Local Methods for a Theorem of Z_3^*-type

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Contents

0 Introduction

In his booklet [18] George Glauberman asks whether it is possible to generalise his Z^*-Theorem [17] to odd primes. The so called “Odd Z_p^*-Theorem” might be stated as:

Let G be a finite group and p an odd prime. Suppose that P is a Sylow p-subgroup with an element $x \in P$ such that, whenever $x^g \in P$ for some $g \in G$, then $g \in C_G(x)$. Then x is an element of $Z_p^*(G)$.

Here $Z_p^*(G)$ denotes the full pre-image of $Z(G/O_{p'}(G))$ and $O_{p'}(G)$ is the largest normal subgroup of G of order prime to p.

Glauberman’s question was answered positively (see for example 7.8.3 of [24]). In order to prove the theorem, a minimal counterexample is investigated. The first step is a reduction to the case where the counterexample is almost simple. Then the Classification of Finite Simple Groups is applied. By running through the list of the 26 sporadic groups and 17 infinite families of finite simple non-abelian groups it is possible to check that none of these groups occur as a counterexample.
But there is still neither a Classification-free proof of the Odd Z_p^*-Theorem nor a proof which provides some structure theoretical insight in terms of the subgroup structure.

Glauberman proved the Z^*-Theorem with modular representation theory. Recently Rebecca Waldecker [37] gave local arguments for a new proof of the Z^*-Theorem under the additional hypothesis that the simple groups involved in the centraliser of an isolated involution are known simple groups.
In 1981 Peter Rowley [32] weakened the hypothesis of the Odd Z_p^*-Theorem by introducing the following concept:

*In a finite group G an element x of a Sylow p-subgroup P is called **p-locally central** in G with respect to G if and only if $N_G(R) \leq C_G(x)$ for all $1 \neq R \leq P$.*

He proved the following theorem using group theoretical arguments.

Let P be a Sylow 3-subgroup of a finite group G. Suppose that $x \in P$ is a 3-locally central element in G with respect to P. Then x is an element of $Z_3^*(G)$.

Rowley’s and Waldecker’s results raise hopes of finding a new proof of the Odd Z_p^*-Theorem for the prime 3 that is independent of the Classification of Finite Simple Groups and provides a better knowledge of the structure of finite groups in general.

In his proof Rowley analyses a minimal counterexample G to his theorem. He reduces G to an almost simple group. The main part of Rowley’s proof is to investigate the components of $C_G(a)/O(C_G(a))$ for all involutions a of G. He shows that they all belong to a list of finite quasisimple groups. Finally he proves that G' is a known simple group.
Altogether he cites many results from the contents of the Classification of Finite Simple Groups which reduce his problem to the case that G' belongs to a list of known simple groups. This list includes six sporadic groups and five infinite families of groups of Lie type, and in the end Rowley says that G' cannot be any simple group of this list, but without explicitly proving this.

A first step to the desired proof of the Odd Z_p^*-Theorem for the prime 3 is a new proof of Rowley’s theorem that avoids arguments in certain finite simple groups and instead gives more structural insight.

The main result of this thesis is such a new proof of Rowley's theorem.

Except for Helmut Bender's classification of finite groups with a strongly embedded subgroup [6], we try to avoid theorems that yield lists of finite simple groups. We use more heavily the property of an element to be 3-locally central and give structural arguments. Moreover we do not need any special knowledge about finite simple groups except for the Suzuki groups, $PSU(3, 2^n)$ for some natural number $n \geq 2$ and $PSL(2, q)$ for some prime power q.

In Part 1 of this thesis we state results that are independent of the concept of 3-locally central elements. Therefore the first chapter includes well-known statements and elementary results. Moreover we collect properties of the above stated simple groups. The results presented here will be used in the subsequent chapters.
The next chapter is an introduction to the important concepts for the proof of the main theorem. We define a notion of balance and get acquainted with the concept of strongly closed elementary abelian subgroups of a finite group. This plays an important role in the following investigation of the structure of finite simple groups with a strongly closed elementary abelian subgroup where the centralisers of many involutions are 3-soluble. For this we use arguments and ideas of Daniel Goldschmidt [20], but not the classification of finite groups with a strongly closed 2-subgroup in its full strength.
Finally we introduce the Bender method. We adopt this and other ideas of Bender [4] to give an alternative proof of the well-known statement that finite simple groups with a Sylow 2-subgroup of order 4 are isomorphic to $PSL(2, q)$, where q is a prime power such that $q \equiv 3$ or $5 \pmod 8$.

Part 2 consists of the proof of Rowley's theorem. Analogously to Rowley, we investigate a minimal counterexample G to the theorem with a 3-locally central element x. The main idea of our proof is to conclude that the minimal counterexample has a strongly embedded subgroup. Then we apply Bender's classification of these groups [6] to deduce that G' is isomorphic to $PSU(3, 2^n)$, $PSL(2, 2^n)$ or $Sz(2^n)$ for some suitable natural number $n \geq 2$. In the beginning of the third chapter we exclude these cases.
Then the real work starts. Similarly to Rowley, we reduce to the case where a minimal counterexample G is almost simple, more precisely $G = G' \cdot \langle x \rangle$ where G' is non-abelian simple and has order divisible by 3 and an index equal to 1 or 3 in G. Moreover we describe properties of the important objects $C_G(x)$ and $\sigma := \{q \in \pi(G) \mid q \nmid |G : C_G(x)|\}$.
The most relevant result of the third chapter is that every non-cyclic elementary abelian 2-subgroup has an involution whose centraliser is not contained in $C_G(x)$. This implies that the centraliser of every non-cyclic elementary abelian 2-subgroup is S_4-free. Moreover we show that our minimal counterexample itself is S_4-free or that we can already see its whole $\{2, 3\}$-structure in $C_G(x)$.
This illustrates that the connection between the 2-structure and the 3-structure of our minimal counterexample is either deep or non-existent. This dichotomy intensively influences the structure of our proof.

Concerning the 2-structure of G we divide our investigation into two cases. The first where G has an elementary abelian 2-subgroup of order at least 8 and the other.

The first case is excluded in the fourth chapter. Using signalizer functors and further arguments of balance, we show that $\langle [x, O_{\sigma'}(C_G(a))] \mid a \in A^\# \rangle$ is trivial, if we have $2 \in \sigma$. On the other hand for every involution of $C_G(x)$ we have $C_G(a) = (C_G(a) \cap C_G(x)) \cdot [x, O_{\sigma'}(C_G(a))]$ in this case. It follows that $C_G(x)$ is strongly embedded.
If there is almost no connection between the 2- and the 3-structure, then G is S_4-free and

possesses a strongly closed elementary abelian 2-subgroup. We use our results about finite simple groups with a strongly closed elementary abelian 2-subgroup, where the centralisers of many involutions are 3-soluble, to obtain a contradiction.

In the second case with small 2-rank we first determine the structure of the Sylow 2-subgroups of G.
With arguments about fusion from Jonathan Alperin [1] we show that a Sylow 2-subgroup T of G is either homocyclic abelian, dihedral or isomorphic to a Sylow 2-subgroup of $U_3(4)$.
If T is abelian, then a result of Richard Brauer [10] forces T to be elementary abelian of order 4. Since we classified these groups before, we find a contradiction.
In the case where T is dihedral we follow Bender's investigation of finite simple groups with dihedral subgroups [4] and we intensively use the Bender method. In our situation many arguments become simplified. Quoting a result of Bender and Glauberman from [7] and a variation [5] of the Brauer-Suzuki-Wall-Theorem due to Bender, the group G' is forced to be isomorphic to $PSL(2, q)$ for some prime power q, which leads to a contradiction.
The case remains where T is isomorphic to a Sylow 2-subgroup of the group $U_3(4)$. From a result of Richard Lyons [31] we could immediately conclude that G' is isomorphic to $U_3(4)$. But this would not deliver the desired structural insights.
We first prove that $C_G(x)$ contains a Sylow 2-subgroup of G and is not soluble. With the additional hypothesis that the theorem of Lyons holds in sections of $C_G(x)$ we adopt ideas of Graham Higman [28] to obtain a final contradiction.

Throughout the proof we often use the "Odd Order Theorem" of Walter Feit and John Thompson [13], and the Z^*-Theorem of Glauberman [17].

Our notation is standard as in [30] or explicitly defined except that we write $U\ max\ G$ if U is a maximal subgroup of G and cyclic groups of order n are denoted by Z_n.

Part I

General Results

1 Preliminaries

1.1 Background Results

This section is a congeries of notation and required results that we often use in this thesis. Most results stated are well-known and we just give a reference.
Throughout this section let p be a prime and G be a finite group.
Furthermore let S be a Sylow p-subgroup of G.

1.1.1 Definition

(a) The **rank** of an elementary abelian finite p-group A is a natural number n such that $|A| = p^n$ holds.

(b) The **rank** of a finite p-group P is equal to the rank of the largest elementary abelian subgroup of P. We denote the rank of P by $r(P)$.

(c) The **p-rank** of G is the rank of a Sylow p-subgroup of G and denoted by $r_p(G)$.

1.1.2 Lemma

Let G be a p-group.
If we have $r(G) = 1$, then G is cyclic or we have $p = 2$ and G is a quaternion group.

Proof
This is Proposition 1.3 of [8]. □

1.1.3 Lemma

If G is a cyclic 2-group, a dihedral group of order at least 8, a quaternion group of order at least 16 or semidihedral, then $Aut(G)$ is a 2-group.
Furthermore we have $Aut(Q_8) \cong S_4$.

Proof
These are I 4.6 of [29], Theorem 34.8 of [8] and 5.3.3 of [30]. □

1.1.4 Lemma

Let G be a p-group. If H is a subgroup of G such that $G = H \cdot \phi(G)$, then $G = H$.
Moreover $G/\phi(G)$ is elementary abelian.

Proof
These are 5.2.3 and 5.2.7 (a) of [30]. □

1.1.5 Lemma (Dedekind Identity)

Let $G = U \cdot V$, where U and V are subgroups of G. Then every subgroup H satisfying $U \leq H \leq G$ admits the factorisation: $H = U \cdot (V \cap H)$.

Proof
This is 1.1.11 of [30]. □

1.1.6 Definition

If G and H are finite groups, then G is called H-**free** if and only if G contains no section isomorphic to H.

1.1.7 Lemma

Let G be a finite group and suppose that G is not S_4-free. Then there exists a non-trivial 2-subgroup T of G such that $N_G(T)$ is not S_4-free.

Proof

This is Lemma 2.3 of [32]. □

1.1.8 Theorem (Hall)

If G is soluble, then there exist Hall π-subgroups for every set π of primes.
Moreover all Hall π-subgroups are conjugate in G and every π-subgroup is contained in some Hall π-subgroup.

Proof

This is VI 1.8 of [29]. □

1.1.9 Focal Subgroup Theorem

We have $S \cap G' = \langle a^{-1}a^g \mid a, a^g \in S \text{ and } g \in G\rangle$.

Proof

This is Theorem 7.3.4 of [22]. □

1.1.10 p-Complement Theorem of Burnside

Suppose that $N_G(S) = C_G(S)$. Then G has a normal p-complement.

Proof

This is Theorem 7.4.3 of [22]. □

1.1.11 p-Complement Theorem of Frobenius

The finite group G possesses a normal p-complement if and only if one of the following conditions holds:

(a) For every non-identity p-subgroup P of G we have that $N_G(P)/C_G(P)$ is a p-group.

(b) For every non-identity p-subgroup P of G the group $N_G(P)$ has a normal p-complement.

Proof

This is Theorem 7.4.5 of [22]. □

1.1.12 Odd Order Theorem of Feit and Thompson

All finite groups of odd order are soluble.

Proof

This is [13]. □

1.1.13 Z^*-Theorem of Glauberman

Let $p = 2$ and suppose that $c \in S$. A necessary and sufficient condition for $c \notin Z^*(G)$ is that there exists an element $a \in C_S(c)$ such that a is conjugate to c in G and $a \neq c$.

Proof

This is Corollary 1 of [17]. □

1.1.14 Lemma (Coprime action)

Let π be a set of primes. Suppose that G is a π'-group and let A be a finite π-group acting on G and let $\rho \subseteq \pi'$. Then the following hold:

(a) If G is normal in H and A also acts on H, then $C_{H/G}(A) = C_H(A) \cdot G/G$.

(b) There exists an A-invariant Sylow q-subgroup of G for every prime q.

(c) If G is soluble, then there exists an A-invariant Hall ρ-subgroup of G.

(d) We have $G = [G, A] \cdot C_G(A)$ and $[G, A] = [G, A, A]$.

(e) If A is elementary abelian and non-cyclic, then we have
$$G = \langle C_G(B) \mid B \ max\ A \rangle = \langle C_G(a) \mid a \in A^\# \rangle.$$

(f) If G is abelian, then $G = C_G(A) \times [G, A]$ holds.

(g) If A centralises some normal (or subnormal) subgroup H of G satisfying $C_G(H) \leq H$, then A centralises G.

Proof

From the Odd Order Theorem 1.1.12 we see that either G or A is soluble. Thus we may apply Chapter 8 of [30]. Parts (a) to (f) are 8.2.2 (a), 8.2.3 (a), 8.2.6 (a), 8.2.7, 8.3.4 (a) and 8.4.2 of [30]. Part (g) is a variation of Thompson $P \times Q$-Lemma 8.2.8 of [30]. To see this we remark that $H \times A$ acts on G. □

1.1.15 Definition

Let p be odd and G be a p-group. A characteristic subgroup R of G of class at most 2 and of exponent p such that every non-trivial p'-automorphism of G induces a non-trivial automorphism of R is called **critical**.

1.1.16 Lemma

If p is odd and G is a p-group, then G possesses a critical subgroup.

Proof

This is Corollary 14.4 of [8]. □

1.1.17 Lemma

Let a be an involution acting on an elementary abelian 2-group A.
Then we have $|C_A(a)|^2 \geq |A|$.

Proof

This is 9.1.1 (b) of [30]. □

1.1.18 Lemma

Let K be a component of G. Then the following hold:

(a) The group $E(G)$ is the central product of the components of G.

(b) If N is a subnormal subgroup of G, then we have $K \leq N$ or $[N, K] = 1$.

(c) If L is a component of G with $K \neq L$, then we have $K \cap L \leq Z(K)$ and, if F is a subgroup of $F(G)$, then $F \cap K \leq Z(K)$ holds.

(d) If g is an element of G such that g normalises some subgroup U of K with $U \nleq Z(K)$, then g normalises K.

(e) Let $n \in \mathbb{N}$ and let $G_1, ..., G_n$ be non-abelian simple groups. If N is a normal subgroup of $G_1 \times ... \times G_n$, then there is a subset $J \subseteq \{1, ..., n\}$ such that $N = \bigtimes_{j \in J} G_j$.

(f) If N is a subnormal subgroup of G, then $E(N)$ is a subset of $E(G)$.

(g) If U is a subgroup of G and K is contained in U, then K is a component of U.

(h) We have $C_G(F^*(G)) = Z(F^*(G))$.

Proof

Part (a) is 6.5.6 (a) of [30] and Part (b) is 6.5.2 of [30].
Part (c) follows from (b), since components of G and subgroups of $F(G)$ are subnormal in G and none of these contains a proper subnormal non-abelian quasisimple subgroup.
For Part (d) we observe that K^g is a component of G and we see that $U \leq K \cap K^g$. Hence we apply (c) to K and K^g to obtain $K^g = K$ from $U \nleq Z(K)$.
Part (e) is 1.6.3 (b) of [30].
If K is a component of N, then K is a quasi simple subnormal subgroup of N and therefore a quasi simple subnormal subgroup of G. Thus Part (f) holds.
If K is contained in $U \leq G$, then $K = K \cap U$ is subnormal in U. As K is quasisimple, the assertion of (g) follows.
Finally Part (h) is 6.5.8 of [30]. □

1.2 Specific Non-Soluble Groups

In this section we collect knowledge about certain non-abelian simple groups that occur specifically in our investigation. We give an explicit proof of every result or refer to the literature.

1.2.1 Definition

A finite group G is **almost simple** if and only if $F^*(G)$ is simple and $G/F^*(G)$ is soluble.

1.2.2 Theorem of Dickson

Let p be a prime and let f be a natural number. The group $\mathrm{PSL}(2, p^f)$ contains exactly the following subgroups:

(a) elementary abelian p-groups of order p^m with $m \leq f$,

(b) cyclic groups of order k where k divides either $\frac{p^f+1}{d}$ or $\frac{p^f-1}{d}$ and $d = (p^f - 1, 2)$,

(c) dihedral groups of order $2 \cdot k$ with k as in (b),

(d) alternating groups A_4, if $p > 2$ or $p = 2$ and f is even,

(e) symmetric groups S_4, if $p^{2 \cdot f} - 1 \equiv 0 \pmod{16}$,

(f) alternating groups A_5, if $p = 5$ or $p^{2 \cdot f} - 1 \equiv 0 \pmod 5$,

(g) semidirect products of elementary abelian groups of order p^m and cyclic groups of order k where k is as in (b) with the additionally condition $k \mid p^m - 1$ and $k \mid p^f - 1$,

(h) groups $\mathrm{PSL}(2, p^m)$ with $m \mid f$ and $\mathrm{PGL}(2, p^m)$ with $2 \cdot m \mid f$.

Proof

This is II 8.27 of [29]. □

1.2.3 Proposition

Let G be a finite group with $O(G) = 1$. Suppose further that $F(G)$ contains exactly one involution and $E(G) \cong \mathrm{PSL}(2,5)$ or $E(G) \cong SL(2,5)$.
Then $O_{\{2,3\}'}(C_G(b)) = 1$ for all involutions $b \in G$.

Proof

Let c be the involution of $F(G)$.
Then we have $c \in Z(G)$ and hence $O_{\{2,3\}'}(C_G(c)) \leq O(C_G(c)) = O(G) = 1$. The outer automorphism group of $E(G)$ has order 2 by Theorem 3.2. (ii) of [39]. Moreover from Lemma 1.1.2 and Lemma 1.1.3 we deduce that $F(G)$ admits only automorphisms that are $\{2,3\}$-elements. Therefore we see with Lemma 1.1.18 (h) that $G/F^*(G)$ is a $\{2,3\}$-group. It follows for all involutions $b \in G$ that the group $O_{\{2,3\}'}(C_G(b))$ is a subgroup of $C_{E(G)}(b)$.
Let $- : G \to G/Z(E(G))$ be the natural epimorphism and let first b be an involution of $F^*(G) \setminus \{c\}$. Then there are elements $e \in E(G)$ and $f \in F(G)$ such that $b = e \cdot f$. From $f \in F(G) \leq C_G(E(G))$ we obtain that $C_{E(G)}(b) = C_{E(G)}(e \cdot f) = C_{E(G)}(e)$. Moreover we see $1 = b^2 = (e \cdot f)^2 = e^2 \cdot f^2$. This shows that $e^2 = (f^{-1})^2 \in E(G) \cap F(G) \leq \langle c \rangle$. We conclude that e is a 2-element of $E(G)$. From $\overline{E(G)} \cong \mathrm{PSL}(2,5) \cong A_5$ it follows that $C_{\overline{E(G)}}(\bar{e})$ is elementary abelian of order 4. Thus we have that $O_{\{2,3\}'}(C_G(b)) = 1$, if $E(G)$ is simple. In the other case, if $E(G)$ is not simple, then c is the unique involution of $E(G)$ and $E(G)$ has quaternion Sylow 2-subgroups of order 8.
Suppose for a contradiction that $e = c$. Then we obtain $1 = b^2 = f^2$ and hence f is trivial or an involution of $F(G)$. Since c is the unique involution of $F(G)$, it follows that $b \in \{1, c\}$. This is a contradiction. Therefore e has order 4 and $\bar{e}$ has order 2. Consequently the group

$$\overline{N_{E(G)}(\langle e \rangle \cdot Z(E(G)))} = N_{\overline{E(G)}}(\langle \bar{e} \rangle) = C_{\overline{E(G)}}(\langle \bar{e} \rangle) = C_{\overline{E(G)}}(\bar{e})$$

is of order 4. Thus its full pre-image $N_{E(G)}(\langle e \rangle \cdot Z(E(G)))$ has order 8. Moreover $C_{E(G)}(e)$ is a subgroup of $N_{E(G)}(\langle e \rangle \cdot Z(E(G)))$. This implies that $C_{E(G)}(b) = C_{E(G)}(e)$ is a 2-group. In particular $O_{\{2,3\}'}(C_G(b)) \leq O_{\{2,3\}'}(C_{E(G)}(b)) = O_{\{2,3\}'}(C_{E(G)}(e)) = 1$.
Let now b be an involution of G not contained in $F^*(G)$. If we have $b \in C_G(E(G))$, then $O_{\{2,3\}'}(C_G(b))$ is a normal subgroup of $E(G)$ of odd order. Thus $O_{\{2,3\}'}(C_G(b))$ is trivial.
We finally suppose that b induces a non trivial automorphism on $E(G)$. Then we have that $\overline{E(G) \cdot \langle b \rangle} \cong S_5$. Since b is an involution, $\bar{b}$ is a transposition in $\overline{E(G) \cdot \langle b \rangle} \cong S_5$. In S_5 it we obtain that $C_{S_5}((4,5)) = \langle (1,2),(4,5),(1,2,3) \rangle \cong \langle (4,5) \rangle \cdot S_3$. Since all transpositions in S_5 are conjugate $C_{\overline{E(G) \cdot \langle b \rangle}}(\bar{b})$ is a $\{2,3\}$-group. Consequently $|C_{E(G)}(b)| = |\overline{C_{E(G)}(b)}| \cdot |Z(E(G))|$ is a divisor of the $\{2,3\}$-number $|C_{\overline{E(G)}}(\bar{b})| \cdot |Z(E(G))|$. In particular $C_{E(G)}(b)$ is a $\{2,3\}$-group and so $O_{\{2,3\}'}(C_G(b)) = 1$. □

1.2.4 Lemma

Suppose that $G = \mathrm{PSL}(2, p^n)$ for some prime p.
Then the outer automorphism group of G is isomorphic to $\langle \alpha \rangle \times \langle \beta \rangle$, where α has order n and induces a field automorphism in G and β is of order 2, if p is odd, or trivial for $p = 2$. Moreover $G \cdot \langle \beta \rangle \cong \mathrm{PGL}(2, p^n)$ and, if β is non trivial, then $C_G(\beta)$ is dihedral order $p^n - 1$. If $\gamma \in \langle \alpha \rangle$ has order i, then $C_G(\gamma) \cong \mathrm{PSL}(2, p^{\frac{n}{i}})$.

Proof

By Theorem 3.2. (ii) of [39] we obtain that $Out(\mathrm{PSL}(2, p^n)) \cong Z_n \times Z_{(2,p^n-1)} = \langle \alpha \rangle \times \langle \beta \rangle$, where α has order n and β is of order 2 for odd p or trivial for $p = 2$.
The analysis in section 3.3.4 in [39] yields that α induces a field automorphism in G and that $G \cdot \langle \beta \rangle \cong \mathrm{PGL}(2, p^n)$. If k is an algebraically closed field such that $GF(p^n) \leq k$, then $\mathrm{PGL}(2, p^n) \leq \mathrm{PGL}(2, k)$ and all involutions of $\mathrm{PGL}(2, p^n) \setminus \mathrm{PSL}(2, p^n)$ are conjugate in $\mathrm{PGL}(2, k)$. In particular we may choose $\beta = Z(GL(2, p^n)) \cdot \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ to obtain the structure of its centraliser.

Let $p \neq 2$ and $L = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in GF(p^n)^\# \right\}$. Then L has $p^n - 1$ elements.

We suppose that $H = SL(2, p^n)$ and $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) \in H$. Then we observe that $Z(H) \cdot \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) \in C_G(\beta)$ if and only if

$$Z(H) \cdot \begin{pmatrix} -a & b \\ -c & d \end{pmatrix} = Z(H) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot Z(H) \cdot \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= Z(H) \cdot \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \cdot Z(H) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = Z(H) \cdot \begin{pmatrix} -a & -b \\ c & d \end{pmatrix}.$$

Equivalently we either have $b = c = 0$ or $a = d = 0$. As we moreover have $a \cdot d - b \cdot c = 1$, this is the case if and only if either

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in L \text{ or}$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} = \begin{pmatrix} 0 & b \\ -b^{-1} & 0 \end{pmatrix} \in L \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Combined we have $Z(H) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in C_G(\beta)$ if and only if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in L \cup L \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

We set $\gamma := \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Then we observe $|L \cup L \cdot \gamma| = 2 \cdot |L| = 2 \cdot (p^n - 1)$. Since for all $z \in L \cup L \cdot \gamma$ we have $z \cdot \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in L \cup L \cdot \gamma$, it follows that $C_G(\beta)$ has order $\frac{2 \cdot (p^n - 1)}{2} = p^n - 1$. From Dickson's Theorem 1.2.2 we conclude that $C_G(\beta)$ is dihedral.

Since α induces field automorphisms in G, the assertion of the lemma follows from Proposition 4.9.1 (a) of [24]. □

1.2.5 Lemma

Let n be a natural number and let T be a Sylow 2-subgroup of $\mathrm{PSL}(3, 2^n)$.
Then there are exactly two elementary abelian subgroups of order 2^{2n} in T and every elementary abelian subgroup of T is contained in one of them.

Proof

By Sylow's Theorem and II 7.1 of [29] we may choose T such that T consists of all 3×3 lower triangular matrices where every diagonal entry is 1. For such an element we have

$$\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 & 0 \\ a + a & 1 & 0 \\ b + a \cdot c + b & c + c & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a \cdot c & 0 & 1 \end{pmatrix}.$$

Thus the set of elements of T of order at most 2 is

$$\tilde{I}(T) := I(T) \cup \{id\} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix} \mid a = 0 \text{ or } c = 0 \right\}.$$

In particular every elementary abelian subgroup of T is a subset of $\tilde{I}(T)$.
Furthermore for all $a, b, x, y \in GF(2^n)$ such that $a \neq 0 \neq y$, the following holds:

$$\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & y & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b + x & y & 1 \end{pmatrix} \notin \tilde{I}(T).$$

For this reason every elementary abelian subgroup of T is a subset of one of the following sets.

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{pmatrix} \mid a, b \in GF(2^n) \right\}, \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b & a & 1 \end{pmatrix} \mid a, b \in GF(2^n) \right\}.$$

As every of this sets is an elementary abelian subgroup of T, the assertion follows. □

1.2.6 Lemma

Let A be an elementary abelian group of order 16 and $G = Aut(A)$.
If H is subgroup of G that acts irreducibly on A and its order is divisible by 7, then H has a section isomorphic to S_4.

Proof

Let H be subgroup of G that acts irreducibly on A and such that H is no $7'$-group.
In [11] all maximal subgroups of G are listed. Since H is neither contained in a point or plane stabiliser, we conclude that H is a subgroup of A_7. We check again in [11], that all maximal subgroups of A_7 of order divisible by 7 are isomorphic to $\mathrm{PSL}(2,7)$. Finally [11] yields that all maximal subgroups of order divisible by 7 of $\mathrm{PSL}(2,7)$ are Frobenius groups of order 21.
Moreover $16-1$ is not divisible by 7. Hence a cyclic group C of order 7 that acts on A does not act irreducible on A. If C is normalised by some group D of order 3 that also acts on A, then $C_A(C)$ is normalised by D. We conclude that a Frobenius group of order 21 does not act irreducible on an elementary abelian group of order 16. Since G, A_7 and $\mathrm{PSL}(2,7)$ are not S_4-free, the assertion follows. □

1.2.7 Theorem

Let G be the outer automorphism group of an extra-special 2-group T of order 2^{2n+1} for an $n \in \{1,2,3,4\}$. Then G is isomorphic to the orthogonal group $O^{\epsilon}(2 \cdot n, 2)$ with $\epsilon \in \{-,+\}$.
Moreover G has no non-soluble section that does not involve a S_3.

Proof

The first part follows from Theorem 1 (c) of [40]. Moreover the same theorem provides an order formula of $O^{\epsilon}(2 \cdot n, 2)$. If we have $n = 1$ or $n = 2$ and $\epsilon = +$, then we compute that G is a 2-group or a $\{2,3\}$-group and therefore soluble by Burnside's $p^{\alpha}q^{\beta}$-Theorem 10.2.1 of [30]. If G is not soluble, then we know from [11] that $O^2(G)$ is isomorphic to one of $O_4^-(2) \cong A_5$, $O_6^+(2) \cong A_8$, $O_6^-(2) \cong \mathrm{PSU}(4,2)$, $O_8^+(2)$ or $O_8^-(2)$. For all this groups we again obtain the assertion from [11].
More precisely the simple section that occur are $O_8^-(2)$, $O_8^+(2)$, $\mathrm{Sp}(6,2)$, A_9, $\mathrm{PSU}(4,2)$, A_8, A_7, $\mathrm{PSL}(2,16)$, A_6, $\mathrm{PSL}(2,8)$, $\mathrm{PSL}(2,7)$, A_5. The minimal non-soluble section are $\mathrm{PSL}(2,8)$ that has a maximal subgroup isomorphic to D_{18}, which contains a S_3, $\mathrm{PSL}(2,7)$ that has a maximal subgroup isomorphic to S_4, that contains a S_3, and A_5 that has a maximal subgroup isomorphic to S_3. □

1.2.8 Theorem

Let G be a finite quasisimple non-abelian group of order prime to 3 and let $- : G \to G/Z(G)$ denote the natural epimorphism.
Then there exists a natural number $n \geq 1$ such that $\bar{G} \cong Sz(2^{2n+1})$. Moreover if T is a Sylow 2-subgroup of G, then the following hold:

(a) The centre of G is trivial except for the case $n = 1$, then it is a subgroup of an elementary abelian group of order 4.

(b) If G is not simple, then G admits no outer automorphism.

(c) If G is simple, then the outer automorphism group of G is cyclic of order $2n+1$ and induces Galois automorphisms on $\Omega_1(T)$.

(d) The group $\bar{T}$ has order $(2^{2n+1})^2$ and we have that $\Omega_1(\bar{T}) = Z(\bar{T}) = \phi(\bar{T})$ is elementary abelian of order $2^{2n+1} \geq 8$. Furthermore $\Omega_1(T)$ is the full pre-image of $\Omega_1(\bar{T})$.

(e) The group $N_{\bar{G}}(\bar{T}) = N_{\bar{G}}(\Omega_1(\bar{T}))$ is a Frobenius group of order $(2^{2n+1})^2 \cdot (2^{2n+1} - 1)$ with kernel $\bar{T}$ and cyclic complement.

(f) All involutions of $\bar{G}$ are conjugate.

(g) If t is an involution of G, then $C_{\bar{G}}(\bar{t}) \in \mathrm{Syl}_2(\bar{G})$ or $t \in Z(G)$.

(h) Whenever we have $\bar{t}^g \in \bar{T}$ for an element $g \in G$ and an involution $\bar{t} \in Z(\bar{T})$, then $\bar{t}^g \in Z(\bar{T})$. Moreover $Z(\bar{T})$ is the only elementary abelian 2-subgroup of $\bar{T}$ with that property.

(i) If U is subgroup of G containing $\Omega_1(T)$, then $O(U)$ is trivial.

(j) If L be a non-soluble subgroup of G, then we have $N_G(L) = L$ and $Z(L) = Z(G)$.

Proof
The first statement of the Theorem is proven in [36].

(a) This follows from Theorem 1 and 2 of [2].

(b) This is Theorem 2 of [2].

(c) This is Theorem 11 of [35] and its proof.

(d) The statement about $\bar{T}$ follows from Theorem 9 of [35] together with Lemma 1 of the same article. We have that $\Omega_1(T)$ is contained in the full pre-image of $\Omega_1(\bar{T})$. Moreover in [11] we check that the pre-images of the elements in $\Omega_1(\bar{T})$ are involutions. Thus $\Omega_1(\bar{T}) = \overline{\Omega_1(T)}$.

(e) This follows from Theorem 9 of [35] together with Lemma 7 of the same article.

(f) The assertion follows by Lemma 1 and Lemma 7 of [35].

(g) This again is a combination of Theorem 9 and Lemma 1 of of [35].

(h) Since $Z(\bar{T}) = \Omega_1(\bar{T})$ is the set of all elements of $\bar{T}$ of order 2 or 1 by (d), the group $Z(\bar{T})$ has the described property. From (f) it follows that $Z(\bar{T})$ is the unique elementary abelian subgroup of $\bar{T}$ with that property.

(i)(j) Both of the last statements follow again from Theorem 9 of [35]. □

1.2.9 Definition
If G is one of the groups in Theorem 1.2.8, then we call G a **Suzuki group**.

1.2.10 Lemma
Let K be a component of the finite group G such that $K/Z(K)$ is a Suzuki group.
Suppose that $y \in G$ is an element of order 3. Then y normalises K or there is a section of $C_G(y)$ isomorphic to $K/Z(K)$. In both cases $C_G(y)$ is of even order.

Proof
We suppose first that $Z(K) = 1$. If y centralises K, then the assertion follows immediately from the fact that K is of even order by Theorem 1.2.8 (d).
If y normalises but does not centralise K, then y induces an automorphism of order 3 on K. Thus we obtain from Theorem 1.2.8 (c) an element $x \in G$ and a Sylow 2-subgroup T of K such that x induces a Galois automorphism on $\Omega_1(T)$ and moreover, such that there is an element $k \in K$ such that $x \cdot k$ and y induce the same automorphism on K. From Proposition

4.9.1 (d) of [24] we see that $x \cdot k$ is conjugate to x in a K-invariant group. In particular $C_K(y) \cong C_K(x \cdot k) \cong C_K(x)$. As x induces a Galois automorphism on $\Omega_1(T)$, the group $C_K(x)$ is of even order. It follows that $C_G(y)$ is of even order.
In the other case, if K is not normalised by y, then K^y is a component of G different from K. Consequently Lemma 1.1.18 (b) implies that $[K, K^y] = 1$. We further consider the map $\psi : K \to G, a \mapsto a \cdot a^y \cdot a^{y^2}$. From $[K, K^y] = 1$ we deduce that the map ψ is a homomorphism. From the Homomorphism Theorem we observe that $\{a \cdot a^y \cdot a^{y^2} \mid a \in K\} = im(\psi) \cong K/ker(\psi)$. For all $a \in K$ we have $(a \cdot a^y \cdot a^{y^2})^y = a^y \cdot a^{y^2} \cdot a = a \cdot a^y \cdot a^{y^2}$. Thus $im(\psi) \subseteq C_G(y)$. Since K is simple and $ker(\psi) \trianglelefteq K$, we have $im(\psi) \cong K$ or $im(\psi) = 1$. In the first case the assertion follows immediately. We assume for a contradiction that $im(\psi) = 1$. Then for all $a \in K$ we obtain that $a^{y^2} = (a \cdot a^y)^{-1} = a^{-1} \cdot (a^{-1})^y$. This shows that the component K^{y^2} is a subgroup of $K \times K^y$. From Lemma 1.1.18 (e) it follows that $K^{y^2} \in \{K, K^y\}$. This implies that $K = K^y$ which is a contradiction.
Let now $Z(K) \neq 1$. Then we have $3 \nmid |Z(K)|$ by Theorem 1.2.8 (a). Denote $Z(\langle K^G \rangle)$ with Z. Then we observe that Z is normal in $\langle K^G \rangle$ and y-invariant. In the factor group $\langle y, K^G \rangle / Z$ all assumptions of our lemma are fulfiled. As we showed above $C_{\langle y, K^G \rangle / Z}(Zy)$ has a section isomorphic to KZ/Z or Zy normalises KZ/Z and $C_{\langle y, K^G \rangle / Z}(Zy)$ is of even order. In the first case the assertion follows from $KZ/Z \cong K/(Z \cap K) = K/Z(K)$ and from Lemma 1.1.14 (a), in particular $C_{\langle y, K^G \rangle / Z}(Zy) = C_{\langle y, K^G \rangle / Z}(y) = C_{\langle y, K^G \rangle}(y)Z/Z$ is a section of $C_G(y)$.
In the second case y normalises $K \cdot Z$ and hence it normalises $E(K \cdot Z) = K$. Moreover from Lemma 1.1.14 (a) we see that $|C_{\langle y, K^G \rangle / Z}(Zy)|$ is a divisor of $|C_G(y)|$. Thus $C_G(y)$ is of even order too. □

1.2.11 Definition

Let G be a finite group. A subgroup H of G of even order is called **strongly embedded** in G if and only if $O^{2'}(G) \nleq H$ and for all involutions $c \in H$ we have $C_G(c) \leq H$.

1.2.12 Theorem

Let G be a quasi simple group. Suppose that G possesses a strongly embedded subgroup. Then there is a power q of 2 such that $G/Z(G)$ is isomorphic $\mathrm{PSL}(2, q)$, $Sz(q)$ or $\mathrm{PSU}(3, q)$. Moreover the following hold:

(a) If we have $G/Z(G) \cong \mathrm{PSL}(2, q)$, then $Z(G)$ is trivial or $q = 4$. In the case $q = 4$ the group $Z(G)$ has order at most 2.

(b) If we have $G/Z(G) \cong \mathrm{PSU}(3, q)$, then $Z(G)$ is cyclic of order 1 or 3.

(c) If G has a section isomorphic to a Suzuki group, then G is a Suzuki group.

(d) If G no $3'$-group and $G/Z(G) \ncong \mathrm{PSL}(2, 4)$ and if moreover $H \leq Aut(G)$ such that $Inn(G) \leq H$ and $s \in H$ is an involution, then we have $O(C_H(s)) = 1$.

(e) The group G has an elementary abelian Sylow 2-subgroup of order 4 if and only if $G \cong \mathrm{PSL}(2, 4)$.

(f) If G is isomorphic to $\mathrm{PSU}(3, 2^4) \cong U_3(4)$, then a Sylow 2-subgroup of G does not involve a Sylow 2-subgroup of a Suzuki group.

(g) Suppose that $G \cong \mathrm{PSU}(3, 2^4)$ and T be a Sylow 2-subgroup of G. If T is isomorphic Sylow 2-subgroup for a finite simple group H, then $N_G(T)/O(N_G(T)) \cong T \cdot \langle \beta \rangle$ where β is a fixed-point-free automorphism of T of order 15.

Proof
The first statement of the theorem is proven in [6].
Parts (a) and (b) are consequences of Lemma 4.2.8 of [25].

(c) This follows by the Theorem of Dickson 1.2.2 together with Theorem 6.5.3 of [24].

(d) In [27] the authors remark this statement directly under Definition 1. We want to convince ourselves.
Let G be of order divisible by 3 and $G/Z(G) \ncong \mathrm{PSL}(2,4)$. Suppose further that there is a subgroup $H \leq Aut(G)$ such that $Inn(G) \leq H$ and let s be an involution of H. As $|G|$ is no $3'$-group G is no Suzuki group. We remark at first that for a Sylow 2-subgroup T of G we have $C_H(T) = Z(T)$ (*) by Lemma 4.3.6 (a) of [25].

Assume that $G \cong \mathrm{PSL}(2,2^n)$ for some natural number $n \geq 3$. Then the group $Out(G)$ is an isomorphic image of the cyclic group of field automorphism of $\mathrm{PSL}(2,2^n)$ of order n by Lemma 4.3.1 (a) of [25].
If we have $s \in G$, then Lemma 4.3.4 (b) of [25] yields that $C_G(s)$ is a Sylow 2-subgroup T. Because of (*) and $T = O_2(C_G(s)) \leq O_2(C_H(s))$, we conclude that $O(C_H(s)) = 1$.
Suppose that $s \notin G$. Then Lemma 4.3.4 (c) of [25] yields that $C_G(s) \cong \mathrm{PSL}(2,2^{n/2})$. Since every field automorphism of odd order centralising s normalises $C_G(s)$ and induces a field automorphism in $C_G(s)$, we conclude that $O(C_H(s)) = 1$.

Assume now that $G \cong \mathrm{PSU}(3,2^n)$ for some natural number $n \geq 2$. Then we obtain from Lemma 4.3.1 (c) of [25] that $Out(G)$ is an extension of a cyclic group $\langle\beta\rangle$ of order $(3, 2^n+1)$ by the isomorphic image of the cyclic group $\langle\alpha\rangle$ of field automorphism of $\mathrm{PSU}(3,2^n)$ of order $2n$.
If we have $s \in G$, then Lemma 4.3.4 (b) of [25] yields that $C_G(s)$ is an extension of Sylow 2-subgroup T by a cyclic group of order $\frac{2^n+1}{(3,2^n+1)}$. As above we conclude that $O(C_H(s))$ is trivial, because of $T = O_2(C_G(s)) \leq O_2(C_H(s))$ and (*).
Let s be no element of G. Then we have $C_G(s) \cong \mathrm{PSL}(2,2^n)$ by Lemma 4.3.4 (c) of [25]. Moreover because of Lemma 4.3.4 (a) of [25] we may suppose that $s \in \langle\alpha\rangle$. Since s does not centralise β and every field automorphism of odd order centralising s normalises $C_G(s)$ and induces a field automorphism in $C_G(s)$, we conclude again $O(C_H(s)) = 1$.

(e) This follows by the Theorem of Dickson 1.2.2, Theorem 1.2.8 (d) and the fact that the Sylow 2-subgroups of $\mathrm{PSU}(3,q)$ for q a power of 2 have at least order 8, as in stated Section 3.6 of [39].

(f) If G is isomorphic to $U_3(4) \cong \mathrm{PSU}(3,2^4)$, then we know from [11] a Sylow 2-subgroup T of G has order $64 = 2^6$ and exactly 3 involutions. By Theorem 1.2.8 (d) a Sylow 2-subgroup S of a simple factor of a Suzuki group has order $\left(2^{2\cdot n+1}\right)^2$ for some $n \in \mathbb{N} \setminus \{0\}$. Suppose for a contradiction that T involves S. Then we observe $n = 1$ and $T = S$. By Theorem 1.2.8 (d) the group S contains seven involutions. This is a contradiction.

(g) This is Lemma 1 of [31]. □

1.2.13 Definition
If G is one of the groups in Theorem 1.2.12, then we call G a **Bender group**.

1.3 Miscellaneous

Here we conglomerate result that do not fit in any of the previous sections.
All the results stated here are quite elementary.

1.3.1 Lemma

Let G be a finite group and suppose that $x \in G$ acts coprimely on $H \leq G$.

(a) Suppose that H is a p-subgroup of G. If there exists a subgroup $K \leq [H,x]$ such that $[H,x] \leq \langle C_{[H,x]}(x), K\rangle$, then we have $[H,x] = K$.

(b) If there is an elementary abelian subgroup $A \leq C_G(x)$ acting also coprimely on H, then we have $[H,x] = \langle C_{[H,x]}(B) \mid B \; max \; A,\; C_G(B) \nsubseteq C_G(x)\rangle$.

Proof

We set $H_0 = [H,x]$.
Since x acts coprimely on H, Lemma 1.1.14 (d) yields $[H_0,x] = [H,x,x] = [H,x] = H_0$.

(a) The element x acts on $H_0/\phi(H_0)$. This group is elementary abelian by Lemma 1.1.4. From Lemma 1.1.14 (a) we observe that $C_{H_0/\phi(H_0)}(x) = C_{H_0}(x) \cdot \phi(H_0)/\phi(H_0)$.
Since the natural epimorphism $H_0 \to H_0/\phi(H_0)$ is a homomorphism it follows that $[H_0/\phi(H_0), x] = [H_0,x] \cdot \phi(H_0)/\phi(H_0) = H_0 \cdot \phi(H_0)/\phi(H_0) = H_0/\phi(H_0)$.

Applying Lemma 1.1.14 (f) we get

$$H_0/\phi(H_0) = [H_0/\phi(H_0), x] \times C_{H_0/\phi(H_0)}(x) = H_0/\phi(H_0) \times C_{H_0}(x) \cdot \phi(H_0)/\phi(H_0).$$

We deduce that $C_{H_0}(x) \leq \phi(H_0)$ and so $H_0 \leq \langle C_{[H,x]}(x), K\rangle = \langle \phi(H_0), K\rangle = \phi(H_0) \cdot K$.
Finally Lemma 1.1.4 implies the assertion.

(b) As A centralises x and normalises H, the group $H_0 = [H,x]$ is $\langle A, x\rangle$-invariant. From Lemma 1.1.14 (b) we obtain an $\langle A,x\rangle$-invariant Sylow q-subgroup Q of H, for all primes q dividing $|H|$. Part (e) of the same lemma shows that

$[Q,x] = \langle C_{[Q,x]}(B) \mid B \; max \; A\rangle \subseteq \langle C_{[Q,x]}(x), \langle C_{[Q,x]}(B) \mid B \; max \; A,\; C_G(B) \nsubseteq C_G(x)\rangle\rangle$.

From Part (a) of our lemma we deduce

$$\begin{aligned}[Q,x] &= \langle C_{[Q,x]}(B) \mid B \; max \; A,\; C_G(B) \nsubseteq C_G(x)\rangle \\ &\leq \langle C_{[H_0,x]}(B) \mid B \; max \; A,\; C_G(B) \nsubseteq C_G(x)\rangle.\end{aligned}$$

Finally Lemma 2.8 of [19] yields that

$$\begin{aligned}H_0 &= \langle [Q,x] \mid Q \text{ is } \langle x, A\rangle\text{-inv. Sylow subgr. of } H_0\rangle \\ &\subseteq \langle C_{[H_0,x]}(B) \mid B \; max \; A,\; C_G(B) \nsubseteq C_G(x)\rangle \\ &= \langle C_{[H,x]}(B) \mid B \; max \; A,\; C_G(B) \nsubseteq C_G(x)\rangle, \text{ as } [H_0,x] = H_0 = [H,x].\end{aligned}$$

□

1.3.2 Lemma

Let V be a finite vector space over $GF(2)$ and B be a basis of V. Then $W = \langle\{b+a \mid a,b \in B\}\rangle$ is the unique hyperplane of V with $W \cap B = \emptyset$.

Proof

Let $M = \{b + a \mid a,b \in B\}$. We define a map $\alpha : V \to GF(2)$ such that $v = \sum_{b\in B} \lambda_b \cdot b$ is mapped to $\sum_{b\in B} \lambda_b$ with $\{\lambda_b \mid b \in B\} \subseteq GF(2)$ suitable. Then α is linear and therefore $ker(\alpha)$ is a hyperplane of V. Since $GF(2)$ has characteristic 2, we conclude that $\langle M\rangle = ker(\alpha)$. From $b^\alpha = 1 \neq 0$ for all $b \in B$ we deduce that $\langle M\rangle \cap B = ker(\alpha) \cap B = \emptyset$.

If W is a hyperplane W of V with $W \cap B = \emptyset$, then we see that $|V/W| = 2$. This implies that $W + a = W + b \neq W$ for every $b \in B$. Hence $\langle B_0\rangle \leq \langle a + b \mid b \in B\rangle \leq W$ yields $W = \langle B_0\rangle$. □

1.3.3 Lemma

Let G be a finite group and suppose that V is a 2-subgroup of G such that $N_G(V)$ is not S_3-free. Then there exist an element y of order 3 and there is a 2-element b of $N_G(V)$ such that $\langle y, b\rangle/\langle b^2\rangle \cong S_3$.
If further V is abelian and the element y does not centralise the 2-group V, then there is an involution a of V such that $\langle a, y, b\rangle/\langle b^2\rangle \cong S_4$. Moreover $\langle a, y\rangle \cong A_4$.

Proof

Let A be a subgroup of $N_G(V)$ and $B \trianglelefteq A$ such that $A/B \cong S_3$. Furthermore let $R \in \mathrm{Syl}_3(A)$. Then we observe that $N_A(R) \neq C_A(R)$ and 2 divides $|N_A(R) : C_A(R)|$. Let $T \in \mathrm{Syl}_2(N_A(R))$ and let $b \in T$ be of minimal order such that $[b, R] \neq 1$. Then there is an element $y_0 \in R$ with $1 \neq [y_0, b] \in R$. The element b^2 centralises R and so y_0. Consequently we have

$$[y_0, b]^b = (y_0^{-1} \cdot y_0^b)^b = (y_0^{-1})^b \cdot y_0^{b^2} = (y_0^{-1})^b \cdot y_0 = [b, y_0] = [y_0, b]^{-1}.$$

Hence b inverts $\langle [y_0, b]\rangle \neq 1$. Let $y \in \langle [y_0, b]\rangle$ be of order 3. Then we have $\langle y, b\rangle/\langle b^2\rangle \cong S_3$, as asserted.
In addition we suppose now that y does not centralise the abelian group V. Then we observe that $1 \neq [V, \langle y\rangle] \leq V$. Moreover $[V, \langle y\rangle]$ is normalised by b, because V and $\langle y\rangle$ are b-invariant. Since V is abelian, Lemma 1.1.14 (f) shows that $[V, \langle y\rangle] \cap C_V(\langle y\rangle) = 1$. Moreover the 2-element b acts on the 2-group $[V, \langle y\rangle]$. This provides an involution $a_0 \in C_{[V,\langle y\rangle]}(b)$. The group $\langle a_0, a_0^y, a_0^{y^2}\rangle$ is a y-invariant subgroup of $[V, \langle y\rangle]$ and hence elementary abelian. Therefore $\langle a_0 \cdot a_0^y, a_0 \cdot a_0^{y^2}\rangle = \{1, a_0 \cdot a_0^y, a_0 \cdot a_0^{y^2}, a_0^y \cdot a_0^{y^2}\}$ is a y-invariant elementary abelian group of order 4 that is not centralised by y. This implies that for $a := a_0 \cdot a_0^y$ the group $\langle a, y\rangle$ is isomorphic to A_4. Furthermore we notice that

$$(a^y)^b = ((a_0 \cdot a_0^y)^y)^b = (a_0^y \cdot a_0^{y^2})^b = a_0^{yb} \cdot a_0^{y^2 b} = a_0^{by^b} \cdot a_0^{b(y^2)^b} = a_0^{y^2} \cdot a_0^y = a_0^y \cdot a_0^{y^2} = a^y$$
$$a^b = (a_0 \cdot a_0^y)^b = a_0^b \cdot a_0^{yb} = a_0 \cdot a_0^{by^b} = a_0 \cdot (a_0^b)^{y^2} = a_0 \cdot a_0^{y^2} = a_0^{y^2} \cdot a_0 = a^{y^2},$$
$$\text{and } (a^{y^2})^b = (a_0 \cdot a_0^{y^2})^b = a_0^b \cdot a_0^{y^2 b} = a_0 \cdot a_0^{b(y^2)^b} = a_0 \cdot a_0^y = a.$$

In particular $b^2 \in C_G(\langle a, a^y\rangle)$ and $\langle a, a^y\rangle$ is normalised by $\langle y, b\rangle$.
Altogether we see that $\langle a, y, b\rangle/\langle b^2\rangle \cong S_4$. □

1.3.4 Lemma

Let T be a finite 2-group and S be a self-centralising subgroup of T.

(a) If $S \cong V_4$, then T is dihedral or semidihedral.

(b) If $S \cong Q_8$, then T is semidihedral or a quaternion group.

Proof

(a) As S is self-centralising it, contains the non-trivial group $Z(T)$. Let $c \in Z(T)^\#$ and $a \in S \setminus \langle c\rangle$. Then we have that $S \leq C_T(a) = C_T(\langle a, c\rangle) = C_T(S) \leq S$, since S is elementary abelian of order 4. This shows that $C_T(a) = S$. Now the assertion follows from 5.3.10 of [30].

(b) Again we have $Z(T) \leq C_G(S) \leq S$. It follows from $1 \neq Z(T) \leq Z(S)$, that $Z(T)$ is of order 2. Let $- : T \to T/Z(T)$ be the natural epimorphism. We consider $C_{\bar{T}}(\bar{S})$. Let $t \in T$ such that $\bar{t} \in C_{\bar{T}}(\bar{S})$. Then we see that $t \in N_T(S \cdot Z(T)) = N_T(S)$.
Suppose for a contradiction that $t \notin S$. Then t induces a non-trivial outer automorphism on S. Every non-trivial element of the outer automorphism group of S permutes the three maximal subgroups of S non-trivially by 5.3.3 of [30]. Thus $\bar{t}$ permutes the involutions of $\bar{S}$ non-trivially. This is a contradiction.
Thus $t \in S$ and hence $\bar{S}$ is self-centralising in $\bar{T}$. Applying (a) we conclude that $\bar{T}$ is

dihedral or semidihedral.
For this reason $\bar{T}$ has exactly one central involution $\bar{c}$ and $\bar{c}$ is contained in $\bar{S}$. Thus its pre-images are elements of order 4. Moreover there is an element $\bar{g} \in \bar{T}$ such that $\langle \bar{g} \rangle$ is a maximal subgroup of $\bar{T}$ and there is some natural number n such that $\bar{g}^n = \bar{c}$. Hence g^n is a pre-image of $\bar{c}$ and has therefore order 4. It follows that g has order $2 \cdot o(\bar{g})$. This implies that T has a maximal cyclic subgroup. The fact that $\bar{T}$ is semidihedral or dihedral together with Theorem 1.2 of [8] finally leads to the assertion. □

1.3.5 Lemma

Let A be an extra-special group of order 27 and of exponent 3 and let Z denote $Z(A)$.
Then we have $C_{Aut(A)}(Z)/Inn(A) \cong \mathrm{SL}(2,3)$. If $\varphi \in Aut(A)$ is of order 3 and normalises every subgroup of order 9 of A, then φ centralises a subgroup of order 9 of A.

Proof
From [40] and II 9.12 of [29] we deduce that $C_{Aut(A)}(Z)/Inn(A) \cong \mathrm{Sp}(2,3) \cong \mathrm{SL}(2,3)$.
Let $\varphi \in Aut(A)$ have order 3 and normalise every subgroup of order 9 of A. Then φ normalises the characteristic subgroup Z of A. Hence the 3-automorphism φ centralises the cyclic group Z of order 3. Suppose further that a and b are elements of A such that $A = \langle a, b \rangle$ and neither $\langle a, Z \rangle$ nor $\langle b, Z \rangle$ is centralised by φ. As φ has order 3 and normalises $\langle b, Z \rangle$ and centralises Z, we may choose a and b such that $b^\varphi = b^a = b \cdot z$ for an element $z \in Z^\#$. In addition we have $a^\varphi = a \cdot z^i$ for a suitable $i \in \{1, 2\}$. It follows that

$$(a \cdot b^{3-i})^\varphi = a \cdot z^i \cdot (b \cdot z)^{3-i} \overset{z \in Z(A)}{=} a \cdot z^i \cdot b^{3-i} \cdot z^{3-i} = a \cdot b^{3-i} \cdot z^{i+3-i} = a \cdot b^{3-i}.$$

This implies that $\langle a \cdot b^{3-i}, Z \rangle$ is a subgroup of order 9 of A centralised by φ. □

1.3.6 Lemma

Let P be a 3-group of rank 2.
Then P has a characteristic non-cyclic subgroup of exponent 3 containing $\Omega_1(Z(P))$.
If R is a subgroup of exponent 3 of P, then R is cyclic, elementary abelian of order 9 or extra-special of order 27.

Proof
Suppose first that $r(Z(P)) \geq 2$. Then we conclude that $2 \leq r(Z(P)) \leq r(P) = 2$. Therefore $\Omega_1(Z(P))$ fulfils the first part of the conclusion of our lemma. Since $r(P) = 2$, we further have $\Omega_1(Z(P)) = \Omega_1(P)$. Hence every subgroup of exponent 3 is a subgroup of the elementary abelian group $\Omega_1(Z(P))$. Thus the lemma holds in this case.
Suppose now that $Z(P)$ is cyclic. Then P is non-abelian because of $r(P) = 2$. Consequently P contains an elementary abelian normal subgroup Y of order 9 by Lemma 1.4 of [8].
If Y is the unique elementary abelian normal subgroup of order 9 of P, then Y is characteristic in P and fulfils the first part of the assertion. From $Y \nleq Z(P)$ we obtain that $|P : C_P(Y)| = |N_P(Y) : C_P(Y)| = 3$. Moreover we have $Y = \Omega_1(C_P(Y))$, since P has rank 2. If R is a non-cyclic subgroup of exponent 3 of P and different from Y, then $R \nleq C_P(Y)$ and so $R \cdot C_P(Y) = P$. Further we have $C_R(Y) = C_P(Y) \cap R \leq \Omega_1(C_P(Y)) = Y$. Together this implies that $R/C_R(Y) = R/(C_P(Y) \cap R) \cong (R \cdot C_P(Y))/C_P(Y) = P/C_P(Y) \cong C_3$. For this reason we conclude that $|R| = 3 \cdot |C_R(Y)| \leq 3 \cdot |Y| = 27$. We deduce from the fact that R is of exponent 3 and rank 2 that R is elementary abelian of order 9 or non-abelian. In the second case Theorem 5.5.1 of [22] forces R to be extraspecial of order 27.
Suppose that there is another elementary abelian normal subgroup X of order 9 of P. Then $R := X \cdot Y$ is a normal subgroup of P and we have $|R| = |X \cdot Y| = \frac{|X| \cdot |Y|}{|X \cap Y|} = \frac{9 \cdot 9}{3} = 27$. Moreover R contains the 8 elements of order 3 of Y and as X is elementary abelian and different from Y there is at least one element of order 3 in $X \setminus Y \subseteq R \setminus Y$. The hypothesis on P having rank

2 forces the group R to be non-abelian. There exist exactly two non-abelian groups of order 27, the extra-special ones, by Theorem 5.5.1 of [22]. The group of order 27 of exponent 9 possesses exactly eight elements of order 3. Hence R is of exponent 3.
Suppose for a contradiction that there exists an element $y \in P \setminus R$ of order 3. Then $\langle R, y\rangle$ is a subgroup of P of order 81. Since P and hence $\langle R, y\rangle = R \cdot \langle y\rangle$ have rank 2, the element y induces a non-trivial automorphism on R and y normalises X and Y. Since R has exactly four maximal subgroups and y has order 3, the element y normalises every elementary abelian subgroup of order 9 of R and centralises $\langle x\rangle = Z(R)$. Now Lemma 1.3.5 provides an elementary abelian subgroup Q of P of order 9 that is centralised by y. Therefore $\langle y, Q\rangle$ is elementary abelian of order 27. This is contradicts $r(P) = 2$. Thus we have $R = \Omega_1(P)$ *char* P. Furthermore every subgroup of exponent 3 is a subgroup of the extra-special group $\Omega_1(P)$ of order 27 and hence it is cyclic elementary abelian of order 9 or $\Omega_1(P)$. □

1.3.7 Lemma

Let $p \in \{2, 3\}$ and H be a finite group with a normal p-complement.
Suppose that $P \in \mathrm{Syl}_p(H)$ and let $X \leq Z(P)$ act faithfully on $O_{p'}(H)$. Furthermore let σ denote the set of primes q such that $|H : C_H(X)|$ is not divisible by q. If we have $2 \in \sigma$, then the following hold:

(a) $[H, X, X] = [H, X] = [O_{p'}(H), X]$,

(b) $H = C_H(X) \cdot O(H) = C_H(X) \cdot [X, O(H)]$ and $[X, H] = [X, O(H)]$ and

(c) $H = C_H(X) \cdot O_{\pi'}(H) = C_H(X) \cdot [X, O_{\pi'}(H)]$ for all $\pi \subseteq \sigma$ with $p \in \pi$.

Proof

(a) Since H has a normal p-complement, we conclude that $H = O_{p'}(H) \cdot P$. The assumption that $X \leq Z(P)$ implies $[H, X] = [O_{p'}(H) \cdot P, X] = [O_{p'}(H), X]$. Finally Lemma 1.1.14 (d) yields

$$[H, X, X] = [[H, X], X] = [[O_{p'}(H), X], X] = [O_{p'}(H), X, X] = [O_{p'}(H), X] = [H, X].$$

Thus Part (a) holds.

(b) Let first p be 2. Then $H = P \cdot O(H)$, since H has a normal 2-complement. Consequently Lemma 1.1.14 (d) and the fact that $P \leq C_H(X)$ imply that

$$H = P \cdot O(H) \leq C_H(X) \cdot O(H) \leq C_H(X) \cdot C_{O(H)}(X) \cdot [O(H), X] = C_H(X) \cdot [O(H), X].$$

Furthermore $[X, H] = [X, O(H)]$ holds by (a).

