# Comparison Principles and Multiple Solutions for Nonlinear Elliptic Problems

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Herrn **Dipl.-Math. Patrick Winkert**

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Gutachter:

1. Prof. Dr. Siegfried Carl, Halle (Saale)
2. Prof. Dr. Dumitru Motreanu, Perpignan

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# Contents

*To Anna-Katharina*

# Chapter 1
# Introduction

This thesis is devoted to the study of elliptic problems with respect to comparison principles as well as multiple solutions. Our approach is mainly based on the method of sub- and supersolutions, which is an extremely useful and effective tool for proving existence and comparison results for a wide range of nonlinear elliptic boundary value problems. It implies the existence of solutions of variational equations within the interval $[\underline{u}, \overline{u}]$ for a given ordered pair $\underline{u}, \overline{u}$ of sub- and supersolutions, i.e., the method yields both existence and bounds of solutions. The sub- and supersolution technique for general classes of elliptic and parabolic variational equations is nowadays an important tool for qualitative analysis of elliptic and parabolic boundary value problems. The aim of this dissertation is the generalization and practical realization of the sub- and supersolution method to suitable classes of variational inequalities, hemivariational inequalities as well as certain mixed types of nonsmooth variational problems. For variational equations the terms sub- and supersolution are a natural generalization of the corresponding classical terms, whereas there are different possibilities to define sub- and supersolution in the case of nonsmooth variational problems. Concerning this, V. K. Le, D. Motreanu and S. Carl provided promising new approaches (cf. [17, 18, 19, 22, 23, 24, 26, 27, 29, 30, 31, 32, 33, 38, 39, 40, 86, 87]) which served as basis and starting point for this dissertation.

This thesis is organized as follows.
In **Chapter 2** we provide the mathematical background as it will be used in later chapters. The objective of the first two parts is the presentation of the notations of Sobolev spaces and operators of monotone type as well as the specification of their main properties. The third section briefly introduces the theory of nonsmooth analysis which will be used in Chapter 4 and 5. The main notion therein is the definition of Clarke's generalized gradient along with its characteristic features. The last section in this chapter lists some important tools like the Mountain-Pass Theorem or Vázquez's strong maximum principle which are needed in the proofs of our main results in later chapters.

In **Chapter 3** we study a class of nonlinear elliptic problems under nonlinear Neumann conditions involving the $p-$Laplacian. This chapter is divided into two parts. In the first one, the investigation of the problem

$$\begin{aligned} -\Delta_p u &= f(x,u) - |u|^{p-2}u && \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= \lambda |u|^{p-2}u + g(x,u) && \text{on } \partial\Omega, \end{aligned} \tag{1.0.1}$$

takes the center of our considerations. The domain $\Omega \subset \mathbb{R}^N$ is supposed to be bounded with a smooth boundary $\partial\Omega$ and the operator $-\Delta_p$ is the negative $p$-Laplacian. Moreover, $\frac{\partial u}{\partial \nu}$ denotes the outer normal derivative of $u$ with respect to $\partial\Omega$, $\lambda$ is a real parameter and the nonlinearities $f : \Omega \times \mathbb{R} \to \mathbb{R}$ as well as $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are some Carathéodory functions. For $u \in W^{1,p}(\Omega)$ (the usual Sobolev space), the functions $u \mapsto \lambda|u|^{p-2}u + g(x,u)$ on $\partial\Omega$ apply to the corresponding traces $u \mapsto \lambda|\gamma(u)|^{p-2}\gamma(u) + g(x,\gamma(u))$, where $\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ is the compact trace operator.

Neumann boundary value problems in the form (1.0.1) arise in different areas of pure and applied mathematics, for example in the theory of quasiregular and quasiconformal mappings in Riemannian manifolds with boundary (see [59],[114]), in the study of optimal constants for the Sobolev trace embedding (see [47], [64], [65], [63]) or in the theory of non-Newtonian fluids, flow through porous media, nonlinear elasticity, reaction diffusion problems and glaciology (see [6], [8], [7], [49]).

We prove the existence of at least three nontrivial solutions of problem (1.0.1). To be more precise, we obtain two extremal constant-sign solutions and one sign-changing solution by using truncation techniques and comparison principles for nonlinear elliptic differential inequalities. In our consideration, the nonlinearities $f$ and $g$ only need to be Carathéodory functions which are bounded on bounded sets whereby their growth does not need to be necessarily polynomial. We only require some growth properties at zero and infinity given by

$$\lim_{s\to 0} \frac{f(x,s)}{|s|^{p-2}s} = \lim_{s\to 0} \frac{g(x,s)}{|s|^{p-2}s} = 0, \qquad \lim_{|s|\to\infty} \frac{f(x,s)}{|s|^{p-2}s} = \lim_{|s|\to\infty} \frac{g(x,s)}{|s|^{p-2}s} = -\infty$$

and we suppose the existence of $\delta_f > 0$ such that $f(x,s)/|s|^{p-2}s \geq 0$ for all $0 < |s| \leq \delta_f$.

Our main idea is the construction of a positive and a negative ordered pair of sub- and supersolutions by using the solutions of the Neumann boundary value problems given by

$$\begin{aligned} -\Delta_p u &= -|u|^{p-2}u && \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= \lambda|u|^{p-2}u && \text{on } \partial\Omega, \end{aligned} \qquad \begin{aligned} -\Delta_p u &= -\varsigma|u|^{p-2}u + 1 && \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= 1 && \text{on } \partial\Omega, \end{aligned}$$

where $\varsigma > 1$ and $\lambda$ are real parameters. The problem on the left-hand side is the well-known Steklov eigenvalue problem which has a positive first eigenvalue $\lambda_1$ corresponding to its first eigenfunction $\varphi_1 > 0$ in $\overline{\Omega}$ (see [95] or [85]). Moreover, the second problem possesses a unique $C^1(\overline{\Omega})$-solution $e > 0$ in $\overline{\Omega}$. With the aid of these solutions and under the assumption that $\lambda > \lambda_1$, we prove that $A_1 = [\varepsilon\varphi_1, \vartheta e]$ (respectively, $A_2 = [-\vartheta e, -\varepsilon\varphi_1]$) is a positive (respectively, negative) ordered pair of sub- and supersolutions of problem (1.0.1) with a positive constant $\vartheta$ and $\varepsilon > 0$ sufficiently small. Based on this result, Theorem 3.1.8 proves the existence of a smallest positive solution $u_+$ in $[0, \vartheta e]$, respectively, the existence of a greatest negative solution $u_-$ in $[-\vartheta e, 0]$ of problem (1.0.1). A variational characterization of these extremal solutions is given in Section 3.1.4 and finally, Theorem 3.1.16 provides the existence of a nontrivial sign-changing solution $u_0$ of (1.0.1) satisfying $u_- \leq u_0 \leq u_+$ provided $\lambda > \lambda_2$. The proof is based on the Mountain-Pass Theorem, the Second Deformation Lemma and a variational characterization of the second eigenvalue $\lambda_2$ of the Steklov eigenvalue problem. We emphasize the regularity problem that arises in the proof of Proposition 3.1.11 ($C^1(\overline{\Omega})$ *versus* $W^{1,p}(\Omega)$ *local minimizers*) which at the end is solved by proving $L^\infty$-bounds where we make use of the Moser iteration technique along with real interpolation theory.

The second part of Chapter 3 extends our results to the more general problem: Find $u \in W^{1,p}(\Omega) \setminus \{0\}$ and constants $a \in \mathbb{R}$, $b \in \mathbb{R}$ such that

$$\begin{aligned} -\Delta_p u &= f(x,u) - |u|^{p-2}u && \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= a(u^+)^{p-1} - b(u^-)^{p-1} + g(x,u) && \text{on } \partial\Omega, \end{aligned} \tag{1.0.2}$$

where $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$ are the positive and negative parts of $u$, respectively. In case $a = b = \lambda$, problem (1.0.2) reduces to the Neumann boundary value problem given in (1.0.1). The existence of extremal constant-sign solutions in the intervals $[0, \vartheta_a e]$ and $[-\vartheta_b e, 0]$ will be shown by similar arguments provided $a, b > \lambda_1$, where $\vartheta_a$ and $\vartheta_b$ are positive constants depending on $a$ and $b$, respectively. However, the proof for the existence of a sign-changing solution proceeds in a different way . We obtain a nontrivial sign-changing solution of problem (1.0.2) provided $(a, b) \in \mathbb{R}^2_+$ is above the curve $\mathcal{C}$ of the Fučik spectrum constructed in [97] (see Figure 1.1).

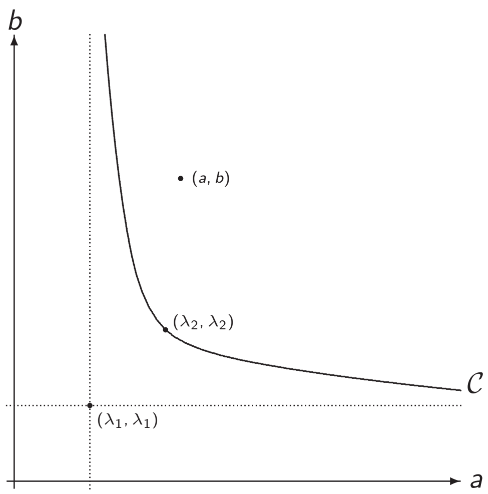


**Figure 1.1.** *Fučik Spectrum*

In addition, the applicability of our results is demonstrated by an example in which functions $f$ and $g$ are given satisfying all the assumptions. Furthermore, the graphs of these functions are presented.

**Chapter 4** is devoted to the study of quasilinear elliptic variational-hemivariational inequalities involving general Leray-Lions operators. Hemivariational inequalities have been introduced by P. D. Panagiotopoulos (cf. [105, 106]) to describe, e.g., problems in mechanics and engineering governed by nonconvex, possibly nonsmooth energy functionals (so-called superpotentials). This kind of energy functionals appears if nonmonotone, possibly multivalued constitutive laws are taken into account. Variational-hemivariational inequalities arise from hemivariational inequalities if in addition some constraints have to be taken into account.
Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$. We deal with the following elliptic variational-hemivariational inequality: Find $u \in K$ such that

$$\langle Au + F(u), v - u\rangle + \int_\Omega j_1^o(\cdot, u; v - u)dx + \int_{\partial\Omega} j_2^o(\cdot, \gamma u; \gamma v - \gamma u)d\sigma \geq 0, \tag{1.0.3}$$

for all $v \in K$, where $j_k^o(x, s; r), k = 1, 2$ denotes the generalized directional derivative of the locally Lipschitz functions $s \mapsto j_k(x, s)$ at $s$ in the direction $r$. The constraints are given by a closed convex subset $K \subset W^{1,p}(\Omega)$, and $A$ is a second-order quasilinear differential operator in divergence form of Leray-Lions type given by

$$Au(x) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, u(x), \nabla u(x)). \tag{1.0.4}$$

Moreover, the operator F stands for the Nemytskij operator associated with some Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$. The novelty is to provide existence and comparison results whereby only a local growth condition on Clarke's generalized gradient is required. More precisely, first we prove the existence of at least one solution between a given ordered pair of sub- and supersolutions. The proof is presented in Theorem 4.3.1. In order to obtain extremal solutions, we drop the $u-$dependence of the operator $A$. This result is stated in Theorem 4.4.3 whose proof is mainly based on an approach developed recently in [38].
In Section 4.5, we will extend our problem (1.0.3) to include discontinuous nonlinearities $f$ of the form $f : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$. The Nemytskij operator $F$ is given by $F(u)(x) = f(x, u(x), u(x), \nabla u(x))$ where we will allow $f$ to depend discontinuously on its third argument. An important tool in extending the previous results to discontinuous Nemytskij operators is a fixed point result given in Lemma 2.4.6. The existence of extremal solution of problem (1.0.3) is the main goal in Section 4.5. This will extend recent results obtained in [120].
In the last part of this chapter, the construction of sub- and supersolutions of (1.0.3) will be demonstrated in case $A$ is the negative $p$-Laplacian. Under additional conditions, the constructed sub- and supersolutions in Chapter 3 are also sub- and supersolutions of problem

(1.0.3) which is an amazing result. Finally, an example is given to show the applicability of our results.

The subject of **Chapter 5** are multivalued quasilinear elliptic problems of hemivariational type in all of $\mathbb{R}^N$. More precisely, we study elliptic differential inclusions of Clarke's gradient type in the form

$$Au + \partial j(\cdot, u) \ni 0 \quad \text{in } \mathcal{D}', \tag{1.0.5}$$

where $A$ is again a second-order quasilinear differential operator as in (1.0.4). The function $j : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is assumed to be measurable in $x \in \mathbb{R}^N$ for all $s \in \mathbb{R}$, and locally Lipschitz continuous in $s \in \mathbb{R}$ for almost all (a.a.) $x \in \mathbb{R}^N$. The multivalued function $s \mapsto \partial j(x, s)$ stands for Clarke's generalized gradient of the locally Lipschitz function $s \mapsto j(x, s)$ and is given by

$$\partial j(x, s) = \{\xi \in \mathbb{R} : j^o(x, s; r) \geq \xi r, \forall r \in \mathbb{R}\}, \tag{1.0.6}$$

for a.a. $x \in \mathbb{R}^N$. We denote by $\mathcal{D} = C_0^\infty(\mathbb{R}^N)$ the space of all infinitely differentiable functions with compact support in $\mathbb{R}^N$ and by $\mathcal{D}'$ its dual space.
This type of hemivariational inequalities has been studied by various authors on bounded domains. Concerning Dirichlet boundary conditions under local growth conditions, we refer e.g. to [34] and for hemivariational inequalities with measure data on the right-hand side see [25]. Single valued problems in the form (1.0.5) for Neumann boundary conditions of Clarke's gradient type are considered in [16]. In [15] the author discusses our problem (1.0.5) with a multivalued term in form of a state-dependent subdifferential in all of $\mathbb{R}^N$ which turns out to be a special case of problem (1.0.5). Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. We consider problem (1.0.5) under zero Dirichlet boundary values as well as $A = -\Delta_p$ which is the negative $p-$Laplacian. Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function. If $j$ is a primitive of $f$, meaning that

$$j(x, s) := \int_0^s f(x, t)dt,$$

then $s \mapsto j(x, s)$ is continuously differentiable and hence, $\partial j(x, s) = \{\partial j(x, s)/\partial s\} = \{f(x, s)\}$. Thus, problem (1.0.5) simplifies to the elliptic boundary value problem

$$u \in W_0^{1,p}(\Omega) : \quad -\Delta_p u + f(\cdot, u) = 0 \quad \text{in } W^{-1,q}(\Omega) \quad (1/p + 1/q = 1) \tag{1.0.7}$$

for which the method of sub- and supersolutions is well known (see [28, Chapter 3]). Comparison principles for general elliptic operators $A$, in particular for the negative $p$-Laplacian $-\Delta_p$ and Clarke's gradient $s \mapsto \partial j(x, s)$ satisfying a one-sided growth condition in the form

$$\xi_1 \leq \xi_2 + c_1(s_2 - s_1)^{p-1}, \tag{1.0.8}$$

for all $\xi_i \in \partial j(x, s_i), i = 1, 2$, for a.a. $x \in \Omega$, and for all $s_1, s_2$ with $s_1 < s_2$, are also studied in [28, Chapter 4]. Recently, a new comparison result for inclusions of the form (1.0.5) for

bounded domains without the condition (1.0.8) has been obtained in [39].
The main goal in this chapter is to show the existence of entire extremal solutions of (1.0.5) by applying the method of sub- and supersolutions without imposing any condition at infinity. Due to the unboundedness of the domain, standard variational methods cannot be applied. The novelty of our approach is on the one hand to obtain entire solutions, and on the other hand that Clarke's generalized gradient only needs to satisfy a natural growth condition without assuming any conditions as in (1.0.8). In the last section conditions are provided that ensure the existence of nontrivial positive solutions. We refer to the paper in [121] studying problem (1.0.5) in case $A = -\Delta_p$.

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# Chapter 2
# Mathematical Preliminaries

In this chapter, we provide the mathematical background as it will be used in later chapters.

## 2.1 Sobolev Spaces

This section is devoted to the introduction of Lebesgue and Sobolev spaces including their main properties.

### 2.1.1 Lebesgue Spaces

Let $\mathbb{R}^N$, $N \geq 1$, be equipped with the Lebesgue measure and let $\Omega \subset \mathbb{R}^N$ be a domain which means that $\Omega$ is an open and connected subset of $\mathbb{R}^N$. For $1 \leq p < \infty$, we denote by $L^p(\Omega)$ the class of all measurable functions $u : \Omega \to \mathbb{R}$ satisfying

$$\|u\|_{L^p(\Omega)} = \left( \int_\Omega |u|^p dx \right)^{\frac{1}{p}} < \infty,$$

for which $L^p(\Omega)$ becomes a Banach space. A measurable function $u : \Omega \to \mathbb{R}$ is called essentially bounded on $\Omega$ if there is a constant $C$ such that $|u(x)| \leq C$ almost everywhere (a.e.) on $\Omega$. The greatest lower bound of such constants $C$ is said to be the essential supremum of $|u|$ on $\Omega$. We put

$$\|u\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x \in \Omega} |u(x)|$$

and denote by $L^\infty(\Omega)$ the Banach space of all measurable functions $u$ satisfying $\|u\|_{L^\infty(\Omega)} < \infty$. Moreover, we also introduce the local $L^p$-spaces denoted by $L^p_{\text{loc}}(\Omega)$. A function $u$ belongs to $L^p_{\text{loc}}(\Omega)$ if it is measurable and

$$\int_K |u|^p dx < \infty$$

for every compact subset $K$ of $\Omega$. Here and also later on, we denote the Lebesgue measure of a measurable subset $\Omega \subset \mathbb{R}^N$ through

$$\text{meas}(\Omega) = |\Omega|.$$

The next theorems present some main results dealing with Lebesgue spaces and their qualities. As for the proofs, we refer to standard textbooks in real analysis and measure theory, for example [78, 111].

**Theorem 2.1.1** (Lebesgue's Dominated Convergence Theorem). *Suppose $(u_n)$ is a sequence in $L^1(\Omega)$ such that*

$$u(x) = \lim_{n\to\infty} u_n(x)$$

*exists a.e. on $\Omega$. If there is a function $g \in L^1(\Omega)$ such that, for almost all (a.a.) $x \in \Omega$, and for all $n = 1, 2, \dots,$*

$$|u_n(x)| \leq g(x),$$

*then $u \in L^1(\Omega)$ and*

$$\lim_{n\to\infty} \int_\Omega |u_n - u| dx = 0.$$

A reverse statement of Theorem 2.1.1 can be given as follows.

**Theorem 2.1.2.** *Let $u_n, u \in L^1(\Omega), n \in \mathbb{N}$, such that*

$$\lim_{n\to\infty} \int_\Omega |u_n - u| dx = 0.$$

*Then a subsequence $(u_{n_k})$ of $(u_n)$ exists with*

$$u_{n_k}(x) \to u(x) \quad \text{for a.a. } x \in \Omega.$$

**Theorem 2.1.3** (Fatou's Lemma). *Let $(u_n)$ be a sequence of measurable functions and let $g \in L^1(\Omega)$. If*

$$u_n \geq g \quad \text{a.e. on } \Omega,$$

*then we obtain*

$$\int_\Omega \liminf_{n\to\infty} u_n dx \leq \liminf_{n\to\infty} \int_\Omega u_n dx.$$

The dual space of $L^p(\Omega)$ is characterized in the following theorem.

**Theorem 2.1.4** (Dual Space). *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and let $\Phi$ be a linear continuous functional on $L^p(\Omega), 1 < p < \infty$. Then a uniquely defined function $g \in L^q(\Omega)$ exists with $q$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$ such that*

$$\langle \Phi, u \rangle = \int_\Omega g u dx \quad \text{for all } u \in L^p(\Omega),$$

*and*

$$\|\Phi\|_{(L^p(\Omega))^*} = \|g\|_{L^q(\Omega)}.$$

*If $\Phi$ is a linear continuous functional on $L^1(\Omega)$, then a uniquely defined function $g \in L^\infty(\Omega)$ exists such that*

$$\langle \Phi, u\rangle = \int_\Omega gudx \quad \textit{for all } u \in L^1(\Omega),$$

*and*

$$\|\Phi\|_{(L^1(\Omega))^*} = \|g\|_{L^\infty(\Omega)}.$$

Theorem 2.1.4 implies that the dual space of $L^p(\Omega)$ is isometrically isomorphic to $L^q(\Omega)$ for $1 \le p < \infty$. In case $p = 1$, we set $q = \infty$. Let us consider some important properties of $L^p$-spaces given in the next theorem.

**Theorem 2.1.5.** *Let $\Omega \subset \mathbb{R}^N$ be a domain.*

*(i) For $1 \le p < \infty$, the spaces $L^p(\Omega)$ are separable.*

*(ii) $L^\infty(\Omega)$ is not separable.*

*(iii) For $1 < p < \infty$, the spaces $L^p(\Omega)$ are reflexive.*

*(iv) $L^1(\Omega)$ and $L^\infty(\Omega)$ are not reflexive.*

*(v) For $1 < p < \infty$, the spaces $L^p(\Omega)$ are uniformly convex.*

### 2.1.2 Definition of Sobolev Spaces

The objective of this subsection is the study and characterization of Sobolev spaces. To this end, let $\alpha = (\alpha_1, \dots, \alpha_N)$ be a multi-index with nonnegative integers $\alpha_1, \dots, \alpha_N$. Its order is denoted by $|\alpha| = \alpha_1 + \cdots + \alpha_N$. We set $D_i = \frac{\partial}{\partial x_i}$, $i = 1, \dots, N$, and $D^\alpha u = D_1^{\alpha_1} \cdots D_N^{\alpha_N} u$, with $D^0 u = u$. Let $\Omega$ be a domain in $\mathbb{R}^N$ with $N \ge 1$. Then $w \in L^1_{\text{loc}}(\Omega)$ is said to be the $\alpha^{th}$ weak or generalized derivative of $u \in L^1_{\text{loc}}(\Omega)$ if and only if

$$\int_\Omega uD^\alpha \varphi dx = (-1)^{|\alpha|} \int_\Omega w\varphi dx, \quad \text{for all } \varphi \in C_0^\infty(\Omega),$$

holds, where $C_0^\infty(\Omega)$ denotes the space of infinitely differentiable functions with compact support in $\Omega$. The generalized derivative $w = D^\alpha u$ is unique up to a change of the values of $w$ on a set of Lebesgue measure zero.

**Definition 2.1.6.** *Let $1 \le p \le \infty$ and $m = 0, 1, 2, \dots$. The Sobolev space $W^{m,p}(\Omega)$ is the space of all functions $u \in L^p(\Omega)$ which have generalized derivatives up to order $m$ such that $D^\alpha u \in L^p(\Omega)$ for all $\alpha$ with $|\alpha| \le m$. We set $W^{0,p}(\Omega) = L^p(\Omega)$ if $m = 0$.*

The space $W^{m,p}(\Omega)$ is a Banach space with respect to the norms

$$\|u\|_{W^{m,p}(\Omega)} = \left( \sum_{|\alpha| \leq m} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}},$$

if $1 \leq p < \infty$, and

$$\|u\|_{W^{m,\infty}(\Omega)} = \max_{|\alpha| \leq m} \|D^\alpha u\|_{L^\infty(\Omega)},$$

if $p = \infty$.

**Definition 2.1.7.** *$W_0^{m,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$.*

Notice that $W_0^{m,p}(\Omega)$ becomes a Banach space with the norm $\|\cdot\|_{W^{m,p}(\Omega)}$.
The definition of the regularity of boundaries reads as follows.

**Definition 2.1.8.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with boundary $\partial\Omega$. The boundary $\partial\Omega$ is of class $C^{k,\lambda}, k \in \mathbb{N}_0, \lambda \in (0,1]$ if there are $m \in \mathbb{N}$ Cartesian coordinate systems $C_j, j = 1, \dots, m$,*

$$C_j = (x_{j,1}, \dots, x_{j,N-1}, x_{j,N}) = (x'_j, x_{j,N})$$

*and real numbers $\alpha$, $\beta > 0$, as well as $m$ functions $a_j$ with*

$$a_j \in C^{k,\lambda}([-\alpha, \alpha]^{N-1}), j = 1, \dots, m,$$

*such that the sets defined by*

$$\begin{aligned} \Lambda^j &= \{(x'_j, x_{j,N}) \in \mathbb{R}^N : |x'_j| \leq \alpha, x_{j,N} = a_j(x'_j)\}, \\ V^j_+ &= \{(x'_j, x_{j,N}) \in \mathbb{R}^N : |x'_j| \leq \alpha, a_j(x'_j) < x_{j,N} < a_j(x'_j) + \beta\}, \\ V^j_- &= \{(x'_j, x_{j,N}) \in \mathbb{R}^N : |x'_j| \leq \alpha, a_j(x'_j) - \beta < x_{j,N} < a_j(x'_j)\}, \end{aligned}$$

*possess the following properties:*

$$\Lambda^j \subset \partial\Omega, \quad V^j_+ \subset \Omega, \quad V^j_- \subset \mathbb{R}^N \setminus \Omega, \quad j = 1, \dots, m,$$

*and*

$$\bigcup_{j=1}^m \Lambda^j = \partial\Omega.$$

The boundary $\partial\Omega$ is said to be a Lipschitz boundary if $\partial\Omega \in C^{0,1}$ which means that $\partial\Omega$ can be locally characterized by a graph of a Lipschitz continuous function. Now, we summarize some basic properties of Sobolev spaces stated in the next theorem. The proofs can be found in [74].

**Theorem 2.1.9.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $N \geq 1$. Then we have the following:*

(i) *$W^{m,p}(\Omega)$ is separable for $1 \leq p < \infty$.*

(ii) *$W^{m,p}(\Omega)$ is reflexive for $1 < p < \infty$.*

(iii) *Let $1 \leq p < \infty$. Then $C^\infty(\Omega) \cap W^{m,p}(\Omega)$ is dense in $W^{m,p}(\Omega)$, and if $\partial\Omega$ is a Lipschitz boundary, then $C^\infty(\overline{\Omega})$ is dense in $W^{m,p}(\Omega)$, where $C^\infty(\Omega)$ and $C^\infty(\overline{\Omega})$ are the spaces of infinitely differentiable functions in $\Omega$ and $\overline{\Omega}$, respectively (cf. [74]).*

Let us briefly recall the definition of an embedding operator. Let $X, Y$ be two normed linear spaces satisfying $X \subset Y$. The operator $i : X \to Y$ defined by $i(u) = u$ for all $u \in X$ is said to be the embedding operator of $X$ into $Y$. The space $X$ is continuously (respectively, compactly) embedded in $Y$ if the embedding operator $i : X \to Y$ is continuous (respectively, compact). The following theorem presents the important Sobolev Embedding Theorem whose proof can be found, e.g. in [74, 123].

**Theorem 2.1.10** (Sobolev Embedding Theorem). *Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded domain with Lipschitz boundary $\partial\Omega$. Then the following holds:*

(i) *If $mp < N$, then the space $W^{m,p}(\Omega)$ is continuously embedded in $L^{p^*}(\Omega)$, $p^* = Np/(N - mp)$, and compactly embedded in $L^q(\Omega)$ for any $q$ with $1 \leq q < p^*$.*

(ii) *If $0 \leq k < m - \frac{N}{p} < k + 1$, then the space $W^{m,p}(\Omega)$ is continuously embedded in $C^{k,\lambda}(\overline{\Omega})$, $\lambda = m - \frac{N}{p} - k$, and compactly embedded in $C^{k,\lambda'}(\overline{\Omega})$ for any $\lambda' < \lambda$.*

(iii) *Let $1 \leq p < \infty$, then the embeddings*

$$L^p(\Omega) \supset W^{1,p}(\Omega) \supset W^{2,p}(\Omega) \supset \cdots$$

*are compact.*

The space $C^{k,\lambda}(\overline{\Omega})$ stands for the Hölder space introduced for example in [74]. In order to define Sobolev functions on the boundary, we make use of the important Trace Theorem.

**Theorem 2.1.11** (Trace Theorem). *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz ($C^{0,1}$) boundary $\partial\Omega$, $N \geq 1$, and $1 \leq p < \infty$. Then exactly one continuous linear operator exists*

$$\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$$

*such that:*

(i) *$\gamma(u) = u|_{\partial\Omega}$ if $u \in C^1(\overline{\Omega})$.*

(ii) *$\|\gamma(u)\|_{L^p(\partial\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}$ with $C$ depending only on $p$ and $\Omega$.*

(iii) *If $u \in W^{1,p}(\Omega)$, then $\gamma(u) = 0$ in $L^p(\partial\Omega)$ if and only if $u \in W_0^{1,p}(\Omega)$.*

We call $\gamma(u)$ the trace (or generalized boundary function) of $u$ on $\partial\Omega$. It should be pointed out that the trace operator

$$\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$$

mentioned in Theorem 2.1.11 is not surjective. Indeed, there exist functions $\vartheta \in L^p(\partial\Omega)$ which are not the traces of functions $u$ from $W^{1,p}(\Omega)$. The next result provides the surjective result (see [83, Theorem 6.8.13, Theorem 6.9.2]).

**Theorem 2.1.12.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$, $N \geq 1$, and $1 < p < \infty$. Then*

$$\gamma(W^{1,p}(\Omega)) = W^{1-\frac{1}{p},p}(\partial\Omega).$$

As we know, the trace operator is compact due to the following theorem.

**Theorem 2.1.13.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$, $N \geq 1$.*

*(i) If $1 < p < N$, then*

$$\gamma : W^{1,p}(\Omega) \to L^q(\partial\Omega)$$

*is completely continuous for any $q$ with $1 \leq q < \frac{Np-p}{N-p}$.*

*(ii) If $p \geq N$, then for any $q \geq 1$,*

$$\gamma : W^{1,p}(\Omega) \to L^q(\partial\Omega)$$

*is completely continuous.*

We refer to [83] verifying the proof of the theorem.

### 2.1.3 Chain Rule and Lattice Structure

In this subsection, we suppose that $\Omega \subset \mathbb{R}^N$ is a bounded domain with a Lipschitz boundary $\partial\Omega$. The important chain rule is stated in the next two lemmas.

**Lemma 2.1.14** (Chain Rule). *Let $f \in C^1(\mathbb{R})$ and $\sup_{s\in\mathbb{R}} |f'(s)| < \infty$. Let $1 \leq p < \infty$ and $u \in W^{1,p}(\Omega)$. Then the composite function $f \circ u \in W^{1,p}(\Omega)$ and its generalized derivatives are given by*

$$D_i(f \circ u) = (f' \circ u)D_i u, \quad i = 1, \dots, N.$$

**Lemma 2.1.15** (Generalized Chain Rule). *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and piecewise continuously differentiable with $\sup_{s\in\mathbb{R}} |f'(s)| < \infty$ and $u \in W^{1,p}(\Omega), 1 \leq p < \infty$. Then $f \circ u \in W^{1,p}(\Omega)$, and its generalized derivative is given by*

$$D_i(f \circ u)(x) = \begin{cases} f'(u(x))D_i u(x) & \text{if } f \text{ is differentiable at } u(x), \\ 0 & \text{otherwise.} \end{cases}$$

In order to extend the chain rule to Lipschitz continuous functions $f$, we refer to [74, 123]. The Sobolev space $W^{1,p}(\Omega)$ satisfies the so-called lattice structure which yields the following result (see [77, Theorem 1.20]).

**Lemma 2.1.16** (Lattice Structure for $W^{1,p}(\Omega)$). *Let $u, v \in W^{1,p}(\Omega), 1 \le p < \infty$. Then* $\max\{u, v\}$ *and* $\min\{u, v\}$ *are in $W^{1,p}(\Omega)$ with generalized derivatives*

$$D_i \max\{u, v\}(x) = \begin{cases} D_i u(x) & \text{if } u(x) > v(x), \\ D_i v(x) & \text{if } v(x) \ge u(x), \end{cases}$$

$$D_i \min\{u, v\}(x) = \begin{cases} D_i u(x) & \text{if } u(x) < v(x), \\ D_i v(x) & \text{if } v(x) \le u(x). \end{cases}$$

From [77, Lemma 1.22], we obtain the next lemma.

**Lemma 2.1.17.** *If $(u_j), (v_j) \subset W^{1,p}(\Omega), 1 \le p < \infty$, are such that $u_j \to u$ and $v_j \to v$ in $W^{1,p}(\Omega)$, then* $\min\{u_j, v_j\} \to \min\{u, v\}$ *and* $\max\{u_j, v_j\} \to \max\{u, v\}$ *in $W^{1,p}(\Omega)$ as $j \to \infty$.*

As a consequence of Lemma 2.1.17, truncation operators defined on $W^{1,p}(\Omega)$ are bounded and continuous.

**Lemma 2.1.18.** *Let $\underline{u}, \overline{u} \in W^{1,p}(\Omega)$ satisfy $\underline{u} \le \overline{u}$, and let $T$ be the truncation operator defined by*

$$Tu(x) = \begin{cases} \overline{u}(x) & \text{if } u(x) > \overline{u}(x), \\ u(x) & \text{if } \underline{u}(x) \le u(x) \le \overline{u}(x), \\ \underline{u}(x) & \text{if } u(x) < \underline{u}(x). \end{cases}$$

*Then $T$ is a bounded continuous mapping from $W^{1,p}(\Omega)$ (respectively, $L^p(\Omega)$) into itself.*

The lattice structure also holds for the subspace $W_0^{1,p}(\Omega) \subset W^{1,p}(\Omega)$, which is proven in [77].

**Lemma 2.1.19** (Lattice Structure for $W_0^{1,p}(\Omega)$). *If $u, v \in W_0^{1,p}(\Omega)$, then* $\max\{u, v\}$ *and* $\min\{u, v\}$ *are in $W_0^{1,p}(\Omega)$.*

In view of Lemma 2.1.19, a partial ordering of traces on $\partial\Omega$ is defined in the following way.

**Definition 2.1.20.** *Let $u \in W^{1,p}(\Omega), 1 \le p < \infty$. Then $u \le 0$ on $\partial\Omega$ if $u^+ \in W_0^{1,p}(\Omega)$.*

### 2.1.4 Some Inequalities

In later chapters, we make use of some well-known inequalities given in this subsection. We refer to standard textbooks (see [60, 83, 123]) reproducing the proofs of the inequalities.

**Young's Inequality**

Let $1 < p, q < \infty$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$ and let $a, b \geq 0$. Then it holds

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

**Young's Inequality with Epsilon**

Let $1 < p, q < \infty$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$ and let $a, b, \varepsilon \geq 0$. Then it holds

$$ab \leq \varepsilon a^p + C(\varepsilon) b^q$$

with the positive constant $C(\varepsilon) = \left(\frac{1}{\varepsilon p}\right)^{\frac{q}{p}} \frac{1}{q}$.

**Monotonicity Inequality**

Let $1 < p < \infty$ and consider the vector-valued function $a : \mathbb{R}^N \to \mathbb{R}^N$ defined by

$$a(\xi) = |\xi|^{p-2}\xi \quad \text{for } \xi \neq 0 \text{ and } a(0) = 0.$$

If $1 < p < 2$, then we obtain

$$(a(\xi) - a(\xi')) \cdot (\xi - \xi') > 0 \text{ for all } \xi, \xi' \in \mathbb{R}^N, \xi \neq \xi'.$$

If $2 \leq p < \infty$, then a constant $c > 0$ exists such that

$$(a(\xi) - a(\xi')) \cdot (\xi - \xi') \geq c|\xi - \xi'|^p \text{ for all } \xi \in \mathbb{R}^N.$$

**Hölder's Inequality**

Let $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $u \in L^p(\Omega), v \in L^q(\Omega)$, then we get

$$\int_\Omega |uv| dx \leq \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}.$$

**Minkowski's Inequality**

Let $1 \leq p \leq \infty$ and $u, v \in L^p(\Omega)$, then it holds

$$\|u + v\|_{L^p(\Omega)} \leq \|u\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)}.$$

## 2.2 Operators of Monotone Type

In this section, we give some results about pseudomonotone and monotone operators acting from $X$ into $X^*$.

### 2.2.1 Main Theorem on Pseudomonotone Operators

First, we denote by $X$ a real, reflexive Banach space equipped with the norm $\|\cdot\|$. Its dual space is identified by $X^*$ and $\langle \cdot, \cdot \rangle$ stands for the duality pairing between them. In order to avoid misunderstandings, we recall that the notation of the norm convergence in $X$ and $X^*$ is denoted by $\to$ and the weak convergence by $\rightharpoonup$.

**Definition 2.2.1.** *Let $A : X \to X^*$ be given. Then $A$ is said to be*

*(i) continuous iff $u_n \to u$ implies $Au_n \to Au$.*

*(ii) weakly continuous iff $u_n \rightharpoonup u$ implies $Au_n \rightharpoonup Au$.*

*(iii) demicontinuous iff $u_n \to u$ implies $Au_n \rightharpoonup Au$.*

*(iv) hemicontinuous iff the real function $t \to \langle A(u + tv), w\rangle$ is continuous on $[0, 1]$ for all $u, v, w \in X$.*

*(v) completely continuous iff $u_n \rightharpoonup u$ implies $Au_n \to Au$.*

*(vi) bounded iff $A$ maps bounded sets into bounded sets.*

*(vii) coercive iff $\displaystyle\lim_{\|u\|\to\infty} \frac{\langle Au, u\rangle}{\|u\|} = +\infty$.*

Next, we recall the definition of operators of monotone type.

**Definition 2.2.2.** *Let $A : X \to X^*$ be given. Then $A$ is called*

*(i) monotone iff $\langle Au - Av, u - v\rangle \geq 0$ for all $u, v \in X$ with $u \neq v$.*

*(ii) strictly monotone iff $\langle Au - Av, u - v\rangle > 0$ for all $u, v \in X$ with $u \neq v$.*

*(iii) strongly monotone iff there is a constant $c > 0$ such that $\langle Au - Av, u - v\rangle \geq c\|u - v\|^2$ for all $u, v \in X$.*

*(iv) uniformly monotone iff $\langle Au - Av, u - v\rangle \geq a(\|u - v\|)\|u - v\|$ for all $u, v \in X$ where $a : [0, \infty) \to [0, \infty)$ is strictly increasing with $a(0) = 0$ and $a(s) \to +\infty$ as $s \to \infty$.*

*(v) pseudomonotone iff $u_n \rightharpoonup u$ and $\limsup_{n\to\infty}\langle Au_n, u_n - u\rangle \leq 0$ implies $\langle Au, u - w\rangle \leq \displaystyle\liminf_{n\to\infty}\langle Au_n, u_n - w\rangle$ for all $w \in X$.*

*(vi) to satisfy $(S_+)$-condition iff $u_n \rightharpoonup u$ and $\displaystyle\limsup_{n\to\infty}\langle Au_n, u_n - u\rangle \leq 0$ imply $u_n \to u$.*

An equivalent definition for the pseudomonotonicity is given as follows.

**Definition 2.2.3.** *The operator $A : X \to X^*$ is called pseudomonotone iff $u_n \rightharpoonup u$ and $\displaystyle\limsup_{n\to\infty}\langle Au_n, u_n - u\rangle \leq 0$ implies $Au_n \rightharpoonup Au$ and $\langle Au_n, u_n\rangle \to \langle Au, u\rangle$.*

The next result plays an important role in our considerations. The proof can be found for example in [123, Proposition 27.6].

**Lemma 2.2.4.** *Let $A, B : X \to X^*$ be given operators on the real reflexive Banach space $X$. Then it holds:*

*(i) If $A$ is monotone and hemicontinuous, then $A$ is pseudomonotone.*

*(ii) If $A$ is completely continuous, then $A$ is pseudomonotone.*

*(iii) If $A$ and $B$ are pseudomonotone, then $A + B$ is pseudomonotone.*

Due to Brézis, the main theorem on pseudomonotone operators reads in the following way (see [123, Theorem 27.A]).

**Theorem 2.2.5** (Main Theorem on Pseudomonotone Operators)**.** *Let $X$ be a real, reflexive Banach space and let $A : X \to X^*$ be a pseudomonotone, bounded, and coercive operator, and $b \in X^*$. Then there exists a solution of the equation $Au = b$.*

### 2.2.2 Leray–Lions Operators

We introduce the so called Leray–Lions operators which stand for an important class of elliptic operators. For more details we refer to [90] and [118].

**Definition 2.2.6** (Leray–Lions Operator)**.** *Let $X$ be a real, reflexive Banach space. We say that $A : X \to X^*$ is a Leray–Lions operator if it is bounded and satisfies*

$$Au = \mathcal{A}(u, u), \quad \text{for } u \in X,$$

*where $\mathcal{A} : X \times X \to X^*$ has the following properties:*

*(i) For any $u \in X$, the mapping $v \mapsto \mathcal{A}(u, v)$ is bounded and hemicontinuous from $X$ to its dual $X^*$ with*

$$\langle \mathcal{A}(u, u) - \mathcal{A}(u, v), u - v \rangle \geq 0, \quad \text{for } v \in X.$$

*(ii) For any $v \in X$, the mapping $u \mapsto \mathcal{A}(u, v)$ is bounded and hemicontinuous from $X$ to its dual $X^*$.*

*(iii) For any $v \in X$, $\mathcal{A}(u_n, v)$ converges weakly to $\mathcal{A}(u, v)$ in $X^*$ if $(u_n) \subset X$ such that $u_n \rightharpoonup u$ in $X$ and*

$$\langle \mathcal{A}(u_n, u_n) - \mathcal{A}(u_n, u), u_n - u \rangle \to 0.$$

*(iv) For any $v \in X$, $\langle \mathcal{A}(u_n, v), u_n \rangle$ converges to $\langle F, u \rangle$ if $(u_n) \subset X$ such that $u_n \rightharpoonup u$ in $X$ and $\mathcal{A}(u_n, v) \rightharpoonup F$ in $X^*$.*

An important result is the following.

**Theorem 2.2.7.** *Every Leray–Lions operator $A : X \to X^*$ is pseudomonotone.*

The proof of this theorem can be found in [118]. Let us consider the mapping properties of superposition operators which are also called Nemytskij operators.

**Definition 2.2.8** (Nemytskij Operator). *Let $\Omega \subset \mathbb{R}^N, N \geq 1$ be a nonempty measurable set and let $f : \Omega \times \mathbb{R}^N \to \mathbb{R}, m \geq 1$ and $u : \Omega \to \mathbb{R}^m$ be a given function. Then the superposition or Nemytskij operator $F$ assigns $u \mapsto f \circ u$ that means $F$ is given by*

$$Fu(x) = (f \circ u)(x) = f(u(x)) \quad \text{for } x \in \Omega.$$

**Definition 2.2.9** (Carathéodory Function). *Let $\Omega \subset \mathbb{R}^N, N \geq 1$ be a nonempty measurable set and let $f : \Omega \times \mathbb{R}^m \to \mathbb{R}, m \geq 1$. The function $f$ is said to be a Carathéodory function if the following two conditions are fulfilled:*

*(i) $x \mapsto f(x, s)$ is measurable in $\Omega$ for all $s \in \mathbb{R}^m$.*

*(ii) $s \mapsto f(x, s)$ is continuous on $\mathbb{R}^m$ for a.a. $x \in \Omega$.*

**Lemma 2.2.10.** *Let $f : \Omega \times \mathbb{R}^m \to \mathbb{R}, m \geq 1$ be a Carathéodory function satisfying a growth condition of the form*

$$|f(x, s)| \leq k(x) + c \sum_{i=1}^{m} |s_i|^{\frac{p_i}{q}}$$

*with some positive constant $c$ and some function $k \in L^q(\Omega)$ and $1 \leq q, p_i < \infty$ for all $i = 1, \ldots, m$. Then the Nemytskij operator $F$ defined by*

$$Fu(x) = f(x, u_1(x), \ldots, u_m(x))$$

*is continuous and bounded from $L^{p_1}(\Omega) \times \cdots \times L^{p_m}(\Omega)$ into $L^q(\Omega)$. Here $u$ denotes the vector function $u = (u_1, \ldots, u_m)$. Moreover, we have*

$$\|Fu\|_{L^q(\Omega)} \leq c \left( \|k\|_{L^q(\Omega)} + \sum_{i=1}^{m} \|u_i\|_{L^{p_i}(\Omega)}^{\frac{p_i}{q}} \right).$$

**Definition 2.2.11.** *Let $\Omega \subset \mathbb{R}^N, N \geq 1$ be a nonempty measurable set. A function $f : \Omega \times \mathbb{R}^m \to \mathbb{R}, m \geq 1$, is called superpositionally measurable (or sup-measurable), if the function $x \mapsto Fu(x)$ is measurable in $\Omega$ whenever the component functions $u_i : \Omega \to \mathbb{R}$ of $u = (u_1, \ldots, u_m)$ are measurable.*

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$. We consider the second order quasilinear differential operator in divergence form given by

$$A_1 u(x) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, u(x), \nabla u(x)),$$

and let $A_0$ denote the operator defined by

$$A_0u(x) = a_0(x, u(x), \nabla u(x)).$$

Let $1 < p < \infty$, $\frac{1}{p}+\frac{1}{q} = 1$, and assume for the coefficients $a_i : \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$, $i = 0, 1, \dots, N$ the following conditions.

(H1) Each $a_i(x, s, \xi)$ satisfies Carathéodory conditions, i.e., is measurable in $x \in \Omega$ for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and continuous in $(s, \xi)$ for a.a. $x \in \Omega$. Furthermore, a constant $c_0 > 0$ and a function $k_0 \in L^q(\Omega)$ exist such that

$$|a_i(x, s, \xi)| \le k_0(x) + c_0(|s|^{p-1} + |\xi|^{p-1}),$$

for a.a. $x \in \Omega$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$, where $|\xi|$ denotes the Euclidian norm of the vector $\xi$.

(H2) The coefficients $a_i$ satisfy a monotonicity condition with respect to $\xi$ in the form

$$\sum_{i=1}^{N}(a_i(x, s, \xi) - a_i(x, s, \xi'))(\xi_i - \xi_i') > 0,$$

for a.a. $x \in \Omega$, for all $s \in \mathbb{R}$, and for all $\xi, \xi' \in \mathbb{R}^N$ with $\xi \ne \xi'$.

(H3) A constant $c_1 > 0$ and a function $k_1 \in L^1(\Omega)$ exist such that

$$\sum_{i=1}^{N} a_i(x, s, \xi)\xi_i \ge c_1|\xi|^p - k_1(x),$$

for a.a. $x \in \Omega$, for all $s \in \mathbb{R}$, and for all $\xi \in \mathbb{R}^N$.

Let $V$ be a closed subspace of $W^{1,p}(\Omega)$ satisfying $W_0^{1,p}(\Omega) \subseteq V \subseteq W^{1,p}(\Omega)$. Due to (H1) the operators $A_1$ and $A_0$ generate mappings from $V$ into its dual space defined by

$$\langle A_1u, \varphi\rangle = \int_\Omega \sum_{i=1}^{N} a_i(x, u, \nabla u)\frac{\partial\varphi}{\partial x_i}dx, \quad \langle A_0u, \varphi\rangle = \int_\Omega a_0(x, u, \nabla u)\varphi dx.$$

We set $A = A_1 + A_0$. The next theorem provides some properties of the operators $A, A_1$ and $A_0$ (see e.g. [123]).

**Theorem 2.2.12.**

*(i) If (H1) is satisfied, then the mappings $A, A_1, A_0 : V \to V^*$ are continuous and bounded.*

*(ii) If (H1) and (H2) are satisfied, then $A : V \to V^*$ is pseudomonotone.*

*(iii) If (H1), (H2), and (H3) are satisfied, then $A$ has the $(S_+)$-property.*

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. A prototype of a nonlinear monotone elliptic operator is the negative $p$-Laplacian $-\Delta_p, 1 < p < \infty$, defined by

$$-\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2}\nabla u), \quad \text{where} \quad \nabla u = (\partial u/\partial x_1, \dots, \partial u/\partial x_N).$$

The coefficients $a_i, i = 1, \dots, N$ are given by

$$a_i(x, s, \xi) = |\xi|^{p-2}\xi_i.$$

Thus, hypothesis (H1) is satisfied with $k_0 = 0$ and $c_0 = 1$. Hypothesis (H2) is a consequence of the inequalities from the vector-valued function $\xi \mapsto |\xi|^{p-2}\xi$ (see Section 2.1) and (H3) is satisfied with $c_1 = 1$ and $k_1 = 0$. In Chapter 3, we make use of the $p-$Laplacian which has the following characteristics.

**Lemma 2.2.13.** *Let $V$ be a closed subspace of $W^{1,p}(\Omega)$ such that $W_0^{1,p}(\Omega) \subseteq V \subseteq W^{1,p}(\Omega)$. Then it holds:*

*(i) $-\Delta_p : V \to V^*$ is continuous, bounded, pseudomonotone and has the $(S_+)-$property.*

*(ii) $-\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,q}(\Omega)$ is*

*(a) strictly monotone if $1 < p < \infty$.*

*(b) strongly monotone if $p = 2$ (which is the well-known Laplace operator).*

*(c) uniformly monotone if $2 < p < \infty$.*

### 2.2.3 Multivalued Pseudomonotone Operators

This section provides some results about pseudomonotone multivalued operators. For completeness we refer to the monographs [104] and [123]. First, we start with some definitions. Let $X$ be a real, reflexive Banach space and $X^*$ denotes its dual space.

**Definition 2.2.14.** *Let $A : X \to 2^{X^*}$ be a multivalued mapping meaning that to each $u \in X$ there is assigned a subset $A(u)$ of $X^*$ which may be empty if $u \notin D(A)$ where $D(A)$ is the domain of $A$ given by*

$$D(A) = \{u \in X : A(u) \neq \emptyset\}.$$

*The graph of $A$ denoted by $\mathrm{Gr}(A)$ is given by*

$$\mathrm{Gr}(A) = \{(u, u^*) \in X \times X^* : u^* \in A(u)\}.$$

**Definition 2.2.15.** *The mapping $A : X \to 2^{X^*}$ is said to be*

*(i) monotone iff*

$$\langle u^* - v^*, u - v\rangle \geq 0 \quad \text{for all } (u, u^*), (v, v^*) \in \mathrm{Gr}(A).$$

*(ii) strictly monotone iff*

$$\langle u^* - v^*, u - v\rangle > 0 \quad \text{for all } (u, u^*), (v, v^*) \in \mathrm{Gr}(A) \text{ with } u \neq v.$$

*(iii) maximal monotone iff $A$ is monotone and there is no monotone mapping $\tilde{A} : X \to 2^{X^*}$ such that $\mathrm{Gr}(A)$ is a proper subset of $\mathrm{Gr}(\tilde{A})$ which is equivalent to the following implication*

$$(u, u^*) \in X \times X^* : \quad \langle u^* - v^*, u - v\rangle \geq 0 \quad \text{for all } (v, v^*) \in \mathrm{Gr}(A)$$

*implies $(u, u^*) \in \mathrm{Gr}(A)$.*

Note that the notions of strongly and uniformly monotone multivalued operators are defined in a similar way as for single-valued operators. The definition of a pseudomonotone operator reads as follows.

**Definition 2.2.16.** *The operator $A : X \to 2^{X^*}$ is called pseudomonotone if the following conditions are satisfied.*

*(i) The set $A(u)$ is nonempty, bounded, closed and convex for all $u \in X$.*

*(ii) $A$ is upper semicontinuous from each finite dimensional subspace of $X$ to the weak topology on $X^*$.*

*(iii) If $(u_n) \subset X$ with $u_n \rightharpoonup u$, and if $u_n^* \in A(u_n)$ such that*

$$\limsup \langle u_n^*, u_n - u\rangle \leq 0,$$

*then for each element $v \in X$ there exists $u^*(v) \in A(u)$ with*

$$\liminf \langle u_n^*, u_n - v\rangle \geq \langle u^*(v), u - v\rangle.$$

The next proposition provides a sufficient condition to prove the pseudomonotonicity of multivalued operators and is an important part of our argumentations. The proof is presented for example in [104, Chapter 2].

**Proposition 2.2.17.** *Let $X$ be a reflexive Banach space, and assume that $A : X \to 2^{X^*}$ satisfies the following conditions:*

*(i) For each $u \in X$ we have that $A(u)$ is a nonempty, closed and convex subset of $X^*$;*

*(ii) $A : X \to 2^{X^*}$ is bounded;*

*(iii) If $u_n \rightharpoonup u$ in $X$ and $u_n^* \rightharpoonup u^*$ in $X^*$ with $u_n^* \in A(u_n)$ and if $\limsup \langle u_n^*, u_n - u\rangle \leq 0$, then $u^* \in A(u)$ and $\langle u_n^*, u_n\rangle \to \langle u^*, u\rangle$.*

*Then the operator $A : X \to 2^{X^*}$ is pseudomonotone.*

The invariance of pseudomonotonicity under addition is stated in the next theorem.

**Theorem 2.2.18.** *Let $A, A_i : X \to 2^{X^*}$, $i = 1, 2$. Then it holds:*

*(i) If $A$ is maximal monotone with $D(A) = X$, then $A$ is pseudomonotone.*

*(ii) If $A_1$ and $A_2$ are two pseudomonotone operators, then the sum $A_1 + A_2 : X \to 2^{X^*}$ is pseudomonotone, too.*

The main theorem on pseudomonotone multivalued operators is given as follows.

**Theorem 2.2.19.** *Let $X$ be a real reflexive Banach space and let $A : X \to 2^{X^*}$ be a pseudomonotone and bounded operator which is coercive in the sense that there exists a real-valued function $c : \mathbb{R}_+ \to \mathbb{R}$ with*

$$c(r) \to +\infty, \quad \text{as } r \to +\infty$$

*such that for all $(u, u^*) \in Gr(A)$, we have*

$$\langle u^*, u - u_0 \rangle \geq c(\|u\|_X)\|u\|_X$$

*for some $u_0 \in X$. Then $A$ is surjective, which means* $\mathrm{range}(A) = X^*$.

An significant instrument is the following surjectivity result for multivalued pseudomonotone mappings perturbed by maximal monotone operators in reflexive Banach spaces.

**Theorem 2.2.20.** *Let $X$ be a real reflexive Banach space with the dual space $X^*$, $\Phi : X \to 2^{X^*}$ a maximal monotone operator and $u_0 \in D(\Phi)$. Let $A : X \to 2^{X^*}$ be a pseudomonotone operator and assume that either $A_{u_0}$ is quasi-bounded or $\Phi_{u_0}$ is strongly quasi-bounded. Assume further that $A : X \to 2^{X^*}$ is $u_0-$coercive, that is, there exists a real-valued function $c : \mathbb{R}_+ \to \mathbb{R}$ with $c(r) \to +\infty$ as $r \to +\infty$ such that for all $(u, u^*) \in \mathrm{Gr}(A)$ one has $\langle u^*, u - u_0 \rangle \geq c(\|u\|_X)\|u\|_X$. Then $A + \Phi$ is surjective, that is,* $\mathrm{range}(A + \Phi) = X^*$.

The proof of the theorem can be found for example in [104, Theorem 2.12]. The notations $A_{u_0}$ and $\Phi_{u_0}$ stand for $A_{u_0}(u) := A(u_0 + u)$ and $\Phi_{u_0}(u) := \Phi(u_0 + u)$, respectively. Note that any bounded operator is, in particular, also quasi-bounded and strongly quasi-bounded. For more details we refer to [104].

## 2.3 Nonsmooth Analysis

In this section, we provide some basic facts of nonsmooth analysis.

### 2.3.1 Clarke's Generalized Gradient

Let $X$ be a real Banach space equipped with the norm $\|\cdot\|$. The dual space of $X$ is denoted by $X^*$ and the notation $\langle \cdot, \cdot \rangle$ means the duality pairing between them.

**Definition 2.3.1** (Lipschitz Condition). *A functional $\Phi : X \to \mathbb{R}$ is said to be locally Lipschitz if for every point $x \in X$ a neighborhood $V$ of $x$ in $X$ and a constant $K > 0$ exist such that*

$$|\Phi(y) - \Phi(z)| \leq K\|y - z\|, \quad \forall y, z \in V.$$

Notice that a convex and continuous function $\Phi : X \to \mathbb{R}$ is locally Lipschitz. More generally, a convex function $\Phi : X \to \mathbb{R}$ which is bounded above on a neighborhood of some point is locally Lipschitz (cf. [42, Proposition 2.2.6]).
The classical theory of differentiability does not work in the case of locally Lipschitz functions. However, a suitable subdifferential calculus approach has been developed by Clarke (see [42]). The definition of the generalized directional derivative is stated as follows.

**Definition 2.3.2** (Generalized Directional Derivative). *Let $\Phi : X \to \mathbb{R}$ be a locally Lipschitz function and fix two points $u, v \in X$. The generalized directional derivative of $\Phi$ at $u$ in the direction $v$ is defined as*

$$\Phi^\circ(u; v) = \limsup_{x \to u, t \downarrow 0} \frac{\Phi(x + tv) - \Phi(x)}{t}.$$

It is clear that $\Phi^\circ(u; v) \in \mathbb{R}$, because $\Phi$ is locally Lipschitz. We also denote $\Phi^\circ$ as Clarke's generalized directional derivative which has the following properties (see [43, Proposition 2.1.1]).

