 - B_6 - \mathbb{R}^q_> \\ &\subset z^{B_6,k_6}(y_2)k_6 - \operatorname{int} B_6 \\ &\overset{(3.32)}{\subset} \{y \in \mathbb{R}^q \mid z^{B_6,k_6}(y) < z^{B_6,k_6}(y_2)\}. \end{aligned}$$

We conclude that $z^{B_6,k_6}(y_1) < z^{B_6,k_6}(y_2)$ and thus z^{B_6,k_6} is strictly $\mathbb{R}^q_>$-monotone. $\square$

3.1.7 Summary

In the following table, we present a short summary of the presented robustness concepts and the stochastic programming approach together with the according parameters B, k and $\mathcal{F}$ that are used to formulate the minimization problem $(P_{k,B,\mathcal{F}})$ (see (2.3)) with the nonlinear scalarizing functional $z^{B,k}$ (see (2.2)) as objective function.

Concept	B	k	$\mathcal{F}$
Weighted R.	$\{y \mid Wy \geqq 0\}$	$k_1^i = w_i^{-1}$	$\{(f(x,\xi_1),\ldots,f(x,\xi_q))^T \mid x \in \mathfrak{A}_1\}$
Deviation R.	$\mathbb{R}^q_{\geqq} - f^0$	1_q	$\{(f(x,\xi_1),\ldots,f(x,\xi_q))^T \mid x \in \mathfrak{A}_2\}$
Reliable R.	$\mathbb{R}^q_{\geqq}$	1_q	$\{(f(x,\xi_1),\ldots,f(x,\xi_q))^T \mid x \in \mathfrak{A}_3\}$
Light R.	B_4	1_m	$\mathcal{F}_4$, see below
Stochastic P.	B_5	1_q	$\{(f(x,u_1,\xi_1),\ldots,f(x,u_q,\xi_q))^T \mid (x,u) \in \mathfrak{A}_5\}$
ϵ-constraint R.	$\mathbb{R}^q_{\geqq} - \bar{b}$	$k_6^i = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$	$\{(f(x,\xi_1),\ldots,f(x,\xi_q))^T \mid x \in \mathfrak{A}_6\}$

We use the following vectors f^0, $\bar{b}$, matrix W and sets B_4, B_5 and $\mathcal{F}_4$.

$$W = \begin{pmatrix} w_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & w_q \end{pmatrix},$$

$$f^0 = (f^0(\xi_1), \dots, f^0(\xi_q))^T, \text{ where } f^0(\xi) \in \mathbb{R} \text{ is the optimal value of problem } (Q(\xi)),$$

$$B_4 = \{(\delta_1, \dots, \delta_m)^T \in \mathbb{R}^m | \sum_{i=1}^{m} w_i \delta_i \geq 0\},\ w_i \geq 0,\ i = 1, \dots, m,\ \sum_{i=1}^{m} w_i = 1,$$

$$B_5 = \{(y_1, \dots, y_q)^T \in \mathbb{R}^q | \sum_{k=1}^{q} p_k y_k \geq 0\},\ p_k \geq 0,\ k = 1, \dots, q,\ \sum_{k=1}^{q} p_k = 1,$$

$$\bar{b} = (\bar{b}^1, \dots, \bar{b}^q)^T, \text{ where } \bar{b}^l = \begin{cases} 0 & \text{for } l = j, \\ \epsilon_l & \text{for } l \neq j, \end{cases}$$

$$\mathcal{F}_4 = \{(\delta_1, \dots, \delta_m)^T | \ \exists\, x \in \mathbb{R}^n: \ F_i(x, \hat{\xi}) \leq 0,\ f(x, \hat{\xi}) \leq (1+\gamma) z^0, \\ \forall\, \xi \in \mathcal{U}: \ F_i(x, \xi) \leq \delta_i,\ \delta_i \in \mathbb{R},\ i = 1, \dots, m\}.$$

The following sets $\mathfrak{A}$ are used:

$$\mathfrak{A}_1 = \mathfrak{A},$$
$$\mathfrak{A}_2 = \mathfrak{A},$$
$$\mathfrak{A}_3 = \{x \in \mathbb{R}^n | \ F_i(x, \hat{\xi}) \leq 0,\ \forall\, \xi \in \mathcal{U}: \ F_i(x, \xi) \leq \delta_i,\ i = 1, \dots, m\},$$
$$\mathfrak{A}_5 = \{(x, u) := (x, u_1, \dots, u_q) \in \mathbb{R}^{n \times \bar{n} \cdot q} | \ \forall\, \xi_k \in \mathcal{U}: \ F_i(x, \xi_k) - \delta_k(u_k) \leq 0, \\ i = 1, \dots, m,\ u_k \in \mathcal{G}(x, \xi_k),\ k = 1, \dots, q\},$$
$$\mathfrak{A}_6 = \mathfrak{A}.$$

Note that we have $\mathfrak{A}_5 = \mathfrak{A}$ and $\mathcal{F}_5 = \mathcal{F}_1$ in the special case of static stochastic programming (i.e., if the second stage decision variable is omitted).

In Remarks 1 - 9, we have already presented some properties of the objective function $z^{B,k}$ for each corresponding robustness concept and the stochastic programming approach. These properties are summarized in the following corollary (compare Theorem 2 and Corollary 1).

Corollary 3. *The following properties hold for $i = 1, 2, 3, 5$ ($i = 1$: weighted robustness, $i = 2$: deviation robustness, $i = 3$: reliable robustness, $i = 5$: stochastic programming): The corresponding functional z^{B_i,k_i} is continuous, finite-valued, convex, $\mathbb{R}^q_{\geqq}$-monotone and strictly $\mathbb{R}^q_{>}$-monotone, and the following properties hold:*

$$\forall\, y \in \mathcal{F}_i,\ \forall\, r \in \mathbb{R}: \ z^{B_i,k_i}(y) \leq r \iff y \in rk_i - B_i, \tag{P1}$$
$$\forall\, y \in \mathcal{F}_i,\ \forall\, r \in \mathbb{R}: \ z^{B_i,k_i}(y + rk_i) = z^{B_i,k_i}(y) + r, \tag{P2}$$
$$\forall\, y \in \mathcal{F}_i,\ \forall\, r \in \mathbb{R}: \ z^{B_i,k_i}(y) = r \iff y \in rk_i - \text{ bd } B_i, \tag{P3}$$
$$\forall\, y \in \mathcal{F}_i,\ \forall\, r \in \mathbb{R}: \ z^{B_i,k_i}(y) < r \iff y \in rk_i - \text{ int } B_i. \tag{P4}$$

For $i = 1, 3, 5$, z^{B_i,k_i} is even sublinear. For $i = 4$ (light robustness), the properties (P1) – (P4) *are fulfilled, and z^{B_4,k_4} is continuous, sublinear and finite-valued. Additionally, z^{B_4,k_4} is $\mathbb{R}^m_{\geqq}$-monotone and strictly $\mathbb{R}^m_{>}$-monotone. Finally, for $i = 6$ (ϵ-constraint robustness), the properties* (P1) – (P4) *are satisfied, and z^{B_6,k_6} is lower semi-continuous, convex, $\mathbb{R}^q_{\geqq}$-monotone and strictly $\mathbb{R}^q_{>}$-monotone.*

Remark 12. *In addition, continuity, translation property and convexity of the functional z^{B_i,k_i} were shown for $i = 1, \ldots, 5$. These properties are present in the theory of risk measures as well and our analysis of robustness hence suggests further research in the theory of financial mathematics. For $i = 1, 3, 4, 5$, the functional z^{B_i,k_i} is sublinear.*

3.1.8 Multiple Objective Counterpart Problems and Relations to Robust Optimization and Stochastic Programming

This section is concerned with analyzing the properties of each nonlinear scalarizing functional $z^{B,k}$ used for formulating several introduced robustness concepts in terms of connections to a multi-objective counterpart problem. In particular, the monotonicity properties of $z^{B,k}$ will play an essential role in the following analysis.

To this end, we connect the uncertain (scalar) optimization problem $(Q(\xi), \xi \in \mathcal{U})$, as introduced in (1.2), see Chapter 1, to its (deterministic) multiple objective counterpart. The general idea is that every scenario $\xi \in \mathcal{U}$ yields its own objective function such that an uncertain scalar optimization problem can be interpreted as a multi-objective optimization problem, which we will refer to as the multiple objective counterpart. The vector of objectives in the multiple objective counterpart then contains the objectives $h_l(x) := f(x, \xi_l)$ for every scenario $\xi_l \in \mathcal{U}, l = 1, \ldots, q$. For the concept of light robustness, however, the roles of objective and constraints are reversed. Following the example of the different robustness concepts discussed above, the multiple objective counterparts formulated below can be distinguished with respect to the solution set $\mathfrak{A}$, i.e., the way in which the (uncertain) constraints are handled. To simplify the following analysis, in the case of stochastic programming we focus on a static model.

Connections between scalar robust optimization and multi-objective optimization have been mentioned by several authors for specific robustness concepts. In Kouvelis and Sayin [63, 87], this relation is used to develop solution methods to solve bicriteria optimization problems while focusing on two classical robustness concepts that we referred to as strict (weighted robustness with weights $w_k = 1,\ k = 1, \ldots, q$) and deviation robustness (see Sections 3.1.1 and 3.1.2). Kouvelis and Sayin exemplarily solve the bicriteria knapsack problem, the bicriteria assignment problem, and the bicriteria minimum cost network flow problem using an algorithm which is based on solution procedures originally introduced to solve uncertain scalar optimization problems (see also [64]).

A detailed analysis of the connections between uncertain scalar optimization and deterministic multi-objective optimization is presented by Ogryczak [78]. He exemplarily mentions expected value optimization and maximum regret models in relation to weighted sums and achievement scalarizing functions, respectively. Based on this analysis, new concepts for decisions under risk like symmetric and equitable optimization (efficiency) are introduced. These are further extended in Orgyczak and Śliwiński [81], where weighted ordered weighted averaging aggregation (WOWA) is used to model both risk aversion and scenario importance. Moreover, in Ogryczak [79, 80], the robust mean solution concept is related to the tail mean concept and to equitable solutions, among others. In order to solve robust shortest path and robust minimal spanning tree problems, Perny et al. [84] propose a multi-objective counterpart where elements are compared with respect to a generalized Lorenz dominance rule.

A critical evaluation of scalar robust optimization and its corresponding multi-objective counterpart is presented in Hites et al. [45]. The authors investigate the robust optimization framework in the context of multicriteria optimization by comparing the two methodologies. One feature tying both approaches together is the goal to obtain solutions that are good in all scenarios (in the robust optimization framework), or in all criteria (for the multi-objective counterpart, respectively). The authors in [45] discuss how both approaches bear a lot of complexity, since usually in real life problems, conflicting goals require the need to compromise. In that regard, robust and deterministic multi-objective optimization have a lack of an optimality notion in common: In robust optimization, it seems to be rare that a solution is optimal for all given scenarios. The same applies to multicriteria optimization: Due to a lack of a total order in $\mathbb{R}^q$, solutions are compared with respect to the natural ordering cone $\mathbb{R}^q_{\geq}$. The authors in [45] conclude, however, that both approaches should not be confused, as Pareto optimality cannot replace comparing solutions according to one just scenario in terms of robustness. However, as will be seen below, there certainly is a strong relation from a theoretical point of view. Iancu and Trichakis [47] argue that the traditional robust optimization approach, namely the weighted robust counterpart (3.1) with $w_k = 1,\ k = 1, \dots, q$, may produce solutions that are not Pareto optimal for the corresponding multi-objective problem. In fact, as will be seen below, a uniqueness assumption on a solution ensures that this solution is Pareto optimal for the multi-objective counterpart.

From the stochastic programming perspective, a multiple objective counterpart for a two-stage stochastic programming problem was introduced in Gast [32] and used to interrelate stochastic programming models with the concept of recoverable robustness, see Stiller [94]. We will in the following focus on the static stochastic programming approach, such that the second stage decision $u_k,\ k = 1, \dots, q$, may be omitted and we use the set of feasible solutions $\mathfrak{A}_5 := \mathfrak{A}$ (compare (3.2)). A **static stochastic counterpart** is formulated as

$$(sSP) \qquad \begin{aligned} &\min \rho_{sSP}(x) \\ &\text{s.t. } \forall\, \xi_k \in \mathcal{U}:\ F_i(x, \xi_k) \leq 0,\ i = 1, \dots, m, \\ &\quad x \in \mathbb{R}^n, \end{aligned} \tag{3.33}$$

with $\rho_{sSP}(x, u) := \sum_{k=1}^{q} p_k f(x, \xi_k)$.

Consider

$$h(x) := \begin{pmatrix} h_1(x) \\ \vdots \\ h_q(x) \end{pmatrix} := \begin{pmatrix} f(x,\xi_1) \\ \vdots \\ f(x,\xi_q) \end{pmatrix}. \tag{3.34}$$

Recall from (3.2) (see also Section 3.1.7) that

$$\mathfrak{A}_1 = \mathfrak{A}_2 = \mathfrak{A}_5 = \mathfrak{A}_6 = \mathfrak{A} = \{x \in \mathbb{R}^n | \ \forall\ \xi \in \mathcal{U}:\ F_i(x,\xi) \leq 0,\ i = 1,\dots,m\}.$$

Then we introduce the **multiple objective strictly robust counterpart** to $(Q(\xi), \xi \in \mathcal{U})$ by

$$(RC') \qquad \mathrm{Min}(h[\mathfrak{A}_1], \mathbb{R}^q_{\geq}), \tag{3.35}$$

where $h[\mathfrak{A}_1] = \mathcal{F}_1$ (see (3.7)).

Similarly, recall from (3.15) that

$$\mathfrak{A}_3 := \{x \in \mathbb{R}^n | \ F_i(x,\hat{\xi}) \leq 0,\ \forall\ \xi \in \mathcal{U}:\ F_i(x,\xi) \leq \delta_i,\ i = 1,\dots,m\}.$$

We propose the **multiple objective reliably robust counterpart** to $(Q(\xi), \xi \in \mathcal{U})$ as

$$(rRC') \qquad \mathrm{Min}(h[\mathfrak{A}_3], \mathbb{R}^q_{\geq}), \tag{3.36}$$

where $h[\mathfrak{A}_3] = \mathcal{F}_3$ (see formula (3.18)).

Now let us introduce a multiple objective counterpart that corresponds to the lightly robust counterpart (lRC) (see (3.19)). Let $\mathcal{F}_4$ be defined by (3.22), i.e.,

$$\begin{aligned} \mathcal{F}_4 = &\{(\delta_1,\dots,\delta_m)^T | \ \exists\ x \in \mathbb{R}^n:\ F_i(x,\hat{\xi}) \leq 0,\ f(x,\hat{\xi}) \leq (1+\gamma)z^0, \\ &\forall\ \xi \in \mathcal{U}:\ F_i(x,\xi) \leq \delta_i,\ \delta_i \in \mathbb{R},\ i = 1,\dots,m\}. \end{aligned}$$

We define the **multiple objective lightly robust counterpart** to $(Q(\xi), \xi \in \mathcal{U})$ by

$$(lRC') \qquad \mathrm{Min}(\mathcal{F}_4, \mathbb{R}^m_{\geq}). \tag{3.37}$$

In the following corollary, the monotonicity properties of the functional z^{B_i,k_i}, $i = 1,\dots,6$, that represent the respective robustness approaches, are used to display relationships between the scalar problem $(P_{k_i,B_i,\mathcal{F}_i})$ and the set of minimal solutions $\mathrm{Min}(h[\mathfrak{A}_i], \mathbb{R}^q_{\geq})$. It implies that problem $(P_{k_i,B_i,\mathcal{F}_i})$, $i = 1,2,5,6$ $((P_{k_3,B_3,\mathcal{F}_3}), (P_{k_4,B_4,\mathcal{F}_4})$, respectively) is a scalarization of the multiple objective counterpart (RC') (see (3.35)) ((rRC') (see (3.36)), (lRC'), (see (3.37)), respectively), taking into account Theorem 1 together with Corollaries 2 and 3.

Corollary 4. *For $i = 1,2,5,6$ ($i = 1$: weighted robustness, $i = 2$: deviation robustness, $i = 5$: static stochastic programming, $i = 6$: ϵ-constraint robustness), it holds:*

$$\begin{aligned} [\forall\ y \in \mathcal{F}_i \setminus \{y^0\}:\ z^{B_i,k_i}(y^0) < z^{B_i,k_i}(y)] \Longrightarrow y^0 \in \mathrm{Min}(h[\mathfrak{A}_1], \mathbb{R}^q_{\geq}), \\ [\forall\ y \in \mathcal{F}_i:\ z^{B_i,k_i}(y^0) \leq z^{B_i,k_i}(y)] \Longrightarrow y^0 \in \mathrm{Min}(h[\mathfrak{A}_1], \mathbb{R}^q_{>}). \end{aligned}$$

In terms of reliable robustness ($i = 3$), we have

$$\begin{aligned}[\forall\, y \in \mathcal{F}_3 \setminus \{y^0\}:\ z^{B_3,k_3}(y^0) < z^{B_3,k_3}(y)] &\Longrightarrow y^0 \in \mathrm{Min}(h[\mathfrak{A}_3], \mathbb{R}^q_{\geq}),\\ [\forall\, y \in \mathcal{F}_3:\ z^{B_3,k_3}(y^0) \leq z^{B_3,k_3}(y)] &\Longrightarrow y^0 \in \mathrm{Min}(h[\mathfrak{A}_3], \mathbb{R}^q_{>}).\end{aligned}$$

For light robustness ($i = 4$), we conclude

$$\begin{aligned}[\forall\, y \in \mathcal{F}_4 \setminus \{y^0\}:\ z^{B_4,k_4}(y^0) < z^{B_4,k_4}(y)] &\Longrightarrow y^0 \in \mathrm{Min}(\mathcal{F}_4, \mathbb{R}^m_{\geq}),\\ [\forall\, y \in \mathcal{F}_4:\ z^{B_4,k_4}(y^0) \leq z^{B_4,k_4}(y)] &\Longrightarrow y^0 \in \mathrm{Min}(\mathcal{F}_4, \mathbb{R}^m_{>}).\end{aligned}$$

The above analysis shows that by using the nonlinear scalarizing functional $z^{B,k}$, connections between scalar robust optimization problems and a multi-objective counterpart are given in a very natural way. Naturally, the finally chosen robust solution will vary on the concept of robustness which a decision maker chooses due to his preferences. We argue that there may exist even more scalar robustness concepts which have yet to be determined and which the decision maker may be unaware of. If only following one approach, he may lose solutions that would have suited him but were not represented by a particular concept. Therefore, it seems reasonable to solve a multi-objective counterpart and let the decision maker choose appropriate solutions.

We illustrate the above results in the following example, which is based on [62, Example 4.1].

Example 1. *We introduce the uncertain optimization problem*

$$(Q(\xi)) \qquad \min_{x \in \mathbb{R}^2} f(x, \xi), \tag{3.38}$$

where $\xi \in \mathcal{U} := \{0, 1\}$ and $f(x, \xi) := \frac{1}{2}x^T(x + 2c(\xi)) + 3$ with $c(\xi) := (1 + \xi,\ 1 - \xi)^T$. To keep the example as simple as possible, $(Q(\xi))$ does not involve any constraints. The corresponding bicriteria optimization problem reads

$$(RC') \qquad \mathrm{Min}(f[\mathbb{R}^2], \mathbb{R}^2_{\geq}) \tag{3.39}$$

with

$$\begin{aligned} f_1(x) &:= f(x, \xi_1) = \frac{1}{2}x^T(x + 2c(\xi_1)) + 3,\\ f_2(x) &:= f(x, \xi_2) = \frac{1}{2}x^T(x + 2c(\xi_2)) + 3.\end{aligned}$$

One can easily find that

$$x_1^0 = \mathrm{argmin}_{x \in \mathbb{R}^2} f(x, 0) = (-1, -1)^T, \qquad x_2^0 = \mathrm{argmin}_{x \in \mathbb{R}^2} f(x, 1) = (-2, 0)^T$$

are the two solutions of the uncertain problems $(Q(\xi))$ for $\xi = \xi_1$ and $\xi = \xi_2$. The respective function values are

$$f^0(\xi_1) = \min_{x \in \mathbb{R}^2} f(x, 0) = 2, \qquad f^0(\xi_2) = \min_{x \in \mathbb{R}^2} f(x, 1) = 1.$$

The weighted robust counterpart of problem $(Q(\xi))$ *with weights* $w_k = 1,\ k = 1,\dots,q,$ *reads*

$$\min_{x\in\mathbb{R}^2}\max_{\xi\in\mathcal{U}} f(x,\xi) = \min_{x\in\mathbb{R}^2}\max_{\xi\in\mathcal{U}}(\frac{1}{2}x^T(x+2c(\xi))+3) = 2$$

with the strictly robust solution $x_{sr}^0 = (-1,-1)^T$. *The function values are*

$$f_{sr} = \begin{pmatrix} f(x_{sr},0) \\ f(x_{sr},1) \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

Thus, from Corollary 4, we have that x_{sr} *is Pareto optimal for the multi-objective optimization problem* (RC') *(see* (3.39)*). For the deviation robust optimization problem we find the solution*

$$\begin{aligned} x_{dr} &= \text{argmin}_{x\in\mathbb{R}^2} \max_{\xi\in\{0,1\}} \{f(x,0) - f^0(0),\ f(x,1) - f^0(1)\} \\ &= \text{argmin}_{x\in\mathbb{R}^2} \max_{\xi\in\{0,1\}} \{f(x,0) - 2,\ f(x,1) - 1\} \\ &= \begin{pmatrix} -\frac{3}{2} \\ -\frac{1}{2} \end{pmatrix}. \end{aligned}$$

The according function values are

$$f_{dr} = \begin{pmatrix} f(x_{dr},0) \\ f(x_{dr},1) \end{pmatrix} = \begin{pmatrix} \frac{9}{4} \\ \frac{5}{4} \end{pmatrix}.$$

Now let $j = 1$ *and choose* $\epsilon_2 = \frac{5}{4}$. *Then the solution* x_{dr} *of the deviation robust problem simultaneously is the solution of the* ϵ*-constraint robust problem with function values denoted by* f_{ϵ_2}. *If, on the other hand, one takes* $j = 2$ *and chooses* $\epsilon_1 = \frac{11}{4}$, *the solution of the* ϵ*-constraint problem is*

$$x_{\epsilon_1} = \text{argmin}_{x\in\mathbb{R}^2,\ f(x,0)\leq\frac{11}{4}} f(x,1) = \begin{pmatrix} \frac{-\sqrt{3}}{2} - 1 \\ \frac{\sqrt{3}}{2} - 1 \end{pmatrix} \approx \begin{pmatrix} -1.866 \\ -0.134 \end{pmatrix}.$$

The function values here are

$$f_{\epsilon_1} = \begin{pmatrix} \frac{11}{4} \\ \frac{11}{4} - \sqrt{3} \end{pmatrix} \approx \begin{pmatrix} 2.75 \\ 1.01795 \end{pmatrix}.$$

Figure 3.1 visualizes the function values f_{sr} *of the strictly robust solution* x_{sr}, *the function values* f_{dr} *of the deviation robust solution* x_{dr} *and, respectively, the values* $f_{\epsilon_i},\ i = 1,2,$ *of the* ϵ*-constraint robust points* $x_{\epsilon_i},\ i = 1,2$. *It can be seen that the solutions of the different robust optimization problems belong to the set of Pareto optimal points of the problem* (RC') *(see* (3.39)*).*

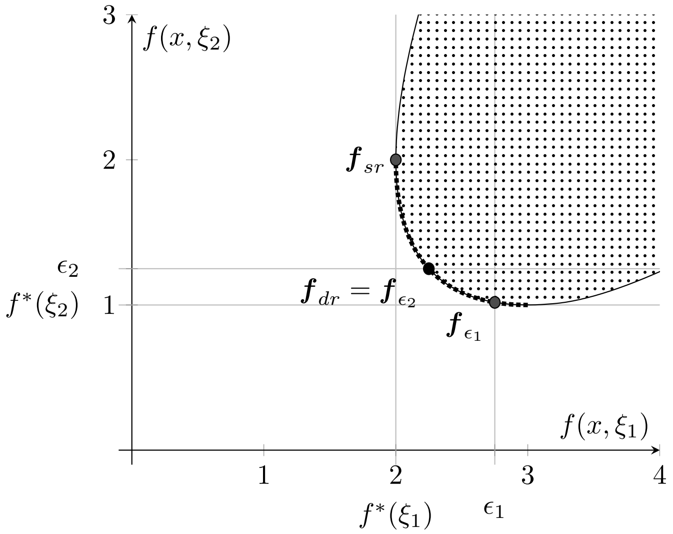

Figure 3.1: Pareto optimal solutions of problem (RC') (see (3.39)) in the objective space with the function value f_{sr} of the strictly robust solution, function value f_{dr} of the deviation robust solution, and the values $f_{\epsilon_i},\ i = 1, 2$ of the ϵ-constraint robust point $x_{\epsilon_i},\ i = 1, 2$. $f^0(\xi_i) \in \mathbb{R}$ are the optimal values of problem $(Q(\xi_i))$ for $\xi_i \in \mathcal{U},\ i = 1, 2$.

3.2 Continuous Compact Uncertainty Set

This section is concerned with examining how the unifying approach for robustness and stochastic programming that is presented above may be extended to continuous compact uncertainty sets $\mathcal{U} \subset \mathbb{R}^N$. So far, we have studied the special case of discrete uncertainty sets $\mathcal{U} = \{\xi_1, \ldots, \xi_q\}$ and were able to characterize robust and stochastic optimization problems by minimizing a nonlinear scalarizing function $z^{B,k}$, which resulted in choosing the involved parameters B, k and the set of feasible solutions $\mathcal{F}$ in a particular way. In order to extend this approach to more general, namely compact uncertainty sets, we present two approaches. First, we suggest a so called **dominance of functions**, which is a rather direct extension from the case of finite uncertainty sets. It will be shown that a robust optimization problem can be expressed using the functional $z^{B,k}$ with a decision variable $y = f(x, \xi)$. The idea is, similar to the finite approach, that each parameter $\xi \in \mathcal{U}$ yields its own objective function $f(x, \xi)$. The second approach deals with **dominance of sets** and suggests that all possible objective values are regarded as a set if solution x is chosen. Specifically, we propose the set $A_x := \{f(x, \xi) |\ \xi \in \mathcal{U}\} =: f_{\mathcal{U}}(x)$. In the following, both approaches will be exemplarily illustrated for the concept of **strict robustness**: For $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$, $F_i : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$, $i = 1, \ldots, m$, the **strictly robust**

counterpart is given as

$$
(RC) \qquad \begin{aligned} &\min \rho_{RC}(x) \\ &\text{s.t. } \forall\, \xi \in \mathcal{U}:\ F_i(x,\xi) \leq 0,\ i = 1,\dots,m, \\ &\quad x \in \mathbb{R}^n, \end{aligned} \tag{3.40}
$$

where $\rho_{RC}(x) := \sup_{\xi \in \mathcal{U}} f(x,\xi)$ and $\mathcal{U} \subset \mathbb{R}^N$ is a compact set. The following Sections 3.2.1 and 3.2.2 are based upon [60], a joint work with K. Klamroth, A. Schöbel and Chr. Tammer.

3.2.1 Approach 1: Dominance of Functions

Throughout this subsection, let Y be the space of all continuous functions $H:\ \mathcal{U} \to \mathbb{R}$. We define for a given $x \in \mathbb{R}^n$ the function $H_x \in Y,\ H_x:\ \mathcal{U} \to \mathbb{R}$ as

$$H_x(\xi) := f(x,\xi).$$

The canonical ordering cone in Y is defined by

$$Y^+ := \{H \in Y |\ \forall\, \xi \in \mathcal{U}:\ H(\xi) \geq 0\}.$$

Furthermore, throughout this subsection, we suppose that $\operatorname{int} Y^+ \neq \emptyset$, which implies that $Y^+ \neq \emptyset$. We need to ensure that these sets are not empty such that they can be used in the formulation of the nonlinear scalarizing functional $z^{B,k}$ (see (2.2)).

The set of feasible solutions in the objective space Y is denoted by

$$\mathcal{F}_7 := \{H_x \in Y |\ x \in \mathfrak{A}\}, \tag{3.41}$$

where $\mathfrak{A}$ is given by (3.2).

Theorem 9. *Consider*

$$B_7 := Y^+, \tag{3.42}$$

$$k_7 \equiv 1 := \{k_7(\xi) \in Y |\ \forall\, \xi \in \mathcal{U}:\ k_7(\xi) = 1\}. \tag{3.43}$$

For $k = k_7,\ B = B_7$, condition (2.1) *is satisfied and with $\mathcal{F} = \mathcal{F}_7$, problem $(P_{k,B,\mathcal{F}})$ (see* (2.3)*) is equivalent to problem (RC) (see* (3.40)*) in the following sense:*

(i) For all $y \in \mathcal{F}_7$ there exists $x \in \mathfrak{A}$ such that $z^{B_7,k_7}(y) = \rho_{RC}(x)$.

(ii) For all $x \in \mathfrak{A}$ there exists $y \in \mathcal{F}_7$ such that $z^{B_7,k_7}(y) = \rho_{RC}(x)$.

Proof. Since $B_7+[0,+\infty)\cdot k_7 = Y^+ + [0,+\infty)\cdot k_7 \subset Y^+ = B_7$, condition (2.1) is satisfied. Now consider a pair $y \in \mathcal{F}_7$, $x \in \mathfrak{A}$ with $y = H_x$. Then

$$\begin{aligned} z^{B_7,k_7}(y) = z^{B_7,k_7}(H_x) &= \inf\{t \in \mathbb{R}|\ H_x \in tk_7 - B_7\} \\ &= \inf\{t \in \mathbb{R}|\ H_x - tk_7 \in -B_7\} \\ &= \inf\{t \in \mathbb{R}|\ \forall\ \xi \in \mathcal{U}:\ H_x(\xi) \le tk_7\} \\ &= \inf\{t \in \mathbb{R}|\ \forall\ \xi \in \mathcal{U}:\ f(x,\xi) \le t\} \\ &= \sup_{\xi\in\mathcal{U}} f(x,\xi) \\ &= \max_{\xi\in\mathcal{U}} f(x,\xi) \qquad (3.44) \\ &= \rho_{RC}(x). \end{aligned}$$

Note that the maximum in (3.44) is attained because $f(x,\cdot) = H_x(\cdot) \in Y$, i.e., H_x is continuous in $\xi \in \mathcal{U}$ for every $x \in \mathbb{R}^n$ on a compact set $\mathcal{U}$.

(i) Let $y \in \mathcal{F}_7$. Hence there exists $x \in \mathfrak{A}$ with $y = H_x$. We conclude $z^{B_7,k_7}(y) = \rho_{RC}(x)$.

(ii) Vice versa, if $x \in \mathfrak{A}$ we define $y := H_x \in Y$ and again obtain $z^{B_7,k_7}(y) = \rho_{RC}(x)$.

□

Some properties of the nonlinear scalarizing functional z^{B_7,k_7} are collected in the following remark, which remains stated but unproven for it follows directly from the properties of B_7 and k_7 as a consequence of Theorem 2.

Remark 13. *Since $B_7 = Y^+$ is a proper closed convex cone and $k_7 \in \operatorname{int} B_7$, the functional z^{B_7,k_7} is continuous, finite-valued, Y^+-monotone, strictly $(\operatorname{int} Y^+)$-monotone and sublinear, taking into account Corollary 1.*

3.2.2 Approach 2: Dominance of Sets

Let Z be the set of all subsets on $\mathbb{R}$. We denote by $A_x := \{f(x,\xi)|\ \xi \in \mathcal{U}\} =: f_{\mathcal{U}}(x) \in Z$ the image of the mapping $f(x,\cdot)$ under $\mathcal{U}$. Note that $A_x \subseteq \mathbb{R}$ is a (maybe unbounded) interval in case that $f(x,\cdot)$ is a continuous function. Furthermore, we define a relation on a set $A \in Z$ for some $r \in \mathbb{R}$ in the following way:

$$A \le r \ :\Longleftrightarrow\ \forall\ a \in A:\ a \le r.$$

Now the canonical ordering cone in Z is given by

$$Z^+ := \{A \in Z|\ A \ge 0\} \qquad (3.45)$$

We denote the set of feasible solutions in the objective space Z by

$$\mathcal{F}_8 := \{A_x \in Z|\ x \in \mathfrak{A}\}, \qquad (3.46)$$

where $\mathfrak{A}$ is given by (3.2).
Now we are able to formulate the following theorem.

Theorem 10. *Consider*

$$B_8 := Z^+, \tag{3.47}$$

$$k_8 = 1. \tag{3.48}$$

For $k = k_8$, $B = B_8$, condition (2.1) is satisfied and with $\mathcal{F} = \mathcal{F}_8$, problem $(P_{k,B,\mathcal{F}})$ (see (2.3)) is equivalent to problem (RC) (see (3.40)) in the following sense:

(i) For all $y \in \mathcal{F}_8$ there exists $x \in \mathfrak{A}$ such that $z^{B_8,k_8} = \rho_{RC}(x)$.

(ii) For all $x \in \mathfrak{A}$ there exists $y \in \mathcal{F}_8$ such that $z^{B_8,k_8} = \rho_{RC}(x)$.

Proof. Because $B_8 + [0, +\infty) \cdot k_8 = Z^+ + [0, +\infty) \cdot k_8 \subset Z^+ = B_8$, (2.1) holds true. Now consider a pair $y \in \mathcal{F}_8$, $x \in \mathfrak{A}$ with $y = A_x$. Then

$$\begin{aligned} z^{B_8,k_8}(y) = z^{B_8,k_8}(A_x) &= \inf\{t \in \mathbb{R} |\ A_x \in tk_8 - B_8\} \\ &= \inf\{t \in \mathbb{R} |\ A_x - tk_8 \in -B_8\} \\ &= \inf\{t \in \mathbb{R} |\ A_x \leq t\} \\ &= \inf\{t \in \mathbb{R} |\ \forall\, \xi \in \mathcal{U} :\ f(x, \xi) \leq t\} \\ &= \sup_{\xi \in \mathcal{U}} f(x, \xi) \\ &= \rho_{RC}(x). \end{aligned}$$

(i) Let $y \in \mathcal{F}_8$. Thus there there is an $x \in \mathfrak{A}$ with $y = A_x$. Hence we conclude $z^{B_8,k_8}(y) = \rho_{RC}(x)$.

(ii) Let $x \in \mathfrak{A}$ and we define $y := A_x \in Z$ and again we conclude $z^{B_8,k_8}(y) = \rho_{RC}(x)$.

$\square$

Theorem 10 verifies that the nonlinear scalarizing functional $z^{B,k}$ may be used to characterize the strictly robust optimization problem (RC) (see (3.40)) for a compact uncertainty set $\mathcal{U}$ when using a set approach.

Remark 14. *Since $B_8 = Z^+$ is a proper closed convex cone and $k_8 \in \operatorname{int} B_8$, Corollary 1 implies that the functional z^{B_8,k_8} is continuous, finite-valued, Z^+-monotone, strictly $(\operatorname{int} Z^+)$-monotone and sublinear.*

3.2.3 Reducing the Uncertainty Set

In this subsection, we will show that under a quasiconvexity assumption on $f(x, \cdot)$ for $x \in \mathbb{R}^n$, robust counterparts with an uncertainty set $\mathcal{U}$ as the convex hull of finitely many scenarios ξ can be reduced to a simple robust optimization problem for which results from Section 3.1 hold for the specific uncertainty set $\widetilde{\mathcal{U}} = \{\xi_1, \ldots, \xi_q\}$. Note that the robust problems (dRC), (rRC) and (lRC) given by (3.8), (3.14) and (3.19), were introduced for finite uncertainty sets $\widetilde{\mathcal{U}} := \{\xi_1, \ldots, \xi_q\}$, but their definitions may easily be extended to the case when the uncertainty set is compact. First we mention the following theorem.

Theorem 11 ([12])**.** *Let $\widetilde{\mathcal{U}} := \{\xi_1, \dots, \xi_q\}$, $\mathcal{U} := \mathrm{conv}(\widetilde{\mathcal{U}})$, the convex hull of $\widetilde{\mathcal{U}}$. Assume that $f:\ \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$ and $F_i:\ \mathbb{R}^n \times \mathcal{U} \to \mathbb{R},\ i = 1, \dots, m$ are quasiconvex in $\xi \in \mathcal{U}$ for each $x \in \mathbb{R}^n$. Then it holds $\max_{\xi \in \mathcal{U}} f(x, \xi) = \max_{\xi \in \widetilde{\mathcal{U}}} f(x, \xi)$ and for every $x \in \mathbb{R}^n$*

$$\forall\ \xi \in \mathcal{U}:\ F_i(x, \xi) \leq 0,\ i = 1, \dots, m \iff \forall\ \xi \in \widetilde{\mathcal{U}}:\ F_i(x, \xi) \leq 0,\ i = 1, \dots, m.$$

Proof. Because $\mathcal{U} := \mathrm{conv}(\widetilde{\mathcal{U}})$, $\xi \in \mathcal{U}$ can be expressed by $\xi = \sum_{k=1}^{q} \lambda_k \xi_k$ for $\xi_k \in \widetilde{\mathcal{U}}$, and $\lambda_k \in [0, 1],\ k = 1, \dots, q,\ \sum_{k=1}^{q} \lambda_k = 1$. It holds

$$\begin{aligned}
\max_{\xi \in \mathcal{U}} f(x, \xi) &= \min\{t \in \mathbb{R} |\ \forall\ \xi \in \mathcal{U}:\ f(x, \xi) \leq t\} \\
&= \min\{t \in \mathbb{R} |\ \forall\ \lambda_k \in [0, 1],\ k = 1, \dots, q,\ \sum_{k=1}^{q} \lambda_k = 1: \\
&\qquad f(x, \sum_{k=1}^{q} \lambda_k \xi_k) \leq \max\{f(x, \xi_1), \dots, f(x, \xi_q)\} \leq t\} \\
&= \max_{\xi \in \widetilde{\mathcal{U}}} f(x, \xi).
\end{aligned}$$

The second part of the theorem can be proved in the exact same way. □

Now we consider the strictly robust optimization problem (RC) given in (3.40). We denote the strictly robust optimization problem with uncertainty set $\widetilde{\mathcal{U}} = \{\xi_1, \dots, \xi_q\}$ by $(RC(\widetilde{\mathcal{U}}))$, while the strictly robust optimization problem with $\mathcal{U} = \mathrm{conv}(\widetilde{\mathcal{U}})$ is denoted by $(RC(\mathcal{U}))$. Using Theorem 11 together with Corollary 3 we obtain the following result.

Corollary 5. *Let $\widetilde{\mathcal{U}} := \{\xi_1, \dots, \xi_q\}$, $\mathcal{U} := \mathrm{conv}(\widetilde{\mathcal{U}})$. Assume that*

$$f:\ \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$$

and

$$F_i:\ \mathbb{R}^n \times \mathcal{U} \to \mathbb{R},\ i = 1, \dots, m$$

are quasiconvex in $\xi \in \mathcal{U}$ for every $x \in \mathbb{R}^n$. Let $\mathcal{F}_1$ be defined by the set of feasible solutions (3.7)*. If x^0 solves $(RC(\mathcal{U}))$, then $y(x^0) := (f(x^0, \xi_1), \dots, f(x^0, \xi_q))^T \in$ $\mathrm{Min}(\mathcal{F}_1, \mathbb{R}^q_{>})$. If x^0 is a unique solution of the strictly robust problem $(RC(\mathcal{U}))$, then $y(x^0) := (f(x^0, \xi_1), \dots, f(x^0, \xi_q))^T \in \mathrm{Min}(\mathcal{F}_1, \mathbb{R}^q_{\geq})$.*

For the deviation robust optimization problem (3.8), we obtain the following conclusion. Note that in order to distinguish between the different problems with respective uncertainty sets, we denote by $(dRC(\mathcal{U}))$ ($(dRC(\widetilde{\mathcal{U}})$, respectively) the deviation robust problem with uncertainty set $\mathcal{U}$ ($\widetilde{\mathcal{U}}$, respectively).

Corollary 6. *Let $\widetilde{\mathcal{U}} := \{\xi_1, \dots, \xi_q\}$, $\mathcal{U} := \mathrm{conv}(\widetilde{\mathcal{U}})$. Assume that*

$$g:\ \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$$

and

$$F_i : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}, \ i = 1, \dots, m$$

are quasiconvex in ξ*, where* $g(x, \xi) := f(x, \xi) - f^0(\xi)$*. Let* $\mathcal{F}_1$ *be defined by the set of feasible solutions* (3.7)*. If* x^0 *solves* $(dRC(\mathcal{U}))$*, then* $y(x^0) := (f(x^0, \xi_1), \dots, f(x^0, \xi_q))^T \in \mathrm{Min}(\mathcal{F}_1, \mathbb{R}^q_>)$*. If* x^0 *is a unique solution of the deviation robust problem* $(dRC(\mathcal{U}))$*, then* $y(x^0) := (f(x^0, \xi_1), \dots, f(x^0, \xi_q))^T \in \mathrm{Min}(\mathcal{F}_1, \mathbb{R}^q_\geq)$*.*

Results analogous to the above conclusions for some of the before mentioned robustness concepts are described below. In the case of reliable robustness, the analysis is strongly related to Corollary 5. We denote the reliably robust optimization problem (compare (3.14)) with uncertainty set $\mathcal{U}$ ($\widetilde{\mathcal{U}}$, respectively) by $(rRC(\mathcal{U}))$ ($(rRC(\widetilde{\mathcal{U}}))$, respectively).

Corollary 7. *Let* $\widetilde{\mathcal{U}} := \{\xi_1, \dots, \xi_q\}$*,* $\mathcal{U} := \mathrm{conv}(\widetilde{\mathcal{U}})$*. Assume that* $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$ *and* $F_i : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}, \ i = 1, \dots, m$ *are quasiconvex in* $\xi \in \mathcal{U}$ *for every* $x \in \mathbb{R}^n$*. Let* $\mathcal{F}_3$ *be defined by the set of feasible solutions* (3.18)*. If* x^0 *solves* $(rRC(\mathcal{U}))$*, then* $y(x^0) := (f(x^0, \xi_1), \dots, f(x^0, \xi_q))^T \in \mathrm{Min}(\mathcal{F}_3, \mathbb{R}^q_>)$*. If* x^0 *is a unique solution of the reliably robust problem* $(rRC(\mathcal{U}))$*, then* $y(x^0) := (f(x^0, \xi_1), \dots, f(x^0, \xi_q))^T \in \mathrm{Min}(\mathcal{F}_3, \mathbb{R}^q_\geq)$*.*

Below we denote by $(lRC(\mathcal{U}))$ ($(lRC(\widetilde{\mathcal{U}}))$, respectively) the lightly robust optimization problem (see (3.19)) with uncertainty set $\mathcal{U}$ ($\widetilde{\mathcal{U}}$, respectively).

Corollary 8. *Let* $\widetilde{\mathcal{U}} := \{\xi_1, \dots, \xi_q\}$*,* $\mathcal{U} := \mathrm{conv}(\widetilde{\mathcal{U}})$*. Assume that* $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$ *and* $F_i : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}, \ i = 1, \dots, m$ *are quasiconvex in* $\xi \in \mathcal{U}$ *for every* $x \in \mathbb{R}^n$*. Since for the lightly robust approach, the uncertainty is only reflected within the constraints, we conclude that* $(lRC(\mathcal{U}))$ *and* $(lRC(\widetilde{\mathcal{U}}))$ *are equivalent, such that their solution sets coincide.*

In this chapter, we presented a unifying framework for robust optimization by means of a nonlinear scalarizing functional. We first considered finite uncertainty sets and then extended our research to compact-valued uncertainty sets. The analysis naturally led to a multi-objective counterpart problem. Finally, we provided connections between both approaches in case the objective function in the considered robustness concept is quasiconvex in the uncertain parameter $\xi \in \mathcal{U}$.

The research conducted in this chapter inspires us to study interrelations between robust scalar problems and an unconstrained vector-valued optimization problem, which will be the main topic in the following chapter.

Chapter 4

Relations Between Scalar Robust Problems and Unconstrained Multicriteria Optimization Problems

In Chapter 3, we showed that by using a nonlinear scalarizing functional optimal elements of certain kinds of robust optimization problems belong to the set of weakly Pareto optimal points of a *constrained* multi-objective optimization problem. Under uniqueness assumptions, these solutions are even Pareto optimal. Now we show that solutions of scalar robust optimization problems belong to the set of weakly Pareto optimal solutions of a particularly chosen *unconstrained* vector-valued optimization problem.

To this end, let $\mathcal{U} := \{\xi_1, \ldots, \xi_q\} \subseteq \mathbb{R}^N$ be again a finite uncertainty set and let $\xi \in \mathcal{U}$ be an uncertain parameter that can take on q different values. Furthermore, let $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$ and $F_i : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R},\ i = 1, \ldots, m$.

We introduce the *unconstrained* multi-objective optimization problem

$$(RC_u) \qquad \operatorname{Min}(h[\mathbb{R}^n], \mathbb{R}^{q+m\cdot q}_{\geq}), \tag{4.1}$$

where $h(x) := (f(x, \xi_1), \ldots, f(x, \xi_q), F_1(x, \xi_1), \ldots, F_m(x, \xi_q))^T$.

We call (RC_u) the vector-valued unconstrained counterpart of an uncertain optimization problem $(Q(\xi))$ (see (1.3)).

In this chapter, we even show that unique solutions of scalar robust optimization problems are Pareto optimal solutions to problem (RC_u).

Steuer [93] observed that solutions of constrained scalar optimization problems are weakly Pareto optimal (and, under uniqueness assumptions to the solution, Pareto optimal) for a vector-valued optimization problem if the constraints of the scalar problem are added as further elements to the vector of objectives of the vector-valued problem and are hence treated as additional objective functions. Chankong and Haimes [18] showed this result for the ϵ-constraint optimization problem. For a summary of this concept applied to more types of optimization problems, see [61]. In the following, we extend this result to several definitions of robust optimization problems. The results presented within this chapter are based on Köbis [62].

In the following we show that the set of weakly Pareto optimal elements of the multi-objective problem (RC_u) (see (4.1)) comprises all solutions of the weighted robust problem (wRC) (see (3.1)).

Theorem 12. *Consider the weighted robust optimization problem* (wRC) *(see* (3.1)*) and let* $w_j > 0,\ j = 1, \ldots, q,\ \mathcal{U} = \{\xi_1, \ldots, \xi_q\},\ f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$ *and* $F_i : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R},\ i = 1, \ldots, m$. *If* x^0 *is a solution of the weighted robust optimization problem* (wRC)*, then* x^0 *is weakly Pareto optimal for the unconstrained multi-objective optimization problem* (RC_u) *(see* (4.1)*).*

Proof. Let $x^0 \in \mathbb{R}^n$ be a solution of (wRC) (see (3.1)), that means

$$\forall\ k = 1, \ldots, q:\ w_k f(x^0, \xi_k) \le \max_{l=1,\ldots,q} w_l f(x^0, \xi_l) \le \max_{p=1,\ldots,q} w_p f(x, \xi_p) \tag{4.2}$$

for all $x \in \mathbb{R}^n$ that satisfy the constraints $F_i(x, \xi) \le 0$ for every $\xi \in \mathcal{U}$ and $i = 1, \ldots, m$. Suppose that x^0 is not weakly Pareto optimal for (RC_u) (see (4.1)), i.e., there exists $\overline{x} \in \mathbb{R}^n$ such that

$$\begin{pmatrix} f(\overline{x}, \xi_1) \\ \ldots \\ f(\overline{x}, \xi_q) \\ F_1(\overline{x}, \xi_1) \\ \ldots \\ F_m(\overline{x}, \xi_q) \end{pmatrix} < \begin{pmatrix} f(x^0, \xi_1) \\ \ldots \\ f(x^0, \xi_q) \\ F_1(x^0, \xi_1) \\ \ldots \\ F_m(x^0, \xi_q) \end{pmatrix}. \tag{4.3}$$

Because x^0 is feasible for (wRC), it follows for each $\xi \in \mathcal{U}:\ F_i(\overline{x}, \xi) < F_i(x^0, \xi) \le 0,\ i = 1, \ldots, m$, therefore $\overline{x}$ is feasible for (wRC). The first q inequalities in (4.3) are a contradiction to x^0 being optimal for (wRC), as stated in (4.2). □

Furthermore, we obtain the following result:

Theorem 13. *Consider the weighted robust optimization problem* (wRC) *(see* (3.1)*) and let* $w_j > 0,\ j = 1, \ldots, q,\ \mathcal{U} = \{\xi_1, \ldots, \xi_q\},\ f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$ *and* $F_i : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R},\ i = 1, \ldots, m$. *If* x^0 *is the unique solution of the weighted robust optimization problem* (wRC)*, then* x^0 *is Pareto optimal for the unconstrained multi-objective optimization problem* (RC_u) *(see* (4.1)*).*

Proof. Let $x^0 \in \mathbb{R}^n$ be the unique solution of (wRC) (see (3.1)), i.e., it holds

$$\forall\ k = 1, \ldots, q:\ w_k f(x^0, \xi_k) \le \max_{l=1,\ldots,q} w_l f(x^0, \xi_l) < \max_{p=1,\ldots,q} w_p f(x, \xi_p) \tag{4.4}$$

for all $x \in \mathbb{R}^n$ fulfilling the constraints for all $\xi \in \mathcal{U}:\ F_i(x, \xi) \le 0,\ i = 1, \ldots, m$. Note that the strict inequality in (4.4) follows from x^0 being the unique solution of problem (wRC). Suppose that x^0 is not Pareto optimal for (RC_u) (see (4.1)). Thus, there exists an $\overline{x} \in \mathbb{R}^n$ with

$$\begin{pmatrix} f(\overline{x}, \xi_1) \\ \dots \\ f(\overline{x}, \xi_q) \\ F_1(\overline{x}, \xi_1) \\ \dots \\ F_m(\overline{x}, \xi_q) \end{pmatrix} \leq \begin{pmatrix} f(x^0, \xi_1) \\ \dots \\ f(x^0, \xi_q) \\ F_1(x^0, \xi_1) \\ \dots \\ F_m(x^0, \xi_q) \end{pmatrix}. \tag{4.5}$$

Since x^0 is feasible for the weighted robust optimization problem (wRC), it follows together with (4.5) that for all $\xi \in \mathcal{U}: \ F_i(\overline{x}, \xi) \leq F_i(x^0, \xi) \leq 0, \ i = 1, \dots, m$. Thus, $\overline{x}$ is feasible for (wRC), too. Hence, we arrive at a contradiction to our assumption (4.4). □

Theorem 13 points out that the unique optimal solution of the weighted robust optimization problem (wRC) is even Pareto optimal for the multi-objective optimization problem (RC_u).

