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ARABISCHER TEIL

THE *KITĀB AL-KĀFĪ FĪ MUKHTAṢAR (AL-ḤISĀB) AL-HINDĪ* OF AL-ṢARDAFĪ

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Yemenite authors of *ḥisāb*-treatises are rare.[1]

However, this finding does not allow us to conclude that Yemenites collectively refrained from mathematics. Rather it is to be assumed that the relative geographical isolation of the southern part of the Arabian peninsula kept much scientific information from being included in the biographical dictionaries of the Islamic heartlands. The confusion about Abū Yaʿqūb Isḥāq b. Yūsuf al-Ṣardafī al-Yamanī and his books seems to confirm this suspicion. The aim of this article is neither to present a complete portrait of this author nor to describe completely his *ḥisāb*-treatise titled *Kitāb al-Kāfī*. Instead, both the biographical and textual aspects will be merged in order to evaluate the composition of a *ḥisāb*-text of marginal scientific significance and its importance for recognizing a local mathematical tradition that has until now escaped our attention.

a) *The Author*

All hitherto knowledge on the life and work of this person has been drawn from a single entry in *Kashf al-ẓunūn* of Ḥājjī Khalīfa (died 1657). From an unspecified source Ḥājjī Khalīfa identifies him as a scholar from Ṣardaf, an area to the East of Janad in Yemen, who had written among other works a *Kāfī fī al-ḥisāb* and a *Kāfī fī al-farā'iḍ* and died 505/1111 [14, I 200/-1f - 201/3]. H. Suter (who slightly misread the *nisba* given by Ḥājjī Khalīfa into "Ṣardī"[2]), C. Brockelmann [10, I 470], G. P. Matvievskaja / B. A. Rozenfeld [23, II no. 260, p. 311] and D. King [20, 53] repeated the dates of *Kashf* with two minor changes: The year of his death is advanced to 500/1106 [10, ibid.; 30, ibid.; 20, ibid.] and his *ḥisāb*-treatise is called *Mukhtaṣar al-Hindī* [10,

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[1] See Appendix A in [20, 53-60], where 16 Yemeni works on arithmetic, surveying and inheritance of 11 authors are listed.

[2] The passage [30, 111, note c] referred to ("H. CH. V 21") does not contain news about a "Ṣardī". Probably a zero in '21' has been lost.

ibid.; 30, ibid.].[3] As to this title, it is evident that H. Suter profited from the reading of *Ma'ūnat al-ṭullāb*, a commentary on the *Kitāb al-Kāfī* by a later Yemenite scholar who calls his original copy *Mukhtaṣar* [3, fol. 2a/9]. The manuscript at hand, though, carries the simple title *Kitāb al-Hindī*.[4] Nine copies of it have survived in European, five in Oriental libraries, four others of the *Kāfī fī al-farā'iḍ*.[5] The number of copies existing is evidently at variance with the silence with which al-Ṣardafī was treated by the medieval collectors of scientific biographies.[6] New evidence contained in *Shadharāt al-dhahab* of Ibn al-'Imād al-Ḥanbalī (died 1679) [10, S II 403] may explain this silence. Ibn al-'Imād cites a certain Ibn al-Ahdal as his authority for the entry on al-Ṣardafī [17, III 411/5]. The biographical work *Ghirbāl al-zamān* of this Ibn al-Ahdal, a Yemenite Muftī who died 1451 [10, S II 238-239], extended the scope of the *Shadharāt* to the Yemenite milieu. The content of the report on al-Ṣardafī reflects the intimacy of the author with his subject. He first praises the artistic composition of the *Kāfī fī al-farā'iḍ*. In fact, the *Kāfī* follows a unique method to teach the rules of *farā'iḍ*. The classes of heirs and their shares are not arranged according to the prescriptions of the *sharī'a* but along arithmetical fractions [25, 223-230]. The widely used epithet of al-Ṣardafī, "al-Faraḍī" – i.e. calculator of inheritance shares –, therefore was possibly less inspired by the legal knowledge of his bearer than his skill to make the *sharī'a*-rules arithmetically transparent. Ibn al-Ahdal yet does not mention al-Ṣardafī's qualification in *ḥisāb*. He talks about the tribal affiliation of the author to the famous Yemenite tribe of Banū Ma'āfir and about his family. Quite obviously, al-Ṣardafī was a bibliophile. He married

[3] [23, II, No. 260]: *Kitāb Mukhtaṣar al-hindī*.

[4] [7, fol. 90a/3]; [20, 53] records three different titles: *K. al-Kāfī fī mukhtaṣar al-Hindī*, *K. al-Ḍarb al-Hindī*, *Mukhtaṣar (al-ḥisāb) al-hindī*, but without relating them to specific manuscripts.