**Proposition 2.3.3.** *Let $\Phi : X \to \mathbb{R}$ be a locally Lipschitz function. Then it holds:*

*(i) The function $\Phi^\circ(u; \cdot) : X \to \mathbb{R}$ is subadditive, positively homogeneous and satisfies the inequality*

$$|\Phi^\circ(u; v)| \leq K\|v\|, \quad \forall v \in X,$$

*where $K > 0$ denotes the Lipschitz constant of $\Phi$ near the point $u \in X$.*

*(ii)* $\Phi^\circ(u; -v) = (-\Phi)^\circ(u; v)$, $\forall v \in X$.

*(iii) The function $(u, v) \in X \times X \mapsto \Phi^\circ(u; v) \in \mathbb{R}$ is upper semicontinuous.*

Now, we point out the relation between Clarke's generalized directional derivative and the usual directional derivative given by

$$\Phi'(u; v) = \lim_{t \downarrow 0} \frac{\Phi(u + tv) - \Phi(u)}{t}.$$

**Definition 2.3.4.** *A locally Lipschitz function $\Phi : X \to \mathbb{R}$ is called regular at a point $u \in X$ if*

*(i) there exists the directional derivative $\Phi'(u; v)$ for every $v \in X$.*

*(ii)* $\Phi^\circ(u; v) = \Phi'(u; v), \quad \forall v \in X.$

For example, every continuous convex function $\Phi : X \to \mathbb{R}$ is regular.
One of the main notations in this subsection is the following.

**Definition 2.3.5** (Generalized Gradient). *The generalized gradient of a locally Lipschitz functional $\Phi : X \to \mathbb{R}$ at a point $u \in X$ is the subset of $X^*$ defined by*

$$\partial\Phi(u) = \{\xi \in X^* : \Phi^o(u; v) \geq \langle \xi, v \rangle, \forall v \in X\}.$$

The Hahn-Banach theorem ensures that $\partial\Phi(u)$ is not empty (cf. [13]). Let us consider some examples.

(i) If $\Phi : X \to \mathbb{R}$ is continuously differentiable, then $\partial\Phi(u) = \{\Phi'(u)\}$ for all $u \in X$, where $\Phi'(u)$ denotes the Fréchet differential of $\Phi$ at $u$.

(ii) If $\Phi : X \to \mathbb{R}$ is convex and continuous, then the generalized gradient $\partial\Phi(u)$ coincides with the subdifferential of $\Phi$ at $u$ in the sense of convex analysis.

(iii) The generalized gradient of a locally Lipschitz functional $\Phi : X \to \mathbb{R}$ at a point $u \in X$ is given by

$$\partial\Phi(u) = \partial(\Phi^o(u; \cdot))(0),$$

where in the right-hand side the subdifferential in the sense of convex analysis is written.

The next proposition presents some important properties of generalized gradients.

**Proposition 2.3.6.** *Let $\Phi : X \to \mathbb{R}$ be a locally Lipschitz function. Then for any $u \in X$ the properties below hold:*

*(i) $\partial\Phi(u)$ is a convex, weak*-compact subset of $X^*$ and*

$$\|\xi\|_{X^*} \leq K, \quad \forall \xi \in \partial\Phi(u),$$

*where $K > 0$ is the Lipschitz constant of $\Phi$ near $u$.*

*(ii) $\Phi^o(u; v) = \max\{\langle \xi, v \rangle : \xi \in \partial\Phi(u)\}, \forall v \in X$.*

*(iii) The mapping $u \mapsto \partial\Phi(u)$ is weak*-closed from $X$ into $2^{X^*}$.*

*(iv) The mapping $u \mapsto \partial\Phi(u)$ is upper semicontinuous from $X$ into $2^{X^*}$, where $X^*$ is equipped with the weak*-topology.*

The proof of Proposition 2.3.6 can be found for example in [28, Proposition 2.171].

### 2.3.2 Basic Calculus

In this subsection, we present some calculus for Clarke's generalized gradient. For the proofs of the following two propositions, we refer to [28, Proposition 2.173 and Proposition 2.174] and [42, Proposition 2.3.1 and Proposition 2.3.3].

**Proposition 2.3.7** (Scalar Multiples). *Let $\Phi : X \to \mathbb{R}$ be a locally Lipschitz function, let $\alpha \in \mathbb{R}$ and let $u \in X$. Then the following formula holds*

$$\partial(\alpha\Phi)(u) = \alpha\partial\Phi(u).$$

*In particular, one has*

$$\partial(-\Phi)(u) = -\partial\Phi(u).$$

**Proposition 2.3.8** (Finite Sums). *Let $\Phi_i : X \to \mathbb{R}$, $i = 1, \dots, m$, be locally Lipschitz functions. Then for every $u \in X$ the following inclusion holds*

$$\partial\left(\sum_{i=1}^{m} \Phi_i\right)(u) \subset \sum_{i=1}^{m} \partial\Phi_i(u).$$

*If all but at most one of the locally Lipschitz functions $\Phi_i$ are strictly differentiable, then the inclusion above becomes an equality.*

Note that the inclusion in Proposition 2.3.8 also becomes an equality if all functions $\Phi_i$ are regular at the point $u \in X$. Then it holds, in particular, that $\sum_{i=1}^{m} \Phi_i$ is regular at $u \in X$.
Now, we give the relationship between local extrema and Clarke's generalized gradient (see [28, 42]).

**Proposition 2.3.9** (Local Extrema). *If $u \in X$ is a local minimum or maximum point for the locally Lipschitz function $\Phi : X \to \mathbb{R}$, then $0 \in \partial\Phi(u)$.*

The Mean-Value theorem for locally Lipschitz functions is presented in the next theorem due to Lebourg (cf. [42, Proposition 2.3.7]).

**Theorem 2.3.10** (Lebourg's Theorem). *Let $\Phi : X \to \mathbb{R}$ be a locally Lipschitz function. Then for all $x, y \in X$, there exist $u = x + t_0(y - x)$, with $0 < t_0 < 1$, and $\xi \in \partial\Phi(u)$ such that*

$$\Phi(y) - \Phi(x) = \langle \xi, y - x \rangle.$$

In our calculations we apply the very useful chain rule given as follows.

**Theorem 2.3.11** (Chain Rule). *Let $F : X \to Y$ be a continuously differentiable mapping between the Banach spaces $X$, $Y$, and let $\Phi : Y \to \mathbb{R}$ be a locally Lipschitz function. Then* *the function $\Phi \circ F : X \to \mathbb{R}$ is locally Lipschitz and for any point $u \in X$ the formula below holds*

$$\partial(\Phi \circ F)(u) \subset \partial\Phi(F(u)) \circ DF(u), \tag{2.3.1}$$

*in the sense that every element $z \in \partial(\Phi \circ F)(u)$ can be expressed as*

$$z = DF(u)^*\xi, \qquad \text{for some } \xi \in \partial\Phi(F(u)),$$

*where $DF(u)^*$ denotes the adjoint of the Fréchet differential $DF(u)$ of $F$ at $u$. If, in addition, $F$ maps every neighborhood of $u$ onto a dense subset of a neighborhood of $F(u)$, then (2.3.1) becomes an equality.*

**Corollary 2.3.12.** *If there exists a (linear) continuous embedding $i : X \to Y$ of the Banach space $X$ into a Banach space $Y$, then for every locally Lipschitz function $\Phi : Y \to \mathbb{R}$ one has*

$$\partial(\Phi \circ i)(u) \subset i^*\partial\Phi(i(u)), \forall u \in X.$$

*If, in addition, $i(X)$ is dense in $Y$, then*

$$\partial(\Phi \circ i)(u) = i^*\partial\Phi(i(u)), \forall u \in X.$$

## 2.4 Variational Tools

This section lists some variational tools which we need in later chapters. The following theorem is an important one to prove the existence of minimum points of weakly coercive functionals (cf. [123, Theorem 25.D]).

**Theorem 2.4.1** (Main Theorem on Weakly Coercive Functionals)**.** *Suppose that the functional $f : M \subseteq X \to \mathbb{R}$ has the following three properties:*

*(i) $M$ is a nonempty closed convex set in the reflexive Banach space $X$.*

*(ii) $f$ is weakly sequentially lower semicontinuous on $M$.*

*(iii) $f$ is weakly coercive.*

*Then $f$ has a minimum on $M$.*

A criterion for the weak sequential lower semicontinuity of $C^1$-functionals can be read as follows. For more details we refer to Zeidler [123, Proposition 25.21].

**Proposition 2.4.2.** *Let $f : M \subseteq X \to \mathbb{R}$ be a $C^1-$functional on the open convex set $M$ of the real Banach space $X$, and let $f'$ be pseudomonotone and bounded. Then, $f$ is weakly sequentially lower semicontinuous on $M$.*

A significant tool in the proof for the existence of a nontrivial sign-changing solution is the following Mountain-Pass Theorem (see [110]). First, we give the definition of the Palais-Smale-Condition.

**Definition 2.4.3** (Palais-Smale-Condition). *Let $E$ be a real Banach space and $I \in C^1(E, \mathbb{R})$. The functional $I$ is said to satisfy the Palais-Smale-Condition if for each sequence $(u_n) \subset E$ that fulfills*

*(i) $I(u_n)$ is bounded,*

*(ii) $I'(u_n) \to 0$ as $n \to \infty$,*

*there exists a strong convergent subsequence of $(u_n)$.*

**Theorem 2.4.4** (Mountain-Pass Theorem). *Let $E$ be a real Banach space and $I \in C^1(E, \mathbb{R})$ satisfying the Palais-Smale-Condition. Suppose*

*($I_1$) there are constants $\rho > 0$ and $\alpha$ as well as an $e_1 \in E$ such that $I_{\partial B_\rho(e_1)} \geq \alpha$, and*

*($I_2$) there is an $e_2 \in E \setminus \overline{B_\rho(e_1)}$ such that $I(e_2) \leq I(e_1) < \alpha$.*

*Then $I$ possesses a critical value $c$ corresponding to a critical point $u_0$ such that $I(u_0) = c \geq \alpha$. Moreover, the critical value $c$ can be characterized as*

$$c = \inf_{g \in \Pi} \max_{u \in g([-1,1])} I(u), \tag{2.4.1}$$

*where*

$$\Pi = \{g \in C([-1,1], E) \mid g(-1) = e_1, g(1) = e_2\}.$$

In our considerations, we make use of the following strong maximum principle due to Vázquez (see [119]).

**Theorem 2.4.5** (Vázquez's strong maximum principle). *Let $u \in C^1(\Omega)$ such that*

*(i) $\Delta_p u \in L^2_{\text{loc}}(\Omega)$,*

*(ii) $u \geq 0$ a.e. in $\Omega$ and $u \not\equiv 0$ in $\Omega$,*

*(iii) $\Delta_p u \leq \beta(u)$ a.e. in $\Omega$ with $\beta : [0, \infty) \to \mathbb{R}$ continuous, nondecreasing, $\beta(0) = 0$ and either*

*(i) $\beta(s) = 0$ for some $s > 0$ or,*

*(ii) $\beta(s) > 0$ for all $s > 0$ with $\int_0^1 (\beta(s)s)^{-1/p} ds = +\infty$.*

*Then it holds*

$$u(x) > 0 \text{ a.e. in } \Omega.$$

*Moreover, if* $u \in C^1(\Omega \cup x_0)$ *for an* $x_0 \in \partial\Omega$ *satisfying an interior sphere condition and* $u(x_0) = 0$, *then*

$$\frac{\partial u}{\partial \nu}(x_0) < 0,$$

*where* $\nu$ *is the outer normal derivative of* $u$ *at* $x_0 \in \partial\Omega$.

We recall that a point $x_0 \in \partial\Omega$ satisfies the interior sphere condition if there exists an open ball $B = B_R(x_1) \subset \Omega$ such that $\overline{B} \cap \partial\Omega = \{x_0\}$. Then one can choose a unit vector

$$\nu = (x_0 - x_1)/|x_0 - x_1|,$$

and $\nu$ is a normal to $\partial B$ at $x_0$ pointing outside. A sufficient condition to satisfy the interior sphere condition is a $C^2-$boundary.
The proof of the following fixed point result is given in [20, Theorem 1.1.1].

**Lemma 2.4.6.** *Let* $P$ *be a subset of an ordered normed space,* $G : P \to P$ *an increasing mapping and* $G[P] = \{Gx \mid x \in P\}$.

*(i) If* $G[P]$ *has a lower bound in* $P$ *and the increasing sequences of* $G[P]$ *converge weakly in* $P$, *then* $G$ *has the least fixed point* $x_*$, *and* $x_* = \min\{x \mid Gx \le x\}$.

*(ii) If* $G[P]$ *has an upper bound in* $P$ *and the decreasing sequences of* $G[P]$ *converge weakly in* $P$, *then* $G$ *has the greatest fixed point* $x^*$, *and* $x^* = \max\{x \mid x \le Gx\}$.

# Chapter 3

# Nonlinear Neumann Boundary Value Problems

This chapter is devoted to the study of a class of nonlinear elliptic problems under Neumann conditions involving the $p-$Laplacian.

## 3.1 Multiple Solutions Depending on Steklov Eigenvalues

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$. We consider the quasilinear elliptic equation

$$\begin{aligned} -\Delta_p u &= f(x,u) - |u|^{p-2}u \qquad && \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= \lambda |u|^{p-2}u + g(x,u) && \text{on } \partial\Omega, \end{aligned} \tag{3.1.1}$$

where $-\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the negative $p$-Laplacian, $\frac{\partial u}{\partial \nu}$ means the outer normal derivative of $u$ with respect to $\partial\Omega$, $\lambda$ is a real parameter and the nonlinearities $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are some Carathéodory functions. For $u \in W^{1,p}(\Omega)$ defined on the boundary $\partial\Omega$, we make use of the trace operator $\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ which is well known to be compact. For easy readability we will drop the notation $\gamma(u)$ and write $u$ for short.
Our main goal is to provide the existence of multiple solutions of (3.1.1) meaning that for all values $\lambda > \lambda_2$, where $\lambda_2$ denotes the second eigenvalue of $(-\Delta_p, W^{1,p}(\Omega))$ known as the Steklov eigenvalue problem (see, e.g., [67, 97, 109]) given by

$$\begin{aligned} -\Delta_p u &= -|u|^{p-2}u \qquad && \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= \lambda |u|^{p-2}u && \text{on } \partial\Omega, \end{aligned} \tag{3.1.2}$$

there exist at least three nontrivial solutions. More precisely, we obtain two constant-sign solutions and one sign-changing solution of problem (3.1.1). This is the main result of the present section and it is formulated in the Theorems 3.1.8 and 3.1.16, respectively. In our consideration, the nonlinearities $f$ and $g$ only need to be Carathéodory functions which are bounded on bounded sets whereby their growth does not need to be necessarily polynomial. We only require some growth properties at zero and infinity given by

$$\lim_{s\to 0}\frac{f(x,s)}{|s|^{p-2}s}=\lim_{s\to 0}\frac{g(x,s)}{|s|^{p-2}s}=0,\qquad \lim_{|s|\to\infty}\frac{f(x,s)}{|s|^{p-2}s}=\lim_{|s|\to\infty}\frac{g(x,s)}{|s|^{p-2}s}=-\infty$$

and we suppose the existence of $\delta_f>0$ such that $f(x,s)/|s|^{p-2}s\geq 0$ for all $0<|s|\leq\delta_f$.
In the past many papers about the existence of Neumann problems like the form (3.1.1) were developed (see, e.g., [5, 46, 62, 66, 96, 125]). Martínez et al. [96] proved the existence of weak solutions of the Neumann boundary problem

$$\begin{aligned} -\Delta_p u &= -|u|^{p-2}u - f(x,u) && \text{in } \Omega,\\ |\nabla u|^{p-2}\frac{\partial u}{\partial\nu} &= \lambda|u|^{p-2}u - h(x,u) && \text{on } \partial\Omega, \end{aligned} \tag{3.1.3}$$

where the perturbations $f:\Omega\times\mathbb{R}\to\mathbb{R}$ and $h:\partial\Omega\times\mathbb{R}\to\mathbb{R}$ are bounded Carathéodory functions satisfying an integral condition of Landesmann-Lazer type. Their main result is given in [96, Theorem 1.2] which yields the existence of a weak solution of (3.1.3) with $\lambda=\lambda_1$, where $\lambda_1$ is the first eigenvalue of the Steklov eigenvalue problem (see (3.1.2)). Moreover, they suppose in their main theorem the boundedness of $f(x,t)$ and $h(x,t)$ by functions $\overline{f}\in L^q(\Omega)$ and $\overline{h}\in L^q(\partial\Omega)$ for all $(x,t)\in\Omega\times\mathbb{R}$ and $(x,t)\in\partial\Omega\times\mathbb{R}$, respectively. A similar work on (3.1.1) can be found in [63]. There the authors get as well three nontrivial solutions for the nonlinear boundary value problem

$$\begin{aligned} -\Delta_p u + |u|^{p-2}u &= f(x,u) && \text{in } \Omega,\\ |\nabla u|^{p-2}\frac{\partial u}{\partial\nu} &= g(x,u) && \text{on } \partial\Omega, \end{aligned} \tag{3.1.4}$$

where they assume among others that the Carathéodory functions $f$ and $g$ are also continuously differentiable in the second argument. The proof is based on the Lusternik-Schnirelmann method for non-compact manifolds. If the Neumann boundary values are defined by a function $f:\mathbb{R}\to\mathbb{R}$ meaning the problem

$$\begin{aligned} -\Delta_p u &= -|u|^{p-2}u && \text{in } \Omega,\\ |\nabla u|^{p-2}\frac{\partial u}{\partial\nu} &= f(u) && \text{on } \partial\Omega, \end{aligned} \tag{3.1.5}$$

we refer to the results of J. Fernández Bonder and J.D. Rossi in [66]. They consider various cases where $f$ has subcritical growth, critical growth and supercritical growth, respectively. In the first two cases the existence of infinitely many solutions under some conditions on the exponents of the growth were demonstrated.
Another result to obtain multiple solutions with nonlinear boundary conditions can be found in the paper of J.H. Zhao and P.-H. Zhao [125]. They study the equation

$$\begin{aligned} -\Delta_p u + \lambda(x)|u|^{p-2}u &= f(x,u) && \text{in } \Omega,\\ |\nabla u|^{p-2}\frac{\partial u}{\partial\nu} &= \eta|u|^{p-2}u && \text{on } \partial\Omega, \end{aligned} \tag{3.1.6}$$

where $\lambda(x) \in L^\infty(\Omega)$ satisfying $\operatorname{ess\,inf}_{x\in\overline{\Omega}} \lambda(x) > 0$ and $\eta$ is a real parameter. They prove the existence of infinitely many solutions when $f$ is superlinear and subcritical with respect to $u$ by using the fountain theorem and the dual fountain theorem, respectively. In case that $f$ has the form $f(x,u) = |u|^{p^*-2}u + |u|^{r-2}u$ they get at least one nontrivial solution when $p < r < p^*$ and infinitely many solutions when $1 < r < p$ by using the Mountain-Pass Theorem and the "concentration-compactness principle", respectively. A similar result of the same authors is also developed in [124]. The existence of multiple solutions and sign-changing solutions for zero Neumann boundary values has been proven in [88, 107, 108, 122] and [125], respectively. Analogous results for the Dirichlet problem have been recently obtained in [35, 36, 37, 41, 57, 99, 101]. An interesting problem about the existence of multiple solutions for both, the Dirichlet problem and the Neumann problem, can be found in [44]. The authors study the existence of multiple solutions to the abstract equation $J_p u = N_f u$, where $J_p$ is the duality mapping on a real reflexive and smooth Banach space $X$, corresponding to the gauge function $\varphi(t) = t^{p-1}, 1 < p < \infty$ and $N_f : L^q(\Omega) \to L^{q'}(\Omega), 1/q + 1/q' = 1$, is the Nemytskij operator generated by a function $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$.
The novelty of our treatment is the fact that we do not need differentiability, polynomial growth or some integral conditions on the mappings $f$ and $g$. In order to prove our main results we make use of variational and topological tools, e.g. critical point theory, Mountain-Pass Theorem, Second Deformation Lemma and variational characterization of the second eigenvalue of the Steklov eigenvalue problem. This section is motivated by recent publications of S. Carl and D. Motreanu in [37] and [36], respectively. In [37] the authors consider the Dirichlet problem $-\Delta_p u = \lambda |u|^{p-2}u + g(x,u)$ in $\Omega$, $u = 0$ on $\partial\Omega$, and show the existence of at least three nontrivial solutions for all values $\lambda > \lambda_2$, where $\lambda_2$ denotes the second eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$. Therein, the main theorem about the existence of a sign-changing solution is also based on the Mountain-Pass Theorem and the Second Deformation Lemma. These results have been extended by the same authors to the equation $-\Delta_p u = a(u^+)^{p-1} - b(u^-)^{p-1} + g(x,u)$ in $\Omega$, $u = 0$ on $\partial\Omega$, where $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$ denote the positive and negative part of $u$, respectively. Carl et al. have shown that at least three nontrivial solutions exist provided the value $(a,b)$ is above the first nontrivial curve $\mathcal{C}$ of the Fŭcik spectrum constructed by Cuesta et al. in [45].

### 3.1.1 Auxiliary Neumann Problems

Let us consider some nonlinear boundary value problems with Neumann conditions involving the $p$-Laplacian. In [95] the authors study the Steklov eigenvalue problem

$$\begin{aligned} -\Delta_p u &= -|u|^{p-2}u && \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= \lambda |u|^{p-2}u && \text{on } \partial\Omega. \end{aligned} \tag{3.1.7}$$

The trace operator $\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ is linear bounded (and even compact), thus a best constant $\lambda_1$ exists such that

$$\lambda_1^{1/p}\|u\|_{L^p(\partial\Omega)} \le \|u\|_{W^{1,p}(\Omega)}.$$

The best Sobolev trace constant $\lambda_1$ can be characterized as

$$\lambda_1 = \inf_{u\in W^{1,p}(\Omega)} \left\{ \int_\Omega [|\nabla u|^p + |u|^p]dx \text{ such that } \int_{\partial\Omega} |u|^p d\sigma = 1 \right\},$$

and $\lambda_1$ is the first eigenvalue of (3.1.7). Martínez et al. showed that the first eigenvalue $\lambda_1 > 0$ is isolated and simple. The corresponding eigenfunction $\varphi_1$ is strictly positive in $\overline{\Omega}$ and belongs to $L^\infty(\Omega)$ (cf. [85, Lemma 5.6 and Theorem 4.3]). Applying the results of Lieberman in [89, Theorem 2] implies $\varphi_1 \in C^{1,\alpha}(\overline{\Omega}), \alpha \in (0,1)$. This fact along with $\varphi_1(x) > 0$ in $\overline{\Omega}$ yields $\varphi_1 \in \text{int}(C^1(\overline{\Omega})_+)$, where $\text{int}(C^1(\overline{\Omega})_+)$ denotes the interior of the positive cone $C^1(\overline{\Omega})_+ = \{u \in C^1(\overline{\Omega}) : u(x) \ge 0, \forall x \in \Omega\}$ in the Banach space $C^1(\overline{\Omega})$, given by

$$\text{int}(C^1(\overline{\Omega})_+) = \left\{u \in C^1(\overline{\Omega}) : u(x) > 0, \forall x \in \overline{\Omega}\right\}.$$

The study of Neumann eigenvalue problems with or without weights are also considered in [46, 58, 81, 85, 115]. Analogous to the results for the Dirichlet eigenvalue problem (see [45]), there also exists a variational characterization of the second eigenvalue of (3.1.7) meaning that $\lambda_2$ can be represented as follows

$$\lambda_2 = \inf_{g\in\Pi} \max_{u\in g([-1,1])} \int_\Omega \Big(|\nabla u|^p + |u|^p\Big) dx, \tag{3.1.8}$$

where

$$\Pi = \{g \in C([-1,1], S) \mid g(-1) = -\varphi_1, g(1) = \varphi_1\}, \tag{3.1.9}$$

and

$$S = \left\{u \in W^{1,p}(\Omega) : \int_{\partial\Omega} |u|^p d\sigma = 1\right\}. \tag{3.1.10}$$

The proof of this result can be found in [97]. Now we consider solutions of the Neumann boundary value problem

$$\begin{aligned} -\Delta_p u &= -\varsigma|u|^{p-2}u + 1 && \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= 1 && \text{on } \partial\Omega, \end{aligned} \tag{3.1.11}$$

where $\varsigma > 1$ is a constant. Let $B : L^p(\Omega) \to L^q(\Omega)$ be the Nemytskij operator defined by $Bu(x) := \varsigma|u(x)|^{p-2}u(x)$. It is well known that $B : L^p(\Omega) \to L^q(\Omega)$ is bounded and continuous. We set $\widehat{B} := i^* \circ B \circ i : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$, where $i^* : L^q(\Omega) \to (W^{1,p}(\Omega))^*$ is the adjoint operator of the compact embedding $i : W^{1,p}(\Omega) \to L^p(\Omega)$. The operator $\widehat{B}$ is bounded, completely continuous and thus, also pseudomonotone. We denote by $\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ the trace operator and by $\gamma^* : L^q(\partial\Omega) \to (W^{1,p}(\Omega))^*$ its adjoint operator. The weak formulation of (3.1.11) is given by

$$u \in W^{1,p}(\Omega): \quad \langle -\Delta_p u + \widehat{B}u - i^*(1) - \gamma^*(1), \varphi \rangle = 0, \quad \forall \varphi \in W^{1,p}(\Omega), \tag{3.1.12}$$

meaning

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla \varphi dx + \varsigma \int_\Omega |u|^{p-2} u \varphi dx - \int_\Omega \varphi dx - \int_{\partial\Omega} \varphi d\sigma = 0, \quad \forall \varphi \in W^{1,p}(\Omega),$$

where $\langle \cdot, \cdot \rangle$ stands for the duality pairing between $W^{1,p}(\Omega)$ and its dual space $(W^{1,p}(\Omega))^*$. The negative $p$-Laplacian $-\Delta_p$ is pseudomonotone and therefore, the sum $-\Delta_p + \widehat{B}$ is pseudomonotone. The coercivity of $-\Delta_p + \widehat{B}$ follows directly and thus, classical existence results imply the existence of a solution of problem (3.1.11). Let $e_1, e_2$ be solutions of (3.1.11) satisfying $e_1 \neq e_2$. Subtracting the corresponding weak formulation of (3.1.11) with respect to $e_1, e_2$ and taking $\varphi = e_1 - e_2$ yields

$$\begin{aligned} &\int_\Omega [|\nabla e_1|^{p-2} \nabla e_1 - |\nabla e_2|^{p-2} \nabla e_2] \nabla (e_1 - e_2) dx \\ &+ \varsigma \int_\Omega [|e_1|^{p-2} e_1 - |e_2|^{p-2} e_2](e_1 - e_2) dx = 0. \end{aligned}$$

As the left-hand side is strictly positive for $e_1 \neq e_2$, we obtain a contradiction and thus, $e_1 = e_2$. Let $e$ be the unique solution of (3.1.11) in the weak sense. Choosing the test function $\varphi = e^- = \max\{-e, 0\} \in W^{1,p}(\Omega)$ results in

$$-\int_{\{x \in \Omega : e(x) < 0\}} |\nabla e|^p dx - \varsigma \int_{\{x \in \Omega : e(x) < 0\}} |e|^p dx = \int_\Omega e^- dx + \int_{\partial\Omega} e^- d\sigma \geq 0,$$

which proves that $e$ is nonnegative. Notice that $e$ is not identically zero. Applying the Moser Iteration (cf. [56],[85] or see the proof of Proposition 3.1.11) yields $e \in L^\infty(\Omega)$ and thus, the regularity results of Lieberman (see [89, Theorem 2]) ensure $e \in C^{1,\alpha}(\overline{\Omega}), \alpha \in (0,1)$. From (3.1.11) we conclude

$$\Delta_p e = \varsigma |e|^{p-2} e - 1 \leq \varsigma e^{p-1} \text{ a.e. in } \Omega.$$

Setting $\beta(s) = \varsigma s^{p-1}$ for $s > 0$ allows us to apply Vázquez's strong maximum principle stated in Theorem 2.4.5 which is possible since $\int_{0^+} \frac{1}{(s\beta(s))^{1/p}} ds = +\infty$. This shows $e(x) > 0$ for all $x \in \Omega$. If there exists $x_0 \in \partial\Omega$ such that $e(x_0) = 0$, we obtain by applying again Vázquez's strong maximum principle that $\frac{\partial e}{\partial \nu}(x_0) < 0$, which is a contradiction since $|\nabla e|^{p-2} \frac{\partial e}{\partial \nu}(x_0) = 1$. Hence, $e(x) > 0$ in $\overline{\Omega}$ and therefore, we get $e \in \operatorname{int}(C^1(\overline{\Omega})_+)$.

### 3.1.2 Notations and Hypotheses

We impose the following conditions on the nonlinearities $f$ and $g$ in problem (3.1.1). The mappings $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions (that is, measurable in the first argument and continuous in the second argument) such that (f1) $\lim_{s\to 0} \frac{f(x,s)}{|s|^{p-2}s} = 0$, uniformly with respect to a.a. $x \in \Omega$.

(f2) $\lim_{|s|\to \infty} \frac{f(x,s)}{|s|^{p-2}s} = -\infty$, uniformly with respect to a.a. $x \in \Omega$.

(f3) $f$ is bounded on bounded sets.

(f4) There exists $\delta_f > 0$ such that $\frac{f(x,s)}{|s|^{p-2}s} \geq 0$ for all $0 < |s| \leq \delta_f$ and for a.a. $x \in \Omega$.

(g1) $\lim_{s\to 0} \frac{g(x,s)}{|s|^{p-2}s} = 0$, uniformly with respect to a.a. $x \in \partial\Omega$.

(g2) $\lim_{|s|\to \infty} \frac{g(x,s)}{|s|^{p-2}s} = -\infty$, uniformly with respect to a.a. $x \in \partial\Omega$.

(g3) $g$ is bounded on bounded sets.

(g4) $g$ satisfies the condition

$$|g(x_1,s_1) - g(x_2,s_2)| \leq L\Big[|x_1 - x_2|^\alpha + |s_1 - s_2|^\alpha\Big],$$

for all pairs $(x_1,s_1),(x_2,s_2)$ in $\partial\Omega \times [-M_0, M_0]$, where $M_0$ is a positive constant and $\alpha \in (0,1]$.

Note that the mapping $\Phi : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ defined by $\Phi(x,s) := \lambda|s|^{p-2}s + g(x,s)$ also fulfills a condition as in (g4). Recall that we write $g(x,u(x)) := g(x,\gamma(u(x)))$ for $u \in W^{1,p}(\Omega)$, where $\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ stands for the trace operator. With a view to the conditions (f1) and (g1), we see at once that $f(x,0) = g(x,0) = 0$ and thus, $u = 0$ is a trivial solution of problem (3.1.1).

**Corollary 3.1.1.** *Let (f1),(f3) and (g1),(g3) be satisfied. Then, for each $a > 0$ there exist constants $b_1, b_2 > 0$ such that*

$$\begin{aligned} |f(x,s)| &\leq b_1|s|^{p-1}, \quad \textit{for a.a. } x \in \Omega \textit{ and all } 0 \leq |s| \leq a,\\ |g(x,s)| &\leq b_2|s|^{p-1}, \quad \textit{for a.a. } x \in \partial\Omega \textit{ and all } 0 \leq |s| \leq a. \end{aligned} \tag{3.1.13}$$

**Proof.** The assumption (f1) implies that for each $c_1 > 0$ there exists $\delta > 0$ such that

$$|f(x,s)| \leq c_1|s|^{p-1}, \quad \text{for a.a. } x \in \Omega \text{ and all } 0 \leq |s| \leq \delta. \tag{3.1.14}$$

Due to condition (f3), there exists a constant $c_2 > 0$ such that for a given $a > 0$ holds

$$|f(x,s)| \leq c_2, \quad \text{for a.a. } x \in \Omega \text{ and all } 0 \leq |s| \leq a. \tag{3.1.15}$$

If $\delta > a$, then inequality (3.1.14), in particular, implies

$$|f(x,s)| \leq b_1|s|^{p-1}, \quad \text{for a.a. } x \in \Omega \text{ and all } 0 \leq |s| \leq a,$$

where $b_1 := c_1$. Let us assume $\delta < a$. From (3.1.15) we obtain

$$|f(x,s)| \leq \frac{c_2}{\delta^{p-1}}|s|^{p-1}, \quad \text{for a.a. } x \in \Omega \text{ and all } \delta \leq |s| \leq a, \tag{3.1.16}$$

and thus, combining (3.1.14) and (3.1.16) yields

$$|f(x,s)| \leq \left(c_1 + \frac{c_2}{\delta^{p-1}}\right)|s|^{p-1}, \quad \text{for a.a. } x \in \Omega \text{ and all } 0 \leq |s| \leq a,$$

where the setting $b_1 := c_1 + \frac{c_2}{\delta^{p-1}}$ proves (3.1.13). In the same way, one shows the assertion for $g$. □

**Example 3.1.2.** *Consider the functions $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ defined by*

$$f(x,s) = \begin{cases} |s|^{p-2}s(1+(s+1)e^{-s}) & \text{if } s \leq -1 \\ \operatorname{sgn}(s)\frac{|s|^p}{2}(|(s-1)\cos(s+1)|+s+1) & \text{if } -1 \leq s \leq 1 \\ s^{p-1}e^{1-s} - |x|(s-1)s^{p-1}e^s & \text{if } s \geq 1, \end{cases}$$

*and*

$$g(x,s) = \begin{cases} |s|^{p-2}s(s+1+e^{s+1}) & \text{if } s \leq -1 \\ |s|^{p-1}se^{(s^2-1)\sqrt{|x|}} & \text{if } -1 \leq s \leq 1 \\ s^{p-1}(\cos(1-s)+(1-s)e^s) & \text{if } s \geq 1. \end{cases}$$

*One verifies that all assumptions (f1)-(f4) and (g1)-(g4) are satisfied.*

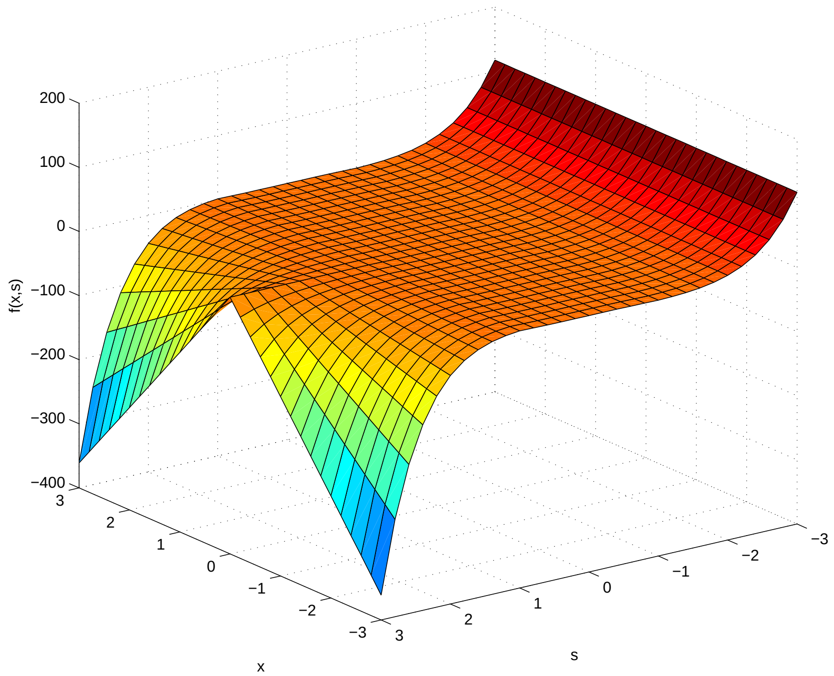


**Figure 3.1.** *The function $f$ in case $\Omega = (-3,3)$ and $p = 2$*

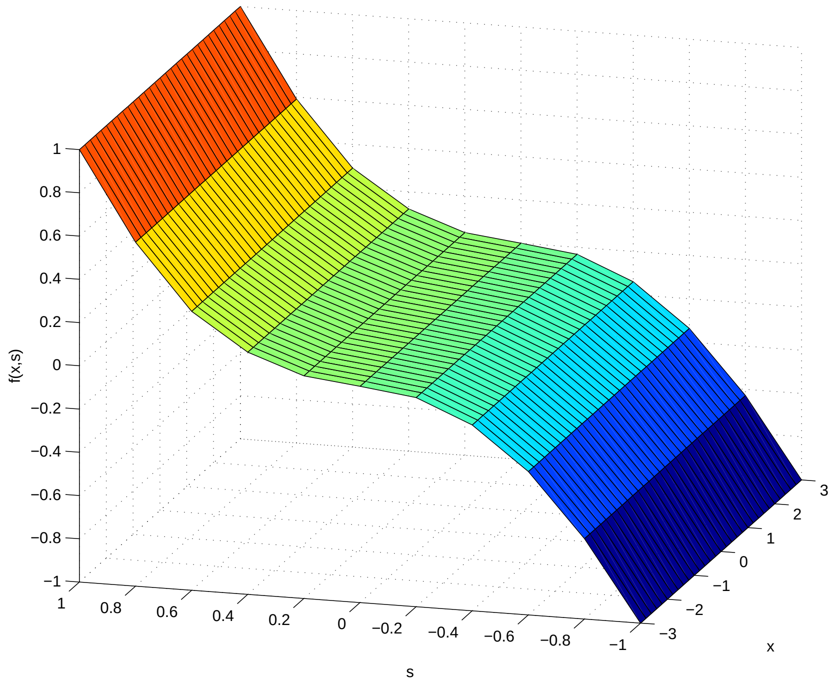


**Figure 3.2.** *The function $f$ near zero in case $\Omega = (-3, 3)$ and $p = 2$*

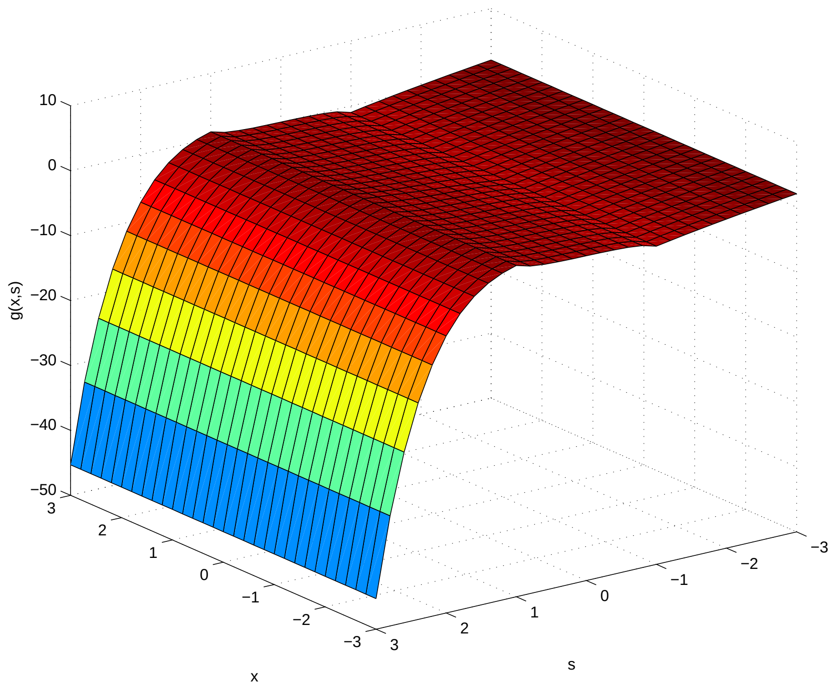


**Figure 3.3.** *The function $g$ in case $\Omega = (-3, 3)$ and $p = 2$*

The definition of a solution of problem (3.1.1) in the weak sense is defined as follows.

**Definition 3.1.3.** *A function $u \in W^{1,p}(\Omega)$ is called a solution of (3.1.1) if the following holds:*

$$\begin{aligned}&\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi dx\\ &= \int_\Omega (f(x,u) - |u|^{p-2}u)\varphi dx + \int_{\partial\Omega}(\lambda|u|^{p-2}u + g(x,u))\varphi d\sigma, \quad \forall\varphi \in W^{1,p}(\Omega).\end{aligned}$$

Next, we recall the notations of sub- and supersolutions of problem (3.1.1).

**Definition 3.1.4.** *A function $\underline{u} \in W^{1,p}(\Omega)$ is called a subsolution of (3.1.1) if the following holds:*

$$\begin{aligned}&\int_\Omega |\nabla \underline{u}|^{p-2}\nabla \underline{u} \nabla\varphi dx\\ &\leq \int_\Omega (f(x,\underline{u}) - |\underline{u}|^{p-2}\underline{u})\varphi dx + \int_{\partial\Omega}(\lambda|\underline{u}|^{p-2}\underline{u} + g(x,\underline{u}))\varphi d\sigma, \quad \forall\varphi \in W^{1,p}(\Omega)_+.\end{aligned}$$

**Definition 3.1.5.** *A function $\overline{u} \in W^{1,p}(\Omega)$ is called a supersolution of (3.1.1) if the following holds:*

$$\begin{aligned}&\int_\Omega |\nabla \overline{u}|^{p-2}\nabla \overline{u} \nabla\varphi dx\\ &\geq \int_\Omega (f(x,\overline{u}) - |\overline{u}|^{p-2}\overline{u})\varphi dx + \int_{\partial\Omega}(\lambda|\overline{u}|^{p-2}\overline{u} + g(x,\overline{u}))\varphi d\sigma, \quad \forall\varphi \in W^{1,p}(\Omega)_+.\end{aligned}$$

Here, $W^{1,p}(\Omega)_+ := \{\varphi \in W^{1,p}(\Omega) : \varphi \geq 0\}$ stands for all nonnegative functions of $W^{1,p}(\Omega)$. Recall that if $u \in W^{1,p}(\Omega)$ satisfies $v \leq u \leq w$, where $v, w$ are some functions in $W^{1,p}(\Omega)$, then it holds $\gamma(v) \leq \gamma(u) \leq \gamma(w)$, where $\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ denotes the trace operator.

### 3.1.3 Extremal Constant-Sign Solutions

We start by generating two ordered pairs of sub- and supersolutions of problem (3.1.1) having constant signs. Here and in the following we denote by $\varphi_1 \in \text{int}(C^1(\overline{\Omega})_+)$ the first eigenfunction of the Steklov eigenvalue problem (3.1.7) corresponding to the first eigenvalue $\lambda_1$.

**Lemma 3.1.6.** *Assume (f1)–(f4), (g1)–(g4) and $\lambda > \lambda_1$ and let e be the unique solution of problem (3.1.11). Then there exists a constant $\vartheta > 0$ such that $\vartheta e$ and $-\vartheta e$ are supersolution and subsolution, respectively, of problem (3.1.1). In addition, $\varepsilon\varphi_1$ is a subsolution and $-\varepsilon\varphi_1$ is a supersolution of problem (3.1.1) provided the number $\varepsilon > 0$ is sufficiently small.*

**Proof.** Let $\underline{u} = \varepsilon\varphi_1$, where $\varepsilon$ is a positive constant to be specified later. In view of the Steklov eigenvalue problem (3.1.7) it holds

$$\begin{aligned}&\int_\Omega |\nabla(\varepsilon\varphi_1)|^{p-2}\nabla(\varepsilon\varphi_1)\nabla\varphi dx\\ &= -\int_\Omega (\varepsilon\varphi_1)^{p-1}\varphi dx + \int_{\partial\Omega}\lambda_1(\varepsilon\varphi_1)^{p-1}\varphi d\sigma, \forall\varphi \in W^{1,p}(\Omega).\end{aligned} \tag{3.1.17}$$

We are going to prove that Definition 3.1.4 is satisfied for $\underline{u} = \varepsilon\varphi_1$ meaning that the inequality

$$\begin{aligned} &\int_\Omega |\nabla(\varepsilon\varphi_1)|^{p-2}\nabla(\varepsilon\varphi_1)\nabla\varphi dx \\ &\leq \int_\Omega (f(x,\varepsilon\varphi_1) - (\varepsilon\varphi_1)^{p-1})\varphi dx + \int_{\partial\Omega} (\lambda(\varepsilon\varphi_1)^{p-1} + g(x,\varepsilon\varphi_1))\varphi d\sigma, \end{aligned} \tag{3.1.18}$$

is valid for all $\varphi \in W^{1,p}(\Omega)_+$. By (3.1.17) we see that (3.1.18) is fulfilled provided the following holds true

$$\int_\Omega -f(x,\varepsilon\varphi_1)\varphi dx + \int_{\partial\Omega} ((\lambda_1 - \lambda)(\varepsilon\varphi_1)^{p-1} - g(x,\varepsilon\varphi_1))\varphi d\sigma \leq 0, \quad \forall \varphi \in W^{1,p}(\Omega)_+.$$

Condition (f4) implies for $\varepsilon \in (0, \delta_f/\|\varphi_1\|_\infty]$

$$\int_\Omega -f(x,\varepsilon\varphi_1)\varphi dx = \int_\Omega -\frac{f(x,\varepsilon\varphi_1)}{(\varepsilon\varphi_1)^{p-1}}(\varepsilon\varphi_1)^{p-1}\varphi dx \leq 0,$$

where $\|\cdot\|_\infty$ stands for the supremum norm. Due to assumption (g1) there exists a number $\delta_\lambda > 0$ such that

$$\frac{|g(x,s)|}{|s|^{p-1}} < \lambda - \lambda_1 \quad \text{for a.a. } x \in \partial\Omega \text{ and all } 0 < |s| \leq \delta_\lambda.$$

If $\varepsilon \in \left(0, \frac{\delta_\lambda}{\|\varphi_1\|_\infty}\right]$, we get

$$\begin{aligned} \int_{\partial\Omega} ((\lambda_1 - \lambda)(\varepsilon\varphi_1)^{p-1} - g(x,\varepsilon\varphi_1))\varphi d\sigma &\leq \int_{\partial\Omega} \left(\lambda_1 - \lambda + \frac{|g(x,\varepsilon\varphi)|}{(\varepsilon\varphi_1)^{p-1}}\right)(\varepsilon\varphi_1)^{p-1}\varphi d\sigma \\ &< \int_{\partial\Omega} (\lambda_1 - \lambda + \lambda - \lambda_1)(\varepsilon\varphi_1)^{p-1}\varphi d\sigma \\ &= 0. \end{aligned}$$

Choosing $0 < \varepsilon \leq \min\{\delta_f/\|\varphi_1\|_\infty, \delta_\lambda/\|\varphi_1\|_\infty\}$ proves that $\underline{u} = \varepsilon\varphi_1$ is a positive subsolution. In a similar way one proves that $\overline{u} = -\varepsilon\varphi_1$ is a negative supersolution.
Let $\overline{u} = \vartheta e$, where $\vartheta$ is a positive constant to be specified later. From the auxiliary problem (3.1.11) we conclude

$$\begin{aligned} &\int_\Omega |\nabla(\vartheta e)|^{p-2}\nabla(\vartheta e)\nabla\varphi dx \\ &= -\varsigma\int_\Omega (\vartheta e)^{p-1}\varphi dx + \int_\Omega \vartheta^{p-1}\varphi dx + \int_{\partial\Omega} \vartheta^{p-1}\varphi d\sigma, \forall \varphi \in W^{1,p}(\Omega). \end{aligned} \tag{3.1.19}$$

In order to fulfill the assertion of the lemma, we have to show the validity of Definition 3.1.5 for $\overline{u} = \vartheta e$ meaning that for all $\varphi \in W^{1,p}(\Omega)_+$ holds

$$\begin{aligned} &\int_\Omega |\nabla(\vartheta e)|^{p-2}\nabla(\vartheta e)\nabla\varphi dx \\ &\geq \int_\Omega (f(x,\vartheta e) - (\vartheta e)^{p-1})\varphi dx + \int_{\partial\Omega} (\lambda(\vartheta e)^{p-1} + g(x,\vartheta e))\varphi d\sigma. \end{aligned} \tag{3.1.20}$$

With a view to (3.1.19) we see at once that inequality (3.1.20) is satisfied if the following holds

$$\begin{aligned}&\int_\Omega (\vartheta^{p-1} - \widetilde{c}(\vartheta e)^{p-1} - f(x,\vartheta e))\varphi dx\\&\quad + \int_{\partial\Omega} (\vartheta^{p-1} - \lambda(\vartheta e)^{p-1} - g(x,\vartheta e))\varphi d\sigma \geq 0,\end{aligned} \tag{3.1.21}$$

where $\widetilde{c} = \varsigma - 1$ with $\widetilde{c} > 0$. By (f2) there exists $s_\varsigma > 0$ such that

$$\frac{f(x,s)}{s^{p-1}} < -\widetilde{c}, \quad \text{for a.a. } x \in \Omega \text{ and all } s > s_\varsigma,$$

and by (f3) we have

$$|-f(x,s) - \widetilde{c}s^{p-1}| \leq |f(x,s)| + \widetilde{c}s^{p-1} \leq c_\varsigma, \quad \text{for a.a. } x \in \Omega \text{ and all } s \in [0, s_\varsigma].$$

Thus, we get

$$f(x,s) \leq -\widetilde{c}s^{p-1} + c_\varsigma, \quad \text{for a.a. } x \in \Omega \text{ and all } s \geq 0. \tag{3.1.22}$$

Applying (3.1.22) to the first integral in (3.1.21) yields

$$\begin{aligned}&\int_\Omega (\vartheta^{p-1} - \widetilde{c}(\vartheta e)^{p-1} - f(x,\vartheta e))\varphi dx\\&\geq \int_\Omega (\vartheta^{p-1} - \widetilde{c}(\vartheta e)^{p-1} + \widetilde{c}(\vartheta e)^{p-1} - c_\varsigma)\varphi dx\\&= \int_\Omega (\vartheta^{p-1} - c_\varsigma)\varphi dx,\end{aligned}$$

which shows that for $\vartheta \geq c_\varsigma^{\frac{1}{p-1}}$ the integral is nonnegative. Due to hypothesis (g2) there is $s_\lambda > 0$ such that

$$\frac{g(x,s)}{s^{p-1}} < -\lambda, \quad \text{for a.a. } x \in \Omega \text{ and all } s > s_\lambda.$$

Assumption (g3) ensures the existence of a constant $c_\lambda > 0$ such that

$$|-g(x,s) - \lambda s^{p-1}| \leq |g(x,s)| + \lambda s^{p-1} \leq c_\lambda, \quad \text{for a.a. } x \in \Omega \text{ and all } s \in [0, s_\lambda].$$

We obtain

$$g(x,s) \leq -\lambda s^{p-1} + c_\lambda, \quad \text{for a.a. } x \in \partial\Omega \text{ and all } s \geq 0. \tag{3.1.23}$$

Using (3.1.23) to the second integral in (3.1.21) provides

$$\begin{aligned}&\int_{\partial\Omega} (\vartheta^{p-1} - \lambda(\vartheta e)^{p-1} - g(x,\vartheta e))\varphi dx\\&\geq \int_{\partial\Omega} (\vartheta^{p-1} - \lambda(\vartheta e)^{p-1} + \lambda(\vartheta e)^{p-1} - c_\lambda)\varphi dx\\&\geq \int_{\partial\Omega} (\vartheta^{p-1} - c_\lambda)\varphi dx.\end{aligned}$$

Choosing $\vartheta := \max\left\{c_\varsigma^{\frac{1}{p-1}}, c_\lambda^{\frac{1}{p-1}}\right\}$ proves that both integrals in (3.1.21) are nonnegative and thus, $\overline{u} = \vartheta e$ is a positive supersolution of problem (3.1.1). In order to prove that $\underline{u} = -\vartheta e$ is a negative subsolution we make use of the following estimates

$$\begin{aligned} f(x,s) &\geq -\widetilde{c}s^{p-1} - c_\varsigma, \quad \text{for a.a. } x \in \Omega \text{ and all } s \leq 0, \\ g(x,s) &\geq -\lambda s^{p-1} - c_\lambda, \quad \text{for a.a. } x \in \partial\Omega \text{ and all } s \leq 0, \end{aligned} \tag{3.1.24}$$

which can be derivated as stated above. With the aid of (3.1.24) one verifies that $\underline{u} = -\vartheta e$ is a negative subsolution of problem (3.1.1). □

According to Lemma 3.1.6 we obtain a positive pair $[\varepsilon\varphi_1, \vartheta e]$ and a negative pair $[-\vartheta e, -\varepsilon\varphi_1]$ of sub- and supersolutions of problem (3.1.1) assumed $\varepsilon > 0$ is sufficiently small.
The next lemma will prove the $C^{1,\alpha}$ regularity of solutions of problem (3.1.1) lying in the order interval $[0, \vartheta e]$ and $[-\vartheta e, 0]$, respectively. Note that $\underline{u} = \overline{u} = 0$ is both, a subsolution and a supersolution due to the assumptions (f1) and (g1). In the following proof we make use of the regularity results of Lieberman (see [89]) and Vázquez in [119]. To obtain regularity results, in particular for elliptic Neumann problems, we also refer to the papers of Tolksdorf in [114] and DiBenedetto in [50].

**Lemma 3.1.7.** *Let the conditions (f1)–(f4) and (g1)–(g4) be satisfied and let $\lambda > \lambda_1$. If $u \in [0, \vartheta e]$ (respectively, $u \in [-\vartheta e, 0]$) is a solution of problem (3.1.1) satisfying $u \not\equiv 0$ in $\Omega$, then it holds $u \in \operatorname{int}(C^1(\overline{\Omega})_+)$ (respectively, $u \in -\operatorname{int}(C^1(\overline{\Omega})_+)$).*

**Proof.** Let $u$ be a solution of (3.1.1) such that $0 \leq u \leq \vartheta e$. Then it follows $u \in L^\infty(\Omega)$ and thus, $u \in C^{1,\alpha}(\overline{\Omega})$ by Lieberman [89, Theorem 2] (see also Fan [61]). The conditions (f1),(f3),(g1) and (g3) (cf. Corollary 3.1.1) imply the existence of constants $c_f, c_g > 0$ such that

$$\begin{aligned} |f(x,s)| &\leq c_f s^{p-1} \qquad \text{for a.a. } x \in \Omega \text{ and all } 0 \leq s \leq \vartheta\|e\|_\infty, \\ |g(x,s)| &\leq c_g s^{p-1} \qquad \text{for a.a. } x \in \partial\Omega \text{ and all } 0 \leq s \leq \vartheta\|e\|_\infty. \end{aligned} \tag{3.1.25}$$

Applying the first line in (3.1.25) along with (3.1.1) yields $\Delta_p u \leq \widetilde{c}u^{p-1}$ a.e. in $\Omega$, where $\widetilde{c}$ is a positive constant. This allows us to apply Vázquez's strong maximum principle (see Theorem 2.4.5). We take $\beta(s) = \widetilde{c}s^{p-1}$ for all $s > 0$ which is possible because $\int_{0^+} \frac{1}{(s\beta(s))^{\frac{1}{p}}} ds = +\infty$. Hence, it holds $u > 0$ in $\Omega$. Let us assume there exists $x_0 \in \partial\Omega$ such that $u(x_0) = 0$. By applying again the maximum principle we obtain $\frac{\partial u}{\partial \nu}(x_0) < 0$. But taking into account $g(x_0, u(x_0)) = g(x_0, 0) = 0$ along with the Neumann condition in (3.1.1) yields $\frac{\partial u}{\partial \nu}(x_0) = 0$, which is a contradiction. Thus, $u > 0$ in $\overline{\Omega}$ which proves $u \in \operatorname{int}(C^1(\overline{\Omega})_+)$. The proof in case $u \in [-\vartheta e, 0]$ can be shown in an analogous manner. □

The result of the existence of extremal constant-sign solutions reads as follows.