The set of weakly Pareto optimal solutions of (RC_u) (see (4.1)) includes also solutions of the deviation robust optimization problem (dRC) (see (3.8)), which is an assertion of the following theorem.

Theorem 14. *Recall that the deviation robust optimization problem (dRC) (see (3.8)) is defined for $\mathcal{U} = \{\xi_1, \dots, \xi_q\}$, $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$, $f^0(\xi) \in \mathbb{R}$ for all $\xi \in \mathcal{U}$ and $F_i : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$, $i = 1, \dots, m$. If x^0 is a (unique) solution of the deviation robust optimization problem (dRC), then x^0 is weakly Pareto optimal (Pareto optimal, respectively) for the unconstrained multi-objective optimization problem (RC_u) (see (4.1)).*

Proof. We first prove that a solution x^0 of (dRC) (see (3.8)) is weakly Pareto optimal for (RC_u) (see (4.1)). Let $x^0 \in \mathbb{R}^n$ be a solution of (dRC), that means

$$\forall\, \xi \in \mathcal{U}: \ f(x^0, \xi) - f^0(\xi) \leq \max_{\xi \in \mathcal{U}} (f(x^0, \xi) - f^0(\xi)) \leq \max_{\xi \in \mathcal{U}} (f(x, \xi) - f^0(\xi)) \tag{4.6}$$

for every $x \in \mathbb{R}^n$ that satisfies the constraints for all $\xi \in \mathcal{U}: \ F_i(x, \xi) \leq 0, \ i = 1, \dots, m$. Specifically, for every $x \in \mathbb{R}^n$ satisfying the above constraints, there is a $j_x \in \{1, \dots, q\}$ where

$$f(x^0, \xi_{j_x}) - f^0(\xi_{j_x}) \leq f(x, \xi_{j_x}) - f^0(\xi_{j_x}). \tag{4.7}$$

Suppose that x^0 is not weakly Pareto optimal for (RC_u). Then there is an $\overline{x} \in \mathbb{R}^n$ such that (4.3) is fulfilled. Because x^0 is feasible for (dRC), it follows for each $\xi \in \mathcal{U}: \ F_i(\overline{x}, \xi) < F_i(x^0, \xi) \leq 0, \ i = 1, \dots, m$, therefore $\overline{x}$ is feasible for (dRC). The first q inequalities in (4.3) are a contradiction to x^0's optimality for (dRC), as formulated by (4.6) and (4.7). In the following, we show that the unique solution x^0 of (dRC) is Pareto optimal for the multi-objective optimization problem (RC_u). Now let $x^0 \in \mathbb{R}^n$ be the unique solution of (dRC), i.e., for all $x \in \mathbb{R}^n$ that satisfy the constraints for all $\xi \in \mathcal{U}: \ F_i(x, \xi) \leq 0, \ i = 1, \dots, m$, it holds:

$$\forall\, \xi \in \mathcal{U}: \ f(x^0, \xi) - f^0(\xi) \leq \max_{\xi \in \mathcal{U}} (f(x^0, \xi) - f^0(\xi)) < \max_{\xi \in \mathcal{U}} (f(x, \xi) - f^0(\xi)). \tag{4.8}$$

The strict inequality in (4.8) follows from the uniqueness assumption. Similar to the first case, for every feasible $x \in \mathbb{R}^n$, there exists a $j_x \in \{1, \ldots, q\}$ with

$$f(x^0, \xi_{j_x}) - f^0(\xi_{j_x}) < f(x, \xi_{j_x}) - f^0(\xi_{j_x}). \tag{4.9}$$

Note that in this case, the inequality in (4.9) is strict. Now assume that x^0 is not Pareto optimal for the multi-objective optimization problem (RC_u), thus there exists an $\overline{x} \in \mathbb{R}^n$ such that (4.5) holds true. Because x^0 is feasible for (dRC), it follows for each $\xi \in \mathcal{U}: \ F_i(\overline{x}, \xi) \leq F_i(x^0, \xi) \leq 0, \ i = 1, \ldots, m$, hence $\overline{x}$ is also feasible for the deviation robust optimization problem (dRC). With (4.5), we arrive at a contradiction to the optimality assumption (4.8), specifically to (4.9). □

Similarly to the above results, here we can also conclude that the set of weakly Pareto optimal solutions of (RC_u) (see (4.1)) contains all solutions of the reliably robust optimization problem (rRC) (see (3.14)).

Theorem 15. *Consider the reliably robust optimization problem* (rRC) *(see* (3.14)*) for* $\mathcal{U} = \{\xi_1, \ldots, \xi_q\}$*,* $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$*,* $F_i : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$ *and* $\delta_i \in \mathbb{R}$*,* $i = 1, \ldots, m$*. If* x^0 *is a (unique) solution of the reliably robust optimization problem* (rRC)*, then* x^0 *is weakly Pareto optimal (Pareto optimal, respectively) for the unconstrained multi-objective optimization problem* (RC_u) *(see* (4.1)*).*

We do not prove Theorem 15 here because it is similar to proving Theorems 12 and 13, since the only difference between (wRC) and (rRC) is the set of feasible points and both objective functions of (wRC) and (rRC) coincide for $w_j = 1, \ j = 1, \ldots, q$.

It is also interesting to discover relations between the lightly robust optimization problem (lRC) (see (3.19)) and the multi-objective optimization problem (RC_u) (see (4.1)):

Theorem 16. *Recall that the lightly robust optimization problem* (lRC) *(see* (3.19)*) is formulated for* $\mathcal{U} = \{\xi_1, \ldots, \xi_q\}$*,* $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$*,* $F_i : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$*,* $i = 1, \ldots, m$*,* $\gamma \in \mathbb{R}_{\geqq}$*,* $z^0 \in \mathbb{R}$*,* $w_i \geq 0$*,* $i = 1, \ldots, m$*,* $\sum_{i=1}^m w_i = 1$*. If* $(x^0, \delta^0)^T$ *is a (unique) solution of the lightly robust optimization problem* (lRC)*, then* x^0 *is weakly Pareto optimal (Pareto optimal, respectively) for the unconstrained multi-objective optimization problem* (RC_u) *(see* (4.1)*).*

Proof. Let $(x^0, \delta^0)^T \in \mathbb{R}^n \times \mathbb{R}^m$ be a solution of (lRC) (see (3.19)), that means

$$\sum_{i=1}^{m} w_i \delta_i^0 \leq \sum_{i=1}^{m} w_i \delta_i \tag{4.10}$$

for all $(x, \delta)^T \in \mathbb{R}^n \times \mathbb{R}^m$ that satisfy the constraints given in the definition of problem (lRC). Suppose that x^0 is not weakly Pareto optimal for (RC_u) (see (4.1)). Then there is an $\overline{x} \in \mathbb{R}^n$ such that (4.3) is fulfilled. Because x^0 is feasible for (lRC), it follows

$$F_i(\overline{x}, \hat{\xi}) < F_i(x^0, \hat{\xi}) \leq 0, \tag{4.11}$$

$$f(\overline{x}, \hat{\xi}) < f(x^0, \hat{\xi}) \leq (1+\gamma)z^0. \tag{4.12}$$

Because $w_i \geq 0$ for every $i = 1, \ldots, m$ and not all w_i are zero, we have for every $\xi \in \mathcal{U}: \ \sum_{i=1}^m w_i F_i(\overline{x}, \xi) \overset{(4.3)}{<} \sum_{i=1}^m w_i F_i(x^0, \xi) \leq \sum_{i=1}^m w_i \delta_i^0$ (the second inequality follows from $(x^0, \delta^0)^T$ being feasible for (lRC)).
Setting $\bar{\delta}_i := \max_{\xi \in \mathcal{U}} F_i(\overline{x}, \xi), \ i = 1, \ldots, m$, we have for all $\xi \in \mathcal{U}: \ F_i(\overline{x}, \xi) \leq \bar{\delta}_i, \ i = 1, \ldots, m$. Together with (4.11) and (4.12), it follows that $(\overline{x}, \bar{\delta})^T$ is feasible for (lRC). Thus we have found an element $(\overline{x}, \bar{\delta})^T$ contradicting $(x^0, \delta^0)^T$'s optimality as stated in (4.10). For a unique solution of (lRC), the proof is similar and therefore omitted. □

Theorem 16 above shows that the set of weakly Pareto optimal solutions of the unconstrained multi-objective optimization problem (RC_u) (see (4.1)) includes all solutions of the lightly robust optimization problem (lRC) (see (3.19)). The unique solution of problem (lRC) is even Pareto optimal for (RC_u). In the next theorem, it is shown that the set of weakly Pareto optimal solutions of (RC_u) contains all solutions of the ϵ-constraint robust optimization problem (ϵRC) (see (3.31)), whereas the unique solution of (ϵRC) even belongs to the set of Pareto optimal points of (RC_u).

Theorem 17. *Consider the ϵ-constraint robust optimization problem (ϵRC) (see (3.31)). Let $\mathcal{U} = \{\xi_1, \ldots, \xi_q\}, \ f: \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}, \ F_i: \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}, \ i = 1, \ldots, m$, and fix $j \in \{1, \ldots, q\}, \ \epsilon_k \in \mathbb{R}, \ k \in \{1, \ldots, q\}, \ k \neq j$. If x^0 is a (unique) solution of the ϵ-constraint robust optimization problem (ϵRC), then x^0 is weakly Pareto optimal (Pareto optimal, respectively) for the unconstrained multi-objective optimization problem (RC_u) (see (4.1)).*

Proof. Let $x^0 \in \mathbb{R}^n$ be a solution of (ϵRC) (see (3.31)), that means for a fixed chosen index $j \in \{1, \ldots, q\}$

$$f(x^0, \xi_j) \leq f(x, \xi_j) \tag{4.13}$$

for all $x \in \mathbb{R}^n$ that fulfill the constraints that were defined by problem (ϵRC). Suppose that there is an $\overline{x} \in \mathbb{R}^n$ that satisfies (4.3). Because x^0 is feasible for (ϵRC), with (4.3) follows feasibility of $\overline{x}$ for problem (ϵRC). Obviously, from (4.3) we get a contradiction to (4.13). Equivalently, one can prove the second part of the theorem. □

Thus, we have shown that the set of weakly Pareto optimal solutions of the unconstrained multi-objective optimization problem (RC_u) (see (4.1)) comprises all solutions of the concepts of robustness that were introduced in Section 3, provided that the uncertainty set $\mathcal{U}$ is discrete. The unique optimal point of each robust optimization problem even belongs to the set of Pareto optimal solutions. These results provide insight into interrelations between robust optimization problems and the multi-objective optimization problem (RC_u) which can facilitate the process when a decision maker is aiming for a choice between different robust solutions. It can also be interpreted as an approach to develop new concepts for robustness whose solution algorithms can be directly deducted from the scalarization results presented within this chapter.

So far, we have shown that solutions of scalar robust optimization problems also belong to the set of weakly Pareto optimal solutions of the unconstrained multi-objective optimization problem (RC_u). Unfortunately, the other direction is, in general, not true: A weakly Pareto optimal solution of (RC_u) is not always a solution to one of the introduced scalar robust optimization problems: Since the constraints of the scalar robust optimization problems are shifted to the vector of objectives in (RC_u), a point x^0 may minimize one constraint and would be weakly Pareto optimal for (RC_u), regardless of the function value $f(x^0, \xi)$. We can, however, derive relations between a *constrained* multi-objective optimization problem and a scalar robust optimization problem that differs a bit from the weighted robust optimization problem (wRC) (see (3.1)). In order to introduce these problems, we need some additional definitions.

Definition 3. *We call a vector* $y^I = (y_1^I, \dots, y_q^I)^T$ *with*

$$y_k^I := \min_{x \in \mathbb{R}^n} f(x, \xi_k),\ k = 1, \dots, q,$$

an ideal point *of the problem*

$$\operatorname{Min}(f[\mathbb{R}^n], \mathbb{R}^q_{\geq}),$$

where $f(x) := (f(x, \xi_1), \dots, f(x, \xi_q))^T$. *The vector* $y^U = (y_1^I - \epsilon_1, \dots, y_q^I - \epsilon_q)^T,\ \epsilon_k > 0,\ k = 1, \dots, q,$ *is called a* utopia point*, compare [24].*

Set $\lambda_k > 0$ for all $k = 1, \dots, q$. Now we introduce the problem

$$(P_\lambda) \qquad \begin{aligned} \min \max_{k=1,\dots,q}\ & \lambda_k (f(x, \xi_k) - y_k^U) \\ \text{s.t.}\ & \forall\, \xi \in \mathcal{U}:\ F_i(x, \xi) \leq 0,\ i = 1, \dots, m, \\ & x \in \mathbb{R}^n. \end{aligned} \tag{4.14}$$

Furthermore, let the *constrained* multi-objective optimization problem

$$(RC') \qquad \operatorname{Min}(f[\mathfrak{A}], \mathbb{R}^q_{\geq}) \tag{4.15}$$

with $\mathfrak{A} := \{x \in \mathbb{R}^n | \forall\, \xi \in \mathcal{U}:\ F_i(x, \xi) \leq 0,\ i = 1, \dots, m\}$ (see (3.2)) and $f(x) = (f(x, \xi_1), \dots, f(x, \xi_q))^T$ be given, compare (3.35), the multiple objective strictly robust counterpart.

Now we extend a theorem proposed by Ehrgott [24, Proposition 5.10] to the concept of robustness. The following theorem points out that a solution x^0 is weakly Pareto optimal for the multi-objective optimization problem (RC') (see (4.15)) if and only if we can find a $\lambda \in \mathbb{R}^q_{>}$ such that x^0 is a solution of the problem (P_λ) (see (4.14)). Note that we do not need any convexity assumptions on f.

Theorem 18. *Let* $\mathcal{U} = \{\xi_1, \dots, \xi_q\},\ f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$ *and* $F_i : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R},\ i = 1, \dots, m.$ x^0 *is weakly Pareto optimal for* (RC') *(see* (4.15)*) if and only if there exists a* $\lambda \in \mathbb{R}^q_{>}$ *such that* x^0 *solves* (P_λ) *(see* (4.14)*).*

Proof. $\Rightarrow$ Let x^0 be weakly Pareto optimal for (RC') (see (4.15)). Let further the weights be given by $\lambda_k = \frac{1}{f(x^0, \xi_k) - y_k^U},\ k = 1, \dots, q$. We have that $\lambda_k > 0,\ k =$

$1, \dots, q$, because $f(x^0, \xi_k) > y_k^U$. Suppose that x^0 is not optimal for (P_λ) (see (4.14)). Then there is $\overline{x} \in \mathbb{R}^n$ that satisfies the constraints $F_i(\overline{x}, \xi) \leq 0$ for every $\xi \in \mathcal{U}$ and $i = 1, \dots, m$ and

$$\begin{aligned}\max_{k=1,\dots,q} \lambda_k(f(\overline{x}, \xi_k) - y_k^U) &< \max_{k=1,\dots,q} \lambda_k(f(x^0, \xi_k) - y_k^U) \\ &= \max_{k=1,\dots,q} \frac{1}{f(x^0, \xi_k) - y_k^U}(f(x^0, \xi_k) - y_k^U).\end{aligned}$$

Thus, we have for all $k = 1, \dots, q: \; \lambda_k(f(\overline{x}, \xi_k) - y_k^U) < 1$. Dividing by $\lambda_k > 0$, we get for all $k = 1, \dots, q: \; f(\overline{x}, \xi_k) - y_k^U < f(x^0, \xi_k) - y_k^U$, and therefore $f(\overline{x}, \xi_k) < f(x^0, \xi_k)$ for every $k = 1, \dots, q$, contradicting x^0's weak Pareto optimality.

$\Leftarrow$ Let now $x^0 \in \mathbb{R}^n$ solve (P_λ), i.e.,

$$\max_{k=1,\dots,q} \lambda_k(f(x^0, \xi_k) - y_k^U) \leq \max_{k=1,\dots,q} \lambda_k(f(x, \xi_k) - y_k^U) \tag{4.16}$$

for all $x \in \mathbb{R}^n$ satisfying the constraints $F_i(x, \xi) \leq 0$ for every $\xi \in \mathcal{U}$ and $i = 1, \dots, m$. Assume x^0 is not weakly Pareto optimal for (RC'). Then there is an $\overline{x} \in \mathbb{R}^n$ that fulfills the constraints for all $\xi \in \mathcal{U}: \; F_i(\overline{x}, \xi) \leq 0, \; i = 1, \dots, m$ and

$$\forall\, k = 1, \dots, q: \; f(\overline{x}, \xi_k) < f(x^0, \xi_k). \tag{4.17}$$

That contradicts (4.16), because there we stated that for every feasible $x \in \mathbb{R}^n$ there exists a $j_x \in \{1, \dots, q\}$ such that

$$\begin{aligned}\max_{k=1,\dots,q} \lambda_k(f(x^0, \xi_k) - y_k^U) &\overset{(4.16)}{\leq} \max_{k=1,\dots,q} \lambda_k(f(x, \xi_k) - y_k^U) \\ &=: \lambda_{j_x}(f(x, \xi_{j_x}) - y_{j_x}^U).\end{aligned}$$

Furthermore, it holds $\lambda_{j_x}(f(x^0, \xi_{j_x}) - y_{j_x}^U) \leq \max_{k=1,\dots,q} \lambda_k(f(x^0, \xi_k) - y_k^U)$, and this yields $f(x^0, \xi_{j_x}) \leq f(x, \xi_{j_x})$ for every feasible $x \in \mathbb{R}^n$ and a $j_x \in \{1, \dots, q\}$, a contradiction to (4.17).

□

The next theorem is a generalization of [24, Proposition 5.11]. It shows that, assuming that the set of optimal solutions of problem (wRC) (see (3.1)) is not empty, it always contains Pareto optimal solutions of the unconstrained multi-objective optimization problem (RC_u) (see (4.1)). Note that since the strictly robust optimization problem is a special case of (wRC) with $w_k = 1, \; k = 1, \dots, q$, this theorem can also be formulated for a strictly robust optimization problem.

Theorem 19. *Let* $\mathcal{U} = \{\xi_1, \dots, \xi_q\}, \; f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}, \; F_i : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}, \; i = 1, \dots, m$ *and* $w_k > 0, \; k = 1, \dots, q$. *Suppose that the set of Pareto optimal solutions of problem* (RC_u) *(see* (4.1)*) is externally stable and there exists a solution of the optimization problem* (wRC) *(see* (3.1)*). Then it holds:*

$$\{x^0 | x^0 \textit{ solves } (wRC)\} \cap \{x^0 | x^0 \textit{ is Pareto optimal for } (RC_u)\} \neq \emptyset.$$

Proof. Let $x^0 \in \mathbb{R}^n$ solve(wRC) (see (3.1)), i.e.,

$$\max_{k=1,\dots,q} w_k f(x^0, \xi_k) \leq \max_{k=1,\dots,q} w_k f(x, \xi_k) \tag{4.18}$$

for every $x \in \mathbb{R}^n$ that fulfills $F_i(x, \xi) \leq 0$ for all $\xi \in \mathcal{U}$ and $i = 1, \dots, m$.

- Case 1: x^0 is Pareto optimal for (RC_u) (compare (4.1)), then there is nothing to prove.
- Case 2: x^0 is not Pareto optimal for (RC_u). Because the set of Pareto optimal solutions of problem (RC_u) is externally stable, there exists an $\overline{x} \in \mathbb{R}^n$ that belongs to the set of Pareto optimal solutions of (4.1) and

$$\forall\ k = 1, \dots, q: \ f(\overline{x}, \xi_k) \leq f(x^0, \xi_k), \tag{4.19}$$

$$\forall\ k = 1, \dots, q,\ \forall\ i = 1, \dots, m: \ F_i(\overline{x}, \xi_k) \leq F_i(x^0, \xi_k). \tag{4.20}$$

  Because x^0 is feasible for (wRC), (4.20) implies that $\overline{x} \in \mathbb{R}^n$ is feasible for (wRC), too. Furthermore, from (4.19), we acquire

$$\max_{k=1,\dots,q} w_k f(\overline{x}, \xi_k) \leq \max_{k=1,\dots,q} w_k f(x^0, \xi_k). \tag{4.21}$$

  Because of (4.18) we have equality in (4.21), i.e., $\overline{x}$ is optimal for (wRC) and $\overline{x}$ is Pareto optimal for (RC_u).

$\square$

The next theorem shows that the intersection of the set of solutions of problem (wRC) (see (3.1)) and the set of Pareto optimal solutions of the *constrained* optimization problem (RC') (see (4.15)) is nonempty, assuming that a solution of (wRC) exists and that the set of Pareto optimal solutions of (RC') is externally stable. The proof is quite similar to proving Theorem 19 and is skipped here for the sake of shortness.

Theorem 20. *Let $\mathcal{U} = \{\xi_1, \dots, \xi_q\}$, $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$, $F_i : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$, $i = 1, \dots, m$ and $w_k > 0$, $k = 1, \dots, q$. Suppose that the set of Pareto optimal solutions of problem (RC') (compare (4.15)) is externally stable and there exists a solution of the optimization problem (wRC) (see (3.1)). Then it holds:*

$$\{x^0 | x^0 \text{ solves } (wRC)\} \cap \{x^0 | x^0 \text{ is Pareto optimal for } (RC')\} \neq \emptyset.$$

Below we provide a sufficient condition for an optimal element of the strictly robust optimization problem (wRC) (i.e., (wRC) with $w_j = 1$, $j = 1, \dots, q$).

Theorem 21. *Let $\mathcal{U} = \{\xi_1, \dots, \xi_q\}$, $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$ and $F_i : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$, $i = 1, \dots, m$. If for an x^0 with $f(x^0, \xi_r) = y_r^I$ and for all $\xi \in \mathcal{U} : F_i(x^0, \xi) \leq 0$, $i = 1, \dots, m$, it holds*

$$f(x^0, \xi_k) \leq y_r^I \text{ for all } k = 1, \dots, q,$$

then x^0 is an optimal solution of the strictly robust optimization problem (i.e., (wRC), (3.1), with $w_j = 1$, $j = 1, \dots, q$) with objective value y_r^I.

Proof. It holds

$$y_r^I = f(x^0, \xi_r) \stackrel{\text{Def. } 3}{=} \min_{x \in \mathbb{R}^n} f(x, \xi_r) \leq \min_{x \in \mathbb{R}^n} \max_{k=1,\ldots,q} f(x, \xi_k) \leq \max_{k=1,\ldots,q} f(x^0, \xi_k). \tag{4.22}$$

Because $f(x^0, \xi_k) \leq y_r^I$ for all $k = 1, \ldots, q$, it follows

$$\max_{k=1,\ldots,q} f(x^0, \xi_k) \leq y_r^I. \tag{4.23}$$

Due to (4.22) we have equality in (4.23), i.e.,

$$y_r^I = f(x^0, \xi_r) = \min_{x \in \mathbb{R}^n} \max_{k=1,\ldots,q} f(x, \xi_k).$$

□

We conclude this chapter with an example concerning robust linear programming for norm-bounded sets of uncertainties.

Example 2. *Let the set of uncertainties be given by* $\mathcal{U} := \{\xi \in \mathbb{R}^N | \ \|\xi\|_\infty \leq 1\}$ *with* $\|\xi\|_\infty := \max_{i=1,\ldots,N} |\xi_i|$. *Note that* $\mathcal{U}$ *is compact in this setting, but as we have already see in Section 3.2.3, for certain classes, this situation may be reduced to the finite case. Now consider the strictly robust optimization problem*

$$(LP) \qquad \min_{x \in \mathcal{X} \subseteq \mathbb{R}^n} \sup_{\xi \in \mathcal{U}} c(\xi)^T x \tag{4.24}$$

with $c_k \in \mathbb{R}^n, \ k = 0, \ldots, N,$ *and* $c(\xi)^T x := c_0^T x + \sum_{k=1}^N \xi_k c_k^T x$ *for* $\xi \in \mathcal{U}$.

Then the objective in (LP) *can be expressed by the scalarizing functional* z^{B_7,k_7} *for* $k_7 \equiv 1, \ B = B_7, \ y = c(\xi)^T x$*:*

$$\begin{aligned} z^{B_7,k_7} = \inf\{t \in \mathbb{R} | y \in tk_7 - B_7\} &= \inf\{t \in \mathbb{R} | y - tk_7 \in -B_7\} \\ &= \inf\{t \in \mathbb{R} | c(\xi)^T x - tk_7 \in -B_7\} \\ &= \inf\{t \in \mathbb{R} | \forall \ \xi \in \mathcal{U} : \ c(\xi)^T x \leq t\} \\ &= \sup_{\xi \in \mathcal{U}} c(\xi)^T x. \end{aligned} \tag{4.25}$$

We already mentioned that z^{B_7,k_7} *is* B_7*-monotone and strictly* $(\operatorname{int} B_7)$*-monotone in Remark 13.*

Furthermore, the detailed examinations in [8, Example 1.3.2] inspire us to simplify formulation (4.25)*:*

$$\begin{aligned}
&\inf\{t \in \mathbb{R} | \forall\ \xi \in \mathcal{U}:\ c(\xi)^T x \le t\} = \inf\{t \in \mathbb{R} | \forall\ \xi \in \mathcal{U}:\ c_0^T x + \sum_{k=1}^{N} c_k^T x \xi_k \le t\} \\
&= \inf\{t \in \mathbb{R} | \max_{\xi \in \mathcal{U}} (c_0^T x + \sum_{k=1}^{N} c_k^T x \xi_k) \le t\} \\
&= \inf\{t \in \mathbb{R} | c_0^T x + \sum_{k=1}^{N} |c_k^T x| \le t\} \\
&= \inf\{t \in \mathbb{R} | \exists\ v_i \in \mathbb{R},\ i = 1, \dots, N:\ -v_i - \sum_{k=1}^{N} c_k^T x \le 0, \\
&\qquad -v_i + \sum_{k=1}^{N} c_k^T x \le 0,\ c_0^T x - \sum_{i=1}^{N} v_i \le t,\ i = 1, \dots, N\} \\
&= \{c_0^T x - \sum_{i=1}^{N} v_i | -v_i - \sum_{k=1}^{N} c_k^T x \le 0,\ -v_i + \sum_{k=1}^{N} c_k^T x \le 0,\ i = 1, \dots, N\}.
\end{aligned}$$

That means the objective function in (LP) can be reduced to a linear optimization problem. Consequently, we arrive at the following conclusion: If x^0 is a solution of (4.24), *then there is a vector $v := (v_1, \dots, v_N)^T \in \mathbb{R}^N$ such that $g(x^0, v) \in \mathrm{Min}(g[\mathcal{X} \times \mathbb{R}^N], \mathbb{R}^{2N+1}_{>})$ with*

$$g(x, v) = \begin{pmatrix} g_1(x, v) \\ \dots \\ g_{2N+1}(x, v) \end{pmatrix} := \begin{pmatrix} c_0^T x - \sum_{i=1}^{N} v_i \\ -v_1 - \sum_{k=1}^{N} c_k^T x \\ -v_1 + \sum_{k=1}^{N} c_k^T x \\ \dots \\ -v_N - \sum_{k=1}^{N} c_k^T x \\ -v_N + \sum_{k=1}^{N} c_k^T x \end{pmatrix}.$$

The research conducted in this chapter shows that many well known results from deterministic optimization can be extended to robust optimization problems, assuming that the set of uncertainties is discrete. The above example suggests future research that consists of investigating the case where the uncertainty set $\mathcal{U}$ takes a compact form.

Chapter 5

Robust Approaches to Vector Optimization

5.1 Literature Review

In many optimization problems, there exist conflicting goals which have to be optimized at the same time. As a prominent example, the classical Markowitz problem consists of minimizing risk and maximizing expected returns of financial assets [74]. These objectives are clearly conflicting. Another example is concerned with tumor radiation treatment, in which the goals are to expose the tumor to as much radiation as is needed to achieve best medical results, but at the same time to ensure that the surrounding organs are not harmed, cf. [26, Chapter 6]. Many more examples can be found in game theory, where two players are trying to optimize their individual objectives and have to reach an agreement, see, for instance, [76]. Hence, to gain realistic insights into a problem in a complex surrounding, contrary objectives play an important role and are thus intensely studied in optimization. In multi-objective optimization one is concerned with comparing elements and one searches for solutions that are not dominated by another element of the set of feasible solutions with respect to a set or a cone. Similar to scalar optimization, many complex multi-objective problems are contaminated with uncertain data. Naturally, the issue of incorporating uncertainties in multi-objective optimization is an important task that needs to be addressed. Therefore, being able to model uncertain vector-valued optimization problems and define robust solutions thereof would be very valuable.

This chapter is devoted to developing solution concepts for uncertain multi-objective optimization problems, specifically, our goal is to obtain *robust* solutions. Only a few approaches to uncertain vector optimization have been mentioned in the literature, of which we briefly summarize the following. Hughes [46] presented a first concept of dealing with uncertain multi-objective optimization by computing the expected value of the errors that occur in the objective functions. The vector of expected errors is then used in the classical concept of Pareto optimality. Teich [96] generalized the concept of Pareto optimality in a probabilistic nature for uncertain vector-valued problems where the objective values are constrained by intervals. Another idea was presented by Li et al. [71]

who develop solution procedures that compare the performance of solutions regarding optimality and its robustness. They propose a biobjective optimization problem, one of the objective functions being a fitness value and the other one containing a robustness index. The considered method in [71] may be beneficial for obtaining solutions that satisfy certain optimality and robustness criteria, and a decision maker may choose depending on his preferences toward uncertainty. Another approach was presented by Deb and Gupta [21] who used an idea by Branke [16], and defined robustness as a kind of sensitivity against perturbations in the decision-space. Branke [16] proposes to replace the objective function f by its mean function $\overline{f}$ which maps any point x to the average function value in a pre-defined neighborhood of x. A minimizer of $\overline{f}$ is then more robust in the sense that the function values in its neighborhood do not change too much. Based on this idea for single objective optimization problems, Deb and Gupta [21] introduced two concepts of robustness for vector-valued optimization problems. The first one replaces all objective functions by their mean functions. Efficient solutions to the resulting optimization problem are called robust solutions of the original problem. Deb and Gupta's second concept minimizes the original objective functions but adds constraints to the problem that restrict the variation between the original objective functions and a perturbed function value (that can be chosen as their mean functions) to a pre-defined limit. This approach proves to be more pragmatic and enables the user to control the desired level of robustness.

Barrico and Antunes [2, 4] consider a multi-objective optimization problem with perturbations in the decision space . In [2, 4], a solution is called *robust* if small perturbations in the decision-space only yield small perturbations in the objective-space. The authors in [2, 4] define a *degree of robustness* that allows the decision maker to specify the level of robustness of the solution. Specifically, the user is able to determine the size of the neighborhood that the solution belongs to. Furthermore, Barrico and Antunes [3] extend the concept of *degree of robustness* to the space of the objective function coefficients, where perturbations are treated in a similar manner as in [2, 4]. For more results on this line of research, compare [41, 28].

The first scenario-based approach to uncertain vector-valued problems was introduced by Kuroiwa and Lee [68] who directly transferred the main idea of scalar robust optimization, meaning minimizing the worst-case objective function, to a multicriteria setting. For an uncertain objective function $f : \mathbb{R}^n \supseteq \mathcal{X} \times \mathcal{U} \to \mathbb{R}^k$, where $\mathcal{U} \subseteq \mathbb{R}^N$ is a given uncertainty set, Kuroiwa and Lee [68] introduce a multi-objective problem

$$\mathrm{Min}(h[\mathcal{X}], \mathbb{R}^k_{\geq}) \tag{5.1}$$

with

$$h(x) := \begin{pmatrix} \max_{\xi \in \mathcal{U}} f_1(x, \xi) \\ \dots \\ \max_{\xi \in \mathcal{U}} f_k(x, \xi) \end{pmatrix}$$

and call (weakly) Pareto optimal solution of problem (5.1) (weakly) robust efficient. The special case for convex functions $f_i,\ i = 1, \dots, k$, is studied in [69]. Because this approach is a rather direct transferral from scalar robust optimization, it may not be sufficient to describe robust solutions of multi-objective optimization problems. To verify this aspect, consider the following small example in Figure 5.1: First we define for $x \in \mathcal{X}$

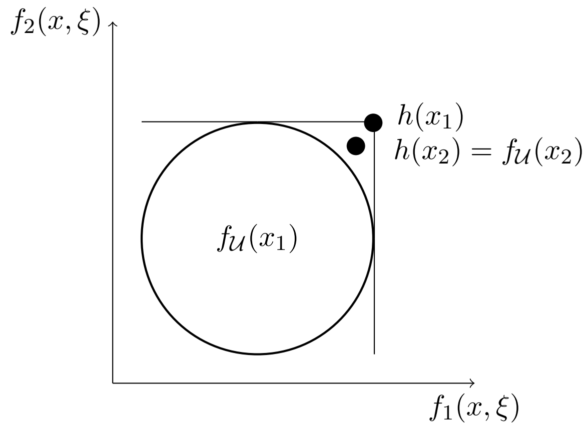

Figure 5.1: Here, x_1 is not weakly Pareto optimal for problem (5.1), while x_2 is Pareto optimal for (5.1).

$f_{\mathcal{U}}(x) := \{f(x,\xi)|\ \xi \in \mathcal{U}\}$. Here, the point $h(x_1)$ is dominated by $h(x_2) = f_{\mathcal{U}}(x_2)$ which, for simplification, is just one point. Hence, x_1 would not be considered robust efficient by the authors in [68], but x_2 is robust efficient. The issue clearly arises because the point $h(x_1)$ does not belong to $f_{\mathcal{U}}(x_1)$. Problem (5.1) still is beneficial and was recently used by Ehrgott et al. [25] to obtain solutions that they call robust in a slightly different setting. The authors in [25] generalize the above approach from Kuroiwa and Lee [68] by considering the whole set that is obtained when analyzing a possible solution x. They call a solution x^0 robust efficient if its set $f_{\mathcal{U}}(x^0)$ is not dominated by any other set $f_{\mathcal{U}}(x)$. For the above example in Figure 5.1 this means that both x_1 and x_2 would be considered robust efficient. The authors in [25] observe that (weakly) Pareto efficient solutions of the above problem (5.1) are also (weakly) robust efficient solutions within their definition of robust efficiency, and the reverse implication holds under the requirement that the uncertainty set takes the form $\mathcal{U} := \mathcal{U}_1 \times \ldots \times \mathcal{U}_k$, i.e., if the uncertainties are independent of each other. The robustness concept introduced in [25] implicitly uses a set order relation to compare solution sets. We will show in this chapter that this approach is closely connected to set optimization, because the objective function considered here is set-valued.

In the literature, two main ways of treating a set-valued problem are reported: Using a vector concept, one wishes to obtain single elements that satisfy a certain efficiency condition (possibly similar to Pareto optimality) for the union of all sets in the objective space. Since having one element that is optimal in some sense does not reveal any information about the performance of the remaining elements in that particular solution set, it can be argued that this approach may not be useful enough in practical applications. The second concept deals with obtaining solution sets out of all possible sets in the objective space. The authors in [25] use the latter approach to define robust solutions to an uncertain multi-objective optimization problem, i.e., every robust solution of that problem is a set.

This chapter is organized as follows: In Section 5.2, we recall set order relations that are well known from set-valued optimization. Section 5.3 is concerned with providing new robustness concepts for uncertain multi-objective optimization problems using the before mentioned set relations. We will see that these order relations are useful to describe *strictly* robust elements with respect to a certain order relation. In order to model (weak, · , respectively) robust solution concepts as well, we will extend the mentioned set order relations that have thus far been introduced for cones to more general sets. For each introduced concept, algorithms for obtaining robust solutions will be provided. We will consider a wide spectrum of robust approaches to uncertain vector-valued optimization, since every real-world problem possesses its own particular requirements and a generic concept needs to adapt to different circumstances.

Sections 5.2, 5.3.1, 5.3.2, 5.3.3 and 5.3.4 are based on Ide, Köbis, Kuroiwa, Schöbel and Tammer [49]. Section 5.3.6 is based on Ide, Köbis [48].

5.2 Set Order Relations

Uncertain vector optimization and set-valued optimization are closely related, as further discussion in Section 5.3 will show. In order to approach uncertain vector-valued optimization, set order relations known from set-valued optimization (compare Jahn [53], Jahn, Ha [55], and Eichfelder, Jahn [27]) will be used. In order to be able to formulate solution concepts for new definitions of robustness for uncertain vector optimization, we mention several known set order relations in this section and some of their properties and interrelations.

Here, let Y be a linear topological space, partially ordered by a proper pointed convex closed cone C, X is a linear space, $G : X \rightrightarrows Y$ is a set-valued mapping, $\mathcal{X}$ is a subset of X. In contrast to the problems discussed before, the objective values of $x \in \mathcal{X}$ are now sets $G(x)$. The graph of the set-valued map G is denoted by $\operatorname{graph} G := \{(x, y) \in X \times Y \mid y \in G(x)\}$. We denote by

$$G(\mathcal{X}) := \bigcup_{x \in \mathcal{X}} G(x)$$

the image set of G.

Our goal is to minimize $G : X \rightrightarrows Y$ and we note the problem

$$(SetP) \qquad \min_{x \in \mathcal{X}} G(x), \tag{5.2}$$

where "min" denotes a solution concept from set-valued optimization that we will discuss in the following. There are three approaches to dealing with $(SetP)$ discussed in the literature: Using a *vector approach* leads to solutions that are not dominated by any other elements of the image set $G(\mathcal{X})$, compare Definition 4 below. A *set approach* uses whole sets $G(x)$ that are compared with respect to a set order relation, compare Definition 13 below. A third concept called *lattice approach* to the set-valued problem $(SetP)$ is investigated in [58].

Using the notation from Definition 2, formula (2.4), we introduce a first solution concept based on a vector approach below.

Definition 4 (Minimizer of $(SetP)$)**.** *Let $x \in \mathcal{X}$ and $(x, y) \in \operatorname{graph} G$. The pair $(x, y) \in \operatorname{graph} G$ is called a* minimizer *of the problem $(SetP)$ if $y \in \operatorname{Min}\big(G(\mathcal{X}), C \setminus \{0\}\big)$.*

The above definition of a minimizer describes a vector approach to solving a set-valued optimization problem $(SetP)$. Generally, one pair $(x, y) \in \operatorname{graph} G$ being a minimizer does not imply that the whole set $G(x)$ is minimal in some sense when compared to other sets $G(\overline{x})$, $\overline{x} \in \mathcal{X}$ (cf. Jahn, Ha [55], Eichfelder, Jahn [27]). Therefore, we will consider a set approach when dealing with a set-valued optimization problem. That way, the problem consists of finding sets $G(x)$ for $x \in \mathcal{X}$ that are not dominated by another set $G(\overline{x})$, $\overline{x} \in \mathcal{X}$, with respect to a set order relation. There are many set order relations known in the literature that can be used to compare sets. In order to discuss several set order relations, let us first mention a relation $\leq_Q$ with respect to a nonempty set $Q \subset Y$ that satisfies $\operatorname{cl} Q \cap (-\operatorname{cl} Q) = \{0\}$. We define

$$a \leq_Q b \;:\Longleftrightarrow\; a \in b - Q \tag{5.3}$$

for $a, b \in Y$.

We will introduce several order relations that are used to formulate corresponding solution concepts for the set-valued problem $(SetP)$ (see (5.2)).

We call

$$\mathcal{P}(Y) := \{A \subseteq Y \mid A \text{ is nonempty}\}$$

the power set of Y, which was introduced by Hamel [43], see also Jahn, Ha [55] and Eichfelder, Jahn [27].

Definition 5. *Let $A, B, D \in Y$ be arbitrarily chosen sets and let $\preceq$ be a binary relation. $\preceq$ is* ***reflexive*** *if $A \preceq A$. Furthermore, $\preceq$ is* ***transitive*** *if $A \preceq B$ and $B \preceq D$ implies that $A \preceq D$. The binary relation $\preceq$ is a* ***pre-order*** *if $\preceq$ is reflexive and transitive.*

Young [100] and Nishnianidze [77] independently introduced the set less order relation $\preceq^s_C$ for the comparison of sets (cf. Eichfelder, Jahn [27]):

Definition 6 (Set less order relation, [100, 77])**.** *Let $C \subset Y$ be a proper closed convex and pointed cone. Furthermore, let $A, B \in \mathcal{P}(Y)$ be arbitrarily chosen sets. Then the* ***set less order relation*** *$\preceq^s_C$ is defined by*

$$\begin{aligned} A \preceq^s_C B :\Longleftrightarrow\;& A \subseteq B - C \text{ and } A + C \supseteq B \\ \Longleftrightarrow\;& (\forall\, a \in A\ \exists\, b \in B : a \leq_C b) \text{ and } (\forall\, b \in B\ \exists\, a \in A : a \leq_C b). \end{aligned}$$

Kuroiwa [67, 65] introduced the following order relations:

Definition 7 (Lower (upper) set less order relation, [67, 65])**.** *Let $C \subset Y$ be a proper closed convex and pointed cone. Furthermore, let $A, B \in \mathcal{P}(Y)$ be arbitrarily chosen sets. Then the **lower set less order relation** $\preceq^l_C$ is defined by*

$$A \preceq^l_C B :\Longleftrightarrow A + C \supseteq B \Longleftrightarrow \forall\, b \in B\ \exists\, a \in A : a \leq_C b$$

*and the **upper set less order relation** $\preceq^u_C$ is defined by*

$$A \preceq^u_C B :\Longleftrightarrow A \subseteq B - C \Longleftrightarrow \forall\, a \in A\ \exists\, b \in B : a \leq_C b.$$

Remark 15. *There is the following relationship between the lower set less order relation $\preceq^l_C$ and the upper set less order relation $\preceq^u_{-C}$:*

$$A \preceq^l_C B :\Longleftrightarrow A + C \supseteq B \Longleftrightarrow B \subseteq A - (-C) \Longleftrightarrow: B \preceq^u_{-C} A.$$

Remark 16. *The following connection between the set less order relation $\preceq^s_C$ and the lower (upper, respectively) set less order relation $\preceq^l_C$ ($\preceq^u_C$, respectively) may be observed:*

$$A \preceq^s_C B \Longrightarrow A \preceq^l_C B,$$
$$A \preceq^s_C B \Longrightarrow A \preceq^u_C B.$$

Remark 17. *Obviously $A \preceq^l_C B$ is equivalent to*

$$A + C \supseteq B + C.$$

Furthermore, note that $A \preceq^u_C B$ is equivalent to

$$A - C \subseteq B - C.$$

It is important to mention that

$$A \preceq^l_C B \text{ and } B \preceq^l_C A \Longleftrightarrow A + C = B + C,$$

compare [55, Proposition 3.1]. Under our assumption that C is a pointed closed convex cone it holds $\mathrm{Min}(A+C, C\setminus\{0\}) = \mathrm{Min}(A, C\setminus\{0\})$ and $\mathrm{Min}(B+C, C\setminus\{0\}) = \mathrm{Min}(B, C\setminus\{0\})$, which implies

$$A \preceq^l_C B \text{ and } B \preceq^l_C A \Longrightarrow \mathrm{Min}(A, C\setminus\{0\}) = \mathrm{Min}(B, C\setminus\{0\}).$$

Under the additional assumptions $A \subset \mathrm{Min}(A, C\setminus\{0\}) + C$ and $B \subset \mathrm{Min}(B, C\setminus\{0\}) + C$, we have

$$\mathrm{Min}(A, C\setminus\{0\}) = \mathrm{Min}(B, C\setminus\{0\}) \Longleftrightarrow A + C = B + C$$

and so

$$A \preceq^l_C B \text{ and } B \preceq^l_C A \Longleftrightarrow \mathrm{Min}(A, C\setminus\{0\}) = \mathrm{Min}(B, C\setminus\{0\}).$$

Similarly,

$$A \preceq^u_C B \text{ and } B \preceq^u_C A \Longleftrightarrow A - C = B - C,$$

compare [55, Proposition 3.1] and because of $\mathrm{Max}(A - C, C \setminus \{0\}) = \mathrm{Max}(A, C \setminus \{0\})$ *and* $\mathrm{Max}(B - C, C \setminus \{0\}) = \mathrm{Max}(B, C \setminus \{0\})$ *it holds*

$$A \preceq_C^u B \text{ and } B \preceq_C^u A \Longrightarrow \mathrm{Max}(A, C \setminus \{0\}) = \mathrm{Max}(B, C \setminus \{0\}).$$

Under the additional assumption $A \subset \mathrm{Max}(A, C \setminus \{0\}) - C$ *and* $B \subset \mathrm{Max}(B, C \setminus \{0\}) - C$ *it holds*

$$\mathrm{Max}(A, C \setminus \{0\}) = \mathrm{Max}(B, C \setminus \{0\}) \Longleftrightarrow A - C = B - C$$

and so

$$A \preceq_C^u B \text{ and } B \preceq_C^u A \Longleftrightarrow \mathrm{Max}(A, C \setminus \{0\}) = \mathrm{Max}(B, C \setminus \{0\}).$$

Furthermore, the minmax less order relation $\preceq_C^m$ is introduced for sets A, B belonging to

$$\mathcal{F}_{\min,\max} := \{A \subset \mathcal{P}(Y) \mid \mathrm{Min}(A, C \setminus \{0\}) \neq \emptyset \text{ and } \mathrm{Max}(A, C \setminus \{0\}) \neq \emptyset\}. \tag{5.4}$$

Note that for instance in a topological real linear space Y for every compact set in $\mathcal{P}(Y)$ minimal and maximal elements exist.

Definition 8 (Minmax less order relation, [55])**.** *Let* $A, B \in \mathcal{F}_{\min,\max}$*. Then the* ***minmax less order relation*** $\preceq_C^m$ *is defined by*

$$\begin{aligned} A \preceq_C^m B :\Longleftrightarrow\ & \mathrm{Min}(A, C \setminus \{0\}) \preceq_C^s \mathrm{Min}(B, C \setminus \{0\}) \\ & \text{and } \mathrm{Max}(A, C \setminus \{0\}) \preceq_C^s \mathrm{Max}(B, C \setminus \{0\}). \end{aligned}$$

In interval analysis there are even more order relations in use, like the certainly less order relation $\preceq_C^{cert}$ (see Jahn, Ha [55], Eichfelder, Jahn [27]):

Definition 9 (Certainly less order relation, [55])**.** *For arbitrary sets* $A, B \in \mathcal{P}(Y)$ *the* ***certainly less order relation*** $\preceq_C^{cert}$ *is defined by*

$$A \preceq_C^{cert} B :\Longleftrightarrow (A = B) \text{ or } (A \neq B, \forall\, a \in A,\ \forall\, b \in B : a \leq_C b).$$

Note that the above definition of the certainly less order relation has been modified from the originally defined form by Chiriaev, Walster [19]. In [19], the authors introduce the certainly less order relation

$$A \preceq_C^{cert} B :\Longleftrightarrow\ \forall\, a \in A,\ \forall\, b \in B : a \leq_C b,$$

but in this definition $\preceq_C^{cert}$ is generally not a pre-order, as it is not reflexive in general.

Moreover, the possibly less order relation $\preceq_C^p$ is given in the following definition:

Definition 10 (Possibly less order relation, [19, 55])**.** *For arbitrary sets* $A, B \in \mathcal{P}(Y)$ *the* ***possibly less order relation*** $\preceq_C^p$ *is given by*

$$A \preceq_C^p B :\Longleftrightarrow (\exists\, a \in A, \exists\, b \in B : a \leq_C b).$$

Note that the possibly less order relation is in general not transitive (compare [55, Proposition 3.3]).

Remark 18. *It is clear that $A \preceq_C^{cert} B$ implies*

$$\exists\ a \in A \quad \text{such that} \quad \forall\ b \in B : a \leq_C b. \tag{5.5}$$

Moreover, (5.5) implies $A \preceq_C^{l} B$ (see Definition 7) such that

$$A \preceq_C^{cert} B \Longrightarrow A \preceq_C^{l} B.$$

Furthermore, $A \preceq_C^{l} B$ implies

$$\exists\ a \in A,\ \exists\ b \in B \text{ such that } a \leq_C b. \tag{5.6}$$

Taking into account Definition 10, we obtain

$$A \preceq_C^{cert} B \quad \Longrightarrow \quad A \preceq_C^{l} B \quad \Longrightarrow \quad A \preceq_C^{p} B. \tag{5.7}$$

Remark 19. *The relation $A \preceq_C^{cert} B$ implies*

$$\exists\ b \in B \quad \text{such that} \quad \forall\ a \in A : a \leq_C b. \tag{5.8}$$

In addition, (5.8) implies $A \preceq_C^{u} B$ (see Definition 7) such that

$$A \preceq_C^{cert} B \Longrightarrow A \preceq_C^{u} B.$$

Furthermore, $A \preceq_C^{u} B$ implies

$$\exists\ a \in A,\ \exists\ b \in B \text{ such that } a \leq_C b, \tag{5.9}$$

such that we acquire

$$A \preceq_C^{cert} B \quad \Longrightarrow \quad A \preceq_C^{u} B \quad \Longrightarrow \quad A \preceq_C^{p} B,$$

taking into account Definition 10. These observations are depicted in Figure 5.2.

$$\begin{tikzcd}
 & \text{certainly less} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
\text{lower set less} \arrow[dr, Rightarrow] & & \text{upper set less} \arrow[dl, Rightarrow] \\
 & \text{possibly less} &
\end{tikzcd}$$

Figure 5.2: Relationships between set order relations.

