On Set Optimization with Set Relations: A Scalarization Approach to Optimality Conditions and Algorithms

Dissertation

Zur Erlangung des Doktorgrades der Naturwissenschaften
(Dr.rer.nat) der

Naturwissenschaftlichen Fakultät II
Chemie, Physik und Mathematik der Martin-Luther-Universität Halle-Wittenberg

vorgelegt von

Herrn Ernest Quintana Aparicio
geboren am 23. Januar 1991 in Cienfuegos, Kuba

Gutachter:

Frau Prof. Dr. Christiane Tammer (Martin-Luther-Universität Halle-Wittenberg)

Frau Prof. Dr. Gemayqzel Bouza (Universität von Havanna)

Herr Prof. Dr. Johannes Jahn (Friedrich-Alexander-Universität Erlangen-Nürnberg)

Tag der Verteidigung: 30.09.2020

Acknowledgments

First, I would like to thank both of my supervisors, Prof. Dr. Christiane Tammer and Prof. Dr. Gemayqzel Bouza, for their guidance, support, and encouragement during this time. Prof. Dr. Bouza introduced me to the field of mathematical optimization during my years as a bachelor student, and her immense knowledge motivated me to keep working on this topic. During my PhD studies, despite the fact that we were far apart, she always found time for discussing new ideas and giving me advice. Working with Prof. Dr. Tammer was also a true pleasure. She warmly welcomed me into her group and supported me from the first day, both in the personal and the research aspect. Her kindness and deep knowledge on the topic always made discussions a very fruitful experience, and that gave me confidence during difficult times.

I also want to thank all the members of the group Optimization and Stochastic at MLU for providing a friendly working atmosphere. Special thanks go to Prof. Dr. Truong Q. Bao and Prof. Dr. Alireza Doagooei for their mentorship during the first years, and to Dr. Vu Anh Tuan, Dr. Le Thanh Tam, Dr. Behnam Soleimani, Dr. Christian Günther, Marcus Hillmann, Bettina Zargini and Niklas Hebestreit for interesting discussions, collaboration, and friendship. I am also thankful to Sylke Sauter for all the help with different bureaucratic procedures during this time.

I am very grateful to the state of Sachsen-Anhalt for funding this research according to the "Graduiertenförderungsgesetz". I am also thankful to the "Stiftung Theoretische Physik/Mathematik" for providing travel support, which allowed me to present the results of the research in different international conferences.

Finally, I want to express my gratitude to my family for being always there for me. Special thanks go to my wife, Anai, for her emotional support during the hardest times. Above all, I am in debt to my parents for their unconditional love. Without their sacrifices, I would not have made it this far.

Contents

List of Symbols and Abbreviations

$\emptyset$	empty set
$\mathbb{N}$	set of positive natural numbers
$\mathbb{R}^n$	n-dimensional Euclidean space
$\mathbb{R}^n_+$	nonnegative orthant of $\mathbb{R}^n$
$\overline{\mathbb{R}}$	set of extended real numbers, $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$
X, Y	real vector spaces or real normed spaces
$\mathcal{L}(X, Y)$	vector space of continuous linear operators from X to Y
$\dim X$	dimension of the real vector space X, if this is finite
$\mathcal{P}(Y)$	class of all nonempty subsets of Y
$\operatorname{conv} A$	convex hull of the set A
$\operatorname{cone} A$	cone generated by the set A
$\\|\cdot\\|$	a norm defined on a vector space
$\mathbb{B}$	unit ball in a normed space
$\mathbb{S}$	unit sphere in a normed space
X^*	the topological dual space of the normed space X
x^*, y^*	generic elements of the dual spaces X^* and Y^* respectively
$\langle x^*, x\rangle$	evaluation of the functional $x^* \in X^*$ at the point $x \in X$
$\\|\cdot\\|_*$	dual norm of $\\|\cdot\\|$
$\operatorname{int} A$	interior of the set A
$\operatorname{cl} A$	closure of the set A
$\operatorname{cl}^* A$	closure of the set A in the w^*- topology
$\operatorname{bd} A$	boundary of the set A
δ_A	indicator functional of a set A
σ_A	support functional of a set A
γ_A	Minkowski functional of a set A
$d_{\\|\cdot\\|}(\cdot, A)$	distance functional associated to a set A.
$A \ominus B$	Hadwiger-Pontryagin difference of the sets A and B
MFCQ	Mangasarian-Fromovitz constraint qualification

K	cone in a vector space Y
$\preceq_K$, $\prec_K$	partial order and strict partial order induced by a cone K
$\preceq_D^{(l)}$	lower set less relation induced by a set D
$\prec_D^{(l)}$	lower set less relation induced by the set $\operatorname{int} D$
$\preceq_D^{(u)}$	upper set less relation induced by a set D
$\prec_D^{(u)}$	upper set less relation induced by the set $\operatorname{int} D$
$\preceq_D^{(s)}$	set less relation induced by a set D
$\prec_D^{(s)}$	set less relation induced by the set $\operatorname{int} D$
K^*	dual cone of K
K^{s*}	quasinterior of the cone K^*
Ψ_e	Gerstewitz-Weidner scalarizing functional associated to $e \in \operatorname{int} K$
$\Psi_{\|\cdot\|}$	Hiriart-Urruty scalarizing functional associated to a norm $\|\cdot\|$
Ψ_G	Drummond- Svaiter scalarizing functional associated to a set G
$\mathrm{WMin}(A, K)$	set of weakly minimal elements of A with respect to K
$\mathrm{Min}(A, K)$	set of minimal elements of A with respect to K
$\mathrm{SMin}(A, K)$	set of strongly minimal elements of A with respect to K
$\mathrm{WMax}(A, K)$	set of weakly maximal elements of A with respect to K
$\mathrm{Max}(A, K)$	set of maximal elements of A with respect to K
$\mathrm{SMax}(A, K)$	set of strongly maximal elements of A with respect to K
f	generic vector-valued function
$\operatorname{dom} f$	domain of a functional f
$\operatorname{epi} f$	epigraph of the functional f
$\operatorname{gph} f$	graph of the functional f
$f\|_A$	restriction of the function f to the set A
F	generic set-valued mapping
$\operatorname{dom} F$	domain of a set-valued mapping F
$\mathcal{E}_F$	epigraphical multifunction of F
$\mathcal{H}_F$	hypographical multifunction of F
$N(x, A)$	limiting normal cone to the set A at the point x
$\widehat{N}_\epsilon(x, A)$	the set ϵ- normal vectors to the set A at the point x
$\widehat{N}(x, A)$	Fréchet normal cone to the set A at the point x
$\partial f(\cdot)$	limiting subdifferential of the functional f at the point x
$f'(x, d)$	directional derivative of the function f at point x in the direction d
$\nabla f(x)$	Fréchet derivative of the function f at the point x
$D^*F(x, y)$	limiting coderivative of F at (x, y)
$\mathcal{OP}(f, \Omega)$	scalar optimization problem $\min\limits_{x \in \Omega} f(x)$
$\mathcal{VOP}(f, K, \Omega)$	vector optimization problem $K - \min\limits_{x \in \Omega} f(x)$
$\mathcal{SOP}(F, K, \Omega)$	set optimization problem $K - \min\limits_{x \in \Omega} F(x)$

List of Figures

List of Tables

Chapter 1

Introduction

This dissertation is devoted to an important branch of optimization theory known as set optimization. Roughly speaking, this is the class of mathematical problems dealing with the minimization of set-valued mappings acting between two normed spaces, where the image space is partially ordered by a closed, convex, and pointed cone. These problems generalize vector optimization models and have received considerable attention during the past decades due to their applications in finance [49, 74], socio-economics [18, 135], robotics [89], and robust multiobjective decision making [46, 86].

There are two main approaches for defining optimal solutions of a set optimization problem, namely the vector approach and the set approach. In the vector approach, one looks for minimal points in the image set of the set-valued objective mapping [89]. Thus, in this case, an optimal set is selected by identifying just one of its elements, without taking into account the rest of the set. However, in some practical scenarios, this feature poses an important drawback from the modelling point of view, therefore restricting the range of applications they can describe. The set approach is an attempt to overcome this problem, and it is the solution concept that we will mainly use in this work. The idea there lies on comparing sets with respect to a binary relation (usually a preorder) defined on the power set of the image space, and to consider minimal solutions accordingly. To the best of our knowledge, the first of these set relations were introduced independently by Young [158] and Nishnianidze [136], and later by Kuroiwa [117, 119]. Moreover, recently new set relations were defined by Jahn and Ha [95], and Karaman et al. [100]. On the other hand, set optimization problems and their solution concepts with respect to the set approach were considered for the first time by Kuroiwa in [115]. Since then, the research in this area has grown intensively, and different directions have been pursued. A non exhaustive list of topics considered are, among others, the existence of solutions [2, 78, 95, 119, 120, 127], duality statements [79, 77, 116, 118], well-posedness [35, 37, 62, 64, 75, 76, 125, 126], optimality conditions [2, 3, 11, 38, 69, 70, 91, 93, 109, 113, 137, 140, 142], and algorithms [46, 48, 65, 66, 85, 86, 90, 94, 97, 107, 110, 112, 146]. We refer the reader to [89, 106, 130] for a comprehensive overview of the field.

On the other hand, scalarization techniques are a fundamental tool in vector and set optimization, both from the theoretical and the computational point of view. Roughly speaking, this technique consists in replacing the vector or set optimization problem by a parametric family of so called scalarization problems. By a scalarization problem, we understand a standard optimization problem whose solution set has connections to that of the initial one. Scalarization problems are typically created by minimizing the composition of a so called scalarizing functional with the vector- (set-)valued objective mapping of the vector (set) optimization problem, and perhaps adding some additional constraints. Moreover, these scalarizing functionals, as well as the additional constraints, depend on certain parameters. Thus, because of their connection with the initial vector optimization or set optimization problem, scalarization techniques have been successfully employed as a solution method (by solving for each value of the admissible parameters the corresponding scalarization problem), or as an intermediate tool to obtain results in other topics. In fact, they have been of utmost importance in some of the lines of research mentioned above.

In vector optimization, there are already several scalarization techniques discussed in the literature, see [47] for a general overview. Furthermore, some of them have been extended to the set-valued context. In particular, many authors have studied generalizations of the nonlinear separating functionals introduced by Gerstewitz [56] (compare [55] and [57]) and by Hiriart-Urruty [81], see [7, 30, 63, 73, 80, 98, 107, 123]. In this dissertation, we derive optimality conditions and algorithms for set optimization problems using scalarizing functionals as the main tool. The highlights are the following:

- We provide a unified characterization of different classes of scalarizing functionals from the literature, and introduce a new class extending the previous ones.
- We obtain new necessary optimality conditions for set optimization problems using tools from variational analysis.
- We propose a steepest descent method for a particular class of set optimization problems.

In the rest of this chapter we discuss the structure of the dissertation. With the intention of making the exposition as self-contained as possible, we start in Chapter 2 by introducing the relevant mathematical concepts and results that will be used in our work.

Chapter 3 deals with different classes of scalarizing functionals. In a first part, we establish relationships between three of the well known classes in the sense of inclusion. The elements in these classes of functionals are sublinear, and we completely determine their relationships. In a second part, we introduce a new class of scalarizing functionals whose elements are not necessarily sublinear, but rather can be expressed as the difference of sublinear functionals. There, we examine relationships of inclusion between this new class and the classes previously considered. Moreover, we discuss important connections with set optimization.

In Chapter 4, we establish the main results concerning optimality conditions for set optimization problems where the solution concept is given by the set approach. Specifically, we deal with the so called lower set less and upper set less preorder relations. Motivated by the results obtained in the literature and in Chapter 3, we introduce two extended real valued functionals that are induced by the so called Gerstewitz-Weidner functional and the set-valued objective mapping. These functionals are associated to each of the set relations. First, we show that they inherit different properties from the set-valued objective mapping. We then study the limiting subdifferential of these functionals. The estimated subdifferential, together with the properties of the functionals, allow us to obtain Fermat rules for set optimization problems with Lipschitzian data.

Chapter 5 examines a first order solution method for a particular class of set optimization problems. In this case, the images of the set-valued objective mapping are of finite and equal cardinality. We further assume that the set-valued mapping can be decomposed into a finite number of selections that are continuously differentiable. In the first part of the chapter, we develop tailored optimality conditions for problems with that particular structure. These optimality conditions differ from the ones studied in Chapter 4, and exploit the assumptions on the set-valued mapping. In the second part, based on the previously derived optimality conditions, we propose the descent method and its convergence analysis. Moreover, we also discuss the implementation of the algorithm, and illustrate some of the results obtained on different instances previously considered in the literature.

Finally, in Chapter 6, we summarize our contributions and establish some further lines of research.

Chapter 2

Mathematical Preliminaries

In this chapter we provide an exposition of the most important theorems and definitions employed in the thesis. We assume that the reader is already familiar with the topological notions of the real line and with the basic operations in set theory. The chapter is organized as follows. Since we will be working in the setting of normed spaces, we start by recalling in Section 2.1 elements of functional analysis. Specifically, the dual space construction, the separation theorem, and the w^*- topology will be presented. In Section 2.2, we consider binary relations and their relationships with cones in a vector space. In particular, we also present the definitions of the set relations we are going to deal with and some basic results related to them. In Section 2.3, we recall different tools from set-valued and variational analysis. In this case, the notions of limiting subdifferential of a functional and coderivatives of a set-valued mapping will prove to be a key concept when deriving optimality conditions for set optimization problems. Section 2.4 formally introduces vector and set optimization problems, together with the different solution concepts. We conclude in Section 2.5 by considering several types of scalarizing functionals and their properties.

2.1 Fundamentals of Functional Analysis

We start with some fundamental notions of sets that will be used very often. As usual, the symbols $\mathbb{R}, \mathbb{R}_+, \mathbb{R}_{++}$ and $\mathbb{N}$ stand for the set of real numbers, the set of nonnegative real numbers, the set of positive real numbers and the set of positive natural numbers respectively. In addition, the set of extended real numbers is $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$.

Given sets A and B, their union, intersection, difference and cartesian product are denoted respectively by $A \cup B, A \cap B, A \setminus B$ and $A \times B$. In addition, if the set A is finite, we denote its cardinality by $|A|$. Furthermore, we denote the class of all nonempty subsets of A by $\mathcal{P}(A)$. We now recall the basic notion of a vector space.

Definition 2.1.1. *Let X be a nonempty set and consider two mappings $+ : X \times X \to X$ and $\cdot : \mathbb{R} \times X \to X$ that will act as sum and multiplication by a scalar respectively. The tuple $(X, +, \cdot)$ is called a real vector space if the following conditions are satisfied:*

(i) $\forall\ x_1, x_2, x_3 \in X\ :\ (x_1 + x_2) + x_3 = x_1 + (x_2 + x_3)$,

(ii) $\forall\ x_1, x_2 \in X\ :\ x_1 + x_2 = x_2 + x_1$,

(iii) $\exists\ 0 \in X :\ \forall\ x \in X\ :\ x + 0 = x$,

(iv) $\forall\ x_1 \in X,\ \exists\ x_2 \in X\ :\ x_1 + x_2 = 0$; *we write* $x_2 = -x_1$,

(v) $\forall\ x_1, x_2 \in X,\ \forall\ t \in \mathbb{R}\ :\ t(x_1 + x_2) = tx_1 + tx_2$,

(vi) $\forall\ x \in X,\ \forall\ t_1, t_2 \in \mathbb{R}\ :\ (t_1 + t_2)x = t_1x + t_2x$,

(vii) $\forall\ x \in X,\ \forall\ t_1, t_2 \in \mathbb{R} :\ t_1(t_2x) = (t_1t_2)x$,

(viii) $\forall\ x \in X\ :\ 1x = x$.

If there is no risk of confusion, we say that X is a vector space, and we omit the internal operations. The element 0 is also called the origin. Furthermore, for $A, B \subseteq X$ and a scalar $t \in \mathbb{R}$, the sum of sets and the multiplication by a scalar are given by $tA := \{ta \mid a \in A\}$ and $A + B := \{a + b \mid a \in A, b \in B\}$ respectively.

Very important concepts under a vector space structure are those related to linearity and convexity, which we define next.

Definition 2.1.2. *Let X be a vector space and A be a nonempty subset of X.*

(i) We say that A is a subspace of X if for every $t_1, t_2 \in \mathbb{R} :\ t_1A + t_2A \subseteq A$.

(ii) The set A is said to be convex if for every $t \in [0, 1] :\ tA + (1 - t)A \subseteq A$.

(iii) The convex hull of A, denoted by $\operatorname{conv} A$, *is defined as*

$$\operatorname{conv} A := \bigcap \{A' \supseteq A \mid A' \textit{ convex } \}.$$

As usual, a functional is understood as an operator whose range of values is contained in $\overline{\mathbb{R}}$. Furthermore, for a set-valued mapping $F : X \rightrightarrows Y$ and nonempty subsets A and B of X and Y respectively, we put

$$F[A] := \bigcup_{x \in A} F(x), \quad F^{-1}[B] := \{x \in X \mid F(x) \cap B \neq \emptyset\}.$$

We also denote by $f|_A$ the restriction of the vector-valued function $f : X \to Y$ to the set A, that is, $f|_A : A \to Y$ is defined as

$$f|_A(x) = y \iff x \in A, y = f(x).$$

Below we establish some of the main algebraic definitions associated to functionals defined on a vector space.

Definition 2.1.3. *Let X be a vector space and $f : X \to \overline{\mathbb{R}}$ be a given functional. The domain and the epigraph of f are defined respectively as*

$$\operatorname{dom} f := \{x \in X \mid f(x) < +\infty\},$$

$$\operatorname{epi} f := \{(x,t) \in X \times \mathbb{R} \mid f(x) \leq t\}.$$

Furthermore, f is said to be proper if $\operatorname{dom} f \neq \emptyset$ *and $f(x) > -\infty$ for every $x \in X$.*

Definition 2.1.4. *Let X be a vector space and $f : X \to \overline{\mathbb{R}}$ be a given functional.*

(i) We call f additive if

$$\forall\, x_1, x_2 \in X : f(x_1 + x_2) = f(x_1) + f(x_2).$$

If

$$\forall\, x_1, x_2 \in X : f(x_1 + x_2) \leq f(x_1) + f(x_2),$$

we say that f is subadditive.

(ii) We call f homogeneous if

$$\forall\, t \in \mathbb{R}, x \in X : f(tx) = tf(x).$$

If the above equality is only satisfied for $t \in \mathbb{R}_+$, we say that f is positive homogeneous.

(iii) We say that f is linear if it is finite valued, additive, and homogeneous. If f is only subadditive and positve homogeneous, we say that it is sublinear.

(iv) Suppose that $A \subseteq X$ is convex. We say that f is convex on A if the inequality

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$$

holds for all $t \in [0,1]$ and $x_1, x_2 \in A$ for which the right hand side is well defined.

Definition 2.1.5. *Let X be a vector space and $A \subseteq X$ be nonempty. The indicator functional of the set A is $\delta_A : X \to \overline{\mathbb{R}}$ defined as*

$$\delta_A(x) := \begin{cases} 0 & \text{if } x \in A, \\ +\infty & \text{otherwise}. \end{cases}$$

Next, we turn our attention to topological spaces.

Definition 2.1.6. *Let X be a nonempty set, and $\mathcal{T}$ be a family of subsets of X. We say that the pair $(X, \mathcal{T})$ is a topological space if $\mathcal{T}$ satisfies the following conditions:*

(i) $\emptyset, X \in \mathcal{T}$,

(ii) *the union of elements in* $\mathcal{T}$ *belongs to* $\mathcal{T}$,

(iii) *the finite intersection of sets in* $\mathcal{T}$ *belongs to* $\mathcal{T}$.

The elements of $\mathcal{T}$ *are called open sets. Furthermore, a subset of* X *is said to be closed if its complement is open.*

We define now some basic topological concepts.

Definition 2.1.7. *Let* $(X, \mathcal{T}_X)$ *and* $(Y, \mathcal{T}_Y)$ *be topological spaces,* $A \subseteq X$, $\bar{x} \in X$, *and* $f : X \to Y$ *be a given vector-valued function.*

(i) *The interior of* A *is the set denoted by* $\operatorname{int} A$ *and given by*

$$\operatorname{int} A := \bigcup \{A' \subseteq A \mid A' \text{ open } \}.$$

(ii) *The closure of* A *is the set denoted by* $\operatorname{cl} A$ *and defined as*

$$\operatorname{cl} A := \bigcap \{A' \supseteq A \mid A' \text{ closed } \}.$$

(iii) *The boundary of* A *is the set denoted by* $\operatorname{bd} A$ *and defined as* $\operatorname{bd} A := (\operatorname{cl} A) \setminus \operatorname{int} A$.

(iv) *We say that* A *is a neighborhood of* $\bar{x}$ *if there exists an open set* U *such that* $\bar{x} \in U \subseteq A$. *The class of all neighborhoods of* $\bar{x}$ *will be denoted by* $\mathcal{N}_{\mathcal{T}_X}(\bar{x})$.

(v) *A collection* $\mathcal{B}_{\bar{x}} \subseteq \mathcal{N}_{\mathcal{T}_X}(\bar{x})$ *is said to be a neighborhood base of* $\bar{x}$ *if for every* $U \in \mathcal{N}_{\mathcal{T}_X}(\bar{x})$ *there is* $B \in \mathcal{B}_{\bar{x}}$ *such that* $B \subseteq U$.

(vi) *We say that* A *is dense on a subset* B *of* X *if* $B \subseteq \operatorname{cl} A$.

(vii) *We say that* A *is compact if for any collection* $\mathcal{T}_1 \subseteq \mathcal{T}_X$ *satisfying* $A \subseteq \bigcup \{A' \mid A' \in \mathcal{T}_1\}$, *there exists* $p \in \mathbb{N}$ *and* $A'_1, \ldots, A'_p \in \mathcal{T}_1$ *such that*

$$A \subseteq \bigcup_{i=1}^{p} A'_i.$$

(viii) *Suppose that* $\bar{x} \in A$. *We say that* A *is locally closed around* $\bar{x}$ *if there exists a neighborhood* U *of* $\bar{x}$ *such that* $A \cap U$ *is a closed set.*

(ix) *A sequence* $\{x_k\}_{k \geq 1} \subset X$ *is said to be convergent to a point* $\bar{x} \in X$ *if the following holds:*

$$\forall\, U \in \mathcal{N}_{\mathcal{T}_X}, \exists\, k_U \in \mathbb{N} : \; x_k \in U \text{ for every } k \geq k_U.$$

This is denoted by $x_k \to \bar{x}$.

(x) *The functional f is said to be continuous at $\bar{x}$ if the following condition is satisfied:*

$$\forall\, V \in \mathcal{N}_{\mathcal{T}_Y}(f(\bar{x})) :\ f^{-1}[V] \in \mathcal{N}_{\mathcal{T}_X}(\bar{x}).$$

We say that f is continuous if it is continuous at every point of X.

(xi) *Suppose that $Y = \mathbb{R}$. The functional f is called lower semicontinuous at $\bar{x}$ if the following assertion holds:*

$$\forall\, \epsilon > 0, \exists\, U \in \mathcal{N}_{\mathcal{T}_X}(\bar{x}) : f(x) > f(\bar{x}) - \epsilon \text{ for every } x \in U.$$

We say that f is lower semicontinuous on the set A if it is lower semicontinuous at every point of A.

The concept of basis will be very useful when defining different topologies later in the text.

Definition 2.1.8. *Let X be a given set and $\mathcal{B}$ be a family of subsets of X.*

(i) *We say that $\mathcal{B}$ is a basis for a topology on X if the following properties are satisfied:*

(a) *X is exactly the union of the elements of $\mathcal{B}$,*

(b) *If $x \in A \cap B$ with $A, B \in \mathcal{B}$, then there is $C \in \mathcal{B}$ such that $x \in C \subseteq A \cap B$.*

(ii) *Suppose that $\mathcal{B}$ is a basis for a topology and consider the family $\mathcal{T}$ of subsets of X defined as follows: $A \subseteq X$ belongs to $\mathcal{T}$ if and only if A can be represented as the union of elements in $\mathcal{B}$. Then, it can be shown that $\mathcal{T}$ is a topology on X. In this case, we say that $\mathcal{T}$ is the topology generated by $\mathcal{B}$.*

At the intersection between vector spaces and topologies lies the following concept.

Definition 2.1.9. *Let X be a vector space and let $\mathcal{T}$ be a topology on X.*

(i) *We say that $(X, \mathcal{T})$ is a topological vector space if the operations of sum and product by a scalar are continuous.*

(ii) *The topological vector space $(X, \mathcal{T})$ is said to be locally convex if there exists a neighborhood base of $0 \in X$ consisting of convex sets.*

The most important class of topological vector spaces that we will consider in this work is that of normed spaces.

Definition 2.1.10. *Let X be a vector space and $\|\cdot\|_X : X \to \mathbb{R}_+$ be a given functional.*

(i) *We say that $\|\cdot\|_X$ is a norm on X if the following conditions are fulfilled for every $x_1, x_2 \in X$ and every $t \in \mathbb{R}$:*

(a) $\|x_1\|_X = 0 \Longleftrightarrow x_1 = 0,$

(b) $\|tx_1\|_X = |t|\|x_1\|_X$,

(c) $\|x_1 + x_2\|_X \leq \|x_1\|_X + \|x_2\|_X$.

(ii) If $\|\cdot\|_X$ *is a norm on* X*, we say that the pair* $(X, \|\cdot\|_X)$ *is a normed space.*

(iii) The unit ball and the unit sphere in a normed space $(X, \|\cdot\|_X)$ *are defined respectively as follows:*

$$\mathbb{B}_X := \{x \in X \mid \|x\|_X \leq 1\}, \ \mathbb{S}_X := \{x \in X \mid \|x\|_X = 1\}$$

(iv) Suppose that $(X, \|\cdot\|_X)$ *is a normed space. The norm topology on* X *is the topology generated by the collection*

$$\mathcal{B} := \{x + \epsilon\,(\mathbb{B}_X \setminus \mathbb{S}_X) \mid x \in X, \epsilon \in \mathbb{R}_{++}\}.$$

It is well known that normed spaces with the norm topology are locally convex topological vector spaces. On the other hand, the norm $\|\cdot\|_X$ induces a metric on X defined as follows: $d(x_1, x_2) := \|x_1 - x_2\|_X$. Hence, in this context, the unit sphere $\mathbb{S}_X$ is the set of elements in X at distance 1 from the origin. In this setting, a set $A \subseteq X$ is said to be bounded if there is an upper bound between the distances of its points and the origin, that is, if we can find a constant $L > 0$ such that $A \subseteq L\mathbb{B}$.

Definition 2.1.11. *Let* $(X, \|\cdot\|_X)$ *be a normed space and consider a set* $A \subseteq X$*. The distance functional associated to* A *is* $d_{\|\cdot\|_X}(\cdot, A) : X \to \mathbb{R}$ *defined as*

$$d_{\|\cdot\|_X}(x, A) := \inf_{x' \in A} \|x - x'\|_X.$$

From now on, when referring to a normed space, we drop the norm in the tuple definition if there is no confusion. We also omit the subscript X in the norm, the unit sphere and the unit ball. Hence, for example, we say that X is a normed space and that $\|\cdot\|$, $\mathbb{S}$ and $\mathbb{B}$ are its norm, unit sphere and unit ball respectively.

Some other notations and conventions used in the text are the following:

- Unless otherwise stated, whenever we consider topological concepts in a normed space, we do so according to the norm topology.

- If X and Y are normed spaces, the cartesian product $X \times Y$ is also a vector space, and we will consider this as a normed space with the norm $\|\cdot\|_{X\times Y}$ given by $\|(x,y)\|_{X\times Y} = \|x\|_X + \|y\|_Y$.

- Two norms $\|\cdot\|$ and $\|\cdot\|'$ in a vector space X are said to be equivalent if we can find constants $\alpha, \beta > 0$ such that

$$\forall\, x \in X : \alpha\|x\| \leq \|x\|' \leq \beta\|x\|.$$

This is denoted by $\|\cdot\| \sim \|\cdot\|'$.

- If $X = \mathbb{R}^n$ for some $n \in \mathbb{N}$, we assume implicitly that vectors are represented as columns and that the Euclidean norm is used, that is, for $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$,
$$\|x\|_2 := \left(\sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}}.$$
Furthermore, if $\{x_k\}_{k\geq 1}$ is a sequence in $\mathbb{R}^n$, we denote by $x_{k,i}$ the i^{th}- component of the vector x_k.

An important class of sequences in normed spaces are those who satisfy Cauchy's property.

Definition 2.1.12. *Let $(X, \|\cdot\|_X)$ be a normed space and $\{x_k\}_{k\geq 1}$ be a sequence in X. We call $\{x_k\}_{k\geq 1}$ a Cauchy sequence if $\|x_n - x_m\| \to 0$.*

Another fundamental notion in a vector space that is closely related to norms is that of Minkowski functionals.

Definition 2.1.13. *Let X be a vector space and consider a nonempty set $U \subseteq X$.*

(i) We say that U is absorbing if for every $x \in X$ there is $\bar{t} > 0$ such that $x \in tU$ whenever $|t| > \bar{t}$.

(ii) We say that U is balanced if $tU \subseteq U$ for every $t \in [-1, 1]$.

(iii) Suppose that U is convex and absorbing. The Minkowski functional $\gamma_U : X \to \mathbb{R}$ of U is defined as
$$\gamma_U(x) := \inf\{t > 0 \mid x \in tU\}.$$

Remark 2.1.14. *It is easy to see that, if X is normed space and U is a neighborhood of 0, then U is absorbing. Furthermore, the unit ball $\mathbb{B}$ is in this case also a balanced set.*

In the next proposition we show the connection between Minkowski functionals and norms.

Proposition 2.1.15. *([144, Theorem 1.39]) Let X be a normed space and $U \subset X$ be a convex, balanced and bounded neighborhood of 0. Then, the Minkowski functional of U is a norm in X.*

Duality will be another fundamental tool in our work. In that direction, we must first recall the space of continuous linear operators.

Definition 2.1.16. *Let X and Y be normed spaces and $T : X \to Y$ be a given operator.*

(i) We say that T is linear if:

$$\forall\, x_1, x_2 \in X,\ t \in \mathbb{R} : T(tx_1 + x_2) = tT(x_1) + T(x_2).$$

Of course, when $Y = \mathbb{R}$, this concept reduces to the one in Definition 2.1.4.

(ii) Suppose that T is a linear operator. We say that T is bounded if there exists a constant $\ell > 0$ such that

$$\forall\, x \in X : \|Tx\| \leq \ell \|x\|.$$

Since it is well known that the boundedness and continuity of a linear operator T are equivalent, we also say that T is continuous.

(iii) The vector space of all continuous linear operators acting on the space X and with range on the space Y is the set

$$\mathcal{L}(X, Y) := \{T : X \to Y \mid T \textit{ is linear and bounded}\}$$

endowed with the addition and product by a scalar defined as follows:

$$\forall\, T_1, T_2 \in \mathcal{L}(X, Y), x \in X, t \in \mathbb{R} : \qquad \begin{aligned} &(T_1 + T_2)(x) := T_1(x) + T_2(x), \\ &(tT_1)(x) := tT_1(x). \end{aligned}$$

(iv) The norm topology on the vector space $\mathcal{L}(X, Y)$ is the norm topology on the normed space $\left(\mathcal{L}(X, Y), \|\cdot\|_{\mathcal{L}(X,Y)}\right)$, where the functional $\|\cdot\|_{\mathcal{L}(X,Y)} : \mathcal{L}(X, Y) \to \mathbb{R}$ is defined as

$$\|T\|_{\mathcal{L}(X,Y)} := \sup_{x \in X \setminus \{0\}} \frac{\|T(x)\|_Y}{\|x\|_X}. \tag{2.1}$$

We can now proceed to define dual spaces.

Definition 2.1.17. *Let $(X, \|\cdot\|)$ be a normed space.*

(i) The topological dual space of X is denoted by X^* and is defined as $X^* := \mathcal{L}(X, \mathbb{R})$.

(ii) Consider the dual space X^* of X. The dual norm of $\|\cdot\|$ is the functional $\|\cdot\|_* : X^* \to \mathbb{R}_+$ given by $\|\cdot\|_* := \|\cdot\|_{\mathcal{L}(X,\mathbb{R})}$, where $\|\cdot\|_{\mathcal{L}(X,\mathbb{R})}$ is given in (2.1), that is,

$$\forall\, x^* \in X^* : \|x^*\|_* = \sup_{x \in X \setminus \{0\}} \frac{|\langle x^*, x\rangle|}{\|x\|}.$$

In the equations above, $\langle x^*, x\rangle$ denotes the evaluation of the functional x^* at the point $x \in X$.

(iii) The norm topology on X^* is the norm topology on the normed space $(X^*, \|\cdot\|_*)$.

We will keep the notation of Definition 2.1.17 throughout the thesis, that is, we will denote an arbitrary element in X^* by x^* and use $\langle \cdot, \cdot \rangle$ as the dual pairing. The only exception to this notation is when we are working in the finite dimensional Euclidean space $\mathbb{R}^n$, for some $n \in \mathbb{N}$. In that case, $(\mathbb{R}^n)^*$ can be identified with $\mathbb{R}^n$, and hence their vectors are denoted by lower case letters. Furthermore, for vectors $u, v \in \mathbb{R}^n$, we have $\langle u, v \rangle = u^\top v$.

The following relation holds between a norm and its dual:

Proposition 2.1.18. *([114, Corollary 4.3-4]) Let $(X, \|\cdot\|)$ be a normed space and let X^* be its topological dual. Then, for every $x \in X$, we have*

$$\|x\| = \sup_{x^* \in X^* \setminus \{0\}} \frac{|\langle x^*, x \rangle|}{\|x^*\|_*}.$$

We will use the concept of adjoint operator a few times in our work. This is defined next.

Definition 2.1.19. *Let X and Y be normed spaces and $T : X \to Y$ be a continuous linear operator. The adjoint operator of T is $T^* : Y^* \to X^*$ defined as*

$$\forall\, y^* \in Y^*, x \in X : \langle T(y^*), x \rangle = \langle y^*, T(x) \rangle.$$

Remark 2.1.20. *It is well known [144, Theorem 4.10] that the adjoint operator T^* of T in the definition above satisfies $T^* \in \mathcal{L}(Y^*, X^*)$, and that $\|T\|_{\mathcal{L}(X,Y)} = \|T^*\|_{\mathcal{L}(Y^*,X^*)}$.*

We recall next the concept of support functional of a set.

Definition 2.1.21. *Let X be a normed space and consider a set $G \subseteq X^*$. The support functional $\sigma_G : X \to \overline{\mathbb{R}}$ of G is defined as*

$$\sigma_G(x) := \sup_{x^* \in G} \langle x^*, x \rangle .$$

Furthermore, for an element $x \in X$, the x- face of G is defined by

$$G^x := \{x^* \in G \mid \langle x^*, x \rangle = \sigma_G(x)\} . \tag{2.2}$$

The dual of the normed space $(X^*, \|\cdot\|_*)$ is called the bidual of X and is denoted by X^{**}(instead of $(X^*)^*$). In order to define the second topology that we will consider in the dual space, we need the following concept:

Definition 2.1.22. *Let X be a normed space and consider its bidual X^{**}. The canonical mapping $J_X : X \to X^{**}$ is defined as follows:*

$$\forall\, x \in X, x^* \in X^* : \langle J_X(x), x^* \rangle = \langle x^*, x \rangle. \tag{2.3}$$

Next, for $\bar{x}^* \in X^*, x_1, \ldots, x_p \in X$ and $\epsilon > 0$, we consider the set:

$$U(\bar{x}^*, x_1, \ldots, x_p, \epsilon) := \bigcap_{i=1}^{p} \{x^* \in X^* \mid |\langle x^*, x_i \rangle - \langle \bar{x}^*, x_i \rangle| < \epsilon\}.$$

Definition 2.1.23. *Let X be a normed space and consider its topological dual space X^*. Then, the class*

$$\mathcal{B} := \{U(x^*, x_1, \ldots, x_p, \epsilon) \mid x^* \in X^*, p \in \mathbb{N}, x_1, \ldots, x_p \in X, \epsilon > 0\}$$

is a basis for a topology on X^*, and the topology generated by $\mathcal{B}$ is called the w^*- topology on X^*.

The closure of a set $G \subseteq X^*$ with respect to the w^*- topology is denoted by $\text{cl}^* G$, in order to differentiate it from the closure in the (dual) norm topology. With the same intention, we also write $x_k^* \xrightarrow{w^*} \bar{x}^*$ when the sequence $\{x_k^*\}_{k\geq 1} \subset X^*$ converges to the point $\bar{x}^*$ in the w^*- topology. We collect some useful results related to the w^*- topology in the next Proposition.

Proposition 2.1.24. *([83]) Let X be a normed space and consider its topological dual space X^*. The following assertions hold:*

(i) The set X^*, together with the w^* topology, is a locally convex topological vector space. Moreover, the set of linear functionals in the topological bidual X^{**} that are continuous with respect to the w^*- topology is exactly $J_X[X]$.

(ii) Let $A \subset X^*$ be convex. Then, A is closed with respect to the norm topology in X^* if and only if it is w^*- closed.

(iii) Let a nonempty set $A \subset X^*$ be bounded and w^*- closed. Then, A is w^*- compact. The converse statement is satisfied if X is a Banach space (see Definition 2.1.25).

We define now the three main classes of normed spaces that will be used in the text. For the concept of Fréchet differentiability, see Subsection 2.3.2.

Definition 2.1.25. *Let X be a normed space.*

(i) We say that X is a Banach space if every Cauchy sequence is convergent.

(ii) Suppose that X is a Banach space. We say that X is reflexive if the canonical mapping J_X is surjective, that is, if $J_X[X] = X^{**}$.

(iii) Suppose that X is a Banach space. We say that X is Asplund if every continuous convex functional f defined on an open convex subset U of X is Fréchet differentiable on a dense subset of U.

We close the section with the fundamental separation theorem for convex sets, which is of supreme importance in functional analysis.

Theorem 2.1.26. *([83]) Let X be a locally convex topological vector space and $A, B \subseteq X$ be closed and convex. The following statements holds:*

(i) Suppose that $\operatorname{int} A \neq \emptyset$*. Then,* $\operatorname{int} A \cap B = \emptyset$ *if and only if there is* $x^* \in X^* \setminus \{0\}$ *such that*

$$\sup_{a \in A} \langle x^*, a \rangle \leq \inf_{b \in B} \langle x^*, b \rangle$$

(ii) Suppose that A *is compact and that* $A \cap B = \emptyset$*. Then, there is* $x^* \in X^* \setminus \{0\}$ *such that*

$$\sup_{a \in A} \langle x^*, a \rangle < \inf_{b \in B} \langle x^*, b \rangle.$$

2.2 Binary Relations and Cones

2.2.1 Properties of Binary Relations

In this section, we introduce binary relations on a set. In order to keep the notation consistent throughout the dissertation, and because binary relations will be considered on an image space, we will use a set Y in the definition.

Definition 2.2.1. *Let* Y *be a nonempty set. A binary relation on* Y *is a set* $\mathcal{R} \subseteq Y \times Y$*. We write* $y_1 \mathcal{R} y_2$ *when* $(y_1, y_2) \in \mathcal{R}$*, and we say that* y_1 *and* y_2 *are related.*

Some basic concepts associated to binary relations are the following:

Definition 2.2.2. *Let* $\mathcal{R}$ *be a binary relation on a set* Y*. We say that* $\mathcal{R}$ *is:*

(i) reflexive, if for every $y \in Y : y \mathcal{R} y$*,*

(ii) symmetric, if for every $y_1, y_2 \in Y$*:* $y_1 \mathcal{R} y_2 \Longrightarrow y_2 \mathcal{R} y_1$*,*

(iii) antisymmetric, if for every $y_1, y_2 \in Y$*:* $y_1 \mathcal{R} y_2$ *and* $y_2 \mathcal{R} x_1 \Longrightarrow y_1 = y_2$*,*

(iv) transitive, if for every $y_1, y_2, y_3 \in Y$*:* $y_1 \mathcal{R} y_2$ *and* $y_2 \mathcal{R} y_3 \Longrightarrow y_1 \mathcal{R} y_3$*.*

Definition 2.2.3. *Let* $\mathcal{R}$ *be a binary relation on a set* Y*. We say that* $\mathcal{R}$ *is:*

(i) total, if for every $y_1, y_2 \in A : y_1 \mathcal{R} y_2$ *or* $y_2 \mathcal{R} y_1$*,*

(ii) a preorder, if $\mathcal{R}$ *is reflexive and transitive,*

(iii) a partial order, if $\mathcal{R}$ *is reflexive, antisymmetric, and transitive.*

(iv) an equivalent relation, if $\mathcal{R}$ *is reflexive, symmetric, and transitive.*

Example 2.2.4. *Let* Y *be an arbitrary nonempty set. Then,*

$$\mathcal{R} := \{(A, B) \in \mathcal{P}(Y) \times \mathcal{P}(Y) \mid A \subseteq B\}$$

defines a binary relation on $\mathcal{P}(Y)$ *that is reflexive, antisymmetric, and transitive. Hence, in particular, it constitutes a partial order on* $\mathcal{P}(Y)$*.*

Definition 2.2.5. *Let $\mathcal{R}$ be a binary relation on a nonempty set Y, and consider $A \subseteq Y$. An element $\bar{a} \in A$ is said to be minimal of A with respect to $\mathcal{R}$ if the following holds:*

$$\forall\, a \in A:\ a\mathcal{R}\bar{a} \Longrightarrow \bar{a}\mathcal{R}a.$$

Similarly, we say that $\bar{a} \in A$ is a maximal element of A with respect to $\mathcal{R}$ if

$$\forall\, a \in A:\ \bar{a}\mathcal{R}a \Longrightarrow a\mathcal{R}\bar{a}.$$

The set of minimal and maximal elements of A are denoted by $\mathrm{Min}(A, \mathcal{R})$ and $\mathrm{Max}(A, \mathcal{R})$, respectively.

In this dissertation, we will be mostly considering binary relations related to an underlying vector space. In that case, we want the binary relation to have properties that are directly linked to the sum and product operations defining the vector space. The following concept is fundamental in that aspect.

Definition 2.2.6. *Let Y be a vector space and $\mathcal{R}$ be a binary relation on Y. We say that $\mathcal{R}$ is compatible with the linear structure of Y if the following properties are fulfilled:*

(i) $\forall\, t \geq 0, y_1, y_2 \in Y : y_1\mathcal{R}y_2 \Longrightarrow ty_1\mathcal{R}ty_2$,

(ii) $\forall\, y, y_1, y_2 \in Y : y_1\mathcal{R}y_2 \Longrightarrow (y + y_1)\mathcal{R}(y + y_2)$.

In the next section we will see how to define partial orders that are compatible with a linear structure.

2.2.2 Cone Properties

In this part, we introduce the notion of a cone together with some of its properties. As we will see, this concept is directly related to Definition 2.2.6.

Definition 2.2.7. *Let Y be a vector space and K be a nonempty subset of Y. We say that K is a cone if $tx \in K$ for every $x \in K$ and every $t \geq 0$. A cone K is called:*

(i) convex, if $K + K \subseteq K$,

(ii) proper, if $K \neq \{0\}$ *and* $K \neq Y$,

(iii) pointed, if $K \cap (-K) = \{0\}$.

Other properties of cones are related to a topology.

Definition 2.2.8. *Let Y be a topological vector space and K be a cone in Y. We say that K is solid if* $\operatorname{int} K \neq \emptyset$,

Subsets of a vector space Y induce a binary relation that is defined below.

Definition 2.2.9. *Let Y be a vector space and K be a nonempty subset of Y. The binary relation on Y induced by K is denoted by $\preceq_K$ and is defined as*

$$\preceq_K := \{(y_1, y_2) \in Y \times Y \mid y_2 - y_1 \in K\}.$$

If in the above definition Y is a topological vector space and $\operatorname{int} K \neq \emptyset$, we write $\prec_K$ in place of $\preceq_{\operatorname{int} K}$. The following theorem shows the importance of the binary relations induced by cones.

Theorem 2.2.10. *([58, Theorem 2.1.13]) Let Y be a vector space and K be a proper cone in Y. Then, the following statements concerning the relation $\preceq_K$ from Definition 2.2.9 hold:*

(i) The relation $\preceq_K$ is reflexive and compatible with the linear structure of Y.

(ii) The cone K is convex if and only if $\preceq_K$ is transitive.

(iii) The cone K is pointed if and only if $\preceq_K$ is antisymmetric.

Hence, in particular, $\preceq_K$ is a partial order if and only if K is a convex and pointed cone. Conversely, if $\mathcal{R}$ is a binary relation on Y that is reflexive and compatible with the linear structure of Y, then the set $K := \{y \in Y \mid 0\mathcal{R}y\}$ is a cone and $\mathcal{R} = \preceq_K$.

Another important concept that we will use is that of generators.

Definition 2.2.11. *Let Y be a topological vector space, $G \subseteq Y$, and consider a proper convex cone $K \subset Y$.*

(i) The cone generated by G is the set denoted by $\operatorname{cone} G$ that is defined as

$$\operatorname{cone} G := \{tx \mid t \geq 0,\ g \in G\}.$$

(ii) We say that a set G is a topological generator of K if the following conditions are fulfilled:

(a) G is convex,

(b) $0 \notin \operatorname{cl} G$,

(c) $\operatorname{cone} G = K$.

(iii) We say that G is a topological base of K if it is a generator of K such that for every $x \in K \setminus \{0\}$ the representation

$$y = tg,\ t > 0,\ g \in G$$

is unique.

Example 2.2.12. *The set $\mathbb{R}^m_+ \subset \mathbb{R}^m$ is a closed, convex, pointed, and solid cone. In the literature, it is often called the natural ordering cone. It is also easy to see that the set*

$$B := \left\{ x \in \mathbb{R}^m_+ \;\middle|\; \sum_{i=1}^{m} x_i = 1 \right\}$$

is a base (and hence a generator) of $\mathbb{R}^m_+$.

Definition 2.2.13. *Let Y be a topological vector space and $K \subset Y$ be a proper cone.*

(i) The dual cone of K is the set K^* defined as

$$K^* := \{y^* \in Y^* \mid \forall\, y \in K : \langle y^*, y\rangle \geq 0\}.$$

(ii) The quasinterior of K^* is the set K^{s*} given by

$$K^{s*} := \{y^* \in Y^* \mid \forall\, y \in K \setminus \{0\} : \langle y^*, y\rangle > 0\}.$$

We establish some properties of convex cones and their duals in the next proposition.

Proposition 2.2.14. *([58, 89]) Let Y be a normed space and $K \subset Y$ be a closed, convex, pointed, and solid cone. The following assertions hold:*

(i) $K + K = K$.

(ii) $K + \operatorname{int} K = \operatorname{int} K$.

(iii) K^* is a w^*- closed and convex cone.

(iv) $K = \{y \in Y \mid \forall\, y^* \in K^* : \langle y^*, y\rangle \geq 0\}$.

(v) $\operatorname{int} K = \{y \in Y \mid \forall\, y^* \in K^* \setminus \{0\} : \langle y^*, y\rangle > 0\}$.

(vi) A set $B \subset Y^*$ is a base of K^* with respect to the w^*- topology if and only if there exists $e \in \operatorname{int} K$ such that

$$B = \{y^* \in K^* \mid \langle y^*, e\rangle = 1\}.$$

2.2.3 Set Relations

In this part of the section, we define preorder relations between the subsets of a vector space, and recall some of its properties. These set relations are the basis of the solution concepts for the set optimization problem that will be studied in this thesis.

Definition 2.2.15 ([41, 117, 136, 158])**.** *Let Y be a vector space and fix a nonempty set $K \subseteq Y$. The following set relations are defined on $\mathcal{P}(Y)$:*

(i) *The lower set less relation* $\preceq_K^{(l)}$ *is defined as*

$$\forall\, A, B \subseteq Y : A \preceq_K^{(l)} B :\Longleftrightarrow B \subseteq A + K.$$

(ii) *The upper set less relation* $\preceq_K^{(u)}$ *is defined as*

$$\forall\, A, B \subseteq Y : A \preceq_K^{(u)} B :\Longleftrightarrow A \subseteq B - K.$$

(iii) *The set less relation* $\preceq_K^{(s)}$ *is defined as*

$$\forall\, A, B \subseteq Y : A \preceq_K^{(s)} B :\Longleftrightarrow A \preceq_K^{(l)} B \text{ and } A \preceq_K^{(u)} B.$$

(iv) *The possibly set less relation* $\preceq_K^{(p)}$ *is defined as*

$$\forall\, A, B \subseteq Y : A \preceq_K^{(p)} B :\Longleftrightarrow B \cap (A + K) \neq \emptyset.$$

(v) *The certainly set less relation* $\preceq_K^{(c)}$ *is defined as*

$$\forall\, A, B \subseteq Y : A \preceq_K^{(c)} B :\Longleftrightarrow B \subseteq \bigcap_{a \in A} (a + K).$$

For simplicity, whenever we consider the set relation with respect to an open set $\operatorname{int} K$, *we write* $\prec_K^{(r)}$ *instead of* $\preceq_{\operatorname{int} K}^{(r)}$, *for* $r \in \{l, u, s, p, c\}$.

Remark 2.2.16. *It is easy to check [95, Proposition 3.9] that the following implications between the set relations hold:*

$$\begin{array}{ccccc} A \preceq_K^{(c)} B & \Longrightarrow & A \preceq_K^{(s)} B & \Longrightarrow & A \preceq_K^{(l)} B \\ & & \Downarrow & & \Downarrow \\ & & A \preceq_K^{(u)} B & \Longrightarrow & A \preceq_K^{(p)} B. \end{array}$$

In addition, it is not difficult to construct examples on which the converse implications are not fulfilled.

It is worth to point out that the set relations in Definition 2.2.15 are not the only ones in the literature. Indeed, new set order relations were defined later by Jahn and Ha in [95], and very recently by Karaman et al. in [100].

We collect some properties of the set relations in the next proposition.

Proposition 2.2.17. *([95]) Let* Y *be a vector space and consider a proper convex cone* $K \subset Y$. *Then, the following assertions hold:*

(i) *For* $r \in \{l, u, s\}$, *the relation* $\preceq_K^{(r)}$ *is a preorder in* $\mathcal{P}(Y)$. *Furthermore,* $\preceq_K^{(r)}$ *is compatible with the vector space operations of* Y, *in the sense that* *(a)* $\forall\, t \geq 0, A, B \subseteq Y : A \preceq_K^{(r)} B \Longrightarrow tA \preceq_K^{(r)} tB,$

(b) $\forall\, A, B, C \subseteq Y : A \preceq_K^{(r)} B \Longrightarrow A + C \preceq_K^{(r)} B + C.$

(ii) *For* $A, B \subseteq Y$*, we have*

$$A \preceq_K^{(l)} B \text{ and } B \preceq_K^{(l)} A \Longleftrightarrow A + K = B + K,$$

$$A \preceq_K^{(u)} B \text{ and } B \preceq_K^{(u)} A \Longleftrightarrow A - K = B - K.$$

(iii) *The set relation* $\preceq_K^{(p)}$ *is reflexive and compatible with the vector space operations of* Y *in the sense indicated in item* (i)*.*

(iv) *The relation* $\preceq_K^{(c)}$ *is antisymmetric and transitive.*

In this work, we will only deal with the relations $\preceq_K^{(l)}, \preceq_K^{(u)}$ and $\preceq_K^{(s)}$ in the context of set optimization. This is because, for optimization purposes, the relations $\preceq_K^{(p)}$ and $\preceq_K^{(c)}$ seem to be too weak and too strong respectively. Intuitively, since the set relation $\preceq_K^{(c)}$ is very strong, it is difficult to have two sets that are comparable in general. Thus, we expect, when finding the minimal elements among a family of sets with respect to the binary relation $\preceq_K^{(c)}$ (see Definition 2.2.5), to have all the family as the solution set. In the same context, we could argue that, since $\preceq_K^{(p)}$ is weak, it is almost symmetric, and hence almost all the elements in the family are minimal. Nevertheless, we believe that our approach in Chapter 4 to optimality conditions could be extended to deal with these set relations.

As a final note, we mention that, taking into account Proposition 2.2.17 (ii), it is possible to show [80] that the binary relation $\sim_K^{(l)}$ defined on $\mathcal{P}(Y)$ as

$$\forall\, A, B \subseteq Y : A \sim_K^{(l)} B \Longleftrightarrow A + K = B + K$$

is an equivalence relation. Thus, the relation $\sim_K^{(l)}$ defines, for any set $A \subseteq Y$, a corresponding equivalence class $[A]^{(l)}$ as follows:

$$[A]^{(l)} := \left\{ B \subseteq Y \mid A \sim_K^{(l)} B \right\}.$$

A similar concept can be considered for the preorder $\preceq_K^{(u)}$.

2.3 Generalized Differentiation

2.3.1 Set Valued Analysis

In this part of the section, we recall some basic notions of set-valued analysis, which lie at the core of our work. We start by defining the domain of a set-valued mapping, together with other associated concepts.

Definition 2.3.1. *Let X and Y be vector spaces and $F : X \rightrightarrows Y$ be a set-valued mapping.*

(i) The domain of F is given by

$$\operatorname{dom} F := \{x \in X \mid F(x) \neq \emptyset\}.$$

(ii) The graph of F is defined by

$$\operatorname{gph} F := \big\{(x, y) \in X \times Y \mid y \in F(x)\big\}.$$

(iii) Let $K \subset Y$ be a cone. Then, the epigraph of F is defined as

$$\operatorname{epi} F := \big\{(x, y) \in X \times Y \mid y \in F(x) + K\big\}.$$

(iv) Let $K \subset Y$ be a convex cone. The epigraphical and hypographical multifunctions associated to F are the set-valued mappings $\mathcal{E}_F, \mathcal{H}_F : X \rightrightarrows Y$ given by

$$\mathcal{E}_F(x) := F(x) + K \tag{2.4}$$

and

$$\mathcal{H}_F(x) := F(x) - K \tag{2.5}$$

respectively.

