Mixed Lattice Polytope Theory with a View towards Sparse Polynomial Systems

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Zusammenfassung

Diese Arbeit befasst sich mit verschiedenen gemischten Fragestellungen aus der Gitterpolytoptheorie. Damit sind Fragestellungen gemeint, die sich auf Tupel von Gitterpolytopen beziehen und für die somit sowohl die Struktur der einzelnen Polytope als auch deren Lage zueinander eine Rolle spielen. Eine zentrale Motivation hierfür ist der berühmte Satz von Bernstein-Khovanskii-Kushnirenko, der die Anzahl an Lösungen eines polynomiellen Gleichungssystems durch das gemischte Volumen des Tupels der Newton-Polytope der Polynome beschränkt. Ziel dieser Arbeit ist es, verschiedene Probleme an der Grenze zwischen algebraischer und diskreter Geometrie aus dem Blickwinkel einer gemischten Gitterpolytoptheorie zu behandeln und damit sowohl die Relevanz dieses Gebietes zu illustrieren als auch die Entwicklung der Grundlagen auf diesem Feld voranzutreiben.

Im ersten Teil der Arbeit führen wir grundlegende Begriffe und Notationen ein.

Im zweiten Teil präsentieren wir Resultate über die Cayley-Summe eines Tupels von Gitterpolytopen, welche wir an verschiedenen anderen Stellen in dieser Arbeit benötigen.

Im dritten Teil widmen wir uns der gemischten Diskriminanten eines Tupels von ganzzahligen Punktkonfigurationen. Diese ist ein Polynom, das enkodiert unter welchen Bedingungen ein polynomielles Gleichungssystem bestimmte mehrfache Nullstellen hat. Wir geben eine hinreichende kombinatorische Bedingung für die Existenz dieser gemischten Diskrimanten und beweisen damit eine Vermutung von Cattani et. al.

Der vierte Teil ist der Entwicklung eines Algorithmus für die Klassifikation von Tripeln von Gitterpolytopen im $\mathbb{R}^3$ mit gegebenem gemischten Volumen gewidmet. Nach dem Satz von BKK ist dies äquivalent zu der Klassifikation von generischen Systemen trivariater Polynome mit gegebener Anzahl von Lösungen. Anhand einer Implementierung dieses Algorithmus erhalten wir eine vollständige Klassifikation dieser Tripel mit gemischtem Volumen höchstens vier.

Im fünften Teil dieser Arbeit untersuchen wir Tupel von Gitterpolytopen, deren gemischter Grad höchstens eins ist. Wir zeigen, dass es in jeder Dimension abgesehen von einer gut verstandenen Familie von Tupeln nur endlich viele exzeptionelle Tupel mit gemischtem Grad eins gibt. Des Weiteren klassifizieren wir solche Tupel in Dimension drei vollständig.

Im sechsten und letzten Teil dieser Arbeit zeigen wir eine obere Schranke an das Volumen der Minkowski-Summe eines Tupels konvexer Körper, dessen gemischtes Volumen gegeben ist. Unsere Schranke ist asymptotisch scharf und in den Spezialfällen von Dimension zwei und drei finden wir darüber hinaus eine scharfe exakte Schranke.

Abstract

This work treats several mixed questions in the theory of lattice polytopes. By that we mean questions that are in terms of tuples of lattice polytopes and for which one has to consider not only the structure of the single lattice polytopes in the tuple, but also their alignment with respect to each other. Central motivation for treating such questions comes from the famous Bernstein-Khovanskii-Kushnirenko theorem. This result bounds the number of solutions of a polynomial system by the mixed volume of the tuple of Newton polytopes of the polynomials. The scope of this work is to treat different problems at the intersection of algebraic and discrete geometry from the point of view of a mixed lattice polytope theory. In the course of this we illustrate the relevance of this field of research and make progress in the development of its foundations.

The first chapter is dedicated to the introduction of basic concepts and notation.

In the second chapter we present results about the Cayley sum of a tuple of lattice polytopes that we make use of in several parts of this work.

The third chapter deals with the mixed discriminant of a tuple of point configurations, which is a polynomial that encodes the conditions for a system of polynomial equations to have a multiple root. We prove a sufficient combinatorial condition for the existence of the mixed discriminant and employ this to solve a conjecture by Cattani et. al.

In the fourth chapter we present an algorithm for the classification of triples of lattice polytopes in $\mathbb{R}^3$ with a given mixed volume. By the BKK-theorem, this is equivalent to the classification of generic systems of trivariate polynomials with a given number of solutions. Via this algorithm, we obtain a complete classification of triples of lattice polytopes with mixed volume at most four.

The fifth chapter treats tuples of lattice polytopes whose mixed degree is at most one. We show that, in dimension at least four, there exist only finitely many exceptional tuples of mixed degree one that are not part of a well-understood family. We furthermore present a complete classification of such tuples in dimension three.

Finally, in chapter six we prove an upper bound on the volume of the Minkowksi sum of a tuple of convex bodies in terms of its mixed volume. Our bound is asymptotically sharp. In dimensions two and three we furthermore prove an exact sharp bound.

Contents

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Introduction

The fundamental objects in this thesis are lattice polytopes. These are polytopes with vertices in the integer lattice $\mathbb{Z}^d \subset \mathbb{R}^d$. A general pattern throughout this thesis is to consider *families* or *tuples* of lattice polytopes instead of single lattice polytopes and to generalize methods and concepts to a *mixed* setting. The main motivation comes from algebraic geometry and is given by the classical Bernstein-Khovanskii-Kushnirenko (short BKK) theorem that relates the number of solutions of a system of (Laurent) polynomials to the so-called mixed volume of the Newton polytopes of the given polynomials. The Newton polytope of a polynomial may hold significantly more information compared to the total degree if the polynomial does not contain all monomials up to a certain degree. Such polynomials are called *sparse*. In particular lately, there has been progress along this intersection of algebraic and discrete geometry, adressing, among others, the classification of systems with a single solution from a polytopal point of view ([EG15]), a generalization of classical Galois theory to systems of polynomials and their solvability ([Est19]), the introduction and study of mixed discriminants of systems of polynomials ([DFS07, CCD+13, DEK14]), and the introduction of the mixed degree of a family of lattice polytopes ([Sop07, Nil20]). This thesis is dedicated to making progress in several of these research directions. In the course of this we contribute to the development of the foundations of a general *mixed lattice polytope theory*.

In Chapter 1 we fix basic notation and introduce the reader to the general concepts of the discrete geometry behind sparse polynomial systems. The fundamental notion is the support of a polynomial $f \in \mathbb{C}[x_1, \ldots, x_d]$:

$$\operatorname{supp}(f) = \left\{(z_1, \ldots, z_d) \in \mathbb{Z}^d \colon x_1^{z_1} \ldots x_d^{z_d} \text{ is a monomial of } f\right\}.$$

This allows us to interpret a finite set $A \subset \mathbb{Z}^d$ as the space of all polynomials that have support inside A. We may identify this space with the space $\mathbb{C}^A$, as choosing a polynomial with a given support set is equivalent to associating a coefficient to each of the monomials corresponding to the points in A. In this thesis we work with an extended view of this to tuples of polynomials $(f_1, \ldots, f_k)$ for which $\operatorname{supp}(f_i) \subseteq A_i$ for all $1 \leq i \leq k$, for certain fixed configurations $A_1, \ldots, A_k \subset \mathbb{Z}^d$. Analogously to the above, such a set can be identified with the vector space $\mathbb{C}^{A_1} \times \cdots \times \mathbb{C}^{A_k}$. After introducing another fundamental notion, the *mixed volume* of a tuple of lattice polytopes, in Section 1.2.1, we present and illustrate the classical BKK-theorem in Section 1.2.2. We furthermore shortly discuss how we treat the algorithmic problem of checking whether two lattice polytopes are equivalent based on joint work with Gennadiy Averkov and Ivan Soprunov ([ABS19]).

In Chapter 2, we introduce and study the construction of the *Cayley sum* (also known as Cayley polytope) of a tuple of configurations or lattice polytopes. There are two perspectives on this construction. On the one hand, given a tuple of lattice polytopes, one can construct its Cayley sum in order to view properties of the tuple as properties of a single, higher-dimensional polytope. On the other hand, one may investigate whether a given lattice polytope has a *Cayley decomposition*, that is, whether it is equivalent to a Cayley sum of lower-dimensional lattice polytopes. The fundamental idea is that understanding the geometry, combinatorial structure and interaction with the lattice of the Cayley sum of a tuple $(P_1, \ldots, P_k)$ is equivalent to understanding the corresponding mixed structures of $(P_1, \ldots, P_k)$. This interaction is made precise by the well-known *combinatorial Cayley trick* (see Proposition 2.2.1). We deduce several basic facts about the structure of Cayley sums that will prove useful in various parts of this thesis, in particular in Chapter 3. This is largely based on joint work with Benjamin Nill ([BN20]). Our main original contribution is Theorem 2.3.1, that provides conditions on the uniqueness of Cayley decompositions of a lattice polytope and can be used to reduce the question of equivalence of tuples to the question of equivalence of single lattice polytopes (see Corollary 2.3.4). This result is generalizing joint work with Gabriele Balletti ([BB20]), and joint work with Gennadiy Averkov and Ivan Soprunov ([ABS19]).

Chapter 3 is dedicated to the study of so-called mixed discriminants, which have been introduced in [CCD+13]. The mixed discriminant of a family of point configurations $(A_1, \ldots, A_k) \subset (\mathbb{Z}^d)^k$ is the defining polynomial $\Delta_{(A_1,\ldots,A_k)}$ of an algebraic hypersurface in the space of tuples $(f_1, \ldots, f_k) \in \mathbb{C}[x_1, \ldots, x_n]$ with fixed support sets $A_1, \ldots, A_k$. The described hypersurface is given by the Zariski closure of all tuples $(f_1, \ldots, f_k)$ in this space for which the system $f_1 = \cdots = f_k = 0$ has a non-degenerate multiple root. Usually it is interpreted as an algebraic hypersurface in the space of coefficients $\mathbb{C}^{A_1} \times \cdots \times \mathbb{C}^{A_k}$. The mixed discriminant generalizes the classical concept of the A-discriminant of a single point configuration introduced in [GKZ94]. In the case in which one considers only a single polynomial, the probably most famous example is the A-discriminant for the set $A = \{0, 1, 2\}$. It is given as $\Delta_A = b^2 - 4ac \in \mathbb{C}[a, b, c]$ and vanishes for all choices of coefficients $a_0, b_0, c_0 \in \mathbb{C}$ for which the quadratic polynomial $a_0x^2 + b_0x + c_0$ has a multiple root. One prominent question about A-discriminants is regarding their *defectivity*. For certain choices of configurations it can happen that the algebraic closure of the set of coefficients leading to multiple roots cannot be described by a single polynomial. Recently, two independent characterizations of such *defective* configurations were given by Esterov [Est10, Est18a] and Furukawa-Ito [FI20]. One encounters an analogous phenomenon for mixed discriminants, which leads to the definition of a *defective tuple of configurations*. While all definitions make sense in the case of a general number of polynomials k, the case in which one has d polynomials in d variables is of particular interest. Our main contribution to this area is the following result, which settles the question about defectivity in this case under the assuption that all configurations involved are full-dimensional.

Theorem (Corollary 3.2.2). *A tuple of full-dimensional configurations* $(A_1, \ldots, A_d) \subset (\mathbb{Z}^d)^d$ *is defective if and only if*

$$(A_1, \ldots, A_d) \cong (\{0, e_1, \ldots, e_d\}, \ldots, \{0, e_1, \ldots, e_d\}).$$

This result has been conjectured by Cattani et al. in [CCD+13] and is saying that, up to an appropriate notion of equivalence, there exists only one defective tuple $(A_1, \ldots, A_d) \subset (\mathbb{Z}^d)^d$. Furthermore, this tuple describes the support sets of d linear polynomials. Such a system can never have isolated multiple roots and therefore defectivity is obvious in this case. We derive this result as a corollary of a more general result on the discrete geometry of defective tuples of arbitrary length (see Theorem 3.2.1). Our main tools are a recent characterization of defective configurations in terms of Cayley sums by Furukawa-Ito [FI20] and combinatorial insights in the interaction between different Cayley decompositions that we deduce. The results of this chapter are joint work with Benjamin Nill and have been published in the article [BN20].

While mixed discriminants are about describing coefficients that lead to exceptional behavior of a system of polynomials, Chapter 4 is devoted to applying computational classification techniques in order to exhaustively enumerate systems of polynomials with a certain generic behavior. We focus on systems that generically have a small number of solutions. By the BKK-theorem, these are exactly those system whose support sets $(A_1, \ldots, A_d)$ lie inside tuples of lattice polytopes $(P_1, \ldots, P_d)$ with small *mixed volume*. A first fundamental result in this direction has been given by Esterov and Gusev [EG15], who completely settled the case of mixed volume one in general dimension d. They showed that any generic system of polynomials with only one solution is build from linear systems. On the polytopal side, they proved a generalization to tuples of the well-known fact that a single lattice polytope has volume one if and only if it is a unimodular simplex. Little has been known about tuples of higher mixed volumes, even in small dimension, apart from the classification of pairs of lattice polygons of mixed volume at most 4 in [EG15]. Tuples of lattice polytopes of mixed volume at most 4 are of particular interest, as Esterov showed in [Est19] that such tuples correspond to systems of polynomials that are solvable by radicals. While for the classification of single lattice polytopes important achievements have been made using computer algorithms (see for example [KS98, AKW17, Bal18, IVS18]), such a computational approach had so far not been taken towards the classification of tuples of lattice polytopes. With this being our point of departure, we develop an algorithm to computationally classify tuples of a given mixed volume in dimensions 2 and 3. We encounter that the complexity of the classification of tuples of a given mixed volume is tremendously greater than of the classification of single polytopes. For example, a lattice polytope occuring inside a tuple of mixed volume m can have volume up to m^d. In fact, the classification of single lattice polytopes of a given volume is a true subproblem of the classification of tuples of a given mixed volume. To tackle this, our algorithm makes use of powerful tools from convex geometry, in particular from *Brunn-Minkowski theory*. This yields to a complete classification of triples of lattice polytopes in dimension 3, whose mixed volume is at most 4 (see Theorem 4.3.1). The results of this chapter are joint work with Gennadiy Averkov and Ivan Soprunov and have been published in form of the preprint [ABS19].

Chapter 5 focuses on studying a generalization of a central concept from single lattice polytopes to tuples of lattice polytopes. This generalization is given by the so-called *mixed degree*, which has recently been introduced by Nill ([Nil20]) and is generalizing the *lattice degree* $\deg(P)$ of a single lattice polytope $P \subset \mathbb{R}^d$. The lattice degree is defined as the smallest integer $0 \leq r \leq d-1$ such that the dilation $(d-r)P$ does not contain an interior lattice point. If P itself contains an interior lattice point, one sets $\deg(P) = d$. An intuitive interpretation is that the degree of a lattice polytopes is its complexity or true dimension. Following this intuition, polytopes of low degree should have a very particular and simple structure and there is a variety of results making this precise (see e.g. [BN07, HNP08, NZ11]). The *mixed degree* $\mathrm{md}(P_1, \dots, P_d)$ of a d-tuple of d-dimensional lattice polytopes $P_1, \dots, P_d \subset \mathbb{R}^d$ is the smallest integer $0 \leq r \leq d-1$ such that the Minkowski sum of any choice of $(d-r)$ polytopes from the tuple $P_1, \dots, P_d$ does not contain an interior lattice point. If any of the polytopes in the tuple already contains an integer point in its interior, we set $\mathrm{md}(P_1, \dots, P_d) = d$. This is a true generalization of the lattice degree as $\mathrm{md}(P, \dots, P) = \deg(P)$. For this to be a reasonable notion one should expect tuples of lattice polytopes of low degree to have a very simple, similar to the situation in the unmixed setting. A first result in this direction has already implicitly been given in [CCD+13] by showing that, up to equivalence, in any dimension there exists a unique tuple of mixed degree zero, consisting of copies of the same unimodular simplex. Our contribution focuses on tuples of mixed degree one. These are of particular interest, as Nill observed in [Nil20] that they are the ones satisfying the following bound with equality:

$$|\operatorname{int}(P_1 + \cdots + P_d) \cap \mathbb{Z}^d| \geq \mathrm{MV}(P_1, \dots, P_d) - 1.$$

This bound was derived by Soprunov in the context of sparse polynomial interpolation in [Sop07]. Soprunov also observed in [BNR+08] that, whenever one has $P_1 = \cdots = P_d = P$, equality in the above formula holds if and only if the lattice degree of P is at most one. Motivated by this, he already introduced the notion *tuples of mixed degree at most one* for tuples attaining the bound above and posed the question of a classification of such tuples before the introduction of the general mixed degree. We partially answer this question by describing a natural class of lattice polytopes of mixed degree one and showing that, in any dimension $d \geq 4$, there exist only finitely many full-dimensional tuples of mixed degree one that are not of this type.

Theorem (Theorem 5.2.3)**.** *Let $d \geq 4$ and $(P_1, \dots, P_d)$ be a d-tuple of full-dimensional lattice polytopes in $\mathbb{R}^d$ of mixed degree at most one. Then $(P_1, \dots, P_d)$ is either equivalent to one of finitely many exceptional tuples in dimension d, or to a tuple*

$$(\mathrm{Cay}(I_1^1, \dots, I_d^1), \dots, \mathrm{Cay}(I_1^d, \dots, I_d^d)),$$

where I_i^j is a lattice segment for all $1 \leq i, j \leq d$. This class can be finitely parametrized as each segment is defined by the choice of two numbers.

With Theorem 5.2.4 we furthermore completely classify tuples of mixed degree one in dimension three computationally. Our classification includes the description of infinite classes of exceptional triples, showing that Theorem 5.2.3 does not hold in dimension three. The results of this chapter are joint work with Gabriele Balletti and have been published in the article [BB20].

Finally, Chapter 6 deals with a question that is motivated by its application to sparse polynomial systems, but which extends naturally to general convex bodies. The initial question is the following. Given a d-tuple of d-dimensional convex bodies $(K_1, \dots, K_d)$ with mixed volume $\mathrm{MV}(K_1, \dots, K_d) = m$, what is the maximal value for the volume of the Minkowski sum $\mathrm{Vol}(K_1 + \dots + K_d)$? In the special case of lattice polytopes $P_1, \dots, P_d$ a non-sharp upper bound on $\mathrm{Vol}(P_1 + \dots + P_d)$ has been used by Esterov in order to show the finiteness of irreducible tuples of any given mixed volume. This is due to the fact that, using a well-known result by Lagarias-Ziegler, one may assume $P_1 + \dots + P_d$, and therefore also P_i for each $1 \leq i \leq d$, to lie inside a hypercube with edge-lengths $d \cdot d! \,\mathrm{Vol}(P_1 + \dots + P_d)$. The main tool in Esterov's proof was the *Aleksandrov-Fenchel inequality* that ensures relations of the following form between mixed volumes.

$$\mathrm{MV}(K_1, K_2, K_3, \dots, K_d)^2 \geq \mathrm{MV}(K_1, K_1, K_3, \dots, K_d)\,\mathrm{MV}(K_2, K_2, K_3, \dots, K_d).$$

The bound that arises this way is of order $O(m^{2^d})$ as $m \to \infty$. Using additional inequalities from [BGL18], we are able to prove the following improvement of this bound in the case of full-dimensional compact convex sets (convex bodies).

Theorem (Theorem 6.5.10)**.** *Among all convex bodies $K_1, \dots, K_d$ in $\mathbb{R}^d$ satisfying*

$$\mathrm{Vol}(K_1) \geq 1, \dots, \mathrm{Vol}(K_d) \geq 1, \quad \textit{and} \quad \mathrm{MV}(K_1, \dots, K_d) = m,$$

the maximum of $\mathrm{Vol}(K_1 + \dots + K_d)$ *is of order $O(m^d)$, as $m \to \infty$.*

We show that this bound is asymptotically sharp. We conjecture the exact sharp upper bound to be equal to $(m+d-1)^d$ and provide evidence for this by proving the bound for $d = 2$ and $d = 3$ with Proposition 6.2.4 and Theorem 6.2.5. The results of this chapter are joint work with Gennadiy Averkov and Ivan Soprunov, and have been published in form of the article [ABS20].

1. Setting

1.1. Basics and notation

We fix the notation $[k] = \{1, \dots, k\}$ and write $\mathbb{Z}_{\geq k}$ for the set of all integers that are greater or equal to $k \in \mathbb{Z}$. Whenever we write $\mathbb{N}$ we mean integers that are greater or equal to 1. Furthermore, we denote by $e_1, \dots, e_d \in \mathbb{R}^d$ the *standard basis vectors* of $\mathbb{R}^d$.

(Lattice) polytopes

A *polytope*

$$P = \operatorname{conv}(x_1, \dots, x_k)$$

is the convex hull of a set of finitely many generating points $x_1, \dots, x_k \in \mathbb{R}^d$. A *hyperplane* $H \subset \mathbb{R}^d$ is a $(d-1)$-dimensional affine subspace of $\mathbb{R}^d$. Any hyperplane $H \subset \mathbb{R}^d$ has a description of the form

$$H = \left\{ x \in \mathbb{R}^d \colon \langle a, x \rangle = b \right\}$$

for a vector $(a, b) \in \mathbb{R}^{d+1}$ that is unique up to scaling. Any hyperplane yields two *half-spaces*

$$H^+ = \left\{ x \in \mathbb{R}^d \colon \langle a, x \rangle \geq b \right\} \text{ and } H^- = \left\{ x \in \mathbb{R}^d \colon \langle a, x \rangle \leq b \right\}.$$

$H \subset \mathbb{R}^d$ is called a *supporting hyperplane* of a polytope $P \subset \mathbb{R}^d$ if either $P \subset H^+$ or $P \subset H^-$. The intersection of P with a supporting hyperplane H is called a *face* of P and is again a polytope. Additionally we consider the polytope P as a face of itself and speak of *proper* faces whenever we specifically want to exclude the polytope itself. The *affine hull* $\operatorname{aff}(P)$ is the smallest affine subspace of $\mathbb{R}^d$ that contains P and the *dimension* $\dim(P)$ is the dimension of this affine subspace. We denote the set of faces of a polytope P as $\mathcal{F}(P)$. The elements of $\mathcal{F}(P)$ of dimension $\dim(P) - 1$ are called *facets*, the ones of dimension 1 are called *edges* and the ones of dimension 0 are called *vertices*. The set of vertices of P is denoted by $\operatorname{vert} P$. Given $X \subseteq \mathbb{R}^d$, we denote by $\mathcal{P}(X)$ the family of all non-empty polytopes P with $\operatorname{vert}(P) \subseteq X$. Throughout this thesis we will mostly be dealing with *lattice polytopes*, by which we mean elements of $\mathcal{P}(\mathbb{Z}^d)$. For any subsets $S_1, \dots, S_k \subset \mathbb{R}^d$, we denote by

$$S_1 + \dots + S_k = \{s_1 + \dots + s_k \colon s_i \in S_i \text{ for all } 1 \leq i \leq k\},$$

the *Minkowski sum* of $S_1, \dots, S_k$. For polytopes $P_1, \dots, P_d \subset \mathbb{R}^d$ one has

$$P_1 + \dots + P_d = \operatorname{conv}(\operatorname{vert}(P_1) + \dots + \operatorname{vert}(P_d)).$$

In particular, the Minkowski sum of polytopes is again a polytope and the Minkowski sum of lattice polytopes is a lattice polytope.

We denote by $\operatorname{int}(X)$ the *interior* of a full-dimensional set $X \subseteq \mathbb{R}^d$ and call a full-dimensional lattice polytope $P \in \mathcal{P}(\mathbb{Z}^d)$ *hollow* if the set of its *interior lattice points* $\operatorname{int}(P) \cap \mathbb{Z}^d$ is empty.

A *point configuration* $A \subset \mathbb{Z}^d$ is a finite set of points in $\mathbb{Z}^d$. In the setting of this thesis we do not consider repeated points in a configuration. We implicitly view $A \subset \mathbb{Z}^d$ as a subset of the lattice $\mathbb{Z}^d$ naturally embedded in $\mathbb{R}^d$. This way it makes sense to talk about the *affine hull* $\operatorname{aff}(A) \subset \mathbb{R}^d$ and the *dimension* $\dim(A)$ of A analogously to the definition for polytopes. A *face* of A is the intersection of A with a face of the polytope $\operatorname{conv}(A)$ and we denote the set of all faces of A by $\mathcal{F}(A)$. Analogously to above, we may talk about *vertices*, *edges*, and *facets* of A.

To improve readability we sometimes write a configuration $A \subset \mathbb{Z}^d$ as a matrix whose columns are the elements of A.

Lattice preserving maps and volumes

We denote by $\mathrm{GL}(\mathbb{Z}^d)$ the group of *linear unimodular transformations*, that is linear bijections $\varphi\colon \mathbb{R}^d \to \mathbb{R}^d$ that satisfy $\varphi(\mathbb{Z}^d) = \mathbb{Z}^d$. By choosing a lattice basis of $\mathbb{Z}^d$ we can identify $\mathrm{GL}(\mathbb{Z}^d)$ with the group of $d \times d$ *unimodular matrices*, which are the matrices $U \in \mathbb{Z}^{d\times d}$ with $|\det(U)| = 1$. Furthermore, we denote by $\mathrm{Aff}(\mathbb{Z}^d)$ the group of *(affine) unimodular transformations*, that is affine bijections $\varphi\colon \mathbb{R}^d \to \mathbb{R}^d$ that satisfy $\varphi(\mathbb{Z}^d) = \mathbb{Z}^d$. Sometimes it is convenient to extend this notion to maps $\varphi\colon L_1 \to L_2$ for rational affine subspaces $L_1 \subseteq \mathbb{R}^{d_1}$ and $L_2 \subseteq \mathbb{R}^{d_2}$. A *rational affine subspace* of $\mathbb{R}^d$ is a rational translate of a linear subspace that can be generated by rational vectors. In this case we call φ an affine unimodular transformation if it is an affine bijection satisfying $\varphi(L_1 \cap \mathbb{Z}^{d_1}) = L_2 \cap \mathbb{Z}^{d_2}$. Affine unimodular transformations yield the standard notion of equivalence of lattice polytopes. We say that two lattice polytopes $P_1, P_2 \in \mathcal{P}(\mathbb{Z}^d)$ are *(unimodularly) equivalent*, and write $P_1 \cong P_2$, if there exists a unimodular affine transformation $\varphi \in \mathrm{Aff}(\mathbb{Z}^d)$ satisfying $\varphi(P_1) = P_2$. Sometimes it is practical to extend this notion of equivalence to lattice polytopes $P_1 \in \mathcal{P}(\mathbb{Z}^{d_1})$ and $P_2 \in \mathcal{P}(\mathbb{Z}^{d_2})$ that live inside different ambient dimensions $d_1 \neq d_2$. We say that P_1 and P_2 are equivalent if one has $\dim(P_1) = \dim(P_2)$ and there exists an affine unimodular transformation $\varphi\colon \operatorname{aff}(P_1) \to \operatorname{aff}(P_2)$ that satisfies $\varphi(P_1) = P_2$. Analogously, two point configurations $A_1 \subset \mathbb{Z}^{d_1}$, $A_2 \subset \mathbb{Z}^{d_2}$ are *equivalent* if the exists an affine unimodular transformation $\varphi\colon \operatorname{aff}(A_1) \to \operatorname{aff}(A_2)$ mapping A_1 onto A_2.

An *(affine) lattice projection* is a surjective affine map $\varphi\colon L_1 \to L_2$, where $L_1 \subseteq \mathbb{R}^{d_1}$ and $L_2 \subseteq \mathbb{R}^{d_2}$ are rational affine subspaces, that satisfies $\varphi(L_1 \cap \mathbb{Z}^{d_1}) = L_2 \cap \mathbb{Z}^{d_2}$. The *kernel* of such a projection is the largest linear space $K \subseteq \mathbb{R}^{d_1}$ satisfying $L_1 + K = L_1$ and $\varphi(x + k) = \varphi(x)$ for all $x \in L_1$ and $k \in K$.

For any compact convex body $K \subset \mathbb{R}^d$, we define its *(normalized) volume* as

$$\mathrm{Vol}_d(K) = d!\,\mathrm{vol}_d(K),$$

where $\mathrm{vol}_d(K)$ is the standard Euclidean volume of K. In particular, $\mathrm{Vol}_d(K) = 0$ whenever K is contained in a lower-dimensional affine subspace of $\mathbb{R}^d$. The normalization with the factor $d!$ guarantees that one has $\mathrm{Vol}_d(P) \in \mathbb{Z}_{\geq 0}$ for any lattice polytope $P \in \mathcal{P}(\mathbb{Z}^d)$. For lattice polytopes we furthermore introduce the notion of a *relative volume*. Let $P \in \mathcal{P}(\mathbb{Z}^d)$ be a lattice polytope. Then $\mathrm{aff}(P)$ is a rational affine subspace of $\mathbb{R}^d$ and there exists an affine unimodular transformation $\varphi\colon \mathrm{aff}(P) \to \mathbb{R}^{\dim(P)}$. We define the relative volume of P as

$$\mathrm{Vol}(P) = \mathrm{Vol}_{\dim(P)}(\varphi(P)).$$

When talking about lattice polytopes, we often simply say "volume" for "relative volume".

Special lattice polytopes

We denote by

$$\Delta_d = \mathrm{conv}(0, e_1, \ldots, e_d)$$

the *standard unimodular simplex* and call a lattice simplex $S \in \mathcal{P}(\mathbb{Z}^d)$ *unimodular*, if it is unimodularly equivalent to $\Delta_{\dim(S)}$. Additionally, we denote by

$$\tilde{\Delta}_{d-1} = \mathrm{conv}(e_1, \ldots, e_d) \in \mathcal{P}(\mathbb{Z}^d)$$

the homogeneous version of the standard simplex. Note that $\tilde{\Delta}_{d-1} \cong \Delta_{d-1}$. Furthermore, we denote by

$$\square_d = \sum_{i \in d} [0, e_i] \subset \mathbb{R}^d$$

the *standard cube*. Given any lattice polytope $P \in \mathcal{P}(\mathbb{Z}^d)$, we denote by

$$\mathrm{Pyr}(P) = \mathrm{conv}(P \times \{0\} \cup \{e_{d+1}\}) \in \mathcal{P}(\mathbb{Z}^{d+1})$$

the *lattice pyramid* over P. We write $\mathrm{Pyr}^k(P)$ for the polytope obtained by performing k iterations of the above construction.

1.1.1. Sparse polynomials

One fundamental connection between discrete geometry and algebraic geometry is given by the construction of the *Newton polytope* or, more generally, the *support* of a polynomial. The basic concept for this construction is identifying d-variate monomials with points in $\mathbb{Z}^d$ via the bijection:

$$\begin{aligned} p\colon \mathbb{Z}^d &\to \mathrm{Mon}[x_1, \ldots, x_d] \\ a &\mapsto x^a := x_1^{a_1} \cdots x_d^{a_d}. \end{aligned}$$

Let $\mathbb{C}[x_1, \ldots, x_d]$ denote the polynomial ring in d variables over $\mathbb{C}$, and let furthermore $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ denote the ring of *Laurent polynomials* in d variables over $\mathbb{C}$. Using the bijection above, any Laurent polynomial $f \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$ can be written as $f = \sum_{a \in \mathbb{Z}^d} c_a x^a$ for coefficients $c_a \in \mathbb{C}$ such that only finitely many of the coefficients are non-zero.

Definition 1.1.1. Let $f = \sum_{a \in \mathbb{Z}^d} c_a x^a \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$ be a Laurent polynomial. We denote by $\operatorname{supp}(f) = \{a \in \mathbb{Z}^d \colon c_a \neq 0\}$ the *support* of f and by $\operatorname{Newt}(f) = \operatorname{conv}(\operatorname{supp}(f))$ the *Newton polytope* of f.

In particular, both the support and the Newton polytope of a Laurent polynomial do not contain any information about the concrete values of the coefficients of the polynomial, apart from whether they are zero or not. This motivates the point of view that a support set is rather associated to a finite-dimensional vector space inside $\mathbb{C}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$, than to a concrete polynomial. The following makes this precise.

Definition 1.1.2. Let $A \subset \mathbb{Z}^d$ be a point configuration and $P \in \mathcal{P}(\mathbb{Z}^d)$ a lattice polytope. We introduce the following finite-dimensional sub-vector spaces of $\mathbb{C}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$:

$$\begin{aligned}\mathbb{C}[A] &= \{f \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}] \colon \operatorname{supp}(f) \subseteq A\},\\ \mathbb{C}[P] &= \{f \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}] \colon \operatorname{supp}(f) \subseteq P \cap \mathbb{Z}^d\}.\end{aligned}$$

Suppose S is one of the vector spaces above. We say that a property holds *generically in* S, if it is true for all polynomials living outside an algebraic hypersurface H in S.

As a polynomial in $\mathbb{C}[A]$ is uniquely determined by its coefficient for each monomial x^a with $a \in A$, there is a natural isomorphism between $\mathbb{C}[A]$ and the space of coefficient vectors $\mathbb{C}^A \cong \mathbb{C}^{|A|}$. Given a coefficient vector $c \in \mathbb{C}^A$, we denote by f_c the corresponding polynomial in $\mathbb{C}[A]$.

The following classical result has been shown in [Kou76] and gives a first impression of how the Newton polytope contains crucial information about the polynomial itself. We call a solution (or root) $p \in (\mathbb{C}^*)^d$ of a system of Laurent polynomials $f_1 = \cdots = f_d = 0$ *isolated*, if there exists an $\varepsilon > 0$ such that p is the unique solution of the system inside an ε-ball around p.

Theorem 1.1.3 (Kushnirenko)**.** *Let $A \subset \mathbb{Z}^d$ be a point configuration and $f_1, \ldots, f_d \in \mathbb{C}[A]$. Then the number of isolated solutions in $(\mathbb{C}^*)^d$ of the system $f_1 = \cdots = f_d = 0$ is at most $\operatorname{Vol}_d(\operatorname{conv}(A))$. Equality is attained generically in $\mathbb{C}[A] \times \cdots \times \mathbb{C}[A]$.*

1.2. The mixed setting: generalizing to tuples of lattice polytopes

In the setting of the theorem of Kushnirenko (Theorem 1.1.3) one actually already deals with a tuple of d Laurent polynomials $f_1, \ldots, f_d$ and only the condition that we consider a common support set for all of them makes the number of solutions depend on a single lattice polytope. However, it seems very natural to consider the case in which one specifies distinct support sets for each of the polynomials involved. Indeed, there exists the famous BKK-theorem generalizing Theorem 1.1.3 to that case. In order to state this theorem in Section 1.2.2, let us introduce a bit more notation.

Definition 1.2.1. Let $(A_1, \dots, A_k) \subset (\mathbb{Z}^d)^k$ be a k-tuple of point configurations and $(P_1, \dots, P_k) \in \mathcal{P}(\mathbb{Z}^d)^k$ a k-tuple of lattice polytopes. We introduce the following finite-dimensional sub-vector spaces of $\mathbb{C}[x_1^{\pm 1}, \dots, x_d^{\pm 1}]^n$:

$$\mathbb{C}[A_1, \dots, A_k] = \left\{ (f_1, \dots, f_k) \in \mathbb{C}[x_1^{\pm 1}, \dots, x_d^{\pm 1}]^k \colon \operatorname{supp}(f_i) \subseteq A_i \text{ for all } i \in [k] \right\},$$
$$\mathbb{C}[P_1, \dots, P_k] = \left\{ (f_1, \dots, f_k) \in \mathbb{C}[x_1^{\pm 1}, \dots, x_d^{\pm 1}]^k \colon \operatorname{supp}(f_i) \subseteq P_i \cap \mathbb{Z}^d \text{ for all } i \in [k] \right\}$$

Suppose S is one of the vector spaces above. We say that a property holds *generically in* S, if it is true for all polynomials living outside an algebraic hypersurface H in S.

Analogously to Definition 1.1.2, there is a natural isomorphism between the vector space $\mathbb{C}[A_1, \dots, A_k]$ and the space of coefficient vectors $\mathbb{C}^{A_1} \times \dots \times \mathbb{C}^{A_k} \cong \mathbb{C}^{|A_1| + \dots + |A_k|}$ and we denote by $(f_{c_1}, \dots, f_{c_k} \in \mathbb{C}[A_1, \dots, A_k]$ the tuple of Laurent polynomials corresponding to the coefficient vector $(c_1, \dots, c_k)$.

Definition 1.2.1 is phrased in a more general way than we need for our formulation of the BKK-theorem, where we only deal with the case of d configurations in $\mathbb{Z}^d$. However, also the geometry of a k-tuple of configurations in $\mathbb{Z}^d$ for $k < d$ holds information about the infinite solution sets of polynomial systems $f_1 = \dots = f_k = 0$ with corresponding supports. In Chapter 3 we work in this generality.

1.2.1. The mixed volume

In order to generalize Theorem 1.1.3 we need an appropriate generalization of the volume of a lattice polytope. This is given by the so-called *mixed volume* of a d-tuple of polytopes in $\mathbb{R}^d$. This notion extends naturally to general compact convex sets and plays a central role in what is called *Brunn-Minkowski theory*, that goes far beyond lattice polytopes. We refer to [Sch14, Chapter 5] for a very thorough treatment of the mixed volume from a convex geometric point of view.

There exists a uniquely defined functional

$$\operatorname{mv} : \mathcal{P}(\mathbb{R}^d)^d \to \mathbb{R},$$

with $\operatorname{mv}(P_1, \dots, P_d)$ being invariant under permutations of $P_1, \dots, P_d \in \mathcal{P}(\mathbb{R}^d)$, such that the equality

$$\operatorname{vol}(\lambda_1 P_1 + \dots + \lambda_k P_k) = \sum_{i_1=1}^{k} \cdots \sum_{i_d=1}^{k} \lambda_{i_1} \cdots \lambda_{i_d} \operatorname{mv}(P_{i_1}, \dots, P_{i_d})$$

holds for all $P_1, \dots, P_k \in \mathcal{P}(\mathbb{R}^d)$, non-negative scalars $\lambda_1, \dots, \lambda_k \geq 0$, and $k \in \mathbb{N}$ (see [Sch14, Theorem and Definition 5.1.7]). The definition of mv extends to the set of d-tuples of non-empty compact convex sets. The value $\operatorname{mv}(P_1, \dots, P_d)$ is called the *Euclidean mixed volume* of the d-tuple $(P_1, \dots, P_d)$. Replacing the Euclidean volume in the above definition with the normalized volume relative to the lattice $\mathbb{Z}^d \subset \mathbb{R}^d$ we obtain the *normalized mixed volume* $\operatorname{MV}(P_1, \dots, P_d)$ relative to $\mathbb{Z}^d$. During this thesis term mixed volume will stand for the normalized mixed volume if not specifically stated otherwise.

The mixed volume satisfies a number of properties. Their proof can be found for example in [Sch14, Sections 5.1, 7.3] and [Ewa96, p. 120].

Proposition 1.2.2. *For all non-empty compact convex sets $K_1, \dots, K_d, L_1, \dots, L_d \subset \mathbb{R}^d$ and non-negative $\lambda, \mu \in \mathbb{R}$, one has*

1. $\mathrm{MV}(K_1, \dots, K_d) \geq 0$.

2. $\mathrm{MV}(\lambda K_1 + \mu L_1, K_2, \dots, K_d) = \lambda \, \mathrm{MV}(K_1, K_2, \dots, K_d) + \mu \, \mathrm{MV}(L_1, K_2, \dots, K_d)$.

3. $\mathrm{MV}(K_1, \dots, K_d) \in \mathbb{Z}$, *whenever $K_1, \dots, K_d$ are lattice polytopes.*

4. *Inclusion-exclusion formula*

$$\mathrm{MV}(K_1, \dots, K_d) = \frac{1}{d!} \sum_{k=1}^{d} (-1)^{d+k} \sum_{i_1 < \dots < i_k} \mathrm{Vol}_d(K_{i_1} + \dots + K_{i_k}). \tag{1.1}$$

5. *Aleksandrov–Fenchel Inequality*

$$\mathrm{MV}(K_1, K_2, K_3 \dots, K_d)^2 \geq \mathrm{MV}(K_1, K_1, K_3, \dots, K_d) \, \mathrm{MV}(K_2, K_2, K_3, \dots, K_d).$$

6. *Monotonicity*

$$\mathrm{MV}(K_1, \dots, K_d) \leq \mathrm{MV}(L_1, \dots, L_d), \quad \text{whenever } K_1 \subseteq L_1, \dots, K_d \subseteq L_d. \tag{1.2}$$

1.2.2. The BKK-theorem

We have now assembled the tools to state the following central result relating systems of d equations defined by d-variate polynomials with the geometry of the tuples of corresponding support sets.

Theorem 1.2.3 (Berstein-Khovanskii-Kushnirenko). *Let $(A_1, \dots, A_d) \subset (\mathbb{Z}^d)^d$ be a d-tuple of point configurations and $(f_1, \dots, f_d) \in \mathbb{C}[A_1, \dots, A_d]$. Then the number of isolated solutions in $(\mathbb{C}^*)^d$ for the system $f_1 = \dots = f_d = 0$ is at most $\mathrm{MV}(\mathrm{conv}(A_1), \dots, \mathrm{conv}(A_d))$. Equality is attained generically in $\mathbb{C}[A_1, \dots, A_d]$.*

Theorem 1.2.3 has first been proven by David Bernstein in 1975 [Ber75]. One year later, Anatoli Kushnirenko presented another proof in [Kus76]. Askold Khovanskii has published strongly related results in [Kho77] and given a variety of independent proofs of the theorem over the years. We refer to [CLO05, Chapter 5] for further algebro-geometric background about the BKK-theorem. Let us illustrate Theorem 1.2.3 with the following example.

Example 1.2.4. Consider equations $f_1 = 0$ and $f_2 = 0$ of vertically and horizontally aligned parabolas given by polynomials

$$f_1(x, y) = c_{1,(0,1)}y + c_{1,(2,0)}x^2 + c_{1,(1,0)}x + c_{1,(0,0)}$$
$$f_2(x, y) = c_{2,(1,0)}x + c_{2,(0,2)}y^2 + c_{2,(0,1)}y + c_{2,(0,0)}.$$

See also Fig. 1.1. Then the tuple (f_1, f_2) lives in $\mathbb{C}[A_1, A_2]$ for the configurations

$$A_1 = \{(0, 0), (1, 0), (2, 0), (0, 1)\}, \qquad A_2 = \{(0, 0), (1, 0), (0, 1), (0, 2)\}$$

Denote $P_1 = \mathrm{conv}(A_1)$ and $P_2 = \mathrm{conv}(A_2)$. By Theorem 1.2.3, if the vector

$$(c_{1,(0,1)}, c_{1,(2,0)}, c_{1,(1,0)}, c_{1,(0,0)}, c_{2,(1,0)}, c_{2,(0,2)}, c_{2,(0,1)}, c_{2,(0,0)}) \in \mathbb{C}^8$$

of all coefficients of the polynomials f_1 and f_2 is generic, then the system $f_1 = f_2 = 0$ has exactly 4 solutions in $(\mathbb{C}^*)^2$, because the normalized mixed volume $\mathrm{MV}(P_1, P_2)$ of P_1 and P_2 equals 4. The value $\mathrm{MV}(P_1, P_2)$ can be computed using formula (1.1) from Proposition 1.2.2 as

$$\begin{aligned}\mathrm{MV}(P_1, P_2) &= (1/2)(\mathrm{Vol}(P_1 + P_2) - \mathrm{Vol}(P_1) - \mathrm{Vol}(P_2))\\ &= (1/2)(12 - 2 - 2) = 4.\end{aligned}$$

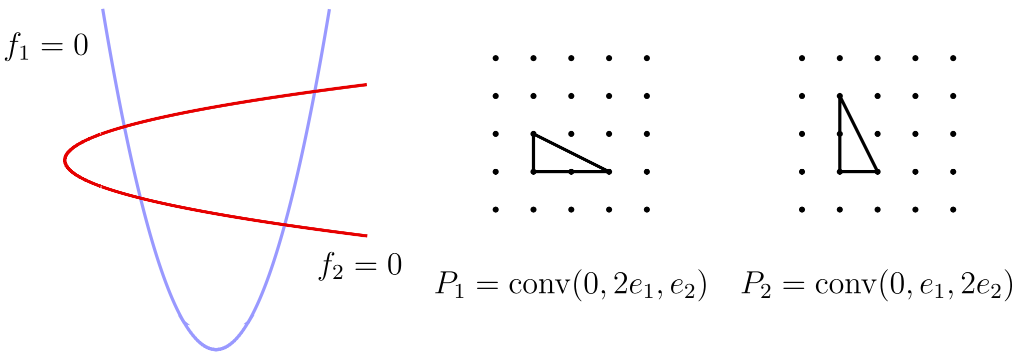


Figure 1.1.: A system $f_1 = f_2 = 0$, with a generic choice of $(f_1, f_2) \in \mathbb{C}[A_1, A_2]$ has 4 solutions in $(\mathbb{C}^*)^2$, because the normalized mixed volume of P_1 and P_2 equals 4.

Remark 1.2.5. Note that, in the setting of Theorem 1.2.3, one may still obtain a bound on the number of isolated solutions of the system $f_1 = \cdots = f_d = 0$ by applying Theorem 1.1.3 to the point configuration $A = A_1 \cup \cdots \cup A_d$. However, one will in general obtain a worse bound in this case. Consider for example the case of $d = 2$ and $A_1 = \{(0, 0), (1, 0), (1, 2), (0, 2)\}$, $A_2 = \{(0, 0), (2, 0), (2, 1)\}$. In this case one has $\mathrm{MV}(\mathrm{conv}(A_1), \mathrm{conv}(A_2)) = 5$. So a system $f_1 = f_2 = 0$ for $(f_1, f_2) \in \mathbb{C}[A_1, A_2]$ has at most 5 isolated solutions in $(\mathbb{C}^*)^2$. If we denote the union of the configurations A_1 and A_2 by $A = A_1 \cup A_2 = \{(0, 0), (1, 0), (2, 0), (2, 1), (1, 2), (0, 2)\}$, we may view the pair (f_1, f_2) as living inside the larger vector space $\mathbb{C}[A, A]$. See also Figure 1.2. Applying Theorem 1.1.3 yields the worse bound of $\mathrm{Vol}(\mathrm{conv}(A)) = 7$. Also recall that

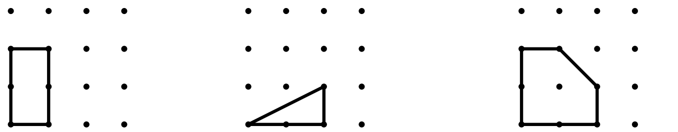


Figure 1.2.: The convex hulls of the configurations A_1, A_2 from Remark 1.2.5 and their union.

both Theorem 1.1.3 and Theorem 1.2.3 contain an additional statement about the bounds being attained generically. What happens here is that a tuple (f_1, f_2) which is generic in $\mathbb{C}[A_1, A_2]$ does not need to be generic inside the larger vector space $\mathbb{C}[A, A]$. Or, more precisely, the whole vector space $\mathbb{C}[A_1, A_2] \subset \mathbb{C}[A, A]$ is contained in an algebraic hyperplane in $\mathbb{C}[A, A]$ (for example the one given by the condition $c_{1,(2,1)} = 0$) and therefore a property that holds generically in $\mathbb{C}[A, A]$ might not hold for any element in $\mathbb{C}[A_1, A_2]$.

1.2.3. Equivalence of tuples

Similarly to the case of single lattice polytopes, also for tuples it often makes sense to only distinguish them up to an appropriate equivalence relation. Let $\mathbb{G}_{d,k}$ denote the set of maps $\psi \colon (\mathbb{R}^d)^k \to (\mathbb{R}^d)^k$ that are of the form

$$(x_1, \dots, x_k) \mapsto (\varphi(x_{\sigma(1)}) + t_{\sigma(1)}, \dots, \varphi(x_{\sigma(k)}) + t_{\sigma(k)}),$$

for a unimodular transformation $\varphi \in \mathrm{Aff}(\mathbb{Z}^d)$, a choice of k lattice vectors $t_1, \dots, t_k \in \mathbb{Z}^d$ and a permutation σ on $[k]$. We say that two k-tuples $(P_1, \dots, P_k), (Q_1, \dots, Q_k) \in (\mathcal{P}(\mathbb{Z}^d))^d$ are *equivalent*, and write $(P_1, \dots, P_k) \cong (Q_1, \dots, Q_k)$, if $\psi(P_1, \dots, P_k) = (Q_1, \dots, Q_k)$ holds for some $\psi \in \mathbb{G}_{d,k}$. This notion of equivalence specializes to the notion of (unimodular) equivalence of single lattice polytopes $P, Q \in \mathcal{P}(\mathbb{Z}^d)$ if one either considers P and Q as 1-tuples, or if one views P and Q as d-tuples $(P, \dots, P), (Q, \dots, Q) \in \mathcal{P}(\mathbb{Z}^d)^d$. Furthermore, it is straightforward to verify that one has $P_1 + \dots + P_k \cong Q_1 + \dots + Q_k$ whenever $(P_1, \dots, P_k) \cong (Q_1, \dots, Q_k)$. By the inclusion-exclusion formula for the mixed volume (1.1), one also has $\mathrm{MV}(P_1, \dots, P_k) = \mathrm{MV}(Q_1, \dots, Q_k)$ in this case. Note, however, that one might have $P_i \cong Q_i$ for all $i \in [k]$ without the tuples $(P_1, \dots, P_k)$ and $(Q_1, \dots, Q_k)$ being equivalent. This is illustrated in the following example.

Example 1.2.6. Set $P = \mathrm{conv}(0, e_1, e_1 + e_2) \in \mathcal{P}(\mathbb{Z}^2)$ and consider the pairs of unimodular triangles (Δ_2, Δ_2) and (Δ_2, P). While $P \cong \Delta_2$, these pairs are not equivalent. One way to see this is to observe that

$$\mathrm{Vol}(\Delta_2 + \Delta_2) = 4 \neq 6 = \mathrm{Vol}(\Delta_2 + P),$$

and therefore $\Delta_2 + \Delta_2 \ncong \Delta_2 + P$. See also Figure 1.3.

Figure 1.3.: The pairs of polygons from Example 1.2.6 and their Minkowski sums.

Analogously to above, we call two tuples $(A_1, \dots, A_k), (B_1, \dots, B_k) \subset (\mathbb{Z}^d)^k$ of point configurations *equivalent* if there exists a map $\psi \in \mathbb{G}_{d,k}$ satisfying $\psi(A_1, \dots, A_k) = (B_1, \dots, B_k)$.

Let us also explain the notion of equivalence of tuples of lattice polytopes from the perspective of systems of Laurent polynomials with fixed Newton polytopes. Consider a tuple of Laurent polynomials $(f_1, \dots, f_k) \in \mathbb{C}[P_1, \dots, P_k]$ for a tuple of lattice polytopes $(P_1, \dots, P_k) \in \mathcal{P}(\mathbb{Z}^d)^k$. A *monomial change of variables* for a Laurent polynomial is given by mapping $(x_1, \dots, x_d) \mapsto (x^{u_1}, \dots, x^{u_d})$, where $x^{u_i} = x_1^{u_{1i}} \cdots x_d^{u_{di}}$ for some unimodular matrix $U = (u_{ij}) \in \mathrm{GL}(\mathbb{Z}^d)$. We call two systems $f_1 = \cdots = f_k = 0$ and $f_1' = \cdots = f_k' = 0$ *monomially equivalent* if, after a possible permutation of the f_i, there is a monomial change of variables which transforms f_i to $x^{a_i} f_i'$ for some monomial x^{a_i} for every $1 \leq i \leq k$. If two systems of polynomials are monomially equivalent, there exists a natural bijection between the solutions of them in the torus $(\mathbb{C}^*)^d$. Our notion of equivalence of tuples of lattice polytopes is the finest one possible ensuring that two monomially equivalent systems have equivalent tuples of Newton polytopes.

1.3. Algorithmic aspects of equivalence testing

A computational task that we encounter throughout the thesis is to decide algorithmically whether two lattice polytopes are equivalent. We may restrict ourselves to full-dimensional polytopes, as we may otherwise choose a unimodular transformation between the affine hulls of the polytopes and some $\mathbb{R}^d$ in which they are full-dimensional. The literature contains several algorithms that test whether two full-dimensional lattice polytopes $P, Q \in \mathcal{P}(\mathbb{Z}^d)$ are equivalent modulo a *linear* unimodular transformation $\varphi \in \mathrm{GL}(\mathbb{Z}^d)$. See for example [KS98] and [GK13], where the latter also provides an overview of existing techniques. The algorithm of Kreuzer and Skarke from [KS98], relying on the so-called *normal form* of a lattice polytope, is implemented both in Sagemath [Sag18] and Magma [BCP97]. The normal form of a lattice polytope P is uniquely determined by P. It encodes a sequence of vertices $(v_1, \dots, v_t)$ of a polytope $\mathrm{conv}(v_1, \dots, v_t)$ that coincides with P up to $\mathrm{GL}(\mathbb{Z}^d)$. Two polytopes $P, Q \in \mathcal{P}(\mathbb{Z}^d)$ coincide up to $\mathrm{GL}(\mathbb{Z}^d)$ if and only if their normal forms are the same. See also Example 3.4 in [GK13]. Using the normal form, each polytope in $\mathcal{P}(\mathbb{Z}^d)$ can be brought into a *normal* $\mathrm{GL}(\mathbb{Z}^d)$*-position*. In other words, in each equivalence class modulo $\mathrm{GL}(\mathbb{Z}^d)$ in $\mathcal{P}(\mathbb{Z}^d)$ a unique representative is chosen. Using such a normal position in enumeration algorithms is convenient because, for avoiding repetitions modulo $\mathrm{GL}(\mathbb{Z}^d)$, it suffices to bring each newly found polytope into its normal position.

Equivalence of polytopes modulo $\mathrm{Aff}(\mathbb{Z}^d)$.

The notion of equivalence that we are interested in is slightly more general, as we want to identify any two polytopes as equivalent for which there exists an *affine* unimodular transformation $\varphi \in \mathrm{Aff}(\mathbb{Z}^d)$ sending one onto the other. Testing equivalence of two polytopes $P, Q \in \mathcal{P}(\mathbb{Z}^d)$ modulo $\mathrm{Aff}(\mathbb{Z}^d)$ can be reduced to testing equivalence modulo $\mathrm{GL}(\mathbb{Z}^{d+1})$. Indeed, $P, Q \in \mathcal{P}(\mathbb{Z}^d)$ are equivalent modulo $\mathrm{Aff}(\mathbb{Z}^d)$ if and only if the respective pyramids $\mathrm{conv}(\{\mathbf{0}\} \cup (P \times \{1\})), \mathrm{conv}(\{\mathbf{0}\} \cup (Q \times \{1\})) \in \mathcal{P}(\mathbb{Z}^{d+1})$ are equivalent modulo $\mathrm{GL}(\mathbb{Z}^{d+1})$.