[5] See [23, II, ibid.], where a third text of the author is registered under the title: *Kifāyat al-muhtadī wa-ijābat al-mujī[b]*, and [20, ibid. and 56] who identifies this treatise (*Kifāyat al-muhtadā*) as a commentary by a certain Abū 'Abdallāh M. b. 'Abdallāh on authority of [10, S II (= I!), 855]. Brockelmann's version of the title seems to be the correct one: *Kifāyat al-muhtadī wa-ijābat al-mahdī*. Some confusion about this work remains yet, since D. King identifies it slightly later as a commentary on the *Kāfī fī al-farā'iḍ*. Several attempts failed to get hold of this MS at the Ambrosiana (grammalogue D 559). O. Löfgren and R. Traini [22,144] mention six copies of the *Kāfī* in the Ambrosiana.

[6] None of Ibn Abī Uṣaibi'a (*'Uyūn al-anbā'*), Ibn al-Qifṭī (*Akhbār al-ḥukamā'*), Ẓahīr ad-Dīn al-Baihaqī (*Ta'rīkh al-ḥukamā'*), or Ibn Khallikān (*Wafayāt al-a'yān*) took notice of him.

his two daughters, one to a prominent *faqīh* and the other to the *Imām* of Janad and so managed to secure the library of the *faqīh* for the son of his one daughter, the later *Imām* of the Friday mosque, and his own library for the son of his other daughter [17, III 410/-3 - 411/5]. Certainly, al-Ṣardafī's concern for books would be more informative if the contents of his shelves were listed. Yet the texts he left, as written by the owner of a library, must now be regarded as being the product of a well-read person whose acquaintance with the literature of his discipline must be anticipated to a certain degree. This short biographical survey should not end without mentioning an irritating mistake of Ibn al-ʿImād. He, in fact, calls al-Ṣardafī "al-Ṣarūfī". According to the recently edited *Muʿjam al-buldān wa al-qabāʾil al-yamanīya* [18, 378 b] this must have slipped into the *Shadharāt* by misreading or misprinting. The author of the dictionary makes clear, that it was Ṣardafī, a mountainous region to the East of the city of al-Janad, which lent its name to the famous local scholar "al-Faraḍī al-Ṣardafī".

b) *The Text*

The *Kitāb al-Kāfī* can roughly be divided into two parts. Part one (fols. 90a-115b/13) contains a concise introduction into the methods of calculation with 'Indian' numerals. The shape of the rarely used numbers, especially the 4,5 and 6, is clearly of Eastern origin [19, 492-495; 26, 12]. After describing the multiplication of units, tens, hundreds separately and mixed, al-Ṣardafī proceeds to a detailed representation of the multiplication of 54.321 by itself. With the verbal elements eliminated, the figure representation runs as follows [7, fol. 94a/1f]:

```
        [scribe's negligence?]
1)              5 4 3 2 1        1)              5 4 3 2 1
        5 4 3 2 1                        5 4 3 2 1

2)      2 7 1 6 0 5 4 3 2 1      2)    2 7 1 6 0 5 4 3 2 1
          5 4 3 2 1                      5 4 3 2 1

3)      2 9 3 3 3 3 4 3 2 1      3)    2 9 3 3 3 3 4 3 2 1
            5 4 3 2 1            ...

4)      2 9 4 9 6 3 0 3 2 1      ...
            5 4 3 2 1
```

```
5)    2 9 5 0 7 1 6 7 2 1        [modern notation:]
              5 4 3 2 1
                                 1)    5 4 3 2 1 · 5 4 3 2 1
6)    2 9 5 0 7 7 1 0 4 1
                5 4 3 2 1        2)     2 7 1 6 0 5
                                         [2 1 7 2 8 4]

                                 3)     2 9 3 3 3 3 4
                                           [1 6 2 9 6 3]

                                 4)     2 9 4 9 6 3 0 3
                                 ...
                                 ...                      ...2
```

Compared with the methods current at this time, as demonstrated for example by al-Uqlīdisī:

```
374·256:          374        374        374        374
                256   →   256   →   256   →    256
                           768       9472       95744
```

a remarkable change can be detected.

Al-Uqlīdisī argues for this scheme with the words: "We may have to multiply two numbers one by the other so that we know the product with the two numbers not rubbed out, but standing safe."[7] But, what he evidently took for unavoidable, the intermediate products had to be rubbed out in order to complete the following multiplication. Apart from this minor blemish the basic problem of all contemporary variants remains untouched: the multiplication starts from the highest place and proceeds to the lower ones. A. Saidan adds: "The major characteristic of it that the Islamic reckoners, including al-Kāshī [– who, by the way, avoids all dust board methods –], could not avoid is that multiplication starts from the highest places, highest times highest, and so on."[8] Obviously, al-Ṣardafī ignored this limitation and presented a method that could do without rubbing out, that required only one gradually enlarged figure and that was composed of no more steps than the multiplicand (or multiplicator) had places – and that could easily be reexamined after the completion.

[7] [26, 149]; cf. the shortly earlier published Arabic version [4, 198/13].

