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# ON DEFECTIVITY OF FAMILIES OF FULL-DIMENSIONAL POINT CONFIGURATIONS

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Abstract. The mixed discriminant of a family of point configurations can be considered as a generalization of the $A$-discriminant of one Laurent polynomial to a family of Laurent polynomials. Generalizing the concept of defectivity, a family of point configurations is called defective if the mixed discriminant is trivial. Using a recent criterion by Furukawa and Ito we give a necessary condition for defectivity of a family in the case that all point configurations are full-dimensional. This implies the conjecture by Cattani, Cueto, Dickenstein, Di Rocco, and Sturmfels that a family of $n$ full-dimensional configurations in $\mathbb{Z}^n$ is defective if and only if the mixed volume of the convex hulls of its elements is 1.

## 1. Introduction

Let us fix some notation. Throughout the paper, a *configuration* $A \subset \mathbb{Z}^n$ denotes a finite subset of $\mathbb{Z}^n$. We write $A_0 + A_1 \coloneqq \{a_0 + a_1 : a_0 \in A_0, a_1 \in A_1\}$ for the *Minkowski sum* of two configurations $A_0, A_1 \subset \mathbb{Z}^n$. We denote by $e_1, \dots, e_n$ the standard basis vectors in $\mathbb{Z}^n$ and in this context also set $e_0 \coloneqq 0 \in \mathbb{Z}^n$. Furthermore we denote by $\Delta_k \coloneqq \{e_0, e_1, \dots, e_k\}$ the vertices of the *standard unimodular simplex*. The *dimension* of $A \subset \mathbb{Z}^n$ is the dimension of its affine hull (which we denote by $\operatorname{aff}(A)$) as an affine subspace of $\mathbb{R}^n$ and is denoted by $\dim(A)$. We call $A$ *full-dimensional* if $\dim(A) = n$. We say that two configurations $A \subset \mathbb{Z}^n$, $B \subset \mathbb{Z}^m$ are *isomorphic* and denote this by $A \cong B$ if there is an affine lattice isomorphism of the ambient lattices $\operatorname{aff}(A) \cap \mathbb{Z}^n \to \operatorname{aff}(B) \cap \mathbb{Z}^m$ mapping $A$ onto $B$. A lattice polytope that is isomorphic to a standard unimodular simplex is called *unimodular simplex*. If a lattice homomorphism $\varphi\colon \mathbb{Z}^n \to \mathbb{Z}^m$ is surjective, we call $\varphi$ a *lattice projection*. For convenience we use the notation $[m] \coloneqq \{0, \dots, m\}$.

Let us recall the definition of the mixed discriminant (see [CCD+13]). Consider a configuration $A \subset \mathbb{Z}^n$. We say that $f \in \mathbb{C}[x, x^{-1}] = \mathbb{C}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$ is *supported on $A$* if it is of the form

$$f = \sum_{a \in A} c_a x^a,$$

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with $c_a \in \mathbb{C}$ for all $a \in A$. We call an isolated solution $u \in (\mathbb{C}^*)^n$ for a system of Laurent polynomials $f_0(x) = \cdots = f_k(x) = 0$ a *non-degenerate multiple root* if the gradients $\nabla f_i(u)$ are linearly dependent, while any $k$ of them are linearly independent. Now consider $A_0, \ldots, A_k \subset \mathbb{Z}^n$. Each polynomial $f_i$ supported on $A_i$ is of the form $f_i = \sum_{a \in A_i} c_{i,a} x^a$, and we define the *discriminantal variety* $\Sigma_{A_0,\ldots,A_k}$ as the closure of the set of coefficients $c_{i,a}$ such that the corresponding system of the Laurent polynomials $f_i$ has a non-degenerate multiple root. If $\Sigma_{A_0,\ldots,A_k}$ is a hypersurface, one defines the *mixed discriminant* $\Delta_{A_0,\ldots,A_k}$ to be the up-to-sign unique irreducible integral polynomial defining it. Otherwise, and this is the case we are going to be interested in, we set $\Delta_{A_0,\ldots,A_k} = 1$ and call the set of configurations $A_0, \ldots, A_k$ *defective*.