This approach has the slight disadvantage that the normal forms that we compare are not equivalent to the polytopes themselves. Therefore we decided to implement a *normal* $\mathrm{Aff}(\mathbb{Z}^d)$*-position* of polytopes in $\mathcal{P}(\mathbb{Z}^d)$ by choosing a representative in each of the equivalence classes modulo $\mathrm{Aff}(\mathbb{Z}^d)$. Our construction is as follows. For a full-dimensional lattice polytope $P \in \mathcal{P}(\mathbb{Z}^d)$, consider

$$c_P := \frac{1}{|\operatorname{vert}(P)|} \sum_{v \in \operatorname{vert}(P)} v,$$

which is the barycenter of the set of vertices of P. Furthermore, we can order points of $\mathbb{R}^d$ lexicographically: $x = (x_1, \ldots, x_d)$ is *lexicographically smaller* than $y = (y_1, \ldots, y_d)$ if, for the smallest $i \in [d]$ with $x_i \neq y_i$, one has $x_i < y_i$. For a compact subset X of $\mathbb{R}^d$, let $\operatorname{lexmin}(X)$ denote the lexicographic minimum of the set X. It is not hard to see that for a polytope $P \in \mathcal{P}(\mathbb{R}^d)$ one has $\operatorname{lexmin}(P) \in \operatorname{vert}(P)$. In particular, if $P \in \mathcal{P}(\mathbb{Z}^d)$ is a lattice polytope then $\operatorname{lexmin}(P)$ is a lattice point. Based on this notions we introduce the following normal $\mathrm{Aff}(\mathbb{Z}^d)$-position.

Proposition 1.3.1. *Let $P \in \mathcal{P}(\mathbb{Z}^d)$ be a full-dimensional polytope with N vertices and let P' be the normal $\mathrm{GL}(\mathbb{Z}^d)$-position of the lattice polytope $N(P - c_P)$. Then the lattice polytope*

$$P'' := \frac{1}{N}\left(P' - \operatorname{lexmin}(P')\right)$$

is ($\mathrm{Aff}(\mathbb{Z}^d)$-)equivalent to P. For any other lattice polytope $Q \in \mathcal{P}(\mathbb{Z}^d)$ with $P \cong Q$, one has $P'' = Q''$.

Proof. Let $\phi \in \mathrm{GL}(\mathbb{Z}^d)$ be a linear unimodular transformation sending P' to $N(P - c_P)$. Using the fact that, for any compact set $X \subset \mathbb{R}^d$, one has $\operatorname{lexmin}(X - x) = \operatorname{lexmin}(X) - x$ and $\operatorname{lexmin}(kX) = k \operatorname{lexmin}(X)$ for all $x \in \mathbb{R}^d$ and $k \in \mathbb{R}_{\geq 0}$ one obtains:

$$\begin{aligned}
\phi(P'') &= \frac{1}{N}\phi(P') - \frac{1}{N}\phi\left(\operatorname{lexmin}(P')\right) \\
&= P - c_P - \phi\left(\operatorname{lexmin}(\phi^{-1}(P - c_P))\right) \\
&= P - \phi\left(\operatorname{lexmin}(\phi^{-1}(P))\right).
\end{aligned}$$

As $\phi(\operatorname{lexmin}(\phi^{-1}(P)))$ is a lattice point, this proves the first claim.

Let now $Q \in \mathcal{P}(\mathbb{Z}^d)$ be another lattice polytope that is equivalent to P. Then the number of vertices of Q also equals N and the polytopes $N(P - c_p)$ and $N(Q - c_Q)$ are equivalent. Let $\varphi \in \operatorname{Aff}(\mathbb{Z}^d)$ be a unimodular transformation satisfying $\varphi(N(P - c_p)) = N(Q - c_Q)$. As the barycenter of both polytopes $N(P - c_p)$ and $N(Q - c_Q)$ is the origin and as the map φ preserves barycenters, we deduce that φ is in fact linear. So $N(P - c_p)$ and $N(Q - c_Q)$ are equivalent modulo $\operatorname{GL}(\mathbb{Z}^d)$ and therefore their normal $\operatorname{GL}(\mathbb{Z}^d)$-positions P' and Q' are equal. This directly implies $P'' = Q''$. $\square$

The polytope P'' in Proposition 1.3.1 is uniquely determined by P. We call P'' the *normal* $\operatorname{Aff}(\mathbb{Z}^d)$*-position* of P. We employ this way of checking for equivalence throughout Chapter 4 and an implementation in SageMath can be found at `https://github.com/christopherborger/mixed_volume_classification`.

Remark 1.3.2. Grinis and Kasprzyk [GK13, §3.3] suggest to use a similar affine normal form. It has the small disadvantage that determining it requires determining distinct linear normal forms for each vertex of the polytope. However, they have a way to make this considerably more efficient than simply computing several linear normal forms from scratch. Their algorithm is implemented in Magma and we use it during the computational parts of Chapter 5.

2. Cayley Constructions

This chapter is devoted to the introduction of a construction called the Cayley sum and the presentation of various of its properties. In Section 2.1 we present the main definitions and show some immediate basic facts. Section 2.2 is devoted to showing the *combinatorial Cayley trick* (Proposition 2.2.1) that relates the Cayley sum construction and Minkowski sums, and to illustrate some implications of this relation to the combinatorial structure (Corollary 2.2.3) and the interaction with the lattice (Corollary 2.2.6) of the Cayley sum. In Section 2.3, we study the question of the uniqueness of Cayley decompositions. In particular, with Theorem 2.3.1 we give criteria that ensure the equivalence of tuples $(P_1, \ldots, P_k)$ and $(Q_1, \ldots, Q_k)$ whose Cayley sums $\operatorname{Cay}(P_1, \ldots, P_k)$ and $\operatorname{Cay}(Q_1, \ldots, Q_k)$ are equivalent.

2.1. Definition and basic properties

The Cayley sum (or Cayley configuration, Cayley polytope) of the elements in a tuple is a reoccurring object throughout this thesis. We define two versions of this construction, as both turn out to be useful in certain cases.

Definition 2.1.1. Let $A_1, \ldots, A_k \subset \mathbb{Z}^d$ be configurations. We define their *Cayley sum* as

$$\operatorname{Cay}(A_1, \ldots, A_k) = (A_1 \times \{e_1\}) \cup \cdots \cup (A_k \times \{e_k\}) \subset \mathbb{Z}^{d+k},$$

and their *affine Cayley sum* as

$$\operatorname{cay}(A_1, \ldots, A_k) = (A_1 \times \{\mathbf{0}\}) \cup (A_2 \times \{e_1\}) \cup \cdots \cup (A_k \times \{e_{k-1}\}) \subset \mathbb{Z}^{d+k-1}.$$

Analogously, for lattice polytopes $P_1, \ldots, P_k \in \mathcal{P}(\mathbb{Z}^d)$, we define their *Cayley sum* as

$$\operatorname{Cay}(P_1, \ldots, P_k) = \operatorname{conv}((P_1 \times \{e_1\}) \cup \cdots \cup (P_k \times \{e_k\})) \in \mathcal{P}(\mathbb{Z}^{d+k-1}),$$

and their *affine Cayley sum* as

$$\operatorname{cay}(P_1, \ldots, P_k) = \operatorname{conv}((P_1 \times \{\mathbf{0}\}) \cup (P_2 \times \{e_1\}) \cup \cdots \cup (P_k \times \{e_{k-1}\})) \in \mathcal{P}(\mathbb{Z}^{d+k}).$$

Let us denote by

$$H_{d+k,k} = \left\{x \in \mathbb{R}^{d+k} \colon x_{d+1} + \cdots + x_{d+k} = 1\right\} \subset \mathbb{R}^{d+k}$$

the affine hyperplane in $\mathbb{R}^{d+k}$ of all vectors whose last k coordinates sum up to one.

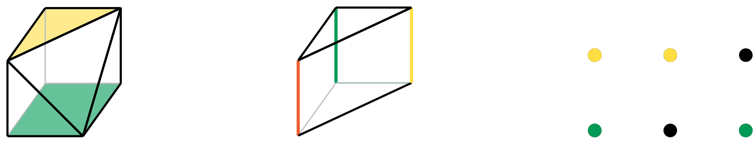


Figure 2.1.: Examples of cayley sums. From left to right this is the lattice polytope $\operatorname{cay}(\square_2, \Delta_2) \subset \mathbb{R}^3$, the lattice polytope $\operatorname{cay}([0,1],[0,1],[0,1]) \subset \mathbb{R}^3$, and the point configuration $\operatorname{cay}(\{0,2\},\{0,1\}) \subset \mathbb{Z}^2$.

Remark 2.1.2. The Cayley sum $\operatorname{Cay}(A_1,\dots,A_k)$ for configurations $A_1,\dots,A_k \subset \mathbb{Z}^d$ (and therefore $\operatorname{Cay}(P_1,\dots,P_k)$ for lattice polytopes $P_1,\dots,P_d \in \mathcal{P}(\mathbb{Z}^d)$) lives inside the affine hyperplane $H_{d+k,k}$. One has $\dim(\operatorname{Cay}(A_1,\dots,A_k)) = \dim(A_1+\dots+A_k)+k-1$ and the analogous statement for lattice polytopes. Furthermore, one has $\operatorname{Cay}(A_1,\dots,A_k) \cong \operatorname{cay}(A_1,\dots,A_k)$ and $\operatorname{Cay}(P_1,\dots,P_k) \cong \operatorname{cay}(P_1,\dots,P_k)$. This can be seen by observing that the map

$$\varphi\colon \mathbb{R}^{d+k} \to \mathbb{R}^{d+k-1}$$
$$\varphi(e_i) = \begin{cases} e_i \text{ if } i \in \{1,\dots,d\}, \\ \mathbf{0} \text{ if } i = d+1, \\ e_{i-1} \text{ if } i \in \{d+2,\dots,d+k\}, \end{cases}$$

defines a unimodular transformation when restricted to the affine hyperplane $H_{d+k,k}$.

The Cayley construction can be used to view problems regarding tuples of polytopes as problems of single polytopes in a higher dimension. For example, with Theorem 2.3.1 and Corollary 2.3.4. we explain how to use the Cayley polytope to reduce the question of equivalence of two tuples to the question of equivalence of a single lattice polytope. Another application is in determining so-called mixed subdivisions of Minkowski sums (see e.g. [DLRS10]) and furthermore in Theorem 3.2.1 we make use of the fact how a mixed discriminant can be viewed as a special A-discriminant, where the configuration A is constructed as a Cayley sum.

Apart from applications for the Cayley sum as a construction, one also encounters special classes of polytopes to be Cayley sums of lower-dimensional ones. For example point configurations that are defective and lattice polytopes of small lattice degree with respect to their dimension have been shown to be certain special Cayley sums (see Theorem 3.3.3 and Theorem 5.1.2).

From the point of view of determining whether a given polytope or configuration has a certain Cayley structure, it is often convenient to talk about *Cayley decompositions*.

Definition 2.1.3 (Cayley Decomposition)**.** Let $A \subset \mathbb{Z}^d$ and $F_1,\dots,F_k \in \mathcal{F}(A)$ faces. We say that $F_1,\dots,F_k$ form a *Cayley decomposition* of A if there exists a lattice projection $\pi\colon \operatorname{aff}(A) \to H_{k,k}$ such that $F_i = \pi^{-1}(e_i) \cap A$ for all $i \in [k]$.
Analogously, let $P \in \mathcal{P}(\mathbb{Z}^d)$ be a lattice polytope and $F_1,\dots,F_k \in \mathcal{F}(P)$ faces of P. We say that $F_1,\dots,F_k$ form a *Cayley decomposition* of P if there exists a lattice projection $\pi\colon \operatorname{aff}(P) \to H_{k,k}$ such that $F_i = \pi^{-1}(e_i) \cap P$ for all $i \in [k]$.

Whenever we construct a Cayley sum we obtain a configuration/lattice polytope with a Cayley decomposition into the summands. The following makes these interactions precise. This has been shown in [BN07] but we include a proof for the convenience of the reader.

Proposition 2.1.4. *Let $A \subset \mathbb{Z}^d$ be a configuration (respectively $P \in \mathcal{P}(\mathbb{Z}^d)$ a lattice polytope). Then the following are equivalent:*

1. *There exist non-empty configurations $A_1, \ldots, A_k \subset \mathbb{Z}^{d-k+1}$ (resp. lattice polytopes $P_1, \ldots, P_k \in \mathcal{P}(\mathbb{Z}^{d-k+1})$) such that $A \cong \operatorname{Cay}(A_1, \ldots, A_k)$ (resp. $P \cong \operatorname{Cay}(P_1, \ldots, P_k)$),*

2. *there exists a lattice projection $\pi\colon \operatorname{aff}(A) \to H_{k,k}$ with $\pi(A) = \tilde{\Delta}_{k-1} \cap \mathbb{Z}^k$ (resp. $\pi(P) = \tilde{\Delta}_{k-1}$),*

3. *there exists a Cayley decomposition of A (resp. P) into non-empty faces $F_1, \ldots, F_k \in \mathcal{F}(A)$ (resp. $\in \mathcal{F}(P)$).*

Proof. We restrict ourselves to presenting the proof for point configurations as the proof for lattice polytopes is analogous. One has (1) $\Rightarrow$ (2), as by construction any point of $\operatorname{Cay}(A_1, \ldots, A_k) \subset H_{d+1,k} \subset \mathbb{Z}^{d+1}$ is of the form (a, s) for $a \in A_i$ for some $i \in [k]$ and $s \in \tilde{\Delta}_{k-1} \cap \mathbb{Z}^k$, and we may therefore choose $\pi = \pi' \circ \varphi$, where $\pi'\colon H_{d+1,k} \to H_{k,k}$ is the lattice projection mapping onto the last k coordinates and $\varphi\colon \operatorname{aff}(A) \to H_{d,k}$ is a unimodular transformation mapping A onto $\operatorname{Cay}(A_1, \ldots, A_k)$. It is also straightforward to see (2) $\Rightarrow$ (3) by setting $F_i = \pi^{-1}(e_i) \cap A$ for all $i \in [k]$. Note that this is using the fact that $\pi^{-1}(F) \cap A$ is a face of A for any face $F \in \mathcal{F}(\pi(A))$. This is a general fact about lattice projections. Let us finally show (3) $\Rightarrow$ (1). Let $F_1, \ldots, F_k \in \mathcal{F}(A)$ form a Cayley decomposition of the configuration A with corresponding lattice projection $\pi\colon \operatorname{aff}(A) \to H_{k,k}$. Denote $d' := \dim(\operatorname{aff}(A))$. One has $\dim(\ker \pi) = d' - \dim(H_{k,k}) = d' - k + 1$. Thus, we may choose $\varphi\colon \operatorname{aff}(A) \to H_{d'+1,k}$ to be a unimodular transformation such that $\pi \circ \varphi^{-1}\colon H_{d'+1,k} \to H_{k,k}$ equals the projection onto the last k coordinates (as this also has codimension $d' - k + 1$). Then one has $\varphi(A) = \operatorname{Cay}(\tilde{\pi}(\varphi(F_1)), \ldots, \tilde{\pi}(\varphi(F_k)))$, where $\tilde{\pi}\colon H_{d'+1,k} \to \mathbb{R}^{d'-k+1}$ is the projection onto the first $d' - k + 1$ coordinates. $\square$

Remark 2.1.5. Note that condition (2) (and therefore also (1) and (3)) is equivalent to the existence of a lattice projection $\pi\colon \operatorname{aff}(A) \to \mathbb{R}^{k-1}$ with $\pi(A) = \Delta_{k-1} \cap \mathbb{Z}^{k-1}$ (resp. $\pi\colon \operatorname{aff}(P) \to \mathbb{R}^{k-1}$ with $\pi(P) = \Delta_{k-1}$).

2.2. Cayley polytopes and Minkowski sums

There is a fundamental, well-known relation between the Cayley polytope of a tuple of lattice polytopes $\operatorname{Cay}(P_1, \ldots, P_k)$ and its Minkowski sum $P_1 + \cdots + P_k$. The main observation for this is the fact that parametrized Minkowski sums occur as intersections of a Cayley polytope with certain linear subspaces. This connection is actually even deeper and extends to a connection between triangulations of the Cayley sum and fine mixed subdivisions of the Minkowski sum. For our purposes we restrict to presenting and proving the connection between the polytopes and refer to [DLRS10] for more details.

Proposition 2.2.1 (Combinatorial Cayley Trick). *Let $P_1, \dots, P_k \in \mathcal{P}(\mathbb{Z}^d)$ be lattice polytopes. For $\lambda \in \mathbb{R}^k$ denote by $H_\lambda \subset \mathbb{R}^{d+k}$ the linear subspace given as*

$$H_\lambda = \{x \in \mathbb{R}^{d+k} \colon x_{d+i} = \lambda_i \text{ for all } i \in [k]\}.$$

Then for any $\lambda \in \mathbb{R}^k$ with $0 \le \lambda_i \le 1$ and $\lambda_1 + \dots + \lambda_k = 1$ one has

$$\operatorname{Cay}(P_1, \dots, P_k) \cap H_\lambda \cong \lambda_1 P_1 + \dots + \lambda_k P_k.$$

Proof. By construction, each point $p \in \operatorname{Cay}(P_1, \dots, P_k)$ is a convex combination

$$p = \mu_1(p_1, e_1) + \dots + \mu_k(p_k, e_k) = (\mu_1 p_1 + \dots + \mu_k p_k, \mu),$$

for points $p_i \in P_i$ and $\mu = (\mu_1, \dots, \mu_k)$ with $0 \le \mu_i \le 1$ for all $i \in [k]$ and $\mu_1 + \dots + \mu_k = 1$. One clearly has $(\mu_1 p_1 + \dots + \mu_k p_k, \mu) \in H_\lambda$ if and only if $\mu = \lambda$. Therefore, for any $\lambda \in \mathbb{R}^k$ with $0 \le \lambda_i \le 1$ and $\lambda_1 + \dots + \lambda_k = 1$, one has

$$\operatorname{Cay}(P_1, \dots, P_d) \cap H_\lambda = \{(\lambda_1 p_1, \dots, \lambda_k p_k, \lambda) \colon p_i \in P_i \text{ for all } i \in [k]\},$$

and therefore $\operatorname{Cay}(P_1, \dots, P_d) \cap H_\lambda \cong \lambda_1 P_1 + \dots + \lambda_k P_k$. □

Remark 2.2.2. Analogously to Proposition 2.2.1, the intersection of the affine Cayley sum $\operatorname{cay}(P_1, \dots, P_k)$ with the hyperplane

$$h_\lambda = \left\{x \in \mathbb{R}^{d+k-1} \colon x_{d+i} = \lambda_{i+1} \text{ for all } i \in [k-1]\right\}$$

is equivalent to the Minkowski sum $\lambda_1 P_1 + \dots + \lambda_k P_k$ for any $\lambda \in \mathbb{R}^k$ with $0 \le \lambda_i \le 1$ and $\lambda_1 + \dots + \lambda_k = 1$.

Proposition 2.2.1 yields the following complete characterization of the faces of a Cayley sum. Recall that we also consider the whole polytope as a face of itself.

Corollary 2.2.3 (Faces of Cayley sums). *Let $P_1, \dots, P_k \in \mathcal{P}(\mathbb{Z}^d)$ be lattice polytopes. Consider faces $F_1, \dots, F_k$ with $F_i \in \mathcal{F}(P_i)$ for all $i \in [k]$. Let $I \subseteq [k]$ be the index set of all $i \in [k]$ for which $F_i \neq \emptyset$. Then $\operatorname{Cay}(F_1, \dots, F_k)$ is a face of $\operatorname{Cay}(P_1, \dots, P_k)$ if and only if $\sum_{i \in I} F_i$ is a face of the Minkowski sum $\sum_{i \in I} P_i$. All faces of $\operatorname{Cay}(P_1, \dots, P_k)$ arise in this way.*
The analogous statement holds if one considers configurations $A_1, \dots, A_k \subset \mathbb{Z}^d$ instead of lattice polytopes.

Proof. Let $F \in \mathcal{F}(\operatorname{Cay}(P_1, \dots, P_k))$ be a face. Assume that F is a non-empty and proper face as otherwise the statement is clear. All vertices of $\operatorname{Cay}(P_1, \dots, P_k)$ are of the form (v_i, e_i) for some $i \in [k]$ and a vertex $v_i \in \operatorname{vert}(P_i)$. Therefore F is of the form $F = \operatorname{Cay}(F_1, \dots, F_k)$ for *subsets* $F_i \subseteq P_i$ for $i \in [k]$. Let H be a supporting hyperplane corresponding to the face F. For any $i \in [k]$, the projection onto the first d coordinates of the intersection of H with H_{e_i} (in the notation of Proposition 2.2.1) is either a supporting hyperplane of P_i or the whole space $\mathbb{R}^d$. Furthermore, the

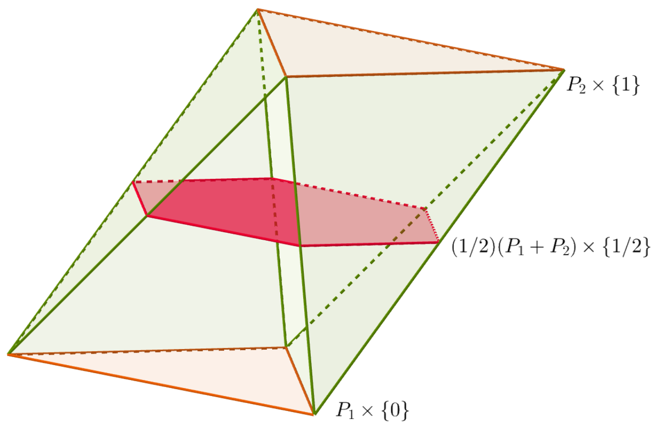


Figure 2.2.: Illustration of Remark 2.2.2 for $P_1 = \operatorname{conv}(\mathbf{0}, -e_1, -e_2)$, $P_2 = \operatorname{conv}(\mathbf{0}, e_1, e_2)$ (in orange). The intersection of $\operatorname{cay}(P_1, P_2)$ with $h_{(1/2,1/2)}$ is the dilated Minkowski sum $(1/2)(P_1 + P_2)$ (in red).

intersection of this projection with P_i is F_i and therefore $F_1, \dots, F_k$ are faces of $P_1, \dots, P_k$, respectively.

Let $I \subseteq [k]$ be the index set of all $i \in [k]$ for which $F_i \neq \emptyset$. Define

$$\lambda = \frac{1}{|I|} \sum_{i \in I} e_i \in \mathbb{R}^k.$$

Let H_λ be defined as in Proposition 2.2.1. Then $F \cap H_\lambda \in \mathcal{F}(\operatorname{Cay}(P_1, \dots, P_k) \cap H_\lambda)$ and, by Proposition 2.2.1, one has

$$\operatorname{Cay}(P_1, \dots, P_k) \cap H_\lambda \cong \sum_{i \in I} P_i.$$

This proves the statement for lattice polytopes. The arguments work analogously for point configurations $A_1, \dots, A_k \subset \mathbb{Z}^d$ as faces of a configuration A are intersections of A with faces of the lattice polytope $\operatorname{conv}(A)$. □

Remark 2.2.4. Corollary 2.2.3 implies in particular that $P_i \times \{e_i\}$ is a face of $\operatorname{Cay}(P_1, \dots, P_k)$. Furthermore, also the *complement*

$$P_i^c = \operatorname{conv}\{v \in \operatorname{vert}(\operatorname{Cay}(P_1, \dots, P_k)) \colon v \notin P_i \times \{e_i\}\},$$

is a face of $\operatorname{Cay}(P_1, \dots, P_k)$ as one has $P_i^c = \operatorname{Cay}(P_1, \dots, P_{i-1}, \emptyset, P_{i+1}, \dots, P_k)$.

Remark 2.2.5. Consider the Cayley sum $\operatorname{Cay}(P_1, \dots, P_k)$, let $F_1, \dots, F_k$ be faces of $P_1, \dots, P_k$, respectively, and denote by $P_i' = P_i \times \{e_i\}$ the faces corresponding to the Cayley summands. As one has $\operatorname{Cay}(F_1, \dots, F_k) = \operatorname{conv}(F_1 \times \{e_1\} \cup \dots \cup F_k \times \{e_k\})$, Corollary 2.2.3 yields that any face $F \in \mathcal{F}(\operatorname{Cay}(P_1, \dots, P_k))$ is of the the form

$$F = \operatorname{conv}((F \cap P_1') \cup \dots \cup (F \cap P_k')).$$

In particular, $\dim(F) = \dim(F \cap P_1') + \dots + \dim(F \cap P_k') + r - 1$, where r is the number of $i \in [k]$ for which one has $F \cap P_i' \neq \emptyset$.

The connection of Proposition 2.2.1 does not depend on $P_1, \dots, P_k$ being lattice polytopes. In fact, one could formulate an analogous statement with general convex bodies (while being careful about using the correct notion of equivalence). Its application to the special case of lattice polytopes, however, offers an interesting perspective on some questions regarding (mixed) lattice properties.

Recall that a lattice polytope $P \in \mathcal{P}(\mathbb{Z}^d)$ has the *integer decomposition property* (or *is IDP*) if one has $kP \cap \mathbb{Z}^d = P \cap \mathbb{Z}^d + (k-1)P \cap \mathbb{Z}^d$ for each number $k \in \mathbb{Z}_{\geq 2}$. We say that a tuple of lattice polytopes $(P_1, \dots, P_k) \in \mathcal{P}(\mathbb{Z}^d)^k$ is *mixed IDP*, if it satisfies

$$(P_1 + \cdots + P_k) \cap \mathbb{Z}^d = (P_1 \cap \mathbb{Z}^d) + \cdots + (P_k \cap \mathbb{Z}^d).$$

Part (1) of the following statement seems to be folklore but we are not aware of a published version of this explicit statement. Part (2) has been shown in [Tsu18].

Corollary 2.2.6 (Cayley and Minkowski sums of lattice polytopes)**.** *Let $P_1, \dots, P_k \in \mathcal{P}(\mathbb{Z}^d)$ be lattice polytopes.*

1. *For every $n \in \mathbb{Z}_{\geq 1}$ one has*

$$|\operatorname{int}(n \cdot \operatorname{Cay}(P_1, \dots, P_k)) \cap \mathbb{Z}^{d+k}| = \sum_{\substack{\lambda_1, \dots, \lambda_k \in \mathbb{Z}_{\geq 1} \\ \lambda_1 + \cdots + \lambda_k = n}} |\operatorname{int}(\lambda_1 P_1 + \cdots + \lambda_k P_k) \cap \mathbb{Z}^d|.$$

   In particular, the dilated Cayley polytope $n \cdot \operatorname{Cay}(P_1, \dots, P_k)$ is hollow if and only if for all choices of $\lambda_1, \dots, \lambda_k \in \mathbb{Z}_{\geq 1}$ with $\lambda_1 + \cdots + \lambda_k = n$ the Minkowski sum $\lambda_1 P_1 + \cdots + \lambda_k P_k$ is hollow.

2. *The Cayley polytope $\operatorname{Cay}(P_1, \dots, P_k)$ is IDP if and only if for all $\lambda_1, \dots, \lambda_k \in \mathbb{Z}_{\geq 0}$ the family $(\underbrace{P_1, \dots, P_1}_{\lambda_1 \text{ times}}, \dots, \underbrace{P_k, \dots, P_k}_{\lambda_k \text{ times}})$ is mixed IDP.*

Proof. We first prove (1). Denote by $\pi \colon H_{d+k,k} \to H_{k,k}$ the lattice projection onto the last k coordinates. The projection π maps every interior lattice point of $n \cdot \operatorname{Cay}(P_1, \dots, P_k)$ to an interior lattice point of $n \cdot \tilde{\Delta}_{k-1}$. It is not hard to verify that the fibers of π over the interior lattice points of $n \cdot \tilde{\Delta}_{k-1}$ are precisely the affine subspaces of the form H_λ for $\lambda \in (\mathbb{Z}_{\geq 1})^k$ satisfying $\lambda_1 + \cdots + \lambda_k = n$. By Proposition 2.2.1, one has

$$n \cdot \operatorname{Cay}(P_1, \dots, P_k) \cap H_\lambda \cong \lambda_1 P_1 + \cdots + \lambda_k P_k,$$

which proves the statement. Let us now proceed to proving (2). Assume that $\operatorname{Cay}(P_1, \dots, P_k)$ is IDP, let $\lambda = (\lambda_1, \dots, \lambda_k) \in (\mathbb{Z}_{\geq 0})^k$ and $p \in (\lambda_1 P_1 + \cdots + \lambda_k P_k) \cap \mathbb{Z}^d$ a lattice point. By the equivalence given in Proposition 2.2.1, the lattice point p corresponds to the lattice point $(p, \lambda) \in (n \cdot \operatorname{Cay}(P_1, \dots, P_k) \cap H_\lambda) \cap \mathbb{Z}^{d+k}$, where $n = \lambda_1 + \cdots + \lambda_k$. As $\operatorname{Cay}(P_1, \dots, P_k)$ is IDP, there exist $(q_1, e_{i_1}), \dots, (q_n, e_{i_n}) \in \operatorname{Cay}(P_1, \dots, P_k) \cap \mathbb{Z}^{d+k}$ satisfying $(p, \lambda) = (q_1, e_{i_1}) + \cdots + (q_n, e_{i_n})$. In particular, as $\lambda = e_{i_1} + \cdots + e_{i_n}$, we may assume the sequence $(e_{i_1}, \dots, e_{i_n})$ to be of the form

$$(\underbrace{e_1, \dots, e_1}_{\lambda_1 \text{ times}}, \dots, \underbrace{e_k, \dots, e_k}_{\lambda_k \text{ times}}),$$

and therefore

$$\begin{aligned} q_1, \ldots, q_{\lambda_1} &\in P_1 \cap \mathbb{Z}^d \\ q_{\lambda_1+1}, \ldots, q_{\lambda_1+\lambda_2} &\in P_2 \cap \mathbb{Z}^d \\ &\vdots \\ q_{\lambda_1+\cdots+\lambda_{k-1}+1}, \ldots, q_{\lambda_1+\cdots+\lambda_k} &\in P_k \cap \mathbb{Z}^d. \end{aligned}$$

This shows that the tuple

$$(\underbrace{P_1, \ldots, P_1}_{\lambda_1 \text{ times}}, \ldots, \underbrace{P_k, \ldots, P_k}_{\lambda_k \text{ times}}),$$

is mixed IDP, finishing the proof of one direction of (2). Let us finally prove the reverse implication. Let $n \in \mathbb{Z}_{\geq 2}$ and let $p \in (n \cdot \operatorname{Cay}(P_1, \ldots, P_k)) \cap \mathbb{Z}^{d+k}$. Then, by construction, p is of the form

$$p = \lambda_1(q_1, e_1) + \cdots + \lambda_k(q_k, e_k),$$

for $\lambda_1, \ldots, \lambda_k \in \mathbb{R}_{\geq 0}$ with $\lambda_1 + \cdots + \lambda_k = n$ and $q_i \in P_i$ for all $i \in [k]$. Since p is a lattice point, one has $\lambda_1, \ldots, \lambda_k \in \mathbb{Z}_{\geq 0}$. Furthermore, by Proposition 2.2.1, $\lambda_1 q_1 + \cdots + \lambda_k q_k$ is a lattice point of the Minkowski sum $\lambda_1 P_1 + \cdots + \lambda_k P_k$. By assumption, the tuple

$$(\underbrace{P_1, \ldots, P_1}_{\lambda_1 \text{ times}}, \ldots, \underbrace{P_k, \ldots, P_k}_{\lambda_k \text{ times}}),$$

is mixed IDP. Therefore one can choose lattice points $r_{i,1}, \ldots, r_{i,\lambda_i} \in P_i \cap \mathbb{Z}^d$, for each $i \in [k]$, such that one has

$$r_{1,1} + \cdots + r_{1,\lambda_1} + \cdots + r_{k,1} + \cdots + r_{k,\lambda_k} = \lambda_1 q_1 + \cdots + \lambda_k q_k,$$

and therefore

$$p = (r_{1,1}, e_1) + \cdots + (r_{1,\lambda_1}, e_1) + \cdots + (r_{k,1}, e_k) + \cdots + (r_{k,\lambda_k}, e_k),$$

which proves the claim as $(r_{i,j}, e_i) \in \operatorname{Cay}(P_1, \ldots, P_k) \cap \mathbb{Z}^{d+k}$ for each $i \in [k]$ and $j \in \{1, \ldots, \lambda_i\}$. □

Questions regarding the hollowness of certain Minkowski sums will play a great role in Chapter 5 in the context of the mixed degree of a tuple of lattice polytopes. We will not be treating the IDP or mixed IDP property in further detail in this thesis and refer to [Tsu18] for further results on the interplay of these notions for special classes of polytopes. In general, the question whether a family is mixed IDP or not seems difficult. Already in dimension 2, the question for a description of all pairs that are mixed IDP, which has been asked in [Oda08], has not been completely answered (cf. [HNPS08]).

2.3. Equivalence of Cayley sums

It is a straightforward observation that two equivalent tuples of lattice polytopes $(P_1, \dots, P_k), (Q_1, \dots, Q_k) \in \mathcal{P}(\mathbb{Z}^d)^k$ yield two equivalent Cayley polytopes. Indeed, if the above tuples are equivalent, then there exists a permutation σ on the index set $[k]$, a unimodular transformation $\varphi \in \operatorname{Aff}(\mathbb{Z}^d)$, and a translation by a tuple of lattice vectors $(t_1, \dots, t_k) \in (\mathbb{Z}^d)^k$ such that successive application of these operations maps $(P_1, \dots, P_k)$ to $(Q_1, \dots, Q_k)$. Each of these operations yields a map in $\operatorname{Aff}(H_{d+k,k})$ (a permutation of the last k coordinates, the unimodular transformation $\varphi \times \mathrm{Id}_k$, and a shearing map, respectively), and the composition of these maps sends $\operatorname{Cay}(P_1, \dots, P_k)$ to $\operatorname{Cay}(Q_1, \dots, Q_k)$.

However, it can in general happen that two non-equivalent tuples have equivalent Cayley sums. In order to avoid this one needs to ensure that the unimodular transformation between the Cayley polytopes "respects the Cayley structure". The following provides certain conditions that ensure this situation.

Theorem 2.3.1. *Let $(P_1, \dots, P_k), (Q_1, \dots, Q_k) \in \mathcal{P}(\mathbb{Z}^d)^k$ with $\dim(P_1 + \cdots + P_k) = d$. If there exists no lattice projection $\pi \colon \mathbb{R}^d \to \mathbb{R}$ satisfying $\pi(P_i) \subseteq \Delta_1 + z_i$ for some $z_i \in \mathbb{Z}$ for all $i \in [k]$, then one has:*

$$(P_1, \dots, P_k) \cong (Q_1, \dots, Q_k) \Leftrightarrow \operatorname{Cay}(P_1, \dots, P_k) \cong \operatorname{Cay}(Q_1, \dots, Q_k). \qquad (2.1)$$

If $Q_1, \dots, Q_k$ are full-dimensional, then (2.1) holds under the weaker assumption that there exists no lattice projection $\pi \colon \mathbb{R}^d \to \mathbb{R}^{k-1}$ satisfying $\pi(P_i) \subseteq \Delta_{k-1} + z_i$ for some $z_i \in \mathbb{Z}^{k-1}$ for all $i \in [k]$.

Proof. The first implication is straightforward, as sketched above. So assume $\varphi \in \operatorname{Aff}(H_{d+k,k})$ is a unimodular transformation sending $\operatorname{Cay}(P_1, \dots, P_k)$ to $\operatorname{Cay}(Q_1, \dots, Q_k)$. Denote $P_i' = \varphi(P_i \times \{e_i\})$ and $\hat{Q}_i = Q_i \times \{e_i\}$ for all $i \in [k]$. Then $P_1', \dots, P_k'$ and $\hat{Q}_1, \dots, \hat{Q}_k$ are both Cayley decompositions of $\operatorname{Cay}(Q_1, \dots, Q_k)$. Denote by $\pi_Q \colon H_{d+k,k} \to H_{k,k}$ the projection corresponding to the Cayley decomposition $\hat{Q}_1, \dots, \hat{Q}_k$ (the projection onto the last k coordinates). Let $\epsilon_i \colon \mathbb{R}^d \to H_{d+k,k}$ be the embedding given by $x \mapsto (x, e_i) \in H_{d+k,k}$ and consider the composition

$$\psi_i \colon \mathbb{R}^d \hookrightarrow H_{d+k,k} \xrightarrow{\sim} H_{d+k,k} \twoheadrightarrow H_{k,k}$$

given by $\psi_i = \pi_Q \circ \varphi \circ \epsilon_i$. As $P_i' \subseteq \operatorname{Cay}(Q_1, \dots, Q_k)$, one has

$$\psi_i(P_i) = \pi_Q(P_i') \subseteq \tilde{\Delta}_{k-1}.$$

As one has $\epsilon_i = \epsilon_j + e_{d+i} - e_{d+j}$ for every $i, j \in [k]$, also the affine maps ψ_i and ψ_j only differ by a constant lattice translation. Therefore there exist $t_1, \dots, t_k \in H_{k,k}$ such that $\psi_1(P_i) \subseteq \tilde{\Delta}_{k-1} + t_i$ for all $i \in [k]$.

We make a case distinction depending on the image $\psi_1(\mathbb{R}^d)$ of the affine lattice-preserving map ψ_1. If $\dim(\psi_1(\mathbb{R}^d)) \geq 1$, one has $\dim(\psi_1(P_1 + \cdots + P_k)) \geq 1$ (as by assumption $\dim(P_1 + \cdots + P_k) = d$). In particular, there exists an index $i_0 \in [k]$ such that $\dim(\psi_1(P_{i_0})) \geq 1$. As $\psi_1(P_{i_0})$ is a lattice polytope inside $\tilde{\Delta}_{k-1} + t_{i_0}$, this implies that it contains an edge I of $\tilde{\Delta}_{k-1} + t_{i_0}$. One may choose a lattice projection

$\pi_I\colon H_{k,k} \to \mathbb{R}$ "onto" the linear subspace $\operatorname{span}_{\mathbb{R}}(I - t_{i_0})$ (such that $\pi_I(H_{k,k}) = \pi_I(\operatorname{span}_{\mathbb{R}}(I - t_{i_0})) = \mathbb{R}$), which satisfies $\pi_I(\tilde{\Delta}_{k-1}) = \pi_I(I - t_{i_0}) = \Delta_1$. Then the map $\pi = \pi_I \circ \psi_1$ is a lattice projection satisfying

$$\pi(P_i) = \pi_I(\psi_1(P_i)) \subseteq \pi_I(\tilde{\Delta}_{k-1} + t_i) = \Delta_1 + \pi_I(t_i),$$

for all $i \in [k]$. In this case the claimed conditions for (2.1) are not satisfied.

So let us assume that $\dim(\psi_1(\mathbb{R}^d)) = 0$. This is equivalent to the existence of a lattice vector $s \in H_{d+k,k}$ such that

$$\varphi(\mathbb{R}^d \times \{e_1\}) + s \subseteq \ker(\pi_Q) = \mathbb{R}^d \times \{\mathbf{0}\}\,.$$

As φ is a unimodular transformation, comparing the dimensions yields that the above containment is in fact an equality and therefore $\varphi(\mathbb{R}^d \times \{e_1\}) + s = \mathbb{R}^d \times \{\mathbf{0}\}$. Thus, φ splits into $\varphi = \varphi_d \times \varphi_k$, where $\varphi_d \in \operatorname{Aff}(\mathbb{Z}^d)$ and $\varphi_k \in \operatorname{Aff}(\mathbb{Z}^k)$. Then φ_k is a symmetry of the simplex $\tilde{\Delta}_{k-1}$ and therefore a coordinate permutation. In particular, there exists a permutation σ on $[k]$ such that φ sends $P_i \times \{e_i\}$ to $Q_{\sigma(i)} \times \{e_{\sigma(i)}\}$ for any $i \in [k]$. The map φ_d provides a unimodular transformation sending each P_i to a lattice translate of $Q_{\sigma(i)}$ and therefore one has $(P_1, \ldots, P_k) \cong (Q_1, \ldots, Q_k)$.

We now show that, whenever $Q_1, \ldots, Q_k$ are full-dimensional, it suffices to pose the claimed weaker conditions on the tuple $(P_1, \ldots, P_k)$ in order to deduce equivalence of $(P_1, \ldots, P_k)$ and $(Q_1, \ldots, Q_k)$. We do so by showing that in this case the mapping ψ_1 is either trivial (and therefore (2.1) holds analogously to above) or it is surjective and therefore a lattice projection. As $\psi_1(P_i) \subseteq \tilde{\Delta}_{k-1} + t_i$ for all $i \in [k]$, the latter case contradicts the assumptions.

Let us assume that the map ψ_1 is non-trivial. As shown above, this implies the existence of an index $i_0 \in [k]$ for which $\dim(\pi_Q(P'_{i_0})) = \dim(\psi_1(P_{i_0})) \geq 1$. This means in particular that $P'_{i_0} \neq \hat{Q}_i$ for all $i \in [k]$, as $\pi_Q(\hat{Q}_i) = \{\mathbf{0}\}$. Note that P'_{i_0} also cannot strictly contain any $\hat{Q}_i$, as by Corollary 2.2.3 any face $F \in \mathcal{F}(\operatorname{Cay}(Q_1, \ldots, Q_k))$ that properly contains $\hat{Q}_i$ is of the form

$$F = \operatorname{Cay}(F_1, \ldots, F_{i-1}, Q_i, F_{i+1}, \ldots, F_d),$$

for faces $F_j \in \mathcal{F}(Q_j)$, at least one of which is non-empty. So $\dim(F) \geq d + 1$ and therefore F cannot equal the face P'_{i_0}, which has dimension at most d. Furthermore, P'_{i_0} cannot be disjoint to any $\hat{Q}_i$ due to the following. If P'_{i_0} is disjoint to, say, $\hat{Q}_1$, its complement $(P'_{i_0})^c$ contains $\hat{Q}_1$. As P'_{i_0} does not fully contain any $\hat{Q}_i$, the complement $(P'_{i_0})^c$ additionally contains at least one point of $\hat{Q}_i$ for each $i \in [k]$. Therefore by Corollary 2.2.3, there exist points $p_2, \ldots, p_k$ in $Q_2, \ldots, Q_k$, respectively, such that $\operatorname{Cay}(Q_1, \{p_2\}, \ldots, \{p_k\}) \subseteq (P'_{i_0})^c$ and therefore

$$\dim((P'_{i_0})^c) \geq \dim(\operatorname{Cay}(Q_1, \{p_2\}, \ldots, \{p_k\})) = d + k - 1.$$

This is a contradiction to Remark 2.2.4, as the $(d + k - 1)$-dimensional Cayley sum $\operatorname{Cay}(Q_1, \ldots, Q_k)$ cannot have a proper face of dimension at least $d + k - 1$. We conclude that P'_{i_0} has non-empty intersection with $\hat{Q}_i$ for each $i \in [k]$. Thus $\psi_1(P_{i_0}) = \tilde{\Delta}_{k-1}$. This implies that the map ψ is surjective and finishes the proof. $\square$

Let us comment on the relation between the two different conditions that ensure that (2.1) holds. Note first that the assumption $\dim(P_1 + \cdots + P_k) = d$ is only a technical one as we may otherwise identify $\operatorname{aff}(P_1+\cdots+P_k)$ with some $\mathbb{R}^{d'}$ in a lattice-preserving way. Requiring that there exists no lattice projection commonly mapping all P_i onto translates of Δ_1 is a necessary condition in the sense that there exist pairs of tuples $(P_1,\ldots,P_k),(Q_1,\ldots,Q_k) \in \mathcal{P}(\mathbb{Z}^d)^k$ not meeting this assumption for which one has $\operatorname{Cay}(Q_1,\ldots,Q_k) \cong \operatorname{Cay}(P_1,\ldots,P_k)$ but $(Q_1,\ldots,Q_k) \not\cong (P_1,\ldots,P_k)$. However, if one has a tuple of lattice polytopes $(P_1,\ldots,P_k) \in \mathcal{P}(\mathbb{Z}^d)^k$ that violates this condition, it is not always true that there exists a non-equivalent tuple with equivalent Cayley sum. On the other hand, consider a tuple $(P_1,\ldots,P_k)$ that also violates the weaker condition of the non-existence of a lattice projection commonly mapping all P_i onto translates of Δ_{k-1}. By Proposition 2.1.4, this implies that each P_i itself is equivalent to a Cayley sum of k polytopes $F_{i1},\ldots,F_{ik}$, and that the unimodular transformations yielding these equivalences are compatible in the sense that one has

$$(P_1,\ldots,P_k) \cong (\operatorname{Cay}(F_{11},\ldots,F_{1k}),\ldots,\operatorname{Cay}(F_{k1},\ldots,F_{kk})).$$

In this situation one has the following general statement, showing that one would in general expect the existence of a non-equivalent tuple $(Q_1,\ldots,Q_k) \not\cong (P_1,\ldots,P_k)$ satisfying $\operatorname{Cay}(Q_1,\ldots,Q_k) \cong \operatorname{Cay}(P_1,\ldots,Q_k)$.

Proposition 2.3.2. *Consider lattice polytopes $P_{11},\ldots,P_{1k},\ldots,P_{k1},\ldots,P_{kk} \in \mathcal{P}(\mathbb{Z}^d)$. Then one has the following equivalence of Cayley sums:*

$$\begin{gathered}\operatorname{Cay}(\operatorname{Cay}(P_{11},\ldots,P_{1k}),\ldots,\operatorname{Cay}(P_{k1},\ldots,P_{kk}))\\ \cong \\ \operatorname{Cay}(\operatorname{Cay}(P_{11},\ldots,P_{k1}),\ldots,\operatorname{Cay}(P_{1k},\ldots,P_{kk})).\end{gathered}$$

Example 2.3.3. Consider two triples of lattice polytopes $(P_1,P_2,P_3),(Q_1,Q_2,Q_3) \in \mathcal{P}(\mathbb{Z}^3)^3$ where $P_1 = P_2 = Q_1 = \Delta_2 \times \Delta_1$, the polytope $P_3 = \Delta_3$, and $Q_2 = Q_3 = \operatorname{conv}(\Delta_2 \cup \{e_3, e_1+e_3\})$ (see Figure 2.3). Note that the lattice projection $\pi_{1,2}\colon \mathbb{R}^3 \to \mathbb{R}^2$ onto the first two coordinates satisfies $\pi(P_i) = \Delta_2$ for all $i \in [3]$. Furthermore, the two tuples are not equivalent. This can be seen for example by observing that none of the Q_i has four vertices and therefore none of them is equivalent to P_3. However, by Proposition 2.1.4, the lattice projection $\pi_{1,2}$ yields a Cayley decomposition of each P_i into three faces (colored in red, blue and orange in Figure 2.3). In particular:

$$P_1 = \operatorname{cay}(I_{11}, I_{12}, I_{13}) \qquad P_2 = \operatorname{cay}(I_{21}, I_{22}, I_{23}) \qquad P_3 = \operatorname{cay}(I_{31}, I_{32}, I_{33}),$$

and therefore

$$\operatorname{cay}(P_1,P_2,P_3) = \operatorname{cay}(\operatorname{cay}(I_{11},I_{12},I_{13}),\operatorname{cay}(I_{21},I_{22},I_{23}),\operatorname{cay}(I_{31},I_{32},I_{33})).$$

By Proposition 2.3.2 one therefore has

$$\begin{aligned}\operatorname{cay}(P_1,P_2,P_3) &\cong \operatorname{cay}(\operatorname{cay}(I_{11},I_{21},I_{31}),\operatorname{cay}(I_{12},I_{22},I_{32}),\operatorname{cay}(I_{13},I_{23},I_{33}))\\ &= \operatorname{cay}(Q_1,Q_2,Q_3).\end{aligned}$$

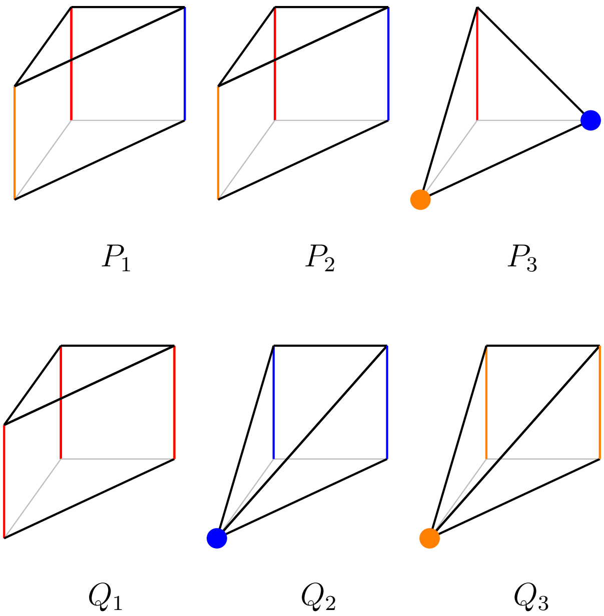


Figure 2.3.: The triples of polytopes (P_1, P_2, P_3) and (Q_1, Q_2, Q_3) from Example 2.3.3.

The following result shows how one may use Theorem 2.3.1 in order to reduce the question whether two tuples of lattice polytopes are equivalent to the question whether the Cayley sums of their second dilates are. Passing to the second dilates when forming the Cayley sums allows us to not have to worry about the original tuples meeting the conditions of Theorem 2.3.1.

Corollary 2.3.4. *Let* $(P_1, \dots, P_k), (Q_1, \dots, Q_k) \in \mathcal{P}(\mathbb{Z}^d)^k$ *with* $\dim(P_1 + \dots + P_k) = d$*. Then one has:*

$$(P_1, \dots, P_k) \cong (Q_1, \dots, Q_k) \Leftrightarrow \operatorname{Cay}(2P_1, \dots, 2P_k) \cong \operatorname{Cay}(2Q_1, \dots, 2Q_k).$$

Proof. It is straightforward to verify that one has $(P_1, \dots, P_k) \cong (Q_1, \dots, Q_k)$ if and only if one has $(2P_1, \dots, 2P_k) \cong (2Q_1, \dots, 2Q_k)$. In order to complete the proof we show that the tuple $(2P_1, \dots, 2P_k)$ satisfies the conditions of Theorem 2.3.1. Let therefore $\pi \colon \mathbb{R}^d \to \mathbb{R}$ be a lattice projection. Then $\pi(2P_1 + \dots + 2P_k) = \pi(2P_1) + \dots + \pi(2P_k) \in \mathcal{P}(\mathbb{Z}^1)$ is a 1-dimensional lattice polytope (as we assumed $\dim(P_1 + \dots + P_k) = d$). This implies $\dim(\pi(2P_{i_0})) = 1$ for some $i_0 \in [k]$ and therefore $\pi(2P_{i_0}) = 2\pi(P_{i_0})$ cannot be contained in Δ_1. $\square$

3. Defectivity of Mixed Discriminants

The core of this chapter is the presentation of a necessary criterion for a tuple $(A_0, \dots, A_k) \subset (\mathbb{Z}^d)^{k+1}$ of support sets for a polynomial system to be defective (Theorem 3.2.1). Defective tuples of point configurations $(A_0, \dots, A_k)$ are essentially tuples for which the condition for a system $f_0 = \cdots = f_k = 0$ with $(f_0, \dots, f_k) \in \mathbb{C}[A_0, \dots, A_k]$ to have a multiple root cannot be described by a single polynomial. In the case of square systems this result solves a conjecture by Cattani et al. saying that such a polynomial always exists and has positive degree except if the system is given as the intersection of affine hyperplanes (Corollary 3.2.2). In Section 3.1 we provide some background and introduce the notion of an A-discriminant and a mixed discriminant. Section 3.2 is devoted to the presentation of the main result and its implications. In Section 3.3 we state the criterion that is the main tool in the deduction of our results (Theorem 3.3.3) and illustrate some direct implications. Section 3.4 is dedicated to presenting the proof of Theorem 3.2.1 and we conclude the chapter with an outlook of further research directions in Section 3.5.

Note that, in the setting of this chapter, it shows convenient to work with the affine Cayley sum $\operatorname{cay}(A_0, \dots, A_k)$ and tuples of configurations indexed by numbers starting with 0. We use the notation $[k]_0 = \{0, \dots, k\}$. As $\operatorname{cay}(A_0, \dots, A_k) \cong \operatorname{Cay}(A_0, \dots, A_k)$, we may still use the results from Chapter 2.

3.1. (Mixed) discriminants and defectivity

The probably most well-known instance of a discriminant is the one for a univariate quadratic polynomial $ax^2 + bx + c \in \mathbb{C}[x]$, which is given as $b^2 - 4ac \in \mathbb{C}[a, b, c]$ and which vanishes for all choices of coefficients $a_0, b_0, c_0 \in \mathbb{C}$ for which the polynomial $a_0x^2 + b_0x + c_0$ has a multiple root. This concept has been generalized by the introduction of *A-discriminants* by Gelfand, Kapranov and Zelevinsky in [GKZ94].

Recall that the *affine variety* $\mathcal{V}(I)$ of a finitely generated ideal $I \subseteq \mathbb{C}[x_1, \dots, x_d]$ with generating polynomials $f_1, \dots, f_k$ is given by

$$\mathcal{V}(I) = \mathcal{V}(f_1, \dots, f_k) = \left\{x \in \mathbb{C}^d \colon f_1(x) = \cdots = f_k(x) = 0\right\}.$$

We call $\mathcal{V}(I)$ an *algebraic hypersurface* if I is generated by a single polynomial. Given an arbitrary set $S \subseteq \mathbb{C}^d$, its *Zariski closure* is the smallest set V containing S and satisfying $V = \mathcal{V}(I)$ for some ideal $I \subseteq \mathbb{C}[x_1, \dots, x_d]$. Furthermore, a *multiple root* of a Laurent polynomial $f \in \mathbb{C}[x_1^{\pm}, \dots, x_d^{\pm}]$ is a point $p \in (\mathbb{C}^*)^d$ satisfying $f(p) = 0$ and $\nabla f(p) = \mathbf{0}$.

Definition 3.1.1 (A-Discriminant)**.** Let $A \subset \mathbb{Z}^d$ be a point configuration. The *discriminantal variety* $\Sigma_A \subset \mathbb{C}^A$ is the Zariski closure of all coefficient vectors $c \in \mathbb{C}^A$ whose corresponding polynomial $f_c \in \mathbb{C}[A]$ has a multiple root in $(\mathbb{C}^*)^d$. If Σ_A is an algebraic hypersurface, the A-*discriminant* Δ_A is the (up to sign) unique irreducible polynomial with $\mathcal{V}(\Delta_A) = \Sigma_A$. Otherwise we set $\Delta_A = 1$ and say that the configuration A is *defective*.

Let us illustrate this definition with an example.

