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Chebyshev–Edgeworth-Type Approximations for Statistics Based on Samples with Random Sizes

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Abstract: Second-order Chebyshev–Edgeworth expansions are derived for various statistics from samples with random sample sizes, where the asymptotic laws are scale mixtures of the standard normal or chi-square distributions with scale mixing gamma or inverse exponential distributions. A formal construction of asymptotic expansions is developed. Therefore, the results can be applied to a whole family of asymptotically normal or chi-square statistics. The random mean, the normalized Student t-distribution and the Student t-statistic under non-normality with the normal limit law are considered. With the chi-square limit distribution, Hotelling's generalized T_0^2 statistics and scale mixture of chi-square distributions are used. We present the first Chebyshev–Edgeworth expansions for asymptotically chi-square statistics based on samples with random sample sizes. The statistics allow non-random, random, and mixed normalization factors. Depending on the type of normalization, we can find three different limit distributions for each of the statistics considered. Limit laws are Student t-, standard normal, inverse Pareto, generalized gamma, Laplace and generalized Laplace as well as weighted sums of generalized gamma distributions. The paper continues the authors' studies on the approximation of statistics for randomly sized samples.

Keywords: second-order expansions; random sample size; asymptotically normal statistics; asymptotically chi-square statistics; Student's t-distribution; normal distribution; inverse Pareto distribution; Laplace and generalized Laplace distribution; weighted sums of generalized gamma distributions

MSC: 62E17 (Primary) 62H10; 60E05 (Secondary)

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1. Introduction

In classical statistical inference, the number of observations is usually known. If observations are collected in a fixed time span or we lack observations the sample size may be a realization of a random variable. The number of failed devices in the warranty period, the number of new infections each week in a flu season, the number of daily customers in a supermarket or the number of traffic accidents per year are all random numbers.

Interest in studying samples with a random number of observations has grown steadily over the past few years. In medical research, the authors of [1–3] examines ANOVA models with unknown sample sizes for the analysis of fixed one-way effects in order to avoid false rejection. Applications of orthogonal mixed models to situations with samples of a random number of observations of a Poisson or binomial distributed random variable are presented. Based a random number of observations [4], Al-Mutairi and Raqab [5] and Barakat et al. [6] examined the mean with known and unknown variances and the variance in the normal model, confidence intervals for quantiles and prediction intervals for the future observations for generalized order statistics. An overview on statistical inference of samples with random sample sizes and some applications are given in [4], see also the references therein.

When the non-random sample size is replaced by a random variable, the asymptotic features of statistics can change radically, as shown by Gnedenko [7]. The monograph by Gnedenko and Korolev [8] deals with below limit distributions for randomly indexed sequences and their applications.

General transfer theorems for asymptotic expansions of the distribution function of statistics based on samples with non-random sample sizes to their analogues for samples of random sizes are proven in [9,10]. In these papers, rates of convergence and first-order expansion are proved for asymptotically normal statistics. The results depend on the rates of convergence with which the distributions of the normalized random sample sizes approach the corresponding limit distribution.

The difficulty of obtaining second-order expansions for the normalized random sample sizes beyond the rates of convergences was overcome by Christoph et al. [11]. Second-order expansions were proved by the authors of [11,12] for the random mean and the median of samples with random sample sizes and the authors of [13,14] for the three geometric statistics of Gaussian vectors, the length of a vector, the distance between two vectors and the angle between two vectors associated with their correlation coefficient when the dimension of the vectors is random.

The classical Chebyshev–Edgeworth expansions strongly influenced the development of asymptotic statistics. The fruitful interactions between Chebyshev Edgeworth expansions and Bootstrap methods are demonstrated in [15]. Detailed reviews of applications of Chebyshev–Edgeworth expansions in statistics were given by, e.g., Bickel [16] and Kolassa [17]. If the arithmetic mean of independent random variables is considered as the statistic, only the expected value and the dispersion are taken into account in the central limit theorem or in the Berry–Esseen inequalities. The two important characteristics of random variables, skewness and kurtosis, have great influence on second order expansions, provided that the corresponding moments exist. The Cornish–Fisher inversion of the Chebyshev–Edgeworth expansion allows the approximation of the quantiles of the test statistics used, for example, in many hypothesis tests. In [11], Theorems 3 and 6, and [12], Corollaries 6.2 and 6.3, Cornish–Fisher expansions for the random mean and median from samples with random sample sizes are obtained. In the same way, Cornish–Fisher expansions for the quantiles of the statistics considered in present paper can be derived from the corresponding Chebyshev–Edgeworth expansions.

In the present paper, we continue our research on approximations if the sample sizes are random. To the best of our knowledge, Chebyshev–Edgeworth-type expansions with asymptotically chi-square statistics have not yet been proven in the literature when the sample sizes are random.

The article is structured as follows. Section 2 describes statistical models with random numbers of observations, the assumptions about statistics and random sample sizes and transfer propositions from samples with non-random to random sample sizes. Section 3 presents statistics with non-random sample sizes with Chebyshev–Edgeworth expansions based on standard normal or chi-square distributions. Corresponding expansions of the negative binomial or discrete Pareto distributions as random sample sizes are considered in Section 4. Section 5 describes the influence of non-random, random or mixed normalization factors on the limit distributions of the examined statistics that are based on samples with random sample sizes. Besides the common Student's t, normal and Laplace distributions, inverse Pareto, generalized gamma and generalized Laplace as well as weighted sums of generalized gamma distributions also occur as limit laws. The main results for statistic families with different normalization factors and examples are given in Section 6. To prove statements about a family of statistics, formal constructions for the expansions are worked out in Section 7, which are used in Section 8 to prove the theorems. Conclusions are drawn in Section 9. We leave four auxiliary lemmas to Appendix A.

2. Statistical Models with a Random Number of Observations

Let $X_1, X_2, \ldots \in \mathbb{R} = (-\infty\ \infty)$ and $N_1, N_2, \ldots \in \mathbb{N}_+ = \{1, 2, \ldots\}$ be random variables defined on a common probability space $(\Omega, \mathbb{A}, \mathbb{P})$. The random variables $X_1, \ldots, X_m$ denote the observations and form the random sample with a non-random sample size $m \in \mathbb{N}_+$. Let

$$T_m := T_m(X_1, \ldots, X_m) \quad \text{with} \quad m \in \mathbb{N}_+$$

be some statistic obtained from the sample $\{X_1, X_2, \ldots, X_m\}$. Consider now the sample $X_1, \ldots, X_{N_n}$. The random variable $N_n \in \mathbb{N}_+$ denotes the random size of the underlying sample, that is the random number of observations, depending on a parameter $n \in \mathbb{N}_+$. We suppose for each $n \in \mathbb{N}_+$ that $N_n \in \mathbb{N}_+$ is independent of random variables $X_1, X_2, \ldots$ and $N_n \to \infty$ in probability as $n \to \infty$.

Let T_{N_n} be a statistic obtained from a random sample $X_1, X_2, \ldots, X_{N_n}$ defined as

$$T_{N_n}(\omega) := T_{N_n(\omega)}\Big(X_1(\omega), \ldots, X_{N_n(\omega)}(\omega)\Big) \quad \text{for all} \quad \omega \in \Omega \quad \text{and every} \quad n \in \mathbb{N}_+.$$

2.1. Assumptions on Statistics T_m and Random Sample Sizes N_n

In further consideration, we restrict ourselves to only those terms in the expansions that are used below.

We assume that the following condition for the statistic T_m with $\mathbb{E}T_m = 0$ from a sample with non-random sample size $m \in \mathbb{N}_+$ is fulfilled:

Assumption 1. *There are differentiable functions for all $x \neq 0$ distribution function $F(x)$ and bounded functions $f_1(x)$, $f_2(x)$ and real numbers $\gamma \in \{0, \pm 1/2, \pm 1\}$, $a > 1$ and $0 < C_1 < \infty$ so that for all integers $m \geq 1$*

$$\sup_x \left| \mathbb{P}\big(m^\gamma T_m \leq x\big) - F(x) - m^{-1/2} f_1(x) - m^{-1} f_2(x) \right| \leq C_1\, m^{-a}. \tag{1}$$

Remark 1. *In contrast to Bening et al. [10], the differentiability of $F(x)$, $f_1(x)$ and $f_2(x)$ is only required for $x \neq 0$. In the present article, in addition to the normal distribution, the chi-square distribution with p degrees of freedom is used as $F(x)$, which is not differentiable in $x = 0$ if $p = 1$ or $p = 2$.*

The distribution functions of the normalized random variables $N_n \in \mathbb{N}_+$ satisfy the following condition:

Assumption 2. *A distribution function $H(y)$ with $H(0+) = 0$, a function of bounded variation $h_2(y)$, a sequence $0 < g_n \uparrow \infty$ and real numbers $b > 0$ and $C_2 > 0$ exist so that for all integers $n \geq 1$*

$$\left.\begin{array}{ll} \sup_{y \geq 0} \left| \mathbb{P}\big(g_n^{-1} N_n \leq y\big) - H(y) \right| \leq C_2 n^{-b}, & \textit{for} \quad 0 < b \leq 1, \\ \sup_{y \geq 0} \left| \mathbb{P}\big(g_n^{-1} N_n \leq y\big) - H(y) - n^{-1} h_2(y) \right| \leq C_2 n^{-b}, & \textit{for} \quad b > 1. \end{array}\right\} \tag{2}$$

2.2. Transfer Proposition from Samples with Non-Random to Random Sample Sizes

Assumptions 1 and 2 allow the construction of expansions for distributions of normalized random-size statistics T_{N_n} based on approximate results for fixed-size normalized statistics T_m in (1) and for the random size N_n in (2).

Proposition 1. *Suppose $\gamma \in \{0, \pm 1/2, \pm 1\}$ and the statistic T_m and the sample size N_n satisfy Assumptions 1 and 2. Then, for all $n \in \mathbb{N}_+$, the following inequality applies:*

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}\Big(g_n^\gamma T_{N_n} \leq x\Big) - G_n(x, 1/g_n) \right| \leq C_1\, \mathbb{E}\big(N_n^{-a}\big) + (C_3 D_n + C_4)\, n^{-b}, \tag{3}$$

where

$$G_n(x, 1/g_n) = \int_{1/g_n}^{\infty} \Big(F(x\,y^{\gamma}) + \frac{f_1(xy^{\gamma})}{\sqrt{g_n y}} + \frac{f_2(xy^{\gamma})}{g_n y} \Big) d\Big(H(y) + \frac{h_2(y)}{n} \Big), \tag{4}$$

$$D_n = \sup_x \int_{1/g_n}^{\infty} \Big| \frac{\partial}{\partial y} \Big(F(xy^{\gamma}) + \frac{f_1(xy^{\gamma})}{\sqrt{g_n y}} + \frac{f_2(xy^{\gamma})}{y g_n} \Big) \Big| dy, \tag{5}$$

$a > 1, b > 0$ and $f_1(z), f_2(z), h_2(y)$ are given in (1) and (2). The constants C_1, C_3, C_4 do not depend on n.

General transfer theorems with more terms are proved in [9,10] for $\gamma \geq 0$.

Remark 2. *The approximation function $G_n(x, 1/g_n)$ is not a polynomial in $g_n^{-1/2}$ and $n^{-1/2}$. The domain $[1/g_n, \infty)$ of integration in (4) depends on g_n. Some of the integrals in (4) could tend to infinity with $1/g_n \to 0$ as $n \to \infty$.*

The following statement clarifies the problem.

Proposition 2. *In addition to the conditions of Proposition 1, let the following conditions be satisfied on the functions $H(.)$ and $h_2(.)$, depending on the rate of convergence $b > 0$ in (2):*

$$H(1/g_n) \leq c_1\, g_n^{-b}, \qquad \text{for} \quad b > 0, \tag{6}$$

$$\int_0^{1/g_n} y^{-1/2} dH(y) \leq c_2\, g_n^{-b+1/2}, \qquad \text{for} \quad b > 1/2, \tag{7}$$

$$\left.\begin{array}{ll} i: & \int_0^{1/g_n} y^{-1} dH(y) \leq c_3\, g_n^{-b+1}, \\ ii: & h_2(0) = 0 \quad \text{and} \quad |h_2(1/g_n)| \leq c_4\, n\, g_n^{-b}, \\ iii: & \int_0^{1/g_n} y^{-1} |h_2(y)| dy \leq c_5\, n\, g_n^{-b}, \end{array}\right\} \qquad \text{for} \quad b > 1. \tag{8}$$

Then, for the function $G_n(x, 1/g_n)$ defined in (4), one has

$$\sup_x \big| G_n(x, 1/g_n) - G_{n,2}(x) - I_1(x,n) - I_2(x,n) - I_3(x,n) - I_4(x,n) \big| \leq C\, g_n^{-b}$$

with

$$G_{n,2}(x) = \left\{ \begin{array}{ll} \int_0^{\infty} F(x\,y^{\gamma}) dH(y), & \text{for} \quad 0 < b \leq 1/2, \\ \int_0^{\infty} \Big(F(x\,y^{\gamma}) + \frac{f_1(x\,y^{\gamma})}{\sqrt{g_n y}} \Big) dH(y) =: G_{n,1}(x), & \text{for} \quad 1/2 < b \leq 1, \\ G_{n,1}(x) + \int_0^{\infty} \frac{f_2(x\,y^{\gamma})}{g_n y} dH(y) + \int_0^{\infty} \frac{F(x\,y^{\gamma})}{n} dh_2(y), & \text{for} \quad b > 1, \end{array} \right\} \tag{9}$$

$$I_1(x,n) = \int_{1/g_n}^{\infty} \frac{f_1(x\,y^{\gamma})}{\sqrt{g_n y}} dH(y) \ \text{for} \ b \leq 1/2, \quad I_2(x,n) = \int_{1/g_n}^{\infty} \frac{f_2(x\,y^{\gamma})}{g_n\, y} dH(y) \ \text{for} \ b \leq 1, \tag{10}$$

$$I_3(x,n) = \int_{1/g_n}^{\infty} \frac{f_1(x\,y^{\gamma})}{n\,\sqrt{g_n y}} dh_2(y) \quad \text{and} \quad I_4(x,n) = \int_{1/g_n}^{\infty} \frac{f_2(x\,y^{\gamma})}{n\, g_n y} dh_2(y) \quad \text{for} \ b > 1. \tag{11}$$

Remark 3. *The lower limit of integration in $I_1(x,n)$ to $I_4(x,n)$ in (10) and (11) depends on g_n. If the sample size $N_n = N_n(r)$ is negative binomial distributed with, e.g., $0 < r < 1/2$ or $1 < r < 2$ and $g_n = r(n-1) + 1$ (see (28) below), then both $I_1(x,n)$ and $I_4(x,n)$ have order n^{-r} and not $n^{1/2}$ or n^{-2}, as it seems at first glance.*

Remark 4. *The additional conditions (6)–(8) guarantee to extend the integration range of the integrals in (9) from $[1/g_n, \infty)$ to $(0, \infty)$.*

Proof of Propositions 1 and 2: Evidence of Proposition 1 follows along the similar arguments of the more general Transfer Theorem 3.1 in [10] for $\gamma \geq 0$. The proof was adapted by Christoph and Ulyanov [13] to negative $\gamma < 0$, too. Therefore, the Proposition 1 applies to $\gamma \in \{0, \pm 1/2, \pm 1\}$.