Of utmost importance in our work is the convexity of a set-valued mapping.

Definition 2.3.2. *([9, 122]) Let X and Y be vector spaces and $F : X \rightrightarrows Y$ be a given set-valued mapping.*

(i) We say that F is convex if $\operatorname{gph} F$ is a convex subset of $X \times Y$.

(ii) Let $\Omega \subseteq X$ be a convex set, $K \subset Y$ be a convex cone, and consider the preorder relation $\preceq_K^{(r)}$ from Definition 2.2.15, where $r \in \{l, u\}$. We say that F is $\preceq_K^{(r)}$- convex on Ω if $\Omega \subseteq \operatorname{dom} F$ and

$$\forall\, x_1, x_2 \in \Omega,\ t \in [0, 1] : F(tx_1 + (1 - t)x_2) \preceq_K^{(r)} tF(x_1) + (1 - t)F(x_2). \tag{2.6}$$

If we omit the set Ω, we assume that $\Omega = \operatorname{dom} F$ and that $\operatorname{dom} F$ is a convex set.

Remark 2.3.3. *It can be shown that F is $\preceq_K^{(l)}$- convex on Ω if and only if $\operatorname{epi} F \cap (\Omega \times Y)$ is a convex set, or equivalently, if the restriction of the epigraphical multifunction $\mathcal{E}_F$ to the set Ω is convex in the sense of Definition 2.3.2 (i). However, a similar result concerning the set relation $\preceq_K^{(u)}$ and the hypographical multifunction $\mathcal{H}_F$ does not hold.*

Remark 2.3.4. *It is easy to verify that the convexity concepts in Definition 2.3.2 (ii) are independent of the set relation in case F is single valued, that is, when $F = f : X \to Y$ is a vector-valued function well defined on Ω. In that case, condition (2.6) is equivalent to*

$$\forall\, x_1, x_2 \in \Omega,\ t \in (0,1) : f(tx_1 + (1-t)x_2) \preceq_K tf(x_1) + (1-t)f(x_2),$$

where $\preceq_K$ is given in Definition 2.2.9. We also say that the function f is K- convex on Ω.

In the next definition we consider different concepts of boundedness for a set-valued mapping.

Definition 2.3.5. *Let X and Y be normed spaces, $F : X \rightrightarrows Y$ be a given set-valued mapping and $K \subset Y$ be a closed, convex, pointed, and solid cone. In addition, let $A \subseteq Y$ be nonempty, and fix an element $e \in \operatorname{int} K$. Furthermore, consider the preorder relations $\preceq_K^{(l)}$ and $\preceq_K^{(u)}$ from Definition 2.2.15. We say that:*

(i) A is K- bounded below (above), if there exists $\alpha > 0$ such that

$$-\alpha e \preceq_K^{(l)} A \ \ (\textit{respectively, } A \preceq_K^{(u)} \alpha e).$$

(ii) F is locally K- bounded below (above) at $\bar{x} \in X$, if there exists a neighborhood U of $\bar{x}$ such that the set $F[U]$ is K- bounded below (above).

(iii) F is locally $\preceq_K^{(r)}$- bounded at $\bar{x} \in X$ for some $r \in \{l, u\}$, if there exists $\alpha > 0$ and a neighborhood U of $\bar{x}$ such that

$$\forall\, x \in U : \ F(x) \cap (-\alpha e + K) \cap (\alpha e - K) \in [F(x)]^{(r)}.$$

(iv) F is locally bounded at $\bar{x}$ if there exists $L > 0$ and a neighborhood U of $\bar{x}$ such that

$$F[U] \subseteq L\mathbb{B}.$$

Remark 2.3.6. *It can be shown that the boundedness concepts introduced in items $(i)-(iii)$ above are independent of the vector e.*

We conclude the subsection with other topological notions associated to set-valued mappings.

Definition 2.3.7. *Let X and Y be normed spaces and $F : X \rightrightarrows Y$ be a given set-valued mapping. In addition, fix a point $\bar{x} \in X$. We say that:*

(i) F is locally compact at $\bar{x}$ if there exists a neighborhood U of $\bar{x}$ and a compact set $C \subset Y$ such that

$$F[U] \subseteq C.$$

(ii) *F is locally Lipschitz-like at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ if there are neighborhoods U of $\bar{x}$ and V of $\bar{y}$ and a constant $\ell \geq 0$ such that*

$$\forall\, x, x' \in U : \qquad F(x) \cap V \subseteq F(x') + \ell\|x - x'\|\mathbb{B}.$$

(iii) *F is locally Lipschitz at $\bar{x}$ if there is a neighborhood U of $\bar{x}$ and a constant $\ell \geq 0$ such that*

$$\forall\, x, x' \in U : \qquad F(x) \subseteq F(x') + \ell\|x - x'\|\mathbb{B}.$$

(iv) *F is inner semicompact at $\bar{x} \in \operatorname{dom} F$ if, for every sequence $\{x_k\}_{k\geq 1} \subset X$ satisfying $x_k \to \bar{x}$, there is a sequence $\{y_k\}_{k\geq 1} \subset Y$ that contains a convergent subsequence and is such that $y_k \in F(x_k)$ for every $k \in \mathbb{N}$. In particular, $\bar{x} \in \operatorname{int} \operatorname{dom} F$.*

(v) *F is closed at $\bar{x}$ if, for any sequence $\{(x_k, y_k)\}_{k\geq 1} \subseteq \operatorname{gph} F$ with $(x_k, y_k) \to (\bar{x}, \bar{y})$, we have $(\bar{x}, \bar{y}) \in \operatorname{gph} F$.*

Remark 2.3.8. *Note that, if F is locally compact at $\bar{x}$, then it is locally bounded at the same point. The converse holds if Y is finite dimensional. Moreover, if F is locally Lipschitz at $\bar{x}$, it is in particular locally Lipschitz-like at every point $(\bar{x}, \bar{y}) \in \operatorname{gph} F$.*

Proposition 2.3.9. *([132, Theorem 1.42]) Let X and Y be Banach spaces and $F : X \rightrightarrows Y$ be a given set-valued mapping. Suppose that F is locally compact and closed at $\bar{x} \in X$ Then, F is locally Lipschitz at $\bar{x}$ if and only if it is locally Lipschitz-like at $(\bar{x}, \bar{y})$ for every $\bar{y} \in F(\bar{x})$.*

2.3.2 Subdifferential and Coderivatives

In this subsection, we introduce the tools from variational analysis that will be fundamental when deriving optimality conditions for set optimization problems in Chapter 4. Most of the material is taken from [132, 133], and we refer the reader to these texts for more details.

We start with some differentiability concepts.

Definition 2.3.10. *Let X and Y be normed spaces, $f : X \to Y$ be a given vector-valued function, and $\bar{x} \in X$.*

(i) *The directional derivative of f at $\bar{x}$ in the direction $d \in X$ is defined as the limit*

$$f'(\bar{x}, d) := \lim_{t \downarrow 0} \frac{f(\bar{x} + td) - f(\bar{x})}{t}, \tag{2.7}$$

if it exists. If this limit exists for every $d \in X$, we say that f is directionally differentiable at $\bar{x}$, and we call the map $f'(\bar{x}, \cdot) : X \to Y$ defined by (2.7) the directional derivative of f at $\bar{x}$.

(ii) *Suppose that $Y = \mathbb{R}$. We say that f is quasidifferentiable at $\bar{x}$ if it is directionally differentiable at $\bar{x}$ and there exists two nonempty, convex and w^*- compact sets $G, H \subset X^*$ such that*

$$f'(\bar{x}, \cdot) = \sigma_G(\cdot) - \sigma_H(\cdot).$$

In that case, the sets G and $-H$ are called the subdifferential and superdifferential of f at $\bar{x}$, respectively. Furthermore, the pair $(G, -H)$ is called the quasidifferential of f at $\bar{x}$.

(iii) *We say that f is Fréchet differentiable at $\bar{x}$ if there exists a bounded linear operator $\nabla f(\bar{x}) : X \to Y$ such that*

$$\lim_{x \to \bar{x}} \frac{\|f(x) - f(\bar{x}) - \nabla f(\bar{x})(x - \bar{x})\|}{\|x - \bar{x}\|} = 0.$$

In that case, we refer to the operator $\nabla f(\bar{x})$ as the Fréchet derivative of f at $\bar{x}$. If in addition the mapping $\nabla f(\cdot) : X \to \mathcal{L}(X, Y)$ is continuous, we say that the function f is continuously Fréchet differentiable at $\bar{x}$.

(iv) *We say that f is strictly Fréchet differentiable at $\bar{x}$ if there exists a bounded linear operator $\nabla f(\bar{x}) : X \to Y$ such that*

$$\lim_{\substack{u \to \bar{x} \\ x \to \bar{x}}} \frac{\|f(x) - f(u) - \nabla f(\bar{x})(x - u)\|}{\|x - u\|} = 0.$$

Remark 2.3.11. *Some important observations are the following:*

- *The class of quasidifferentiable functionals is very important since it includes the class of D.C.-functionals (difference of convex functionals) and the class of locally convex functionals. For a comprehensive review on quasidifferentiability, see also [39].*
- *It can be verified that, if f is continuously Fréchet differentiable at $\bar{x}$, then it is in particular strictly Fréchet differentiable at that point.*

The main class of functionals that will be used in the nonsmooth setting is that of locally Lipschitz. We recall this concept below.

Definition 2.3.12. *Let X and Y be normed spaces and $f : X \to Y$ be a given vector-valued function. In addition, fix a point $\bar{x} \in X$.*

(i) *We say that f is Lipschitz on a set $A \subseteq X$, provided that there exists $\ell \geq 0$ such that*

$$\forall\, x_1, x_2 \in A : \ \|f(x_1) - f(x_2)\| \leq \ell \|x_1 - x_2\|.$$

This is also referred to as a Lipschitz condition of rank ℓ. We say that f is locally Lipschitz at $\bar{x}$ if there is a neighborhood U of $\bar{x}$ such that f is Lipschitz on U. Moreover, f is said to be locally Lipschitz on A, if f is locally Lipschitz at every point $x \in A$.

(ii) *The function f is said to be strictly Lipschitz at $\bar{x}$ if f is locally Lipschitz at $\bar{x}$ and, for every $u \in X$ and sequences $\{x_k\}_{k \geq 1} \subset X$, $\{t_k\}_{k \geq 1} \subset \mathbb{R}$ with $x_k \to \bar{x}$ and $t_k \downarrow 0$, the sequence $\{y_k\}_{k \geq 1} \subset Y$ defined as*

$$\forall\, k \in \mathbb{N} : y_k := \frac{f(x_k + t_k u) - f(x_k)}{t_k}$$

contains a convergent subsequence.

Remark 2.3.13. *The notion of strict Lipschitzianity was introduced in [132, Definition 3.25]. Although this concept is less known in the literature, it is useful when considering relationships between coderivatives and subdifferentials of a vector valued functional, as we will see later in this subsection. It is easy to see that the Lipschitz and strictly Lipschitz property of a function are equivalent if Y is finite dimensional.*

We recall next a notion of limits of sets that is necessary for the forthcoming definitions.

Definition 2.3.14. *Let X be a normed space and $F : X \rightrightarrows X^*$ be a given set-valued mapping. The Painlevé-Kuratowski outer limit of F at $\bar{x}$ with respect to the norm topology of X and the w^*- topology of X^* is defined by*

$$\limsup_{x \to \bar{x}} F(x) := \{x^* \in X^* \mid \exists \ \{(x_k, x_k^*)\}_{k \geq 1} \subseteq \operatorname{gph} F : x_k \to \bar{x}, \ x_k^* \xrightarrow{w^*} x^*\}.$$

We consider next two fundamental concepts.

Definition 2.3.15. *Let X be a Banach space and consider a set $\Omega \subseteq X$.*

(i) Given $x \in X$ and $\epsilon \geq 0$, the set of ϵ- normals to Ω at x is defined by

$$\widehat{N}_\epsilon(x, \Omega) := \left\{ x^* \in X^* \;\middle|\; \limsup_{u \xrightarrow{\Omega} x} \frac{\langle x^*, u - x \rangle}{\|u - x\|} \leq \epsilon \right\}, \tag{2.8}$$

where $u \xrightarrow{\Omega} x$ means that $u \to x$ with $u \in \Omega$. When $\epsilon = 0$, the set defined by (2.8) is called the Fréchet normal cone to Ω at x, and is denoted by $\widehat{N}(x, \Omega)$. If $x \notin \Omega$, we put $\widehat{N}_\epsilon(x, \Omega) := \emptyset$ for all $\epsilon \geq 0$.

(ii) The limiting normal cone to Ω at $\bar{x} \in X$ is defined by

$$N(\bar{x}, \Omega) := \limsup_{\substack{x \to \bar{x} \\ \epsilon \downarrow 0}} \widehat{N}_\epsilon(x, \Omega). \tag{2.9}$$

We also put $N(\bar{x}, \Omega) := \emptyset$ for $\bar{x} \notin \Omega$.

Definition 2.3.16. *Let X be a Banach space and consider a functional $f : X \to \overline{\mathbb{R}}$, together with a point $\bar{x} \in X$ such that $|f(\bar{x})| < +\infty$. The limiting subdifferential of f at $\bar{x}$ is defined by*

$$\partial f(\bar{x}) := \big\{x^* \in X^* \mid (x^*, -1) \in N\big((\bar{x}, f(\bar{x})), \operatorname{epi} f\big)\big\}.$$

We put $\partial f(\bar{x}) := \emptyset$ if $|f(\bar{x})| = +\infty$.

Remark 2.3.17. *It is well known [132, Theorem 1.93] that, if f is convex and finite at $\bar{x}$, then*

$$\partial f(\bar{x}) = \{x^* \in X^* \mid \forall \, x \in X : f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x} \rangle \ \},$$

and hence $\partial f(\bar{x})$ coincides with Fenchel's subdifferential from convex analysis. In case Ω is a convex set, we also have [132, Proposition 1.5]:

$$N(\bar{x}, \Omega) = \{x^* \in X^* \mid \forall\, x \in \Omega : \langle x^*, x - \bar{x}\rangle \leq 0\},$$

and hence $N(\bar{x}, \Omega)$ equals the normal cone in the sense of convex analysis. For this reason, in this dissertation we make no distinction in the notation of the subdifferential or the normal cone when considering the convex case.

The following proposition summarizes some useful facts about convex functionals and their subdifferentials.

Proposition 2.3.18. *([145]) Let X be a Banach space, $f : X \to \mathbb{R}$ be a convex functional, and fix $\bar{x} \in X$. Suppose that f is bounded above in a neighborhood U of $\bar{x}$, that is, there is $L > 0$ such that*

$$\forall\, x \in U : f(x) < L.$$

Then, the following assertions hold:

(i) The functional f is locally Lipschitz and directionally differentiable at $\bar{x}$.

(ii) The set $\partial f(\bar{x})$ is a nonempty, convex, bounded, and w^*- closed subset of X^*. In particular, $\partial f(\bar{x})$ is w^*- compact.

(iii) The directional derivative $f'(\bar{x}, \cdot)$ is sublinear, continuous, and satisfies

(a) $f'(\bar{x}, \cdot) = \sigma_{\partial f(\bar{x})}(\cdot)$,

(b) $\partial f'(\bar{x}, \cdot)(0) = \partial f(\bar{x})$.

(iv) Suppose that $f = \sigma_G$, where $G \subset X^*$ is convex and w^*- compact. Then,

$$\partial f(\bar{x}) = G^{\bar{x}},$$

where $G^{\bar{x}}$ is the $\bar{x}$- face of G, see Definition 2.1.21.

Proposition 2.3.19. *([32, 132]) Let X and Y be Asplund spaces, and consider a point $\bar{x} \in X$. The following statements hold:*

(i) Assume that $f_1, f_2 : X \to \overline{\mathbb{R}}$ are given functionals, that f_1 is locally Lipschitz at $\bar{x}$, and that f_2 is lower semicontinuous at every point in a neighborhood of $\bar{x}$. Then,

$$\partial(f_1 + f_2)(\bar{x}) \subseteq \partial f_1(\bar{x}) + \partial f_2(\bar{x}).$$

Furthermore, the equality holds if both f_1 and f_2 are convex or strictly differentiable, even without the Asplund assumption.

(ii) Assume that $f : X \to Y$ is strictly Lipschitz at $\bar{x}$ and that $\psi : Y \to \mathbb{R}$ is locally Lipschitz at $f(\bar{x})$. Then,

$$\partial(\psi \circ f)(\bar{x}) \subseteq \bigcup_{y^* \in \partial\psi(f(\bar{x}))} \partial(y^* \circ f)(\bar{x}).$$

Moreover, the equality holds if ψ is convex and $f = T \in \mathcal{L}(X, Y)$ (see Definition 2.1.16), even without the Asplund assumption. In that case, we have

$$\partial(\psi \circ T)(\bar{x}) = T^* \left[\partial\psi(T(\bar{x}))\right],$$

where T^* is the adjoint operator of T, see Definition 2.1.19.

The following proposition relates the subdifferential of a functional and that of its opposite:

Proposition 2.3.20. *Let X be an Asplund space and suppose that the functional $f : X \to \overline{\mathbb{R}}$ is locally Lipschitz at $\bar{x}$. Then,*

$$\partial(-f)(\bar{x}) \subseteq -\operatorname{cl}^* \operatorname{conv}\left(\partial f(\bar{x})\right).$$

Proof. Let ∂° denote Clarke's subdifferential operator, see [33] for details. Then, we have

$$\begin{aligned}
\partial(-f)(\bar{x}) &\subseteq \operatorname{cl}^* \operatorname{conv}\left(\partial(-f)(\bar{x})\right) \\
&\overset{([132,\text{Theorem 3.57}])}{=} \partial^\circ(-f)(\bar{x}) \\
&\overset{([33,\text{Proposition 2.3.1}])}{=} -\partial^\circ f(\bar{x}) \\
&\overset{([132,\text{Theorem 3.57}])}{=} -\operatorname{cl}^* \operatorname{conv}\left(\partial f(\bar{x})\right).
\end{aligned}$$

□

We continue by defining the basic coderivative of a set-valued mapping at a point of its graph.

Definition 2.3.21. *Let X and Y be Banach spaces and $F : X \rightrightarrows Y$ be a set-valued mapping. The limiting coderivative of F at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ is the multifunction $D^*F(\bar{x}, \bar{y}) : Y^* \rightrightarrows X^*$ defined by*

$$D^*F(\bar{x}, \bar{y})(y^*) = \left\{x^* \in X^* \mid (x^*, -y^*) \in N\big((\bar{x}, \bar{y}), \operatorname{gph} F\big)\right\}. \tag{2.10}$$

We put $D^*F(\bar{x}, \bar{y})(y^*) := \emptyset$ for all $y^* \in Y^*$ if $(\bar{x}, \bar{y}) \notin \operatorname{gph} F$.

As a convention, we omit the value of $\bar{y}$ in the coderivative notation above if $F : X \rightrightarrows Y$ is given by $F(x) := \{f(x)\}$ and $f : X \to Y$ is a given vector-valued function. Hence, we write $D^*f(\bar{x})$ instead of $D^*f(\bar{x}, f(\bar{x}))$. The following proposition shows two useful properties of coderivatives of functionals.

Proposition 2.3.22. *([132, Theorem 1.38, Theorem 3.28]) Let X and Y be Banach spaces, $f : X \to Y$ be a given vector-valued function, and fix $\bar{x} \in X$. The following statements hold:*

(i) Suppose that f is strictly Fréchet differentiable at $\bar{x}$. Then,

$$\forall\, y^* \in Y : D^* f(\bar{x})(y^*) = \big\{\nabla f(\bar{x})^*(y^*)\big\}.$$

(ii) Assume that X satisfies the Asplund property and that f is strictly Lipschitz at $\bar{x}$. Then,

$$\forall\, y^* \in Y : D^* f(\bar{x})(y^*) = \partial(y^* \circ f)(\bar{x}).$$

Next, we recall two useful constraint qualifications from nonlinear programming.

Definition 2.3.23. *Let $f : \mathbb{R}^m \to \mathbb{R}^p$ be a given vector-valued function and consider the set*

$$C := \{y \in \mathbb{R}^p \mid f_i(y) \le 0, i = 1, \ldots, p\}. \tag{2.11}$$

(i) We say that C satisfies the Mangasarian-Fromovitz constraint qualification (MFCQ for short) at $\bar{y} \in C$ with respect to the representation (2.11) *if each f_i is continuously differentiable and the vectors $\{\nabla f_i(\bar{y})\}_{i \in I(\bar{y})}$ are positively linearly independent, that is, if*

$$\nabla f(\bar{y})\mu = 0,\ \mu^\top f(\bar{y}) = 0, \mu \in \mathbb{R}^p_+ \Longrightarrow \mu = 0.$$

Here, $I(\bar{y}) := \{i \in \{1, \ldots, p\} \mid f_i(\bar{y}) = 0\}$ is the set of active indexes at $\bar{y}$.

(ii) We say that C satisfies Slater's constraint qualification with respect to the representation (2.11) *if there exists $\bar{y} \in C$ such that $\max\limits_{i=1,\ldots,p} f_i(\bar{y}) < 0$.*

Remark 2.3.24. *It is well known [20] that, if C satisfies Slater's constraint qualification and each f_i is convex and continuously differentiable, then C satisfies MFCQ at every $\bar{y} \in C$.*

We close the subsection with two propositions that show how to compute normal cones to particular types of sets.

Proposition 2.3.25. *([132, Corollary 4.35]) Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be defined as*

$$F(x) := \{y \in \mathbb{R}^m \mid f_i(x, y) \le 0,\ i = 1, \ldots, p\},$$

where each $f_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is continuously differentiable. In addition, let $\bar{y} \in F(\bar{x})$ and consider the set of active indexes

$$I(\bar{x}, \bar{y}) := \{i \in \{1, \ldots, p\} \mid f_i(\bar{x}, \bar{y}) = 0\}. \tag{2.12}$$

Furthermore, suppose that $\operatorname{gph} F$ *satisfies MFCQ at $(\bar{x}, \bar{y})$. Then,*

$$N((\bar{x},\bar{y}), \operatorname{gph} F) = \operatorname{cone}\operatorname{conv}\{\nabla f_i(\bar{x},\bar{y})\}_{i\in I(\bar{x},\bar{y})}$$

and the basic coderivative of F at $(\bar{x},\bar{y})$ is given by

$$D^*F(\bar{x},\bar{y})(v) = \{u \in \mathbb{R}^n \mid (u,-v) \in \operatorname{cone}\operatorname{conv}\{\nabla f_i(\bar{x},\bar{y})\}_{i\in I(\bar{x},\bar{y})}\}.$$

Proposition 2.3.26. *([8, Proposition 9.6.1]) Let X be a normed space and $f : X \to \mathbb{R}$ be convex and continuous. Furthermore, suppose that the set*

$$C := \{x \in X \mid f(x) \leq 0\}$$

satisfies Slater's condition with respect to that representation and that $\bar{x} \in C$ is such that $f(\bar{x}) = 0$. Then,

$$N(\bar{x}, C) = \operatorname{cone} \partial f(\bar{x}).$$

2.3.3 Marginal Functions

In the last part of this section, we present some properties of marginal functions and their subdifferentials. This problem is closely related to the computation of the subdifferentials of the functionals we will see in Chapter 4.

Suppose that X and Y are Banach spaces. Recall that, for a functional $f : X \times Y \to \overline{\mathbb{R}}$ and a set-valued mapping $F : X \rightrightarrows Y$, the associated marginal function $\varphi : X \to \overline{\mathbb{R}}$ is defined as

$$\varphi(x) := \inf_{y\in F(x)} f(x,y). \tag{2.13}$$

In this setting, we can also consider the so called solution map $S : X \rightrightarrows Y$ given by

$$S(x) := \{y \in F(x) \mid f(x,y) = \varphi(x)\}. \tag{2.14}$$

The following proposition establishes a sufficient condition in order to have the Lipschitz property of marginal functions. Although this is a basic result, we couldn't find a proof in the literature, and hence we provide it.

Proposition 2.3.27. *Let X and Y be Banach spaces, $F : X \rightrightarrows Y$ be a set-valued mapping, and $f : X \times Y \to \overline{\mathbb{R}}$ be a given functional. Moreover, consider the marginal function φ given by* (2.13)*. Suppose that F is Lipschitz on a set $U \subseteq X$ with constant $\ell > 0$, and that f is Lipschitz on the set $(U \times Y) \cap \operatorname{gph} F$ with constant $\ell' > 0$. Furthermore, assume that φ is finite at some point $\bar{x} \in U$. Then, φ is Lipschitz on U with constant $\ell'(1 + \ell)$.*

Proof. Take $x_1, x_2 \in U$ and let $\ell, \ell' > 0$ be the Lipschitz constants of F and f respectively. Then, because F is Lipschitz on U, we have:

$$\forall\, y_2 \in F(x_2), \exists\, y_1 \in F(x_1) : \|y_1 - y_2\| \leq \ell \|x_1 - x_2\|.$$

Taking this into account, together with the Lipschitz continuity of f on $(U \times Y) \cap \operatorname{gph} F$, we get

$$\begin{aligned} \forall\, y_2 \in F(x_2), \exists\, y_1 \in F(x_1) : f(x_1, y_1) &\leq f(x_2, y_2) + \ell'(\|x_1 - x_2\| + \|y_1 - y_2\|) \\ &\leq f(x_2, y_2) + \ell'(1+\ell)\|x_1 - x_2\|. \end{aligned}$$

This implies

$$\varphi(x_1) \leq \varphi(x_2) + \tilde{\ell}\|x_1 - x_2\|, \tag{2.15}$$

with $\tilde{\ell} := \ell'(1+\ell)$. Since $\varphi(\bar{x}) > -\infty$, we can substitute $x_1 = \bar{x}$ in (2.15) to obtain that $\varphi(x_2) > -\infty$ for every $x_2 \in U$. From this, it follows that φ is Lipschitz on U.

□

We now present two results concerning the subdifferential of φ. The first one treats the case in which F and f are assumed to be convex.

Theorem 2.3.28. *([6, Theorem 4.2]) Let X and Y be Banach spaces, $F : X \rightrightarrows Y$ be a set-valued mapping, and $f : X \times Y \to \overline{\mathbb{R}}$ be a given functional. Consider the marginal function φ given by* (2.13)*, together with the solution map S defined by* (2.14)*. Furthermore, suppose that F is convex, that f is a proper and convex function, and that at least one of the following regularity conditions is satisfied:*

(i) $\operatorname{int} \operatorname{gph} F \cap \operatorname{dom} f \neq \emptyset$*,*

(ii) f is continuous at a point $(\tilde{x}, \tilde{y}) \in \operatorname{gph} F$.

Then, φ is convex and, for any $\bar{x} \in \operatorname{dom} \varphi$ with $\varphi(\bar{x}) \neq -\infty$ and any $\bar{y} \in S(\bar{x})$, we have

$$\partial\varphi(\bar{x}) = \bigcup_{(x^*, y^*) \in \partial f(\bar{x}, \bar{y})} \Big[x^* + D^*F(\bar{x}, \bar{y})(y^*) \Big].$$

For the case in which F is not supposed to be convex, many results already exist in the literature. We conclude by establishing a weaker version of [132, Theorem 3.38 (ii)], which will be enough for our purposes.

Theorem 2.3.29. *([132, Theorem 3.38]) Let X and Y be Asplund spaces, $F : X \rightrightarrows Y$ be a set-valued mapping, and $f : X \times Y \to \overline{\mathbb{R}}$ be a given functional. Consider now the marginal function φ given by* (2.13)*, the solution map S defined by* (2.14)*, and a point $\bar{x} \in X$. Furthermore, assume that:*

(i) F is closed at $\bar{x}$,

(ii) S is inner semicompact at $\bar{x}$,

(iii) there exists a neighborhood U of $\bar{x}$ such that f is Lipschitz on $U \times Y$,

(iv) $\operatorname{gph} F$ *is locally closed around every point of the set* $\{\bar{x}\} \times S(\bar{x})$.

Then,

$$\partial\varphi(\bar{x}) \subseteq \bigcup_{\substack{\bar{y}\in S(\bar{x}) \\ (x^*,y^*)\in\partial f(\bar{x},\bar{y})}} \Big[x^* + D^*F(\bar{x},\bar{y})(y^*)\Big]. \tag{2.16}$$

2.4 Vector and Set Optimization

In this section, we provide an overview of set optimization problems, which are the main topic of the thesis. We will formally introduce the different solution concepts, and recall some important results and particular cases. We start by defining the standard optimization problem, and later we move to the more general setting.

Definition 2.4.1. *Let* X *be a normed space,* $f : X \to \overline{\mathbb{R}}$ *be a given functional, and* $\Omega \subseteq X$*. The optimization problem associated to this data is defined as*

$$\min_{x\in\Omega} f(x), \tag{$\mathcal{OP}(f,\Omega)$}$$

where the solution concepts are given as follows:

(i) We say that a point $\bar{x} \in \Omega$ *is a minimum of* $\mathcal{OP}(f,\Omega)$ *if*

$$\forall\, x \in \Omega : f(\bar{x}) \le f(x).$$

(ii) We call a point $\bar{x} \in \Omega$ *a strict minimum of* $\mathcal{OP}(f,\Omega)$ *if*

$$\forall\, x \in \Omega : f(\bar{x}) < f(x).$$

If in the above definition we replace Ω *by* $\Omega \cap U$*, with* U *being a neighborhood of* $\bar{x}$*, we say that* $\bar{x}$ *is a local minimum and a local strict minimum respectively.*

We recall necessary conditions for $\mathcal{OP}(f,\Omega)$ in the following theorem. Since in our thesis we only treat problems with Lipschitzian data, we restrict ourselves to consider only optimality conditions for problems were the objective functional is locally Lipschitz at the point of interest. However, it is worth mentioning that this condition can be weakened, at the expense of additional assumptions [133].

Theorem 2.4.2. *([133, Proposition 5.3]) Let* X *be an Asplund space,* $f : X \to \overline{\mathbb{R}}$ *be a given functional, and* $\Omega \subseteq X$ *be closed. Suppose that* $\bar{x} \in \Omega$ *is a local minimum of* $\mathcal{OP}(f,\Omega)$ *and that* f *is locally Lipschitz* $\bar{x}$*. Then,*

$$0 \in \partial f(\bar{x}) + N(\bar{x},\Omega). \tag{2.17}$$

Moreover, even without the Asplund assumption, condition (2.17) *is both necessary and sufficient for optimality if* f *is convex and continuous at* $\bar{x}$*, and* Ω *is a convex set.*

Definition 2.4.3. *Let X be a Banach space, $f : X \to \overline{\mathbb{R}}$ be a given functional, and $\Omega \subseteq X$. We say that the point $\bar{x} \in \Omega$ is stationary for $\mathcal{OP}(f,\Omega)$ if* (2.17) *holds.*

Remark 2.4.4. *In the literature, statements like the one in Theorem 2.4.2 have been considered with respect to other type of subdifferentials. In particular, for the case $\Omega = X$, it is well known that the inclusion*

$$0 \in \widehat{\partial} f(\bar{x}) \tag{2.18}$$

is also necessary for optimality in $\mathcal{OP}(f,X)$. Here, $\widehat{\partial} f(\bar{x})$ is the so called Fréchet subdifferential of f at $\bar{x}$, see [131, 132] and the references therein for details. In fact, we always have $\widehat{\partial} f(x) \subseteq \partial f(x)$ for every $x \in X$, so that (2.18) *is a stronger necessary condition for $\mathcal{OP}(f,X)$ than* (2.17). *However, the Fréchet subdifferential has other limitations, like poor calculus rules, that restricts its applications in optimization.*

Next, we consider vector optimization problems. In order to do this, one must define first the concept of minimal elements of a set in a partially ordered space.

Definition 2.4.5. *Let Y be a normed space and $K \subset Y$ be a proper, closed, convex, and pointed cone. Furthermore, consider a set $A \subseteq Y$.*

(i) Suppose that $\operatorname{int} K \neq \emptyset$. *The set of weakly minimal elements of A with respect to K is defined as*

$$\mathrm{WMin}(A,K) := \{y \in A \mid (y - \operatorname{int} K) \cap A = \emptyset\}.$$

(ii) The set of minimal elements of A with respect to K is given by

$$\mathrm{Min}(A,K) := \{y \in A \mid (y - K) \cap A = \{y\}\}.$$

(iii) Suppose that $\operatorname{int} K \neq \emptyset$. *The set of weakly maximal elements of A with respect to K is defined as*

$$\mathrm{WMax}(A,K) := \mathrm{WMin}(A,-K).$$

(iv) The set of maximal elements of A with respect to K is given by

$$\mathrm{Max}(A,K) := \mathrm{Min}(A,-K).$$

(v) The set of strongly minimal elements of A with respect to K is defined as

$$\mathrm{SMin}(A,K) := \{y \in A \mid A \subseteq y + K\}.$$

(vi) The set of strongly maximal elements of A with respect to K is given by

$$\mathrm{SMax}(A, K) := \mathrm{SMin}(A, -K).$$

Remark 2.4.6. *If we consider the binary relations $\preceq_K$ and $\prec_K$ from Definition 2.2.9, it is possible to check that $\mathrm{WMin}(A, K) = \mathrm{Min}(A, \prec_K)$ and that $\mathrm{Min}(A, K) = \mathrm{Min}(A, \preceq_K)$, where the sets $\mathrm{Min}(A, \prec_K)$ and $\mathrm{Min}(A, \preceq_K)$ are understood in the sense of Definition 2.2.5. A similar statement can be made about the sets $\mathrm{WMax}(A, K)$ and $\mathrm{Max}(A, K)$.*

The following proposition shows the relationships between the different solution concepts and provides an existence result.

Proposition 2.4.7. *([89]) Let Y be a normed space and $K \subset Y$ be a closed, convex, pointed, and solid cone. Furthermore, consider a set $A \subseteq Y$. The following statements hold:*

(i) If A is compact, then $\mathrm{Min}(A, K) \neq \emptyset$ and $A + K = \mathrm{Min}(A, K) + K$.

(ii) The following inclusions are true:

$$\mathrm{SMin}(A, K) \subseteq \mathrm{Min}(A, K) \subseteq \mathrm{WMin}(A, K).$$

Similarly,

$$\mathrm{SMax}(A, K) \subseteq \mathrm{Max}(A, K) \subseteq \mathrm{WMax}(A, K).$$

Definition 2.4.8. *Let X and Y be normed spaces and $K \subset Y$ be a closed, convex, and pointed cone. Let $f : X \to Y$ be a given vector-valued function, and consider $\Omega \subseteq X$. The vector optimization problem associated to this data is defined as*

$$K\text{-} \min_{x \in \Omega} f(x), \tag{$\mathcal{VOP}(f, K, \Omega)$}$$

where the solution concepts are given as follows:

(i) Suppose that $\operatorname{int} K \neq \emptyset$. We say that a point $\bar{x} \in \Omega$ is a weakly minimal solution of $\mathcal{VOP}(f, K, \Omega)$ if $f(\bar{x}) \in \mathrm{WMin}(f[\Omega], K)$, or equivalently, if

$$\nexists\, x \in \Omega : f(x) \prec_K f(\bar{x}).$$

(ii) We say that a point $\bar{x} \in \Omega$ is a minimal solution of $\mathcal{VOP}(f, K, \Omega)$ if $f(\bar{x}) \in \mathrm{Min}(f[\Omega], K)$, or equivalently, if

$$\nexists\, x \in \Omega : f(x) \preceq_K f(\bar{x}) \text{ and } f(x) \neq f(\bar{x}).$$

If in the above definitions we replace Ω by $\Omega \cap U$, with U being a neighborhood of $\bar{x}$, we call $\bar{x}$ a local weakly minimal and a local minimal solution respectively.

Remark 2.4.9. *It is easy to verify that every minimal of solution of $\mathcal{VOP}(f, K, \Omega)$ is also weakly minimal. Hence, in particular, optimality conditions for weakly minimal solutions are also necessary for minimal ones.*

On the other hand, note that when $Y = \mathbb{R}$, the problem $\mathcal{VOP}(f, \mathbb{R}_+, \Omega)$ is equivalent to $\mathcal{OP}(f, \Omega)$, in the sense that both solution concepts collapse into the concept of minimum of $\mathcal{OP}(f, \Omega)$.

Necessary and sufficient optimality conditions for weakly minimal solutions of $\mathcal{VOP}(f, K, \Omega)$ are considered in the following theorem. As for problem $\mathcal{OP}(f, \Omega)$, we analyze only the case on which the vector-valued function is locally Lipschitz. The statement will be an immediate consequence of other results in the literature, but the formulation below is more suitable for our purposes.

Theorem 2.4.10. *([151, 152]) Let X and Y be Banach spaces, $\Omega \subseteq X$ be nonempty and closed, and $K \subset Y$ be a closed, convex, pointed, and solid cone. Let $f : X \to Y$ be a given vector-valued function, and consider a point $\bar{x} \in \Omega$. The following statements hold:*

(i) Suppose that X and Y satisfy the Asplund property, and that f is locally Lipschitz at $\bar{x}$. If $\bar{x}$ is a local weakly minimal solution of $\mathcal{VOP}(f, K, \Omega)$, then

$$\exists\, k^* \in K^* \setminus \{0\} : 0 \in D^* f(\bar{x})(k^*) + N(\bar{x}, \Omega). \tag{2.19}$$

Furthermore, if f is strictly Fréchet differentiable at $\bar{x}$ or f is strictly Lipschitz at $\bar{x}$, then (2.19) implies

$$\exists\, k^* \in K^* \setminus \{0\} : 0 \in \nabla f(\bar{x})^*(k^*) + N(\bar{x}, \Omega) \tag{2.20}$$

and

$$\exists\, k^* \in K^* \setminus \{0\} : 0 \in \partial(k^* \circ f)(\bar{x}) + N(\bar{x}, \Omega) \tag{2.21}$$

respectively.

(ii) Suppose that Ω is a convex set and that f is K- convex on Ω and continuous at $\bar{x}$. Then, condition (2.21) is both necessary and sufficient for the weak minimality of $\bar{x}$ in $\mathcal{VOP}(f, K, \Omega)$.

Proof. (i) From [152, Theorem 5], we have

$$0 \in D^* \mathcal{E}_f(\bar{x}, f(\bar{x}))[K^* \setminus \{0\}] + N(\bar{x}, \Omega),$$

where $\mathcal{E}_f$ is the epigraphical multifunction of f from Definition 2.3.1 (iv). Statement (2.19) follows then from [16, Proposition 4.3], where it is established the fact that $D^* \mathcal{E}_f(\bar{x}, f(\bar{x}))(y^*) \subseteq$

$D^*f(\bar{x})(y^*)$ for every $y^* \in Y^*$. The implications (2.20) and (2.21) are deduced from Proposition 2.3.22.

(*ii*) The necessity is a direct consequence of [151, Theorem 3]. In order to show the sufficiency, we argue by contradiction. Suppose that $\bar{x}$ is not a weakly minimal solution of $\mathcal{VOP}(f, K, \Omega)$. Then, we could find $x \in \Omega$ such that

$$f(x) \prec_K f(\bar{x}). \tag{2.22}$$

According to (2.21), we can find k^* in $K^* \setminus \{0\}$ and $x^* \in X^*$ such that

$$x^* \in \partial(k^* \circ f)(\bar{x}) \cap -N(\bar{x}, \Omega). \tag{2.23}$$

Furthermore, it is straightforward to verify that, since $k^* \in K^*$ and f is K- convex, the functional $k^* \circ f$ is convex and continuous at $\bar{x}$. Hence, we obtain

$$\begin{aligned} \langle x^*, x - \bar{x} \rangle &\overset{((2.23)\,+\,\text{Remark } 2.3.17)}{\leq} \langle k^*, f(x) - f(\bar{x}) \rangle \\ &\overset{((2.22)\,+\,(k^* \in K^*))}{<} 0. \end{aligned}$$

However, according to Remark 2.3.17, this would contradicts the fact that $x^* \in -N(\bar{x}, \Omega)$ in (2.23). $\square$

Remark 2.4.11. *According to Remark 2.4.9, when $Y = \mathbb{R}$, the problem $\mathcal{VOP}(f, \mathbb{R}_+, \Omega)$ reduces to the standard optimization problem $\mathcal{OP}(f, \Omega)$. In that case, note that the condition* (2.21) *is now equivalent to* (2.17) *in Theorem 2.4.2.*

Definition 2.4.12. *Let X and Y be Banach spaces, and $K \subset Y$ be a proper, closed, convex, pointed, and solid cone. Let $f : X \to Y$ be a given vector-valued function, and $\Omega \subseteq X$. We say that the point $\bar{x} \in \Omega$ is stationary for $\mathcal{VOP}(f, K, \Omega)$ if* (2.19) *holds.*

A direct consequence of Theorem 2.4.10 (*ii*) is the following corollary:

Corollary 2.4.13. *Consider problem $\mathcal{VOP}(f, \mathbb{R}^m_+, \mathbb{R}^n)$ for a given vector-valued function $f : \mathbb{R}^n \to \mathbb{R}^m$. Suppose that each component functional f_i is convex and strictly differentiable at $\bar{x} \in \mathbb{R}^n$. Then, f is $\mathbb{R}^m_+$- convex, and $\bar{x}$ is a weakly minimal solution of $\mathcal{VOP}(f, \mathbb{R}^m_+, \mathbb{R}^n)$ if and only if there exists $\mu \in \mathbb{R}^m_+ \setminus \{0\}$ such that $\nabla f(\bar{x})\mu = 0$.*

We can now define formally set optimization problems.

Definition 2.4.14. *Let X and Y be normed spaces and $K \subset Y$ be a closed, convex, pointed, and solid cone. Let $\Omega \subseteq X$ be nonempty and consider a set-valued mapping $F : X \rightrightarrows Y$ such that $\Omega \subseteq \operatorname{int} \operatorname{dom} F$. The set optimization problem is defined as*

$$K\text{-} \min_{x \in \Omega} F(x), \tag{$\mathcal{SOP}(F, K, \Omega)$}$$

and its minimal solutions are defined according to the following two approaches:

(i) *(Vector Approach) We say that a point $\bar{x} \in \Omega$ is a weak minimizer of $\mathcal{SOP}(F, K, \Omega)$ if*

$$F(\bar{x}) \cap \mathrm{WMin}(F[\Omega], K) \neq \emptyset.$$

(ii) *(Set Approach) Consider, for $r \in \{l, u, s\}$, the set relations $\preceq_K^{(r)}$ from Definition 2.2.15. We say that a point $\bar{x} \in \Omega$ is a*

(a) *$\preceq_K^{(r)}$- weakly minimal solution of $\mathcal{SOP}(F, K, \Omega)$, if*

$$\nexists\, x \in \Omega \setminus \{\bar{x}\} : F(x) \prec_K^{(r)} F(\bar{x}),$$

(b) *$\preceq_K^{(r)}$- minimal solution of $\mathcal{SOP}(F, K, \Omega)$, if for every $x \in \Omega$ the following implication holds:*

$$F(x) \preceq_K^{(r)} F(\bar{x}) \Longrightarrow F(\bar{x}) \preceq_K^{(r)} F(x),$$

(c) *$\preceq_K^{(r)}$- strictly minimal solution of $\mathcal{SOP}(F, K, \Omega)$, if*

$$\nexists\, x \in \Omega \setminus \{\bar{x}\} : F(x) \preceq_K^{(r)} F(\bar{x}).$$

If in the above definition we replace Ω by $\Omega \cap U$, with U being a neighborhood of $\bar{x}$, we say that $\bar{x}$ is a local weak minimizer in (i) and a local ($\preceq_K^{(r)}$- weakly, $\preceq_K^{(r)}$- , $\preceq_K^{(r)}$- strictly) minimal solution respectively in (ii).

Remark 2.4.15. *It is easy to verify that the concepts of weak minimizers and $\preceq_K^{(r)}$- weak minimality ($r \in \{l, u, s\}$) in Definition 2.4.14 are all equivalent in case F is a vector-valued function $f : X \to Y$. In that case, they all coincide with the concept of weakly minimal solutions of $\mathcal{VOP}(f, K, \Omega)$ in Definition 2.4.8 (i).*

Remark 2.4.16. *In Definition 2.4.14 (i), one could also consider the concept of minimizer and strong minimizer of $\mathcal{SOP}(F, K, \Omega)$ by replacing* WMin *by* Min *and* SMin *respectively. On the other hand, additional set relations have been considered in the literature by Jahn and Ha [95] and Karaman et.al [100]. For each of these set relations, corresponding minimal solutions could also be defined. In the above definition we just established those that will be used in the dissertation.*

Remark 2.4.17. *Let $\Omega \subseteq \mathbb{R}^n$ and $\mathcal{U} \subseteq \mathbb{R}^q$ be nonempty sets, $K \subseteq \mathbb{R}^m$ be proper, closed, convex, pointed, and solid cone, and $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^m$ be a given function. Furthermore, consider the set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ given by*

$$F(x) = \{f(x, u) \mid u \in \mathcal{U}\}. \tag{2.24}$$

Then, the set optimization problem $\mathcal{SOP}(F, K, \Omega)$ associated to this data arises when modelling uncertainty in a vector optimization problem. In that case, the cost function f of the vector *problem depends not only on the decision variable x, but also on an uncertain parameter u which belongs to the so called uncertainty set $\mathcal{U}$. In this setting, the $\preceq_K^{(l)}$- weakly minimal and $\preceq_K^{(u)}$- weakly minimal solutions of $\mathcal{SOP}(F,K,\Omega)$ model optimistic and pessimistic solutions of the so called robust counterpart problem to the vector optimization problem under uncertainty, respectively. We refer the reader to [86] and the references therein for an overview of the topic.*

The solution concept with the vector approach in Definition 2.4.14 (i) was the first one considered in the literature and hence it is already well studied, see [89, 106, 130] and the references therein for an extensive treatment on the topic. It is worth pointing out that, although in this dissertation we deal only with set approach solutions, the vector approach is still useful for us as important connections between these concepts can be made. Some of these relationships are collected in the next proposition.

Proposition 2.4.18. *Let X and Y be normed spaces and $K \subset Y$ be a closed, convex, pointed, and solid cone. In addition, let $\Omega \subseteq X$ be nonempty and closed, and fix $\bar{x} \in \Omega$. Furthermore, let $F : X \rightrightarrows Y$ be a set-valued mapping such that $\Omega \subseteq \operatorname{int} \operatorname{dom} F$, and consider the set optimization problem $\mathcal{SOP}(F,K,\Omega)$. The following assertions hold:*

(i) Suppose that $\bar{x}$ is a $\preceq_K^{(r)}$- strictly minimal solution of $\mathcal{SOP}(F,K,\Omega)$ for some $r \in \{l,u,s\}$. Then, $\bar{x}$ is both a $\preceq_K^{(r)}$- minimal and a $\preceq_K^{(r)}$- weakly minimal solution of $\mathcal{SOP}(F,K,\Omega)$.

(ii) Assume that $\bar{x}$ is a $\preceq_K^{(l)}$- minimal solution of $\mathcal{SOP}(F,K,\Omega)$ and that $\operatorname{WMin}(F(\bar{x}),K) \neq \emptyset$. Then, $\bar{x}$ is a $\preceq_K^{(l)}$- weakly minimal solution of $\mathcal{SOP}(F,K,\Omega)$. Similarly, if $\bar{x}$ is a $\preceq_K^{(u)}$- minimal solution of $\mathcal{SOP}(F,K,\Omega)$ and $\operatorname{WMax}(F(\bar{x}),K) \neq \emptyset$, then $\bar{x}$ is a $\preceq_K^{(u)}$- weakly minimal solution of $\mathcal{SOP}(F,K,\Omega)$.

(iii) Suppose that $\operatorname{WMin}(F(\bar{x}),K) \neq \emptyset$. Then, $\bar{x}$ is a $\preceq_K^{(l)}$- weakly minimal solution of $\mathcal{SOP}(F,K,\Omega)$ if and only if $F(\bar{x}) \in \operatorname{Min}\left(\mathcal{F}, \prec_K^{(l)}\right)$, where $\mathcal{F} := \{F(x) \mid x \in \Omega\}$ and the term $\operatorname{Min}\left(\mathcal{F}, \prec_K^{(l)}\right)$ is understood in the sense of Definition 2.2.5.

(iv) Assume that $\operatorname{WMin}(F(\bar{x}),K) \neq \emptyset$ and that $\bar{x}$ is a weak minimizer of $\mathcal{SOP}(F,K,\Omega)$. Then, $\bar{x}$ is also a $\preceq_K^{(l)}$- weakly minimal solution. Conversely, suppose that $\bar{x}$ is a $\preceq_K^{(l)}$- weakly minimal solution of $\mathcal{SOP}(F,K,\Omega)$ and that $\operatorname{SMin}(F(\bar{x}),K) \neq \emptyset$. Then, $\bar{x}$ is a weak minimizer of $\mathcal{SOP}(F,K,\Omega)$.

Proof. (i) This follows directly from the definition and the reflexivity of the set relation $\preceq_K^{(r)}$.

(ii) We only prove the statement for the lower set less relation as the other one is similar. Assume that $\bar{x}$ is not a $\preceq_K^{(l)}$- weakly minimal solution of $\mathcal{SOP}(F,K,\Omega)$. Then, we could find $x \in \Omega$ such that $F(x) \prec_K^{(l)} F(\bar{x})$. In particular, we deduce that $F(x) \preceq_K^{(l)} F(\bar{x})$. Since $\bar{x}$ is a $\preceq_K^{(l)}$- minimal solution, we obtain $F(\bar{x}) \preceq_K^{(l)} F(x)$. From this, it follows that

$$F(\bar{x}) \preceq_K^{(l)} F(x) \prec_K^{(l)} F(\bar{x}),$$

a contradiction to the fact that $\operatorname{WMin}(F(\bar{x}),K) \neq \emptyset$.

(iii) Suppose first that $\bar{x}$ is a $\preceq_K^{(l)}$- weakly minimal solution of $\mathcal{SOP}(F, K, \Omega)$, and fix $x \in \Omega$. Then, it suffices to show that $F(x) \not\prec_K^{(l)} F(\bar{x})$. Indeed, otherwise it would follow from the $\preceq_K^{(l)}$-weak minimality of $\bar{x}$ that $x = \bar{x}$. Then, we would obtain $F(\bar{x}) \prec_K^{(l)} F(\bar{x})$, a contradiction to the fact that $\text{WMin}(F(\bar{x}), K) \neq \emptyset$.

On the other hand, suppose now that $F(\bar{x}) \in \text{Min}\left(\mathcal{F}, \prec_K^{(l)}\right)$ and that $\bar{x}$ is not a $\preceq_K^{(l)}$- weakly minimal solution of $\mathcal{SOP}(F, K, \Omega)$. Then, we could find $x \in \Omega$ such that $F(x) \prec_K^{(l)} F(\bar{x})$. From the minimality of $F(\bar{x})$ on the family $\mathcal{F}$, we deduce that $F(\bar{x}) \prec_K^{(l)} F(x)$. Hence, similarly to the proof of (ii), we have $F(\bar{x}) \prec_K^{(l)} F(\bar{x})$, a contradiction to $\text{WMin}(F(\bar{x}), K) \neq \emptyset$.

(iv) The first part of the statement follows from (iii) and [80, Proposition 2.10]. The converse is just [109, Proposition 3.9]. $\square$

Remark 2.4.19. *A similar statement to Proposition 2.4.18* (iv) *can be stated for the upper set less relation, see [80, Remark 2.11].*

Optimality conditions for weak minimizers of set optimization problems are established below, see [14, 15, 16, 19, 43, 68] for similar results.

Theorem 2.4.20. *([17, Theorem 5.3]) Let X and Y be Asplund spaces and $K \subset Y$ be a closed, convex, pointed, and solid cone. Let $F : X \rightrightarrows Y$ be a given set valued mapping and $\Omega \subseteq X$ be nonempty and closed. Suppose that $\bar{x} \in \Omega$ is a weak minimizer of $\mathcal{SOP}(F, K, \Omega)$ and let $\bar{y} \in F(\bar{x}) \cap \text{WMin}(F(\Omega \cap U), K)$, where U is the neighborhood of $\bar{x}$ on which $\bar{x}$ is optimal. Furthermore, assume that $\text{gph}\, F$ is locally closed around $(\bar{x}, \bar{y})$, and that F is locally Lipschitz-like at $(\bar{x}, \bar{y})$. Then,*

$$\exists\, k^* \in K^* \setminus \{0\} : 0 \in D^*F(\bar{x}, \bar{y})(k^*) + N(\bar{x}, \Omega). \tag{2.25}$$

Remark 2.4.21. *It is easy to see that, when F is a vector-valued function $f : X \to Y$, the statement of Theorem 2.4.20 is equivalent to that of* (2.19) *in Theorem 2.4.10. This makes sense since, according to Remark 2.4.15, $\bar{x}$ would be a weakly minimal solution of $\mathcal{VOP}(f, K, \Omega)$ in that case.*

Definition 2.4.22. *Let X and Y be Banach spaces, and $K \subset Y$ be a closed, convex, pointed, and solid cone. Let $F : X \rightrightarrows Y$ be a given set-valued mapping, and $\Omega \subseteq X$ be nonempty. We say that a point $\bar{x} \in \Omega$ is stationary for $\mathcal{SOP}(F, K, \Omega)$ if there exists $\bar{y} \in F(\bar{x})$ such that* (2.25) *holds.*

We close this section by mentioning that, in Chapter 4, we will derive optimality conditions for set optimization problems with respect to the set approach, along the lines of Theorem 2.4.20.