[8] [26, 391]; cf. the considerably shorter and slightly different commentary of the Arabic version [26, 470-472].

On one of the next pages al-Ṣardafī explains that "to multiply 4 by 4 *murabba'a*[9] means 4^{16} = 4.296.967.296. By the term *murabba'a* he illustrates the consecutive squaring of the products, that is the doubling of the powers. Furthermore, when listing different *mīzān*-methods (casting-out) with whole numbers and fractions, he uses the completely unusual word *jubūr* (sing. *jabr*) for whole numbers [7, fol. 108a f.]. For *mīzān* he also uses the nominal derivations *ṭarḥ* and *taṭarruḥ* of the verb *ṭaraḥa*, to subtract [7, fol. 115b/13f]. The next chapter is dedicated to the extracting of roots (*tajdhīr*) [7, fol. 110a f.]. Three cases are differentiated.

1. Whole numbers $\sqrt{N^2} = N$
2. *Aṣamm* numbers $\sqrt{N^2 - 1}$
3. Composite numbers $\sqrt{\left(N+\frac{p}{q}\right)^2}$

Before proceeding to the explanation a new terminology is introduced. Starting from the right the first place (*manzila*), i.e. the units, is called *bait*^{u} *witr*^{in} (even 'house'), the second, i.e. the tens, is called *bait*^{u} *shaf*^{in} (odd 'house'), the third again *witr*, the forth again *shaf*' and so on.[10] Later on (fol. 112a/15) the rule is added that the root-extraction must always be started with the last *witr*-place from the right.

The first case is quickly dealt with.[11]

To the second case[12] al-Ṣardafī adds a curious amendment which he calls *ḥīla* (ruse) [7, fol. 111a/-5]. This *ḥīla* will be discussed immediately, but in a different context.

9 [7, fol. 95a/1]: represented as $\overset{(4)}{4}$.

10 Cf. [26, 80-81, note 8.12, p. 439] and [4, 106/5f.] who still uses the unspecific terms "is" (*yakūn*^{u}) and "is not" (*lā yakūn*^{u}) which were to be replaced by the specific terms *zauj* and *fard*.

11 [7, fol. 110b/9f.]:

$\sqrt{2^2}$, $\sqrt{3^2}$, $\sqrt{10} \approx 3\frac{1}{6}$, $\sqrt{11} \approx 3\frac{1}{3}$, $\sqrt{12} \approx 3\frac{1}{2}$, $\sqrt{13} \approx 3\frac{2}{3}$, $\sqrt{14} \approx 3\frac{5}{6}$;

[7, fol. 111b/9f.]:

$\sqrt{16}$, $\sqrt{25}$, $\sqrt{36}$, ... $\sqrt{144}$, $\sqrt{841}$, $\sqrt{576}$, $\sqrt{1024}$, $\sqrt{6889}$ $\sqrt{10004004}$;

12 [7, fol. 111a/9]: $\sqrt{15}$, $\sqrt{24}$, $\sqrt{35}$, ... $\sqrt{143}$;

[7, fol. 113a/15]: $\sqrt{9035} \approx 95\frac{1}{19}$;

[7, fol. 114b/6]: $\sqrt{1440044} \approx 1200\frac{44}{2400}$, $\sqrt{7777777} \approx 2788\frac{4933}{5576}$.

Evidently, aṣ-Ṣardafī mixed his two cases with squares of the general form $\sqrt{a^2 + b}$.

On folio 115b/13 the second and smaller part of the *Kitāb al-Kāfī* begins. Its main characteristic consists of demonstrating how variously disguised everyday problems can be resolved by reducing their conditions to general procedural steps.

It is introduced by a simple geometrical problem dealing with the measuring of areas. The units of the lengths and widths to be multiplied with each other are defined as *labin*, or *libna* (bricks), with the base 1 *dhirāʿ* times 1 *dhirāʿ*. A remarkable parallel is to be found in the *Kitāb al-Misāḥa* of Abū ʿAbdallāh M. b. Ibrāhīm al-Ḥalabī al-Ḥanbalī (died 1563)[13], a commentary on Aḥmad b. Ṯabāt's *Ghunyat al-ḥussāb* [8]. al-Ḥalabī quotes Ibn Ṯabāt who himself restricted the term *labinī* to rectangular solids [8, 127]. Since Ibn Ṯabāt does not mention his sources a direct relationship between the *Ghunya* and the *Kitāb al-Kāfī* cannot be drawn. It rather seems plausible to argue that *labin*, the word for the basic construction element, was gradually applied for terminological purposes when describing and measuring areas and solids.[14] Thus, the terminological peculiarities of the *Kitāb al-Kāfī* turn out to be representative both of the variety of terms and of their tendency to be standardized in practical *ḥisāb*-treatises. Neither the general language dictionaries nor the only existing historical dictionary of Arabic mathematical terms [27] suffice for the understanding of these texts.