The present Propositions 1 and 2 differ from Theorems 1 and 2 in [13] only by the additional term $f_1(xy^\gamma)\,(g_n y)^{-1/2}$ and the added condition (7) to estimate this additional term. Therefore, the details are omitted her. □

Remark 5. *In Appendix 2 of the monograph by Gnedenko and Korolev [8], asymptotic expansions for generalized Cox processes are proved (see Theorems A2.6.1–A2.6.3). As random sample size, the authors considered a Cox process $N(t)$ controlled by a Poisson process $\Lambda(t)$ (also known as a doubly stochastic Poisson process) and proved asymptotic expansions for the random sum $S(t) = \sum_{k=1}^{N(t)} X_k$, where $X_1, X_2, \ldots$ are independent identically distributed random variables. For each $t \geq 0$, the random variables $N(t), X_1, X_2, \ldots$ are independent. The above-mentioned theorems are close to Proposition 1. The structure of the functions $G_{2;n}(.)$ in (4) and the bounds on the right-hand side of inequality (3) in Proposition 1 differ from the corresponding terms in Theorems A2.6.1–A2.6.3. Thus, the bounds contain little o-terms.*

3. Chebyshev–Edgeworth Expansions Based on Standard Normal and Chi-Square Distributions

We consider two classes of statistics which are asymptotically normal or chi-square distributed.

3.1. Examples for Asymptotically Normally Distributed Statistics

Let $X, X_1, X_2, \ldots$ be independent identically distributed random variables with

$$\left.\begin{array}{c} \mathbb{E}|X|^5 < \infty,\ \ \mathbb{E}(X) = \mu,\ \ 0 < \mathrm{Var}(X) = \sigma^2, \\ \text{skewness } \lambda_3 = \sigma^{-3}\,\mathbb{E}(X-\mu)^3 \ \text{ and kurtosis } \lambda_4 = \sigma^{-4}\,\mathbb{E}(X-\mu)^4. \end{array}\right\} \tag{12}$$

The random variable X is assumed to satisfy Cramér's condition

$$\limsup_{|t|\to\infty} \left|\mathbb{E}e^{itX}\right| < 1. \tag{13}$$

Consider the asymptotically normal sample mean:

$$\overline{X}_m = (X_1 + \cdots + X_m)/m \quad m = 1, 2, \ldots, \tag{14}$$

It follows from Petrov [18], Theorem 5.18 with $k = 5$, that

$$\sup_x \left|\mathbb{P}(\sigma^{-1}\sqrt{m}(\overline{X}_m - \mu) \leq x) - \Phi_{2;m}(x)\right| \leq C m^{-3/2}, \tag{15}$$

with C being independent of m and second order expansion

$$\Phi_{2;m}(x) = \Phi(x) - \left(\frac{\lambda_3}{6\sqrt{m}} H_2(x) + \frac{1}{m}\left(\frac{\lambda_4}{24} H_3(x) + \frac{\lambda_3^2}{72} H_5(x)\right)\right)\varphi(x), \tag{16}$$

where $\Phi(x)$ and $\varphi(x)$ are standard normal distribution function and its density and $H_k(x)$ are the Chebyshev–Hermite polynomials

$$H_2(x) = x^2 - 1 \quad H_3(x) = x^3 - 3x \quad \text{and} \quad H_5(x) = x^5 - 10x^3 + 15x.$$

Let the random variable χ_d^2 be *chi-square distributed with d degrees of freedom* having distribution function $G_d(x)$ and density function $g_d(x)$:

$$G_d(x) = \mathbb{P}(\chi_d^2 \le x) = \int_0^x g_d(y)dy \quad \text{and} \quad g_d(y) = \frac{1}{2^{d/2}\,\Gamma(d/2)}\, y^{(d-2)/2}\, e^{-y/2}, \quad y > 0. \tag{17}$$

Next, we examine the scale-mixed normalized statistic $T_m = \sqrt{m}\,Z/\sqrt{\chi_m^2}$, where Z and χ_m^2 are independent random variables with the standard normal distribution $\Phi(x)$ and the chi-square distribution $G_m(x)$, respectively. Then, the statistic $T_m = \sqrt{m}\,Z/\sqrt{\chi_m^2}$ follows the Student's t-distribution with m degrees of freedom. Example 2.1 in [19] indicates

$$\left| \mathbb{P}\left(\frac{\sqrt{m}\,Z}{\sqrt{\chi_m^2}} \le x \right) - \Phi(x) - \frac{(x^3 + x)\varphi(x)}{4\,m} \right| \le \frac{\sup_x\{|x^5 + 2x^3 + 3x|\varphi(x)\}}{6m^2} + \frac{6(m+4)}{m^3} \le \frac{30.5}{m^2}. \tag{18}$$

Chebyshev–Edgeworth expansions of Student's t-statistic under non-normality are well investigated, but only Hall [20] proved these under minimal moment condition. Let conditions (12) and (13) are satisfied for independent identically distributed random variables $X_1, X_2, \ldots$ Define $T_m^* = m^{1/2}(\overline{X}_m - \mu)/\hat{\sigma}_m$ with sample mean $\overline{X}_n$ and biased sample variance $\hat{\sigma}_m^2 = m^{-1}\sum_{i=1}^m (X_i - \overline{X}_m)^2$. It follows from Hall [20] that for Student's t-statistic T_m^*:

$$R_m(x) = \left| \mathbb{P}\left(m^{1/2}\frac{\overline{X}_m - \mu}{\hat{\sigma}_m} \le x \right) - \Phi(x) - \varphi(x)\left(\frac{P_1(x)}{\sqrt{m}} + \frac{P_2(x)}{m} \right) \right| \le C\, m^{-3/2}(1 + u(m)) \tag{19}$$

uniformly in x, where $u(m) \to 0$ as $m \to \infty$,

$$P_1(x) = \lambda_3(2x^2+1)/6 \quad \text{and} \quad P_2(x) = x\left\{ \frac{\lambda_4}{12}\,(x^2-3) - \frac{\lambda_3^2}{18}\,(x^4 + 2x^2 - 3) - \frac{1}{4}\,(x^2+3) \right\}. \tag{20}$$

Remark 6. *The estimate (19) does not satisfy (1) in Assumption 1 because we do not have a computable error bound U with $|u(m)| \le U < \infty$ for all $m \in \mathbb{N}_+$. The estimate (19) does not satisfy (1) in Assumption 1 because we do not have a computable constant C with $|u(m)| \le C < \infty$ for all $m \in \mathbb{N}_+$, if all parameter are given. The remainder in (19) meets order condition $R_m(x) = \mathcal{O}(m^{-3/2})$ as $m \to \infty$, but in the equivalent condition $\sup_x R_m(x) \le Cm^{-3/2}$ for all $m \ge M$ the values $C > 0$ and $M > 0$ are unknown. About non-asymptotic bounds and order conditions, see the work of Fujikoshi and Ulyanov [19] (Section 1.1).*

In [21], an inequality for a first order approximation is proved:

$$\sup_x \left| \mathbb{P}\left(m^{1/2}\frac{\overline{X}_m - \mu}{\hat{\sigma}_m} \le x \right) - \Phi(x) - \frac{P_1(x)\,\varphi(x)}{\sqrt{m}} \right| \le C\, m^{-1}, \tag{21}$$

where $\mathbb{E}|X|^{4+\varepsilon} < \infty$ is required for arbitrary $\varepsilon > 0$ and $P_1(x)$ is defined in (20).

3.2. Examples for Asymptotically Chi-Square Distributed Statistics

The baseline distribution of the second order expansions is now the chi-square distribution $G_d(x)$ occurring as limit distribution in different multivariate tests (see [22], Chapters 5 and 8–10, [19,23]).

At first, we consider statistic $T_m = T_0^2 = m\,\mathrm{tr}\,\mathbf{S}_q\mathbf{S}_m^{-1}$, where $\mathbf{S}_q$ and $\mathbf{S}_m$ are random matrices independently distributed as Wishart distributions $W_p(q, \mathbf{I}_p)$ and $W_p(m, \mathbf{I}_p)$, respectively, with identity operator $\mathbf{I}_p$ in $\mathbb{R}_p$. Note that $\mathbf{W}$ has Wishart distribution $W_p(q, \mathbf{\Sigma})$ if $q \ge p$ and its density is

$$\frac{1}{2^{pq/2}\,\Gamma_p(q/2)\,|\mathbf{\Sigma}|^{q/2}}\exp\left\{ -\frac{1}{2}\mathrm{tr}\left(\mathbf{\Sigma}^{-1}\mathbf{W} \right) \right\} |\mathbf{W}|^{(q-p-1)/2},$$

where $\Gamma_p(q/2) = \pi^{p(p-1)/4}\,\Pi_{k=1}^p\,\Gamma\big((q-k+1)/2\big)$ (see [23], Chapter 2, for some basic properties).

Hotelling's generalized T_0^2 distribution allows approximation

$$\sup_x \Big| \mathbb{P}\Big(m \,\mathrm{tr}\Big(\mathbf{S}_q \mathbf{S}_m^{-1}\Big) \le x\Big) - G_d(x) - \frac{d}{4\,m}\Big(a_0 G_d(x) + a_1 G_{d+2}(x) + a_2 G_{d+4}(x)\Big)\Big| \le C\, m^{-2} \quad (22)$$

(see [24], Theorem 4.1), where

$$d = pq, \quad a_0 = q - p - 1, \quad a_1 = -2q \quad \text{and} \quad a_2 = q + p + 1 \quad \text{with} \quad a_0 + a_1 + a_2 = 0. \quad (23)$$

If $T_m = \chi_d^2/\chi_m^2$ is a scale mixture, where χ_d^2 and χ_m^2 are independent, T_m allows asymptotic expansion

$$\sup_x \Big| \mathbb{P}\Big(m\, \chi_d^2/\chi_m^2 \le x\Big) - G_d(x) - \frac{d}{4\,m}\Big(a_0 G_d(x) + a_1 G_{d+2}(x) + a_2 G_{d+4}(x)\Big)\Big| \le C\, m^{-2} \quad (24)$$

(see [25], Section 5), where now

$$a_0 = 2 - d, \quad a_1 = 2d \quad \text{and} \quad a_2 = -(2+d) \quad \text{with} \quad a_0 + a_1 + a_2 = 0. \quad (25)$$

Integration by parts gives $G_{k+2}(x) = -2\, g_{k+2}(x) + G_k(x)$. Moreover, $g_{k+2}(x) = (x/k)\, g_k(x)$ for $k = d$ and $k = d+2$. Then, it follows for both statistics $Z_m = \mathrm{tr}\big(\mathbf{S}_q \mathbf{S}_m^{-1}\big)$ in (22) and $Z_m = \chi_d^2/\chi_m^2$ in (24) that

$$\sup_x \Big| \mathbb{P}(m\, Z_m \le x) - G_d(x) + \frac{g_d(x)}{m}\left(\frac{(a_1 + a_2)\, x}{2} + \frac{a_2\, x^2}{2\,(d+2)}\right)\Big| \le C\, m^{-2} \quad (26)$$

where the coefficients a_1 and a_2 are defined in (23) and (25).

The scaled mixture $T_m = m\chi_4^2/\chi_m^2$ is considered in the works by Fujikoshi et al. [23] (Example 13.2.2) and Fujikoshi and Ulyanov [19] (Example 2.2). The estimation given there leads to a computable error bound:

$$\sup_x \left| \mathbb{P}\left(m\, \frac{\chi_4^2}{\chi_m^2} \le x\right) - G_4(x) + \frac{(2\,x - x^2) g_4(x)}{2m}\right| \le \frac{x^2 |x^2 - 4| e^{-x/2}}{12 m^2} + \frac{12(m+4)}{m^3} \le \frac{65.9}{m^2}. \quad (27)$$

Remark 7. *The statistics T_m in (15), (18) and (21) satisfy Assumption 1 with the normal limit distribution $\Phi(x)$ and in (26) and (27) with chi-square distributions $G_d(x)$ and $G_4(x)$, respectively.*

4. Chebyshev–Edgeworth Expansions for Distributions of Normalized Random Sample Sizes

As in the articles by, e.g., Bening et al. [9,10], Christoph et al. [11,12] and Christoph and Ulyanov [13] and Christoph and Ulyanov [14], we consider as random sample sizes N_n the negative binomial random variable $N_n(r)$ and the maximum of n independent discrete Pareto random variables $N_n(s)$ where $r > 0$ and $s > 0$ are parameters.

"The negative binomial distribution is one of the two leading cases for count models, it accommodates the overdispersion typically observed in count data (which the Poisson model cannot)" [26]. Moreover, $\mathbb{E}N_n(r) < \infty$ and $\mathbb{P}(N_n(r)/\mathbb{E}N_n(r) \le y)$ tends to the gamma distribution $G_{r\,r}(y)$ with identical shape and rate parameters $r > 0$.

On the other hand, the mean for the discrete Pareto-like variable $N_n(s)$ does not exist, yet $\mathbb{P}(N_n(s)/n \le x)$ tends to the inverse exponential distribution $W_s(y) = e^{-s/y}$ with scale parameter $s > 0$.

Remark 8. *The authors of [1–4,27], among others, considered the binomial or Poisson distributions as random number N of observations. If $N = N_n$ is binomial (with parameters n and $0 < p < 1$) or Poisson (with rate $\lambda\, n$, $0 < \lambda < \infty$) distributed, then $\mathbb{P}(N_n \le \mathbb{E}N_n x)$ tends to the degenerated in 1 distribution as $n \to \infty$. Therefore, Assumption 2 for the Transfer Proposition 1 is not fulfilled. On the other hand, since binomial or Poisson sample sizes are asymptotically normally distributed and if the statistic T_m is also asymptotically normally distributed, so is the statistic T_{N_n}, too (see [28]).*

Chebyshev–Edgeworth expansions for lattice distributed random variables exist so far only with bounds of small-o or large-$\mathcal{O}$ order (see [29]). For (2) in Assumption 2, computable error bounds C_2 are required because the constant C_3 in (3) depends on C_2 (see also Remark 6 on large-$\mathcal{O}$-bounds and computable error bounds).

4.1. The Random Sample Size $N_n = N_n(r)$ Has Negative Binomial Distribution with Success Probability $1/n$

The sample size $N_n(r)$ has a negative binomial distribution shifted by 1 with the parameters $1/n$ and $r > 0$, the probability mass function

$$\mathbb{P}(N_n(r) = k) = \frac{\Gamma(k+r-1)}{\Gamma(k)\,\Gamma(r)} \left(\frac{1}{n}\right)^r \left(1 - \frac{1}{n}\right)^{k-1}, \quad k = 1, 2, \ldots \tag{28}$$

and $g_n = \mathbb{E}(N_n(r)) = r\,(n-1) + 1$. Bening and Korolev [30] and Schluter and Trede [26] showed

$$\lim_{n\to\infty} \sup_y |\mathbb{P}(N_n(r)/g_n \le y) - G_{r,r}(y)| = 0, \tag{29}$$

where $G_{r,r}(y)$ is the gamma distribution function with its density

$$g_{r,r}(y) = \frac{r^r}{\Gamma(r)} y^{r-1} e^{-ry}\, \mathbb{I}_{(0\ \infty)}(y), \quad y \in \mathbb{R}. \tag{30}$$

In addition to the expansion of $N_n(r)$, a bound of the negative moment $\mathbb{E}(N_n(r))^{-a}$ in (3) is required, where m^{-a} is rate of convergence of the Chebyshev–Edgeworth expansion for T_m in (1).