2.5 Scalarizing Functionals

As mentioned in the introduction, scalarization techniques are a fundamental tool in vector and set optimization, and play a mayor role in this dissertation. In this section, we recall three types of scalarizing functionals that will be used throughout the text, together with some of their properties. We start by defining some fundamental conditions that a functional must fulfill in order to be a good candidate for scalarization, see [106, 129, 153, 154].

Definition 2.5.1. *Let Y be a normed space and $\psi : Y \to \mathbb{R}$ be a given functional. Furthermore, let $K \subset Y$ be a closed, convex, pointed, and solid cone, and consider the binary relation $\preceq_K$ from Definition 2.2.9. We say that ψ is*

(i) K- monotone, if $y \preceq_K z \Longrightarrow \psi(y) \le \psi(z)$,

(ii) strictly K- monotone, if $y \prec_K z \Longrightarrow \psi(y) < \psi(z)$,

(iii) strongly K- monotone, if $y \preceq_K z,\ y \neq z \Longrightarrow \psi(y) < \psi(z)$.

We simply call ψ (strictly, strongly) monotone if there is no risk of confusion with the cone K.

Definition 2.5.2. *Let Y be a normed space and $\psi : Y \to \mathbb{R}$ be a given continuous functional. Furthermore, let $K \subset Y$ be a closed, convex, pointed, and solid cone.*

(i) We say that ψ satisfies the monotonicity property if ψ is strictly K- monotone.

(ii) We say that ψ satisfies the representability property if $\{y \in Y \mid \psi(y) < 0\} \subseteq -\operatorname{int} K$.

Remark 2.5.3. *Note that, as a consequence of continuity, the monotonicity property implies the K- monotonicity of ψ and the representability property implies that $\{y \in Y \mid \psi(y) \le 0\} \subseteq -K$. Using this, it is straightforward to verify that a functional ψ satisfying both the monotonicity and representability property give a robust representation of $-K$, that is,*

$$\{y \in Y \mid \psi(y) \le 0\} = -K,\ \{y \in Y \mid \psi(y) < 0\} = -\operatorname{int} K. \tag{2.26}$$

Another important concept related to scalarizing functionals is that of translativity along a direction [58, Theorem 2.3.1].

Definition 2.5.4. *Let Y be a normed space. A functional $\psi : Y \to \mathbb{R}$ is said to be translation invariant along $e \in Y$ if*

$$\forall\, y \in Y,\ t \in \mathbb{R} : \quad \psi(y + te) = \psi(y) + t. \tag{2.27}$$

We can now introduce the main functionals employed in the text.

Definition 2.5.5. *([54, 55, 57, 81, 60]) Let Y be a normed space and $K \subset Y$ be a closed, convex, pointed, and solid cone. The following functionals are introduced:*

(i) Let $e \in \operatorname{int} K$. The Gerstewitz-Weidner functional $\psi_e : Y \to \mathbb{R}$ associated to e is defined as

$$\psi_e(y) := \min\{t \in \mathbb{R} \mid te \in y + K\}. \tag{2.28}$$

(ii) *Let* $\|\cdot\|'$ *be a norm equivalent to* $\|\cdot\|$. *The Hiriart-Urruty functional* $\psi_{\|\cdot\|'}: Y \to \mathbb{R}$ *associated to* $\|\cdot\|'$ *is given by*

$$\psi_{\|\cdot\|'}(y) := d_{\|\cdot\|'}(y, -K) - d_{\|\cdot\|'}(y, Y \setminus -K). \tag{2.29}$$

(iii) *Let* G *be a* w^**- compact generator of* K^*. *The Drummond-Svaiter functional* $\psi_G : Y \to \mathbb{R}$ *associated to* G *is defined as*

$$\psi_G(y) := \sigma_G(y). \tag{2.30}$$

The following proposition shows that the scalarizing functionals in Definition 2.5.5 share some common properties, including those of monotonicity and representability.

Proposition 2.5.6. *([54, 60, 81]) Let* Y *be a normed space and* $K \subset Y$ *be a closed, convex, pointed, and solid cone. Let* $\psi : Y \to \mathbb{R}$ *be a given, where* ψ *stands for the functionals* (2.28), (2.29), *and* (2.30) *in Definition 2.5.5. Then,*

(i) ψ *is continuous and sublinear,*

(ii) ψ *is strictly* K*- monotone (and hence* K*- monotone according to Remark 2.5.3),*

(iii) ψ *satisfies*

$$-K = \{y \in Y \mid \psi(y) \leq 0\}, \quad -\operatorname{int} K = \{y \in Y \mid \psi(y) < 0\}.$$

Some specific properties of Gerstewitz-Weidner functionals are given next.

Proposition 2.5.7. *([44, 106]) Let* Y *be a Banach space and* $K \subset Y$ *be a closed, convex, pointed, and solid cone. For* $e \in \operatorname{int} K$, *consider the Gerstewitz-Weidner functional* ψ_e *from Definition 2.5.5. Then:*

(i) ψ_e *is Lipschitz on* Y *with constant*

$$\rho := \max_{y^* \in \partial\psi_e(0)} \|y^*\|_*, \tag{2.31}$$

(ii) ψ_e *is translation invariant along* e,

(iii) $\partial\psi_e(\bar{y}) = \{k^* \in K^* \mid \langle k^*, e\rangle = 1, \psi_e(\bar{y}) = \langle k^*, \bar{y}\rangle\}$.

Remark 2.5.8. *According to Proposition 2.5.6* (*iii*), *for any* $\bar{y} \in -\operatorname{bd} K$ *we have* $\psi_e(\bar{y}) = 0$. *Then, it follows from* (*iii*) *that* $\partial\psi_e(\bar{y}) = \{k^* \in K^* \mid \langle k^*, e\rangle = 1, \langle k^*, \bar{y}\rangle = 0\}$.

We close this chapter with the following result, that characterizes different set relations by means of a Gerstewitz-Weidner scalarizing functional.

Theorem 2.5.9. *([107, 108, 111]) Let Y be a normed space and $K \subset Y$ be a closed, convex, pointed, and solid cone. Furthermore, consider nonempty sets $A, B \subseteq Y$ and the set relations $\preceq_K^{(l)}, \preceq_K^{(u)}$ from Definition 2.2.15. Then,*

(i) $A \preceq_K^{(l)} B \Longrightarrow \sup_{b \in B} \inf_{a \in A} \psi_e(a - b) \leq 0.$

(ii) $A \preceq_K^{(u)} B \Longrightarrow \sup_{a \in A} \inf_{b \in B} \psi_e(a - b) \leq 0.$

Chapter 3

Unified Characterization of Nonlinear Scalarizing Functionals

In this chapter we derive, in a specific sense, relationships between three main types of scalarizing functionals, namely, those from Definition 2.5.5. In addition, an extended class of quasidifferentiable scalarizing functionals is introduced, which also has important connections with set optimization. The results are based on the paper by Bouza, Quintana and Tammer [25], and are derived in the following setting:

Assumption 1. *Let Y be a Banach space and $K \subset Y$ be a proper, closed, convex, pointed, and solid cone.*

There are two main reasons for considering relationships between well known classes of scalarizing functionals and their generalizations:

- The functionals from Definition 2.5.5 have been employed in the setting of vector and set optimization for the same purpose (deriving optimality conditions and algorithms). However, it is not at all clear whether there is an advantage on using one scalarization or the other.
- Understanding the set of all scalarizing functionals and being able to generate all or part of them could have both theoretical and practical applications. For example, for robust counterpart problems to vector optimization problems under uncertainty (see Remark 2.4.17), it is known [24] that there are minimal solutions that can not be recovered with any convex scalarizing functional. Thus, as a first step, it is also of interest to consider simpler classes of nonconvex scalarizations.

In this chapter, we provide a partial solution to both of these problems. The structure is as follows. In Section 3.1, we briefly discuss previous work in the literature that is related to ours in this chapter. Section 3.2 studies the relationships between the classes of scalarizing functionals by Gerstewitz-Weidner, Huriart-Urruty and Drummond-Svaiter in the sense of inclusion. Based on these results, in Section 3.3 we introduce an extended class of scalarizing functionals, whose elements can be represented as the difference of two sublinear functionals. The relationships with the previous classes are also discussed.

3.1 Literature Review

In this section, we review different works in the literature that are related to the main two topics of this chapter: relationships between scalarization techniques, and generalized classes of scalarization. We start by recalling, as mentioned in the introduction, that a scalarization problem is usually determined by an underlying scalarizing functional and possibly a set of constraints, and that in turn these depend on different parameters.

Many comparisons between different scalarization problems in vector optimization where the ordering cone is the standard Pareto cone (that is, the nonnegative orthant) can be found in the literature, see for example [128] and the references therein. However, these comparisons are either experimental, or are made with the aid of a decision maker. A mathematically rigorous treatment of the relationships between different methods is more rare. In fact, to the best of our knowledge, the main references in this case are [47, 104]. Furthermore, some isolated results can also be found in [36, 63, 124].

- In the context of vector optimization, Eichfelder proved in [47] that several well known types of scalarization problems could be reformulated as a Pascoletti-Serafini problem [57, 138] if the corresponding parameters are carefully chosen. The methods examined included, among others: the linear scalarization technique [52], the ϵ- constraint method [72], the weighted Chebyshev scalarization [27], the Gourion and Luc problem [59], and the method by Kaliszewski [99].

  Moreover, it is also well known [54] that, under different assumptions, Pascoletti-Serafini problems can be reformulated using Gerstewitz-Weidner scalarizing functionals, see Definition 2.5.5. Thus, the class of Gerstewitz-Weidner functionals already unifies some of the well known scalarization approaches.

- In [104], also for vector optimization problems, both a qualitative and quantitative comparison between different scalarization methods and the so called conic scalarization problem [51, 102, 103] was studied. Differently to [47], the quantitative comparison in this case was mostly focused on inclusions between the solution sets of each scalarization problem, and not on the equivalence of them under reformulation. The main results stated in this paper are the following:

  - Every linear scalarization problem can be reformulated as a conic scalarization problem with an appropriate parameter.
  - Every solution of a Benson's scalar problem [23] can be obtained as a solution of a conic scalarization problem with a specified parameter.

  - There exists a subset of Pascoletti-Serafini problems for which there are conic scalarization problems with smaller optimal value.

- In [36, 124], the authors proved that Gerstewitz-Weidner functionals are a particular case of Hiriart-Urruty functionals. Later, in [63], this result was extended to the set optimization context. In this chapter we derive a similar result in a slightly different setting.

On the other hand, the topic of generalized scalarizing functionals was first addressed in the earlier papers by Wierzbicki [153, 154, 155], and later by Miglierina and Molho in [129]. In these works, scalarizing functionals were introduced in an axiomatic way, and it was shown that these axioms are necessary and sufficient in order to characterize the sets of minimal and weakly minimal points of a vector optimization problem. Specifically, the axioms stated are those of monotonicity and order representability introduced in Definition 2.5.2. However, the discussion on how to generate all the functionals satisfying the axioms is very limited. In Section 3.3, some first steps in that direction are analyzed.

3.2 Relationships Among the Main Classes of Scalarizing Functionals

In this section, based on Definition 2.5.5, we consider different classes of scalarizing functionals that have been previously studied in the literature and show relationships between them in the sense of inclusion. We start by formally introducing these classes.

Definition 3.2.1. *Let Assumption 1 be fulfilled.*

(i) The class of Gerstewitz-Weidner scalarizing functionals is denoted by $\mathcal{S}_{GW}$ and is defined as

$$\mathcal{S}_{GW} := \{\psi_e \mid e \in \operatorname{int} K\},$$

where ψ_e is given by (2.28) *in Definition 2.5.5* (*i*).

(ii) The class of Hiriart-Urruty functionals is denoted by $\mathcal{S}_{HU}$ and is defined as

$$\mathcal{S}_{HU} := \{\psi_{\|\cdot\|'} \mid \|\cdot\|' \textit{ is a norm in } Y,\ \|\cdot\|' \sim \|\cdot\|\},$$

where $\psi_{\|\cdot\|'}$ is given by (2.29) *in Definition 2.5.5* (*ii*).

(iii) The class of Drummond-Svaiter functionals is denoted by $\mathcal{S}_{DS}$ and is defined as

$$\mathcal{S}_{DS} := \{\psi_G \mid G \textit{ is } w^*\textit{- compact generator of } K^*\},$$

where ψ_G is given by (2.30) *in Definition 2.5.5* (*iii*).

In Section 2.5, specifically in Proposition 2.5.6, we saw that the functionals belonging to the set $\mathcal{S}_{GW} \cup \mathcal{S}_{HU} \cup \mathcal{S}_{DS}$ are sublinear, continuous, and satisfy both the monotonicity and representability properties. It turns out that these properties are inherent of the class $\mathcal{S}_{DS}$, as our next theorem shows.

Theorem 3.2.2. *Let Assumption 1 be fulfilled and consider any continuous and sublinear functional $\psi : Y \to \mathbb{R}$ that satisfies the monotonicity and representability properties. Then, $G := \partial\psi(0)$ is a w^*- compact generator of K^* such that*

$$\psi = \psi_G \in \mathcal{S}_{DS}.$$

In particular, this implies that $\mathcal{S}_{GW} \subseteq \mathcal{S}_{DS}$ and $\mathcal{S}_{HU} \subseteq \mathcal{S}_{DS}$.

Proof. It follows from the second part of Remark 2.5.3 that the resulting cone $-K$ has the representation given in (2.26), and thus Slater's condition is satisfied. By noticing that $\psi(0) = 0$ and that $N(0, K) = -K^*$, we deduce from Proposition 2.3.26 that

$$K^* = \operatorname{cone}(\partial\psi(0)). \tag{3.1}$$

Furthermore, because of Slater's condition, it follows that 0 is not an optimal solution of $\mathcal{OP}(\psi, Y)$. Thus, Theorem 2.4.2 implies that $0 \notin \partial\psi(0)$. This, together with (3.1), implies that G is a w^*- compact generator of K^*. Finally, from the sublinearity of ψ and Proposition 2.3.18 (iii), we deduce that

$$\psi(y) = \psi'(0, y) = \sigma_{\partial\psi(0)}(y) = \sigma_G(y).$$

$\square$

The following proposition shows that $\mathcal{S}_{GW}$ is the subset of functionals in $\mathcal{S}_{DS}$ associated to the bases of K^*.

Proposition 3.2.3. *Let Assumption 1 be fulfilled and, for some w^*- generator G of K^*, consider the corresponding element $\psi_G \in \mathcal{S}_{DS}$. Then, $\psi_G \in \mathcal{S}_{GW}$ if and only if G is a base of K^* in the w^*- topology.*

Proof. Suppose first that $\psi_G \in \mathcal{S}_{GW}$. Then, there exists $e \in \operatorname{int} K$ such that $\psi_G = \psi_e$. By Theorem 3.2.2, it is sufficient to show now that $\partial\psi_e(0)$ is a base of K^*. The statement is then a consequence of Proposition 2.5.7 (iii) and Proposition 2.2.14 (vi).

Conversely, assume that G is a base of K^*. Then, by Proposition 2.2.14 (vi), there exists $e \in \operatorname{int} K$ such that

$$G = \{y^* \in K^* \mid \langle y^*, e\rangle = 1\}. \tag{3.2}$$

Consider now the functional $\psi_e \in \mathcal{S}_{GW}$. Then, from Proposition 2.5.7 (iii) and (3.2), we get $\partial\psi_e(0) = G$. Thus, Theorem 3.2.2 implies that $\psi_e = \psi_G$. $\square$

Next proposition provides another characterization of the class $\mathcal{S}_{GW}$ related to the translation invariance property.

Proposition 3.2.4. *Let Assumption 1 be fulfilled and consider a functional $\psi_G \in \mathcal{S}_{DS}$, together with an element $e \in \operatorname{int} K$. Then, ψ_G satisfies the translation property with respect to e if and only if $\psi_G = \psi_e \in \mathcal{S}_{GW}$.*

Proof. Suppose first that ψ_G satisfies the translation invariance property with respect to e. Then, by substituting $y = 0$ and $t = \pm 1$ in (2.27), we get $\sigma_G(e) = 1$ and $\sigma_G(-e) = -1$ respectively. The definition of the support functional now implies:

$$\forall\, g^* \in G : \langle g^*, e\rangle \leq 1, \ \ \langle g^*, -e\rangle \leq -1.$$

From this, we deduce that

$$\forall\, g^* \in G : \langle g^*, e\rangle = 1.$$

Furthermore, since G is a generator of K^*, it follows that

$$G = \{y^* \in K^* \mid \langle g^*, e\rangle = 1\}.$$

Consider now the functional $\psi_e \in \mathcal{S}_{GW}$. Then, according to Proposition 2.5.7 (*iii*), we get $G = \partial\psi_e(0)$. Te first part of the statement follows from Theorem 3.2.2. The sufficiency is just Proposition 2.5.7 (*ii*). □

Remark 3.2.5. *Proposition 3.2.4 establishes that $\mathcal{S}_{GW}$ is exactly the set of elements in $\mathcal{S}_{DS}$ that satisfy the translation invariance property. It is worth to point out that it was recently established in [50, Lemma 3.2] that an element $\psi_{\|\cdot\|} \in \mathcal{S}_{HU}$ satisfies the translation property with respect to $e \in K$, provided that*

$$d_{\|\cdot\|}(e, -K) = d_{\|\cdot\|}(-e, Y \setminus -K) = 1. \tag{3.3}$$

However, it turns out that this condition already implies that $\psi_{\|\cdot\|} = \psi_e \in \mathcal{S}_{GW}$. Indeed, from (3.3) *we deduce that $e \in \operatorname{int} K$. Thus, the statement follows from Theorem 3.2.2 and Proposition 3.2.4.*

In the rest of this section, we address the question of the relationships between the classes $\mathcal{S}_{GW}$ and $\mathcal{S}_{HU}$. In the following lemma, we compute the subdifferential of an arbitrary element $\psi_{\|\cdot\|'} \in \mathcal{S}_{HU}$ at the point $\bar{y} = 0$. For a finite dimensional version of this result, see [34, Theorem 4.2]. For general characterizations (for the case that $K = A$ and A is a subset of Y without convexity assumptions concerning the involved set A) of the approximate subdifferential by Ioffe of $\psi_{\|\cdot\|'}$, see [67, Proposition 21.11].

Lemma 3.2.6. *Let Assumption 1 be fulfilled and consider the functional $\psi_{\|\cdot\|} \in \mathcal{S}_{HU}$. Then,*

$$\partial\psi_{\|\cdot\|}(0) = \mathrm{cl}^* \, \mathrm{conv} \left(K^* \cap \mathbb{S}^*\right),$$

where $\mathbb{S}^*$ *is the unit sphere in the dual space with respect to* $\|\cdot\|_*$.

Proof. Let $f : Y \to \overline{\mathbb{R}}$ be defined as

$$f(y) := \left\{ \begin{array}{ll} -d_{\|\cdot\|}(y, Y \setminus -K) & \text{if } y \in -K, \\ +\infty & \text{if } y \notin -K \end{array} \right\}.$$

Then, by [81, Proposition 5], we get

$$\partial\psi_{\|\cdot\|}(0) = \partial f(0) \cap \mathbb{B}. \tag{3.4}$$

Furthermore, from [28, Proposition 3.1], we also have

$$d_{\|\cdot\|}(y, Y \setminus -K) = \inf_{\|y^*\|_* \geq 1} \left\{\sigma_{(-K)}(y^*) - \langle y^*, y\rangle\right\},$$

or equivalently,

$$- d_{\|\cdot\|}(y, Y \setminus -K) = \sup_{\|y^*\|_* \geq 1} \left\{\langle y^*, y\rangle - \sigma_{(-K)}(y^*)\right\}. \tag{3.5}$$

Since K is a cone, it is easy to verify that $\sigma_{(-K)} = \delta_{K^*}$, the indicator functional of K^*. Hence, by defining $G := K^* \cap \{y^* \in Y^* \mid \|y^*\|_* \geq 1\}$, we get from (3.5) that

$$- d_{\|\cdot\|}(y, Y \setminus -K) = \sigma_G(y). \tag{3.6}$$

- **Claim 1**: $\forall\, y \in -K : -d_{\|\cdot\|}(y, Y \setminus -K) = \sigma_{(K^* \cap\, \mathbb{S}^*)}(y)$.

  Indeed, since $K^* \cap \mathbb{S}^* \subseteq G$ and (3.6) holds, we have $-d_{\|\cdot\|}(y, Y \setminus -K) \geq \sigma_{(K^* \cap\, \mathbb{S}^*)}(y)$. Choose now any $y^* \in G$ and $y \in -K$. Then, $\langle y^*, y\rangle \leq 0$, which implies

  $$(\|y^*\|_* - 1)\langle y^*, y\rangle \leq 0.$$

  From this we deduce that $\frac{1}{\|y^*\|_*} y^* \in K^* \cap \mathbb{S}^*$ and that $\frac{\langle y^*, y\rangle}{\|y^*\|_*} \geq \langle y^*, y\rangle$, which proves our claim.

Set

$$D := \mathrm{cl}^* \, \mathrm{conv} \left(K^* \cap \mathbb{S}^*\right).$$

Then, taking into account (3.4) and Claim 1 just proved, we have that $y^* \in \partial\psi_{\|\cdot\|}(0)$ if and only if $\|y^*\|_* \leq 1$ and

$$\forall\, y \in -K : \langle y^*, y\rangle \leq \sigma_{(K^* \cap \mathbb{S}^*)}(y). \tag{3.7}$$

By convexity and the w^*- closedness of $\partial\psi_{\|\cdot\|}(0)$, it is easy to verify that $D \subseteq \partial\psi_{\|\cdot\|}(0)$. Thus, in order to finish the proof, we only need to show that the reverse inclusion also holds. We proceed now to prove that assertion.

Suppose that $\partial\psi_{\|\cdot\|}(0) \nsubseteq D$ and fix an element $\bar{y}^* \in \partial\psi_{\|\cdot\|}(0) \setminus D$. The following claim holds:

- **Claim 2**: $\bar{y}^* \notin [1, +\infty)D$.

  Since $\bar{y}^* \notin D$, we can apply Theorem 2.1.26 (*ii*) and Proposition 2.1.24 (*i*) to obtain an element $\hat{y} \in Y$ such that

$$\langle \bar{y}^*, \hat{y} \rangle < \inf_{d^* \in D} \langle d^*, \hat{y} \rangle. \tag{3.8}$$

  Taking into account (3.8) and the fact that $\frac{1}{\|\bar{y}^*\|_*}\bar{y}^* \in D$, we find that $\langle \bar{y}^*, \hat{y} \rangle < \frac{1}{\|\bar{y}^*\|_*}\langle \bar{y}^*, \hat{y} \rangle$. Equivalently, we have

$$(\|\bar{y}^*\|_* - 1)\langle \bar{y}^*, \hat{y} \rangle < 0. \tag{3.9}$$

  Because of (3.4) and (3.9), we get that $\|\bar{y}^*\|_* < 1$. From this and (3.9), we also obtain $\langle \bar{y}^*, \hat{y} \rangle > 0$. Thus, we deduce that

$$0 < \langle \bar{y}^*, \hat{y} \rangle < \inf_{d^* \in D} \langle d^*, \hat{y} \rangle = \inf_{d^* \in [1, +\infty)D} \langle d^*, \hat{y} \rangle.$$

  In particular, this implies that $\bar{y}^* \notin [1, +\infty)D$, which proves the claim.

Now, note that $0 \notin D$. Otherwise, we would have $0 \in \partial\psi_{\|\cdot\|}(0)$, which would contradict the fact that $\partial\psi_{\|\cdot\|}(0)$ is a generator of K^*, see Theorem 3.2.2. Applying now [32, Lemma in page 218], we obtain that the set $[1, +\infty)D$ is w^*- closed and convex. Furthermore, by Theorem 2.1.26 (*ii*) and Proposition 2.1.24 (*i*), we find $\bar{y} \in Y$ such that $\langle \bar{y}^*, \bar{y} \rangle > \sigma_{([1,+\infty)D)}(\bar{y})$. The following claim is true:

- **Claim 3**: $\forall\, y^* \in D : \langle y^*, \bar{y} \rangle \leq 0$.

  Indeed, otherwise there exists $y^* \in D$ such that $\langle y^*, \bar{y} \rangle > 0$. Hence, we get that $ty^* \in [1, +\infty)D$ for every $t \geq 1$, which implies

$$\langle \bar{y}^*, \bar{y} \rangle > \sigma_{([1,+\infty)D)}(\bar{y}) \geq t\langle y^*, \bar{y} \rangle > 0.$$

  By letting $t \to +\infty$ in the above inequality, we obtain $\langle \bar{y}^*, \bar{y} \rangle > +\infty$, a contradiction.

Now, from Claim 3 and Proposition 2.2.14 (*iv*), we find that $\bar{y} \in -K$. Thus, we deduce that

$$\langle \bar{y}^*, \bar{y} \rangle > \sigma_{([1,+\infty)D)}(\bar{y}) \geq \sigma_D(\bar{y}) \geq \sigma_{(K^* \cap \mathbb{S}^*)}(\bar{y}),$$

a contradiction to (3.7). This completes the proof. □

We can now establish sufficient conditions to guarantee that $\mathcal{S}_{GW} \subseteq \mathcal{S}_{HU}$.

Theorem 3.2.7. *In addition to Assumption 1, suppose that Y is reflexive and consider, for $e \in \operatorname{int} K$, the corresponding element $\psi_e \in \mathcal{S}_{GW}$. Then, there exists a norm $\|\cdot\|'$ in Y such that $\|\cdot\|' \sim \|\cdot\|$ and $\psi_{\|\cdot\|'} = \psi_e$. In particular, we have that $\mathcal{S}_{GW} \subseteq \mathcal{S}_{HU}$.*

Proof. Set

$$V := \{y^* \in Y^* \mid |\langle y^*, e\rangle| \leq 1\}$$

and $B := \partial\psi_e(0)$. Then, according to Proposition 2.5.7 (iii),

$$B = \{y^* \in K^* \mid \langle y^*, e\rangle = 1\}.$$

Furthermore, from Proposition 2.3.18 (ii), we have that B is bounded. Hence, there exists $L > 0$ such that

$$\forall\, y^* \in B : \|y^*\|_* \leq L.$$

Consider the set

$$U := V \cap L\mathbb{B}.$$

It is then easy to see that U is a convex and balanced neighborhood of the origin. In Figure 3.1 we illustrate this construction. Furthermore, consider now the Minkowski functional associated to U, that is, the functional $\gamma_U : Y^* \to \mathbb{R}$ given by

$$\gamma_U(y^*) := \inf\{t > 0 \mid y^* \in tU\}.$$

Then, by Proposition 2.1.15, it follows that γ_U is a norm in Y^*. Moreover, by construction, it is straightforward to verify that $\gamma_U \sim \|\cdot\|_*$. Let $\mathbb{S}^*_U$ denote the unit sphere in Y^* with respect to the norm γ_U. Then, the following claim is true

- **Claim 1**: $B = \mathbb{S}^*_U \cap K^*$.

  Indeed, take any $y^* \in B$. Then, $y^* \in K^* \cap U$, which implies $\gamma_U(y^*) \leq 1$. If this inequality is strict, we could find $t > 1$ such that $ty^* \in U$. Hence, we would have

  $$1 \overset{(ty^* \in V)}{\geq} t\langle y^*, e\rangle \overset{(y^* \in B)}{=} t > 1,$$

  a contradiction. It follows then that $\gamma_U(y^*) = 1$, which implies $B \subseteq \mathbb{S}^*_U \cap K^*$.

  In order to see the reverse inclusion, take $y^* \in \mathbb{S}^*_U \cap K^*$. Then, since $e \in \operatorname{int} K$, we have $\langle y^*, e\rangle > 0$. Hence, we find that $\frac{1}{\langle y^*, e\rangle} y^* \in B \subseteq \mathbb{S}^*_U$, which implies

  $$\gamma_U\left(\frac{1}{\langle y^*, e\rangle} y^*\right) = 1.$$

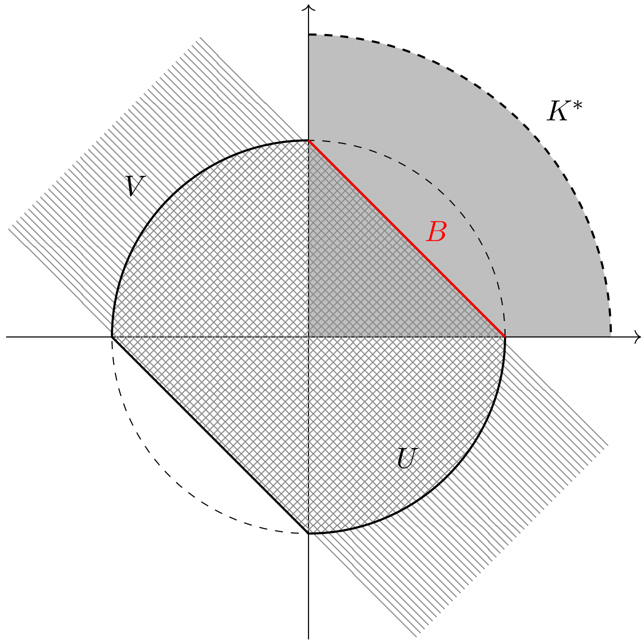

Figure 3.1: Geometrical construction in the proof of Theorem 3.2.7

Thus, we get that $\langle y^*, e\rangle = 1$. Since $y^* \in K^*$, this is equivalent to $y^* \in B$, which shows that

$$\mathbb{S}_U^* \cap K^* \subseteq B.$$

Next, consider the canonical mapping J_Y from Definition 2.1.22, and let γ_{U*} be the dual norm of γ_U in Y^{**}. We define next $\|\cdot\|' : Y \longrightarrow \mathbb{R}$ as

$$\|y\|' := (\gamma_{U*} \circ J_Y)(y). \tag{3.10}$$

Then, it follows from the definition that $\|\cdot\|'$ is a norm in Y. Furthermore, since $\gamma_U \sim \|\cdot\|_*$, we also obtain $\|\cdot\|' \sim \|\cdot\|$. We deduce now that

$$\begin{aligned}
\forall\, y^* \in Y^* : \gamma_U(y^*) &\overset{\text{(Proposition 2.1.18)}}{=} \sup_{y^{**}\in Y^{**}\setminus\{0\}} \frac{|\langle y^{**}, y^*\rangle|}{\gamma_{U*}(y^{**})} \\
&\overset{(Y \text{reflexive})}{=} \sup_{y\in Y\setminus\{0\}} \frac{|\langle J_Y(y), y^*\rangle|}{(\gamma_{U*}\circ J_Y)(y)} \\
&\overset{\text{(Definition 2.1.22 + (3.10))}}{=} \sup_{y\in Y\setminus\{0\}} \frac{|\langle y^*, y\rangle|}{\|y\|'} \\
&\overset{\text{(Definition 2.1.17 }(ii))}{=} \|y^*\|'_*,
\end{aligned}$$

so that $\|\cdot\|'_* = \gamma_U$. Considering now the functional $\psi_{\|\cdot\|'} \in \mathcal{S}_{HU}$, we find that

$$\partial\psi_{\|\cdot\|'}(0) \overset{\text{(Lemma 3.2.6)}}{=} \mathrm{cl}^* \operatorname{conv}(K^* \cap \mathbb{S}_U^*) \overset{\text{(Claim 3)}}{=} \mathrm{cl}^* \operatorname{conv} B = B.$$

The statement follows now from Theorem 3.2.2.

□

Remark 3.2.8. *As mentioned in Section 3.1, a result like Theorem 3.2.7 was first stated in [36, Proposition 2], extended in [63] to the context of set optimization, and rediscovered in [124, Theorem 4.2]. Our argument was derived independently and uses a similar idea to that of [36]. The main difference between our proof and that stated in [36] is that we work towards constructing a norm in the primal space for which the subdifferential of the corresponding Hiriart-Urruty functional at the origin coincides with that of the Gerstewitz-Weidner functional. This working scheme has the advantage of being applicable when analyzing relationships among other types of scalarizations.*

In Theorem 3.2.2 and Theorem 3.2.7, we have shown, under mild assumptions, inclusions between the classes $\mathcal{S}_{GW}, \mathcal{S}_{HU}$ and $\mathcal{S}_{DS}$. The following example illustrates that these inclusions are strict in general.

Example 3.2.9. *In Assumption 1, let $Y = \mathbb{R}^2$ and $K = \mathbb{R}^2_+$, so that $K^* = \mathbb{R}^2_+$. Recall that in finite dimensions we always assume the Euclidean norm $\|\cdot\|_2$, and that $\mathbb{S}$ denotes the corresponding unit sphere. Next, consider the sets*

$$G_1 := \operatorname{conv}(\mathbb{S} \cap K^*), \quad G_2 := G_1 \cup \operatorname{conv}\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \right\}.$$

Then, it is clear that G_1 and G_2 are compact generators of K^*, and hence $\sigma_{G_1}, \sigma_{G_2} \in \mathcal{S}_{DS}$. The following holds:

(i) $\sigma_{G_1} \in \mathcal{S}_{HU} \setminus \mathcal{S}_{GW}$.

According to Lemma 3.2.6 and Theorem 3.2.2, it is easy to see that $\sigma_{G_1} \in \mathcal{S}_{HU}$. Assume that $\sigma_{G_1} \in \mathcal{S}_{GW}$. Then, according to Proposition 2.5.7 (iii), we could find an element $e \in \operatorname{int} K$ such that $G_1 = \{v \in \mathbb{R}^2_+ \mid e^\top v = 1\}$. Since the interior of G_1 is nonempty, this would be a contradiction.

(ii) $\sigma_{G_2} \in \mathcal{S}_{DS} \setminus \mathcal{S}_{HU}$.

Since we already know that $\sigma_{G_2} \in \mathcal{S}_{DS}$, it remains to show that $\sigma_{G_2} \notin \mathcal{S}_{HU}$. Assume otherwise. Then, we can find a norm $\|\cdot\|'$ equivalent to $\|\cdot\|_2$ such that $\sigma_{G_2} = \psi_{\|\cdot\|'}$. According to Lemma 3.2.6, we now have

$$G_2 = \partial \psi_{\|\cdot\|'}(0) = \operatorname{conv}(\mathbb{S}' \cap \mathbb{R}^2_+),$$

where $\mathbb{S}'$ is the unit sphere in $\mathbb{R}^2$ with respect to $\|\cdot\|'_*$. Let $v := \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$. Then, because $(\mathbb{S} \cap \mathbb{R}^2_+) \cup \{v\}$ are extreme points of G_2, it follows that $(\mathbb{S} \cap \mathbb{R}^2_+) \cup \{v\} \subseteq \mathbb{S}' \cap \mathbb{R}^2_+$. From this, we then obtain $v \in \mathbb{S}'$ and $\sqrt{2}v \in \mathbb{S} \cap \mathbb{R}^2_+ \subseteq \mathbb{S}'$, a contradiction.

Taking into account Theorem 3.2.2, Theorem 3.2.7, and Example 3.2.9, we get the following corollary.

Corollary 3.2.1. *Let Assumption 1 be fulfilled and suppose in addition that Y is reflexive. Then,*

$$\mathcal{S}_{GW} \subseteq \mathcal{S}_{HU} \subseteq \mathcal{S}_{DS}$$

and these inclusions are, in general, strict.

Remark 3.2.10. *Corollary 3.2.1 shows that, since the class $\mathcal{S}_{GW}$ is the smallest, its elements can only have additional properties. In particular, according to Proposition 3.2.4, the translation invariance property is one that only the functionals in this class enjoy. Furthermore, this type of functionals are exploited in the context of risk measures in mathematical finance [106]. We conclude that the class $\mathcal{S}_{GW}$ has more advantages from both the theoretical and practical point of view.*

3.3 Generalized Class of Scalarizing Functionals

In this section, we further elaborate on the idea of generators of dual cones to extend the class $\mathcal{S}_{DS}$. Specifically, given w^*- compact sets $G, H \subset Y^*$, we consider scalarizing functionals $\psi : Y \to \mathbb{R}$ of the form

$$\psi(y) := \sigma_G(y) - \sigma_H(y). \tag{3.11}$$

These functionals are quasidifferentiable at any point as a consequence of the quasidifferentiability of the involved support functionals. Furthermore, they are also continuous, since in Banach spaces the w^*- compact sets are also bounded, see Proposition 2.1.24. In the following, we study necessary and sufficient geometrical conditions on G and H under which ψ satisfies the two main axioms of scalarizations: monotonicity and order representability. These conditions will motivate the definition of the new class of quasidifferentiable scalarizing functionals.

First, we focus on monotonicity properties based on set relations between the faces of the subdifferential and the superdifferential. Our starting point is the following lemma, that can be seen as a generalization of Hörmander's Theorem [145, Theorem 2.3.1].

Lemma 3.3.1. *Let Assumption 1 be fulfilled and let $G, H \subseteq Y^*$ be convex and w^*- compact. Then,*

(i) The equivalence

$$H \preceq^{(u)}_{K^*} G \quad \Longleftrightarrow \quad \sigma_G|_K \geq \sigma_H|_K$$

holds.

(ii) The implication

$$H \preceq^{(u)}_{K^*\setminus\{0\}} G \quad \Longrightarrow \quad \sigma_G|_{\operatorname{int} K} > \sigma_H|_{\operatorname{int} K}$$

holds. The converse is also true if $H \cap \operatorname{Min}(G, K^*) = \emptyset$.

(iii) Assume that $K^{s*} \neq \emptyset$. *Then,*

$$H \preceq^{(u)}_{K^{s*}} G \quad \Longrightarrow \quad \sigma_G|_{K\setminus\{0\}} > \sigma_H|_{K\setminus\{0\}}.$$

The converse holds if Y *is reflexive and* $\operatorname{int} K^* \neq \emptyset$.

Proof. We only prove (ii) and (iii), since (i) is a particular case of [91, Lemma 2.1].

(ii) Suppose first that $H \preceq^{(u)}_{K^*\setminus\{0\}} G$, and fix elements $y \in \operatorname{int} K$ and $h^* \in H$ such that $\langle h^*, y\rangle = \sigma_H(y)$. Then, we can find $g^* \in G$ such that $g^* - h^* \in K^* \setminus \{0\}$. Hence, by Proposition 2.2.14 (v), we have $\langle g^* - h^*, y\rangle > 0$. Thus, it follows that

$$\sigma_G(y) \geq \langle g^*, y\rangle = \langle g^* - h^*, y\rangle + \langle h^*, y\rangle = \sigma_H(y) + \langle g^* - h^*, y\rangle > \sigma_H(y),$$

which proves the first part of the statement.

In order to verify the second part, assume now that $H \cap \operatorname{Min}(G, K^*) = \emptyset$ and that $\sigma_G|_{\operatorname{int} K} > \sigma_H|_{\operatorname{int} K}$. Then, since K is convex, we have that $\operatorname{cl} \operatorname{int} K = K$, see [83]. This fact, together with our assumption, imply that

$$\sigma_G|_K \geq \sigma_H|_K.$$

Hence, according to statement (i), we have $H \subseteq G - K^*$. Assume that, on the contrary, $H \nsubseteq G - K^* \setminus \{0\}$. Then, there exists an element

$$h^* \in H \cap ((G - K^*) \setminus (G - K^* \setminus \{0\})).$$

From this, we deduce that $h^* \in G \cap H$, and that $h^* \notin y^* - K^* \setminus \{0\}$ for any $y^* \in G$. However, by definition, this means that $h^* \in H \cap \operatorname{Min}(G, K^*)$, a contradiction.

(iii) To this end, let $K^{*s} \neq \emptyset$ and assume that $H \subseteq G - K^{*s}$. Furthermore, fix $y \in K \setminus \{0\}$ and $h^* \in H$ such that $\langle h^*, y\rangle = \sigma_H(y)$. Analogous to the proof of (ii), there exists $g^* \in G$ such that $k^* := g^* - h^* \in K^{*s}$. Then, it follows that

$$\sigma_H(y) = \langle h^*, y\rangle = \langle g^*, y\rangle - \langle k^*, y\rangle \leq \sigma_G(y) - \langle k^*, y\rangle < \sigma_G(y),$$

where the last inequality holds because $k^* \in K^{*s}$ and $y \in K \setminus \{0\}$. Hence, the first implication is proved.

Assume now that Y is reflexive and that $\operatorname{int} K^* \neq \emptyset$. It is well known that we always have $\operatorname{int} K^* = \operatorname{int} K^{*s}$. Furthermore, the reflexivity implies that

$$\operatorname{int} K^* = \operatorname{int} K^{*s} = K^{*s}.$$

In fact, this is a characterization of reflexive spaces [29, Theorem 3.6].

If

$$H \nsubseteq G - K^{*s} = G - \operatorname{int} K^*,$$

then there exists $h^* \in H,\ h^* \notin G - \operatorname{int} K^*$. By Theorem 2.1.26 (ii) and Proposition 2.1.24 (i), we can find $y \in Y \setminus \{0\}$ such that

$$\forall\, g^* \in G, k^* \in K^* : \langle h^*, y\rangle \geq \langle g^* - k^*, y\rangle.$$

From this, it is easy to deduce that $y \in K$ and that $\sigma_H(y) \geq \sigma_G(y)$, as desired. □

For the forthcoming results we recall that, for an element $y \in Y$ and a set $A \subseteq Y^*$, the y- face of A is denoted by A^y, see Definition 2.1.21.

Proposition 3.3.2. *Let Assumption 1 be fulfilled and let $G, H \subseteq Y^*$ be convex and w^*- compact. Then, the following assertions are equivalent:*

(i) $\forall\, y \in Y :\ H^y \preceq^{(s)}_{K^*} G^y$,

(ii) $\forall\, y \in Y :\ H^y \preceq^{(u)}_{K^*} G^y$,

(iii) $\forall\, y \in Y :\ H^y \preceq^{(l)}_{K^*} G^y$.

Proof. According to Remark 2.2.16, we always have $(i) \implies (ii)$ and $(i) \implies (iii)$. Hence, by the definition of the set less relation, it suffices to show that (ii) and (iii) are equivalent. We now proceed to prove this statement. The key lies in the following claim:

- **Claim**: $\forall\, y \in Y : H^y \preceq^{(u)}_{K^*} G^y \implies H \preceq^{(l)}_{K^*} G.$

  Indeed, assume otherwise. Then, there exists $\bar{g}^* \in G \setminus (H + K^*)$. Consider the sets

  $$S := G \cap (H + K^*),\ M := (\bar{g}^* - K^*) \cap G.$$

  It is easy to see that M is w^*- compact. Furthermore, the definition of $\bar{g}^*$ also implies that $M \cap (H + K^*) = \emptyset$. We can now apply Theorem 2.1.26 (ii) to strongly separate the sets M and $H + K^*$ and obtain an element $\bar{y} \in Y$ such that

  $$\sigma_G(\bar{y}) \geq \inf_{g^* \in M} \langle g^*, \bar{y}\rangle > \sup_{y^* \in H + K^*} \langle y^*, \bar{y}\rangle \geq \sigma_S(\bar{y}). \tag{3.12}$$

  Now, consider the set $G^{\bar{y}}$. By (3.12), we have that $G^{\bar{y}} \subseteq G \setminus S$, which is equivalent to $G^{\bar{y}} \cap (H + K^*) = \emptyset$. However, this means in particular that $H^{\bar{y}} \nsubseteq G^{\bar{y}} - K^*$, a contradiction. This proves the claim.

Assume now that (ii) holds and that there exists $y \in Y$ such that $H^y \not\preceq^{(l)}_{K^*} G^y$, or equivalently, that $G^y \nsubseteq H^y + K^*$. Then, we can find $\bar{g}^* \in G^y \setminus (H^y + K^*)$. By Theorem 2.1.26 (ii) and Proposition 2.1.24 (i), there exists $\bar{y} \in Y$ such that

$$\langle \bar{g}^*, \bar{y}\rangle > \sup_{(h^*,k^*)\in H\times K^*} \langle h^* + k^*, \bar{y}\rangle.$$

In particular, this implies that $\langle k^*, \bar{y}\rangle \leq 0$ for every $k^* \in K^*$, which, in virtue of Proposition 2.2.14 (iv), means that $\bar{y} \in -K$. Since $\bar{g}^* \in G$, our claim gives us the existence of $\bar{h}^* \in H$ and $\bar{k}^* \in K^*$ such that $\bar{g}^* = \bar{h}^* + \bar{k}^*$. Then, we deduce

$$\sigma_H(\bar{y}) \geq \langle \bar{h}^*, \bar{y}\rangle \geq \langle \bar{h}^*, \bar{y}\rangle + \langle \bar{k}^*, \bar{y}\rangle = \langle \bar{g}^*, \bar{y}\rangle > \sup_{(h^*,k^*)\in H\times K^*} \langle h^* + k^*, \bar{y}\rangle \geq \sigma_H(\bar{y}),$$

a contradiction. This proves that $(ii) \implies (iii)$. By interchanging G and H and considering $(-K^*)$ instead of K^*, a similar analysis proves that $(iii) \implies (ii)$. The proof is complete. $\square$

The following result completely characterizes K- monotone functionals of the form (3.11) with respect to the corresponding y- faces of the sets G and H.

Lemma 3.3.3. *Let Assumption 1 be fulfilled and let $G, H \subseteq Y^*$ be convex and w^*- compact. Consider the functional $\psi : Y \to \mathbb{R}$ defined by* (3.11). *Then:*

(i) The functional ψ is K- monotone if and only if

$$\forall\, y \in Y : H^y \preceq^{(u)}_{K^*} G^y.$$

(ii) If $H^y \preceq^{(u)}_{K^*\setminus\{0\}} G^y$ for every $y \in Y$, the functional ψ is strictly K- monotone.

(iii) If $K^{*s} \neq \emptyset$ and $H^y \preceq^{(u)}_{K^{s*}} G^y$ for every $y \in Y$, the functional ψ is strongly K- monotone.

Proof. (i) Suppose that ψ is K- monotone and consider, for $y \in Y$ and $z \in K$, the functional $\zeta_{y,z} : \mathbb{R} \to \mathbb{R}$ given by

$$\zeta_{y,z}(t) := \psi(y + tz).$$

Then, it is easy to verify that the K- monotonicity of ψ is equivalent to having that, for every $y \in Y,\ z \in K$, the point $\bar{t} = 0$ is a solution of the problem $\mathcal{OP}(\zeta_{y,z}, \mathbb{R}_+)$, that is,

$$\min_{t\in\mathbb{R}_+} \quad \zeta_{y,z}(t).$$

Applying now the optimality conditions of [88, Theorem 3.8 (a)] for every $y \in Y, z \in K$, we get

$$\forall\, y \in Y, z \in K, t \geq 0 :\ \zeta'_{y,z}(0, t) \geq 0. \tag{3.13}$$

Fix now $y \in Y, z \in K$, and consider the operator $T \in \mathcal{L}(\mathbb{R}, Y)$ defined as

$$T(t) = tz.$$

Furthermore, let $g : \mathbb{R} \to \mathbb{R}$ be defined as

$$g(t) = \sigma_G(y + T(t)).$$

Note that, because f is convex and continuous at $\bar{t}$, we can apply Proposition 2.3.18 (iii) (a) to obtain

$$\forall\, t \in \mathbb{R} : g'(\bar{t}, t) = \sigma_{\partial g(\bar{t})}(t). \tag{3.14}$$

Then, by considering the adjoint operator T^* of T, we deduce that

$$\begin{aligned} \partial g(\bar{t}) &\overset{\text{(Proposition 2.3.19 } (ii))}{=} T^* \left[\partial \sigma_G(y + \bar{t}z)\right] \\ &= T^* \left[\partial \sigma_G(y)\right] \\ &\overset{\text{(Proposition2.3.18 } (iv))}{=} T^* \left[G^y\right]. \end{aligned} \tag{3.15}$$

On the other hand, it is straightforward to verify that

$$\forall\, y^* \in Y : T^*(y^*) = \langle y^*, z\rangle. \tag{3.16}$$

Thus, from (3.14), (3.15), and (3.16), we find that

$$g'(0, t) = t\sigma_{G^y}(z). \tag{3.17}$$

In a similar fashion, we can show that the functional $h : \mathbb{R} \to \mathbb{R}$ given by $h(t) = \sigma_H(y + tz)$, satisfies

$$h'(0, t) = t\sigma_{H^y}(z). \tag{3.18}$$

Hence, from (3.17) and (3.18), we find that (3.13) is equivalent to

$$\forall\, y \in Y,\ z \in K :\ \sigma_{G^y}(z) - \sigma_{H^y}(z) \geq 0.$$

Applying now Lemma 3.3.1 (i), we obtain

$$\forall\, y \in Y : H^y \subseteq G^y - K^*,$$

and the necessity follows.

In order to prove sufficiency, assume that $H^y \preceq^{(s)}_{K^*} G^y$ for every $y \in Y$. Then, according to what we just saw, this is equivalent to

$$\forall\, y \in Y, z \in K, t \geq 0 : \zeta'_{y,z}(0, t) \geq 0.$$

In particular, this implies that $\zeta_{y,z}$ is increasing along any $z \in K$. Hence, it follows that $\bar{t} = 0$ is a global minimum of $\zeta_{y,z}$, from which the monotonicity is deduced.

(ii) Assume now that $H^y \subseteq G^y - K^* \setminus \{0\}$ for every $y \in Y$. By Lemma 3.3.1 *(ii)*, this implies

$$\sigma_{G^y}|_{\operatorname{int} K} > \sigma_{H^y}|_{\operatorname{int} K}.$$

Hence, for any $y \in Y$, $z \in \operatorname{int} K$ and $t > 0$, we have

$$\zeta'_{y,z}(0, t) > 0.$$

Thus, it follows that $\bar{t} = 0$ is a strict local minimum of $\mathcal{OP}(\zeta_{y,z}, \mathbb{R}_+)$. Hence, ψ is strictly monotone.

(iii) Assume in addition that $H^y \subseteq G^y - K^{*s}$ for each $y \in Y$. By Lemma 3.3.1 *(iii)*, this implies

$$\sigma_{G^y}|_{K\setminus\{0\}} > \sigma_{H^y}|_{K\setminus\{0\}}.$$

Then, we have $\zeta'_{y,z}(0, t) > 0$ for every $y \in Y$, $z \in K \setminus \{0\}$, $t > 0$. Hence, in this case $\bar{t} = 0$ is a strict minimum of $\mathcal{OP}(\zeta_{y,z}, \mathbb{R}_+)$, which gives us the strong monotonicity of ψ.

□

Corollary 3.3.4. *Let Assumption 1 be fulfilled and let $G, H \subseteq Y^*$ be convex and w^*- compact such that $G \cap H = \emptyset$. Suppose that the functional $\psi : Y \to \mathbb{R}$ defined by* (3.11) *is K- monotone. Then, it is also strictly K- monotone.*

Proof. Because of the monotonicity assumption on ψ and Lemma 3.3.3 (i), we have in particular $H^y \subseteq G^y - K^*$ for every $y \in Y$. Moreover, because $G \cap H = \emptyset$, we actually get $H^y \subseteq G^y - K^* \setminus \{0\}$ for every $y \in Y$, or equivalently,

$$\forall\, y \in Y : H^y \preceq^{(u)}_{K^*\setminus\{0\}} G^y.$$

Applying now Lemma 3.3.3 (ii), we obtain the strict monotonicity of ψ. □

Now, we focus on the conditions that G and H must fulfill in order to guarantee the order representability axiom. To this aim, we introduce the set-valued mapping $P : Y \rightrightarrows Y^*$ defined by

$$P(y) := \{y^* \in Y^* \mid \langle y^*, y\rangle \geq 0\} = (\operatorname{cone}\{y\})^*.$$

Note that

$$\operatorname{int} P(y) = (\operatorname{cone}\{y\})^{s*} \neq \emptyset,$$

for $y \neq 0$. With this definition, it is immediate how to find a geometrical condition on G and H that is equivalent to the order representability, as the following lemma shows.

Lemma 3.3.5. *Let Assumption 1 be fulfilled and let $G, H \subseteq Y^*$ be convex and w^*- compact. Consider the functional $\psi : Y \to \mathbb{R}$ defined by* (3.11)*. Then,* *(i) The implication*

$$H \subseteq \bigcap_{y \notin -K} (G - \operatorname{int} P(y)) \Longrightarrow \{y \in Y \mid \psi(y) \leq 0\} \subseteq -K$$

holds. The converse is true if Y is reflexive.

(ii) If ψ is K- monotone and $G \cap H = \emptyset$,

$$H \subseteq \bigcap_{y \notin -K} (G - \operatorname{int} P(y)) \Longrightarrow -\operatorname{int} K = \{y \in Y \mid \psi(y) < 0\}.$$

Proof. (i) We have

$$\begin{aligned}
H \subseteq \bigcap_{y \notin -K} (G - \operatorname{int} P(y)) \quad &\Longleftrightarrow \quad \forall\, y \notin -K : H \prec^{(u)}_{P(y)} G \\
&\Longleftrightarrow \quad \forall\, y \notin -K : H \preceq^{(u)}_{(\operatorname{cone}\{y\})^{s*}} G \\
&\overset{(\text{Lemma 3.3.1 } (iii))}{\Longrightarrow} \quad \forall\, y \notin -K :\ \sigma_G|_{\operatorname{cone}\{y\}\setminus\{0\}} > \sigma_H|_{\operatorname{cone}\{y\}\setminus\{0\}} \\
&\Longleftrightarrow \quad \forall\, y \notin -K :\ \sigma_G(y) > \sigma_H(y) \\
&\Longleftrightarrow \quad \forall\, y \notin -K :\ \psi(y) > 0 \\
&\Longleftrightarrow \quad \{y \in Y \mid \psi(y) \leq 0\} \subseteq -K,
\end{aligned}$$

which proves the first part of the statement.