Example 3.1.2. The polynomial $b^2 - 4ac$ is the A-discriminant of the 1-dimensional configuration $A = \{0, 1, 2\} \subset \mathbb{Z}$. In particular, the configuration $\{0, 1, 2\}$ is not defective. Also note that it is important to keep in mind that we are forming a Zariski closure in the definition of Σ_A. For example the point $(a_0, b_0, c_0) = (0, 0, 1)$ corresponds to the constant polynomial $f = 1$ (which does not have any root at all), although $(0, 0, 1) \in \Sigma_A$.
As another example consider the vector space of polynomials $\mathbb{C}[A']$ where

$$A' = \begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \subset \mathbb{Z}^2.$$

Polynomials in $\mathbb{C}[A']$ are of the form $f = ax^2 + bx + c + dy$ for coefficients $a, b, c, d \in \mathbb{C}[A']$. A multiple root of f is a point $(x_0, y_0) \in (\mathbb{C}^*)^2$ that satisfies

$$\begin{aligned} f(x_0, y_0) &= ax_0^2 + bx_0 + c + dy_0 = 0 \\ \nabla f(x_0, y_0) &= (2ax_0 + b, d) = 0. \end{aligned}$$

Clearly, f can only have such a multiple root whenever $d = 0$. It remains to compute for which choices of a, b, c the terms $x_0^2 + bx_0 + c$ and $2ax_0 + b$ can vanish for the same $x_0 \in \mathbb{C}^*$. This, however, is exactly given by the condition that $b^2 - 4ac = 0$. So we conclude that the discriminantal variety of A' is given by

$$\Sigma_{A'} = \mathcal{V}(d) \cap \mathcal{V}(b^2 - 4ac) = \mathcal{V}(d, b^2 - 4ac).$$

In particular, $\Sigma_{A'}$ is not an algebraic hypersurface and therefore the configuration A' is defective. In other words, the condition of a polynomial in $\mathbb{C}[A']$ to have a multiple root is given by the vanishing of two independent polynomials. Therefore we cannot find an A-discriminant $\Delta_{A'}$ satisfying $\mathcal{V}(\Delta_{A'}) = \Sigma_{A'}$.

There have been intensive studies on connecting the (discrete) geometry of the point configuration A to invariants of the corresponding discriminant. One direction of research has been to find a combinatorial characterization of defective configurations (see for example [DR06, CDR08, DDRP09, DN10, DNV12]). Recently two independent characterizations were given by Esterov [Est10, Est18a] and Furukawa-Ito [FI20]. We refer to the survey article [Pie15] for additional background on A-discriminants.

Our main contribution is regarding a modification of the A-discriminant to systems of polynomials. This has been introduced as the mixed discriminant by Cattani et al. in [CCD+13]. Instead of describing the choices of coefficients leading to multiple roots among a vector space of polynomials, it describes tuples of polynomials in $\mathbb{C}[A_0,\ldots,A_k]$ for which the system $f_0 = \cdots = f_k = 0$ has so-called non-degenerate multiple roots. Let us make this definition precise.

Definition 3.1.3. Let $f_0,\ldots,f_k \in \mathbb{C}[x_1^{\pm 1},\ldots,x_d^{\pm 1}]$. A *multiple root* of the system $f_1 = \cdots = f_k = 0$ is a point $p \in (\mathbb{C}^*)^d$ such that $f_1(p) = \cdots = f_k(p) = 0$ and such that the gradient vectors $\nabla f_0(p),\ldots,\nabla f_k(p)$ are linearly dependent. A multiple root p is called *non-degenerate* if any proper subset of $\{\nabla f_0(p),\ldots,\nabla f_k(p)\}$ is linearly independent.

The following definition of the mixed discriminant is a slightly generalized version of the one in [CCD+13] to the case of k polynomials in d variables. This definition is due to personal communication with Alicia Dickenstein and Sandra Di Rocco regarding the announced paper [DDRM20] and also appears in [Est19].

Definition 3.1.4 (Mixed Discriminant)**.** Let $(A_0,\ldots,A_k) \subset (\mathbb{Z}^d)^{k+1}$ be a tuple of point configurations. The *discriminantal variety* $\Sigma_{A_0,\ldots,A_k} \subset \mathbb{C}^{A_0} \times \cdots \times \mathbb{C}^{A_k}$ is the Zariski closure of all coefficient vectors $(c_1,\ldots,c_k)$ for which the system of corresponding polynomials $f_{c_1} = \cdots = f_{c_k} = 0$ has a non-degenerate multiple root. If $\Sigma_{A_0,\ldots,A_k}$ is an algebraic hypersurface, the *mixed discriminant* $\Delta_{A_0,\ldots,A_k}$ is the (up to sign) unique irreducible polynomial that satisfies $\mathcal{V}(\Delta_{A_0,\ldots,A_k}) = \Sigma_{A_0,\ldots,A_k}$. Otherwise we set $\Delta_{A_0,\ldots,A_k} = 1$ and call the tuple $(A_0,\ldots,A_k)$ *defective*.

Let us illustrate the above definition with an example.

Example 3.1.5. Consider the pair of configurations $(A_0, A_1) \subset (\mathbb{Z}^2)^2$ given by:

$$A_0 = \begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

A system of polynomials $f_0 = f_1 = 0$ with $(f_0, f_1) \in \mathbb{C}[A_0, A_1]$ is the intersection of a parabola with a line. Therefore the mixed discriminant Δ_{A_0,A_1} should provide the conditions for the line $f_1 = 0$ to be tangent to the parabola $f_0 = 0$. So let us consider polynomials:

$$\begin{aligned} f_0 &= a_{00} + a_{10}x + a_{20}x^2 + a_{01}y, \\ f_1 &= b_{00} + b_{10}x + b_{01}y, \end{aligned}$$

Already computing Δ_{A_0,A_1} in this special case would become very tedious by hand. We therefore use Macaulay2 [GS] in order to compute

$$\Delta_{A_0,A_1} = a_{01}^2 b_{10}^2 + 4a_{20}a_{01}b_{00}b_{01} - 2a_{10}a_{01}b_{10}b_{01} + a_{10}^2 b_{01}^2 - 4a_{00}a_{20}b_{01}^2.$$

The following result is soon to appear in an announced paper by Di Rocco, Dickenstein and Morrison [DDRM20] (see also [CCD+13] for the special case where $k = d$). It shows how to reduce the computation of a mixed discriminant to that of a certain special A-discriminant using the construction of the Cayley sum.

Theorem 3.1.6. *Let $(A_0, \dots, A_k) \subset (\mathbb{Z}^d)^{k+1}$ be a tuple of configurations. If the Cayley sum $\operatorname{cay}(A_0, \dots, A_k) \subset \mathbb{Z}^{d+k}$ is not defective, then $\Delta_{A_0,\dots,A_k} = \Delta_{\operatorname{cay}(A_0,\dots,A_k)}$. In particular, if the tuple $(A_0, \dots, A_k)$ is defective, then the configuration $\operatorname{cay}(A_0, \dots, A_k)$ is defective.*

Example 3.1.7. Consider the pair of configurations $(A_0, A_1) \subset (\mathbb{Z}^2)^2$ from Example 3.1.5. The Cayley sum of A_0 and A_1 is given by

$$\operatorname{cay}(A_0, A_1) = \begin{pmatrix} 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \subset \mathbb{Z}^3.$$

Let us illustrate the basic intuition behind Theorem 3.1.6. Any polynomial $f \in \mathbb{C}[\operatorname{cay}(A_0, A_1)]$ is of the form

$$f = f_0 + z f_1,$$

for a pair $(f_0, f_1) \in \mathbb{C}[A_0, A_1]$. Furthermore, we have

$$\nabla f = \begin{pmatrix} \nabla f_0 + z \nabla f_1 \\ f_1 \end{pmatrix}.$$

Let $(x_0, y_0) \in (\mathbb{C}^*)^3$ be a multiple root of the system $f_0 = f_1 = 0$. This implies $\nabla f_0(x_0, y_0)$ and $\nabla f_1(x_0, y_0)$ to be linearly dependent and thus there exists $z_0 \in \mathbb{C}^*$ satisfying $\nabla f_0(x_0, y_0) + z_0 \nabla f_1(x_0, y_0) = (0, 0)$. In particular, one has $\nabla f(x_0, y_0, z_0) = (0, 0, 0)$. As (x_0, y_0, z_0) is clearly a root of f we conclude that $\Sigma_{A_0,A_1} \subseteq \Sigma_{\operatorname{cay}(A_0,A_1)}$. By Theorem 3.1.6, this and also the reversed inclusion hold in general (whenever $\operatorname{cay}(A_0, A_1)$ is not defective). However, the reversed inclusion is less direct as one has to be careful about the fact that the mixed discriminant by definition gives conditions for a system having a multiple root which is non-degenerate.

3.2. A necessary condition

The main contribution of this chapter is the following necessary condition for mixed defectivity in the case that all configurations of a family are full-dimensional. For $A \subset \mathbb{Z}^d$, we denote by $\langle A - A \rangle$ the subgroup of $\mathbb{Z}^d$ generated by the set $\{a_1 - a_2 \colon a_1, a_2 \in A\}$ and say that $A \subset \mathbb{Z}^d$ is spanning if $\langle A - A \rangle = \mathbb{Z}^d$. More generally, we say that a family $(A_0, \dots, A_k) \subset \mathbb{Z}^d$ is *spanning* if $\langle A_0 - A_0 \rangle + \dots + \langle A_k - A_k \rangle = \mathbb{Z}^d$.

Theorem 3.2.1. *Let $k \leq d$ and $A_0, \dots, A_k \subset \mathbb{Z}^d$ be full-dimensional configurations that form a spanning family. If $(A_0, \dots, A_k)$ is defective, then the convex hull of the Minkowski sum $A_0 + \dots + A_k$ does not have any interior lattice points, i.e.,*

$$\operatorname{int}(\operatorname{conv}(A_0 + \dots + A_k)) \cap \mathbb{Z}^d = \emptyset.$$

We refer to Section 3.4 for the proof of Theorem 3.2.1. The following result is a conjecture in [CCD+13]. It has been proven before in the 2-dimensional case as well as under additional smoothness assumptions.

Corollary 3.2.2. *Let $(A_0,\dots,A_{d-1}) \subset (\mathbb{Z}^d)^d$ be a spanning family of full-dimensional configurations. The tuple $(A_0,\dots,A_{d-1})$ is defective if and only if one has*

$$(A_0,\dots,A_{d-1}) \cong (\Delta_d \cap \mathbb{Z}^d,\dots,\Delta_d \cap \mathbb{Z}^d).$$

Proof. For any tuple $(f_1,\dots,f_d) \in \mathbb{C}[\Delta_d \cap \mathbb{Z}^d,\dots,\Delta_d \cap \mathbb{Z}^d]$, the system $f_1 = \cdots = f_d = 0$ is the intersection of d affine hyperplanes and one readily sees that such a system cannot have an isolated non-degenerate multiple root. Thus the tuple $(\Delta_d \cap \mathbb{Z}^d,\dots,\Delta_d \cap \mathbb{Z}^d)$, and therefore any equivalent tuple, is trivially defective. On the other hand, by Corollary 3.2 of [Nil20], the mixed volume of $(\operatorname{conv}(A_0),\dots,\operatorname{conv}(A_{d-1}))$ can be computed as

$$1 + \sum_{\emptyset \neq I \subseteq [d-1]} (-1)^{d-|I|} |\operatorname{int}(\operatorname{conv}(\sum_{i\in I} A_i)) \cap \mathbb{Z}^d|.$$

If $(A_0,\dots,A_{d-1})$ is defective, Theorem 3.2.1 implies $\operatorname{conv}(A_0+\cdots+A_{d-1})$ and therefore (as all A_i are full-dimensional) also $\operatorname{conv}(\sum_{i\in I} A_i)$ to have no interior lattice points for any $I \subseteq [d-1]_0$. This shows that the mixed volume of $(\operatorname{conv}(A_0),\dots,\operatorname{conv}(A_{k-1}))$ is 1. The claimed equivalence follows then from Proposition 2.7 of [CCD+13] (see also Proposition 5.1.5). □

Note that for given $(A_0,\dots,A_k) \subset (\mathbb{Z}^d)^{k+1}$ one may always choose a spanning family whose mixed discriminantal variety equals $\Sigma_{(A_0,\dots,A_k)}$ (see [GKZ94, Chapter 5, Proposition 1.2]). By applying a suitable transformation, this implies the following slightly more general version of Theorem 3.2.1.

Corollary 3.2.3. *Let $k \le d$ and $(A_0,\dots,A_k) \in (\mathbb{Z}^d)^{k+1}$ be full-dimensional configurations. Define $\Lambda = \langle A_0 - A_0\rangle + \cdots + \langle A_k - A_k\rangle$ the lattice spanned by these configurations. If $(A_0,\dots,A_k)$ is defective then*

$$\operatorname{int}((A_0 - a_0) + \cdots + (A_k - a_k)) \cap \Lambda = \emptyset,$$

for all choices $a_0,\dots,a_k$ such that $a_i \in A_i$ for all $i \in [k]_0$.

Remark 3.2.4. The statement of Theorem 3.2.1 is in general not true if we do not pose sufficient restrictions on the dimensions of the configurations. A counterexample is provided by choosing $(A_0,A_1) \subset (\mathbb{Z}^2)^2$ as

$$A_0 = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$

It is straightforward to verify that the corresponding system

$$f_0 = c_{0,00} + c_{0,10}x_1 + c_{0,20}x_1^2, \;\; f_1 = c_{1,00} + c_{1,01}x_2 + c_{1,02}x_2^2,$$

does not have a non-degenerate multiple root for any choice of coefficients. So the variety $\Sigma_{(A_0,A_1)}$ is empty and therefore (A_0,A_1) is a defective family, while $\operatorname{conv}(A_0+A_1)$ contains $(1,1)$ as an interior lattice point.

In general, the criterion for defectivity given in Theorem 3.2.1 is not sufficient. An easy class of counterexamples is given for $k = 0$ by $A_0 = \mathrm{conv}(d\Delta_d) \cap \mathbb{Z}^d$ for $d > 1$. Clearly, $\mathrm{conv}(A_0)$ does not have any interior lattice points but cannot be defective since its lattice width is $d > 1$ and it therefore does not have any non-trivial Cayley decomposition (see Theorem 3.3.3). One mixed class of counterexamples is the following.

Example 3.2.5. Consider a pair $(A_0, A_1) \subset (\mathbb{Z}^3)^2$, with $A_0 = \mathrm{cay}(I_1, I_2, I_3)$ and $A_1 = \mathrm{cay}(J_1, J_2, J_3)$ for 1-dimensional configurations $I_1, \dots, I_3, J_1, \dots, J_3 \subset \mathbb{Z}^1$. It is explained in Chapter 5 that one has $\mathrm{int}(\mathrm{conv}(A_0 + A_1)) = \emptyset$ (see Proposition 5.2.2). However, as can be seen using Theorem 3.1.6 and the criterion by Esterov (Conjecture 3.20 in [Est10], which is proven in [Est18a]), the tuple (A_0, A_1) is not mixed defective.

3.3. A characterization of defective configurations in terms of special Cayley sums

The crucial tool in proving Theorem 3.2.1 is a characterization of defective configurations by Furukawa-Ito. The central notion in this characterization is the following special kind of Cayley sum.

Definition 3.3.1 (Cayley sums of join type)**.** Let $A_0, \dots, A_k \subset \mathbb{Z}^d$ be configurations. We say that the Cayley sum $\mathrm{cay}(A_0, \dots, A_k)$ is *of join type* if the homomorphism

$$\begin{gathered}\langle A_0 - A_0\rangle \oplus \cdots \oplus \langle A_k - A_k\rangle \to \langle A_0 - A_0\rangle + \cdots + \langle A_k - A_k\rangle \subset \mathbb{Z}^d \\ (a_0, \dots, a_k) \mapsto a_0 + \cdots + a_k,\end{gathered}$$

is injective.

Remark 3.3.2. As one has $\mathrm{aff}(\langle A - A\rangle) = \mathrm{aff}(A)$, we deduce

$$\begin{aligned}\dim(\langle A_0 - A_0\rangle \oplus \cdots \oplus \langle A_k - A_k\rangle) &= \dim(A_0) + \cdots + \dim(A_k),\\ \dim(\langle A_0 - A_0\rangle + \cdots + \langle A_k - A_k\rangle) &= \dim(A_0 + \cdots + A_k).\end{aligned}$$

Therefore a spanning Cayley sum $\mathrm{cay}(A_0, \dots, A_k)$ is of join type if and only if

$$\dim(A_0) + \cdots + \dim(A_k) = \dim(A_0 + \cdots + A_k).$$

See Figure 3.1 for two examples of Cayley sums of join type. Let us now present the characterization by Furukawa-Ito.

Theorem 3.3.3 ([FI20, Theorem 1.3])**.** *Let $A \subset \mathbb{Z}^d$ be a spanning configuration. Then A is defective if and only if there exist natural numbers $c < r$ and a lattice projection $\pi\colon \mathbb{R}^d \to \mathbb{R}^{d-c}$ such that $\pi(A) \cong \mathrm{cay}(B_0, \dots, B_r)$ where the Cayley sum $\mathrm{cay}(B_0, \dots, B_r)$ is of join type and $B_i \neq \emptyset$ for all $i \in [r]_0$.*

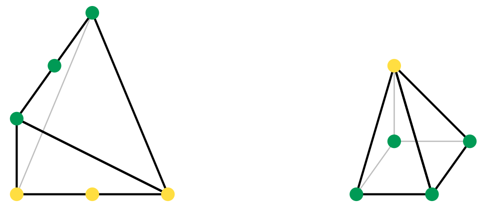


Figure 3.1.: The Cayley sums of join type given by $\mathrm{cay}(A_0, A_1)$ and $\mathrm{cay}(B_0, B_1)$ for the configurations A_0, A_1 from Remark 3.2.4 and $B_1 = \{(0,0),(1,0),(0,1),(1,1)\}$, $B_0 = \{0\}$.

Remark 3.3.4. Let $A \subset \mathbb{Z}^d$ be a spanning configuration that is defective. If π is a lattice projection as in Theorem 3.3.3, then $\pi(A)$ has a lattice projection onto $\Delta_r \cap \mathbb{Z}^r$ by Proposition 2.1.4. Composing π and this lattice projection yields that A itself has a Cayley decomposition of length $r+1$ and therefore is of the form $A \cong \mathrm{cay}(A_0, \ldots, A_r)$.

In order to give an intuition for Theorem 3.3.3, let us present some examples of classes of point configurations that can be shown to be defective using this characterization.

Example 3.3.5.

1. Consider the spanning configuration $\Delta_d \cap \mathbb{Z}^d$. It holds that $\Delta_d \cap \mathbb{Z}^d \cong \mathrm{cay}(\{z_0\}, \ldots, \{z_d\})$ for any choice of integers $z_0, \ldots, z_d \in \mathbb{Z}$ and this Cayley sum is trivially of join type as $\dim(\{z_0\}) + \cdots + \dim(\{z_d\}) = 0 = \dim(\{z_0\} + \cdots + \{z_d\})$. Therefore, the condition of Theorem 3.3.3 is satisfied for $\pi\colon \mathbb{R}^d \to \mathbb{R}^d$ being the identity, $c = 0$, and $r = d$. Hence, $\Delta_d \cap \mathbb{Z}^d$ is a defective configuration.

2. More generally, any lattice pyramid $\mathrm{cay}(A', \{\mathbf{0}\})$ over a spanning configuration $A' \subset \mathbb{Z}^d$ is a Cayley sum of join type and therefore defective, and so is the *join* $B * B' = \mathrm{cay}(B \times \{\mathbf{0}\}, \{\mathbf{0}\} \times B')$ of any two spanning configurations $B \subset \mathbb{Z}^{d_1}$ and $B' \subset \mathbb{Z}^{d_2}$.

3. Consider a spanning configuration satisfying $A \cong \mathrm{cay}(A_0, \ldots, A_k)$, where $A_0, \ldots, A_k \subset \mathbb{Z}^d$ for some $d < k$. Then there exists a lattice projection $\pi\colon \mathbb{Z}^{d+k} \to \mathbb{Z}^k$ with $\pi(A) = \Delta_k \cap \mathbb{Z}^k$. As $\Delta_k \cap \mathbb{Z}^k$ is a Cayley sum of join type, the condition of Theorem 3.3.3 is satisfied with $c = d$ and $r = k$. Therefore A is defective.

4. Let $I_0, \ldots, I_{d-1} \subset \mathbb{Z}^1$ be 1-dimensional configurations and consider the spanning configuration $A = \mathrm{cay}(I_0, \ldots, I_{d-1}) \subset \mathbb{Z}^d$. The lattice projection $\pi\colon \mathbb{R}^d \to \mathbb{R}^{d-1}$ onto all but the first coordinate satisfies $\pi(A) = \Delta_{d-1}$ and therefore the condition of Theorem 3.3.3 is satisfied with $c = 1$ and $r = 2$. So A is defective.

3.4. Proof of Theorem 3.2.1

The overall idea of the proof is the following. On the one hand, we may use Theorem 3.1.6 to reduce the question of defectivity of a family to the question of defectivity of the Cayley sum of its elements. On the other hand, the characterization of Theorem 3.3.3 tells us that a configuration is defective if and only if it has a Cayley decomposition with certain special properties. So the question boils down to asking whether a configuration that is constructed as the Cayley sum of full-dimensional configurations (namely one, where the Minkowski sum of the summands contains an interior lattice point) can have another special Cayley decomposition as required by Theorem 3.3.3. As it turns out, the following lemma yields exactly the required restrictions on configurations with two different Cayley decompositions.

Lemma 3.4.1. *Let $A_0, \ldots, A_k \subset \mathbb{Z}^d$ be full-dimensional configurations and let $B_0, \ldots, B_r \subset \mathbb{Z}^{d+k-r}$ be non-empty configurations such that*

$$\operatorname{cay}(A_0, \ldots, A_k) \cong \operatorname{cay}(B_0, \ldots, B_r) \subset \mathbb{Z}^{d+k}.$$

(a) One has $\dim(B_i) \geq \min(k, d)$ for all $i \in [r]_0$.

(b) If furthermore $\dim(B_i) < d$ for all $i \in [r]_0$, also the following inequality holds:

$$\dim(B_0) + \cdots + \dim(B_r) \geq d - r + (r+1)k.$$

Proof. For $i \in [r]_0$, denote by $\hat{B}_i = B_i \times \{e_i\}$ (where we set $e_0 = \mathbf{0} \in \mathbb{Z}^r$) the faces of $\operatorname{cay}(B_0, \ldots, B_r)$ corresponding to the Cayley summands. For $k = 0$ and for $r = 0$, one can directly verify that both statements hold. So we may assume $k, r \geq 1$, in which case $\hat{B}_i$ is a proper face. Denote by $B_i' \in \mathcal{F}(\operatorname{cay}(A_0, \ldots, A_k))$ the corresponding proper face of $\operatorname{cay}(A_0, \ldots, A_k)$ under the equivalence between $\operatorname{cay}(A_0, \ldots, A_k)$ and $\operatorname{cay}(B_0, \ldots, B_r)$. The faces $B_0', \ldots, B_r'$ form a Cayley decomposition of $\operatorname{cay}(A_0, \ldots, A_k)$ (since the $\hat{B}_i$ form a Cayley decomposition of $\operatorname{cay}(B_0, \ldots, B_r)$). Furthermore, also the complement $(B_i')^c = \operatorname{cay}(A_0, \ldots, A_k) \setminus B_i'$ of each of the B_i' is again a proper face of $\operatorname{cay}(A_0, \ldots, A_k)$ (see Remark 2.2.4). Let now $i \in [r]_0$ be arbitrary and assume $\dim(B_i) < d$ (otherwise (a) holds trivially). This assumption implies that B_i' cannot contain $\hat{A}_j$ for any $j \in [k]_0$ and thus $(B_i')^c$ has non-empty intersection with each of the $\hat{A}_j$. As, by Remark 2.2.5, one has $(B_i')^c = ((B_i')^c \cap \hat{A}_0) \cup \cdots \cup ((B_i')^c \cap \hat{A}_k)$, that implies $\dim(B_i')^c \geq \dim((B_i')^c \cap \hat{A}_j) + k$ for each $j \in [k]_0$. If now $(B_i')^c$ contained one of the $\hat{A}_j$, this inequality would imply $\dim(B_i')^c \geq d + k$ in contradiction to $(B_i')^c$ being a proper face of the $(d+k)$-dimensional Cayley sum $\operatorname{cay}(A_0, \ldots, A_k)$. Hence, also B_i' has non-empty intersection with each of the $\hat{A}_j$ and, as we have

$$B_i' = (\hat{A}_0 \cap B_i') \cup \cdots \cup (\hat{A}_k \cap B_i'),$$

this implies

$$\dim(\hat{A}_j \cap B_i') \leq \dim(B_i') - k, \tag{3.1}$$

for all $j \in [k]_0$ (see Remark 2.2.5). This in particular implies $\dim(B_i) = \dim(B_i') \geq k \geq \min(k, d)$, proving part (a).

Moreover, as also the B_i' form a Cayley decomposition of $\operatorname{cay}(A_0, \dots, A_k)$, Remark 2.2.5 yields

$$\hat{A}_j = (\hat{A}_j \cap B_0') \cup \dots \cup (\hat{A}_j \cap B_r'),$$

for each $j \in [k]_0$. Therefore assuming $\dim(B_i) < d$ for all $i \in [r]_0$ and applying (3.1) yields

$$\begin{aligned} d = \dim(\hat{A}_j) &\leq r + \dim(\hat{A}_j \cap B_0') + \dots + \dim(\hat{A}_j \cap B_r') \\ &\leq r + \dim(B_0') - k + \dots + \dim(B_r') - k. \end{aligned}$$

□

Note that the proof of Lemma 3.4.1 is not actually using the property of having configurations of lattice points anywhere. Therefore the result above remains true in the more general setting of point configurations in $\mathbb{R}^d$ and the notion of isomorphy induced by affine bijections.

Proof of Theorem 3.2.1. It is a straightforward computation to show that a tuple $(A_0, \dots, A_k) \subset (\mathbb{Z}^d)^{k+1}$ is spanning if and only if the Cayley sum $\operatorname{cay}(A_0, \dots, A_k) \subset \mathbb{Z}^{d+k}$ is spanning. Combining this with Theorem 3.1.6, we see that $\operatorname{cay}(A_0, \dots, A_k) \subset \mathbb{Z}^{d+k}$ is a spanning defective configuration. By Theorem 3.3.3 there exist $c < r$ and a lattice projection $\pi \colon \mathbb{R}^{d+k} \to \mathbb{R}^{d+k-c}$ such that $\pi(\operatorname{cay}(A_0, \dots, A_k))$ has a Cayley decomposition of join type into non-empty faces $F_0, \dots, F_r \in \mathcal{F}(\pi(\operatorname{cay}(A_0, \dots, A_k))$. Let us assume that $\operatorname{conv}(A_0 + \dots + A_k)$ has interior lattice points. By Corollary 2.2.6 (1), this is equivalent to $(k+1) \cdot \operatorname{conv}(\operatorname{cay}(A_0, \dots, A_k))$ having an interior point in $\mathbb{Z}^{d+k}$. By Proposition 2.1.4 we have a projection $\pi_r \colon \mathbb{R}^{d+k-c} \to \mathbb{R}^r$ that maps $\pi(\operatorname{cay}(A_0, \dots, A_k))$ surjectively onto $\Delta_r \cap \mathbb{Z}^r$. As any lattice projection maps interior lattice points of a polytope to interior lattice points of its image, we obtain that $(k+1)\Delta_r = \pi_r(\pi(\operatorname{cay}(A_0, \dots, A_k)))$ has interior lattice points. This implies

$$k \geq r, \tag{3.2}$$

as $(k+1)\Delta_r \cap \mathbb{Z}^r \neq \emptyset$ if and only if $k+1 \geq r+1$ (see e.g. Theorem 5.1.2 (2)). We observe now that the lifts

$$\tilde{F}_i := \pi^{-1}(F_i) \cap \operatorname{cay}(A_0, \dots, A_k)$$

define a Cayley decomposition (in general not of join type) of $\operatorname{cay}(A_0, \dots, A_k)$. As π is a projection of codimension c, we see

$$\dim(\tilde{F}_i) \leq \dim(F_i) + c, \tag{3.3}$$

for all $i \in [r]_0$. Combining this with the fact that the F_i form a Cayley decomposition of join type and using Remark 3.3.2, one obtains

$$\begin{aligned} \dim(\tilde{F}_0) + \dots + \dim(\tilde{F}_r) &\leq \dim(F_0) + \dots + \dim(F_r) + c(r+1) \\ &= \dim(F_0 + \dots + F_r) + c(r+1) \\ &= d + k - c - r + c(r+1) \\ &= d + k + r(c-1). \end{aligned}$$

Let us first assume $\dim(\tilde{F}_j) \geq d$ for some $j \in [r]_0$. Then $\dim(F_j) \geq d - c$. Without loss of generality let $j = 0$. As the F_i form a Cayley decomposition of join type of the $(d+k-c)$-dimensional configuration $\pi(\operatorname{cay}(A_0, \dots, A_k))$ we have the following inequality for the remaining summands:

$$\begin{aligned} \dim(F_1) + \cdots + \dim(F_r) &= \dim(F_0 + \cdots + F_r) - \dim(F_0) \\ &= d + k - c - r - \dim(F_0) \\ &\leq d + k - c - r - (n - c) \\ &= k - r. \end{aligned}$$

However, on the other hand, Lemma 3.4.1 (a) implies $\dim(\tilde{F}_i) \geq k$ for all $i \in [r]_0$ (since we assumed $k \leq d$). So by (3.3) we have $\dim(F_i) \geq k - c$ which yields another inequality for the remaining summands:

$$\dim(F_1) + \cdots + \dim(F_r) \geq r(k - c).$$

These inequalities contradict each other since $r(k-c) > k - r$, which can be seen by observing that r is strictly positive and c is strictly smaller than r.

So we only need to deal with the case in which $\dim(\tilde{F}_j) < d$ for all $j \in [r]_0$. We may apply part (b) of Lemma 3.4.1 and obtain $d - r + (r+1)k \leq \dim(\tilde{F}_0) + \cdots + \dim(\tilde{F}_r)$. Hence,

$$d - r + (r+1)k \leq d + k + r(c-1),$$

which is (since r is strictly positive) equivalent to $k \leq c < r$; a contradiction. □

3.5. Outlook

Let us conclude this chapter by presenting questions and suggestions for further research.

Theorem 3.2.1 for irreducible tuples

There has lately been given another proof of Corollary 3.2.2 by Esterov, which is in fact more general. We say that a tuple $(A_0, \dots, A_{d-1}) \subset (\mathbb{Z}^d)^d$ is *irreducible* if it contains no l distinct configurations that can be shifted to the same l-dimensional affine space in $\mathbb{R}^d$ for any $l \in \{1, \dots, d-1\}$ (see Definition 4.2.2 for another equivalent definition).

Theorem 3.5.1 ([Est19, Corollary 3.23])**.** *Let $(A_0, \dots, A_{d-1}) \subset (\mathbb{Z}^d)^d$ be a spanning irreducible tuple. Then $(A_0, \dots, A_{d-1})$ is defective if and only if it is contained in the tuple $(\Delta_d \cap \mathbb{Z}^d, \dots, \Delta_d \cap \mathbb{Z}^d)$, up to equivalence.*

However, Theorem 3.5.1 does not generalize Theorem 3.2.1, as it only treats the case of $k = d - 1$. It would be interesting to investigate whether the assumption of full-dimensionality in Theorem 3.2.1 can always be replaced by irreducibility of the family. Irreducibility for a general tuple $(A_0, \dots, A_k) \subset (\mathbb{Z}^d)^{k+1}$, where $k \leq d$ is defined as follows. We call a tuple $(A_0, \ldots, A_k) \subset (\mathbb{Z}^d)^{k+1}$ *irreducible* if no l distinct members can be shifted to a common $(l + (d - 1 - k))$-dimensional affine subspace of $\mathbb{R}^d$ for any $l \in \{1, \ldots, d-1\}$.

Question 3.5.2. Let $(A_0, \ldots, A_k) \subset (\mathbb{Z}^d)^{k+1}$ be a spanning irreducible tuple which is defective. Is it true that this implies

$$\operatorname{int}(\operatorname{conv}(A_0 + \cdots + A_k)) \cap \mathbb{Z}^d = \emptyset?$$

Note that Remark 3.2.4 shows that one cannot expect Theorem 3.2.1 to hold without any restrictions on the dimensions of the involved configurations.

Different Cayley decompositions

It turns out that the main combinatorial work for deriving Theorem 3.2.1 is to provide restrictions on different Cayley decompositions of one point configuration. The crucial result in this regard is Lemma 3.4.1. Given a configuration A that has a Cayley decomposition into $k+1$ faces $F_0, \ldots, F_k$ satisfying $\dim(F_i) = \dim(F_0 + \cdots + F_k)$ for all $i \in [k]_0$, Lemma 3.4.1 provides restrictions on any *other* Cayley decomposition that A can have. More concretely, if $G_0, \ldots, G_r$ is another Cayley decomposition of A, there exists a lower bound on the dimension of each single face G_i and also a lower bound on the sum $\dim(B_0) + \cdots + \dim(B_r)$. Theorem 2.3.1 is in the same spirit, providing conditions for a Cayley decomposition to be the unique one of a given length. It would be interesting to provide more results along those lines.

Apart from being a combinatorially interesting question in itself, progress in this direction may lead to a generalization of Theorem 3.2.1 to weaker assumptions on the configurations then all of them being full-dimensional (in particular, to an answer for Question 3.5.2).

Another motivation for this direction of research comes from the study of the *Fano scheme* $\mathbf{F}_k(X_A)$ of the projective toric variety associated to a configuration A. In [IZ17], Ilten and Zotine show that the irreducible components of $\mathbf{F}_k(X_A)$ correspond to *maximal Cayley structures* of length at least k of the configuration A. What they call a Cayley structure of A is in our notation a Cayley decomposition *of a face* F of A into non-empty faces. Maximality of a Cayley structure roughly means that the face F cannot be extended to another face G in a way that respects the Cayley decomposition of F, and that the Cayley decomposition of F cannot be refined. As a first step it would be interesting to investigate the implications of Lemma 3.4.1 and Theorem 2.3.1 from this point of view. For example, it seems reasonable to expect sufficient conditions for $\mathbf{F}_k(X_A)$ to be irreducible from Theorem 2.3.1.

4. Classification of Tuples of Small Mixed Volume

In this chapter we present an algorithm to classify triples of lattice polytopes of a given mixed volume and present the results of an implementation for mixed volume at most 4. In Section 4.1 we provide some background on the more classical problem of the classification of certain classes of single lattice polytopes. Section 4.2 introduces fundamental preliminaries for the classification of tuples of lattice polytopes. In particular, we illustrate how the classification of tuples of lattice polytopes can be reduced to the enumeration of so-called irreducible tuples. We furthermore present the result of Esterov that the number of such tuples is finite up to equivalence once we fix an upper bound on the mixed volume (Theorem 4.2.7). We then proceed to introduce two different notions of maximality of tuples of polytopes (Definition 4.2.8), illustrate their relations and show criteria for maximality in terms of mixed area measures. Having introduced these foundations, we present the result of an implementation of our algorithm in Section 4.3 (Theorem 4.3.1), which is a complete list of maximal irreducible triples of mixed volume at most 4. Furthermore, we describe the structure of three special classes that cover almost all maximal tuples of our enumeration that are full-dimensional (Proposition 4.3.4). Sections 4.4 and 4.5 are dedicated to presenting the algorithm for our enumeration in the special case of full-dimensional triples and its extension to general irreducible triples, respectively. Detailed descriptions of how we carry out certain steps of the algorithms are given in Section 4.6. Section 4.7 contains the description of an algorithm for the enumeration of lattice polytopes of a given volume in fixed dimension. This is due to the fact that our enumeration relies on having a list of all lattice polytopes in dimensions 2 and 3 of volume at most 4. In Section 4.8 we conclude with an outlook about ideas for making partial classification tasks tractable also for larger mixed volume and in higher dimension.

4.1. Background: classification of lattice polytopes

Classification results comprise an important part of algebraic geometry, and in a number of cases, especially in toric geometry, such results have been established by accomplishing related classification tasks in the context of the theory of lattice polytopes (see, for example, [KS98, Kas10, NØ10, BHH+15, IVS18]). Classification usually means enumeration of a finite number of equivalence classes and at the foundation of such classifications there usually stands a finiteness result for a certain set of lattice polytopes. One of the most fundamental ones is that, up to equivalence, there exist only finitely many lattice polytopes of volume lower than a given constant.

Theorem 4.1.1 ([LZ91, Theorem 2])**.** *Let $P \in \mathcal{P}(\mathbb{Z}^d)$ be a full-dimensional lattice polytope with volume* $\mathrm{Vol}(P) \leq K$ *for some positive number K. Then there exists a unimodular transformation $\varphi \in \mathrm{Aff}(\mathbb{Z}^d)$, such that $\varphi(P) \subseteq (d \cdot d!K)\square_d$.*

So, whenever one can show a bound on the volume of a certain set of lattice polytopes, Theorem 4.1.1 yields that there are only finitely many equivalence classes of lattice polytopes to enumerate. Classical examples for such sets are the ones of full-dimensional lattice polytopes $\mathcal{P}(\mathbb{Z}^d)$ with a fixed positive number of interior lattice points. Here, the boundedness of the volume among each of these sets has been shown by Hensley in [Hen83]. Moreover, Theorem 4.1.1 provides a concrete bounding box in which at least one representative of each equivalence class of a given volume is contained. So in order to enumerate a certain set of lattice polytopes for which one knows a volume bound, one could theoretically go through all lattice polytopes inside a large enough cube. This naive approach is usually computationally infeasible.

What has been shown to be a more effective approach for the enumeration of lattice polytopes in a given dimension up to a certain volume are algorithms of the following structure. One starts with a (easily classifiable) subset of minimal polytopes, determines a set of candidate points for each such polytope (that is points that can be added to the polytope without exceeding a certain volume bound) and then iteratively build up all lattice polytopes up to a certain volume. One such algorithm has been developed by Gabriele Balletti and has been used to exhaustively classify lattice polytopes of small volumes in dimension up to 6 ([Bal18]). In Section 4.7, we present an independent algorithm with a similar structure that we also modify in order to enumerate lattice polytopes with further restrictions.

Theorem 4.1.1 shows to also play an important role in the classification of tuples of lattice polytopes of a given mixed volume, which is the central topic of this chapter. However, there are various aspects in which classification questions for tuples differ from the setting of single lattice polytopes. The following section is devoted to providing theoretical foundations for the classification of tuples of lattice polytopes of a given mixed volume.

4.2. Theoretical foundations

Motivated by the idea of classifying all supports $(A_1, \ldots, A_d)$ of generic systems with exactly m solutions, Esterov and Gusev started carrying out classifications for tuples of lattice polytopes of a given mixed volume [EG15, EG16]. This task is a true generalization of the classification of single lattice polytopes of a given volume, as $\mathrm{MV}(P, \ldots, P) = \mathrm{Vol}(P)$, given that $P \in \mathcal{P}(\mathbb{Z}^d)$ is full-dimensional. The general classification problem can be stated in the following way.

Classification Problem 4.2.1. Given $d, m \in \mathbb{Z}_{\geq 1}$, describe all d-tuples of lattice polytopes $(P_1, \ldots, P_d) \in \mathcal{P}(\mathbb{Z}^d)^d$ whose normalized mixed volume equals m.

In general, the number of tuples of lattice polytopes of a given mixed volume in fixed dimension is not finite. Yet, there exists a finite number of so-called *irreducible* tuples and every other tuple can be constructed from a finite number of irreducible ones. In order to make this precise, let us introduce some definitions.

Definition 4.2.2. We say that a d-tuple $(P_1, \dots P_d) \in \mathcal{P}(\mathbb{Z}^d)^d$ is *non-degenerate* if for every $I \subseteq [d]$ with $1 \le |I| \le d$ the dimension of $\sum_{i \in I} P_i$ is at least $|I|$.

We say that a d-tuple $(P_1, \dots P_d) \in \mathcal{P}(\mathbb{Z}^d)^d$ is *irreducible* if for every $I \subseteq [d]$ with $1 \le |I| < d$ the dimension of $\sum_{i \in I} P_i$ is at least $|I| + 1$.

Note that both the notion of being non-degenerate and the one of being irreducible merely depend on the affine hulls of the polytopes $P_1, \dots, P_k$ in a tuple. The reason that we introduce the notion of being non-degenerate is the following.

Proposition 4.2.3 ([Sch14, Theorem 5.1.8])**.** *Let* $(P_1, \dots, P_d) \in \mathcal{P}(\mathbb{Z}^d)^d$*. Then* $\mathrm{MV}(P_1, \dots, P_d) \ge 1$ *if and only if* $(P_1, \dots, P_d)$ *is non-degenerate.*

Let us make precise in which way irreducible tuples form the building blocks of all tuples.

Proposition 4.2.4 ([Sch14, Theorem 5.3.1])**.** *Let* $(P_1, \dots, P_d) \in \mathcal{P}(\mathbb{Z}^d)^d$ *be a tuple such that, for some* $k \in [d]$*, the polytopes* $P_1, \dots, P_k$ *can be shifted to a* k*-dimensional linear subspace* $L \subseteq \mathbb{R}^d$ *and let* $\pi_L : \mathbb{R}^d \to \mathbb{R}^{d-k}$ *be a lattice projection with* $\ker \pi_L = L$*. Then*

$$\mathrm{MV}(P_1, \dots, P_k, P_{k+1}, \dots, P_d) = \mathrm{MV}(P_1, \dots, P_k)\,\mathrm{MV}(\pi_L(P_{k+1}), \dots, \pi_L(P_d)). \quad (4.1)$$

Note that in [Sch14] the above theorem is proven for euclidean mixed volumes and orthogonal projections. It is straightforward to derive our normalized version from this (see also [Est19, Theorem 1.10]).

Remark 4.2.5. Note that a tuple $(P_1, \dots, P_d) \in \mathcal{P}(\mathbb{Z}^d)^d$ is non-irreducible if and only if there exists an index set $\{i_1, \dots, i_k\} \subsetneq [d]$, such that $P_{i_1}, \dots, P_{i_k}$ can be shifted to a common k-dimensional linear subspace of $\mathbb{R}^d$. Any non-irreducible tuple therefore satisfies the conditions of Proposition 4.2.4, up to reordering, and its mixed volume can be expressed as a product of two lower-dimensional mixed volumes. Proposition 4.2.4 also makes clear that one cannot expect the number of tuples of lattice polytopes of a given mixed volume to be finite without excluding non-irreducible ones. Indeed, if we can decompose a tuple $(P_1, \dots, P_d) \in \mathcal{P}(\mathbb{Z}^d)^d$ as in (4.1), we can extend any of the polytopes $P_{k+1}, \dots, P_d$ with arbitrarily many lattice points along L without changing the image under π_L and therefore without changing the mixed volume of the whole tuple.

Let us illustrate Proposition 4.2.4 and Remark 4.2.5 with the following example.

Example 4.2.6. Consider the triple $(P_1, P_2, P_3) \in \mathcal{P}(\mathbb{Z}^3)^3$, where P_1, P_2 are the 2-dimensional polytopes $P_1 = \mathrm{conv}(0, 2e_1, e_2)$, $P_2 = \mathrm{conv}(0, e_1, 2e_2)$ in $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$, and P_3 is a 3-dimensional lattice polytope that is contained in the slab $\mathbb{R}^2 \times [0,1]$ of height 1 (see Fig. 4.1). The triple (P_1, P_2, P_3) is not irreducible since the sum $P_1 + P_2$ is 2-dimensional. By Proposition 4.2.4, the normalized mixed volume of (P_1, P_2, P_3) is the product of the normalized mixed volume of the pair (P_1, P_2), which is equal to 4 (see Example 1.2.4), multiplied by the height 1 of the polytope P_3. Thus we obtain a triple of mixed volume 4 independent of the concrete choice of P_3.

This calculation can also be interpreted in the light of the BKK-theorem. The triple (P_1, P_2, P_3) corresponds to systems $f_1(x, y) = f_2(x, y) = f_3(x, y, z) = 0$ of polynomials $(f_1, f_2, f_3) \in \mathbb{C}[P_1, P_2, P_3]$, where $f_1(x, y) = f_2(x, y) = 0$ is a sub-system depending only on x and y. Since the equation $f_3(x, y, z) = 0$ is linear in z, it can be written as

$$f_3(x, y, z) = a(x, y)z + b(x, y) = 0$$

for some $a, b \in \mathbb{C}[x, x^{-1}, y, y^{-1}]$. If the triple (f_1, f_2, f_3) is generic, the sub-system $f_1(x, y) = f_2(x, y) = 0$ has 4 solutions in $(\mathbb{C}^*)^2$ (see Example 1.2.4). Furthermore, plugging each of these four solutions into $f_3(x, y, z) = 0$ yields univariate linear equations in z, each generically having 1 solution in $\mathbb{C}^*$. Thus, we arrive at the system $f_1(x, y) = f_2(x, y) = f_3(x, y, z) = 0$ having 4 solutions in $(\mathbb{C}^*)^3$.

Iterated application of Proposition 4.2.4 shows how the classification of d-tuples of lattice polytopes in $\mathbb{R}^d$ of a given mixed volume m reduces to the classification of irreducible tuples in dimensions d' and mixed volume m' for all $d' \in [d]$ and all m' being divisors of m.

The following result by Esterov shows that the number of irreducible tuples in fixed dimension is finite, making Classification Problem 4.2.1 a finite problem.

Theorem 4.2.7 ([Est19, Theorem 1.7]). *Given $m \in \mathbb{N}$, there exist finitely many irreducible d-tuples $(P_1, \dots, P_d) \in \mathcal{P}(\mathbb{Z}^d)^d$ with $\mathrm{MV}(P_1, \dots, P_d) = m$, up to equivalence.*

Sketch of proof. The proof of this result uses inequalities between different mixed volumes to bound the volume of the Minkowski sum $P = P_1 + \dots + P_d$ in terms of the mixed volume $\mathrm{MV}(P_1, \dots, P_d)$. [1] We refer to Chapter 6 for a detailed treatment of the question for a sharp bound for this volume in the case in which all polytopes are full-dimensional. By Theorem 4.1.1, there exists a unimodular transformation $\varphi \in \mathrm{Aff}(\mathbb{Z}^d)$ such that $\varphi(P)$ is contained in a cube depending only on the volume of P. This implies $\varphi(P_1) + t_1, \dots, \varphi(P_d) + t_d$ to also be contained in this cube, where $t_1, \dots, t_d \in \mathbb{Z}^d$ are lattice translations. As one has

$$(\varphi(P_1) + t_1, \dots, \varphi(P_d) + t_d) \cong (P_1, \dots, P_d),$$

this proves the claim. □

[1] In the version of [Est19] available at the moment of the final review of this thesis it is claimed that Aleksandrov-Fenchel inequalities (see Proposition 1.2.2 (5.)) suffice to bound the volume of the Minkowski sum. The author, whom we would like to thank for his prompt help and collaboration, was not able to reconstruct the details of this argument on request and made a different proof of an upper bound available as an addendum in the arXiv version of the paper ([Est18b]). Note that the new proof additionally uses what we call square inequalities (see Lemma 6.5.1) and that it remains an interesting open question whether the Minkowski sum can be bounded in terms of the mixed volume only using Aleksandrov-Fenchel inequalities.

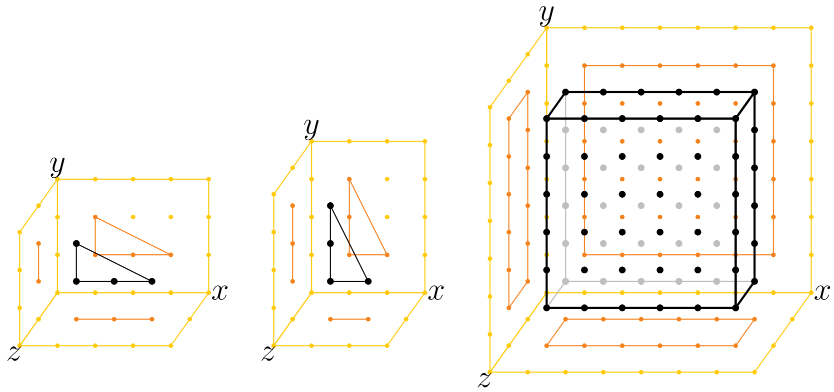


Figure 4.1.: An example of a non-irreducible triple in dimension three.

4.2.1. $\mathbb{Z}$-maximal and $\mathbb{R}$-maximal tuples

Recall that the mixed volume is monotonic with respect to inclusion, see Proposition 1.2.2 (1.2). Unlike the volume of a single polytope, however, it is not strictly monotonic. So, given a tuple of lattice polytopes $(P_1, \ldots, P_d) \in \mathcal{P}(\mathbb{Z}^d)^d$ of mixed volume $\mathrm{MV}(P_1, \ldots, P_d) = m$ there might exist a larger tuple $(Q_1, \ldots, Q_d) \in \mathcal{P}(\mathbb{Z}^d)^d$ with $P_1 \subsetneq Q_1, P_2 \subseteq Q_2, \ldots, P_d \subseteq Q_d$ of the same mixed volume $\mathrm{MV}(Q_1, \ldots, Q_d) = m$. One example of this situation in dimension 2 is shown in Figure 4.2. We want to investigate the structure of tuples that are maximal, in the sense that there does not exist such a larger tuple of the same mixed volume. As by Theorem 4.2.7 the number of irreducible d-tuples of lattice polytopes of a given mixed volume is finite, every irreducible tuple has to be contained in such a maximal irreducible one. This provides an interesting perspective for the classification problem of tuples of a given mixed volume. In a certain sense, enumeration of irreducible d-tuples can be reduced to enumeration of maximal irreducible d-tuples. We will see that maximal d-tuples tend to have an easier structure than general ones. Furthermore, in Algorithms 4.4.4 and 4.5.3, we make use of the maximality assumption and classify specifically such tuples. Let us define the exact notion of maximality in some more generality.

$P_1 = \mathrm{conv}(0, 2e_1, e_2)$ $\qquad$ $P_2 = \mathrm{conv}(0, e_1, 2e_2) \subset 2P_1$

Figure 4.2.: Embedding the pair (P_1, P_2) from Example 1.2.4 into a maximal pair by enlarging P_2 to $2P_1$. Since $\mathrm{MV}(P_1, 2P_1) = 2\,\mathrm{MV}(P_1, P_1) = 2\,\mathrm{Vol}(P_1) = 4$, the normalized mixed volume remains unchanged.

Definition 4.2.8. Let $i \in [d]$ and $R \in \{\mathbb{R}, \mathbb{Z}\}$. A d-tuple $(P_1, \dots, P_d) \in \mathcal{P}(\mathbb{Z}^d)^d$ is called *R-maximal in P_i*, if for all $Q_i \in \mathcal{P}(R^d)$ with $P_i \subseteq Q_i$ the equality

$$\mathrm{MV}(P_1, \dots, P_{i-1}, P_i, P_{i+1}, \dots, P_d) = \mathrm{MV}(P_1, \dots, P_{i-1}, Q_i, P_{i+1}, \dots, P_d) \qquad (4.2)$$

implies $P_i = Q_i$. We call $(P_1, \dots, P_d)$ *R-maximal* if it is R-maximal in each of the polytopes $P_1, \dots, P_d$.

In view of the inclusion $\mathbb{Z} \subset \mathbb{R}$, whenever a d-tuple of lattice polytopes is $\mathbb{R}$-maximal, then it is also $\mathbb{Z}$-maximal. The converse is not true in general as Figs. 4.3 and 4.4 illustrate.

$A = \mathrm{conv}(0, 2e_1, e_2)$ $\qquad$ $B = \mathrm{conv}(0, 3e_1, e_1 + e_2, e_2)$

Figure 4.3.: A pair (A, B) of lattice polygons, which is $\mathbb{R}$-maximal in A and $\mathbb{Z}$-maximal but not $\mathbb{R}$-maximal in B. The dashed lines depict how B can be enlarged to a non-lattice polygon $B' = \mathrm{conv}(0, 3e_1, \frac{3}{2}e_2)$ such that the pair (A, B') has the same mixed volume as (A, B).

$A = \mathrm{conv}(0, 4e_1, 3e_1 + e_2, 3e_2)$ $\qquad$ $B = \mathrm{conv}(0, 4e_1, e_1 + 2e_2, 2e_2)$

Figure 4.4.: A $\mathbb{Z}$-maximal pair (A, B) of lattice polygons for which both A and B can be enlarged to non-lattice polygons A' and B' such that (A', B') has the same mixed volume as (A, B).

While $\mathbb{Z}$-maximality is the natural notion of maximality among tuples of lattice polytopes, the reason that we also introduce the stronger notion of $\mathbb{R}$-maximality is that we have a better understanding of it which is only in terms of the affine structures of the polytopes in a tuple (see Lemma 4.2.15). Additionally, we know that every $\mathbb{Z}$-maximal tuple can be written as the integer hulls of the polytopes of a (in general non-integral) $\mathbb{R}$-maximal tuple.

In order to provide structural results both on $\mathbb{Z}$- and $\mathbb{R}$-maximal tuples, let us introduce some additional notions.

Definition 4.2.9. Given a polytope $P \in \mathcal{P}(\mathbb{R}^d)$, its *support function* $h_P : \mathbb{R}^d \to \mathbb{R}$ is defined by

$$h_P(\xi) = \max\{\langle \xi, x\rangle : x \in P\}.$$

Given $\xi \in \mathbb{R}^d$ we use P^ξ to denote the *face of P corresponding to ξ* defined by

$$P^\xi = \{x \in P : \langle \xi, x\rangle = h_P(\xi)\}.$$

Clearly, when $\xi = 0$ we get $P^\xi = P$. For ξ not equal to zero, P^ξ depends only on the direction of ξ. Let $\mathcal{S}_{d-1} \subset \mathbb{R}^d$ denote the set of all primitive (non-zero) lattice vectors. If $P \in \mathcal{P}(\mathbb{Z}^d)$ is a lattice polytope, each proper face $F \in \mathcal{F}(\mathcal{P})$ is of the form P^u for some $u \in \mathcal{S}_{d-1}$. If P is full-dimensional and F is a facet of P, the choice of u is unique and we say that u is a *primitive outer facet normal* of the polytope P. For the purpose of this chapter, we extend this notion to general (possibly lower-dimensional) lattice polytopes by calling a vector $u \in \mathcal{S}_{d-1}$ a *primitive outer facet normal* of the polytope $P \in \mathcal{P}(\mathbb{Z}^d)$ if one has $\dim(P^u) = d - 1$. In particular if $\dim(P) \le d - 2$, the polytope P does not have any primitive outer facet normals in this setting.

Proposition 4.2.10. *Let $P \in \mathcal{P}(\mathbb{Z}^d)$ be a lattice polytope and $\{u_1, \dots, u_N\}$ the set of primitive outer facet normals of P. Then one has*

$$\mathrm{Vol}_d(P) = \sum_{i=1}^{N} h_P(u_i)\,\mathrm{Vol}(P^{u_i}). \tag{4.3}$$

Proposition 4.2.10 is a normalized version of [Sch14, Lemma 5.1.1] and we refer to there for a proof. In [Sch14, Section 5.1] it is also explained how (4.3) can be reformulated as an integral

$$\mathrm{Vol}_d(P) = \int_{\mathcal{S}^{d-1}} h_P(u)dS_P(u), \tag{4.4}$$

over the support function of P with respect to the *(normalized) surface area measure* S_P on the set of primitive lattice vectors $\mathcal{S}^{d-1}$ with support $\operatorname{supp} S_P = \{u_1, \dots, u_N\}$ and values $S_P(u_i) = \mathrm{Vol}(P^{u_i})$. In particular, the measure S_P maps

$$U \mapsto \sum_{u_i \in U} \mathrm{Vol}(P^{u_i}),$$

for any subset $U \subseteq \mathcal{S}^{d-1}$. Note, that in [Sch14, Section 5.1] formula (4.4) is phrased for general polytopes and unit normal vectors, and generalized to convex bodies using approximation.

For our purposes, we need the following generalization of Proposition 4.2.10 to a mixed setting (see [Sch14, Theorem 5.1.7]).