Next come several algebraic riddles of a stereotype form known since the Chinese *Chin Chang* [11, 86]. Worth mentioning is their application to Islamic tax calculations called *ṣadaqāt*-problems [7, fol. 117b/17f]. Then the author poses a question which turns out to be a badly disguised form of the well-known duplication of the squares of the chess-board [19, 460-461]. "Suppose an employer hires a labourer for 1 *dirham* wage for the first day, two *dirham* for the second, four *dirham* for the third and so on. What's his wage on the 33rd day?" The solution from al-Ṣardafī (or the scribe's representation) is in fact wrong, but the procedure used is interesting. By proposing to equate n with $2k$ he decomposes the completely unworkable relation

$$a_n = a_1 \cdot q^{n-1}$$

into three different steps, where starting with

$$k = 5,$$

[13] [23, II No. 464, 556] with the title *Maḥāyil al-malāḥa fī masāʾil al-misāḥa*, [2, fols. 1b-53b, fol. 46b].

[14] In a similar sense *labinī* is used by Abū 'l Wafāʾ al-Būzjānī [6, 262/10] and [1, 336/11].

the exponent is doubled each time so that with

$$a_{n_1} = q^{2k-1},\ a_{n_2} = q^{2(2k-1)-1} \text{ and so forth}$$

the calculator arrives from 5 by way of 9 and 17 to the exponent 33, having to square only three times. This is certainly a very practical interpretation of the rules of geometric progressions. But, again, more interesting is the fact that the only Arabic reckoner, I know of, who used this method was ʿAbdalqāhir al-Baghdādī in his *Takmila fī al-ḥisāb* written 150 years before. Al-Baġdādī explains the method but not the result [1, 178 and note 29]. There is no direct indication that al-Ṣardafī borrowed this method from the *Takmila* (section: *bāb fī ḥisāb al-yad!*). ʿAbdalqāhir lived and wrote in Isfarā'in. But the impression is justified that by the end of the 11th century this type of mathematical knowledge was 'domesticated' and introduced into the practical *ḥisāb* treatises. It not only begins to be included into the fund worthy of teaching, but it is also developed and diversified. At the end of the 14th century a certain Taqīyaddīn al-Ḥanbalī already mentions in his *K. Ḥāwī al-lubāb min ʿilm al-ḥisāb* [31, fols. 1a-44a, fols. 4b-5b] three methods to calculate the sum of a geometric progression and freely operates with the composition of n in q^{n-1} [31, fol. 5a/-4f].

c) *The Continuation of the Text*

As for the purely mathematical techniques of interest one could as well stop here with the description of the manuscript. But an almost unique fact justifies continuing the description, though indirectly. The *Mukhtaṣar* of Ṣardafī turns out to be one of the rare texts on applied reckoning that did not disappear in libraries or – at best – got worn out in the hands of practitioners – but rather experienced a literary career since a certain Sirāj al-Dīn Abū Bakr b. ʿAlī b. Mūsā al-Hāmilī [*MṬ* 25a/10], another Yemenite who died in 769/1367, composed a commentary on it.[15] The casual reader will be caught, presumably, first by the problem noted on the title page: Suppose a hermaphrodite marries a free woman, who delivers him a child; next, our hermaphrodite gives birth to a child; then he dies; who inherits what? asks the author. The solution given is less sexual than one would imagine. On the next page al-Hāmilī reports that his students had asked him whether he could compose a commentary on the *K. Mukhtaṣar al-Hindī* of a certain 'al-Ṣardafī'. Al-Hāmilī's response, the text called *Maʿūnat al-ṭullāb fī*

[15] [10, S. II, 240-241]; [23, II No. 420b, p. 468]: 7394 (sic!) = 796 h, misread from [30, 111, no. 260]: dies 769/1367-8; [3, fols. 2a-29a, fol. 25a/10].