**Theorem 3.1.8.** *Assume (f1)–(f4) and (g1)–(g4). Then for every $\lambda > \lambda_1$ there exists a smallest positive solution $u_+ = u_+(\lambda) \in \operatorname{int}(C^1(\overline{\Omega})_+)$ in the order interval $[0, \vartheta e]$ and a greatest negative solution $u_- = u_-(\lambda) \in -\operatorname{int}(C^1(\overline{\Omega})_+)$ in the order interval $[-\vartheta e, 0]$ with $\vartheta > 0$ stated in Lemma 3.1.6.*

**Proof.** We fix $\lambda > \lambda_1$. On the basis of Lemma 3.1.6, there exists an ordered pair of a positive supersolution $\overline{u} = \vartheta e \in \operatorname{int}(C^1(\overline{\Omega})_+)$ and a positive subsolution $\underline{u} = \varepsilon\varphi_1 \in \operatorname{int}(C^1(\overline{\Omega})_+)$ of problem (3.1.1) assuming $\varepsilon > 0$ is sufficiently small such that $\varepsilon\varphi_1 \leq \vartheta e$. The method of sub- and supersolution (see [18]) with respect to the order interval $[\varepsilon\varphi_1, \vartheta e]$ implies the existence of a smallest positive solution $u_\varepsilon = u_\varepsilon(\lambda)$ of problem (3.1.1) satisfying $\varepsilon\varphi_1 \leq u_\varepsilon \leq \vartheta e$ which ensures $u_\varepsilon \in \operatorname{int}(C^1(\overline{\Omega})_+)$ (see Lemma 3.1.7). Hence, for every positive integer $n$ sufficiently large there exists a smallest solution $u_n \in \operatorname{int}(C^1(\overline{\Omega})_+)$ of problem (3.1.1) in the order interval $[\frac{1}{n}\varphi_1, \vartheta e]$ and therefore, we have

$$u_n \downarrow u_+ \text{ for a.a. } x \in \Omega, \tag{3.1.26}$$

where $u_+ : \Omega \to \mathbb{R}$ is some function satisfying $0 \leq u_+ \leq \vartheta e$. We are going to show that $u_+$ is a solution of problem (3.1.1). Since $u_n$ belongs to the order interval $[\frac{1}{n}\varphi_1, \vartheta e]$, it follows that $u_n$ is bounded in $L^p(\Omega)$. Moreover, we obtain the boundedness of $u_n$ in $L^p(\partial\Omega)$ because $\gamma(u_n) \leq \gamma(\vartheta e)$. As $u_n$ solves (3.1.1) in the weak sense, one has by setting $\varphi = u_n$ in Definition 3.1.3

$$\begin{aligned}
\|\nabla u_n\|^p_{L^p(\Omega)} &\leq \int_\Omega |f(x,u_n)|u_n dx + \|u_n\|^p_{L^p(\Omega)} + \lambda\|u_n\|^p_{L^p(\partial\Omega)} + \int_\Omega |g(x,u_n)|u_n d\sigma \\
&\leq \|u_n\|^p_{L^p(\Omega)} + a_1\|u_n\|_{L^p(\Omega)} + \lambda\|u_n\|^p_{L^p(\partial\Omega)} + a_2\|u_n\|_{L^p(\partial\Omega)} \\
&\leq a_3,
\end{aligned}$$

where $a_i, i = 1, \dots, 3$, are some positive constants independent of $n$. Thus, $u_n$ is bounded in $W^{1,p}(\Omega)$. The reflexivity of $W^{1,p}(\Omega), 1 < p < \infty$, ensures the existence of a weakly convergent subsequence of $u_n$. Because of the compact embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$, the monotony of $u_n$ and the compactness of the trace operator $\gamma$, we get for the entire sequence $u_n$

$$\begin{aligned}
u_n &\rightharpoonup u_+ && \text{in } W^{1,p}(\Omega), \\
u_n &\to u_+ && \text{in } L^p(\Omega) \text{ and for a.a. } x \in \Omega, \\
u_n &\to u_+ && \text{in } L^p(\partial\Omega) \text{ and for a.a. } x \in \partial\Omega.
\end{aligned} \tag{3.1.27}$$

Due to the fact that $u_n$ solves problem (3.1.1), one has for all $\varphi \in W^{1,p}(\Omega)$

$$\begin{aligned}
&\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla\varphi dx \\
&= \int_\Omega (f(x,u_n) - u_n^{p-1})\varphi dx + \int_{\partial\Omega} (\lambda u_n^{p-1} + g(x,u_n))\varphi d\sigma.
\end{aligned} \tag{3.1.28}$$

The choice $\varphi = u_n - u_+ \in W^{1,p}(\Omega)$ is admissible in equation (3.1.28) which implies

$$\begin{aligned}&\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u_+)dx\\&= \int_\Omega (f(x,u_n) - u_n^{p-1})(u_n - u_+)dx + \int_{\partial\Omega} (\lambda u_n^{p-1} + g(x,u_n))(u_n - u_+)d\sigma.\end{aligned} \tag{3.1.29}$$

Applying (3.1.27) and the conditions (f3), (g3) results in

$$\limsup_{n\to\infty} \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u_+)dx \le 0, \tag{3.1.30}$$

which ensures by the $(S_+)$-property of $-\Delta_p$ on $W^{1,p}(\Omega)$ combined with (3.1.27)

$$u_n \to u_+ \text{ in } W^{1,p}(\Omega). \tag{3.1.31}$$

Taking into account the uniform boundedness of the sequence $(u_n)$ in combination with the strong convergence in (3.1.31) and the assumptions (f3) and (g3) allows us to pass to the limit in (3.1.28) which proves that $u_+$ is a weak solution of problem (3.1.1).
As $u_+$ is a solution of (3.1.1) belonging to $[0, \vartheta e]$, we can use Lemma 3.1.7 provided $u_+ \not\equiv 0$. We argue by contradiction and assume that $u_+ \equiv 0$ which in view of (3.1.26) results in

$$u_n(x) \downarrow 0 \quad \text{for all } x \in \Omega. \tag{3.1.32}$$

We set

$$\widetilde{u}_n = \frac{u_n}{\|u_n\|_{W^{1,p}(\Omega)}} \quad \text{for all } n. \tag{3.1.33}$$

Obviously, the sequence $(\widetilde{u}_n)$ is bounded in $W^{1,p}(\Omega)$ which implies the existence of a weakly convergent subsequence of $\widetilde{u}_n$, not relabeled, such that

$$\begin{aligned}\widetilde{u}_n &\rightharpoonup \widetilde{u} && \text{in } W^{1,p}(\Omega),\\ \widetilde{u}_n &\to \widetilde{u} && \text{in } L^p(\Omega) \text{ and for a.a. } x \in \Omega,\\ \widetilde{u}_n &\to \widetilde{u} && \text{in } L^p(\partial\Omega) \text{ and for a.a. } x \in \partial\Omega,\end{aligned} \tag{3.1.34}$$

where $\widetilde{u} : \Omega \to \mathbb{R}$ is some function belonging to $W^{1,p}(\Omega)$. Moreover, we may suppose there are functions $z_1 \in L^p(\Omega)_+, z_2 \in L^p(\partial\Omega)_+$ such that

$$\begin{aligned}|\widetilde{u}_n(x)| &\le z_1(x) && \text{for a.a. } x \in \Omega,\\ |\widetilde{u}_n(x)| &\le z_2(x) && \text{for a.a. } x \in \partial\Omega.\end{aligned} \tag{3.1.35}$$

By means of (3.1.28), we get for $\widetilde{u}_n$ the following variational equation

$$\begin{aligned}\int_\Omega |\nabla \widetilde{u}_n|^{p-2}\nabla \widetilde{u}_n \nabla\varphi dx &= \int_\Omega \left(\frac{f(x,u_n)}{u_n^{p-1}}\widetilde{u}_n^{p-1} - \widetilde{u}_n^{p-1}\right)\varphi dx + \int_{\partial\Omega} \lambda \widetilde{u}_n^{p-1}\varphi d\sigma\\&\quad + \int_{\partial\Omega} \frac{g(x,u_n)}{u_n^{p-1}}\widetilde{u}_n^{p-1}\varphi d\sigma, \qquad \forall \varphi \in W^{1,p}(\Omega).\end{aligned} \tag{3.1.36}$$

Choosing $\varphi = \widetilde{u}_n - \widetilde{u} \in W^{1,p}(\Omega)$ in (3.1.36), we obtain

$$\begin{aligned}
&\int_\Omega |\nabla \widetilde{u}_n|^{p-2} \nabla \widetilde{u}_n \nabla (\widetilde{u}_n - \widetilde{u}) dx \\
&= \int_\Omega \left( \frac{f(x,u_n)}{u_n^{p-1}} \widetilde{u}_n^{p-1} - \widetilde{u}_n^{p-1} \right) (\widetilde{u}_n - \widetilde{u}) dx + \int_{\partial\Omega} \lambda \widetilde{u}_n^{p-1} (\widetilde{u}_n - \widetilde{u}) d\sigma \\
&\quad + \int_{\partial\Omega} \frac{g(x,u_n)}{u_n^{p-1}} \widetilde{u}_n^{p-1} (\widetilde{u}_n - \widetilde{u}) d\sigma.
\end{aligned} \tag{3.1.37}$$

Using (3.1.25) along with (3.1.35) implies

$$\frac{|f(x,u_n(x))|}{u_n^{p-1}(x)} \widetilde{u}_n^{p-1}(x) |\widetilde{u}_n(x) - \widetilde{u}(x)| \leq c_f z_1(x)^{p-1} (z_1(x) + |\widetilde{u}(x)|), \tag{3.1.38}$$

respectively,

$$\frac{|g(x,u_n(x))|}{u_n^{p-1}(x)} \widetilde{u}_n^{p-1}(x) |\widetilde{u}_n(x) - \widetilde{u}(x)| \leq c_g z_2(x)^{p-1} (z_2(x) + |\widetilde{u}(x)|). \tag{3.1.39}$$

The right-hand sides of (3.1.38) and (3.1.39) are in $L^1(\Omega)$ and $L^1(\partial\Omega)$, respectively, which allows us to apply Lebesgue's Dominated Convergence Theorem (cf. Theorem 2.1.1). This fact and the convergence properties in (3.1.34) show

$$\begin{aligned}
\lim_{n\to\infty} \int_\Omega \frac{f(x,u_n)}{u_n^{p-1}} \widetilde{u}_n^{p-1} (\widetilde{u}_n - \widetilde{u}) dx = 0, \\
\lim_{n\to\infty} \int_{\partial\Omega} \frac{g(x,u_n)}{u_n^{p-1}} \widetilde{u}_n^{p-1} (\widetilde{u}_n - \widetilde{u}) d\sigma = 0.
\end{aligned} \tag{3.1.40}$$

From (3.1.34), (3.1.37), (3.1.40) we conclude

$$\limsup_{n\to\infty} \int_\Omega |\nabla \widetilde{u}_n|^{p-2} \nabla \widetilde{u}_n \nabla (\widetilde{u}_n - u_n) dx = 0.$$

Taking into account the $(S_+)$-property of $-\Delta_p$ with respect to $W^{1,p}(\Omega)$, we have

$$\widetilde{u}_n \to \widetilde{u} \quad \text{in } W^{1,p}(\Omega). \tag{3.1.41}$$

Notice that $\|\widetilde{u}\|_{W^{1,p}(\Omega)} = 1$. The statements in (3.1.32), (3.1.41) and (3.1.36) yield along with the conditions (f1),(g1)

$$\int_\Omega |\nabla \widetilde{u}|^{p-2} \nabla \widetilde{u} \nabla \varphi dx = -\int_\Omega \widetilde{u}^{p-1} \varphi dx + \int_{\partial\Omega} \lambda \widetilde{u}^{p-1} \varphi d\sigma, \quad \forall \varphi \in W^{1,p}(\Omega). \tag{3.1.42}$$

Due to $\widetilde{u} \not\equiv 0$, the equation (3.1.42) is the Steklov eigenvalue problem in (3.1.7), where $\widetilde{u} \geq 0$ is the eigenfunction corresponding to the eigenvalue $\lambda > \lambda_1$. The fact that $\widetilde{u} \geq 0$ is nonnegative in $\overline{\Omega}$ yields a contradiction to the results of Martínez et al. in [95, Lemma 2.4] because $\widetilde{u}$ must change sign on $\partial\Omega$. Thus, $u_+ \not\equiv 0$ and we obtain by applying Lemma 3.1.7 that $u_+ \in \operatorname{int}(C^1(\overline{\Omega})_+)$.

Now we need to show that $u_+$ is the smallest positive solution of (3.1.1) within $[0, \vartheta e]$. Let

$u \in W^{1,p}(\Omega)$ be a positive solution of (3.1.1) lying in the order interval $[0, \vartheta e]$. Lemma 3.1.7 implies $u \in \operatorname{int}(C^1(\overline{\Omega})_+)$. Then there exists an integer $n$ sufficiently large such that $u \in [\frac{1}{n}\varphi_1, \vartheta e]$. On the basis that $u_n$ is the smallest solution of (3.1.1) in $[\frac{1}{n}\varphi_1, \vartheta e]$ it holds $u_n \le u$. This yields by passing to the limit $u_+ \le u$. Hence, $u_+$ must be the smallest positive solution of (3.1.1). In similar way one proves the existence of the greatest negative solution of (3.1.1) within $[-\vartheta e, 0]$. This completes the proof of the theorem. $\square$

### 3.1.4 Variational Characterization of Extremal Solutions

Theorem 3.1.8 implies the existence of extremal positive and negative solutions of (3.1.1) for all $\lambda > \lambda_1$ denoted by $u_+ = u_+(\lambda) \in \operatorname{int}(C^1(\overline{\Omega})_+)$ and $u_- = u_-(\lambda) \in -\operatorname{int}(C^1(\overline{\Omega})_+)$, respectively. Now, we introduce truncation functions $T_+, T_-, T_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ and $T_+^{\partial\Omega}, T_-^{\partial\Omega}, T_0^{\partial\Omega} : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ as follows.

$$T_+(x,s) = \begin{cases} 0 & \text{if } s \le 0 \\ s & \text{if } 0 < s < u_+(x) \\ u_+(x) & \text{if } s \ge u_+(x) \end{cases}, \quad T_+^{\partial\Omega}(x,s) = \begin{cases} 0 & \text{if } s \le 0 \\ s & \text{if } 0 < s < u_+(x) \\ u_+(x) & \text{if } s \ge u_+(x) \end{cases}$$

$$T_-(x,s) = \begin{cases} u_-(x) & \text{if } s \le u_-(x) \\ s & \text{if } u_-(x) < s < 0 \\ 0 & \text{if } s \ge 0 \end{cases}, \quad T_-^{\partial\Omega}(x,s) = \begin{cases} u_-(x) & \text{if } s \le u_-(x) \\ s & \text{if } u_-(x) < s < 0 \\ 0 & \text{if } s \ge 0 \end{cases}$$

$$T_0(x,s) = \begin{cases} u_-(x) & \text{if } s \le u_-(x) \\ s & \text{if } u_-(x) < s < u_+(x) \\ u_+(x) & \text{if } s \ge u_+(x) \end{cases}, \quad T_0^{\partial\Omega}(x,s) = \begin{cases} u_-(x) & \text{if } s \le u_-(x) \\ s & \text{if } u_-(x) < s < u_+(x) \\ u_+(x) & \text{if } s \ge u_+(x) \end{cases}$$

For $u \in W^{1,p}(\Omega)$ the truncation operators on $\partial\Omega$ apply to the corresponding traces $\gamma(u)$. We just write for simplification $T_+^{\partial\Omega}(x,u)$, $T_+^{\partial\Omega}(x,u)$, $T_+^{\partial\Omega}(x,u)$ without $\gamma$. Furthermore, the truncation operators are continuous and uniformly bounded on $\mathbb{R}$ and they are Lipschitz continuous with respect to the second argument (see, e.g. [77]). By means of these truncations, we define the following associated functionals given by

$$\begin{aligned} E_+(u) = &\frac{1}{p}[\|\nabla u\|_{L^p(\Omega)}^p + \|u\|_{L^p(\Omega)}^p] - \int_\Omega \int_0^{u(x)} f(x, T_+(x,s))dsdx \\ &- \int_{\partial\Omega} \int_0^{u(x)} \left[\lambda T_+^{\partial\Omega}(x,s)^{p-1} + g(x, T_+^{\partial\Omega}(x,s))\right] dsd\sigma, \end{aligned} \tag{3.1.43}$$

$$\begin{aligned} E_-(u) = &\frac{1}{p}[\|\nabla u\|_{L^p(\Omega)}^p + \|u\|_{L^p(\Omega)}^p] - \int_\Omega \int_0^{u(x)} f(x, T_-(x,s))dsdx \\ &- \int_{\partial\Omega} \int_0^{u(x)} \left[\lambda |T_-^{\partial\Omega}(x,s)|^{p-2} T_-^{\partial\Omega}(x,s) + g(x, T_-^{\partial\Omega}(x,s))\right] dsd\sigma, \end{aligned} \tag{3.1.44}$$

$$\begin{aligned} E_0(u) =&\frac{1}{p}[\|\nabla u\|^p_{L^p(\Omega)} + \|u\|^p_{L^p(\Omega)}] - \int_\Omega \int_0^{u(x)} f(x, T_0(x,s))dsdx \\ &- \int_{\partial\Omega} \int_0^{u(x)} \left[\lambda |T_0^{\partial\Omega}(x,s)|^{p-2} T_0^{\partial\Omega}(x,s) + g(x, T_0^{\partial\Omega}(x,s))\right] dsd\sigma, \end{aligned} \tag{3.1.45}$$

which are well-defined and belong to $C^1(W^{1,p}(\Omega))$.

**Lemma 3.1.9.** *The functionals $E_+, E_-, E_0 : W^{1,p}(\Omega) \to \mathbb{R}$ are coercive and weakly sequentially lower semicontinuous.*

**Proof.** First, we introduce the Nemytskij operators $F, F^\Omega : L^p(\Omega) \to L^q(\Omega)$ and $G, F^{\partial\Omega} : L^p(\partial\Omega) \to L^q(\partial\Omega)$ by

$$\begin{aligned} &Fu(x) = f(x, T_+(x, u(x))), &&F^\Omega u(x) = |u(x)|^{p-2}u(x), \\ &Gu(x) = g(x, T_+^{\partial\Omega}(x, u(x))), &&F^{\partial\Omega} u(x) = \lambda |T_+^{\partial\Omega}(x, u(x))|^{p-2} T_+^{\partial\Omega}(x, u(x)). \end{aligned}$$

It is clear that $E_+ \in C^1(W^{1,p}(\Omega))$. The embedding $i : W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ and the trace operator $\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ are compact. We set

$$\begin{aligned} &\widehat{F} := i^* \circ F \circ i : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*, \\ &\widehat{F}^\Omega := i^* \circ F^\Omega \circ i : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*, \\ &\widehat{G} := \gamma^* \circ G \circ \gamma : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*, \\ &\widehat{F}^{\partial\Omega} := \gamma^* \circ F^{\partial\Omega} \circ \gamma : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*, \end{aligned}$$

where $i^* : L^q(\Omega) \to (W^{1,p}(\Omega))^*$ and $\gamma^* : L^q(\partial\Omega) \to (W^{1,p}(\Omega))^*$ denote the adjoint operators. With a view to (3.1.43) we obtain

$$\langle E_+'(u), \varphi \rangle = \langle -\Delta_p u, \varphi \rangle + \langle \widehat{F}^\Omega u, \varphi \rangle - \langle \widehat{F} u, \varphi \rangle - \langle \widehat{F}^{\partial\Omega} u + \widehat{G} u, \varphi \rangle, \tag{3.1.46}$$

where $\langle \cdot, \cdot \rangle$ stands for the duality pairing between $W^{1,p}(\Omega)$ and its dual space $(W^{1,p}(\Omega))^*$. The operators $\widehat{F}, \widehat{F}^\Omega, \widehat{F}^{\partial\Omega}$ and $\widehat{G}$ are bounded, completely continuous and hence also pseudomonotone. Since the sum of pseudomonotone operators is also pseudomonotone, we obtain that $E_+' : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ is pseudomonotone. Note that the negative $p$-Laplacian $-\Delta_p : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ is bounded and pseudomonotone for $1 < p < \infty$. Using Proposition 2.4.2 shows that $E_+$ is weakly sequentially lower semicontinuous. Applying the assumptions in (f3),(g3), the boundedness of the truncation operators and the trace operator $\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$, we obtain for a positive constant $c$

$$\frac{E_+(u)}{\|u\|_{W^{1,p}(\Omega)}} \geq \frac{\frac{1}{p}\|u\|^p_{W^{1,p}(\Omega)} - c\|u\|_{W^{1,p}(\Omega)}}{\|u\|_{W^{1,p}(\Omega)}} \to \infty \text{ as } \|u\|_{W^{1,p}(\Omega)} \to \infty,$$

which proves the coercivity. In the same manner, one shows this lemma for $E_-$ and $E_0$, respectively. $\square$

**Lemma 3.1.10.** *Let $u_+$ and $u_-$ be the extremal constant-sign solutions of (3.1.1). Then the following holds:*

*(i) A critical point $v \in W^{1,p}(\Omega)$ of $E_+$ is a (nonnegative) solution of (3.1.1) satisfying $0 \leq v \leq u_+$.*

*(ii) A critical point $v \in W^{1,p}(\Omega)$ of $E_-$ is a (nonpositive) solution of (3.1.1) satisfying $u_- \leq v \leq 0$.*

*(iii) A critical point $v \in W^{1,p}(\Omega)$ of $E_0$ is a solution of (3.1.1) satisfying $u_- \leq v \leq u_+$.*

**Proof.** Let $v$ be a critical point of $E_+$, that is, it holds $E_+'(v) = 0$. In view of (3.1.46) we obtain

$$\begin{aligned}
&\int_\Omega |\nabla v|^{p-2}\nabla v \nabla\varphi dx \\
&= \int_\Omega [f(x, T_+(x,v)) - |v|^{p-2}v]\varphi dx \\
&\quad + \int_{\partial\Omega} [\lambda T_+^{\partial\Omega}(x,v)^{p-1} + g(x, T_+^{\partial\Omega}(x,v))]\varphi d\sigma, \quad \forall \varphi \in W^{1,p}(\Omega).
\end{aligned} \tag{3.1.47}$$

Since $u_+$ is a positive solution of (3.1.1) we have by Definition 3.1.3

$$\begin{aligned}
\int_\Omega |\nabla u_+|^{p-2}\nabla u_+ \nabla\varphi dx = &\int_\Omega [f(x,u_+) - u_+^{p-1}]\varphi dx \\
&+ \int_{\partial\Omega} [\lambda u_+^{p-1} + g(x,u_+)]\varphi d\sigma, \quad \forall \varphi \in W^{1,p}(\Omega).
\end{aligned} \tag{3.1.48}$$

Choosing $\varphi = (v-u_+)^+ \in W^{1,p}(\Omega)$ in (3.1.48) and (3.1.47) and subtracting (3.1.48) from (3.1.47) results in

$$\begin{aligned}
&\int_\Omega [|\nabla v|^{p-2}\nabla v - |\nabla u_+|^{p-2}\nabla u_+]\nabla(v-u_+)^+ dx + \int_\Omega [|v|^{p-2}v - u_+^{p-1}](v-u_+)^+ dx \\
&= \int_\Omega [f(x, T_+(x,v)) - f(x,u_+)](v-u_+)^+ dx \\
&\quad + \int_{\partial\Omega} [\lambda T_+^{\partial\Omega}(x,v)^{p-1} - \lambda u_+^{p-1} + g(x, T_+^{\partial\Omega}(x,v)) - g(x,u_+)](v-u_+)^+ d\sigma \\
&= 0,
\end{aligned}$$

by the definition of $T_+$ and $T_+^{\partial\Omega}$, respectively. The monotonicity inequalities in Section 2.1.4 provide for $v > u_+$

$$0 = \int_\Omega [|\nabla v|^{p-2}\nabla v - |\nabla u_+|^{p-2}\nabla u_+]\nabla(v-u_+)^+ dx + \int_\Omega [|v|^{p-2}v - u_+^{p-1}](v-u_+)^+ dx > 0,$$

which is a contradiction. This implies $(v-u_+)^+ = 0$ and thus, $v \leq u_+$. Taking $\varphi = v^- = \max(-v, 0)$ in (3.1.47) yields

$$\int_{\{x: v(x)<0\}} |\nabla v|^p dx + \int_{\{x: v(x)<0\}} |v|^p \varphi dx = 0,$$

consequently, it holds $\|v^-\|^p_{W^{1,p}(\Omega)} = 0$ and equivalently $v^- = 0$, that is, $v \geq 0$. By the definition of the truncation operators we see at once that $T_+(x, v) = v$, $T_+^{\partial\Omega}(x, v) = v$ and therefore, $v$ is a solution of (3.1.1) satisfying $0 \leq v \leq u_+$. The statements in (ii) and (iii) can be shown in a similar way. $\square$

The next result matches $C^1(\overline{\Omega})$ and $W^{1,p}(\Omega)$-local minimizers for a large class of $C^1-$ functionals. We will show that every local $C^1-$minimizer of $E_0$ is a local $W^{1,p}(\Omega)$-minimizer of $E_0$. This result was first proven for the Dirichlet problem by Brezis and Nirenberg [14] if $p = 2$ and was extended by García Azorero et al. in [72] for $p \neq 2$ (see also [76] when $p > 2$). For the zero Neumann problem we refer to the recent results of Motreanu et al. in [100] for $1 < p < \infty$. In case of nonsmooth functionals the authors in [102] and [11] prove the same result for the Dirichlet problem and the zero Neumann problem when $p \geq 2$. We give the proof for the nonlinear nonzero Neumann problem for any $1 < p < \infty$.

**Proposition 3.1.11.** *If $z_0 \in W^{1,p}(\Omega)$ is a local $C^1(\overline{\Omega})$-minimizer of $E_0$ meaning that there exists $r_1 > 0$ such that*

$$E_0(z_0) \leq E_0(z_0 + h) \quad \textit{for all } h \in C^1(\overline{\Omega}) \textit{ with } \|h\|_{C^1(\overline{\Omega})} \leq r_1,$$

*then $z_0$ is a local minimizer of $E_0$ in $W^{1,p}(\Omega)$ meaning that there exists $r_2 > 0$ such that*

$$E_0(z_0) \leq E_0(z_0 + h) \quad \textit{for all } h \in W^{1,p}(\Omega) \textit{ with } \|h\|_{W^{1,p}(\Omega)} \leq r_2.$$

**Proof.** Let $h \in C^1(\overline{\Omega})$. If $\beta > 0$ is small, we have

$$0 \leq \frac{E_0(z_0 + \beta h) - E_0(z_0)}{\beta},$$

meaning that the directional derivative of $E_0$ at $z_0$ in direction $h$ satisfies

$$0 \leq E_0'(z_0; h) \qquad \text{for all } h \in C^1(\overline{\Omega}).$$

We recall that $h \mapsto E_0'(z_0; h)$ is continuous on $W^{1,p}(\Omega)$ and the density of $C^1(\overline{\Omega})$ in $W^{1,p}(\Omega)$ results in

$$0 \leq E_0'(z_0; h) \qquad \text{for all } h \in W^{1,p}(\Omega).$$

Therefore, setting $-h$ instead of $h$, we get

$$0 = E_0'(z_0),$$

which yields

$$\begin{aligned} 0 = &\int_\Omega |\nabla z_0|^{p-2} \nabla z_0 \nabla \varphi dx - \int_\Omega (f(x, z_0) - |z_0|^{p-2} z_0) \varphi dx \\ &- \int_{\partial\Omega} \lambda |z_0|^{p-2} z_0 \varphi d\sigma - \int_{\partial\Omega} g(x, z_0) \varphi d\sigma, \quad \forall \varphi \in W^{1,p}(\Omega). \end{aligned} \tag{3.1.49}$$

By means of Lemma 3.1.10, we obtain $u_- \leq z_0 \leq u_+$ and thus, $z_0 \in L^\infty(\Omega)$. As before, the regularity results of Lieberman [89] imply $z_0 \in C^{1,\alpha}(\overline{\Omega}), \alpha \in (0,1)$.
Let us assume that the proposition is not valid. The functional $E_0 : W^{1,p}(\Omega) \to \mathbb{R}$ is weakly sequentially lower semicontinuous (cf. Lemma 3.1.9 ) and the set $\overline{B}_\varepsilon = \{y \in W^{1,p}(\Omega) : \|y\|_{W^{1,p}(\Omega)} \leq \varepsilon\}$ is weakly compact in $W^{1,p}(\Omega)$. Thus, for any $\varepsilon > 0$ we can find $y_\varepsilon \in \overline{B}_\varepsilon$ such that

$$E_0(z_0 + y_\varepsilon) = \min\{E_0(z_0 + y) : y \in \overline{B}_\varepsilon)\} < E_0(z_0). \tag{3.1.50}$$

Obviously, $y_\varepsilon$ is a solution of the following minimum-problem

$$\begin{cases} \min E_0(z_0 + y) \\ y \in \overline{B}_\varepsilon, g_\varepsilon(y) := \frac{1}{p}(\|y\|^p_{W^{1,p}(\Omega)} - \varepsilon^p) \leq 0. \end{cases}$$

Applying the Lagrange multiplier rule (see, e.g., [92] or [42]) yields the existence of a multiplier $\lambda_\varepsilon > 0$ such that

$$E_0'(z_0 + y_\varepsilon) + \lambda_\varepsilon g_\varepsilon'(y_\varepsilon) = 0, \tag{3.1.51}$$

which results in

$$\begin{aligned} &\int_\Omega |\nabla(z_0 + y_\varepsilon)|^{p-2}\nabla(z_0 + y_\varepsilon)\nabla\varphi dx \\ &- \int_\Omega (f(x, T_0(x, z_0 + y_\varepsilon)) - |z_0 + y_\varepsilon|^{p-2}(z_0 + y_\varepsilon))\varphi dx \\ &- \int_{\partial\Omega} (\lambda |T_0^{\partial\Omega}(x, z_0 + y_\varepsilon)|^{p-2} T_0^{\partial\Omega}(x, z_0 + y_\varepsilon) + g(x, T_0^{\partial\Omega}(x, z_0 + y_\varepsilon)))\varphi d\sigma \\ &+ \lambda_\varepsilon \int_\Omega |\nabla y_\varepsilon|^{p-2}\nabla y_\varepsilon \nabla\varphi dx + \lambda_\varepsilon \int_\Omega |y_\varepsilon|^{p-2} y_\varepsilon \varphi dx = 0, \end{aligned} \tag{3.1.52}$$

for all $\varphi \in W^{1,p}(\Omega)$. Notice that $\lambda_\varepsilon$ cannot be zero since the constraints guarantee that $y_\varepsilon$ belongs to $\overline{B}_\varepsilon$. Let $0 < \lambda_\varepsilon \leq 1$ for all $\varepsilon \in (0,1]$. We multiply (3.1.49) with $\lambda_\varepsilon$, set $v_\varepsilon = z_0 + y_\varepsilon$ in (3.1.52) and add these new equations. One obtains

$$\begin{aligned} &\int_\Omega |\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon \nabla\varphi dx + \lambda_\varepsilon \int_\Omega |\nabla z_0|^{p-2}\nabla z_0 \nabla\varphi dx \\ &+ \lambda_\varepsilon \int_\Omega |\nabla(v_\varepsilon - z_0)|^{p-2}\nabla(v_\varepsilon - z_0)\nabla\varphi dx \\ &= \int_\Omega (\lambda_\varepsilon f(x, z_0) + f(x, T_0(x, v_\varepsilon)))\varphi dx \\ &\quad - \int_\Omega (\lambda_\varepsilon |z_0|^{p-2} z_0 + |v_\varepsilon|^{p-2} v_\varepsilon + \lambda_\varepsilon |v_\varepsilon - z_0|^{p-2}(v_\varepsilon - z_0))\varphi dx \\ &\quad + \int_{\partial\Omega} \lambda(\lambda_\varepsilon |z_0|^{p-2} z_0 + |T_0^{\partial\Omega}(x, v_\varepsilon)|^{p-2} T_0^{\partial\Omega}(x, v_\varepsilon))\varphi d\sigma \\ &\quad + \int_{\partial\Omega} (\lambda_\varepsilon g(x, z_0) + g(x, T_0^{\partial\Omega}(x, v_\varepsilon)))\varphi d\sigma. \end{aligned} \tag{3.1.53}$$

Now, we introduce the maps $\mathcal{A}_\varepsilon : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$, $\mathcal{B}_\varepsilon : \Omega \times \mathbb{R} \to \mathbb{R}$ and $\Phi_\varepsilon : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$\begin{aligned}
\mathcal{A}_\varepsilon(x,\xi) &= |\xi|^{p-2}\xi + \lambda_\varepsilon |H|^{p-2}H + \lambda_\varepsilon|\xi - H|^{p-2}(\xi - H),\\
-\mathcal{B}_\varepsilon(x,\psi) &= \lambda_\varepsilon f(x,z_0) + f(x, T_0(x,\psi))\\
&\quad - (\lambda_\varepsilon |z_0|^{p-2}z_0 + |\psi|^{p-2}\psi + \lambda_\varepsilon|\psi - z_0|^{p-2}(\psi - z_0)),\\
\Phi_\varepsilon(x,\psi) &= \lambda(\lambda_\varepsilon|z_0|^{p-2}z_0 + |T_0^{\partial\Omega}(x,\psi)|^{p-2}T_0^{\partial\Omega}(x,\psi)) + \lambda_\varepsilon g(x,z_0) + g(x, T_0^{\partial\Omega}(x,\psi)),
\end{aligned}$$

where $H(x) = \nabla z_0(x)$ and $H \in (C^\alpha(\overline{\Omega}))^N$ for some $\alpha \in (0,1]$. Apparently, the operator $\mathcal{A}_\varepsilon(x,\xi)$ belongs to $C(\overline{\Omega} \times \mathbb{R}^N, \mathbb{R}^N)$. For $x \in \Omega$ we have

$$\begin{aligned}
&(\mathcal{A}_\varepsilon(x,\xi),\xi)_{\mathbb{R}^N}\\
&= \|\xi\|^p + \lambda_\varepsilon(|\xi - H|^{p-2}(\xi - H) - |-H|^{p-2}(-H), \xi - H - (-H))_{\mathbb{R}^N} \qquad (3.1.54)\\
&\geq \|\xi\|^p \quad \text{for all } \xi \in \mathbb{R}^N,
\end{aligned}$$

where $(\cdot,\cdot)_{\mathbb{R}^N}$ stands for the inner product in $\mathbb{R}^N$. (3.1.54) shows that $\mathcal{A}_\varepsilon$ satisfies a strong ellipticity condition. Hence, the equation in (3.1.53) is the weak formulation of the elliptic Neumann problem

$$\begin{aligned}
-\operatorname{div}\mathcal{A}_\varepsilon(x,\nabla v_\varepsilon) + \mathcal{B}_\varepsilon(x,v_\varepsilon) &= 0 \qquad \text{in } \Omega,\\
\frac{\partial v_\varepsilon}{\partial \nu} &= \Phi_\varepsilon(x,v_\varepsilon) \qquad \text{on } \partial\Omega,
\end{aligned} \qquad (3.1.55)$$

where $\frac{\partial v_\varepsilon}{\partial \nu}$ denotes the conormal derivative of $v_\varepsilon$.
To prove the $L^\infty-$regularity of $v_\varepsilon$, we will use the Moser iteration technique (see e.g. [53], [54], [55], [56], [85]). It suffices to consider the proof in case $1 \leq p \leq N$, otherwise we would be done. First we are going to show that $v_\varepsilon^+ = \max\{v_\varepsilon, 0\}$ belongs to $L^\infty(\Omega)$. For $M > 0$ we define $v_M(x) = \min\{v_\varepsilon^+(x), M\}$. Letting $K(t) = t$ if $t \leq M$ and $K(t) = M$ if $t > M$, it follows by [85, Theorem B.3] that $K \circ v_\varepsilon^+ = v_M \in W^{1,p}(\Omega)$ and hence $v_M \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. For real $k \geq 0$ we choose $\varphi = v_M^{kp+1}$, then $\nabla\varphi = (kp+1)v_M^{kp}\nabla v_M$ and $\varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. Notice that $v_\varepsilon(x) \leq 0$ implies directly $v_M(x) = 0$. Testing (3.1.53) with $\varphi = v_M^{kp+1}$, one gets

$$\begin{aligned}
&(kp+1)\int_\Omega |\nabla v_\varepsilon^+|^{p-2}\nabla v_\varepsilon^+ \nabla v_M v_M^{kp} dx + \int_\Omega |v_\varepsilon^+|^{p-2} v_\varepsilon^+ v_M^{kp+1} dx\\
&+ \lambda_\varepsilon(kp+1)\int_\Omega \Big[|\nabla(v_\varepsilon^+ - z_0)|^{p-2}\nabla(v_\varepsilon^+ - z_0) - |-\nabla z_0|^{p-2}(-\nabla z_0)\Big]\\
&\quad \times (\nabla v_M - \nabla z_0 - (-\nabla z_0)) v_M^{kp} dx\\
&= \int_\Omega (\lambda_\varepsilon f(x,z_0) + f(x, T_0(x,v_\varepsilon^+)))v_M^{kp+1} dx \qquad (3.1.56)\\
&\quad - \int_\Omega (\lambda_\varepsilon|z_0|^{p-2}z_0 + \lambda_\varepsilon|v_\varepsilon^+ - z_0|^{p-2}(v_\varepsilon^+ - z_0))v_M^{kp+1} dx\\
&\quad + \int_{\partial\Omega} \lambda(\lambda_\varepsilon|z_0|^{p-2}z_0 + |T_0^{\partial\Omega}(x,v_\varepsilon^+)|^{p-2}T_0^{\partial\Omega}(x,v_\varepsilon^+)))v_M^{kp+1} d\sigma\\
&\quad + \int_{\partial\Omega} (\lambda_\varepsilon g(x,z_0) + g(x, T_0^{\partial\Omega}(x,v_\varepsilon^+)))v_M^{kp+1} d\sigma.
\end{aligned}$$

Since $z_0 \in [u_-, u_+]$, $\gamma(z_0) \in [\gamma(u_-), \gamma(u_+)]$, $T_0(x, v_\varepsilon) \in [u_-, u_+]$ and $T_0^{\partial\Omega}(x, v_\varepsilon) \in [\gamma(u_-), \gamma(u_+)]$ we get for the right-hand side of (3.1.56) by using (f3) and (g3)

$$\begin{aligned}
&(1)\ \int_\Omega (\lambda_\varepsilon f(x, z_0) + f(x, T_0(x, v_\varepsilon^+)))v_M^{kp+1} dx \le e_1 \int_\Omega (v_\varepsilon^+)^{kp+1} dx \\
&(2)\ -\int_\Omega (\lambda_\varepsilon |z_0|^{p-2} z_0 + \lambda_\varepsilon |v_\varepsilon^+ - z_0|^{p-2}(v_\varepsilon^+ - z_0))v_M^{kp+1} dx \\
&\qquad \le e_2 \int_\Omega |v_\varepsilon^+|^{p-1}(v_\varepsilon^+)^{kp+1} dx + e_3 \int_\Omega |z_0|^{p-1}(v_\varepsilon^+)^{kp+1} dx \\
&\qquad \le \int_\Omega e_2(v_\varepsilon^+)^{(k+1)p} dx + e_4 \int_\Omega (v_\varepsilon^+)^{kp+1} dx \\
&(3)\ \int_{\partial\Omega} \lambda(\lambda_\varepsilon |z_0|^{p-2} z_0 + |T_0^{\partial\Omega}(x, v_\varepsilon^+)|^{p-2} T_0^{\partial\Omega}(x, v_\varepsilon^+)))v_M^{kp+1} d\sigma \\
&\qquad \le e_5 \int_{\partial\Omega} (v_\varepsilon^+)^{kp+1} d\sigma \\
&(4)\ \int_{\partial\Omega} (\lambda_\varepsilon g(x, z_0) + g(x, T_0^{\partial\Omega}(x, v_\varepsilon^+)))v_M^{kp+1} d\sigma \\
&\qquad \le e_6 \int_{\partial\Omega} (v_\varepsilon^+)^{kp+1} d\sigma.
\end{aligned} \tag{3.1.57}$$

The left-hand side of (3.1.56) can be estimated to obtain

$$\begin{aligned}
&(kp+1)\int_\Omega |\nabla v_\varepsilon^+|^{p-2}\nabla v_\varepsilon^+ \nabla v_M v_M^{kp} dx + \int_\Omega |v_\varepsilon^+|^{p-2} v_\varepsilon^+ v_M^{kp+1} dx \\
&+ \lambda_\varepsilon(kp+1)\int_\Omega \Big[|\nabla(v_\varepsilon^+ - z_0)|^{p-2}\nabla(v_\varepsilon^+ - z_0) - |-\nabla z_0|^{p-2}(-\nabla z_0)\Big] \\
&\quad \times (\nabla v_M - \nabla z_0 - (-\nabla z_0))v_M^{kp} dx \\
&\ge (kp+1)\int_\Omega |\nabla v_M|^p v_M^{kp} dx + \int_\Omega (v_\varepsilon^+)^{p-1} v_M^{kp+1} dx \\
&\ge \frac{kp+1}{(k+1)^p}\left[\int_\Omega |\nabla v_M^{k+1}|^p dx + \int_\Omega (v_\varepsilon^+)^{p-1} v_M^{kp+1} dx\right].
\end{aligned} \tag{3.1.58}$$

Using the Hölder inequality we see at once

$$\int_\Omega 1\cdot(v_\varepsilon^+)^{kp+1} dx \le |\Omega|^{\frac{p-1}{(k+1)p}} \left(\int_\Omega (v_\varepsilon^+)^{(k+1)p} dx\right)^{\frac{kp+1}{(k+1)p}}, \tag{3.1.59}$$

and analogous for the boundary integral

$$\int_{\partial\Omega} 1\cdot(v_\varepsilon^+)^{kp+1} d\sigma \le |\partial\Omega|^{\frac{p-1}{(k+1)p}} \left(\int_{\partial\Omega} (v_\varepsilon^+)^{(k+1)p} d\sigma\right)^{\frac{kp+1}{(k+1)p}}. \tag{3.1.60}$$

Applying the estimates (3.1.57)–(3.1.60) to (3.1.56) one gets

$$\begin{aligned}
&\frac{kp+1}{(k+1)^p}\left[\int_\Omega |\nabla v_M^{k+1}|^p dx + \int_\Omega (v_\varepsilon^+)^{p-1} v_M^{kp+1} dx\right] \\
&\le e_2 \int_\Omega (v_\varepsilon^+)^{(k+1)p} dx + e_7 \left(\int_\Omega (v_\varepsilon^+)^{(k+1)p} dx\right)^{\frac{kp+1}{(k+1)p}} + e_8 \left(\int_{\partial\Omega} (v_\varepsilon^+)^{(k+1)p} d\sigma\right)^{\frac{kp+1}{(k+1)p}}.
\end{aligned}$$

We have $\lim_{M\to\infty} v_M(x) = v_\varepsilon^+(x)$ for a.a. $x \in \Omega$ and can apply Fatou's Lemma which results in

$$\begin{aligned} &\frac{kp+1}{(k+1)^p}\left[\int_\Omega |\nabla (v_\varepsilon^+)^{k+1}|^p dx + \int_\Omega |(v_\varepsilon^+)^{k+1}|^p dx\right] \\ &\leq e_2 \int_\Omega (v_\varepsilon^+)^{(k+1)p} dx + e_7 \left(\int_\Omega (v_\varepsilon^+)^{(k+1)p} dx\right)^{\frac{kp+1}{(k+1)p}} \\ &\quad + e_8 \left(\int_{\partial\Omega} (v_\varepsilon^+)^{(k+1)p} d\sigma\right)^{\frac{kp+1}{(k+1)p}}. \end{aligned} \tag{3.1.61}$$

We have either

$$\left(\int_\Omega (v_\varepsilon^+)^{(k+1)p} dx\right)^{\frac{kp+1}{(k+1)p}} \leq 1 \quad \text{or} \quad \left(\int_\Omega (v_\varepsilon^+)^{(k+1)p} dx\right)^{\frac{kp+1}{(k+1)p}} \leq \int_\Omega (v_\varepsilon^+)^{(k+1)p} dx,$$

respectively, either

$$\left(\int_{\partial\Omega} (v_\varepsilon^+)^{(k+1)p} d\sigma\right)^{\frac{kp+1}{(k+1)p}} \leq 1 \quad \text{or} \quad \left(\int_{\partial\Omega} (v_\varepsilon^+)^{(k+1)p} d\sigma\right)^{\frac{kp+1}{(k+1)p}} \leq \int_{\partial\Omega} (v_\varepsilon^+)^{(k+1)p} d\sigma.$$

From (3.1.61) we obtain

$$\begin{aligned} &\frac{kp+1}{(k+1)^p}\left[\int_\Omega |\nabla (v_\varepsilon^+)^{k+1}|^p dx + \int_\Omega |(v_\varepsilon^+)^{k+1}|^p dx\right] \\ &\leq e_9 \int_\Omega (v_\varepsilon^+)^{(k+1)p} dx + e_{10} \int_{\partial\Omega} (v_\varepsilon^+)^{(k+1)p} d\sigma + e_{11}. \end{aligned} \tag{3.1.62}$$

Next we want to estimate the boundary integral by an integral in the domain $\Omega$. To this end, we need the following continuous embeddings

$$\begin{aligned} &T_1 : B^s_{pp}(\Omega) \to B^{s-\frac{1}{p}}_{pp}(\partial\Omega), \quad \text{with } s > \frac{1}{p}, \\ &T_2 : B^{s-\frac{1}{p}}_{pp}(\partial\Omega) = F^{s-\frac{1}{p}}_{pp}(\partial\Omega) \to F^0_{p2}(\partial\Omega) = L^p(\partial\Omega), \quad \text{with } s > \frac{1}{p}, \end{aligned}$$

where $\Omega$ is a bounded $C^\infty$-domain (see [112, Page 75 and Page 82], [116, 2.3.1 and 2.3.2] and [117, 3.3.1]). Let $s = m + \iota$ with $m \in \mathbb{N}_0$ and $0 \leq \iota < 1$. Then the embeddings are also valid if $\partial\Omega \in C^{m,1}$ ([113]). In [51, Satz 9.40] a similar proof is given for $p = 2$, however, it can be extended to $p \in (1, \infty)$ by using the Fourier transformation in $L^p(\Omega)$ ([52]).
Here $B^s_{pq}$ and $F^s_{pq}$ denote the Besov and Lizorkin-Triebel spaces, respectively, which are equal in case $p = q$ with $1 < p < \infty$ and $-\infty < s < \infty$. We set $s = \frac{1}{p} + \widetilde{\varepsilon}$, where $\widetilde{\varepsilon} > 0$ is arbitrarily fixed such that $s = \frac{1}{p} + \widetilde{\varepsilon} < 1$. Thus the embeddings are valid for a Lipschitz boundary $\partial\Omega$. This yields the continuous embedding

$$T_3 : B^{\frac{1}{p}+\widetilde{\varepsilon}}_{pp}(\Omega) \to L^p(\partial\Omega). \tag{3.1.63}$$

The real interpolation theory implies

$$\left(F^0_{p2}(\Omega), F^1_{p2}(\Omega)\right)_{\frac{1}{p}+\widetilde{\varepsilon},p} = \left(L^p(\Omega), W^{1,p}(\Omega)\right)_{\frac{1}{p}+\widetilde{\varepsilon},p} = B_{pp}^{\frac{1}{p}+\widetilde{\varepsilon}}(\Omega),$$

(for more details see [2], [116], [117]) which ensures the norm estimate

$$\|v\|_{B_{pp}^{\frac{1}{p}+\widetilde{\varepsilon}}(\Omega)} \leq e_{12}\|v\|^{\frac{1}{p}+\widetilde{\varepsilon}}_{W^{1,p}(\Omega)}\|v\|^{1-\frac{1}{p}-\widetilde{\varepsilon}}_{L^p(\Omega)}, \quad \forall v \in W^{1,p}(\Omega) \tag{3.1.64}$$

with a positive constant $e_{12}$. Using (3.1.63), (3.1.64) and Young's inequality yields

$$\begin{aligned}
&\int_{\partial\Omega}((v_\varepsilon^+)^{k+1})^p d\sigma \\
&= \|(v_\varepsilon^+)^{k+1}\|^p_{L^p(\partial\Omega)} \\
&\leq e_{13}^p\|(v_\varepsilon^+)^{k+1}\|^p_{B_{pp}^{\frac{1}{p}+\widetilde{\varepsilon}}(\Omega)} \\
&\leq e_{13}^p e_{12}^p\|(v_\varepsilon^+)^{k+1}\|^{\left(\frac{1}{p}+\widetilde{\varepsilon}\right)p}_{W^{1,p}(\Omega)}\|(v_\varepsilon^+)^{k+1}\|^{\left(1-\frac{1}{p}-\widetilde{\varepsilon}\right)p}_{L^p(\Omega)} \\
&\leq e_{13}^p e_{12}^p(\delta\|(v_\varepsilon^+)^{k+1}\|^{(1+\widetilde{\varepsilon}p)\widetilde{q}}_{W^{1,p}(\Omega)} + C(\delta)\|(v_\varepsilon^+)^{k+1}\|^{(p-1-\widetilde{\varepsilon}p)\widetilde{q}'}_{L^p(\Omega)}) \\
&= e_{13}^p e_{12}^p(\delta\|(v_\varepsilon^+)^{k+1}\|^p_{W^{1,p}(\Omega)} + C(\delta)\|(v_\varepsilon^+)^{k+1}\|^p_{L^p(\Omega)}),
\end{aligned} \tag{3.1.65}$$

where $\widetilde{q} = \frac{p}{1+\widetilde{\varepsilon}p}$ and $\widetilde{q}' = \frac{p}{p-1-\widetilde{\varepsilon}p}$ satisfy $\frac{1}{\widetilde{q}} + \frac{1}{\widetilde{q}'} = 1$ and $\delta$ is a free parameter to be specified later. Note that the positive constant $C(\delta)$ depends only on $\delta$. Applying (3.1.65) to (3.1.62) shows

$$\begin{aligned}
&\frac{kp+1}{(k+1)^p}\left[\int_\Omega |\nabla(v_\varepsilon^+)^{k+1}|^p dx + \int_\Omega |(v_\varepsilon^+)^{k+1}|^p dx\right] \\
&\leq e_9\int_\Omega (v_\varepsilon^+)^{(k+1)p}dx + e_{10}\int_{\partial\Omega}(v_\varepsilon^+)^{(k+1)p}d\sigma + e_{11} \\
&\leq e_9\int_\Omega (v_\varepsilon^+)^{(k+1)p}dx + e_{14}\delta\|(v_\varepsilon^+)^{k+1}\|^p_{W^{1,p}(\Omega)} + e_{14}C(\delta)\|(v_\varepsilon^+)^{k+1}\|^p_{L^p(\Omega)} + e_{11},
\end{aligned}$$

where $e_{14} = e_{10}e_{13}^p e_{12}^p$ is a positive constant. We take $\delta = \frac{kp+1}{e_{14}2(k+1)^p}$ to get

$$\begin{aligned}
&\left(\frac{kp+1}{(k+1)^p} - e_{14}\frac{kp+1}{e_{14}2(k+1)^p}\right)\left[\int_\Omega |\nabla(v_\varepsilon^+)^{k+1}|^p dx + \int_\Omega |(v_\varepsilon^+)^{k+1}|^p)dx\right] \\
&\leq e_9\int_\Omega (v_\varepsilon^+)^{(k+1)p}dx + e_{14}C(\delta)\|(v_\varepsilon^+)^{k+1}\|^p_{L^p(\Omega)} + e_{11},
\end{aligned} \tag{3.1.66}$$

where it holds

$$C(\delta) = \left(\frac{2e_{14}}{p}\right)^{\frac{q}{p}} \cdot \left(\frac{(k+1)^p}{kp+1}\right)^{\frac{q}{p}} \cdot \frac{1}{q} \leq e_{15}(kp+1)^{\frac{p}{p-1}}.$$

This yields

$$\begin{aligned}
&\frac{kp+1}{2(k+1)^p}\left[\int_\Omega |\nabla(v_\varepsilon^+)^{k+1}|^p dx + \int_\Omega |(v_\varepsilon^+)^{k+1}|^p)dx\right] \\
&\leq e_{16}(kp+1)^{\frac{p}{p-1}}\left[\int_\Omega (v_\varepsilon^+)^{(k+1)p}dx + 1\right],
\end{aligned}$$

equivalently

$$\|(v_\varepsilon^+)^{k+1}\|^p_{W^{1,p}(\Omega)} \leq e_{17}(kp+1)^{\frac{1}{p-1}}(k+1)^p\left[\int_\Omega (v_\varepsilon^+)^{(k+1)p}dx+1\right].$$

By Sobolev's Embedding Theorem a positive constant $e_{18}$ exists such that

$$\|(v_\varepsilon^+)^{k+1}\|_{L^{p^*}(\Omega)} \leq e_{18}\|(v_\varepsilon^+)^{k+1}\|_{W^{1,p}(\Omega)}, \tag{3.1.67}$$

where $p^* = \frac{Np}{N-p}$ if $1 < p < N$ and $p^* = 2p$ if $p = N$. We have

$$\begin{aligned}
&\|v_\varepsilon^+\|_{L^{(k+1)p^*}(\Omega)}\\
&= \|(v_\varepsilon^+)^{k+1}\|^{\frac{1}{k+1}}_{L^{p^*}(\Omega)}\\
&\leq e_{18}^{\frac{1}{k+1}}\|(v_\varepsilon^+)^{k+1}\|^{\frac{1}{k+1}}_{W^{1,p}(\Omega)}\\
&\leq e_{18}^{\frac{1}{k+1}}\left((kp+1)^{\frac{1}{(p-1)p}}(k+1)\right)^{\frac{1}{k+1}} e_{17}^{\frac{1}{(k+1)p}}\left[\int_\Omega (v_\varepsilon^+)^{(k+1)p}dx+1\right]^{\frac{1}{(k+1)p}}.
\end{aligned}$$

Since $\left((kp+1)^{\frac{1}{(p-1)p}}(k+1)\right)^{\frac{1}{\sqrt{k+1}}} \geq 1$ and $\lim_{k\to\infty}\left((kp+1)^{\frac{1}{(p-1)p}}(k+1)\right)^{\frac{1}{\sqrt{k+1}}} = 1$, there exists a constant $e_{19} > 1$ such that $\left((kp+1)^{\frac{1}{(p-1)p}}(k+1)\right)^{\frac{1}{k+1}} \leq e_{19}^{\frac{1}{\sqrt{k+1}}}$. This implies

$$\|v_\varepsilon^+\|_{L^{(k+1)p^*}(\Omega)} \leq e_{18}^{\frac{1}{k+1}} e_{19}^{\frac{1}{\sqrt{k+1}}} e_{17}^{\frac{1}{(k+1)p}}\left[\int_\Omega (v_\varepsilon^+)^{(k+1)p}dx+1\right]^{\frac{1}{(k+1)p}}. \tag{3.1.68}$$

Now, we will use the bootstrap arguments similarly as in the proof of [56, Lemma 3.2] starting with $(k_1+1)p = p^*$ to get

$$\|v_\varepsilon^+\|_{L^{(k+1)p^*}(\Omega)} \leq c(k)$$

for any finite number $k > 0$ which shows that $v_\varepsilon^+ \in L^r(\Omega)$ for any $r \in (1,\infty)$. To prove the uniform estimate with respect to $k$ we argue as follows. If there is a sequence $k_n \to \infty$ such that

$$\int_\Omega (v_\varepsilon^+)^{(k_n+1)p}dx \leq 1,$$

we immediately have

$$\|v_\varepsilon^+\|_{L^\infty(\Omega)} \leq 1,$$

(cf. the proof of [56, Lemma 3.2]). In the opposite case there exists $k_0 > 0$ such that

$$\int_\Omega (v_\varepsilon^+)^{(k+1)p}dx > 1$$

for any $k \geq k_0$. Then we conclude from (3.1.68)

$$\|v_\varepsilon^+\|_{L^{(k+1)p^*}(\Omega)} \leq e_{18}^{\frac{1}{k+1}} e_{19}^{\frac{1}{\sqrt{k+1}}} e_{20}^{\frac{1}{(k+1)p}}\|v_\varepsilon^+\|_{L^{(k+1)p}}, \text{ for any } k \geq k_0, \tag{3.1.69}$$

where $e_{20} = 2e_{17}$. Choosing $k := k_1$ such that $(k_1+1)p = (k_0+1)p^*$ yields

$$\|v_\varepsilon^+\|_{L^{(k_1+1)p^*}(\Omega)} \le e_{18}^{\frac{1}{k_1+1}} e_{19}^{\frac{1}{\sqrt{k_1+1}}} e_{20}^{\frac{1}{(k_1+1)p}} \|v_\varepsilon^+\|_{L^{(k_1+1)p}(\Omega)}. \tag{3.1.70}$$

Next, we can choose $k_2$ in (3.1.69) such that $(k_2+1)p = (k_1+1)p^*$ to get

$$\begin{aligned}\|v_\varepsilon^+\|_{L^{(k_2+1)p^*}(\Omega)} &\le e_{18}^{\frac{1}{k_2+1}} e_{19}^{\frac{1}{\sqrt{k_2+1}}} e_{20}^{\frac{1}{(k_2+1)p}} \|v_\varepsilon^+\|_{L^{(k_2+1)p}(\Omega)}\\ &= e_{18}^{\frac{1}{k_2+1}} e_{19}^{\frac{1}{\sqrt{k_2+1}}} e_{20}^{\frac{1}{(k_2+1)p}} \|v_\varepsilon^+\|_{L^{(k_1+1)p^*}(\Omega)}.\end{aligned} \tag{3.1.71}$$