Proposition 4.2.11. *Let $(P_1, \dots, P_d) \in \mathcal{P}(\mathbb{Z}^d)^d$ be a tuple of lattice polytopes and $\{u_1, \dots, u_N\}$ the set of primitive outer facet normals of the Minkowski sum $P_1 + \dots + P_{d-1}$. Then one has*

$$\mathrm{MV}(P_1, \dots, P_d) = \sum_{i=1}^{N} h_{P_d}(u_i)\,\mathrm{MV}(P_1^{u_i}, \dots, P_{d-1}^{u_i}). \tag{4.5}$$

Note that the faces $P_1^{u_i}, \dots, P_{d-1}^{u_i}$ in (4.5) can be shifted to a common rational $(d-1)$-dimensional subspace of $\mathbb{R}^d$. As the mixed volume is translation-invariant, it is well-defined to set the mixed volume of $P_1^{u_i}, \dots, P_{d-1}^{u_i}$ to be the mixed volume of appropriate translates of the faces relative to the corresponding sublattice.

Analogously to the unmixed case we can we can view (4.5) as an integral

$$\mathrm{MV}(P_1, \dots, P_{d-1}, P_d) = \int_{\mathcal{S}^{d-1}} h_{P_d}(u) dS_{P_1,\dots,P_{d-1}}(u), \tag{4.6}$$

over the support function of P_d with respect to the *(normalized) mixed area measure* $S_{P_1,\dots,P_{d-1}}$ on the set of primitive vectors $\mathcal{S}^{d-1}$ with support $\operatorname{supp} S_{P_1,\dots,P_{d-1}} = \{u_1, \dots, u_N\}$ and values $S_{P_1,\dots,P_{d-1}}(u) = \mathrm{MV}(P_1^u, \dots, P_{d-1}^u)$. Note that, similarly to the mixed volume, the mixed area measure is Minkowski linear in each of its arguments.

Let us now come to showing the connection between (mixed) area measures and the different notions of maximality of tuples. The following result describes $\mathbb{Z}$-maximal tuples in terms of mixed area measures.

Lemma 4.2.12. *Let $(P_1, \dots, P_d) \in \mathcal{P}(\mathbb{Z}^d)^d$ be an irreducible tuple which is $\mathbb{Z}$-maximal in P_d. Let $\{u_1, \dots, u_r\}$ be the support of the mixed area measure $S_{P_1,\dots,P_{d-1}}$. Then*

$$P_d = \operatorname{conv}\{x \in \mathbb{Z}^d : \langle u_i, x\rangle \le h_i, i \in [r]\} \tag{4.7}$$

for some $h_1, \dots, h_r \in \mathbb{Z}_{\ge 0}$ satisfying

$$\sum_{i=1}^{r} h_i S_{P_1,\dots,P_{d-1}}(u_i) = \mathrm{MV}(P_1, \dots, P_d). \tag{4.8}$$

Proof. Let $h_i = h_{P_d}(u_i)$, the value of the support function of P_d at u_i. Then (4.8) follows directly from (4.6). Consider

$$Q = \{x \in \mathbb{R}^d : \langle u_i, x\rangle \le h_i \text{ for all } i \in [r]\}.$$

This is a rational polytope. (Its boundedness follows from the fact that the volume of $P_1 + \dots + P_d$ is bounded by Theorem 4.2.7.) Clearly, $P_d \subseteq Q$ and h_{P_d} coincides with h_Q on the support of $S_{P_1,\dots,P_{d-1}}$. Therefore, by (4.6) and (4.8),

$$\mathrm{MV}(P_1, \dots, P_{d-1}, Q) = \sum_{i=1}^{r} h_i S_{P_1,\dots,P_{d-1}}(u_i) = \mathrm{MV}(P_1, \dots, P_d).$$

Now the $\mathbb{Z}$-maximality in P_d and the monotonicity of the mixed volume imply that P_d must contain all lattice points of Q, i.e. (4.7) holds. $\square$

Lemma 4.2.12 provides an algorithmic way of finding all possible $P_d \in \mathcal{P}(\mathbb{Z}^d)$ such that $\mathrm{MV}(P_1, \dots, P_d) = m$ and $(P_1, \dots, P_d)$ is $\mathbb{Z}$-maximal in P_d, given a value of m and a tuple $(P_1, \dots, P_{d-1}) \in \mathcal{P}(\mathbb{Z}^d)^{d-1}$ that can be extended to an irreducible one (see Algorithm 4.6.2).

Remark 4.2.13. The values for the h_i in Lemma 4.2.12 are given by $h_{P_d}(u_i)$, the values of the support function of P_d at the support vector u_i of the mixed area measure $S_{P_1,\dots,P_{d-1}}$. This fact can be used to determine whether a given d-tuple $(P_1,\dots,P_d) \in (\mathcal{P}(\mathbb{Z}^d))^d$ is $\mathbb{Z}$-maximal in P_d by checking whether the equality $P_d = \operatorname{conv}\{x \in \mathbb{Z}^d : \langle u_i, x\rangle \le h_{P_d}(u_i), i \in [r]\}$ holds.

A similar argument provides the corresponding statement for tuples $(P_1,\dots,P_d) \in (\mathcal{P}(\mathbb{Z}^d))^d$ that are $\mathbb{R}$-maximal in P_d.

Lemma 4.2.14. *Let $(P_1,\dots,P_d) \in (\mathcal{P}(\mathbb{Z}^d))^d$ be an irreducible tuple which is $\mathbb{R}$-maximal in P_d. Let $\{u_1,\dots,u_r\}$ be the support of the mixed area measure $S_{P_1,\dots,P_{d-1}}$. Then*

$$P_d = \{x \in \mathbb{R}^d : \langle u_i, x\rangle \le h_i \text{ for all } i \in [r]\} \tag{4.9}$$

for some $h_1,\dots,h_r \in \mathbb{Z}_{\ge 0}$ satisfying

$$\sum_{i=1}^{r} h_i S_{P_1,\dots,P_{d-1}}(u_i) = \operatorname{MV}(P_1,\dots,P_d). \tag{4.10}$$

For $\mathbb{R}$-maximality we also have the following simple criterion based on comparing supports of (mixed) area measures. This criterion ensures specific combinatorial structures among $\mathbb{R}$-maximal tuples and is the main reason that we introduce this stronger notion of maximality.

Lemma 4.2.15. *Let $(P_1,\dots,P_d) \in \mathcal{P}(\mathbb{Z}^d)^d$ be an irreducible tuple and assume that P_d is full-dimensional. Then $(P_1,\dots,P_d)$ is $\mathbb{R}$-maximal in P_d if and only if $\operatorname{supp} S_{P_d} \subseteq \operatorname{supp} S_{P_1,\dots,P_{d-1}}$. In particular, $(P_1,\dots,P_d)$ is $\mathbb{R}$-maximal if and only if*

$$\operatorname{supp} S_{P_i} \subseteq \operatorname{supp} S_{P_1,\dots,P_{i-1},P_{i+1},\dots,P_d},$$

for all $i \in [d]$.

Proof. To simplify notation, let $s_1 = \operatorname{supp} S_{P_d}$ and $s_2 = \operatorname{supp} S_{P_1,\dots,P_{d-1}}$. Suppose $s_1 \subseteq s_2$. Consider a full-dimensional polytope $Q_d \in \mathcal{P}(\mathbb{Z}^d)$ such that $P_d \subseteq Q_d$ and

$$\operatorname{MV}(P_1,\dots,P_{d-1},P_d) = \operatorname{MV}(P_1,\dots,P_{d-1},Q_d).$$

By Proposition 4.2.11 we have $h_{P_d}(u) = h_{Q_d}(u)$ for every $u \in s_2$. Then

$$Q_d \subseteq \bigcap_{u\in s_2}\{x \in \mathbb{R}^d : \langle u, x\rangle \le h_{P_d}(u)\} \subseteq \bigcap_{u\in s_1}\{x \in \mathbb{R}^d : \langle u, x\rangle \le h_{P_d}(u)\} = P_d,$$

since s_1 coincides with the set of outer facet normals of P_d. (Here we use the assumption that P_d is d-dimensional.) Therefore $Q_d = P_d$.

Conversely, let $(P_1,\dots,P_d)$ be $\mathbb{R}$-maximal in P_d. Consider

$$Q_d = \bigcap_{u\in s_2}\{x \in \mathbb{R}^d : \langle u, x\rangle \le h_{P_d}(u)\}.$$

Clearly, $P_d \subseteq Q_d$ and $h_{P_d}(u) = h_{Q_d}(u)$ for every $u \in s_2$. Therefore, by Proposition 4.2.11, we have

$$\operatorname{MV}(P_1,\dots,P_{d-1},P_d) = \operatorname{MV}(P_1,\dots,P_{d-1},Q_d).$$

This implies that $P_d = Q_d$, and, hence, the set s_1 of primitive facet normals of P_d is contained in s_2. $\square$

Lemma 4.2.15 provides a more efficient computational test for $\mathbb{R}$-maximality of d-tuples than using Lemma 4.2.14.

Remark 4.2.16. Applying Lemma 4.2.15 in the 2-dimensional case yields that a pair of polygons $(P_1, P_2) \in \mathcal{P}(\mathbb{Z}^2)^2$ is $\mathbb{R}$-maximal if and only if P_1 and P_2 have the same set of primitive facet normals. It would be interesting to understand the restrictiveness of the conditions of Lemma 4.2.15 also in higher dimensions, as this might yield interesting tools for the classification of tuples of lattice polytopes. For $d = 3$ we obtain the conditions

$$\begin{aligned}
\operatorname{supp} S_{P_1} &\subseteq \operatorname{supp} S_{P_2,P_3},\\
\operatorname{supp} S_{P_2} &\subseteq \operatorname{supp} S_{P_1,P_3},\\
\operatorname{supp} S_{P_3} &\subseteq \operatorname{supp} S_{P_1,P_2}.
\end{aligned}$$

Our classification results show that all full-dimensional $\mathbb{R}$-maximal triples of lattice polytopes (P_1, P_2, P_3) with $\operatorname{MV}(P_1, P_2, P_3) \leq 4$ satisfy (up to reordering) $\operatorname{supp} S_{P_1} \subseteq \operatorname{supp} S_{P_2} = \operatorname{supp} S_{P_3}$. It would be interesting to know whether this can be derived for general mixed volumes.

Remark 4.2.17. A better understanding of the conditions in Lemma 4.2.15 may also shed a new light on the questions that we are tackling in Chapter 6. There we conjecture certain tuples of convex bodies of mixed volume m for which the Minkowski sum has maximal volume. If we restrict to polytopes, it is straightforward that such maximizers have to be $\mathbb{R}$-maximal. Further insights into $\mathbb{R}$-maximal tuples would allow us to only have to show the maximality of the conjectured tuples among a more specific class.

4.3. Results

The following is the answer to Classification Problem 4.2.1 for $d = 3$ and $m \in [4]$. The case of mixed volume at most 4 is of particular interest as it has been shown by Esterov that these are precisely those tuples of lattice polytopes that correspond to systems of polynomial equations which are *solvable in radicals* (see [Est19]). This essentially means that these are precisely the systems for which one can determine a formula for the solutions in terms of the coefficients of the polynomials. Parts (1) and (2) of Theorem 4.3.1 below correspond to triples that are not irreducible, whose classification reduces to the enumeration of pairs of lattice polygons of mixed volume at most 4. These pairs have already been enumerated in [EG16]. Part (3) corresponds to triples that are irreducible and whose enumeration is our main contribution.

Theorem 4.3.1 (Classification of triples with normalized mixed volume at most 4)**.** *Let (P_1, P_2, P_3) be a triple of lattice polytopes in $\mathbb{R}^3$ with normalized mixed volume $m \in \{1, 2, 3, 4\}$. Let $\pi_{1,2} : \mathbb{R}^3 \to \mathbb{R}^2$ and $\pi_3 : \mathbb{R}^3 \to \mathbb{R}^1$ be the lattice projections given by $\pi_{1,2}(x_1, x_2, x_3) = (x_1, x_2)$ and $\pi_3(x_1, x_2, x_3) = x_3$. Then, up to equivalence of triples, (P_1, P_2, P_3) has one of the following forms:*

1. *For some* $m_1, m_2 \in \{1,2,3,4\}$ *satisfying* $m = m_1 m_2$*,* P_3 *is a lattice segment* $\{0\}^2 \times [0, m_1]$*, while* $\pi_{1,2}(P_1) \subseteq Q_1$ *and* $\pi_{1,2}(P_2) \subseteq Q_2$ *for some pair* (Q_1, Q_2) *appearing in the list of pairs of lattice polytopes of normalized mixed volume* m_2 *given in Appendix B.*

2. *For some* $m_1, m_2 \in \{1,2,3,4\}$ *satisfying* $m = m_1 m_2$*,* $\pi_3(P_3) = [0, m_1]$*, while* $P_1 \subseteq Q_1 \times \{0\}$*,* $P_2 \subseteq Q_2 \times \{0\}$ *for some pair* (Q_1, Q_2) *appearing in the list of pairs of lattice polytopes of normalized mixed volume* m_2 *given in Appendix B.*

3. $P_1 \subseteq Q_1, P_2 \subseteq Q_2$*, and* $P_3 \subseteq Q_3$ *for some triple* (Q_1, Q_2, Q_3) *appearing in the list of irreducible triples of lattice polytopes of normalized mixed volume* m *given in Appendix A.*

Furthermore, no triple (P_1, P_2, P_3) *in* $\mathbb{R}^3$ *with normalized mixed volume larger than* m *satisfies any of the above three conditions.*

Viewing Theorem 4.3.1 in the light of the BKK-theorem yields the following result on systems of Laurent polynomials in three variables.

Corollary 4.3.2 (Classification of trivariate Laurent polynomial systems with at most 4 solutions)**.** *Let* $f_1, f_2, f_3 \in \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}]$ *be three-variate Laurent polynomials with respective support sets* $A_1, A_2, A_3 \subset \mathbb{Z}^3$*. Suppose that the system* $f_1 = f_2 = f_3 = 0$ *has* $m \in \{1,2,3,4\}$ *solutions in* $(\mathbb{C}^*)^3$ *and that the triple* (f_1, f_2, f_3) *is generic in* $\mathbb{C}[A_1, A_2, A_3]$*. Then, up to equivalence, the tuple of support sets* $(A_1, A_2, A_3) \subset (\mathbb{Z}^3)^3$ *has one of the following forms:*

1. *For some* $m_1, m_2 \in \{1,2,3,4\}$ *satisfying* $m = m_1 m_2$*,* $A_3 \subseteq \{0\}^2 \times \{0, \dots, m_1\}$*, while* $\pi_{1,2}(A_1) \subseteq Q_1$ *and* $\pi_{1,2}(A_2) \subseteq Q_2$ *for some pair* (Q_1, Q_2) *appearing in the list of pairs of lattice polytopes of normalized mixed volume* m_2 *given in Appendix B.*

2. *For some* $m_1, m_2 \in \{1,2,3,4\}$ *satisfying* $m = m_1 m_2$*,* $\pi_3(A_3) \subseteq \{0, \dots, m_1\}$*, while* $A_1 \subseteq Q_1 \times \{0\}$*,* $A_2 \subseteq Q_2 \times \{0\}$ *for some pair* (Q_1, Q_2) *appearing in the list of pairs of lattice polytopes of normalized mixed volume* m_2 *given in Appendix B.*

3. $A_1 \subseteq Q_1, A_2 \subseteq Q_2$ *and* $A_3 \subseteq Q_3$ *for some triple* (Q_1, Q_2, Q_3) *appearing in the list of triples of lattice polytopes of normalized mixed volume* m *given in Appendix A.*

Furthermore, for no system of Laurent polynomials $f_1 = f_2 = f_3 = 0$ *satisfying the above conditions for* $m > 4$ *any of the above three is satisfied.*

Examples 1.2.4 and 4.2.6 provide an illustration to Theorem 4.3.1 and Corollary 4.3.2. The triple (P_1, P_2, P_3) from Example 4.2.6 is classified by Case (1) of our result with $m_1 = 1$, $m_2 = 4$ and the pair (Q_1, Q_2) coinciding with the pair $(\operatorname{conv}(0, 2e_1, e_2), 2\operatorname{conv}(0, 2e_1, e_2))$, up to equivalence.

Theorem 4.3.3. *Let $N_3(m)$ (resp. $N'_3(m)$) be the number of equivalence classes of triples of all (resp. full-dimensional) lattice polytopes in $\mathbb{R}^3$ of mixed volume m that are irreducible and $\mathbb{Z}$-maximal. We have the following table of values for $1 \le m \le 4$:*

m	$N_3(m)$	$N'_3(m)$
1	1	1
2	7	4
3	21	10
4	92	30

In particular, our classification verifies the list of $\mathbb{Z}$-maximal irreducible triples of lattice polytopes with mixed volume 2 proposed in [EG15].

In order to make further structural observations regarding our classification results, let us give some sufficient conditions for $\mathbb{R}$-maximality (and therefore for $\mathbb{Z}$-maximality). Recall that a polytope P is a *combinatorial pyramid* if P has a facet containing all but one vertex of P.

Proposition 4.3.4. *A triple $(P_1, P_2, P_3) \in \mathcal{P}(\mathbb{Z}^3)^3$ of full-dimensional lattice polytopes is $\mathbb{R}$-maximal, whenever:*

(0) the polytopes are the same, $P_1 = P_2 = P_3$ (in this case $\mathrm{MV}(P_1, P_2, P_3) = \mathrm{Vol}(P_1)$)

or the polytopes are not all the same and one of the following holds:

(1) there exists a lattice polytope P such that $P_1 = \alpha P$, $P_2 = \beta P$, and $P_3 = \gamma P$ for some $\alpha, \beta, \gamma \in \mathbb{Z}_{\ge 1}$ (in this case $\mathrm{MV}(P_1, P_2, P_3) = \alpha\beta\gamma \,\mathrm{Vol}(P)$),

(2) there exists a lattice polytope P and a primitive lattice segment I in $\mathbb{R}^3$, as well as $\alpha, \beta \in \mathbb{Z}_{\ge 1}$ and $\gamma \in \mathbb{Z}_{\ge 0}$ such that

$$P_1 = P + \alpha I, P_2 = P + \beta I \text{ and } P_3 = P + \gamma I,$$

(in this case $\mathrm{MV}(P_1, P_2, P_3) = \mathrm{Vol}(P) + (\alpha + \beta + \gamma)\,\mathrm{Vol}(\pi_I(P))$, where π_I is a lattice projection parallel to I),

(3) there exists a primitive lattice segment I and a lattice polytope P, which is a combinatorial pyramid with base having two edges parallel to I, such that

$$P_1 = P_2 = P \text{ and } P_3 = P + \alpha I,$$

for some $\alpha \in \mathbb{Z}_{\ge 1}$ (in this case $\mathrm{MV}(P_1, P_2, P_3) = \mathrm{Vol}(P) + \alpha \,\mathrm{Vol}(\pi_I(P))$ where π_I is a lattice projection parallel to I).

Proof. We show that under each of the assumptions (0)–(3) the conditions for $\mathbb{R}$-maximality from Lemma 4.2.15 are satisfied. For case (0) and (1) this is straightforward, since these conditions imply that the polytopes P_1, P_2, P_3 have the same sets of facet normal rays. So let us assume that the triple $(P_1, P_2, P_3) \in \mathcal{P}(\mathbb{Z}^3)^3$ satisfies the conditions of case (2). One computes

$$S_{P_1,P_2} = S_{P+\alpha I, P+\beta I} = S_P + (\alpha + \beta) S_{P,I},$$

using the linearity, $S_{P,P} = S_P$, and the fact that the measure $S_I = S_{I,I}$ is zero. Furthermore, one obtains

$$S_{P_3} = S_{P_3,P_3} = S_P + 2\gamma S_{P,I}.$$

This shows $\operatorname{supp} S_{P_3} \subseteq \operatorname{supp} S_{P_1,P_2}$ and therefore (P_1, P_2, P_3) is $\mathbb{R}$-maximal in P_3. The fact that the triple is also $\mathbb{R}$-maximal in P_1 and P_2 can be computed analogously. Let us now assume that the triple $(P_1, P_2, P_3) \in \mathcal{P}(\mathbb{Z}^3)^3$ satisfies the conditions of case (3). We are going to show that P and $P+I$ have the same sets of facet normals. Let $P = \operatorname{conv}(P' \cup \{v\})$ for some $v \in \mathbb{Z}^3$ and $P' \in \mathcal{P}(\mathbb{Z}^3)$ with $\dim(P') = 2$, and assume $I = [0, w]$ for some $w \in \mathbb{Z}^3$. Let J_1 and J_2 be two edges of P' parallel to I. Note that each edge of $P' + I$ is equal to either an edge of P', or the sum of w and an edge of P', or $J_i + I$ for $i = 1, 2$. Similarly, each facet of $P + I$ is equal to either a facet of P, or the sum of w and a facet of P, or $\operatorname{conv}((J_i + I) \cup (v + I))$, for $i = 1, 2$. This implies P and $P + I$ have the same facet normals.

The respective values for the mixed volumes follow directly from multilinearity of the mixed volume and Proposition 4.2.4. □

Theorem 4.3.5. *Among all the $\mathbb{Z}$-maximal triples $(P_1, P_2, P_3) \in \mathcal{P}(\mathbb{Z}^3)^3$ of full-dimensional lattice polytopes with mixed volume at most 4, only the following three exceptional triples do not fall into one of the categories from Proposition 4.3.4:*

- $P_1 = P_2 = \operatorname{conv}(0, 2e_1, e_2, e_3)$, $P_3 = P_1 + [0, e_1]$ *(where* $\operatorname{MV}(P_1, P_2, P_3) = 3$*)*,
- $P_1 = P_2 = \operatorname{conv}(0, 3e_1, e_2, e_3)$, $P_3 = P_1 + [0, e_1]$ *(where* $\operatorname{MV}(P_1, P_2, P_3) = 4$*)*,
- $P_1 = P_2 = \operatorname{conv}(0, e_1, e_2, e_1 + e_2 + 2e_3)$, $P_3 = P_1 + [0, e_1 + e_3]$ *(where* $\operatorname{MV}(P_1, P_2, P_3) = 4$*)*.

Furthermore, all three exceptional triples are $\mathbb{Z}$-, but not $\mathbb{R}$-maximal. The number of triples that are of types (0)–(3) and of mixed volume m is given as follows:

m	type (0)	type (1)	type (2)	type (3)
1	1	0	0	0
2	3	1	0	0
3	6	1	1	1
4	17	5	3	3

Examples of triples of polytopes of types (1)–(3) are presented in Fig. 4.5. We refer to Appendix A for a more detailed presentation and analysis of the results.

Parts of our algorithm also provide a direct way to carry out a classification of pairs of polygons of given mixed volume. We have carried out this classification for mixed volume up to 10 and obtained the following result.

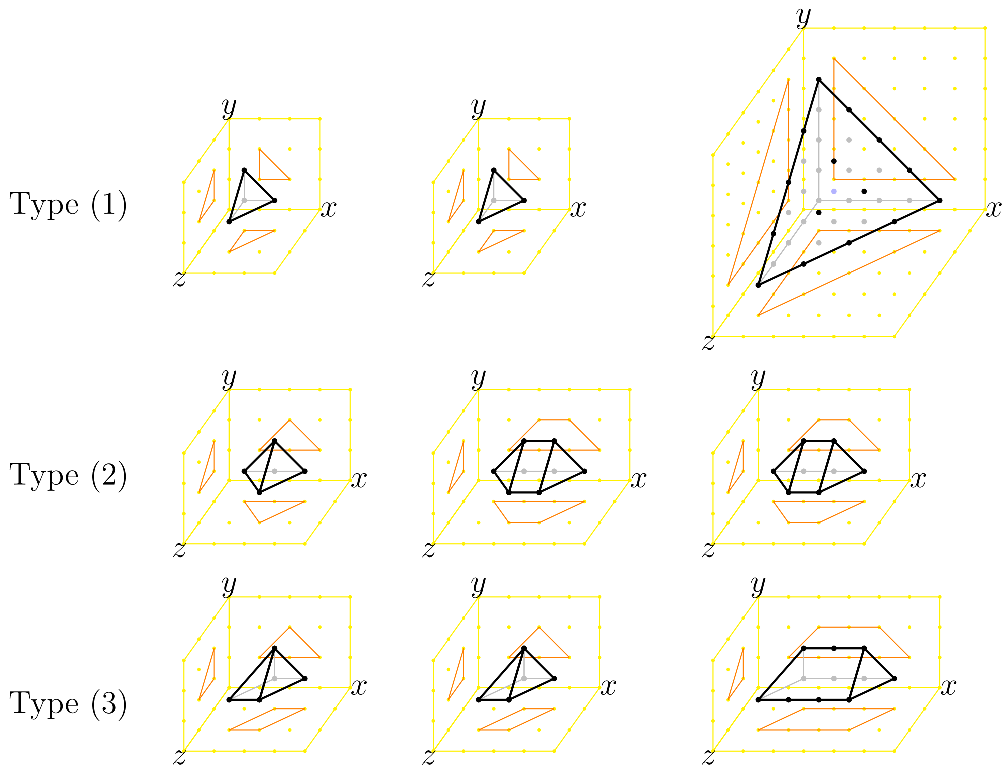


Figure 4.5.: Examples of triples of each of the types (1)–(3) of mixed volume 4.

Theorem 4.3.6. *Let $N_2(m)$ be the number of equivalence classes of pairs of 2-dimensional lattice polytopes in $\mathbb{R}^2$ of mixed volume m that are irreducible and $\mathbb{Z}$-maximal. We have the following table of values for $1 \leq m \leq 10$.*

m	$N_2(m)$
1	1
2	3
3	6
4	13
5	18
6	38
7	46
8	87
9	118
10	202

Our computations have been carried out using Sagemath [Sag18] and an implementation of our classification algorithm as well as data files containing the classification results can be found at `https://github.com/christopherborger/mixed_volume_classification`.

4.4. The algorithm for full-dimensional polytopes in dimension three

In this section we present an algorithm for solving the following enumeration problem. Throughout this and subsequent sections the word "maximal" will mean "$\mathbb{Z}$-maximal".

Enumeration Problem 4.4.1. Given $m \in \mathbb{N}$, enumerate, up to equivalence, all maximal triples $(P_1, P_2, P_3) \in \mathcal{P}(\mathbb{Z}^3)^3$ of full-dimensional lattice polytopes satisfying $\mathrm{MV}(P_1, P_2, P_3) = m$.

The main idea for making Enumeration Problem 4.4.1 computationally tractable is to produce upper bounds for the mixed volumes $\mathrm{MV}(P_i, P_j, P_k)$ for all choices $i, j, k \in [3]$ instead of an upper bound for the volume of the Minkowski sum $P_1+P_2+P_3$. As an illustration, consider the triple $(P_1, P_2, P_3) = (\Delta_3, \Delta_3, m\Delta_3)$, which has mixed volume m. While the volume of P_3 and the volume of $P_1 + P_2 + P_3$ are large, one has $\mathrm{MV}(P_1, P_1, P_2) = 1$, i.e. some of the mixed volumes $\mathrm{MV}(P_i, P_j, P_k)$ are small. Such relations are enforced by the Aleksandrov-Fenchel inequality (Proposition 1.2.2 (5.)). Proposition 4.4.2 below characterizes this phenomenon in general.

Proposition 4.4.2. *Let $(P_1, P_2, P_3) \in \mathcal{P}(\mathbb{Z}^3)^3$ be a triple of full-dimensional lattice polytopes satisfying $\mathrm{MV}(P_1, P_2, P_3) = m$ for a given $m \in \mathbb{N}$. Then, up to relabeling, either $\mathrm{MV}(P_1, P_1, P_2) < m$, or $\mathrm{MV}(P_i, P_i, P_j) = m$ for all $i, j \in [3]$ with $i \neq j$ and furthermore $\mathrm{Vol}(P_i) \leq m$ for all $i \in [3]$.*

Proof. Suppose there are $i, j \in [3]$ with $i \neq j$ such that $\mathrm{MV}(P_i, P_i, P_j) \neq m$. After possibly reordering we may assume $\mathrm{MV}(P_1, P_1, P_2) \neq m$. If $\mathrm{MV}(P_1, P_1, P_2)$ is strictly smaller than m, we have proven the claim. So let us assume $\mathrm{MV}(P_1, P_1, P_2) > m$. In this case, the Aleksandrov-Fenchel inequality given by $\mathrm{MV}(P_1, P_2, P_3)^2 \geq \mathrm{MV}(P_1, P_1, P_2)\,\mathrm{MV}(P_3, P_3, P_2)$ yields that $\mathrm{MV}(P_2, P_3, P_3) < m$, so that the claim holds for the ordering (P_3, P_2, P_1).

It is left to prove that, if $\mathrm{MV}(P_i, P_i, P_j) = m$ for all $i, j \in [3]$ with $i \neq j$, then $\mathrm{Vol}(P_i) \leq m$ for all $i \in [3]$. This is a direct consequence of the Aleksandrov-Fenchel inequality, as $\mathrm{MV}(P_i, P_i, P_j)^2 \geq \mathrm{MV}(P_i, P_j, P_j)\,\mathrm{Vol}(P_i)$ holds for any $i, j \in [3]$ with $i \neq j$. ☐

Remark 4.4.3. Note that in the case $\mathrm{MV}(P_i, P_i, P_j) = m$ for all $i, j \in [3]$ with $i \neq j$ and $\mathrm{Vol}(P_i) = m$ for all $i \in [3]$, the Aleksandrov-Fenchel inequalities $\mathrm{MV}(P_i, P_i, P_j) \geq \mathrm{MV}(P_i, P_j, P_j)\,\mathrm{Vol}(P_i)$ become equalities, which implies that $P_1 = P_2 = P_3$. This is due to the characterization of the equality case in Minkowski's inequality, see [Sch14, Theorem 7.2.1].

Let us now present an algorithm to solve Enumeration Problem 4.4.1. Note that Proposition 4.4.2 justifies the restriction in Step 2 to a case a. relying on an inductive classification of maximal triples of lower mixed volume (see Step 1) and a very specific case b.

Algorithm 4.4.4 (Classification of full-dimensional triples)**.**

Input: A number $m \in \mathbb{N}$.

Output: A list of all maximal triples of full-dimensional lattice polytopes $(P_1, P_2, P_3) \in \mathcal{P}(\mathbb{Z}^3)^3$ with $\mathrm{MV}(P_1, P_2, P_3) = m$, up to equivalence.

Step 1: If $m = 1$, return the triple $(\Delta_3, \Delta_3, \Delta_3)$. Else, iteratively run Algorithm 4.4.4 for all input values $m' < m$ in order to obtain a list of all maximal triples of full-dimensional lattice polytopes $(P_1, P_2, P_3) \in \mathcal{P}(\mathbb{Z}^3)^3$ with $\mathrm{MV}(P_1, P_2, P_3) < m$, up to equivalence.

Step 2:

a. Classify, up to equivalence, all full-dimensional pairs $(P_1, P_2) \in \mathcal{P}(\mathbb{Z}^3)^2$ such that $\mathrm{MV}(P_1, P_1, P_2) < m$ (see Remark 4.4.5).

b. Classify, up to equivalence, all full-dimensional pairs $(P_1, P_2) \in \mathcal{P}(\mathbb{Z}^3)^2$ such that $\mathrm{MV}(P_1, P_1, P_2) = \mathrm{MV}(P_2, P_2, P_1) = m$ (see Algorithm 4.4.6).

Step 3: For a given pair $(P_1, P_2) \in \mathcal{P}(\mathbb{Z}^3)^2$ as in Step 2 enumerate, up to translations, all full-dimensional lattice polytopes $P_3 \in \mathcal{P}(\mathbb{Z}^3)$ such that $\mathrm{MV}(P_1, P_2, P_3) = m$ and such that the triple (P_1, P_2, P_3) is maximal in P_3 (see Algorithm 4.6.2).

Step 4: Given a full-dimensional triple $(P_1, P_2, P_3) \in \mathcal{P}(\mathbb{Z}^3)^3$ as in Step 3 check whether it is maximal in P_1 and P_2 and, if so, add it modulo equivalence to the final list of maximal triples of mixed volume m.

Remark 4.4.5. We may obtain the pairs of Step 2.a from the list of all maximal triples of full-dimensional lattice polytopes $(P_1, P_2, P_3) \in \mathcal{P}(\mathbb{Z}^3)^3$ that satisfy $\mathrm{MV}(P_1, P_2, P_3) < m$, as inductively obtained in Step 1. Note that we need to consider not only those pairs $(P_1, P_2) \in \mathcal{P}(\mathbb{Z}^3)^2$ for which the triple (P_1, P_1, P_2) is maximal, but all pairs such that $\mathrm{MV}(P_1, P_1, P_2) = m' < m$. These can be obtained by iteratively pealing off vertices of the maximal triples of mixed volume less than m and searching among them for triples of the form (P_1, P_1, P_2) up to permutations and translations. The running time of this task is very reasonable for values $m' \in \{1, 2, 3\}$ but is growing very fast in m' and would also be growing extensively if we were to consider higher dimensions.

Dealing with Step 2.b is more involved and we employ the following algorithm:

Algorithm 4.4.6 (Step 2.b of Algorithm 4.4.4)**.**

Input: A number $m \in \mathbb{N}$.

Output: A list of all full-dimensional pairs $(P_1, P_2) \in \mathcal{P}(\mathbb{Z}^3)^2$ that satisfy $\mathrm{MV}(P_1, P_1, P_2) = \mathrm{MV}(P_1, P_2, P_2) = m$, up to equivalence.

Step 1: Classify, up to equivalence, all full-dimensional $P_1 \in \mathcal{P}(\mathbb{Z}^3)$ with $\mathrm{Vol}(P_1) \leq m$ (Enumeration Problem 4.7.1).

Step 2: Given a full-dimensional lattice polytope $P_1 \in \mathcal{P}(\mathbb{Z}^3)$ with $\mathrm{Vol}(P_1) \leq m$, determine, up to translations, all full-dimensional lattice polytopes $Q \in \mathcal{P}(\mathbb{Z}^3)$ such that $\mathrm{MV}(P_1, P_1, Q) = m$ and the triple (P_1, P_1, Q) is maximal in Q (see Algorithm 4.6.2).

Step 3: Given a pair $(P_1, Q) \in \mathcal{P}(\mathbb{Z}^3)^2$ of full-dimensional lattice polytopes as in Step 2, determine all subpolytopes $P_2 \subseteq Q$ such that $\mathrm{Vol}(P_2) \leq m$ and $\mathrm{MV}(P_1, P_1, P_2) = \mathrm{MV}(P_2, P_2, P_1) = m$ (see Algorithm 4.7.7).

Step 4: Given a pair $(P_1, P_2) \in \mathcal{P}(\mathbb{Z}^3)^2$ with $P_2 \subseteq Q$ as above, add it modulo equivalence to the final list.

4.5. Extension to the general case in dimension three

In this section we present an extension of Algorithm 4.4.4 allowing us to classify general maximal irreducible triples $(P_1, P_2, P_3) \in P(\mathbb{Z}^3)^3$ without assuming that all P_i are full-dimensional, thus solving the following enumeration problem.

Enumeration Problem 4.5.1. Given $m \in \mathbb{N}$, enumerate, up to equivalence, all maximal irreducible triples $(P_1, P_2, P_3) \in \mathcal{P}(\mathbb{Z}^3)^3$ satisfying $\mathrm{MV}(P_1, P_2, P_3) = m$.

While we may still formulate a statement analogous to Proposition 4.4.2, the existence of $i, j \in [3]$ with $i \neq j$ such that $\mathrm{MV}(P_i, P_i, P_j) < m$ does not necessarily allow us to build upon the classification for lower mixed volumes as in Remark 4.4.5. The reason is that, if one has $\dim(P_i) = 2$, the triple (P_i, P_i, P_j) is not irreducible anymore and therefore may not be contained in one of the maximal irreducible triples of lower mixed volume. Hence, we carry out a different case distinction for triples that contain at least one polytope of dimension 2.

Proposition 4.5.2. *Let $(P_1, P_2, P_3) \in \mathcal{P}(\mathbb{Z}^3)^3$ be an irreducible triple such that $\mathrm{MV}(P_1, P_2, P_3) = m$ and at least one of the P_i is 2-dimensional. Then there exist distinct indices $i, j \in [3]$ such that one of the following holds:*

(a.) (P_i, P_i, P_j) *is irreducible and satisfies* $\mathrm{MV}(P_i, P_i, P_j) < m$,

(b.) $\dim(P_i) = 2$, $\dim(P_j) = 3$, $\mathrm{MV}(P_i, P_i, P_j) \leq m$, *and* $\mathrm{MV}(P_j, P_j, P_i) = m$,

(c.) $\dim(P_i) = \dim(P_j) = 2$, $\mathrm{MV}(P_i, P_i, P_j) \leq m$, *and* $\mathrm{MV}(P_j, P_j, P_i) \leq m^2$.

Proof. Without loss of generality we may assume $\dim(P_1) \leq \dim(P_2) \leq \dim(P_3)$. We distinguish cases based on the dimensions of the polytopes in the tuple. Assume first $\dim(P_1) = 2$ and $\dim(P_2) = \dim(P_3) = 3$. Consider the Aleksandrov-Fenchel inequality $m^2 = \mathrm{MV}(P_1, P_2, P_3)^2 \geq \mathrm{MV}(P_2, P_2, P_1)\,\mathrm{MV}(P_3, P_3, P_1)$. If $\mathrm{MV}(P_2, P_2, P_1) < m$ or $\mathrm{MV}(P_3, P_3, P_1) < m$, one has (a) for $(i, j) = (2, 1)$ or $(i, j) = (3, 1)$, respectively. Otherwise, one has $\mathrm{MV}(P_2, P_2, P_1) = \mathrm{MV}(P_3, P_3, P_1) = m$. Now, if (a) does not hold for $(i, j) = (2, 3)$ then $\mathrm{MV}(P_2, P_2, P_3) \geq m$ and the Aleksandrov-Fenchel inequality $m^2 \geq \mathrm{MV}(P_1, P_1, P_3)\,\mathrm{MV}(P_2, P_2, P_3)$ implies $\mathrm{MV}(P_1, P_1, P_3) \leq m$, i.e. (b) holds for $(i, j) = (1, 3)$. Similarly, we show that either (a) holds for $(i, j) = (3, 2)$ or (b) holds for $(i, j) = (1, 2)$.

Assume now that $\dim(P_1) = \dim(P_2) = 2$, and $\dim(P_3) = 3$. Consider the Aleksandrov-Fenchel inequality $m^2 \geq \mathrm{MV}(P_2, P_2, P_1)\,\mathrm{MV}(P_3, P_3, P_1)$. If one has $\mathrm{MV}(P_3, P_3, P_1) < m$, then (a) holds for $(i, j) = (3, 1)$. Otherwise, $\mathrm{MV}(P_2, P_2, P_1) \leq$

m. As, additionally, $\mathrm{MV}(P_1, P_1, P_2)\,\mathrm{MV}(P_3, P_3, P_2) \leq m^2$, in this case (c) holds for $(i, j) = (2, 1)$.

Let us finally assume that $\dim(P_1) = \dim(P_2) = \dim(P_3) = 2$. Then the inequality $m^2 \geq \mathrm{MV}(P_1, P_1, P_2)\,\mathrm{MV}(P_3, P_3, P_2)$ yields that either $\mathrm{MV}(P_1, P_1, P_2) \leq m$ or $\mathrm{MV}(P_3, P_3, P_2) \leq m$. Analogously to the above, one also has $\mathrm{MV}(P_2, P_2, P_1) \leq m^2$ and $\mathrm{MV}(P_2, P_2, P_3) \leq m^2$. Therefore, case (c) holds for $(i, j) = (1, 2)$ or $(i, j) = (3, 2)$. $\square$

Let us now present an extension of Algorithm 4.4.4 that allows us to solve Enumeration Problem 4.5.1. In particular, Algorithm 4.5.3 is used to classify maximal irreducible triples of a given mixed volume containing at least one 2-dimensional lattice polytope. Note that Proposition 4.5.2 justifies the restriction to the three cases a., b., and c. in Step 2.

Algorithm 4.5.3 (Extension of Algorithm 4.4.4 to general maximal irreducible triples)**.**

> **Input:** A number $m \in \mathbb{N}$.
>
> **Output:** A list of all maximal irreducible triples $(P_1, P_2, P_3) \in \mathcal{P}(\mathbb{Z}^3)^3$ with $\mathrm{MV}(P_1, P_2, P_3) = m$ and $\dim(P_i) = 2$ for at least one $i \in [3]$, up to equivalence.
>
> **Step 1**: If $m = 1$, return an empty list (as the only maximal irreducible triple of mixed volume 1 is $(\Delta_3, \Delta_3, \Delta_3)$ and therefore full-dimensional). Else, iteratively run Algorithm 4.4.4 and Algorithm 4.5.3 for input values $m' < m$ in order to obtain a list of all maximal irreducible triples $(P_1, P_2, P_3) \in \mathcal{P}(\mathbb{Z}^3)^3$ with $\mathrm{MV}(P_1, P_2, P_3) < m$, up to equivalence.
>
> **Step 2**:
>
> **a.** Classify, up to equivalence, all pairs $(P_1, P_2) \in \mathcal{P}(\mathbb{Z}^3)^2$ such that the triple (P_1, P_1, P_2) is irreducible with $\mathrm{MV}(P_1, P_1, P_2) < m$ (see Remark 4.5.4).
>
> **b.** Classify, up to equivalence, all pairs $(P_1, P_2) \in \mathcal{P}(\mathbb{Z}^3)^2$ with $\dim(P_1) = 2$ and $\dim(P_2) = 3$ and where $\mathrm{MV}(P_1, P_1, P_2) \leq m$ and $\mathrm{MV}(P_2, P_2, P_1) = m$ (see Algorithm 4.5.5).
>
> **c.** Classify, up to equivalence, all pairs $(P_1, P_2) \in \mathcal{P}(\mathbb{Z}^3)^2$ where $\dim(P_1) = \dim(P_2) = 2$, $\mathrm{MV}(P_1, P_1, P_2) \leq m$ and $\mathrm{MV}(P_1, P_2, P_2) \leq m^2$ (see Algorithm 4.5.5).
>
> **Step 3**: For a given pair $(P_1, P_2) \in \mathcal{P}(\mathbb{Z}^3)^2$ as in Step 2, enumerate, up to translations, all $P_3 \in \mathcal{P}(\mathbb{Z}^3)$ such that $\mathrm{MV}(P_1, P_2, P_3) = m$ and the triple (P_1, P_2, P_3) is irreducible and maximal in P_3 (see Algorithm 4.6.2).
>
> **Step 4**: Given an irreducible triple $(P_1, P_2, P_3) \in \mathcal{P}(\mathbb{Z}^3)^3$ as in Step 3, check whether it is also maximal in P_1 and P_2 and, if so, add it modulo equivalence to the final list of maximal triples of mixed volume m.

Remark 4.5.4. The classification in Step 2.a can be obtained from the list of maximal triples of mixed volume $m' < m$ of Step 1 analogously to the procedure described in Remark 4.4.5.

In order to treat Step 2.b and Step 2.c of Algorithm 4.5.3 we apply the following algorithm. While we treat both cases in a similar way, the separation in some of the steps has an important computational advantage. This is because it allows us to have relatively small bounding boxes in which one has to perform a rather expensive search for full-dimensional subpolytopes (Step 2.b), while we may restrict the search inside larger bounding boxes to lower-dimensional subpolytopes (Step 2.c).

Algorithm 4.5.5 (Step 2.b and Step 2.c of Algorithm 4.5.3)**.**

Input: A number $m \in \mathbb{N}$.

Output:

for b: A list of all pairs $(P_1, P_2) \in \mathcal{P}(\mathbb{Z}^3)^2$, up to equivalence, with $\dim(P_1) = 2$ and $\dim(P_2) = 3$, and where $\mathrm{MV}(P_1, P_1, P_2) \leq m$ and $\mathrm{MV}(P_2, P_2, P_1) = m$.

for c: A list of all pairs $(P_1, P_2) \in \mathcal{P}(\mathbb{Z}^3)^2$, up to equivalence, with $\dim(P_1) = \dim(P_2) = 2$ and where $\mathrm{MV}(P_1, P_1, P_2) \leq m$ and $\mathrm{MV}(P_2, P_2, P_1) \leq m^2$.

Step 1: Classify all $P_1 = P_1' \times \{0\} \in \mathcal{P}(\mathbb{Z}^3)$ for $P_1' \in \mathcal{P}(\mathbb{Z}^2)$ being a full-dimensional lattice polygon with $\mathrm{Vol}(P_1) \leq m$, up to equivalence (Enumeration Problem 4.7.1).

Step 2:

for b: Given P_1 as above, determine bounding boxes $Q_1, \ldots, Q_r \in \mathcal{P}(\mathbb{Z}^3)$ containing, up to shearing along the affine span of P_1 and translations, all P_2 which satisfy $\mathrm{MV}(P_1, P_1, P_2) \leq m$ and $\mathrm{MV}(P_1, P_2, P_2) = m$ (see Lemma 4.6.5).

for c: Given P_1 as above, determine bounding boxes $R_1, \ldots, R_s \in \mathcal{P}(\mathbb{Z}^3)$ containing, up to shearing along the affine span of P_1 and translations, all P_2 satisfying $\mathrm{MV}(P_1, P_1, P_2) \leq m$ and $\mathrm{MV}(P_1, P_2, P_2) \leq m^2$ (see Lemma 4.6.5).

Step 3:

for b: Determine all full-dimensional subpolytopes $P_2 \in \mathcal{P}(\mathbb{Z}^3)$ of the bounding boxes $Q_1, \ldots, Q_r$ that satisfy $\mathrm{MV}(P_1, P_1, P_2) \leq m$ and $\mathrm{MV}(P_1, P_2, P_2) = m$ (see Remark 4.6.6 and Algorithm 4.7.7).

for c: Determine all 2-dimensional subpolytopes $P_2 \in \mathcal{P}(\mathbb{Z}^3)$ of the bounding boxes $R_1, \ldots, R_s$ that satisfy $\mathrm{MV}(P_1, P_1, P_2) \leq m$ and $\mathrm{MV}(P_1, P_2, P_2) \leq m^2$ (see Remark 4.6.6 and Algorithm 4.7.7).

Step 4: Given P_1 and P_2 as above, add the pair $(P_1, P_2) \in \mathcal{P}(\mathbb{Z}^3)^2$ modulo equivalence to the final list.

4.6. Details of the algorithms

In this section we provide further details of the classification algorithms presented in the previous sections.

4.6.1. Finding maximal P_3

A problem that we have to solve in various steps of the classification algorithm is the following.

Enumeration Problem 4.6.1. Let $(P_1, P_2) \in \mathcal{P}(\mathbb{Z}^3)^2$ be a pair of lattice polytopes satisfying $\dim(P_1), \dim(P_2) \geq 2$, $\dim(P_1 + P_2) = 3$. Given $m \in \mathbb{N}$, enumerate, up to translations, all lattice polytopes $P_3 \in \mathcal{P}(\mathbb{Z}^3)$ such that $\mathrm{MV}(P_1, P_2, P_3) = m$ and such that the triple (P_1, P_2, P_3) is irreducible, and $\mathbb{Z}$-maximal in P_3.

Note that the conditions $\dim(P_1), \dim(P_2) \geq 2$ and $\dim(P_1+P_2) = 3$ are equivalent to the fact that the pair (P_1, P_2) can be extended to an irreducible triple (P_1, P_2, P_3). Indeed, if any of the conditions is not satisfied, a triple (P_1, P_2, P_3) is not irreducible by definition. On the other hand, if we choose P_3 full-dimensional and the above conditions are satisfied, the triple (P_1, P_2, P_3) is irreducible.

We solve Enumeration Problem 4.6.1 by making use of Lemma 4.2.12 in form of the following Algorithm.

Algorithm 4.6.2 (Finding maximal P_3)**.**

> **Input:** A pair $(P_1, P_2) \in \mathcal{P}(\mathbb{Z}^3)^2$ of lattice polytopes that satisfy $\dim(P_1) \geq 2$, $\dim(P_2) \geq 2$, and $\dim(P_1 + P_2) = 3$, and a number $m \in \mathbb{N}$.
>
> **Output:** A list of all lattice polytopes $P_3 \in \mathcal{P}(\mathbb{Z}^3)$, up to translations, such that $\mathrm{MV}(P_1, P_2, P_3) = m$ and the triple (P_1, P_2, P_3) is irreducible and $\mathbb{Z}$-maximal in P_3.
>
> **Step 1**: Compute the mixed area measure of P_1 and P_2. In particular, compute the normalized mixed areas $\mathrm{MV}(P_1^u, P_2^u)$ for all $u \in \mathbb{Z}^3$ that are primitive outer normal vectors of a facet of the Minkowski sum $P_1 + P_2$. Obtain a vector $a = (a_1, \ldots, a_r) \in \mathbb{Z}^r_{\geq 1}$ of the mixed areas of P_1, P_2 with respect to those primitive normal vectors $u_1, \ldots, u_r \in \mathbb{Z}^3$ that yield a positive mixed area.
>
> **Step 2**: Determine all vectors $h = (h_1, \ldots, h_r) \in (\mathbb{Z}_{\geq 0})^r$ satisfying $\sum_{i=1}^r h_i a_i = m$.
>
> **Step 3**: Given a vector $h \in (\mathbb{Z}_{\geq 0})^r$ as above, compute
>
> $$P_3 = \mathrm{conv}\left\{x \in \mathbb{Z}^3 \colon \langle u_i, x\rangle \leq h_i \text{ for all } i \in [r]\right\},$$
>
> and check whether the triple (P_1, P_2, P_3) is irreducible and $\mathrm{MV}(P_1, P_2, P_3) = m$. If this is true, append P_3 modulo translations to the final list.

Remark 4.6.3. Algorithm 4.6.2 allows us to benefit from the restriction to maximal triples (or triples that are maximal in at least one polytope) in our classification. For example, fixing the pair $(\Delta_3, \Delta_3) \in \mathcal{P}_3(\mathbb{Z}^3)^2$ and mixed volume $m = 4$, Algorithm 4.6.2 directly determines $Q = 4\Delta_3$ as the unique maximal lattice polytope such that $\mathrm{MV}(\Delta_3, \Delta_3, Q) = 4$.

Remark 4.6.4. A slight modification of Algorithm 4.6.2 can be used in order to classify maximal pairs of polygons $(P_1, P_2) \in \mathcal{P}(\mathbb{Z}^2)$ of a given mixed volume m. By the Aleksandrov-Fenchel inequality in the two-dimensional case one may assume without loss of generality that $\mathrm{Vol}(P_1) \leq m$. For any fixed full-dimensional lattice polytope $P_1 \in \mathcal{P}(\mathbb{Z}^2)$ one may compute the area measure and, analogously to Step 2 and Step 3 of Algorithm 4.6.2, determine a list of all full-dimensional $P_2 \in \mathcal{P}(\mathbb{Z}^2)$ such that (P_1, P_2) is $\mathbb{Z}$-maximal in P_2 and $\mathrm{MV}(P_1, P_2) = m$.

4.6.2. Bounding P_2 given a lower-dimensional P_1

In the lemma below $K - K$ denotes the *difference set* $K + (-K)$ of a convex set $K \subset \mathbb{R}^d$ and $K^* = \{y \in \mathbb{R}^d : \langle x, y\rangle \leq 1 \text{ for all } x \in K\}$ denotes the *polar dual* convex set.

Lemma 4.6.5. *Let $(P_1, P_2) \in \mathcal{P}(\mathbb{Z}^3)^2$ be a pair of lattice polytopes such that P_1 is 2-dimensional of the form $P_1 = P' \times \{0\} \subset \mathbb{R}^2 \times \{0\}$ and P_2 is 2-dimensional with positive width w in direction e_3. Assume $\mathrm{MV}(P_1, P_1, P_2) = m_1$ and $\mathrm{MV}(P_2, P_2, P_1) \leq m_2$ for some $m_1, m_2 \in \mathbb{N}$. Then, up to a shearing along $\mathbb{R}^2 \times \{0\}$ and a lattice translation, P_2 is contained in the bounding polytope*

$$R_{q_1,q_2} := \mathrm{conv}\left(\left\{\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \in \mathbb{Z}^2 \times \{0, \dots, w-1\} \, : \, \begin{pmatrix}x_1\\x_2\end{pmatrix} \in Q' + \frac{1}{w}\begin{pmatrix}q_1x_3\\q_2x_3\end{pmatrix}\right\}\right),$$

where $q_1, q_2 \in \{0, \dots, w-1\}$ and

$$Q' := \begin{pmatrix}0 & \frac{m_2}{w}\\ -\frac{m_2}{w} & 0\end{pmatrix}(P' - P')^*.$$

Proof. We may assume $0 \in P_2$ and $h_{P_2}(-e_3) = 0$ and therefore Proposition 4.2.4 yields $m_1 = \mathrm{MV}(P_1, P_1, P_2) = \mathrm{Vol}_2(P')h_{P_2}(e_3)$. Then P_2 contains a lattice point (q_1, q_2, w) at height w and, up to shearing, we may assume that $q_1, q_2 \in \{0, \dots, w-1\}$. Let $x = (x_1, x_2, x_3) \in P_2 \cap \mathbb{Z}^3$ be another lattice point of P_2 and consider the triangle $T_x := \mathrm{conv}((0,0,0), (q_1, q_2, w), (x_1, x_2, x_3)) \subseteq P_2$. Let $(n_1, n_2, n_3) = (q_1, q_2, w) \times (x_1, x_2, x_3)$ be the normal vector to $\mathrm{aff}(T_x)$ with lattice length equal to $\mathrm{Vol}(T_x)$. Then (4.5) yields

$$\begin{aligned}\mathrm{MV}(T_x, T_x, P_1) &= h_{P_1}((n_1, n_2, n_3)) + h_{P_1}(-(n_1, n_2, n_3))\\ &= h_{P'}((n_1, n_2)) + h_{P'}(-(n_1, n_2))\\ &= h_{P'-P'}((n_1, n_2)).\end{aligned}$$

One has $(n_1, n_2) = (-wx_2, wx_1) + (q_2x_3, -q_1x_3)$ by explicit computation of the cross product. By the monotonicity of the mixed volume one obtains $m_2 \geq \mathrm{MV}(P_2, P_2, P_1) \geq \mathrm{MV}(T_x, T_x, P_1)$ and therefore,

$$(-wx_2, wx_1) + (q_2x_3, -q_1x_3) = (n_1, n_2) \in m_2(P' - P')^*.$$

This is equivalent to

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \begin{pmatrix} 0 & \frac{m_2}{w} \\ -\frac{m_2}{w} & 0 \end{pmatrix} (P' - P')^* + \frac{1}{w} \begin{pmatrix} q_1 x_3 \\ q_2 x_3 \end{pmatrix},$$

which shows the assertion. □

Remark 4.6.6. Note that, in the setting of Lemma 4.6.5, the bounding box R_{q_1,q_2} is actually constructed under the assumption that P_2 contains the segment $I_{q_1,q_2} = \operatorname{conv}((0,0,0),(q_1,q_2,w))$. Therefore in order to enumerate the set of all suitable P_2 we may restrict to searching for all $q_1, q_2 \in \{0,\dots,w-1\}$ for lattice polytopes inside R_{q_1,q_2} that contain I_{q_1,q_2}. We use this fact when we apply Algorithm 4.7.7. Also note that any lattice polytope $P \in \mathcal{P}(\mathbb{Z}^3)$ with $I_{q_1,q_2} \subset P \subseteq R_{q_1,q_2}$ satisfies $\operatorname{MV}(P_1,P_1,P) = m_1$ by construction of R_{q_1,q_2}, while the upper bound of m_2 on $\operatorname{MV}(P,P,P_1)$ may in general not be satisfied for some subpolytope $P \subseteq R_{q_1,q_2}$.