Proposition 3. *Suppose that $r > 0$ and the discrete random variables $N_n(r)$ have probability mass function (28) with $g_n := \mathbb{E}N_n(r) = r(n-1) + 1$. Then,*

$$\sup_{y\ge 0}\left|\mathbb{P}\left(\frac{N_n(r)}{g_n} \le y\right) - G_{r,r}(y) - \frac{h_{2;r}(y)}{n}\right| \le C_2(r)\, n^{-\min\{r,2\}}, \tag{31}$$

for all $n \in \mathbb{N}_+$, where the constant $C_2(r) > 0$ does not depent on n and

$$h_{2;r}(y) = \begin{cases} 0, & for\ r < 1, \\ (2\,r)^{-1}\, g_{r,r}(y)\big((y-1)(2-r) + 2Q_1(g_n\, y)\big), & for\ r \ge 1. \end{cases} \tag{32}$$

$$Q_1(y) = 1/2 - (y - [y]) \quad \text{and} \quad [.] \text{ denotes the integer part of a number.} \tag{33}$$

Moreover, negative moments $\mathbb{E}(N_n(r))^{-a}$ fulfill the estimate for all $r > 0$, $\alpha > 0$

$$\mathbb{E}\big(N_n(r)\big)^{-\alpha} \le C(r) \begin{cases} n^{-\min\{r,\alpha\}}, r \ne \alpha \\ \ln(n)\, n^{-\alpha}, r = \alpha \end{cases} \tag{34}$$

and the convergence rate in case $r = \alpha$ cannot be improved.

Proof. In [10] (Formula (21)) and in [31] (Formula (11)), the convergence rate is reported for the case $r < 1$. In [11] (Theorem 1), the Chebyshev–Edgeworth expansion for $r > 1$ is proved. In the case $r = 1$, for geometric distributed random variable $N_n(1) \in \mathbb{N}_+$ with success probability $1/n$ the proof is straightforward:

$$\mathbb{P}(N_n \le n\,y) = 1 - \mathbb{P}(N_n \ge [ny] + 1) = 1 - \left(1 - \frac{1}{n}\right)^{ny-\tau} = \left(1 - e^{-y} + \frac{e^{-y}}{n}\left(\frac{y}{2} - \tau\right)\right) + r_n(y),$$

where $\sup_y |r_n(y)| \le C\,n^{-2}$ and $\tau = ny - [ny] = 1/2 - Q_1(ny) \in [0\ 1)$. Hence, (31) holds for $r = 1$.

In [12] (Corollary 4.2), leading terms for the negative moments of $\mathbb{E}\big(N_n(r)\big)^{-p}$ are derived, which lead to (34). $\square$

Remark 9. *The negative binomial random variables $N_n(r)$ satisfy (2) in Assumption 2 and the additional conditions (6), (7) and (8) in Proposition 2 with $H(y) = G_{r,r}(y)$, $h_2(y) = h_{2;r}(y)$, $g_n = \mathbb{E}N_n(r) = r(n-1)+1$ and $b = \min\{r\ 2\}$. The jumps of the distribution function $\mathbb{P}(N_n(r) \le g_n\, y)$ only affect the function $Q_1(.)$ in the term $h_{2;r}(.)$.*

4.2. The Random Sample Size $N_n = N_n(s)$ Is the Maximum of n Independent Discrete Pareto Variables

We consider the continuous Pareto Type II (Lomax) distribution function

$$F_{Y^*}(x) = 1 - (1 + (x-1)/s)^{-1} \quad \text{for} \quad x \ge 1.$$

The discrete Pareto II distribution $F_{Y(s)}$ is obtained by discretizing the continuous Pareto distribution $F_{Y^*}(x)$, : $\mathbb{P}(Y(s) = k) = F_{Y^*}(k) - F_{Y^*}(k-1)$, $\quad k \in \mathbb{N}_+$. The random variable $Y(s)$ is the discrete counterpart on the positive integers to the continuous random variable Y^*. Both random variables Y^* and $Y(s)$ have shape parameter 1 and scale parameter $s > 0$ (see [32]). The discrete Pareto distributed $Y(s)$ has probability mass and distribution functions:

$$\mathbb{P}(Y(s) = k) = \frac{s}{s+k-1} - \frac{s}{s+k} \quad \text{and} \quad \mathbb{P}\big(Y(s) \le k\big) = \frac{k}{s+k}, \quad \text{for} \quad k \in \mathbb{N}_+. \tag{35}$$

Let $Y_1(s), Y_2(s), \dots$ be a sequence of independent random variables with the common distribution function (35). Define

$$N_n(s) = \max\nolimits_{1\le j\le n} Y_j(s) \quad \text{with} \quad \mathbb{P}(N_n(s) \le k) = \left(\frac{k}{s+k}\right)^n, \quad n \in \mathbb{N}_+, \quad k \in \mathbb{N}_+ \quad s > 0. \tag{36}$$

The random variable $N_n(s)$ is extremely spread over the positive integers.

Proposition 4. *Consider the discrete random variable $N_n(s)$ with distribution function (36). Then,*

$$\sup\nolimits_{y>0}\left|\mathbb{P}\left(\frac{N_n(s)}{n} \le y\right) - W_s(y) - \frac{h_{2;s}(y)}{n}\right| \le \frac{C_3(s)}{n^2} \quad \textit{for all} \quad n \in \mathbb{N}_+ \quad \textit{and fixed} \quad s > 0, \tag{37}$$

$$W_s(y) = \mathrm{e}^{-s/y} \quad \textit{and} \quad h_{2;s}(y) = s\,\mathrm{e}^{-s/y}\,\big(s - 1 + 2Q_1(n\,y)\big)/\big(2\,y^2\big),\ y > 0 \tag{38}$$

where $C_3(s) > 0$ does not depend on n and $Q_1(y)$ is defined in (33). Moreover,

$$\mathbb{E}\big(N_n(s)\big)^{-p} \le C(p)\, n^{-\min\{p,2\}}, \tag{39}$$

where for $0 < p \le 2$ the order of the bound is optimal.

The Chebyshev–Edgeworth expansion (37) is proved in [11] (Theorem 4). In [12] (Corollary 5.2), leading terms for the negative moments $\mathbb{E}\big(N_n(s)\big)^{-p}$ are derived for the negative moments that lead to (39).

Remark 10. *Let the random variable $V(s)$ is exponentially distributed with rate parameter $s > 0$. Then, $W(s) = 1/V(s)$ is an inverse exponentially distributed random variable with the continuous distribution function $W_s(y) = e^{-s/y}\, \mathbb{I}_{(0\ \infty)}(y)$. Both $W_s(y)$ and $\mathbb{P}(N_n(s) \le y)$ are heavy tailed with shape parameter 1.*

Remark 11. *Since $\mathbb{E}\big(W(s)\big) = \infty$ and $\mathbb{E}\big(N_n(s)\big) = \infty$ for all $n \in \mathbb{N}_+$, we choose $g_n = n$ as normalizing factor for $N_n(s)$ in (37).*

Remark 12. *The random sample sizes $N_n(s)$ satisfy (2) in Assumption 2 and the additional conditions (6)–(8) in Proposition 2 with $W_s(y) = \mathrm{e}^{-s/y}$, $h_2(y) = h_{2;s}(y)$, $g_n = n$ and $b = 2$. The jumps of the distribution function $\mathbb{P}(N_n(s) \le n\,y)$ only affects the function $Q_1(.)$ in the term $h_{2;s}(.)$.*

Remark 13. *Lyamin [33] proved a bound $|\mathbb{P}(N_n(s) \le n\,y) - W_s(y)| \le 0.37\,n^{-1}$ for integers $s \ge 1$.*

5. Limit Distributions of Statistics with Random Sample Sizes Using Different Scaling Factors

The statistic T_m from a sample with non-random sample size $m \in \mathbb{N}_+$ fulfills condition (1) in Assumption 1. Instead of the non-random sample size m, we consider a random sample size $N_n \in \mathbb{N}_+$ satisfying condition (2) in Assumption 2. Let g_n be a sequence with $g_n \uparrow \infty$ as $n \to \infty$. Consider the scaling factor $g_n^\gamma N_n^{\gamma^*-\gamma}$ by the statistics T_{N_n} with $\gamma \in \{0, \pm 1/2\}$ if $F(x) = \Phi(x)$ and $\gamma^* = 1/2$ or $\gamma \in \{0, \pm 1\}$ if $F(x) = G_u(x)$ and $\gamma^* = 1$. Then, conditioning on N_n and using (1) and (2), we have

$$\mathbb{P}\Big(g_n^\gamma N_n^{\gamma^*-\gamma} T_{N_n} \le x\Big) = \mathbb{P}\Big(N_n^\gamma T_{N_n} \le x\,(N_n/g_n)^\gamma\Big) = \sum_{m=1}^{\infty} \mathbb{P}\Big(m^\gamma T_m \le x(m/g_n)^\gamma\Big)\,\mathbb{P}(N_n = m)$$
$$\overset{(1)}{\approx} \mathbb{E}\Big(F(x(N_n/g_n)^\gamma)\Big) = \int_{1/g_n}^{\infty} F(xy^\gamma) d\mathbb{P}(N_n/g_n \le y) \overset{(2)}{\approx} \int_{1/g_n}^{\infty} F(xy^\gamma) dH(y). \tag{40}$$

If there exists a limit distribution of $\mathbb{P}\Big(g_n^\gamma N_n^{\gamma^*-\gamma}\,T_{N_n} \le x\Big)$ as $n \to \infty$, then it has to be a scale mixture of parent distribution $F(x)$ and positive mixing parameter $H(y)$: $\int_0^\infty F(xy^\gamma)dH(y)$ (see, e.g., [23,34], Chapter 13, and [19] and the references therein).

Remark 14. *Formula (40) shows that different normalization factors at T_{N_n} lead to different scale mixtures of the limit distribution of the normalized statistics T_{N_n}.*

5.1. The Case $F(x) = \Phi(x)$ and $H(y) = G_{r,r}(y)$

The statistics (15), (18) and (21) considered in Section 3.1 have normal approximations $\Phi(x)$. The limit distribution for the normalized random sample size $N_n(r)/\mathbb{E}N_n(r)$ is the gamma distribution $G_{r,r}(y)$ with density (30). We investigate the dependence of the limit distributions in $\mathbb{P}\Big(g_n^\gamma N_n(r)^{1/2-\gamma} T_{N_n(r)} \le x\Big) \to \int_0^\infty \Phi(xy^\gamma)dG_{r,r}(y)$ as $n \to \infty$ for $\gamma \in \{1/2,\, 0,\, -1/2\}$.

(i) If $\gamma = 1/2$, then the limit distribution is Student-s t distribution $S_{2r}(x)$ having density

$$s_{2r}(x) = \frac{\Gamma((r+1/2)}{\sqrt{2r\pi}\,\Gamma(r)} \left(1 + \frac{x^2}{2r}\right)^{-r+1/2}, \quad r > 0, \quad x \in \mathbb{R}. \tag{41}$$

(ii) If $\gamma = 0$, the standard normal law $\Phi(x)$ is the limit one with density $\varphi(x)$.

(iii) For $\gamma = -1/2$, the generalized Laplace distributions $L_r(x)$ occur with density (see [13], Section 5.1.3):

$$l_r(x) = \frac{r^r}{\Gamma(r)} \int_0^\infty \varphi(xy^{-1/2})\, y^{r-3/2} e^{-ry} dy = \frac{2\,r^r}{\Gamma(r)\,\sqrt{2\,\pi}} \left(\frac{|x|}{\sqrt{2\,r}}\right)^{r-1/2} K_{r-1/2}(\sqrt{2\,r}\,|x|). \tag{42}$$

where $K_\alpha(u)$ is the *Macconald function of order α* or modified Bessel function of the third kind with index α. The function $K_\alpha(u)$ is also sometimes called a modified Bessel function of the second kind of order α. For properties of these functions, see, e.g., Chapter 51 in [35] or the Appendix on Bessel functions in [36].

If $r \in N_+$, the so-called *Sargan densities* $l_1(x), l_2(x), \ldots$ and their distribution functions are computable in closed forms (see Formulas (63)–(65) below in Section 7):

$$\left.\begin{array}{lll} l_1(x) = \frac{1}{\sqrt{2}}\, e^{-\sqrt{2}\,|x|} & \text{and} & L_1(x) = 1 - \frac{1}{2} e^{-\sqrt{2}\,|x|}, \quad x > 0 \\ l_2(x) = \left(\frac{1}{2} + |x|\right) e^{-2\,|x|} & \text{and} & L_2(x) = 1 - \frac{1}{2}(1+x)\, e^{-2\,|x|}, \quad x > 0 \\ l_3(x) = \frac{3\sqrt{6}}{16}\left(1 + \sqrt{6}\,|x| + 2\,x^2\right) e^{-\sqrt{6}\,|x|} & \text{and} & L_3(x) = 1 - \left(\frac{1}{2} + \frac{5\sqrt{6}\,x}{16} + \frac{3\,x^2}{8}\right) e^{-\sqrt{6}\,|x|}, \end{array}\right\} \quad (43)$$

where $L_r(-x) = 1 - L_r(x)$ for $x \geq 0$.

The double exponential or standard Laplace density is $l_1(x)$ with variance 1 and distribution function $L_1(x)$ given in (43). The Sargan distributions are therefore a generalisation of the standard Laplace distribution.

5.2. The Case $F(x) = G_d(x)$ and $H(y) = W_s(y) = e^{-s/y}$

The statistics considered in Section 3.2 asymptotically approach chi-square distribution $G_d(x)$. The limit distribution for the normalized random sample size $N_n(s)/n$ is the inverse exponential distribution $W_s(y) = e^{-s/y}\,\mathbb{I}_{(0,\infty)}(y)$.

(i) If $\gamma = 1$, then the generalized gamma distribution $W_d(x;2s)$ occurs with density $w_d(x;2s)$:

$$w_d(x;2s) = s\left(\frac{sx}{2}\right)^{d/4-1/2} K_{d/2-1}(\sqrt{2s\,x})\,\mathbb{I}_{(0,\infty)}(x) \quad (44)$$

where the Macconald function $K_\alpha(u)$ already appears in Formula (42) with different α and argument. For $\alpha = m + 1/2$, where m is an integer, the Macconald function $K_{m+1/2}(u)$ has a closed form (see Formulas (63)–(65) below in Section 7). Therefore, if $d = 1,3,5\ldots$ is an odd number, then the density $w_d(x;2s)$ may be calculated in closed form. The distribution functions $W_d(x;2s)$ with density functions $w_d(x;2s)$ for $d = 1,3,5$ and $x > 0$ are

$$w_1(x;2s) = s\,(2sx)^{-1/2}\,e^{-\sqrt{2\,s\,x}} \quad \text{and} \quad W_1(x;2s) = 1 - e^{-\sqrt{2\,s\,x}}, \quad (45)$$

$$w_3(x;2s) = s\,e^{-\sqrt{2\,s\,x}} \quad \text{and} \quad W_3(x;2s) = 1 - e^{-\sqrt{2\,s\,x}}\,(\sqrt{2sx}+1), \quad (46)$$

$$w_5(x;2s) = \frac{s}{3}\left(1 + 2\sqrt{2sx}\right) e^{-\sqrt{2\,s\,x}} \quad \text{and} \quad W_5(x;2s) = 1 - e^{-\sqrt{2\,s\,x}}\left(\sqrt{2sx} + \frac{2sx}{3}\right). \quad (47)$$

Remark 15. *Functions in (45) are Weibull density and distribution functions, in (46) there are density and distribution functions of a generalized gamma distribution, but $w_5(x)$ and $W_5(x)$ are even more general.*

The family of generalized gamma distributions contains many absolutely continuous distributions concentrated on the non-negative half-line.