By induction we obtain

$$\begin{aligned}\|v_\varepsilon^+\|_{L^{(k_n+1)p^*}(\Omega)} &\le e_{18}^{\frac{1}{k_n+1}} e_{19}^{\frac{1}{\sqrt{k_n+1}}} e_{20}^{\frac{1}{(k_n+1)p}} \|v_\varepsilon^+\|_{L^{(k_n+1)p}(\Omega)}\\ &= e_{18}^{\frac{1}{k_n+1}} e_{19}^{\frac{1}{\sqrt{k_n+1}}} e_{20}^{\frac{1}{(k_n+1)p}} \|v_\varepsilon^+\|_{L^{(k_{n-1}+1)p^*}(\Omega)},\end{aligned} \tag{3.1.72}$$

where the sequence $(k_n)$ is chosen such that $(k_n+1)p = (k_{n-1}+1)p^*$ with $k_0 > 0$. One easily verifies that $k_n + 1 = \left(\frac{p^*}{p}\right)^n$. Thus

$$\|v_\varepsilon^+\|_{L^{(k_n+1)p^*}(\Omega)} = e_{18}^{\sum_{i=1}^n \frac{1}{k_i+1}} e_{19}^{\sum_{i=1}^n \frac{1}{\sqrt{k_i+1}}} e_{20}^{\sum_{i=1}^n \frac{1}{(k_i+1)p}} \|v_\varepsilon^+\|_{L^{(k_0+1)p^*}(\Omega)}, \tag{3.1.73}$$

with $r_n = (k_n+1)p^* \to \infty$ as $n \to \infty$. Since $\frac{1}{k_i+1} = (\frac{p}{p^*})^i$ and $\frac{p}{p^*} < 1$ there is a constant $e_{21} > 0$ such that

$$\|v_\varepsilon^+\|_{L^{(k_n+1)p^*}(\Omega)} \le e_{21}\|v_\varepsilon^+\|_{L^{(k_0+1)p^*}(\Omega)} < \infty. \tag{3.1.74}$$

Let us assume that $v_\varepsilon^+ \notin L^\infty(\Omega)$. Then there exist $\eta > 0$ and a set $A$ of positive measure in $\Omega$ such that $v_\varepsilon^+(x) \ge e_{21}\|v_\varepsilon^+\|_{L^{(k_0+1)p^*}(\Omega)} + \eta$ for $x \in A$. It follows that

$$\begin{aligned}\|v_\varepsilon^+\|_{L^{(k_n+1)p^*}(\Omega)} &\ge \left(\int_A |v_\varepsilon^+(x)|^{(k_n+1)p^*}\right)^{\frac{1}{(k_n+1)p^*}}\\ &\ge (e_{21}\|v_\varepsilon^+\|_{L^{(k_0+1)p^*}(\Omega)} + \eta)|A|^{\frac{1}{(k_n+1)p^*}}.\end{aligned}$$

Passing to the limes inferior in the inequality above yields

$$\liminf_{n\to\infty} \|v_\varepsilon^+\|_{L^{(k_n+1)p^*}(\Omega)} \ge e_{21}\|v_\varepsilon^+\|_{L^{(k_0+1)p^*}(\Omega)} + \eta,$$

which is a contradiction to (3.1.74) and hence, $v_\varepsilon^+ \in L^\infty(\Omega)$. In a similar way one shows that $v_\varepsilon^- = \max\{-v_\varepsilon, 0\} \in L^\infty(\Omega)$. This proves $v_\varepsilon = v_\varepsilon^+ - v_\varepsilon^- \in L^\infty(\Omega)$.
In order to show some structure properties of $\mathcal{A}_\varepsilon$ note that its derivative has the form

$$\begin{aligned}D_\xi \mathcal{A}_\varepsilon(x,\xi) =&|\xi|^{p-2}I + (p-2)|\xi|^{p-4}\xi\xi^T\\ &+ \lambda_\varepsilon|\xi - H|^{p-2}I + \lambda_\varepsilon(p-2)|\xi - H|^{p-4}(\xi - H)(\xi - H)^T,\end{aligned} \tag{3.1.75}$$

where $I$ is the unit matrix and $\xi^T$ stands for the transpose of $\xi$. The use of (3.1.75) implies

$$\|D_\xi \mathcal{A}_\varepsilon(x,\xi)\|_{\mathbb{R}^N} \le a_1 + a_2|\xi|^{p-2}, \tag{3.1.76}$$

where $a_1, a_2$ are some positive constants. We also obtain

$$\begin{aligned}
&(D_\xi \mathcal{A}_\varepsilon(x,\xi)y,y)_{\mathbb{R}^N}\\
&= |\xi|^{p-2}\|y\|^2_{\mathbb{R}^N} + (p-2)|\xi|^{p-4}(\xi,y)^2_{\mathbb{R}^N}\\
&\quad + \lambda_\varepsilon|\xi-H|^{p-2}\|y\|^2_{\mathbb{R}^N} + \lambda_\varepsilon(p-2)|\xi-H|^{p-4}(\xi-H,y)^2_{\mathbb{R}^N}\\
&\ge \begin{cases} |\xi|^{p-2}\|y\|^2_{\mathbb{R}^N} & \text{if } p \ge 2\\ (p-1)|\xi|^{p-2}\|y\|^2_{\mathbb{R}^N} & \text{if } 1<p<2\end{cases}\\
&\ge \min\{1,p-1\}|\xi|^{p-2}\|y\|^2_{\mathbb{R}^N}.
\end{aligned} \tag{3.1.77}$$

For the case $1<p<2$ in (3.1.77) we have used the estimate $|\xi|^{p-2}\|y\|^2_{\mathbb{R}^N} + (p-2)|\xi|^{p-4}(\xi,y)^2_{\mathbb{R}^N} \ge (p-1)|\xi|^{p-2}\|y\|^2_{\mathbb{R}^N}$. Because of (3.1.76) and (3.1.77), the operators $\mathcal{A}_\varepsilon, \mathcal{B}_\varepsilon$ and $\Phi_\varepsilon$ satisfy the assumptions (0.3a-d) and (0.6) of Lieberman in [89] and thus, Theorem 2 in [89] ensures the existence of $\alpha \in (0,1)$ and $M>0$, both independent of $\varepsilon \in (0,1]$, such that

$$v_\varepsilon \in C^{1,\alpha}(\overline{\Omega}) \quad \text{and} \quad \|v_\varepsilon\|_{C^{1,\alpha}(\overline{\Omega})} \le M, \quad \text{for all } \varepsilon \in (0,1]. \tag{3.1.78}$$

Due to $y_\varepsilon = v_\varepsilon - z_0$ and the fact that $v_\varepsilon, z_0 \in C^{1,\alpha}(\overline{\Omega})$, one immediately realizes that $y_\varepsilon$ satisfies (3.1.78), too. Next, we assume $\lambda_\varepsilon > 1$ with $\varepsilon \in (0,1]$. Multiplying (3.1.49) with $-1$ and adding this new equation to (3.1.52) yields

$$\begin{aligned}
&\int_\Omega |\nabla(z_0+y_\varepsilon)|^{p-2}\nabla(z_0+y_\varepsilon)\nabla\varphi dx - \int_\Omega |\nabla z_0|^{p-2}\nabla z_0\nabla\varphi dx\\
&+\lambda_\varepsilon\int_\Omega |\nabla y_\varepsilon|^{p-2}\nabla y_\varepsilon\nabla\varphi dx\\
&=\int_\Omega (f(x,T_0(x,z_0+y_\varepsilon)) - f(x,z_0))\varphi dx\\
&\quad+\int_\Omega (|z_0|^{p-2}z_0 - |z_0+y_\varepsilon|^{p-2}(z_0+y_\varepsilon) - \lambda_\varepsilon|y_\varepsilon|^{p-2}y_\varepsilon)\varphi dx\\
&\quad+\int_{\partial\Omega}\lambda(|T_0^{\partial\Omega}(x,z_0+y_\varepsilon)|^{p-2}T_0^{\partial\Omega}(x,z_0+y_\varepsilon) - |z_0|^{p-2}z_0)\varphi d\sigma\\
&\quad+\int_{\partial\Omega}(g(x,T_0^{\partial\Omega}(x,z_0+y_\varepsilon)) - g(x,z_0))\varphi d\sigma.
\end{aligned} \tag{3.1.79}$$

Defining again

$$\begin{aligned}
\mathcal{A}_\varepsilon(x,\xi) &= \frac{1}{\lambda_\varepsilon}(|H+\xi|^{p-2}(H+\xi) - |H|^{p-2}H) + |\xi|^{p-2}\xi\\
-\mathcal{B}_\varepsilon(x,\psi) &= f(x,T_0(x,z_0+\psi)) - f(x,z_0) + |z_0|^{p-2}z_0\\
&\qquad - |z_0+\psi|^{p-2}(z_0+\psi) - \lambda_\varepsilon|\psi|^{p-2}y_\varepsilon\\
\Phi_\varepsilon(x,\psi) &= \lambda(|T_0^{\partial\Omega}(x,z_0+\psi)|^{p-2}T_0^{\partial\Omega}(x,z_0+\psi) - |z_0|^{p-2}z_0)\\
&\qquad + g(x,T_0^{\partial\Omega}(x,z_0+\psi)) - g(x,z_0),
\end{aligned} \tag{3.1.80}$$

and rewriting (3.1.79) yields the Neumann equation

$$\begin{aligned} -\operatorname{div}\mathcal{A}_\varepsilon(x,\nabla y_\varepsilon)+\frac{1}{\lambda_\varepsilon}\mathcal{B}_\varepsilon(x,y_\varepsilon)&=0 && \text{in } \Omega,\\ \frac{\partial v_\varepsilon}{\partial \nu}&=\frac{1}{\lambda_\varepsilon}\Phi_\varepsilon(x,y_\varepsilon) && \text{on } \partial\Omega, \end{aligned} \tag{3.1.81}$$

where $\frac{\partial v_\varepsilon}{\partial \nu}$ denotes the conormal derivative of $v_\varepsilon$. As above, we have the following estimate

$$\begin{aligned} (\mathcal{A}_\varepsilon(x,\xi),\xi)_{\mathbb{R}^N}&=\frac{1}{\lambda_\varepsilon}(|H+\xi|^{p-2}(H+\xi)-|H|^{p-2}H,H+\xi-H)_{\mathbb{R}^N}+\|\xi\|^p\\ &\geq \|\xi\|^p \quad \text{for all } \xi\in\mathbb{R}^N, \end{aligned} \tag{3.1.82}$$

and can write the derivative $D_\xi\mathcal{A}_\varepsilon(x,\xi)$ as

$$\begin{aligned} D_\xi\mathcal{A}_\varepsilon(x,\xi)=&\frac{1}{\lambda_\varepsilon}(|H+\xi|^{p-2}I+(p-2)|H+\xi|^{p-4}(H+\xi)(H+\xi)^T\\ &|\xi|^{p-2}I+(p-2)|\xi|^{p-4}\xi\xi^T. \end{aligned} \tag{3.1.83}$$

We have again the following estimate

$$\|D_\xi\mathcal{A}_\varepsilon(x,\xi)\|_{\mathbb{R}^N}\leq a_1+a_2|\xi|^{p-2}, \tag{3.1.84}$$

where $a_1,a_2$ are some positive constants. One also gets

$$\begin{aligned} &(D_\xi\mathcal{A}_\varepsilon(x,\xi)y,y)_{\mathbb{R}^N}\\ &=\frac{1}{\lambda_\varepsilon}(|H+\xi|^{p-2}\|y\|^2_{\mathbb{R}^N}+(p-2)|H+\xi|^{p-4}(H+\xi,y)^2_{\mathbb{R}^N})\\ &\quad+|\xi|^{p-2}\|y\|^2_{\mathbb{R}^N}+(p-2)|\xi|^{p-4}(\xi,y)^2_{\mathbb{R}^N}\\ &\geq\begin{cases}|\xi|^{p-2}\|y\|^2_{\mathbb{R}^N} & \text{if } p\geq 2\\ (p-1)|\xi|^{p-2}\|y\|^2_{\mathbb{R}^N} & \text{if } 1<p<2\end{cases}\\ &\geq \min\{1,p-1\}|\xi|^{p-2}\|y\|^2_{\mathbb{R}^N}. \end{aligned} \tag{3.1.85}$$

As before, the nonlinear regularity theory implies the existence of $\alpha\in(0,1)$ and $M>0$, both independent of $\varepsilon\in(0,1)$ such that (3.1.78) holds for $y_\varepsilon$.
Let $\varepsilon\downarrow 0$. Using the compact embedding $C^{1,\alpha}(\overline{\Omega})\hookrightarrow C^1(\overline{\Omega})$ (cf. [83, p. 38] or [1, p. 11]), we may assume $y_\varepsilon\to\widetilde{y}$ in $C^1(\overline{\Omega})$ for a subsequence. By construction we have $y_\varepsilon\to 0$ in $W^{1,p}(\Omega)$ and thus, it holds $\widetilde{y}=0$ which implies $\|y_\varepsilon\|_{C^1(\overline{\Omega})}\leq r_1$ for a subsequence. Hence, one has

$$E_0(z_0)\leq E_0(z_0+y_\varepsilon),$$

which is a contradiction to (3.1.50). This completes the proof of the proposition. □

**Lemma 3.1.12.** *Let $\lambda>\lambda_1$. Then the extremal positive solution $u_+$ (respectively, negative solution $u_-$) of (3.1.1) is the unique global minimizer of the functional $E_+$ (respectively, $E_-$). Moreover, $u_+$ and $u_-$ are local minimizers of $E_0$.*

**Proof.** By Lemma 3.1.9 we know that $E_+ : W^{1,p}(\Omega) \to \mathbb{R}$ is coercive and weakly sequentially lower semicontinuous. Therefore, by Theorem 2.4.1 there exists a global minimizer $v_+ \in W^{1,p}(\Omega)$ of $E_+$. Since $v_+$ is a critical point of $E_+$, Lemma 3.1.10 implies that $v_+$ is a nonnegative solution of (3.1.1) satisfying $0 \le v_+ \le u_+$. By (g1) we infer

$$|g(x,s)| \le (\lambda - \lambda_1)s^{p-1}, \qquad \forall s : 0 < s \le \delta_\lambda. \tag{3.1.86}$$

Using (f4), (3.1.86) and the Steklov eigenvalue problem in (3.1.7), we conclude for $\varepsilon < \min\left\{\frac{\delta_f}{\|\varphi_1\|_\infty}, \frac{\delta_\lambda}{\|\varphi_1\|_\infty}\right\}$

$$\begin{aligned}
E_+(\varepsilon\varphi_1) &= -\int_\Omega \int_0^{\varepsilon\varphi_1(x)} f(x,s)dsdx + \frac{\lambda_1 - \lambda}{p}\varepsilon^p \|\varphi_1\|^p_{L^p(\partial\Omega)} \\
&\quad - \int_{\partial\Omega}\int_0^{\varepsilon\varphi_1(x)} g(x,s)dsd\sigma \\
&< \frac{\lambda_1 - \lambda}{p}\varepsilon^p \|\varphi_1\|^p_{L^p(\partial\Omega)} + \int_{\partial\Omega}\int_0^{\varepsilon\varphi_1(x)} (\lambda - \lambda_1)s^{p-1}dsd\sigma \\
&= 0.
\end{aligned}$$

This shows $E_+(v_+) < 0$ and we obtain $v_+ \neq 0$. Applying Lemma 3.1.7 implies $v_+ \in \operatorname{int}(C^1(\overline{\Omega})_+)$. Since $u_+$ is the smallest positive solution of (3.1.1) in $[0, \vartheta e]$ and $0 \le v_+ \le u_+$, it holds $v_+ = u_+$. Thus, $u_+$ is the unique global minimizer of $E_+$. In the same way one verifies that $u_-$ is the unique global minimizer of $E_-$.

Now we want to show that $u_+$ and $u_-$ are local minimizers of the functional $E_0$. As $u_+ \in \operatorname{int}(C^1(\overline{\Omega})_+)$ there exists a neighborhood $V_{u_+}$ of $u_+$ in the space $C^1(\overline{\Omega})$ such that $V_{u_+} \subset C^1(\overline{\Omega})_+$. Hence, $E_+ = E_0$ on $V_{u_+}$ which ensures that $u_+$ is a local minimizer of $E_0$ on $C^1(\overline{\Omega})$. In view of Proposition 3.1.11, we obtain that $u_+$ is also a local minimizer of $E_0$ on the space $W^{1,p}(\Omega)$. By the same arguments as above one may prove that $u_-$ is a local minimizer of $E_0$. □

**Lemma 3.1.13.** *The functional $E_0 : W^{1,p}(\Omega) \to \mathbb{R}$ has a global minimizer $v_0$ which is a nontrivial solution of (3.1.1) satisfying $u_- \le v_0 \le u_+$.*

**Proof.** The functional $E_0 : W^{1,p}(\Omega) \to \mathbb{R}$ is coercive and weakly sequentially lower semicontinuous (see Lemma 3.1.9). Hence, a global minimizer $v_0$ of $E_0$ exists. Since $v_0$ is a critical point of $E_0$ we know by Lemma 3.1.10 that $v_0$ is a solution of (3.1.1) satisfying $u_- \le v_0 \le u_+$. Due to $E_0(u_+) = E_+(u_+) < 0$ (cf. the proof of Lemma 3.1.12) we obtain that $v_0$ is nontrivial meaning $v_0 \neq 0$. □

### 3.1.5 Existence of Sign-Changing Solutions

First, we are going to show that our functionals introduced in Section (3.1.4) satisfy the Palais-Smale condition. In order to prove this result, we will need a preliminary lemma which can be found in [96, Lemma 2.1-Lemma 2.3] in similar form.

**Lemma 3.1.14.** *Let $A, B, C : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ be given by*

$$\begin{aligned}
\langle A(u), v\rangle &:= \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx + \int_\Omega |u|^{p-2} uv dx,\\
\langle B(u), v\rangle &:= \int_{\partial\Omega} \lambda |T_0^{\partial\Omega}(x,u)|^{p-2} T_0^{\partial\Omega}(x,u) v dx,\\
\langle C(u), v\rangle &:= \int_\Omega f(x, T_0(x,u)) v dx + \int_{\partial\Omega} g(x, T_0^{\partial\Omega}(x,u)) v dx,
\end{aligned}$$

*then $A$ is continuous, continuously invertible and the operators $B, C$ are continuous and compact.*

**Proof.** According to Lemma 3.1.9 we introduce again the Nemytskij operators $F, F^\Omega : L^p(\Omega) \to L^q(\Omega)$ and $G, F^{\partial\Omega} : L^p(\partial\Omega) \to L^q(\partial\Omega)$ by

$$\begin{aligned}
&Fu(x) = f(x, T_0(x, u(x))), &&F^\Omega u(x) = |u(x)|^{p-2} u(x),\\
&Gu(x) = g(x, T_0^{\partial\Omega}(x, u(x))), &&F^{\partial\Omega} u(x) = \lambda |T_0^{\partial\Omega}(x, u(x))|^{p-2} T_0^{\partial\Omega}(x, u(x)).
\end{aligned}$$

We set

$$\begin{aligned}
&\widehat{F} := i^* \circ F \circ i : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*,\\
&\widehat{F}^\Omega := i^* \circ F^\Omega \circ i : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*,\\
&\widehat{G} := \gamma^* \circ G \circ \gamma : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*,\\
&\widehat{F}^{\partial\Omega} := \gamma^* \circ F^{\partial\Omega} \circ \gamma : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*.
\end{aligned}$$

The operators $\widehat{F}, \widehat{F}^\Omega, \widehat{F}^{\partial\Omega}$ and $\widehat{G}$ are bounded, completely continuous and hence also compact. Thus, $B = \widehat{F}^{\partial\Omega}$ and $C = \widehat{F} + \widehat{G}$ are bounded, continuous and compact. Since the negative $p$-Laplacian is bounded and continuous for $1 < p < \infty$, we obtain that $A = -\Delta_p + \widehat{F}^\Omega$ is bounded and continuous.

Finally, we have to show that $A$ is continuously invertible. By Lemma 2.1 in [66] there exists for every fixed $\phi \in (W^{1,p}(\Omega))^*$ a unique solution $u \in W^{1,p}(\Omega)$ of the equation

$$Au = -\Delta_p u + \widehat{F}^\Omega u = \phi, \tag{3.1.87}$$

which is a consequence of the Browder theorem (e.g. in [71]) since $A$ is bounded, continuous, coercive and strictly monotone. This implies the surjectivity of $A$ and since $A$ is also injective, the mapping $A^{-1}$ exists. To prove that $A^{-1}$ is continuous, we make use of the following estimates

$$(|x|^{p-2}x - |y|^{p-2}y, x-y)_{\mathbb{R}^m} \geq \begin{cases} C(p)|x-y|^p & \text{if } p \geq 2,\\ C(p)\frac{|x-y|^2}{(|x|+|y|)^{2-p}} & \text{if } p \leq 2, (x,y) \neq (0,0), \end{cases} \tag{3.1.88}$$

where $(\cdot,\cdot)_{\mathbb{R}^m}$ denotes the usual scalar product in $\mathbb{R}^m$. Let $\phi_1, \phi_2 \in (W^{1,p}(\Omega))^*$ be given and let $u_1 = A^{-1}(\phi_1)$, $u_2 = A^{-1}(\phi_2)$ be the corresponding solutions of (3.1.87). Testing the related weak formulation with $\varphi = u_1 - u_2$, subtracting the equations and using (3.1.88) for $p \geq 2$ yields

$$\begin{aligned}
&\|\phi_1 - \phi_2\|_{(W^{1,p}(\Omega))^*}\|u_1 - u_2\|_{W^{1,p}(\Omega)} \\
&\geq \int_\Omega (|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2)(\nabla u_1 - \nabla u_2)dx \\
&\quad + \int_\Omega (|u_1|^{p-2}u_1 - |u_2|^{p-2}u_2)(u_1 - u_2)dx \\
&\geq C(p)\int_\Omega (|\nabla u_1 - \nabla u_2|^p + |u_1 - u_2|^p)dx \\
&= C(p)\|u_1 - u_2\|^p_{W^{1,p}(\Omega)}.
\end{aligned}$$

Consequently,

$$\|A^{-1}(\phi_1) - A^{-1}(\phi_2)\|_{W^{1,p}(\Omega)} \leq C\|\phi_1 - \phi_2\|^{\frac{1}{p-1}}_{(W^{1,p}(\Omega))^*}.$$

Let us consider the case $p < 2$. We have

$$\begin{aligned}
|\nabla(u_1 - u_2)|^p &= \frac{|\nabla(u_1 - u_2)|^p}{(|\nabla u_1| + |\nabla u_2|)^{\frac{2-p}{2}p}}(|\nabla u_1| + |\nabla u_2|)^{\frac{2-p}{2}p}, \\
|u_1 - u_2|^p &= \frac{|u_1 - u_2|^p}{(|u_1| + |u_2|)^{\frac{2-p}{2}p}}(|u_1| + |u_2|)^{\frac{2-p}{2}p},
\end{aligned}$$

and obtain by applying the Hölder inequality

$$\begin{aligned}
\int_\Omega |\nabla(u_1 - u_2)|^p dx &\leq \left(\int_\Omega \frac{|\nabla(u_1 - u_2)|^2}{(|\nabla u_1| + |\nabla u_2|)^{2-p}}dx\right)^{\frac{p}{2}} \left(\int_\Omega (|\nabla u_1| + |\nabla u_2|)^p dx\right)^{\frac{2-p}{2}}, \\
\int_\Omega |u_1 - u_2|^p dx &\leq \left(\int_\Omega \frac{|u_1 - u_2|^2}{(|u_1| + |u_2|)^{2-p}}dx\right)^{\frac{p}{2}} \left(\int_\Omega (|u_1| + |u_2|)^p dx\right)^{\frac{2-p}{2}}.
\end{aligned}$$

From (3.1.88) and the estimates above we get

$$\frac{\|u_1 - u_2\|_{W^{1,p}(\Omega)}}{(\|u_1\|_{W^{1,p}(\Omega)} + \|u_2\|_{W^{1,p}(\Omega)})^{2-p}} \leq C\|\phi_1 - \phi_2\|_{(W^{1,p}(\Omega))^*}, \tag{3.1.89}$$

where $C$ is a positive constant. The weak formulation of (3.1.87) implies for $u = u_i$ and $\varphi = u_i$

$$\|u_i\|^p_{W^{1,p}(\Omega)} \leq \|\phi_i\|_{(W^{1,p}(\Omega))^*}\|u_i\|_{W^{1,p}(\Omega)}, \; i = 1, 2,$$

and thus, (3.1.89) provides

$$\begin{aligned}
&\|A^{-1}(\phi_1) - A^{-1}(\phi_2)\|_{W^{1,p}(\Omega)} \\
&\leq C\left(\|\phi_1\|^{\frac{1}{p-1}}_{(W^{1,p}(\Omega))^*} + \|\phi_2\|^{\frac{1}{p-1}}_{(W^{1,p}(\Omega))^*}\right)^{2-p} \|\phi_1 - \phi_2\|_{(W^{1,p}(\Omega))^*},
\end{aligned}$$

which completes the proof. $\square$

By means of this lemma, we can prove the following.

**Lemma 3.1.15.** *The functionals $E_+, E_-, E_0 : W^{1,p}(\Omega) \to \mathbb{R}$ satisfy the Palais-Smale condition.*

**Proof.** We show this Lemma only for $E_0$. The proof for $E_+, E_-$ is very similar. Let $(u_n) \subset W^{1,p}(\Omega)$ be a sequence such that $E_0(u_n)$ is bounded and $E_0'(u_n) \to 0$ as $n$ tends to infinity. Since $|E_0(u_n)| \le M$ for all $n$, we obtain by using Young's inequality and the compact embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$

$$\begin{aligned} M &\ge E_0(u_n) \\ &= \frac{1}{p}\left[\|\nabla u_n\|^p_{L^p(\Omega)} + \|u_n\|^p_{L^p(\Omega)}\right] - \int_\Omega \int_0^{u_n(x)} f(x, T_0(x,s))dsdx \\ &\quad - \int_{\partial\Omega}\int_0^{u_n(x)} \left[\lambda |T_0^{\partial\Omega}(x,s)|^{p-2}T_0^{\partial\Omega}(x,s) + g(x, T_0^{\partial\Omega}(x,s))\right] dsd\sigma \\ &\ge (1/p - \varepsilon_1 - \varepsilon_2 - \varepsilon_3)||u_n\|^p_{W^{1,p}(\Omega)} - C. \end{aligned}$$

Choosing $\varepsilon_i, i = 1,2,3$ sufficiently small yields the boundedness of $u_n$ in $W^{1,p}(\Omega)$, and thus, we get $u_n \rightharpoonup u$ for a subsequence of $u_n$ still denoted with $u_n$. We have

$$A(u_n) - \lambda B(u_n) - C(u_n) = E_0'(u_n) \to 0,$$

which implies the existence of a sequence $(\delta_n) \subset (W^{1,p}(\Omega))^*$ converging to zero such that

$$u_n = A^{-1}(\lambda B(u_n) + C(u_n) + \delta_n).$$

By Lemma 3.1.14 we know that $B, C$ are compact and $A^{-1}$ is continuous. Passing to the limit in the previous equality yields

$$u_n \to A^{-1}(\lambda B(u) + C(u)) =: u,$$

meaning that $u_n \to u$ strongly in $W^{1,p}(\Omega)$. $\square$

Now, we can formulate our main result about the existence of a nontrivial solution of problem (3.1.1).

**Theorem 3.1.16.** *Under hypotheses (f1)–(f4), (g1)–(g4) and for every number $\lambda > \lambda_2$, problem (3.1.1) has a nontrivial sign-changing solution $u_0 \in C^1(\overline{\Omega})$.*

**Proof.** Lemma 3.1.10 implies that every critical point of $E_0$ is a solution of problem (3.1.1) in $[u_-, u_+]$. The coercivity and the weakly sequentially lower semicontinuity of $E_0$ ensure along with $\inf_{W^{1,p}(\Omega)} E_+(u) < 0$ (cf. the proof of Lemma 3.1.12) the existence of a global minimizer $v_0 \in W^{1,p}(\Omega)$ satisfying $v_0 \neq 0$. This means that $v_0$ is a nontrivial solution of (3.1.1) belonging to $[u_-, u_+]$. If $v_0 \neq u_-$ and $v_0 \neq u_+$, then $u_0 := v_0$ must be a sign-changing solution since

$u_-$ is the greatest negative solution and $u_+$ is the smallest positive solution of (3.1.1) which proves the theorem in this case. So, we still have to show that the theorem is also true in case that either $v_0 = u_-$ or $v_0 = u_+$. Without loss of generality we suppose $v_0 = u_+$. The function $u_-$ can be assumed to be a strict local minimizer. Otherwise we would be done. Now, we can find a $\rho \in (0, \|u_+ - u_-\|_{W^{1,p}(\Omega)})$ such that

$$E_0(u_+) \leq E_0(u_-) < \inf\{E_0(u) : u \in \partial B_\rho(u_-)\}, \tag{3.1.90}$$

where $\partial B_\rho = \{u \in W^{1,p}(\Omega) : \|u - u_-\|_{W^{1,p}(\Omega)} = \rho\}$. Due to (3.1.90) along with the fact that $E_0$ satisfies the Palais-Smale condition (see Lemma 3.1.15) we may apply the Mountain-Pass Theorem to $E_0$ (cf. Theorem 2.4.4) which yields the existence of $u_0 \in W^{1,p}(\Omega)$ satisfying $E_0'(u_0) = 0$ and

$$\inf\{E_0(u) : u \in \partial B_\rho(u_-)\} \leq E_0(u_0) = \inf_{\pi \in \Pi} \max_{t \in [-1,1]} E_0(\pi(t)), \tag{3.1.91}$$

where

$$\Pi = \{\pi \in C([-1,1], W^{1,p}(\Omega)) : \pi(-1) = u_-, \pi(1) = u_+\}.$$

We see at once that (3.1.90) and (3.1.91) show $u_0 \neq u_-$ and $u_0 \neq u_+$, and therefore, $u_0$ is a sign-changing solution provided $u_0 \neq 0$. In order to prove $u_0 \neq 0$ we are going to show that $E_0(u_0) < 0$, which is satisfied if there exists a path $\tilde{\pi} \in \Pi$ such that

$$E_0(\tilde{\pi}(t)) < 0, \quad \forall t \in [-1,1]. \tag{3.1.92}$$

Let $S = W^{1,p}(\Omega) \cap \partial B_1^{L^p(\partial\Omega)}$, where $\partial B_1^{L^p(\partial\Omega)} = \{u \in L^p(\partial\Omega) : \|u\|_{L^p(\partial\Omega)} = 1\}$, and $S_C = S \cap C^1(\overline{\Omega})$ be equipped with the topologies induced by $W^{1,p}(\Omega)$ and $C^1(\overline{\Omega})$, respectively. Furthermore, we set

$$\Pi_0 = \{\pi \in C([-1,1], S) : \pi(-1) = -\varphi_1, \pi(1) = \varphi_1\},$$
$$\Pi_{0,C} = \{\pi \in C([-1,1], S_C) : \pi(-1) = -\varphi_1, \pi(1) = \varphi_1\}.$$

In view of assumption (g1) there exists a constant $\delta_2 > 0$ such that

$$\frac{|g(x,s)|}{|s|^{p-1}} \leq \mu, \quad \text{for a.a. } x \in \partial\Omega \text{ and all } 0 < |s| \leq \delta_2, \tag{3.1.93}$$

where $\mu \in (0, \lambda - \lambda_2)$. We select $\rho_0 \in (0, \lambda - \lambda_2 - \mu)$. Thanks to the results of Martínez and Rossi in [97] we have the following variational characterization of $\lambda_2$ given by (see (3.1.8)-(3.1.10) in Chapter 2)

$$\lambda_2 = \inf_{\pi \in \Pi_0} \max_{u \in \pi([-1,1])} \int_\Omega \left[|\nabla u|^p + |u|^p\right] dx. \tag{3.1.94}$$

Since (3.1.94) there exists a $\pi \in \Pi_0$ such that

$$\max_{t \in [-1,1]} \|\pi(t)\|^p_{W^{1,p}(\Omega)} < \lambda_2 + \frac{\rho_0}{2}.$$

It is well known that $S_C$ is dense in $S$. Let $\pi \in \Pi_0$ meaning $\pi : [-1,1] \to S$ is continuous and let $t_0 \in [-1,1]$ fixed. The continuity of $\pi$ implies the existence of $\delta^1 > 0$ such that for $\varepsilon > 0$ it holds

$$\|\pi(t) - \pi(t_0)\| \leq \frac{\varepsilon}{3}, \quad \forall t \in B(t_0, \delta^1),$$

where $B(t_0, \delta^1)$ stands for the open ball around $t_0$ with radius $\delta^1$. Since $S_C$ is dense in $S$, we find $\pi_c \in \Pi_{0,C}$ such that

$$\|\pi_c(t_0) - \pi(t_0)\| \leq \frac{\varepsilon}{3}.$$

Applying the continuity argument again guarantees the existence of $\delta^2 > 0$ such that

$$\|\pi_c(t_0) - \pi(t)\| \leq \frac{\varepsilon}{3}, \quad \forall t \in B(t_0, \delta^2).$$

Let $\delta^3 := \min\{\delta^1, \delta^2\}$. Then we obtain

$$\begin{aligned}
&\|\pi_c(t) - \pi(t)\| \\
&\leq \|\pi_c(t) - \pi_c(t_0)\| + \|\pi_c(t_0) - \pi(t_0)\| + \|\pi(t_0) - \pi(t)\| \\
&\leq \varepsilon, \quad \forall t \in B(t_0, \delta^3).
\end{aligned}$$

Hence, we have found an open cover of $[-1,1]$ such that

$$[-1,1] \subset \bigcup_{t_i \in [-1,1]} B(t_i, \delta(t_i)),$$

and due to the compactness of $[-1,1]$, there exists a finite open cover meaning

$$[-1,1] \subset \bigcup_{\substack{t_i \in [-1,1] \\ i=1,\ldots,k}} B(t_i, \delta(t_i)),$$

which implies

$$\|\pi_c(t) - \pi(t)\| \leq k\varepsilon =: \widetilde{\varepsilon}, \quad \forall t \bigcup_{\substack{t_i \in [-1,1] \\ i=1,\ldots,k}} B(t_i, \delta(t_i)) \supset [-1,1].$$

This proves the density of $\Pi_{0,C}$ in $\Pi_0$ and thus, for a fixed number $r$ satisfying $0 < r \leq (\lambda_2 + \rho_0)^{\frac{1}{p}} - (\lambda_2 + \frac{\rho_0}{2})^{\frac{1}{p}}$, there is a $\pi_0 \in \Pi_{0,C}$ such that

$$\max_{t \in [-1,1]} \|\pi(t) - \pi_0(t)\|^p_{W^{1,p}(\Omega)} < r.$$

This yields

$$\max_{t \in [-1,1]} \|\pi_0(t)\|^p_{W^{1,p}(\Omega)} < \lambda_2 + \rho_0.$$

Let $\delta := \min\{\delta_f, \delta_2\}$, where $\delta_f$ is the constant in condition (f4). The boundedness of the set $\pi_0([-1,1])(\overline{\Omega})$ in $\mathbb{R}$ ensures the existence of $\varepsilon_0 > 0$ such that

$$\varepsilon_0|u(x)| \leq \delta \quad \text{for all } x \in \Omega \text{ and all } u \in \pi_0([-1,1]). \tag{3.1.95}$$

Lemma 3.1.8 ensures that $u_+, -u_- \in \operatorname{int}(C^1(\overline{\Omega})_+)$. Thus, for every $u \in \pi_0([-1,1])$ and any bounded neighborhood $V_u$ of $u$ in $C^1(\overline{\Omega})$ there exist positive numbers $h_u$ and $j_u$ satisfying

$$u_+ - \frac{1}{h}v \in \operatorname{int}(C^1(\overline{\Omega})_+) \quad \text{and} \quad -u_- + \frac{1}{j}v \in \operatorname{int}(C^1(\overline{\Omega})_+), \tag{3.1.96}$$

if $h \geq h_u, j \geq j_u, v \in V_u$. By a compactness argument along with (3.1.96) we conclude the existence of $\varepsilon_1 > 0$ such that

$$u_-(x) \leq \widetilde{\varepsilon}u(x) \leq u_+(x) \quad \text{for all } x \in \Omega, u \in \pi_0([-1,1]) \text{ and } \forall \widetilde{\varepsilon} \in (0,\varepsilon_1). \tag{3.1.97}$$

Let $0 < \varepsilon < \min\{\varepsilon_0, \varepsilon_1\}$. Now, we consider the continuous path $\varepsilon\pi_0$ in $C^1(\overline{\Omega})$ joining $-\varepsilon\varphi_1$ and $\varepsilon\varphi_1$. Using hypothesis (f4) yields

$$-\int_\Omega \int_0^{\varepsilon\pi_0(t)(x)} f(x, T_0(x,s))dsdx \leq 0. \tag{3.1.98}$$

Applying (3.1.93), (3.1.95), (3.1.96), (3.1.97), (3.1.98) and the fact that $\pi_0([-1,1]) \subset \partial B_1^{L^p(\partial\Omega)}$ we have

$$\begin{aligned}
&E_0(\varepsilon\pi_0(t)) \\
&= \frac{\varepsilon^p}{p}[\|\nabla\pi_0(t)\|^p_{L^p(\Omega)} + \|\pi_0(t)\|^p_{L^p(\Omega)}] - \int_\Omega \int_0^{\varepsilon\pi_0(t)(x)} f(x, T_0(x,s))dsdx \\
&\quad - \int_{\partial\Omega} \int_0^{\varepsilon\pi_0(t)(x)} \left[\lambda|T_0^{\partial\Omega}(x,s)|^{p-2}T_0^{\partial\Omega}(x,s) + g(x, T_0^{\partial\Omega}(x,s))\right] dsd\sigma \\
&< \frac{\varepsilon^p}{p}(\lambda_2 + \rho_0) - \frac{\varepsilon^p}{p}\lambda - \int_{\partial\Omega} \int_0^{\varepsilon\pi_0(t)(x)} g(x,s)dsd\sigma \\
&< \frac{\varepsilon^p}{p}(\lambda_2 + \rho_0 - \lambda + \mu) \\
&< 0 \quad \text{for all } t \in [-1,1].
\end{aligned} \tag{3.1.99}$$

In the next step we are going to construct continuous paths $\pi_+, \pi_-$ which join $\varepsilon\varphi_1$ and $u_+$, respectively, $u_-$ and $-\varepsilon\varphi_1$. We denote

$$\begin{aligned}
c_+ &= c_+(\lambda) = E_+(\varepsilon\varphi_1), \\
m_+ &= m_+(\lambda) = E_+(u_+), \\
E_+^{c_+} &= \{u \in W^{1,p}(\Omega) : E_+(u) \leq c_+\}.
\end{aligned}$$

Since $u_+$ is a global minimizer of $E_+$, we see at once that $m_+ < c_+$. Using Lemma 3.1.10 yields the nonexistence of critical values in the interval $(m_+, c_+]$. Due to the coercivity of $E_+$ along with its property to satisfy the Palais-Smale condition (see Lemma 3.1.15), we can apply the Second Deformation Lemma (see, e.g. [73, p. 366]) to $E_+$. This guarantees the existence of a continuous mapping $\eta \in C([0,1] \times E_+^{c_+}, E_+^{c_+})$ with the following properties:

(i) $\eta(0,u) = u$ for all $u \in E_+^{c_+}$,

(ii) $\eta(1,u) = u_+$ for all $u \in E_+^{c_+}$,

(iii) $E_+(\eta(t,u)) \le E_+(u)$, $\forall t \in [0,1]$ and $\forall u \in E_+^{c_+}$.

We introduce the path $\pi_+ : [0,1] \to W^{1,p}(\Omega)$ given by $\pi_+(t) = \eta(t,\varepsilon\varphi_1)^+ = \max\{\eta(t,\varepsilon\varphi_1),0\}$ for all $t \in [0,1]$. Apparently, $\pi_+$ is continuous in $W^{1,p}(\Omega)$ and joins $\varepsilon\varphi_1$ and $u_+$. Moreover, we have

$$E_0(\pi_+(t)) = E_+(\pi_+(t)) \le E_+(\eta(t,\varepsilon\varphi_1)) \le E_+(\varepsilon\varphi_1) < 0 \quad \text{for all } t \in [0,1]. \tag{3.1.100}$$

Analogously, we can apply the Second Deformation Lemma to the functional $E_-$ and obtain a continuous path $\pi_- : [0,1] \to W^{1,p}(\Omega)$ between $-\varepsilon\varphi_1$ and $u_-$ such that

$$E_0(\pi_-(t)) < 0 \quad \text{for all } t \in [0,1]. \tag{3.1.101}$$

Putting the paths together, $\pi_-, \varepsilon\pi_0$ and $\pi_+$ yield a continuous path $\widetilde{\pi} \in \Pi$ joining $u_-$ and $u_+$. In view of (3.1.99), (3.1.100) and (3.1.101) it holds $u_0 \neq 0$. So, we have found a nontrivial sign-changing solution $u_0$ of problem (3.1.1) satisfying $u_- \le u_0 \le u_+$. This completes the proof. $\square$

## 3.2 Multiple Solutions Depending on the Fučik Spectrum

In this section, we consider the following nonlinear elliptic boundary value problem. Find $u \in W^{1,p}(\Omega) \setminus \{0\}$ and constants $a \in \mathbb{R}, b \in \mathbb{R}$ such that

$$\begin{aligned} -\Delta_p u &= f(x,u) - |u|^{p-2}u && \text{in } \Omega,\\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= a(u^+)^{p-1} - b(u^-)^{p-1} + g(x,u) && \text{on } \partial\Omega, \end{aligned} \tag{3.2.1}$$

where $-\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the negative $p$-Laplacian, $\frac{\partial u}{\partial \nu}$ means the outer normal derivative of $u$ with respect to $\partial\Omega$, and $u^+ = \max\{u,0\}$ and $u^- = \max\{-u,0\}$ are the positive and negative parts of $u$, respectively. The domain $\Omega \subset \mathbb{R}^N$ is supposed to be bounded with a smooth boundary $\partial\Omega$ and the nonlinearities $f : \Omega \times \mathbb{R} \to \mathbb{R}$ as well as $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are some Carathéodory functions which are bounded on bounded sets. As before, we will drop the notation $\gamma(u)$ and write $u$ for the sake of simplicity.

In Section 3.1, we investigated problem (3.2.1) in case $a = b$ for proving multiple solutions. Now, we extend these results to show the existence of multiple solutions of (3.2.1). More precisely, we are going to show the existence of at least three nontrivial solutions of (3.2.1) meaning two extremal constant-sign solutions and at least one nontrivial sign-changing solution. The conditions for the nonlinearities $f$ and $g$ are the same as in Section 3.1.

First, we have to make an analysis of the associated spectrum of (3.2.1). The Fučik spectrum for the $p$-Laplacian with a nonlinear boundary condition is defined as the set $\widetilde{\Sigma}_p$ of $(a, b) \in \mathbb{R} \times \mathbb{R}$ such that

$$\begin{aligned} -\Delta_p u &= -|u|^{p-2}u && \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= a(u^+)^{p-1} - b(u^-)^{p-1} && \text{on } \partial\Omega, \end{aligned} \tag{3.2.2}$$

has a nontrivial solution. In view of the identity

$$|u|^{p-2}u = |u|^{p-2}(u^+ - u^-) = (u^+)^{p-1} - (u^-)^{p-1},$$

we see at once that for $a = b = \lambda$ problem (3.2.2) reduces to the Steklov eigenvalue problem

$$\begin{aligned} -\Delta_p u &= -|u|^{p-2}u && \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= \lambda |u|^{p-2}u && \text{on } \partial\Omega. \end{aligned} \tag{3.2.3}$$

We say that $\lambda$ is an eigenvalue if (3.2.3) has nontrivial solutions. As we pointed out in Section 3.1, the first eigenvalue $\lambda_1 > 0$ is isolated and simple as well as its corresponding first eigenfunction $\varphi_1$ belongs to $\mathrm{int}(C^1(\overline{\Omega})_+)$ meaning

$$\mathrm{int}(C^1(\overline{\Omega})_+) = \left\{u \in C^1(\overline{\Omega}) : u(x) > 0, \forall x \in \overline{\Omega}\right\}.$$

Let us recall some properties of the Fučik spectrum. If $\lambda$ is an eigenvalue for (3.2.3) then the point $(\lambda, \lambda)$ belongs to $\widetilde{\Sigma}_p$. Since the first eigenfunction of (3.2.3) is positive, $\widetilde{\Sigma}_p$ clearly contains the two lines $\mathbb{R} \times \{\lambda_1\}$ and $\{\lambda_1\} \times \mathbb{R}$. A first nontrivial curve $\mathcal{C}$ in $\widetilde{\Sigma}_p$ through $(\lambda_2, \lambda_2)$ was constructed and variationally characterized by a mountain-pass procedure by Martínez and Rossi [97]. This yields the existence of a continuous path in $\{u \in W^{1,p}(\Omega) : I^{(a,b)}(u) < 0, \|u\|_{L^p(\partial\Omega)} = 1\}$ joining $-\varphi_1$ and $\varphi_1$ provided $(a, b)$ is above the curve $\mathcal{C}$. The functional $I^{(a,b)}$ on $W^{1,p}(\Omega)$ is given by

$$I^{(a,b)} = \int_\Omega \Big(|\nabla u|^p + |u|^p\Big) dx - \int_{\partial\Omega} \Big(a(u^+)^p + b(u^-)^p\Big) d\sigma. \tag{3.2.4}$$

Due to the fact that $\lambda_2$ belongs to $\mathcal{C}$, there also exists a variational characterization of the second eigenvalue of (3.2.3) meaning that $\lambda_2$ can be represented as follows

$$\lambda_2 = \inf_{g \in \Pi} \max_{u \in g([-1,1])} \int_\Omega \Big(|\nabla u|^p + |u|^p\Big) dx, \tag{3.2.5}$$

where

$$\Pi = \{g \in C([-1, 1], S) \mid g(-1) = -\varphi_1, g(1) = \varphi_1\}, \tag{3.2.6}$$

and

$$S = \left\{u \in W^{1,p}(\Omega) : \int_{\partial\Omega} |u|^p d\sigma = 1\right\}. \tag{3.2.7}$$

An important part in our considerations plays again the following Neumann boundary value problem defined by

$$\begin{aligned} -\Delta_p u &= -\varsigma|u|^{p-2}u + 1 && \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= 1 && \text{on } \partial\Omega, \end{aligned} \tag{3.2.8}$$

where $\varsigma > 1$ is a constant. As proved in Section 3.1, there exists a unique solution $e \in \operatorname{int}(C^1(\overline{\Omega})_+)$ of problem (3.2.8) which is required for the construction of supersolutions of problem (3.2.1).

### 3.2.1 Notations and Hypotheses

Now, we impose the following conditions on the Carathéodory functions $f$ and $g$ as well as the real parameters $a$ and $b$ in problem (3.2.1).

(H1) (f1) $\lim\limits_{s\to 0}\dfrac{f(x,s)}{|s|^{p-2}s} = 0,$ uniformly with respect to a.a. $x \in \Omega$.

(f2) $\lim\limits_{|s|\to \infty}\dfrac{f(x,s)}{|s|^{p-2}s} = -\infty,$ uniformly with respect to a.a. $x \in \Omega$.

(f3) $f$ is bounded on bounded sets.

(f4) There exists $\delta_f > 0$ such that $\dfrac{f(x,s)}{|s|^{p-2}s} \geq 0$ for all $0 < |s| \leq \delta_f$ and for a.a. $x \in \Omega$.

(H2) (g1) $\lim\limits_{s\to 0}\dfrac{g(x,s)}{|s|^{p-2}s} = 0,$ uniformly with respect to a.a. $x \in \partial\Omega$.

(g2) $\lim\limits_{|s|\to \infty}\dfrac{g(x,s)}{|s|^{p-2}s} = -\infty,$ uniformly with respect to a.a. $x \in \partial\Omega$.

(g3) $g$ is bounded on bounded sets.

(g4) $g$ satisfies the condition

$$|g(x_1,s_1) - g(x_2,s_2)| \leq L\Big[|x_1 - x_2|^{\alpha} + |s_1 - s_2|^{\alpha}\Big],$$

for all pairs $(x_1,s_1),(x_2,s_2)$ in $\partial\Omega \times [-M_0, M_0]$, where $M_0$ is a positive constant and $\alpha \in (0,1]$.

(H3) Let $(a,b) \in \mathbb{R}^2_+$ be above the curve of the Fučik spectrum constructed in [97] (see Figure 1.1).

We see at once that $u = 0$ is a trivial solution of problem (3.2.1) because of the conditions (H1)(f1) and (H2)(g1) implying that $f(x,0) = g(x,0) = 0$. It should be noted that hypothesis (H3) include that $a, b > \lambda_1$ (see Figure 1.1). Let us briefly recall some definitions.

**Definition 3.2.1.** *A function $u \in W^{1,p}(\Omega)$ is called a solution of (3.2.1) if the following holds:*

$$\begin{aligned}\int_\Omega |\nabla u|^{p-2}\nabla u \nabla \varphi dx = &\int_\Omega (f(x,u) - |u|^{p-2}u)\varphi dx \\ &+ \int_{\partial\Omega}(a(u^+)^{p-1} - b(u^-)^{p-1} + g(x,u))\varphi d\sigma, \quad \forall \varphi \in W^{1,p}(\Omega).\end{aligned}$$

**Definition 3.2.2.** *A function $\underline{u} \in W^{1,p}(\Omega)$ is called a subsolution of (3.2.1) if the following holds:*

$$\begin{aligned}\int_\Omega |\nabla \underline{u}|^{p-2}\nabla \underline{u} \nabla \varphi dx \le &\int_\Omega (f(x,\underline{u}) - |\underline{u}|^{p-2}\underline{u})\varphi dx \\ &+ \int_{\partial\Omega}(a(\underline{u}^+)^{p-1} - b(\underline{u}^-)^{p-1} + g(x,\underline{u}))\varphi d\sigma, \quad \forall \varphi \in W^{1,p}(\Omega)_+.\end{aligned}$$

**Definition 3.2.3.** *A function $\overline{u} \in W^{1,p}(\Omega)$ is called a supersolution of (3.2.1) if the following holds:*

$$\begin{aligned}\int_\Omega |\nabla \overline{u}|^{p-2}\nabla \overline{u} \nabla \varphi dx \ge &\int_\Omega (f(x,\overline{u}) - |\overline{u}|^{p-2}\overline{u})\varphi dx \\ &+ \int_{\partial\Omega}(a(\overline{u}^+)^{p-1} - b(\overline{u}^-)^{p-1} + g(x,\overline{u}))\varphi d\sigma, \quad \forall \varphi \in W^{1,p}(\Omega)_+.\end{aligned}$$

### 3.2.2 Extremal Constant-Sign Solutions

For the rest of the section we denote by $\varphi_1 \in \operatorname{int}(C^1(\overline{\Omega})_+)$ the first eigenfunction of the Steklov eigenvalue problem (3.2.3) related to its first eigenvalue $\lambda_1$. Furthermore, the function $e \in \operatorname{int}(C^1(\overline{\Omega})_+)$ stands for the unique solution of the auxiliary Neumann boundary value problem defined in (3.2.8). Our first lemma reads as follows.

**Lemma 3.2.4.** *Let the conditions (H1)–(H2) be satisfied and let $a, b > \lambda_1$. Then there exist constants $\vartheta_a, \vartheta_b > 0$ such that $\vartheta_a e$ and $-\vartheta_b e$ are a positive supersolution and a negative subsolution, respectively, of problem (3.2.1).*

**Proof.** Setting $\overline{u} = \vartheta_a e$ with a positive constant $\vartheta_a$ to be specified and considering the auxiliary problem (3.2.8), we obtain

$$\int_\Omega |\nabla(\vartheta_a e)|^{p-2}\nabla(\vartheta_a e)\nabla\varphi dx = -\varsigma\int_\Omega (\vartheta_a e)^{p-1}\varphi dx + \int_\Omega \vartheta_a^{p-1}\varphi dx + \int_{\partial\Omega}\vartheta_a^{p-1}\varphi d\sigma,$$

for all $\varphi \in W^{1,p}(\Omega)$. In order to satisfy Definition 3.2.3 for $\overline{u} = \vartheta_a e$, we have to show that the following inequality holds true meaning

$$\begin{aligned}&\int_\Omega (\vartheta_a^{p-1} - \widetilde{c}(\vartheta_a e)^{p-1} - f(x,\vartheta_a e))\varphi dx \\ &\quad + \int_{\partial\Omega}(\vartheta_a^{p-1} - a(\vartheta_a e)^{p-1} - g(x,\vartheta_a e))\varphi d\sigma \ge 0,\end{aligned} \tag{3.2.9}$$

where $\widetilde{c} = \varsigma - 1$ with $\widetilde{c} > 0$. Condition (H1)(f2) implies the existence of $s_\varsigma > 0$ such that

$$\frac{f(x,s)}{s^{p-1}} < -\widetilde{c}, \quad \text{for a.a. } x \in \Omega \text{ and all } s > s_\varsigma,$$

and due to (H1)(f3) we have

$$|-f(x,s) - \widetilde{c}s^{p-1}| \le |f(x,s)| + \widetilde{c}s^{p-1} \le c_\varsigma, \quad \text{for a.a. } x \in \Omega \text{ and all } s \in [0, s_\varsigma].$$

Hence, we get

$$f(x,s) \le -\widetilde{c}s^{p-1} + c_\varsigma, \quad \text{for a.a. } x \in \Omega \text{ and all } s \ge 0. \tag{3.2.10}$$

Because of hypothesis (H2)(g2) there exists $s_a > 0$ such that

$$\frac{g(x,s)}{s^{p-1}} < -a, \quad \text{for a.a. } x \in \Omega \text{ and all } s > s_a,$$

and in consequence of condition (H2)(g3) we find a constant $c_a > 0$ to get

$$|-g(x,s) - as^{p-1}| \le |g(x,s)| + as^{p-1} \le c_a, \quad \text{for a.a. } x \in \Omega \text{ and all } s \in [0, s_a].$$

Finally, we have

$$g(x,s) \le -as^{p-1} + c_a, \quad \text{for a.a. } x \in \partial\Omega \text{ and all } s \ge 0. \tag{3.2.11}$$

Applying the inequality in (3.2.10) to the first integral in (3.2.9) yields

$$\begin{aligned}
&\int_\Omega (\vartheta_a^{p-1} - \widetilde{c}(\vartheta_a e)^{p-1} - f(x, \vartheta_a e))\varphi dx \\
&\ge \int_\Omega (\vartheta_a^{p-1} - \widetilde{c}(\vartheta_a e)^{p-1} + \widetilde{c}(\vartheta_a e)^{p-1} - c_\varsigma)\varphi dx \\
&= \int_\Omega (\vartheta_a^{p-1} - c_\varsigma)\varphi dx,
\end{aligned}$$

which proves its nonnegativity if $\vartheta_a \ge c_\varsigma^{\frac{1}{p-1}}$. Equally, we apply (3.2.11) to the second integral in (3.2.9) to obtain

$$\begin{aligned}
&\int_{\partial\Omega} (\vartheta_a^{p-1} - a(\vartheta_a e)^{p-1} - g(x, \vartheta_a e))\varphi dx \\
&\ge \int_{\partial\Omega} (\vartheta_a^{p-1} - a(\vartheta_a e)^{p-1} + a(\vartheta_a e)^{p-1} - c_a)\varphi dx \\
&\ge \int_{\partial\Omega} (\vartheta_a^{p-1} - c_a)\varphi dx.
\end{aligned}$$

We take $\vartheta_a := \max\left\{c_\varsigma^{\frac{1}{p-1}}, c_a^{\frac{1}{p-1}}\right\}$ to verify that both integrals in (3.2.9) are nonnegative. Hence, the function $\overline{u} = \vartheta_a e$ is in fact a positive supersolution of problem (3.2.1). In similar way one proves that $\underline{u} = -\vartheta_b e$ is a negative subsolution, where we apply the following estimates:

$$\begin{aligned}
f(x,s) &\ge -\widetilde{c}s^{p-1} - c_\varsigma, \quad \text{for a.a. } x \in \Omega \text{ and all } s \le 0, \\
g(x,s) &\ge -bs^{p-1} - c_b, \quad \text{for a.a. } x \in \partial\Omega \text{ and all } s \le 0.
\end{aligned} \tag{3.2.12}$$

This completes the proof. $\square$

The next two lemmas demonstrate that constant multipliers of $\varphi_1$ may be sub- and supersolution of (3.2.1). More precisely, we have the following result.