ma'rifat al-ḥisāb (The help for the students to study arithmetic), turned out to be – in fact – a selective, but close commentary of al-Ṣardafī's text, drawn probably from different sources. This conclusion can only be explained by a certain proliferation of the *Mukhtaṣar* between the author's death and the student's dialogue with his teacher. During these two and a half centuries the text must have circulated to some extent at least. Otherwise Ḥājjī Khalīfa, the only biographer of al-Ṣardafī who mentioned the *Mukhtaṣar* [14, V, 200], could not have gained knowledge of its existence. Perhaps it was less the contents of the *Mukhtaṣar* that let it survive than the reputation of its author as a specialist on inheritance law. His before-mentioned treatise on *ḥisāb al-farā'iḍ*, no biographer has failed to mention. The peculiar character of this text is described elsewhere [25, 223-230]. Its distinguishing feature is the stress laid on the facilitation of calculating juridically defined lots by analogy to arithmetical proportions. Bearing this rather one-sided representation of al-Ṣardafī in the biographical collections as an inheritance specialist in mind, one is surprised by the before-mentioned fact that while the *Kāfī fī al-farā'iḍ* has only survived in four Mss, the *Mukhtaṣar* has come down to us in more than a dozen known texts between Sanaa and Manchester. The key to understanding the misproportion of factual survival and biographical representation can be found in al-Hāmilī's commentary. The selection he made to explain the *Mukhtaṣar* to his student must be interpreted as having been intentional. Scrutinized under this aspect the *Ma'ūna* discloses a striking imbalance. Whereas al-Ṣardafī dedicated almost 60 of total 68 pages to more or less theoretical and methodical problems and demonstrated their application only briefly in the last part, called *masā'il* (practical solutions of questions posed, i.e. questions), al-Hāmilī dedicated almost two-thirds of his *Ma'ūna* to these *masā'il*. In his introduction he argues explicitly for this selection: "Our Shaikh has not mentioned methods of *mawāzīn* (weights), *nisba* (proportion) and *ma'rifat al-kusūr* (fractions) which the scholars often use for solving problems of 'extension', 'inheritance' and *mu'āmalāt* (everyday calculation)" [3, fol. 2a/2]. The intention to refine and enlarge al-Ṣardafī's text for practical purposes also dominates the first third, the introductory part.

– On fol. 3b/6f. he explains how to multiply economically numbers that can be decomposed in factors easy to handle:

$$12 \cdot 13 = [(3 \cdot 13) \cdot 4]$$

– On fol. 7a/18f. he describes a method to square sums of whole numbers – which again are called *jubūr* and not *ṣiḥāḥ*[16] – and fractions

$$\left[6\tfrac{1}{x}\right]^2$$

– On fol. 9a he reproaches al-Ṣardafī for not having explained the extraction of the root of the sum of whole numbers and fractions.

– On fol. 10b, he shows that by repeating the procedure the approximation of a composite square-root can be made more precise. [See next page.]

His concern appears to be much more 'practical' – as opposed to 'abstract' – than al-Ṣardafī's. Among his examples to make his method transparent we find (fol. 9b):

$$\sqrt{45+\frac{1}{2}+\frac{\frac{1}{2}}{8}} \text{ and } \sqrt{496+\frac{1}{5}+\frac{1}{10}}.$$

While the first root represents the class of fractions that "may have a root" (*qad yumkin*[u] *an yakūna lahū jidhr*) because both denominator and numerator end on one of the 'possible' numbers 6 and 9[17], that is

$$\sqrt{\frac{729}{16}},$$

and because the number of places, i.e. 3, is 'a possible one', he demonstrates with the second that the smallest common denominator,

$$\text{i.e. } 10: \sqrt{\frac{4963}{10}},$$

causes the square to belong to the group of the 'unextractable ones' (*ghair majdhūr*). These numbers are called 'mute' (*aṣamm*) and can only be extracted by approximation. It is this approximation which is dealt with next (*M*, fol. 10a*f.*, see the foldout facing p. 198):

The development is peculiar in two aspects: first the approximation converges rapidly to the root – and second: the structure of r_n guarantees the steady decrease of the subtrahend and prevents at the same time the value of r_n from reaching zero.

It is worth mentioning that al-Hāmilī does not bother about what al-Ṣardafī (fol. 111a/13) explains to be a trick (*ḥīla*) with square-roots that display the peculiarity (a<b), for instance the root $\sqrt{15}$.

Since in this case $\frac{b}{2a}$ yields 1, which – added to 3[2] – gives $\sqrt{16}$, the mistake is evident.

[16] See note 9.

[17] It is noteworthy that al-Hāmilī does not take up the *witr-shaf*' terminology of aṣ-Ṣardafī.

Therefore, the trick is to add two to the denominator and one to the numerator so that the root:

$$a + \frac{b+1}{2a+2} = 3\frac{7}{8}$$

is found. Perhaps he did not mention this *ḥīla* because it had – long before – stopped being one. That by the time of al-Hāmilī this approximation variant was well-known has been shown by A. Saʿīdān in his edition of Yaʿīš b. Ibrāhīm al-Umawī al-Andalusī's *Marāsim al-intisāb fī ʿilm al-ḥisāb.*[18] Al-Umawī, before turning to the cube root, is slightly more precise than al-Ṣardafī by distinguishing *three* cases

$b<a$, $b>a$, and $b\geq a$

but seems to be unaware of what his Yemenite contemporary al-Hāmilī proposed. In fact, the *ḥīla* of al-Ṣardafī is reported by older and modern historians of Arabic mathematics – I am referring to Heinrich Suter's paraphrased translation of the *ḥisāb*-treatise of Abū Zakarīyā' Muḥammad al-Ḥaṣṣār [29, 115-143; 21, 56-60] and Driss Lamrabet's *Introduction à l'histoire des mathématiques Maghrébines* – to have been introduced by the Maroccan al-Ḥaṣṣār [29, 140-141; 21, 195] and repeated by Ibn al-Bannā' [29, 141; 16, 64/3 (ar.); 15, 286/10, 162; 24, 79/10]. Where al-Ṣardafī drew his knowledge, almost two generations earlier than al-Ḥaṣṣār, we do not know.[19] The same holds for al-Hāmilī's approximation method, although its chronological setting is less exciting. We shall see that – though relatively isolated in East-Yemen – this and the following efforts of al-Hāmilī must not necessarily be regarded as those of a lone wolf. There are indications that he must have been familiar with parts of the older Northern and perhaps Eastern mathematical traditions too.