Suppose now that $p = 3$. Then we obtain from Lemma 2.7 of [32] that

$$O_{3'}(H) = C_{O_{3'}(H)}(X) \cdot O(O_{3'}(H)) \leq C_H(X) \cdot O(H).$$

From $P \leq C_H(X)$ we deduce that $H = P \cdot O_{3'}(H) \leq C_H(X) \cdot O(H)$ and Lemma 1.1.14(d) shows that

$$O_{3'}(H) = C_{O_{3'}(H)}(X) \cdot [O_{3'}(H), X] \leq C_{O_{3'}(H)}(X) \cdot [C_H(X) \cdot O(H), X] \leq C_H(X) \cdot [O(H), X].$$

Consequently we also obtain $H = P \cdot O_{3'}(H) \leq C_H(X) \cdot [O(H), X]$. In particular we have $[H, X] = [C_H(X) \cdot [O(H), X], X] = [[O(H), X], X] \leq [O(H), X] \leq [H, X]$.

(c) For Part (c) we suppose that π is a subset of σ with $p \in \pi$. Moreover suppose that H is a minimal counterexample to $H = C_H(X) \cdot O_{\pi'}(H)$.
From Lemma 1.1.14 (d) an the fact that H has a normal p-complement we obtain that $H = C_H(X) \cdot O_{p'}(H) = C_H(X) \cdot [X, O_{p'}(H)]$. Thus the minimal choice of H implies that $H = [O_{p'}(H), X] \cdot X$. In particular we observe that X is a Sylow p-subgroup of H.

From $[O_{p'}(H), X] \leq [H, X] = [O(H), X] \leq O(H)$ by (b) we deduce that $[O_{p'}(H), X]$ is a normal subgroup of odd order of H with a p-factor group. Hence H is soluble by the Odd Order Theorem 1.1.12.
Suppose for a contradiction that $O_{\pi'}(H) \neq 1$. Then the minimality of H implies that the assertion is true for $H/O_{\pi'}(H)$ and $H/O_{\pi'}(H) = C_{H/O_{\pi'}(H)}(X) \cdot O_{\pi'}(H/O_{\pi'}(H))$. Since p is an element of π, Lemma 1.1.14 (a) implies that

$$H/O_{\pi'}(H) = C_{H/O_{\pi'}(H)}(X) \cdot O_{\pi'}(H/O_{\pi'}(H)) = C_{H/O_{\pi'}(H)}(X) = C_H(X) \cdot O_{\pi'}(H)/O_{\pi'}(H).$$

Altogether $H = C_H(X) \cdot O_{\pi'}(H)$ contradicting the choice of H as a counterexample.
Therefore the group $O_{\pi'}(H)$ is trivial and we conclude that $O_\pi(H) \geq F(H) \neq 1$. Suppose for a contradiction that the order $F(H)$ is divisible by p. Then we have $[O_p(H), O_{p'}(H)] \leq O_p(H) \cap O_{p'}(H) = 1$ and $1 \neq O_p(H) \leq X$. This contradicts the assumption that X acts faithfully on $O_{p'}(H)$.
This shows that X acts coprimely on $O_\pi(H)$ and we conclude from the minimal choice of H and Lemma 1.1.14 (a), that

$$\begin{aligned} H/O_\pi(H) &= C_{H/O_\pi(H)}(X) \cdot O_{\pi'}(H/O_\pi(H)) = (C_H(X) \cdot O_\pi(H))/O_\pi(H) \cdot (O_{\pi'}(H/O_\pi(H))) \\ &= (C_H(X) \cdot O_\pi(H))/O_\pi(H) \cdot O_{\pi,\pi'}(H)/O_\pi(H) = (C_H(X) \cdot O_{\pi,\pi'}(H))/O_\pi(H). \end{aligned}$$

Consequently we have $H = C_H(X) \cdot O_{\pi,\pi'}(H)$. Since X acts coprimely on $O_\pi(H)$, it acts coprimely on $O_{\pi,\pi'}(H)$. Therefore Lemma 1.1.14 (c) yields that $O_{\pi,\pi'}(H)$ has a X-invariant Hall π'-subgroup K. This implies that $O_{\pi,\pi'}(H) = O_\pi(H) \cdot K$. The assumption that $\pi \subseteq \sigma$ forces $O_\pi(H) \leq C_H(X)$.
We finally conclude that $[X, H] = [X, C_H(X) \cdot O_{\pi,\pi'}(H)] = [X, C_H(X) \cdot K] = [X, K] \leq K$. As we have $[X, H] \trianglelefteq H$ and $O_{\pi'}(H) = 1$, it follows that $[X, H] = 1$.
This is a contradiction. In conclusion Lemma 1.1.14 (d) yields that
$H = C_H(X) \cdot O_{\pi'}(H) = C_H(X) \cdot C_{[X,O_{\pi'}(H)]}(X) \cdot [X, O_{\pi'}(H)] = C_H(X) \cdot [X, O_{\pi'}(H)]$. □

Remark

The proof of Lemma 2.7 of [32] uses heavily the fact that $O_{3'}(H)$ is a 3'-group and that all its components are Suzuki groups (compare Theorem 1.2.8). This implies that the Lemma 1.3.7 is dependent on the primes 2 and 3.

1.3.8 Lemma

Let G be a finite S_4-free group with $O(G) = 1$ that has no normal 3-complement.
Suppose for every involution $t \in G$ that $C_G(t)$ is a $3'$-group. Then $F^*(G)$ is simple and no $3'$-group. Further $G = F^*(G) \cdot N_G(R)$ for a Sylow 3-subgroup R of $F^*(G)$ and $r_2(N_G(R)) \leq 1$. Moreover if t is an involution of G, then $C_G(t)/C_{F^*(G)}(t)$ is soluble.

Proof

It follows from the assumption that the centraliser of every 3-group is of odd order. (*)
Assume for a contradiction that $O_2(G) \neq 1$. Then Lemma 1.1.14 (e) forces the Sylow 3-subgroups of G to be cyclic. Since G is not 3-nilpotent, Burnside's p-Complement Theorem 1.1.10 provides a section of $G = N_G(O_2(G))$ isomorphic to S_3. But the elements of order 3 act non trivial on $Z(O_2(G))$, so Lemma 1.3.3 forces a contradiction to the assumption on G to be S_4-free.
From $O(G) = 1 = O_2(G)$ we deduce that $F^*(G) = E(G)$ is semi-simple. Moreover the group G contains a 3-element. Therefore Lemma 1.2.10 and (*) show that no component of G is a Suzuki group. In particular $F^*(G)$ is no $3'$-group. Since components have even order by the Odd Order Theorem 1.1.12 and since different components commute by Lemma 1.1.18 (b), the Statement (*) forces $F^*(G)$ to be simple.
Let R be a Sylow 3-subgroup of $F^*(G)$. Then we obtain from a Frattini argument that

$G = F^*(G) \cdot N_G(R)$. If $N_G(R)$ had a non-cyclic elementary abelian 2-subgroup, then Lemma 1.1.14 (e) and (*) would contradict each other. Thus we have $r_2(N_G(R)) \leq 1$. More precisely $N_G(R)$ has cyclic or quaternion Sylow 2-subgroups by Lemma 1.1.2. Consequently Theorem 1.2.8 (d) implies that $N_G(R)$ does not involve a Suzuki group. If t is an involution of G, then we have that

$$\begin{aligned} C_G(t)/C_{F^*(G)}(t) &= C_G(t)/(C_G(t) \cap F^*(G)) \cong C_G(t) \cdot F^*(G)/F^*(G) \leq G/F^*(G) \\ &= N_G(R) \cdot F^*(G)/F^*(G) \cong N_G(R)/N_{F^*(G)}(R) \end{aligned}$$

Since $C_G(t)$ is a $3'$-group and $N_G(R)$ does not involve a Suzuki group, $C_G(t)/C_{F^*(G)}(t)$ is soluble. □

1.3.9 Lemma

Let G be a finite group and t be an involution of G such that $C_{F(G)}(t)$ is Hall subgroup of $F(G)$. Then $\{g \in G \mid g^t = g^{-1} \text{ and } 2 \nmid o(g)\}$ is a subset of $C_G(F(G))$.
If in addition $F^*(G) = F(G)$, then $\{g \in G \mid g^t = g^{-1} \text{ and } 2 \nmid o(g)\} = [F(G), t]$.

Proof

As $C_{F(G)}(t)$ is a Hall subgroup of the nilpotent group $F(G)$, it is a characteristic subgroup of $F(G)$. Moreover $C_{F(G)}(t)$ has a unique complement K in $F(G)$, which is also characteristic in $F(G)$. Since $K \cap C_G(t) = 1$, the group K is inverted by t. Furthermore we observe that $K = [K, t] \leq [F(G), t] = [C_{F(G)}(t) \cdot K, t] = [K, t] = K$.
We fix elements $h \in \{g \in G \mid g^t = g^{-1} \text{ and } 2 \nmid o(g)\}$, $c \in C_{F(G)}(t)$ and $k \in K$. Then we have $c^h \in C_{F(G)}(t)$ and $k^h \in K$, since $C_{F(G)}(t)$ and K are characteristic subgroups of the normal subgroup $F(G)$ of G. It follows that

$$c^h = (c^h)^t = c^{ht} = c^{th^{-1}} = c^{h^{-1}} \text{ and } k^h = ((k^h)^t)^{-1} = (k^{th^{-1}})^{-1} = ((k^{-1})^{h^{-1}})^{-1} = k^{h^{-1}}.$$

Altogether we conclude that $h^2 \in C_G(F(G))$. Thus we have $h \in C_G(F(G))$, because h has odd order. In addition we assume now that $F^*(G) = F(G)$.
Then we have that $h \in C_G(F(G)) \leq F(G)$ by Lemma 1.1.18 (h) and hence

$$h \in \langle h^2 \rangle = \langle h \cdot (h^{-1})^{-1} \rangle = \langle h \cdot (h^{-1})^t \rangle = \langle [h^{-1}, t] \rangle \leq [F(G), t].$$

From $[F(G), t] = K \subseteq \{g \in G \mid g^t = g^{-1} \text{ and } 2 \nmid o(g)\}$ the assertion follows. □

2 Specific Preparatory Results

2.1 Balance

The notion of balance and signalizer functors was developed in the last century. In the literature there are several concepts of balance. The definition of a balanced group of Gorenstein and Walter in [27] differs for example with the concept given in Section F of [23].
To avoid any possible misunderstanding, we explicitly define a notion of balance in this section. Moreover we introduce signalizer functors as in [3].

2.1.1 Definition

Let G be a finite group and p be a prime.
For some elements $a \in G$ of order p let $\theta(a)$ be a p'-subgroup of $C_G(a)$.

(a) Suppose that $a \in G$ is an element of order p.

Then a is called **θ-balanced in G** if and only if for all $b \in C_G(a)$ of order p the group $\theta(b)$ is defined and normal in $C_G(b)$ and we have

$$\theta(b) \cap C_G(a) \leq \theta(a).$$

(b) The group G is called **θ-balanced** if and only if all elements of order p are θ-balanced in G.

(c) An elementary abelian non-cyclic p-subgroup A of G is said to be **θ-balanced in G** if and only if for all elements $a \in A^{\#}$ the group $\theta(a)$ is defined and A-invariant and such that for all $a, b \in A^{\#}$ the following holds:

$$\theta(b) \cap C_G(a) \leq \theta(a).$$

2.1.2 Remark

If $p = 2$ and for all involutions a in G we have $\theta(a) = O(C_G(a))$, then we omit the θ and say that a is balanced in G instead of a is θ-balanced. Respectively we say that G is balanced and A is balanced in G in this case.

2.1.3 Lemma

Let G be a finite group and p be a prime.
For all elements b of order p in G we set $\theta(b) := O_{p'}(C_G(b))$. If $b \in G$ is an element of order p such that $C_G(b)$ is p-constrained, then b is θ-balanced in G.

Proof
We set $H := C_G(b)$ and assume that $a \in H$ is an element of order p. We suppose further that $D := O_{p'}(C_G(a)) \cap H$ and let $- : H \to H/O_{p'}(H)$ denote the natural epimorphism.
Then $\bar{D}$ acts on $O_p(\bar{H})$ and Lemma 1.1.14 (a) yields

$$[C_{O_p(\bar{H})}(\bar{a}), \bar{D}] = [\overline{C_{O_{p',p}(H)}(a)}, \bar{D}] = \overline{[C_{O_{p',p}(H)}(a), D]} \leq \overline{O_{p',p}(H) \cap D} = 1.$$

We remark that $\bar{D}$ moreover centralises $\bar{a}$ to obtain that $\bar{D}$ centralises the subnormal subgroup $C_{\langle\bar{a}\rangle \cdot O_p(\bar{H})}(\bar{a})$ of the $\bar{D}$-invariant group $\langle\bar{a}\rangle \cdot O_p(\bar{H})$. Hence Lemma 1.1.14 (g) implies that $\bar{D}$ acts trivially on $O_p(\bar{H})$. Since H is p-constrained and D has p'-order, this leads to $D \leq O_{p'}(C_G(b))$. □

2.1.4 Lemma

Let G be a finite group and suppose that b is an involution in G such that the only non-abelian composition factors of $C_G(b)$ are Suzuki groups. Then b is balanced in G.

Proof

This is Lemma 1.6. of [36]. □

2.1.5 Definition

Let G be a finite group, let p be a prime and suppose that A is an elementary abelian p-subgroup of G.

(a) An **A-signalizer functor** on G is a map θ from $A^\#$ into the set of all A-invariant p'-subgroups of G such that A is θ-balanced in G.

(b) An A-signalizer functor on G is **complete** if and only if there exists an A-invariant p'-subgroup $\theta(G)$ of G such that $\theta(a) = C_{\theta(G)}(a)$ for all $a \in A^\#$.

(c) An A-signalizer functor on G is **soluble** if and only if $\theta(a)$ is soluble for all $a \in A^\#$.

(d) A soluble A-signalizer functor on G is **solubly complete** if and only if θ is complete and $\theta(G)$ is soluble.

2.1.6 Theorem (Soluble Signalizer Functor Theorem)

Let G be a finite group, p be a prime and suppose that A is an elementary abelian p-subgroup of G of order at least p^3.
Then each soluble A-signalizer functor θ on G is solubly complete.
In particular $\langle \theta(a) \mid a \in A^\# \rangle$ is a soluble p'-group.

Proof

This can be found in Chapter 15 of [3]. □

2.1.7 Lemma

Let G be a finite group such that $E(G)$ is quasisimple and $F(G) = O_2(G)$ has at most one involution. Suppose that B_1 and B_2 are non-cyclic elementary abelian 2-subgroups of G such that $B_1 \cap Z(G) = 1 = B_2 \cap Z(G)$ and $[B_1, B_2] \leq Z(G)$.
Furthermore assume that for every involution b of B_1 and B_2 the group $C_G(b)$ is 3-soluble.
Then we have $\langle O(C_{G/Z(G)}(Z(G)b)) \mid b \in B_1^\# \rangle = \langle O(C_{G/Z(G)}(Z(G)b)) \mid b \in B_2^\# \rangle$.

Proof

Let $- : G \to G/Z(G)$ denote the natural epimorphism and set $\langle z \rangle := \Omega_1(Z(G))$.
For all $b \in B_1^\#$ we observe $\bar{B}_2 \leq C_{\bar{G}}(\bar{b})$ from $[B_1, B_2] \leq Z(G)$. Thus $\bar{B}_2$ acts coprimely on $O(C_{\bar{G}}(\bar{b}))$. From $B_2 \cap Z(G) = 1$ we obtain that $\bar{B}_2$ is non-cyclic and as B_2 is elementary abelian, we see that also $\bar{B}_2$ is elementary abelian. Thus Lemma 1.1.14 (e) yields $O(C_{\bar{G}}(\bar{b})) = \langle C_{O(C_{\bar{G}}(\bar{b}))}(\bar{c}) \mid \bar{c} \in \bar{B}_2^\# \rangle = \langle C_{O(C_{\bar{G}}(\bar{b}))}(\bar{c}) \mid c \in B_2^\# \rangle$.
Moreover for all $a \in B_1^\# \cup B_2^\#$ the group $N_G(\langle a, Z(G) \rangle) = N_G(\Omega_1(\langle a, Z(G) \rangle)) = N_G(\langle a, z \rangle)$ is the full pre-image of $C_{\bar{G}}(\bar{a})$ in G. Since $\langle a, z \rangle$ is elementary abelian of order 4 and $z \in Z(G)$, it follows that $C_{\bar{G}}(\bar{a})/\overline{C_G(a)} \cong N_G(\langle a, z \rangle)/C_G(a) = N_G(\langle a, z \rangle)/C_G(\langle a, z \rangle) \lesssim Z_2$. In particular the assumption that $C_G(a)$ is 3-soluble implies that $C_{\bar{G}}(\bar{a})$ is 3-soluble.
Consequently Lemma 2.1.4 forces the involutions of $\bar{B}_1$ and $\bar{B}_2$ to be balanced in $\bar{G}$. More precisely, for all $b \in B_1^\#$ and $c \in B_2^\#$ we have $C_{O(C_{\bar{G}}(\bar{b}))}(\bar{c}) = C_{\bar{G}}(\bar{c}) \cap O(C_{\bar{G}}(\bar{b})) \leq O(C_{\bar{G}}(\bar{c}))$.
This shows that

$$\begin{aligned} \langle O(C_{\bar{G}}(\bar{b})) \mid b \in B_1^\# \rangle &= \langle \langle C_{O(C_{\bar{G}}(\bar{b}))}(\bar{c}) \mid c \in B_2^\# \rangle \mid b \in B_1^\# \rangle \\ &\leq \langle \langle O(C_{\bar{G}}(\bar{c})) \mid c \in B_2^\# \rangle \mid b \in B_1^\# \rangle = \langle O(C_{\bar{G}}(\bar{c})) \mid c \in B_2^\# \rangle \end{aligned}$$

As all conditions on B_1 and B_2 are symmetric, we also obtain the other inclusion. □

2.2 Strongly Closed Abelian Subgroups

In this section we want to become acquainted with the concept of strongly closed subgroups. The strength of this concept will arise in the following section.
Our definition of a strongly closed subgroup differs slightly from the general literature but the concept is the same.

2.2.1 Definition

Let G be a finite group, let p be a prime and suppose that $S \in Syl_p(G)$.
A subgroup A of S is **strongly closed in G with respect to S** if and only if, whenever we have $g \in G$, then $A^g \cap S \leq A$ holds.

2.2.2 Lemma

Let G be a finite group, let p be a prime and suppose that $S \in Syl_p(G)$.
Furthermore assume that $A \leq S$ is elementary abelian and strongly closed in G with respect to S. Then the following hold:

(a) For all subgroups S_0 of S with $A \leq S_0$ we have $N_G(S_0) \leq N_G(A)$.

(b) For all $S_0 \in Syl_p(G)$ such that $A \leq S_0$ the group A is strongly closed in G with respect to S_0.

(c) For all $U \leq G$ such that $S \cap U \in Syl_p(U)$ the group $A \cap U$ is strongly closed in U with respect to $S \cap U$.

(d) If we have $A \leq U \leq G$, then there exists a Sylow p-subgroup S_0 of U such that A is strongly closed in U with respect to S_0.

(e) The group $N_G(A)$ controls the fusion of its p-elements.

(f) For all $N \trianglelefteq G$ the group $A \cdot N/N$ is strongly closed in G/N with respect to $S \cdot N/N$.

(g) We have that $G = N_G(A) \cdot \langle A^G \rangle$ holds.

(h) If we have $N \leq O_p(G)$, then $A \cap N$ is normal in G.

(i) Let N be a normal p'-subgroup of G and suppose that G/N has a subgroup B/N that is strongly closed in G/N with respect to SN/N. If B is the full pre-image of B/N, then $B \cap S$ is strongly closed in G with respect to S.

(j) For all $B \leq A$ with $N_G(B) \geq N_G(A)$ the group B is strongly closed in G with respect to S.

(k) For all $B \leq A$ the group $\langle B^{N_G(A)} \rangle$ is strongly closed in G with respect to S.

(l) If $Z \leq Z(G)$ is a p-subgroup of G, then $A \cdot Z$ is strongly closed in G with respect to S.

Proof

(a) Suppose that $S_0 \leq S$ with $A \leq S_0$ and let g be an element of $N_G(S_0)$.
Then we have $A^g \leq S_0 \leq S$. Since A is strongly closed in G with respect to S, it follows that $A^g = A^g \cap S \leq A$. Thus we have that $g \in N_G(A)$.

(b) Let $S_0 \in Syl_p(G)$. Then Sylow's Theorem provides an element $h \in S_0$ such that $S_0^h = S$ and we have $A^h \leq S_0^h \leq S$. Since A is strongly closed in G with respect to S, we see that $A^h = A^h \cap S \leq A$ and consequently h is an element of $N_G(A)$. Let $g \in G$

and $a \in A$ such that $a^g \in S_0$. Then we have $a^{g \cdot h} \in S_0^h = S$. The assumption that A is strongly closed in G with respect to S forces $a^{g \cdot h} \in A$. As $h \in N_G(A)$, we finally see that $a^g = (a^{g \cdot h})^{h^{-1}} \in A$.

(c) Suppose that $U \leq G$ with $A \leq U$. Let $u \in U$ and $a \in A \cap U$ such that $a^u \in S \cap U$. Then we observe that $a^u \in S$ and, as A is strongly closed in G with respect to S, we conclude that $a^u \in A$. Altogether $a^u \in A \cap S \cap U \leq A \cap U$.

(d) Let S_0 be a Sylow p-subgroup of $U \leq G$ containing A. Suppose that $S_1 \in Syl_p(G)$ such that $S_0 \leq S_1$. By (b) the group A is strongly closed in G with respect to S_1. Finally $U \cap S_1 = S_0 \in Syl_p(U)$ implies together with (c) the assertion.

(e) By Theorem 6.1 of [16] the group A controls strong fusion in S with respect to G. Moreover every p-element of $N_G(A)$ is contained in a Sylow p-subgroup of $N_G(A)$ by Sylow's Theorem. Thus the assertion follows from (b).

(f) Let N be a normal subgroup of G. Suppose that $g \in G$ and $a \in A$ are elements of G such that $(Na)^g \in S \cdot N/N$. Then we have $a^g \in S \cdot N$. Since S is a Sylow p-subgroup of $S \cdot N$ and a^g is a p-element, Sylow's Theorem provides an element $h \in N$ with $a^{g \cdot h} \in S$. From the assumption that A is strongly closed in G with respect to S we deduce that $a^{g \cdot h} \in A$. Hence we have $a^{g \cdot h} \in A \cdot N$ and consequently we obtain that $(Na)^g = (Na)^{g \cdot h} \in A \cdot N/N$.

(g) Let $S_0 \in Syl_p(\langle A^G \rangle)$ such that $A \leq S_0$. From (d) and (b) we conclude that A is strongly closed in $\langle A^G \rangle$ with respect to S_0. Thus (a) implies that $N_G(S_0) \leq N_G(A)$. Finally a Frattini arguments shows that $G = \langle A^G \rangle \cdot N_G(S_0) \leq \langle A^G \rangle \cdot N_G(A)$.

(h) Let $N \leq O_p(G)$. Then we observe that $N \leq S$. This implies for all $g \in G$ and $a \in A \cap N$ that $a^g \in N \leq S$. Again the assumption that A is strongly closed in G with respect to S implies $a^g \in A$. Altogether we have that $a^g \in A \cap N$.

(i) Let B be the full pre-image of B/N. Since B/N is a subgroup of SN/N, Lemma 1.1.5 yields $B = B \cap SN = (B \cap S)N$. Let $g \in G$ and $b \in B \cap S$ such that $b^g \in S$. Then we see that $Nb \in B/N$ and $Nb^g \in SN/N$. As B/N is strongly closed in G/N with respect to $S \cdot N/N$, it follows that $b^g \in BN = B$. This shows that $b^g \in B \cap S$.

(j) Suppose that $B \leq A$ with $N_G(B) \geq N_G(A)$. Let $g \in G$ and $b \in B$ such that $b^g \in S$. Since $N_G(A)$ controls by (e) the fusion of S, there is an element $h \in N_G(A) \leq N_G(B)$ such that $b^g = b^h$. Thus $b^g = b^h \in B$ holds.

(k) We have $\langle B^{N_G(A)} \rangle \leq A$ and $N_G(A) \leq N_G(\langle B^{N_G(A)} \rangle)$. The assertion follows from (j).

(l) Let $Z \leq Z(G)$ be a p-subgroup of G. Then Z is a subgroup of S. Suppose that there are elements $g \in G$ and $c \in A \cdot Z$ such that $c^g \in S$. Then there is an element $a \in A$ and an element $z \in Z$ such that $c = a \cdot z$ and we obtain $a^g = a^g \cdot z^{-1} \cdot z = a^g \cdot (z^{-1})^g \cdot z = c^g \cdot z \in S$. Since A is strongly closed in G with respect to S, it follows that $a^g \in A$. This shows that $c^g = a^g \cdot z^g = a^g \cdot z \in A \cdot Z$. □

2.2.3 Remark

Part (b) of Lemma 2.2.2 shows that the property of being a strongly closed elementary abelian p-subgroup of a finite group with respect to a Sylow subgroup does not depend on the choice of the Sylow subgroup. For this reason in the remainder of this thesis we omit the "respect"-part and say that an elementary abelian p-subgroup A of a finite group G is strongly closed in G, if it is strongly closed in G with respect to one and therefore all Sylow p-subgroups of G containing A.

2.2.4 Proposition (Glauberman)

Let G be S_4-free and suppose that every composition factor of every 2-constrained section of G is a $3'$-group or abelian. Let further T be a Sylow 2-subgroup of G.
Then $\langle \Omega_1(Z(T))^{N_G(J(T))} \rangle$ is a strongly closed elementary abelian 2-subgroup of G.

Proof
This is a consequence of Proposition II 6.1 of [18]. □

2.2.5 Proposition (Goldschmidt)

Let A be an elementary abelian 2-subgroup of a 3-soluble group G. Suppose that A is strongly closed in G.
Furthermore let T be a Sylow 2-subgroup of $O^2(\langle A^G \rangle)$ containing $A \cap O^2(\langle A^G \rangle)$.
Then $\overline{\langle A^G \rangle} := \langle A^G \rangle / O(\langle A^G \rangle)$ is a central product of an elementary abelian 2-subgroup and Suzuki groups with possibly trivial factors. Moreover we have $\overline{A} = O_2(\overline{\langle A^G \rangle}) \cdot \overline{\Omega_1(T)}$.

Proof *Compare with (4.2) of [20].*
Let G be a minimal counterexample and let A be an elementary abelian strongly closed 2-subgroup of G and T be a Sylow 2-subgroup of $O^2(\langle A^G \rangle)$ containing $A \cap O^2(\langle A^G \rangle)$. We choose A of minimal order such that our proposition false for A and G. Then A is non-trivial.

(1) We have $O(G) = 1$, $G = \langle A^G \rangle$ and $Z(G) = 1$.

Proof. By Lemma 2.2.2 (f) the group $A \cdot O(G)/O(G)$ is strongly closed in $G/O(G)$. From $O(\langle A^G \rangle) \leq O(G)$ and the minimal choice of G we deduce that $O(G)$ is trivial.
Moreover we see that $\langle A^G \rangle = \langle A^{N_G(A) \cdot \langle A^G \rangle} \rangle = \langle A^{\langle A^G \rangle} \rangle$ by Lemma 2.2.2 (g). The strong closure of A in $\langle A^G \rangle$ by Lemma 2.2.2 (d) and the minimal choice of G imply that $\langle A^G \rangle = G$.
Let now $- : G \to G/Z(G)$ be the natural epimorphism and let U denote the full pre-image of $O(\bar{G})$. Then U has a central Sylow 2-subgroup and Burnside's p-Complement Theorem 1.1.10 implies that U has a normal 2-complement. We observe that $O(U)$ char $U \trianglelefteq G$ to conclude that $O(U) = 1$ by $O(G) = 1$. This shows that $U = Z(G) \leq O_2(G)$. Suppose for a contradiction that $Z(G) \neq 1$. Then the minimal choice of G and the fact that $\bar{A}$ is strongly closed in $\bar{G}$ by Lemma 2.2.2 (f) imply that our proposition holds for $\bar{G}$ and $\bar{A}$.
From $O(\bar{G}) = 1$ it follows that $O(\langle \bar{A}^{\bar{G}} \rangle) = 1$. In particular $\langle \bar{A}^{\bar{G}} \rangle = \overline{\langle A^G \rangle}$ is a central product of an elementary abelian 2-subgroup and Suzuki groups. This implies that $\langle A^G \rangle \cdot Z(G)$ is a central product of an abelian 2-subgroup and Suzuki groups. But A is elementary abelian and $Z(G)$ centralises G. Consequently $\langle A^G \rangle$ is a central product of an elementary abelian 2-subgroup and Suzuki groups. Moreover A intersects each of this factors non-trivially and the intersection is strongly closed in the factor by Lemma 2.2.2 (c). From Theorem 1.2.8 (h) it follows that $A = O_2(\langle A^G \rangle) \cdot \Omega_1(T)$. This is a contradiction, as G is a counterexample. We conclude that $Z(G) = 1$. ■

(2) We have $F(G) = 1$ and $E(G) \neq 1$. In particular every component of G is simple.

Proof. Suppose for a contradiction that $O_2(G) \neq 1$. Then the 2-group $O_2(G)$ normalises the 2-group A by Lemma 2.2.2 (a) and hence $C := C_{O_2(G)}(A) \neq 1$. Let $g \in G$ and $c \in C$. Then we have $c^g \in O_2(G) \leq N_G(A)$. Consequently Lemma 2.2.2 (e) provides an element $h \in N_G(A)$ such that $c^g = c^h$. Since $N_G(A)$ normalises C, we conclude that $c^g = c^h \in C$. This shows that C is a normal subgroup of G. Moreover for every $g \in G$ we have $[A^g, C] = [A, C]^g = 1$. Altogether we have $G = \langle A^G \rangle \leq C_G(C)$ and hence $1 \neq C \leq Z(G) = 1$ by (1). This is a contradiction.
From $O_2(G) = 1 = O(G)$ by (1) we deduce that $F(G) = 1$ and therefore we have $E(G) \neq 1$ and every component of G is simple. ■

(3) All components of G are normal in G.

Proof. Let $E := E(G)$ and set $E_1 := \langle (A \cap E)^E \rangle$. Suppose that K is a component of E_1. Then we have $A \cap K \neq 1 = Z(K)$ and Lemma 1.1.18 (d) shows that A normalises K.
Let E_0 be the product of all components of G which are not contained in E_1. Then E_0 is A-invariant. Let T_0 be a Sylow 2-subgroup of E_0 that is normalised by A. Then $\langle A, T_0 \rangle$ is a 2-subgroup of G. From Lemma 2.2.2 (a) we deduce that $[A, T_0] \leq T_0 \cap A \leq E_0 \cap A = 1$. We apply again Lemma 1.1.18 (d) to observe that A normalises every component of E_0. Altogether A normalises every component of G. Because of $G = \langle A^G \rangle$, it follows that every component of G is normal G. ■

(4) We have $A \leq E(G)$.

Proof. Let K be a component of G. Then A normalises K by (3). From the fact that G is 3-soluble, Theorem 1.2.8 and Part (c) of the same theorem we conclude that A induces inner automorphisms in K. It follows that $A \leq K \cdot C_G(K)$ for every component K of G. This implies that $A \leq E(G) \cdot C_G(E(G))$. From (2) we deduce that $E(G) = F^*(G)$ and Lemma 1.1.18 (h) yields that $A \leq E(G) \cdot C_G(E(G)) \leq F^*(G) \cdot C_G(F^*(G)) \leq F^*(G) = E(G)$. ■

Finally from Theorem 1.2.8 we deduce that the 3-soluble group $G = \langle A^G \rangle = E(G)$ is a central product of Suzuki groups. Moreover A intersects each of the components non-trivially and the intersection is strongly closed in the component by Lemma 2.2.2 (c). From Theorem 1.2.8 (h) it follows that $A = \Omega_1(T)$. This contradiction, as G is a counterexample.□

2.2.6 Lemma (Goldschmidt)

Let G be a finite simple group and $T \in Syl_2(G)$. If $A \leq T$ is a strongly closed elementary abelian subgroup of G, then we have $G = \langle C_G(a) \mid a \in A^\# \rangle$ or G is a Bender group.

Proof
This is (4.4) of [20]. □

2.3 Finite Groups with many Involutions having a 3-Soluble Centraliser

In [20] Goldschmidt showed that the appearance of a strongly closed abelian 2-subgroup has a strong influence on the structure of finite groups. In this section we capture and develop his ideas in finite groups where the centralisers of almost all involutions are 3-soluble.

2.3.1 Definition

Let G be a finite group and p be a prime.
A p-subgroup A of G is **minimal strongly closed in** G if and only if A is strongly closed in G and A has no proper non-trivial subgroup which is strongly closed in G.

2.3.2 Lemma

Let G be a finite group with a non-cyclic elementary abelian subgroup A that is minimal strongly closed in G. For $|A| = 16$ let $N_G(A)$ be S_4-free. Suppose that W is a subgroup of G of odd order and normalised by $\Gamma^* := \langle N_G(B) \mid B \leq A \text{ and } r(B) \geq 2 \rangle$.
Furthermore assume that $a \in A^\#$ is such that $C_G(a)$ is 3-soluble and $O(\langle A^{C_G(a)} \rangle) \leq W$.
Then $C_G(a)$ is contained in $W \cdot \Gamma^*$.

Proof
For all $b \in A^\#$ we set $C_b := C_G(b)$.
By Lemma 2.2.2 (d) the group A is strongly closed in the 3-soluble group C_a. (*)
Suppose for a contradiction that $C_a \nsubseteq W \cdot \Gamma^*$.

(1) There is a $n \in \mathbb{N}$ such that $\langle A^{C_a} \rangle / O(\langle A^{C_a} \rangle) \cong C * Sz(2^{2n+1})$ and C is cyclic of order 2.

Proof. Since A is not cyclic, it follows that $N_G(A) \leq \Gamma^*$. The Statement (*), Proposition 2.2.5 and Lemma 2.2.2 (g) imply that $C_a = N_{C_a}(A) \cdot \langle A^{C_a} \rangle$ and $\langle A^{C_a} \rangle / O(\langle A^{C_a} \rangle)$ is a central product of an elementary abelian 2-subgroup and Suzuki groups.
Let $- : C_a \to C_a / O(\overline{\langle A^{C_a} \rangle)}$ be the natural epimorphism and let S denote a Sylow 2-subgroup of a pre-image of $O_2(\overline{\langle A^{C_a} \rangle})$ in C_a. By Proposition 2.2.5 we may choose S such that $S \leq A$. Suppose for a contradiction that S has an elementary abelian subgroup of order 4. Then we observe $N_G(S) \leq \Gamma^*$. Further a Frattini argument leads to
$C_a = N_{C_a}(A) \cdot \langle A^{C_a} \rangle = N_{C_a}(A) \cdot N_{\langle A^{C_a} \rangle}(S) \cdot O_{2',2}(\langle A^{C_a} \rangle) \subseteq N_G(A) \cdot N_G(S) \cdot O(\langle A^{C_a} \rangle) \subseteq \Gamma^* \cdot W$.
This is a contradiction. From $a \in S$ we conclude that $S = \langle a \rangle$. It follows that $\overline{\langle A^{C_a} \rangle}$ is a central product of $\langle \bar{a} \rangle$ and Suzuki groups.
From $N_{C_a}(A) \cdot O(\langle A^{C_a} \rangle) \leq W \cdot \Gamma^*$ we obtain a component of $\overline{\langle A^{C_a} \rangle}$, which we denote by $\bar{L}$. Let S_1 be a Sylow 2-subgroup of a pre-image of $\bar{L}$. Again by Proposition 2.2.5 we may choose S_1 such that $\Omega_1(S_1) \leq A$. Then S_1 is not of rank 1. Thus $N_G(S_1) \leq N_G(\Omega_1(S_1)) \leq \Gamma^*$. In addition a Frattini argument yields:

$$C_a = N_{C_a}(A) \cdot \langle A^{C_a} \rangle = N_{C_a}(A) \cdot N_{C_a}(S_1) \cdot L \cdot O(\langle A^{C_a} \rangle) \subseteq \Gamma^* \cdot W \cdot L.$$

Suppose for a contradiction that $\overline{\langle A^{C_a} \rangle}$ has a second component $\bar{K}$. Then we similarly conclude that $C_a \subseteq \Gamma^* \cdot W \cdot K$ holds. Furthermore Lemma 1.1.14 (a) and Lemma 1.1.18 (b) lead to $\bar{K} \leq C_{\bar{C}_a}(\bar{L}) \leq C_{\bar{C}_a}(\bar{S}_1) = \overline{C_{C_a}(S_1)}$. This implies $K \leq C_{C_a}(S_1) \cdot O(\langle A^{C_a} \rangle) \subseteq \Gamma^* \cdot W$. Altogether $C_a \subseteq \Gamma^* \cdot W \cdot K \subseteq \Gamma^* \cdot Q$ is a contradiction. ∎

Let K be the full pre-image of $E(\langle A^{C_a} \rangle / O(\langle A^{C_a} \rangle))$.

(2) $E(\langle A^{C_a} \rangle / O(\langle A^{C_a} \rangle))$ is simple and $K \cap A$ is the centre of a Sylow 2-subgroup of K.

Proof. Suppose for a contradiction that $E(\langle A^{C_a} \rangle / O(\langle A^{C_a} \rangle))$ is not simple then (1) and Theorem 1.2.8 (a), (d) and (c) imply that A has order 16 and $N_G(A)$ is divisible by 7. Since A is minimal strongly closed in G, Lemma 2.2.2 (j) yields that $N_G(A)$ acts irreducibly on A. From the assumption $N_G(A)$ is S_4-free in this case we obtain a contradiction with Lemma 1.2.6. Consequently $E(\langle A^{C_a} \rangle / O(\langle A^{C_a} \rangle))$ is simple.
Moreover $K \cap A$ is strongly closed in K by Lemma 2.2.2 (d). Finally Theorem 1.2.8 (h) forces $K \cap A$ to be the centre of some Sylow 2-subgroup of K. ∎

(3) The group $N_G(A)$ acts transitively on $A^\#$.

Proof. Since $\langle a \rangle < A$ is not strongly closed in G, Lemma 2.2.2 (j) provides an element $g \in N_G(A) \leq W \cdot \Gamma^*$ such that $a \neq a^g \in A$. Hence we have $C_a \cong (C_a)^g = C_{a^g}$ and $C_{a^g} \nleq W \cdot \Gamma^*$. Moreover (1) and (2) yield that $a \notin K$. We set $B := K \cap B$. By (2) and Theorem 1.2.8 (f) all involution of B are conjugate. Thus $\{a\}$, $B^\#$ and $a \cdot B \setminus \{a\}$ are exactly the conjugacy classes of A in $N_{C_a}(A)$. Consequently $a^g \in B^\#$ or $a^g \in a \cdot B \setminus \{a\}$.
Suppose for a contradiction (3) is false and let $C := a^G$. If we have $a^g \in B^\#$, then $\{a\} \cup B^\#$ is contained in C. More precisely we obtain that $(a \cdot B \setminus \{a\}) \cap C = \emptyset$. The groups B and B^g are maximal subgroups of A. Moreover, as A has order at least 16, we conclude that $B \cap B^g \neq 1$. Further we obtain that all involutions of B^g are conjugate in G from the fact that all involutions of B are conjugate in G. This implies that $(B^g)^\# \subseteq C$. Since $a^g \in B \subseteq C$, it follows that $\{a\} \cup B^\# = C \cap A = \{a^g\} \cup (B^g)^\#$. Thus we conclude $a \cdot B \setminus \{a\} = A^\# \setminus C = a^g \cdot B^g \setminus \{a^g\}$.

Let finally $1 \neq c \in B \cap B^g$. Then we have $a^g \cdot c \in B \cap (a^g \cdot B^g \setminus \{a^g\}) = B \cap (a \cdot B \setminus \{a\})$. This implies the contradiction $(a \cdot B \setminus \{a\}) \cap C \neq \emptyset$.
Consequently we obtain that $a^g \notin B^\#$ and hence $\{a\} \cup (a \cdot B \setminus \{a\})$ is contained in C and $B^\# \cap C = \emptyset$. Thus we obtain that $N_G(B) \geq N_G(A)$. Finally Lemma 2.2.2 (j) yields that B is strongly closed in G this is a contradiction. ■

(4) The group $N_G(A)/C_G(A)$ is of odd order.

Proof. Let t be a 2-element such that $t \in N_G(A)$. Since A is a 2-group, there exists an element $b \in A^\#$ such that $t \in C_b$. By (3) we may assume that $b = a$. Then t normalises K by (1). The outer automorphism group of Suzuki groups is of odd order by Theorem 1.2.8 (c). Moreover (2) yields that the elementary abelian group $K \cap A$ is contained in the centre of a Sylow 2-subgroup of K. Thus we conclude that $O(K)t \in C_{C_a/O(K)}(K \cap A) = C_{C_a}(K \cap A) \cdot O(K)/O(K)$ from Lemma 1.1.14 (a). Since t is a 2-element that normalises $A \cap K$, we deduce that t centralises $A \cap K$. Altogether $t \in C_G(A \cap K) \cap C_a \leq C_G(A)$. ■

Let $H := N_G(A)$. Let $- : H \to H/C_G(A)$ be the natural Epimorphism and let $\bar{N}$ be a minimal normal subgroup of $\overline{N_G(A)}$ such that N is the full pre-image of $\bar{N}$ in H.

(5) The group $\bar{N}$ is cyclic of prime order and acts fixed-point-freely on $A^\#$.

Proof. By (4) and the Odd Order Theorem 1.1.12 we see that $N_G(A)/C_G(A)$ is soluble and of odd order. This implies that $\bar{N}$ is an elementary abelian group of odd order.
Since all elements of $A^\#$ are conjugate by (3), Lemma 2.2.2 (e) implies that they are conjugate by $H = N_G(A)$. As the kernel of $-$ is $C_G(A)$, there is a natural action from $\bar{H}$ on $A^\#$ that is also transitive.
Suppose for a contradiction that $\bar{N}$ does not act elementwise fixed point freely on $A^\#$. Then there is an element $b \in A^\#$ such that $C_{\bar{N}}(b) \neq 1$. Since $\bar{H}$ is transitive on $A^\#$ we may suppose that $b = a$. Suppose for a contradiction that $C_{\bar{N}}(a) = \bar{N}$. Then it follows from the fact $\bar{H}$ normalises $\bar{N}$ and acts transitively on $A^\#$ that $C_{\bar{N}}(b) = \bar{N}$ for all $b \in A^\#$. But this is a contradiction. Thus $\bar{N} \neq C_{\bar{N}}(a)$ and there is an element $\bar{h} \in \bar{N}$ such that $a^{\bar{h}} \neq a$. Consequently we have $C_{\bar{N}}(a) = (C_{\bar{N}}(a))^{\bar{h}} = C_{\bar{N}^{\bar{h}}}(a^{\bar{h}}) = C_{\bar{N}}(a^{\bar{h}})$. In particular $C_{\bar{N}}(a)$ has more than one fixed point in $A^\#$. Altogether we observe that $C_G(A) < C_N(a) < N$ and $C_A(C_N(a)) > \langle a \rangle$. Moreover we deduce from (1) we deduce that $A = \langle a \rangle \times (A \cap K)$ and hence we have $C_A(C_N(a)) \cap K \neq 1$.
Let b be an involution of $C_A(C_N(a)) \cap K$. Then we have $C_N(a) \leq C_G(b) \cap C_a = C_{C_a}(b)$. Let further $\wedge$ denote the natural epimorphism from C_a onto $C_a/O(K)$. Since N is a normal subgroup of H, the group $C_N(a)$ is normal in $C_H(a)$. Thus $\widehat{K \cap H}$ normalises $\widehat{C_N(a)}$. This implies together with the fact that K is normal in C_a by (1) and Lemma 1.1.14 (a) that

$$[\widehat{C_N(a)}, \widehat{K \cap H}] \leq \widehat{C_N(a)} \cap \hat{K} \leq \widehat{C_{C_a}(b)} \cap \hat{K} = C_{\hat{C}_a}(\hat{b}) \cap \hat{K} = C_{\hat{K}}(\hat{b}).$$

Theorem 1.2.8 (g) forces $C_{\hat{K}}(\hat{b})$ to be a 2-group. Since $\widehat{C_N(a)}$ is of odd order, it follows that $[\widehat{C_N(a)}, \widehat{H \cap K}] = \widehat{H \cap K} \cap \widehat{C_N(a)} \leq \widehat{C_N(a)} \cap C_{\hat{K}}(\hat{b}) = 1$.
By (1) and Theorem 1.2.8 (c) there is a cyclic subgroup $\hat{Q}$ of $\hat{K}$ that acts transitively on $\widehat{K \cap A}$. Then $\hat{Q}$ acts like the "Singer-cycle" on $\widehat{K \cap A}$. By II 7.3 (a) of [29] we see that

$$\widehat{C_N(a)} \leq C_{\widehat{C_H(a)}}(\widehat{K \cap H}) \leq C_{\widehat{C_H(a)}}(\hat{Q}) \leq \hat{Q} \leq \widehat{H \cap K}.$$

But $\widehat{H \cap K}$ intersects $\widehat{C_N(a)}$ trivially. Hence we conclude that $C_N(a)$ is a subgroup of $O(K)$. Finally $[C_N(a), A] \leq O(K) \cap A = 1$ leads to a contradiction.
For this reason $\bar{N}$ acts elementwise fixed point freely on $A^\#$. Since $\bar{N}$ is elementary abelian Lemma 1.1.14 (e) forces $\bar{N}$ to be cyclic. In particular $\bar{N}$ has prime order, because $\bar{N}$ is a minimal normal subgroup of $\bar{H}$. ■

By (2) and Theorem 1.2.8 (d) there is a $n \in \mathbb{N}$ such that $n \geq 1$ and $|A \cap K| = 2^{2n+1}$, hence $|A| = 2^{2(n+1)}$ by (1).

(6) $2^{2n+1} - 1$ divides $|N| - 1$

Proof. Theorem 1.2.8 (e) and (1) imply that $(H \cap K) \cdot O(K)/O(K)$ is soluble. Thus $H \cap K$ is soluble by the Odd Order Theorem 1.1.12.
Let Q be a Hall $2'$-subgroup of $H \cap K \leq C_H(a)$. Then $[O(K) \cap Q, A] \leq O(K) \cap A = 1$. This shows that $O(K) \cap Q \leq C_G(A) \cap Q = C_Q(A)$. Theorem 1.2.8 (e) yields together with Lemma 1.1.14 (a) that $1 = C_{Q \cdot O(K)/O(K)}(A) = C_Q(A) \cdot O(K)/O(K)$. Altogether we have $O(K) \cap Q = C_Q(A) = Q \cap C_G(A)$ and so

$$\bar{Q} = Q \cdot C_G(A)/C_G(A) \cong Q/(Q \cap C_G(A)) = Q/(Q \cap O(K)) \cong Q \cdot O(K)/O(K).$$

Consequently we have $|\bar{Q}| = |Q \cdot O(K)/O(K)| = 2^{2n+1} - 1$ by Theorem 1.2.8 (e).
Moreover the same statement yields that $Q \cdot O(K)/O(K)$ acts elementwise fixed-point freely on $((K \cap A) \cdot O(K)/O(K))^{\#}$. This implies for every element $g \in Q \setminus O(K)$ that we have $C_A(g) = \langle a \rangle$.
Let now $g \in Q \leq C_G(a)$ such that $\bar{g} \in C_{\bar{H}}(\bar{N})$ and let $h \in N^{\#}$. Then (5) yields that $a^h \neq a$ and there is an element $c \in C_G(A)$ such that $g^h = g \cdot c$. We have $g^h \in C_G(a^h)$ and hence $g = g^h \cdot c \in C_G(a^h)$. This implies that g centralises the two different involutions a and a^h and therefore we conclude that $g \in O(K)$. From $Q \cap O(K) = Q \cap C_G(A)$ we deduce that $\bar{Q} \cap C_{\bar{H}}(\bar{N}) = 1$. For that reason we see that

$$\bar{Q} \cong \bar{Q}/\bar{Q} \cap C_{\bar{H}}(\bar{N}) \cong \bar{Q} \cdot C_{\bar{H}}(\bar{N})/C_{\bar{H}}(\bar{N}) \leq N_{\bar{H}}(\bar{N})/C_{\bar{H}}(\bar{N}) \lesssim Aut(\bar{N}).$$

In particular $2^{2n+1} - 1 = |Q|$ divides $|Aut(N)| = |N| - 1$, since $|N|$ is a prime by (5). ■

We further have $\bar{N} \trianglelefteq \bar{H} = N_G(A)/C_G(A) \lesssim GL(2^{2n+2}, 2)$.
Thus $|\bar{N}|$ is an odd prime divisor of $|GL(2^{2n+2}, 2)| = \prod_{i=0}^{2n+1}(2^{2n+2} - 2^i)$. Consequently there is an $i \in \{0, ..., 2n+1\}$ such that $|\bar{N}|$ is an odd prime divisor of $2^{2n+2} - 2^i = 2^i(2^{2n+2-i} - 1)$.
By (6) we have $2^{2n+1} - 1$ divides $|\bar{N}| - 1$. This yields that $2^{2n+1} \leq |\bar{N}| \leq 2^{2n+2-i} - 1$. This implies that $i = 0$.
Altogether $|\bar{N}|$ is a prime divisor of $2^{2n+2} - 1 = (2^{n+1} - 1)(2^{n+1} + 1)$. In particular we conclude that $2^{2n+1} \leq |\bar{N}| \leq 2^{n+1} + 1$. This forces $n \leq 0$ contradicting $n \geq 1$. □

2.3.3 Theorem

Let G be a finite group with $O(G) = 1$. Suppose that $r(O_2(G)) \leq 1$ and $E(G)$ is quasi-simple. For a Sylow 2-subgroup T let $\Omega_1(Z(T)) > \Omega_1(Z(G))$ and let the possibly trivial group $\Omega_1(Z(G))$ be generated by c.
Furthermore assume that $E(G)$ has an elementary abelian subgroup A_0 such that A_0 is strongly closed in G and $\langle A_0, c \rangle / \langle c \rangle$ is not cyclic. Finally suppose that for all involutions $b \in G \setminus \langle c \rangle$ the group $C_G(b)$ is 3-soluble. Then one of the following holds:

(a) We have $E(G) \cong 2.Sz(8)$.

(b) There is an elementary abelian subgroup of order 4 of A_0 that is strongly closed in G.

(c) We have that $Z(E(G)) \neq 1$ and there is an elementary abelian subgroup of order 8 of A_0 that is strongly closed in G and contains c.

(d) The group $E(G)$ is a simple Bender group but not isomorphic to $\mathrm{PSL}(2,4)$.

(e) There is an element $x_0 \in G$ with $x_0^3 = 1$ such that for all involutions $b \in T$ we have $O(C_G(b)) \leq \langle x_0 \rangle$ and $O(C_{E(G)}(b)) = 1$.

(f) The group $G/Z(G)$ has a strongly closed elementary abelian subgroup $A/Z(G)$ order 16 such that $N_{G/Z(G)}(A/Z(G))/C_{G/Z(G)}(A/Z(G))$ is not S_4-free.

Proof

We set $K := E(G)$ and $Z := Z(G)$. Then Z is an abelian 2-group of rank 1 and Lemma 1.1.2 force Z to be cyclic.

Let G be a minimal counterexample and let $A \leq A_0$ be an elementary abelian subgroup of K that is strongly closed in G and such that $A \nleq Z$. Moreover we choose A such that $|A|$ is minimal with this properties.

Then by Lemma 2.2.2 (d) the group A is strongly closed in K. Since K is quasi-simple and $O_2(G)$ has at most one involution, we observe that $\Omega_1(Z(K)) \leq Z$. Thus the fact that $A \nleq Z$ together with the Z^*-Theorem 1.1.13 shows that A is not cyclic. Moreover we have $|A| \geq 8$, since (b) is false, as G is a counterexample. We further obtain from the fact (c) is false that $|\langle A, c\rangle| \geq 16$ in the case that $Z \neq 1$.

Furthermore let $- : G \to G/Z$ be the natural epimorphism.

(1) We have $O(\bar{G}) = 1$, $\bar{K} = E(\bar{G})$ and $O_2(\bar{G}) = \overline{O_2(G)}$ is either cyclic or dihedral.

Proof. Let U denote the full pre-image of $O(\bar{G})$. Then U is normal in G. Since $O(\bar{G})$ is of odd order and $Z = Z(G)$ is 2-group, Z is a central Sylow 2-subgroup of U. We conclude that U has a normal 2-complement U_1 and observe that $U_1 = O(U)\mathrm{char}U \trianglelefteq G$. Therefore U_1 is a normal subgroup of odd order of G. This implies that $U_1 \leq O(G) = 1$ and $O(\bar{G}) = \bar{U} = \bar{Z} = 1$.

Let F be the full pre-image of $O_2(\bar{G})$. Then F is normal in G. Moreover, as Z is a 2-group, F is a 2-group. For this reason we conclude that $F \leq O_2(G)$. It follows from $O_2(G) \leq F$ that $F = O_2(G)$. Consequently we have $r_2(F) = r(O_2(G)) \leq 1$ and Lemma 1.1.2 forces F to be either cyclic or a quaternion group. If we have $F \neq 1$, then $\Omega_1(F)$ is a normal subgroup of order 2 of G. Thus $\Omega_1(F) \leq Z(G) = Z$. This shows that $O_2(\bar{G})$ is a proper factor group of F in this case. Since cyclic groups and generalised quaternion groups have only cyclic or dihedral proper factors, the assertion about $O_2(\bar{G})$ follows.

Let finally $\bar{L}$ be a component of $\bar{G}$ with full pre-image L. Then L is, as a full pre-image of a subnormal group of $\bar{G}$, subnormal in G. In particular L' is subnormal in G. The facts that Z is cyclic and $\bar{L}$ is perfect, force L' to be perfect and $L = L' \cdot Z$. Thus L' is a component of G. Since $E(G)$ is quasi-simple, we observe $L' = K$ and $\bar{K} = \bar{L}' = \overline{L' \cdot Z} = \bar{L}$.

Altogether $\bar{G}$ has exactly one component and hence we conclude $\bar{K} = \bar{L} = E(\bar{G})$. ■

(2) We have $\bar{K} \leq \langle C_{\bar{G}}(\bar{a}) \mid \bar{a} \in \bar{A}^{\#}\rangle$.

Proof. Suppose for a contradiction that $\bar{K} \neq \langle C_{\bar{K}}(\bar{a}) \mid \bar{a} \in \bar{A}^{\#}\rangle \leq \langle C_{\bar{G}}(\bar{a}) \mid \bar{a} \in \bar{A}^{\#}\rangle$. Then Lemma 2.2.6 forces $\bar{K}$ to be a Bender group. Since G is a counterexample the failure of (d) yields that either $K \cong \mathrm{PSL}(2,4)$ or $Z \geq Z(K) \neq 1$ holds. In the first case K has elementary abelian Sylow 2-subgroups of order 4. This contradicts $|A| \geq 8$. We conclude that $Z \geq Z(K) \neq 1$. But $Z(K)$ is a cyclic 2-group. This implies that $\bar{K}$ is not isomorphic to $PSU(3,q)$ by Theorem 1.2.12 (b). Part (a) of the same theorem yields that $\bar{K}$ is a non-simple Suzuki group or $K \cong SL(2,4)$. By Theorem 1.2.8 (a) the first case yields (a). This contradiction shows that $K \cong SL(2,4)$. But then again a Sylow 2-subgroup of $\bar{K}$ has order 4 and $\bar{A}$ has order at least 8, since $Z \neq 1$. This is a final contradiction. ■

For all involutions $\bar{t} \in \bar{G}$ we set $\theta(t) := O(C_{\bar{G}}(\bar{t}))$ and for every elementary abelian subgroup B of T we set $W_{\bar{B}} := \langle \theta(\bar{b}) \mid \bar{b} \in \bar{B}^{\#}\rangle$.

(3) For every involution $b \in G \setminus Z$ we have that $\bar{b} = \overline{b \cdot c}$ is balanced in $\bar{G}$. Moreover $C_{\bar{G}}(\bar{b})$ is 3-soluble and has the full pre-image $N_G(\langle b, c\rangle)$ and $O(C_G(b)) = O(N_G(\langle b, c\rangle))$ holds.

Proof. Let $b \in G \setminus Z$ be an involution. Then we observe from $c \in Z$ that $\bar{b} = \overline{b \cdot c}$.

Moreover we have $N_G(\langle b, c\rangle) \geq Z$. Since $N_G(\langle b, c\rangle)$ is a pre-image of $C_{\bar{G}}(\bar{b})$ that contains

Z, the group $N_G(\langle b, c\rangle)$ is the full pre-image of $C_{\bar{G}}(\bar{b})$.
Furthermore we obtain $|N_G(\langle b, c\rangle) : C_G(b)| \le 2$ from $c \in \Omega_1(Z) \le Z(G)$. This implies that $O(C_G(b)) = O(N_G(\langle b, c\rangle))$ and, as $C_G(b)$ is 3-soluble, $N_G(\langle b, c\rangle)$ is 3-soluble. In particular $\bar{b}$ is balanced in $\bar{G}$ by Lemma 2.1.4. ∎

(4) The group $W_{\bar{A}}$ has odd order and for all non-cyclic elementary abelian subgroups B of T with $B \cap Z = 1$ we have $W_{\bar{A}} = W_{\bar{B}}$.

Proof. From (3) and the Odd Order Theorem 1.1.12 we deduce that θ is a soluble $\bar{A}$-signalizer functor in $\bar{G}$. Therefore the Soluble Signalizer Functor Theorem 2.1.6 yields that $\langle \theta(a) \mid a \in \bar{A}^{\#}\rangle = W_{\bar{A}}$ has odd order. We further observe together with (3) that $W_{\bar{B}} = \langle \theta(\bar{b}) \mid \bar{b} \in \bar{B}^{\#}\rangle = \langle \theta(\bar{b}) \mid b \in B \setminus \langle c\rangle\rangle$ for every elementary abelian subgroup B of T.
Let B be a non-cyclic elementary abelian subgroups B of T with $B \cap Z = 1$. From the hypothesis of our theorem, we have $\Omega_1(Z(T)) \nleq Z$. Thus we observe that $1 \neq \overline{\Omega_1(Z(T))}$. This provides, together with $|B| \ge 4$ and $B \cap Z = 1$, an element $b \in B^{\#}$ such that $\overline{\langle \Omega_1(Z(T)), b\rangle}$ is an elementary abelian group of order at least 4. In addition the pre-image $\langle \Omega_1(Z(T)), b\rangle$ is also elementary abelian. Thus there is a subgroup C of order 4 of $\langle \Omega_1(Z(T)), b\rangle$ such that $C \cap Z = 1$. From $B \subseteq C_G(\langle \Omega_1(Z(T)), b\rangle)$ we obtain that $[B, C] = 1$ and Lemma 2.1.7 implies that $W_{\bar{B}} = W_{\bar{C}}$.
From Lemma 1.1.17 we deduce that $|C_{\bar{A}}(\bar{b})|^2 \ge |\bar{A}| \ge 8$ and hence $|C_{\bar{A}}(\bar{b})| \ge 4$. Let $A_1 \le A$ be a subgroup of the full pre-image of $C_{\bar{A}}(\bar{b})$ such that $c \notin A_1$ and $\bar{A}_1 = C_{\bar{A}}(\bar{b})$. Then A_1 is not cyclic and $A_1 \cap Z = 1$. Moreover we have that $[A_1, C] \le [A_1, \langle \Omega_1(Z(T)), b\rangle] = [A_1, b] \le Z$. Again Lemma 2.1.7 yields $W_{\bar{A}_1} = W_{\bar{C}} = W_{\bar{B}}$.
Let finally A_2 be a complement of $\langle c\rangle \cap A$ in A. Then we have $[A_1, A_2] = 1$, since A is abelian. Thus Lemma 2.1.7 yields $W_{\bar{A}_1} = W_{\bar{A}_2}$.
Since we have $\bar{A}_2 = \bar{A}$, we finally conclude that $W_{\bar{A}} = W_{\bar{A}_2} = W_{\bar{A}_1} = W_{\bar{B}}$. ∎

We set $\Gamma^* := \langle N_{\bar{G}}(\bar{B}) \mid \bar{B} \le \bar{A} \text{ und } r(\bar{B}) \ge 2\rangle$. Furthermore we denote by $\mathfrak{T}$ the set all 2-subgroups of G that contain an elementary abelian subgroup of order at least 4 and intersect Z trivially and we set $\Gamma := \langle N_{\bar{G}}(\bar{U}) \mid U \le T \text{ and } U \in \mathfrak{T}\rangle$.