**Lemma 3.2.5.** *Assume (H1)–(H2) are satisfied. If $a > \lambda_1$, then for $\varepsilon > 0$ sufficiently small and any $b \in \mathbb{R}$ the function $\varepsilon\varphi_1$ is a positive subsolution of problem (3.2.1).*

**Proof.** The Steklov eigenvalue problem (3.2.3) implies

$$\begin{aligned}
&\int_\Omega |\nabla(\varepsilon\varphi_1)|^{p-2}\nabla(\varepsilon\varphi_1)\nabla\varphi dx \\
&= -\int_\Omega (\varepsilon\varphi_1)^{p-1}\varphi dx + \int_{\partial\Omega} \lambda_1(\varepsilon\varphi_1)^{p-1}\varphi d\sigma, \forall\varphi \in W^{1,p}(\Omega).
\end{aligned} \tag{3.2.13}$$

Definition 3.2.2 is satisfied for $\underline{u} = \varepsilon\varphi_1$ provided the following inequality

$$\int_\Omega -f(x,\varepsilon\varphi_1)\varphi dx + \int_{\partial\Omega}((\lambda_1 - a)(\varepsilon\varphi_1)^{p-1} - g(x,\varepsilon\varphi_1))\varphi d\sigma \leq 0,$$

is valid for all $\varphi \in W^{1,p}(\Omega)_+$. With regard to hypothesis (H1)(f4) we obtain for $\varepsilon \in (0, \delta_f/\|\varphi_1\|_\infty]$

$$\int_\Omega -f(x,\varepsilon\varphi_1)\varphi dx = \int_\Omega -\frac{f(x,\varepsilon\varphi_1)}{(\varepsilon\varphi_1)^{p-1}}(\varepsilon\varphi_1)^{p-1}\varphi dx \leq 0,$$

where $\|\cdot\|_\infty$ denotes the usual supremum norm. Thanks to condition (H2)(g1) there exists a number $\delta_a > 0$ such that

$$\frac{|g(x,s)|}{|s|^{p-1}} < a - \lambda_1 \quad \text{for a.a. } x \in \partial\Omega \text{ and all } 0 < |s| \leq \delta_a.$$

In case $\varepsilon \in \left(0, \frac{\delta_a}{\|\varphi_1\|_\infty}\right]$ it holds

$$\begin{aligned}
\int_{\partial\Omega}((\lambda_1 - a)(\varepsilon\varphi_1)^{p-1} - g(x,\varepsilon\varphi_1))\varphi d\sigma &\leq \int_{\partial\Omega}\left(\lambda_1 - a + \frac{|g(x,\varepsilon\varphi)|}{(\varepsilon\varphi_1)^{p-1}}\right)(\varepsilon\varphi_1)^{p-1}\varphi d\sigma \\
&< \int_{\partial\Omega}(\lambda_1 - a + a - \lambda_1)(\varepsilon\varphi_1)^{p-1}\varphi d\sigma \\
&= 0.
\end{aligned}$$

Selecting $0 < \varepsilon \leq \min\{\delta_f/\|\varphi_1\|_\infty, \delta_\lambda/\|\varphi_1\|_\infty\}$ guarantees that $\underline{u} = \varepsilon\varphi_1$ is a positive subsolution. □

In a similar way the following lemma on the existence of a negative subsolution can be proven.

**Lemma 3.2.6.** *Assume (H1)–(H2) are satisfied. If $b > \lambda_1$, then for $\varepsilon > 0$ sufficiently small and any $a \in \mathbb{R}$ the function $-\varepsilon\varphi_1$ is a negative supersolution of problem (3.2.1).*

Concerning Lemma 3.2.4-3.2.6, we obtain a positive pair $[\varepsilon\varphi_1, \vartheta_a e]$ and a negative pair $[-\vartheta_b e, -\varepsilon\varphi_1]$ of sub- and supersolutions of problem (3.2.1) assumed $\varepsilon > 0$ sufficiently small.

In the next step we are going to prove the $C^{1,\alpha}$-regularity of solutions of problem (3.2.1) belonging to the order interval $[0, \vartheta_a e]$ and $[-\vartheta_b e, 0]$, respectively. We also point out that $\underline{u} = \overline{u} = 0$ is both, a subsolution and a supersolution because of the hypotheses (H1)(f1) as well as (H2)(g1).

**Lemma 3.2.7.** *Assume (H1)–(H2) and let $a, b > \lambda_1$. If $u \in [0, \vartheta_a e]$ (respectively, $u \in [-\vartheta_b e, 0]$) is a solution of problem (3.2.1) satisfying $u \not\equiv 0$ in $\Omega$, then it holds $u \in \operatorname{int}(C^1(\overline{\Omega})_+)$ (respectively, $u \in -\operatorname{int}(C^1(\overline{\Omega})_+)$).*

**Proof.** We just show the first case, the other case acts in the same way. Let $u$ be a solution of (3.2.1) satisfying $0 \le u \le \vartheta_a e$. We directly obtain the $L^\infty-$boundedness, and hence, the regularity results of Lieberman in [89, Theorem 2] imply $u \in C^{1,\alpha}(\overline{\Omega})$ with $\alpha \in (0,1)$. By reason of the assumptions (H1)(f1),(H1)(f3),(H2)(g1) and (H2)(g3), we obtain the existence of constants $c_f, c_g > 0$ fulfilling

$$\begin{aligned} |f(x,s)| &\le c_f s^{p-1} && \text{for a.a. } x \in \Omega \text{ and all } 0 \le s \le \vartheta_a \|e\|_\infty, \\ |g(x,s)| &\le c_g s^{p-1} && \text{for a.a. } x \in \partial\Omega \text{ and all } 0 \le s \le \vartheta_a \|e\|_\infty, \end{aligned} \tag{3.2.14}$$

(cf. Corollary 3.1.1). Applying (3.2.14) to (3.2.1) provides

$$\Delta_p u \le \widetilde{c} u^{p-1} \text{ a.e. in } \Omega,$$

where $\widetilde{c}$ is a positive constant. We set $\beta(s) = \widetilde{c} s^{p-1}$ for all $s > 0$ and use Vázquez's strong maximum principle (cf. [119]) which is possible because $\int_{0^+} \frac{1}{(s\beta(s))^{\frac{1}{p}}} ds = +\infty$. Hence, it holds $u > 0$ in $\Omega$. Finally, we suppose the existence of a $x_0 \in \partial\Omega$ satisfying $u(x_0) = 0$. Applying again the maximum principle yields $\frac{\partial u}{\partial \nu}(x_0) < 0$. However, because of $g(x_0, u(x_0)) = g(x_0, 0) = 0$ in combination with the Neumann condition in (3.2.1) we get $\frac{\partial u}{\partial \nu}(x_0) = 0$. This is a contradiction and hence, $u > 0$ in $\overline{\Omega}$ which demonstrates $u \in \operatorname{int}(C^1(\overline{\Omega})_+)$. $\square$

The main result in this subsection about the existence of extremal constant-sign solutions is given in the following way.

**Theorem 3.2.8.** *Assume (H1)–(H2). For every $a > \lambda_1$ and $b \in \mathbb{R}$ there exists a smallest positive solution $u_+ = u_+(a) \in \operatorname{int}(C^1(\overline{\Omega})_+)$ of (3.2.1) in the order interval $[0, \vartheta_a e]$ with the constant $\vartheta_a$ as in Lemma 3.2.10. For every $b > \lambda_1$ and $a \in \mathbb{R}$ there exists a greatest solution $u_- = u_-(b) \in -\operatorname{int}(C^1(\overline{\Omega})_+)$ in the order interval $[-\vartheta_b e, 0]$ with the constant $\vartheta_b$ as in Lemma 3.2.10.*

**Proof.** Let $\lambda > \lambda_1$. Lemma 3.2.4 and Lemma 3.2.5 guarantee that $\underline{u} = \varepsilon\varphi_1 \in \operatorname{int}(C^1(\overline{\Omega})_+)$ is a subsolution of problem (3.2.1) and $\overline{u} = \vartheta_a e \in \operatorname{int}(C^1(\overline{\Omega})_+)$ is a supersolution of problem (3.2.1). Moreover, we choose $\varepsilon > 0$ sufficiently small such that $\varepsilon\varphi_1 \le \vartheta_a e$. Applying the method of sub- and supersolution (see [18]) corresponding to the order interval $[\varepsilon\varphi_1, \vartheta_a e]$ provides the existence of a smallest positive solution $u_\varepsilon = u_\varepsilon(\lambda)$ of problem (3.2.1) fulfilling

$\varepsilon\varphi_1 \leq u_\varepsilon \leq \vartheta_a e$. We have $u_\varepsilon \in \operatorname{int}(C^1(\overline{\Omega})_+)$ (see Lemma 3.2.7). Hence, for every positive integer $n$ sufficiently large there exists a smallest solution $u_n \in \operatorname{int}(C^1(\overline{\Omega})_+)$ of problem (3.2.1) in the order interval $[\frac{1}{n}\varphi_1, \vartheta_a e]$. We obtain

$$u_n \downarrow u_+ \text{ for a.a. } x \in \Omega, \tag{3.2.15}$$

with some function $u_+ : \Omega \to \mathbb{R}$ satisfying $0 \leq u_+ \leq \vartheta_a e$.

**Claim 1:** $u_+$ is a solution of problem (3.2.1).

As $u_n \in [\frac{1}{n}\varphi_1, \vartheta_a e]$ and $\gamma(u_n) \in [\gamma(\frac{1}{n}\varphi_1), \gamma(\vartheta_a e)]$, we obtain the boundedness of $u_n$ in $L^p(\Omega)$ and $L^p(\partial\Omega)$, respectively. Definition 3.2.1 holds, in particular, for $u = u_n$ and $\varphi = u_n$ which results in

$$\begin{aligned}
\|\nabla u_n\|^p_{L^p(\Omega)} &\leq \int_\Omega |f(x, u_n)| u_n dx + \|u_n\|^p_{L^p(\Omega)} + a\|u_n\|^p_{L^p(\partial\Omega)} + \int_\Omega |g(x, u_n)| u_n d\sigma \\
&\leq \|u_n\|^p_{L^p(\Omega)} + a_1\|u_n\|_{L^p(\Omega)} + a\|u_n\|^p_{L^p(\partial\Omega)} + a_2\|u_n\|_{L^p(\partial\Omega)} \\
&\leq a_3
\end{aligned}$$

with some positive constants $a_i, i = 1, \dots, 3$, independent of $n$. Consequently, $u_n$ is bounded in $W^{1,p}(\Omega)$ and due to the reflexivity of $W^{1,p}(\Omega), 1 < p < \infty$, we obtain the existence of a weakly convergent subsequence of $u_n$ . Because of the compact embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$, the monotony of $u_n$ and the compactness of the trace operator $\gamma$, we get for the entire sequence $u_n$

$$\begin{aligned}
u_n &\rightharpoonup u_+ && \text{in } W^{1,p}(\Omega), \\
u_n &\to u_+ && \text{in } L^p(\Omega) \text{ and for a.a. } x \in \Omega, \\
u_n &\to u_+ && \text{in } L^p(\partial\Omega) \text{ and for a.a. } x \in \partial\Omega.
\end{aligned} \tag{3.2.16}$$

Since $u_n$ solves problem (3.2.1), one obtains for all $\varphi \in W^{1,p}(\Omega)$

$$\begin{aligned}
&\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla\varphi dx \\
&= \int_\Omega (f(x, u_n) - u_n^{p-1})\varphi dx + \int_{\partial\Omega} (a u_n^{p-1} + g(x, u_n))\varphi d\sigma.
\end{aligned} \tag{3.2.17}$$

Setting $\varphi = u_n - u_+ \in W^{1,p}(\Omega)$ in (3.2.17) results in

$$\begin{aligned}
&\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u_+) dx \\
&= \int_\Omega (f(x, u_n) - u_n^{p-1})(u_n - u_+) dx + \int_{\partial\Omega} (a u_n^{p-1} + g(x, u_n))(u_n - u_+) d\sigma.
\end{aligned} \tag{3.2.18}$$

Using (3.2.18) and the hypotheses (H1)(f3) as well as (H2)(g3) yields

$$\limsup_{n\to\infty} \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u_+) dx \leq 0, \tag{3.2.19}$$

which provides by the $(S_+)$-property of $-\Delta_p$ on $W^{1,p}(\Omega)$ along with (3.2.16)

$$u_n \to u_+ \text{ in } W^{1,p}(\Omega). \tag{3.2.20}$$

The uniform boundedness of the sequence $(u_n)$ in conjunction with the strong convergence in (3.2.20) and the conditions (H1)(f3) and (H2)(g3) admit us to pass to the limit in (3.2.17). This shows that $u_+$ is a solution of problem (3.2.1).

**Claim 2:** $u_+ \in \operatorname{int}(C^1(\overline{\Omega})_+)$.

In order to apply Lemma 3.2.7, we have to prove that $u_+ \not\equiv 0$. Let us assume this assertion is not valid meaning $u_+ \equiv 0$. From (3.2.15) it follows

$$u_n(x) \downarrow 0 \quad \text{for all } x \in \Omega. \tag{3.2.21}$$

We set

$$\widetilde{u}_n = \frac{u_n}{\|u_n\|_{W^{1,p}(\Omega)}} \quad \text{for all } n. \tag{3.2.22}$$

It is clear that the sequence $(\widetilde{u}_n)$ is bounded in $W^{1,p}(\Omega)$ which ensures the existence of a weakly convergent subsequence of $\widetilde{u}_n$, denoted again by $\widetilde{u}_n$, such that

$$\begin{aligned} &\widetilde{u}_n \rightharpoonup \widetilde{u} && \text{in } W^{1,p}(\Omega),\\ &\widetilde{u}_n \to \widetilde{u} && \text{in } L^p(\Omega) \text{ and for a.a. } x \in \Omega,\\ &\widetilde{u}_n \to \widetilde{u} && \text{in } L^p(\partial\Omega) \text{ and for a.a. } x \in \partial\Omega, \end{aligned} \tag{3.2.23}$$

with some function $\widetilde{u} : \Omega \to \mathbb{R}$ belonging to $W^{1,p}(\Omega)_+$. In addition, we may suppose there are functions $z_1 \in L^p(\Omega)_+, z_2 \in L^p(\partial\Omega)_+$ such that

$$\begin{aligned} &|\widetilde{u}_n(x)| \le z_1(x) \quad \text{for a.a. } x \in \Omega,\\ &|\widetilde{u}_n(x)| \le z_2(x) \quad \text{for a.a. } x \in \partial\Omega. \end{aligned} \tag{3.2.24}$$

With the aid of (3.2.17), we obtain for $\widetilde{u}_n$ the following variational equation

$$\begin{aligned} \int_\Omega |\nabla \widetilde{u}_n|^{p-2}\nabla\widetilde{u}_n \nabla\varphi dx = \int_\Omega \left(\frac{f(x,u_n)}{u_n^{p-1}}\widetilde{u}_n^{p-1} - \widetilde{u}_n^{p-1}\right)\varphi dx + \int_{\partial\Omega} a\widetilde{u}_n^{p-1}\varphi d\sigma \\ + \int_{\partial\Omega} \frac{g(x,u_n)}{u_n^{p-1}}\widetilde{u}_n^{p-1}\varphi d\sigma, \qquad \forall \varphi \in W^{1,p}(\Omega). \end{aligned} \tag{3.2.25}$$

We select $\varphi = \widetilde{u}_n - \widetilde{u} \in W^{1,p}(\Omega)$ in the last equality to get

$$\begin{aligned} &\int_\Omega |\nabla \widetilde{u}_n|^{p-2}\nabla\widetilde{u}_n \nabla(\widetilde{u}_n - \widetilde{u}) dx \\ &= \int_\Omega \left(\frac{f(x,u_n)}{u_n^{p-1}}\widetilde{u}_n^{p-1} - \widetilde{u}_n^{p-1}\right)(\widetilde{u}_n - \widetilde{u}) dx + \int_{\partial\Omega} a\widetilde{u}_n^{p-1}(\widetilde{u}_n - \widetilde{u}) d\sigma \\ &\quad + \int_{\partial\Omega} \frac{g(x,u_n)}{u_n^{p-1}}\widetilde{u}_n^{p-1}(\widetilde{u}_n - \widetilde{u}) d\sigma. \end{aligned} \tag{3.2.26}$$

Making use of (3.2.14) in combination with (3.2.24) results in

$$\frac{|f(x, u_n(x))|}{u_n^{p-1}(x)} \widetilde{u}_n^{p-1}(x)|\widetilde{u}_n(x) - \widetilde{u}(x)| \leq c_f z_1(x)^{p-1}(z_1(x) + |\widetilde{u}(x)|), \tag{3.2.27}$$

respectively,

$$\frac{|g(x, u_n(x))|}{u_n^{p-1}(x)} \widetilde{u}_n^{p-1}(x)|\widetilde{u}_n(x) - \widetilde{u}(x)| \leq c_g z_2(x)^{p-1}(z_2(x) + |\widetilde{u}(x)|). \tag{3.2.28}$$

We see at once that the right-hand sides of (3.2.27) and (3.2.28) belong to $L^1(\Omega)$ and $L^1(\partial\Omega)$, respectively, which allows us to apply Lebesgue's dominated convergence theorem. This fact and the convergence properties in (3.2.23) show

$$\begin{aligned} \lim_{n\to\infty} \int_\Omega \frac{f(x, u_n)}{u_n^{p-1}} \widetilde{u}_n^{p-1}(\widetilde{u}_n - \widetilde{u}) dx = 0, \\ \lim_{n\to\infty} \int_{\partial\Omega} \frac{g(x, u_n)}{u_n^{p-1}} \widetilde{u}_n^{p-1}(\widetilde{u}_n - \widetilde{u}) d\sigma = 0. \end{aligned} \tag{3.2.29}$$

From (3.2.23), (3.2.26), (3.2.29) we infer

$$\limsup_{n\to\infty} \int_\Omega |\nabla \widetilde{u}_n|^{p-2} \nabla \widetilde{u}_n \nabla(\widetilde{u}_n - u_n) dx = 0,$$

and the $(S_+)$-property of $-\Delta_p$ corresponding to $W^{1,p}(\Omega)$ implies

$$\widetilde{u}_n \to \widetilde{u} \quad \text{in } W^{1,p}(\Omega). \tag{3.2.30}$$

Remark that $\|\widetilde{u}\|_{W^{1,p}(\Omega)} = 1$ which means $\widetilde{u} \not\equiv 0$. Applying (3.2.21), (3.2.30) and (3.2.25) along with the conditions (H1)(f1),(H2)(g1) provides

$$\int_\Omega |\nabla \widetilde{u}|^{p-2} \nabla \widetilde{u} \nabla \varphi dx = -\int_\Omega \widetilde{u}^{p-1} \varphi dx + \int_{\partial\Omega} a\widetilde{u}^{p-1} \varphi d\sigma, \quad \forall \varphi \in W^{1,p}(\Omega). \tag{3.2.31}$$

The equation above is the Steklov eigenvalue problem in (3.2.3), where $\widetilde{u} \geq 0$ is the eigenfunction with respect to the eigenvalue $a > \lambda_1$. As $\widetilde{u} \geq 0$ is nonnegative in $\overline{\Omega}$, we get a contradiction to the results of Martínez et al. in [95, Lemma 2.4] because $\widetilde{u}$ must change sign on $\partial\Omega$. Hence, $u_+ \not\equiv 0$. Applying Lemma 3.2.7 yields $u_+ \in \operatorname{int}(C^1(\overline{\Omega})_+)$.

**Claim 3:** $u_+ \in \operatorname{int}(C^1(\overline{\Omega})_+)$ is the smallest positive solution of (3.2.1) in $[0, \vartheta_a e]$.

Let $u \in W^{1,p}(\Omega)$ be a positive solution of (3.2.1) satisfying $0 \leq u \leq \vartheta_a e$. Lemma 3.2.7 immediately implies $u \in \operatorname{int}(C^1(\overline{\Omega})_+)$. Then there exists an integer $n$ sufficiently large such that $u \in [\frac{1}{n}\varphi_1, \vartheta_a e]$. However, we already know that $u_n$ is the smallest solution of (3.2.1) in $[\frac{1}{n}\varphi_1, \vartheta_a e]$ which yields $u_n \leq u$. Passing to the limit proves $u_+ \leq u$. Hence, $u_+$ must be the smallest positive solution of (3.2.1). In a similar way one proves the existence of the greatest negative solution of (3.2.1) within $[-\vartheta_b e, 0]$. This completes the proof of the theorem. $\square$

### 3.2.3 Variational Characterization of Extremal Solutions

Theorem 3.2.8 ensures the existence of extremal positive and negative solutions of (3.2.1) for all $a > \lambda_1$ and $b > \lambda_1$ denoted by $u_+ = u_+(a) \in \operatorname{int}(C^1(\overline{\Omega})_+)$ and $u_- = u_-(b) \in -\operatorname{int}(C^1(\overline{\Omega})_+)$, respectively. At the moment, we introduce truncation functions $T_+, T_-, T_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ and $T_+^{\partial\Omega}, T_-^{\partial\Omega}, T_0^{\partial\Omega} : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ as follows:

$$T_+(x,s) = \begin{cases} 0 & \text{if } s \le 0 \\ s & \text{if } 0 < s < u_+(x) \\ u_+(x) & \text{if } s \ge u_+(x) \end{cases}, \quad T_+^{\partial\Omega}(x,s) \begin{cases} 0 & \text{if } s \le 0 \\ s & \text{if } 0 < s < u_+(x) \\ u_+(x) & \text{if } s \ge u_+(x) \end{cases}$$

$$T_-(x,s) = \begin{cases} u_-(x) & \text{if } s \le u_-(x) \\ s & \text{if } u_-(x) < s < 0 \\ 0 & \text{if } s \ge 0 \end{cases}, \quad T_-^{\partial\Omega}(x,s) = \begin{cases} u_-(x) & \text{if } s \le u_-(x) \\ s & \text{if } u_-(x) < s < 0 \\ 0 & \text{if } s \ge 0 \end{cases}$$

$$T_0(x,s) = \begin{cases} u_-(x) & \text{if } s \le u_-(x) \\ s & \text{if } u_-(x) < s < u_+(x) \\ u_+(x) & \text{if } s \ge u_+(x) \end{cases}, \quad T_0^{\partial\Omega}(x,s) = \begin{cases} u_-(x) & \text{if } s \le u_-(x) \\ s & \text{if } u_-(x) < s < u_+(x) \\ u_+(x) & \text{if } s \ge u_+(x) \end{cases}$$

As before, the truncation operators on $\partial\Omega$ apply to the corresponding traces $\gamma(u)$. With the aid of these truncations, we introduce the associated functionals

$$\begin{aligned} E_+(u) =& \frac{1}{p}[\|\nabla u\|_{L^p(\Omega)}^p + \|u\|_{L^p(\Omega)}^p] - \int_\Omega \int_0^{u(x)} f(x, T_+(x,s))dsdx \\ & - \int_{\partial\Omega} \int_0^{u(x)} \left[ a T_+^{\partial\Omega}(x,s)^{p-1} + g(x, T_+^{\partial\Omega}(x,s)) \right] dsd\sigma, \end{aligned} \tag{3.2.32}$$

$$\begin{aligned} E_-(u) =& \frac{1}{p}[\|\nabla u\|_{L^p(\Omega)}^p + \|u\|_{L^p(\Omega)}^p] - \int_\Omega \int_0^{u(x)} f(x, T_-(x,s))dsdx \\ & + \int_{\partial\Omega} \int_0^{u(x)} \left[ b |T_-^{\partial\Omega}(x,s)|^{p-1} - g(x, T_-^{\partial\Omega}(x,s)) \right] dsd\sigma, \end{aligned} \tag{3.2.33}$$

$$\begin{aligned} E_0(u) =& \frac{1}{p}[\|\nabla u\|_{L^p(\Omega)}^p + \|u\|_{L^p(\Omega)}^p] - \int_\Omega \int_0^{u(x)} f(x, T_0(x,s))dsdx \\ & - \int_{\partial\Omega} \int_0^{u(x)} \left[ a T_+^{\partial\Omega}(x,s)^{p-1} - b |T_-^{\partial\Omega}(x,s)|^{p-1} \right] dsd\sigma \\ & - \int_{\partial\Omega} \int_0^{u(x)} g(x, T_0^{\partial\Omega}(x,s))dsd\sigma, \end{aligned} \tag{3.2.34}$$

which are well-defined and belong to $C^1(W^{1,p}(\Omega))$. Due to the truncations, one can easily show that these functionals are coercive and weakly lower semicontinuous which implies that their global minimizers exist (see Lemma 3.1.9 and Theorem 2.4.1).

**Lemma 3.2.9.** *Let $u_+$ and $u_-$ be the extremal constant-sign solutions of (3.2.1). Then the following holds:*

*(i) A critical point $v \in W^{1,p}(\Omega)$ of $E_+$ is a (nonnegative) solution of (3.2.1) satisfying $0 \leq v \leq u_+$.*

*(ii) A critical point $v \in W^{1,p}(\Omega)$ of $E_-$ is a (nonpositive) solution of (3.2.1) satisfying $u_- \leq v \leq 0$.*

*(iii) A critical point $v \in W^{1,p}(\Omega)$ of $E_0$ is a solution of (3.2.1) satisfying $u_- \leq v \leq u_+$.*

**Proof.** Let $v$ be a critical point of $E_0$ meaning $E_0'(v) = 0$. With a view to (3.2.34), we have

$$\begin{aligned} &\int_\Omega |\nabla v|^{p-2}\nabla v \nabla \varphi dx \\ &= \int_\Omega [f(x, T_0(x,v)) - |v|^{p-2}v]\varphi dx + \int_{\partial\Omega} aT_+^{\partial\Omega}(x,v)^{p-1}\varphi d\sigma \\ &\quad + \int_{\partial\Omega} [-b|T_-^{\partial\Omega}(x,v)|^{p-1} + g(x, T_0^{\partial\Omega}(x,v))]\varphi d\sigma, \quad \forall \varphi \in W^{1,p}(\Omega). \end{aligned} \tag{3.2.35}$$

As $u_+$ is a positive solution of (3.2.1) it satisfies

$$\begin{aligned} \int_\Omega |\nabla u_+|^{p-2}\nabla u_+ \nabla \varphi dx &= \int_\Omega [f(x,u_+) - u_+^{p-1}]\varphi dx \\ &\quad + \int_{\partial\Omega} [au_+^{p-1} + g(x,u_+)]\varphi d\sigma, \quad \forall \varphi \in W^{1,p}(\Omega). \end{aligned} \tag{3.2.36}$$

Subtracting (3.2.36) from (3.2.35) and setting $\varphi = (v-u_+)^+ \in W^{1,p}(\Omega)$ provides

$$\begin{aligned} &\int_\Omega [|\nabla v|^{p-2}\nabla v - |\nabla u_+|^{p-2}\nabla u_+]\nabla (v-u_+)^+ dx + \int_\Omega [|v|^{p-2}v - u_+^{p-1}](v-u_+)^+ dx \\ &= \int_\Omega [f(x, T_0(x,v)) - f(x,u_+)](v-u_+)^+ dx \\ &\quad + \int_{\partial\Omega} [aT_+^{\partial\Omega}(x,v)^{p-1} - b|T_-^{\partial\Omega}(x,v)|^{p-1} - au_+^{p-1}](v-u_+)^+ d\sigma \\ &\quad + \int_{\partial\Omega} [g(x, T_0^{\partial\Omega}(x,v)) - g(x,u_+)](v-u_+)^+ d\sigma. \end{aligned}$$

Based on the definition of the truncation operators, we see that the right-hand side of the equality above is equal to zero. On the other hand the integrals on the left-hand side are strictly positive in case $v > z_+$ (cf. Section 2.1.4) which is a contradiction. Thus, we get $(v-u_+)^+ = 0$ and hence, $v \leq u_+$. The proof for $v \geq u_-$ acts in similar way which shows that $T_0(x,v) = v$, $T_+^{\partial\Omega}(x,v) = v^+$, $T_-^{\partial\Omega}(x,v) = v^-$ and therefore, $v$ is a solution of (3.2.1) satisfying $u_- \leq v \leq u_+$. The statements in (i) and (ii) can be shown in an analogous manner.

□

An important tool in our considerations is the relation between $C^1(\overline{\Omega})$ and $W^{1,p}(\Omega)$-local minimizers for $C^1-$ functionals. Fact is that every local $C^1-$minimizer of $E_0$ is a local $W^{1,p}(\Omega)$-minimizer of $E_0$ which is proven in similar form in Proposition 3.1.11. This result reads as follows.

**Proposition 3.2.10.** *If $z_0 \in W^{1,p}(\Omega)$ is a local $C^1(\overline{\Omega})$-minimizer of $E_0$ meaning that there exists $r_1 > 0$ such that*

$$E_0(z_0) \leq E_0(z_0 + h) \quad \text{for all } h \in C^1(\overline{\Omega}) \text{ with } \|h\|_{C^1(\overline{\Omega})} \leq r_1,$$

*then $z_0$ is a local minimizer of $E_0$ in $W^{1,p}(\Omega)$ meaning that there exists $r_2 > 0$ such that*

$$E_0(z_0) \leq E_0(z_0 + h) \quad \text{for all } h \in W^{1,p}(\Omega) \text{ with } \|h\|_{W^{1,p}(\Omega)} \leq r_2.$$

With the aid of Proposition 3.2.10, we can formulate the next lemma about the existence of local and global minimizers with respect to the functionals $E_+$, $E_-$ and $E_0$.

**Lemma 3.2.11.** *Let $a > \lambda_1$ and $b > \lambda_1$. Then the extremal positive solution $u_+$ of (3.2.1) is the unique global minimizer of the functional $E_+$ and the extremal negative solution $u_-$ of (3.2.1) is the unique global minimizer of the functional $E_-$. In addition, both $u_+$ and $u_-$ are local minimizers of the functional $E_0$.*

**Proof.** As $E_+ : W^{1,p}(\Omega) \to \mathbb{R}$ is coercive and weakly sequentially lower semicontinuous, its global minimizer $v_+ \in W^{1,p}(\Omega)$ exists (cf. [123, Theorem 25.D]) meaning that $v_+$ is a critical point of $E_+$. Concerning Lemma 3.2.9 we know that $v_+$ is a nonnegative solution of (3.2.1) satisfying $0 \leq v_+ \leq u_+$. Due to condition (H2)(g1) there exists a number $\delta_a > 0$ such that

$$|g(x,s)| \leq (a - \lambda_1)s^{p-1}, \qquad \forall s : 0 < s \leq \delta_a. \tag{3.2.37}$$

Choosing $\varepsilon < \min\left\{\frac{\delta_f}{\|\varphi_1\|_\infty}, \frac{\delta_a}{\|\varphi_1\|_\infty}\right\}$ and applying assumption (H1)(f4), inequality (3.2.37) along with the Steklov eigenvalue problem implies

$$\begin{aligned}
E_+(\varepsilon\varphi_1) = &- \int_\Omega \int_0^{\varepsilon\varphi_1(x)} f(x,s)dsdx + \frac{\lambda_1 - a}{p}\varepsilon^p \|\varphi_1\|^p_{L^p(\partial\Omega)} \\
&- \int_{\partial\Omega} \int_0^{\varepsilon\varphi_1(x)} g(x,s)dsd\sigma \\
&< \frac{\lambda_1 - a}{p}\varepsilon^p \|\varphi_1\|^p_{L^p(\partial\Omega)} + \int_{\partial\Omega} \int_0^{\varepsilon\varphi_1(x)} (a - \lambda_1)s^{p-1} dsd\sigma \\
&= 0.
\end{aligned}$$

From the calculations above, we see at once that $E_+(v_+) < 0$ which means that $v_+ \neq 0$. This allows us to apply Lemma 3.2.7 to get $v_+ \in \operatorname{int}(C^1(\overline{\Omega})_+)$. Since $u_+$ is the smallest positive solution of (3.2.1) in $[0, \vartheta_a e]$ fulfilling $0 \leq v_+ \leq u_+$, it must hold $v_+ = u_+$ which proves that $u_+$ is the unique global minimizer of $E_+$. The same considerations show that $u_-$ is the unique global minimizer of $E_-$. In order to complete the proof, we are going to demonstrate that $u_+$ and $u_-$ are local minimizers of the functional $E_0$, too. The extremal positive solution $u_+$ belongs to $\operatorname{int}(C^1(\overline{\Omega})_+)$ which means that there is a neighborhood $V_{u_+}$ of $u_+$ in the space $C^1(\overline{\Omega})$ satisfying $V_{u_+} \subset C^1(\overline{\Omega})_+$. Therefore $E_+ = E_0$ on $V_{u_+}$ proving that $u_+$ is a local minimizer of $E_0$ on $C^1(\overline{\Omega})$. Applying Proposition 3.2.10 yields that $u_+$ is also a local minimizer of $E_0$ on the space $W^{1,p}(\Omega)$. Similarly we see that $u_-$ is a local minimizer of $E_0$ which completes the proof. □

**Lemma 3.2.12.** *The functional $E_0 : W^{1,p}(\Omega) \to \mathbb{R}$ has a global minimizer $v_0$ which is a nontrivial solution of (3.2.1) satisfying $u_- \le v_0 \le u_+$.*

**Proof.** As we know, the functional $E_0 : W^{1,p}(\Omega) \to \mathbb{R}$ is coercive and weakly sequentially lower semicontinuous. Hence, it has a global minimizer $v_0$. More precisely, $v_0$ is a critical point of $E_0$ which is a solution of (3.2.1) satisfying $u_- \le v_0 \le u_+$ (see Lemma 3.2.9). The fact that $E_0(u_+) = E_+(u_+) < 0$ (see the proof of Lemma 3.2.11) proves that $v_0$ is nontrivial meaning $v_0 \neq 0$. □

### 3.2.4 Existence of Sign-Changing Solutions

In order to prove the existence of a sign-changing solution of problem (3.2.1), we have to show first that the functional $E_0$ introduced in Section 3.2.3 satisfies the Palais-Smale condition. The following result is proven in similar form in Lemma 3.1.14.

**Lemma 3.2.13.** *Let $A, B, C : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ be given by*

$$\begin{aligned}
\langle A(u), v\rangle &:= \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx + \int_\Omega |u|^{p-2} uv dx, \\
\langle B(u), v\rangle &:= \int_{\partial\Omega} (a T_+^{\partial\Omega}(x,u)^{p-1} - b|T_-^{\partial\Omega}(x,u)|^{p-1}) v dx, \\
\langle C(u), v\rangle &:= \int_\Omega f(x, T_0(x,u)) v dx + \int_{\partial\Omega} g(x, T_0^{\partial\Omega}(x,u)) v dx,
\end{aligned}$$

*then $A$ is continuous, continuously invertible and the operators $B, C$ are continuous and compact.*

We obtain the following lemma.

**Lemma 3.2.14.** *The functional $E_0 : W^{1,p}(\Omega) \to \mathbb{R}$ fulfills the Palais-Smale condition.*

**Proof.** Let $(u_n) \subset W^{1,p}(\Omega)$ be a sequence such that $E_0(u_n)$ is bounded and $E_0'(u_n) \to 0$ as $n$ tends to infinity. Applying Young's inequality and the compact embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$

along with the premise $|E_0(u_n)| \leq M$ for all $n$ provides

$$
\begin{aligned}
M &\geq E_0(u_n) \\
&= \frac{1}{p}\left[\|\nabla u_n\|_{L^p(\Omega)}^p + \|u_n\|_{L^p(\Omega)}^p\right] - \int_\Omega \int_0^{u_n(x)} f(x, T_0(x,s))dsdx \\
&\quad - \int_{\partial\Omega} \int_0^{u_n(x)} \left[a T_+^{\partial\Omega}(x,s)^{p-1} - b|T_-^{\partial\Omega}(x,s)|^{p-1} + g(x, T_0^{\partial\Omega}(x,s))\right] dsd\sigma \\
&\geq (1/p - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)||u_n\|_{W^{1,p}(\Omega)}^p - C.
\end{aligned}
$$

We select $\varepsilon_i$, $i = 1, \dots, 4$ sufficiently small to have $1/p - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 > 0$. This yields the boundedness of $u_n$ in $W^{1,p}(\Omega)$, and thus, we get $u_n \rightharpoonup u$ for a subsequence of $u_n$ still denoted with $u_n$. We obtain

$$A(u_n) - \lambda B(u_n) - C(u_n) = E_0'(u_n) \to 0,$$

which implies the existence of a sequence $(\delta_n) \subset (W^{1,p}(\Omega))^*$ converging to zero such that

$$u_n = A^{-1}(\lambda B(u_n) + C(u_n) + \delta_n).$$

Lemma 3.2.13 shows that $B$, $C$ are compact and $A^{-1}$ is continuous. Passing to the limit in the previous equality yields

$$u_n \to A^{-1}(\lambda B(u) + C(u)) =: u,$$

meaning that $u_n \to u$ strongly in $W^{1,p}(\Omega)$. □

The main result in this subsection about the existence of a nontrivial solution of problem (3.2.1) is phrased as follows.

**Theorem 3.2.15.** *Under hypotheses (H1)–(H3) problem (3.2.1) has a nontrivial sign-changing solution $u_0 \in C^1(\overline{\Omega})$.*

**Proof.** In view of Lemma 3.2.12 the existence of a global minimizer $v_0 \in W^{1,p}(\Omega)$ satisfying $v_0 \neq 0$ has been proven. This means that $v_0$ is a nontrivial solution of (3.2.1) belonging to $[u_-, u_+]$. If $v_0 \neq u_-$ and $v_0 \neq u_+$, then $u_0 := v_0$ must be a sign-changing solution since $u_-$ is the greatest negative solution and $u_+$ is the smallest positive solution of (3.2.1) which proves the theorem in this case. We still have to prove the theorem in case that either $v_0 = u_-$ or $v_0 = u_+$. Let us only consider the case $v_0 = u_+$ because the case $v_0 = u_-$ can be proven similar. The function $u_-$ is a local minimizer of $E_0$. Without loss of generality we suppose that $u_-$ is a strict local minimizer, otherwise we would obtain infinitely many critical points $v$ of $E_0$ which are sign-changing solutions due to $u_- \leq v \leq u_+$ and the extremality of the solutions $u_-, u_+$. Under these assumptions, there exists a $\rho \in (0, \|u_+ - u_-\|_{W^{1,p}(\Omega)})$ such that

$$E_0(u_+) \leq E_0(u_-) < \inf\{E_0(u) : u \in \partial B_\rho(u_-)\}, \tag{3.2.38}$$

where $\partial B_\rho = \{u \in W^{1,p}(\Omega) : \|u - u_-\|_{W^{1,p}(\Omega)} = \rho\}$. Now we may apply the Mountain-Pass theorem to $E_0$ (cf. Theorem 2.4.4) thanks to (3.2.38) along with the fact that $E_0$ satisfies the Palais-Smale condition (see Lemma 3.2.14). This yields the existence of $u_0 \in W^{1,p}(\Omega)$ satisfying $E_0'(u_0) = 0$ and

$$\inf\{E_0(u) : u \in \partial B_\rho(u_-)\} \le E_0(u_0) = \inf_{\pi\in\Pi} \max_{t\in[-1,1]} E_0(\pi(t)), \tag{3.2.39}$$

where

$$\Pi = \{\pi \in C([-1,1], W^{1,p}(\Omega)) : \pi(-1) = u_-, \pi(1) = u_+\}.$$

One easily verifies that (3.2.38) and (3.2.39) imply $u_0 \neq u_-$ and $u_0 \neq u_+$. Hence, $u_0$ is a nontrivial sign-changing solution of (3.2.1) provided $u_0 \neq 0$. We have to show that $E_0(u_0) < 0$ which is fulfilled if there exists a path $\widetilde{\pi} \in \Pi$ such that

$$E_0(\widetilde{\pi}(t)) < 0, \quad \forall t \in [-1,1]. \tag{3.2.40}$$

Let $S = W^{1,p}(\Omega) \cap \partial B_1^{L^p(\partial\Omega)}$, where $\partial B_1^{L^p(\partial\Omega)} = \{u \in L^p(\partial\Omega) : \|u\|_{L^p(\partial\Omega)} = 1\}$, and $S_C = S \cap C^1(\overline{\Omega})$ be equipped with the topologies induced by $W^{1,p}(\Omega)$ and $C^1(\overline{\Omega})$, respectively. Furthermore, we set

$$\Pi_0 = \{\pi \in C([-1,1], S) : \pi(-1) = -\varphi_1, \pi(1) = \varphi_1\},$$
$$\Pi_{0,C} = \{\pi \in C([-1,1], S_C) : \pi(-1) = -\varphi_1, \pi(1) = \varphi_1\}.$$

Because of the results of Martínez and Rossi in [97] there exists a continuous path $\pi \in \Pi_0$ satisfying $t \mapsto \pi(t) \in \{u \in W^{1,p}(\Omega) : I^{(a,b)}(u) < 0, \|u\|_{L^p(\partial\Omega)} = 1\}$ provided $(a,b)$ is above the curve $\mathcal{C}$. Recall that the functional $I^{(a,b)}$ is given by

$$I^{(a,b)} = \int_\Omega (|\nabla u|^p + |u|^p)dx - \int_{\partial\Omega} (a(u^+)^p + b(u^-)^p)d\sigma. \tag{3.2.41}$$

This implies the existence of $\mu > 0$ such that

$$I^{(a,b)}(\pi(t)) \le -\mu < 0, \quad \forall t \in [-1,1]. \tag{3.2.42}$$

It is well known that $S_C$ is dense in $S$ which implies the density of $\Pi_{0,C}$ in $\Pi_0$. Thus, a continuous path $\pi_0 \in \Pi_{0,C}$ exists such that

$$|I^{(a,b)}(\pi(t)) - I^{(a,b)}(\pi_0(t))| < \frac{\mu}{2}, \quad \forall t \in [-1,1]. \tag{3.2.43}$$

The boundedness of the set $\pi_0([-1,1])(\overline{\Omega})$ in $\mathbb{R}$ ensures the existence of $M > 0$ such that

$$|\pi_0(t)(x)| \le M \quad \text{for all } x \in \overline{\Omega} \text{ and for all } t \in [-1,1]. \tag{3.2.44}$$

Lemma 3.2.8 yields that $u_+, -u_- \in \operatorname{int}(C^1(\overline{\Omega})_+)$. Thus, for every $u \in \pi_0([-1,1])$ and any bounded neighborhood $V_u$ of $u$ in $C^1(\overline{\Omega})$ there exist positive numbers $h_u$ and $j_u$ satisfying

$$u_+ - hv \in \operatorname{int}(C^1(\overline{\Omega})_+) \quad \text{and} \quad -u_- + jv \in \operatorname{int}(C^1(\overline{\Omega})_+), \tag{3.2.45}$$

for all $h : 0 \le h \le h_u$, for all $j : 0 \le j \le j_u$, and for all $v \in V_u$. Using (3.2.45) along with a compactness argument implies the existence of $\varepsilon_0 > 0$ such that

$$u_-(x) \le \varepsilon \pi_0(t)(x) \le u_+(x), \tag{3.2.46}$$

for all $x \in \Omega$, for all $t \in [-1,1]$ and for all $\varepsilon \le \varepsilon_0$. Representing $E_0$ in terms of $I^{(a,b)}$ we obtain

$$\begin{aligned} E_0(u) =& \frac{1}{p} I^{(a,b)}(u) + \int_{\partial\Omega} (a(u^+)^p + b(u^-)^p) d\sigma - \int_\Omega \int_0^{u(x)} f(x, T_0(x,s)) ds dx \\ & - \int_{\partial\Omega} \int_0^{u(x)} (a T_+^{\partial\Omega}(x,s)^{p-1} - b|T_-^{\partial\Omega}(x,s)|^{p-1}) ds d\sigma \\ & - \int_{\partial\Omega} \int_0^{u(x)} g(x, T_0^{\partial\Omega}(x,s)) ds d\sigma. \end{aligned} \tag{3.2.47}$$

In view of (3.2.46) we get for all $\varepsilon \le \varepsilon_0$ and all $t \in [-1,1]$

$$\begin{aligned} & E_0(\varepsilon\pi_0(t)) \\ & = \frac{1}{p} I^{(a,b)}(\varepsilon\pi_0(t)) - \int_\Omega \int_0^{\varepsilon\pi_0(t)(x)} f(x,s) ds dx - \int_{\partial\Omega} \int_0^{\varepsilon\pi_0(t)(x)} g(x,s) ds d\sigma \\ & = \varepsilon^p \left[ \frac{1}{p} I^{(a,b)}(\pi_0(t)) - \frac{1}{\varepsilon^p} \int_\Omega \int_0^{\varepsilon\pi_0(t)(x)} f(x,s) ds dx \right. \\ & \qquad \left. - \frac{1}{\varepsilon^p} \int_{\partial\Omega} \int_0^{\varepsilon\pi_0(t)(x)} g(x,s) ds d\sigma \right] \\ & < \varepsilon^p \left[ -\frac{\mu}{2p} + \frac{1}{\varepsilon^p} \int_\Omega \left| \int_0^{\varepsilon\pi_0(t)(x)} f(x,s) ds \right| dx \right. \\ & \qquad \left. + \frac{1}{\varepsilon^p} \int_{\partial\Omega} \left| \int_0^{\varepsilon\pi_0(t)(x)} g(x,s) ds \right| d\sigma \right]. \end{aligned} \tag{3.2.48}$$

Due to hypotheses (H1)(f1) and (H2)(g1) there exist positive constants $\delta_1, \delta_2$ such that

$$\begin{aligned} |f(x,s)| &\le \frac{\mu}{5M^p} |s|^{p-1}, \quad \text{for a.a. } x \in \Omega \text{ and all } s : |s| \le \delta_1, \\ |g(x,s)| &\le \frac{\mu}{5M^p} |s|^{p-1}, \quad \text{for a.a. } x \in \partial\Omega \text{ and all } s : |s| \le \delta_2. \end{aligned} \tag{3.2.49}$$

Choosing $\varepsilon > 0$ such that $\varepsilon < \min\{\varepsilon_0, \frac{\delta_1}{M}, \frac{\delta_2}{M}\}$ one obtains by using (3.2.49)

$$\begin{aligned} \frac{1}{\varepsilon^p} \int_\Omega \left| \int_0^{\varepsilon\pi_0(t)(x)} f(x,s) ds \right| dx &\le \frac{\mu}{5p}, \\ \frac{1}{\varepsilon^p} \int_{\partial\Omega} \left| \int_0^{\varepsilon\pi_0(t)(x)} g(x,s) ds \right| d\sigma &\le \frac{\mu}{5p}. \end{aligned} \tag{3.2.50}$$

Applying (3.2.50) to (3.2.48) yields

$$E_0(\varepsilon\pi_0(t)) \le \varepsilon^p \left(-\frac{\mu}{2p} + \frac{\mu}{5p} + \frac{\mu}{5p}\right) < 0, \quad \text{for all } t \in [-1,1]. \tag{3.2.51}$$

We have constructed a continuous path $\varepsilon\pi_0$ joining $-\varepsilon\varphi_1$ and $\varepsilon\varphi_1$. In order to construct continuous paths $\pi_+, \pi_-$ connecting $\varepsilon\varphi_1$ and $u_+$, respectively, $u_-$ and $-\varepsilon\varphi_1$, we first denote

$$c_+ = E_+(\varepsilon\varphi_1), \qquad m_+ = E_+(u_+), \qquad E_+^{c_+} = \{u \in W^{1,p}(\Omega) : E_+(u) \leq c_+\}.$$

It holds $m_+ < c_+$ because $u_+$ is a global minimizer of $E_+$. By Lemma 3.2.9 the functional $E_+$ has no critical values in the interval $(m_+, c_+]$. The coercivity of $E_+$ along with its property to satisfy the Palais-Smale condition (see Lemma 3.2.14) allows us to apply the Second Deformation Lemma (see, e.g. [73, p. 366]) to $E_+$. This ensures the existence of a continuous mapping $\eta \in C([0,1] \times E_+^{c_+}, E_+^{c_+})$ satisfying the following properties:

(i) $\eta(0,u) = u \quad$ for all $u \in E_+^{c_+}$,

(ii) $\eta(1,u) = u_+ \quad$ for all $u \in E_+^{c_+}$,

(iii) $E_+(\eta(t,u)) \leq E_+(u), \quad \forall t \in [0,1]$ and $\forall u \in E_+^{c_+}$.

Next, we introduce the path $\pi_+ : [0,1] \to W^{1,p}(\Omega)$ given by $\pi_+(t) = \eta(t, \varepsilon\varphi_1)^+ = \max\{\eta(t,\varepsilon\varphi_1), 0\}$ for all $t \in [0,1]$ which is obviously continuous in $W^{1,p}(\Omega)$ joining $\varepsilon\varphi_1$ and $u_+$. Additionally, one has

$$E_0(\pi_+(t)) = E_+(\pi_+(t)) \leq E_+(\eta(t,\varepsilon\varphi_1)) \leq E_+(\varepsilon\varphi_1) < 0 \qquad \text{for all } t \in [0,1]. \tag{3.2.52}$$

Similarly, the Second Deformation Lemma can be applied to the functional $E_-$. We get a continuous path $\pi_- : [0,1] \to W^{1,p}(\Omega)$ connecting $-\varepsilon\varphi_1$ and $u_-$ such that

$$E_0(\pi_-(t)) < 0 \quad \text{for all } t \in [0,1]. \tag{3.2.53}$$

In the end, we combine the curves $\pi_-, \varepsilon\pi_0$ and $\pi_+$ to obtain a continuous path $\widetilde{\pi} \in \Pi$ joining $u_-$ and $u_+$. Taking into account (3.2.51), (3.2.52) and (3.2.53) we see $u_0 \neq 0$. This yields the existence of a nontrivial sign-changing solution $u_0$ of problem (3.2.1) satisfying $u_- \leq u_0 \leq u_+$ which completes the proof. $\square$

# Chapter 4

# General Comparison Principle for Variational-Hemivariational Inequalities

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded domain with Lipschitz boundary $\partial\Omega$. By $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega), 1 < p < \infty$, we denote the usual Sobolev spaces with their dual spaces $(W^{1,p}(\Omega))^*$ and $W^{-1,q}(\Omega)$, respectively, where $q$ is the Hölder conjugate satisfying $\frac{1}{p} + \frac{1}{q} = 1$. We consider the following elliptic variational-hemivariational inequality: Find $u \in K$ such that

$$\langle Au + F(u), v - u \rangle + \int_\Omega j_1^\circ(\cdot, u; v - u)dx + \int_{\partial\Omega} j_2^\circ(\cdot, \gamma u; \gamma v - \gamma u)d\sigma \geq 0, \tag{4.0.1}$$

for all $v \in K$ where $j_k^\circ(x, s; r), k = 1, 2$ denotes the generalized directional derivative of the locally Lipschitz functions $s \mapsto j_k(x, s)$ at $s$ in the direction $r$ given by

$$j_k^\circ(x, s; r) = \limsup_{y \to s, t \downarrow 0} \frac{j_k(x, y + tr) - j_k(x, y)}{t}, k = 1, 2, \tag{4.0.2}$$

(cf. Section 2.3). We denote by $K$ a closed convex subset of $W^{1,p}(\Omega)$ and $A$ is a second-order quasilinear differential operator in divergence form of Leray-Lions type given by

$$Au(x) = -\sum_{i=1}^N \frac{\partial}{\partial x_i} a_i(x, u(x), \nabla u(x)).$$

The operator F stands for the Nemytskij operator associated with some Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ defined by

$$F(u)(x) = f(x, u(x), \nabla u(x)). \tag{4.0.3}$$

Furthermore, we denote the trace operator by $\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ which is known to be linear, bounded and even compact.
The aim of this chapter is to establish the method of sub- and supersolutions for problem (4.0.1). We prove the existence of solutions between a given pair of sub-supersolution assuming only a local growth condition on Clarke's generalized gradient which extends results recently obtained

by S. Carl in [19]. To complete our findings, we also present the proof for the existence of extremal solutions of problem (4.0.1) for a fixed ordered pair of sub- and supersolutions in case $A$ has the form

$$Au(x) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, \nabla u(x)). \tag{4.0.4}$$

In the second part we consider (4.0.1) with a discontinuous Nemytskij operator $F$ involved, which extends results in [120] and partly of [21]. Finally, we construct some sub- and supersolution of problem (4.0.1) where we make use of the special case $A = -\Delta_p$ which is the negative $p$-Laplacian. Let us consider some special cases of problem (4.0.1) where we suppose $A = -\Delta_p$, too.

(1) If $K = W^{1,p}(\Omega)$ and $j_k$ are smooth, problem (4.0.1) reduces to

$$\langle -\Delta_p u + F(u), v\rangle + \int_\Omega j_1'(\cdot, u) v dx + \int_{\partial\Omega} j_2'(\cdot, \gamma u)\gamma v d\sigma = 0, \quad \forall v \in W^{1,p}(\Omega),$$

which is equivalent to the weak formulation of the nonlinear boundary value problem

$$\begin{aligned} -\Delta_p u + F(u) + j_1'(u) &= 0 \qquad \text{in } \Omega,\\ \frac{\partial u}{\partial \nu} + j_2'(\gamma u) &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

where $\frac{\partial u}{\partial \nu}$ denotes the conormal derivative of $u$. The method of sub- and supersolution for this kind of problems is a special case of [18].

(2) For $f \in W^{-1,q}(\Omega)$, $K \subset W_0^{1,p}(\Omega)$ and $j_2 = 0$, (4.0.1) corresponds to the variational-hemivariational inequality given by

$$\langle -\Delta_p u + f, v - u\rangle + \int_\Omega j_1^\circ(\cdot, u; v - u) dx \geq 0, \quad \forall v \in K,$$

which has been discussed in detail in [17].

(3) If $K \subset W_0^{1,p}(\Omega)$ and $j_k = 0$, then (4.0.1) is a classical variational inequality of the form

$$u \in K : \quad \langle -\Delta_p u + F(u), v - u\rangle \geq 0, \quad \forall v \in K,$$

whose method of sub- and supersolution has been developed in [28, Chapter 5].

(4) Let $K = W_0^{1,p}(\Omega)$ or $K = W^{1,p}(\Omega)$ and $j_k$ not necessarily smooth. Then problem (4.0.1) is a hemivariational inequality which contains for $K = W_0^{1,p}(\Omega)$ as a special case the following Dirichlet problem for the elliptic inclusion

$$\begin{aligned} -\Delta_p u + F(u) + \partial j_1(\cdot, u) &\ni 0 \qquad \text{in } \Omega,\\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{4.0.5}$$

and for $K = W^{1,p}(\Omega)$ the elliptic inclusion

$$\begin{aligned} -\Delta_p u + F(u) + \partial j_1(\cdot, u) &\ni 0 \qquad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + \partial j_2(\cdot, u) &\ni 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{4.0.6}$$

where the multivalued functions $s \mapsto \partial j_k(x, s)$, $k = 1, 2$ stand for Clarke's generalized gradient of the locally Lipschitz functions $s \mapsto j_k(x, s)$, $k = 1, 2$ given by

$$\partial j_k(x, s) = \{\xi \in \mathbb{R} : j_k^o(x, s; r) \geq \xi r, \forall r \in \mathbb{R}\}. \tag{4.0.7}$$

Problems of the form (4.0.5) and (4.0.6) have been studied in [39] and [18], respectively.

Existence results for variational-hemivariational inequalities with or without the method of sub- and supersolutions have been obtained under different structure and regularity conditions on the nonlinear functions by various authors. We refer for example to [12, 29, 30, 69, 75, 82, 91, 94]. In case that $K$ is the whole space $W_0^{1,p}(\Omega)$ or $W^{1,p}(\Omega)$, respectively, problem (4.0.1) reduces to a hemivariational inequality which has been treated in [10, 40, 48, 68, 70, 80, 84, 93, 101]. Comparison principles for general elliptic operators $A$ including the negative $p$-Laplacian $-\Delta_p$ and Clarke's generalized gradient $s \mapsto \partial j(x, s)$ satisfying a one-sided growth condition in the form

$$\xi_1 \leq \xi_2 + c_1(s_2 - s_1)^{p-1}, \tag{4.0.8}$$

for all $\xi_i \in \partial j(x, s_i)$, $i = 1, 2$, for a.a. $x \in \Omega$, and for all $s_1, s_2$ with $s_1 < s_2$, can be found in [28]. Inspired by results recently obtained in [39] and [38], we prove the existence of (extremal) solutions for the variational-hemivariational inequality (4.0.1) within a sector of an ordered pair of sub- and supersolutions $\underline{u}, \overline{u}$ without assuming a one-sided growth condition on Clarke's generalized gradient of the form (4.0.8).

## 4.1 Notation of Sub- and Supersolutions

For functions $u, v : \Omega \to \mathbb{R}$ we use the notation $u \wedge v = \min(u, v)$, $u \vee v = \max(u, v)$, $K \wedge K = \{u \wedge v : u, v \in K\}$, $K \vee K = \{u \vee v : u, v \in K\}$, and $u \wedge K = \{u\} \wedge K$, $u \vee K = \{u\} \vee K$ and introduce the following definitions.

**Definition 4.1.1.** *A function $\underline{u} \in W^{1,p}(\Omega)$ is said to be a subsolution of (4.0.1) if the following hold:*

*(i)* $F(\underline{u}) \in L^q(\Omega)$,

*(ii)* $\langle A\underline{u} + F(\underline{u}), w - \underline{u}\rangle + \int_\Omega j_1^o(\cdot, \underline{u}; w - \underline{u})dx + \int_{\partial\Omega} j_2^o(\cdot, \gamma\underline{u}; \gamma w - \gamma\underline{u})d\sigma \geq 0, \quad \forall w \in \underline{u} \wedge K.$

**Definition 4.1.2.** *A function $\overline{u} \in W^{1,p}(\Omega)$ is said to be a supersolution of (4.0.1) if the following hold:*

*(i)* $F(\overline{u}) \in L^q(\Omega)$,

*(ii)* $\langle A\overline{u} + F(\overline{u}), w - \overline{u}\rangle + \int_\Omega j_1^o(\cdot, \overline{u}; w - \overline{u})dx + \int_{\partial\Omega} j_2^o(\cdot, \gamma\overline{u}; \gamma w - \gamma\overline{u})d\sigma \geq 0, \quad \forall w \in \overline{u} \vee K.$

In order to prove our main results, we additionally assume the following

$$\underline{u} \vee K \subset K, \qquad \overline{u} \wedge K \subset K. \tag{4.1.1}$$

# 4.2 Preliminaries and Hypotheses

Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, and assume for the coefficients $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, $i = 1, \dots, N$ the following conditions.