[18] [33, 54 and note 66, p. 94]; cf. [26, 444-5]; [10, S II, 379]: "schrieb um 1489"; [23, II, Nr. 453, p. 538]: 15th century; Saʿīdān [33, 4, 9, where a misprint slipped in] mentions an *ijāza* from the hand of al-Umawī in the year 774/1373 which would make him at least one century older.

[19] The most extensive comparative study on the (approximative) extraction of square-roots is contained in [4, 441-455]. After citing Heron's extraction of $\sqrt{720}$ (*Metrica* I. 8, cf. [26, 454]) A. Saʿīdān points to the fact that "the Babylonians" used this process which produced the mean of the first and second approximation:

$$\sqrt{a^2+r} - a + \frac{r+1}{2a+2}.$$

To my knowledge, the only Arabic source, the undated and anonymous *al-Hindī al-muntazaʿ min al-Kāfī*, that contains this approximation, operates with the special case $\sqrt{a+2a} \approx a + \frac{2a+1}{2a+2}$ [26, 448].

10a/7: $\sqrt{40} = \sqrt{36+4} \approx 6+\frac{4}{2\cdot 6} = 6+\frac{1}{3}$ $\left(\sqrt{N} = \sqrt{a^2+b} \approx a+\frac{b}{2a}\right)$

/11: al-Hāmilī: "al-Ṣardafī": $\approx a+\frac{\frac{b}{2}}{a}$	1. $\sqrt{N} \approx \sqrt{a^2+b} \approx a+\frac{b}{2a}$ (= 6.333333)
/13: "then square $6+\frac{1}{3}$ " $\sqrt{\left(6+\frac{1}{3}\right)^2} = \sqrt{40\frac{1}{9}}$	2. $\left(a+\frac{b}{2a}\right)^2 = a^2+2a\frac{b}{2a}+\frac{b^2}{4a^2}$ $= a^2+b+\frac{b^2}{4a^2}$
/15: "If you want to extract the root more exactly:" 1. "Double" $\left(6+\frac{1}{3}\right) \to 12\frac{2}{3}$	3. $2\left(a+\frac{b}{2a}\right)$
2. "Divide the remainder": $\left(\frac{1}{3}\right)^2$ $\left[=\sqrt{\left(\frac{1}{9}\right)}\right]^2$ $\frac{1}{9} : 12\frac{2}{3} = \frac{1}{114}$ $\left[F : \frac{1}{124}\right]$	4. $(a^2+b)+R_1 = (a^2+b)+\frac{b^2}{4a^2}$ $R_1 = \frac{b^2}{4a^2}$ 4. a) $\frac{\frac{b^2}{4a^2}}{2\left(a+\frac{b}{2a}\right)}$
$= \sqrt{\frac{1}{12996}}$	4. b) $\sqrt{a^2+b} \approx a+\frac{b}{2a} - \frac{\frac{b^2}{4a^2}}{2\left(a+\frac{b}{2a}\right)}$ (= 6.326389)
/18 3. "If you want the result to be more exact (*adaqq*), proceed as before!"	5. (See below)

5.

$$\sqrt{a^2+b} \approx a+\frac{b}{2a} - \frac{\frac{b^2}{4a^2}}{2\left(a+\frac{b}{2a}\right)} - \frac{\frac{\frac{b^2}{4a^2}}{2\left(a+\frac{b}{2a}\right)}}{2\left(a+\frac{b}{2a} - \frac{\frac{b^2}{4a^2}}{2\left(a+\frac{b}{2a}\right)}\right)}$$

(= 6,325676;
$\sqrt{40}$ = 6,324555;
D = 0,001121)