(5) We have $\Gamma^* \le \Gamma \le N_{\bar{G}}(W_{\bar{A}})$

Proof. Let $B \le A$ such that $r(\bar{B}) \ge 2$. Then B is elementary abelian and has a subgroup B_1 of order 4 that intersects Z trivially. This shows that $\Gamma^* \le \Gamma$.
Let $U \in \mathfrak{T}$ be a subgroup of T. Then U has an elementary abelian subgroup B of order 4 such that $B \cap Z = 1$. Hence (4) yields $W_{\bar{B}} = W_{\bar{A}}$.
Let $g \in G$ such that $\bar{g} \in N_{\bar{G}}(\bar{U})$. Then $g \in N_G(U \cdot Z)$ and so B^g is an elementary abelian subgroup of order 4 of T. Moreover we see that $B^g \cap Z = B^g \cap Z^g = (B \cap Z)^g = 1$. Thus $W_{\bar{A}} = W_{\overline{B^g}}$ by (4). Altogether we conclude

$$\begin{aligned} W_{\bar{A}}^{\bar{g}} = W_{\bar{B}}^{\bar{g}} &= \left(\langle \theta(\bar{b}) \mid \bar{b} \in \bar{B}^{\#}\rangle\right)^{\bar{g}} = \langle \theta(\bar{b})^{\bar{g}} \mid \bar{b} \in \bar{B}^{\#}\rangle = \langle \theta(\bar{b}^{\bar{g}}) \mid \bar{b} \in \bar{B}^{\#}\rangle = \langle \theta(\overline{b^g}) \mid \bar{b} \in \bar{B}^{\#}\rangle \\ &= \langle \theta(\bar{d}) \mid \bar{d} \in (\overline{B^g})^{\#}\rangle = W_{\overline{B^g}} = W_{\bar{A}}. \end{aligned}$$ ∎

(6) We have $[\bar{K}, W_{\bar{A}}] \le \bar{K} \cap W_{\bar{A}} = 1$.

Proof. By the minimal choice of A, we see that $\bar{A}$ is minimal strongly closed in $\bar{G}$. As (f) is false, the group $N_{\bar{G}}(\bar{A})/C_{\bar{G}}(\bar{A})$ is S_4-free in the case that $|\bar{A}| = 16$. Moreover we deduce from (4) and (5) that $W_{\bar{A}}$ has odd order and is normalised by Γ^*. So we may apply Lemma 2.3.2 to $\bar{G}$. Let $\bar{a} \in \bar{A}^{\#}$. Then $C_{\bar{G}}(\bar{a})$ is 3-soluble by (3) and $O(\langle \bar{A}^{C_{\bar{G}}(\bar{a})}\rangle) \le O(C_{\bar{G}}(\bar{a})) = \theta(\bar{a}) \le W_{\bar{A}}$. Thus Lemma 2.3.2 yields $C_{\bar{G}}(\bar{a}) \le W_{\bar{A}} \cdot \Gamma^*$.
Further (2) shows that $\bar{K} \le \langle C_{\bar{G}}(\bar{a}) \mid \bar{a} \in \bar{A}^{\#}\rangle \le W_{\bar{A}} \cdot \Gamma^*$. In particular (5) implies that $\bar{K}$ normalises $W_{\bar{A}}$. Consequently we obtain that $[\bar{K}, W_{\bar{A}}] \le \bar{K} \cap W_{\bar{A}} \trianglelefteq \bar{K}$. Since $W_{\bar{A}} \cap \bar{K}$ is a

normal subgroup of odd order of the simple group $\bar{K}$, the Odd Order Theorem 1.1.12 yields that $W_{\bar{A}} \cap \bar{K}$ is trivial. ■

By (1) we have that $O(\bar{G}) = 1$ and $\bar{K} = E(\bar{G})$. This implies with Lemma 1.1.18 (h) that
$$C_{W_{\bar{A}}}(O_2(\bar{G})) = C_{W_{\bar{A}}}(F^*(W_{\bar{A}})) \leq W_{\bar{A}} \cap F^*(\bar{G}) = 1.$$
Thus $W_{\bar{A}}$ acts faithfully on $O_2(\bar{G})$. The group $O_2(\bar{G})$ is by (1) either cyclic or dihedral. Thus $O_2(\bar{G})$ admits no automorphism of odd order except for the case where $O_2(\bar{G})$ is elementary abelian of order 4 by Lemma 1.1.3. In this case it admits an automorphism of order 3. In every case we have $W_{\bar{A}} \lesssim O^2(Aut(O_2(\bar{G}))) \lesssim Z_3$. Let W denote the full pre-image of $W_{\bar{A}}$ in G. Then W is a $\{2,3\}$-subgroup of G and there is an possibly trivial element $x_0 \in G$ such that $\langle x_0 \rangle \times Z = W$ and $x_0^3 = 1$.
For all involutions $b \in T \setminus Z$, we observe that $\overline{\langle b \rangle \cdot \Omega_1(Z(T))}$ has order at least 4 or we have that $b \in \Omega_1(Z(T))$ and $\overline{\langle b \rangle \cdot A}$ has at least order 4. In every case b is contained in a non-cyclic elementary abelian subgroup B_b of G with $B_b \cap Z = 1$. Hence (3) and (4) imply that $\overline{O(C_G(b))} = \overline{O(N_G(\langle b, c \rangle))} \leq O(\overline{N_G(\langle b, c \rangle)}) = O(C_{\bar{G}}(\bar{b})) \leq W_{\bar{A}} = \bar{W}$. This shows that $O(C_G(b)) \leq O^2(W) = \langle x_0 \rangle$. Since $C_K(b)$ is normal in $C_G(b)$, we conclude that $O(C_K(b)) \leq O(C_G(b)) \cap K \leq \langle x_0 \rangle \cap K = 1$. Finally (e) holds. This is a contradiction. □

2.3.4 Lemma

Let H be a finite group 3-soluble group such that $O(H) = 1$ and let A be an elementary abelian 2-subgroup that is strongly closed in H. Suppose further that $U = A \cdot E(U)$ is a $C_H(a)$-invariant subgroup of H for an element $a \in A^\#$.
If L is a component of $\langle A^U \rangle$, then L is a component of $\langle A^H \rangle$.

Proof *Compare with 3.7(2) of [20].*
Let L be a component of $\langle A^U \rangle$. By assumption the group U is 3-soluble and Lemma 2.2.2 (d) implies that A is strongly closed in $\langle A^U \rangle$. We set $E := E(\langle A^U \rangle)$ and apply Proposition 2.2.5 to conclude that E is a central product of Suzuki groups. Furthermore the same proposition implies that $A \cap E = \Omega_1(S)$ for a Sylow 2-subgroup S of E and hence $\langle (E \cap A)^E \rangle = E$. We notice from Theorem 1.2.8 (d) that $(L \cap A)/Z(L)$ determines the size of $L/Z(L)$.
Moreover we deduce from Proposition 2.2.5 and $O(H) = 1$ that
$$E = \langle (E \cap A)^E \rangle \leq \langle A^H \rangle = A \cdot E(\langle A^H \rangle).$$
Since $E = E'$ is perfect, we conclude that $E \leq E(\langle A^H \rangle)$. The fact that $Z(E(\langle A^H \rangle))$ is abelian and L is not abelian provides a component K of $\langle A^H \rangle$, such that $[K, L] \neq 1$. We see that E normalises $K \trianglelefteq E(\langle A^H \rangle)$. This implies that $C_L(K) \trianglelefteq L$. Hence, as L is simple, we deduce from $[K, L] \neq 1$ that $C_L(K) \leq Z(L)$. This implies that L induces inner automorphism on K, since the outer automorphism group of K is soluble by Theorem 1.2.8 (b) and (c).
In addition we observe that $C_K(a) \geq A \cap K \nleq Z(K)$ from Proposition 2.2.5. Furthermore we notice from Theorem 1.2.8 (d) that $(K \cap A)/Z(K)$ determines the size of $K/Z(K)$.
Since E is $C_H(a)$-invariant, we see that $[E, C_K(a)] \leq K \cap E$. If we had $E \cap K = 1$, then L would centralise $C_K(a) \geq A \cap K$. Consequently we would have a contradiction, because $C_K(A \cap K)$ is a 2-group by Theorem 1.2.8 (g) and L induces inner automorphism in K.
We conclude that $1 \neq E \cap K \trianglelefteq E$ and from Lemma 1.1.18 (e) we deduce that $(E \cap K) \cdot (E)/Z(E)$ is a direct product of components of $E/Z(E)$. Since different components of E commute by Part (b) of the same lemma, we conclude that $L \leq K$. Moreover Theorem 1.2.8 (j) yields that $Z(K) = Z(L)$ and $N_K(L) = L$.
In addition we have $K \cap A \leq K \cap ((A \cap L) \cdot C_A(L)) \leq K \cap N_A(L) \leq N_K(L) \cap A = L \cap A \leq K \cap A$. The fact that the sizes of $L/Z(K)$ and $K/Z(K)$ are determined by $(L \cap A)/Z(K) = (K \cap A)/Z(K)$ implies $L = K$. □

2.3.5 Theorem

Let G be a finite simple group and let A be an elementary abelian 2-subgroup of G of order at least 8 that is minimal strongly closed in G. Suppose that the centralisers of the involutions of A are 3-soluble. If A has order 16, then assume further that $N_G(A)$ is S_4-free.
Moreover, for all $a \in A^\#$, suppose that $\langle A^{C_G(a)} \rangle$ is soluble or that $C_G(a)$ is contained in a 3-soluble maximal subgroup of G.
Then G is a Bender group.

Proof *Compare with 4.3 and paragraph 7 of [20].*
Let G be minimal counterexample. Then the assumption that the centralisers of the involutions in A are 3-soluble and Lemma 2.1.4 force involutions of A to be balanced in G.

(1) We have $G = \langle C_G(a) \mid a \in A^\# \rangle$. Moreover $O(C_G(a))$ is trivial for all $a \in A^\#$.
If further H is a 3-soluble subgroup of G containing A with $O(H) = 1$, then we have
$H = N_H(A) \cdot E(\langle A^H \rangle)$ and $\langle A^H \rangle = A \cdot E(\langle A^H \rangle) \subseteq F^*(H)$.

Proof. We first obtain $G = \langle C_G(a) \mid a \in A^\# \rangle$ from Lemma 2.2.6, as G is not a Bender group. For all involutions $a \in A^\#$ the group $O(C_G(a))$ is soluble by the Odd Order Theorem 1.1.12. Moreover $|A| \geq 8$ and A is balanced in G. Altogether we may apply the Soluble Signalizer Functor Theorem 2.1.6 to conclude that $W = \langle O(C_G(a)) \mid a \in A^\# \rangle$ has odd order. Let B be a non-cyclic subgroup of A. Then B acts coprimely on $O(C_G(a))$ for every $a \in A^\#$. Since A is balanced in G, Lemma 1.1.14 (e) yields that

$$O(C_G(a)) = \langle O(C_G(a)) \cap C_G(b) \mid b \in B^\# \rangle \leq \langle O(C_G(b)) \mid b \in B^\# \rangle.$$

It follows that $W \leq \langle O(C_G(b)) \mid b \in B^\# \rangle \leq \langle O(C_G(a)) \mid a \in A^\# \rangle = W$ and consequently we have $W = \langle O(C_G(b)) \mid b \in B^\# \rangle$. This implies that W is normalised by the group $\Gamma^* = \langle N_G(B) \mid B \leq A,\ r(B) \geq 2 \rangle$. Moreover we have $O(A^{C_G(a)}) \leq O(C_G(a)) \leq W$ for all $a \in A^\#$. Applying Lemma 2.3.2 we deduce that $G = \langle C_G(a) \mid a \in A^\# \rangle \leq N_G(W)$. Consequently W is trivial, as G is simple. In particular we have $O(C_G(a)) = 1$ for all $a \in A^\#$.
Additionally let H be a 3-soluble subgroup of G containing A with $O(H) = 1$. Then A is strongly closed in H by Lemma 2.2.2 (d). Moreover we have $O(\langle A^H \rangle) \leq O(H) = 1$. Hence we deduce from Lemma 2.2.2 (g) and Proposition 2.2.5 that
$H = N_H(A) \cdot \langle A^H \rangle = N_H(A) \cdot E(\langle A^H \rangle)$ and that $\langle A^H \rangle = A \cdot E(\langle A^H \rangle)$. ■

(2) The group $N_G(A) = O^2(N_G(A))$ acts irreducibly on A and there exists an element $a \in A^\#$ such that $C_G(a)$ is not soluble.

Proof. By Lemma 2.2.2 (e) the group $N_G(A)$ controls the fusion of its 2-elements. Thus for all Sylow 2-subgroups T of the simple group G such that $A \leq T$, the Focal Subgroup Theorem 1.1.9 leads to

$$\begin{aligned} T = G \cap T = G' \cap T &= \langle t^{-1}t^g \mid t \in T, g \in G \text{ and } t^g \in T \rangle \\ &= \langle t^{-1}t^g \mid t \in T, g \in N_G(A) \text{ and } t^g \in T \rangle = T \cap (N_G(A))'. \end{aligned}$$

It follows that $N_G(A) = O^2(N_G(A))$. As A is minimal strongly closed in G, the group $N_G(A)$ acts irreducibly on A by Lemma 2.2.2 (j). Finally suppose for a contradiction that $C_G(a)$ is soluble for all $a \in A^\#$. Then (1) implies that $C_G(a) \subseteq N_G(A)$ for all $a \in A^\#$. Thus again (1) leads to $G = \langle C_G(a) \mid a \in A^\# \rangle \subseteq N_G(A)$. This is a contradiction. ■

(3) If $b \in A^\#$ and $C_G(b) \leq H \leq G$, then $O(H) = 1$.

Proof. Let $b \in A^\#$ and suppose that H is a subgroup of G containing $C_G(b)$. Then (1) yields that $O(H) \cap C_G(b) \leq O(C_G(b)) = 1$. In particular b acts fixed-point-freely on $O(H)$ and hence b inverts $O(H)$. Let $d \in A^\#$. Then we have:

$$O(H) \cap C_G(d) = [O(H) \cap C_G(d), b] \leq O(H) \cap \langle A^{C_{O(H)}(d)} \rangle$$
$$\leq O(H) \cap A \cdot E(\langle A^{C_G(d)} \rangle) = O(H) \cap E(\langle A^{C_G(d)} \rangle).$$

The group $E(\langle A^{C_G(d)} \rangle)$ is 3-soluble and A is a strongly closed subgroup of $E(\langle A^{C_G(d)} \rangle)$ by Lemma 2.2.2 (d). Consequently Proposition 2.2.5 yields that $E(\langle A^{C_G(d)} \rangle)$ is a central product Suzuki groups such that $E(\langle A^{C_G(d)} \rangle) \cap A = \Omega_1(S)$ for a Sylow 2-subgroup S of $E(\langle A^{C_G(d)} \rangle)$. From Theorem 1.2.8 (i) we obtain that $O(H \cap E(\langle A^{C_G(d)} \rangle)) = 1$.
Altogether it follows that $O(H) \cap C_G(d) \leq O(H) \cap E(\langle A^{C_G(d)} \rangle) \leq O(H \cap E(\langle A^{C_G(d)} \rangle)) = 1$.
Finally Lemma 1.1.14 (e) implies $O(H) = \langle O(H) \cap C_H(d) \mid d \in A^\# \rangle = 1$. ■

Let $a \in A^\#$ such that G has a 3-soluble maximal subgroup H containing $C_G(a)$. Such an element exists by the assumption and (2). Moreover let K denote a component of $\langle A^H \rangle$.

(4) We have $E(H) = E(K \cdot C_G(K))$.

Proof. We obtain from (3) that $O(H) = 1$ and so (1) implies that $H = N_H(A) \cdot E(\langle A^H \rangle)$ and $A \leq F^*(H)$. This shows that $A \cdot K$ is a subgroup of the 3-soluble group H. Moreover from our choice of K and Proposition 2.2.5 we obtain an element $b \in A \cap K^\#$ and we observe that $C_G(K) \leq C_G(b)$. Additionally we remark that $O(C_G(b)) = 1$ by (1). From the fact that $\langle K^H \rangle$ is normalised by $C_G(a) \leq H$ we deduce that $E(\langle K^H \rangle \cap C_G(b))$ is $C_{C_G(b)}(a)$-invariant. Moreover every component L of $A \cdot \langle K^H \rangle$ that is different from K is a component from $E(\langle K^H \rangle \cap C_G(b))$ by Lemma 1.1.18 (g). We apply Lemma 2.3.4 to observe that L is a component of $C_G(b)$. From $L \leq C_G(K)$ and Lemma 1.1.18 (f) we conclude that L is a component of $C_G(K)$.
Altogether $\langle K^H \rangle$ is a product of components of $K \cdot C_G(K)$ and so $\langle K^H \rangle$ is normalised by $E(K \cdot C_G(K))$. Since H is a maximal subgroup of the simple group G, we conclude that $E(K \cdot C_G(K)) \leq N_G(\langle K^H \rangle) = H$. Thus we see from Lemma 1.1.18 (g) that we have $E(C_G(K)) \trianglelefteq E(C_H(K))$ is a subnormal subgroup of H and Lemma 1.1.18 (f) implies that $E(C_G(K)) \leq E(H)$.
On the other hand from the fact that K is a component of $\langle A^H \rangle$ and hence of H we deduce that $E(H) \leq K \cdot C_G(K)$.
Finally Lemma 1.1.18 (g) yields that $E(H) \leq E(K \cdot C_G(K)) = K \cdot E(C_G(K)) \leq E(H)$. ■

(5) For all $b \in C_A(K)^\#$ the unique maximal 3-soluble subgroup of G containing $C_G(b)$ is H.

Proof. Let $b \in C_A(K)^\#$ and let H_1 be a maximal 3-soluble subgroup of G containing $C_G(b)$. Then we have $K \leq H_1$ and $O(H_1) = 1$ by (3). Moreover K s a component of the $C_{H_1}(a)$-invariant group $E(H_1 \cap E(H))$. Thus K is a component of H_1 by Lemma 2.3.4. Furthermore (4) yields that $E(H_1) = E(K \cdot C_G(K)) = E(H)$ is normalised by H_1 and by H. As G is simple and H and H_1 are maximal subgroups of G, it follows that $H_1 = H$. ■

We further choose a, H and K such that $|C_A(K)|$ is minimal.

(6) $N_G(A)$ is not contained in H.

Proof. Suppose for a contradiction that $N_G(A) \leq H$.
We know from (2) that $O^2(N_G(A)) = N_G(A)$ and that $N_G(A)$ acts irreducibly on A.
Assume first that $A \leq K$. Then we may apply Proposition 2.2.5, since H is 3-soluble and A is strongly closed in H by Lemma 2.2.2 (d), to observe that $K \trianglelefteq H$. Since K is a Suzuki group either $1 \neq A \cap Z(K) \leq Z(K) \leq Z(H)$ and $A \cap Z(K) \neq 1$ or all elements of $A^\#$ are conjugate in $K \leq H$ by Theorem 1.2.8 (f). The first case is not possible, as $N_G(A) = N_H(A)$ acts irreducibly on A. In the second case we have $C_G(b) \leq H$ for all $b \in A^\#$ and hence (1) yields that $G = \langle C_G(b) \mid b \in A^\# \rangle \leq H$. This is a contradiction.
Thus $1 \neq A \cap K \neq A$. The irreducible action of $N_G(A)$ on A implies that $N_G(A)$ does not normalise K. From $O^2(N_G(A)) = N_G(A)$ it follows that $\langle A^H \rangle$ has at least three distinct

components that are isomorphic to K. This implies $|C_A(K)| > \sqrt{|A|}$. We fix an element $b \in A^\#$. If we have $E(\langle A^{C_G(b)}\rangle) = 1$, then we obtain from (1) that $C_G(b) \leq N_G(A) \leq H$.
If we have $E(\langle A^{C_G(b)}\rangle) \neq 1$, then $\langle A^{C_G(b)}\rangle$ is not soluble. Thus by the assumption of our theorem $C_G(b)$ is contained in a 3-soluble maximal subgroup of G. Let H_1 be a 3-soluble maximal subgroup of G containing $C_G(b)$ and let E be a component of $\langle A^{H_1}\rangle$. From the choice of K it follows that $|C_A(E)| \geq |C_A(K)| > \sqrt{|A|}$ and so $C_A(K) \cap C_A(E) \neq 1$. Let $c \in C_A(K) \cap C_A(E)^\#$. Then we have $c \in C_A(K)$ and so (5) yields that $C_G(c) \leq H$. Thus $E \leq C_G(c) \leq H$. Altogether we conclude that $E(\langle A^{H_1}\rangle) \leq H$. Moreover, as $O(H_1) = 1$ by (3), Part (1) implies that $C_G(b) \leq H_1 = N_{H_1}(A) \cdot E(\langle A^{H_1}\rangle) \leq H$.
Finally we have shown that $\langle C_G(b) \mid b \in A^\#\rangle \leq H$. This contradicts (1). ∎

(7) $C_A(K) \cap C_A(K)^g = 1$ for all $g \in N_G(A) \backslash N_H(K)$.

Proof. Let $g \in N_G(A)$ be such that $C_A(K) \cap C_A(K)^g \neq 1$ and let $b \in C_A(K)^\# \cap C_A(K)^g$. From $b^{g^{-1}}, b \in C_A(K)$ and (5) we deduce that H is the unique maximal subgroup of G containing $C_G(b)$ and $C_G(b)^{g^{-1}}$. Hence we have $H^{g^{-1}} = H$ and so $g \in H$.
Applying (6) we choose an element $h \in N_G(A) \backslash H$. Then we have $C_A(K) \cap C_A(K)^h = 1$. This forces $(|C_A(K)|)^2 = |C_A(K)| \cdot |C_A(K)^h| = |C_A(K) \cdot C_A(K)^h| \leq |A|$. Since $A = C_A(K) \cdot (A \cap K)$ it follows that $|K \cap A| \geq \sqrt{|A|}$.
We suppose for a contradiction that $g \notin N_H(K)$. Then $[K, K^g] = 1$ by Lemma 1.1.18 (b) and so we have $\sqrt{|A|} \leq |K \cap A| \leq |C_A(K^g)| = |C_A(K)| \leq \sqrt{|A|}$. This implies that $K \cap A = C_A(K^g)$ and similar we conclude $K^g \cap A = C_A(K)$.
From all this we deduce

$$\begin{aligned} |K \cap A| \cdot |K^g \cap A| &= |K \cap A| \cdot |(K \cap A)^g| = |K \cap A|^2 = (\sqrt{|A|})^2 = |A| = |C_A(K) \cdot (A \cap K)| \\ &= |(A \cap K^g) \cdot (A \cap K)| = \tfrac{|A \cap K| \cdot |A \cap K^g|}{|A \cap K \cap K^g|}. \end{aligned}$$

It follow that $1 = K \cap A \cap K^g = K \cap A \cap A \cap K^g = C_A(K^g) \cap C_A(K)$
$= (C_A(K))^g \cap C_A(K) \neq 1$.
But this is a contradiction. ∎

Let $X := N_G(A)/C_G(A)$, let Y_0 be a complement of the Sylow 2-subgroup of $N_K(A)$ (the group $N_K(A)$ is a Frobenius group by Theorem 1.2.8 (e)) and $Y = Y_0 \cdot C_G(A)/C_G(A)$.

(8) The groups X and Y satisfy hypothesis (2.9) of [20] in their action on A, and $Z(K) \neq 1$.

Proof. We recall that $N_G(A)$ acts irreducibly on A and $O^2(N_G(A)) = N_G(A)$ by (2). Hence X acts faithfully and irreducibly on A and $O^2(X) = X$. By Theorem 1.2.8 (e) the group Y is cyclic of odd order and acts transitively on $[A, Y]^\#$. By (7) the distinct elements of $\{C_A(Y)^{\tilde{h}} \mid \tilde{h} \in X\}$ intersect pair-wise trivially. Thus hypothesis (2.9) of [20] is fulfiled.
Assume that $Z(K) = 1$. Let $\tilde{h} \in N_X(C_A(K))$ and let $g \in N_G(A)$ be a pre-image of $\tilde{h}$ in G. Then $C_A(K) \cap C_A(K)^g \neq 1$, so (7) implies that $g \in N_H(K) \cap N_G(A) \leq N_H(N_K(A))$. By the definition of Y_0 it follows that $g \in N_H(Y_0 \cdot C_K(A))$ and so $\tilde{h} \in N_X(Y)$. Applying (2.11) of [20], we conclude that $|A| = 8$ and $|C_A(K)| = 2$. This is a contradiction to Theorem 1.2.8 (d). ∎

(9) We have $|C_A(K)| = 2$.

Proof. We want to apply (2.10) of [20]. Thus, it remains to show that Y acts semi-regularly on the set $\{C_A(Y)^h \mid h \in X\} \backslash \{C_A(K)\}$.
Suppose for a contradiction that Y does not act semi-regularly on $\{C_A(Y)^h \mid h \in X\} \backslash \{C_A(K)\}$. Then there exists an element $g \in N_G(A) \backslash C_G(A)$ and there is an element $y \in Y^\#$ such that $C_A(K)^g \neq C_A(K)$ and $C_A(K)^g$ is fixed by y. Since $Z(K) \neq 1$ we conclude $K/Z(K) \cong Sz(8)$ from Theorem 1.2.8 (a). This forces y to have order 7 and hence $C_A(K)^g$ is Y- and so Y_0-invariant. Since $C_A(K)^g \cap C_A(K) = 1$ by (7) and since Y_0 acts transitively on $[A, Y_0]^\#$, we

conclude that $C_A(K)^g = [A, Y_0]$.
Now we have $|C_A(K)|^2 = |C_A(K)| \cdot |[A, Y_0]| = |A|$ and K is not simple by (8), so $K \trianglelefteq H$ and $C_A(K) = O_2(\langle A^H \rangle) \trianglelefteq H$. The fact that H is a maximal subgroup of the simple group G yields that $Y_0^{g^{-1}} \leq N_G(C_A(K)) = H$ and that $1 \neq C_A(K) \cap K$ is a proper H-invariant subgroup of $C_A(K)$. But $Y_0^{g^{-1}}$ acts transitively on $[A, Y_0]^{g^{-1}} = C_A(K)$. This is a contradiction.
Now (2.10) of [20] forces $|C_A(K)| = 2$ and thus $C_A(K) = Z(K)$. ■

Finally $A = 2 \cdot 8 = 16$ and $8 \cdot 7$ divides $|N_G(A)/C_G(A)|$. Bearing in mind that $N_G(A)$ acts irreducibly on A, Lemma 1.2.6 contradicts the assumption that in this case $N_G(A)$ is S_4-free. □

2.3.6 Lemma

Let G be a finite S_4-free group such that $Z^*(G) = 1$.
Suppose that A is an elementary abelian 2-subgroup of G of order at least 8 that is strongly closed in G. For every involution $t \in G$ suppose that $C_G(t)$ is a $3'$-group. Moreover let $a \in A^\#$ such that $\langle A^{C_G(a)} \rangle$ is not soluble.
Then every subgroup of A that is minimal strongly closed in G has order at least 8.

Additionally if N is a normal subgroup of G such that $Z^*(N) = 1$ and $\langle A^{C_G(a)} \rangle \cap N$ is not soluble, then all subgroups of $A \cap N$ that are minimal strongly closed in N and of order at least 8. Moreover there exists at least one.

Proof
Let B be a non-trivial subgroup of A that is minimal strongly closed in G.
Suppose for a contradiction that B has order 4. From Lemma 2.2.2 (a) we observe that $N_G(A) \leq N_G(B)$. Moreover the assumptions that $Z^*(G) = 1$ and that G is S_4-free force $N_G(B)/C_G(B)$ to be cyclic of order 3. We further know that $C_G(B)$ is a $3'$-group. It follows that $N_G(B)$ is 3-soluble and possesses cyclic Sylow 3-subgroups of order 3. Let $x \in N_G(B)$ be an element of order 3. Then $B = [B, x]$ and hence $x \notin O(N_G(B))$. Moreover $C_G(x)$ is of odd order, since the centralisers of involutions are $3'$-groups by assumption. We set $H := N_G(B)$ and let $- : H \to H/O(\langle A^H \rangle)$ denote the natural epimorphism. Then we deduce from Lemma 1.1.14 (a) that $C_{\bar{C}}(\bar{x})$ is of odd order.
In addition we observe that the group A is strongly closed in the 3-soluble group $N_G(B)$ by Lemma 2.2.2 (d). Hence we may apply Proposition 2.2.5 to obtain that $\overline{\langle A^H \rangle}$ is a central product of an abelian 2-group and quasi-simple Suzuki groups. Since $C_{\bar{C}}(\bar{x})$ is of odd order, Lemma 1.2.10 yields that $\overline{\langle A^H \rangle}$ is a 2-group. In particular $\langle A^{C_G(B)} \rangle$ is soluble.

Now we set $C := \langle A^{C_G(a)} \rangle$ and let $\wedge : C \to C/O(C)$ be the natural epimorphism. Moreover we observe from Lemma 2.2.2 (d) that $B \leq C$ is strongly closed in C. From the assumption that C is a $3'$-group and from $N_G(B)/C_G(B) \cong Z_3$ we deduce together with the Z^*-Theorem 1.1.13 that $B \in Z^*(C)$. We further obtain from our above investigation and Lemma 1.1.14 (a) that $\hat{C} = C_{\hat{C}}(\hat{B}) = C_C(B) \cdot O(C)/O(C)$ is soluble. The Odd Order Theorem 1.1.12 finally forces $C = \langle A^{C_G(a)} \rangle$ to be soluble. This is a contradiction.

Let additionally N be a normal subgroup of G such that $Z^*(N) = 1$ and such that $\langle A^{C_G(a)} \rangle \cap N$ is not soluble. We set $X := \langle a \rangle \cdot N$. Since X is a subgroup of G, it is S_4-free and for every involution $t \in X$ we have that $C_X(t)$ is a $3'$-group. Moreover we observe that $A \cap X$ is strongly closed in X and A is strongly closed in the 3-soluble group $\langle A^{C_G(a)} \rangle$ from Lemma 2.2.2 (d). Applying Proposition 2.2.5 we deduce that $(\langle A^{C_G(a)} \rangle \cap N) \cdot O(\langle A^{C_G(a)} \rangle)/O(\langle A^{C_G(a)} \rangle)$ contains a Suzuki group. Let L_1 be its full pre-image and $L = L_1 \cap N$. Then $L \cap A$ has at least 8 elements by Theorem 1.2.8 (d). Moreover $\langle (L \cap A)^L \rangle \leq \langle (A \cap L)^{C_X(a)} \rangle$ is not soluble.
Finally we apply our Lemma to X and deduce that every elementary abelian subgroup B of

$A \cap X$ that is minimal strongly closed in X has order at least 8 or that $Z^*(X) \neq 1$. In the first case such a B is centralised by $\langle a \rangle$ and hence $B \cap N$ is strongly closed in $N\langle a \rangle$. The minimality of B implies that $B \leq N$. Moreover the fact that $\langle a \rangle$ centralises every subgroup of B yields that B is also minimal strongly closed in N. This is the assertion.
In the second case we deduce from $Z^*(N) = 1$ that there is an element $n \in N$ such that $a \cdot n \in Z^*(X) = Z(X)$ and $(a \cdot n)^2 = 1$. The assumption forces N to be a $3'$-group. Let B be a subgroup of $A \cap N$ that is minimal strongly closed in N. Then the fact that N is a $3'$-group with $Z^*(N) = 1$ implies together with the Z^*-Theorem 1.1.13 that B has order at least 8. □

2.3.7 Proposition

Let G be a finite S_4-free group such that $O(G) = 1$ and G has no normal 3-complement.
Let further A be an elementary abelian 2-subgroup of G of order at least 8 that is minimal strongly closed in G. For every involution $t \in G$ let $C_G(t)$ be a $3'$-group.
Then $\langle A^{C_G(a)} \rangle$ is soluble for all $a \in A^\#$.

Proof

Let G be a minimal counterexample. Since the centralisers of involutions are $3'$-groups, the centraliser of every 3-group is of odd order. (*)
By assumption we have $O(G) = 1$ and G is S_4-free and has no normal 3-complement. Hence Lemma 1.3.8 implies that $F^*(G) = E(G)$ is simple and no $3'$-group. In particular we have $Z^*(G) = 1$ and $Z^*(E(G)) = 1$.
Moreover there is an element $a \in A^\#$ such that $\langle A^{C_G(a)} \rangle$ is not soluble. Lemma 1.3.8 implies that $C_G(a)/C_{E(G)}(a)$ is soluble. Consequently $\langle A^{C_G(a)} \rangle \cap E(G)$ is not soluble. We apply Lemma 2.3.6 to observe that all subgroups of A which is minimal strongly closed in $E(G)$ are of order at least 8.
Now the simple group $E(G)$ fulfils the hypothesis of our proposition. From our choice of G as a minimal counterexample we conclude that $\langle B^{C_{E(G)}(b)} \rangle$ is soluble for all $b \in B^\#$ or $G = E(G)$ is simple. In the first case Theorem 2.3.5 forces $E(G)$ to be a Bender group. The centraliser of the involutory automorphism of $E(G)$ that is induced by a involves a Suzuki group. Therefore Theorem 1.2.12 (c) forces $E(G)$ to be a Suzuki group. This is a contradiction, as $E(G)$ is not a $3'$-group.
Consequently G is no Bender group and hence $G = E(G)$ is simple. Again by Theorem 2.3.5 there exists an involution $b \in A$ such that neither $\langle A^{C_G(b)} \rangle$ is soluble nor $C_G(b)$ is contained in a 3-soluble maximal subgroup of G.
Let H be a maximal subgroup containing $C_G(b)$ and let $- : H \to H/O(H)$ be the natural epimorphism. From Lemma 1.3.8 we obtain that $F^*(\bar{H}) = E(\bar{H})$ is simple and no $3'$-group and we see that $Z^*(\bar{H}) = Z^*(E(\bar{H})) = 1$.
Further we get $A \leq H$ and $\bar{A}$ is strongly closed in $\bar{H}$ by Lemma 2.2.2 (d) and (f). Since we have $C_G(b) \leq H$, we conclude that $C_G(b) = C_H(b)$. The fact that $\langle A^{C_H(b)} \rangle$ is not soluble yields together with Lemma 1.1.14 (a) that $\overline{\langle A^{C_H(b)} \rangle} = \langle \bar{A}^{\overline{C_H(b)}} \rangle = \langle \bar{A}^{C_{\bar{H}}(\bar{b})} \rangle$ is not soluble. But $C_{\bar{H}}(\bar{b})/C_{E(\bar{H})}(\bar{b})$ is soluble by Lemma 1.3.8. Thus the group $\langle \bar{A}^{C_{\bar{H}}(\bar{b})} \rangle \cap E(\bar{H})$ is not soluble. We apply Lemma 2.3.6 to observe that $\bar{A}$ has a subgroup $\bar{B}$ of order at least 8 that is minimal strongly closed in $E(\bar{H})$.
Altogether $E(\bar{H})$ fulfils the assumption of our proposition and hence $\langle \bar{B}^{C_{E(\bar{H})}(\bar{b})} \rangle$ is soluble for every $\bar{b} \in \bar{B}$. For that reason $E(\bar{H})$ fulfils the hypothesis of Theorem 2.3.5 and so $E(\bar{H})$ is a Bender group. Since b induces an involutory automorphism on $E(H)$ such that its centraliser involves a Suzuki group, again Theorem 1.2.12 (c) and the fact that $E(\bar{H}) = F^*(\bar{H})$ is no $3'$-group lead to a contradiction. □

2.4 The Bender Method

With the Bender method we analyse maximal subgroups of almost simple groups, where the generalised Fitting subgroup has at most prime index. Bender elaborated this method for finite simple groups. In [38] Waldecker showed that it can be adopted to almost simple groups with a simple normal maximal subgroup of index 2. We need the results for almost simple groups with a simple normal maximal subgroup of index 3. Therefore we need to prove some of the standard results with our additional assumption again. The ideas are due to Bender.

2.4.1 Definition

Let G be a group then G is called **m-simple** if and only if G is a finite almost simple and 1, $E(G)$ and G are all normal subgroups of G.

Remark

Every simple group is m-simple.

2.4.2 Lemma

Let G be a m-simple group. Then $|G : E(G)|$ is 1 or a prime.
Suppose further that H is a maximal subgroup of G and N is a normal subgroup of H. Then we have $H = N_G(N)$ or $N = 1$ or $E(G) = N = H$.

Proof

Assume first that $G \neq E(G)$. Then $G/E(G)$ is soluble and simple, since G is m-simple. As all simple soluble groups are cyclic of prime order, there exists a prime p such that $|G : E(G)| = p$.
Assume now that G is arbitrary. If we have $H = E(G)$, then H is simple and we have $H = E(G) = N$. Otherwise 1 is the unique normal subgroup of G contained in H. If we have $H \neq N_G(N)$, then we observe that $H < N_G(N)$, since N is a normal subgroup of H. The maximality in G of H implies that $N_G(N) = G$. This forces N to be trivial. □

2.4.3 Definition

Let G be a finite group. Suppose that H is a maximal subgroup of G and let U be a proper subgroup of G.
The group H **infects** the group U if and only if there is a subgroup $A \leq F(H)$ such that $N_{F^*(H)}(A) \leq U$. If H infects U, then we write $H \rightsquigarrow U$.

2.4.4 Lemma

Let G be a m-simple group. Suppose further that $H \neq E(G)$ is a maximal subgroup of G that infects the proper subgroup $U \neq E(G)$ of G. Moreover we set $\pi := \pi(F(H))$.
Then the following hold:

(a) $O_\pi(F(U)) \leq H$ or $F^*(H)$ is a p-group.

(b) $O_{\pi'}(F(U)) \cap H = 1$.

(c) If $U \rightsquigarrow H$, then $H = U$ or $F^*(H)$ and $F^*(U)$ are p-groups for the same prime p.

Proof *Compare with 6.1 and 6.2 of [38].*
Let A be a subgroup of $F(H)$ such that $N_{F^*(H)}(A) \leq U$. Then we have

$$(*)\ E(H) \leq C_{F^*(H)}(A) \leq U \text{ and } Z(F(H)) \leq C_{F^*(H)}(A) \leq U.$$

We further observe that $C_{F(H)}(U \cap F(H)) \leq C_{F(H)}(A) \leq U$ and Lemma 2.4.2 yields that $N_U(X) \leq H$ for every non-trivial characteristic subgroup X of H. Now Part (a) follows from Lemma X 15.6 (b) of [9].

If we additionally have $U \rightsquigarrow H$, then we similarly observe that $E(U)$ is a subgroup of H and $C_{F(U)}(H \cap F(U)) \leq H$ and Lemma 2.4.2 yields also that $N_H(X) \leq U$ for every non-trivial characteristic subgroup X of U. So we may apply Theorem X 15.7 of [9] to obtain (c).

It remains to prove Part (b). Suppose that $p \in \pi'$. Then we observe

$$[O_p(U) \cap H, A \cdot C_{F(H)}(A)] \leq O_p(U) \cap F(H) = 1.$$

Moreover $O_p(U) \cap H$ acts coprimely on $F(H)$. We apply Lemma 1.1.14 (g) to the group $O_p(U) \cap H$ that acts on $F(H)$ and the self-centralising subnormal subgroup $A \cdot C_{F(H)}(A)$. Then we conclude that $O_p(U) \cap H$ centralises $F(H)$. Moreover we obtain from (*) that $E(H) \leq U$ normalises $O_p(U)$ and so $H \cap O_p(U)$. Since $H \cap O_p(U)$ normalises $E(H)$, we deduce from Lemma 1.1.18 (b) that $[O_p(U) \cap H, E(H)] = 1$.

Finally Part (h) of the same lemma implies together with Lemma 2.4.2 that

$$H \cap O_p(H) \leq C_G(F^*(H)) = C_H(F^*(H)) = Z(F^*(H)) \leq F(H).$$

Now the fact that $F(H)$ has p'-order leads to the assertion. □

2.4.5 Lemma

Let G be a m-simple group and let H and U be distinct maximal subgroups of G both different from $E(G)$.

Suppose that both $F^*(H)$ and $F^*(U)$ are p-subgroups of G for an odd prime p.

If we have $H \rightsquigarrow U$, then H or U is not $SL(2, p)$-free.

Proof *Compare with 2.4 of [4].*

Suppose for a contradiction that H and U are $SL(2, p)$-free groups and fix a Sylow p-subgroup S of H and a Sylow p-subgroup Q of U.

Since H is $SL(2, p)$-free, 1.4 of [15] yields $Z(J(S)) \trianglelefteq H$. Hence Lemma 2.4.2 implies that $H = N_G(Z(J(S)))$. Moreover $N_G(S)$ is a subgroup of $N_G(Z(J(S))) = H$. As U is $SL(2, p)$-free, we also observe $N_G(Q) \leq N_G(Z(J(Q))) = U$. Thus S and Q are Sylow p-subgroups of G and therefore conjugate in G. This implies that $H = N_G(Z(J(S)))$ and $U = N_G(Z(J(Q)))$ are conjugate in G. Let $g \in G$ with $H = U^g$.

From the assumption $H \rightsquigarrow U$ we deduce that $Z(O_p(H))^g$ is a subgroup of H. Because of Sylow's Theorem we may choose g such that $Z(O_p(H))^g \leq S$. Now 1.5 of [15] yields that $g = c \cdot n$ with $c \in N_G(Z(O_p(H))) = H$ and $n \in N_G(Z(J(S))) = H$. This means $g \in H$ and hence we have $H = U$. This is a contradiction. □

2.4.6 Lemma

Let G be a m-simple group and suppose that $H \neq E(G)$ is a maximal subgroup of G such that either $F^*(H)$ is no p-group or $F^*(H)$ is a p-group for an odd prime p and H is $SL(2, q)$-free. If we have $F(H) = F^*(H)$ and if there is a $g \in G$ with $H \rightsquigarrow H^g$, then g is an element of H or $O(H)$ is trivial.

Proof *Compare with 2.5 of [4].*

Assume that $O(H) \neq 1$. Because of Lemma 2.4.5 we may assume that $F^*(H)$ is not a p-group. Then we have $F(H^g) = O_{\pi(F(H))}(H^g) \leq H$ by Lemma 2.4.4 (a). As $F(H^g) = F^*(H^g)$ it follows that $H^g \rightsquigarrow H$. Now Lemma 2.4.4 (c) yields $H = H^g$ and so $g \in N_G(H) = H$. □

2.4.7 Definition

Let G be a finite group, q be a prime and t be an involution. We say that **t commutes q down in G** if and only if $[t, Q] \leq F(G)$ for every $C_G(t)$-invariant q-subgroup Q of G.

2.4.8 Lemma

Let G be a m-group and suppose that A is an elementary abelian subgroup of order 4 of G such that there exists an element y acting transitively on $A^{\#}$. Assume further that there is an involution c of A and a $C_G(c)$-invariant q-subgroup Q of G for an odd prime q such that Q is not centralised by c but every proper $C_G(c)$-invariant subgroup of Q lies in $C_G(c)$.
Moreover let $H \neq E(G)$ be a maximal subgroup of G such that $C_G(c) \leq N_G(Q) \leq H$ and every element of $A^{\#}$ commutes q down in H. Then one of the following holds:

(a) The element y normalises H.

(b) There is an element $a \in A \setminus \langle c \rangle$, such that $[c, C_Q(a)] \neq 1$ and there is no $SL(2, q)$-free, maximal subgroup $M \neq E(G)$ of G containing $N_G([c, C_Q(a)])$ such that a commutes q down in M.

(c) There is a maximal subgroup of G containing $C_G(c)$ that is not $SL(2, q)$-free and different from $E(G)$.

Proof

Suppose that (b) and (c) are false and set $C := C_G(c)$.
The group Q does not centralise c. This implies that the C-invariant group $[Q, c]$ is non-trivial. From the assumption on Q we deduce that $Q = [Q, c]$. The further assumption that $c \in A^{\#}$ commutes q down in H immediately forces Q to be a contained in $F(H)$.
Moreover $O_q(C) \cdot Q$ is a c-invariant q-group. We see that $N_Q(O_q(C)) \neq 1$. Suppose for a contradiction that c centralises $K_0 := N_Q(O_q(C))$. Then Lemma 1.1.14 (g) implies that $[O_q(C) \cdot Q, c] = 1$. This contradicts $1 \neq [Q, c]$. Hence c does not centralises $N_Q(O_q(C))$. That leads to $Q = N_Q(O_q(C))$.
Suppose for a contradiction that c centralises the C-invariant group $C_Q(O_q(C))$. Then we have $C_Q(O_q(C)) \leq C$ and hence we conclude that

$$[C_Q(O_q(C)), O_{q'}(F(C))] \subseteq Q \cap O_{q'}(F(C)) = 1.$$

Therefore $C_Q(O_q(C))$ is a subgroup of $O_q(C)$. Again Lemma 1.1.14 (g) yields a contradiction to $[c, Q] = 1$. Thus we have $[C_Q(O_q(C)), c] \neq 1$ and conclude $Q = C_Q(O_q(C))$.

We further observe from $c \in A$ that the abelian group A is a subgroup of C. Hence Lemma 1.1.14 (e) implies that $Q = \langle C_Q(t) \mid t \in A^{\#} \rangle$ and we obtain an involution $a \in A$ such that $[c, C_Q(a)] \neq 1$. Moreover there exists an element $i \in \{1, 2\}$ such that $a = c^{y^i}$. We want to show that $y \in H$. We notice from $y^3 \in C_G(c) \leq H$ that it suffices to show that $y^i \in H$. Hence we may assume that $y^i = y$.
We set $Q_0 := [c, C_Q(a)] \subseteq C_G(a) = C^y \subseteq H^y$. From $Q \leq F(H)$ it follows that Q_0 is a subgroup of $[F(H), c] \leq F(H)$. The group $H^y \cap Q$ is $C_{H^y}(c)$-invariant. Since every involution of A commutes q down in H, every involution of $A = A^y$ commutes q down in H^y. Consequently we observe $Q_0 = [c, C_Q(a)] \leq [c, H^y \cap Q] \leq F(H^y)$.
Altogether we have $Q_0 \leq O_q(H^y) \cap C^y \leq O_q(C^y) = O_q(C)^y \leq C_G(Q)^y$. This implies that $Q^y \leq C_G(Q_0) \leq N_G(Q_0) \leq M$ for every maximal subgroup M of G, containing $N_G(Q_0)$. Since Q is normalised by C, the group Q^y is $C_H(c^y)$-invariant. As (b) is false there is a maximal subgroup $M \neq E(G)$ containing $N_G(Q_0)$ such that a commutes q down in M and furthermore M is $SL(2, q)$-free. We see that Q^y is $C_M(a)$-invariant and hence we deduce that $Q^y = [Q, c]^y = [Q^y, a] \leq F(M)$.

Finally we have $M \rightsquigarrow H^y$ via Q^y, $H^y \rightsquigarrow M$ via Q_0 and $H \rightsquigarrow M$ via Q_0.
Thus Lemma 2.4.4 implies that $M = H^y$ or $F^*(M)$ and $F^*(H^y)$ are both q-subgroups of G. In the second case Lemma 2.4.5 implies that $M = H^y$ since M and H^y are $SL(2, q)$-free by the failure of (c). For this reason $H \rightsquigarrow M = H^y$ holds. Now Lemma 2.4.6 yields $H = H^y$. Finally Part (a) follows, because H is a maximal non-normal subgroup of G. □

2.5 Simple Groups with Small Sylow 2-subgroups

In this section we will prove the following well-known theorem.

2.5.1 Theorem

Let G be a finite simple group such that a Sylow 2-subgroup of G is elementary abelian of order 4. Then $G \cong PSL(2,q)$ for some $q \equiv 3$ or 5 mod 8.

In [4] Bender gave a new proof for the classification of finite simple groups with dihedral Sylow 2-subgroups that is due to Gorenstein and Walter. The ideas of the second section in Bender's article are sufficient to verify Theorem 2.5.1

For the remainder of this section let G be a minimal counterexample to Theorem 2.5.1. Moreover suppose that T is a Sylow 2-group of G and c is an involution of T. Let further C be a maximal subgroup of G containing $C_G(c)$.

2.5.2 Lemma

All involution of G are conjugate in G.
Moreover if U is a proper subgroup of G containing $C_G(c)$, then U has exactly three conjugation-classes of involutions. Moreover U has a normal 2-complement and every involution of T commutes every odd prime down in H.

Proof
If H is a of G such that $C_G(c) \leq H$, then the index $|N_H(T) : C_H(T)|$ is odd, since T is an abelian Sylow 2-subgroup of G contained in $C_G(c) \leq H$.
Suppose first that $H = G$. The group G has no normal 2-complement and hence Burnside's p-Complement Theorem 1.1.10 implies that $N_G(T) \neq C_G(T)$. As T is elementary abelian of order 4, we conclude that $N_G(T)/C_G(T) \cong Z_3$. Thus T has exactly one class of involutions in G.
Assume now that $H = U$ is a proper subgroup of G and suppose for a contradiction that all involutions of T are conjugate in U. Then all involutions of U are conjugate in U by Sylow's Theorem. It follows that U contains all G-centralisers of its involutions. This means that U is strongly embedded in G. Finally Theorem 1.2.12 (e) yields a contradiction.
It follows that $N_U(T) = C_U(T)$ and the p-Complement Theorem of Burnside 1.1.10 together with Lemma 3.6 of [38] imply the assertion. $\square$

2.5.3 Lemma

We have $C = C_G(c)$ and $F^*(C) = F(C) > O_2(C) = \langle c \rangle$.

Proof *Compare with Theorem 2.6 of [4]*
We first observe that the group G and hence every maximal subgroup of G is $SL(2,q)$-free for every odd prime q, because 8 divides $|SL(2,q)|$ for every odd prime q but not the order of G. From Lemma 2.5.2 we deduce that C has a normal 2-complement. The Odd Order Theorem 1.1.12 forces C to be soluble. In particular we have $F^*(C) = F(C)$.
Suppose for a contradiction that there is a prime q and a $C_G(c)$-invariant q-subgroup of G which is not centralised by c. Let Q be a q-group of minimal order such that Q is normalised by $C_G(c)$ but not centralised by c. We may choose C such that $C_G(c) \leq N_G(Q) \leq C$. Then Lemma 2.5.2 shows that every involution of C commutes q down in C. Thus Lemma 2.4.8 is applicable. The involutions of T are not conjugate in C by Lemma 2.5.2 and every maximal subgroup of G is $SL(2,q)$-free. So Lemma 2.4.8 provides an element $a \in T \setminus \langle c \rangle$, such that $[c, C_Q(a)] \neq 1$ and there is no maximal subgroup H of G containing $N_G([c, C_Q(a)])$ such that a commutes q down. As G is a minimal counterexample, Lemma 1.10 (i) of [4] leads to a contradiction.

It follows that $c \in C_G(T \cdot O(F(H))) \leq C(F(H)) \leq F(H)$ from Lemma 1.1.18 (h). Suppose for a contradiction that $T = O_2(C)$. Then we obtain $T \leq Z(C)$, because C has a normal 2-complement. Consequently we have $C = C_G(a)$ for every involution $a \in C$. This means that C is a strongly embedded subgroup of G contradicting Theorem 1.2.12 (e). Finally we conclude $O_2(C) = \langle c \rangle$ and observe that $C = N_G(O_2(C)) = N_G(\langle c \rangle) = C_G(c)$. Altogether we deduce from Lemma 1.1.18 (h) that $F(C) \geq \langle c \rangle$. □

2.5.4 Lemma

Suppose that $H \neq C$ is a subgroup of G containing T and such that $N_G(Q) \leq H$ for some subgroup Q of $F(C)$ and such that $c \notin E(H)$.
Then we have that $[H, c] \leq O_{\pi(F(C))'}(F(H))$ and $[T, O(C)] \nleq F(O(C))$.

Proof *Compare with Lemma 2.7 of [4]*
We set $\pi := \pi(F(C))$ and $F := F(H)$. Then Lemma 2.5.3 implies that $|\pi| \geq 2$. We observe that $N_{F^*(C)}(Q) \leq H$ and hence C infects H.
Since $c \notin E(H)$, the group $E(H)$ has cyclic Sylow 2-subgroups and is therefore trivial by Burnside's p-Complement Theorem 1.1.10 and the Odd Order Theorem 1.1.12.
Suppose for a contradiction that $[F, c] = 1$. Then we have $F^*(H) \leq F \leq C$ and hence $H \rightsquigarrow C$. From Lemma 2.4.4 (c) and $|\pi| \geq 2$ we deduce the contradiction $H = C$. From Lemma 2.4.4 (a) we see that $O_\pi(F)$ is centralised by c. We conclude that $1 \neq O_{\pi'}(F(H))$. Moreover (b) of the same lemma implies that $O_{\pi'}(F(H))$ is inverted by c.
Thus Lemma 1.3.9 is applicable. Since $[h, c]^c = (h^{-1} \cdot h^c)^c = [c, h] = [h, c]^{-1}$ for all $h \in H$, Lemma 1.3.9 yields $[O^2(H), c] \leq \langle [F, c]^{O^2(H)} \rangle \leq \langle O_{\pi'}(F)^{O^2(H)} \rangle = O_{\pi'}(F)$. We conclude that $[H, c] = [O^2(C) \cdot T, c] = [O^2(H), c] \leq O_{\pi'}(F)$.
Furthermore Lemma 1.1.14 (e) implies that $O_{\pi'}(F) = \langle C_{O_{\pi'}(F)}(t) \mid t \in T^\# \rangle$. Therefore there is an element $a \in T^\#$ such that $1 \neq C_{O_{\pi'}(F)}(a) = [C_{O_{\pi'}(F)}(a), c] \leq [O(C_G(a)), T]$. We obtain that $F(O(C_G(a)))$ is a π-group, since $F(O(C))$ is a π-group since a and c are conjugate in G. We finally conclude that $[T, O(C_G(a))] \nleq F(O(C_G(a)))$. As a and c are conjugate in $N_G(T)$, the assertion follows. □

2.5.5 Lemma

Let $a \in T \setminus \langle c \rangle$ and set $N := \{g \in O(C) \mid g^a = g^{-1}\}$.
Then N is a normal subgroup of $F(O(C))$ and $N_G(X) \leq C$ for all $1 \neq X \leq N$.

Proof *Compare with Theorem 2.10 of [4]*
Suppose for a contradiction that there is a subgroup H of G with $T \leq H$ and $c \in E(H)$ and a T-invariant q-subgroup Q for some odd prime q that is not centralised by T. Then we deduce from Burnside's p-Complement Theorem 1.1.10 and the Odd Order Theorem 1.1.12 that $T \leq E(H)$. We conclude that $E(H)$ is isomorphic to $PSL(2, r)$ for some prime power r from our choice that G is a minimal counterexample. We further see that $C_{E(H)}(T) \neq N_{E(H)}(T)$ and, since $N_G(T)/C_G(T)$ is cyclic of order 3, we obtain that $N_H(T) \cdot E(H) = C_H(T) \cdot E(H)$. Moreover Frattini argument yields that $H = E(H) \cdot N_H(T) = E(H) \cdot C_H(T)$. It follows from $[T, Q] \neq 1$ that $[T, Q] \cap E(H) \neq 1$. But now $[T, Q] \cap E(H)$ is an T-invariant q-subgroup of $E(H)$, which is impossible by Dickson's Theorem 1.2.2.
Suppose for a contradiction that there is a prime q such that $1 \neq X := C_{O_q(C)}(a) \neq O_q(C)$. From $O_2(C) = \langle c \rangle \leq C_G(a)$ we obtain that q is odd.
The group $C_G(a)$ has a normal 2-complement by Lemma 2.5.2. Hence Lemma 1.1.14 (b) provides a T-invariant Sylow q-subgroup Q of $C_G(a)$ containing X. Since a and c are conjugate in G, we have $Q \nleq C$. In particular from Lemma 1.1.14 (g) we deduce that $1 \neq [c, C_Q(X)] \leq [c, N_Q(X)]$.
Let $H < G$ be such that $N_G(X) \leq H$. Then we observe that $[c, N_Q(X)] \leq [c, H] \cap Q$. This

forces $[c, H]$ to be no q'-groups. Now Lemma 2.5.4 implies that $c \in E(H)$. But $[c, N_Q(X)]$ is a non-trivial T-invariant q-subgroup of H, which is impossible by the above investigation. Consequently we have that $C_{O_q(C)}(a) = O_q(C)$ or $C_{O_q(C)}(a) = 1$ for all primes q. In particular $C_{F(H)}(a)$ is a Hall subgroup of $F(H)$. We apply Lemma 1.3.9 to observe that we have $N = [F(C), a] \leq F(O(C))$.
Suppose for a contradiction that there is a subgroup $X \neq 1$ of N such that $N_G(X) \nleq C$. Then X is T-invariant, as it is centralised by c and inverted by a. Let H be a proper subgroup of G that contains $N_G(X)$. Then the above investigation shows that $c \notin E(H)$. Finally Lemma 2.5.4 implies that $F(O(C)) \ngeq [T, O(C)] = [a, O(C)] = N$. This is a contradiction. □

Proof of Theorem 2.5.1

Lemma 2.5.5 yields exactly the configuration from Lemma 2.2 of [4]. Thus Theorem 3 of [7] yields that $C = T \cdot N = \langle a \rangle \cdot N$ with $N = \{g \in O(C) \mid g^a = g^{-1}\}$ for an involution $a \in T \setminus \langle a \rangle$. Now we can apply [5] to conclude that G is considered as permutation group a Zassenhaus group of degree q for some odd natural number q, that has order $(q+1) \cdot q \cdot (q-1)$. Moreover there is a subgroup Q of order q such that $N_G(Q)$ is a stabiliser of a point and has the form $Q \cdot D$, where D is an abelian group of order $\frac{q-1}{2}$. Theorem 13.1.1 of [22] yields that $N_G(Q)$ is a Frobenius group with abelian Frobenius complement D. Finally 10.3.1 (iv) of [22] implies that D is cyclic. Thus 13.3.5 of [22] forces G to be isomorphic to $SL(2, q)$. Since the order of G is $(q + 1) \cdot q \cdot (q - 1)$ and divisible by 4 but not by 8, we conclude that $q \equiv 3$ or $5 \bmod 8$. This is a contradiction. □

Part II

Finite Groups with 3-Locally Central Elements

3 A First Approach

3.1 The Main Theorem

In this section we present the main theorem and its minimal counterexample.
We examine some first cases and introduce the most important objects.

3.1.1 Definition

Let G be a finite group, p be a prime and suppose that P is a Sylow p-subgroup of G.

(a) The group $Z_p^*(G)$ is the full pre-image of $Z(G/O_{p'}(G))$ in G.

(b) An element $x \in P$ is **p-locally central in G with respect to P** if and only if for all non-trivial subgroups R of P we have $x \in Z(N_G(R))$.

3.1.2 Lemma

Let G be a finite group and p be a prime and suppose that P is a Sylow p-subgroup of G. Then $x \in P$ is p-locally central with respect to P if an only if for all non-trivial subgroups R of P we have $N_G(R) \leq C_G(x)$.
In particular, if $x \in P$ is p-locally central, then $P \leq C_G(x)$ and further the following hold:

(a) The element x is strongly closed in G with respect to P. This means, if $g \in G$ such that $x^g \in P$, then $g \in C_G(x)$.

(b) If $x \in H$ and H is a subgroup of G, then x is p-locally central in H with respect to $H \cap P^g$ for some $g \in C_G(x)$.

(c) If $N \trianglelefteq G$, then Nx is p-locally central in G/N with respect to PN/N.

(d) If $x \in S$ and $S \in \mathrm{Syl}_p(G)$, then x is p-locally central in G with respect to S.

Proof

Let $1 \neq R \leq P$ and x be an element of P. If we have $N_G(R) \leq C_G(x)$, then x centralises R and $N_G(R)$. Thus we conclude $x \in C_G(R) \cap C_G(N_G(R)) \leq N_G(R) \cap C_G(N_G(R)) = Z(N_G(R))$. On the other hand $x \in Z(N_G(R))$ implies that $N_G(R) \leq C_G(x)$.