4.7. An algorithm for the enumeration of lattice polytopes

Enumeration Problem 4.7.1. Given $m \in \mathbb{N}$ and $d \in \mathbb{N}$, enumerate all full-dimensional polytopes $P \in \mathcal{P}(\mathbb{Z}^d)$ with $1 \le \operatorname{Vol}(P) \le m$, up to equivalence.

4.7.1. Sandwich-factory based approach

We present a relatively simple algorithm to Enumeration Problem 4.7.1 which we also found to lead to very reasonable running times. The running time of the SageMath [Sag18] implementation of our algorithm was just a few minutes for $d = 3$ and $m = 4$. For $d = 3$, $m = 6$ our implementation terminates within an hour. For $d = 2$, much larger values of m can be handled within an hour. Even more important in the context of our original enumeration problem about mixed volumes is the fact that we use our algorithm for solving Enumeration Problem 4.7.1 as a template for solving further similar enumeration problems by appropriately modifying the basic steps of the algorithm (see Algorithm 4.7.7).

We also refer to [Bal18] for an alternative approach with a similar structure to enumeration of lattice polytopes by their volume. Note also that [EG16] provides an explicit description of lattice polytopes of arbitrary dimension d with the normalized volume at most 4.

We call a pair (A,B) of full-dimensional polytopes $A, B \in \mathcal{P}(\mathbb{Z}^d)$ a *sandwich* if A is a subset of B. The basic principle of our algorithm is to capture all possible polytopes in a set of sandwiches (A,B). If for $P \in \mathcal{P}(\mathbb{Z}^d)$ the inclusion $A \subseteq P \subseteq B$ holds, we say that P *occurs in the sandwich* (A,B). In our context, A will be a lattice polytope with $\operatorname{Vol}(A) \le m$ and B will be a lattice polytope containing all points $p \in \mathbb{Z}^d$ that one can add to A without exceeding the volume bound m. Our algorithm maintains a *sandwich factory*, which is a set of sandwiches with the property that each P in question occurs in some of the sandwiches from the set.

We call the difference $\mathrm{Vol}(B) - \mathrm{Vol}(A)$ the *volume gap* of a sandwich (A, B). The algorithm starts with a sandwich factory containing sandwiches with a large volume gap and iteratively replaces sandwiches with a large volume gap by sandwiches with a smaller volume gap. Eventually, only sandwiches with volume gap 0 remain; such sandwiches correspond to polytopes P with $\mathrm{Vol}(P) \leq m$. Thus, as soon as there are no sandwiches with positive volume gap, the enumeration task is completed.

4.7.2. Initialization of the sandwich factory.

It is clear that every full-dimensional lattice polytope $P \in \mathcal{P}(\mathbb{Z}^d)$ contains an empty lattice simplex A, that is a simplex with exactly $d+1$ lattice points. Also, if $\mathrm{Vol}(P) \leq m$ then, clearly, $\mathrm{Vol}(A) \leq m$. Thus, we start with a set of sandwiches (A, B) which involves all possible empty simplices A with $1 \leq \mathrm{Vol}(A) \leq m$. In dimension $d = 2$ there is only one empty simplex up to equivalence, namely the triangle $A = \Delta_2$. In dimension $d = 3$, by White's classification (see [Whi64] or [Rez06, Theorem 5]), every empty 3-dimensional simplex is equivalent to either the standard simplex Δ_3 or to $\mathrm{conv}(0, e_1, e_3, e_3 + ae_1 + be_2)$ with $a, b \in \mathbb{N}$, $a < b$, and $\gcd(a, b) = 1$. Thus, it suffices to determine such simplices A with $1 \leq \mathrm{Vol}(A) \leq m$. To complete the initialization of the sandwich factory, one needs to choose an appropriate B for each A so that (A, B) is a sandwich, which contains all lattice polytopes P with $1 \leq \mathrm{Vol}(P) \leq m$ and the property $A \subseteq P$. It is intuitively clear that if a point x is far away from A, then the volume $\mathrm{conv}(A \cup \{x\})$ must be large. This informal idea is expressed explicitly in the following lemma.

Lemma 4.7.2. *Let A be a d-dimensional simplex and let $m \geq \mathrm{Vol}(A)$. Then*

$$\left\{x \in \mathbb{R}^d \,:\, \mathrm{Vol}(\mathrm{conv}(A \cup \{x\})) \leq m\right\} \subseteq \lambda A + (1 - \lambda) c_A,$$

where c_A is the barycenter of A and $\lambda = (d+1)\left(\frac{m}{\mathrm{Vol}(A)} - 1\right) + 1$.

Proof. The proof for $d = 3$ can be found in [AKW17, Lemma 13]. The proof extends directly to the case of an arbitrary dimension $d \in \mathbb{N}$. □

In view of Lemma 4.7.2, one can fix B to be the integral hull of $\lambda A + (1 - \lambda)c_A$, that is

$$B = \mathrm{conv}((\lambda A + (1 - \lambda)c_A) \cap \mathbb{Z}^d).$$

It may still be the case that B is chosen to be too large in the sense that B may contain vertices v with the property that $\mathrm{Vol}(\mathrm{conv}(A \cup \{v\})) > m$. Clearly, if a polytope $P \in \mathcal{P}(\mathbb{Z}^d)$ occurs in (A, B) and has the property $\mathrm{Vol}(P) \leq m$ then P cannot contain v as above. This means that one can iteratively make B smaller by removing vertices v as above, as long as such vertices exist. More precisely, while v as above exists, one iteratively replaces B by $\mathrm{conv}((B \cap \mathbb{Z}^d) \setminus \{v\})$. We call this procedure the *reduction* of B relative to A. Having carried out the above reduction of B for each A, we complete the initialization of the sandwich factory.

4.7.3. Iterative updates of sandwich factory

The purpose of the iterative procedure is to reduce the maximum volume gap occurring in a sandwich factory. That is, as long as there are sandwiches with positive volume gap, one considers those sandwiches (A, B) in the sandwich factory, for which the volume gap $\mathrm{Vol}(B) - \mathrm{Vol}(A)$ is maximized. For each such sandwich (A, B) one picks a vertex v of B not belonging to A. Every polytope $P \in \mathcal{P}(\mathbb{Z}^d)$ with $\mathrm{Vol}(P) \le m$ occurring in (A, B) may or may not contain v. If P contains v, we can enclose P into the sandwich $(\mathrm{conv}(A \cup \{v\}), B)$ with a smaller volume gap. If P does not contain v, we can enclose P into the sandwich $(A, \mathrm{conv}((B \cap \mathbb{Z}^d) \setminus \{v\}))$, whose volume gap is also smaller. Thus, we can remove the sandwich (A, B) from the factory and replace it by two other sandwiches (see also Fig. 4.6).

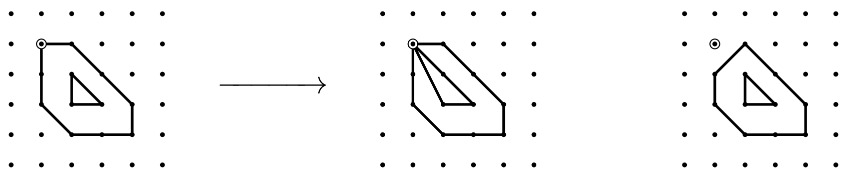


Figure 4.6.: Replacing a sandwich by two other sandwiches with a smaller volume gap.

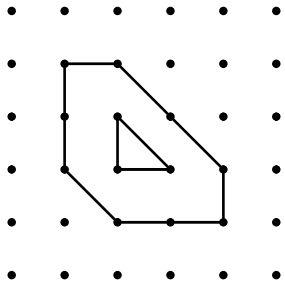
Here it should also be noticed that, when we let A grow, by considering $(\mathrm{conv}(A \cup \{v\}, B)$, we can make B smaller. Indeed, B may contain vertices w with the property that $\mathrm{Vol}(\mathrm{conv}(A \cup \{v, w\})) > m$. Then rather than adding the sandwich $(\mathrm{conv}(A \cup \{v\}, B)$, we first reduce B relative $\mathrm{conv}(A \cup \{v\})$ to a potentially smaller polytope B' and then add $(\mathrm{conv}(A \cup \{v\}), B')$ to the sandwich factory.

4.7.4. Equivalent sandwiches.

While the above algorithmic steps can already be used for finding all full-dimensional polytopes $P \in \mathcal{P}(\mathbb{Z}^d)$ with $\mathrm{Vol}(P) \le m$, its efficiency would not be very good as one would generate many polytopes that are equivalent. When for two sandwiches (A, B) and (A', B') there exists an affine unimodular transformation φ with $\varphi(A) = A'$ and $\varphi(B) = B'$, then, up to affine unimodular transformations, the lattice polytopes occurring in (A, B) also occur in (A', B') and vice versa. We call such sandwiches (A, B) and (A', B') *equivalent*. Thus, if a sandwich (A, B) is already present in the sandwich factory, there is no need to add (A', B'). Based on this idea, we add a new sandwich (A, B) to the sandwich factory only if it does not already contain a sandwich equivalent to (A, B). The test for equivalence of sandwiches can be reduced to the test for equivalence of lattice polytopes as follows. If (A, B) is a sandwich then, by embedding $3A$ into $\mathbb{R}^d \times \mathbb{R}$ at heights 1 and -1 and $3B$ at height 0, we obtain the polytope

$$P_{A,B} = \mathrm{conv}\Big(\underbrace{(3A) \times \{1\}}_{\text{height } 1} \cup \underbrace{(3B) \times \{0\}}_{\text{height } 0} \cup \underbrace{(3A) \times \{-1\}}_{\text{height } -1}\Big) \in \mathcal{P}(\mathbb{Z}^d \times \mathbb{Z}),$$

see also Fig. 4.7.



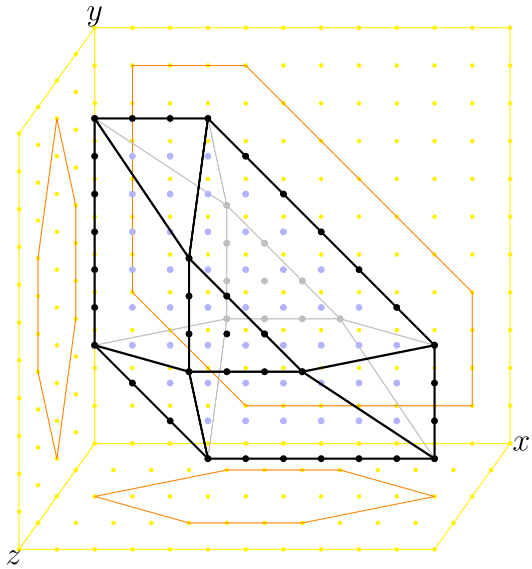


Figure 4.7.: An example of a sandwich (A, B) in dimension two (left) and the three-dimensional lattice polytope $P_{A,B}$ assigned to this sandwich (right), whose affine normal form is used to distinguish sandwiches up to affine unimodular transformations.

Lemma 4.7.3. *Two sandwiches (A, B) and (A', B') are equivalent if and only if the polytopes $P_{A,B}$ and $P_{A',B'}$ are equivalent.*

Proof. The first implication is direct. If (A, B) and (A', B') are equivalent, then there exists an affine unimodular transformation $\varphi \in \operatorname{Aff}(\mathbb{Z}^d)$ such that $\varphi(A) = A'$ and $\varphi(B) = B'$. The map $\varphi \times \operatorname{Id} \in \operatorname{Aff}(\mathbb{Z}^d \times \mathbb{Z})$ then satisfies $(\varphi \times \operatorname{Id})(P_{A,B}) = P_{A',B'}$. In order to show the reverse implication assume that $P_{A,B}$ and $P_{A',B'}$ are equivalent and let $\psi \in \operatorname{Aff}(\mathbb{Z}^d \times \mathbb{Z})$ be an affine unimodular transformation such that $\psi(P_{A,B}) = P_{A',B'}$. Note that for both $P = P_{A,B}$ and $P = P_{A',B'}$ the vector $v = e_{d+1}$ is the unique direction such that $|h_P(v) - h_P(-v)| = 2$. Therefore ψ maps the intersection of $P_{A,B}$ with any of the hyperplanes $\mathbb{R}^d \times \{-1\}$, $\mathbb{R}^d \times \{0\}$ and $\mathbb{R}^d \times \{1\}$ to the intersection of $P_{A',B'}$ with the respective hyperplane. Here we use that, as $P_{A,B}$ is symmetric with respect to the hyperplane $\mathbb{R}^d \times \{0\}$, we may assume that the intersections with the hyperplanes $\mathbb{R}^d \times \{-1\}$ and $\mathbb{R}^d \times \{1\}$ are not permuted by ψ. In particular, $\psi(3B \times \{0\}) = 3B' \times \{0\}$ and, as B is full-dimensional, $\psi(\mathbb{R}^d \times \{h\}) = \mathbb{R}^d \times \{h\}$ for any $h \in \mathbb{R}$. Furthermore, we may assume that ψ is linear and, hence, with respect to the standard basis of $\mathbb{R}^{d+1}$ to be of the form

$$\begin{pmatrix} U & t \\ 0 & 1 \end{pmatrix} \in \operatorname{GL}(\mathbb{Z}^{d+1}),$$

for a unimodular matrix $U \in \operatorname{GL}(\mathbb{Z}^d)$ and an integer vector $t \in \mathbb{Z}^d$. Denote by $\varphi \in \operatorname{GL}(\mathbb{Z}^d)$ the linear unimodular transformation corresponding to U. Then $\psi((3A \times \{-1\})) = (\varphi(3A) - t) \times \{-1\}$ and $\psi((3A \times \{1\})) = (\varphi(3A) + t) \times \{1\}$. In particular $\varphi(3A) - t = 3A' = \varphi(3A) + t$ and therefore one has $t = 0$. So φ is a unimodular transformation such that $\varphi(A) = A'$ and $\varphi(B) = B'$ and hence the sandwiches (A, B) and (A', B') are equivalent. □

Remark 4.7.4. The construction of $P_{A,B}$ is similar to the construction of the Cayley sum $\mathrm{Cay}(2A, 2B)$ in Corollary 2.3.4. However, the notion of equivalence of two sandwiches is more restrictive than the one for general pairs of lattice polytopes, as the position of A inside of B is important and we therefore do not want to allow independent translation.

4.7.5. Summary of the sandwich-factory algorithm.

We give a complete description of the algorithm we have developed above.

Algorithm 4.7.5 (Sandwich-factory algorithm)**.**

Input: Dimension $d \in \mathbb{N}$ and volume bound $m \in \mathbb{N}$.

Output: A list of all full-dimensional lattice polytopes $P \in \mathcal{P}(\mathbb{Z}^d)$ with $\mathrm{Vol}(P) \leq m$, up to equivalence.

Step 1: Enumerate, up to equivalence, all empty lattice simplices A with $\mathrm{Vol}(A) \leq m$.

- For each A as above, choose B to be the integral hull

$$\mathrm{conv}((\lambda A + (1 - \lambda)c_A) \cap \mathbb{Z}^d),$$

where c_A is the barycenter of A and

$$\lambda = (d + 1)\left(\frac{m}{\mathrm{Vol}(A)} - 1\right) + 1,$$

and then reduce B relative to A.

- Initialize the sandwich factory $\mathcal{F}$ with all pairs (A, B) obtained as above.

Step 2: While $\mathcal{F}$ contains sandwiches with a positive volume gap, carry out the following steps for sandwiches (A, B) whose volume gap is maximized:

- pick a vertex v of B, not contained in A,
- fix $A' = \mathrm{conv}(A \cup \{v\})$,
- compute the reduction B' of B relative to A',
- fix $B'' = \mathrm{conv}((B \cap \mathbb{Z}^d) \setminus \{v\})$,
- add (A', B') to $\mathcal{F}$, unless $\mathcal{F}$ already contains a sandwich equivalent to (A', B'),
- add (A, B'') to $\mathcal{F}$, unless $\mathcal{F}$ already contains a sandwich equivalent to (A, B''),
- remove (A, B) from $\mathcal{F}$.

Step 3: In this step, all sandwiches (A, B) in $\mathcal{F}$ have the form $A = B$. Return the set of all A, with $(A, B) \in \mathcal{F}$. Up to equivalence, this is the set of all polytopes $P \in \mathcal{P}(\mathbb{Z}^d)$ with $1 \leq \mathrm{Vol}(P) \leq m$.

Remark 4.7.6. In Section 4.7.2 we described an efficient implementation of Step 1 for dimension two and three (the dimensions we are interested in in the context of this paper). For higher dimensions we do not specify how to implement Step 1 and only notice that it can be implemented algorithmically. We also note that rather than using empty lattice simplices of normalized volume at most m, one can start with *all* lattice simplices of normalized volume at most m. This is the approach that is taken in [Bal18].

4.7.6. Sandwich type search for subpolytopes

In this section we describe our approach towards the task of finding all subpolytopes fulfilling certain conditions inside a given bounding polytope. For our purposes we found it computationally efficient to employ an algorithm similar to the one presented in Section 4.7. Note that we do not restrict ourselves to full-dimensional lattice polytopes in this modification. In our classification there occur three different variations of this task that we solve using three different variations a., b., and c. of Algorithm 4.7.7. In particular, variation a. is employed for the search for full-dimensional subpolytopes inside of a maximal polytope as obtained using Algorithm 4.6.2, while variation b. and c. deal with the search for full-dimensional or 2-dimensional subpolytopes of a bounding polytope as in Lemma 4.6.5. Note that a sandwich type search seems particularly natural for the search inside bounding polytopes of the form obtained using Lemma 4.6.5, as by Remark 4.6.6 it suffices to search for those subpolytopes that contain a given segment I depending on the bounding polytope.

For a sandwich $(A, B) \in \mathcal{P}(\mathbb{Z}^3)^2$ we call the nonnegative number $|B \cap \mathbb{Z}^3| - |A \cap \mathbb{Z}^3|$ the *lattice point gap* of (A, B). Generalizing the concept of the reduction of a lattice polytope $B \supseteq A$ relative to A used in Algorithm 4.7.5, we define the *reduction of B relative to A with respect to the conditions* $\mathrm{Vol}(\,\cdot\,) \leq m_1$ *and* $\mathrm{MV}(\,\cdot\,, \,\cdot\,, P_1) \leq m_2$ to be the polytope

$$B' = \mathrm{conv}\{x \in B \cap \mathbb{Z}^3 : A_x = \mathrm{conv}(A \cup \{x\}) \text{ satisfies} \\ \mathrm{Vol}(A_x) \leq m_1 \text{ and } \mathrm{MV}(A_x, A_x, P_1) \leq m_2\}.$$

Algorithm 4.7.7 (Sandwich approach to subpolytopes)**.**

Input:

for a: A bounding box $M \in \mathcal{P}(\mathbb{Z}^3)$, a lattice polytope $P_1 \in \mathcal{P}(\mathbb{Z}^3)$, and a bound $m \in \mathbb{N}$.

for b/c: A bounding box $M \in \mathcal{P}(\mathbb{Z}^3)$, a segment $I \subset M$, a lattice polytope $P_1 \in \mathcal{P}(\mathbb{Z}^3)$, and bounds $m_1, m_2 \in \mathbb{N}$.

Output:

for a: A list of all full-dimensional lattice polytopes P_2, up to translations, with $P_2 \subseteq M$ such that $\mathrm{Vol}(P_2) \leq m$ and $\mathrm{MV}(P_2, P_2, P_1) \leq m$.

for b: A list of all full-dimensional lattice polytopes P_2, up to translations, with $I \subset P_2 \subseteq M$ such that $\mathrm{Vol}(P_2) \leq m_1$ and $\mathrm{MV}(P_2, P_2, P_1) \leq m_2$.

for c: A list of all 2-dimensional lattice polytopes P_2, up to translations, with $I \subset P_2 \subset M$ such that $\mathrm{Vol}(P_2) \leq m_1$ and $\mathrm{MV}(P_2, P_2, P_1) \leq m_2$.

Step 1:

for a: Initialize the sandwich factory $\mathcal{F}$ with pairs (S, M) where S ranges over all empty simplices in M satisfying the bounding conditions (in particular, $\mathrm{MV}(S, S, P_1) \leq m$). Set $m_1 = m_2 = m$.

for b/c: Initialize the sandwich factory $\mathcal{F}$ with the pair (I, M'), where M' is the reduction of M relative to I with respect to the conditions $\mathrm{Vol}_r(\,\cdot\,) \leq m_1$ and $\mathrm{MV}(\,\cdot\,,\,\cdot\,, P_1) \leq m_2$.

Step 2: While $\mathcal{F}$ contains sandwiches with positive lattice point gap, carry out the following steps for sandwiches (A, B) having the maximal lattice point gap among the sandwiches in $\mathcal{F}$:

- pick a vertex v of B, not contained in A,
- fix $A' = \mathrm{conv}(A \cup \{v\})$ (note that, as B is reduced relative to A with respect to the conditions $\mathrm{Vol}(\,\cdot\,) \leq m_1$ and $\mathrm{MV}(\,\cdot\,,\,\cdot\,, P_1) \leq m_2$, the polytope A' is ensured to satisfy the bounding conditions),
- (for c:) if $\dim(A') = 2$ and $\dim(B) = 3$, set $B := B \cap \mathrm{aff}(A')$,
- compute the reduction B' of B relative to A with respect to the conditions $\mathrm{Vol}_r(\,\cdot\,) \leq m_1$ and $\mathrm{MV}(\,\cdot\,,\,\cdot\,, P) \leq m_2$,
- fix $B'' = \mathrm{conv}((B \cap \mathbb{Z}^d) \setminus \{v\})$,
- add (A', B') to $\mathcal{F}$, unless $\mathcal{F}$ already contains a translate of (A', B'),
- add (A, B'') to $\mathcal{F}$, unless $\mathcal{F}$ already contains a translate of (A, B''),
- remove (A, B) from $\mathcal{F}$.

Step 3: In this step all sandwiches (A, B) have lattice point gap 0 and therefore fulfill $A = B$. Return a list of A for all sandwiches $(A, B) \in \mathcal{F}$.

Remark 4.7.8. While the overall structure of Algorithm 4.7.7 above is very similar to Algorithm 4.7.5, there are some modifications. In Algorithm 4.7.7 we also work with sandwiches (A, B) for which $\dim(A) < \dim(B)$ and therefore the volume gap is not necessarily strictly decreasing in our iterative steps. We deal with this by considering the lattice point gap of a sandwich instead. Furthermore, we only identify two sandwiches (A, B) and (A', B') if there is a translation vector $t \in \mathbb{Z}^3$ such that $(A', B') = (A + t, B + t)$. Also note that, in addition to a volume bound for P_2, we have a bound for the mixed volume $\mathrm{MV}(P_2, P_2, P_1)$ and therefore perform a slightly different reduction step.

4.8. Outlook: classification for larger mixed volume or dimension

Recall that, apart from 3 exceptions, the full-dimensional triples of lattice polytopes up to mixed volume 4 fall into one of 4 types of $\mathbb{R}$-maximal triples presented in Proposition 4.3.4. We do not know how the number of exceptional tuples behaves when we regard higher mixed volumes or higher dimensions. Nevertheless, it seems to be a reasonable approach towards further enumerations to concentrate on full-dimensional tuples that are $\mathbb{R}$-maximal. A first step would be the understanding of $\mathbb{R}$-maximal triples of general mixed volume.

Question 4.8.1. Does every $\mathbb{R}$-maximal triple of full-dimensional lattice polytopes $(P_1, P_2, P_3) \in (\mathcal{P}(\mathbb{Z}^3))^3$ satisfy $\operatorname{supp} S_{P_1} \subseteq \operatorname{supp} S_{P_2} = \operatorname{supp} S_{P_3}$, up to reordering? Is every such triple of one of the types of Proposition 4.3.4?

If the answer to these questions is yes, this would significantly reduce the complexity of the classification of $\mathbb{R}$-maximal triples of higher mixed volume, as one could tailor a classification algorithm precisely for these types.

For higher dimensions, we have to pose the question in an even more open way. We suspect a steep increase of the number of different types of $\mathbb{R}$-maximal tuples. However, in dimension 4, one might be able to make some progress.

Question 4.8.2. Can one deduce a more specific structure for $\mathbb{R}$-maximal 4-tuples of full dimensional lattice polytopes $(P_1, P_2, P_3, P_4) \in (\mathcal{P}(\mathbb{Z}^4))^4$ from Lemma 4.2.15? What is a reasonable generalization of the list of types from Proposition 4.3.4?

5. Tuples of Lattice Polytopes of Mixed Degree One

This chapter is devoted to presenting structural classification results regarding d-tuples of d-dimensional lattice polytopes whose so-called mixed degree is at most one. In Section 5.1 we give some background on the definition of the mixed degree of a tuple of lattice polytopes as a generalization of the concept of the lattice degree of a single lattice polytope. In the course of this we also recall some classification results on lattice polytopes with a small degree (Theorem 5.1.2), which provide an outline of what kind of results we would like to generalize to a mixed setting. In Section 5.2 we present and illustrate our main structural results on tuples of lattice polytopes of mixed degree one. These are, on the one hand, a finiteness result on non-trivial classes of such tuples in general dimension at least 4 (Theorem 5.2.3), and the complete classification of such tuples in dimension 3 (Theorem 5.2.4). Sections 5.3 and 5.4 deal with the proofs of these two results respectively. Finally, in Section 5.5 we present additional insights regarding the non-trivial tuples in dimension 3, which motivate a more explicit question about what one could expect in higher dimensions.

5.1. The mixed degree of a tuple of lattice polytopes

It has been shown by Ehrhart that, for any lattice polytope $P \in \mathcal{P}(\mathbb{Z}^d)$, the function $t \mapsto |tP \cap \mathbb{Z}^d|$ is given by a polynomial of degree $\dim(d)$. This polynomial is called the *Ehrhart polynomial* of P and is denoted by $\mathrm{Ehr}_P(t)$. This has first been shown in [Ehr62]. By writing the Ehrhart polynomial in a binomial basis as

$$\mathrm{Ehr}_P(t) = h_0^* \binom{t+d}{d} + h_1^* \binom{t+d-1}{d} + \cdots + h_d^* \binom{t}{d},$$

we obtain the *h^*-polynomial* of P as

$$h_P^*(t) = h_0^* + h_1^* t + \cdots + h_d^* t^d.$$

We refer to [BR15] for a detailed treatment of the theory behind counting lattice points in polyhedra. The *degree* of a lattice polytope P is defined as the degree of its h^*-polynomial $\deg(P) = \deg(h_P^*)$. In order to provide another, more geometric, point of view towards the degree, let us also introduce the *codegree* of a lattice polytope P as $\operatorname{codeg}(P) = 1 + \dim(P) - \deg(P)$.

Proposition 5.1.1. *Let $P \in \mathcal{P}(\mathbb{Z}^d)$. Then* $\operatorname{codeg}(P)$ *equals the smallest natural number $n \in \mathbb{Z}_{\geq 1}$ for which nP is not hollow.*

See for example [BR15, Theorem 4.5] for a proof of Proposition 5.1.1. The degree of a lattice polytope can be seen as describing the complexity of the polytope with respect to the lattice. Having this view in mind one expects polytopes of low degree to have a specifically easy structure and indeed one has the following results supporting that view.

Theorem 5.1.2 (Lattice polytopes of small degree)**.** *Let $P \in \mathcal{P}(\mathbb{Z}^d)$ be a lattice polytope with degree* $\deg(P) = r$*.*

1. *The lattice pyramid* $\mathrm{Pyr}(P) \in \mathcal{P}(\mathbb{Z}^{d+1})$ *satisfies* $\deg(\mathrm{Pyr}(P)) = r$*.*
2. *One has $r = 0$ if and only if $P \cong \Delta_d$.*
3. *One has $r \leq 1$ if and only if one of the following holds:*
   a) *P is a* Lawrence prism*, that is it is of the form* $\mathrm{cay}(I_1, \ldots, I_d)$ *for lattice segments $I_1, \ldots, I_d \in \mathcal{P}(\mathbb{Z}^1)$, or*
   b) *P is an* exceptional simplex*, that is $P \cong \mathrm{Pyr}^{n-2}(2\Delta_2)$.*
4. *P is the (possibly trivial) Cayley sum of lattice polytopes in $\mathcal{P}(\mathbb{Z}^q)$ for some $q \leq (r^2 + 19r - 4)/2$.*

Parts (1) and (2) are well known facts. Part (3) has been shown in [BN07] and part (4) has been shown in [HNP08].

Recently the following generalization of the degree to tuples of lattice polytopes has been introduced by Nill ([Nil20]).

Definition 5.1.3 (Mixed degree)**.** Let $(P_1, \ldots, P_k) \in (\mathcal{P}(\mathbb{Z}^d))^k$ be a k-tuple of lattice polytopes in $\mathbb{R}^d$. The *mixed codegree* $\mathrm{mcd}(P_1, \ldots, P_k)$ is the smallest natural number $n \in \mathbb{Z}_{\geq 1}$ such that for any $I \subseteq [k]$ with $|I| = n$ the Minkowski sum $\sum_{i \in I} P_i$ has a lattice point in its relative interior. If such a number does not exist (that is if $P_1 + \cdots + P_k$ is hollow) we set $\mathrm{mcd}(P_1, \ldots, P_k) := k + 1$. The *mixed degree* is defined as $\mathrm{md}(P_1, \ldots, P_k) = 1 + \dim(P_1 + \cdots + P_k) - \mathrm{mcd}(P_1, \ldots, P_k)$.

Remark 5.1.4. The mixed degree generalizes the degree of a single lattice polytope. Consider a lattice polytope $P \in \mathcal{P}(\mathbb{Z}^d)$. Then Proposition 5.1.1 implies that one has

$$\deg(P) = \mathrm{md}(\underbrace{P, \ldots, P}_{\dim(P) \text{ times}}).$$

A first research direction regarding the mixed degree is to investigate in which sense the intuition of the mixed degree presenting a measure of the complexity of a tuple can be made precise. In particular, we are interested in generalizing parts of Theorem 5.1.2 to the mixed degree.

A first result in this direction is the following generalization of Theorem 5.1.2 (2) which has been shown in [Nil20]

Proposition 5.1.5 ([Nil20, Theorem 2.2])**.** *Let $(P_1, \ldots, P_d) \in \mathcal{P}(\mathbb{Z}^d)^d$ such that $\dim(P_1 + \cdots + P_d) = d$ and $\dim(P_i) \geq 1$ for all $i \in [d]$. Then the following are equivalent:*

1. $\operatorname{md}(P_1, \ldots, P_d) = 0$,
2. $\operatorname{MV}(P_1, \ldots, P_d) = 1$.

In particular, if one assumes $\dim(P_1) = \cdots = \dim(P_d) = d$ *then* $\operatorname{md}(P_1, \ldots, P_d) = 0$ *if and only if* $(P_1, \ldots, P_d) \cong (\Delta_d, \ldots, \Delta_d)$.

The fact that $\operatorname{MV}(P_1, \ldots, P_d) = 1$ implies $(P_1, \ldots, P_d) \cong (\Delta_d, \ldots, \Delta_d)$ if all P_i are full-dimensional has first been shown by Cattani et al. in [CCD+13]. It can also be deduced as a special case from the more general classification result of Esterov and Gusev ([EG15]).

Our contribution focuses on generalizing part (3) of Theorem 5.1.2. We restrict to d-tuples $(P_1, \ldots, P_d) \in \mathcal{P}(\mathbb{Z}^d)^d$ of lattice polytopes in $\mathbb{R}^d$ that are full-dimensional. This subclass of tuples is of particular interest by the following result that Soprunov showed by combining the BKK-theorem with a generalization of the Euler-Jacobi theorem due to Khovanskii ([Kho78]) in the context of sparse polynomial interpolation.

Theorem 5.1.6 ([Sop07, Nil20])**.** *Let* $(P_1, \ldots, P_d) \in \mathcal{P}(\mathbb{Z}^d)^d$ *be a* d*-tuple of full-dimensional lattice polytopes. Then the following inequality holds:*

$$|\operatorname{int}(P_1 + \cdots + P_d) \cap \mathbb{Z}^d| \geq \operatorname{MV}(P_1, \ldots, P_d) - 1.$$

Furthermore, equality holds if and only if $\operatorname{md}(P_1, \ldots, P_d) \leq 1$.

Those tuples for which equality holds in the above theorem have already been called tuples *of mixed degree at most one* by Soprunov in [BNR+08], where a characterization of such tuples has been posed as an open problem ([BNR+08, Section 5, Problem 2]).

5.2. Results

As we are aiming towards a generalization of Theorem 5.1.2 (3), let us define the following generalization of a Lawrence prism to tuples. As Lawrence prisms present a generic class of lattice polytopes of degree one, these tuples present a generic class of mixed degree one.

Definition 5.2.1 (Mixed Lawrence Prism)**.** A *mixed Lawrence prism* is a tuple $(P_1, \ldots, P_d) \in \mathcal{P}(\mathbb{Z}^d)^d$ that satisfies:

$$(P_1, \ldots, P_d) \cong (\operatorname{cay}(I_1^1, \ldots, I_d^1), \ldots, \operatorname{cay}(I_1^d, \ldots, I_d^d)),$$

where $I_i^j \in \mathcal{P}(\mathbb{Z}^1)$ is a lattice segment for all $i, j \in [d]$.

Proposition 5.2.2. *A tuple* $(P_1, \ldots, P_d) \in \mathcal{P}(\mathbb{Z}^d)^d$ *is a mixed Lawrence prism if and only if there exists a lattice projection* $\pi \colon \mathbb{R}^d \to \mathbb{R}^{d-1}$ *satisfying* $\pi(P_i) = \Delta_{d-1} + t_i$ *for some translation vector* $t_i \in \mathbb{Z}^{d-1}$ *for all* $i \in [d]$*. In particular, if* $(P_1, \ldots, P_d)$ *is a mixed Lawrence prism, then* $\operatorname{md}(P_1, \ldots, P_d) \leq 1$.

Proof. The first statement follows from Proposition 2.1.4. If a lattice projection π as in the statement exists, then for any $I \subseteq [d]$ one has $\pi(P_I) = |I|\Delta_{d-1} + \sum_{i \in I} t_i$. If $|I| \leq d-1$, this implies that $\pi(P_I)$ and therefore also P_I is hollow. So $\operatorname{md}(P_1, \ldots, P_d) \leq 1$. ∎

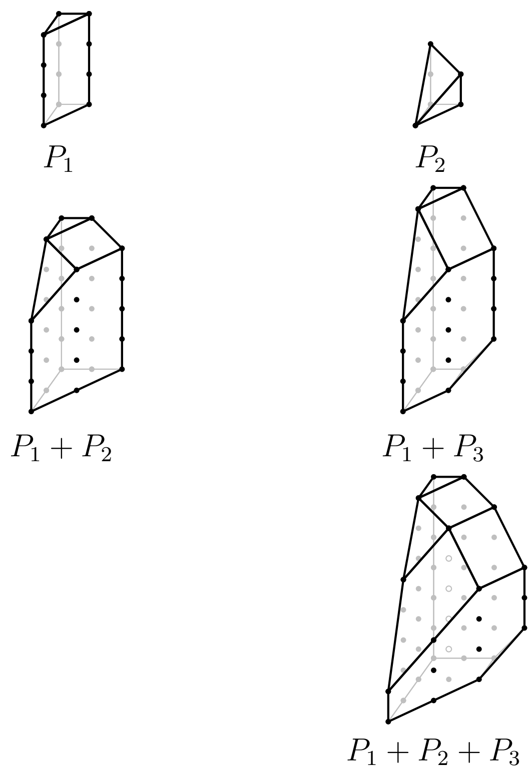


Figure 5.1.: A triple $P_1, P_2, P_3 \subset \mathbb{R}^3$ having mixed degree one, where P_1, P_2, P_3 all project onto Δ_2 under the projection along the vertical axis.

An example of a mixed Lawrence prism in dimension $n = 3$ is shown in Figure 5.1.

Clearly, we cannot expect this to be the only class of tuples of mixed degree one, as already the unmixed setting of Theorem 5.1.2 (3) additionally yields tuples of copies of the same exceptional simplex as having mixed degree one. Unlike in the unmixed case, there actually exist many more such non-trivial examples (see our classification result for $n = 3$ in Theorem 5.2.4, one example is shown in Figure 5.2).

This raises the question whether there is any chance of obtaining a reasonable generalization of Theorem 5.1.2 (3) (and an answer to Soprunov's question) at all. Our main result is to provide a positive answer to this question by showing that, for any dimension d at least 4, all but finitely many exceptions of d-tuples of mixed degree one are actually mixed Lawrence prisms.

Theorem 5.2.3. *Fix $d \geq 4$ and let $(P_1, \ldots, P_d) \in \mathcal{P}(\mathbb{Z}^d)^d$ be a tuple of full-dimensional lattice polytopes satisfying* $\operatorname{md}(P_1, \ldots, P_d) = 1$. *Then, up to equivalence, the tuple $(P_1, \ldots, P_d)$ either belongs to a finite list of exceptions or there is a lattice projection $\pi \colon \mathbb{R}^d \to \mathbb{R}^{d-1}$ such that $\pi(P_i) = \Delta_{d-1}$ for all $i \in [d]$.*

We refer the reader to Section 3 for the proof of Theorem 5.2.3.

Theorem 5.2.3 is not true for dimension $d \in \{2, 3\}$. This fact is straightforward to see for $d = 2$, as a pair of lattice polygons $(P_1, P_2) \in \mathcal{P}(\mathbb{Z}^2)^2$ is of mixed degree (at most) one if and only if both P_1 and P_2 are hollow. Fixing P_1 to be any hollow polygon and letting P_2 range through all polygons that are equivalent to a fixed

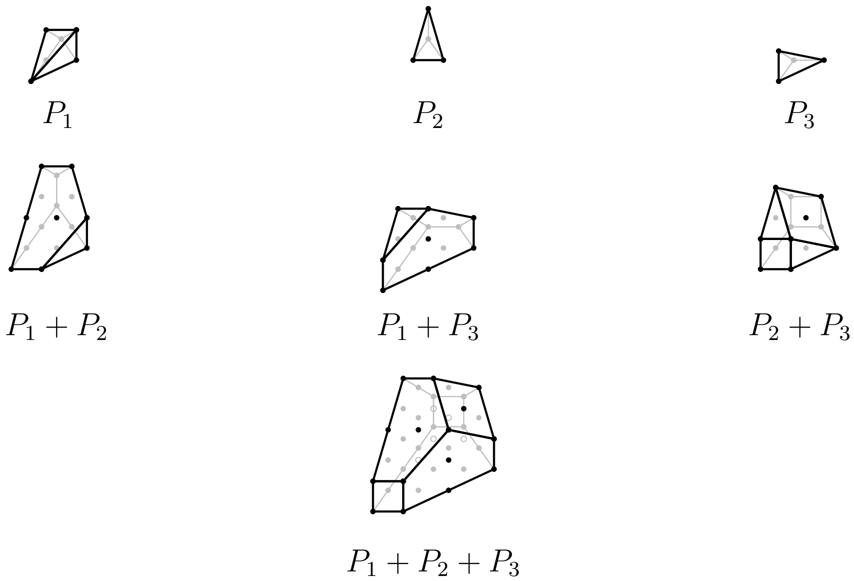


Figure 5.2.: A triple $P_1, P_2, P_3 \subset \mathbb{R}^3$ having mixed degree one for which no lattice projection exists commonly mapping P_1, P_2, P_3 onto translates of Δ_2 (see (d) of Corollary 5.5.4).

hollow polygon will clearly yield infinitely many non-equivalent pairs of mixed degree one without there being a projection commonly mapping both polytopes onto the segment Δ_1.

For $d = 3$, however, we find that only a very specific class of triples of mixed degree one contains an infinite number of exceptions and we can explicitly describe a finite number of 1-parameter families covering this class. This is part of the following classification result, which essentially gives a complete answer to Soprunov's problem for dimension $d = 3$. We say that a k-tuple of d-dimensional lattice polytopes $(P_1, \ldots, P_k) \in \mathcal{P}(\mathbb{Z}^d)^k$ *admits a common lattice projection* onto translates of an $(d-1)$-dimensional lattice polytope Q if there exists a lattice projection $\pi \colon \mathbb{R}^d \to \mathbb{R}^{d-1}$ satisfying $\pi(P_i) = Q + t_i$ for all $i \in [k]$ and some $t_i \in \mathbb{Z}^{d-1}$.

Theorem 5.2.4. *Let $(P_1, P_2, P_3) \in \mathcal{P}(\mathbb{Z}^3)^3$ be a triple of full-dimensional lattice polytopes that satisfies $\operatorname{md}(P_1, P_2, P_3) = 1$. Then either there is a lattice projection $\pi \colon \mathbb{R}^3 \to \mathbb{R}^2$ such that $\pi(P_i) = \Delta_2$ for all $i \in [3]$, or one of the following holds.*

1. *There is no pair in (P_1, P_2, P_3) admitting a common lattice projection onto translates of Δ_2 and (P_1, P_2, P_3) is equivalent to one out of 29 possible triples.*

2. *There is exactly one pair in (P_1, P_2, P_3) admitting a common lattice projection onto translates of Δ_2 and (P_1, P_2, P_3) is equivalent to one out of 141 possible triples.*

3. *There are exactly two pairs in (P_1, P_2, P_3) admitting a common lattice projection onto translates of Δ_2 and (P_1, P_2, P_3) is equivalent to one out of 82 possible triples.*

4. *All pairs in (P_1, P_2, P_3) admit a common lattice projection onto translates of Δ_2 and*
   a) *the kernels of the projections cannot be shifted into a common hyperplane, and the triple (P_1, P_2, P_3) is equivalent to one out of 27 possible triples.*
   b) *the kernels of the projections can be shifted into a common hyperplane, and (P_1, P_2, P_3) belongs, up to equivalence, to one out of finitely many infinite 1-parameter families of triples.*

We refer the reader to Section 4 for a proof of Theorem 5.2.4.

In the following example we present one of the 1-parameter families from Theorem 5.2.4 (4).

Example 5.2.5. Let $(P_1^k, P_2^k, P_3) \in \mathcal{P}(\mathbb{Z}^3)^3$ be the triple given by

$$P_1^k := \operatorname{conv}(e_1, e_2, e_1 + e_2, 2e_2, ke_2 + e_3),$$

$$P_2^k := \operatorname{conv}(e_1, e_2, e_1 + e_2, 2e_1, ke_1 + e_3),$$

$$P_3 := \operatorname{conv}(\mathbf{0}, e_1, e_2, e_1 + e_2, e_3),$$

for some $k \in \mathbb{Z}_{\geq 0}$. Then $\operatorname{md}(P_1^k, P_2^k, P_3) = 1$ and, while all pairs in (P_1^k, P_2^k, P_3) admit a common lattice projection onto translates of Δ_2, there is no lattice projection commonly mapping the whole triple (P_1^k, P_2^k, P_3) onto translates of Δ_2. Note that P_1^k, P_2^k and P_3 as single lattice polytopes are all equivalent to $\operatorname{Pyr}(\square_2)$ for all $k \in \mathbb{Z}_{\geq 0}$.

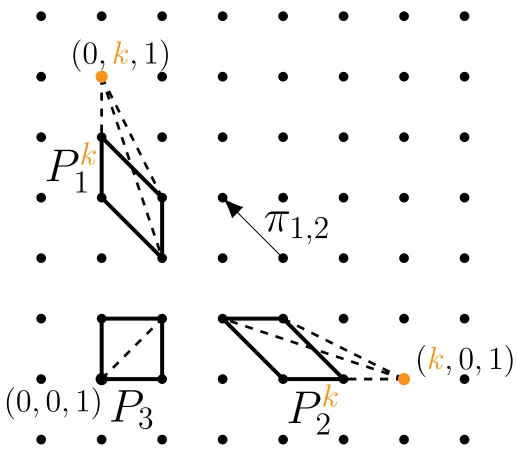


Figure 5.3.: Top view of the infinite family from Example 5.2.5. The arrow labeled $\pi_{1,2}$ shows the direction of the common projection of P_1^k and P_2^k onto translates of Δ_2. The common projections of P_3 and P_1^k as well as P_3 and P_2^k are given by the projection along the second and the first coordinate respectively.

All computations have been carried out using Magma [BCP97] and the code can be found at `https://github.com/christopherborger/mixed_degree_one`.

5.3. Proof of Theorem 5.2.3

From Proposition 2.1.4 we can easily deduce the following lemma.

Lemma 5.3.1. *Let $P \in \mathcal{P}(\mathbb{Z}^d)$ be a full-dimensional lattice polytope and $\pi\colon \mathbb{R}^d \to \mathbb{R}^{d-1}$ a linear lattice projection that projects P onto Δ_{d-1}. Then $\ker \pi = \mathbb{R}e$ where e is a vector parallel to an edge between vertices v_1, v_2, where $v_1 \in F_1$ and $v_2 \in F_2$ for two different unimodular facets $F_1 \neq F_2$ of P.*

We now study d-dimensional polytopes projecting onto Δ_{d-1} along multiple directions. In this section we will several times use the terms of two projections being "the same" or "different". Note that we only consider two lattice projections to be different if they do not have the same kernel. In particular, if $\pi\colon \mathbb{R}^{d_1} \to \mathbb{R}^{d_2}$ is a lattice projection and $\varphi \in \operatorname{Aff}(\mathbb{Z}^{d_2})$ an affine unimodular transformation, we consider π and $\varphi \circ \pi$ to not be different projections.

Recall that we denote by $\operatorname{Pyr}^{d-2}(\square_2)$ the $(d-2)$-fold lattice pyramid formed over the square $\square_2 = \operatorname{conv}(\mathbf{0}, e_1, e_2, e_1 + e_2) \in \mathcal{P}(\mathbb{Z}^2)$.

Lemma 5.3.2. *Let $P \in \mathcal{P}(\mathbb{Z}^d)$ be a full-dimensional lattice polytope such that there are different lattice projections $\pi_1, \pi_2\colon \mathbb{R}^d \to \mathbb{R}^{d-1}$ that map P onto Δ_{d-1}. Then P is equivalent either to the unimodular simplex Δ_d or to $\operatorname{Pyr}^{d-2}(\square_2)$. If there exists another projection $\pi_3\colon \mathbb{R}^d \to \mathbb{R}^{d-1}$ mapping P onto Δ_{d-1}, then P is necessarily equivalent to Δ_d.*

Proof. As P has one projection onto Δ_{d-1}, by Proposition 2.1.4 we may assume that P is of the form $P = \operatorname{cay}(I_1, \dots, I_d)$ for d segments $I_i = [0, a_i]$ with $a_i \in \mathbb{Z}_{\geq 0}$. Two facets of P are given by $\Delta_{d-1} \times \{0\}$ and $\operatorname{cay}(\{a_1\}, \dots, \{a_d\})$. All other facets of P are of a form we denote by F_k for $k \in [d]$, that is they are the Cayley sum of all I_i excluding I_k. As there exists another lattice projection π_2 mapping P onto Δ_{d-1}, by Lemma 5.3.1 the facet F_k has to be unimodular for some $1 \leq k \leq d$. Assume without loss of generality that F_1 is unimodular and $a_2 = 1$ and $a_3 = \cdots = a_d = 0$. Furthermore, a_1 cannot be greater than one as otherwise P would have an edge of lattice length at least 2. Therefore any projection which is not along this edge direction could not be projecting P onto Δ_{d-1}. If $a_1 = 0$, then P is equivalent to Δ_d, otherwise $a_1 = 1$ and P is equivalent to $\operatorname{Pyr}^{d-2}(\square_2)$. One easily verifies that $\operatorname{Pyr}^{d-2}(\square_2)$ does not have more than two different projections onto Δ_{d-1}. $\square$

Lemma 5.3.3. *Let $S_1, S_2 \in \mathcal{P}(\mathbb{Z}^d)$ be two unimodular full-dimensional simplices, $u_1, u_2 \in \mathbb{Z}^d$ be linearly independent edge directions for S_1 and S_2 respectively, and $C_1, C_2 \subset \mathbb{R}^d$ be the infinite prisms $S_1 + \mathbb{R}u_1$ and $S_2 + \mathbb{R}u_2$ respectively. Given $z \in \mathbb{Z}^d$, denote by P_z the intersection $\operatorname{conv}(C_1 \cap (C_2 + z) \cap \mathbb{Z}^d)$. Let $v, w \in \mathbb{Z}^d$ such that P_v and P_w are both d-dimensional. Then P_v and P_w are the same lattice polytope up to translation.*

Proof. By Lemma 5.3.2 we know that, if there exists $v \in \mathbb{Z}^d$ such that P_v is d-dimensional, then P_v is equivalent either to Δ_d or to $\operatorname{Pyr}^{d-2}(\square_2)$ having two edges parallel to the directions u_1 and u_2. In either of the two cases, we can assume P_v to be exactly Δ_d or $\operatorname{Pyr}^{d-2}(\square_2)$.

If $P_v = \mathrm{Pyr}^{d-2}(\square_2)$ then, up to reordering and changes of signs, $u_1 = e_1$ and $u_2 = e_2$. In particular, $C_1 = \mathrm{conv}(\mathbf{0}, e_2, \ldots, e_d) + \mathbb{R}e_1$ and $C_2 + v = \mathrm{conv}(\mathbf{0}, e_1, e_3, \ldots, e_d) + \mathbb{R}e_2$. One easily verifies that $C_1 \cap (C_2 + w)$ is full-dimensional if and only if $w - v \in \mathbb{Z}e_1 + \mathbb{Z}e_2$. In all these cases $C_1 \cap (C_2 + w)$ is a translation of P_v.

On the other hand, if $P_v = \Delta_d$, then there is another case distinction. If u_1 and u_2 are parallel to adjacent edges of D_d, then we can assume $u_1 = e_1$ and $u_2 = e_2$. But in this case C_1 and $C_2 + v$ must intersect in $\mathrm{Pyr}^{d-2}(\square_2)$ instead of in Δ_d, hence we have a contradiction. Therefore u_1 and u_2 are parallel to non-adjacent edges of Δ_d and we can assume $u_1 = e_1$ and $u_2 = e_2 - e_3$. In particular, $C_1 = \mathrm{conv}(\mathbf{0}, e_2, \ldots, e_d) + \mathbb{R}e_1$ and $C_2 + v = \mathrm{conv}(\mathbf{0}, e_1, e_3, \ldots, e_d) + \mathbb{R}(e_2 - e_3)$. Again, one easily verifies that $C_1 \cap (C_2 + w)$ is full-dimensional if and only if $w - v \in \mathbb{Z}e_1 + \mathbb{Z}(e_2 - e_3)$. In all these cases $C_1 \cap (C_2 + w)$ is a translation of Δ_d. $\square$

Lemma 5.3.4. *Let $P \in \mathcal{P}(\mathbb{Z}^d)$ be a unimodular d-simplex and $\pi_1, \pi_2 \colon \mathbb{R}^d \to \mathbb{R}^{d-1}$ be two different lattice projections such that, for each $i \in [2]$, the images $\pi_i(P)$ and $\pi_i(\Delta_d)$ are translates of Δ_{d-1}. Then, up to translation and coordinate permutation, P is contained in $\mathrm{Pyr}^{d-2}(\square_2)$. If there exists another projection $\pi_3 \colon \mathbb{R}^d \to \mathbb{R}^{d-1}$ mapping P and Δ_d onto translates of Δ_{d-1}, then P is necessarily a translate of Δ_d.*

Proof. By Lemma 5.3.1, π_1 and π_2 are projections along the directions u_1 and u_2 of two edges of Δ_d. If u_1 and u_2 are the directions of two adjacent edges of Δ_d, we can suppose that $u_1 = e_1$ and $u_2 = e_2$. Then P is contained in the intersection $(C_1 + z_1) \cap (C_2 + z_2)$ where $C_1 := \Delta_d + \mathbb{R}e_1$ and $C_2 := \Delta_d + \mathbb{R}e_2$, for some $z_1, z_2 \in \mathbb{Z}^d$. By Lemma 5.3.3, P is, up to translation, contained in $C_1 \cap C_2 = \mathrm{Pyr}^{d-2}(\square_2)$. If u_1 and u_2 are the directions of two non-adjacent edges of Δ_d then we can suppose that $u_1 = e_1$ and $u_2 = e_2 - e_3$. Then P is contained in the intersection $(C_1 + z_1) \cap (C_2 + z_2)$ where $C_1 := \Delta_d + \mathbb{R}e_1$ and $C_2 := \Delta_d + \mathbb{R}(e_2 - e_3)$, for some $z_1, z_2 \in \mathbb{Z}^d$. By Lemma 5.3.3, P is, up to translation, contained in $C_1 \cap C_2 = \Delta_d$, therefore P is a translate of $\Delta_d \subset \mathrm{Pyr}^{d-2}(\square_2)$. This proves the first part of the statement.

For the second part of the statement we note that π_3 must also be a projection along the direction u_3 of an edge of Δ_d. The only case we need to check is when the edges parallel to u_1, u_2 and u_3 form a triangle in Δ_d. Indeed in all the other cases two of these edges are non-adjacent and P must be a translate of Δ_d as above. Indeed, if this is not the case either two of these edges are non-adjacent and P must be a translate of Δ_n as above, or u_1,u_2 and u_3 share a vertex. In the latter case we may assume $u_i = \mathbf{e}_i$ for $1 \le i \le 3$. As deduced above from Lemma 5.3.3, this in particular yields that P is contained in the intersection of a translation of the square pyramid $\mathrm{conv}(\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_3, \ldots, \mathbf{e}_n)$ with the flipped square pyramid $\mathrm{conv}(\mathbf{0}, \mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_4, \ldots, \mathbf{e}_n)$. This implies that P is a translate of Δ_n. Let us therefore assume that $u_1 = e_1$, $u_2 = e_2$ and $u_3 = e_1 - e_2$. In this case P is a translate of one of the four d-dimensional subpolytopes of $\mathrm{Pyr}^{d-2}(\square_2)$. It is easy to verify that Δ_d is the only one of them that is projected by π_3 onto a translate of $\pi_3(\Delta_d)$. $\square$

Definition 5.3.5. Let $P_1, \ldots, P_{d-1} \in \mathcal{P}(\mathbb{Z}^d)$ be d-dimensional polytopes with the Minkowski sum $P_1 + \cdots + P_{d-1}$ being hollow. We call the tuple $(P_1, \ldots, P_{d-1})$ *exceptional*, if there exists no lattice projection $\pi \colon \mathbb{R}^d \to \mathbb{R}^{d-1}$ such that $\pi(P_1 + \cdots + P_{d-1}) \subset \mathbb{R}^{d-1}$ is a hollow polytope.

Remark 5.3.6. By [NZ11, Theorem 1.2] there exist only finitely many d-dimensional lattice polytopes not admitting a lattice projection onto a hollow $(d-1)$-dimensional lattice polytope, up to equivalence. So in particular, up to equivalence, there exist only finitely many exceptional $(d-1)$-tuples of d-dimensional lattice polytopes.

Furthermore, by Proposition 5.1.5, for any non-exceptional tuple $P_1, \ldots, P_{d-1} \in \mathcal{P}(\mathbb{Z}^{d-1})^{d-1}$ of d-dimensional lattice polytopes there exists a lattice projection $\pi \colon \mathbb{R}^d \to \mathbb{R}^{d-1}$ mapping all P_i onto translates of Δ_{d-1}.

