De Giorgi-Nash-Moser estimates for evolutionary partial integro-differential equations

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Chapter 1

Introduction

1.1 Evolutionary partial integro-differential equations

The present contribution is devoted to the study of some classes of linear and quasilinear partial integro-differential equations which can be all written in the general form

$$\partial_t \int_0^t k(t-\tau)\Big(u(\tau,x)-u_0(x)\Big)\,d\tau - \operatorname{div} a\big(t,x,u(t,x),Du(t,x)\big)$$
$$= b\big(t,x,u(t,x),Du(t,x)\big), \quad t\in(0,T),\, x\in\Omega. \tag{1.1}$$

Here $T>0$, Ω is a bounded domain in $\mathbb{R}^N$, $u:[0,T]\times\Omega\to\mathbb{R}$ is the unknown, and Du stands for the gradient of u with respect to the spatial variables. The function u_0 is given and plays the role of the initial data for u, that is $u|_{t=0}=u_0$ in Ω.

The kernel $k\in L_{1,\,loc}(\mathbb{R}_+)$ is assumed to be of type $\mathcal{PC}$ by which we mean the following:

(K) k is nonnegative and nonincreasing, and there exists a kernel $l\in L_{1,\,loc}(\mathbb{R}_+)$ such that $k*l=1$ in $(0,\infty)$.

Here and in what follows $k*v$ denotes the convolution on the positive halfline with respect to the time variable, that is, $(k*v)(t)=\int_0^t k(t-\tau)v(\tau)\,d\tau$, $t\geq 0$. If $k\in L_{1,\,loc}(\mathbb{R}_+)$ satisfies (K) we also write $(k,l)\in\mathcal{PC}$. Note that $(k,l)\in\mathcal{PC}$ implies that l is completely positive, in particular l is nonnegative, cf. [15, Theorem 2.2 and Proposition 2.1].

An important example is given by

$$k(t)=g_{1-\alpha}(t)e^{-\gamma t} \quad \text{and} \quad l(t)=g_\alpha(t)e^{-\gamma t}+\gamma(1*[g_\alpha(\cdot)e^{-\gamma\cdot}])(t), \quad t>0, \tag{1.2}$$

where $\alpha\in(0,1)$, $\gamma\geq 0$, and g_β denotes the Riemann-Liouville kernel

$$g_\beta(t)=\frac{t^{\beta-1}}{\Gamma(\beta)},\;\; t>0,\;\beta>0.$$

Both kernels in (1.2) are strictly positive and decreasing; observe that $\dot{l}(t)=\dot{g}_\alpha(t)e^{-\gamma t}<0$, $t>0$. Their Laplace transforms are given by

$$\hat{k}(\lambda)=\frac{1}{(\lambda+\gamma)^{1-\alpha}},\quad \hat{l}(\lambda)=\frac{1}{(\lambda+\gamma)^{\alpha}}\Big(1+\frac{\gamma}{\lambda}\Big),\quad \operatorname{Re}\lambda>0,$$

which shows that $k*l=1$ on $(0,\infty)$. Hence we have both $(k,l)\in\mathcal{PC}$, and $(l,k)\in\mathcal{PC}$.

In the special case (1.2) equation (1.1) amounts to a *time fractional* equation of order $\alpha \in (0,1)$. Specializing further, by putting $\gamma = 0$ in (2.1), we obtain the pair $(k,l) = (g_{1-\alpha}, g_\alpha)$, $\alpha \in (0,1)$. In this situation the integro-differential operator in (1.1) becomes the Riemann-Liouville fractional derivation operator of order $\alpha \in (0,1)$ defined by

$$\partial_t^\alpha v = \partial_t (g_{1-\alpha} * v), \ \ t > 0,$$

for sufficiently smooth functions v. It is the composition of the fractional integration operator $g_{1-\alpha}*$, which has order $-(1-\alpha)$, and the derivation operator ∂_t.

The linear version of (1.1) takes the form

$$\partial_t \Big(k * (u - u_0) \Big) - \mathcal{L}u = f + \operatorname{div} g, \ t \in (0,T), \, x \in \Omega, \tag{1.3}$$

where

$$\mathcal{L}u = \operatorname{div} \Big(A(t,x)Du + b(t,x)u \Big) + \big(c(t,x) | Du \big) + d(t,x)u.$$

Here $A = (a_{ij})$ is $\mathbb{R}^{N \times N}$-valued, b and c take values in $\mathbb{R}^N$, and d is a real-valued function. Further, $(\cdot|\cdot)$ denotes the scalar product in $\mathbb{R}^N$ and the functions $u_0 = u_0(x)$, $f = f(t,x)$, and $g = g(t,x)$ are given data.

A large part of this contribution deals with the time fractional diffusion equation

$$\partial_t^\alpha (u - u_0) - \operatorname{div} \big(A(t,x)Du \big) = f, \quad t \in (0,T), \, x \in \Omega. \tag{1.4}$$

The main point here is that concerning the coefficients a_{ij} we merely assume measurability, boundedness, and a uniform parabolicity condition.

In (1.1), the functions $a : (0,T) \times \Omega \times \mathbb{R}^{N+1} \to \mathbb{R}^N$ and $b : (0,T) \times \Omega \times \mathbb{R}^{N+1} \to \mathbb{R}$ are measurable and satisfy suitable structure conditions, see Section 4.3 below.

Before describing the main objectives and results we discuss some applications of the above equations in mathematical physics.

1.2 Applications in physics

A strong motivation for studying equations of the type (1.1), in particular (1.4), comes from physics. Time fractional diffusion equations are used to model anomalous diffusion, see e.g. the survey [55]. In this context, equations of the type (1.4) are termed *subdiffusion equations* (the time order α lies in $(0,1)$; in the case $\alpha \in (1,2)$, which is not considered here, one speaks of *superdiffusion equations*. While in normal diffusion (described by the heat equation or more general parabolic equations), the mean squared displacement of a diffusive particle behaves like const$\cdot t$ for $t \to \infty$, equation (1.4), usually with $A = Id$, exhibits the behaviour const$\cdot t^\alpha$, for which there is evidence in a diverse number of systems, see [55] and the references therein. Time fractional diffusion equations of time order $\alpha \in (0,1)$ are also closely related to a class of Montroll-Weiss continuous time random walk models where the waiting time density behaves like const$\cdot t^{-\alpha-1}$ for $t \to \infty$, see e.g. [35], [36], [55]. Problems of the type (1.4) are further used to describe diffusion on fractals ([55], [65]), and they also appear in mathematical finance, see e.g. [68].

Another context where equations of the type (1.1) appear is the modelling of dynamic processes in materials with *memory*. They typically arise by combining the usual conservation laws such as balance of energy or balance of momentum with some constitutive laws pertaining to materials with memory. An example is given by the theory of heat conduction with memory, see

[63] and the references therein. Another application is the following special case of a model for the diffusion of fluids in porous media with memory, which has been introduced in [7]:

$$\begin{aligned} \partial_t^\alpha(p-p_0) - \operatorname{div}\big(\kappa(p)Dp\big) &= f,\ t\in(0,T),\ x\in\Omega, \\ p &= 0,\ t\in(0,T),\ x\in\Gamma, \\ p|_{t=0} &= p_0,\ x\in\Omega. \end{aligned} \tag{1.5}$$

Here $\alpha\in(0,1)$, $p=p(t,x)$ denotes the pressure of the fluid, $\kappa=\kappa(p)$ stands for the permeability of the porous medium, and f is related to external sources in the equation of balance of mass. Model (1.5) is obtained by combining the latter equation with a modified version of Darcy's law for the mass flux q which reads

$$q = -\partial_t^{1-\alpha}\big(\kappa(p)Dp\big),$$

and by assuming that the (average) mass of the fluid is proportional to the pressure. We refer to [37], where a more general model is considered.

In the physical literature one also finds models of the type (1.1) where k enjoys property (K) but the dynamics is not fractional in time. An important example is a class of *ultraslow diffusion equations*. In the context of anomalous diffusion these equations are used to model the case where the mean squared displacement has a logarithmic growth. This so-called 'strong anomaly' is encountered in several systems, e.g. in polymer physics, see the recent papers [9], [10], [46], [59], [73]. In [54] ultra slow diffusion equations are obtained when looking at scaling limits of certain continuous time random walk models with random waiting times between jumps. Mathematically, ultra slow diffusion can be described by means of equations of the form (1.3) where the operator $\partial_t(k*u)$ is a so-called *distributed order derivative*, that is, the kernel k takes the form

$$k(t) = \int_0^1 g_{1-\alpha}(t)\mu(\alpha)\,d\alpha, \quad t>0. \tag{1.6}$$

Here $\mu:[0,1]\to[0,\infty)$ is a weight function, it is different from zero on a set of positive measure. It can be shown that under appropriate conditions on μ the kernel k in (1.6) is of type $\mathcal{PC}$, see [46].

1.3 Main objectives and literature

The main goal of this work is to develop a theory of weak solutions for problems of the form (1.3) and (1.1) and to study the regularity problem in the time fractional case.

To be more specific, let us consider the linear time fractional diffusion equation (1.4), which is the prototypical example. As already mentioned before, we only want to assume that the coefficients a_{ij} be measurable and bounded, that is, $A\in L_\infty((0,T)\times\Omega;\mathbb{R}^{N\times N})$, and that they satisfy a uniform parabolicity condition, that is, there exists a constant $\nu>0$ such that

$$\big(A(t,x)\xi|\xi\big)\ge\nu|\xi|^2, \quad \text{for a.a. } (t,x)\in(0,T)\times\Omega, \text{ and all } \xi\in\mathbb{R}^N.$$

A first important question is to find a suitable notion of weak solution for equation (1.4). One problem is to understand the role of the initial condition $u|_{t=0}=u_0$ in Ω. If this condition cannot be given a direct meaning within the weak setting, then at least one should require that it is automatically satisfied whenever the weak solution is smooth.

Having found an appropriate weak setting, the next task then consists in proving existence and uniqueness for (1.4), together with a suitable boundary condition like, e.g., in the associated Dirichlet problem. To achieve this, suitable energy estimates are required.

In the elliptic and classical parabolic case it is well-known that for second-order problems the maximum principle is valid. Moreover, there exists a powerful theory of a priori estimates, often referred to as De Giorgi-Nash-Moser theory, which provides local and global estimates for weak solutions of the respective equations such as local and global boundedness, the Harnack and the weak Harnack inequality, and Hölder continuity of weak solutions, see [29], [34] for the elliptic and [49], [51] for the parabolic case. Hölder estimates are of utmost significance for the study of quasilinear problems. In fact, in the elliptic case their discovery opened up the theory of quasilinear equations in higher dimensions; in the parabolic case they among others allow to prove global in time existence.

Hölder continuity of weak solutions to elliptic equations in divergence form with discontinuous coefficients was proved by De Giorgi [19] and, independently, by Nash [60]. Nash also obtained the corresponding result for parabolic equations. Another seminal contribution was made by Moser [56], who found a new proof of the De Giorgi-Nash theorem by means of the Harnack inequality. Later, Moser ([57], [58]) also established the parabolic version of the Harnack inequality for positive weak solutions of equations in divergence form, and using this result he was able to give a different proof for Nash's regularity result in the parabolic case. In the non-divergence case corresponding results were obtained by Krylov and Safonov [47], [48]. Concerning Harnack and Hölder estimates for degenerate and singular parabolic equations we refer to [20], [22], [23], [24], and [25].

Since the time fractional case with $\alpha \in (0,1)$ can be viewed in some sense as an intermediate case between the elliptic ($\alpha = 0$) and the classical parabolic case ($\alpha = 1$), one might conjecture that corresponding results can also be obtained in the time fractional situation. However, there is a significant difference to the cases $\alpha = 0$ and $\alpha = 1$: the time fractional equations are *non-local*, due to the non-local nature of the operator ∂_t^α in time. This feature complicates the matter considerably, as the theory described above essentially relies on *local* estimates. Another difficulty consists in the lack of a simple calculus for integro-differential operators like ∂_t^α. In particular one needs a kind of chain rule for such operators in order to use the test-function method, the latter being the basic tool for deriving a priori bounds for weak solutions of equations in divergence form.

One of the main objectives of this contribution is to prove a time fractional analogue of the classical parabolic version of the celebrated De Giorgi-Nash theorem, that is, we want to show that under appropriate assumptions on the data f and u_0 any weak solution of the time fractional diffusion equation (1.4) with arbitrary $\alpha \in (0,1)$ is Hölder continuous in the interior of the parabolic cylinder $(0,T)\times\Omega$. We are further interested in conditions which ensure Hölder continuity up to the parabolic boundary. Having achieved this it is another goal to demonstrate the strength of these regularity results by establishing global strong well-posedness for a quasilinear time fractional problem.

Besides the regularity problem we are further interested in Harnack estimates for time fractional diffusion equations. Does there hold a weak Harnack inequality for nonnegative supersolutions, and if so, what is the optimal exponent? In the purely time-dependent case, that is, for scalar equations of the form

$$\partial_t^\alpha (u - u_0) + \sigma u = 0, \quad t \in (0,T),$$

with $\sigma \geq 0$, a weak Harnack inequality with optimal exponent $1/(1-\alpha)$ has been proved by the author in [78]. Much more difficult seems to be the question whether a full Harnack inequality holds for nonnegative solutions of the time fractional equation (1.4) with $f = 0$. Note that in the literature, neither the weak nor the full Harnack inequality are known to hold in the time fractional case even with the Laplacian, that is, when $A(t,x) = Id$.

Concerning the case of rough coefficients, to the author's knowledge nothing seems to be known for time fractional diffusion equations like (1.4). The main obstruction for the existing results to be applicable is that we merely assume measurability with respect to t of the leading coefficients. On the other hand there exist many papers where equations of the type (1.4), as well as nonlinear or abstract variants of them are studied in a *strong(er)* setting, assuming more smoothness on the coefficients and nonlinearities, see e.g. [3], [13], [14], [17], [27], [31], [63], [81], [82]. Note that (1.4) can be rewritten as

$$u - g_\alpha * \operatorname{div}\big(ADu\big) = u_0 + g_\alpha * f, \quad t \in (0,T),\, x \in \Omega,$$

which is a parabolic Volterra equation, so that for regular coefficients the theory in, e.g., [63] and [81] yields results on existence, uniqueness and regularity. In particular there exists a fully developed L_p-theory (see also Theorem 2.8.1 below) as well as optimal Schauder estimates. In some papers generalized solutions are constructed for problems of the type (1.1) where the quasilinear term is not allowed to depend explicitly on t, see [31] and [38]. These results are based on the theory of accretive operators.

We further remark that concerning non-local operators there exists a very active field of research which deals with integro-differential operators the prototype of which is $(-\Delta)^\alpha$, $\alpha \in (0,1)$. These operators are closely connected to purely jump processes. We refer to [1], [2], [6], [39], [40], [43], [70], [72] and the references given therein for Harnack and Hölder estimates for harmonic functions with respect to this type of operators.

1.4 Overview

We give now an overview of the contents of this contribution and describe the main results and some of the principal ideas behind them.

Chapter 2 is devoted to preliminaries and collects the basic tools needed for our investigation of the problems to be studied. After fixing some notation we first discuss some basic properties of kernels of type $\mathcal{PC}$ and give some more examples. An important issue is how such kernels can be suitably approximated by more regular kernels which are also nonnegative and nonincreasing. This can be achieved by means of the Yosida approximations B_n, $n \in \mathbb{N}$, of the operator B defined by $Bu = \frac{d}{dt}(k * u)$, e.g. in $L_2([0,T])$. In fact, one can show that $B_n u = \frac{d}{dt}(k_n * u)$, where k_n has the desired properties and $k_n \to k$ in $L_1([0,1])$ as $n \to \infty$. A crucial tool for our approach is provided by the *fundamental identity* (2.6), stated in Lemma 2.3.1. This identity gives a formula for expressions of the form $H'(u)\frac{d}{dt}(k * u)$ and can be viewed as a kind of 'chain rule' for the operator B. In Section 2.4 we state two lemmas on the geometrically fast convergence to zero of sequences of numbers satisfying certain recursive inequalities. These lemmas are needed for the De Giorgi technique, which will be used in the proofs of L_∞-bounds and regularity of weak solutions. Section 2.5 provides the basic tools of Moser's iteration technique including an abstract lemma of Bombieri and Giusti which is very useful for accomplishing the 'crossover at zero'. This method will be later used in the proof of the weak Harnack inequality. The remaining part of Chapter 2 is devoted to weighted Poincaré inequalities and parabolic embeddings, and we state a result about maximal L_p-regularity for linear time fractional initial-boundary value problems of second order.

In *Chapter 3* we address the problem of existence of weak solutions to linear equations in the case of rough coefficients. This is done in the canonical Hilbert space setting. Let $\mathcal{V}$ and $\mathcal{H}$ be real separable Hilbert spaces such that $\mathcal{V} \hookrightarrow H$ densely and identify $\mathcal{H}$ with its dual $\mathcal{H}'$, that is, $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}'$. We study the abstract problem

$$\frac{d}{dt}\Big([k * (u - x)](t), v\Big)_{\mathcal{H}} + a\big(t, u(t), v\big) = \big\langle f(t), v\big\rangle_{\mathcal{V}' \times \mathcal{V}}, \quad v \in \mathcal{V},\ \text{a.a.}\, t \in (0,T), \tag{1.7}$$

where d/dt means the generalized derivative of real functions on $(0,T)$, k is a kernel of type $\mathcal{PC}$, $x \in \mathcal{H}$ and $f \in L_2([0,T]; \mathcal{V}')$ are given data, and $a : (0,T) \times \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ is measurable w.r.t.

t and a bounded $\mathcal{V}$-coercive bilinear form w.r.t. the second and third argument. By means of the Galerkin method and the Hilbert space version of a special case of the fundamental identity, relation (2.8), we prove existence and uniqueness of a solution u of (1.7) in the class

$$W(x, \mathcal{V}, \mathcal{H}) = \{u \in L_2([0,T]; \mathcal{V}) : \; k * (u - x) \in {}_0H_2^1([0,T]; \mathcal{V}')\},$$

where the zero means vanishing trace at $t = 0$, see Theorem 3.3.1. This result can be viewed as the analogue of the well-known existence and uniqueness result for the corresponding abstract parabolic equation

$$\begin{cases} \frac{d}{dt}\big(u(t), v\big)_{\mathcal{H}} + a\big(t, u(t), v\big) = \big\langle f(t), v\big\rangle_{\mathcal{V}' \times \mathcal{V}}, \quad v \in \mathcal{V}, \text{ a.a. } t \in (0,T), \\ u(0) = x \;\in \mathcal{H}, \\ u \in H_2^1([0,T]; \mathcal{V}') \cap L_2([0,T]; \mathcal{V}), \end{cases}$$

see e.g. Theorem 4.1 and Remark 4.3 in Chapter 4 in [52] or [87, Section 23].

Besides the existence result we also prove some useful interpolation results for functions in the class $W(x, \mathcal{V}, \mathcal{H})$ with $x \in \mathcal{H}$ and $(k,l) \in \mathcal{PC}$. It is shown that for $u \in W(x, \mathcal{V}, \mathcal{H})$ the function $k * u$ belongs to the space $C([0,T]; \mathcal{H})$, and if in addition $l \in L_{p,w}([0,T])$, the weak L_p space, for some $p > 1$ then $u \in L_{2p,w}([0,T]; \mathcal{H})$. The corresponding statement without 'weak' is true as well. We finally apply these results to time fractional diffusion equations like (1.4) with measurable coefficients and establish the unique (weak) solvability of the corresponding Dirichlet problem in the class

$$u \in L_{\frac{2}{1-\alpha}, w}([0,T]; L_2(\Omega)) \cap L_2([0,T]; \mathring{H}_2^1(\Omega)), \text{ with } u - u_0 \in {}_0H_2^\alpha([0,T]; H_2^{-1}(\Omega))\}.$$

For the solution we additionally obtain $g_{1-\alpha} * u \in C([0,T]; L_2(\Omega))$ with $g_{1-\alpha} * u|_{t=0} = 0$.

In *Chapter 4* we study linear and quasilinear second-order equations of the type (1.3) and (1.1), respectively. The leading coefficients of $\mathcal{L}$ in (1.3) are merely assumed to be measurable and bounded, and they satisfy a uniform parabolicity condition. The main purpose of the chapter is to show that under appropriate conditions on the data and nonlinearities, respectively, any weak solution is essentially bounded on $(0,T) \times \Omega$, provided it is bounded on the parabolic boundary, see Corollary 4.2.1 and Theorem 4.3.1. Concerning the kernel, the crucial assumption is that $(k,l) \in \mathcal{PC}$ where $l \in L_p([0,T])$ for some $p > 1$. This condition allows to gain higher integrability w.r.t. time from the energy estimates. Our proofs of the L_∞-bounds use De Giorgi's iteration technique and rely on truncated energy estimates, which are derived by means of the fundamental identity (2.6). The latter is only possible after reformulating the problems in a different weak form where the singular kernel k is replaced with the more regular kernel k_n ($n \in \mathbb{N}$) resulting from the Yosida approximation method described above. This is an important technical detail as our method resolves the time regularization problem for equations of the type (1.1) in the weak setting. In the classical parabolic case this can be achieved by means of Steklov averages, a method which no longer works in the case of (1.1).

Chapter 5 deals with the weak Harnack inequality for nonnegative weak supersolutions of the time fractional diffusion equation

$$\partial_t^\alpha (u - u_0) - \operatorname{div}\big(A(t,x)Du\big) = 0, \quad t \in (0,T), \, x \in \Omega, \tag{1.8}$$

where $\alpha \in (0,1)$, $u_0 \in L_2(\Omega)$, and A is as described at the beginning of Section 1.3. To formulate the result, let $B(x,r)$ denote the open ball with radius $r > 0$ centered at $x \in \mathbb{R}^N$, and let λ_N be the Lebesgue measure in $\mathbb{R}^N$. For $\delta \in (0,1)$, $t_0 \geq 0$, $\tau > 0$, and a ball $B(x_0, r)$, define the parabolic cylinders

$$\begin{aligned} Q_-(t_0, x_0, r) &= (t_0, t_0 + \delta\tau r^{2/\alpha}) \times B(x_0, \delta r), \\ Q_+(t_0, x_0, r) &= (t_0 + (2 - \delta)\tau r^{2/\alpha}, t_0 + 2\tau r^{2/\alpha}) \times B(x_0, \delta r). \end{aligned}$$

We will prove that for any $\eta > 1$, $\tau > 0$, $t_0 \geq 0$ and $r > 0$ with $t_0 + 2\tau r^{2/\alpha} \leq T$, any ball $B(x_0, \eta r) \subset \Omega$, any $0 < p < \frac{2+N\alpha}{2+N\alpha-2\alpha}$, and any nonnegative weak supersolution u of (1.8) in $(0, t_0 + 2\tau r^{2/\alpha}) \times B(x_0, \eta r)$ with $u_0 \geq 0$ in $B(x_0, \eta r)$, we have the inequality

$$\Big(\frac{1}{\lambda_{N+1}\big(Q_-(t_0, x_0, r)\big)} \int_{Q_-(t_0,x_0,r)} u^p \, d\lambda_{N+1}\Big)^{1/p} \leq C \operatorname*{ess\,inf}_{Q_+(t_0,x_0,r)} u, \tag{1.9}$$

where the constant $C = C(\nu, \Lambda, \delta, \tau, \eta, \alpha, N, p)$. The proof uses Moser's iteration technique and relies on rather intricate local estimates for powers of u and $\log u$. Once again, the fundamental identity (2.6) is the principal tool in the derivation of these a priori estimates. We also show that the critical exponent $\frac{2+N\alpha}{2+N\alpha-2\alpha}$ is optimal. Note that by sending $\alpha \to 1$ we recover the critical exponent $1 + \frac{2}{N}$ of the weak Harnack inequality in the classical parabolic case. As a simple consequence of the weak Harnack inequality (1.9) we obtain the strong maximum principle for weak subsolutions of (1.8). Another application is a theorem of Liouville type, which says that any bounded weak solution of (1.8) on $\mathbb{R}_+ \times \mathbb{R}^N$ with $u_0 = 0$ vanishes a.e. on $\mathbb{R}_+ \times \mathbb{R}^N$. It will be further shown that in the case $u_0 = 0$ any bounded weak solution u of (1.8) is continuous at $(0, x_0)$ for all $x_0 \in \Omega$ and $\lim_{(t,x)\to(0,x_0)} u(t,x) = 0$, that is, the initial condition $u|_{t=0} = 0$ is satisfied in the classical sense. We point out that the weak Harnack inequality described above is not strong enough to show Hölder continuity for weak solutions of (1.8), the main obstruction being the global positivity assumption (in time). This is a significant difference to the case $\alpha = 1$, where the weak Harnack inequality implies a Hölder estimate for weak solutions.

The regularity problem for weak solutions of (1.4) with $\alpha \in (0, 1)$ and rough coefficients as before is addressed in *Chapter 6*. Here we prove the main result of this contribution, Theorem 6.1.1. It states that for $u_0 \in L_\infty(\Omega)$ and $f \in L_r([0, T]; L_q(\Omega))$ with r and q sufficiently large any bounded weak solution of (1.4) is Hölder continuous in the interior of the parabolic cylinder $(0, T) \times \Omega$. The proof uses De Giorgi's technique and the method of *non-local growth lemmas*, which has been recently developed in [70] for integro-differential operators like the fractional Laplacian. In contrast to the proof of the weak Harnack inequality memory terms that result from time-shifts in the equation cannot be dropped but must be estimated carefully to make the proof work. As before, the fundamental identity (2.6) is frequently used to derive various a priori estimates for u and certain logarithmic expressions involving u. In Chapter 6 we also give sufficient conditions for Hölder continuity up to the parabolic boundary. Here we do not aim at high generality but we are content with finding some simple conditions which are satisfied in the strong L_p-setting considered in the following chapter.

In *Chapter 7* we apply the regularity results from the preceding chapter to prove the unique *global* strong solvability of the quasilinear problem

$$\begin{aligned} \partial_t^\alpha (u - u_0) - \operatorname{div}\big(A(u)Du\big) &= f, \; t \in (0, T), \, x \in \Omega, \\ u &= g, \; t \in (0, T), \, x \in \Gamma, \\ u|_{t=0} &= u_0, \; x \in \Omega, \end{aligned} \tag{1.10}$$

which is a generalization of the model (1.5). Here $\alpha \in (0, 1)$, $T > 0$, $N \geq 2$, and $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^2-smooth boundary Γ. Concerning the nonlinearity it will be assumed that $A \in C^1(\mathbb{R}; \mathbb{R}^{N\times N})$ is symmetric and that there exists $\nu > 0$ such that $(A(y)\xi|\xi) \geq \nu|\xi|^2$ for all $y \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$. Assuming $p > N + \frac{2}{\alpha}$ it will be shown that under suitable regularity and compatibility conditions on the data u_0, f, g problem (1.10) possesses a unique strong solution u in the class

$$u \in H_p^\alpha([0, T]; L_p(\Omega)) \cap L_p([0, T]; H_p^2(\Omega)),$$

see Theorem 7.1.1. The point here is that $T > 0$ can be given arbitrarily large. Note that short-time existence of strong or classical solutions to problems like (1.10) can be established by means of maximal regularity and the contraction mapping principle. This has been known before, see e.g. [14] and [82].

Chapter 8 is devoted to the full Harnack inequality for the Riemann-Liouville fractional derivation operator. So in this part we restrict ourselves to the purely time-dependent case. To describe the main result of this chapter, Theorem 8.1.1, let $t_* \geq 0$, $0 < \sigma_1 < \sigma_2 < \sigma_3$, and $\rho > 0$. Suppose that $\alpha \in (0,1)$ and $u_0 \geq 0$. We will prove that for any sufficiently smooth, nonnegative function u on $(0, t_* + \sigma_3 \rho)$ that satisfies

$$\partial_t^\alpha (u - u_0)(t) = 0, \quad \text{a.a. } t \in (t_*, t_* + \sigma_3 \rho),$$

we have the Harnack inequality

$$\sup_{W_-} u \leq \frac{\sigma_3}{\sigma_1} \inf_{W_+} u, \tag{1.11}$$

where

$$W_- = (t_* + \sigma_1 \rho, t_* + \sigma_2 \rho), \quad W_+ = (t_* + \sigma_2 \rho, t_* + \sigma_3 \rho).$$

We will further show that, similarly to the case of the fractional Laplacian, the Harnack inequality is no longer valid if the global positivity assumption is replaced by a local one. Furthermore, the Harnack estimate fails to hold if the relation $\partial_t^\alpha (u - u_0) = 0$ is only satisfied on the smaller interval $(t_* + \sigma_1 \rho, t_* + \sigma_3 \rho)$. In the last section of the chapter (1.11) is generalized to nonnegative solutions of a class of nonhomogenous fractional differential equations. The results indicate that a full Harnack inequality should also hold for time fractional diffusion equations like (1.8) with $\alpha \in (0,1)$.

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Chapter 2

Preliminaries

2.1 Some notation

We begin by fixing some notation. For $T > 0$ and a bounded domain $\Omega \subset \mathbb{R}^N$ with boundary $\Gamma := \partial\Omega$ let Ω_T denote the cylindrical domain $(0,T) \times \Omega$ and put also $\Gamma_T = (0,T) \times \Gamma$. For $x_0 \in \mathbb{R}^N$ and $r > 0$, by $B(x_0, r)$ and $B_r(x_0)$ we mean the open ball of radius r centered at x_0. The Lebesgue measure in $\mathbb{R}^N$ will be denoted by λ_N or μ_N.

The boundary Γ is said to satisfy the property of *positive geometric density*, if there exist $\beta \in (0,1)$ and $\rho_0 > 0$ such that for any $x_0 \in \Gamma$, any ball $B(x_0,\rho)$ with $\rho \le \rho_0$ we have that $\lambda_N(\Omega \cap B(x_0,\rho)) \le \beta\lambda_N(B(x_0,\rho))$, cf. e.g. [20, Section I.1].

By y_+ we mean the positive part of $y \in \mathbb{R}$, i.e. $y_+ := \max\{y, 0\}$. Note that $y_+^2 := (y_+)^2$. Further, $\mathrm{Sym}\{N\}$ stands for the space of N-dimensional real symmetric matrices. For $s > 0$ and $1 < p < \infty$ the symbols H^s_p and B^s_{pp} refer to Bessel potential (Sobolev spaces for integer s) and Sobolev-Slobodeckij spaces, respectively. Recall that $\mathring{H}^1_2(\Omega) := \overline{C_0^\infty(\Omega)}^{H^1_2(\Omega)}$. The derivative of a kernel $k \in H^1_1([0,T])$ will be usually denoted by $\dot{k}$.

2.2 Kernels of type $\mathcal{PC}$

The following class of kernels has been introduced in [85] and is basic to our treatment of (1.1).

Definition 2.2.1 *A kernel $k \in L_{1,\,loc}(\mathbb{R}_+)$ is called to be* of type $\mathcal{PC}$ *if it is nonnegative and nonincreasing, and there exists a kernel $l \in L_{1,\,loc}(\mathbb{R}_+)$ such that $k * l = 1$ in $(0,\infty)$. In this case, we say that (k,l) is a $\mathcal{PC}$* pair *and write $(k,l) \in \mathcal{PC}$.*

From $(k,l) \in \mathcal{PC}$ it follows that l is completely positive (see e.g. Theorem 2.2 in [15]), in particular l is nonnegative, cf. [15, Proposition 2.1].

Example 2.2.1 An important example is given by

$$k(t) = g_{1-\alpha}(t)e^{-\mu t} \quad \text{and} \quad l(t) = g_\alpha(t)e^{-\mu t} + \mu(1 * [g_\alpha e^{-\mu\cdot}])(t), \quad t > 0, \tag{2.1}$$

with $\alpha \in (0,1)$ and $\mu \ge 0$. As already shown in Section 1.1, we have both $(k,l) \in \mathcal{PC}$, and $(l,k) \in \mathcal{PC}$.

Example 2.2.2 There exist pairs $(k,l) \in \mathcal{PC}$ with the property that for any $p > 1$ and $T > 0$ the kernel l does not belong to $L_p([0,T])$. To construct such a pair, let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers such that $\sum_{n=1}^{\infty}\gamma_n<\infty$. Let further $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of numbers in $(0,1)$ that converges to 0 as $n\to\infty$. We then set

$$l(t)=\sum_{n=1}^{\infty}\gamma_n g_{\alpha_n}(t)e^{-t},\quad t>0.$$

By Euler's integral for the Gamma function,

$$|g_{\alpha_n}(\cdot)e^{-\cdot}|_{L_1(\mathbb{R}_+)}=1,\quad n\in\mathbb{N},$$

and therefore $l\in L_1(\mathbb{R}_+)$ with $|l|_{L_1(\mathbb{R}_+)}=\sum_{n=1}^{\infty}\gamma_n$. Moreover, for every $n\in\mathbb{N}$, $g_{\alpha_n}(t)e^{-t}$ is completely monotone, that is $(-1)^j(g_{\alpha_n}e^{-\cdot})^{(j)}(t)\geq 0$, $t>0$, for $j=0,1,2,\ldots$ Consequently, l enjoys the same property. Furthermore, by Theorem 5.4 in Chapter 5 of [32], the kernel l has a resolvent $k\in L_{1,loc}(\mathbb{R}_+)$ of the first kind, that is $k*l=1$ on $(0,\infty)$, and this resolvent is completely monotone as well. In particular, k is nonnegative and nonincreasing, and so $(k,l)\in\mathcal{PC}$. Since $\alpha_n\to 0$, there do not exist $p>1$ and $T>0$ such that $l\in L_p([0,T])$.

$\mathcal{PC}$ pairs enjoy a useful stability property with respect to exponential shifts. Writing $k_\mu(t)=k(t)e^{-\mu t}$, $t>0$, $\mu\geq 0$, we have

$$(k,l)\in\mathcal{PC}\quad\Rightarrow\quad(k_\mu,l_\mu+\mu(1*l_\mu))\in\mathcal{PC},\ \ \mu\geq 0. \tag{2.2}$$

To prove (2.2), we first note that for any $\mu\geq 0$, k_μ is evidently nonnegative and nonincreasing, and $l_\mu+\mu(1*l_\mu)$ is nonnegative. Multiplying $k*l=1$ by $1_\mu(t)=e^{-\mu t}$ gives $k_\mu*l_\mu=1_\mu$, which in turn implies that $\mu k_\mu*1*l_\mu=\mu 1*1_\mu=1-1_\mu$. Adding these relations, we see that $k_\mu*[l_\mu+\mu(1*l_\mu)]=1$.

We next discuss an important method of approximating kernels of type $\mathcal{PC}$. Let $1\leq p<\infty$, $(k,l)\in\mathcal{PC}$, $T>0$, and X be a real Banach space. Then the operator B defined by

$$Bu=\frac{d}{dt}\,(k*u),\ \ D(B)=\{u\in L_p([0,T];X):\ k*u\in {}_0H^1_p([0,T];X)\},$$

where the zero means vanishing at $t=0$, is known to be m-accretive in $L_p([0,T];X)$, cf. [11], [16], and [31]. Its Yosida approximations B_n, defined by $B_n=nB(n+B)^{-1}$, $n\in\mathbb{N}$, enjoy the property that for any $u\in D(B)$, one has $B_nu\to Bu$ in $L_p([0,T];X)$ as $n\to\infty$. Further, one has the representation

$$B_nu=\frac{d}{dt}\,(k_n*u),\quad u\in L_p([0,T];X),\ n\in\mathbb{N}, \tag{2.3}$$

where $k_n=ns_n$, and s_n is the unique solution of the scalar-valued Volterra equation

$$s_n(t)+n(s_n*l)(t)=1,\quad t>0,\ n\in\mathbb{N},$$

see e.g. [75]. Denoting by $h_n\in L_{1,\,loc}(\mathbb{R}_+)$ the resolvent kernel associated with nl, we have

$$h_n(t)+n(h_n*l)(t)=nl(t),\quad t>0,\ n\in\mathbb{N}, \tag{2.4}$$

and hence, by convolving (2.4) with k,

$$(k*h_n)(t)+n(k*h_n*l)(t)=n,\quad t>0,\ n\in\mathbb{N},$$

which shows that

$$k_n=ns_n=k*h_n,\quad n\in\mathbb{N}. \tag{2.5}$$

Note that complete positivity of l implies that h_n is nonnegative, and that the kernels s_n are nonnegative and nonincreasing for all $n \in \mathbb{N}$, see e.g. [63, Proposition 4.5] and [15, Proposition 2.1]. From $s_n = 1 - 1 * h_n$ we further see that $s_n \in H^1_1([0,T])$. In view of (2.5) we conclude that the kernels k_n, $n \in \mathbb{N}$, are also nonnegative and nonincreasing, and that they belong to $H^1_1([0,T])$.

Note that for any function $f \in L_p([0,T];X)$, $1 \le p < \infty$, there holds $h_n * f \to f$ in $L_p([0,T];X)$ as $n \to \infty$. In fact, defining $u = l * f$, we have $u \in D(B)$, and

$$B_n u = \frac{d}{dt}\,(k_n * u) = \frac{d}{dt}\,(k * l * h_n * f) = h_n * f \;\to\; Bu = f \quad \text{in } L_p([0,T];X)$$

as $n \to \infty$. In particular, $k_n \to k$ in $L_1([0,T])$ as $n \to \infty$.

2.3 A fundamental identity for integro-differential operators of the form $\frac{d}{dt}(k * u)$

We next state a fundamental identity for integro-differential operators of the form $\frac{d}{dt}(k * u)$, cf. also [78], [84]. It can be viewed as the analogue to the chain rule $(H(u))' = H'(u)u'$.

Lemma 2.3.1 *Let $T > 0$ and U be an open subset of $\mathbb{R}$. Let further $k \in H^1_1([0,T])$, $H \in C^1(U)$, and $u \in L_1([0,T])$ with $u(t) \in U$ for a.a. $t \in (0,T)$. Suppose that the functions $H(u)$, $H'(u)u$, and $H'(u)(\dot{k} * u)$ belong to $L_1([0,T])$ (which is the case if, e.g., $u \in L_\infty([0,T])$). Then we have for a.a. $t \in (0,T)$,*

$$\begin{aligned} H'(u(t))\frac{d}{dt}\,(k * u)(t) = &\; \frac{d}{dt}\,\big(k * H(u)\big)(t) + \Big(- H(u(t)) + H'(u(t))u(t)\Big)k(t) \\ &+ \int_0^t \Big(H(u(t-s)) - H(u(t)) - H'(u(t))[u(t-s) - u(t)]\Big)[-\dot{k}(s)]\,ds. \end{aligned} \tag{2.6}$$

The lemma follows from a straightforward computation. Note that in particular identity (2.6) applies to the Yosida approximations of the operator $\frac{d}{dt}(k * u)$ discussed in Section (2.2). We remark that an integrated version of (2.6) can be found in [32, Lemma 18.4.1]. Observe that the last term in (2.6) is nonnegative in case H is convex and k is nonincreasing.

An important example with regard to truncated energy estimates is given by $H(y) = \frac{1}{2}(y_+)^2$, $y \in \mathbb{R}$. Evidently, $H \in C^1(\mathbb{R})$ with derivative $H'(y) = y_+$, $y \in \mathbb{R}$. Assume in addition that the kernel $k \in H^1_1([0,T])$ is nonnegative and nonincreasing. Then it follows from (2.6) and the convexity of H that for any function $u \in L_2([0,T])$,

$$\begin{aligned} u(t)_+\frac{d}{dt}\,(k * u)(t) &\ge \frac{1}{2}\,\frac{d}{dt}\,\Big(k * (u_+)^2\Big)(t) + \frac{1}{2}\,k(t)(u_+)^2(t) \\ &\ge \frac{1}{2}\,\frac{d}{dt}\,\Big(k * (u_+)^2\Big)(t), \quad \text{a.a. } t \in (0,T). \end{aligned} \tag{2.7}$$

The following identity is basic to energy estimates in the Hilbert space setting. For $\mathcal{H} = \mathbb{R}$ it coincides with (2.6) with $H(y) = \frac{1}{2}y^2$, $y \in \mathbb{R}$.

Lemma 2.3.2 *Let $\mathcal{H}$ be a real Hilbert space with scalar product $(\cdot,\cdot)_{\mathcal{H}}$ and $T > 0$. Then for any $k \in H^1_1([0,T])$ and any $v \in L_2([0,T];\mathcal{H})$ there holds*

$$\begin{aligned} \Big(\frac{d}{dt}\,(k * v)(t), v(t)\Big)_{\mathcal{H}} = &\; \frac{1}{2}\,\frac{d}{dt}\,(k * |v(\cdot)|^2_{\mathcal{H}})(t) + \frac{1}{2}\,k(t)|v(t)|^2_{\mathcal{H}} \\ &+ \frac{1}{2}\int_0^t [-\dot{k}(s)]\,|v(t) - v(t-s)|^2_{\mathcal{H}}\,ds, \quad a.a.\ t \in (0,T). \end{aligned} \tag{2.8}$$

The subsequent two lemmas are also obtained by simple algebra. The first one can be viewed as a product rule for expressions of the form $\frac{d}{dt}(k * [u_1 u_2])$.

Lemma 2.3.3 *Let $T > 0$, $k \in H_1^1([0,T])$, $v \in L_1([0,T])$, and $\varphi \in C^1([0,T])$. Then*

$$\varphi(t)\,\frac{d}{dt}\,(k * v)(t) = \frac{d}{dt}\,\big(k * [\varphi v]\big)(t) + \int_0^t \dot{k}(t-\tau)\big(\varphi(t)-\varphi(\tau)\big)v(\tau)\,d\tau, \quad a.a.\ t \in (0,T).$$

Lemma 2.3.4 *Let $T > 0$ and $\alpha \in (0,1)$. Suppose that $v \in {}_0H_1^1([0,T])$ and $\varphi \in C^1([0,T])$. Then*

$$\big(g_\alpha * (\varphi \dot{v})\big)(t) = \varphi(t)(g_\alpha * \dot{v})(t) + \int_0^t v(\sigma)\partial_\sigma\big(g_\alpha(t-\sigma)[\varphi(t)-\varphi(\sigma)]\big)\,d\sigma, \quad a.a.\ t \in (0,T).$$

If in addition v is nonnegative and φ is nondecreasing there holds

$$\big(g_\alpha * (\varphi \dot{v})\big)(t) \ge \varphi(t)(g_\alpha * \dot{v})(t) - \int_0^t g_\alpha(t-\sigma)\dot{\varphi}(\sigma)v(\sigma)\,d\sigma, \quad a.a.\ t \in (0,T).$$

2.4 Auxiliary lemmas on fast geometric convergence

The following lemmas concerning the geometric convergence of sequences of numbers will be needed for the De Giorgi iteration arguments in Chapter 4 and 6. The first can be found, e.g., in [49, Chapter II, Lemma 5.6] and [20, Chapter I, Lemma 4.1]. Its proof is by induction.

Lemma 2.4.1 *Let $\{Y_n\}$, $n = 0,1,2,\ldots$, be a sequence of positive numbers, satisfying the recursion inequality*

$$Y_{n+1} \le C b^n Y_n^{1+\gamma}, \quad n = 0,1,2,\ldots,$$

where C, $b > 1$ and $\gamma > 0$ are given numbers. If

$$Y_0 \le C^{-1/\gamma} b^{-1/\gamma^2},$$

then $Y_n \to 0$ as $n \to \infty$.

The next result has been taken from [20, Chap. I, Lemma 4.2], see also [49].

Lemma 2.4.2 *Let $\{Y_n\}$ and $\{Z_n\}$, $n = 0,1,2,\ldots$, be sequences of positive numbers, satisfying the recursive inequalities*

$$\begin{aligned} Y_{n+1} &\le C b^n \big(Y_n^{1+\beta} + Y_n^\beta Z_n^{1+\gamma}\big), \\ Z_{n+1} &\le C b^n \big(Y_n + Z_n^{1+\gamma}\big) \end{aligned}$$

for $n = 0,1,2,\ldots$, where C, $b > 1$ and $\beta, \gamma > 0$ are given numbers. If

$$Y_0 + Z_0^{1+\gamma} \le (2C)^{-\frac{1+\gamma}{\delta}} b^{-\frac{1+\gamma}{\delta^2}}, \quad \text{with } \delta = \min\{\beta,\gamma\},$$

then $\{Y_n\}$ and $\{Z_n\}$ tend to zero as $n \to \infty$.

2.5 Moser iterations and an abstract lemma of Bombieri and Giusti

Throughout this section U_σ, $0<\sigma\le 1$, will denote a collection of measurable subsets of a fixed finite measure space endowed with a measure μ, such that $U_{\sigma'}\subset U_\sigma$ if $\sigma'\le\sigma$. For $p\in(0,\infty)$ and $0<\sigma\le 1$, $L_p(U_\sigma)$ stands for the Lebesgue space $L_p(U_\sigma,d\mu)$ of all μ-measurable functions $f:U_\sigma\to\mathbb{R}$ with $|f|_{L_p(U_\sigma)}:=(\int_{U_\sigma}|f|^p\,d\mu)^{1/p}<\infty$.

The following two lemmas are basic to Moser's iteration technique. The arguments in their proofs have been repeatedly used in the literature (see e.g. [29], [51], [56], [57], [66], [74]), so it is worthwhile to formulate them as lemmas in abstract form, also for future reference. We provide proofs for the sake of completeness.

The first Moser iteration result reads as follows, see also [18, Lemma 2.3].

Lemma 2.5.1 *Let $\kappa>1$, $\bar p\ge 1$, $C\ge 1$, and $\gamma>0$. Suppose f is a μ-measurable function on U_1 such that*

$$|f|_{L_{\beta\kappa}(U_{\sigma'})}\le\Big(\frac{C(1+\beta)^\gamma}{(\sigma-\sigma')^\gamma}\Big)^{1/\beta}|f|_{L_\beta(U_\sigma)},\quad 0<\sigma'<\sigma\le 1,\ \beta>0. \tag{2.9}$$

Then there exist constants $M=M(C,\gamma,\kappa,\bar p)$ and $\gamma_0=\gamma_0(\gamma,\kappa)$ such that

$$\operatorname*{ess\,sup}_{U_\delta}|f|\le\Big(\frac{M}{(1-\delta)^{\gamma_0}}\Big)^{1/p}|f|_{L_p(U_1)}\quad \textit{for all}\ \delta\in(0,1),\ p\in(0,\bar p].$$

Proof: For $q>0$ and $0<\sigma\le 1$, let

$$\Phi(q,\sigma)=(\int_{U_\sigma}|f|^q\,d\mu)^{1/q}.$$

Let $0<p\le\bar p$ and $\delta\in(0,1)$. Set $p_i=p\kappa^i$, $i=0,1,\ldots$ and define the sequence $\{\sigma_i\}$, $i=0,1,\ldots$, by $\sigma_0=1$ and $\sigma_i=1-\sum_{j=1}^i 2^{-j}(1-\delta)$, $i=1,2,\ldots$; observe that $1=\sigma_0>\sigma_1>\ldots>\sigma_i>\sigma_{i+1}>\delta$ as well as $\sigma_{i-1}-\sigma_i=2^{-i}(1-\delta)$, $i\ge 1$. Suppose now $n\in\mathbb{N}$. By using (2.9) with $\beta=p_i$, $i=0,1,\ldots,n-1$, we obtain

$$\begin{aligned}\Phi(p_n,\delta)\ &\le\Phi(p_n,\sigma_n)\ =\ \Phi(p_{n-1}\kappa,\sigma_n)\ \le\ \Big(\frac{C(1+p\kappa^{n-1})^\gamma}{[2^{-n}(1-\delta)]^\gamma}\Big)^{\frac1p\kappa^{-(n-1)}}\Phi(p_{n-1},\sigma_{n-1})\\
&\le\Big(\frac{C(2\bar p\kappa^{n-1})^\gamma}{[2^{-n}(1-\delta)]^\gamma}\Big)^{\frac1p\kappa^{-(n-1)}}\Phi(p_{n-1},\sigma_{n-1})\\
&\le\Big(\frac{\tilde C(C,\bar p,\gamma)^n\kappa^{\gamma(n-1)}}{(1-\delta)^\gamma}\Big)^{\frac1p\kappa^{-(n-1)}}\Phi(p_{n-1},\sigma_{n-1})\ \le\ \ldots\\
&\le\Big(\tilde C^{\sum_{j=0}^{n-1}(j+1)\kappa^{-j}}\kappa^{\gamma\sum_{j=0}^{n-1}j\kappa^{-j}}(1-\delta)^{-\gamma\sum_{j=0}^{n-1}\kappa^{-j}}\Big)^{1/p}\Phi(p_0,\sigma_0)\\
&\le\Big(\frac{M(C,\bar p,\gamma,\kappa)}{(1-\delta)^{\frac{\gamma\kappa}{\kappa-1}}}\Big)^{1/p}\Phi(p,1).\end{aligned}$$

We let now n tend to ∞ and use the fact that

$$\lim_{n\to\infty}\Phi(p_n,\delta)=\operatorname*{ess\,sup}_{U_\delta}|f|$$

to get

$$\operatorname*{ess\,sup}_{U_\delta}|f|\le\Big(\frac{M(C,\bar p,\gamma,\kappa)}{(1-\delta)^{\frac{\gamma\kappa}{\kappa-1}}}\Big)^{1/p}|f|_{L_p(U_1)}.$$

Hence the proof is complete. □

The second Moser iteration result is the following, see also [18, Lemma 2.5].

Lemma 2.5.2 *Assume that $\mu(U_1) \leq 1$. Let $\kappa > 1$, $0 < p_0 < \kappa$, and $C \geq 1$, $\gamma > 0$. Suppose f is a μ-measurable function on U_1 such that*

$$|f|_{L_{\beta\kappa}(U_{\sigma'})} \leq \Big(\frac{C}{(\sigma-\sigma')^\gamma}\Big)^{1/\beta} |f|_{L_\beta(U_\sigma)}, \quad 0 < \sigma' < \sigma \leq 1,\ 0 < \beta \leq \frac{p_0}{\kappa} < 1. \tag{2.10}$$

Then there exist constants $M = M(C, \gamma, \kappa)$ and $\gamma_0 = \gamma_0(\gamma, \kappa)$ such that

$$|f|_{L_{p_0}(U_\delta)} \leq \Big(\frac{M}{(1-\delta)^{\gamma_0}}\Big)^{1/p-1/p_0} |f|_{L_p(U_1)} \quad \textit{for all } \delta \in (0,1),\ p \in (0, \frac{p_0}{\kappa}].$$

Proof: Set $p_i = p_0\kappa^{-i}$, $i = 1, 2, \ldots$. Given $\delta \in (0,1)$ we take again the sequence $\{\sigma_i\}$, $i = 0, 1, 2, \ldots$, defined by $\sigma_0 = 1$ and $\sigma_i = 1 - \sum_{j=1}^i 2^{-j}(1-\delta)$, $i \geq 1$. Suppose now $n \in \mathbb{N}$. By using (2.10) with $\beta = p_i$, $i = 1, \ldots, n$, we obtain

$$\begin{aligned}
\Phi(p_0, \delta) \leq \Phi(p_0, \sigma_n) \ &= \ \Phi(p_1\kappa, \sigma_n) \ \leq \ \frac{C^{\kappa/p_0}}{[2^{-n}(1-\delta)]^{\gamma\kappa/p_0}}\, \Phi(p_1, \sigma_{n-1}) \\
&\leq \ \frac{C^{\kappa/p_0}}{[2^{-n}(1-\delta)]^{\gamma\kappa/p_0}}\ \frac{C^{\kappa^2/p_0}}{[2^{-(n-1)}(1-\delta)]^{\gamma\kappa^2/p_0}}\, \Phi(p_2, \sigma_{n-2}) \ \leq \ \ldots \\
&\leq \ \frac{C^{\frac{1}{p_0}(\kappa+\kappa^2+\ldots+\kappa^n)}}{2^{-\frac{\gamma}{p_0}(n\kappa+(n-1)\kappa^2+\ldots+2\kappa^{n-1}+\kappa^n)}(1-\delta)^{\frac{\gamma}{p_0}(\kappa+\kappa^2+\ldots+\kappa^n)}}\, \Phi(p_n, \sigma_0).
\end{aligned}$$

Since $p_i = p_0\kappa^{-i}$, we have

$$\frac{1}{p_0}\sum_{j=1}^n \kappa^j = \frac{\kappa(\kappa^n-1)}{p_0(\kappa-1)} = \frac{\kappa}{p_0(\kappa-1)}\,(\frac{p_0}{p_n}-1) = \frac{\kappa}{\kappa-1}\,(\frac{1}{p_n}-\frac{1}{p_0}).$$

Employing the formula

$$\sum_{j=1}^n j\kappa^{j-1} = \frac{1-(n+1)\kappa^n + n\kappa^{n+1}}{(\kappa-1)^2}$$

we have further

$$\begin{aligned}
\sum_{j=1}^n (n+1-j)\kappa^j \ &= \ (n+1)\sum_{j=1}^n \kappa^j - \sum_{j=1}^n j\kappa^j \\
&= \ (n+1)\kappa\,\frac{\kappa^n-1}{\kappa-1} - \kappa\,\frac{1-(n+1)\kappa^n+n\kappa^{n+1}}{(\kappa-1)^2} \\
&= \ \kappa\,\frac{\kappa^{n+1}-(n+1)\kappa+n}{(\kappa-1)^2} \ \leq \ \frac{\kappa}{(\kappa-1)^2}\kappa^{n+1} \\
&\leq \ \frac{\kappa^3}{(\kappa-1)^3}\,(\kappa^n-1) \ \leq \ \frac{\kappa^3}{(\kappa-1)^3}\,(\frac{p_0}{p_n}-1),
\end{aligned}$$

which yields

$$\frac{1}{p_0}\sum_{j=1}^n (n+1-j)\kappa^j \leq \frac{\kappa^3}{(\kappa-1)^3}\,(\frac{1}{p_n}-\frac{1}{p_0}).$$

Therefore

$$\Phi(p_0,\delta) \le \Big[\frac{2^{\frac{\gamma\kappa^3}{(\kappa-1)^3}} C^{\frac{\kappa}{\kappa-1}}}{(1-\delta)^{\frac{\gamma\kappa}{\kappa-1}}}\Big]^{\frac{1}{p_n}-\frac{1}{p_0}} \Phi(p_n,\sigma_0).$$

Given $p \in (0, p_0/\kappa]$ there exists $n \ge 2$ such that $p_n < p \le p_{n-1}$. We then have

$$\begin{aligned}\frac{1}{p_n} - \frac{1}{p_0} &= \frac{\kappa^n - 1}{p_0} \le \frac{\kappa^n + \kappa^{n-1} - \kappa - 1}{p_0} = \frac{(1+\kappa)(\kappa^{n-1}-1)}{p_0} \\ &= (1+\kappa)(\frac{1}{p_{n-1}} - \frac{1}{p_0}) \le (1+\kappa)(\frac{1}{p} - \frac{1}{p_0}),\end{aligned}$$

as well as

$$\Phi(p_n,\sigma_0) = \Phi(p_n,1) \le \Phi(p,1),$$

by Hölder's inequality and the assumption $\mu(U_1) \le 1$. All in all, we obtain

$$\Phi(p_0,\delta) \le \Big[\frac{2^{\frac{\gamma\kappa^3}{(\kappa-1)^3}} C^{\frac{\kappa}{\kappa-1}}}{(1-\delta)^{\frac{\gamma\kappa}{\kappa-1}}}\Big]^{(1+\kappa)(\frac{1}{p}-\frac{1}{p_0})} \Phi(p,1),$$

which proves the lemma. □

The following abstract lemma is due to Bombieri and Giusti [4]. For a proof we also refer to [66, Lemma 2.2.6] and [18, Lemma 2.6]

Lemma 2.5.3 *Let δ, $\eta \in (0,1)$, and let γ, C be positive constants and $0 < \beta_0 \le \infty$. Suppose f is a positive μ-measurable function on U_1 which satisfies the following two conditions:*

(i)

$$|f|_{L_{\beta_0}(U_{\sigma'})} \le [C(\sigma-\sigma')^{-\gamma}\mu(U_1)^{-1}]^{1/\beta - 1/\beta_0}|f|_{L_\beta(U_\sigma)},$$

for all σ, σ', β such that $0 < \delta \le \sigma' < \sigma \le 1$ and $0 < \beta \le \min\{1, \eta\beta_0\}$.

(ii)

$$\mu(\{\log f > \lambda\}) \le C\mu(U_1)\lambda^{-1}$$

for all $\lambda > 0$.

Then

$$|f|_{L_{\beta_0}(U_\delta)} \le M\mu(U_1)^{1/\beta_0},$$

where M depends only on δ, η, γ, C, and β_0.

2.6 Weighted Poincaré inequalities

The following result can be found in [57, Lemma 3], see also [51, Lemma 6.12].

Proposition 2.6.1 *Let $\varphi \in C(\mathbb{R}^N)$ with non-empty compact support of diameter d and assume that $0 \le \varphi \le 1$. Suppose that the domains $\{x \in \mathbb{R}^N : \varphi(x) \ge a\}$ are convex for all $a \le 1$. Then for any function $u \in H^1_2(\mathbb{R}^N)$,*

$$\int_{\mathbb{R}^N} \big(u(x) - u_\varphi\big)^2 \varphi(x)\,dx \le \frac{2d^2\mu_N(\operatorname{supp}\varphi)}{|\varphi|_{L_1(\mathbb{R}^N)}} \int_{\mathbb{R}^N} |Du(x)|^2\varphi(x)\,dx,$$

where

$$u_\varphi = \frac{\int_{\mathbb{R}^N} u(x)\varphi(x)\,dx}{\int_{\mathbb{R}^N}\varphi(x)\,dx}.$$

The next Poincaré-type inequality has been taken from [20, Chap. I, Prop. 2.1], see also [49].