But let us first reconsider his approximation. When discussing the extraction of mute roots and the methods of approximation at the beginning of the 14^{th} century, one immediately thinks of the achievements of the famous Maghribī scholar Ibn al-Bannā' (died 1321)[20] preserved in his *Talkhīṣ a'māl al-ḥisāb* and its commentary *Raf' al-ḥijāb*. Neither Suter nor Tropfke [32, 277-278], neither of whom knew of the *Raf' al-ḥijāb*, were fully aware of the eminent role of Ibn al-Bannā' as a central link of a Maghribī school of mathematicians that had been started by al-Ḥaṣṣār at the end of the 12^{th} century [23, II Nr. 325d, p. 361; 13; 25, 41] and flourished down to al-Qalaṣādī (died 1486) [23, II, Nr. 444, 510-2; 25, 53, 231-2]. For our purpose these connecting lines are of interest insofar as both texts of Ibn al-Bannā' lack what al-Ḥaṣṣār had already fully described and al-Qalaṣādī uncompletely repeated, i.e. the repetitive subtraction of the correcting link as proposed by al-Hāmilī [16, 64/9; 15, 285-7, 160-2]. Al-Ḥaṣṣār, when extracting $\sqrt{5}$, subtracts from $\sqrt{N}<\sqrt{(a_1+r)^2}$ a second and a third remainder and adds literally: "[to make the result more precise] you can continue with that as long as you want".[21]

As to al-Qalaṣādī, his approximation method (*tadqīq al-taqrīb*) [24, 81f] resembles the second step of al-Hāmilī (= 4). After having regularly extracted

$$\sqrt{6}\approx 2\tfrac{1}{2}$$

and found out "a fourth of approximation" (*taqrīb bi-rub'*), he orders this fourth to be divided by the double of the root – this is 4 of al-Hāmilī – and to subtract the result of the root (= 4b) which gives the more exact root (*adaqq*)

$$2+\frac{1}{2}-\frac{1}{20}=2\frac{2}{5}\frac{1}{20}.$$

But he stops here by squaring the root found in order to check the difference between this and $\sqrt{6}$, that is $\frac{1}{400}$, while al-Hāmilī continues with 5 by instructing one to repeat 4a and 4b (see above) to minimize this difference.

This short methodological and chronological comparison justifies

[20] For biographical data, see [23, II Nr. 399, p. 443]; [9, Nr. 51, 83-90] preserves an extensive article on this eminent personality. D. Lamrabet [21, 79-90] has summarized the numerous recent contributions to the knowledge of vita and opus of Ibn al-Bannā'.

[21] [29, 141]; cf. [21, 196-197], where in formula 6 the denominator has to be corrected to $2\left(x+\frac{a}{2x}\right)$.

two remarks. The first is to qualify the mathematical importance of this repetitive approximation and its partial neglect by Ibn al-Bannā'. Since he, after all, mentions the one-fold subtraction of the remainder [21, 195] and since its repetition does not change the procedure essentially, one should not overvalue these divergencies.

The second refers to the possible background of al-Hāmilī's method. Driss Lamrabet pronounces explicitly what can now be contradicted: "La formule d'al-Ḥaṣṣār était inconnue des mathématiciens musulmans orientaux." [21, 197] Obviously, al-Hāmilī had known it and – given the complete unfamiliarity with al-Ḥaṣṣār outside the confines of the Maghrib and given the neglect of Ibn al-Bannā' – had most probably not learnt it from North-African sources. Tropfke preserves an interesting note that could turn our attention into a wholly different direction [10, I 272]. The "Bakhshalī"-manuscript, a text from North-West India dated at the latest to the 12th century, contains three examples of root approximations among which $\sqrt{41}$ is dealt with very similarly to the way al-Hāmilī took. Like al-Hāmilī, the author gives two repetitive values of which the second is arrived at by subtracting

$$\frac{\frac{b^2}{4a^2}}{2\left(a+\frac{b}{2a}\right)}.$$

But the indication that this process could be repeated is lacking.[22] The similarity is striking but too isolated for speculation.

Before returning to al-Hāmilī's commentary on the second part of al-Ṣardafī's *Mukhtaṣar*, one impression can be anticipated: al-Hāmilī is not trying to correct al-Ṣardafī but to complement him. On several occasions he adds general methods and specific solutions al-Ṣardafī did not deduce (*lam yadhkurhu al-Shaikh*). This is the case with the problem of a man who arrives in a city with a certain amount of capital which he manages to double each day, donates 4 *dirham* in exchange for his success as *ṣadaqa* and ends up on the fifth day with empty hands [3, fol. 23a/-1f; 7, fol. 118a/8]. The solution given is lengthy but very transparent:

	ṣadaqa 4	$4\cdot2+4$	$12\cdot2+4$	$28\cdot2+4$	$60\cdot2+4=\frac{124}{32}=3\frac{7}{8}$
"always" 1	double 2	4	8	16	32

which represents a geometric progression

[22] See Saʿīdān [26, 445] who refers to [28, 92] and adds that "this formula has not been traced in Arabic texts".

with $a_k = 2a_{k-1} - 4 \Rightarrow a_1 = \frac{2a_{k-1} - 4}{q^{k-1}}$ $(k \in \{2,3,4,5\})$

The indication "always to start with one and double consecutively in the lower line" makes clear that al-Hāmilī is not only concerned about the proper solution of this merchant's misfortune but also about the general explication of the method to solve this type of problem. He therefore adds several similar *masā'il* in order to practice this method.