The minmax certainly less order relation $\preceq_C^{mc}$ is introduced in the next definition:

Definition 11 (Minmax certainly less order relation, [55, 27]). *For arbitrary $A, B \in \mathcal{F}_{\min,\max}$ the **minmax certainly less order relation** $\preceq_C^{mc}$ is given by*

$$\begin{aligned} A \preceq_C^{mc} B :\Longleftrightarrow\ & (A = B) \text{ or} \\ & (A \neq B,\ \operatorname{Min}(A, C \setminus \{0\}) \preceq_C^{cert} \operatorname{Min}(B, C \setminus \{0\}) \text{ and} \\ & \qquad \operatorname{Max}(A, C \setminus \{0\}) \preceq_C^{cert} \operatorname{Max}(B, C \setminus \{0\})). \end{aligned}$$

Finally, we introduce the minmax certainly nondominated order relation $\preceq_C^{mn}$ (see Jahn, Ha [55]).

Definition 12 (Minmax certainly nondominated order relation, [55, 27]). *The **minmax certainly nondominated order relation** $\preceq_C^{mn}$ is defined for arbitrary $A, B \in \mathcal{F}_{\min,\max}$ by*

$$A \preceq_C^{mn} B :\Longleftrightarrow (A = B) \text{ or } (A \neq B, \operatorname{Max}(A, C \setminus \{0\}) \preceq_C^{s} \operatorname{Min}(B, C \setminus \{0\})).$$

The set less order relation $\preceq_C^s$ and the order relations $\preceq_C^l$, $\preceq_C^u$, $\preceq_C^m$, $\preceq_C^{mc}$ and $\preceq_C^{mn}$ are pre-orders. If $\preceq$ denotes one of these order relations, then we can define optimal solutions with respect to the pre-order $\preceq$ and the corresponding set-valued optimization problem is given by

$$(SP - \preceq) \qquad \preceq -\text{minimize} \quad G(x) \quad \text{s.t.} \quad x \in \mathcal{X},$$

where we assume again (compare $(SetP)$, (5.2)) that Y is a linear topological space, partially ordered by a proper pointed convex closed cone C, X is a linear space, $G : X \rightrightarrows Y$, $\mathcal{X}$ is a subset of X.

Definition 13 (Minimal solutions of $(SP - \preceq)$ w.r.t. the pre-order $\preceq$). *An element $x \in \mathcal{X}$ is called a **minimal solution** of problem $(SP - \preceq)$ w.r.t. the pre-order $\preceq$ if*

$$G(\overline{x}) \preceq G(x) \quad \text{for some} \quad \overline{x} \in \mathcal{X} \Longrightarrow G(x) \preceq G(\overline{x}).$$

Remark 20. *If we use the set relation $\preceq_C^l$ introduced in Definition 7 in the formulation of the solution concept, i.e., we study the set-valued optimization problem $(SP - \preceq_C^l)$, we observe that this solution concept is based on comparisons among sets of minimal points of values of G. Furthermore, considering the upper set less order relation $\preceq_C^u$ (Definition 7), i.e., considering the problem $(SP - \preceq_C^u)$, we recognize that this solution concept is based on comparisons of maximal points of values of G. When $x \in \mathcal{X}$ is a minimal solution of problem $(SP - \preceq_C^l)$ there does not exist $\overline{x} \in \mathcal{X}$ such that $G(\overline{x})$ is smaller than $G(x)$ with respect to the set order $\preceq_C^l$.*

In the following we give three examples (see Kuroiwa [66]) of set-valued optimization problems in order to illustrate the different solution concepts introduced in Definitions 4 and 13.

Example 3. *Consider the set-valued optimization problem*

$$(SP - \preceq^l_C) \qquad \preceq^l_C -\textit{minimize} \quad G_1(x) \quad \textit{s.t.} \quad x \in \mathcal{X},$$

with $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_{\geqq}$, $\mathcal{X} = [0,1]$ *and* $G_1 : \mathcal{X} \rightrightarrows Y$ *is given by*

$$G_1(x) := \begin{cases} [(1,0),(0,1)] & \textit{if } x = 0 \\ [(1-x,x),(1,1)] & \textit{if } x \in (0,1], \end{cases}$$

where $[(a,b),(c,d)]$ *is the line segment between* (a,b) *and* (c,d). *Only the element* $\overline{x} = 0$ *is a minimal solution of* $(SP - \preceq^l_C)$. *However, all elements* $(\overline{x},\overline{y}) \in \operatorname{graph} G_1$ *with* $\overline{x} \in [0,1]$, $\overline{y} = (1-\overline{x},\overline{x})$ *for* $\overline{x} \in (0,1]$ *and* $\overline{y} = (1,0)$ *for* $\overline{x} = 0$ *are minimizers of the set-valued optimization problem in the sense of Definition 4. This example shows that the solution concept with respect to the set relation* $\preceq^l_C$ *(see Definition 13) is more natural and useful than the concept of minimizers introduced in Definition 4.*

Example 4. *Now we discuss the set-valued optimization problem*

$$(SP - \preceq^l_C) \qquad \preceq^l_C -\textit{minimize} \quad G_2(x) \quad \textit{s.t.} \quad x \in \mathcal{X},$$

with $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_{\geqq}$, $\mathcal{X} = [0,1]$ *and* $G_2 : \mathcal{X} \rightrightarrows Y$ *is given by*

$$G_2(x) := \begin{cases} \left[(1,\frac{1}{3}),(\frac{1}{3},1)\right] & \textit{if } x = 0 \\ [(1-x,x),(1,1)] & \textit{if } x \in (0,1]. \end{cases}$$

The set of minimal solutions of $(SP - \preceq^l_C)$ *in the sense of Definition 13 is the interval* $[0,1]$, *but the set of minimizers in the sense of Definition 4 is given by*

$$\{(\overline{x},\overline{y}) \in \operatorname{graph} G_2 \mid \overline{x} \in (0,1],\ \overline{y} = (1-\overline{x},\overline{x})\}.$$

Here we observe that $\overline{x} = 0$ *is a* $\preceq^l_C$*-minimal solution, but the set* $G(\overline{x})$ *(*$\overline{x} = 0$*) has no Pareto minimal points.*

Example 5. *In this example we are looking for minimal solutions of a set-valued optimization problem with respect to the set relation* $\preceq^u_C$ *introduced in Definition 7.*

$$(SP - \preceq^u_C) \qquad \preceq^u_C -\textit{minimize} \quad G_3(x) \quad \textit{s.t.} \quad x \in \mathcal{X},$$

with $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_{\geqq}$, $\mathcal{X} = [0,1]$ *and* $G_3 : \mathcal{X} \rightrightarrows Y$ *is given by*

$$G_3(x) := \begin{cases} [[(1,1),(2,2)]] & \textit{if } x = 0 \\ [[(0,0),(3,3)]] & \textit{if } x \in (0,1], \end{cases}$$

where $[[(a,b),(c,d)]] := \{(y_1,y_2) \mid a \leq y_1 \leq c,\ b \leq y_2 \leq d\}$. *Then a minimal solution of* $(SP - \preceq^u_C)$ *in the sense of Definition 13 is only* $\overline{x} = 0$. *On the other hand,* $x \in (0,1]$ *are not minimal solutions of* $(SP - \preceq^u_C)$ *in the sense of Definition 13, but for all* $\overline{x} \in (0,1]$ *there are* $\overline{y} \in G_3(\overline{x})$ *such that* $(\overline{x},\overline{y})$ *are minimizers in the sense of Definition 4.*

A visualization of the above discussed examples is given in Figure 5.3.

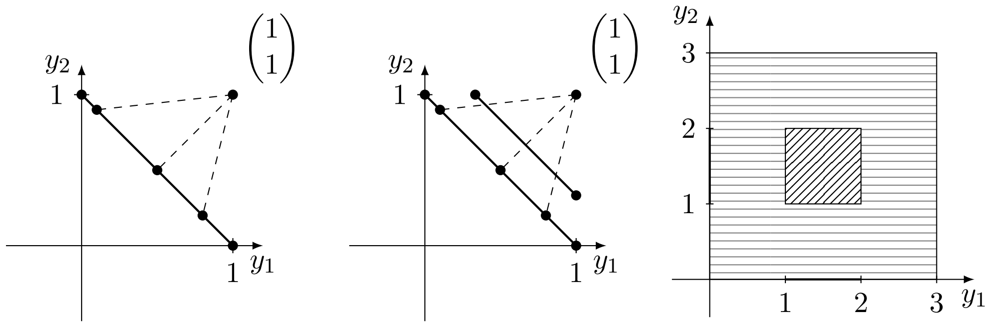

Figure 5.3: Feasible solution sets of $G_1,\ G_2,\ G_3$, described in Example 3, 4 and 5.

5.3 New Concepts for Robustness in Multi-Objective Optimization Using Set-Order Relations

In Ehrgott et al. [25] the concept of strict robustness (compare problem (3.1) with weights $w_i = 1,\ i = 1, \dots, q$, in Section 3.1.1) is extended to **vector-valued optimization problems**. We derive corresponding results in more general settings using the set order relations introduced in Section 5.2.

Recall the ***robust counterpart*** to an uncertain minimization problem from scalar robust optimization

$$(RC) \qquad \min_{x\in\mathcal{X}} \sup_{\xi\in\mathcal{U}} f(x,\xi) \tag{5.10}$$

with $f:\ \mathbb{R}^n \times \mathcal{U} \to \mathbb{R},\ \mathcal{X} \subseteq \mathbb{R}^n,\ \mathcal{U} \subseteq \mathbb{R}^N$. We call (5.10) a strictly robust optimization problem (see Section 3.1.1). Note that here we suppose that the set of feasible solutions $\mathcal{X}$ remains unchanged for every realization of the uncertain parameter $\xi \in \mathcal{U}$. Thus we only focus on the uncertainty in the objective function during this chapter. The corresponding ***optimistic counterpart*** to an uncertain minimization problem is given as

$$(OC) \qquad \min_{x\in\mathcal{X}} \inf_{\xi\in\mathcal{U}} f(x,\xi). \tag{5.11}$$

(OC) was used in Beck and Ben-Tal [6] to develop duality results in robust optimization. Based on [6] and a work on robust duality by Jeyakumar and Li [56], Goberna et al. [36] present strong duality results between a robust counterpart of a linear program and an optimistic counterpart of its Lagrangian dual. They show that strong duality holds under convexity and closedness assumptions on a particularly defined “robust moment cone”.

We will see that the upper and lower set less order relation, originally introduced by Kuroiwa [67, 65], play an important role in developing corresponding multi-objective counterparts to (RC) (see (5.10)) and (OC) (see (5.11)). Furthermore, we derive new concepts of multicriteria robustness using different set order relations.

Throughout this chapter – otherwise will be mentioned – we suppose that X is a linear space, Y a linear topological space and $C \subset Y$ is a proper closed convex and pointed cone.

Consider a **deterministic vector optimization problem**, where all parameters involved in the objective function are known and certain,

$$(VOP) \qquad \qquad \mathrm{Min}(f(\mathcal{X}), C)$$

with a vector-valued function $f : X \to Y$, and a feasible set $\mathcal{X} \subseteq X$.

Naturally, as in the single objective case, the assumption that all parameters are given is not realistic and thus uncertainty should be modeled into the objective function. For that reason, we assume that the objective function f may depend on scenarios ξ which are unknown or uncertain. As in uncertain single objective optimization, given an uncertainty set $\mathcal{U} \subseteq \mathbb{R}^N$, an **uncertain vector-valued optimization problem** $P(\mathcal{U})$ is given as the family

$$(P(\xi); \xi \in \mathcal{U})$$

of vector-valued optimization problems

$$(P(\xi)) \qquad \qquad \mathrm{Min}(f(\mathcal{X}, \xi), C)$$

with the objective function $f : X \times \mathcal{U} \to Y$, a feasible set $\mathcal{X} \subseteq X$ and the notation (for $\xi \in \mathcal{U}$)

$$f(\mathcal{X}, \xi) := \{f(x, \xi) \mid x \in \mathcal{X}\}.$$

We call $\xi \in \mathcal{U}$ a scenario and $(P(\xi))$ an instance of $P(\mathcal{U})$ (note that the analogue was introduced for *scalar* problems in Chapter 1).

Given an uncertain vector-valued optimization problem $P(\mathcal{U})$, the next step would be to evaluate feasible solutions $x \in \mathcal{X}$. The maxmin-approach from scalar uncertain optimization (compare (RC) in (5.10)) cannot be easily transferred to the multi-objective case, since we obtain a vector of objective values for each scenario $\xi \in \mathcal{U}$. For every $x \in \mathcal{X}$ the set of objective values of x is given by

$$f_{\mathcal{U}}(x) := \{f(x, \xi) \mid \xi \in \mathcal{U}\} \subseteq Y. \tag{5.12}$$

Note that in general, for $|\mathcal{U}| > 1$, $f_{\mathcal{U}}(x)$ is not a singleton, but a set. Throughout this chapter, we suppose that the set-valued map $f_{\mathcal{U}}$ is compact-valued.

Dealing with an uncertain vector-valued optimization problem $P(\mathcal{U})$ leads to the following set-valued optimization problem with an objective map $f_{\mathcal{U}} : X \rightrightarrows Y$ given in (5.12) and an order relation $\preceq$:

$$(SP- \preceq) \qquad \qquad \preceq -\text{minimize} \quad f_{\mathcal{U}}(x), \quad \text{subject to} \quad x \in \mathcal{X}.$$

Ehrgott et al. [25, Definition 3.1] introduced robust efficient elements for the case $X = \mathbb{R}^n$, $Y = \mathbb{R}^k$, $C = \mathbb{R}^k_{\geqq}$. In [25], a feasible solution $x^0 \in \mathcal{X}$ is called robust efficient if $f_{\mathcal{U}}(x^0) - \mathbb{R}^k_{\geq}$ does not contain any other set $f_{\mathcal{U}}(\overline{x})$ with $\overline{x} \neq x^0 \in \mathcal{X}$. Furthermore, Ehrgott et al. formulated in [25, Definition 3.1] robust **weakly** efficient and robust **strictly** efficient elements:

Definition 14 ([25, Definition 3.1]). *Given an uncertain vector-valued optimization problem $P(\mathcal{U})$. A feasible element x^0 is called [strictly, weakly, $\cdot$] robust efficient if there does not exist $\overline{x} \in \mathcal{X}$ s.t.*

$$f_{\mathcal{U}}(\overline{x}) \subseteq f_{\mathcal{U}}(x^0) - \mathbb{R}^k_{[\geqq / > / \geq]}.$$

Definition 14 above shows that the authors in [25] implicitly use the upper set less order relation by Kuroiwa [67, 65] which is defined for proper closed convex and pointed cones $C \subset Y$:

$$f_{\mathcal{U}}(\overline{x}) \subseteq f_{\mathcal{U}}(x^0) - \mathbb{R}^k_{\geqq} \Longleftrightarrow f_{\mathcal{U}}(\overline{x}) \preceq^u_C f_{\mathcal{U}}(x^0)$$

for $C = \mathbb{R}^k_{\geqq}$ (compare Definition 7). It is important to mention that the order relations discussed in the previous section are defined for proper closed convex and pointed cones $C \subset Y$. In order to model weak ($\cdot$, respectively) robust efficiency notions, we extend existing order relations to more general sets $Q \subset Y$.

In a first step, we generalize the concepts given in [25] to more general sets Q using the upper set less order relation $\preceq^u_Q$.

5.3.1 Upper Set Less Ordered Robustness

Using the upper set less order relation $\preceq^u_s$ (see Definition 7), we are able to introduce the concept of *upper set less ordered robustness* for uncertain vector-valued optimization problems for a proper closed convex and pointed cone C. In accordance with [25], we also wish to define **weakly** ($\cdot$, respectively) upper set less ordered robust elements.

Suppose that $Q \subset Y$ is a proper set with $C \subset \operatorname{cl} Q$ and $\operatorname{cl} Q \cap (-\operatorname{cl} Q) = \{0\}$, where $\operatorname{cl} Q$ denotes the closure of a set Q. Under these assumptions we introduce an order relation with respect to Q analogously to the upper set less order relation $\preceq^u_s$ introduced in Definition 7 for arbitrary nonempty sets $A, B \subset Y$:

$$A \preceq^u_Q B :\Longleftrightarrow A \subseteq B - Q. \tag{5.13}$$

If we are dealing with $Q = \operatorname{int} C$, we suppose $\operatorname{int} C \neq \emptyset$.

We collect the definitions of strictly (weakly, $\cdot$, respectively) *upper set less ordered robust* elements of an uncertain multi-objective optimization problem:

Definition 15. *A solution x^0 of $P(\mathcal{U})$ is called strictly (weakly, $\cdot$, respectively)* **upper set less ordered robust** *if there is no $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ such that $f_{\mathcal{U}}(\overline{x}) \preceq^u_Q f_{\mathcal{U}}(x^0)$, which is equivalent to*

$$\nexists\, \overline{x} \in \mathcal{X} \setminus \{x^0\} : \quad f_{\mathcal{U}}(\overline{x}) \subseteq f_{\mathcal{U}}(x^0) - Q$$

for $Q = C$ ($Q = \operatorname{int} C$, $Q = C \setminus \{0\}$, respectively).

We exemplarily illustrate this definition in the figure below for $Q = \mathbb{R}^2_{\geq}$.

As in deterministic vector-valued optimization, there is a relationship between strictly (weakly, $\cdot$, respectively) upper set less ordered robust solutions, which may be observed in the following lemma.

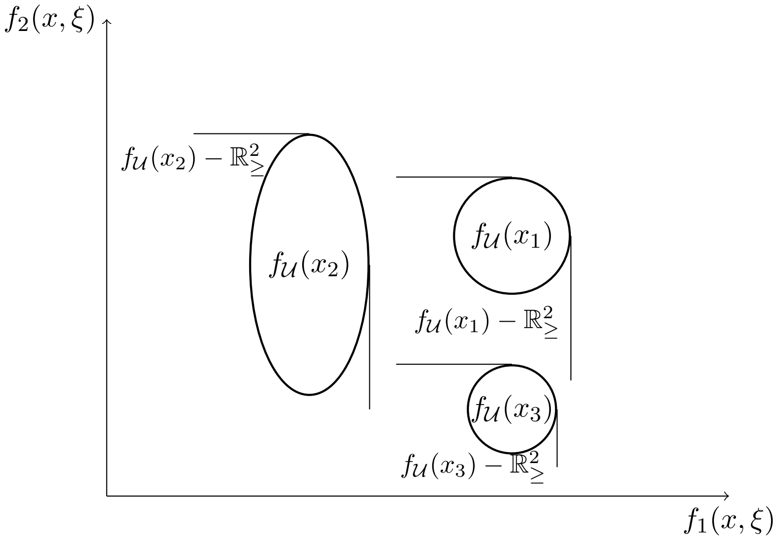

Figure 5.4: x_1 is not upper set less ordered robust, x_2 and x_3 are upper set less ordered robust.

Lemma 1. *Let $P(\mathcal{U})$ be an uncertain vector-valued optimization problem, where $C \subset Y$ is a proper closed convex and pointed cone with* $\operatorname{int} C \neq \emptyset$*. Then we have:*

$$x^0 \text{ is strictly upper set less ordered robust} \Longrightarrow x^0 \text{ is upper set less ordered robust} \Longrightarrow x^0 \text{ is weakly upper set less ordered robust.}$$

The robust counterpart of an uncertain vector-valued optimization problem is the problem of identifying all $x \in \mathcal{X}$ which are weakly upper set less ordered robust, upper set less ordered robust, or strictly upper set less ordered robust.

Next we show the essential result that for $|\mathcal{U}| = 1$, i.e., in the deterministic multi-objective case, Definition 15 coincides with the definition of C-minimality (compare Definition 2 in Chapter 2). For the special case $Y = \mathbb{R}^k$, $X = \mathbb{R}^n$, $C = \mathbb{R}^k_{\geqq}$, this relation was observed by Ehrgott et al. [25].

Lemma 2. *Given $P(\mathcal{U})$ with $|\mathcal{U}| = 1$. Then $x^0 \in \mathcal{X}$ is strictly (weakly, $\cdot$, respectively) upper set less ordered robust if and only if $f(x^0)$ is strictly (weakly, $\cdot$, respectively) minimal.*

Proof. Let $Q = C$ ($Q = \operatorname{int} C$, $Q = C \setminus \{0\}$, respectively). x^0 is strictly (weakly, $\cdot$, respectively) upper set less ordered robust

$$\begin{aligned}
&\iff \nexists\, \overline{x} \in \mathcal{X} \setminus \{x^0\} : \; f_{\mathcal{U}}(\overline{x}) \preceq_Q^u f_{\mathcal{U}}(x^0) \\
&\iff \nexists\, \overline{x} \in \mathcal{X} \setminus \{x^0\} : \; f_{\mathcal{U}}(\overline{x}) \subseteq f_{\mathcal{U}}(x^0) - Q \\
&\overset{|\mathcal{U}|=1}{\iff} \nexists\, \overline{x} \in \mathcal{X} \setminus \{x^0\} : \; f(\overline{x}) \in f(x^0) - Q \\
&\iff f(x^0) \text{ is strictly (weakly, } \cdot \text{ , respectively) minimal.}
\end{aligned}$$

□

Ehrgott et al. [25] prove the following lemma.

Lemma 3 ([25, Lemma 3.6]). *Given $P(\mathcal{U})$ with $Y = \mathbb{R}$, $X = \mathbb{R}^n$ and $C = \mathbb{R}_{\geqq}$.*

(a) x^0 is weakly upper set less ordered robust $\iff$ x^0 is upper set less ordered robust.

(b) If x^0 is uniquely optimal for the robust counterpart (RC) (see (5.10)), then x^0 is strictly upper set less ordered robust.

(c) Suppose $\max_{\xi \in \mathcal{U}} f(x', \xi)$ exists for every $x' \in \mathcal{X}$. Then it holds: If x^0 is strictly upper set less ordered robust, then x^0 is uniquely optimal for the robust counterpart (RC) (see (5.10)).

(d) Suppose $\max_{\xi \in \mathcal{U}} f(x', \xi)$ exists for every $x' \in \mathcal{X}$. Then it holds: If x^0 is optimal for the robust counterpart (RC) (see (5.10)), then x^0 is upper set less ordered robust.

(e) If x^0 is weakly upper set less ordered robust, then x^0 is optimal for the robust counterpart (RC) (see (5.10)).

Scalarization techniques for deriving upper set less ordered robust elements are presented below.

Scalarization Approach for Computing Upper Set Less Ordered Robust Points: Weighted Sum Scalarization

In the following we will see that for special structures of set-valued optimization problems we are able to use scalarization methods in order to derive algorithms for computing upper set less ordered robust solutions.

For the special case $Y = \mathbb{R}^k$, $X = \mathbb{R}^n$, $C = \mathbb{R}^k_{\geqq}$ Ehrgott et al. [25] propose solution procedures for uncertain vector optimization problems. In this section, we derive algorithms to obtain upper set less ordered robust solutions to an uncertain vector optimization problem $P(\mathcal{U})$. The most common approach to computing minimal solutions for a deterministic vector optimization problem in finite dimensional image spaces is the weighted sum scalarization. The general idea is to form a scalar optimization problem by multiplying each objective function with a positive weight and summing up the weighted objectives. The weighted sum problem (P_{y^*}) for a given (deterministic) vector optimization problem $\mathrm{Min}(f(\mathcal{X}), C)$ and a weight vector $y^* \in C^* \setminus \{0\}$ is

$$(P_{y^*}) \qquad \min_{x\in\mathcal{X}} \; y^* \circ f(x).$$

We now reduce the uncertain vector-valued optimization problem to a single objective optimization problem in order to be able to obtain upper set less ordered robust solutions. To this end, we introduce the robust version of the weighted sum scalarization problem of an uncertain vector-valued optimization problem $P(\mathcal{U})$ in a general setting as

$$(P\mathcal{U}^{ur}_{y^*}) \qquad \min_{x\in\mathcal{X}} \sup_{\xi\in\mathcal{U}} \; y^* \circ f(x,\xi). \tag{5.14}$$

Theorem 22. *Consider an uncertain vector optimization problem $P(\mathcal{U})$. The following statements hold:*

(a) If $x^0 \in \mathcal{X}$ is a unique optimal solution of $(P\mathcal{U}^{ur}_{y^*})$ for some $y^* \in C^* \setminus \{0\}$, then x^0 is strictly upper set less ordered robust for $P(\mathcal{U})$.

(b) If $x^0 \in \mathcal{X}$ is an optimal solution of $(P\mathcal{U}^{ur}_{y^*})$ for some $y^* \in C^* \setminus \{0\}$ and $\max_{\xi\in\mathcal{U}} y^* \circ f(x',\xi)$ exists for all $x' \in \mathcal{X}$, then x^0 is weakly upper set less ordered robust for $P(\mathcal{U})$.

(c) If $x^0 \in \mathcal{X}$ is an optimal solution of $(P\mathcal{U}^{ur}_{y^*})$ for some $y^* \in C^\#$ and $\max_{\xi\in\mathcal{U}} y^* \circ f(x',\xi)$ exists for all $x' \in \mathcal{X}$ and the chosen weight $y^* \in C^\#$, then x^0 is upper set less ordered robust for $P(\mathcal{U})$.

Proof. Let $Q = C$ ($Q = \operatorname{int} C$, $Q = C \setminus \{0\}$, respectively). Suppose that x^0 is not strictly (weakly, $\cdot$, respectively) upper set less ordered robust. Then there exists an element $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ such that

$$f_{\mathcal{U}}(\overline{x}) \subseteq f_{\mathcal{U}}(x^0) - Q. \tag{5.15}$$

This implies

$$\forall\, \xi \in \mathcal{U}\ \exists\, \eta \in \mathcal{U}:\ f(\overline{x},\xi) \in f(x^0,\eta) - Q.$$

Choose now $y^* \in C^* \setminus \{0\}$ for $Q = C$ ($y^* \in C^* \setminus \{0\}$ for $Q = \operatorname{int} C$, $y^* \in C^\#$ for $Q = C \setminus \{0\}$, respectively) arbitrary, but fixed. This implies

$$\begin{aligned}
&\Longrightarrow\ \forall\, \xi \in \mathcal{U}\ \exists\, \eta \in \mathcal{U}:\ y^* \circ f(\overline{x},\xi)\ [\leq / < / <]\ y^* \circ f(x^0,\eta)\\
&\Longrightarrow\ \forall\, \xi \in \mathcal{U}:\ y^* \circ f(\overline{x},\xi)\ [\leq / < / <]\ \sup_{\eta\in\mathcal{U}} y^* \circ f(x^0,\eta)\\
&\Longrightarrow\ \sup_{\xi\in\mathcal{U}} y^* \circ f(\overline{x},\xi)\ [\leq / < / <]\ \sup_{\eta\in\mathcal{U}} y^* \circ f(x^0,\eta).
\end{aligned}$$

The last two inequalities hold because for (b) and (c) $\max_{\xi\in\mathcal{U}} y^* \circ f(x',\xi)$ exists for every $x' \in \mathcal{X}$. But this means that x^0 is not the unique optimal (an optimal, an optimal, respectively) solution of $(P\mathcal{U}^{ur}_{y^*})$ for $y^* \in C^* \setminus \{0\}$ ($y^* \in C^* \setminus \{0\}$, $y^* \in C^\#$, respectively). $\square$

Remark 21. *In Theorem 22 (c) we consider $y^* \in C^\#$. Under our assumptions concerning the cone C and if we assume additionally $Y = \mathbb{R}^k$ we have $C^\# \neq \emptyset$ (compare [39, Theorem 2.2.12], [39, Example 2.2.16]). Moreover, if Y is a Hausdorff locally convex space, $C \subset Y$ is a proper convex cone and C has a base B with $0 \notin \operatorname{cl} B$, then $C^\# \neq \emptyset$ (compare [39, Theorem 2.2.12]).*

Based on Theorem 22 we derive the following algorithms for computing strictly (weakly, $\cdot$, respectively) upper set less ordered robust solutions of the uncertain vector-valued optimization problem $P(\mathcal{U})$. In the following algorithms the set of strictly (weakly, $\cdot$, respectively) upper set less ordered robust solutions is denoted by $\mathrm{Opt}_{\mathrm{sur}}$ ($\mathrm{Opt}_{\mathrm{wur}}$, $\mathrm{Opt}_{\mathrm{ur}}$, respectively).

Algorithm 1 for solving $P(\mathcal{U})$ based on weighted sum scalarization:

Input: Uncertain vector-valued problem $P(\mathcal{U})$, solution sets $\mathrm{Opt}_{\mathrm{sur}} = \mathrm{Opt}_{\mathrm{ur}} = \mathrm{Opt}_{\mathrm{wur}} = \emptyset$.

Step 1: Choose a set $\overline{C} \subset C^* \setminus \{0\}$.

Step 2: If $\overline{C} = \emptyset$: **STOP. Output:** Set of strictly upper set less ordered robust solutions $\mathrm{Opt}_{\mathrm{sur}}$, set of upper set less ordered robust solutions $\mathrm{Opt}_{\mathrm{ur}}$, set of weakly upper set less ordered robust solutions $\mathrm{Opt}_{\mathrm{wur}}$.

Step 3: Choose $y^* \in \overline{C}$. Set $\overline{C} := \overline{C} \setminus \{y^*\}$. Find a set of optimal solutions $SOL(P\mathcal{U}^{ur}_{y^*})$ of $(P\mathcal{U}^{ur}_{y^*})$ (see (5.14)).

Step 4: If $SOL(P\mathcal{U}^{ur}_{y^*}) = \emptyset$, then go to Step 2.

Step 5: **(a)** If $|SOL(P\mathcal{U}^{ur}_{y^*})| = 1$, then $SOL(P\mathcal{U}^{ur}_{y^*})$ consists of a strictly upper set less ordered robust solution for $P(\mathcal{U})$, thus

$$\mathrm{Opt}_{\mathrm{sur}} := \mathrm{Opt}_{\mathrm{sur}} \cup SOL(P\mathcal{U}^{ur}_{y^*}).$$

(b) If $\max_{\xi \in \mathcal{U}} y^* \circ f(x', \xi)$ exists for all $x' \in \mathcal{X}$, then $SOL(P\mathcal{U}^{ur}_{y^*})$ consists of weakly upper set less ordered robust elements of $P(\mathcal{U})$, thus

$$\mathrm{Opt}_{\mathrm{wur}} := \mathrm{Opt}_{\mathrm{wur}} \cup SOL(P\mathcal{U}^{ur}_{y^*}).$$

(c) If $\max_{\xi \in \mathcal{U}} y^* \circ f(x', \xi)$ exists for all $x' \in \mathcal{X}$ and $y^* \in C^\#$, then $SOL(P\mathcal{U}^{ur}_{y^*})$ consists of upper set less ordered robust solutions of $P(\mathcal{U})$, thus

$$\mathrm{Opt}_{\mathrm{ur}} := \mathrm{Opt}_{\mathrm{ur}} \cup SOL(P\mathcal{U}^{ur}_{y^*}).$$

Go to Step 2.

The following algorithm represents an interactive procedure for solving the uncertain vector-valued optimization problem $P(\mathcal{U})$. In this approach we reduce the scalarized problem to a one-parametric optimization problem by altering the weights $y^* \in C^* \setminus \{0\}$ chosen in the scalarized problem $(P\mathcal{U}^{ur}_{y^*})$.

Algorithm 2 for computing upper set less ordered robust solutions to the uncertain vector-valued optimization problem $P(\mathcal{U})$:

Input: Uncertain vector-valued problem $P(\mathcal{U})$, solution sets $\mathrm{Opt}_{\mathrm{sur}} = \mathrm{Opt}_{\mathrm{ur}} = \mathrm{Opt}_{\mathrm{wur}} = \emptyset$.

Step 1: Choose a set $\overline{C} \subset C^* \setminus \{0\}$ with at least two distinct elements. Set $j := 0$. Choose $y_0^* \in \overline{C}$, set $\overline{C} := \overline{C} \setminus \{y_0^*\}$.

Step 2: If $\overline{C} = \emptyset$ or if $\mathrm{Opt}_{\mathrm{sur}}$, $\mathrm{Opt}_{\mathrm{ur}}$, $\mathrm{Opt}_{\mathrm{wur}}$ are accepted by the decision maker: **STOP. Output:** Set of strictly upper set less ordered robust solutions $\mathrm{Opt}_{\mathrm{sur}}$, set of upper set less ordered robust solutions $\mathrm{Opt}_{\mathrm{ur}}$, set of weakly upper set less ordered robust solutions $\mathrm{Opt}_{\mathrm{wur}}$.

Step 3: Choose $y_{j+1}^* \in \overline{C}$. Set $\overline{C} := \overline{C} \setminus \{y_{j+1}^*\}$. Set $t = 0$.

Step 4: Set $\hat{y}^* := y_j^* + t(y_{j+1}^* - y_j^*)$.

Step 5: Find a set of optimal solutions $SOL(P\mathcal{U}^{ur}_{\hat{y}^*})$ of problem $(P\mathcal{U}^{ur}_{\hat{y}^*})$ (see (5.14)).

Step 6: If $SOL(P\mathcal{U}^{ur}_{\hat{y}^*}) = \emptyset$, go to Step 8.

Step 7: **(a)** If $|SOL(P\mathcal{U}^{ur}_{\hat{y}^*})| = 1$, then $SOL(P\mathcal{U}^{ur}_{\hat{y}^*})$ consists of a strictly upper set less ordered robust solution for $P(\mathcal{U})$, thus

$$\mathrm{Opt}_{\mathrm{sur}} := \mathrm{Opt}_{\mathrm{sur}} \cup SOL(P\mathcal{U}^{ur}_{\hat{y}^*}).$$

(b) If $\max_{\xi \in \mathcal{U}} \hat{y}^* \circ f(x', \xi)$ exists for all $x' \in \mathcal{X}$, then all elements in $SOL(P\mathcal{U}^{ur}_{\hat{y}^*})$ are weakly upper set less ordered robust solutions for $P(\mathcal{U})$, thus

$$\mathrm{Opt}_{\mathrm{wur}} := \mathrm{Opt}_{\mathrm{wur}} \cup SOL(P\mathcal{U}^{ur}_{\hat{y}^*}).$$

(c) If $\max_{\xi \in \mathcal{U}} \hat{y}^* \circ f(x', \xi)$ exists for all $x' \in \mathcal{X}$ and $\hat{y}^* \in C^\#$, then all elements in $SOL(P\mathcal{U}^{ur}_{\hat{y}^*})$ are upper set less ordered robust for $P(\mathcal{U})$, thus

$$\mathrm{Opt}_{\mathrm{ur}} := \mathrm{Opt}_{\mathrm{ur}} \cup SOL(P\mathcal{U}^{ur}_{\hat{y}^*}).$$

Step 8: If $t = 1$, then set $j := j + 1$ and go to Step 2. Otherwise, choose $t :\in (t, 1]$ and go to Step 4.

In the following, we mention another scalarization method that will be useful to derive upper set less ordered robust solutions.

Scalarization Approach for Computing Upper Set Less Ordered Robust Points: ϵ-Constraint Scalarization

In this subsection, we focus on the case where $Y = \mathbb{R}^k$, $X = \mathbb{R}^n$ and $C = \mathbb{R}^k_{\geqq}$. We now use the ϵ-constraint approach to reduce a vector-valued uncertain optimization problem $P(\mathcal{U})$ to a single objective optimization problem. To this end we define the robust ϵ-constraint version $(P\mathcal{U}_{\epsilon,i})$ of $P(\mathcal{U})$, thus an ϵ-constraint problem for an uncertain vector-valued optimization problem:

$$(P\mathcal{U}^{ur}_{\epsilon,i}) \qquad \min \sup_{\xi \in \mathcal{U}} f_i(x,\xi) \qquad \text{s.t.} \quad \forall\, j \neq i,\ \forall\, \xi \in \mathcal{U}:\ f_j(x,\xi) \leq \epsilon_j,\ x \in \mathcal{X}. \tag{5.16}$$

In the following theorem the relationships between solutions of $(P\mathcal{U}^{ur}_{\epsilon,i})$ and strictly upper set less ordered robustness (weakly upper set less ordered robustness, respectively) are presented (see [25]).

Theorem 23 ([25, Theorem 4.6])**.** *Consider an uncertain vector-valued optimization problem $P(\mathcal{U})$ with $Y = \mathbb{R}^k,\ X = \mathbb{R}^n,\ C = \mathbb{R}^k_{\geqq}$. The following statements hold.*

(a) If x^0 is the unique optimal solution of $(P\mathcal{U}^{ur}_{\epsilon,i})$ for some $\epsilon \in \mathbb{R}^k$ and some $i \in \{1,\dots,k\}$, then x^0 is strictly upper set less ordered robust for $P(\mathcal{U})$.

(b) If $x^0 \in \mathcal{X}$ is an optimal solution of $(P\mathcal{U}^{ur}_{\epsilon,i})$ for some $\epsilon \in \mathbb{R}^k$ and $\max_{\xi\in\mathcal{U}} f_j(x',\xi)$ exists for all $x' \in \mathcal{X}$ and $j = 1,\dots,k$, then x^0 is weakly upper set less ordered robust for $P(\mathcal{U})$.

The proof can be found in [25] and is omitted here.

This leads to the following algorithm for computing strictly (weakly, respectively) upper set less ordered robust solutions of the uncertain vector-valued optimization problem $P(\mathcal{U})$ with $Y = \mathbb{R}^k$, $X = \mathbb{R}^n$ and $C = \mathbb{R}^k_{\geqq}$ (see also [25]).

Algorithm 3 for solving $P(\mathcal{U})$ based on an ϵ-constraint method:

Input: Uncertain vector-valued problem $P(\mathcal{U})$, solution sets $\mathrm{Opt}_{\mathrm{sur}} = \mathrm{Opt}_{\mathrm{wur}} = \emptyset$.

Step 1: Choose a set $\mathcal{E} \subset \mathbb{R}^k$.

Step 2: If $\mathcal{E} = \emptyset$: **STOP. Output:** Set of strictly upper set less ordered robust solutions $\mathrm{Opt}_{\mathrm{sur}}$ and set of weakly upper set less ordered robust solutions $\mathrm{Opt}_{\mathrm{wur}}$.

Step 3: Choose $\epsilon \in \mathcal{E}$. Set $\mathcal{E} := \mathcal{E} \setminus \{\epsilon\}$. Find a set of optimal solutions $SOL(P\mathcal{U}^{ur}_{\epsilon,i})$ of $(P\mathcal{U}^{ur}_{\epsilon,i})$ (see (5.16)).

Step 4: If $SOL(P\mathcal{U}^{ur}_{\epsilon,i}) = \emptyset$, go to Step 2.

Step 5: **(a)** If $|SOL(P\mathcal{U}^{ur}_{\epsilon,i})| = 1$, then $SOL(P\mathcal{U}^{ur}_{\epsilon,i})$ consists of a strictly upper set less ordered robust solution of $P(\mathcal{U})$, thus

$$\mathrm{Opt}_{\mathrm{sur}} := \mathrm{Opt}_{\mathrm{sur}} \cup SOL(P\mathcal{U}^{ur}_{\epsilon,i}).$$

(b) If $\max_{\xi\in\mathcal{U}} f_j(x',\xi)$ exists for all $x' \in \mathcal{X}$ and $j = 1,\dots,k$, then $SOL(P\mathcal{U}^{ur}_{\epsilon,i})$ consists of weakly upper set less ordered robust solutions for $P(\mathcal{U})$, thus

$$\mathrm{Opt}_{\mathrm{wur}} := \mathrm{Opt}_{\mathrm{wur}} \cup SOL(P\mathcal{U}^{ur}_{\epsilon,i}).$$

Go to Step 2.

Scalarization Approach for Computing Upper Set Less Ordered Robust Points: Max-Ordering Scalarization

In this subsection, we provide a characterization of robust strictly (weakly, respectively) upper set less ordered robust solutions via max-ordering scalarization for the special case $Y = \mathbb{R}^k,\ X = \mathbb{R}^n,\ C = \mathbb{R}^k_{\geqq}$. Consider the problem

$$(P\mathcal{U}^{max,ur}_{y^*}) \qquad \min_{x\in\mathcal{X}} \max_{i=1,\dots,k} \sup_{\xi\in\mathcal{U}} y^*_i f_i(x,\xi) \tag{5.17}$$

with a weight vector $y^* \in \mathbb{R}^k_>$.

Theorem 24. *Consider an uncertain vector optimization problem $P(\mathcal{U})$ with $Y = \mathbb{R}^k$, $X = \mathbb{R}^n$, $C = \mathbb{R}^k_{\geqq}$. The following statements hold:*

(a) If $x^0 \in \mathcal{X}$ is a unique optimal solution of $(P\mathcal{U}^{max,ur}_{y^*})$ for some $y^* \in \mathbb{R}^k_>$, then x^0 is strictly upper set less ordered robust for $P(\mathcal{U})$.

(b) If $x^0 \in \mathcal{X}$ is an optimal solution of $(P\mathcal{U}^{max,ur}_{y^*})$ for some $y^* \in \mathbb{R}^k_>$ and $\max_{\xi\in\mathcal{U}} y^*_i f_i(x',\xi)$ exists for all $x' \in \mathcal{X},\ i = 1,\dots,k$, then x^0 is weakly upper set less ordered robust for $P(\mathcal{U})$.

Proof. Let $Q = \mathbb{R}^k_{\geqq}$ ($Q = \mathbb{R}^k_>$, respectively). Suppose x^0 is not strictly (weakly, respectively) upper set less ordered robust. Then there exists an element $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ such that

$$f_{\mathcal{U}}(\overline{x}) \subseteq f_{\mathcal{U}}(x^0) - Q.$$

This implies

$$\forall\ \xi \in \mathcal{U}\ \exists\ \eta \in \mathcal{U}:\ f(\overline{x},\xi) \in f(x^0,\eta) - Q.$$

It follows

$$\forall\ \xi \in \mathcal{U}\ \exists\ \eta \in \mathcal{U}:\ f_i(\overline{x},\xi)\ [\leq / <]\ f_i(x^0,\eta),\ i = 1,\dots,k.$$

Choose now $y^* \in \mathbb{R}^k_>$ arbitrary, but fixed. Then

$$\begin{aligned} &\forall\ \xi \in \mathcal{U}\ \exists\ \eta \in \mathcal{U}:\ y^*_i f_i(\overline{x},\xi)\ [\leq / <]\ y^*_i f_i(x^0,\eta),\ i = 1,\dots,k,\\ &\Longrightarrow\ \forall\ \xi \in \mathcal{U}:\ y^*_i f_i(\overline{x},\xi)\ [\leq / <]\ \sup_{\eta\in\mathcal{U}} y^*_i f_i(x^0,\eta),\ i = 1,\dots,k,\\ &\Longrightarrow\ \sup_{\xi\in\mathcal{U}} y^*_i f_i(\overline{x},\xi)\ [\leq / <]\ \sup_{\eta\in\mathcal{U}} y^*_i f_i(x^0,\eta),\ i = 1,\dots,k. \end{aligned}$$

The last implication holds because $\max_{\xi\in\mathcal{U}} y^*_i f_i(x',\xi)$ exists for all $x' \in \mathcal{X},\ i = 1,\dots,k$. But this means that x^0 is not the unique optimal (optimal, respectively) solution for $(P\mathcal{U}^{max,ur}_{y^*})$ for $y^* \in \mathbb{R}^k_>$. □

Remark 22. *The results presented in Theorem 24 also hold in case X is a linear space, although the objective space $Y = \mathbb{R}^k$ is finite dimensional.*

Based on the scalarization results obtained in Theorem 24, we provide an algorithm for solving $P(\mathcal{U})$ via max-ordering scalarization below.

Algorithm 4 for computing upper set less ordered robust solutions based on max-ordering scalarization:

Input: Uncertain vector-valued problem $P(\mathcal{U})$, solution sets $\mathrm{Opt}_{\mathrm{sur}} = \mathrm{Opt}_{\mathrm{wur}} = \emptyset$.

Step 1: Choose a set $\overline{C} \subset \mathbb{R}^k_>$.

Step 2: If $\overline{C} = \emptyset$: **STOP. Output:** Set of strictly upper set less ordered robust solutions $\mathrm{Opt}_{\mathrm{sur}}$, set of weakly upper set less ordered robust solutions $\mathrm{Opt}_{\mathrm{wur}}$.

Step 3: Choose $y^* \in \overline{C}$. Set $\overline{C} := \overline{C} \setminus \{y^*\}$. Now find a set of optimal solutions $SOL(P\mathcal{U}^{max,ur}_{y^*})$ of $(P\mathcal{U}^{max,ur}_{y^*})$ (see (5.17)).

Step 4: If $SOL(P\mathcal{U}^{max,ur}_{y^*}) = \emptyset$, then go to Step 2.

Step 5: **(a)** If $|SOL(P\mathcal{U}^{max,ur}_{y^*})| = 1$, then $SOL(P\mathcal{U}^{max,ur}_{y^*})$ consists of a strictly upper set less ordered robust solution of $P(\mathcal{U})$, thus

$$\mathrm{Opt}_{\mathrm{sur}} := \mathrm{Opt}_{\mathrm{sur}} \cup SOL(P\mathcal{U}^{max,ur}_{y^*}).$$

(b) If $\max_{\xi \in \mathcal{U}} y^*_i f_i(x', \xi)$ exists for all $x' \in \mathcal{X}$, $i = 1, \ldots, k$, then $SOL(P\mathcal{U}^{max,ur}_{y^*})$ consists of weakly upper set less ordered robust elements of $P(\mathcal{U})$, thus

$$\mathrm{Opt}_{\mathrm{wur}} := \mathrm{Opt}_{\mathrm{wur}} \cup SOL(P\mathcal{U}^{max,ur}_{y^*}).$$

Go to Step 2.

In the following, as has been done in Algorithm 2, we offer an interactive algorithm to obtain solutions of the uncertain vector-valued optimization problem $P(\mathcal{U})$ by reducing the problem to a one-parametric max-ordering optimization problem.

Algorithm 5 for for computing upper set less ordered robust solutions based on max-ordering scalarization:

Input: Uncertain vector-valued problem $P(\mathcal{U})$, solution sets $\mathrm{Opt}_{\mathrm{sur}} = \mathrm{Opt}_{\mathrm{wur}} = \emptyset$.

Step 1: Choose a set $\overline{C} \subset \mathbb{R}^k_>$ with at least two distinct elements. Set $j := 0$. Choose $y^*_0 \in \overline{C}$, set $\overline{C} := \overline{C} \setminus \{y^*_0\}$.

Step 2: If $\overline{C} = \emptyset$ or if $\mathrm{Opt}_{\mathrm{sur}}$, $\mathrm{Opt}_{\mathrm{wur}}$ are accepted by the decision maker: **STOP.** **Output:** Set of strictly upper set less ordered robust solutions $\mathrm{Opt}_{\mathrm{sur}}$, set of weakly upper set less ordered robust solutions $\mathrm{Opt}_{\mathrm{wur}}$.

Step 3: Choose $y^*_{j+1} \in \overline{C}$. Set $\overline{C} := \overline{C} \setminus \{y^*_{j+1}\}$. Set $t = 0$.

Step 4: Set $\hat{y}^* := y^*_j + t(y^*_{j+1} - y^*_j)$.

Step 5: Find a set of optimal solutions $SOL(P\mathcal{U}^{max,ur}_{\hat{y}^*})$ of problem $(P\mathcal{U}^{max,ur}_{\hat{y}^*})$ (see (5.17)).

Step 6: If $SOL(P\mathcal{U}^{max,ur}_{\hat{y}^*}) = \emptyset$, go to Step 8.

Step 7: **(a)** If $|SOL(P\mathcal{U}^{max,ur}_{\hat{y}^*})| = 1$, then $SOL(P\mathcal{U}^{max,ur}_{\hat{y}^*})$ consists of a strictly upper set less ordered robust solution for $P(\mathcal{U})$, thus

$$\mathrm{Opt}_{\mathrm{sur}} := \mathrm{Opt}_{\mathrm{sur}} \cup SOL(P\mathcal{U}^{max,ur}_{\hat{y}^*}).$$

(b) If $\max_{\xi\in\mathcal{U}} \hat{y}^*_i f_i(x', \xi)$ exists for all $x' \in \mathcal{X}$, $i = 1, \ldots, k$, then all elements in $SOL(P\mathcal{U}^{max,ur}_{\hat{y}^*})$ are weakly upper set less ordered robust elements for $P(\mathcal{U})$, thus

$$\mathrm{Opt}_{\mathrm{wur}} := \mathrm{Opt}_{\mathrm{wur}} \cup SOL(P\mathcal{U}^{max,ur}_{\hat{y}^*}).$$

Step 8: If $t = 1$, then set $j := j + 1$ and go to Step 2. Otherwise, choose $t :\in (t, 1]$ and go to Step 4.

Remark 23. *Ehrgott et al. [25] propose a vectorization approach for computing upper set less ordered robust solutions. Instead of repeating this concept here, we derive similar results for an approach to uncertain vector-valued optimization involving the lower set less order relation $\preceq^l_C$ (see Definition 7) in the following section.*

5.3.2 Lower Set Less Ordered Robustness

Contrary to the upper set less order relation, the lower set less order relation (Definition 7) compares sets while focusing on the lower bounds. We now use the lower set less order relation $\preceq^l_C$ for deriving a concept to approach uncertain multicriteria optimization problems. To this end, we define a solution concept to uncertain problems $P(\mathcal{U})$ that relies on the comparison of sets via the lower set less order relation.

In the following, we extend Definition 7 of the *lower set less order relation* to general nonempty sets $Q \subset Y$. To this end, let $C \subset Y$ be a proper closed convex and pointed cone. Suppose $C \subset \mathrm{cl}\, Q$ and $\mathrm{cl}\, Q \cap (-\mathrm{cl}\, Q) = \{0\}$. Then we define the *lower set less order relation* $\preceq^l_Q$ for two nonempty sets $A, B \subset Y$

$$A \preceq^l_Q B :\Longleftrightarrow A + Q \supseteq B \Longleftrightarrow \forall\, b \in B\ \exists\, a \in A : a \leq_Q b.$$

When dealing with $Q = \mathrm{int}\, C$, we assume that $\mathrm{int}\, C \neq \emptyset$.

Definition 16. *A solution x^0 of $P(\mathcal{U})$ is called strictly (weakly, $\cdot$, respectively)* **lower set less ordered robust** *if there is no $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ such that $f_{\mathcal{U}}(\overline{x}) \preceq^l_Q f_{\mathcal{U}}(x^0)$, which is equivalent to*

$$\nexists\, \overline{x} \in \mathcal{X} \setminus \{x^0\} : \quad f_{\mathcal{U}}(\overline{x}) + Q \supseteq f_{\mathcal{U}}(x^0)$$

for $Q = C$ ($Q = \operatorname{int} C$, $Q = C \setminus \{0\}$, respectively).