In the specific case of a single configuration $A \subset \mathbb{Z}^n$ the mixed discriminant $\Delta_A$ agrees with the *A-discriminant* as introduced in [GKZ94]. Let us recall the relation of defectivity of a point configuration to defectivity of projective varieties. Let $A = \{a_0, \ldots, a_k\} \subset \mathbb{Z}^n$ and denote by $X_A \subseteq \mathbb{P}^k$ the toric variety obtained as the closure of the image of the morphism

$$\varphi_A \colon (\mathbb{C}^*)^n \to \mathbb{P}^k \qquad t \mapsto [t^{a_0} \colon \ldots \colon t^{a_k}].$$

Then the variety $X_A^*$ projectively dual to $X_A$ is the same as the projectivization of the variety $\Sigma_A$. The *dual defect* $\delta_{X_A}$ of $X_A$ is defined as $\delta_{X_A} := \operatorname{codim}(X_A^*) - 1$, and the variety $X_A$ is called *defective* if $\delta_{X_A} > 0$. In particular, $X_A$ is defective if and only if $A$ is a defective configuration, or equivalently, the degree of the $A$-discriminant is zero. The $A$-discriminant, especially its degree, has been studied intensively starting with the book [GKZ94]. We refer to the survey article [Pie15] for background and references. In particular, a special focus has been on the question of defectivity when $A$ is the set of all lattice points of its convex hull ([DR06], [CDR08], [DDRP09], [DN10], [DNV12]). In more general situations, conditions for defectivity were given in [CC07], [DFS07], [Est10], [Ito15]. In particular, a complete characterization in terms of so-called *iterated circuits* was presented by Esterov [Est10] and proven in [Est18] (see also [For19] for a more general version). Recently, a different characterization was obtained by Furukawa and Ito [FI16] phrased in terms of so-called *Cayley sums* (we refer the reader to Section 2 for the definition of Cayley sums).

The study of defectivity of a family of point configurations has so far been addressed in [CCD+13], [DEK14], [Est19] and, using a slightly different definition of defectivity of a family, in [Est10]. By the so-called Cayley trick, their defectivity can be reduced to defectivity of their Cayley sum if all point configurations are full-dimensional (see Theorem 3.1). Using the recent results by Furukawa and Ito, this allows us to deduce a necessary condition for defectivity of a family. For this, let us introduce some notation. For $A \subset \mathbb{Z}^n$ we denote by $\langle A - A \rangle$ the subgroup of $\mathbb{Z}^n$ generated by the set $\{a_1 - a_2 \colon a_1, a_2 \in A\}$ and say that $A \subset \mathbb{Z}^n$ is spanning if $\langle A - A \rangle = \mathbb{Z}^n$. More generally we say that a family $A_0, \ldots, A_k \subset \mathbb{Z}^n$ is *spanning* if $\langle A_0 - A_0 \rangle + \cdots + \langle A_k - A_k \rangle = \mathbb{Z}^n$.

**Theorem 1.1.** *Let $k \leq n$ and $A_0, \ldots, A_k \subset \mathbb{Z}^n$ be full-dimensional configurations that form a spanning family. If $A_0, \ldots, A_k$ is defective, then the convex hull of the Minkowski sum $A_0 + \cdots + A_k$ does not have any interior lattice points, i.e.,*

$$\operatorname{int}(\operatorname{conv}(A_0 + \cdots + A_k)) \cap \mathbb{Z}^n = \emptyset.$$

As a consequence, we get the following result, which was conjectured in [CCD+13], where it was proven in the 2-dimensional case as well as under additional smoothness assumptions.

**Corollary 1.2.** *Let $A_0, \dots, A_{n-1} \subset \mathbb{Z}^n$ be a spanning family of full-dimensional configurations. Then $A_0, \dots, A_{n-1}$ is defective if and only if it has mixed volume 1. In this case, $A_0, \dots, A_{n-1}$ are all translates of the vertex set of the same unimodular simplex.*

*Proof.* Clearly, having mixed volume one implies defectivity. By Theorem 1 in [Hov78] (or Corollary 3.2 of [Nil20]) the mixed volume of $\operatorname{conv}(A_0), \dots, \operatorname{conv}(A_{n-1})$ can be computed as

$$1 + \sum_{\emptyset \neq I \subseteq [n-1]} (-1)^{n-|I|} |\operatorname{int}(\operatorname{conv}(\sum_{i\in I} A_i)) \cap \mathbb{Z}^n|.$$