Proof of Theorem 5.2.3. Let $d \geq 4$. Given $k \in [d]$, denote by I_k the set $\{1, ..., d\} \backslash \{k\}$, and by $[P]_k \in \mathcal{P}(\mathbb{Z}^d)^{d-1}$ the $(d-1)$-tuple given by all P_i for $i \in I_k$. Denote furthermore by P_{I_k} the Minkowski sum $\sum_{i \in I_k} P_i$ of the polytopes in $[P]_k$. Since $\operatorname{md}(P_1, \ldots, P_d) = 1$, the Minkowski sum P_{I_k} is hollow for any $k \in [d]$. Recall that, if $[P]_k$ is not exceptional, then by Remark 5.3.6 there exists a projection $\pi : \mathbb{R}^d \to \mathbb{R}^{d-1}$ mapping all polytopes in $[P]_k$ onto translates of Δ_{d-1}. We treat cases separately, depending on the number of exceptional $(d-1)$-subtuples of the tuple $(P_1, \ldots, P_d)$.

(0) If $(P_1, \ldots, P_d)$ has no exceptional $(d-1)$-subtuples then either there exists a projection $\pi : \mathbb{R}^d \to \mathbb{R}^{d-1}$ mapping $(P_1, \ldots, P_d)$ onto translates of Δ_{d-1} (and in this case there is nothing to prove), or each of the P_i admits $d-1$ pairwise different projections onto Δ_{d-1}. Indeed if two of these projections were the same, then we would be in the previous case. Suppose there exist $d-1$ pairwise different projections. As $d \geq 4$, Lemma 5.3.2 yields that each of the P_i is a unimodular d-dimensional simplex. Without loss of generality we assume $P_1 = \Delta_d$. Given $2 \leq i \leq d$, there exist $d-2$ pairwise different projections mapping P_1 and P_i onto translates of Δ_{d-1}. If $d \geq 5$, by Lemma 5.3.4, we can immediately deduce that, up to translations, $P_1 = P_2 = \ldots = P_d = \Delta_d$. If $d = 4$, Lemma 5.3.4 only ensures that $P_2, \ldots, P_d$ are, up to translation and coordinate permutation, contained in $\operatorname{Pyr}^{d-2}(\square_2)$. This yields finitely many cases and checking them computationally we find among them only 4-tuples admitting a common projection onto Δ_3.

(1) $(P_1, \ldots, P_d)$ has exactly one exceptional $(d-1)$-subtuple, which we can assume to be $[P]_d$. As $[P]_d$ is an exceptional $(d-1)$-tuple, the Minkowski sum P_{I_d} belongs to a finite list of hollow d-dimensional polytopes. This means that there are, up to equivalence, finitely many exceptional tuples to choose $[P]_d$ from. We now show, that given $[P]_d$ there are finitely many possible choices for P_d that lead to the d-tuple $(P_1, \ldots, P_d)$ having exactly $[P]_d$ as an exceptional $(d-1)$-subtuple, which shows the finiteness of this case.

Let therefore $\pi_1 : \mathbb{R}^d \to \mathbb{R}^{d-1}$ be a lattice projection mapping the lattice polytopes in $[P]_2$ to translates of Δ_{d-1}. Similarly, let $\pi_2 : \mathbb{R}^d \to \mathbb{R}^{d-1}$ be a lattice projection mapping the lattice polytopes in $[P]_1$ to translates of Δ_{d-1}. The existence of such projections follows from the fact that $[P]_2$ and $[P]_1$ are non-exceptional. We remark that there exist finitely many such projections. Let C_i be the infinite prism $P_i + \ker \pi_i$, for $i \in [2]$. Then we know that any possible choice of P_d is contained in $(C_1 + z_1) \cap (C_2 + z_2)$ for some $z_1, z_2 \in \mathbb{Z}^d$. By Lemma 5.3.3, for any choices of lattice points $z_1, z_2, z_1', z_2' \in \mathbb{Z}^d$ such that $\dim((C_1 + z_1) \cap (C_2 + z_2) \cap \mathbb{Z}^n) = \dim((C_1 + z_1') \cap (C_2 + z_2') \cap \mathbb{Z}^d) = d$ we find that $(C_1 + z_1') \cap (C_2 + z_2')$ is a translate of $(C_1 + z_1) \cap (C_2 + z_2)$. Therefore, up to translations, all possible choices for P_d are contained in $(C_1 + z_1) \cap (C_2 + z_2)$ for fixed $z_1, z_2 \in \mathbb{Z}^d$. Note that the intersection $(C_1 + z_1) \cap (C_2 + z_2)$ is either equivalent to Δ_d or $\mathrm{Pyr}^{d-2}(\square_2)$ by Lemma 5.3.2, where the choice of the equivalence class depends entirely on $[P]_d$. This implies that P_d must be one element of a finite list of lattice polytopes fully determined by $[P]_d$.

(2+) If $(P_1, \dots, P_d)$ has two or more exceptional $(d-1)$-subtuples, then we can suppose that $[P]_d$ and $[P]_{d-1}$ are exceptional. In particular, there exists an upper bound depending only on d for the volume of the Minkowski sums P_{I_d} and $P_{I_{d-1}}$ and therefore (since $d > 2$) for the volume of $P_1 + P_i$ for any $2 \le i \le d$. Recall that, by Theorem 4.1.1, there are, up to equivalence, only finitely many lattice polytopes of any fixed volume $K \in \mathbb{Z}_{\ge 0}$. Therefore, as in particular the volume of P_1 is bounded, there exist only finitely many choices for P_1 up to equivalence. Furthermore, fixing P_1 determines, up to translation, finitely many possibilities for each P_i with $2 \le i \le d$ due to the volume bound on $P_1 + P_i$. This yields that there are only finitely many tuples $(P_1, \dots, P_d)$ in this case, up to equivalence.

□

Note, that the assumption $d > 3$ is only used in case (0) of the previous proof.

The unmixed result of Theorem 5.1.2 (3) also gives an explicit description of lattice polytopes of degree one that are not Lawrence prisms, in fact, up to equivalence and the lattice pyramid construction, there exists only one such exception over all dimensions. Such an explicit description of the list of exceptions from the statement of Theorem 5.2.3 is not known in dimension $d \ge 4$.

Question 5.3.7. For dimension $d \ge 4$, what are the tuples of d-dimensional lattice polytopes $(P_1, \dots, P_d) \in \mathcal{P}(\mathbb{Z}^d)^d$ with $\mathrm{md}(P_1, \dots, P_d) = 1$ that are not mixed Lawrence prisms? Is there a finite description over all dimensions as there is in the unmixed case?

Our result actually allows for a few straightforward generalisations regarding the number of lattice polytopes in a tuple. Recall that a k-tuple of lattice polytopes $(P_1, \dots, P_k) \in \mathcal{P}(\mathbb{Z}^d)^k$ satisfies $\mathrm{md}(P_1, \dots, P_k) \le 1$ if and only if $k \ge d-1$ and the Minkowski sum of each $(d-1)$-subtuple is hollow. For $k = d-1$ we obtain an analogous result to Theorem 5.2.3 (even for $d \in \{2, 3\}$) immediately from [NZ11, Theorem 1.2]. We remark that Theorem 5.2.3 also inductively extends to the case of $k > d$ as follows.

Remark 5.3.8. Fix $d \ge 4$ and let $(P_1, \dots, P_{d+m}) \in \mathcal{P}(\mathbb{Z}^d)^{d+m}$ be a tuple full-dimensional lattice polytopes with $\mathrm{md}(P_1, \dots, P_{d+m}) = 1$. Then, up to equivalence, the $(d+m)$-tuple $(P_1, \dots, P_{d+m})$ either belongs to a finite list of exceptions or there is a lattice projection $\pi \colon \mathbb{R}^d \to \mathbb{R}^{d-1}$ such that $\pi(P_i) = \Delta_{d-1}$ for all $i \in [d+m]$.

One can see this with an induction argument on m, where the base case is given by Theorem 5.2.3. Indeed, let $(P_1, \dots, P_{d+m+1}) \in \mathcal{P}(\mathbb{Z}^d)^{d+m+1}$ be a tuple of d-dimensional lattice polytopes with $\mathrm{md}(P_1, \dots, P_{d+m+1}) = 1$. One easily verifies that this implies that any $(d+m)$-subtuple of $P_1, \dots, P_{d+m+1}$ has mixed degree at most 1. Analogously to the proof of Theorem 5.2.3 one can distinguish three cases depending on how many $(d+m)$-subtuples of $(P_1, \dots, P_{d+m+1})$ do not admit a common lattice projection onto translates of Δ_{d-1}, and use the induction hypothesis.

5.4. The 3-dimensional case

This section is devoted to the proof of Theorem 5.2.4, giving a classification of triples of d-dimensional lattice polytopes $(P_1, P_2, P_3) \in \mathcal{P}(\mathbb{Z}^3)^3$ of mixed degree one. Note that from the proof of Theorem 5.2.3 it follows that the number of such triples is finite, if we assume at least one of the subpairs of (P_1, P_2, P_3) to be exceptional. Here we first classify, up to equivalence, these finitely many triples. In Proposition 5.4.8 we show that there are non-trivial infinite 1-parameter families of triples.

As an intermediate step towards the classification of triples of lattice polytopes of mixed degree one with at least one exceptional subpair we calculate all (equivalence classes of) exceptional pairs of 3-dimensional lattice polytopes. In order to do that we consider the list of maximal hollow 3-dimensional lattice polytopes classified by Averkov–Wagner–Weismantel [AWW11] and Averkov–Krümpelmann–Weltge [AKW17], and compute all subpolytopes of the maximal hollow lattice polytopes that have lattice width greater than one.

Proposition 5.4.1 ([AKW17, Corollary 2])**.** *Let $P \in \mathcal{P}(\mathbb{Z}^3)$ be a hollow full-dimensional lattice polytope of width at least two. Then, up to equivalence, P is contained either in the unbounded polyhedron $2\Delta_2 \times \mathbb{R}$ or in one of 12 maximal hollow lattice polytopes.*

As we are interested in obtaining a list of exceptional pairs $(P, Q) \in \mathcal{P}(\mathbb{Z}^3)^2$ we use an implementation in Magma in order to compute the decompositions of all subpolytopes of the 12 maximal hollow polytopes into Minkowski sums of two 3-dimensional lattice polytopes. Afterwards we determine those pairs that actually do not admit a common projection onto translates of Δ_2 and then determine equivalent pairs using Theorem 2.3.1.

Corollary 5.4.2. *There are, up to equivalence, 32 pairs of 3-dimensional lattice polytopes whose Minkowski sum is hollow and that do not admit a common projection onto translates of Δ_2.*

We use this classification in order to compute all triples of lattice polytopes $(P_1, P_2, P_3) \in \mathcal{P}(\mathbb{Z}^3)^3$ of mixed degree one with at least two exceptional subpairs as follows.

Assume that (P_1, P_2) and (P_1, P_3) are exceptional pairs. Then there exist two pairs (A, B) and (C, D) out of the 32 of Corollary 5.4.2 such that (A, B) is equivalent to (P_1, P_2) and (C, D) is equivalent to (P_1, P_3). We can suppose that P_1 is equal to A and equivalent to C. Thus there exists an affine lattice-preserving transformation φ mapping C to $A = P_1$ such that the triple (P_1, P_2, P_3) is equivalent to the triple $(A, B, \varphi(D))$.

This justifies the following algorithm to construct all the triples (P_1, P_2, P_3) containing at least two exceptional subpairs: we iterate over all the pairs of ordered pairs (A, B) and (C, D) of Corollary 5.4.2, and, whenever there exists an affine lattice-preserving transformation φ mapping C to A, check if the triple $(A, B, \varphi(\psi(D)))$ has mixed degree one, where ψ ranges among all the possible affine automorphisms of C (and therefore $\varphi \circ \psi$ ranges among all affine lattice-preserving transformations sending C to A). Equivalent triples can be removed using the criterion following from Theorem 2.3.1. An implementation in Magma yields the following result proving parts (i)-(ii) of Theorem 5.2.4.

Proposition 5.4.3. *There are, up to equivalence, 170 triples of* 3*-dimensional lattice polytopes of mixed degree one having two or three exceptional subpairs. In the first case there are 29 triples, in the latter there are 141.*

We now discuss the case of triples of lattice polytopes of mixed degree one having exactly one exceptional subpair. Specifically, $(P_1, P_2, P_3) \in \mathcal{P}(\mathbb{Z}^3)^3$ is a triple of 3-dimensional lattice polytopes with $\operatorname{md}(P_1, P_2, P_3) = 1$ and (without loss of generality) there are two different lattice projections $\pi_2 \colon \mathbb{R}^3 \to \mathbb{R}^2$ and $\pi_3 \colon \mathbb{R}^3 \to \mathbb{R}^2$ where π_k maps P_i and P_j to translates of Δ_2 whenever $i, j \neq k$. In particular P_1 is a lattice polytope with two different lattice projections onto Δ_2 (and therefore by Lemma 5.3.2 is equivalent either to Δ_3 or to $\operatorname{Pyr}(\square_2)$) and P_2, P_3 is an exceptional pair. Note that P_1 must be contained in both the infinite prisms $C_2 \coloneqq P_3 + \ker \pi_2 + u$ and $C_3 \coloneqq P_2 + \ker \pi_3 + v$, for some translation vectors $u, v \in \mathbb{Z}^3$.

In order to classify all such triples we use the fact that we may choose (P_2, P_3) from the list of 32 exceptional pairs of Corollary 5.4.2. Given an exceptional pair (P_2, P_3), we iterate over all the possible pairs of lattice projections (π_3, π_2), such that $\pi_3(P_2)$ and $\pi_2(P_3)$ are unimodular triangles. Each such choice determines two infinite prisms $C_3 \coloneqq P_2 + \ker \pi_3$ and $C_2 \coloneqq P_3 + \ker \pi_2$. We know that any lattice polytope $P_1 \in \mathcal{P}(\mathbb{Z}^3)$, such that $\pi_3(P_1)$ and $\pi_2(P_1)$ are translates of $\pi_3(P_2)$ and $\pi_2(P_3)$ respectively, is contained in both the infinite prisms $C_2 \coloneqq P_3 + \ker \pi_2 + u$ and $C_3 \coloneqq P_2 + \ker \pi_3 + v$, for some translation vectors $u, v \in \mathbb{Z}^3$. Up to translation of P_1 we may assume $u = \mathbf{0}$. By Lemma 5.3.3 it suffices to find one choice of $v \in \mathbb{Z}^3$ such that C_2 and C_3 intersect in a full-dimensional lattice polytope, in order to determine the inclusion-maximal choice for P_1 up to translation. Furthermore, there are only finitely many choices for $v \in \mathbb{Z}^3$ to check for the existence of a full-dimensional intersection of C_2 and C_3 as we may suppose P_2 and P_3 to have a common vertex. This is due to the fact that, if C_2 and C_3 intersect in a full-dimensional lattice polytope, then one may translate P_2 along $\ker \pi_3$ and P_3 along $\ker \pi_2$ without changing the infinite prisms. It therefore suffices to restrict to translation vectors v that map a vertex of P_2 to a vertex of P_3. Thus we can determine, up to equivalence, all inclusion-maximal P_1 as above, form triples for all subpolytopes of P_1 and remove equivalent triples using Theorem 2.3.1. An implementation in Magma yields the following result proving part (iii) of Theorem 5.2.4.

Proposition 5.4.4. *There are, up to equivalence, 82 triples of* 3*-dimensional lattice polytopes of mixed degree one having exactly one exceptional subpair.*

In the remaining part of this section we are going to deal with non-trivial triples not having any exceptional subpair in order to prove part (iv) of Theorem 5.2.4.

Lemma 5.4.5. *Let $P_1, P_2, P_3 \in \mathcal{P}(\mathbb{Z}^3)$ be lattice polytopes, and $\pi_1, \pi_2, \pi_3 \colon \mathbb{R}^3 \to \mathbb{R}^2$ be lattice projections such that, for all $i, j, k \in [3]$, the images $\pi_k(P_i)$ and $\pi_k(P_j)$ are translates of Δ_2 if and only if $i, j \neq k$. Let $v_i \in \mathbb{Z}^3$ be the projection direction of π_i for $i \in [3]$. Then v_i and v_j are part of a lattice basis of $\mathbb{Z}^3$, for all $i, j \in [3]$. Moreover, if v_1, v_2, v_3 linearly span $\mathbb{R}^3$, then they form a lattice basis of $\mathbb{Z}^3$.*

Proof. For $k \in [3]$ let C_k be the infinite prism $\pi_k(P_i) + \mathbb{R}v_k$, for some $i \neq k$. Note that, up to translation, this does not depend on the choice of i as both P_i and P_j are contained in different translates of C_k, whenever $i, j \neq k$. We now fix any of the infinite prisms, say C_1. For simplicity we suppose $C_1 = (\{0\} \times \Delta_2) + \mathbb{R}e_1$ and $P_2, P_3 \subset C_1$. In this way we avoid dealing with translations. Note that v_2 is parallel to an edge of P_3, and v_3 is parallel to an edge of P_2. Since both edges are contained in C_1, they project along e_1 either to the same side of the triangle $\pi_1(P_2) = \pi_1(P_3) = \Delta_2$, or to two adjacent sides. In the second case e_1, v_2 and v_3 linearly span $\mathbb{R}^3$ and it is easy to verify that they form a lattice basis of $\mathbb{Z}^3$. In the first case e_1, v_2 and v_3 span a plane, and from Lemma 5.3.2 it follows that any two of them are part of a lattice basis of $\mathbb{Z}^3$. $\square$

Proposition 5.4.6. *There are, up to equivalence, 27 triples of 3-dimensional lattice polytopes $(P_1, P_2, P_3) \in \mathcal{P}(\mathbb{Z}^3)$ satisfying the hypothesis of Lemma 5.4.5 for projection directions $v_1, v_2, v_3 \in \mathbb{Z}^3$ that linearly span $\mathbb{R}^3$. All of them are, up to equivalence, contained in one of the following three inclusion-maximal triples of mixed degree one:*

- *the maximal triple given by the following three reflections of* $\mathrm{Pyr}(\square_2)$

$$\mathrm{conv}(\boldsymbol{0}, e_2, e_3, e_2 + e_3, e_1),$$
$$\mathrm{conv}(\boldsymbol{0}, e_1, e_3, e_1 + e_3, e_2),$$
$$\mathrm{conv}(\boldsymbol{0}, e_1, e_2, e_1 + e_2, e_3) = \mathrm{Pyr}(\square_2),$$

- *the maximal triple*

$$\mathrm{conv}(\boldsymbol{0}, e_1, e_3, e_1 + e_2),$$
$$\mathrm{conv}(\boldsymbol{0}, e_1, e_3, e_1 + e_2, e_1 + e_3),$$
$$\mathrm{conv}(\boldsymbol{0}, e_1, e_2, e_1 + e_2, e_3) = \mathrm{Pyr}(\square_2),$$

- *and the maximal triple*

$$\mathrm{conv}(e_1, e_2, e_1 + e_2, e_2 + e_3),$$
$$\mathrm{conv}(e_1, e_2, e_1 + e_2, e_1 + e_3),$$
$$\mathrm{conv}(\boldsymbol{0}, e_1, e_2, e_1 + e_2, e_3) = \mathrm{Pyr}(\square_2).$$

Proof. By Lemma 5.4.5 we may assume v_1, v_2, v_3 to be e_1, e_2, e_3 respectively, and that two primitive segments parallel to the directions e_2 and e_3 are contained in C_1. This restricts C_1 to be, up to translation, one of the four infinite prisms of the form $\mathrm{conv}(\mathbf{0}, \pm e_2, \pm e_3) + \mathbb{R}e_1$. In particular, up to translation, C_1 is contained in the infinite prism $\operatorname{conv}(\mathbf{0}, e_2) + \operatorname{conv}(\mathbf{0}, e_3) + \mathbb{R}e_1$. Similarly, $C_2 \subset \operatorname{conv}(\mathbf{0}, e_1) + \operatorname{conv}(\mathbf{0}, e_3) + \mathbb{R}e_2$ and $C_3 \subset \operatorname{conv}(\mathbf{0}, e_1) + \operatorname{conv}(\mathbf{0}, e_2) + \mathbb{R}e_3$. In particular all the P_i are, up to translations, subpolytopes of the unit cube $\square_3 = \operatorname{conv}(\mathbf{0}, e_1) + \operatorname{conv}(\mathbf{0}, e_2) + \operatorname{conv}(\mathbf{0}, e_3)$, which leaves finitely many cases that we check computationally. $\square$

Remark 5.4.7. From the proof of Proposition 5.4.6 it is clear that all the maximal triples from Proposition 5.4.6 are actually contained inside the triple consisting of three copies of the unit cube $\square_3$. Note however that one has $\operatorname{md}(\square_3, \square_3, \square_3) > 1$, as the Minkowski sum $\square_3 + \square_3$ has an interior lattice point.

Proposition 5.4.8. *There are infinitely many equivalence classes of triples of 3-dimensional lattice polytopes $(P_1, P_2, P_3) \in \mathcal{P}(\mathbb{Z}^3)^3$ satisfying the hypothesis of Lemma 5.4.5 for projection directions $v_1, v_2, v_3 \in \mathbb{Z}^3$ that linearly span $\mathbb{R}^2 \times \{0\}$. All of them, up to equivalence, are contained in one of the following triples of mixed degree two given by the parallelepipeds $Q_k, R_k, \square_3$ for some $k \in \mathbb{Z}_{\geq 0}$, where*

$$\begin{aligned} Q_k &:= \operatorname{conv}(\mathbf{0}, e_1 - e_2) + \operatorname{conv}(\mathbf{0}, e_2) + \operatorname{conv}(\mathbf{0}, ke_2 + e_3), \\ R_k &:= \operatorname{conv}(\mathbf{0}, e_2 - e_1) + \operatorname{conv}(\mathbf{0}, e_1) + \operatorname{conv}(\mathbf{0}, ke_1 + e_3), \\ \square_3 &:= \operatorname{conv}(\mathbf{0}, e_1) + \operatorname{conv}(\mathbf{0}, e_2) + \operatorname{conv}(\mathbf{0}, e_3). \end{aligned}$$

They can be covered by a finite number of 1-parameter families. In particular, if we denote by ψ_1^k the shearing $(x, y, z) \mapsto (x, y + kz, z)$ and by ψ_2^k the shearing $(x, y, z) \mapsto (x + kz, y, z)$, one may choose 1-parameter families of the form $\left\{\psi_1^k(P_1^0), \psi_2^k(P_2^0), P_3\right\}_{k \in \mathbb{Z}_{\geq 0}}$ for all subpolytopes $P_1^0 \subset Q_0, P_2^0 \subset R_0$ and $P_3 \subset \square_3$ satisfying $\operatorname{md}(P_1^0, P_2^0, P_3) = 1$.

Proof. By Lemma 5.4.5 we may assume v_1, v_2, v_3 to be $e_1, e_2, e_1 - e_2$. Here, the assumption $v_3 = e_1 - e_2$ is justified by fact that both the pairs e_1, v_3 and e_2, v_3 need to be part of a lattice basis of $\mathbb{Z}^3$, and the projection directions v_i may be chosen with arbitrary sign. By Lemma 5.3.2 the polytope P_3, which projects onto Δ_2 along the directions e_1 and e_2, can be fixed to be in the unit cube $\square_3$. Consequently we can assume C_1, C_2 to be in the infinite prisms $\square_3 + \mathbb{R}e_1$ and $\square_3 + \mathbb{R}e_2$, respectively. Finally, we assume P_1 and P_2 to be in the infinite prisms C_2 and C_1, respectively. Now consider the linear functional f defined by $(x, y, z) \mapsto x + y$. Consider a lattice point $v_0 \in P_1 \cap (\mathbb{R}^2 \times \{0\})$ minimizing f. Since P_1 projects onto Δ_2 along the direction $e_1 - e_2$, one verifies that for any other point $u_0 \in P_1 \cap (\mathbb{R}^2 \times \{0\})$ one has $f(v_0) \leq f(u_0) \leq f(v_0) + 1$. Analogously, if v_1 is a lattice point in $P_1 \cap (\mathbb{R}^2 \times \{1\})$ minimizing f, then $f(v_1) \leq f(u_1) \leq f(v_1) + 1$ for all $u_1 \in P_1 \cap (\mathbb{R}^2 \times \{1\})$. Since we are free to translate P_1 along e_2, we can suppose $f(v_0) = 0$ and we denote $k = f(v_1)$. As a consequence, $P_1 \cap (\mathbb{R}^2 \times \{0\})$ is contained in the parallelogram $q_0 := \operatorname{conv}(\mathbf{0}, e_1 - e_2) + \operatorname{conv}(\mathbf{0}, e_2)$. Analogously $P_1 \cap (\mathbb{R}^2 \times \{1\})$ is contained in the parallelogram $q_1 := q_0 + ke_2 + e_3$. In particular P_1 is contained in the parallelepiped $\operatorname{conv}(q_0 \cup q_1) = Q_k$. Therefore C_3 is contained in the infinite prism $Q_k + \mathbb{R}(e_1 - e_2)$. This completely determines the parallelepiped $R_k = C_1 \cap C_3$, satisfying $R_k \supset P_2$. It is easy to verify that the triple $(Q_k, R_k, \square_3)$ is equivalent to the triple $(Q_{-k}, R_{-k}, \square_3)$, so one can always assume $k \in \mathbb{Z}_{\geq 0}$.

In order to see that the set of triples that are subtriples of $(Q_k, R_k, \square_3)$ for some $k \in \mathbb{Z}_{\geq 0}$ can be covered by 1-parameter families as claimed it suffices to notice that any subtriple of $(Q_k, R_k, \square_3)$ can be written as $(\psi_1^k(P_1^0), \psi_2^k(P_2^0), \square_3)$ for subpolytopes $P_1^0 \subset Q_0, P_2^0 \subset R_0$ and $P_3 \subset \square_3$. The fact that any family $\left\{\psi_1^k(P_1^0), \psi_2^k(P_2^0), P_3\right\}_{k \in \mathbb{Z}_{\geq 0}}$ for subpolytopes $P_1^0 \subset Q_0, P_2^0 \subset R_0$ and $P_3 \subset \square_3$ actually contains infinitely many non-equivalent triples can be verified by picking edges $E_1 \subset P_1^0$, $E_2 \subset P_2^0$ and $E_3 \subset \square_3$ between vertices on height 0 and 1, and noticing that the volume of the parallelepiped $\psi_1^k(E_1) + \psi_2^k(E_2) + E_3$ grows quadratically in k. An example of one of these infinite 1-parameter families is given in Example 5.2.5. $\square$

A computer assisted search for mixed degree one triples in $Q_k, R_k, \square_3$ for small values of k shows that there are 51 non-equivalent triples when $k = 0$, and 36 for larger values of k, where, for each k, the overlaps that occur for preceding values of k are excluded.

5.5. Outlook: exceptional tuples in general dimension

Let us make some comments regarding the list of exceptions from Theorem 5.2.3 for $d > 3$. In these cases we have shown that this list is finite. In particular, there exist no d-tuples analogous to type (iv) of Theorem 5.2.4. However, our approach for obtaining the list in dimension 3 heavily relied on the non-trivial classification of hollow lattice polytopes in dimension 3 by Averkov et al. (see Proposition 5.4.1). For any $d > 3$ such a list is not known and it seems far out of reach to obtain it. In order to make further progress it still seems helpful to have a structural conjecture regarding Question 5.3.7. For this it is useful to introduce the definition of md-maximality of a tuples.

Definition 5.5.1. We call a tuple $(P_1, \dots, P_d) \in \mathcal{P}(\mathbb{Z}^d)^d$ of full-dimensional lattice polytopes md-*maximal* if for any tuple $(Q_1, \dots, Q_d) \in \mathcal{P}(\mathbb{Z}^d)^d$ with $(P_1, \dots, P_d) \neq (Q_1, \dots, Q_d)$ and $P_i \subseteq Q_i$ for all $i \in [d]$ one has:

$$\mathrm{md}(P_1, \dots, P_d) < \mathrm{md}(Q_1, \dots, Q_d).$$

Remark 5.5.2. Recall that the mixed degree is monotonous among full-dimensional tuples and that the number of exceptional tuples of mixed degree one is finite for all $d > 3$. This implies that every exceptional tuple of mixed degree one is contained in an md-maximal tuple of degree one.

Let us give a short example that illustrates how restricting one's view to md-maximal tuples reduces the complexity of the classification task.

Example 5.5.3. Consider the tuple $(\mathrm{Pyr}(2\Delta_2), \mathrm{Pyr}(2\Delta_2), \mathrm{Pyr}(2\Delta_2)) \in \mathcal{P}(\mathbb{Z}^3)^3$. As $\deg(\mathrm{Pyr}(2\Delta_2)) = 1$ one has $\mathrm{md}(\mathrm{Pyr}(2\Delta_2), \mathrm{Pyr}(2\Delta_2), \mathrm{Pyr}(2\Delta_2)) = 1$ and it is straightforward to verify that the tuple is no mixed Lawrence prism and therefore exceptional. Furthermore, explicit computations show that $(\mathrm{Pyr}(2\Delta_2), \mathrm{Pyr}(2\Delta_2), \mathrm{Pyr}(2\Delta_2))$ is md-maximal (see Corollary 5.5.4) and that it contains 249 non-equivalent exceptional triples of mixed degree one (233 if we exclude the ones coming from the infinite families of Proposition 5.4.8).

Investigations of the 252 triples classified in Theorem 5.2.4 *(i) –(iii)* yield that they can all be found as subtriples of six md-maximal ones. We have verified this computationally by enumerating all subtriples of these six and finding all 252 triples among them.

Corollary 5.5.4. *All triples* $(P_1, P_2, P_3) \in \mathcal{P}(\mathbb{Z}^3)^3$ *of full-dimensional lattice polytopes of mixed degree one that are of types (i) –(iii) from Theorem 5.2.4 are, up to equivalence, contained in one of the following* 6 *maximal triples:*

(a) the maximal triple $\operatorname{Pyr}(2\Delta_2), \operatorname{Pyr}(2\Delta_2), \operatorname{Pyr}(2\Delta_2)$,

(b) the maximal triple $2\Delta_3, \Delta_3, \Delta_3$,

(c) the maximal triple $\{\operatorname{conv}(\boldsymbol{0}, 2e_i, e_j, e_k) \colon i, j, k \in [3] \text{ pairwise different}\}$,

(d) the maximal triple

$$\begin{gathered}\operatorname{conv}(e_1, e_2, -e_2, e_3, e_1 + e_3),\\ \operatorname{conv}(\boldsymbol{0}, e_1, -e_2, -e_2 + e_3),\\ \operatorname{conv}(\boldsymbol{0}, e_1, e_2, e_2 + e_3),\end{gathered}$$

(e) the maximal triple

$$\begin{gathered}\operatorname{conv}(\boldsymbol{0}, 2e_2, e_3, e_1 + e_3),\\ \operatorname{conv}(\boldsymbol{0}, -e_1, -e_1 - e_2, -e_1 - 2e_2 + e_3),\\ \operatorname{conv}(\boldsymbol{0}, e_2, -e_1, e_1 + e_3),\end{gathered}$$

(f) the maximal triple

$$\begin{gathered}\operatorname{conv}(\boldsymbol{0}, 2e_2, e_3, e_1 + e_3),\\ \operatorname{conv}(\boldsymbol{0}, -e_1, -e_1 - e_2, -e_1 - 2e_2 + e_3),\\ \operatorname{conv}(\boldsymbol{0}, -e_2, -e_1, e_1 - 2e_2 + e_3).\end{gathered}$$

The maximal triples (a) and (b) of Corollary 5.5.4 admit direct generalizations to an arbitrary dimension d that are of mixed degree one. Furthermore, we have verified that the straightforward generalization of the maximal family (c) to dimension $d \geq 4$ does not yield a d-tuple of mixed degree one. This motivates the following more explicit version of Question 5.3.7.

Question 5.5.5. Is there a natural number N, such that for each $d \geq N$, every md-maximal tuple of full dimensional lattice polytopes $(P_1, \ldots, P_d) \in \mathcal{P}(\mathbb{Z}^d)^d$ with $\operatorname{md}(P_1, \ldots, P_d) = 1$ is equivalent either to $(\operatorname{Pyr}^{d-2}(2\Delta_2), \ldots, \operatorname{Pyr}^{d-2}(2\Delta_2))$ or to $(2\Delta_d, \Delta_d, \ldots, \Delta_d)$? Is this true for $N = 4$?

6. Inequalities between Mixed Volumes

In this chapter we treat the question of bounding the volume of the Minkowski sum of a tuple of lattice polytopes in terms of its mixed volume. As our results turn out to rely only on more general metric inequalities, the vast majority of this chapter is phrased for tuples of general convex bodies. In Section 6.1 we illustrate our initial motivation, and present our main results in Section 6.2. These are an asymptotically sharp bound on the volume of the Minkowski sum in general dimension (Theorem 6.2.1) and exact sharp bounds in dimensions 2 and 3 (Proposition 6.2.4 and Theorem 6.2.5). In Section 6.3 we introduce the notion of mixed volume configurations, which is the fundamental concept for our proofs. Section 6.4 is devoted to employing the Aleksandrov-Fenchel inequalities to prove an upper bound for the volume of the Minkowski sum of a tuple of fixed mixed volume in general dimension (Theorem 6.4.3). Furthermore, we show that the relations providing this bound are best possible if we consider only the Aleksandrov-Fenchel inequalities (Proposition 6.4.6). In Section 6.5 we use additional inequalities between mixed volumes in order to obtain a stronger bound proving Theorem 6.2.1. Section 6.6 is devoted to computationally proving Theorem 6.2.5 and, finally, in Section 6.7 we present ideas for further research directions.

6.1. Motivation

The initial motivation for the studies that are presented in this chapter is very much related to Chapter 4. Recall that Theorem 4.2.7 ensures the finiteness of the number of irreducible d-tuples of lattice polytopes of a given mixed volume. The proof of this theorem by Esterov relies on providing an upper bound on the volume of the Minkowski sum $P_1 + \cdots + P_d$ among all irreducible tuples of lattice polytopes $(P_1, \ldots, P_d) \in \mathcal{P}(\mathbb{Z}^d)^d$ with fixed mixed volume $m = \mathrm{MV}(P_1, \ldots, P_d)$. In particular, Esterov [Est19] has shown that the volume of the Minkowski sum

$$\Sigma(P) := P_1 + \cdots + P_d$$

has the asymptotic order at most $O(m^{2^d})$, as $m \to \infty$. This bound, however, seems far from being sharp. Indeed, already in the course of showing this bound, Esterov [Est19] raised the question of determining a sharper bound for the volume of $\Sigma(P)$. The approach of Esterov was based on the fact that one can write the volume of the Minkowski sum $\mathrm{Vol}_d(P_1 + \cdots + P_d)$ in terms of different mixed volumes as follows

$$\mathrm{Vol}_d(\Sigma(P)) = \sum_{i_1,\ldots,i_d \in \{1,\ldots,d\}} \mathrm{MV}(P_{i_1}, \ldots, P_{i_d}). \tag{6.1}$$

This boils the question for an upper bound of $\mathrm{Vol}_d(\Sigma(K))$ down to bounding the different values for $\mathrm{MV}(P_{i_1}, \ldots, P_{i_d})$, given that the value $\mathrm{MV}(P_1, \ldots, P_d)$ is fixed.

In this chapter we treat the question for a sharper bound for the case of tuples of full-dimensional polytopes. While our motivation comes from the theory of Newton polytopes, we do not exploit any combinatorial properties of lattice polytopes in this chapter. In fact, our approach works in the more general context of *convex bodies*, by which we mean compact convex sets that are full-dimensional. Therefore most of this chapter is phrased in terms of convex bodies. As it is common in the theory of Newton polytopes, we prefer to stick with the normalized (mixed) volume as in the other parts of this thesis. We remark that any other rescaling of the Euclidean volume would work just as well.

6.2. Results

The following is our main asymptotical result (see Theorem 6.5.10 for an explicit bound and the proof).

Theorem 6.2.1. *Let $m \in \mathbb{R}_{\geq 1}$. Among all convex bodies $K_1, \ldots, K_d$ in $\mathbb{R}^d$ satisfying*

$$\mathrm{Vol}_d(K_1) \geq 1, \ldots, \mathrm{Vol}_d(K_d) \geq 1, \quad \textit{and} \quad \mathrm{MV}(K_1, \ldots, K_d) = m,$$

the maximum of $\mathrm{Vol}_d(K_1 + \cdots + K_d)$ is of order $O(m^d)$, as $m \to \infty$.

Remark 6.2.2. Consider a tuple of d copies of the same d-dimensional convex body $K_1 = \cdots = K_d$ with $\mathrm{Vol}_d(K_1) = 1$. Then one has $\mathrm{MV}(mK_1, K_2, \ldots, K_d) = m$ and

$$\mathrm{Vol}_d(mK_1 + K_2 + \cdots + K_d) = \mathrm{Vol}_d((m + d - 1)K_1) = (m + d - 1)^d.$$

As $(m + d - 1)^d$ is of order $O(m^d)$ one the bound provided in Theorem 6.2.1 is asymptotically strict. Note that, if one restricts to d-tuples of full-dimensional lattice polytopes, there exists a unique tuple of this form (up to equivalence), given by $(m\Delta_d, \Delta_d, \ldots, \Delta_d)$.

Interpreting Theorem 6.2.1 in terms of the BKK-theorem allows to derive the following corollary for generic system of polynomial equations.

Corollary 6.2.3. *Let $(f_1, \ldots, f_d) \in \mathbb{C}[P_1, \ldots, P_d]$ be generic Laurent polynomials for a tuple of full-dimensional lattice polytopes $(P_1, \ldots, P_d)$ and let m be the number of solutions of the system $f_1 = \cdots = f_d = 0$ in $(\mathbb{C}^*)^d$. Then the Newton polytope of the product $f_1 \cdots f_d$ has volume at most $O(m^d)$, as $m \to \infty$. In particular, the product $f_1 \ldots f_d$ contains at most $O(m^d)$ monomials, as $m \to \infty$.*

As a further contribution we present the exact bounds in dimensions 2 and 3.

Proposition 6.2.4. *Let $m \in \mathbb{R}_{\geq 1}$. Consider 2-dimensional convex bodies K_1, K_2 in $\mathbb{R}^2$ satisfying*

$$\mathrm{Vol}_d(K_1) \geq 1,\ \mathrm{Vol}_d(K_2) \geq 1, \quad \textit{and} \quad \mathrm{MV}(K_1, K_2) = m.$$

Among all such bodies,

1. *the maximum of* $\mathrm{Vol}_d(K_1)$ *is* m^2 *and*
2. *the maximum of* $\mathrm{Vol}_d(K_1+K_2)$ *is* $(m+1)^2$.

Both maxima are attained when $K_1 = mK_2$ *and* $\mathrm{Vol}_d(K_2) = 1$.

Theorem 6.2.5. *Let* $m \in \mathbb{R}_{\geq 1}$. *Consider* 3*-dimensional convex bodies* $K_1, K_2, K_3 \subset \mathbb{R}^3$ *satisfying*

$$\mathrm{Vol}_d(K_1) \geq 1,\ \mathrm{Vol}_d(K_2) \geq 1,\ \mathrm{Vol}_d(K_3) \geq 1, \quad \text{and} \quad \mathrm{MV}(K_1, K_2, K_3) = m.$$

Among all such bodies,

1. *the maximum of* $\mathrm{Vol}_d(K_1)$ *is* m^3,
2. *the maximum of* $\mathrm{Vol}_d(K_1+K_2)$ *is* $(m+1)^3$, *and*
3. *the maximum of* $\mathrm{Vol}_d(K_1+K_2+K_3)$ *is* $(m+2)^3$.

All three maxima are attained when $K_1 = mK_2 = mK_3$ *and* $\mathrm{Vol}_d(K_3) = 1$.

Based on the above evidence and the asymptotic behavior of $\mathrm{Vol}_d(\Sigma(P))$ presented in Theorem 6.2.1, we propose the following conjecture.

Conjecture 6.2.6. *Let* $m \in \mathbb{R}_{\geq 1}$. *Among all convex bodies* $K_1, \ldots, K_d$ *in* $\mathbb{R}^d$ *satisfying*

$$\mathrm{Vol}_d(K_1) \geq 1, \ldots, \mathrm{Vol}_d(K_d) \geq 1, \quad \text{and} \quad \mathrm{MV}(K_1, \ldots, K_d) = m,$$

for any $1 \leq \ell \leq d$, *the maximum of* $\mathrm{Vol}_d(K_1 + \cdots + K_\ell)$ *equals* $(m+\ell-1)^d$ *and is attained when* $K_1 = mK_2 = \cdots = mK_d$ *with* $\mathrm{Vol}_d(K_d) = 1$.

We know that the conjecture is true for $\ell = 1$ (Remark 6.4.2) and for $d \leq 3$ (Proposition 6.2.4 and Theorem 6.2.5). All other cases are open. Note that the bounds in Conjecture 6.2.6 would be sharp as they are all attained for tuples as presented in Remark 6.2.2.

6.3. Mixed volume configurations

Define the sets

$$\Delta_{n,d} = \left\{(x_1, \ldots, x_n) \in \mathbb{Z}^n_{\geq 0} \,:\, x_1 + \cdots + x_n = d\right\}, \tag{6.2}$$

for $n, d \in \mathbb{Z}_{\geq 1}$. Let $\mathbb{R}^{\Delta_{n,d}}$ be the set of all functions from $\Delta_{n,d}$ to $\mathbb{R}$. In what follows, we have two points of view for the elements of $\mathbb{R}^{\Delta_{n,d}}$. On one hand, $\mathbb{R}^{\Delta_{n,d}}$ is a vector space over $\mathbb{R}$ and we can treat its elements as vectors of $\mathbb{R}^N$ with $N = |\Delta_{n,d}|$. On the other hand, since $\Delta_{n,d}$ is a subset of $\mathbb{R}^d$, we can talk about elements of $\mathbb{R}^{\Delta_{n,d}}$ as functions on $\Delta_{n,d}$ which may or may not poses some discrete concavity properties. Because of this, we will call the elements of $\mathbb{R}^{\Delta_{n,d}}$ functions or vectors depending on the context.

Let us denote by $\mathcal{K}_d$ the set of all *convex bodies* in $\mathbb{R}^d$, that is compact convex subsets of $\mathbb{R}^d$ which are full-dimensional, and let $\mathcal{K}_{d,1} \subset \mathcal{K}_d$ denote the subset of those convex bodies, whose normalized volume is at least 1. Fix $n > 0$ and a family $\mathcal{K}$ of compact convex sets in $\mathbb{R}^d$, and consider an ordered n-tuple $K = (K_1, \dots, K_n)$ of elements in $\mathcal{K}$. It defines a collection of n^d mixed volumes

$$\Big(\mathrm{MV}(K_{i_1}, \dots, K_{i_d}) : i_1, \dots, i_d \in [n]\Big).$$

Since the mixed volume $\mathrm{MV}(K_{i_1}, \dots, K_{i_d})$ is invariant under permutation of the indices, we introduce an alternative notation

$$\mathrm{V}_K(p_1, \dots, p_n) = \mathrm{MV}(\underbrace{K_1, \dots, K_1}_{p_1}, \dots, \underbrace{K_n, \dots, K_n}_{p_n}).$$

In this notation, the *mixed volume configuration* of an n-tuple $K = (K_1, \dots, K_n)$ is the vector

$$\Big(\mathrm{V}_K(p)\Big)_{p \in \Delta_{n,d}} \in \mathbb{R}_{\geq 0}^{\Delta_{n,d}}. \tag{6.3}$$

So any n-tuple of convex bodies in $\mathbb{R}^d$ defines a vector in $\mathbb{R}_{\geq 0}^{\Delta_{n,d}}$ or, equivalently, a function from $\Delta_{n,d}$ to $\mathbb{R}_{\geq 0}$. For example, in the case $d = 3$, $n = 2$, the mixed volume configuration of a pair $K = (K_1, K_2)$ of 3-dimensional convex bodies consists of the following four mixed volumes:

$$\begin{aligned}
\mathrm{V}_K(3,0) &= \mathrm{MV}(K_1, K_1, K_1) = \mathrm{Vol}_d(K_1),\\
\mathrm{V}_K(2,1) &= \mathrm{MV}(K_1, K_1, K_2),\\
\mathrm{V}_K(1,2) &= \mathrm{MV}(K_1, K_2, K_2),\\
\mathrm{V}_K(0,3) &= \mathrm{MV}(K_2, K_2, K_2) = \mathrm{Vol}_d(K_2).
\end{aligned}$$

Furthermore, given a family of compact convex sets $\mathcal{K}$, we define the *m*ixed volume configuration space

$$\mathrm{V}(\mathcal{K}, \Delta_{n,d}) := \left\{ \Big(\mathrm{V}_K(p)\Big)_{p \in \Delta_{n,d}} \;:\; K \in \mathcal{K}^n \right\}, \tag{6.4}$$

which represents all possible sets of values of the different mixed volumes indexed by $p \in \Delta_{n,d}$ built for convex sets from $\mathcal{K}$. When all sets from $\mathcal{K}$ are full-dimensional, we also introduce the *logarithmic mixed volume configuration space*

$$\mathrm{v}(\mathcal{K}, \Delta_{n,d}) := \left\{ \Big(\mathrm{v}_K(p)\Big)_{p \in \Delta_{n,d}} \;:\; K \in \mathcal{K}^n \right\}, \quad \text{where} \quad \mathrm{v}_K(p) = \log \mathrm{V}_K(p). \tag{6.5}$$

It shows that the concrete base for the logarithm log is not important for our results. We choose to set $\log := \log_2$ as this makes some constants more compact to write down.

The following is formula (6.1) in this new notation.

Proposition 6.3.1. *Let $K \in (\mathcal{K}_d)^n$. Then we have the following formula for the volume of the Minkowski sum of the elements in K:*

$$\mathrm{Vol}_d(\Sigma(K)) = \sum_{p\in\Delta_{n,d}} \binom{d}{p} \mathrm{V}_K(p), \tag{6.6}$$

where $\binom{d}{p} = \frac{d!}{p_1!\cdots p_n!}$ denotes the multinomial coefficient for $p = (p_1, \dots, p_n)$.

For our purposes it suffices to work in the special case of d-tuples K of convex bodies in $\mathbb{R}^d$. In this case, the mixed volume configuration V_K is indexed by points $p \in \Delta_{d,d}$. We denote by $\mathbf{1} \in \Delta_{d,d}$ the vector $(1, \dots, 1)$.

Remark 6.3.2. Proposition 6.3.1 shows that the volume of the Minkowski sum can be interpreted as a function on a mixed volume configuration space. In particular, we can rephrase the problem of bounding the volume of the Minkowski sum among all d-tuples of d-dimensional convex bodies $(K_1, \dots, K_d)$ with $\mathrm{MV}(K_1, \dots, K_d) = m$ and $\mathrm{Vol}_d(K_i) \geq 1$ for all $i \in [d]$ as determining the value

$$\max\left\{\sum_{p\in\Delta_{d,d}} \binom{d}{p} w(p) \colon w \in \mathrm{V}(\mathcal{K}_{d,1}, \Delta_{d,d}) \text{ and } w(\mathbf{1}) = m\right\}.$$

Recall the Aleksandrov-Fenchel inequalities from Proposition 1.2.2 (5.)) The following is a reformulation of them in the notation of mixed volume configurations.

Theorem 6.3.3 (Aleksandrov-Fenchel Inequalities). *Let $i, j \in [d]$ with $i \neq j$ and $p = (p_1, \dots, p_d) \in \Delta_{d,d}$ a point satisfying $p_i, p_j \geq 1$. Then, for every d-tuple K of d-dimensional convex bodies in $\mathbb{R}^d$, one has*

$$\mathrm{V}_K(p)^2 \geq \mathrm{V}_K(p + e_i - e_j)\, \mathrm{V}_K(p - e_i + e_j). \tag{AF}$$

Equivalently, in the log*-notation, one has*

$$2\, \mathrm{v}_K(p) \geq \mathrm{v}_K(p + e_i - e_j) + \mathrm{v}_K(p - e_i + e_j). \tag{log AF}$$

Recall that a sequence $r_0, r_1 \dots, r_n$ of non-negative real numbers is *l*og-concave if $r_i^2 \geq r_{i-1}r_{i+1}$ holds for all $0 < i < n$. Furthermore, a sequence $r_0, \dots, r_n$ of arbitrary real numbers is called concave if $2r_i \geq r_{i-1} + r_{i+1}$ for all $0 < i < n$. In this terminology, (AF) is the discrete log-concavity property of the function $\mathrm{V}_K \in \mathbb{R}^{\Delta_{d,d}}$ along the direction $e_i - e_j$ for every $i, j \in [d]$ and $i \neq j$. Equivalently, (log AF) describes the concavity of $\mathrm{v}_K \in \mathbb{R}^{\Delta_{d,d}}$ in the direction $e_i - e_j$. See also Fig. 6.1 for an illustration in the case $d = 3$.

Concave and log-concave sequences are well studied in convex analysis and combinatorics. In Section 6.5 we will work with relations of the more general type

$$\frac{1}{2}r_{i-1} + \frac{1}{2}r_{i+1} \leq r_i + C \tag{6.7}$$

that depend on a constant $C \geq 0$. We informally refer to inequalities of the form (6.7) as *weak concavity relations*. In the following lemma we include basic properties of sequences satisfying such weak concavity relations which mimic basic properties of concave sequences.

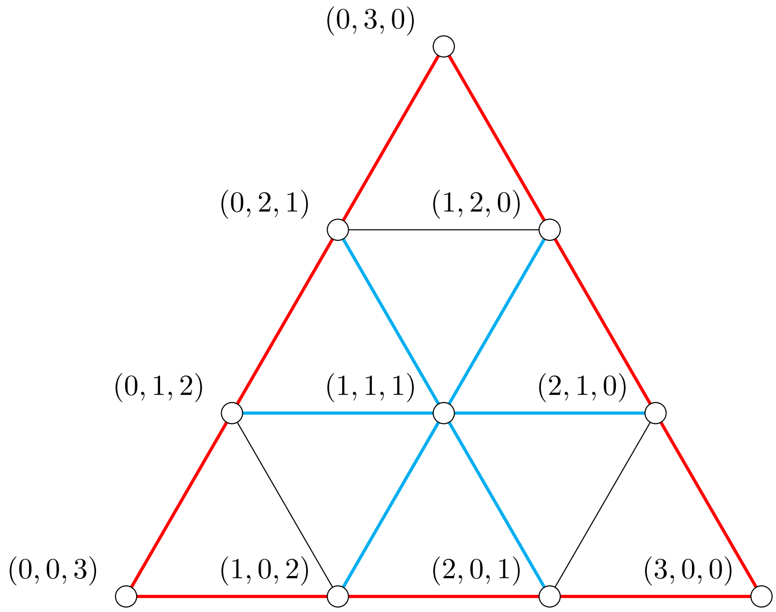


Figure 6.1.: Illustration of (log AF) for $d = 3$. Elements of $\mathbb{R}^{\Delta_{3,3}}$ are real-valued functions on the ten lattice points of the triangle with the vertices $3e_1, 3e_2, 3e_3$. The restriction of $\mathrm{v}_K \in \mathbb{R}^{\Delta_{3,3}}$ to any of the red and any of the cyan segments generates a concave sequence.

Lemma 6.3.4. *Let $r_0, r_1 \dots, r_n$ be a sequence of non-negative real numbers satisfying (6.7) for some constant $C \geq 0$. Then*

(i) $\frac{1}{2}r_{i-1} + \frac{1}{2}r_{j+1} \leq \frac{1}{2}r_i + \frac{1}{2}r_j + (j-i+1)C$ *for all* $0 < i \leq j < n$,

(ii) $\frac{n-1}{n}r_0 + \frac{1}{n}r_n \leq r_1 + (n-1)C$,

(iii) $\frac{n-k}{n}r_0 + \frac{k}{n}r_n \leq r_k + k(n-k)C$ *for all* $1 \leq k \leq n$.

Proof. (i) This follows by adding (and simplifying) the inequalities $\frac{1}{2}r_{k-1} + \frac{1}{2}r_{k+1} \leq r_k + C$ for $i \leq k \leq j$.

(ii) For every $0 < i < n$ we have $\frac{n-i}{2}r_{i-1} + \frac{n-i}{2}r_{i+1} \leq (n-i)r_i + (n-i)C$. Adding these inequalities and simplifying we obtain the required inequality.

(iii) We use induction on k. For $k = 1$ this is the statement of part (ii). Assume

$$\frac{n-k}{n}r_0 + \frac{k}{n}r_n \leq r_k + k(n-k)C.$$

Applying this to the sequence $r_1, \dots, r_n$ we get

$$\frac{n-k-1}{n-1}r_1 + \frac{k}{n-1}r_n \leq r_{k+1} + k(n-1-k)C. \tag{6.8}$$

Applying part (ii) to the sequence $r_k, \dots, r_n$ we get

$$\frac{n-k-1}{n-k}r_k + \frac{1}{n-k}r_n \leq r_{k+1} + (n-k-1)C. \tag{6.9}$$

From part (i) we have

$$r_0 + r_{k+1} \leq r_1 + r_k + 2kC. \tag{6.10}$$

Finally, multiplying (6.8) by $n-1$, (6.9) by $n-k$, and (6.10) by $n-k-1$ and adding the results we obtain

$$(n-k-1)r_0 + (k+1)r_n \leq nr_{k+1} + n(k+1)(n-k-1)C,$$

as required. □

Remark 6.3.5. It is convenient to restate (iii) in Lemma 6.3.4 in a more symmetric form:

$$\frac{k}{k+l} r_{p-l} + \frac{l}{k+l} r_{p+k} \leq r_p + klC, \tag{6.11}$$

for any $0 \leq l \leq p$ and $0 \leq k \leq n-p$.

6.4. The asymptotics derived from the Aleksandrov-Fenchel inequalities

The goal of this section is to investigate the relations among mixed volumes that follow from the Aleksandrov-Fenchel inequalities and to study the sharpness of such relations.

6.4.1. Relations and bounds coming from Aleksandrov-Fenchel inequalities

The following lemma shows how Aleksandrov-Fenchel inequalities yield certain higher-order log-concavity relations on the function $\mathrm{V}_K \in \mathbb{R}^{\Delta_{d,d}}$.

Lemma 6.4.1 (Concavity Relations from Aleksandrov-Fenchel)**.** *For $n, k \in [d]$, consider a "copy" of $\Delta_{n,k}$ in $\Delta_{d,d}$ given by*

$$S = \{c_1 e_{i_1} + \cdots + c_n e_{i_n} + t \,:\, (c_1, \ldots, c_n) \in \Delta_{n,k}\},$$

where $1 \leq i_1 < \cdots < i_n \leq d$ and $t \in \mathbb{Z}^d_{\geq 0}$ satisfies $t_1 + \cdots + t_d = d-k$. Denote the vertices of $\operatorname{conv}(S)$ by $b_j = ke_{i_j} + t \in \Delta_{d,d}$ for $j \in [n]$. Then, for every $K \in (\mathcal{K}_d)^d$ and every $p \in S$, the mixed volume configuration V_K satisfies the log-concavity relation

$$\mathrm{V}_K(p)^k \geq \mathrm{V}_K(b_1)^{c_1} \cdots \mathrm{V}_K(b_n)^{c_n}, \tag{6.12}$$

where $(c_1, \ldots, c_n) \in \Delta_{n,k}$ is the unique vector satisfying $kp = c_1 b_1 + \cdots + c_n b_n$.