Now, assume that Y is reflexive. In order to prove the converse, it suffices to show that the converse of the one-way implication in the previous proof is true. But this is a consequence of the second part of Lemma 3.3.1 (iii) by noticing that $\operatorname{int} P(y) \neq \emptyset$ for every $y \in Y$. This finishes the proof of (i).

(ii) By Corollary 3.3.4, the functional ψ is strictly monotone. This already implies

$$-\operatorname{int} K \subseteq \{y \in Y \mid \psi(y) < 0\}.$$

On the other hand, if $\psi(y) < 0$, then $y \in -K$ by statement (i) in this lemma. Hence, it follows from the continuity of ψ that $y \in -\operatorname{int} K$. The proof is complete. $\square$

The results on the previous section motivates the following definition:

Definition 3.3.6. *Let Assumption 1 be fulfilled and let G, H be convex and w^*- compact subsets of Y^*. We say that the pair (G, H) is a scalarization pair if:*

(i) $\forall\, y \in Y : H^y \preceq^{(s)}_{K^*} G^y$,

(ii) $H \cap G = \emptyset$,

(iii) $H \subseteq \bigcap_{y \notin -K} (G - \operatorname{int} P(y))$.

The class of all scalarization pairs is denoted by $\mathcal{D}$. Furthermore, we define the class of quasidifferentiable and positively homogeneous scalarizing functionals as the set

$$\mathcal{S}_{QD} := \{\psi : Y \to \mathbb{R} \mid \exists\, (G, H) \in \mathcal{D} : \psi = \sigma_G - \sigma_H\}.$$

Our next theorem is an immediate consequence of the previous lemmas. It shows that $\mathcal{S}_{QD}$ is a class of functionals whose elements fulfill the monotonicity and order representability conditions.

Theorem 3.3.7. *Let Assumption 1 be fulfilled and consider $\psi \in \mathcal{S}_{QD}$. Then, ψ is strictly K-monotone and satisfies the representability property.*

Proof. By Lemma 3.3.3, the functional ψ is monotone. Hence, the strict monotonicity follows from Corollary 3.3.4. The representability property is a consequence of Lemma 3.3.5 $(i), (ii)$, the convexity of K, and the continuity of ψ. □

The following result confirms that $\mathcal{S}_{QD}$ extends the class $\mathcal{S}_{DS}$. First, we recall that, for sets $G, H \subseteq Y^*$, the set $G \ominus H$ defined as $G \ominus H := \{y^* \in Y^* \mid y^* + H \subseteq G\}$ is the so called Hadwiger-Pontryagin difference of sets, see [71, 141].

Theorem 3.3.8. *Let Assumption 1 be fulfilled and let G, H be w^*- compact convex subsets of Y^*. Then,*

(i) The set G is a w^*- compact generator of $K^* \iff (G, \{0\}) \in \mathcal{D}$. In particular, $\mathcal{S}_{DS} \subseteq \mathcal{S}_{QD}$.

(ii) Assume that $(G, H) \in \mathcal{D}$ and let the functional ψ be defined by (3.11). Then,

$$\psi \in \mathcal{S}_{DS} \iff H + G \ominus H = G.$$

In particular, the set $G \ominus H$ is necessarily a generator of K^*.

Proof. (i) Let us assume first that G is a generator of K^*. We now prove that $(G, \{0\}) \in \mathcal{D}$. Indeed, it follows from the definition that $0 \notin G$, or equivalently, $\{0\} \cap G = \emptyset$. Furthermore, by Proposition 3.3.2, the condition $\{0\} \preceq^{(s)}_{K^*} G^y$ for every $y \in Y$ is equivalent to $\{0\} \preceq^{(l)}_{K^*} G^y$ for every $y \in Y$. This just means that $G^y \subseteq K^*$ for all y, which is trivially satisfied by the definition of the generator. In order to finish this first part, it remains to show that

$$0 \in \bigcap_{y \notin -K} (G - \operatorname{int} P(y)) .$$

Assume otherwise. Then, we could find $y \notin -K$ such that $0 \notin G - \operatorname{int} P(y)$. This is equivalent to $G \subseteq -P(y)$, and hence

$$\forall\, g^* \in G : \langle g^*, y \rangle \leq 0.$$

Because G is a generator of K^*, we can apply Proposition 2.2.14 (iv) to obtain that $y \in -K$, a contradiction. This proves the first implication.

Now, assume that $(G, \{0\}) \in \mathcal{D}$. By Theorem 3.3.7, the functional $\psi := \sigma_G - \sigma_{\{0\}} = \sigma_G$ satisfies both the monotonicity and the order representability axiom. Hence, from the proof of Theorem 3.2.2, we get that G is in fact a generator of K^*.

(ii) We have $\psi \in \mathcal{S}_{DS}$ if and only if there exists a generator D of K^* such that $\sigma_G - \sigma_H = \sigma_D$. Adding σ_H to both members we get

$$\sigma_G = \sigma_D + \sigma_H = \sigma_{D+H}.$$

Then, applying Hörmander's Theorem [145, Theorem 2.3.1], we find that $G = D + H$. Therefore, by the definition of $G \ominus H$, we deduce that $G \ominus H = D$.

□

By now, we know that $\mathcal{S}_{DS} \subseteq \mathcal{S}_{QD}$. However, it is not clear whether these classes are equivalent. We close this section by showing that this inclusion is actually strict under natural assumptions.

Theorem 3.3.9. *In addition to Assumption 1, suppose that* $\operatorname{int} K^* \neq \emptyset$. *Then,*

$$\mathcal{S}_{QD} \setminus \mathcal{S}_{DS} \neq \emptyset.$$

Proof. First, let us note that, if $\dim Y = 1$, the result is trivial. Indeed, in this case, we can assume without loss of generality that $Y = \mathbb{R}$ and $K = \mathbb{R}_+$. Then, it is easy to see that the sets $G = [a, b]$ and $H = [c, d]$ form a scalarization pair if and only if $a > d$. By choosing them so that $b - d < a - c$, we ensure that $G \ominus H = \emptyset$ and hence, by Theorem 3.3.8 (ii), the associated functional to this sets will be nonconvex.

For the rest of the proof, we assume that $\dim Y > 1$. Here, the argument will be divided in several steps:

<u>**Step 1**</u>: Definition of suitable sets G and H of Y^*.

Fix elements $e \in \operatorname{int} K$ and $v^* \in \operatorname{int} K^*$, and consider the basis

$$B := \{y^* \in K^* \mid \langle y^*, e\rangle = 1\}$$

of K^*. Then, we can find $\epsilon > 0$ such that

$$v^* + \epsilon\mathbb{B} \subset K^*. \tag{3.19}$$

On the other hand, according to Proposition 2.5.7 (iii), we have $B = \partial\psi_e(0)$. Hence, from Proposition 2.3.18 (ii), we deduce that B is bounded. Thus,

$$L := \sup_{b^* \in B} \|b^*\|_* < +\infty.$$

Consider now the point $p^* = \frac{L}{\epsilon} v^*$. Then, for any $b^* \in B$, we have

$$\epsilon = \frac{\epsilon}{L} L \geq \frac{\epsilon}{L} \|b^*\|_* = \left\| v^* - \frac{\epsilon}{L} b^* - v^* \right\|_* .$$

By (3.19), we now have $v^* - \frac{\epsilon}{L} b^* \in K^*$, which is equivalent to $b^* \in p^* - K^*$. Since b^* was chosen arbitrarily in B, it follows that the constructed point p^* satisfies

$$B \subseteq p^* - K^*.$$

Consider now the sets

$$G := (B + K^*) \cap (p^* - K^*), \quad C := \{y \in Y \mid \langle p^*, y \rangle < \sigma_G(y)\}.$$

Note that $-e \in C$, and hence $C \neq \emptyset$. Indeed, assume the contrary. In case $G \neq \{p^*\}$, we could find $g^* \in G \setminus \{p^*\}$, Then, we would have $p^* - g^* \in K^* \setminus \{0\}$, such that $\langle p^* - g^*, -e \rangle < 0$. However, this implies

$$\langle p^*, -e \rangle < \langle g^*, -e \rangle \leq \sigma_G(-e),$$

a contradiction. If on the other hand $G = \{p^*\}$, then we also get $B = \{p^*\}$. Since $\operatorname{int} K^* \neq \emptyset$ and B is a basis of K^*, we deduce that $\dim Y^* = 1$, which implies $\dim Y = 1$, again a contradiction.

Let us define next

$$\tilde{H} := \mathrm{cl}^* \operatorname{conv} \left(\bigcup_{y \in C} G^y \cup B \right).$$

Furthermore, let

$$\beta := \inf_{b^* \in B} \|b^*\|_*, \quad \eta := \sup_{h^* \in \tilde{H}} \|h^*\|_*.$$

Finally, put

$$H := \frac{\beta}{2\eta} \tilde{H}.$$

Step 2: Proving that (G, H) is a scalarization pair.

This will be a consequence of the following claims:

- **Claim 1**: G and H are convex and w^*- compact.

  We have that G is convex because it is the intersection of convex sets. By definition, $\tilde{H}$ is convex, so H is convex too. Since $\tilde{H}$ is a w^*- closed subset of G, in order to prove w^*-compactness of H it suffices to prove the w^*- compactness of G. However, from Proposition 2.1.24 (*iii*), this is equivalent to show that G is w^*- closed and $\| \cdot \|_*$- bounded.

  The w^*- closedness of G is easy to see: G is the intersection of two w^*- closed sets. In order to see that G is $\| \cdot \|_*$- bounded, note that $\langle g^*, e \rangle \leq \langle p^*, e \rangle$ for every $g^* \in G$. Now,

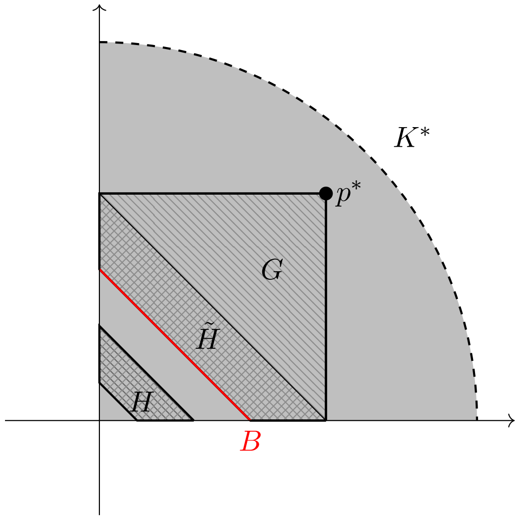

Figure 3.2: Idea of the construction in the proof of Theorem 3.3.9

since B is a basis of K^*, every element $g^* \in G$ can be written as $g^* = tb^*$, where $t > 0$ and $b^* \in B$. Then, we have the inequality $\langle tb^*, e\rangle \leq \langle p^*, e\rangle$, from which we find that

$$t \leq \frac{\langle p^*, e\rangle}{\langle b^*, e\rangle} = \langle p^*, e\rangle.$$

Thus, it follows that

$$\|g^*\|_* = t\|b^*\|_* \leq \langle p^*, e\rangle \|b^*\|_* \leq L\langle p^*, e\rangle < +\infty,$$

as we wanted.

- **Claim 2**: $G \cap H = \emptyset$.

  By construction, we have

$$\forall\, h^* \in H : \|h^*\|_* \leq \frac{\beta}{2\eta}\eta = \frac{\beta}{2} < \beta,$$

  and

$$\forall\, g^* \in G : \|g^*\|_* \geq \beta.$$

  In particular, this implies that $G \cap H = \emptyset$.

- **Claim 3**: $\forall\, y \in Y : H^y \preceq^{(s)}_{K^*} G^y$.

  From Proposition 3.3.2, we only need to show that $H^y \subseteq G^y - K^*$ for every $y \in Y$. Before proceeding, note that the definition of $\tilde{H}$ implies

$$\forall\, y \in C : G^y \subseteq \tilde{H} \subseteq G.$$

  Hence, we have

$$\forall\, y \in C : \tilde{H}^y = G^y. \tag{3.20}$$

Now, taking into account (3.20) and the fact that $\eta \geq \beta$, we get

$$\forall\, y \in C : H^y = \frac{\beta}{2\eta}\tilde{H}^y = \frac{\beta}{2\eta}G^y \subseteq G^y - K^*.$$

On the other hand, for $y \notin C$, we have by definition that $p^* \in G^y$. Thus, we find that

$$H^y \subset H \subseteq p^* - K^* \subseteq G^y - K^*,$$

as desired.

- **Claim 4**: $H \subseteq \bigcap\limits_{y \notin -K} (G - \operatorname{int} P(y))$.

  Assume otherwise. Then,

$$\begin{aligned}
\exists\, y \notin -K,\ h^* \in H \quad &: \quad h^* \notin G - \operatorname{int} P(y) \\
&\iff \quad h^* \notin G^y - \operatorname{int} P(y) \\
&\iff \quad \langle h^*, y\rangle \geq \sigma_G(y).
\end{aligned}$$

  Since $y \notin -K$ and $G \subseteq K^*$, we have $\sigma_G(y) > 0$. Then, taking into account that $\frac{2\eta}{\beta} > 1$ and that $\frac{2\eta}{\beta}h^* \in \tilde{H} \subseteq G$, we get

$$0 < \sigma_G(y) \leq \langle h^*, y\rangle < \langle \underbrace{\frac{2\eta}{\beta}h^*}_{\in \tilde{H} \subseteq G}, y\rangle \leq \sigma_G(y),$$

  a contradiction.

This proves that (G, H) is a scalarization pair and hence, by Theorem 3.3.7, it follows that $\psi := \sigma_G - \sigma_H \in \mathcal{S}_{QD}$.

<u>**Step 3**</u>: Proving that ψ is nonconvex.

Assume that ψ is convex. By Theorem 3.3.8 (ii), this is true if and only if $G \ominus H + H = G$. In particular, since $p^* \in G$, it follows that there exists $\bar{y}^* \in Y^*$ such that

$$p^* \in \bar{y}^* + H,\ \bar{y}^* + H \subseteq G.$$

Let $u^* := p^* - \bar{y}^* \in H$. In order to arrive at a contradiction, we use the following claims:

- **Claim 5**: The inclusion $H \subseteq u^* - K^*$ holds.

  Indeeed, take any $h^* \in H$. Then, by hypothesis, we have $\bar{y}^* + h^* \in G \subseteq p^* - K^*$. Hence, we can find $k^* \in K^*$ such that $\bar{y}^* + h^* = p^* - k^*$. This implies that

$$h^* = p^* - \bar{y}^* - k^* = u^* - k^* \in u^* - K^*,$$

  which justifies the claim.

- **Claim 6**: $u^* \notin B' = \frac{\beta}{2\eta} B$.

  Otherwise, note that $B' \subseteq H \subseteq u^* - K^*$ by Claim 5. Take any $b^* \in B'$. Then, we have $\langle b^*, e\rangle = \langle u^*, e\rangle$ and $u^* - b^* \in K^*$. Thus,

$$\langle u^*, e\rangle = \langle b^*, e\rangle + \langle u^* - b^*, e\rangle \geq \langle b^*, e\rangle$$

  and, since $e \in \operatorname{int} K$, the equality holds if and only if $u^* = b^*$. Hence $B' = \{u^*\}$, which implies that $\dim K^* = 1$. Since $\operatorname{int} K^* \neq \emptyset$, this in particular means that $\dim Y^* = 1$, and hence $\dim Y = 1$, a contradiction.

- **Claim 7**: $u^* \in \operatorname{int} K^*$.

  Assume otherwise. Then, we could apply Theorem 2.1.26 (i) to obtain a functional $y^{**} \in Y^{**} \setminus \{0\}$ such that

$$\forall\, k^* \in K^* : \langle y^{**}, u^*\rangle \leq \langle y^{**}, k^*\rangle.$$

  From this we deduce that $y^{**} \in (K^*)^*$ and that $\langle y^{**}, u^*\rangle \leq 0$. This, together with Claim 5, the fact that B' is a basis of K^* and that $B' \subseteq H$, gives us

$$\forall\, b^* \in B' : \langle y^{**}, b^*\rangle \leq \langle y^{**}, u^*\rangle \leq 0.$$

  On the other hand, since B' is in particular a generator of K^*, this implies $\langle y^{**}, k^*\rangle = 0$ for any $k^* \in K^*$. Since $\operatorname{int} K^* \neq \emptyset$, this would imply that $y^{**} = 0$, a contradiction.

- **Claim 8**: $\left(\bigcup_{y \in C} G^y \cup B\right) \subseteq B \cup \operatorname{bd} K^*$.

  Assume otherwise. Then, we can find $y \in C$ and $y^* \in G^y$ such that $y^* \notin B \cup \operatorname{bd} K^*$. Since $G^y \subseteq K^*$, we have that $y^* \in \operatorname{int} K^*$. Since $y \in C$, we also get $\langle y^*, y\rangle > \langle p^*, y\rangle$. By the definition of G, we have that $k^* := p^* - y^* \in K^*$, which implies in particular that $\langle k^*, y\rangle < 0$. Then, taking into account that $y^* \notin B$, we find that $y^* - tk^* \in y^* - K^* \subseteq p^* - K^*$ and $y^* - tk^* \in B + K^*$ for $t > 0$ small enough. By the definition of G, this means that $y^* - tk^* \in G$ for $t > 0$ small enough. Thus, we deduce that

$$\langle y^* - tk^*, y\rangle = \langle y^*, y\rangle - t\langle k^*, y\rangle > \langle y^*, y\rangle,$$

  a contradiction to the fact that $y^* \in G^y$. The claim is true.

- **Claim 9**: $\exists\, \alpha > 0$ such that $\langle k^*, e\rangle \leq \langle u^*, e\rangle - \alpha$ for every $k^* \in B' \cup [(u^* - K^*) \cap \operatorname{bd} K^*]$.

  Indeed, because of Claims 6 and 7, we have $u^* \notin B'$ and $u^* \in \operatorname{int} K^*$. Hence, we can find $\delta > 0$ such that $u^* + \delta\mathbb{B} \subseteq \operatorname{int} K^*$ and $(u^* + \delta\mathbb{B}) \cap B' = \emptyset$. On the other hand, by Proposition 2.3.18 *(ii)* and Proposition 2.5.7 *(iii)*, for any $\alpha > 0$ the set

$$B_\alpha := \{y^* \in K^* \mid \langle y^*, e\rangle = \alpha\}$$

  is bounded. This implies the existence of $\alpha > 0$ such that $B_\alpha \subseteq \delta\mathbb{B}$. From Claim 5 and Claim 6 we get that $\frac{\beta}{2\eta} < \langle u^*, e\rangle$ and, hence, we can choose α small enough such that

$$\frac{\beta}{2\eta} \leq \langle u^*, e\rangle - \alpha. \tag{3.21}$$

  In particular, this means that $\langle k^*, e\rangle \leq \langle u^*, e\rangle - \alpha$ for any $k^* \in B'$. Then, we deduce that

$$\{y^* \in u^* - K^* \mid \langle y^*, e\rangle = \langle u^*, e\rangle - \alpha\} = u^* + B_\alpha \subseteq u^* + \delta\mathbb{B} \subseteq \operatorname{int} K^*. \tag{3.22}$$

  Take now any $k^* \in (u^* - K^*) \cap \operatorname{bd} K^*$. Then, we have the existence of $t \geq 0$ and $b^*_\alpha \in B_\alpha$ such that $k^* = u^* + t b^*_\alpha$. In fact, because of (3.22), we get that $t > 1$. Thus,

$$\langle k^*, e\rangle = \langle u^*, e\rangle + t\langle b^*_\alpha, e\rangle \leq \langle u^*, e\rangle - \alpha,$$

  as desired.

Finally, choose any $y \in C$. Then, according to Claim 8, we have $G^y \subseteq B \cup \operatorname{bd} K^*$, so that $\frac{\beta}{2\eta} G^y \subseteq [B' \cup \operatorname{bd} K^*] \cap H$. Furthermore, because of Claim 5, this implies

$$\frac{\beta}{2\eta} G^y \subseteq \left(B' \cup \operatorname{bd} K^*\right) \cap H \subseteq \left(B' \cup \operatorname{bd} K^*\right) \cap (u^* - K^*) \subseteq B' \cup [(u^* - K^*) \cap \operatorname{bd} K^*]. \tag{3.23}$$

Now, taking α as in Claim 9, we get $\langle h^*, e\rangle \leq \langle u^*, e\rangle - \alpha$ for any $h^* \in \frac{\beta}{2\eta} G^y \cup B'$. Since, by definition, $H = \operatorname{cl}^* \operatorname{conv}\left(\bigcup_{y \in C} \frac{\beta}{2\eta}\left(G^y \cup B\right)\right)$, we find that

$$\forall\, h^* \in H : \langle h^*, e\rangle \leq \langle u^*, e\rangle - \alpha.$$

However, this is a contradiction to the fact that $u^* \in H$. Therefore, the functional ψ is nonconvex.

□

Chapter 4

Optimality Conditions in Set Optimization

In this chapter, we study necessary optimality conditions for the general problem $\mathcal{SOP}(F, K, \Omega)$, where the solution concept is given by the set approach. Specifically, we deal with the preorders $\preceq_K^{(l)}$ and $\preceq_K^{(u)}$, although our methodology could be extended to other set relations. Our results generalize Theorem 2.4.10 for set optimization problems with respect to the set approach, and are based on the paper by Bouza, Quintana, Tuan and Tammer [26]. The main assumption throughout this part is the following:

Assumption 2. *Let X and Y be Banach spaces, $K \subset Y$ be a proper, closed, convex, pointed, and solid cone, and $e \in \operatorname{int} K$. Let $\Omega \subseteq X$ be nonempty and closed, and fix $\bar{x} \in \Omega$. Furthermore, let $F : X \rightrightarrows Y$ be a set-valued mapping such that $\Omega \subseteq \operatorname{int} \operatorname{dom} F$.*

The chapter is organized as follows. In Section 4.1, we present a literature review on optimality conditions for set optimization problems, and comment on some of the advantages that our results have over the existing ones. In Section 4.2, we derive properties of convexity and Lipschitzianity of suitable scalarizing functionals under the same assumptions on the set-valued objective mapping. Sections 4.3 and 4.4 are devoted to obtaining upper estimates of the limiting subdifferential of these scalarizing functionals. These upper estimates are then employed in Section 4.5 to derive the optimality conditions for set optimization problems. In Section 4.6, we derive Karush-Kuhn-Tucker type optimality conditions for a class of convex problems given by functional constraints.

4.1 Literature Review

The literature on the topic of optimality conditions for set optimization problems with respect to the set approach is very rich and different results have been obtained using objects of generalized differentiation lying in both the primal and dual spaces. The techniques employed in the primal space are mainly based on some type of directional derivatives and can be roughly separated into the following classes:

- Directional derivatives based on set differences [38, 91, 101, 140].

  The main idea is to consider a suitable operation that resembles subtraction in the power set of the image space. These operations are based on the well known differences of sets of Minkowski and Demyanov [71, 143], but usually slight modifications are introduced in order to make it useful in set optimization. Then, with the help of the set difference, a directional derivative is defined as a limit of an associated incremental quotient. Furthermore, the optimality conditions obtained in this setting establish the nonnegativity of the directional derivative, according to the treated set relation.

- Directional derivatives based on a distance type functional [69, 70].

  In contrast to the previous technique, a directional derivative is introduced in [69] with the help of the standard algebraic difference of sets and a distance type functional. The distance functional is a modification of the well known Hausdorff distance for sets and is based on the classical Hiriart-Urruty functional studied in Chapter 3. The directional derivative is, in this case, defined as the minimal set of some compact set to which the incremental quotient converges (in the sense of the modified distance). A similar idea is used in [70] to introduce a concept of slope for a set-valued mapping at a given point, together with necessary conditions for minimal solutions of the set optimization problem in the convex case.

- Directional derivatives based on embedding [11, 121].

  The idea in [11, 121] is to embed the class of convex and bounded sets (with respect to the ordering cone) into a suitable normed space. With this construction, the original set optimization problem is equivalent to a standard vector optimization problem having as a target function the composition of the embedding map and the set-valued objective mapping. Hence, a directional derivative of the set-valued mapping is defined in a standard way as the directional derivative of this composition.

- Directional derivatives of selections of the set-valued objective mapping [2, 3].

  In this approach, there is no explicit definition of directional derivatives for a set-valued mapping, but rather they use those of its continuous selections. Roughly speaking, the optimality conditions establish the nonnegativity, in the sense of the ordering cone, of these directional derivatives.

- Contingent derivatives and variations [109, 113, 137, 142].

  Contingent derivatives and epiderivatives have been successfully employed in obtaining optimality conditions for set optimization problems with respect to the vector approach [89]. Consequently, it was a natural idea to apply these tools also in the set approach setting. In this direction, other modifications of the derivatives were also studied, like those of Shi [147] and Studniarski [148].

On the other hand, to the best of our knowledge, optimality conditions using objects of generalized differentiation lying in the dual space have been considered only twice in the literature [93, 109]. In particular:

- In [93], the case in which the set-valued mapping is given by functional constraints was analyzed. Using a vectorization result by Jahn [92], the set-valued problem was transformed into a vector-valued one (with an infinite dimensional image space), and hence classical optimality conditions for vector optimization problems were applied.

- In [109] the idea is that, under different assumptions, set approach solutions of the set-valued problem are also solutions based on the vector approach. Hence, under these assumptions, the optimality conditions in Theorem 2.4.20 for vector approach solutions are also applicable in the context of the set approach.

We want to mention however, that some of these optimality conditions are derived under somewhat strong assumptions on the set-valued objective mapping. For example, in [2, 3, 109, 113, 137, 142], it is required that the optimal set has a strongly minimal element in order to verify optimality. In addition, either the convexity or compactness (mostly both) of the images of the set-valued objective mapping are needed in [38, 69, 70, 91, 93, 140].

Recently, it also caught our attention that, independently, Amahroq and Oussarhan in [4, 5] and Huerga, Jiménez and Novo in [84] were working with similar ideas to ours for deriving optimality conditions in set optimization. The main differences between the results derived in these papers and our optimality conditions are the following:

- In [4, 5, 84], the authors studied only solution concepts based on the lower set less relation. In this chapter, we also examine solution concepts based on the upper set less relation.

- In [5, 84], the case in which the set-valued objective mapping is convex was analyzed under stronger assumptions to ours. In addition, the optimality conditions in [5] require that the optimal set has a strongly minimal element. In our results for the convex case, we have no assumption on the structure of the minimal elements of the optimal set.

- In [4], the case in which the set-valued objective mapping is locally Lipschitzian is studied. However, the authors assume the compactness of the images of the set-valued objective mapping. Furthermore, the optimality conditions are not established using the initial data, but rather they are expressed in a limiting form. Also in [84], certain compactness assumptions concerning the involved set-valued mappings are supposed. In this chapter, we derive our main results in terms of the initial data, and we do not impose any convexity or compactness condition on the set-valued objective mapping.

4.2 Properties of Two Classes of Scalarizing Functionals in Set Optimization

With the purpose of deriving optimality conditions for set optimization problems we introduce in this section, for a given set-valued mapping F, two associated functionals. We then proceed to show that these functionals inherit the convexity and Lipschitz property from F. It is natural to think that the scalarizing functionals for set optimization problems are based on those for vector optimization. In that case, according to our discussion in Chapter 3 (specifically, Remark 3.2.10), it makes sense to consider, under Assumption 2, the functional ψ_e defined by (2.28). This, together with the characterization of the set relations presented in Theorem 2.5.9, motivates our next definition.

Definition 4.2.1. *Let Assumption 2 be fulfilled.*

(i) The lower inner function $g_l : X \times Y \to \overline{\mathbb{R}}$ is defined as

$$g_l(x, z) := \inf_{y \in F(x)} \psi_e(y - z). \tag{4.1}$$

(ii) The upper inner function $g_{u,\bar{x}} : Y \to \overline{\mathbb{R}}$ is defined as

$$g_{u,\bar{x}}(y) := \inf_{\bar{y} \in F(\bar{x})} \psi_e(y - \bar{y}). \tag{4.2}$$

(iii) For $r \in \{l, u\}$, the functional $f_{r,\bar{x}} : X \to \overline{\mathbb{R}}$ is defined as follows:

$$f_{r,\bar{x}}(x) := \begin{cases} \sup\limits_{\bar{y} \in F(\bar{x})} g_l(x, \bar{y}) = \sup\limits_{\bar{y} \in F(\bar{x})} \inf\limits_{y \in F(x)} \psi_e(y - \bar{y}) & \text{if } r = l, \\ \sup\limits_{y \in F(x)} g_{u,\bar{x}}(y) = \sup\limits_{y \in F(x)} \inf\limits_{\bar{y} \in F(\bar{x})} \psi_e(y - \bar{y}) & \text{if } r = u. \end{cases} \tag{4.3}$$

As mentioned at the beginning of the section, we now show that for preorders $\preceq_K^{(l)}$ and $\preceq_K^{(u)}$, the corresponding scalarizing functional inherits the convexity property of the set-valued mapping. We start with a simple proposition.

Proposition 4.2.2. *Let Assumption 2 be fulfilled and consider the functionals given in Definition 4.2.1. Then, the following statements are true:*

(i) For every $x \in X$, the functional $g_l(x, \cdot)$ is $-K$- monotone. Furthermore, for $\bar{y} \in F(\bar{x})$, we have that $g_l(\bar{x}, \bar{y}) = 0$ if and only if $\bar{y} \in \mathrm{WMin}(F(\bar{x}), K)$.

(ii) The functional $g_{u,\bar{x}}$ is K- monotone. Furthermore, for $y \in F(\bar{x})$, we have that $g_{u,\bar{x}}(y) = 0$ if and only if $y \in \mathrm{WMax}(F(\bar{x}), K)$.

(iii) For any $r \in \{l, u\}$, we have $f_{r,\bar{x}}(\bar{x}) \leq 0$. Equality holds if $r = l$ and $\mathrm{WMin}(F(\bar{x}), K) \neq \emptyset$, or $r = u$ and $\mathrm{WMax}(F(\bar{x}), K) \neq \emptyset$.

Proof. (i) The monotonicity of $g_l(x,\cdot)$ follows directly from the monotonicity of ψ_e. Now fix $\bar{y} \in F(\bar{x})$. Then, we have $g_l(\bar{x},\bar{y}) \leq 0$ and hence

$$\begin{aligned} g_l(\bar{x},\bar{y}) = 0 &\iff \inf_{y \in F(\bar{x})} \psi_e(y-\bar{y}) \geq 0 \\ &\iff \forall\, y \in F(\bar{x}):\ \psi_e(y-\bar{y}) \geq 0 \\ &\iff \forall\, y \in F(\bar{x}): y-\bar{y} \notin -\operatorname{int} K \\ &\iff \bar{y} \in \operatorname{WMin}(F(\bar{x}),K). \end{aligned}$$

(ii) The monotonicity of $g_{u,\bar{x}}$ is easily deduced from the monotonicity of ψ_e. Now take $y \in F(\bar{x})$. Then, we always have $g_{u,\bar{x}}(y) \leq 0$. Analogous to (i) we get

$$\begin{aligned} g_{u,\bar{x}}(y) = 0 &\iff \inf_{\bar{y} \in F(\bar{x})} \psi_e(y-\bar{y}) \geq 0 \\ &\iff \forall\, \bar{y} \in F(\bar{x}):\ \psi_e(y-\bar{y}) \geq 0 \\ &\iff \forall\, \bar{y} \in F(\bar{x}): y-\bar{y} \notin -\operatorname{int} K \\ &\iff y \in \operatorname{WMax}(F(\bar{x}),K), \end{aligned}$$

as desired.

(iii) The fact that $f_{r,\bar{x}}(\bar{x}) \leq 0$ is trivial. If $r = l$ and $\tilde{y} \in \operatorname{WMin}(F(\bar{x}),K) \neq \emptyset$ then, by statement (i), we get $g_l(\bar{x},\tilde{y}) = 0$. From this we deduce that $f_{l,\bar{x}}(\bar{x}) \geq g_l(\bar{x},\tilde{y}) = 0$, and hence the equality holds. Analogously, if $r = u$ and $\tilde{y} \in \operatorname{WMax}(F(\bar{x}),K) \neq \emptyset$ then, by statement (ii), we get $g_{u,\bar{x}}(\tilde{y}) = 0$. Again, this implies that $f_{u,\bar{x}}(\bar{x}) \geq 0$, and hence the equality.

□

Remark 4.2.3. *Proposition 4.2.2 (i) together with Proposition 2.5.6 (ii) gives us monotonicity properties of the functionals $g_l(x,\cdot)$ and ψ_e. From this, it easily follows that the functionals g_l and $f_{l,\bar{x}}$ are invariant under replacement of F by any set-valued mapping of the form $F_A := F + A$, with $A \subseteq K$ and $0 \in A$. In particular, this is true when we replace F by $\mathcal{E}_F$.*

Similarly, from Proposition 4.2.2 (ii) and Proposition 2.5.6 (ii) we deduce that the functionals $f_{u,\bar{x}}$ and $g_{u,\bar{x}}$ are invariant under replacement of F by any set-valued mapping of the form $F_A := F - A$, with $A \subseteq K$ and $0 \in A$.

The next lemma proves the convexity of the inner functions given by (4.1) and (4.2) under different convexity assumptions on F.

Lemma 4.2.4. *Let Assumption 2 be fulfilled and consider the lower and upper inner functions given in Definition 4.2.1. The following statements hold:*

(i) If F is $\preceq_K^{(l)}$- convex, then $g_l(\cdot,\bar{y})$ is convex for every $\bar{y} \in F(\bar{x})$. Furthermore, if F is locally $\preceq_K^{(l)}$- bounded at $\bar{x}$, then $\bar{x} \in \operatorname{int}\operatorname{dom} g_l(\cdot,\bar{y})$ and $g_l(\cdot,\bar{y})$ is continuous at $\bar{x}$.

(ii) If the set $\mathcal{H}_F(\bar{x})$ is convex and K- bounded above, then $g_{u,\bar{x}}$ is a convex K- monotone functional that is continuous on Y.

Proof. (i) Take $\bar{y} \in F(\bar{x})$, $x_1, x_2 \in X$, and $t \in (0,1)$. Let $x_t := tx_1 + (1-t)x_2$ and $F_t := tF(x_1) + (1-t)F(x_2)$. Since F is $\preceq_K^{(l)}$- convex, we have

$$F_t \subseteq F(x_t) + K. \tag{4.4}$$

We now have

$$\begin{aligned}
g_l(tx_1 + (1-t)x_2, \bar{y}) &= \inf_{y \in F(x_t)} \psi_e(y - \bar{y}) \\
&\overset{(\text{Proposition 2.5.6 } (ii))}{=} \inf_{y \in F(x_t)+K} \psi_e(y - \bar{y}) \\
&\overset{((4.4))}{\leq} \inf_{y \in F_t} \psi_e(y - \bar{y}) \\
&= \inf_{(y_1,y_2) \in F(x_1) \times F(x_2)} \psi_e(ty_1 + (1-t)y_2 - \bar{y}) \\
&= \inf_{(y_1,y_2) \in F(x_1) \times F(x_2)} \psi_e(t(y_1 - \bar{y}) + (1-t)(y_2 - \bar{y})) \\
&\overset{(\text{Proposition 2.5.6 } (i))}{\leq} \inf_{(y_1,y_2) \in F(x_1) \times F(x_2)} t\psi_e(y_1 - \bar{y}) + (1-t)\psi_e(y_2 - \bar{y}) \\
&= tg_l(x_1, \bar{y}) + (1-t)g_l(x_2, \bar{y}).
\end{aligned}$$

Now, let us assume that F is locally $\preceq_K^{(l)}$- bounded at $\bar{x}$. Hence, we can find $\alpha > 0$ and a neighborhood U of $\bar{x}$ such that

$$\forall\, x \in U: \; F(x) \cap (-\alpha e + K) \cap (\alpha e - K) + K = F(x) + K.$$

By the monotonicity of ψ_e in Proposition 2.5.6 (ii) we have, for every $x \in U$:

$$\begin{aligned}
-\infty &< \psi_e(-\alpha e - \bar{y}) \\
&= \inf_{y \in -\alpha e + K} \psi_e(y - \bar{y}) \\
&\leq \inf_{y \in F(x)+K} \psi_e(y - \bar{y}) \\
&= g_l(x, \bar{y}) \\
&= \inf_{y \in F(x) \cap (\alpha e - K)} \psi_e(y - \bar{y}) \\
&\leq \sup_{y \in F(x) \cap (\alpha e - K)} \psi_e(y - \bar{y}) \\
&\leq \psi_e(\alpha e - \bar{y}) \\
&< +\infty.
\end{aligned} \tag{4.5}$$

This shows that $g_l(\cdot, \bar{y})$ is finite and bounded above around $\bar{x}$. The continuity follows then from Proposition 2.3.18 (i).

(ii) The monotonicity of $g_{u,\bar{x}}$ was already established in Proposition 4.2.2 (ii). In order to show the convexity, we check that $\operatorname{epi} g_{u,\bar{x}}$ is convex. Indeed, take $(y_1, t_1), (y_2, t_2) \in \operatorname{epi} g_{u,\bar{x}}$ and $t \in (0,1)$. Hence, $g_{u,\bar{x}}(x_1) \leq t_1$ and $g_{u,\bar{x}}(x_2) \leq t_2$. Then, for any $\epsilon > 0$, we have

$$g_{u,\bar{x}}(x_1) < t_1 + \epsilon, \quad g_{u,\bar{x}}(x_2) < t_2 + \epsilon.$$

We can now find $\bar{y}_1, \bar{y}_2 \in F(\bar{x})$ such that

$$\psi_e(y_1 - \bar{y}_1) < t_1 + \epsilon, \quad \psi_e(y_2 - \bar{y}_2) < t_2 + \epsilon.$$

From this, we get

$$\begin{aligned} \psi_e((ty_1 + (1-t)y_2) - (t\bar{y}_1 + (1-t)\bar{y}_2)) &= \psi_e(t(y_1 - \bar{y}_1) + (1-t)(y_2 - \bar{y}_2)) \\ &\leq t\psi_e(y_1 - \bar{y}_1) + (1-t)\psi_e(y_2 - \bar{y}_2) \\ &\leq t(t_1 + \epsilon) + (1-t)(t_2 + \epsilon) \\ &= tt_1 + (1-t)t_2 + \epsilon. \end{aligned}$$

Now, because $F(\bar{x}) - K$ is convex, we have

$$t\bar{y}_1 + (1-t)\bar{y}_2 \in \operatorname{conv}(F(\bar{x})) \subseteq \mathcal{H}_F(\bar{x}),$$

and hence we can find $\bar{y} \in F(\bar{x})$ such that $t\bar{y}_1 + (1-t)\bar{y}_2 \in \bar{y} - K$. Then, by monotonicity of ψ_e, we get

$$\begin{aligned} g_{u,\bar{x}}(ty_1 + (1-t)y_2) &\leq \psi_e(ty_1 + (1-t)y_2 - \bar{y}) \\ &\leq \psi_e((ty_1 + (1-t)y_2) - (t\bar{y}_1 + (1-t)\bar{y}_2)) \\ &\leq tt_1 + (1-t)t_2 + \epsilon. \end{aligned}$$

Since $\epsilon > 0$ was chosen arbitrarily, we conclude that $(ty_1 + (1-t)y_2, tt_1 + (1-t)t_2) \in \operatorname{epi} g_{u,\bar{x}}$. This means that $\operatorname{epi} g_{u,\bar{x}}$ is a convex set, as desired.

Now, since $\mathcal{H}_F(\bar{x})$ is K- bounded above, we have

$$-\infty < \psi(y - \alpha e) = \inf_{\bar{y} \in \alpha e - K} \psi_e(y - \bar{y}) \leq \inf_{\bar{y} \in \mathcal{H}_F(\bar{x})} \psi_e(y - \bar{y}) \overset{\text{(Remark 4.2.3)}}{=} g_{u,\bar{x}}(y).$$

This means that $g_{u,\bar{x}}$ is finite on Y. The continuity of $g_{u,\bar{x}}$ is now deduced by fixing $\bar{y} \in F(\bar{x})$ and noticing that $g_{u,\bar{x}}(\cdot) \leq \psi_e(\cdot - \bar{y})$, a continuous convex functional. □

We are now ready to establish the convexity of the functionals $f_{l,\bar{x}}$ and $f_{u,\bar{x}}$, under the assumption that F is $\preceq_K^{(l)}$- convex ($\preceq_K^{(u)}$- convex, respectively), see Definition 2.3.2.

Theorem 4.2.5. *Let Assumption 2 be fulfilled and, for $r \in \{l, u\}$, consider the functional $f_{r,\bar{x}}$ given in Definition 4.2.1 (iii). The following statements hold:*

(i) If F is $\preceq_K^{(l)}$- convex then $f_{l,\bar{x}}$ is convex. Furthermore, if F is locally $\preceq_K^{(l)}$- bounded at $\bar{x}$, then $\bar{x} \in \operatorname{int}\operatorname{dom} f_{l,\bar{x}}$ and $f_{l,\bar{x}}$ is continuous at $\bar{x}$.

(ii) If F is $\preceq_K^{(u)}$- convex and $\mathcal{H}_F(\bar{x})$ is a convex set, then $f_{u,\bar{x}}$ is convex. Furthermore, if F is locally K- bounded above at $\bar{x}$, then $\bar{x} \in \operatorname{int}\operatorname{dom} f_{u,\bar{x}}$ and $f_{u,\bar{x}}$ is continuous at $\bar{x}$.

Proof. (i) We have

$$f_{l,\bar{x}}(x) = \sup_{\bar{y} \in F(\bar{x})} g_l(x, \bar{y}).$$

By Lemma 4.2.4 (i), for every $\bar{y} \in F(\bar{x})$, the functional $g_l(\cdot, \bar{y})$ is convex. Hence, $f_{l,\bar{x}}$ is convex as it is the supremum of convex functionals. According to Proposition 2.3.18 (i), to prove the second part it suffices to show that $f_{l,\bar{x}}$ is finite and upper bounded on a neighborhood of $\bar{x}$. In order see this, note that the assumptions on the second part of Lemma 4.2.4 (i) are fulfilled. Hence, from (4.5) we get the existence of $\alpha > 0$ and neighborhood U of $\bar{x}$ on which

$$\forall\, x \in U : -\infty < g_l(x, \bar{y}) \leq \psi_e(\alpha e - \bar{y}). \tag{4.6}$$

Taking the supremum over $\bar{y} \in F(\bar{x})$ in (4.6), we get

$$\forall\, x \in U : -\infty < f_{l,\bar{x}}(x) \leq \sup_{\bar{y} \in F(\bar{x})} \psi_e(\alpha e - \bar{y}). \tag{4.7}$$

Since F is locally $\preceq_K^{(l)}$- bounded at $\bar{x}$, it is easy to verify in particular that $F(\bar{x}) \subseteq -\alpha e + K$. By the monotonicity of ψ_e in Proposition 2.5.6 (ii), we now obtain

$$\sup_{\bar{y} \in F(\bar{x})} \psi_e(\alpha e - \bar{y}) \leq \psi_e(2\alpha e) = 2\alpha.$$

This, together with (4.7), implies that $f_{l,\bar{x}}$ is finite and upper bounded on U. The statement follows.

(ii) Let us now prove that $f_{u,\bar{x}}$ is convex. Indeed, take any $x_1, x_2 \in X$ and $t \in (0,1)$, Again, by denoting $x_t = tx_1 + (1-t)x_2$ and $F_t = tF(x_1) + (1-t)F(x_2)$, we have

$$\begin{aligned}
f_{u,\bar{x}}(x_t) &= \sup_{y \in F(x_t)} g_{u,\bar{x}}(y) \\
&\overset{\text{(Convexity of } F)}{\leq} \sup_{y \in F_t} g_{u,\bar{x}}(y) \\
&= \sup_{(y_1,y_2) \in F(x_1) \times F(x_2)} g_{u,\bar{x}}(ty_1 + (1-t)y_2) \\
&\overset{\text{(Convexity of } g_{u,\bar{x}})}{\leq} \sup_{(y_1,y_2) \in F(x_1) \times F(x_2)} tg_{u,\bar{x}}(y_1) + (1-t)g_{u,\bar{x}}(y_2) \\
&= tf_{u,\bar{x}}(x_1) + (1-t)f_{u,\bar{x}}(x_2),
\end{aligned}$$

as desired.

Now, assume that F is locally K- bounded above at $\bar{x}$ and let U be the neighborhood on which the boundedness property holds. Again, in order to prove the second part it suffices to show that $f_{u,\bar{x}}$ is finite and upper bounded on a neighborhood of $\bar{x}$. We proceed as follows: since $\bar{x} \in \operatorname{int}\operatorname{dom} F$, we can assume without loss of generality that $U \subseteq \operatorname{int}\operatorname{dom} F$. Moreover, since in particular the assumptions of Lemma 4.2.4 (ii) are fulfilled, we get that $g_{u,\bar{x}}(y) > -\infty$ for every $y \in Y$. Taking any selection θ of F on U, we deduce that

$$\forall\, x \in U: \ -\infty < g_{u,\bar{x}}(\theta(x)) \leq f_{u,\bar{x}}(x).$$

On the other hand, recall that from Lemma 4.2.4 (ii) the functional $g_{u,\bar{x}}$ is K- monotone and finite. Taking this into account and the fact that $F(x) - K \subseteq \alpha e - K$ for every $x \in U$, we obtain

$$\forall\, x \in U: \ f_{u,\bar{x}}(x) \leq \sup_{y \in \alpha e - K} g_{u,\bar{x}}(y) = g_{u,\bar{x}}(\alpha e) < +\infty.$$

The theorem is proved. □

Next, we show that the Lipschitz properties of the set-valued mapping are also transfered to the corresponding functionals. The result will be an immediate consequence of Proposition 2.3.27, Proposition 2.5.7 (i) and Proposition 4.2.2.

Lemma 4.2.6. *Let Assumption 2 be fulfilled. Consider the lower and upper inner functions g_l and $g_{u,\bar{x}}$ given in Definition 4.2.1, and let ρ be the Lipschitz constant of ψ_e given by* (2.31). *The following statements hold:*

(i) If F is Lipschitz with constant $\ell > 0$ on a neighborhood U of $\bar{x}$ and there exists $\bar{y} \in Y$ with $g_l(\bar{x}, \bar{y}) > -\infty$, then g_l is Lipschitz on $U \times Y$ with constant $\rho(1 + \ell)$. In particular, the condition $g_l(\bar{x}, \bar{y}) > -\infty$ can be replaced by $\bar{y} \in \operatorname{WMin}(F(\bar{x}), K)$.

(ii) The functional $g_{u,\bar{x}}$ is Lipschitz on Y with constant ρ if and only if $g_{u,\bar{x}}(\bar{y}) > -\infty$ for some $\bar{y} \in Y$. In particular, this is true if $\operatorname{WMax}(F(\bar{x}), K) \neq \emptyset$.

Proof. (i) Consider the set-valued mapping $\tilde{F} : X \times Y \rightrightarrows Y$ and the functional $\tilde{f} : X \times Y \times Y \to \mathbb{R}$ defined as

$$\tilde{F}(x, y) := F(x), \quad \tilde{f}(x, y, z) := \psi_e(z - y).$$

Apply now Proposition 2.3.27 (i) with $\varphi := g_l$, $F := \tilde{F}$ and $f := \tilde{f}$ to obtain the Lipschitz property of g_l. If $\bar{y} \in \operatorname{WMin}(F(\bar{x}), K)$, then it follows from Proposition 4.2.2 (i) that $g_l(\bar{x}, \bar{y}) = 0 > -\infty$.

(ii) Follows easily from the fact that $g_{u,\bar{x}}$ is the finite infimum of a fixed family of Lipschitz functionals on Y. Of course, when $\bar{y} \in \operatorname{WMax}(F(\bar{x}), K) \neq \emptyset$, we get $g_{u,\bar{x}}(\bar{y}) = 0 > -\infty$ from Proposition 4.2.2 (ii). □

We can now establish the Lipschitz property of the functionals $f_{l,\bar{x}}$ and $f_{u,\bar{x}}$.

Theorem 4.2.7. *Let Assumption 2 be fulfilled. For $r \in \{l, u\}$, consider the functional $f_{r,\bar{x}}$ given by* (4.3) *and suppose that F is locally Lipschitz at $\bar{x}$. The following statements hold:*

(i) If $\mathrm{WMin}(F(\bar{x}), K) \neq \emptyset$*, then* $f_{l,\bar{x}}$ *is locally Lipschitz at* $\bar{x}$*.*

(ii) If $\mathrm{WMax}(F(\bar{x}), K) \neq \emptyset$*, then* $f_{u,\bar{x}}$ *is locally Lipschitz at* $\bar{x}$*.*

Proof. (i) Consider the constant set-valued mapping $\tilde{F} : X \rightrightarrows Y$ given by $\tilde{F}(x) := F(\bar{x})$ for every $x \in X$. By Lemma 4.2.6 (i), we know that g_l is Lipschitz on $U \times Y$, where U is a neighborhood of $\bar{x}$ on which F is Lipschitz. Furthermore, according to Proposition 4.2.2 (iii), we have $f_{l,\bar{x}}(\bar{x}) = 0 < +\infty$. Hence, the Lipschitz property of $f_{l,\bar{x}}$ around $\bar{x}$ follows from Proposition 2.3.27 (ii) with $\varphi := f_{l,\bar{x}}, F := \tilde{F}$ and $f := g_l$.

(ii) Similarly, consider the functional $\tilde{f} : X \times Y \to \mathbb{R}$ given by $\tilde{f}(x, y) := g_{u,\bar{x}}(y)$ for every $(x, y) \in X \times Y$. From 4.2.6 (ii), we get that $\tilde{f}$ is Lipschitz on $X \times Y$. In addition, Proposition 4.2.2 (iii) implies that $f_{u,\bar{x}}(\bar{x}) = 0 < +\infty$. Hence, the Lipschitz property of $f_{u,\bar{x}}$ around $\bar{x}$ follows from Proposition 2.3.27 (ii) with $\varphi := f_{u,\bar{x}}, F := F$ and $f := \tilde{f}$.

□

4.3 Subdifferential of the Functional Associated to the Lower Set Less Relation

In this part, we derive upper estimates for the limiting subdifferential of the functional $f_{l,\bar{x}}$ studied in Section 4.2. Our upper estimates are given in terms of the coderivative of the set-valued objective map F and are based in Theorem 2.3.28 and Theorem 2.3.29. These motivates the definition of the following solution maps.

Definition 4.3.1. *Let Assumption 2 be fulfilled.*

(i) The lower inner solution map $S_F^{l,1} : X \times Y \rightrightarrows Y$ *is defined as*

$$S_F^{l,1}(x, y) := \{z \in F(x) \mid \psi_e(z - y) = g_l(x, y)\}.$$

(ii) The lower outer solution map $S_F^{l,2} : X \rightrightarrows Y$ *is defined as*

$$S_F^{l,2}(x) := \{y \in F(\bar{x}) \mid f_{l,\bar{x}}(x) = g_l(x, y)\}.$$

Remark 4.3.2. *According to Remark 4.2.3, the functionals* g_l *and* $f_{l,\bar{x}}$ *are invariant under replacement of* F *by* $\mathcal{E}_F$*. However, although the set-valued mappings* $S_F^{l,i}$ *and* $S_{\mathcal{E}_F}^{l,i}$ *are based on the same functionals (*$i = 1, 2$*), we always have* $S_F^{l,i}(\cdot) \subseteq S_{\mathcal{E}_F}^{l,i}(\cdot)$ *and the inclusions can be strict.*

We divide the analysis in two cases, corresponding to whether F is $\preceq_K^{(l)}$- convex or locally Lipschitz at $\bar{x}$. We start the study with the convex case. The next lemma shows an exact formula for the subdifferential of the inner function given in Definition 4.2.1 (i). It is worth mentioning that a similar version of this result was recently obtained in [70, Lemma 2], but assuming the separability of X.