Proposition 2.6.2 *Let Ω be a bounded convex set in $\mathbb{R}^N$ and let $w \in C(\bar{\Omega})$ with values in $[0,1]$ be such that the sets $\{x \in \Omega : w(x) > c\}$ are convex for all $c \in (0,1)$. Let $v \in H_2^1(\Omega)$ and assume that the set $\mathcal{E}_0 := \{v = 0\} \cap \{w = 1\}$ has positive measure. Then*

$$\Big(\int_\Omega v^2 w\,dx\Big)^{1/2} \le C\,\frac{(\operatorname{diam}\Omega)^N}{\lambda_N(\mathcal{E}_0)^{\frac{N-1}{N}}}\Big(\int_\Omega |Dv|^2 w\,dx\Big)^{1/2},$$

where the constant C only depends on N.

2.7 Parabolic embeddings

We next state an interpolation result which will be frequently used throughout this contribution.

Let $T > 0$ and Ω be a bounded domain in $\mathbb{R}^N$. For $1 < p \le \infty$ we define the space

$$V_p := V_p([0,T]\times\Omega) = L_{2p}([0,T]; L_2(\Omega)) \cap L_2([0,T]; H_2^1(\Omega)), \tag{2.11}$$

endowed with the norm

$$|u|_{V_p([0,T]\times\Omega)} := |u|_{L_{2p}([0,T];L_2(\Omega))} + |Du|_{L_2([0,T];L_2(\Omega;\mathbb{R}^N))}.$$

Suppose that

$$p'\Big(1-\frac{2}{r}\Big) + N\Big(\frac{1}{2}-\frac{1}{q}\Big) = 1, \tag{2.12}$$

where $p' = p/(p-1)$, and

$$\left.\begin{array}{lll} r \in [2, 2p], & q \in \Big[2, \frac{2N}{N-2}\Big] & \text{for } N > 2 \\ r \in (2, 2p], & q \in [2, \infty) & \text{for } N = 2 \\ r \in \Big[\frac{4p}{p+1}, 2p\Big], & q \in [2,\infty] & \text{for } N = 1. \end{array}\right\}$$

Assuming that $\partial\Omega$ satisfies the property of positive density we have that $V_p \hookrightarrow L_r([0,T]; L_q(\Omega))$. Moreover, there exists a constant $C = C(N,q)$ such that

$$|u|_{L_r([0,T];L_q(\Omega))} \le C|u|_{V_p([0,T]\times\Omega)}, \tag{2.13}$$

for all $u \in V_p \cap L_2([0,T]; \mathring{H}_2^1(\Omega))$. This is a consequence of the Gagliardo-Nirenberg and Hölder's inequality. For a proof we refer to [76]. The case $p = \infty$ is contained, e.g., in [49, p. 74 and 75].

Setting

$$\kappa := \kappa_p := \frac{2p + N(p-1)}{2 + N(p-1)} \tag{2.14}$$

with $\kappa_\infty = 1 + 2/N$, it follows from (2.13) that

$$|u|_{L_{2\kappa}([0,T]\times\Omega)} \le C(N,p)|u|_{V_p([0,T]\times\Omega)}, \tag{2.15}$$

for all $u \in V_p \cap L_2([0,T]; \mathring{H}_2^1(\Omega))$.

2.8 L_p-estimates for linear equations

We conclude this preliminary part with a maximal L_p-regularity result, which is a special case of [82, Theorem 3.4] on linear boundary value problems in the context of parabolic Volterra equations.

Let $T > 0$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain with C^2-boundary Γ, and $N \geq 2$. We consider the problem

$$\begin{aligned} \partial_t^\alpha\big(u - u_0(x)\big) - a_{ij}(t,x)D_iD_ju = f,\;\; t \in (0,T),\, x \in \Omega, \\ u = g,\;\; t \in (0,T),\, x \in \Gamma, \\ u|_{t=0} = u_0,\;\; x \in \Omega, \end{aligned} \tag{2.16}$$

where we use the sum convention.

Theorem 2.8.1 *Let $\alpha \in (0,1)$ and $p > \frac{1}{\alpha} + \frac{N}{2}$. Suppose that $A = (a_{ij})_{i,j=1,\ldots,N} \in C([0,T] \times \overline{\Omega}; \mathrm{Sym}\{N\})$, and there exists $\nu > 0$ such that $a_{ij}(t,x)\xi_i\xi_j \geq \nu|\xi|^2$ for all $(t,x) \in [0,T] \times \overline{\Omega}$ and $\xi \in \mathbb{R}^N$. Then the problem (2.16) has a unique solution u in the class*

$$Z := H_p^\alpha([0,T]; L_p(\Omega)) \cap L_p([0,T]; H_p^2(\Omega)) \hookrightarrow C([0,T] \times \overline{\Omega})$$

if and only if the following conditions are satisfied.

(i) $f \in L_p([0,T]; L_p(\Omega))$, $g \in Y_D := B_{pp}^{\alpha(1-\frac{1}{2p})}([0,T]; L_p(\Gamma)) \cap L_p([0,T]; B_{pp}^{2-\frac{1}{p}}(\Gamma))$, *and* $u_0 \in Y_\gamma := B_{pp}^{2-\frac{2}{p\alpha}}(\Omega)$;

(ii) $u_0 = g|_{t=0}$ *on* Γ.

In this case one has an estimate of the form

$$|u|_Z \leq C\big(|f|_{L_p(\Omega_T)} + |g|_{Y_D} + |u_0|_{Y_\gamma}\big),$$

where C only depends on $\alpha, p, N, T, \Omega, A$.

Chapter 3

Abstract equations in Hilbert spaces

3.1 Setting and introductory remarks

Let $\mathcal{V}$ and $\mathcal{H}$ be real separable Hilbert spaces such that $\mathcal{V}$ is densely and continuously embedded into $\mathcal{H}$. Identifying $\mathcal{H}$ with its dual $\mathcal{H}'$ we have $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}'$, and

$$(h,v)_{\mathcal{H}} = \langle h,v\rangle_{\mathcal{V}'\times\mathcal{V}}, \quad h\in\mathcal{H},\, v\in\mathcal{V}, \tag{3.1}$$

where $(\cdot,\cdot)_{\mathcal{H}}$ and $\langle\cdot,\cdot\rangle_{\mathcal{V}'\times\mathcal{V}}$ denote the scalar product in $\mathcal{H}$ and the duality pairing between $\mathcal{V}'$ and $\mathcal{V}$, respectively.

In this chapter we study the abstract problem

$$\frac{d}{dt}\Big([k*(u-x)](t),v\Big)_{\mathcal{H}} + a\big(t,u(t),v\big) = \big\langle f(t),v\big\rangle_{\mathcal{V}'\times\mathcal{V}}, \quad v\in\mathcal{V},\ \text{a.a.}\, t\in(0,T), \tag{3.2}$$

where d/dt means the generalized derivative of real functions on $(0,T)$, $k \in L_{1,\,loc}(\mathbb{R}_+)$ is a scalar kernel of type $\mathcal{PC}$, see Definition 2.2.1, $k*u$ stands for the convolution on the positive halfline, and $a:(0,T)\times\mathcal{V}\times\mathcal{V}\to\mathbb{R}$ is a bounded $\mathcal{V}$-coercive bilinear form. Further, $x\in\mathcal{H}$ and $f\in L_2([0,T];\mathcal{V}')$ are given data.

We seek a solution u of (3.2) in the regularity class

$$W(x,\mathcal{V},\mathcal{H}) := \{u\in L_2([0,T];\mathcal{V}):\ k*(u-x)\in {}_0H_2^1([0,T];\mathcal{V}')\},$$

where the zero means vanishing trace at $t=0$. The vector x can be regarded as initial data for u, at least in a weak sense. If e.g. u, and $\frac{d}{dt}(k*[u-x])$ belong to $C([0,T];\mathcal{V}')$, then the condition $k*(u-x)(0)=0$ implies $u(0)=x$, see Section 3.3.

In the special case

$$k(t) \,=\, g_{1-\alpha}(t)e^{-\mu t}, \quad t>0,\ \alpha\in(0,1),\ \mu\geq 0, \tag{3.3}$$

(3.2) amounts to an abstract differential equation of fractional order $\alpha\in(0,1)$.

In this chapter we will prove that problem (3.2) possesses exactly one solution in the class $W(x,\mathcal{V},\mathcal{H})$, see Theorem 3.3.1 below. This result can be regarded as the analogue of the well-known existence and uniqueness result for the corresponding abstract parabolic equation

$$\begin{cases} \frac{d}{dt}\,(u(t),v)_{\mathcal{H}} + a(t,u(t),v) = \langle f(t),v\rangle_{\mathcal{V}'\times\mathcal{V}}, \quad v\in\mathcal{V},\ \text{a.a.}\, t\in(0,T),\\ u(0) = x \ \in\mathcal{H},\\ u\in H_2^1([0,T];\mathcal{V}')\cap L_2([0,T];\mathcal{V}), \end{cases} \tag{3.4}$$

see e.g. Theorem 4.1 and Remark 4.3 in Chapter 4 in [52] or [87, Section 23]. We point out that concerning time regularity the bilinear form a is *only* assumed to be *measurable* in t. This allows, e.g., to treat parabolic partial integro-differential equations in divergence form with merely bounded and measurable coefficients, see Section 3.4.

The proof of Theorem 3.3.1 is based on the Galerkin method and suitable a priori estimates for solutions of (3.2). These estimates are derived by means of the basic identity (2.8). It has been known before but does not seem to appear in the literature in the context of problems of the form (3.2). We remark that recently ([75]) the identity (2.8) was successfully employed to construct Lyapunov functions for certain nonlinear differential equations of fractional order between 0 and 2.

In order to be able to apply (2.8), we approximate the kernel k by the sequence (k_n) which is obtained from the Yosida approximation of the operator B defined by $Bv = \frac{d}{dt}(k * v)$, e.g. in $L_2([0,T])$, see Section 2.2. This method was already used in [75], we also refer to [31], where a more general class of integro-differential operators (in time) is studied.

Note that (3.2) is equivalent to the equation

$$\frac{d}{dt}[k * (u - x)](t) + A(t)u(t) = f(t), \quad \text{a.a.}\, t \in (0,T), \tag{3.5}$$

in $\mathcal{V}'$, where the operator $A(t) : \mathcal{V} \to \mathcal{V}'$ is defined by

$$\langle A(t)u, v\rangle_{\mathcal{V}' \times \mathcal{V}} = a(t, u, v), \quad u,\, v \in \mathcal{V}. \tag{3.6}$$

For equations of the form (3.5) with $A(t) \equiv A$ there exists a vast literature, even in general Banach spaces, see e.g. [31], and [63] and the references given therein. However, in the case of time-dependent A and without smoothness assumption nothing seems to be known in the literature concerning existence and uniqueness, except for [85], which forms the basis for this chapter.

3.2 A basic interpolation result

Let $\mathcal{V}$ and $\mathcal{H}$ be real Hilbert spaces as described above, that is $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}'$. In the theory of abstract parabolic equations the continuous embedding

$$H^1_2([0,T]; \mathcal{V}') \cap L_2([0,T]; \mathcal{V}) \hookrightarrow C([0,T]; \mathcal{H}) \tag{3.7}$$

is well-known, see e.g. Proposition 2.1 and Theorem 3.1 in Chapter 1 of [52], or Proposition 23.23 in [87]. The following theorem provides the analogue of (3.7) in the case of the space $W(x, \mathcal{V}, \mathcal{H})$.

Theorem 3.2.1 *Let $\mathcal{V}$ and $\mathcal{H}$ be real Hilbert spaces as described above ($\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}'$). Let further $T > 0$, and $k \in L_{1,\,loc}(\mathbb{R}_+)$ be of type $\mathcal{PC}$. Suppose that $x \in \mathcal{H}$, and $u \in W(x, \mathcal{V}, \mathcal{H})$. Then $k * (u - x)$ and $k * u$ belong to the space $C([0,T]; \mathcal{H})$ (after possibly being redefined on a set of measure zero). The mapping $\{t \mapsto |k * u|^2_{\mathcal{H}}(t)\}$ is absolutely continuous on $[0,T]$, with*

$$\frac{d}{dt}\,|k * u|^2_{\mathcal{H}}(t) = 2\Big\langle [k * (u - x)]'(t), [k * u](t)\Big\rangle_{\mathcal{V}' \times \mathcal{V}} + 2k(t)\Big(x, (k * u)(t)\Big)_{\mathcal{H}} \tag{3.8}$$

for a.a. $t \in [0,T]$. Furthermore,

$$|k * u|_{C([0,T];\mathcal{H})} \le C\Big(\Big|\frac{d}{dt}[k * (u - x)]\Big|_{L_2([0,T];\mathcal{V}')} + |u|_{L_2([0,T];\mathcal{V})} + |x|_{\mathcal{H}}\Big), \tag{3.9}$$

the constant C depending only on T, $|k|_{L_1([0,T])}$, and the constant of the embedding $\mathcal{V} \hookrightarrow \mathcal{H}$.

We remark that in the case $x = 0$, the property $k * u \in C([0,T];\mathcal{H})$ follows immediately from the embedding (3.7). In fact, $u \in L_2([0,T];\mathcal{V})$ implies $k * u \in L_2([0,T];\mathcal{V})$, by Young's inequality, and so

$$k * u \in H_2^1([0,T];\mathcal{V}') \cap L_2([0,T];\mathcal{V}) \hookrightarrow C([0,T];\mathcal{H}).$$

We point out that for $x \neq 0$ this simple reduction is not possible.

Proof of Theorem 3.2.1. Note first that $(k*x)(\cdot) = (1*k)(\cdot)x \in H_1^1([0,T];\mathcal{H}) \hookrightarrow C([0,T];\mathcal{H})$. Thus $k*(u-x) \in C([0,T];\mathcal{H})$ if and only if $k*u \in C([0,T];\mathcal{H})$.

Let $k_n \in H_1^1([0,T])$, $n \in \mathbb{N}$, be the kernel associated with the Yosida approximation B_n of the operator

$$Bv = \frac{d}{dt}(k*v), \quad D(B) = \{v \in L_2([0,T];\mathcal{V}') : \, k*v \in {}_0H_2^1([0,T];\mathcal{V}')\}. \tag{3.10}$$

Then $k_n * u \in H_2^1([0,T];\mathcal{V})$, and we have for n, $m \in \mathbb{N}$,

$$\frac{d}{dt}\Big|(k_n*u)(t) - (k_m*u)(t)\Big|_{\mathcal{H}}^2 = 2\Big([k_n*u]'(t) - [k_m*u]'(t), [k_n*u](t) - [k_m*u](t)\Big)_{\mathcal{H}}.$$

Thus, in view of (3.1) and Young's inequality,

$$\begin{aligned}
&\Big|(k_n*u)(t) - (k_m*u)(t)\Big|_{\mathcal{H}}^2 = \Big|(k_n*u)(s) - (k_m*u)(s)\Big|_{\mathcal{H}}^2\\
&\quad + 2\int_s^t \Big\langle [k_n*(u-x)]'(\tau) - [k_m*(u-x)]'(\tau), [k_n*u](\tau) - [k_m*u](\tau)\Big\rangle_{\mathcal{V}'\times\mathcal{V}}\, d\tau\\
&\quad + 2\int_s^t [k_n(\tau) - k_m(\tau)]\Big(x, [k_n*u](\tau) - [k_m*u](\tau)\Big)_{\mathcal{H}}\, d\tau\\
&\le \Big|(k_n*u)(s) - (k_m*u)(s)\Big|_{\mathcal{H}}^2 + \Big|\frac{d}{dt}[k_n*(u-x)] - \frac{d}{dt}[k_m*(u-x)]\Big|_{L_2([0,T];\mathcal{V}')}^2\\
&\quad + \Big|k_n*u - k_m*u\Big|_{L_2([0,T];\mathcal{V})}^2 + 2|x|_{\mathcal{H}}^2|k_n - k_m|_{L_1([0,T])}^2 + \frac{1}{2}\Big|k_n*u - k_m*u\Big|_{C([0,T];\mathcal{H})}^2
\end{aligned} \tag{3.11}$$

for all s, $t \in [0,T]$. Since $k_n \to k$ in $L_1([0,T])$ as $n \to \infty$, we have $k_n*u \to k*u$ in $L_2([0,T];\mathcal{H})$ as well as in $L_2([0,T];\mathcal{V})$. Further, $u - x \in D(B)$ implies that $\frac{d}{dt}[k_n*(u-x)] \to \frac{d}{dt}[k*(u-x)]$ in $L_2([0,T];\mathcal{V}')$.

We fix now a point $s \in (0,T)$ for which

$$(k_n*u)(s) \to (k*u)(s) \quad \text{in } \mathcal{H} \text{ as } n \to \infty.$$

Taking then in (3.11) the maximum over all $t \in [0,T]$ and absorbing the last term, it follows that (k_n*u) is a Cauchy sequence in $C([0,T];\mathcal{H})$. Thus k_n*u converges in $C([0,T];\mathcal{H})$ to some $v \in C([0,T];\mathcal{H})$. Since we also know that $k_n*u \to k*u$ in $L_2([0,T];\mathcal{H})$, we deduce $k*u = v$ a.e. in $[0,T]$, proving the first part of the theorem.

Similarly as above we see that

$$\begin{aligned}
|(k_n*u)(t)|_{\mathcal{H}}^2 = |(k_n*u)(s)|_{\mathcal{H}}^2 &+ 2\int_s^t \Big\langle [k_n*(u-x)]'(\tau), [k_n*u](\tau)\Big\rangle_{\mathcal{V}'\times\mathcal{V}}\, d\tau\\
&+ 2\int_s^t k_n(\tau)\Big(x, (k_n*u)(\tau)\Big)_{\mathcal{H}}\, d\tau
\end{aligned}$$

for all $s, t \in [0, T]$, and $n \in \mathbb{N}$. Taking the limits as $n \to \infty$ we obtain

$$|(k * u)(t)|_{\mathcal{H}}^2 = |(k * u)(s)|_{\mathcal{H}}^2 + 2 \int_s^t \Big\langle [k * (u - x)]'(\tau), [k * u](\tau) \Big\rangle_{\mathcal{V}' \times \mathcal{V}} d\tau$$
$$+ 2 \int_s^t k(\tau) \Big(x, (k * u)(\tau) \Big)_{\mathcal{H}} d\tau \tag{3.12}$$

for all $s, t \in [0, T]$. Hence $\{t \mapsto |k * u|_{\mathcal{H}}^2(t)\}$ is absolutely continuous on $[0, T]$, and (3.8) holds true.

To obtain (3.9), we estimate the integral terms in (3.12) similarly as for (3.11) and integrate with respect to s. This yields

$$|(k * u)(t)|_{\mathcal{H}}^2 \le \frac{1}{T} |(k * u)|_{L_2([0,T];\mathcal{H})}^2 + \Big| \frac{d}{dt} [k * (u - x)] \Big|_{L_2([0,T];\mathcal{V}')}^2$$
$$+ |k * u|_{L_2([0,T];\mathcal{V})}^2 + 2|x|_{\mathcal{H}}^2 |k|_{L_1([0,T])}^2 + \frac{1}{2} |k * u|_{C([0,T];\mathcal{H})}^2 \tag{3.13}$$

for all $t \in [0, T]$. We then take the maximum over all $t \in [0, T]$, absorb the last term, and use Young's inequality for convolutions, to the result

$$\frac{1}{2} |k * u|_{C([0,T];\mathcal{H})}^2 \le \frac{1}{T} |k|_{L_1([0,T])}^2 |u|_{L_2([0,T];\mathcal{H})}^2 + \Big| \frac{d}{dt} [k * (u - x)] \Big|_{L_2([0,T];\mathcal{V}')}^2$$
$$+ |k|_{L_1([0,T])}^2 |u|_{L_2([0,T];\mathcal{V})}^2 + 2|x|_{\mathcal{H}}^2 |k|_{L_1([0,T])}^2,$$

which implies (3.9). □

3.3 The main existence and uniqueness result

In this section we are concerned with existence and uniqueness for the abstract problem (3.2). Recall that $\mathcal{V}$ and $\mathcal{H}$ are real separable Hilbert spaces such that $\mathcal{V}$ is densely and continuously embedded into $\mathcal{H}$. Identifying $\mathcal{H}$ with its dual $\mathcal{H}'$, we have $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}'$, and the relation (3.1) holds. It will be assumed that $\dim \mathcal{V} = \infty$.

We will suppose that the following assumptions are satisfied.

(Hk) $(k, l) \in \mathcal{PC}$ for some $l \in L_{1,\,loc}(\mathbb{R}_+)$.

(Hd) $x \in \mathcal{H}$, $f \in L_2([0, T]; \mathcal{V}')$.

(Ha) For a.a. $t \in (0, T)$, the mapping $a(t, \cdot, \cdot) : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ is bilinear, and there exist constants $M > 0$, $c > 0$, and $d \ge 0$, which are independent of t, such that

$$|a(t, u, v)| \le M |u|_{\mathcal{V}} |v|_{\mathcal{V}}, \tag{3.14}$$
$$a(t, u, u) \ge c|u|_{\mathcal{V}}^2 - d|u|_{\mathcal{H}}^2, \tag{3.15}$$

for all $u, v \in \mathcal{V}$ and a.a. $t \in (0, T)$. Moreover, the function $\{t \mapsto a(t, u, v)\}$ is measurable on $(0, T)$ for all $u, v \in \mathcal{V}$.

We seek a solution of (3.2) in the space

$$W(x, \mathcal{V}, \mathcal{H}) = \{u \in L_2([0, T]; \mathcal{V}) : \ k * (u - x) \in {}_0H_2^1([0, T]; \mathcal{V}')\}.$$

Note that the vector x plays the role of the initial data for u, at least in a weak sense. If e.g. u, and $\frac{d}{dt}(k * [u-x]) =: \tilde{f}$ belong to $C([0,T];\mathcal{V}')$, then the assumption (Hk) and the condition $k*(u-x)(0) = 0$ entail that

$$u - x = \frac{d}{dt}\,(l * k * [u-x]) = l * \tilde{f}$$

in $C([0,T];\mathcal{V}')$, and therefore $u(0) = x$.

In order to construct a solution in the desired class, we will use the Galerkin method. We will assume that

(Hb) $\{w_1, w_2, \ldots\}$ is a basis in $\mathcal{V}$, and (x_m) is a sequence in $\mathcal{H}$ such that $x_m \in \operatorname{span}\{w_1,\ldots,w_m\}$, $m \in \mathbb{N}$, and $x_m \to x$ in $\mathcal{H}$ as $m \to \infty$.

Setting

$$u_m(t) = \sum_{j=1}^{m} c_{jm}(t) w_j, \qquad x_m = \sum_{j=1}^{m} \beta_{jm} w_j,$$

and replacing u, x, and v in (3.2) by u_m, x_m, and w_i, respectively, we formally obtain for every $m \in \mathbb{N}$, the system of Galerkin equations

$$\sum_{j=1}^{m} \frac{d}{dt}\,[k * (c_{jm} - \beta_{jm})](t)(w_j, w_i)_{\mathcal{H}} + \sum_{j=1}^{m} c_{jm}(t) a(t, w_j, w_i) = \langle f(t), w_i\rangle_{\mathcal{V}'\times\mathcal{V}}, \tag{3.16}$$

for a.a. $t \in (0,T)$, where i runs through the set $\{1,\ldots,m\}$.

The main result in this section is the following.

Theorem 3.3.1 *Let $T > 0$, and $\mathcal{V}$ and $\mathcal{H}$ be real Hilbert spaces as described above. Suppose the assumptions (Hk), (Hd), (Ha), and (Hb) hold. Then the problem (3.2) has exactly one solution u in the space $W(x,\mathcal{V},\mathcal{H})$. The mapping $(x,f) \mapsto u$ is linear, and there exists a constant $M_0 > 0$ such that*

$$|k * (u-x)|_{H_2^1([0,T];\mathcal{V}')} + |u|_{L_2([0,T];\mathcal{V})} \le M_0\Big(|x|_{\mathcal{H}} + |f|_{L_2([0,T];\mathcal{V}')}\Big) \tag{3.17}$$

for all $x \in \mathcal{H}$ and $f \in L_2([0,T];\mathcal{V}')$. Moreover, for every $m \in \mathbb{N}$, the Galerkin equation (3.16) possesses precisely one solution $u_m \in W(x_m,\mathcal{V},\mathcal{H})$. The sequence (u_m) converges weakly to u in $L_2([0,T];\mathcal{V})$ as $m \to \infty$.

Proof. **Uniqueness.** Suppose that u_1, $u_2 \in W(x,\mathcal{V},\mathcal{H})$ are solutions of (3.2). The difference $u = u_1 - u_2$ then belongs to the space $W(0,\mathcal{V},\mathcal{H})$ and satisfies the equation

$$\langle (k*u)'(t), v\rangle_{\mathcal{V}'\times\mathcal{V}} + a(t,u(t),v) = 0, \quad v \in V, \text{ a.a.}\, t \in (0,T).$$

We may take $v = u(t)$, thereby getting

$$\langle (k*u)'(t), u(t)\rangle_{\mathcal{V}'\times\mathcal{V}} + a(t,u(t),u(t)) = 0, \quad \text{a.a.}\, t \in (0,T). \tag{3.18}$$

Let $k_n \in H_1^1([0,T])$, $n \in \mathbb{N}$, be the kernel associated with the Yosida approximation B_n of the operator B defined in (3.10). Then (3.18) is equivalent to

$$\langle (k_n*u)'(t), u(t)\rangle_{\mathcal{V}'\times\mathcal{V}} + a(t,u(t),u(t)) = h_n(t), \quad \text{a.a.}\, t \in (0,T), \tag{3.19}$$

where

$$h_n(t) = \langle (k_n*u)'(t) - (k*u)'(t), u(t)\rangle_{\mathcal{V}'\times\mathcal{V}}, \quad \text{a.a.}\, t \in (0,T).$$

Since $k_n * u \in H^1_2([0,T];\mathcal{H})$, we may apply (3.1) to the first term in (3.19), to the result

$$\Big(\frac{d}{dt}\,(k_n * u)(t), u(t)\Big)_{\mathcal{H}} + a(t,u(t),u(t)) = h_n(t), \quad \text{a.a.}\, t \in (0,T), \tag{3.20}$$

for all $n \in \mathbb{N}$.

The kernels k_n are nonnegative and nonincreasing. Thus, by Lemma 2.3.2,

$$\frac{1}{2}\,\frac{d}{dt}\,(k_n * |u(\cdot)|^2_{\mathcal{H}})(t) \,\le\, \Big(\frac{d}{dt}\,(k_n * u)(t), u(t)\Big)_{\mathcal{H}}, \quad \text{a.a.}\, t \in (0,T).$$

The second term in (3.20) is estimated by means of the abstract Gårding inequality (3.15) in (Ha). Proceeding this way, it follows from (3.20) that

$$\frac{d}{dt}\,(k_n * |u(\cdot)|^2_{\mathcal{H}})(t) \le 2d|u(t)|^2_{\mathcal{H}} + 2h_n(t), \quad \text{a.a.}\, t \in (0,T). \tag{3.21}$$

Observe that all terms in (3.21) viewed as functions of t belong to $L_1([0,T])$. Therefore we may convolve (3.21) with the kernel l from assumption (Hk). Letting then n go to ∞ and selecting an appropriate subsequence, if necessary, we arrive at

$$|u(t)|^2_{\mathcal{H}} \le 2d\,(l * |u(\cdot)|^2_{\mathcal{H}})(t), \quad \text{a.a.}\, t \in (0,T). \tag{3.22}$$

Here we use the fact that $h_n \to 0$ in $L_1([0,T])$, which entails $l * h_n \to 0$ in $L_1([0,T])$, and that

$$l * \frac{d}{dt}\,(k_n * |u(\cdot)|^2_{\mathcal{H}}) = \frac{d}{dt}\,(k_n * l * |u(\cdot)|^2_{\mathcal{H}}) \to \frac{d}{dt}\,(k * l * |u(\cdot)|^2_{\mathcal{H}}) = |u(\cdot)|^2_{\mathcal{H}}$$

in $L_1([0,T])$ as $n \to \infty$.

Since l is nonnegative, (3.22) implies that $|u(t)|^2_{\mathcal{H}} = 0$ a.e. in $(0,T)$, by the abstract Gronwall lemma [86, Prop. 7.15], i.e. $u = 0$.

Existence. 1. We show first that for every $m \in \mathbb{N}$, the system of Galerkin equations (3.16) admits a unique solution $\psi := \psi_m := (c_{1m}, \ldots, c_{mm})^T$ on $[0,T]$ in the class $W(\xi, \mathbb{R}^m, \mathbb{R}^m)$, where $\xi := \xi_m := (\beta_{1m}, \ldots, \beta_{mm})^T$.

Since the vectors $w_1, \ldots, w_m$ are linearly independent, the matrix $((w_j, w_i)_{\mathcal{H}}) \in \mathbb{R}^{m\times m}$ is invertible. Hence (3.16) can be solved for $\frac{d}{dt}\,[k * (c_{jm} - \beta_{jm})]$, which leads to an equivalent system of the form

$$\frac{d}{dt}\,[k * (\psi - \xi)](t) = B(t)\psi(t) + g(t), \quad \text{a.a.}\, t \in (0,T), \tag{3.23}$$

where $B \in L_\infty([0,T];\mathbb{R}^{m\times m})$, and $g \in L_2([0,T];\mathbb{R}^m)$, by the assumptions (Ha) and (Hd). In order to solve (3.23), we transform it into the system of Volterra equations

$$\psi(t) = \xi + l * [B(\cdot)\psi(\cdot)](t) + (l * g)(t), \quad \text{a.a.}\, t \in (0,T),$$

which has a unique solution $\psi \in L_2([0,T];\mathbb{R}^m)$, see e.g. [32, Chapter 9]. But then $\psi \in W(\xi, \mathbb{R}^m, \mathbb{R}^m)$, and hence it is also a solution of (3.23). This shows that for every $m \in \mathbb{N}$, the Galerkin equation (3.16) has exactly one solution $u_m \in W(x_m, \mathcal{V}, \mathcal{H})$.

2. We next derive a priori estimates for the Galerkin solutions. The Galerkin equations (3.16) are equivalent to

$$\Big(\frac{d}{dt}\,[k * (u_m - x_m)](t), w_i\Big)_{\mathcal{H}} + a(t, u_m(t), w_i) = \langle f(t), w_i\rangle_{\mathcal{V}'\times\mathcal{V}}, \quad \text{a.a.}\, t \in (0,T), \tag{3.24}$$

$i = 1, \dots, m$. Multiplying (3.24) by c_{im} and summing over i, we obtain

$$\Big(\frac{d}{dt}\,[(k * (u_m - x_m)](t), u_m(t)\Big)_{\mathcal{H}} + a(t, u_m(t), u_m(t)) = \langle f(t), u_m(t)\rangle_{\mathcal{V}'\times\mathcal{V}}. \tag{3.25}$$

Let $k_n \in H_1^1([0,T])$, $n \in \mathbb{N}$, be as in the uniqueness part above. Then (3.25) can be written as

$$\begin{aligned}&\Big(\frac{d}{dt}\,(k_n * u_m)(t), u_m(t)\Big)_{\mathcal{H}} + a(t, u_m(t), u_m(t))\\ &\qquad = k_n(t)(x_m, u_m(t))_{\mathcal{H}} + \langle f(t), u_m(t)\rangle_{\mathcal{V}'\times\mathcal{V}} + h_{mn}(t), \quad \text{a.a.}\, t \in (0,T),\end{aligned} \tag{3.26}$$

with

$$h_{mn}(t) = \langle [k_n * (u_m - x_m)]'(t) - [k * (u_m - x_m)]'(t), u_m(t)\rangle_{\mathcal{V}'\times\mathcal{V}}.$$

Using Lemma 2.3.2 and inequality (3.15), we find that

$$\begin{aligned}&\frac{1}{2}\,\frac{d}{dt}\,(k_n * |u_m(\cdot)|^2_{\mathcal{H}})(t) + \frac{1}{2}\,k_n(t)|u_m(t)|^2_{\mathcal{H}} + c|u_m(t)|^2_{\mathcal{V}}\\ &\qquad \le d|u_m(t)|^2_{\mathcal{H}} + k_n(t)(x_m, u_m(t))_{\mathcal{H}} + \langle f(t), u_m(t)\rangle_{\mathcal{V}'\times\mathcal{V}} + h_{mn}(t),\end{aligned}$$

which, by Young's inequality, yields the estimate

$$\frac{d}{dt}\,(k_n * |u_m(\cdot)|^2_{\mathcal{H}})(t) + c|u_m(t)|^2_{\mathcal{V}} \le 2d|u_m(t)|^2_{\mathcal{H}} + k_n(t)|x_m|^2_{\mathcal{H}} + \frac{1}{c}\,|f(t)|^2_{\mathcal{V}'} + 2h_{mn}(t). \tag{3.27}$$

Similarly as in the uniqueness part, we see that $l * h_{mn} \to 0$ and

$$l * \frac{d}{dt}\,(k_n * |u_m(\cdot)|^2_{\mathcal{H}}) \to |u_m(\cdot)|^2_{\mathcal{H}}$$

in $L_1([0,T])$ as $n \to \infty$. Consequently, if we convolve (3.27) with l, and let n tend to ∞, selecting an appropriate subsequence, if necessary, we obtain the estimate

$$|u_m(t)|^2_{\mathcal{H}} \le 2d\,(l * |u_m(\cdot)|^2_{\mathcal{H}})(t) + |x_m|^2_{\mathcal{H}} + \frac{1}{c}\,(l * |f(\cdot)|^2_{\mathcal{V}'})(t) \tag{3.28}$$

for a.a. $t \in (0,T)$, and all $m \in \mathbb{N}$. By positivity of l, it follows from (3.28) that

$$|u_m|_{L_2([0,T];\mathcal{H})} \le C\Big(|x_m|_{\mathcal{H}} + |f|_{L_2([0,T];\mathcal{V}')}\Big), \tag{3.29}$$

where the constant C depends only on c, d, l, T.

Returning to (3.27), we may integrate from 0 to T - note that $(k_n * |u_m(\cdot)|^2_{\mathcal{H}})(0) = 0$ - and then let n go to ∞ to find that

$$c\int_0^T |u_m(t)|^2_{\mathcal{V}}\,dt \le 2d\int_0^T |u_m(t)|^2_{\mathcal{H}}\,dt + |k|_{L_1([0,T])}|x_m|^2_{\mathcal{H}} + \frac{1}{c}\int_0^T |f(t)|^2_{\mathcal{V}'}\,dt.$$

This, together with (3.29) and the assumption $x_m \to x$ in $\mathcal{H}$, yields the *a priori* bound

$$|u_m|_{L_2([0,T];\mathcal{V})} \le C_1\Big(|x|_{\mathcal{H}} + |f|_{L_2([0,T];\mathcal{V}')}\Big), \quad m \in \mathbb{N}, \tag{3.30}$$

with some $C_1 > 0$ being independent of $m \in \mathbb{N}$.

3. By (3.30) there exists a subsequence of (u_m), which we will again denote by (u_m), such that

$$u_m \rightharpoonup u \quad \text{in } L_2([0,T];\mathcal{V}) \quad \text{as } m \to \infty, \tag{3.31}$$

for some $u \in L_2([0,T];\mathcal{V})$. We will show that $u \in W(x,\mathcal{V},\mathcal{H})$, and that u is a solution of (3.2).

Let $\varphi \in C^1([0,T];\mathbb{R})$ with $\varphi(T) = 0$. Multiplying (3.24) by φ and using integration by parts, we obtain

$$-\int_0^T \varphi'(t)([k*(u_m - x_m)](t), w_i)_{\mathcal{H}}\,dt + \int_0^T \varphi(t)a(t,u_m(t),w_i)\,dt \\ = \int_0^T \varphi(t)\langle f(t), w_i\rangle_{\mathcal{V}'\times\mathcal{V}} \tag{3.32}$$

for all $m \ge i$, because $[k*(u_m - x_m)](0) = 0$. We apply then the limits (3.31), and $x_m \to x$ in $\mathcal{H}$ to equation (3.32). By means of (3.14), the embedding $\mathcal{V} \hookrightarrow \mathcal{H}$, and Young's and Hölder's inequality, one easily verifies that this leads to

$$-\int_0^T \varphi'(t)([k*(u-x)](t), w_i)_{\mathcal{H}}\,dt + \int_0^T \varphi(t)a(t,u(t),w_i)\,dt = \int_0^T \varphi(t)\langle f(t), w_i\rangle_{\mathcal{V}'\times\mathcal{V}} \tag{3.33}$$

for all $i \in \mathbb{N}$. Observe that $([k*(u-x)](t), w_i)_{\mathcal{H}} = \langle [k*(u-x)](t), w_i\rangle_{\mathcal{V}'\times\mathcal{V}}$, by (3.1). It is not difficult to see that the terms in (3.33) represent linear continuous functionals on the space $\mathcal{V}$, with respect to w_i. Consequently, in light of (Hb), (3.33) implies

$$-\int_0^T \varphi'(t)\langle [k*(u-x)](t), v\rangle_{\mathcal{V}'\times\mathcal{V}}\,dt + \int_0^T \varphi(t)a(t,u(t),v)\,dt = \int_0^T \varphi(t)\langle f(t), v\rangle_{\mathcal{V}'\times\mathcal{V}} \tag{3.34}$$

for all $v \in \mathcal{V}$.

Since (3.34) holds in particular for all $\varphi \in C_0^\infty(0,T)$, we infer that $k*(u-x)$ has a generalized derivative on $(0,T)$ with

$$\frac{d}{dt}[k*(u-x)](t) + A(t)u(t) = f(t), \quad \text{a.a. } t \in (0,T), \tag{3.35}$$

where the operator $A(t) : \mathcal{V} \to \mathcal{V}'$ is defined as in (3.6). From $u \in L_2([0,T];\mathcal{V})$ and $|A(t)u(t)|_{\mathcal{V}'} \le M|u(t)|_{\mathcal{V}}$ for a.a. $t \in (0,T)$, we deduce that $A(\cdot)u \in L_2([0,T];\mathcal{V}')$. Since $f \in L_2([0,T];\mathcal{V}')$, too, it follows that $[k*(u-x)]' \in L_2([0,T];\mathcal{V}')$.

To see that $u \in W(x,\mathcal{V},\mathcal{H})$, it remains to show that $[k*(u-x)](0) = 0$. We set $z := k*(u-x)$. Then $z \in H_2^1([0,T];\mathcal{V}') \hookrightarrow C([0,T];\mathcal{V}')$, and by (3.34) and (3.35), there holds

$$-\int_0^T \varphi'(t)\langle z(t), v\rangle_{\mathcal{V}'\times\mathcal{V}}\,dt = \int_0^T \varphi(t)\langle z'(t), v\rangle_{\mathcal{V}'\times\mathcal{V}} \tag{3.36}$$

for all $v \in \mathcal{V}$, and all $\varphi \in C^1([0,T];\mathbb{R})$ with $\varphi(T) = 0$. Choosing φ such that $\varphi(0) = 1$, and approximating z in $H_2^1([0,T];\mathcal{V}')$ by a sequence of functions $z_n \in C^1([0,T];\mathcal{V}')$, it follows from (3.36) and the formula of integration by parts that $\langle z(0), v\rangle_{\mathcal{V}'\times\mathcal{V}} = 0$ for all $v \in \mathcal{V}$. Hence $z(0) = 0$.

Summarizing, we have found a function $u \in W(x,\mathcal{V},\mathcal{H})$ that solves the operator equation (3.35). Since (3.35) is equivalent to (3.2), the existence proof is complete.

Moreover, (3.35) has exactly one solution in the class $W(x,\mathcal{V},\mathcal{H})$. Consequently, all subsequences of the *original* sequence (u_m) that are weakly convergent in $L_2([0,T];\mathcal{V})$ have the same limit u. Hence, the original sequence (u_m) converges weakly to u in $L_2([0,T];\mathcal{V})$.

Continuous dependence on the data. From $u_m \rightharpoonup u$ in $L_2([0,T];\mathcal{V})$ and the estimate (3.30), it follows by means of the theorem of Banach and Steinhaus, that

$$|u|_{L_2([0,T];\mathcal{V})} \le \liminf_{m\to\infty} |u_m|_{L_2([0,T];\mathcal{V})} \le C_1\Big(|x|_{\mathcal{H}} + |f|_{L_2([0,T];\mathcal{V}')}\Big).$$

Using this estimate, together with $|A(\cdot)u|_{L_2([0,T];\mathcal{V}')} \leq M|u|_{L_2([0,T];\mathcal{V})}$, (Hd), and (3.35), we obtain the desired estimate (3.17). $\square$

If not only the kernel l in (Hk) but also some p-th power of it with $p > 1$ belongs to $L_1([0,T])$, then one can get an additional estimate for solutions of (3.2). This is the consequence of the first part of the subsequent interpolation result for functions in the space $W(x,\mathcal{V},\mathcal{H})$. It also contains the analogue of

$$\int_0^T t^{-1}|u(t)|_{\mathcal{H}}^2\,dt < \infty \quad \text{for all } u \in {}_0H_2^1([0,T];\mathcal{V}') \cap L_2([0,T];\mathcal{V}),$$

see [52, Chap. 3, Prop. 5.3 and Prop. 5.4], in the case of the space $W(x,\mathcal{V},\mathcal{H})$.

By $L_{p,w}([0,T])$, $p \in [1,\infty)$ we mean the weak L_p space of Lebesgue measurable functions on $(0,T)$.

Theorem 3.3.2 *Let $\mathcal{V}$ and $\mathcal{H}$ be real Hilbert spaces as described above ($\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}'$). Let further $T > 0$, $(k,l) \in \mathcal{PC}$, and suppose that $x \in \mathcal{H}$, and $u \in W(x,\mathcal{V},\mathcal{H})$. Then the following statements hold.*

(i) If $l \in L_{p,w}([0,T])$ for some $p > 1$ then $u \in L_{2p,w}([0,T];\mathcal{H})$, and there holds

$$|u|_{L_{2p,w}([0,T];\mathcal{H})} \leq C\Big(\Big|\frac{d}{dt}[k*(u-x)]\Big|_{L_2([0,T];\mathcal{V}')} + |u|_{L_2([0,T];\mathcal{V})} + |x|_{\mathcal{H}}\Big), \tag{3.37}$$

the constant C depending only on T, and $|l|_{L_{p,w}([0,T])}$. If $l \in L_p([0,T])$ for some $p > 1$ then $u \in L_{2p}([0,T];\mathcal{H})$, and the estimate corresponding to (3.37) holds.

(ii) There holds the estimate

$$\Big(\int_0^T k(t)|u(t)|_{\mathcal{H}}^2\,dt\Big)^{1/2} \leq C_1\Big(\Big|\frac{d}{dt}[k*(u-x)]\Big|_{L_2([0,T];\mathcal{V}')} + |u|_{L_2([0,T];\mathcal{V})} + |x|_{\mathcal{H}}\Big),$$

where the constant C_1 only depends on $|k|_{L_1([0,T])}$.

Proof. We proceed similarly as in the proof of the previous result. The key idea again is to apply the identity (2.8) from Lemma 2.3.2.

Let (k_n) be the sequence of approximating kernels used above, and put

$$g(t) = \langle (k*[u-x])'(t), u(t)\rangle_{\mathcal{V}'\times\mathcal{V}}, \quad t \in (0,T),$$

and

$$h_n(t) = \langle (k_n*[u-x])'(t) - (k*[u-x])'(t), u(t)\rangle_{\mathcal{V}'\times\mathcal{V}}, \quad t \in (0,T).$$

Then

$$\Big(\frac{d}{dt}\,(k_n*u)(t), u(t)\Big)_{\mathcal{H}} = k_n(t)(x,u(t))_{\mathcal{H}} + g(t) + h_n(t), \quad \text{a.a. } t \in (0,T),$$

with each term being in $L_1([0,T])$. Using (2.8) and the inequality $ab \leq \frac{1}{4}a^2 + b^2$ it follows that

$$\frac{1}{2}\,\frac{d}{dt}\,(k_n*|u(\cdot)|_{\mathcal{H}}^2)(t) + \frac{1}{4}\,k_n(t)|u(t)|_{\mathcal{H}}^2 \leq k_n(t)|x|_{\mathcal{H}}^2 + g(t) + h_n(t), \quad \text{a.a. } t \in (0,T). \tag{3.38}$$

To prove (i), we drop the second term on the left, which is nonnegative, convolve the resulting inequality with l, and send n to ∞. Arguing as in the proof of Theorem 3.3.1 we obtain

$$|u(t)|_{\mathcal{H}}^2 \leq 2\Big(|x|_{\mathcal{H}}^2 + (l*g)(t)\Big), \quad \text{a.a. } t \in (0,T).$$

Young's inequality for weak type L_p spaces (see e.g. [30, Theorem 1.2.13]) then gives

$$|u|^2_{L_{2p,w}([0,T];\mathcal{H})} = \Big||u(\cdot)|^2_{\mathcal{H}}\Big|_{L_{p,w}([0,T])} \le 2\Big(|l|_{L_{p,w}([0,T])}|g|_{L_1([0,T])} + T^{1/p}|x|^2_{\mathcal{H}}\Big),$$

which together with

$$2|g|_{L_1([0,T])} \le \Big|\frac{d}{dt}[k*(u-x)]\Big|^2_{L_2([0,T];\mathcal{V}')} + |u|^2_{L_2([0,T];\mathcal{V})}$$

implies the first desired bound in (i). If $l \in L_p([0,T])$ one may apply Young's classical inequality for convolutions to establish the asserted estimate in (i).

As to (ii), we integrate (3.38) from 0 to T, and drop the term $k_n * |u(\cdot)|^2_{\mathcal{H}}(T)$, to the result

$$\int_0^T k_n(t)|u(t)|^2_{\mathcal{H}}\,dt \le 4\Big((1*k_n)(T)|x|^2_{\mathcal{H}} + |g|_{L_1([0,T])} + |h_n|_{L_1([0,T])}\Big).$$

Observe that $(1*k_n)(T) \to (1*k)(T)$, and $|h_n|_{L_1([0,T])} \to 0$ as $n \to \infty$. Therefore, for sufficiently large n we have

$$\int_0^T k_n(t)|u(t)|^2_{\mathcal{H}}\,dt \le 8\Big((1*k)(T)|x|^2_{\mathcal{H}} + |g|_{L_1([0,T])}\Big).$$

Since $k_n \to k$ in $L_1([0,1])$, the assertion then follows from Fatou's lemma. □

In the case of fractional evolution equations we have the following corollary. Here we set

$${}_0H^\alpha_2([0,T];\mathcal{V}') := \{v|_{[0,T]} : v \in H^\alpha_2(\mathbb{R};\mathcal{V}') \text{ and } \operatorname{supp} v \subseteq \mathbb{R}_+\},$$

where $H^\alpha_2(\mathbb{R};\mathcal{V}')$ stands for the Bessel potential space of order α of $\mathcal{V}'$-valued functions on the line.

Corollary 3.3.1 *Let $T > 0$, and $\mathcal{V}$ and $\mathcal{H}$ be real Hilbert spaces as described above. Suppose (Hd), (Ha), and (Hb), and assume that $k(t) = g_{1-\alpha}(t)e^{-\mu t}$, $t > 0$, with $\alpha \in (0,1)$, and $\mu \ge 0$. Then the problem (3.2) admits exactly one solution u in the space*

$$W(\alpha; x, \mathcal{V}, \mathcal{H}) := \{u \in L_2([0,T];\mathcal{V}) : u - x \in {}_0H^\alpha_2([0,T];\mathcal{V}')\}.$$

Furthermore, we have

$$(g_{1-\alpha}e^{-\mu\cdot}) * u \in C([0,T];\mathcal{H}), \ \ u \in L_{\frac{2}{1-\alpha},w}([0,T];\mathcal{H}), \ \ \text{and} \ \int_0^T t^{-\alpha}|u(t)|^2_{\mathcal{H}}\,dt < \infty.$$

There exists a constant $M = M(\alpha,\mu,T) > 0$ such that

$$|u-x|_{{}_0H^\alpha_2([0,T];\mathcal{V}')} + |u|_{L_2([0,T];\mathcal{V})} + |(g_{1-\alpha}e^{-\mu\cdot}) * u|_{C([0,T];\mathcal{H})} + |u|_{L_{\frac{2}{1-\alpha},w}([0,T];\mathcal{H})}$$
$$+\Big(\int_0^T t^{-\alpha}|u(t)|^2_{\mathcal{H}}\,dt\Big)^{1/2} \le M\Big(|x|_{\mathcal{H}} + |f|_{L_2([0,T];\mathcal{V}')}\Big),$$

for all $x \in \mathcal{H}$ and $f \in L_2([0,T];\mathcal{V}')$.

Proof. Let l as in (2.1). Then

$$\begin{aligned}{}_0H^\alpha_2([0,T];\mathcal{V}') &= \{l * v : v \in L_2([0,T];\mathcal{V}')\} \\ &= \{v \in L_2([0,T];\mathcal{V}') : (g_{1-\alpha}e^{-\mu\cdot}) * v \in {}_0H^1_2([0,T];\mathcal{V}'),\end{aligned}$$

for the first equals sign see e.g. [81, Corollary 2.1]; the second one follows from $k*l=1$. Note further that $l\in L_{1/(1-\alpha),w}([0,T])$. So the assertions of the corollary follow immediately from the previous results. □

Note that taking *formally* the limit $\alpha\to 1$ in the above estimates (with $\mu=0$) we recover the well-known estimates for solutions of the abstract parabolic equation (3.4).

3.4 Second order problems

Let $T>0$, and Ω be a bounded domain in $\mathbb{R}^N$ with $N\geq 3$. We consider the problem

$$\begin{cases} \partial_t(k*(u-u_0)) - \operatorname{div}(A\,Du) + (b|Du) + cu = g, & t\in(0,T),\ x\in\Omega,\\ u(t,x)=0, & t\in(0,T),\ x\in\partial\Omega,\\ u(0,x)=u_0(x), & x\in\Omega. \end{cases} \tag{3.39}$$

Here $(\cdot|\cdot)$ denotes the scalar product in $\mathbb{R}^N$. We make the following assumptions on the kernel k, the coefficients, and the data.

(Hk) $(k,l)\in\mathcal{PC}$ for some $l\in L_{1,\,loc}(\mathbb{R}_+)$.

(Hd) $u_0\in L_2(\Omega)$, $g\in L_2([0,T];L_{\frac{2N}{N+2}}(\Omega))$.

(HA) $A\in L_\infty((0,T)\times\Omega;\mathbb{R}^{N\times N})$, and $\exists\nu>0$ such that

$$(A(t,x)\xi|\xi)\geq\nu|\xi|^2,\quad \text{for a.a. } t\in(0,T),\ x\in\Omega, \text{ and all}\,\xi\in\mathbb{R}^n.$$

(Hc) $b\in L_\infty((0,T)\times\Omega;\mathbb{R}^N)$, $c\in L_\infty((0,T)\times\Omega)$.

We set $\mathcal{V}=\mathring{H}^1_2(\Omega)$, and $\mathcal{H}=L_2(\Omega)$, endowed with the inner product $(u,v)_{\mathcal{H}}=\int_\Omega uv\,dx$. Define

$$a(t,u,v)=\int_\Omega\Big(\big(A(t,x)Du(x)|Dv(x)\big)+\big(b(t,x)|Du(x)\big)v(x)+c(t,x)u(x)v(x)\Big)\,dx$$

and

$$\langle f(t),v\rangle_{\mathcal{V}'\times\mathcal{V}}=\int_\Omega g(t,x)v(x)\,dx,\quad \text{a.a. } t\in(0,T).$$

Then the weak formulation of (3.39) reads

$$\frac{d}{dt}\Big([k*(u-u_0)](t),v\Big)_{\mathcal{H}}+a(t,u(t),v)=\langle f(t),v\rangle_{\mathcal{V}'\times\mathcal{V}},\quad v\in\mathcal{V},\ \text{a.a.}\,t\in(0,T), \tag{3.40}$$

and we seek a solution in the class

$$W(u_0,\mathring{H}^1_2(\Omega),L_2(\Omega))=\{u\in L_2([0,T];\mathring{H}^1_2(\Omega)):k*(u-u_0)\in{}_0H^1_2([0,T];H^{-1}_2(\Omega))\}.$$

It is folklore that in the described setting the assumptions (Hd), and (Ha) in Theorem 3.3.1 are satisfied. Concerning (Hb), we could take $\{w_1,w_2,\ldots\}$ to be the complete set of eigenfunctions for $-\Delta$ in $\mathring{H}^1_2(\Omega)$. Consequently, we obtain

Corollary 3.4.1 *Suppose the assumptions (Hk), (Hd), (HA), and (Hc) hold. Then the problem (3.39) has a unique weak solution* $u\in W(u_0,\mathring{H}^1_2(\Omega),L_2(\Omega))$ *in the sense that (3.40) is satisfied. Further,* $k*u\in C([0,T];L_2(\Omega))$*. In the case* $k(t)=g_{1-\alpha}(t)e^{-\mu t}$, $t>0$, *with* $\alpha\in(0,1)$, *and* $\mu\geq 0$, *we have*

$$u\in L_{\frac{2}{1-\alpha},\,w}([0,T];L_2(\Omega))\cap L_2([0,T];\mathring{H}^1_2(\Omega)),\ \text{and}\ u-u_0\in{}_0H^\alpha_2([0,T];H^{-1}_2(\Omega))\}.$$

Of course, a corresponding result also holds in the case $N\leq 2$ with the assumption on g appropriately modified. In any case, $g\in L_2([0,T]\times\Omega)$ is sufficient.

Chapter 4

Boundedness of weak solutions and the maximum principle

4.1 Introductory remarks

Let $T > 0$, and Ω be a bounded domain in $\mathbb{R}^N$. In this chapter we investigate linear partial integro-differential equations of the form

$$\partial_t\Big(k * (u - u_0)\Big) - \mathcal{L}u = f + \operatorname{div} g,\ t \in (0,T),\ x \in \Omega, \tag{4.1}$$

as well as related quasilinear problems

$$\partial_t\Big(k * (u - u_0)\Big) - \operatorname{div} a(t,x,u,Du) = b(t,x,u,Du),\ t \in (0,T),\ x \in \Omega, \tag{4.2}$$

where in both cases $k \in L_{1,\,loc}(\mathbb{R}_+)$ is a kernel of type $\mathcal{PC}$.

In (4.1), $\mathcal{L}$ is a second order operator w.r.t. the spatial variables in divergence form:

$$\mathcal{L}u = \operatorname{div}\Big(A(t,x)Du + b(t,x)u\Big) + (c(t,x)|Du) + d(t,x)u.$$

Here A is $\mathbb{R}^{N\times N}$-valued, b and c take values in $\mathbb{R}^N$, and d is a real-valued function. Recall that $(\cdot|\cdot)$ denotes the scalar product in $\mathbb{R}^N$. The functions $u_0 = u_0(x)$, $f = f(t,x)$, and $g = g(t,x)$ are given data; the function u_0 plays the role of the initial data for u.

Concerning the leading coefficients of $\mathcal{L}$ we merely assume measurability, boundedness, and a uniform parabolicity condition. The coefficients of the lower order terms are assumed to belong to certain Lebesgue spaces of mixed type, so they need not be bounded.

In (4.2), the functions $a : (0,T) \times \Omega \times \mathbb{R}^{N+1} \to \mathbb{R}^N$ and $b : (0,T) \times \Omega \times \mathbb{R}^{N+1} \to \mathbb{R}$ are measurable and satisfy suitable structure conditions, see (Q1)–(Q5) in Section 4.3.

One of the main objectives of this chapter is to derive results asserting the boundedness on Ω_T of appropriately defined weak solutions of (4.1) and (4.2), respectively, that are bounded on Γ_T. We further establish the analogue of the well-known weak maximum principle for weak solutions of the parabolic equation corresponding to (4.1), i.e. $\partial_t u - \mathcal{L}u = f + \operatorname{div} g$, see e.g. [49, Theorem 7.2, p. 188].

Our proofs of the global boundedness results use De Giorgi's iteration technique and are based on suitable a priori estimates for weak solutions of (4.1) and (4.2), respectively. These estimates, which by partly standard arguments (c.p. [49, Chapters III and V]) lead to suitable Caccioppoli type inequalities, are derived by means of the basic inequality (2.7).

One of the technical difficulties in deriving the desired estimates in the weak setting is to find an appropriate time regularization of the equation. In the classical parabolic theory this is achieved by means of Steklov averages in time. In the case of equations (4.1) and (4.2) this method does not work any more, since Steklov average operators and convolution do not commute. It turns out that instead one can again use the Yosida approximation of the operator B defined by $Bv = \partial_t(k * v)$, e.g. in $L_2([0,T])$, which leads to a regularization of the kernel k (not of u!).

4.2 Linear equations

In this section we study the linear equation (4.1). Let $T > 0$, and Ω be a bounded domain in $\mathbb{R}^N$ such that $\partial\Omega$ satisfies the property of positive density, the latter will be assumed throughout this chapter. In what follows (except for Theorem 4.2.2 and Theorem 4.2.3) we will assume that

(H1) There exists $l \in L_{1,\,loc}(\mathbb{R}_+)$ such that $(k,l) \in \mathcal{PC}$. Further, $l \in L_p([0,T])$ for some $p > 1$.