Example 6. *Figure 5.5 shows that x_1 is strictly lower set less ordered robust with $Q = \mathbb{R}^2_{\geqq}$, while it is not upper set less ordered robust.*

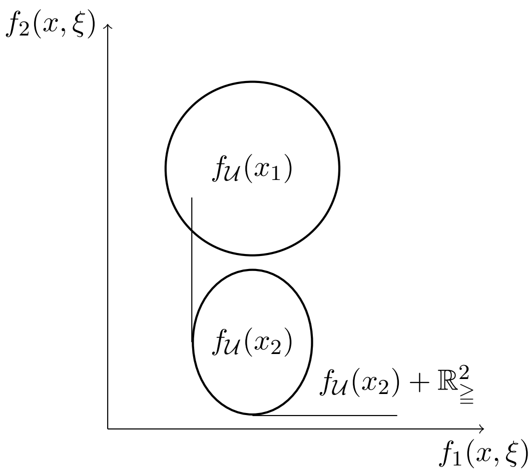

Figure 5.5: x_1 is strictly lower set less ordered robust.

The lower set less ordered robustness concept is introduced here to give the decision maker an alternative tool for obtaining robust solutions of an uncertain multicriteria optimization problem. Contrary to the upper set less ordered robustness approach, lower set less ordered robustness is not a worst-case concept, since this robustness concept focuses on the lower bound of a set $f_{\mathcal{U}}(\overline{x})$. This optimistic concept hedges against perturbations in the best-case scenarios. Thus the decision maker is considered to be risk affine. A possible explanation for such an optimistic approach may be some knowledge that the decision maker has about the future: He may be sure that the worst case is very unlikely to happen and thus wishes to consider other solutions as well, namely those with smaller objectives in the best-case scenario.

The following lemma shows the essential result that for deterministic multi-objective optimization the concept of lower set less ordered robustness is equivalent to deterministic minimality as introduced in Definition 2 in Chapter 2.

Lemma 4. *Given $P(\mathcal{U})$ with $|\mathcal{U}| = 1$. Then x^0 is strictly (weakly, $\cdot$, respectively) lower set less ordered robust if and only if $f(x^0)$ is strictly (weakly, $\cdot$, respectively) minimal.*

Proof. The following holds for $Q = C$ ($Q = \operatorname{int} C$, $Q = C \setminus \{0\}$, respectively):

$$\begin{aligned}
& x^0 \text{ is strictly (weakly, } \cdot \text{ , respectively) lower set less ordered robust} \\
& \iff \nexists\, \overline{x} \in \mathcal{X} \setminus \{x^0\} : \; f_{\mathcal{U}}(\overline{x}) + Q \supseteq f_{\mathcal{U}}(x^0) \\
& \iff \nexists\, \overline{x} \in \mathcal{X} \setminus \{x^0\} : \; \forall\, \eta \in \mathcal{U}\ \exists\, \xi \in \mathcal{U} : \; f(\overline{x}, \xi) + Q \ni f(x^0, \eta) \\
& \overset{|\mathcal{U}|=1}{\iff} \nexists\, \overline{x} \in \mathcal{X} \setminus \{x^0\} : \; f(\overline{x}) + Q \ni f(x^0) \\
& \iff \nexists\, \overline{x} \in \mathcal{X} \setminus \{x^0\} : \; f(\overline{x}) \in f(x^0) - Q \\
& \iff f(x^0) \text{ is strictly (weakly, } \cdot \text{ , respectively) minimal.}
\end{aligned}$$

□

Next we verify that for the scalar case ($Y = \mathbb{R}$, $X = \mathbb{R}^n$) and with the natural ordering cone $C = \mathbb{R}_{\geqq}$ the lower set less ordered robustness concept coincides with the optimistic counterpart (OC) (compare (5.11)).

Lemma 5. *Given $P(\mathcal{U})$ with $Y = \mathbb{R}$, $X = \mathbb{R}^n$ and $C = \mathbb{R}_{\geqq}$.*

(a) x^0 is weakly lower set less ordered robust $\iff$ x^0 is lower set less ordered robust.

(b) If x^0 is uniquely optimal for the optimistic counterpart (OC) (see (5.11)), then x^0 is strictly lower set less ordered robust.

(c) Suppose $\min_{\xi \in \mathcal{U}} f(x', \xi)$ exists for every $x' \in \mathcal{X}$. Then it holds: If x^0 is strictly lower set less ordered robust, then x^0 is uniquely optimal for the optimistic counterpart (OC) (see (5.11)).

(d) Suppose $\min_{\xi \in \mathcal{U}} f(x', \xi)$ exists for every $x' \in \mathcal{X}$. Then it holds: If x^0 is optimal for the optimistic counterpart (OC) (see (5.11)), then x^0 is lower set less ordered robust.

(e) If x^0 is weakly lower set less ordered robust, then x^0 is optimal for the optimistic counterpart (OC) (see (5.11)).

Proof. (a) Holds due to $\mathbb{R}_{>} = \mathbb{R}_{\geq}$.

(b) x^0 is uniquely optimal for the optimistic counterpart (OC) (see (5.11))

$$\iff \nexists\, \overline{x} \in \mathcal{X} \setminus \{x^0\} : \; \inf_{\xi \in \mathcal{U}} f(\overline{x}, \xi) \leq \inf_{\xi \in \mathcal{U}} f(x^0, \xi). \tag{5.18}$$

Now suppose that x^0 is not strictly lower set less ordered robust. Then there exists $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ s.t.

$$\begin{aligned}
& f_{\mathcal{U}}(\overline{x}) + \mathbb{R}_{\geqq} \supseteq f_{\mathcal{U}}(x^0) \\
& \iff \forall\, \eta \in \mathcal{U}\ \exists\, \xi \in \mathcal{U} : \; f(\overline{x}, \xi) + \mathbb{R}_{\geqq} \ni f(x^0, \eta) \\
& \iff \forall\, \eta \in \mathcal{U}\ \exists\, \xi \in \mathcal{U} : \; f(\overline{x}, \xi) \leq f(x^0, \eta) \\
& \implies \inf_{\xi \in \mathcal{U}} f(\overline{x}, \xi) \leq \inf_{\xi \in \mathcal{U}} f(x^0, \xi),
\end{aligned}$$

in contradiction to (5.18).

(c) x^0 is strictly lower set less ordered robust

$$
\begin{aligned}
&\Longleftrightarrow \nexists\, \overline{x} \in \mathcal{X} \setminus \{x^0\} : \; f_{\mathcal{U}}(\overline{x}) + \mathbb{R}_{\geqq} \supseteq f_{\mathcal{U}}(x^0) \\
&\Longleftrightarrow \nexists\, \overline{x} \in \mathcal{X} \setminus \{x^0\} : \; \forall\, \eta \in \mathcal{U}\ \exists\, \xi \in \mathcal{U} : \; f(\overline{x}, \xi) + \mathbb{R}_{\geqq} \ni f(x^0, \eta) \\
&\Longleftrightarrow \nexists\, \overline{x} \in \mathcal{X} \setminus \{x^0\} : \; \forall\, \eta \in \mathcal{U}\ \exists\, \xi \in \mathcal{U} : \; f(\overline{x}, \xi) \leq f(x^0, \eta) \\
&\Longleftrightarrow \forall\, \overline{x} \in \mathcal{X} \setminus \{x^0\} : \; \exists\, \eta \in \mathcal{U}\ \forall\, \xi \in \mathcal{U} : \; f(\overline{x}, \xi) > f(x^0, \eta) \\
&\Longrightarrow \forall\, \overline{x} \in \mathcal{X} \setminus \{x^0\} : \; \min_{\xi \in \mathcal{U}} f(\overline{x}, \xi) > \min_{\xi \in \mathcal{U}} f(x^0, \xi) \\
&\Longleftrightarrow x^0 \text{ is uniquely optimal for the optimistic counterpart } (OC) \text{ (see (5.11)).}
\end{aligned}
$$

(d) x^0 is optimal for the optimistic counterpart (OC) (see (5.11))

$$
\Longleftrightarrow \nexists\, \overline{x} \in \mathcal{X} \setminus \{x^0\} : \; \min_{\xi \in \mathcal{U}} f(\overline{x}, \xi) < \min_{\xi \in \mathcal{U}} f(x^0, \xi). \tag{5.19}
$$

Now suppose that x^0 is not lower set less ordered robust. Then there exists $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ s.t.

$$
\begin{aligned}
& f_{\mathcal{U}}(\overline{x}) + \mathbb{R}_{\geq} \supseteq f_{\mathcal{U}}(x^0) \\
\Longleftrightarrow \;\; & \forall\, \eta \in \mathcal{U}\ \exists\, \xi \in \mathcal{U} : \; f(\overline{x}, \xi) + \mathbb{R}_{\geq} \ni f(x^0, \eta) \\
\Longleftrightarrow \;\; & \forall\, \eta \in \mathcal{U}\ \exists\, \xi \in \mathcal{U} : \; f(\overline{x}, \xi) < f(x^0, \eta) \\
\Longrightarrow \;\; & \min_{\xi \in \mathcal{U}} f(\overline{x}, \xi) < \min_{\xi \in \mathcal{U}} f(x^0, \xi),
\end{aligned}
$$

in contradiction to (5.19).

(e) x^0 is weakly lower set less ordered robust

$$
\begin{aligned}
&\Longleftrightarrow \nexists\, \overline{x} \in \mathcal{X} \setminus \{x^0\} : \; f_{\mathcal{U}}(\overline{x}) + \mathbb{R}_{>} \supseteq f_{\mathcal{U}}(x^0) \\
&\Longleftrightarrow \nexists\, \overline{x} \in \mathcal{X} \setminus \{x^0\} : \; \forall\, \eta \in \mathcal{U}\ \exists\, \xi \in \mathcal{U} : \; f(\overline{x}, \xi) + \mathbb{R}_{>} \ni f(x^0, \eta) \\
&\Longleftrightarrow \nexists\, \overline{x} \in \mathcal{X} \setminus \{x^0\} : \; \forall\, \eta \in \mathcal{U}\ \exists\, \xi \in \mathcal{U} : \; f(\overline{x}, \xi) < f(x^0, \eta) \\
&\Longleftrightarrow \forall\, \overline{x} \in \mathcal{X} \setminus \{x^0\} : \; \exists\, \eta \in \mathcal{U}\ \forall\, \xi \in \mathcal{U} : \; f(\overline{x}, \xi) \geq f(x^0, \eta) \\
&\Longrightarrow \forall\, \overline{x} \in \mathcal{X} \setminus \{x^0\} : \; \inf_{\xi \in \mathcal{U}} f(\overline{x}, \xi) \geq \inf_{\xi \in \mathcal{U}} f(x^0, \xi) \\
&\Longleftrightarrow x^0 \text{ is optimal for the optimistic counterpart } (OC) \text{ (compare (5.11)).}
\end{aligned}
$$

$\square$

Scalarization Approach for Computing Lower Set Less Ordered Robust Points: Weighted Sum Scalarization

In the following we derive scalarization results based on weighted sums that will be helpful to obtain lower set less ordered robust solutions.

Let $y^* \in C^* \setminus \{0\}$ ($y^* \in C^\#$, respectively). Consider the weighted sum scalarization problem

$$(P\mathcal{U}^{lr}_{y^*}) \qquad \min_{x\in\mathcal{X}} \inf_{\xi\in\mathcal{U}} \; y^* \circ f(x,\xi). \tag{5.20}$$

The following theorem is presented in [48, Theorem 2.4] for the special case $Y = \mathbb{R}^k$, $X = \mathbb{R}^n$ and $C = \mathbb{R}^k_{\geqq}$.

Theorem 25. *Consider an uncertain vector optimization problem $P(\mathcal{U})$. The following statements hold:*

(a) If x^0 is a unique optimal solution of $(P\mathcal{U}^{lr}_{y^*})$ for some $y^* \in C^* \setminus \{0\}$, then x^0 is strictly lower set less ordered robust.

(b) If x^0 is an optimal solution of $(P\mathcal{U}^{lr}_{y^*})$ for some $y^* \in C^* \setminus \{0\}$ and $\min_{\xi\in\mathcal{U}} y^* \circ f(x',\xi)$ exists for all $x' \in \mathcal{X}$, then x^0 is weakly lower set less ordered robust.

(c) If x^0 is an optimal solution of $(P\mathcal{U}^{lr}_{y^*})$ for some $y^* \in C^\#$ and $\min_{\xi\in\mathcal{U}} y^* \circ f(x',\xi)$ exists for all $x' \in \mathcal{X}$ and $y^* \in C^\#$, then x^0 is lower set less ordered robust.

Proof. Let $Q = C$ ($Q = \operatorname{int} C$, $Q = C \setminus \{0\}$, respectively). Suppose that x^0 is not strictly (weakly, $\cdot$, respectively) lower set less ordered robust. Consequently, there exists an element $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ s.t. $f_{\mathcal{U}}(\overline{x}) + Q \supseteq f_{\mathcal{U}}(x^0)$. That is equivalent to

$$\begin{aligned} &\forall\, \xi \in \mathcal{U}\ \exists\, \eta \in \mathcal{U}:\ f(\overline{x},\eta) + Q \ni f(x^0,\xi) \\ \Longleftrightarrow\ &\forall\, \xi \in \mathcal{U}\ \exists\, \eta \in \mathcal{U}:\ f(\overline{x},\eta) \in f(x^0,\xi) - Q. \end{aligned} \tag{5.21}$$

Now choose $y^* \in C^* \setminus \{0\}$ for $Q = C$ ($y^* \in C^* \setminus \{0\}$ for $Q = \operatorname{int} C$, $y^* \in C^\#$ for $Q = C \setminus \{0\}$, respectively) arbitrary, but fixed. Hence, we obtain from (5.21)

$$\begin{aligned} &\Longrightarrow\ \forall\, \xi \in \mathcal{U}\ \exists\, \eta \in \mathcal{U}:\ y^* \circ f(\overline{x},\eta)\ [\leq / < / <]\ y^* \circ f(x^0,\xi) \\ &\Longrightarrow\ \inf_{\eta\in\mathcal{U}}\ y^* \circ f(\overline{x},\eta)\ [\leq / < / <]\ \inf_{\xi\in\mathcal{U}}\ y^* \circ f(x^0,\xi), \end{aligned}$$

in contradiction to the assumption. □

Based on these results, we are able to present the following algorithm that computes lower set less ordered robust solutions of an uncertain multicriteria optimization problem. We use the notation $\mathrm{Opt}_{\mathrm{slr}}$, ($\mathrm{Opt}_{\mathrm{wlr}}$, $\mathrm{Opt}_{\mathrm{lr}}$, respectively) for the set of strictly (weakly, $\cdot$, respectively) lower set less ordered robust solutions.

Algorithm 6 for deriving lower set less ordered robust solutions based on weighted sum scalarization:

Input & Steps 1-5: Analogous to Algorithm 1, only replacing $\mathrm{Opt}_{\mathrm{sur}}$, $\mathrm{Opt}_{\mathrm{ur}}$, $\mathrm{Opt}_{\mathrm{wur}}$ by $\mathrm{Opt}_{\mathrm{slr}}$, $\mathrm{Opt}_{\mathrm{lr}}$, $\mathrm{Opt}_{\mathrm{wlr}}$, $(P\mathcal{U}^{ur}_{y^*})$ (see (5.14)) by $(P\mathcal{U}^{lr}_{y^*})$ (see (5.20)), $\max_{\xi\in\mathcal{U}} y^* \circ f(x',\xi)$ by $\min_{\xi\in\mathcal{U}} y^* \circ f(x',\xi)$ in Step 5 (b) and (c) and replacing "upper" by "lower" in Step 5.

The next algorithm computes lower set less ordered robust solutions via weighted sum scalarization by altering the weights.

Algorithm 7 for computing lower set less ordered robust elements of an uncertain vector-valued optimization problem $P(\mathcal{U})$:

Input & Steps 1-8: Analogous to Algorithm 2, only replacing $\mathrm{Opt}_{\mathrm{sur}}$, $\mathrm{Opt}_{\mathrm{ur}}$, $\mathrm{Opt}_{\mathrm{wur}}$ by $\mathrm{Opt}_{\mathrm{slr}}$, $\mathrm{Opt}_{\mathrm{lr}}$, $\mathrm{Opt}_{\mathrm{wlr}}$, $(P\mathcal{U}^{ur}_{\hat{y}^*})$ (see (5.14)) by $(P\mathcal{U}^{lr}_{\hat{y}^*})$ (see (5.20)), $\max_{\xi\in\mathcal{U}} \hat{y}^* \circ f(x',\xi)$ by $\min_{\xi\in\mathcal{U}} \hat{y}^* \circ f(x',\xi)$ in Step 7 (b) and (c) and replacing "upper" by "lower" in Step 7.

Scalarization Approach for Computing Lower Set Less Ordered Robust Points: ϵ-Constraint Scalarization

In the following we provide a solution procedure for obtaining lower set less ordered robust solutions via ϵ-constraint scalarization for the special case $Y=\mathbb{R}^k$, $X=\mathbb{R}^n$, $C=\mathbb{R}^k_{\geqq}$. For this purpose, consider the ϵ-constraint minimization problem

$$(P\mathcal{U}^{lr}_{\epsilon,i}) \qquad \min \inf_{\xi\in\mathcal{U}} f_i(x,\xi) \qquad \text{s.t.} \quad \forall\, j\neq i:\ \inf_{\xi\in\mathcal{U}} f_j(x,\xi)\le \epsilon_j,\ x\in\mathcal{X}. \tag{5.22}$$

The next theorem highlights the connections between solutions of $(P\mathcal{U}^{lr}_{\epsilon,i})$ and strictly (weakly, respectively) lower set less ordered robustness.

Theorem 26. *Consider an uncertain vector-valued optimization problem $P(\mathcal{U})$ with $Y=\mathbb{R}^k$, $X=\mathbb{R}^n$, $C=\mathbb{R}^k_{\geqq}$. Then the following statements hold.*

(a) If x^0 is the unique optimal solution of $(P\mathcal{U}^{lr}_{\epsilon,i})$ for $\epsilon\in\mathbb{R}^k$ and some $i\in\{1,\dots,k\}$, then x^0 is strictly lower set less ordered robust for $P(\mathcal{U})$.

(b) If $x^0\in\mathcal{X}$ is an optimal solution of $(P\mathcal{U}^{lr}_{\epsilon,i})$ for $\epsilon\in\mathbb{R}^k$ and $\min_{\xi\in\mathcal{U}} f_j(x',\xi)$ exists for all $x'\in\mathcal{X}$ and $j=1,\dots,k$, then x^0 is weakly lower set less ordered robust for $P(\mathcal{U})$.

Proof. Suppose that x^0 is not strictly (weakly, respectively) lower set less ordered robust for $P(\mathcal{U})$. Thus there is an element $\overline{x}\in\mathcal{X}\setminus\{x^0\}$ with

$$f_{\mathcal{U}}(\overline{x}) \preceq^l_Q f_{\mathcal{U}}(x^0),$$

where $\preceq^l_Q$ is given with respect to $Q = \mathbb{R}^k_{\geqq}$ ($Q = \mathbb{R}^k_{>}$, respectively). This is equivalent to

$$f_{\mathcal{U}}(x^0) \subseteq f_{\mathcal{U}}(\overline{x}) + Q.$$

In other words,

$$\begin{aligned}
&\forall\, \xi \in \mathcal{U}\ \exists\, \eta \in \mathcal{U}:\ f(\overline{x},\eta) \in f(x^0,\xi) - Q\\
\Longleftrightarrow\ &\forall\, \xi \in \mathcal{U}\ \exists\, \eta \in \mathcal{U}:\ f(\overline{x},\eta)\ [\leqq / <]\ f(x^0,\xi)\\
\Longleftrightarrow\ &\forall\, \xi \in \mathcal{U}\ \exists\, \eta \in \mathcal{U}:\ f_j(\overline{x},\eta)\ [\leq / <]\ f_j(x^0,\xi),\ j = 1,\dots,k\\
\Longleftrightarrow\ &\exists\, \eta \in \mathcal{U}: f_j(\overline{x},\eta)\ [\leq / <]\ \inf_{\xi\in\mathcal{U}} f_j(x^0,\xi),\ j = 1,\dots,k
\end{aligned}$$

and this yields

$$\inf_{\xi\in\mathcal{U}} f_j(\overline{x},\xi)\ [\leq / <]\ \inf_{\xi\in\mathcal{U}} f_j(x^0,\xi),\ j = 1,\dots,k. \tag{5.23}$$

Because $\inf_{\xi\in\mathcal{U}} f_j(x^0,\xi) \leq \epsilon_j$ for every $j = 1,\dots,k,\ j \neq i$, $\overline{x}$ is also feasible and has an equal or better objective function value (a better objective function value, respectively) than x^0, contradicting the assumption that x^0 is the unique optimal (an optimal, respectively) solution of $(P\mathcal{U}^{lr}_{\epsilon,i})$. Note that the strict inequality in (5.23) holds because the existence of $\min_{\xi\in\mathcal{U}} f_j(x',\xi)$ for all $x' \in \mathcal{X}$ and $j = 1,\dots,k$ is presumed. □

The previous theorem enables us to present the following algorithm.

Algorithm 8 for solving $P(\mathcal{U})$ based on ϵ-constraint method:

Input, Steps 1-5: Analogous to Algorithm 3, only replacing Opt_{sur}, Opt_{wur} by Opt_{slr}, Opt_{wlr}, $(P\mathcal{U}^{ur}_{\epsilon,i})$ (see (5.16)) by $(P\mathcal{U}^{lr}_{\epsilon,i})$ (see (5.22)) and $\max_{\xi\in\mathcal{U}} f_j(x',\xi)$ by $\min_{\xi\in\mathcal{U}} f_j(x',\xi)$ in Step 5 (b), and replacing "upper" by "lower" in Step 5.

Scalarization Approach for Computing Lower Set Less Ordered Robust Points: Max-Ordering Scalarization

In this subsection, we focus once more on the special case $Y = \mathbb{R}^k,\ X = \mathbb{R}^n,\ C = \mathbb{R}^k_{\geqq}$ and present a scalarization method for computing lower set less ordered robust solutions of $P(\mathcal{U})$. Let $y^* \in \mathbb{R}^k_{>}$. Consider the max-ordering problem

$$(P\mathcal{U}^{max,lr}_{y^*}) \qquad \min_{x\in\mathcal{X}} \max_{i=1,\dots,k} \inf_{\xi\in\mathcal{U}}\ y^*_i f_i(x,\xi). \tag{5.24}$$

The following theorem describes the relationship between solutions of $(P\mathcal{U}^{max,lr}_{y^*})$ and lower set less ordered robust elements.

Theorem 27. *Consider an uncertain vector optimization problem $P(\mathcal{U})$ with $Y = \mathbb{R}^k$, $X = \mathbb{R}^n$, $C = \mathbb{R}^k_{\geqq}$. The following statements hold:*

(a) If x^0 is a unique optimal solution of $(P\mathcal{U}^{max,lr}_{y^*})$ for some $y^* \in \mathbb{R}^k_{>}$, then x^0 is strictly lower set less ordered robust.

(b) If x^0 is an optimal solution of $(P\mathcal{U}_{y^*}^{max,lr})$ for some $y^* \in \mathbb{R}^k_>$ and $\min_{\xi\in\mathcal{U}} y_i^* f_i(x',\xi)$ exists for all $x' \in \mathcal{X}$, $i = 1,\dots,k$, then x^0 is weakly lower set less ordered robust.

Proof. Suppose x^0 is not strictly (weakly, respectively) lower set less ordered robust. Then there is an element $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ s.t. $f_{\mathcal{U}}(\overline{x}) + Q \supseteq f_{\mathcal{U}}(x^0)$ for $Q = \mathbb{R}^k_\geqq$ ($Q = \mathbb{R}^k_>$, respectively). That is equivalent to

$$\begin{aligned} &\forall\, \xi \in \mathcal{U}\ \exists\, \eta \in \mathcal{U}:\ f(\overline{x},\eta) + Q \ni f(x^0,\xi) \\ \iff &\forall\, \xi \in \mathcal{U}\ \exists\, \eta \in \mathcal{U}:\ f(\overline{x},\eta) \in f(x^0,\xi) - Q. \end{aligned} \tag{5.25}$$

Now choose $y^* \in \mathbb{R}^k_>$ arbitrary, but fixed. Hence, we obtain from (5.25)

$$\begin{aligned} &\Longrightarrow\ \forall\, \xi \in \mathcal{U}\ \exists\, \eta \in \mathcal{U}:\ y_i^* f_i(\overline{x},\eta)\ [\leq / <]\ y_i^* f_i(x^0,\xi),\ i = 1,\dots,k, \\ &\Longrightarrow\ \exists\, \eta \in \mathcal{U}:\ y_i^* f_i(\overline{x},\eta)\ [\leq / <]\ \inf_{\xi\in\mathcal{U}}\ y_i^* f_i(x^0,\xi),\ i = 1,\dots,k, \\ &\Longrightarrow\ \inf_{\eta\in\mathcal{U}}\ y_i^* f_i(\overline{x},\eta)\ [\leq / <]\ \inf_{\xi\in\mathcal{U}}\ y_i^* f_i(x^0,\xi),\ i = 1,\dots,k, \end{aligned}$$

in contradiction to the assumption. □

Theorem 27 enables us to provide the following algorithm for deriving strictly and weakly lower set less ordered robust solutions of an uncertain vector optimization problem.

Algorithm 9 for computing lower set less ordered robust solutions based on max-ordering scalarization:

Input, Steps 1-5: Analogous to Algorithm 4, only replacing Opt_{sur}, Opt_{wur} by Opt_{slr}, Opt_{wlr}, $(P\mathcal{U}_{y^*}^{max,ur})$ (see (5.17)) by $(P\mathcal{U}_{y^*}^{max,lr})$ (see (5.24)) and $\max_{\xi\in\mathcal{U}} y_i^* f_i(x',\xi)$ by $\min_{\xi\in\mathcal{U}} y_i^* f_i(x',\xi)$ in Step 5 (b), and replacing "upper" by "lower" in Step 5.

The following algorithm provides an interactive procedure for deriving (weakly, strictly) lower set less ordered robust solutions.

Algorithm 10 for computing lower set less ordered robust solutions based on max-ordering scalarization:

Input, Steps 1-8: Analogous to Algorithm 5, only replacing Opt_{sur}, Opt_{wur} by Opt_{slr}, Opt_{wlr}, $(P\mathcal{U}_{\hat{y}^*}^{max,ur})$ (see (5.17)) by $(P\mathcal{U}_{\hat{y}^*}^{max,lr})$ (see (5.24)) and $\max_{\xi\in\mathcal{U}} \hat{y}_i^* f_i(x',\xi)$ by $\min_{\xi\in\mathcal{U}} \hat{y}_i^* f_i(x',\xi)$ in Step 7 (b), and replacing "upper" by "lower".

Vectorization Approach for Computing Lower Set Less Ordered Robust Points

In addition to deriving scalarization techniques for obtaining lower set less ordered robust solutions to $P(\mathcal{U})$, we provide the following vectorization method for the special case $Y = \mathbb{R}^k,\ X = \mathbb{R}^n,\ C = \mathbb{R}^k_{\geqq}$. Here, vectorization means that the uncertain vector-valued optimization problem $P(\mathcal{U})$ is reduced to a deterministic vector-valued problem which can be used to determine lower set less ordered robust solutions.

Consider the multi-objective problem

$$(VOP^{lr}) \qquad \operatorname{Min}(f[\mathcal{X}], \mathbb{R}^k_{\geqq}) \tag{5.26}$$

with $f(x) := (\inf_{\xi\in\mathcal{U}} f_1(x,\xi), \dots, \inf_{\xi\in\mathcal{U}} f_k(x,\xi))^T$ for $x \in \mathcal{X}$.

In the following theorem we observe that strictly (weakly, respectively) Pareto optimal solutions of the above problem (VOP^{lr}) are strictly (weakly, respectively) lower set less ordered robust elements of the uncertain multi-objective optimization problem $P(\mathcal{U})$.

Theorem 28. *Given an uncertain vector-valued optimization problem $P(\mathcal{U})$ with $Y = \mathbb{R}^k,\ X = \mathbb{R}^n,\ C = \mathbb{R}^k_{\geqq}$. The following statements hold.*

(a) If x^0 is strictly Pareto optimal for (VOP^{lr}), then x^0 is strictly lower set less ordered robust.

(b) If x^0 is weakly Pareto optimal for (VOP^{lr}) and $\min_{\xi\in\mathcal{U}} f_i(x',\xi)$ exist for all $x' \in \mathcal{X}$ and $i = 1,\dots,k$, then x^0 is weakly lower set less ordered robust.

Proof. Suppose that x^0 is not strictly (weakly, respectively) lower set less ordered robust. Then there exists $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ s.t. for $Q = \mathbb{R}^k_{\geq}$ ($Q = \mathbb{R}^k_{>}$, respectively)

$$\begin{aligned}
f_{\mathcal{U}}(\overline{x}) + Q \supseteq f_{\mathcal{U}}(x^0) &\iff \forall\, \xi \in \mathcal{U}\ \exists\, \eta \in \mathcal{U}:\ f(\overline{x},\eta) + Q \ni f(x^0,\xi)\\
&\iff \forall\, \xi \in \mathcal{U}\ \exists\, \eta \in \mathcal{U}:\ f_i(\overline{x},\eta)\ [\leq / <]\ f_i(x^0,\xi),\ i = 1,\dots,k,\\
&\implies \exists\, \eta \in \mathcal{U}:\ f_i(\overline{x},\eta)\ [\leq / <]\ \inf_{\xi\in\mathcal{U}} f_i(x^0,\xi),\ i = 1,\dots,k,\\
&\implies \inf_{\xi\in\mathcal{U}} f_i(\overline{x},\xi)\ [\leq / <]\ \inf_{\xi\in\mathcal{U}} f_i(x^0,\xi),\ i = 1,\dots,k,
\end{aligned}$$

in contradiction to the assumption of x^0 being strictly (weakly, respectively) Pareto optimal for (VOP^{lr}). $\square$

Algorithm 11 for computing lower set less ordered robust solutions using a vectorization approach:

Input: Uncertain vector-valued problem $P(\mathcal{U})$, solution sets $\mathrm{Opt}_{\mathrm{slr}} = \mathrm{Opt}_{\mathrm{wlr}} = \emptyset$.

Step 1: Find a set $\mathrm{SOL}_{\mathrm{we}}(VOP^{lr})$ of weakly Pareto optimal solutions of (VOP^{lr}) (see (5.26)).

Step 2: If $\mathrm{SOL}_{\mathrm{we}}(VOP^{lr}) = \emptyset$, then **STOP**. **Output**: Set of strictly lower set less ordered robust solutions $\mathrm{Opt}_{\mathrm{slr}}$ and set of weakly lower set less ordered robust solutions $\mathrm{Opt}_{\mathrm{wlr}}$.

Step 3: Choose $x \in \mathrm{SOL}_{\mathrm{we}}(VOP^{lr})$. Set $\mathrm{SOL}_{\mathrm{we}}(VOP^{lr}) := \mathrm{SOL}_{\mathrm{we}}(VOP^{lr}) \setminus \{x\}$.

a) If x is a strictly Pareto optimal solution of (VOP^{lr}) (see (5.26)), then x is strictly lower set less ordered robust for $P(\mathcal{U})$, thus

$$\mathrm{Opt}_{\mathrm{slr}} := \mathrm{Opt}_{\mathrm{slr}} \cup \{x\}.$$

b) If $\min_{\xi \in \mathcal{U}} \ f_i(x', \xi)$ exists for all $x' \in \mathcal{X}, \ i = 1, \dots, k$, then x is weakly lower set less ordered robust for $P(\mathcal{U})$, thus

$$\mathrm{Opt}_{\mathrm{wlr}} := \mathrm{Opt}_{\mathrm{wlr}} \cup \{x\}.$$

Go to Step 2.

Interestingly, applying the weighted sum / ϵ-constraint / max-ordering scalarization to problem (VOP^{lr}) (see (5.26)) yields the same scalarization approach as $(P\mathcal{U}_{y^*}^{lr})$ (see (5.20)) / $(P\mathcal{U}_{\epsilon,i}^{lr})$ (see 5.22) / $(P\mathcal{U}_{y^*}^{max,lr})$ (see (5.24)).

Remark 24. *Applying weighted sum / ϵ-constraint / max-ordering scalarization to (VOP^{lr}) (see (5.26)) yields problem $(P\mathcal{U}_{y^*}^{lr})$ (see (5.20)) / $(P\mathcal{U}_{\epsilon,i}^{lr})$ (see 5.22) / $(P\mathcal{U}_{y^*}^{max,lr})$ (compare (5.24)). In addition, Theorem 28 implies that solutions of scalarizations of (VOP^{lr}) are connected to lower set less ordered robust elements of $P(\mathcal{U})$. For example, the nonlinear scalarization functional z (compare (2.2)) as discussed in Chapter 2 may be used to scalarize (VOP^{lr}) using different involved parameters, thus enabling numerous scalarization techniques to be applied to (VOP^{lr}). Furthermore, note that for $|\mathcal{U}| = 1$, (VOP^{lr}) reduces to a classical deterministic vector-valued optimization problem. In case $Y = \mathbb{R}$, (VOP^{lr}) is equivalent to the optimistic counterpart (OC) (see (5.11)).*

In the following we investigate under which conditions the inverse direction in Theorem 28 is satisfied. To this end, we introduce *object-wise uncertainty*, which is motivated by the notion of constraint-wise uncertainty (see Ben-Tal et al. [8]) and has first been introduced by Ehrgott et al. [25, Definition 5.1].

Definition 17 (Object-wise uncertainty, [25])**.** *An uncertain multi-objective optimization problem $P(\mathcal{U})$ with objective function $f: \ \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^k$ is called* object-wise uncertain *if the uncertainties in the objective function are independent of each other, i.e., if $\mathcal{U} := \mathcal{U}_1, \dots, \mathcal{U}_k$ with $\mathcal{U}_i \in \mathbb{R}^{N_i}, \ i = 1, \dots, k, \ \sum_{i=1}^k N_i = N$ such that*

$$f(x, \xi) = \begin{pmatrix} f_1(x, \xi_1) \\ \dots \\ f_k(x, \xi_k) \end{pmatrix}$$

with $\xi_i \in \mathcal{U}_i, \ i = 1, \dots, k$.

Note that constraint-wise uncertainty plays an important role in applications, for instance in the case where the objective coefficients are noisy.

We first mention the essential property that the existence of a best-case scenario for every objective function $\min_{\xi_i \in \mathcal{U}_i} f_i(x, \xi_i),\ i = 1, \ldots, k$, implies that a best-case scenario exists for the uncertain vector-valued optimization problem.

Lemma 6. *Let $P(\mathcal{U})$ be object-wise uncertain and suppose that $\min_{\xi_i \in \mathcal{U}_i} f_i(x, \xi_i)$ exists for all $x \in \mathcal{X}$ and $i = 1, \ldots, k$. Then*

$$\xi^{min}(x) := \begin{pmatrix} \operatorname{argmin}_{\xi_1 \in \mathcal{U}_1} f_1(x, \xi_1) \\ \cdots \\ \operatorname{argmin}_{\xi_k \in \mathcal{U}_k} f_k(x, \xi_k) \end{pmatrix} \in \mathcal{U}.$$

Proof. We have that for every $x \in \mathcal{X}$ and $i = 1, \ldots, k$:

$$\operatorname{argmin}_{\xi_i \in \mathcal{U}_i} f_i(x, \xi_i) \in \mathcal{U}_i,$$

thus $\xi^{min}(x) \in \mathcal{U}_1 \times \ldots \times \mathcal{U}_k = \mathcal{U}$. □

Note that Ehrgott et al. [25] study the property of object-wise uncertainty for uncertain vector-valued optimization problems to obtain upper set less ordered robust solutions. Therefore, the authors in [25] consider

$$\xi^{max}(x) := (\operatorname{argmax}_{\xi_1 \in \mathcal{U}_1} f_1(x, \xi_1), \ldots, \operatorname{argmax}_{\xi_k \in \mathcal{U}_k} f_k(x, \xi_k))^T$$

and obtain a result similar to Lemma 6.

Next we derive the important result that object-wise uncertain vector-valued optimization problems are equivalent to deterministic multi-objective optimization.

Theorem 29. *Assume that the uncertain multi-objective problem $P(\mathcal{U})$ is object-wise uncertain and $\min_{\xi_i \in \mathcal{U}_i} f_i(x', \xi_i)$ exist for all $x' \in \mathcal{X}$ and $i = 1, \ldots, k$. Then x^0 is strictly (· , weakly, respectively) Pareto optimal for (VOP^{lr}) (see (5.26)) if and only if x^0 is strictly (· , weakly, respectively) lower set less ordered robust.*

Proof. Due to Lemma 6

$$f(x, \xi^{min}) := \begin{pmatrix} \min_{\xi_1 \in \mathcal{U}_1} f_1(x, \xi_1) \\ \cdots \\ \min_{\xi_k \in \mathcal{U}_k} f_k(x, \xi_k) \end{pmatrix} \in f_{\mathcal{U}}(x)$$

for all $x \in \mathcal{X}$. Thus

$$f(x, \xi^{min}) + Q \subseteq f_{\mathcal{U}}(x) + Q \tag{5.27}$$

for $Q = \mathbb{R}^k_{[\geqq/\geq/>]}$. We now prove "$\supseteq$" in (5.27). Consider $y \in f_{\mathcal{U}}(x) + Q$. Thus, there exists $\xi \in \mathcal{U}$ such that $y \in f(x, \xi) + Q$. Due to the definition of ξ^{min}, we have $f(x, \xi) \in f(x, \xi^{min}) + \mathbb{R}^k_{\geqq}$, and this implies $y \in f(x, \xi^{min}) + Q$, since $Q + \mathbb{R}^k_{\geqq} \subseteq Q$. □

Ehrgott [24, Theorem 4.5] showed that a feasible solution $f(x)$ is minimal for a vector-valued optimization problem if and only if there exists an ϵ such that x solves the corresponding ϵ-constraint scalarized problem. From that result, we deduce the following lemma.

Lemma 7. *Assume that the uncertain multi-objective problem $P(\mathcal{U})$ is object-wise uncertain and $\min_{\xi_i \in \mathcal{U}_i} f_i(x', \xi_i)$ exists for all $x' \in \mathcal{X}$ and $i = 1, \dots, k$. Then x^0 is lower set less ordered robust if and only if there exists an $\epsilon \in \mathbb{R}^k$ such that x^0 is an optimal solution to problem $(P\mathcal{U}^{lr}_{\epsilon,i})$ (see (5.22)) for all $i = 1, \dots, k$.*

Note that an equivalent result has been obtained by Ehrgott et al. [25, Corollary 5.6] for upper set less ordered robust solutions.

5.3.3 Set Less Ordered Robustness

Definition 6 motivates us to extend the set less order relation to general nonempty sets $Q \subset Y$. To this end, let $C \subset Y$ be a proper closed convex and pointed cone. Suppose $C \subset \operatorname{cl} Q$ and $\operatorname{cl} Q \cap (-\operatorname{cl} Q) = \{0\}$. Under these assumptions, we define the *set less order relation* $\preceq^s_Q$ for two sets $A, B \subset Y$

$$
\begin{aligned}
A \preceq^s_Q B :&\Longleftrightarrow A \subseteq B - Q \text{ and } A + Q \supseteq B \\
&\Longleftrightarrow (\forall\, a \in A\ \exists\, b \in B : a \leq_Q b) \text{ and } (\forall\, b \in B\ \exists\, a \in A : a \leq_Q b).
\end{aligned}
$$

In case we are dealing with $Q = \operatorname{int} C$, we suppose that $\operatorname{int} C \neq \emptyset$.

Definition 18. *A solution x^0 of $P(\mathcal{U})$ is called strictly (weakly, $\cdot$, respectively)* **set less ordered robust** *if there is no $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ such that $f_\mathcal{U}(\overline{x}) \preceq^s_Q f_\mathcal{U}(x^0)$, which is equivalent to*

$$
\nexists\, \overline{x} \in \mathcal{X} \setminus \{x^0\} : \quad f_\mathcal{U}(\overline{x}) + Q \supseteq f_\mathcal{U}(x^0) \text{ and } f_\mathcal{U}(\overline{x}) \subseteq f_\mathcal{U}(x^0) - Q
$$

for $Q = C$ ($Q = \operatorname{int} C$, $Q = C \setminus \{0\}$, respectively).

Example 7. *In Figure 5.6, x_1 is set less ordered robust with $Q = \mathbb{R}^2_\geq$.*

The motivation of introducing this concept of robustness is the following: Consider Example 7. Although x_1 is not weakly upper set less ordered robust, there are scenarios $\xi \in \mathcal{U}$ for which the objective $f(x_1, \xi)$ is smaller than $f(x_2, \xi)$ in at least one component f_i. Thus, the definition of set less ordered robustness would reflect a decision maker's strategy if he is not only interested in minimizing the worst-case, but does not want to neglect solutions that are smaller in at least one scenario. Consequently, this approach could possibly yield a larger number of solutions than the upper set less ordered robustness approach, which could give the decision maker a wider choice of solutions.

First we verify that in case the uncertainty set $\mathcal{U}$ contains only one element, set less ordered robustness is equivalent to the minimality concept introduced in Definition 2.

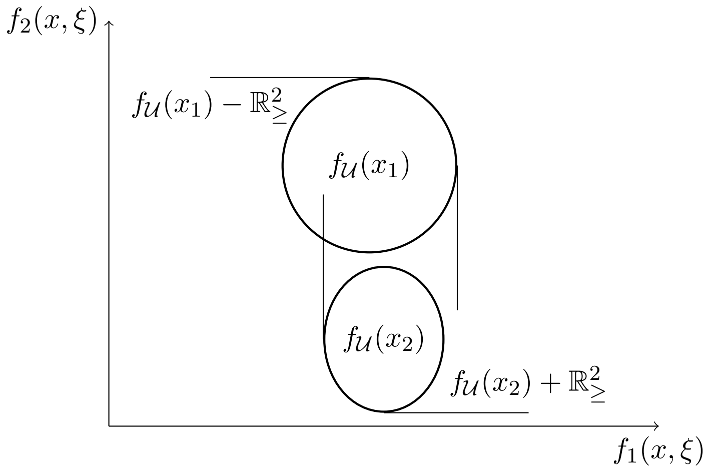

Figure 5.6: x_1 and x_2 are set less ordered robust.

Lemma 8. *Given $P(\mathcal{U})$ with $|\mathcal{U}| = 1$. Then x^0 is strictly (weakly, $\cdot$, respectively) set less ordered robust if and only if $f(x^0)$ is strictly (weakly, $\cdot$, respectively) minimal.*

Proof. The following holds for $Q = C$ ($Q = \operatorname{int} C$, $Q = C \setminus \{0\}$, respectively): x^0 is strictly (weakly, $\cdot$, respectively) set less ordered robust

$$\begin{aligned}
&\iff \nexists\, \overline{x} \in \mathcal{X} \setminus \{x^0\} : \ f_{\mathcal{U}}(\overline{x}) + Q \supseteq f_{\mathcal{U}}(x^0) \text{ and } f_{\mathcal{U}}(\overline{x}) \subseteq f_{\mathcal{U}}(x^0) - Q \\
&\iff \nexists\, \overline{x} \in \mathcal{X} \setminus \{x^0\} : \ \forall\, \eta_1, \eta_2 \in \mathcal{U}\ \exists\, \xi_1,\ \xi_2 \in \mathcal{U} : \ f(\overline{x}, \xi_1) + Q \ni f(x^0, \eta_1) \\
&\qquad \text{and } f(\overline{x}, \eta_2) \in f(x^0, \xi_2) - Q \\
&\overset{|\mathcal{U}|=1}{\iff} \nexists\, \overline{x} \in \mathcal{X} \setminus \{x^0\} : \ f(\overline{x}) \in f(x^0) - Q \\
&\iff f(x^0) \text{ is strictly (weakly, } \cdot \text{ , respectively) minimal.}
\end{aligned}$$

□

Next we check for consistency of the introduced set less ordered robustness concept in the scalar case $Y = \mathbb{R}$, $X = \mathbb{R}^n$ and $C = \mathbb{R}_{\geqq}$.

Lemma 9. *Given $P(\mathcal{U})$ with $Y = \mathbb{R}$, $X = \mathbb{R}^n$ and $C = \mathbb{R}_{\geqq}$.*

(a) x^0 is weakly set less ordered robust $\iff$ x^0 is set less ordered robust.

(b) If x^0 is uniquely optimal for the robust counterpart (RC) (see (5.10)) or for the optimistic counterpart (OC) (see (5.11)), then x^0 is strictly set less ordered robust.

(c) Suppose $\max_{\xi \in \mathcal{U}} f(x', \xi)$ and $\min_{\xi \in \mathcal{U}} f(x', \xi)$ exist for every $x' \in \mathcal{X}$. Then it holds: If x^0 is strictly set less ordered robust, then x^0 is uniquely optimal for the robust counterpart (RC) (see (5.10)) or for the optimistic counterpart (OC) (see (5.11)).

(d) *Suppose* $\max_{\xi\in\mathcal{U}} f(x',\xi)$ *and* $\min_{\xi\in\mathcal{U}} f(x',\xi)$ *exist for every* $x' \in \mathcal{X}$. *Then it holds: If* x^0 *is optimal for the robust counterpart* (RC) *(see* (5.10)*) or for the optimistic counterpart* (OC) *(see* (5.11)*), then* x^0 *is set less ordered robust.*

(e) *If* x^0 *is weakly set less ordered robust, then* x^0 *is optimal for the robust counterpart* (RC) *(see* (5.10)*) or for the optimistic counterpart* (OC) *(see* (5.11)*).*

Proof. (a) Holds due to $\mathbb{R}_> = \mathbb{R}_\geq$.

(b) x^0 is uniquely optimal for the robust counterpart (RC) (compare (5.10))

$$\Longleftrightarrow \nexists\ \overline{x} \in \mathcal{X} \setminus \{x^0\}: \ \sup_{\xi\in\mathcal{U}} f(\overline{x},\xi) \leq \sup_{\xi\in\mathcal{U}} f(x^0,\xi) \tag{5.28}$$

or for the optimistic counterpart (OC) (see (5.11))

$$\Longleftrightarrow \nexists\ \overline{x} \in \mathcal{X} \setminus \{x^0\}: \ \inf_{\xi\in\mathcal{U}} f(\overline{x},\xi) \leq \inf_{\xi\in\mathcal{U}} f(x^0,\xi). \tag{5.29}$$

Now suppose that x^0 is not strictly set less ordered robust. Then there exists $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ s.t.

$$\begin{aligned}
& f_{\mathcal{U}}(\overline{x}) + \mathbb{R}_{\geqq} \supseteq f_{\mathcal{U}}(x^0) \text{ and } f_{\mathcal{U}}(\overline{x}) \subseteq f_{\mathcal{U}}(x^0) - \mathbb{R}_{\geqq} \\
\Longleftrightarrow\ & \forall\ \eta_1,\eta_2 \in \mathcal{U}\ \exists\ \xi_1,\xi_2 \in \mathcal{U}: \ f(\overline{x},\xi_1) + \mathbb{R}_{\geqq} \ni f(x^0,\eta_1) \\
& \quad \text{and } f(\overline{x},\eta_2) \in f(x^0,\xi_2) - \mathbb{R}_{\geqq} \\
\Longleftrightarrow\ & \forall\ \eta_1,\eta_2 \in \mathcal{U}\ \exists\ \xi_1,\xi_2 \in \mathcal{U}: \ f(\overline{x},\xi_1) \leq f(x^0,\eta_1) \text{ and } f(\overline{x},\eta_2) \leq f(x^0,\xi_2) \\
\Longrightarrow\ & \sup_{\xi\in\mathcal{U}} f(\overline{x},\xi) \leq \sup_{\xi\in\mathcal{U}} f(x^0,\xi) \text{ and } \inf_{\xi\in\mathcal{U}} f(\overline{x},\xi) \leq \inf_{\xi\in\mathcal{U}} f(x^0,\xi),
\end{aligned}$$

in contradiction to (5.28) and (5.29).