Part (a), (b) and (c) are Lemma 3.1 and Lemma 3.2 of [32].
For Part (d) let $x \in S$ and $S \in \mathrm{Syl}_p(G)$. Then we apply Part (b) to conclude that x is p-locally central in S with respect to $S \cap P^g$ for some $g \in C_G(x)$. In particular we observe that $S, S \cap P^g \in \mathrm{Syl}_p(S)$. It follows that $S = P^g$. Let $1 \neq R \leq S$. Then $R^{g^{-1}}$ is a non-trivial subgroup of P. Consequently we obtain $N_G(R)^{g^{-1}} = N_G(R^{g^{-1}}) \leq C_G(x)$, since x is p-locally central in G with respect to P. From $g \in C_G(x)$ we conclude that $N_G(R) \leq C_G(x)$. □

3.1.3 Remark

From Lemma 3.1.2 (d) we obtain that the property of being p-locally central in a finite group with respect to a Sylow subgroup does not depend on the choice of the Sylow subgroup. Therefore we omit the "respect"-part and say that an element x of a finite group G is p-locally central in G, if it is p-locally central in G with respect to one and therefore all Sylow p-subgroups of G containing x.

Now we state the main theorem.

3.1.4 Main Theorem

Let G be a finite group and P be a Sylow 3-subgroup of G. Suppose that $x \in P$ is a 3-locally central element in G with respect to P. Then x is an element of $Z_3^*(G)$.

For the remainder of this part let G be a minimal counterexample to the main theorem. Then G is a finite group and there exists a 3-locally central element $x \in G$ such that $x \notin Z^*(G)$. We choose first $|G|$ minimal and then x of minimal order.

We set $M := C_G(x)$ and let P be a Sylow 3-subgroup of G contained in M.
We further define

$$\mathfrak{M} := \{H \mid H\ max\ G,\ x \in H \text{ and } H \neq M\},$$
$$\sigma := \{q \in \pi(G) \mid q \nmid |G : M|\},$$
$$D^*(M) := \{y \in M \mid y^3 = 1 \text{ and } y \notin \langle x \rangle\} \text{ and } I^*(M) := \{a \in M^\# \mid a^2 = 1 \text{ and } C_G(a) \leq M\}.$$

3.1.5 Lemma

If we have $G = G' \cdot \langle x \rangle$ such that $|G : G'| \in \{1, 3\}$, then G' is not isomorphic to $\mathrm{PSL}(2, q)$ for any prime power q.

Proof

Suppose for a contradiction that we have $G' \cong \mathrm{PSL}(2, q)$ for some prime power q and that $G = G' \cdot \langle x \rangle$ with $|G : G'| \in \{1, 3\}$.
If x is not contained in G', then there is an element $g \in G'$ such that $x \cdot g$ induces field automorphisms in G' by Lemma 1.2.4. Thus $x \cdot g$ is a conjugate of x by Proposition 4.9.1 of [24]. Moreover $q = r^3$ for some prime power r and $C_{G'}(x) \cong \mathrm{PSL}(2, r)$ by Lemma 1.2.4. Thus we obtain

$$|G' : C_{G'}(x)| = \frac{|\mathrm{PSL}(2,r^3)|}{|\mathrm{PSL}(2,r)|} = \frac{(r^3)^2 \cdot (r^3+1) \cdot (r^3-1)}{r^2 \cdot (r+1) \cdot (r-1)} r^4 \cdot (r^2 - r + 1) \cdot (r^2 + r + 1).$$

If we have $r \equiv 0 \pmod 3$, then 3 divides r^4. If we have $r \equiv 1 \pmod 3$, then we conclude $r^2 + r + 1 \equiv 1^2 + 1 + 1 \equiv 0 \pmod 3$. Finally, if we have $r \equiv 2 \pmod 3$, then we observe $r^2 - r + 1 \equiv 4 - 2 + 1 \equiv 0 \pmod 3$. In all cases 3 divides $|G' : C_{G'}(x)|$. This is a contradiction, because x is 3-locally central in G and therefore centralises a Sylow 3-subgroup of G' by Lemma 3.1.2.
We conclude that $G = G'$. If G had cyclic Sylow 3-subgroups, then the property of x to be 3-locally central would force $N_G(P)$ to be $C_G(P)$. This would contradict the p-Complement Theorem of Burnside. Now Dickson's Theorem 1.2.2 implies that q is a power of 3. By the same theorem G has elementary abelian Sylow 3-subgroups of order q and $N_G(P)/C_G(P)$ is cyclic of order $q - 1$. In particular $N_G(P)$ is transitive on $P^\#$. This contradicts Lemma 3.1.2 (a). □

3.1.6 Proposition

If we have $G = G' \cdot \langle x \rangle$ such that $|G : G'| \in \{1, 3\}$ and $C_G(x)$ is a maximal subgroup of G, then G' is not a simple Bender group.

Proof

Suppose for a contradiction that the Lemma is false.
If G' is a Suzuki group, then $G' = O_{3'}(G)$ and $G'x \in G/G' = \langle x \rangle \cdot G'/G' = Z(G/O_{3'}(G))$. In particular G is not a counterexample. This is a contradiction.
Thus G' is not a Suzuki group. Moreover G' is not isomorphic to $\mathrm{PSL}(2, q)$ for any prime power q by Lemma 3.1.5. Hence Theorem 1.2.12 forces G' to be isomorphic to $\mathrm{PSU}(3, 2^n)$ for some natural number $n \geq 2$.
Suppose for a contradiction that $x \notin G'$, then by 3.6.3 of [39] either $x \in Inndiag(G)$ or we obtain from Proposition 4.9.1 (d) of [24] an element $z \in G$ that is conjugate to x and induces

a field automorphism of G'.
In the first case $x \in \mathrm{PGU}(3, 2^n)$. In particular 2^n+1 is divisible by 3. Moreover Lemma 4.3.6 of [25] yields that $C_{G'}(x)$ is isomorphic to the direct product $\mathrm{PSL}(2, 2^n)$ with a cyclic group of order $\frac{2^n+1}{3}$. We use order formulas of the sections 3.3.1 and 3.6 of [39] to conclude that:

$$|G' : C_{G'}(x)| = \frac{(2^n)^3\cdot((2^n)^3+1)\cdot((2^n)^2-1)\cdot 3}{3\cdot 2^n\cdot((2^n)^2-1)\cdot(2^n+1)} \quad = 2^{2n}\cdot\frac{2^{3n}+1}{2^n+1} = 2^{2n}\cdot(2^{2n}-2^n+1).$$

In addition we observe that $2^{2n} - 2^n + 1 \equiv 1 - (-1) + 1 \equiv 0 \pmod 3$.
This contradicts the fact that $P \leq M$, as x is 3-locally central.
In the second case Lemma 4.3.10 of [25] yields that $C_{G'}(x) \cong \mathrm{PGU}(3, 2^{n/3})$, if n is odd. For even n Proposition 4.9.1 (a) of [24] implies that $O^{2'}(C_{G'}(x)) \cong \mathrm{PSU}(3, 2^{n/3})$. From Theorem 6.5.3 of [24] we obtain that $C_{G'}(x) \cong \mathrm{PSU}(3, 2^{n/3}) \cong \mathrm{PGU}(3, 2^{n/3})$ in this case too. We observe that:

$$\begin{aligned}|G' : C_{G'}(x)| = &\quad \frac{(2^n)^3\cdot((2^n)^3+1)\cdot((2^n)^2-1)}{(3,2^n+1)\cdot(2^{n/3})^3\cdot(2^n+1)\cdot((2^{n/3})^2-1)} = 2^{2n}\cdot\frac{2^{3n}+1}{(3,2^n+1)\cdot 2^n+1}\cdot\frac{2^{2n}-1}{2^{2n/3}-1}\\ = &\quad 2^{2n}\cdot\frac{2^{2n}-2^n+1}{(3,2^n+1)}\cdot(2^{4n/3}+2^{2n/3}+1).\end{aligned}$$

As above $2^{2n} - 2^n + 1$ is divisible by 3 if n is odd. Therefore $\frac{2^{2n}-2^n+1}{(3,2^n+1)}$ is a natural number. Moreover we observe $2^{4n/3} + 2^{2n/3} + 1 \equiv 1 + 1 + 1 \equiv 0 \pmod 3$. This again contradicts the fact that $P \leq C_G(x)$.
This contradiction shows that $G = G'$ is isomorphic to $\mathrm{PSU}(3, 2^n)$. We apply Theorem 6.5.3 of [24] to obtain from $x \in Z(C_G(x))$ and $C_G(x)$ *max* G that $C_G(x)$ is the image of the stabiliser in $\mathrm{SU}(3, 2^n)$ of a non-degenerate subspace of the natural module of $\mathrm{SU}(3, 2^n)$ in its natural action.
By section 3.6.2 of [39] the stabiliser in $\mathrm{GU}(3, 2^n)$ of a non-degenerate subspace of the natural module of $\mathrm{GU}(3, 2^n)$ in its natural action is isomorphic to $\mathrm{GU}(1, q) \times \mathrm{GU}(2, q)$. Moreover $x \in Z(C_G(x))$ is a 3-element. It follows that 3 divides $2^n + 1$. Additionally $\mathrm{GU}(1, q) \times \mathrm{GU}(2, q)$ has order $(2^n + 1)^2 \cdot (2^{2n} - 1) \cdot 2^n$. Thus its image in $\mathrm{PGU}(3, 2^n)$ has order $(2^n+1)\cdot(2^{2n}-1)\cdot 2^n$. In particular $C_G(x)$ has order $\frac{(2^n+1)\cdot(2^{2n}-1)\cdot 2^n}{(3,2^n+1)}$. Finally we calculate

$$|G : C_G(x)| = \frac{(2^n)^3\cdot((2^n)^3+1)\cdot((2^n)^2-1)\cdot(3,2^n+1)}{(3,2^n+1)\cdot(2^n+1)\cdot(2^{2n}-1)\cdot 2^n} \quad = 2^{2n}\cdot\frac{2^{3n}+1}{2^n+1} = 2^{2n}\cdot(2^{2n}-2^n+1).$$

As 3 divides $2^n + 1$, we conclude that 3 divides $|G : C_G(x)|$.
This finally and again contradicts $P \leq C_{G'}(x)$. □

3.2 The Reduction and First Results of Rowley

In this section we start to investigate our minimal counterexample.
We go along Rowley's reduction in Section 3 of [32].

3.2.1 Lemma

We have:

(a) We have $O_{3'}(G) = F(G) = Z^*(G) = Z_3^*(G) = 1$.

(b) The group G has no non-trivial normal soluble subgroup.

(c) The element x is of order 3.

(d) The Sylow 3-subgroups of G are not cyclic.

(e) The group G' is non-abelian simple and $G = \langle x \rangle \cdot G'$ holds.
In particular $|G : G'| \in \{1, 3\}$.

Proof

This is Lemma 3.3 of [32]. As the first part of Rowley's lemma is here divided into four statements, we prove part (a).

By Lemma 3.1.2 (c) the element $O_{3'}(G)x$ is 3-locally central in $G/O_{3'}(G)$. Suppose for a contradiction that $O_{3'}(G) \neq 1$. Then the minimal choice of $|G|$ together with the fact that $O_{3'}(G/O_{3'}(G))$ is trivial, imply that

$$O_{3'}(G)x \in Z_3^*(G/O_{3'}(G)) = Z(G/O_{3'}(G)).$$

Therefore x is an element of $Z_3^*(G)$ contradicting the choice of G as a counterexample. Thus $O_{3'}(G)$ is trivial and for all primes $p \in \mathbb{P} \setminus \{3\}$ we obtain that $O_p(G) \leq O_{3'}(G) = 1$. Suppose for a contradiction that $O_3(G) \neq 1$. Then we have $x \in Z(N_G(O_3(G)))$, because x is 3-locally central in G with respect to P and $O_3(G) \leq P$. Hence $N_G(O_3(G)) = G$ leads to $x \in Z(G) \leq Z_3^*(G)$. This contradicts again the fact that G is a counterexample. Therefore we have $F(G) = 1$. It follows that $O(G) = 1$ and hence we obtain $Z^*(G) = Z(G) \leq F(G) = 1$. From $O_{3'}(G) = 1$ we similarly observe that $Z_3^*(G) = Z(G) \leq F(G) = 1$. □

3.2.2 Lemma

The following hold:

(a) If H is a maximal subgroup of G containing x, then H contains no non-trivial normal subgroup of G.

(b) Suppose that $g \in G$ is such that $\emptyset \neq Y \subseteq M \cap M^g$.
Then $g = m \cdot c$ for an element $c \in C_G(Y)$ and an element $m \in M$.

(c) If L is a non-trivial subgroup of M, then $N_G(L) = N_M(L) \cdot C_G(L)$ holds.

(d) Let H be a maximal subgroup of G containing x. Then we have one of

(1) $H = C_G(x) = M$ or

(2) $H = R \cdot O_{3'}(H)$, where R is a cyclic 3-group with $\Omega_1(R) = \langle x \rangle$.

(e) We have $\mathfrak{M} \neq \emptyset$.

(f) Suppose that $x \in U < G$. If U has non-cyclic Sylow 3-subgroups or no normal 3-complement, then U is a subgroup of M.

(g) If we have $a \in I^*(M)$, then $a^G \cap M$ is a subset of $I^*(M)$.

Proof

Part (a) to (e) are Lemmas 3.4. to 3.7. of [32].
Part (f) follows directly from Part (d), since every proper subgroup of G is contained in at least one maximal subgroup of G.
Part (g) is Lemma 3.9 (ii) of [32]. □

3.2.3 Corollary

The group G' is neither a Bender group nor isomorphic to $\mathrm{PSL}(2, q)$ for any prime power 2.

Proof

By Lemma 3.2.1 (e) we have that $G = G' \cdot \langle x \rangle$ such that $|G : G'| \in \{1, 3\}$. Moreover G' is non-abelian simple and $C_G(x)$ is a maximal subgroup of G by Lemma 3.2.1 (d) and Lemma 3.2.2 (d). Hence Lemma 3.1.5 and Proposition 3.1.6 yield the assertion. □

3.3 The Set σ

In this section we consider Sylow subgroups of M. Moreover we show that we either obtain the whole $\{2,3\}$-structure of G in M or that G is S_4-free.

3.3.1 Lemma

Let $q \in \pi(G)$ and suppose there is a non-trivial x-invariant q-subgroup of G.
Then there exists a x-invariant Sylow q-subgroup of G. In addition all those x-invariant Sylow q-subgroups of G are conjugate in M.

Proof

Let Q be a x-invariant q-subgroup of G with maximal order. Then we have $x \in N_G(Q)$ and Lemma 3.2.2 (f) yields that $N_G(U) \leq M$ or $N_G(U)$ has a normal 3-complement. In both cases $N_G(Q)$ has a x-invariant Sylow q-subgroup, in the second case by Lemma 1.1.14 (b). From the maximal choice of Q it follows that Q is a Sylow q-subgroup of $N_G(Q)$. Therefore Q is a Sylow q-subgroup of G.

Let now Q_1 and Q_2 be x-invariant Sylow q-subgroups of G. By Sylow's Theorem there exists an element $g \in G$ such that $Q_1^g = Q_2$. This leads to $x, x^g \in N_G(Q_2)$.
If we have $N_G(Q_2) \nleq M$, then $N_G(Q)$ has cyclic Sylow 3-subgroups by Lemma 3.2.2 (f). Hence Sylow's Theorem provides an element $h \in N_G(Q_2)$ such that $\langle x \rangle = \langle x^g \rangle^h = \langle x \rangle^{gh}$. In particular, as x is 3-locally central in G, we see that $g \cdot h \in N_G(\langle x \rangle) \leq M$ and $Q_1^{gh} = Q_2^h = Q_2$.
If we have $N_G(Q_2) \leq M$, then we conclude that $x \in N_G(Q_2) \cap N_G(Q_2)^{g^{-1}} \leq M \cap M^{g^{-1}}$ and Lemma 3.2.2 (b) forces g^{-1} to be an element of $M \cdot C_G(x) = M$. □

3.3.2 Lemma

Let $q \in \pi(M) \setminus \{3\}$ and $y \in D^*(M)$. Then all of the following conditions lead to $q \in \sigma$.

(a) There is a non-trivial q-subgroup Q of M such that $C_G(Q) \leq M$.

(b) There is a q-element $g \in M$ such that $C_G(g) \leq M$.

(c) There is a non-trivial $\langle x, y \rangle$-invariant q-subgroup Q of G.

(d) We have $q \in \pi(C_G(y))$.

(e) There is a $\langle x, y \rangle$-invariant subgroup H of G such that H has a normal 3-complement and $q \in \pi(H)$.

Proof

(a) Suppose condition (a) holds and let S be a Sylow q-subgroup of M such that $Q \leq S$. Then Lemma 3.2.2 (c) shows that $N_G(S) = N_M(S) \cdot C_G(S) \subseteq N_M(S) \cdot C_G(Q) \subseteq M$. Therefore S is a Sylow q-subgroup of G.

(b) *(Compare with 3.10 of [32]).*
This follows immediately from (a) since $C_G(g) = C_G(\langle g \rangle)$ for every element $g \in G$.

(c) Suppose condition (c) holds. Then we have $\langle x, y \rangle \leq N_G(Q)$. Therefore $N_G(Q)$ has non-cyclic Sylow 3-subgroups and contains x. By Lemma 3.2.2 (a) the group $N_G(Q)$ is a proper subgroup of G. Thus Part (f) of the same lemma yields that we have $C_G(Q) \leq N_G(Q) \leq M$. Now the assertion follows by Part (a).

(d) Suppose condition (d) holds. If we have $q = 3$, then we obtain $q \in \sigma$ from the fact that x is 3-locally central. Suppose that $q \neq 3$ and let $1 \neq Q$ be a q-subgroup of $C_G(y)$. Then Q is $\langle x, y \rangle$-invariant, since $C_G(y) \leq N_G(\langle y \rangle) \leq M$. Therefore Part (c) forces $q \in \sigma$.

(e) Suppose condition (e) holds. If we have $q = 3$, then we observe again $q \in \sigma$. Suppose that $q \neq 3$. Then, as H has a normal 3-complement, we have $q \in \pi(O_{3'}(H))$. Since H is $\langle x, y\rangle$-invariant, $\langle x, y\rangle$ acts coprimely on $O_{3'}(H)$. Consequently we obtain from Lemma 1.1.14 (b) a $\langle x, y\rangle$-invariant Sylow q-subgroup of $O_{3'}(H)$. Now Part (c) yields that we have $q \in \sigma$. □

3.3.3 Proposition (Rowley)

Suppose that $I^*(M) \neq \emptyset$. If $V = \langle a, b\rangle$ is an elementary abelian subgroup of G of order 4 for some involutions a and b of G, then $(V^h)^\# \not\subseteq I^*(M)$ for all $h \in G$.
Moreover, if $g \in G$ and H is a maximal subgroup of G that contains $C_G(ab)$ and $\langle a, b\rangle^g$, then we have $g \in H$. In particular we observe that $N_G(T) \leq H$ for some Sylow 2-subgroup T of H and thereby of G.

Proof
This result is basically Lemma 4.2, Lemma 4.7 and Lemma 4.8 of [32].

As G' is no Bender group by Corollary 3.2.3 and since $I^*(M) \neq \emptyset$, Lemma 4.2 of [32] provides some elements $c, d \in I^*(M)$ such that $[c, d] = 1$ and $c \cdot d \notin I^*(M)$.
Let H be a maximal subgroup of G that contains $C_G(cd)$.
Since $x \in C_G(c) \cap C_G(d) \leq C_G(cd)$ and $C_G(cd)$ is not a subgroup of M, Lemma 3.2.2 (f) forces H to be an element of $\mathfrak{M}$. We obtain $2 \in \sigma$ from Lemma 3.3.2 (b). From this and Lemma 1.3.7 (b) we deduce that $H = C_H(x) \cdot O(H)$. Applying Lemma 1.3.1 (b) we get

$$[H, x] = [O(H), x] \leq \langle C_{[O(H),x]}(B) \mid B\ max\ \langle c, d\rangle,\ C_G(B) \not\subseteq C_G(x)\rangle = C_{[O(H),x]}(cd).$$

Consequently (*): $H = C_H(x) \cdot [O(H), x] = C_H(x) \cdot C_{O(H)}(cd)$ holds.

Let T be a x-invariant Sylow 2-subgroup of H containing $\langle c, d\rangle$. Then x centralises T by Lemma 3.3.1. Let further g be an element of $N_G(T)$. Then $c^g \in \langle c, d\rangle^g \leq T \cap M^g \leq M \cap M^g$. According to this Lemma 3.2.2 (b) yields that $g \in M \cdot C_G(c) = M$ and we have $c^g, d^g \in I^*(M)$ by Lemma 3.2.2 (g). Analogously to (*) we get

$$H = C_H(x) \cdot C_{O(H)}((cd)^g) \text{ and } [H, x] \leq C_{O(H)}((cd)^g) \leq C_G(cd)^g \leq H^g.$$

Applying Lemma 1.3.7 (a) we observe that

$$[H, x] = [O_{3'}(H), x] = [[O_{3'}(H), x], x] = [[H, x], x] \leq [H^g, x].$$

From $g \in M$ we deduce $[H, x] \leq [H^g, x] = [H, x]^g$ and so $[H, x] = [H, x]^g = [H^g, x]$. Since $[H^g, x]$ is normal in H^g, it follows that $H^g \leq N_G([H, x]) = H$. Now the fact that H is a maximal subgroup of G together with Lemma 3.2.2 (a) implies $g \in H$. Therefore T is a Sylow subgroup of G and we have $N_G(T) \leq H$.

Let now $a, b \in I^*(M)$ such that $\langle a, b\rangle$ is elementary abelian of order 4. By Sylow's Theorem there is some $g \in M$ such that $\langle a, b\rangle^g \leq T \leq H = C_H(x) \cdot O(H)$. Since $\langle a, b\rangle$ is elementary abelian Lemma 1.1.14 (e) leads to $O(H) = \langle C_{O(H)}(v) \mid v \in \{a^g, b^g, (ab)^g\}\rangle$. As we have $O(H) \nleq M$ and $a^g, b^g \in I^*(M)$ by Lemma 3.2.2 (g), it follows that $(ab)^g \notin I^*(M)$. Finally the same lemma implies $a \cdot b \notin I^*(M)$. □

3.3.4 Corollary

The following hold:

(a) For all elementary abelian non-cyclic 2-subgroups V of G the centraliser $C_G(V)$ has cyclic Sylow 3-subgroups R such that $\Omega_1(R)$ is conjugate to a possibly trivial subgroup of $\langle x\rangle$.

(b) For all $y \in D^*(M)$ we have $r_2(C_G(y)) \leq 1$.

(c) For all elementary abelian non-cyclic 2-subgroups V of G such that all involutions of V are conjugate in G, we have $V \cap I^*(M) = \emptyset$.

Proof

(a) If $C_G(V)$ is a $3'$-subgroup of G, then the assertion is true.
Suppose that $1 \neq R \in \mathrm{Syl}_3(C_G(V))$. By Sylow's Theorem there is an element $g \in G$ such that $R \leq P^g$. As x^g is 3-locally central in G with respect to P^g, it follows that $V \leq C_G(R) \leq N_G(R) \leq M^g$. In particular we observe that $x^g \in C_G(V)$. Since V is non-cyclic, Proposition 3.3.3 provides an element $v \in V^\#$ such that $v \notin (I^*(M))^g$. From $v \notin Z(G)$ we deduce with Lemma 3.2.2 (f) that $C_G(v^{g^{-1}})$ has cyclic Sylow 3-subgroups. Due to $C_G(V) \leq C_G(v) = C_G(v^{g^{-1}})^g$ also $C_G(V)$ has cyclic Sylow 3-subgroups and contains x^g. This is the assertion.

(b) Suppose for a contradiction that $r_2(C_G(y)) \geq 2$. Then there is an elementary abelian non-cyclic 2-subgroup V of $C_G(y)$. We obtain that $V \leq C_G(y) \leq N_G(\langle y \rangle) \leq M$, as x is 3-locally central in G. Now $\langle x, y \rangle \leq C_G(V)$ contradicts (a).

(c) If we have $V \cap I^*(M) \neq \emptyset$, then we conclude $V^\# \subseteq I^*(M)$ from Lemma 3.2.2 (g).
As V is non-cyclic, this is a contradiction to Proposition 3.3.3. □

3.3.5 Lemma

(a) Suppose that H is a $\{2,3\}$-subgroup of G. If we have $r_2(H) \geq 3$, then $O_3(H)$ is cyclic and $\Omega_1(O_3(H))$ is conjugate to a subgroup of $\langle x \rangle$.

(b) Suppose that H is a subgroup of G with cyclic Sylow 3-subgroups and $x \in H$. Then H has a normal 3-complement.

Proof

(a) If we have $O_3(H) = 1$, then the assertion holds.
Let $O_3(H)$ be non-trivial. By Sylow's Theorem we may suppose that $H \cap P \in \mathrm{Syl}_3(H)$. Then we have $O_3(H) \leq H \cap P = P$ and so we obtain $x^G \cap O_3(H) \subseteq x^G \cap P \subseteq \{x\}$ from Part (a) of Lemma 3.1.2. Suppose that $r_2(H) \geq 3$. Then there is an elementary abelian subgroup A of order 8 of H. Corollary 3.3.4 (a) implies for every Sylow 3-subgroup R of the centraliser of every non-cyclic subgroup V of A that $\Omega_1(R)$ is conjugate to a subgroup of $\langle x \rangle$.
By Lemma 1.1.16 there exists a critical subgroup K of $O_3(H)$. The exponent of K is 3. Altogether Lemma 1.1.14 (e) implies $K = \langle C_K(V) \mid V \; max \; A \rangle \leq \langle x \rangle$. From $x \in Z(N_G(\langle x \rangle))$ we obtain that A acts trivially on K. As K is critical in $O_3(H)$, the group A acts trivially on $O_3(H)$. According to Corollary 3.3.4 (a) the group $C_G(A)$ has cyclic Sylow 3-subgroup containing x.

(b) Let R be a Sylow 3-subgroup of H containing x. Then every non-trivial $3'$-automorphism of R acts non-trivially on $\Omega_1(R) = \langle x \rangle$. Since x is 3-locally central, x is not inverted in G. Thus G does not induce any non-trivial automorphisms on R. Hence we observe that $C_H(R) = N_H(R)$, as R is a Sylow 3-subgroup of H. For the same reason Burnside's p-Complement Theorem 1.1.10 finishes the proof of Part (b). □

3.3.6 Lemma

Let S be a non-trivial 2-subgroup of G such that $N_G(S)$ involves a S_4.
Then we have $2 \in \sigma$ or $N_G(S)$ contains a subgroup that is isomorphic to S_4.

Proof
Since $N_G(S)$ is not S_4-free, $N_G(Z(S)) \geq N_G(S)$ involves a S_3. Therefore Lemma 1.3.3 provides elements y and b of $N_G(Z(S))$ such that y has order 3 and b is a 2-element and $\langle y, b\rangle/\langle b^2\rangle \cong S_3$. In particular b inverts y. As x is 3-locally central, x is not inverted in G and so we see $y \neq x$. Via conjugation in G we may choose S and y such that $y \in P$. Then $y \in D^*(M)$. If b is not an involution or if y acts trivially on $Z(S)$, then $C_G(y)$ is of even order and Lemma 3.3.2 (d) leads to $2 \in \sigma$. Otherwise Lemma 1.3.3 provides an involution $a \in Z(S)$ such that $S_4 \cong \langle a, y, b\rangle/\langle b^2\rangle \cong \langle a, y, b\rangle \leq N_G(S)$. □

3.3.7 Theorem (Stellmacher and Rowley)

If we have $2 \notin \sigma$, then G is S_4-free.

Proof *This is almost Lemma 5.2. of [32]. But we do not need the full strength of [34].*
Suppose for a contradiction that $2 \notin \sigma$ and that G is not S_4-free.
Then there is a non-trivial 2-subgroup S of G such that its normaliser $N_G(S)$ is not S_4-free by Lemma 1.1.7.
Let S be of maximal order with that property and such that $N_P(S) \in \mathrm{Syl}_3(N_G(S))$. Then $N_G(S)$ has a subgroup U isomorphic to S_4 by Lemma 3.3.6. We set $H := O^{2'}(N_G(S))$.
Then U is a subgroup of H. By Sylow's Theorem we may choose U such that $U \cap N_P(S) \neq 1$.
Let finally $T \in \mathrm{Syl}_2(H) = \mathrm{Syl}_2(N_G(S))$ be such that $U \cap T \in \mathrm{Syl}_2(U)$.

(1) We have $S = O_2(H)$ and $O_{2',2}(H) = S \cdot O(H)$.

Proof. The maximal choice of S implies that $S = O_2(N_G(S)) = O_2(H)$.
If we have $S_0 \in \mathrm{Syl}_2(O_{2',2}(H))$, then $S = O_2(H)$ is contained in S_0. A Frattini argument yields that $H = N_H(S_0)\cdot O_{2',2}(H) = N_H(S_0)\cdot O(H)$. From $O(H)\cap U \leq O(U) = 1$ we deduce that
$$S_4 \cong U \cong U/(O(H) \cap U) \cong U \cdot O(H)/O(H) \leq H/O(H) = N_H(S_0) \cdot O(H)/O(H) \cong N_H(S_0)/N_{O(H)}(S_0).$$
In particular $N_G(S_0)$ is not S_4-free and the maximal choice of S forces S to be S_0. ■

(2) No conjugate of x is contained in $N_G(S)$ but x normalises a Sylow 2-subgroup of G. Moreover S is not characteristic in T.

Proof. Suppose for a contradiction that $x^{g^{-1}} \in N_G(S)$ for some $g \in G$. Then we have that $x \in N_G(S)^g$ and $N_G(S)^g$ has no normal 3-complement, as it is not S_4-free. Hence Lemma 3.2.2 (f) implies that $N_G(S^g) = N_G(S)^g \leq M$. Now Lemma 3.3.2 (a) leads to the contradiction $2 \in \sigma$.
From $S_4 \cong U \leq N_G(S)$ we obtain that the Sylow 3-subgroup $N_P(S)$ has a non-trivial subgroup R such that $N_G(R) \leq C_G(x) = M$ is of even order. This and Lemma 3.3.1 yield that x normalises some Sylow 2-subgroup of G.
Assume for a contradiction that S *char* T. Then we see that $N_G(S) \geq N_G(T)$ and conclude that T is a Sylow 2-subgroup of G. Thus T is normalised by some conjugate of x. But now $N_G(T) \leq N_G(S)$ leads to a contradiction. ■

(3) The group H has no element of order 6 and $N_P(S)$ is cyclic and acts fixed-point-freely on S. In particular $O(H)$ is a $3'$-group.

Proof. By (2) we have $x^G \cap N_G(S) = \varnothing$. It follows that every element of order 3 in H is conjugate to an element of $D^*(M)$. Since every element of $D^*(M)$ has a centraliser of odd order by Lemma 3.3.2 (d), the group H has no element of order 6. This also shows that $N_P(S)$ acts elementwise fixed-point-freely on S. Consequently Lemma 1.1.14 (e) forces $N_P(S)$ to be cyclic. From $[O(H), S] \leq S \cap O(H) = 1$ we see that $N_P(S) \cap O(H) \leq C_P(S) = 1$. Hence $O(H)$ is a $3'$-group. ■

Let $R \leq N_P(S)$ be a Sylow 3-subgroup of $H \trianglelefteq N_G(S)$ and $y \in R$ be of order 3. Then we have $y \in U$, as $U \cap P \in \mathrm{Syl}_3(U)$. Moreover (2) implies that $y \in D^*(M)$.

(4) The group S is abelian.

Proof. Since y acts fixed-point-freely on S, the group $U \cdot S$ fulfils "Voraussetzung A" of [14]. Let $t \in U$ be an involution that inverts y. Then we have for every maximal abelian subgroup A of S that $\langle y, t\rangle \leq \langle y^U\rangle \leq \langle y^{U \cdot S}\rangle \leq N_{U \cdot S}(A)$ by Lemma 1.1 (c) of [14]. So we may apply Lemma 1.2 of [14] that forces S to be abelian. ■

(5) If H is soluble, then we have $\langle y\rangle \cdot T = U$ and $O_2(U) = S$.

Proof. If H is soluble, then we may choose T such that $R \cdot T$ is a Hall $\{2, 3\}$-subgroup of H. We observe that

$$\langle y\rangle \cdot O_2(R \cdot T)/O_2(R \cdot T) = \Omega_1(O_3(R \cdot T/O_2(R \cdot T))) \trianglelefteq R \cdot T/O_2(R \cdot T).$$

This shows that the product $\langle y\rangle \cdot T$ is a subgroup of H. Since $O(H)$ is a $\{2, 3\}'$-group by (3), we may apply Lemma (3.2) of [34] to $H/O(H)$. By (1) and (2) we have $T \neq S = O_2(H)$. This implies together with (1) that $T \cdot O(H)/O(H) \neq S \cdot O(H)/O(H) = O_2(H/O(H))$ holds. Combined we get

$$\begin{aligned}\langle y\rangle \cdot T/S &\cong \langle y\rangle \cdot T/(\langle y\rangle \cdot T \cap S \cdot O(H))\\ &\cong \langle y\rangle \cdot T \cdot O(H)/S \cdot O(H)\\ &\cong (\langle y\rangle \cdot T \cdot O(H)/O(H))/O_2(H/O(H)) \quad\cong\quad S_3.\end{aligned}$$

It follows that $S = O_2(\langle y\rangle T)$. As S is not characteristic in T by (2), Statement 8.1(iii) of [28] shows that $|S| = 4$. Now $24 = |U| \leq |\langle y\rangle \cdot T| = |\langle y\rangle \cdot T/S| \cdot |S| = 6 \cdot 4 = 24$ yields $U = \langle y\rangle \cdot T$. ■

(6) The group H is not soluble.

Proof. Assume for a contradiction that H is soluble.
Then S is an elementary abelian group of order 4 by (5). Let $c \in Z(T)^\#$ and $t \in S$ be such that $S = \langle t, c\rangle$. Then (5) shows that c is a square in T. Furthermore let $T_0 \in \mathrm{Syl}_2(G)$ be such that $T \leq T_0$ and take an involution $a \in C_{T_0}(t)$.
Suppose for a contradiction that $a \notin C_G(c)$. Then $D := \langle a, c\rangle \leq C_{T_0}(t)$ is a dihedral group of order at least 8. Let b be the central involution of D. From $a \notin C_G(c)$ we conclude that $b \neq c$. Moreover $b \in C_G(c) \cap C_{T_0}(t) = C_{T_0}(S) = C_{T_0}(S) \cap N_{T_0}(S) = C_{T_0}(S) \cap T = C_T(S) = S$. Consequently we have $S = \langle b, c\rangle$ and S is a subgroup of D. From $|D| \geq 8$ it follows that $S < N_D(S)$ and that $T \leq D$ holds. This implies that c is also a square in D. Being a dihedral group D has only one involution that is a square. This is the central involution. That contradicts $c \neq b$. Consequently we have $\Omega_1(C_{T_0}(t)) \leq C_G(c)$ and

$$\Omega_1(C_{T_0}(t)) \leq C_G(c) \cap C_{T_0}(t) = C_{T_0}(S) = C_T(S) = S \leq \Omega_1(C_{T_0}(t)).$$

This yields that $\Omega_1(C_{T_0}(t)) = S$. Hence we obtain $C_{T_0}(t) \leq N_{T_0}(\Omega_1(C_{T_0}(t))) = N_{T_0}(S) = T$. As t is not central in T, we see that $C_{T_0}(t) = C_T(t) = S$ is elementary abelian of order 4. Lemma 1.3.4 (a) implies that T_0 is dihedral or semi-dihedral. Thus having order at least 8 the group T_0 admits no automorphism of order 3 by Lemma 1.1.3. From the existence of a conjugate of x that normalises T_0 by (2) we deduce that T_0 is a subgroup of a conjugate of M. This contradicts $2 \notin \sigma$. ■

(7) The group $O_{2,2'}(H)$ is of order prime to 3.

Proof. Suppose for a contradiction that 3 divides $|O_{2,2'}(H)|$.
Then a Frattini argument shows for some Sylow 3-subgroup R_0 of $O_{2,2'}(H)$ contained in R, that $T \cdot O_{2,2'}(H) = O_{2,2'}(H) \cdot N_{T \cdot O_{2,2'}(H)}(R_0)$. From (3) we know that $R_0 \leq R \leq N_P(S)$ is cyclic. We conclude that $N_H(R_0)/C_H(R_0)$ is of order at most 2. As y is an element of order 3 of R, it is contained in R_0. Hence $C_{T \cdot O_{2,2'}(H)}(R_0) \leq C_G(y)$ is of odd order by (3). Consequently we have

$$\begin{aligned}2 \geq |N_H(R_0)|_2 \geq |N_{T \cdot O_{2,2'}(H)}(R_0)|_2 \geq |O_{2,2'}(H) \cdot N_{T \cdot O_{2,2'}(H)}(R_0) : O_{2,2'}(H)|_2\\ = |T \cdot O_{2,2'}(H) : O_{2,2'}(H)|.\end{aligned}$$

Therefore $H/O_{2,2'}(H)$ has cyclic Sylow 2-subgroups and Burnside's p-Complement Theorem and the Odd Order Theorem 1.1.12 force $H/O_{2,2'}(H)$ to be soluble. The conclusion that H is soluble contradicts (6). ∎

Let $- : H \to H/O_{2,2'}(G)$ be the natural epimorphism.

(8) The group $\bar{H}$ is S_4-free.

Proof. Suppose for a contradiction that $\bar{H}$ is not S_4-free. Then we obtain from Lemma 1.1.7 a non-trivial 2-subgroup $\bar{S}_0$ of $\bar{H}$ such that $N_{\bar{H}}(\bar{S}_0)$ has a section isomorphic to S_4. Let L be the full pre-image of $N_{\bar{H}}(\bar{S}_0)$ in H and S_1 be a Sylow 2-subgroup of the full pre-image $\bar{S}_0$. Then $S_0 := S_1 \cdot O_{2,2'}(H)$ is a normal subgroup of L and from a Frattini argument it follows that $L = S_0 \cdot N_L(S_1) = O_{2,2'}(H) \cdot N_L(S_1)$. As $\bar{L} = \overline{N_L(S_1)}$ has a section isomorphic to S_4, the maximal choice of S forces $S = S_1$. This is a contradiction to $\bar{S}_0 \neq 1$. ∎

(9) For all $t \in T \setminus S$ with $t^2 \in S$ we have $|C_{\Omega_1(S)}(t)|^2 = |\Omega_1(S)|$.

Proof. (Compare with 1.3 (d) of [14].)
Let $t \in T \setminus S$ such that $t^2 \in S$. From (4) we obtain that t induces an automorphism of order at most 2 of $\Omega_1(A)$. We apply Lemma 1.1.17 to see that $|C_{\Omega_1(S)}(t)|^2 \geq |\Omega_1(S)|$. Suppose for a contradiction that $|C_{\Omega_1(S)}(t)|^2 > |\Omega_1(S)|$. Then $C_{\Omega_1(S)}(t) \cap C_{\Omega_1(S)}(t)^y \neq 1$ and there is an element $a \in \Omega_1(S)$ such that $a \in C_{\Omega_1(S)}(t)$ and $a^y \in C_{\Omega_1(S)}(t)$. From (4) we observe that $A = \langle a, a^y, a^{y^2} \rangle$ is an elementary abelian y-invariant group. Moreover $C_H(A)$ is a $3'$-group by (3). Thus Lemma 1.1.14 (b) provides a $\langle y \rangle$-invariant Sylow 2-subgroup T_1 of $C_H(A)$. By Sylow's Theorem there is an element $h \in C_H(A) \leq N_H(\Omega_1(S))$ such that $t^h \in T_1$, as we have $t \in C_H(A)$. We may suppose that $t \in T_1$, since we have $|C_{\Omega_1(S)}(t)| = |(C_{\Omega_1(S)}(t))^h| = |C_{\Omega_1(S)}(t^h)|$.
Let $d \in U \leq H$ be an involution that inverts y. Then we have $|C_{\Omega_1(S)}(d)|^2 \geq |\Omega_1(S)|$ by Lemma 1.1.17. From this, $|C_{\Omega_1(S)}(t)|^2 > |\Omega_1(S)|$ and the fact that $\Omega_1(S)$ is elementary abelian by (4) we obtain an element $b \in C_{\Omega_1(S)}(d) \cap C_{\Omega_1(S)}(t) \leq \Omega_1(S) \leq T_1$. Applying Theorem 8.1 (ii) of [28] on $\langle y \rangle T_1$ we observe that $\langle \langle t, b \rangle, \langle t, b \rangle^y \rangle$ is abelian. In particular we have $t \in C_H(\langle b, b^y \rangle)$. We set $B := \langle b, b^y, b^{y^2} \rangle$. Then y normalises the by (4) abelian group B. Since y does not centralise any involution of H by (3), we conclude that B has order 4. Moreover d centralises b and normalises $\langle y \rangle$. Hence B is d-invariant. Moreover we obtain that $t \in C_H(\langle b, b^y \rangle) = C_H(B)$. Altogether we have $S_3 \cong \langle y, d \rangle \leq N_H(B)$ and $\overline{C_H(B)}$ is of even order.
Let $T_2 \in \mathrm{Syl}_2(C_H(B))$. Then a Frattini argument yields that $N_H(B) = C_H(B) \cdot N_{N_H(B)}(T_2)$. The group $C_H(B)$ is of order prime to 3 by (3). Consequently $N_H(T_2) \leq N_{N_H(B)}(T_2)$ is not S_3-free. From $3 \nmid |O_{2',2}(H)|$ by (7) we observe that $\overline{N_H(T_2)} \leq N_{\bar{H}}(\overline{T_2})$ is not S_3-free and Lemma 1.1.14 (a) and (3) yield that every element of order 3 in $N_{\bar{H}}(Z(\overline{T_2}))$ acts fixed-point-freely on the non-trivial abelian group $Z(\overline{T_2})$. Finally Lemma 1.3.3 implies that $N_{\bar{H}}(Z(\overline{T_2})) \geq N_{\bar{H}}(\overline{T_2})$ is not S_4-free. This contradicts (8). ∎

(10) The group $F^*(\bar{H})$ is simple and has an order divisible by 3.

Proof. Let $O_{2,2'}(G) \leq N \leq H$ be such that $\bar{N}$ is a minimal normal subgroup of $\bar{H}$.
Suppose for a contradiction that $\bar{N}$ is soluble. Then $\bar{N}$ is an elementary abelian 2-group. Let $T_1 \in \mathrm{Syl}_2(N)$. Then a Frattini argument shows that $H = N \cdot N_H(T_1)$. As 3 does not divide $|N|$ and $S_4 = U \leq H$, the group $H/N = N_H(T_1) \cdot N/N$ is not S_3-free. Consequently $N_H(T_1)$ is not S_3-free. Since every 3-element of $N_H(T_1)$ acts fixed-point-freely on T_1 by (3), it acts non-trivially on $Z(T_1)$. We apply Lemma 1.3.3 to obtain that $N_H(T_1)$ is not S_4-free. The maximal choice of S forces T_1 to be S. This contradicts $\bar{N} \neq 1$.
Suppose now for a contradiction $\bar{N}$ is a $3'$-group. Then $\bar{N}$ is a direct product of simple

Suzuki groups by Theorem 1.2.8. As 3 divides $|\bar{H}| = |N_{\bar{H}}(\bar{N})|$, the group $\bar{N}$ is normalised by some element $\bar{y}$ of order 3. Thus Lemma 1.2.10 implies that $C_{\bar{H}}(\bar{y})$ is of even order. Since $O_{2,2'}(H)$ is of order prime to 3 by (7), Lemma 1.1.14 (a) yields $C_{\bar{H}}(y) = \overline{C_H(y)}$. This contradicts (3).
Finally suppose for a contradiction that $F^*(\bar{H})$ is not simple. Then $F^*(\bar{H})$ is the direct product of at least two simple groups with order divisible by 3. Let K_1 and K_2 be the full pre-images of two of those groups such that $K_1 \neq K_2$. And let z be an element of order 3 in K_1. Then $[z, K_2] \leq [K_1, K_2] \leq O_{2,2'}(H)$. In particular $\bar{K}_2 \leq C_{\bar{H}}(z) = \overline{C_H(z)}$, the last equation holds by Lemma 1.1.14 (a). This implies that $C_G(z)$ is of even order which again contradicts (3). ■

(11) We have $C_H(\Omega_1(S)) \leq O_{2,2'}(H)$ and $J(T) \nleq S$.

Proof. From (3) we observe that $C_H(\Omega_1(S))$ has order prime to 3. So $C_H(\Omega_1(S))$ is a normal $3'$-subgroup of H. Now (10) forces $C_H(\Omega_1(S)) \leq O_{2,2'}(H)$.
Suppose for a contradiction that $J(T) \leq S$. Then we see from (4) that $\Omega_1(S) = J(S) = J(T)$. Furthermore we deduce that $H \leq N_G(S) \leq N_G(J(S)) = N_G(J(T))$. Let T_1 be a Sylow 2-subgroup of $C_G(J(T)) = C_G(\Omega_1(S))$ such that $S \leq T_1$. If S is properly contained in T_1, then we have $S < N_{T_1}(S) \leq C_H(J(T)) = C_H(\Omega_1(S)) \leq O_{2,2'}(H)$. This is a contradiction since $S \in \mathrm{Syl}_2(O_{2',2}(H))$ by (1). Thus S is a Sylow 2 subgroup of $C_G(J(T))$.
Let now $T_2 \in \mathrm{Syl}_2(N_G(J(T)))$ such that $S \leq T_2$. Then we obtain from the fact that $C_G(J(T))$ is normal in $N_G(J(T))$ that we have $S = C_G(J(T)) \cap T_2 \trianglelefteq T_2$. Consequently we see that $T_2 \leq N_G(S)$ and T_2 is a subgroup of H. From $T \leq N_H(J(T))$ we obtain that $T = T_2$. Therefore T is a Sylow 2-subgroup of $N_G(J(T)) \geq N_G(T)$. This implies that T is a Sylow 2-subgroup of G. Part (2) provides an element $g \in G$ such that $x^g \in N_G(T)$. In particular we have $x^g \in N_G(J(T))$. From Lemma 3.3.2 (a) and our assumption $2 \notin \sigma$ we observe that $N_G(J(T))$ is no subgroup of M^g. So Lemma 3.2.2 (d) implies that $N_G(J(T))$ has a normal 3-complement. This finally contradicts $S_4 \cong U \leq N_G(S) \leq N_G(J(T))$. ■

(12) We have $\bar{H} \cong \mathrm{PSL}(2, 2^n)$ and S is elementary abelian of order 2^{2n} for some $n \in \mathbb{N}$, $n \geq 2$. Furthermore T is isomorphic to some Sylow 2-subgroup of $\mathrm{PSL}(3, 2^n)$.

Proof. We want to apply Lemma 1.4 of [14] to H. "Voraussetzung B" of [14] holds by (1), (3), (11) and the choice of H. Furthermore (6) and (11) yield the remaining assumptions of the lemma. Its proof uses Lemma 1.3 of the same article which is proven by using non-elementary results. But the assertion of Lemma 1.3 in [14] holds in our case by (3), (4), (8), (9) and (10). ■

(13) The group T is a chracteristic subgroup of some Sylow 2-subgroup T_0 of $N_G(T)$.

Proof. We show that $T = J(T_0)$.
From (12) and Lemma 1.2.5 we conclude that T possesses exactly two elementary abelian subgroups S and S_0 of order 2^{2n}. Moreover S is not normalised by any element of $T_0 \setminus T$. This shows that every element of $T_0 \setminus T$ interchanges S and S_0. For all $s, t \in T_0 \setminus T$ we have $S^{ts} = S_0^s = S$. This leads to $t \cdot s \in N_{T_0}(S) = T$. Therefore we see that $|T_0 : T| = 2$. In particular S is a maximal elementary abelian subgroup of T_0.
Let A be an elementary abelian subgroup of order 2^{2n} of T_0 and suppose for a contradiction that $A \nleq T$. Then $A \cap T$ is elementary abelian of order 2^{2n-1}. By Lemma 1.2.5 we conclude that $A \cap T$ is contained in S or S_0. Since T is normal in T_0, we may suppose that $A \cap T \leq S$ (else we investigate A^s instead of A for some $s \in T_0 \setminus T$). For all $t \in A \setminus (A \cap T)$ we have shown that $S^t = S_0$.
Therefore we have $A \cap S = A \cap T = A^t \cap T^t = (A \cap T)^t = (A \cap S)^t = A^t \cap S^t = A \cap S_0$ for all $t \in A \setminus (A \cap T)$. This shows that $A \cap T \leq A \cap S \cap S_0 = A \cap Z(T)$. Although we have

$|Z(T)| = 2^n$. Together we get $2^{2n-1} = 2^n$. This implies $n = 1$ contradicting (12).
In total we have $J(T_0) \leq T = J(T)$. This leads to our assertion. ■

The fact that $N_G(T_0) \leq N_G(T)$ forces T_0 to be as Sylow 2-subgroup of G. Finally (2) provides an element $g \in G$ such that $x^g \in N_G(T_0) \leq N_G(T)$. As T has exactly two elementary abelian subgroups of order 2^{2n} by Lemma 1.2.5, they are normalised by x^g. This implies $x^g \in N_G(S)$ contradicting (2). □

4 The Big Rank Case

4.1 Subgroups of M

In this section we investigate elementary abelian 2-subgroups of order 8 of G and their appearance in M and other maximal subgroups.

4.1.1 Lemma

Suppose that $c \in M$ is involution and that V is an elementary abelian subgroup of $C_G(c)$ of order 4. Assume further that V is normalised by an element $y \in D^*(M) \cap C_G(c)$. Then the following hold:

(a) We have $c \in I^*(M)$.

(b) The set $\pi(C_G(V))$ is contained in σ.

(c) For all $v \in V^\#$ the group $O_{\sigma'}(C_G(v))$ is abelian and $[x, O_{\sigma'}(C_G(cv))]$ is non-abelian.

(d) We have $\langle V, c\rangle \cap I^*(M) = \{c\}$.

(e) The element c is not balanced in G.

(f) There is no elementary abelian subgroup of order 8 of G which contains V and is contained in $C_G(z) \setminus \{z\}$.

Proof

We set $C := C_G(c)$. From $c \in M = C_G(x)$ we deduce $x \in C$. Further C contains $\langle x, y\rangle$ and has therefore non-cyclic Sylow 3-subgroups. Altogether Lemma 3.2.2 (f) yields that $C \leq M$. This implies that $c \in I^*(M)$. This is (a).

By Corollary 3.3.4 (a) the group V is not centralised by y. From $V \leq C \leq M$ we observe that the element x centralises V. Moreover we have $C_G(V) \trianglelefteq N_G(V)$. Therefore $C_G(V)$ is $\langle x, y\rangle$-invariant. By Corollary 3.3.4 (a) and Lemma 3.3.5 (b) the group $C_G(V)$ has a normal 3-complement. Finally Lemma 3.3.2 (e) forces $\pi(C_G(V)) \subseteq \sigma$. This is(b).

(*) There is no elementary abelian subgroup A of G of order at least 8 such that A contains V and admits a soluble A-signalizer functor θ in G with $O_{\sigma'}(C_G(v_0)) \cap \theta(v_0) \neq 1$ for some $v_0 \in V^\#$.

Proof. Assume that Statement (*) is false. Then the Soluble Signalizer Functor Theorem 2.1.6 implies that $W_A := \langle \theta(a) \mid a \in A^\#\rangle$ is a subgroup of G of odd order. This implies that also $W_V := \langle \theta(v) \mid v \in V^\#\rangle \leq W_A$ is a subgroup of G of odd order. Moreover W_A is normalised by A and Theorem 1.1.8 provides a Hall $\{2, 3\}$-subgroup of the soluble group $A \cdot W_A$. From both parts of Lemma 3.3.5 we obtain that $A \cdot W_A$ has a normal 3-complement. It follows that W_A has a normal 3-complement and hence W_V has a normal 3-complement too. Since $\langle x, y\rangle$ normalises V, the group $\langle x, y\rangle$ normalises W_V. Thus Lemma 3.3.2 (e) yields $\pi(W_V) \subseteq \sigma$. But now the assumption $1 \neq O_{\sigma'}(C_G(v_0)) \cap \theta(v_0) \subseteq W_V$ leads to a contradiction. ∎

We set $A := \langle V, c\rangle$. Then A is elementary abelian and has order 8.

The Statements (c) and (d) are true.

Proof. Let $v \in V^{\#}$ and $w \in V \setminus \langle v\rangle$. Then from $C_{C_G(v)}(w) \subseteq C_G(V)$ and (b) it follows that $\pi(C_{C_G(v)}(w)) \subseteq \sigma$. Thus the involution w acts fixed point freely on $O_{\sigma'}(C_G(v))$. This implies that $O_{\sigma'}(C_G(v))$ is abelian. For all involutions $a \in A$ we set $\gamma(a) := [O_{\sigma'}(C_G(a)), x]$.
Assume for a contradiction that $\gamma(a)$ is abelian for all $a \in A^{\#}$. We will show that γ is a soluble A-signalizer functor. Since $A \leq M$ for all $a \in A^{\#}$ the group $\gamma(a)$ is A-invariant and soluble by the Odd Order Theorem 1.1.12. We fix an element $a \in A^{\#}$. Since $x \in C_G(a)$ and $3 \in \sigma$, the element x acts coprimely on $\gamma(a)$. From Lemma 1.1.14 (f) and (d) we deduce

$$\begin{aligned}\gamma(a) &= C_{\gamma(a)}(x) \times [\gamma(a), x] = C_{\gamma(a)}(x) \times [O_{\sigma'}(C_G(a)), x, x]\\ &= C_{\gamma(a)}(x) \times [O_{\sigma'}(C_G(a)), x] = C_{\gamma(a)}(x) \times \gamma(a).\end{aligned}$$

This implies $C_{\gamma(a)}(x) = 1$ and hence the element x acts fixed-point-freely on $\gamma(a)$. In particular we have that $\gamma(a) = \{[g, x] \mid g \in \gamma(a)\}$ by Lemma 10.1.1 of [22].
For all $d \in A \cap I^*(M)$ we have $\gamma(d) = \gamma(d) \leq [M, x] = 1$. If additionally $b \in A^{\#}$, then $C_G(b) \cap \gamma(d) = 1 \leq \gamma(b)$ and we observe that

$$C_G(d) \cap \gamma(b) \leq M \cap \gamma(b) \leq C_{\gamma(b)}(x) = 1 = \gamma(d).$$

Let now $d, b \in A^{\#} \setminus I^*(M)$ and suppose that $h \in C_G(b) \cap \gamma(d)$.
Then $h \in \gamma(d) = \{[g, x] \mid g \in \gamma(d)\}$. This provides an element $g \in \gamma(d) = [O_{\sigma'}(C_G(d)), x]$ such that $h = [g, x]$. In particular we obtain that $[g, x] = h = h^b = [g, x]^b = [g^b, x]$. We conclude that $g^{-1}g^b \in C_G(x)$. Since $\gamma(d)$ is A-invariant, we moreover have $g, g^b \in \gamma(d)$ and hence $g^{-1}g^b \in C_{\gamma(d)}(x) = 1$. Consequently we observe $g = g^b$ and so $g \in C_G(b)$. This shows together with Lemma 1.3.7 (c) that $h \in [C_G(b), x] = [O_{\sigma'}(C_G(b)), x] = \gamma(b)$.
Altogether γ is a soluble A-signalizer functor in G. Finally (*) yields a contradiction.

The contradiction provides an element $a \in A^{\#}$ such that $\gamma(a)$ is not abelian. For all $v \in V^{\#}$ the group $\gamma(v)$ is abelian, because it is a subgroup of the abelian group $O_{\sigma'}(C_G(v))$. From $1 = [O_{\sigma'}(C), x] = \gamma(c)$ and we obtain an element $v_0 \in V^{\#}$ such that $a = v_0 \cdot c$. As y acts transitively on $V^{\#}$ and centralises c, we conclude that $\gamma(vc) = [O_{\sigma'}(C_G(vc)), x]$ is non-abelian for all $v \in V^{\#}$. This implies (c).
In particular we obtain that $v \cdot c \notin I * (M)$ for all $v \in V$ an thus $(A \setminus V) \cap I^*(M) = \{c\}$. Moreover, since all involutions of V are conjugate by $y \in G$, Corollary 3.3.4 (c) implies that $V \cap I^*(M) = \emptyset$. This leads to (d). ■

For all involutions $a \in G$ we set $\theta(a) := O(C_G(a))$.

The statement of (e) holds.

Proof. From (d) and Lemma 3.2.2 (d) we see that all involutions of $A \setminus \{c\}$ have a 3-soluble centraliser in G. This shows together with Lemma 2.1.4 and Theorem 1.2.8 that all involutions of $A \setminus \{c\}$ are balanced in G. Suppose for a contradiction that there is no element $a \in A \setminus \langle c\rangle$ such that $O(C_G(a)) \cap C_G(c) \nleq O(C_G(c))$. Then A is balanced in G and θ is a soluble A-signalizer functor in G. But this contradicts (*). In particular (e) holds. ■

(**) For all $a \in A \setminus \langle c\rangle$ we have $\pi(O(C_C(a))) \nsubseteq \sigma$.

Proof. Let $v \in V^{\#}$. Then (c) yields that $[x, O_{\sigma'}(C_G(cv))]$ is not abelian. In particular the group $O_{\sigma'}(C_G(cv))$ is not abelian. This implies that c does not act fixed-point-freely on $O_{\sigma'}(C_G(cv))$. We conclude that $1 \neq O_{\sigma'}(C_G(cv)) \cap C \leq O(C_C(cv))$. Consequently we have $\pi(O(C_C(cv))) \nsubseteq \sigma$.
Moreover we see that $C_C(v) = C_G(v) \cap C = C_G(\langle c, v\rangle) = C_G(cv) \cap C = C_C(cv)$. This leads to $\pi(O(C_C(v))) = \pi(O(C_C(cv))) \nsubseteq \sigma$.
Finally the assertion follows from $A \setminus \langle c\rangle = V^{\#} \cup \{cv \mid v \in V^{\#}\}$. ■

Suppose for a final contradiction that (f) is false. Then there is an elementary abelian subgroup B of G of order at least 8 such that $V \leq B$ and such that B is a subset of $C \setminus \{c\}$. Corollary 3.3.4 and Lemma 3.3.5 (b) imply that $C_G(\langle b, c\rangle) = C_C(b)$ has a normal 3-complement for all involutions b of $C \setminus \{c\}$. Thus we conclude from Lemma 2.1.4 that all involutions of C except for c are balanced in C. For all $a \in B^\#$ let $\tilde{\theta}(a) := O(C_C(a))$. Then $\tilde{\theta}$ is a soluble B-signalizer functor in C by the Odd Order Theorem 1.1.12. The Soluble Signalizer Functor Theorem 2.1.6 forces $W_B := \langle O(C_C(b)) \mid b \in B^\#\rangle$ to have odd order. As $x \in \tilde{\theta}(a) \leq W_B$ for all $a \in B^\#$ and W_B is normalised by B, both parts of Lemma 3.3.5 imply that $B \cdot W_B$ has a normal 3-complement. It follows that $W_V := \langle O(C_C(v)) \mid v \in V^\#\rangle \leq W_B \cdot B$ has a normal 3-complement too. Since V is normalised by $\langle x, y\rangle$, the group W_V is $\langle x, y\rangle$-invariant. Finally Lemma 3.3.2 (e) yields $\pi(W_V) \subseteq \sigma$. In particular $\pi(O(C_C(v))) \subseteq \sigma$ for all $v \in V^\#$. This contradicts (**). □

4.1.2 Lemma

Suppose that A is an elementary abelian subgroup of M of order 8. Assume that A has a maximal subgroup V such that $I^*(M) \cap V = \varnothing$ but $|A \cap I^*(M)| \geq 3$. Then the following hold:

(a) For all $H \in \mathfrak{M}$ such that $A \leq H$, we have $H = C_H(x) \cdot C_H(V)$ and $[H, x] \leq C_H(V)$.

(b) The group $C_G(V)$ is contained in an unique element H_V of $\mathfrak{M}$.

(c) If U is a subgroup of $H_V \cap M$ such that $A \leq U$, then $N_G(U) \leq H_V$ holds.

(d) The group $\langle O(C_G(v)) \mid v \in V^\#\rangle$ is of odd order and a subgroup of H_V. Moreover for every involution $b \in C_G(V)$ we have $O(C_G(b)) \leq \langle O(C_G(v)) \mid v \in V^\#\rangle$.