(A1) Each $a_i(x, s, \xi)$ satisfies Carathéodory conditions, i.e., is measurable in $x \in \Omega$ for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and continuous in $(s, \xi)$ for a.a. $x \in \Omega$. Furthermore, a constant $c_0 > 0$ and a function $k_0 \in L^q(\Omega)$ exist such that

$$|a_i(x, s, \xi)| \leq k_0(x) + c_0(|s|^{p-1} + |\xi|^{p-1}),$$

for a.a. $x \in \Omega$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$, where $|\xi|$ denotes the Euclidian norm of the vector $\xi$.

(A2) The coefficients $a_i$ satisfy a monotonicity condition with respect to $\xi$ in the form

$$\sum_{i=1}^N (a_i(x, s, \xi) - a_i(x, s, \xi'))(\xi_i - \xi_i') > 0,$$

for a.a. $x \in \Omega$, for all $s \in \mathbb{R}$, and for all $\xi, \xi' \in \mathbb{R}^N$ with $\xi \neq \xi'$.

(A3) A constant $c_1 > 0$ and a function $k_1 \in L^1(\Omega)$ exist such that

$$\sum_{i=1}^N a_i(x, s, \xi)\xi_i \geq c_1|\xi|^p - k_1(x),$$

for a.a. $x \in \Omega$, for all $s \in \mathbb{R}$, and for all $\xi \in \mathbb{R}^N$.

Condition (A1) implies that $A : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ is bounded continuous and along with (A2) it holds that $A$ is pseudomonotone. Due to (A1) the operator $A$ generates a mapping from $W^{1,p}(\Omega)$ into its dual space defined by

$$\langle Au, \varphi\rangle = \int_\Omega \sum_{i=1}^N a_i(x, u, \nabla u)\frac{\partial\varphi}{\partial x_i}dx,$$

where $\langle\cdot,\cdot\rangle$ stands for the duality pairing between $W^{1,p}(\Omega)$ and $(W^{1,p}(\Omega))^*$. Assumption (A3) is a coercivity type condition.

Let $[\underline{u}, \overline{u}]$ be an ordered pair of sub- and supersolutions of problem (4.0.1). We impose the following hypotheses on $j_k$ and the nonlinearity $f$ in problem (4.0.1):

(j1) $x \mapsto j_1(x,s)$ and $x \mapsto j_2(x,s)$ are measurable in $\Omega$ and $\partial\Omega$, respectively, for all $s \in \mathbb{R}$.

(j2) $s \mapsto j_1(x,s)$ and $s \mapsto j_2(x,s)$ are locally Lipschitz continuous in $\mathbb{R}$ for a.a. $x \in \Omega$ and for a.a. $x \in \partial\Omega$, respectively.

(j3) There are functions $L_1 \in L^q_+(\Omega)$ and $L_2 \in L^q_+(\partial\Omega)$ such that for all $s \in [\underline{u}(x), \overline{u}(x)]$ the following local growth conditions hold:

$$\begin{aligned} &\eta \in \partial j_1(x,s) : |\eta| \le L_1(x), \quad \text{for a.a. } x \in \Omega,\\ &\xi \in \partial j_2(x,s) : |\xi| \le L_2(x), \quad \text{for a.a. } x \in \partial\Omega. \end{aligned} \tag{4.2.1}$$

(F1) (i) $x \mapsto f(x,s,\xi)$ is measurable in $\Omega$ for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$.

(ii) $(s,\xi) \mapsto f(x,s,\xi)$ is continuous in $\mathbb{R} \times \mathbb{R}^N$ for a.a. $x \in \Omega$.

(iii) There exist a constant $c_2 > 0$ and a function $k_3 \in L^q_+(\Omega)$ such that

$$|f(x,s,\xi)| \le k_3(x) + c_2|\xi|^{p-1},$$

for a.a. $x \in \Omega$, for all $\xi \in \mathbb{R}^N$, and for all $s \in [\underline{u}(x), \overline{u}(x)]$.

Note that the associated Nemytskij operator $F$ defined by $F(u)(x) = f(x,u(x),\nabla u(x))$ is continuous and bounded from $[\underline{u}, \overline{u}] \subset W^{1,p}(\Omega)$ to $L^q(\Omega)$ (cf. [123]). We recall that the normed space $L^p(\Omega)$ is equipped with the natural partial ordering of functions defined by $u \le v$ if and only if $v - u \in L^p_+(\Omega)$ where $L^p_+(\Omega)$ is the set of all nonnegative functions of $L^p(\Omega)$. Based on an approach in [39], the main idea in our considerations is to modify the functions $j_k$. First we set for $k = 1, 2$

$$\alpha_k(x) := \min\{\xi : \xi \in \partial j_k(x, \underline{u}(x))\}, \qquad \beta_k(x) := \max\{\xi : \xi \in \partial j_k(x, \overline{u}(x))\}. \tag{4.2.2}$$

By means of (4.2.2) we introduce the mappings $\widetilde{j}_1 : \Omega \times \mathbb{R} \to \mathbb{R}$ and $\widetilde{j}_2 : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$\widetilde{j}_k(x,s) = \begin{cases} j_k(x,\underline{u}(x)) + \alpha_k(x)(s - \underline{u}(x)) & \text{if } s < \underline{u}(x),\\ j_k(x,s) & \text{if } \underline{u}(x) \le s \le \overline{u}(x),\\ j_k(x,\overline{u}(x)) + \beta_k(x)(s - \overline{u}(x)) & \text{if } s > \overline{u}(x). \end{cases} \tag{4.2.3}$$

The following lemma provides some properties of the functions $\widetilde{j}_1$ and $\widetilde{j}_2$.

**Lemma 4.2.1.** *Let the assumptions (j1)–(j3) be satisfied. Then the modified functions $\widetilde{j}_1 : \Omega \times \mathbb{R} \to \mathbb{R}$ and $\widetilde{j}_2 : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ have the following characteristics:*

($\widetilde{j}1$) *$x \mapsto \widetilde{j}_1(x,s)$ and $x \mapsto \widetilde{j}_2(x,s)$ are measurable in $\Omega$ and $\partial\Omega$, respectively, for all $s \in \mathbb{R}$ and $s \mapsto \widetilde{j}_1(x,s)$ and $s \mapsto \widetilde{j}_2(x,s)$ are locally Lipschitz continuous in $\mathbb{R}$ for a.a. $x \in \Omega$ and for a.a. $x \in \partial\Omega$, respectively.*

($\widetilde{j}2$) *Let $\partial\widetilde{j}_k(x,s)$ be Clarke's generalized gradient of $s \mapsto \widetilde{j}_k(x,s)$. Then for all $s \in \mathbb{R}$ the following estimates hold true:*

$$\begin{aligned} &\eta \in \partial\widetilde{j}_1(x,s) : |\eta| \le L_1(x), \quad \text{for a.a. } x \in \Omega, \\ &\xi \in \partial\widetilde{j}_2(x,s) : |\xi| \le L_2(x), \quad \text{for a.a. } x \in \partial\Omega. \end{aligned} \tag{4.2.4}$$

($\widetilde{j}3$) *Clarke's generalized gradient of $s \mapsto \widetilde{j}_k(x,s)$, $k = 1,2$, is given by*

$$\partial\widetilde{j}_k(x,s) = \begin{cases} \alpha_k(x) & \text{if } s < \underline{u}(x), \\ \partial\widetilde{j}_k(x,\underline{u}(x)) & \text{if } s = \underline{u}(x), \\ \partial j_k(x,s) & \text{if } \underline{u}(x) < s < \overline{u}(x), \\ \partial\widetilde{j}_k(x,\overline{u}(x)) & \text{if } s = \overline{u}(x), \\ \beta_k(x) & \text{if } s > \overline{u}(x), \end{cases} \tag{4.2.5}$$

*and the inclusions $\partial\widetilde{j}_k(x,\underline{u}(x)) \subset \partial j_k(x,\underline{u}(x))$ and $\partial\widetilde{j}_k(x,\overline{u}(x)) \subset \partial j_k(x,\overline{u}(x))$ are valid for $k = 1,2$.*

**Proof.** With a view to the assumptions (j1)–(j3) and the definition of $\widetilde{j}_k$ in (4.2.3), one verifies the lemma in few steps. □

With the aid of Lemma 4.2.1, we introduce the integral functionals $J_1$ and $J_2$ defined on $L^p(\Omega)$ and $L^p(\partial\Omega)$, respectively, given by

$$\begin{aligned} J_1(u) &= \int_\Omega \widetilde{j}_1(x,u(x))dx, \quad u \in L^p(\Omega), \\ J_2(v) &= \int_{\partial\Omega} \widetilde{j}_2(x,v(x))d\sigma, \quad v \in L^p(\partial\Omega). \end{aligned} \tag{4.2.6}$$

Due to the conditions ($\widetilde{j}1$)–($\widetilde{j}2$) and Lebourg's mean value theorem (see [43, Chapter 2]), the functionals $J_1 : L^p(\Omega) \to \mathbb{R}$ and $J_2 : L^p(\partial\Omega) \to \mathbb{R}$ are well-defined and Lipschitz continuous on bounded sets of $L^p(\Omega)$ and $L^p(\partial\Omega)$, respectively. This implies among others that Clarke's generalized gradients $\partial J_1 : L^p(\Omega) \to 2^{L^q(\Omega)}$ and $\partial J_2 : L^p(\partial\Omega) \to 2^{L^q(\partial\Omega)}$ are well-defined, too. Furthermore, by means of Aubin-Clarke's theorem (see [43]), for $u \in L^p(\Omega)$ and $v \in L^p(\partial\Omega)$ we get

$$\begin{aligned} &\eta \in \partial J_1(u) \Longrightarrow \eta \in L^q(\Omega) \text{ with } \eta(x) \in \partial\widetilde{j}_1(x,u(x)) \text{ for a.a. } x \in \Omega, \\ &\xi \in \partial J_2(v) \Longrightarrow \xi \in L^q(\partial\Omega) \text{ with } \xi(x) \in \partial\widetilde{j}_2(x,v(x)) \text{ for a.a. } x \in \partial\Omega. \end{aligned} \tag{4.2.7}$$

We denote by $i^* : L^q(\Omega) \to (W^{1,p}(\Omega))^*$ and $\gamma^* : L^q(\partial\Omega) \to (W^{1,p}(\Omega))^*$ the adjoint operators of the imbedding $i : W^{1,p}(\Omega) \to L^p(\Omega)$ and the trace operator $\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$, respectively, given by

$$\langle i^*\eta, \varphi\rangle = \int_\Omega \eta\varphi dx, \quad \forall\varphi \in W^{1,p}(\Omega), \qquad \langle \gamma^*\xi, \varphi\rangle = \int_{\partial\Omega} \xi\gamma\varphi d\sigma, \quad \forall\varphi \in W^{1,p}(\Omega).$$

Next, we introduce the following multivalued operators

$$\Phi_1(u) := (i^* \circ \partial J_1 \circ i)(u), \qquad \Phi_2(u) := (\gamma^* \circ \partial J_2 \circ \gamma)(u), \tag{4.2.8}$$

where $i, i^*, \gamma, \gamma^*$ are defined as mentioned above. The operators $\Phi_k$, $k = 1, 2$, have the following properties (see e.g. [18, Lemma 3.1 and Lemma 3.2]).

**Lemma 4.2.2.** *The multivalued operators $\Phi_1 : W^{1,p}(\Omega) \to 2^{(W^{1,p}(\Omega))^*}$ and $\Phi_2 : W^{1,p}(\Omega) \to 2^{(W^{1,p}(\Omega))^*}$ are bounded and pseudomonotone.*

Let $b : \Omega \times \mathbb{R} \to \mathbb{R}$ be the cut-off function related to the given ordered pair $\underline{u}, \overline{u}$ of sub- and supersolutions defined by

$$b(x,s) = \begin{cases} (s - \overline{u}(x))^{p-1} & \text{if } \ s > \overline{u}(x), \\ 0 & \text{if } \ \underline{u}(x) \le s \le \overline{u}(x), \\ -(\underline{u}(x) - s)^{p-1} & \text{if } \ s < \underline{u}(x). \end{cases} \tag{4.2.9}$$

It is clear that the mapping $b$ is a Carathéodory function satisfying the growth condition

$$|b(x,s)| \le k_4(x) + c_3|s|^{p-1}, \tag{4.2.10}$$

for a.a. $x \in \Omega$, for all $s \in \mathbb{R}$, where $k_4 \in L^q_+(\Omega)$ and $c_3 > 0$. Furthermore, elementary calculations show the following estimate

$$\int_\Omega b(x,u(x))u(x)dx \ge c_4\|u\|^p_{L^p(\Omega)} - c_5, \quad \forall u \in L^p(\Omega), \tag{4.2.11}$$

where $c_4$ and $c_5$ are some positive constants. Due to (4.2.10) the associated Nemytskij operator $B : L^p(\Omega) \to L^q(\Omega)$ defined by

$$Bu(x) = b(x,u(x)), \tag{4.2.12}$$

is bounded and continuous. Since the embedding $i : W^{1,p}(\Omega) \to L^p(\Omega)$ is compact, the composed operator $\widehat{B} := i^* \circ B \circ i : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ is completely continuous.
For $u \in W^{1,p}(\Omega)$, we define the truncation operator $T$ with respect to the functions $\underline{u}$ and $\overline{u}$ given by

$$Tu(x) = \begin{cases} \overline{u}(x) & \text{if } u(x) > \overline{u}(x), \\ u(x) & \text{if } \underline{u}(x) \le u(x) \le \overline{u}(x), \\ \underline{u}(x) & \text{if } u(x) < \underline{u}(x). \end{cases} \tag{4.2.13}$$

The mapping $T$ is continuous and bounded from $W^{1,p}(\Omega)$ into $W^{1,p}(\Omega)$ which follows from the fact that the functions $\min(\cdot,\cdot)$ and $\max(\cdot,\cdot)$ are continuous from $W^{1,p}(\Omega)$ to themselves and that $T$ can be represented as $Tu = \max(u, \underline{u}) + \min(u, \overline{u}) - u$ (cf. [77]). Let $F \circ T$ be the composition of the Nemytskij operator $F$ given by

$$(F \circ T)(u)(x) = f(x, Tu(x), \nabla Tu(x)).$$

Due to hypothesis (F1)(iii), the mapping $F \circ T : W^{1,p}(\Omega) \to L^q(\Omega)$ is bounded and continuous. We set $\widehat{F} : i^* \circ (F \circ T) : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ and consider the multivalued operator

$$\widetilde{A} = A_T u + \widehat{F} + \lambda \widehat{B} + \Phi_1 + \Phi_2 : W^{1,p}(\Omega) \to 2^{(W^{1,p}(\Omega))^*}, \tag{4.2.14}$$

where $\lambda$ is a constant specified later and the operator $A_T$ is given by

$$\langle A_T u, \varphi \rangle = -\sum_{i=1}^{N} \int_\Omega a_i(x, Tu, \nabla u) \frac{\partial \varphi}{\partial x_i} dx.$$

We are going to prove the following properties for the operator $\widetilde{A}$.

**Lemma 4.2.3.** *The operator $\widetilde{A} : W^{1,p}(\Omega) \to 2^{(W^{1,p}(\Omega))^*}$ is bounded, pseudomonotone and coercive for $\lambda$ sufficiently large.*

**Proof.** The boundedness of $\widetilde{A}$ follows directly from the boundedness of the specific operators $A_T$, $\widehat{F}$, $\widehat{B}$, $\Phi_1$ and $\Phi_2$. As seen above, the operator $\widehat{B}$ is completely continuous and thus pseudomonotone. The elliptic operator $A_T + \widehat{F}$ is pseudomonotone because of hypotheses (A1), (A2), and (F1), and in view of Lemma 4.2.2 the operators $\Phi_1$ and $\Phi_2$ are bounded and pseudomonotone as well. Since pseudomonotonicity is invariant under addition, we conclude that $\widetilde{A} : W^{1,p}(\Omega) \to 2^{(W^{1,p}(\Omega))^*}$ is bounded and pseudomonotone. To prove the coercivity of $\widetilde{A}$, we have to find a real-valued function $c : \mathbb{R}_+ \to \mathbb{R}$ satisfying

$$\lim_{s \to +\infty} c(s) = +\infty, \tag{4.2.15}$$

such that for all $u \in W^{1,p}(\Omega)$ and $u^* \in \widetilde{A}(u)$ the following holds

$$\langle u^*, u - u_0 \rangle \geq c(\|u\|_{W^{1,p}(\Omega)}) \|u\|_{W^{1,p}(\Omega)}, \tag{4.2.16}$$

for some $u_0 \in K$. Let $u^* \in \widetilde{A}(u)$, that is, $u^*$ is of the form

$$u^* = (A_T + \widehat{F} + \lambda \widehat{B})(u) + i^* \eta + \gamma^* \xi,$$

where $\eta \in L^q(\Omega)$ with $\eta(x) \in \partial \widetilde{j}_1(x, u(x))$ for a.a. $x \in \Omega$ and $\xi \in L^q(\partial\Omega)$ with $\xi(x) \in \partial \widetilde{j}_2(x, u(x))$ for a.a. $x \in \partial\Omega$. Applying (A1), (A3), (F1)(iii), (4.2.11), ($\widetilde{j}$2), the trace operator $\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ and Young's inequality yields

$$\begin{aligned}
&\langle u^*, u - u_0 \rangle \\
&= \langle (A_T + \widehat{F} + \lambda \widehat{B})(u) + i^* \eta + \gamma^* \xi, u - u_0 \rangle \\
&= \int_\Omega \sum_{i=1}^{N} a_i(x, Tu, \nabla u) \frac{\partial u - \partial u_0}{\partial x_i} dx + \int_\Omega (f(\cdot, Tu, \nabla Tu)(u - u_0) + \lambda b(x, u)(u - u_0)) dx \\
&\quad + \int_\Omega \eta(u - u_0)) dx + \int_{\partial\Omega} \xi \gamma(u - u_0) d\sigma \\
&\geq c_1 \|\nabla u\|^p_{L^p(\Omega)} - \|k_1\|_{L^1(\Omega)} - d_1 \|u\|^{p-1}_{L^p(\Omega)} - d_2 \|\nabla u\|^{p-1}_{L^p(\Omega)} - d_3 - \varepsilon \|\nabla u\|^p_{L^p(\Omega)} - c(\varepsilon) \|u\|^p_{L^p(\Omega)}
\end{aligned}$$

$$
\begin{aligned}
&- d_5\|u\|_{L^p(\Omega)} - d_6\|\nabla u\|^{p-1}_{L^p(\Omega)} - d_7 + \lambda c_4\|u\|^p_{L^p(\Omega)} - \lambda c_5 - d_8 - d_9\|u\|^{p-1}_{L^p(\Omega)}\\
&- d_{10}\|u\|_{L^p(\Omega)} - d_{11} - d_{12}\|u\|_{L^p(\partial\Omega)} - d_{13}\\
= {}& (c_1-\varepsilon)\|\nabla u\|^p_{L^p(\Omega)} + (\lambda c_4 - c(\varepsilon))\|u\|^p_{L^p(\Omega)} - d_{14}\|\nabla u\|^{p-1}_{L^p(\Omega)} - d_{15}\|u\|^{p-1}_{L^p(\Omega)}\\
&- d_{16}\|u\|_{L^p(\Omega)} - d_{17},
\end{aligned}
$$

where $d_j$ are some positive constants. Choosing $\varepsilon < c_1$ and $\lambda$ such that $\lambda > \frac{c(\varepsilon)}{c_4}$ yields the estimate

$$
\langle u^*, u - u_0\rangle \geq d_{18}\|u\|^p_{W^{1,p}(\Omega)} - d_{19}\|u\|^{p-1}_{W^{1,p}(\Omega)} - d_{20}\|u\|_{W^{1,p}(\Omega)} - d_{21}.
$$

Setting $c(s) = d_{18}s^{p-1} - d_{19}s^{p-2} - d_{20} - \frac{d_{21}}{s}$ for $s > 0$ and $c(0) = 0$ it follows that (4.2.15) and (4.2.16) are satisfied. This proves the coercivity of $A$ and completes the proof of the lemma. $\square$

## 4.3 Existence and Comparison Results

**Theorem 4.3.1.** *Let hypotheses (A1)–(A3), (j1)–(j3) and (F1) be satisfied and assume the existence of sub- and supersolutions $\underline{u}$ and $\overline{u}$, respectively, satisfying $\underline{u} \leq \overline{u}$ and (4.1.1). Then, there exists a solution of (4.0.1) in the order interval $[\underline{u}, \overline{u}]$.*

**Proof.** Let $I_K : W^{1,p}(\Omega) \to \mathbb{R} \cup \{+\infty\}$ be the indicator function corresponding to the closed convex set $K \neq \emptyset$ given by

$$
I_K(u) = \begin{cases} 0 & \text{if} \quad u \in K,\\ +\infty & \text{if} \quad u \notin K, \end{cases} \tag{4.3.1}
$$

which is known to be proper, convex and lower semicontinuous. The variational-hemivariational inequality (4.0.1) can be rewritten as follows. Find $u \in K$ such that

$$
\begin{aligned}
\langle Au + F(u), v - u\rangle + I_K(v) - I_K(u) + \int_\Omega j_1^\circ(\cdot, u; v-u)dx&\\
+ \int_{\partial\Omega} j_2^\circ(\cdot, \gamma u; \gamma v - \gamma u)d\sigma \geq 0, \quad \forall v \in W^{1,p}(\Omega).&
\end{aligned} \tag{4.3.2}
$$

By using the operators $A_T, \widehat{F}, \widehat{B}$ and the functions $\widetilde{j}_1, \widetilde{j}_2$ introduced in Section 4.2, we consider the following auxiliary problem. Find $u \in K$ such that

$$
\begin{aligned}
\langle A_T u + \widehat{F}(u) + \lambda\widehat{B}(u), v - u\rangle + I_K(v) - I_K(u) + \int_\Omega \widetilde{j}_1^\circ(\cdot, u; v-u)dx&\\
+ \int_{\partial\Omega} \widetilde{j}_2^\circ(\cdot, \gamma u; \gamma v - \gamma u)d\sigma \geq 0, \quad \forall v \in W^{1,p}(\Omega).&
\end{aligned} \tag{4.3.3}
$$

Consider now the multivalued operator

$$
\widetilde{A} + \partial I_K : W^{1,p}(\Omega) \to 2^{(W^{1,p}(\Omega))^*}, \tag{4.3.4}
$$

where $\widetilde{A}$ is as in (4.2.14) and $\partial I_K : W^{1,p}(\Omega) \to 2^{(W^{1,p}(\Omega))^*}$ is the subdifferential of the indicator function $I_K$ which is known to be a maximal monotone operator (cf. [104, Page 20]). Lemma 4.2.3 provides that $\widetilde{A}$ is bounded, pseudomonotone and coercive. Applying Theorem 2.2.20 proves the surjectivity of $\widetilde{A} + \partial I_K$ meaning that range$(\widetilde{A} + \partial I_K) = (W^{1,p}(\Omega))^*$. Since $0 \in (W^{1,p}(\Omega))^*$, there exists a solution $u \in K$ of the inclusion

$$\widetilde{A}(u) + \partial I_K(u) \ni 0. \tag{4.3.5}$$

This implies the existence of $\eta^* \in \Phi_1(u), \xi^* \in \Phi_2(u)$, and $\theta^* \in \partial I_K(u)$ such that

$$A_T u + \widehat{F}(u) + \lambda \widehat{B}(u) + \eta^* + \xi^* + \theta^* = 0, \qquad \text{in } (W^{1,p}(\Omega))^*, \tag{4.3.6}$$

where it holds in view of (4.2.7) and (4.2.8) that $\eta^* = i^*\eta$ and $\xi^* = \gamma^*\xi$ with $\eta \in L^q(\Omega)$ and $\eta(x) \in \partial\widetilde{j}_1(x, u(x))$ as well as $\xi \in L^q(\partial\Omega)$ and $\xi(x) \in \partial\widetilde{j}_2(x, \gamma u(x))$. Due to the Definition of Clarke's generalized gradient $\partial\widetilde{j}_k(\cdot, u)$, $k = 1, 2$, one gets

$$\begin{aligned}
\langle \eta^*, \varphi \rangle &= \int_\Omega \eta(x)\varphi(x)dx \le \int_\Omega \widetilde{j}_1^o(x, u(x); \varphi(x))dx, \quad \forall \varphi \in W^{1,p}(\Omega), \\
\langle \xi^*, \varphi \rangle &= \int_{\partial\Omega} \xi(x)\gamma\varphi(x)d\sigma \le \int_{\partial\Omega} \widetilde{j}_2^o(x, \gamma u(x); \gamma\varphi(x))d\sigma, \quad \forall \varphi \in W^{1,p}(\Omega).
\end{aligned} \tag{4.3.7}$$

Moreover, we have by the definition of $\partial I_K$

$$\langle \theta^*, v - u \rangle \le I_K(v) - I_K(u), \quad \forall v \in W^{1,p}(\Omega). \tag{4.3.8}$$

From (4.3.6) we conclude

$$\langle A_T u + \widehat{F}(u) + \lambda \widehat{B}(u) + \eta^* + \xi^* + \theta^*, \varphi \rangle = 0, \qquad \forall \varphi \in W^{1,p}(\Omega). \tag{4.3.9}$$

Applying the estimates in (4.3.7) and (4.3.8) to the equation above where $\varphi$ is replaced by $v - u$ yields for all $v \in W^{1,p}(\Omega)$

$$\begin{aligned}
0 &= \langle A_T - \Delta_p u + \widehat{F}(u) + \lambda \widehat{B}(u) + \eta^* + \xi^* + \theta^*, v - u \rangle \\
&\le \langle A_T u + \widehat{F}(u) + \lambda \widehat{B}(u), v - u \rangle + I_K(v) - I_K(u) \\
&\quad + \int_\Omega \widetilde{j}_1^o(\cdot, u; v - u)dx + \int_{\partial\Omega} \widetilde{j}_2^o(\cdot, \gamma u; \gamma v - \gamma u)d\sigma.
\end{aligned}$$

Hence, we obtain a solution $u$ of the auxiliary problem (4.3.3) which is equivalent to the problem below: Find $u \in K$ such that

$$\begin{aligned}
&\langle A_T u + \widehat{F}(u) + \lambda \widehat{B}(u), v - u \rangle + \int_\Omega \widetilde{j}_1^o(\cdot, u; v - u)dx \\
&\quad + \int_{\partial\Omega} \widetilde{j}_2^o(\cdot, \gamma u; \gamma v - \gamma u)d\sigma \ge 0, \quad \forall v \in K.
\end{aligned} \tag{4.3.10}$$

In the next step we have to show that any solution $u$ of (4.3.10) belongs to $[\underline{u}, \overline{u}]$. By Definition 4.1.2 and by choosing $w = \overline{u} \vee u = \overline{u} + (u - \overline{u})^+ \in \overline{u} \vee K$, we obtain

$$\langle A\overline{u} + F(\overline{u}), (u - \overline{u})^+ \rangle + \int_\Omega j_1^o(\cdot, \overline{u}; (u - \overline{u})^+)dx + \int_{\partial\Omega} j_2^o(\cdot, \gamma\overline{u}; \gamma(u - \overline{u})^+)d\sigma \ge 0,$$

and selecting $v = \overline{u} \wedge u = u - (u - \overline{u})^+ \in K$ in (4.3.10) provides

$$\langle A_T u + \widehat{F}(u) + \lambda \widehat{B}(u), -(u-\overline{u})^+\rangle + \int_\Omega \widetilde{j}_1^\circ(\cdot, u; -(u-\overline{u})^+)dx$$
$$+ \int_{\partial\Omega} \widetilde{j}_2^\circ(\cdot, \gamma u; -\gamma(u-\overline{u})^+)d\sigma \geq 0.$$

Adding these inequalities yields

$$\begin{aligned}
&\sum_{i=1}^N \int_\Omega (a_i(x, \overline{u}, \nabla \overline{u}) - a_i(x, Tu, \nabla u))\frac{\partial (u-\overline{u})^+}{\partial x_i}dx \\
&+ \int_\Omega (F(\overline{u}) - (F\circ T)(u))(u-\overline{u})^+ dx \\
&+ \int_\Omega (j_1^\circ(\cdot, \overline{u}; 1) + \widetilde{j}_1^\circ(\cdot, u; -1))(u-\overline{u})^+ dx \\
&+ \int_{\partial\Omega} (j_2^\circ(\cdot, \gamma\overline{u}; 1) + \widetilde{j}_2^\circ(\cdot, \gamma u; -1))\gamma(u-\overline{u})^+ d\sigma \\
&\geq \lambda \int_\Omega B(u)(u-\overline{u})^+ dx.
\end{aligned} \tag{4.3.11}$$

Let us analyze the specific integrals in (4.3.11). By using (A2) and the definition of the truncation operator, we obtain

$$\begin{aligned}
&\int_\Omega (a_i(x, \overline{u}, \nabla \overline{u}) - a_i(x, Tu, \nabla u))\frac{\partial (u-\overline{u})^+}{\partial x_i}dx \leq 0, \\
&\int_\Omega (F(\overline{u}) - (F\circ T)(u))(u-\overline{u})^+ dx = 0.
\end{aligned} \tag{4.3.12}$$

Furthermore, we consider the third integral of (4.3.11) in case $u > \overline{u}$. Otherwise it would be zero. Applying (4.2.3) proves

$$\begin{aligned}
&\widetilde{j}_1^\circ(x, u(x); -1) \\
&= \limsup_{s\to u(x), t\downarrow 0} \frac{\widetilde{j}_1(x, s-t) - \widetilde{j}_1(x,s)}{t} \\
&= \limsup_{s\to u(x), t\downarrow 0} \frac{j_1(x, \overline{u}(x)) + \beta_1(x)(s-t-\overline{u}(x)) - j_1(x, \overline{u}(x)) - \beta_1(x)(s-\overline{u}(x))}{t} \\
&= \limsup_{s\to u(x), t\downarrow 0} \frac{-\beta_1(x)t}{t} \\
&= -\beta_1(x).
\end{aligned}$$

Proposition 2.3.6 along with (4.2.2) shows

$$j_1^\circ(x, \overline{u}(x); 1) = \max\{\xi : \xi \in \partial j_1(x, \overline{u}(x))\} = \beta_1(x).$$

We obtain

$$\int_\Omega (j_1^\circ(\cdot, \overline{u}; 1) + \widetilde{j}_1^\circ(\cdot, u; -1))(u-\overline{u})^+ dx = \int_\Omega (\beta_1(x) - \beta_1(x))(u-\overline{u})^+ dx = 0, \tag{4.3.13}$$

and analogous to this calculation

$$\int_{\partial\Omega} (j_2^{\circ}(\cdot, \gamma\overline{u}; 1) + \widetilde{j}_2^{\circ}(\cdot, \gamma u; -1))\gamma(u - \overline{u})^+ d\sigma = 0. \tag{4.3.14}$$

Due to (4.3.12), (4.3.13), and (4.3.14), we immediately realize that the left-hand side in (4.3.11) is nonpositive. Thus, we have

$$\begin{aligned} 0 &\geq \lambda \int_{\Omega} B(u)(u - \overline{u})^+ dx \\ &= \lambda \int_{\Omega} b(\cdot, u)(u - \overline{u})^+ dx \\ &= \lambda \int_{\{x:u(x)>\overline{u}(x)\}} (u - \overline{u})^p dx \\ &= \lambda \int_{\Omega} ((u - \overline{u})^+)^p dx \\ &\geq 0, \end{aligned}$$

which implies $(u - \overline{u})^+ = 0$ and hence, $u \leq \overline{u}$. The proof for $\underline{u} \leq u$ is done in a similar way. So far we have shown that any solution of the inclusion (4.3.5) (which is a solution of (4.3.3) as well) belongs to the interval $[\underline{u}, \overline{u}]$. The latter implies $A_T u = Au$, $B(u) = 0$, and $(F \circ T)(u) = F(u)$, and thus from (4.3.5) it follows

$$\langle Au + F(u) + i^*\eta + \gamma^*\xi, v - u\rangle \geq 0, \quad \forall v \in K,$$

where $\eta(x) \in \partial\widetilde{j}_1(x, u(x)) \subset \partial j_1(x, u(x))$ and $\xi(x) \in \partial\widetilde{j}_2(x, \gamma u(x)) \subset \partial j_2(x, \gamma u(x))$ which proves that $u \in [\underline{u}, \overline{u}]$ is also a solution of our original problem (4.0.1). This completes the proof of the theorem. □

## 4.4 Compactness and Extremality Results

Let $\mathcal{S}$ denote the set of all solutions of (4.0.1) within the order interval $[\underline{u}, \overline{u}]$. In addition, we will assume that $K$ has lattice structure, that is, $K$ fulfills

$$K \vee K \subset K, \qquad K \wedge K \subset K. \tag{4.4.1}$$

We are going to show that $\mathcal{S}$ possesses the smallest and greatest element with respect to the given partial ordering.

**Theorem 4.4.1.** *Let the hypothesis of Theorem 4.3.1 be satisfied. Then the solution set $\mathcal{S}$ is compact.*

**Proof.** First, we are going to show that $\mathcal{S}$ is bounded in $W^{1,p}(\Omega)$. Let $u \in \mathcal{S}$ be a solution of (4.3.2) and notice that $\mathcal{S}$ is $L^p(\Omega)$-bounded because of $\underline{u} \leq u \leq \overline{u}$. This implies $\gamma\underline{u} \leq \gamma u \leq \gamma\overline{u}$

and thus, $u$ is also bounded in $L^p(\partial\Omega)$. Choosing a fixed $v = u_0 \in K$ in (4.3.2) gives

$$\langle Au + F(u), u_0 - u\rangle + \int_\Omega j_1^o(\cdot, u; u_0 - u)dx + \int_{\partial\Omega} j_2^o(\cdot, \gamma u; \gamma u_0 - \gamma u)d\sigma \geq 0.$$

Using (A1), (j3), (F1)(iii), Proposition 2.3.6 and Young's inequality yields

$$\begin{aligned}
\langle Au, u\rangle &\leq \int_\Omega \sum_{i=1}^N |a_i(x, u, \nabla u)| \left|\frac{\partial u_0}{\partial x_i}\right| dx + \int_\Omega |f(x, u, \nabla u)||u_0 - u|dx \\
&\quad + \int_\Omega \max\{\eta(u_0 - u) : \eta \in \partial j_1(x, u)\}dx + \int_{\partial\Omega} \max\{\xi(u_0 - u) : \xi \in \partial j_2(x, u)\}d\sigma \\
&\leq \int_\Omega \sum_{i=1}^N (k_0 + c_0|u|^{p-1} + c_0|\nabla u|^{p-1})|\nabla u_0|dx + \int_\Omega (k_3 + c_2|\nabla u|^{p-1})|u_0 - u|dx \\
&\quad + \int_\Omega L_1|u_0 - u|dx + \int_{\partial\Omega} L_2|\gamma u_0 - \gamma u|d\sigma \\
&\leq e_1 + e_2\|u\|_{L^p(\Omega)}^{p-1} + e_3\|\nabla u\|_{L^p(\Omega)}^{p-1} + e_4 + e_5\|u\|_{L^p(\Omega)} + e_6\|\nabla u\|_{L^p(\Omega)}^{p-1} + \varepsilon\|\nabla u\|_{L^p(\Omega)}^p \\
&\quad + c(\varepsilon)\|u\|_{L^p(\Omega)}^p + e_7 + e_8\|u\|_{L^p(\Omega)} + e_9 + e_{10}\|u\|_{L^p(\partial\Omega)} \\
&\leq \varepsilon\|\nabla u\|_{L^p(\Omega)}^p + e_{11}\|\nabla u\|_{L^p(\Omega)}^{p-1} + e_{12}\|\nabla u\|_{L^p(\Omega)} + e_{13},
\end{aligned}$$

where the left-hand side fulfills the estimate

$$\langle Au, u\rangle \geq c_1\|\nabla u\|_{L^p(\Omega)}^p - k_1.$$

Thus, one has

$$(c_1 - \varepsilon)\|\nabla u\|_{L^p(\Omega)}^p \leq e_{11}\|\nabla u\|_{L^p(\Omega)}^{p-1} + e_{13},$$

where the choice $\varepsilon < c_1$ proves that $\|\nabla u\|_{L^p(\Omega)}$ is bounded. Hence, we obtain the boundedness of $u$ in $W^{1,p}(\Omega)$. Let $(u_n) \subset \mathcal{S}$. Since $W^{1,p}(\Omega), 1 < p < \infty$, is reflexive, there exists a weakly convergent subsequence, not relabelled, which yields along with the compact imbedding $i : W^{1,p}(\Omega) \to L^p(\Omega)$ and the trace operator $\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$

$$\begin{aligned}
&u_n \rightharpoonup u \ \text{ in } W^{1,p}(\Omega), \\
&u_n \to u \ \text{ in } L^p(\Omega) \ \text{ and a.e. pointwise in } \Omega, \\
&\gamma u_n \to \gamma u \ \text{ in } L^p(\partial\Omega) \ \text{ and a.e. pointwise in } \partial\Omega.
\end{aligned} \tag{4.4.2}$$

As $u_n$ solves (4.3.2), in particular, for $v = u \in K$, we obtain

$$\begin{aligned}
&\langle Au_n, u_n - u\rangle \\
&\leq \langle F(u_n), u - u_n\rangle + \int_\Omega j_1^o(\cdot, u_n; u - u_n)dx + \int_{\partial\Omega} j_2^o(\cdot, \gamma u_n; \gamma u - \gamma u_n)d\sigma.
\end{aligned} \tag{4.4.3}$$

Since $(s,r) \mapsto j_k^o(x,s;r)$, $k=1,2$, is upper semicontinuous and due to Fatou's Lemma, we get from (4.4.3)

$$\begin{aligned}\limsup_{n\to\infty}\langle Au_n, u_n-u\rangle \le \underbrace{\limsup_{n\to\infty}\langle F(u_n), u-u_n\rangle}_{\to 0} + \int_\Omega \underbrace{\limsup_{n\to\infty} j_1^o(\cdot,u_n;u-u_n)}_{\le j_1^o(\cdot,u,0)=0}\,dx \\ + \int_{\partial\Omega}\underbrace{\limsup_{n\to\infty} j_2^o(\cdot,\gamma u_n;\gamma u-\gamma u_n)}_{\le j_2^o(\cdot,\gamma u,\gamma 0)=0}\,d\sigma \le 0.\end{aligned} \tag{4.4.4}$$

The elliptic operator $A$ satisfies the $(S_+)$-property, which due to (4.4.4) and (4.4.2) implies

$$u_n \to u \quad \text{in } W^{1,p}(\Omega).$$

Replacing $u$ by $u_n$ in (4.0.1) yields the following inequality

$$\begin{aligned}\langle Au_n + F(u_n), v-u_n\rangle + \int_\Omega j_1^o(\cdot,u_n;v-u_n)dx \\ + \int_{\partial\Omega} j_2^o(\cdot,\gamma u_n;\gamma v-\gamma u_n)d\sigma \ge 0, \quad \forall v\in K.\end{aligned} \tag{4.4.5}$$

Passing to the limes superior in (4.4.5) and using Fatou's Lemma, the strong convergence of $(u_n)$ in $W^{1,p}(\Omega)$ and the upper semicontinuity of $(s,r)\to j_k^o(x,s;r)$, $k=1,2$, we obtain

$$\langle Au+F(u), v-u\rangle + \int_\Omega j_1^o(\cdot,u;v-u)dx + \int_{\partial\Omega} j_2^o(\cdot,\gamma u;\gamma v-\gamma u)d\sigma \ge 0, \quad \forall v\in K.$$

Hence, $u\in\mathcal{S}$. This shows the compactness of the solution set $\mathcal{S}$. □

In order to prove the existence of extremal elements of the solution set $\mathcal{S}$, we drop the $u-$dependence of the operator $A$. Then, our assumptions can be read as follows.

(A1') Each $a_i(x,\xi)$ satisfies Carathéodory conditions, i.e., is measurable in $x\in\Omega$ for all $\xi\in\mathbb{R}^N$ and continuous in $\xi$ for a.a. $x\in\Omega$. Furthermore, a constant $c_0>0$ and a function $k_0\in L^q(\Omega)$ exist such that

$$|a_i(x,\xi)| \le k_0(x) + |\xi|^{p-1},$$

for a.a. $x\in\Omega$ and for all $\xi\in\mathbb{R}^N$, where $|\xi|$ denotes the Euclidian norm of the vector $\xi$.

(A2') The coefficients $a_i$ satisfy a monotonicity condition with respect to $\xi$ in the form

$$\sum_{i=1}^N (a_i(x,\xi)-a_i(x,\xi'))(\xi_i-\xi_i') > 0,$$

for a.a. $x\in\Omega$, and for all $\xi,\xi'\in\mathbb{R}^N$ with $\xi\ne\xi'$.

(A3') A constant $c_1 > 0$ and a function $k_1 \in L^1(\Omega)$ exist such that

$$\sum_{i=1}^{N} a_i(x,\xi)\xi_i \geq c_1|\xi|^p - k_1(x),$$

for a.a. $x \in \Omega$, and for all $\xi \in \mathbb{R}^N$.

Then the operator $A : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ acts in the following way

$$\langle Au, \varphi\rangle = \int_\Omega \sum_{i=1}^{N} a_i(x, \nabla u)\frac{\partial \varphi}{\partial x_i}\,dx.$$

Let us recall the definition of a directed set.

**Definition 4.4.2.** *Let $(\mathcal{P}, \leq)$ be a partially ordered set. A subset $\mathcal{C}$ of $\mathcal{P}$ is said to be upward directed if for each pair $x, y \in \mathcal{C}$ there is a $z \in \mathcal{C}$ such that $x \leq z$ and $y \leq z$. Similarly, $\mathcal{C}$ is downward directed if for each pair $x, y \in \mathcal{C}$ there is a $w \in \mathcal{C}$ such that $w \leq x$ and $w \leq y$. If $\mathcal{C}$ is both upward and downward directed it is called directed.*

**Theorem 4.4.3.** *Let hypotheses (A1')–(A3') and (j1)–(j3) be fulfilled and assume that (F1) and (4.4.1) are valid. Then the solution set $\mathcal{S}$ of problem (4.0.1) is a directed set.*

**Proof.** By Theorem 4.3.1, we have $\mathcal{S} \neq \emptyset$. Let $u_1, u_2 \in \mathcal{S}$ be solutions of (4.0.1) and let $u_0 = \max\{u_1, u_2\}$. We have to show that there is an $u \in \mathcal{S}$ such that $u_0 \leq u$. Our proof is mainly based on an approach developed recently in [38] which relies on a properly constructed auxiliary problem. Let the operator $\widehat{B}$ be given basically as in (4.2.9)-(4.2.12) with the following slight change:

$$b(x,s) = \begin{cases} (s - \overline{u}(x))^{p-1} & \text{if} \quad s > \overline{u}(x), \\ 0 & \text{if} \quad \underline{u}(x) \leq s \leq \overline{u}(x), \\ -(u_0(x) - s)^{p-1} & \text{if} \quad s < u_0(x). \end{cases} \tag{4.4.6}$$

We introduce truncation operators $T_j$ associated with $u_j$, and modify the truncation operator $T$ as follows: For $j = 1, 2$, we define

$$T_j u(x) = \begin{cases} \overline{u}(x) & \text{if} \quad u(x) > \overline{u}(x), \\ u(x) & \text{if} \quad u_j(x) \leq u(x) \leq \overline{u}(x), \\ u_j(x) & \text{if} \quad u(x) < u_j(x), \end{cases}$$

$$Tu(x) = \begin{cases} \overline{u}(x) & \text{if} \quad u(x) > \overline{u}(x), \\ u(x) & \text{if} \quad u_0(x) \leq u(x) \leq \overline{u}(x), \\ u_0(x) & \text{if} \quad u(x) < u_0(x), \end{cases}$$

and we set

$$Gu(x) = f(x, Tu(x), \nabla Tu(x)) - \sum_{j=1}^{2} |f(x, Tu(x), \nabla Tu(x)) - f(x, T_j u(x), \nabla T_j u(x))|,$$

as well as

$$\widehat{F} : i^* \circ G : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*.$$

Moreover, we define

$$\alpha_{k,j}(x) := \min\{\xi : \xi \in \partial j_k(x, u_j(x))\}, \qquad \beta_k(x) := \max\{\xi : \xi \in \partial j_k(x, \overline{u}(x))\}, \tag{4.4.7}$$

and

$$\alpha_{k,0}(x) := \begin{cases} \alpha_{k,1}(x) & \text{if } x \in \{u_1 \geq u_2\}, \\ \alpha_{k,2}(x) & \text{if } x \in \{u_2 > u_1\}, \end{cases}$$

for $k, j = 1, 2$, and introduce the functions $\widetilde{j_1} : \Omega \times \mathbb{R} \to \mathbb{R}$ and $\widetilde{j_2} : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$\widetilde{j_k}(x, s) = \begin{cases} j_k(x, u_0(x)) + \alpha_{k,0}(x)(s - u_0(x)) & \text{if } s < u_0(x), \\ j_k(x, s) & \text{if } u_0(x) \leq s \leq \overline{u}(x), \\ j_k(x, \overline{u}(x)) + \beta_k(x)(s - \overline{u}(x)) & \text{if } s > \overline{u}(x). \end{cases} \tag{4.4.8}$$

Furthermore, we define the functions $h_{1,j} : \Omega \times \mathbb{R} \to \mathbb{R}$ and $h_{2,j} : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ for $j = 0, 1, 2$ as follows:

$$h_{k,0}(x, s) = \begin{cases} \alpha_{k,0}(x) & \text{if } s \leq u_0(x), \\ \alpha_{k,0}(x) + \frac{\beta_k(x) - \alpha_{k,0}(x)}{\overline{u}(x) - u_0(x)}(s - u_0(x)) & \text{if } u_0(x) < s < \overline{u}(x), \\ \beta_k(x) & \text{if } s \geq \overline{u}(x), \end{cases}$$

and for $j = 1, 2$

$$h_{k,j}(x, s) = \begin{cases} \alpha_{k,j}(x) & \text{if } s \leq u_j(x), \\ \alpha_{k,j}(x) + \frac{\alpha_{k,0}(x) - \alpha_{k,j}(x)}{u_0(x) - u_k(x)}(s - u_j(x)) & \text{if } u_j(x) < s < u_0(x), \\ h_{k,0}(x, s) & \text{if } s \geq u_0(x), \end{cases}$$

where $k = 1, 2$. (Note that for $k = 2$ we understand the functions above being defined on $\partial\Omega$.) Apparently, the mappings $(x, s) \mapsto h_{k,j}(x, s)$ are Carathéodory functions which are piecewise linear with respect to $s$. Let us introduce the Nemytskij operators $H_1 : L^p(\Omega) \to L^q(\Omega)$ and $H_2 : L^p(\partial\Omega) \to L^q(\partial\Omega)$ defined by

$$H_1 u(x) = \sum_{j=1}^{2} |h_{1,j}(x, u(x)) - h_{1,0}(x, u(x))|,$$

$$H_2 u(x) = \sum_{j=1}^{2} |h_{2,j}(x, \gamma(u(x))) - h_{2,0}(x, \gamma(u(x)))|.$$

Due to the compact imbedding $i : W^{1,p}(\Omega) \to L^p(\Omega)$ and the compactness of the trace operator $\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$, the operators $\widetilde{H}_1 = i^* \circ H_1 \circ i : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ and $\widetilde{H}_2 = \gamma^* \circ H_2 \circ \gamma : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ are bounded and completely continuous, and thus pseudomonotone. Now, we consider the following auxiliary variational-hemivariational inequality: Find $u \in K$ such that

$$\begin{aligned} &\langle Au + \widehat{F}(u) + \lambda \widehat{B}(u), v-u\rangle + \int_\Omega \widetilde{j}_1^o(\cdot, u; v-u)dx - \langle \widetilde{H}_1 u, v-u\rangle \\ &+ \int_{\partial\Omega} \widetilde{j}_2^o(\cdot, \gamma u; \gamma v - \gamma u)d\sigma - \langle \widetilde{H}_2 \gamma u, \gamma v - \gamma u\rangle \geq 0, \end{aligned} \tag{4.4.9}$$

for all $v \in K$. The construction of the auxiliary problem (4.4.9) including the functions $H_k$ and $G$ is inspired by a very recent approach introduced by Carl and Motreanu in [38]. The first part of the proof of Theorem 4.3.1 yields the existence of a solution $u$ of (4.4.9), since all calculations in Section 4.2 are still valid. In order to show that the solution set $\mathcal{S}$ of (4.0.1) is upward directed, we have to verify that a solution $u$ of (4.4.9) satisfies $u_l \leq u \leq \overline{u}$, $l = 1, 2$. By assumption $u_l \in \mathcal{S}$, that is, $u_l$ solves

$$\langle Au_l + F(u_l), v - u_l\rangle + \int_\Omega j_1^o(\cdot, u_l; v - u_l)dx + \int_{\partial\Omega} j_2^o(\cdot, \gamma u_l; \gamma v - \gamma u_l)d\sigma \geq 0, \tag{4.4.10}$$

for all $v \in K$. Selecting $v = u \wedge u_l = u_l - (u_l - u)^+ \in K$ in the inequality above yields

$$\begin{aligned} &\langle Au_l + F(u_l), -(u_l-u)^+\rangle + \int_\Omega j_1^o(\cdot, u_l; -(u_l-u)^+)dx \\ &+ \int_{\partial\Omega} j_2^o(\cdot, \gamma u_l; -\gamma(u_l-u)^+)d\sigma \geq 0. \end{aligned} \tag{4.4.11}$$

Taking the special test function $v = u \vee u_l = u + (u_l - u)^+ \in K$ in (4.4.9), we get

$$\begin{aligned} &\langle Au + \widehat{F}(u) + \lambda \widehat{B}(u), (u_l-u)^+\rangle + \int_\Omega \widetilde{j}_1^o(\cdot, u; (u_l-u)^+)dx - \langle \widetilde{H}_1, (u_l-u)^+\rangle \\ &+ \int_{\partial\Omega} \widetilde{j}_2^o(\cdot, \gamma u; \gamma(u_l-u)^+)d\sigma - \langle \widetilde{H}_2\gamma u, \gamma(u_l-u)^+\rangle \geq 0. \end{aligned} \tag{4.4.12}$$

Adding (4.4.11) and (4.4.12) yields

$$\begin{aligned} &\int_\Omega \sum_{i=1}^N (a_i(x, \nabla u) - a_i(x, \nabla u_l)) \frac{\partial (u_l-u)^+}{\partial x_i} dx \\ &+ \int_\Omega \Big[ f(x, Tu), \nabla Tu) - f(x, u_l, \nabla u_l) \\ &\qquad - \sum_{j=1}^2 |f(x, Tu, \nabla Tu) - f(x, T_j u, \nabla T_j u)| \Big] (u_l-u)^+ dx \\ &+ \int_\Omega \Big[ \widetilde{j}_1^o(\cdot, u; 1) + j_1^o(\cdot, u_l; -1) \\ &\qquad - \sum_{j=1}^2 |h_{1,j}(x, u) - h_{1,0}(x, u)| \Big] (u_l-u)^+ dx \end{aligned} \tag{4.4.13}$$

$$
\begin{aligned}
&+\int_{\partial\Omega}\Bigg[\widetilde{j}_2^{\circ}(\cdot,\gamma u;1)+j_2^{\circ}(\cdot,\gamma u_l;-1)\\
&\qquad -\sum_{j=1}^{2}|h_{2,j}(x,\gamma u)-h_{2,0}(x,\gamma u)|\Bigg]\gamma(u_l-u)^+d\sigma\\
&\geq -\lambda\int_{\Omega}B(u)(u_l-u)^+dx.
\end{aligned}
$$

The condition (A2') implies directly

$$
\int_{\Omega}\sum_{i=1}^{N}(a_i(x,\nabla u)-a_i(x,\nabla u_l))\frac{\partial(u_l-u)^+}{\partial x_i}dx\leq 0, \tag{4.4.14}
$$

and the second integral can be estimated to obtain

$$
\begin{aligned}
&\int_{\Omega}\Bigg[f(x,Tu,\nabla Tu)-f(x,u_l,\nabla u_l)\\
&\qquad -\sum_{j=1}^{2}|f(x,Tu,\nabla Tu)-f(x,T_ju,\nabla T_ju)|\Bigg](u_l-u)^+dx\\
&\leq \int_{\Omega}\Bigg[f(x,Tu,\nabla Tu)-f(x,u_l,\nabla u_l)\\
&\qquad -|f(x,Tu,\nabla Tu)-f(x,T_lu,\nabla T_lu)|\Bigg](u_l-u)^+dx\\
&=\int_{\{x\in\Omega:u_l(x)>u(x)\}}\Bigg[f(x,Tu,\nabla Tu)-f(x,u_l,\nabla u_l)\\
&\qquad -|f(x,Tu,\nabla Tu)-f(x,u_l,\nabla u_l)|\Bigg](u_l-u)dx\\
&\leq 0.
\end{aligned} \tag{4.4.15}
$$

In order to investigate the third integral, we make use of some auxiliary calculations. In view of (4.4.8) we have for $u_l(x)>u(x)$

$$
\begin{aligned}
&\widetilde{j}_1^{\circ}(x,u(x);1)\\
&=\limsup_{s\to u(x),t\downarrow 0}\frac{\widetilde{j}_1(x,s+t)-\widetilde{j}_1(x,s)}{t}\\
&=\limsup_{s\to u(x),t\downarrow 0}\frac{j_1(x,u_0(x))+\alpha_{1,0}(x)(s+t-u_0(x))-j_1(x,u_0(x))-\alpha_{1,0}(x)(s-u_0(x))}{t}\\
&=\limsup_{s\to u(x),t\downarrow 0}\frac{\alpha_{1,0}(x)t}{t}\\
&=\alpha_{1,0}(x).
\end{aligned}
$$

Applying Proposition 2.3.6 and (4.4.7) results in

$$
\begin{aligned}
j_1^{\circ}(x,u_l(x);-1)&=\max\{-\xi:\xi\in\partial j_1(x,u_l(x))\}\\
&=-\min\{\xi:\xi\in\partial j_1(x,u_l(x))\}\\
&=-\alpha_{1,l}(x).
\end{aligned}
$$

Furthermore, we have in case $u_l(x) > u(x)$

$$h_{1,l}(x, u(x)) = \alpha_{1,l}(x),$$
$$h_{1,0}(x, u(x)) = \alpha_{1,0}(x).$$

Thus, we get

$$\begin{aligned}
&\int_\Omega \Big(\widetilde{j_1^\circ}(\cdot, u; 1) + j_1^\circ(\cdot, u_l; -1) - \sum_{j=1}^{2} |h_{1,j}(x, u) - h_{1,0}(x, u)|\Big)(u_l - u)^+ dx \\
&\le \int_\Omega \Big(\widetilde{j_1^\circ}(\cdot, u; 1) + j_1^\circ(\cdot, u_l; -1) - |h_{1,l}(x, u) - h_{1,0}(x, u)|\Big)(u_l - u)^+ dx \\
&= \int_{\{x\in\Omega: u_l(x)>u(x)\}} (\alpha_{1,0}(x) - \alpha_{1,l}(x) - |\alpha_{1,l}(x) - \alpha_{1,0}(x)|)(u_l - u)^+ dx \\
&\le 0.
\end{aligned} \tag{4.4.16}$$

The corresponding estimate can be proven for the boundary integral that is

$$\begin{aligned}
\int_{\partial\Omega} \Big[&\widetilde{j_2^\circ}(\cdot, \gamma u; 1) + j_2^\circ(\cdot, \gamma u_l; -1) \\
&- \sum_{j=1}^{2} |h_{2,j}(x, \gamma u) - h_{2,0}(x, \gamma u)|\Big]\gamma(u_l - u)^+ d\sigma \le 0.
\end{aligned} \tag{4.4.17}$$

Applying (4.4.14)–(4.4.17) to (4.4.13) yields

$$\begin{aligned}
0 &\ge -\lambda \int_\Omega B(u)(u_l - u)^+ dx \\
&= -\lambda \int_{\{x\in\Omega: u_l(x)>u(x)\}} -(u_0 - u)^{p-1}(u_l - u) dx \\
&\ge \lambda \int_\Omega ((u_l - u)^+)^p dx \\
&\ge 0,
\end{aligned}$$

and hence, $(u_l - u)^+ = 0$ showing that $u_l \le u$ for $l = 1, 2$. This proves $u_0 = \max\{u_1, u_2\} \le u$. The proof for $u \le \overline{u}$ can be done in a similar way. More precisely, we obtain a solution $u \in K$ of (4.4.9) satisfying $\underline{u} \le u_0 \le u \le \overline{u}$ which implies $\widehat{F}(u) = f(\cdot, u, \nabla u)$, $\widehat{B}(u) = 0$ and $H_1(u) = H_2(\gamma u) = 0$. The same arguments as at the end of the proof of Theorem 4.3.1 apply, which shows that $u$ is in fact a solution of problem (4.0.1) belonging to the interval $[u_0, \overline{u}]$. Thus, the solution set $\mathcal{S}$ is upward directed. Analogously, one proves that $\mathcal{S}$ is downward directed. □

Theorem 4.4.1 and Theorem 4.4.3 allow us to formulate the next theorem about the existence of extremal solutions.