If $A_0, \dots, A_{n-1}$ is defective, Theorem 1.1 implies that $\operatorname{conv}(A_0 + \dots + A_{n-1})$, and therefore (as all $A_i$ are full-dimensional) also $\operatorname{conv}(\sum_{i\in I} A_i)$ for any $I \subseteq [n-1]$, has no interior lattice points. This shows that the mixed volume of $\operatorname{conv}(A_0), \dots, \operatorname{conv}(A_{n-1})$ is 1. The last statement follows from Proposition 2.7 of [CCD+13] (see also [EG15]). ☐

*Remark* 1.3. After the first version of this paper was made available, there was another proof of Corollary 1.2 given by Esterov (Corollary 3.23 in [Est19]). Esterov's result is more general in the sense that it only makes the weaker assumption of $A_0, \dots, A_{n-1}$ forming a so-called irreducible family instead of all configurations being full-dimensional. However, it does not generalize Theorem 1.1, as it only treats the case of $k = n - 1$. It would be interesting to investigate whether the assumption of full-dimensionality in Theorem 1.1 can always be replaced by irreducibility of the family. We call a family $A_0, \dots, A_k \subset \mathbb{Z}^n$ *irreducible* if no $l$ distinct members can be shifted to a common $(l + (n - 1 - k))$-dimensional affine subspace for any $l \in \{1, \dots, k\}$.

Note that for given $A_0, \dots, A_k \subset \mathbb{Z}^n$ one may always choose a spanning family whose mixed discriminantal variety equals $\Sigma_{A_0,\dots,A_k}$. By applying a suitable transformation, this implies the following slightly more general version of Theorem 1.1.

**Corollary 1.4.** *Let $k \le n$ and let $A_0, \dots, A_k \subset \mathbb{Z}^n$ be full-dimensional configurations. Define by $\Lambda := \langle A_0 - A_0\rangle + \dots + \langle A_k - A_k\rangle$ the lattice spanned by these configurations. If $A_0, \dots, A_k$ is defective, then*

$$\operatorname{int}((A_0 - a_0) + \dots + (A_k - a_k)) \cap \Lambda = \emptyset,$$

*for all choices $a_0, \dots, a_k$ such that $a_i \in A_i$ for all $i \in [k]$.*

*Remark* 1.5. The statement of Theorem 1.1 is in general not true if we do not pose sufficient restrictions on the dimensions of the configurations. A counterexample is provided by choosing $A_0, A_1 \subset \mathbb{Z}^2$ as

$$A_0 = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$

It is straightforward to verify that the corresponding system

$$f_0 = c_{0,00} + c_{0,10}x_1 + c_{0,20}x_1^2, \;\; f_1 = c_{1,00} + c_{1,01}x_2 + c_{1,02}x_2^2$$

does not have a non-degenerate multiple root for any choice of coefficients. Therefore the variety $\Sigma_{A_0,A_1}$ is empty, in particular $A_0, A_1$ is a defective family, while $\operatorname{conv}(A_0 + A_1)$ contains $(1,1)$ as an interior lattice point.

*Remark* 1.6. Note that the criterion for defectivity given in Theorem 1.1 is not sufficient. An easy class of counterexamples is given for $k = 0$ by $A_0 \coloneqq \operatorname{conv}(n\Delta_n) \cap \mathbb{Z}^n$ for $n > 1$. Clearly $\operatorname{conv}(A_0)$ does not have any interior lattice points but cannot be defective since its lattice width is $n > 1$.

**Organization of the paper.** In Section 2 we introduce Cayley sums and recall some basic results. Section 3 contains the proof of Theorem 1.1.

## 2. Basics of Cayley sums

As Cayley sums are going to play a crucial role in our proof, let us recall some basic facts.

**Definition 2.1.** Let $A_0, \dots, A_k \subset \mathbb{Z}^n$ be configurations. We define the *Cayley sum* $A_0 * \cdots * A_k$ as

$$A_0 * \cdots * A_k \coloneqq (A_0 \times \{e_0\}) \cup (A_1 \times \{e_1\}) \cup \cdots \cup (A_k \times \{e_k\}) \subset \mathbb{Z}^{n+k}.$$

We call a Cayley sum $A_0 * \cdots * A_k$ *proper* if all $A_i$ are non-empty. In this case one has $\dim(A_0 * \cdots * A_k) = \dim(A_0 + \cdots + A_k) + k$.

Let $F \subseteq A$ be a subconfiguration of a configuration $A \subset \mathbb{Z}^n$. We denote by $F^c = \{x \in A \colon x \notin F\}$ the *complement of $F$ in $A$*. Furthermore, we call $F$ a *face* of $A$ if it is the intersection of a face of the lattice polytope $\operatorname{conv}(A)$ with $A$ and denote by $\mathcal{F}(A)$ the set of all faces of $A$. We call a face $F \in \mathcal{F}(A)$ *proper* if $F \neq A$.