(H2) $A \in L_\infty((0,T)\times\Omega;\mathbb{R}^{N\times N})$, and $\exists\nu > 0$ such that

$$(A(t,x)\xi|\xi) \geq \nu|\xi|^2, \quad \text{for a.a. } (t,x) \in \Omega_T, \text{ and all } \xi \in \mathbb{R}^N.$$

(H3) $u_0 \in L_2(\Omega)$, and

$$\Big|\, |b|^2 + |g|^2 + |c|^2 + |d| + |f| \,\Big|_{L_r([0,T];L_q(\Omega))} =: C_D < \infty,$$

where

$$\frac{p'}{r} + \frac{N}{2q} = 1 - \beta,$$

and

$$r \in \left[\tfrac{p'}{(1-\beta)}, \infty\right],\ q \in \left[\tfrac{N}{2(1-\beta)}, \infty\right],\ \beta \in (0,1) \quad \text{for} \quad N \geq 2,$$
$$r \in \left[\tfrac{p'}{(1-\beta)}, \tfrac{2p'}{(1-2\beta)}\right],\ q \in [1,\infty],\ \beta \in \left(0, \tfrac{1}{2}\right) \quad \text{for} \quad N = 1.$$

We say that a function u is a *weak solution (subsolution, supersolution)* of (4.1) in Ω_T, if u belongs to the space

$$\tilde{V}_p := \{\, v \in L_{2p}([0,T];L_2(\Omega)) \cap L_2([0,T];H_2^1(\Omega)) \text{ such that } k * v \in C([0,T];L_2(\Omega)),\ \text{and } (k*v)|_{t=0} = 0\},$$

and for any nonnegative test function

$$\eta \in \overset{\circ}{H}{}_2^{1,1}(\Omega_T) := H_2^1([0,T];L_2(\Omega)) \cap L_2([0,T];\overset{\circ}{H}{}_2^1(\Omega))$$

with $\eta|_{t=T} = 0$ there holds

$$\int_0^T \int_\Omega \Big(-\eta_t[k*(u-u_0)] + (ADu + bu|D\eta) - (c|Du)\eta - du\eta \Big)\, dxdt = (\leq, \geq) \int_0^T \int_\Omega \Big(f\eta - (g|D\eta) \Big)\, dxdt. \tag{4.3}$$

It is not difficult to verify, by means of Hölder's inequality and the interpolation inequality (2.13), that under conditions (H1)-(H3) the integrals in (4.3) are finite, see also the proof of Theorem

4.2.1 below. We point out that (4.1) is considered without any boundary conditions, in this sense weak solutions of (4.1) as defined above are *local* ones (w.r.t. x). Note that for an energy estimate for weak solutions $u \in \tilde{V}_p$ of (4.1) one can work with a weaker version of condition (H3), see e.g. Theorem 4.2.2 below. We further remark that weak solutions of (4.1) in the class $\tilde{V}_p$ have already been constructed in Chapter 3 under the assumptions (H1), (H2), and a stronger variant of (H3), see Section 3.4. Notice also that the function u_0 plays the role of the initial data for u, at least in a weak sense. In case of sufficiently smooth functions u and $k * (u - u_0)$ the condition $(k * u)|_{t=0} = 0$ implies $u|_{t=0} = u_0$, see Section 3.3.

The following lemma is basic to deriving a priori estimates for weak (sub-/super-) solutions of (4.1) as it provides an equivalent weak formulation of (4.1) where the kernel k is replaced with the more regular kernel k_n ($n \in \mathbb{N}$) defined in (2.5). In what follows the kernels h_n, $n \in \mathbb{N}$, are as in Section 2.2.

Lemma 4.2.1 *Let the assumptions (H1)–(H3) be satisfied. Then $u \in \tilde{V}_p$ is a weak solution (subsolution, supersolution) of (4.1) if and only if for any nonnegative function $\psi \in \mathring{H}^1_2(\Omega)$ one has*

$$\int_\Omega \Big(\psi \partial_t [k_n {*} (u - u_0)] + (h_n * [ADu + bu]|D\psi) - (h_n * [(c|Du) + du])\psi\Big)\, dx$$
$$= (\le, \ge) \int_\Omega \Big([h_n * f]\psi - (h_n * g|D\psi)\Big)\, dx, \quad a.a.\ t \in (0,T),\ n \in \mathbb{N}. \tag{4.4}$$

Proof. We may restrict ourselves to the subsolution case as the remaining cases can be treated analogously.

The 'if' part is readily seen as follows. Given an arbitrary nonnegative $\eta \in \mathring{H}^{1,1}_2(\Omega_T)$ satisfying $\eta|_{t=T} = 0$, we take in (4.4) $\psi(x) = \eta(t,x)$ for any fixed $t \in (0,T)$, integrate from $t = 0$ to $t = T$, and integrate by parts w.r.t. the time variable. Relation (4.3) then follows by sending $n \to \infty$; here we use the approximating properties of the kernels h_n described in Section 2.2.

To show the 'only–if' part, we choose the test function

$$\eta(t,x) = \int_t^T h_n(\sigma - t)\varphi(\sigma, x)\, d\sigma = \int_0^{T-t} h_n(\sigma)\varphi(\sigma + t, x)\, d\sigma, \quad t \in (0,T),\ x \in \Omega, \tag{4.5}$$

with arbitrary $n \in \mathbb{N}$ and nonnegative $\varphi \in \mathring{H}^{1,1}_2(\Omega_T)$ satisfying $\varphi|_{t=T} = 0$; η is nonnegative since φ and h_n are so (see Section 2.2). Then

$$\eta_t(t,x) = \int_t^T h_n(\sigma - t)\varphi_\sigma(\sigma, x)\, d\sigma, \quad \text{a.a. } (t,x) \in \Omega_T.$$

By Fubini's theorem, we have

$$\int_0^T \Big(\int_t^T h_n(\sigma - t)\psi_1(\sigma)\, d\sigma\Big)\psi_2(t)\, dt = \int_0^T \psi_1(t)\Big(\int_0^t h_n(t - \sigma)\psi_2(\sigma)\, d\sigma\Big)\, dt,$$

for all ψ_1, $\psi_2 \in L_2([0,T])$. So it follows from (4.3) and $k_n = h_n * k$ (c.p. (2.5)) that

$$\int_0^T \int_\Omega \Big(-\varphi_t [k_n {*} (u - u_0)] + (h_n * [ADu + bu]|D\varphi) - (h_n * [(c|Du) + du])\varphi\Big)\, dx\, dt$$
$$\le \int_0^T \int_\Omega \Big([h_n * f]\varphi - (h_n * g|D\varphi)\Big)\, dx\, dt, \quad n \in \mathbb{N}.$$

Observe that $k_n * (u - u_0) \in {}_0H_2^1([0,T]; L_2(\Omega))$. Therefore, integrating by parts and using $\varphi|_{t=T} = 0$ yields

$$\int_0^T \int_\Omega \Big(\varphi\partial_t[k_n*(u-u_0)] + (h_n * [ADu + bu]|D\varphi) - (h_n * [(c|Du) + du])\varphi\Big)\, dx\, dt$$
$$\leq \int_0^T \int_\Omega \Big([h_n * f]\varphi - (h_n * g|D\varphi)\Big)\, dx\, dt \qquad (4.6)$$

for all $n \in \mathbb{N}$ and $\varphi \in \mathring{H}_2^{1,1}(\Omega_T)$ with $\varphi|_{t=T} = 0$. By means of a simple approximation argument, we infer that (4.6) holds true for any φ of the form $\varphi(t,x) = \chi_{(t_1,t_2)}(t)\psi(x)$, where $\chi_{(t_1,t_2)}$ denotes the characteristic function of the time-interval (t_1,t_2), $0 < t_1 < t_2 < T$, and $\psi \in \mathring{H}_2^1(\Omega)$ is nonnegative. Appealing to the Lebesgue differentiation theorem, we then obtain the desired relation (4.4). ☐

In what follows we say that a function $u \in \tilde{V}_p$ satisfies $u \leq K$ a.e. on Γ_T for some number $K \in \mathbb{R}$ if $(u-K)_+ \in L_2([0,T]; \mathring{H}_2^1(\Omega))$, likewise for lower bounds on Γ_T.

Theorem 4.2.1 *Let $T > 0$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain. Let further the assumptions (H1) – (H3) be satisfied. Suppose $K \geq 0$ is such that $u_0 \leq K$ a.e. in Ω. Then there exists a constant $C = C(p, q, r, |l|_{L_p([0,T])}, T, N, \nu, \Omega, C_D)$ such that for any weak subsolution $u \in \tilde{V}_p$ of (4.1) in Ω_T satisfying $u \leq K$ a.e. on Γ_T there holds $u \leq C(1+K)$ a.e. in Ω_T.*

Remarks 4.2.1 (i) There is a corresponding result for weak supersolutions u of (4.1) in the situation where $u_0 \geq K$ a.e. in Ω, and $u \geq K$ a.e. on Γ_T, for some $K \leq 0$. This follows immediately from Theorem 4.2.1 by replacing u with $-u$, and u_0 with $-u_0$.

(ii) The statement of Theorem 4.2.1 remains true if r and q in (H3) are different for different coefficients and data, that is when $|b|^2 \in L_{r_1}([0,T]; L_{q_1}(\Omega))$, $|g|^2 \in L_{r_2}([0,T]; L_{q_2}(\Omega))$, and so forth with r_i and q_i satisfying the same conditions as r and q in (H3). This can been seen by working with several functions $\mu_{\kappa,i}$ and by generalizing the iteration argument for the function ϕ, see below. In the classical parabolic case this issue is discussed in [49, Chapter III, Remark 7.2].

Proof of Theorem 4.2.1. Suppose $u \in \tilde{V}_p$ is a weak subsolution of (4.1) in Ω_T. Then, by Lemma 4.2.1, for any nonnegative function $\psi \in \mathring{H}_2^1(\Omega)$ relation (4.4) holds with the '$\leq$' sign. For $t \in (0,T)$ we take in (4.4) the test function $\psi = u_\kappa^+ := (u_\kappa)_+$, where $u_\kappa := u - \kappa$, and $\kappa \in \mathbb{R}$ satisfying the condition

$$\kappa \geq \kappa_0 := \max\{0, \operatorname*{ess\,sup}_{\Omega} u_0, \operatorname*{ess\,sup}_{\Gamma_T} u\}. \qquad (4.7)$$

The resulting inequality can be written in the form

$$\int_\Omega \Big(u_\kappa^+ \partial_t(k_n * u_\kappa) + (h_n * [ADu + bu]|Du_\kappa^+) - (h_n * [(c|Du) + du])u_\kappa^+\Big)\, dx$$
$$\leq \int_\Omega \Big([h_n * f]u_\kappa^+ - (h_n * g|Du_\kappa^+) + u_\kappa^+(u_0 - \kappa)k_n\Big)\, dx, \quad \text{a.a. } t \in (0,T). \qquad (4.8)$$

Clearly,

$$\int_\Omega u_\kappa^+(u_0 - \kappa)k_n\, dx \leq 0, \quad \text{a.a. } t \in (0,T),$$

by positivity of k_n and (4.7). Thanks to (2.7) we further have

$$u_\kappa^+ \partial_t(k_n * u_\kappa) \geq \frac{1}{2}\,\partial_t\Big(k_n * (u_\kappa^+)^2\Big), \quad \text{a.a. } (t,x) \in \Omega_T. \qquad (4.9)$$

Using these relations it follows from (4.8) that

$$\int_\Omega \Big(\frac{1}{2}\,\partial_t[k_n*(u_\kappa^+)^2]+(h_n*[ADu+bu]|Du_\kappa^+)-(h_n*[(c|Du)+du])u_\kappa^+\Big)\,dx$$
$$\leq \int_\Omega \Big([h_n*f]u_\kappa^+-(h_n*g|Du_\kappa^+)\Big)\,dx,\quad \text{a.a. } t\in(0,T). \tag{4.10}$$

Next we convolve (4.10) with the nonnegative kernel l from assumption (H1), and observe that in view of

$$k_n*(u_\kappa^+)^2\in {}_0H_1^1([0,T];L_1(\Omega))$$

and $k_n=k*h_n$ we have

$$l*\partial_t\Big(k_n*(u_\kappa^+)^2\Big)=\partial_t\Big(l*k_n*(u_\kappa^+)^2\Big)=h_n*(u_\kappa^+)^2.$$

Sending then $n\to\infty$, and selecting an appropriate subsequence, if necessary, we thus arrive at

$$\frac{1}{2}\int_\Omega (u_\kappa^+)^2\,dx+l*\int_\Omega (ADu|Du_\kappa^+)\,dx\leq l*F,\quad \text{a.a. } t\in(0,T), \tag{4.11}$$

where

$$F(t)=\int_\Omega \Big(-(bu+g|Du_\kappa^+)+[(c|Du)+du+f]u_\kappa^+\Big)\,dx.$$

By (H2), we have

$$\int_\Omega (ADu|Du_\kappa^+)\,dx=\int_\Omega (ADu_\kappa^+|Du_\kappa^+)\,dx\geq \nu\int_\Omega |Du_\kappa^+|^2\,dx, \tag{4.12}$$

and thus

$$\int_\Omega (u_\kappa^+)^2\,dx\leq 2\,l*F,\quad \text{a.a. } t\in(0,T).$$

Young's inequality for convolutions then gives

$$|u_\kappa^+|^2_{L_{2p}([0,t_1];L_2(\Omega))}=|(u_\kappa^+)^2|_{L_p([0,t_1];L_1(\Omega))}$$
$$\leq 2|l|_{L_p([0,t_1])}|F|_{L_1([0,t_1])}\leq 2|l|_{L_p([0,T])}|F|_{L_1([0,t_1])} \tag{4.13}$$

for all $t_1\in(0,T]$.

Returning to (4.11), we may also drop the first term, convolve the resulting inequality with k, and use $k*l=1$ as well as (4.12), thereby obtaining

$$\nu|Du_\kappa^+|^2_{L_2([0,t_1];L_2(\Omega))}\leq |F|_{L_1([0,t_1])}. \tag{4.14}$$

In order to estimate $|F|_{L_1([0,t_1])}$, which appears on the right side of both (4.13) and (4.14), we proceed similarly as in [49, p. 184]. We denote the Lebesgue measure in $\mathbb{R}^N$ by λ_N and set

$$A_\kappa(t)=\{x\in\Omega: u(t,x)>\kappa\},\quad t\in(0,T).$$

Then

$$|F|_{L_1([0,t_1])}\leq \varepsilon|Du_\kappa^+|^2_{L_2([0,t_1];L_2(\Omega))}$$
$$+C(\varepsilon)\int_0^{t_1}\int_{A_\kappa(t)}\Big(|b|^2u^2+|g|^2+|c|^2(u_\kappa)^2+|du|u_\kappa+|f|u_\kappa\Big)\,dx\,dt,$$

for all $\varepsilon > 0$. Selecting ε sufficiently small and assuming $\kappa \geq 1$, this together with (4.13), and (4.14) gives

$$|u_\kappa^+|^2_{V_p([0,t_1]\times\Omega)} \leq C(\nu, |l|_p, T, p) \int_0^{t_1} \int_{A_\kappa(t)} \mathcal{D}(t,x)\Big((u_\kappa)^2 + \kappa^2\Big)\, dx\, dt, \tag{4.15}$$

where $|l|_p := |l|_{L_p([0,T])}$, and

$$\mathcal{D}(t,x) = |b(t,x)|^2 + |g(t,x)|^2 + |c(t,x)|^2 + |d(t,x)| + |f(t,x)|,$$

and $V_p([0,t_1]\times\Omega)$ is defined as in (2.11). Using Hölder's inequality and (H3) we thus have with $1/r + 1/r' = 1$ and $1/q + 1/q' = 1$ that

$$|u_\kappa^+|^2_{V_p([0,t_1]\times\Omega)} \leq C|\mathcal{D}|_{L_r([0,t_1];L_q(\Omega))}|(u_\kappa^+)^2 + \kappa^2\chi_{\{u>\kappa\}}|_{L_{r'}([0,t_1];L_{q'}(\Omega))}; \tag{4.16}$$

here C is as in (4.15), and $\chi_{\{u>\kappa\}}$ denotes the characteristic function of the set of points $(t,x) \in (0,t_1)\times\Omega$ at which $u(t,x) > \kappa$. We may then estimate, using again Hölder's inequality,

$$|(u_\kappa^+)^2|_{L_{r'}([0,t_1];L_{q'}(\Omega))} \leq |u_\kappa^+|^2_{L_{2r'(1+\delta)}([0,t_1];L_{2q'(1+\delta)}(\Omega))}\mu_\kappa^{\frac{\delta}{r'(1+\delta)}}, \tag{4.17}$$

with

$$\mu_\kappa = \begin{cases} \int_0^{t_1} \lambda_N(A_\kappa(t))^{\frac{r'}{q'}}\, dt & : q > 1 \\ \lambda_1(\{t \in (0,t_1) : \lambda_N(A_\kappa(t)) > 0\}) & : q = 1, \end{cases}$$

and

$$\delta = \frac{2\beta}{2(p'-1)+N}.$$

It is not difficult to verify that, by virtue of (H3), the numbers $\tilde{r} := 2r'(1+\delta)$ and $\tilde{q} := 2q'(1+\delta)$ are subject to the condition (2.12) with (r,q) being replaced by $(\tilde{r},\tilde{q})$. Therefore, using inequality (2.13), it follows from (4.17) that

$$|(u_\kappa^+)^2|_{L_{r'}([0,t_1];L_{q'}(\Omega))} \leq C(N,q)|u_\kappa^+|^2_{V_p([0,t_1]\times\Omega)}\mu_\kappa^{\frac{\delta}{r'(1+\delta)}}. \tag{4.18}$$

We may further write

$$|\kappa^2\chi_{\{u>\kappa\}}|_{L_{r'}([0,t_1];L_{q'}(\Omega))} = \kappa^2\mu_\kappa^{\frac{1}{r'}}. \tag{4.19}$$

Combining (4.16), (4.18), and (4.19) we obtain

$$|u_\kappa^+|^2_{V_p([0,t_1]\times\Omega)} \leq C_1|\mathcal{D}|_{L_r([0,t_1];L_q(\Omega))}\Big(|u_\kappa^+|^2_{V_p([0,t_1]\times\Omega)}\mu_\kappa^{\frac{\delta}{r'(1+\delta)}} + \kappa^2\mu_\kappa^{\frac{1}{r'}}\Big), \tag{4.20}$$

with $C_1 = C_1(\nu, |l|_p, T, p, N, q)$.

We now choose $t_1 = T/n$ where $n \in \mathbb{N}$ is so large that

$$C_1|\mathcal{D}|_{L_r([0,T];L_q(\Omega))} t_1^{\frac{\delta}{r'(1+\delta)}} \lambda_N(\Omega)^{\frac{\delta}{q'(1+\delta)}} \leq \frac{1}{2}. \tag{4.21}$$

Setting $C_2^2 = 2C_1|\mathcal{D}|_{L_r([0,T];L_q(\Omega))}$, inequality (4.20) then implies

$$|u_\kappa^+|^2_{V_p([0,t_1]\times\Omega)} \leq C_2^2\kappa^2\mu_\kappa^{\frac{1}{r'}}, \quad \kappa \geq \tilde{\kappa}_0 := \max\{\kappa_0, 1\}. \tag{4.22}$$

Define the function

$$\phi(\kappa) = \mu_\kappa^{\frac{1}{\tilde{r}}}, \quad \kappa \geq \tilde{\kappa}_0.$$

We will show that $\phi(2M) = 0$ provided $M \geq \tilde{\kappa}_0$ is sufficiently large. The argument is analogous to the proof of Theorem 6.1 in Chapter II of [49]. For the sake of completeness we give the details.

By virtue of inequalities (2.13) and (4.22), we have for any $\kappa_2 > \kappa_1 \geq \tilde{\kappa}_0$

$$(\kappa_2 - \kappa_1)\phi(\kappa_2) \leq |u^+_{\kappa_1}|_{L_{\tilde{r}}([0,t_1];L_{\tilde{q}}(\Omega))} \leq C(N,q)|u^+_{\kappa_1}|_{V_p([0,t_1]\times\Omega)} \leq C_3\kappa_1\phi(\kappa_1)^{1+\delta}, \qquad (4.23)$$

where $C_3 = CC_2$. We take $\kappa_2 = \xi_{n+1}$ and $\kappa_1 = \xi_n$ with $\xi_n = M(2-2^{-n})$, $n = 0, 1, 2, \ldots$, and $M \geq \tilde{\kappa}_0$ being fixed. This gives

$$\phi(\xi_{n+1}) \leq \frac{C_3\xi_n}{\xi_{n+1}-\xi_n}\,\phi(\xi_n)^{1+\delta} \leq 4C_3 2^n \phi(\xi_n)^{1+\delta},$$

which, together with Lemma 2.4.1, shows that the sequence $Y_n = \phi(\xi_n)$, $n = 0, 1, \ldots$, will go to zero as $n \to \infty$, provided $\phi(\xi_0)$ is sufficiently small, namely

$$\phi(\xi_0) = \phi(M) \leq (4C_3)^{-1/\delta}2^{-1/\delta^2}. \qquad (4.24)$$

By taking in (4.23) $\kappa_2 = M = m\tilde{\kappa}_0$ and $\kappa_1 = \tilde{\kappa}_0$, we obtain

$$\phi(M) \leq \frac{C_3}{m-1}\,\phi(\tilde{\kappa}_0)^{1+\delta} \leq \frac{C_3}{m-1}\,t_1^{(1+\delta)/\tilde{r}}\lambda_N(\Omega)^{(1+\delta)/\tilde{q}}.$$

Hence (4.24) is satisfied for

$$m = 1 + C_3 t_1^{(1+\delta)/\tilde{r}}\lambda_N(\Omega)^{(1+\delta)/\tilde{q}}(4C_3)^{1/\delta}2^{1/\delta^2}.$$

It follows that for this m

$$\operatorname*{ess\,sup}_{[0,t_1]\times\Omega} u \leq 2M = 2m\tilde{\kappa}_0. \qquad (4.25)$$

To obtain a bound on the whole time-interval $[0,T]$, we proceed by induction. Using (4.25) we next derive an estimate on $[t_1, 2t_1]$, which together with (4.25) is then employed to find an upper bound on $[2t_1, 3t_1]$, and so forth until we reach T after finitely many steps. Due to the nonlocalness of the integro-differential operator in time, in each step we have to use the bounds established in all of the previous steps, that is up to $t = 0$.

Let $T_0 \in (0,T)$ and suppose that $u \in \tilde{V}_p$ is a weak subsolution of (4.1) in Ω_T which is bounded above on $[0,T_0]\times\Omega$. Then as above we have

$$\int_\Omega \Big(\psi\partial_t(k_n * u_\kappa) + (h_n * [ADu + bu]|D\psi) - (h_n * [(c|Du) + du])\psi\Big)\,dx$$
$$\leq \int_\Omega \Big([h_n * f]\psi - (h_n * g|D\psi) + \psi(u_0 - \kappa)k_n\Big)\,dx, \quad \text{a.a. } t \in (T_0,T), \qquad (4.26)$$

for any nonnegative $\psi \in \mathring{H}^1_2(\Omega)$, $\kappa \in \mathbb{R}$, and $n \in \mathbb{N}$. Recall that $k_n \in H^1_1([0,T])$ with derivative $\dot{k}_n \leq 0$. We define

$$H_{\kappa,n}(t,x) = \int_0^{T_0} [-\dot{k}_n(t-\tau)]u_\kappa(\tau,x)\,d\tau, \quad t \in (T_0,T),\ x \in \Omega. \qquad (4.27)$$

By Jensen's inequality,

$$|H_{\kappa,n}(t,x)|^2 \leq \Big(k_n(t-T_0) - k_n(t)\Big)\int_0^{T_0} [-\dot{k}_n(t-\tau)]\,|u_\kappa(\tau,x)|^2\,d\tau, \qquad (4.28)$$

which shows that $H_{k,n} \in L_2([T_0,T]\times\Omega)$. Therefore we may use the decomposition

$$(k_n * u_\kappa)(t,x) = \int_{T_0}^{t} k_n(t-\tau)u_\kappa(\tau,x)\,d\tau + \int_0^{T_0} k_n(t-\tau)u_\kappa(\tau,x)\,d\tau, \quad t\in(T_0,T),$$

to rewrite (4.26) as

$$\int_\Omega \Big(\psi\partial_t \int_{T_0}^{t} k_n(t-\tau)u_\kappa(\tau,x)\,d\tau + (h_n*[ADu+bu]|D\psi) - (h_n*[(c|Du)+du])\psi\Big)\,dx$$
$$\le \int_\Omega \Big([h_n*f]\psi - (h_n*g|D\psi) + \psi(u_0-\kappa)k_n + \psi H_{\kappa,n}\Big)\,dx, \quad \text{a.a. } t\in(T_0,T). \tag{4.29}$$

We then shift the time according to $s=t-T_0$. Employing the notation $\tilde v(s) = v(s+T_0)$, $s\in(0,T-T_0)$, for functions v defined on (T_0,T), (4.29) becomes

$$\int_\Omega \Big(\psi\partial_s(k_n*\tilde u_\kappa) + ((h_n*[ADu+bu])\tilde{}\,|D\psi) - (h_n*[(c|Du)+du])\tilde{}\,\psi\Big)\,dx$$
$$\le \int_\Omega \Big([h_n*f]\tilde{}\,\psi - ((h_n*g)\tilde{}\,|D\psi) + \psi(u_0-\kappa)\tilde k_n + \psi\tilde H_{\kappa,n}\Big)\,dx, \quad \text{a.a. } s\in(0,T-T_0). \tag{4.30}$$

Setting $T_0=t_1$, we can now argue as above to get an upper bound for u on $[t_1,2t_1]\times\Omega$. We restrict κ to

$$\kappa \ge \tilde\kappa_1 := \max\{\tilde\kappa_0, \operatorname*{ess\,sup}_{[0,t_1]\times\Omega} u\} = \max\{\tilde\kappa_0, 2m\tilde\kappa_0\} = 2m\tilde\kappa_0,$$

which entails that $u_0-\kappa\le 0$ as well as $\tilde H_{\kappa,n}\le 0$. Consequently, the terms involving these functions can be dropped in (4.30). We take $\psi=\tilde u_\kappa^+$ and use the analogue of (4.9). Convolving the resulting inequality with l, and sending $n\to\infty$ then yields

$$\frac12\int_\Omega (\tilde u_\kappa^+)^2\,dx + l*\int_\Omega (\tilde A D\tilde u|D\tilde u_\kappa^+)\,dx \le l*\tilde F, \quad \text{a.a. } s\in(0,T-t_1),$$

which is the time shifted version of (4.11). We conclude that

$$\operatorname*{ess\,sup}_{[t_1,t_2]\times\Omega} u \le 2m\tilde\kappa_1 = 4m^2\tilde\kappa_0. \tag{4.31}$$

These arguments can now be repeated for the time-intervals $[jt_1,(j+1)t_1]$, $j=2,\ldots,n-1$, thereby obtaining a bound

$$\operatorname*{ess\,sup}_{\Omega_T} u \le C\tilde\kappa_0,$$

with a constant $C=C(p,q,r,|l|_p,T,N,\nu,\lambda_N(\Omega),C_D)$. □

As an immediate consequence of Theorem 4.2.1 and Remark 4.2.1(i) we obtain the global boundedness of weak solutions of (4.1) that are bounded on the parabolic boundary of Ω_T.

Corollary 4.2.1 *Let $T>0$ and $\Omega\subset\mathbb{R}^N$ be a bounded domain. Assume that the assumptions (H1) – (H3) are satisfied. Suppose $K\ge 0$ is such that $|u_0|\le K$ a.e. in Ω. Then there exists a constant $C=C(p,q,r,|l|_{L_p([0,T])},T,N,\nu,\Omega,C_D)$ such that for any weak solution $u\in\tilde V_p$ of (4.1) in Ω_T satisfying $|u|\le K$ a.e. on Γ_T there holds $|u|\le C(1+K)$ a.e. in Ω_T.*

For weak subsolutions (supersolutions) of (4.1) the maximum (minimum) principle is valid in the subsequent form. Let (H3') stand for

$$u_0 \in L_2(\Omega), \qquad \Big| |c|^2 + |d| \Big| \in L_r([0,T]; L_q(\Omega)),$$

where

$$\frac{p'}{r} + \frac{N}{2q} = 1,$$

and

$$\begin{array}{ll} r \in [p', \infty),\ q \in \left[\frac{N}{2}, \infty\right] & \text{for} \quad N \geq 2, \\ r \in [p', 2p'],\ q \in [1, \infty] & \text{for} \quad N = 1. \end{array}$$

Theorem 4.2.2 *Let $T > 0$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain. Suppose that the conditions (H1),(H2), and (H3') are fulfilled, and assume that $b \equiv g \equiv 0$, $f \equiv 0$, and $d \leq 0$ in Ω_T. Then for any weak subsolution (supersolution) $u \in \tilde{V}_p$ of (4.1), we have for a.a. $(t,x) \in \Omega_T$*

$$u(t,x) \leq \max\Big\{0, \operatorname*{ess\,sup}_{\Omega} u_0, \operatorname*{ess\,sup}_{\Gamma_T} u\Big\} \qquad \Big(u(t,x) \geq \min\Big\{0, \operatorname*{ess\,inf}_{\Omega} u_0, \operatorname*{ess\,inf}_{\Gamma_T} u\Big\} \Big),$$

provided this maximum (minimum) is finite.

Proof. It suffices to consider the subsolution case. Note first that Lemma 4.2.1 also holds under the conditions of Theorem 4.2.2. We take

$$\kappa = \max\Big\{0, \operatorname*{ess\,sup}_{\Omega} u_0, \operatorname*{ess\,sup}_{\Gamma_T} u\Big\}$$

in (4.11), assuming that this quantity is finite. By the assumptions on the coefficients and data, we have

$$F(t) \leq G(t) := \int_{\Omega} (c|Du) u_\kappa^+ \, dx, \quad \text{a.a. } t \in (0,T).$$

We may then argue similarly as in the proof of Theorem 4.2.1 to find that for any $t_1 \in (0,T]$

$$|u_\kappa^+|^2_{V_p([0,t_1]\times\Omega)} \leq C(\nu, |l|_{L_p([0,T])}, p, T) |G|_{L_1([0,t_1])},$$

and thus

$$|u_\kappa^+|^2_{V_p([0,t_1]\times\Omega)} \leq \tilde{C}(\nu, |l|_p, p, T) \Big| |c|^2 \Big|_{L_r([0,t_1];L_q(\Omega))} |u_\kappa^+|^2_{L_{2r'}([0,t_1];L_{2q'}(\Omega))}. \tag{4.32}$$

By (H3'), the numbers $2r'$ and $2q'$ are subject to the condition (2.12). Therefore, using inequality (2.13), we deduce that

$$|u_\kappa^+|^2_{V_p([0,t_1]\times\Omega)} \leq C_0 \Big| |c|^2 \Big|_{L_r([0,t_1];L_q(\Omega))} |u_\kappa^+|^2_{V_p([0,t_1]\times\Omega)},$$

with a positive constant $C_0 = C_0(\nu, |l|_p, p, T, N, q)$. For t_1 satisfying the condition

$$C_0 \Big| |c|^2 \Big|_{L_r([0,t_1];L_q(\Omega))} < 1$$

we then obtain

$$|u_\kappa^+|^2_{V_p([0,t_1]\times\Omega)} \leq 0,$$

that is $u \le \kappa$ a.e. in $(0,t_1)\times\Omega$. To establish this inequality on Ω_T we proceed by induction as in the proof of Theorem 4.2.1, using the fact that the function $H_{\kappa,n}$ defined in (4.27) is nonpositive on (T_0,T) whenever $u \le \kappa$ a.e. in $(0,T_0)\times\Omega$. $\square$

We remark that in the time fractional case the maximum principle stated above was recently reproved in [53] for classical solutions.

In all of the previous results we assumed that the kernel l belongs to $L_p([0,T])$ for some $p>1$. It turns out that the maximum principle still holds when this assumption is dropped and in addition we have $c \equiv 0$.

Theorem 4.2.3 *Let $T>0$ and $\Omega\subset\mathbb{R}^N$ be a bounded domain. Suppose there exists $l\in L_{1,\,loc}(\mathbb{R}_+)$ such that $(k,l)\in\mathcal{PC}$. Let further (H2) be satisfied, and assume that $u_0\in L_2(\Omega)$, $b\equiv c\equiv g\equiv 0$, $f\equiv 0$, and $0\ge d\in L_\infty([0,T];L_q(\Omega))$, where $q\in[N/2,\infty]$ for $N\ge 3$, $q\in(1,\infty]$ for $N=2$, and $q\in[1,\infty]$ for $N=1$. Then for any weak subsolution (supersolution) $u\in\tilde{V}_1$ of (4.1), we have for a.a. $(t,x)\in\Omega_T$*

$$u(t,x)\le\max\Big\{0,\operatorname*{ess\,sup}_{\Omega}u_0,\operatorname*{ess\,sup}_{\Gamma_T}u\Big\}\qquad\Big(u(t,x)\ge\min\Big\{0,\operatorname*{ess\,inf}_{\Omega}u_0,\operatorname*{ess\,inf}_{\Gamma_T}u\Big\}\Big),$$

provided this maximum (minimum) is finite.

Proof. We proceed as in the proof of the preceding theorem. Observe that the assumptions on d ensure that $duu_\kappa^+\in L_1(\Omega_T)$. Since $c\equiv 0$, we have this time $F\le 0$ a.e. in $(0,T)$, and hence $(V_1(\Omega_T)=L_2([0,T];H_2^1(\Omega)))$

$$|u_\kappa^+|^2_{V_1([0,T]\times\Omega)}\le 0,\quad \text{with } \kappa=\max\Big\{0,\operatorname*{ess\,sup}_{\Omega}u_0,\operatorname*{ess\,sup}_{\Gamma_T}u\Big\},$$

which immediately implies the assertion. $\square$

We remind the reader that in Example 2.2.2 it was shown that the case $p=1$ can occur, that is, there exist pairs $(k,l)\in\mathcal{PC}$ such that for any $p>1$ and $T>0$ the kernel l does not belong to $L_p([0,T])$.

4.3 Quasilinear equations

In this section we extend the previous results to quasilinear equations of the form (4.2) with suitable structure conditions. This is possible, as also known from the elliptic and parabolic case, since the test function method used above does not depend so much on the linearity of the operator $\mathcal{L}$ but on a certain nonlinear structure.

Let (H1) hold, and $u_0\in L_2(\Omega_T)$. We will assume that the functions $a:\Omega_T\times\mathbb{R}^{N+1}\to\mathbb{R}^N$ and $b:\Omega_T\times\mathbb{R}^{N+1}\to\mathbb{R}$ are measurable and satisfy

(Q1) $$(a(t,x,\xi,\eta)|\eta)\ge C_0|\eta|^2-c_0|\xi|^\gamma-\varphi_0(t,x),$$

(Q2) $$|a(t,x,\xi,\eta)|\le C_1|\eta|+c_1|\xi|^{\tilde\gamma}+\varphi_1(t,x),$$

(Q3) $$|b(t,x,\xi,\eta)|\le C_2|\eta|^{\frac{2(\gamma-1)}{\gamma}}+c_2|\xi|^{\gamma-1}+\varphi_2(t,x),$$

for a.a. $(t,x)\in\Omega_T$, and all $\xi\in\mathbb{R}$, $\eta\in\mathbb{R}^N$. Here C_i,c_i, $i=0,1,2$, are positive constants, and

(Q4) The parameter γ lies in the range

$$2 \leq \gamma < 2\tilde{\gamma}, \quad \text{with } \tilde{\gamma} := \frac{2p' + N}{2p' + N - 2}.$$

(Q5) The functions φ_i, $i = 0, 1, 2$, defined on Ω_T are nonnegative, $\varphi_1 \in L_2(\Omega_T)$, and $\varphi_0, \varphi_2 \in L_{\hat{q}}(\Omega_T)$, where

$$\frac{1}{\hat{q}}\Big(p' + \frac{N}{2}\Big) = 1 - \hat{\beta}, \quad \hat{\beta} \in (0, 1].$$

A function $u \in \tilde{V}_p$ is called a *weak solution (subsolution, supersolution)* of equation (4.2) in Ω_T, if $a(t, x, u, Du)$ and $b(t, x, u, Du)$ are measurable, and for any nonnegative test function $\eta \in \mathring{H}^{1,1}_2(\Omega_T)$ with $\eta|_{t=T} = 0$ there holds

$$\int_0^T \int_\Omega \Big(-\eta_t[k * (u - u_0)] + (a(t, x, u, Du)|D\eta) - b(t, x, u, Du)\eta\Big)\, dx\, dt = (\leq, \geq)\ 0. \quad (4.33)$$

One verifies using (2.13), which shows $V_p \hookrightarrow L_{2\tilde{\gamma}}(\Omega_T)$, and Hölder's inequality that under the above structure conditions this definition makes sense, see also the estimates below.

Theorem 4.3.1 *Let $T > 0$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain. Let $u_0 \in L_2(\Omega)$, and assume that (H1), (Q1)–(Q5) are satisfied. Let q be a fixed positive number such that*

$$(\gamma - 2)\Big(p' + \frac{N}{2}\Big) < q \leq 2\tilde{\gamma}.$$

Suppose further that $K \geq 0$ is such that $u_0 \leq K$ a.e. in Ω. Then any weak subsolution $u \in \tilde{V}_p$ of (4.2) satisfying $u \leq K$ a.e. on Γ_T is essentially bounded above in Ω_T by a constant C depending only on the data, q, and $|u|_{L_q(\Omega_T)}$. In the case $\gamma = 2$, the constant C depends only on the data.

An analogous result holds for supersolutions that are bounded below on the parabolic boundary, c.p. Remark 4.2.1(i) in the linear case.

Proof of Theorem 4.3.1. We proceed as in the linear case. Note first that one can easily prove a result analogous to Lemma 4.2.1. Following the lines in the proof of Theorem 4.2.1 we obtain for $\kappa \geq \kappa_0$ (see (4.7)), by means of the assumed structure conditions,

$$|u^+_\kappa|^2_{V_p([0,t_1]\times\Omega)} \leq C \int_0^{t_1} \int_{A_\kappa(t)} \Big(\Big[|Du|^{\frac{2(\gamma-1)}{\gamma}} + |u|^{\gamma-1} + \varphi_2\Big] u^+_\kappa + |u|^\gamma + \varphi_0 \Big)\, dx\, dt, \quad (4.34)$$

where the constant C depends only on $|l|_p, T, p$ and the constants appearing in (Q1) and (Q3). The first term on the right is estimated using Young's inequality,

$$|Du|^{\frac{2(\gamma-1)}{\gamma}} u^+_\kappa \leq \varepsilon |Du|^2 + C(\varepsilon)(u^+_\kappa)^\gamma, \quad \varepsilon > 0.$$

Hence, choosing ε sufficiently small, the gradient term can be absorbed by the left hand side in (4.34). Setting $\mu_\kappa := |\lambda_N(A_\kappa(\cdot))|_{L_1([0,t_1])}$,

$$\beta := 1 - \frac{1}{q}(\gamma - 2)\Big(p' + \frac{N}{2}\Big) \in (0, 1], \quad \text{and } \delta := \frac{2\beta}{2(p' - 1) + N},$$

we further have (c.p. [49, p. 425, 426])

$$\int_0^{t_1} \int_{A_\kappa(t)} |u|^\gamma\, dx\, dt \leq |u|^{\gamma-2}_{L_q(\Omega_T)} |u\chi_{\{u>\kappa\}}|^2_{L_{\frac{2q}{q-(\gamma-2)}}([0,t_1]\times\Omega)}$$
$$\leq C(N, q)|u|^{\gamma-2}_{L_q(\Omega_T)} \Big(|u^+_\kappa|^2_{V_p([0,t_1]\times\Omega)} \mu_\kappa^{\frac{\delta(q-\gamma+2)}{(1+\delta)q}} + \kappa^2 \mu_\kappa^{\frac{q-(\gamma-2)}{q}} \Big). \quad (4.35)$$

Recall that $V_p \hookrightarrow L_{2\bar{\gamma}}(\Omega_T)$, so $|u|_{L_q(\Omega_T)}$ is finite.

As in the proof of Theorem 4.2.1 we may estimate, with the aid of (Q5),

$$\int_0^{t_1}\int_{A_\kappa(t)} (\varphi_2 u_\kappa^+ + \varphi_0)\,dx\,dt \le C(N,\hat{q})|\varphi_2+\varphi_0|_{L_{\hat{q}}(\Omega_T)}\Big(|u_\kappa^+|^2_{V_p([0,t_1]\times\Omega)}\mu_\kappa^{\frac{\hat{\delta}}{\hat{q}'(1+\hat{\delta})}} + \kappa^2\mu_\kappa^{\frac{1}{\hat{q}'}}\Big), \quad (4.36)$$

provided that $\kappa \ge 1$; here $\hat{\delta}$ is defined as δ with β replaced by $\hat{\beta}$. From (4.34)– (4.36) and the trivial inequality $\mu_\kappa \le t_1\lambda_N(\Omega)$ we then infer that

$$|u_\kappa^+|^2_{V_p([0,t_1]\times\Omega)} \le C\Big(|u_\kappa^+|^2_{V_p([0,t_1]\times\Omega)} t_1^\rho + \kappa^2\mu_\kappa^{\min\left\{\frac{q-(\gamma-2)}{q},\frac{1}{\hat{q}'}\right\}}\Big), \quad (4.37)$$

where

$$\rho = \min\Big\{\frac{\delta(q-\gamma+2)}{(1+\delta)q}, \frac{\hat{\delta}}{\hat{q}'(1+\hat{\delta})}\Big\},$$

and C depends on the data (including $\lambda_N(\Omega)$), q, and on $|u|_{L_q(\Omega_T)}$; in the case $\gamma = 2$ the constant C depends only on the data. Choose t_1 so small that $Ct_1^\rho \le \frac{1}{2}$. Then

$$|u_\kappa^+|^2_{V_p([0,t_1]\times\Omega)} \le 2C\kappa^2\mu_\kappa^{\min\left\{\frac{q-(\gamma-2)}{q},\frac{1}{\hat{q}'}\right\}}, \quad \kappa \ge \tilde{\kappa}_0 = \max\{\kappa_0, 1\}.$$

Defining $\phi(\kappa) = \mu_\kappa^{1/\tilde{q}}$, $\kappa \ge \tilde{\kappa}_0$, with

$$\tilde{q} = \begin{cases} \frac{2(1+\delta)q}{q-(\gamma-2)} & : \frac{q-(\gamma-2)}{q} < \frac{1}{\hat{q}'} \\ 2\hat{q}'(1+\hat{\delta}) & : \frac{q-(\gamma-2)}{q} \ge \frac{1}{\hat{q}'}, \end{cases}$$

we may then proceed exactly as in the proof of Theorem 4.2.1, thereby establishing first an upper bound on $(0,t_1)\times\Omega$, and then also on Ω_T, by an analogous induction argument. $\square$

The maximum principle holds in the following form.

Theorem 4.3.2 *Let $T > 0$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain. Suppose there exists $l \in L_{1,loc}(\mathbb{R}_+)$ such that $(k,l) \in \mathcal{PC}$. Suppose further $u_0 \in L_2(\Omega)$, (Q1) with $c_0 = 0$ and $\varphi_0 \equiv 0$, as well as (Q2) with $\varphi_1 \in L_2(\Omega_T)$, and assume that $b \equiv 0$. Then for any weak subsolution (supersolution) $u \in \tilde{V}_1$ of (4.2), we have for a.a. $(t,x) \in \Omega_T$*

$$u(t,x) \le \max\Big\{0, \operatorname*{ess\,sup}_\Omega u_0, \operatorname*{ess\,sup}_{\Gamma_T} u\Big\} \qquad \Big(\, u(t,x) \ge \min\Big\{0, \operatorname*{ess\,inf}_\Omega u_0, \operatorname*{ess\,inf}_{\Gamma_T} u\Big\}\,\Big),$$

provided this maximum (minimum) is finite.

Proof. The proof is analogous to that of Theorem 4.2.3. $\square$

Finally we consider the case of 'natural' or Hadamard growth conditions with respect to $|Du|$. Suppose for simplicity that

(Q) $\qquad (a(t,x,\xi,\eta)|\eta) \ge C_0|\eta|^2, \quad |a(t,x,\xi,\eta)| \le C_1|\eta|, \quad |b(t,x,\xi,\eta)| \le C_2|\eta|^2,$

for a.a. $(t,x) \in \Omega_T$, and all $\xi \in \mathbb{R}$, $\eta \in \mathbb{R}^N$, where C_i, $i = 0,1,2$ are positive constants. In the classical parabolic case one knows that weak solutions of the corresponding problem under the conditions (Q) are in general not bounded. However there exist results (also in a more general situation) providing L_∞ bounds in terms of the data under the additional assumption that the weak solution is bounded, see e.g. [49, Chapter V, Theorem 2.2]. It turns out that analogous results can be proved for (4.2). Here we only formulate such a result in the case where (Q) holds.

Theorem 4.3.3 *Let $T > 0$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain. Suppose there exists $l \in L_{1,\,loc}(\mathbb{R}_+)$ such that $(k,l) \in \mathcal{PC}$. Suppose further $u_0 \in L_\infty(\Omega)$, and that (Q) is satisfied. Then for any bounded weak solution $u \in \tilde{V}_1$ of (4.2),*

$$|u|_{L_\infty(\Omega_T)} \le \max\Big\{|u_0|_{L_\infty(\Omega)}, \operatorname*{ess\,sup}_{\Gamma_T} |u|\Big\}.$$

Proof. We proceed as in the proof of [20, Theorem 17.1]. Set

$$\kappa_0 = \Big\{|u_0|_{L_\infty(\Omega)}, \operatorname*{ess\,sup}_{\Gamma_T} |u|\Big\},$$

and assume that $K := \operatorname{ess\,sup}_{\Omega_T} u > \kappa_0$. We then take test functions u_κ^+ where $\kappa = K - \varepsilon \ge \kappa_0$, $\varepsilon > 0$, and estimate as above. By (Q) we obtain

$$|u_\kappa^+|^2_{V_1(\Omega_T)} \le C(C_0, C_2)\Big||Du_\kappa^+|^2 u_\kappa^+\Big|_{L_1(\Omega_T)} \le \varepsilon C(C_0, C_2)\Big||Du_\kappa^+|^2\Big|_{L_1(\Omega_T)}.$$

Thus if ε is sufficiently small, we have $|u_\kappa^+|^2_{V_1(\Omega_T)} \le 0$, that is $u \le \kappa < K$ a.e. in Ω_T, a contradiction. Hence, $u \le \kappa_0$ a.e. in Ω_T. The lower bound is proved analogously. ☐

4.4 Degenerate and singular problems

We conclude this chapter by stating a very recent result obtained in [76] to demonstrate that the theory developed in this chapter can be extended to quasilinear problems with a so-called *p-structure*. For the sake of simplicity we restrict ourselves to the situation with fractional dynamics. An important special case then is the following time fractional p-Laplace equation

$$\partial_t^\alpha (u - u_0) - \operatorname{div}\big(|Du|^{p-2} Du\big) = f \quad \text{in } \Omega_T, \tag{4.38}$$

with $\alpha \in (0,1)$ and $p > 1$. The following result was proved in [76, Theorem 1.2].

Theorem 4.4.1 *Let $\alpha \in (0,1)$, $p > 1$, $T > 0$, and $\Omega \subset \mathbb{R}^N$ be a bounded domain. Suppose that $u_0 \in L_\infty(\Omega)$ and that $f \in L_s(\Omega_T)$ with $s > \frac{N}{p} + \frac{1}{\alpha}$. Let further $q > 1$ be a fixed number satisfying*

$$s > \frac{N}{p} + q' > \frac{N}{p} + \frac{1}{\alpha}. \tag{4.39}$$

Then for any (appropriately defined) weak solution u of (4.38) in Ω_T which is essentially bounded on Γ_T there holds

$$|u|_{L_\infty(\Omega_T)} \le C\big(N, p, \alpha, s, |f|_{L_s(\Omega_T)}, T, |\Omega|, \max\{|u_0|_{L_\infty(\Omega)}, \operatorname*{ess\,sup}_{\Gamma_T} |u|\}\big).$$

Corresponding results in the case $\alpha = 1$ are well-known, see the monograph [20].

Chapter 5

The weak Harnack inequality

5.1 Introductory remarks and the weak Harnack estimate

Let $T > 0$ and Ω be a bounded domain in $\mathbb{R}^N$. In this chapter we will prove a weak Harnack inequality for nonnegative weak supersolutions of the time fractional diffusion equation

$$\partial_t^\alpha (u - u_0) - \operatorname{div}\big(A(t,x)Du\big) = 0, \quad t \in (0,T),\, x \in \Omega, \tag{5.1}$$

where $\alpha \in (0,1)$. We will assume that

(H1) $A \in L_\infty(\Omega_T; \mathbb{R}^{N\times N})$, and

$$\sum_{i,j=1}^{N} |a_{ij}(t,x)|^2 \le \Lambda^2, \quad \text{for a.a. } (t,x) \in \Omega_T.$$

(H2) There exists $\nu > 0$ such that

$$\big(A(t,x)\xi|\xi\big) \ge \nu|\xi|^2, \quad \text{for a.a. } (t,x) \in \Omega_T, \text{ and all } \xi \in \mathbb{R}^N.$$

(H3) $u_0 \in L_2(\Omega)$.

We say that a function u is a *weak solution (subsolution, supersolution)* of (5.1) in Ω_T, if u belongs to the space

$$\begin{aligned}\mathcal{S}_\alpha := \{\, & v \in L_{\frac{2}{1-\alpha},w}([0,T]; L_2(\Omega)) \cap L_2([0,T]; H_2^1(\Omega)) \text{ such that} \\ & g_{1-\alpha} * v \in C([0,T]; L_2(\Omega)), \text{ and } (g_{1-\alpha} * v)|_{t=0} = 0\},\end{aligned}$$

and for any nonnegative test function

$$\eta \in \overset{\circ}{H}{}_2^{1,1}(\Omega_T) = H_2^1([0,T]; L_2(\Omega)) \cap L_2([0,T]; \overset{\circ}{H}{}_2^1(\Omega))$$

with $\eta|_{t=T} = 0$ there holds

$$\int_0^T \int_\Omega \Big(-\eta_t [g_{1-\alpha} * (u - u_0)] + (ADu|D\eta)\Big)\, dxdt = (\le, \ge)\, 0. \tag{5.2}$$

Recall that $L_{p,w}$ stands for the weak L_p space.

Existence of weak solutions of (5.1) in the class $\mathcal{S}_\alpha$ follows from Corollary 3.4.1. Notice that the regularity class for weak solutions differs slightly from the one considered in Section 4.2. For the specific kernels $k = g_{1-\alpha}$ and $l = g_\alpha$ it is more natural to work with the weak L_p space as $g_\alpha \in L_{\frac{1}{1-\alpha},w}([0,T])$ but $g_\alpha \notin L_{\frac{1}{1-\alpha}}([0,T])$. We have $\mathcal{S}_\alpha \subset \tilde{V}_{\frac{1}{1-\alpha}-\varepsilon}$ for all $0 < \varepsilon \le \frac{\alpha}{1-\alpha}$.

To formulate the main result of this chapter, let $B(x,r)$ denote the open ball with radius $r > 0$ centered at $x \in \mathbb{R}^N$. In this chapter, the Lebesgue measure in $\mathbb{R}^N$ is denoted by μ_N. For $\delta \in (0,1)$, $t_0 \ge 0$, $\tau > 0$, and a ball $B(x_0,r)$, define the boxes

$$Q_-(t_0,x_0,r) = (t_0, t_0 + \delta\tau r^{2/\alpha}) \times B(x_0,\delta r),$$
$$Q_+(t_0,x_0,r) = (t_0 + (2-\delta)\tau r^{2/\alpha}, t_0 + 2\tau r^{2/\alpha}) \times B(x_0,\delta r).$$

Theorem 5.1.1 *Let $\alpha \in (0,1)$, $T > 0$, and $\Omega \subset \mathbb{R}^N$ be a bounded domain. Suppose the assumptions (H1)–(H3) are satisfied. Let further $\delta \in (0,1)$, $\eta > 1$, and $\tau > 0$ be fixed. Then for any $t_0 \ge 0$ and $r > 0$ with $t_0 + 2\tau r^{2/\alpha} \le T$, any ball $B(x_0,\eta r) \subset \Omega$, any $0 < p < \frac{2+N\alpha}{2+N\alpha-2\alpha}$, and any nonnegative weak supersolution u of (5.1) in $(0, t_0 + 2\tau r^{2/\alpha}) \times B(x_0,\eta r)$ with $u_0 \ge 0$ in $B(x_0,\eta r)$, there holds*

$$\Big(\frac{1}{\mu_{N+1}\big(Q_-(t_0,x_0,r)\big)} \int_{Q_-(t_0,x_0,r)} u^p \, d\mu_{N+1}\Big)^{1/p} \le C \operatorname*{ess\,inf}_{Q_+(t_0,x_0,r)} u, \tag{5.3}$$

where the constant $C = C(\nu, \Lambda, \delta, \tau, \eta, \alpha, N, p)$.

Theorem 5.1.1 will be proved in Sections 5.2–5.5. It states that nonnegative weak supersolutions of (5.1) satisfy a weak form of Harnack inequality in the sense that we do not have an estimate for the supremum of u on $Q_-(t_0,x_0,r)$ but only an L_p estimate. In Section 5.6 we will show that the critical exponent $\frac{2+N\alpha}{2+N\alpha-2\alpha}$ is optimal, i.e. the inequality fails to hold for $p \ge \frac{2+N\alpha}{2+N\alpha-2\alpha}$.

Theorem 5.1.1 can be viewed as the time fractional analogue of the corresponding result in the classical parabolic case $\alpha = 1$, see e.g. [51, Theorem 6.18] and [74]. Sending $\alpha \to 1$ in the expression for the critical exponent yields $1 + 2/N$, which is the well-known critical exponent for the heat equation. We would like to point out that the statement of Theorem 5.1.1 remains valid for (appropriately defined) weak supersolutions of (5.1) on $(t_0, t_0 + 2\tau r^{2/\alpha}) \times B(x_0,\eta r)$ which are nonnegative on $(0, t_0 + 2\tau r^{2/\alpha}) \times B(x_0,\eta r)$. Here the global positivity assumption cannot be replaced by a local one, as simple examples show, see Section 8.3. This significant difference to the case $\alpha = 1$ is due to the non-local nature of ∂_t^α. The same phenomenon is known for integro-differential operators like $(-\Delta)^\alpha$ with $\alpha \in (0,1)$, see e.g. [41].

As a simple consequence of the weak Harnack inequality we derive the strong maximum principle for weak subsolutions of (5.1), see Theorem 5.7.1 below. As a further application of the weak Harnack inequality we obtain a theorem of Liouville type, see Corollary 5.7.1 below. It states that any bounded weak solution of (5.1) on $\mathbb{R}_+ \times \mathbb{R}^N$ with $u_0 = 0$ vanishes a.e. on $\mathbb{R}_+ \times \mathbb{R}^N$.

In the classical parabolic case boundedness and the weak (or full) Harnack inequality imply an Hölder estimate for weak solutions, cf. [20], [49], [51], [57]. We also refer to [29] and [56] for the elliptic case. In the present situation one cannot argue anymore as in the classical parabolic case, due to the global positivity assumption in Theorem 5.1.1. The same problem arises for the fractional Laplacian, see [70]. However, in our case it is possible to establish at least continuity at $t = 0$. This is done in Theorem 5.7.2 in the case $u_0 = 0$. It is shown that in this case any bounded weak solution u of (5.1) is continuous at $(0,x_0)$ for all $x_0 \in \Omega$ and $\lim_{(t,x)\to(0,x_0)} u(t,x) = 0$. Thus for such weak solutions the initial condition $u|_{t=0} = 0$ is satisfied in the classical sense.

Our proof of Theorem 5.1.1 relies on a priori estimates for time fractional problems, which are derived by means of the fundamental identity (2.6) for the regularized fractional derivative.

It further uses Moser's iteration technique and Lemma 2.5.3 of Bombieri and Giusti [4], which allows to avoid the rather technically involved approach via BMO-functions. This simplification is already of great significance in the classical parabolic case, see Moser [58] and Saloff-Coste [66].

We point out that the results obtained in this chapter can be easily generalized to quasilinear equations of the form

$$\partial_t^\alpha (u-u_0) - \operatorname{div} a(t,x,u,Du) = b(t,x,u,Du),\ t\in(0,T),\ x\in\Omega, \tag{5.4}$$

with suitable structure conditions on the functions a and b. This is possible, as also known from the elliptic and the classical parabolic case, since the test function method used in the proof of Theorem 5.1.1 does not depend so much on the linearity of the differential operator w.r.t. the spatial variables but on a certain nonlinear structure, cf. [29], [51], and [74].

We further remark that in the purely time-dependent case, that is for scalar equations of the form

$$\partial_t^\alpha (u-u_0) + \sigma u = 0, \quad t\in(0,T),$$

with $\sigma \geq 0$, a weak Harnack inequality with optimal exponent $\frac{1}{1-\alpha}$ was proved in [83] for nonnegative supersolutions, see Section 8.1 for the precise statement. As a curiosity, note that putting $N=0$ in the expression $\frac{2+N\alpha}{2+N\alpha-2\alpha}$ results in the critical exponent from the purely time-dependent case.

5.2 The regularized weak formulation, time shifts, and scalings

The following lemma provides the regularized weak formulation of (5.1), with the singular kernel $g_{1-\alpha}$ being replaced by the kernel $g_{1-\alpha,n}$ ($n\in\mathbb{N}$) given by

$$g_{1-\alpha,n} = n s_{\alpha,n} = h_{\alpha,n} * g_{1-\alpha}.$$

Here $s_{\alpha,n}$ and $h_{\alpha,n}$ are the unique solutions of the scalar-valued Volterra equations

$$\begin{aligned} s_{\alpha,n}(t) + n(s_{\alpha,n}*g_\alpha)(t) &= 1, \quad t>0,\ n\in\mathbb{N},\\ h_{\alpha,n}(t) + n(h_{\alpha,n}*g_\alpha)(t) &= n g_\alpha(t), \quad t>0,\ n\in\mathbb{N}, \end{aligned}$$

cf. Section 2.2. In what follows $\alpha\in(0,1)$ is fixed, so we may put $h_n := h_{\alpha,n}$, $n\in\mathbb{N}$.