(c) x^0 is strictly set less ordered robust

$$\begin{aligned}
\Longleftrightarrow\ & \nexists\ \overline{x} \in \mathcal{X} \setminus \{x^0\}: \ f_{\mathcal{U}}(\overline{x}) + \mathbb{R}_{\geqq} \supseteq f_{\mathcal{U}}(x^0) \text{ and } f_{\mathcal{U}}(\overline{x}) \subseteq f_{\mathcal{U}}(x^0) - \mathbb{R}_{\geqq} \\
\Longleftrightarrow\ & \nexists\ \overline{x} \in \mathcal{X} \setminus \{x^0\}: \ \forall\ \eta_1,\eta_2 \in \mathcal{U}\ \exists\ \xi_1,\xi_2 \in \mathcal{U}: \ f(\overline{x},\xi_1) + \mathbb{R}_{\geqq} \ni f(x^0,\eta_1) \\
& \quad \text{and } f(\overline{x},\eta_2) \in f(x^0,\xi_2) - \mathbb{R}_{\geqq} \\
\Longleftrightarrow\ & \nexists\ \overline{x} \in \mathcal{X} \setminus \{x^0\}: \ \forall\ \eta_1,\eta_2 \in \mathcal{U}\ \exists\ \xi_1,\xi_2 \in \mathcal{U}: \ f(\overline{x},\xi_1) \leq f(x^0,\eta_1) \\
& \quad \text{and } f(\overline{x},\eta_2) \leq f(x^0,\xi_2) \\
\Longleftrightarrow\ & \forall\ \overline{x} \in \mathcal{X} \setminus \{x^0\}: \ \exists\ \eta_1,\eta_2 \in \mathcal{U}\ \forall\ \xi_1,\xi_2 \in \mathcal{U}: \ f(\overline{x},\xi_1) > f(x^0,\eta_1) \\
& \quad \text{or } f(\overline{x},\eta_2) > f(x^0,\xi_2) \\
\Longrightarrow\ & \forall\ \overline{x} \in \mathcal{X} \setminus \{x^0\}: \ \max_{\xi\in\mathcal{U}} f(\overline{x},\xi) > \max_{\xi\in\mathcal{U}} f(x^0,\xi) \text{ or } \min_{\xi\in\mathcal{U}} f(\overline{x},\xi) > \min_{\xi\in\mathcal{U}} f(x^0,\xi) \\
\Longleftrightarrow\ & x^0 \text{ is uniquely optimal for the robust counterpart } (RC) \text{ (see (5.10))} \\
& \quad \text{or for the optimistic counterpart } (OC) \text{ (see (5.11))}.
\end{aligned}$$

(d) x^0 is optimal for the robust counterpart (RC) (see (5.10))

$$\Longleftrightarrow \nexists\ \overline{x} \in \mathcal{X} \setminus \{x^0\}:\ \max_{\xi\in\mathcal{U}} f(\overline{x},\xi) < \max_{\xi\in\mathcal{U}} f(x^0,\xi) \tag{5.30}$$

or for the optimistic counterpart (OC) (see (5.11))

$$\Longleftrightarrow \nexists\ \overline{x} \in \mathcal{X} \setminus \{x^0\}:\ \min_{\xi\in\mathcal{U}} f(\overline{x},\xi) < \min_{\xi\in\mathcal{U}} f(x^0,\xi). \tag{5.31}$$

Now suppose x^0 is not set less ordered robust. Then there exists $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ s.t.

$$\begin{aligned}
& f_{\mathcal{U}}(\overline{x}) + \mathbb{R}_{\geq} \supseteq f_{\mathcal{U}}(x^0) \text{ and } f_{\mathcal{U}}(\overline{x}) \subseteq f_{\mathcal{U}}(x^0) - \mathbb{R}_{\geq} \\
\Longleftrightarrow\ & \forall\ \eta_1,\eta_2 \in \mathcal{U}\ \exists\ \xi_1,\xi_2 \in \mathcal{U}:\ f(\overline{x},\xi_1) + \mathbb{R}_{\geq} \ni f(x^0,\eta_1) \\
& \quad \text{and } f(\overline{x},\eta_2) \in f(x^0,\xi_2) - \mathbb{R}_{\geq} \\
\Longleftrightarrow\ & \forall\ \eta_1,\eta_2 \in \mathcal{U}\ \exists\ \xi_1,\xi_2 \in \mathcal{U}:\ f(\overline{x},\xi_1) < f(x^0,\eta_1) \text{ and } f(\overline{x},\eta_2) < f(x^0,\xi_2) \\
\Longrightarrow\ & \max_{\xi\in\mathcal{U}} f(\overline{x},\xi) < \max_{\xi\in\mathcal{U}} f(x^0,\xi) \text{ and } \min_{\xi\in\mathcal{U}} f(\overline{x},\xi) < \min_{\xi\in\mathcal{U}} f(x^0,\xi),
\end{aligned}$$

in contradiction to (5.30) and (5.31).

(e) x^0 is weakly set less ordered robust

$$\begin{aligned}
\Longleftrightarrow\ & \nexists\ \overline{x} \in \mathcal{X} \setminus \{x^0\}:\ f_{\mathcal{U}}(\overline{x}) + \mathbb{R}_{>} \supseteq f_{\mathcal{U}}(x^0) \text{ and } f_{\mathcal{U}}(\overline{x}) \subseteq f_{\mathcal{U}}(x^0) - \mathbb{R}_{>} \\
\Longleftrightarrow\ & \nexists\ \overline{x} \in \mathcal{X} \setminus \{x^0\}:\ \forall\ \eta_1,\eta_2 \in \mathcal{U}\ \exists\ \xi_1,\xi_2 \in \mathcal{U}:\ f(\overline{x},\xi_1) + \mathbb{R}_{>} \ni f(x^0,\eta_1) \\
& \quad \text{and } f(\overline{x},\eta_2) \in f(x^0,\xi_2) - \mathbb{R}_{>} \\
\Longleftrightarrow\ & \nexists\ \overline{x} \in \mathcal{X} \setminus \{x^0\}:\ \forall\ \eta_1,\eta_2 \in \mathcal{U}\ \exists\ \xi_1,\xi_2 \in \mathcal{U}:\ f(\overline{x},\xi_1) < f(x^0,\eta_1) \\
& \quad \text{and } f(\overline{x},\eta_2) < f(x^0,\xi_2) \\
\Longleftrightarrow\ & \forall\ \overline{x} \in \mathcal{X} \setminus \{x^0\}:\ \exists\ \eta_1,\eta_2 \in \mathcal{U}\ \forall\ \xi_1,\xi_2 \in \mathcal{U}:\ f(\overline{x},\xi_1) \geq f(x^0,\eta_1) \\
& \quad \text{or } f(\overline{x},\eta_2) \geq f(x^0,\xi_2) \\
\Longrightarrow\ & \forall\ \overline{x} \in \mathcal{X} \setminus \{x^0\}:\ \sup_{\xi\in\mathcal{U}} f(\overline{x},\xi) \geq \sup_{\xi\in\mathcal{U}} f(x^0,\xi) \text{ or } \inf_{\xi\in\mathcal{U}} f(\overline{x},\xi) \geq \inf_{\xi\in\mathcal{U}} f(x^0,\xi) \\
\Longleftrightarrow\ & x^0 \text{ is optimal for the robust counterpart } (RC) \text{ (see (5.10))} \\
& \quad \text{or for the optimistic counterpart } (OC) \text{ (see (5.11)).}
\end{aligned}$$

□

The following theorem reveals an important connection between the set of strictly (weakly, · , respectively) upper / lower set less ordered robust solutions and the set of strictly (weakly, · , respectively) set less ordered robust points.

Theorem 30. *If x^0 is strictly (weakly, · , respectively) upper / lower set less ordered robust, then x^0 is strictly (weakly, · , respectively) set less ordered robust.*

Proof. Let $Q = C$ ($Q = \operatorname{int} C$, $Q = C \setminus \{0\}$, respectively), and let $x^0 \in \mathcal{X}$ be strictly (weakly, $\cdot$, respectively) upper set less ordered robust. Then there exists no $\overline{x} \in \mathcal{X} \setminus \{x^0\}$: $f_{\mathcal{U}}(\overline{x}) \preceq^u_Q f_{\mathcal{U}}(x^0)$

$$\iff \nexists\, \overline{x} \in \mathcal{X} \setminus \{x^0\} : \; f_{\mathcal{U}}(\overline{x}) \subseteq f_{\mathcal{U}}(x^0) - Q. \tag{5.32}$$

Suppose that x^0 is not strictly (weakly, $\cdot$, respectively) set less ordered robust. Then there exists $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ such that

$$f_{\mathcal{U}}(\overline{x}) \preceq^s_Q f_{\mathcal{U}}(x^0) \iff f_{\mathcal{U}}(\overline{x}) + Q \supseteq f_{\mathcal{U}}(x^0) \text{ and } f_{\mathcal{U}}(\overline{x}) \subseteq f_{\mathcal{U}}(x^0) - Q,$$

in contradiction to (5.32). The second part of the proof, for lower set less ordered robust elements, works in the exact same way. $\square$

Example 8. *Theorem 30 verifies that the union of lower and upper set less ordered robust points belongs to the set of set less ordered robust solutions. The inverse inclusion is in general not fulfilled, which the example in Figure 5.7 shows. We have depicted a solution* x_1 *that is neither lower set less ordered robust nor upper set less ordered robust, while it is set less ordered robust.*

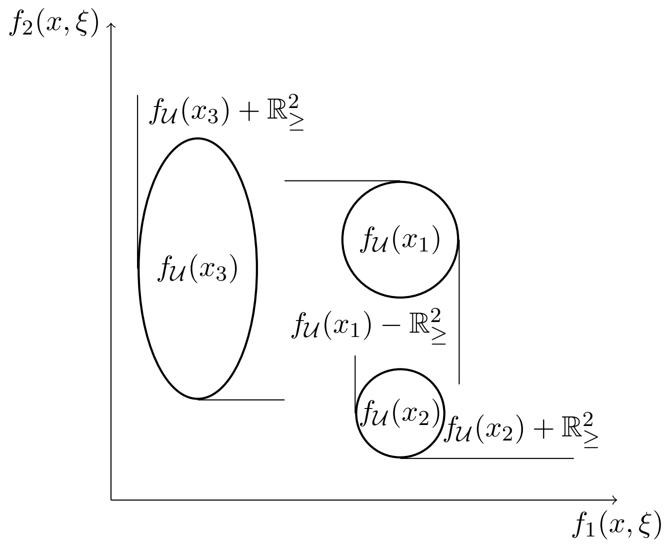

Figure 5.7: x_1 is set less ordered robust, but neither lower set less ordered robust nor upper set less ordered robust.

Computing Set Less Ordered Robust Solutions

After introducing the concept of set less ordered robustness, we now present an approach to compute set less ordered robust solutions via vectorization. To this end, consider the bicriteria optimization problem

$$(VOP^{sl}(y^*, y^{**})) \qquad \mathrm{Min}(h[\mathcal{X}], \mathbb{R}^2_{\geqq}) \tag{5.33}$$

with $h(x) := (\inf_{\xi\in\mathcal{U}} y^* \circ f(x,\xi),\ \sup_{\xi\in\mathcal{U}} y^{**} \circ f(x,\xi))^T,\ y^*, y^{**} \in C^* \setminus \{0\}$ ($y^*, y^{**} \in C^\#$, respectively). Here we minimize h with respect to the natural ordering cone $\mathbb{R}^2_{\geqq}$, i.e., we call the decision variable x of minimal elements $f(x)$ in $\mathbb{R}^2$ Pareto optimal. Note that the selection $y^* = y^{**}$ is entirely possible and not excluded here. Now we have the following theorem:

Theorem 31. *Given an uncertain vector-valued optimization problem $P(\mathcal{U})$. The following statements hold.*

(a) If x^0 is strictly Pareto optimal for problem $(VOP^{sl}(y^*, y^{**}))$ for some $y^*, y^{**} \in C^* \setminus \{0\}$, then x^0 is strictly set less ordered robust.

(b) If x^0 is weakly Pareto optimal for problem $(VOP^{sl}(y^*, y^{**}))$ for some $y^*, y^{**} \in C^* \setminus \{0\}$ and $\min_{\xi\in\mathcal{U}} y^* \circ f(x',\xi)$ and $\max_{\xi\in\mathcal{U}} y^{**} \circ f(x',\xi)$ exist for all $x' \in \mathcal{X}$ and the chosen weights $y^*, y^{**} \in C^* \setminus \{0\}$, then x^0 is weakly set less ordered robust.

(c) If x^0 is weakly Pareto optimal for problem $(VOP^{sl}(y^*, y^{**}))$ for some $y^*, y^{**} \in C^\#$ and $\min_{\xi\in\mathcal{U}} y^* \circ f(x',\xi)$ and $\max_{\xi\in\mathcal{U}} y^{**} \circ f(x',\xi)$ exist for all $x' \in \mathcal{X}$ and the chosen weights $y^*, y^{**} \in C^\#$, then x^0 is set less ordered robust.

Proof. Let x^0 be strictly Pareto optimal (weakly Pareto optimal, weakly Pareto optimal, respectively) for problem $(VOP^{sl}(y^*, y^{**}))$ with some weights $y^*, y^{**} \in C^* \setminus \{0\}$ ($y^*, y^{**} \in C^* \setminus \{0\}$, $y^*, y^{**} \in C^\#$, respectively), i.e., there is no $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ such that

$$\begin{aligned} \inf_{\xi\in\mathcal{U}} y^* \circ f(\overline{x},\xi) \ [\leq / < / <] \ \inf_{\xi\in\mathcal{U}} y^* \circ f(x^0,\xi) \\ \text{and } \sup_{\xi\in\mathcal{U}} y^{**} \circ f(\overline{x},\xi) \ [\leq / < / <] \ \sup_{\xi\in\mathcal{U}} y^{**} \circ f(x^0,\xi). \end{aligned}$$

Let $Q = C$ ($Q = \operatorname{int} C,\ Q = C \setminus \{0\}$). Now suppose that x^0 is not strictly (weakly, $\cdot$, respectively) set less ordered robust. Then there exists an $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ such that

$$f_{\mathcal{U}}(\overline{x}) + Q \supseteq f_{\mathcal{U}}(x^0) \text{ and } f_{\mathcal{U}}(\overline{x}) \subseteq f_{\mathcal{U}}(x^0) - Q.$$

That implies

$$\begin{aligned} \exists\, \overline{x} \in \mathcal{X} \setminus \{x^0\} :\ \forall\, \xi_1, \xi_2 \in \mathcal{U}\ \exists\, \eta_1, \eta_2 \in \mathcal{U} :\ f(\overline{x}, \eta_1) + Q \ni f(x^0, \xi_1) \\ \text{and } f(\overline{x}, \xi_2) \in f(x^0, \eta_2) - Q. \end{aligned} \tag{5.34}$$

Choose now $y^*, y^{**} \in C^* \setminus \{0\}$ ($y^*, y^{**} \in C^* \setminus \{0\}$, $y^*, y^{**} \in C^\#$, respectively) as in problem $(VOP^{sl}(y^*, y^{**}))$. We obtain from (5.34)

$$\begin{aligned} \exists\, \overline{x} \in \mathcal{X} \setminus \{x^0\} :\ \forall\, \xi_1, \xi_2 \in \mathcal{U}\ \exists\, \eta_1, \eta_2 \in \mathcal{U} :\ y^* \circ f(\overline{x}, \eta_1)\ [\leq / < / <]\ y^* \circ f(x^0, \xi_1) \\ \text{and } y^{**} \circ f(\overline{x}, \xi_2)\ [\leq / < / <]\ y^{**} \circ f(x^0, \eta_2) \end{aligned}$$

$$\Longrightarrow \inf_{\xi\in\mathcal{U}} y^* \circ f(\overline{x},\xi) \; [\leq / < / <] \; \inf_{\xi\in\mathcal{U}} y^* \circ f(x^0,\xi) \tag{5.35}$$

$$\text{and } \sup_{\xi\in\mathcal{U}} y^{**} \circ f(\overline{x},\xi) \; [\leq / < / <] \; \sup_{\xi\in\mathcal{U}} y^{**} \circ f(x^0,\xi). \tag{5.36}$$

The last two strict inequalities in (5.35) and (5.36) hold because the minimum and maximum exists. But this is a contradiction to the assumption. □

Based on these observations, we provide the following algorithm for computing set less ordered robust solutions of an uncertain vector-valued optimization problem. The sets of strictly (weakly, · , respectively) set less ordered robust solutions are denoted by $\text{Opt}_{\text{sslor}}$ ($\text{Opt}_{\text{wslor}}$, Opt_{slor}, respectively).

Algorithm 12 for computing set less ordered robust solutions using a family of problems $(VOP^{sl}(y^*, y^{**}))$ **(see** (5.33)**):**

Input: Uncertain vector-valued problem $P(\mathcal{U})$, solution sets

$$\text{Opt}_{\text{sslor}} = \text{Opt}_{\text{slor}} = \text{Opt}_{\text{wslor}} = \emptyset.$$

Step 1: Choose a set $\overline{C} \subset C^* \setminus \{0\}$.

Step 2: If $\overline{C} = \emptyset$: **STOP. Output:** Set of strictly set less ordered robust solutions $\text{Opt}_{\text{sslor}}$, set of set less ordered robust solutions Opt_{slor}, set of weakly set less ordered robust solutions $\text{Opt}_{\text{wslor}}$.

Step 3: Choose $y^*, y^{**} \in \overline{C}$. Set $\overline{C} := \overline{C} \setminus \{y^*, y^{**}\}$.

Step 4: Find a set of weakly Pareto optimal solutions $SOL_{we}(y^*, y^{**})$ of $(VOP^{sl}(y^*, y^{**}))$ (see (5.33)).

Step 5: If $SOL_{we}(y^*, y^{**}) = \emptyset$, then go to Step 2.

Step 6: Choose $x \in SOL_{we}(y^*, y^{**})$. Set $SOL_{we}(y^*, y^{**}) := SOL_{we}(y^*, y^{**}) \setminus \{x\}$.

a) If x is a strictly Pareto optimal solution of $(VOP^{sl}(y^*, y^{**}))$ (see (5.33)), then x is strictly set less ordered robust for $P(\mathcal{U})$, thus

$$\text{Opt}_{\text{sslor}} := \text{Opt}_{\text{sslor}} \cup \{x\}.$$

b) If x is a weakly Pareto optimal solution of $(VOP^{sl}(y^*, y^{**}))$ (see (5.33)) and $\max_{\xi\in\mathcal{U}} y^{**} \circ f(x', \xi)$ and $\min_{\xi\in\mathcal{U}} y^* \circ f(x', \xi)$ exist for all $x' \in \mathcal{X}$, then x is weakly set less ordered robust for $P(\mathcal{U})$, thus

$$\text{Opt}_{\text{wslor}} := \text{Opt}_{\text{wslor}} \cup \{x\}.$$

c) If x is weakly Pareto optimal for problem $(VOP^{sl}(y^*, y^{**}))$ (compare (5.33)) and $y^*, y^{**} \in C^\#$ and $\min_{\xi \in \mathcal{U}} \; y^* \circ f(x', \xi)$ and $\max_{\xi \in \mathcal{U}} \; y^{**} \circ f(x', \xi)$ exist for all $x' \in \mathcal{X}$ and the chosen weight $y^*, y^{**} \in C^\#$, then x is set less ordered robust for $P(\mathcal{U})$, thus

$$\mathrm{Opt}_{\mathrm{slor}} := \mathrm{Opt}_{\mathrm{slor}} \cup \{x\}.$$

Step 7: Go to Step 5.

In the following we present an algorithm that computes set less ordered robust solutions while varying the weights in the vector of objectives of problem $(VOP^{sl}(y^*, y^{**}))$ (see (5.33)).

Algorithm 13 for computing set less ordered robust solutions using a family of problems $(VOP^{sl}(y^*, y^{**}))$ **(see** (5.33)**):**

Input: Uncertain vector-valued problem $P(\mathcal{U})$, solution sets

$$\mathrm{Opt}_{\mathrm{sslor}} = \mathrm{Opt}_{\mathrm{slor}} = \mathrm{Opt}_{\mathrm{wslor}} = \emptyset.$$

Step 1: Choose a set $\overline{C} \subset C^* \setminus \{0\}$ with at least four distinct elements. Set $j := 0$. Choose $y_0^*, \; y_0^{**} \in \overline{C}$, and set $\overline{C} := \overline{C} \setminus \{y_0^*, \; y_0^{**}\}$.

Step 2: If $\overline{C} = \emptyset$ or if $\mathrm{Opt}_{\mathrm{sslor}}, \; \mathrm{Opt}_{\mathrm{slor}}, \; \mathrm{Opt}_{\mathrm{wslor}}$ are accepted by the decision maker: **STOP. Output:** Set of strictly set less ordered robust solutions $\mathrm{Opt}_{\mathrm{sslor}}$, set of set less ordered robust solutions $\mathrm{Opt}_{\mathrm{slor}}$, set of weakly set less ordered robust solutions $\mathrm{Opt}_{\mathrm{wslor}}$.

Step 3: Choose $y_{j+1}^*, y_{j+1}^{**} \in \overline{C}$. Set $\overline{C} := \overline{C} \setminus \{y_{j+1}^*, y_{j+1}^{**}\}$. Set $t := 0$.

Step 4: Set $\hat{y}^* := y_j^* + t(y_{j+1}^* - y_j^*)$ and $\hat{y}^{**} := y_j^{**} + t(y_{j+1}^{**} - y_j^{**})$.

Step 5: Find a set of weakly Pareto optimal solutions $SOL_{we}(\hat{y}^*, \hat{y}^{**})$ of $(VOP^{sl}(\hat{y}^*, \hat{y}^{**}))$.

Step 6: If $SOL_{we}(\hat{y}^*, \hat{y}^{**}) = \emptyset$, then go to Step 8.

Step 7: Choose $x \in SOL_{we}(\hat{y}^*, \hat{y}^{**})$. Set $SOL_{we}(\hat{y}^*, \hat{y}^{**}) := SOL_{we}(\hat{y}^*, \hat{y}^{**}) \setminus \{x\}$.

a) If x is a strictly Pareto optimal solution of $(VOP^{sl}(\hat{y}^*, \hat{y}^{**}))$, then x is strictly set less ordered robust for $P(\mathcal{U})$, thus

$$\mathrm{Opt}_{\mathrm{sslor}} := \mathrm{Opt}_{\mathrm{sslor}} \cup \{x\}.$$

b) If x is a weakly Pareto optimal solution of $(VOP^{sl}(\hat{y}^*, \hat{y}^{**}))$ and $\max_{\xi \in \mathcal{U}} \; \hat{y}^{**} \circ f(x', \xi)$ and $\min_{\xi \in \mathcal{U}} \; \hat{y}^* \circ f(x', \xi)$ exist for all $x' \in \mathcal{X}$, then x is weakly set less ordered robust for $P(\mathcal{U})$, thus

$$\mathrm{Opt}_{\mathrm{wslor}} := \mathrm{Opt}_{\mathrm{wslor}} \cup \{x\}.$$

c) If x is weakly Pareto optimal for problem $(VOP^{sl}(\hat{y}^*, \hat{y}^{**}))$ with $\hat{y}^*,\ \hat{y}^{**} \in C^{\#}$ and $\min_{\xi \in \mathcal{U}}\ \hat{y}^* \circ f(x', \xi)$ and $\max_{\xi \in \mathcal{U}}\ \hat{y}^{**} \circ f(x', \xi)$ exist for all $x' \in \mathcal{X}$ and the chosen weights $\hat{y}^*, \hat{y}^{**} \in C^{\#}$, then x is set less ordered robust for $P(\mathcal{U})$, thus

$$\mathrm{Opt}_{\mathrm{slor}} := \mathrm{Opt}_{\mathrm{slor}} \cup \{x\}.$$

Go to Step 6.

Step 8: If $t = 1$, set $j := j + 1$ and go to Step 2. Otherwise, choose $t :\in (t, 1]$ and go to Step 4.

5.3.4 Alternative Set Less Ordered Robustness

In Sections 5.3.1 and 5.3.2 we investigated upper and lower set less ordered robust solutions of uncertain vector-valued problems. For the special case $Y = \mathbb{R}$, $X = \mathbb{R}^n$, $C = \mathbb{R}_{\geqq}$ we revealed connections between the robust counterpart (RC) (compare (5.10)) and the concept of upper set less ordered robustness (between the optimistic counterpart (OC) (see (5.11)) and the lower set less ordered robustness approach, respectively). We noted that a risk averse decision maker would be likely to choose the upper set less ordered robust approach, while a person who is risk affine may be eligible to choose the latter concept. It is not clear, however, which decision strategy a person should follow in case his attitude toward risk has not been revealed. In this section, we wish to combine both approaches such that a decision maker who is not sure which concept to choose may be presented with an alternative concept.

To this end, we define the *alternative set less order relation* $\preceq^a_Q$ for a nonempty set $Q \subset Y$ in the following way. Let $C \subset Y$ be a proper closed convex and pointed cone. Suppose $C \subset \mathrm{cl}\, Q$ and $\mathrm{cl}\, Q \cap (-\,\mathrm{cl}\, Q) = \{0\}$. Then we define for two nonempty sets $A, B \subset Y$

$$\begin{aligned} A \preceq^a_Q B :&\Longleftrightarrow A \subseteq B - Q \text{ or } A + Q \supseteq B \\ &\Longleftrightarrow (\forall\ a \in A\ \exists\ b \in B:\ a \leq_Q b) \text{ or } (\forall\ b \in B\ \exists\ a \in A:\ a \leq_Q b). \end{aligned}$$

As usual, we assume that $\mathrm{int}\, C \neq \emptyset$ if we are dealing with $Q = \mathrm{int}\, C$.

Note that

$$A \preceq^a_Q B \Longleftrightarrow (A \preceq^u_Q B \text{ or } A \preceq^l_Q B),$$

taking into account the definition of the upper and lower set less order relation (see Sections 5.3.1 and 5.3.2). Furthermore, it is important to mention that $\preceq^a_C$ is in general not a pre-order. Consider, for example, the illustration in Figure 5.8. It shows that $A \preceq^a_C B$ and $B \preceq^a_C D$, but $A \not\preceq^a_C D$, thus $\preceq^a_C$ is not transitive and hence not a pre-order.

Definition 19. *A solution x^0 of $P(\mathcal{U})$ is called strictly (weakly, $\cdot$, respectively)* **alternative set less ordered robust** *if there is no $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ such that $f_{\mathcal{U}}(\overline{x}) \preceq^a_Q f_{\mathcal{U}}(x^0)$, which is equivalent to*

$$\nexists\ \overline{x} \in \mathcal{X} \setminus \{x^0\} : \quad f_{\mathcal{U}}(\overline{x}) + Q \supseteq f_{\mathcal{U}}(x^0) \text{ or } f_{\mathcal{U}}(\overline{x}) \subseteq f_{\mathcal{U}}(x^0) - Q$$

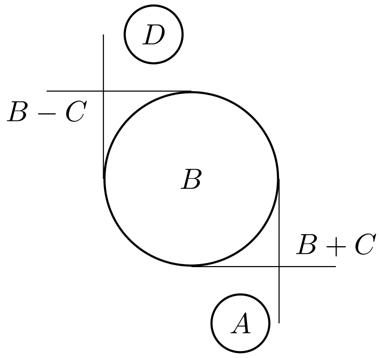

Figure 5.8: $\preceq_C^a$ is not transitive and consequently not a pre-order.

for $Q = C$ *(*$Q = \operatorname{int} C$*,* $Q = C \setminus \{0\}$*, respectively).*

Example 9. *In Figure 5.9, using* $Q = \mathbb{R}^2_{\geqq}$*, we see that* x_1 *is alternative set less ordered robust.*

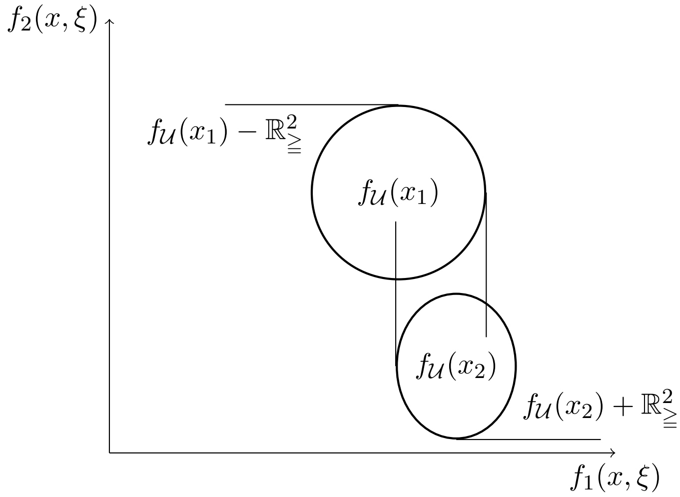

Figure 5.9: x_1 is alternative set less ordered robust.

This approach is extremely restrictive. Using the alternative set less ordered robustness concept may produce a smaller number of solutions than the above mentioned concepts for multicriteria robustness. A decision maker whose preferences reflect this approach is considered to be risk averse and risk affine at the same time. The following lemma verifies that for $|\mathcal{U}| = 1$, alternative set less ordered robustness is equivalent to deterministic minimality.

Lemma 10. *Given* $P(\mathcal{U})$ *with* $|\mathcal{U}| = 1$*. Then* x^0 *is strictly (weakly,* $\cdot$ *, respectively)* *alternative set less ordered robust if and only if $f(x^0)$ is strictly (weakly, $\cdot$, respectively) minimal.*

Proof. The following holds for $Q = C$ ($Q = \operatorname{int} C$, $Q = C \setminus \{0\}$, respectively):

$$
\begin{aligned}
& x^0 \text{ is strictly (weakly, } \cdot \text{ , respectively) alternative set less ordered robust} \\
\iff & \nexists\, \overline{x} \in \mathcal{X} \setminus \{x^0\} : \; f_{\mathcal{U}}(\overline{x}) + Q \supseteq f_{\mathcal{U}}(x^0) \text{ or } f_{\mathcal{U}}(\overline{x}) \subseteq f_{\mathcal{U}}(x^0) - Q \\
\iff & \nexists\, \overline{x} \in \mathcal{X} \setminus \{x^0\} : \; \forall\, \eta_1, \eta_2 \in \mathcal{U}\ \exists\, \xi_1,\ \xi_2 \in \mathcal{U} : \; f(\overline{x}, \xi_1) + Q \ni f(x^0, \eta_1) \\
& \quad \text{or } f(\overline{x}, \eta_2) \in f(x^0, \xi_2) - Q \\
\overset{|\mathcal{U}|=1}{\iff} & \nexists\, \overline{x} \in \mathcal{X} \setminus \{x^0\} : \; f(\overline{x}) \in f(x^0) - Q \\
\iff & f(x^0) \text{ is strictly (weakly, } \cdot \text{ , respectively) minimal.}
\end{aligned}
$$

□

Lemma 11. *Given $P(\mathcal{U})$ with $Y = \mathbb{R}$, $X = \mathbb{R}^n$ and $C = \mathbb{R}_{\geqq}$.*

(a) x^0 is weakly alternative set less ordered robust $\iff$ x^0 is alternative set less ordered robust.

(b) If x^0 is uniquely optimal for the robust counterpart (RC) (see (5.10)) and for the optimistic counterpart (OC) (see (5.11)), then x^0 is strictly alternative set less ordered robust.

(c) Suppose $\max_{\xi \in \mathcal{U}} f(x', \xi)$ and $\min_{\xi \in \mathcal{U}} f(x', \xi)$ exist for every $x' \in \mathcal{X}$. Then it holds: If x^0 is strictly alternative set less ordered robust, then x^0 is uniquely optimal for the robust counterpart (RC) (see (5.10)) and for the optimistic counterpart (OC) (see (5.11)).

(d) Suppose $\max_{\xi \in \mathcal{U}} f(x', \xi)$ and $\min_{\xi \in \mathcal{U}} f(x', \xi)$ exist for every $x' \in \mathcal{X}$. Then it holds: If x^0 is optimal for the robust counterpart (RC) (see (5.10)) and for the optimistic counterpart (OC) (see (5.11)), then x^0 is weakly alternative set less ordered robust.

(e) If x^0 is weakly alternative set less ordered robust, then x^0 is optimal for the robust counterpart (RC) (see (5.10)) and for the optimistic counterpart (OC) (see (5.11)).

Proof. (a) Holds due to $\mathbb{R}_{>} = \mathbb{R}_{\geq}$.

(b) Let x^0 be uniquely optimal for the robust counterpart (RC) (compare (5.10)), thus

$$\nexists\, \overline{x} \in \mathcal{X} \setminus \{x^0\} : \; \sup_{\xi \in \mathcal{U}} f(\overline{x}, \xi) \leq \sup_{\xi \in \mathcal{U}} f(x^0, \xi). \tag{5.37}$$

In addition, x^0 is uniquely optimal for the optimistic counterpart (OC) (see (5.11)), i.e.,

$$\nexists\, \overline{x} \in \mathcal{X} \setminus \{x^0\} : \; \inf_{\xi \in \mathcal{U}} f(\overline{x}, \xi) \leq \inf_{\xi \in \mathcal{U}} f(x^0, \xi). \tag{5.38}$$

Now suppose that x^0 is not strictly alternative set less ordered robust. Thus there exists $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ such that

$$\begin{aligned}
&f_{\mathcal{U}}(\overline{x}) + \mathbb{R}_{\geqq} \supseteq f_{\mathcal{U}}(x^0) \text{ or } f_{\mathcal{U}}(\overline{x}) \subseteq f_{\mathcal{U}}(x^0) - \mathbb{R}_{\geqq} \\
&\Longrightarrow \forall\ \xi_1, \xi_2 \in \mathcal{U}\ \exists\ \eta_1, \eta_2 \in \mathcal{U}:\ f(\overline{x}, \eta_1) + \mathbb{R}_{\geqq} \ni f(x^0, \xi_1) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{or } f(\overline{x}, \xi_2) \in f(x^0, \eta_2) - \mathbb{R}_{\geqq} \\
&\Longrightarrow \forall\ \xi_1, \xi_2 \in \mathcal{U}\ \exists\ \eta_1, \eta_2 \in \mathcal{U}:\ f(\overline{x}, \eta_1) \leq f(x^0, \xi_1) \text{ or } f(\overline{x}, \xi_2) \leq f(x^0, \eta_2) \\
&\Longrightarrow \inf_{\xi \in \mathcal{U}} f(\overline{x}, \xi) \leq \inf_{\xi \in \mathcal{U}} f(x^0, \xi) \text{ or } \sup_{\xi \in \mathcal{U}} f(\overline{x}, \xi) \leq \sup_{\xi \in \mathcal{U}} f(x^0, \xi),
\end{aligned}$$

in contradiction to (5.38) and (5.37).

(c) x^0 is strictly alternative set less ordered robust

$$\begin{aligned}
&\Longleftrightarrow \nexists\ \overline{x} \in \mathcal{X} \setminus \{x^0\}:\ f_{\mathcal{U}}(\overline{x}) + \mathbb{R}_{\geqq} \supseteq f_{\mathcal{U}}(x^0) \text{ or } f_{\mathcal{U}}(\overline{x}) \subseteq f_{\mathcal{U}}(x^0) - \mathbb{R}_{\geqq} \\
&\Longleftrightarrow \nexists\ \overline{x} \in \mathcal{X} \setminus \{x^0\}:\ \forall\ \eta_1, \eta_2 \in \mathcal{U}\ \exists\ \xi_1, \xi_2 \in \mathcal{U}:\ f(\overline{x}, \xi_1) + \mathbb{R}_{\geqq} \ni f(x^0, \eta_1) \\
&\qquad \text{or } f(\overline{x}, \eta_2) \in f(x^0, \xi_2) - \mathbb{R}_{\geqq} \\
&\Longleftrightarrow \nexists\ \overline{x} \in \mathcal{X} \setminus \{x^0\}:\ \forall\ \eta_1, \eta_2 \in \mathcal{U}\ \exists\ \xi_1, \xi_2 \in \mathcal{U}:\ f(\overline{x}, \xi_1) \leq f(x^0, \eta_1) \\
&\qquad \text{or } f(\overline{x}, \eta_2) \leq f(x^0, \xi_2) \\
&\Longleftrightarrow \forall\ \overline{x} \in \mathcal{X} \setminus \{x^0\}:\ \exists\ \eta_1, \eta_2 \in \mathcal{U}\ \forall\ \xi_1, \xi_2 \in \mathcal{U}:\ f(\overline{x}, \xi_1) > f(x^0, \eta_1) \\
&\qquad \text{and } f(\overline{x}, \eta_2) > f(x^0, \xi_2) \\
&\Longrightarrow \forall\ \overline{x} \in \mathcal{X} \setminus \{x^0\}:\ \max_{\xi \in \mathcal{U}} f(\overline{x}, \xi) > \max_{\xi \in \mathcal{U}} f(x^0, \xi) \text{ and } \min_{\xi \in \mathcal{U}} f(\overline{x}, \xi) > \min_{\xi \in \mathcal{U}} f(x^0, \xi) \\
&\Longleftrightarrow x^0 \text{ is uniquely optimal for the robust counterpart } (RC) \text{ (see (5.10))} \\
&\qquad \text{and for the optimistic counterpart } (OC) \text{ (see (5.11)).}
\end{aligned}$$

(d) Let x^0 be optimal for the robust counterpart (RC) (see (5.10)), thus

$$\nexists\ \overline{x} \in \mathcal{X} \setminus \{x^0\}:\ \max_{\xi \in \mathcal{U}} f(\overline{x}, \xi) < \max_{\xi \in \mathcal{U}} f(x^0, \xi). \tag{5.39}$$

Furthermore, let x^0 be optimal for the optimistic counterpart (OC) (see (5.11)), i.e.,

$$\nexists\ \overline{x} \in \mathcal{X} \setminus \{x^0\}:\ \min_{\xi \in \mathcal{U}} f(\overline{x}, \xi) < \min_{\xi \in \mathcal{U}} f(x^0, \xi). \tag{5.40}$$

Now suppose that x^0 is not weakly alternative set less ordered robust. This implies that there exists $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ such that

$$\begin{aligned}
&f_{\mathcal{U}}(\overline{x}) + \mathbb{R}_{>} \supseteq f_{\mathcal{U}}(x^0) \text{ or } f_{\mathcal{U}}(\overline{x}) \subseteq f_{\mathcal{U}}(x^0) - \mathbb{R}_{>} \\
&\Longleftrightarrow \forall\ \xi_1, \xi_2 \in \mathcal{U}\ \exists\ \eta_1, \eta_2 \in \mathcal{U}:\ f(\overline{x}, \eta_1) + \mathbb{R}_{>} \ni f(x^0, \xi_1) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{or } f(\overline{x}, \xi_2) \in f(x^0, \eta_2) - \mathbb{R}_{>} \\
&\Longleftrightarrow \forall\ \xi_1, \xi_2 \in \mathcal{U}\ \exists\ \eta_1, \eta_2 \in \mathcal{U}:\ f(\overline{x}, \eta_1) < f(x^0, \xi_1) \text{ or } f(\overline{x}, \xi_2) < f(x^0, \eta_2) \\
&\Longrightarrow \min_{\xi \in \mathcal{U}} f(\overline{x}, \xi) < \min_{\xi \in \mathcal{U}} f(x^0, \xi) \text{ or } \max_{\xi \in \mathcal{U}} f(\overline{x}, \xi) < \max_{\xi \in \mathcal{U}} f(x^0, \xi),
\end{aligned}$$

in contradiction to (5.40) and (5.39).

(e) x^0 is weakly alternative set less ordered robust

$$
\begin{aligned}
&\iff \nexists\, \overline{x} \in \mathcal{X} \setminus \{x^0\}: \; f_{\mathcal{U}}(\overline{x}) + \mathbb{R}_> \supseteq f_{\mathcal{U}}(x^0) \text{ or } f_{\mathcal{U}}(\overline{x}) \subseteq f_{\mathcal{U}}(x^0) - \mathbb{R}_> \\
&\iff \nexists\, \overline{x} \in \mathcal{X} \setminus \{x^0\}: \; \forall\, \eta_1, \eta_2 \in \mathcal{U}\ \exists\, \xi_1,\ \xi_2 \in \mathcal{U}: \; f(\overline{x}, \xi_1) + \mathbb{R}_> \ni f(x^0, \eta_1) \\
&\qquad \text{or } f(\overline{x}, \eta_2) \in f(x^0, \xi_2) - \mathbb{R}_> \\
&\iff \nexists\, \overline{x} \in \mathcal{X} \setminus \{x^0\}: \; \forall\, \eta_1, \eta_2 \in \mathcal{U}\ \exists\, \xi_1,\ \xi_2 \in \mathcal{U}: \; f(\overline{x}, \xi_1) < f(x^0, \eta_1) \\
&\qquad \text{or } f(\overline{x}, \eta_2) < f(x^0, \xi_2) \\
&\iff \forall\, \overline{x} \in \mathcal{X} \setminus \{x^0\}: \; \exists\, \eta_1, \eta_2 \in \mathcal{U}\ \forall\, \xi_1,\ \xi_2 \in \mathcal{U}: \; f(\overline{x}, \xi_1) \geq f(x^0, \eta_1) \\
&\qquad \text{and } f(\overline{x}, \eta_2) \geq f(x^0, \xi_2) \\
&\Longrightarrow \forall\, \overline{x} \in \mathcal{X} \setminus \{x^0\}: \; \inf_{\xi \in \mathcal{U}} f(\overline{x}, \xi) \geq \inf_{\xi \in \mathcal{U}} f(x^0, \xi) \text{ and } \sup_{\xi \in \mathcal{U}} f(\overline{x}, \xi) \geq \sup_{\xi \in \mathcal{U}} f(x^0, \xi) \\
&\iff x^0 \text{ is optimal for the robust counterpart } (RC) \text{ (compare (5.10))} \\
&\qquad \text{and for the optimistic counterpart } (OC) \text{ (see (5.11)).}
\end{aligned}
$$

□

The following theorem reveals important interrelations between alternative set less ordered robust solutions and upper / lower set less ordered robust points.

Theorem 32. *x^0 is strictly (weakly, · , respectively) alternative set less ordered robust if and only if x^0 is strictly (weakly, · , respectively) lower set less ordered robust and strictly (weakly, · , respectively) upper set less ordered robust.*

Proof. Let $Q = C$ ($Q = \operatorname{int} C$, $Q = C \setminus \{0\}$, respectively).

$\Longrightarrow$ Let $x^0 \in \mathcal{X}$ be strictly (weakly, · , respectively) alternative set less ordered robust. Then there exists no $\overline{x} \in \mathcal{X} \setminus \{x^0\}$: $f_{\mathcal{U}}(\overline{x}) \preceq_Q^a f_{\mathcal{U}}(x^0)$

$$\iff \nexists\, \overline{x} \in \mathcal{X} \setminus \{x^0\}: \; f_{\mathcal{U}}(\overline{x}) + Q \supseteq f_{\mathcal{U}}(x^0) \text{ or } f_{\mathcal{U}}(\overline{x}) \subseteq f_{\mathcal{U}}(x^0) - Q.$$

In particular there exists no $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ such that $f_{\mathcal{U}}(\overline{x}) + Q \supseteq f_{\mathcal{U}}(x^0)$, therefore x^0 is strictly (weakly, · , respectively) lower set less ordered robust. Analogously there exists no $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ such that $f_{\mathcal{U}}(\overline{x}) \subseteq f_{\mathcal{U}}(x^0) - Q$, therefore x^0 is strictly (weakly, · , respectively) upper set less ordered robust.

$\Longleftarrow$ Let $x^0 \in \mathcal{X}$ be strictly (weakly, · , respectively) lower set less ordered robust, i.e., there is no $\overline{x} \in \mathcal{X} \setminus \{x^0\}: \; f_{\mathcal{U}}(\overline{x}) + Q \supseteq f_{\mathcal{U}}(x^0)$. Furthermore, let x^0 be upper set less ordered robust, i.e., there does not exist $\overline{x} \in \mathcal{X} \setminus \{x^0\}: \; f_{\mathcal{U}}(\overline{x}) \subseteq f_{\mathcal{U}}(x^0) - Q$. Suppose that x^0 is not strictly (weakly, · , respectively) alternative set less ordered robust. That means there is an $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ such that $f_{\mathcal{U}}(\overline{x}) + Q \supseteq f_{\mathcal{U}}(x^0)$ or $f_{\mathcal{U}}(\overline{x}) \subseteq f_{\mathcal{U}}(x^0) - Q$, a contradiction.

□

Example 10. *This example in Figure 5.10 shows that there are upper (lower, respectively) set less ordered robust solutions that are not alternative set less ordered robust.*

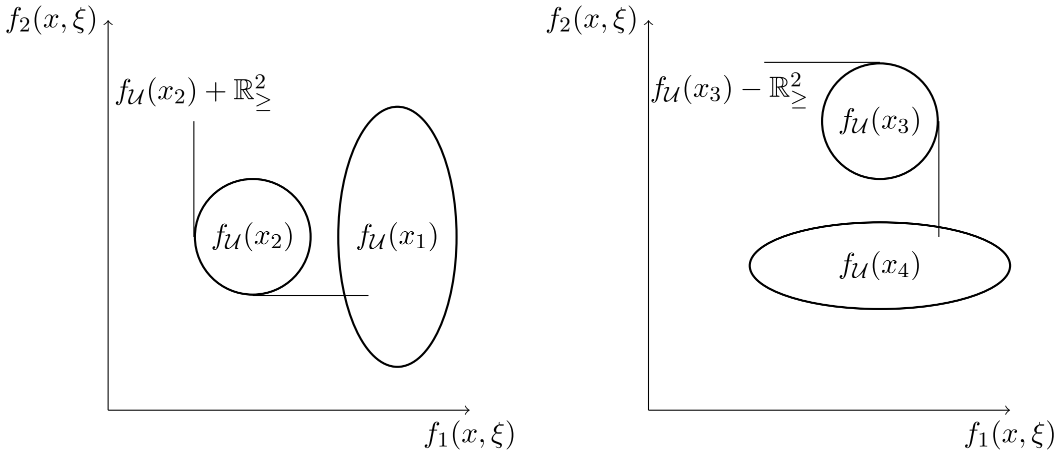

Figure 5.10: Left: x_1 is lower set less ordered robust, while it is not alternative set less ordered robust. Right: x_3 is upper set less ordered robust, but not alternative set less ordered robust.

Computing Alternative Set Less Ordered Robust Solutions

Theorem 32 proves to be beneficial for computing alternative set less ordered robust elements of an uncertain multi-objective optimization problem, which the following algorithm illustrates. The sets of strictly (weakly, · , respectively) alternative set less ordered robust solutions are denoted by ($\mathrm{Opt}_{\mathrm{sar}}$, $\mathrm{Opt}_{\mathrm{war}}$, $\mathrm{Opt}_{\mathrm{ar}}$).

Algorithm 14 for deriving alternative set less ordered robust solutions:

Input: Uncertain vector-valued problem $P(\mathcal{U})$, solution sets $\mathrm{Opt}_{\mathrm{sar}} = \mathrm{Opt}_{\mathrm{war}} = \mathrm{Opt}_{\mathrm{ar}} = \emptyset$.

Step 1: Compute the set of strictly (weakly, · , respectively) lower set less ordered robust points $\mathrm{Opt}_{\mathrm{slr}}$, ($\mathrm{Opt}_{\mathrm{wlr}}$, $\mathrm{Opt}_{\mathrm{lr}}$, respectively) using Algorithm 6, 7, 8, 9, 10 or 11.

Step 2: Compute the set of strictly (weakly, · , respectively) upper set less ordered robust points $\mathrm{Opt}_{\mathrm{sur}}$, ($\mathrm{Opt}_{\mathrm{wur}}$, $\mathrm{Opt}_{\mathrm{ur}}$, respectively) using Algorithm 1, 2, 3, 4 or 5.

Output: Set of strictly (weakly, · , respectively) alternative set less ordered robust points

$$\mathrm{Opt}_{\mathrm{sar}} = \mathrm{Opt}_{\mathrm{sur}} \cap \mathrm{Opt}_{\mathrm{slr}},$$
$$\mathrm{Opt}_{\mathrm{war}} = \mathrm{Opt}_{\mathrm{wur}} \cap \mathrm{Opt}_{\mathrm{wlr}},$$
$$\mathrm{Opt}_{\mathrm{ar}} = \mathrm{Opt}_{\mathrm{ur}} \cap \mathrm{Opt}_{\mathrm{lr}}\,.$$

5.3.5 Minmax Less Ordered Robustness

We extend the definition of the *minmax less order relation* (Definition 8) to a nonempty set $Q \subset Y$ in the following way. Let $C \subset Y$ be a proper closed convex and pointed cone. Suppose $C \subset \operatorname{cl} Q$ and $\operatorname{cl} Q \cap (-\operatorname{cl} Q) = \{0\}$. Under these assumptions we extend $\mathcal{F}_{\min,\max}$ given by (5.4) to sets Q:

$$\mathcal{F}^Q_{\min,\max} := \{A \in \mathcal{P}(Y) | \operatorname{Min}(A,Q) \neq \emptyset \text{ and } \operatorname{Max}(A,Q) \neq \emptyset\}. \tag{5.41}$$

Then we define for two sets $A, B \in \mathcal{F}^Q_{\min,\max}$

$$A \preceq^m_Q B :\Longleftrightarrow \big(\operatorname{Min}(A,Q) \preceq^s_Q \operatorname{Min}(B,Q) \text{ and } \operatorname{Max}(A,Q) \preceq^s_Q \operatorname{Max}(B,Q)\big).$$

In this section, we assume that $f_\mathcal{U}(x) \in \mathcal{F}^Q_{\min,\max}$ is satisfied for every $x \in \mathcal{X}$ and for $Q = C$ ($Q = \operatorname{int} C$, $Q = C \setminus \{0\}$, respectively). In case we have $Q = \operatorname{int} C$, we suppose that $\operatorname{int} C \neq \emptyset$.

Definition 20. *A solution x^0 of $P(\mathcal{U})$ is called strictly (weakly, $\cdot$, respectively)* **minmax less ordered robust** *if there is no $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ such that $f_\mathcal{U}(\overline{x}) \preceq^m_Q f_\mathcal{U}(x^0)$, which is equivalent to: There does not exist $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ s.t.*

$$\begin{aligned} &\operatorname{Min}(f_\mathcal{U}(\overline{x}),Q) + Q \supseteq \operatorname{Min}(f_\mathcal{U}(x^0),Q) \\ \textit{and } &\operatorname{Min}(f_\mathcal{U}(\overline{x}),Q) \subseteq \operatorname{Min}(f_\mathcal{U}(x^0),Q) - Q \\ \textit{and } &\operatorname{Max}(f_\mathcal{U}(\overline{x}),Q) + Q \supseteq \operatorname{Max}(f_\mathcal{U}(x^0),Q) \\ \textit{and } &\operatorname{Max}(f_\mathcal{U}(\overline{x}),Q) \subseteq \operatorname{Max}(f_\mathcal{U}(x^0),Q) - Q \end{aligned} \tag{5.42}$$

for $Q = C$ ($Q = \operatorname{int} C$, $Q = C \setminus \{0\}$, respectively).

This approach is appealing for a decision maker because its definition contains comparisons of minimal as well as maximal elements of sets. In that manner it reflects optimism about the future as well as the risk averse nature of the approaches containing maximal elements. Contrary to upper / lower set less ordered robustness, the decision maker is now able to hedge against the strictly (weakly, $\cdot$) minimal / maximal solutions of sets $f_\mathcal{U}(x)$ instead of the whole lower / upper bound. This enables a user to specify his wishes during the decision process even more. This concept is less restrictive than set less ordered robustness, which Theorem 33 below will show.