Remark 16. *The generalized gamma distribution $G^*(x;r,\alpha,\lambda)$ corresponds to the density*

$$g^*(x;r,\alpha,\lambda) = \frac{|\alpha|\,\lambda^r}{\Gamma(r)}\,x^{\alpha r-1}\,e^{-\lambda x^\alpha}, \quad x \geq 0, \quad |\alpha| > 0,\ r > 0,\ \lambda > 0, \quad (48)$$

where α and r are the two shape parameters and λ the scale parameter. The density representation (48) is suggested in the work of Korolev and Zeifman [37] or Korolev and Gorshenin [38], and many special cases are listed therein. In addition to, e.g., Gamma and Weibull distributions (with $a > 0$), inverse Gamma, Lévy and Fréche distributions (with $a < 0$) also belong to that family of generalized gamma distributions.

Remark 17. *The Weibull density in (45) is* $w_1(x;2s) = g^*(x;1,1/2,\sqrt{2s})$*. Moreover,* $w_3(x;2s) = g^*(x;2,1/2,\sqrt{2s})$*. The densities* $w_5(x),\ w_7(x), w_9(x), \ldots$ *are weighted sums of generalized gamma distribution with different shape parameters r, e.g.,*

$$\begin{aligned} w_5(x;2s) &= \frac{1}{3}\, g^*(x;2,\frac{1}{2},\sqrt{2s}) + \frac{2}{3}\, g^*(x;3,\frac{1}{2},\sqrt{2s}) \\ w_9(x;2s) &= \frac{5}{35}\, g^*(x;2,\frac{1}{2},\sqrt{2s}) + \frac{10}{35}\, g^*(x;3,\frac{1}{2},\sqrt{2s}) + \frac{12}{35}\, g^*(x;4,\frac{1}{2},\sqrt{2s}) + \frac{8}{35}\, g^*(x;5,\frac{1}{2},\sqrt{2s}). \end{aligned}$$

(i) If $\gamma = 1$. For better readability I have introduced for (i) and after (ii).

(ii) If $\gamma = 0$, the standard normal law $\Phi(x)$ is the limit distribution with density $\varphi(x)$.

(iii) If $\gamma = -1$, as limit distribution the *inverse Pareto distribution* occurs $V_{d/2}(x;2s)$ with shape parameter $d/2$, scale parameter $2\,s$ and density $v_{d/2}(x;2s)$:

$$V_{d/2}(x;2s) = \left(\frac{x}{2\,s+x}\right)^{d/2} \quad \text{and} \quad v_{d/2}(x;2s) = \frac{s\,d\,x^{d/2-1}}{(x+2s)^{d/2+1}} \quad \text{for} \quad x \geq 0 \tag{49}$$

In [39], a robust and efficient estimator for the shape parameter of the inverse Pareto distribution and applications are given.

6. Main Results

We examine asymptotic approximations of $\mathbb{P}\left(g_n^{\gamma} N_n^{\gamma^*-\gamma} T_{N_n} \leq x\right)$ depending on the scaling factor $g_n^{\gamma} N_n^{\gamma^*-\gamma}$ for $\gamma \in \{0, \pm 1/2, \pm 1\}$, for $\gamma^* = 1/2$ if the statistic T_m is asymptotically normal or $\gamma^* = 1$ if T_m is asymptotically chi-square distributed.

6.1. Asymptotically Normal Distributed Statistics with Negative Binomial Distributed Sample Sizes

Consider first the statistics estimated in (15), (18) and (21) with the normal limiting distribution $\Phi(x)$. They have the form

$$\left|\mathbb{P}(\sqrt{m}Z_m \leq x) - \Phi(x) - \left(m^{-1/2}(p_0 + p_2x^2) + m^{-1}(p_1x + p_3x^3 + p_5x^5)\mathbb{I}_{a>1}(a)\right)\varphi(x)\right| \leq m^{-a}\,. \tag{50}$$

The sample size is negative binomial $N_n = N_n(r)$ with probability mass function (28).

Theorem 1. *Let* $r > 0$*. If inequality (50) for the statistic* Z_m *and inequality (31) for random sample size* $N_n(r)$ *with* $g_n = \mathbb{E}N_n(r) = r(n-1)+1$ *hold, then, for all* $n \in \mathbb{N}_+$*, the following expansions apply:*

i: *The non-random scaling factor* $\sqrt{g_n}$ *by statistic* $Z_{N_n(r)}$ *leads to Student's t-approximation.*

$$\sup_x \left|\mathbb{P}\left(\sqrt{g_n}\, Z_{N_n(r)} \leq x\right) - S_{2r;n}(x)\right| \leq C_r \begin{cases} n^{-r}\ \ln(n), & r \in \{1/2,\, 3/2,\, 2\}, \\ n^{-\min\{r,2\}}, & r \notin \{1/2,\, 3/2,\, 2\} \end{cases}$$

where

$$\begin{aligned} S_{2r;n}(x) &= S_{2r}(x) + \frac{s_{2r}(x)}{\sqrt{g_n}}\left(p_0\,\frac{x^2+2r}{2r-1} + p_2x^2\right)\mathbb{I}_{\{r>1/2\}}(r) \\ &+ \frac{s_{2r}(x)}{g_n}\left(p_1\,\frac{x(x^2+2r)}{(2r-1)} + p_3x^3 + p_5\,\frac{x^5(2r+1)}{x^2+2r} + \frac{(2-r)x(x^2+1)}{4\,(2r-1)}\right)\mathbb{I}_{\{r>1\}}(r), \end{aligned} \tag{51}$$

$S_{2r}(x)$ *is Student's t-distribution having density* $s_{2r}(x)$*, defined in (41), and* p_k *are the coefficients in (50).*

ii: *The standard normal approximation occurs at random scaling factor* $\sqrt{N_n(r)}$ *by statistic* $Z_{N_n(r)}$*:*

$$\sup_x \left| \mathbb{P}(\sqrt{N_n(r)}\, Z_{N_n(r)} \le x) - \Phi_{n\,2}(x) \right| \le C_r \begin{cases} n^{-\min\{r,2\}}, & r \ne 2, \\ \ln(n)\, n^{-2}, & r = 2, \end{cases} \tag{52}$$

where

$$\begin{aligned} \Phi_{n\,2}(x) &= \Phi(x) + \frac{\sqrt{r}\,\Gamma(r-1/2)}{\Gamma(r)\,\sqrt{g_n}}\,(p_0 + p_2 x^2)\varphi(x)\,\mathbb{I}_{\{r>1/2\}}(r) \\ &\quad + \frac{(p_1 x + p_3\,x^3 + p_5\,x^5)\varphi(x)}{g_n}\left(\ln n\,\mathbb{I}_{\{r=1\}}(r) + \frac{r}{r-1}\,\mathbb{I}_{\{r>1\}}(r)\right), \end{aligned} \tag{53}$$

iii: *If* $r = 2$*, the mixed scaling factor* $g_n^{-1/2}\, N_n(2)$ *by statistic* $Z_{N_n(2)}$ *leads to generalized Laplace approximation:*

$$\sup_x \left| \mathbb{P}\left(g_n^{-1/2}\, N_n(2)\, Z_{N_n(2)} \le x\right) - L_2(x) - l_{n;2}(x) \right| \le C_2\, \ln(n)\, n^{-2} \tag{54}$$

where $L_2(x) = 1 - \frac{1}{2}(1+x)\,e^{-2\,|x|}$, $L_2(-x) = 1 - L_2(x)$ *for* $x \ge 0$ *and*

$$l_{n;2}(x) = \frac{e^{-2|x|}}{\sqrt{g_n}}\left(p_0(|x| + 1/2) + 2p_2 x^2\right) - \frac{e^{-2|x|}}{g_n}(p_1\,x + 4\,p_3\,|x|\,x + 4\,p_5\,(2x^3 + |x|\,x))\,. \tag{55}$$

Remark 18. *Analogous to (54) and (55), expansions for all* $r > 0$ *can be derived from Formulas (42) and (63)–(66) below in Section 7, whereby closed forms can be presented only for* $r \in \{1, 2, 3, \ldots\}$.

The statistics from Section 3.1 are considered with different normalization factors as applications of Theorem 1:

Corollary 1. *Let the conditions of Theorem 1 be satisfied:*

i: *In the case of the Student's t-statistic* $Z/\sqrt{\chi^2_m}$ *estimated in (18), one has (51) with* $p_0 = p_2 = p_5 = 0$ *and* $p_1 = p_3 = 1/4$ *using non-random scaling factor* $\sqrt{g_n}$*:*

$$\sup_x \left| \mathbb{P}\left(\frac{\sqrt{g_n}\, Z}{\sqrt{\chi^2_{N_n(r)}}} \le x \right) - S_{2r}(x; n) \right| \le C_r \begin{cases} n^{-\min\{r,2\}}, & r \ne 2, \\ \ln(n)\, n^{-2}, & r = 2, \end{cases}$$

where

$$S_{2r}(x;n) = S_{2r}(x) - s_{2r}(x)\,\frac{2r(x + x^3) - (2-r)x(x^2+1)}{4(2r-1)\,g_n}\mathbb{I}_{\{r>1\}}(r)$$

ii: *In the case of Student's one-sample t-test statistic under non-normality* $T_m = (\overline{X}_m - \mu)/\hat{\sigma}_m$ *estimated in (21) with* $a = 1$*, the first-order approximation defined in (52) for* $0 < r \le 1$ *and (53) with* $p_0 = \lambda_3/6$ *and* $p_2 = \lambda_3/3$ *using random scaling factor* $\sqrt{N_n(r)}$ *leads uniformly in x to:*

$$\left| \mathbb{P}(\sqrt{N_n(r)}\, T_{N_n(r)} \le x) - \Phi(x) + \frac{\sqrt{r}\,\Gamma(r-1/2)\lambda_3(2x^2+1)}{6\,\Gamma(r)\,\sqrt{g_n}}\,\varphi(x) \right| \le C_r \begin{cases} n^{-\min\{r,1\}}, & r \ne 1, \\ \ln(n)\, n^{-1}, & r = 1. \end{cases}$$

iii: *Considering sample mean* $\overline{X}_m$ *estimated in (15), one has (55) with* $p_0 = -p_2 = \lambda_3/6$, $p_1 = \lambda_4/8 - 5\lambda_3^2/24$, $p_3 = -\lambda_4/24 + 5\lambda_3^2/36$*, and* $p_5 = -\lambda_3^2/72$ *using mixed scaling factor* $g_n^{-1/2}\, N_n(2)$*:*

$$\sup_x \left| \mathbb{P}\left(g_n^{-1/2}\, N_n(2)\, \overline{X}_{N_n(2)} \le x\right) - L_2(x) - l_{2;n}(x) \right| \le C_2\, \ln(n)\, n^{-2},$$

where the generalized Laplace distributions $L_2(x)$ is defined in (43) and

$$l_{2;n}(x) = \frac{\lambda_3\, e^{-2|x|}}{6\,\sqrt{g_n}}\left(2x^2 - |x| - 1/2\right) + \frac{e^{-2|x|}}{36\, g_n}\left(\lambda_4\,(9\,x - 6\,x\,|x|) + 12\lambda_3^2\,(4\,x^3 + 18\,x\,|x| - 15x)\right).$$

Remark 19. *The approximating functions in the expansions for $\mathbb{P}\left(g_n^{\gamma} N_n(r)^{1/2-\gamma} T_{N_n(r)} \le x\right)$ with the statistics estimated in (15), (18) and (21) can only be given in closed form for all $r > 0$ in the case of non-random ($\gamma = 1/2$) or random ($\gamma = 0$) normalization factors. In the case of the mixed ($\gamma = -1/2$) normalization factor, only for positive integer r closed forms are available, while in the other cases Macconald functions are involved.*

6.2. Asymptotically Chi-Square Distributed Statistics with Pareto-Like Distributed Sample Sizes

Consider now the statistics, estimated in (26) and (27) with limit chi-square distributions. They have the form

$$\left|\mathbb{P}(mT_m \le x) - G_d(x) - m^{-1}(q_1 x + q_2 x^2) g_d(x)\right| \le C(s)\, m^{-2}. \tag{56}$$

The sample size is the Pareto-like random variable $N_n = N_n(s)$ with probability mass function (35).

Theorem 2. *Let $s > 0$ and (36) be the distribution function of the random sample size $N_n = N_n(s)$. If for the statistic T_m the inequality (56) with limiting chi-square distribution $G_d(x)$ and the inequality (37) with $g_n = n$ for the random sample size $N_n(s)$ hold, then for all $n \in \mathbb{N}_+$ one has the following approximation:*

i: *The non-random scaling factor n by $T_{N_n(s)}$ leads to the limiting generalized gamma distributions.*

$$\sup_{x>0}\left|\mathbb{P}\left(n\, T_{N_n(s)} \le x\right) - W_{d\,;n}(x;2s)\right| \le C(s)\, n^{-2}\, \ln\, n \quad \textit{for} \quad d = 1 \quad \textit{and} \qquad d = 3, \tag{57}$$

$$W_{1\,;n}(x;2s) = W_1(x;2s) + n^{-1}\, w_1(x;2s)\left(q_1\, x\left(1 + \sqrt{2sx}\right) + q_2\, x^2 - \frac{(s-1)\,x\,(\sqrt{2sx}+1)}{4\,s}\right) \tag{58}$$

and

$$W_{3\,;n}(x;2s) = W_3(x;2s) + n^{-1}\, w_3(x;2s)\left(q_1\, \frac{x\,\sqrt{2sx}}{2s} + q_2\, x^2 - \frac{(s-1)\,x^2}{2\,\sqrt{2sx}}\right). \tag{59}$$

where the limit law $W_d(x;2s)$ with density $w_d(x;2s)$ for $d = 1$ and $d = 3$ are given in (45) and (46).

ii: *The random scaling factor $N_n(s)$ by $T_{N_n(s)}$ induces the limiting chi-square distribution.*

$$\sup_x\left|\mathbb{P}(N_n(s)\, T_{N_n(s)} \le x) - G_d(x) - \frac{g_d(x)}{s\, n}\left(q_1\, x + q_2\, x^2\right)\right| \le C(s)\, \frac{\ln\, n}{n^2}, \tag{60}$$

iii: *Limiting inverse Pareto distributions occur at mixed scaling factor $n^{-1}\, N_n^2(s)$ by $T_{N_n(s)}$.*

$$\sup_{x>0}\left|\mathbb{P}\left(\frac{N_n^2(s)}{n}\, T_{N_n(s)} \le x\right) - V_{d/2}(x;2s) - \frac{1}{n}\, v_{d/2;n}(x;2s)\right| \le C(s)\, \frac{\ln\, n}{n^2},$$

where

$$v_{d/2;n}(x;2s) = v_{d/2}(x;2s)\left(q_1\, \frac{x(d+2)}{x+2s} + q_2\, \frac{x^2(d+4)(d+2)}{(x+2s)^2} + \frac{(s-1)\,x\,(2+d)}{2\,(x+2s)}\right) \tag{61}$$

with inverse Pareto distribution $V_{d/2}(x;2s)$ having shape parameter $d/2$, scale parameter $2\,s$ and density $v_{d/2}(x;2s)$ defined in (49).

Remark 20. *Analogous to (57), expansions for all $d \in \mathbb{N}_+$ can be derived from Formulas (44), (63)–(65) and (69) below in Section 7, whereby closed forms can be given for $d \in \{1, 3, 5, \dots\}$.*

The statistics from Section 3.2 are considered with different normalization factors as applications of Theorem 2.