Lemma 5.2.1 *Let $\alpha\in(0,1)$, $T>0$, and $\Omega\subset\mathbb{R}^N$ be a bounded domain. Suppose the assumptions (H1)–(H3) are satisfied. Then $u\in\mathcal{S}_\alpha$ is a weak solution (subsolution, supersolution) of (5.1) in Ω_T if and only if for any nonnegative function $\psi\in \mathring{H}^1_2(\Omega)$ one has*

$$\int_\Omega \Big(\psi\partial_t[g_{1-\alpha,n}*(u-u_0)] + (h_n*[ADu]|D\psi)\Big)\,dx = \ (\leq,\geq)\,0, \quad \textit{a.a. } t\in(0,T),\ n\in\mathbb{N}.$$

Lemma 5.2.1 is a special case of Lemma 4.2.1 except for the slightly different regularity class for weak solutions. The proof is the same.

Let $u\in\mathcal{S}_\alpha$ be a weak supersolution of (5.1) in Ω_T and assume that $u_0\geq 0$ in Ω. Then Lemma 5.2.1 and positivity of $g_{1-\alpha,n}$ imply that

$$\int_\Omega \Big(\psi\partial_t(g_{1-\alpha,n}*u) + (h_n*[ADu]|D\psi)\Big)\,dx \geq\ 0, \quad \text{a.a. } t\in(0,T),\ n\in\mathbb{N}, \tag{5.5}$$

for any nonnegative function $\psi \in \mathring{H}^1_2(\Omega)$.

Let now $t_1 \in (0,T)$ be fixed. For $t \in (t_1,T)$ we introduce the shifted time $s = t - t_1$ and set $\tilde{f}(s) = f(s+t_1)$, $s \in (0, T-t_1)$, for functions f defined on (t_1,T). From the decomposition

$$(g_{1-\alpha,n} * u)(t,x) = \int_{t_1}^{t} g_{1-\alpha,n}(t-\tau)u(\tau,x)\,d\tau + \int_0^{t_1} g_{1-\alpha,n}(t-\tau)u(\tau,x)\,d\tau, \quad t \in (t_1,T),$$

we then deduce that

$$\partial_t(g_{1-\alpha,n} * u)(t,x) = \partial_s(g_{1-\alpha,n} * \tilde{u})(s,x) + \int_0^{t_1} \dot{g}_{1-\alpha,n}(s+t_1-\tau)u(\tau,x)\,d\tau. \tag{5.6}$$

Assuming in addition that $u \geq 0$ on $(0,t_1) \times \Omega$ it follows from (5.5), (5.6), and the positivity of ψ and of $-\dot{g}_{1-\alpha,n}$ that

$$\int_\Omega \Big(\psi \partial_s(g_{1-\alpha,n} * \tilde{u}) + \big((h_n * [ADu])\tilde{\ } | D\psi\big)\Big)\,dx \geq 0, \quad \text{a.a. } s \in (0, T-t_1),\ n \in \mathbb{N}, \tag{5.7}$$

for any nonnegative function $\psi \in \mathring{H}^1_2(\Omega)$. This relation will be the starting point for all of the estimates in the next two sections.

We conclude this section with a remark on the scaling properties of equation (5.1). Let $t_0, r > 0$ and $x_0 \in \mathbb{R}^N$. Suppose $u \in \mathcal{S}_\alpha$ is a weak solution (subsolution, supersolution) of (5.1) in $(0, t_0 r^{2/\alpha}) \times B(x_0,r)$. Changing the coordinates according to $s = t/r^{2/\alpha}$ and $y = (x-x_0)/r$ and setting $v(s,y) = u(sr^{2/\alpha}, x_0 + yr)$, $v_0(y) = u_0(x_0+yr)$, and $\tilde{A}(s,y) = A(sr^{2/\alpha}, x_0+yr)$, the problem for u is transformed to a problem for v in $(0,t_0) \times B(0,1)$, namely there holds with $D = D_y$ (also in the weak sense)

$$\partial_s^\alpha (v - v_0) - \operatorname{div}\big(\tilde{A}(s,y)Dv\big) = (\leq, \geq)\, 0, \quad s \in (0,t_0),\ y \in B(0,1). \tag{5.8}$$

5.3 Mean value inequalities

For $\sigma > 0$ we put $\sigma B(x,r) := B(x, \sigma r)$. Recall that μ_N denotes the Lebesgue measure in $\mathbb{R}^N$.

Theorem 5.3.1 *Let $\alpha \in (0,1)$, $T > 0$, and $\Omega \subset \mathbb{R}^N$ be a bounded domain. Suppose the assumptions (H1)–(H3) are satisfied. Let $\eta > 0$ and $\delta \in (0,1)$ be fixed. Then for any $t_0 \in (0,T]$ and $r > 0$ with $t_0 - \eta r^{2/\alpha} \geq 0$, any ball $B = B(x_0,r) \subset \Omega$, and any weak supersolution $u \geq \varepsilon > 0$ of (5.1) in $(0,t_0) \times B$ with $u_0 \geq 0$ in B , there holds*

$$\operatorname*{ess\,sup}_{U_{\sigma'}} u^{-1} \leq \Big(\frac{C\mu_{N+1}(U_1)^{-1}}{(\sigma - \sigma')^{\tau_0}}\Big)^{1/\gamma} |u^{-1}|_{L_\gamma(U_\sigma)}, \quad \delta \leq \sigma' < \sigma \leq 1,\ \gamma \in (0,1].$$

Here $U_\sigma = (t_0 - \sigma\eta r^{2/\alpha}, t_0) \times \sigma B$, $0 < \sigma \leq 1$, $C = C(\nu, \Lambda, \delta, \eta, \alpha, N)$ and $\tau_0 = \tau_0(\alpha, N)$.

Proof: We may assume that $r = 1$ and $x_0 = 0$. In fact, in the general case we change coordinates as $t \to t/r^{2/\alpha}$ and $x \to (x-x_0)/r$, thereby transforming the equation to a problem of the same type on $(0, t_0/r^{2/\alpha}) \times B(0,1)$, cf. Section 5.2.

Fix σ' and σ such that $\delta \leq \sigma' < \sigma \leq 1$ and put $B_1 = \sigma B$. For $\rho \in (0,1]$ we set $V_\rho = U_{\rho\sigma}$. Given $0 < \rho' < \rho \leq 1$, let $t_1 = t_0 - \rho\sigma\eta$ and $t_2 = t_0 - \rho'\sigma\eta$. Then $0 \leq t_1 < t_2 < t_0$. We introduce further the shifted time $s = t - t_1$ and set $\tilde{f}(s) = f(s+t_1)$, $s \in (0, t_0 - t_1)$, for functions f defined on (t_1, t_0). Since $u_0 \geq 0$ in B and u is a positive weak supersolution of (5.1) in $(0,t_0) \times B$, we have (cf. (5.7))

$$\int_B \Big(v \partial_s(g_{1-\alpha,n} * \tilde{u}) + \big((h_n * [ADu])\tilde{\ } | Dv\big)\Big)\,dx \geq 0, \quad \text{a.a. } s \in (0, t_0 - t_1),\ n \in \mathbb{N}, \tag{5.9}$$

for any nonnegative function $v \in \mathring{H}^1_2(B)$. For $s \in (0, t_0 - t_1)$ we choose the test function $v = \psi^2 \tilde{u}^\beta$ with $\beta < -1$ and $\psi \in C^1_0(B_1)$ so that $0 \le \psi \le 1$, $\psi = 1$ in $\rho' B_1$, $\operatorname{supp} \psi \subset \rho B_1$, and $|D\psi| \le 2/[\sigma(\rho - \rho')]$. By the fundamental identity (2.6) applied to $k = g_{1-\alpha,n}$ and the convex function $H(y) = -(1+\beta)^{-1} y^{1+\beta}$, $y > 0$, there holds for a.a. $(s, x) \in (0, t_0 - t_1) \times B$

$$\begin{aligned} -\tilde{u}^\beta \partial_s(g_{1-\alpha,n} * \tilde{u}) \ge & -\frac{1}{1+\beta}\,\partial_s(g_{1-\alpha,n} * \tilde{u}^{1+\beta}) + \Big(\frac{\tilde{u}^{1+\beta}}{1+\beta} - \tilde{u}^{1+\beta}\Big) g_{1-\alpha,n} \\ = & -\frac{1}{1+\beta}\,\partial_s(g_{1-\alpha,n} * \tilde{u}^{1+\beta}) - \frac{\beta}{1+\beta}\,\tilde{u}^{1+\beta} g_{1-\alpha,n}. \end{aligned} \tag{5.10}$$

We further have

$$Dv = 2\psi D\psi\, \tilde{u}^\beta + \beta \psi^2 \tilde{u}^{\beta-1} D\tilde{u}.$$

Using this and (5.10) it follows from (5.9) that for a.a. $s \in (0, t_0 - t_1)$

$$\begin{aligned} -\frac{1}{1+\beta} \int_{B_1} \psi^2 \partial_s(g_{1-\alpha,n} * \tilde{u}^{1+\beta})\, dx + |\beta| \int_{B_1} \big((h_n * [ADu])\tilde{\ }|\psi^2 \tilde{u}^{\beta-1} D\tilde{u}\big)\, dx \\ \le 2 \int_{B_1} \big((h_n * [ADu])\tilde{\ }|\psi D\psi\, \tilde{u}^\beta\big)\, dx + \frac{\beta}{1+\beta} \int_{B_1} \psi^2 \tilde{u}^{1+\beta} g_{1-\alpha,n}\, dx. \end{aligned} \tag{5.11}$$

Next, choose $\varphi \in C^1([0, t_0 - t_1])$ such that $0 \le \varphi \le 1$, $\varphi = 0$ in $[0, (t_2 - t_1)/2]$, $\varphi = 1$ in $[t_2 - t_1, t_0 - t_1]$, and $0 \le \dot{\varphi} \le 4/(t_2 - t_1)$. Multiplying (5.11) by $-(1+\beta) > 0$ and by $\varphi(s)$, and convolving the resulting inequality with g_α yields

$$\begin{aligned} \int_{B_1} g_\alpha * \big(\varphi \partial_s(g_{1-\alpha,n} * [\psi^2 \tilde{u}^{1+\beta}])\big)\, dx + \beta(1+\beta)\, g_\alpha * \int_{B_1} \big((h_n * [ADu])\tilde{\ }|\psi^2 \tilde{u}^{\beta-1} D\tilde{u}\big)\varphi\, dx \\ \le 2|1+\beta|\, g_\alpha * \int_{B_1} \big((h_n * [ADu])\tilde{\ }|\psi D\psi\, \tilde{u}^\beta\big)\varphi\, dx + |\beta|\, g_\alpha * \int_{B_1} \psi^2 \tilde{u}^{1+\beta} g_{1-\alpha,n} \varphi\, dx, \end{aligned} \tag{5.12}$$

for a.a. $s \in (0, t_0 - t_1)$. By Lemma 2.3.4,

$$\begin{aligned} \int_{B_1} g_\alpha * \big(\varphi \partial_s(g_{1-\alpha,n} * [\psi^2 \tilde{u}^{1+\beta}])\big)\, dx \ge \int_{B_1} \varphi g_\alpha * \big(\partial_s(g_{1-\alpha,n} * [\psi^2 \tilde{u}^{1+\beta}])\big)\, dx \\ - \int_0^s g_\alpha(s-\sigma) \dot{\varphi}(\sigma) \big(g_{1-\alpha,n} * \int_{B_1} \psi^2 \tilde{u}^{1+\beta}\, dx\big)(\sigma)\, d\sigma. \end{aligned} \tag{5.13}$$

Furthermore, by virtue of

$$g_{1-\alpha,n} * [\psi^2 \tilde{u}^{1+\beta}] \in {}_0H^1_1([0, t_0 - t_1]; L_1(B_1))$$

and $g_{1-\alpha,n} = g_{1-\alpha} * h_n$ as well as $g_\alpha * g_{1-\alpha} = 1$ we have

$$g_\alpha * \partial_s(g_{1-\alpha,n} * [\psi^2 \tilde{u}^{1+\beta}]) = \partial_s(g_\alpha * g_{1-\alpha,n} * [\psi^2 \tilde{u}^{1+\beta}]) = h_n * (\psi^2 \tilde{u}^{1+\beta}). \tag{5.14}$$

Combining (5.12), (5.13), and (5.14), sending $n \to \infty$, and selecting an appropriate subsequence, if necessary, we thus obtain

$$\begin{aligned} \int_{B_1} \varphi \psi^2 \tilde{u}^{1+\beta}\, dx + \beta(1+\beta)\, g_\alpha * \int_{B_1} \big(\tilde{A} D\tilde{u}|\psi^2 \tilde{u}^{\beta-1} D\tilde{u}\big)\varphi\, dx \\ \le 2|1+\beta|\, g_\alpha * \int_{B_1} \big(\tilde{A} D\tilde{u}|\psi D\psi\, \tilde{u}^\beta\big)\varphi\, dx + |\beta|\, g_\alpha * \int_{B_1} \psi^2 \tilde{u}^{1+\beta} g_{1-\alpha} \varphi\, dx \\ + \int_0^s g_\alpha(s-\sigma)\dot{\varphi}(\sigma)\big(g_{1-\alpha} * \int_{B_1} \psi^2 \tilde{u}^{1+\beta}\, dx\big)(\sigma)\, d\sigma, \quad \text{a.a. } s \in (0, t_0 - t_1). \end{aligned} \tag{5.15}$$

Put $w=\tilde u^{\frac{\beta+1}{2}}$. Then $Dw=\frac{\beta+1}{2}\tilde u^{\frac{\beta-1}{2}}D\tilde u$. By assumption (H2), we have

$$\beta(1+\beta)\,g_\alpha*\int_{B_1}\big(\tilde A D\tilde u|\psi^2\tilde u^{\beta-1}D\tilde u\big)\varphi\,dx\;\ge\nu\beta(1+\beta)\,g_\alpha*\int_{B_1}\varphi\psi^2\tilde u^{\beta-1}|D\tilde u|^2\,dx$$
$$=\;\frac{4\nu\beta}{1+\beta}\,g_\alpha*\int_{B_1}\varphi\psi^2|Dw|^2\,dx. \qquad (5.16)$$

Using (H1) and Young's inequality we may estimate

$$\begin{aligned}2\big|\big(\tilde A D\tilde u|\psi D\psi\,\tilde u^\beta\big)\varphi\big| &\le 2\Lambda\psi|D\psi|\,|D\tilde u|\tilde u^\beta\varphi=2\Lambda\psi|D\psi|\,|D\tilde u|\tilde u^{\frac{\beta-1}{2}}\tilde u^{\frac{\beta+1}{2}}\varphi\\ &\le\;\frac{\nu|\beta|}{2}\,\psi^2\varphi|D\tilde u|^2\tilde u^{\beta-1}+\;\frac{2}{\nu|\beta|}\,\Lambda^2|D\psi|^2\varphi\tilde u^{\beta+1}\\ &=\;\frac{2\nu|\beta|}{(1+\beta)^2}\,\psi^2\varphi|Dw|^2+\;\frac{2}{\nu|\beta|}\,\Lambda^2|D\psi|^2\varphi w^2. \end{aligned}\qquad (5.17)$$

From (5.15), (5.16), and (5.17) we conclude that

$$\int_{B_1}\varphi\psi^2w^2\,dx+\frac{2\nu|\beta|}{|1+\beta|}\,g_\alpha*\int_{B_1}\varphi\psi^2|Dw|^2\,dx\le g_\alpha*F,\quad \text{a.a. } s\in(0,t_0-t_1), \qquad (5.18)$$

where

$$\begin{aligned}F(s)=\frac{2\Lambda^2|1+\beta|}{\nu|\beta|}\;&\int_{B_1}|D\psi|^2\varphi w^2\,dx+|\beta|\varphi(s)g_{1-\alpha}(s)\int_{B_1}\psi^2w^2\,dx\\ &+\dot\varphi(s)\big(g_{1-\alpha}*\int_{B_1}\psi^2w^2\,dx\big)(s)\ge0,\quad \text{a.a. } s\in(0,t_0-t_1).\end{aligned}$$

We may drop the second term in (5.18), which is nonnegative. By Young's inequality for convolutions and the properties of φ we then infer that for all $p\in(1,1/(1-\alpha))$

$$\Big(\int_{t_2-t_1}^{t_0-t_1}\big(\int_{B_1}[\psi(x)w(s,x)]^2\,dx)^p\,ds\Big)^{1/p}\le|g_\alpha|_{L_p([0,t_0-t_1])}\int_0^{t_0-t_1}F(s)\,ds, \qquad (5.19)$$

where

$$|g_\alpha|_{L_p([0,t_0-t_1])}=\;\frac{(t_0-t_1)^{\alpha-1+1/p}}{\Gamma(\alpha)[(\alpha-1)p+1]^{1/p}}\;\le\;\frac{\eta^{\alpha-1+1/p}}{\Gamma(\alpha)[(\alpha-1)p+1]^{1/p}}\;=:C_1(\alpha,p,\eta). \qquad (5.20)$$

We choose any of these p and fix it.

Returning to (5.18), we may also drop the first term, convolve the resulting inequality with $g_{1-\alpha}$ and evaluate at $s=t_0-t_1$, thereby obtaining

$$\int_{t_2-t_1}^{t_0-t_1}\int_{B_1}\psi^2|Dw|^2\,dx\,ds\le\;\frac{|1+\beta|}{2\nu|\beta|}\;\int_0^{t_0-t_1}F(s)\,ds. \qquad (5.21)$$

Using

$$\int_{t_2-t_1}^{t_0-t_1}\int_{B_1}|D(\psi w)|^2\,dx\,ds\le2\int_{t_2-t_1}^{t_0-t_1}\int_{B_1}\big(\psi^2|Dw|^2+|D\psi|^2w^2\big)\,dx\,ds$$

we infer from (5.19)–(5.21) that

$$\begin{aligned}|\psi w|^2_{V_p([t_2-t_1,t_0-t_1]\times B_1)}\le2\Big(C_1(\alpha,p,\eta)+\;\frac{|1+\beta|}{\nu|\beta|}\Big)&\int_0^{t_0-t_1}F(s)\,ds\\ +4\int_0^{t_0-t_1}\;&\int_{B_1}|D\psi|^2w^2\,dx\,ds.\end{aligned}\qquad (5.22)$$

We will next estimate the right-hand side of (5.22). By the assumptions on ψ and φ, and since $|\beta| > 1$, we have

$$\int_0^{t_0-t_1}\int_{B_1} |D\psi|^2 w^2\,dx\,ds \le \frac{4}{\sigma^2(\rho-\rho')^2}\int_0^{t_0-t_1}\int_{\rho B_1} w^2\,dx\,ds$$

and

$$F(s) \le \Big(\frac{8\Lambda^2|1+\beta|}{\nu\sigma^2(\rho-\rho')^2} + |\beta| g_{1-\alpha}((t_2-t_1)/2)\Big)\int_{\rho B_1} w^2\,dx$$
$$+ \frac{4}{t_2-t_1}\big(g_{1-\alpha} * \int_{\rho B_1} w^2\,dx\big)(s), \quad \text{a.a. } s\in(0,t_0-t_1).$$

Recall that $\sigma \ge \delta > 0$. So we have

$$\int_0^{t_0-t_1} F(s)\,ds \le \Big(\frac{8\Lambda^2|1+\beta|}{\nu\sigma^2(\rho-\rho')^2} + \frac{2^\alpha|\beta|}{\Gamma(1-\alpha)(\rho-\rho')^\alpha(\sigma\eta)^\alpha}\Big)\int_0^{t_0-t_1}\int_{\rho B_1} w^2\,dx\,ds$$
$$+ \frac{4}{(\rho-\rho')\sigma\eta}\int_0^{t_0-t_1} g_{2-\alpha}(t_0-t_1-\tau)\int_{\rho B_1} w(\tau,x)^2\,dx\,d\tau$$
$$\le C(\nu,\Lambda,\delta,\eta,\alpha)\,\frac{1+|1+\beta|}{(\rho-\rho')^2}\int_0^{t_0-t_1}\int_{\rho B_1} w^2\,dx\,ds.$$

Combining these estimates and (5.22) yields

$$|\psi w|_{V_p([t_2-t_1,t_0-t_1]\times B_1)} \le C(\nu,\Lambda,\delta,\eta,\alpha,p)\,\frac{1+|1+\beta|}{\rho-\rho'}\,|w|_{L_2([0,t_0-t_1]\times\rho B_1)}.$$

We apply next the interpolation inequality (2.15) to the function ψw and make use of $\psi=1$ in $\rho' B_1$ to deduce that

$$|w|_{L_{2\kappa}([t_2-t_1,t_0-t_1]\times\rho' B_1)} \le C(\nu,\Lambda,\delta,\eta,\alpha,p,N)\,\frac{1+|1+\beta|}{\rho-\rho'}\,|w|_{L_2([0,t_0-t_1]\times\rho B_1)}, \tag{5.23}$$

where the number $\kappa > 1$ is given in (2.14). Since $w = \tilde{u}^{\frac{\beta+1}{2}}$ and by transforming back to the time t, we see that (5.23) is equivalent to

$$\Big(\int_{V_{\rho'}} u^{-|1+\beta|\kappa}\,d\mu_{N+1}\Big)^{\frac{1}{2\kappa}} \le \frac{\tilde{C}(1+|1+\beta|)}{\rho-\rho'}\Big(\int_{V_\rho} u^{-|1+\beta|}\,d\mu_{N+1}\Big)^{\frac12}$$

with $\tilde{C} = \tilde{C}(\nu,\Lambda,\delta,\eta,\alpha,p,N)$. Hence, with $\gamma = |1+\beta|$,

$$|u^{-1}|_{L_{\gamma\kappa}(V_{\rho'})} \le \Big(\frac{\tilde{C}^2(1+\gamma)^2}{(\rho-\rho')^2}\Big)^{1/\gamma}|u^{-1}|_{L_\gamma(V_\rho)}, \quad 0<\rho'<\rho\le 1,\ \gamma>0.$$

Employing the first Moser iteration, Lemma 2.5.1 (with $\bar{p}=1$), it follows that there exist constants $M_0 = M_0(\nu,\Lambda,\delta,\eta,\alpha,p,N)$ and $\tau_0=\tau_0(\kappa)$ such that

$$\operatorname*{ess\,sup}_{V_\theta} u^{-1} \le \Big(\frac{M_0}{(1-\theta)^{\tau_0}}\Big)^{1/\gamma}|u^{-1}|_{L_\gamma(V_1)} \quad \text{for all } \theta\in(0,1),\ \gamma\in(0,1].$$

Thus if we take $\theta = \sigma'/\sigma$ and notice that

$$\frac{1}{1-\theta} = \frac{\sigma}{\sigma-\sigma'} \le \frac{1}{\sigma-\sigma'},$$

we obtain

$$\operatorname*{ess\,sup}_{U_{\sigma'}} u^{-1} \le \Big(\frac{M_0}{(\sigma-\sigma')^{\tau_0}}\Big)^{1/\gamma} |u^{-1}|_{L_\gamma(U_\sigma)}, \quad \gamma\in(0,1].$$

Hence the proof is complete. □

We put (cf. (2.14))

$$\tilde\kappa := \kappa_{1/(1-\alpha)} = \frac{2+N\alpha}{2+N\alpha-2\alpha}.$$

Theorem 5.3.2 *Let $\alpha\in(0,1)$, $T>0$, and $\Omega\subset\mathbb{R}^N$ be a bounded domain. Suppose the assumptions (H1)–(H3) are satisfied. Let $\eta>0$ and $\delta\in(0,1)$ be fixed. Then for any $t_0\in[0,T)$ and $r>0$ with $t_0+\eta r^{2/\alpha}\le T$, any ball $B=B(x_0,r)\subset\Omega$, any $p_0\in(0,\tilde\kappa)$, and any nonnegative weak supersolution u of (5.1) in $(0,t_0+\eta r^{2/\alpha})\times B$ with $u_0\ge 0$ in B, there holds*

$$|u|_{L_{p_0}(U'_{\sigma'})} \le \Big(\frac{C\mu_{N+1}(U'_1)^{-1}}{(\sigma-\sigma')^{\tau_0}}\Big)^{1/\gamma-1/p_0} |u|_{L_\gamma(U'_\sigma)}, \quad \delta\le\sigma'<\sigma\le 1,\ 0<\gamma\le p_0/\tilde\kappa.$$

Here $U'_\sigma=(t_0,t_0+\sigma\eta r^{2/\alpha})\times\sigma B$, $C=C(\nu,\Lambda,\delta,\eta,\alpha,N,p_0)$, and $\tau_0=\tau_0(\alpha,N)$.

Proof: We proceed similarly as in the previous proof. Without restriction of generality we may assume that $p_0>1$ and $r=1$. By replacing u with $u+\varepsilon$ and u_0 with $u_0+\varepsilon$ and eventually letting $\varepsilon\to 0+$ we may further assume that u is bounded away from zero.

Fix σ', σ such that $\delta\le\sigma'<\sigma\le 1$ and put $B_1=\sigma B$. For $\rho\in(0,1]$ we set $V'_\rho=U'_{\rho\sigma}$. Given $0<\rho'<\rho\le 1$, let $t_1=t_0+\rho'\sigma\eta$ and $t_2=t_0+\rho\sigma\eta$, so $0\le t_0<t_1<t_2$. We shift the time by means of $s=t-t_0$ and set $\tilde f(s)=f(s+t_0)$, $s\in(0,t_2-t_0)$, for functions f defined on (t_0,t_2).

We then repeat the first steps of the preceding proof, the only difference being that now we take $\beta\in(-1,0)$. Note that, as a consequence of this, (5.10) simplifies to

$$-\tilde u^\beta\partial_s(g_{1-\alpha,n}*\tilde u) \ge -\frac{1}{1+\beta}\,\partial_s(g_{1-\alpha,n}*\tilde u^{1+\beta}), \quad \text{a.a. } (s,x)\in(0,t_2-t_0)\times B,$$

hence we obtain with $\psi\in C_0^1(B_1)$ as above

$$\begin{aligned} -\frac{1}{1+\beta}\int_{B_1}\psi^2\partial_s(g_{1-\alpha,n}*\tilde u^{1+\beta})\,dx + |\beta|\int_{B_1}\big((h_n*[ADu])\tilde{}\,|\psi^2\tilde u^{\beta-1}D\tilde u\big)\,dx \\ \le 2\int_{B_1}\big((h_n*[ADu])\tilde{}\,|\psi D\psi\,\tilde u^\beta\big)\,dx, \qquad \text{a.a. } s\in(0,t_2-t_0). \end{aligned} \tag{5.24}$$

Next, choose $\varphi\in C^1([0,t_2-t_0])$ such that $0\le\varphi\le 1$, $\varphi=1$ in $[0,t_1-t_0]$, $\varphi=0$ in $[t_1-t_0+(t_2-t_1)/2,t_2-t_0]$, and $0\le-\dot\varphi\le 4/(t_2-t_1)$. Multiplying (5.24) by $1+\beta>0$ and by $\varphi(s)$, and applying Lemma 2.3.3 to the first term gives

$$\begin{aligned} -\int_{B_1}\partial_s\big(g_{1-\alpha,n}*[\varphi\psi^2\tilde u^{1+\beta}]\big)\,dx + |\beta|(1+\beta)\int_{B_1}\big(\tilde AD\tilde u|\psi^2\tilde u^{\beta-1}D\tilde u\big)\varphi\,dx \\ \le \int_0^s \dot g_{1-\alpha,n}(s-\sigma)\big(\varphi(s)-\varphi(\sigma)\big)\Big(\int_{B_1}\psi^2\tilde u^{1+\beta}\,dx\Big)(\sigma)\,d\sigma \\ + 2(1+\beta)\int_{B_1}\big(\tilde AD\tilde u|\psi D\psi\,\tilde u^\beta\big)\varphi\,dx + \mathcal{R}_n(s), \quad \text{a.a. } s\in(0,t_2-t_0), \end{aligned} \tag{5.25}$$

where

$$\mathcal{R}_n(s) = -|\beta|(1+\beta)\int_{B_1}\big((h_n*[ADu])\tilde{} - \tilde{A}D\tilde{u}|\psi^2\tilde{u}^{\beta-1}D\tilde{u}\big)\varphi\,dx$$
$$+2(1+\beta)\int_{B_1}\big((h_n*[ADu])\tilde{} - \tilde{A}D\tilde{u}|\psi D\psi\,\tilde{u}^{\beta}\big)\varphi\,dx,\quad \text{a.a. } s\in(0,t_2-t_0).$$

We set again $w=\tilde{u}^{\frac{\beta+1}{2}}$ and estimate exactly as in the preceding proof, using (H1), (H3) and (5.17), to the result

$$\begin{aligned}-\int_{B_1}\partial_s\big(g_{1-\alpha,n}*[\varphi\psi^2w^2]\big)\,dx+\frac{2\nu|\beta|}{1+\beta}\int_{B_1}\varphi\psi^2|Dw|^2\,dx\\ \le\int_0^s\dot{g}_{1-\alpha,n}(s-\sigma)\big(\varphi(s)-\varphi(\sigma)\big)\Big(\int_{B_1}\psi^2w^2\,dx\Big)(\sigma)\,d\sigma\\ +\frac{2\Lambda^2(1+\beta)}{\nu|\beta|}\int_{B_1}|D\psi|^2\varphi w^2\,dx+\mathcal{R}_n(s),\quad \text{a.a. } s\in(0,t_2-t_0).\end{aligned}\tag{5.26}$$

Recall that $g_{1-\alpha,n}=g_{1-\alpha}*h_n$. Putting

$$W(s)=\int_{B_1}\varphi(s)\psi(x)^2w(s,x)^2\,dx$$

and denoting the right-hand side of (5.26) by $F_n(s)$, it follows from (5.26) that

$$G_n(s):=\partial_s^\alpha(h_n*W)(s)+F_n(s)\ge 0,\qquad \text{a.a. } s\in(0,t_2-t_0).$$

By (5.14) and positivity of h_n, we have

$$0\le h_n*W=g_\alpha*\partial_s^\alpha(h_n*W)\le g_\alpha*G_n+g_\alpha*[-F_n(s)]_+$$

a.e. in $(0,t_2-t_0)$, where $[y]_+$ stands for the positive part of $y\in\mathbb{R}$. For any $p\in(1,1/(1-\alpha))$ and any $t_*\in[t_2-t_0-(t_2-t_1)/4,t_2-t_0]$ we thus obtain by Young's inequality

$$|h_n*W|_{L_p([0,t_*])}\le|g_\alpha|_{L_p([0,t_*])}\big(|G_n|_{L_1([0,t_*])}+|[-F_n]_+|_{L_1([0,t_*])}\big).\tag{5.27}$$

Since $t_*\le t_2-t_0\le\eta$, we have $|g_\alpha|_{L_p([0,t_*])}\le C_1(\alpha,p,\eta)$ with the same constant as in (5.20). By positivity of G_n,

$$|G_n|_{L_1([0,t_*])}=(g_{1-\alpha,n}*W)(t_*)+\int_0^{t_*}F_n(s)\,ds.$$

Observe that $\mathcal{R}_n\to 0$ in $L_1([0,t_2-t_0])$ as $n\to\infty$. Hence $|[-F_n]_+|_{L_1([0,t_*])}\to 0$ as $n\to\infty$. Further,

$$\begin{aligned}&\int_0^{t_*}\!\!\int_0^s\dot{g}_{1-\alpha,n}(s-\sigma)\big(\varphi(s)-\varphi(\sigma)\big)\Big(\int_{B_1}\psi^2w^2\,dx\Big)(\sigma)\,d\sigma\,ds\\ &=\int_0^{t_*}g_{1-\alpha,n}(t_*-\sigma)\big(\varphi(t_*)-\varphi(\sigma)\big)\Big(\int_{B_1}\psi^2w^2\,dx\Big)(\sigma)\,d\sigma\\ &\quad-\int_0^{t_*}\dot{\varphi}(s)\int_0^s g_{1-\alpha,n}(s-\sigma)\Big(\int_{B_1}\psi^2w^2\,dx\Big)(\sigma)\,d\sigma\,ds\\ &\le-\int_0^{t_*}\dot{\varphi}(s)\int_0^s g_{1-\alpha,n}(s-\sigma)\Big(\int_{B_1}\psi^2w^2\,dx\Big)(\sigma)\,d\sigma\,ds,\end{aligned}$$

since φ is nonincreasing. We also know that $g_{1-\alpha,n} * W \to g_{1-\alpha} * W$ in $L_1([0, t_2 - t_0])$. Hence we can fix some $t_* \in [t_2 - t_0 - (t_2 - t_1)/4, t_2 - t_0]$ such that for some subsequence $(g_{1-\alpha,n_k} * W)(t_*) \to (g_{1-\alpha} * W)(t_*)$ as $k \to \infty$. Sending $k \to \infty$ it follows then from (5.27), the preceding estimates, and from $\varphi = 1$ in $[0, t_1 - t_0]$ that

$$\Big(\int_0^{t_1-t_0} (\int_{B_1} [\psi(x)w(s,x)]^2\,dx)^p\,ds\Big)^{1/p} \leq C_1(\alpha,p,\eta)\Big((g_{1-\alpha} * W)(t_*) + |F|_{L_1([0,t_2-t_0])}\Big), \quad (5.28)$$

with

$$F(s) = \frac{2\Lambda^2(1+\beta)}{\nu|\beta|} \int_{B_1} |D\psi|^2 \varphi w^2\,dx - \dot{\varphi}(s)\big(g_{1-\alpha} * \int_{B_1} \psi^2 w^2\,dx\big)(s).$$

On the other hand, we can integrate (5.26) over $(0, t_*)$ and take the limit as $k \to \infty$ for the same subsequence as before, thereby getting

$$\int_0^{t_1-t_0}\!\!\int_{B_1} \psi^2 |Dw|^2\,dx\,ds \leq \frac{1+\beta}{2\nu|\beta|}\Big((g_{1-\alpha} * W)(t_*) + |F|_{L_1([0,t_2-t_0])}\Big). \quad (5.29)$$

Arguing as above (cf. the lines before (5.22)), we conclude from (5.28) and (5.29) that

$$\begin{aligned}|\psi w|^2_{V_p([0,t_1-t_0]\times B_1)} \leq\ & 4\int_0^{t_2-t_0}\int_{B_1} |D\psi|^2 w^2\,dx\,ds \\ & + 2\Big(C_1(\alpha,p,\eta) + \frac{1+\beta}{\nu|\beta|}\Big)\Big((g_{1-\alpha} * W)(t_*) + |F|_{L_1([0,t_2-t_0])}\Big). \quad (5.30)\end{aligned}$$

Since $\varphi = 0$ in $[t_1 - t_0 + (t_2 - t_1)/2, t_2 - t_0]$ and $t_* \in [t_2 - t_0 - (t_2 - t_1)/4, t_2 - t_0]$, we have

$$\begin{aligned}(g_{1-\alpha} * W)(t_*) &\leq g_{1-\alpha}\big((t_2 - t_1)/4\big)\int_0^{t_2-t_0}\!\!\int_{B_1} \psi^2 w^2\,dx\,ds \\ &= \frac{4^\alpha}{\Gamma(1-\alpha)(\rho-\rho')^\alpha(\sigma\eta)^\alpha}\int_0^{t_2-t_0}\!\!\int_{\rho B_1} w^2\,dx\,ds.\end{aligned}$$

Further,

$$\int_0^{t_2-t_0}\int_{B_1} |D\psi|^2 w^2\,dx\,ds \leq \frac{4}{\sigma^2(\rho-\rho')^2}\int_0^{t_2-t_0}\!\!\int_{\rho B_1} w^2\,dx\,ds.$$

The term $|F|_{L_1([0,t_2-t_0])}$ is estimated similarly as in the proof of Theorem 5.3.1 (cf. the lines that follow (5.22)). We obtain

$$|F|_{L_1([0,t_2-t_0])} \leq C(\nu,\Lambda,\delta,\eta,\alpha)\,\frac{1+(1+\beta)}{|\beta|(\rho-\rho')^2}\int_0^{t_2-t_0}\!\!\int_{\rho B_1} w^2\,dx\,ds.$$

Notice the additional factor $|\beta|$ in the denominator. Combining these estimates we deduce from (5.30) that

$$|\psi w|_{V_p([0,t_1-t_0]\times B_1)} \leq C(\nu,\Lambda,\delta,\eta,\alpha,p)\,\frac{1+(1+\beta)}{|\beta|(\rho-\rho')}\,|w|_{L_2([0,t_2-t_0]\times\rho B_1)}.$$

By the interpolation inequality (2.15) and since $\psi = 1$ in $\rho' B_1$, this implies for all $\beta \in (-1, 0)$

$$|w|_{L_{2\kappa}([0,t_1-t_0]\times\rho' B_1)} \leq C(\nu,\Lambda,\delta,\eta,\alpha,p,N)\,\frac{1+|1+\beta|}{|\beta|(\rho-\rho')}\,|w|_{L_2([0,t_2-t_0]\times\rho B_1)}, \quad (5.31)$$

where

$$\kappa = \kappa_p = \frac{2p + N(p-1)}{2 + N(p-1)} \in (1, \tilde{\kappa}).$$

We now fix $1 < p < 1/(1-\alpha)$ such that $\kappa_p = (p_0 + \tilde{\kappa})/2$. This is possible because $\kappa_p \nearrow \tilde{\kappa}$ as $p \nearrow 1/(1-\alpha)$.

Next, we set $\gamma = 1 + \beta \in (0,1)$ and transform back to u to get

$$|u|_{L_{\gamma\kappa}(V'_{\rho'}, d\mu)} \leq \Big(\frac{\tilde{C}}{(\rho - \rho')^2}\Big)^{1/\gamma} |u|_{L_\gamma(V'_\rho, d\mu)}, \quad 0 < \rho' < \rho \leq 1,\ 0 < \gamma \leq p_0/\kappa. \tag{5.32}$$

Here, $\mu = (\eta\omega_N)^{-1}\mu_{N+1}$, ω_N the volume of the unit ball in $\mathbb{R}^N$, and $\tilde{C} = \tilde{C}(\nu, \Lambda, \delta, \eta, \alpha, N, p_0)$ is independent of $\gamma \in (0, p_0/\kappa]$, since $|\beta|$ is bounded away from zero. Note that $\mu(V'_1) \leq 1$.

Finally, we employ the second Moser iteration scheme, Lemma 2.5.2, to conclude from (5.32) that there are constants $M_0 = M_0(\nu, \Lambda, \delta, \eta, \alpha, N, p_0)$ and $\tau_0 = \tau_0(\kappa)$ such that

$$|u|_{L_{p_0}(V'_\theta, d\mu)} \leq \Big(\frac{M_0}{(1-\theta)^{\tau_0}}\Big)^{1/\gamma - 1/p_0} |u|_{L_\gamma(V'_1, d\mu)}, \quad 0 < \theta < 1,\ 0 < \gamma \leq p_0/\kappa. \tag{5.33}$$

If we take $\theta = \sigma'/\sigma$ and translate (5.33) back to the measure μ_{N+1}, we obtain

$$|u|_{L_{p_0}(U'_{\sigma'})} \leq \Big(\frac{M_0(\eta\omega_N)^{-1}}{(\sigma - \sigma')^{\tau_0}}\Big)^{1/\gamma - 1/p_0} |u|_{L_\gamma(U'_\sigma)}, \quad 0 < \gamma \leq p_0/\kappa. \tag{5.34}$$

Since $\kappa < \tilde{\kappa}$, (5.34) holds in particular for all $\gamma \in (0, p_0/\tilde{\kappa}]$. This finishes the proof. □

5.4 Logarithmic estimates

Theorem 5.4.1 *Let $\alpha \in (0,1)$, $T > 0$, and $\Omega \subset \mathbb{R}^N$ be a bounded domain. Suppose the assumptions (H1)–(H3) are satisfied. Let $\tau > 0$ and $\delta, \eta \in (0,1)$ be fixed. Then for any $t_0 \geq 0$ and $r > 0$ with $t_0 + \tau r^{2/\alpha} \leq T$, any ball $B = B(x_0, r) \subset \Omega$, and any weak supersolution $u \geq \varepsilon > 0$ of (5.1) in $(0, t_0 + \tau r^{2/\alpha}) \times B$ with $u_0 \geq 0$ in B, there is a constant $c = c(u)$ such that*

$$\mu_{N+1}\big(\{(t,x) \in K_- : \log u(t,x) > c + \lambda\}\big) \leq C r^{2/\alpha} \mu_N(B) \lambda^{-1}, \quad \lambda > 0, \tag{5.35}$$

and

$$\mu_{N+1}\big(\{(t,x) \in K_+ : \log u(t,x) < c - \lambda\}\big) \leq C r^{2/\alpha} \mu_N(B) \lambda^{-1}, \quad \lambda > 0, \tag{5.36}$$

where $K_- = (t_0, t_0 + \eta\tau r^{2/\alpha}) \times \delta B$ and $K_+ = (t_0 + \eta\tau r^{2/\alpha}, t_0 + \tau r^{2/\alpha}) \times \delta B$. Here the constant C depends only on $\delta, \eta, \tau, N, \alpha, \nu$, and Λ.

Proof: Since $u_0 \geq 0$ in B and u is a positive weak supersolution we may assume without loss of generality that $u_0 = 0$ and $t_0 = 0$. In fact, in the case $t_0 > 0$ we shift the time as $t \to t - t_0$, thereby obtaining an inequality of the same type on the time-interval $J := [0, \tau r^{2/\alpha}]$. Observe that the property $g_{1-\alpha} * u \in C([0, t_0 + \tau r^{2/\alpha}]; L_2(B))$ implies $g_{1-\alpha} * \tilde{u} \in C(J; L_2(B))$ for the shifted function $\tilde{u}(s,x) = u(s + t_0, x)$. So we have

$$\int_B \Big(v\partial_t(g_{1-\alpha,n} * u) + (h_n * [ADu]|Dv)\Big)\,dx \geq 0, \quad \text{a.a. } t \in J,\ n \in \mathbb{N}, \tag{5.37}$$

for any nonnegative test function $v \in \mathring{H}^1_2(B)$.

For $t \in J$ we choose the test function $v = \psi^2 u^{-1}$ with $\psi \in C_0^1(B)$ such that $\operatorname{supp}\psi \subset B$, $\psi = 1$ in δB, $0 \le \psi \le 1$, $|D\psi| \le 2/[(1-\delta)r]$ and the domains $\{x \in B : \psi(x)^2 \ge b\}$ are convex for all $b \le 1$. We have

$$Dv = 2\psi D\psi\, u^{-1} - \psi^2 u^{-2} Du,$$

so that by substitution into (5.37) we obtain for a.a. $t \in J$

$$-\int_B \psi^2 u^{-1} \partial_t (g_{1-\alpha,n} * u)\, dx + \int_B \big(ADu|u^{-2}Du\big)\psi^2\, dx \le 2\int_B \big(ADu|u^{-1}\psi D\psi\big)\, dx + \mathcal{R}_n(t), \tag{5.38}$$

where

$$\mathcal{R}_n(t) = \int_B \big(h_n * [ADu] - ADu|Dv\big)\, dx.$$

By (H1) and Young's inequality,

$$\big|2\big(ADu|u^{-1}\psi D\psi\big)\big| \le 2\Lambda\psi|D\psi|\,|Du|u^{-1} \le \frac{\nu}{2}\,\psi^2|Du|^2u^{-2} + \frac{2}{\nu}\,\Lambda^2|D\psi|^2.$$

Using this, (H2) and $|D\psi| \le 2/[(1-\delta)r]$, we infer from (5.38) that for a.a. $t \in J$

$$-\int_B \psi^2 u^{-1} \partial_t (g_{1-\alpha,n} * u)\, dx + \frac{\nu}{2}\int_B |Du|^2 u^{-2}\psi^2\, dx \le \frac{8\Lambda^2 \mu_N(B)}{\nu(1-\delta)^2 r^2} + \mathcal{R}_n(t). \tag{5.39}$$

Setting $w = \log u$ we have $Dw = u^{-1}Du$. The weighted Poincaré inequality of Proposition 2.6.1 with weight ψ^2 yields

$$\int_B (w-W)^2\psi^2 dx \le \frac{8r^2\mu_N(B)}{\int_B \psi^2 dx}\int_B |Dw|^2\psi^2 dx, \quad \text{a.a. } t \in J, \tag{5.40}$$

where

$$W(t) = \frac{\int_B w(t,x)\psi(x)^2 dx}{\int_B \psi(x)^2 dx}, \qquad \text{a.a. } t \in J.$$

From (5.39) and (5.40) we deduce that

$$-\int_B \psi^2 u^{-1}\partial_t(g_{1-\alpha,n} * u)\, dx + \frac{\nu\int_B \psi^2 dx}{16r^2\mu_N(B)}\int_B (w-W)^2\psi^2 dx \le \frac{8\Lambda^2\mu_N(B)}{\nu(1-\delta)^2r^2} + \mathcal{R}_n(t),$$

which in turn implies

$$\frac{-\int_B \psi^2 u^{-1}\partial_t(g_{1-\alpha,n} * u)\, dx}{\int_B \psi^2 dx} + \frac{\nu}{16r^2\mu_N(B)}\int_{\delta B}(w-W)^2 dx \le \frac{C_1}{r^2} + S_n(t), \tag{5.41}$$

for a.a. $t \in J$, with some constant $C_1 = C_1(\delta, N, \nu, \Lambda)$ and $S_n(t) = \mathcal{R}_n(t)/\int_B \psi^2 dx$.

The fundamental identity (2.6) with $H(y) = -\log y$ reads (with the spatial variable x being suppressed)

$$-u^{-1}\partial_t(g_{1-\alpha,n} * u) = -\partial_t(g_{1-\alpha,n} * \log u) + (\log u - 1)g_{1-\alpha,n}(t) + \int_0^t \Big(-\log u(t-s) + \log u(t) + \frac{u(t-s)-u(t)}{u(t)}\Big)[-\dot{g}_{1-\alpha,n}(s)]\, ds.$$

In terms of $w = \log u$ this means that

$$-u^{-1}\partial_t(g_{1-\alpha,n} * u) = -\,\partial_t(g_{1-\alpha,n} * w) + (w-1)g_{1-\alpha,n}(t) + \int_0^t \Psi\big(w(t-s) - w(t)\big)[-\dot g_{1-\alpha,n}(s)]\,ds, \tag{5.42}$$

where $\Psi(y) = e^y - 1 - y$. Since Ψ is convex, it follows from Jensen's inequality that

$$\frac{\int_B \psi^2 \Psi\big(w(t-s,x) - w(t,x)\big)\,dx}{\int_B \psi^2 dx} \ge \Psi\Big(\frac{\int_B \psi^2\big(w(t-s,x) - w(t,x)\big)\,dx}{\int_B \psi^2 dx}\Big).$$

Using this and (5.42) we obtain

$$\begin{aligned}\frac{-\int_B \psi^2 u^{-1}\partial_t(g_{1-\alpha,n} * u)\,dx}{\int_B \psi^2 dx} &\ge -\partial_t(g_{1-\alpha,n} * W) + (W-1)g_{1-\alpha,n}(t)\\ &\quad + \int_0^t \Psi\big(W(t-s) - W(t)\big)[-\dot g_{1-\alpha,n}(s)]\,ds\\ &= -e^{-W}\partial_t(g_{1-\alpha,n} * e^W),\end{aligned} \tag{5.43}$$

where the last equals sign holds again by (5.42) with u replaced by e^W. From (5.41) and (5.43) we conclude that

$$\frac{\nu}{16r^2\mu_N(B)}\int_{\delta B}(w-W)^2 dx \le e^{-W}\partial_t(g_{1-\alpha,n} * e^W) + \frac{C_1}{r^2} + S_n(t), \quad \text{a.a. } t \in J. \tag{5.44}$$

We choose

$$c(u) = \log\Big(\frac{(g_{1-\alpha} * e^W)(\eta\tau r^{2/\alpha})}{g_{2-\alpha}(\eta\tau r^{2/\alpha})}\Big). \tag{5.45}$$

This definition makes sense, since $g_{1-\alpha} * e^W \in C(J)$. The latter is a consequence of $g_{1-\alpha} * u \in C(J; L_2(B))$ and

$$e^{W(t)} \le \frac{\int_B u(t,x)\psi(x)^2 dx}{\int_B \psi(x)^2 dx}, \qquad \text{a.a. } t \in J,$$

where we apply again Jensen's inequality.

To prove (5.35) and (5.36), one of the key ideas is to use the inequalities

$$\begin{aligned}&\mu_{N+1}(\{(t,x) \in K_- :\ w(t,x) > c(u) + \lambda\})\\ &\quad \le \mu_{N+1}(\{(t,x) \in K_- : w(t,x) > c(u) + \lambda \text{ and } W(t) \le c(u) + \lambda/2\})\\ &\qquad + \mu_{N+1}(\{(t,x) \in K_- :\ W(t) > c(u) + \lambda/2\}) =: I_1 + I_2, \quad \lambda > 0,\end{aligned} \tag{5.46}$$

$$\begin{aligned}&\mu_{N+1}(\{(t,x) \in K_+ :\ w(t,x) < c(u) - \lambda\})\\ &\quad \le \mu_{N+1}(\{(t,x) \in K_+ : w(t,x) < c(u) - \lambda \text{ and } W(t) \ge c(u) - \lambda/2\})\\ &\qquad + \mu_{N+1}(\{(t,x) \in K_+ :\ W(t) < c(u) - \lambda/2\}) =: I_3 + I_4, \quad \lambda > 0,\end{aligned} \tag{5.47}$$

and to estimate each of the four terms I_j separately.

We begin with the estimates for W. To estimate I_2 and I_4 we adopt some of the ideas developed in [83]. We set $J_- := (0, \eta\tau r^{2/\alpha})$, $J_+ := (\eta\tau r^{2/\alpha}, \tau r^{2/\alpha})$, and introduce for $\lambda > 0$ the sets $J_-(\lambda) := \{t \in J_- : W(t) > c(u) + \lambda\}$ and $J_+(\lambda) := \{t \in J_+ : W(t) < c(u) - \lambda\}$.

Interestingly, positivity and integrability of the function e^W are sufficient to derive the desired estimate for I_2, cf. also [83, Theorem 2.3]. In fact, with $\rho = \tau r^{2/\alpha}$ we have

$$\begin{aligned} e^\lambda \mu_1\big(J_-(\lambda)\big) &= e^\lambda \mu_1\big(\{t \in J_- : e^{W(t)} > e^{c(u)} e^\lambda\}\big) = \int_{J_-(\lambda)} e^\lambda \, dt \\ &\le \int_{J_-(\lambda)} e^{W(t)-c(u)} \, dt \le \int_{J_-} e^{W(t)-c(u)} \, dt \\ &= \frac{g_{2-\alpha}(\eta\rho)}{(g_{1-\alpha} * e^W)(\eta\rho)} \int_0^{\eta\rho} e^{W(t)} \, dt \\ &\le \frac{g_{2-\alpha}(\eta\rho)}{(g_{1-\alpha} * e^W)(\eta\rho)} \cdot \frac{1}{g_{1-\alpha}(\eta\rho)} \int_0^{\eta\rho} g_{1-\alpha}(\eta\rho - t) e^{W(t)} \, dt \\ &= \frac{\Gamma(1-\alpha)}{\Gamma(2-\alpha)} \, \eta\rho = \frac{\eta\tau r^{2/\alpha}}{1-\alpha}, \end{aligned}$$

and therefore

$$I_2 = \mu_1\big(J_-(\lambda/2)\big)\mu_N(\delta B) \le \frac{2\eta\tau\delta^N}{(1-\alpha)\lambda} \, r^{2/\alpha} \mu_N(B), \quad \lambda > 0. \tag{5.48}$$

We come now to I_4. For $m > 0$ define the function H_m on $\mathbb{R}$ by $H_m(y) = y$, $y \le m$, and $H_m(y) = m + (y-m)/(y-m+1)$, $y \ge m$. Then H_m is increasing, concave, and bounded above by $m+1$. Further, we have $H_m \in C^1(\mathbb{R})$, and so by concavity

$$0 \le y H_m'(y) \le H_m(y) \le m+1, \quad y \ge 0. \tag{5.49}$$

Multiplying (5.44) by $e^W H_m'\big(e^W\big)$ and employing (5.49) as well as the fundamental identity (2.6), we infer that

$$\partial_t\Big(g_{1-\alpha,n} * H_m\big(e^W\big)\Big) + \frac{C_1}{r^2} \, H_m\big(e^W\big) \ge -S_n e^W H_m'\big(e^W\big), \quad \text{a.a. } t \in J. \tag{5.50}$$

For $t \in J_+$ we shift the time by setting $s = t - \eta\tau r^{2/\alpha} = t - \eta\rho$ and put $\tilde{f}(s) = f(s + \eta\rho)$, $s \in (0, (1-\eta)\rho)$, for functions f defined on J_+. By the time-shifting identity (5.6), (5.50) implies that for a.a. $s \in (0, (1-\eta)\rho)$

$$\partial_s\Big(g_{1-\alpha,n} * H_m\big(e^{\tilde{W}}\big)\Big) + \frac{C_1}{r^2} \, H_m\big(e^{\tilde{W}}\big) \ge \Upsilon_{n,m}(s) - \tilde{S}_n e^{\tilde{W}} H_m'\big(e^{\tilde{W}}\big), \tag{5.51}$$

with the history term

$$\Upsilon_{n,m}(s) = \int_0^{\eta\rho} \big[-\dot{g}_{1-\alpha,n}(s + \eta\rho - \sigma)\big] H_m\big(e^{W(\sigma)}\big) \, d\sigma.$$

For $\theta \ge 0$ define the kernel $r_{\alpha,\theta} \in L_{1,loc}(\mathbb{R}_+)$ by means of

$$r_{\alpha,\theta}(t) + \theta(r_{\alpha,\theta} * g_\alpha)(t) = g_\alpha(t), \quad t > 0.$$

Observe that $r_{\alpha,0} = g_\alpha$. Since g_α is completely monotone, $r_{\alpha,\theta}$ enjoys the same property (cf. [32, Chap. 5]), in particular $r_{\alpha,\theta}(s) > 0$ for all $s > 0$. Moreover, we have (see e.g. [83])

$$r_{\alpha,\theta}(s) = \Gamma(\alpha) g_\alpha(s) \, E_{\alpha,\alpha}(-\theta s^\alpha), \quad s > 0, \tag{5.52}$$

where $E_{\alpha,\beta}$ denotes the generalized Mittag-Leffler-function (see e.g. [28, p. 210] and [44, Section 1.8]) defined by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + \beta)} \, , \quad z \in \mathbb{C}. \tag{5.53}$$

We put $\theta = C_1/r^2$ and convolve (5.51) with $r_{\alpha,\theta}$. We have a.e. in $(0,(1-\eta)\rho)$

$$\begin{aligned} r_{\alpha,\theta} * \partial_s\Big(g_{1-\alpha,n} * H_m\big(e^{\tilde W}\big)\Big) &= \partial_s\Big(r_{\alpha,\theta} * g_{1-\alpha,n} * H_m\big(e^{\tilde W}\big)\Big) \\ &= \partial_s\Big([g_\alpha - \theta(r_{\alpha,\theta} * g_\alpha)] * g_{1-\alpha,n} * H_m\big(e^{\tilde W}\big)\Big) \\ &= h_n * H_m\big(e^{\tilde W}\big) - \theta r_{\alpha,\theta} * h_n * H_m\big(e^{\tilde W}\big), \end{aligned}$$

and so we obtain a.e. in $(0,(1-\eta)\rho)$

$$\begin{aligned} h_n * H_m\big(e^{\tilde W}\big) \geq\; & r_{\alpha,\theta} * \Upsilon_{n,m} - r_{\alpha,\theta} * \big[\tilde S_n e^{\tilde W} H_m'\big(e^{\tilde W}\big)\big] \\ & + \theta h_n * r_{\alpha,\theta} * H_m\big(e^{\tilde W}\big) - \theta r_{\alpha,\theta} * H_m\big(e^{\tilde W}\big). \end{aligned} \tag{5.54}$$

Sending $n \to \infty$ and selecting an appropriate subsequence, if necessary, it follows that

$$H_m\big(e^{\tilde W}\big) \geq r_{\alpha,\theta} * \Upsilon_m, \qquad \text{a.a. } s \in (0,(1-\eta)\rho), \tag{5.55}$$

where

$$\Upsilon_m(s) = \int_0^{\eta\rho} \big[-\dot g_{1-\alpha}(s+\eta\rho-\sigma)\big] H_m\big(e^{W(\sigma)}\big)\,d\sigma;$$

in fact, this can be seen by using the approximation property of the kernels h_n.