Lemma 12. *Given $P(\mathcal{U})$ with $|\mathcal{U}| = 1$. Then x^0 is strictly (weakly, $\cdot$, respectively) minmax less ordered robust if and only if $f(x^0)$ is strictly (weakly, $\cdot$, respectively) minimal.*

Proof. Note first that for $|\mathcal{U}| = 1$, $f_\mathcal{U}(x) = f(x)$ is just one point and hence $\operatorname{Min}(f(x),Q) = \operatorname{Max}(f(x),Q) = f(x)$ for $Q = C$ ($Q = \operatorname{int} C$, $Q = C \setminus \{0\}$). Now the following holds for

$Q = C$ ($Q = \operatorname{int} C$, $Q = C \setminus \{0\}$, respectively):

$$\begin{aligned}
& x^0 \text{ is strictly (weakly, } \cdot \text{ , respectively) minmax less ordered robust} \\
\iff\ & \nexists\, \overline{x} \in \mathcal{X} \setminus \{x^0\} : \ (5.42) \text{ holds} \\
\overset{|\mathcal{U}|=1}{\iff}\ & \nexists\, \overline{x} \in \mathcal{X} \setminus \{x^0\} : \ f(\overline{x}) + Q \ni f(x^0) \text{ and } f(\overline{x}) \in f(x^0) - Q \\
\iff\ & \nexists\, \overline{x} \in \mathcal{X} \setminus \{x^0\} : \ f(\overline{x}) \in f(x^0) - Q \\
\iff\ & f(x^0) \text{ is strictly (weakly, } \cdot \text{ , respectively) minimal.}
\end{aligned}$$

□

Below we define the *domination property* which was given in [55] (also compare [72]) for convex cones C. We will use this property in Theorem 33 below.

Definition 21 (Domination property)**.** *If for some $A \in \mathcal{F}^Q_{\min,\max}$ and a set $Q \subset Y$, it holds*

$$A \subseteq \operatorname{Min}(A, Q) + Q, \tag{5.43}$$

then A fulfills the domination property according to the minimum*. If*

$$A \subseteq \operatorname{Max}(A, Q) - Q \tag{5.44}$$

is satisfied, then A fulfills the domination property according to the maximum*.*

Note that the domination property according to the minimum / maximum is introduced in [55] as *quasi domination property.*

Remark 25. *Suppose $A \subset Y$ fulfills the domination property according to the minimum* (5.43) *and assume $Q + Q \subseteq Q$ holds. With* (5.43)*, we obtain*

$$A + Q \subseteq \operatorname{Min}(A, Q) + Q + Q \subseteq \operatorname{Min}(A, Q) + Q. \tag{5.45}$$

Similarly, assuming that A satisfies the domination property according to the maximum (5.44) *and $-Q - Q \subseteq -Q$, we have*

$$A - Q \subseteq \operatorname{Max}(A, Q) - Q - Q \subseteq \operatorname{Max}(A, Q) - Q. \tag{5.46}$$

Theorem 33. *Let the domination property according to the minimum and maximum* (5.43), (5.44) *be satisfied for every $f_{\mathcal{U}}(x)$, $x \in \mathcal{X}$ and $Q = C$ for a proper closed convex and pointed cone C. Then it holds: If x^0 is strictly set less ordered robust, then x^0 is strictly minmax less ordered robust.*

Proof. First note that since C is a convex cone, the assumption $C + C \subseteq C$ that is used in Remark 25 is fulfilled. Assume that $x^0 \in \mathcal{X}$ is not strictly minmax less ordered robust. Then there exists $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ such that

$$\begin{aligned}
& \operatorname{Min}(f_{\mathcal{U}}(\overline{x}), C) + C \supseteq \operatorname{Min}(f_{\mathcal{U}}(x^0), C) && (5.47) \\
\text{and } & \operatorname{Min}(f_{\mathcal{U}}(\overline{x}), C) \subseteq \operatorname{Min}(f_{\mathcal{U}}(x^0), C) - C \\
\text{and } & \operatorname{Max}(f_{\mathcal{U}}(\overline{x}), C) + C \supseteq \operatorname{Max}(f_{\mathcal{U}}(x^0), C) \\
\text{and } & \operatorname{Max}(f_{\mathcal{U}}(\overline{x}), C) \subseteq \operatorname{Max}(f_{\mathcal{U}}(x^0), C) - C. && (5.48)
\end{aligned}$$

From (5.45), we conclude

$$f_{\mathcal{U}}(x) + C = \mathrm{Min}(f_{\mathcal{U}}(x), C) + C \tag{5.49}$$

for any $x \in \mathcal{X}$. Together with (5.47), we deduce

$$\begin{aligned} f_{\mathcal{U}}(x^0) &\subseteq f_{\mathcal{U}}(x^0) + C \\ &\overset{(5.49)}{=} \mathrm{Min}(f_{\mathcal{U}}(x^0), C) + C \\ &\overset{(5.47)}{\subseteq} \mathrm{Min}(f_{\mathcal{U}}(\overline{x}), C) + C + C \\ &\subseteq \mathrm{Min}(f_{\mathcal{U}}(\overline{x}), C) + C \\ &\subseteq f_{\mathcal{U}}(\overline{x}) + C. \end{aligned}$$

In the same way we show $f_{\mathcal{U}}(\overline{x}) \subseteq f_{\mathcal{U}}(x^0) - C$: From (5.46), we obtain

$$f_{\mathcal{U}}(x) - C = \mathrm{Max}(f_{\mathcal{U}}(x), C) - C \tag{5.50}$$

for any $x \in \mathcal{X}$. Together with (5.48), we acquire

$$\begin{aligned} f_{\mathcal{U}}(\overline{x}) &\subseteq f_{\mathcal{U}}(\overline{x}) - C \\ &\overset{(5.50)}{=} \mathrm{Max}(f_{\mathcal{U}}(\overline{x}), C) - C \\ &\overset{(5.48)}{\subseteq} \mathrm{Max}(f_{\mathcal{U}}(x^0), C) - C - C \\ &\subseteq \mathrm{Max}(f_{\mathcal{U}}(x^0), C) - C \\ &\subseteq f_{\mathcal{U}}(x^0) - C. \end{aligned}$$

Therefore $f_{\mathcal{U}}(\overline{x}) + C \supseteq f_{\mathcal{U}}(x^0)$ and $f_{\mathcal{U}}(\overline{x}) \subseteq f_{\mathcal{U}}(x^0) - C$, thus x^0 is not strictly set less ordered robust, in contradiction to the assumption. □

Example 11. *In Theorem 33, we have seen that strictly set less ordered robust solutions are also strictly minmax less ordered robust. By considering the example in Figure 5.11, we observe that the inverse direction is not true. Here, using $Y = \mathbb{R}^2$, $X = \mathbb{R}^n$ and $C = \mathbb{R}^2_{\geqq}$, x_1 is strictly minmax less ordered robust, while it is not strictly set less ordered robust.*

Lemma 13. *Given $P(\mathcal{U})$ with $Y = \mathbb{R}$, $X = \mathbb{R}^n$ and $C = \mathbb{R}_{\geqq}$. Then x^0 is minmax less ordered robust if and only if x^0 is weakly minmax less ordered robust. Furthermore, if x^0 is uniquely optimal for the robust counterpart (RC) (compare (5.10)) or for the optimistic counterpart (OC) (see (5.11)), then x^0 is strictly minmax set less ordered robust.*

Proof. The first part obviously holds because $\mathbb{R}_{\geq} = \mathbb{R}_{>}$. The second assertion follows from Lemma 9 (b) applied to Theorem 33. □

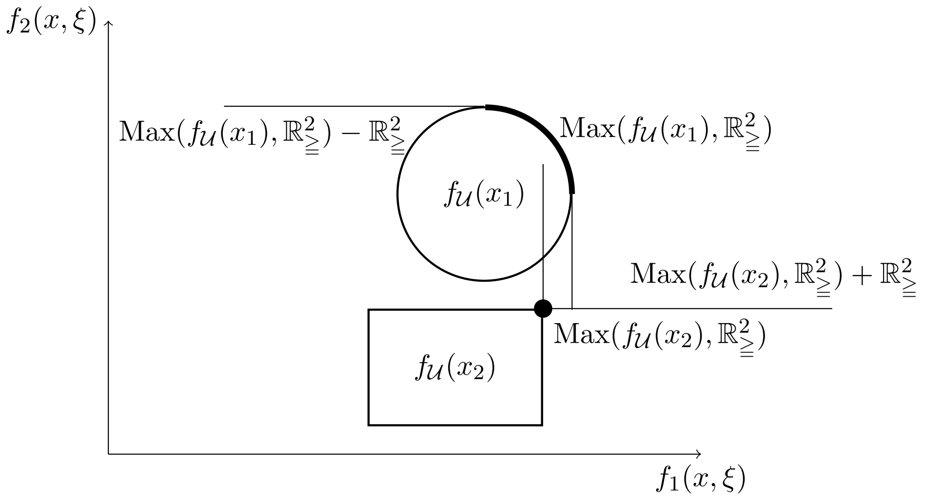

Figure 5.11: x_1 is strictly minmax less ordered robust, but x_1 is not strictly set less ordered robust.

Computing Minmax Less Ordered Robust Solutions

In order to find minmax less ordered robust solutions of an uncertain multicriteria optimization problem, we suggest the following method. Consider for $Q = C$ ($Q = \operatorname{int} C$, $Q = C \setminus \{0\}$, respectively) and $y^* := (y_1^*, \dots, y_4^*)^T$, $y_j^* \in C^* \setminus \{0\}$, $j = 1, \dots, 4$, the problem

$$(VOP^{ml}(y^*)) \qquad \mathrm{Min}(h[\mathcal{X}], \mathbb{R}^4_{\geqq}), \tag{5.51}$$

where

$$h(x) := \begin{pmatrix} \inf\{y_1^* \circ f(x,\xi) \mid \xi \in \mathcal{U},\ f(x,\xi) \in \mathrm{Min}(f_{\mathcal{U}}(x), Q)\} \\ \sup\{y_2^* \circ f(x,\xi) \mid \xi \in \mathcal{U},\ f(x,\xi) \in \mathrm{Min}(f_{\mathcal{U}}(x), Q)\} \\ \inf\{y_3^* \circ f(x,\xi) \mid \xi \in \mathcal{U},\ f(x,\xi) \in \mathrm{Max}(f_{\mathcal{U}}(x), Q)\} \\ \sup\{y_4^* \circ f(x,\xi) \mid \xi \in \mathcal{U},\ f(x,\xi) \in \mathrm{Max}(f_{\mathcal{U}}(x), Q)\} \end{pmatrix}.$$

Theorem 34. *Consider an uncertain vector optimization problem $P(\mathcal{U})$. The following statements hold:*

(a) If x^0 is strictly Pareto optimal for problem $(VOP^{ml}(y^*))$ with $Q = C$ for some $y_j^* \in C^* \setminus \{0\}$, $j = 1, \dots, 4$, then x^0 is strictly minmax less ordered robust.

(b) If x^0 is weakly Pareto optimal for problem $(VOP^{ml}(y^*))$ with $Q = \operatorname{int} C$ for some $y_j^* \in C^* \setminus \{0\}$, $j = 1, \dots, 4$, and the minima and maxima in each of the components in the vector of objectives exist for all $x \in \mathcal{X}$ and the chosen weights $y_j^* \in C^* \setminus \{0\}$, $j = 1, \dots, 4$, then x^0 is weakly minmax less ordered robust.

(c) *If x^0 is weakly Pareto optimal for problem $(VOP^{ml}(y^*))$ with $Q = C \setminus \{0\}$ for some $y_j^* \in C^{\#},\ j = 1, \dots, 4$, and the minima and maxima in each of the components in the vector of objectives exist for all $x \in \mathcal{X}$ and the chosen weights $y_j^* \in C^{\#},\ j = 1, \dots, 4$, then x^0 is minmax less ordered robust.*

Proof. Suppose x^0 is not strictly (weakly, $\cdot$, respectively) minmax less ordered robust. Consequently, there exists $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ such that for $Q = C$ ($Q = \operatorname{int} C$, $Q = C \setminus \{0\}$, respectively) (5.42) holds. Thus, it follows

$$\begin{aligned}
\forall\ f(x^0, \xi) \in \operatorname{Min}(f_{\mathcal{U}}(x^0), Q)\ \exists\ f(\overline{x}, \eta) \in \operatorname{Min}(f_{\mathcal{U}}(\overline{x}), Q) :\ & f(\overline{x}, \eta) + Q \ni f(x^0, \xi),\\
\forall\ f(\overline{x}, \xi) \in \operatorname{Min}(f_{\mathcal{U}}(\overline{x}), Q)\ \exists\ f(x^0, \eta) \in \operatorname{Min}(f_{\mathcal{U}}(x^0), Q) :\ & f(\overline{x}, \xi) \in f(x^0, \eta) - Q,\\
\forall\ f(x^0, \xi) \in \operatorname{Max}(f_{\mathcal{U}}(x^0), Q)\ \exists\ f(\overline{x}, \eta) \in \operatorname{Max}(f_{\mathcal{U}}(\overline{x}), Q) :\ & f(\overline{x}, \eta) + Q \ni f(x^0, \xi),\\
\forall\ f(\overline{x}, \xi) \in \operatorname{Max}(f_{\mathcal{U}}(\overline{x}), Q)\ \exists\ f(x^0, \eta) \in \operatorname{Max}(f_{\mathcal{U}}(x^0), Q) :\ & f(\overline{x}, \xi) \in f(x^0, \eta) - Q.
\end{aligned}$$

For chosen $y_j^* \in C^* \setminus \{0\}$ ($y_j^* \in C^* \setminus \{0\}$, $y_j^* \in C^{\#}$, respectively) for $j = 1, \dots, 4$, we acquire

$$\begin{aligned}
\forall\ f(x^0, \xi) \in \operatorname{Min}(f_{\mathcal{U}}(x^0), Q)\ \exists\ f(\overline{x}, \eta) \in \operatorname{Min}(f_{\mathcal{U}}(\overline{x}), Q) :\ & y_1^* \circ f(\overline{x}, \eta)\ [\le / < / <]\ y_1^* \circ f(x^0, \xi),\\
\forall\ f(\overline{x}, \xi) \in \operatorname{Min}(f_{\mathcal{U}}(\overline{x}), Q)\ \exists\ f(x^0, \eta) \in \operatorname{Min}(f_{\mathcal{U}}(x^0), Q) :\ & y_2^* \circ f(\overline{x}, \xi)\ [\le / < / <]\ y_2^* \circ f(x^0, \eta),\\
\forall\ f(x^0, \xi) \in \operatorname{Max}(f_{\mathcal{U}}(x^0), Q)\ \exists\ f(\overline{x}, \eta) \in \operatorname{Max}(f_{\mathcal{U}}(\overline{x}), Q) :\ & y_2^* \circ f(\overline{x}, \eta)\ [\le / < / <]\ y_3^* \circ f(x^0, \xi),\\
\forall\ f(\overline{x}, \xi) \in \operatorname{Max}(f_{\mathcal{U}}(\overline{x}), Q)\ \exists\ f(x^0, \eta) \in \operatorname{Max}(f_{\mathcal{U}}(x^0), Q) :\ & y_4^* \circ f(\overline{x}, \xi)\ [\le / < / <]\ y_4^* \circ f(x^0, \eta).
\end{aligned}$$

We obtain

$$\begin{aligned}
&\inf\{y_1^* \circ f(\overline{x}, \eta) |\ \eta \in \mathcal{U},\ f(\overline{x}, \eta) \in \operatorname{Min}(f_{\mathcal{U}}(\overline{x}), Q)\}\\
&\qquad [\le / < / <] \inf\{y_1^* \circ f(x^0, \xi) |\ \xi \in \mathcal{U},\ f(x^0, \xi) \in \operatorname{Min}(f_{\mathcal{U}}(x^0), Q)\}, \qquad (5.52)\\
&\sup\{y_2^* \circ f(\overline{x}, \xi) |\ \xi \in \mathcal{U},\ f(\overline{x}, \xi) \in \operatorname{Min}(f_{\mathcal{U}}(\overline{x}), Q)\}\\
&\qquad [\le / < / <] \sup\{y_2^* \circ f(x^0, \eta) |\ \eta \in \mathcal{U},\ f(x^0, \eta) \in \operatorname{Min}(f_{\mathcal{U}}(x^0), Q)\}, \qquad (5.53)\\
&\inf\{y_3^* \circ f(\overline{x}, \eta) |\ \eta \in \mathcal{U},\ f(\overline{x}, \eta) \in \operatorname{Max}(f_{\mathcal{U}}(\overline{x}), Q)\}\\
&\qquad [\le / < / <] \inf\{y_3^* \circ f(x^0, \xi) |\ \xi \in \mathcal{U},\ f(x^0, \xi) \in \operatorname{Max}(f_{\mathcal{U}}(x^0), Q)\}, \qquad (5.54)\\
&\sup\{y_4^* \circ f(\overline{x}, \xi) |\ \xi \in \mathcal{U},\ f(\overline{x}, \xi) \in \operatorname{Max}(f_{\mathcal{U}}(\overline{x}), Q)\}\\
&\qquad [\le / < / <] \sup\{y_4^* \circ f(x^0, \eta) |\ \eta \in \mathcal{U},\ f(x^0, \eta) \in \operatorname{Max}(f_{\mathcal{U}}(x^0), Q)\}. \qquad (5.55)
\end{aligned}$$

The last two strict inequalities in (5.52) – (5.55) hold because the maxima and minima exist. But (5.52) – (5.55) are a contradiction to the assumption. □

The following algorithm computes minmax less ordered robust solutions based on the vectorization results presented in Theorem 34. We denote the sets of strictly ($\cdot$, weakly) minmax less ordered robust solutions by $\mathrm{Opt}_{\mathrm{smlr}}$ ($\mathrm{Opt}_{\mathrm{mlr}}$, $\mathrm{Opt}_{\mathrm{wmlr}}$, respectively).

Algorithm 15 for computing minmax less ordered robust solutions using a family of problems $(VOP^{ml}(y^*))$ **(see** (5.51)**):**

Input & Steps 1-7: Analogous to Algorithm 12, only replacing the sets $\mathrm{Opt}_{\mathrm{sslor}}$, $\mathrm{Opt}_{\mathrm{slor}}$, $\mathrm{Opt}_{\mathrm{wslor}}$ by $\mathrm{Opt}_{\mathrm{smlr}}$, $\mathrm{Opt}_{\mathrm{mlr}}$, $\mathrm{Opt}_{\mathrm{wmlr}}$, y^*, y^{**} by $y_1^*, \ldots, y_4^*$, $(VOP^{sl}(y^*, y^{**}))$ by $(VOP^{ml}(y_1^*, \ldots, y_4^*))$, and replacing $\min_{\xi\in\mathcal{U}}\ y^* \circ f(x', \xi)$ and $\max_{\xi\in\mathcal{U}}\ y^{**} \circ f(x', \xi)$ by $\min\{y_1^* \circ f(x', \xi)|\ \xi \in \mathcal{U},\ f(x', \xi) \in \mathrm{Min}(f_{\mathcal{U}}(x'), Q)\}$, $\max\{y_2^* \circ f(x', \xi)|\ \xi \in \mathcal{U},\ f(x', \xi) \in \mathrm{Min}(f_{\mathcal{U}}(x'), Q)\}$, $\min\{y_3^* \circ f(x', \xi)|\ \xi \in \mathcal{U},\ f(x', \xi) \in \mathrm{Max}(f_{\mathcal{U}}(x'), Q)\}$, $\max\{y_4^* \circ f(x', \xi)|\ \xi \in \mathcal{U},\ f(x', \xi) \in \mathrm{Max}(f_{\mathcal{U}}(x'), Q)\}$, and replacing "set less ordered robust" by "minmax less ordered robust".

In accordance with prior algorithms, it is also possible here to provide an interactive procedure to obtain minmax less ordered robust solutions. In order to keep it simple and short, we use the same weight $\hat{y}^*$ in every objective function in problem $(VOP^{ml}(\hat{y}^*))$ (see (5.51)), but of course, as seen in Theorem 34, it is entirely possible to use different weights in every objective.

Algorithm 16 for computing minmax less ordered robust solutions using a family of problems $(VOP^{ml}(y^*))$ **(see** (5.51)**):**

Input & Steps 1-8: Analogous to Algorithm 13, only replacing the solution sets $\mathrm{Opt}_{\mathrm{sslor}}$, $\mathrm{Opt}_{\mathrm{slor}}$, $\mathrm{Opt}_{\mathrm{wslor}}$ by $\mathrm{Opt}_{\mathrm{smlr}}$, $\mathrm{Opt}_{\mathrm{mlr}}$, $\mathrm{Opt}_{\mathrm{wmlr}}$, $\hat{y}^*$, $\hat{y}^{**}$ by $\hat{y}^*$, $(VOP^{sl}(\hat{y}^*, \hat{y}^{**}))$ by $(VOP^{ml}(\hat{y}^*))$, and replacing $\min_{\xi\in\mathcal{U}}\ \hat{y}^* \circ f(x', \xi)$ and $\max_{\xi\in\mathcal{U}}\ \hat{y}^{**} \circ f(x', \xi)$ by $\min\{\hat{y}^* \circ f(x', \xi)|\ \xi \in \mathcal{U},\ f(x', \xi) \in \mathrm{Min}(f_{\mathcal{U}}(x'), Q)\}$, $\max\{\hat{y}^* \circ f(x', \xi)|\ \xi \in \mathcal{U},\ f(x', \xi) \in \mathrm{Min}(f_{\mathcal{U}}(x'), Q)\}$, $\min\{\hat{y}^* \circ f(x', \xi)|\ \xi \in \mathcal{U},\ f(x', \xi) \in \mathrm{Max}(f_{\mathcal{U}}(x'), Q)\}$, $\max\{\hat{y}^* \circ f(x', \xi)|\ \xi \in \mathcal{U},\ f(x', \xi) \in \mathrm{Max}(f_{\mathcal{U}}(x'), Q)\}$, and replacing "set less ordered robust" by "minmax less ordered robust".

5.3.6 Certainly Less Ordered Robustness

The so far introduced methods for solving uncertain multi-objective optimization problems are, due to our investigations, useful for acquiring solutions which are of good quality in regard to a decision maker's preferences. In some cases, however, it might be beneficial for the decision process to sort out those solutions which are of obviously bad quality beforehand, without considering a certain concept of robustness. In this section, we present an approach which is able to filter out those feasible elements which are obviously of bad quality for all possible interpretations of robustness. We will see below that the *certainly less order relation* mentioned in Definition 9 will be useful to obtain such a concept.

In the following, we extend the definition of the *certainly less order relation* to general nonempty sets $Q \subset Y$ with $\mathrm{cl}\, Q \cap (-\,\mathrm{cl}\, Q) = \{0\}$, $C \subset \mathrm{cl}\, Q$, $C \subset Y$ being a proper closed convex and pointed cone. The *certainly less order relation* $\preceq_Q^{cert}$ is introduced as

$$A \preceq_Q^{cert} B :\Longleftrightarrow (\forall\ a \in A,\ \forall\ b \in B : a \leq_Q b).$$

If we are dealing with $Q = \operatorname{int} C$, we suppose that $\operatorname{int} C \neq \emptyset$. Note that in the definition we use, $\preceq_C^{cert}$ is generally not a pre-order.

Definition 22. *A solution x^0 of $P(\mathcal{U})$ is called strictly (weakly, $\cdot$, respectively)* **certainly less ordered robust** *if there is no $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ such that $f_{\mathcal{U}}(\overline{x}) \preceq_Q^{cert} f_{\mathcal{U}}(x^0)$, which is equivalent to*

$$\nexists\, \overline{x} \in \mathcal{X} \setminus \{x^0\} : \quad \forall\, f(\overline{x}, \xi) \in f_{\mathcal{U}}(\overline{x}),\ \forall\, f(x^0, \eta) \in f_{\mathcal{U}}(x^0) : f(\overline{x}, \xi) \leq_Q f(x^0, \eta)$$

for $Q = C$ ($Q = \operatorname{int} C$, $Q = C \setminus \{0\}$, respectively).

This approach is, compared to the before introduced robustness concepts, the least restrictive and may yield more solutions for a decision maker to choose from. A solution obtained from this robustness concept may not be "robust" in its general meaning, but this approach will lead to more variety in optimal points which gives the decision maker more flexibility. Furthermore, this concept can serve as a pre-selection in order to sort out those solutions which are of bad quality.

In order to be consistent with the literature, we check if the concept at hand would be useful in case $|\mathcal{U}| = 1$.

Lemma 14. *Given $P(\mathcal{U})$ with $|\mathcal{U}| = 1$. Then x^0 is strictly (weakly, $\cdot$, respectively) certainly less ordered robust if and only if $f(x^0)$ is strictly (weakly, $\cdot$, respectively) minimal.*

The proof of the above lemma can be led analogously to that of Lemma 8.

To verify whether this concept would be helpful in the case of scalar robust optimization, we check for consistency for $Y = \mathbb{R}$, $X = \mathbb{R}^n$ and $C = \mathbb{R}_{\geqq}$.

Lemma 15. *Given $P(\mathcal{U})$ with $Y = \mathbb{R}$, $X = \mathbb{R}^n$ and $C = \mathbb{R}_{\geqq}$.*

(a) x^0 is weakly certainly less ordered robust $\iff$ x^0 is certainly less ordered robust.

(b) If x^0 is uniquely optimal for the robust counterpart (RC) (compare (5.10)) or for the optimistic counterpart (OC) (compare (5.11)), then x^0 is strictly certainly less ordered robust.

(c) Suppose $\max_{\xi \in \mathcal{U}} f(x', \xi)$ and $\min_{\xi \in \mathcal{U}} f(x', \xi)$ exist for every $x' \in \mathcal{X}$. Then it holds: If x^0 is optimal for the robust counterpart (RC) (compare (5.10)) or for the optimistic counterpart (OC) (compare (5.11)), then x^0 is certainly less ordered robust.

Proof. (a) Holds since $\mathbb{R}_{>} = \mathbb{R}_{\geq}$.

(b) x^0 is uniquely optimal for the robust counterpart (RC) (compare (5.10))

$$\iff \nexists\, \overline{x} \in \mathcal{X} \setminus \{x^0\} : \sup_{\xi \in \mathcal{U}} f(\overline{x}, \xi) \leq \sup_{\xi \in \mathcal{U}} f(x^0, \xi). \tag{5.56}$$

Alternatively, x^0 may be uniquely optimal for the optimistic counterpart (OC) (see (5.11))

$$\iff \nexists\, \overline{x} \in \mathcal{X} \setminus \{x^0\} : \inf_{\xi \in \mathcal{U}} f(\overline{x}, \xi) \leq \inf_{\xi \in \mathcal{U}} f(x^0, \xi). \tag{5.57}$$

Now suppose that x^0 is not strictly certainly less ordered robust. Then there exists $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ s.t.

$$\begin{aligned}
&\forall\ f(\overline{x},\xi) \in f_{\mathcal{U}}(\overline{x}),\ \forall\ f(x^0,\eta) \in f_{\mathcal{U}}(x^0) : f(\overline{x},\xi) \in f(x^0,\eta) - \mathbb{R}_{\geqq} \\
&\Longrightarrow \sup_{\xi\in\mathcal{U}} f(\overline{x},\xi) \leq \inf_{\xi\in\mathcal{U}} f(x^0,\xi) \\
&\Longrightarrow \inf_{\xi\in\mathcal{U}} f(\overline{x},\xi) \leq \sup_{\xi\in\mathcal{U}} f(\overline{x},\xi) \leq \inf_{\xi\in\mathcal{U}} f(x^0,\xi) \leq \sup_{\xi\in\mathcal{U}} f(x^0,\xi),
\end{aligned}$$

in contradiction to (5.56), or alternatively, contradicting (5.57).

(c) Suppose that x^0 is not certainly less ordered robust. Then there exists $\overline{x} \in \mathcal{X}\setminus\{x^0\}$ s.t.

$$\begin{aligned}
&\forall\ f(\overline{x},\xi) \in f_{\mathcal{U}}(\overline{x}),\ \forall\ f(x^0,\eta) \in f_{\mathcal{U}}(x^0) : f(\overline{x},\xi) \in f(x^0,\eta) - \mathbb{R}_{\geq} \\
&\Longrightarrow \max_{\xi\in\mathcal{U}} f(\overline{x},\xi) < \min_{\xi\in\mathcal{U}} f(x^0,\xi) \\
&\Longrightarrow \min_{\xi\in\mathcal{U}} f(\overline{x},\xi) \leq \max_{\xi\in\mathcal{U}} f(\overline{x},\xi) < \min_{\xi\in\mathcal{U}} f(x^0,\xi) \leq \max_{\xi\in\mathcal{U}} f(x^0,\xi),
\end{aligned}$$

contradicting x^0's optimality for (RC) (compare (5.10)) ((OC), see (5.11), respectively).

□

The following theorem verifies that set less ordered robust solutions to $P(\mathcal{U})$ are certainly less ordered robust.

Theorem 35. *If x^0 is strictly (weakly, $\cdot$, respectively) set less ordered robust, then x^0 is strictly (weakly, $\cdot$, respectively) certainly less ordered robust.*

Proof. Let $Q = C$ ($Q = \operatorname{int} C$, $Q = C \setminus \{0\}$, respectively), and let $x^0 \in \mathcal{X}$ be strictly (weakly, $\cdot$, respectively) set less ordered robust. Then there exists no $\overline{x} \in \mathcal{X} \setminus \{x^0\}$: $f_{\mathcal{U}}(\overline{x}) \preceq^s_Q f_{\mathcal{U}}(x^0)$

$$\iff \nexists\ \overline{x} \in \mathcal{X} \setminus \{x^0\} :\ f_{\mathcal{U}}(\overline{x}) \subseteq f_{\mathcal{U}}(x^0) - Q \text{ and } f_{\mathcal{U}}(\overline{x}) + Q \supseteq f_{\mathcal{U}}(x^0). \tag{5.58}$$

Suppose that x^0 is not strictly (weakly, $\cdot$, respectively) certainly less ordered robust. Then there exists $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ such that

$$\begin{aligned}
f_{\mathcal{U}}(\overline{x}) \preceq^{cert}_Q f_{\mathcal{U}}(x^0) &\iff \forall\ f(\overline{x},\xi) \in f_{\mathcal{U}}(\overline{x}),\ \forall\ f(x^0,\eta) \in f_{\mathcal{U}}(x^0) : f(\overline{x},\xi) \leq_Q f(x^0,\eta) \\
&\Longrightarrow f_{\mathcal{U}}(\overline{x}) \subseteq f_{\mathcal{U}}(x^0) - Q \text{ and } f_{\mathcal{U}}(\overline{x}) + Q \supseteq f_{\mathcal{U}}(x^0),
\end{aligned}$$

in contradiction to (5.58). □

In contrast to the work line that we followed before, we will not continue working with the above definition for certainly less ordered robust elements. Instead, Definition 22 rather serves as an inspiration to motivate the following robustness approach for the special case $Y = \mathbb{R}^k$, $X = \mathbb{R}^n$, $C = \mathbb{R}^k_{\geqq}$.

Definition 23. *Given an uncertain multi-objective optimization problem $P(\mathcal{U})$ with $Y = \mathbb{R}^k$, $X = \mathbb{R}^n$ and $C = \mathbb{R}^k_{\geqq}$. For all $x \in \mathcal{X}$ we define*

$$\begin{aligned}\mathrm{CMax}\, f_{\mathcal{U}}(x) &:= (\sup_{\xi\in\mathcal{U}} f_1(x,\xi), \ldots, \sup_{\xi\in\mathcal{U}} f_k(x,\xi))^T, \\ \mathrm{CMin}\, f_{\mathcal{U}}(x) &:= (\inf_{\xi\in\mathcal{U}} f_1(x,\xi), \ldots, \inf_{\xi\in\mathcal{U}} f_k(x,\xi))^T.\end{aligned}$$

A solution x^0 to $P(\mathcal{U})$ is called [strictly, weakly, $\cdot$] **certainly less alternative ordered robust***, if there is no $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ such that:*

$$\mathrm{CMax}\, f_{\mathcal{U}}(\overline{x}) \in \mathrm{CMin}\, f_{\mathcal{U}}(x^0) - \mathbb{R}^k_{[\geqq/>/\geq]}.$$

The relation between the *certainly less order relation* $\preceq_Q^{cert}$ for $Q = C = \mathbb{R}^k_{\geqq}$ and certainly less *alternative* ordered robust solutions is given below.

Lemma 16. *Given an uncertain multi-objective optimization problem $P(\mathcal{U})$ with $Y = \mathbb{R}^k$, $X = \mathbb{R}^n$. Then for all $x, \overline{x} \in \mathcal{X}$:*

$$f_{\mathcal{U}}(\overline{x}) \preceq_Q^{cert} f_{\mathcal{U}}(x) \textit{ with respect to } Q = \mathbb{R}^k_{\geqq} \Longleftrightarrow \mathrm{CMax}\, f_{\mathcal{U}}(\overline{x}) \in \mathrm{CMin}\, f_{\mathcal{U}}(x) - \mathbb{R}^k_{\geqq}.$$

Proof.

$$\begin{aligned}& f_{\mathcal{U}}(\overline{x}) \preceq_Q^{cert} f_{\mathcal{U}}(x) \text{ with respect to } Q = \mathbb{R}^k_{\geqq} \\ \Longleftrightarrow\ & \forall\, \xi \in \mathcal{U},\ \forall\, \eta \in \mathcal{U}:\ f(\overline{x},\xi) \leqq f(x,\eta) \\ \Longleftrightarrow\ & \sup_{\xi\in\mathcal{U}} f_i(\overline{x},\xi) \leq \inf_{\eta\in\mathcal{U}} f_i(x,\eta),\ i = 1,\ldots,k \\ \Longleftrightarrow\ & \mathrm{CMax}\, f_{\mathcal{U}}(\overline{x}) \in \mathrm{CMin}\, f_{\mathcal{U}}(x) - \mathbb{R}^k_{\geqq}.\end{aligned}$$

□

Remark 26. *Note that the analogous of Lemma 16 for $\mathbb{R}^k_{[>/\geq]}$ only hold for the direction $\Longleftarrow$. The inverse only holds if $\min_{\xi\in\mathcal{U}} f_i(x',\xi)$ and $\max_{\xi\in\mathcal{U}} f_i(x',\xi)$ exist for all $x' \in \mathcal{X},\ i = 1,\ldots,k$.*

In the sense of robustness, the *certainly less alternative ordered* robustness concept is able to filter out solutions that cannot be optimal in some sense, since the best cases of one solution are dominated by the worst cases of another element. Thus, this approach may serve as a pre-selection before a finite decision strategy has been made by the decision maker. Note that the reason the certainly less *alternative* ordered robustness concept is introduced here is to derive algorithms in order to compute strictly certainly less ordered robust solutions.

Example 12. *The left side of Figure 5.12 shows an example for a strictly certainly less alternative ordered robust solution x_1. On the right side of Figure 5.12, x_3 is not weakly certainly less alternative ordered robust.*

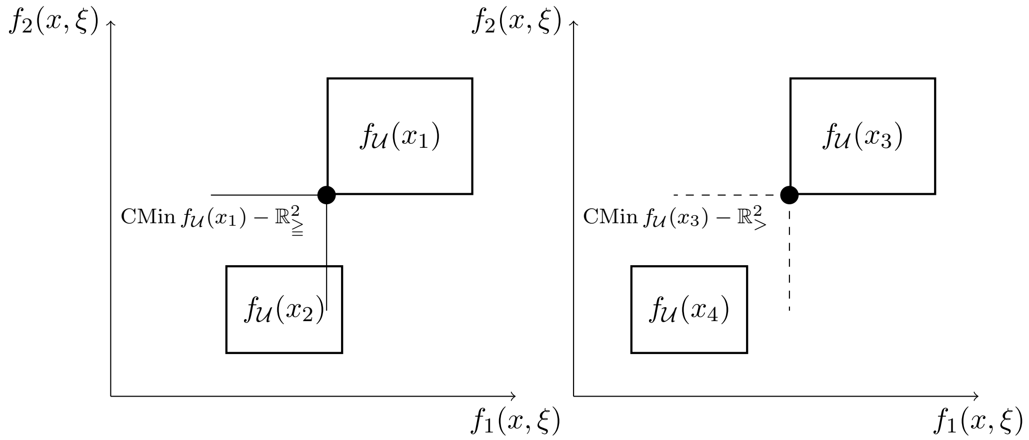

Figure 5.12: Left: x_1 is strictly certainly less alternative ordered robust. Right: x_3 is not weakly certainly less alternative ordered robust.

Computing Certainly Less Alternative Ordered Robust Solutions

This subsection is devoted to analyzing how one may obtain certainly less alternative ordered robust solutions to an uncertain multi-objective optimization problem. Recall that certainly less *alternative* ordered robustness has only been defined for the special case $Y = \mathbb{R}^k$, $X = \mathbb{R}^n$, $C = \mathbb{R}^k_{\geqq}$. To this end, we introduce a bicriteria minimization problem and show that weakly Pareto optimal solutions of this problem are at least weakly certainly less alternative ordered robust for $P(\mathcal{U})$. Consider the problem

$$(VOP^{cert}(y^*)) \qquad \qquad \mathrm{Min}(h[\mathcal{X}], \mathbb{R}^2_{\geqq}) \tag{5.59}$$

where $y^* \in \mathbb{R}^k_{\geq}$ and $h(x) := (\sum_{i=1}^k y_i^* \inf_{\xi\in\mathcal{U}} f_i(x,\xi),\ \sum_{i=1}^k y_i^* \sup_{\xi\in\mathcal{U}} f_i(x,\xi))^T$. Note that we use the same weight y^* in both objectives.

Theorem 36. *Consider an uncertain vector-valued optimization problem $P(\mathcal{U})$ with $Y = \mathbb{R}^k$, $X = \mathbb{R}^n$, $C = \mathbb{R}^k_{\geqq}$. The following statements hold:*

(a) If x^0 is strictly Pareto optimal for problem $(VOP^{cert}(y^*))$ for some $y^* \in \mathbb{R}^k_{\geq}$, then x^0 is strictly certainly less alternative ordered robust.

(b) If x^0 is weakly Pareto optimal for problem $(VOP^{cert}(y^*))$ for some $y^* \in \mathbb{R}^k_{\geq}$, then x^0 is weakly certainly less alternative ordered robust.

(c) If x^0 is weakly Pareto optimal for problem $(VOP^{cert}(y^*))$ for some $y^* \in \mathbb{R}^k_{>}$, then x^0 is certainly less alternative ordered robust.

Proof. Let x^0 be strictly Pareto optimal (weakly Pareto optimal, weakly Pareto optimal, respectively) for problem $(VOP^{cert}(y^*))$ with some $y^* \in \mathbb{R}^k_{\geq}$ ($y^* \in \mathbb{R}^k_{\geq}$, $y^* \in \mathbb{R}^k_{>}$, respectively), i.e., there is no $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ such that

$$\sum_{i=1}^{k} y_i^* \inf_{\xi \in \mathcal{U}} f_i(\overline{x}, \xi) \; [\leq / < / <] \; \sum_{i=1}^{k} y_i^* \inf_{\xi \in \mathcal{U}} f_i(x^0, \xi)$$
$$\text{and } \sum_{i=1}^{k} y_i^* \sup_{\xi \in \mathcal{U}} f_i(\overline{x}, \xi) \; [\leq / < / <] \; \sum_{i=1}^{k} y_i^* \sup_{\xi \in \mathcal{U}} f_i(x^0, \xi).$$

Now suppose that x^0 is not [strictly, weakly, $\cdot$] certainly less alternative ordered robust. Then there exists $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ such that

$$\text{CMax}\, f_{\mathcal{U}}(\overline{x}) \in \text{CMin}\, f_{\mathcal{U}}(x^0) - \mathbb{R}^k_{[\geqq/>/\geq]}.$$

Thus, for $y^* \in \mathbb{R}^k_{\geq}$ ($y^* \in \mathbb{R}^k_{\geq}$, $y^* \in \mathbb{R}^k_{>}$, respectively),

$$\sum_{i=1}^{k} y_i^* \sup_{\xi \in \mathcal{U}} f_i(\overline{x}, \xi) \; [\leq / < / <] \; \sum_{i=1}^{k} y_i^* \inf_{\xi \in \mathcal{U}} f_i(x^0, \xi).$$

This implies

$$\begin{aligned} \sum_{i=1}^{k} y_i^* \inf_{\xi \in \mathcal{U}} f_i(\overline{x}, \xi) \leq & \sum_{i=1}^{k} y_i^* \sup_{\xi \in \mathcal{U}} f_i(\overline{x}, \xi) \\ & [\leq / < / <] \; \sum_{i=1}^{k} y_i^* \inf_{\xi \in \mathcal{U}} f_i(x^0, \xi) \\ & \leq \sum_{i=1}^{k} y_i^* \sup_{\xi \in \mathcal{U}} f_i(x^0, \xi), \end{aligned}$$

a contradiction. □

The following algorithm is based on the vectorization results in Theorem 36. The sets of strictly (weakly, $\cdot$, respectively) certainly less alternative ordered robust solutions are denoted by $\text{Opt}_{\text{sclor}}$ ($\text{Opt}_{\text{wclor}}$, Opt_{clor}, respectively).

Algorithm 17 for computing certainly less alternative ordered robust solutions using a family of problems $(VOP^{cert}(y^*))$ (see (5.59)):

Input, Steps 1-7: Analogous to Algorithm 12, only replacing $C^* \setminus \{0\}$ by $\mathbb{R}^k_{\geq}$, $C^{\#}$ by $\mathbb{R}^k_{>}$, y^*, y^{**} by one single y^*, $(VOP^{sl}(y^*, y^{**}))$ (see (5.33)) by $(VOP^{cert}(y^*))$, $\text{Opt}_{\text{sslor}}$, $\text{Opt}_{\text{wslor}}$, Opt_{slor} by $\text{Opt}_{\text{sclor}}$, $\text{Opt}_{\text{wclor}}$, Opt_{clor}, and replacing "set less ordered robust" by "certainly less alternative ordered robust". Note that the existence of $\min_{\xi \in \mathcal{U}} f_i(x', \xi)$ and $\max_{\xi \in \mathcal{U}} f_i(x', \xi)$ is not required for any $i = 1, \ldots, k$.

An interactive procedure for obtaining certainly less alternative ordered robust solutions is given below.

Algorithm 18 for computing certainly less alternative ordered robust solutions using a family of problems $(VOP^{cert}(y^*))$ (see (5.59)) by altering the weights:

Input, Steps 1-8: Analogous to Algorithm 13, only replacing $C^* \setminus \{0\}$ by $\mathbb{R}^k_{\geq}$, $C^{\#}$ by $\mathbb{R}^k_{>}$, $y^*_j, y^*_{j+1}, y^{**}_j, y^{**}_{j+1}$ by one single pair y^*_j, y^*_{j+1}, $\hat{y}^*, \hat{y}^{**}$ by one single $\hat{y}^*$, $(VOP^{sl}(\hat{y}^*, \hat{y}^{**}))$ (see (5.33)) by $(VOP^{cert}(\hat{y}^*))$ (see (5.59)), $\mathrm{Opt}_{\mathrm{sslor}}$, $\mathrm{Opt}_{\mathrm{wslor}}$, $\mathrm{Opt}_{\mathrm{slor}}$ by $\mathrm{Opt}_{\mathrm{sclor}}$, $\mathrm{Opt}_{\mathrm{wclor}}$, $\mathrm{Opt}_{\mathrm{clor}}$, and replacing "set less ordered robust" by "certainly less alternative ordered robust". Notice again that we do not need to assume that $\min_{\xi \in \mathcal{U}} f_i(x', \xi)$ and $\max_{\xi \in \mathcal{U}} f_i(x', \xi)$ exist for any $i = 1, \ldots, k$.

5.3.7 Possibly Less Ordered Robustness

We extend the characterization of the *possibly less order relation* given in Definition 10 to general nonempty sets $Q \subset Y$. To do this, let $C \subset Y$ be a proper closed convex and pointed cone and assume $C \subset \operatorname{cl} Q$ and $\operatorname{cl} Q \cap (-\operatorname{cl} Q) = \{0\}$. Under these requirements, we define the *possibly less order relation* $\preceq^p_Q$ for two sets $A, B \subset Y$ by

$$A \preceq^p_Q B :\Longleftrightarrow (\exists\, a \in A,\ \exists\, b \in B:\ a \leq_Q b)\,.$$

We assume that $\operatorname{int} C$ is nonempty in case we are dealing with $Q = \operatorname{int} C$.

Definition 24. *A solution x^0 of $P(\mathcal{U})$ is called strictly (weakly, $\cdot$, respectively)* **possibly less ordered robust** *if there is no $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ such that $f_{\mathcal{U}}(\overline{x}) \preceq^p_Q f_{\mathcal{U}}(x^0)$, which is equivalent to*

$$\nexists\, \overline{x} \in \mathcal{X} \setminus \{x^0\}: \quad f_{\mathcal{U}}(\overline{x}) \cap \big(f_{\mathcal{U}}(x^0) - Q\big) \neq \emptyset$$

for $Q = C$ ($Q = \operatorname{int} C$, $Q = C \setminus \{0\}$, respectively).

Example 13. *In Figure 5.13, we have depicted an example for a possibly less ordered robust solution x_1 and a solution x_3 which is not possibly less ordered robust, where $Q = \mathbb{R}^2_{\geq}$.*

This approach is actually the most restrictive concept for robustness which we consider. A solution is called possibly less ordered robust if there is no other point whose objective value for some scenario ξ and some objective is smaller with respect to the ordering set Q. This concept may be appropriate for a decision maker who is looking for a solution which is not dominated by any other points for all objectives. Due to these extreme restrictions to a solution, the set of possibly less ordered robust points may be empty.

Lemma 17. *Given $P(\mathcal{U})$ with $|\mathcal{U}| = 1$. Then x^0 is strictly (weakly, $\cdot$, respectively) possibly less ordered robust if and only if $f(x^0)$ is strictly (weakly, $\cdot$, respectively) minimal.*

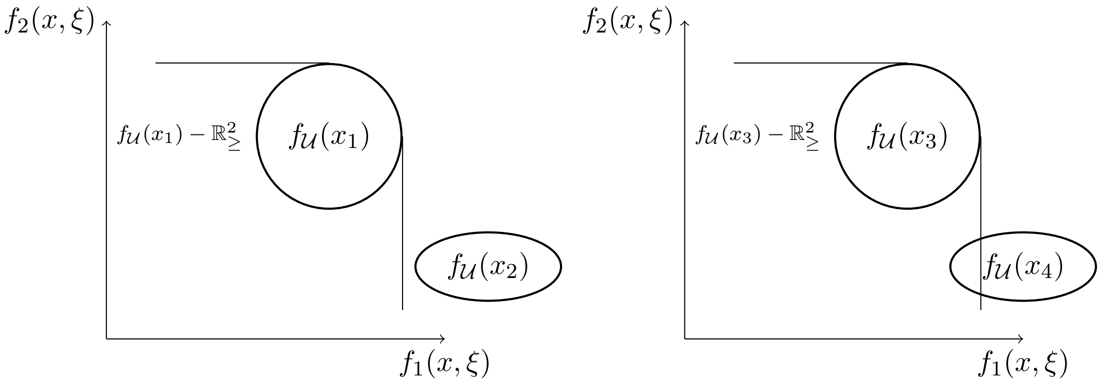

Figure 5.13: Left: x_1 is possibly less ordered robust. Right: x_3 is not possibly less ordered robust.

Proof. The following holds for $Q = C$ ($Q = \operatorname{int} C$, $Q = C \setminus \{0\}$, respectively):

$$\begin{aligned}
&x^0 \text{ is strictly (weakly, } \cdot \text{ , respectively) possibly less ordered robust}\\
&\iff \nexists\, \overline{x} \in \mathcal{X} \setminus \{x^0\}: \; f_{\mathcal{U}}(\overline{x}) \cap (f_{\mathcal{U}}(x^0) - Q) \neq \emptyset\\
&\overset{|\mathcal{U}|=1}{\iff} \nexists\, \overline{x} \in \mathcal{X} \setminus \{x^0\}: \; f(\overline{x}) \cap (f(x^0) - Q) \neq \emptyset\\
&\iff \forall\, \overline{x} \in \mathcal{X} \setminus \{x^0\}: \; f(\overline{x}) \cap (f(x^0) - Q) = \emptyset\\
&\iff f(x^0) \text{ is strictly (weakly, } \cdot \text{ , respectively) minimal.}
\end{aligned}$$

□

The assertions in the lemma below are easy to verify.

Lemma 18. *Given $P(\mathcal{U})$ with $Y = \mathbb{R}$, $X = \mathbb{R}^n$ and $C = \mathbb{R}_{\geqq}$. Then x^0 is possibly less ordered robust if and only if x^0 is weakly possibly less ordered robust. Furthermore, x^0 is strictly ($\cdot$, respectively) possibly less ordered robust if and only if there does not exist any $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ such that $f(\overline{x}, \xi)$ $[\leq / <]$ $f(x^0, \eta)$ for some $\xi, \eta \in \mathcal{U}$.*

Interrelations between possibly and alternative set less ordered robust solutions of an uncertain vector-valued optimization problem $P(\mathcal{U})$ are presented in the following theorem.

Theorem 37. *If x^0 is strictly (weakly, $\cdot$, respectively) possibly less ordered robust, then x^0 is strictly (weakly, $\cdot$, respectively) alternative set less ordered robust.*

Proof. Let $Q = C$ ($Q = \operatorname{int} C$, $Q = C \setminus \{0\}$, respectively). Assume that $x^0 \in \mathcal{X}$ is not strictly (weakly, $\cdot$, respectively) alternative set less ordered robust. Then there exists an $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ such that $f_{\mathcal{U}}(\overline{x}) + Q \supseteq f_{\mathcal{U}}(x^0)$ or $f_{\mathcal{U}}(\overline{x}) \subseteq f_{\mathcal{U}}(x^0) - Q$.

Case 1: $f_{\mathcal{U}}(\overline{x}) + Q \supseteq f_{\mathcal{U}}(x^0)$: Then for all $y^0 \in f_{\mathcal{U}}(x^0)$ there exists $\overline{y} \in f_{\mathcal{U}}(\overline{x})$ such that $y^0 \in \{\overline{y}\} + Q$, i.e. $\overline{y} \leq_Q y^0$. Therefore x^0 is not strictly (weakly, $\cdot$, respectively) possibly less ordered robust.