**Definition 2.2.** Let $A \subset \mathbb{Z}^n$ and $F_0, \dots, F_k \in \mathcal{F}(A)$ be faces that cover $A$. We say that $F_0, \dots, F_k$ form a *Cayley decomposition* of $A$ if there exists a lattice projection $\pi \colon \mathbb{Z}^n \to \mathbb{Z}^k$ such that $\pi(F_i) \subseteq \{e_i\}$ for all $i \in [k]$.

*Remark* 2.3. Clearly, a Cayley sum $A_0 * \cdots * A_k$ has a Cayley decomposition into the faces $(A_0 \times \{e_0\}), \dots, (A_k \times \{e_k\})$, and we denote them by $\tilde{A}_i \coloneqq A_i \times \{e_i\}$.

**Proposition 2.4.** *Let $A \subset \mathbb{Z}^n$ be a configuration. Then the following are equivalent.*

1. *There exists a Cayley decomposition of $A$ into non-empty faces $F_0, \dots, F_k \in \mathcal{F}(A)$.*
2. *There exists a lattice projection $\pi \colon \mathbb{Z}^n \to \mathbb{Z}^k$ with $\pi(A) = \Delta_k$.*
3. *There exist configurations $A_0, \dots, A_k \subset \mathbb{Z}^{n-k}$ such that $A \cong A_0 * \cdots * A_k$.*

The proof is left to the reader (cf. [BN08, Proposition 2.3]).

*Remark* 2.5. Let $A \subset \mathbb{Z}^n$ be a configuration, let $F_0, \dots, F_k \in \mathcal{F}(A)$ be a Cayley decomposition of $A$, and let $F \in \mathcal{F}(A)$ be an arbitrary face. Then we have a Cayley decomposition

$$F \cong (F_0 \cap F) * \cdots * (F_k \cap F).$$

In particular, any face of a Cayley sum $A_0 * \cdots * A_k$ is isomorphic to a Cayley sum of (maybe empty) faces of each of the $A_i$.

**Definition 2.6.** Let $A_0, \dots, A_k \subset \mathbb{Z}^n$ be configurations. We say that the Cayley sum $A_0 * \cdots * A_k$ is *of join type* if the homomorphism $\langle A_0 - A_0 \rangle \oplus \cdots \oplus \langle A_k - A_k \rangle \to \langle A_0 - A_0 \rangle + \cdots + \langle A_k - A_k \rangle \subset \mathbb{Z}^n$ given by $(a_0, \dots, a_k) \mapsto a_0 + \cdots + a_k$ is injective.

*Remark* 2.7. As $\dim(\langle A_0 - A_0\rangle \oplus \cdots \oplus \langle A_k - A_k\rangle) = \dim(A_0) + \cdots + \dim(A_k)$ and $\dim(\langle A_0 - A_0\rangle + \cdots + \langle A_k - A_k\rangle) = \dim(A_0 + \cdots + A_k)$, a Cayley sum $A_0 * \cdots * A_k$ is of join type if and only if $\dim(A_0) + \cdots + \dim(A_k) = \dim(A_0 + \cdots + A_k)$. In particular, the dimension of a proper Cayley sum $A_0 * \cdots * A_k$ of join type equals $\dim(A_0) + \cdots + \dim(A_k) + k$, which is the maximal Cayley dimension for given dimensions of the summands $A_0, \ldots, A_k$.

## 3. Proof of main theorem

The following result was presented by Di Rocco in a talk in June 2016 at the Fields Institute for Research in Mathematical Sciences and is soon to appear in an announced paper by Di Rocco, Dickenstein, and Morrison [DDRM18] (see also [CCD+13] for the special case where $k = n - 1$).

**Theorem 3.1.** *If a family of configurations $A_0, \ldots, A_k \subset \mathbb{Z}^n$ is defective, then the Cayley sum $A_0 * \cdots * A_k \subset \mathbb{Z}^{n+k}$ is defective.*

This identification allows us to apply the following characterization of defective configurations by Furukawa and Ito [FI16] as the main tool in proving our statement about defectivity of a family of configurations.