Lemma 4.3.3. *Let Assumption 2 be fulfilled and, for* $\bar{y} \in \mathrm{WMin}(\mathcal{E}_F(\bar{x}), K)$*, consider the functional* $g_{l,\bar{y}} := g_l(\cdot, \bar{y})$*. Assume in addition that* F *is* $\preceq_K^{(l)}$*- convex and locally* $\preceq_K^{(l)}$*- bounded at* $\bar{x}$*. Then,*

$$\partial g_{l,\bar{y}}(\bar{x}) = D^* \mathcal{E}_F(\bar{x}, \bar{y}) \left[\partial \psi_e(0)\right]. \tag{4.8}$$

Proof. The result will be a simple consequence of Theorem 2.3.28. Indeed, note that according to Remark 4.2.3 we can write

$$g_{l,\bar{y}}(x) = \inf_{y \in \mathcal{E}_F(x)} f(x, y),$$

where $f : X \times Y \to \mathbb{R}$ is defined as $f(x, y) = \psi_e(y - \bar{y})$. Since F is $\preceq_K^{(l)}$- convex, we have that $\mathcal{E}_F$ is a convex set-valued mapping. It is also obvious that f is proper and convex. Moreover, by Proposition 4.2.2 (i), we have that $g_{l,\bar{y}}(\bar{x}) = 0 \neq -\infty$. According to Proposition 2.5.7 (i), f is Lipschitz on $X \times Y$ and hence the regularity condition (ii) in Theorem 2.3.28 is satisfied. In this case, the solution map is just $S_{\mathcal{E}_F}^{l,1}(\cdot, \bar{y})$. According to Proposition 4.2.2 (i) and Proposition 2.5.6 (iii), we get

$$S_{\mathcal{E}_F}^{l,1}(\bar{x}, \bar{y}) = \{y \in \mathcal{E}_F(\bar{x}) \mid \ \psi_e(y - \bar{y}) = 0\} = \mathcal{E}_F(\bar{x}) \cap (\bar{y} - \mathrm{bd}\, K). \tag{4.9}$$

Since $0 \in \mathrm{bd}\, K$, it follows that $\bar{y} \in S_{\mathcal{E}_F}^{l,1}(\bar{x}, \bar{y})$. Applying now Theorem 2.3.28, we obtain

$$\begin{aligned}
\partial g_{l,\bar{y}}(\bar{x}) &= \bigcup_{(x^*, y^*) \in \partial f(\bar{x}, \bar{y})} \left[x^* + D^* \mathcal{E}_F(\bar{x}, \bar{y})(y^*)\right] \\
&= \bigcup_{(x^*, y^*) \in \{0\} \times \partial \psi_e(0)} \left[x^* + D^* \mathcal{E}_F(\bar{x}, \bar{y})(y^*)\right] \\
&= \bigcup_{y^* \in \partial \psi_e(0)} D^* \mathcal{E}_F(\bar{x}, \bar{y})(y^*),
\end{aligned}$$

which proves the statement.

□

Lemma 4.3.4. *Let Assumption 2 be fulfilled and take points* $(\bar{x}, \bar{y}_1), (\bar{x}, \bar{y}_2) \in \mathrm{gph}\, \mathcal{E}_F$ *such that* $\bar{y}_1 \preceq_K \bar{y}_2$*. If* F *is* $\preceq_K^{(l)}$*- convex, then:*

$$\forall\, y^* \in K^* : \ D^* \mathcal{E}_F(\bar{x}, \bar{y}_2)(y^*) \subseteq D^* \mathcal{E}_F(\bar{x}, \bar{y}_1)(y^*).$$

Proof. Fix $y^* \in K^*$ and $x^* \in D^* \mathcal{E}_F(\bar{x}, \bar{y}_2)(y^*)$. Since $\bar{y}_1 - \bar{y}_2 \in -K$, we have that $\langle y^*, \bar{y}_1 - \bar{y}_2 \rangle \leq 0$. Then, for every $(x, y) \in \mathrm{gph}\, \mathcal{E}_F$, we have

$$\begin{aligned}
\langle x^*, x - \bar{x} \rangle &\leq \langle y^*, y - \bar{y}_2 \rangle \\
&= \langle y^*, y - \bar{y}_1 \rangle + \langle y^*, \bar{y}_1 - \bar{y}_2 \rangle \\
&\leq \langle y^*, y - \bar{y}_1 \rangle,
\end{aligned}$$

which implies that $(x^*, -y^*) \in N((\bar{x}, \bar{y}_1), \operatorname{gph} \mathcal{E}_F)$. The statement is proved. $\square$

The following concept was introduced in [149].

Definition 4.3.5. *Let Assumption 2 be fulfilled and consider $A \subseteq Y$. We say that A is strongly K- compact if there exists a compact set $B \subseteq A$ such that $B \in [A]^{(l)}$.*

Theorem 4.3.6. *Let Assumption 2 be fulfilled and suppose that F is $\preceq_K^{(l)}$- convex and locally $\preceq_K^{(l)}$- bounded at $\bar{x}$, and that $F(\bar{x})$ is strongly K- compact. Then,*

$$\partial f_{l,\bar{x}}(\bar{x}) = \operatorname{cl}^* \operatorname{conv} \left(\bigcup_{\bar{y} \in \operatorname{Min}(F(\bar{x}),K)} D^* \mathcal{E}_F(\bar{x}, \bar{y}) \left[\partial \psi_e(0) \right] \right).$$

Proof. Under the assumptions of the theorem we can apply Theorem 4.2.5 (i) to obtain that the functional $f_{l,\bar{x}}$ is convex and continuous at $\bar{x}$. Hence, by [139, Proposition 1.11], we have $\partial f_{l,\bar{x}}(\bar{x}) \neq \emptyset$. Since $F(\bar{x})$ is strongly K- compact, there exists a compact set $A \subseteq F(\bar{x})$ such that $A + K = F(\bar{x}) + K$. Applying [89, Lemma 4.7], we get that

$$\operatorname{Min}(F(\bar{x}), K) = \operatorname{Min}(A, K) \neq \emptyset. \tag{4.10}$$

As in Lemma 4.3.3, we consider, for $\bar{y} \in F(\bar{x})$, the functional $g_{l,\bar{y}} := g_l(\cdot, \bar{y})$. Then, according to Proposition 4.2.2 (i), the functional $g_l(x, \cdot)$ is $-K$ monotone for any $x \in X$. This implies

$$f_{l,\bar{x}}(x) = \sup_{\bar{y} \in F(\bar{x})} g_l(x, \bar{y}) = \sup_{\bar{y} \in F(\bar{x})+K} g_l(x, \bar{y}) = \sup_{\bar{y} \in A+K} g_l(x, \bar{y}) = \sup_{\bar{y} \in A} g_l(x, \bar{y}) = \sup_{\bar{y} \in A} g_{l,\bar{y}}(x).$$

The above equation implies that $f_{l,\bar{x}}$ can be expressed as the pointwise supremum of the parametric family $\{g_{l,\bar{y}}\}_{\bar{y} \in A}$. In this context, it is stated in [145, Proposition 4.5.2] an exact formula for the subdifferential of the maximum of convex functions. In order to apply this proposition, it is sufficient to verify the following statements:

- $(A, \|\cdot\|)$ is a compact Hausdorff space.

  This is obviously fulfilled because of our compactness assumption.

- For any $\bar{y} \in A$, the functional $g_{l,\bar{y}}$ is convex and continuous at $\bar{x}$.

  Since $A \subseteq F(\bar{x})$, the statement follows directly from Lemma 4.2.4 (i).

- For every $x \in X$, the functional $g_l(x, \cdot)$ is u.s.c at every point of A.

  Indeed, fix $x \in X$ and take $\bar{y} \in A, \alpha \in \mathbb{R}$ such that $g_l(x, \bar{y}) < \alpha$. This is equivalent to

  $$\inf_{y \in F(x)} \psi_e(y - \bar{y}) < \alpha,$$

  and hence we can find $y' \in F(x)$ such that $\psi_e(y' - \bar{y}) < \alpha$. Because of the continuity of ψ_e, we can find a neighborhood $V(\bar{y})$ of $\bar{y}$ such that for every $z \in V(\bar{y})$, the inequality $\psi_e(y' - z) < \alpha$ holds. This, together with the definition of $g_l(x, \cdot)$, gives us

$$\forall\, z \in V(\bar{y}) \cap A : \; g_l(x,z) \leq \psi_e(y' - z) < \alpha,$$

as desired.

Applying now [145, Proposition 4.5.2], we obtain that

$$\partial f_{l,\bar{x}}(\bar{x}) = \mathrm{cl}^* \operatorname{conv} \left(\bigcup_{\bar{y} \in \tilde{S}} \partial g_{l,\bar{y}}(\bar{x}) \right), \tag{4.11}$$

where

$$\tilde{S} = \{\bar{y} \in A \mid g_{l,\bar{y}}(\bar{x}) = f_{l,\bar{x}}(\bar{x})\}.$$

Recall that $\mathrm{WMin}(F(\bar{x}), K) \neq \emptyset$ according to (4.10). Then, by Proposition 4.2.2 (iii), we know that $f_{l,\bar{x}}(\bar{x}) = 0$. Hence, $\bar{y} \in \tilde{S}$ if and only if $g_{l,\bar{y}}(\bar{x}) = 0$. Fix $\bar{y} \in A$. Note that, because of the monotonicity of ψ_e, we have

$$g_{l,\bar{y}}(\bar{x}) = \inf_{y \in F(\bar{x})} \psi_e(y - \bar{y}) = \inf_{y \in F(\bar{x})+K} \psi_e(y - \bar{y}) = \inf_{y \in A+K} \psi_e(y - \bar{y}) = \inf_{y \in A} \psi_e(y - \bar{y}).$$

Then, following the same lines in the proof of Proposition 4.2.2 (i), we get

$$\inf_{y \in A} \psi_e(y - \bar{y}) = 0 \Longleftrightarrow \bar{y} \in \mathrm{WMin}(A, K).$$

This shows that

$$\tilde{S} = \mathrm{WMin}(A, K). \tag{4.12}$$

Furthermore, since A is compact, we can apply Proposition 2.4.7 (i) to obtain

$$A \subseteq \mathrm{Min}(A, K) + K. \tag{4.13}$$

Hence, taking into account (4.11), (4.12) and Lemma 4.3.3, we obtain

$$\partial f_{l,\bar{x}}(\bar{x}) = \mathrm{cl}^* \operatorname{conv} \left(\bigcup_{\bar{y} \in \mathrm{WMin}(A,K)} D^*\mathcal{E}_F(\bar{x}, \bar{y}) \left[\partial \psi_e(0)\right] \right). \tag{4.14}$$

By (4.13), for every $\bar{y} \in \mathrm{WMin}(A, K)$ there exists $\bar{y}_1 \in \mathrm{Min}(A, K)$ such that $\bar{y}_1 \preceq_K \bar{y}$. This, together with the fact that $\partial \psi_e(0) \subseteq K^*$, allows us to apply Lemma 4.3.4 to obtain

$$D^*\mathcal{E}_F(\bar{x}, \bar{y}) \left[\partial \psi_e(0)\right] \subseteq D^*\mathcal{E}_F(\bar{x}, \bar{y}_1) \left[\partial \psi_e(0)\right]. \tag{4.15}$$

Combining equations (4.14) and (4.15), we have

$$\begin{aligned} \partial f_{l,\bar{x}}(\bar{x}) &= \mathrm{cl}^* \operatorname{conv} \left(\bigcup_{\bar{y} \in \mathrm{WMin}(A,K)} D^*\mathcal{E}_F(\bar{x}, \bar{y}) \left[\partial \psi_e(0)\right] \right) \\ &\subseteq \mathrm{cl}^* \operatorname{conv} \left(\bigcup_{\bar{y}_1 \in \mathrm{Min}(A,K)} D^*\mathcal{E}_F(\bar{x}, \bar{y}_1) \left[\partial \psi_e(0)\right] \right). \end{aligned}$$

Since the reverse inclusion is obviously true, we obtain

$$\partial f_{l,\bar{x}}(\bar{x}) = \mathrm{cl}^* \operatorname{conv} \left(\bigcup_{\bar{y} \in \mathrm{Min}(A,K)} D^* \mathcal{E}_F(\bar{x}, \bar{y}) \left[\partial \psi_e(0) \right] \right).$$

The desired result follows from (4.10). □

Next we analyze the case on which F is locally Lipschitz at $\bar{x}$. Similar to the convex case, we start by establishing an upper estimate of the limiting subdifferential of the inner function.

Lemma 4.3.7. *Let Assumption 2 be fulfilled with X and Y being Asplund spaces, and let $\bar{y} \in \mathrm{WMin}(F(\bar{x}), K)$. Suppose also that:*

(i) F *is closed at* $\bar{x}$,

(ii) $S_F^{l,1}(x, y)$ *is inner semicompact at* $(\bar{x}, \bar{y})$,

(iii) $\mathrm{gph}\, F$ *is locally closed around every point in the set* $\{\bar{x}\} \times F(\bar{x}) \cap (\bar{y} - \mathrm{bd}\, K)$.

Then,

$$\partial g_l(\bar{x}, \bar{y}) \subseteq \bigcup_{\substack{\bar{z} \in F(\bar{x}) \cap (\bar{y} - \mathrm{bd}\, K) \\ z^* \in \partial \psi_e(\bar{z} - \bar{y})}} D^* F(\bar{x}, \bar{z})(z^*) \times \{-z^*\}. \tag{4.16}$$

Proof. Consider the set-valued mapping $\tilde{F} : X \times Y \rightrightarrows Y$ and the functional $f : X \times Y \times Y \to \mathbb{R}$ defined as

$$\tilde{F}(x, y) := F(x), \quad f(x, y, z) := \psi_e(z - y).$$

Thus, we have

$$g_l(x, y) = \inf_{z \in \tilde{F}(x,y)} f(x, y, z).$$

Now, we check that it is possible to apply Theorem 2.3.29. First, note that the associated solution map in this case is just $S_F^{l,1}$, from Definition 4.3.1. Next, observe that $g_l(\bar{x}, \bar{y}) = 0$ by Proposition 4.2.2 (i). Hence, from the representability property of ψ_e in Proposition 2.5.6 (iii), we get

$$S_F^{l,1}(\bar{x}, \bar{y}) = \{z \in F(\bar{x}) \mid \psi_e(z - \bar{y}) = 0\} = F(\bar{x}) \cap (\bar{y} - \mathrm{bd}\, K) \supseteq \{\bar{y}\} \neq \emptyset. \tag{4.17}$$

We proceed to check that the hypothesis of the theorem are fulfilled.

- $\tilde{F}$ is closed at $(\bar{x}, \bar{y})$.

  This is obviously fulfilled by the definition of $\tilde{F}$ and condition (i) above.

- $S_F^{l,1}$ is inner semicompact at $(\bar{x}, \bar{y})$.

  This is precisely condition (ii) in the lemma.

- There is a neighborhood U' of $(\bar{x}, \bar{y})$ such that f is Lipschitz on $U' \times Y$.

  This follows directly from the definition of f and Proposition 2.5.7 (i).

- $\operatorname{gph} \tilde{F}$ is locally closed around every point in the set $\{(\bar{x}, \bar{y})\} \times S_F^{l,1}(\bar{x}, \bar{y})$.

  Taking into account (4.17), the statement follows from condition (iii).

Applying now Theorem 2.3.29 we obtain

$$\partial g_l(\bar{x}, \bar{y}) \subseteq \bigcup_{\substack{\bar{z} \in F(\bar{x}) \cap (\bar{y} - \operatorname{bd} K) \\ (x^*, y^*, z^*) \in \partial f(\bar{y}, \bar{y}, \bar{z})}} \Big\{ (x^*, y^*) + D^* \tilde{F}(\bar{x}, \bar{y}, \bar{z})(z^*) \Big\}. \tag{4.18}$$

We now simplify the above inclusion. The first step will be to examine $D^* \tilde{F}(\bar{x}, \bar{y}, \bar{z})$. Note that

$$\operatorname{gph} \tilde{F} = \{(x, y, z) \mid z \in F(x)\}.$$

Hence, we obtain

$$N((\bar{x}, \bar{y}, \bar{z}), \operatorname{gph} \tilde{F}) = \{(x^*, 0, z^*) \in X^* \times Y^* \times Y^* \mid (x^*, z^*) \in N((\bar{x}, \bar{z}), \operatorname{gph} F)\}.$$

From this we deduce that

$$\begin{aligned} D^* \tilde{F}(\bar{x}, \bar{y}, \bar{z})(z^*) &= \{(x^*, 0) \in X^* \times Y^* \mid (x^*, -z^*) \in N((\bar{x}, \bar{z}), \operatorname{gph} F)\} \\ &= D^* F(\bar{x}, \bar{z})(z^*) \times \{0\}. \end{aligned} \tag{4.19}$$

Next, we compute $\partial f(\bar{x}, \bar{y}, \bar{z})$. For this, we first note that f is convex and continuous at every point. Considering the operator $T : X \times Y \to Y$ defined as $T(x, y, z) := z - y$, we get $f = \psi_e \circ T$. Since T is linear and bounded, and ψ_e is convex and continuous, we can apply the chain rule in Proposition 2.3.19 (ii) to obtain

$$\partial f(\bar{x}, \bar{y}, \bar{z}) = \partial f(\bar{x}, \bar{y}, \bar{z}) = T^*[\partial \psi_e(\bar{z} - \bar{y})] = T^*[\partial \psi_e(\bar{z} - \bar{y})].$$

Moreover, it is easy to check that $T^*(z^*) = (0, -z^*, z^*)$. Hence, we get

$$\partial f(\bar{x}, \bar{y}, \bar{z}) = \{0\} \times \bigcup_{z^* \in \partial \psi_e(\bar{z} - \bar{y})} (-z^*, z^*). \tag{4.20}$$

Substituting now (4.19) and (4.20) into (4.18), the desired estimate is obtained.

□

Theorem 4.3.8. *In addition to Assumption 2, let X and Y be Asplund spaces. Suppose also that:*

(i) *F is locally Lipschitz at $\bar{x}$,*

(ii) $\operatorname{WMin}(F(\bar{x}), K) \neq \emptyset$,

(iii) *F is closed at $\bar{x}$,*

(iv) $S_F^{l,1}$ *is inner semicompact at every point of* $\{\bar{x}\} \times \mathrm{WMin}(F(\bar{x}), K)$,

(v) $S_F^{l,2}$ *is inner semicompact at* $\bar{x}$,

(vi) $\operatorname{gph} F$ *is locally closed around every point in the set* $\{\bar{x}\} \times \mathrm{WMin}(F(\bar{x}), K)$.

Then,

$$\partial f_{l,\bar{x}}(\bar{x}) \subseteq \mathrm{cl}^* \operatorname{conv}\left(\bigcup_{\bar{y} \in \mathrm{WMin}(F(\bar{x}),K)} \left\{x^* \in X^* \mid \exists\, y^* \in N(\bar{y}, F(\bar{x})) : (x^*, y^*) \in G_{(\bar{x},\bar{y})}\right\}\right), \tag{4.21}$$

where

$$G_{(\bar{x},\bar{y})} = \mathrm{cl}^* \operatorname{conv}\left(\bigcup_{\substack{\bar{z} \in F(\bar{x}) \cap (\bar{y} - \operatorname{bd} K) \\ z^* \in \partial \psi_e(\bar{z} - \bar{y})}} D^* F(\bar{x}, \bar{z})(z^*) \times \{-z^*\}\right).$$

Proof. Consider the (constant) set-valued mapping $\tilde{F} : X \rightrightarrows Y$ defined as $\tilde{F}(x) := F(\bar{x})$ for every $x \in X$. Then, we can write

$$f_{l,\bar{x}}(x) = \sup_{y \in \tilde{F}(x)} g_l(x, y).$$

Next, note that the solution map in this case is $S_F^{l,2}$. Furthermore, as a consequence of (ii) and Proposition 4.2.2 (iii), we obtain $f_{l,\bar{x}}(\bar{x}) = 0$. The definition of $\tilde{F}$ and g_l allow us then to apply Proposition 4.2.2 (i) to obtain that

$$S_F^{l,2}(\bar{x}) = \mathrm{WMin}(F(\bar{x}), K).$$

We now check that it is possible to apply Theorem 2.3.29 to obtain an upper estimate of Mordukhovich's subdifferential of $f_{l,\bar{x}}$ at $\bar{x}$.

- $\tilde{F}$ is closed at $\bar{x}$.

  It is easy to see that the closedness of $\tilde{F}$ at $\bar{x}$ is equivalent to the closedness of the set $F(\bar{x})$. The statement follows from condition (iii).

- $S_F^{l,2}$ is inner semicompact at $\bar{x}$.

  This is precisely condition (v).

- There is a neighborhood U of $\bar{x}$ such that g_l is Lipschitz on $U \times Y$.

  This follows from conditions $(i), (ii)$ and Lemma 4.2.6 (i).

- $\operatorname{gph} \tilde{F}$ is locally closed around every point of the set $\{\bar{x}\} \times S_F^{l,2}(\bar{x})$.

  Again, this is deduced from the fact that $F(\bar{x})$ is a closed set, which is implied by (iii).

Hence, taking into account the Lipschitz property of $f_{l,\bar{x}}$ from Theorem 4.2.7 (i), we obtain:

$$\begin{aligned}
\partial f_{l,\bar{x}}(\bar{x}) &= \partial\left(-\inf_{y\in\tilde{F}(\cdot)} -g_l(\cdot,y)\right)(\bar{x}) \\
&\overset{\text{(Proposition 2.3.20)}}{\subseteq} -\mathrm{cl}^*\,\mathrm{conv}\left(\partial\left(\inf_{y\in\tilde{F}(\cdot)} -g_l(\cdot,y)\right)(\bar{x})\right) \\
&\overset{\text{(Theorem 2.3.29)}}{\subseteq} -\mathrm{cl}^*\,\mathrm{conv}\left(\bigcup_{\substack{\bar{y}\in S_{\tilde{F}}^{l,2}(\bar{x})\\ (x^*,y^*)\in\partial(-g_l)(\bar{x},\bar{y})}} \left[x^* + D^*\tilde{F}(\bar{x},\bar{y})(y^*)\right]\right). \qquad (4.22)
\end{aligned}$$

Now we examine $D^*\tilde{F}(\bar{x},\bar{y})$ for any $(\bar{x},\bar{y})\in X\times Y$. Since $\mathrm{gph}\,\tilde{F} = X\times F(\bar{x})$, we get in this case $N((\bar{x},\bar{y}),\mathrm{gph}\,\tilde{F}) = \{0\}\times N(\bar{y},F(\bar{x}))$. From this, we deduce that

$$D^*\tilde{F}(\bar{x},\bar{y})(y^*) = \begin{cases} \{0\} & \text{if } y^*\in -N(\bar{y},F(\bar{x})), \\ \emptyset & \text{otherwise}. \end{cases}$$

Plugging this back into (4.22) and taking into account that $S_{\tilde{F}}^{l,2}(\bar{x}) = \mathrm{WMin}(F(\bar{x}),K)$, we obtain

$$\partial f_{l,\bar{x}}(\bar{x}) \subseteq \mathrm{cl}^*\,\mathrm{conv}\left(\bigcup_{\bar{y}\in\mathrm{WMin}(F(\bar{x}),K)} \left\{x^*\in X^* \mid \exists\, y^*\in N(\bar{y},F(\bar{x})) : -(x^*,y^*)\in\partial(-g_l)(\bar{x},\bar{y})\right\}\right). \qquad (4.23)$$

On the other hand, taking into account the Lipschitz property of g_l from Lemma 4.2.6 (i), for every $\bar{y}\in\mathrm{WMin}(F(\bar{x}),K)$ we also have:

$$\begin{aligned}
\partial(-g_l)(\bar{x},\bar{y}) &\overset{\text{(Proposition 2.3.20)}}{\subseteq} -\mathrm{cl}^*\,\mathrm{conv}\,(\partial g_l(\bar{x},\bar{y})) \\
&\overset{\text{(Lemma 4.3.3)}}{\subseteq} -\mathrm{cl}^*\,\mathrm{conv}\left(\bigcup_{\substack{\bar{z}\in F(\bar{x})\cap(\bar{y}-\mathrm{bd}\,K)\\ z^*\in\partial\psi_e(\bar{z}-\bar{y})}} D^*F(\bar{x},\bar{z})(z^*)\times\{-z^*\}\right) \qquad (4.24)
\end{aligned}$$

Finally, by putting (4.24) back into (4.23), we obtain our desired estimate. □

Remark 4.3.9. *According to Remark 4.2.3, the scalarizing functional $f_{l,\bar{x}}$ would remain unchanged if we substitute F by a set-valued mapping $\tilde{F} : X \rightrightarrows Y$ of the form $\tilde{F}(x) = F(x) + A$, with $A\subseteq K$ and $0\in A$. Hence, in Theorem 4.3.8 we can substitute F by any other set-valued mapping $\tilde{F}$ of the above form. By doing this, we can obtain different (maybe sharper) upper estimates of $\partial f_{l,\bar{x}}(\bar{x})$. This is worth keeping in mind when obtaining optimality conditions for set optimization problems, as these are based on the subdifferential of $f_{l,\bar{x}}(\bar{x})$, see Section 4.5.*

Remark 4.3.10. *Note that, since the upper estimate of $\partial f_{l,\bar{x}}(\bar{x})$ obtained in (4.21) is convex, it also constitutes an upper estimate of $\partial^\circ f_{l,\bar{x}}(\bar{x})$ according to [132, Theorem 3.57]. However, as we will see in Example 4.5.7, when applying this result to optimality conditions for set optimization problems, the convexity of the upper estimate can not be removed very easily.*

The following corollary shows that if Y is finite dimensional our assumptions in Theorem 4.3.8 are natural.

Corollary 4.3.11. *Let Assumption 2 be fulfilled with X being an Asplund space. Suppose that Y is finite dimensional and that* $\operatorname{gph} F$ *is closed. Furthermore, assume that F is locally Lipschitz and locally bounded at $\bar{x}$. Then, inclusion* (4.21) *holds.*

Proof. Since $\operatorname{gph} F$ is closed, in particular we have that F is closed valued. This, together with the local boundedness at $\bar{x}$ and the finite dimensionality of Y, gives us the compactness of $F(\bar{x})$. Hence, according to Proposition 2.4.7, we have $\operatorname{WMin}(F(\bar{x}), K) \neq \emptyset$. Furthermore, the local boundedness of F at $\bar{x}$ also implies that of the set-valued mappings $S_F^{l,1}$ and $S_F^{l,2}$ in the statement of Theorem 4.3.8. This, together with the fact that Y is finite dimensional gives us the inner semicompactness of $S_F^{l,1}$ and $S_F^{l,2}$. Thus, all the conditions of Theorem 4.3.8 are satisfied. The statement follows.

□

4.4 Subdifferential of the Functional Associated to the Upper Set Less Relation

In this section, we compute an approximation of the limiting subdifferential of the functional $f_{u,\bar{x}}$ given in Definition 4.2.1 at the point $\bar{x}$. We start again by defining two useful solution maps.

Definition 4.4.1. *Let Assumption 2 be fulfilled.*

(i) The upper inner solution map $S_F^{u,1} : Y \rightrightarrows Y$ is defined as

$$S_F^{u,1}(y) := \{z \in F(\bar{x}) \mid g_{u,\bar{x}}(y) = \psi_e(y - z)\}.$$

(ii) The upper outer solution map $S_F^{u,2} : X \rightrightarrows Y$ is defined as

$$S_F^{u,2}(x) := \{y \in F(x) \mid f_{u,\bar{x}}(x) = g_{u,\bar{x}}(y)\}.$$

In the next lemma, we obtain upper estimates for the subdifferentials of the inner function in both the convex and Lipschitz cases.

Lemma 4.4.2. *Let Assumption 2 be fulfilled. The following statements hold:*

(i) Let $\bar{y} \in \operatorname{WMax}(\mathcal{H}_F(\bar{x}), K)$ and suppose that $\mathcal{H}_F(\bar{x})$ is convex and K- bounded above. Then $g_{u,\bar{x}}$ is convex, continuous at $\bar{x}$ and

$$\partial g_{u,\bar{x}}(\bar{y}) = \partial \psi_e(0) \cap N(\bar{y}, \mathcal{H}_F(\bar{x})). \tag{4.25}$$

(ii) Let X and Y be Asplund and fix $\bar{y} \in \operatorname{WMax}(F(\bar{x}), K)$. Suppose that:

(a) $F(\bar{x})$ *is closed,*

(b) $S_F^{u,1}$ *is inner semicompact at* $\bar{y}$.

Then,

$$\partial g_{u,\bar{x}}(\bar{y}) \subseteq \bigcup_{\bar{z}\in F(\bar{x})\cap(\bar{y}+\operatorname{bd}K)} \partial\psi_e(\bar{y}-\bar{z})\cap N(\bar{z},F(\bar{x})). \tag{4.26}$$

Proof. Our statements will follow from Theorem 2.3.28 and Theorem 2.3.29, respectively. In order to see this, we consider $T : Y\times Y\to Y$ and $f : Y\times Y\to\mathbb{R}$ defined respectively as

$$T(y,z) := y-z,\ f(y,z) := (\psi_e\circ T)(y,z).$$

Furthermore, we define the set-valued maps $\tilde{F},\hat{F} : Y\rightrightarrows Y$ respectively as $\tilde{F}(y) = \mathcal{H}_F(\bar{x})$ and $\hat{F}(y) = F(\bar{x})$ for every $y\in Y$. We can then write

$$g_{u,\bar{x}}(y) = \inf_{z\in\tilde{F}(y)} f(y,z),$$

with corresponding solution map $S_{\mathcal{H}_F}^{u,1}$, and

$$g_{u,\bar{x}}(y) = \inf_{z\in\hat{F}(y)} f(y,z),$$

with corresponding solution map $S_F^{u,1}$. By Proposition 4.2.2 *(ii)* and Proposition 2.5.6 *(iii)*, we get

$$\begin{aligned} S_{\mathcal{H}_F}^{u,1}(\bar{y}) &= \{z\in\mathcal{H}_F(\bar{x}) \mid \psi_e(\bar{y}-z) = g_{u,\bar{x}}(\bar{y})\} \\ &= \{z\in\mathcal{H}_F(\bar{x}) \mid \psi_e(\bar{y}-z) = 0\} \\ &= \mathcal{H}_F(\bar{x})\cap(\bar{y}+\operatorname{bd}K). \end{aligned}$$

In particular, we deduce that $\bar{y}\in S_{\mathcal{H}_F}^{u,1}(\bar{x})$. Similarly, we obtain

$$S_F^{u,1}(\bar{y}) = F(\bar{x})\cap(\bar{y}+\operatorname{bd}K). \tag{4.27}$$

On the other hand, it is obvious that f is convex and continuous. Moreover, for any $\bar{z}\in Y$, the chain rule in Proposition 2.3.19 *(ii)* implies

$$\partial f(\bar{y},\bar{z}) = T^*[\partial\psi_e(T(\bar{y},\bar{z}))] = T^*[\partial\psi_e(\bar{y}-\bar{z})].$$

It is easy to verify that in this case $T^*(y^*) = (y^*,-y^*)$. Hence, we get

$$\partial f(\bar{y},\bar{z}) = \bigcup_{y^*\in\partial\psi_e(\bar{y}-\bar{z})} (y^*,-y^*). \tag{4.28}$$

We proceed now to analyze each case separately.

(i) The convexity and continuity follows from Lemma 4.2.4 (ii). The subdifferential formula will be a simple application of Theorem 2.3.28 and to do so, we check that the hypothesis are fulfilled. Indeed, by assumption, $\mathcal{H}_F(\bar{x})$ is a convex set and hence $\tilde{F}$ is a convex set-valued mapping. Moreover, from Proposition 2.5.6 (i) and Proposition 2.5.7 (i), it follows that f is a proper convex function that is continuous at any point of $\operatorname{gph}\tilde{F}$ and hence, in particular, the regularity condition (ii) in Theorem 2.3.28 is satisfied. As a consequence of Proposition 4.2.2 (ii), we also have that $\bar{y} \in \operatorname{dom} g_{u,\bar{x}}$ and $\operatorname{dom} g_{u,\bar{x}}(\bar{y}) = 0 < +\infty$.

Since $\bar{y} \in S^{u,1}_{\mathcal{H}_F}(\bar{x})$, we can apply now Theorem 2.3.28 to obtain

$$\partial g_{u,\bar{x}}(\bar{y}) = \bigcup_{(y^*,z^*)\in\partial f(\bar{y},\bar{y})} \Big[y^* + D^*\tilde{F}(\bar{y},\bar{y})(z^*) \Big]. \tag{4.29}$$

Next, we examine the term $D^*\tilde{F}(\bar{y},\bar{y})(z^*)$ in the above formula. Note that $\operatorname{gph}\tilde{F} = Y \times \mathcal{H}_F(\bar{x})$. Hence, we get $N((\bar{y},\bar{y}), \operatorname{gph}\tilde{F}) = \{0\} \times N(\bar{y}, \mathcal{H}_F(\bar{x}))$ and from this it follows that, for any $y^* \in Y^*$:

$$\begin{aligned} D^*\tilde{F}(\bar{y},\bar{y})(-y^*) &= \{z^* \in Y^* \mid (z^*, y^*) \in N(\bar{y}, \mathcal{H}_F(\bar{x}))\} \\ &= \{z^* \in Y^* \mid (z^*, y^*) \in \{0\} \times N(\bar{y}, \mathcal{H}_F(\bar{x}))\} \\ &= \begin{cases} \{0\} & \text{if } y^* \in N(\bar{y}, \mathcal{H}_F(\bar{x})), \\ \emptyset & \text{otherwise.} \end{cases} \end{aligned}$$

Taking this into account together with (4.28), we obtain the following in (4.29):

$$\begin{aligned} \partial g_{u,\bar{x}}(\bar{y}) &= \bigcup_{y^*\in\partial\psi_e(0)} \Big[y^* + D^*\tilde{F}(\bar{y},\bar{y})(-y^*) \Big] \\ &= \bigcup_{y^*\in\partial\psi_e(0)} \left[y^* + \begin{cases} \{0\} & \text{if } y^* \in N(\bar{y}, \mathcal{H}_F(\bar{x})), \\ \emptyset & \text{otherwise} \end{cases} \right] \\ &= \partial\psi_e(0) \cap N(\bar{y}, \mathcal{H}_F(\bar{x})), \end{aligned}$$

as expected.

(ii) In this case, we will apply Theorem 2.3.29 to obtain an upper estimate of $\partial g_{u,\bar{x}}(\bar{y})$. We check that all the conditions of the theorem are fulfilled:

- $\hat{F}$ is closed at $\bar{y}$.

  This follows from condition (a).

- $S^{u,1}_F$ is inner semicompact at $\bar{y}$.

  This is just condition (b).

- There exists a neighborhood V of $\bar{y}$ such that f is Lipschitz on $V \times Y$.

  Follows directly from the Lipschitz property of ψ_e in Proposition 2.5.7 (i).

- $\operatorname{gph} \tilde{F}$ is locally closed around every point in the set $\{\bar{y}\} \times S_F^{u,1}(\bar{y})$.

  This is a consequence of (a).

Theorem 2.3.29 together with (4.27) gives us now

$$\partial g_{u,\bar{x}}(\bar{y}) \subseteq \bigcup_{\substack{\bar{z} \in F(\bar{x}) \cap (\bar{y} + \operatorname{bd} K) \\ (y^*, z^*) \in \partial f(\bar{y}, \bar{z})}} \left[y^* + D^* \hat{F}(\bar{y}, \bar{z})(z^*) \right]. \tag{4.30}$$

Analogous to the proof of statement (i), we obtain

$$D^* \hat{F}(\bar{y}, \bar{z})(z^*) = \begin{cases} \{0\} & \text{if } z^* \in -N(\bar{z}, F(\bar{x})), \\ \emptyset & \text{otherwise}. \end{cases}$$

Finally, by substituting this and (4.28) into (4.30), the desired estimate is obtained. □

Next, we state the main result of the section.

Theorem 4.4.3. *In addition to Assumption 2, let X and Y be Asplund spaces. Suppose also that:*

(i) *F is locally Lipschitz at $\bar{x}$,*

(ii) $\operatorname{WMax}(F(\bar{x}), K) \neq \emptyset$,

(iii) *F is closed at $\bar{x}$,*

(iv) *$S_F^{u,1}$ is inner semicompact at every point in the set* $\operatorname{WMax}(F(\bar{x}), K)$,

(v) *$S_F^{u,2}$ is inner semicompact at $\bar{x}$.*

(vi) $\operatorname{gph} F$ *is locally closed around any point in the set* $\{\bar{x}\} \times \operatorname{WMax}(F(\bar{x}), K)$.

Then,

$$\partial f_{u,\bar{x}}(\bar{x}) \subseteq -\operatorname{cl}^* \operatorname{conv} \left(\bigcup_{\bar{y} \in \operatorname{WMax}(F(\bar{x}), K)} D^* F(\bar{x}, \bar{y}) \left[H_{(\bar{x}, \bar{y})} \right] \right), \tag{4.31}$$

where

$$H_{(\bar{x}, \bar{y})} := -\operatorname{cl}^* \operatorname{conv} \left(\bigcup_{\bar{z} \in F(\bar{x}) \cap (\bar{y} + \operatorname{bd} K)} \partial \psi_e(\bar{y} - \bar{z}) \cap N\left(\bar{z}, F(\bar{x})\right) \right).$$

Proof. Consider the function $f : X \times Y \to Y$ defined as $f(x, y) = g_{u,\bar{x}}(y)$. By definition, we have

$$f_{u,\bar{x}}(x) = \sup_{y \in F(x)} f(x, y).$$

We verify that we can apply Theorem 2.3.29. First, note that the solution map in this case is just $S_F^{u,2}$. Hence, Proposition 4.2.2 (iii) can be applied to obtain $f_{u,\bar{x}}(\bar{x}) = 0$. Then, from Proposition 4.2.2 (ii) we get

$$S_F^{u,2}(\bar{x}) = \mathrm{WMax}(F(\bar{x}), K) \neq \emptyset. \tag{4.32}$$

We proceed to check the rest of the assumptions:

- F is closed at $\bar{x}$,

  This is just condition (iii) in the theorem.

- $S_F^{u,2}$ is inner semicompact at $\bar{x}$,

  This is exactly condition (v) in our theorem.

- There is a neighborhood U of $\bar{x}$ such that f is Lipschitz on $U \times Y$.

  Follows directly from condition (ii) and Lemma 4.2.6 (ii).

- $\operatorname{gph} F$ is locally closed around every point in the set $\{\bar{x}\} \times S_F^{u,2}(\bar{x})$.

  This follows from (4.32) and condition (vi) in the theorem.

Hence, taking into account the Lipschitz property of $f_{u,\bar{x}}$ from Theorem 4.2.7 (ii), we obtain:

$$\begin{aligned} \partial f_{u,\bar{x}}(\bar{x}) &= \partial\left(-\inf_{y\in F(\cdot)} -f(\cdot,y)\right)(\bar{x}) \\ &\overset{\text{(Proposition 2.3.20)}}{\subseteq} -\mathrm{cl}^*\operatorname{conv}\left(\partial\left(\inf_{y\in F(\cdot)} -f(\cdot,y)\right)(\bar{x})\right) \\ &\overset{\text{(Theorem 2.3.29 + (4.32))}}{\subseteq} -\mathrm{cl}^*\operatorname{conv}\left(\bigcup_{\substack{\bar{y}\in \mathrm{WMax}(F(\bar{x}),K)\\ (x^*,y^*)\in\partial(-f)(\bar{x},\bar{y})}} \left[x^* + D^*F(\bar{x},\bar{y})(y^*)\right]\right). \end{aligned} \tag{4.33}$$

Note that f is independent of the argument in the space X. Furthermore, since F is closed at $\bar{x}$, we also have that $F(\bar{x})$ is a closed set. Hence, together with condition (iv), it is easy to see that the assumptions of Lemma 4.4.2 are satisfied. Then, for any $\bar{y} \in \mathrm{WMax}(F(\bar{x}), K)$, we get:

$$\begin{aligned} \partial(-f)(\bar{x},\bar{y}) &= \{0\} \times \partial(-g_{u,\bar{x}})(\bar{y}) \\ &\overset{\text{(Proposition 2.3.20)}}{\subseteq} -\{0\} \times \mathrm{cl}^*\operatorname{conv}\left(\partial g_{u,\bar{x}}(\bar{y})\right) \\ &\overset{\text{(Lemma 4.4.2 }(ii))}{\subseteq} -\{0\} \times \mathrm{cl}^*\operatorname{conv}\left(\bigcup_{\bar{z}\in F(\bar{x})\cap(\bar{y}+\operatorname{bd} K)} \partial\psi_e(\bar{y}-\bar{z}) \cap N(\bar{z}, F(\bar{x}))\right). \end{aligned} \tag{4.34}$$

Taking into account (4.34) and (4.33), we obtain the desired estimate. □

Remark 4.4.4. *Similar to Remark 4.3.9, the functional $f_{u,\bar{x}}$ remains unchanged if we substitute F by $\tilde{F} : X \rightrightarrows Y$ of the form $\tilde{F}(x) = F(x) - A$, with $A \subseteq K$ and $0 \in A$. Hence, in Theorem 4.4.3 we can substitute F by any other set-valued mapping $\tilde{F}$ of the above form. From this, we can obtain different (maybe sharper) upper estimates of $\partial f_{u,\bar{x}}(\bar{x})$, which can be translated into sharper optimality conditions set optimization problems, see Section 4.5.*

Remark 4.4.5. *Similarly to Remark 4.3.10, we mention that, although the upper estimate in* (4.31) *is convex (and hence we are also estimating $\partial^\circ f_{u,\bar{x}}(\bar{x})$), Example 4.5.7 illustrates that convexity is necessary.*

The proof of the following corollary is similar to that of Corollary 4.3.11, and it is hence omitted.

Corollary 4.4.6. *Let Assumption 2 be fulfilled with X being Asplund. Suppose that Y is finite dimensional and that* $\operatorname{gph} F$ *is closed. Furthermore, assume that F is locally Lipschitz and locally bounded at $\bar{x}$. Then, inclusion* (4.31) *holds.*

We conclude this section with a sharper result in the convex case.

Theorem 4.4.7. *In addition to Assumption 2, let X and Y be Asplund spaces. Suppose also that*

(i) *F is $\preceq_K^{(u)}$- convex and locally K- bounded above at $\bar{x}$,*

(ii) *$\mathcal{H}_F$ is convex valued in a neighborhood of $\bar{x}$,*

(iii) *$\mathcal{H}_F$ is closed at $\bar{x}$,*

(iv) $\mathrm{WMax}(\mathcal{H}_F(\bar{x}), K) \neq \emptyset$,

(v) *$S^{u,2}_{\mathcal{H}_F}(x)$ is inner semicompact at $\bar{x}$.*

(vi) $\operatorname{gph} \mathcal{H}_F$ *is locally closed around any point in the set* $\{\bar{x}\} \times \mathrm{WMax}(\mathcal{H}_F(\bar{x}), K)$.

Then,

$$\partial f_{u,\bar{x}}(\bar{x}) \subseteq -\mathrm{cl}^* \operatorname{conv} \left(\bigcup_{\bar{y} \in \mathrm{WMax}(\mathcal{H}_F(\bar{x}), K)} D^*\mathcal{H}_F(\bar{x}, \bar{y}) \left[-\partial \psi_e(0) \cap N(\bar{y}, \mathcal{H}_F(\bar{x})) \right] \right).$$

Proof. Because of conditions (i) and (ii), we can apply [150, Theorem 7.4.9] to obtain that $\mathcal{H}_F$ is locally Lipschitz at $\bar{x}$. Then, it is easy to see that assumptions $(i)-(iii), (v)-(vi)$ of Theorem 4.4.3 are satisfied if we replace F by $\mathcal{H}_F$. Since these assumptions are the only ones needed to obtain (4.33), we can take into account Remark 4.4.4 to get in this case

$$\partial f_{u,\bar{x}}(\bar{x}) \subseteq -\mathrm{cl}^* \operatorname{conv} \left(\bigcup_{\substack{\bar{y} \in \mathrm{WMax}(\mathcal{H}_F(\bar{x}), K) \\ (x^*, y^*) \in \partial(-f)(\bar{x}, \bar{y})}} \left[x^* + D^*\mathcal{H}_F(\bar{x}, \bar{y})(y^*) \right] \right), \tag{4.35}$$

where f is the same function defined in Theorem 4.4.3. Similar to (4.34), but applying Lemma 4.4.2 (i) instead, we obtain

$$\partial(-f)(\bar{x}, \bar{y}) \subseteq -\{0\} \times \left(\partial \psi_e(0) \cap N(\bar{y}, \mathcal{H}_F(\bar{x})) \right). \tag{4.36}$$

The estimate is then obtained by replacing the term $\partial(-f)(\bar{x}, \bar{y})$ in (4.35) by the upper estimate obtained in (4.36). □

4.5 Fermat Rules in Set Optimization

In this section, we obtain the optimality conditions for set optimization problems based on our previous results. We start by establishing that, as in the scalar case, local solutions are also global under convexity.

Proposition 4.5.1. *Let Assumption 2 be fulfilled and, for $r \in \{l, u\}$, consider the preorder relation $\preceq_K^{(r)}$ in Definition 2.2.15. Suppose that Ω is convex, that F is $\preceq_K^{(r)}$- convex and that $\bar{x}$ is a local $\preceq_K^{(r)}$- weakly minimal solution of $\mathcal{SOP}(F, K, \Omega)$. The following statements are true:*

(i) If $r = l$, then $\bar{x}$ is also a global $\preceq_K^{(l)}$- weakly minimal solution.

(ii) If $r = u$ and $\mathcal{H}_F(\bar{x})$ is convex, then $\bar{x}$ is also a global $\preceq_K^{(u)}$- weakly minimal solution.

Proof. Since the proofs are similar and resemble the one in the scalar case, we only show (ii). See also [70, Proposition 5] for a proof of (i) with a slightly different optimality concept. Let U be the neighborhood of $\bar{x}$ such that

$$\forall\, x \in \Omega \cap U \setminus \{\bar{x}\} :\; F(x) \not\prec_K^{(r)} F(\bar{x})$$

and suppose that $\bar{x}$ is not a global $\preceq_K^{(u)}$- weakly minimal solution of $\mathcal{SOP}(F, K, \Omega)$. Then, we can find $\tilde{x} \in \Omega \setminus \{\bar{x}\}$ such that $F(\tilde{x}) \prec_K^{(u)} F(\bar{x})$. Hence, we get the existence of $t \in (0, 1]$ such that $x_t := t\tilde{x} + (1-t)\bar{x} \in \Omega \cap U \setminus \{\bar{x}\}$. It follows that

$$\begin{aligned}
F(x_t) &\subseteq F(x_t) - K \\
&\overset{(F \text{ is } \preceq_K^{(u)}-\text{convex})}{\subseteq} tF(\tilde{x}) + (1-t)F(\bar{x}) - K \\
&\subseteq t\mathcal{H}_F(\tilde{x}) + (1-t)\mathcal{H}_F(\bar{x}) - K \\
&\overset{(F(\tilde{x}) \prec_K^{(u)} F(\bar{x}))}{\subseteq} t\mathcal{H}_F(\bar{x}) + (1-t)\mathcal{H}_F(\bar{x}) - \operatorname{int} K \\
&\overset{(\mathcal{H}_F(\bar{x}) \text{ is convex})}{=} \mathcal{H}_F(\bar{x}) - \operatorname{int} K \\
&= F(\bar{x}) - \operatorname{int} K,
\end{aligned}$$

which is equivalent to $F(x_t) \prec_K^{(u)} F(\bar{x})$. This contradicts to the local minimality of F at $\bar{x}$.

□

In the following theorem we establish relationships between the set-valued problem and a corresponding scalar problem. We want to mention that a similar statement to (i) below have been established in [80, Corollary 4.11] for the case $r = l$.

Theorem 4.5.2. *Let Assumption 2 be fulfilled and, for $r \in \{l, u\}$, consider the preorder relation $\preceq_K^{(r)}$ in Definition 2.2.15 and the functional $f_{r,\bar{x}}$ in Definition 4.2.1 (iii). The following assertions are true:*

(i) If $\bar{x}$ is a local $\preceq_K^{(r)}$- weakly minimal solution of $\mathcal{SOP}(F, K, \Omega)$, then $\bar{x}$ is a local solution of the problem $\mathcal{OP}(f_{r,\bar{x}}, \Omega)$, that is,

$$\min_{x \in \Omega} f_{r,\bar{x}}(x). \tag{$\mathcal{OP}(f_{r,\bar{x}}, \Omega)$}$$

(ii) Conversely, suppose that $\bar{x}$ is a local strict solution of $\mathcal{OP}(f_{r,\bar{x}}, \Omega)$ and either $r = l$ and $\mathrm{WMin}(F(\bar{x}), K) \neq \emptyset$, or $r = u$ and $\mathrm{WMax}(F(\bar{x}), K) \neq \emptyset$. Then, $\bar{x}$ is a local $\preceq_K^{(r)}$- strictly minimal solution of $\mathcal{SOP}(F, K, \Omega)$.

Proof. (i) Assume that $\bar{x}$ is not a local solution of $\mathcal{OP}(f_{r,\bar{x}}, \Omega)$. Then, for every neighborhood U of $\bar{x}$ we can find $\tilde{x} \in \Omega \cap U$ such that

$$f_{r,\bar{x}}(\tilde{x}) < f_{r,\bar{x}}(\bar{x}) \leq 0. \tag{4.37}$$

We just analyze the case $r = u$ since the other one is similar. From the definition of $f_{u,\bar{x}}$ and (4.37), we deduce that for every $\tilde{y} \in F(\tilde{x})$, the inequality $g_{u,\bar{x}}(\tilde{y}) < 0$ holds. Equivalently, we obtain

$$\forall\, \tilde{y} \in F(\tilde{x})\ \exists\, \bar{y} \in F(\bar{x}) : \psi_e(\tilde{y} - \bar{y}) < 0.$$

Again, by Proposition 2.5.6 (iii), we obtain $F(\tilde{x}) \prec_K^{(u)} F(\bar{x})$, a contradiction.

(ii) By Proposition 4.2.2 (iii) we know that $f_{r,\bar{x}}(\bar{x})$ is finite. Assume that $\bar{x}$ is not a local $\preceq_K^{(r)}$- strictly minimal solution of $\mathcal{SOP}(F, K, \Omega)$. Then, for any neighborhood U of $\bar{x}$ we can find $\tilde{x} \in (\Omega \cap U) \setminus \{\bar{x}\}$ such that

$$F(\tilde{x}) \preceq_K^{(r)} F(\bar{x}).$$

Hence, according to Theorem 2.5.9, we get

$$f_{r,\bar{x}}(\tilde{x}) \leq f_{r,\bar{x}}(\bar{x}).$$

This contradicts the fact that $\bar{x}$ is a local strict solution of $\mathcal{OP}(f_{r,\bar{x}}, \Omega)$. $\square$

Necessary optimality conditions for $\mathcal{SOP}(F, K, \Omega)$ with respect to the relation $\preceq_K^{(l)}$ are established in the next theorem.

Theorem 4.5.3. *Let Assumption 2 be fulfilled and consider the preorder relation $\preceq_K^{(l)}$ in Definition 2.2.15. Suppose that $\bar{x}$ is a local $\preceq_K^{(l)}$- weakly minimal solution of $\mathcal{SOP}(F, K, \Omega)$. The following statements are true:*

(i) Suppose that Ω is convex, that F is $\preceq_K^{(l)}$- convex and locally $\preceq_K^{(l)}$- bounded at $\bar{x}$, and that $F(\bar{x})$ is strongly K- compact. Then,

$$0 \in \mathrm{cl}^* \operatorname{conv} \left(\bigcup_{\bar{y} \in \mathrm{Min}(F(\bar{x}), K)} D^* \mathcal{E}_F(\bar{x}, \bar{y}) \left[\partial \psi_e(0) \right] \right) + N(\bar{x}, \Omega). \tag{4.38}$$

This condition is sufficient for optimality provided that, in addition, F is strongly K- compact valued in Ω.

(ii) Suppose that X and Y are Asplund spaces, that F is locally Lipschitz at $\bar{x}$, and that the rest of the conditions in Theorem 4.3.8 are fulfilled. Then,

$$0 \in \mathrm{cl}^* \operatorname{conv} \left(\bigcup_{\bar{y} \in \mathrm{WMin}(F(\bar{x}),K)} \left\{ x^* \in X^* \mid \exists\, y^* \in N(\bar{y}, F(\bar{x})) : (x^*, y^*) \in G_{(\bar{x},\bar{y})} \right\} \right) + N(\bar{x}, \Omega), \tag{4.39}$$

where

$$G_{(\bar{x},\bar{y})} = \mathrm{cl}^* \operatorname{conv} \left(\bigcup_{\substack{\bar{z} \in F(\bar{x}) \cap (\bar{y} - \mathrm{bd}\, K) \\ z^* \in \partial \psi_e(\bar{z} - \bar{y})}} D^* F(\bar{x}, \bar{z})(z^*) \times \{-z^*\} \right).$$

Proof. First note that, by Theorem 4.5.2, $\bar{x}$ is a solution of $\mathcal{OP}(f_{l,\bar{x}}, \Omega)$.

(i) Because of Theorem 4.2.5 (i), we know that $f_{l,\bar{x}}$ is convex and continuous at $\bar{x}$. Then, according to Theorem 2.4.2, the inclusion $0 \in \partial f_{l,\bar{x}}(\bar{x}) + N(\bar{x}, \Omega)$ is both necessary and sufficient for the optimality of $\bar{x}$ in $\mathcal{OP}(f_{l,\bar{x}}, \Omega)$. Hence, the first part of the statement follows from Theorem 4.3.6.

Suppose now that F is strongly K- compact valued in Ω and that $\bar{x}$ is not a $\preceq_K^{(l)}$- weakly minimal solution of $\mathcal{SOP}(F, K, \Omega)$. Then, without loss of generality we can assume that F is compact valued and that there exists $\tilde{x} \in \Omega$ such that

$$F(\tilde{x}) \prec_K^{(l)} F(\bar{x}). \tag{4.40}$$

We claim that $f_{l,\bar{x}}(\tilde{x}) < 0 = f_{l,\bar{x}}(\bar{x})$, which contradicts (4.38). Indeed, note that because $F(\tilde{x})$ is compact, the functional $g_l(\tilde{x}, \cdot)$ is finite. It is also upper semicontinuous in Y because it is the infimum of continuous functionals. Since $F(\bar{x})$ is compact, it follows from the classical Weierstrass's theorem that $\mathcal{OP}\left(-g_l(\tilde{x}, \cdot), F(\bar{x})\right)$ has a solution $\bar{y}$. According to (4.40), we can find $\tilde{y} \in F(\tilde{x})$ such that $\tilde{y} \prec_K \bar{y}$. Hence, we get

$$f_{l,\bar{x}}(\tilde{x}) = g_l(\tilde{x}, \bar{y}) \leq \psi_e(\tilde{y} - \bar{y}) < 0,$$

as desired.