Case 2: $f_{\mathcal{U}}(\overline{x}) \subseteq f_{\mathcal{U}}(x^0) - Q$: Then for all $\overline{y} \in f_{\mathcal{U}}(\overline{x})$ there exists $y^0 \in f_{\mathcal{U}}(x^0)$ such that $\overline{y} \in \{y^0\} - Q$, i.e. $\overline{y} \leq_Q y^0$. But then x^0 is not strictly (weakly, $\cdot$, respectively) possibly less ordered robust, a contradiction to the assumption. □

Remark 27. *The inverse direction in Theorem 37 is generally not fulfilled: Consider, for instance, the right hand side in Figure 5.13. x_3 is not possibly less ordered robust, but x_3 is an alternative set less ordered robust solution.*

Theorem 38. *If x^0 is strictly (weakly, $\cdot$, respectively) possibly less ordered robust, then x^0 is strictly (weakly, $\cdot$, respectively) minmax less ordered robust, provided that $f_{\mathcal{U}} \in \mathcal{F}^Q_{\min,\max}$ for $Q = C$ ($Q = \operatorname{int} C$, $Q = C \setminus \{0\}$, respectively) (compare* (5.41)*).*

Proof. Suppose that x^0 is not strictly (weakly, $\cdot$, respectively) minmax less ordered robust. Consequently, there exists $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ such that for $Q = C$ ($Q = \operatorname{int} C$, $Q = C \setminus \{0\}$, respectively) (5.42) holds. Thus, it follows

$$\begin{aligned}
\forall\ f(x^0,\xi) \in \operatorname{Min}(f_{\mathcal{U}}(x^0),Q)\ \exists\ f(\overline{x},\eta) \in \operatorname{Min}(f_{\mathcal{U}}(\overline{x}),Q):\ & f(\overline{x},\eta) + Q \ni f(x^0,\xi),\\
\forall\ f(\overline{x},\xi) \in \operatorname{Min}(f_{\mathcal{U}}(\overline{x}),Q)\ \exists\ f(x^0,\eta) \in \operatorname{Min}(f_{\mathcal{U}}(x^0),Q):\ & f(\overline{x},\xi) \in f(x^0,\eta) - Q,\\
\forall\ f(x^0,\xi) \in \operatorname{Max}(f_{\mathcal{U}}(x^0),Q)\ \exists\ f(\overline{x},\eta) \in \operatorname{Max}(f_{\mathcal{U}}(\overline{x}),Q):\ & f(\overline{x},\eta) + Q \ni f(x^0,\xi),\\
\forall\ f(\overline{x},\xi) \in \operatorname{Max}(f_{\mathcal{U}}(\overline{x}),Q)\ \exists\ f(x^0,\eta) \in \operatorname{Max}(f_{\mathcal{U}}(x^0),Q):\ & f(\overline{x},\xi) \in f(x^0,\eta) - Q.
\end{aligned}$$

But this is a contradiction to $f_{\mathcal{U}}(\overline{x}) \cap \big(f_{\mathcal{U}}(x^0) - Q\big) = \emptyset$ for every $\overline{x} \in \mathcal{X} \setminus \{x^0\}$. □

Since the possibly less ordered robustness concept is extremely restrictive and its application may result in an empty solution set, we will not investigate this concept any further and neglect to provide any algorithms. Of course, deriving algorithms for computing possibly less ordered robust solutions may be a topic for future research.

5.3.8 Minmax Certainly Less Ordered Robustness

The *minmax certainly less order relation* given in Definition 11 inspires us to extend it to general nonempty sets $Q \subset Y$. To this end, let $C \subset Y$ be a proper closed convex and pointed cone. Suppose $C \subset \operatorname{cl} Q$ and $\operatorname{cl} Q \cap (-\operatorname{cl} Q) = \{0\}$. Recall that $\mathcal{F}^Q_{\min,\max}$ was given by (5.41). Under these assumptions, we define the *minmax certainly less order relation* for two sets $A, B \in \mathcal{F}^Q_{\min,\max}$ by

$$A \preceq^{mc}_Q B :\Longleftrightarrow (\operatorname{Min}(A,Q) \preceq^{cert}_Q \operatorname{Min}(B,Q) \text{ and } \operatorname{Max}(A,Q) \preceq^{cert}_Q \operatorname{Max}(B,Q)).$$

If we are considering $Q = \operatorname{int} C$, we suppose that $\operatorname{int} C \neq \emptyset$.

Throughout this section, we assume $f_{\mathcal{U}}(x) \in \mathcal{F}^Q_{\min,\max}$ for each $x \in \mathcal{X}$ and $Q = C$ ($Q = \operatorname{int} C$, $Q = C \setminus \{0\}$, respectively).

Definition 25. *A solution x^0 of $P(\mathcal{U})$ is called strictly (weakly, $\cdot$, respectively)* **minmax certainly less ordered robust** *if there is no $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ such that $f_{\mathcal{U}}(\overline{x}) \preceq_Q^{mc} f_{\mathcal{U}}(x^0)$, which is equivalent to: There does not exist $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ such that*

$$\begin{aligned} &\operatorname{Min}(f_{\mathcal{U}}(\overline{x}), Q) \preceq_Q^{cert} \operatorname{Min}(f_{\mathcal{U}}(x^0), Q) \\ &\quad \textit{and } \operatorname{Max}(f_{\mathcal{U}}(\overline{x}), Q) \preceq_C^{cert} \operatorname{Max}(f_{\mathcal{U}}(x^0), Q) \\ \iff &\forall\ \overline{y} \in \operatorname{Min}(f_{\mathcal{U}}(\overline{x}), Q),\ \forall\ y^0 \in \operatorname{Min}(f_{\mathcal{U}}(x^0), Q):\ \overline{y} \leq_Q y^0 \\ &\quad \textit{and } \forall\ \overline{y}' \in \operatorname{Max}(f_{\mathcal{U}}(\overline{x}), Q),\ \forall\ y^{0'} \in \operatorname{Max}(f_{\mathcal{U}}(x^0), Q):\ \overline{y}' \leq_Q y^{0'} \end{aligned} \tag{5.60}$$

for $Q = C$ ($Q = \operatorname{int} C$, $Q = C \setminus \{0\}$, respectively).

Example 14. *The left hand side of Figure 5.14 shows an example for a strictly minmax certainly less ordered robust solution x_1. On the right hand side, we have depicted an element x_3 that is not weakly minmax certainly less ordered robust.*

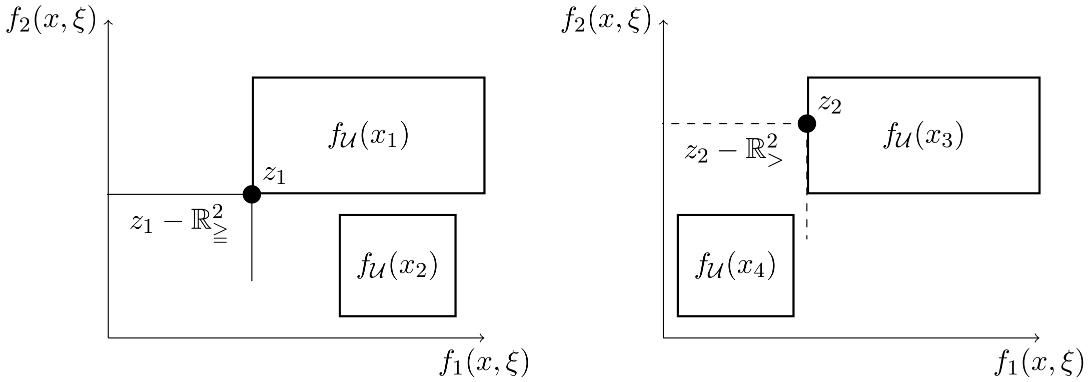

Figure 5.14: Left: x_1 is strictly minmax certainly less ordered robust, where $z_1 \in \operatorname{Min}(f_{\mathcal{U}}(x_1), \mathbb{R}^2_{\geqq})$. Right: x_3 is not weakly minmax certainly less ordered robust, where $z_2 \in \operatorname{Min}(f_{\mathcal{U}}(x_3), \mathbb{R}^2_{>})$.

This concept compares the values $f(x,\xi)$ that belong to the set $\operatorname{Min}(f_{\mathcal{U}}(x), Q)$ ($\operatorname{Max}(f_{\mathcal{U}}(x), Q)$, respectively) to those of another point $f(\overline{x},\xi) \in \operatorname{Min}(f_{\mathcal{U}}(\overline{x}), Q)$ ($\operatorname{Max}(f_{\mathcal{U}}(\overline{x}), Q)$, respectively). Thus, this approach would reflect a decision maker's preferences if he is risk-averse or optimistic about the future.

The following lemma states that for $|\mathcal{U}| = 1$, the minmax certainly less ordered robustness concept is equivalent to deterministic minimality. The proof is omitted here as it is quite similar to that of Lemma 12.

Lemma 19. *Given $P(\mathcal{U})$ with $|\mathcal{U}| = 1$. Then x^0 is strictly (weakly, $\cdot$, respectively) minmax certainly less ordered robust if and only if $f(x^0)$ is strictly (weakly, $\cdot$, respectively) minimal.*

Lemma 20. *Given $P(\mathcal{U})$ with $Y = \mathbb{R}$, $X = \mathbb{R}^n$ and $C = \mathbb{R}_{\geqq}$.*

(a) x^0 is weakly minmax certainly less ordered robust $\iff$ x^0 is minmax certainly less ordered robust.

(b) If x^0 is uniquely optimal for the robust counterpart (RC) (see (5.10)) or for the optimistic counterpart (OC) (see (5.11)), then x^0 is strictly minmax certainly less ordered robust.

(c) Suppose $\max_{\xi\in\mathcal{U}} f(x',\xi)$ and $\min_{\xi\in\mathcal{U}} f(x',\xi)$ exist for every $x' \in \mathcal{X}$. Then it holds: If x^0 is optimal for the robust counterpart (RC) (see (5.10)) or for the optimistic counterpart (OC) (see (5.11)), then x^0 is minmax certainly less ordered robust.

Proof. (a) Holds due to $\mathbb{R}_{>} = \mathbb{R}_{\geq}$.

(b) x^0 is uniquely optimal for the robust counterpart (RC) (see (5.10))

$$\iff \nexists\, \overline{x} \in \mathcal{X} \setminus \{x^0\} : \sup_{\xi\in\mathcal{U}} f(\overline{x},\xi) \leq \sup_{\xi\in\mathcal{U}} f(x^0,\xi)$$

or for the optimistic counterpart (OC) (see (5.11))

$$\iff \nexists\, \overline{x} \in \mathcal{X} \setminus \{x^0\} : \inf_{\xi\in\mathcal{U}} f(\overline{x},\xi) \leq \inf_{\xi\in\mathcal{U}} f(x^0,\xi).$$

Assume that x^0 is not strictly minmax certainly less ordered robust. Then there exists $\overline{x} \in \mathcal{X}\setminus\{x^0\}$ s.t. (5.60) is fulfilled for $Q = \mathbb{R}_{\geqq}$. But then we acquire $f(\overline{x},\overline{\xi}) \leq f(x^0,\xi^0)$ for every $f(\overline{x},\overline{\xi}) \in \mathrm{Min}(f_{\mathcal{U}}(\overline{x}),\mathbb{R}_{\geqq})$ and $f(x^0,\xi^0) \in \mathrm{Min}(f_{\mathcal{U}}(x^0),\mathbb{R}_{\geqq})$. $f(x^0,\xi^0) \in \mathrm{Min}(f_{\mathcal{U}}(x^0),\mathbb{R}_{\geqq})$ implies that $f(x^0,\xi^0) \leq f(x^0,\xi)$ for all $\xi \in \mathcal{U}$, thus $f(\overline{x},\overline{\xi}) \leq f(x^0,\xi)$ for every $\xi \in \mathcal{U}$, and hence $\inf_{\xi\in\mathcal{U}} f(\overline{x},\xi) \leq \inf_{\xi\in\mathcal{U}} f(x^0,\xi)$. An analogous analysis can be performed for $f(\overline{x},\overline{\xi}) \leq f(x^0,\xi^0)$ for every $f(\overline{x},\overline{\xi}) \in \mathrm{Max}(f_{\mathcal{U}}(\overline{x}),\mathbb{R}_{\geqq})$ and $f(x^0,\xi^0) \in \mathrm{Max}(f_{\mathcal{U}}(x^0),\mathbb{R}_{\geqq})$. But then we arrive at a contradiction.

(c) x^0 is optimal for the robust counterpart (RC) (see (5.10))

$$\iff \nexists\, \overline{x} \in \mathcal{X} \setminus \{x^0\} : \max_{\xi\in\mathcal{U}} f(\overline{x},\xi) < \max_{\xi\in\mathcal{U}} f(x^0,\xi) \tag{5.61}$$

or for the optimistic counterpart (OC) (see (5.11))

$$\iff \nexists\, \overline{x} \in \mathcal{X} \setminus \{x^0\} : \min_{\xi\in\mathcal{U}} f(\overline{x},\xi) < \min_{\xi\in\mathcal{U}} f(x^0,\xi). \tag{5.62}$$

Now suppose x^0 is not minmax certainly less ordered robust. Then there exists $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ s.t. (5.60) is satisfied for $Q = \mathbb{R}_{>}$. This implies $f(\overline{x},\overline{\xi}) < f(x^0,\xi^0)$ for every $f(\overline{x},\overline{\xi}) \in \mathrm{Min}(f_{\mathcal{U}}(\overline{x}),\mathbb{R}_{\geq})$ and $f(x^0,\xi^0) \in \mathrm{Min}(f_{\mathcal{U}}(x^0),\mathbb{R}_{\geq})$. $f(x^0,\xi^0) \in \mathrm{Min}(f_{\mathcal{U}}(x^0),\mathbb{R}_{\geq})$ implies that $f(x^0,\xi^0) \leq f(x^0,\xi)$ for all $\xi \in \mathcal{U}$, consequently, $f(\overline{x},\overline{\xi}) < f(x^0,\xi)$ for every $\xi \in \mathcal{U}$, and thus $\min_{\xi\in\mathcal{U}} f(\overline{x},\xi) < \min_{\xi\in\mathcal{U}} f(x^0,\xi)$.

The analogous follows for $f(\overline{x}, \overline{\xi}) \leq f(x^0, \xi^0)$ for every $f(\overline{x}, \overline{\xi}) \in \mathrm{Max}(f_{\mathcal{U}}(\overline{x}), \mathbb{R}_{\geq})$ and $f(x^0, \xi^0) \in \mathrm{Max}(f_{\mathcal{U}}(x^0), \mathbb{R}_{\geq})$, contradicting the assumptions (5.61) ((5.62), respectively).

$\square$

Connections between the minmax less ordered robustness concept and minmax certainly less ordered robustness are revealed below.

Theorem 39. *If x^0 is strictly (weakly, $\cdot$, respectively) minmax less ordered robust, then x^0 is strictly (weakly, $\cdot$, respectively) minmax certainly less ordered robust.*

Proof. Let $Q = C$ ($Q = \operatorname{int} C$, $Q = C \setminus \{0\}$, respectively). Assume that $x^0 \in \mathcal{X}$ is not strictly (weakly, $\cdot$, respectively) minmax certainly less ordered robust. Then there exists an $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ s.t.

$$\begin{aligned} &\forall\, \overline{y} \in \mathrm{Min}(f_{\mathcal{U}}(\overline{x}), Q),\ \forall\, y^0 \in \mathrm{Min}(f_{\mathcal{U}}(x^0), Q) :\ \overline{y} \leq_Q y^0 \\ \text{and } &\forall\, \overline{y}' \in \mathrm{Max}(f_{\mathcal{U}}(\overline{x}), Q),\ \forall\, y^{0'} \in \mathrm{Max}(f_{\mathcal{U}}(x^0), Q) :\ \overline{y}' \leq_Q y^{0'}. \end{aligned}$$

Consequently, we acquire (5.42), a contradiction. $\square$

Example 15. *The inverse direction in Theorem 39 is generally not satisfied. Consider, as an example, the left hand side of Figure 5.15: Here, x_1 is minmax certainly less ordered robust, but it is not minmax less ordered robust.*

Theorem 40. *If x^0 is strictly (weakly, $\cdot$, respectively) minmax certainly less ordered robust, then x^0 is strictly (weakly, $\cdot$, respectively) certainly less ordered robust.*

Proof. Let $Q = C$ ($Q = \operatorname{int} C$, $Q = C \setminus \{0\}$, respectively). Assume that $x^0 \in \mathcal{X}$ is not strictly (weakly, $\cdot$, respectively) certainly less ordered robust. Then there exists an $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ such that for every $\overline{y} \in f_{\mathcal{U}}(\overline{x})$ and for all $y^0 \in f_{\mathcal{U}}(x^0) :\ \overline{y} \leq_Q y^0$. Because $\mathrm{Min}(f_{\mathcal{U}}(\overline{x}), Q) \subseteq f_{\mathcal{U}}(\overline{x})$ and $\mathrm{Min}(f_{\mathcal{U}}(x^0), Q) \subseteq f_{\mathcal{U}}(x^0)$, it holds for all $\overline{y} \in \mathrm{Min}(f_{\mathcal{U}}(\overline{x}), Q)$ and for every $y^0 \in \mathrm{Min}(f_{\mathcal{U}}(x^0), Q) :\ \overline{y} \leq_Q y^0$, in contradiction to the assumption. $\square$

Example 16. *The inverse direction in Theorem 40 is generally not fulfilled, as the example at the right hand side in Figure 5.15 verifies.*

Deriving algorithms for obtaining minmax certainly less ordered robust solutions shall remain a future challenge. In order to deal with that concept in the special case $Y = \mathbb{R}^k,\ X = \mathbb{R}^n,\ C = \mathbb{R}^k_{\geqq}$, we present an alternative definition of minmax certainly less ordered robust elements, along with a solution procedure.

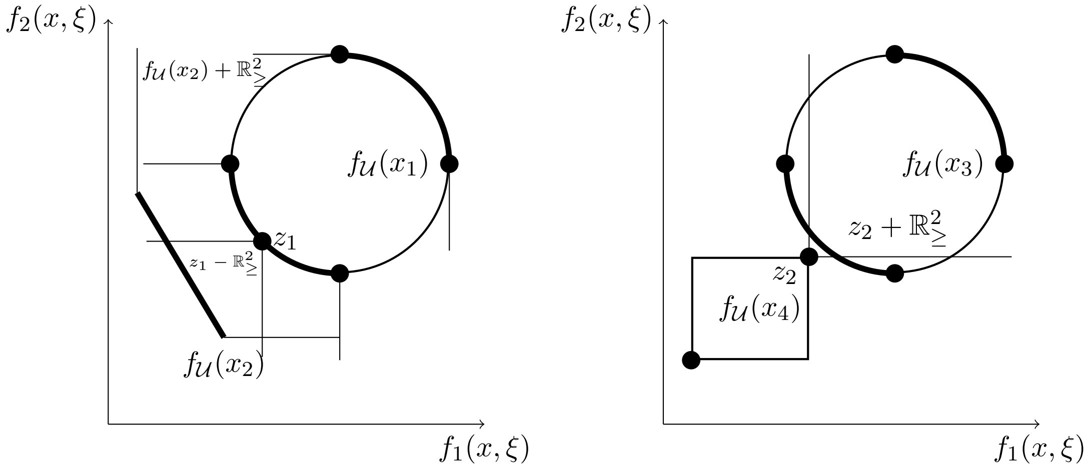

Figure 5.15: Left: x_1 is minmax certainly less ordered robust, but it is not minmax less ordered robust, where $z_1 \in \operatorname{Min}(f_{\mathcal{U}}(x_1), \mathbb{R}^2_{\geq})$. Right: Here, x_3 is certainly less ordered robust, while it is not minmax certainly less ordered robust, where $z_2 \in \operatorname{Max}(f_{\mathcal{U}}(x_4), \mathbb{R}^2_{\geq})$.

Definition 26. *Let $Y = \mathbb{R}^k$, $X = \mathbb{R}^n$. Consider for $Q = \mathbb{R}^k_{[\geqq/>/\geq]}$*

$$\operatorname{CInf}_Q^{\mathrm{Min}} f_{\mathcal{U}}(x) := \begin{pmatrix} \inf\{f_1(x,\xi)|\ \xi \in \mathcal{U}:\ (f_1(x,\xi),\dots,f_k(x,\xi))^T \in \operatorname{Min}(f_{\mathcal{U}}(x),Q)\} \\ \dots \\ \inf\{f_k(x,\xi)|\ \xi \in \mathcal{U}:\ (f_1(x,\xi),\dots,f_k(x,\xi))^T \in \operatorname{Min}(f_{\mathcal{U}}(x),Q)\} \end{pmatrix},$$
$$\operatorname{CInf}_Q^{\mathrm{Max}} f_{\mathcal{U}}(x) := \begin{pmatrix} \inf\{f_1(x,\xi)|\ \xi \in \mathcal{U}:\ (f_1(x,\xi),\dots,f_k(x,\xi))^T \in \operatorname{Max}(f_{\mathcal{U}}(x),Q)\} \\ \dots \\ \inf\{f_k(x,\xi)|\ \xi \in \mathcal{U}:\ (f_1(x,\xi),\dots,f_k(x,\xi))^T \in \operatorname{Max}(f_{\mathcal{U}}(x),Q)\} \end{pmatrix},$$
$$\operatorname{CSup}_Q^{\mathrm{Min}} f_{\mathcal{U}}(x) := \begin{pmatrix} \sup\{f_1(x,\xi)|\ \xi \in \mathcal{U}:\ (f_1(x,\xi),\dots,f_k(x,\xi))^T \in \operatorname{Min}(f_{\mathcal{U}}(x),Q)\} \\ \dots \\ \sup\{f_k(x,\xi)|\ \xi \in \mathcal{U}:\ (f_1(x,\xi),\dots,f_k(x,\xi))^T \in \operatorname{Min}(f_{\mathcal{U}}(x),Q)\} \end{pmatrix},$$
$$\operatorname{CSup}_Q^{\mathrm{Max}} f_{\mathcal{U}}(x) := \begin{pmatrix} \sup\{f_1(x,\xi)|\ \xi \in \mathcal{U}:\ (f_1(x,\xi),\dots,f_k(x,\xi))^T \in \operatorname{Max}(f_{\mathcal{U}}(x),Q)\} \\ \dots \\ \sup\{f_k(x,\xi)|\ \xi \in \mathcal{U}:\ (f_1(x,\xi),\dots,f_k(x,\xi))^T \in \operatorname{Max}(f_{\mathcal{U}}(x),Q)\} \end{pmatrix}.$$

A feasible solution $x^0 \in \mathcal{X}$ is called [strictly, weakly, $\cdot$] **minmax certainly less alternative ordered robust** *if there does not exist $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ such that*

$$\operatorname{CSup}_Q^{\mathrm{Min}} f_{\mathcal{U}}(\overline{x}) \in \operatorname{CInf}_Q^{\mathrm{Min}} f_{\mathcal{U}}(x^0) - \mathbb{R}^k_{[\geqq/>/\geq]}$$
$$\text{and}\quad \operatorname{CSup}_Q^{\mathrm{Max}} f_{\mathcal{U}}(\overline{x}) \in \operatorname{CInf}_Q^{\mathrm{Max}} f_{\mathcal{U}}(x^0) - \mathbb{R}^k_{[\geqq/>/\geq]}$$

for $Q = \mathbb{R}^k_{[\geqq/>/\geq]}$.

The relation between the *minmax certainly less order relation* $\preceq_Q^{mc}$ for $Q = C = \mathbb{R}^k_{\geqq}$ and minmax certainly less *alternative* ordered robust solutions is given below.

Lemma 21. *Given an uncertain multi-objective optimization problem $P(\mathcal{U})$ with $Y = \mathbb{R}^k$, $X = \mathbb{R}^n$ and $Q = \mathbb{R}^k_{\geqq}$. Then for all $x, \overline{x} \in \mathcal{X}$:*

$$f_{\mathcal{U}}(\overline{x}) \preceq^{mc}_{Q} f_{\mathcal{U}}(x) \text{ with respect to } Q = \mathbb{R}^k_{\geqq} \Longleftrightarrow \mathrm{CSup}^{\mathrm{Min}}_{Q} f_{\mathcal{U}}(\overline{x}) \in \mathrm{CInf}^{\mathrm{Min}}_{Q} f_{\mathcal{U}}(x) - \mathbb{R}^k_{\geqq}$$
$$\text{and } \mathrm{CSup}^{\mathrm{Max}}_{Q} f_{\mathcal{U}}(\overline{x}) \in \mathrm{CInf}^{\mathrm{Max}}_{Q} f_{\mathcal{U}}(x) - \mathbb{R}^k_{\geqq}.$$

Proof.

$$\begin{aligned}
& f_{\mathcal{U}}(\overline{x}) \preceq^{mc}_{Q} f_{\mathcal{U}}(x) \text{ with respect to } Q = \mathbb{R}^k_{\geqq} \\
& \Longleftrightarrow \mathrm{Min}(f_{\mathcal{U}}(x), \mathbb{R}^k_{\geqq}) \preceq^{cert}_{Q} \mathrm{Min}(f_{\mathcal{U}}(\overline{x}), \mathbb{R}^k_{\geqq}) \text{ and } \mathrm{Max}(f_{\mathcal{U}}(x), \mathbb{R}^k_{\geqq}) \preceq^{cert}_{Q} \mathrm{Max}(f_{\mathcal{U}}(\overline{x}), \mathbb{R}^k_{\geqq}) \\
& \Longleftrightarrow \forall\ f(x,\xi) \in \mathrm{Min}(f_{\mathcal{U}}(x), \mathbb{R}^k_{\geqq})\ \forall\ f(\overline{x},\overline{\xi}) \in \mathrm{Min}(f_{\mathcal{U}}(\overline{x}), \mathbb{R}^k_{\geqq}) : f(x,\xi) \leq_Q f(\overline{x},\overline{\xi}) \\
& \text{and } \forall\ f(x,\eta) \in \mathrm{Max}(f_{\mathcal{U}}(x), \mathbb{R}^k_{\geqq})\ \forall\ f(\overline{x},\overline{\eta}) \in \mathrm{Max}(f_{\mathcal{U}}(\overline{x}), \mathbb{R}^k_{\geqq}) : f(x,\eta) \leq_Q f(\overline{x},\overline{\eta}) \\
& \Longleftrightarrow \mathrm{CSup}^{\mathrm{Min}}_{Q} f_{\mathcal{U}}(\overline{x}) \in \mathrm{CInf}^{\mathrm{Min}}_{Q} f_{\mathcal{U}}(x) - \mathbb{R}^k_{\geqq} \text{ and } \mathrm{CSup}^{\mathrm{Max}}_{Q} f_{\mathcal{U}}(\overline{x}) \in \mathrm{CInf}^{\mathrm{Max}}_{Q} f_{\mathcal{U}}(x) - \mathbb{R}^k_{\geqq}.
\end{aligned}$$

□

Computing Minmax Certainly Less Alternative Ordered Robust Solutions

To this end, we are able to derive solution procedures for computing minmax certainly less alternative ordered robust elements.

Consider the vector-valued optimization problem

$$(VOP^{mca}(y^*, y^{**})) \qquad \mathrm{Min}(h[\mathcal{X}], \mathbb{R}^4_{\geqq}), \tag{5.63}$$

where

$$h(x) := \begin{pmatrix} \sum_{i=1}^k y_i^* \inf\{f_i(x,\xi) |\ \xi \in \mathcal{U},\ f(x,\xi) \in \mathrm{Min}(f_{\mathcal{U}}(x), Q)\} \\ \sum_{i=1}^k y_i^{**} \inf\{f_i(x,\xi) |\ \xi \in \mathcal{U},\ f(x,\xi) \in \mathrm{Max}(f_{\mathcal{U}}(x), Q)\} \\ \sum_{i=1}^k y_i^* \sup\{f_i(x,\xi) |\ \xi \in \mathcal{U},\ f(x,\xi) \in \mathrm{Min}(f_{\mathcal{U}}(x), Q)\} \\ \sum_{i=1}^k y_i^{**} \sup\{f_i(x,\xi) |\ \xi \in \mathcal{U},\ f(x,\xi) \in \mathrm{Max}(f_{\mathcal{U}}(x), Q)\} \end{pmatrix}$$

for $y^*, y^{**} \in \mathbb{R}^k_{\geq}$ and $Q = \mathbb{R}^k_{[\geqq/>/\geq]}$. Note that the selection $y^* = y^{**}$ is possible here.

Now we have the following connection between $(VOP^{mca}(y^*, y^{**}))$ and minmax certainly less alternative ordered robust elements of an uncertain multi-objective optimization problem.

Theorem 41. *Given $P(\mathcal{U})$ with $Y = \mathbb{R}^k$, $X = \mathbb{R}^n$ and $C = \mathbb{R}^k_{\geqq}$. The following statements holds:*

(a) If x^0 is a strictly Pareto optimal solution to $(VOP^{mca}(y^*, y^{**}))$ for some y^*, $y^{**} \in \mathbb{R}^k_{\geq}$, then x^0 is strictly minmax certainly less alternative ordered robust.

(b) If x^0 is a weakly Pareto optimal solution to $(VOP^{mca}(y^*, y^{**}))$ for some y^*, $y^{**} \in \mathbb{R}^k_{\geq}$, then x^0 is weakly minmax certainly less alternative ordered robust.

(c) If x^0 is a weakly Pareto optimal solution to $(VOP^{mca}(y^*, y^{**}))$ for some $y^*,\ y^{**} \in \mathbb{R}^k_>$, then x^0 is minmax certainly less alternative ordered robust.

Proof. Set $Q = \mathbb{R}^k_{[\geqq/>/\geq]}$. Let x^0 be strictly Pareto optimal (weakly Pareto optimal, weakly Pareto optimal, respectively) for problem $(VOP^{mca}(y^*, y^{**}))$ with some $y^*,\ y^{**} \in \mathbb{R}^k_\geq$ ($y^*,\ y^{**} \in \mathbb{R}^k_\geq$, $y^*,\ y^{**} \in \mathbb{R}^k_>$, respectively), i.e., there is no $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ such that

$$\begin{pmatrix} \sum_{i=1}^k y_i^* \inf\{f_i(\overline{x},\xi)|\ \xi \in \mathcal{U},\ f(\overline{x},\xi) \in \mathrm{Min}(f_\mathcal{U}(\overline{x}), Q)\} \\ \sum_{i=1}^k y_i^{**} \inf\{f_i(\overline{x},\xi)|\ \xi \in \mathcal{U},\ f(\overline{x},\xi) \in \mathrm{Max}(f_\mathcal{U}(\overline{x}), Q)\} \\ \sum_{i=1}^k y_i^* \sup\{f_i(\overline{x},\xi|\ \xi \in \mathcal{U},\ f(\overline{x},\xi) \in \mathrm{Min}(f_\mathcal{U}(\overline{x}), Q)\}) \\ \sum_{i=1}^k y_i^{**} \sup\{f_i(\overline{x},\xi)|\ \xi \in \mathcal{U},\ f(\overline{x},\xi) \in \mathrm{Max}(f_\mathcal{U}(\overline{x}), Q)\} \end{pmatrix}$$
$$\in \begin{pmatrix} \sum_{i=1}^k y_i^* \inf\{f_i(x^0,\xi)|\ \xi \in \mathcal{U},\ f(x^0,\xi) \in \mathrm{Min}(f_\mathcal{U}(x^0), Q)\} \\ \sum_{i=1}^k y_i^{**} \inf\{f_i(x^0,\xi)|\ \xi \in \mathcal{U},\ f(x^0,\xi) \in \mathrm{Max}(f_\mathcal{U}(x^0), Q)\} \\ \sum_{i=1}^k y_i^* \sup\{f_i(x^0,\xi)|\ \xi \in \mathcal{U},\ f(x^0,\xi) \in \mathrm{Min}(f_\mathcal{U}(x^0), Q)\} \\ \sum_{i=1}^k y_i^{**} \sup\{f_i(x^0,\xi)|\ \xi \in \mathcal{U},\ f(x^0,\xi) \in \mathrm{Max}(f_\mathcal{U}(x^0), Q)\} \end{pmatrix} - \mathbb{R}^k_{[\geqq/>/>]}.$$

Now suppose that x^0 is not strictly (weakly, $\cdot$, respectively) minmax certainly less alternative ordered robust. Then there exists $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ such that

$$\begin{aligned} &\mathrm{CSup}_Q^{\mathrm{Min}} f_\mathcal{U}(\overline{x}) \in \mathrm{CInf}_Q^{\mathrm{Min}} f_\mathcal{U}(x) - Q \\ \text{and}\quad &\mathrm{CSup}_Q^{\mathrm{Max}} f_\mathcal{U}(\overline{x}) \in \mathrm{CInf}_Q^{\mathrm{Max}} f_\mathcal{U}(x) - Q, \end{aligned}$$

and this implies for $y^*, y^{**} \in \mathbb{R}^k_\geq$ ($y^*, y^{**} \in \mathbb{R}^k_\geq$, $y^*, y^{**} \in \mathbb{R}^k_>$, respectively)

$$\begin{aligned} &\sum_{i=1}^k y_i^* \sup\{f_i(\overline{x},\xi)|\ \xi \in \mathcal{U},\ f(\overline{x},\xi) \in \mathrm{Min}(f_\mathcal{U}(\overline{x}), Q)\} \\ &\qquad [\leq / < / <] \sum_{i=1}^k y_i^* \inf\{f_i(x^0,\xi)|\ \xi \in \mathcal{U},\ f(x^0,\xi) \in \mathrm{Min}(f_\mathcal{U}(x^0), Q)\}, \\ &\sum_{i=1}^k y_i^{**} \sup\{f_i(\overline{x},\xi)|\ \xi \in \mathcal{U},\ f(\overline{x},\xi) \in \mathrm{Max}(f_\mathcal{U}(\overline{x}), Q)\} \\ &\qquad [\leq / < / <] \sum_{i=1}^k y_i^{**} \inf\{f_i(x^0,\xi)|\ \xi \in \mathcal{U},\ f(x^0,\xi) \in \mathrm{Max}(f_\mathcal{U}(x^0), Q)\}. \end{aligned}$$

Furthermore,

$$\begin{aligned}
&\sum_{i=1}^{k} y_i^* \inf\{f_i(\overline{x},\xi) |\ \xi \in \mathcal{U},\ f(\overline{x},\xi) \in \mathrm{Min}(f_{\mathcal{U}}(\overline{x}), Q)\} \\
&\le \sum_{i=1}^{k} y_i^* \sup\{f_i(\overline{x},\xi) |\ \xi \in \mathcal{U},\ f(\overline{x},\xi) \in \mathrm{Min}(f_{\mathcal{U}}(\overline{x}), Q)\} \\
&[\le / < / <] \sum_{i=1}^{k} y_i^* \inf\{f_i(x^0,\xi) |\ \xi \in \mathcal{U},\ f(x^0,\xi) \in \mathrm{Min}(f_{\mathcal{U}}(x^0), Q)\} \\
&\le \sum_{i=1}^{k} y_i^* \sup\{f_i(x^0,\xi) |\ \xi \in \mathcal{U},\ f(x^0,\xi) \in \mathrm{Min}(f_{\mathcal{U}}(x^0), Q)\}.
\end{aligned}$$

In addition,

$$\begin{aligned}
&\sum_{i=1}^{k} y_i^{**} \inf\{f_i(\overline{x},\xi) |\ \xi \in \mathcal{U},\ f(\overline{x},\xi) \in \mathrm{Max}(f_{\mathcal{U}}(\overline{x}), Q)\} \\
&\le \sum_{i=1}^{k} y_i^{**} \sup\{f_i(\overline{x},\xi) |\ \xi \in \mathcal{U},\ f(\overline{x},\xi) \in \mathrm{Max}(f_{\mathcal{U}}(\overline{x}), Q)\} \\
&[\le / < / <] \sum_{i=1}^{k} y_i^{**} \inf\{f_i(x^0,\xi) |\ \xi \in \mathcal{U},\ f(x^0,\xi) \in \mathrm{Max}(f_{\mathcal{U}}(x^0), Q)\} \\
&\le \sum_{i=1}^{k} y_i^{**} \sup\{f_i(x^0,\xi) |\ \xi \in \mathcal{U},\ f(x^0,\xi) \in \mathrm{Max}(f_{\mathcal{U}}(x^0), Q)\}.
\end{aligned}$$

But this is a contradiction. □

Applying Theorem 41 leads to the following algorithm for deriving minmax certainly less alternative ordered robust solutions. The sets of strictly (weakly, · , respectively) minmax certainly less alternative ordered robust solutions are denoted by $\mathrm{Opt}_{\mathrm{smcr}}$ ($\mathrm{Opt}_{\mathrm{wmcr}}$, $\mathrm{Opt}_{\mathrm{mcr}}$, respectively).

Algorithm 19 for computing minmax certainly less alternative ordered robust solutions using a family of problems $(VOP^{mca}(y^*, y^{**}))$ (see (5.63)):

Input, Steps 1-7: Analogous to Algorithm 12, only replacing $C^* \setminus \{0\}$ by $\mathbb{R}^k_{\geq}$, $C^{\#}$ by $\mathbb{R}^k_{>}$, $(VOP^{sl}(y^*, y^{**}))$ (see (5.33)) by $(VOP^{mca}(y^*, y^{**}))$ (see (5.63)), $\mathrm{Opt}_{\mathrm{sslor}}$, $\mathrm{Opt}_{\mathrm{wslor}}$, $\mathrm{Opt}_{\mathrm{slor}}$ by $\mathrm{Opt}_{\mathrm{smcr}}$, $\mathrm{Opt}_{\mathrm{wmcr}}$, $\mathrm{Opt}_{\mathrm{mcr}}$, and replacing "set less ordered robust" by "minmax certainly less alternative ordered robust". Note that the existence of $\min_{\xi\in\mathcal{U}} y^* \circ f(x', \xi)$ and $\max_{\xi\in\mathcal{U}} y^{**} \circ f(x', \xi)$ is not required for the present concept.

The following algorithm computes minmax certainly less alternative ordered robust solutions by varying the weights in $(VOP^{mca}(y^*, y^{**}))$.

Algorithm 20 for computing minmax certainly less alternative ordered robust solutions using a family of problems $(VOP^{mca}(y^*, y^{**}))$ **(see** (5.63)**) by altering the weights:**

Input, Steps 1-8: Analogous to Algorithm 13, only replacing $C^* \setminus \{0\}$ by $\mathbb{R}^k_{\geq}$, $C^{\#}$ by $\mathbb{R}^k_{>}$, $(VOP^{sl}(\hat{y}^*, \hat{y}^{**}))$ by $(VOP^{mca}(\hat{y}^*, \hat{y}^{**}))$, $\mathrm{Opt}_{\mathrm{sslor}}$, $\mathrm{Opt}_{\mathrm{wslor}}$, $\mathrm{Opt}_{\mathrm{slor}}$ by $\mathrm{Opt}_{\mathrm{smcr}}$, $\mathrm{Opt}_{\mathrm{wmcr}}$, $\mathrm{Opt}_{\mathrm{mcr}}$, and replacing "set less ordered robust" by "minmax certainly less alternative ordered robust". Notice again that $\min_{\xi\in\mathcal{U}} \hat{y}^* \circ f(x', \xi)$ and $\max_{\xi\in\mathcal{U}} \hat{y}^{**} \circ f(x', \xi)$ do not need to exist for this concept.

5.3.9 Further Relationships Between the Concepts

The following corollary shows the essential result that for deterministic multi-objective optimization the introduced concepts of robustness are equivalent to deterministic C-minimality.

Corollary 9. *Given $P(\mathcal{U})$ with $|\mathcal{U}| = 1$. Suppose that for all $x \in \mathcal{X}$, it holds $f_{\mathcal{U}}(x) \in \mathcal{F}^Q_{\mathrm{min,max}}$ (compare* (5.41)*) for $Q = C$, ($Q = \operatorname{int} C$, $Q = C \setminus \{0\}$, respectively). Then $f(x)$ is strictly (weakly, $\cdot$, respectively) minimal if and only if x is strictly (weakly, $\cdot$, respectively)*

- *upper set less ordered robust;*
- *lower set less ordered robust;*
- *set less ordered robust;*
- *alternative set less ordered robust;*
- *minmax less ordered robust;*
- *certainly less ordered robust;*
- *possibly set less ordered robust;*
- *minmax certainly less ordered robust.*

In addition, we have the following corollary.

Corollary 10. *Given $P(\mathcal{U})$ with $Y = \mathbb{R}$, $X = \mathbb{R}^n$ and $Q = \mathbb{R}_{\geq}$. Suppose that for all $x \in \mathcal{X}$ $f_{\mathcal{U}}(x) \in \mathcal{F}^Q_{\mathrm{min,max}}$ holds (compare* (5.41)*). Then it holds due to $\mathbb{R}_{>} = \mathbb{R}_{\geq}$:*

- *x is weakly upper set less ordered robust $\Longleftrightarrow$ x is upper set less ordered robust;*

- *x is weakly lower set less ordered robust $\Longleftrightarrow$ x is lower set less ordered robust;*
- *x is weakly set less ordered robust $\Longleftrightarrow$ x is set less ordered robust;*
- *x is weakly alternative set less ordered robust $\Longleftrightarrow$ x is alternative set less ordered robust;*
- *x is weakly minmax less ordered robust $\Longleftrightarrow$ x is minmax less ordered robust;*
- *x is weakly certainly less ordered robust $\Longleftrightarrow$ x is certainly less ordered robust;*
- *x is weakly possibly set less ordered robust $\Longleftrightarrow$ x is possibly set less ordered robust;*
- *x is weakly minmax certainly less ordered robust $\Longleftrightarrow$ x is minmax certainly less ordered robust.*

From Theorems 37, 32, 30, 33, 40 and 39 we deduce a summary for interrelations of robust solutions in Figure 5.16.
Furthermore, Table 5.1 summarizes the definitions of all introduced concepts for uncertain multi-objective problems.
The diagram in Figure 5.17 shows the relationships between some of the introduced robustness concepts for uncertain multi-objective optimization.
Links between scalar problems and corresponding robustness concepts for uncertain vector-valued optimization problems in the special case of scalar optimization ($Y = \mathbb{R}$, $X = \mathbb{R}^n$, $C = \mathbb{R}_{\geqq}$) are summarized below in Table 5.2.
In order to give some insight to the problem structure and the concepts of robustness we introduced, we present an example that illustrates the different robustness concepts. This example can be found in Ide, Köbis [48].

Example 17. *([48]) Imagine the decision process of choosing a hotel with respect to two objective functions a decision maker might value the most: silence and weather conditions. Since it is not entirely known which weather conditions will occur, it is also not known how many tourists will stay at the specific hotel and therefore how noisy it will be there during the stay.*

The following result for possible scenarios is considered: For the sake of simplicity, we restrict ourselves to four weather scenarios, each of which yielding a different score on weather conditions and noise for every hotel. The score is estimated in grades from 1 to 20, 1 being perfect and 20 being very bad.

The decision maker now has to choose a suitable hotel due to his preferences. Since the problem is multi-objective, the decision maker has to choose the trade-off he is willing to pay between the two objective functions. Furthermore, since the problem is also uncertain, he has to define what would be a suitable solution considering not just one but all four scenarios. Thus, he has to decide in a definition of what is called robust in this context.

We now discuss this example for the case $C = \mathbb{R}^2_{\geqq}$ and use it to validate the different concepts of robustness. For this we plot the objective values of the above Table 5.3.

First we note that in terms of upper set less ordered robustness, one is searching for solutions where the set of worst cases is non-dominated by any other set of worst cases.

$$\begin{tikzcd}
& \text{possibly less ordered robust} \arrow[d, Rightarrow, "\text{Thm. 37}"'] & \\
& \text{alternative set less ordered robust} \arrow[dl, Rightarrow, "\text{Thm. 32}"] \arrow[dr, Rightarrow, "\text{Thm. 32}"'] & \\
\text{lower set less ordered robust} \arrow[dr, Rightarrow, "\text{Thm. 30}"'] & & \text{upper set less ordered robust} \arrow[dl, Rightarrow, "\text{Thm. 30}"] \\
& \text{set less ordered robust} \arrow[d, Rightarrow, "\text{Thm. 35}"'] & \\
& \text{certainly less ordered robust} &
\end{tikzcd}$$

$$\begin{tikzcd}
\text{possibly less ordered robust} \arrow[d, Rightarrow, "\text{Thm. 38}"'] \\
\text{minmax less ordered robust} \arrow[d, Rightarrow, "\text{Thm. 39}"'] \\
\text{minmax certainly less ordered robust} \arrow[d, Rightarrow, "\text{Thm. 40}"'] \\
\text{certainly less ordered robust}
\end{tikzcd}$$

Figure 5.16: Interrelations between robust solutions.

Concept	**Section**	**Definition**, x^0 robust if $\nexists\, \overline{x} \in \mathcal{X} \setminus \{x^0\}$ s.t. ...
Upper set less ordered robustness	5.3.1	$f_{\mathcal{U}}(\overline{x}) \subseteq f_{\mathcal{U}}(x^0) - Q$
Lower set less ordered robustness	5.3.2	$f_{\mathcal{U}}(\overline{x}) + Q \supseteq f_{\mathcal{U}}(x^0)$
Set less ordered robustness	5.3.3	$f_{\mathcal{U}}(\overline{x}) \subseteq f_{\mathcal{U}}(x^0) - Q$ and $f_{\mathcal{U}}(\overline{x}) + Q \supseteq f_{\mathcal{U}}(x^0)$
Alternative set less ordered robustness	5.3.4	$f_{\mathcal{U}}(\overline{x}) \subseteq f_{\mathcal{U}}(x^0) - Q$ or $f_{\mathcal{U}}(\overline{x}) + Q \supseteq f_{\mathcal{U}}(x^0)$
Minmax less ordered robustness	5.3.5	$\mathrm{Min}(f_{\mathcal{U}}(\overline{x}), Q) + Q \supseteq \mathrm{Min}(f_{\mathcal{U}}(x^0), Q)$ and $\mathrm{Min}(f_{\mathcal{U}}(\overline{x}), Q) \subseteq \mathrm{Min}(f_{\mathcal{U}}(x^0), Q) - Q$ and $\mathrm{Max}(f_{\mathcal{U}}(\overline{x}), Q) + Q \supseteq \mathrm{Max}(f_{\mathcal{U}}(x^0), Q)$ and $\mathrm{Max}(f_{\mathcal{U}}(\overline{x}), Q) \subseteq \mathrm{Max}(f_{\mathcal{U}}(x^0), Q) - Q$
Certainly less ordered robustness	5.3.6	$\forall\, f(\overline{x}, \xi) \in f_{\mathcal{U}}(\overline{x}),\ \forall\, f(x^0, \eta) \in f_{\mathcal{U}}(x^0):$ $f(\overline{x}, \xi) \leq_Q f(x^0, \eta)$
Possibly less ordered robustness	5.3.7	$f_{\mathcal{U}}(\overline{x}) \cap \big(f_{\mathcal{U}}(x^0) - Q\big) \neq \emptyset$
Minmax certainly less ordered robustness	5.3.8	$\forall\, \overline{y} \in \mathrm{Min}(f_{\mathcal{U}}(\overline{x}), Q),\ \forall\, y^0 \in \mathrm{Min}(f_{\mathcal{U}}(x^0), Q):$ $\overline{y} \leq_Q y^0$ and $\forall\, \overline{y}' \in \mathrm{Max}(f_{\mathcal{U}}(\overline{x}), Q),\ \forall\, y^{0'} \in \mathrm{Max}(f_{\mathcal{U}}(x^0), Q):$ $\overline{y}' \leq_Q y^{0'}$

Table 5.1: Summary of all introduced concepts, using $Q = C$ ($Q = \operatorname{int} C$, $Q = C \setminus \{0\}$, respectively) for strict (weak, $\cdot$, respectively) robustness.

Concept	**For** $Y = \mathbb{R}$, $X = \mathbb{R}^n$, $C = \mathbb{R}_{\geqq}$ **related to:**	**See Lemma:**
Upper set less ordered robustness	(RC) $\min_{x\in\mathcal{X}} \sup_{\xi\in\mathcal{U}} f(x,\xi)$	3
Lower set less ordered robustness	(OC) $\min_{x\in\mathcal{X}} \inf_{\xi\in\mathcal{U}} f(x,\xi)$	5
Set less ordered robustness	(RC) or (OC)	9
Alternative set less ordered robustness	(RC) and (OC)	11
Minmax less ordered robustness	(RC) or (OC)	13
Certainly less ordered robustness	(RC) or (OC)	15
Possibly less ordered robustness	x^0 is possibly less ordered robust if there is no $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ s.t. $f(\overline{x},\xi) \leq f(x^0,\eta)$ for some $\xi,\eta \in \mathcal{U}$	18
Minmax certainly less ordered robustness	(RC) or (OC)	20

Table 5.2: Relations of introduced robustness notions in the scalar case are described. (RC) ((OC), respectively) is given by (5.10) ((5.11), respectively).

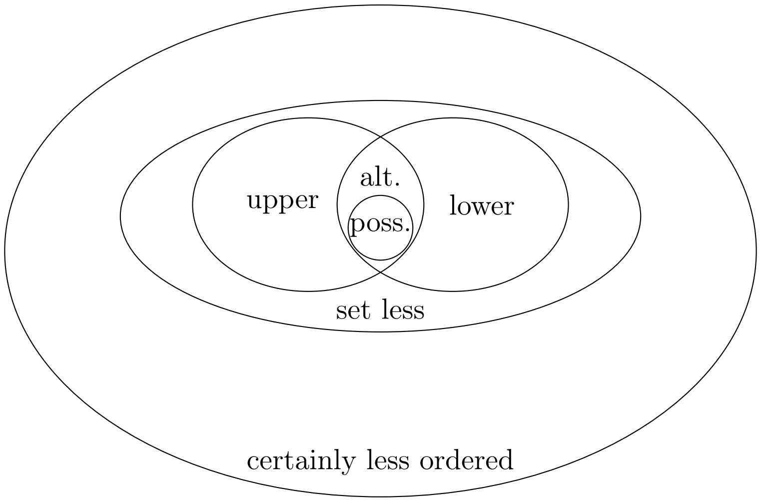

Figure 5.17: Scheme of robust solutions to an uncertain multicriteria optimization problem.

The left hand side in Figure 5.18 illustrates that Hotels No. 1 and No. 4 are upper set less ordered robust as their worst cases are non-dominated by another set of worst cases. At the same time Hotels No. 2, 3, 5 and 6 are not upper set less ordered robust as their sets of worst cases are dominated by the set of worst cases of Hotel No. 4.