**Theorem 3.2** (Furukawa, Ito)**.** *Let $A \subset \mathbb{Z}^n$ be a spanning configuration. Then $A$ is defective if and only if there exist natural numbers $c < r$ and a lattice projection $\pi\colon \mathbb{Z}^n \to \mathbb{Z}^{n-c}$ such that $\pi(A) \cong B_0 * \cdots * B_r$ where this Cayley sum $B_0 * \cdots * B_r$ is of join type and $B_i \neq \emptyset$ for all $i \in [r]$.*

It is a straightforward computation to show that $A_0, \ldots, A_k \subset \mathbb{Z}^n$ form a spanning family if and only if their Cayley sum $A_0 * \cdots * A_k \subset \mathbb{Z}^{n+k}$ is spanning.

The following technical lemma is crucial for the proof of the main theorem.

**Lemma 3.3.** *Let $A_0, \ldots, A_k \subset \mathbb{Z}^n$ be full-dimensional configurations and let $B_0, \ldots, B_r \subset \mathbb{Z}^{n+k-r}$ be non-empty configurations such that*

$$A_0 * \cdots * A_k \cong B_0 * \cdots * B_r \subset \mathbb{Z}^{n+k}.$$

(a) *One has $\dim(B_i) \geq \min(k, n)$ for all $i \in [r]$.*
(b) *If furthermore $\dim(B_i) < n$ for all $i \in [r]$, also the following inequality holds:*

$$\dim(B_0) + \cdots + \dim(B_r) \geq n - r + (r + 1)k.$$

*Proof.* For $k = 0$ or $r = 0$ one can directly verify that both statements hold. So we may assume that $k, r \geq 1$ and observe that in this case each of the $\tilde{B}_i \subseteq B_0 * \cdots * B_r$ (see Remark 2.3) is isomorphic to a proper face $B'_i \subseteq A_0 * \cdots * A_k$ and $B'_0, \ldots, B'_r$ form a Cayley decomposition of $A_0 * \cdots * A_k$ (since the $\tilde{B}_i$ form a Cayley decomposition of $B_0 * \cdots * B_r$). The complement $(B'_i)^c$ of each of the $B'_i$ is again a proper face of $A_0 * \cdots * A_k$ (since this is true for the complement of $\tilde{B}_i$). Now let $i \in [r]$ be arbitrary and assume that $\dim(B_i) < n$ (otherwise (a) is trivial). Then $B'_i$ cannot contain $\tilde{A}_j$ for any $j \in [k]$ and $(B'_i)^c$ has non-empty intersection with each of the $\tilde{A}_j$. Therefore by Remark 2.5 in particular $\dim(B'_i)^c \geq \dim((B'_i)^c \cap \tilde{A}_j) + k$ for all $j \in [k]$. Now if $(B'_i)^c$ contained one of the $\tilde{A}_j$, this inequality would imply that $\dim(B'_i)^c \geq n + k$ in contradiction to $(B'_i)^c$ being a proper face of $A_0 * \cdots * A_k$.

So also $B'_i$ has non-empty intersection with all of the $\tilde{A}_j$, and by Remark 2.5 we have

$$B'_i \cong (\tilde{A}_0 \cap B'_i) * \cdots * (\tilde{A}_k \cap B'_i),$$

which implies that

$$\dim(\tilde{A}_j \cap B'_i) \leq \dim(B'_i) - k \tag{1}$$

for all $j \in [k]$ and all $i \in [r]$ with $\dim(B_i) < n$. This in particular implies that $\dim(B_i) = \dim(B'_i) \geq k \geq \min(k, n)$. Moreover, since the $B'_i$ also form a Cayley decomposition of $A_0 * \cdots * A_k$, we obtain

$$\tilde{A}_j \cong (\tilde{A}_j \cap B'_0) * \cdots * (\tilde{A}_j \cap B'_r),$$

and therefore assuming $\dim(B_i) < n$ for all $i \in [r]$, applying (1) yields

$$\begin{aligned} n = \dim(\tilde{A}_j) &\leq r + \dim(\tilde{A}_j \cap B'_0) + \cdots + \dim(\tilde{A}_j \cap B'_r) \\ &\leq r + \dim(B'_0) - k + \cdots + \dim(B'_r) - k. \end{aligned}$$

□

Note that the result above remains true in the more general setting of point configurations in $\mathbb{R}^n$ and the notion of isomorphy induced by affine bijections.