Corollary 2. *Let the conditions of Theorem 2 be satisfied.*

i: *Let χ^2_d/χ^2_m be scale mixture, estimated in (24), where χ^2_d and χ^2_m are independent. Then, using non-random scaling factor, n limiting generalized gamma distributions occur with $q_1 = (d-2)/2$ and $q_2 = -1/2$ in (58) and (59):*

$$\sup_{x>0}\left|\mathbb{P}\left(n\ \mathrm{tr}\left(\chi^2_d/\chi^2_{N_n(s)}\right) \le x\right) - W_{d\,;\,n}(x;2s)\right| \le C(s)\, n^{-2}\ln n, \quad \text{for} \quad d=1 \quad \text{and} \quad d=3,$$

$$W_{1\,;\,n}(x;2s) = W_1(x;2s) + n^{-1}\, w_1(x;2s)\left(-\,x\left(1+\sqrt{2sx}\right) - \frac{x^2}{2} - \frac{(s-1)\,x\,(\sqrt{2sx}+1)}{4\,s}\right)$$

$x > 0$ and

$$W_{3\,;\,n}(x;2s) = W_3(x;2s) + n^{-1}\, w_3(x;2s)\left(\frac{x}{4s}\sqrt{2sx} - \frac{x^2}{2} - \frac{(s-1)\,x^2}{2\sqrt{2sx}}\right), \quad x > 0,$$

where the limit law $W_d(x;2s)$ with density $w_d(x;2s)$ for $d=1$ and $d=3$ are given in (45) and (46).

ii: *For the scaled mixture χ^2_4/χ^2_m estimated in (27), one gets the limiting chi-square distribution with a random scaling factor $N_n(s)$ in (60) with $q_1 = 1$ and $q_2 = 1/2$:*

$$\sup_x\left|\mathbb{P}(N_n(s)\,\chi^2_4/\chi^2_{N_n(s)} \le x) - G_d(x) - \frac{g_d(x)}{s\,n}\left(x - x^2/2\right)\right| \le C(s)\,\frac{\ln n}{n^2},$$

iii: *In the case of the Hotelling's generalized T^2_0 statistic $T^2_0 = m\ \mathrm{tr}\left(\mathbf{S}_q\mathbf{S}_m^{-1}\right)$ estimated in (22), one has the limiting inverse Pareto distributions with mixed scaling factor $n^{-1}\,N_n^2(s)$ by $\mathrm{tr}\left(\mathbf{S}_q\mathbf{S}^{-1}_{N_n(s)}\right)$. Here, (61) holds with $q_1 = (p+1-q)/2$ and $q_2 = (p+1+q)/(2d+4)$.*

$$\sup_{x>0}\left|\mathbb{P}\left(\frac{N_n^2(s)}{n}\ \mathrm{tr}\left(\mathbf{S}_q\mathbf{S}^{-1}_{N_n(s)}\right) \le x\right) - V_{d/2}(x;2s) - \frac{1}{n}\, v_{d/2;n}(x;2s)\right| \le C(s)\,\frac{\ln n}{n^2},$$

and

$$v_{d/2;n}(x;2s) = v_{d/2}(x;2s)\left(\frac{(p+1-q)\,x(d+2)}{2(x+2s)} + \frac{(p+1+q)x^2(d+4)}{2(x+2s)^2} + \frac{(s-1)\,x\,(2+d)}{2(x+2s)}\right)$$

where the inverse Pareto distribution $V_{d/2}(x;2s)$ with shape parameter $d/2$, scale parameter $2\,s$ and density $v_{d/2}(x;2s)$ is defined in (49).

Remark 21. *For the statistics estimated in (26) and (27), the approximating functions in the expansions for $\mathbb{P}\left(g_n^{\gamma} N_n(s)^{1-\gamma} T_{N_n(s)} \le x\right)$ can only be given in closed form for all integer d in the case of non-random ($\gamma = 1$) or random ($\gamma = 0$) normalization factors. In the case of the mixed ($\gamma = -1$) normalization factor, only for odd integer d in closed form can be presented; for even integer d, the Macconald functions are involved.*

7. Formal Construction of the Expansions

Expansions of the statistics considered in (15), (18), (21), (26) and (27) have the structure:

$$G(x) + g(x)\left(m^{-1/2}\, P_1(x; j_1^*) + m^{-1}\, P_2(x; j_2^*)\right)$$

with $g(x) = G'(x)$ and polynomials $P_1(x; j_1^*)$, $P_2(x; j_2^*)$ of degrees j_1^* and j_2^*, respectively. Here, $G(x) = \Phi(x)$ or $G(x) = \mathbb{P}(\chi_d^2 \leq x)$.

We calculate the integrals with $k = 1, 2$ and $j = 0, 1, \dots, j_k^*$:

$$J_1(x;\gamma) = \int_0^\infty G(xy^\gamma)dH(y) \quad \text{and} \quad J_2(x;\gamma,k,j) = x^j \int_0^\infty y^{\gamma j - k/2} g(xy^\gamma)dH(y).$$

The limit distributions of the random sizes N_n are $H(y) = G_{r,r}(y)$ and $H(y) = W_s(y) = e^{-s/y}$ with corresponding second approximation $h_2(y)$.

We use the following formulas several times: Formula 2.3.3.1 in [40]

$$M_\alpha(p) = \int_0^\infty y^{\alpha-1} \mathrm{e}^{-py} dy \overset{y=1/z}{=} \int_0^\infty z^{-\alpha-1} \mathrm{e}^{-p/z} dz = \Gamma(\alpha)\, p^{-\alpha} \quad \alpha > 0, \quad p > 0. \tag{62}$$

and Formula 2.3.16.1 in [40] with real α and $p, q > 0$:

$$K_\alpha^*(p,q) = \int_0^\infty y^{\alpha-1}\, e^{-py-q/y} dy = 2\left(\frac{q}{p}\right)^{\alpha/2} K_\alpha(2\sqrt{p\,q}), \tag{63}$$

where the Macconald function $K_\alpha(u)$ already appears in Formula (42) with different α and argument.

For $\alpha = m + 1/2$, where m is an integer, the Macdonald function $K_{m+1/2}(u)$ has a closed form (see Formulas 2.3.16.2 and 2.3.16.3 in [40] with $p, q > 0$):

$$K_{m+1/2}^*(p,q) = K_m^{**}(p,q) \quad \text{if} \quad m \quad \text{is an integer}, \tag{64}$$

where

$$K_m^{**}(p,q) = \int_0^\infty y^{m-1/2}\, e^{-py-q/y} dy = \begin{cases} (-1)^m \sqrt{\pi}\, \dfrac{\partial^m}{\partial p^m} \left(p^{-1/2}\, e^{-2\sqrt{p\,q}}\right), & m = 0, 1, 2, \dots, \\ (-1)^{-m} \sqrt{\dfrac{\pi}{p}}\, \dfrac{\partial^{-m}}{\partial q^{-m}}\, e^{-2\sqrt{p\,q}}, & m = 0, -1, -2, \dots \end{cases} \tag{65}$$

7.1. The Case $G(x) = \Phi(x)$ and $H(y) = G_{r,r}(y)$

Consider statistics that meet the condition (50).

Let $J_1(x;\gamma) = \int_0^\infty \Phi(x\, y^\gamma) dG_{r,r}(y)$ with $\gamma \in \{0, \pm 1/2\}$. Then, $J_1(x;\gamma) = \Phi(x)$ for $\gamma = 0$ and

$$\begin{aligned} \frac{\partial}{\partial x} J_1(x;\gamma) &= \frac{r^r}{\Gamma(r)\,\sqrt{2\pi}} \int_0^\infty y^{r+\gamma-1} e^{-(x^2 y^{2\gamma}/2 + r\,y)} dy \\ &= \frac{r^r}{\Gamma(r)\,\sqrt{2\,\pi}} \begin{cases} M_{r+1/2}\Big(r\big(1 + x^2/(2r)\big)\Big), & \text{for} \quad \gamma = 1/2, \\ K_{r-1/2}^*(r, x^2/2) = 2\left(\dfrac{x^2}{2r}\right)^{r/2-1/4} K_{r-1/2}(\sqrt{2r}x), & \text{for} \quad \gamma = -1/2. \end{cases} \end{aligned} \tag{66}$$

If $r > 0$ is an integer number then using (64) with $m = r - 1$, the density of $J_1(x; -1/2)$ can be calculated with (65) in a closed form.

Let $\gamma \in \{0, \pm 1/2\}$. Let $k = 1, 2$ and $j = 0, 1, \dots, 5$ be the exponents at $m^{-k/2}$ and x^j in (50), respectively.

$$\begin{aligned} J_2(x;\gamma,k,j) &= \frac{r^r\, x^j}{\Gamma(r)\,\sqrt{2\,\pi}} \int_0^\infty y^{j\gamma + r - 1 - k/2} e^{-(x^2 y^{2\gamma}/2 + ry)} dy \\ &= \frac{r^r\, x^j}{\Gamma(r)\,\sqrt{2\pi}} \begin{cases} M_{j/2+r-k/2}\Big(r\big(1 + x^2/(2r)\big)\Big), & \text{for} \quad \gamma = 1/2, \\ e^{-x^2/2}\, M_{r-k/2}(r), & \text{for} \quad \gamma = 0, \\ K_{r-(j+k)/2}^*(r, x^2/2), & \text{for} \quad \gamma = -1/2. \end{cases} \end{aligned} \tag{67}$$

In (50) $k+j$ are odd integers. If $r>0$ is an integer, then $K^*_{r-(j+k)/2}(r,x^2/2) = K^{**}_{r-(j+k+1)/2}(r,x^2/2)$.

Define $p_j^* = p_j\, I_2(x;\gamma,k,j)$ with coefficient p_j from (50) and calculate the terms in (67):

$$\left.\begin{array}{l} \gamma=1/2,\quad k=1: p_0^* = \dfrac{p_0(2r+x^2)}{2r-1}\, s_{2r}(x),\;\; p_2^* = p_2 x^2\, s_{2r}(x), \\ \qquad\qquad\quad k=2: p_1^* = \dfrac{p_1 x(x^2+2r)}{2r-1}\, s_{2r}(x),\;\; p_3^* = p_3 x^3\, s_{2r}(x),\;\; p_5^* = p_5 \dfrac{x^5(2r+1)}{x^2+2r}\, s_{2r}(x), \\ \gamma=-1/2, k=1, r=2:\;\; p_0^* = p_0\,(|x|+1/2)\, e^{-2|x|},\;\; p_2^* = p_2\, 2\, x^2\, e^{-2|x|}, \\ \qquad k=2,\, r=2:\;\; p_1^* = p_1\, 2\, x\, e^{-2|x|},\;\; p_3^* = p_3\, 4\, |x|\, x\, e^{-2|x|},\;\; p_5^* = p_5\, 4\,(2x^3 + x|x|)\, e^{-2|x|}, \\ \gamma=0,\qquad k=1,2,\;\; j=0,1,2,3,5: p_j^* = p_j\, x^j \varphi(x)\, r^{k/2}\, \Gamma(r-k/2)/\Gamma(r), \end{array}\right\} \tag{68}$$

7.2. *The Case $G(x)=G_d(x)$ and $H(y)=W_s(y)=e^{-s/y}$*

Consider statistics that meet the condition (56). Let $J_1(x;\gamma) = \int_0^\infty G_u(x\,y^\gamma) s y^{-2} e^{-s/y} dy$, $\gamma \in \{0, \pm 1\}$. Then, $J_1(x;0) = G_u(x)$ and

$$\frac{\partial}{\partial x} J_1(x;\gamma) = s\int_0^\infty y^{\gamma-2}\, g_u(x\,y^\gamma)\, e^{-s/y} dy = \frac{s\,x^{d/2-1}}{2^{d/2}\Gamma(d/2)} \int_0^\infty y^{\gamma d/2-2} e^{-x\,y^\gamma/2-s/y} dy,\;\; \gamma=\pm 1\,.$$

Let $\gamma=1$. Using (63) with $\alpha = d/2-1$, $p=x/2$ and $q=s$, we find

$$\begin{aligned} \frac{\partial}{\partial x} J_1(x;1) &= s\int_0^\infty y\, g_u(x\,y)\, e^{-s/y} dy = \frac{s\,x^{d/2-1}}{2^{d/2}\Gamma(d/2)} \int_0^\infty y^{d/2-2} e^{-x\,y/2-s/y} dy \\ &= \frac{s\,x^{d/2}}{2^{d/2}\Gamma(d/2)} K^*_{d/2-1}(x/2,s) = s\left(\frac{sx}{2}\right)^{d/4-1/2} K_{d/2-1}(\sqrt{2s\,x}). \end{aligned}$$

If $d=1,3,5,\ldots$ is an odd number, using the closed form $K^{**}_m(p,q)$ in (65) with $m=(d-3)/2$, $p=x/2$ and $q=s$, then $J_1(x;1) = W_d(x,2s)$ and its density $w_d(x;2s)$ may be calculated in closed form:

$$w_d(x;2s) = \frac{\partial}{\partial x} J_1(x;1) = \frac{s\,x^{d/2-1}}{2^{d/2-1}\Gamma(d/2)} K^{**}_{(d-3)/2}(x/2,s) \quad \text{for} \quad d=1,3,5,\ldots \tag{69}$$

The distribution functions $W_d(x;2s)$ and their densities $w_d(x;2s)$ for $d=1,3,5$ are given in (45)–(47).

If $\gamma=-1$, we use (62) with $\alpha = d/2+1$, $p=(x+2s)/2$, $q=s$ and the substitution $y=1/z$:

$$\begin{aligned} \frac{\partial}{\partial x} J_1(x;-1) &= \frac{s\,x^{d/2-1}}{2^{d/2}\Gamma(d/2)} \int_0^\infty y^{-d/2-2} e^{-(x/2+s)/y} dy \overset{y=1/z}{=} \frac{s\,x^{d/2-1}}{2^{d/2}\Gamma(d/2)} \int_0^\infty z^{d/2} e^{-(x+2s)\,z/2} dz \\ &= \frac{s\,x^{d/2-1}\,\Gamma(d/2+1)\,2^{d/2+1}}{2^{d/2}\Gamma(d/2)(x+2s)^{d/2+1}} = \frac{s\,d\,x^{d/2-1}}{(x+2s)^{d/2+1}} = \frac{s\,d}{x^2}\left(1+\frac{2\,s}{x}\right)^{-d/2-1} = v_{d/2}(x;2s). \end{aligned} \tag{70}$$

where $v_{d/2}(x;2s)$ is the density of the *inverse Pareto distribution* defined in (49).

Suppose $\gamma \in \{0, \pm 1\}$. Let $j=1,2$ be the exponent at x^j in (56). Then, by (65) for positive odd numbers d with $\alpha = j+(d-7)/2$, $p=x/2, q=s$ if $\gamma=1$, by (62) with $\alpha=2$, $p=s$ for $\gamma=0$ and with $\alpha = j+d/2-1$, $p=(x+2s)/2$ for $\gamma=-1$:

$$J_2(x;\gamma,2,j) = \frac{s\,x^j}{2^{d/2}\Gamma(d/2)} \int_0^\infty y^{j\gamma-3}(xy^\gamma)^{d/2-1}\, e^{-(xy^\gamma/2+s/y)}dy$$
$$= \frac{s\,x^{j+d/2-1}}{2^{d/2}\Gamma(d/2)} \int_0^\infty \frac{y^{\gamma(j+d/2-1)-3}}{e^{xy^\gamma/2+s/y}}dy = \frac{s\,x^{j+d/2-1}}{2^{d/2}\Gamma(d/2)} \begin{cases} K^{**}_{j+(d-7)/2}(x/2,s), & \gamma=1, \\ e^{-x/2}\,M_2(s), & \gamma=0, \\ M_{j+1+d/2}((x+2s)/2), & \gamma=-1. \end{cases} \quad (71)$$

If d is not an odd number, $K^{**}_{j+(d-7)/2}(x/2,s)$ in (71) has to be replaced by $K^*_{j+d/2-3}(x/2,s)$, which may be calculated with (63) where the Macdonald functions $K_{j+d/2-3}(\sqrt{2sx})$ are involved.