**Theorem 4.4.4.** *Let the hypotheses of Theorem 4.4.3 be satisfied. Then the solution set $\mathcal{S}$ possesses extremal elements.*

**Proof.** Since $\mathcal{S} \subset W^{1,p}(\Omega)$ and $W^{1,p}(\Omega)$ is separable, $\mathcal{S}$ is also separable, that is, there exists a countable, dense subset $Z = \{z_n : n \in \mathbb{N}\}$ of $\mathcal{S}$. We construct an increasing sequence $(u_n) \subset \mathcal{S}$ as follows. Let $u_1 = z_1$ and select $u_{n+1} \in \mathcal{S}$ such that

$$\max(z_n, u_n) \leq u_{n+1} \leq \overline{u}.$$

By Theorem 4.4.3, the element $u_{n+1}$ exists because $\mathcal{S}$ is upward directed. Moreover, we can choose, by Theorem 4.4.1, a convergent subsequence (denoted again by $u_n$) with $u_n \to u$ in $W^{1,p}(\Omega)$ and $u_n(x) \to u(x)$ a. e. in $\Omega$. Since $(u_n)$ is increasing, the entire sequence converges in $W^{1,p}(\Omega)$ and further, $u = \sup u_n$. One sees at once that $Z \subset [\underline{u}, u]$ which follows from

$$\max(z_1, \dots, z_n) \leq u_{n+1} \leq u, \quad \forall n,$$

and the fact that $[\underline{u}, u]$ is closed in $W^{1,p}(\Omega)$ implies

$$\mathcal{S} \subset \overline{Z} \subset \overline{[\underline{u}, u]} = [\underline{u}, u].$$

Therefore, as $u \in \mathcal{S}$, we conclude that $u$ is the greatest element in $\mathcal{S}$. The existence of the smallest solution of (4.0.1) in $[\underline{u}, \overline{u}]$ can be proven in a similar way. □

**Remark 4.4.5.** *If $A$ depends on $s$, we have to require additional assumptions. For example, if $A$ satisfies in $s$ a monotonicity condition, the existence of extremal solutions can be shown, too. In case $K = W^{1,p}(\Omega)$, a Lipschitz condition with respect to $s$ is sufficient for proving extremal solutions. For more details we refer to [28].*

## 4.5 Generalization to Discontinuous Nemytskij Operators

In this section, we will extend our problem in (4.0.1) to include discontinuous nonlinearities $f$ of the form $f : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$. We consider again the elliptic variational-hemivariational inequality

$$\begin{aligned} \langle Au + F(u), v - u\rangle &+ \int_\Omega j_1^o(\cdot, u; v-u)dx \\ &+ \int_{\partial\Omega} j_2^o(\cdot, \gamma u; \gamma v - \gamma u)d\sigma \geq 0, \quad \forall v \in K, \end{aligned} \tag{4.5.1}$$

where all denotations of the previous sections are valid. Here, $F$ denotes the Nemytskij operator given by

$$F(u)(x) = f(x, u(x), u(x), \nabla u(x)). \tag{4.5.2}$$

where we will allow $f$ to depend discontinuously on its third argument. The aim of this section is to deal with discontinuous Nemytskij operators $F : [\underline{u}, \overline{u}] \subset W^{1,p}(\Omega) \to L^q(\Omega)$ by combining the results of Section 4.3 with an abstract fixed point result for not necessarily continuous operators, cf. Lemma 2.4.6. This will extend recent results obtained in [120]. Let us recall the definitions of sub- and supersolutions.

**Definition 4.5.1.** *A function $\underline{u} \in W^{1,p}(\Omega)$ is called a subsolution of (4.5.1) if the following hold:*

(i) $F(\underline{u}) \in L^q(\Omega)$,

(ii) $\langle A\underline{u} + F(\underline{u}), w - \underline{u}\rangle + \int_\Omega j_1^\circ(\cdot, \underline{u}; w - \underline{u})dx + \int_{\partial\Omega} j_2^\circ(\cdot, \gamma\underline{u}; \gamma w - \gamma\underline{u})d\sigma \geq 0 \quad \forall w \in \underline{u} \wedge K.$

**Definition 4.5.2.** *A function $\overline{u} \in W^{1,p}(\Omega)$ is called a supersolution of (4.5.1) if the following hold:*

(i) $F(\overline{u}) \in L^q(\Omega)$,

(ii) $\langle A\overline{u} + F(\overline{u}), w - \overline{u}\rangle + \int_\Omega j_1^\circ(\cdot, \overline{u}; w - \overline{u})dx + \int_{\partial\Omega} j_2^\circ(\cdot, \gamma\overline{u}; \gamma w - \gamma\overline{u})d\sigma \geq 0, \quad \forall v \in \overline{u} \vee K.$

The conditions for Clarke's generalized gradient $s \mapsto \partial j_k(x, s)$ and the functions $j_k : \Omega \times \mathbb{R} \to \mathbb{R}$, $k = 1, 2$, are the same as in (j1)–(j3). We only change the property (F1) to the following:

(F2) (i) $x \mapsto f(x, r, u(x), \xi)$ is measurable for all $r \in \mathbb{R}$, for all $\xi \in \mathbb{R}^N$, and for all measurable functions $u : \Omega \to \mathbb{R}$.

(ii) $(r, \xi) \mapsto f(x, r, s, \xi)$ is continuous in $\mathbb{R} \times \mathbb{R}^N$ for all $s \in \mathbb{R}$ and for a.a. $x \in \Omega$.

(iii) $s \mapsto f(x, r, s, \xi)$ is decreasing for all $(r, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and for a.a. $x \in \Omega$.

(iv) There exist a constant $c_2 > 0$ and a function $k_2 \in L_+^q(\Omega)$ such that

$$|f(x, r, s, \xi)| \leq k_2(x) + c_0|\xi|^{p-1}, \tag{4.5.3}$$

for a.a. $x \in \Omega$, for all $\xi \in \mathbb{R}^N$, and for all $r, s \in [\underline{u}(x), \overline{u}(x)]$.

By [4] the mapping $x \mapsto f(x, u(x), u(x), \nabla u(x))$ is measurable for $x \mapsto (u(x), \nabla u(x))$ measurable, however, the associated Nemytskij operator $F : W^{1,p}(\Omega) \subset L^p(\Omega) \to L^q(\Omega)$ is not necessarily continuous.

One of our main results is the following theorem.

**Theorem 4.5.3.** *Let hypotheses (A1')–(A3'),(j1)–(j3),(F2) and (4.4.1) be satisfied and assume the existence of sub- and supersolutions $\underline{u}$ and $\overline{u}$ satisfying $\underline{u} \leq \overline{u}$ and (4.1.1). If $f$ is right-continuous (respectively, left-continuous) in the third argument, then there exists a greatest solution $u^*$ (respectively, a smallest solution $u_*$) of (4.5.1) in the order interval $[\underline{u}, \overline{u}]$.*

**Proof.** We choose a fixed element $z \in [\underline{u}, \overline{u}]$ which is a supersolution of (4.5.1) satisfying $z \wedge K \subset K$ and consider the following auxiliary problem

$$\begin{aligned} u \in K : \quad & \langle Au + F_z(u), v - u\rangle + \int_\Omega j_1^\circ(\cdot, u; v - u)dx \\ & + \int_{\partial\Omega} j_2^\circ(\cdot, \gamma u; \gamma v - \gamma u)d\sigma \geq 0, \quad \forall v \in K, \end{aligned} \tag{4.5.4}$$

where $F_z(u)(x) = f(x, u(x), z(x), \nabla u(x))$. It is readily seen that the mapping $(x, u, \nabla u) \mapsto f(x, u, z(x), \nabla u)$ is a Carathéodory function satisfying some growth condition as in (4.5.3). Since $F_z(z) = F(z)$, $z$ is also a supersolution of (4.5.4). By Definition 4.5.1, we have for a given subsolution $\underline{u}$ of (4.5.1)

$$\langle A\underline{u} + F(\underline{u}), w - \underline{u}\rangle + \int_\Omega j_1^o(\cdot, \underline{u}; w - \underline{u})dx + \int_{\partial\Omega} j_2^o(\cdot, \gamma\underline{u}; \gamma w - \gamma\underline{u})d\sigma \geq 0, \quad \forall w \in \underline{u} \wedge K.$$

Setting $w = \underline{u} - (\underline{u} - v)^+$ for all $v \in K$ and using the monotonicity of $f$ with respect to $s$ we get

$$\begin{aligned} 0 &\geq \langle A\underline{u} + F(\underline{u}), (\underline{u} - v)^+\rangle - \int_\Omega j_1^o(\cdot, \underline{u}; -(\underline{u} - v)^+)dx - \int_{\partial\Omega} j_2^o(\cdot, \gamma\underline{u}; -\gamma(\underline{u} - v)^+)d\sigma \\ &\geq \langle A\underline{u} + F_z(\underline{u}), (\underline{u} - v)^+\rangle - \int_\Omega j_1^o(\cdot, \underline{u}; -(\underline{u} - v)^+)dx - \int_{\partial\Omega} j_2^o(\cdot, \gamma\underline{u}; -\gamma(\underline{u} - v)^+)d\sigma, \end{aligned}$$

for all $v \in K$, which shows that $\underline{u}$ is also a subsolution of (4.5.4). Theorem 4.4.4 implies the existence of a greatest solution $u^* \in [\underline{u}, z]$ of (4.5.4). Now we introduce the set $H$ given by $H := \{z \in W^{1,p}(\Omega) : z \in [\underline{u}, \overline{u}]$ and $z$ is a supersolution of (4.5.1) satisfying $z \wedge K \subset K\}$ and define the operator $L : H \to K$ by $z \mapsto u^* =: Lz$. This means that the operator $L$ assigns to each $z \in H$ the greatest solution $u^*$ of (4.5.4) in $[\underline{u}, z]$. Due to the lattice structure of the closed convex set $K$, $Lz$ is also a supersolution of problem (4.5.1). In the next step we construct a decreasing sequence as follows:

$$\begin{aligned} &u_0 := \overline{u} \\ &u_1 := Lu_0 \quad \text{with} \quad u_1 \in [\underline{u}, u_0] \\ &u_2 := Lu_1 \quad \text{with} \quad u_2 \in [\underline{u}, u_1] \\ &\vdots \\ &u_n := Lu_{n-1} \quad \text{with} \quad u_n \in [\underline{u}, u_{n-1}] \end{aligned} \tag{4.5.5}$$

As $u_n \in [\underline{u}, u_{n-1}]$, we get $u_n(x) \searrow u(x)$ a. e. $x \in \Omega$. Furthermore, the sequence $u_n$ is bounded in $W^{1,p}(\Omega)$, that is, $\|u_n\|_{W^{1,p}(\Omega)} \leq C$ for all $n$ (see the proof of Theorem 4.4.1). Due to the monotony of $u_n$ and the compact embedding $i : W^{1,p}(\Omega) \to L^p(\Omega)$ and the trace operator $\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ we obtain

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{in } W^{1,p}(\Omega), \\ &u_n \to u \quad \text{in } L^p(\Omega) \text{ and a.e. pointwise in } \Omega, \\ &\gamma u_n \to \gamma u \quad \text{in } L^p(\partial\Omega) \text{ and a.e. pointwise in } \partial\Omega. \end{aligned} \tag{4.5.6}$$

The fact that $u_n$ is a solution of (4.5.4) with $z = u_{n-1}$ and $v = u \in K$ results in

$$\langle Au_n, u_n - u\rangle \leq \langle F_{u_{n-1}}(u_n), u - u_n\rangle + \int_\Omega j_1^o(\cdot, u_n; u - u_n)dx + \int_{\partial\Omega} j_2^o(\cdot, \gamma u_n; \gamma u - \gamma u_n)d\sigma.$$

Applying Fatou's Lemma, (4.5.6) and the upper semicontinuity of $(s,r) \to j_k^o(x,s;r)$, $k=1,2$, yields

$$\begin{aligned}
\limsup_{n\to\infty}\langle Au_n, u_n - u\rangle &\le \underbrace{\limsup_{n\to\infty}\langle F_{u_{n-1}}(u_n), u-u_n\rangle}_{\to 0} + \int_\Omega \underbrace{\limsup_{n\to\infty} j_1^o(\cdot, u_n; u-u_n)}_{\le j_1^o(\cdot,u;0)=0}\,dx \\
&\quad + \int_{\partial\Omega} \underbrace{\limsup_{n\to\infty} j_2^o(\cdot,\gamma u_n;\gamma u - \gamma u_n)}_{\le j_2^o(\cdot,\gamma u;\gamma 0)=0}\,d\sigma \\
&\le 0,
\end{aligned}$$

which implies by the $(S_+)$-property of $A$ on $W^{1,p}(\Omega)$ along with (4.5.6)

$$u_n \to u \quad \text{in } W^{1,p}(\Omega).$$

The right-continuity of $f$ and the strong convergence of the decreasing sequence $(u_n)$ along with the upper semicontinuity of $j_k^o(x,\cdot;\cdot)$ allow us to pass to the lim sup in (4.5.4) where $u$ (respectively, $z$) is replaced by $u_n$ (respectively, $u_{n-1}$). We have

$$\begin{aligned}
0 &\le \limsup_{n\to\infty}\langle Au_n + F_{u_{n-1}}(u_n), v-u_n\rangle + \limsup_{n\to\infty}\int_\Omega j_1^o(\cdot,u_n;v-u_n)dx \\
&\quad + \limsup_{n\to\infty}\int_{\partial\Omega} j_2^o(\cdot,\gamma u_n;\gamma v-\gamma u_n)d\sigma \\
&\le \lim_{n\to\infty}\langle Au_n + F_{u_{n-1}}(u_n), v-u_n\rangle + \int_\Omega \limsup_{n\to\infty} j_1^o(\cdot,u_n;v-u_n)dx \\
&\quad + \int_{\partial\Omega}\limsup_{n\to\infty} j_2^o(\cdot,\gamma u_n;\gamma v-\gamma u_n)d\sigma \\
&\le \langle Au + F_u(u), v-u\rangle + \int_\Omega j_1^o(\cdot,u;v-u)dx + \int_{\partial\Omega} j_2^o(\cdot,\gamma u;\gamma v-\gamma u)d\sigma, \quad \forall v\in K.
\end{aligned}$$

This shows that $u$ is a solution of (4.5.1) in the order interval $[\underline{u},\overline{u}]$. Now, we still have to prove that $u$ is the greatest solution of (4.5.1) in $[\underline{u},\overline{u}]$. Let $\widetilde{u}$ be any solution of (4.5.1) in $[\underline{u},\overline{u}]$. Because of the fact that $K$ has lattice structure, $\widetilde{u}$ is also a subsolution of (4.5.1), respectively, a subsolution of (4.5.4). By the same construction as in (4.5.5) we obtain:

$$\begin{aligned}
\widetilde{u_0} &:= \overline{u} \\
\widetilde{u_1} &:= Lu_0 \quad \text{with} \quad \widetilde{u_1}\in[\widetilde{u},u_0] \\
\widetilde{u_2} &:= Lu_1 \quad \text{with} \quad \widetilde{u_2}\in[\widetilde{u},u_1] \\
&\vdots \\
\widetilde{u_n} &:= Lu_{n-1} \quad \text{with} \quad \widetilde{u_n}\in[\widetilde{u},u_{n-1}]
\end{aligned}\tag{4.5.7}$$

Obviously, the sequences in (4.5.5) and (4.5.7) create the same extremal solutions $u_n$ and $\widetilde{u_n}$ which implies that $\widetilde{u}\le\widetilde{u_n}=u_n$ for all $n$. Passing to the limit yields the assertion. The existence of a smallest solution can be shown in a similar way. $\square$

In the next theorem we will prove that only the monotony of $f$ in the third argument is sufficient for the existence of extremal solutions. The function $f$ needs neither be right-continuous nor left-continuous. An important tool in extending the results in the previous sections to discontinuous Nemytskij operators is the fixed point result given in Lemma 2.4.6 to obtain the following.

**Theorem 4.5.4.** *Assume that hypotheses (A1')–(A3'), (j1)–(j3), (F2) and (4.4.1) are valid and let $\underline{u}$ and $\overline{u}$ be sub- and supersolutions of (4.5.1) satisfying $\underline{u} \le \overline{u}$ and (4.1.1). Then there exist extremal solutions $u^*$ and $u_*$ of (4.5.1) with $\underline{u} \le u_* \le u^* \le \overline{u}$.*

**Proof.** As in the proof of Theorem 4.5.3 we consider the following auxiliary problem

$$\begin{aligned} u \in K: \quad & \langle Au + F_z(u), v-u\rangle + \int_\Omega j_1^\circ(\cdot, u; v-u)dx \\ & + \int_{\partial\Omega} j_2^\circ(\cdot, \gamma u; \gamma v - \gamma u)d\sigma \ge 0, \quad \forall v \in K, \end{aligned} \tag{4.5.8}$$

where $F_z(u)(x) = f(x, u(x), z(x), \nabla u(x))$. We define again the set $H := \{z \in W^{1,p}(\Omega) : z \in [\underline{u}, \overline{u}]$ and $z$ is a supersolution of (4.5.1) satisfying $z \wedge K \subset K\}$ and introduce the fixed point operator $L : H \to K$ by $z \mapsto u^* =: Lz$. For a given supersolution $z \in H$, the element $Lz$ is the greatest solution of (4.5.8) in $[\underline{u}, z]$ and thus, it holds $\underline{u} \le Lz \le z$ for all $z \in H$ which implies $L : H \to [\underline{u}, \overline{u}] \cap K$. Because of (4.4.1), $Lz$ is also a supersolution of (4.5.8) satisfying

$$\langle ALz + F_z(Lz), w - Lz\rangle + \int_\Omega j_1^\circ(\cdot, Lz; w - Lz)dx + \int_{\partial\Omega} j_2^\circ(\cdot, \gamma Lz; \gamma w - \gamma Lz)d\sigma \ge 0,$$

for all $w \in Lz \vee K$. By the monotonicity of $f$ with respect to its third argument, $Lz \le z$ and using the representation $w = Lz + (v - Lz)^+$ for any $v \in K$ we obtain

$$\begin{aligned} 0 &\le \langle ALz + F_z(Lz), (v-Lz)^+\rangle + \int_\Omega j_1^\circ(\cdot, Lz; (v-Lz)^+)dx + \int_{\partial\Omega} j_2^\circ(\cdot, \gamma Lz; \gamma(v-Lz)^+)d\sigma \\ &\le \langle ALz + F_{Lz}(Lz), (v-Lz)^+\rangle + \int_\Omega j_1^\circ(\cdot, Lz; (v-Lz)^+)dx + \int_{\partial\Omega} j_2^\circ(\cdot, \gamma Lz; \gamma(v-Lz)^+)d\sigma, \end{aligned}$$

for all $v \in K$. Consequently, $Lz$ is a supersolution of (4.5.1). This shows $L : H \to H$.
Let $v_1, v_2 \in H$ and assume that $v_1 \le v_2$. Then we have

$Lv_1 \in [\underline{u}, v_1]$ is the greatest solution of

$$\begin{aligned} & \langle Au + F_{v_1}(u), v-u\rangle + \int_\Omega j_1^\circ(\cdot, u; v-u)dx \\ & + \int_{\partial\Omega} j_2^\circ(\cdot, \gamma u; \gamma v - \gamma u)d\sigma \ge 0, \quad \forall v \in K \end{aligned} \tag{4.5.9}$$

and

$Lv_2 \in [\underline{u}, v_2]$ is the greatest solution of

$$\begin{aligned} & \langle Au + F_{v_2}(u), v-u\rangle + \int_\Omega j_1^\circ(\cdot, u; v-u)dx \\ & + \int_{\partial\Omega} j_2^\circ(\cdot, \gamma u; \gamma v - \gamma u)d\sigma \ge 0, \quad \forall v \in K. \end{aligned} \tag{4.5.10}$$

Since $v_1 \le v_2$ it follows that $Lv_1 \le v_2$, and due to (4.4.1), $Lv_1$ is also a subsolution of (4.5.9), that is, (4.5.9) holds, in particular, for $v \in Lv_1 \wedge K$ meaning

$$0 \ge \langle ALv_1 + F_{v_1}(Lv_1), (Lv_1 - v)^+\rangle - \int_\Omega j_1^o(\cdot, Lv_1; -(Lv_1 - v)^+)dx \\ - \int_{\partial\Omega} j_2^o(\cdot, \gamma Lv_1; -\gamma(Lv_1 - v)^+)d\sigma, \quad \forall v \in K.$$

Using the monotonicity of $f$ with respect to its third argument yields

$$\begin{aligned} 0 &\ge \langle ALv_1 + F_{v_1}(Lv_1), (Lv_1 - v)^+\rangle - \int_\Omega j_1^o(\cdot, Lv_1; -(Lv_1 - v)^+)dx \\ &\quad - \int_{\partial\Omega} j_2^o(\cdot, \gamma Lv_1; -\gamma(Lv_1 - v)^+)d\sigma \\ &\ge \langle ALv_1 + F_{v_2}(Lv_1), (Lv_1 - v)^+\rangle - \int_\Omega j_1^o(\cdot, Lv_1; -(Lv_1 - v)^+)dx \\ &\quad - \int_{\partial\Omega} j_2^o(\cdot, \gamma Lv_1; -\gamma(Lv_1 - v)^+)d\sigma, \end{aligned}$$

for all $v \in K$. Hence, $Lv_1$ is a subsolution of (4.5.10). By Theorem 4.4.4, we know there exists a greatest solution of (4.5.10) in $[Lv_1, v_2]$. But $Lv_2$ is the greatest solution of (4.5.10) in $[\underline{u}, v_2] \supseteq [Lv_1, v_2]$ and therefore, $Lv_1 \le Lv_2$. This shows that $L$ is increasing.
In the last step we have to prove that any decreasing sequence of $L(H)$ converges weakly in $H$. Let $(u_n) = (Lz_n) \subset L(H) \subset H$ be a decreasing sequence. Then $u_n(x) \searrow u(x)$ for a.a. $x \in \Omega$ and for some $u \in [\underline{u}, \overline{u}]$. The boundedness of $u_n$ in $W^{1,p}(\Omega)$ can be shown similarly as in Section 4.4. Thus the compact imbedding $i : W^{1,p}(\Omega) \to L^p(\Omega)$ along with the monotony of $u_n$ as well as the compactness of the trace operator $\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ imply

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{in } W^{1,p}(\Omega), \\ &u_n \to u \quad \text{in } L^p(\Omega) \text{ and a.e. pointwise in } \Omega, \\ &\gamma u_n \to \gamma u \quad \text{in } L^p(\partial\Omega) \text{ and a.e. pointwise in } \partial\Omega. \end{aligned} \tag{4.5.11}$$

Since $u_n \in K$, it follows $u \in K$. From (4.5.8) with $u$ being replaced by $u_n$ and $v$ by $u$, and using the fact that $(s, r) \mapsto j_k^o(x, s; r)$, $k = 1, 2$, is upper semicontinuous, we obtain by applying Fatou's Lemma

$$\begin{aligned} &\limsup_{n\to\infty}\langle Au_n, u_n - u\rangle \\ &\le \limsup_{n\to\infty}\langle F_{z_n}(u_n), u - u_n\rangle + \limsup_{n\to\infty}\int_\Omega j_1^o(\cdot, u_n; u - u_n)dx \\ &\quad + \limsup_{n\to\infty}\int_{\partial\Omega} j_2^o(\cdot, \gamma u_n; \gamma u - \gamma u_n)d\sigma \\ &\le \underbrace{\limsup_{n\to\infty}\langle F_{z_n}(u_n), u - u_n\rangle}_{\to 0} + \int_\Omega \underbrace{\limsup_{n\to\infty} j_1^o(\cdot, u_n; u - u_n)}_{\le j_1^o(\cdot, u; 0) = 0} dx \end{aligned}$$

$$+\int_{\partial\Omega}\underbrace{\limsup_{n\to\infty} j_2^\circ(\cdot,\gamma u_n;\gamma u-\gamma u_n)}_{\le j_2^\circ(\cdot,\gamma u;\gamma 0)=0}\,d\sigma$$
$$\le 0.$$

The $(S_+)$-property of $A$ provides the strong convergence of $(u_n)$ in $W^{1,p}(\Omega)$. As $Lz_n = u_n$ is also a supersolution of (4.5.8) Definition 4.5.2 yields

$$\langle Au_n + F_{z_n}(u_n),(v-u_n)^+\rangle + \int_\Omega j_1^\circ(\cdot,u_n;(v-u_n)^+)dx + \int_{\partial\Omega} j_2^\circ(\cdot,\gamma u_n;\gamma(v-u_n)^+)d\sigma \ge 0,$$

for all $v\in K$. Due to $z_n \ge u_n \ge u$ and the monotonicity of $f$ we get

$$\begin{aligned}0 &\le \langle Au_n + F_{z_n}(u_n),(v-u_n)^+\rangle + \int_\Omega j_1^\circ(\cdot,u_n;(v-u_n)^+)dx + \int_{\partial\Omega} j_2^\circ(\cdot,\gamma u_n;\gamma(v-u_n)^+)d\sigma\\ &\le \langle Au_n + F_{u}(u_n),(v-u_n)^+\rangle + \int_\Omega j_1^\circ(\cdot,u_n;(v-u_n)^+)dx + \int_{\partial\Omega} j_2^\circ(\cdot,\gamma u_n;\gamma(v-u_n)^+)d\sigma,\end{aligned}$$

for all $v\in K$. Since the mapping $u\mapsto u^+ = \max(u,0)$ is continuous from $W^{1,p}(\Omega)$ to itself (cf. [77]), we can pass to the upper limit on the right hand side for $n\to\infty$. This yields

$$\begin{aligned}&\langle Au + F_u(u),(v-u)^+\rangle + \int_\Omega j_1^\circ(\cdot,u;(v-u)^+)dx\\ &\quad+\int_{\partial\Omega} j_2^\circ(\cdot,\gamma u;\gamma(v-u)^+)dx \ge 0,\quad \forall v\in K,\end{aligned}$$

which shows that $u$ is a supersolution of (4.5.1), that is, $u\in H$. As $\overline{u}$ is an upper bound of $L(H)$, we can apply Lemma 2.4.6, which yields the existence of a greatest fixed point $u^*$ of $L$ in $H$. This implies that $u^*$ must be the greatest solution of (4.5.1) in $[\underline{u},\overline{u}]$. By analogous reasoning, one shows the existence of a smallest solution $u_*$ of (4.5.1). This completes the proof of the theorem. □

## 4.6 Construction of Sub- and Supersolutions

In this section we are going to construct sub- and supersolutions of problem (4.5.1) under the conditions (A1')–(A3'), (j1)–(j2) and (F2)(i)–(F2)(iii), where we drop the gradient dependence of $f$ meaning $f(x,r,s) := f(x,r,s,\xi)$. Further, we set $A=-\Delta_p$ which is the negative $p$-Laplacian defined by

$$-\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2}\nabla u) \quad \text{where} \quad \nabla u = (\partial u/\partial x_1,\dots,\partial u/\partial x_N). \tag{4.6.1}$$

The coefficients $a_i$, $i=1,\dots,N$ are given by

$$a_i(x,s,\xi) = |\xi|^{p-2}\xi_i.$$

Thus, hypothesis (A1') is satisfied with $k_0=0$ and $c_0=1$. Hypothesis (A2') is a consequence of the inequalities from the vector-valued function $\xi\mapsto|\xi|^{p-2}\xi$ (see [28, p. 37]) and (A3') is

satisfied with $c_1 = 1$ and $k_1 = 0$.
First, we suppose that $u = 0$ on $\partial\Omega$ and $j_2 = 0$ and denote for simplification $j := j_1$. Then, our variational-hemivariational inequality gets the form

$$\langle -\Delta_p u + F(u), v - u\rangle + \int_\Omega j^o(\cdot, u; v - u)dx \geq 0, \quad \forall v \in K, \tag{4.6.2}$$

where $K$ is a closed convex subset of $W_0^{1,p}(\Omega)$. Furthermore, we denote by $\lambda_1 > 0$ the first eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$ and by $\varphi_1$ the first eigenfunction of $(-\Delta_p, W_0^{1,p}(\Omega))$ corresponding to $\lambda_1$ satisfying $\varphi_1 \in \text{int}(C_0^1(\overline{\Omega})_+)$ and $\|\varphi\|_{L^p(\Omega)} = 1$ (cf. [3]). Here, $\text{int}(C_0^1(\overline{\Omega})_+)$ describes the interior of the positive cone $C_0^1(\overline{\Omega})_+ = \{u \in C_0^1(\overline{\Omega}) : u(x) \geq 0, \forall x \in \Omega\}$ in the Banach space $C_0^1(\overline{\Omega})$ given by

$$\text{int}(C_0^1(\overline{\Omega})_+) = \left\{u \in C_0^1(\overline{\Omega}) : u(x) > 0, \forall x \in \Omega, \text{ and } \frac{\partial u}{\partial \nu}(x) < 0, \forall x \in \partial\Omega\right\}, \tag{4.6.3}$$

where $\frac{\partial u}{\partial \nu}(x)$ is the outer normal derivative of $u$ in the point $x \in \partial\Omega$. The definitions of sub- and supersolution in this case are defined as follows.

**Definition 4.6.1.** *A function $\underline{u} \in W^{1,p}(\Omega)$ is said to be a subsolution of (4.6.2) if the following hold:*

*(i)* $F(\underline{u}) \in L^q(\Omega), \underline{u} \leq 0$ *on* $\partial\Omega$,

*(ii)* $\langle -\Delta_p \underline{u} + F(\underline{u}), w - \underline{u}\rangle + \int_\Omega j^o(\cdot, \underline{u}; w - \underline{u})dx \geq 0, \quad \forall w \in \underline{u} \wedge K.$

**Definition 4.6.2.** *A function $\overline{u} \in W^{1,p}(\Omega)$ is said to be a supersolution of (4.6.2) if the following hold:*

*(i)* $F(\overline{u}) \in L^q(\Omega), \overline{u} \geq 0$ *on* $\partial\Omega$,

*(ii)* $\langle -\Delta_p \overline{u} + F(\overline{u}), w - \overline{u}\rangle + \int_\Omega j^o(\cdot, \overline{u}; w - \overline{u})dx \geq 0, \quad \forall w \in \overline{u} \vee K.$

We suppose the following conditions for $f$ and Clarke's generalized gradient of $s \mapsto j(x, s)$, where $\lambda > \lambda_1$ is any fixed constant.

(F3) (i) $\lim\limits_{|s|\to\infty} \frac{f(x, s, s)}{|s|^{p-2}s} = +\infty$, uniformly with respect to a.a. $x \in \Omega$.

(ii) $\lim\limits_{s\to 0} \frac{f(x, s, s)}{|s|^{p-2}s} = -\lambda$, uniformly with respect to a.a. $x \in \Omega$.

(iii) $\lim\limits_{|s|\to\infty} \frac{\xi}{|s|^{p-2}s} = +\infty$, uniformly with respect to a.a. $x \in \Omega$, for all $\xi \in \partial j(x, s)$.

(iv) $\lim\limits_{s\to 0} \frac{\xi}{|s|^{p-2}s} = 0$, uniformly with respect to a.a. $x \in \Omega$, for all $\xi \in \partial j(x, s)$.

**Proposition 4.6.3.** *Assume hypotheses (j1)–(j2), (F2)(i)–(F2)(iii) and (F3). Then there exists a constant* $a_\lambda$ *such that* $a_\lambda e_1$ *and* $-a_\lambda e_1$ *are supersolution and subsolution of problem (4.6.2), where* $e_1 \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ *is the unique solution of* $-\Delta_p u = 1$ *in* $W_0^{1,p}(\Omega)$*. Moreover,* $-\varepsilon\varphi_1$ *is a supersolution and* $\varepsilon\varphi_1$ *is a subsolution of (4.6.2) provided that* $\varepsilon > 0$ *is sufficiently small.*

**Proof.** A sufficient condition for a subsolution $\underline{u} \in W^{1,p}(\Omega)$ of problem (4.6.2) is $\underline{u} \le 0$ on $\partial\Omega$, $F(\underline{u}) \in L^q(\Omega)$ and

$$-\Delta_p \underline{u} + F(\underline{u}) + \xi \le 0 \text{ in } W^{-1,q}(\Omega), \text{ for all } \xi \in \partial j(\cdot, \underline{u}). \tag{4.6.4}$$

Multiplying (4.6.4) with $(\underline{u} - v)^+ \in W_0^{1,p}(\Omega) \cap L_+^p(\Omega)$ and using the fact $j^o(\cdot, \underline{u}; -1) \ge -\xi$, for all $\xi \in \partial j(\cdot, \underline{u})$, yields

$$\begin{aligned} 0 &\ge \langle -\Delta_p \underline{u} + F(\underline{u}) + \xi, (\underline{u} - v)^+ \rangle \\ &= \langle -\Delta_p \underline{u} + F(\underline{u}), (\underline{u} - v)^+ \rangle + \int_\Omega \xi (\underline{u} - v)^+ dx \\ &\ge \langle -\Delta_p \underline{u} + F(\underline{u}), (\underline{u} - v)^+ \rangle - \int_\Omega j^o(\cdot, \underline{u}; -1)(\underline{u} - v)^+ dx \\ &= \langle -\Delta_p \underline{u} + F(\underline{u}), (\underline{u} - v)^+ \rangle - \int_\Omega j^o(\cdot, \underline{u}; -(\underline{u} - v)^+) dx, \qquad \forall v \in K, \end{aligned}$$

and thus, $\underline{u}$ is a subsolution of (4.6.2). Analogously, $\overline{u} \in W^{1,p}(\Omega)$ is a supersolution of problem (4.6.2) if $\overline{u} \ge 0$ on $\partial\Omega$, $F(\overline{u}) \in L^q(\Omega)$ and if the following inequality is satisfied

$$-\Delta_p \overline{u} + F(\overline{u}) + \xi \ge 0 \text{ in } W^{-1,q}(\Omega), \text{ for all } \xi \in \partial j(\cdot, \overline{u}).$$

The main idea of this proof is to show the applicability of [36, Lemmas 2.1–2.3]. We put $g(x,s) = f(x,s,s) + \xi + \lambda |s|^{p-2}s$ for $\xi \in \partial j(x,s)$ and notice that in our considerations the nonlinearity $g$ does not need to be a continuous function. The condition (F3) yields the following limit values

$$\lim_{|s| \to \infty} \frac{g(x,s)}{|s|^{p-2}s} = +\infty, \qquad \text{and} \qquad \lim_{s \to 0} \frac{g(x,s)}{|s|^{p-2}s} = 0.$$

By [36, Lemmas 2.1–2.3] we obtain a pair of positive sub- and supersolutions given by $\underline{u} = \varepsilon\varphi_1$ and $\overline{u} = a_\lambda e_1$, respectively, a pair of negative sub- and supersolutions given by $\underline{u} = -a_\lambda e_1$ and $\overline{u} = -\varepsilon\varphi_1$. □

With the aid of these constructed sub- and supersolutions, we see at once that the assumptions (j3) and (F2)(iv) are satisfied, too.

**Example 4.6.4.** *The function* $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *defined by*

$$f(x,r,s) = \begin{cases} -(\lambda+1)|s|^{p-2}s + e^{|r|+|x|}|r|^{p-1}r & \text{for } s < -1, \\ -\lambda |s|^{p-2}s + e^{|r|+|x|}|r|^{p-1}r & \text{for } -1 \le s \le 1, \\ -(\lambda+1)|s|^{p-2}s + e^{|r|+|x|}|r|^{p-1}r & \text{for } s > 1, \end{cases} \tag{4.6.5}$$

*fulfills the assumption (F2) and (F3), where $\lambda > \lambda_1$ is fixed. Clarke's generalized gradient can be given by*

$$\partial j(x,s) = \begin{cases} (e^{-(s+1)} - 2)|s|^{p-2}s & \text{if } s < -1, \\ [-1,1] & \text{if } s = -1, \\ -|s|^p & \text{if } -1 < s < 1, \\ [-1,1] & \text{if } s = 1, \\ e^{(s-1)(|x|+1)}s^{p-1} & \text{if } s > 1, \end{cases} \tag{4.6.6}$$

*where all conditions in (j1)-(j3) and (F3) are satisfied.*

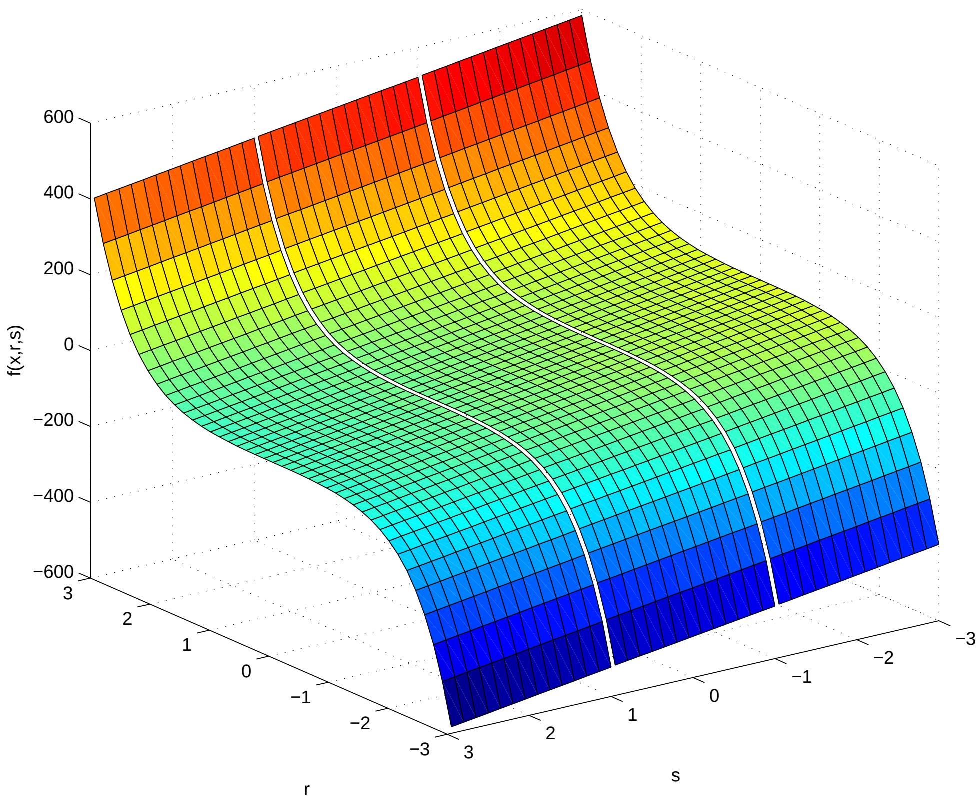


**Figure 4.1.** *The function $f$ with respect to $\Omega = (0,\pi), \varphi_1(x) = \sin(x), \lambda_1 = 1$, $\lambda = 2$, $p = 2$ and $x = 1$*

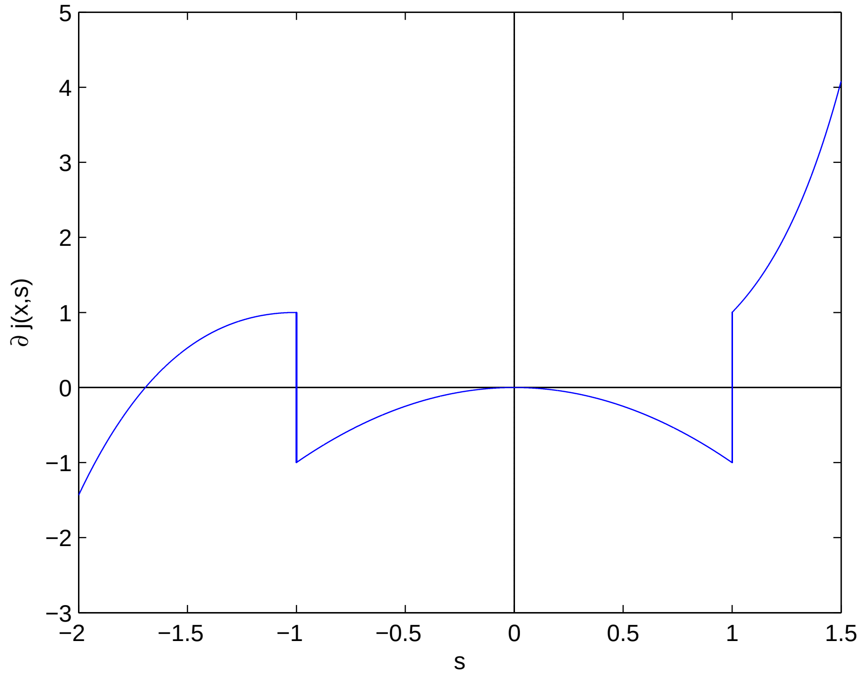


**Figure 4.2.** *Clarke's generalized gradient with respect to $p = 2$ and $x = 1$*

**Remark 4.6.5.** *In order to apply Theorem 4.5.4 we need to satisfy the assumptions*

$$\underline{u} \vee K \subset K, \quad \overline{u} \wedge K \subset K, \quad K \vee K \subset K, \quad K \wedge K \subset K, \tag{4.6.7}$$

*which depends on the specific $K$. For example the obstacle problem is given by*

$$K = \{v \in W_0^{1,p}(\Omega) : v(x) \le \psi(x) \text{ for a.a. } x \in \Omega\}, \quad \psi \in L^\infty(\Omega), \psi \ge C > 0, \tag{4.6.8}$$

*where $C$ is a positive constant. One can show that for the positive pair of sub- and supersolutions in Proposition 4.6.3 all these conditions in (4.6.7) with respect to the closed convex set $K$ defined in (4.6.8) can be satisfied.*

Next, we are going to construct sub- and supersolutions for our main problem (4.5.1) meaning

$$\begin{aligned} &\langle -\Delta_p u + F(u), v - u\rangle + \int_\Omega j_1^\circ(\cdot, u; v - u)dx \\ &+ \int_{\partial\Omega} j_2^\circ(\cdot, \gamma u; \gamma v - \gamma u)d\sigma \ge 0, \quad \forall v \in K. \end{aligned} \tag{4.6.9}$$

To this end, we study some auxiliary problems in form of differential equations with Neumann boundary values. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$. We consider the quasilinear elliptic equation

$$\begin{aligned} -\Delta_p u &= h_1(x, u) - \beta|u|^{p-2}u \qquad \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= \lambda|u|^{p-2}u + h_2(x, u) \qquad \text{on } \partial\Omega, \end{aligned} \tag{4.6.10}$$

where $\frac{\partial u}{\partial \nu}$ means the outer normal derivative of $u$ with respect to $\partial\Omega$, $\beta, \lambda$ are real parameters and the nonlinearities $h_1 : \Omega \times \mathbb{R} \to \mathbb{R}$ and $h_2 : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are some Carathéodory functions which satisfy the following conditions:

(H') (a) $\displaystyle\lim_{s\to 0} \frac{h_1(x,s)}{|s|^{p-2}s} = 0,$ uniformly with respect to a.a. $x \in \Omega$.

(b) $\displaystyle\lim_{|s|\to\infty} \frac{h_1(x,s)}{|s|^{p-2}s} = -\infty,$ uniformly with respect to a.a. $x \in \Omega$.

(c) $h_1$ is bounded on bounded sets.

(d) $\displaystyle\lim_{s\to 0} \frac{h_2(x,s)}{|s|^{p-2}s} = 0,$ uniformly with respect to a.a. $x \in \partial\Omega$.

(e) $\displaystyle\lim_{|s|\to\infty} \frac{h_2(x,s)}{|s|^{p-2}s} = -\infty,$ uniformly with respect to a.a. $x \in \partial\Omega$.

(f) $h_2$ is bounded on bounded sets.

In order to obtain subsolutions of the auxiliary problem (4.6.10), we make use of the Steklov eigenvalue problem again, meaning

$$\begin{aligned} -\Delta_p u &= -|u|^{p-2}u && \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= \lambda |u|^{p-2}u && \text{on } \partial\Omega. \end{aligned} \tag{4.6.11}$$

Note once more that the first eigenvalue $\lambda_1 > 0$ is isolated and simple and the related eigenfunction $\varphi_1$ belongs to $\operatorname{int}(C^1(\overline{\Omega})_+)$. Analogously, we use the unique solution $e \in \operatorname{int}(C^1(\overline{\Omega})_+)$ of the following boundary value problem

$$\begin{aligned} -\Delta_p u &= -\varsigma |u|^{p-2}u + 1 && \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= 1 && \text{on } \partial\Omega, \end{aligned} \tag{4.6.12}$$

with a constant $\varsigma > 1$ (see Section 3.1).
Next, we recall the notations of sub- and supersolutions of problem (4.6.10).

**Definition 4.6.6.** *A function $\underline{u} \in W^{1,p}(\Omega)$ is called a subsolution of (4.6.10) if the following holds*

$$\int_\Omega |\nabla \underline{u}|^{p-2}\nabla \underline{u} \nabla \varphi dx \le \int_\Omega (h_1(x,\underline{u}) - \beta|\underline{u}|^{p-2}\underline{u})\varphi dx + \int_{\partial\Omega} (\lambda |\underline{u}|^{p-2}\underline{u} + h_2(x,\underline{u}))\varphi d\sigma,$$

*for all $\varphi \in W^{1,p}(\Omega)_+$.*

**Definition 4.6.7.** *A function $\overline{u} \in W^{1,p}(\Omega)$ is called a supersolution of (4.6.10) if the following holds*

$$\int_\Omega |\nabla \overline{u}|^{p-2}\nabla \overline{u} \nabla \varphi dx \ge \int_\Omega (h_1(x,\overline{u}) - \beta|\overline{u}|^{p-2}\overline{u})\varphi dx + \int_{\partial\Omega} (\lambda |\overline{u}|^{p-2}\overline{u} + h_2(x,\overline{u}))\varphi d\sigma,$$

*for all $\varphi \in W^{1,p}(\Omega)_+$.*

With the aid of (4.6.11) and (4.6.12) we start by generating two ordered pairs of sub- and supersolutions of problem (4.6.10) having constant signs.

**Lemma 4.6.8.** *Assume (H'), $\lambda > \lambda_1$, $\beta \in (0,1)$ and let e be the unique solution of problem (4.6.12). Then there exists a constant $\vartheta > 0$ such that $\vartheta e$ and $-\vartheta e$ are supersolution and subsolution, respectively, of problem (4.6.10). In addition, $\varepsilon\varphi_1$ is a subsolution and $-\varepsilon\varphi_1$ is a supersolution of problem (4.6.10) provided the number $\varepsilon > 0$ is sufficiently small.*

**Proof.** First, we prove that $\underline{u} = \varepsilon\varphi_1$ is a positive subsolution, where the positive constant $\varepsilon$ is stated later. Thanks to the auxiliary eigenvalue problem (4.6.11), we get

$$\begin{aligned}&\int_\Omega |\nabla(\varepsilon\varphi_1)|^{p-2}\nabla(\varepsilon\varphi_1)\nabla\varphi dx\\&= -\int_\Omega (\varepsilon\varphi_1)^{p-1}\varphi dx + \int_{\partial\Omega}\lambda_1(\varepsilon\varphi_1)^{p-1}\varphi d\sigma, \forall\varphi\in W^{1,p}(\Omega).\end{aligned}\tag{4.6.13}$$

In order to show that $\underline{u}$ is a subsolution, we have to indicate the validity of Definition 4.6.6 for $\underline{u}$ which means that the inequality

$$\begin{aligned}&\int_\Omega |\nabla(\varepsilon\varphi_1)|^{p-2}\nabla(\varepsilon\varphi_1)\nabla\varphi dx\\&\le \int_\Omega (h_1(x,\varepsilon\varphi_1) - \beta(\varepsilon\varphi_1)^{p-1})\varphi dx + \int_{\partial\Omega}(\lambda(\varepsilon\varphi_1)^{p-1} + h_2(x,\varepsilon\varphi_1))\varphi d\sigma,\end{aligned}\tag{4.6.14}$$

is satisfied for all $\varphi \in W^{1,p}(\Omega)_+$. In view of (4.6.13), the inequality (4.6.14) is complied if the following holds

$$\int_\Omega ((\beta-1)(\varepsilon\varphi_1)^{p-1} - h_1(x,\varepsilon\varphi_1))\varphi dx + \int_{\partial\Omega}((\lambda_1-\lambda)(\varepsilon\varphi_1)^{p-1} - h_2(x,\varepsilon\varphi_1))\varphi d\sigma \le 0,$$

for all $\varphi \in W^{1,p}(\Omega)_+$, where $0 < \beta < 1$ and $\lambda > \lambda_1$. Because of (H')(a) and (H')(d) there are numbers $\delta_\beta, \delta_\lambda > 0$ such that

$$\begin{aligned}&\frac{|h_1(x,s)|}{|s|^{p-1}} < 1-\beta \quad \text{for a.a. } x\in\Omega \text{ and all } 0<|s|\le\delta_\beta,\\&\frac{|h_2(x,s)|}{|s|^{p-1}} < \lambda-\lambda_1 \quad \text{for a.a. } x\in\partial\Omega \text{ and all } 0<|s|\le\delta_\lambda.\end{aligned}$$

We select $0 < \varepsilon < \min\{\delta_\beta/\|\varphi_1\|_\infty, \delta_\lambda/\|\varphi_1\|_\infty\}$, where $\|\cdot\|_\infty$ stands for the supremum norm, to get

$$\begin{aligned}&\int_\Omega ((\beta-1)(\varepsilon\varphi_1)^{p-1} - h_1(x,\varepsilon\varphi_1))\varphi dx\\&\le \int_\Omega \left(\beta - 1 + \frac{|h_1(x,\varepsilon\varphi)|}{(\varepsilon\varphi_1)^{p-1}}\right)(\varepsilon\varphi_1)^{p-1}\varphi dx\\&< \int_\Omega (\beta-1+1-\beta)(\varepsilon\varphi_1)^{p-1}\varphi dx = 0,\end{aligned}\tag{4.6.15}$$

respectively,

$$\begin{aligned}
&\int_{\partial\Omega}((\lambda_1-\lambda)(\varepsilon\varphi_1)^{p-1}-h_2(x,\varepsilon\varphi_1))\varphi d\sigma\\
&\leq \int_{\partial\Omega}\left(\lambda_1-\lambda+\frac{|h_2(x,\varepsilon\varphi)|}{(\varepsilon\varphi_1)^{p-1}}\right)(\varepsilon\varphi_1)^{p-1}\varphi d\sigma\\
&< \int_{\partial\Omega}(\lambda_1-\lambda+\lambda-\lambda_1)(\varepsilon\varphi_1)^{p-1}\varphi d\sigma=0.
\end{aligned}\tag{4.6.16}$$

Applying (4.6.15) and (4.6.16) to (4.6.14) implies directly that $\underline{u}=\varepsilon\varphi_1$ is a positive subsolution. In order to prove that $\overline{u}=-\varepsilon\varphi_1$ is a negative supersolution, we argue in much the same manner. Let $\overline{u}=\vartheta e$ with a positive constant $\vartheta$. Due to the auxiliary problem (4.6.12) we obtain

$$\begin{aligned}
&\int_{\Omega}|\nabla(\vartheta e)|^{p-2}\nabla(\vartheta e)\nabla\varphi dx\\
&=-\varsigma\int_{\Omega}(\vartheta e)^{p-1}\varphi dx+\int_{\Omega}\vartheta^{p-1}\varphi dx+\int_{\partial\Omega}\vartheta^{p-1}\varphi d\sigma, \forall\varphi\in W^{1,p}(\Omega).
\end{aligned}\tag{4.6.17}$$

Definition 4.6.7 is fulfilled for $\overline{u}=\vartheta e$ if the following inequality is satisfied

$$\begin{aligned}
&\int_{\Omega}|\nabla(\vartheta e)|^{p-2}\nabla(\vartheta e)\nabla\varphi dx\\
&\geq\int_{\Omega}(h_1(x,\vartheta e)-\beta(\vartheta e)^{p-1})\varphi dx+\int_{\partial\Omega}(\lambda(\vartheta e)^{p-1}+h_2(x,\vartheta e))\varphi d\sigma,
\end{aligned}\tag{4.6.18}$$

for all $\varphi\in W^{1,p}(\Omega)_+$. Using (4.6.17) to (4.6.18) yields an equivalent formulation to satisfy Definition 4.6.7 resulting in

$$\begin{aligned}
&\int_{\Omega}(\vartheta^{p-1}-\widetilde{c}(\vartheta e)^{p-1}-h_1(x,\vartheta e))\varphi dx\\
&+\int_{\partial\Omega}(\vartheta^{p-1}-\lambda(\vartheta e)^{p-1}-h_2(x,\vartheta e))\varphi d\sigma\geq 0,
\end{aligned}\tag{4.6.19}$$

where $\widetilde{c}=\varsigma-\beta$ with $\widetilde{c}>0$. Because of (H')(b) there exists $s_\varsigma>0$ such that

$$\frac{h_1(x,s)}{s^{p-1}}<-\widetilde{c},\qquad \text{for a.a. } x\in\Omega \text{ and all } s>s_\varsigma,$$

and by (H')(c) we get

$$|-h_1(x,s)-\widetilde{c}s^{p-1}|\leq|h_1(x,s)|+\widetilde{c}s^{p-1}\leq c_\varsigma,\qquad \text{for a.a. } x\in\Omega \text{ and all } s\in[0,s_\varsigma].$$

Therefore, we obtain

$$h_1(x,s)\leq-\widetilde{c}s^{p-1}+c_\varsigma,\quad \text{for a.a. } x\in\Omega \text{ and all } s\geq 0.\tag{4.6.20}$$

Applying (4.6.20) to the first integral in (4.6.19) provides

$$\begin{aligned}
\int_{\Omega}(\vartheta^{p-1}-\widetilde{c}(\vartheta e)^{p-1}-h_1(x,\vartheta e))\varphi dx&\geq\int_{\Omega}(\vartheta^{p-1}-\widetilde{c}(\vartheta e)^{p-1}+\widetilde{c}(\vartheta e)^{p-1}-c_\varsigma)\varphi dx\\
&=\int_{\Omega}(\vartheta^{p-1}-c_\varsigma)\varphi dx,
\end{aligned}$$

which proves that for $\vartheta \geq c_\varsigma^{\frac{1}{p-1}}$ the integral is nonnegative. Hypothesis (H')(e) implies the existence of a $s_\lambda > 0$ such that

$$\frac{h_2(x,s)}{s^{p-1}} < -\lambda, \qquad \text{for a.a. } x \in \Omega \text{ and all } s > s_\lambda.$$

Because of (H')(f) there exists a constant $c_\lambda > 0$ such that

$$|-h_2(x,s) - \lambda s^{p-1}| \leq |h_2(x,s)| + \lambda s^{p-1} \leq c_\lambda, \qquad \text{for a.a. } x \in \Omega \text{ and all } s \in [0, s_\lambda].$$

Finally, we obtain

$$h_2(x,s) \leq -\lambda s^{p-1} + c_\lambda, \quad \text{for a.a. } x \in \partial\Omega \text{ and all } s \geq 0. \tag{4.6.21}$$

Applying (4.6.21) to the second integral in (4.6.19) yields

$$\begin{aligned}\int_{\partial\Omega} (\vartheta^{p-1} - \lambda(\vartheta e)^{p-1} - h_2(x,\vartheta e))\varphi dx &\geq \int_{\partial\Omega} (\vartheta^{p-1} - \lambda(\vartheta e)^{p-1} + \lambda(\vartheta e)^{p-1} - c_\lambda)\varphi dx \\ &\geq \int_{\partial\Omega} (\vartheta^{p-1} - c_\lambda)\varphi dx.\end{aligned}$$

Choosing $\vartheta \geq \max\left\{c_\varsigma^{\frac{1}{p-1}}, c_\lambda^{\frac{1}{p-1}}\right\}$ proves that both integrals in (4.6.19) are nonnegative and thus, $\overline{u} = \vartheta e$ is a positive supersolution of problem (4.6.10). In order to prove that $\underline{u} = -\vartheta e$ is a negative subsolution we make use of the following estimates

$$\begin{aligned} h_1(x,s) &\geq -\tilde{c} s^{p-1} - c_\varsigma, \quad \text{for a.a. } x \in \Omega \text{ and all } s \leq 0,\\ h_2(x,s) &\geq -\lambda s^{p-1} - c_\lambda, \quad \text{for a.a. } x \in \partial\Omega \text{ and all } s \leq 0,\end{aligned} \tag{4.6.22}$$

which can be derivated as stated above. With the aid of (4.6.22) one verifies that $\underline{u} = -\vartheta e$ is a negative subsolution of problem (4.6.10). □

According to Lemma 4.6.8, we obtain a positive pair $[\varepsilon\varphi_1, \vartheta e]$ and a negative pair $[-\vartheta e, -\varepsilon\varphi_1]$ of sub- and supersolutions of problem (4.6.10) assumed $\varepsilon > 0$ is sufficiently small.
Now, we will use this result to our variational-hemivariational inequality in (4.6.9). First, we suppose the following conditions on $f$ and Clarke's generalized gradient of $s \mapsto j_k(x,s)$, $k = 1, 2$, where $\lambda > \lambda_1$ and $\beta \in (0,1)$ are some fixed constants:

(F4) (i) $\lim\limits_{|s|\to\infty} \dfrac{f(x,s,s)}{|s|^{p-2}s} = +\infty$, uniformly with respect to a.a. $x \in \Omega$

(ii) $\lim\limits_{s\to 0} \dfrac{f(x,s,s)}{|s|^{p-2}s} = \beta$, uniformly with respect to a.a. $x \in \Omega$

(iii) $\lim\limits_{|s|\to\infty} \dfrac{\xi}{|s|^{p-2}s} = +\infty$, uniformly with respect to a.a. $x \in \Omega$, for all $\xi \in \partial j_1(x,s)$

(iv) $\lim\limits_{s\to 0} \dfrac{\xi}{|s|^{p-2}s} = 0$, uniformly with respect to a.a. $x \in \Omega$, for all $\xi \in \partial j_1(x,s)$

(v) $\lim\limits_{|s|\to\infty} \dfrac{\eta}{|s|^{p-2}s} = +\infty$ , uniformly with respect to a.a. $x \in \partial\Omega$, for all $\eta \in \partial j_2(x,s)$

(vi) $\lim\limits_{s\to 0} \dfrac{\eta}{|s|^{p-2}s} = \lambda$, uniformly with respect to a.a. $x \in \partial\Omega$, for all $\eta \in \partial j_2(x,s)$

**Proposition 4.6.9.** *Let the conditions (j1)–(j2), (F2)(i)–(F2)(iii) and (F4) be satisfied. Then there is a constant $\vartheta$ such that $\vartheta e$ and $-\vartheta e$ are supersolution and subsolution of problem (4.6.9), where $e \in \operatorname{int}(C^1(\overline{\Omega})_+)$ is the unique solution of (4.6.12). Moreover, $-\varepsilon\varphi_1$ is a supersolution and $\varepsilon\varphi_1$ is a subsolution of (4.6.9) provided that $\varepsilon > 0$ is sufficiently small.*

**Proof.** As in the proof of Proposition 4.6.3, we give first a sufficient condition for a subsolution of (4.6.9). A function $\underline{u} \in W^{1,p}(\Omega)$ is a subsolution of problem (4.6.9) if $F(\underline{u}) \in L^q(\Omega)$ and if the inequality

$$-\Delta_p \underline{u} + F(\underline{u}) + \xi + \eta \le 0 \text{ in } (W^{1,p}(\Omega))^*, \text{ for all } \xi \in \partial j_1(\cdot,\underline{u}) \text{ and all } \eta \in \partial j_2(\cdot,\underline{u}),$$

is fulfilled. To prove this, we multiply the inequality above with $(\underline{u}-v)^+ \in W^{1,p}(\Omega) \cap L^p_+(\Omega)$ and we use the fact $j_1^o(\cdot,\underline{u};-1) \ge -\xi$, for all $\xi \in \partial j_1(\cdot,\underline{u})$ and $j_2^o(\cdot,\gamma\underline{u};-\gamma 1) \ge -\eta$, for all $\eta \in \partial j_2(\cdot,\underline{u})$, to obtain

$$\begin{aligned}
0 &\ge \langle -\Delta_p \underline{u} + F(\underline{u}) + \xi + \eta, (\underline{u}-v)^+\rangle \\
&= \langle -\Delta_p \underline{u} + F(\underline{u}), (\underline{u}-v)^+\rangle + \int_\Omega \xi(\underline{u}-v)^+ dx + \int_{\partial\Omega} \eta\gamma(\underline{u}-v)^+ d\sigma \\
&\ge \langle -\Delta_p \underline{u} + F(\underline{u}), (\underline{u}-v)^+\rangle - \int_\Omega j_1^o(\cdot,\underline{u};-1)(\underline{u}-v)^+ dx - \int_{\partial\Omega} j_2^o(\cdot,\gamma\underline{u};-\gamma 1)\gamma(\underline{u}-v)^+ d\sigma \\
&= \langle -\Delta_p \underline{u} + F(\underline{u}), (\underline{u}-v)^+\rangle - \int_\Omega j_1^o(\cdot,\underline{u};-(\underline{u}-v)^+) dx - \int_{\partial\Omega} j_2^o(\cdot,\gamma\underline{u};-\gamma(\underline{u}-v)^+) d\sigma,
\end{aligned}$$

for all $v \in K$, which shows that $\underline{u}$ is a subsolution of (4.6.9). By the same calculation, we obtain that $\overline{u} \in W^{1,p}(\Omega)$ is a supersolution of problem (4.6.9) if $F(\overline{u}) \in L^q(\Omega)$ and if the following inequality is satisfied

$$-\Delta_p \overline{u} + F(\overline{u}) + \xi + \eta \ge 0 \text{ in } (W^{1,p}(\Omega))^*, \text{ for all } \xi \in \partial j_1(\cdot,\overline{u}) \text{ and for all } \eta \in \partial j_2(\cdot,\overline{u}).$$

Now, we set $h_1(x,s) = \beta|s|^{p-2}s - f(x,s,s) - \xi$ for $\xi \in \partial j_1(x,s)$ and $h_2(x,s) = -\lambda|s|^{p-2}s - \eta$ for $\eta \in \partial j_2(x,s)$ and notice that in our considerations the nonlinearities $h_1$ and $h_2$ do not need to be continuous functions. Applying assumption (F3) provides

$$\begin{aligned}
\lim_{|s|\to\infty} \frac{h_1(x,s)}{|s|^{p-2}s} &= -\infty, \quad \text{and} \quad \lim_{s\to 0} \frac{h_1(x,s)}{|s|^{p-2}s} = 0, \\
\lim_{|s|\to\infty} \frac{h_2(x,s)}{|s|^{p-2}s} &= -\infty, \quad \text{and} \quad \lim_{s\to 0} \frac{h_2(x,s)}{|s|^{p-2}s} = 0.
\end{aligned}$$

With a view to Lemma 4.6.8, we see that the assumptions therein are satisfied. This yields an ordered pair of positive sub- and supersolutions given by $\underline{u} = \varepsilon\varphi_1$ and $\overline{u} = \vartheta e$, respectively, a pair of negative sub- and supersolutions given by $\underline{u} = -\vartheta e$ and $\overline{u} = -\varepsilon\varphi_1$ of problem (4.6.9). $\square$

Note again that in order to apply the existence and comparison result the constructed sub-supersolutions have to satisfy additional conditions related to $K$ meaning that

$$\underline{u} \vee K \subset K, \qquad \overline{u} \wedge K \subset K$$

are fulfilled, too.