Let us recall that the *codegree* $\operatorname{codeg}(P)$ of a lattice polytope $P \subset \mathbb{R}^n$ is the smallest natural number $c \geq 1$ such that $\operatorname{int}(cP) \cap \mathbb{Z}^n \neq \emptyset$ (see e.g. [DN10]).

*Proof of Theorem* 1.1. As remarked above, Theorem 3.1 implies that $A_0 * \cdots * A_k \subset \mathbb{Z}^{n+k}$ is a spanning defective configuration. By Theorem 3.2 there exist $c < r$ and a lattice projection $\pi\colon \mathbb{Z}^{n+k} \to \mathbb{Z}^{n+k-c}$ such that $\pi(A_0 * \cdots * A_k)$ has a Cayley decomposition of join type into non-empty faces $F_0, \ldots, F_r \in \mathcal{F}(\pi(A_0 * \cdots * A_k))$. Let us assume that $\operatorname{conv}(A_0 + \cdots + A_k)$ has interior lattice points. By the well-known connection between Cayley sums and weighted Minkowski sums (see e.g. [HRS00]) this is equivalent to $(k+1) \cdot \operatorname{conv}(A_0 * \cdots * A_k)$ having an interior point in $\mathbb{Z}^{n+k}$, which implies that $\operatorname{codeg}(\operatorname{conv}(A_0 * \cdots * A_k)) \leq k+1$. By Proposition 2.4 we have a projection $\pi_r\colon \mathbb{Z}^{n+k-c} \to \mathbb{Z}^r$ that maps $\pi(A_0 * \cdots * A_k)$ surjectively onto $\Delta_r$. Since under lattice projections the codegree of a lattice polytope cannot increase we get inequalities

$$k+1 \geq \operatorname{codeg}(A_0 * \cdots * A_k) \geq \operatorname{codeg}(F_0 * \cdots * F_r) \geq \operatorname{codeg}(\Delta_r) = r+1, \text{ hence}$$

$$k \geq r. \tag{2}$$

We observe that the lifts

$$\hat{F}_i := \pi^{-1}(F_i) \cap (A_0 * \cdots * A_k)$$

define a Cayley decomposition (in general not of join type) of $A_0 * \cdots * A_k$. As $\pi$ is a projection of codimension $c$, we see that

$$\dim(\hat{F}_i) \leq \dim(F_i) + c, \tag{3}$$

for all $i \in [r]$. Combining this with the fact that the $F_i$ form a Cayley decomposition of join type and using Remark 2.7 one obtains

$$\begin{aligned}\dim(\hat{F}_0) + \cdots + \dim(\hat{F}_r) &\leq \dim(F_0) + \cdots + \dim(F_r) + c(r+1) \\ &= \dim(F_0 + \cdots + F_r) + c(r+1) \\ &= n + k - c - r + c(r+1) \\ &= n + k + r(c-1).\end{aligned}$$

Let us assume that $\dim(\hat{F}_j) \geq n$ for some $j \in [r]$ . Therefore $\dim(F_j) \geq n - c$. Without loss of generality let $j = 0$. As the $F_i$ form a Cayley decomposition of join type of the $(n+k-c)$-dimensional configuration $\pi(A_0 * \cdots * A_k)$ we have the following inequality for the remaining summands:

$$\begin{aligned}\dim(F_1) + \cdots + \dim(F_r) &= \dim(F_0 + \cdots + F_r) - \dim(F_0) \\ &= n + k - c - r - \dim(F_0) \\ &\leq n + k - c - r - (n - c) \\ &= k - r.\end{aligned}$$

However, on the other hand Lemma 3.3(a) implies that $\dim(\hat{F}_i) \geq k$ for all $i \in [r]$ (since we assumed $k \leq n$). So by (3) we have $\dim(F_i) \geq k - c$, which yields another inequality for the remaining summands:

$$\dim(F_1) + \cdots + \dim(F_r) \geq r(k-c).$$

These inequalities contradict each other since $r(k-c) > k - r$, which can be seen by observing that $r$ is strictly positive and $c$ is strictly smaller than $r$.

Therefore $\dim(\hat{F}_j) < n$ for all $j \in [r]$. So we may apply part (b) of Lemma 3.3 and obtain $n - r + (r+1)k \leq \dim(\hat{F}_0) + \cdots + \dim(\hat{F}_r)$. Hence,

$$n - r + (r+1)k \leq n + k + r(c-1),$$

which is (since $r$ is strictly positive) equivalent to $k \leq c < r$, a contradiction. ☐

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