Proof

(a) The assumptions imply for all $v \in V^\#$ that we have $C_G(v) \not\leq M$ and $\langle A, x\rangle \leq C_G(v)$. Thus we obtain from Lemma 3.2.2 (d) an element $H \in \mathfrak{M}$ that contains A. Since $I^*(M)$ is non-empty, Lemma 3.3.2 (b) implies that $2 \in \sigma$. By Lemma 1.3.7 (b) we obtain that $H = C_H(x) \cdot O(H)$ and hence that $[H, x] = [C_H(x) \cdot O(H), x] = [O(H), x]$. Moreover $|I^*(M) \cap A| \geq 3$ forces A to be equal to $\langle I^*(M) \cap A\rangle$. Thus from Lemma 1.3.2 and $I^*(M) \cap V = \varnothing$ we deduce that V is the unique maximal subgroup of A containing no involution of $I^*(M)$. If $B \neq V$ is a maximal subgroup of A and $b \in B \cap I^*(M)$, then $C_G(B) \subseteq C_G(b) \subseteq C_G(x)$. Furthermore $\langle x\rangle \times A$ acts coprimely on $O(H)$. Finally Lemma 1.3.1 (b) yields

$$[H, x] = [O(H), x] \leq \langle C_{[O(H),x]}(B) \mid B\ max\ A,\ C_G(B) \nsubseteq C_G(x)\rangle \leq C_H(V).$$

Altogether we get $H = C_H(x) \cdot C_H(V)$.

(b) The element x centralises V and for all $v \in V^\#$ we observe $C_G(V) \leq C_G(v)$ and $v \notin I^*(M)$. Therefore there exists an element $H_V \in \mathfrak{M}$ such that $C_G(V) \leq H_V$. We apply (a) to obtain $H_V = C_{H_V}(x) \cdot C_{H_V}(V) = C_{H_V}(x) \cdot C_G(V)$.
Suppose that $H \in \mathfrak{M}$ such that $A \leq C_G(V) \leq H$. Then (a) and Lemma 1.3.7 (a) imply that $[H, x] = [H, x, x] \leq [C_H(V), x] = [C_G(V), x] \leq [H_V, x]$ and similarly we conclude $[H_V, x] = [H_V, x, x] \leq [C_G(V), x] \leq [H, x]$. Altogether we observe that $[H_V, x] = [H, x] \trianglelefteq \langle H, H_V\rangle$. Consequently, as H and H_V are maximal in G, Lemma 3.2.2 (a) leads to $H = H_V$.

(c) From $I^*(M) = \varnothing$ and $|A \cap I^*(M)| \geq 3$ we obtain an involution $v \in V^\#$ and elements $a, b \in I^*(M)$ such that $v = a \cdot b$. Moreover we have $C_G(V) \leq C_G(v)$. This implies together with (b) that $C_G(v) \leq H_V$. Let U be a subgroup of M such that $A \leq U \leq H_V$ and suppose that $g \in N_G(U)$. Then we have $\langle a, b\rangle^g \leq U^g = U \leq H_V$. Consequently Proposition 3.3.3 yields that $g \in H_V$.

(d) For all $a \in A^\#$ we set $\theta(a) := O(C_{H_V}(a))$. Then $\theta(a)$ is soluble for all $a \in A^\#$ because of the Odd Order Theorem 1.1.12. Since H_V is 3-soluble by Lemma 3.2.2 (d), Lemma 2.1.4 implies that θ is a soluble A-signalizer functor in H_V. As $r(A) = 3$, the Soluble Signalizer Functor Theorem 2.1.6 forces $\langle O(C_{H_V}(a)) \mid a \in A^\#\rangle$ to have odd order. For all $v \in V^\#$ we deduce from (b) and $x \in C_G(V) \le C_G(v)$ that $C_G(v) = C_{H_V}(v)$. Moreover Lemma 3.2.2 (d) yields that $C_{H_V}(v) = C_G(v)$ has a normal 3-complement for all $v \in V^\#$. In particular the centralisers of the involutions of V are 3-soluble. Thus the involutions of V are balanced in G by Lemma 2.1.4.
Altogether we conclude with Lemma 1.1.14 (e) that

$$\begin{aligned} O(C_G(b)) &= \langle O(C_G(b)) \cap C_G(v) \mid v \in V^\#\rangle \le \langle O(C_G(v)) \mid v \in V^\#\rangle \\ &= \langle O(C_{H_V}(v)) \mid v \in V^\#\rangle \le \langle O(C_{H_V}(a)) \mid a \in A^\#\rangle. \end{aligned}$$

In particular $\langle O(C_G(v)) \mid v \in V^\#\rangle$ has odd order. □

4.1.3 Proposition

Let $c \in I^*(M)$ be an involution such that $\langle c\rangle = \Omega_1(Z(T))$ for some $T \in \mathrm{Syl}_2(G)$.
Then $\langle [O_{\sigma'}(C_G(b)), x] \mid b \in T^\#,\ b^2 = 1\rangle$ is of odd order.

Proof

By Lemma 3.2.1 (a) we have $Z^*(G) = 1$. Thus the Z^*-Theorem 1.1.13 provides an element $g \in G \setminus C_G(c)$ such that $d := z^g \in C_T(c) \le M$. We conclude that $d \notin Z(T) = Z^*(T)$ and by the same theorem there is an element $t \in T$ such that $d^t \in C_T(d) \le M$ and $d^t \neq d$.
We set $A := \langle c, d, d^t\rangle$ and $V := \langle c \cdot d, c \cdot d^t\rangle$. Then A is an elementary abelian subgroup of M of order 8 and hence V is elementary abelian. From $c, d, d^t \in I^*(M)$ and Proposition 3.3.3 it follows that $V \cap I^*(M) = \varnothing$. Since we have $|A \cap I^*(M)| \ge |\{c, d, d^t\}| = 3$, we may apply Lemma 4.1.2. By Lemma 4.1.2 (b) the group $C_G(V)$ is contained in a unique element $H \in \mathfrak{M}$. For all $v \in V^\#$ we have $C_G(v) \ge C_G(V)$. The uniqueness of H implies that $C_G(v) \le H$ and hence $C_G(v) = C_H(v)$ for all $v \in V^\#$.
Since A is contained in $C_G(d)$, there is a Sylow 2-subgroup T_0 of $C_G(d)$ such that $A \le T_0$. As d and c are conjugate in G, we conclude that $T_0 \in \mathrm{Syl}_2(G)$. Moreover by Lemma 3.2.2 (g) we have $C_G(d) \le M$ and hence $T_0 \in \mathrm{Syl}_2(M)$. From Lemma 4.1.2 (c) we observe that H contains $N_G(H \cap T_0)$ and $N_G(H \cap T)$. This leads to $T, T_0 \le H$, since T and T_0 are Sylow 2-subgroups. We further obtain from Sylow's Theorem an element $h \in H$ such that $T^h = T_0$. In particular we have $\langle d\rangle = \Omega_1(Z(T_0)) = \Omega_1(Z(T^h)) = \Omega_1(Z(T))^h = \langle c\rangle^h$. For that reason we conclude $c^h = d$ and so $c \notin Z^*(H)$. Finally we deduce $Z^*(H) = O(H)$ from $T \cap Z^*(H) \le Z(T)$.

We set $Z := \langle \Omega_1(Z(T))^{N_H(J(T))}\rangle = \langle c^{N_H(J(T))}\rangle$ and $W_Z := \langle O(C_H(a)) \mid a \in Z^\#\rangle$.

(*) We have $W_Z = \langle O(C_G(v)) \mid v \in V^\#\rangle \ge \langle O(C_G(a)) \mid a \in Z^\#\rangle$ and W_Z has odd order.

Proof. As H has a normal 3-complement by Lemma 3.2.2 (d), the group Z is elementary abelian and strongly closed in H by Theorem 2.2.4. Thus Lemma 2.2.2 (g) yields that $H = N_H(Z) \cdot \langle Z^H\rangle$. In particular we have $d = c^h \in Z$ and $d^t \in Z$. This implies $A \le Z$.
For all $a \in Z^\#$ we set $\theta(a) := O(C_H(a))$. Since H has a normal 3-complement, Lemma 2.1.4 yields that H is balanced. Furthermore the Odd Order Theorem forces $\theta(a)$ to be soluble for all $a \in A^\#$. Therefore θ is a soluble A-signalizer functor in H. Consequently the Soluble Signalizer Functor Theorem 2.1.6 forces $W_Z = \langle O(C_H(a)) \mid a \in Z^\#\rangle$ to have odd order. Moreover for all non-trivial elements a of the abelian group Z the subgroup V of Z acts coprimely on $O(C_H(a))$. From Lemma 1.1.14 (e) we obtain

$$O(C_H(a)) = \langle O(C_H(a)) \cap C_H(v) \mid v \in V^\#\rangle \le \langle O(C_H(v)) \mid v \in V^\#\rangle.$$

This yields $\langle O(C_H(v)) \mid v \in V^\#\rangle \le W_Z \le \langle O(C_H(v)) \mid v \in V^\#\rangle$. Since we have $C_G(v) = C_H(v)$ for all $v \in V^\#$, we obtain that $W_Z = \langle O(C_H(v)) \mid v \in V^\#\rangle = \langle O(C_G(v)) \mid v \in V^\#\rangle$.
Moreover, as H and hence $C_G(v) = C_H(v)$ is 3-soluble for all $v \in V^\#$, Lemma 2.1.4 yields that all $v \in V^\#$ are balanced in G. In addition V acts on $O(C_G(a))$ coprimely for all $a \in Z^\#$. From Lemma 1.1.14 (e) we conclude that

$$O(C_G(a)) = \langle O(C_G(a)) \cap C_G(v) \mid v \in V^\#\rangle \le \langle O(C_G(v)) \mid v \in V^\#\rangle = W_Z. \qquad \blacksquare$$

For all involutions $b \in T$ we set $\gamma(b) := [O_{\sigma'}(C_G(b)), x]$.
If for every involution $b \in T$ the group $\gamma(b)$ is contained in W_Z, then the proposition holds. So it suffices to prove $\gamma(b) \le W_Z$ for all involutions $b \in T$. Let $b \in T$ be an involution.

1. Case: Suppose that $|C_Z(b)| \ge 8$.
From $T \le M$, Lemma 3.2.2 (f) and Lemma 2.1.4 we deduce that the involutions of $T \setminus I^*(M)$ are balanced in G. If there are involutions $a_1, a_2, a_3 \in C_Z(b)$ such that $\{a_1, a_2, a_3\} \subseteq I^*(M)$ and $|\{a_1, a_2, a_3\}| = 3$, then $B := \langle a_1 \cdot a_2, a_1 \cdot a_3\rangle$ is an elementary abelian subgroup of order 4 of $C_Z(b)$. Furthermore all involutions of B are not contained in $I^*(M)$ by Proposition 3.3.3 and so balanced in G. In the other case, if $|C_Z(b) \cap I^*(M)| \le 2$, then there also exists an elementary abelian subgroup B of order 4 of $C_Z(b)$ such that $B \cap I^*(M) = \emptyset$.
In both cases Lemma 1.1.14 (e) and (*) imply that

$$O(C_G(b)) = \langle O(C_G(b)) \cap C_G(a) \mid a \in B^\#\rangle \le \langle O(C_G(a)) \mid a \in B^\#\rangle \le W_Z.$$

We finally recall $2 \in \sigma$ to conclude that $\gamma(b) = [O_{\sigma'}(C_G(b)), x] \le O(C_G(b)) \le W_Z$.

2. Case: Suppose that $|C_Z(b)| \le 4$.
Then we have $8 = |A| \le |Z| \le |C_Z(b)|^2$ by Lemma 1.1.17. It follows that $|C_Z(b)| = 4$.
From Lemma 3.2.2 (g) and the fact that $Z \le C_G(c) \le M$, we see that $I^*(M) \cap Z$ is an union of G-conjugacy classes of elements in Z.
Since A is generated by $\{c, d, d^t\} \subseteq I^*(M)$ and $V \cap I^*(M) = \emptyset$. Lemma 1.3.2 forces V to be the unique maximal subgroup of A containing no element of $I^*(M)$. Therefore V is not conjugate to another subgroup of A. It follows that $N_G(A) \le N_G(V)$. If $Z = A$ held, then we would have $N_G(V) \ge N_G(A) = N_G(Z) \ge T$ and from $V \cap Z(T) = 1$ would follow a contradiction. Thus we have $A < Z$. Hence we conclude that $|Z| = 16$, because of $8 = |A| < |Z| \le |C_G(b)|^2 \le 16$. If $A = Z \cap \langle c^{N_G(Z)}\rangle$ held, then we would have $N_G(Z) \le N_G(A)$ and again $T \le N_G(Z) \le N_G(A) \le N_G(V)$ would imply together with $V \cap Z(T) = 1$ a contradiction.
Consequently we have that $A < Z \cap \langle c^{N_G(Z)}\rangle$ and as $|Z| = 16 = 2 \cdot |A|$, we conclude that $Z = Z \cap \langle c^{N_G(Z)}\rangle \le \langle c^{N_G(Z)}\rangle \le Z$. For this reason $Z = \langle c^{N_G(Z)}\rangle$ is generated by a subset $\{a_1, a_2, a_3, a_4\}$ of $I^*(M)$ with four elements. By Lemma 1.3.2 there is a unique maximal subgroup B of Z such that $B \cap \{a_1, a_2, a_3, a_4\} = \emptyset$.
Suppose for a contradiction that $B \cap I^*(M) = \emptyset$. Then B is the unique maximal subgroup of Z such that $B \cap I^*(M) = \emptyset$. From the fact that $I^*(M) \cap Z$ is an union of G-conjugacy classes of elements in Z we deduce that $N_G(Z) \le N_G(B)$. In particular b normalises B. By Lemma 1.1.17 we see $|C_B(b)|^2 \ge |B| = 8$. Therefore we have that $|C_B(b)| \ge 4 = |C_Z(b)|$ holds. It follows that $c \in C_Z(b) = C_B(b) \le B$, which is a contradiction. Consequently $B \cap I^*(M)$ is not empty. From Lemma 1.3.2 we see that

$$B = \langle \{a_i a_j \mid i, j \in \{1, 2, 3, 4\}, i \ne j\}\rangle = \{a_i a_j \mid i, j \in \{1, 2, 3, 4\}, i \ne j\} \cup \{a_1 a_2 a_3 a_4, 1\}.$$

Therefore Proposition 3.3.3 leads to $a_1 a_2 a_3 a_4 \in I^*(M)$. We set $a_5 := a_1 a_2 a_3 a_4$. Then Proposition 3.3.3 finally implies that $Z \cap I^*(M) = \{a_1, a_2, a_3, a_4, a_5\}$.
The element b permutes the elements of $I^*(M) \cap Z$ and so b centralises an odd number of elements of $I^*(M) \cap Z$. Since the product of two elements in $I^*(M) \cap Z$ is not an element of $I^*(M) \cap Z$ by Proposition 3.3.3 and since we have $|C_Z(b)| = 4$, the element b fixes exactly one element of $Z \cap I^*(M)$. This unique element is c, because of $b \in T = C_T(c)$. Let $v, w \in Z \setminus I^*(M)$ be such that $C_Z(b) = \{1, c, v, w\}$. Then $\langle v, w, x\rangle$ acts coprimely on $O_{\sigma'}(C_G(b))$. By Lemma 1.3.1 we conclude that

$$\gamma(b) = [O_{\sigma'}(C_G(b)), x] \leq \langle C_{\gamma(b)}(D) \mid D \; max \; \langle v, w\rangle, \; C_G(D) \nsubseteq C_G(x)\rangle$$
$$= \langle C_{\gamma(b)}(e) \mid e \in \langle v, w\rangle \setminus I^*(M)\rangle \subseteq \langle C_{\gamma(b)}(w), C_{\gamma(b)}(v)\rangle.$$

Assume that $C_G(v)$ and $C_G(w)$ are subgroups of H. Then $\gamma(b) \leq \langle C_{\gamma(b)}(w), C_{\gamma(b)}(v)\rangle \leq H$. According to this Lemma 1.1.14 (d) implies

$$\gamma(b) = [O_{\sigma'}(C_G(b)), x] = [O_{\sigma'}(C_G(b)), x, x] = [\gamma(b), x] \subseteq [H, x].$$

From $2 \in \sigma$ and Lemma 1.3.7 (c) we deuce that that $[H, x] = [O_{\sigma'}(H), x]$. Therefore we obtain that $\gamma(b) \leq [O_{\sigma'}(H), x] \trianglelefteq H$.

Finally for all $a \in Z^\#$ the group $C_{[O_{\sigma'}(H),x]}(a) = [O_{\sigma'}(H), x] \cap C_H(a)$ is a normal subgroup of odd order of $C_H(a)$. Now Lemma 1.1.14 (e) yields

$$\gamma(b) \leq [O_{\sigma'}(H), x] \leq \langle C_{[O_{\sigma'}(H),x]}(a) \mid a \in Z^\#\rangle \leq \langle O(C_H(a)) \mid a \in Z^\#\rangle = W_Z.$$

Consequently it suffices to show that $C_G(v), C_G(w) \leq H$.

For this let $u \in \{v, w\}$. Then we have $u = a_i a_j$ for some $i, j \in \{1, 2, 3, 4, 5\}$ with $i \neq j$. We further choose $k \in \{1, 2, 3, 4, 5\} \setminus \{i, j\}$ and set $U := \langle u, a_i a_k\rangle = \{1, a_i a_j, a_i a_k, a_j a_k\}$. Then we have $U \cap I^*(M) = \emptyset$ by Proposition 3.3.3. We set $C := \langle a_i, a_j, a_k\rangle \leq M$. Then $U \leq C$ and we may apply Lemma 4.1.2 for C and U. Part (b) provides a unique element $K \in \mathfrak{M}$ such that $C_G(U) \leq K$. It follows that $A \leq C_G(U) \leq K$ and from $x \in C_G(U) \leq C_G(u) \nleq M$ we deduce that $C_G(u)$ is a subgroup of K. We further obtain from Lemma 4.1.2 (a) that $[H, x] \leq C_G(U)$. Analogously Lemma 4.1.2 (a) applied to A and V with K yields that $[K, x] \leq C_G(V)$. As H and K have normal 3-complements by Lemma 3.2.2 (d), Lemma 1.3.7 (a) shows

$$[H, x] = [H, x, x] \leq [C_G(U), x] \leq [K, x] = [K, x, x] \leq [C_G(V), x] \leq [H, x].$$

This implies that $[H, x] = [K, x] \trianglelefteq \langle H, K\rangle$. Since H and K are maximal subgroups of G, Lemma 3.2.2 (a) yields $H = K$. This shows $C_G(u) \leq K = H$. □

4.1.4 Proposition

Let $c \in I^*(M)$ and suppose that $C_G(c)$ is S_4-free. Then for every section $\tilde{E}$ of $C_G(c)$ of even order and $T_{\tilde{E}} \in \mathrm{Syl}_2(\tilde{E})$ the group $\langle \Omega_1(Z(T_{\tilde{E}}))^{N_{\tilde{E}}(J(T_{\tilde{E}}))}\rangle$ is a strongly closed elementary abelian 2-subgroup of $\tilde{E}$.

Proof

We set $C := C_G(c)$ and let further $\tilde{E}$ be a 2-constrained section of C and $T_{\tilde{E}} \in \mathrm{Syl}_2(\tilde{E})$.

If every non-abelian composition factor of $\tilde{E}$ is a $3'$-group, then Theorem 2.2.4 yields the assertion of our proposition.

Suppose for a contradiction that this is not the case. Let $H \leq C$ be of minimal order such that there is a normal subgroup N of H with $O_2(H/N) = F^*(H/N)$ and H/N has a non-abelian composition factor of order divisible by 3. Moreover we choose N in H of maximal order. Further let $- : H \to H/N$ be the natural epimorphism, U be the full pre-image of $O_2(\bar{H})$ and choose $T \in \mathrm{Syl}_2(U)$.

(1) The group $\bar{H}$ is perfect and $\bar{T}$ is elementary abelian of order at least 2^3.

Proof. By the minimal choice of H we immediately see that $\bar{H}$ is perfect.

A Frattini argument shows that $H = U \cdot N_H(T) = N \cdot N_H(T)$. Furthermore $\bar{T}$ is non-cyclic and hence $\bar{T}/\phi(\bar{T})$ is non-cyclic by a result of Burnside III 3.15 of [29]. Additionally $\phi(\bar{T})$ *char* $\bar{T} \trianglelefteq \bar{H}$ and so $\phi(\bar{T})$ is normal in $\bar{H}$. As we have $C_{\bar{H}}(\bar{T}) \leq \bar{T}$, there is no element of odd order in $\bar{H}$ acting trivially on $\bar{T}$. From Lemma 1.1.14 (a) and Lemma 1.1.4 we conclude that there is no element of odd order in $\bar{H}/\phi(\bar{T})$ acting non-trivially on $\bar{T}/\phi(\bar{T})$. Therefore $C_{\bar{H}/\phi(\bar{T})}(\bar{T}/\phi(\bar{T}))$ is a 2-group. Since $\bar{T}/\phi(\bar{T}) \leq O_2(\bar{H}/\phi(\bar{T}))$ holds, it follows that $F^*(\bar{H}/\phi(\bar{T})) \leq C_{\bar{H}/\phi(\bar{T})}(O_2(\bar{H}/\phi(\bar{T}))) \cdot O_2(\bar{H}/\phi(\bar{T})) \leq C_{\bar{H}/\phi(\bar{T})}(\bar{T}/\phi(\bar{T})) \cdot O_2(\bar{H}/\phi(\bar{T}))$ is a 2-group. By the maximal choice of N, we conclude that $\bar{T}$ is elementary abelian.

As $\bar{H}$ is non-soluble, $\bar{T}$ has oder at least 2^3. ■

Let $R \in \mathrm{Syl}_3(N_H(T))$.

(2) The group $\bar{H}$ is S_3-free and R is non-cyclic.

Proof. Suppose for a contradiction that $\bar{H}$ involves some S_3.
Then $\bar{H}$ has an element $\bar{y}$ of order 3 and a 2-element $\bar{b}$ such that $\langle \bar{y}, \bar{b}\rangle / \langle \bar{b}^2 \rangle \cong S_3$ by Lemma 1.3.3. Since $\bar{y}$ has odd order, it acts non-trivially on $\bar{T}$. From the same lemma we see that $\bar{H}$ and hence C is not S_4-free. This is a contradiction.
Further $\bar{R} \in \mathrm{Syl}_3(\bar{H})$. Suppose for a contradiction that R is cyclic. Then, as $\bar{H}$ is perfect by (1), the p-Complement Theorem of Burnside 1.1.10 implies that $\bar{R}$ is inverted in $\bar{H}$. This provides some S_3 involved in $\bar{H}$, which is a contradiction. ∎

(3) There is an elementary abelian subgroup Y of order 9 of R such that $x \notin Y$.

Proof. Suppose for a contradiction that (3) is false. Then R is of rank 2. From the fact that $\bar{H}$ has no normal 3-complement we obtain a subgroup $\bar{R}_0$ of $\bar{R}$ such that $N_{\bar{H}}(\bar{R}_0)/C_{\bar{H}}(\bar{R}_0)$ is no 3-group by Frobenius' p-Complement Theorem 1.1.11. Let R_0 denote the full pre-image of $\bar{R}_0$ in R.
Assume for a contradiction that $\bar{t} \in N_{\bar{H}}(\bar{R}_0) \setminus C_{\bar{H}}(\bar{R}_0)$ is a 2-element with $\overline{t^2} \in C_{\bar{H}}(\bar{R}_0)$. Then $\bar{t}$ acts non-trivially on $\bar{R}_0/\phi(\bar{R}_0)$. Thus there is an element $\bar{y} \in \bar{R}_0$ such that $[\bar{y}, \bar{t}] \notin \phi(\bar{R}_0)$ and $\bar{y}^{\overline{t^2}} = \bar{y} \cdot \bar{z}$ for a suitable $\bar{z} \in \phi(\bar{R}_0)$. It follows that:

$$[\bar{y}, \bar{t}]^{\bar{t}} = (\bar{y}^{-1} \cdot \bar{y}^{\bar{t}})^{\bar{t}} = (\bar{y}^{-1})^{\bar{t}} \cdot \bar{y}^{\overline{t^2}} = (\bar{y}^{-1})^{\bar{t}} \cdot \bar{y} \cdot \bar{z} = [\bar{t}, \bar{y}] \cdot \bar{z} = [\bar{y}, \bar{t}]^{-1} \cdot \bar{z}.$$

But this shows that $\bar{H}$ involves $\langle \bar{t}, [\bar{y}, \bar{t}]\rangle \phi(\bar{R})/\phi(\bar{R}) \cong S_3$ which contradicts (2).
Consequently $N_{\bar{H}}(\bar{R}_0)/C_{\bar{H}}(\bar{R}_0)$ is of odd order and there exists a prime $q \in \pi(H)$ with $q \geq 5$ such that q divides the order of $N_{\bar{H}}(\bar{R}_0)/C_{\bar{H}}(\bar{R}_0)$. Since we have $N_{\bar{H}}(\bar{R}_0) = \overline{N_H(R_0 \cdot N)}$ and $C_{\bar{H}}(\bar{R}_0) \geq \overline{C_H(R_0)}$, it follows that q divides $|\overline{N_H(R_0 \cdot N)} : \overline{C_H(R_0)}| = |N_H(R_0 \cdot N) : C_H(R_0) \cdot N|$. A Frattini argument yields that $|N_{N_H(R_0 \cdot N)}(R_0) \cdot N : C_H(R_0) \cdot N|$ is divisible by q. Altogether q divides the order of $N_H(R_0)/C_H(R_0)$.
By Lemma 1.1.16 there exists a critical subgroup R_1 of R_0. Hence R_1 admits a non-trivial q-automorphism induced by an element of H. Therefore R_1 is neither cyclic nor we have that R_1 contains x and is elementary abelian of order 9 or extraspecial of order 27 by Lemma 1.3.5. From $r(R_1) \leq r(R_0) \leq r(R) = 2$ and Lemma 1.3.6 we obtain a contradiction. ∎

(4) For all $y \in Y^\#$ we have $c \in C_T(y) \cong Q_8$ or $C_T(y) = \langle c \rangle$ and we have that $C_T(y)$, $\langle c \rangle$ and 1 are all Y-invariant subgroups of $C_T(y)$.

Proof. The group Y acts coprimely on T. Thus Lemma 1.1.14 (e) yields that we have $T = \langle C_T(y) \mid y \in Y^\# \rangle$. In particular $C_T(z) \neq 1$ for an element $z \in Y^\#$. Let $y \in Y$. From $H \leq C$ we deduce that y centralises the element c. From Corollary 3.3.4 (b) and (3) we see that $r_2(C_G(y)) \leq 1$. This implies that c is the unique involution of $C_C(y)$. From $C_T(z) \neq 1$ we conclude that $c \in T$ and we have $c \in C_T(y)$ for all $y \in Y^\#$.
Moreover $C_T(y)$ is either cyclic or a quaternion group by Lemma 1.1.2. Since Y is abelian, $C_T(y)$ is Y-invariant.
Suppose for a contradiction that there is a $z \in Y^\#$ such that $C_T(z)$ has a Y-invariant subgroup Q which is neither a subgroup of $\langle c \rangle$ nor isomorphic to Q_8. Choose z and Q among those such that Q has the largest order. All subgroups of cyclic 2-groups and quaternion groups are cyclic 2-groups or quaternion groups. Hence Q is one of those. Since Q is not isomorphic to Q_8, the group Q admits no automorphism of order 3 by Lemma 1.1.3. It follows that $Y \leq C_G(Q)$ and equivalently that $Q \leq C_T(y)$ for all $y \in Y$. If the order of Q was at least 8, then the maximal choice of Q would imply that $Q = C_T(y)$ for all $y \in Y$ and hence $T = \langle C_T(y) \mid y \in Y^\# \rangle = Q$ would contradict (1).
This contradiction shows that $Q \cong Z_4$. Let $y \in Y^\#$. Then we observe from the maximal choice of Q that $C_T(y)$ is isomorphic to some subgroup of Q_8. Since the group Q is Y-invariant and a subgroup of $C_T(y)$, we conclude that $C_T(y)$ is centralised by Y. Altogether we have $T = \langle C_T(y) \mid y \in Y^\# \rangle \leq C_T(Y)$. This contradicts the choice of H.
Consequently we have either $C_T(y) \cong Q_8$ or $C_T(y) = \langle c \rangle$ for all $y \in Y^\#$. ∎

(5) The group T is extraspecial of order $2^{2\cdot n+1}$ for some $n \in \{1,2,3,4\}$.

Proof. From Theorem 5.3.16 of [22] we observe that $T = \prod_{y \in Y^\#} C_T(y)$.
Because of $C_T(y) = C_T(\langle y \rangle) = C_T(y^{-1})$ for all $y \in Y^\#$, there exist elements $y_1, y_2, y_3, y_4 \in Y$ such that $T = \prod_{i \in \{1,2,3,4\}} C_T(y_i)$ (the y_i are just some representatives of the four cyclic subgroup of order 3 of Y). For every $i \in \{1,2,3,4\}$ we set $C_i := C_T(y_i)$. Since $C_T(y)$ is Y-invariant for all $y \in Y^\#$, we see that for every subset I of $\{1,2,3,4\}$ the group $\prod_{i \in I} C_i$ is normalised by Y. Hence $(\prod_{i \in I} C_i) \cap C_j$ is a Y-invariant subgroup of C_j for all $j \in \{1,2,3,4\}$. Moreover for all $j \in \{1,2,3,4\}$ we have that $c \in (\prod_{i \in I} C_i) \cap C_j$ and (4) implies that $|(\prod_{i \in I} C_i) \cap C_j| \in \{2,8\}$. Consequently we have

$$|T| = |\prod_{i \in \{1,2,3,4\}} C_i| = \frac{|C_4| \cdot |\prod_{i \in \{1,2,3\}} C_i|}{|C_4 \cap \prod_{i \in \{1,2,3\}} C_i|} = \frac{|C_4| \cdot \frac{|C_3| \cdot |\prod_{i \in \{1,2\}} C_i|}{|C_3 \cap \prod_{i \in \{1,2\}} C_i|}}{|C_4 \cap \prod_{i \in \{1,2,3\}} C_i|}$$
$$= \frac{|C_4| \cdot |C_3| \cdot |\prod_{i \in \{1,2\}} C_i|}{|C_4 \cap \prod_{i \in \{1,2,3\}} C_i| \cdot |C_3 \cap \prod_{i \in \{1,2\}} C_i|}$$
$$= \frac{|C_4| \cdot |C_3| \cdot \frac{|C_2| \cdot |C_1|}{|C_2 \cap C_1|}}{|C_4 \cap \prod_{i \in \{1,2,3\}} C_i| \cdot |C_3 \cap \prod_{i \in \{1,2\}} C_i|}$$
$$= \frac{|C_4| \cdot |C_3| \cdot |C_2| \cdot |C_1|}{|C_4 \cap \prod_{i \in \{1,2,3\}} C_i| \cdot |C_3 \cap \prod_{i \in \{1,2\}} C_i| \cdot |C_2 \cap C_1|}.$$

Every factor of the numerator and every factor of the denominator in this fraction is either 2 or 8. As the numerator has four factors and the denominator has three factors, we conclude that $|T| = 2^{2\cdot n+1}$ for some suitable $n \in \{1,2,3,4\}$.
Additionally Y normalises every abelian characteristic subgroup A of T. This implies $\Omega_1(A) = \langle C_{\Omega_1(A)}(y) \mid y \in Y^\# \rangle$ by Lemma 1.1.14 (e). Since we have $\langle c \rangle = \Omega_1(C_T(y))$ and since A is abelian, we obtain $C_{\Omega_1(A)}(y) \leq \Omega_1(C_T(y)) = \langle c \rangle$ for all $y \in Y^\#$. Thus it follows that $\Omega_1(A) = \langle C_{\Omega_1(A)}(y) \mid y \in Y^\# \rangle \leq \langle c \rangle$. This forces A to be cyclic.
P. Hall's Theorem III 13.10 of [29] yields that T is extra-special or a central product of an extra-special group and a cyclic, dihedral, semi-dihedral or generalised quaternion group. (As T has an elementary abelian section of order at least 2^3 by (1), the group T is not isomorphic to any dihedral, semi-dihedral or generalised quaternion group.) In particular $\phi(T)$ is non-trivial and cyclic. For all $y \in Y^\#$ we have that $C_{\phi(T)}(y)$ is an abelian Y-invariant subgroup of $C_T(y)$. Therefore (4) yields that $C_{\phi(T)}(y) \leq \langle c \rangle$ for all $y \in Y^\#$. From Lemma 1.1.14 (e) we obtain that $1 \neq \phi(T) = \langle C_{\phi(T)}(y) \mid y \in Y^\# \rangle \leq \langle c \rangle$. In particular $|T/\phi(T)| = \frac{|T|}{|\langle c \rangle|} = 2^{2n}$.
This implies that T is not a central product of an extra-special group with a cyclic group of order at least 4, because otherwise $|T/\phi(T)|$ would be a power of 2 with an odd exponent. Suppose for a contradiction that T is the central product of an extra-special group and dihedral, semi-dihedral or generalised quaternion group, such that the second factor has order at least 16. Then $\phi(T) = \langle c \rangle$ is cyclic of order at least 4. This is a contradiction.
It remains that T is extraspecial. ∎

We recall that U is the full pre-image of $O_2(H)$.
The Frattini argument $H = U \cdot N_H(T) = N \cdot N_H(T)$ shows that $\bar{H}$ is isomorphic to a section of $\overline{N_H(T)}$. But $\bar{T}$ is self-centralising in $\bar{H}$. Altogether we observe from (5) that $\bar{H}$ is isomorphic to a section of the automorphism group of an extra-special group of order $2^{2\cdot n+1}$. Hence from Theorem 1.2.7 we deduce that $\bar{H}$ is isomorphic to a section of an orthogonal group over a vector space of order $2^{2\cdot n} \leq 2^8$. Since $\bar{H}$ is non-soluble, the same theorem implies that $\bar{H}$ involves a S_3. This contradicts (2). □

4.2 Failure of Balance

In this section we obtain a subgroup of odd order generated by specific normal subgroups of the centralisers of involutions in an elementary abelian 2-group of order 8. In order to do this we consider several possible signalizer functors and especially analyse involutions that are not balanced in G.
Throughout this section we fix $T \in \mathrm{Syl}_2(M)$ and suppose that there is an involution $c \in T$ such that $C_G(c)$ is neither 3-soluble nor 2-constrained. Additionally we set $C := C_G(c)$ and denote by $- : C \to C/O(C)$ the natural epimorphism.

4.2.1 Lemma

The following hold:

(a) There is an element $d \in c^G \cap C$ and an element $s \in C$ such that $\langle c, d, d^s \rangle$ is elementary abelian of order 8.

(b) We have $c \in I^*(M)$. Moreover $x \in O(C)$ and $O(C) = O_{\{2,3\}'}(C) \cdot R$ for a cyclic Sylow 3-subgroup R of $O(C)$.

(c) Every component of $E(C/O(C))$ is not 3-soluble. Furthermore there is a unique component or the Sylow 2-subgroups of every component are quaternion groups.

(d) The element c is the unique involution of $O_{2',2}(C)$.

Proof
From $x \in C$, the assumption that C is not 3-soluble and Lemma 3.2.2 (f) we deduce that $C \leq M$.

(a) From $Z^*(G) = 1$ by Lemma 3.2.1 (a) and the Z^*-Theorem 1.1.13 we obtain an element $d \in C \setminus \{c\}$ that is conjugate to c in G. Suppose for a contradiction that $d \in Z^*(C)$. Then we have $\langle c, d \rangle \leq Z^*(C)$. By Corollary 3.3.4 (a) and Lemma 3.3.5 (b) the group $C_G(\langle z, d \rangle)$ has a normal 3-complement and is 3-soluble. Now the Odd Order Theorem 1.1.12 yields that $C = C_C(\langle c, d \rangle) \cdot O(C)$ is 3-soluble. This is e a contradiction.
Thus we have $d \notin Z^*(C)$ and the Z^*-Theorem 1.1.13 provides an element $s \in C \backslash C_G(d)$ such that d and d^s commute. From $s \in C$ we observe that $d^s \neq c$. Moreover, by Proposition 3.3.3 there is no elementary abelian subgroup of order 4 such that all its involutions are contained in $I^*(M)$. For that reason $\langle z, d, d^c \rangle$ is elementary abelian of order 8.

(b) Since C is a subgroup of M, we have $c \in I^*(M)$ and $x \in O(C)$. From (a) we obtain that C has an elementary abelian subgroup of order 8. Hence Lemma 3.3.5 (a) implies that $O(C)$ has cyclic Sylow 3-subgroups containing x. By Part (b) of the same lemma the group $O(C)$ has a normal 3-complement.

$$(*) \text{ For all } y \in D^*(M) \cap C \text{ we have } r_2(C_{\bar{C}}(\bar{y})) = 1.$$

Proof. Let $y \in D^*(M) \cap C$ and let U be the full pre-image of $C_{\bar{C}}(\bar{y})$. Further let $\bar{a} \in \bar{U}$ be an involution. Then, as $O(C)$ has odd order, we may choose a pre-image a of $\bar{a}$ in U also as an involution. The group $\langle y \rangle \cdot O(C)$ is normal subgroup of odd order of U. By Lemma 1.1.14 (b) there is an a-invariant Sylow 3-subgroup R of $\langle y \rangle \cdot O(C)$. From (b) we obtain an element $x_0 \in R$ such that $x \in \langle x_0 \rangle$. So there exists an element $g \in O(C)$ such that $R = \langle y^g, x_0 \rangle$. We remark that $\bar{U} = C_{\bar{C}}(\bar{y}^g)$. Moreover $\langle x_0 \rangle = R \cap O(C)$ is a-invariant. Since $x \in \langle x_0 \rangle$ and x is centralised by a, also $x_0^a = x_0$ holds. We know that $(y^g)^a \in (O(C) \cdot y^g) \cap R$. This provides a natural number i such that $(y^g)^a = x_0^i \cdot y^g$. It follows that

$$y^g = (y^g)^{a^2} = (x_0^i \cdot y^g)^a = (x_0^a)^i \cdot (y^g)^a = x_0^i \cdot x_0^i \cdot y^g.$$

This forces x_0^{2i} to be trivial. We conclude that $x_0^i = 1$, since x_0 is a 3-element, and hence we have $a \in C_C(y^g)$. From $y \in D^*(M)$ we observe that $y^g \in D^*(M)$. By Corollary 3.3.4 (b) we have $r_2(C_C(y^g)) = 1$ and we know $c \in Z(C_C(y^g))$. This shows that $a = c$. Altogether $\bar{c}$ is the unique involution of $\bar{U}$. ■

(c) Since C is not 3-soluble and the 3-locally central element x is contained in $Z(C)$, the p-Complement Theorem of Burnside 1.1.10 implies that C has non-cyclic Sylow 3-subgroups. In particular there is an element $y \in D^*(M) \cap C$.
Moreover there is a component in the non 2-constrained group $\bar{C}$. Let $\bar{K}$ be such a component and let K denote its full pre-image. Suppose for a contradiction that $\bar{K}$ is of order prime to 3. Then K is a Suzuki group by Theorem 1.2.8. Thus Lemma 1.2.10 implies that $C_{\bar{K}}(\bar{y})$ is of even order. From $r_2(C_{\bar{C}}(y)) \leq 1$ by (*) and $\bar{c} \in C_{\bar{C}}(\bar{y})$ we deduce that $\bar{c} \in Z(\bar{K})$. Finally Theorem 1.2.8 (a) forces $\bar{K}$ to be a central extension of $Sz(8)$ and Part (c) of the same theorem yields that $\bar{K}$ has an automorphism group of order prime to 3. Hence $C_{\bar{C}}(\bar{y})$ has a section isomorphic to $\bar{K}/\langle\bar{c}\rangle$ by Lemma 1.2.10. In particular $C_{\bar{C}}(\bar{y})$ has an elementary abelian section of order 8 by Theorem 1.2.8 (d). This contradicts (*) and Lemma 1.1.2.
Thus 3 divides $|\bar{K}|$ and K has no normal 3-complement. It follows that K has non-cyclic Sylow 3-subgroups from Lemma 3.3.5 (b) and $x \in K$. For that reason there is an element $y_K \in K \cap D^*(M)$.
If we have $\bar{K} = E(\bar{C})$, then the assertion of (c) is true. Suppose that $\bar{L}$ be a component of $\bar{C}$ different from $\bar{K}$. Then we get $\bar{L} \leq C_{\bar{C}}(\bar{K}) \leq C_{\bar{C}}(\overline{y_K})$ by Lemma 1.1.18 (b). As $r_2(C_{\bar{C}}(\overline{y_K})) \leq 1$ by (*) and a Sylow 2-subgroup of $\langle\bar{c}, \bar{L}\rangle$ is contained in $C_{\bar{C}}(\overline{y_K})$, we conclude that $c \in L$. The quasi-simple group L has non-cyclic Sylow 2-groups by Burnside's p-Complement Theorem 1.1.10 and the Odd Order Theorem 1.1.12. Consequently Lemma 1.1.2 forces $\bar{L}$ to have Sylow 2-subgroups that are quaternion groups. Similarly this holds for $\bar{K}$.

(d) Let K and y_K be as in (c).
For $T_0 \in \mathrm{Syl}_2(O_{2',2}(C))$ we have that $\bar{T}_0 = O_2(\bar{C}) \leq C_{\bar{C}}(\bar{K}) \subseteq C_{\bar{C}}(\bar{y}_K)$. From $r_2(C_{\bar{C}}(\bar{y}_K)) \leq 1$ by (*) we conclude that $r(T_0) = r(\bar{T}_0) \leq 1$. Finally the fact that $c \in T_0$ implies (d). □

4.2.2 Lemma

Suppose that C has a strongly closed elementary abelian subgroup A_0 such that $\langle c\rangle < A_0$. Let E be a pre-image of a component of $\bar{C}$. Then $A_0 \cap E$ is not contained in $\langle c\rangle$.

Proof

We may suppose that $A_0 \leq T$. Since A_0 is strongly closed in C, Lemma 2.2.2 (a) and (b) imply that A_0 is normal in every 2-subgroup of C containing A_0.
Suppose for a contradiction that $A_0 \cap E \leq \langle c\rangle$. Then for every Sylow 2-subgroup T_E of E such that $A_0 \cdot T_E \leq T$ we observe that $[A_0, T_E] \leq A_0 \cap E \leq \langle c\rangle$. Thus A_0 normalises $T_E\langle c\rangle$ and Lemma 1.1.18 (d) implies that $\bar{A}_0 \leq N_{\bar{C}}(\bar{E})$ and consequently $\bar{A}_0/\langle\bar{c}\rangle$ normalises $\bar{E}/\langle\bar{c}\rangle$. Additionally Lemma 2.2.2 (f) yields that $\bar{A}_0/\langle\bar{c}\rangle$ is strongly closed in $\bar{C}/\langle\bar{c}\rangle$.
Moreover we observe that $\bar{E}/\langle\bar{c}\rangle \cap \bar{A}_0/\langle\bar{c}\rangle = 1$. Since we have $O(\bar{E}/\langle\bar{c}\rangle) = 1$, we may apply (2.4) of [21] to conclude that $[\bar{E}, \bar{A}]/\langle\bar{c}\rangle = [\bar{E}/\langle\bar{c}\rangle, \bar{A}_0/\langle\bar{c}\rangle] = 1$. Consequently we have $[\bar{E}, \bar{A}_0] \leq \langle\bar{c}\rangle$ and hence $[E, A_0] \leq \langle c\rangle \cdot O(C)$. As E is not 3-soluble by Lemma 4.2.1 (c), there exists an element $y \in D^*(M) \cap E$. We obtain that $[y, A_0] \leq [E, A_0] \leq \langle c\rangle \cdot O(C)$ and so it follows that $\langle y\rangle \cdot A_0 \cdot O(C)$ is soluble from the fact that $O(C)$ is soluble by the Odd Order Theorem 1.1.12.

By Theorem 1.1.8 there exists a Hall $\{2,3\}$-subgroup H of $\langle y\rangle \cdot A_0 \cdot O(C)$ such that A_0 is a Sylow 2-subgroup of H. The group $O(C)$ has cyclic Sylow 3-subgroups containing x by Lemma 4.2.1 (b). This provides an element $x_0 \in O(C)$ with $H \cap O(C) = \langle x_0\rangle \in \mathrm{Syl}_3(O(C))$. Let further $y_0 \in H$ be such that y_0 is conjugate to y in $\langle c\rangle \cdot O(C)$. Then $H = \langle x_0, y_0\rangle \cdot A_0$ and $y_0 \in D^*(M)$. Since c centralises y_0, Corollary 3.3.4 implies that $r_2(C_H(y_0)) = 1$. Moreover $O_2(H) \leq A_0$ is elementary abelian. Hence Lemma 1.1.14 (f) and the Dedekind Identity Lemma 1.1.5 imply that

$$\begin{aligned} O_2(H) &= C_{O_2(H)}(y_0) \times [O_2(H), y_0] = \langle c\rangle \times [O_2(H), y_0] \leq \langle c\rangle \cdot [A_0, y_0] \\ &\leq \langle c\rangle \cdot ((\langle c\rangle \cdot O(C)) \cap H) = \langle c\rangle \cdot (O(C) \cap H) = \langle c, x_0\rangle. \end{aligned}$$

This shows that $O_2(H) = \langle c\rangle$. From $\langle x_0\rangle = O(C) \cap H \trianglelefteq H$ and $\Omega_1(\langle x_0\rangle) = \langle x\rangle \leq Z(H)$ we deduce that $A_0 \leq C_H(x_0)$ and $x_0 \in O_3(H)$. Suppose for a contradiction that y_0 is not contained in $O_3(H)$. Then we have $O_3(H) = \langle x_0\rangle$. Hence we conclude that

$$A_0 \leq C_H(O_2(H)) \cap C_H(O_3(H)) = C_H(F^*(H)) \leq F^*(H)$$

from Lemma 1.1.18 (h). This contradicts the assumption $A_0 > \langle c\rangle = O_2(H)$.

Therefore we have $y_0 \in O_3(H)$ and $O_3(H) = \langle x_0, y_0\rangle \in \mathrm{Syl}_3(H)$. Since $\langle x_0, y_0\rangle$ is a 3-group with a cyclic maximal subgroup, we obtain that $\Omega_1(\langle O_3(H)\rangle) = \langle x, y_0\rangle$ is abelian from Theorem 1.2 (a) of [8]. Again Lemma 1.1.14 (f) yields that

$$\langle x, y_0\rangle = C_{\langle x,y_0\rangle}(A_0) \times [\langle x\rangle \cdot \langle y_0\rangle, A_0] = C_{\langle x,y_0\rangle}(A_0) \times [y_0, A_0].$$

From $[y_0, A_0] \leq ((\langle c\rangle \cdot O(C)) \cap O_3(H) = \langle x_0\rangle \leq C_{\langle x,y_0\rangle}(A_0)$ and Lemma 1.1.14 (a) we conclude that $[y_0, A_0] = [\langle x\rangle \cdot \langle y_0\rangle, A_0] = [\Omega_1(O_3(H)), A_0] = [\Omega_1(O_3(H)), A_0, A_0] = [y_0, A_0, A_0] = 1$. This contradicts $r_2(C_H(y_0)) = 1$, because we have $r(A_0) \geq 2$. □

4.2.3 Lemma

One of the following holds:

(a) The group $\langle [O_{\sigma'}(C_G(b)), x] \mid b \in T^\#$ and $b^2 = 1\rangle$ is of odd order and $\Omega_1(Z(T))^\# \subseteq I^*(M)$.

(b) The group C is S_4-free.

Proof

From $c \in I^*(M)$ by Lemma 4.2.1 (b) we observe that $2 \in \sigma$ by Lemma 3.3.2 (b) and $T \leq M$ by Lemma 3.3.1. If we have $\langle c\rangle = \Omega_1(Z(T_0))$ for a Sylow 2-subgroup T_0 of G, then (a) follows from Proposition 4.1.3.

Suppose for a contradiction that the assertion is false.

Then $\langle c\rangle \neq \Omega_1(Z(T_0))$ for all Sylow 2-subgroups T_0 of G and C is not S_4-free. By Lemma 1.1.7 there is a 2-subgroup of C such that its normaliser contains a section isomorphic to S_4. Let S be such a 2-subgroup of C of maximal order.

Moreover let $T_1 \in \mathrm{Syl}_2(C_C(S))$, $T_2 \in \mathrm{Syl}_2(C)$ and $T_3 \in \mathrm{Syl}_2(G)$ be such that $T_1 = C_{T_2}(S)$ and $T_2 = C_{T_3}(c)$. Then we have

$$\begin{aligned} \langle c, Z(T_3)\rangle &\leq C_{T_3}(c) \cap C_C(C_{T_3}(c)) = Z(C_{T_3}(c)) = Z(T_2) \\ &\leq C_{T_2}(S) \cap C_{T_2}(C_{T_2}(S)) = Z(C_{T_2}(S)) = Z(T_1). \end{aligned}$$

It follows that $r(Z(T_1)) \geq r(Z(T_2)) \geq r(\langle c, Z(T_3)\rangle) \geq 2$.

A Frattini argument shows that $N_C(S) = N_{N_C(S)}(T_1) \cdot C_C(S)$. As $N_C(S)/C_C(S)$ is not S_3-free, we deduce that $N_C(T_1)$ is also not S_3-free. Altogether the group $N_C(Z(T_1))$ is not S_3-free.

By Lemma 1.3.3 there is a subgroup $\langle y, b\rangle$ of $N_C(Z(T_1))$ such that $\langle y, b\rangle/\langle b^2\rangle \cong S_3$, the element y has order 3 and b is a 2-element. Additionally we see that $\langle y, b\rangle$ has cyclic Sylow 3-subgroups and no normal 3-complement. For that reason $y \in D^*(M)$. Moreover the element y acts by Corollary 3.3.4 (a) non-trivially on $Z(T_1)$, since $r(Z(T_1)) \geq 2$. Consequently

Lemma 1.3.3 provides an involution $a \in Z(T_0)$ such that $\langle a, b, y\rangle/\langle b^2\rangle \cong S_4$. In particular Lemma 1.3.3 yields that $V := \langle a, a^y, a^{y^{-1}}\rangle = \langle a, a^y\rangle$ is elementary abelian of order 4.

Let now $T_0 \in \mathrm{Syl}_2(C)$ be such that $b \in N_{T_0}(V) \in \mathrm{Syl}_2(N_C(V))$. Then, as y permutes the involutions of V transitively, Corollary 3.3.4 (c) yields $V \cap I^*(M) = \emptyset$. Therefore we obtain that $c \notin V$. Moreover by Lemma 4.1.1 (f) there is no elementary abelian subgroup of order 8 of C contained in $C \setminus \{c\}$ that contains V. For that reason we have $\Omega_1(C_{T_0}(V)) = \langle V, c\rangle$. This forces $\Omega_1(Z(T_0)) \leq \langle V, c\rangle$. Since $T_0 \cong T_2$ and $r(\Omega_1(Z(T_2))) \geq 2$ hold, the group $\Omega_1(Z(T_0)) \cap V$ is non-trivial. From the fact that $b \in T_0$ acts non-trivially on V we conclude that $\Omega_1(Z(T_0)) \cap V$ is cyclic of order 2. More precisely, as $C_V(b) = \langle a\rangle$, we see that $\Omega_1(Z(T_0)) \cap V = \langle a\rangle$. Altogether we conclude that $\Omega_1(Z(T_0)) = \langle a, c\rangle$ and that $\langle V, c\rangle = \Omega_1(C_{T_0}(V)) = \Omega_1(C_{T_0}(a^y))$ hold.
Furthermore Lemma 4.1.1 (d) leads to $\langle V, c\rangle \cap I^*(M) = \{c\}$ and for all $v \in V^\#$ Part (c) of the same lemma implies that

$$(*)\ O_{\sigma'}(C_G(v)) \text{ is abelian and } [x, O_{\sigma'}(C_G(c \cdot v))] \text{ is not abelian.}$$

From Lemma 4.2.1 (a) we obtain an element $d \in c^G \cap C \setminus \{c\}$ and from Lemma 3.2.2 (g) we see that $d \in I^*(M)$. Hence $d \notin \langle V, c\rangle = \Omega_1(C_{T_0}(a^y))$. This implies that $\langle d, a^y\rangle$ is a non-abelian dihedral group. Let e denote the central involution of $\langle d, a^y\rangle$. Then we observe that $e \in \Omega_1(C_{T_0}(a^y)) = \langle V, c\rangle$. Since d is conjugate to $d \cdot e$ in $\langle d, a^y\rangle$, Lemma 3.2.2 (g) implies that $d \cdot e \in I^*(M)$. Thus Proposition 3.3.3 shows that $e = d \cdot (d \cdot e) \notin I^*(M)$. In particular we have $e \neq c$ and hence, from $c \in C_G(\langle d, a^y\rangle)$ it follows that $\langle c\rangle \cap \langle d, a^y\rangle = 1$. Since $C_{\langle d, a^y\rangle}(a^y)$ is elementary abelian of order 4, we conclude that $\langle V, c\rangle = \Omega_1(C_{T_0}(a^y)) = \langle e, a^y, c\rangle$. This shows that $\langle c, d, V\rangle = \langle c, \langle d, a^y\rangle\rangle = \langle c\rangle \times \langle d, a^y\rangle \cong Z_2 \times D_{2^n}$ for a suitable $n \in \mathbb{N}$.
Consequently we have

$$\langle a, c\rangle = \Omega_1(Z(T_0)) \leq \langle c, V\rangle \cap C_{T_0}(\langle c, d, V\rangle) \leq \langle c, d, V\rangle \cap C_{T_0}(\langle c, d, V\rangle) = Z(\langle c, d, V\rangle) = \langle e, c\rangle.$$

It follows that $e \in \langle a, c\rangle$. More precisely, we have that $e \in \{a, ac\}$.
In addition $\langle x\rangle \cdot \langle d, e\rangle$ acts coprimely on $O_{\sigma'}(C_G(ce))$. From Lemma 1.3.1 we conclude that

$$\begin{aligned}[O_{\sigma'}(C_G(ce)), x] &\leq \langle C_{[O_{\sigma'}(C_G(ce)),x]}(B) \mid B \text{ max } \langle d, e\rangle,\ C_G(B) \nsubseteq C_G(x)\rangle \\ &= \langle C_{[O_{\sigma'}(C_G(ce)),x]}(b) \mid b \in \langle d, e\rangle^\# \setminus I^*(M)\rangle \\ &\subseteq C_G(e),\end{aligned}$$

since $d, d \cdot e \in I^*(M)$.

Let $g \in [O_{\sigma'}(C_G(c \cdot e)), x] \subseteq C_G(e)$. From $2 \in \sigma$ and Lemma 1.3.7 we deduce that

$$[g, x] \in [C_G(e), x] \overset{1.3.7}{=} [C_{C_G(e)}(x) \cdot O_{\sigma'}(C_G(e)), x] = [O_{\sigma'}(C_G(e)), x] \subseteq O_{\sigma'}(C_G(e)).$$

This shows together with Lemma 1.1.14 (d) that

$$[O_{\sigma'}(C_G(ce)), x] = [[O_{\sigma'}(C_G(ce)), x], x] = \langle [g, x] \mid g \in [O_{\sigma'}(C_G(ce)), x]\rangle \subseteq O_{\sigma'}(C_G(e)).$$

Suppose first that $e = a$. Then $O_{\sigma'}(C_G(e))$ is abelian by (*) although it follows that also the group $[O_{\sigma'}(C_G(ce)), x] = [O_{\sigma'}(C_G(ca)), x]$ is abelian. But this contradicts (*).
Suppose now that $e = c \cdot a$. Then a^y is conjugate to $e \cdot a^y = c \cdot a \cdot a^y$ in $\langle d, a^y\rangle \subseteq C_G(x)$. As $[O_{\sigma'}(C_G(a^y)), x]$ is abelian by $(*)$ this shows that also $[O_{\sigma'}(C_G(c \cdot a \cdot a^y)), x]$ is abelian contradicting (*) as well. □

4.2.4 Proposition

Suppose that $\Omega_1(Z(T))^\# \nsubseteq I^*(M)$.
Then $E(\bar{C})$ is quasi-simple and one of the following holds:

(a) The group $E(\bar{C})$ is a simple Bender group or isomorphic to $\mathrm{SL}(2, 5)$.

(b) For every involution $b \in C$ we have $O_{\{2,3\}'}(C_G(b)) \cap C \leq O_{\{2,3\}'}(C)$.

(c) The group C is balanced.

Proof

Let $T_0 \in \mathrm{Syl}_2(C)$ and let $T_1 \in \mathrm{Syl}_2(G)$ be such that $T_0 \leq T_1$.
Then $Z(T_1) \leq C_{T_1}(c) \cap C_{T_1}(T_0) = T_0 \cap C_{T_1}(T_0) = Z(T_0)$. By assumption we have $\Omega_1(Z(T))^\# \nsubseteq I^*(M)$. This implies that $\Omega_1(Z(T_1))^\# \neq \{c\} \subseteq I^*(M)$. Hence $\Omega_1(Z(T_0))$ is non-cyclic.
Moreover we deduce from Lemma 4.2.3 that C is S_4-free. Thus Proposition 4.1.4 implies that the group defined by $A_0 := \langle \Omega_1(Z(T_0))^{N_C(J(T_0))}\rangle$ is a strongly closed elementary abelian 2-subgroup of C. As $\Omega_1(Z(T_0))$ is non-cyclic, also A_0 is non-cyclic and contains c.
Let E denote the full pre-image of a component of $E(\bar{C})$ in C. Then Lemma 4.2.2 yields $\overline{A_0 \cap E} \nleq \langle c\rangle$. From $1 \neq E \cap T_0 \trianglelefteq T_0$ we obtain that $1 \neq \Omega_1(Z(T_0)) \cap E \leq A_0 \cap E$ and further $\overline{A_0 \cap E}$ is an elementary abelian strongly closed subgroup of $\bar{E}$ by Lemma 2.2.2 (c) and (f).
Moreover c is the unique involution of $O_{2',2}(C)$ by Lemma 4.2.1 (d). Hence we conclude from $Z(\bar{C}) \leq O_2(\bar{C}) = \overline{O_{2',2}(C)}$ that $\overline{A_0 \cap E} \nleq Z(\bar{E})$. Applying the Z^*-Theorem 1.1.13 we see that $(\overline{A_0 \cap E}) \cdot \langle \bar{c}\rangle/\langle \bar{c}\rangle$ is non-cyclic. In particular the Sylow 2-subgroups of $\bar{E}$ are not quaternion groups. Finally Part (c) of Lemma 4.2.1 yields that $\bar{E} = E(\bar{C})$ is the unique component of $\bar{C}$. It follows that $E(\bar{C})$ is quasi-simple.

We need to verify one of (a), (b) or (c). Suppose for a contradiction that none of the statements is true.

(1) There is no $x_0 \in C$ such that $x_0^3 \in O(C)$ and $O(C_{\bar{C}}(\bar{b})) \leq \langle \bar{x}_0\rangle$ for all involutions $b \in T_0$.

Proof. Assume for a contradiction that there is an element $x_0 \in C$ with $x_0^3 \in O(C)$ and such that $O(C_{\bar{C}}(\bar{b})) \leq \langle \bar{x}_0\rangle$ holds for all involutions $b \in T_0$. Let further $b \in T_0$ be an involution. Then $O_{\{2,3\}'}(C_G(b)) \cap C$ is a normal subgroup of $C_C(b)$ of odd order and so $O_{\{2,3\}'}(C_G(b)) \cap C \leq O(C_C(b)) \leq \langle x_0, O(C)\rangle$. By Lemma 4.2.1 (b) the group $O(C)$ has a normal 3-complement. Thus we have:

$$O_{\{2,3\}'}(C_G(b)) \cap C \leq O_{\{2,3\}'}(C_G(b)) \cap \langle x_0, O(C)\rangle \leq O^{\{2,3\}}(\langle x_0\rangle \cdot O(C)) = O_{\{2,3\}'}(C).$$

This is Part (c) and therefore a contradiction. ∎

We set $F := E\langle c\rangle$ and let Z be the full pre-image of $Z(\bar{F})$. Then $\bar{Z} \leq O_2(\bar{C})$ and hence by Lemma 4.2.1 (d) the abelian group $\bar{Z}$ is cyclic. Furthermore let $\wedge : C \to C/\langle c\rangle$ be the natural epimorphism.

(2) The group F has an elementary abelian subgroup A of order 8 with $c \in A$ and such that A is strongly closed in C.