Observe that for $s \in (0,(1-\eta)\rho)$ we have

$$0 \leq \theta s^\alpha \leq \frac{C_1}{r^2}(1-\eta)^\alpha\big(\tau r^{2/\alpha}\big)^\alpha = C_1(1-\eta)^\alpha\tau^\alpha =: \omega,$$

and thus by continuity and strict positivity of $E_{\alpha,\alpha}$ in $(-\infty,0]$,

$$r_{\alpha,\theta}(s) \geq \Gamma(\alpha) g_\alpha(s) \min_{y\in[0,\omega]} E_{\alpha,\alpha}(-y) =: C_2(\alpha,\omega)\Gamma(\alpha)g_\alpha(s), \quad s \in (0,(1-\eta)\rho).$$

We may then argue as in [83, Section 2.1] to obtain

$$H_m\big(e^{\tilde W(s)}\big) \geq C_2(\alpha,\omega)\,\frac{\alpha(s/[\eta\rho])^\alpha}{1+(s/[\eta\rho])}\,(\eta\rho)^{\alpha-1}\big(g_{1-\alpha} * H_m\big(e^W\big)\big)(\eta\rho), \quad \text{a.a. } s \in (0,(1-\eta)\rho).$$

Evidently, $H_m(y) \nearrow y$ as $m \to \infty$ for all $y \in \mathbb{R}$. Thus by sending $m \to \infty$ and applying Fatou's lemma we conclude that

$$e^{\tilde W(s)} \geq C_2(\alpha,\omega)\,\frac{\alpha(s/[\eta\rho])^\alpha}{1+(s/[\eta\rho])}\,(\eta\rho)^{\alpha-1}\big(g_{1-\alpha} * e^W\big)(\eta\rho), \quad \text{a.a. } s \in (0,(1-\eta)\rho). \tag{5.56}$$

We then employ (5.56) to estimate as follows.

$$\begin{aligned} e^\lambda \mu_1\big(J_+(\lambda)\big) &= e^\lambda \mu_1\big(\{t \in J_+ : e^{W(t)} < e^{c(u)}e^{-\lambda}\}\big) = \int_{J_+(\lambda)} e^\lambda\,dt \\ &\leq \int_{J_+(\lambda)} e^{c(u)-W(t)}\,dt \leq \int_{J_+} e^{c(u)-W(t)}\,dt \\ &= \frac{(g_{1-\alpha} * e^W)(\eta\rho)}{g_{2-\alpha}(\eta\rho)} \int_0^{(1-\eta)\rho} e^{-\tilde W(s)}\,ds \\ &\leq \frac{C_2(\alpha,\omega)^{-1}(\eta\rho)^{1-\alpha}}{\alpha g_{2-\alpha}(\eta\rho)} \int_0^{(1-\eta)\rho} (1+s/\eta\rho)(s/\eta\rho)^{-\alpha}\,ds \\ &= \frac{\Gamma(2-\alpha)\eta\rho}{\alpha C_2(\alpha,\omega)} \int_0^{\frac{1-\eta}{\eta}} \sigma^{-\alpha}(1+\sigma)\,d\sigma = C_3(\alpha,\eta,\omega)\rho. \end{aligned}$$

Hence

$$I_4 = \mu_1\big(J_+(\lambda/2)\big)\mu_N(\delta B) \le \frac{2C_3(\alpha,\eta,\omega)\delta^N}{\lambda}\, r^{2/\alpha}\mu_N(B), \quad \lambda > 0. \tag{5.57}$$

We come now to I_1. Set $J_1(\lambda) = \{t \in J_- : c - W(t) + \lambda/2 \ge 0\}$ and $\Omega_t^-(\lambda) = \{x \in \delta B : w(t,x) > c + \lambda\}$, $t \in J_1(\lambda)$, where $c = c(u)$ is given by (5.45). For $t \in J_1(\lambda)$, we have

$$w(t,x) - W(t) > c - W(t) + \lambda \ge \lambda/2, \quad x \in \Omega_t^-(\lambda),$$

and thus we deduce from (5.44) that a.e. in $J_1(\lambda)$

$$\frac{\nu}{16r^2\mu_N(B)}\, \mu_N\big(\Omega_t^-(\lambda)\big) \le \frac{1}{(c - W + \lambda)^2}\Big(e^{-W}\partial_t(g_{1-\alpha,n} * e^W) + \frac{C_1}{r^2} + S_n\Big). \tag{5.58}$$

Set $\chi(t,\lambda) = \mu_N\big(\Omega_t^-(\lambda)\big)$, if $t \in J_1(\lambda)$, and $\chi(t,\lambda) = 0$ in case $t \in J_- \setminus J_1(\lambda)$. Let further $H(y) = (c - \log y + \lambda)^{-1}$, $0 < y \le y_* := e^{c+\lambda/2}$. Clearly, $H'(y) = (c - \log y + \lambda)^{-2}y^{-1}$ as well as

$$H''(y) = \frac{1}{(c - \log y + \lambda)^2 y^2}\Big(\frac{2}{c - \log y + \lambda} - 1\Big), \quad 0 < y \le y_*,$$

which shows that H is concave in $(0, y_*]$ whenever $\lambda \ge 4$. We will assume this in what follows.

We next choose a C^1 extension $\bar H$ of H on $(0,\infty)$ such that $\bar H$ is concave, $0 \le \bar H'(y) \le \bar H'(y_*)$, $y_* \le y \le 2y_*$, and $\bar H'(y) = 0$, $y \ge 2y_*$. Then

$$0 \le y\bar H'(y) \le \frac{2}{\lambda}, \quad y > 0. \tag{5.59}$$

In fact, for $y \in (0, y_*]$ we have

$$y\bar H'(y) = \frac{1}{(c - \log y + \lambda)^2} \le \frac{1}{(c - \log y_* + \lambda)^2} \le \frac{4}{\lambda^2} \le \frac{1}{\lambda}, \tag{5.60}$$

while in case $y \in [y_*, 2y_*]$ we may simply estimate

$$y\bar H'(y) \le 2y_*\bar H'(y_*) \le \frac{2}{\lambda}.$$

It is clear that $\bar H$ is bounded above. There holds

$$\bar H(y) \le \frac{3}{\lambda}, \quad y > 0. \tag{5.61}$$

To see this, note that since $\bar H$ is nondecreasing with $\bar H'(y) = 0$ for all $y \ge 2y_*$, the claim follows if the inequality is valid for all $y \in [y_*, 2y_*]$. For such y we have by (5.60) and by concavity of $\bar H$

$$\bar H(y) \le \bar H(y_*) + \bar H'(y_*)(y - y_*) \le \bar H(y_*) + y_*\bar H'(y_*) \le \frac{3}{\lambda}.$$

Observe also that

$$e^{W(t)}H'(e^{W(t)}) = \frac{1}{(c - W(t) + \lambda)^2}, \quad \text{a.a. } t \in J_1(\lambda).$$

Since $\bar H' \ge 0$, and $e^{-W}\partial_t(g_{1-\alpha,n} * e^W) + C_1 r^{-2} + S_n \ge 0$ on J_- by virtue of (5.44), we infer from (5.58) and (5.59) that

$$\begin{aligned}\frac{\nu}{16r^2\mu_N(B)}\, \chi(t,\lambda) &\le e^W\bar H'(e^W)\Big(e^{-W}\partial_t(g_{1-\alpha,n} * e^W) + \frac{C_1}{r^2} + S_n\Big)\\ &\le \bar H'(e^W)\partial_t(g_{1-\alpha,n} * e^W) + \frac{2C_1}{\lambda r^2} + \frac{2|S_n(t)|}{\lambda}, \quad \text{a.a. } t \in J_-. \end{aligned} \tag{5.62}$$

Since $\bar{H}$ is concave, the fundamental identity (2.6) yields

$$\bar{H}'(e^W)\partial_t(g_{1-\alpha,n} * e^W) \le \partial_t\Big(g_{1-\alpha,n} * \bar{H}\big(e^W\big)\Big) + \Big(-\bar{H}(e^W) + \bar{H}'(e^W)e^W\Big)g_{1-\alpha,n}$$
$$\le \partial_t\Big(g_{1-\alpha,n} * \bar{H}\big(e^W\big)\Big) + \frac{2}{\lambda}\, g_{1-\alpha,n}, \quad \text{a.a. } t \in J_-,$$

which, together with (5.62), gives a.e. in J_-

$$\frac{\nu}{16r^2\mu_N(B)}\;\chi(t,\lambda) \le \partial_t\Big(\big(g_{1-\alpha,n} * \bar{H}\big(e^W\big)\Big) + \frac{2}{\lambda}\, g_{1-\alpha,n} + \frac{2C_1}{\lambda r^2} + \frac{2|S_n(t)|}{\lambda}\,. \tag{5.63}$$

We then integrate (5.63) over $J_- = (0,\eta\rho)$ and employ (5.61) for the estimate

$$\Big(g_{1-\alpha,n} * \bar{H}\big(e^W\big)\Big)(\eta\rho) \le \frac{3}{\lambda}\int_0^{\eta\rho} g_{1-\alpha,n}(t)\,dt.$$

By sending $n \to \infty$, this leads to

$$\int_{J_1(\lambda)} \mu_N\big(\Omega_t^-(\lambda)\big)\,dt = \int_0^{\eta\rho} \chi(t,\lambda)\,dt \le \frac{16r^2\mu_N(B)}{\nu}\Big(\frac{5}{\lambda}\, g_{2-\alpha}(\eta\rho) + \frac{2C_1\eta\rho}{\lambda r^2}\Big)$$
$$= \frac{16r^{2/\alpha}\mu_N(B)}{\nu\lambda}\big(5g_{2-\alpha}(\eta\tau) + 2C_1\eta\tau\big) =: C_4\,\frac{r^{2/\alpha}\mu_N(B)}{\lambda}, \quad \lambda \ge 4.$$

Hence with $C_5 = \max\{4\tau, C_4\}$ we find that

$$I_1 \le \frac{C_5 r^{2/\alpha}\mu_N(B)}{\lambda}, \quad \lambda > 0. \tag{5.64}$$

It remains to derive the desired estimate for I_3. To this purpose we shift again the time by putting $s = t - \eta\rho$, and denote the corresponding transformed functions as above by $\tilde{W}$, $\tilde{w}$, ... and so forth. Set further $\tilde{J}_+ := (0,(1-\eta)\rho)$. By the time-shifting property (5.6) and by positivity of e^W, relation (5.44) then implies

$$\frac{\nu}{16r^2\mu_N(B)}\int_{\delta B}(\tilde{w} - \tilde{W})^2 dx \le e^{-\tilde{W}}\partial_s(g_{1-\alpha,n} * e^{\tilde{W}}) + \frac{C_1}{r^2} + \tilde{S}_n(s), \quad \text{a.a. } s \in \tilde{J}_+. \tag{5.65}$$

Next, set $J_2(\lambda) = \{s \in \tilde{J}_+ : \tilde{W}(s) - c + \lambda/2 \ge 0\}$ and $\Omega_s^+(\lambda) = \{x \in \delta B : \tilde{w}(s,x) < c - \lambda\}$, $s \in J_2(\lambda)$. For $s \in J_2(\lambda)$, we have

$$\tilde{W}(s) - \tilde{w}(s,x) \ge \tilde{W}(s) - c + \lambda \ge \lambda/2, \quad x \in \Omega_s^+(\lambda),$$

and thus (5.65) yields that a.e. in $J_2(\lambda)$

$$\frac{\nu}{16r^2\mu_N(B)}\;\mu_N\big(\Omega_s^+(\lambda)\big) \le \frac{1}{(\tilde{W} - c + \lambda)^2}\Big(e^{-\tilde{W}}\partial_s(g_{1-\alpha,n} * e^{\tilde{W}}) + \frac{C_1}{r^2} + \tilde{S_n}\Big). \tag{5.66}$$

We proceed now similarly as above for the term I_1. Set $\chi(s,\lambda) = \mu_N\big(\Omega_s^+(\lambda)\big)$, if $s \in J_2(\lambda)$, and $\chi(s,\lambda) = 0$ in case $s \in \tilde{J}_+ \setminus J_1(\lambda)$. We consider this time the convex function $H(y) = (\log y - c + \lambda)^{-1}$ for $y \ge y_* := e^{c-\lambda/2}$ with derivative $H'(y) = -(\log y - c + \lambda)^{-2}y^{-1} < 0$. We define a C^1 extension $\bar{H}$ of H on $[0,\infty)$ by means of

$$\bar{H}(y) = \begin{cases} H'(y_*)(y - y_*) + H(y_*) : 0 \le y < y_* \\ H(y) \qquad\qquad\qquad\qquad : y \ge y_*. \end{cases}$$

Evidently, $-\bar{H}$ is concave in $[0,\infty)$ and

$$0 \le -\bar{H}'(y)y \le \frac{1}{(\log y_* - c + \lambda)^2} \le \frac{1}{(\lambda/2)^2} \le \frac{4}{\lambda}, \quad y \ge 0,\ \lambda \ge 1. \tag{5.67}$$

We will assume $\lambda \ge 1$ in the subsequent lines.

Observe that

$$-e^{\tilde{W}(s)} H'(e^{\tilde{W}(s)}) = \frac{1}{(\tilde{W}(s) - c + \lambda)^2}, \quad \text{a.a. } s \in J_2(\lambda).$$

Since $-\bar{H}' \ge 0$, and $e^{-\tilde{W}}\partial_s(g_{1-\alpha,n} * e^{\tilde{W}}) + C_1 r^{-2} + \tilde{S_n} \ge 0$ on $\tilde{J}_+$ due to (5.65), it thus follows from (5.66) and (5.67) that

$$\begin{aligned}\frac{\nu}{16r^2\mu_N(B)}\,\chi(s,\lambda) &\le -e^{\tilde{W}}\bar{H}'(e^{\tilde{W}})\Big(e^{-\tilde{W}}\partial_s(g_{1-\alpha,n} * e^{\tilde{W}}) + \frac{C_1}{r^2} + \tilde{S}_n\Big)\\ &\le -\bar{H}'(e^{\tilde{W}})\partial_s(g_{1-\alpha,n} * e^{\tilde{W}}) + \frac{4C_1}{\lambda r^2} + \frac{4|\tilde{S}_n(s)|}{\lambda}, \quad \text{a.a. } s \in \tilde{J}_+. \end{aligned} \tag{5.68}$$

By concavity of $-\bar{H}$, the fundamental identity (2.6) provides the estimate

$$\begin{aligned}-\bar{H}'(e^{\tilde{W}})\partial_s(g_{1-\alpha,n} * e^{\tilde{W}}) &\le -\partial_s\Big(g_{1-\alpha,n} * \bar{H}\big(e^{\tilde{W}}\big)\Big) + \Big(\bar{H}(e^{\tilde{W}}) - \bar{H}'(e^{\tilde{W}})e^{\tilde{W}}\Big)g_{1-\alpha,n}\\ &\le -\partial_s\Big(g_{1-\alpha,n} * \bar{H}\big(e^{\tilde{W}}\big)\Big) + \bar{H}(0)g_{1-\alpha,n} \le -\partial_s\Big(g_{1-\alpha,n} * \bar{H}\big(e^{\tilde{W}}\big)\Big) + \frac{6}{\lambda}\,g_{1-\alpha,n},\end{aligned}$$

a.e. in $\tilde{J}_+$, which when combined with (5.68) leads to

$$\frac{\nu}{16r^2\mu_N(B)}\,\chi(s,\lambda) \le -\partial_s\Big(g_{1-\alpha,n} * \bar{H}\big(e^{\tilde{W}}\big)\Big) + \frac{6}{\lambda}\,g_{1-\alpha,n} + \frac{4C_1}{\lambda r^2} + \frac{4|\tilde{S}_n(s)|}{\lambda},$$

for a.a. $s \in \tilde{J}_+$. We integrate this estimate over $\tilde{J}_+$ and send $n \to \infty$ to the result

$$\begin{aligned}\int_{J_2(\lambda)} \mu_N\big(\Omega_s^+(\lambda)\big)\,ds &= \int_0^{(1-\eta)\rho} \chi(s,\lambda)\,ds \le \frac{16r^2\mu_N(B)}{\nu}\Big(\frac{6}{\lambda}\,g_{2-\alpha}\big((1-\eta)\rho\big) + \frac{4C_1(1-\eta)\rho}{\lambda r^2}\Big)\\ &= \frac{16r^{2/\alpha}\mu_N(B)}{\nu\lambda}\big(6g_{2-\alpha}\big((1-\eta)\tau\big) + 4C_1(1-\eta)\tau\big) =: C_6\,\frac{r^{2/\alpha}\mu_N(B)}{\lambda}, \quad \lambda \ge 1.\end{aligned}$$

Hence with $C_7 = \max\{\tau, C_6\}$ we obtain that

$$I_3 \le \frac{C_7 r^{2/\alpha}\mu_N(B)}{\lambda}, \quad \lambda > 0. \tag{5.69}$$

Finally, combining (5.46), (5.47), and (5.48), (5.57), (5.64), (5.69) establishes the theorem. □

5.5 The final step of the proof

We are now in position to prove Theorem 5.1.1. Without loss of generality we may assume that $u \ge \varepsilon$ for some $\varepsilon > 0$; otherwise replace u by $u + \varepsilon$, which is a supersolution of (5.1) with $u_0 + \varepsilon$ instead of u_0, and eventually let $\varepsilon \to 0+$.

For $0<\sigma\le 1$, we set $U_\sigma=(t_0+(2-\sigma)\tau r^{2/\alpha},t_0+2\tau r^{2/\alpha})\times\sigma B$ and $U'_\sigma=(t_0,t_0+\sigma\tau r^{2/\alpha})\times\sigma B$. Clearly, $Q_-(t_0,x_0,r)=U'_\delta$ and $Q_+(t_0,x_0,r)=U_\delta$.

By Theorem 5.3.1,

$$\operatorname*{ess\,sup}_{U_{\sigma'}} u^{-1}\le\Big(\frac{C\mu_{N+1}(U_1)^{-1}}{(\sigma-\sigma')^{\tau_0}}\Big)^{1/\gamma}|u^{-1}|_{L_\gamma(U_\sigma)},\quad \delta\le\sigma'<\sigma\le 1,\ \gamma\in(0,1].$$

Here $C=C(\nu,\Lambda,\delta,\tau,\alpha,N)$ and $\tau_0=\tau_0(\alpha,N)$. This shows that the first hypothesis of Lemma 2.5.3 is satisfied by any positive constant multiple of u^{-1} with $\beta_0=\infty$.

Consider now $f_1=u^{-1}e^{c(u)}$ where $c(u)$ is the constant from Theorem 5.4.1 with $K_-=U'_1$ and $K_+=U_1$. Since $\log f_1=c(u)-\log u$, we see from Theorem 5.4.1, estimate (5.36), that

$$\mu_{N+1}(\{(t,x)\in U_1:\ \log f_1(t,x)>\lambda\})\le M\mu_{N+1}(U_1)\lambda^{-1},\quad \lambda>0,$$

where $M=M(\nu,\Lambda,\delta,\tau,\eta,\alpha,N)$. Hence we may apply Lemma 2.5.3 with $\beta_0=\infty$ to f_1 and the family U_σ; thereby we obtain

$$\operatorname*{ess\,sup}_{U_\delta} f_1\le M_1$$

with $M_1=M_1(\nu,\Lambda,\delta,\tau,\eta,\alpha,N)$. In terms of u this means that

$$e^{c(u)}\le M_1\ \operatorname*{ess\,inf}_{U_\delta} u. \tag{5.70}$$

On the other hand, Theorem 5.3.2 yields

$$|u|_{L_p(U'_{\sigma'})}\le\Big(\frac{C\mu_{N+1}(U'_1)^{-1}}{(\sigma-\sigma')^{\tau_1}}\Big)^{1/\gamma-1/p}|u|_{L_\gamma(U'_\sigma)},\quad \delta\le\sigma'<\sigma\le 1,\ 0<\gamma\le p/\tilde\kappa.$$

Here $C=C(\nu,\Lambda,\delta,\tau,\alpha,N,p)$ and $\tau_1=\tau_1(\alpha,N)$. Thus the first hypothesis of Lemma 2.5.3 is satisfied by any positive constant multiple of u with $\beta_0=p$ and $\eta=1/\tilde\kappa$. Taking $f_2=ue^{-c(u)}$ with $c(u)$ from above, we have $\log f_2=\log u-c(u)$ and so Theorem 5.4.1, estimate (5.35), gives

$$\mu_{N+1}(\{(t,x)\in U'_1:\ \log f_2(t,x)>\lambda\})\le M\mu_{N+1}(U'_1)\lambda^{-1},\quad \lambda>0,$$

where M is as above. Therefore we may again apply Lemma 2.5.3, this time to the function f_2 and the sets U'_σ, and with $\beta_0=p$ and $\eta=1/\tilde\kappa$; we get

$$|f_2|_{L_p(U'_\delta)}\le M_2\mu_{N+1}(U'_1)^{1/p},$$

where $M_2=M_2(\nu,\Lambda,\delta,\tau,\eta,\alpha,N,p)$. Rephrasing then yields

$$\mu_{N+1}(U'_1)^{-1/p}|u|_{L_p(U'_\delta)}\le M_2e^{c(u)}. \tag{5.71}$$

Finally, we combine (5.70) and (5.71) to the result

$$\mu_{N+1}(U'_1)^{-1/p}|u|_{L_p(U'_\delta)}\le M_1M_2\ \operatorname*{ess\,inf}_{U_\delta} u,$$

which proves the assertion. □

5.6 Optimality of the exponent $\frac{2+N\alpha}{2+N\alpha-2\alpha}$ in the weak Harnack inequality

In this section we will show that the exponent $\frac{2+N\alpha}{2+N\alpha-2\alpha}$ in Theorem 5.1.1 is optimal.

To this purpose consider the nonhomogeneous fractional diffusion equation on $\mathbb{R}^N$

$$\partial_t^\alpha u - \Delta u = f, \quad t \in (0,T],\, x \in \mathbb{R}^N, \tag{5.72}$$

with initial condition

$$u(0,x) = 0, \quad x \in \mathbb{R}^n. \tag{5.73}$$

Following [27], we say that a function $u \in C([0,T]\times\mathbb{R}^N) \cap C((0,T]; C^2(\mathbb{R}^N))$ with $g_{1-\alpha} * u \in C^1((0,T]; C(\mathbb{R}^N))$ is a classical solution of the problem (5.72), (5.73) if u satisfies (5.72) and (5.73). For any bounded continuous function f that is locally Hölder continuous in x, there exists a unique classical solution u of the problem (5.72), (5.73), and it is of the form

$$u(t,x) = \int_0^t \int_{\mathbb{R}^N} Y(t-\tau, x-y) f(\tau,y)\, dy\, d\tau, \tag{5.74}$$

where

$$Y(t,x) = c(N)|x|^{-N} t^{\alpha-1} H^{20}_{12}\Big(\frac{1}{4}\, t^{-\alpha}|x|^2 \Big|^{(\alpha,\alpha)}_{(N/2,1),\,(1,1)}\Big),$$

cf. [27]. Here $H^{20}_{12}(z|^{(\alpha,\alpha)}_{(N/2,1),\,(1,1)})$ denotes a special H function (also termed Fox's H function), see [44, Section 1.12] and [27] for its definition. It is differentiable for $z > 0$, the asymptotic behaviour for $z \to \infty$ and $z \to +0$, respectively, is described in [27, formulae (3.9) and (3.14)]. It has been also proved in [27] that Y is nonnegative.

We choose a smooth and nonnegative approximation of unity $\{\phi_n(t,x)\}_{n\in\mathbb{N}}$ in $\mathbb{R}_+ \times \mathbb{R}^N$ such that each ϕ_n is bounded. Put $f = \phi_n$ in (5.72) and denote the corresponding classical solution of (5.72), (5.73) by u_n. Evidently, u_n is nonnegative and satisfies

$$\partial_t^\alpha u_n - \Delta u_n = \phi_n \geq 0, \quad t \in (0,T],\, x \in \mathbb{R}^N.$$

Hence u_n is a nonnegative supersolution of (5.72) with $f = 0$ for all $n \in \mathbb{N}$.

Suppose the weak Harnack inequality (5.3) holds for some $p \geq \frac{2+N\alpha}{2+N\alpha-2\alpha}$. Then, by taking $Q_- = (0,1)\times B(0,1)$ and $Q_+ = (2,3)\times B(0,1)$ it follows that

$$\Big(\int_{Q_-} u_n^p \, d\mu_{N+1}\Big)^{1/p} \leq C \inf_{Q_+} u_n, \quad n \in \mathbb{N}, \tag{5.75}$$

where the constant C is independent of n. Since $u_n \to Y$ in the distributional sense as $n \to \infty$, we have

$$\inf_{Q_+} u_n \leq \frac{1}{\mu_{N+1}(Q_+)} \int_{Q_+} u_n \, d\mu_{N+1} \leq 1 + \frac{1}{\mu_{N+1}(Q_+)} \int_{Q_+} Y \, d\mu_{N+1} < \infty, \quad n \geq n_0,$$

for a sufficiently large n_0. On the other hand, the left-hand side of (5.75) cannot stay bounded, since $Y \notin L_p(Q_-)$ for $p \geq \frac{2+N\alpha}{2+N\alpha-2\alpha}$. In fact, writing $H^{20}_{12}(z) = H^{20}_{12}(z|^{(\alpha,\alpha)}_{(N/2,1),\,(1,1)})$ for short, we have

$$
\begin{aligned}
|Y|^p_{L_p(Q_-)} &= \int_0^1 \int_{B(0,1)} c(N)^p |x|^{-Np} t^{(\alpha-1)p} H^{20}_{12}\big(t^{-\alpha}|x|^2/4\big)^p \, dx\, dt \\
&= c_1 \int_0^1 \int_0^1 r^{N-1-Np} t^{(\alpha-1)p} H^{20}_{12}\big(t^{-\alpha} r^2/4\big)^p \, dr\, dt \\
&= c_1 \int_0^1 \int_0^{t^{-\alpha/2}} \big(\rho t^{\alpha/2}\big)^{N-1-Np} t^{(\alpha-1)p+\alpha/2} H^{20}_{12}\big(\rho^2/4\big)^p \, d\rho\, dt \\
&\geq c_1 \int_0^1 t^{\alpha(N-Np)/2+(\alpha-1)p}\, dt \int_0^1 \rho^{N-1-Np} H^{20}_{12}\big(\rho^2/4\big)^p \, d\rho \\
&\geq c_2 \int_0^1 t^{\alpha(N-Np)/2+(\alpha-1)p}\, dt,
\end{aligned}
$$

with some positive constant c_2. The last integral diverges for all $p \geq \frac{2+N\alpha}{2+N\alpha-2\alpha}$. Hence (5.75) yields a contradiction.

5.7 Applications of the weak Harnack inequality

The strong maximum principle for weak subsolutions of (5.1) may be easily derived as a consequence of the weak Harnack inequality.

Theorem 5.7.1 *Let $\alpha \in (0,1)$, $T > 0$, and $\Omega \subset \mathbb{R}^N$ be a bounded domain. Suppose the assumptions (H1)–(H3) are satisfied. Let $u \in \mathcal{S}_\alpha$ be a weak subsolution of (5.1) in Ω_T and assume that $0 \leq \operatorname{ess\,sup}_{\Omega_T} u < \infty$ and that $\operatorname{ess\,sup}_{\Omega} u_0 \leq \operatorname{ess\,sup}_{\Omega_T} u$. Then, if for some cylinder $Q = (t_0, t_0 + \tau r^{2/\alpha}) \times B(x_0, r) \subset \Omega_T$ with $t_0, \tau, r > 0$ and $\overline{B(x_0,r)} \subset \Omega$ we have*

$$
\operatorname*{ess\,sup}_{Q} u \;=\; \operatorname*{ess\,sup}_{\Omega_T} u, \tag{5.76}
$$

the function u is constant on $(0, t_0) \times \Omega$.

Proof: Let $M = \operatorname{ess\,sup}_{\Omega_T} u$. Then $v := M - u$ is a nonnegative weak supersolution of (5.1) with u_0 replaced by $v_0 := M - u_0 \geq 0$. For any $0 \leq t_1 < t_1 + \eta r^{2/\alpha} < t_0$ the weak Harnack inequality with $p = 1$ applied to v yields an estimate of the form

$$
r^{-(N+2/\alpha)} \int_{t_1}^{t_1+\eta r^{2/\alpha}} \int_{B(x_0,r)} (M-u)\, dx\, dt \leq C \operatorname*{ess\,inf}_{Q} (M-u) \;=\; 0.
$$

This shows that $u = M$ a.e. in $(0, t_0) \times B(x_0, r)$. As in the classical parabolic case (cf. [51]) the assertion now follows by a chaining argument. □

We next apply the weak Harnack inequality to establish continuity at $t = 0$ for weak solutions.

Theorem 5.7.2 *Let $\alpha \in (0,1)$, $T > 0$, and $\Omega \subset \mathbb{R}^N$ be a bounded domain. Suppose the assumptions (H1) and (H2) are satisfied. Let $u \in \mathcal{S}_\alpha$ be a bounded weak solution of (5.1) in Ω_T with $u_0 = 0$. Then u is continuous at $(0, x_0)$ for all $x_0 \in \Omega$ and $\lim_{(t,x)\to(0,x_0)} u(t,x) = 0$. Moreover, letting $\eta > 0$ we have for any cylinder $Q(x_0, r_0) := (0, \eta r_0^{2/\alpha}) \times B(x_0, r_0) \subset \Omega_T$ and $r \in (0, r_0]$*

$$
\operatorname*{ess\,osc}_{Q(x_0,r)} u \leq C \Big(\frac{r}{r_0}\Big)^{\delta} |u|_{L_\infty(\Omega_T)}, \tag{5.77}
$$

with $\operatorname{ess\,osc}_{Q(x_0,r)} = \operatorname{ess\,sup}_{Q(x_0,r)} - \operatorname{ess\,inf}_{Q(x_0,r)}$ *and constants* $C = C(\nu, \Lambda, \eta, \alpha, N) > 0$ *and* $\delta = \delta(\nu, \Lambda, \eta, \alpha, N) \in (0, 1)$.

Proof: Let $u \in \mathcal{S}_\alpha$ be a bounded weak solution of (5.1) in Ω_T with $u_0 = 0$. Set $u(t,x) = 0$ and $A(t,x) = Id$ for $t < 0$ and $x \in \Omega$. For $T_0 > 0$ we shift the time by setting $s = t + T_0$ and put $\tilde{f}(s) = f(s - T_0)$, $s \in (0, T + T_0)$, for functions f defined on $(-T_0, T)$. Since $Du(t, \cdot) = 0$ for $t < 0$ and

$$\partial_t (g_{1-\alpha,n} * u)(t,x) = \partial_t \int_{-T_0}^{t} g_{1-\alpha,n}(t-\tau) u(\tau, x)\, d\tau = \partial_s (g_{1-\alpha,n} * \tilde{u})(s,x),$$

the function $\tilde{u}$ is a bounded weak solution of

$$\partial_s^\alpha \tilde{u} - \operatorname{div}\big(\tilde{A}(s,x) D\tilde{u}\big) = 0, \quad s \in (0, T+T_0),\ x \in \Omega.$$

Next, assuming $r \in (0, r_0/2]$ we introduce the cylinders

$$Q_*(x_0, r) = \big(-\eta r^{2/\alpha}, \eta r^{2/\alpha}\big) \times B(x_0, r),$$
$$Q_-(x_0, r) = \big(-\eta (2r)^{2/\alpha}, -\eta (3r/2)^{2/\alpha}\big) \times B(x_0, r),$$

and denote by $\tilde{Q}_*(x_0,r)$ resp. $\tilde{Q}_-(x_0,r)$ the corresponding cylinders in the (s,x) coordinate system. Let us write $M_i = \operatorname{ess\,sup}_{\tilde{Q}_*(x_0, ir)} \tilde{u}$ and $m_i = \operatorname{ess\,inf}_{\tilde{Q}_*(x_0, ir)} \tilde{u}$ for $i = 1, 2$. Choosing $T_0 \ge \eta (2r)^{2/\alpha}$, we may apply Theorem 5.1.1 with $p = 1$ to the functions $M_2 - \tilde{u}$, $\tilde{u} - m_2$, which are nonnegative in $(0, \eta(2r)^{2/\alpha} + T_0) \times B(x_0, 2r)$, thereby obtaining

$$r^{-N+2/\alpha} \int_{\tilde{Q}_-(x_0,r)} (M_2 - \tilde{u})\, d\mu_{N+1} \le C(M_2 - M_1),$$
$$r^{-N+2/\alpha} \int_{\tilde{Q}_-(x_0,r)} (\tilde{u} - m_2)\, d\mu_{N+1} \le C(m_1 - m_2),$$

where $C > 1$ is a constant independent of u and r. By addition, it follows that

$$M_2 - m_2 \le C(M_2 - m_2 + m_1 - M_1).$$

Writing $\omega(x_0, r) = \operatorname{ess\,sup}_{\tilde{Q}_*(x_0,ir)} \tilde{u} - \operatorname{ess\,inf}_{\tilde{Q}_*(x_0,ir)} \tilde{u}$, this yields

$$\omega(x_0, r) \le \theta \omega(x_0, 2r), \quad r \le r_0/2, \tag{5.78}$$

where $\theta = 1 - C^{-1} \in (0,1)$. Iterating (5.78) as in the proof of [29, Lemma 8.23] we obtain

$$\omega(x_0, r) \le \frac{1}{\theta} \Big(\frac{r}{r_0}\Big)^{\log\theta / \log(1/2)} \omega(x_0, r_0), \quad r \le r_0.$$

The estimate (5.77) then follows by transforming back to the function u and using that $u = 0$ for negative times. In particular, we also see that u is continuous at $(0, x_0)$ for all $x_0 \in \Omega$ and that $\lim_{(t,x)\to(0,x_0)} u(t,x) = 0$. $\square$

The last application is a theorem of Liouville type. We say that a function u on $\mathbb{R}_+ \times \mathbb{R}^N$ is a *global weak solution* of

$$\partial_t^\alpha u - \operatorname{div}\big(A(t,x) Du\big) = 0, \tag{5.79}$$

if it is a weak solution of (5.79) in $(0,T) \times B(0,r)$ for all $T > 0$ and $r > 0$.

Corollary 5.7.1 *Let $\alpha \in (0,1)$. Assume that $A \in L_\infty(\mathbb{R}_+ \times \mathbb{R}^N; \mathbb{R}^{N\times N})$ and that there exists $\nu > 0$ such that*

$$\big(A(t,x)\xi|\xi\big) \geq \nu|\xi|^2, \quad \textit{for a.a. } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^N, \textit{ and all } \xi \in \mathbb{R}^N.$$

Suppose that u is a global bounded weak solution of (5.79). Then $u = 0$ a.e. on $\mathbb{R}_+ \times \mathbb{R}^N$.

Proof: For $r > 0$ and $x_0 = 0$ it follows from the proof of Theorem 5.7.2 that

$$\omega(0,r) \leq \theta\omega(0,2r), \quad r > 0, \tag{5.80}$$

where $\theta \in (0,1)$ is independent of r and u. By induction, (5.80) yields

$$\omega(0,r) \leq \theta^n \omega(0,2^n r) \leq 2\theta^n |u|_{L_\infty(\mathbb{R}_+\times\mathbb{R}^N)}, \quad r > 0,\, n \in \mathbb{N}.$$

Sending $n \to \infty$ shows that u is constant. The claim then follows by Theorem 5.7.2. □

Chapter 6

Hölder estimates for weak solutions of fractional evolution equations

6.1 The main regularity theorem

Let $T>0$ and Ω be a bounded domain in $\mathbb{R}^N$. The main purpose of this chapter is to study the regularity of weak solutions to fractional evolution equations of the form

$$\partial_t^\alpha (u-u_0) - D_i\big(a_{ij}(t,x)D_j u\big) = f, \;\; t\in(0,T),\, x\in\Omega. \tag{6.1}$$

Here $\alpha\in(0,1)$, $u_0=u_0(x)$ is a given initial data for u, $A=(a_{ij})\in L_\infty((0,T)\times\Omega;\mathbb{R}^{N\times N})$, and we use the sum convention.

We assume that

(H1) $A\in L_\infty(\Omega_T;\mathbb{R}^{N\times N})$, and

$$\sum_{i,j=1}^N |a_{ij}(t,x)|^2 \le \Lambda^2, \quad \text{for a.a. } (t,x)\in\Omega_T.$$

(H2) There exists $\nu>0$ such that

$$a_{ij}(t,x)\xi_i\xi_j \ge \nu|\xi|^2, \quad \text{for a.a. } (t,x)\in\Omega_T, \text{ and all } \xi\in\mathbb{R}^N.$$

(H3) $u_0\in L_\infty(\Omega)$; $f\in L_r([0,T];L_q(\Omega))$, where $r,q\ge 1$ fulfill

$$\frac{1}{\alpha r}+\frac{N}{2q}=1-\kappa,$$

and

$$\begin{array}{ll} r\in\left[\frac{1}{\alpha(1-\kappa)},\infty\right],\; q\in\left[\frac{N}{2(1-\kappa)},\infty\right],\; \kappa\in(0,1) & \text{for} \quad N\ge 2,\\ r\in\left[\frac{1}{\alpha(1-\kappa)},\frac{2}{\alpha(1-2\kappa)}\right],\; q\in[1,\infty],\; \kappa\in\left(0,\frac12\right) & \text{for} \quad N=1. \end{array}$$

For weak solutions we choose the same regularity class as in Chapter 5, thus u is a weak solution of (6.1) in Ω_T, if u belongs to the space

$$\mathcal{S}_\alpha = \{ v \in L_{2/(1-\alpha),w}([0,T]; L_2(\Omega)) \cap L_2([0,T]; H_2^1(\Omega)) \text{ such that } \\ g_{1-\alpha} * v \in C([0,T]; L_2(\Omega)), \text{ and } (g_{1-\alpha} * v)|_{t=0} = 0\},$$

and for any test function

$$\eta \in \mathring{H}_2^{1,1}(\Omega_T) = H_2^1([0,T]; L_2(\Omega)) \cap L_2([0,T]; \mathring{H}_2^1(\Omega))$$

with $\eta|_{t=T} = 0$ there holds

$$\int_0^T \int_\Omega \Big(-\eta_t \big[g_{1-\alpha} * (u - u_0)\big] + a_{ij} D_j u D_i \eta \Big) \, dxdt = \int_0^T \int_\Omega f \eta \, dxdt.$$

For $\beta_1, \beta_2 \in (0,1)$ and $Q \subset \Omega_T$ we set

$$[u]_{C^{\beta_1,\beta_2}(Q)} := \sup_{(t,x),(s,y) \in Q,\, (t,x) \neq (s,y)} \Big\{ \frac{|u(t,x) - u(s,y)|}{|t-s|^{\beta_1} + |x-y|^{\beta_2}} \Big\}.$$

The main regularity theorem reads as follows.

Theorem 6.1.1 *Let $\alpha \in (0,1)$, $T > 0$ and Ω be a bounded domain in $\mathbb{R}^N$. Let the assumptions (H1)-(H3) be satisfied and suppose that $u \in \mathcal{S}_\alpha$ is a bounded weak solution of (6.1) in Ω_T. Then there holds for any $Q \subset \Omega_T$ separated from Γ_T by a positive distance d,*

$$[u]_{C^{\frac{\alpha\epsilon}{2},\epsilon}(\bar{Q})} \le C\Big(|u|_{L_\infty(\Omega_T)} + |u_0|_{L_\infty(\Omega)} + |f|_{L_r([0,T];L_q(\Omega))} \Big)$$

with positive constants $\epsilon = \epsilon(\Lambda, \nu, \alpha, r, q, N, diam\,\Omega, \inf_{(\tau,z)\in Q} \tau)$ and $C = C(\Lambda, \nu, \alpha, r, q, N, diam\,\Omega, \lambda_{N+1}(Q), d)$.

Theorem 6.1.1 gives an interior Hölder estimate for bounded weak solutions of (6.1) in terms of the data and the L_∞-bound of the solution. It can be viewed as the time fractional analogue of the classical parabolic version ($\alpha = 1$) of the well-known De Giorgi-Nash theorem on the Hölder continuity of weak solutions to elliptic equations in divergence form (De Giorgi [19], Nash [60]), see also [29],[34],[56] for the elliptic and [49],[51], as well as the seminal contributions by Moser [57],[58] for the parabolic case. In the non-divergence case corresponding results were obtained by Krylov and Safonov [47], [48]. Concerning parabolic degenerate and singular equations we refer to [20] and [23].

The significance of Theorem 6.1.1 lies among others in providing the key a priori estimate for certain quasilinear time fractional diffusion equations to establish global strong (or classical) well-posedness for these problems, see Chapter 7. In order to succeed there, we also have to find conditions which ensure Hölder continuity of weak solutions of (6.1) up to the parabolic boundary. This is achieved by means of Theorem 6.1.1 and suitable extensions to larger domains within the framework of maximal L_p-regularity, see Theorem 6.7.1.

The proof of Theorem 6.1.1, which is contained in Sections 6.2–6.5, relies on local a priori estimates, which are derived by means of the fundamental identity (2.6) for the regularized fractional derivative. In the proof we further use De Giorgi's technique (employed similarly as in [49, Section II.6 and Section V.10], see also [26]) combined with the method of *non-local growth lemmas*, which has been recently developed in [70] for integro-differential operators like the fractional Laplacian, see also [40]. Concerning growth lemmas for partial differential equations we also refer to the work of Landis [50]. The adaption of this method to equations with memory requires an additional condition on the memory term, see (6.11) and (6.27) below, which naturally appears when shifting the time in the equation. To derive a suitable oscillation estimate for sequences of nested and shrinking cylinders exhibiting the natural scaling behaviour, we proceed by induction. The key idea here is to use the induction hypothesis, that is the oscillation estimate on all larger cylinders, to obtain the required estimate for the memory term.

As already pointed out, boundedness and the weak Harnack inequality, Theorem 5.1.1, are not strong enough for proving regularity of weak solutions. The different quality of the weak Harnack estimate and Theorem 6.1.1 is also reflected by the fact that nonnegative supersolutions of (6.1) are also nonnegative supersolutions of localizations of (6.1) when shifting the time, that is, the memory terms resulting from time-shifts have the right sign and can be dropped in the estimates, see Section 5.2. This is no longer possible in the case of the Hölder estimate, making the proof of the latter substantially more involved.

We further remark that Theorem 6.1.1 can be generalized without much effort to quasilinear equations of the form

$$\partial_t^\alpha (u-u_0) - D_i\big(a_i(t,x,u,Du)\big) = b(t,x,u,Du),\ t\in(0,T),\ x\in\Omega,$$

with appropriate structure conditions on the functions a_i and b, which correspond to the ones given in [49] in the case $\alpha=1$.

6.2 A basic nonlocal growth lemma

We begin with the regularized weak formulation of (6.1). The proof is the same as for Lemma 4.2.1.

Lemma 6.2.1 *Let $\alpha\in(0,1)$, $T>0$, and $\Omega\subset\mathbb{R}^N$ be a bounded domain. Suppose the assumptions (H1)–(H3) are satisfied. Then $u\in\mathcal{S}_\alpha$ is a weak solution of (6.1) in Ω_T if and only if for any test function $\psi\in\mathring{H}^1_2(\Omega)$ and any $n\in\mathbb{N}$ one has*

$$\int_\Omega \Big(\psi\partial_t[g_{1-\alpha,n}*(u-u_0)] + \big(h_n*[a_{ij}D_ju]\big)D_i\psi\Big)\,dx = \int_\Omega (h_n*f)\psi\,dx,\qquad a.a.\ t\in(0,T). \tag{6.2}$$

In order to derive local (in time) estimates for (6.1) it is necessary, as in the two previous chapters, to shift the time in the equation, which gives rise to an additional history term. Let $T_0\in(0,T)$ be fixed. For $t\in(T_0,T)$ we introduce the shifted time $s=t-T_0$ and set $\tilde v(s)=v(s+T_0)$, $s\in(0,T-T_0)$, for functions v defined on (T_0,T). Arguing as in Section 5.2 it follows from (6.2) that for any test function $\psi\in\mathring{H}^1_2(\Omega)$

$$\int_\Omega \Big(\psi\partial_s(g_{1-\alpha,n}*\tilde u)+\big(h_n*[a_{ij}D_ju]\big)\tilde{}\,D_i\psi\Big)\,dx = \int_\Omega \Big((h_n*f)\tilde{}+\tilde g_{1-\alpha,n}(s)u_0(x)\Big)\psi(x)\,dx$$
$$+\int_\Omega \psi(x)\int_0^{T_0}\big[-\dot g_{1-\alpha,n}(s+T_0-\tau)\big]u(\tau,x)\,d\tau\,dx, \tag{6.3}$$

for a.a. $s\in(0,T-T_0)$.

Let now u be a bounded weak solution of (6.1) and $(t_1,x_1)\in(0,T]\times\Omega$ be a fixed point. We consider the cylinders

$$Q(\rho):=Q(t_1,x_1,\theta,\rho)=(t_1-\theta\rho^{2/\alpha},t_1)\times B_\rho(x_1),$$

with scaling parameter $\rho>0$ and parameter $\theta>0$. We also write $B_\rho=B_\rho(x_1)$ for short.

Suppose $Q(2\rho) \subset \Omega_T$, that is $\rho \leq \rho_0 := \max\{1, \operatorname{diam}\Omega/4\}$. We put

$$t_0 = t_1 - \theta(2\rho)^{2/\alpha}, \quad \bar{t} = t_1 - \theta\theta_1\rho^{2/\alpha}, \ \ \theta_1 \in (1, 2^{2/\alpha}),$$

and assume that $t_0 \geq \tau_0$, where $\tau_0 \in (0,T)$ is a fixed number.

Let $\sigma_1 \in (0,1)$ and $k \in \mathbb{R}$. For $t \in (0,t_1)$ we choose in (6.2) the test function $\psi = (u-k)_+\eta^2$ with $\eta \in C_0^1(\Omega)$ so that $0 \leq \eta \leq 1$, $\eta = 1$ in $B_{(1-\sigma_1)\rho}$, $\operatorname{supp}\eta \subset B_\rho$, and $|D\eta| \leq 2/(\sigma_1\rho)$. We have

$$D\psi = \eta^2 D(u-k)_+ + 2\eta D\eta\,(u-k)_+,$$

and thus by substitution into (6.2) and inequality (2.7) we obtain

$$\begin{aligned}
&\int_{B_\rho}\Big(\frac{1}{2}\,\eta^2\partial_t[g_{1-\alpha,n} * (u-k)_+^2] + \frac{1}{2}\,\eta^2 g_{1-\alpha,n}(u-k)_+^2\Big)\,dx \\
&\quad + \int_{B_\rho}\eta^2(h_n * [a_{ij}D_ju])\,D_i(u-k)_+\,dx \leq -\int_{B_\rho}2\eta D_i\eta\,(h_n * [a_{ij}D_ju])(u-k)_+\,dx \\
&\qquad + \int_{B_\rho}\Big(h_n * f + g_{1-\alpha,n}(u_0-k)\Big)\eta^2(u-k)_+\,dx, \qquad (6.4)
\end{aligned}$$

for a.a. $t \in (0,t_1)$. Since

$$(u_0-k)(u-k)_+ \leq \frac{1}{2}\,(u_0-k)^2 + \frac{1}{2}\,(u-k)_+^2,$$

it follows that

$$\begin{aligned}
&\int_{B_\rho}\Big(\frac{1}{2}\,\eta^2\partial_t[g_{1-\alpha,n} * (u-k)_+^2] + \eta^2(h_n * [a_{ij}D_ju])\,D_i(u-k)_+\Big)\,dx \\
&\quad \leq \int_{B_\rho}\Big(-2\eta D_i\eta\,(h_n * [a_{ij}D_ju])(u-k)_+ + (h_n * f)\eta^2(u-k)_+\Big)\,dx \\
&\qquad + \frac{1}{2}\int_{B_\rho}\eta^2 g_{1-\alpha,n}(u_0-k)^2\,dx, \quad \text{a.a. } t \in (0,t_1). \qquad (6.5)
\end{aligned}$$

Suppose now that $t \in (\bar{t}, t_1)$ and shift the time by setting $s = t - \bar{t}$. Employing the same notation as in (6.3) with $T_0 = \bar{t}$, that is $\tilde{v}(s) = v(s+\bar{t})$ for functions v defined on $(\bar{t}, T)$, we have

$$\begin{aligned}
&\int_{B_\rho}\Big(\frac{1}{2}\,\eta^2\partial_s[g_{1-\alpha,n} * (\tilde{u}-k)_+^2] + \eta^2(h_n * [a_{ij}D_ju])\tilde{\ }D_i(\tilde{u}-k)_+\Big)\,dx \\
&\quad \leq \int_{B_\rho}\Big(-2\eta D_i\eta\,(h_n * a_{ij}D_ju)\tilde{\ }(\tilde{u}-k)_+ + (h_n * f)\tilde{\ }\eta^2(\tilde{u}-k)_+\Big)\,dx \\
&\qquad + \frac{1}{2}\int_{B_\rho}\Big(\eta^2\tilde{g}_{1-\alpha,n}(u_0-k)^2 + \eta^2\tilde{H}_{k,n}\Big)\,dx, \quad \text{a.a. } s \in (0, t_1-\bar{t}), \qquad (6.6)
\end{aligned}$$

where

$$H_{k,n}(t,x) = \int_0^{\bar{t}}[-\dot{g}_{1-\alpha,n}(t-\tau)](u(\tau,x)-k)_+^2\,d\tau, \ \ t \in (\bar{t}, t_1),\ x \in \Omega.$$

We next convolve (6.6) with g_α and observe that in view of

$$g_{1-\alpha,n} * (\tilde{u}-k)_+^2 \in {}_0H_1^1([0, t_1-\bar{t}]; L_1(\Omega))$$

and $g_{1-\alpha,n} = g_\alpha * h_n$ we have

$$g_\alpha * \partial_s\Big(g_{1-\alpha,n} * (\tilde u - k)_+^2\Big) = \partial_s\Big(g_\alpha * g_{1-\alpha,n} * (\tilde u - k)_+^2\Big) = h_n * (\tilde u - k)_+^2.$$

Sending then $n \to \infty$ and selecting an appropriate subsequence, if necessary, we obtain

$$\begin{aligned}\frac{1}{2}\int_{B_\rho} \eta^2(\tilde u - k)_+^2\,dx &+ g_\alpha * \int_{B_\rho} \eta^2 \tilde a_{ij} D_j \tilde u\, D_i(\tilde u - k)_+\,dx\\ &\le g_\alpha * \int_{B_\rho}\Big(-2\eta D_i\eta\, \tilde a_{ij} D_j \tilde u(\tilde u - k)_+ + \tilde f \eta^2(\tilde u - k)_+\Big)\,dx\\ &\quad + \frac{1}{2}\, g_\alpha * \int_{B_\rho} \eta^2\Big(\tilde g_{1-\alpha}(u_0 - k)^2 + \int_0^{\bar t} [-\dot g_{1-\alpha}(\bar t + \cdot - \tau)]\,(u(\tau,x) - k)_+^2\,d\tau\Big)\,dx, \quad (6.7)\end{aligned}$$

for a.a. $s \in (0, t_1 - \bar t)$. Note that by boundedness of u, the last integral in (6.7) is well-defined.

By (H2), we have

$$\begin{aligned}\int_{B_\rho} \eta^2 \tilde a_{ij} D_j \tilde u D_i(\tilde u - k)_+\,dx &= \int_{B_\rho} \eta^2 \tilde a_{ij} D_j(\tilde u - k)_+ D_i(\tilde u - k)_+\,dx\\ &\ge \nu \int_{B_\rho} \eta^2 |D(\tilde u - k)_+|^2\,dx.\end{aligned}$$

(H1) and Young's inequality imply that

$$2\eta|D_i\eta|\,|\tilde a_{ij}|\,|D_j\tilde u|(\tilde u - k)_+ \le \nu\,\eta^2|D(\tilde u - k)_+|^2 + \frac{\Lambda^2}{\nu}\,|D\eta|^2(\tilde u - k)_+^2.$$

From these relations, relation (6.7), and the properties of η we deduce that

$$\begin{aligned}\int_{B_{(1-\sigma_1)\rho}} (\tilde u - k)_+^2\,dx &\le g_\alpha * \int_{B_\rho}\Big(\frac{8\Lambda^2}{\nu(\sigma_1\rho)^2}\,(\tilde u - k)_+^2 + 2|\tilde f|(\tilde u - k)_+\Big)\,dx\\ &\quad + g_\alpha * \int_{B_\rho}\Big(\tilde g_{1-\alpha}(u_0 - k)^2 + \int_0^{\bar t}[-\dot g_{1-\alpha}(\bar t + \cdot - \tau)]\,(u(\tau,x) - k)_+^2\,d\tau\Big)\,dx, \quad (6.8)\end{aligned}$$

a.e. in $(0, t_1 - \bar t)$. This estimate is the starting point for the following nonlocal growth lemma.