(ii) Similarly to the previous case, by Theorem 4.2.7 (i) we obtain that $f_{l,\bar{x}}$ is locally Lipschitz at $\bar{x}$. Hence, all the assumptions for the necessary optimality conditions in Theorem 2.4.2 are satisfied. From this, we get $0 \in \partial f_{l,\bar{x}}(\bar{x}) + N(\bar{x}, \Omega)$. The result follows then from Theorem 4.3.8. $\square$

With a similar argument to the one in the previous theorem, we can obtain the optimality conditions for problems with the relation $\preceq_K^{(u)}$. The proof is hence omitted.

Theorem 4.5.4. *In addition to Assumption 2, suppose that X and Y are Asplund spaces and consider the preorder relation $\preceq_K^{(u)}$ in Definition 2.2.15. Furthermore, assume that $\bar{x}$ is a local $\preceq_K^{(u)}$- weakly minimal solution of $\mathcal{SOP}(F,K,\Omega)$. The following assertions hold:*

(i) Suppose that F is $\preceq_K^{(u)}$- convex and locally K- bounded above at $\bar{x}$, and that the conditions in Theorem 4.4.7 are fulfilled. Then,

$$0 \in -\mathrm{cl}^* \operatorname{conv}\left(\bigcup_{\bar{y}\in \mathrm{WMax}(\mathcal{H}_F(\bar{x}),K)} D^*\mathcal{H}_F(\bar{x},\bar{y})\left[-\partial\psi_e(0)\cap N(\bar{y},\mathcal{H}_F(\bar{x}))\right]\right) + N(\bar{x},\Omega).$$

(ii) Suppose that the F is locally Lipschitz at $\bar{x}$ and that the conditions in Theorem 4.4.3 are fulfilled. Then

$$0 \in -\mathrm{cl}^* \operatorname{conv}\left(\bigcup_{\bar{y}\in \mathrm{WMax}(F(\bar{x}),K)} D^*F(\bar{x},\bar{y})\left[H_{(\bar{x},\bar{y})}\right]\right) + N(\bar{x},\Omega), \tag{4.41}$$

where

$$H_{(\bar{x},\bar{y})} := -\mathrm{cl}^* \operatorname{conv}\left(\bigcup_{\bar{z}\in F(\bar{x})\cap(\bar{y}+\mathrm{bd}\,K)} \partial\psi_e(\bar{y}-\bar{z})\cap N\left(\bar{z},F(\bar{x})\right)\right).$$

Remark 4.5.5. *Under the assumptions of both Theorem 4.5.3 (ii) and Theorem 4.5.4 (ii), it is possible to derive optimality conditions for $\preceq_K^{(s)}$- weakly minimal solutions of $\mathcal{SOP}(F,K,\Omega)$. Indeed, let $\bar{x}$ be a $\preceq_K^{(s)}$- weakly minimal solution of $\mathcal{SOP}(F,K,\Omega)$, and consider $f_{s,\bar{x}}: X\to\overline{\mathbb{R}}$ given by*

$$f_{s,\bar{x}}(x) := \max\{f_{l,\bar{x}}(x), f_{u,\bar{x}}(x)\}.$$

Then, from Proposition 4.2.2 (iii), we deduce that $f_{s,\bar{x}}(\bar{x}) = 0$. Moreover, by Theorem 4.2.7, we get that $f_{s,\bar{x}}$ is locally Lipschitz at $\bar{x}$. Furthermore, similarly to the proof of Theorem 4.5.2 (i), we deduce that $\bar{x}$ is a solution of $\mathcal{OP}(f_{s,\bar{x}},\Omega)$. Thus, according to Theorem 2.4.2,

$$0 \in \partial f_{s,\bar{x}}(\bar{x}) + N(\bar{x},\Omega). \tag{4.42}$$

Let C_l and C_u be the convex upper estimates of $\partial f_{l,\bar{x}}(\bar{x})$ and $\partial f_{u,\bar{x}}(\bar{x})$ in (4.21) and (4.31) respectively. Then, we deduce that

$$\begin{aligned}
\partial f_{s,\bar{x}}(\bar{x}) &\overset{([132, \textit{Theorem 3.46}])}{\subseteq} \bigcup_{t\in[0,1]} \partial\left(tf_{l,\bar{x}} + (1-t)f_{u,\bar{x}}\right)(\bar{x}) \\
&\overset{(\textit{Proposition } 2.3.19\ (i))}{\subseteq} \bigcup_{t\in[0,1]} \partial\left(tf_{l,\bar{x}}\right)(\bar{x}) + \partial\left((1-t)f_{u,\bar{x}}\right)(\bar{x}) \\
&= \bigcup_{t\in[0,1]} t\partial f_{l,\bar{x}}(\bar{x}) + (1-t)\partial f_{u,\bar{x}}(\bar{x}) \\
&\overset{(\textit{Theorem 4.3.8 + Theorem 4.4.3})}{\subseteq} \bigcup_{t\in[0,1]} tC_l + (1-t)C_u \\
&\overset{(C_l, C_u \textit{ convex})}{\subseteq} \operatorname{conv}(C_l \cup C_u).
\end{aligned}$$

Hence, from this and (4.42)*, we find that the inclusion*

$$0 \in \operatorname{conv}(C_l \cup C_u) + N(\bar{x}, \Omega)$$

is necessary for $\preceq_K^{(s)}$*- weak minimality.*

Theorem 4.5.3 and Theorem 4.5.4 motivates the following definition of stationary points for $\mathcal{SOP}(F, K, \Omega)$. Since in the next chapter we will derive stronger optimality conditions for a particular class of set optimization problems, we refer to the concepts below as weak.

Definition 4.5.6. *Let Assumption 2 be fulfilled. We say that* $\bar{x}$ *is a*

(i) weak $\preceq_K^{(l)}$*- stationary point of* $\mathcal{SOP}(F, K, \Omega)$*, if* (4.39) *is fulfilled,*

(ii) weak $\preceq_K^{(u)}$*- stationary point of* $\mathcal{SOP}(F, K, \Omega)$*, if* (4.41) *is fulfilled.*

We conclude this section with the following example, that illustrate our results and compare them with other results obtained for the vector approach.

Example 4.5.7. *In Assumption 2, let* $X = \Omega = \mathbb{R}$, $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $e = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, *and* $\bar{x} = 0$. *Furthermore, let the functional* $f : \mathbb{R} \to \mathbb{R}^2$ *and the set-valued mapping* $F : \mathbb{R} \rightrightarrows \mathbb{R}^2$ *be defined as*

$$f(x) := \begin{pmatrix} x+1 \\ x-1 \end{pmatrix}$$

and

$$F(x) := \{f(x), -f(x)\}$$

respectively, and consider $\mathcal{SOP}(F, K, \Omega)$*. Thus, in particular, we have* $\nabla f(\bar{x}) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$*. Then,* *(i)* F *is locally Lipschitz at* $\bar{x}$.

(ii) $\bar{x}$ *is both a local* $\preceq_K^{(l)}, \preceq_K^{(u)}$*- weakly minimal solution of* $\mathcal{SOP}(F, K, \Omega)$*:*

Indeed, it is easy to verify that

$$\forall\, x \in (-1, 1):\; F(x) \not\prec_K^{(l)} F(\bar{x}), \quad F(x) \not\prec_K^{(u)} F(\bar{x}).$$

(iii) $\bar{x}$ *is not a local weak minimizer of* $\mathcal{SOP}(F, K, \Omega)$*:*

Indeed, note that in any neighborhood U *of* $\bar{x}$ *we can find* $x \in U \setminus \{\bar{x}\}$ *such that* $-x \in U$. *Then, it is easy to check that*

$$F(\bar{x}) \subset \big(F(x) + \operatorname{int} K\big) \cup \big(F(-x) + \operatorname{int} K\big).$$

(iv) $\bar{x}$ *is not a stationary point in the sense of Definition 2.4.22 (vector approach):*

Since $f(\bar{x}) \neq -f(\bar{x})$ *and* $\operatorname{gph} F = \operatorname{gph} f \cup \operatorname{gph}(-f)$, *we have that* $\operatorname{gph} F = \operatorname{gph} f$ *and* $\operatorname{gph} F = \operatorname{gph}(-f)$ *around* $(\bar{x}, f(\bar{x}))$ *and* $(\bar{x}, -f(\bar{x}))$ *respectively. By the differentiability of* f *and Proposition 2.3.22, we obtain*

$$\forall\, v \in \mathbb{R}^2:\; D^*F(\bar{x}, f(\bar{x}))(v) = \{\nabla f(\bar{x})v\} = \{v_1 + v_2\}, \tag{4.43}$$

$$\forall\, v \in \mathbb{R}^2:\; D^*F(\bar{x}, -f(\bar{x}))(v) = \{-\nabla f(\bar{x})v\} = \{-(v_1 + v_2)\}. \tag{4.44}$$

According to Definition 2.4.22 and Theorem 2.4.20, $\bar{x}$ *is a stationary point of* F *in the sense of the vector approach if and only if there exists* $\bar{y} \in F(\bar{x})$ *and* $v \in K^* \setminus \{0\}$ *such that*

$$0 \in D^*F(\bar{x}, \bar{y})(v).$$

Since $K^* = K$ *in our context, it is then easy to check that*

$$0 \in D^*F(\bar{x}, f(\bar{x}))(v),\; v \in K^* \Longleftrightarrow v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Similarly, we obtain that

$$0 \in D^*F(\bar{x}, -f(\bar{x}))(v),\; v \in K^* \Longleftrightarrow v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

It follows that $\bar{x}$ *is not a stationary point in the sense of Definition 2.4.22.*

(v) $\bar{x}$ *is both weak* $\preceq_K^{(l)}$*- and weak* $\preceq_K^{(u)}$*- stationary:*

Of course, this is a direct consequence of Theorem 4.5.3 and Theorem 4.5.4, but we show the calculus for completeness. First, we note that $\operatorname{WMin}(F(\bar{x}), K) = \operatorname{WMax}(F(\bar{x}), K) = F(\bar{x})$. *Because* $F(\bar{x})$ *consists of isolated points, we obtain*

$$N(f(\bar{x}), F(\bar{x})) = N(-f(\bar{x}), F(\bar{x})) = \mathbb{R}^2. \tag{4.45}$$

On the other hand, from Proposition 2.5.7 (iii) we have

$$\partial\psi_e(0) = \{v \in \mathbb{R}^2_+ \mid v_1 + v_2 = 1\}. \tag{4.46}$$

The weak $\preceq^{(l)}_K$*- stationarity of* $\bar{x}$ *is now equivalent to* $0 \in \operatorname{cl}\operatorname{conv}(A_1 \cup A_2)$*, where*

$$A_1 := \left\{ u \in \mathbb{R} \mid \exists\, v \in \mathbb{R}^2 : \begin{pmatrix} u \\ v \end{pmatrix} \in G_{(\bar{x}, f(\bar{x}))} \right\}, \tag{4.47}$$

$$A_2 := \left\{ u \in \mathbb{R} \mid \exists\, v \in \mathbb{R}^2 : \begin{pmatrix} u \\ v \end{pmatrix} \in G_{(\bar{x}, -f(\bar{x}))} \right\}. \tag{4.48}$$

We have

$$\begin{aligned} G_{(\bar{x}, f(\bar{x}))} &\overset{(4.43)}{=} \operatorname{cl}\operatorname{conv}\left(\bigcup_{v \in \partial\psi_e(0)} \{v_1 + v_2\} \times \{-v\} \right) \\ &\overset{(4.46)}{=} \operatorname{cl}\operatorname{conv}\left(\bigcup_{v \in \partial\psi_e(0)} \{1\} \times \{-v\} \right) \\ &= \{1\} \times (-\partial\psi_e(0)). \end{aligned}$$

From this, we deduce that $A_1 = \{1\}$*. Using a similar argument we can obtain* $G_{(\bar{x}, -f(\bar{x}))} = \{-1\} \times (-\partial\psi_e(0))$*, from which we obtain* $A_2 = \{-1\}$*. Hence, we have*

$$0 \in [-1, 1] = \operatorname{cl}\operatorname{conv}(A_1 \cup A_2),$$

and the weak $\preceq^{(l)}_K$*- stationarity of* $\bar{x}$ *follows.*

Next, we show that $\bar{x}$ *is also weak* $\preceq^{(u)}_K$*- stationary. Similarly to the previous case, this is equivalent to* $0 \in \operatorname{cl}\operatorname{conv}(B_1 \cup B_2)$*, where*

$$B_1 := -D^*F(\bar{x}, f(\bar{x}))\left[H_{(\bar{x}, f(\bar{x}))}\right], \tag{4.49}$$

$$B_2 := -D^*F(\bar{x}, f(\bar{x}))\left[H_{(\bar{x}, -f(\bar{x}))}\right]. \tag{4.50}$$

We now have

$$\begin{aligned} H_{(\bar{x}, f(\bar{x}))} &= -\operatorname{cl}\operatorname{conv}\left(\partial\psi_e(0) \cap N\left(f(\bar{x}), F(\bar{x})\right)\right) \\ &\overset{(4.45)}{=} -\partial\psi_e(0). \end{aligned}$$

From this, we deduce that

$$B_1 \overset{(4.49)}{=} -D^*F(\bar{x}, f(\bar{x}))[-\partial\psi_e(0)] \overset{(4.43)}{=} \{1\}.$$

Similarly, we can obtain $H_{(\bar{x},-f(\bar{x}))} = -\partial\psi_e(0)$, *from which we get*

$$B_2 \overset{(4.50)}{=} -D^*F(\bar{x}, -f(\bar{x}))[-\partial\psi_e(0)] \overset{(4.44)}{=} \{-1\}.$$

Hence, we have $0 \in [-1, 1] = \operatorname{cl}\operatorname{conv}(B_1 \cup B_2)$, *and* $\bar{x}$ *is weak* $\preceq_K^{(u)}$*- stationary.*

4.6 Application to Convex Problems Involving Functional Constraints

In this last section, by means of the optimality conditions developed in Section 4.5, we derive Karush - Kuhn- Tucker conditions for set optimization problems where the solution concept is given by the $\preceq_K^{(l)}$ relation and the set-valued objective mapping is defined by convex inequality constraints. Specifically, we work with the following set of assumptions:

Assumption 3. *Let Assumption 2 holds with* $X = \Omega = \mathbb{R}^n$, $Y = \mathbb{R}^m$, *and* $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ *given by*

$$F(x) := \{y \in \mathbb{R}^m \mid f_i(x, y) \leq 0, \ i = 1, \dots, p\}, \tag{4.51}$$

where each $f_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ *is convex and continuously Fréchet differentiable at every point.*

It is worth to point out that, to the best of our knowledge, the set optimization problem where the set-valued objective mapping F has the structure (4.51) has only been studied once in the literature by Jahn [93]. In fact, in that case, even equality constraints can be considered, as long as the sets $F(x)$ are compact and the sets $F(x) + K$ and $F(x) - K$ are convex for every feasible point $x \in \mathbb{R}^n$. It is also worth mentioning that, although only $\preceq_K^{(s)}$- minimal solutions are studied in that reference, the results can be easily adapted to $\preceq_K^{(l)}$- weakly minimal solutions.

Our main result is the following:

Theorem 4.6.1. *Let Assumption 3 be fulfilled and consider problem* $\mathcal{SOP}(F, K, \Omega)$. *Suppose that* F *is strongly* K*- compact valued and locally* $\preceq_K^{(l)}$*- bounded at* $\bar{x}$. *Furthermore, assume that* $F(\bar{x})$ *satisfies MFCQ at every* $\bar{y} \in \operatorname{Min}(F(\bar{x}), K)$. *Then,* $\bar{x}$ *is a* $\preceq_K^{(l)}$*- weakly minimal solution of* $\mathcal{SOP}(F, K, \Omega)$ *if and only if there exists* $\lambda \in \mathbb{R}^{n+1}_+$, $\bar{y}_1, \dots, \bar{y}_{n+1} \in \operatorname{Min}(F(\bar{x}), K)$ *and* $\mu_1, \dots, \mu_{n+1} \in \mathbb{R}^p_+$ *such that*

$$\sum_{i=1}^{n+1} \lambda_i \nabla_x f(\bar{x}, \bar{y}_i)\mu_i = 0,$$

$$\sum_{i=1}^{n+1} \lambda_i = 1,$$

$$\nabla_y f(\bar{x}, \bar{y}_i)\mu_i \in -K^*, \quad i = 1, \dots, n+1,$$

$$e^\top \nabla_y f(\bar{x}, \bar{y}_i)\mu_i = -1, \quad i = 1, \dots, n+1,$$

$$\mu_i^\top f(\bar{x}, \bar{y}_i) = 0, \quad i = 1, \dots, n+1.$$

Here, $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ *is given by* $f(x) := \begin{pmatrix} f_1(x) \\ \vdots \\ f_p(x) \end{pmatrix}$.

Proof. The proof will be divided in several steps.

Step 1: We prove that for any $v \in K^*$ and $\bar{y} \in F(\bar{x}) : D^*F(\bar{x}, \bar{y})(v) = D^*\mathcal{E}_F(\bar{x}, \bar{y})(v)$.

Note that, under Assumption 3, $\operatorname{gph} F = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid f_i(x, y) \leq 0,\ i = 1, \dots, p\}$ is a convex set. Moreover, from the definition of $\mathcal{E}_F$ it follows that

$$\operatorname{gph} \mathcal{E}_F = \operatorname{gph} F + \{0\} \times K. \tag{4.52}$$

Hence, $\operatorname{gph} \mathcal{E}_F$ is also convex and, according to Remark 2.3.3, it follows that F is $\preceq_K^{(l)}$- convex. From the definition of the coderivative, the statement of the claim is equivalent to

$$N\big((\bar{x}, \bar{y}), \operatorname{gph} \mathcal{E}_F\big) = N\big((\bar{x}, \bar{y}), \operatorname{gph} F\big) \cap (\mathbb{R}^m \times -K^*). \tag{4.53}$$

We proceed then to prove (4.53). First, suppose that $(u, v) \in N\big((\bar{x}, \bar{y}), \operatorname{gph} \mathcal{E}_F\big)$. According to (4.52), the convexity of $\operatorname{gph} \mathcal{E}_F$ and Remark 2.3.17, this is the same as

$$\forall\ (x, y) \in \operatorname{gph} F, k \in K : u^\top(x - \bar{x}) + v^\top(y + k - \bar{y}) \leq 0. \tag{4.54}$$

Taking into account that $\operatorname{gph} F$ is also convex and Remark 2.3.17, we can put $k = 0$ in (4.54) to obtain that $(u, v) \in N\big((\bar{x}, \bar{y}), \operatorname{gph} F\big)$. On the other hand, by substituting $x = \bar{x}, y = \bar{y}$ in (4.54), we get $v^\top k \leq 0$ for every $k \in K$. According to the definition of the dual cone, this implies that $v \in -K^*$. Hence,

$$N\big((\bar{x}, \bar{y}), \operatorname{gph} \mathcal{E}_F\big) \subseteq N\big((\bar{x}, \bar{y}), \operatorname{gph} F\big) \cap (\mathbb{R}^m \times -K^*).$$

In order to see the reverse inclusion, choose $(u, v) \in N\big((\bar{x}, \bar{y}), \operatorname{gph} F\big) \cap (\mathbb{R}^m \times -K^*)$. Then, $v \in -K^*$ and

$$\forall\ (x, y) \in \operatorname{gph} F : u^\top(x - \bar{x}) + v^\top(y - \bar{y}) \leq 0.$$

It is then easy to see that this implies (4.54), and the statement follows.

Step 2: We prove that the set

$$A := \bigcup_{\bar{y} \in \operatorname{Min}(F(\bar{x}), K)} D^*F(\bar{x}, \bar{y})\,[\partial\psi_e(0)]$$

is compact.

It suffices to show that A is both closed and bounded. In order to see the closedness of A, let $\{u_k\}_{k \geq 1} \subseteq A$ be such that $u_k \to \bar{u}$. Hence, taking into account the definition of the coderivative, there are sequences $\{y_k\}_{k \geq 1} \subseteq \operatorname{Min}(F(\bar{x}), K)$ and $\{v_k\}_{k \geq 1} \subseteq \partial\psi_e(0)$ such that

$$(u_k, -v_k) \in N\big((\bar{x}, y_k), \operatorname{gph} F\big). \tag{4.55}$$

Since $F(\bar{x})$ and $\partial\psi_e(0)$ are compact, we can assume without loss of generality that $y^k \to \hat{y} \in F(\bar{x})$ and that $v_k \to \bar{v} \in \partial\psi_e(0)$. Furthermore, it is well known that the set-valued mapping $N(\cdot, \operatorname{gph} F)$ is closed at any point. Therefore, taking the limit when $k \to +\infty$ in (4.55), we obtain

$$(\bar{u}, -\bar{v}) \in N\big((\bar{x}, \hat{y}), \operatorname{gph} F\big),$$

or equivalently, $\bar{u} \in D^*F(\bar{x}, \hat{y})(\bar{v})$. Next, using the compactness of $F(\bar{x})$ is compact, we can apply Proposition 2.4.7 (i) to obtain an element $\bar{y} \in \operatorname{Min}(F(\bar{x}), K)$ such that $\bar{y} \preceq_K \hat{y}$. By replacing $\operatorname{gph} \mathcal{E}_F$ by $\operatorname{gph} F$ in the proof of Lemma 4.3.4 and taking into account that $\bar{v} \in \partial\psi_e(0) \subseteq K^*$, we deduce that $D^*F(\bar{x}, \hat{y})(\bar{v}) \subseteq D^*F(\bar{x}, \bar{y})(\bar{v})$. Thus, we have that $\bar{u} \in D^*F(\bar{x}, \bar{y})(\bar{v})$, which implies the closedness of A.

Suppose now that A is not bounded. Then, we could find an unbounded sequence $\{u_k\}_{k\geq 1} \subseteq A$. Let $\{y_k\}_{k\geq 1} \subseteq \operatorname{Min}(F(\bar{x}), K)$ and $\{v_k\}_{k\geq 1} \subseteq \partial\psi_e(0)$ be the sequences that satisfy (4.55) and, as before, denote by $\bar{y}$ and $\bar{v}$ their respective limits. Then, we also have

$$\left(\frac{u_k}{\|u_k\|}, -\frac{v_k}{\|u_k\|}\right) \in N\big((\bar{x}, y_k), \operatorname{gph} F\big). \tag{4.56}$$

Without loss of generality we can now assume that $\frac{u_k}{\|u_k\|} \to \bar{u}$. Then, taking the limit when $k \to +\infty$ in (4.56), we get

$$(\bar{u}, 0) \in N\big((\bar{x}, \bar{y}), \operatorname{gph} F\big).$$

According to Proposition 2.3.25, we now have

$$N((\bar{x}, \bar{y}), \operatorname{gph} F) = \operatorname{cone}\operatorname{conv}\{\nabla f_i(\bar{x}, \bar{y})\}_{i \in I(\bar{x}, \bar{y})}, \tag{4.57}$$

where $I(\bar{x}, \bar{y}) = \{i \in \{1, \dots, p\} \mid f_i(\bar{x}, \bar{y}) = 0\}$ is the set of active indexes. In particular, there exists $\alpha \in \mathbb{R}^p_+$ such that

$$\nabla_x f(\bar{x}, \bar{y})\alpha = \bar{u}, \tag{4.58}$$

$$\nabla_y f(\bar{x}, \bar{y})\alpha = 0, \tag{4.59}$$

$$\alpha^\top f(\bar{x}, \bar{y}) = 0. \tag{4.60}$$

Since $F(\bar{x})$ satisfies MFCQ at $\bar{y}$, equation (4.59) implies $\alpha = 0$. Hence, from (4.58) we deduce that $\bar{u} = 0$, a contradiction to the fact that $\|\bar{u}\| = 1$.

Step 3: We obtain the multipliers.

By Theorem 4.5.3 (i), we know that the $\preceq_K^{(l)}$- weak minimality of $\bar{x}$ for $\mathcal{SOP}(F, K, \Omega)$ is equivalent to

$$0 \in \operatorname{cl}\operatorname{conv}\left(\bigcup_{\bar{y}\in\operatorname{Min}(F(\bar{x}),K)} D^*F(\bar{x},\bar{y})\,[\partial\psi_e(0)]\right) = \operatorname{cl}\operatorname{conv} A.$$

Note that we replaced $D^*\mathcal{E}_F(\bar{x},\bar{y})$ by $D^*F(\bar{x},\bar{y})$ in the above equation because of the result in the first step of this proof. Since A is compact, so is $\operatorname{conv} A$ and hence the necessary and sufficient condition is read as $0 \in \operatorname{conv} A$. By Carathéodory's Theorem [20, Theorem 2.1.6], we now get the existence of $\bar{y}_1,\ldots,\bar{y}_{n+1} \in \operatorname{Min}(F(\bar{x}),K)$ and $\lambda \in \mathbb{R}^{n+1}_+$ such that $\sum\limits_{i=1}^{n+1} \lambda_i = 1$ and

$$0 \in \sum_{i=1}^{n+1} \lambda_i D^*F(\bar{x},\bar{y}_i)[\partial\psi_e(0)]. \tag{4.61}$$

From (4.61), it follows the existence of $u_1,\ldots,u_{n+1} \in \mathbb{R}^n$ and $v_1,\ldots v_{n+1} \in \partial\psi_e(0)$ such that $u_i \in D^*F(\bar{x},\bar{y}_i)(v_i)$ for every $i \in \{1,\ldots,n+1\}$ and

$$0 = \sum_{i=1}^{n+1} \lambda_i u_i. \tag{4.62}$$

On the other hand, taking into account that $F(\bar{x})$ satisfies MFCQ at every $\bar{y}_i$, we can verify that $\operatorname{gph} F$ satisfies MFCQ at every $(\bar{x},\bar{y}_i)$. Hence, according to Proposition 2.3.25, we get the existence of $\mu_1,\ldots,\mu_{n+1} \in \mathbb{R}^p_+$ such that for each $i \in \{1,\ldots,n+1\}$:

$$u_i = \nabla_x f(\bar{x},\bar{y}_i)\mu_i, \tag{4.63}$$

$$-v_i = \nabla_y f(\bar{x},\bar{y}_i)\mu_i, \tag{4.64}$$

$$\mu_i^\top f(\bar{x},\bar{y}_i) = 0. \tag{4.65}$$

The statement of the theorem follows now from (4.62), (4.63), (4.64), (4.65) and Proposition 2.5.7 (*iii*).

□

Remark 4.6.2. *The main difference between our result (Theorem 4.6.1) and those derived in [93, Theorem 3.1] and [93, Theorem 4.3] lie on the type of multipliers obtained for the set-valued objective mapping F. Indeed, in [93, Theorem 3.1], these multipliers are Radon measures [156], and in [93, Theorem 4.3] they are positive functionals defined on the elements of K^* with unit length. On the other hand, our multipliers are really objects lying in $\mathbb{R}^n$ and $\mathbb{R}^m$, and hence we make no use of infinite dimensional constructions. This shows that our approach is very promising, since the optimality conditions obtained are more tractable from the computational point of view. However, we must mention that the results in [93] were derived under slightly weaker assumptions.*

We analyze next some consequences of Theorem 4.6.1.

Corollary 4.6.3. *Let Assumption 3 be fulfilled and suppose that F is compact valued and locally bounded at $\bar{x}$. Furthermore, assume that $F(\bar{x})$ satisfies Slater's condition. Then, the statement of Theorem 4.6.1 holds.*

Proof. Since F is compact valued and locally bounded at $\bar{x}$, it is in particular strongly K-compact valued and locally $\preceq_K^{(l)}$- bounded at $\bar{x}$. In addition, according to Remark 2.3.24, the fact that $F(\bar{x})$ satisfies Slater's condition implies that $F(\bar{x})$ satisfies MFCQ at every point. The statement follows then from Theorem 4.6.1. □

The following corollary shows that with Theorem 4.6.1 we can recover Corollary 2.4.13.

Corollary 4.6.4. *Let $\tilde{f} : \mathbb{R}^n \to \mathbb{R}^m$ be continuously differentiable and suppose that each $\tilde{f}_i$ is convex. Moreover, consider the multiobjective problem $\mathcal{VOP}(\tilde{f}, \mathbb{R}^m_+, \mathbb{R}^n)$, that is*

$$\mathbb{R}^m_+\text{-} \min_{x \in \mathbb{R}^n} \tilde{f}(x). \qquad (\mathcal{VOP}(\tilde{f}, \mathbb{R}^m_+, \mathbb{R}^n))$$

Then, $\bar{x}$ is a weakly minimal solution of $\mathcal{VOP}(\tilde{f}, \mathbb{R}^m_+, \mathbb{R}^n)$ if and only if there exists $\mu \in \mathbb{R}^m_+ \setminus \{0\}$ such that $\nabla \tilde{f}(\bar{x})\mu = 0$.

Proof. This is an immediate consequence of Theorem 4.6.1 with $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ being defined as $f(x) := \tilde{f}(x) - y$. □

Our last example shows necessary and sufficient conditions for problems where the graph of the set-valued objective mapping is a polyhedral.

Corollary 4.6.5. *Consider $\mathcal{SOP}(F, K, \Omega)$ associated to the set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ given by*

$$F(x) := \{y \in \mathbb{R}^m \mid Ax + By \leq c\},$$

where $A \in \mathbb{R}^{p \times n}, B \in \mathbb{R}^{p \times m}, c \in \mathbb{R}^p$ and B is assumed to have full row rank. Furthermore, let $\bar{x} \in \operatorname{int} \operatorname{dom} F$ and suppose that F is compact valued and locally bounded at $\bar{x}$. Then, $\bar{x}$ is a $\preceq_K^{(l)}$-weakly minimal solution of $\mathcal{SOP}(F, K, \Omega)$ if and only if there exists $\lambda \in \mathbb{R}^{n+1}_+$, $\bar{y}_1, \ldots, \bar{y}_{n+1} \in \operatorname{Min}(F(\bar{x}), K)$ and $\mu_1, \ldots, \mu_{n+1} \in \mathbb{R}^p_+$ such that

$$\begin{aligned}
\sum_{i=1}^{n+1} \lambda_i A^\top \mu_i &= 0, \\
\sum_{i=1}^{n+1} \lambda_i &= 1, \\
B^\top \mu_i &\in -K^*, \quad i = 1, \ldots, n+1, \\
e^\top B^\top \mu_i &= -1, \quad i = 1, \ldots, n+1, \\
\mu_i^\top (A\bar{x} + B\bar{y}_i - c) &= 0, \quad i = 1, \ldots, n+1.
\end{aligned}$$

Chapter 5

Steepest Descent Method for Set-Valued Mappings of Finite Cardinality

In this chapter, we derive a new steepest descent method for $\mathcal{SOP}(F, K, \Omega)$ where the set-valued objective mapping is identified by a finite number of continuously Fréchet differentiable selections. Formally, we consider the problem of finding $\preceq_K^{(l)}$- weakly minimal solutions of

$$K\text{-} \min_{x \in \mathbb{R}^n} F(x) \qquad (\mathcal{SOP}(F, K, \mathbb{R}^n))$$

under the following assumption:

Assumption 4. *Let $f_1, f_2, \ldots, f_p : \mathbb{R}^n \to \mathbb{R}^m$ be continuously Fréchet differentiable at every point and let $K \subset \mathbb{R}^m$ be a proper, closed, convex, pointed and solid cone. Furthermore, let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be defined as*

$$F(x) = \Big\{ f_1(x), f_2(x), \ldots, f_p(x) \Big\}, \tag{5.1}$$

and fix points $\bar{x} \in \mathbb{R}^n, e \in \operatorname{int} K$.

The motivation for considering $\mathcal{SOP}(F, K, \mathbb{R}^n)$ under Assumption 4 is twofold:

- In this setting, $\mathcal{SOP}(F, K, \mathbb{R}^n)$ is equivalent to the robust counterpart of a vector optimization problem with a finite uncertainty set (see Remark 2.4.17). Indeed, if in Remark 2.4.17 the uncertainty set $\mathcal{U} := \{u_1, \ldots, u_p\} \subset \mathbb{R}^q$ is considered, we can define $f_i(x) := f(x, u_i)$ to obtain a formulation as $\mathcal{SOP}(F, K, \mathbb{R}^n)$ with F given by (5.1). Conversely, if $\mathcal{SOP}(F, K, \mathbb{R}^n)$ is given and F has the structure (5.1), we can set $\mathcal{U} := \{1, \ldots, p\}$ and $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^m$ as

  $$f(x, i) := f_i(x)$$

  to recover the formulation (2.24).

- Also within the context of vector optimization under uncertainty, we believe that the treatment of robust counterpart problems with a finite uncertainty set is very useful when deriving methods for dealing with the general case, see for example [134] for a cutting plane strategy. In fact, under different assumptions, solving a problem with respect to a finite subset of a general uncertainty set can be enough to guarantee that a solution for the original problem was found, see [46, Theorem 5.9] and [22, Proposition 2.1].

The chapter is structured as follows. In Section 5.1, we present a short survey on the different methods for the solution of set optimization problems in the literature. In Section 5.2, we derive optimality conditions for $\mathcal{SOP}(F, K, \mathbb{R}^n)$ that are independent of those studied in Chapter 4. These optimality conditions constitute the basis of the descent method described in Section 5.3, where the full convergence of the algorithm is also obtained. Finally, in Section 5.4, we present the performance of the method on different test instances.

5.1 Literature Review

There are only a few methods for the solution of set optimization problems with respect to the set approach. These algorithms can be roughly clustered into three different groups:

- Algorithms for unconstrained problems with no particular structure of the set-valued objective mapping [90, 94, 107].

  In this setting, the algorithms derived are descent methods and use a derivative free strategy. First, a discretization of the unit sphere is chosen in advance. Then, at every iteration, an element in this discretization is labelled as a descent direction, and an initial initial step size is determined. This is achieved by comparing the images of the set-valued objective mapping both at the current iterate and at displacements of the current iterate in the directions of the discretization (by a specified step size). Then, in a second step, a line search is performed in the selected direction in order to refine the initial step size.

  In [90], a method for finding $\preceq_K^{(s)}$- minimal solutions was described. There, the case in which both the epigraphical and hypographical multifunctions of the set-valued objective mapping have convex values was analyzed. Furthermore, it is straightforward to deduce, from this reference, a method for each of the relations $\preceq_K^{(l)}$ and $\preceq_K^{(u)}$. The convexity assumption was then relaxed in [107], where the approach was extended for the preorder $\preceq_K^{(u)}$. Finally, in [94], the proposed method deals with the so called minmax order relation that was introduced in [95]. The main feature of this paper is that, instead of choosing only one descent direction at every iteration, it considers several of them at the same time. Thus, the method generates a tree with the initial point as the root, and the possible solutions as leaves.

- Algorithms for problems with a finite feasible set [65, 66, 110, 112].

The algorithms in this class are extensions, to the set-valued setting, of corresponding methods for vector optimization problems.

The first method proposed to find the minimal elements of a finite family of vectors in a partially ordered space can be found in [157]. This algorithm uses a so called forward reduction procedure that, in practice, avoids making many comparisons between vectors in the family. Therefore, this method performs more efficiently than a naive implementation in which every pair of vectors must be compared. The main drawback in that approach was that the output of the method would give only a superset of the set of minimal elements. This problem was later fixed by Jahn in [87, 96] by adding an extra reduction procedure after the first forward step (known in the literature as the backward iteration), thus obtaining exactly the set of minimal elements. The algorithms described in [110, 112] extend those of Jahn [87, 96] to deal with set-valued problems.

Very recently, Günther and Popovici [61] introduced a new strategy for the solution of the problem in the vector case. The idea now is to, first, find an enumeration of the elements in the family whose images by a strongly monotone functional are increasing. In a second step, the backward iteration procedure of Jahn [87, 96] is performed. The method guarantees to find all the minimal elements in the family. Moreover, the remarkable feature of that algorithm is the computational complexity, since it is superior to the other method derived by Jahn. This presorting idea was then extended to set optimization problems in [65, 66].

- Algorithms for robust counterpart problems arising in vector optimization under uncertainty [46, 48, 85, 86, 97, 146].

  The methods in this group are derived for problems where the set-valued objective mapping has the particular structure (2.24), and hence they are of most interest to us in this chapter. Except for the branch and bound scheme described in [48], the algorithms are based on some type of scalarization. Thus, as in any scalarization process, the idea is to substitute the set optimization problem by a scalar one and solve it. Typically, the resulting scalar problem is the robust counterpart of an optimization problem under uncertainty [21], and their solutions are $\preceq_K^{(u)}$- minimal or $\preceq_K^{(l)}$- minimal for the initial set optimization problem.

  In [46, 85, 86], a linear scalarization was employed for the solution of the set optimization problem. Furthermore, the ϵ- constraint method was extended too in [46, 85], for the particular case in which the ordering cone is the nonnegative orthant. Weighted Chebyshev scalarization and some of its variants (augmented, min-ordering) were also studied in [85, 97, 146].

  The main drawback of the scalarization- based methods in this class is that, in general, they are not able to recover all the solutions of the set optimization problem. In fact, the ϵ-constraint method, which is known to overcome this difficulty in standard multiobjective optimization, will fail in this setting. Thus, algorithms that are able to deal with this problem are of interest.

Our approach in this chapter is different. We propose a first order method for the solution of $\mathcal{SOP}(F, K, \mathbb{R}^n)$ under Assumption 4, that is a natural extension of those analyzed in [31] and [60] for vector optimization problems. The accumulation points of the sequence generated by the algorithm satisfy (under different assumptions) some type of necessary optimality conditions, and is able to detect whether a given point is already stationary or not. To the best of our knowledge, this would be the first method to have such property in the context of set optimization.

5.2 Optimality Conditions

In this section, we study optimality conditions for $\preceq_K^{(l)}$- weakly minimal solutions of $\mathcal{SOP}(F, K, \mathbb{R}^n)$. These conditions will be the foundation on which the proposed algorithm will be built. Although a natural idea would be to consider the result from Theorem 4.5.3, the computation of the coderivative of the set-valued objective mapping is very difficult in this particular case. In fact, because of the representation (5.1), for any point $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ we have the existence of neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$\operatorname{gph} F \cap (U \times V) = \left(\bigcup_{\{i \in \{1,\dots,p\} \mid \; f_i(\bar{x}) = \bar{y}\}} \operatorname{gph} f_i \right) \cap (U \times V). \tag{5.2}$$

Hence, in order to compute $D^*F(\bar{x}, \bar{y})$ we need to find $N((\bar{x}, \bar{y}), \operatorname{gph} F)$ which, according to (5.2), is equivalent to computing $N\left((\bar{x}, \bar{y}), \bigcup_{\{i \in \{1,\dots,p\} \mid \; f_i(\bar{x}) = \bar{y}\}} \operatorname{gph} f_i\right)$. However, to the best of our knowledge, there is no exact formula for finding this cone in terms of the initial data. Thus, instead of considering Theorem 4.5.3, we exploit the particular structure of F and the differentiability of the functionals f_i to deduce necessary conditions.

We start by defining some index-related set-valued mappings that will be of importance.

Definition 5.2.1. *Let Assumption 4 be fulfilled. The following set-valued mappings are defined:*

(i) The active index of minimal elements associated to F is $I : \mathbb{R}^n \rightrightarrows \{1, \dots, p\}$ given by

$$I(x) := \big\{ i \in \{1, \dots, p\} \mid f_i(x) \in \operatorname{Min}(F(x), K) \big\}.$$

(ii) The active index of weakly minimal elements associated to F is $I_0 : \mathbb{R}^n \rightrightarrows \{1, \dots, p\}$ defined as

$$I_0(x) := \big\{ i \in \{1, \dots, p\} \mid f_i(x) \in \operatorname{WMin}(F(x), K) \big\}.$$

(iii) For a vector $v \in \mathbb{R}^m$, we define $I_v : \mathbb{R}^n \rightrightarrows \{1, \dots, p\}$ as

$$I_v(x) := \{ i \in I(x) \mid f_i(x) = v \}.$$

It follows directly from the definition that $I_v(x) = \emptyset$ whenever $v \notin \operatorname{Min}(F(x), K)$ and that

$$\forall\, x \in \mathbb{R}^n : I(x) = \bigcup_{v \in \operatorname{Min}(F(x),K)} I_v(x). \tag{5.3}$$

Definition 5.2.2. *Let Assumption 4 be fulfilled. The map $\omega : \mathbb{R}^n \to \mathbb{R}$ is defined as the cardinality of the set of minimal elements of F, that is,*

$$\omega(x) := |\operatorname{Min}(F(x), K)|.$$

Furthermore, we set $\bar{\omega} := \omega(\bar{x})$.

From now on we consider that, for any point $x \in \mathbb{R}^n$, an enumeration $\{v_1^x, \ldots, v_{\omega(x)}^x\}$ of the set $\operatorname{Min}(F(x), K)$ has been chosen in advance.

Definition 5.2.3. *Let Assumption 4 be fulfilled and, for a given point $x \in \mathbb{R}^n$, consider the enumeration $\{v_1^x, \ldots, v_{\omega(x)}^x\}$ of the set $\operatorname{Min}(F(x), K)$. The partition set of x is defined as*

$$P_x := \prod_{j=1}^{\omega(x)} I_{v_i^x}(x),$$

where $I_{v_j^x}(x)$ is given in Definition 5.2.1 (iii) for $j \in \{1, \ldots, \omega(x)\}$.

The optimality conditions for $\mathcal{SOP}(F, K, \mathbb{R}^n)$ we will present are based on the following idea: from the particular structure of F, we will construct a family of vector optimization problems that, together, locally represent $\mathcal{SOP}(F, K, \mathbb{R}^n)$ (in a sense to be specified) around the point which must be checked for optimality. Then, standard, optimality conditions (via Theorem 2.4.10) are applied to the family of vector optimization problems. The following lemma is the key step in that direction.

Lemma 5.2.4. *Let Assumption 4 be fulfilled and let the cone $\tilde{K} \subset \prod_{j=1}^{\bar{\omega}} \mathbb{R}^m$ be defined as*

$$\tilde{K} := \prod_{j=1}^{\bar{\omega}} K. \tag{5.4}$$

Furthermore, consider the partition set $P_{\bar{x}}$ associated to $\bar{x}$ and define, for every $a \in P_{\bar{x}}$, the functional $\tilde{f}_a : \mathbb{R}^n \to \prod_{j=1}^{\bar{\omega}} \mathbb{R}^m$ as

$$\tilde{f}_a(x) := \begin{pmatrix} f_{a_1}(x) \\ \vdots \\ f_{a_{\bar{\omega}}}(x) \end{pmatrix}. \tag{5.5}$$

Then, $\bar{x}$ is a local $\preceq_K^{(l)}$- weakly minimal solution of $\mathcal{SOP}(F, K, \mathbb{R}^n)$ if and only if, for every $a \in P_{\bar{x}}$, $\bar{x}$ is a local weakly minimal solution of $\mathcal{VOP}(\tilde{f}_a, \tilde{K}, \mathbb{R}^n)$, that is,

$$\tilde{K}\text{-} \min_{x \in \mathbb{R}^n} \tilde{f}_a(x). \tag{$\mathcal{VOP}(\tilde{f}_a, \tilde{K}, \mathbb{R}^n)$}$$

Proof. We argue by contradiction in both cases. First, assume that $\bar{x}$ is a local $\preceq_K^{(l)}$- weakly minimal solution of $\mathcal{SOP}(F, K, \mathbb{R}^n)$ and that, for some $a \in P_{\bar{x}}$, $\bar{x}$ is not a local weakly minimal solution of $\mathcal{VOP}(\tilde{f}_a, \tilde{K}, \mathbb{R}^n)$. Then, we could find a sequence $\{x_k\}_{k\geq 1} \subset \mathbb{R}^n$ such that $x_k \to \bar{x}$ and

$$\forall\, k \in \mathbb{N} : \tilde{f}_a(x_k) \prec_{\tilde{K}} \tilde{f}_a(\bar{x}). \tag{5.6}$$

Hence, we deduce that

$$\begin{aligned}
\forall\, k \in \mathbb{N} : F(\bar{x}) &\overset{\text{(Proposition 2.4.7 }(i))}{\subseteq} \{f_{a_1}(\bar{x}), \ldots, f_{a_{\bar{\omega}}}(\bar{x})\} + K \\
&\overset{(5.6)}{\subseteq} \{f_{a_1}(x_k), \ldots, f_{a_{\bar{\omega}}}(x_k)\} + \operatorname{int} K + K \\
&\overset{\text{(Proposition 2.2.14 }(ii))}{\subseteq} F(x_k) + \operatorname{int} K.
\end{aligned}$$

Since this is equivalent to $F(x_k) \prec_K^{(l)} F(\bar{x})$ for every $k \in \mathbb{N}$ and $x_k \to \bar{x}$, we contradict the $\preceq_K^{(l)}$-weak minimality of $\bar{x}$.

Next, suppose that $\bar{x}$ is a local weakly minimal solution of $\mathcal{VOP}(\tilde{f}_a, \tilde{K}, \mathbb{R}^n)$ for every $a \in P_{\bar{x}}$, but not a local $\preceq_K^{(l)}$- weakly minimal solution of $\mathcal{SOP}(F, K, \mathbb{R}^n)$. Then, we could find a sequence $\{x_k\}_{k\geq 1} \subset \mathbb{R}^n$ such that $x_k \to \bar{x}$ and $F(x_k) \prec_K^{(l)} F(\bar{x})$ for every $k \in \mathbb{N}$. Consider the enumeration $\{v_1^{\bar{x}}, \ldots, v_{\bar{\omega}}^{\bar{x}}\}$ of the set $\operatorname{Min}(F(\bar{x}), K)$. Then,

$$\forall\, j \in \{1, \ldots, \bar{\omega}\}, k \in \mathbb{N}, \exists\, i_{(j,k)} \in \{1, \ldots, p\} : f_{i_{(j,k)}}(x_k) \prec_K v_j^{\bar{x}}. \tag{5.7}$$

Since the indexes $i_{(j,k)}$ are being chosen on the finite set $\{1, \ldots, p\}$, we can assume without loss of generality that $i_{(j,k)}$ is independent of k, that is, $i_{(j,k)} = \bar{i}_j$ for every $k \in \mathbb{N}$ and some $\bar{i}_j \in \{1, \ldots, p\}$. Hence, taking the limit in (5.7) when $k \to +\infty$, we get

$$\forall\, j \in \{1, \ldots, \bar{\omega}\} : f_{\bar{i}_j}(\bar{x}) \preceq_K v_j^{\bar{x}}. \tag{5.8}$$

Because $v_j^{\bar{x}} \in \operatorname{Min}(F(\bar{x}), K)$, it follows from (5.8) that $f_{\bar{i}_j}(\bar{x}) = v_j^{\bar{x}}$ and that $\bar{i}_j \in I(\bar{x})$ for every $j \in \{1, \ldots, \bar{\omega}\}$. Consider now the tuple $\bar{a} := (\bar{i}_1, \ldots, \bar{i}_{\bar{\omega}})$. Then, it can be verified that $\bar{a} \in P_{\bar{x}}$. Moreover, from (5.7) we deduce that $\tilde{f}_{\bar{a}}(x_k) \prec_{\tilde{K}} \tilde{f}_{\bar{a}}(\bar{x})$ for every $k \in \mathbb{N}$. Since $x_k \to \bar{x}$, we contradict the weak minimality of $\bar{x}$ for $\mathcal{VOP}(\tilde{f}_a, \tilde{K}, \mathbb{R}^n)$ when $a = \bar{a}$. □

We can now establish the necessary optimality condition for $\mathcal{SOP}(F, K, \mathbb{R}^n)$ that will be used in our descent method.

Theorem 5.2.5. *Let Assumption 4 be fulfilled and suppose that $\bar{x}$ is a local $\preceq_K^{(l)}$- weakly minimal solution of $\mathcal{SOP}(F, K, \mathbb{R}^n)$. Then,*

$$\forall\, a \in P_{\bar{x}},\ \exists\, \mu_1, \mu_2, \ldots, \mu_{\bar{\omega}} \in K^* : \sum_{j=1}^{\bar{\omega}} \nabla f_{a_j}(\bar{x})\mu_j = 0,\ (\mu_1, \ldots, \mu_{\bar{\omega}}) \neq 0. \tag{5.9}$$

Furthermore, if every f_i is K- convex for each $i \in I(\bar{x})$, this condition is also sufficient for the local $\preceq_K^{(l)}$- weak optimality of $\bar{x}$.

Proof. By Lemma 5.2.4, we get that $\bar{x}$ is a local weakly minimal solution of $\mathcal{VOP}(\tilde{f}_a, \tilde{K}, \mathbb{R}^n)$ for every $a \in P_{\bar{x}}$. Applying Theorem 2.4.10 (i) (in the Fréchet differentiable case) for every $a \in P_{\bar{x}}$, we get

$$\forall\, a \in P_{\bar{x}}, \exists\, \mu \in \tilde{K}^* \setminus \{0\} : \nabla \tilde{f}_a(\bar{x})\mu = 0. \tag{5.10}$$

Since $\tilde{K}^* = \prod_{j=1}^{\bar{\omega}} K^*$, it is easy to verify that (5.10) is equivalent to the first part of the statement

In order to see the sufficiency under convexity, assume that $\bar{x}$ satisfies (5.9). Note that for any $a \in P_{\bar{x}}$, the function $\tilde{f}_a$ is $\tilde{K}$- convex provided that each f_i is K- convex for every $i \in I(\bar{x})$. Hence, from Theorem 2.4.10 (ii), we deduce that $\bar{x}$ is a local weakly minimal solution of $\mathcal{VOP}(\tilde{f}_a, \tilde{K}, \mathbb{R}^n)$ for every $a \in P_{\bar{x}}$. Applying now Lemma 5.2.4, we obtain that $\bar{x}$ is a local $\preceq_K^{(l)}$- weakly minimal solution of $\mathcal{SOP}(F, K, \mathbb{R}^n)$. □

Based on Theorem 5.2.5, we define the following concepts of stationarity for $\mathcal{SOP}(F, K, \mathbb{R}^n)$.

Definition 5.2.6. *Let Assumption 4 be fulfilled. We say that $\bar{x}$ is a $\preceq_K^{(l)}$- stationary point of $\mathcal{SOP}(F, K, \mathbb{R}^n)$ if there exists a nonempty set $Q \subseteq P_{\bar{x}}$ such that the following assertion holds:*

$$\forall\, a \in Q,\ \exists\, \mu_1, \mu_2, \ldots, \mu_{\bar{\omega}} \in K^* : \sum_{j=1}^{\bar{\omega}} \nabla f_{a_j}(\bar{x})\mu_j = 0,\ (\mu_1, \ldots, \mu_{\bar{\omega}}) \neq 0. \tag{5.11}$$

In that case, we also say that $\bar{x}$ is $\preceq_K^{(l)}$- stationary with respect to Q. If, in addition, we can chose $Q = P_{\bar{x}}$ in (5.11), we simply call $\bar{x}$ a strongly $\preceq_K^{(l)}$- stationary point.

Remark 5.2.7. *It follows from Definition 5.2.6 that a strongly $\preceq_K^{(l)}$- stationary point of $\mathcal{SOP}(F, K, \mathbb{R}^n)$ is also $\preceq_K^{(l)}$- stationary with respect to Q for every $Q \subseteq P_{\bar{x}}$. In addition, from the proof of Theorem 5.2.5, we know that (5.11) is equivalent to $\bar{x}$ being a stationary point (in the sense of Theorem 2.4.10 (i)) of $\mathcal{VOP}(\tilde{f}_a, \tilde{K}, \mathbb{R}^n)$ for every $a \in Q$. Hence, in particular, $\preceq_K^{(l)}$-stationarity is also a necessary optimality condition for $\mathcal{SOP}(F, K, \mathbb{R}^n)$.*

Remark 5.2.8. *Let Assumption 4 be fulfilled and suppose that $m = 1, K = \mathbb{R}_+$. Furthermore, consider the functional $f : \mathbb{R}^n \to \mathbb{R}$ defined as*

$$f(x) := \min_{i=1,\ldots,p} f_i(x)$$

and problem $\mathcal{SOP}(F, K, \mathbb{R}^n)$ associated to this data. Hence, in this case,

$$I(\bar{x}) = \{i \in \{1, \ldots, p\} \mid f_i(\bar{x}) = f(\bar{x})\}.$$

It is then easy to verify that the following statements hold:

(i) $\bar{x}$ is strongly $\preceq_K^{(l)}$- stationary for $\mathcal{SOP}(F, K, \mathbb{R}^n)$ if and only if

$$\forall\, i \in I(\bar{x}) : \nabla f_i(\bar{x}) = 0.$$

(ii) $\bar{x}$ *is* $\preceq_K^{(l)}$*- stationary for* $\mathcal{SOP}(F, K, \mathbb{R}^n)$ *if and only if*

$$\exists\, i \in I(\bar{x}) : \nabla f_i(\bar{x}) = 0.$$

On the other hand, one could also check that $\bar{x}$ *is a* $\preceq_K^{(l)}$*- weakly minimal solution of* $\mathcal{SOP}(F, K, \mathbb{R}^n)$ *if and only if* $\bar{x}$ *solves* $\mathcal{OP}(f, \mathbb{R}^n)$*, see Definition 2.4.1. Then, taking into account Theorem 2.4.2 and Remark 2.4.4, we find that the inclusions*

$$0 \in \widehat{\partial} f(\bar{x}) \tag{5.12}$$

and

$$0 \in \partial f(\bar{x}) \tag{5.13}$$

are necessary for $\bar{x}$ *being a* $\preceq_K^{(l)}$*- weakly minimal solution of* $\mathcal{SOP}(F, K, \mathbb{R}^n)$*. Furthermore, from [45, Proposition 5] and [132, Proposition 1.113], we have*

$$\widehat{\partial} f(\bar{x}) = \bigcap_{i \in I(\bar{x})} \{\nabla f_i(\bar{x})\} \tag{5.14}$$

and

$$\partial f(\bar{x}) \subseteq \bigcup_{i \in I(\bar{x})} \{\nabla f_i(\bar{x})\} \tag{5.15}$$

respectively. Thus, from (5.12), (5.14) *and (i), we deduce that*

(iii) $\bar{x}$ *is strongly* $\preceq_K^{(l)}$*- stationary for* $\mathcal{SOP}(F, K, \mathbb{R}^n)$ *if and only if* $\bar{x}$ *is stationary for* $\mathcal{OP}(f, \mathbb{R}^n)$ *in the sense of Fréchet, see Remark 2.4.4.*

Similarly, from (5.13), (5.15) *and (ii), we find that*

(iii) If $\bar{x}$ *is stationary for* $\mathcal{OP}(f, \mathbb{R}^n)$ *in the sense of Definition 2.4.3, then* $\bar{x}$ *is* $\preceq_K^{(l)}$*- stationary for* $\mathcal{SOP}(F, K, \mathbb{R}^n)$*.*

We close the section with the following proposition, that presents an alternative characterization of $\preceq_K^{(l)}$- stationary points.