Define $q_j^* = q_j\, I_2(x;\gamma,2,j)$ for $j=1,2$ with the coefficient q_j from (56). Calculating the corresponding terms in (71) we find

$$\left.\begin{array}{lll} \gamma=1,\, d=1: & q_1^* = q_1\, x\,(2s)^{-1}\left(1+\sqrt{2sx}\right) w_1(x), & q_2^* = q_2\, x^2 w_1(x), \\ \gamma=1,\, d=3: & q_1^* = q_1\,(2s)^{-1}\, x\,\sqrt{2sx}\; w_3(x), & q_2^* = q_2\, x^2\, w_3(x), \\ \gamma=-1: & q_1^* = q_1 \dfrac{x(d+2)}{x+2s}\, v_{d/2}(x;2s), & q_2^* = q_2\, \dfrac{x^2(d+4)(d+2)}{(x+2s)^2}\, v_{d/2}(x;2s), \\ \gamma=0: & q_1^* = q_1\, x\, s^{-1}\, g_{d/2}(x) & q^*21 = q_2\, x^2\, s^{-1}\, g_{d/2}(x) \end{array}\right\} \quad (72)$$

8. Proof of Theorems

We find from Lemmas A1 and A2 that D_n in (5) in Proposition 1 is bounded and the integrals in (10) and (11) in Proposition 2 have the necessary convergence rates. It remains to calculate the integrals in (9).

Proof of Theorem 1. Let $F(x) = \Phi(x)$, $H(y) = G_{r,r}(y)$ and $h_2(y) = h_{2;r}(y)$ defined in (32).

Suppose $J_1(x;\gamma) = \int_0^\infty \Phi(x\,y^\gamma)dG_{r,r}(y)$ with $\gamma \in \{0, \pm 1/2\}$, which are the limit distributions in (9) for $\mathbb{P}\left(g_n^\gamma\, N_n(r)^{1/2-\gamma}\, Z_{N_n(r)} \le x\right)$ under the condition of Theorem 1. Then, $J_1(x;\gamma) = \Phi(x)$ for $\gamma = 0$. It follows from (66), (62) for $\gamma = 1/2$ and (65) for $\gamma = -1/2$ that

$$\frac{\partial}{\partial x} J_1(x;\gamma) = \begin{cases} s_{2r}(x) = \dfrac{\Gamma(r+1/2)}{\sqrt{2r\pi}\,\Gamma(r)}\left(1+\dfrac{x^2}{2r}\right)^{-(r+1/2)}, & \gamma=1/2 & \text{with } J_1(x;1/2) = S_{2r}(x), \\ \varphi(x) = \dfrac{1}{\sqrt{2\pi}}\, e^{-x^2/2}, & \gamma=0 & \text{with } J_1(x;0) = \Phi(x), \\ l_2(x) = \left(\dfrac{1}{2}+|x|\right) e^{-2|x|}, \quad r=2, & \gamma=-1/2, & \text{with } J_1(x;-1/2) = L_2(x), \end{cases} \quad (73)$$

where $s_{2\,r}(x)$ is the density of Student's t-distribution with $2\,r$ degrees of freedom and $l_2(x)$ is the density of a generalized Laplace distribution.

Integral $J_2(x;\gamma) = \int_0^\infty y^{-1/2}\,(p_0 + p_2\,x^2\,y^{2\gamma})\,\varphi(x\,y^\gamma)\,dG_{r,r}(y)$ is the integral by $g_n^{-1/2}$ in the expansion (9). Then, using (67) and (68) with $k=1$, we obtain

$$J_2(x;\gamma) = p_0\, J_2(x;\gamma,1,0) + p_2\, J_2(x;\gamma,1,2) = p_0^* + p_2^*, \quad (74)$$

Integral $J_3(x;\gamma) = \int_0^\infty y^{-1}\,(p_1 x\,y^\gamma + p_3 x^3\,y^{3\gamma} + p_5 x^5\,y^{5\gamma})\,\varphi(x\,y^\gamma)\,dG_{r,r}(y)$ is the integral by g_n^{-1} in the expansion (9). Then, using again (67) and (68) with $k=2$, we obtain

$$J_3(x;\gamma) = p_1\, J_2(x;\gamma,2,1) + p_3\, J_2(x;\gamma,2,3) + p_5\, J_2(x;\gamma,2,5) = p_1^* + p_3^* + p_5^* \quad (75)$$

Integration by parts in the last integral by n^{-1} in (9) for $\gamma = \pm 1/2$ and $r > 1$ leads to

$$J_4(x;\gamma) = \int_0^\infty \Phi(xy^\gamma)dh_{2,r}(y) = -\frac{\gamma\,x\,r^r}{2\,r\,\sqrt{2\,\pi}\,\Gamma(r)} \int_0^\infty \frac{y^{\gamma+r-2}}{e^{x^2y^{2\gamma}/2+r\,y}}\left((y-1)(2-r) + 2Q_1(g_n\,y)\right)dy.$$

Suppose $\gamma = 1/2$. We find from (62)

$$\begin{aligned} J_4(x;1/2) &= \frac{x\,r^r\,(2-r)}{4r\,\sqrt{2\,\pi}\,\Gamma(r)}\Big(M_{r-1/2}(r+x^2/2) - M_{r+1/2}(r+x^2/2)\Big) - J_4^*(x;1/2) \\ &= \frac{(2-r)x(x^2+1)}{4\,r\,(2r-1)}\,s_{2r}(x) - J_4^*(x;1/2). \end{aligned}$$

with

$$J_4^*(x;1/2) = \frac{x r^{r-1}}{2\sqrt{2\pi}\Gamma(r)} \int_0^\infty y^{r-3/2}\,e^{-(r+x^2/2)y}\,Q_1\big(g_n y\big)\,dy, \tag{76}$$

where $Q_1(y)$ is defined in (33). It follows from Lemma A3 that for $r > 1$

$$\sup_x n^{-1}|J_4^*(x;1/2)| \le c(r)\,n^{-r}.$$

Hence, because of $0 \le g_n^{-1} - (rn)^{-1} \le (ng_n)^{-1}$ for $r \ge 1$, we obtain

$$\left|\frac{1}{n}\int_0^\infty \Phi(x\sqrt{y})dh_2(y) - \frac{(2-r)x(x^2+1)}{4\,(2r-1)g_n}\,s_{2r}(x)\right| \le \frac{1}{n}|J_4^*| + \frac{C(r)}{n\,g_n} \le c_1(r)\,n^{-\min\{r\ 2\}}. \tag{77}$$

For $\gamma = -1/2$, we only consider the case $r = 2$, which results in $J_4(x;-1/2) = 0$ and

$$J_4^*(x;-1/2) = \frac{x}{2\sqrt{2\pi}} \int_0^\infty y^{-1/2}\,Q_1\big(g_n y\big)\,e^{-(2y+x^2/(2y))}\,dy, \tag{78}$$

where $\sup_x n^{-1} J_4^*(x;-1/2) \le C\,n^{-2}$ is proved in Lemma A3.

If $\gamma = 0$, then $J_4(x;0) = \Phi(x)\,(h_{2;r}(\infty) - h_{2;r}(0)) = 0$ since $Q_1(0) = 1/2$.

The proof of Theorem 1 follows from (73)–(75) and (77) and Lemma A3. □

Proof of Theorem 2. Let $F(x) = G_d(x)$, $H(y) = W_s(y) = e^{-s/y}$ and $h_2(y) = h_{2;s}(y)$ defined in (32).

Suppose $J_1(x;\gamma) = \int_0^\infty G_d(x\,y^\gamma)s\,y^{-2}e^{-s/y}dy$ with $\gamma \in \{0, \pm 1\}$ which are the limit distributions in (9) for $\mathbb{P}\Big(g_n^\gamma\,N_n(s)^{1/2-\gamma}\,Z_{N_n(s)} \le x\Big)$ under the condition of Theorem 2.

Then, $J_1(x;\gamma) = G_d(x)$ for $\gamma = 0$. It follows from (69) and (65) for $\gamma = 1$ and (70) for $\gamma = -1$ that

$$\frac{\partial}{\partial x}\,J_1(x;\gamma) = \begin{cases} w_1(x;2s) = s\,(2sx)^{-1/2}\,e^{-\sqrt{2\,s\,x}} & \gamma = 1 \quad \text{with } J_1(x;1) = W_1(x), \\ w_3(x;2s) = s\,e^{-\sqrt{2\,s\,x}} & \gamma = 1 \quad \text{with } J_1(x;1) = W_3(x), \\ \varphi(x) = \dfrac{1}{\sqrt{2\,\pi}}\,e^{-x^2/2}, & \gamma = 0 \quad \text{with } J_1(x;0) = \Phi(x), \\ v_{d/2}(x;2s) = \dfrac{s\,d\,x^{d/2-1}}{(x+2s)^{d/2+1}} & \gamma = -1, \quad \text{with } J_1(x;-1/2) = V_{d/2}(x;2s), \end{cases}$$

where $w_1(x;2s)$ is the Weibull density (see (45)), $w_3(x;2s)$ is the generalized gamma density0 (see (46)) and $v_{d/2}(x;2s)$ is the density of the inverse Pareto distribution $V_{d/2}(x;2s)$ defined in (49).

Integral $J_2(x;\gamma) = \int_0^\infty y^{-1}\,(q_1 x\,y^\gamma + q_2 x_2\,y^{2\gamma})\,g_{d/2}(x\,y^\gamma)\,sy^{-2}e^{-s/y}dy$ is the integral by g_n^{-1} in the expansion (9). Then, the use of (71) and (72) leads to

$$J_2(x;\gamma) = q_1\,J_2(x;\gamma,2,1) + q_3\,J_2(x;\gamma,2,3) = q_1^* + q_3^*.$$

Integration by parts in the last integral by n^{-1} in (9) for $\gamma = \pm 1$ leads to

$$
\begin{aligned}
J_3(x;\gamma,d) &= \int_0^\infty G_d(xy^\gamma) dh_{2;s}(y) = J_4(x;\gamma,d) + J_4^*(x;\gamma,d), \\
J_4(x;\gamma,d) &= -\frac{s\,(s-1)\,\gamma\, x^{d/2}}{2^{d/2+1}\,\Gamma(d/2)} \int_0^\infty y^{\gamma\, d/2-3}\, e^{-xy^\gamma/2-s/y} dy = -\frac{s\,(s-1)\,\gamma\, x^{d/2}}{2^{d/2+1}\,\Gamma(d/2)}\, K^{**}_{(\gamma d-5)/2}(x/2,s), \\
J_4^*(x;\gamma,d) &= -\frac{s\,\gamma\, x^{d/2}}{2^{d/2}\,\Gamma(d/2)} \int_0^\infty y^{\gamma\, d/2-3}\, e^{-xy^\gamma/2-s/y} Q_1(n\,y) dy, \qquad (79)
\end{aligned}
$$

where $Q_1(y)$ is defined in (33). Suppose $\gamma = 1$. We get with (65)

$$
J_4(x;1,1) = -\frac{(s-1)\,x\,(\sqrt{2sx}+1)}{4\,s}\, w_1(x;2s) \quad \text{and} \quad J_4(x;1,3) = -\frac{(s-1)\,x^2}{2\,\sqrt{2sx}}\, w_3(x;2s)
$$

For $\gamma = -1$ using (65), we see that

$$
J_4(x;-1,d) = \frac{s\,(s-1)\,x^{d/2}\,\sqrt{2sx}}{2^{1+d/2}\,\Gamma(d/2)} M_{(d+4)/2}\big((x+2s)/2\big) = \frac{(s-1)\,x\,(2+d)}{2\,(x+2s)}\, v_{d/2}(x;2s).
$$

In Lemma A4, $\sup_x\, n^{-1}\, J_4^*(x;\gamma,d) \le c(s) n^{-2}$ for $\gamma = \pm 1$ is proved.

If $\gamma = 0$, then $J_3(x;0,d) = G_d(x)\,(h_{2;s}(\infty) - \lim_{y\to 0} h_{2;s}(y)) = 0$.

Combining the above estimates proves Theorem 2. □

9. Conclusions

Chebyshev–Edgeworth expansions are derived for the distributions of various statistics from samples with random sample sizes. The construction of these asymptotic expansions is based on the given asymptotic expansions for the distributions of statistics of samples with a fixed sample sizes as well as those of the distributions of the random sample sizes.

The asymptotic laws are scale mixtures of the underlying standard normal or chi-square distributions with gamma or inverse exponential mixing distributions. The results hold for a whole family of asymptotically normal or chi-squared statistics since a formal construction of asymptotic expansions are developed. In addition to the random sample size, a normalization factor for the examined statistics also has a significant influence on the limit distribution. As limit laws, Student, standard normal, Laplace, inverse Pareto, generalized gamma, generalized Laplace and weighted sums of generalized gamma distributions occur. As statistica the random mean, the scale-mixed normalized Student t-distribution and the Student's t-statistic under non-normality with normal limit law, as well as Hotelling's generalized T_0^2 and scale mixture of chi-squared statistics with chi-square limit laws, are considered. The bounds for the corresponding residuals are presented in terms of inequalities.

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Appendix A. Auxiliary Statements and Lemmas

In Section 3, we consider statistics satisfying (1) in Assumption 1 and in Section 4 random sample sizes satisfying (2) in Assumption 2. The statistics T_m in (15), (18) and (21) satisfy Assumption 1 with the normal limit distribution $\Phi(x)$ and in (26) and (27) with chi-square limit distributions $G_d(x)$ and $G_4(x)$, defined in (17), respectively.

Further, we estimate the functions $f_k(xy^\gamma)$ and $\frac{\partial}{\partial y}\left(\frac{f_k(xy^\gamma)}{y^{k/2}}\right)$, $k = 1, 2$ that appear in (1) of Assumption 1 and in the term D_n in (5). Since the functions $f_k(z)$ are products of a polynomial $P_k(z)$ and a density function $p(z)$ with $p(z) = \varphi(z)$ or $p(z) = g_{r,r}(z)$, it follows for $\gamma \in \{\pm 1/2, \pm 1\}$ that, if

$$f_k(xy^\gamma) = P_k(xy^\gamma)\, p(xy^\gamma) \quad \text{then} \quad \frac{\partial}{\partial y}\left(\frac{P_k(xy^\gamma)\, p(xy^\gamma)}{y^{k/2}}\right) = \frac{Q_k(xy^\gamma)\, p(xy^\gamma)}{y^{1+k/2}}, \tag{A1}$$

with some polynomial $Q_k(z)$. For $\gamma = 0$, we have $f_k(xy^\gamma) = f_k(x)$ and $Q_k(x) = -(k/2)\, f_k(x)$. Hence, (A1) also holds for $\gamma = 0$.