Proof. We want to apply Theorem 2.3.3 to $\bar{C}$.
We know that $\bar{C}$ is a finite group with $O(\bar{C}) = 1$. Moreover we have $r_2(O_2(\bar{C})) = 1$ by Lemma 4.2.1 (d). The group $E(\bar{C})$ is quasi-simple. Since $\Omega_1(Z(T_0))$ is non-cyclic and $O(C)$ has odd order, we see that $\Omega_1(Z(\bar{T}_0)) > \langle \bar{c}\rangle = \Omega_1(F(\bar{C})) = \Omega_1(Z(\bar{C}))$. The group $E(\bar{C})$ contains the elementary abelian subgroup $\overline{A_0 \cap E}$. Furthermore we already observed that $(\overline{A_0 \cap E}) \cdot \langle \bar{c}\rangle/\langle \bar{c}\rangle$ is a non-cyclic group.
Moreover for all $\bar{g} \in N_{\bar{C}}(\bar{A}_0)$ we see that $\overline{A_0 \cap E}^{\bar{g}} = \overline{(A_0 \cap E)^g} = \overline{A_0^g \cap E^g} = \overline{A_0 \cap E}$. This shows that $N_{\bar{C}}(\bar{A}_0) \leq N_{\bar{C}}(\overline{A_0 \cap E})$. Hence Lemma 2.2.2 (j) implies that $\overline{A_0 \cap E}$ is strongly closed in $\bar{C}$.
In addition let $\bar{b} \neq \bar{c}$ be an involution of $\bar{C}$. Then, as $O(C)$ has odd order, we may choose a pre-image b as an involution in C. Then $C_C(b) = C_G(\langle b, c\rangle)$ has cyclic Sylow 3-subgroups containing x by Corollary 3.3.4 (a). Thus Lemma 3.3.5 (b) implies that $C_C(b)$ has a normal 3-complement. In particular $C_C(b)$ is 3-soluble. Since $O(C)$ is of odd order, we may apply Lemma 1.1.14 (a) to conclude that also $C_{\bar{C}}(\bar{b}) = \overline{C_C(b)}$ is 3-soluble.

Altogether the assumptions on Theorem 2.3.3 are satisfied. Thus one of (a)-(f) of Theorem 2.3.3 is true.
Since $\bar{E}$ is not 3-soluble by Lemma 4.2.1 (c), Part (a) of Theorem 2.3.3 false. As we supposed that the assertion (a) of our proposition is false, also Part (d) of Theorem 2.3.3 is not true. Moreover Part (e) of the theorem is false by (1). Also Part (f) of the theorem is not valid, because C is S_4-free. Consequently we are left with two cases:

Case 1: There is a strongly closed elementary abelian subgroup $\bar{A}_1$ of order 4 of $\bar{C}$ contained in $\bar{E}$. Then, as $O(C)$ has odd order, Lemma 2.2.2 (i) implies that C has a strongly closed elementary abelian subgroup A_1 of order 4 that is contained in E and a pre-image of $\bar{A}_1$. Suppose for a contradiction that $c \in A_1$. Then $c \in E$ and the cyclic group $\bar{A}_1/\langle\bar{c}\rangle$ is by Lemma 2.2.2 (c) and (f) strongly closed in $\bar{E}/\langle\bar{c}\rangle$. This contradicts the Z^*-Theorem 1.1.13, since $\bar{E}\cdot\langle\bar{c}\rangle/\bar{Z}$ is simple and $\Omega_1(\bar{Z}) = \langle\bar{c}\rangle$ is cyclic.
Thus $c \notin A_1$ and as $\langle c\rangle$ is obviously strongly closed in $C = C_G(c)$, Lemma 2.2.2 (l) yields that $\langle c, A_1\rangle =: A$ is strongly closed in C. Moreover A is elementary abelian of order 8 and $c \in A$.

Case 2: We have $Z(\bar{E}) \neq 1$ and $\bar{E}$ has an elementary abelian subgroup $\bar{A}$ of order 8 that is strongly closed in $\bar{C}$ and such that $\Omega_1(Z(\bar{E})) \leq \bar{A}$.
Then Lemma 2.2.2 (i) provides a strongly closed elementary abelian subgroup A of order 8 of C that is contained in E and a pre-image of $\bar{A}$, since $O(C)$ has odd order. From $Z(\bar{E}) \neq 1$ we see that $\langle\bar{c}\rangle = \Omega_1(\bar{Z}) \leq \Omega_1(Z(\bar{E}))$. We conclude that $c \in A$, as $\bar{A}$ contains $\Omega_1(Z(\bar{E}))$. ■

Let V be a complement of $\langle c\rangle$ in A.

(3) Let $v \in V^\#$ and let b be an involution of T_0. Then following hold:

(i) $N_C(\langle b, c\rangle)$ is the full pre-image of $C_{\hat{C}}(\hat{b})$,

(ii) $N_F(\langle v, c\rangle)\cdot O(C)$ is the full pre-image of $C_{F/Z}(Zv)$,

(iii) $C_C(\langle b, c\rangle) = C_C(b) = C_C(bc)$,

(iv) $O^2(N_C(\langle v, c\rangle)) = O^2(C_C(\langle v, c\rangle))$ and

(v) $N_C(\langle v, c\rangle) \subseteq N_C(A)\cdot O(C_C(v))$.

Proof. Since b is an involution, we observe that $C_{\hat{C}}(\hat{b}) = N_{\hat{C}}(\langle\hat{b}\rangle) = N_{\hat{C}}(\widehat{\langle b, c\rangle}) = \widehat{N_C(\langle b, c\rangle)}$. Thus ($i$) holds, as $\langle c\rangle \subseteq N_C(\langle v, c\rangle)$.

For Part (ii) we first observe that

$$C_{F/Z}(Zv) = N_{F/Z}(\langle Zv\rangle) = N_{F/Z}(\langle v\rangle\cdot Z/Z) = N_{F/Z}(\langle v, c\rangle\cdot Z/Z) = N_F(\langle v, c\rangle\cdot Z)/Z.$$

Hence $N_F(\langle v, c\rangle\cdot Z)$ is the full pre-image of $C_{F/Z}(Zv)$ in C. Moreover we have

$$\overline{N_F(\langle v, c\rangle\cdot Z)} = N_{\bar{F}}(\overline{\langle v, c\rangle\cdot Z}) \leq N_{\bar{F}}(\Omega_1(\overline{\langle v, c\rangle\cdot Z})) = N_{\bar{F}}(\overline{\langle v, c\rangle}) = \overline{N_F(\langle v, c\rangle\cdot O(C))}.$$

This implies together with a Frattini argument that

$$N_F(\langle v, c\rangle\cdot Z) \leq N_F(\langle v, c\rangle\cdot O(C)) = \langle v, c\rangle\cdot O(C)\cdot N_{N_F(\langle v,c\rangle\cdot O(C))}(\langle v, c\rangle) \leq O(C)\cdot N_F(\langle v, c\rangle).$$

In particular $O(C)\cdot N_F(\langle v, c\rangle)$ contains the full pre-image of $C_{F/Z}(Zv)$ in C. Finally the assertion (ii) follows by $O(C)\cdot N_F(\langle v, c\rangle)/Z = N_F(\langle v, c\rangle)\cdot Z/Z \leq N_F(\langle v, c\rangle\cdot Z)/Z = C_{F/Z}(Zv)$.

From $c \in Z(C)$ we obtain $C_C(b) = C_C(\langle b, c\rangle) = C_C(bc)$. This implies (iii).

Moreover we observe that $|N_C(\langle v, c\rangle)/C_C(\langle v, c\rangle)| \leq 2$ from $c \in Z(C)$. Therefore we have $O^2(C_C(\langle v, c\rangle)) = O^2(N_C(\langle v, c\rangle))$ and (iv) is valid.

For (v) we first recall that A is strongly closed in C. This implies that $T_0 \in N_C(A)$ and that A is strongly closed in $C_C(\langle v, c\rangle)$ by Lemma2.2.2 (a) and (d). We further deduce from (iv) that $N_C(\langle v, c\rangle) \subseteq C_C(\langle v, c\rangle) \cdot T \subseteq C_C(\langle v, c\rangle) \cdot N_C(A)$. By Corollary 3.3.4 (a) and Lemma 3.3.5 (b) the group $C_C(\langle v, c\rangle)$ has a normal 3-complement and is therefore 3-soluble. Consequently we may apply Proposition 2.2.5 to obtain that $\langle A^{C_C(\langle v,c\rangle)}\rangle/O(\langle A^{C_C(\langle v,c\rangle)}\rangle)$ is a central product of Suzuki groups with an elementary abelian 2-group such that $\Omega_1(S) \leq A$ for a Sylow 2-subgroup of $\langle A^{C_C(\langle v,c\rangle)}\rangle$. From $|A/\langle c\rangle| = 4$ and Theorem 1.2.8 (d) we deduce that $\langle A^{C_C(\langle v,c\rangle)}\rangle$ is soluble. Finally Lemma 2.2.2 (g) yields that $C_C(\langle v, c\rangle) = N_{C_C(\langle v,c\rangle)}(A) \cdot O(\langle A^{C_C(\langle v,c\rangle)}\rangle)$.
Moreover we observe from (iii) that $O(\langle A^{C_C(\langle v,c\rangle)}\rangle) = O(\langle A^{C_C(\langle v\rangle)}\rangle) \leq O(C_G(v))$. Altogether we conclude $N_C(\langle v, c\rangle) \subseteq N_C(A) \cdot C_C(\langle v, c\rangle) \subseteq N_C(A) \cdot O(\langle A^{C_C(\langle v,c\rangle)}\rangle) \subseteq N_C(A) \cdot O(C_C(v))$. ∎

(4) We have $F = \langle N_F(\langle c, v\rangle) \mid v \in V^\#\rangle \cdot O(C)$. Moreover $\hat{A} \leq Z(\hat{T}_0)$ and $N_{\hat{C}}(\hat{A})/C_{\hat{C}}(\hat{A}) \cong Z_3$.

Proof. Since F/Z is a simple non-Bender group and $A \cdot Z/Z$ is an elementary abelian strongly closed 2-subgroup of F/Z, Lemma 2.2.6 yields that $F/Z = \langle C_{F/Z}(Zv) \mid Zv \in (AZ/Z)^\#\rangle$. We conclude by Part (3)(ii) that

$$\begin{aligned} F/Z &= \langle C_{F/Z}(Zv) \mid Zv \in (A \cdot Z/Z)^\#\rangle \overset{(ii)}{=} \langle N_F(\langle v, c\rangle) \cdot O(C)/Z \mid v \in A \setminus Z\rangle \\ &= \langle N_F(\langle v, c\rangle) \cdot O(C)/Z \mid v \in V^\#\rangle = \langle N_F(\langle v, c\rangle) \cdot O(C) \mid v \in V^\#\rangle/Z \end{aligned}$$

As for all $v \in V^\#$ we have $Z \leq N_F(\langle v, c\rangle) \cdot O(C)$, we finally obtain that

$$F = \langle N_F(\langle v, c\rangle) \cdot O(C) \mid v \in V^\#\rangle = \langle N_F(\langle v, c\rangle) \mid v \in V^\#\rangle \cdot O(C).$$

Further we observe that $|\hat{A}| = 4$ and that $N_{\hat{C}}(\hat{A})/C_{\hat{C}}(\hat{A})$ is S_3-free by Lemma 1.3.3 and the fact that C is S_4-free. Moreover we obtain that $\hat{A} \cap Z^*(\hat{C})$ is trivial from $A \cap O_{2',2}(C) \leq \langle c\rangle$ and $\langle c\rangle \leq Z(C)$. Altogether $N_{\hat{C}}(\hat{A})/C_{\hat{C}}(\hat{A})$ is a cyclic group of order 3. ∎

(5) We have $A \neq \Omega_1(T_0)$.

Proof. Suppose for a contradiction that $A = \Omega_1(T_0)$. Then there are elements $d \in c^G \cap C$ and $s \in C$ by Lemma 4.2.1 (a) such that $A = \langle c, d, d^s\rangle$. Moreover Proposition 3.3.3 yields that $c \cdot d$, $c \cdot d^c$ and $d \cdot d^c$ are not elements of $I^*(M)$, as $d, d^c, c \in I^*(M)$ by Lemma 3.2.2 (g). From $N_{\hat{C}}(\hat{A})/C_{\hat{C}}(\hat{A}) \cong Z_3$ by (4), we see that there is an element $y \in C$ such that $\hat{y}$ permutes the elements of $\{\langle c\rangle d, \langle c\rangle d^s, \langle c\rangle(d \cdot d^s)\}$ transitively. As d is neither conjugated to $c \cdot d$ nor $c \cdot d^s$ nor $d \cdot d^s$ by Lemma 3.2.2 (g), we conclude that no element of $\{c \cdot d, c \cdot d^s, d \cdot d^s\}$ is conjugate to its product with c. Further we may choose $V = \langle d \cdot c, d^s \cdot c\rangle$. Then we have $V = A \setminus I^*(M)$ and $|A \cap I^*(M)| \geq |\{c, d, d^s\}| = 3$. Thus we can apply Lemma 4.1.2. Part (b) of the lemma provides a subgroup $H \in \mathfrak{M}$ such that H is the unique maximal subgroup of G containing $C_G(V)$. In particular we conclude that $C_G(v) \leq H$ for all $v \in V^\#$.
Altogether we observe that $N_C(\langle v, c\rangle) = C_C(v)$ for all $v \in V^\#$. By (4) we have

$$F = \langle N_F(\langle v, c\rangle) \mid v \in V^\#\rangle \cdot O(C) = \langle C_F(v) \mid v \in V^\#\rangle \cdot O(C) \leq C_H(c) \cdot O(C).$$

Since F/Z is not 3-soluble by Lemma 4.2.1 (c), there is a contradiction to the fact that H has a normal 3-complement by Lemma 3.2.2 (d). ∎

(6) We have $W := \langle O(C_C(v)) \mid v \in V^\#\rangle \leq O(C) \cdot \langle x_0\rangle$, for some $x_0 \in C$ with $x_0^3 \in O(C)$.

Proof. If we have $A \leq Z(T_0)$, then we obtain from (5) an involution $e \in T_0$ such that $c \notin \langle V, e\rangle$. In this case we set $D := \langle V, e\rangle$.
If we have $A \nleq Z(T_0)$, then we deduce from $3 \mid |N_C(A)/C_C(A)|$ and $\hat{A} \leq Z(\hat{T}_0)$ by (4) that $N_C(A)/C_C(A) \cong A_4$. This implies that $A \cap Z(T_0) = \langle c\rangle < \Omega_1(Z(T_0))$. In this case we let D be a complement of $\langle c\rangle$ in the elementary abelian group $A \cdot \Omega_1(Z(T_0))$ containing V. Then D has order at least 8, since $\Omega_1(Z(T_0))$ is non-cyclic.
In both cases D is an elementary abelian group of order at least 8 such that $c \notin D$. This implies that the involutions of D have a 3-soluble centraliser in C by Corollary 3.3.4 (a) and Lemma 3.3.5 (b). In particular the involutions of D are balanced in C by Lemma 2.1.4.

We set $\theta(a) := O(C_C(a))$ for all $a \in D^\#$. Then $\theta(a)$ is soluble for all $a \in D^\#$ by the Odd Order Theorem 1.1.12. Consequently θ is a soluble D-signalizer functor and the Soluble Signalizer Functor Theorem 2.1.6 implies that $\langle O(C_C(a)) \mid a \in D^\#\rangle$ has odd order As $\langle O(C_C(a)) \mid a \in D^\#\rangle \geq \langle O(C_C(v)) \mid v \in V^\#\rangle = W$, we conclude that W has odd order.
By (3)(ii) we have for all $v \in V^\#$ that $C_C(v) = C_C(vc)$. Thus we see that

$$W = \langle O(C_C(v)) \mid v \in V^\#\rangle = \langle O(C_C(v)) \mid v \in A \setminus \langle c\rangle\rangle.$$

This and the fact that $c \in Z(C)$ is strongly closed in C show that W is normalised by $N_C(A)$. Moreover (3)(v) yields for all $v \in V^\#$ that $N_C(\langle v, c\rangle) \subseteq N_C(A) \cdot O(C_C(v))$. Since we have $O(C_C(v)) \leq W$ for all elements $v \in V^\#$, the group W is also normalised by the group $N_C(\langle v, c\rangle) \subseteq N_C(A) \cdot O(C_C(v))$. Altogether $W \trianglelefteq \langle N_C(\langle c, v\rangle) \mid v \in V^\#\rangle$. Now (4) together with the facts that F/Z is simple and $\bar{Z}$ is a 2-group imply that $[\bar{W}, \bar{F}] \leq \bar{W} \cap \bar{F}$ is trivial.
But we already showed that $F^*(\bar{C}) = O_2(\bar{C}) \cdot \bar{E} = O_2(\bar{C}) \cdot \bar{F}$. From Lemma 1.1.18 (h) we deduce that $\bar{W}$ acts faithfully on $O_2(\bar{C})$. As $r_2(O_2(\bar{C})) = 1$ by Lemma 4.2.1 (d), the group $O_2(\bar{C})$ only admits in the case $O_2(\bar{C}) \cong Q_8$ an automorphism of odd order by Lemma 1.1.2 and Lemma 1.1.3. In this case there is an element $x_0 \in C$ with $x_0^3 \in O(C)$ and such that $W = \langle O(C_C(v)) \mid v \in V^\#\rangle \leq O(C) \cdot \langle x_0\rangle$. In the other case $W \leq O(C)$ and we set $x_0 = 1$ to verify (6). ■

Let finally $b \in T_0$ be an involution different from c.
By (4) the element $\hat{b}$ acts trivially on $\hat{A}$. Thus $\hat{V}$ is contained in $C_{\hat{C}}(\hat{b}) = \widehat{N_C(\langle b, c\rangle)}$ by (3)(i). Since $O(\widehat{C_C(b)})$ *char* $\widehat{C_C(b)} = \widehat{C_C(\langle b, c\rangle)} \trianglelefteq \widehat{N_C(\langle b, c\rangle)}$ by 3(iii), the group $\hat{V}$ acts coprimely on $O(\widehat{C_C(b)})$.
Moreover $O(C_C(b)) \cdot \langle c\rangle/\langle c\rangle$ is a normal subgroup of odd order of $\widehat{C_C(\langle c\rangle)}$ and so it is contained in $O(\widehat{C_C(b)})$. Since the full pre-image of $O(\widehat{C_C(b)})$ has $\langle c\rangle$ as a Sylow 2-subgroup, it has a central Sylow 2-subgroup and by Burnside's p-Complement Theorem 1.1.10 a normal 2-complement. Consequently we have $O(\widehat{C_C(b)}) = O(C_C(b)) \cdot \langle c\rangle/\langle c\rangle$.
By Lemma 1.1.14 (e), Part (3) and Part (6) we get:

$$\begin{aligned}
O(\widehat{C_C(b)}) &\overset{1.1.14\,(e)}{=} \langle O(\widehat{C_C(b)}) \cap C_{\hat{C}}(\hat{v}) \mid v \in V^\#\rangle \\
&\overset{(i)}{=} \langle O(C_C(b)) \cdot \langle c\rangle/\langle c\rangle \cap N_C(\langle v, c\rangle)/\langle c\rangle \mid v \in V^\#\rangle \\
&\overset{(+)}{=} \langle\, (O(C_C(b)) \cdot \langle c\rangle \cap N_C(\langle v, c\rangle))\,/\langle c\rangle \mid v \in V^\#\rangle \\
&= \langle O(C_C(b)) \cdot \langle c\rangle \cap N_C(\langle v, c\rangle) \mid v \in V^\#\rangle/\langle c\rangle \\
&\overset{Ded}{=} \langle (O(C_C(b)) \cap N_C(\langle v, c\rangle)) \cdot \langle c\rangle \mid v \in V^\#\rangle/\langle c\rangle \\
&= \langle \big(O(C_C(b)) \cap O^2(N_C(\langle v, c\rangle))\big) \cdot \langle c\rangle \mid v \in V^\#\rangle/\langle c\rangle \\
&\overset{(iv)}{=} \langle \big(O(C_C(b)) \cap O^2(C_C(\langle v, c\rangle))\big) \cdot \langle c\rangle \mid v \in V^\#\rangle/\langle c\rangle \\
&\overset{(iii)}{=} \langle \big(O(C_C(b)) \cap O^2(C_C(v))\big) \cdot \langle c\rangle \mid v \in V^\#\rangle/\langle c\rangle \\
&\leq \langle\, (O(C_C(b)) \cap C_C(v)) \cdot \langle c\rangle \mid v \in V^\#\rangle/\langle c\rangle \\
&\overset{(++)}{\leq} \langle O(C(v)) \cdot \langle c\rangle \mid v \in V^\#\rangle/\langle c\rangle \\
&= \langle O(C_C(v)) \mid v \in V^\#\rangle \cdot \langle c\rangle/\langle c\rangle \\
&\overset{(6)}{\leq} \widehat{O(C)} \cdot \langle \hat{x}_0\rangle.
\end{aligned}$$

The equation (+) holds as $O(C_C(b)) \cdot \langle c\rangle$ and $N_C(\langle v, c\rangle)$ are full pre-images of $O(\widehat{C_C(b)})$ respectively $\widehat{N_C(\langle v, c\rangle)}$. The inclusion (++) is valid as b and the involutions of V are balanced in C by Corollary 3.3.4 (a), Lemma 3.3.5 (b) and Lemma 2.1.4. Moreover *Ded* denotes the Dedekind Identity Lemma 1.1.5.
Altogether $O(C_C(b)) \cdot \langle c\rangle \leq O(C) \cdot \langle x_0\rangle \cdot \langle c\rangle$.
But $O(C_C(b)) = O^2(O(C_C(b)) \cdot \langle c\rangle) \leq O^2(O(C) \cdot \langle x_0\rangle \cdot \langle c\rangle) = O(C) \cdot \langle x_0\rangle$ contradicts (1). □

4.2.5 Lemma

The following hold:

(a) If $E(\bar{C})$ is a simple Bender group but not isomorphic to $\mathrm{PSL}(2,5)$, then C is balanced.

(b) If $E(\bar{C}) \cong \mathrm{PSL}(2,5)$ or $E(\bar{C}) \cong \mathrm{SL}(2,5)$,
then $O_{\{2,3\}'}(C_G(b)) \cap C \leq O_{\{2,3\}'}(C)$ for every involution $b \in C$.

Proof

(a) By Lemma 4.2.1 (c) the group $E(\bar{C})$ is not 3-soluble. Thus $E(\bar{C})$ is no Suzuki group. Theorem 1.2.12 (d) shows that for all Bender groups L with order divisible by 3 except for $\mathrm{PSL}(2,5)$, for all subgroups H of $Aut(L)$ containing $Inn(L)$ and for all involutions s of H, the group $O(C_H(s))$ is trivial. Finally Proposition 2 of [27] yields for every two commuting involutions $a, b \in C$, that $O(C_C(a)) \cap C_C(b) \leq O(C_C(b))$ holds. This is, that C is balanced.

(b) Obviously we have that $O(\bar{C}) = 1$. By Lemma 4.2.1 (d) it follows that $O_2(\bar{C}) = F(\bar{C})$ contains exactly one involution. Thus we may apply Proposition 1.2.3 to $\bar{C}$. This yields the assertion. □

4.2.6 Theorem

Let $2 \in \sigma$ and let A be an elementary abelian 2-subgroup of M of order at least 8.

Then $\langle [O_{\sigma'}(C_G(a)), x] \mid a \in A^\# \rangle$ has odd order.

Additionally suppose that $\Omega_1(Z(T))^\# \not\subseteq I^*(M)$ for all x-invariant Sylow 2-subgroup T of G and set $\theta(a) := O(C_G(a))$ and $\rho(a) := O_{\{2,3\}'}(C_G(a))$ for all $a \in A^\#$.

Then θ or ρ is a solubly complete A-signalizer functor in G.

Proof

If $\Omega_1(Z(T))^\# \subseteq I^*(M)$ for a x-invariant Sylow 2-subgroup T of G, then the assertion follows by Lemma 4.1.3.

Suppose that $\Omega_1(Z(T))^\# \not\subseteq I^*(M)$ for all x-invariant Sylow 2-subgroups T of G.
We set $\theta(a) := O(C_G(a))$ and $\rho(a) := O_{\{2,3\}'}(C_G(a))$ for all $a \in A^\#$.
If $\zeta \in \{\theta, \rho\}$ is a soluble A-signalizer functor in G, then we deduce for all $a \in A^\#$ from our assumption $\{2,3\} \subseteq \sigma$ that $[O_{\sigma'}(C_G(a)), x] \leq \zeta(a)$ holds. Thus the Soluble Signalizer Functor Theorem 2.1.6 implies the assertion.

By Lemma 2.1.3 and Lemma 2.1.4 all involutions of G with 3-soluble or 2-constrained centraliser are balanced in G. If all $a \in A^\#$ are balanced in G, then θ is a soluble A-signalizer functor in G.
Suppose that there exists an involution $c_1 \in A^\#$ such that c_1 is not balanced in G. Then we may apply Proposition 4.2.4 and Lemma 4.2.5 to observe that $C_G(c_1)$ is balanced or c_1 is ρ-balanced in G. Let $b \in C_G(c_1)$ be an involution. Since $O(C_G(b)) \cap C_G(c_1)$ is a normal subgroup of odd order of $C_{C_G(c_1)}(b)$, we conclude in the first case that

$$O(C_G(b)) \cap C_G(c_1) \leq O(C_{C_G(c_1)}(b)) \cap C_{C_G(c_1)}(c_1) \leq O(C_G(c_1)).$$

This is a contradiction because c_1 is not balanced in G.
Thus c_1 is ρ-balanced in G. For this reason every element $a \in A^\#$ is θ-balanced or ρ-balanced in G. Further for all $a \in A^\#$ we have $A \leq C_G(a)$, as A is abelian. Since A has order at least 8, Lemma 1.1.14 (b) and both parts of Lemma 3.3.5 imply that $O(C_G(a))$ has a normal 3-complement for all $a \in A^\#$.
Let $a \in A^\#$ be θ-balanced and $b \in A^\#$ be an involution. Then the following holds:

$$\begin{aligned}\rho(b) \cap C_G(a) = \;& O_{\{2,3\}'}(C_G(b)) \cap C_G(a)\\ =\;& O_{\{2,3\}'}(C_G(b)) \cap O(C_G(b)) \cap C_G(a)\\ \le\;& O_{\{2,3\}'}(C_G(b)) \cap O(C_G(a))\\ \le\;& O_{\{2,3\}'}(C_G(b)) \cap O^3(O(C_G(a)))\\ =\;& O_{\{2,3\}'}(C_G(b)) \cap O_{3'}(O(C_G(a)))\\ \le\;& O_{\{2,3\}'}(C_G(a)) = \rho(a)\end{aligned}$$

This means that a is ρ-balanced. Finally ρ is a soluble A-signalizer functor in G. □

4.3 Excluding the Big Rank

In this section we show that G is S_4-free in the case of $r_2(G) \ge 3$. Moreover we apply the results of Section 2.3 to force the rank of G to be 2.

4.3.1 Lemma

Suppose that there is a 2-subgroup T_1 of M of rank at least 2, which is normalised by an element $y \in D^*(M)$. Then we have $r_2(G) \le 2$.

Proof

Suppose for a contradiction that the lemma is false. Then we have $r_2(G) \ge 3$ and there is a subgroup T_1 of M such that $N_G(T) \cap D^*(M) \ne \varnothing$ and $r(T_1) \ge 2$. We choose $T_1 \le M$ of minimal order with these properties. Then we observe that $T_1 = \Omega_1(T_1)$. Moreover T_1 is $\langle x, y\rangle$-invariant and hence we deduce from Lemma 3.3.2 (c) that $2 \in \sigma$.
Let T be a Sylow 2-subgroup of G in M such that $C_T(T_1) \in \mathrm{Syl}_2(C_G(T_1))$ and $T_1 \le T$. Then $C_G(T_1)$ has a normal 3-complement by Corollary 3.3.4 (a) and Lemma 3.3.5 (b), since T_1 has rank at least 2. Consequently Lemma 1.1.14 (b) provides a y-invariant Sylow 2-subgroup of $C_G(T_1)$. As we have $y^M \subseteq D^*(M)$, we may choose y such that $C_T(T_1)$ is normalised by y.

We set $S := C_T(T_1) \cdot T_1$ and for all $t \in T$ let $\gamma(t) := [O_{\sigma'}(C_G(t)), x]$.
Furthermore for all subgroups U of T we set $W_U := \langle \gamma(u) \mid u \in A^\#,\ u^2 = 1\rangle$.

(1) For $U \le T$ with $r(U) \ge 2$ and $|W_U|$ odd we have $D^*(M) \cap N_G(W_U) = \varnothing$.

Proof. Let $U \le T$ with $r(U) \ge 2$ and such that W_U has odd order. Then we obtain from Proposition 3.3.3 an involution $v \in U \setminus I^*(M)$ and we observe from Lemma 1.3.7 that $C_G(v) = C_M(v) \cdot \gamma(v)$. It follows that $\gamma(v) \nleq M$. Thus W_U is not a subgroup of M. Since U is centralised by x, the group W_U is normalised by x. This implies that $x \in N_G(W_U) \nleq M$. Moreover Lemma 3.2.2 (a) forces $N_G(W_U)$ to be a proper subgroup of G. Consequently Part (d) of the same lemma yields that $N_G(W_U)$ has a cyclic Sylow 3-subgroup containing x and a normal 3-complement. Finally (1) follows from $x, x^{-1} \notin D^*(M)$. ■

(2) We have $r(S) = r(Z(S)) = 2$, $\Omega_1(S) = \Omega_1(Z(S)) = T_1$ and $r(Z(T)) = 1$.

Proof. Suppose for a contradiction that $Z(S)$ is cyclic. Then, as $Z(S)$ is y-invariant, we obtain from Lemma 1.1.3 that $Z(S)$ is centralised by y. Thus Lemma 3.2.2 (f) implies that the involution of $Z(S)$ is contained in $I^*(M)$. Moreover from Lemma 4.1.3 and from $Z(T) \le C_T(T_1) \cap C_T(S) \le S \cap C_T(S) = Z(S)$ we obtain that W_T has odd order. For this reason also $W_S \le W_T$ has odd order. Since y normalises S, we have $y \in N_G(W_S)$. This contradicts (1) and so $r(S) \ge r(Z(S)) \ge 2$ holds.
Suppose now for a contradiction that $r(S) \ge 3$. Then there is an elementary abelian subgroup B of order at least 8 of S. We choose B such that $\Omega_1(Z(S)) \le B$. Then we observe that

$W_{\Omega_1(Z(S))} \leq W_B$. By Theorem 4.2.6 the group W_B has odd order. Consequently $W_{\Omega_1(Z(S))}$ has odd order. From $y \in N_G(S) \leq N_G(\Omega_1(Z(S)))$ we deduce that y normalises $W_{\Omega_1(Z(S))}$. This contradicts (1). Therefore $2 \leq r(Z(S)) \leq r(S) \leq 2$ is true.
We also conclude that $\Omega_1(S) = \Omega_1(Z(S))$ is elementary abelian of order 4. In particular we observe that $T_1 = \Omega_1(T_1) \leq \Omega_1(S)$. The assumption $r(T_1) \geq 2$ implies that $\Omega_1(S) = T_1$. From $3 \leq r_2(G) = r(T)$ and $S = C_T(T_1) \cdot T_1$ it follows that $\Omega_1(Z(T)) < \Omega_1(Z(S)) = T_1$. In particular $Z(T)$ is cyclic. ■

Let c be the involution of $Z(T)$ and let $a \in T$ be an involution such that $T_1 = \Omega_1(S) = \langle a, c\rangle$. As y normalises T_1, we may choose a such that $c^y = a$.

(3) The group T has no elementary abelian normal subgroup of order at least 8.

Proof. Suppose for a contradiction that there is a elementary abelian normal subgroup A of T of order at least 8. Then we have $C_A(a) = C_A(\langle a, c\rangle) = C_A(T_1) \leq S$. Since A centralises c and is elementary abelian, it follows that $C_A(a) = C_A(T_1) \leq \Omega_1(C_T(T_1)) \leq \Omega_1(S) = T_1$ by (2). The result of (2) that $r(S) = r(C_T(T_1)) = 2$ implies that $T_1 \not\leq A$. Thus we have $C_A(a) < T_1$ and it follows that $|C_A(a)| \leq 2$. Applying Lemma 1.1.17 we conclude that $4 = 2^2 \geq |C_A(a)|^2 \geq |A| \geq 8$, which is a contradiction. ■

From (3) and $r(T) \geq 3$ and Lemma 1.4 of [8] it follows that T has an elementary abelian normal subgroup N of order 4. Since $Z(T)$ is cyclic with $c \in Z(T)$, we conclude that $T \neq C_T(N)$. This shows that $|T : C_T(N)| = 2$ and $c \in N$ hold.

(4) The involution c is no square in $C_T(a)$.

Proof. Suppose for a contradiction that there is an element $t \in C_T(a)$ such that $t^2 = c$.
Let $T_0 \in \mathrm{Syl}_2(C_M(a))$ be such that $t \in T_0$. Then we deduce from $c^y = a$ and $c \in Z(T)$ that $T_0 \in \mathrm{Syl}_2(M) \subseteq \mathrm{Syl}_2(G)$ and $\langle a\rangle = \Omega_1(Z(T_0))$. By Sylow's Theorem there exists an element $g \in M$ such that $T_0^g = T$. In particular we observe that $\langle a\rangle^g = \Omega_1(Z(T_0))^g = \Omega_1(Z(T)) = \langle c\rangle$ and hence $a^g = c$. Additionally we obtain $T_1^g = \langle a, c\rangle^g = \langle a, t^2\rangle^g \leq T_0^g$ and T_1^g is normalised by $y^g \in D^*(M)$. Moreover we have that $t^g \in C_{T_0^g}(T_1^g) = C_T(T_1^g)$ and $(t^g)^2 = (t^2)^g = c^g \in T_1^g$. Consequently c^g and $c = a^g$ are both squares in T. From $|T : C_T(N)| = 2$ we conclude that a^g and c^g are elements of $C_T(N)$. This implies $T_1^g = \langle a^g, c^g\rangle \leq C_T(N)$. From $a^g = c$ we obtain that $N \cdot T_1^g = N \cdot \langle c^g\rangle$ is elementary abelian. Furthermore Sylow's Theorem provides an element $h \in C_G(T_1^g)$ with $(\Omega_1(C_T(T_1^g)))^h \leq (C_T(T_1))^g \leq S^g$, since $C_T(T_1) \in \mathrm{Syl}_2(C_G(T_1))$. Altogether we get that

$$N \cdot T_1^g \leq \Omega_1(C_T(T_1^g)) \leq \Omega_1(S)^{gh^{-1}} = T_1^{gh^{-1}} = T_1^g$$

From this we deduce that $N = T_1^g = \Omega_1(C_T(T_1^g))$. By assumption there is an elementary abelian subgroup A of T of order at least 8. We observe that A normalises the group N and so $|A : A \cap C_T(N)| = |(A \cdot C_T(N)) : C_T(N)| \leq |T : C_T(N)| = 2$. It follows that $|A \cap C_T(N)| \geq 4$. Moreover we have that $A \cap C_T(N) = A \cap C_T(T_1^g) \leq \Omega_1(C_T(T_1^g)) = N$. From $|N| = 4$ we conclude that $A \geq A \cap C_T(N) = N$. Since A is elementary abelian we finally observe the contradiction $A \leq \Omega_1(C_T(N)) = \Omega_1(C_T(T_1^g)) = N$. ■

(5) We have that $T_1 = S$.

Proof. Assume for a contradiction that $T_1 < S$. Then at least one of the involutions in T_1 is a square in $S \leq C_T(a)$. From (4) we see that a or ac is a square in S. Since we have $|T : C_T(N)| = 2$ and $c \in \Omega_1(Z) \leq N$, we conclude that both, a and ca, are elements of $C_T(N)$. In particular $T_1 \leq C_T(N)$. This implies that $N \cdot T_1 = N \cdot \langle a\rangle$ is elementary abelian. We deuce that $N \leq \Omega_1(C_T(T_1)) = T_1$ and hence that $T_1 = N$. The choice of S and (2) imply that $N = T_1 = \Omega_1(C_T(T_1)) = \Omega_1(C_T(N))$. Let A be an elementary abelian subgroup of T order at least 8. Again we see that $|A \cap C_T(N)| \geq 4$. From $A \cap C_T(N) \leq \Omega_1(C_T(N)) = N$

it follows that $N \leq A$. As A is elementary abelian, we conclude that $A \leq \Omega_1(C_T(N)) = N$. This is a contradiction. ■

Thus we have $T_1 = S$ and deduce that $V_4 \cong T_1 = S = C_T(T_1) = C_T(\langle a, c\rangle) = C_T(a)$ from $c \in Z(T)$. Hence we may apply Lemma 1.3.4 (a) to T and conclude that T is dihedral or semidihedral. This is a final contradiction to $r(T) \geq 3$. □

4.3.2 Lemma

If we have $r_2(G) \neq 2$, then there is a minimal strongly closed elementary abelian subgroup A of G of order at least 8 such that $\Omega_1(Z(T)) \leq A$ for a suitable Sylow 2-subgroup T of G with $A \leq T$. Moreover G is S_4-free.

Proof

Suppose that $r_2(G) \neq 2$. By Lemma 3.2.1 (e) the group G' is simple and contains a Sylow 2-subgroup of G. Since non-abelian simple groups have no Sylow 2-subgroups of rank at most 1 by the Z^*-Theorem 1.1.13, we conclude that $r_2(G) \geq r_2(G') > 1$. Hence we obtain $r_2(G) = r_2(G') \geq 3$. Let $T \in \mathrm{Syl}_2(G)$ and if possible choose T such that T is x-invariant.

(1) The group G is S_4-free.

Proof. Suppose for a contradiction that G is not S_4-free. Then Theorem 3.3.7 yields $2 \in \sigma$. Thus we have $T \in \mathrm{Syl}_2(M)$ by Lemma 3.3.1. For all subgroups S of T we observe that $x \in C_G(S) \leq N_G(S)$. Moreover by Lemma 1.1.7 there is a non-trivial 2-subgroup S of G such that $N_G(S)$ is not S_4-free. We choose S of maximal order such that $S \leq T$ and $N_G(S)$ has a section isomorphic to S_4. In particular $N_G(S)$ has a section isomorphic to S_3.
Suppose for a contradiction that $N_G(S)/(S \cdot C_G(S))$ is not S_4-free. Let A/B be a section of $N_G(S)$ isomorphic to S_4 with $S \cdot C_G(S) \leq B$. Then a Sylow 2-subgroup S_1 of the full pre-image of $O_2(A/B)$ contains S properly. Moreover $N_G(S_1)$ is not S_4-free. This contradiction shows that $N_G(S)/(S \cdot C_G(S))$ is S_4-free.
Thus Lemma 1.3.3 provides an element y of order 3 of $N_G(S)$ that is inverted by a 2-element of $N_G(S)$ and acts non-trivially on S. This yields $y \in D^*(M) \cap N_G(S)$. Now Lemma 4.3.1 forces $r(S)$ to be 1. This shows that $S \neq T$, since we already observed $r_2(G) \geq 3$. As S admits an automorphism of order 3 which is induced by y, we conclude that $S \cong Q_8$ by Lemma 1.1.2 and Lemma 1.1.3. For this reason we observe that $N_G(S)/C_G(S) \cong S_4$. Let T_0 be a Sylow 2-subgroup of $C_G(S)$. Then a Frattini argument shows that

$$N_G(S) = N_{N_G(S)}(T_0) \cdot C_G(S) = (N_G(S) \cap N_G(T_0)) \cdot C_G(S).$$

It follows that $N_G(S \cdot T_0) \geq N_G(S) \cap N_G(T_0)$ has a section isomorphic to $N_G(S)/C_G(S) \cong S_4$. The maximal choice of S leads to $T_0 \cdot S = S$. Now we apply Lemma 1.3.4 (b) to conclude that T is either semidihedral or a quaternion group. This again contradicts $r_2(G) \geq 3$. ■

(2) There is a strongly closed elementary abelian subgroup A of G of order at least 4 such that $A \leq T$ and $\Omega_1(Z(T)) \leq A$.

Proof. The Z^*-Theorem 1.1.13 implies that a strongly closed abelian subgroup of G is not cyclic, since G is almost simple with $|G : G'| \in \{1, 3\}$ by Lemma 3.2.1 (e).
If for all 2-constrained sections of G all non-abelian composition-factors are $3'$-groups, then Theorem 2.2.4 provides a strongly closed elementary abelian subgroup A of G such that $\Omega_1(Z(T)) \leq A \leq T$.
Suppose for a contradiction that there is a 2-constrained section H^* of G such that H^* has a non-abelian composition factor with order divisible by 3. Let T_1 be a Sylow 2-subgroup of the full pre-image of $O_2(H^*)$ in G. Then H^* is isomorphic to a subgroup of the automorphism group of a factor group of T_1. The automorphism groups of cyclic 2-groups, dihedral

groups, and quaternion groups are soluble by Lemma 1.1.3 and every section of a 2-group of rank 1 is of one of theses types. Consequently we see that $r(T_1) \geq 2$. Thus we may apply Lemma 4.3.1 to obtain that $N_G(T_1) \cap D^*(M) = \emptyset$.
Moreover $N_G(T_1)$ does not have a normal 3-complement. Suppose for a contradiction that the Sylow 3-subgroups of $N_G(T_1)$ are cyclic. Then they are inverted in $N_G(T_1)$ by Burnside's p-Complement Theorem. This implies that $N_G(T_1)$ is not S_3-free. Since G is S_4-free by (1), also $N_{N_G(T_1)/\phi(T_1)}(T_1/\phi(T_1))$ is S_4-free but involves a section isomorphic to S_3. Furthermore $T_1/\phi(T_1)$ is abelian by Lemma 1.1.4. Consequently Lemma 1.3.3 provides an element of order 3 acting trivially on $T_1/\phi(T_1)$. From Lemma 1.1.14 (a) and Lemma 1.1.4 we deduce that the element of order 3 centralises T_1. From the fact that x is 3-locally central and Sylow's Theorem we obtain an element $g \in G$ such that $x \in C_G(T_1)^g \leq N_G(T_1)^g$. The Sylow 3-subgroups of $N_G(T_1)^g$ are cyclic. Therefore Lemma 3.3.5 (b) implies that $N_G(T_1)^g$ has a normal 3-complement. This is a contradiction, since
$N_G(T_1) \cong N_G(T_1)^g$. Hence there is an elementary abelian subgroup U of order 9 of $N_G(T_1)$. In particular we have $D^*(M) \cap N_G(T_1) \neq \emptyset$. This is a contradiction. ∎

(3) The group G has no strongly closed elementary abelian subgroup of order 4.

Proof. Suppose for a contradiction that there is a strongly closed elementary abelian subgroup A of order 4 of G. Then $N_G(A)/C_G(A)$ has a cyclic subgroup of order 3 by the Z^*-Theorem 1.1.13. It follows from (1) that $N_G(A)/C_G(A) \cong Z_3$. Now Corollary 3.3.4 (c) implies that $A \cap I^*(M) = \emptyset$. As A is normal in T by Lemma 2.2.2 (a) and $N_G(A)/C_G(A)$ is a $2'$-group, we observe that that $A \leq Z(T)$. From $r_2(G) \geq 3$ we obtain an involution $b \in T$ such that $A \cdot \langle b \rangle$ is elementary abelian of order 8.
For all $a \in (A \cdot \langle b \rangle)^\#$ we set $\theta(a) := O(C_G(a))$ and $\rho(a) := O_{\{2,3\}'}(C_G(a))$.
Let $a \in A^\#$. Since A is of order 4, the group A does not contain $\Omega_1(S)$ for a Sylow 2-subgroup S of a Suzuki group by Theorem 1.2.8 (d). Therefore Lemma 2.2.5 implies that

$$C_G(a) = N_{C_G(a)}(A) \cdot O(\langle A^{C_G(a)} \rangle) \subseteq N_G(A) \cdot O(C_G(a)) = N_G(A) \cdot \theta(a). \ (*)$$

Since we have $A \cap I^*(M) = \emptyset$, either $C_G(a)$ is a $3'$-group or Lemma 3.2.2 (f) yields that $C_G(a)$ has a normal 3-complement. In both cases the group $\theta(a)$ has a normal 3-complement and possibly trivial cyclic Sylow 3-subgroups. So there is a 3-element $x_a \in M$ with either $x \in \langle x_a \rangle$ or $x_a = 1$ such that we have $\theta(a) = \langle x_a \rangle \cdot \rho(a)$. By Lemma 1.1.14 (b) we may choose x_a such that $\langle x_a \rangle$ is A-invariant. As $\Omega_1(\langle x_a \rangle) \leq \langle x \rangle$ and x is 3-locally central and of order 3, we see that A centralises $\langle x_a \rangle$. We conclude that

$$C_G(a) \subseteq N_G(A) \cdot \theta(a) = N_G(A) \cdot \langle x_a \rangle \cdot \rho(a) = N_G(A) \cdot \rho(a). \ (**)$$

We set $V_A := \langle \theta(a) \mid a \in A^\# \rangle$ and $W_A := \langle \rho(a) \mid a \in A^\# \rangle$. Then both groups, V_A and W_A, are normalised by the normaliser $N_G(A)$ and we have for all elements $a \in A^\#$ that $C_G(a) \leq \langle N_G(A), V_A \rangle = N_G(A) \cdot V_A \leq N_G(V_A)$ and $C_G(a) \leq \langle N_G(A), W_A \rangle = N_G(A) \cdot W_A \leq N_G(W_A)$. Consequently we obtain $\langle C_G(a) \mid a \in A^\# \rangle \leq N_G(V_A)$ and $\langle C_G(a) \mid a \in A^\# \rangle \leq N_G(W_A)$.
Furthermore G' is simple and by Corollary 3.2.3 not a Bender group and we have $G = G'\langle x \rangle$. We apply Lemma 2.2.6 to observe that that

$$G' = \langle C_{G'}(a) \mid a \in A^\# \rangle \leq N_{G'}(W_A) \cap N_{G'}(V_A). \ (***)$$

If we have $2 \notin \sigma$, then the centraliser of an involution of G is not contained in M by Lemma 3.3.2 (b). Hence it is a $3'$-group or it is by Lemma 3.2.2 (f) a 3-nilpotent group. In both cases the centraliser is 3-soluble. Applying Lemma 2.1.4 we conclude that G is balanced in this case. Thus θ is a solubly complete $(A \cdot \langle b \rangle)$-signalizer functor of G.
In the other case, if $2 \in \sigma$, we deduce from $A \leq \Omega_1(Z(T)) \nsubseteq I^*(M)$ and Theorem 4.2.6 that θ or ρ is a solubly complete $(A \cdot \langle b \rangle)$-signalizer functor of G.
We further observe that $V_A \leq \langle \theta(a) \mid a \in (A \cdot \langle b \rangle)^\# \rangle$ and $W_A \leq \langle \rho(a) \mid a \in (A \cdot \langle b \rangle)^\# \rangle$. Finally the Soluble Signalizer Functor Theorem 2.1.6 yields that one of the groups W_A or V_A has

odd order. Moreover (***) implies that it is normalised by G. The Odd Order Theorem consequently forces one of the groups W_A or V_A to be trivial. But then (*) or (**) yields that $C_{G'}(a) \le N_{G'}(A)$ for all $a \in A^{\#}$. This implies that $G' = \langle C_{G'}(a) \mid a \in A^{\#}\rangle \le N_{G'}(A)$ but this is a contradiction, as G' is simple and $A \neq 1$. ■ □

4.3.3 Theorem

The group G has 2-rank equal to 2.

Proof

Suppose for a contradiction that $r(G) \neq 3$.
Then we obtain from Lemma 4.3.2 that G is S_4-free and that there is a strongly closed subgroup A of G. We choose A of minimal order. The same lemma forces A to have order at least 8.
Moreover from Lemma 2.2.6 and Corollary 3.2.3 we conclude that $G' \le \langle C_G(a) \mid a \in A^{\#}\rangle$.

(1) We have $A \cap I^*(M) = \varnothing$.

Proof. Suppose for a contradiction that there is an involution $c \in A \cap I^*(M)$.
By Lemma 2.2.2 (k) the group $\langle c^{N_G(A)}\rangle$ is strongly closed in G. Therefore the minimal choice of A implies that $A = \langle c^{N_G(A)}\rangle$. Thus there is a minimal set of generators $D \subseteq c^{N_G(A)}$ of A. By Lemma 3.2.2 (g) in we see that $D \subseteq I^*(M)$. As A is a 2-group Lemma 1.3.2 implies that there is a unique maximal subgroup B_0 of A with $D \cap B_0 = \varnothing$ and that $B_0 := \langle a \cdot b \mid a, b \in D\rangle$. Suppose for a contradiction that $B_0 \cap I^*(M) = \varnothing$. Then B_0 is the unique maximal subgroup of A that contains no element of $I^*(M)$. According to this Lemma 3.2.2 (g) implies that $N_G(A) \le N_G(B_0)$. Now the minimal choice of A and Lemma 2.2.2 (j) lead to a contradiction. Thus we have that $B_0 \cap I^*(M) \neq \varnothing$. Consequently for all maximal subgroups B of A we observe from $I^*(M) \cap B \neq \varnothing$ that $C_G(B) \le M$.
In addition we obtain from $c \in I^*(M)$ and Lemma 3.3.2 (b) that $2 \in \sigma$. For all $a \in A^{\#}$ we set $\gamma(a) := [O_{\sigma'}(C_G(a_0)), x]$. Then we deduce from Theorem 4.2.6 that $W = \langle \gamma(a) \mid a \in A^{\#}\rangle$ has odd order. Moreover Lemma 1.1.14 (e) yields $W = \langle C_W(B) \mid B \; max \; A\rangle \le M$, since W is normalised by A. By Lemma 1.3.7 we have for all $a \in A^{\#}$ that $C_G(a) = C_M(a) \cdot \gamma(a) \le M$. Finally we obtain $G' \le \langle C_G(a) \mid a \in A^{\#}\rangle \le M$. This is a contradiction. ■

(2) For all $a \in A^{\#}$ we have $O(C_G(a)) = 1$ and $C_G(a) = N_{C_G(a)}(A) \cdot E(\langle A^{C_G(a)}\rangle)$ and $\langle A^{C_G(a)}\rangle = A \cdot E(\langle A^{C_G(a)}\rangle)$.

Proof. By (1) and Lemma 3.2.2 (f) the centralisers of the involutions in A have a normal 3-complement. Applying Lemma 2.1.4 we see that A is balanced in G. The Soluble Signalizer Functor Theorem 2.1.6 yields that $U := \langle O(C_G(a)) \mid a \in A^{\#}\rangle$ has odd order. Furthermore if B is a non-cyclic subgroup of A, then for all $a \in A^{\#}$ Lemma 1.1.14 (e) leads to

$$O(C_G(a)) = \langle O(C_G(a)) \cap C_G(b) \mid b \in B^{\#}\rangle \le \langle O(C_G(b)) \mid b \in B^{\#}\rangle.$$

Thus we have $U \le \langle O(C_G(b)) \mid b \in B^{\#}\rangle \le U$. In particular U is normalised by $N_G(B)$. Moreover for all $a \in A^{\#}$ the group $O(\langle A^{C_G(a)}\rangle)$ is contained in $O(C_G(a)) \le U$. From Lemma 2.3.2 we deduce that $G' \le \langle C_G(a) \mid a \in A^{\#}\rangle \le N_G(U)$. The fact that G' is simple forces $G' \cap U$ to be trivial. Since $\langle x\rangle$ is not normalised by G', we conclude $U = 1$ and hence $O(C_G(a)) = 1$ for all $a \in A^{\#}$.
Further we deduce from $A \le C_G(a)$ that A is strongly closed in $C_G(a)$ by Lemma 2.2.2 (d). Finally Proposition 2.2.5 yields (2). ■

(3) We have $2 \notin \sigma$ but $2 \in \pi(M)$ and there exists an $a \in A^{\#}$ such that $\langle A^{C_G(a)}\rangle$ is not soluble and every maximal subgroup of G' containing $C_G(a)$ is not 3-soluble.

Proof. If 2 was an element of σ, then Lemma 1.3.7 (b) and (2) would imply that $C_G(a) \leq M$ for all $a \in A^{\#}$, contradicting (1).
Suppose for a contradiction that for all $a \in A^{\#}$ the group $\langle A^{C_G(a)}\rangle$ is soluble or that $C_G(a)$ is contained in a 3-soluble maximal subgroup of G'. Then Theorem 2.3.5 forces G' to be a Bender group. This contradicts Corollary 3.2.3.
Finally Proposition 2.3.7 provides an involution $t \in G$ such that $C_G(t)$ is not a $3'$-group. Since x is 3-locally central, we conclude that $2 \in \pi(M)$. ■

Let $a \in A^{\#}$ be such that $\langle A^{C_G(a)}\rangle$ is not soluble and $C_G(a)$ is not contained in a 3-soluble maximal subgroup. Let further $H \max G$ contain $C_G(a)$.

(4) We have $O(H) = 1$ and $C_H(t)$ is a $3'$-group for every involution $t \in H$.

Proof. From (2) we obtain that $O(H) \cap C_G(a) \leq O(C_G(a)) = 1$. Thus a acts fixed-point-freely on $O(H)$. This shows that $O(H)$ is abelian.
Let $b \in A^{\#}$. Then (1) implies

$$\begin{aligned} O(H) \cap C_G(b) &= [O(H) \cap C_G(b), a] \leq O(H) \cap \langle A^{C_{O(H)}(b)}\rangle \\ &\leq O(H) \cap A \cdot E(\langle A^{C_G(b)}\rangle) = O(H) \cap E(\langle A^{C_G(b)}\rangle). \end{aligned}$$

From Proposition 2.2.5 we know that $E(\langle A^{C_G(b)}\rangle)$ is a central product of Suzuki groups and that there is a Sylow 2-subgroup T_1 of $E(\langle A^{C_G(b)}\rangle)$ such that the following hold:

$$\Omega_1(T_1) = E(\langle A^{C_G(b)}\rangle) \cap A \leq E(\langle A^{C_G(b)}\rangle) \cap H.$$

Part (i) of Theorem 1.2.8 yields that $O(E(\langle A^{C_G(b)}\rangle) \cap H) = 1$. Altogether it follows that $O(H) \cap C_G(b) \leq O(H) \cap E(\langle A^{C_G(b)}\rangle) \leq O(E(\langle A^{C_G(b)}\rangle) \cap H) = 1$. Finally Lemma 1.1.14 (e) leads to $O(H) = \langle O(H) \cap C_H(b) \mid b \in A^{\#}\rangle = 1$.
Since H is non-soluble and $a \notin I^*(M)$, Lemma 3.2.2 (f) shows that there is no conjugate of x in H. Now Lemma 3.3.2 (d) forces the orders of the centralisers in H of involutions of H to be coprime to 3, because of $2 \notin \sigma$. ■

(5) The group $E(H)$ is a simple Bender group and $E(H) \cap \langle A^{C_G(a)}\rangle$ is not soluble.

Proof. Since G is S_4-free, also H is S_4-free. Moreover H is not 3-soluble and hence it has no normal 3-complement. Applying Lemma 1.3.8 we observe that the group $F^*(H) = E(H)$ is simple and $C_H(a)/C_{E(H)}(a)$ is soluble. In particular $Z^*(H) = Z^*(E(H))$ is trivial.
Furthermore we have $C_G(a) = C_H(a)$, as $C_G(a) \leq H$, and $\langle A^{C_G(a)}\rangle$ is non-soluble. Hence we conclude that $E(H) \cap \langle A^{C_H(a)}\rangle$ is non-soluble. From $A \leq C_G(a) \leq H$ and Lemma 2.2.2 (d) we obtain that A is strongly closed in H. Thus we may apply Lemma 2.3.6. The lemma provides a minimal strongly closed subgroup B of $E(H)$ such that $B \leq A$ and B has order at least 8. By Proposition 2.3.7 the group $\langle B^{C_H(b)}\rangle$ is soluble for all $b \in B^{\#}$. These are exactly the conditions of Theorem 2.3.5. Hence the theorem forces $E(H)$ to be a Bender group. ■

Finally $E(H) \cap \langle A^{C_H(a)}\rangle$ is a non soluble $3'$-group. By Theorem 1.2.8 the group $E(H)$ involves a Suzuki group. Now Theorem 1.2.12 (c) forces $E(H)$ to be a Suzuki group. Since Suzuki groups have order prime to 3, the group H is 3-soluble by (5). This contradicts the choice of a. □

5 The Small Rank Case

5.1 The Structure of a Sylow 2-Group

In this section we show that a Sylow 2-subgroup of G is either dihedral or isomorphic to a Sylow 2-subgroup of the unitary group $U_3(4)$.
In order to do this we use arguments about the control of Fusion.

5.1.1 Proposition

Let T be a Sylow 2-subgroup of G. Then the following hold:

(a) The group T is dihedral, semidihedral, a wreathed product of a cyclic 2-group of order at least 4 with a cyclic group of order 2, homocyclic abelian or isomorphic to a Sylow 2-subgroup of $U_3(4)$.

(b) The group G has only one class of involutions.

(c) If A is an elementary abelian subgroup of order 4 of T, then we have $N_G(A)/C_G(A) \cong Z_3$ if $A = \Omega_1(T)$, and $N_G(A)/C_G(A) \cong S_3$ otherwise.

Proof
This is Proposition 2.1 of chapter 3 in [26].
The proof of this theorem uses the Z^*-Theorem 1.1.13, Lemma 2.2 and Lemma 2.3 of [26] and Theorem 16.1 of [23] which is a theorem about the control of Fusion by Alperin and Goldschmidt, its proof can be found in X 4.8 and X 4.12 of [9].
In particular the proof of the proposition uses local arguments and no $\mathcal{K}$-hypothesis. □

5.1.2 Lemma

The set $I^*(M)$ is empty and for every $y \in D^*(M)$ the group $C_G(y)$ has odd order.
Moreover if Q is a 2-subgroup of G isomorphic to Q_8 such that $N_G(Q)/(Q \cdot C_G(Q))$ is of even order, then $N_G(Q)/(Q \cdot C_G(Q))$ is cyclic of order 2.

Proof
By Proposition 5.1.1 (b) the group G has exactly one class of involutions. If $I^*(M) \neq \varnothing$ then Lemma 3.3.2 (b) implies that $2 \in \sigma$. Therefore Lemma 3.2.2 (g) yields that either $I^*(M) = \varnothing$ or $I^*(M) = \{a \in M^\# \mid a^2 = 1\}$. In the second case we have for every involution $a \in M \cap G'$ that $C_{G'}(a) \leq G' \cap C_G(a) \leq G' \cap M$. It follows that $M \cap G'$ is a strongly embedded subgroup of G'. Hence Theorem 1.2.12 forces G' to be a Bender group. This contradicts Corollary 3.2.3.
Suppose for a contradiction that there is an element $y \in D^*(M)$ such that $C_G(y)$ is of even order. Then y centralises an involution $a \in G$. So we have $a \in C_G(y) \leq C_G(x)$, because x is 3-locally central and $y \in D^*(M)$. Thus $C_G(a)$ contains $\langle x, y\rangle$ and so $C_G(a)$ has non-cyclic Sylow 3-subgroups. By Lemma 3.2.2 (f) we conclude that $a \in I^*(M)$. This is a contradiction.
Moreover let Q be a 2-subgroup of G isomorphic to Q_8 such that $N_G(Q)/(Q \cdot C_G(Q))$ has even order. Then we observe that $N_G(Q)/C_G(Q) \lesssim Aut(Q_8) = S_4$ from Lemma 1.1.3 and hence we have $N_G(Q)/(Q \cdot C_G(Q)) \lesssim S_3$.

Suppose for a contradiction that $N_G(Q)/(Q \cdot C_G(Q)) \cong S_3$. Then $N_G(Q)$ has no normal 3-complement. Thus Lemma 3.3.5 (b) provides an element $y \in (D^*(M))^g \cap N_G(Q)$ for a suitable element $g \in G$. We obtain that $y \in N_G(\Omega_1(Q)) = C_G(\Omega_1(Q))$. As we have shown above $C_G(y) \cong C_G(y^{g^{-1}})$ is of odd order. This is a contradiction. □

5.1.3 Lemma

Let T be a Sylow 2-subgroup of G. Then T is not semidihedral.