We set

$$\begin{aligned}A_{k,\rho}(t) &= \{x \in B_\rho :\, u(t,x) > k\}, \quad t \in (t_0, t_1),\\ \tilde A_{k,\rho}(s) &= \{x \in B_\rho :\, \tilde u(s,x) > k\}, \quad s \in (0, t_1 - \bar t),\end{aligned}$$

and introduce the number

$$t_* = t_1 - \theta\theta_2\rho^{2/\alpha}, \quad \theta_2 \in (1, \theta_1).$$

Proposition 6.2.1 *Let u be a bounded weak solution of (6.1) and $M_0 = |u|_{L_\infty(\Omega_T)}$. Suppose further that the above assumptions are satisfied. Let $M = \operatorname{ess\,sup}_{Q(2\rho)} u$ and $|k| \le M_0$. Then there exist constants $1 < \theta_2 < \theta_1 < 2^{2/\alpha}$, $\mu, \xi, \sigma_1, \beta \in (0,1)$, and $\theta, \gamma_0 > 0$ that depend only on $\Lambda, \nu, \alpha, r, q, N, \tau_0, \rho_0, M_0, |u_0|_{L_\infty(\Omega)}, |f|_{L_r([0,T];L_q(\Omega))}$ and are such that for any $\gamma \in (0, \gamma_0]$ the following implication holds true: If*

$$\max\{\mu(M - k), \rho^\kappa\} < \operatorname*{ess\,sup}_{[t_0,t_1]\times B_\rho} u - k =: Z_k \quad (6.9)$$

and

$$\lambda_{N+1}\Big(\{(t,x)\in(t_0,\bar t)\times B_\rho:\, u(t,x)\geq k\}\Big)\leq \frac12\,\lambda_{N+1}\Big((t_0,\bar t)\times B_\rho\Big), \tag{6.10}$$

as well as

$$u(t,x)-k\leq (M-k)\Big(2\Big[2^{2/\alpha}(\frac{t_1-t}{t_1-t_0})\Big]^\gamma-1\Big),\quad a.a.\ (t,x)\in(0,t_0)\times B_\rho, \tag{6.11}$$

then

$$\lambda_N\Big(A_{k+\xi Z_k,\rho}(t)\Big)\leq \beta\lambda_N(B_\rho),\quad a.a.\ t\in(t_*,t_1). \tag{6.12}$$

Proof. Let $\xi\in(0,1)$ and suppose that $t\in(t_*,t_1)$, i.e. $s\in(t_*-\bar t,t_1-\bar t)$. Then

$$\begin{aligned}\lambda_N\Big(\tilde A_{k+\xi Z_k,\rho}(s)\Big) &\leq \lambda_N\Big(\tilde A_{k+\xi Z_k,(1-\sigma_1)\rho}(s)\Big)+\lambda_N(B_\rho)-\lambda_N(B_{(1-\sigma_1)\rho})\\ &\leq \lambda_N\Big(\tilde A_{k+\xi Z_k,(1-\sigma_1)\rho}(s)\Big)+\sigma_1 N\lambda_N(B_\rho).\end{aligned} \tag{6.13}$$

By the definition of Z_k, which is positive, by assumption (6.9), we have

$$(\xi Z_k)^2\lambda_N\Big(\tilde A_{k+\xi Z_k,(1-\sigma_1)\rho}(s)\Big)\leq \int_{\tilde A_{k,(1-\sigma_1)\rho}(s)}(\tilde u-k)^2\,dx.$$

This, together with (6.8), yields that

$$\begin{aligned}\lambda_N\Big(\tilde A_{k+\xi Z_k,(1-\sigma_1)\rho}(s)\Big) &\leq \frac{8\Lambda^2}{\nu(\xi Z_k)^2(\sigma_1\rho)^2}\,g_\alpha*\int_{B_\rho}(\tilde u-k)_+^2\,dx\\ &+\frac{2}{(\xi Z_k)^2}\,g_\alpha*\int_{B_\rho}|\tilde f|(\tilde u-k)_+\,dx+\frac{1}{(\xi Z_k)^2}\,g_\alpha*\int_{B_\rho}\tilde g_{1-\alpha}(u_0-k)^2\,dx\\ &+\frac{1}{(\xi Z_k)^2}\,g_\alpha*\int_{B_\rho}\int_0^{t_0}[-\dot g_{1-\alpha}(\bar t+\cdot-\tau)]\,(u(\tau,x)-k)_+^2\,d\tau\,dx\\ &+\frac{1}{(\xi Z_k)^2}\,g_\alpha*\int_{B_\rho}\int_{t_0}^{\bar t}[-\dot g_{1-\alpha}(\bar t+\cdot-\tau)]\,(u(\tau,x)-k)_+^2\,d\tau\,dx\\ &=: \mathcal M_1+\mathcal M_2+\mathcal M_3+\mathcal M_4+\mathcal M_5.\end{aligned} \tag{6.14}$$

We now estimate the $\mathcal M_i$ terms one after the other. Evidently,

$$\begin{aligned}\mathcal M_1 &\leq \frac{C(\Lambda,\nu)}{(\xi\sigma_1\rho)^2}\,g_{1+\alpha}(s)\lambda_N(B_\rho)\leq \frac{C(\Lambda,\nu)}{(\xi\sigma_1\rho)^2}\,g_{1+\alpha}(t_1-\bar t)\lambda_N(B_\rho)\\ &\leq \frac{C(\Lambda,\nu,\alpha)}{(\xi\sigma_1\rho)^2}\Big(\theta\theta_1\rho^{2/\alpha}\Big)^\alpha\lambda_N(B_\rho)\leq C_1(\Lambda,\nu,\alpha)\,\frac{\theta^\alpha}{(\xi\sigma_1)^2}\,\lambda_N(B_\rho).\end{aligned} \tag{6.15}$$

Using (H3), (6.9), and Hölder's inequality, we have with $q'=q/(q-1)$

$$\begin{aligned}\mathcal M_2 &\leq \frac{4M_0}{\Gamma(\alpha)(\xi Z_k)^2}\int_0^s(s-\tau)^{(\alpha-1)}|\tilde f(\tau,\cdot)|_{L_q(B_\rho)}\,d\tau\,\lambda_N(B_\rho)^{\frac{1}{q'}}\\ &\leq \frac{C(\alpha,r)M_0 s^{\alpha-\frac1r}}{(\xi Z_k)^2}\,|\tilde f|_{L_r([0,t_1-\bar t];L_q(B_\rho))}\lambda_N(B_\rho)^{\frac{1}{q'}}\\ &\leq \frac{C(\alpha,r,|f|_{L_r(L_q)})M_0}{\xi^2\rho^{2\kappa}}\,(\theta\theta_1\rho^{2/\alpha})^{\alpha-\frac1r}\lambda_N(B_\rho)^{1-\frac1q}\\ &\leq C(\alpha,r,q,N,|f|_{L_r(L_q)})M_0\,\frac{\theta^{\alpha-\frac1r}}{\xi^2}\,\lambda_N(B_\rho).\end{aligned} \tag{6.16}$$

Further,

$$\begin{aligned}\mathcal{M}_3 \;&\le\; \frac{|u_0-k|^2_{L_\infty(\Omega)}}{(\xi Z_k)^2}\,\lambda_N(B_\rho)\int_0^s g_\alpha(s-\sigma)g_{1-\alpha}(\sigma+\bar t)\,d\sigma\\ &\le\; \frac{|u_0-k|^2_{L_\infty(\Omega)}}{\xi^2\rho^{2\kappa}}\,\lambda_N(B_\rho)g_{1-\alpha}(\tau_0)g_{1+\alpha}(t_1-\bar t)\\ &\le\; C(\alpha,\tau_0,\rho_0)\Big(M_0^2+|u_0|^2_{L_\infty(\Omega)}\Big)\,\frac{\theta^\alpha}{\xi^2}\,\lambda_N(B_\rho).\end{aligned} \tag{6.17}$$

We come now to the short-term memory term $\mathcal{M}_5$. By means of assumption (6.10) and the fact that $-\dot g_{1-\alpha}$ is positive and decreasing, we may estimate for $\sigma\in(0,t_1-\bar t)$,

$$\begin{aligned}&\int_{t_0}^{\bar t}\int_{B_\rho}[-\dot g_{1-\alpha}(\bar t+\sigma-\tau)]\,(u(\tau,x)-k)_+^2\,dxd\tau \le Z_k^2\int_{t_0}^{\bar t}[-\dot g_{1-\alpha}(\bar t+\sigma-\tau)]\lambda_N(A_{k,\rho}(\tau))\,d\tau\\ &\quad\le Z_k^2\lambda_N(B_\rho)\int_{\frac{t_0+\bar t}{2}}^{\bar t}[-\dot g_{1-\alpha}(\bar t+\sigma-\tau)]\,d\tau \le Z_k^2\lambda_N(B_\rho)\Big(g_{1-\alpha}(\sigma)-g_{1-\alpha}(\sigma+\,\frac{\bar t-t_0}{2}\,)\Big).\end{aligned}$$

Hence

$$\begin{aligned}\mathcal{M}_5 &\le \frac{\lambda_N(B_\rho)}{\xi^2}\,\int_0^s g_\alpha(s-\sigma)\Big(g_{1-\alpha}(\sigma)-g_{1-\alpha}(\sigma+\,\frac{\bar t-t_0}{2}\,)\Big)\,d\sigma\\ &= \frac{\lambda_N(B_\rho)}{\xi^2}\,\Big(1-\int_0^1 g_\alpha(1-\varsigma)g_{1-\alpha}(\varsigma+\,\frac{\bar t-t_0}{2s}\,)\,d\varsigma\Big)\\ &\le \frac{\lambda_N(B_\rho)}{\xi^2}\,\Big(1-\int_0^1 g_\alpha(1-\varsigma)g_{1-\alpha}(\varsigma+\,\frac{\bar t-t_0}{2(t_*-\bar t)}\,)\,d\varsigma\Big)\\ &= \frac{\lambda_N(B_\rho)}{\xi^2}\,\Big(1-\int_0^1 g_\alpha(1-\varsigma)g_{1-\alpha}(\varsigma+\,\frac{2^{2/\alpha}-\theta_1}{2(\theta_1-\theta_2)}\,)\,d\varsigma\Big).\end{aligned} \tag{6.18}$$

For the long-term memory term $\mathcal{M}_4$ we use the assumptions (6.9), (6.11). For a.a. $\sigma\in(0,t_1-\bar t)$ we have

$$\begin{aligned}&\int_{B_\rho}\int_0^{t_0}[-\dot g_{1-\alpha}(\bar t+\sigma-\tau)]\,(u(\tau,x)-k)_+^2\,d\tau\,dx\\ &\le \frac{Z_k^2\lambda_N(B_\rho)}{\mu^2}\,\int_0^{t_0}[-\dot g_{1-\alpha}(\bar t+\sigma-\tau)]\,\Big(2\Big[2^{2/\alpha}(\frac{t_1-\tau}{t_1-t_0})\Big]^\gamma-1\Big)^2\,d\tau\\ &\le \frac{Z_k^2\lambda_N(B_\rho)}{\mu^2}\,(t_1-t_0)\int_1^\infty[-\dot g_{1-\alpha}(\sigma+\bar t-t_1+\tau(t_1-t_0))]\,\Big(2[2^{2/\alpha}\tau]^\gamma-1\Big)^2\,d\tau.\end{aligned}$$

Observe that the last integral is finite provided that $2\gamma<\alpha$. We will assume this in what follows. Setting

$$\psi_\gamma(\tau)=2[2^{2/\alpha}\tau]^\gamma-1,\quad \tau\ge 1,$$

we then obtain

$$
\begin{aligned}
\mathcal{M}_4 \leq\; & \frac{\lambda_N(B_\rho)}{(\mu\xi)^2}\,(t_1-t_0)\int_0^s g_\alpha(s-\sigma)\int_1^\infty [-\dot g_{1-\alpha}(\sigma+\bar t-t_1+\tau(t_1-t_0))]\,\psi_\gamma^2(\tau)\,d\tau d\sigma\\
=\; & \frac{\lambda_N(B_\rho)}{(\mu\xi)^2}\cdot\frac{t_1-t_0}{s}\int_0^1 g_\alpha(1-\varsigma)\int_1^\infty \Big[-\dot g_{1-\alpha}(\varsigma+\frac{\bar t-t_1+\tau(t_1-t_0)}{s})\Big]\,\psi_\gamma^2(\tau)\,d\tau d\varsigma\\
=\; & \frac{\lambda_N(B_\rho)(t_1-t_0)}{(\mu\xi)^2 s}\int_0^1 g_\alpha(1-\varsigma)\int_1^\infty \Big[-\dot g_{1-\alpha}(\varsigma+\frac{\bar t-t_1+\tau(t_1-t_0)}{s})\Big]\,(\psi_\gamma^2(\tau)-1)\,d\tau d\varsigma\\
& +\frac{\lambda_N(B_\rho)}{(\mu\xi)^2}\int_0^1 g_\alpha(1-\varsigma)g_{1-\alpha}(\varsigma+\frac{\bar t-t_0}{s})\,d\varsigma =:\mathcal{M}_{4,1}+\mathcal{M}_{4,2}. \qquad (6.19)
\end{aligned}
$$

To estimate the second term we use the monotonicity of $g_{1-\alpha}$ and $s\leq t_1-\bar t$, thereby getting

$$\mathcal{M}_{4,2}\leq \frac{\lambda_N(B_\rho)}{(\mu\xi)^2}\int_0^1 g_\alpha(1-\varsigma)g_{1-\alpha}(\varsigma+\frac{2^{2/\alpha}-\theta_1}{\theta_1})\,d\varsigma. \qquad (6.20)$$

For $\mathcal{M}_{4,1}$ we need both the lower and the upper bound for s. We conclude that

$$\mathcal{M}_{4,1}\leq \frac{2^{2/\alpha}\lambda_N(B_\rho)}{(\mu\xi)^2(\theta_1-\theta_2)}\int_0^1 g_\alpha(1-\varsigma)\int_1^\infty \Big[-\dot g_{1-\alpha}(\varsigma+\frac{2^{2/\alpha}\tau-\theta_1}{\theta_1})\Big]\big(\psi_\gamma^2(\tau)-1\big)\,d\tau d\varsigma. \qquad (6.21)$$

Combining (6.13) – (6.21) and enlarging $\mathcal{M}_5$ by the factor $1/\mu^2$ we obtain with some constant

$$C=C(\Lambda,\nu,\alpha,r,q,N,\tau_0,\rho_0,M_0,|u_0|_{L_\infty(\Omega)},|f|_{L_r(L_q)})$$

the inequality

$$
\begin{aligned}
\lambda_N\Big(\tilde A_{k+\xi Z_k,\rho}(s)\Big) \leq\; & \lambda_N(B_\rho)\Big\{\sigma_1 N + C\Big[\frac{\theta^\alpha}{(\xi\sigma_1)^2}+\frac{\theta^{\alpha-\frac{1}{r}}}{\xi^2}+\frac{\theta^\alpha}{\xi^2}\Big]\\
& +\frac{1}{\mu^2\xi^2}\Big[1-\int_0^1 g_\alpha(1-\varsigma)\Big(g_{1-\alpha}(\varsigma+\frac{2^{2/\alpha}-\theta_1}{2(\theta_1-\theta_2)})-g_{1-\alpha}(\varsigma+\frac{2^{2/\alpha}-\theta_1}{\theta_1})\Big)\,d\varsigma\Big]\\
& +\frac{2^{2/\alpha}}{(\theta_1-\theta_2)\mu^2\xi^2}\int_0^1 g_\alpha(1-\varsigma)\int_1^\infty\Big[-\dot g_{1-\alpha}(\varsigma+\frac{2^{2/\alpha}\tau-\theta_1}{\theta_1})\Big]\big(\psi_\gamma^2(\tau)-1\big)\,d\tau d\varsigma\Big\}\\
=:\; & \lambda_N(B_\rho)\Big(\sigma_1 N+\beta_1(\theta,\xi,\sigma_1)+\beta_2(\theta_1,\theta_2,\mu,\xi)+\beta_3(\theta_1,\theta_2,\mu,\xi,\gamma)\Big).
\end{aligned}
$$

Observe that the integral occurring in β_2 is strictly positive if and only if

$$\frac{2^{2/\alpha}-\theta_1}{2(\theta_1-\theta_2)} < \frac{2^{2/\alpha}-\theta_1}{\theta_1}\,, \qquad (6.22)$$

by monotonicity of $g_{1-\alpha}$. (6.22) is equivalent to the condition $2\theta_2<\theta_1$, which can be satisfied by suitable θ_1, θ_2 subject to $1<\theta_2<\theta_1<2^{2/\alpha}$. Notice as well that with $\xi,\sigma_1,\mu,\theta_1,\theta_2$ being fixed, we have $\beta_1\to 0$ as $\theta\to 0$, and $\beta_3\to 0$ as $\gamma\to 0$. The last assertion follows from Lebesgue's dominated convergence theorem.

We will now fix the parameters as follows. 1. Choose θ_1 and θ_2 such that $2<2\theta_2<\theta_1<2^{2/\alpha}$. 2. Select then $\xi,\mu\in(0,1)$ both close enough to 1 so that $\beta_2(\theta_1,\theta_2,\mu,\xi)<1$. 3. Choose $\sigma_1>0$ sufficiently small so that $\sigma_1 N+\beta_2(\theta_1,\theta_2,\mu,\xi)<1$. 4. Finally fix sufficiently small $\theta>0$ and $\gamma_0\in(0,\alpha/2)$ such that

$$\beta:=\sigma_1 N+\beta_1(\theta,\xi,\sigma_1)+\beta_2(\theta_1,\theta_2,\mu,\xi)+\beta_3(\theta_1,\theta_2,\mu,\xi,\gamma_0)<1.$$

Since β_3 is non-increasing in γ, the assertion of the proposition follows immediately. $\square$

Remark 6.2.1 Note that the assertion of Proposition 6.2.1 remains valid when equation (6.1) has a second term on the right-hand side of the form $-D_i g^i$, where $|g|^2 \in L_r([0,T]; L_q(\Omega))$. In fact, this would result in the additional term

$$g_\alpha * \int_{B_\rho} \tilde{g}^i D_i \psi \, dx = g_\alpha * \int_{B_\rho} \tilde{g}^i \big(\eta^2 D_i (u-k)_+ + 2\eta (u-k)_+ D_i \eta\big) \, dx$$

on the right of (6.7). Using $\tilde{g}^i D_i (u-k)_+ \leq \epsilon_0 |D(u-k)_+|^2 + C(\epsilon_0)|\tilde{g}|^2$ with sufficiently small $\epsilon_0 > 0$ the term containing $|D(u-k)_+|^2$ can be absorbed and the $|\tilde{g}|^2$-term is estimated similarly as the $\tilde{f}$-term above. Further, $\tilde{g}^i \eta (u-k)_+ D_i \eta$ can be estimated by Young's inequality, too, leading to suitable terms.

From now on we will assume that

(P) The set of parameters $\theta, \theta_1, \theta_2, \mu, \xi, \sigma_1, \beta, \gamma_0$ is fixed such that the implication of Proposition 6.2.1 holds true.

Recall that $M = \operatorname{ess\,sup}_{Q(2\rho)} u$. Let $m = \operatorname{ess\,inf}_{Q(2\rho)} u$ and $\bar{m} = (M+m)/2$. Suppose u satisfies (6.9), (6.10), and (6.11) for $k = \bar{m}$. Then, by Proposition 6.2.1,

$$\lambda_N \Big(A_{\bar{m}+\xi Z_{\bar{m}}, \rho}(t) \Big) \leq \beta \lambda_N (B_\rho), \quad \text{a.a. } t \in (t_*, t_1).$$

Consequently,

$$\begin{aligned} \lambda_N \Big(\{x \in B_\rho : \, u(t,x) - \bar{m} \leq \xi (M - \bar{m})\} \Big) &\geq \lambda_N \Big(\{x \in B_\rho : \, u(t,x) - \bar{m} \leq \xi Z_{\bar{m}})\} \Big) \\ &\geq (1-\beta) \lambda_N (B_\rho), \qquad \text{a.a. } t \in (t_*, t_1). \end{aligned} \tag{6.23}$$

Consider then in $(t_*, t_1) \times B_{2\rho}$ the function

$$v = \log \Big(\frac{(1-\xi)(M-\bar{m})}{M - u + \varepsilon (M - \bar{m}) + \rho^\kappa} \Big), \tag{6.24}$$

where $\varepsilon \in (0,1)$ is a parameter. Observe that we have $v \leq 0$ whenever $u - \bar{m} \leq \xi (M - \bar{m})$. In view of (6.23), the set of points in B_ρ for which this holds constitutes a certain portion of B_ρ, for a.a. $t \in (t_*, t_1)$. This property will later allow us to apply the weighted Poincaré inequality, Proposition 2.6.2, with suitable weight to v_+ in $(t_*, t_1) \times B_{2\rho}$.

Notice as well that the estimate

$$\operatorname*{ess\,sup}_{Q(\rho)} v \leq M_1 \tag{6.25}$$

(with $M_1 \geq 0$ independent of ρ, ε) implies the inequality

$$u \leq M - [e^{-M_1}(1-\xi) - \varepsilon](M - \bar{m}) + \rho^\kappa \quad \text{a.e. in } Q(\rho).$$

Setting $\omega(\rho) = \operatorname{ess\,sup}_{Q(\rho)} u - \operatorname{ess\,inf}_{Q(\rho)} u$ and choosing ε sufficiently small, it follows that

$$\tilde{\mu} := \max \big\{ 1 - e^{-M_1}(1-\xi)/2 + \varepsilon/2, (1+\mu)/2 \big\} \in (1/2, 1),$$

and

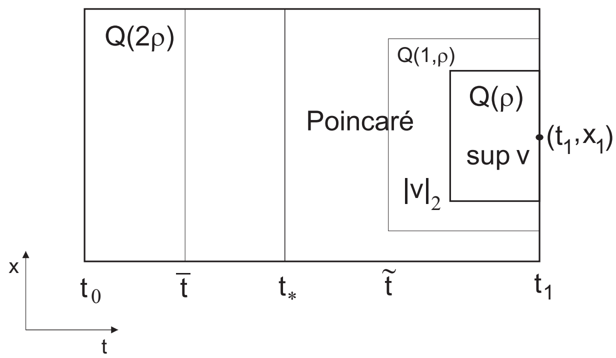
$$\omega(\rho) \leq \tilde{\mu} \omega(2\rho) + \rho^\kappa. \tag{6.26}$$

If the condition (6.9) with $k = \bar{m}$ is violated, then

$$\omega(\rho) \leq \operatorname*{ess\,sup}_{[t_0,t_1] \times B_\rho} u - m \leq \mu (M - \bar{m}) + \bar{m} - m + \rho^\kappa = \frac{\mu + 1}{2} (M - m) + \rho^\kappa,$$

and hence the oscillation estimate(6.26) holds in this case as well.

Our next objective is to establish (6.25). We proceed in two steps. First we derive an L_2-estimate for v on a slightly larger cylinder. Using De Giorgi's iteration technique this estimate then allows us to establish a sup-bound for v on $Q(\rho)$.



6.3 An L_2-estimate for v on a cylinder containing $Q(\rho)$

Let $\tilde{t} = t_1 - \theta\theta_3\rho^{2/\alpha}$ with $\theta_3 \in (1, \theta_2)$ being fixed. Our goal is to derive an L_2-estimate for v on the cylinder $(\tilde{t}, t_1) \times B_{3\rho/2}(x_1)$.

Proposition 6.3.1 *Let $\varepsilon \in (0,1)$ and u be a bounded weak solution of (6.1). Suppose that the assumptions formulated at the beginning of Section 6.2 are satisfied. Suppose that (P) holds and that the conditions (6.9) and (6.10) are satisfied with $k = \bar{m}$. Suppose further that*

$$u(t,x) - \bar{m} \le (M - \bar{m})\Big(2\Big[2^{2/\alpha}(\frac{t_1 - t}{t_1 - t_0})\Big]^{\gamma} - 1\Big), \quad a.a.\ (t,x) \in (0, t_0) \times B_{2\rho}, \tag{6.27}$$

where $\gamma \in (0, \gamma_0]$ is such that

$$\varepsilon \ge \varepsilon_\gamma := \int_1^\infty [-\dot{g}_{1-\alpha}(\sigma - \frac{\theta_2}{2^{2/\alpha}})](\psi_\gamma(\sigma) - 1)\, d\sigma. \tag{6.28}$$

Then

$$\int_{\tilde{t}}^{t_1} \int_{B_{\frac{3\rho}{2}}} v(t,x)^2\, dx\, dt \le C\, \rho^{N + \frac{2}{\alpha}}.$$

where $C = C(\Lambda, \nu, \alpha, r, q, N, \tau_0, \rho_0, \xi, \beta, \theta, \theta_2, \theta_3, M_0, |u_0|_{L_\infty(\Omega)}, |f|_{L_r([0,T];L_q(\Omega))})$, in particular C does not depend on ρ, ε, and γ.

Proof. Let $t \in (t_*, t_1)$ and shift the time by setting $s = t - t_*$. Letting $u_{\bar{m}} = u - \bar{m}$ and using the same notation as in (6.3) with $T_0 = t_*$, we have for all $\psi \in \mathring{H}^1_2(\Omega)$,

$$\int_\Omega \Big(\psi \partial_s (g_{1-\alpha,n} * \tilde{u}_{\bar{m}}) + (h_n * [a_{ij} D_j u])\tilde{}\, D_i \psi\Big)\, dx = \int_\Omega \Big((h_n * f)\tilde{} + \tilde{g}_{1-\alpha,n}(u_0 - \bar{m}) + \tilde{\mathcal{H}}_{\bar{m},n}\Big)\psi\, dx, \quad \text{a.a. } s \in (0, t_1 - t_*), \tag{6.29}$$

where

$$\mathcal{H}_{\bar{m},n}(t,x) = \int_0^{t_*} [-\dot{g}_{1-\alpha,n}(t - \tau)](u(\tau, x) - \bar{m})\, d\tau, \quad t \in (t_*, t_1),\ x \in \Omega.$$

Define

$$\Phi(y) = -\log\Big(\frac{(1+\varepsilon)(M - \bar{m}) - y + \rho^\kappa}{(1-\xi)(M - \bar{m})}\Big), \quad y \in [m - \bar{m}, M - \bar{m}]. \tag{6.30}$$

Then Φ is bounded and

$$\Phi(y) \ge -\log\Big(\frac{4}{1-\xi}\Big) =: -C_\infty, \tag{6.31}$$

since (6.9) with $k = \bar{m}$ implies

$$\rho^\kappa < Z_{\bar{m}} \le M - \bar{m}.$$

We further have

$$\Phi'(y) = \frac{1}{(1+\varepsilon)(M-\bar{m}) - y + \rho^\kappa}$$

and

$$\Phi''(y) = \frac{1}{((1+\varepsilon)(M-\bar{m}) - y + \rho^\kappa)^2} = \Phi'(y)^2 > 0,$$

for all $y \in [m - \bar{m}, M - \bar{m}]$. In particular, Φ is a convex function.

For $s \in (0, t_1 - t_*)$ we choose in (6.29) the test function $\psi = \Phi'(\tilde{u}_{\bar{m}}) w^2$ with $w \in C_0^1(B_{2\rho})$ such that $\operatorname{supp} w \subset B_{2\rho}$, $w = 1$ in $B_{3\rho/2}$, $0 \le w \le 1$, $|Dw| \le 4/\rho$, and the domains $\{x \in B_{2\rho} : w(x)^2 \ge c\}$ are convex for all $c \le 1$.

By the fundamental identity (2.6) and since $\tilde{v} = \Phi(\tilde{u}_{\bar{m}})$, we have (with the spatial variable x being suppressed)

$$\begin{aligned}
&\Phi'(\tilde{u}_{\bar{m}}(s))\partial_s(g_{1-\alpha,n} * \tilde{u}_{\bar{m}})(s) \\
&= \partial_s\Big(g_{1-\alpha,n} * \Phi(\tilde{u}_{\bar{m}})\Big)(s) + \Big(-\Phi(\tilde{u}_{\bar{m}}(s)) + \tilde{u}_{\bar{m}}(s)\Phi'(\tilde{u}_{\bar{m}}(s))\Big) g_{1-\alpha,n}(s) \\
&\quad + \int_0^s \Big(\Phi(\tilde{u}_{\bar{m}}(s-\sigma)) - \Phi(\tilde{u}_{\bar{m}}(s)) - \Phi'(\tilde{u}_{\bar{m}}(s))[\tilde{u}_{\bar{m}}(s-\sigma) - \tilde{u}_{\bar{m}}(s)]\Big)[-\dot{g}_{1-\alpha,n}(\sigma)]\,d\sigma \\
&\ge \partial_s\Big(g_{1-\alpha,n} * \tilde{v}\Big)(s) + \Big(-\tilde{v}(s) - 1 + (M-\bar{m})\Phi'(\tilde{u}_{\bar{m}}(s))\Big) g_{1-\alpha,n}(s) \\
&\quad + \int_0^s \Upsilon\big(\tilde{v}(s) - \tilde{v}(s-\sigma)\big)[-\dot{g}_{1-\alpha,n}(\sigma)]\,d\sigma, \qquad (6.32)
\end{aligned}$$

where $\Upsilon(y) = e^y - 1 - y$, $y \in \mathbb{R}$.

We next consider the history term on the right of (6.29). We write

$$\begin{aligned}
\mathcal{H}_{\bar{m},n}(t,x) &= \int_0^{t_0} [-\dot{g}_{1-\alpha,n}(t-\tau)](u(\tau,x) - \bar{m})\,d\tau + \int_{t_0}^{t_*} [-\dot{g}_{1-\alpha,n}(t-\tau)](u(\tau,x) - \bar{m})\,d\tau \\
&=: \mathcal{H}^{(1)}_{\bar{m},n}(t,x) + \mathcal{H}^{(2)}_{\bar{m},n}(t,x).
\end{aligned}$$

Since $u \le M$ in $Q(2\rho)$, the short-term memory term can be estimated as follows.

$$\tilde{\mathcal{H}}^{(2)}_{\bar{m},n}(s,x) \le (M-\bar{m})[g_{1-\alpha,n}(s) - g_{1-\alpha,n}(s + t_* - t_0)], \quad \text{a.a. } (s,x) \in (0, t_1 - t_*) \times B_{2\rho}. \qquad (6.33)$$

Thanks to (6.27) we further have

$$\begin{aligned}
\tilde{\mathcal{H}}^{(1)}_{\bar{m},n}(s,x) &\le (M-\bar{m}) \int_0^{t_0} [-\dot{g}_{1-\alpha,n}(s + t_* - \tau)]\Big(\psi_\gamma\big(\frac{t_1 - \tau}{t_1 - t_0}\big) - 1\Big)\,d\tau \\
&\quad + (M-\bar{m}) \int_0^{t_0} [-\dot{g}_{1-\alpha,n}(s + t_* - \tau)]\,d\tau \\
&\le \varepsilon_{\gamma,n}(M-\bar{m})(t_1 - t_0)^{-\alpha} + (M-\bar{m}) g_{1-\alpha,n}(s + t_* - t_0), \qquad (6.34)
\end{aligned}$$

for a.a. $(s,x) \in (0, t_1 - t_*) \times B_{2\rho}$, where

$$\varepsilon_{\gamma,n} = (t_1 - t_0)^{1+\alpha} \int_1^{\frac{t_0}{t_1 - t_0}} [-\dot{g}_{1-\alpha,n}(t_* - t_1 + \sigma(t_1 - t_0))]\big(\psi_\gamma(\sigma) - 1\big)\,d\sigma. \qquad (6.35)$$

By means of integration by parts and the approximation property of the kernels h_n one verifies that as $n \to \infty$,

$$\varepsilon_{\gamma,n} \to \varepsilon_{\gamma,\infty} := \int_1^{\frac{t_0}{t_1-t_0}} [-\dot{g}_{1-\alpha}(\sigma - \frac{\theta_2}{2^{2/\alpha}})](\psi_\gamma(\sigma) - 1)\, d\sigma$$
$$\leq \int_1^{\infty} [-\dot{g}_{1-\alpha}(\sigma - \frac{\theta_2}{2^{2/\alpha}})](\psi_\gamma(\sigma) - 1)\, d\sigma = \varepsilon_\gamma. \tag{6.36}$$

Setting

$$\mathcal{R}_n(s) = \int_{B_{2\rho}} \Big((h_n * f)\tilde{} - \tilde{f} + (u_0 - \bar{m})(\tilde{g}_{1-\alpha,n} - \tilde{g}_{1-\alpha}) \Big) \psi\, dx$$
$$+ \int_{B_{2\rho}} \Big((\varepsilon_{\gamma,n} - \varepsilon_{\gamma,\infty})(M - \bar{m})(t_1 - t_0)^{-\alpha} \psi + \Big[\tilde{a}_{ij} D_j \tilde{u} - (h_n * [a_{ij} D_j u])\tilde{}\, \Big] D_i \psi \Big)\, dx,$$

for $s \in (0, t_1 - t_*)$, it follows from (6.29), (6.32), (6.33), (6.34), and (6.36) that with $\psi = \Phi'(\tilde{u}_{\bar{m}}) w^2$

$$\int_{B_{2\rho}} w^2 \Big(\partial_s (g_{1-\alpha,n} * \tilde{v}) - (\tilde{v} + 1) g_{1-\alpha,n} \Big)\, dx$$
$$+ \int_{B_{2\rho}} w^2 \int_0^s \Upsilon(\tilde{v}(s,x) - \tilde{v}(s-\sigma,x))[-\dot{g}_{1-\alpha,n}(\sigma)]\, d\sigma\, dx + \int_{B_{2\rho}} \tilde{a}_{ij} D_j \tilde{u} D_i \psi\, dx$$
$$\leq \int_{B_{2\rho}} \Big(\tilde{f} + (u_0 - \bar{m}) \tilde{g}_{1-\alpha} + \varepsilon_\gamma (M - \bar{m})(t_1 - t_0)^{-\alpha} \Big) \psi\, dx + \mathcal{R}_n(s), \tag{6.37}$$

for a.a. $s \in (0, t_1 - t_*)$.

By (H1), (H2), and since $D\tilde{v} = \Phi'(\tilde{u}_{\bar{m}}) D\tilde{u}_{\bar{m}}$ and $\Phi''(y) = \Phi'(y)^2$, we have

$$\int_{B_{2\rho}} \tilde{a}_{ij} D_j \tilde{u} D_i \psi\, dx = \int_{B_{2\rho}} \tilde{a}_{ij} D_j \tilde{u} \Big(w^2 \Phi''(\tilde{u}_{\bar{m}}) D_i \tilde{u}_{\bar{m}} + 2w \Phi'(\tilde{u}_{\bar{m}}) D_i w \Big)\, dx$$
$$= \int_{B_{2\rho}} w^2 \tilde{a}_{ij} D_j \tilde{v} D_i \tilde{v}\, dx + 2 \int_{B_{2\rho}} w \tilde{a}_{ij} D_j \tilde{v} D_i w\, dx$$
$$\geq \frac{\nu}{2} \int_{B_{2\rho}} w^2 |D\tilde{v}|^2\, dx - \frac{2\Lambda^2}{\nu} \int_{B_{2\rho}} |Dw|^2\, dx$$
$$\geq \frac{\nu}{2} \int_{B_{2\rho}} w^2 |D\tilde{v}|^2\, dx - C(\nu, \Lambda, N) \rho^{N-2}, \quad \text{a.a. } s \in (0, t_1 - t_*). \tag{6.38}$$

Since u satisfies (6.23), we may apply Proposition 2.6.2 with weight w^2 to $\tilde{v}_+$ in $B_{2\rho}$ pointwise for a.a. $s \in (0, t_1 - t_*)$. In view of $\tilde{v}^2 \leq (\tilde{v}_+)^2 + C_\infty^2$ (c.p. (6.31)), and $|D\tilde{v}_+|^2 \leq |D\tilde{v}|^2$ we obtain

$$\int_{B_{2\rho}} w^2\, \tilde{v}^2\, dx \leq C_\infty^2 \int_{B_{2\rho}} w^2\, dx + \int_{B_{2\rho}} w^2\, [\tilde{v}_+]^2\, dx$$
$$\leq C_\infty^2 \lambda_N(B_{2\rho}) + C(N) \frac{(4\rho)^{2N}}{([1-\beta]\lambda_N(B_\rho))^{\frac{2(N-1)}{N}}} \int_{B_{2\rho}} w^2\, |D\tilde{v}_+|^2\, dx$$
$$\leq C_\infty^2 \lambda_N(B_{2\rho}) + C(N, \beta) \rho^2 \int_{B_{2\rho}} w^2\, |D\tilde{v}|^2\, dx, \quad \text{a.a. } s \in (0, t_1 - t_*). \tag{6.39}$$

Turning to the first term on the right of (6.37), we may estimate for a.a. $s \in (0, t_1 - t_*)$,

$$\int_{B_{2\rho}} \tilde{f} \psi\, dx \leq \frac{1}{\rho^\kappa} |\tilde{f}(s,\cdot)|_{L_q(\Omega)} \lambda_N(B_{2\rho})^{\frac{1}{q'}} = C(N,q) |\tilde{f}(s,\cdot)|_{L_q(\Omega)} \rho^{2\kappa + \frac{2}{\alpha r} + N - 2 - \kappa}$$
$$\leq C(N, q, \rho_0) |\tilde{f}(s,\cdot)|_{L_q(\Omega)} \rho^{\frac{2}{\alpha r} + N - 2}. \tag{6.40}$$

Further, since $g_{1-\alpha}(s+t_*) \le g_{1-\alpha}(\tau_0)$,

$$\int_{B_{2\rho}} (u_0-\bar m)\tilde g_{1-\alpha}\psi\,dx \le \frac{1}{\rho^\kappa}\,(|u_0|_{L_\infty(\Omega)}+M_0)\lambda_N(B_{2\rho})g_{1-\alpha}(s+t_*)$$
$$\le C(|u_0|_{L_\infty(\Omega)},M_0,N,\alpha,\tau_0,\rho_0)\rho^{N-2}. \tag{6.41}$$

Note that $\psi \le [\varepsilon(M-\bar m)]^{-1}$. By assumption (6.28) we also have $\varepsilon_\gamma \le \varepsilon$. Therefore

$$\int_{B_{2\rho}} \varepsilon_\gamma (M-\bar m)(t_1-t_0)^{-\alpha}\psi\,dx \le (t_1-t_0)^{-\alpha}\lambda_N(B_{2\rho}) = C(\alpha,\theta,N)\rho^{N-2}. \tag{6.42}$$

From (6.37)–(6.42) we infer that

$$\int_{B_{2\rho}} w^2\Big(\partial_s(g_{1-\alpha,n}*\tilde v)-(\tilde v+1)g_{1-\alpha,n}\Big)\,dx$$
$$+\int_{B_{2\rho}} w^2\int_0^s \Upsilon(\tilde v(s,x)-\tilde v(s-\sigma,x))[-\dot g_{1-\alpha,n}(\sigma)]\,d\sigma\,dx+\frac{\nu_0}{\rho^2}\int_{B_{2\rho}} w^2\tilde v^2\,dx$$
$$\le C\rho^{N-2}\Big(1+|\tilde f(s,\cdot)|_{L_q(\Omega)}\rho^{\frac{2}{\alpha r}}\Big)+\mathcal{R}_n(s),\quad \text{a.a. } s\in(0,t_1-t_*), \tag{6.43}$$

where $\nu_0=\nu_0(\nu,N,\beta)$ and $C=C(|u_0|_{L_\infty(\Omega)},M_0,N,\alpha,\tau_0,\rho_0,\nu,\Lambda,q,\theta,\beta,C_\infty)$ are positive constants.

We next introduce the function

$$V(s)=\frac{\int_{B_{2\rho}} w(x)^2\,\tilde v(s,x)\,dx}{\int_{B_{2\rho}} w(x)^2\,dx},\quad s\in(0,t_1-t_*).$$

By Jensen's inequality,

$$V(s)^2 \le \frac{\int_{B_{2\rho}} w(x)^2\,\tilde v(s,x)^2\,dx}{\int_{B_{2\rho}} w(x)^2\,dx},\quad \text{a.a. } s\in(0,t_1-t_*). \tag{6.44}$$

Since Υ is convex, Jensen's inequality also yields

$$\int_0^s \Upsilon\Big(V(s)-V(s-\sigma)\Big)[-\dot g_{1-\alpha,n}(\sigma)]\,d\sigma$$
$$\le \int_0^s \frac{1}{\int_{B_{2\rho}} w(x)^2\,dx}\int_{B_{2\rho}} w(x)^2\Upsilon(\tilde v(s,x)-\tilde v(s-\sigma,x))[-\dot g_{1-\alpha,n}(\sigma)]\,d\sigma\,dx, \tag{6.45}$$

a.e. in $(0,t_1-t_*)$. By the fundamental identity (2.6) with $H(y)=-\log y$ applied to the function e^{-V}, we further have

$$-e^{V(s)}\,\partial_s\Big(g_{1-\alpha,n}*e^{-V}\Big)(s)=\partial_s(g_{1-\alpha,n}*V)(s)-\Big(V(s)+1\Big)g_{1-\alpha,n}(s)$$
$$+\int_0^s \Upsilon\Big(V(s)-V(s-\sigma)\Big)[-\dot g_{1-\alpha,n}(\sigma)]\,d\sigma,\quad \text{a.a. } s\in(0,t_1-t_*). \tag{6.46}$$

Dividing then (6.43) by $\int_{B_{2\rho}} w^2\,dx$, and setting

$$F(s)=1+|\tilde f(s,\cdot)|_{L_q(\Omega)}\rho^{\frac{2}{\alpha r}},\quad \bar{\mathcal{R}}_n(s)=\frac{|\mathcal{R}_n(s)|}{\int_{B_{2\rho}} w^2\,dx}$$

it follows by virtue of (6.44)–(6.46) that

$$-e^{V(s)}\,\partial_s\Big(g_{1-\alpha,n}{*}e^{-V}\Big)(s)+\frac{\nu_1}{\rho^2}\,V(s)^2+\frac{\nu_1}{\rho^{N+2}}\int_{B_{2\rho}}w(x)^2\,\tilde v(s,x)^2\,dx$$
$$\le\ \frac{C_1}{\rho^2}\,F(s)+\bar{\mathcal{R}}_n(s),\quad \text{a.a. } s\in(0,t_1-t_*),\tag{6.47}$$

with $\nu_1=\nu_1(\nu,N,\beta)>0$ and $C_1=C_1(|u_0|_{L_\infty(\Omega)},M_0,N,\alpha,\tau_0,\rho_0,\nu,\Lambda,q,\theta,\beta,C_\infty)>0$.

We next put

$$W(s)=V(s)+C_\infty+1,\quad s\in(0,t_1-t_*).$$

Since $\tilde v\ge -C_\infty$ (see (6.31)), we have $W\ge 1$. Furthermore, (6.47) implies

$$-e^{W(s)}\,\partial_s\Big(g_{1-\alpha,n}{*}e^{-W}\Big)(s)+\frac{\nu_1}{2\rho^2}\,W(s)^2+\frac{\nu_1}{\rho^{N+2}}\int_{B_{2\rho}}w(x)^2\,\tilde v(s,x)^2\,dx$$
$$\le\ \frac{C_2}{\rho^2}\,F(s)+\bar{\mathcal{R}}_n(s),\quad \text{a.a. } s\in(0,t_1-t_*),\tag{6.48}$$

where $C_2=C_1+(C_\infty+1)^2\nu_1$. We then fix a negative number ζ such that

$$-1<\zeta<\frac{1}{\alpha}-2.\tag{6.49}$$

Multiplying (6.48) by $W^\zeta(\le 1)$ and dropping the third term on the left-hand side gives

$$-W^\zeta e^W\partial_s\Big(g_{1-\alpha,n}*e^{-W}\Big)+\frac{\nu_1}{2\rho^2}\,W^{2+\zeta}\le\frac{C_2}{\rho^2}\,F+\bar{\mathcal{R}}_n.\tag{6.50}$$

By the fundamental identity (2.6) applied to the function $H(y)=(-\log y)^{1+\zeta}/(1+\zeta)$, where $y\in(0,1/e]$, we have

$$-W^\zeta e^W\partial_s\Big(g_{1-\alpha,n}*e^{-W}\Big)\ge\partial_s\Big(g_{1-\alpha,n}*\frac{W^{1+\zeta}}{1+\zeta}\Big)-\Big(\frac{W^{1+\zeta}}{1+\zeta}+W^\zeta\Big)g_{1-\alpha,n}.$$

In fact, H is convex since

$$H''(y)=\frac{1}{y^2}\,(-\log y)^{\zeta-1}\big(\zeta-\log y\big)\ge 0,\quad y\in(0,1/e].$$

From (6.50) and $W^\zeta\le 1$ we thus conclude that for a.a. $s\in(0,t_1-t_*)$,

$$\partial_s\Big(g_{1-\alpha,n}*\frac{W^{1+\zeta}}{1+\zeta}\Big)+\frac{\nu_1}{2\rho^2}\,W^{2+\zeta}\le\Big(\frac{W^{1+\zeta}}{1+\zeta}+1\Big)g_{1-\alpha,n}+\frac{C_2}{\rho^2}\,F+\bar{\mathcal{R}}_n.$$

Integrating from 0 to t_1-t_* and sending $n\to\infty$ yields

$$\frac{\nu_1}{2\rho^2}\int_0^{t_1-t_*}W^{2+\zeta}\,ds\ \le\int_0^{t_1-t_*}\Big(\frac{W^{1+\zeta}}{1+\zeta}\ +1\Big)g_{1-\alpha}\,ds+\ \frac{C_2}{\rho^2}\,|F|_{L_1([0,t_1-t_*])}$$
$$\le\int_0^{t_1-t_*}\Big(\tilde\varepsilon\,W^{2+\zeta}+\tilde\varepsilon^{-(1+\zeta)}C(\zeta)g_{1-\alpha}^{2+\zeta}\Big)\,ds+|g_{1-\alpha}|_{L_1([0,t_1-t_*])}+\ \frac{C_2}{\rho^2}\,|F|_{L_1([0,t_1-t_*])}$$

for all $\tilde\varepsilon>0$. By Hölder's inequality,

$$\frac{1}{\rho^2}\,|F|_{L_1([0,t_1-t_*])}\le\theta\theta_2\rho^{\frac{2}{\alpha}-2}+\rho^{\frac{2}{\alpha r}-2}|\tilde f|_{L_r([0,t_1-t_*];L_q(\Omega))}(\theta\theta_2)^{\frac{1}{r'}}\rho^{\frac{2}{\alpha r'}}.$$

Choosing $\tilde{\varepsilon} = \nu_1/(4\rho^2)$ it then follows that

$$\int_0^{t_1-t_*} W\,ds \le \int_0^{t_1-t_*} W^{2+\zeta}\,ds \le C_3\rho^{2/\alpha}\Big(1+|f|_{L_r([0,T];L_q(\Omega))}\Big), \tag{6.51}$$

where $C_3 = C_3(\nu_1, \theta, \theta_2, \alpha, r, \zeta, C_2)$. This L_1-estimate for the function W on $(0, t_1 - t_*)$ is crucial for what follows.

We return to (6.48) and use the identity (6.46) (with V replaced by W) to reformulate the first term. Note that the last term in (6.46) is nonnegative, due to the convexity of $H(y) = -\log y$. Therefore

$$\begin{aligned}\partial_s(g_{1-\alpha,n} * W)(s) - \Big(W(s)+1\Big)g_{1-\alpha,n}(s) + \frac{\nu_1}{\rho^{N+2}}\int_{B_{2\rho}} w(x)^2\,\tilde{v}(s,x)^2\,dx \\ \le \frac{C_2}{\rho^2}F(s) + \bar{\mathcal{R}}_n(s), \quad \text{a.a. } s \in (0, t_1 - t_*).\end{aligned}$$

We choose $\varphi \in C^1([0, t_1 - t_*])$ such that $0 \le \varphi \le 1$, $\varphi = 0$ in $[0, (\tilde{t} - t_*)/2]$, $\varphi = 1$ in $[\tilde{t} - t_*, t_1 - t_*]$, and $0 \le \dot{\varphi} \le 4/(\tilde{t} - t_*)$. Multiplying the last inequality by $\varphi(s)$ and using Lemma 2.3.3 yields

$$\begin{aligned}\partial_s(g_{1-\alpha,n} * [\varphi W])(s) - \Big(W(s)+1\Big)\varphi(s)g_{1-\alpha,n}(s) + \frac{\nu_1}{\rho^{N+2}}\int_{B_{2\rho}} \varphi(s)w(x)^2\,\tilde{v}(s,x)^2\,dx \\ \le \int_0^s [-\dot{g}_{1-\alpha,n}(s-\sigma)]\big(\varphi(s) - \varphi(\sigma)\big)W(\sigma)\,d\sigma + \frac{C_2}{\rho^2}F(s) + \bar{\mathcal{R}}_n(s) \\ \le \frac{4}{\tilde{t} - t_*}\int_0^s [-\dot{g}_{1-\alpha,n}(s-\sigma)](s-\sigma)W(\sigma)\,d\sigma + \frac{C_2}{\rho^2}F(s) + \bar{\mathcal{R}}_n(s).\end{aligned}$$

We next integrate from 0 to $t_1 - t_*$ and send $n \to \infty$. Selecting an appropriate subsequence, if necessary, and employing (6.51) we obtain

$$\begin{aligned}\frac{\nu_1}{\rho^{N+2}}\int_0^{t_1-t_*}\int_{B_{2\rho}} \varphi(s)w(x)^2\,\tilde{v}(s,x)^2\,dx\,ds \le \int_0^{t_1-t_*} \big(W(s)+1\big)\varphi(s)g_{1-\alpha}(s)\,ds \\ + \frac{4}{\tilde{t} - t_*}\int_0^{t_1-t_*}\int_0^s [-\dot{g}_{1-\alpha}(s-\sigma)](s-\sigma)W(\sigma)\,d\sigma\,ds + \frac{C_2}{\rho^2}|F|_{L_1([0,t_1-t_*])} \\ \le g_{1-\alpha}\big(\frac{\tilde{t} - t_*}{2}\big)|W+1|_{L_1([0,t_1-t_*])} + \frac{4\alpha}{\tilde{t} - t_*}\big(g_{2-\alpha} * W\big)(t_1 - t_*) + \frac{C_2}{\rho^2}|F|_{L_1([0,t_1-t_*])} \\ \le g_{1-\alpha}\big(\frac{\tilde{t} - t_*}{2}\big)\big(t_1 - t_* + |W|_{L_1([0,t_1-t_*])}\big) + \frac{4\alpha g_{2-\alpha}(t_1 - t_*)}{\tilde{t} - t_*}|W|_{L_1([0,t_1-t_*])} \\ + \frac{C_2}{\rho^2}|F|_{L_1([0,t_1-t_*])} \\ \le C_4(\alpha, \theta, \theta_2, \theta_3, C_3)\,\rho^{\frac{2}{\alpha}-2}\Big(1+|f|_{L_r([0,T];L_q(\Omega))}\Big),\end{aligned}$$

which in turn implies

$$\int_{\tilde{t}}^{t_1}\int_{B_{\frac{3\rho}{2}}} v(t,x)^2\,dx\,dt \le C_5(\nu_1, C_4)\,\rho^{N+\frac{2}{\alpha}}\Big(1+|f|_{L_r([0,T];L_q(\Omega))}\Big).$$

This completes the proof of Proposition 6.3.1. □

Remark 6.3.1 Note that the assertion of Proposition 6.3.1 remains true when equation (6.1) has a second term on the right-hand side of the form $-D_i g^i$, where $|g|^2 \in L_r([0,T];L_q(\Omega))$. In fact, recall Remark 6.2.1 and note that for a.a. $s \in (0, t_1 - t_*)$ and any $\epsilon_0 > 0$ we have

$$\int_{B_{2\rho}} \tilde{g}^i D_i \psi \, dx = \int_{B_{2\rho}} \tilde{g}^i \big(w^2 \Phi'(\tilde{u}_{\bar{m}}) D_i \tilde{v} + 2w \Phi'(\tilde{u}_{\bar{m}}) D_i w\big)\, dx$$
$$\leq \int_{B_{2\rho}} \big(\epsilon_0 w^2 |D\tilde{v}|^2 + C(\epsilon_0)|\tilde{g}|^2 \Phi'(\tilde{u}_{\bar{m}})^2 + |Dw|^2\big)\, dx.$$

The first term can be absorbed for sufficiently small ϵ_0, while the second term is estimated similarly to the $\tilde{f}$-term above:

$$\int_{B_{2\rho}} \tilde{g}^i D_i \psi \, dx \leq C(N,q) \big| |\tilde{g}(s,\cdot)|^2 \big|_{L_q(\Omega)} \rho^{\frac{2}{\alpha r} + N - 2}.$$

Observe that this is possible, since in the line before (6.40) we have an extra factor ρ^κ at disposal which is just good enough to control the additional factor $\Phi'(\tilde{u}_{\bar{m}})$.

6.4 A sup-bound for v on the cylinder $Q(\rho)$

Proposition 6.4.1 *Let the assumptions of Proposition 6.3.1 be satisfied. Then*

$$\operatorname*{ess\,sup}_{Q(\rho)} v \leq M_1, \tag{6.52}$$

where $0 \leq M_1 = M_1(\Lambda, \nu, \alpha, r, q, N, \tau_0, \rho_0, \xi, \beta, \theta, \theta_2, \theta_3, M_0, |u_0|_{L_\infty(\Omega)}, |f|_{L_r([0,T];L_q(\Omega))})$, *in particular* M_1 *does not depend on* ρ, ε, *and* γ.

Proof. **1. Local truncated energy estimates.** We introduce the family of nested cylinders

$$Q(\vartheta, \rho) = (t_1 - [1 + \vartheta(\theta_3 - 1)]\theta \rho^{2/\alpha}, t_1) \times B_{\rho(1+\vartheta/2)}(x_1)$$

with parameter $\vartheta \in [0,1]$. Note that $Q(0,\rho) = Q(\rho)$ and $Q(1,\rho) = (\tilde{t}, t_1) \times B_{3\rho/2}$.

Let $0 \leq \vartheta_1 < \vartheta_2 \leq 1$ be fixed. We set

$$t' = t_1 - [1 + \vartheta_2(\theta_3 - 1)]\theta\rho^{2/\alpha}, \quad \hat{t} = t_1 - [1 + \vartheta_1(\theta_3 - 1)]\theta\rho^{2/\alpha}.$$

Let $t \in (t', t_1)$ and shift the time by setting $s = t - t'$. Employing the same notation as in (6.3) with $T_0 = t'$, we have for all $\psi \in \mathring{H}^1_2(\Omega)$,

$$\int_\Omega \Big(\psi \partial_s (g_{1-\alpha,n} * \tilde{u}_{\bar{m}}) + (h_n * [a_{ij} D_j u])\tilde{\ } D_i \psi\Big)\, dx =$$
$$\int_\Omega \Big((h_n * f)\tilde{\ } + \tilde{g}_{1-\alpha,n}(u_0 - \bar{m}) + \tilde{\mathcal{H}}_{\bar{m},n}\Big)\psi \, dx, \quad \text{a.a. } s \in (0, t_1 - t'), \tag{6.53}$$

where

$$\mathcal{H}_{\bar{m},n}(t,x) = \int_0^{t'} [-\dot{g}_{1-\alpha,n}(t-\tau)](u(\tau,x) - \bar{m})\, d\tau, \;\; t \in (\tilde{t}, t_1),\, x \in \Omega.$$

Define the function Ψ by means of

$$\Psi(y) = \frac{1}{2}\,(\Phi(y) - k)_+^2, \quad y \in [m - \bar{m}, M - \bar{m}],$$

with Φ as in (6.30) and $k > 0$. Then Ψ is a C^1 function,

$$\Psi'(y) = \Phi'(y)(\Phi(y) - k)_+ = \frac{(\Phi(y) - k)_+}{(1 + \varepsilon)(M - \bar{m}) - y + \rho^\kappa},$$

and Ψ is convex, by virtue of $\Phi', \Phi'' > 0$.

For $s \in (0, t_1 - t')$ we choose in (6.53) the test function $\psi = \Psi'(\tilde{u}_{\bar{m}})\eta^2$ where $\eta \in C_0^1(B_{2\rho})$ has values only in $[0, 1]$, $\operatorname{supp}\eta \subset B_{\rho(1+\vartheta_2/2)}$, $\eta = 1$ in $B_{\rho(1+\vartheta_1/2)}$, and $|D\eta| \leq 4/(\rho[\vartheta_2 - \vartheta_1])$. Since

$$\begin{aligned} -\Psi(y) + \Psi'(y)y &= -\frac{1}{2}(\Phi(y) - k)_+^2 + (\Phi(y) - k)_+\Phi'(y)y \\ &\geq -\frac{1}{2}(\Phi(y) - k)_+^2 + \Big([M - \bar{m}]\Phi'(y) - 1\Big)(\Phi(y) - k)_+ \end{aligned}$$

and $\tilde{v} = \Phi(\tilde{u}_{\bar{m}})$, the fundamental identity (2.6) yields

$$\begin{aligned} &\Psi'(\tilde{u}_{\bar{m}})\partial_s(g_{1-\alpha,n} * \tilde{u}_{\bar{m}}) \geq \partial_s\Big(g_{1-\alpha,n} * \Psi(\tilde{u}_{\bar{m}})\Big) + \Big(-\Psi(\tilde{u}_{\bar{m}}) + \tilde{u}_{\bar{m}}\Psi'(\tilde{u}_{\bar{m}})\Big)g_{1-\alpha,n} \\ &\geq \frac{1}{2}\partial_s\Big(g_{1-\alpha,n} * (\tilde{v} - k)_+^2\Big) + \Big(-\frac{1}{2}(\tilde{v} - k)_+^2 + \big([M - \bar{m}]\Phi'(\tilde{u}_{\bar{m}}) - 1\big)(\tilde{v} - k)_+\Big)g_{1-\alpha,n}. \quad (6.54) \end{aligned}$$

We split again the history term on the right of (6.53) into a long- and a short-term memory term:

$$\begin{aligned} \mathcal{H}_{\bar{m},n}(t, x) &= \int_0^{t_0} [-\dot{g}_{1-\alpha,n}(t - \tau)](u(\tau, x) - \bar{m})\,d\tau + \int_{t_0}^{t'} [-\dot{g}_{1-\alpha,n}(t - \tau)](u(\tau, x) - \bar{m})\,d\tau \\ &=: \mathcal{H}^{(1)}_{\bar{m},n}(t, x) + \mathcal{H}^{(2)}_{\bar{m},n}(t, x). \end{aligned}$$

Similarly as in the proof of Proposition 6.3.1 (c.p. (6.33) and (6.34)) we obtain

$$\begin{aligned} \tilde{\mathcal{H}}^{(2)}_{\bar{m},n}(s, x) &\leq (M - \bar{m})[g_{1-\alpha,n}(s) - g_{1-\alpha,n}(s + t' - t_0)], \\ \tilde{\mathcal{H}}^{(1)}_{\bar{m},n}(s, x) &\leq \varepsilon_{\gamma,n}(M - \bar{m})(t_1 - t_0)^{-\alpha} + (M - \bar{m})g_{1-\alpha,n}(s + t' - t_0), \end{aligned}$$

for a.a. $(s, x) \in (0, t_1 - t') \times B_{2\rho}$, and hence

$$\tilde{\mathcal{H}}_{\bar{m},n}(s, x) \leq (M - \bar{m})g_{1-\alpha,n}(s) + \varepsilon_{\gamma,n}(M - \bar{m})(t_1 - t_0)^{-\alpha}, \quad (6.55)$$

for a.a. $(s, x) \in (0, t_1 - t') \times B_{2\rho}$, where $\varepsilon_{\gamma,n}$ is defined as in (6.35).

Inserting $\psi = \Psi'(\tilde{u}_{\bar{m}})\eta^2$ into (6.53) and using (6.54) and (6.55) we obtain with $\rho_2 := \rho(1 + \vartheta_2/2)$

$$\begin{aligned} &\frac{1}{2}\int_{B_{\rho_2}} \eta^2\,\partial_s\Big(g_{1-\alpha,n} * (\tilde{v} - k)_+^2\Big)\,dx - \frac{1}{2}\int_{B_{\rho_2}} \eta^2(\tilde{v} - k)_+^2 g_{1-\alpha,n}\,dx \\ &\quad + \int_{B_{\rho_2}} \eta^2(M - \bar{m})\Phi'(\tilde{u}_{\bar{m}})(\tilde{v} - k)_+ g_{1-\alpha,n}\,dx - \int_{B_{\rho_2}} \eta^2(\tilde{v} - k)_+ g_{1-\alpha,n}\,dx \\ &\quad + \int_{B_{\rho_2}} (h_n * [a_{ij}D_j u])\tilde{}\,D_i\big[\Psi'(\tilde{u}_{\bar{m}})\eta^2\big]\,dx \\ &\leq \int_{B_{\rho_2}} \Big((h_n * f)\tilde{} + \tilde{g}_{1-\alpha,n}(u_0 - \bar{m}) + \tilde{\mathcal{H}}_{\bar{m},n}\Big)\Phi'(\tilde{u}_{\bar{m}})(\tilde{v} - k)_+\eta^2\,dx \\ &\leq \int_{B_{\rho_2}} \Big((h_n * f)\tilde{} + \tilde{g}_{1-\alpha,n}(u_0 - \bar{m})\Big)\Phi'(\tilde{u}_{\bar{m}})(\tilde{v} - k)_+\eta^2\,dx \\ &\quad + \int_{B_{\rho_2}} \Big((M - \bar{m})g_{1-\alpha,n} + \varepsilon_{\gamma,n}(M - \bar{m})(t_1 - t_0)^{-\alpha}\Big)\Phi'(\tilde{u}_{\bar{m}})(\tilde{v} - k)_+\eta^2\,dx, \quad (6.56) \end{aligned}$$

for a.a. $s \in (0, t_1 - t')$.