For example, with $f_1(z) = \frac{\lambda_3}{6}(z^2 - 1)\varphi(z)$ occurring in (16) for the sample mean $\overline{X}_m$ and $f_2(z) = (Az - Bz^2)\, g_d(z)$ occurring in (27) with $d = 4$ for scale mixture of chi-square statistics, we obtain

$$\frac{\partial}{\partial y}\left(\frac{f_1(x\,y^\gamma)}{y^{1/2}}\right) = \frac{Q_1(x\,y^\gamma)\,\varphi(x\,y^\gamma)}{y^{3/2}} \quad \text{and} \quad \frac{\partial}{\partial y}\left(\frac{f_2(x\,y^\gamma)}{y}\right) = \frac{Q_2(x\,y^\gamma)\, g_d(x\,y^\gamma)}{y^2}$$

with $Q_1(z) = \lambda_3\,(1/2 + (2\gamma - 1/2)\,z^2 - \gamma\, z^4)/6$ and $Q_2(z) = (Bz^3 - (Bd - 2)\,z^2 + A(d-2)z)/4$.

Remark A1. *If $P_k(0) \neq 0$, i.e., the absolute term of the polynomial $P_k(z)$ is not equal to zero, then it is also the absolute term of $Q_k(z)$, i.e., $Q_k(0) \neq 0$.*

The functions $\varphi(z)$ and e^{-rz} in $g_{r,r}(z)$ allow obtaining the estimates for

$$c_k^* = \sup_z |f_k(z)| < \infty \quad \text{and} \quad c_k^{**} = \sup_z |Q_k(z)|\, p(z) < \infty, \qquad k = 1, 2. \tag{A2}$$

Appendix A.1. Lemmas A1 and A2

Lemma A1. *Consider the statistics estimated in (15), (18), (21), (26) and (27). Let g_n be a sequence with $0 < g_n \uparrow \infty$ as $n \to \infty$ and $\gamma \in \{-1/2, 0, 1/2, 1\}$. Then, with some computable constant $0 < C^*(\gamma) < \infty$, we obtain*

$$D_n = \sup_x \int_{1/g_n}^\infty \left|\frac{\partial}{\partial y}\left(F(xy^\gamma) + \frac{f_1(xy^\gamma)}{\sqrt{g_n y}} + \frac{f_2(xy^\gamma)}{y g_n}\right)\right| dy \le C^*(\gamma),$$

where $F(x)$, $f_1(x)$ and $f_2(x)$ are defined in the approximation estimates (15), (18), (21), (26) and (27).

Proof of Lemma A1. The statistics in (15), (18) and (21) satisfy Assumption 1 with the normal limit distribution $\Phi(x)$. To estimate $D_n = \sup_x |D_n(x)|$, we consider the cases $x \neq 0$ and $x = 0$.

Let $x \neq 0$. Since $\frac{\partial}{\partial y}\Phi(x\,y^\gamma) = 0$ for $\gamma = 0$ and $\frac{\partial}{\partial y}\Phi(x\,y^\gamma) = \gamma\, x\, y^{\gamma-1}\varphi(x\,y^\gamma)dy$ has constant sign$(\gamma\, x)$ for $\gamma = \pm 1/2$ and $y > 0$, we find

$$\int_{1/g_n}^\infty \left|\frac{\partial}{\partial y}\Phi(x\,y^\gamma)\right| dy = \left|\int_{1/g_n}^\infty \gamma\, x\, y^{\gamma-1}\varphi(x\,y^\gamma)dy\right| = \begin{cases} 1 - \Phi(x\,g_n^{-\gamma}) \le 1/2 & \text{for} \quad x > 0. \\ \Phi(x\,g_n^{-\gamma}) \le 1/2 & \text{for} \quad x < 0. \end{cases}$$

From (A1) and (A2), it follows that $\left|\dfrac{\partial}{\partial y}\left(\dfrac{f_k(x\,y^\gamma)}{(g_n\,y)^{k/2}}\right)\right| \le 2c_k^{**}/k$, which proves the first case. Moreover, $D_n(0) = |Q_1(0)|$ since $Q_2(0) = 0$ for the considered statistics.

Consider now the statistics estimated in (26) and (27) with limit chi-square distributions. We only need to examine $x > 0$ and $\gamma \in \{0, \pm 1\}$. In the cases now under review, we have $f_1(x) = 0$ and $f_2(z) = (Az + Bz^2)g_d(z)$ with some real constants A and B. The proof is completed with (A1) and (A2),
$\int_{1/g_n}^{\infty}\left|\dfrac{\partial}{\partial y}G_d(xy\gamma)\right|dy = \left|\int_{1/g_n}^{\infty}\dfrac{\partial}{\partial y}G_d(xy\gamma)dy\right| \le 1$ and $\int_{1/g_n}^{\infty}\left|\dfrac{\partial}{\partial y}\left(\dfrac{f_2(xy^\gamma)}{g_n\,y}\right)\right|dy \le c_2^{**}$. □

Next, the integrals in (10) and (11) in Proposition 2 for the gamma limit distributions $H(y) = G_{r,r}(y)$ and the inverse exponential limit distribution $H(y) = \exp\{-s/y\}$ are estimated.

Lemma A2. *(i) The conditions (6), (7) and (8) in Proposition 2 are satisfied for $G_{r,r}(y)$ and $W_s(y) = e^{-s/y}$.*

(ii) Let $\gamma \in \{0, \pm 1/2\}$. Consider $f_1(x)$ given in (16) and $f_2(z)$ is given in (16) or (18) for statistics occurring in (15), (18) and (21) with limiting distribution $\Phi(x)$.

(iia) Let the mixing distribution be $H(y) = G_{r,r}(y)$ with $g_n = r(n-1)+1$, $b = \min\{r, 2\}$ and $h_2(y) = h_{2;r}(y) = g_{r,r}(y)\big((y-1)(2-r) + 2Q_1(g_n\,y)\big)$, $y > 0$. Then, we obtain with $k = 1, 2$

$$\sup_x |I_1(x,n)| \le \sup_x \int_{1/g_n}^{\infty}\left|\frac{f_1(x\,y^\gamma)}{(g_n y)^{1/2}}\right| dG_{r,r}(y) \le c_1^*\,g_n^{-r} \quad \textit{for} \quad 0<r<1/2,\ \gamma\in\{0,\pm1/2\}, \tag{A3}$$

$$\sup_x |I_1(x,n)| \le c_1^*\,g_n^{-1/2}\ln g_n \quad \textit{for} \quad r=1/2, \quad \gamma = \pm\frac{1}{2}, \tag{A4}$$

$$\sup_x |I_2(x,n)| \le \sup_x \int_{1/g_n}^{\infty}\left|\frac{f_2(x\,y^\gamma)}{g_n y}\right| dG_{r,r}(y) \le c_2^*\,g_n^{-r} \quad \textit{for} \quad 0<r<1,\ \gamma\in\{0,\pm1/2\}, \tag{A5}$$

$$\sup_x |I_2(x,n)| \le c_3\,g_n^{-1} \quad \textit{for} \quad r=1, \quad \gamma=\pm1/2, \tag{A6}$$

$$\sup_x \left|I_k(x,n) - g_n^{-k/2} f_k(x)\ln g_n\right| \le c_4 g_n^{-k/2} \quad \textit{for} \quad r=k/2, \quad \gamma=0, \tag{A7}$$

$$\sup_x |I_{2+k}(x,n)| = \sup_x\left|\int_{1/g_n}^{\infty}\frac{f_k(x\,y^\gamma)}{n\,g_n^{k/2}\,y}dh_{2;r}(y)\right| \le \begin{cases} c_5(r)g_n^{-r}, & r>1, r\ne 1+k/2,\\ c_6(r)\,g_n^{-1-k/2}\ln n, & r=1+k/2.\end{cases} \tag{A8}$$

(iib) Now, consider the mixing distribution $H(y) = W_s(y) = e^{-s/y}$ with $g_n = n$, $b = 2$ and $h_2(y) = h_{2;s}(y) = s\,e^{-s/y}\,(s - 1 + 2Q_1(n\,y))/(2\,y^2)$, $y > 0$. Then, apply it to $I_4(x,n)$ in (11)

$$\sup_x |I_4(x,n)| = \sup_x\left|\int_{1/n}^{\infty}\frac{f_2(x\,y^\gamma)}{n^2\,y}\,dh_{2;s}(y)\right| \le c_6(s)n^{-2}, \quad \textit{for} \quad s>0,\ \gamma\in\{0,\pm1/2\}. \tag{A9}$$

(iii) Put $\gamma \in \{0, \pm 1\}$. Consider $f_2(x) = (Ax + Bx^2)g_d(x)$ for statistics occurring in (26) and (27) satisfying Assumption 1 with the chi-square distribution $G_d(x)$.

(iiia) The mixing distribution is $H(y) = G_{r,r}(y)$ as in Case (ia) above. Then, for $\gamma \in \{0, \pm 1\}$,

$$\sup_x |I_2(x,n)| \le \sup_x \int_{1/g_n}^{\infty}\left|\frac{f_2(x\,y^\gamma)}{g_n y}\right| dG_{r,r}(y) \le \frac{c_2^*\,r^r}{(r-1)\,\Gamma(r)}\,g_n^{-r} \quad \textit{for} \quad r<1, \tag{A10}$$

$$\left.\begin{array}{ll} \sup_x |I_2(x,n)| \le c_2^*\,n^{-1} & \textit{with} \quad \gamma=\pm1/2,\\ \sup_x \left|I_2(x,n) - n^{-1}f_2(x)\ln n\right| \le c_2^* n^{-1} & \textit{with} \quad \gamma = 0, \end{array}\right\} \quad \textit{for} \quad r=1, \tag{A11}$$

$$\sup_x |I_4(x,n)| = \sup_x\left|\int_{1/g_n}^{\infty}\frac{f_2(x\,y^\gamma)}{n\,g_n\,y}dh_{2;r}(y)\right| \le \begin{cases} c_3(r)g_n^{-\min\{r,2\}}, & \textit{for } r>1, r\ne 2,\\ c_4(r)\,g_n^{-2}\ln n, & \textit{for } r=2.\end{cases} \tag{A12}$$

(iiib) The mixing distribution is $H(y) = W_s(y) = e^{-s/y}$ with $g_n = n$ and $b = 2$ as in Case (iib). Then,

$$\sup_x |I_4(x,n)| = \sup_x \left| \int_{1/g_n}^{\infty} \frac{f_2(x\,y^{\gamma})}{n^2 y} dh_{2;s}(y) \right| \leq C_5(r) n^{-2}. \quad \textit{for} \quad s > 0, \ \gamma \in \{0, \pm 1/2\}. \tag{A13}$$

Proof of Lemma A2. (i) Insertion of $G_{r,r}(y)$ with $h_{2,r}(y)$ and $W_s(y) = e^{-s/y}$ with $h_{2,s}(y)$ and simple calculation result in the necessary estimates in (6)–(8). In the case of $W_s(y) = e^{-s/y}$, one even gets for all terms exponentially fast decrease.

(ii) The limit distribution of the considered statistics is standard normal $\Phi(x)$.

(iia) Let $H(x) = G_{r,r}(x)$. Using (A2), the estimations (A3) and (A5) for $r < k/2$, with $k = 1, 2$, are

$$\sup_x |I_k(x,n)| \leq \frac{c_k^* \, r^r}{g_n^{k/2}\,\Gamma(r)} \int_{1/g_n}^{\infty} y^{r-1-k/2} dy \leq \frac{c_k^* \, r^r}{(k/2 - r)\,\Gamma(r)}\, g_n^{-r}.$$

Taking into account

$$0 \leq \ln g_n - \int_{1/g_n}^{1} \frac{e^{-ry}}{y} dy = \int_{1/g_n}^{1} \frac{1 - e^{-ry}}{y} dy \leq r \quad \text{and} \quad \int_1^{\infty} \frac{e^{-ry}}{y}\, dy \leq e^{-r}/r \quad \text{for} \quad r > 0, \tag{A14}$$

the bound (A4) follows from

$$|I_1(x,n)| \leq \frac{c_1^*}{(2\,g_n)^{1/2}\,\Gamma(1/2)} \left(\int_1^{\infty} \frac{e^{-y/2}}{y}\, dy + \int_{1/g_n}^{1} \frac{e^{-y/2}}{y} dy \right) \leq \frac{c_1^*(2e^{-1/2} + \ln g_n)}{\Gamma(1/2)\,(2\,g_n)^{1/2}}.$$

If $r = 1$ with $d_2^* = \sup_z\{|z^{-1} f_2(z)| \varphi(z/\sqrt{2})\}$, we find $|f_2(z)| \leq d_2^* |z|\, \varphi(z/\sqrt{2})$ and the bound (A6) follows from

$$|I_2(x,n)| \leq \frac{d_2^*\,|x|}{\sqrt{2\,\pi}\,n} \int_{1/n}^{\infty} y^{\gamma-1} e^{-\,(y + x^2\,y^{2\,\gamma}/4)}\, dy \quad \text{with} \quad \gamma = \pm 1/2,$$

where for $\gamma = 1/2$ using $|x|\,(1 + x^2/4)^{-1/2} \leq 2$, we obtain

$$|I_2(x,n)| \leq \frac{d_2^*\,|x|}{\sqrt{2\,\pi}\,n} \int_{1/n}^{\infty} y^{1/2-1} e^{-\,(1+x^2/4)\,y}\, dy \leq \frac{d_2^*\,|x|\Gamma(1/2)}{\sqrt{2\,\pi}\,(1 + x^2/4)^{1/2}} n^{-1} \leq \frac{\sqrt{2}\,d_2^*}{n}$$

and, in the case of $\gamma = -1/2$, the substitution $z = x^2/(4\,y)$ for $x \neq 0$ leads to

$$I_2(x,n) \leq \frac{c_2^*\,|x|}{\sqrt{2\,\pi}\,n} \int_{1/n}^{\infty} y^{-1-1/2}\, e^{-(y + x^2/(4\,y))} dy \leq \frac{2\,c_2^*}{\sqrt{2\,\pi}\,n} \int_0^{\infty} z^{-1/2}\, e^{-z} dz \leq \frac{\sqrt{2}\,d_2^*}{n}.$$

Finally, if $\gamma = 0$, then $f_k(x\,y^{\gamma}) = f_k(x)$ does not depend on y. Then, (A7) follows from (A1), (A2), and (A14) for $r = k/2$.

Let $r > 1$. Integration by parts for Lebesgue–Stieltjes integrals $I_{k+2}(x,n)$, $k = 1, 2$, in (11) leads to

$$\sup_x I_{k+2}(x,n) \leq \frac{1}{n\,g_n^{k/2}} \left(c_k^*\,|h_{2;r}(1/g_n)| + \sup_x \int_{1/g_n}^{\infty} \frac{|Q_k(xy^{\gamma})|}{y^{1+k/2}}\, |h_{2;r}(y)| dy \right) \tag{A15}$$

with bound c_k^* given in (A2). Defining $C_r^* = \dfrac{r^r}{2r\,\Gamma(r)} \sup_y\{e^{-r\,y}\,(|y-1||2-r| + 1)\} < \infty$, we find

$$\int_{1/g_n}^{\infty} \frac{|h_{2;r}(y)|}{y^{1+k/2}} dy \leq C_r^* \int_{1/g_n}^{\infty} y^{r-2-k/2} dy = \frac{C_r^*}{(1 + k/2 - r)}\, g_n^{-r+1+k/2} \quad \text{for} \quad 1 < r < 1 + k/2$$

and, with $C_r^{**} = \dfrac{r^{r-1}}{2\,\Gamma(r)}\ \sup_y\{(e^{-r\,y/2}\,(|y-1|\,|2-r|+1)\} < \infty$, we obtain

$$\int_{1/g_n}^{\infty} \frac{|h_{2;r}(y)|}{y^{1+k/2}}dy \le C_r^{**} \int_{1/g_n}^{\infty} y^{r-2-k/2}e^{-r\,y/2}dy \le \frac{C_r^{**}\Gamma(r-1-k/2)}{(r/2)^{r-1-k/2}} \quad \text{for} \quad r > 1+k/2.$$

Hence, using $g_n \le r\,n$ for $r > 1$, we obtain (A8) and, for $r > 1$, $r \ne 1+k/2$.