Proof

Suppose for a contradiction that T is semidihedral.
Then all subgroups of T are cyclic, dihedral, semidihedral or generalised quaternion groups. If S is a subgroup of T such that $N_G(S)/C_G(S)$ is no 2-group, then Lemma 1.1.3 implies that S is elementary abelian of order 4 or $S \cong Q_8$. Moreover we observe that $N_G(S)/C_G(S)$ is of even order. Thus Lemma 5.1.2 forces S to be not isomorphic to Q_8. Altogether we conclude that S is elementary abelian of order 4.
Furthermore G/G' is of odd order by Lemma 3.2.1 (e). Consequently T is a subgroup of G'. By Theorem 4.2. of [1] the group $T \cap G' = T$ is generated by $[T, N_G(T)]$ together with all the subgroups $[H, g]$ where H ranges over all the non-identity subgroups of T such that $N_T(H) \in \mathrm{Syl}_2(N_G(H))$ and g runs over all 2-elements of $N_G(H)$.
In particular we have that $T \leq \langle [H, N_G(H)] \mid 1 \neq H \leq T \text{ and } N_T(H) \in \mathrm{Syl}_2(H)\rangle$.
Let H be a non-trivial subgroup of T such that $N_T(H) \in \mathrm{Syl}_2(N_G(H))$ and let D denote the maximal subgroup of T that is a dihedral group.
If H is elementary abelian of order 4, then we have $H \leq D$ and $[H, N_G(H)] \leq H \leq D$.
Suppose now that H is not elementary abelian of order 4. Then the investigation above implies that $N_G(H)/C_G(H)$ is a 2-group. It follows that $N_G(H) = N_T(H) \cdot C_G(H)$, because of $N_T(H) \in \mathrm{Syl}_2(N_G(H))$. We conclude that $[H, N_G(H)] = [H, N_T(H)] \leq [T, T] \leq D$.
Altogether we obtain $T \leq \langle [H, N_G(H)] \mid 1 \neq H \leq T \text{ and } N_T(H) \in \mathrm{Syl}_2(H)\rangle \leq D$. This is a contradiction. □

5.1.4 Lemma

Let T be a Sylow 2-subgroup of G. Then T is not a wreathed product of a cyclic 2-group of order at least 4 with a cyclic group of order 2.

Proof

Suppose for a contradiction that $T \cong Z_{2^n} \wr Z_2$ for a natural number $n \geq 2$. Let N denote the maximal subgroup of T that is normal and homocyclic.
If $s \in T \backslash N$, then $s^2 \in C_T(N)$ and there is an element t of order 2^n of N such that $N = \langle t \rangle \times \langle t^s \rangle$. In particular if $a \in C_N(s)$, then there are some natural numbers i and j such that we have $t^i(t^s)^j = a = (t^i(t^s)^j)^s = (t^i)^s t^j = t^j(t^s)^i$ and so $i = j$. It follows that $a \in \langle t \cdot t^s \rangle \leq C_N(s)$. Altogether $C_N(s)$ is cyclic of order n. (*)
If $Q \leq T$ is isomorphic to Q_8, then $Q \cap N$ is a maximal abelian subgroup of Q. Moreover, if we have $s \in Q \setminus N$, then we observe that $\Omega_1(Q) \leq C_N(Q) \leq C_N(s)$. Since Q is not abelian, we conclude that $C_T(Q) = C_N(Q)$ is cyclic. (**)

As in the lemma before we are interested in the non-trivial subgroups S of T such that $N_G(S)/C_G(S)$ is no 2-group.
Let S be such a subgroup of T. Then $S \cong Q_8$ or $r(S) \geq 2$ and we apply Theorem 1.3 of [12] to observe from $r(S) \leq r(T) \leq 2$ that and (**) that S is homocyclic abelian, a central product of Q_8 and a cyclic group of order 2^m for some natural number $m \geq 2$ or isomorphic to a Sylow 2-subgroup of $U_3(4)$ from Section 1 and Lemma 1 of [31] we obtain that the last case is not possible.

(1) The group S is homocyclic abelian.

Proof. Suppose for a contradiction that S is non-abelian and choose S of maximal order. Then $N_G(S)$ contains an element y that induces an automorphism of order q in S for some odd prime q.
If S is a quaternion group of order 8, then we have $q = 3$ and we set $Q := S$.
We investigate the case that S is a central product of Q_8 and a cyclic group of order 2^m for some natural number $m \geq 2$. If we have $m = 2$, then S contains exactly four subgroups of order 4. One of those is contained in the centre of S. The other three cyclic subgroups of order 4 generate a group Q isomorphic to Q_8. In particular we see that Q *char* S. If we have $m \geq 3$, then $S_0 := \{s \in S \mid o(s) \leq 4\} \cong Q_8 * Z_4$ is a characteristic subgroup of S. We conclude that S has characteristic subgroup $Q \cong Q_8$. The element $y \in N_G(S)$ normalises every characteristic subgroup of S. In particular it normalises the cyclic group $Z(S)$. As cyclic 2-groups admit no automorphism of odd order by Lemma 1.1.3, we conclude that y centralises $Z(S)$. Moreover we observe that $S = Z(S) * Q$. Thus y acts non-trivially on Q and $q = 3$.

In all cases S has a characteristic subgroup $Q \cong Q_8$ such that $y \in N_G(Q) \setminus C_G(Q)$ is a 3-element. In particular $y \in N_G(Q)$ Lemma 5.1.2 forces $N_G(Q)/(Q \cdot C_G(Q))$ to have odd order. Let a denote the involution of Q. Then we have $Q \cdot C_G(Q) \leq C_G(a)$. From Lemma 3.2.2 (f) and Lemma 5.1.2 we observe that $C_G(a)$ and hence $Q \cdot C_G(Q)$ has a normal 3-complement. This is normalised by y and hence Lemma 1.1.14 (b) provides a y-invariant Sylow 2-subgroup T_0 of $Q \cdot C_G(Q)$.
The group T_0 is not abelian and normalised but not centralised by y. The Theorem of Sylow implies together with the maximal choice of S that $S = T_0$. In addition we obtain that $S < N_T(S) \leq N_T(Q) = Q \cdot C_T(Q) \leq O_{3'}(Q \cdot C_T(Q))$. This is a contradiction. ■

Assume that $N_T(S) \in \mathrm{Syl}_2(N_G(S))$.

(2) We have $S \leq N$.

Proof. Since $C_G(S)$ is normal in $N_G(T)$, the group $C_T(S)$ is a Sylow 2-subgroup of $C_G(S)$. Suppose for a contradiction that $S \nleq N$ and let $s \in S \setminus N$. Then $C_N(s) = C_N(S)$ is cyclic of order 2^n by (*) and so $\Omega_1(S) \nleq N$. It follows that S has order 4. Moreover Proposition 5.1.1 (c) yields that $N_G(S)/C_G(S) \cong S_3$.
Consequently $N_G(S)$ has no normal 3-complement and Lemma 3.3.5 (b) provides an element $y \in (D^*(M))^g \cap N_G(S)$ for some suitable $g \in G$. In addition Corollary 3.3.4 (a) implies together with Lemma 3.3.5 (b) that $C_G(S)$ has a normal 3-complement and so there is a y-invariant Sylow 2-subgroup of $O_{3'}(C_G(T))$ by Lemma 1.1.14 (a). This is isomorphic to $C_T(S) = C_N(s) \cdot S \cong Z_{2^n} \times Z_2$. We observe that $C_T(S)$ admits no non-trivial automorphism of order 3. It follows that y centralises a Sylow 2-subgroup of $O_{3'}(C_G(T))$. This contradicts Lemma 5.1.2. ■

Like in the lemma before we want to apply Theorem 4.2. of [1].
Again we conclude that $T = T \cap G' \leq \langle [H, N_G(H)] \mid 1 \neq H \leq T \text{ and } N_T(H) \in \mathrm{Syl}_2(H) \rangle$.

Let H be a non-trivial subgroup of T such that $N_T(H) \in \mathrm{Syl}_2(N_G(H))$.
If we have $N_G(H) \neq N_T(H) \cdot C_G(H)$, then $N_G(H)/C_G(H)$ is no 2-group. Therefore we conclude that $H \leq N$ by (1) and (2). This implies that $[H, N_G(H)] \leq H \leq N$.
If $N_G(H) = N_T(H) \cdot C_G(H)$, then we have $[H, N_G(H)] = [H, N_T(H)] \leq [T, T] \leq N$.
Altogether we obtain $T \leq \langle [H, N_G(H)] \mid 1 \neq H \leq T \text{ and } N_T(H) \in \mathrm{Syl}_2(H) \rangle \leq N$. This is a contradiction. □

5.1.5 Lemma

Let T be a Sylow 2-subgroup of G. Then T is not abelian.

Proof

Suppose for a contradiction that T is abelian.
Then T is homocyclic by Proposition 5.1.1 (a). A result of Brauer, Theorem 1 of [10], forces T to have order 4.
Finally we apply Theorem 2.5.1 to get $G' \cong \mathrm{PSL}(2, q)$ for some prime power q. This contradicts Corollary 3.2.3. □

Remark

The proof of Theorem 1 of [10] is based on modular representation theory in similar complexity to Glauberman's proof of the Z^*-Theorem.

5.1.6 Corollary

A Sylow 2-subgroup of G is either dihedral of order at least 8 or isomorphic to a Sylow 2-subgroup of $U_3(4)$.

Proof

By Proposition 5.1.1 (a) the Sylow 2-subgroups of G are dihedral, semidihedral, wreathed, homocyclic abelian or isomorphic to a Sylow 2-subgroup of $U_3(4)$. By Lemma 5.1.3, Lemma 5.1.4 and Lemma 5.1.5 only the asserted possibilities are left. □

5.1.7 Lemma

Either M has odd order and there is a Sylow 2-subgroup of G that is isomorphic to a Sylow 2-subgroup of $U_3(4)$ and normalised by x or we have $2 \in \sigma$.

Proof

Let T be a Sylow 2-subgroup of G and let $A \leq T$ elementary abelian of order 4.
Suppose that we have $2 \notin \sigma$. Then Theorem 3.3.7 forces G to be S_4-free. In particular we observe that $N_G(A)/C_G(A)$ is not isomorphic to S_3. Thus Proposition 5.1.1 (c) implies that $A = \Omega_1(T)$ and so T is not dihedral of order at least 8. Now Corollary 5.1.6 yields that T is isomorphic to a Sylow 2-subgroup of $U_3(4)$ and therefore we have $\Omega_1(T) = A = Z(T)$.
Theorem 1.2.12 (g) yields that T is normalised by some 3-element. Suppose for a contradiction that $y \in N_G(A) \cap (D^*(M))^g \neq \emptyset$ for some $g \in G$. Then we have $C_G(y) \leq M^g$, since x^g is 3-locally central. By Theorem 1.2.12 (g) there is an element $u \in C_{N_G(T)}(y) \leq C_G(y) \leq M$ of order 5. It follows that $x^g, y \in C_G(u)$ and so $C_G(u)$ is a proper subgroup of G containing x^g with non-cyclic Sylow 3-subgroups. Lemma 3.2.2 (f) implies that $C_G(u) \leq M^g$. From $N_G(T) \leq N_G(A)$ and $N_G(A)/C_G(A) \cong Z_3$ we obtain that $u \in C_G(A)$. We conclude that $A \leq M^g$ is $\langle x, y\rangle$-invariant and Lemma 3.3.2 (c) leads to a contradiction.
Therefore $x^h \in N_G(T) \leq N_G(A)$ for some $h \in G$. We observe that $x^h \notin C_G(T)$ by $2 \notin \sigma$. Again Theorem 1.2.12 (g) forces x^h to act fixed-point-freely on $T^\#$. This implies together with Lemma 3.3.1 that $2 \notin \pi(M)$ and the assertion holds. □

5.2 Centralisers of Involutions

In this section we analyse the case where M contains a Sylow 2-subgroup of G. We show that the centraliser C of an involution is a maximal subgroup of G. Moreover C contains the normaliser of all non-trivial subgroups of $F(C)$ is contained.
In order to show this we intensively use the Bender method and Section 2.4.
Many of the ideas in arise from [4].

Throughout this section we assume that $2 \in \sigma$. Let T be a Sylow 2-subgroup of M and $c \in Z(T)$ be an involution. In addition let C be a maximal subgroup of G containing $C_G(c)$. For all in involutions $a \in T^{\#}$ we further set $K_a := \{g \in C \mid g^a = g^{-1} \text{ and } 2 \nmid o(g)\}$.

5.2.1 Lemma

The following hold:

(a) If T is dihedral and U is a $3'$-subgroup of G, then U has a normal 2-complement.

(b) If U is a 3-soluble subgroup of G, then U is soluble.

(c) If U is a proper subgroup of G containing $C_G(c)$, then U is soluble.

(d) If U is a proper subgroup of G' containing $C_{G'}(c)$, then we have $c \in Z^*(U)$.
In particular c is not conjugate to any involution of T in U.

Proof

(a) Let T be a dihedral group. Then all subgroups of T are dihedral or cyclic. The fact that the automorphism group of an elementary abelian group of order 4 is isomorphic to S_3, together with Lemma 1.1.3 and the p-Complement Theorem of Frobenius 1.1.11, implies that $3'$-subgroups of G have a normal 2-complement.

(b) Suppose for a contradiction that $U \leq G$ is not soluble but 3-soluble. Then Theorem 1.2.8 forces U to involve a Suzuki group. In particular T involves a Sylow 2-subgroup of a Suzuki group. Theorem 1.2.8 (d) implies that T has a section of rank at least 3. In particular T is not dihedral. By Corollary 5.1.6 we observe that T is isomorphic to a Sylow 2-subgroup of $U_3(4)$ but this contradicts Theorem 1.2.12 (f).

(c) Let $U < G$ contain $C_G(c)$. Then Lemma 5.1.2 implies that $x \in C_G(c) \leq U \nleq M$ and so U is 3-soluble by Lemma 3.2.2 (f). The assertion follows by Part (b).

(d) Let $U < G'$ contain $C_{G'}(c)$ and suppose for a contradiction that $c \notin Z^*(U)$.
Then the Z^*-Theorem provides an element $u \in U$ such that $c^u \in C_T(c) \setminus \langle c\rangle = T \setminus \langle c\rangle$.
If T is dihedral, then c^u is conjugate in $T \leq U$ to $c \cdot c^u$. Hence all involutions of $A := \langle c, c^u\rangle$ are contained in c^U.
If T is not dihedral, then Corollary 5.1.6 forces T to be isomorphic to a Sylow 2-subgroup of $U_3(4)$. Then $A := \Omega_1(T)$ is a strongly closed elementary abelian subgroup of U and by Lemma 2.2.2 (e) we may assume that $u \in N_U(A) \setminus C_U(A)$. Moreover Proposition 5.1.1 implies in this case that $N_G(A)/C_G(A)$ is cyclic of order 3. Hence we have $\langle u\rangle \cdot C_G(A) = N_G(A)$ and u acts transitively on $A^{\#}$. Altogether we have $A^{\#} \subseteq c^{\langle u\rangle} \subseteq c^U$ in this case. In both cases we deduce

$$H := \langle C_{G'}(a) \mid a \in A^{\#}\rangle \leq \langle C_{G'}(c^g) \mid g \in U\rangle = \langle C_{G'}(c)^g \mid g \in U\rangle \leq U.$$

Consequently Lemma 2.2.6 and Corollary 3.2.3 imply that A is not strongly closed in G'. It follows that $A \neq \Omega_1(T)$ and Proposition 5.1.1 (c) leads to $N_G(A)/C_G(A) \cong S_3$. This shows, that $N_G(A)$ has no normal 3-complement.
We remark that $N_G(A)$ normalises H to conclude that $N_G(H)$ has no normal 3-complement. As $x \in C_G(c) \leq H \leq N_G(H)$ and $c \notin I^*(M)$, the group $N_G(H)$ is no subgroup of M. Altogether we conclude with Lemma 3.2.2 (d) that $N_G(H)$ is not contained in any maximal subgroup of G. This implies that $N_G(H) = G$. Since x normalises H, Part (a) of the same lemma yields that $H \cdot \langle x\rangle = G$. Finally the Dedekind Identity 1.1.5 implies $G' = G \cap G' = (\langle x\rangle \cdot H) \cap G' = H \cdot (\langle x\rangle \cap G') = H$, because H is a subgroup of G'. This is a contradiction as $H \leq U < G'$. □

5.2.2 Lemma

If U is a subgroup of T, then $C_G(U)$ has a normal 3-complement and cyclic Sylow 3-subgroups.
Moreover suppose that R is a r-subgroup of G for some prime r such that $C_G(R)$ has cyclic Sylow 3-subgroups. If $N_G(R)$ has no normal 3-complement, then $N_G(R)/C_G(R)$ has no normal 3-complement.

Proof

Let $d \in U$ be an involution. Then d is a conjugate of c by Proposition 5.1.1 (b). Hence there exists an element $g \in G$ such that $C_G(U) \leq C_G(d) = C_G(c)^g$. By the assumption of this section we have $2 \in \sigma$. In particular we see that $x \in C_G(c)$. Now Lemma 5.1.2 and Lemma 3.2.2 (d) imply that $C_G(c)$ has a normal 3-complement and cyclic Sylow 3-subgroups. Thus also $C_G(U)$ has this property.
In addition we assume that R is a r-subgroup of G for some prime r such that $C_G(R)$ has cyclic Sylow 3-subgroups and $N_G(R)/C_G(R)$ possesses a normal 3-complements. Let K be the full pre-image of $O_{3'}(N_G(R)/C_G(R))$ in $N_G(R)$. If $C_G(R)$ is a $3'$-group, then K is a normal 3-complement of $N_G(R)$. Suppose that $C_G(R)$ is not a $3'$-group. Then we have $R \leq M^g$ for some element $g \in G$, because x is 3-locally central. By Sylow's Theorem we may suppose that $R \leq M$. We observe that $x \in C_G(R) \leq K$. Moreover the Sylow 3-subgroups of K are those of $C_G(R)$ and hence cyclic. Therefore Lemma 3.3.5 (b) implies that K has a normal 3-complement. As $O_{3'}(K)$ is characteristic in the normal subgroup K of $N_G(R)$ and $|N_G(R) : K|$ is a 3-number, $N_G(R)$ has a normal 3-complement. □

5.2.3 Lemma

Let $A \leq T$ be an elementary abelian subgroup of order 4.
If $N_G(A)$ contains an element of $D^*(M)$, then we have $\pi(C_G(A)) \subseteq \sigma$.

Proof

Let $y \in N_G(A) \cap D^*(M)$. Then $C_G(A)$ is $\langle x, y\rangle$-invariant, because of $A \leq T \leq M$. If we have $a \in A^\#$, then Lemma 5.1.2 yields that $C_G(a)$ is not a subgroup of M. Since x centralises every $a \in A^\#$, Lemma 3.2.2 (f) implies that $C_G(a)$ has a normal 3-complement. In particular $C_G(A)$ has a normal 3-complement. Finally Lemma 3.3.2 (e) leads to the assertion. □

We recall Definition 2.4.7 of Section 2.4.

5.2.4 Lemma

Let U be a subgroup of G such that U has a normal 3-complement and let S be a Sylow 2-subgroup of U.
Then we have $\Omega_1(S) \leq O_{2',2}(U)$ and for all primes $q \geq 5$ every involution a of U commutes q down in U.

Proof

By Sylow's Theorem we may suppose that $S \leq T$.
Suppose first that $\Omega_1(S)$ is abelian. Then $\Omega_1(S)$ is elementary abelian of order at most 4. This shows that $\Omega_1(S) = \{s \in S \mid s^2 = 1\}$ and hence $\Omega_1(S)$ strongly closed in U. As U has a normal 3-complement and quasi-simple $3'$-groups have a 2-rank at least 3 by Theorem 1.2.8 (d), Lemma 2.2.5 yields that $\Omega(S) \leq O_{2',2}(C)$.
If $\Omega_1(S)$ is non-abelian, then $\Omega_1(T) \geq \Omega_1(S)$ is also non-abelian. Thus Corollary 5.1.6 forces T to be dihedral of order at least 8. This implies that $O_{3'}(U)$ has a normal 2-complement by Lemma 5.2.1 (a). The assumption that U has a normal 3-complement implies that $\Omega_1(S) \leq O_{2',2}(U)$.
In both cases Lemma 3.6 of [38] forces every involution of S to commute q down in U. □

5.2.5 Proposition

We have $C = C_G(c) = C_C(x) \cdot O_{\sigma'}(C)$.
Moreover C has a normal 3-complement and cyclic Sylow 3-subgroups. Furthermore $F(C)$ is a $3'$-group but no 2-group and we have $r_2(F(C)) = 1$. The group $O_{\sigma'}(C)$ is not contained in M and if we additionally assume $N_G(A) \cap D^*(M) \neq \emptyset$, then $O_{\sigma'}(C)$ is abelian.

Proof

Let $A \leq T$ be an elementary abelian subgroup of order 4 and set $F := O_{\sigma'}(C)$. Then $c \in A$ because T is dihedral or isomorphic to a subgroup of $U_3(4)$ by Corollary 5.1.6.

(1) $C = C_C(x) \cdot F$ has a normal 3-complement and cyclic Sylow 3-subgroups. Further F is not contained in M and $F(C)$ is a $3'$-group but no 2-group.

Proof. From $2 \in \sigma$ and Lemma 1.3.7 we observe that $C = C_C(x) \cdot F$. Further Lemma 5.1.2 implies that $c \notin I^*(M)$. It follows that $F \nleq M$. Thus we conclude from Lemma 3.2.2 (d) that C has cyclic Sylow 3-subgroups and a normal 3-complement. If $O_3(C)$ was non-trivial, then x would be an element of $O_3(C)$ and $[x, F] \leq O_3(C) \cap F = 1$. This would be a contradiction. Finally F is a normal subgroup of C of odd order. Consequently the Odd Order Theorem 1.1.12 implies that $1 \neq F^*(F)$ is a nilpotent characteristic subgroup of F. We conclude that $F^*(F) \leq F(C)$ and that $F(C)$ is no 2-group. ∎

(2) If Q is a q-subgroup of G for a prime $q \geq 5$ such that Q is normalised by $C_G(c)$ but not centralised by c and such that every proper $C_G(c)$-invariant subgroup of Q lies in $C_G(c)$, then there is an element $a \in A \setminus \langle c \rangle$ such that $[c, C_Q(a)] \neq 1$ and $N_G([c, C_Q(a)]) \leq M$.

Proof. We want to apply Lemma 2.4.8 to A. By Lemma 3.2.1 (e) the group G is almost simple and the only non-trivial normal proper subgroup of G is $G' = E(G)$. Moreover Proposition 5.1.1 implies that 3 divides $|N_G(A)/C_G(A)|$. Hence the assumptions on A of Lemma 2.4.8 hold. Further we may choose C such that $C_G(c) \leq N_G(Q) \leq C$, since the assertion of (2) does not depend on C. Finally Lemma 5.2.4 yields the remaining conditions to apply Lemma 2.4.8, because A and x are contained in the proper subgroup $C \neq G'$ of G. The involutions of A are not conjugate in C by Lemma 5.2.1 (d). Moreover Part (b) of the same Lemma forces C to be soluble and hence $SL(2, q)$-free. Thus neither Part (a) nor Part (c) of Lemma 2.4.8 holds. Consequently Part (b) applies and there is an element $a \in A \setminus \langle c \rangle$ such that $[c, C_Q(a)] \neq 1$ and there is no $SL(2, q)$-free, maximal subgroup $H_1 \neq G'$ of G containing $N_G([c, C_Q(a)])$ such that a commutes q down in H_1. We set $Q_0 := [c, C_Q(a)]$. As Q is $C_G(c)$-invariant and $x \in C_G(c)$, the element x normalises Q and centralises c and a. Therefore $Q_0^x = [c^x, C_{Q^x}(a^x)] = Q_0$ implies that $x \in N_G(Q_0)$. Let H be a maximal subgroup of G such that $N_G(Q_0) \leq H_2$. Then Lemma 3.2.2 (d) yields that either H has a normal 3-complement or $H = M$. In the first case H is $SL(2, q)$-free and by Lemma 5.2.4 every involution of A commutes q down in H. This is a contradiction. Thus we have $N_G([c, C_Q(a)]) = N_G(Q_0) \leq M$. ∎

(3) Every non-trivial $C_G(c)$-invariant σ'-subgroup of C is centralised by c.

Proof. Suppose for a contradiction that there is a prime $q \in \sigma'$ and a $C_G(c)$-invariant q-subgroup of C which is not centralised by c. Let Q be a q-subgroup of C of minimal order such that Q is normalised by $C_G(c)$ but not centralised by c. Then we have $q \geq 5$, because of $\{2, 3\} \subseteq \sigma$. Additionally (2) provides an element $a \in A \setminus \langle c \rangle$ such that $[c, C_Q(a)] \neq 1$ and $N_G([c, C_Q(a)]) \leq M$. Consequently Lemma 3.3.2 (a) leads to the contradiction $q \in \sigma$. ∎

(4) If we have $N_G(A) \cap D^*(M) \neq \emptyset$, then F is abelian and inverted by an involution of A.

Proof. If $N_G(A) \cap D^*(M) \neq \emptyset$, then $\pi(C_G(A)) \subseteq \sigma$ by Lemma 5.2.3. Since (3) implies that F is centralised by c, it follows that every involution $a \in A \setminus \langle c \rangle$ acts fixed-point-freely on F. Thus F is abelian in this case. ∎

(5) We have $C = C_G(c)$.

Proof. From (3) we deduce that $c \in C_C(F) \trianglelefteq C$. By Lemma 5.2.1 (d) we moreover have that $c \in Z^*(C)$. Thus $c \in Z^*(C_C(F))$.
Suppose for a contradiction that $c \notin Z(C_C(F))$.
Then Lemma 1.1.14 (b) provides a prime $q \in \pi(O(C_C(F)))$ and a $C_C(c)$-invariant Sylow q-subgroup of $O(C_C(F))$ which is not centralised by c. Moreover (1) implies that x does not centralise F and that the Sylow 3-subgroups of C are cyclic. Altogether $q \neq 3$ and so $q \geq 5$. Let Q be a q-subgroup of $O(C_C(F))$ of minimal order such that Q is normalised by $C_C(c) = C_G(c)$ but not centralised by c. Then (2) provides an element $a \in A \setminus \langle c \rangle$ such that $Q_0 := [c, C_Q(a)] \neq 1$ and $N_G(Q_0) \leq M$. We observe that $Q_0 \leq Q$ and so $M \geq C_G(Q_0) \geq C_G(Q) \geq F$. This contradicts (1).
Thus we have $c \in Z(C_C(F))$. Moreover $\langle c \rangle$ is strongly closed in C by Lemma 5.2.1 (d). Hence we obtain $N_C(C_T(F)) \leq N_C(\langle c \rangle) = C_C(c)$ by Lemma 2.2.2 (a). Finally a Frattini argument leads to $C = C_G(F) \cdot N_C(C_T(F)) \leq C_G(c)$. ■

It remains to show that $F(C)$ has 2-rank 1.
Suppose for a contradiction that $\Omega_1(O_2(C))$ is not cyclic. Then $O_2(C)$ has an elementary abelian subgroup B of order 4 containing c. It follows that $[B, F] \leq O_2(C) \cap F = 1$.
Suppose for a contradiction that T is dihedral. Then Proposition 5.1.1 implies that $N_G(B)$ has no normal 3-complement. Since B is centralised by x, Lemma 3.2.2 (f) forces $N_G(B)$ to be a subgroup of M. We conclude that $D^*(M) \cap N_G(B) \neq \emptyset$. Thus Lemma 5.2.3 yields that $\pi(F) \subseteq \pi(C_G(B)) \subseteq \sigma$. This contradicts (1). Consequently we have that T is not dihedral.
Corollary 5.1.6 implies that $B = A = \Omega_1(O_2(C))$. From Lemma 2.5.2 (c) we observe that the involutions of A are not conjugate in C. This shows that $A \leq Z(C)$. In particular $C_G(a) = C$ for all $a \in A^\#$. Moreover $A = \Omega_1(T)$ is a strongly closed elementary abelian subgroup of G.
Finally $\langle C_G(a) \mid a \in A^\# \rangle = C$ and Lemma 2.2.6 yield that G' is a Bender group. This is a contradiction to Corollary 3.2.3. □

5.2.6 Lemma

For all involutions $a \in T \setminus \langle c \rangle$ we have $C = T \cdot C_C(a) \cdot K_a$ and K_a is a normal Hall subgroup of $F(C)$. Moreover $C_C(T)$ contains a Sylow 3-subgroup of C.
If further $y \in D^*(M) \cap N_G(\langle a, c \rangle)$, then $O_{\sigma'}(C)$ is an abelian Hall subgroup of G.

Proof

Let $a \in T \setminus \langle c \rangle$ be an involution and let F be a the full pre-image of $F(C/O_2(C))$.

(1) We have $F = F(G)$.

Proof. We first observe that $F(G) \leq F$ and $O_2(C/O_2(c)) = 1$. Thus F is a normal subgroup of C such that $O_2(C) \in \mathrm{Syl}_2(F)$. Moreover every Hall $2'$-subgroup of F is nilpotent.
By Proposition 5.2.5 the group $O_\sigma(C)$ is not centralised by x and the Sylow 3-subgroups are cyclic. Suppose for a contradiction that F is no $3'$-group. Then x is contained in F and normalises $O_{\sigma'}(C) \cap F$. It follows that x centralises $O_{\sigma'}(C) \cap F$. By Lemma 1.1.14 (d) we conclude that $[O_{\sigma'}(C), x] = [O_{\sigma'}(C), x, x] \leq [O_{\sigma'}(C) \cap F, x] = 1$. This is a contradiction.
Moreover Proposition 5.2.5 forces $O_2(C)$ to be of rank 1. In particular $O_2(C)$ contains no subgroup that admits an automorphism of $\{2, 3\}'$-order by Lemma 1.1.2 and Lemma 1.1.3.
Applying the p-Complement Theorem of Frobenius 1.1.11 we obtain that F has a normal 2-complement. This forces $O(F)$ to be a Hall $2'$-subgroup of F. Altogether we conclude that $O(F)$ is a nilpotent normal subgroup of C and hence $F(G) \leq F = O(F) \cdot O_2(F) \leq F(G)$. ■

(2) For all odd primes $q \in \pi(F(C))$ we have $O_q(C) \leq C_G(a)$ or $C_{O_q(C)}(a) = 1$.

Proof. Suppose for a contradiction that (2) is false. Then there is a prime $q \in \pi(F(C)) \setminus \{2\}$ such that $1 \neq C_{O_q(C)}(a) \neq O_q(C)$. Proposition 5.2.5 yields that $q \neq 3$.
Moreover from $x \in C \cap C_G(a)$ we deduce that $Q_0 := C_{O_q(C)}(a)$ is x-invariant.
By Proposition 5.1.1 (c) there is an element $y \in N_G(A)$ such that $a = c^y$. It follows that $Q_0 \leq C_G(a) = C_G(c^y) = C^y$. As C^y is soluble by Lemma 5.2.1 (c), Theorem 1.1.8 provides a Hall $\{2, q\}$-subgroup H_1 of C^y. If T is dihedral, then H_1 has a normal 2-complement by Lemma 5.2.1 (a). If T is not dihedral, then T is isomorphic to a Sylow 2-subgroup of $U_3(4)$. This shows that $A = \Omega_1(T) = Z(T)$. Together with Lemma 5.2.4 we conclude that $A \cdot O(H)/O(H) = \Omega_1(O_2(H/O(H)))$. The property of H_1 to be a $3'$-group implies that $A \leq Z^*(T)$ in this case.
Consequently C^y has a c-invariant Sylow q-subgroup Q containing Q_0 in both cases.
If Q was a subgroup of C, then Q would be a Sylow q-subgroup of C and therefore we would have $O_q(C) \leq Q \leq C^y = C_G(a)$ contradicting the choice of Q_0. Moreover we see that $C_Q(N_Q(Q_0)) \leq C_Q(Q_0) \leq N_Q(Q_0)$. Altogether Lemma 1.1.14 (g) yields $[c, N_Q(Q_0)] \neq 1$.
We set $H := N_G(Q_0)$. Then we observe that $C \rightsquigarrow H$. We further set $\pi := \pi(F(C))$. Then we have $|\pi| \geq 2$ by Proposition 5.2.5. Furthermore Parts (a) and (b) of Lemma 2.4.4 force $O_\pi(F(H)) \subseteq C = C_G(c)$ and $O_{\pi'}(F(H)) \cap C = 1$. In particular c inverts $O_{\pi'}(F(H))$ and $C_{F(H)}(c)$ is a Hall subgroup of $F(H)$.
Suppose for a contradiction that H is a subgroup of M. Then we have $x \in O_3(H)$. Therefore Proposition 5.2.5 leads to the contradiction $x \in O_{\pi'}(F(H)) \cap C = 1$. This shows that H is not contained in M. Hence, as $x \in H$, Lemma 3.2.2 implies that H has a normal 3-complement. Moreover Lemma 5.2.1 (b) forces H to be soluble and in particular we have $F(H) = F^*(H)$. Altogether we may apply Lemma 1.3.9 to H and c. If h is an element of H, then $[h, c]^c = (h^{-1} \cdot h^c)^c = (h^{-1})^c \cdot h = [c, h] = [h, c]^{-1}$ holds. Thus the lemma shows that $1 \neq [c, N_Q(Q_0)] \leq \langle \{g \in H \mid g^c = g^{-1} \text{ and } 2 \nmid o(g)\} \rangle \leq [F(H), c] = O_{\pi'}(F(H))$. This contradicts the choice of $q \in \pi$. ■

Let $- : C \to C/O_2(C)$ be the natural epimorphism.

(3) The group $C_{F(\bar{C})}(\bar{a})$ is a Hall subgroup of $F(\bar{C})$.

Proof. By (2) we have that $C_{O(F(C))}(a)$ is a Hall subgroup of $O(F(C))$. Moreover (1) shows that $\bar{F} = F(\bar{C}) = \overline{F(C)} = \overline{O(F(C))}$. So we conclude that $\overline{C_{O(F(C))}(a)}$ is a Hall subgroup of $F(\bar{C})$. Furthermore we observe that $\overline{C_{O(F(C))}(a)} = \overline{C_{O(F(C))}(a)} \leq C_{\overline{O(F(C))}}(\bar{a}) = C_{\bar{F}}(\bar{a})$. Let $\bar{g}$ be an element of $C_{\bar{F}}(\bar{a})$. Then $\bar{g}$ has odd order and so it has a pre-image g of odd order. Moreover there is an element $c_0 \in O_2(C)$ such that $g^a = g \cdot c_0$. Let n be the order of c_0, then $\langle g \rangle = \langle g^n \rangle$, since g has odd order and c_0 is a 2-element. In addition $g \in O(F)$ and $c_0 \in O_2(F)$. Hence the fact that F is nilpotent by (1) forces c_0 and g to commute. Finally $g^n = g^n \cdot c_0^n = (g \cdot c_0)^n = (g^a)^n = (g^n)^a$ implies that $a \in C_G(g^n) = C_G(g)$ and we conclude $\bar{g} \in \overline{C_{O(F(C))}(a)}$. Altogether $C_{\bar{F}}(\bar{a}) = \overline{C_{O(F(C))}(a)}$ is a Hall subgroup of $F(\bar{C})$. ■

(4) The set K_a is a normal subgroup of $F(C)$.

Proof. Statement (3) shows that we may apply Lemma 1.3.9 to $\bar{a}$ and $\bar{C}$. From the fact that C is soluble by Lemma 5.2.1 (b) it follows that $\{\bar{g} \in \bar{C} \mid \bar{g}^{\bar{a}} = \bar{g}^{-1} \text{ and } 2 \nmid (\bar{g})\} = \overline{K_a}$ is equal to $[F(\bar{C}), \bar{a}]$. By (1) we conclude that $\overline{K_a} = [F(\bar{C}), \bar{a}] = [\bar{F}, \bar{a}] = [\overline{O(F(C))}, \bar{a}] = \overline{[O(F(C)), a]}$. Taking pre-images we see that $K_a \cdot O_2(C) = [O(F(C)), a] \cdot O_2(C)$. Since every element of K_a has odd order, this implies $K_a = O([O(F(C)), a] \cdot O_2(C)) = [O(F(C)), a]$. Moreover (2) implies that $K_a = [O(F(C)), a]$ is a Hall subgroup of $F(C)$. Since $F(C)$ characteristic in C, the group K_a is a normal subgroup of $F(C)$. ■

(5) We have that $C = T \cdot C_C(a) \cdot K_a$ and $C_C(T)$ contains a Sylow 3-subgroup of C.

Proof. If T is dihedral, then Lemma 5.2.1 (a) implies that $O_{3'}(C) \leq T \cdot O(C)$.
If T is not dihedral, then T is by Corollary 5.1.6 isomorphic to a Sylow 2-subgroup of $U_3(4)$. From Lemma 5.2.4 we deduce that $\langle a, c\rangle = \Omega_1(T) \leq O_{2',2}(C)$. In this case it follows that $O_{3'}(C) \leq T \cdot C_C(\langle a, c\rangle) \cdot O(C) = T \cdot C_C(c) \cdot O(C)$.
Let $g \in C$ be an element of odd order. Then $[g, a]^a = a \cdot a^g \cdot a^2 = [a, g] = [g, a]^{-1}$. This shows that $[O(C), a] \leq K_a$. We apply Lemma 1.1.14 (d) to conclude that

$$O_{3'}(C) \leq T \cdot C_C(a) \cdot O(C) = T \cdot C_C(a) \cdot [O(C), a] \leq T \cdot C_C(a) \cdot K_a.$$

Proposition 5.2.5 provides an element $x_0 \in C$ such that $\langle x_0\rangle$ is a Sylow 3-subgroup of C that normalises T and centralises c. From Lemma 1.1.3 and Theorem 1.2.12 (g) we deduce that x_0 centralises T. In particular $C_C(T)$ contains a Sylow 3-subgroup of C. Moreover it follows that $C = \langle x_0\rangle \cdot O_{3'}(c) \leq T \cdot C_C(a) \cdot K_a \leq C$. ■

Suppose finally that $y \in D^*(M) \cap N_G(\langle a, c\rangle)$. Then $O_{\sigma'}(C)$ is abelian by Proposition 5.2.5 and Lemma 5.2.3 yields $\pi(C_G(\langle a, c\rangle)) \subseteq \sigma$. From the definition of K_a, (4) and (5) it follows that $O_{\sigma'}(C) \subseteq K_a \leq F(C)$ and that $|C : K_a|$ is a σ-number.
We see $C = N_G(O_q(C))$ for all $q \in \pi(O_{\sigma'}(C))$, since C is a maximal subgroup of G. This implies that every Sylow subgroup of $O_{\sigma'}(C)$ is one of G. Finally $O_{\sigma'}(C)$ is a Hall subgroup of G and a normal Hall subgroup of C. □

5.2.7 Lemma

Suppose that K is a non-trivial subgroup of $F(C)$. Then $N_G(K) \leq C$.

Proof

Suppose for a contradiction that $N := N_G(K)$ is not contained in C and set $\pi := \pi(F(C))$. Then K is a nilpotent normal subgroup of N and hence we have $K \leq F(N)$. Moreover C infects N and we have $3 \notin \pi$ but $2 \in \pi$ and $|\pi| \geq 2$ by Proposition 5.2.5.
Suppose for a contradiction that $E(N) \neq 1$. Then $E(N)$ is not 3-soluble by Lemma 5.2.1 (b). In particular N has no normal 3-complement. Let R be a Sylow 3-subgroup of $E(N)$. Then R is non-trivial and so we have $[R, K] \leq [E(N), F(N)] = 1$. This provides an element $g \in G$ such that $K \leq C_G(R) \leq M^g$, since x is 3-locally central. Therefore we have $x^g \in N$ and Lemma 3.2.2 (f) yields that $c \in N \leq M^g$, as N is not 3-nilpotent. We observe that $x^g \in C$ and $x^g \in O_3(N)$ to conclude $x^g \in C \cap O_{\pi'}(F(C))$, from $3 \in \pi'$ with Proposition 5.2.5. This contradicts Lemma 2.4.4 (b).
It follows that $E(N) = 1$ and that $F(N) = F^*(N)$. Part (a) and (b) of Lemma 2.4.4 yield that we may apply Lemma 1.3.9. For all $n \in N$ we have $[n, c]^c = c \cdot c^n \cdot c^2 = [c, n] = [n, c]^{-1}$, since c is an involution. According to Lemma 1.3.9 this implies

$$[O^2(N), c] \leq \langle [F(N), c]^{O^2(N)}\rangle \leq \langle O_{\pi'}(F(N))^{O^2(N)}\rangle = O_{\pi'}(F(N)).$$

Moreover Lemma 2.4.4 (a) shows that $O_2(N) \leq O_\pi(F(N)) \leq C$. Consequently we see that $[O_2(N), N_{O(F(C))}(K)] \leq O_2(N) \cap O(F(C)) = 1$. Since $K \leq F(C)$ and $F(C)$ is nilpotent, we have $\pi(O(F(C))) = \pi(N_{O(F(C))}(K))$.
Suppose for a contradiction that $O_2(N)$ contains an involution a different from c. Then a inverts no Sylow q-subgroup of $F(C)$ for any prime $q \in \pi(N_{O(F(C))}(K)) = \pi(O(F(C)))$. Moreover Lemma 5.2.6 forces $a \in C_G(O(F(C)))$. This implies $a \in F(C)$, since a is a 2-element of C. But this contradicts Proposition 5.2.5. Consequently $\Omega_1(O_2(N)) \leq \langle c\rangle$ is cyclic and hence $N \leq N_G(\Omega_1(O_2(N))) = C_G(\Omega_1(O_2(N)))$. We conclude that $O_2(N) = 1$, as we assumed $N \nleq C$.
Altogether we have $1 \neq [O^2(N), c] \leq O_{\pi'}(F(N))$. Let A be an elementary abelian subgroup of T. Then we have $c \in A$, because of $r(T) = 2$. Moreover Lemma 1.1.14 (e) yields $O_{\pi'}(F(N)) = \langle C_{O_{\pi'}(F(N))}(a) \mid a \in A^\#\rangle$. Hence there is an element $a \in A^\#$ such

that the group $[C_{O_{\pi'}(F(N))}(a), c] = C_{O_{\pi'}(F(N))}(a)$ is non-trivial. As $F(C)$ is a π-group and a and c are conjugate by Proposition 5.1.6, also $F(C_G(a))$ is a π-group. It follows that $[C_{O_{\pi'}(F(N))}(a), c] \nleq F(C_G(a))$. This leads to $[O(C_G(a)), A] = [O(C_G(a)), c] \nleq F(C_G(a))$. This contradicts Lemma 5.2.6 and the fact that a and c are conjugate in G. □

5.2.8 Proposition

The group T is isomorphic to a Sylow 2-subgroup of the group $U_3(4)$.

Proof

Suppose for a contradiction that the proposition is false. Then T is dihedral by Corollary 5.1.6 of order at least 8. Let S denote the maximal cyclic subgroup of T.

If we have $O_2(C) = S$, then G fulfils the assumptions of Lemma 2.2 of [4]. Thus Theorem 3 of [7] implies that $C = T \cdot K_a = \langle a \rangle \cdot K_a$ for every involution $a \in T \setminus \{c\}$. Hence we can apply [5] to conclude that G, considered as permutation group, is a Zassenhaus group of degree q for some odd natural number q and that the order of G is $(q+1) \cdot q \cdot (q-1)$. Moreover there is a subgroup Q of order q such that $N_G(Q)$ is a stabiliser of a point and has the form $Q \cdot D$, where D is an abelian group of order $\frac{q-1}{2}$. Theorem 13.1.1 (i) of [22] yields that $N_G(Q)$ is a Frobenius group with Frobenius complement D. Theorem 10.3.1 (iv) of [22] implies that D is cyclic, because D is abelian. Finally Theorem 13.3.5 of [22] forces G to be isomorphic to $\mathrm{PSL}(2, q)$. This contradicts Corollary 3.2.3.

Thus it remains to show $O_2(C) = S$. By Proposition 5.2.5 we have $r_2(O(C)) = 1$. Therefore it suffices to verify $O_2(C) \geq S$, as every normal subgroup of T of rank 1 is contained in S. There are involutions $a, b \in T$ with $\langle a \cdot b \rangle = S$. We set $A := \langle a, c \rangle$ and $B := \langle b, c \rangle$. Since T is dihedral, Proposition 5.1.1 yields that neither $N_G(A)$ nor $N_G(B)$ has a normal 3-complement. Moreover x centralises T and so x centralises A and B. Applying Lemma 3.2.2 (f) we observe that $N_G(A)$ and $N_G(B)$ are subgroups of M and hence we have

$$N_G(A) \cap D^*(M) \neq \emptyset \neq N_G(B) \cap D^*(M).$$

Suppose for a contradiction that $S > O_2(C)$. Lemma 5.2.1 (b) implies that $F(C) = F^*(C)$ and hence Lemma 1.1.18 (h) provides a prime $q \in \pi(F(C))$ such that $O_q(C) \nleq C_G(S)$. We see that $q \neq 2$, since S is cyclic and $O_2(C) \leq S$. Moreover Proposition 5.2.5 implies $O_3(C) = 1$. Altogether the prime q is at least 5. By Lemma 5.2.6 each element of a and b either inverts or centralises $O_q(C)$. Since $O_q(C)$ is not centralised by $a \cdot b$ we may assume that a inverts $O_q(C)$ and b centralises $O_q(C)$.

(1) The group $O_q(C)$ is cyclic.

Proof. Suppose for a contradiction that $O_q(C) \leq C_G(b)$ is not cyclic. Then $O_q(C)$ does not act elementwise fixed-point-freely on $O_{\sigma'}(C_G(b))$ by Lemma 1.1.14 (e). So there is an element $g \in O_q(C)$ such that $1 \neq U := O_{\sigma'}(C_G(b)) \cap C_G(g)$. We apply Lemma 5.2.7 to obtain that $U \leq O_{\sigma'}(C_G(b)) \cap N_G(\langle g \rangle) \leq O_{\sigma'}(C_G(b)) \cap C$. Since $O_{\sigma'}(C)$ is a normal Hall subgroup of C by Lemma 5.2.6, we conclude that $U \leq O_{\sigma'}(C)$. Moreover we have $N_G(A) \cap D^*(M) \neq \emptyset$ and so Proposition 5.2.5 forces $O_{\sigma'}(C)$ to be abelian.

Altogether Lemma 5.2.7 yields that $O_{\sigma'}(C) \leq N_G(U) \leq C_G(b)$. Consequently the fact that the involutions of G are conjugate implies $O_{\sigma'}(C) = O_{\sigma'}(C_G(b))$. Again it follows from Lemma 5.2.7 that $C = N_G(O_{\sigma'}(C)) = N_G(O_{\sigma'}(C_G(b))) = C_G(b)$. Hence we have $C = C_G(B)$. This contradicts Proposition 5.2.5 ■

(2) We have $C = F(C) \cdot C_C(T) \cdot T$.

Proof. Let $- : C \to C/O(F(C))$ be the natural epimorphism.

According to Lemma 5.2.6 we obtain from the fact that C has a normal 3-complement by

Proposition 5.2.5 that $C = C_C(T) \cdot O_{3'}(C) = C_C(T) \cdot T \cdot C_{O_{3'}(C)}(d) \cdot F(C)$ for all $d \in \{a, b\}$. Moreover Lemma 5.2.1 (a) yields that for every $d \in \{a, b\}$ that $C_{O_{3'}(C)}(d)$ has a normal 2-complement. From Lemma 1.1.14 (a) we further observe that for all $d \in \{a, b\}$ we have $\overline{C_{O_{3'}(C)}(d)} = C_{\overline{O_{3'}(C)}}(\bar{d})$. Altogether we obtain that $O(C_{\overline{O_{3'}(C)}}(\bar{a})) = O(\overline{C_{O_{3'}(C)}(a)}) = O(\overline{C_{O_{3'}(C)}(b)}) = O(C_{\overline{O_{3'}(C)}}(\bar{b}))$.
Therefore it follows again from Lemma 1.1.14 (a) that

$$O(\overline{C_{O_{3'}(C)}(a)}) \leq C_{\bar{C}}(\langle \bar{a}, \bar{b} \rangle) = C_{\bar{C}}(\bar{T}) = \overline{C_C(T)}.$$

This implies $\bar{C} = \overline{C_C(T)} \cdot \bar{T} \cdot \overline{C_{O_{3'}(C)}(a)} \leq \bar{T} \cdot \overline{C_C(T)} = \overline{T \cdot C_C(T)}$. ■

The group $C_G(T)$ has a normal 3-complement by Lemma 5.2.2. From $x \in C_G(T)$ we obtain a x-invariant Sylow q-subgroup Q of $C_G(T)$.
We conclude that $Q \in \mathrm{Syl}_q(C_G(A))$ and $Q \cdot O_q(C) \in \mathrm{Syl}_q(C_G(B)) \subseteq \mathrm{Syl}_q(C)$. Furthermore the group $Q_1 := Q \cdot O_q(C)$ is x-invariant.

(3) The groups $N_G(A)$ and $N_G(B)$ are soluble subgroups of M and have Sylow 3-subgroups R such that $\Omega_1(R)$ is elementary abelian of order 9.
Moreover there is an element $y \in D^*(M) \cap N_G(A)$ that is inverted by an involution $t \in T$ and such that $\langle N_T(A), y \rangle \cong S_4$ and Q is $\langle N_T(A), y \rangle$-invariant.
In addition there is an $z \in D^*(M) \cap N_G(A)$ that is inverted by an involution $s \in T$ and such that $\langle N_T(B), z \rangle \cong S_4$ and Q_1 is $\langle N_T(B), z \rangle$-invariant.

Proof. Let $D \in \{A, B\}$.
Let T_0 be a Sylow 2-subgroup of $N_G(D)$ that contains $N_T(D)$. Then $T_0 = N_T(D)$ is dihedral of order 8, because T is dihedral of order at least 8. Further Proposition 5.1.1 (c) yields that $N_G(D)/C_G(D) \cong S_3$. Being a subgroup of C, the group $C_G(D)$ has a normal 3-complement and is soluble by Proposition 5.2.5 and Lemma 5.2.1 (a). This also forces $N_G(D)$ to be soluble. Moreover D is a Sylow 2-subgroups of $C_G(D)$. Altogether $C_G(D)$ has a normal $\{2, 3\}$-complement.
By Theorem 1.1.8 there is a Hall $\{2, 3\}$-subgroup H of $N_G(D)$. Consequently group H is not S_3-free. Thus Lemma 1.3.3 and Sylow's Theorem provide an element $v \in H$ of order 3 and an element $t \in N_T(D)$ such that $\langle v, t \rangle / \langle t^2 \rangle$ is isomorphic to S_3. We obtain that t inverts v and so $v \notin \langle x \rangle$. In particular we have $v \in (D^*(M))^g \cap N_G(D)$ for a suitable element $g \in G$. Lemma 5.1.2 yields that $C_G(v)$ is of odd order, so v acts non-trivially on D. Consequently Lemma 1.3.3 shows that H has a subgroup U such that $t, v \in U$ and $U/\langle t \rangle \cong S_4$. From $|H|_2 = 8$ we conclude that $\langle N_T(D), y \rangle = U \cong S_4$.
Furthermore Lemma 5.2.2 implies that $C_G(D)$ has cyclic Sylow 3-subgroups. We conclude that a Sylow 3-subgroup R of $N_G(D)$ has a cyclic subgroup of index 3 and thus $|\Omega_1(R)| = 9$ by Theorem 1.2 of [8] or R is abelian. In the second case also $|\Omega_1(R)| = 9$ holds, as $x \in C_G(D)$ and $N_G(D) \cap D^*(M) \neq \emptyset$. This implies that $N_G(D)$ has no cyclic Sylow 3-subgroup and it follows from Lemma 3.2.2 (f) and $x \in N_G(D)$ that $N_G(D) \leq M$.
In addition H acts coprimely on $O_{\{2,3\}'}(C_G(D))$. From Lemma 1.1.14 (a) we deduce that there is a H-invariant Sylow q-subgroup Q_0 of $O_{\{2,3\}'}(C_G(A))$. Since $C_G(D)$ has a normal $\{2, 3\}$-complement, we have $Q_0 \in \mathrm{Syl}_2(C_G(D))$. In particular $Q_0 \cdot H$ is a Hall $\{2, 3, q\}$-subgroup of $N_G(D)$. By Theorem 1.1.8 every $\{2, 3, q\}$-subgroup of $N_G(D)$ is contained in a Hall $\{2, 3, q\}$-subgroup of $N_G(D)$ and all Hall $\{2, 3, q\}$-subgroups of $N_G(D)$ are conjugate in $N_G(D)$. This shows that we may assume that $\langle x \rangle \cdot N_T(D) \leq H$. Since 2 and 3 are elements of σ, we also choose $H \leq M$. In particular we observe that $v \in M$ and consequently $v \in D^*(M) \cap N_G(D) \cap N_G(Q_0)$.
For $D = A$ we may moreover choose $Q_0 = Q$, since Q is $\langle x \rangle \cdot N_T(A)$-invariant and for $D = B$ we may choose $Q_0 = Q_1$, since Q_1 is $\langle x \rangle \cdot N_T(B)$-invariant. ■

(4) The group Q is cyclic.

Proof. By (3) we observe that

$$y = (y^{-1})^2 = y^{-1} \cdot y^t = [y, t] \in [y, T] \leq [y, C_G(Q)] \leq C_G(Q).$$

This implies that $Q \leq C_G(y) \leq M$. For all $g \in Q$ it follows that $\langle x, y \rangle$ is a non-cyclic 3-subgroup of $C_G(g)$. We deduce $C_G(g) \leq M$ from Lemma 3.2.2 (f). In particular Q acts elementwise fixed-point-freely on the abelian group $[O_{\sigma'}(C), x] \neq 1$. Now Lemma 1.1.14 (e) yields the assertion. ∎

(5) T has order 8.

Proof. Since $\langle z, T \rangle$ is contained in $N_G(Q_1)$, the group $N_G(Q_1)$ does not have a normal 3-complement. Moreover we see by Lemma 5.2.7 that $C_G(Q_1) \leq C_G(O_Q(C)) \leq C$ and so $C_G(Q_1)$ has cyclic Sylow 3-subgroups by Proposition 5.2.5. Thus we apply Lemma 5.2.2 to observe that $N_G(Q_1)/C_G(Q_1)$ has no normal 3-complement. Let $T_0 \in \mathrm{Syl}_2(C_G(Q_1))$ with $C_T(Q_1) \leq T_0$. Then $N_G(Q_1) = C_G(Q_1) \cdot N_{N_G(Q_1)}(T_0)$ by a Frattini argument. Therefore $N_G(T_0)$ has no normal 3-complement and Lemma 5.2.2 yields that $N_G(T_0)/C_G(T_0)$ has no normal 3-complement. In particular T_0 admits an automorphism of order 3. The assumption that T is dihedral together with Lemma 1.1.3 force T_0 to be elementary abelian of order 4. Furthermore we have $N_T(O_q(C)) = T$ and so $B \leq \langle b^T \rangle \leq C_T(O_q(C)) = C_T(Q_1) \leq T_0$. This implies $\langle b^T \rangle = T_0$ has order 4. Since T is a dihedral group, we finally conclude that T has order 8. ∎

Let $Q_0 = \Omega_1(Q)$ and set $F := F(N_G(Q_0))$ and $E := E(N_G(Q_0))$. Then the cyclic group Q_0 of order q is centralised by every q-subgroup of $N_G(Q_0)$.

(6) We have $Q_1 \leq N_G(Q_0)$ and $E \neq 1$.

Proof. By (1) and (4) we observe that $Q_1 = O_q(C) \cdot Q$ is the product of two cyclic groups. Further Q normalises $O_q(C)$ and hence we have that $r(Q_1) = 2$ and $\Omega_1(O_q(C)) \leq Z(Q_1)$. The element z normalises Q_1 but, by Lemma 5.2.7, it does not normalise $O_q(C)$. Therefore we conclude that $\Omega_1(Z((Q_1))) = \langle \Omega_1(O_q(C)), \Omega_1(O_q(C))^z \rangle = \Omega_1(Q_1) \leq C_G(Q)$ is elementary abelian of order q^2. In particular we observe that $Q_0 \leq Z(Q_1)$ and hence we have that $Q_1 \leq C_G(Q_0) \leq N_G(Q_0)$.

From the choice of a we deduce that $O_q(C)$ is inverted by a. This shows that the element a is not contained in F. Part (3) and (4) show that

$$S_4 \cong \langle N_T(A), y \rangle = \langle T, y \rangle \leq N_G(Q) \leq N_G(Q_0) \leq N_G(O_2(N_G(Q_0))).$$

We remark that A is the unique normal subgroup of T that is normalised by the element y. Altogether this implies that F is of odd order. In addition Lemma 1.1.14 (e) yields that $F = \langle C_F(a), C_F(a \cdot c), C_F(c) \rangle$ and, as all involutions of A are conjugate by $y \in N_G(F)$, we obtain that $|C_F(a)| = |C_F(ac)| = |C_F(c)|$.

Suppose for a contradiction that $1 \neq [a, C_F(c)] \leq [a, F]$.

Then $[a, C_F(c)]$ is contained in $F(C)$ by Lemma 5.2.6 and hence Lemma 5.2.7 implies that $C_G([a, C_F(c)]) \leq N_G([a, C_F(c)]) \leq C$. According to Lemma 1.1.14 (g) we conclude that the nilpotent group F is centralised by c. This implies that $|F| = |C_F(c)| = |C_F(a)|$ and so we have $F \leq C_G(A)$. This contradicts $[a, F] \neq 1$.

Altogether we deduce that $C_F(c) \leq C_F(a)$. From $|C_F(a)| = |C_F(c)|$ we further conclude that $C_F(c) = C_F(a) \leq C_F(A) \leq C_F(ac)$ and again that $C_F(c) = C_F(ac)$. Finally it follows that $F \leq \langle C_F(a), C_F(a \cdot c), C_F(c) \rangle = C_F(A)$. In conclusion from $O_2(F) = 1$ we obtain that $A \leq C_{N_G(Q_0)}(F) \nleq F$ and so $E \neq 1$ by Lemma 1.1.18 (h). ∎

(7) We have $B \nleq E$.

Proof. Suppose for a contradiction that $B \leq E$. Then Thompson's Transfer Lemma 12.1.1 of [30] implies that $N_E(B)/C_E(B) \cong S_3$. By (3) we have $N_E(B) \leq N_G(B) \leq M$. Altogether Lemma 3.3.5 (b) provides an element $z_0 \in N_E(B) \cap D^*(M)$. Since $C_G(z_0)$ has odd order by Lemma 5.1.2, the element z_0 acts transitively on $B^\#$.

Thus $\langle z_0, Q_0\rangle$ and $\langle z_1, Q_1\rangle$ are $\{3, q\}$-subgroups of $N_G(B)$. The group $N_G(B)$ is soluble by (3) and so Theorem 1.1.8 provides Hall $\{3, q\}$-subgroup H_0 and H_1 of $N_G(B)$ such that $\langle z_0, Q_0\rangle \leq H_0$ and $\langle z_1, Q_1\rangle \leq H_1$. If R_0 is a critical subgroup of subgroup of a Sylow 3-subgroup of $N_G(B)$, then R has exponent 3 and (3) forces R to be cyclic or elementary abelian of order 9. This implies that R admits no automorphism of order q. Finally Frobenius' p-Complement Theorem 1.1.11 implies that both groups H_0 and H_1 have a normal 3-complement.

Let $R_0 \in \mathrm{Syl}_3(H_0)$ with $z_0 \in R_0$ and $R_1 \in \mathrm{Syl}_3(H_1)$ with $z_1 \in R_1$. Then $\Omega_1(R_i)$ normalises $O_q(H_i)$ for all $i \in \{0, 1\}$. From the choice of z_0, we see that $[Q_0, z_0] \leq [F, E] = 1$. In addition we have $N_G(B) \leq M$ and (3) implies that $\Omega_1(R_0) = \langle z_0, x\rangle$. We conclude that $\Omega_1(R)$ centralises the cyclic subgroup Q_0 of $Z(O_q(H_0))$, because x is 3-locally central.