Proof. For the sake of readability we pass to proving an equivalent logarithmic version of (6.12), that is, we show the inequality

$$\mathrm{v}_K(p) \geq \frac{c_1}{k}\,\mathrm{v}_K(b_1) + \cdots + \frac{c_n}{k}\,\mathrm{v}_K(b_n).$$

We will prove the statement by induction on the number n of vertices of S. For $n = 2$ the statement follows directly from Remark 6.3.5 together with Theorem 6.3.3. Let n now be an arbitrary positive integer and assume without loss of generality that the vertices of S are of the form $b_i = ke_i + t$ for all $i \in [n]$. We may assume that p is an interior point of $\mathrm{conv}(S)$, as otherwise we can pass to the face of $\mathrm{conv}(S)$ containing p and obtain the statement by induction. It is straightforward to verify that the line $p + \mathbb{R}(e_1 - e_2)$ intersects the two facets $F_1 = \mathrm{conv}(b_2, \dots, b_n)$ and $F_2 = \mathrm{conv}(b_1, b_3, \dots, b_n)$ of $\mathrm{conv}(S)$ in lattice points a_1, a_2 in the relative interior of F_1 and F_2, respectively. Then $\mathrm{v}_K(p) = \mathrm{v}_K(a_1 + \tau(e_1 - e_2))$ for some $\tau \in \mathbb{Z}_{\geq 1}$ and, by Theorem 6.3.3, the logarithmic mixed volumes

$$\mathrm{v}_K(a_1), \mathrm{v}_K(a_1 + (e_1 - e_2)), \dots, \mathrm{v}_K(a_1 + \tau(e_1 - e_2)), \dots, \mathrm{v}_K(a_2)$$

form a concave sequence. By Remark 6.3.5, this implies

$$\mathrm{v}_K(p) \geq \sigma_1 \, \mathrm{v}_K(a_1) + \sigma_2 \, \mathrm{v}_K(a_2), \tag{6.13}$$

for unique rational positive numbers $\sigma_1, \sigma_2 \in \mathbb{Q}_{>0}$ with $\sigma_1 + \sigma_2 = 1$ and $p = \sigma_1 a_1 + \sigma_2 a_2$. As a_1 and a_2 are lattice points in the relative interior of the facets F_1 and F_2, respectively, one has

$$a_1 = \mu_2^1 b_2 + \mu_3^1 b_3 + \cdots + \mu_n^1 b_n, \qquad a_2 = \mu_1^2 b_1 + \mu_3^2 b_3 + \cdots + \mu_n^2 b_n,$$

for some positive rational numbers $\mu_2^1, \mu_3^1, \dots, \mu_n^1, \mu_1^2, \mu_3^2, \dots, \mu_n^2 \in \mathbb{Q}_{>0}$. By the induction hypothesis this implies

$$\begin{aligned} \mathrm{v}_K(a_1) &\geq \mu_2^1 \, \mathrm{v}_K(b_2) + \mu_3^1 \, \mathrm{v}_K(b_3) + \cdots + \mu_n^1 \, \mathrm{v}_K(b_n), \\ \mathrm{v}_K(a_2) &\geq \mu_1^2 \, \mathrm{v}_K(b_1) + \mu_3^2 \, \mathrm{v}_K(b_3) + \cdots + \mu_n^2 \, \mathrm{v}_K(b_n). \end{aligned}$$

Combining this with (6.13) one obtains

$$\begin{aligned} \mathrm{v}_K(p) \geq{} & \left(\sigma_2 \mu_1^2\right) \mathrm{v}_K(b_1) + \left(\sigma_1 \mu_2^1\right) \mathrm{v}_K(b_2) \\ & + \left(\sigma_1 \mu_3^1 + \sigma_2 \mu_3^2\right) \mathrm{v}_K(b_3) + \cdots + \left(\sigma_1 \mu_n^1 + \sigma_2 \mu_n^2\right) \mathrm{v}_K(b_n). \end{aligned}$$

By construction, the coefficients on the right hand-side satisfy

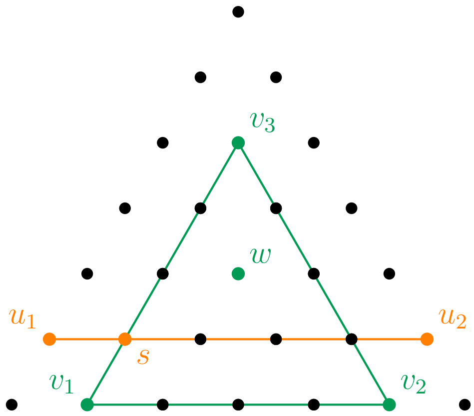
$$p = \left(\sigma_2 \mu_1^2\right) b_1 + \left(\sigma_1 \mu_2^1\right) b_2 + \left(\sigma_1 \mu_3^1 + \sigma_2 \mu_3^2\right) b_3 + \cdots + \left(\sigma_1 \mu_n^1 + \sigma_2 \mu_n^2\right) b_n, \tag{6.14}$$

which proves the claim as the barycentric coordinates of p with respect to the vertices $b_1, \dots, b_n$ are unique (in particular, all coefficients in (6.14) are integral multiples of $\frac{1}{k}$ by construction of S). $\square$

Remark 6.4.2. The particular case of Lemma 6.4.1 when $S = \Delta_{d,d}$ and $p = \mathbf{1}$ provides the following bound for the product of the volumes of the K_i:

$$\mathrm{V}_K(\mathbf{1})^d \geq \mathrm{Vol}_d(K_1) \cdots \mathrm{Vol}_d(K_d).$$

This inequality can also be found in [Sch14, (7.64)]. In particular, we see that if all K_i have volume at least 1 then $\mathrm{Vol}_d(K_i) \leq \mathrm{V}_K(\mathbf{1})^d$ for every $i \in [d]$.



$$\mathrm{v}_K(w) \geq \tfrac{1}{4}\,\mathrm{v}_K(v_1) + \tfrac{1}{4}\,\mathrm{v}_K(v_2) + \tfrac{2}{4}\,\mathrm{v}_K(v_3)$$

$$\mathrm{v}_K(s) \geq \tfrac{4}{5}\,\mathrm{v}_K(u_1) + \tfrac{1}{5}\,\mathrm{v}_K(u_2)$$

Figure 6.2.: Two examples of concavity relations of the type shown in Lemma 6.4.1.

The following is the main statement of this section which provides bounds that the Aleksandrov-Fenchel relations yield for the mixed volume $\mathrm{V}_K(p)$ for any $p \in \Delta_{d,d}$ when $\mathrm{V}_K(\mathbf{1})$ is fixed.

Theorem 6.4.3 (Bounds from Aleksandrov-Fenchel inequalities)**.** *Let $K \in (\mathcal{K}_{d,1})^d$ be a d-tuple of d-dimensional convex bodies of volume at least 1 and $p \in \Delta_{d,d}$. Then it holds that:*

$$\mathrm{v}_K(p) \leq \mathrm{v}_K(\mathbf{1}) \prod_{i\,:\,p_i>0} p_i. \tag{6.15}$$

Furthermore, given that $\mathrm{V}_K(\mathbf{1}) = m$, one obtains the following bound:

$$\mathrm{Vol}_d(\Sigma(K)) \leq m^{3^q 2^r} d^d, \tag{6.16}$$

where $n = 3q + 2r$ with $q \in \mathbb{Z}$ and $r \in \{0, 1, 2\}$.

Proof. We prove (6.15) by inductively making use of Lemma 6.4.1. The induction is over the number of zero entries of p which we denote by k. Let us without loss of generality restrict to the case that p is decreasing, that is $p_1 \geq \cdots \geq p_d$.

As $k = 0$ implies $p = \mathbf{1}$ the statement is trivially fulfilled in this case. Now let $k \in [d-1]$ be arbitrary. Assume $p = (p_1, \ldots, p_{d-k}, 0, \ldots, 0)$ has exactly k zero entries and assume that the statement is true for any vector with at most $k-1$ zero entries. Consider the vector

$$p' = (1, p_2, \ldots, p_{d-k}, \underbrace{1, \ldots, 1}_{p_1-1 \text{ times}}, 0, \ldots, 0).$$

Clearly p' has fewer zero entries than p and, therefore,

$$\mathrm{v}_K(p') \leq \mathrm{v}_K(\mathbf{1}) \prod_{i\,:\,p'_i>0} p'_i = \mathrm{v}_K(\mathbf{1})\, p_2 \cdots p_{d-k}.$$

However, if one writes p' as the barycenter of a suitable (p_1-1)-simplex, Lemma 6.4.1 yields

$$\begin{aligned}\mathrm{v}_K(p') \geq &\frac{1}{p_1}\,\mathrm{v}_K(p_1,p_2,\ldots,p_{d-k},0,\ldots,0)\\ &+\frac{1}{p_1}\,\mathrm{v}_K(0,p_2,\ldots,p_{d-k},p_1,0,\ldots,0)\\ &+\frac{1}{p_1}\,\mathrm{v}_K(0,p_2,\ldots,p_{d-k},0,p_1,0,\ldots,0)\\ &+\cdots\\ &+\frac{1}{p_1}\,\mathrm{v}_K(0,p_2,\ldots,p_{d-k},0,\ldots,0,p_1,0,\ldots,0).\end{aligned}$$

In particular, $\mathrm{v}_K(p) = \mathrm{v}_K(p_1,\ldots,p_{d-k},0,\ldots,0) \leq p_1\,\mathrm{v}_K(p')$, as we assumed the volumes of the K_i to be at least 1 and therefore all terms on the right hand-side of the above inequality are non-negative. This proves (6.15).

We now proceed to using (6.15) in order to show the bound (6.16). Let r be the remainder of the division of d by 3. Write $n = 3q+2r$ for unique integers q,r with $r \in \{0,1,2\}$. We first show that the maximal value of $\prod_{i\,:\,p_i>0} p_i =: g(p)$ is attained at a point $p_{\max}$ with q entries equal to 3, r entries equal to 2, and the remaining entries equal to 0.

Note first that $2^{\lfloor k/2\rfloor} > k$ for all $k \geq 6$. Therefore, for any point $p \in \Delta_{d,d}$ with one coordinate being $k \geq 6$, we can construct another point p' by replacing the entry with value k with $\lfloor k/2 \rfloor$ entries with value 2 and obtain $g(p') > g(p)$. Similarly, any entry with value 5 in p can be replaced by two entries with values 2 and 3 respectively to increase the value of g. As also any entry with value 4 can be replaced by two entries both with value 2 without changing the value of g, this shows that there exists a point p maximizing g with $p_i \leq 3$ for all $i \in [d]$. If p has an entry with value 1, one can construct a point increasing the value of g by replacing $1,3$ with $2,2$, or $2,1$ with 3, or $1,1$ with 2. One of these replacements is always possible and therefore a point p maximizing g can be chosen such that $p_i \in \{0,2,3\}$ for all $i \in [d]$. Finally the observation that $2\cdot 2\cdot 2 < 3\cdot 3$ shows that the maximum of g is actually attained by $p_{\max}$.

Combining this insight with Proposition 6.3.1 one obtains that, for any tuple $K \in (\mathcal{K}_{d,1})^d$, one has

$$\mathrm{Vol}_d(\Sigma(K)) = \sum_{p\in\Delta_{d,d}} \binom{d}{p} 2^{\mathrm{v}_K(p)} \leq \sum_{p\in\Delta_{d,d}} \binom{d}{p} 2^{\mathrm{v}_K(\mathbf{1})g(p_{\max})} = d^d m^{g(p_{\max})},$$

where $m = 2^{\mathrm{v}_K(\mathbf{1})} = \mathrm{V}_K(\mathbf{1})$. This shows (6.16). □

6.4.2. On the optimality of Theorem 6.4.3

This subsection is devoted to showing that Theorem 6.4.3 actually provides the best bounds that one can get by using Aleksandrov-Fenchel inequalities in what we call *black-box style*. In order to make this term precise we need to define the set of all positive-valued functions on $\Delta_{d,d}$ that satisfy all linearized Aleksandrov-Fenchel inequalities (log AF). A statement that is obtained in black-box style from the Aleksandrov-Fenchel inequalities is a statement that holds for each function in this set.

Definition 6.4.4. We define the *A*leksandrov-Fenchel cone $\mathrm{AFC}_d \subset \mathbb{R}^{\Delta_{d,d}}$ as the set of all $\mathrm{v} \in \mathbb{R}^{\Delta_{d,d}}$ satisfying

$$\begin{aligned} \mathrm{v}(p) &\geq 0 && \text{for all } p \in \{de_1, \ldots, de_d\}, \text{ and} \\ 2\,\mathrm{v}(p) &\geq \mathrm{v}(p + e_i - e_j) + \mathrm{v}(p - e_i + e_j) && \text{for all } p, p \pm (e_i - e_j) \in \Delta_{d,d} \text{ with } i, j \in [d]. \end{aligned}$$

We also define the *Aleksandrov-Fenchel polytope* AFP_d to be the following hyperplane section of AFC_d:

$$\mathrm{AFP}_d := \{\mathrm{v} \in \mathrm{AFC}_d \,:\, \mathrm{v}(\mathbf{1}) = 1\}\,.$$

The Aleksandrov-Fenchel inequality implies

$$\mathrm{v}(\mathcal{K}_{d,1}, \Delta_{d,d}) \subseteq \mathrm{AFC}_d\,.$$

Furthermore, for all d-tuples $K \in (\mathcal{K}_{d,1})^d$ with $\mathrm{V}_K(\mathbf{1}) = m$, we have $\mathrm{v}_K(\mathbf{1}) = \log m$ and, hence, $\mathrm{v}_K \in (\log m)\,\mathrm{AFP}_d$.

Remark 6.4.5. It is straightforward to verify that Theorem 6.4.3 and in particular Lemma 6.4.1 are proven by iterated linear combination of inequalities (log AF). This means that Theorem 6.4.3 actually proves relations and bounds that follow for a function from being inside the polytope $(\log m)\,\mathrm{AFP}_d$ and is therefore obtained by black-box application of the Aleksandrov-Fenchel inequalities.

The following proposition shows that Theorem 6.4.3 provides the best possible bounds that can be deduced from Aleksandrov-Fenchel inequalities in a black-box style.

Proposition 6.4.6. *Let $p^* \in \Delta_{d,d}$. Then*

$$\max_{w \in \mathrm{AFP}_d} w(p^*) = \prod_{i\,:\,p_i^* > 0} p_i^*.$$

Proof. Let $p^* = (p_1^*, \ldots, p_d^*)$. Without loss of generality, we can assume that the entries of p are sorted in descending order. Let $r \in [d]$ be the largest number satisfying $p_r^* > 0$.

The fact that $\prod_{i\,:\,p_i > 0} p_i = p_1 \cdots p_r$ is an upper bound is true by Theorem 6.4.3. It remains to confirm that this value is indeed the maximum. To this end, consider $w \in \mathbb{R}^{\Delta_{d,d}}$ given by

$$w(p) = p_1 \cdots p_r \quad \text{for } p \in \Delta_{d,d}.$$

Under this assumption, we see that for the chosen w one has $w(p^*) = \prod_{i\,:\,p_i^* > 0} p_i^*$.

It remains to verify $w \in \mathrm{AFP}_d$. We need to show that $w \in \mathbb{R}^{\Delta_{d,d}}$ is discretely concave in the directions $e_i - e_j$ with $i \neq j$ in the variables $p = (p_1, \ldots, p_d)$. The function w is a product of some of the variables $p_1, \dots, p_d$. If neither p_i nor p_j occurs in the product, $w(p)$ is constant therefore concave in direction $e_i - e_j$. If exactly one of the variables p_i and p_j occurs in the product, then the function is linear in direction $e_i - e_j$. Now consider the case that both p_i and p_j occur in the product. For simplicity, let $i = 1$ and $j = 2$, so

$$w(p) = p_1 p_2 u,$$

where $u = \prod_{i=3}^{r} p_i \geq 0$ is independent of p_1 and p_2 and so is constant when we change p along the direction $e_1 - e_2$. Changing p along the direction $e_1 - e_2$ means, fixing $p \in \Delta_{d,d}$, and considering the discrete function $\phi : \{-p_1, \dots, p_2\} \to \mathbb{Z}$ given by

$$\phi(s) := w(p + se_1 - se_2) = (p_1 + s)(p_2 - s)u.$$

If $u = 0$, ϕ is identically equal to 0. Otherwise it is immediately clear that ϕ is concave, because it is given by an expression that defines a concave quadratic polynomial. $\square$

6.5. An asymptotically sharp bound derived from square inequalities

One of the main tools in proving the asymptotically sharp bound in Theorem 6.2.1 is the following inequality which expresses a log-concavity property of V_K over a "square" in $\Delta_{d,d}$ whose edge directions are the standard directions $e_i - e_j$.

Lemma 6.5.1 (Square Inequalities)**.** *Let $K \in (\mathcal{K}_d)^d$ be a d-tuple of d-dimensional convex bodies. Let $u_1 = e_{i_1} - e_j$ and $u_2 = e_{i_2} - e_j$ for pairwise different $i_1, i_2, j \in [d]$. Then*

$$\mathrm{V}_K(p)\,\mathrm{V}_K(p + u_1 + u_2) \leq 2\,\mathrm{V}_K(p + u_1)\,\mathrm{V}_K(p + u_2),$$

for any $p \in \Delta_{d,d}$ satisfying $p_j \geq 2$.

Proof. This result appears in [BGL18, Lemma 5.1]. For the sake of completeness we outline a proof which also appears in the proof of [Sch14, Lemma 7.4.1]. For simplicity we assume that $u_1 = e_1 - e_3$, $u_2 = e_2 - e_3$, and $p + u_1 + u_2 = \mathbf{1}$. Then in the standard notation the above statement becomes

$$\mathrm{MV}(K_1, K_2, K')\,\mathrm{MV}(K_3, K_3, K') \leq 2\,\mathrm{MV}(K_1, K_3, K')\,\mathrm{MV}(K_2, K_3, K'), \quad (6.17)$$

where K' denotes the $(d-2)$-tuple $(K_3, \dots, K_d)$. Consider a family of d-tuples of convex bodies $(K_1 + sK_3, K_2 + tK_3, K')$ for positive real s, t. It follows by the Aleksandrov-Fenchel inequality applied to this tuple that the quadratic form $At^2 + 2Bst + Cs^2$, where

$$\begin{aligned}
A &= \mathrm{MV}(K_1, K_3, K')^2 - \mathrm{MV}(K_1, K_1, K')\,\mathrm{MV}(K_3, K_3, K')\\
B &= \mathrm{MV}(K_1, K_2, K')\,\mathrm{MV}(K_3, K_3, K') - \mathrm{MV}(K_1, K_3, K')\,\mathrm{MV}(K_2, K_3, K')\\
C &= \mathrm{MV}(K_2, K_3, K')^2 - \mathrm{MV}(K_2, K_2, K')\,\mathrm{MV}(K_3, K_3, K')
\end{aligned}$$

is non-negative for all positive s, t. Similarly, applying the Aleksandrov-Fenchel inequality to the tuple $(tK_1 + sK_2, K_3, K')$ we see that the quadratic form $At^2 - 2Bst + Cs^2$ is non-negative for all positive s, t. This implies that the discriminant of both forms must be non-positive, i.e. $B^2 \leq AC$. Ignoring the negative terms in A and C, this produces:

$$\begin{aligned}(\mathrm{MV}(K_1, K_2, K')\,\mathrm{MV}(K_3, K_3, K') - \mathrm{MV}(K_1, K_3, K')\,\mathrm{MV}(K_2, K_3, K'))^2 \\ \leq \mathrm{MV}(K_1, K_3, K')^2\,\mathrm{MV}(K_2, K_3, K')^2.\end{aligned}$$

Finally, taking the square root of both sides and rearranging, we obtain (6.17). □

The square inequalities indeed give relations that do not follow as combinations of Aleksandrov-Fenchel inequalities as the following shows.

Corollary 6.5.2. *Let $d \in \mathbb{Z}_{\geq 3}$. There exist functions $f\colon \Delta_{d,d} \to \mathbb{R}_{\geq 0}$ that satisfy all Aleksandrov-Fenchel relations but that are not of the form V_K for any $K \in (\mathcal{K}_d)^d$.*

Proof. We will explicitly construct one such function f. Set $f(\mathbf{1}) = 3$ and $f(p) = 1$ for all $\mathbf{1} \neq p \in \Delta_{d,d}$. It is easy to verify that f satisfies all Aleksandrov-Fenchel relations. However, one has

$$3 = f(3, 0, 0, 1 \ldots, 1) f(1, 1, 1, 1, \ldots, 1) > 2 f(2, 1, 0, 1 \ldots, 1) f(2, 0, 1, 1, \ldots, 1) = 2.$$

By Lemma 6.5.1, there exists no $K \in (\mathcal{K}_d)^d$ that satisfies $\mathrm{V}_K = f$. □

Remark 6.5.3. It may seem curious that the square inequalities yield relations that do not follow from Aleksandrov-Fenchel inequalities in black-box style, while the main tool in the proof of Lemma 6.5.1 is precisely the Aleksandrov-Fenchel inequality. Note that when we apply Aleksandrov-Fenchel inequalities in a black-box style we always derive relations between the values of the function $\mathrm{V}_K \in \mathbb{R}^{\Delta_{d,d}}$ that hold for any *fixed* tuple of convex bodies $K \in \mathcal{K}_d$. The proof of Lemma 6.5.1, however, applies the Aleksandrov-Fenchel inequality to a whole family of tuples $(K_1 + sK_3, K_2 + tK_3, K')$. Therefore it implicitly uses relations between the values of functions V_K and V_L for *different* convex bodies $K, L \in \mathcal{K}_d$.

For our later purposes we need a slight generalization of Lemma 6.5.1 that can be obtained by combining different square inequalities. It is convenient to introduce the following notation. Consider a subset $I \subset [d]$ and an element $j \in [d] \setminus I$. Denote

$$u_{I,j} = \sum_{i \in I} (e_i - e_j).$$

When $I = \{i\}$ we write $u_{i,j}$ for $u_{\{i\},j} = e_i - e_j$.

Lemma 6.5.4 (Generalized Square Inequalities)**.** *Let $K \in (\mathcal{K}_d)^d$ be a d-tuple of d-dimensional convex bodies. Let $I \subset [d]$ and $i, j \in [d] \setminus I$. Then*

$$\mathrm{V}_K(p)\,\mathrm{V}_K(p + u_{I,j} + u_{i,j}) \leq 2^{|I|}\,\mathrm{V}_K(p + u_{I,j})\,\mathrm{V}_K(p + u_{i,j}).$$

for any $p \in \Delta_{d,d}$ satisfying $p_j > |I|$.

Proof. We will prove the statement by induction on $|I|$. Note that for $|I| = 1$ the statement is given by Lemma 6.5.1. Assume $|I| > 1$. Pick $k \in I$ and let $I' = I \setminus \{k\}$. By the induction hypothesis

$$\mathrm{V}_K(p)\,\mathrm{V}_K(p + u_{I',j} + u_{i,j}) \leq 2^{|I'|}\,\mathrm{V}_K(p + u_{I',j})\,\mathrm{V}_K(p + u_{i,j}).$$

Applying Lemma 6.5.1 where we replace p by $p + u_{I',j}$ and set $u_1 = u_{i,j}$ and $u_2 = u_{k,j}$, we obtain

$$\mathrm{V}_K(p + u_{I',j})\,\mathrm{V}_K(p + u_{I',j} + u_{i,j} + u_{k,j}) \leq 2\,\mathrm{V}_K(p + u_{I',j} + u_{i,j})\,\mathrm{V}_K(p + u_{I',j} + u_{k,j}).$$

Multiplying the above two inequalities and noting that $u_{I',j} + u_{k,j} = u_{I,j}$ we obtain the claim. $\square$

The following lemma shows that the functions $\mathrm{V}_K \in \mathbb{R}^{\Delta_{d,d}}$ satisfy certain weak log-concavity relations in any direction of the form $u_{I,j}$ for $I \subset [d]$ and $j \in [d] \setminus I$.

Lemma 6.5.5. *Let $K \in (\mathcal{K}_d)^d$ be a d-tuple of d-dimensional convex bodies. Let $I \subset [d]$ and $j \in [d] \setminus I$. Then*

$$\mathrm{V}_K(p + k u_{I,j})^{\frac{l}{k+l}}\,\mathrm{V}_K(p - l u_{I,j})^{\frac{k}{k+l}} \leq 2^{kl\binom{|I|}{2}}\,\mathrm{V}_K(p)$$

for any $k, l \in \mathbb{N}$ and $p \in \Delta_{d,d}$ satisfying $p + k u_{I,j}, p - l u_{I,j} \in \Delta_{d,d}$.

Proof. We will prove the special case of $k = l = 1$ and the general case follows from (6.11) in Remark 6.3.5. The proof of the special case is again via induction on $|I|$. For $|I| = 1$ we recover the Aleksandrov-Fenchel inequality.

Assume $|I| > 1$. Pick $i \in I$ and let $I' = I \setminus \{i\}$. Then $u_{I,j} = u_{i,j} + u_{I',j}$. By the induction hypothesis, replacing p by $p + u_{i,j}$, we have

$$\mathrm{V}_K(p + u_{i,j} + u_{I',j})^{\frac{1}{2}}\,\mathrm{V}_K(p + u_{i,j} - u_{I',j})^{\frac{1}{2}} \leq 2^{\binom{|I'|}{2}}\,\mathrm{V}_K(p + u_{i,j}).$$

Furthermore, by the Aleksandrov-Fenchel inequality we have

$$\mathrm{V}_K(p + u_{i,j} - u_{I',j})^{\frac{1}{2}}\,\mathrm{V}_K(p - u_{i,j} - u_{I',j})^{\frac{1}{2}} \leq \mathrm{V}_K(p - u_{I',j}).$$

Finally, by Lemma 6.5.4, where we replace I by I' and p by $p - u_{I',j}$, we have

$$\mathrm{V}_K(p - u_{I',j})\,\mathrm{V}_K(p + u_{i,j}) \leq 2^{|I'|}\,\mathrm{V}_K(p)\,\mathrm{V}_K(p + u_{i,j} - u_{I',j}).$$

It remains to multiply the three inequalities above and note that $\binom{|I'|}{2} + |I'| = \binom{|I|}{2}$. $\square$

Our next result (Theorem 6.5.7) provides a method for bounding mixed volumes in directions of the form $\sum_{i \in I} e_i - \sum_{j \in J} e_j$ for some disjoint subsets $I, J \subset [d]$ with $|I| = |J|$. Similar to above we introduce special notation for such directions:

$$u_{I,J} = \sum_{i \in I} e_i - \sum_{j \in J} e_j.$$

We will first illustrate the statement and the proof of Theorem 6.5.7 with an example.

Example 6.5.6. Let $K \in (\mathcal{K}_{6,1})^6$ be a 6-tuple of 6-dimensional convex bodies of volume at least 1. We will show that

$$\mathrm{v}_K(2,2,2,0,0,0) \leq 2\,\mathrm{v}_K(\mathbf{1}) + 6. \tag{6.18}$$

First, by Lemma 6.4.1, in logarithmic notation we have

$$\frac{1}{3}\left(\mathrm{v}_K(1,1,1,3,0,0) + \mathrm{v}_K(1,1,1,0,3,0) + \mathrm{v}_K(1,1,1,0,0,3)\right) \leq \mathrm{v}_K(\mathbf{1}). \tag{6.19}$$

The corresponding 2-simplex is depicted in blue in Figure 6.3. Now, for each of the summands in the left hand side of (6.19), we use the weak log-concavity relations in the directions $(1,1,1,-3,0,0)$, $(1,1,1,0,-3,0)$, and $(1,1,1,0,0,-3)$ given by Lemma 6.5.5 and obtain

$$\begin{aligned}
\frac{1}{2}\,\mathrm{v}_K(2,2,2,0,0,0) + \frac{1}{2}\,\mathrm{v}_K(0,0,0,6,0,0) &\leq 3 + \mathrm{v}_K(1,1,1,3,0,0)\\
\frac{1}{2}\,\mathrm{v}_K(2,2,2,0,0,0) + \frac{1}{2}\,\mathrm{v}_K(0,0,0,0,6,0) &\leq 3 + \mathrm{v}_K(1,1,1,0,3,0)\\
\frac{1}{2}\,\mathrm{v}_K(2,2,2,0,0,0) + \frac{1}{2}\,\mathrm{v}_K(0,0,0,0,0,6) &\leq 3 + \mathrm{v}_K(1,1,1,0,0,3).
\end{aligned}$$

In Figure 6.3 these directions are shown in green. These inequalities, together with (6.19), provide the bound (6.18), as $\mathrm{v}_K(0,0,0,6,0,0)$, $\mathrm{v}_K(0,0,0,0,6,0)$, and $\mathrm{v}_K(0,0,0,0,0,6)$ are non-negative.

Theorem 6.5.7. *Let $K \in (\mathcal{K}_{d,1})^d$ be a d-tuple of d-dimensional convex bodies of volume at least 1. Let $I, J \subset [d]$ be disjoint subsets with $|I| = |J|$. Then*

$$\mathrm{v}_K(p + u_{I,J}) \leq \frac{\mu+1}{\mu}\,\mathrm{v}_K(p) + (\mu+1)\binom{\lfloor d/2 \rfloor}{2},$$

for any $p \in \Delta_{d,d}$ such that $p \pm u_{I,J} \in \Delta_{d,d}$, where $\mu = \min(p_i : i \in I)$.

Proof. First we write p as the barycenter of a simplex with vertices $b_j = p - u_{J\setminus\{j\},j}$, for $j \in J$. Applying Lemma 6.4.1 we get

$$\frac{1}{|J|}\sum_{j \in J} \mathrm{v}_K(p - u_{J\setminus\{j\},j}) \leq \mathrm{v}_K(p). \tag{6.20}$$

In order to establish the required bound for $\mathrm{v}_K(p + u_{I,J})$ we estimate each summand $\mathrm{v}_K(p - u_{J\setminus\{j\},j})$ from below using the weak concavity relations along $u_{I,j}$ given by Lemma 6.5.5. Indeed, applying Lemma 6.5.5 with p replaced by $p - u_{J\setminus\{j\},j}$ and $(k,l) = (1,\mu)$, in the logarithmic notation we get

$$\frac{\mu}{\mu+1}\,\mathrm{v}_K(p - u_{J\setminus\{j\},j} + u_{I,j}) + \frac{1}{\mu+1}\,\mathrm{v}_K(p - u_{J\setminus\{j\},j} - \mu u_{I,j}) \leq \mu\binom{|I|}{2} + \mathrm{v}_K(p - u_{J\setminus\{j\},j}).$$

Note that $-u_{J\setminus\{j\},j} + u_{I,j} = u_{I,J}$. Also, since we assumed that the K_i have volume at least 1, the second term in the left-hand side is non-negative and, hence, can be dropped. We thus obtain

$$\frac{\mu}{\mu+1}\,\mathrm{v}_K(p + u_{I,J}) - \mu\binom{|I|}{2} \leq \mathrm{v}_K(p - u_{J\setminus\{j\},j}).$$

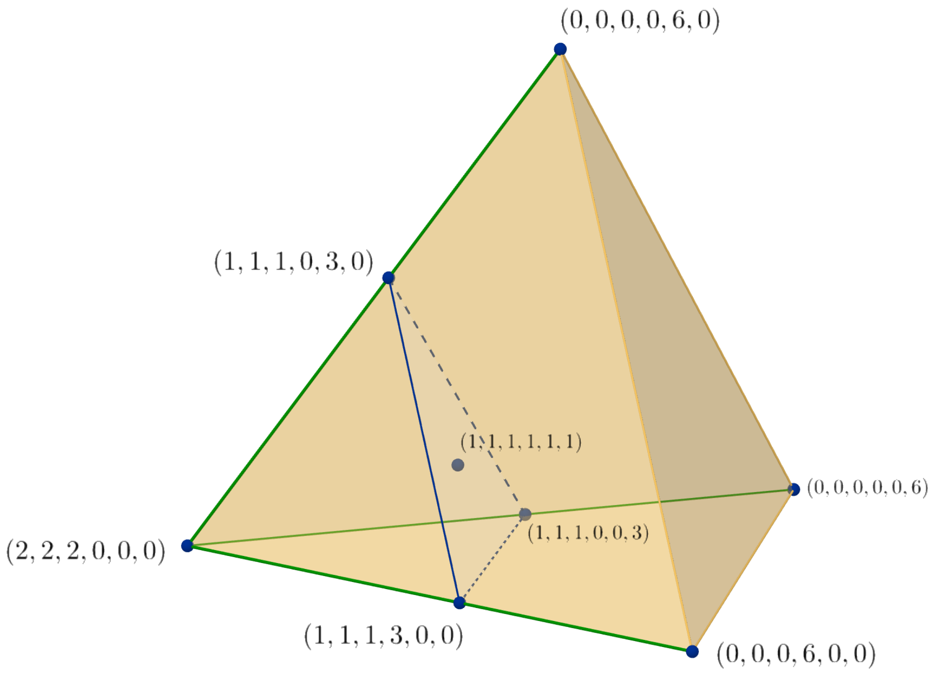


Figure 6.3.: Bounding $\mathrm{v}_K(2, 2, 2, 0, 0, 0)$ in terms of $\mathrm{v}_K(1, 1, 1, 1, 1, 1)$. We use the fact that all points that we draw live inside the 3-dimensional slice $\{p \in \Delta_{6,6} \colon p_1 = p_2 = p_3\}$ of the 5-dimensional simplex $\Delta_{6,6}$.

Plugging these estimates into (6.20) yields

$$\mathrm{v}_K(p+u_{I,J}) \leq \frac{\mu+1}{\mu}\,\mathrm{v}_K(p) + (\mu+1)\binom{|I|}{2}.$$

Finally, using $|I| \leq \lfloor d/2 \rfloor$ we get the claim. □

Remark 6.5.8. One may be led to think that functions V_K satisfy certain weak log-concavity relations in any direction $u_{I,J}$. Indeed, the bound of Theorem 6.5.7 is exactly what one would get from relations of the form

$$\frac{\mu}{\mu+1}\,\mathrm{v}_K(p+u_{I,J}) + \frac{1}{\mu+1}\,\mathrm{v}_K(p-u_{I,J}) \leq \mathrm{v}_K(p) + C.$$

for $C = \binom{|I|}{2}$. In Proposition 6.5.9 below we show that for $|I| = |J| = 2$ we indeed almost have such relations but for a slightly larger constant $C = 2$. However, according to our computations, for $|I| = |J| > 2$ our methods cannot show such weak concavity relations along $u_{I,J}$ anymore no matter the constant C.

Proposition 6.5.9. *Let $K \in (\mathcal{K}_d)^d$ be a d-tuple of d-dimensional convex bodies and I, J disjoint subsets of $[d]$ with $|I| = |J| = 2$. Then*

$$\mathrm{V}_K(p+u_{I,J})\,\mathrm{V}_K(p-u_{I,J}) \leq 2^4\,\mathrm{V}_K(p)^2,$$

for any $p \in \Delta_{d,d}$ satisfying $p \pm u_{I,J} \in \Delta_{d,d}$.

Proof. For simplicity, we assume $I = \{1,2\}$, $J = \{3,4\}$. Applying Lemma 6.5.1 with p replaced by $p - e_1 + e_3$ and $u_1 = e_1 - e_3$, $u_2 = e_2 - e_3$ we get

$$\mathrm{V}_K(p-e_1+e_3)\,\mathrm{V}_K(p+e_2-e_3) \leq 2\,\mathrm{V}_K(p)\,\mathrm{V}_K(p-e_1+e_2).$$

Next, applying Lemma 6.5.1 with p replaced by $p+e_1+e_2-e_3-e_4$ and $u_1 = e_3 - e_1$, $u_2 = e_4 - e_1$ we get

$$\mathrm{V}_K(p+e_1+e_2-e_3-e_4)\,\mathrm{V}_K(p-e_1+e_2) \leq 2\,\mathrm{V}_K(p+e_2-e_4)\,\mathrm{V}_K(p+e_2-e_3).$$

Multiplying the above inequalities we obtain

$$\mathrm{V}_K(p+e_1+e_2-e_3-e_4)\,\mathrm{V}_K(p-e_1+e_3) \leq 4\,\mathrm{V}_K(p)\,\mathrm{V}_K(p+e_2-e_4).$$

Similarly, switching the indices $1 \leftrightarrow 4$ and $2 \leftrightarrow 3$, we obtain

$$\mathrm{V}_K(p-e_1-e_2+e_3+e_4)\,\mathrm{V}_K(p-e_4+e_2) \leq 4\,\mathrm{V}_K(p)\,\mathrm{V}_K(p+e_3-e_1).$$

Finally, the product of the last two inequalities provides the result. □

The following is our key result regarding bounds on mixed volumes in general dimension.

Theorem 6.5.10. *Let $K \in (\mathcal{K}_{d,1})^d$ be a d-tuple of d-dimensional convex bodies of volume at least* 1 *and $p \in \Delta_{d,d}$. Then one has:*

$$\mathrm{v}_K(p) \leq \max(p)\left(\mathrm{v}_K(\mathbf{1}) + (\max(p)-1)\binom{\lfloor d/2 \rfloor}{2}\right), \tag{6.21}$$

Consequently,

$$\mathrm{v}_K(p) \leq \max(p)\,\mathrm{v}_K(\mathbf{1}) + C(d),$$

where $C(d)$ is a constant only depending on the dimension d.

Furthermore, given that $\mathrm{V}_K(\mathbf{1}) = m$, one obtains the following bound:

$$\mathrm{Vol}_d(\Sigma(K)) \leq 2^{d(d-1)\binom{\lfloor d/2 \rfloor}{2}} d^d\, m^d. \tag{6.22}$$

In particular, the maximum of $\mathrm{Vol}_d(\Sigma(K))$ is of order $O(m^d)$.

Proof. We will show that there is a sequence of inequalities of the type shown in Theorem 6.5.7 that yields (6.21). Let us, without loss of generality, assume that p is a decreasing vector, that is $p_1 \geq \cdots \geq p_d$. Hence $\max(p) = p_1$.

Let us define the set of *admissible vectors* $\mathcal{S}_p$ at a point $p \in \Delta_{d,d}$ to be

$$\mathcal{S}_p := \left\{ \sum_{i=1}^{n} e_i - \sum_{j=l+1}^{l+n} e_j \text{ for } l \geq n \geq 1, l+n \leq d \text{ and } n \text{ satisfying } p_1 = \cdots = p_n \right\}.$$

We claim that there is a sequence of decreasing vectors $a_1, \ldots, a_{p_1} \in \Delta_{d,d}$ starting at $a_1 = \mathbf{1}$ and ending at $a_{p_1} = p$ such that $a_{i+1} - a_i \in \mathcal{S}_{a_i}$ for $1 \leq i < p_1$ and, hence, $\max(a_i) = i$ for all $1 \leq i \leq p_1$. We call such a sequence an *a*dmissible path from $\mathbf{1}$ to p. The existence of such a path can be easily seen by induction on p_1. If $p_1 = 1$ then $p = \mathbf{1}$ and there is nothing to show, so let $p_1 \geq 2$. Let n be the maximal index satisfying $p_n = p_1$ and l be the maximal index satisfying $p_l > 0$. Consider the vector

$$p' = (p_1 - 1, \ldots, p_n - 1, p_{n+1}, \ldots, p_l, \underbrace{1, \ldots, 1}_{n \text{ times}}, 0, \ldots, 0).$$

One can check that $p' \in \Delta_{d,d}$ exists and is decreasing by construction. By the induction hypothesis there is an admissible path from $\mathbf{1}$ to p' of length $p_1 - 1$. Moreover, $p - p' \in \mathcal{S}_{p'}$, and therefore there exists an admissible path from $\mathbf{1}$ to p of length p_1.

Let us now show how the existence of such an admissible path implies (6.21). Let a_{i+1} and a_i be two terms in an admissible path from $\mathbf{1}$ to p. By Theorem 6.5.7 we have

$$\mathrm{v}_K(a_{i+1}) \leq \frac{\mu+1}{\mu}\,\mathrm{v}_K(a_i) + (\mu+1)\binom{\lfloor d/2 \rfloor}{2},$$

where μ is the minimum of those entries of a_i which increase when we pass to a_{i+1}. But all these entries are equal to i by the construction of the admissible sequence. Hence, we can write

$$\mathrm{v}_K(a_{i+1}) \leq \frac{i+1}{i}\,\mathrm{v}_K(a_i) + (i+1)\binom{\lfloor d/2 \rfloor}{2}.$$

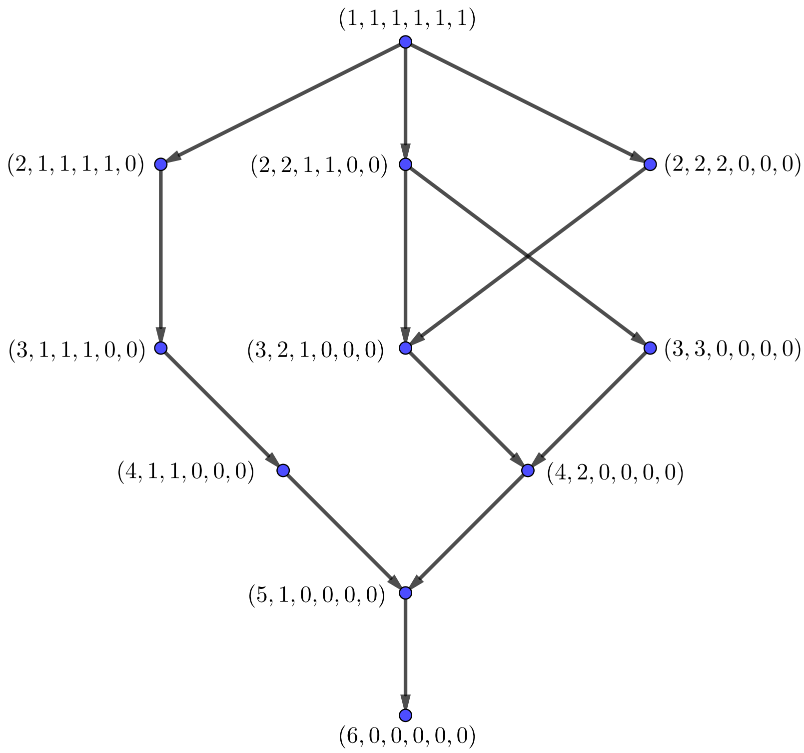


Figure 6.4.: Admissible paths from $\mathbf{1} \in \Delta_{6,6}$ to any decreasing point $p \in \Delta_{6,6}$.

Applying this repeatedly we obtain

$$\begin{aligned}
\mathrm{v}_K(p) &\le \left(\frac{2}{1}\right)\left(\frac{3}{2}\right)\cdots\left(\frac{p_1}{p_1-1}\right)\mathrm{v}_K(\mathbf{1}) + p_1(p_1-1)\binom{\lfloor d/2 \rfloor}{2} \\
&= p_1\,\mathrm{v}_K(\mathbf{1}) + p_1(p_1-1)\binom{\lfloor d/2 \rfloor}{2},
\end{aligned}$$

which concludes the proof of (6.21). The inequality using a constant $C(d)$ only depending on the dimension d follows directly from (6.21) and the observation that $\max(p)$ is bounded by d.

Assume now $\mathrm{V}_K(\mathbf{1}) = m$. Combining Proposition 6.3.1 with the observation that the maximum of the bounds from (6.21) is attained e.g. at $p = (d, 0, \ldots, 0)$, one obtains

$$\mathrm{Vol}_d(\Sigma(K)) = \sum_{p \in \Delta_{d,d}} \binom{d}{p} 2^{\mathrm{v}_K(p)} \le \sum_{p \in \Delta_{d,d}} \binom{d}{p} 2^{\mathrm{v}_K(d,0,\ldots,0)} = d^d 2^{\mathrm{v}_K(d,0,\ldots,0)}.$$

Explicitly plugging in the bound from (6.21) for $\mathrm{v}_K(d, 0, \ldots, 0)$ yields (6.22). □

Remark 6.5.11. Note that the bound from Theorem 6.5.10 shows that, for any $p \in \Delta_{d,d}$, the maximum of $\mathrm{V}_K(p)$ among all d-tuples $(\mathcal{K}_{d,1})^d$ of d-dimensional convex bodies of volume at least 1 that satisfy $\mathrm{V}_K(\mathbf{1}) = m$ is of order $O(m^{\max(p)})$ as $m \to \infty$.

To see that the order of this bound is sharp, fix $p \in \Delta_{d,d}$ and let $i \in [d]$ be an index satisfying $p_i = \max(p)$. Then any tuple $K \in (\mathcal{K}_{d,1})^d$ of the form $K_i = mA$ and $K_j = A$ for every $j \in [d] \setminus \{i\}$ for a convex body A with $\mathrm{Vol}_d(A) = 1$ yields $\mathrm{V}_K(\mathbf{1}) = m$, while $\mathrm{V}_K(p) = m^{p_i} = m^{\max(p)}$.

6.6. Confirmation of Conjecture 6.2.6 in dimension 3

In this section we use a computer-assisted approach to prove Theorem 6.2.5, which establishes Conjecture 6.2.6 in dimension 3. The high level description of the approach is as follows. In the setting of Conjecture 6.2.6, we know that $\mathrm{v}_K \in (\log m)\,\mathrm{AFP}_d$. So we calculate the vertices of the Aleksandrov-Fenchel polytope AFP_3 using a computer. Since $\mathrm{Vol}_d(K_1 + \cdots + K_\ell)$ is a linear combination of mixed volumes, we conclude that $\mathrm{Vol}_d(K_1 + \cdots + K_\ell) = F(\mathrm{v}_K)$, where F is an explicitly given convex function. Since F is convex, the maximum of F on $(\log m)\,\mathrm{AFP}_3$ is attained at the vertices of $(\log m)\,\mathrm{AFP}_3$. The values of F at the vertices of $(\log m)\,\mathrm{AFP}_3$ are functions of m given by rather simple algebraic expressions. It turns out that one can bound all such expressions from above by $(m + \ell - 1)^3$ for $m \in \mathbb{R}_{\geq 1}$.

While the Aleksandrov-Fenchel polytope has rather many vertices (there are 24 vertices in total), the amount of algebraic computations that we need to carry out can be significantly reduced by taking into account the symmetries. On $\mathbb{R}^{\Delta_{3,3}}$ we introduce the action of the symmetric group S_3 on three elements. We introduce the action of S_3 on $\mathbb{R}^{\Delta_{3,3}}$ by defining σv as

$$(\sigma v)(p_1, p_2, p_3) = v(p_{\sigma(1)}, p_{\sigma(2)}, p_{\sigma(3)})$$

for $\sigma \in S_3$ and $v \in \mathbb{R}^{\Delta_{3,3}}$. It is clear that AFP_3 is invariant under the action of S_3 on $\mathbb{R}^{\Delta_{3,3}}$, which means that $\sigma v \in \mathrm{AFP}_3$ holds for all $\sigma \in S_3$ and all $v \in \mathrm{AFP}_3$.

In the following proposition, we use e_p with $p \in \Delta_{d,d}$ to denote the standard basis vectors of $\mathbb{R}^{\Delta_{d,d}}$. This means, $e_p(q) \in \{0, 1\}$ with $e_p(q) = 1$ if and only if $p = q$.

Proposition 6.6.1 (Vertices of AFP_3)**.** *The polytope AFP_3 has 24 vertices, which are split into 7 orbits under the action of S_3 on AFP_3, with the orbits generated by the following seven vertices*

$$\begin{aligned}
v_1 &= e_{(1,1,1)},\\
v_2 &= 2e_{(2,1,0)} + e_{(1,2,0)} + e_{(1,1,1)},\\
v_3 &= 2e_{(2,1,0)} + 2e_{(1,2,0)} + e_{(1,1,1)},\\
v_4 &= 2e_{(2,1,0)} + e_{(1,2,0)} + \frac{1}{2}e_{(2,0,1)} + e_{(1,0,2)} + e_{(1,1,1)},\\
v_5 &= 2e_{(2,1,0)} + e_{(1,2,0)} + 2e_{(2,0,1)} + e_{(1,0,2)} + e_{(1,1,1)},\\
v_6 &= 2e_{(2,1,0)} + e_{(1,2,0)} + 2e_{(2,0,1)} + e_{(1,0,2)} + 3e_{(3,0,0)} + e_{(1,1,1)},\\
v_7 &= \frac{2}{3}e_{(2,1,0)} + \frac{4}{3}e_{(1,2,0)} + \frac{4}{3}e_{(2,0,1)} + \frac{2}{3}e_{(1,0,2)} + \frac{2}{3}e_{(0,2,1)} + \frac{4}{3}e_{(0,1,2)} + e_{(1,1,1)}.
\end{aligned}$$

Proof. We use SageMath [Sag18] to determine the vertices of AFP_3, given by a system of linear inequalities. SageMath is one of the many possibilities to do computations with polytopes over the field of rational numbers. $\square$

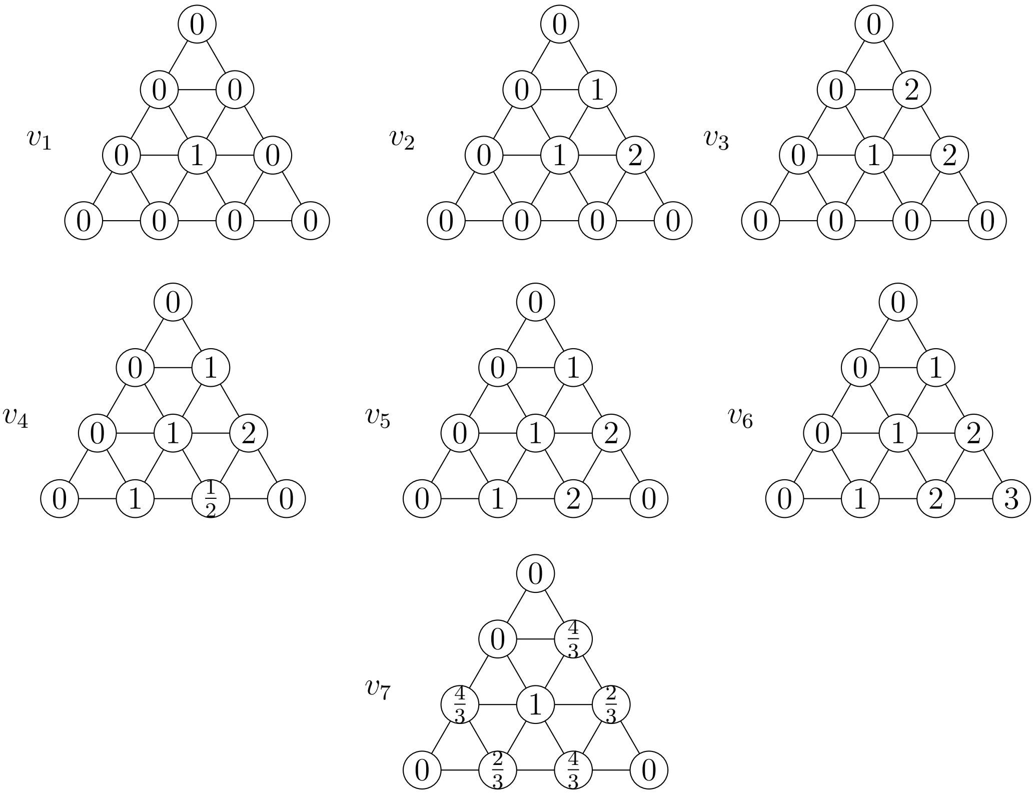


Figure 6.5.: Illustration to Proposition 6.6.1. The Aleksandrov-Fenchel polytope AFP_3 has 24 vertices that are split into 7 orbits under the action of S_3. The diagrams present the coordinates $v_i(p)$ of the seven vertices $v_1, \ldots, v_7$.

Proof of Theorem 6.2.5. For all three assertions, the equality case is verified in a straightforward way. We prove the respective inequalities.

By Remark 6.4.2, $\mathrm{Vol}_d(K_1) \le m^3$, so (1) follows. For the verification of assertions (2) and (3), we use Proposition 6.6.1. We fix the standard component-wise partial order $\le$ on $\mathbb{R}^{\Delta_{3,3}}$, that is, $\mathrm{v} \le \mathrm{w}$ if and only if $\mathrm{v}(p) \le \mathrm{w}(p)$ holds for every $p \in \Delta_{3,3}$. It is clear that the vertices $v_1, \ldots, v_6$ of AFP_3 are related by

$$v_1 \le v_2 \le v_3 \tag{6.23}$$
$$v_4 \le v_5 \le v_6 \tag{6.24}$$

For (2) we have

$$\mathrm{Vol}_d(K_1 + K_2) = \sum_{i=0}^{3} \binom{3}{i} \mathrm{V}_K(i, 3-i, 0) = \sum_{i=0}^{3} \binom{3}{i} 2^{\mathrm{v}_K(i,3-i,0)},$$

where $\mathrm{v}_K \in (\log m)\,\mathrm{AFP}_3$. Changing the base from 2 to m, we see that $\mathrm{Vol}_d(K_1+K_2)$ is bounded by the maximum of the function $g_m : \mathbb{R}^{\Delta_{3,3}} \to \mathbb{R}$

$$g_m(\mathrm{v}) := m^{\mathrm{v}(3,0,0)} + 3m^{\mathrm{v}(2,1,0)} + 3m^{\mathrm{v}(1,2,0)} + m^{\mathrm{v}(0,3,0)}$$

over $\mathrm{v} \in \mathrm{AFP}_3$. The function $g_m(\mathrm{v})$ is convex so that the maximum is attained at one of the vertices of AFP_3. By Proposition 6.6.1, the vertices of AFP_3 have the form σv_i with $\sigma \in S_3$ and $i \in \{1, \ldots, 7\}$. Taking into account (6.23) and (6.24), it follows that it is enough to check the cases $i \in \{3, 6, 7\}$. First, we detect the maximum of g_m in the orbits generated by v_3, v_6 and v_7. It is straightforward to check that

$$\begin{aligned}
\phi_3(m) &:= \max_{\sigma \in S_3} f_m(\sigma v_3) = 2 + 6m^2,\\
\phi_6(m) &:= \max_{\sigma \in S_3} f_m(\sigma v_6) = 1 + 3m + 3m^2 + m^3 = (m+1)^3\\
\phi_7(m) &:= \max_{\sigma \in S_3} f_m(\sigma v_7) = 2 + 3m^{2/3} + 3m^{4/3}.
\end{aligned}$$

Clearly, $\phi_7(m) \le \phi_3(m) \le \phi_6(m)$, where $\phi_3(m) \le \phi_6(m)$ holds since $\phi_6(m) - \phi_3(m) = (m-1)^3$. Thus, $(m+1)^3$ is an upper bound for $\mathrm{Vol}_d(K_1 + K_2)$.