For $r = 1+k/2$, the second integral in the line above is an exponential integral. Therefore, with (A14), we find (A8) for $r = 1+k/2$, too.

(iib) The mixing distribution is $H(y) = W_s(y) = e^{-s/y}$ with $g_n = n$. Since $b = 2$, only $I_4(x,n)$ has to be estimated. Integration by parts for $I_4(x,n)$ in (11) leads to (A15) with $k = 2$, $g_n = n$ and $h_{2,s}(y)$ instead of $h_{2,r}(y)$. Hence, (A9) follows from

$$\int_{1/n}^{\infty} \frac{|h_{2;s}(y)|}{y^2}dy \le s(s+2)\int_{1/n}^{\infty} y^{-4}e^{-s/y}dy \le \frac{s+2}{s^2}\int_0^{s\,n} z^2e^{-z}dz \le \frac{(s+2)\Gamma(3)}{s^2}. \qquad \text{(A16)}$$

(iii) The limit distribution of statistics in (26) and (27) is chi-square distribution $G_u(x)$ defined in (17). In the considered cases $f_1(x) = 0$. Let $\gamma \in \{0, \pm 1\}$. Consider $f_2(x) = (Ax+Bx^2)g_d(x)$ with chi-square density $g_d(x)$.

(iiia) Let $H(y) = G_{r,r}(y)$. We have to estimate $I_2(x,n)$ for $r \le 1$ and $I_4(x,n)$ for $r > 1$. The bound (A10) for $0 < r < 1$ follows from (A2) and

$$\sup_x |I_2(x,n)| \le \frac{c_2^*\,r^r}{g_n\,\Gamma(r)}\int_{1/g_n}^{\infty} y^{r-2}e^{-ry}dy \le \frac{c_2^*\,r^r}{(r-1)\,\Gamma(r)}\,g_n^{-r} \quad \text{for} \quad \gamma \in \{0, \pm 1\}.$$

If $r = 1$ with $C_2^* = \sup_z\left\{|A+Bz|\dfrac{1}{2^{d/2}\Gamma(d/2)}\,e^{-z/4}\right\} < \infty$, we find $|f_2(z)| \le C_2^* z^{d/2}\,e^{-z/4}$ and

$$|I_2(x,n)| \le \frac{C_2^*\,x^{d/2}}{g_n}\int_{1/g_n}^{\infty} y^{-1+d/2}\,e^{-\,(1+x/4)\,y}\,dy \le \frac{C_2^*\,x^{d/2}}{g_n\,(1+x/4)^{d/2}} \le \frac{4^{d/2}\,C_2^*}{g_n} \quad \text{for} \quad \gamma = 1.$$

in the case of $\gamma = -1$ using variable transformation $z = x/(4y)$ for $x > 0$ one has

$$|I_2(x,n)| \le \frac{C_2^*\,x^{d/2}}{g_n}\int_{1/g_n}^{\infty} \frac{e^{-x/(4y)}}{y^{1+d/2}}\,dy \le \frac{C_2^*\,x^{d/2}}{g_n\,(x/4)^{d/2}}\int_0^{\infty} z^{-1+d/2}e^{-z}\,dz \le \frac{C_2^*\,4^{d/2}\,\Gamma(d/2)}{g_n}.$$

If $\gamma = 0$ then $f_2(x\,y^{\gamma}) = f_2(x)$, noting (A14) and $g_n = n$ for $r = 1$, we prove (A11).

Let now $r > 1$. It remains to estimate $I_4(x,n)$. Using (A15) with $k = 2$, remembering (A2), we obtain (*A12*) in the same way as for $r > 1$ with $k = 2$ in case (iia) above.

(iiib) The limit distribution $H(y) = W_s(y) = e^{s/y}$ with $g_n = n$ and $b = 2$. As in Case (iib), taking into consideration (A16), we obtain (A13). □

Appendix A.2. Lemmas A3 and A4

We show that the integrals $J_4^*(x;\gamma)$ and $J_4^*(x;\gamma,d)$ in the proofs of Theorems 1 and 2 have the order of the remaining terms. Therefore, the involved jump correcting function $Q_1(y) = 1/2 - (y-[y])$ occurring in (32) and (38) has no effect on the second approximation. The function $Q_1(y)$ is periodic with period 1. The Fourier series expansion of $Q_1(y)$ at all non-integer points y is

$$Q_1(y) = 1/2 - (y-[y]) = \sum_{k=1}^{\infty} \frac{\sin(2\,\pi\,k\,y)}{k\,\pi} \qquad y \ne [y] \qquad \text{(A17)}$$

(see formula 5.4.2.9 in [40] with $a = 0$).

Lemma A3. *Let $J_4^*(x;\pm 1)$ be defined by (76) and (78), respectively. Then, $n^{-1}\, J_4^*(x;1/2) \le C\, n^{-r}$ for $r > 1$ and $n^{-1}\, J_4^*(x;-1/2) \le C\, n^{-2}$.*

Proof of Lemma A3. We begin by considering $Q_1(y)$ in $J_4^*(x;1/2)$ defined in (A17) following the estimate of $J_4^*(x)$ in the proof of Theorem 2 in [11]. Inserting Fourier series expansion of $Q_1(y)$ into the integral $J_4^*(x;1/2)$, interchanging the integral and sum and applying formula (2.5.31.4) in [40] with $\alpha = r - 1/2$, $p = (r + x^2/2)$ and $b = 2\pi k g_n$, then

$$J_4^*(x;1/2) = \frac{x r^{r-1}}{(2\pi)^{3/2}\Gamma(r)} \sum_{k=1}^{\infty} \frac{1}{k} \int_0^{\infty} y^{r-3/2}\, e^{-(r+x^2/2)y} \sin\left(2\pi k g_n y\right) dy$$

$$= \frac{x r^{r-1}\Gamma(r-1/2)}{(2\pi)^{3/2}\Gamma(r)} \sum_{k=1}^{\infty} \frac{\sin\Big((r-1/2)\arctan\big(4\pi k g_n/(x^2+2r)\big)\Big)}{k\Big(\left(2\pi k g_n\right)^2 + (r+x^2/2)^2\Big)^{(r-1/2)/2}} = \frac{r^{r-1}\Gamma(r-1/2)}{2\pi\sqrt{2\pi}\Gamma(r)} \sum_{k=1}^{\infty} \frac{a_k(x;n)}{k}.$$

Now, we split the exponent $(r-1/2)/2 = (r-1)/2 + 1/4$ and obtain

$$|a_k(x;n)| \le \frac{|x|}{\Big(\left(2\pi k g_n\right)^2 + (r+x^2/2)^2\Big)^{(r-1)/2+1/4}} \le \frac{|x|}{(2\pi k g_n)^{r-1}\,(r+x^2/2)^{1/2}} \le \frac{\sqrt{2}}{(2\pi\, k\,(n-1))^{r-1}}.$$

Since $r > 1$ and $n \ge 2$, the first statement in Lemma A3 follows:

$$\sup_x n^{-1}\, |J_4^*(x;1/2)| \le c(r)\, n^{-r} \textstyle\sum_{k=1}^{\infty} k^{-r} = c_1(r)\, n^{-r}\,.$$

To prove the second statement about $J_4^*(x;-1/2)$, we insert again the Fourier series expansion of $Q_1(y)$ given in (A17) into $J_4^*(x;-1/2)$ and interchange the integral and sum

$$J_4^*(x;-1/2) = \frac{x}{2\sqrt{2\pi}} \textstyle\sum_{k=1}^{\infty} \displaystyle\frac{1}{\pi k} \int_0^{\infty} y^{-1/2}\, e^{-(2y+x^2/(2y))} \sin(2\,\pi\, k\, g_n\, y)\, dy\,.$$

Further, we use formula 2.5.37.4 in [40]

$$\int_0^{\infty} y^{-1/2}\, e^{-p\,y-q/y} \sin(b\,y) dy = \frac{\sqrt{\pi}}{\sqrt{p^2+b^2}}\, e^{-2\sqrt{q}\, z_+}(z_+ \sin(2\sqrt{q}\, z_-) + z_- \cos(2\sqrt{q}\, z_-))$$

with $2\, z_{\pm}^2 = \sqrt{p^2+b^2} \pm p$, $p = 2 > 0$, $q = x^2/2 > 0$ and $b = 2\,\pi\, g_n\, k > 0$. Use of the estimates

$$0 < z_- \le z_+, \quad |x| z_+ e^{-|x| z_+} \le e^{-1}, \quad \sqrt{p^2+b^2} \ge b = 2\pi g_n k \quad \text{and} \quad \textstyle\sum_{k=1}^{\infty} k^{-2} = \pi^2/6$$

leads to the inequalities

$$\sup_x \frac{1}{n}\, |J_4^*(x;-1/2)| \le \sup_x \frac{1}{2\sqrt{2\pi}} \textstyle\sum_{k=1}^{\infty} \displaystyle\frac{\sqrt{\pi}\, 2\, |x|\, z_+}{2\pi^2\, g_n\, n\, k^2}\, e^{-\sqrt{2}\,|x|} \le \frac{1}{e\,\sqrt{2}\, 12\, g_n\, n} = C n^{-2} \tag{A18}$$

and Lemma A3 is proven. □

Lemma A4. *Let $J_4^*(x;\gamma,d)$ be defined by (79), then $n^{-1}\, J_4^*(x;\gamma,d) \le C\, n^{-2}$ for $\gamma = \pm 1$.*

Proof of Lemma A4. Using the Fourier series expansion (A17) of the periodic function $Q_1(y)$, given in (33), and interchange integral and sum, we find

$$J_4^*(x;\gamma,d) = -\frac{s\,\gamma\, x^{d/2}}{2^{d/2}\,\Gamma(d/2)} \textstyle\sum_{k=1}^{\infty} \displaystyle\frac{1}{k\,\pi} \int_0^{\infty} y^{(\gamma\, d-6)/2}\, e^{-x y^{\gamma}/2 - s/y} \sin(2\pi\, k\, n\, y) dy. \tag{A19}$$

We begin by estimating $J_4^*(x;1,3)$, i.e., the exponent by y in (A19) is $-3/2$. Thus, we can use formula 2.5.37.3 in [40]

$$\int_0^\infty y^{-3/2}\, e^{-p\,y-s/y}\, \sin(b\,y)dy = \frac{\sqrt{\pi}}{\sqrt{s}}\, e^{-2\sqrt{s}\,z_+}\, \sin(2\sqrt{s}\,z_-) \tag{A20}$$

where $p = x/2 > 0, s > 0, b = 2\pi\,k\,n > 0$ and $2\,z_\pm^2 = \sqrt{p^2+b^2} \pm p = \sqrt{x^2/4 + (2\pi\,k\,n)^2} \pm x/2$. Since

$$z_+ = \frac{1}{2}\left(\left(x^2/4 + 2\pi kn\right)^{1/2} + x/2\right)^{1/2} \geq \frac{1}{4}x^{1/2} + \frac{1}{8}(2\pi)^{1/4}(k^{1/4} + n^{1/4}) \tag{A21}$$

it results in

$$\frac{1}{n}|J_4^*(x:1,3| \leq \frac{(s\,x)^{3/2}}{n\,s\,2^{1/2}\,\pi} \textstyle\sum_{k=1}^\infty \exp\Big\{-(s\,x)^{1/2}/2 - (2\pi)^{1/4}\Big(k^{1/4}+n^{1/4}\Big)/4\Big\} \leq \frac{C(s)}{n^2}\,.$$

Let now $\gamma = 1$ and $d = 1$. The main difference compared with the previous estimate of $J_4^*(x;1,3)$ is that we are facing more technical trouble in order to estimate $J_4^*(x;1,1)$. The exponent by y in (A19) is $-5/2$ and we cannot find a closed formula similar to (A19) for this case. To estimate J_4^* in the proof of Theorem 5 in [11], we show that differentiation with respect to s under the integral sign in (A20) is allowed. Hence,

$$\begin{aligned}\int_0^\infty y^{-5/2}\, e^{-p\,y-s/y}\, \sin(b\,y)dy \quad &= \quad (\sqrt{\pi}/2)\; e^{-2\sqrt{s}\,z_+}\Big(s^{-3/2}\, \sin(2\sqrt{s}\,z_-) \\ + \; 2\,s^{-1}\,z_+\, \sin(2\sqrt{s}\,z_-) \quad &- \quad 2\,s^{-1}\,z_-\, \cos(2\sqrt{s}\,z_-)\Big)\,.\end{aligned}$$

with the same coefficients $p,\ s,\ b$ and $z_\pm$ as in (A20). The use of (A21) and the obvious inequalities $0 < z_- \leq z_+$ and $z_+ \geq \frac{1}{2}z_+ + \frac{1}{8}x^{1/2} + \frac{1}{16}(2\pi)^{1/4}(k^{1/4}$ leads to

$$\frac{1}{n}|J_4^*(x;1,1)| \leq \frac{(sx)^{1/2}}{n\,2^{1/2}\,\pi} e^{-(sx)^{1/2}/4-(2\pi)^{1/4}n^{1/4}/8} \textstyle\sum_{k=1}^\infty \frac{1}{k}\, e^{-(2\pi)^{1/4}k^{1/4}/8} \leq \frac{C(s)}{n^2}.$$

Finally, let now $\gamma = -1$.

$$J_4^*(x;-1,d) = \frac{s\,x^{d/2}}{2^{d/2}\,\Gamma(d/2)} \textstyle\sum_{k=1}^\infty \frac{1}{k\,\pi} \int_0^\infty y^{-(d/2+3)}\, e^{-(x/2+s)/y}\, \sin(2\pi\,k\,n\,y)dy.$$

Partial integration in the integral with $A = d/2+3$, $B = x/2+s$ and $C = 2\pi\,k\,n$ leads to

$$\int_0^\infty y^{-A}\, e^{-B/y}\, \sin(Cy)dy = -\int_0^\infty \frac{1}{C}\Big(Ay^{-(A+1)} + B\,y^{-(A+2)}\Big)\, e^{-B/y}\, \cos(C\,y)dy$$

and using (62) to

$$\int_0^\infty \frac{1}{C}\Big(Ay^{-(A+1)} + B\,y^{-(A+2)}\Big)\, e^{-B/y}dy \leq \frac{1}{C}\left(\frac{A\,\Gamma(A)}{B^A} + \frac{B\,\Gamma(A+1)}{B^{A+1}}\right) = \frac{\Gamma(A+1)}{C\,B^A}\,.$$

Therefore,

$$\frac{1}{n}|J_4^*(x;-1,d)| \leq \frac{s\,x^{d/2}}{n\,2^{d/2}\,\Gamma(d/2)} \textstyle\sum_{k=1}^\infty \frac{\Gamma(d/2+4)}{k^2\,2\,\pi^2\,n\,(x/2+s)^{d/2+3}} \leq \frac{C(s)}{n^2},$$

and Lemma A4 is proven. □

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