Proposition 5.2.9. *Let Assumption 4 be fulfilled and let* $Q \subseteq P_{\bar{x}}$*. Then,* $\bar{x}$ *is* $\preceq_K^{(l)}$*- stationary for* $\mathcal{SOP}(F, K, \mathbb{R}^n)$ *with respect to* Q *if and only if*

$$\forall\, a \in Q, u \in \mathbb{R}^n, \exists\, j \in \{1, \ldots, \bar{\omega}\} : \nabla f_{a_j}(\bar{x})^\top u \notin -\operatorname{int} K. \tag{5.16}$$

Proof. Suppose first that $\bar{x}$ is $\preceq_K^{(l)}$- stationary with respect to Q. Fix now $a \in Q, u \in \mathbb{R}^n$, and consider the vectors $\mu_1, \mu_2, \ldots, \mu_{\bar{\omega}} \in K^*$ that satisfy (5.11). We argue by contradiction. Assume that

$$\forall\, j \in \{1, \ldots, \bar{\omega}\} : \nabla f_{a_j}(\bar{x})^\top u \in -\operatorname{int} K. \tag{5.17}$$

From (5.17) and the fact that $(\mu_1, \ldots, \mu_{\bar{\omega}}) \in \left(\prod_{j=1}^{\bar{\omega}} K^*\right) \setminus \{0\}$, we deduce that

$$\left(\mu_1^\top \left(\nabla f_{a_1}(\bar{x})^\top u\right), \ldots, \mu_j^\top \left(\nabla f_{a_j}(\bar{x})^\top u\right)\right) \in -\mathbb{R}_+^{\bar{\omega}} \setminus \{0\}. \tag{5.18}$$

Hence, we get

$$0 \overset{(5.11)}{=} \left(\sum_{j=1}^{\bar{\omega}} \nabla f_{a_j}(\bar{x}) \mu_j\right)^\top u = \sum_{j=1}^{\bar{\omega}} \mu_j^\top \left(\nabla f_{a_j}(\bar{x})^\top u\right) \overset{(5.18)}{<} 0,$$

a contradiction.

Suppose now that (5.16) holds, and fix $a \in Q$. Consider the functional $\tilde{f}_a$ and the cone $\tilde{K}$ from Lemma 5.2.4, together with the set

$$A := \left\{\nabla \tilde{f}_a(\bar{x})^\top u \mid u \in \mathbb{R}^n\right\}.$$

Then, we deduce from (5.16) that

$$A \cap \operatorname{int} \tilde{K} = \emptyset.$$

Applying Theorem 2.1.26 (i), we obtain $(\mu_1, \ldots, \mu_{\bar{\omega}}) \in \left(\prod_{j=1}^{\bar{\omega}} \mathbb{R}^m\right) \setminus \{0\}$ such that

$$\forall\, u \in \mathbb{R}^n, v_1, \ldots, v_{\bar{\omega}} \in K : \left(\sum_{j=1}^{\bar{\omega}} \nabla f_{a_j}(\bar{x}) \mu_j\right)^\top u \leq \sum_{j=1}^{\bar{\omega}} \mu_j^\top v_j. \tag{5.19}$$

By fixing $\bar{j} \in \{1, \ldots, \bar{\omega}\}$ and substituting $u = 0, v_j = 0$ for $j \neq \bar{j}$ in (5.19), we obtain

$$\forall\, v_{\bar{j}} \in K : \mu_{\bar{j}}^\top v_{\bar{j}} \geq 0.$$

Hence, $\mu_{\bar{j}} \in K^*$. Since $\bar{j}$ was chosen arbitrarily, we get that $(\mu_1, \ldots, \mu_{\bar{\omega}}) \in \left(\prod_{j=1}^{\bar{\omega}} K^*\right) \setminus \{0\}$. Define now

$$\bar{u} := \sum_{j=1}^{\bar{\omega}} \nabla f_{a_j}(\bar{x}) \mu_j.$$

Then, to finish the proof, we need to show that $\bar{u} = 0$. In order to see this, substitute $u = \bar{u}$ and $v_j = 0$ for each $j \in \{1, \ldots, \bar{\omega}\}$ in (5.19) to obtain

$$\left\| \sum_{j=1}^{\bar{\omega}} \nabla f_{a_j}(\bar{x}) \mu_j \right\|^2 \leq 0.$$

Hence, it can only be $\bar{u} = 0$, and statement (5.11) is true.

$\square$

5.3 Descent Method and its Convergence Analysis

Now we present the solution approach. It is clearly based on the result shown in Lemma 5.2.4. At every iteration, an element a in the partition set of the current iterate point is selected, and then a descent direction for $\mathcal{VOP}(\tilde{f}_a, \tilde{K}, \mathbb{R}^n)$ will be found using ideas from [31, 60]. However, one must be careful with the selection process of the element a in order to guarantee convergence. Thus, we propose a specific way to achieve this. After the descent direction is determined, we follow a classical backtracking procedure of Armijo type to determine a suitable step size, and we update the iterate in the desired direction. Formally, the method is the following:

Algorithm 1: Descent Method in Set Optimization

Step 0. Choose $x_0 \in \mathbb{R}^n, \beta, \nu \in (0,1)$, and set $k := 0$.

Step 1. Compute

$$M_k := \operatorname{Min}(F(x_k), K), \;\; P_k := P_{x_k}, \;\; \omega_k := |\operatorname{Min}(F(x_k), K)|.$$

Step 2. Find

$$(a_k, u_k) \in \underset{(a,u)\in P_k\times\mathbb{R}^n}{\operatorname{argmin}} \max_{j=1,\ldots,\omega_k} \left\{\psi_e\big(\nabla f_{a_j}(x_k)^\top d\big)\right\} + \frac{1}{2}\|u\|^2.$$

Step 3. If $u_k = 0$, Stop. Otherwise, go to Step 4.

Step 4. Compute

$$t_k := \max_{q\in\mathbb{N}\cup\{0\}} \left\{\nu^q \mid \forall\, j \in \{1,\ldots,\omega_k\} : f_{a_{k,j}}(x_k+\nu^q u_k) \preceq_K f_{a_{k,j}}(x_k)+\beta\nu^q\nabla f_{a_{k,j}}(x_k)^\top u_k\right\}.$$

Step 5. Set $x_{k+1} := x_k + t_k u_k,\; k := k+1$ and go to Step 2.

Remark 5.3.1. *It is possible to verify that, when $p = 1$, Algorithm 1 reduces to the methods described in [31, 60] for vector optimization problems. Indeed, the only difference in that case would be the scalarizing functional employed in Steps 2 and 4: in [60], an element $\psi_G \in \mathcal{S}_{DS}$, and in [31], an element $\psi_{\|\cdot\|} \in \mathcal{S}_{HU}$, see Definition 3.2.1. However, by Corollary 3.2.1, $\psi_e \in \mathcal{S}_{HU} \subset \mathcal{S}_{DS}$, and hence our assertion is true.*

Now, we start the convergence analysis of Algorithm 1. Our first lemma describes local properties of the active indexes.

Lemma 5.3.2. *Let Assumption 4 be fulfilled. Then, there exists a neighborhood U of $\bar{x}$ such that the following properties are satisfied (some of them under additional conditions to be established below) for every $x \in U$:*

(i) $I_0(x) \subseteq I_0(\bar{x})$,

(ii) $I(x) \subseteq I(\bar{x})$, *provided that* $\operatorname{Min}(F(\bar{x}), K) = \operatorname{WMin}(F(\bar{x}), K)$,

(iii) $\forall\, v \in \operatorname{Min}(F(\bar{x}), K) : \operatorname{Min}\left(\{f_i(x)\}_{i \in I_v(\bar{x})}, K\right) \subseteq \operatorname{Min}(F(x), K)$,

(iv) $\forall\, v_1, v_2 \in \operatorname{Min}(F(\bar{x}), K), v_1 \neq v_2 : \operatorname{Min}\left(\{f_i(x)\}_{i \in I_{v_1}(\bar{x})}, K\right) \cap \operatorname{Min}\left(\{f_i(x)\}_{i \in I_{v_2}(\bar{x})}, K\right) = \emptyset$,

(v) $\omega(x) \geq \omega(\bar{x})$.

Proof. It suffices to show the existence of the neighborhood U for each item independently, as we could later take the intersection of them to satisfy all the properties.

(i) Assume that this is not satisfied in any neighborhood U of $\bar{x}$. Then, we could find a sequence $\{x_k\}_{k \geq 1} \subset \mathbb{R}^n$ such that $x_k \to \bar{x}$ and

$$\forall\, k \in \mathbb{N} : I_0(x_k) \setminus I_0(\bar{x}) \neq \emptyset. \tag{5.20}$$

Because of the finite cardinality of all possible differences in (5.20), we can assume without loss of generality that there exists a common $\bar{i} \in \{1, \ldots, p\}$ such that

$$\forall\, k \in \mathbb{N} : \bar{i} \in I_0(x_k) \setminus I_0(\bar{x}). \tag{5.21}$$

In particular, (5.21) implies that $\bar{i} \in I_0(x_k)$. Hence, we get

$$\forall\, k \in \mathbb{N},\ i \in \{1, \ldots, p\} :\ f_i(x_k) - f_{\bar{i}}(x_k) \in -\left(\mathbb{R}^m \setminus \operatorname{int} K\right).$$

Since $\mathbb{R}^m \setminus \operatorname{int} K$ is closed, taking the limit when $k \to +\infty$ we obtain

$$\forall\, i \in \{1, \ldots, p\} :\ f_i(\bar{x}) - f_{\bar{i}}(\bar{x}) \in -\left(\mathbb{R}^m \setminus \operatorname{int} K\right).$$

Hence, we deduce that $f_{\bar{i}}(\bar{x}) \in \operatorname{WMin}(F(\bar{x}), K)$ and $\bar{i} \in I_0(\bar{x})$, a contradiction to (5.20).

(ii) Consider the same neighborhood U on which statement (i) holds. Note that, under the given assumption, we have $I_0(\bar{x}) = I(\bar{x})$. This, together with statement (i), implies:

$$\forall\, x \in U :\ I(x) \subseteq I_0(x) \subseteq I_0(\bar{x}) = I(\bar{x}).$$

(iii) For this statement, it is also sufficient to show that the neighborhood U can be chosen for any point in the set $\operatorname{Min}(F(\bar{x}), K)$. Hence, fix $v \in \operatorname{Min}(F(\bar{x}), K)$ and assume that there is no neighborhood U of $\bar{x}$ on which the statement is satisfied. Then, we could find sequences $\{x_k\}_{k \geq 1} \subset \mathbb{R}^n$ and $\{i_k\}_{k \geq 1} \subseteq I_v(\bar{x})$ such that $x_k \to \bar{x}$ and

$$\forall\, k \in \mathbb{N} : f_{i_k}(x_k) \in \operatorname{Min}(\{f_i(x_k)\}_{i \in I_v(\bar{x})}, K) \setminus \operatorname{Min}(F(x_k), K). \tag{5.22}$$

Since $I_v(\bar{x})$ is finite, we deduce that the elements in the sequence $\{i_k\}$ can only take a finite number values. Hence, we can assume without loss of generality that there exists $\bar{i} \in I_v(\bar{x})$ such that $i_k = \bar{i}$ for every $k \in \mathbb{N}$. Then, (5.22), is equivalent to

$$\forall\, k \in \mathbb{N} : f_{\bar{i}}(x_k) \in \operatorname{Min}(\{f_i(x_k)\}_{i \in I_v(\bar{x})}, K) \setminus \operatorname{Min}(F(x_k), K). \tag{5.23}$$

From (5.23) we get in particular that $f_{\bar{i}}(x_k) \notin \operatorname{Min}(F(x_k), K)$ for every $k \in \mathbb{N}$. This, together with the domination property in Proposition 2.4.7 (i) and the fact that the sets $I(x_k)$ are contained in the finite set $\{1, \ldots, p\}$, allow us to obtain without loss of generality the existence of $\tilde{i} \in I(\bar{x})$ such that

$$\forall\, k \in \mathbb{N} : \; f_{\bar{i}}(x_k) \preceq f_{\tilde{i}}(x_k), \; f_{\bar{i}}(x_k) \neq f_{\tilde{i}}(x_k). \tag{5.24}$$

Taking the limit in (5.24) now when $k \to +\infty$ we obtain $f_{\tilde{i}}(\bar{x}) \preceq f_{\bar{i}}(\bar{x}) = v$. Since v is a minimal element of $F(\bar{x})$, it can only be $f_{\tilde{i}}(\bar{x}) = v$ and, hence, $\tilde{i} \in I_v(\bar{x})$. From this, the first inequality in (5.24), and the fact that $f_{\bar{i}}(x_k) \in \operatorname{Min}(\{f_i(x_k)\}_{i \in I_v(\bar{x})}, K)$ for every $k \in \mathbb{N}$, we get that $f_{\bar{i}}(x_k) = f_{\tilde{i}}(x_k)$ for all $k \in \mathbb{N}$. This contradicts the second part of (5.24), and hence our statement is true.

(iv) It follows directly from the continuity of the functionals f_i, $i = 1, \ldots, p$.

(v) The statement is an immediate consequence of (iii) and (iv). □

For the main convergence theorem of our method, we will need the notion of regularity of a point for a set-valued mapping.

Definition 5.3.3. *Let Assumption 4 be fulfilled. We say that $\bar{x}$ is a regular point of F if the following conditions are satisfied:*

(i) $\operatorname{Min}(F(\bar{x}), K) = \operatorname{WMin}(F(\bar{x}), K)$,

(ii) the functional ω is constant in a neighborhood of $\bar{x}$.

Remark 5.3.4. *Since we will analyze the stationarity of the regular limit points of the sequence generated by Algorithm 1, the following points must be addressed:*

- *Notice that, by definition, the regularity property of a point is independent of the optimality concept in Definition 2.4.14 (ii). Thus, by only knowing that a point is regular, we can not infer anything about whether it is optimal or not.*
- *The concept of regularity seems to be linked to the complexity of comparing sets in a high dimensional space. For example, in case $m = 1$ or $p = 1$, every point in $\mathbb{R}^n$ is regular for any set-valued mapping F of the form (5.1). Indeed, in these cases, we have $\omega(x) = 1$ and*

$$\operatorname{Min}(F(x), K) = \operatorname{WMin}(F(x), K) = \begin{cases} \left\{ \min\limits_{i=1,\ldots,p} f_i(x) \right\} & \text{if } m = 1, \\ \{f_1(x)\} & \text{if } p = 1 \end{cases}$$

for all $x \in \mathbb{R}^n$.

A natural question is whether regularity is a strong assumption to impose on a point. In that sense, given the finite structure of the sets $F(x)$, the condition (i) in Definition 5.3.3 seems to be very reasonable. In fact, we would expect that, for most practical cases, this condition is fulfilled at almost every point. For condition (ii), a formalized statement is derived in Proposition 5.3.5 below.

Proposition 5.3.5. *Let Assumption 4 be fulfilled. Then, the set*

$$S := \{x \in \mathbb{R}^n \mid \omega \text{ is locally constant at } x\}$$

is open and dense in $\mathbb{R}^n$.

Proof. (i) The openness is trivial. Suppose now that S is not dense in $\mathbb{R}^n$. Then, $\mathbb{R}^n \setminus (\operatorname{cl} S)$ is nonempty and open. Furthermore, since ω is bounded above, the real number

$$p_0 := \max_{x \in \mathbb{R}^n \setminus (\operatorname{cl} S)} \omega(x)$$

is well defined. Consider the set

$$A := \left\{ x \in \mathbb{R}^n \mid \omega(x) \leq p_0 - \frac{1}{2} \right\}.$$

From Lemma 5.3.2 (v), it follows that ω is lower semicontinuous. Hence, A is closed as it is the sublevel set of a lower semicontinuous functional, see [145, Lemma 1.7.2]. Consider now the set

$$U := (\mathbb{R}^n \setminus (\operatorname{cl} S)) \cap (\mathbb{R}^n \setminus A).$$

Then, U is a nonempty open subset of $\mathbb{R}^n \setminus (\operatorname{cl} S)$. This, together with the definition of A, gives us $\omega(x) = p_0$ for every $x \in U$. However, this contradicts the fact that ω is not locally constant at any point of $\mathbb{R}^n \setminus (\operatorname{cl} S)$. Hence, S is dense in $\mathbb{R}^n$.

□

An essential property of regular points of a set-valued mapping is described in the next proposition.

Proposition 5.3.6. *Let Assumption 4 be fulfilled and suppose that $\bar{x}$ is a regular point of F. Then, there exists a neighborhood U of $\bar{x}$ such that the following properties hold for every $x \in U$:*

(i) $\omega(x) = \bar{\omega}$,

(ii) there is an enumeration $\{w_1^x, \ldots, w_{\bar{\omega}}^x\}$ *of* $\operatorname{Min}(F(x), K)$ *such that*

$$\forall\, j \in \{1, \ldots, \bar{\omega}\} : I_{w_j^x}(x) \subseteq I_{v_j^{\bar{x}}}(\bar{x}).$$

In particular, without loss of generality, we have $P_x \subseteq P_{\bar{x}}$ for every $x \in U$.

Proof. Let U be the neighborhood of $\bar{x}$ from Lemma 5.3.2. Since $\bar{x}$ is a regular point of F, we can assume without loss of generality that ω is constant on U. Hence, property (i) is fulfilled. Fix now $x \in U$ and consider the enumeration $\{v_1^{\bar{x}}, \ldots, v_{\bar{\omega}}^{\bar{x}}\}$ of $\operatorname{Min}(F(\bar{x}), K)$. Then, from properties (iii) and (iv) in Lemma 5.3.2 and the fact that $\omega(x) = \bar{\omega}$, we deduce that

$$\forall\, j \in \{1, \ldots, \bar{\omega}\} : \left| \operatorname{Min}\left(\{f_i(x)\}_{i \in I_{v_j^{\bar{x}}}(\bar{x})}, K \right) \right| = 1. \tag{5.25}$$

Next, for $j \in \{1, \ldots, \bar{\omega}\}$, we define w_j^x as the unique element of the set $\operatorname{Min}\left(\{f_i(x)\}_{i \in I_{v_j^{\bar{x}}}(\bar{x})}, K\right)$. Then, from (5.25), property (iii) in Lemma 5.3.2 and the fact that ω is constant on U, we obtain that $\{w_1^x, \ldots, w_{\bar{\omega}}^x\}$ is an enumeration of the set $\operatorname{Min}(F(x), K)$.

It remains to show now that this enumeration satisfies (ii). In order to see this, fix $j \in \{1, \ldots, \bar{\omega}\}$ and $\bar{i} \in I_{w_j^x}(x)$. Then, from the regularity of $\bar{x}$ and property (ii) in Lemma 5.3.2, we get that $I(x) \subseteq I(\bar{x})$. In particular, this implies $\bar{i} \in I(\bar{x})$. From this and (5.3), we have the existence of $j' \in \{1, \ldots, \bar{\omega}\}$ such that $\bar{i} \in I_{v_{j'}^{\bar{x}}}(\bar{x})$. Hence, we deduce that

$$w_j^x = f_{\bar{i}}(x) \in \{f_i(x)\}_{i \in I_{v_{j'}^{\bar{x}}}(\bar{x})}. \tag{5.26}$$

Then, from (5.25), (5.26) and the definition of $w_{j'}^x$, we find that $w_{j'}^x \preceq_K w_j^x$. Moreover, because $w_{j'}^x, w_j^x \in \operatorname{Min}(F(x), K)$, it can only be $w_{j'}^x = w_j^x$. Thus, it follows that $j = j'$, since $\{w_1^x, \ldots, w_{\bar{\omega}}^x\}$ is an enumeration of the set $\operatorname{Min}(F(x), K)$. This shows that $\bar{i} \in I_{v_j^{\bar{x}}}(\bar{x})$, as desired.

□

For the rest of the analysis we need to introduce the parametric family of functionals $\{\varphi_x\}_{x \in \mathbb{R}^n}$, whose elements $\varphi_x : P_x \times \mathbb{R}^n \to \mathbb{R}$ are defined as follows:

$$\forall\, a \in P_x, u \in \mathbb{R}^n : \varphi_x(a, u) := \max_{j=1,\ldots,\omega(x)} \left\{\psi_e\big(\nabla f_{a_j}(x)^\top u\big)\right\} + \frac{1}{2}\|u\|^2. \tag{5.27}$$

It is easy to see that, for every $x \in \mathbb{R}^n$ and $a \in P_x$, the functional $\varphi_x(a, \cdot)$ is strongly convex in $\mathbb{R}^n$, that is, there exists a constant $\alpha > 0$ such that the inequality

$$\varphi_x\left(a, tu + (1-t)u'\right) + \alpha t(1-t)\|u - u'\|^2 \leq t\varphi_x(a, u) + (1-t)\varphi_x\left(a, u'\right)$$

is satisfied for every $u, u' \in \mathbb{R}^n$ and $t \in [0, 1]$. According to [53, Lemma 3.9], the functional $\varphi_x(a, \cdot)$ attains its minimum over $\mathbb{R}^n$, and this minimum is unique. In particular, we can check that

$$\forall\, x \in \mathbb{R}^n, a \in P_x : \min_{u \in \mathbb{R}^n} \varphi_x(a, u) \leq 0 \tag{5.28}$$

and that, if $u_a \in \mathbb{R}^n$ is such that $\varphi_x(a, u_a) = \min_{u \in \mathbb{R}^n} \varphi_x(a, u)$, then

$$\varphi_x(a, u_a) = 0 \Longleftrightarrow u_a = 0. \tag{5.29}$$

Taking into account that P_x is finite, we also obtain that φ_x attains its minimum over the set $P_x \times \mathbb{R}^n$. Hence, we can consider the functional $\phi : \mathbb{R}^n \to \mathbb{R}$ given by

$$\phi(x) := \min_{(a,u) \in P_x \times \mathbb{R}^n} \varphi_x(a, u). \tag{5.30}$$

Then, because of (5.28), it can be verified that

$$\forall\, x \in \mathbb{R}^n : \phi(x) \leq 0. \tag{5.31}$$

Furthermore, if $(a,u) \in P_x \times \mathbb{R}^n$ is such that $\phi(x) = \varphi_x(a,u)$, it follows from (5.29) (see also [60]) that

$$\phi(x) = 0 \Longleftrightarrow u = 0. \tag{5.32}$$

In the following two propositions we show that Algorithm 1 is well defined. We start by proving that, if Algorithm 1 stops in Step 3, a $\preceq^{(l)}_K$- stationary point was found.

Proposition 5.3.7. *Let Assumption 4 be fulfilled and consider the functionals $\varphi_{\bar{x}}$ and ϕ given in* (5.27) *and* (5.30) *respectively. Furthermore, let $(\bar{a},\bar{u}) \in P_{\bar{x}} \times \mathbb{R}^n$ be such that $\phi(\bar{x}) = \varphi_{\bar{x}}(\bar{a},\bar{u})$. Then, the following statements are equivalent:*

(i) $\bar{x}$ is a strongly $\preceq^{(l)}_K$- stationary point of $\mathcal{SOP}(F,K,\mathbb{R}^n)$,

(ii) $\phi(\bar{x}) = 0$,

(iii) $\bar{u} = 0$.

Proof. The result will be a consequence of [31, Proposition 2.2] where, using an Hiriart- Urruty functional, a similar statement is proved for vector optimization problems. Consider the cone $\tilde{K}$ given by (5.4) , the vector $\tilde{e} := \begin{pmatrix} e \\ \vdots \\ e \end{pmatrix} \in \operatorname{int} \tilde{K}$, and the scalarizing functional $\psi_{\tilde{e}}$ associated to $\tilde{e}$ and $\tilde{K}$, see Definition 2.5.5 (i). Then, for any $v_1, \ldots, v_{\bar{\omega}} \in \mathbb{R}^m$ and $v := \begin{pmatrix} v_1 \\ \vdots \\ v_{\bar{\omega}} \end{pmatrix}$, we get

$$\begin{aligned} \psi_{\tilde{e}}(v) &= \min\{t \in \mathbb{R} \mid t\tilde{e} \in v + \tilde{K}\} \\ &= \min\{t \in \mathbb{R} \mid \forall\, j \in \{1,\ldots,\bar{\omega}\} : te \in v_j + K\} \\ &= \max_{j=1,\ldots,\bar{\omega}} \psi_e(v_j). \end{aligned} \tag{5.33}$$

From Theorem 3.2.7, we know that $\psi_{\tilde{e}} \in \mathcal{S}_{HU}$, the class of Hiriart-Urruty functionals. Hence, for a fixed $a \in P_{\bar{x}}$ we can apply [31, Proposition 2.2] to $\mathcal{VOP}(\tilde{f}_a, \tilde{K}, \mathbb{R}^n)$ to obtain that

$$\bar{x} \text{ is a stationary point of } \mathcal{VOP}(\tilde{f}_a, \tilde{K}, \mathbb{R}^n) \Longleftrightarrow \min_{u \in \mathbb{R}^n} \ \psi_{\tilde{e}}\left(\nabla \tilde{f}_a(\bar{x})^\top u\right) + \frac{1}{2}\|u\|^2 = 0 \tag{5.34}$$

Thus, we deduce that

$$
\begin{aligned}
\bar{x} \text{ is strongly } \preceq_K^{(l)}\text{- stationary} \quad &\overset{\text{(Remark 5.2.7)}}{\Longleftrightarrow} \quad \forall\, a \in P_{\bar{x}} : \bar{x} \text{ is stationary for } \mathcal{VOP}(\tilde{f}_a, \tilde{K}, \mathbb{R}^n) \\
&\overset{(5.34)}{\Longleftrightarrow} \quad \forall\, a \in P_{\bar{x}} : \min_{u \in \mathbb{R}^n} \psi_{\tilde{e}}\left(\nabla \tilde{f}_a(\bar{x})^\top u\right) + \frac{1}{2}\|u\|^2 = 0 \\
&\overset{((5.27)+(5.33))}{\Longleftrightarrow} \quad \forall\, a \in P_{\bar{x}} : \min_{u \in \mathbb{R}^n} \varphi_{\bar{x}}(a, u) = 0 \\
&\Longleftrightarrow \quad \min_{(a,u) \in P_{\bar{x}} \times \mathbb{R}^n} \varphi_{\bar{x}}(a, u) = 0 \\
&\overset{(5.30)}{\Longleftrightarrow} \quad \phi(\bar{x}) = 0 \\
&\overset{(5.32)}{\Longleftrightarrow} \quad \bar{u} = 0,
\end{aligned}
$$

as desired.

□

Remark 5.3.8. *A similar statement to the one in Proposition 5.3.7 can be made for* $\preceq_K^{(l)}$*-stationary points of* $\mathcal{SOP}(F, K, \mathbb{R}^n)$*. Indeed, for a set* $Q \subseteq P_{\bar{x}}$*, consider a point* $(\bar{a}_Q, \bar{u}_Q) \in Q \times \mathbb{R}^n$ *such that* $\varphi_{\bar{x}}(\bar{a}_Q, \bar{u}_Q) = \min_{(a,u) \in Q \times \mathbb{R}^n} \varphi_{\bar{x}}(a, u)$*. Then, by replacing* $P_{\bar{x}}$ *by* Q *in the proof of Proposition 5.3.7, we can show that the following statements are equivalent:*

(i) $\bar{x}$ *is* $\preceq_K^{(l)}$*- stationary for* $\mathcal{SOP}(F, K, \mathbb{R}^n)$ *with respect to* Q*,*

(ii) $\min_{(a,u) \in Q \times \mathbb{R}^n} \varphi_{\bar{x}}(a, u) = 0,$

(iii) $\bar{u}_Q = 0.$

Next, we show that the line search in Step 4 of Algorithm 1 terminates in finitely many steps.

Proposition 5.3.9. *Let Assumption 4 be fulfilled and fix* $\beta \in (0, 1)$*. Consider the functionals* $\varphi_{\bar{x}}$ *and* ϕ *given in* (5.27) *and* (5.30) *respectively. Furthermore, let* $(\bar{a}, \bar{u}) \in P_{\bar{x}} \times \mathbb{R}^n$ *be such that* $\phi(\bar{x}) = \varphi_{\bar{x}}(\bar{a}, \bar{u})$ *and suppose that* $\bar{x}$ *is not a strongly* $\preceq_K^{(l)}$*- stationary point of* $\mathcal{SOP}(F, K, \mathbb{R}^n)$*. The following assertions hold:*

(i) There exists $\tilde{t} > 0$ *such that*

$$\forall\, t \in (0, \tilde{t}\,], j \in \{1, \ldots, \bar{\omega}\} : f_{\bar{a}_j}(\bar{x} + t\bar{u}) \preceq_K f_{\bar{a}_j}(\bar{x}) + \beta t \nabla f_{\bar{a}_j}(\bar{x})^\top \bar{u}.$$

(ii) Let $\tilde{t}$ *be the parameter in statement* (i)*. Then,*

$$\forall\, t \in (0, \tilde{t}\,] : F(\bar{x} + t\bar{u}) \preceq_K^{(l)} \left\{ f_{\bar{a}_j}(\bar{x}) + \beta t \nabla f_{\bar{a}_j}(\bar{x})^\top \bar{u} \right\}_{j \in \{1, \ldots, \bar{\omega}\}} \prec_K^{(l)} F(\bar{x}),$$

In particular, $\bar{u}$ *is a descent direction of* F *at* $\bar{x}$ *with respect to the preorder* $\preceq_K^{(l)}$ *.*

Proof. (i) Assume otherwise. Then, we could find a sequence $\{t_k\}_{k\geq 1}$ and $\bar{j} \in \{1, \ldots, \bar{\omega}\}$ such that $t_k \to 0$ and

$$\forall\, k \in \mathbb{N} : f_{\bar{a}_{\bar{j}}}(\bar{x} + t_k \bar{u}) - f_{\bar{a}_{\bar{j}}}(\bar{x}) - \beta t_k \nabla f_{\bar{a}_{\bar{j}}}(\bar{x})^\top \bar{u} \notin -K. \tag{5.35}$$

As $(\mathbb{R}^m \setminus -K) \cup \{0\}$ is a cone, we can multiply (5.35) by $\frac{1}{t_k}$ for each $k \in \mathbb{N}$ to obtain

$$\forall\, k \in \mathbb{N} : \frac{f_{\bar{a}_{\bar{j}}}(\bar{x} + t_k \bar{u}) - f_{\bar{a}_{\bar{j}}}(\bar{x})}{t_k} - \beta \nabla f_{\bar{a}_{\bar{j}}}(\bar{x})^\top \bar{u} \notin -K. \tag{5.36}$$

Taking now the limit in (5.36) when $k \to +\infty$ we get

$$(1-\beta)\nabla f_{\bar{a}_{\bar{j}}}(\bar{x})^\top \bar{u} \notin -\operatorname{int} K.$$

Since $\beta \in (0,1)$, this is equivalent to

$$\nabla f_{\bar{a}_{\bar{j}}}(\bar{x})^\top \bar{u} \notin -\operatorname{int} K. \tag{5.37}$$

On the other hand, since $\bar{x}$ is not strongly $\preceq_K^{(l)}$- stationary, we can apply Proposition 5.3.7 to obtain that $\bar{u} \neq 0$ and that $\phi(\bar{x}) < 0$. This implies that $\varphi_{\bar{x}}(\bar{a}, \bar{u}) < 0$, and hence

$$\max_{j=1,\ldots,\bar{\omega}} \left\{ \psi_e \left(\nabla f_{\bar{a}_j}(\bar{x})^\top \bar{u} \right) \right\} < -\frac{1}{2}\|\bar{u}\|^2 < 0.$$

From this, we deduce that

$$\forall\, j \in \{1, \ldots, \bar{\omega}\} : \psi_e \left(\nabla f_{\bar{a}_j}(\bar{x})^\top \bar{u} \right) < 0$$

and, by Proposition 2.5.6 (iii),

$$\forall\, j \in \{1, \ldots, \bar{\omega}\} : \nabla f_{\bar{a}_j}(\bar{x})^\top \bar{u} \in -\operatorname{int} K. \tag{5.38}$$

However, this is a contradiction with (5.37), and hence the statement is proved.

(ii) From (5.38), we know that

$$\forall\, j \in \{1, \ldots, \bar{\omega}\}, t \in (0, \tilde{t}\,] : f_{\bar{a}_j}(\bar{x}) + \beta t \nabla f_{\bar{a}_j}(\bar{x})^\top \bar{u} \prec_K f_{\bar{a}_j}(\bar{x}). \tag{5.39}$$

Then, it follows that

$$\begin{aligned}
\forall\, t \in (0, \bar{t}\,] : F(\bar{x}) &\overset{\text{(Proposition 2.4.7 } (i))}{\subset} \{f_{\bar{a}_1}(\bar{x}), \ldots, f_{\bar{a}_{\bar{\omega}}}(\bar{x})\} + K \\
&\overset{(5.39)}{\subset} \left\{ \nabla f_{\bar{a}_j}(\bar{x}) + \beta t \nabla f_{\bar{a}_j}(\bar{x})^\top \bar{u} \right\}_{j\in\{1,\ldots,\bar{\omega}\}} + \operatorname{int} K \\
&\overset{\text{(Statement } (i))}{\subseteq} \left\{ f_{\bar{a}_j}(\bar{x} + t\bar{a}_1), \ldots, f_{\bar{a}_j}(\bar{x} + t\bar{a}_{\bar{\omega}}) \right\}_{j\in\{1,\ldots,\bar{\omega}\}} + K + \operatorname{int} K \\
&\subseteq F(\bar{x} + t\bar{u}) + \operatorname{int} K,
\end{aligned}$$

as desired. $\square$

We are now ready to establish the convergence of Algorithm 1.

Theorem 5.3.10. *Let Assumption 4 be fulfilled and suppose that Algorithm 1 generates an infinite sequence. Furthermore, assume that $\bar{x}$ is an accumulation point of this sequence that is regular for F. Then, $\bar{x}$ is $\preceq^{(l)}_K$- stationary. In particular, when $|P_{\bar{x}}| = 1$, we get that $\bar{x}$ is strongly $\preceq^{(l)}_K$- stationary.*

Proof. Consider the functional $\zeta : \mathcal{P}(\mathbb{R}^m) \to \overline{\mathbb{R}}$ defined as

$$\forall\, A \subseteq \mathbb{R}^m : \ \zeta(A) := \inf_{y \in A} \psi_e(y).$$

The proof will be divided in several steps:

<u>**Step 1**</u>: We show the following sufficient decrease result:

$$\forall\, k \in \mathbb{N} \cup \{0\} : \ (\zeta \circ F)(x_{k+1}) \leq (\zeta \circ F)(x_k) + \beta t_k \left[\phi(x_k) - \frac{1}{2}\|u_k\|^2\right]. \tag{5.40}$$

Indeed, because of the K- monotonicity of ψ_e in Proposition 2.5.6 (ii), the functional ζ is $\preceq^{(l)}_K$- monotone, that is, $A \preceq^{(l)}_K B \Longrightarrow \zeta(A) \leq \zeta(B)$. On the other hand, from Proposition 5.3.9 (ii), we deduce that

$$\forall\, k \in \mathbb{N} \cup \{0\} : F(x_k + t_k u_k) \preceq^{(l)}_K \left\{ f_{a_{k,j}}(x_k) + \beta t_k \nabla f_{a_{k,j}}(x_k)^\top u_k \right\}_{i \in \{1,\ldots,\omega_k\}}.$$

Hence, using the monotonicity of ζ, we obtain for any $k \in \mathbb{N} \cup \{0\}$:

$$\begin{aligned}
(\zeta \circ F)(x_{k+1}) &\leq \min_{j=1,\ldots,\omega_k} \left\{ \psi_e\left(f_{a_{k,j}}(x_k) + \beta t_k \nabla f_{a_{k,j}}(x_k)^\top u_k \right) \right\} \\
&\overset{\text{(Proposition 2.5.6 } (i))}{\leq} \min_{j=1,\ldots,\omega_k} \left\{ \psi_e\left(f_{a_{k,j}}(x_k) \right) + \beta t_k \psi_e\left(\nabla f_{a_{k,j}}(x_k)^\top u_k \right) \right\} \\
&\leq \min_{j=1,\ldots,\omega_k} \left\{ \psi_e\left(f_{a_{k,j}}(x_k) \right) + \beta t_k \max_{j'=1,\ldots,\omega_k} \left\{ \psi_e\left(\nabla f_{a_{k,j'}}(x_k)^\top u_k \right) \right\} \right\} \\
&= (\zeta \circ F)(x_k) + \beta t_k \max_{j=1,\ldots,\omega_k} \left\{ \psi_e\left(\nabla f_{a_{k,j}}(x_k)^\top u_k \right) \right\}.
\end{aligned}$$

The above inequality, together with the definition of ϕ in (5.30), implies (5.40).

On the other hand, since $\bar{x}$ is an accumulation point of the sequence $\{x_k\}_{k \geq 0}$, we can find a subsequence $\mathcal{K} \subseteq \mathbb{N}$ such that $x_k \xrightarrow{\mathcal{K}} \bar{x}$.

<u>**Step 2**</u>: The following inequality holds

$$\forall\, k \in \mathbb{N} \cup \{0\} : \ F(\bar{x}) \preceq^{(l)}_K F(x_k). \tag{5.41}$$

Indeed, from Proposition 5.3.9 (ii), we can guarantee that the sequence $\{F(x_k)\}_{k \geq 0}$ is $\preceq^{(l)}_K$- decreasing, that is,

$$\forall\, k \in \mathbb{N} \cup \{0\} : F(x_{k+1}) \preceq^{(l)}_K F(x_k). \tag{5.42}$$

Fix now $k \in \mathbb{N}$, and $i \in \{1, \ldots, p\}$. Then, according to (5.42), we have

$$\forall\, k' \in \mathcal{K}, k' \geq k, \exists\, i_{k'} \in \{1, \ldots, p\} : f_{i_{k'}}(x_{k'}) \preceq_K f_i(x_k). \tag{5.43}$$

Since there are only a finite number of possible values for $i_{k'}$, we can assume without loss of generality that there is $\bar{i} \in \{1, \ldots, p\}$ such that $i_{k'} = \bar{i}$ for every $k' \in \mathcal{K}, k' \geq k$. Hence, (5.43) is equivalent to

$$\forall\, k' \in \mathcal{K}, k' \geq k : f_{\bar{i}}(x_{k'}) - f_i(x_k) \in -K. \tag{5.44}$$

Taking the limit now in (5.44) when $k' \xrightarrow{\mathcal{K}} +\infty$, we find that

$$f_i(x_k) \in f_{\bar{i}}(\bar{x}) + K.$$

Since i was chosen arbitrarily in $\{1, \ldots, p\}$, this implies the statement.

Step 3: We prove that the sequence $\{u_k\}_{k \in \mathcal{K}}$ is bounded.

In order to see this, note that, since x_k is not a stationary point, we have by Proposition 5.3.7 that $\phi(x_k) < 0$ for every $k \in \mathbb{N} \cup \{0\}$. By the definition of a_k and u_k, we then have

$$\forall\, k \in \mathbb{N} \cup \{0\} : \varphi_{x_k}(a_k, u_k) < 0. \tag{5.45}$$

Recall that ρ is the Lipschitz constant of ψ_e given in Proposition 2.5.7 (i). Thus, we deduce that

$$\begin{aligned}
\forall\, k \in \mathbb{N} \cup \{0\} : \|u_k\|^2 &\overset{((5.45)\,+\,(5.27))}{<} -2 \max_{j=1,\ldots,\omega_k} \left\{ \psi_e \left(\nabla f_{a_{k,j}}(\bar{x})^\top u_k \right) \right\} \\
&= 2 \max_{j=1,\ldots,\omega_k} \left\{ \left| \psi_e \left(\nabla f_{a_{k,j}}(\bar{x})^\top u_k \right) \right| \right\} \\
&\overset{(\text{Proposition 2.5.7 } (i))}{\leq} 2\rho \max_{j=1,\ldots,\omega_k} \left\{ \left\| \nabla f_{a_{k,j}}(\bar{x})^\top u_k \right\| \right\} \\
&\leq 2\rho \|u_k\| \max_{j=1,\ldots,\omega_k} \left\{ \|\nabla f_{a_{k,j}}(x_k)\| \right\}.
\end{aligned}$$

Hence,

$$\forall\, k \in \mathbb{N} \cup \{0\} : \|u_k\| \leq 2\rho \max_{j=1,\ldots,\omega_k} \left\{ \|\nabla f_{a_{k,j}}(x_k)\| \right\}. \tag{5.46}$$

Since $\{x_k\}_{k \in \mathcal{K}}$ is bounded, the statement follows from (5.46).

Step 4: We show that $\bar{x}$ is $\preceq_K^{(l)}$- stationary.

Fix $\kappa \in \mathbb{N}$. Then, it follows from (5.40) that

$$\forall\, k \in \mathbb{N} :\ -\beta t_k \max_{j=1,\ldots,\omega_k} \left\{ \psi_e \left(\nabla f_{a_{k,j}}(x_k)^\top u_k \right) \right\} \leq (\zeta \circ F)(x_k) - (\zeta \circ F)(x_{k+1}). \tag{5.47}$$

Adding this inequality for $k = 0, \ldots, \kappa$, we obtain

$$-\beta\sum_{k=0}^{\kappa} t_k \max_{j=1,\ldots,\omega_k}\left\{\psi_e\left(\nabla f_{a_{k,j}}(x_k)^\top u_k\right)\right\} \leq (\zeta\circ F)(x^0)-(\zeta\circ F)(x^{\kappa+1}). \tag{5.48}$$

On the other hand, similarly to (5.38) in the proof of Proposition 5.3.9 (i), we can obtain that

$$\forall\, k\in\mathbb{N}\cup\{0\}, j\in\{1,\ldots,\omega_k\}: \nabla f_{a_{k,j}}(x_k)^\top u_k \in -\operatorname{int} K. \tag{5.49}$$

In particular, applying Proposition 2.5.6 (iii) in (5.49), we find that

$$\forall\, k\in\mathbb{N}\cup\{0\}, j\in\{1,\ldots,\omega_k\}: \psi_e\left(\nabla f_{a_{k,j}}(x_k)^\top u_k\right) < 0. \tag{5.50}$$

We then have

$$\begin{aligned} 0 \overset{(5.50)}{<} -\sum_{k=0}^{\kappa} t_k \max_{j=1,\ldots,\omega_k}\left\{\psi_e\left(\nabla f_{a_{k,j}}(x_k)^\top u_k\right)\right\} &\overset{(5.48)}{\leq} \frac{(\zeta\circ F)(x^0)-(\zeta\circ F)(x^{\kappa+1})}{\beta} \\ &\overset{(\zeta \text{ monotone } + (5.41))}{\leq} \frac{(\zeta\circ F)(x^0)-(\zeta\circ F)(\bar{x})}{\beta}. \end{aligned}$$

Taking now the limit in the previous inequality when $\kappa\to+\infty$, we deduce that

$$0 \leq -\sum_{k=0}^{\infty} t_k \max_{j=1,\ldots,\omega_k}\left\{\psi_e\left(\nabla f_{a_{k,j}}(x_k)^\top u_k\right)\right\} < +\infty.$$

In particular, this implies

$$\lim_{k\to\infty} t_k \max_{j=1,\ldots,\omega_k}\left\{\psi_e\left(\nabla f_{a_{k,j}}(x_k)^\top u_k\right)\right\} = 0. \tag{5.51}$$

Since there are only a finite number of subsets of $\{1,\ldots,p\}$ and $\bar{x}$ is regular for F, we can apply Proposition 5.3.6 to obtain, without loss of generality, the existence of $Q\subseteq P_{\bar{x}}$ and $\bar{a}\in Q$ such that

$$\forall\, k\in\mathcal{K}:\ \omega_k=\bar{\omega},\ P_{x_k}=Q,\ a_k=\bar{a}. \tag{5.52}$$

Furthermore, since the sequences $\{t_k\}_{k\geq 1}, \{u_k\}_{k\in\mathcal{K}}$ are bounded, we can also assume without loss of generality the existence of $\bar{t}\in\mathbb{R}, \bar{u}\in\mathbb{R}^n$ such that

$$t_k \overset{\mathcal{K}}{\to} \bar{t},\ \ u_k \overset{\mathcal{K}}{\to} \bar{u}. \tag{5.53}$$

The rest of the proof is devoted to show that $\bar{x}$ is a $\preceq^{(l)}_K$- stationary point with respect to Q. First, observe that by (5.52) and the definition of a_k, we have

$$\forall\, a\in Q,\ k\in\mathcal{K},\ u\in\mathbb{R}^n: \phi(x_k)=\varphi_{x_k}(\bar{a},u_k)\leq\varphi_{x_k}(a,u).$$

Then, taking into account that $\omega_k=\bar{\omega}$ in (5.52), we can take the limit when $k\overset{\mathcal{K}}{\to}+\infty$ in the above expression to obtain

$$\forall\, a \in Q,\ u \in \mathbb{R}^n : \varphi_{\bar{x}}(\bar{a}, \bar{u}) \leq \varphi_{\bar{x}}(a, u).$$

Equivalently, we have

$$(\bar{a}, \bar{u}) \in \operatorname*{argmin}_{(a,u)\in Q\times\mathbb{R}^n} \varphi_{\bar{x}}(a, u). \tag{5.54}$$

Next, we analyze two cases:

Case 1: $\bar{t} > 0$.

According to (5.51) and (5.52), we have in this case

$$\lim_{k \xrightarrow{\mathcal{K}} +\infty} \max_{j=1,\dots,\bar{\omega}} \left\{ \psi_e \left(\nabla f_{\bar{a}_j}(x_k)^\top u_k \right) \right\} = 0. \tag{5.55}$$

Then, it follows that

$$\begin{aligned}
0 \quad &\leq \quad \frac{1}{2}\|\bar{u}\|^2 \\
&\overset{((5.52)+(5.53)+(5.55))}{=} \quad \lim_{k \xrightarrow{\mathcal{K}} +\infty} \max_{j=1,\dots,\bar{\omega}} \left\{ \psi_e \left(\nabla f_{\bar{a}_j}(x_k)^\top u_k \right) \right\} + \frac{1}{2}\|u_k\|^2 \\
&= \quad \lim_{k \xrightarrow{\mathcal{K}} +\infty} \phi(x_k) \\
&\overset{(5.31)}{\leq} \quad 0,
\end{aligned}$$

from which we deduce $\bar{u} = 0$. This, together with (5.54) and Remark 5.3.8, imply that $\bar{x}$ is a $\preceq_K^{(l)}$- stationary point with respect to Q.

Case 2: $\bar{t} = 0$.

Fix an arbitrary $\kappa \in \mathbb{N}$. Since $t_k \xrightarrow{\mathcal{K}} 0$, for $k \in \mathcal{K}$ large enough ν^κ does not satisfy Armijo's line search criteria in Step 4 of Algorithm 1. By (5.52) and the finiteness of $\bar{\omega}$, we can assume without loss of generality the existence of $\bar{j} \in \{1, \dots, \bar{w}\}$ such that

$$\forall\, k \in \mathcal{K} :\ f_{\bar{a}_{\bar{j}}}(x_k + \nu^\kappa u_k) \npreceq_K f_{\bar{a}_{\bar{j}}}(x_k) + \beta\nu^\kappa \nabla f_{\bar{a}_{\bar{j}}}(x_k)^\top u_k.$$

From this, it follows that

$$\forall\, k \in \mathcal{K} :\ \frac{f_{\bar{a}_{\bar{j}}}(x_k + \nu^\kappa u_k) - f_{\bar{a}_{\bar{j}}}(x_k)}{\nu^\kappa} - \beta\nabla f_{\bar{a}_{\bar{j}}}(x_k)^\top u_k \notin -K.$$

Taking the limit now when $k \xrightarrow{\mathcal{K}} +\infty$, we obtain

$$\frac{f_{\bar{a}_{\bar{j}}}(\bar{x} + \nu^\kappa \bar{u}) - f_{\bar{a}_{\bar{j}}}(\bar{x})}{\nu^\kappa} - \beta\nabla f_{\bar{a}_{\bar{j}}}(\bar{x})^\top \bar{u} \notin -\operatorname{int} K.$$

Next, taking the limit when $\kappa \to +\infty$, we get

$$(1-\beta)\nabla f_{\bar{a}_{\bar{j}}}(\bar{x})^\top \bar{u} \notin -\operatorname{int} K.$$

Since $\beta \in (0,1)$, we deduce that $\nabla f_{\bar{a}_{\bar{j}}}(\bar{x})^\top \bar{u} \notin -\operatorname{int} K$ and, according to Proposition 2.5.6 (iii), this is equivalent to

$$\psi_e(\nabla f_{\bar{a}_{\bar{j}}}(\bar{x})^\top \bar{u}) \geq 0. \tag{5.56}$$

Finally, we find that

$$0 \overset{(5.56)}{\leq} \psi_e(\nabla f_{\bar{a}_{\bar{j}}}(\bar{x})^\top \bar{u}) \leq \varphi_{\bar{x}}(\bar{a},\bar{u}) \overset{(5.54)}{=} \min_{(a,u)\in Q\times\mathbb{R}^n} \varphi_{\bar{x}}(a,u) \overset{(5.28)}{\leq} 0,$$

which implies

$$\min_{(a,u)\in Q\times\mathbb{R}^n} \varphi_{\bar{x}}(a,u) = 0. \tag{5.57}$$

The $\preceq_K^{(l)}$- stationarity of $\bar{x}$ follows then from (5.57) and Remark 5.3.8. The proof is complete.

□

5.4 Implementation and Numerical Illustrations

In this section, we report some preliminary numerical experience with the proposed method. Algorithm 1 was implemented in Python 3 and the experiments were done in a PC with an Intel(R) Core(TM) i5-4200U CPU processor and 4.0 GB of RAM. We describe below some details of the implementation and the experiments:

- We considered instances of problem $\mathcal{SOP}(F,K,\mathbb{R}^n)$ only for the case in which K is the standard ordering cone, that is, $K=\mathbb{R}^m_+$. In addition, we chose the parameter $e \in \operatorname{int} K$ for the scalarizing functional ψ_e as $e=(1,\ldots,1)^\top$.

- The parameters β and ν for the line search in Step 4 of the method were chosen as $\beta = 0.0001,\ \nu = 0.500$.

- The stopping criteria employed was that $\|u_k\| < 0.0001$, or a maximum number of 200 iterations was reached.

- For finding the set $\operatorname{Min}(F(x_k),K)$ at the k^{th}- iteration in Step 1 of the algorithm, we implemented the method developed by Günther and Popovici in [61]. This procedure requires a strongly K- monotone functional $\psi:\mathbb{R}^m\to\mathbb{R}$ for a so called presorting phase. In our implementation, we used ψ defined as follows:

$$\forall\, v \in \mathbb{R}^m : \psi(v) = \sum_{i=1}^m v_i.$$

The other possibility for finding the set $\mathrm{Min}(F(x_k), K)$ would have been to use the method introduced by Jahn in [87, 89, 96] with ideas from [157]. However, as mentioned in Section 5.1, the first approach has better computational complexity. Thus, the algorithm proposed in [61] was a clear choice.

- At the k^{th}- iteration in Step 2 of the algorithm, we worked with the modelling language CVXPY 1.0 [1, 40] for the solution of the problem $\mathcal{OP}(\varphi_{x_k}, P_k \times \mathbb{R}^n)$, that is,

$$\min_{(a,u)\in P_k\times\mathbb{R}^n} \varphi_{x_k}(a, u).$$

  Since the variable a is constrained to be in the discrete set P_k, we proceeded as follows: using the solver ECOS [42] within CVXPY, we compute for every $a \in P_k$ the unique solution u_a of the strongly convex problem $\mathcal{OP}(\varphi_{x_k}(a, \cdot), \mathbb{R}^n)$, that is,

$$\min_{u\in\mathbb{R}^n} \varphi_{x_k}(a, u).$$

  Then, we set

$$(a_k, u_k) = \operatorname*{argmin}_{a\in P_k} \varphi_{x_k}(a, u_a).$$

- For each test instance considered in the experimental part, we generated initial points randomly on a specific set and run the algorithm. We define as solved those experiments in which the algorithm stopped because $\|u_k\| < 0.0001$, and declared that a strongly $\preceq^{(l)}_K$-stationary point was found. For a given experiment, its final error is the value of $\|u_k\|$ at the last iteration. The following variables are collected for each test instance:

  - **Solved**: this value indicates the number of initial points for which the problem was solved.
  - **Iterations**: this is a 3-tuple (min, mean, max) that indicates the minimum, the mean, and the maximum of the number of iterations in those instances reported as solved.
  - **Mean CPU Time**: Mean of the CPU time(in seconds) among the solved cases.