As all Hall $\{3, q\}$-subgroups of $N_G(B)$ are conjugate by Theorem 1.1.8, also z_1 centralises a non-trivial cyclic subgroup of $Z(O_q(H_1))$. Let Q_2 be a Sylow q-subgroup of $N_G(Q_0)$ containing Q_1. Then we obtain that $Z(Q_2) \leq C_{Q_2}(O_q(C)) \leq Q_2 \cap C$ by Lemma 5.2.7. We conclude that $\Omega_1(Z(Q_2)) \leq \Omega_1(Q_1)$, since Q_1 is a Sylow q-subgroup of C. This shows that z_1 centralises a cyclic subgroup of $\Omega_1(Q_1)$. But z_1 does not normalise $\Omega_1(O_q(C))$ by Lemma 5.2.7 and Proposition 5.2.5. It follows that $[\Omega_1(Q_1), z_1] \neq 1$. By Lemma 1.1.14 (f) we have $C_{\Omega_1(Q_1)}(z_1) \cap [\Omega_1(Q_1), z_1] = 1$.

Let $g \in [\Omega_1(Q_1), z_1]^\#$. Then we have $\Omega_1(O_q(C)) \neq [\Omega_1(Q_1), z_1] = \langle g\rangle$, because z_1 does not normalise the abelian group $\Omega_1(O_q(C))$ of order q^2. Finally there is an involution $s_1 \in T$ that inverts z_1 and normalises Q_1. Therefore $\langle g\rangle$ is s_1-invariant. Since $\langle g, z_1\rangle$ is not abelian and s_1 inverts z_1, the element s_1 does not invert g. Hence s_1 centralises $\langle g\rangle$ and we see that

$$g^{z_1} = (g^{z_1})^{s_1} = g^{z_1 \cdot s_1} = (g^{s_1^{-1}})^{z_1 \cdot s_1} = g^{s_1^{-1} \cdot z_1 \cdot s_1} = g^{z_1^{-1}}.$$

This implies $z_1 = (z_1^{-1})^2 \in C_G(g)$. That is a contradiction. ■

The group T is dihedral of order 8 by (5). From (7) it follows that A is a Sylow 2-subgroup of E, since simple groups do not have cyclic 2-subgroups by Burnside's p-Complement Theorem 1.1.10 and the Odd Order Theorem 1.1.12. According to Theorem 2.5.1 there is a prime power such that $E/Z(E) \cong \mathrm{PSL}(2, r^n)$.

Moreover we obtain $O_q(C) = [O_q(C), a] \leq [N_G(Q_0), E] \leq E$.

Then b induces an involutory automorphism on E centralising the cyclic group $O_q(C) \cdot \langle c\rangle$ but no subgroup of order 4.

Suppose for a contradiction that $E \cdot \langle b\rangle$ is not isomorphic to $\mathrm{PGL}(2, r^n)$. Then Lemma 1.2.4 provides an element $e \in E$ such that $e \cdot b$ induces a field automorphism of order 2 in E. We apply Proposition 4.9.1 (d) of [24] to conclude that $C_E(b) \cong C_E(e \cdot b)$. Again Lemma 1.2.4 implies that $C_E(b) \cong \mathrm{PGL}(2, r^{n/2})$. Since $C_E(b)$ is soluble by (3), it follows that $n = 2$ and $r \in \{2, 3\}$. But this forces $\mathrm{PGL}(2, r^{n/2})$ to be a $\{2, 3\}$-group and hence of order prime to q. This is a contradiction, because b centralises $O_q(C) \leq E$.

Thus $E \cdot \langle b\rangle$ is isomorphic to $\mathrm{PGL}(2, r^n)$. Furthermore $C_E(b)$ is dihedral of order $r^n - 1$ and r is odd by Lemma 1.2.4. Hence either $r^n - 1$ is divisible by 4 or $\frac{r^n-1}{2}$ is odd. As b centralises the cyclic group $O_q(C) \cdot \langle c\rangle$ but no subgroup of order 4, this is a final contradiction. □

5.3 The Last Group Standing

We could apply a theorem of Lyons [31] that implies in our situation that G' isomorphic to $U_3(4)$ and therefore a Bender group which contradicts Corollary 3.2.3.

Lyons shows that a finite simple group with a Sylow 2-subgroup isomorphic to a Sylow 2-subgroup of $U_3(4)$ has a rational representation of degree 12. Then he applies a theorem of Schur [33] to bound the order of G by $2^6 \cdot 3^8 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$. Finally he shows that the finite simple group has a strongly embedded subgroup and applies [6].
His proof does not provide a local insight. But the philosophy of this thesis is to reveal the local structure of the minimal counterexample.

In his first Lemma in [31] Lyons shows with local arguments that $N_G(T)/T$ is of order 15. The third prime 5 helps to connect more precisely the 2-structure of G with its 3-structure to obtain the following Proposition.

5.3.1 Proposition

The group M is not soluble and we have $2 \in \sigma$.

Proof
By Lemma 5.1.7 the group M has odd order if $2 \notin \sigma$. Thus the Odd Order Theorem 1.1.12 implies that it suffices to show that M is not soluble. Moreover again by Lemma 5.1.7 respectively by Lemma 3.3.1 we may choose T such that T is x-invariant. Let further A denote $\Omega_1(T)$. Then $A = Z(T)$ and so $N_G(A)/C_G(A) \cong Z_3$ by Proposition 5.1.1 (c). This further implies that A is strongly closed in G.

Suppose for a contradiction that M is soluble.

$$\text{(1) For all } a \in A^{\#} \text{ we have } C_G(a) = N_{C_G(a)}(A) \cdot O(A^{C_G(a)}).$$

Proof. Let a be an involution of $A^{\#}$. Then $C_G(a)$ is either of order prime to 3 or as x is 3-locally central $C_G(a)$ contains a conjugate of x. We have $I^*(M) = \varnothing$ by Lemma 5.1.2. Hence Lemma 3.2.2 (d) implies in the second case that $C_G(a)$ has a normal 3-complement. In both cases $C_G(a)$ is 3-soluble. Moreover Theorem 1.2.12 (f) yields that G does not involve a Suzuki group. It follows that $C_G(a)$ is soluble by Theorem 1.2.8. In addition from Lemma 2.2.2 (d) we see that A is strongly closed in $C_G(a)$. Thus Proposition 2.2.5 yields $C_G(a) = N_{C_G(a)}(A) \cdot O(A^{C_G(a)})$. ■

$$\text{(2) We have } |N_G(T)/(C_G(T) \cdot T)| = 15 \text{ and } 5 \in \pi(M).$$

Proof. By Theorem 1.2.12 (g) we obtain that $|N_G(T)/(C_G(T) \cdot T)| = 15$. Moreover the group $N_G(T)/(C_G(T) \cdot T)$ is cyclic. We conclude that there is some non-trivial 3-subgroup R of $N_G(T)$ such that $N_G(R)$ has order divisible by 5. Since x is 3-locally central, there is an element u of order 5 in $N_M(T)$. In particular we have $5 \in \pi(M)$. ■

(3) If H is a Hall $\{3, q\}$-subgroup of M for some $q \in \pi(M)$, then one of the following holds:

(i) There is an element g of order q such that $C_G(g) \leq M$ and $q \in \sigma$.

(ii) We have $O_q(H) \neq 1$, $q \in \sigma$ and $r(P) = 2$.

(iii) The Sylow q-subgroups of M are cyclic and, if $q \geq 5$, then we have $r(P) \geq 3$.

Proof. Suppose first that the Sylow q-subgroups of M are not cyclic and let $Q \in Syl_q(H)$. If there is an element g of order q such that $C_G(g) \leq M$, then by Lemma 3.3.2 (b) we conclude $q \in \sigma$. So if (i) is false, then Lemma 3.2.2 (f) implies that the centraliser of every element g

of order q of H has cyclic Sylow 3-subgroups containing x. Moreover there is an elementary abelian non-cyclic q-subgroup V of H such that $g \in V$, since Q is not cyclic. Lemma 1.1.16 provides a critical subgroup R of $O_3(H)$. Since R is of exponent 3 Lemma 1.1.14 (e) implies that $R = \langle C_R(v) \mid v \in V^\# \rangle \leq \langle x \rangle$. It follows that $V \leq C_G(R)$ and, as R is a critical subgroup of $O_3(H)$, we conclude that $V \leq C_G(O_3(H))$. The group H has exactly two prime divisors. Altogether we deduce $O_q(H) \neq 1$ from Lemma 1.1.18 (h) and Burnside's $p^\alpha q^\beta$-Theorem 10.2.1 of [30]. Since the Sylow 3-subgroups of G are not cyclic, Lemma 3.3.2 (c) yields that $q \in \sigma$. Moreover Lemma 1.1.14 (e) leads to $r(P) = 2$, since every element of order q in $O_q(H)$ is not centralised by any element of $D^*(M)$ by our assumption that (*i*) is false and Lemma 3.2.2 (f). This is (*ii*).
Assume now that the Sylow q-subgroups of M are cyclic. If we have $q \leq 3$, then (iii) holds. Suppose that $q \geq 5$ and let further g be an element of order q in H. If g acts trivially on $O_3(H)$, then (*i*) or (*ii*) holds. Otherwise we recall that an elementary abelian subgroup of order 9 admits no automorphism of prime order at least 5 and an extraspecial group of order 27 also admits no automorphism of prime order at least 5 that centralises the centre by Lemma 1.3.5. This implies together with Lemma 1.3.6 that $r(P) \geq 3$. Thus (*iii*) holds. ∎

(4) The group M has odd order. The element x acts transitively on $A^\#$ and $\langle x \rangle$ is a Sylow 3-subgroup of $N_G(A)$. Moreover we have $r_3(G) \geq 3$.

Proof. Suppose for a contradiction that M is a Hall subgroup of G and let Q be a Sylow subgroup of M. Then the Focal Subgroup Theorem 1.1.9 and Lemma 3.2.2 (b) imply that

$$Q \cap G' = \langle h^{-1} \cdot h^g \mid h, h^g \in Q \text{ and } g \in G \rangle = \langle h^{-1} \cdot h^m \mid h, h^m \in Q \text{ and } m \in M \rangle = Q \cap M'.$$

It follows that $M = M' \cdot \langle x \rangle$. Thus we have $M/\langle x \rangle = M' \cdot \langle x \rangle/\langle x \rangle = (M/\langle x \rangle)'$. This means that $M/\langle x \rangle$ is perfect contradicting our assumption that M is soluble.
Suppose now for a contradiction that M has even order. Then from Lemma 5.1.7 it follows that $2 \in \sigma$. Thus, as M is soluble, there is a Hall $\{2,3\}$-subgroup H of M by Theorem 1.1.8. Since T is not cyclic, (3)(iii) does not apply. Moreover Lemma 5.1.2 exclude 3(i). Consequently we have $r(P) = 2$. For $q \in \pi(M) \setminus \{2,3\}$ let H_q denote a Hall $\{3,q\}$-subgroup. Then 3(iii) does not apply and both parts (i) and (ii) forces q to be an element of σ. Altogether M is a Hall subgroup of G, because of $\{2,3\} \subseteq \sigma$. This is a contradiction.
We conclude that M has odd order. Since M is no Hall subgroup of G, there is a prime $q \in \pi(M)$ such that $q \notin \sigma$ and consequently (3)(iii) applies for a Hall $\{3,q\}$-subgroup of M. Then $q \geq 5$, as P is not cyclic and it follows that $r_3(G) = r(P) \geq 3$.
Furthermore $C_G(A)$ is a $3'$-group, because M has odd order and x is 3-locally central. Moreover Proposition 5.1.1 yields that $N_G(A)/C_G(A)$ has order 3. Thus $N_G(A)$ has cyclic Sylow 3-subgroups of order 3. Since we have $x \in N_G(A)$, the element x permutes the involutions of A transitively and the assertion is true. ∎

Let $q \in \pi(G) \setminus \{2,3\}$ be a prime and let Q be a q-subgroup of G.

(5) If Q is A-invariant, then $N_G(Q)$ is soluble.

Proof. Let Q be A-invariant and suppose for a contradiction that $N_G(Q)$ is not 3-soluble. Then $N_G(Q)$ has a non-trivial Sylow 3-subgroup R. Since M is soluble, Lemma 3.2.2 (d) yields that $x^g \notin N_G(Q)$ for all $g \in G$. If R is cyclic, then Burnside's p-complement Theorem 1.1.10 implies that $N_G(R)$ has even order and Sylow's Theorem leads to $2 \in \pi(M)$, as x is 3-locally central. This contradicts (4). Thus R is non-cyclic and Lemma 1.1.14 (e) and Sylow's Theorem provide an element $y \in D^*(M)$ and a q-element g such that y centralises g. Then g is also centralised by x, because x is 3-locally central. Therefore Lemma 3.2.2 (f) forces $C_G(g)$ to be a subgroup of M. In addition Lemma 3.3.2 (b) implies $q \in \sigma$. Again by Sylow's Theorem there is an element $h \in G$ such that $x^h \in N_G(Q)$. This is a contradiction.
Finally $N_G(Q)$ is 3-soluble. From Theorem 1.2.12 (f) we conclude that G does not involve a Suzuki group. In particular Theorem 1.2.8 forces $N_G(Q)$ to be soluble. ∎

(6) The maximal A-invariant q-subgroups of G are trivial or Sylow subgroups of G. If Q is A- and x^g-invariant for some element $g \in G$, then there is an element $h \in N_G(A)$ such that $\langle x, A\rangle \leq N_G(Q^h)$.

Proof. Suppose first that Q is a non-trivial maximal A-invariant q-subgroup of G. Then $N_G(Q)$ is soluble by (5). Since A is strongly closed in G, Lemma 2.2.2 (d) implies that A is strongly closed in $N_G(Q)$. So we may apply Proposition 2.2.5 to obtain the factorisation $N_G(Q) = N_{N_G(Q)}(A) \cdot O(\langle A^{N_G(Q)}\rangle)$. Since we have $|N_G(A)/C_G(A)| = 3 \neq q$, there exists an A-invariant Sylow q-subgroup of $N_G(Q)$. From the maximal choice of Q we deduce that $Q \in Syl_q(G)$. We further assume that there is an element $g \in G$ such that

$$x^g \in N_G(Q) = N_{N_G(Q)}(A) \cdot O(\langle A^{N_G(Q)}\rangle)$$

and suppose for a contradiction that $O(\langle A^{N_G(Q)}\rangle)$ is no $3'$-group. Then Lemma 1.1.14 (b), (e) applied to A and an A-invariant Sylow 3-subgroup of $O(\langle A^{N_G(Q)}\rangle)$ provide an involution $a \in A^\#$ such that $C_G(a)$ is no $3'$-group. Hence we conclude that M has even order, since x is 3-locally central. This contradicts (4). Thus $N_{N_G(Q)}(A)$ contains a Sylow 3-subgroup of $N_G(Q)$ and by Sylow's Theorem there is an element $h \in N_G(Q)$ such that $x^{g\cdot h}$ normalises A. It follows that $x, x^{g\cdot h} \in N_G(A)$. From (4) and Sylow's Theorem we obtain an element $c \in N_G(A)$ such that $x^{g\cdot h} \in \langle x^c\rangle$. Moreover Lemma 3.1.2 (a) yields that $x^{g\cdot h} = x^c$. Thus Q is normalised by $x^g = x^{c\cdot h^{-1}}$ and h. This implies that Q is x^c-invariant and $Q^{c^{-1}}$ is normalised by x and by $A^{c^{-1}} = A$. ■

(7) If $\langle x, T\rangle \leq H < G$, then $H \leq N_G(A)$

Proof. Let $\langle x, T\rangle \leq H < G$. Then we obtain from $T \nleq M$ by (4) and Lemma 3.2.2 (d) that H has a normal 3-complement. Moreover A is strongly closed in H by Lemma 2.2.2 (d). Thus Theorem 1.2.12 (f) and Proposition 2.2.5 yield that $H = N_H(A) \cdot O(\langle A^H\rangle)$. If N is a normal subgroup of H of even order, then we have that $1 \neq A \cap N$. Since x normalises N and acts transitively on $A^\#$, it follows that $A \leq N$ (*).
Suppose for a contradiction that $O(\langle A^H\rangle) \neq 1$. Then $A \neq O_2(\langle A^H\rangle)$ and hence (*) implies $O_2(H) = 1$. Lemma 1.1.18 (h) shows that $A \nleq C_H(F(O(\langle A^H\rangle)))$. In particular there is a prime q such that $A \nleq C_H(O_q(\langle A^H\rangle))$. We set $Q_1 := O_q(\langle A^H\rangle)$. Then we observe that $C_H(Q_1)$ and $\phi(Q_1)$ are normal in H. Statement (*) implies that $C_A(Q_1) = 1$. Moreover A acts coprimely on Q_1. So Lemma 1.1.14 (a) yields $C_A(Q_1/\phi(Q_1)) = 1$.
Let $- : H \to H/\phi(Q_1)$ be the natural epimorphism. Then $\overline{Q_1}$ is abelian by Lemma 1.1.4 and Lemma 1.1.14 (f) yields that $\overline{Q_1} = C_{\overline{Q_1}}(\bar{A}) \times [\bar{A}, \overline{Q_1}]$. In particular we observe that $[\bar{A}, \overline{Q_1}] \neq 1$. We set $\bar{Q} := [\bar{A}, \overline{Q_1}]$. Then $\bar{Q}$ is $N_{\bar{H}}(\bar{A})$-invariant and $C_{\bar{Q}}(\bar{A}) = 1$.
Suppose for a contradiction that also $\bar{x}$ acts fixed-point freely on $\bar{Q}$. Then (4) implies that $\bar{x}$ acts fixed-point freely on the semi-direct product $\bar{Q} \rtimes \bar{A}$. Thus the product is direct by Theorem 10.1.5 of [22]. This is a contradiction and implies $C_{\bar{Q}}(\bar{x}) \neq 1$. Moreover $N_G(Q_1/C_G(Q_1))$ is of even order. This shows together with Lemma 3.2.2 (c) and (4), that we have $q \notin \sigma$ and (3) implies that M has cyclic Sylow q-subgroups. In particular $C_{Q_1}(x)$ is cyclic and the non-trivial group $C_{\bar{Q}}(x)$ is cyclic.
By Maschke's Theorem 3.3.1 of [22] the group $\bar{Q}$ is a direct product of subgroups $\bar{Q}_i$ such that $\bar{A} \cdot \langle \bar{x}\rangle$ acts irreducibly on every $\bar{Q}_i$. In particular we observe $\bar{Q}_i = [\bar{Q}_i, \bar{A}]$ from $C_{\bar{Q}}(\bar{A}) = 1$. Further Lemma 1.1.14 (e) implies that $\bar{Q}_i = \langle C_{\bar{Q}_i}(\bar{a}) \mid \bar{a} \in \bar{A}^\#\rangle$. Let $\bar{a} \in \bar{A}^\#$ such that $1 \neq C_{\bar{Q}_i}(\bar{a})$. Then $C_{\bar{Q}_i}(\bar{a})$ is $\bar{A}$-invariant and from $C_{\bar{A}}(\bar{Q}_i) = 1$ we deduce that the two involutions of $\bar{A}$ different from $\bar{a}$ act fixed-point freely on $C_{\bar{Q}_i}(\bar{a})$. In particular they invert the group $C_{\bar{Q}_i}(\bar{a})$. This shows that for all $\bar{g} \in C_{\bar{Q}_i}(\bar{a})$ the group $\bar{A}$ normalises $\langle \bar{g}\rangle$.
Let $\bar{g}$ be an element of $C_{\bar{Q}_i}(\bar{a})$. Then we have $\bar{g}^{\bar{x}} \in (C_{\bar{Q}_i}(\bar{a}))^{\bar{x}} = C_{\bar{Q}_i}(\overline{a^x})$ and $\bar{g}^{\bar{x}^2} \in C_{\bar{Q}_i}(\overline{a^{x^2}})$. Thus $\bar{A}$ normalises each of $\langle \bar{g}\rangle$, $\langle \bar{g}^{\bar{x}}\rangle$ and $\langle \bar{g}^{\bar{x}^2}\rangle$. In particular $\langle \bar{g}, \bar{g}^x, \bar{g}^{x^2}\rangle$ is $\langle \bar{x}\rangle \cdot \bar{A}$-invariant.

Since $\langle \bar{x} \rangle \cdot \bar{A}$ acts irreducible on $\bar{Q}_i$ we conclude that $\langle \bar{g}, \bar{g}^x, \bar{g}^{x^2} \rangle = \bar{Q}_i$. Suppose for a contradiction that $\bar{Q}_i$ has not order q^3. Then we have $\bar{Q}_i = \langle \bar{g}, \bar{g}^{\bar{x}} \rangle$ and $[\bar{Q}_i, \overline{a^{x^2}}] = \bar{Q}_i$. This contradicts Lemma 1.1.14 (f) together with $\bar{g}^{\bar{x}^2} \in C_{\bar{Q}_i}(a^{x^2})$.
Finally $\bar{Q}_i$ is of order q^3 and $\bar{g} \cdot \bar{g}^x \cdot \bar{g}^{x^2} \neq 1$ and centralised by $\bar{x}$. Since $C_{\bar{Q}}(\bar{x})$ is cyclic, we deduce that Q_i is the unique direct factor in the factorisation obtain from Maschke above. More precisely we have $\bar{Q} = \bar{Q}_i$. But $\bar{T}$ also normalises $\bar{Q}$. This is a contradiction, since $|\bar{Q}| = q^3$ and $C_{\bar{A}}(\bar{Q}) = 1$. This contradiction shows that $O(\langle A^H \rangle) = 1$ and $H = N_H(A) \leq N_G(A)$. ■

(8) For all $a \in A^\#$ we have $(|O(\langle A^{C_G(a)} \rangle)|, |C_G(A)|) = 1$.

Proof. Let $a \in A^\#$. By (1) we have that $C_G(a) = N_{C_G(a)}(A) \cdot O(\langle A^{C_G(a)} \rangle) = C_G(A) \cdot O(\langle A^{C_G(a)} \rangle)$ and $T \leq C_G(A)$. Let Q be a non-trivial T-invariant Sylow q-subgroup of $O(\langle A^{C_G(a)} \rangle)$. Suppose for a contradiction that $Q \cap C_G(A) \neq 1$. Then, as x acts coprimely on $C_G(A)$, we obtain from Lemma 1.1.14 (b) a q-subgroup of G that is x-invariant. Since x is 3-locally central and $2 \notin \pi(M)$ by (4), we obtain that $q \neq 3$. Moreover there is an x-invariant Sylow q-subgroup of G by Lemma 3.3.1. Thus (6) provides an element $c \in N_G(A)$ such that Q^c is x-invariant. Moreover Q^c is normalised by $T^c \cdot \langle x \rangle$ and so (7) yields $Q^c \leq N_G(A^c)$. Hence we have $Q \leq N_G(A)$. It follows from $q \neq 3$ and $|N_G(A)/C_G(A)| = 3$ (by Proposition 5.1.1 (c)) that $Q \leq C_G(A)$.
Applying Lemma 1.3.7 (c), we conclude that $C_G(a) = C_G(A) \cdot O_{\{2,q\}'}(C_G(a))$. Moreover we have that $A \cdot O_{\{2,q\}'}(C_G(a)) \trianglelefteq C_G(A) \cdot O_{\{2,q\}'}(C_G(a)) = C_G(a)$. For this reason we observe that $\langle A^{C_G(a)} \rangle \leq A \cdot O_{\{2,q\}'}(C_G(a))$ is a q'-group. This is a contradiction.
Altogether we obtain $(|O(\langle A^{C_G(a)} \rangle)|, |C_G(A)|) = 1$. ■

(9) The group $C_G(A)$ is a Hall subgroup of G.

Proof. Let q be a prime divisor of $|C_G(A)|$. Then the fact that x is 3-locally central together with (4) imply that $q \neq 3$. Further (6) provides an A-invariant Sylow q-subgroup Q of G. Since A is elementary abelian, Lemma 1.1.14 (e) yields that $Q = \langle C_Q(a) \mid a \in A^\# \rangle$. From (8) we see that $O(\langle A^{C_G(a)} \rangle)$ has order prime to q for every $a \in A^\#$. Thus (1) forces $N_G(A)$ to contain a Sylow q-subgroup of $C_G(a)$. Theorem 1.1.8 implies that every Hall $\{2, q\}$-subgroup normalises a conjugate of A. It follows that the $\{q, 2\}$-subgroup $C_Q(a) \cdot A$ of $C_G(A)$ is contained in $N_{C_G(a)}(A)$. Altogether we have $Q = \langle C_Q(a) \mid a \in A^\# \rangle \leq N_G(A)$ and conclude that $Q \leq C_G(A)$, since $r \neq 3$ and $|N_G(A)/C_G(A)| = 3$ by Proposition 5.1.1 (c). ■

(10) The group $C_G(A)$ contains no elementary abelian p-group of rank at least 3.

Proof. Suppose for a contradiction that there is a prime q and an elementary abelian q-subgroup $V \leq C_G(A)$ of order at least q^3. For all $v \in V^\#$ we set $\theta(v) := O_{q'}(C_G(v))$.
Let $v \in V^\#$. Then $C_G(v) \leq N_G(\langle v \rangle)$ is soluble by (5). Thus Lemma 2.1.3 implies that θ is a soluble V-signalizer functor. Consequently the Soluble Signalizer Functor Theorem 2.1.6 yields that $W_V := \langle \theta(v) \mid v \in V^\# \rangle$ is a soluble q'-subgroup of G. Since $A \leq C_G(v)$ is strongly closed in $C_G(v)$ by Lemma 2.2.2 (d), we deduce from Proposition 2.2.5 that $C_G(v) = N_{C_G(v)}(A) \cdot O(\langle A^{C_G(v)} \rangle)$.
From $q \in \pi(C_G(A))$ and (9) we obtain that $C_G(A)$ contains a Sylow q-subgroup of G and (8) implies for all $a \in A^\#$ that $O(\langle A^{C_G(a)} \rangle)$ is a q'-subgroup of G. Moreover we apply Lemma 1.3.7 (c) to see that $O(\langle A^{C_G(v)} \rangle) \leq C_{C_G(v)}(A) \cdot O_{\pi'}(\langle A^{C_G(v)} \rangle)$ for $\pi := \pi(C_G(A))$. It follows for every $v \in V^\#$ that $O^{\pi'}(C_G(v)) \leq O^{\pi'}(C_{C_G(v)}(A) \cdot O_{\pi'}(\langle A^{C_G(v)} \rangle)) = O_{\pi'}(\langle A^{C_G(v)} \rangle)$.
From (8) we have $O(\langle A^{C_G(a)} \rangle) = O_{\pi'}(\langle A^{C_G(a)} \rangle)$ and so we conclude that

$$C_{O(\langle A^{C_G(a)} \rangle)}(v) = C_G(v) \cap O(\langle A^{C_G(a)} \rangle) = O^{\pi'}(C_G(v)) \cap O_{\pi'}(\langle A^{C_G(a)} \rangle) \leq O_{\pi'}(\langle A^{C_G(v)} \rangle) \leq \theta(v).$$

This shows that $O(\langle A^{C_G(a)} \rangle) = \langle C_{O(\langle A^{C_G(a)} \rangle)}(v) \mid v \in V^\# \rangle \leq W_V$ is of order prime to q. In particular $\langle O(A^{C_G(a)}) \mid a \in A \rangle$ is a $N_G(A)$-invariant q'-subgroup of G. We further deduce that $\langle x \rangle \cdot T \leq N_G(\langle O(\langle A^{C_G(a)} \rangle) \mid a \in A \rangle) < G$.

Therefore (7) leads to $N_G(\langle O(\langle A^{C_G(a)}\rangle) \mid a \in A\rangle) \leq N_G(A)$. We conclude that for all $a \in A^\#$ we have $C_G(a) = C_G(A)$. More precisely $C_G(A)$ is strongly embedded in G. Since G' is no Bender group by Corollary 3.2.3, this contradicts Theorem 1.2.12. ∎

Let $U \in Syl_5(C_G(A))$. Then by (9) we have $U \in Syl_5(G)$. From (6), $5 \in \pi(M)$ by (2) and Lemma 3.3.1 we observe that we may choose U such that U is normalised by x.

(11) We have $U \cong E * \langle w\rangle$, where E is extra-special of order 125 and exponent 5 and $\langle w\rangle = U \cap M$. Moreover every x-invariant abelian subgroup of U is cyclic.

Proof. Suppose for a contradiction that U is cyclic. Then $\Omega_1(U) \leq M$. Moreover Burnside's p-complement Theorem 1.1.10 provides an automorphism of order prime to 5 of U and hence one of $\Omega_1(U)$. We observe that $|Aut(\Omega_1(U))| = 5 - 1 = 4$. Altogether it follows that 2 divides the order of $N_G(\Omega_1(U))/C_G(\Omega_1(U))$. But now Lemma 3.2.2 (c) forces M to be of even order. This contradicts (4). Thus we see that U is non-cyclic.
From $U \leq C_G(A)$, (10) and the fact that U is non-cyclic we deduce that U has rank 2.
Suppose for a contradiction that $U \cap M$ is non-cyclic. Then we apply (3) to a Hall $\{3, 5\}$-subgroup of M. From (4) we see that Statement 3(ii) is false and our assumption implies that 3(iii) is also not true. It follows that 3(i) holds. Thus we have $5 \in \sigma$ and there exists an element $u \in U$ such that $C_G(u) \leq M$. Since $U \leq C_G(A)$ it follows that $A \leq C_G(u) \leq M$ contradicting $2 \notin \pi(M)$ by (4).
Suppose for a contradiction that $Z(U)$ is non-cyclic. Then we deduce from $r(U) = 2$ that $\Omega_1(Z(U)) = \Omega_1(U)$. Since $C_U(x) \neq 1$, we have $C_{Z(U)}(x) \neq 1$. The group $[\Omega_1(Z(U)), x]$ is not cyclic, as 3 does not divide $4 = 5 - 1$. From $r(\Omega_1(U)) = 2$ we deduce that $\Omega_1(U) \leq C_U(x)$. This is a contradiction.
Thus $Z(U)$ is cyclic and x centralises $Z(U)$, since 3 does not divide $4 \cdot 5^n$ for any $n \in \mathbb{N}$.
Let V be an abelian x-invariant subgroup of U. Then we have $V \cap Z(U) \neq 1$ and V is of rank at most 2, since U has rank 2. From $Z(U) \leq C_U(x)$, the fact that $C_U(x)$ is cyclic and Lemma 1.1.14 (f) we deduce that $[V, x]$ is cyclic and normalised by x. Since V is a 5-subgroup, it follows that $[V, x]$ is trivial. Altogether V is cyclic.
We apply III 13.10 of [29] to conclude that $U \cong E*\langle w\rangle$, where E is extra-special of exponent 5 and $\langle w\rangle = Z(U)$, since every characteristic subgroup of U is x-invariant. It finally follows from $r(U) = 2$ and Theorem 5.5.3 of [22] that $|E| = 5^3 = 125$. ∎

(12) We have $U = \Omega_1(U)$.

Proof. We want to apply the Focal Subgroup Theorem 1.1.9.
Let $g \in N_G(A)$ and $v \in U$ such that $v^g \in U$. If we have $v \in \Omega_1(U)$, then $v^g \in \Omega_1(U)$ and hence $v \cdot (v^{-1})^g \in \Omega_1(U)$. Assume now that $o(v) \geq 25$. Then (11) provides elements $u_1, u_2 \in \Omega_1(U)$ and $n, m \in \mathbb{N}$ such that $v = u_1 \cdot w^n$ and $(v^g)^{-1} \cdot v = u_2 \cdot w^m$. This shows that $v^g = v \cdot (u_2 \cdot w^m)^{-1} = u_1 \cdot u_2^{-1} \cdot w^{n-m}$. In particular we deduce from $exp(\Omega_1(U)) = 5$ that $1 \neq (v^g)^5 = (u_1 \cdot u_2^{-1})^5 \cdot w^{5(n-m)} \in \langle w\rangle$ and $v^5 = (u_1 \cdot w^n)^5 = w^{5n} \in \langle w\rangle$. Therefore g normalises $\langle v^5\rangle \leq \langle w\rangle$.
Statement (11) yields that $\langle v^5\rangle \leq \langle w\rangle \leq M$. So we may apply Lemma 3.2.2 (c) to obtain $N_G(\langle v^5\rangle) = N_M(\langle v^5\rangle) \cdot C_G(v^5)$. Since M has cyclic Sylow 5-subgroups we deduce that $|N_M(\langle v^5\rangle)/C_M(\langle v^5\rangle)|$ is not divisible by 5. Moreover (4) implies that $N_G(\langle v^5\rangle) = C_G(v^5)$. Consequently we have $g \in C_G(v^5)$ and so $w^{5(n-m)} = (v^g)^5 = v^5 = w^{5n}$. This implies $w^{5m} = 1$ and hence we have $w^m \in \Omega_1(U)$. Altogether we conclude that $(v^{-1})^g \cdot v = u_2 \cdot w^m \in \Omega_1(U)$ and hence we observe that $v^{-1} \cdot v^g = ((v^{-1})^g \cdot v)^{-1} \in \Omega_1(U)$.
Finally the Focal Subgroup Theorem 1.1.9 yields

$$U \cap N_G(A)' = \langle v^{-1}v^g \mid v, v^g \in U \text{ and } g \in N_G(A)\rangle \leq \Omega_1(U).$$

On the other hand, for all $g \in G$ and $b \in C_G(A)$ such that $b^g \in C_G(A)$, the groups A and $A^{g^{-1}}$ are subgroups of $C_G(b)$. Since $A = \Omega_1(T)$ and $T \in Syl_2(G)$, we observe that $A = \Omega_1(T_0)$ for some Sylow 2-subgroup T_0 of $C_G(b)$. Hence by Sylow's Theorem there is an element $c \in C_G(b)$ such that $A^c = A^{g^{-1}}$. It follows that $c \cdot g \in N_G(A)$ and $b^g = b^{cg}$. In particular $N_G(A)$ controls fusion of $C_G(A)$. Applying again the Focal Subgroup Theorem 1.1.9 we deduce that $U \cap G' = \langle v^{-1}v^g \mid v, v^g \in U \text{ and } g \in G\rangle$

$$= \langle v^{-1}v^g \mid v, v^g \in U \text{ and } g \in N_G(A)\rangle = U \cap N_G(A)' = \Omega_1(U).$$

Finally Lemma 3.2.1 (e) implies that $U = U \cap G' = \Omega_1(U)$. ■

Since $O_{3'}(N_G(A)) = C_G(A)$ is soluble, there is a Hall $\{2,5\}$-subgroup H of $C_G(A)$ that is x-invariant by Lemma 1.1.14 (c). We may choose notation such that $H = U \cdot T$. By (2) we have $|N_G(T)/C_G(T) \cdot T| = 15$. Thus there is an element $u \in U \setminus C_G(T)$ that is centralised by x. Statements (11) implies that $u \in Z(U)$. Therefore $\langle u\rangle = Z(U)$ by (12). Moreover $125 \nmid 15 = |N_G(T)/C_G(T) \cdot T|$ leads to $1 \neq O_5(H)$. Since $O_5(H) \trianglelefteq U$, it follows that $u \in O_5(H)$ contradicting $[T, u] = T$. □

5.3.2 Theorem

The group M has a simple section with a Sylow 2-subgroup isomorphic to T that is not isomorphic to $U_3(4)$.

Proof

Suppose for a contradiction that the theorem is false.

By Proposition 5.3.1 the group M is not soluble and we have $2 \in \sigma$. Hence we may choose a Sylow 2-subgroup T of M and refer to section 5.2.

Moreover we set $A = \Omega_1(T) = Z(T)$. Then we have $A \leq M$ and all involutions of A are conjugate in M by Proposition 5.1.1 (c) and Lemma 3.2.2 (c). Additionally from Proposition 5.2.5 we deduce that $C_G(A)$ has cyclic Sylow 3-subgroups and a normal 3-complement.

(1) We have $O_{2',2}(M) = O(M)$.

Proof. Suppose for a contradiction that $O_{2',2}(M) \neq O(M)$ and let $T_0 \in Syl_2(O_{2',2}(M))$ with $T_0 \leq T$. Then $A \cap T_0 \neq 1$ and as all involutions of A are conjugate in M, we obtain that $A = \Omega_1(T_0)$. A Frattini argument yields that $M = O_{2',2}(M) \cdot N_M(T_0) = O(M) \cdot N_M(A)$. In particular the Odd Order Theorem 1.1.12 forces $M/O(M)$ to be a non-soluble section of $N_G(A)$. Thus Lemma 5.2.1 (b) implies that $N_G(A)$ is not 3-soluble. This contradicts Proposition 5.1.1 (c) together with the fact that $C_G(A)$ has a normal 3-complement. ■

(2) The group $O(M)$ has cyclic Sylow 3-subgroups containing x. Moreover $\pi(O(M)) \subseteq \sigma$.

Proof. Since $A \leq M$, Lemma 1.1.14 (b) provides an A-invariant Sylow 3-subgroup R_0 of $O(M)$. By Lemma 1.1.16 the group R_0 has a critical subgroup R. Then R is of exponent 3. Moreover we have $C_G(a) \cap D^*(M) = \varnothing$ for every $a \in A^\#$ by Lemma 5.1.2. Altogether Lemma 1.1.14 (e) yields that $R = \langle C_R(a) \mid a \in A^\#\rangle \leq \langle x\rangle$. It follows that R_0 is centralised by A, because R is a critical subgroup of R_0. From the property of $C_G(A)$ to have cyclic Sylow 3-subgroups the first assertion follows, since we have $x \in Z(M)$. Finally Lemma 3.3.2 (e) yields the assertion, because $O(M)$ is P-invariant and $r(P) \geq 2$. ■

Let $- : M \to M/O(M)$ be the natural epimorphism.

(3) If $N_G(A) \cap D^*(M)$ is not empty, then $M/\langle x\rangle$ is perfect.

Proof. Suppose that we have $N_G(A) \cap D^*(M) \neq \varnothing$. From (1) we conclude that $E(\bar{M}) \neq 1$. Let E be the full pre-image of $E(\bar{M})$ in M. Then E is normal in M and has even order. Thus $A \cap E \neq 1$ and, as all involutions of A are conjugate in $M = N_M(E)$, it follows that $A \leq E$. Let T_0 be a Sylow 2-subgroup of E such that $A = \Omega_1(T_0)$. Then a Frattini argument yields

$\bar{M} = E(\bar{M}) \cdot N_{\bar{M}}(\bar{T}_0) = E(\bar{M}) \cdot N_{\bar{M}}(\bar{A})$. Since A is a elementary abelian group of order 4, we conclude that $O^3(N_{E(\bar{M})}(\bar{A})) \leq C_{E(\bar{M})}(\bar{A})$. Therefore we obtain $O^3(\bar{M}) \leq E(\bar{M}) \cdot C_{E(\bar{M})}(\bar{A})$. From our assumption and Lemma 5.2.3 we observe that $\pi(C_G(A)) \subseteq \sigma$. Finally, as $3 \in \sigma$, Lemma 1.1.14 (a) shows that $\pi(\bar{M}/E(\bar{M})) \subseteq \sigma$.
Let $q \in \pi(M)$ and let Q be a Sylow q-subgroup of $M \cap G'$. If $q \in \sigma$, then the Focal Subgroup Theorem 1.1.9 yields together with Lemma 3.2.2 (c) that
$Q = G' \cap Q = \langle g^{-1} \cdot g^h \mid g, g^h \in Q \text{ and } h \in G \rangle = \langle g^{-1} \cdot g^h \mid g, g^h \in Q \text{ and } h \in M \rangle = Q \cap M'$.
If we have $q \notin \sigma$, then we see $Q \cong \bar{Q} \leq E(\bar{M})$ by (2) and the above investigation. Hence we have $Q \leq M'$, as $E(\bar{M})$ is perfect. Altogether it follows that $M = M' \cdot \langle x \rangle$. Thus we have $M/\langle x \rangle = M' \cdot \langle x \rangle / \langle x \rangle = (M/\langle x \rangle)'$. ■

(4) The group $E(\bar{M})$ is isomorphic to $U_3(4)$. Moreover $D^*(M) \cap N_G(A) \neq \emptyset$.

Proof. Together with (1) we conclude that $F^*(\bar{M}) = E(\bar{M})$. Since T is isomorphic to a Sylow 2-subgroup of $U_3(4)$, it follows from the Odd Order Theorem 1.1.12 and Burnside's p-complement Theorem 1.1.10 that $E(\bar{M})$ is simple. From Lemma 2.6 of [26] we obtain that the Sylow 2-subgroup of $E(\bar{M})$ is either isomorphic to T or to A. In the first case our assumption that the theorem is false forces $E(\bar{M})$ to be isomorphic to $U_3(4)$.
In the other case $E(\bar{M})$ has Sylow 2-groups of order 4. Thus Theorem 2.5.1 yields that $E(\bar{M})$ is isomorphic to some $PSL(2,q)$ for some prime power q. Moreover we observe with (1) and Lemma 1.1.18 (c) that $\bar{M}/E(\bar{M}) \leq Out(E(\bar{M}))$ is soluble by Lemma 1.2.4. It follows from $\bar{T} \nleq E(\bar{M})$ that $\bar{M}$ is not perfect. Hence from (3) we deduce that $D^*(M) \cap N_G(A) = \emptyset$ in this case.
In both cases $E(\bar{M})$ has cyclic Sylow 3-subgroups that are inverted in $E(\bar{M})$. Let $y \in M$ be a 3-element such that $\langle \bar{y} \rangle$ is a Sylow 3-subgroup of $E(\bar{M})$ and let d be an involution in M such that $\bar{d}$ inverts $\bar{y}$. Then d normalises $\langle y \rangle \cdot O(M)$ and hence, by Lemma 1.1.14 (a), there exists a d-invariant Sylow 3-subgroup R of $\langle y \rangle \cdot O(M)$. Then $x \in R$ and d does not centralise R. Thus Lemma 3.3.5 (b) forces R to be non-cyclic.
The full pre-image of $N_{E(\bar{M})}(\bar{A})$ is soluble, as $O(M)$ and $N_{E(\bar{M})}(\bar{A})$ are soluble. Hence there is a Hall $\{2,3\}$-subgroup H of that full pre-image. Since $\bar{A}$ is normalised by a Sylow 3-subgroup of $E(\bar{M})$, we may choose notation such that $H = A \cdot R$. By (2) the group $O(M)$ has cyclic Sylow 3-subgroups that contain x. Moreover we have $x \in Z(M)$. Thus A acts trivially on $O(M) \cap H$.
Finally we conclude that $R \in N_G(O_2(H)) = N_G(A)$. It follows that $N_G(A) \cap D^*(M) \neq \emptyset$, since R is non-cyclic. In particular $E(\bar{M})$ is not isomorphic to $PSL(2,q)$ and the above investigation implies that $E(\bar{M}) \cong U_3(4)$ holds. ■

(5) The group $E(M)$ is non trivial.

Proof. Let c be an involution of A and let $y \in D^*(M) \cap N_G(A)$. It follows from Proposition 5.2.5 that $O_{\sigma'}(C_G(c))$ is abelian. Moreover by (1) the group A acts coprimely on $F(M)$. Suppose for a contradiction that there is a prime q of $\pi(F(M))$ such that $A \nleq C_G(O_q(M))$.
Let R be an abelian characteristic subgroup of $O_q(M)$. Then R is A-invariant and for all $a \in A^\#$ we have $R = C_R(a) \times [R,a]$ by Lemma 1.1.14 (f). For all involutions $a, b \in A$ with $a \neq b$ we apply Lemma 5.2.6 to see that $[C_R(a), b] \leq F(C_G(a))$ and Lemma 5.2.7 to observe that $R \leq N_G([C_R(a), b]) \leq C_G(a)$. From the fact that the involutions of M are conjugate in $M = N_G(O_q(M)) \leq N_G(R)$ it follows that A centralises R. In particular we have $O_q(M) \neq R$ and so $O_q(M)$ is not abelian. Since R is characteristic in $O_q(M)$, it is normal in M. Therefore we observe that $y \in \langle A^M \rangle \leq C_M(R)$. Thus, for all $g \in R^\#$, the group $C_G(g)$ has non-cyclic Sylow 3-subgroups. Hence Lemma 3.2.2 (f) forces $C_G(g)$ to be a subgroup of M for all $g \in R^\#$. This implies that R acts elementwise fixed-point-freely on the abelian group $[x, O_{\sigma'}(C_G(c))]$. Since $[x, O_{\sigma'}(C_G(c))]$ is non-trivial by Proposition 5.2.5, Lemma 1.1.14 (e)

yields that R is cyclic. Altogether we may apply III 13.10 of [29] to deduce that $O_q(M)$ is a central product of an extra-special group of exponent q and the cyclic group $Z(O_q(M))$.
We set $Q := \Omega_1(O_q(M))$. Then Q is extra-special of exponent q. Since Q is not centralised by A, there is an element $g \in [C_Q(b), a]^\#$ for involutions $a, b \in A^\#$ with $a \neq b$. From Lemma 5.2.7 we deduce that $C_Q(g) \leq C_G(b)$. In particular we see that $g \notin Z(Q)$. From $[Q, g] = Z(Q)$ it follows that $|Q : C_G(g)| = q$. Thus $C_Q(b)$ and hence $C_Q(a)$ are maximal subgroups of Q. Therefore we conclude that $|Q : C_Q(A)| = |Q : C_Q(a) \cap C_Q(b)| = q^2$. Moreover $C_Q(A) \trianglelefteq Q$, as $Z(Q) \leq C_Q(A)$. But finally $a \cdot b$ inverts $C_Q(a)/C_Q(A)$ and $C_Q(b)/C_Q(A)$, so $a \cdot b$ inverts $Q/C_Q(A)$. This is a contradiction as a is conjugate to $a \cdot b$ in $M = N_G(Q)$ and so $C_Q(a \cdot b) = C_Q(a^m) = (C_Q(a))^m$ for some $m \in M$. We conclude that $A \leq C_G(F(M))$ and Lemma 1.1.18 (h) and (1) force $E(M)$ to be non-trivial. ■

(6) We have $M = E(M) \times \langle x \rangle$ and $E(M) \cong U_3(4)$.
Moreover the assumption of Theorem 11 and Statements of Lemmas 11.1 - 11.5 in [28] hold with a suitable change of notation.

Proof. From (4) and (5) we have that $M/E(M)$ is of odd order. Now (3) and the Odd Order Theorem yield that $M = E(M) \cdot \langle x \rangle$. From [11] we know that $Z(E(M)) = 1$. Thus $M = E(M) \times \langle x \rangle$ and $M' = E(M) \cong U_3(4)$ by (3) and (4). As before the Focal Subgroup Theorem 1.1.9 and Lemma 3.2.2 (b) imply $G' \cap P = M' \cap P = E(M) \cap P$. This shows that x is no element of G'.
From [11] the we know the structure of $E(M) = M \cap G'$. We first observe that there is an element y of order 3 and an element g of order 5 in $E(M)$ that commute. The fact that x is 3-locally central implies that $N_{G'}(\langle y \rangle) = N_{M'}(\langle y \rangle)$ and hence $N_{G'}(\langle y \rangle) \cong S_3 \times Z_5$ is the Statement of Lemma 11.1 of [28]. Furthermore $C_G(g)$ has non-cyclic Sylow 3-subgroup and Lemma 3.2.2 (f) forces $C_G(g)$ to be a subgroup of M. Therefore we conclude that the group $C_{G'}(g) = C_{M'}(g) \cong Z_5 \times A_5$ is not soluble. Thus the assumptions of Theorem 11 and Statements of Lemma 11.2 and 11.5 in [28] are satisfied. As all involutions of T are conjugate by y also Statement 11.3 of [28] is true. The group $N_G(T)$ has non-cyclic Sylow 3-subgroups. Hence Lemma 3.2.2 (f) implies that $N_G(T) \leq M$. It follows that $N_{G'}(T) = N_{E(M)}(T) = \langle y, g \rangle \cdot T$. In particular the statement of Lemma 11.4 in [28] holds. ■

Finally we apply Lemmas 11.6, 11.7 and 11.8 of [28] to obtain that
$$|G'| = 62400 = |U_3(4)| = |G' \cap M|.$$
Thus $G' = G' \cap M$ provides the final contradiction. □

5.3.3 Remark

Theorem 5.3.2 shows that wee need to prove Lyon's theorem [31] for simple sections of M. The 3-locally central element x centralises all these sections and does not influence their structure. Moreover the 2-structure in M is very small. So we nearly observe no interaction between the 2- and the 3-elements in M. It seems that a proof of Lyon's theorem for sections of M is similar to a new proof of his theorem. This should not be Part of this thesis.

5.3.4 Hypothesis (A weak $\mathcal{L}_3$-hypothesis)

Suppose that in every section of M the theorem of Lyons hold. More precisely that every simple section of M with a Sylow 2-subgroup isomorphic to T is isomorphic to $U_3(4)$.

Proof of the Main Theorem

Assuming Hypothesis 5.3.4 we immediately get a contradiction to Theorem 5.3.2. □

6 Conclusion and Outlook

The thesis describes the connection between the 2- and the 3-structure in finite simple groups with a large 3-local subgroup.
From Section 2.3 and Proposition 2.2.4 we even know that the information that there is almost no connection between the 2- and 3-structure determines the structure of a simple group.

We recall the Z_3^*-Theorem.

Theorem

Let G be a finite group and P be a Sylow 3-subgroup with an element $x \in P$ such that, whenever $x^g \in P$ for some $g \in G$, then $g \in C_G(x)$.
Then x is an element of $Z_3^*(G)$.

Let G be a minimal counterexample to the the theorem and let P be a Sylow 3-subgroup. Suppose further that x is an element of P such that, whenever $x^g \in P$ for some $g \in G$, then $g \in C_G(x)$ and such that $x \notin Z_3^*(G)$.
If we have no connection between 2- and 3-elements, then we can adopt our methods form this thesis.
In addition every proper subgroup H that contains x has the form $H = C_H(x) \cdot O_{3'}(H)$. Consequently the centraliser of the element x is also a large 3-local subgroup of G. This illustrates that the methods used in this thesis will also provides some results in an investigation of G.

Moreover we observe the following.

Lemma

Let G be a minimal counterexample to the the theorem and let P be a Sylow 3-subgroup. Suppose further that x is an element of P such that, whenever $x^g \in P$ for some $g \in G$, then $g \in C_G(x)$ and such that $x \notin Z_3^*(G)$.
Then $O_{3'}(G) = 1 = O_3(G)$ and there is an element $y \in P$ of order 3 such that $O_{3'}(C_G(y))$ is not centralised by x.

Proof

First we observe that $x^z \in P$ for all $z \in P$ and hence $x \in Z(P)$.
If we have $O_{3'}(G) \neq 1$, then the minimal choice of G and Lemma 2.2.2 (f) yield that $G/O_{3'}(G) = C_{G/O_{3'}(G)}(x) \cdot O_{3'}(G/O_{3'}(G)) = C_{G/O_{3'}(G)}(x)$. From Lemma 1.1.14 (a) we therefore obtain that $G = C_G(x) \cdot O_{3'}(G)$. This is a contradiction.
Moreover the condition on x implies that $C_{G/O_3(G)}(x) = C_G(x)/O_3(G)$. Consequently, if $O_3(G) \neq 1$, then we observe from the minimal choice of G and Lemma 2.2.2 (f) that $G/O_3(G) = C_{G/O_3(G)}(x) \cdot O_{3'}(G/O_3(G)) = (C_G(x) \cdot O_{3,3'}(G))/O_3(G)$. Finally Theorem 6.3.2 of [22], $O_{3'}(G) = 1$ and the fact that x centralises P implies that $x \in O_3(\langle x \rangle \cdot O_{3,3'}(G)) \trianglelefteq G$. Thus for all $g \in G$ we have $x^g \in O_3(\langle x \rangle \cdot O_{3,3'}(G)) \leq P$ and hence $x^g = x$. This again is a contradiction. For this reason the group G has no normal 3-subgroup.
We obtain further that x is not 3-locally central and hence there is a non-trivial subgroup R of $P \leq C_G(x)$ such that $C_G(R) \nleq C_G(x)$. Let $y \in R$ be an element of order 3. The minimality of G forces $N_G(R) \subseteq C_G(x) \cdot O_{3'}(N_G(R))$. From $[O_{3'}(N_G(R)), R] \leq O_{3'}(N_G(R)) \cap R = 1$ we deduce that $O_{3'}(N_G(R)) \leq C_G(R) \leq C_G(y) \subseteq C_G(x) \cdot O_{3'}(C_G(y))$. □

This result shows that we have to deal with $O_{3'}(C_G(y))$ for elements $y \in P$ of order 3.
We recommend signalizer functors for the prime 3 for this. For all elements z of order 3 in G let $\theta(z) = O_{3'}(C_G(z))$. If θ is a soluble signalizer functor in G and $r(G) \geq 3$, then we obtain from the Soluble Signalizer Functor Theorem 2.1.6 a non-trivial subgroup of G of odd order that is normalised by many subgroups.
Furthermore, if it is possible to show that $C_G(x)$ contains a Sylow 2-subgroup of G, then Lemma 1.3.7 provides the solubility of $\theta(y)$ for elements $y \in P$ of order 3.

As we saw in Chapter 5, small 2-ranks require some extra work. Here also the small 3-rank causes problems and needs extra ideas. In this special topic we hope to get help from the representation theory.

Another challenge is that $C_G(x)$ might not be a maximal subgroup of G. Moreover, possibly there are several maximal subgroups in G that contain x and have Sylow 3-subgroups that are not cyclic.
Here we recommend the Bender method. It helps to obtain uniqueness results that could replace Lemma 3.2.2 (f).

Altogether the author thinks that local methods, especially the methods presented in this thesis, provides results towards the Z_3^*-Theorem. But in their actual development stage they won't suffice to prove the Z_3^*-Theorem.

7 References

[1] J. L. Alperin. *Sylow Intersections and Fusion*. J. Algebra 6 (1967), 222-241.

[2] J. L. Alperin, D. Gorenstein. *THE MULTIPLICATORS OF CERTAIN SIMPLE GROUPS*. Proc. Amer. Math. Soc., 17 (1966), 515-519.

[3] M. Aschbacher. *Finite group theory*. Cambridge Studies in Advanced Mathematics, 10. Cambridge University Press, Cambridge, 1986.

[4] H. Bender. *Finite Groups with Dihedral Sylow 2-Subgroups*. J. Algebra 70 (1981), no. 1, 216-228.

[5] H. Bender. *The Brauer-Suzuki-Wall Theorem*. Illinois J. Math. 18 (1974), 229-235.

[6] H. Bender. *Transitive Gruppen gerader Ordnung, in denen jede Involution genau einen Punkt festläßt*. J. Algebra 17 (1971), 527-554.

[7] H. Bender, G. Glauberman. *Characters of Finite Groups with Dihedral Sylow 2-Subgroups*. J. Algebra 70 (1981), no. 1, 200-215.

[8] Y. Berkovich. *Groups of Prime Power Order. Volume 1*. De Gruyter Expositions in Mathematics, 46. Walter de Gruyter GmbH & Co. KG, Berlin, 2008.

[9] N. Blackburn, B. Huppert. *Finite Groups III*. Grundlehren der Mathematischen Wissenschaften, 242. AMD, 44. Springer-Verlag, Berlin-New York, 1982.

[10] R. Brauer. *Some Applications of the Theory of Blocks of Characters of Finite Groups. II*. J. Algebra 1 (1964), 307-334.

[11] H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, R. A. Wilson. *ATLAS of Finite Groups*. Maximal Subgroups and Ordinary Characters for Simple Groups. Clarendon press, Oxford, 1985.

[12] D. A. Craven, A. Glesser. *FUSION SYSTEMS ON SMALL p-GROUPS*. Trans. Amer. Math. Soc. 364 (2012), no. 11, 5945-5967.

[13] W. Feit, J. G. Thompson. *SOLVABILITY OF GROUPS OF ODD ORDER*. Pacific J. Math. 13 (1963), 775-1029.

[14] L. R. Fletcher, B. Stellmacher, W. B. Stewart. *ENDLICHE GRUPPEN, DIE KEIN ELEMENT DER ORDNUNG 6 ENTHALTEN*. Quart. J. Math. Oxford Ser. (2) 28 (1977), no. 110, 143-154.

[15] G. Glauberman. *A CHARACTERISTIC SUBGROUP OF A p-STABLE GROUP.* Canad. J. Math. 20 (1968), 1101-1135.

[16] G. Glauberman. *A SUFFICIENT CONDITION FOR p-STABILITY.* Proc. London Math. Soc., (2) 25 (1972), 253-287.

[17] G. Glauberman. *Central Elements in Core-Free Groups.* J. Algebra 4 (1966), 403-420.

[18] G. Glauberman. *FACTORIZATIONS IN LOCAL SUBGROUPS OF FINITE GROUPS*. Regional Conference Series in Mathematics, No. 33. American Mathematical Society, Providence, R.I., 1977.

[19] G. Glauberman. *ON SOLVABLE SIGNALIZER FUNCTORS IN FINITE GROUPS.* Proc. London Math. Soc., (3) 33 (1976), 1-27.

[20] D. M. Goldschmidt. *2-Fusion in finite groups.* The Annals of Mathematics, Second Series, Voll 99, no. 1 (1974), 70-117.

[21] D. M. Goldschmidt. *Strongly closed* 2*-subgroups of finite groups.* The Annals of Mathematics Second Series, Vol. 102, no. 3 (1975), 475-489.

[22] D. Gorenstein. *FINITE GROUPS*. Harper & Row, Publishers, New York-London 1968.

[23] D. Gorenstein, R. Lyons, R. Solomon. *The Classification of the Finite Simple Groups. Number 2. Part I. Chapter G. General Group Theory*. Mathematical Surveys and Monographs, 40.2. American Mathematical Society, Providence, RI, 1996.

[24] D. Gorenstein, R. Lyons, R. Solomon. *The Classification of the Finite Simple Groups. Number 3. Part I. Chapter A. Almost Simple $\mathcal{K}$-Groups*. Mathematical Surveys and Monographs, 40.3. American Mathematical Society, Providence, RI, 1998.

[25] D. Gorenstein, R. Lyons, R. Solomon. *The Classification of the Finite Simple Groups. Number 4. Part II, Chapters 1-4: Uniqueness Theorems*. Mathematical Surveys and Monographs, 40.4. American Mathematical Society, Providence, RI, 1999.

[26] D. Gorenstein, R. Lyons, R. Solomon. *The Classification of the Finite Simple Groups. Number 6. Part IV. The Special Odd Case*. Mathematical Surveys and Monographs, 40.6. American Mathematical Society, Providence, RI, 2005.

[27] D. Gorenstein, J. H. Walter *Centralizers of Involutions in Balanced Groups*. J. Algebra 20 (1972), 284-319.

[28] G. Higman. *Odd Characterisations of Finite Simple Groups*. University of Michigan 1968.

[29] B. Huppert. *Endliche Gruppen I. Die Grundlehren der Mathematischen Wissenschaften, Band 134*. Springer-Verlag, Berlin-New York 1967.

[30] H. Kurzweil, B. Stellmacher. *The Theory of Finite Groups. An Introduction*. Springer-Verlag, New York, 2004.

[31] R. Lyons. *A CHARACTERIZATION OF THE GROUP $U_3(4)$*. Trans. Amer. Math. Soc. 164 (1972), 371-387.

[32] P. Rowley. *3-LOCALLY CENTRAL ELEMENTS IN FINITE GROUPS*. Proc. London Math. Soc. (3) 43 (1981), no. 2, 357-384.

[33] I. Schur. *Über eine Klasse von endlichen Gruppen linearer Substitutionen*. In A. Brauer, H. Rohrbach *GESAMMELTE ABHANDLUNGEN. BAND I*. Springer-Verlag, Berlin-New York, 1973.

[34] B. Stellmacher. *Über endliche Gruppen mit einer 2-lokalen Untergruppe, die kein Element der Ordnung 6 enthält*. J. Algebra 50 (1978), no. 1, 175-189.

[35] M. Suzuki. *ON A CLASS OF DOUBLY TRANSITIVE GROUPS*. The Annals of Mathematics, Second Series, Vol. 75, no. 1 (Jan., 1962), 105-145.

[36] I. Toborg, R. Waldecker. *Finite simple $3'$-groups are cyclic or Suzuki groups*, submitted, 11 pages.

[37] R. Waldecker. *Isolated involutions in finite groups*. Memoirs of the American Mathematical Society, Vol 226, (2013), no. 1061.

[38] R. Waldecker. *Isolated involutions whose centraliser is soluble*. J. Algebra 321 (2009), no. 5, 1561-1592.

[39] R. A. Wilson. *The Finite Simple Groups*. Springer-Verlag, London, 2009.

[40] D. L. Winter. *THE AUTOMORPHISM GROUP OF AN EXTRASPECIAL p-GROUP*. Rocky Mountain J. Math. 2 (1972), no. 2, 159-168.

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2. I. Toborg. *M_8-freie Gruppen.* Diploma thesis (2010), 62 pages.
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