**Example 4.6.10.** *Let $p \geq 2$ and let the function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be given by*

$$f(x, r, s) = \begin{cases} -2|s|^{p-2}s + (\beta + 1)e^{|r|(|x|+1)}|r|^{p-2}r & \text{for } s < -1, \\ -|s|^{p-2}s + (\beta + 1)e^{|r|(|x|+1)}|r|^{p-2}r & \text{for } -1 \leq s \leq 1, \\ -2|s|^{p-2}s + (\beta + 1)e^{|r|(|x|+1)}|r|^{p-2}r & \text{for } s > 1. \end{cases} \tag{4.6.23}$$

*One easily verifies the validity of the assumptions (F2) and (F4), where $0 < \beta < 1$ is fixed. Moreover, Clarke's gradient $s \mapsto \partial j(x, s)$ from Example 4.6.4 can also be used as example for $\partial j_1(x, \cdot)$. The function*

$$\partial j_2(x, s) = \begin{cases} (e^{-(s+1)} - 2)|s|^{p-2}s & \text{if } s < -1, \\ [-\lambda, 1] & \text{if } s = -1, \\ \lambda|s|^{p-2}s & \text{if } -1 < s < 1, \\ [-1, \lambda] & \text{if } s = 1, \\ (e^{(s-1)(|x|+1)} + s - 3)s^{p-1} & \text{if } s > 1, \end{cases} \tag{4.6.24}$$

*satisfies the assumptions (j1)–(j3) for some fixed $\lambda > \lambda_1$.*

To obtain extremal solutions of problem (4.0.1), it is required that the given closed convex set $K$ fulfills the lattice structure conditions as stated in (4.4.1). In Remark 4.6.5 the one-sided obstacle problem is presented as a closed convex set in $W_0^{1,p}(\Omega)$ satisfying these assumptions. The same holds true in $W^{1,p}(\Omega)$ meaning that

$$K = \{v \in W^{1,p}(\Omega) : v(x) \leq \psi(x) \text{ for a.a. } x \in \Omega\}, \quad \psi \in L^\infty(\Omega), \psi \geq C > 0, \tag{4.6.25}$$

is a closed convex set in $W^{1,p}(\Omega)$ having lattice structure. Other interest closed convex sets in $W^{1,p}(\Omega)$ are the following

$$\begin{aligned} K_1 &= \{v \in W^{1,p}(\Omega) : \alpha(x) \leq v(x) \ \text{ a.e. } x \in \Omega\}, \\ K_2 &= \{v \in W^{1,p}(\Omega) : \beta_1(x) \leq v(x) \leq \beta_2(x) \ \text{ a.e. } x \in \Omega\}, \\ K_3 &= \{v \in W^{1,p}(\Omega) : |\nabla v(x)| \leq C \ \text{ a.e. } x \in \Omega\}, \\ K_4 &= \{v \in W^{1,p}(\Omega) : \int_\Omega v(x)dx \geq a_1\}, \\ K_5 &= \{v \in W^{1,p}(\Omega) : \int_\Omega v(x)dx \leq a_2\}, \end{aligned} \tag{4.6.26}$$

where $\beta_1, \beta_2$ and $\alpha$ are given functions and $a_1, a_2$ and $C$ are some constants. One sees at once that $K_1, K_2$ and $K_3$ fulfill the assumptions in (4.4.1), however, the sets $K_4$ and $K_5$ just satisfy $K \wedge K \subset K$ and $K \vee K \subset K$, respectively.

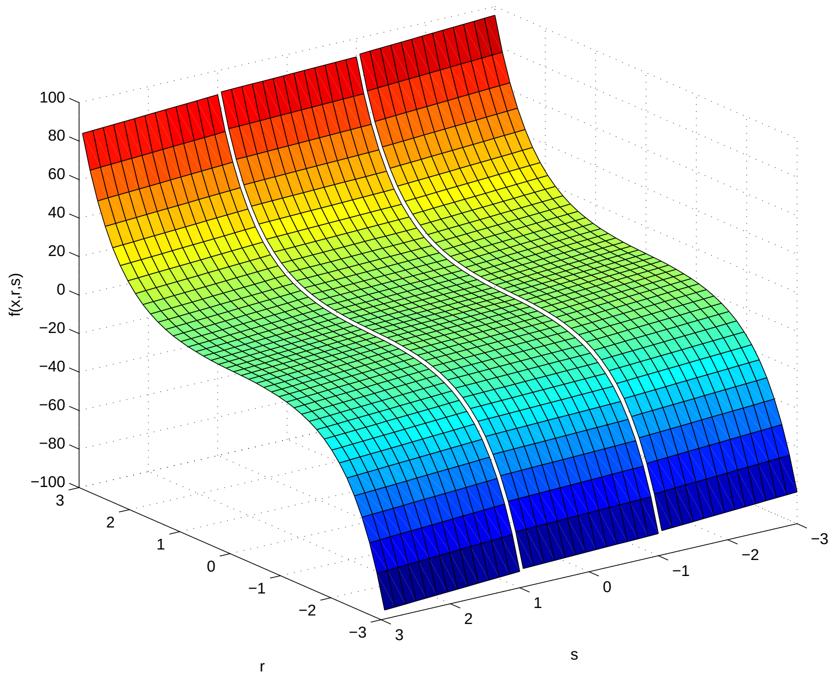


**Figure 4.3.** *The function $f$ with respect to $p = 2$, $\beta = 0.1$ and $x = 0.1$*

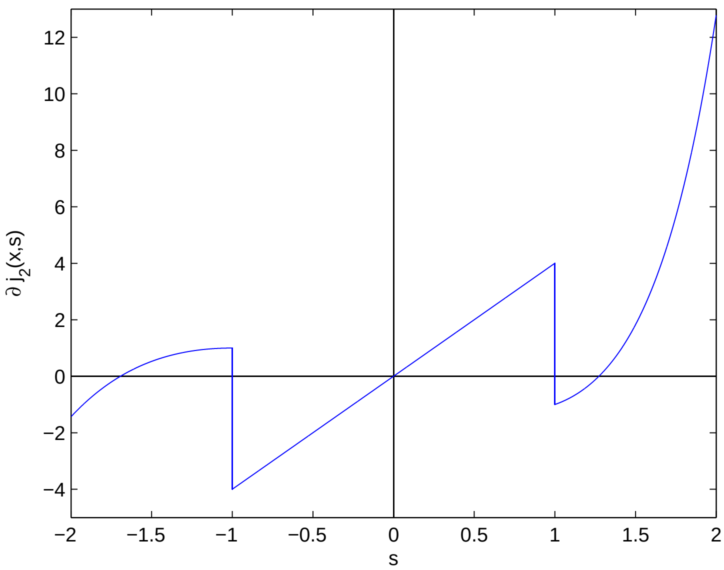


**Figure 4.4.** *Clarke's generalized gradient $s \mapsto j_2(x, s)$ with respect to $\Omega = (0, \pi)$, $\varphi_1(x) = e^x$, $\lambda_1 = 1$, $\lambda = 4$, $p = 2$ and $x = 1$*

# Chapter 5
# Entire Extremal Solutions for Elliptic Inclusions of Clarke's Gradient Type

In this chapter, we deal with quasilinear elliptic differential inclusions of Clarke's gradient type defined in all of $\mathbb{R}^N$ in the form

$$Au + \partial j(\cdot, u) \ni 0 \quad \text{in } \mathcal{D}', \tag{5.0.1}$$

where $A$ is a second-order quasilinear differential operator in divergence form of Leray-Lions type given by

$$Au(x) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, u(x), \nabla u(x)) \qquad \text{with } \nabla u = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N}\right). \tag{5.0.2}$$

The function $j : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is assumed to be measurable in $x \in \mathbb{R}^N$ for all $s \in \mathbb{R}$, and locally Lipschitz continuous in $s \in \mathbb{R}$ for a.a. $x \in \mathbb{R}^N$. The multivalued function $s \mapsto \partial j(x, s)$ stands for Clarke's generalized gradient of the locally Lipschitz function $s \mapsto j(x, s)$ and is given by

$$\partial j(x, s) = \{\xi \in \mathbb{R} : j^o(x, s; r) \geq \xi r, \forall r \in \mathbb{R}\}, \tag{5.0.3}$$

for a.a. $x \in \mathbb{R}^N$, where $j^o(x, s; r)$ is the generalized directional derivative of $j$ at $s$ in the direction $r$ defined by

$$j^o(x, s; r) = \limsup_{y \to s, t \downarrow 0} \frac{j(x, y + tr) - j(x, y)}{t}. \tag{5.0.4}$$

We denote by $\mathcal{D} = C_0^\infty(\mathbb{R}^N)$ the space of all infinitely differentiable functions with compact support in $\mathbb{R}^N$ and by $\mathcal{D}'$ its dual space.

## 5.1 Notations and Hypotheses

Let $\mathcal{W} = W^{1,p}_{\text{loc}}(\mathbb{R}^N)$ be the local Sobolev space of all functions $u : \mathbb{R}^N \to \mathbb{R}$, which belong to the Sobolev space $W^{1,p}(\Omega)$ for every compact domain $\Omega \subset \mathbb{R}^N$. The topology of the locally convex space $\mathcal{W}$ is described by the family of seminorms $\{h_k : k = 1, 2, ...\}$ given by $h_k(u) = \|u\|_{W^{1,p}(B_k)}$, where $B_k \subset \mathbb{R}^N$ is the ball of radius $k$. A sequence $(u_n) \subset \mathcal{W}$ converges to $u$ if and only if

$$h_k(u_n - u) \to 0, \qquad \text{as } n \to \infty, \text{ for all } k = 1, 2, \dots . \tag{5.1.1}$$

Since the space $\mathcal{W}$ has a countable fundamental system of seminorms, there exists a metric $d$ on $\mathcal{W}$ for which $(\mathcal{W}, d)$ is a complete metric vector space. Such spaces are called Frechét spaces (see [98, Theorem 25.1, Corollary 25.2]). For fixed $k$ we denote $\mathcal{W}_k = W^{1,p}(B_k)$ and by $i_k : \mathcal{W} \to \mathcal{W}_k$ the mapping defined by $\mathcal{W} \ni u \mapsto u|_{B_k} \in \mathcal{W}_k$, where $u|_{B_k}$ denotes the restriction of $u$ to $B_k$. Analogously, we define the local Lebesgue space $\mathcal{L}^q := L^q_{\text{loc}}(\mathbb{R}^N)$, where $q$ satisfies the equation $\frac{1}{p} + \frac{1}{q} = 1$. Note that $\mathcal{L}^q$ is equipped with the natural partial ordering $\leq$ defined by $u \leq v$ iff $v - u \in \mathcal{L}^q_+ := L^q_{\text{loc},+}(\mathbb{R}^N)$ which stands for the set of all nonnegative functions of $\mathcal{L}^q$. We impose the following hypotheses on the operator $A$ and its coefficients, where $1 < p < \infty$.

(A1) Each $a_i(x, s, \xi)$ satisfies Carathéodory conditions, i.e., is measurable in $x \in \Omega$ for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and continuous in $(s, \xi)$ for a.a. $x \in \mathbb{R}^N$. Furthermore, a constant $c_0 > 0$ and a function $k_0 \in \mathcal{L}^q$ exist so that

$$|a_i(x, s, \xi)| \leq k_0(x) + c_0(|s|^{p-1} + |\xi|^{p-1}),$$

for a.a. $x \in \mathbb{R}^N$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$, where $|\xi|$ denotes the Euclidian norm of the vector $\xi$.

(A2) The coefficients $a_i$ satisfy a monotonicity condition with respect to $\xi$ in the form

$$\sum_{i=1}^{N} (a_i(x, s, \xi) - a_i(x, s, \xi'))(\xi_i - \xi'_i) > 0,$$

for a.a. $x \in \mathbb{R}^N$, for all $s \in \mathbb{R}$, and for all $\xi, \xi' \in \mathbb{R}^N$ with $\xi \neq \xi'$.

(A3) A constant $c_1 > 0$ and a function $k_1 \in \mathcal{L}^1$ exist such that

$$\sum_{i=1}^{N} a_i(x, s, \xi)\xi_i \geq c_1|\xi|^p - k_1(x),$$

for a.a. $x \in \mathbb{R}^N$, for all $s \in \mathbb{R}$, and for all $\xi \in \mathbb{R}^N$.

(A4) There is a function $k_2 \in \mathcal{L}^q_+$ and a continuous function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|a_i(x, s, \xi) - a_i(x, s', \xi)| \leq [k_2(x) + |s|^{p-1} + |s'|^{p-1} + |\xi|^{p-1}]\omega(|s - s'|),$$

holds for a.a. $x \in \Omega$, for all $s, s' \in \mathbb{R}$ and for all $\xi \in \mathbb{R}^N$, where $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies

$$\int_{0^+} \frac{dr}{\omega(r)} = +\infty,$$

which means that for every $\varepsilon > 0$ the integral taken over $[0, \varepsilon]$ diverges, that is,

$$\int_0^{\varepsilon} \frac{dr}{\omega(r)} = +\infty.$$

Note that hypothesis (A4) is satisfied for example in case $\omega(|s-s'|) = C|s-s'|^{\frac{1}{q}}$ with a positive constant $C$ meaning that the coefficients $a_i(x, s, \xi)$ fulfill a Hölder condition with respect to $s$. It should be mentioned that the operator $A$ is well-defined, that is,

$$a(u, \varphi) = \int_{\mathbb{R}^N} \sum_{i=1}^{N} a_i(x, u, \nabla u) \frac{\partial \varphi}{\partial x_i} dx$$

is well-defined for all $u \in \mathcal{W}$ and all $\varphi \in \mathcal{D}$, where $a$ denotes the semilinear form associated with $A$.

**Definition 5.1.1.** *A function $u \in \mathcal{W}$ is said to be a solution of (5.0.1), if there exists a function $\gamma \in \mathcal{L}^q$ such that*

*(i)* $\gamma(x) \in \partial j(x, u(x)), \qquad$ *for a.a.* $x \in \mathbb{R}^N$,

*(ii)* $\displaystyle\int_{\mathbb{R}^N} \sum_{i=1}^{N} a_i(x, u, \nabla u) \frac{\partial \varphi}{\partial x_i} dx + \int_{\mathbb{R}^N} \gamma \varphi dx = 0, \qquad$ *for all* $\varphi \in \mathcal{D}$.

**Definition 5.1.2.** *A function $\underline{u} \in \mathcal{W}$ is said to be a subsolution of (5.0.1), if there exists a function $\underline{\gamma} \in \mathcal{L}^q$ such that*

*(i)* $\underline{\gamma}(x) \in \partial j(x, \underline{u}(x)), \qquad$ *for a.a.* $x \in \mathbb{R}^N$,

*(ii)* $\displaystyle\int_{\mathbb{R}^N} \sum_{i=1}^{N} a_i(x, \underline{u}, \nabla \underline{u}) \frac{\partial \varphi}{\partial x_i} dx + \int_{\mathbb{R}^N} \underline{\gamma} \varphi dx \leq 0, \quad \forall \varphi \in \mathcal{D}_+.$

**Definition 5.1.3.** *A function $\overline{u} \in \mathcal{W}$ is said to be a supersolution of (5.0.1), if there exists a function $\overline{\gamma} \in \mathcal{L}^q$ such that*

*(i)* $\overline{\gamma}(x) \in \partial j(x, \overline{u}(x)), \qquad$ *for a.a.* $x \in \mathbb{R}^N$,

*(ii)* $\displaystyle\int_{\mathbb{R}^N} \sum_{i=1}^{N} a_i(x, \overline{u}, \nabla \overline{u}) \frac{\partial \varphi}{\partial x_i} dx + \int_{\mathbb{R}^N} \overline{\gamma} \varphi dx \geq 0, \quad \forall \varphi \in \mathcal{D}_+.$

Here, $\mathcal{D}_+ := \{\varphi \in \mathcal{D} : \varphi \geq 0\}$ stands for all nonnegative functions of $\mathcal{D}$. Let $[\underline{u}, \overline{u}]$ be an ordered pair of sub- and supersolutions of problem (5.0.1). We impose the following hypotheses on $j$ and its Clarke's generalized gradient $s \mapsto \partial j(x, s)$.

(j1) $x \mapsto j(x, s)$ is measurable in $\mathbb{R}^N$ for all $s \in \mathbb{R}$.

(j2) $s \mapsto j(x, s)$ is locally Lipschitz continuous in $\mathbb{R}$ for a.a. $x \in \mathbb{R}^N$.

(j3) There exists a function $L \in \mathcal{L}^q_+$ such that for all $s \in [\underline{u}(x), \overline{u}(x)]$ holds

$$\eta \in \partial j(x,s) : |\eta| \le L(x), \quad \text{for a.a. } x \in \mathbb{R}^N.$$

Now, we are going to prove that the assumptions above are sufficient to ensure the existence of entire extremal solutions of (5.0.1) within the interval $[\underline{u}, \overline{u}]$.

## 5.2 Entire Extremal Solutions

**Theorem 5.2.1.** *Let the conditions (A1)–(A4) and (j1)–(j3) be satisfied and let $\underline{u}, \overline{u}$ be a pair of sub- and supersolutions of problem (5.0.1) satisfying $\underline{u} \le \overline{u}$. Then there exist extremal solutions of (5.0.1) belonging to the interval $[\underline{u}, \overline{u}]$.*

**Proof.** First we select a sequence of open balls $(B_k) \subset \mathbb{R}^N$, $k = 1, 2, \dots$, whose union is equal to $\mathbb{R}^N$, that is, $\bigcup_{k=1}^{\infty} B_k = \mathbb{R}^N$. We construct a sequence $(U_k, \Gamma_k) \subset \mathcal{W} \times \mathcal{L}^q$ as follows: By means of the given supersolution according to Definition 5.1.3, one defines

$$\begin{aligned} U_0 &:= \overline{u}, \qquad U_k(x) = \begin{cases} u_k(x) & \text{for } x \in B_k, \\ \overline{u}(x) & \text{for } x \in \mathbb{R}^N \setminus B_k, \end{cases} \\ \Gamma_0 &:= \overline{\gamma}, \qquad \Gamma_k(x) = \begin{cases} \gamma_k(x) & \text{for } x \in B_k, \\ \overline{\gamma}(x) & \text{for } x \in \mathbb{R}^N \setminus B_k, \end{cases} \end{aligned} \tag{5.2.1}$$

where the pair $(u_k, \gamma_k) \in W^{1,p}(B_k) \times L^q(B_k)$ denotes the greatest solution of the differential inclusion

$$\begin{aligned} Au_k + \partial j(\cdot, u_k) &\ni 0 \qquad \text{in } B_k, \\ u_k &= \overline{u} \qquad \text{on } \partial B_k, \end{aligned} \tag{$P_k$}$$

in the order interval $[\underline{u}|_{B_k}, \overline{u}|_{B_k}]$. We recall that a pair $(u_k, \gamma_k) \in W^{1,p}(B_k) \times L^q(B_k)$ is a solution of $(P_k)$ if the following holds:

(1) $u_k = \overline{u}$, on $\partial B_k$,

(2) $\gamma_k(x) \in \partial j(x, u_k(x))$, for a.a. $x \in B_k$,

(3) $\displaystyle\int_{B_k} \sum_{i=1}^{N} a_i(x, u_k, \nabla u_k) \frac{\partial \varphi}{\partial x_i} dx + \int_{B_k} \gamma_k \varphi dx = 0,$ for all $\varphi \in C_0^\infty(B_k)$.

Obviously, the functions $\underline{u}|_{B_k}, \overline{u}|_{B_k}$ form an ordered pair of sub- and supersolutions of the auxiliary problem $(P_k)$ and the existence of a greatest solution $u_k \in [\underline{u}|_{B_k}, \overline{u}|_{B_k}]$ of $(P_k)$ follows directly from [38]. Therein, the homogeneous parabolic case was considered, however, the elliptic case acts by the same arguments. In addition, nonhomogeneous boundary conditions can be reduced to homogeneous ones by translation without changing the class of problems.

Notice that the extensions $(U_k, V_k)$ of $(u_k, v_k)$ are well-defined and belong to $\mathcal{W} \times \mathcal{L}^q$.
By the construction of $U_k$ one immediately sees that $U_1 \leq U_0$ is true. The function $u_2 \in W^{1,p}(B_2)$ is the greatest solution of $(P_2)$ in the interval $[\underline{u}|_{B_2}, \overline{u}|_{B_2}]$. Furthermore, $u_2|_{B_1}$ is a subsolution of $(P_1)$ in $B_1$, and $\overline{u}|_{B_1}$ is a supersolution of $(P_1)$ in $B_1$ satisfying $u_2|_{B_1} \leq \overline{u}|_{B_1}$. Since $u_1 \in W^{1,p}(B_1)$ is the greatest solution of $(P_1)$ in $[\underline{u}|_{B_1}, \overline{u}|_{B_1}] \supset [u_2|_{B_1}, \overline{u}|_{B_1}]$, we obtain $u_2|_{B_1} \leq u_1$ and therefore, $U_2 \leq U_1$. In order to generalize this result, we argue per induction and have by definition of $U_k$ that $u_{k+1}|_{B_k}$ is a subsolution of $(P_k)$ and $u_k$ is the greatest solution in $[\underline{u}|_{B_k}, \overline{u}|_{B_k}] \supset [u_{k+1}|_{B_k}, \overline{u}|_{B_k}]$. This yields

$$\underline{u} \leq \ldots \leq U_{k+1} \leq U_k \leq \ldots \leq U_1 \leq U_0 = \overline{u}, \tag{5.2.2}$$

consequently,

$$\lim_{k\to\infty} U_k(x) = U^*(x), \qquad \text{for almost all } x \in \mathbb{R}^N. \tag{5.2.3}$$

To show that $U^*$ belongs to $\mathcal{W}$, let $\Omega \subset \mathbb{R}^N$ be any compact set, which implies the existence of an open ball $B_k$ satisfying $\Omega \subset B_k$. Due to the fact that $\underline{u}, \overline{u}$ generate lower and upper bounds for $U_l$, we obtain the boundedness of $U_l$ with respect to the norm in $L^p(B_k)$, that is,

$$\|U_l\|_{L^p(B_k)} \leq c_k, \qquad \text{for all l=1,2,}\ldots, \tag{5.2.4}$$

where $c_k$ are some positive constants depending only on $k$. Now we are going to prove the boundedness of $\nabla U_l$ in $L^p(B_k)$. Observe that each $U_l$ with $l \geq k+1$ fulfills in $B_{k+1}$

$$AU_l + \partial j(\cdot, U_l) \ni 0, \tag{5.2.5}$$

which by Definition 5.1.1 means

$$\int_{B_{k+1}} \sum_{i=1}^N a_i(x, U_l, \nabla U_l) \frac{\partial \varphi}{\partial x_i} dx + \int_{B_{k+1}} \Gamma_l \varphi dx = 0, \quad \text{for all } \varphi \in C_0^\infty(B_{k+1}), \tag{5.2.6}$$

where we have

$$\Gamma_l(x) \in \partial j(x, U_l(x)), \qquad \text{for almost all } x \in \mathbb{R}^N. \tag{5.2.7}$$

Since $W_0^{1,p}(B_{k+1})$ is the closure of $C_0^\infty(B_{k+1})$ in $W^{1,p}(B_{k+1})$ (see [1]), the validity of (5.2.6) for all $\varphi \in W_0^{1,p}(B_{k+1})$ can be proven easily by using completion techniques. With the aid of [79, Theorem 1.2.2] we introduce a function $\vartheta \in \mathcal{D}$ given by the following properties:

(i) $0 \leq \vartheta(x) \leq 1$ for all $x \in \mathbb{R}^N$,

(ii) $\vartheta(x) = 0$ for all $x \in \mathbb{R}^N \setminus B_{k+1}$,

(iii) $\vartheta(x) = 1$ for all $x \in \overline{B_k}$.

Additionally, it holds

$$\max\left(\sup_{B_{k+1}} \vartheta, \sup_{B_{k+1}} |\nabla\vartheta|^p\right) \le c, \tag{5.2.8}$$

where $c$ is a positive constant. By using the special test function $\varphi = U_l \cdot \vartheta^p \in W_0^{1,p}(B_{k+1})$ in the left term of (5.2.6), one has along with Young's inequality

$$\begin{aligned}
&\int_{B_{k+1}} \sum_{i=1}^{N} a_i(x, U_l, \nabla U_l) \frac{\partial U_l \vartheta^p}{\partial x_i} \\
&= \int_{B_{k+1}} \sum_{i=1}^{N} a_i(x, U_l, \nabla U_l) \left[\frac{\partial U_l}{\partial x_i}\vartheta^p + p\vartheta^{p-1} U_l \frac{\partial \vartheta}{\partial x_i}\right] \\
&\ge \int_{B_{k+1}} (c_1\vartheta^p |\nabla U_l|^p - k_1\vartheta^p)dx - \int_{B_{k+1}} (k_0 + c_0(|U_l|^{p-1} + |\nabla U_l|^{p-1}))p\vartheta^{p-1}|U_l||\nabla\vartheta|dx \\
&\ge \int_{B_{k+1}} c_1\vartheta^p|\nabla U_l|^p dx - a_1 - a_2 - a_3 - \int_{B_{k+1}} \varepsilon p |\nabla U_l|^p \vartheta^p dx - \int_{B_{k+1}} C(\varepsilon)|\nabla\vartheta|^p|U_l|^p dx \\
&\ge \int_{B_{k+1}} (c_1 - p\varepsilon)\vartheta^p|\nabla U_l|^p dx - p\int_{B_{k+1}} C(\varepsilon)|U_l|^p|\nabla\vartheta|^p dx - a_4,
\end{aligned}$$

where $\varepsilon$ is selected such that $\varepsilon < \frac{c_1}{p}$. Applying (j3) along with (5.2.8) and (5.2.4) yields

$$\begin{aligned}
a_5\int_{B_{k+1}} \vartheta^p|\nabla U_l|^p dx &\le p\int_{B_{k+1}} C(\varepsilon)|U_l|^p|\nabla\vartheta|^p dx + \int_{B_{k+1}} |\Gamma_l||U_l|\vartheta^p dx + a_4 \\
&\le p\int_{B_{k+1}} C(\varepsilon)|U_l|^p|\nabla\vartheta|^p dx + \int_{B_{k+1}} L|U_l|\vartheta^p dx + a_4 \\
&\le a_6
\end{aligned}$$

with a positive constant $a_6$ only depending on $k$. The boundedness of the gradient $\nabla U_l$ in $L^p(B_k)$ follows directly by the estimate

$$a_5\int_{B_k} |\nabla U_l|^p dx \le a_5\int_{B_{k+1}} \vartheta^p|\nabla U_l|^p dx \quad \text{for any } l \ge k+1,$$

which implies along with (5.2.4)

$$\|U_l\|_{W^{1,p}(B_k)} \le \widehat{c}_k, \qquad \text{for all } l = 1, 2, \ldots .$$

The reflexivity of $W^{1,p}(B_k), 1 < p < \infty$, ensures the existence of a weakly convergent subsequence of $U_l$. Due to the compact imbedding $W^{1,p}(B_k) \hookrightarrow L^p(B_k)$ and the monotony of $U_l$ we get for the entire sequence $U_l$

$$U_l\mid_{B_k} \rightharpoonup U^*\mid_{B_k} \text{ in } W^{1,p}(B_k) \quad \text{and} \quad U_l\mid_{B_k} \to U^*\mid_{B_k} \text{ in } L^p(B_k).$$

We have $U^* \in W^{1,p}(B_k)$ and since $\Omega \subset B_k$ it follows $U^* \in W^{1,p}(\Omega)$. As $\Omega$ is a freely selected compact domain in $\mathbb{R}^N$, we obtain $U^* \in \mathcal{W}$. Our aim is to show that $U^*$ is the greatest solution of (5.0.1) in $[\underline{u}, \overline{u}]$. Due to (5.2.1) it holds

$$\Gamma_k \in \partial j(x, U_k(x)) \text{ a.e. in } \mathbb{R}^N \text{ and for all } k. \tag{5.2.9}$$

Immediately, the boundedness of $\Gamma_k$ in $\mathcal{L}^q$ is a consequence of condition (j3) and by using the diagonal process of Cantor one shows the existence of a weakly convergent subsequence of $(\Gamma_k)$, still denoted by $(\Gamma_k)$. In fact, since $\mathcal{L}^q$ is a reflexive Fréchet space for $1 < q < \infty$ (see [98, Theorem 25.15]), we have

$$\int_{\mathbb{R}^N} \Gamma_k \varphi dx \to \int_{\mathbb{R}^N} \Gamma^* \varphi dx \quad \forall \varphi \in \mathcal{D} \text{ as } k \to \infty. \tag{5.2.10}$$

Due to (5.2.9) we get for any ball $B_k$

$$\Gamma_l(x) \in \partial j(x, U_l(x)) \quad \text{for a.a. } x \in B_k, l = 1, 2, \dots,$$

which implies

$$\int_{B_k} \Gamma_l \varphi dx \le \int_{B_k} j^o(x, U_l; \varphi) dx, \quad \forall \varphi \in C_0^\infty(B_k).$$

Using Fatou's Lemma and the upper semicontinuity of $j^o$ yields

$$\begin{aligned}\limsup_{l\to\infty} \int_{B_k} \Gamma_l \varphi dx &\le \limsup_{l\to\infty} \int_{B_k} j^o(x, U_l; \varphi) dx \\ &\le \int_{B_k} \limsup_{l\to\infty} j^o(x, U_l; \varphi) dx \\ &\le \int_{B_k} j^o(x, U^*; \varphi) dx,\end{aligned}$$

which proves in view of (5.2.10)

$$\int_{B_k} \Gamma^* \varphi dx \le \int_{B_k} j^o(x, U^*; \varphi) dx, \quad \forall \varphi \in C_0^\infty(B_k). \tag{5.2.11}$$

We are going to show that (5.2.11) implies $\Gamma^*(x) \in \partial j(x, U^*(x))$ for a.a. $x \in B_k$. The mapping $r \mapsto j^o(x, s; r)$ is positively homogeneous and inequality (5.2.11) holds, in particular, for all $\varphi \in C_0^\infty(B_k)_+$. We obtain

$$\int_{B_k} \Gamma^* \varphi dx \le \int_{B_k} j^o(x, U^*; 1) \varphi dx, \quad \forall \varphi \in C_0^\infty(B_k)_+. \tag{5.2.12}$$

Because of [43, Proposition 2.1.2] Clarke's generalized directional derivative $j^o$ fulfills

$$j^o(x, s; r) = \max\{\xi r : \xi \in \partial j(x, s)\}, \tag{5.2.13}$$

and since $\partial j(x, s)$ is a nonempty, convex, and compact subset of $\mathbb{R}$, there exists a function $\Gamma_1^* : B_k \to \mathbb{R}$ such that

$$j^o(x, U^*(x); 1) = \Gamma_1^*(x), \quad \text{for a.a. } x \in B_k, \tag{5.2.14}$$

where

$$\Gamma_1^*(x) = \max\{\xi : \xi \in \partial j(x, U^*(x))\}. \tag{5.2.15}$$

Applying the general approximation results in [9] for lower (respectively, upper) semicontinuous functions in Hilbert spaces yields a sequence of locally Lipschitz functions converging pointwise to $j^o$. This implies that $s \mapsto j^o(x, s; 1)$ is superpositionally measurable meaning that the mapping $x \mapsto j^o(x, u(x); 1)$ is measurable for all measurable functions $u : B_k \to \mathbb{R}$. Due to (5.2.14) and (j2) we infer $\Gamma_1^* \in L^q(B_k)$. Using (5.2.11) proves

$$\int_{B_k} \Gamma^* \varphi dx \le \int_{B_k} \Gamma_1^* \varphi dx, \quad \forall \varphi \in C_0^\infty(B_k)_+, \tag{5.2.16}$$

which implies

$$\Gamma^*(x) \le \Gamma_1^*, \quad \text{ for a.a. } x \in B_k. \tag{5.2.17}$$

Testing (5.2.11) with nonpositve functions $\varphi = -\psi$, where $\psi \in C_0^\infty(B_k)_+$, we have

$$-\int_{B_k} \Gamma^* \psi dx \le \int_{B_k} j^o(x, U^*; -1)\psi dx, \quad \forall \psi \in C_0^\infty(B_k)_+. \tag{5.2.18}$$

The same arguments as above yield the existence of a function $\tau \in L^q(B_k)$ such that

$$\tau(x) = \max\{-\xi : \xi \in \partial j(x, U^*(x))\} = -\min\{\xi : \xi \in \partial j(x, U^*(x))\}, \tag{5.2.19}$$

which by setting $\Gamma_2^* = -\tau$ in (5.2.18) implies

$$-\int_{B_k} \Gamma^* \psi dx \le -\int_{B_k} \Gamma_2^* \psi dx, \quad \forall \psi \in C_0^\infty(B_k)_+. \tag{5.2.20}$$

Therefore, one gets

$$\int_{B_k} \Gamma^* \psi dx \ge \int_{B_k} \Gamma_2^* \psi dx, \quad \forall \psi \in C_0^\infty(B_k)_+. \tag{5.2.21}$$

From the last inequality we infer

$$\Gamma^*(x) \ge \Gamma_2^*, \quad \text{ for a.a. } x \in B_k. \tag{5.2.22}$$

In view of (5.2.15), (5.2.17), (5.2.19), (5.2.22) and $\Gamma_2^* = -\tau$ we see at once that

$$\Gamma^*(x) \in \partial j(x, U^*(x)) \quad \text{ for a.a. } x \in B_k. \tag{5.2.23}$$

Let $\varphi \in \mathcal{D}$ be arbitrarily fixed. Then there exists an index $k$ such that the support of $\varphi$ fulfills $\operatorname{supp} \varphi \subset B_k$. The approximations above yield for any $l \ge k$

$$\int_{\mathbb{R}^N} a_i(x, U_l, \nabla U_l) \frac{\partial \varphi}{\partial x_i} dx + \int_{\mathbb{R}^N} \Gamma_l \varphi dx = 0,$$

or equivalently

$$\int_{B_k} a_i(x, U_l, \nabla U_l) \frac{\partial \varphi}{\partial x_i} dx + \int_{B_k} \Gamma_l \varphi dx = 0. \tag{5.2.24}$$

The assumptions (A1) and (A2) imply that $A : W^{1,p}(B_k) \to (W^{1,p}(B_k))^*$ is continuous, bounded, and pseudomonotone (see [103]). We have $U_l \rightharpoonup U^*$ in $W^{1,p}(B_k)$ and $\limsup_{n\to\infty}\langle AU_l, U_l - U^*\rangle \le 0$. Due to the pseudomonotonicity it holds $AU_l \rightharpoonup AU^*$ in $(W^{1,p}(B_k))^*$. Along with the weak convergence of $\Gamma_l$ in $L^q(B_k)$ we can pass to the limit in (5.2.24) and obtain

$$\int_{\mathbb{R}^N} a_i(x, U^*, \nabla U^*)\frac{\partial\varphi}{\partial x_i}dx + \int_{\mathbb{R}^N} \Gamma^*\varphi dx = 0. \tag{5.2.25}$$

The statements in (5.2.23) and (5.2.25) show that the pair $(U^*, \Gamma^*)$ is a solution of the problem (5.0.1) in $[\underline{u}, \overline{u}]$. In order to complete the proof we have to prove that $U^*$ is the greatest solution of (5.0.1) in $[\underline{u}, \overline{u}]$. Let $\widetilde{u}$ be any solution of (5.0.1) in the order interval $[\underline{u}, \overline{u}]$. Obviously, the solution $\widetilde{u}$ is also a subsolution of (5.0.1), which implies by the construction in (5.2.1) that the inequality $\widetilde{u} \le U_l \le \overline{u}$ is valid for all $l = 1, 2, \ldots$. This yields $\widetilde{u} \le U_l$, which shows that $U^*$ must be the greatest solution of (5.0.1) in $[\underline{u}, \overline{u}]$. In the same way one can show the existence of a smallest solution. □

**Remark 5.2.2.** *The elliptic inclusion problem with state-dependent subdifferentials investigated by Carl in [15] has the form*

$$Au + \beta(\cdot, u, u) \ni 0 \quad \text{in } \mathcal{D}', \tag{5.2.26}$$

*where $A$ is a general operator of the Leray-Lions type as in (5.0.2) and $\beta(x, u, \cdot) : \mathbb{R} \to 2^{\mathbb{R}} \setminus \emptyset$ is a maximal monotone graph in $\mathbb{R}^2$ depending continuously on the unknown $u$. The multifunction $\beta$ is generated by $f : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ which satisfies the following conditions:*

*(f1)* $(x, r) \to f(x, r, s)$ *is a Carathéodory function uniformly with respect to $s$, which means that $f$ is measurable in $x$ for all $(r, s) \in \mathbb{R} \times \mathbb{R}$ and continuous in $r$ for a.a. $x \in \mathbb{R}^N$ uniformly with respect to $s$.*

*(f2)* $s \to f(x, r, s)$ *is nondecreasing (possibly discontinuous) for a.a. $x \in \mathbb{R}^N$ and for each $r \in \mathbb{R}$, and it is related to the maximal monotone graph $\beta$ by*

$$\beta(x, r, s) = [f(x, r, s - 0), f(x, r, s + 0)], \tag{5.2.27}$$

*where*

$$f(x, r, s \pm 0) = \lim_{\varepsilon\downarrow 0} f(x, r, s \pm \varepsilon).$$

*(f3)* $(x, s) \to f(x, r, s)$ *is measurable in $\mathbb{R}^N \times \mathbb{R}$ for each $r \in \mathbb{R}$.*

*(f4) For a given pair of sub- and supersolutions $\underline{u}, \overline{u}$ satisfying $\underline{u} \le \overline{u}$, there exist a function $k \in \mathcal{L}^q_+$ and a constant $\alpha > 0$ such that*

$$|f(x, r, s)| \le k(x),$$

*for a.a. $x \in \mathbb{R}^N$ and for all $r \in [\underline{u}(x), \overline{u}(x)]$ and $s \in [\underline{u}(x) - \alpha, \overline{u}(x) + \alpha]$.*

*The function $f$ is continuous in the second argument and nondecreasing (possibly discontinuous) in the third argument. Thus, $f \in L^\infty_{\text{loc}}(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R})$ and we can set*

$$j(x, s) = \int_0^s f(x, t, t)dt, \tag{5.2.28}$$

*which yields that the function $s \mapsto j(x, s)$ is locally Lipschitz and Clarke's generalized gradient can be represented by $\partial j(x, s) = \beta(x, s, s)$ (for more details see [39]). Hence, this chapter extends the results in [15] for more general multifunction in form of Clarke's generalized gradients in all of $\mathbb{R}^N$.*

## 5.3 Construction of Sub- and Supersolutions

In this section we give some conditions to find sub- and supersolutions of problem (5.0.1). As a special case, we consider problem (5.0.1) for $A = -\Delta_p$, where $-\Delta_p$ stands for the negative $p$-Laplacian. The main idea is to use the eigenvalues and the corresponding eigenfunctions of the $p$-Laplacian on bounded domains with Dirichlet boundary values. We denote by $\lambda_1$ the first eigenvalue of the $p$-Laplacian on the ball $B_r$ with radius $r$ related to its eigenfunction $\varphi_1$. This means, $\varphi_1$ satisfies the equation

$$\begin{aligned} -\Delta_p u &= \lambda_1 |u|^{p-2}u \quad && \text{in } B_r, \\ u &= 0 && \text{on } \partial B_r. \end{aligned} \tag{5.3.1}$$

In view of the results of Anane in [3], it is well known that $\lambda_1$ is positive and $\varphi_1 \in \text{int}(C_0^1(\overline{B_r})_+)$, where the interior of the positive cone $C_0^1(\overline{B_r})_+$ is given by

$$\text{int}(C_0^1(\overline{B_r})_+) = \left\{ u \in C_0^1(\overline{B_r}) : u(x) > 0, \forall x \in B_r, \text{ and } \frac{\partial u}{\partial \nu}(x) < 0, \forall x \in \partial B_r \right\},$$

where $\frac{\partial u}{\partial \nu}(x)$ means the outer normal derivative. Now, we formulate the hypotheses on Clarke's generalized gradient as follows.

(j4) There exists a Carathéodory function $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, which fulfills

$$\xi \le g(x, s), \quad \forall s \in \mathbb{R}, \text{ for a.a. in } \mathbb{R}^N, \text{ and for all } \xi \in \partial j(x, s), \tag{5.3.2}$$

and has the property

$$\liminf_{s \to +0} \left( -\frac{g(x, s)}{s^{p-1}} \right) > \lambda_1, \tag{5.3.3}$$

uniformly with respect to a.a. $x \in \mathbb{R}^N$. Furthermore, there exists $\widetilde{s} > 0$ such that

$$\partial j(x, \widetilde{s}) \ge 0, \qquad \text{for a.a. } x \in \mathbb{R}^N. \tag{5.3.4}$$

**Proposition 5.3.1.** *Let the conditions (j1), (j2) and (j4) be satisfied. Then there exists a positive ordered pair of sub- and supersolutions of problem (5.0.1) given by*

$$\underline{u}(x) = \begin{cases} \varepsilon\varphi_1(x) & \text{if } x \in B_r \\ 0 & \text{if } x \in \mathbb{R}^N \setminus B_r, \end{cases} \qquad \overline{u}(x) = \widetilde{s}, \quad \text{for a.a. } x \in \mathbb{R}^N, \tag{5.3.5}$$

*provided that $\varepsilon > 0$ is sufficiently small.*

**Proof.** The eigenfunction $\varphi_1$ of (5.3.1) belongs to $\operatorname{int}(C_0^1(\overline{B_r})_+)$, that means in particular, the outer normal derivative $\partial\varphi_1/\partial\nu$ on $\partial B_r$ has a negative sign. By the Divergence Theorem we have for $\varphi \in \mathcal{D}_+$

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla \underline{u}|^{p-2}\nabla \underline{u}\nabla\varphi dx \\
&\quad = \int_{B_r} |\nabla(\varepsilon\varphi_1)|^{p-2}\nabla(\varepsilon\varphi_1)\nabla\varphi dx \\
&\quad = \int_{\partial B_r} |\nabla(\varepsilon\varphi_1)|^{p-2}(\partial(\varepsilon\varphi_1)/\partial\nu)\varphi dx + \int_{B_r} \lambda_1(\varepsilon\varphi_1)^{p-1}\varphi dx \\
&\quad \le \int_{B_r} \lambda_1(\varepsilon\varphi_1)^{p-1}\varphi dx \\
&\quad = \int_{\mathbb{R}^N} \lambda_1 \underline{u}^{p-1}\varphi dx.
\end{aligned}$$

This calculation along with (5.3.2) and (5.3.3) yields for $\underline{\gamma} \in \partial j(\cdot, \varepsilon\varphi_1)$

$$-\Delta_p(\varepsilon\varphi_1) + \underline{\gamma} \le \lambda_1(\varepsilon\varphi_1)^{p-1} + g(\cdot, \varepsilon\varphi_1) \le 0$$

assumed $\varepsilon$ is sufficiently small. Due to (5.3.4) it follows directly that $\overline{u} = \widetilde{s}$ is a positive constant supersolution of (5.0.1). Choosing $\varepsilon$ small enough such that $\underline{u} \le \overline{u}$ completes the proof. □

Notice that the sub- and supersolutions obtained in Proposition 5.3.1 guarantee that condition (j3) is satisfied, too. Hence, Theorem 5.2.1 is applicable and provides the existence of a nontrivial extremal solution $u$ of (5.0.1) belonging to the order interval $[\underline{u}, \overline{u}]$ of sub- and supersolutions given in (5.3.5).

**Example 5.3.2.** *Let $\lambda > \lambda_1$ be fixed and let $j(x, \cdot) : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function satisfying (j1) and (j2) given by*

$$j(x,s) = \begin{cases} -\lambda e^{s-2} - \lambda s - \operatorname{sgn}(s)\dfrac{|x|+2}{p(|x|+1)}|s|^p, & \text{if } s \le 2, \\[2ex] -\dfrac{1}{2}\lambda s^2 + 4\lambda s - 9\lambda - \dfrac{|x|+2}{p(|x|+1)}s^p, & \text{if } 2 \le s \le 3, \\[2ex] -\lambda e^{-s+3} + \lambda s - \dfrac{7}{2}\lambda - \dfrac{|x|+2}{p(|x|+1)}s^p, & \text{if } s \ge 3. \end{cases} \tag{5.3.6}$$

*Its generalized Clarke's gradient has the form*

$$\partial j(x,s)=\begin{cases}-\lambda e^{s-2}-\lambda-\dfrac{|x|+2}{|x|+1}|s|^{p-1}, & \text{if } s<2,\\[2mm] \left[-2\left(\lambda+\dfrac{|x|+2}{|x|+1}2^{p-2}\right),2\left(\lambda-\dfrac{|x|+2}{|x|+1}2^{p-2}\right)\right], & \text{if } s=2,\\[2mm] -\lambda s+4\lambda-\dfrac{|x|+2}{|x|+1}s^{p-1}, & \text{if } 2<s<3,\\[2mm] \left[\lambda-\dfrac{|x|+2}{|x|+1}3^{p-1},2\lambda-\dfrac{|x|+2}{|x|+1}3^{p-1}\right], & \text{if } s=3,\\[2mm] \lambda e^{-s+3}+\lambda-\dfrac{|x|+2}{|x|+1}s^{p-1}, & \text{if } s>3.\end{cases} \tag{5.3.7}$$

*One easily verifies that $\partial j(x,\cdot)$ satisfies the condition (j3) and is bounded above by a Carathéodory function $g:\Omega\times\mathbb{R}\to\mathbb{R}$ defined as*

$$g(x,s)=\begin{cases}|s|-\dfrac{|x|+2}{|x|+1}|s|^{p-1}, & \text{if } s\le 0,\\[2mm] -\left(\lambda+\dfrac{|x|+2}{|x|+1}\right)s^{p-1}, & \text{if } 0\le s\le 1,\\[2mm] 3\lambda s-4\lambda-\dfrac{|x|+2}{|x|+1}s^{p-1}, & \text{if } 1\le s\le 2,\\[2mm] s+2(\lambda-1)-\dfrac{|x|+2}{|x|+1}s^{p-1}, & \text{if } s\ge 2.\end{cases} \tag{5.3.8}$$

*Since $g$ fulfills property (5.3.3), there exists a positive pair of sub- and supersolutions given by (5.3.5) and thus, we obtain a nontrivial positive solution $u\in[\underline{u},\overline{u}]$ of problem (5.0.1).*

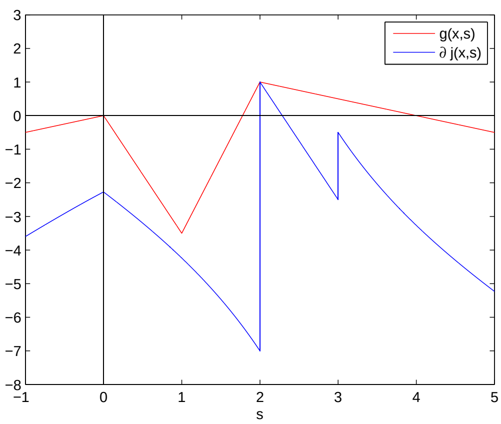


**Figure 5.1.** *The function $g$ and Clarke's generalized gradient $\partial j(x,s)$ in case $\Omega=(0,\pi)$, $\varphi_1(x)=sin(x)$, $\lambda_1=1$, $\lambda=2$, $p=2$ and $x=1$*

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## Personal Data

| | |
|---|---|
| Name | Patrick Winkert |
| Date of Birth | 8th April 1981 |
| Place of Birth | Halle (Saale) |
| Nationality | German |
| Address | Windthorststraße 25, 06114 Halle (Saale) |

## Education

| | |
|---|---|
| 1987 - 1991 | Primary school *Heide Nord* in Halle (Saale) |
| 1991 - 1999 | Grammar school *Thomas-Müntzer* in Halle (Saale), A-level |
| 2000 - 2001 | Civilian service at the Institute of Pathology, *Martin-Luther-University Halle-Wittenberg* |
| 2001 - 2006 | Studies in mathematics (major subject) and informatics (minor subject) *Martin-Luther-University Halle-Wittenberg* |
| 2006 | Diploma (Master) |
| 2006 - 2008 | Ph.D student at the Institute of Mathematics *Martin-Luther-University Halle-Wittenberg* Graduate scholarship assigned by the federal state *Sachsen-Anhalt* |
| 2008 - 2009 | Research assistant at the Institute of Mathematics *Martin-Luther-University Halle-Wittenberg* |

## Awards

| | |
|---|---|
| 2003 | Awarded by the *German Mathematical Society* for the intermediate diploma (master) in mathematics |
| 2006 | Award of the *German Mathematical Society* for course achievement |
| 2007 | Promotion price, awarded by the *Georg Cantor Society* for the diploma (master) in mathematics |

## Publications


1. S. Carl, P. Winkert, *General Comparison Principle For Variational-Hemivariational Inequalities*, to appear in Journal of Inequalities and Applications, 2009.
2. P. Winkert, *Entire Extremal Solutions for Elliptic Inclusions of Clarke's Gradient Type*, to appear in Journal for Analysis and its Applications, 2009.
3. P. Brückmann, P. Winkert, *T-symmetrical Tensor Differential Forms with Logarithmic Poles along a Hypersurface Section*, International Journal of Pure and Applied Mathematics 46 (2008), no. 1, 111-136.
4. P. Winkert, *Discontinuous Variational-Hemivariational Inequalities involving the p-Laplacian*, Journal of Inequalities and Applications, vol. 2007, Article ID 13579, 11 pages, 2007.

## Erklärung

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Arbeit selbständig, ohne unzulässige Hilfe Dritter und ohne Benutzung anderer als der angegebenen Quellen und Hilfsmittel angefertigt habe. Die aus anderen Werken wörtlich oder inhaltlich entnommenen Daten, Fakten und Konzepte sind unter Angabe der entprechenden Quelle als solche gekennzeichnet.

Diese Arbeit wurde bisher weder im In- noch im Ausland in gleicher oder ähnlicher Form in einem anderen Prüfungsverfahren vorgelegt.

Halle (Saale), 15. April 2009

Patrick Winkert