  Furthermore, for clarity, all the numerical values will be displayed for up to four decimal places.

Now, we proceed to the different instances on which our algorithm was tested. Our first test instance can be seen as a continuous version of an example in [65].

Test Instance 5.4.1. *We consider $F : \mathbb{R} \rightrightarrows \mathbb{R}^2$ defined as*

$$F(x) := \{f_1(x), \ldots, f_5(x)\}$$

where, for $i = 1, \ldots, 5$, $f_i : \mathbb{R} \to \mathbb{R}^2$ *is given as*

$$f_i(x) := \begin{pmatrix} x \\ \frac{x}{2}\sin(x) \end{pmatrix} + \sin^2(x)\left[\frac{(i-1)}{4}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \left(1 - \frac{(i-1)}{4}\right)\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right].$$

The objective values in this case are discretized segments moving around a curve and being contracted (dilated) by a factor dependent on the argument. We generated 100 initial points x_0 randomly on the interval $[-5\pi, 5\pi]$ and run our algorithm. Some of the metrics are collected in Table 5.1. As we can see, in this case all the runs terminated finding a strongly $\preceq_K^{(l)}$- stationary point. Moreover, we observed that for this problem not too many iterations were needed.

Test Instance 5.4.1		
Solved	**Iterations**	**Mean CPU Time**
100	(0, 13.97, 71)	0.1872

Table 5.1: Performance of Algorithm 1 in Test Instance 5.4.1

In Figure 5.1, the sequence $\{F(x_k)\}_{k\in\{0,\ldots,7\}}$ generated by Algorithm 1 for a selected starting point is shown. In this case, strong stationarity was declared after 7 iterations. The traces of the curves f_i for $i \in \{1, \ldots, 5\}$ are displayed, with arrows indicating their direction of movement. Moreover, the sets $F(x_0)$ and $F(x_7)$ are represented by black and red points respectively, and the elements of the sets $F(x_k)$ with $k \in \{1, \ldots, 6\}$ are in gray color. The improvements of the objective values after every iteration are clearly observed.

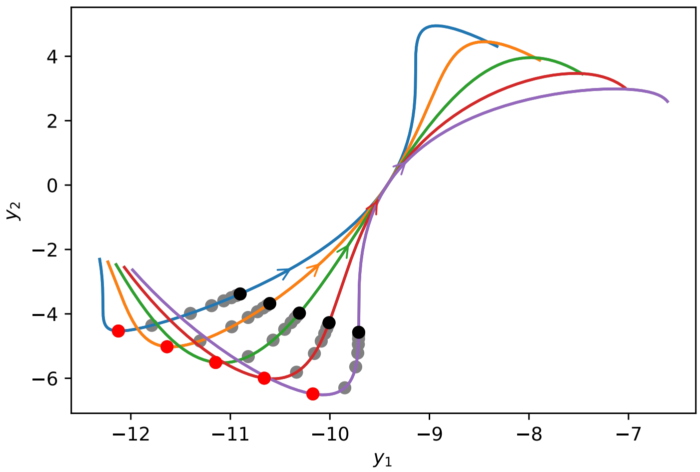

Figure 5.1: Sequence generated in the image space by Algorithm 1 for a selected starting point in Test Instance 5.4.1

Test Instance 5.4.2. *In this example, we start by taking a uniform partition* $\mathcal{U}_1$ *of 10 points of the interval* $[-1, 1]$ *that is,*

$$\mathcal{U}_1 = \{-1, -0.7778, -0.5556, -0.3333, -0.1111, 0.1111, 0.3333, 0.5556, 0.7778, 1\}.$$

Then, the set $\mathcal{U} := \mathcal{U}_1 \times \mathcal{U}_1$ *is a mesh of 100 points of the square* $[-1,1] \times [-1,1]$. *Let* $\{u_1, \ldots, u_{100}\}$ *be an enumeration of* $\mathcal{U}$ *and consider the points*

$$l_1 := \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \; l_2 := \begin{pmatrix} 8 \\ 0 \end{pmatrix}, \; l_3 := \begin{pmatrix} 0 \\ 8 \end{pmatrix}.$$

We define, for $i \in \{1, \ldots, 100\}$, *the functional* $f_i : \mathbb{R}^2 \to \mathbb{R}^3$ *as*

$$f_i(x) := \frac{1}{2} \begin{pmatrix} \|x - l_1 - u_i\|^2 \\ \|x - l_2 - u_i\|^2 \\ \|x - l_3 - u_i\|^2 \end{pmatrix}.$$

Finally, the set-valued mapping $F : \mathbb{R}^2 \rightrightarrows \mathbb{R}^3$ *is defined by*

$$F(x) := \{f_1(x), \ldots, f_{100}(x)\}.$$

Note that problem $\mathcal{SOP}(F, K, \mathbb{R}^n)$ corresponds in this case to the robust counterpart of a vector location problem under uncertainty, where $\mathcal{U}$ represents the uncertainty set on the location facilities l_1, l_2, l_3. Furthermore, with the aid of Theorem 5.2.5, it is possible to show that a point $\bar{x}$ is a local $\preceq_K^{(l)}$- weakly minimal solution if and only if

$$\bar{x} \in \operatorname{conv}\{l_j + u_i \mid (i,j) \in I(\bar{x}) \times \{1,2,3\}\}.$$

Thus, in particular, the local $\preceq_K^{(l)}$- weakly minimal solutions lie on the set

$$C := \operatorname{conv}((l_1 + \mathcal{U}) \cup (l_2 + \mathcal{U}) \cup (l_3 + \mathcal{U})). \tag{5.58}$$

In this test instance, 100 initial points x_0 were generated in the square $[-50, 50] \times [-50, 50]$, and Algorithm 1 was ran in each case. A summary of the results are presented in Table 5.2. Again, for any initial point the sequence generated by the algorithm stopped with a local solution to our problem. Perhaps the most noticeable parameter recorded in this case is the number of iterations required to declare the solution. Indeed, in most cases, only 1 iteration was enough, even when the starting point was far away from the locations l_1, l_2, l_3.

Test Instance 5.4.2		
Solved	**Iterations**	**Mean CPU Time**
100	(0,1.32,2)	0.0637

Table 5.2: Performance of Algorithm 1 in Test Instance 5.4.2

In Figure 5.2, the set of solutions found in this experiment are shown in red. The locations l_1, l_2, l_3 are represented by black points and the elements of the set $(l_1 + \mathcal{U}) \cup (l_2 + \mathcal{U}) \cup (l_3 + \mathcal{U})$ are colored in gray. We can observe, as expected, that all the local solutions found are contained in the set C given in (5.58).

Our next example was studied in [107].

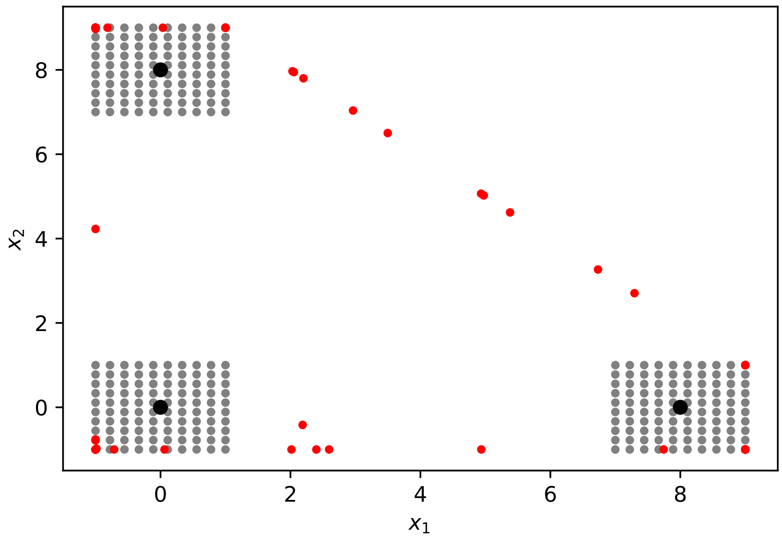

Figure 5.2: Solutions found (in red) in the argument space for Test Instance 5.4.2

Test Instance 5.4.3. *For $i \in \{1, \ldots, 14\}$, we consider the functional $f_i : \mathbb{R}^2 \to \mathbb{R}^2$ defined as*

$$f_i(x) := \begin{pmatrix} x_1^2 + x_2^2 \\ 2(x_1 + x_2) \end{pmatrix} + \frac{1}{4} \begin{pmatrix} \cos\left(\frac{2\pi(i-1)}{14}\right) \\ \sin\left(\frac{2\pi(i-1)}{14}\right) \end{pmatrix}.$$

Hence, $F : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ is given by

$$F(x) := \{f_1(x), \ldots, f_{14}(x)\}.$$

In this case, the images of the set-valued mapping are discretized circumferences of radius $\frac{1}{4}$ centered at a point depending on the argument. Moreover, we can apply Theorem 5.2.5 to show that the set of local $\preceq_K^{(l)}$- weakly minimal solutions is given by

$$D := \{x \in \mathbb{R}^2 \mid x_1 = x_2,\ x_2 \leq 0\}. \tag{5.59}$$

We generated randomly 100 initial points in the square $[-10, 10] \times [-10, 10]$ and ran Algorithm 1. A summary of the results is collected in Table 5.3, and the solutions found in the argument space are illustrated in Figure 5.3. Again, as expected, in Figure 5.3 every point belongs to the set D defined in (5.59).

Test Instance 5.4.3		
Solved	**Iterations**	**Mean CPU Time**
100	(1, 35.6,89)	0.4807

Table 5.3: Performance of Algorithm 1 in Test Instance 5.4.3.

In Figure 5.4, the sequence $\{F(x_k)\}_{k\in\{0,\ldots,20\}}$ generated by Algorithm 1 for a selected starting point in this test instance is presented. In this case, a solution was found after 20 iterations. The

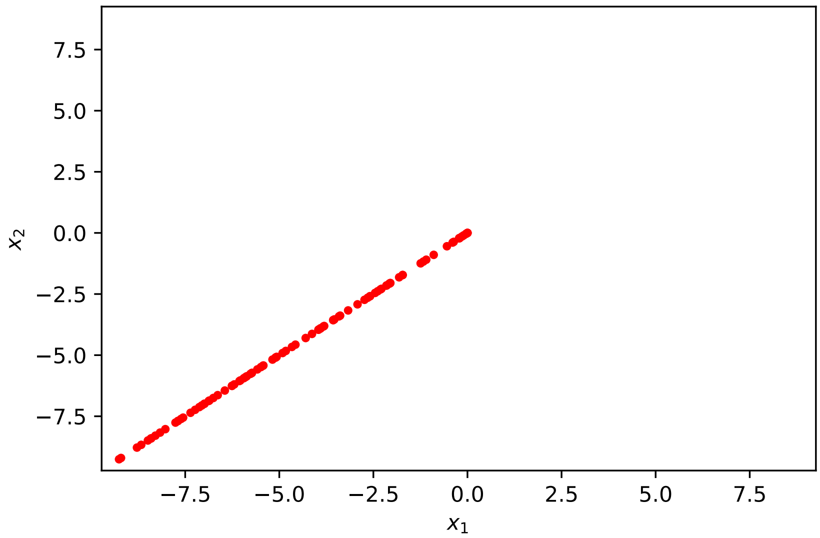

Figure 5.3: Solutions found in the argument space for Test Instance 5.4.3

sets $F(x_0)$ and $F(x_{20})$ are represented by black and red points respectively, and the elements of the sets $F(x_k)$ with $k \in \{1, \ldots, 19\}$ are colored in gray. As in Test Instance 5.4.1, the decrease of the images at every iteration is observed.

Our last test example comes from [94].

Test Instance 5.4.4. *For $i \in \{1, \ldots, 100\}$, we consider the functional $f_i : \mathbb{R}^2 \to \mathbb{R}^2$ defined as*

$$f_i(x) := \begin{pmatrix} e^{\frac{x_1}{2}} \cos(x_2) + x_1 \cos(x_2) \cos^3\left(\frac{2\pi(i-1)}{100}\right) - x_2 \sin(x_2) \sin^3\left(\frac{2\pi(i-1)}{100}\right) \\ e^{\frac{x_2}{20}} \sin(x_1) + x_1 \sin(x_2) \cos^3\left(\frac{2\pi(i-1)}{100}\right) + x_2 \cos(x_2) \sin^3\left(\frac{2\pi(i-1)}{100}\right) \end{pmatrix}.$$

Hence, $F : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ is given by

$$F(x) := \{f_1(x), \ldots, f_{100}(x)\}.$$

The images of the set-valued mapping in this example are discretized, shifted, rotated, and deformated rhombuses, see Figure 5.5. We generated randomly 100 initial points in the square $[-10\pi, 10\pi] \times [-10\pi, 10\pi]$ and ran our algorithm. A summary of the results is collected in Table 5.4. In this case, only for 88 initial points a solution was found. In the rest of the occasions, the algorithm stopped because the maximum number of iterations was reached. Further examination in these unsolved cases revealed that, except for two of the initial points, the final error was of the order of 10^{-1} (even 10^{-3} and 10^{-4} in half of the cases). Thus, perhaps only a few more iterations were needed in order to declare strong stationarity.

Figure 5.5 illustrates the sequence $\{F(x_k)\}_{k \in \{0, \ldots, 18\}}$ generated by Algorithm 1 for a selected starting point. Strong stationarity was declared after 18 iterations in this experiment. The sets $F(x_0)$ and $F(x_{18})$ are represented by black and red points respectively, and the elements of the sets $F(x_k)$ with $k \in \{1, \ldots, 17\}$ are in gray color. Similarly to the other test instances, we can observe that at every iteration the images decrease with respect to the preorder $\preceq_K^{(l)}$.

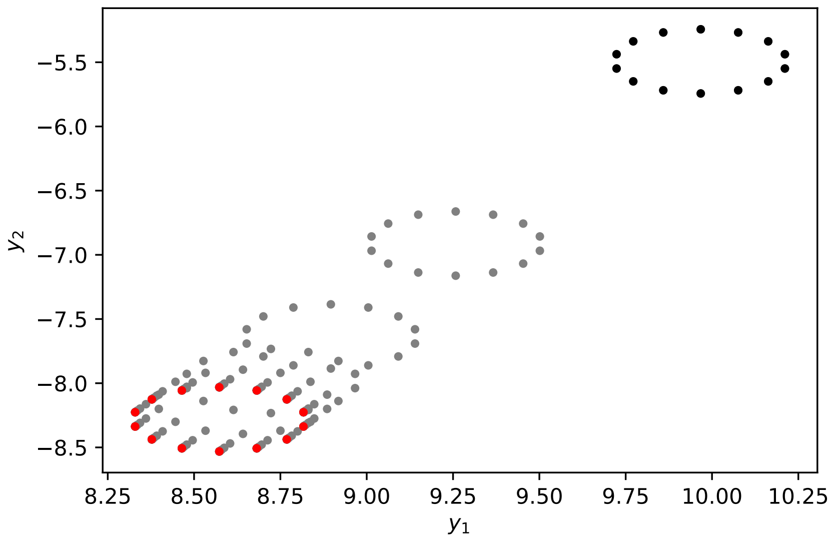

Figure 5.4: Sequence generated in the image space by Algorithm 1 for a selected starting point in Test Instance 5.4.3

Test Instance 5.4.4		
Solved	**Iterations**	**Mean CPU Time**
88	(0, 11.9091 , 110)	0.8492

Table 5.4: Performance of Algorithm 1 in Test Instance 5.4.4.

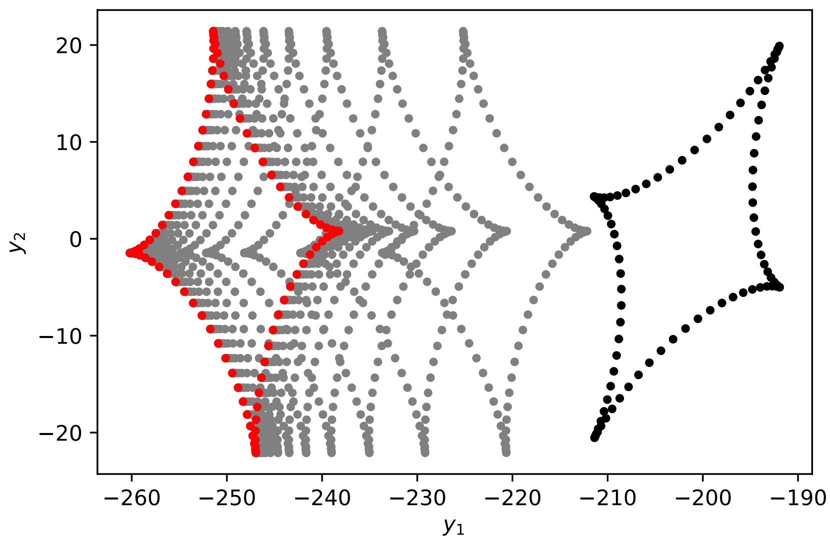

Figure 5.5: Sequence generated in the image space by Algorithm 1 for a selected starting point in Test Instance 5.4.4.

Chapter 6

Conclusions and Outlook

In this dissertation, we considered set optimization problems with respect to the set approach and their interplay with suitable scalarized problems. A summary of our contributions is the following:

- We derived an exact formula for the subdifferential of Hiriart-Urruty functionals, see Lemma 3.2.6. To the best of our knowledge, this representation is new in the infinite dimensional context. Other representations and approximations have been given in the literature, for example in [67, Proposition 21.11], [81, Proposition 5] and [82, Theorem 3].

- We provided several relationships in the sense of inclusion between three mayor classes of scalarizing functionals known today, namely that of Gerstewitz-Weidner ($\mathcal{S}_{GW}$), that of Hiriart-Urruty ($\mathcal{S}_{HU}$) and that of Drummond-Svaiter ($\mathcal{S}_{DS}$). Our results show that, under natural assumptions, $\mathcal{S}_{GW} \subseteq \mathcal{S}_{HU} \subseteq \mathcal{S}_{DS}$, see Theorem 3.2.7 and Theorem 3.2.2. Furthermore, we have shown that $\mathcal{S}_{DS}$ is exactly the set of sublinear scalarizing functionals that satisfy the required axioms to be useful in vector optimization: monotonicity and order representability.

- We introduced a new class of scalarizing functionals that are not necessarily sublinear, but can be represented as the difference of support functionals, see Definition 3.3.6. Thus, the elements in this class are in particular quasidifferentiable and positively homogeneous. To achieve this, we found geometrical conditions on the quasidifferential of the functionals in order to guarantee the fulfillment of the monotonicity and order representability axioms. Furthermore, we proved that this class is strictly larger than $\mathcal{S}_{DS}$ if the dual cone has nonempty interior, see Theorem 3.3.9.

- We derived different properties of two types of functionals that are associated to the preoders $\preceq_K^{(l)}$ and $\preceq_K^{(u)}$, respectively. Roughly speaking, these functionals can be seen as the composition of two well known scalarizing functionals in set optimization with a set-valued mapping. Specifically, it was shown that they inherit the convexity and the

Lipschitz property from the set-valued mapping, see Theorem 4.2.5 and Theorem 4.2.7. Furthermore, upper estimates of the limiting subdifferential of the functionals were also studied in Theorem 4.4.3 and Theorem 4.3.8.

- We obtained, using generalized differentiation objects in the dual space, new optimality conditions for set optimization problems with respect to the preorders $\preceq_K^{(l)}$ and $\preceq_K^{(u)}$, see Theorem 4.5.3 and Theorem 4.5.4. Perhaps the most attractive feature of these new necessary conditions is that classical assumptions such as convexity, compactness of the images of the set-valued objective mapping, and the existence of a strongly minimal element in the optimal set, are not required. The necessity of these assumptions are some of the drawbacks of the other approaches in the literature [2, 3, 11, 38, 69, 70, 91, 93, 109, 113, 137, 140, 142].

- The optimality conditions obtained for weakly $\preceq_K^{(l)}$- minimal solutions of set optimization problems were also shown to be sufficient in the case on which both, the epigraphical multifunction of the set-valued objective mapping and the feasible set, are convex, see Theorem 4.5.3 (i). Moreover, we also examined the particular case on which the graph of the set-valued objective mapping was represented by finitely many inequality constraints involving convex and continuously differentiable functionals. There, under the $MFCQ$, we derived Karush-Kuhn-Tucker type necessary and sufficient optimality conditions, see Theorem 4.6.1. Compared to the previous result in the literature [93], our optimality condition seems to be more tractable from a computational point of view because we also obtain the multipliers in the dual space.

- We studied a first order method for finding $\preceq_K^{(l)}$- weakly minimal solutions of set optimization problems in which the images of the set-valued objective mapping have finite cardinality, see Algorithm 1. Solving a problem with this type of set-valued objective mapping is equivalent to find optimistic solutions to a vector optimization problem under uncertainty (see Remark 2.4.17) with a finite uncertainty set. The main convergence result stated that, under mild assumptions, the accumulation points of the sequence generated by the proposed descent method satisfies some type of first order necessary conditions, see Theorem 5.3.10. For set optimization problems, the proposed algorithm seems to be the first in the literature with this desirable property.

On the other hand, our results also open several ideas for further research. Some of these are the following:

- Derive optimality conditions for set optimization problems with respect to other set relations. Indeed, we believe that the methodology employed in Chapter 4 could be extended to deal with other more complex set relations like those described in [95, 100, 117, 119].

- Relax the Lipschitz assumption on the set-valued objective mapping for the optimality conditions. In this case, perhaps a replacement by some type of lower semicontinuity property could be examined.

- Apply the obtained optimality conditions to set optimization problems were the set-valued objective mapping has a particular structure. In this direction, we have already studied the case in which the set-valued mappings is given by convex functional constraints in Theorem 4.6.1. We believe that this result could be extended to deal with nonconvex descriptions. Another possibility would be to study problems of the form (2.24) that arise in vector optimization under uncertainty.

- Extend the algorithmic strategy from Chapter 5 to other set relations. Of particular interest in this case would be the preorder $\preceq_K^{(u)}$, since it is the one that models pessimistic solutions in vector optimization under uncertainty. A second step in this direction could be to integrate the developed methods with a cutting plane strategy like the one described in [134]. This would allow us to obtain a family of algorithms for general vector optimization problems under uncertainty.

Bibliography

[1] A. Agrawal, R. Verschueren, S. Diamond, and S. Boyd. A rewriting system for convex optimization problems. *Journal of Control and Decision*, 5(1):42–60, 2018.

[2] M. Alonso and L. Rodríguez-Marín. Set-relations and optimality conditions in set-valued maps. *Nonlinear Analysis: Theory, Methods & Applications*, 63(8):1167 – 1179, 2005.

[3] M. Alonso and L. Rodríguez-Marín. Optimality conditions for set-valued maps with set optimization. *Nonlinear Anal.*, 70(9):3057–3064, 2009.

[4] T. Amahroq and A. Oussarhan. Lagrange multiplier rules for weakly minimal solutions of compact-valued set optimization problems. *Asia-Pac. J. Oper. Res.*, 36(4):1950021, 22, 2019.

[5] T. Amahroq, A. Oussarhan, and A. Syam. On Lagrange multiplier rules for set-valued optimization problems in the sense of set criterion. *Numerical Functional Analysis and Optimization*, 0(0):1–20, 2019.

[6] D. T. V. An and N. D. Yen. Differential stability of convex optimization problems under inclusion constraints. *Appl. Anal.*, 94(1):108–128, 2015.

[7] Y. Araya. Four types of nonlinear scalarizations and some applications in set optimization. *Nonlinear Anal.*, 75(9):3821–3835, 2012.

[8] H. Attouch, G. Buttazzo, and G. Michaille. Variational analysis in Sobolev and BV spaces. Applications to PDEs and optimization. Second edition. *SIAM Rev.*, 58(4):800–802, 2016.

[9] J.P. Aubin and I. Ekeland. *Applied nonlinear analysis.* Pure and Applied Mathematics (New York). John Wiley & Sons, Inc., New York, 1984. A Wiley-Interscience Publication.

[10] J. Baier and J. Jahn. On subdifferentials of set-valued maps. *J. Optim. Theory Appl.*, 100(1):233–240, 1999.

[11] R. Baier, G. Eichfelder, and T. Gerlach. Optimality conditions for set optimization using a directional derivative based on generalized steiner sets. *Preprint-Series of the Institute for Mathematics, Technische Universität Ilmenau, Germany*, 2019.

[12] R. Baier and E. Farkhi. Regularity of set-valued maps and their selections through set differences. Part 1: Lipschitz continuity. *Serdica Math. J.*, 39(3-4):365–390, 2013.

[13] R. Baier and E. Farkhi. Regularity of set-valued maps and their selections through set differences. Part 2: One-sided Lipschitz properties. *Serdica Math. J.*, 39(3-4):391–422, 2013.

[14] T.Q. Bao and B.S. Mordukhovich. Existence of minimizers and necessary conditions in set-valued optimization with equilibrium constraints. *Appl. Math.*, 52(6):453–472, 2007.

[15] T.Q. Bao and B.S. Mordukhovich. Variational principles for set-valued mappings with applications to multiobjective optimization. *Control Cybernet.*, 36(3):531–562, 2007.

[16] T.Q. Bao and B.S. Mordukhovich. Necessary conditions for super minimizers in constrained multiobjective optimization. *J. Global Optim.*, 43(4):533–552, 2009.

[17] T.Q. Bao and B.S. Mordukhovich. Relative Pareto minimizers for multiobjective problems: existence and optimality conditions. *Math. Program.*, 122(2, Ser. A):301–347, 2010.

[18] T.Q. Bao and B.S. Mordukhovich. Set-valued optimization in welfare economics. In *Advances in mathematical economics. Volume 13*, volume 13 of *Adv. Math. Econ.*, pages 113–153. Springer, Tokyo, 2010.

[19] T.Q. Bao and C. Tammer. Lagrange necessary conditions for Pareto minimizers in Asplund spaces and applications. *Nonlinear Anal.*, 75(3):1089–1103, 2012.

[20] M. S. Bazaraa, H. D. Sherali, and C. M. Shetty. *Nonlinear programming.* Wiley-Interscience [John Wiley & Sons], Hoboken, NJ, third edition, 2006. Theory and algorithms.

[21] A. Ben-Tal, L. El Ghaoui, and A. Nemirovski. *Robust optimization.* Princeton Series in Applied Mathematics. Princeton University Press, Princeton, NJ, 2009.

[22] A. Ben-Tal and A. Nemirovski. Robust convex optimization. *Math. Oper. Res.*, 23(4):769–805, 1998.

[23] H. P. Benson. Existence of efficient solutions for vector maximization problems. *J. Optim. Theory Appl.*, 26(4):569–580, 1978.

[24] R. Bokrantz and A. Fredriksson. Necessary and sufficient conditions for Pareto efficiency in robust multiobjective optimization. *European J. Oper. Res.*, 262(2):682–692, 2017.

[25] G. Bouza, E. Quintana, and C. Tammer. A unified characterization of nonlinear scalarizing functionals in optimization. *Vietnam J. Math.*, 47(3):683–713, 2019.

[26] G. Bouza, E. Quintana, V.A. Tuan, and C. Tammer. The Fermat rule for set optimization problems with Lipschitzian set-valued mappings. *Journal of Nonlinear and Convex Analysis*, 21(5):1137–1174, 2020.

[27] V. J. Bowman. On the relationship of the Tchebycheff norm and the efficient frontier of multiple-criteria objectives. In *Lecture Notes in Economics and Mathematical Systems*, pages 76–86. Springer Berlin Heidelberg, 1976.

[28] W. Briec. Minimum distance to the complement of a convex set: duality result. *J. Optim. Theory Appl.*, 93(2):301–319, 1997.

[29] E. Casini and E. Miglierina. Cones with bounded and unbounded bases and reflexivity. *Nonlinear Anal.*, 72(5):2356–2366, 2010.

[30] J. Chen, Q.H. Ansari, and J.-C. Yao. Characterizations of set order relations and constrained set optimization problems via oriented distance function. *Optimization*, 66(11):1741–1754, 2017.

[31] T.D. Chuong and J.-C. Yao. Steepest descent methods for critical points in vector optimization problems. *Appl. Anal.*, 91(10):1811–1829, 2012.

[32] F. Clarke. *Functional analysis, calculus of variations and optimal control*, volume 264 of *Graduate Texts in Mathematics*. Springer, London, 2013.

[33] F. H. Clarke. *Optimization and nonsmooth analysis*, volume 5 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, second edition, 1990.

[34] A. Coulibaly and J.-P. Crouzeix. Condition numbers and error bounds in convex programming. *Math. Program.*, 116(1-2, Ser. B):79–113, 2009.

[35] G. P. Crespi, M. Dhingra, and C. S. Lalitha. Pointwise and global well-posedness in set optimization: a direct approach. *Ann. Oper. Res.*, 269(1-2):149–166, 2018.

[36] G. P. Crespi, I. Ginchev, and M. Rocca. Points of efficiency in vector optimization with increasing-along-rays property and Minty variational inequalities. In *Generalized convexity and related topics*, volume 583 of *Lecture Notes in Econom. and Math. Systems*, pages 209–226. Springer, Berlin, 2007.

[37] G. P. Crespi, D. Kuroiwa, and M. Rocca. Convexity and global well-posedness in set-optimization. *Taiwanese J. Math.*, 18(6):1897–1908, 2014.

[38] S. Dempe and M. Pilecka. Optimality conditions for set-valued optimisation problems using a modified Demyanov difference. *J. Optim. Theory Appl.*, 171(2):402–421, 2016.

[39] V. Demyanov and A. Rubinov, editors. *Quasidifferentiability and related topics*, volume 43 of *Nonconvex Optimization and its Applications*. Kluwer Academic Publishers, Dordrecht, 2000.

[40] S. Diamond and S. Boyd. CVXPY: A Python-embedded modeling language for convex optimization. *Journal of Machine Learning Research*, 17(83):1–5, 2016.

[41] C. Dmitri and G. W. Walster. Interval arithmetic specification. Technical report, 1998.

[42] A. Domahidi, E. Chu, and S. Boyd. ECOS: An SOCP solver for embedded systems. In *European Control Conference (ECC)*, pages 3071–3076, 2013.

[43] M. Durea and R. Strugariu. On some Fermat rules for set-valued optimization problems. *Optimization*, 60(5):575–591, 2011.

[44] M. Durea and C. Tammer. Fuzzy necessary optimality conditions for vector optimization problems. *Optimization*, 58(4):449–467, 2009.

[45] A. Eberhard, V. Roshchina, and T. Sang. Outer limits of subdifferentials for min-max type functions. *Optimization*, 68(7):1391–1409, 2019.

[46] M. Ehrgott, J. Ide, and A. Schöbel. Minmax robustness for multi-objective optimization problems. *European J. Oper. Res.*, 239(1):17–31, 2014.

[47] G. Eichfelder. *Adaptive scalarization methods in multiobjective optimization.* Vector Optimization. Springer-Verlag, Berlin, 2008.

[48] G. Eichfelder, J. Niebling, and S. Rocktäschel. An algorithmic approach to multiobjective optimization with decision uncertainty. *Journal of Global Optimization*, 7 2019.

[49] Z. Feinstein and B. Rudloff. A comparison of techniques for dynamic multivariate risk measures. In *Set optimization and applications—the state of the art*, volume 151 of *Springer Proc. Math. Stat.*, pages 3–41. Springer, Heidelberg, 2015.

[50] Y. Gao and X.-M. Yang. Properties of the nonlinear scalar functional and its applications to vector optimization problems. *Journal of Global Optimization*, Nov 2018.

[51] R. N. Gasimov. Characterization of the Benson proper efficiency and scalarization in nonconvex vector optimization. In *Multiple criteria decision making in the new millennium (Ankara, 2000)*, volume 507 of *Lecture Notes in Econom. and Math. Systems*, pages 189–198. Springer, Berlin, 2001.

[52] S. Gass and T. Saaty. The computational algorithm for the parametric objective function. *Naval Res. Logist. Quart.*, 2:39–45, 1955.

[53] C. Geiger and C. Kanzow. Numerische verfahren zur lösung unrestringierter optimierungsaufgaben. 1999.

[54] C. Gerstewitz. Nichtkonvexe Dualität in der Vektoroptimierung. *Wiss. Z. Tech. Hochsch. Leuna-Merseburg*, 25(3):357–364, 1983.

[55] C. Gerstewitz and E. Iwanow. Dualität für nichtkonvexe Vektoroptimierungsprobleme. *Wiss. Z. Tech. Hochsch. Ilmenau*, 31(2):61–81, 1985. Workshop on vector optimization (Plaue, 1984).

[56] Christiane Gerstewitz (Tammer). Beiträge zur Dualitätstheorie der nichtlinearen Vektoroptimierung [Contributions to duality theory in nonlinear vector optimization]. PhD Thesis, Technische Hochschule Leuna-Merseburg, 1984.

[57] C. Gerth and P. Weidner. Nonconvex separation theorems and some applications in vector optimization. *J. Optim. Theory Appl.*, 67(2):297–320, 1990.

[58] A. Göpfert, H. Riahi, C. Tammer, and C. Zălinescu. *Variational methods in partially ordered spaces*, volume 17 of *CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC*. Springer-Verlag, New York, 2003.

[59] D. Gourion and D. T. Luc. Generating the weakly efficient set of nonconvex multiobjective problems. *J. Global Optim.*, 41(4):517–538, 2008.

[60] L. M. Graña Drummond and B. F. Svaiter. A steepest descent method for vector optimization. *J. Comput. Appl. Math.*, 175(2):395–414, 2005.

[61] C. Günther and N. Popovici. New algorithms for discrete vector optimization based on the Graef-Younes method and cone-monotone sorting functions. *Optimization*, 67(7):975–1003, 2018.

[62] M. Gupta and M. Srivastava. Well-posedness and scalarization in set optimization involving ordering cones with possibly empty interior. *J. Global Optim.*, 73(2):447–463, 2019.

[63] C. Gutiérrez, B. Jiménez, E. Miglierina, and E. Molho. Scalarization in set optimization with solid and nonsolid ordering cones. *J. Global Optim.*, 61(3):525–552, 2015.

[64] C. Gutiérrez, E. Miglierina, E. Molho, and V. Novo. Pointwise well-posedness in set optimization with cone proper sets. *Nonlinear Anal.*, 75(4):1822–1833, 2012.

[65] C. Günther, E. Köbis, and N. Popovici. Computing minimal elements of finite families of sets w.r.t. preorder relations in set optimization. *J. Appl. Numer. Optim*, 1(2):131– 144, 2019.

[66] C. Günther, E. Köbis, and N. Popovici. On strictly minimal elements w.r.t. preorder relations in set-valued optimization. *Appl. Set-Valued Anal. and Opt*, 1(3):205–219, 2019.

[67] T. X. D. Ha. Optimality conditions for several types of efficient solutions of set-valued optimization problems. In *Nonlinear analysis and variational problems*, volume 35 of *Springer Optim. Appl.*, pages 305–324. Springer, New York, 2010.

[68] T.X.D. Ha. Lagrange multipliers for set-valued optimization problems associated with coderivatives. *J. Math. Anal. Appl.*, 311(2):647–663, 2005.

[69] T.X.D. Ha. A Hausdorff-type distance, a directional derivative of a set-valued map and applications in set optimization. *Optimization*, 67(7):1031–1050, 2018.

[70] T.X.D. Ha. A new concept of slope for set-valued maps and applications in set optimization studied with Kuroiwa's set approach. *Math. Methods Oper. Res.*, 91(1):137–158, 2020.

[71] H. Hadwiger. Minkowskische Addition und Subtraktion beliebiger Punktmengen und die Theoreme von Erhard Schmidt. *Math. Z.*, 53:210–218, 1950.

[72] Y. Y. Haimes, L. S. Lasdon, and D. A. Wismer. On a bicriterion formulation of the problems of integrated system identification and system optimization. *IEEE Trans. Systems Man Cybernet.*, SMC-1:296–297, 1971.

[73] A. Hamel and A. Löhne. Minimal element theorems and Ekeland's principle with set relations. *J. Nonlinear Convex Anal.*, 7(1):19–37, 2006.

[74] A. H. Hamel, F. Heyde, A. Löhne, B. Rudloff, and C. Schrage. Set optimization—a rather short introduction. In *Set optimization and applications—the state of the art*, volume 151 of *Springer Proc. Math. Stat.*, pages 65–141. Springer, Heidelberg, 2015.

[75] Y. Han and N.-J. Huang. Well-posedness and stability of solutions for set optimization problems. *Optimization*, 66(1):17–33, 2017.

[76] Y. Han, K. Zhang, and N.-J. Huang. The stability and extended well-posedness of the solution sets for set optimization problems via the Painlevé–Kuratowski convergence. *Math. Methods Oper. Res.*, 91(1):175–196, 2020.

[77] E. Hernández and L. Rodríguez-Marín. Duality in set optimization with set-valued maps. *Pac. J. Optim.*, 3(2):245–255, 2007.

[78] E. Hernández and L. Rodríguez-Marín. Existence theorems for set optimization problems. *Nonlinear Anal.*, 67(6):1726–1736, 2007.

[79] E. Hernández and L. Rodríguez-Marín. Lagrangian duality in set-valued optimization. *J. Optim. Theory Appl.*, 134(1):119–134, 2007.

[80] E. Hernández and L. Rodríguez-Marín. Nonconvex scalarization in set optimization with set-valued maps. *Journal of Mathematical Analysis and Applications*, 325(1):1 – 18, 2007.

[81] J.-B. Hiriart-Urruty. New concepts in nondifferentiable programming. *Bull. Soc. Math. France Mém.*, (60):57–85, 1979. Analyse non convexe (Proc. Colloq., Pau, 1977).

[82] J.-B. Hiriart-Urruty. Tangent cones, generalized gradients and mathematical programming in Banach spaces. *Math. Oper. Res.*, 4(1):79–97, 1979.

[83] R. B. Holmes. *Geometric functional analysis and its applications.* Springer-Verlag, New York-Heidelberg, 1975. Graduate Texts in Mathematics, No. 24.

[84] L. Huerga, B. Jiménez, and V. Novo. Lagrange multipliers in convex set optimization with the set and vector criteria. *Vietnam Journal of Mathematics*, Submitted, 2020.

[85] J. Ide and E. Köbis. Concepts of efficiency for uncertain multi-objective optimization problems based on set order relations. *Math. Methods Oper. Res.*, 80(1):99–127, 2014.

[86] J. Ide, E. Köbis, D. Kuroiwa, A. Schöbel, and C. Tammer. The relationship between multi-objective robustness concepts and set-valued optimization. *Fixed Point Theory Appl.*, pages 2014:83, 20, 2014.

[87] J. Jahn. Multiobjective search algorithm with subdivision technique. *Comput. Optim. Appl.*, 35(2):161–175, 2006.

[88] J. Jahn. *Introduction to the theory of nonlinear optimization.* Springer, Berlin, third edition, 2007.

[89] J. Jahn. *Vector Optimization: Theory, Applications, and Extensions.* Springer-Verlag, Berlin Heidelberg, 2 edition, 2011.

[90] J. Jahn. A derivative-free descent method in set optimization. *Comput. Optim. Appl.*, 60(2):393–411, 2015.

[91] J. Jahn. Directional derivatives in set optimization with the set less order relation. *Taiwanese J. Math.*, 19(3):737–757, 2015.

[92] J. Jahn. Vectorization in set optimization. *J. Optim. Theory Appl.*, 167(3):783–795, 2015.

[93] J. Jahn. Karush–Kuhn–Tucker conditions in set optimization. *Journal of Optimization Theory and Applications*, 172(3):707–725, 2017.

[94] J. Jahn. A derivative-free rooted tree method in nonconvex set optimization. *Pure Appl. Funct. Anal.*, 3(4):603–623, 2018.

[95] J. Jahn and T.X.D. Ha. New order relations in set optimization. *J. Optim. Theory Appl.*, 148(2):209–236, 2011.

[96] J. Jahn and U. Rathje. Graef-Younes Method with Backward Iteration. In K.H. Küfer, H. C. Rommelfanger, C . Tammer, and K. Winkler, editors, *Multicriteria Decision Making and Fuzzy Systems - Theory, Methods and Applications*, pages 75 – 81, Aachen, 2006. Shaker Verlag.

[97] L. Jiang, J. Cao, and L. Xiong. Generalized multiobjective robustness and relations to set-valued optimization. *Appl. Math. Comput.*, 361:599–608, 2019.

[98] B. Jiménez, V. Novo, and A. Vílchez. Six set scalarizations based on the oriented distance: properties and application to set optimization. *Optimization*, 69(3):437–470, 2020.

[99] I. Kaliszewski. *Quantitative Pareto analysis by cone separation technique.* Kluwer Academic Publishers, Boston, MA, 1994.

[100] E. Karaman, M. Soyertem, Atasever G.I, D. Tozkan, M. Küçük, and Y. Küçük. Partial order relations on family of sets and scalarizations for set optimization. *Positivity*, 22(3):783–802, 2018.

[101] E. Karaman, M. Soyertem, and I. A. Guevenc. Optimality conditions in set-valued optimization problem with respect to a partial order relation via directional derivative. *Taiwanese Journal of Mathematics*, 2019. Advance publication.

[102] R. Kasimbeyli. A nonlinear cone separation theorem and scalarization in nonconvex vector optimization. *SIAM J. Optim.*, 20(3):1591–1619, 2009.

[103] R. Kasimbeyli. A conic scalarization method in multi-objective optimization. *J. Global Optim.*, 56(2):279–297, 2013.

[104] R. Kasimbeyli, Z. K. Ozturk, N. Kasimbeyli, G. D. Yalcin, and B. I. Erdem. Comparison of some scalarization methods in multiobjective optimization: comparison of scalarization methods. *Bull. Malays. Math. Sci. Soc.*, 42(5):1875–1905, 2019.

[105] R. Kasimbeyli, Z.K. Ozturk, N. Kasimbeyli, G.D. Yalcin, and B. Icmen. Conic scalarization method in multiobjective optimization and relations with other scalarization methods. In Hoai An Le Thi, Tao Pham Dinh, and Ngoc Thanh Nguyen, editors, *Modelling, Computation and Optimization in Information Systems and Management Sciences*, pages 319–329, Cham, 2015. Springer International Publishing.

[106] A.A. Khan, C. Tammer, and C. Zălinescu. *Set-valued optimization.* Vector Optimization: An introduction with applications. Springer, Heidelberg, 2015.

[107] E. Köbis and M. A. Köbis. Treatment of set order relations by means of a nonlinear scalarization functional: a full characterization. *Optimization*, 65(10):1805–1827, 2016.

[108] E. Köbis, M. A. Köbis, and J.-C. Yao. Generalized upper set less order relation by means of a nonlinear scalarization functional. *J. Nonlinear Convex Anal.*, 17(4):725–734, 2016.

[109] E. Köbis, M. A. Köbis, and J.-C. Yao. Optimality conditions for set-valued optimization problems based on set approach and applications in uncertain optimization. *J. Nonlinear Convex Anal.*, 18(6):1001–1014, 2017.

[110] E. Köbis, D. Kuroiwa, and C. Tammer. Generalized set order relations and their numerical treatment. *Appl. Anal. Optim.*, 1(1):45–65, 2017.

[111] E. Köbis and C. Tammer. Characterization of set relations by means of a nonlinear scalarization functional. In *Le Thi H., Pham Dinh T., Nguyen N. (eds) Modelling, Computation and Optimization in Information Systems and Management Sciences. Advances in Intelligent Systems and Computing*, volume 359 of *Advances in Intelligent Systems and Computing*, pages 491–503. Springer, Cham, 2015.

[112] E. Köbis and T. Thanh Le. Numerical procedures for obtaining strong, strict and ideal minimal solutions of set optimization problems. *Appl. Anal. Optim.*, 2(3):423–440, 2018.

[113] X. Kong, G. Yu, and W. Liu. Optimality for set-valued optimization in the sense of vector and set criteria. *J. Inequal. Appl.*, pages Paper No. 47, 11, 2017.

[114] E. Kreyszig. *Introductory functional analysis with applications.* Wiley Classics Library. John Wiley & Sons, Inc., New York, 1989.

[115] D. Kuroiwa. Some criteria in set-valued optimization. Number 985, pages 171–176. 1997. Investigations on nonlinear analysis and convex analysis (Japanese) (Kyoto, 1996).

[116] D. Kuroiwa. Lagrange duality of set-valued optimization with natural criteria. Number 1068, pages 164–170. 1998. Theory and applications of mathematical optimization (Japanese) (Kyoto, 1998).

[117] D. Kuroiwa. The natural criteria in set-valued optimization. *Sūrikaisekikenkyūsho Kōkyūroku*, (1031):85–90, 1998. Research on nonlinear analysis and convex analysis (Japanese) (Kyoto, 1997).

[118] D. Kuroiwa. Some duality theorems of set-valued optimization with natural criteria. In *Nonlinear analysis and convex analysis (Niigata, 1998)*, pages 221–228. World Sci. Publ., River Edge, NJ, 1999.

[119] D. Kuroiwa. On set-valued optimization. In *Proceedings of the Third World Congress of Nonlinear Analysts, Part 2 (Catania, 2000)*, volume 47, pages 1395–1400, 2001.

[120] D. Kuroiwa. Existence theorems of set optimization with set-valued maps. *J. Inf. Optim. Sci.*, 24(1):73–84, 2003.

[121] D. Kuroiwa. On derivatives of set-valued maps and optimality conditions for set optimization. *J. Nonlinear Convex Anal.*, 10(1):41–50, 2009.

[122] D. Kuroiwa, T. Tanaka, and T.X.D. Ha. On cone convexity of set-valued maps. *Nonlinear Anal.*, 30(3):1487–1496, 1997.

[123] I. Kuwano, T. Tanaka, and S. Yamada. Inherited properties of nonlinear scalarizing functions for set-valued maps. In *Nonlinear analysis and convex analysis*, pages 161–177. Yokohama Publ., Yokohama, 2010.

[124] G. H. Li, S. J. Li, and M. X. You. Relationships between the oriented distance functional and a nonlinear separation functional. *J. Math. Anal. Appl.*, 466(1):1109–1117, 2018.

[125] X.-J. Long and J. W. Peng. Generalized B-well-posedness for set optimization problems. *J. Optim. Theory Appl.*, 157(3):612–623, 2013.

[126] X.-J. Long, J.-W. Peng, and Z.-Y. Peng. Scalarization and pointwise well-posedness for set optimization problems. *J. Global Optim.*, 62(4):763–773, 2015.

[127] T. Maeda. On optimization problems with set-valued objective maps: existence and optimality. *J. Optim. Theory Appl.*, 153(2):263–279, 2012.

[128] K. Miettinen and M. M. Mäkelä. On scalarizing functions in multiobjective optimization. *OR Spectrum*, 24(2):193–213, 2002.

[129] E. Miglierina and E. Molho. Scalarization and stability in vector optimization. *J. Optim. Theory Appl.*, 114(3):657–670, 2002.

[130] B. S. Mordukhovich. *Variational analysis and applications.* Springer Monographs in Mathematics. Springer, Cham, 2018.

[131] B. S. Mordukhovich, N. M. Nam, and N. D. Yen. Fréchet subdifferential calculus and optimality conditions in nondifferentiable programming. *Optimization*, 55(5-6):685–708, 2006.

[132] B.S. Mordukhovich. *Variational analysis and generalized differentiation. I*, volume 330 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, 2006. Basic theory.

[133] B.S. Mordukhovich. *Variational analysis and generalized differentiation. II*, volume 331 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, 2006. Applications.

[134] A. Mutapcic and S. Boyd. Cutting-set methods for robust convex optimization with pessimizing oracles. *Optim. Methods Softw.*, 24(3):381–406, 2009.

[135] N. Neukel. Order relations of sets and its application in socio-economics. *Appl. Math. Sci. (Ruse)*, 7(113-116):5711–5739, 2013.

[136] Z. G. Nishnianidze. Fixed points of monotone multivalued operators. *Soobshch. Akad. Nauk Gruzin. SSR*, 114(3):489–491, 1984.

[137] A. Oussarhan and I. Daidai. Necessary and sufficient conditions for set-valued maps with set optimization. *Abstr. Appl. Anal.*, pages Art. ID 5962049, 6, 2018.

[138] A. Pascoletti and P. Serafini. Scalarizing vector optimization problems. *J. Optim. Theory Appl.*, 42(4):499–524, 1984.

[139] R.R. Phelps. *Convex functions, monotone operators and differentiability*, volume 1364 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 1989.

[140] M. Pilecka. Optimality conditions in set-valued programming using the set criterion. Preprint 2014-02. Germany: Technical University of Freiberg, 2014.

[141] L. Pontryagin. Linear differential games. I, II. *Dokl. Akad. Nauk SSSR 174 (1967), 1278–1280; ibid.*, 175:764–766, 1967.

[142] L. Rodríguez-Marín and M. Sama. (Λ, c)-contingent derivatives of set-valued maps. *Journal of Mathematical Analysis and Applications*, 335(2):974 – 989, 2007.

[143] A. M. Rubinov and I. S. Akhundov. Difference of compact sets in the sense of Demyanov and its application to nonsmooth analysis. *Optimization*, 23(3):179–188, 1992.

[144] W. Rudin. *Functional analysis.* McGraw-Hill Book Co., New York, 1973. McGraw-Hill Series in Higher Mathematics.

[145] W. Schirotzek. *Nonsmooth analysis.* Universitext. Springer, Berlin, 2007.

[146] M. Schmidt, A. Schöbel, and L. Thom. Min-ordering and max-ordering scalarization methods for multi-objective robust optimization. *European J. Oper. Res.*, 275(2):446–459, 2019.

[147] D. S. Shi. Contingent derivative of the perturbation map in multiobjective optimization. *Journal of Optimization Theory and Applications*, 70(2):385–396, Aug 1991.

[148] M. Studniarski. Necessary and sufficient conditions for isolated local minima of nonsmooth functions. *SIAM Journal on Control and Optimization*, 24, 07 1986.

[149] X. D. H. Truong. Existence and density results for proper efficiency in cone compact sets. *J. Optim. Theory Appl.*, 111(1):173–194, 2001.

[150] V.A. Tuan and L.T. Tam. The Lipschitz properties of cone convex set-valued functions. *Variational Analysis and Set Optimization: Developments and Applications in Decision Making*, accepted 2018.

[151] V.A. Tuan, L.T. Tam, and C. Tammer. The Lagrange multipliers for convex vector functions in Banach spaces. *Investigación Oper.*, 39(3):411–425, 2018.

[152] V.A. Tuan, C. Tammer, and C. Zălinescu. The Lipschitzianity of convex vector and set-valued functions. *TOP*, 24(1):273–299, 2016.

[153] A. P. Wierzbicki. Basic properties of scalarizing functionals for multiobjective optimization. *Math. Operationsforsch. Statist. Ser. Optimization*, 8(1):55–60, 1977.

[154] A. P. Wierzbicki. The use of reference objectives in multiobjective optimization. In *Multiple criteria decision making theory and application (Proc. Third Conf., Hagen/Königswinter, 1979)*, volume 177 of *Lecture Notes in Econom. and Math. Systems*, pages 468–486. Springer, Berlin-New York, 1980.

[155] A. P. Wierzbicki. On the completeness and constructiveness of parametric characterizations to vector optimization problems. *OR Spektrum*, 8(2):73–87, 1986.

[156] J. Yeh. *Real Analysis: Theory of Measure and Integration Third Edition*. World Scientific Publishing Company Pte Limited, 2014.

[157] YM. Younes. *Studies on discrete vector optimization*. Dissertation, University of Demiatta, 1993.

[158] R.C. Young. The algebra of many-valued quantities. *Math. Ann.*, 104(1):260–290, 1931.

Selbständigkeitserklärung

Ich erkläre an Eides statt, dass ich die vorliegende Arbeit

On Set Optimization with Set Relations: A Scalarization Approach to Optimality Conditions and Algorithms

selbstständig und ohne fremde Hilfe verfasst, keine anderen als die von mir angegebenen Quellen und Hilfsmittel benutzt und die den benutzten Werken wörtlich oder inhaltlich entnommenen Stellen als solche kenntlich gemacht habe.

Halle (Saale), April 29, 2021

(Ernest Quintana Aparicio)

Curriculum Vitae

Ernest Quintana Aparicio, born on 23 January 1991

Oct. 2016 - May. 2020	Doctoral Studies in Applied Mathematics Martin-Luther-Universität Halle-Wittenberg
May. 2016	Master of Science in Mathematics
Sep. 2014 - May. 2016	Master Studies in Applied Mathematics University of Havana
Jun. 2014	Bachelor of Science in Mathematics
Sep. 2010 - Jun. 2014	Bachelor Studies in Mathematics University "Marta Abreu" of Las Villas
Jun. 2009	Abitur
Sep. 2006 - Jun. 2009	Carlos Roloff Highschool (gymnasium), Cienfuegos (Cuba)

Halle (Saale), April 29, 2021

(Ernest Quintana Aparicio)

Publications

1. G. Bouza, E. Quintana, and C. Tammer. A unified characterization of nonlinear scalarizing functionals in optimization. *Vietnam J. Math.*, 47(3):683–713, 2019

2. G. Bouza, E. Quintana, V.A. Tuan, and C. Tammer. The Fermat rule for set optimization problems with Lipschitzian set-valued mappings. *Journal of Nonlinear and Convex Analysis* (accepted)