In terms of lower set less ordered robustness, we can see at the right hand side in Figure 5.18 that Hotels No. 4 and 5 are lower set less ordered robust, while Hotel No. 1 is not lower set less ordered robust since it is dominated by Hotel No.5 (although Hotel No. 1 is weakly lower set less ordered robust). Furthermore, Hotels No. 2, 3 and 6 are dominated by Hotel No. 5 and are thus not lower set less ordered robust.

Theorem 30 implies that Hotels No. 1 and 4 (Hotel No. 5, respectively) are set less ordered robust, since they are upper (lower, respectively) set less ordered robust. We can see that Hotels No. 2 and 3 are both dominated by Hotels No. 4, 5 and 6, thus they cannot be set less ordered robust. Note that Hotel No. 6 is neither upper nor lower set less ordered robust, but still set less ordered robust, see Figure 5.19. This verifies that the inverse implication in Theorem 30 is in general not fulfilled.

At this point, we can use Theorem 32 to conclude that the only Hotel which is alternative set less ordered robust is Hotel No. 4. Thus, a decision maker who acts both risk averse and risk affine with regard to the future would choose this hotel. At the same time, Hotel No. 1 is weakly alternative set less ordered robust, since it is both weakly upper and lower set less ordered robust.

Hotels No. 1, 4, 5 and 6 are minmax less ordered robust. Hotels No. 2 and 3 are not minmax less ordered robust, since they are dominated by Hotel No. 4.

Note that Theorem 35 implies that Hotels No. 1, 4, 5, 6 are certainly less ordered robust. As can be seen in the plot, Hotel No. 3 is certainly less ordered robust. However,

$\binom{\text{noise}}{\text{weather}}$	H_1	H_2	H_3	H_4	H_5	H_6
Scenario 1	$\binom{12}{8}$	$\binom{15}{13}$	$\binom{15}{10}$	$\binom{13}{6}$	$\binom{15}{7}$	$\binom{14}{9}$
Scenario 2	$\binom{9}{3}$	$\binom{15}{13}$	$\binom{10}{8}$	$\binom{6}{5}$	$\binom{7}{3}$	$\binom{8}{4}$
Scenario 3	$\binom{4}{9}$	$\binom{15}{13}$	$\binom{10}{8}$	$\binom{4}{7}$	$\binom{3}{8}$	$\binom{5}{10}$
Scenario 4	$\binom{10}{14}$	$\binom{15}{13}$	$\binom{11}{13}$	$\binom{6}{10}$	$\binom{7}{15}$	$\binom{8}{12}$

Table 5.3: Hotels 1–6 and scenarios 1–4.

we note that Hotel No. 2 is not certainly less ordered robust, and one would want to exclude Hotel No. 2 beforehand because obviously Hotels No. 4 and 6 dominate it in every scenario, see Figure 5.19

Note that in this example, the set of possibly less ordered robust solutions is empty. Due to Theorem 39, Hotels No. 1, 4, 5 and 6 are minmax certainly less ordered robust. Since Hotels No. 2 and 3 are dominated by Hotels No. 4 and 6, they are not minmax certainly less ordered robust. We conclude that Hotel No. 2 is not considered robust for any robustness concept.

5.4 Robustness vs. Set Optimization

As observed in the previous section, there is a strong connection between robustness as it was introduced by Ehrgott et al. [25] for uncertain multi-objective optimization and set optimization. This observation inspired us to introduce new concepts for robust solutions of uncertain vector optimization problems based on various well known set order relations.

Recalling Definition 13 of a minimal solution of $(SP-\preceq)$ w.r.t. a pre-order $\preceq$ (see Definition 5), we are able to mention the following relationship between our definition of robust solutions and minimal solutions. To this end, we define optimal solutions of a set-valued problem $(SP-\preceq)$, where $G : X \supseteq \mathcal{X} \rightrightarrows Y$ is a set-valued mapping.

Definition 27. *x^0 is called an optimal solution of $(SP-\preceq)$ if there does not exist $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ s.t. $G(\overline{x}) \preceq G(x^0)$.*

The following lemma describes links between the optimality notion in Definition 27 and minimal solutions in the sense of Definition 13 of a set-valued problem $(SP-\preceq)$.

Lemma 22. *If x^0 is an optimal solution of $(SP-\preceq)$ w.r.t. a pre-order $\preceq$, then x^0 is a minimal solution of $(SP-\preceq)$ w.r.t. the same pre-order $\preceq$.*

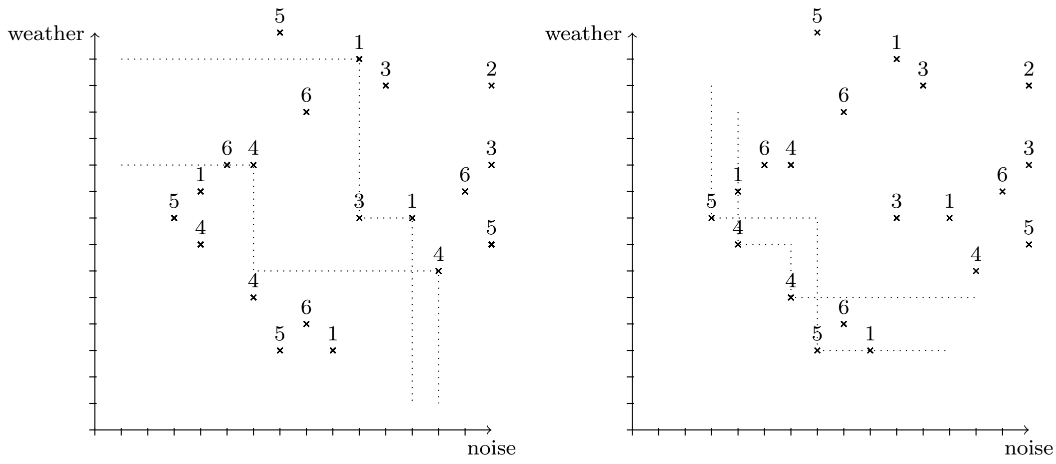

Figure 5.18: Left: Hotels No. 1 and 4 are upper set less ordered robust. Right: Hotels No. 4 and 5 are lower set less ordered robust.

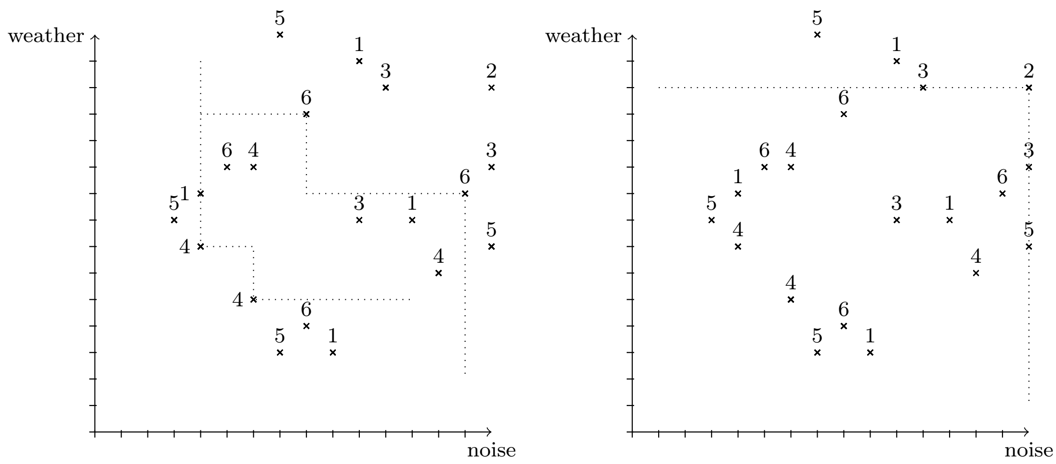

Figure 5.19: Left: Hotel No. 6 is, in addition to Hotels No, 1, 4 and 5, set less ordered robust. Right: Hotel No. 2 is not certainly less ordered robust.

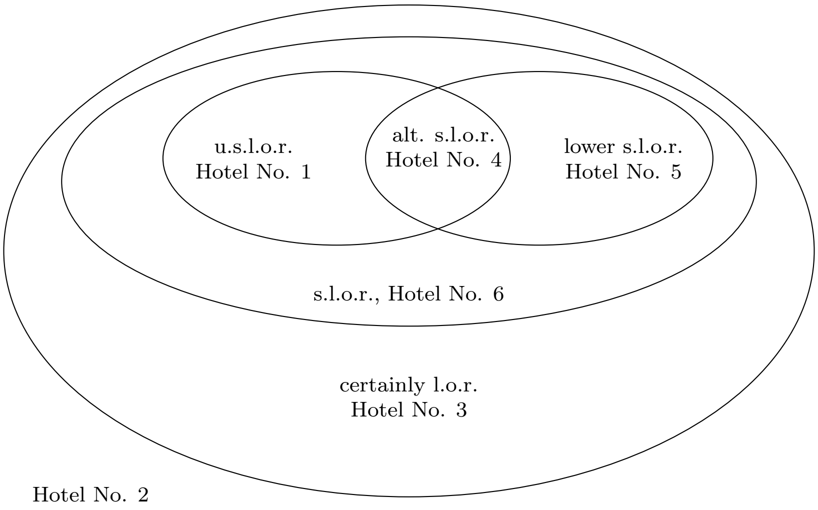

Figure 5.20: Summary of hotels in a robust classification framework. "Set less ordered robust" is abbreviated by "s.l.o.r.".

Proof. Let x^0 be an optimal solution of $(SP-\preceq)$ w.r.t. the pre-order $\preceq$. Then there does not exist $\overline{x} \in \mathcal{X} \setminus \{x^0\}$ s.t. $G(\overline{x}) \preceq G(x^0)$. Suppose x^0 is not a minimal solution of $(SP-\preceq)$. Thus, there exists $\overline{x} \in \mathcal{X}$ s.t.

$$G(\overline{x}) \preceq G(x^0) \text{ and } \neg\big(G(x^0) \preceq G(\overline{x})\big).$$

Due to x^0's optimality, we deduce $\overline{x} = x^0$. But this means that $\neg\big(G(x^0) \preceq G(x^0)\big)$, a contradiction as $\preceq$ was assumed to be a pre-order. □

Hence, due to our analysis of robust solutions to uncertain multi-objective optimization problems, we indirectly provided algorithms for solving set optimization problems using various set order relations. The above lemma shows that the algorithms we provided for obtaining strictly

- lower set less ($\preceq^l_C$);
- upper set less ($\preceq^u_C$);
- set less ($\preceq^s_C$) and
- minmax less ($\preceq^m_C$)

ordered robust solutions can be used for computing minimal solutions in the sense of Definition 13 of $(SP-\preceq)$ w.r.t. the according pre-order $\preceq$. Notice that the introduced order relations $\preceq^l_Q$, $\preceq^u_Q$, $\preceq^s_Q$ and $\preceq^m_Q$ are not pre-orders in general for $Q = C \setminus \{0\}$ and for $Q = \operatorname{int} C$. Note that we redefined the order relations $\preceq^{cert}_C$, $\preceq^{mc}_C$ such that they are in general no longer pre-orders. Of course, this can be done differently in future research.

The algorithms we obtained for these remaining robust solution concepts can be used to obtain optimal solutions w.r.t. the corresponding order relation $\preceq$ in the sense of Definition 27.

Furthermore, note that this suggested approach is only possible if the set-valued mapping $G : \mathcal{X} \rightrightarrows Y$ can be reformulated as

$$G(x) =: f_{\mathcal{U}}(x) = f(x,\mathcal{U}) = \{f(x,\xi)|\ \xi \in \mathcal{U}\}$$

for each $x \in \mathcal{X}$, a set $\mathcal{U} \subseteq \mathbb{R}^N$ and a function $f:\ \mathcal{X} \times \mathcal{U} \to Y$. Determining which classes of set-valued problems satisfy this condition is a topic for future research.

Scalarization and vectorization techniques for set-valued optimization based on the lower, upper and set less order relation have also been investigated by Jahn [54]. The research conducted in this chapter shows that the framework along with solution procedures introduced in [54] can be directly applied to uncertain vector-valued optimization, hence uncertain multi-objective optimization is an important application of set optimization.

Future research could include providing algorithms for solving the uncertain vector-valued problem for possibly less ordered robust solutions. Furthermore, studying applications of uncertain vector-valued problems along with numerical examples would be interesting to investigate.

Chapter 6

Optimality Conditions for Robust Optimization Problems

In this chapter, we study optimality conditions for the strictly robust optimization problem as a special case of weighted robustness that was introduced in Chapter 3 by means of abstract subdifferentials. By an abstract subdifferential we mean a subdifferential that satisfies certain axioms. For an introduction of abstract subdifferentials, we follow Durea and Tammer [22]. Although in [22], "exact" and "fuzzy" calculus rules are considered, we confine ourselves to "exact" calculus rules in the following. Let χ be a class of Banach spaces. An abstract subdifferential is a map that associates to every lower semi-continuous function $h:\ \chi \ni X \to \overline{\mathbb{R}}$ and to every $x \in X$ a subset $\partial h(x) \subset X^*$. For $X, Y \in \chi$, $\mathcal{G}(X,Y)$ is the class of functions mapping from X to Y with the property that by composition at left with a lower semi-continuous function from Y to $\overline{\mathbb{R}}$, the resulting function is again lower semi-continuous. The indicator function I_S of a set $S \subset X$ is defined by

$$I_S(x) := \begin{cases} 0, & x \in S, \\ +\infty, & \text{otherwise.} \end{cases}$$

It is well known that the subdifferential of the indicator function of a convex set $S \subset X$ coincides with the normal cone, defined by

$$N(S,x^0) := \begin{cases} \{p \in L(X,\mathbb{R}) | \forall x \in S:\ p(x-x^0) \leq 0\}, & x^0 \in S, \\ \emptyset, & \text{otherwise,} \end{cases}$$

where $L(X,\mathbb{R})$ is the class of all continuous linear operators from X to $\mathbb{R}$. The following properties are assumed to be satisfied by an abstract subdifferential.

(H1) If h is a convex functional, then $\partial h(x)$ coincides with the Fenchel subdifferential.

(H2) If x_0 is a local minimum for h, then $0 \in \partial h(x_0)$. Furthermore, if $x \notin \operatorname{dom} h$, then $\partial h(x) = \emptyset$.

(H3) If $z: Y \to \overline{\mathbb{R}}$ is a convex functional and $\psi \in \mathcal{G}(X,Y)$, then it holds for all $x \in X$:

$$\partial(z \circ \psi)(x) \subset \bigcup_{y^* \in \partial z(\psi(x))} \partial(y^* \circ \psi)(x).$$

(H4) If $z: Y \to \overline{\mathbb{R}}$ is a convex functional, $\psi \in \mathcal{G}(X,Y)$, and S is a closed subset of X containing x, then it holds

$$\partial(z \circ \psi + I_S)(x) \subset \partial(z \circ \psi)(x) + \partial I_S(x).$$

As usual, we define the normal cone using the subdifferential of the indicator function $\partial I_S(x)$ for a set $S \subset X$:

$$N_\partial(S,x) := \partial I_S(x).$$

Note that (H1) and (H2) are very natural assumptions on a subdifferential. Properties (H3) and (H4) are exact calculus rules for compositions. Examples for subdifferentials that satisfy these requirements include (compare [22])

- the limiting (or Mordukhovich) subdifferential, when χ is the class of Asplund spaces, Y is finite dimensional and with $\mathcal{G}(X,Y)$ as the class of Lipschitz functions from X to Y [75];
- the approximate (or Ioffe) subdifferential when χ is the class of Banach spaces and $\mathcal{G}(X,Y)$ being the class of strongly compact Lipschitz functions from X to Y [50].

Let $C \subset Y$ be a proper closed convex and pointed cone. Recall from (5.3) that for y_1, $y_2 \in Y$:

$$y_1 \leq_C y_2 :\Longleftrightarrow y_1 \in y_2 - C.$$

At this point it is interesting to mention an important result by Valadier [98]. For this we need some additional notations. Adding a greatest element $+\infty$ ($\notin Y$) to Y, we obtain $Y^\bullet := Y \cup \{+\infty\}$. For a function $f: X \to Y^\bullet$ we define the subdifferential $\partial^{\leq_C}$ of f at $x^0 \in \operatorname{dom} f$ by

$$\partial^{\leq_C} f(x^0) := \{T \in L(X,Y) | \forall\, x \in X:\ T(x - x^0) \leq_C f(x) - f(x^0)\},$$

where $L(X,Y)$ is the class of all continuous linear operators from X to Y. If there is no confusion, we write $\partial^{\leq}$ instead of $\partial^{\leq_C}$. Furthermore, f is C-convex for a convex cone C if

$$\forall\, x_1, x_2 \in X,\ \forall\, \lambda \in [0,1]:\ f(\lambda x_1 + (1-\lambda)x_2) \leq_C \lambda f(x_1) + (1-\lambda) f(x_2).$$

Now we are ready to mention a result by Valadier [98], which stated that, under certain assumptions on f and the ordering cone C, it is possible to extract y^* from the subdifferential.

Theorem 42 ([98]). *Let $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$ be real reflexive Banach spaces and $C \subset Y$ a proper convex cone with a weakly compact base. If $f : X \to Y^\bullet$ is a C-convex operator, continuous at some point of its domain, then*

$$\forall\, x \in \operatorname{int}(\operatorname{dom} f),\ \forall\, y^* \in C^* : \quad y^* \circ \partial^{\leq_C} f(x) = \partial(y^* \circ f(x)),$$

using the convention that $y^*(+\infty) = +\infty$ for $y^* \in C^*$.

We now recall the definition of the nonlinear scalarizing functional $z^{B,k}$ (see (2.2) in Chapter 2),

$$z^{B,k}(y) := \inf\{t \in \mathbb{R} | y \in tk - B\}, \tag{6.1}$$

under the requirement that (compare (2.1))

$$B + [0, +\infty) \cdot k \subset B \tag{6.2}$$

for $B \subset Y,\ k \in Y \setminus \{0\}$. The nonlinear scalarizing functional can now be minimized on a set of feasible solutions $\mathcal{F} \subset Y$, resulting in the nonlinear scalarization approach as discussed in Chapter 2 (compare (2.3)):

$$(P_{k,B,\mathcal{F}}) \qquad \inf_{y \in \mathcal{F}} z^{B,k}(y). \tag{6.3}$$

In order to provide optimality conditions for a strictly robust optimization problem, the subdifferential of the nonlinear scalarizing functional $z^{B,k}$ is of importance. The following theorem describes the structure of the subdifferential of $z^{B,k}$ when this functional is convex and proper. Here, ∂ denotes the classical (Fenchel) subdifferential.

Theorem 43 ([22, Theorem 2.2]). *Let $B \subset Y$ be a closed proper set and $k \in Y \setminus \{0\}$ such that* (6.2) *holds and for each $y \in Y$ there exists $t \in \mathbb{R}$ such that $y + tk \notin B$. Then for $\overline{y} \in \operatorname{dom} z^{B,k}$, we have*

$$\partial z^{B,k}(\overline{y}) = \{v^* \in Y^* | v^*(k) = 1,\ \forall\, y \in B :\ v^*(y) + v^*(\overline{y}) - z^{B,k}(\overline{y}) \geq 0\}.$$

The next result presents further insight into the classical (Fenchel) subdifferential of $z^{B,k}$ when B is a cone. We will use the following lemma to provide a necessary optimality condition of a strictly robust optimization problem which was introduced in Section 3.1.1 as a special case of weighted robustness.

Lemma 23 ([22, Lemma 2.4]). *Let $B \subset Y$ be a proper closed convex cone. Then for every $k \in \operatorname{int} B$ and for each $\overline{y} \in Y$, $\partial z^{B,k}(\overline{y}) \neq \emptyset$ and $\partial z^{B,k}(\overline{y}) = \{v^* \in B^* | v^*(k) = 1,\ v^*(\overline{y}) = z^{B,k}(\overline{y})\}$.*

Before using the above lemma to present an optimality condition by means of abstract subdifferentials, we first recall a scalar uncertain optimization problem, where the uncertain parameter ξ is assumed to belong to a finite uncertainty set $\mathcal{U} := \{\xi_1, \ldots, \xi_q\}$. Let $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R},\ F_i : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R},\ i = 1, \ldots, m$. Now an *uncertain* optimization problem is defined as a family of parametrized optimization problems

$$(Q(\xi), \xi \in \mathcal{U}). \tag{6.4}$$

For a fixed $\xi \in \mathcal{U}$, the optimization problem $(Q(\xi))$ is given by

$$(Q(\xi)) \qquad \begin{aligned} \min\ & f(x,\xi) \\ \text{s.t. }\ & F_i(x,\xi) \leq 0,\ i = 1,\dots,m, \\ & x \in \mathbb{R}^n. \end{aligned}$$

Now the **strictly robust counterpart** to the family of uncertain optimization problems $(Q(\xi), \xi \in \mathcal{U})$ is defined for $\mathfrak{A} := \{x \in \mathbb{R}^n | \forall\, \xi \in \mathcal{U} :\ F_i(x,\xi) \leq 0,\ i = 1,\dots,m\}$ (compare (3.2)) as

$$(RC) \qquad \min_{x\in\mathfrak{A}} \sup_{\xi\in\mathcal{U}} f(x,\xi). \tag{6.5}$$

Recall that (RC) may be formulated using $z^{B,k}$ for a specific choice of parameters B, k on a set of feasible solutions $\mathcal{F}$ (compare problem (wRC) in Chapter 3 with weights $w_k = 1$, $k = 1,\dots,q$). For this we redefine the objective function as $\overline{f} : \mathbb{R}^n \to \mathbb{R}^q$ s.t. for every $x \in \mathfrak{A}$

$$\overline{f}(x) = \begin{pmatrix} \overline{f}_1(x) \\ \dots \\ \overline{f}_q(x) \end{pmatrix} := \begin{pmatrix} f(x,\xi_1) \\ \dots \\ f(x,\xi_q) \end{pmatrix}.$$

We use the following notation:

$$B := \mathbb{R}^q_{\geqq}, \tag{6.6}$$

$$k := 1_q = (1,\dots,1)^T, \tag{6.7}$$

$$\mathcal{F} := \{(f(x,\xi_1),\dots,f(x,\xi_q))^T | x \in \mathfrak{A}\}. \tag{6.8}$$

Note that $\mathcal{F}$ is not necessarily a convex set. With the above parameters B, k and $\mathcal{F}$, we obtain for $y = \overline{f}(x) = (f(x,\xi_1),\dots,f(x,\xi_q))^T$

$$\min_{y\in\mathcal{F}} z^{B,k}(y) = \min_{x\in\mathfrak{A}} \max_{\xi\in\mathcal{U}} f(x,\xi),$$

taking into account Theorem 3 of the weighted robust problem with weights $w_k = 1$, $k = 1,\dots,q$. Thus, both problems can be regarded as equivalent and the following connections between problems (RC) (see (6.5)) and $(P_{k,B,\mathcal{F}})$ (see (6.3)) are to mention:

$$\min_{x\in\mathfrak{A}} \sup_{\xi\in\mathcal{U}} f(x,\xi) \Longleftrightarrow \min_{y\in\mathcal{F}} z^{B,k}(y) \Longleftrightarrow \min_{x\in\mathfrak{A}} z^{B,k}(\overline{f}(x)).$$

Now we are able to use the special structure of the subdifferential of the nonlinear scalarizing functional $z^{B,k}$ to characterize optimal solutions of $(P_{k,B,\mathcal{F}})$ by means of abstract subdifferentials. Since both problems $(P_{k,B,\mathcal{F}})$ and (RC) are equivalent, as demonstrated above, optimality conditions that hold for $(P_{k,B,\mathcal{F}})$ apply to (RC) as well. In the following theorem we provide a necessary optimality condition for feasible solutions of the strictly robust optimization problem (RC).

Theorem 44. *Let the nonlinear functional $z^{B,k}$ be defined by* (6.1) *for the parameters B given in* (6.6) *and k as in* (6.7)*. Assume ∂ satisfies $(H1)-(H4)$. If x^0 solves (RC) (see* (6.5)*), then there exists a $v^* \in \mathbb{R}^q_{\geqq}$ with $v^* \cdot 1_q = 1$ and $v^*(\overline{f}(x^0)) = z(\overline{f}(x^0))$ for $\overline{f}(x^0) = (f(x^0,\xi_1),\dots,f(x^0,\xi_q))^T$ and*

$$0 \in \partial(v^* \circ \overline{f})(x^0) + N_\partial(\mathfrak{A}, x^0). \tag{6.9}$$

Proof. For simplification, we set $z := z^{B,k}$. Let x^0 solve (RC) (see (6.5)), i.e.,

$$\overline{f}(x^0) = (f(x^0,\xi_1),\dots,f(x^0,\xi_q))^T$$

solves $(P_{k,B,\mathcal{F}})$ and

$$y = \overline{f}(x) = (f(x,\xi_1),\dots,f(x,\xi_q))^T.$$

Because the subdifferential ∂ satisfies $(H1)-(H4)$, we arrive at

$$\begin{aligned} 0 &\in \partial\big((z\circ\overline{f}) + I_{\mathfrak{A}}\big)(x^0) \\ &\subset \partial(z\circ\overline{f})(x^0) + \partial I_{\mathfrak{A}}(x^0) \\ &= \partial(z\circ\overline{f})(x^0) + N_\partial(\mathfrak{A}, x^0) \\ &\subset \cup_{y^*\in\partial z(\overline{f}(x^0))}\partial(y^*\circ\overline{f})(x^0) + N_\partial(\mathfrak{A}, x^0). \end{aligned}$$

Since $B = \mathbb{R}^q_{\geqq}$ is a proper closed convex cone, it holds for the subdifferential of z:

$$\partial z(\overline{f}(x^0)) = \{v^* \in \mathbb{R}^q_{\geqq} | v^*\cdot 1_q = 1,\ v^*(\overline{f}(x^0)) = z(\overline{f}(x^0))\}$$

(compare Lemma 23) and the proof is complete. □

If $\overline{f} : \mathbb{R}^n \to \mathbb{R}^q$ is an $\mathbb{R}^q_{\geqq}$-convex operator and continuous at some point of its domain, then Theorem 42 is applicable and

$$\forall\ x \in \mathrm{int}(\mathrm{dom}\,\overline{f}),\ \forall\ v^* \in \mathbb{R}^q_{\geqq}:\quad v^*\circ\partial^{\leq}\overline{f}(x) = \partial(v^*\circ\overline{f}(x))$$

holds, such that we get in (6.9) in Theorem 44

$$0 \in v^*\circ\partial^{\leq}\overline{f}(x^0) + N_\partial(\mathfrak{A}, x^0).$$

Note that the above analysis may be performed analogously for the remaining robustness concepts that are described in Chapter 3 for a suitable choice of parameters B, k and $\mathcal{F}$, yielding necessary optimality conditions for these approaches.

Chapter 7

Conclusions

This thesis is devoted to providing and analyzing scalar as well as various new vector-valued approaches to uncertain optimization.

In the first main part of the thesis (Chapter 3), we presented scalar robustness concepts in a unifying framework by means of a nonlinear scalarizing functional. We showed that new concepts for robustness may be deducted from this approach by varying the parameters B, k involved in the functional $z^{B,k}$ and by changing the set of feasible solutions $\mathcal{F}$ on which $z^{B,k}$ is minimized. Several properties of this functional were studied and compared for different robustness concepts. Specifically, the monotonicity property that $z^{B,k}$ fulfills under certain assumptions on B and k allows for links to multi-objective optimization, which led to an investigation of multiple objective robust counterpart problems. We showed that several robustness concepts are scalarizations of particularly chosen vector-valued problems.

Future research interests include the relation to coherent risk measures, since the functional $z^{B,k}$ is an important tool in the field of financial mathematics (compare Heyde [44] and a short note in [59]). It can be used as a coherent risk measure of an investment. One can show that

$$\mu(y) = \inf\{t \in \mathbb{R} | y + tk \in B\}$$

is a coherent risk measure. Obviously, we have (cf. Heyde [44])

$$\mu(y) = z^{B,k}(-y).$$

A risk measure induces a set B_μ of acceptable risks (dependent on μ)

$$B_\mu = \{y \in Y | \mu(y) \leq 0\}.$$

The following interpretation of coherent risk measures is possible: If $Y = \mathbb{R}^q$ (there are q states of the future), $B_1 = \mathbb{R}^q_{\geqq}$ and $k_1 = 1_q$, then

$$\mu(y) = z^{B_1,k_1}(-y) = \max_{\xi \in \mathcal{U}}(-f(x,\xi)) = -\min_{\xi \in \mathcal{U}} f(x,\xi)$$

is a coherent risk measure. Specifically, the risk measure $\max_{\xi \in \mathcal{U}}(-f(x,\xi))$ is the objective function of the strictly robust counterpart (compare (wRC), (3.1), with weights

$w_k = 1$, $k = 1, \dots, q$) with negative values of f. Because $\mu(y) = -\min_{\xi\in\mathcal{U}} f(x,\xi)$, negative payments f of an investment in the future result in a positive risk measure, and positive payments result in a negative risk measure. This seems very reasonable since negative payments (losses) are riskier than investments with only positive payments (bonds). This approach can analogously be performed for other concepts of robustness and may be analyzed in terms of financial theory. Interrelations between robustness and coherent risk measures have also been studied by Quaranta and Zaffaroni in [86]: They minimized the conditional value at risk (which is a coherent risk measure) of a portfolio of shares using concepts of robust optimization.

The second main part of the thesis (Chapter 5) was devoted to analyzing new concepts for robustness for uncertain vector-valued optimization problems. Since for each $x \in \mathcal{X}$, $f_{\mathcal{U}}(x)$ is a set, we used set-valued approaches for dealing with uncertain vector-valued problems. We deducted new concepts for multicriteria robustness using different set orders $\preceq$ in order to identify elements that are immunized against perturbations. In particular, we concluded that the upper set less ordered robustness concept, that uses the order relation $\preceq^u_C$ and hedges against perturbations in the worst-case scenarios, is applicable if a decision maker acts risk averse. Lower set less ordered robustness on the other hand (with the order relation $\preceq^l_C$) hedges against perturbations in the best-case scenarios and is thus useful for a risk affine decision maker. If a decision maker is both risk averse and risk affine at the same time, we introduced the alternative set less ordered robustness concept, which produces solutions that are upper and lower set less ordered robust in parallel. If a user does not know whether to hedge against the worse or best cases, he may rely upon the set less ordered robustness approach (using the relation $\preceq^s_C$), which unifies lower and upper set less ordered robust solutions. More set order relations $\preceq$ are known from the literature and were investigated in this thesis in relation to robustness.

For each concept, we provided solution methods for obtaining robust elements. We pointed out that these methods can be used to handle other set-valued optimization problems as well, provided that they may be transformed in a way such that $G(x) = f_{\mathcal{U}}(x)$ for a set-valued map $G : X \rightrightarrows Y$. We omitted to provide a solution procedure for the possibly less ordered robustness concept, since this concept revealed itself to be extremely restrictive. Of course, this may be a topic for future research. Every new robustness concept was analyzed and compared to other approaches. We concluded Chapter 5 with a simple example to illustrate the different robustness concepts.

The presented results on connections between robust solutions of uncertain multi-objective optimization and set optimization suggest further research. One aspect of interest are different solutions concepts that exist in set-valued optimization. One may wish to investigate if the suggested methods for obtaining minimal solutions of set-valued optimization problems hold true for other solution notions as well, or if they can be adjusted. Further research could include providing applications for these concepts. An extension which would be interesting to investigate is analyzing the presented approaches for a variable ordering cone, i.e., a cone $C(x)$ that depends on the decision variable.

By means of abstract subdifferentials and using the special structure of the nonlinear scalarizing functional $z^{B,k}$, we finally provided an optimality condition for a strictly robust optimization problem, which was introduced in Chapter 3 as a special case of weighted robustness. We noted that this analysis may be performed for the remaining robustness concepts as well in the future.

Bibliography

[1] A. Atamtürk and M. Zhang. Two-stage robust network flow and design under demand uncertainty. *Oper. Res.*, 55(4):662–673, 2007.

[2] C. Barrico and C.H. Antunes. A new approach to robustness analysis in multi-objective optimization. In *Proceedings of the 7th International Conference on Multi-Objective Programming and Goal Programming (MOPGP)*, Loire Valley (Tours), France, 2006.

[3] C. Barrico and C.H. Antunes. Robustness analysis in evolutionary multi-objective optimization - with a case study in electrical distribution networks. Presented at the II European-Latin-American Workshop on Engineering Systems (SELASI II), Porto, Portugal, 2006.

[4] C. Barrico and C.H. Antunes. Robustness analysis in multi-objective optimization using a degree of robustness concept. In *IEEE Congress on Evolutionary Computation. CEC 2006.*, pages 1887 –1892. IEEE Computer Society, 2006.

[5] E.M.L. Beale. On minimizing a convex function subject to linear inequalities. *Journal of the Royal Statistical Society, Series B*, 17:173–184, 1955.

[6] A. Beck and A. Ben-Tal. Duality in robust optimization: Primal worst equals dual best. *Operations Research Letters*, 37:1–6, 2009.

[7] A. Ben-Tal, S. Boyd, and A. Nemirovski. Extending scope of robust optimization: Comprehensive robust counterparts of uncertain problems. *Math. Program.*, Ser. B 107:63–89, 2006.

[8] A. Ben-Tal, L. El Ghaoui, and A. Nemirovski. *Robust Optimization.* Princeton University Press, Princeton and Oxford, 2009.

[9] A. Ben-Tal, A. Goryashko, E. Guslitzer, and A. Nemirovski. Adjustable robust solutions of uncertain linear programs. *Math. Programming A*, 99:351–376, 2003.

[10] A. Ben-Tal and A. Nemirovski. Robust convex optimization. *Mathematics of Operations Research*, 23(4):769–805, 1998.

[11] A. Ben-Tal and A. Nemirovski. Robust solutions of uncertain linear programs. *Operations Research Letters*, 25:1–13, 1999.

[12] A. Ben-Tal and A. Nemirovski. Robust solutions of linear programming problems contaminated with uncertain data. *Mathematical Programming*, 88:411–424, 2000.

[13] D. Bertsimas and D.B. Brown. Constructing uncertainty sets for robust linear optimization. *Operations Research*, 57(6):1483–1495, 2009.

[14] D. Bertsimas and M. Sim. The price of robustness. *Operations Research*, 52(1):35–53, 2004.

[15] J.R. Birge and F.V. Louveaux. *Introduction to Stochastic Programming*. Springer Verlag, New York, 1997.

[16] J. Branke. Creating robust solutions by means of evolutionary algorithms. In E.A. Eiben, T. Bäck, M. Schenauer, and H.-P. Schwefel, editors, *Parallel Problem Solving from Nature – PPSNV*, volume 1498 of *Lecture Notes in Computer Science*, pages 119–128. Springer, Berlin, Heidelberg, 1998.

[17] D.B. Brown. Risk and robust optimization. PhD-Thesis, Massachusetts Institute of Technology, 2006.

[18] V. Chankong and Y.Y. Haimes. *Multiobjective Decision Making: Theory and Methodology*. Elsevier Science Publishing, 1983.

[19] A. Chiriaev and G.W. Walster. Interval arithmetic specification. Technical Report, 1998.

[20] G.B. Dantzig. Linear programming under uncertainty. *Management Science*, 1:197–206, 1955.

[21] K. Deb and H. Gupta. Introducing robustness in multi-objective optimization. *Evolutionary Computation*, 14(4):463–494, 2006.

[22] M. Durea and Chr. Tammer. Fuzzy necessary optimality conditions for vector optimization problems. *Optimization*, 58:449–467, 2009.

[23] F.Y. Edgeworth. *Mathematical Psychics*. Kegan Paul, 1831.

[24] M. Ehrgott. *Multicriteria Optimization*. Springer, New York, 2005.

[25] M. Ehrgott, J. Ide, and A. Schöbel. Minmax robustness for multi-objective optimization problems (submitted).

[26] G. Eichfelder. *Adaptive Scalarization Methods in Multiobjective Optimization*. Springer, Berlin, Heidelberg, 2008.

[27] G. Eichfelder and J. Jahn. Vector optimization problems and their solution concepts. In *Recent developments in vector optimization*, Vector Optim., pages 1–27. Springer, Berlin, 2012.

[28] T. Erfani and S. Utyuzhnikov. Control of robust design in multiobjective optimization under uncertainties. *Structural and Multidisciplinary Optimization*, 45:247–256, 2012.

[29] J.E. Falk. Exact solutions of inexact linear programs. *Operations Research*, 24, 1976.

[30] M. Fischetti and M. Monaci. Light robustness. In R. K. Ahuja, R. Moehring, and C. Zaroliagis, editors, *Robust and Online Large-Scale Optimization*, number 5868 in Lecture Notes in Computer Science, pages 61–84. Springer, Berlin Heidelberg, 2009.

[31] M. Fischetti, D. Salvagnin, and A. Zanette. Fast approaches to improve the robustness of a railway timetable. *Transportation Science*, 43:321–335, 2009.

[32] S. Gast. *Network Flow Problems with Uncertain Input Data in the Context of Supply Chain Management Applications.* PhD thesis, University of Wuppertal, 2010.

[33] Chr. Gerstewitz (Tammer). Nichtkonvexe Dualitität in der Vektoroptimierung. *Wiss. Zeitschr. TH Leuna-Merseburg*, 25:357–364, 1983.

[34] Chr. Gerth (Tammer) and P. Weidner. Nonconvex separation theorems and some applications in vector optimization. *J. Optim. Theory Appl.*, 67:297–320, 1990.

[35] L. El Ghaoui and H. Lebret. Robust solutions to least-squares problems with uncertain data. *SIAM Journal of Matrix Anal. Appl.*, 18:1034–1064, 1997.

[36] M.A. Goberna, V. Jeyakumar, G. Li, and M.A. López. Robust linear semi-infinite programming duality under uncertainty. *Math. Program., Ser. B*, 139:185–203, 2013.

[37] M. Goerigk. *Algorithms and Concepts for Robust Optimization.* Universität Göttingen (Dissertation), 2012.

[38] M. Goerigk, M. Knoth, M. Müller-Hannemann, M. Schmidt, and A. Schöbel. The price of robustness in timetable information. volume 20 of *OpenAccess Series in Informatics (OASIcs)*, pages 76–87, Schloss Dagstuhl-Leibniz-Zentrum für Informatik, Dagstuhl, Germany, 2011.

[39] A. Göpfert, H. Riahi, Chr. Tammer, and C. Zălinescu. *Variational Methods in Partially Ordered Spaces.* Springer, New York, 2003.

[40] A. Göpfert, Chr. Tammer, and C. Zălinescu. A new minimal point theorem in product spaces. *Z. Anal. Anwendungen*, 18:767–770, 1999.

[41] S. Gunawan and S. Azarm. Multi-objective robust optimization using a sensitivity region concept. *Structural and Multidisciplinary Optimization*, 29(1):50–60, 2005.

[42] Y.Y. Haimes, L. S. Lasdon, and D. A. Wismer. On a bicriterion formulation of the problems of integrated system identification and system optimization. *IEEE Trans. Syst. Man Cybern.*, 1:296–297, 1971.

[43] A. Hamel. Variational principles on metric and uniform spaces. Habilitation thesis, University of Halle–Wittenberg, Germany, 2005.

[44] F. Heyde. *Coherent risk measures and vector optimization*, pages 3–12. In K.-H. Küfer (ed.) et al., Multicriteria decision making and fuzzy systems. Theory, methods and applications. Shaker Verlag, Aachen, 2006.

[45] R. Hites, Y. De Smet, N. Risse, M. Salazar-Neumann, and P. Vincke. About the applicability of MCDA to some robustness problems. *European Journal of Operational Research*, 174:322–332, 2006.

[46] E.J. Hughes. Evolutionary multi-objective ranking with uncertainty and noise. In *Proceedings of the First International Conference on Evolutionary Multi-Criterion Optimization (EMO-2001)*, pages 329–343, 2001.

[47] D.A. Iancu and N. Trichakis. Pareto efficiency in robust optimization. *Management Science (forthcoming)*, 2013.

[48] J. Ide and E. Köbis. Concepts of robustness for multi-objective optimization problems based on set order relations. *submitted*, 2013.

[49] J. Ide, E. Köbis, D. Kuroiwa, A. Schöbel, and Chr. Tammer. The relationship between multicriteria robustness concepts and set-valued optimization. *Fixed Point Theory and Applications, to appear*, 2013.

[50] A.D. Ioffe. Metric regularity and subdifferential calculus. *Russian Math. Surveys*, 55:501–558, 2000.

[51] J. Jahn. Scalarization in vector optimization. *Math. Program.*, 29:203–218, 1984.

[52] J. Jahn. *Mathematical vector optimization in partially ordered linear spaces*, volume 31 of *Methoden und Verfahren der mathematischen Physik [Methods and Procedures in Mathematical Physics]*. Verlag Peter D. Lang, Frankfurt am Main, 1986.

[53] J. Jahn. *Vector Optimization - Introduction, Theory, and Extensions*. Springer, Berlin, Heidelberg, 2011. 2nd Edition.

[54] J. Jahn. Vectorization in set optimization. Workshop *Vektor- und mengenwertige Optimierung*, Wittenberg, 2013.

[55] J. Jahn and T.X.D. Ha. New order relations in set optimization. *J. Optim. Theory Appl.*, 148(2):209–236, 2011.

[56] V. Jeyakumar and G. Li. Strong duality in robust convex programming: Complete characterizations. *SIAM J. Optim.*, 20:3384–3407, 2010.

[57] P. Kall, S. Wallace, and W. Stein. *Stochastic programming.* Wiley - Interscience Series in Systems and Optimization. Chichester: John Wiley & Sons, 1994.

[58] A. Khan, Chr. Tammer, and C. Zălinescu. *Set-Valued Optimization – An Introduction with Applications.* Springer, 2014. In preparation.

[59] K. Klamroth, E. Köbis, A. Schöbel, and Chr. Tammer. A unified approach for different concepts of robustness and stochastic programming via nonlinear scalarizing functionals. *Optimization*, 62(5):649–671, 2013.

[60] K. Klamroth, E. Köbis, A. Schöbel, and Chr. Tammer. A unified approach to robustness with a compact uncertainty set. Draft, 2013.

[61] K. Klamroth and J. Tind. Constrained optimization using multiple objective programming. *Journal of Global Optimization*, 37:325–355, 2007.

[62] E. Köbis. On robust optimization: Relations between scalar robust optimization and unconstrained multicriteria optimization. *J. Optim. Theory Appl.*, 2013.

[63] P. Kouvelis and S. Sayin. Algorithm robust for the bicriteria discrete optimization problem. *Annals of Operations Research*, 147:71–85, 2006.

[64] P. Kouvelis and G. Yu. *Robust Discrete Optimization and its Applications.* Kluwer Academic Publishers, 1997.

[65] D. Kuroiwa. Some duality theorems of set-valued optimization with natural criteria. In *Proceedings of the International Conference on Nonlinear Analysis and Convex Analysis.* World Scientific, 221-228, 1999.

[66] D. Kuroiwa. Existence theorems of set optimization with set-valued maps. *J. Inf. Optim. Sci*, 24(1):73–84, 2003.

[67] D. Kuroiwa. The natural criteria in set-valued optimization. *Sūrikaisekikenkyūsho Kōkyūroku*, (1031):85–90, Research on nonlinear analysis and convex analysis. Kyoto, 1997.

[68] D. Kuroiwa and G.M. Lee. On robust multiobjective optimization. *Vietnam Journal of Mathematics*, 40(2&3):305–317, 2012.

[69] D. Kuroiwa and G.M. Lee. On robust convex multiobjective optimization. *accepted to Journal of Nonlinear and Convex Analysis*, 2013.

[70] F. Oustry L. El Ghaoui and H. Lebret. Robust solutions to uncertain semidefinite programs. *SIAM J. Optim.*, 9(1):33–52, 1998.

[71] M. Li, S. Azarm, and V. Aute. A multi-objective genetic algorithm for robust design optimization. In *Proceedings of the Genetic and Evolutionary Computation Conference (GECCO'05)*, pages 771–778, 2005.

[72] D.T. Luc. *Theory of Vector Optimization.* Lecture Notes in Econ. and Math. Syst., 319, Springer, 1989.

[73] M. Mani, A.K. Sing, and M. Orshansky. Joint design-time and postsilicon minimization of parametric yield loss using adjustable robust optimization. In *Proceedings of the 2006 IEEE/ACM international conference on Computer-aided design*, pages 19–26, New York, NY, USA, 2006.

[74] H.M. Markowitz. Portfolio selection. *The Journal of Finance*, 7(1):77–91, 1952.

[75] B.S. Mordukhovich and Y. Shao. Nonsmooth sequential analysis in Asplund spaces. *Trans. Am. Math. Soc.*, 348:1235–1280, 1996.

[76] J. Nash. Equilibrium points in n-person games. In *Proceedings of the National Academy of Sciences of the United States of America*, volume 36, pages 48–49, 1950.

[77] Z.G. Nishnianidze. Fixed points of monotone multivalued operators. *Soobshch. Akad. Nauk Gruzin. SSR*, 114(3):489–491, 1984.

[78] W. Ogryczak. Multiple criteria optimization and decisions under risk. *Control and Cybernetics*, 31:975–1003, 2002.

[79] W. Ogryczak. Robust decisions under risk for imprecise probabilities. In Y. Ermoliev, M. Makowski, and K. Marti, editors, *Managing Safety of Heterogeneous Systems*, volume 658 of *Lecture Notes in Economics and Mathematical Systems*, pages 51–66. Springer, 2012.

[80] W. Ogryczak. Tail mean and related robust solution concepts. *International Journal of Systems Science*, 2013. To appear.

[81] W. Ogryczak and T. Śliwiński. On efficient WOWA optimization for decision support under risk. *International Journal of Approximate Reasoning*, 50:915–928, 2009.

[82] V. Pareto. Manuale di economia politica. (societa editrice libraria, milano, italy, 1906). English translation: V. Pareto, *Manual of political economy*, translated by A.S. Schwier, M. Augustus, Kelley Publishers, New York (1971).

[83] A. Pascoletti and P. Serafini. Scalarizing vector optimization problems. *J. Optim. Theory Appl.*, 42:499–524, 1984.

[84] P. Perny, O. Spanjaard, and L.-X. Storme. A decision-theoretic approach to robust optimization. *Annals of Operations Research*, 147:317–341, 2006.

[85] A. Prekopa. *Stochastic Programming.* Kluwer Academic Publishers, Dordrecht, 1995.

[86] A.G. Quaranta and A. Zaffaroni. Robust optimization of conditional value at risk and portfolio selection. *Journal of Banking & Finance*, 32:2046–2056, 2008.

[87] S. Sayin and P. Kouvelis. The multiobjective discrete optimization problem: A weighted min-max two-stage optimization approach and a bicriteria algorithm. *Management Science*, 51:1572–1581, 2005.

[88] A. Schöbel. Light robustness and the trade-off between robustness and nominal quality. Technical report, Preprint-Reihe, Institut für Numerische und Angewandte Mathematik, Georg-August Universität Göttingen, 2012. submitted.

[89] A. Shapiro, D. Dentcheva, and A. Ruszczyński. *Lectures on Stochastic Programming: Modeling and Theory.* MOS-SIAM Series on Optimization 9, SIAM, 2009.

[90] A. Shapiro and A. Ruszczyński, editors. *Stochastic Programming*, volume 10 of *Handbooks in Operations Research and Management Science.* Elsevier, 2003.

[91] C. Singh. Convex programming with set-inclusive constraints and its applications to generalized linear and fractional programming. *J. of Optimization Theory and Applications*, 38 (1):33–42, 1982.

[92] A.L. Soyster. Convex programming with set-inclusive constraints and applications to inexact linear programming. *Oper. Res.*, 21:1154–1157, 1973.

[93] R.E. Steuer. *Multiple Criteria Optimization: Theory, Computation, and Application.* Wiley & Sons, 1986.

[94] S. Stiller. *Extending Concepts of Reliability: Network Creation Games, Real-time Scheduling, and Robust Optimization.* PhD thesis, TU Berlin, 2009.

[95] Chr. Tammer and K. Winkler. A new scalarization approach and applications in multicriteria d.c. optimization. *Journal of Nonlinear and Convex Analysis*, 4(3):365–380, 2003.

[96] J. Teich. Pareto-front exploration with uncertain objectives. In *Proceedings of the First International Conference on Evolutionary Multi-Criterion Optimization (EMO-2001)*, pages 314–328, 2001.

[97] G. Tintner. Stochastic linear programming with applications to agricultural economics. In H.A. Antosiewicz, editor, *Proceedings of the 2nd Symposium on Linear Programming*, pages 197–228. National Bureau of Standards, 1955.

[98] M. Valadier. Sous-différentiabilité de fonctions convexes à valeurs dans un espace vectoriel ordonné. *Math. Scand.*, 30, 1972.

[99] P. Weidner. *Ein Trennungskonzept und seine Anwendung auf Vektoroptimierungsverfahren.* Martin-Luther-Universität Halle-Wittenberg, 1990 (Dissertation B).

[100] R.C. Young. The algebra of many-valued quantities. *Math. Ann.*, 104(1):260–290, 1931.

Selbständigkeitserklärung

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Dissertation selbständig und ohne fremde Hilfe angefertigt habe. Ich habe keine anderen als die angegebenen Quellen und Hilfsmittel benutzt und die den benutzten Werken wörtlich oder inhaltlich entnommenen Stellen als solche kenntlich gemacht.

Halle (Saale), 19. Dezember 2013

Lebenslauf

	Elisabeth Anna Sophia Köbis, geb. am 31.07.1986
Okt. 2011 - Dez. 2013	Reisestipendium des Allgemeinen Stiftungsfonds Theoretische Physik/Mathematik der Martin-Luther-Universität Halle-Wittenberg
Apr. 2011 - Dez. 2013	Promotionsstudium der Mathematik an der Martin-Luther-Universität Halle-Wittenberg
Apr. 2011	Master of Science in Wirtschaftsmathematik
Okt. 2009 - Apr. 2011	Masterstudium der Wirtschaftsmathematik an der Martin-Luther-Universität Halle-Wittenberg
Sept. 2009	Bachelor of Science in Wirtschaftsmathematik
Okt. 2006 - Sept. 2009	Bachelorstudium der Wirtschaftsmathematik an der Martin-Luther-Universität Halle-Wittenberg
Juli 2006	Abitur
Aug. 1999 - Juli 2006	Landesgymnasium und Europaschule Latina August Hermann Francke