Next, choose $\varphi \in C^1([0, t_1 - t'])$ such that $0 \le \varphi \le 1$, $\varphi = 0$ in $[0, (\hat{t} - t')/2]$, $\varphi = 1$ in $[\hat{t} - t', t_1 - t']$, and $0 \le \dot{\varphi} \le 4/(\hat{t} - t')$. We multiply (6.56) by φ^2 and apply Lemma 2.3.3. Estimating the commutator term similarly as above we get

$$\begin{aligned}
&\frac{1}{2} \int_{B_{\rho_2}} \eta^2 \, \partial_s \Big(g_{1-\alpha,n} * \big[\varphi(\tilde{v} - k)_+\big]^2 \Big) \, dx + \int_{B_{\rho_2}} (h_n * [a_{ij} D_j u])\tilde{\ } D_i \big[\Psi'(\tilde{u}_{\bar{m}})\eta^2\big] \varphi^2 \, dx \\
\le &\int_{B_{\rho_2}} \varphi^2 \eta^2 \Big((\tilde{v} - k)_+ + \frac{1}{2} (\tilde{v} - k)_+^2 \Big) g_{1-\alpha,n} \, dx \\
&+ \int_{B_{\rho_2}} \Big((h_n * f)\tilde{\ } + \tilde{g}_{1-\alpha,n}(u_0 - \bar{m}) + \varepsilon_{\gamma,n}(M - \bar{m})(t_1 - t_0)^{-\alpha} \Big) \Phi'(\tilde{u}_{\bar{m}})(\tilde{v} - k)_+ \eta^2 \varphi^2 \, dx \\
&+ \frac{4}{\hat{t} - t'} \int_0^s [-\dot{g}_{1-\alpha,n}(s - \sigma)](s - \sigma) \int_{B_{\rho_2}} \eta(x)^2 \big(\tilde{v}(\sigma, x) - k\big)_+^2 \, dx \, d\sigma.
\end{aligned}$$

Convolve this last inequality with g_α and send $n \to \infty$. Selecting an appropriate subsequence, if necessary, this gives

$$\frac{1}{2} \int_{B_{\rho_2}} \varphi^2 \eta^2 (\tilde{v} - k)_+^2 \, dx + g_\alpha * \int_{B_{\rho_2}} \tilde{a}_{ij} D_j \tilde{u} \, D_i \big[\Psi'(\tilde{u}_{\bar{m}})\eta^2\big] \varphi^2 \, dx \le g_\alpha * G_1, \tag{6.57}$$

for a.a. $s \in (0, t_1 - t')$, where

$$\begin{aligned}
G_1(s) = &\int_{B_{\rho_2}} \varphi^2 \eta^2 \Big((\tilde{v} - k)_+ + \frac{1}{2} (\tilde{v} - k)_+^2 \Big) g_{1-\alpha} \, dx + \frac{4\alpha}{\hat{t} - t'} \, g_{1-\alpha} * \int_{B_{\rho_2}} \eta^2 (\tilde{v} - k)_+^2 \, dx \\
&+ \int_{B_{\rho_2}} \Big(|\tilde{f}| + \tilde{g}_{1-\alpha}|u_0 - \bar{m}| + \varepsilon_\gamma (M - \bar{m})(t_1 - t_0)^{-\alpha} \Big) \Phi'(\tilde{u}_{\bar{m}})(\tilde{v} - k)_+ \eta^2 \varphi^2 \, dx.
\end{aligned}$$

Since $\tilde{v} = \Phi(\tilde{u}_{\bar{m}})$ and $\Phi''(y) = \Phi'(y)^2$, we further have

$$\begin{aligned}
D\big[\Psi'(\tilde{u}_{\bar{m}})\eta^2\big] &= \eta^2 \Phi'(\tilde{u}_{\bar{m}}) D(\tilde{v} - k)_+ + \eta^2 (\tilde{v} - k)_+ \Phi''(\tilde{u}_{\bar{m}}) D\tilde{u}_{\bar{m}} + 2\eta D\eta \, \Phi'(\tilde{u}_{\bar{m}})(\tilde{v} - k)_+ \\
&= \eta^2 \Phi'(\tilde{u}_{\bar{m}}) D(\tilde{v} - k)_+ + \eta^2 \Phi'(\tilde{u}_{\bar{m}})(\tilde{v} - k)_+ D\tilde{v} + 2\eta D\eta \, \Phi'(\tilde{u}_{\bar{m}})(\tilde{v} - k)_+,
\end{aligned}$$

and therefore by (H1) and (H2)

$$\begin{aligned}
\int_{B_{\rho_2}} \tilde{a}_{ij} D_j \tilde{u} \, D_i \big[\Psi'(\tilde{u}_{\bar{m}})\eta^2\big] \varphi^2 \, dx &= \int_{B_{\rho_2}} \varphi^2 \eta^2 \tilde{a}_{ij} D_j \tilde{v} \big(D_i (\tilde{v} - k)_+ + (\tilde{v} - k)_+ D_i \tilde{v} \big) \, dx \\
&\quad + \int_{B_{\rho_2}} \varphi^2 \tilde{a}_{ij} (\tilde{v} - k)_+ D_j \tilde{v} \, 2\eta D_i \eta \, dx \\
&\ge \nu \int_{B_{\rho_2}} \varphi^2 \eta^2 \Big(|D(\tilde{v} - k)_+|^2 + (\tilde{v} - k)_+ |D\tilde{v}|^2 \Big) \, dx \\
&\quad - 2\Lambda \int_{B_{\rho_2}} \varphi^2 (\tilde{v} - k)_+ |D(\tilde{v} - k)_+| \, |D\eta| \eta \, dx \\
&\ge \frac{\nu}{2} \int_{B_{\rho_2}} \varphi^2 \eta^2 |D(\tilde{v} - k)_+|^2 \, dx - \frac{2\Lambda^2}{\nu} \int_{B_{\rho_2}} \varphi^2 |D\eta|^2 (\tilde{v} - k)_+^2 \, dx.
\end{aligned}$$

Combining this and (6.57) we infer that

$$\frac{1}{2} \int_{B_{\rho_2}} \varphi^2 \eta^2 (\tilde{v} - k)_+^2 \, dx + g_\alpha * \frac{\nu}{2} \int_{B_{\rho_2}} \varphi^2 \eta^2 |D(\tilde{v} - k)_+|^2 \, dx \le g_\alpha * G, \tag{6.58}$$

for a.a. $s \in (0, t_1 - t')$, where G is defined by

$$G(s) = G_1(s) + \frac{2\Lambda^2}{\nu} \int_{B_{\rho_2}} \varphi^2 |D\eta|^2 (\tilde{v} - k)_+^2 \, dx.$$

We may drop the second term in (6.58), which is nonnegative. By Young's inequality for convolutions we conclude that for all $p \in (1, 1/(1-\alpha))$

$$|\varphi\eta(\tilde{v} - k)_+|^2_{L_{2p}([0,t_1-t'];L_2(B_{\rho_2}))} \leq 2|g_\alpha|_{L_p([0,t_1-t'])} |G|_{L_1([0,t_1-t'])}. \tag{6.59}$$

Here

$$|g_\alpha|_{L_p([0,t_1-t'])} = \frac{(t_1 - t')^{\alpha-1+1/p}}{\Gamma(\alpha)[(\alpha-1)p+1]^{1/p}} \leq C_1(\alpha, p, \theta) \rho^{\frac{2}{\alpha}(\alpha-1+\frac{1}{p})}. \tag{6.60}$$

We may also drop the first term in (6.58), convolve the resulting inequality with $g_{1-\alpha}$ and evaluate at $s = t_1 - t'$, thereby obtaining

$$\int_0^{t_1-t'} \int_{B_{\rho_2}} \varphi^2 \eta^2 |D(\tilde{v} - k)_+|^2 \, dx \, ds \leq \frac{2}{\nu} |G|_{L_1([0,t_1-t'])}. \tag{6.61}$$

Using

$$\int_0^{t_1-t'} \int_{B_{\rho_2}} |D\big(\varphi\eta(\tilde{v} - k)_+\big)|^2 \, dx \, ds \leq$$
$$2\int_0^{t_1-t'} \int_{B_{\rho_2}} \Big(\varphi^2\eta^2 |D(\tilde{v} - k)_+|^2 + \varphi^2 |D\eta|^2 (\tilde{v} - k)_+^2\Big) \, dx \, ds$$

we deduce from (6.59)–(6.61) and the definition of G that

$$\rho^{-\frac{2}{\alpha}(\alpha-1+\frac{1}{p})} |\varphi\eta(\tilde{v} - k)_+|^2_{L_{2p}([0,t_1-t'];L_2(B_{\rho_2}))} + \big|D[\varphi\eta(\tilde{v} - k)_+]\big|^2_{L_2([0,t_1-t']\times B_{\rho_2})}$$
$$\leq C_2(\nu, \Lambda, C_1)\Big(\int_0^{t_1-t'} G_1(s) \, ds + \int_0^{t_1-t'} \int_{B_{\rho_2}} \varphi^2 |D\eta|^2 (\tilde{v} - k)_+^2 \, dx \, ds\Big). \tag{6.62}$$

We will next estimate the right-hand side of (6.62). By the assumptions on φ and η we have

$$\int_0^{t_1-t'} \int_{B_{\rho_2}} \varphi^2 |D\eta|^2 (\tilde{v} - k)_+^2 \, dx \, ds \leq \frac{16}{\rho^2(\vartheta_2 - \vartheta_1)^2} \int_0^{t_1-t'} \int_{B_{\rho_2}} (\tilde{v} - k)_+^2 \, dx \, ds.$$

Turning to G_1 we set

$$\tilde{A}_{k,\rho_2}(s) = \{x \in B_{\rho_2} : \tilde{v}(s, x) > k\}, \quad s \in (0, t_1 - t').$$

Then we have for a.a. $s \in (0, t_1 - t')$

$$\int_{B_{\rho_2}} \varphi^2 \eta^2 (\tilde{v} - k)_+ g_{1-\alpha} \, dx \leq g_{1-\alpha}\big(\frac{\hat{t} - t'}{2}\big) \int_{\tilde{A}_{k,\rho_2}(s)} \Big((\vartheta_2 - \vartheta_1)^\alpha + \frac{(\tilde{v} - k)_+^2}{(\vartheta_2 - \vartheta_1)^\alpha}\Big) \, dx$$
$$\leq \frac{C_3(\alpha, \theta, \theta_3)}{\rho^2} \int_{\tilde{A}_{k,\rho_2}(s)} \Big(1 + \frac{(\tilde{v} - k)_+^2}{(\vartheta_2 - \vartheta_1)^{2\alpha}}\Big) \, dx,$$

and also

$$\frac{1}{2}\int_{B_{\rho_2}} \varphi^2\eta^2(\tilde v-k)_+^2 g_{1-\alpha}\,dx \le \frac{C_4(\alpha,\theta,\theta_3)}{\rho^2(\vartheta_2-\vartheta_1)^\alpha}\int_{B_{\rho_2}}(\tilde v-k)_+^2\,dx.$$

Further,

$$\begin{aligned}\int_0^{t_1-t'}\frac{4\alpha}{\hat t-t'}\,g_{1-\alpha}*\int_{B_{\rho_2}}\eta^2(\tilde v-k)_+^2\,dx\,ds &\le \frac{4\alpha g_{2-\alpha}(t_1-t')}{\hat t-t'}\int_0^{t_1-t'}\!\!\int_{B_{\rho_2}}(\tilde v-k)_+^2\,dx\,ds\\ &\le \frac{C_5(\alpha,\theta,\theta_3)}{\rho^2(\vartheta_2-\vartheta_1)}\int_0^{t_1-t'}\!\!\int_{B_{\rho_2}}(\tilde v-k)_+^2\,dx\,ds.\end{aligned}$$

By (H3), there exists $p\in(1,1/(1-\alpha))$ such that for r and q from (H3) we have

$$\frac{p'}{r}+\frac{N}{2q}=1-\tilde\kappa \quad \text{with } \tilde\kappa\in(0,1).$$

Here $p'=p/(p-1)$ as usual. We fix this p and set

$$\delta=\frac{2\tilde\kappa}{2(p'-1)+N}$$

and $\tilde r=2r'(1+\delta)$ and $\tilde q=2q'(1+\delta)$. Observing that $\Phi'(\tilde u_{\bar m})\le\rho^{-\kappa}$ on $[0,t_1-t']\times B_{\rho_2}$ and using Hölder's inequality, we then have for any $\varepsilon_1>0$

$$\begin{aligned}\int_0^{t_1-t'}\!\!\int_{B_{\rho_2}}|\tilde f|\Phi'(\tilde u_{\bar m})(\tilde v-k)_+\eta^2\varphi^2\,dx\,ds &\le \rho^{-\kappa}|\tilde f|_{L_r([0,t_1-t'];L_q(B_{\rho_2}))}\\ &\quad\times|\varphi\eta(\tilde v-k)_+|_{L_{\tilde r}([0,t_1-t'];L_{\tilde q}(B_{\rho_2}))}\mu_{k,\rho_2}^{(1+2\delta)/\tilde r}\\ \le\varepsilon_1|\varphi\eta(\tilde v-k)_+|^2_{L_{\tilde r}([0,t_1-t'];L_{\tilde q}(B_{\rho_2}))}\rho^{-z}&+\frac{1}{\varepsilon_1}\mu_{k,\rho_2}^{2(1+2\delta)/\tilde r}\rho^{z-2\kappa}|f|^2_{L_r([0,T];L_q(\Omega))},\end{aligned}$$

with

$$\mu_{k,\rho_2}=\begin{cases}\int_0^{t_1-t'}\lambda_N(\tilde A_{k,\rho_2})^{\frac{r'}{q'}}\,ds & : q>1\\ \lambda_1(\{s\in(0,t_1-t'):\lambda_N(\tilde A_{k,\rho_2}(s))>0\}) & : q=1,\end{cases}$$

and

$$z=2\Big(\frac{2}{\alpha\tilde r}+\frac{N}{\tilde q}\Big)+2-\frac{2}{\alpha}-N.$$

Further

$$\mu_{k,\rho_2}^{\frac{2(1+2\delta)}{\tilde r}}\rho^{z-2\kappa}=\mu_{k,\rho_2}^{\frac{1}{r'}+\frac{\delta}{r'(1+\delta)}}\rho^{z-2\kappa}\le C_6(\alpha,\theta,\theta_3,r,\delta)\mu_{k,\rho_2}^{\frac{1}{r'}}\rho^{\frac{\delta}{1+\delta}\left(\frac{N}{q'}+\frac{2}{\alpha r'}\right)+z-2\kappa}$$

and

$$\begin{aligned}\frac{\delta}{1+\delta}\Big(\frac{N}{q'}+\frac{2}{\alpha r'}\Big)+z-2\kappa &=\frac{\delta}{1+\delta}\Big(\frac{N}{q'}+\frac{2}{\alpha r'}\Big)+\frac{2}{1+\delta}\Big(\frac{1}{\alpha r'}+\frac{N}{2q'}\Big)+2-\frac{2}{\alpha}-N-2\kappa\\ &=\Big(\frac{2}{\alpha r'}+\frac{N}{q'}\Big)+2-\frac{2}{\alpha}-N+4(1-\kappa)-4+2\kappa\\ &=\Big(\frac{2}{\alpha r'}+\frac{N}{q'}\Big)-2-\frac{2}{\alpha}-N+4\Big(\frac{1}{\alpha r}+\frac{N}{2q}\Big)+2\kappa\\ &=-\Big(\frac{2}{\alpha r'}+\frac{N}{q'}\Big)+\frac{2}{\alpha}+N-2+2\kappa.\end{aligned}\tag{6.63}$$

Therefore

$$\int_0^{t_1-t'}\!\!\int_{B_{\rho_2}} |\tilde f|\Phi'(\tilde u_{\bar m})(\tilde v-k)_+\eta^2\varphi^2\,dx\,ds \le \varepsilon_1|\varphi\eta(\tilde v-k)_+|^2_{L_{\tilde r}([0,t_1-t'];L_{\tilde q}(B_{\rho_2}))}\rho^{-z} + \frac{C_6}{\varepsilon_1}\,\mu_{k,\rho_2}^{\frac{1}{r'}}\rho^{-\left(\frac{2}{\alpha r'}+\frac{N}{q'}\right)}\rho^{\frac{2}{\alpha}+N-2}\rho_0^{2\kappa}|f|^2_{L_r([0,T];L_q(\Omega))}.$$

Using $\tilde g_{1-\alpha}(s) < g_{1-\alpha}(\tau_0)$, the same argument yields

$$\int_0^{t_1-t'}\int_{B_{\rho_2}} \tilde g_{1-\alpha}\,|u_0-\bar m|\,\Phi'(\tilde u_{\bar m})(\tilde v-k)_+\eta^2\varphi^2\,dx\,ds \le \varepsilon_1|\varphi\eta(\tilde v-k)_+|^2_{L_{\tilde r}([0,t_1-t'];L_{\tilde q}(B_{\rho_2}))}\rho^{-z} + \frac{C_7}{\varepsilon_1}\,\mu_{k,\rho_2}^{\frac{1}{r'}}\rho^{-\left(\frac{2}{\alpha r'}+\frac{N}{q'}\right)}\rho^{\frac{2}{\alpha}+N-2}\rho_0^{2\kappa}$$

with $C_7 = C_7(\alpha, N, \theta, \theta_3, r, q, C_6, \tau_0, |u_0|_{L_\infty(\Omega)}, M_0)$.

Finally, since $\varepsilon_\gamma \le \varepsilon$, we may estimate

$$\int_{B_{\rho_2}} \varepsilon_\gamma(M-\bar m)(t_1-t_0)^{-\alpha}\Phi'(\tilde u_{\bar m})(\tilde v-k)_+\eta^2\varphi^2\,dx \le \frac{1}{4\theta^\alpha\rho^2}\int_{B_{\rho_2}}(\tilde v-k)_+\eta^2\varphi^2\,dx \le \frac{1}{4\theta^\alpha\rho^2}\int_{\tilde A_{k,\rho_2}(s)}\big(1+(\tilde v-k)_+^2\big)\,dx.$$

From (6.62) and the previous estimates it follows that for any $\varepsilon_1 > 0$

$$\begin{aligned}&\rho^{-\frac{2}{\alpha}(\alpha-1+\frac{1}{p})}|\varphi\eta(\tilde v-k)_+|^2_{L_{2p}([0,t_1-t'];L_2(B_{\rho_2}))} + \big|D[\varphi\eta(\tilde v-k)_+]\big|^2_{L_2([0,t_1-t']\times B_{\rho_2})}\\ &\le 2\varepsilon_1|\varphi\eta(\tilde v-k)_+|^2_{L_{\tilde r}([0,t_1-t'];L_{\tilde q}(B_{\rho_2}))}\rho^{-z} + C_8\Big(\frac{1}{\rho^2(\vartheta_2-\vartheta_1)^2}\,|(\tilde v-k)_+|\big|^2_{L_2([0,t_1-t']\times B_{\rho_2})}\\ &\quad + \frac{1}{\varepsilon_1}\,\mu_{k,\rho_2}^{\frac{1}{r'}}\rho^{-\left(\frac{2}{\alpha r'}+\frac{N}{q'}\right)}\rho^{\frac{2}{\alpha}+N-2} + \frac{1}{\rho^2}\int_0^{t_1-t'}\lambda_N(\tilde A_{k,\rho_2})\,ds\Big),\end{aligned} \tag{6.64}$$

where $C_8 = C_8(\alpha, \theta, \theta_3, \nu, \Lambda, \tau_0, p, r, q, \delta, \rho_0, |f|_{L_r(L_q)}, |u_0|_{L_\infty(\Omega)}, M_0)$.

Observe that (6.64) is scaling invariant with respect to $\rho > 0$, in fact, each of the terms behaves like $\rho^{N-2+2/\alpha}$ when changing coordinates as $s \to s/\rho^{2/\alpha}$ and $x \to x_1 + (x-x_1)/\rho$. Notice as well that the numbers $\tilde r$ and $\tilde q$ are subject to conditions (2.12) with (r, q) replaced by $(\tilde r, \tilde q)$. Therefore, choosing ε_1 sufficiently small, the first term on the right-hand side of (6.64) is dominated by one half of the sum on the left-hand side.

Further, it is appropriate to reformulate the last term in (6.64) in order to obtain the same structure as for the term involving μ_{k,ρ_2}. To this end, we put

$$\hat r = \frac{\frac{1}{\alpha}+\frac{N}{2}}{1-\kappa},$$

which means that the pair $(\hat r, \hat q)$ with $\hat q := \hat r$ satisfies the same condition as (r, q) in (H3). Then we have

$$\begin{aligned}\hat\mu_{k,\rho_2} &:= \int_0^{t_1-t'}\lambda_N(\tilde A_{k,\rho_2})^{\frac{\hat r'}{\hat q'}}\,ds = \int_0^{t_1-t'}\lambda_N(\tilde A_{k,\rho_2})\,ds\\ &\le \hat\mu_{k,\rho_2}^{\frac{1}{\hat r'}}\Big(\int_0^{t_1-t'}\!\!\int_{B_{\rho_2}} dx\,ds\Big)^{\frac{1}{\hat r}} \le C_9(\alpha, N, \theta, \theta_3, \hat r)\,\hat\mu_{k,\rho_2}^{\frac{1}{\hat r'}}\rho^{-\left(\frac{2}{\alpha\hat r'}+\frac{N}{\hat q'}\right)}\rho^{\frac{2}{\alpha}+N}.\end{aligned}$$

We conclude that

$$\rho^{-\frac{2}{\alpha}(\alpha-1+\frac{1}{p})}|\varphi\eta(\tilde v-k)_+|^2_{L_{2p}([0,t_1-t'];L_2(B_{\rho_2}))}+\big|D[\varphi\eta(\tilde v-k)_+]\big|^2_{L_2([0,t_1-t']\times B_{\rho_2})}$$
$$\le C_{10}\Big(\frac{1}{\rho^2(\vartheta_2-\vartheta_1)^2}|(\tilde v-k)_+|^2_{L_2([0,t_1-t']\times B_{\rho_2})}$$
$$+\rho^{\frac{2}{\alpha}+N-2}\Big[\mu_{k,\rho_2}^{\frac{1}{r'}}\rho^{-\left(\frac{2}{\alpha r'}+\frac{N}{q'}\right)}+\hat\mu_{k,\rho_2}^{\frac{1}{\hat r'}}\rho^{-\left(\frac{2}{\alpha\hat r'}+\frac{N}{\hat q'}\right)}\Big]\Big), \tag{6.65}$$

with $C_{10}=C_{10}(C_8,C_9,N,\hat r)$. Returning to the function v and using the properties of the cut-off functions we obtain with $J(\vartheta,\rho)=(t_1-[1+\vartheta(\theta_3-1)]\theta\rho^{2/\alpha},t_1)$

$$\rho^{-\frac{2}{\alpha}(\alpha-1+\frac{1}{p})}|(v-k)_+|^2_{L_{2p}(J(\vartheta_1,\rho);L_2(B_{\rho(1+\vartheta_1/2)}))}+\big|D(\tilde v-k)_+\big|^2_{L_2(Q(\vartheta_1,\rho))}$$
$$\le C_{10}\Big(\frac{1}{\rho^2(\vartheta_2-\vartheta_1)^2}|(v-k)_+|^2_{L_2(Q(\vartheta_2,\rho))}$$
$$+\rho^{\frac{2}{\alpha}+N-2}\Big[\mu(k,\vartheta_2,\rho)^{\frac{1}{r'}}\rho^{-\left(\frac{2}{\alpha r'}+\frac{N}{q'}\right)}+\hat\mu(k,\vartheta_2,\rho)^{\frac{1}{\hat r'}}\rho^{-\left(\frac{2}{\alpha\hat r'}+\frac{N}{\hat q'}\right)}\Big]\Big) \tag{6.66}$$

for all $0\le\vartheta_1<\vartheta_2\le 1$ and $k>0$, where

$$\mu(k,\vartheta,\rho)=\begin{cases}\int_{J(\vartheta,\rho)}\lambda_N(A_{k,\rho(1+\vartheta/2)}(t))^{\frac{r'}{q'}}\,dt & : q>1\\ \lambda_1(\{t\in J(\vartheta,\rho):\lambda_N(A_{k,\rho(1+\vartheta/2)}(t))>0\}) & : q=1,\end{cases}$$

$$A_{k,\tilde\rho}(t)=\{x\in B_{\tilde\rho}: v(t,x)>k\},\quad t\in(\tilde t,t_1),\ \tilde\rho>0,$$

and $\hat\mu(k,\vartheta,\rho)$ is defined correspondingly.

2. Iterative Inequalities. We next normalize to $\rho=1$. To this purpose we change variables as $t\to t_1+(t-t_1)/\rho^{2/\alpha}$ and $x\to x_1+(x-x_1)/\rho$. Denoting the new variables again by t and x and the transformed function again by v the inequalities (6.66) take the form

$$|(v-k)_+|^2_{V_p(Q(\vartheta_1,1))}\le 2C_{10}\Big(\frac{1}{(\vartheta_2-\vartheta_1)^2}|(v-k)_+|^2_{L_2(Q(\vartheta_2,1))}$$
$$+\mu(k,\vartheta_2,1)^{\frac{1}{r'}}+\hat\mu(k,\vartheta_2,1)^{\frac{1}{\hat r'}}\Big). \tag{6.67}$$

We consider the family of boxes $Q_n=Q(\zeta_n,1)=J(\zeta_n,1)\times B_{1+\zeta_n/2}$ with $\zeta_n=2^{-(n+1)}$, $n=0,1,2,\ldots$, and introduce the sequence of increasing levels

$$k_n=K\Big(2-\frac{1}{2^n}\Big),$$

where $K\ge 1$ is a number to be chosen. For $n=0,1,2,\ldots$ we set

$$Y_n=\frac{1}{K^2}\big|(v-k_n)_+\big|^2_{L_2(Q_n)}$$

and

$$Z_n=\mu(k_n,\zeta_n,1)^{\frac{1}{r'(1+\delta)}}+\hat\mu(k_n,\zeta_n,1)^{\frac{1}{\hat r'(1+\delta)}}.$$

Let $\tilde\zeta_n=(\zeta_n+\zeta_{n+1})/2$. Choose a cutoff-function $\eta_n\in C^1(B_{1+\zeta_n/2})$ that has values only in $[0,1]$, such that $\operatorname{supp}\eta_n\subset B_{1+\tilde\zeta_n/2}$, $\eta_n=1$ in $B_{1+\zeta_{n+1}/2}$, and $|D\eta_n|\le 32\cdot 2^n$. Let

$$\chi=\frac{2p+N(p-1)}{2+N(p-1)}.$$

Then we have with $J_n = J(\zeta_n, 1)$

$$
\begin{aligned}
Y_{n+1} &\le K^{-2} \int_{J_{n+1}} \int_{A_{k_{n+1},1+\tilde{\zeta}_n/2}} (v-k_{n+1})^2 \eta_n^2 \, dx\, dt \\
&\le K^{-2} \Big(\int_{J_{n+1}} \lambda_N(A_{k_{n+1},1+\tilde{\zeta}_n/2})\, dt \Big)^{\frac{1}{\chi'}} \Big(\int_{J_{n+1}} \int_{A_{k_{n+1},1+\tilde{\zeta}_n/2}} \big[(v-k_{n+1})\eta_n\big]^{2\chi} \, dx\, dt \Big)^{\frac{1}{\chi}} \\
&\le C(p,N) K^{-2} \Big(\int_{J_{n+1}} \lambda_N(A_{k_{n+1},1+\tilde{\zeta}_n/2})\, dt \Big)^{\frac{1}{\chi'}} \big|(v-k_{n+1})_+ \eta_n\big|^2_{V_p(J_{n+1}\times B_{1+\tilde{\zeta}_n/2})},
\end{aligned}
$$

where we use the embedding $V_p \hookrightarrow L_{2\chi}$ from (2.13).

Further,

$$
\begin{aligned}
\int_{J_{n+1}} \lambda_N(A_{k_{n+1},1+\tilde{\zeta}_n/2})\, dt &\le (k_{n+1}-k_n)^{-2} \int_{J_{n+1}} \int_{A_{k_{n+1},1+\tilde{\zeta}_n/2}} (v-k_n)_+^2 \, dx\, dt \\
&\le (k_{n+1}-k_n)^{-2} K^2 Y_n \le 4^{n+1} Y_n.
\end{aligned}
$$

Using the properties of η_n and applying (6.67) with $\vartheta_2 = \zeta_n > \tilde{\zeta}_n = \vartheta_1$ and $k = k_{n+1}$ we also have

$$
\begin{aligned}
\big|(v-k_{n+1})_+ \eta_n\big|^2_{V_p(J_{n+1}\times B_{1+\tilde{\zeta}_n/2})} &\le 2\big|(v-k_{n+1})_+\big|^2_{V_p(J_{n+1}\times B_{1+\tilde{\zeta}_n/2})} \\
&\quad + 2\int_{J_{n+1}} \int_{A_{k_{n+1},1+\tilde{\zeta}_n/2}} (v-k_{n+1})_+^2 |D\eta_n|^2 \, dx\, dt \\
&\le 4C_{10}\Big(4^{n+3} |(v-k_{n+1})_+|^2_{L_2(Q_n)} + \mu(k_{n+1},\zeta_n,1)^{\frac{1}{r'}} + \hat{\mu}(k_{n+1},\zeta_n,1)^{\frac{1}{\hat{r}'}} \Big) \\
&\quad + 2\cdot 4^{n+5} |(v-k_{n+1})_+|^2_{L_2(Q_n)} \\
&\le 4^{n+4}(C_{10}+8) K^2 Y_n + 4C_{10} Z_n^{1+\delta}.
\end{aligned} \tag{6.68}
$$

It follows that

$$
Y_{n+1} \le C_{11} 16^n \Big(Y_n^{1+\frac{1}{\chi'}} + Y_n^{\frac{1}{\chi'}} Z_n^{1+\delta} \Big). \tag{6.69}
$$

Turning to Z_{n+1} note that

$$
\begin{aligned}
(k_{n+1}-k_n)^2 Z_{n+1} &= (k_{n+1}-k_n)^2 \big(\mu(k_{n+1},\zeta_{n+1},1)^{\frac{1}{r'(1+\delta)}} + \hat{\mu}(k_{n+1},\zeta_{n+1},1)^{\frac{1}{\hat{r}'(1+\delta)}} \big) \\
&\le |(v-k_n)_+ \eta_n|^2_{L_{2r'(1+\delta)}(J_{n+1}; L_{2q'(1+\delta)}(B_{1+\tilde{\zeta}_n/2}))} \\
&\quad + |(v-k_n)_+ \eta_n|^2_{L_{2\hat{r}'(1+\delta)}(J_{n+1}; L_{2\hat{q}'(1+\delta)}(B_{1+\tilde{\zeta}_n/2}))} \\
&\le 2C(p,N) \big|(v-k_n)_+ \eta_n\big|^2_{V_p(J_{n+1}\times B_{1+\tilde{\zeta}_n/2})}.
\end{aligned} \tag{6.70}
$$

The right-hand side is estimated in the same way as in (6.68) replacing k_{n+1} by k_n. This yields

$$
Z_{n+1} \le 2C(p,N) 4^{n+1} K^{-2} \big(4^{n+4}(C_{10}+8) K^2 Y_n + 4C_{10} Z_n^{1+\delta} \big),
$$

and therefore

$$
Z_{n+1} \le C_{11} 16^n \Big(Y_n + Z_n^{1+\delta} \Big), \tag{6.71}
$$

where we may take the same constant $C_{11} = C_{11}(C_{10}, N, p)$ as in (6.69).

In view of (6.69) and (6.71), Lemma 2.4.2 implies that Y_n and Z_n tend to zero as $n \to \infty$ provided that

$$Y_0 + Z_0^{1+\delta} \le (2C_{11})^{-\frac{1+\delta}{\delta_1}} 16^{-\frac{1+\delta}{\delta_1^2}}, \quad \text{with } \delta_1 := \min\Big\{\delta, \frac{1}{\chi'}\Big\}. \tag{6.72}$$

This condition is satisfied whenever K is sufficiently large. In fact, for Y_0 we have

$$Y_0 = K^{-2}\big|(v-K)_+\big|_{L_2(J(1/2,1)\times B_{5/4})} \le K^{-2}|v|_{L_2(Q(1,1))}.$$

Z_0 can be estimated as in (6.70) if one assumes that $k_{-1} = K/2$ and $\zeta_{-1} = 1$. This gives

$$Z_0 \le \frac{8C(p,N)}{K^2}\big|(v-k_{-1})\,\eta_{-1}\big|^2_{V_p(J_0\times B_{1+\tilde{\zeta}_{-1}/2})}.$$

Using (6.68) with k_{n+1} replaced by k_{-1} we then obtain for some $C_{12} = C_{12}(p, N, C_{10})$

$$\begin{aligned} Z_0 \le\, & C_{12}K^{-2}\Big(|v|_{L_2(Q(1,1))} + \mu(K/2),1,1)^{\frac{1}{r'}} + \hat{\mu}(K/2,1,1)^{\frac{1}{\hat{r}'}}\Big) \\ \le\, & C_{12}K^{-2}\Big(|v|_{L_2(Q(1,1))} + \big(\theta_3\theta\lambda_N(B_{3/2})\big)^{\frac{1}{r'}} + \big(\theta_3\theta\lambda_N(B_{3/2})\big)^{\frac{1}{\hat{r}'}}\Big). \end{aligned}$$

Normalizing the L_2-estimate from Proposition 6.3.1 to $\rho = 1$ yields an a priori bound for $|v|_{L_2(Q(1,1))}$. Hence there exists a number $K \ge 1$ depending only on $\Lambda, \nu, \alpha, r, q, N, \tau_0, \rho_0, \xi, \beta, \theta, \theta_2, \theta_3, M_0$, $|u_0|_{L_\infty(\Omega)}, |f|_{L_r(L_q)}$ such that (6.72) is fulfilled.

Since $k_n \to 2K$ and $Y_n \to 0$ as $n \to \infty$, it follows that $\operatorname{ess\,sup}_{Q(0,1)} v \le 2K$. Scaling back to the original variables, this means that (6.52) holds with $M_1 = 2K$. □

Remark 6.4.1 Note that the assertion of Proposition 6.3.1 remains true when equation (6.1) has a second term on the right-hand side of the form $-D_i g^i$, where $|g|^2 \in L_r([0,T]; L_q(\Omega))$. In fact, recall Remark 6.3.1 and note that for any $\epsilon_0 > 0$ we have

$$\begin{aligned} \int_0^{t_1-t'}\!\!\int_{B_{\rho_2}} \varphi^2 g^i D_i\big[\Psi'(\tilde{u}_{\bar{m}})\eta^2\big]\,dx\,ds \le & \int_0^{t_1-t'}\!\!\int_{B_{\rho_2}} \Big(\epsilon_0\varphi^2\eta^2\big[|D(\tilde{v}-k)_+|^2 + (\tilde{v}-k)_+|D\tilde{v}|^2\big] \\ & + C(\epsilon_0)\big[|\tilde{g}|^2\Phi'(\tilde{u}_{\bar{m}})^2(\tilde{v}-k)_+ + |D\eta|^2(\tilde{v}-k)_+^2\big]\Big)\,dx\,ds \\ & + C(\epsilon_0)\int_0^{t_1-t'}\!\!\int_{A_{k,\rho_2}} |\tilde{g}|^2\Phi'(\tilde{u}_{\bar{m}})^2\,dx\,ds. \end{aligned}$$

The terms with factor ϵ_0 can be absorbed if ϵ_0 is selected small enough. The term containing $|\tilde{g}|^2\Phi'(\tilde{u}_{\bar{m}})^2(\tilde{v}-k)_+$ is estimated similarly as the $\tilde{f}$-term above; note that the additional factor $\Phi'(\tilde{u}_{\bar{m}})$ can be controlled thanks due the last summand in (6.63). Finally, by Hölder's inequality,

$$\int_0^{t_1-t'}\!\!\int_{A_{k,\rho_2}} |\tilde{g}|^2\Phi'(\tilde{u}_{\bar{m}})^2\,dx\,ds \le \big||g|^2\big|_{L_r(L_q)}\mu_{k,\rho_2}^{\frac{1}{r'}}\rho^{-\left(\frac{2}{\alpha r'}+\frac{N}{q'}\right)}\rho^{\frac{2}{\alpha}+N-2},$$

which is the right estimate to proceed as above.

6.5 Oscillation estimates

Suppose u is a bounded weak solution of (6.1). Without restriction of generality we may assume that $\operatorname{ess\,osc}_{\Omega_T} u \le 1$.

As in the previous sections we let $\tau_0 \in (0,T)$ be fixed and $(t_1, x_1) \in (\tau_0, T] \times \Omega$. We consider the family of nested cylinders

$$Q(\rho) := Q(t_1, x_1, \theta, \rho) = (t_1 - \theta\rho^{2/\alpha}, t_1) \times B_\rho(x_1),$$

with scaling parameter $\rho = 2^{-l}$, $l \in \mathbb{Z}$, and the fixed parameter $\theta > 0$ (cf. assumption (P)). By $\tilde{l}$ we denote the integer that corresponds to the largest of those cylinders $Q(2^{-l})$ that are properly contained in $(\tau_0, T) \times \Omega$. Let further $l_0 \geq \tilde{l}$ be an integer, which will be fixed later appropriately. For $l \leq l_0$ we set $a_l := \operatorname{ess\,inf}_{\Omega_T} u$ and $b_l := a_l + 2^{-(l-l_0)\kappa_1}$, where $\kappa_1 > 0$ is another parameter which will be chosen later. Then trivially we have for all $j \leq l_0$, $j \in \mathbb{Z}$,

$$\text{(i)} \quad a_j \leq u \leq b_j \quad \text{a.e. in } Q(2^{-j}) \cap \Omega_T, \quad \text{(ii)} \quad b_j - a_j = 2^{-(j-l_0)\kappa_1}. \tag{6.73}$$

Let now $l \geq l_0$ and assume that there exist sequences $(a_j), (b_j) \subset \mathbb{R}$ such that (6.73) is satisfied for all $j \leq l$. The objective is then to construct two numbers a_{l+1} and b_{l+1} such that (i) and (ii) in (6.73) hold also true for $j = l+1$.

The first observation is that the induction hypothesis implies an estimate of power type for the memory term. More precisely, we have the following

Lemma 6.5.1 *Let $m_l = (a_l + b_l)/2$, and $t_0 = t_1 - \theta 2^{-2l/\alpha}$. Then*

$$|u(t,x) - m_l| \leq (b_l - m_l)\Big(2\Big[2^{2/\alpha}\big(\frac{t_1 - t}{t_1 - t_0}\big)\Big]^{\frac{\alpha\kappa_1}{2}} - 1\Big), \tag{6.74}$$

for a.a. $(t,x) \in (0, t_0) \times B_{2^{-l}}(x_1)$.

Proof. Let $x \in B_{2^{-l}}(x_1)$. Given $t \in (0, t_0)$ we find an integer $l_* \leq l$ so that

$$t_1 - \theta 2^{-2(l_*-1)/\alpha} \;<\; t \;\leq t_1 - \theta 2^{-2l_*/\alpha}, \tag{6.75}$$

which means that $(t,x) \in Q(2^{-(l_*-1)})$. By the induction hypothesis,

$$\begin{aligned} u(t,x) - m_l &\leq b_{l_*-1} - m_l = b_{l_*-1} - a_{l_*-1} + a_{l_*-1} - m_l \\ &\leq b_{l_*-1} - a_{l_*-1} + a_l - m_l = 2^{-(l_*-1-l_0)\kappa_1} - \frac{1}{2}\,2^{-(l-l_0)\kappa_1} \\ &= (b_l - m_l)\Big(2 \cdot 2^{-(l_*-1-l)\kappa_1} - 1\Big). \end{aligned}$$

From (6.75) and $t_1 - t_0 = \theta 2^{-2l/\alpha}$ we deduce that

$$2^{-2(l_*-l)/\alpha} = \frac{\theta 2^{-2l_*/\alpha}}{\theta 2^{-2l/\alpha}} \;\leq\; \frac{t_1 - t}{t_1 - t_0}\,.$$

Hence

$$\begin{aligned} u(t,x) - m_l &\leq (b_l - m_l)\Big(2 \cdot 2^{\kappa_1} 2^{-(l_*-l)\kappa_1} - 1\Big) \\ &\leq (b_l - m_l)\Big(2\Big[2^{2/\alpha}\big(\frac{t_1 - t}{t_1 - t_0}\big)\Big]^{\frac{\alpha\kappa_1}{2}} - 1\Big). \end{aligned}$$

The desired lower estimate for $u - m_l$ is proved analogously. □

From now on we will assume that

$$\gamma := \frac{\alpha\kappa_1}{2} \in (0, \gamma_0] \quad \text{and} \quad \varepsilon_\gamma \leq \varepsilon. \tag{6.76}$$

In the next step we distinguish two cases. With $\bar{t} = t_1 - \theta\theta_1 2^{-2l/\alpha}$ at least one of the following inequalities is satisfied:

$$\textbf{(A)} \quad \lambda_{N+1}\Big(\{(t,x) \in (t_0,\bar{t}) \times B_{2^{-l}}(x_1) :\ u(t,x) \geq m_l\}\Big) \leq \frac{1}{2}\,\lambda_{N+1}\Big((t_0,\bar{t}) \times B_{2^{-l}}(x_1)\Big),$$

or

$$\textbf{(B)} \quad \lambda_{N+1}\Big(\{(t,x) \in (t_0,\bar{t}) \times B_{2^{-l}}(x_1) :\ u(t,x) \leq m_l\}\Big) \leq \frac{1}{2}\,\lambda_{N+1}\Big((t_0,\bar{t}) \times B_{2^{-l}}(x_1)\Big).$$

Suppose (A) holds. We again distinguish two cases. Let us first assume that

$$\max\Big\{\mu(b_l - m_l), 2^{-(l+1)\kappa}\Big\} < \operatorname*{ess\,sup}_{[t_0,t_1]\times B_{2^{-(l+1)}}} u - m_l. \tag{6.77}$$

From Lemma 6.5.1 and Lemma 6.4.1 we then infer that the function v defined in (6.24) with $\rho = 2^{-(l+1)}$, $M = b_l$, and $\bar{m} = m_l$ is subject to

$$\operatorname*{ess\,sup}_{Q(2^{-(l+1)})} v \leq M_1,$$

where $M_1 \geq 0$ is the a priori bound from (6.52). This in turn implies

$$u \leq b_l - \big[e^{-M_1}(1-\xi) - \varepsilon\big](b_l - m_l) + 2^{-(l+1)\kappa} \quad \text{a.e. in } Q(2^{-(l+1)}).$$

If (6.77) is violated, we have

$$\begin{aligned} u &\leq m_l + \mu(b_l - m_l) + 2^{-(l+1)\kappa} \\ &\leq b_l - [1-\mu](b_l - m_l) + 2^{-(l+1)\kappa} \quad \text{a.e. in } Q(2^{-(l+1)}). \end{aligned}$$

We next set

$$\varepsilon = \frac{1}{2}\,e^{-M_1}(1-\xi) \quad \text{and} \quad \varepsilon_* = \frac{1}{2}\,\min\{\varepsilon, 1-\mu\}.$$

Then the previous estimates show that if condition (A) holds, we have in any case

$$u \leq b_l - \varepsilon_*(b_l - a_l) + 2^{-(l+1)\kappa} \quad \text{a.e. in } Q(2^{-(l+1)}). \tag{6.78}$$

Define now

$$a_{l+1} = a_l \quad \text{and} \quad b_{l+1} = a_l + 2^{-(l+1-l_0)\kappa_1}.$$

Then by the induction hypothesis,

$$\begin{aligned} & b_{l+1} \geq b_l - \varepsilon_*(b_l - a_l) + 2^{-(l+1)\kappa} \\ \Leftrightarrow\quad & 2^{-(l+1-l_0)\kappa_1} \geq (1-\varepsilon_*)2^{-(l-l_0)\kappa_1} + 2^{-(l+1)\kappa} \\ \Leftrightarrow\quad & 1 \geq 2^{\kappa_1}(1-\varepsilon_*) + 2^{-(l-l_0+1)(\kappa-\kappa_1)}2^{-l_0\kappa}. \end{aligned}$$

Assuming $\kappa_1 \leq \kappa$ and recalling that $l \geq l_0$, the last condition certainly follows from

$$2^{\kappa_1}(1-\varepsilon_*) + 2^{-l_0\kappa} \leq 1. \tag{6.79}$$

We choose now $\kappa_1 \in (0, \min\{\kappa, 2\gamma_0/\alpha\}]$ so small and $l_0 \geq \tilde{l}$ so large that $\varepsilon_\gamma \leq \varepsilon$ and (6.79) are satisfied. In particular κ_1 and l_0 are independent of $l \geq l_0$. In view of (6.78) and the definition of a_{l+1} and b_{l+1} it is then evident that (6.73) holds for $j = l+1$.

Suppose now that (B) holds. Setting

$$\hat{u} = b_l + a_l - u \quad \text{and} \quad \hat{u}_0 = b_l + a_l - u_0,$$

$\hat{u}$ is a bounded weak solution of

$$\partial_t^\alpha(\hat{u} - \hat{u}_0) - D_i(a_{ij}D_j\hat{u}) = -f,\ t \in (0,T),\ x \in \Omega. \tag{6.80}$$

We further put

$$\hat{a}_j = b_l + a_l - b_j, \quad \hat{b}_j = b_l + a_l - a_j, \quad j \le l.$$

By the induction hypothesis, we have for all $j \le l$, $j \in \mathbb{Z}$,

$$\hat{a}_j \le \hat{u} \le \hat{b}_j \quad \text{a.e. in } Q(2^{-j}) \cap \Omega_T \quad \text{and} \quad \hat{b}_j - \hat{a}_j = b_j - a_j = 2^{-(j-l_0)\kappa_1}.$$

Note also that $\hat{a}_l = a_l$, $\hat{b}_l = b_l$, and

$$\hat{m}_l := \frac{\hat{b}_l + \hat{a}_l}{2} = m_l.$$

Moreover,

$$u \ge m_l \quad \Leftrightarrow \quad \hat{u} \le m_l.$$

Hence

$$\lambda_{N+1}\Big(\{(t,x) \in (t_0,\bar{t}) \times B_{2^{-l}}(x_1) :\ \hat{u}(t,x) \ge m_l\}\Big) \le \frac{1}{2}\,\lambda_{N+1}\Big((t_0,\bar{t}) \times B_{2^{-l}}(x_1)\Big),$$

so that we are in the situation of case (A). We may now argue as above, replacing u, a_j, and b_j by $\hat{u}$, $\hat{a}_j$, and $\hat{b}_j$, respectively, and using equation (6.80). Since $|\hat{u}_0 - \hat{m}_l|_{L_\infty(\Omega)} = |u_0 - m_l|_{L_\infty(\Omega)}$ the corresponding estimates in Propositions 6.2.1, 6.3.1, and 6.4.1 hold with the same constants as above. Thus we obtain

$$\hat{u} \le \hat{b}_l - \varepsilon_*(\hat{b}_l - \hat{a}_l) + 2^{-(l+1)\kappa} \quad \text{a.e. in } Q(2^{-(l+1)}),$$

where ε_* is as above. It follows that

$$\hat{a}_l \le \hat{u} \le \hat{a}_l + 2^{-(l+1-l_0)\kappa_1} \quad \text{a.e. in } Q(2^{-(l+1)}),$$

where κ_1 and l_0 are the same as before. In terms of u this means that

$$a_{l+1} := b_l - 2^{-(l+1-l_0)\kappa_1} \le u \le b_l =: b_{l+1} \quad \text{a.e. in } Q(2^{-(l+1)}).$$

Hence in case (B), this is an admissible choice of a_{l+1} and b_{l+1} in order to satisfy (6.73) for $j = l+1$.

Summarizing, we obtain the following result.

Proposition 6.5.1 *Let u be a bounded weak solution of (6.1). Let $\tau_0 \in (0,T)$ be fixed and $(t_1,x_1) \in (\tau_0,T] \times \Omega$. Let further $Q(\rho) = (t_1 - \theta\rho^{2/\alpha}, t_1) \times B_\rho(x_1)$, where $\theta > 0$ is the fixed parameter from assumption (P). Then*

$$\operatorname*{ess\,osc}_{Q(2^{-j})\cap\Omega_T} u \le C\,2^{-j\kappa_1}, \quad j \in \mathbb{Z},$$

where $\kappa_1 = \kappa_1(\Lambda, \nu, \alpha, r, q, N, \tau_0, diam\,\Omega, |u|_{L_\infty(\Omega_T)}, |u_0|_{L_\infty(\Omega)}, |f|_{L_r([0,T];L_q(\Omega))})$ and $C = C(\Lambda, \nu,$ $\alpha, r, q, N, \tau_0, diam\,\Omega, |u|_{L_\infty(\Omega_T)}, |u_0|_{L_\infty(\Omega)}, |f|_{L_r([0,T];L_q(\Omega))}, dist\,(x_1, \partial\Omega), |t_1 - \tau_0|)$.

Note that here C, compared with κ_1, also depends on $t_1-\tau_0$ and $\mathrm{dist}\,(x_1,\partial\Omega)$. C explodes when $\min\{t_1-\tau_0,\mathrm{dist}\,(x_1,\partial\Omega)\}\to 0$. This comes from fixing first $\kappa_1\in(0,\min\{\kappa,2\gamma_0/\alpha\}]$ so small that $2^{\kappa_1}(1-\varepsilon_*)<1$ as well as $\varepsilon_\gamma\le\varepsilon$ and choosing then $l_0\ge\tilde{l}$ such that (6.79) is satisfied. With κ_1 being fixed, the factor $2^{l_0\kappa_1}$ increases with l_0, and by definition of $\tilde{l}$ we have

$$\Big(\frac{\theta}{t_1-\tau_0}\Big)^{\frac{\alpha}{2}},\,\frac{1}{\mathrm{dist}\,(x_1,\partial\Omega)}\le 2^{\tilde{l}}\le 2^{l_0}.$$

We further remark that the case $\omega:=\operatorname{ess\,osc}_{\Omega_T}u>1$ can be simply reduced to the case considered above by means of the normalization $u\to u/\omega$, $u_0\to u_0/\omega$, $f\to f/\omega$ and the trivial estimate $\operatorname{ess\,osc}_{\Omega_T}u\le 2|u|_{L_\infty(\Omega_T)}$.

The estimate in Proposition 6.5.1 can be improved in that one can derive a bound for the oscillation of u on $Q(2^{-j})\cap\Omega_T$ that depends linearly on the sum

$$\mathcal{D}:=|u|_{L_\infty(\Omega_T)}+|u_0|_{L_\infty(\Omega)}+|f|_{L_r([0,T];L_q(\Omega))}. \tag{6.81}$$

To this purpose we normalize by setting $\hat{u}=u/2\mathcal{D}$, $\hat{u}_0=u_0/2\mathcal{D}$, and $\hat{f}=f/2\mathcal{D}$. Then $|u(t,x)|,|u_0(x)|\le 1/2$ for a.a. $t\in(0,T)$ and $x\in\Omega$, $\operatorname{ess\,osc}_{\Omega_T}u\le 1$ as well as $|\hat{f}|_{L_r([0,T];L_q(\Omega))}\le 1/2$. Since $\hat{u}$ is a weak solution of

$$\partial_t^\alpha(\hat{u}-\hat{u}_0)-D_i(a_{ij}D_j\hat{u})=\hat{f},\ t\in(0,T),\,x\in\Omega,$$

we may apply Propositions 6.2.1, 6.3.1, and 6.4.1 to $\hat{u}$ and obtain corresponding estimates which do *not* depend on $|\hat{u}|_{L_\infty(\Omega_T)}$, $|\hat{u}_0|_{L_\infty(\Omega)}$, and $|\hat{f}|_{L_r([0,T];L_q(\Omega))}$, by normalization and the above proofs. In particular the geometric parameter θ and the bound M_1 do not depend on these quantities. We may then argue as in the proof of Proposition 6.5.1 to obtain corresponding oscillation estimates for $\hat{u}$. Finally, rescaling to the original function u yields the following result.

Theorem 6.5.1 *Let $\tau_0\in(0,T)$ be fixed and $(t_1,x_1)\in(\tau_0,T]\times\Omega$. Then there exist positive numbers θ, κ_1 depending only on $\Lambda,\nu,\alpha,r,q,N,\mathit{diam}\,\Omega,\tau_0$, and there is a positive constant $C=C(\Lambda,\nu,\alpha,r,q,N,\mathit{diam}\,\Omega,\tau_0,\mathit{dist}\,(x_1,\partial\Omega),|t_1-\tau_0|)$ such that for any bounded weak solution u of (6.1) in Ω_T there holds*

$$\operatorname*{ess\,osc}_{Q(2^{-j})\cap\Omega_T}u\le C\,2^{-j\kappa_1}\Big(|u|_{L_\infty(\Omega_T)}+|u_0|_{L_\infty(\Omega)}+|f|_{L_r([0,T];L_q(\Omega))}\Big),\quad j\in\mathbb{Z},$$

with $Q(\rho)=(t_1-\theta\rho^{2/\alpha},t_1)\times B_\rho(x_1)$.

As a simple consequence we obtain the subsequent oscillation estimate for general cylinders with continuous scaling parameter.

Corollary 6.5.1 *Let $\tau_0\in(0,T)$ be fixed and $(t_1,x_1)\in(\tau_0,T]\times\Omega$. Let further $\tau_1>0$ and $\tilde{Q}(\rho)=(t_1-\tau_1\rho^{2/\alpha},t_1)\times B_\rho(x_1)$ for $\rho>0$. Then there exist positive numbers $\kappa_1=\kappa_1(\Lambda,\nu,\alpha,r,q,N,\mathit{diam}\,\Omega,\tau_0)$ and $C=C(\Lambda,\nu,\alpha,r,q,N,\mathit{diam}\,\Omega,\tau_0,\tau_1,\mathit{dist}\,(x_1,\partial\Omega),|t_1-\tau_0|)$ such that for any bounded weak solution u of (6.1) there holds*

$$\operatorname*{ess\,osc}_{\tilde{Q}(\rho)\cap\Omega_T}u\le C\,\rho^{\kappa_1}\Big(|u|_{L_\infty(\Omega_T)}+|u_0|_{L_\infty(\Omega)}+|f|_{L_r([0,T];L_q(\Omega))}\Big),\quad \rho>0.$$

In particular, any bounded weak solution u of (6.1) in Ω_T is Hölder continuous in Ω_T.

Proof. Suppose that $\tau_1 \geq \theta$, where θ is the same as in Theorem 6.5.1. Given $\rho > 0$ we put $\hat{\rho} = \rho(\tau_1/\theta)^{\alpha/2}$. There exists $j_* \in \mathbb{Z}$ such that $2^{-(j_*+1)} \leq \hat{\rho} < 2^{-j_*}$. By Theorem 6.5.1, we then have with D as in (6.81)

$$\operatorname*{ess\,osc}_{\tilde{Q}(\rho)\cap\Omega_T} u \leq \operatorname*{ess\,osc}_{Q(\hat{\rho})\cap\Omega_T} u \leq \operatorname*{ess\,osc}_{Q(2^{-j_*})\cap\Omega_T} u \leq C\,2^{-j_*\kappa_1} D$$
$$\leq C 2^{\kappa_1} \hat{\rho}^{\kappa_1} D \leq C 2^{\kappa_1} \left(\frac{\tau_1}{\theta}\right)^{\frac{\alpha\kappa_1}{2}} \rho^{\kappa_1} D.$$

The case $\tau_1 < \theta$ is treated similarly. □

The interior Hölder estimate stated in Theorem 6.1.1 follows now from Corollary 6.5.1 by means of a standard covering argument.

Remark 6.5.1 All results of this section can be generalized to the case where the right-hand side of equation (6.1) has the form

$$\sum_{k=1}^{k_f} f_k - \sum_{k=1}^{k_g} D_i g_k^i,$$

with $f_k \in L_{r_k}([0,T]; L_{q_k}(\Omega))$, $k = 1, \ldots, k_f$, $\sum_{i=1}^{N}(g_k^i)^2 \in L_{r^{(k)}}([0,T]; L_{q^{(k)}}(\Omega))$, $k = 1, \ldots, k_g$, and all pairs of exponents (r_k, q_k) and $(r^{(k)}, q^{(k)})$, respectively, are subject to the condition in (H3). This follows from Remark 6.4.1 and by straightforward modifications of the proofs given above.

6.6 Regularity up to $t = 0$

The objective of this and the following section is to find conditions on the data which ensure Hölder continuity up to the parabolic boundary. Here we do not aim at great generality but at results which are sufficient for the quasilinear problem to be studied in Chapter 7. Since there we will work with the setting of maximal L_p-regularity, it is natural (and also not so difficult) to use corresponding regularity results to achieve the goal.

We first discuss regularity up to $t = 0$.

Theorem 6.6.1 *Let $\alpha \in (0,1)$, $T > 0$ and Ω be a bounded domain in $\mathbb{R}^N$. Let the assumptions (H1)-(H3) be satisfied. Let further $\Omega' \subset \Omega$ be an arbitrary subdomain and assume that*

$$u_0|_{\tilde{\Omega}} \in B_{pp}^{2-\frac{2}{p\alpha}}(\tilde{\Omega}) \quad \textit{with} \quad p > \frac{1}{\alpha} + \frac{N}{2},$$

for some C^2-smooth domain $\tilde{\Omega}$ such that $\Omega' \subset \tilde{\Omega} \subset \Omega$ and Ω' is separated from $\partial\tilde{\Omega}$ by a positive distance d. Assume in addition that $p \geq 2$ if $N = 1$. Then, for any bounded weak solution u of (6.1) in Ω_T, there holds

$$[u]_{C^{\frac{\alpha\epsilon}{2},\epsilon}([0,T]\times\overline{\Omega'})} \leq C\Big(|u|_{L_\infty(\Omega_T)} + |u_0|_{L_\infty(\Omega)} + |u_0|_{B_{pp}^{2-\frac{2}{p\alpha}}(\tilde{\Omega})} + |f|_{L_r([0,T];L_q(\Omega))}\Big) \tag{6.82}$$

with positive constants $\epsilon = \epsilon(\Lambda, \nu, \alpha, p, r, q, N, diam\,\Omega)$ and $C = C(\Lambda, \nu, \alpha, p, r, q, N, d, diam\,\Omega, T, \lambda_N(\Omega'))$.

Proof. The basic idea of the proof is to extend u to $[-1,T]\times\Omega$ such that u is Hölder continuous on $[-1,0]\times\overline{\Omega'}$ and to apply Theorem 6.1.1.

To this purpose we first extend $u_0|_{\tilde\Omega}\in B_{pp}^{2-\frac{2}{p\alpha}}(\tilde\Omega)$ to a function $\hat u_0\in B_{pp}^{2-\frac{2}{p\alpha}}(\mathbb{R}^N)$. By [82, Theorem 3.1], the problem

$$\begin{aligned}\partial_t^\alpha\big(w-\hat u_0\big)-\Delta w=0,\ \ t\in(0,1),\, x\in\mathbb{R}^N,\\ w|_{t=0}=\hat u_0,\ \ x\in\mathbb{R}^N,\end{aligned}$$

possesses a unique solution w in the class

$$Z:=H_p^\alpha([0,1];L_p(\mathbb{R}^N))\cap L_p([0,1];H_p^2(\mathbb{R}^N)),$$

and one has an estimate of the form

$$|w|_Z\le C_0|\hat u_0|_{B_{pp}^{2-\frac{2}{p\alpha}}(\mathbb{R}^N)}\le \tilde C_0|u_0|_{B_{pp}^{2-\frac{2}{p\alpha}}(\tilde\Omega)}.$$

Note that by the mixed derivative theorem (cf. [71]),

$$Z\hookrightarrow H_p^{\alpha(1-\varsigma)}([0,1];H_p^{2\varsigma}(\mathbb{R}^N)),\quad \varsigma\in[0,1],$$

and thus $Z\hookrightarrow BUC^\delta([0,1]\times\mathbb{R}^N)$ for some sufficiently small $\delta\in(0,\alpha/2)$. In fact, the assumption $p>\frac{1}{\alpha}+\frac{N}{2}$ ensures existence of some $\varsigma\in(0,1)$ with $\alpha(1-\varsigma)-\frac{1}{p}>\delta$ and $2\varsigma-\frac{N}{p}>\delta$.