Together with the explicable completion of al-Ṣardafī's *Mukhtaṣar* by al-Hāmilī, we realize here the second peculiarity of al-Hāmilī's commentary: the demonstration of general methods of solution.

In order to stress the uniformity of the procedure hidden behind the formulation of the problem, al-Hāmilī operates with variations. For demonstrating the solution of equations of the type: $x + ay = P(rice)$; $y + bz = P$; $z + cx = P$, he camouflages the problem as *masā'il al-širka* (problems of financial partnership), *masā'il al-laila* (day and night-problems), *masā'il al-ṭuyūr* (purchase of birds), *masā'il al-ṯaub* (purchase of cloth) and *masā'il al-ṣandūq* (problems of the box, where the box stands for a kind of savings bank book). Many of these *masā'il* possess an original and methodological setting. In view of the fact that the *Maʿūnat al-ṭullāb* contains – until now – the only indication that Indian arithmetical elements might have penetrated the Arabic *ḥisāb* by way of Yemen, the careful comparison of its 'problems' with Indian material seems to be a promising point of departure for further investigations.

It is this variability of application, which exceeds that of al-Ṣardafī by far, that delivers the third peculiarity of al-Hāmilī's commentary. Almost inevitably this variability is only achieved by tapping common and traditional *ḥisāb* texts. So al-Hāmilī cites the riddling love-poem mentioned by al-Ṣūlī, the teacher of the ʿAbbāsid princes al-Rāḍī, al-Muqtadir and al-Muktafī in his *Adab al-kuttāb* [3, fol. 24a/15, p. 243, 3f.; 25, 61-3]. Shortly thereafter he cites verses of a scholar named Abū Bakr ʿAbdallāh al-Ḥ-ʿ-r-m-l whom I could not identify. The same holds true for a certain Yūsuf al-Muhalhal (Yūsuf 'the Slim'), a jurist whose verses on the method of *mīzān al-taṭarruḥ* are inserted [3, fol. 8b/9]. Perhaps they belong to a local Yemenite tradition which has obviously contributed actively and passively to the text. According to al-Hāmilī, scholars from Shujaina and from Wādī Zabīd submitted mathematical problems to him which he introduced into the commentary [3, fol. 24b/9]. At the end of the text when explaining enlarged versions of the before-mentioned equations of the type $x + ay + bz = P$ etc. he explicitly refers to Abū Kāmil Shujāʿ b. Aslam. The wording of this reference displays intimacy with Abū Kāmil's "Algebra". The problem tackled by al-Hāmilī is similar to a *ṯaub*-problem Abū Kāmil dealt with [12, 73-74]. The text closes with the remark of the writer that the author (*al-muṣannaf*) finished with it in the year 724/1324.

Though the *Mukhtaṣar* ends here, we are not at the end of the manuscript. One and a half pages follow, written in the same ductus by a person who calls himself once more *mu'allif hādhā al-kitāb* (author of this book). This appendix contains rediscussions of problems of the *Mukhtaṣar*. Among them, this author shortly resumes what al-Hāmilī himself evidently had not regarded worth being included in his commentary on al-Ṣardafī's *Mukhtaṣar*: the calculation of the final wage of our hired labourer after 33 days. But, again, the conditions are changed and varied insofar as to the labourer's wage the double of what he had earned the previous day is added, which results in a progression with $q = 3$ and the correct values[23]:

k = 5	k = 9	k = 17	k = 33
18 [sic!]	6.561	43.046.721	1.853.020.188.851.841

c) *Conclusion*

Two observations can be summarized and perhaps be used for a more general conclusion.

First, the existence of a local literary *ḥisāb* tradition, widely ignored by the Islamic biographers and, hitherto, the historians of *ḥisāb*, must definitely be included as an important factor in research on applied *ḥisāb* down to the 12th century. Through the identification of textual ramifications, we not only learn more about the significance of *ḥisāb* as a practical science exercised in local school traditions, but we also are occasionally put into a position to observe the creative circulation of more or less standard knowledge with the effect of modest but informative innovations.

Second, the direction of change is always pragmatical. Transparency, economy and simplicity are its motives. The degree of originality, though, is difficult to detect since the authors of this type of *ḥisāb* treatise display proof of a certain familiarity with the widely accepted 'scientific' literature. The seam between this type of knowledge and its reception and transformation for practical purposes – be it just teaching

[23] [3, fol. 28a/14f]. Since the four sums as given in the text indicate each the basic wage of the following day (= *k*), *k* must in fact be reduced by 1.

or even the practitioner's application – is more permeable than has been taken for granted until now.

Both observations combined, the conclusion suggests itself: Without the evaluation of the hundreds of untouched *ḥisāb* manuscripts in European and Oriental libraries not only the cours and development of every-day *ḥisāb* in the Islamic medieval societies will remain spurious and incomplete but also the scope and role of the academical discipline *ʿilm al-ḥisāb.*

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