Multiplying w by a suitable smooth cut-off function $\varphi(t)$ we can construct a function $\hat w\in Z$ with $\hat w|_{t=0}=\hat u_0$ and $\hat w|_{t=1}=0$. We then extend u to $[-1,T]\times\Omega$ by setting $u(t,x)=\hat w(-t,x)$ for $t\in[-1,0)$ and $x\in\Omega$.

Next, we shift the time by setting $\tau=t+1$. Put $\hat u(\tau,x)=u(\tau-1,x)$, $\tau\in(0,T+1)$, $x\in\tilde\Omega$. Define further

$$g:=\partial_\tau^\alpha\hat u-\Delta\hat u,\quad \tau\in(0,1),\, x\in\tilde\Omega.$$

Then $g\in L_p([0,1]\times\tilde\Omega)$, since $\hat u|_{\tau\in(0,1)}\in H_p^\alpha([0,1];L_p(\tilde\Omega))\cap L_p([0,1];H_p^2(\tilde\Omega))$ and $\hat u|_{\tau=0}=0$. Furthermore we have for any test function $\eta\in\mathring{H}_2^{1,1}([0,T+1]\times\tilde\Omega)$,

$$\int_0^1\int_{\tilde\Omega}\Big(-\eta_\tau(g_{1-\alpha}*\hat u)+D_j\hat uD_i\eta\Big)\,dx\,d\tau=\int_0^1\int_{\tilde\Omega}g\eta\,dx\,d\tau-\int_{\tilde\Omega}\eta(g_{1-\alpha}*\hat u)\,dx\Big|_{\tau=1}.\tag{6.83}$$

On the other hand, we have for a.a. $(\tau,x)\in(1,T+1)\times\tilde\Omega$,

$$\begin{aligned}(g_{1-\alpha}*\hat u)(\tau,x)&=(g_{1-\alpha}*u)(\tau-1,x)+\int_0^1 g_{1-\alpha}(\tau-\sigma)\hat u(\sigma,x)\,d\sigma\\&=\big(g_{1-\alpha}*(u-u_0)\big)(\tau-1,x)+g_{2-\alpha}(\tau)u_0(x)\\&\quad+\int_0^1 g_{1-\alpha}(\tau-\sigma)\big(\hat u(\sigma,x)-u_0(x)\big)\,d\sigma.\end{aligned}$$

Set

$$h(\tau,x)=g_{1-\alpha}(\tau)u_0(x)+\int_0^1\dot g_{1-\alpha}(\tau-\sigma)\big(\hat u(\sigma,x)-u_0(x)\big)\,d\sigma=:h_1(\tau,x)+h_2(\tau,x),$$

$\hat a_{ij}(\tau,x)=a_{ij}(\tau-1,x)$, and $\hat f(\tau,x)=f(\tau-1,x)$ for $(\tau,x)\in(1,T+1)\times\tilde\Omega$. Since u is a weak solution of (6.1) in Ω_T, we thus obtain after a short computation that for any $\eta\in$

$\mathring{H}_2^{1,1}([0,T+1]\times\tilde{\Omega})$ with $\eta|_{\tau=T+1}=0$

$$\int_1^{T+1}\int_{\tilde{\Omega}}\Big(-\eta_\tau(g_{1-\alpha}*\hat{u})+\hat{a}_{ij}D_j\hat{u}D_i\eta\Big)\,dx\,d\tau = \int_1^{T+1}\int_{\tilde{\Omega}}(\hat{f}+h)\eta\,dx\,d\tau+\int_{\tilde{\Omega}}\eta(g_{1-\alpha}*\hat{u})\,dx\Big|_{\tau=1}. \tag{6.84}$$

Adding (6.83) and (6.84) shows that $\hat{u}$ is a weak solution of

$$\partial_\tau^\alpha\hat{u}-D_i(b_{ij}D_j\hat{u})=\tilde{f},\ \tau\in(0,T+1),\ x\in\tilde{\Omega},$$

where

$$b_{ij}(\tau,x)=\chi_{[0,1]}(\tau)+\chi_{(1,T+1]}(\tau)\hat{a}_{ij}(\tau,x)$$

and

$$\tilde{f}(\tau,x)=\chi_{[0,1]}(\tau)g(\tau,x)+\chi_{(1,T+1]}(\tau)\big(\hat{f}+h\big)(\tau,x).$$

Evidently, $\chi_{[0,1]}(\tau)g\in L_p([0,T+1]\times\tilde{\Omega})$ and $\chi_{(1,T+1]}(\tau)\hat{f}\in L_r([0,T+1];L_q(\tilde{\Omega}))$. Concerning the h-term we clearly have $\chi_{(1,T+1]}(\tau)h_1\in L_\infty([0,T+1]\times\tilde{\Omega})$. To estimate $\chi_{(1,T+1]}(\tau)h_2$, we employ the Hölder estimate

$$|\hat{u}(\sigma,x)-u_0(x)|=|\hat{u}(\sigma,x)-\hat{u}(1,x)|\le C_1(1-\sigma)^\delta,\quad \sigma\in[0,1],\ x\in\tilde{\Omega},$$

which results from the embedding $Z\hookrightarrow BUC^\delta([0,1]\times\mathbb{R}^N)$ and the construction of $\hat{u}$. It follows that for $1<\tau=t+1\le 1+T$ and $x\in\tilde{\Omega}$

$$\begin{aligned}|h_2(\tau,x)|&\le C_1\int_0^1[-\dot{g}_{1-\alpha}(\tau-\sigma)](1-\sigma)^\delta\,d\sigma\\ &=\frac{\alpha C_1}{\Gamma(1-\alpha)}\int_0^1(t+\sigma)^{-1-\alpha}\sigma^\delta\,d\sigma.\end{aligned}$$

Assuming that $t=\tau-1\in(0,1)$ we then have

$$\begin{aligned}|h_2(\tau,x)|\le&\ \frac{\alpha C_1}{\Gamma(1-\alpha)}\Big(\int_0^t(t+\sigma)^{-1-\alpha}\sigma^\delta\,d\sigma+\int_t^1(t+\sigma)^{-1-\alpha}\sigma^\delta\,d\sigma\Big)\\ \le&\ \frac{\alpha C_1}{\Gamma(1-\alpha)}\Big(\int_0^t(t+\sigma)^{-1-\alpha}t^\delta\,d\sigma+\int_t^1\sigma^{-1-\alpha+\delta}\,d\sigma\Big)\\ \le&\ \frac{\alpha C_1}{\Gamma(1-\alpha)}\,t^{-\alpha+\delta}\Big(\frac{1}{\alpha}+\frac{1}{\alpha-\delta}\Big)\\ \le&\ 3C_1(\tau-1)^\delta g_{1-\alpha}(\tau-1).\end{aligned}$$

This shows that $\chi_{(1,T+1]}(\tau)h_2\in L_{r_0}([0,T+1];L_\infty(\tilde{\Omega}))$ for all $1\le r_0<\frac{1}{\alpha-\delta}$. In particular we find some $\hat{r}>\frac{1}{\alpha}$ such that $\chi_{(1,T+1]}(\tau)h_2\in L_{\hat{r}}([0,T+1];L_\infty(\tilde{\Omega}))$.

All in all we see that $\tilde{f}$ is of the form $\tilde{f}=\sum_{i=1}^4\tilde{f}_i$, where $\tilde{f}_i\in L_{r_i}([0,T+1];L_{q_i}(\tilde{\Omega}))$ with

$$\frac{1}{\alpha r_i}+\frac{N}{2q_i}<1,\quad i=1,2,3,4.$$

Hence Theorem 6.1.1 and Remark 6.5.1 imply that $\hat{u}$ is Hölder continuous in $[1/2,T+1]\times\overline{\Omega'}$. This in turn yields Hölder continuity of u in $[0,T]\times\overline{\Omega'}$, and it is not difficult to see that u is subject to the estimate (6.82). $\square$

Remark 6.6.1 It follows from Remark 6.5.1 and the proof above, that Theorem 6.6.1 can be generalized to the case where the right-hand side of equation (6.1) has the form

$$\sum_{k=1}^{k_f} f_k - \sum_{k=1}^{k_g} D_i g_k^i,$$

with f_k and g_k^i as in Remark 6.5.1.

6.7 Regularity up to the parabolic boundary

The following result gives conditions on the data which are sufficient for Hölder continuity on $[0,T]\times\overline{\Omega}$.

Theorem 6.7.1 *Let $\alpha\in(0,1)$, $T>0$, $N\ge 2$, and $\Omega\subset\mathbb{R}^N$ be a bounded domain with C^2-smooth boundary Γ. Let the assumptions (H1)-(H3) be satisfied. Suppose further that*

$$u_0\in B_{pp}^{2-\frac{2}{p\alpha}}(\Omega),\quad g\in Y_D:=B_{pp}^{\alpha(1-\frac{1}{2p})}([0,T];L_p(\Gamma))\cap L_p([0,T];B_{pp}^{2-\frac{1}{p}}(\Gamma))$$

with $p>\frac{1}{\alpha}+\frac{N}{2}$, and that the compatibility condition

$$u_0=g|_{t=0}\quad on\ \ \Gamma$$

is satisfied. Then for any bounded weak solution u of (6.1) in Ω_T such that $u=g$ a.e. on $(0,T)\times\Gamma$, there holds

$$[u]_{C^{\frac{\alpha\epsilon}{2},\epsilon}([0,T]\times\overline{\Omega})}\le C\Big(|u|_{L_\infty(\Omega_T)}+|u_0|_{B_{pp}^{2-\frac{2}{p\alpha}}(\Omega)}+|f|_{L_r([0,T];L_q(\Omega))}+|g|_{Y_D}\Big) \tag{6.85}$$

with positive constants $\epsilon=\epsilon(\Lambda,\nu,\alpha,p,r,q,N,\Omega)$ and $C=C(\Lambda,\nu,\alpha,p,r,q,N,\Omega,T)$.

Proof. By Theorem 2.8.1, the problem

$$\begin{aligned}\partial_t^\alpha(v-u_0)-\Delta v&=0,\ t\in(0,T),\ x\in\Omega\\ v&=g,\ t\in(0,T),\ x\in\Gamma,\\ v|_{t=0}&=u_0,\ x\in\Omega,\end{aligned}$$

admits a unique strong solution v in the class

$$v\in Z:=H_p^\alpha([0,T];L_p(\Omega))\cap L_p([0,T];H_p^2(\Omega))$$

and

$$|v|_Z\le C_0(|u_0|_{B_{pp}^{2-\frac{2}{p\alpha}}(\Omega)}+|g|_{Y_D}),$$

where C_0 only depends on α,p,N,T,Ω. As in the proof of Theorem 6.6.1 we see that $v\in C^\delta([0,T]\times\overline{\Omega})$ for some $\delta>0$. Furthermore, the mixed derivative theorem implies that

$$D_iv\in H_p^{\frac{\alpha}{2}}([0,T];L_p(\Omega))\cap L_p([0,T];H_p^1(\Omega))\hookrightarrow H_p^{\frac{\alpha\varsigma}{2}}([0,T];H_p^{1-\varsigma}(\Omega))$$

for all $\varsigma\in[0,1]$. Without restriction of generality we may assume that $p\in(\frac{1}{\alpha}+\frac{N}{2},\frac{2}{\alpha}+N)$. With

$$\tilde{p}:=\frac{\frac{1}{\alpha}+\frac{N}{2}}{\frac{2}{\alpha p}+\frac{N}{p}-1}>\frac{1}{\alpha}+\frac{N}{2}$$

and $\varsigma := \frac{2}{\alpha p} - \frac{1}{\alpha \tilde{p}} \in (0,1)$ we then have $H_p^{\frac{\alpha\varsigma}{2}}([0,T]; H_p^{1-\varsigma}(\Omega)) \hookrightarrow L_{2\tilde{p}}(\Omega_T)$, which shows that $|D_i v|^2 \in L_{\tilde{p}}(\Omega_T)$ with $\frac{1}{\alpha\tilde{p}} + \frac{N}{2\tilde{p}} < 1$.

Setting $w = u - v$, w is a bounded weak solution of

$$\partial_t^\alpha w - D_i(a_{ij} D_j w) = f + D_i(a_{ij} D_j v) - \Delta v,\ t \in (0,T),\, x \in \Omega,$$

and $w = 0$ a.e. on $(0,T) \times \Gamma$.

Next, let Ω_0 be an arbitrary bounded domain containing Ω such that $\mathrm{dist}(\Omega, \partial\Omega_0) > 0$. We extend w, f, a_{ij} and $\varphi_i := D_i v$ to $[0,T] \times \Omega_0$ by setting $w, f, \varphi_i = 0$ and $a_{ij} = \delta_{ij}$ on $[0,T] \times (\Omega_0 \setminus \Omega)$. Then w solves

$$\partial_t^\alpha w - D_i(a_{ij} D_j w) = f + D_i(a_{ij}\varphi_j - \varphi_i),\ t \in (0,T),\, x \in \Omega_0,$$

in the weak sense, and thus Theorem 6.6.1 and Remark 6.6.1 imply that w is Hölder continuous on $[0,T] \times \overline{\Omega}$. Since $u = v + w$, the assertion of Theorem 6.7.1 follows. □

Chapter 7

Global solvability of a quasilinear problem

7.1 The global solvability theorem

In this chapter we want to show how the Hölder estimates derived in the previous chapter can be used to prove global solvability of certain quasilinear problems.

Let $\alpha \in (0,1)$, $T>0$, $N \geq 2$, and $\Omega \subset \mathbb{R}^N$ be a bounded domain with C^2-smooth boundary Γ. We consider the quasilinear problem

$$\begin{aligned} \partial_t^\alpha (u-u_0) - D_i(a_{ij}(u)D_j u) = f,\ t\in(0,T),\ x\in\Omega,\\ u = g,\ t\in(0,T),\ x\in\Gamma,\\ u|_{t=0} = u_0,\ x\in\Omega, \end{aligned} \tag{7.1}$$

where we use again the sum convention. Letting $p > N + \frac{2}{\alpha}$ we will assume that

(Q1) $f \in X^T := L_p([0,T];L_p(\Omega))$, $g \in Y_D^T := B_{pp}^{\alpha(1-\frac{1}{2p})}([0,T];L_p(\Gamma)) \cap L_p([0,T];B_{pp}^{2-\frac{1}{p}}(\Gamma))$, $u_0 \in Y_\gamma := B_{pp}^{2-\frac{2}{p\alpha}}(\Omega)$, and $u_0 = g|_{t=0}$ on Γ;

(Q2) $A = (a_{ij})_{i,j=1,\dots,N} \in C^1(\mathbb{R};\mathrm{Sym}\{N\})$, and there exists $\nu > 0$ such that $a_{ij}(y)\xi_i\xi_j \geq \nu|\xi|^2$ for all $y \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$.

The main result concerning the problem (7.1) reads as follows.

Theorem 7.1.1 *Let $T>0$ be an arbitrary number, $p > N+\frac{2}{\alpha}$, and suppose that the assumptions (Q1) and (Q2) are satisfied. Then the problem (7.1) possesses a unique strong solution u in the class*

$$u \in Z^T := H_p^\alpha([0,T];L_p(\Omega)) \cap L_p([0,T];H_p^2(\Omega)).$$

7.2 Proof of the theorem

The proof of theorem 7.1.1 is divided into three parts, devoted respectively to local well-posedness, existence of a maximally defined solution, and to a priori estimates which lead to global existence.

1. Local well-posedness. Short-time existence and uniqueness in the regularity class Z^δ can be established by means of the contraction mapping principle and maximal L_p-regularity for an appropriate linearized problem. We proceed similarly as in [82], see also [12] and [64].

We first define a reference function $w \in Z^T$ as the unique solution of the linear problem

$$\begin{aligned} \partial_t^\alpha (w - u_0) - a_{ij}(u_0) D_i D_j w &= f + a'_{ij}(u_0) D_i u_0 \, D_j u_0, \; t \in (0,T), \, x \in \Omega, \\ w &= g, \; t \in (0,T), \, x \in \Gamma, \\ w|_{t=0} &= u_0, \; x \in \Omega, \end{aligned}$$

see Theorem 2.8.1. Note that the condition $p > N + \frac{2}{\alpha}$ ensures the embedding

$$u_0 \in Y_\gamma = B_{pp}^{2-\frac{2}{p\alpha}}(\Omega) \hookrightarrow C^1(\overline{\Omega}),$$

and thus we also have

$$Z^T \hookrightarrow C([0,T]; Y_\gamma) \hookrightarrow C([0,T]; C^1(\overline{\Omega})).$$

For $\delta \in (0,T]$ and $\rho > 0$ let

$$\Sigma(\delta,\rho) = \{ v \in Z^\delta : v|_{t=0} = u_0, \, |v - w|_{Z^\delta} \le \rho \},$$

which is a closed subset of Z^δ. By Theorem 2.8.1, we may define the mapping $F : \Sigma(\delta,\rho) \to Z^\delta$ which assigns to $u \in \Sigma(\delta,\rho)$ the unique solution $v = F(u)$ of the linear problem

$$\begin{aligned} \partial_t^\alpha (v - u_0) - a_{ij}(u_0) D_i D_j v &= f + h(u, Du, D^2 u), \; t \in (0,\delta), \, x \in \Omega, \\ v &= g, \; t \in (0,\delta), \, x \in \Gamma, \\ v|_{t=0} &= u_0, \; x \in \Omega, \end{aligned} \tag{7.2}$$

where

$$h(u, Du, D^2u) = \big(a_{ij}(u) - a_{ij}(u_0)\big) D_i D_j u + a'_{ij}(u) D_i u \, D_j u.$$

Observe that every fixed point u of F is a local solution of (7.1) and vice versa, at least for some small time interval $[0,\delta]$.

Since $Z^\delta \hookrightarrow C([0,\delta]; C^1(\overline{\Omega}))$ we may set

$$\mu_w(\delta) := \max\{ |w(t,x) - u_0(x)| + |Dw(t,x) - Du_0(x)| : t \in [0,\delta], \, x \in \overline{\Omega} \}.$$

Evidently, $\mu_w(\delta) \to 0$ as $\delta \to 0$, due to $w|_{t=0} = u_0$. Letting $u \in \Sigma(\delta,\rho)$ we then have for all $t \in [0,\delta]$ and $x \in \overline{\Omega}$

$$\begin{aligned} |u(t,x) - u_0(x)| + |Du(t,x) - Du_0(x)| &\le |u - w|_{C([0,\delta];C^1(\overline{\Omega}))} + \mu_w(\delta) \\ &\le M_0 |u - w|_{Z^\delta} + \mu_w(\delta) \le M_0 \rho + \mu_w(\delta), \end{aligned} \tag{7.3}$$

where the embedding constant $M_0 > 0$ does not depend on u and $\delta \in (0,T]$; the latter is true since $u - w$ belongs to the space ${}_0Z^\delta := \{ \varphi \in Z^\delta : \varphi|_{t=0} = 0 \}$. (7.3) yields for any $u \in \Sigma(\delta,\rho)$ the bound

$$|u(t,x) - u_0(x)| + |Du(t,x) - Du_0(x)| \le M_0 \rho_0 + \mu_w(T), \quad t \in [0,\delta], \, x \in \overline{\Omega}, \tag{7.4}$$

where we assume $\rho \in (0,\rho_0]$.

Let now $u \in \Sigma(\delta,\rho)$ and $v = F(u)$. Then $v - w \in {}_0Z^\delta$ solves the problem

$$\begin{aligned} \partial_t^\alpha (v - w) - a_{ij}(u_0) D_i D_j (v - w) &= h(u, Du, D^2u) - a'_{ij}(u_0) D_i u_0 \, D_j u_0, \; t \in (0,\delta), \, x \in \Omega, \\ v - w &= 0, \; t \in (0,\delta), \, x \in \Gamma, \\ (v - w)|_{t=0} &= 0, \; x \in \Omega. \end{aligned}$$

Consequently, it follows from Theorem 2.8.1 that for some constant $M_1 > 0$ which is independent of $\delta \in (0, T]$ there holds

$$\begin{aligned} |v - w|_{Z^\delta} &\leq M_1 |h(u, Du, D^2u) - a'_{ij}(u_0) D_i u_0 \, D_j u_0|_{X^\delta} \\ &\leq M_1 \big| \big(a_{ij}(u) - a_{ij}(u_0)\big) D_i D_j u|_{X^\delta} + M_1 |a'_{ij}(u) D_i u \, D_j u - a'_{ij}(u_0) D_i u_0 \, D_j u_0|_{X^\delta}. \end{aligned}$$

Using (7.3) and (7.4) we may estimate the first term as follows.

$$\begin{aligned} |\big(a_{ij}(u) - a_{ij}(u_0)\big) D_i D_j u|_{X^\delta} &\leq \big(|A(u) - A(w)|_{(L_\infty)^{N^2}} + |A(w) - A(u_0)|_{(L_\infty)^{N^2}}\big) \\ &\quad \times \big(|D^2 u - D^2 w|_{(X^\delta)^{N^2}} + |D^2 w|_{(X^\delta)^{N^2}}\big) \\ &\leq M_2 \big(\rho + \mu_w(\delta)\big)\big(\rho + |D^2 w|_{(X^\delta)^{N^2}}\big), \end{aligned}$$

where $M_2 > 0$ does not depend on δ and ρ. Similarly we obtain

$$|a'_{ij}(u) D_i u \, D_j u - a'_{ij}(u_0) D_i u_0 \, D_j u_0|_{X^\delta} \leq M_3 \big(\rho + \mu_w(\delta)\big)\big(\rho + \delta^{\frac{1}{p}}\big),$$

with $M_3 > 0$ being independent of δ and ρ; here the factor $\delta^{\frac{1}{p}}$ comes from the estimate $|z|_{X^\delta} \leq \big(\lambda_N(\Omega)\delta\big)^{1/p} |z|_\infty$. We conclude that

$$|v - w|_{Z^\delta} \leq M \big((\rho + \mu(\delta)\big)^2, \tag{7.5}$$

where M and $\mu(\delta)$ are constants, which do not depend on ρ, M is independent of δ, and $\mu(\delta)$ is non-decreasing with $\mu(\delta) \to 0$ as $\delta \to 0$.

Next let $u_i \in \Sigma(\delta, \rho)$ and $v_i = F(u_i)$, $i = 1, 2$. Then $v_1 - v_2 \in {}_0Z^\delta$ solves the problem

$$\begin{aligned} \partial_t^\alpha (v_1 - v_2) - a_{ij}(u_0) D_i D_j (v - w) &= h(u_1, Du_1, D^2 u_1) - h(u_2, Du_2, D^2 u_2), \;\; t \in (0, \delta), \, x \in \Omega, \\ v_1 - v_2 &= 0, \; t \in (0, \delta), \, x \in \Gamma, \\ (v_1 - v_2)|_{t=0} &= 0, \; x \in \Omega, \end{aligned}$$

hence

$$|v_1 - v_2|_{Z^\delta} \leq M_1 |h(u_1, Du_1, D^2 u_1) - h(u_2, Du_2, D^2 u_2)|_{X^\delta}.$$

Estimating similarly as above we obtain

$$|v_1 - v_2|_{Z^\delta} \leq M \big((\rho + \mu(\delta)\big) |u_1 - u_2|_{Z^\delta}, \tag{7.6}$$

where M and $\mu(\delta)$ are like those in (7.5).

Finally, the estimates (7.5) and (7.6) show that for sufficiently small ρ and δ the mapping F is a strict contraction which leaves the set $\Sigma(\delta, \rho)$ invariant. Local existence and uniqueness of strong solutions to (7.1) follows now by the contraction mapping principle.

2. The maximally defined solution. The local solution $u \in Z^\delta$ obtained in the first part can be continued to some larger interval $[0, \delta + \delta_1] \subset [0, T]$. In fact, let $u_\delta := u|_{t=\delta} \in Y_\gamma$ and define the set

$$\Sigma(\delta, \delta_1, \rho) := \{v \in Z^{\delta + \delta_1} : v|_{[0,\delta]} = u, \, |v - w|_{Z^{\delta+\delta_1}} \leq \rho\},$$

where the reference function $w \in Z^T$ is now defined as the solution of the linear problem

$$\begin{aligned} \partial_t^\alpha (w - u_0) - a_{ij}(u_\delta) D_i D_j w &= f + h_1 + \chi_{(\delta, T]}(t) a'_{ij}(u_\delta) D_i u_\delta \, D_j u_\delta, \;\; t \in (0, T), \, x \in \Omega, \\ w &= g, \; t \in (0, T), \, x \in \Gamma, \\ w|_{t=0} &= u_0, \; x \in \Omega, \end{aligned}$$

with

$$h_1 = \chi_{[0,\delta]}(t)\Big(\big(a_{ij}(u) - a_{ij}(u_\delta)\big)D_iD_ju + a'_{ij}(u)D_iu\,D_ju\Big).$$

Observe that $w|_{[0,\delta]} = u$, by uniqueness. So $\Sigma(\delta, \delta_1, \rho)$ is not empty and it becomes a complete metric space when endowed with the metric induced by the norm of $Z^{\delta+\delta_1}$.

Define next the mapping $F : \Sigma(\delta, \delta_1, \rho) \to Z^{\delta+\delta_1}$ which assigns to $\tilde{u} \in \Sigma(\delta, \delta_1, \rho)$ the solution $v = F(\tilde{u})$ of the linear problem

$$\begin{aligned} \partial_t^\alpha(v - u_0) - a_{ij}(u_\delta)D_iD_jv = f + \tilde{h}(\tilde{u}, D\tilde{u}, D^2\tilde{u}),\ t \in (0, \delta+\delta_1),\ x \in \Omega, \\ w = g,\ t \in (0, \delta+\delta_1),\ x \in \Gamma, \\ w|_{t=0} = u_0,\ x \in \Omega, \end{aligned}$$

where

$$\tilde{h}(\tilde{u}, D\tilde{u}, D^2\tilde{u}) = \big(a_{ij}(\tilde{u}) - a_{ij}(u_\delta)\big)D_iD_j\tilde{u} + a'_{ij}(\tilde{u})D_i\tilde{u}\,D_j\tilde{u}.$$

Since $\tilde{u}|_{[0,\delta]} = u$ we have also $v|_{[0,\delta]} = u$, by uniqueness.

Observe that $h_1 = \tilde{h}(\tilde{u}, D\tilde{u}, D^2\tilde{u})$ on $[0, \delta]$ and thus

$$|v - w|_{Z^{\delta+\delta_1}} \le M_1|\tilde{h}(\tilde{u}, D\tilde{u}, D^2\tilde{u}) - a'_{ij}(u_\delta)D_iu_\delta\,D_ju_\delta|_{L_p([\delta,\delta+\delta_1]\times\Omega)}.$$

Further,

$$|F(\tilde{u}_1) - F(\tilde{u}_2)|_{Z^{\delta+\delta_1}} \le M_1|\tilde{h}(\tilde{u}_1, D\tilde{u}_1, D^2\tilde{u}_1) - \tilde{h}(\tilde{u}_2, D\tilde{u}_2, D^2\tilde{u}_2)|_{L_p([\delta,\delta+\delta_1]\times\Omega)},$$

for $\tilde{u}_1, \tilde{u}_2 \in \Sigma(\delta, \delta_1, \rho)$. Therefore we may estimate analogously to the first step to see that for sufficiently small δ_1 and ρ we have $F(\Sigma(\delta, \delta_1, \rho)) \subset \Sigma(\delta, \delta_1, \rho)$ and F is a strict contraction. Hence the contraction mapping principle yields existence of a unique fixed point of F in $\Sigma(\delta, \delta_1, \rho)$, which is the unique solution of (7.1) on $[0, \delta + \delta_1]$.

Repeating this argument we obtain a maximal interval of existence $[0, T_{max})$ with $T_{max} \le T$, that is T_{max} is the supremum of all $\tau \in (0, T)$ such that the problem (7.1) has a unique solution $u \in Z^\tau$.

3. A priori bounds and global well-posedness. In order to establish global existence we will show that $|u|_{Z^\tau}$ stays bounded as $\tau \nearrow T_{max}$.

Let $\tau \in (0, T_{max})$ and $u \in Z^\tau$ be the unique solution of (7.1). Setting $b_{ij}(t, x) = a_{ij}(u(t, x))$, it is evident that u is a weak solution of

$$\partial_t^\alpha(u - u_0) - D_i(b_{ij}D_ju) = f,\ t \in (0, \tau),\ x \in \Omega.$$

Since $Y_\gamma \hookrightarrow C(\overline{\Omega})$ and $Y_D^\tau \hookrightarrow C([0, \tau] \times \Gamma)$, Corollary 4.2.1 implies a uniform sup-bound for $|u|$, namely

$$|u(t, x)| \le C_1, \quad t \in [0, \tau],\ x \in \overline{\Omega},$$

where the constant C_1 depends only on the data $|f|_{X^\tau}, |g|_\infty, |u_0|_\infty, \Omega, T, \alpha, N$, and ν, not on τ. It follows then from Theorem 6.7.1 that for some $\varepsilon > 0$ we have

$$|u|_{C^\varepsilon([0,\tau]\times\overline{\Omega})} \le C_2,$$

where the number $C_2 \ge 1$ depends only on $|f|_{X^T}, |g|_{Y_D^T}, |u_0|_{Y_\gamma}, \Omega, T, \alpha, N$, and ν, not on τ. In particular, we obtain a uniform Hölder estimate for the coefficients b_{ij}, $i, j = 1, \dots, N$.

The first equation of (7.1) can be rewritten as

$$\partial_t^\alpha(u - u_0) - b_{ij}D_iD_ju = f + a'_{ij}(u)D_iu\,D_ju.$$

By Theorem 2.8.1, the linear problem

$$\begin{aligned}\partial_t^\alpha(v-u_0)-b_{ij}D_iD_jv=f,\ \ t\in(0,\tau),\ x\in\Omega,\\ v=g,\ t\in(0,\tau),\ x\in\Gamma,\\ v|_{t=0}=u_0,\ x\in\Omega,\end{aligned}$$

has a unique solution $v\in Z^\tau$ and there exists a constant $M_1>0$ independent of τ such that

$$\begin{aligned}|u-v|_{Z^\tau}&\le M_1|a_{ij}'(u)D_iu\,D_ju|_{X^\tau}\\ &\le M_1\sum_{i,j=1}^N\max_{|y|\le C_1}|a_{ij}'(y)|\,\big||Du|^2\big|_{X^\tau}.\end{aligned}\tag{7.7}$$

The assumption on p implies $p>\frac{N}{2}$ and thus

$$H_p^2(\Omega)\hookrightarrow H_{2p}^1(\Omega)\hookrightarrow C^{\varepsilon_0}(\overline{\Omega})$$

for some $\varepsilon_0\in(0,\varepsilon]$. By the Gagliardo-Nirenberg inequality, there exists then $\theta\in(0,\frac{1}{2})$ such that

$$|Du(t,\cdot)|_{L_{2p}(\Omega;\mathbb{R}^N)}\le C|u(t,\cdot)|_{H_p^2(\Omega)}^\theta|u(t,\cdot)|_{C^\varepsilon(\overline{\Omega})}^{1-\theta}\le CC_2|u(t,\cdot)|_{H_p^2(\Omega)}^\theta,\quad t\in[0,\tau],$$

and hence by Hölder's and Young's inequality

$$\begin{aligned}\big||Du|^2\big|_{X^\tau}&\le\tilde{C}|Du|_{L_{2p}([0,\tau]\times\Omega;\mathbb{R}^N)}^2\le C_3|u|_{L_p([0,\tau];H_p^2(\Omega))}^{2\theta}\tau^{\frac{1-2\theta}{p}}\\ &\le C_4|u|_{Z^\tau}^{2\theta}\le\varepsilon_1|u|_{Z^\tau}+C_5(\varepsilon_1,\theta,C_4),\end{aligned}$$

for all $\varepsilon_1>0$. This together with (7.7) yields a bound for $|u-v|_{Z^\tau}$ which is uniform w.r.t. τ. Since $|v|_{Z^\tau}$ stays bounded as $\tau\nearrow T_{max}$, it follows that $|u|_{Z^\tau}$ enjoys the same property. Hence we have global existence. □

Chapter 8

The Harnack inequality for the fractional derivation operator

8.1 Harnack inequalities and the main result

Harnack inequalities have been proved to be an important tool in the theory of linear and nonlinear partial differential equations. We refer to the recent survey [42] for an introduction into this subject. A variant of the classical Harnack inequality for the Laplace operator can be stated as follows. Denote by $B_\rho(y)$ the open ball in $\mathbb{R}^n$ with radius $\rho > 0$ and center $y \in \mathbb{R}^n$. Suppose that u is a nonnegative harmonic function in $B_{4\rho}(y)$. Then

$$\sup_{B_\rho(y)} u \leq 3^n \inf_{B_\rho(y)} u,$$

see e.g. [29, Section 2.3]. The classical parabolic Harnack inequality (i.e. for the heat operator) is due to Hadamard [33] and Pini [61]. The following version was introduced by Moser [57] in a more general context, see also [21]. Letting $\rho > 0$, $\sigma \in (0,1)$, and $y \in \mathbb{R}^n$ we define the boxes

$$Q_- = (-\rho^2, -\sigma\rho^2) \times B_\rho(y), \quad Q_+ = (\sigma\rho^2, \rho^2) \times B_\rho(y).$$

Then there exists a constant $M > 0$ depending only on n and σ such that for any nonnegative and sufficiently smooth function u in $(-4\rho^2, \rho^2) \times B_{4\rho}(y)$ satisfying

$$\partial_t u - \Delta u = 0 \quad \text{in} \quad (-4\rho^2, \rho^2) \times B_{4\rho}(y),$$

there holds the inequality

$$\sup_{Q_-} u \leq M \inf_{Q_+} u.$$

For more general results on Harnack inequalities in the elliptic and parabolic case we refer to [20], [29], [42] [51], and the references given therein.

Concerning *non-local* operators it is known that the Harnack inequality also holds for fractional powers of the negative Laplacian. Let $\alpha \in (0,1)$ and suppose that u is a sufficiently smooth function on $\mathbb{R}^n$ that is nonnegative *everywhere* and satisfies $(-\Delta)^\alpha u = 0$ in $B_{4\rho}(y)$. Then

$$\sup_{B_\rho(y)} u \leq M \inf_{B_\rho(y)} u,$$

where the constant M depends only on α and n, cf. [5, Theorem 5.1]. We point out that here the Harnack inequality fails, if the global positivity assumption is replaced by a local one, cf.

[40]. This is due to the non-local nature of $(-\Delta)^\alpha$. More general results on Harnack estimates for integro-differential operators like $(-\Delta)^\alpha$ can be found in [2].

The main objective of this chapter is to show that a Harnack inequality also holds for the Riemann-Liouville fractional derivation operator ∂_t^α with $\alpha \in (0,1)$.

To state the main result we need some notation. Given $0 \le t_1 < t_2$ we define the space $Z(t_1, t_2)$ by

$$Z(t_1,t_2) = \{u \in C([0,t_2]) : g_{1-\alpha} * u|_{[t_1,t_2]} \in H_1^1([t_1,t_2])\}.$$

For $t_* \ge 0$, $0 < \sigma_1 < \sigma_2 < \sigma_3$, and $\rho > 0$ we introduce the intervals

$$W_- = (t_* + \sigma_1\rho, t_* + \sigma_2\rho), \quad W_+ = (t_* + \sigma_2\rho, t_* + \sigma_3\rho).$$

Then the main result of this chapter is the following.

Theorem 8.1.1 *Let $t_* \ge 0$, $0 < \sigma_1 < \sigma_2 < \sigma_3$, and $\rho > 0$. Let further $\alpha \in (0,1)$ and $u_0 \ge 0$. Then for any function $u \in Z(t_*, t_* + \sigma_3\rho)$ that is nonnegative on $(0, t_* + \sigma_3\rho)$ and that satisfies*

$$\partial_t^\alpha (u - u_0)(t) = 0, \quad a.a.\ t \in (t_*, t_* + \sigma_3\rho), \tag{8.1}$$

there holds the inequality

$$\sup_{W_-} u \le \frac{\sigma_3}{\sigma_1} \inf_{W_+} u. \tag{8.2}$$

Note that in Theorem 8.1.1 we do not assume that $u(0) = u_0$. So by setting $u_0 = 0$ we obtain the Harnack inequality for the Riemann-Liouville fractional derivative. If we assume in addition that $u(0) = u_0$ then Theorem 8.1.1 yields the Harnack inequality for the so-called Caputo fractional derivation operator, which is a regularized version of the Riemann-Liouville fractional derivative, cf. the monographs [44] and [67].

In Section 8.3 we will show that, similarly to the case of the fractional Laplacian, the Harnack inequality fails if the global positivity assumption is replaced by a local one. Furthermore, we will demonstrate that the above Harnack estimate breaks down if the relation $\partial_t^\alpha(u - u_0) = 0$ is only satisfied on the smaller interval $(t_* + \sigma_1\rho, t_* + \sigma_3\rho)$.

In the last section of this chapter we generalize Theorem 8.1.1 to nonnegative solutions of the fractional differential equation

$$\partial_t^\alpha (u - u_0)(t) + \mu u(t) = f(t), \quad \text{a.a. } t \in (t_*, t_* + \sigma_3\rho), \tag{8.3}$$

where $u_0, \mu \ge 0$ and $f \in L_p([t_*, t_* + \sigma_3\rho])$ for some $p > 1/\alpha$, see Theorem 8.4.1 below.

It is highly desirable to have a Harnack inequality also for nonnegative solutions of time fractional diffusion equations the prototype of which reads

$$\partial_t^\alpha (u - u_0)(t,x) - \Delta u(t,x) = 0, \quad t \in (0,T),\ x \in \Omega, \tag{8.4}$$

where $T > 0$, Ω is a domain in $\mathbb{R}^N$, $\alpha \in (0,1)$, and $u_0 = u_0(x)$ is a given function. This is an open problem, even for the Laplacian. However, the results of this contribution indicate that a Harnack inequality should also hold in this situation. The author believes that the estimates obtained in Chapter 6 are potentially very useful to solve this problem.

Before giving the proof of Theorem 8.1.1, let us remark again (cf. Section 5.1) that a weak Harnack inequality is valid for nonnegative supersolutions of (8.3), see [83]. Adopting the notation of the present note and assuming for simplicity that $f = 0$ and $\mu = 0$ it is shown in [83] that for any function $u \in Z(t_*, t_* + \sigma_3\rho)$ that is nonnegative on $(0, t_* + \sigma_3\rho)$ and that satisfies

$$\partial_t^\alpha (u - u_0)(t) \ge 0, \text{ a.a. } t \in (t_*, t_* + \sigma_3\rho), \quad u(0) = u_0,$$

we have

$$\rho^{-1/p}|u|_{L_p((t_*,t_*+\sigma_1\rho))} \le C \inf_{W_+} u, \tag{8.5}$$

for all $0 < p < \frac{1}{1-\alpha}$, where the constant $C > 0$ depends only on $0 < \sigma_1 < \sigma_2 < \sigma_3$, p, and $\alpha \in (0,1)$. The critical exponent $\frac{1}{1-\alpha}$ is optimal. Notice that on the left of (8.5) we have the interval $(t_*, t_* + \sigma_1\rho)$, not W_- as in (8.2).

8.2 Proof of the Harnack inequality

Suppose $u \in Z(t_*, t_* + \sigma_3\rho)$ is nonnegative on $(0, t_* + \sigma_3\rho)$ and satisfies (8.1). We introduce the shifted time $s = t - t_*$ and define the function $\tilde{u}$ by means of $\tilde{u}(s) = u(s+t_*)$, $s \in (0, \sigma_3\rho)$. Then (8.1) implies that

$$\partial_s^\alpha \tilde{u}(s) = g_{1-\alpha}(t_* + s)u_0 + h(s), \quad s \in (0, \sigma_3\rho), \tag{8.6}$$

where the history term $h(s)$ is given by

$$h(s) = \int_0^{t_*} [-\dot{g}_{1-\alpha}(t_* + s - \tau)]u(\tau)\, d\tau, \quad s \in (0, \sigma_3\rho). \tag{8.7}$$

Here, by $\dot{g}$ we mean the derivative of the function g.

Since $(g_{1-\alpha} * \tilde{u})(0) = 0$ and $g_\alpha * g_{1-\alpha} = 1$, we have

$$g_\alpha * \partial_s^\alpha \tilde{u} = g_\alpha * \partial_s(g_{1-\alpha} * \tilde{u}) = \partial_s(g_\alpha * g_{1-\alpha} * \tilde{u}) = \tilde{u}.$$

Therefore convolving (8.6) with g_α yields

$$\tilde{u}(s) = u_0\big(g_\alpha * g_{1-\alpha}(\cdot + t_*)\big)(s) + (g_\alpha * h)(s), \quad s \in (0, \sigma_3\rho). \tag{8.8}$$

The first term on the right-hand side of (8.8) can be rewritten by the use of the identity

$$\begin{aligned}\big(g_\alpha * g_{1-\alpha}(\cdot + t_*)\big)(s) &= \int_0^s g_\alpha(s-\sigma)g_{1-\alpha}(t_* + \sigma)\, d\sigma \\ &= s\int_0^1 g_\alpha(s - rs)g_{1-\alpha}(t_* + rs)\, dr \\ &= \int_0^1 g_\alpha(1-r)g_{1-\alpha}(r + \frac{t_*}{s})\, dr \\ &=: \varphi(s), \quad s \in (0, \sigma_3\rho).\end{aligned} \tag{8.9}$$

Similarly, we have for the second term

$$\begin{aligned}(g_\alpha * h)(s) &= \int_0^s g_\alpha(s-\sigma)\int_0^{t_*} [-\dot{g}_{1-\alpha}(t_* + \sigma - \tau)]u(\tau)\, d\tau\, d\sigma \\ &= \frac{1}{s}\int_0^1 g_\alpha(1-r)\int_0^{t_*} [-\dot{g}_{1-\alpha}(r + \frac{t_* - \tau}{s})]u(\tau)\, d\tau\, dr \\ &=: \psi(s), \quad s \in (0, \sigma_3\rho).\end{aligned} \tag{8.10}$$

Consequently, (8.8) is equivalent to

$$\tilde{u}(s) = u_0\varphi(s) + \psi(s), \quad s \in (0, \sigma_3\rho).$$

Let now $s \in (\sigma_1\rho, \sigma_2\rho)$ and $\bar{s} \in (\sigma_2\rho, \sigma_3\rho)$. Since $g_{1-\alpha}$ is nonincreasing, we evidently have $\varphi(s) \le \varphi(\bar{s})$. As to ψ, we use the positivity of u on $(0, t_*)$ and the monotonicity of $-\dot{g}_{1-\alpha}$ to estimate as follows.

$$\begin{aligned}\psi(s) &\le \frac{1}{\sigma_1\rho}\int_0^1 g_\alpha(1-r)\int_0^{t_*}[-\dot{g}_{1-\alpha}(r+\frac{t_*-\tau}{\sigma_2\rho})]u(\tau)\,d\tau\,dr \\ &\le \frac{\sigma_3}{\sigma_1\bar{s}}\int_0^1 g_\alpha(1-r)\int_0^{t_*}[-\dot{g}_{1-\alpha}(r+\frac{t_*-\tau}{\bar{s}})]u(\tau)\,d\tau\,dr \\ &= \frac{\sigma_3}{\sigma_1}\,\psi(\bar{s}).\end{aligned}$$

By positivity of u_0, we thus obtain

$$\tilde{u}(s) \le \frac{\sigma_3}{\sigma_1}\,\tilde{u}(\bar{s}),$$

which immediately implies inequality (8.2). This completes the proof of Theorem 8.1.1.

Remark 8.2.1 Note that in case $t_* = 0$ relation (8.1) implies $u(t) = u_0$ for all $t \in [0, \sigma_3\rho]$, thus the Harnack inequality (8.2) trivially holds with the constant $\frac{\sigma_3}{\sigma_1} > 1$ being replaced by 1.

8.3 Counterexamples

Example 8.3.1 We show first that the Harnack inequality fails for nonnegative functions $u \in Z(t_* + \sigma_1\rho, t_* + \sigma_3\rho)$ satisfying the relation $\partial_t^\alpha(u - u_0) = 0$ only on the smaller interval $(t_* + \sigma_1\rho, t_* + \sigma_3\rho)$.

To this purpose fix $W_- = (1, 2)$ and $W_+ = (2, 3)$ and consider the family of functions u_ε, $\varepsilon \in (0, 1]$, defined by

$$u_\varepsilon(t) = \begin{cases} 0 & : 0 \le t \le 1-\varepsilon \\ \frac{1}{\varepsilon}(t - 1 + \varepsilon) & : 1-\varepsilon \le t \le 1, \end{cases} \tag{8.11}$$

and

$$\partial_t^\alpha u_\varepsilon = 0, \quad \text{a.a. } t \in (1, 3). \tag{8.12}$$

Apparently $u_\varepsilon|_{[0,1]} \in H_1^1([0, 1])$ so that (8.12) means that with $s = t - 1$ and $\tilde{u}_\varepsilon(s) = u_\varepsilon(s + 1)$ we have

$$\tilde{u}_\varepsilon(s) = (g_\alpha * h_\varepsilon)(s), \quad s \in (0, 2), \tag{8.13}$$

where

$$h_\varepsilon(s) = \int_0^1 [-\dot{g}_{1-\alpha}(1 + s - \tau)]u_\varepsilon(\tau)\,d\tau, \quad s \in (0, 2).$$

Observe that u_ε is nonnegative on $(0, 3)$ and that $u_\varepsilon \in Z(1, 3)$ for all $\varepsilon \in (0, 1]$. From $u_\varepsilon = 0$ in $[0, 1 - \varepsilon]$ and $u_\varepsilon \le 1$ in $[1 - \varepsilon, 1]$ we infer the estimate

$$h_\varepsilon(s) \le \int_{1-\varepsilon}^1 [-\dot{g}_{1-\alpha}(1 + s - \tau)]\,d\tau = g_{1-\alpha}(s) - g_{1-\alpha}(s + \varepsilon), \quad s \in (0, 2).$$

In view of (8.13) this gives for $s \in (1, 2)$

$$\begin{aligned}\tilde{u}(s) &\le \int_0^s g_\alpha(s-\sigma)[g_{1-\alpha}(\sigma) - g_{1-\alpha}(\sigma+\varepsilon)]\,d\sigma \\ &= \int_0^1 g_\alpha(1-r)[g_{1-\alpha}(r) - g_{1-\alpha}(r + \frac{\varepsilon}{s})]\,dr \\ &\le \int_0^1 g_\alpha(1-r)[g_{1-\alpha}(r) - g_{1-\alpha}(r+\varepsilon)]\,dr =: \delta(\varepsilon).\end{aligned}$$

By the dominated convergence theorem, $\delta(\varepsilon)$ vanishes as $\varepsilon \to 0+$. Hence

$$\lim_{\varepsilon\to 0+} \inf_{W_+} u_\varepsilon = 0.$$

On the other hand we have $u_\varepsilon(1) = \tilde{u}(0) = 1$ for all $\varepsilon \in (0, 1]$, and therefore

$$\sup_{W_-} u_\varepsilon \geq 1, \quad \varepsilon \in (0, 1].$$

This shows that an estimate of the form

$$\sup_{W_-} u \leq M \inf_{W_+} u$$

with M independent of u cannot hold.

Example 8.3.2 We next show that the Harnack inequality fails if the positivity assumptions $u_0 \geq 0$ and $u \geq 0$ in $(0, t_*)$ are dropped.

Fix $t_* > 0$ and consider the family of functions u_ε, $\varepsilon > 0$, defined by

$$u_\varepsilon(t) = \frac{1}{\varepsilon}(t - t_* + \varepsilon), \quad 0 \leq t \leq t_*,$$

and

$$\partial_t^\alpha (u_\varepsilon - u_{0,\varepsilon}) = 0, \quad \text{a.a. } t > t_*, \tag{8.14}$$

where

$$u_{0,\varepsilon} = u_\varepsilon(0) = 1 - \frac{t_*}{\varepsilon}.$$

Observe that u_ε has negative values in $[0, t_*]$ if and only if $\varepsilon \in (0, t_*)$. Setting $s = t - t_*$ and $\tilde{u}_\varepsilon(s) = u_\varepsilon(s + t_*)$, $s \geq 0$, (8.14) is equivalent to

$$\tilde{u}_\varepsilon(s) = u_{0,\varepsilon}\big(g_\alpha * g_{1-\alpha}(\cdot + t_*)\big)(s) + (g_\alpha * h_\varepsilon)(s), \quad s > 0, \tag{8.15}$$

where

$$\begin{aligned} h_\varepsilon(s) &= \int_0^{t_*} [-\dot{g}_{1-\alpha}(t_* + s - \tau)] u_\varepsilon(\tau)\, d\tau \\ &= \Big[g_{1-\alpha}(t_* + s - \tau) u_\varepsilon(\tau)\Big]_{\tau=0}^{\tau=t_*} - \int_0^{t_*} g_{1-\alpha}(t_* + s - \tau)\dot{u}_\varepsilon(\tau)\, d\tau \\ &= g_{1-\alpha}(s) - g_{1-\alpha}(t_* + s) u_{0,\varepsilon} + \frac{1}{\varepsilon}\big(g_{2-\alpha}(s) - g_{2-\alpha}(s + t_*)\big), \quad s > 0. \end{aligned}$$

Inserting the last identity into (8.15) yields

$$\tilde{u}_\varepsilon(s) = 1 + \frac{1}{\varepsilon}\,(g_\alpha * [g_{2-\alpha} - g_{2-\alpha}(\cdot + t_*)])(s), \quad s \geq 0.$$

In particular $\tilde{u}_\varepsilon$ is differentiable in $(0, \infty)$ and we have

$$\begin{aligned} \dot{\tilde{u}}_\varepsilon(s) &= \frac{1}{\varepsilon}\,(g_\alpha * [g_{1-\alpha} - g_{1-\alpha}(\cdot + t_*)])(s) - \frac{1}{\varepsilon}\, g_\alpha(s) g_{2-\alpha}(t_*) \\ &< \frac{1}{\varepsilon}\big(1 - g_\alpha(s) g_{2-\alpha}(t_*)\big), \quad s > 0. \end{aligned}$$

This shows that $\tilde{u}_\varepsilon$ is strictly decreasing in the interval $[0, s_*]$ with

$$s_* = \frac{t_*}{[\Gamma(\alpha)\Gamma(2-\alpha)]^{1/(1-\alpha)}}.$$

Selecting

$$\varepsilon = (g_\alpha * [g_{2-\alpha}(\cdot + t_*) - g_{2-\alpha}])(s_*),$$

we have

$$\tilde{u}_\varepsilon(s_*) = 0 \quad \text{and} \quad \tilde{u}_\varepsilon(s) > 0,\ s \in [0, s_*). \tag{8.16}$$

Note that $\varepsilon < t_*$, for otherwise we would have $u_{0,\varepsilon} \geq 0$ and $u_\varepsilon > 0$ in $(0, t_*]$, which by (8.15), entails strict positivity of $\tilde{u}_\varepsilon$, a contradiction.

Choosing the parameters in such a way that $s_* = t_* + \sigma_3\rho$, (8.16) shows that an estimate of the form

$$\sup_{W_-} u_\varepsilon \leq M \inf_{W_+} u_\varepsilon$$

cannot hold.

8.4 Nonhomogeneous fractional differential equations

In this section we derive a Harnack estimate for nonnegative solutions of the more general equation

$$\partial_t^\alpha (u - u_0)(t) + \mu u(t) = f(t), \quad \text{a.a. } t \in (t_*, t_* + \sigma_3\rho), \tag{8.17}$$

here $\mu \geq 0$ is another parameter and we assume that $f \in L_p([t_*, t_* + \sigma_3\rho])$ for some $p > 1/\alpha$. The other parameters are as before.

Suppose $u \in Z(t_*, t_* + \sigma_3\rho)$ is nonnegative on $(0, t_* + \sigma_3\rho)$ and satisfies (8.17). Setting $s = t - t_*$ and $\tilde{u}(s) = u(s + t_*)$, $\tilde{f}(s) = f(s + t_*)$, $\tilde{g}_{1-\alpha}(s) = g_{1-\alpha}(s + t_*)$, $s \in (0, \sigma_3\rho)$, we infer from (8.17) that

$$\partial_s^\alpha \tilde{u}(s) + \mu\tilde{u}(s) = \tilde{g}_{1-\alpha}(s)u_0 + h(s) + \tilde{f}(s), \quad s \in (0, \sigma_3\rho), \tag{8.18}$$

where $h(s)$ is given by (8.7). Let $r_{\alpha,\mu}$ denote the resolvent kernel corresponding to (8.17), that is

$$r_{\alpha,\mu}(s) + \mu(r_{\alpha,\mu} * g_\alpha)(s) = g_\alpha(s), \quad s > 0.$$

Equation (8.18) then implies

$$\tilde{u}(s) = (r_{\alpha,\mu} * [\tilde{g}_{1-\alpha}u_0 + h + \tilde{f}])(s), \quad s \in (0, \sigma_3\rho). \tag{8.19}$$

Recall that (cf. (5.52))

$$r_{\alpha,\mu}(s) = \Gamma(\alpha)g_\alpha(s)\, E_{\alpha,\alpha}(-\mu s^\alpha), \quad s > 0,$$

where $E_{\alpha,\beta}$ denotes the generalized Mittag-Leffler-function defined in (5.53).

Let now $\omega > 0$ be a fixed constant and assume that

$$\mu\rho^\alpha \leq \omega.$$

By continuity and strict positivity of $E_{\alpha,\alpha}$ in $(-\infty, 0]$ we then have

$$0 < c_1 := \min_{z \in [0, \omega\sigma_3^\alpha]} E_{\alpha,\alpha}(-z) \leq E_{\alpha,\alpha}(-\mu s^\alpha) \leq \max_{z \in [0, \omega\sigma_3^\alpha]} E_{\alpha,\alpha}(-z) =: c_2, \quad s \in (0, \sigma_3\rho).$$

Setting $C_i = C_i(\alpha, \omega, \sigma_3) = c_i \Gamma(\alpha)$, $i = 1, 2$, we thus have

$$C_1 g_\alpha(s) \le r_{\alpha,\,\mu}(s) \le C_2 g_\alpha(s), \quad s \in (0, \sigma_3 \rho). \tag{8.20}$$

Further,

$$\max_{s\in[0,\sigma_3\rho]} (g_\alpha * |\tilde{f}|)(s) \le |g_\alpha|_{L_{p'}([0,\sigma_3\rho])} |\tilde{f}|_{L_p([0,\sigma_3\rho])} = C_3 \rho^{\alpha-\frac{1}{p}} |\tilde{f}|_{L_p([0,\sigma_3\rho])}, \tag{8.21}$$

with

$$C_3 = \frac{\sigma_3^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)[(\alpha-1)p'+1]^{1/p'}}.$$

Using the functions φ and ψ from Section 8.2, we infer from (8.19), (8.20), and (8.21) that

$$\tilde{u}(s) \le C_2 \big(\varphi(s) u_0 + \psi(s) + C_3 \rho^{\alpha-\frac{1}{p}} |\tilde{f}|_{L_p([0,\sigma_3\rho])}\big), \quad s \in (0, \sigma_3\rho), \tag{8.22}$$

as well as

$$\tilde{u}(s) \ge C_1 \big(\varphi(s) u_0 + \psi(s)\big) - C_2 C_3 \rho^{\alpha-\frac{1}{p}} |\tilde{f}|_{L_p([0,\sigma_3\rho])}, \quad s \in (0, \sigma_3\rho). \tag{8.23}$$

Suppose now that $s \in (\sigma_1\rho, \sigma_2\rho)$ and $\bar{s} \in (\sigma_2\rho, \sigma_3\rho)$. Employing (8.22), (8.23), and the estimates for φ and ψ from Section 8.2, we have

$$\begin{aligned}
\tilde{u}(s) &\le C_2 \big(\varphi(\bar{s}) u_0 + \frac{\sigma_3}{\sigma_1}\, \psi(\bar{s}) + C_3 \rho^{\alpha-\frac{1}{p}} |\tilde{f}|_{L_p([0,\sigma_3\rho])}\big) \\
&\le \frac{C_2\sigma_3}{C_1\sigma_1} \big(C_1[\varphi(\bar{s}) u_0 + \psi(s)] - C_2 C_3 \rho^{\alpha-\frac{1}{p}} |\tilde{f}|_{L_p([0,\sigma_3\rho])}\big) \\
&\quad + C_2 C_3 \big(1 + \frac{C_2\sigma_3}{C_1\sigma_1}\big) \rho^{\alpha-\frac{1}{p}} |\tilde{f}|_{L_p([0,\sigma_3\rho])} \\
&\le \frac{C_2\sigma_3}{C_1\sigma_1}\, \tilde{u}(\bar{s}) + C_2 C_3 \big(1 + \frac{C_2\sigma_3}{C_1\sigma_1}\big) \rho^{\alpha-\frac{1}{p}} |\tilde{f}|_{L_p([0,\sigma_3\rho])}.
\end{aligned}$$

We have thus proved the following result.

Theorem 8.4.1 *Let $\omega > 0$ be fixed. Let $t_*, \mu \ge 0$, $0 < \sigma_1 < \sigma_2 < \sigma_3$, and $\rho > 0$. Let further $\alpha \in (0,1)$, $u_0 \ge 0$, and $f \in L_p([t_*, t_*+\sigma_3\rho])$ for some $p > 1/\alpha$. Assume that $\mu\rho^\alpha \le \omega$. Then there exists a positive constant $M = M(\alpha, p, \sigma_1, \sigma_3, \omega)$ such that for any function $u \in Z(t_*, t_* + \sigma_3\rho)$ that is nonnegative on $(0, t_* + \sigma_3\rho)$ and that satisfies (8.17) there holds the inequality*

$$\sup_{W_-} u \le M \Big(\inf_{W_+} u + \rho^{\alpha-\frac{1}{p}} |f|_{L_p([t_*, t_*+\sigma_3\rho])} \Big). \tag{8.24}$$

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Erklärung an Eides statt

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