Similarly, for (3) we have

$$\mathrm{Vol}_d(K_1 + K_2 + K_3) = \sum_{p \in \Delta_{3,3}} \binom{3}{p} \mathrm{V}_K(p) = \sum_{p \in \Delta_{3,3}} \binom{3}{p} 2^{\mathrm{v}_K(p)},$$

where $\mathrm{v}_K \in (\log m)\,\mathrm{AFP}_3$. To obtain the desired upper bound for $\mathrm{Vol}_d(K_1+K_2+K_3)$ we maximize the function $f_m : \mathbb{R}^{\Delta_{3,3}} \to \mathbb{R}$

$$f_m(\mathrm{v}) := \sum_{p \in \Delta_{3,3}} \binom{3}{p} m^{\mathrm{v}(p)}$$

over $\mathrm{v} \in \mathrm{AFP}_3$. Again, the function f_m is convex and so its maximum is necessarily attained in one of the vertices of AFP_3. On the other hand, it is clear that the function is invariant under the action of S_3 on AFP_3, as one clearly has $f_m(\sigma v) = f_m(v)$

for every $v \in \mathrm{AFP}_3$ and $\sigma \in S_3$. It follows that it is enough to compare the values of f_m on the vertices $v_1, \dots, v_7$ from Proposition 6.6.1. That means $\mathrm{v} \leq \mathrm{w}$ implies $f_m(\mathrm{v}) \leq f_m(\mathrm{w})$ for all $\mathrm{v}, \mathrm{w} \in \mathbb{R}^{\Delta_{3,3}}$. The latter property follows from the assumption $m \geq 1$ and the non-negativity of multinomial coefficients. In view of (6.23) and (6.24) it suffices to compare $f_m(v_3), f_m(v_6)$ and $f_m(v_7)$. The non-negativity of $f_m(v_6) - f_m(v_3)$ for $m \geq 1$ can be phrased as the non-negativity of $f_{m+1}(v_6) - f_{m+1}(v_3)$ for $m \geq 0$. It turns out that $f_{m+1}(v_6) - f_{m+1}(v_3)$ is a polynomial in m all of whose coefficients are non-negative. Hence $f_{m+1}(v_6) - f_{m+1}(v_3) \geq 0$ holds for every $m \geq 0$, which implies $f_m(v_6) - f_m(v_3) \geq 0$ for $m \geq 1$.

Comparing $f_m(v_7)$ to $f_m(v_6)$ can be carried out in a similar fashion, but note that v_7 is a fractional point. We can still reduce the verification to the polynomial setting by noticing that $3v_7$ is an integral point. The validity of $f_m(v_6) \geq f_m(v_7)$ for all $m \geq 1$ can be rephrased as the inequality $f_{(m+1)^3}(v_6) - f_{(m+1)^3}(v_7) \geq 0$ for all $m \geq 0$. The latter is true since $f_{(m+1)^3}(v_6) - f_{(m+1)^3}(v_7)$ is a polynomial all of whose coefficients are non-negative. Summarizing, we conclude that $f_m(v_6) = (m+2)^3$ is the maximum of $f_m(\mathrm{v})$ for $\mathrm{v} \in \mathrm{AFP}_3$ and, hence, an upper bound on $\mathrm{Vol}_d(K_1 + K_2 + K_3)$. $\square$

6.7. Outlook

Bounds for irreducible tuples of lattice polytopes

While we have concentrated on bounding the volume of the Minkowski sum of a tuple of convex bodies, which in particular are full-dimensional, the bound that Esterov has shown in [Est19] is phrased for general irreducible tuples of lattice polytopes. It would be interesting to derive a stricter upper bound also for irreducible tuples of lattice polytopes. Note that one would have to modify our approach to do so. Consider for example an irreducible triple of lattice polytopes $(P_1, P_2, P_3) \in \mathcal{P}(\mathbb{Z}^3)^3$, where $\dim(P_1) = \dim(P_2) = 2$ and $\dim(P_3) = 3$. Then $\mathrm{Vol}_d(P_1) = \mathrm{Vol}_d(P_2) = 0$ and the Aleksandrov-Fenchel inequality

$$\mathrm{MV}(P_1, P_2, P_3)^2 \geq \mathrm{Vol}_d(P_1)\,\mathrm{Vol}_d(P_2)\,\mathrm{Vol}_d(P_3)$$

does not yield any bound on $\mathrm{Vol}_d(P_3)$. So, while we know that there exists an upper bound on $\mathrm{Vol}_d(P_3)$ in terms of $\mathrm{MV}(P_1, P_2, P_3)$, we cannot use the same paths of inequalities as in our proofs in general.

Tuples maximizing the volume of the Minkowski sum

The following proposition reduces Conjecture 6.2.6 to a more specific situation by showing that one may assume the maximizers to have a specific structure.

Proposition 6.7.1. *Let $m \in \mathbb{R}_{\geq 1}$ and let $\ell \in [d]$. Consider a tuple $K = (K_1, \dots, K_d)$ of convex bodies satisfying*

$$\mathrm{Vol}_d(K_1) \geq 1, \dots, \mathrm{Vol}_d(K_d) \geq 1, \quad \textit{and} \quad \mathrm{MV}(K_1, \dots, K_d) = m$$

and maximizing $\mathrm{Vol}_d(K_1 + \cdots + K_\ell)$. *Then*

1. *For each such optimal tuple,* $\mathrm{Vol}_d(K_i) = 1$ *holds for all except possibly one choice of* $i \in [\ell]$*, and for every* $i > \ell$*.*

2. *For* $\ell \leq d-1$*, there exists an optimal tuple that satisfies* $K_{\ell+1} = \cdots = K_d$*.*

Proof. (1) If $\alpha_i = \mathrm{Vol}_d(K_i)^{1/d} > 1$ holds for some $i > \ell$ then the tuple is not optimal since changing K_1 to $\alpha_i K_1$ and K_i to $\frac{1}{\alpha_i} K_i$ we obtain a new tuple $K' = (K'_1, \dots, K'_d)$ of mixed volume m with $\mathrm{Vol}_d(K'_1 + \cdots + K'_\ell) > \mathrm{Vol}_d(K_1 + \cdots + K_\ell)$.

Now, assume that $\ell \geq 2$ and that for at least two choices of $i \in [\ell]$ one has $\mathrm{Vol}_d(K_i) > 1$. We can assume $\mathrm{Vol}_d(K_1) > 1$ and $\mathrm{Vol}_d(K_2) > 1$. We consider the tuple $(\frac{1}{t}K_1, tK_2, K_3, \dots, K_d)$, depending on $t > 0$. Clearly, $\mathrm{MV}(\frac{1}{t}K_1, tK_2, K_3, \dots, K_d) = \mathrm{MV}(K_1, \dots, K_d)$. Furthermore, the function $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ given by $f(t) = \mathrm{Vol}_d(\frac{1}{t}K_1 + tK_2 + K_3 + \cdots + K_\ell)$ is a strictly convex function. This can be seen by writing $f(t)$ as a non-negative linear combination of functions t^p, which are strictly convex for every $p \in \mathbb{Z} \setminus \{0, 1\}$. For $\epsilon > 0$ small enough and every $t \in [1-\epsilon, 1+\epsilon]$, the volumes of $\frac{1}{t}K_1$ and tK_2 are at least one. Since $f(t)$ is strictly convex, its maximum on $[1-\epsilon, 1+\epsilon]$ is attained at the boundary and is strictly larger than $f(1)$. This contradicts the optimality of the tuple K and shows that $\mathrm{Vol}_d(K_i) = 1$ for all except possible one choice of $i \in [\ell]$.

(2) In view of Lemma 6.4.1,

$$m = \mathrm{MV}(K_1, \dots, K_d) \geq \prod_{i=\ell+1}^{d} \mathrm{MV}(K_1, \dots, K_\ell, K_i, \dots, K_i)^{\frac{1}{d-\ell}}$$
$$\geq \min_{i \in \{\ell+1, \dots, d\}} \mathrm{MV}(K_1, \dots, K_\ell, K_i, \dots, K_i) =: m'.$$

So, taking the i for which the above minimum is attained and replacing the tuple $(K_1, \dots, K_d)$ by the tuple $(\frac{m}{m'}K_1, K_2, \dots, K_\ell, K_i, \dots, K_i)$, we keep the mixed volume of the tuple unchanged without decreasing the volume of the Minkowski sum of its first ℓ bodies (as $\frac{m}{m'} \geq 1$). □

The quest for tight inequalities and a complete description.

The work on the problem of bounding $\mathrm{Vol}_d(\Sigma(K))$ has taught us that the current knowledge of the relations between mixed volumes is still rather limited and the literature might miss some important inequalities beyond the classical ones. Such new inequalities would probably be of interest to a broader community of experts, including researchers interested in metric aspects of convex sets, as well as researchers working on combinatorial aspects of algebraic geometry. The problem of describing the relationship between mixed volumes goes back to the 1960 work [She60] of Shephard (see also Problems 6.1 in [Gru07, p. 109] for a similar problem for the so-called Quermassintegrals). In [She60] Shephard provided a complete description of mixed volume configurations for two d-dimensional convex bodies.

Theorem 6.7.2 (Shephard [She60, Thm. 4])**.** *The mixed-volume configuration space* $\mathrm{V}(\mathcal{K}_d, \Delta_{2,d})$ *is the set of all* $\mathrm{V} \in \mathbb{R}_{>0}^{\Delta_{2,d}}$ *that satisfy the Aleksandrov-Fenchel inequalities*

$$\mathrm{V}(i, d-i)^2 \geq \mathrm{V}(i+1, d-i-1)\,\mathrm{V}(i-1, d-i+1) \quad \forall i \in [d-1].$$

Equivalently, the logarithmic mixed volume configuration space $\mathrm{v}(\mathcal{K}_d, \Delta_{2,d})$ *is a polyhedral cone, described by the linearized Aleksandrov-Fenchel inequalities*

$$2\,\mathrm{v}(i, d-i) \geq \mathrm{v}(i+1, d-i-1) + \mathrm{v}(i-1, d-i+1) \quad \forall i \in [d-1].$$

A refined version of Theorem 6.7.2 can be found in [HHCS12, Lemma 2.1]. This brings us to the following natural question about mixed volume configuration spaces in general.

Problem 6.7.3. *Let* $n, d \in \mathbb{Z}_{\geq 2}$ *and let* $\mathcal{K}$ *be the family of all compact convex sets in* $\mathbb{R}^d$. *Is* $\mathrm{V}(\mathcal{K}, \Delta_{n,d})$ *a semialgebraic set? That is, can* $\mathrm{V}(\mathcal{K}, \Delta_{n,d})$ *be described by a boolean combination of polynomial inequalities?*

Problem 6.7.3 is open for all choices of n and d except for the case $n = 2$, covered by Theorem 6.7.2, and the case $(n, d) = (3, 2)$, solved by Heine [Hei38].

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A. Enumeration data for dimension 3

We present the complete list of maximal irreducible triples of lattice polytopes with mixed volume at most 4. We omit explicitly writing down triples of the form (P, P, P) for a lattice polytope $P \in \mathcal{P}(\mathbb{Z}^3)$, that is, triples of type (0) of Theorem 4.3.5. Instead, we present lists of full-dimensional lattice polytopes $P \in \mathcal{P}(\mathbb{Z}^3)$ of normalized volume up to 4. The lists of full-dimensional $\mathbb{R}$-maximal triples are further subdivided into the types (1)–(3) as in Theorem 4.3.5.

The layout of the figures of polytopes is explained in Fig. A.1.

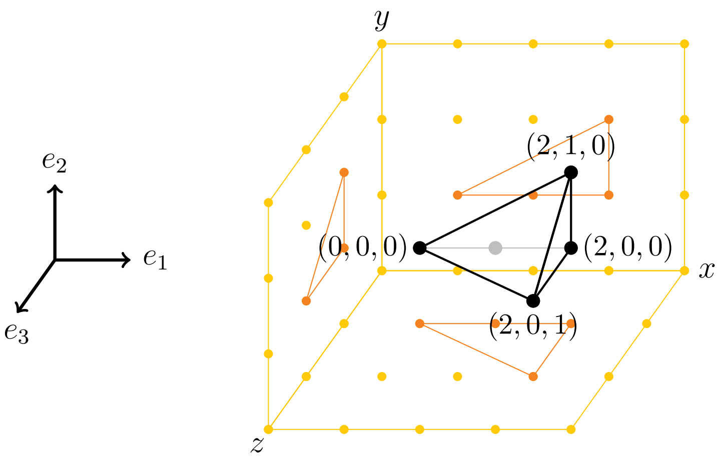
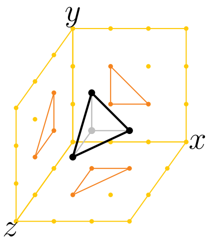


Figure A.1.: An example explaining how polytopes in $\mathbb{R}^3$ are visualized in our figures. The x-axis is directed to the right, the y-axis is directed upwards and the z-axis is directed towards the observer. Each figure depicts appropriately chosen planes orthogonal to the coordinate axes and the orthogonal projections of the polytope onto these planes. In this figure, the planes are given by equations $x = -1$, $y = -1$ and $z = -1$.

We present each polytope as the Minkowski sum of indecomposable lattice polytopes. These representations have been obtained using Magma [BCP97].

Volume 1

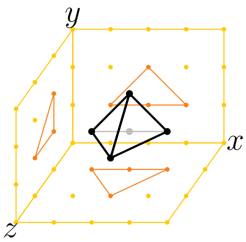
1.



$\mathrm{conv}(0, e_1, e_2, e_3)$

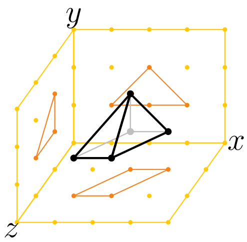
Volume 2

1.



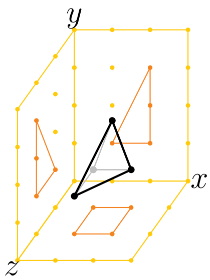
$\mathrm{conv}(0, 2e_1, e_1 + e_2, e_1 + e_3)$

2.



$\mathrm{conv}(0, e_1, 2e_1 - e_3, e_1 - e_3, e_1 + e_2 - e_3)$

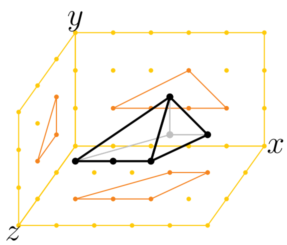
3.



$\mathrm{conv}(0, e_1, e_3, e_1 + 2e_2 + e_3)$

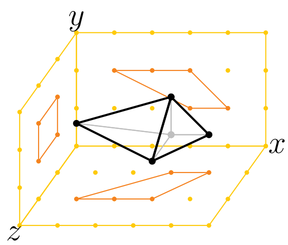
Volume 3

1.



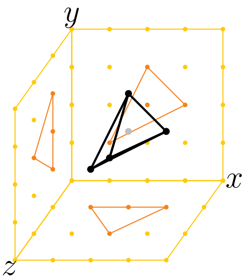
$\mathrm{conv}(0, 2e_1, 3e_1 - e_3, 2e_1 - e_3, 2e_1 + e_2 - e_3)$

2.



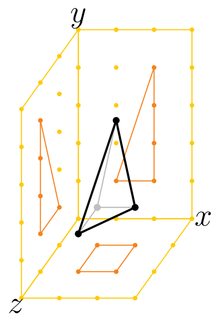
$\mathrm{conv}(0, 2e_1 - e_3, 2e_1 - e_2, 3e_1 - e_2 - e_3, 2e_1 - e_2 - e_3)$

3.



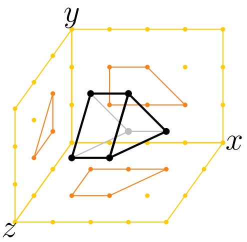
$\mathrm{conv}(0, 2e_1{+}e_2, e_1{+}2e_2, e_1{+}e_2{+}e_3)$

4.



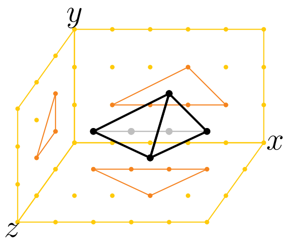
$\mathrm{conv}(0, e_1, e_3, e_1 + 3e_2 + e_3)$

5.



$\mathrm{conv}(0, e_1{-}e_3, e_2{-}e_3){+}\mathrm{conv}(0, e_1)$

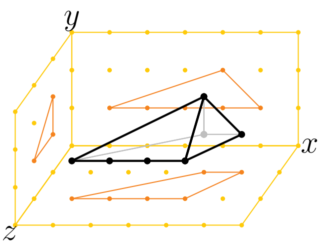
6.



$\mathrm{conv}(0, 3e_1, 2e_1 + e_2, 2e_1 + e_3)$

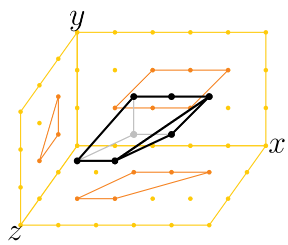
Volume 4

1.



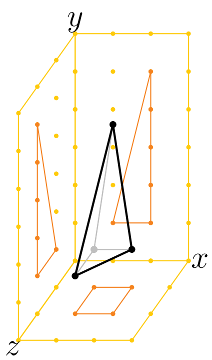
$\mathrm{conv}(0, 3e_1, 4e_1 - e_3, 3e_1 - e_3, 3e_1 + e_2 - e_3)$

2.



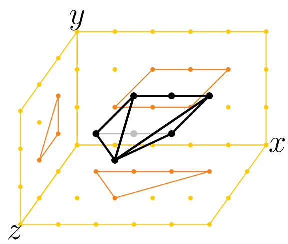
$\mathrm{conv}(0, e_1 - e_3, 2e_1 + e_2 - e_3, e_1 + e_2 - e_3) + \mathrm{conv}(0, e_1)$

3.

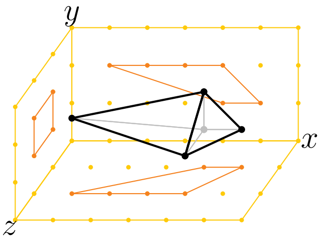


$\mathrm{conv}(0, e_1, e_3, e_1 + 4e_2 + e_3)$

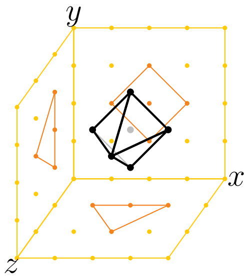
4.



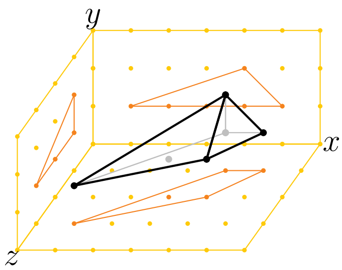
$\mathrm{conv}(0, 2e_1, 3e_1{+}e_2, e_1{+}e_2, e_1{+}e_3)$

5. 

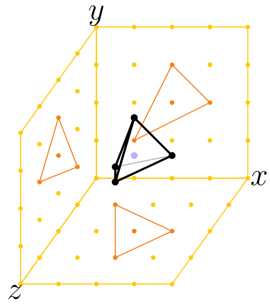
$\mathrm{conv}(0, 3e_1 - e_3, 3e_1 - e_2, 4e_1 - e_2 - e_3, 3e_1 - e_2 - e_3)$

6. 

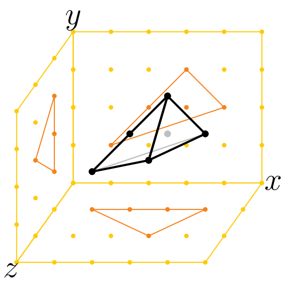
$\mathrm{conv}(0, 2e_1, e_1 + e_2, e_1 + e_3, e_1 - e_2)$

7. 

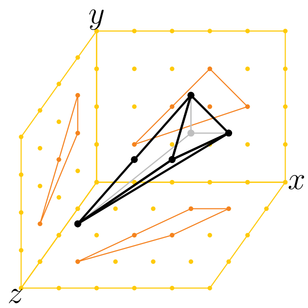
$\mathrm{conv}(0, 4e_1 - 2e_3, 3e_1 - e_3, 3e_1 - 2e_3, 3e_1 + e_2 - 2e_3)$

8. 

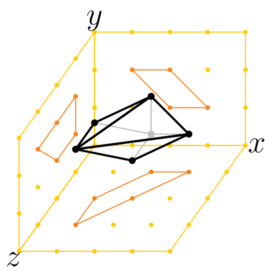
$\mathrm{conv}(0, 2e_1 + e_2 + e_3, e_1 + 2e_2 + e_3, e_1 + e_2 + 2e_3)$

9. 

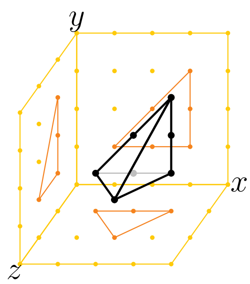
$\mathrm{conv}(0, 3e_1 + e_2, 2e_1 + 2e_2, 2e_1 + e_2 + e_3)$

10. 

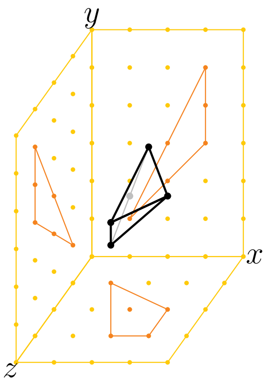
$\mathrm{conv}(0, 3e_1 + e_2 - 2e_3, 2e_1 + 2e_2 - 2e_3, 2e_1 + e_2 - e_3, 2e_1 + e_2 - 2e_3)$

11. 

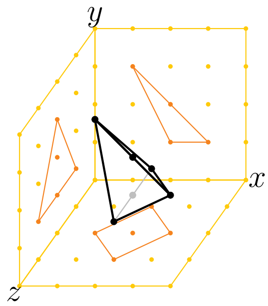
$\mathrm{conv}(0, e_3, e_1 - e_3, e_1 - e_2, 2e_1 - e_2 - e_3, e_1 - e_2 - e_3)$

12. 

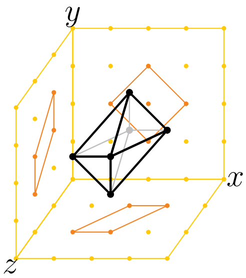
$\mathrm{conv}(0, 2e_1, 2e_1 + 2e_2, e_1 + e_3)$

13. 

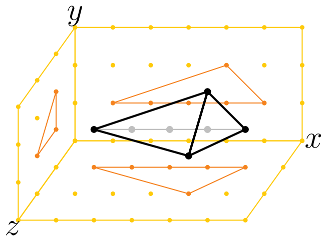
conv$(0, 2e_1 + 4e_2 + 2e_3, 2e_1 + 2e_2 + e_3, e_1 + 2e_2 + 2e_3)$

14. 

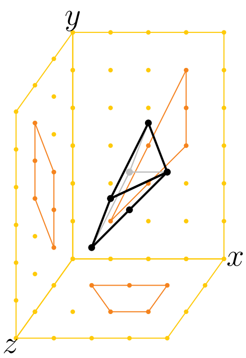
conv$(0, 2e_1 - 2e_2, e_1 - 2e_2 + e_3, e_1 - 2e_2 - e_3)$

15. 

conv$(0, e_1, 2e_1 - e_3, e_1 - e_3, e_1 - e_2, e_1 + e_2 - e_3)$

16. 

conv$(0, 4e_1, 3e_1 + e_2, 3e_1 + e_3)$

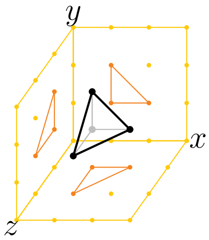
17. 

conv$(0, 2e_1 + 2e_2, e_1 + 2e_2, 2e_1 + 4e_2 + e_3, e_1 + 2e_2 + e_3)$

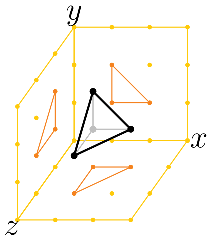
Mixed Volume 2, full-dimensional, $\mathbb{R}$-maximal

Type (1)

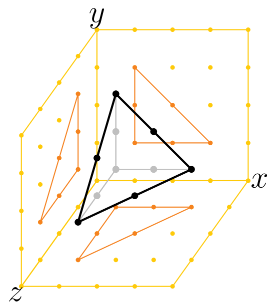
1.



$\operatorname{conv}(0, e_1, e_2, e_3)$



$\operatorname{conv}(0, e_1, e_2, e_3)$



$2\operatorname{conv}(0, e_1, e_2, e_3)$

Mixed Volume 2, lower-dimensional, $\mathbb{R}$-maximal

1.	$\operatorname{conv}(0, e_1, e_2)$	$\operatorname{conv}(0, e_1, e_2) + \operatorname{conv}(0, e_3)$	$\operatorname{conv}(0, e_1, e_2) + \operatorname{conv}(0, e_3)$
2.	$\operatorname{conv}(0, e_2) + \operatorname{conv}(0, e_1 - e_2)$	$\operatorname{conv}(0, e_3) + \operatorname{conv}(0, e_2)$	$\operatorname{conv}(0, e_3) + \operatorname{conv}(0, e_1 - e_2)$

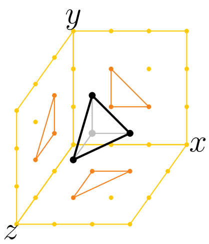
Mixed Volume 2, lower-dimensional, $\mathbb{Z}$-maximal but not $\mathbb{R}$-maximal

1.	$\operatorname{conv}(0, e_1, e_2)$	$\operatorname{conv}(0, 2e_1, 2e_2, e_1 - e_3)$	$\operatorname{conv}(0, 2e_1, 2e_2, e_1 - e_3)$

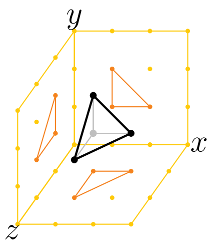
Mixed Volume 3, full-dimensional, $\mathbb{R}$-maximal

Type (1)

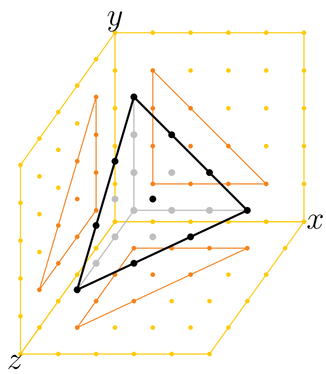
1.



$\operatorname{conv}(0, e_1, e_2, e_3)$



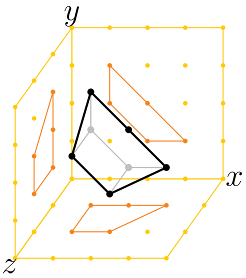
$\operatorname{conv}(0, e_1, e_2, e_3)$



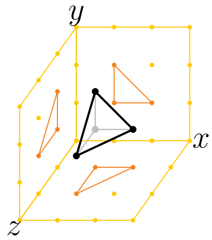
$3\operatorname{conv}(0, e_1, e_2, e_3)$

Type (2)

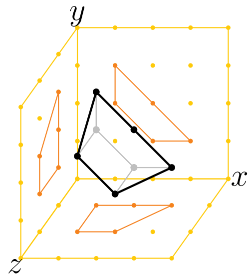
2.



$\mathrm{conv}(0, e_1, e_2, e_3) +$
$\mathrm{conv}(0, e_1 - e_2)$



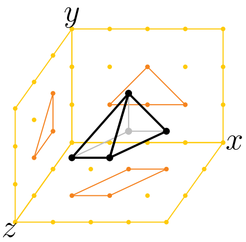
$\mathrm{conv}(0, e_1, e_2, e_3)$



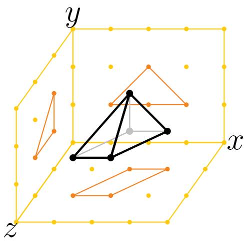
$\mathrm{conv}(0, e_1, e_2, e_3) +$
$\mathrm{conv}(0, e_1 - e_2)$

Type (3)

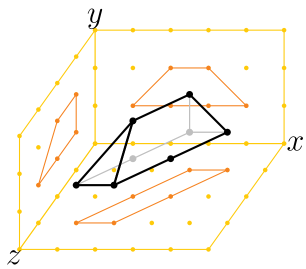
3.



$\mathrm{conv}(0, e_1, 2e_1 -$
$e_3, e_1 - e_3, e_1 + e_2 - e_3)$



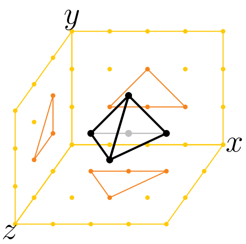
$\mathrm{conv}(0, e_1, 2e_1 -$
$e_3, e_1 - e_3, e_1 + e_2 - e_3)$



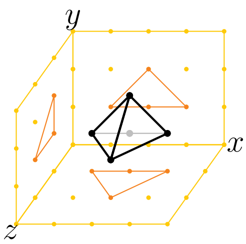
$\mathrm{conv}(0, e_1, 2e_1 -$
$e_3, e_1 - e_3, e_1 + e_2 -$
$e_3) + \mathrm{conv}(0, e_1 - e_3)$

Mixed Volume 3, full-dimensional, $\mathbb{Z}$-maximal but not $\mathbb{R}$-maximal

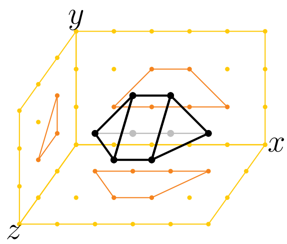
1.



$\mathrm{conv}(0, 2e_1, e_1 +$
$e_2, e_1 + e_3)$



$\mathrm{conv}(0, 2e_1, e_1 +$
$e_2, e_1 + e_3)$



$\mathrm{conv}(0, 2e_1, e_1 + e_2, e_1 +$
$e_3) + \mathrm{conv}(0, e_1)$

Mixed Volume 3, lower-dimensional, $\mathbb{R}$-maximal

1.	$\mathrm{conv}(0, e_1, e_2)$	$\mathrm{conv}(0, e_1, e_2) + \mathrm{conv}(0, e_1, e_2, e_1 + e_3)$	$\mathrm{conv}(0, e_1, e_2) + \mathrm{conv}(0, e_1, e_2, e_1 + e_3)$
2.	$\mathrm{conv}(0, e_1, e_2) + \mathrm{conv}(0, e_3)$	$\mathrm{conv}(0, e_1, e_2)$	$\mathrm{conv}(0, e_1, e_2) + 2\,\mathrm{conv}(0, e_3)$
3.	$\mathrm{conv}(0, e_1, e_2) + \mathrm{conv}(0, e_3)$	$\mathrm{conv}(0, e_1, e_2)$	$2\,\mathrm{conv}(0, e_1, e_2) + \mathrm{conv}(0, e_3)$
4.	$\mathrm{conv}(0, e_2) + \mathrm{conv}(0, e_1 - e_2)$	$\mathrm{conv}(0, 2e_1 - 2e_2 + e_3, e_1 - e_2 + e_3) + \mathrm{conv}(0, e_1 - e_2)$	$\mathrm{conv}(0, 2e_1 - 2e_2 + e_3, e_1 - e_2 + e_3) + \mathrm{conv}(0, e_2)$
5.	$\mathrm{conv}(0, e_2) + \mathrm{conv}(0, e_1 - e_2)$	$2\,\mathrm{conv}(0, e_1 - e_2) + \mathrm{conv}(0, 2e_1 - 2e_2 - e_3)$	$\mathrm{conv}(0, e_2) + \mathrm{conv}(0, 2e_1 - 2e_2 - e_3)$
6.	$\mathrm{conv}(0, e_2) + \mathrm{conv}(0, e_1 - e_2)$	$\mathrm{conv}(0, e_2 - e_3) + \mathrm{conv}(0, e_2)$	$\mathrm{conv}(0, e_2) + \mathrm{conv}(0, e_2 - e_3) + \mathrm{conv}(0, e_1 - e_2)$

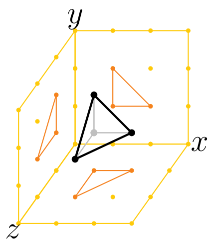
Mixed Volume 3, lower-dimensional, $\mathbb{Z}$-maximal but not $\mathbb{R}$-maximal

1.	$\mathrm{conv}(0, e_1, e_2)$	$\mathrm{conv}(0, 3e_1, 3e_2, 3e_2 - e_3)$	$\mathrm{conv}(0, 3e_1, 3e_2, 3e_2 - e_3)$
2.	$\mathrm{conv}(0, e_2) + \mathrm{conv}(0, e_1 - e_2)$	$\mathrm{conv}(0, e_2, 2e_1 - e_2, 2e_1 - 2e_2, e_1 - e_2 + e_3)$	$\mathrm{conv}(0, e_2, 2e_1 - e_2, 2e_1 - 2e_2, e_1 - e_2 + e_3)$
3.	$\mathrm{conv}(0, e_1, 2e_2)$	$\mathrm{conv}(0, e_1, 3e_2, e_1 + e_2, e_1 - e_3)$	$\mathrm{conv}(0, e_1, 3e_2, e_1 + e_2, e_1 - e_3)$
4.	$\mathrm{conv}(0, 2e_1, 2e_2, e_1 - e_3)$	$\mathrm{conv}(0, e_1, e_2)$	$\mathrm{conv}(0, e_1, e_2) + \mathrm{conv}(0, 2e_1, 2e_2, e_1 - e_3)$
5.	$\mathrm{conv}(0, e_2) + \mathrm{conv}(0, e_1 - e_2)$	$\mathrm{conv}(0, e_3, 2e_2 + e_3)$	$\mathrm{conv}(0, e_3, 2e_2 + e_3) + \mathrm{conv}(0, e_1 - e_2)$

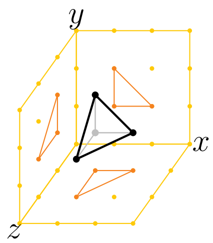
Mixed Volume 4, full-dimensional, $\mathbb{R}$-maximal

Type (1)

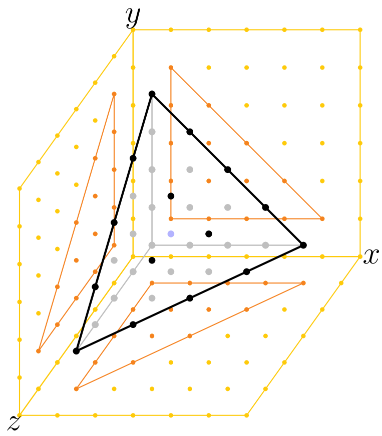
1.



$\operatorname{conv}(0, e_1, e_2, e_3)$

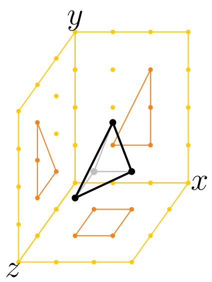


$\operatorname{conv}(0, e_1, e_2, e_3)$

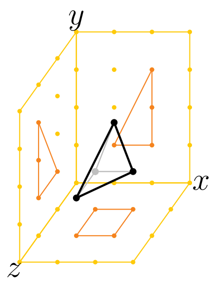


$4\operatorname{conv}(0, e_1, e_2, e_3)$

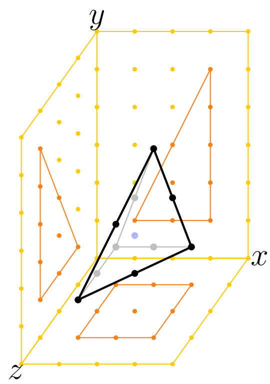
2.



$\operatorname{conv}(0, e_1, e_3, e_1 + 2e_2 + e_3)$

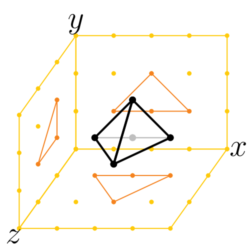


$\operatorname{conv}(0, e_1, e_3, e_1 + 2e_2 + e_3)$

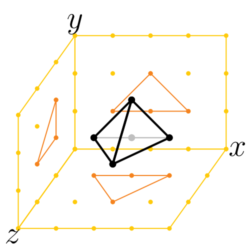


$2\operatorname{conv}(0, e_1, e_3, e_1 + 2e_2 + e_3)$

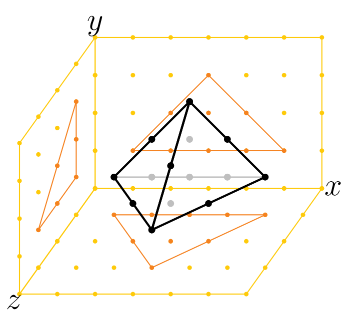
3.



$\operatorname{conv}(0, 2e_1, e_1 + e_2, e_1 + e_3)$

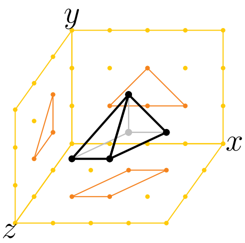


$\operatorname{conv}(0, 2e_1, e_1 + e_2, e_1 + e_3)$

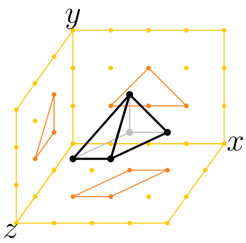


$2\operatorname{conv}(0, 2e_1, e_1 + e_2, e_1 + e_3)$

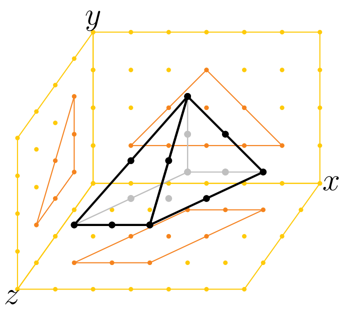
4.



$\mathrm{conv}(0, e_1, 2e_1 - e_3, e_1 - e_3, e_1 + e_2 - e_3)$

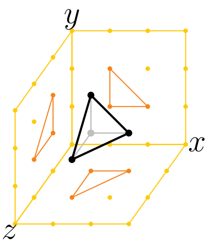


$\mathrm{conv}(0, e_1, 2e_1 - e_3, e_1 - e_3, e_1 + e_2 - e_3)$

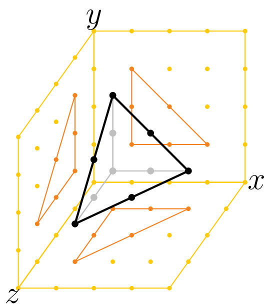


$2\,\mathrm{conv}(0, e_1, 2e_1 - e_3, e_1 - e_3, e_1 + e_2 - e_3)$

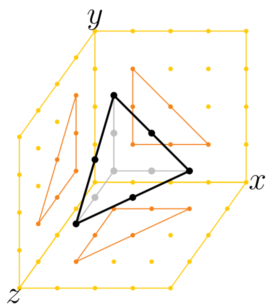
5.



$\mathrm{conv}(0, e_1, e_2, e_3)$



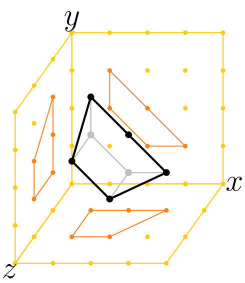
$2\,\mathrm{conv}(0, e_1, e_2, e_3)$



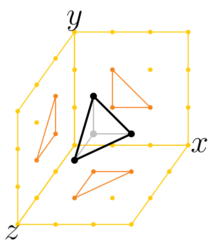
$2\,\mathrm{conv}(0, e_1, e_2, e_3)$

Type (2)

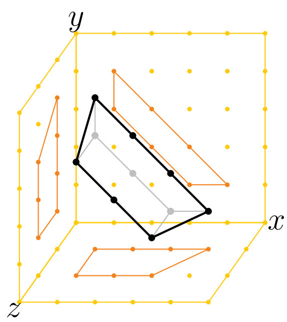
6.



$\mathrm{conv}(0, e_1, e_2, e_3) + \mathrm{conv}(0, e_1 - e_2)$

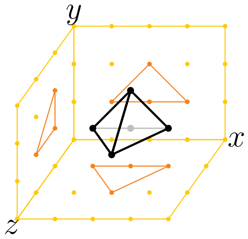


$\mathrm{conv}(0, e_1, e_2, e_3)$

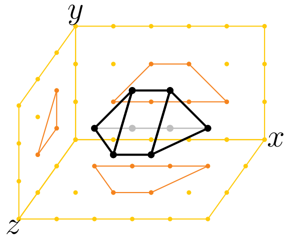


$\mathrm{conv}(0, e_1, e_2, e_3) + 2\,\mathrm{conv}(0, e_1 - e_2)$

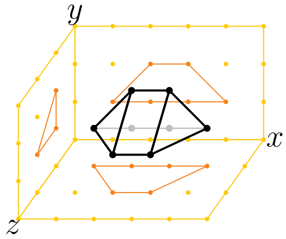
7.



$\mathrm{conv}(0, 2e_1, e_1 + e_2, e_1 + e_3)$

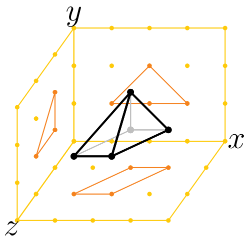


$\mathrm{conv}(0, 2e_1, e_1 + e_2, e_1 + e_3) + \mathrm{conv}(0, e_1)$

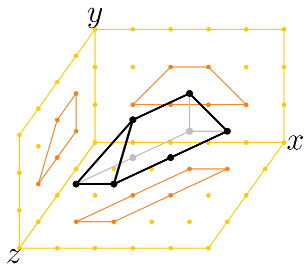


$\mathrm{conv}(0, 2e_1, e_1 + e_2, e_1 + e_3) + \mathrm{conv}(0, e_1)$

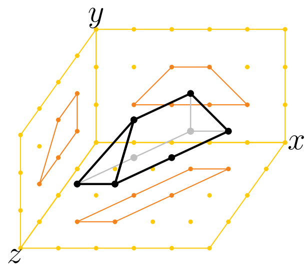
8.



$\mathrm{conv}(0, e_1, 2e_1 - e_3, e_1 - e_3, e_1 + e_2 - e_3)$



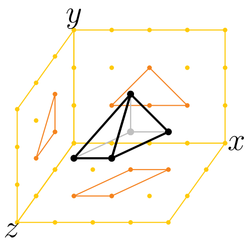
$\mathrm{conv}(0, e_1, 2e_1 - e_3, e_1 - e_3, e_1 + e_2 - e_3) + \mathrm{conv}(0, e_1 - e_3)$



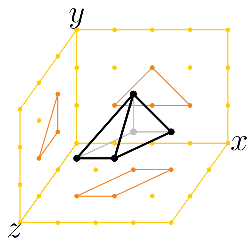
$\mathrm{conv}(0, e_1, 2e_1 - e_3, e_1 - e_3, e_1 + e_2 - e_3) + \mathrm{conv}(0, e_1 - e_3)$

Type (3)

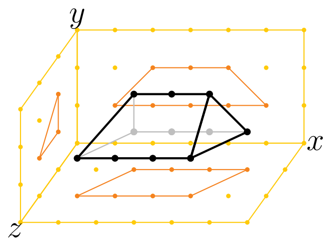
9.



$\mathrm{conv}(0, e_1, 2e_1 - e_3, e_1 - e_3, e_1 + e_2 - e_3)$

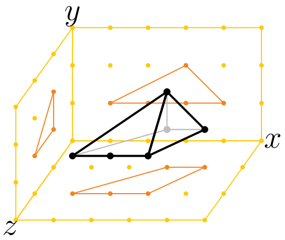


$\mathrm{conv}(0, e_1, 2e_1 - e_3, e_1 - e_3, e_1 + e_2 - e_3)$

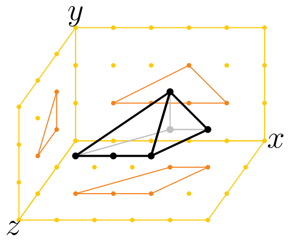


$\mathrm{conv}(0, e_1, 2e_1 - e_3, e_1 - e_3, e_1 + e_2 - e_3) + 2\,\mathrm{conv}(0, e_1)$

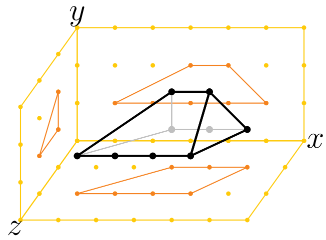
10.



conv$(0, 2e_1, 3e_1 - e_3, 2e_1 - e_3, 2e_1 + e_2 - e_3)$

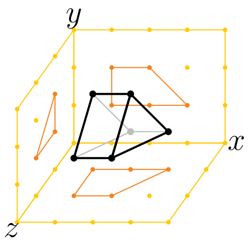


conv$(0, 2e_1, 3e_1 - e_3, 2e_1 - e_3, 2e_1 + e_2 - e_3)$

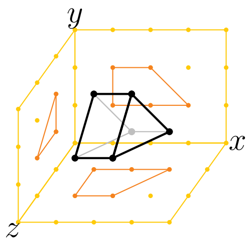


conv$(0, 2e_1, 3e_1 - e_3, 2e_1 - e_3, 2e_1 + e_2 - e_3) + $ conv$(0, e_1)$

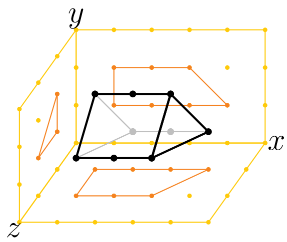
11.



conv$(0, e_1 - e_3, e_2 - e_3) + $ conv$(0, e_1)$



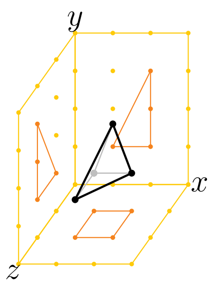
conv$(0, e_1 - e_3, e_2 - e_3) + $ conv$(0, e_1)$



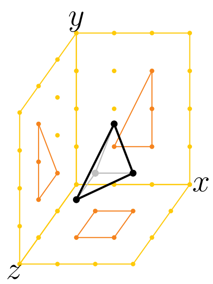
conv$(0, e_1 - e_3, e_2 - e_3) + 2\,$conv$(0, e_1)$

Mixed Volume 4, full-dimensional, $\mathbb{Z}$-maximal but not $\mathbb{R}$-maximal

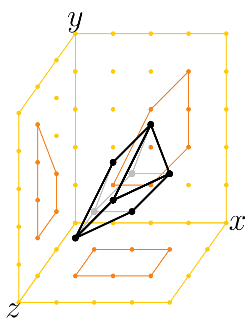
1.



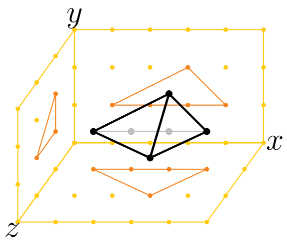
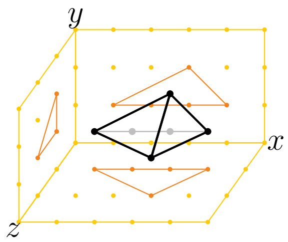
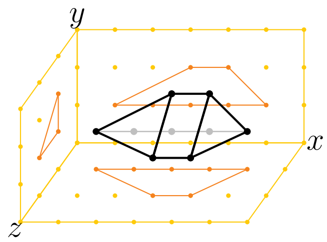
conv$(0, e_1, e_3, e_1 + 2e_2 + e_3)$



conv$(0, e_1, e_3, e_1 + 2e_2 + e_3)$



conv$(0, e_1, e_3, e_1 + 2e_2 + e_3) + $ conv$(0, e_1 + e_2)$

2.	 $\operatorname{conv}(0, 3e_1, 2e_1 + e_2, 2e_1 + e_3)$	 $\operatorname{conv}(0, 3e_1, 2e_1 + e_2, 2e_1 + e_3)$	 $\operatorname{conv}(0, 3e_1, 2e_1 + e_2, 2e_1 + e_3) + \operatorname{conv}(0, e_1)$

Mixed Volume 4, lower-dimensional, $\mathbb{R}$-maximal

1.	$\operatorname{conv}(0, e_1, e_2)$	$2\operatorname{conv}(0, e_1, e_2) + \operatorname{conv}(0, e_1 - e_2 - e_3)$	$2\operatorname{conv}(0, e_1, e_2) + \operatorname{conv}(0, e_1 - e_2 - e_3)$
2.	$\operatorname{conv}(0, e_1, e_2)$	$\operatorname{conv}(0, e_1, e_2) + \operatorname{conv}(0, 2e_1, 2e_2, e_1 - e_2 - e_3)$	$\operatorname{conv}(0, e_1, e_2) + \operatorname{conv}(0, 2e_1, 2e_2, e_1 - e_2 - e_3)$
3.	$\operatorname{conv}(0, e_1, e_2)$	$\operatorname{conv}(0, e_1, e_2) + \operatorname{conv}(0, e_1 + 2e_3)$	$\operatorname{conv}(0, e_1, e_2) + \operatorname{conv}(0, e_1 + 2e_3)$
4.	$\operatorname{conv}(0, e_1, e_2)$	$\operatorname{conv}(0, e_1, e_2) + 2\operatorname{conv}(0, e_3)$	$\operatorname{conv}(0, e_1, e_2) + 2\operatorname{conv}(0, e_3)$
5.	$\operatorname{conv}(0, e_2) + \operatorname{conv}(0, e_1 - e_2)$	$\operatorname{conv}(0, e_2, 2e_1 - 2e_2 - e_3, e_1 - e_2 - e_3) + \operatorname{conv}(0, e_1 - e_2)$	$\operatorname{conv}(0, e_2, 2e_1 - 2e_2 - e_3, e_1 - e_2 - e_3) + \operatorname{conv}(0, e_2)$
6.	$\operatorname{conv}(0, e_2) + \operatorname{conv}(0, e_1 - e_2)$	$\operatorname{conv}(0, e_1, e_2, e_1 - e_2, e_2 - e_3) + \operatorname{conv}(0, e_1 - e_2)$	$\operatorname{conv}(0, e_1, e_2, e_1 - e_2, e_2 - e_3) + \operatorname{conv}(0, e_2)$
7.	$\operatorname{conv}(0, e_2) + \operatorname{conv}(0, e_1 - e_2)$	$\operatorname{conv}(0, e_1 - 2e_2 - e_3) + \operatorname{conv}(0, e_2) + \operatorname{conv}(0, e_1 - e_2)$	$\operatorname{conv}(0, e_1 - 2e_2 - e_3) + \operatorname{conv}(0, e_2) + \operatorname{conv}(0, e_1 - e_2)$
8.	$\operatorname{conv}(0, e_1, 2e_2)$	$\operatorname{conv}(0, e_1, 2e_2) + \operatorname{conv}(0, e_3)$	$\operatorname{conv}(0, e_1, 2e_2) + \operatorname{conv}(0, e_3)$
9.	$2\operatorname{conv}(0, e_2, e_1 + e_2) + \operatorname{conv}(-e_2)$	$\operatorname{conv}(0, e_2, e_1 + e_2) + \operatorname{conv}(0, e_3)$	$\operatorname{conv}(0, e_2, e_1 + e_2) + \operatorname{conv}(0, e_3)$
10.	$\operatorname{conv}(0, e_1, e_2, e_3)$	$\operatorname{conv}(0, e_3) + \operatorname{conv}(0, e_1 - e_2)$	$\operatorname{conv}(0, e_1, e_2, e_3) + \operatorname{conv}(0, e_3) + \operatorname{conv}(0, e_1 - e_2)$
11.	$\operatorname{conv}(0, e_1, e_2) + \operatorname{conv}(0, e_3)$	$\operatorname{conv}(0, e_1, e_2)$	$3\operatorname{conv}(0, e_1, e_2) + \operatorname{conv}(0, e_3)$
12.	$\operatorname{conv}(0, e_1, e_2) + \operatorname{conv}(0, e_3)$	$\operatorname{conv}(0, e_1, e_2)$	$\operatorname{conv}(0, e_1, e_2) + 3\operatorname{conv}(0, e_3)$

13.	$\mathrm{conv}(0,e_1,e_2)+\mathrm{conv}(0,e_3)$	$\mathrm{conv}(0,e_1,e_2)$	$2\,\mathrm{conv}(0,e_1,e_2)+2\,\mathrm{conv}(0,e_3)$
14.	$\mathrm{conv}(0,e_1-e_3,e_2-e_3)+\mathrm{conv}(0,e_1)$	$\mathrm{conv}(0,e_1-e_2)+\mathrm{conv}(0,e_1)$	$\mathrm{conv}(0,e_1-e_3,e_2-e_3)+\mathrm{conv}(0,e_1)+\mathrm{conv}(0,e_1-e_2)$
15.	$\mathrm{conv}(0,e_1,e_2)+\mathrm{conv}(0,e_1,e_2,e_1+e_3)$	$\mathrm{conv}(0,e_1,e_2)$	$2\,\mathrm{conv}(0,e_1,e_2)+\mathrm{conv}(0,e_1,e_2,e_1+e_3)$
16.	$\mathrm{conv}(0,e_2)+\mathrm{conv}(0,e_1-e_2)$	$\mathrm{conv}(0,2e_2,2e_2+e_3)+\mathrm{conv}(0,e_2)$	$\mathrm{conv}(0,2e_2,2e_2+e_3)+\mathrm{conv}(0,e_1-e_2)$
17.	$\mathrm{conv}(0,e_2)+\mathrm{conv}(0,e_1-e_2)$	$2\,\mathrm{conv}(0,e_1-e_2)+\mathrm{conv}(0,2e_1-2e_2-e_3)$	$2\,\mathrm{conv}(0,e_2)+\mathrm{conv}(0,2e_1-2e_2-e_3)$
18.	$\mathrm{conv}(0,e_2)+\mathrm{conv}(0,e_1-e_2)$	$2\,\mathrm{conv}(0,e_1-e_2)+\mathrm{conv}(0,2e_1-2e_2-e_3)$	$\mathrm{conv}(0,e_1-e_2)+\mathrm{conv}(0,e_2)+\mathrm{conv}(0,2e_1-2e_2-e_3)$
19.	$\mathrm{conv}(0,e_2)+\mathrm{conv}(0,e_1-e_2)$	$\mathrm{conv}(0,e_2-2e_3)+\mathrm{conv}(0,e_2)$	$\mathrm{conv}(0,e_2-2e_3)+\mathrm{conv}(0,e_1-e_2)$
20.	$\mathrm{conv}(0,e_2)+\mathrm{conv}(0,e_1-e_2)$	$2\,\mathrm{conv}(0,e_3)+\mathrm{conv}(0,e_1-e_2)$	$2\,\mathrm{conv}(0,e_3)+\mathrm{conv}(0,e_2)$
21.	$\mathrm{conv}(0,e_2)+\mathrm{conv}(0,e_1-e_2)$	$\mathrm{conv}(-e_1+2e_3)+\mathrm{conv}(0,e_2)+\mathrm{conv}(0,e_1-2e_3)$	$\mathrm{conv}(-e_1+2e_3)+\mathrm{conv}(0,e_1-e_2)+\mathrm{conv}(0,e_1-2e_3)$
22.	$\mathrm{conv}(0,e_1,2e_1-e_2)+\mathrm{conv}(-2e_1+e_2)+\mathrm{conv}(0,e_1)$	$\mathrm{conv}(0,e_1-e_3)+2\,\mathrm{conv}(0,e_1)$	$\mathrm{conv}(0,e_1,2e_1-e_2)+\mathrm{conv}(0,e_1-e_3)$
23.	$\mathrm{conv}(0,e_2)+\mathrm{conv}(0,e_1-e_2)$	$\mathrm{conv}(0,2e_2-e_3,e_2-e_3)+2\,\mathrm{conv}(0,e_2)$	$\mathrm{conv}(0,2e_2-e_3,e_2-e_3)+\mathrm{conv}(0,e_1-e_2)$
24.	$\mathrm{conv}(0,e_2)+\mathrm{conv}(0,e_1-e_2)$	$\mathrm{conv}(0,e_1-e_2,e_1-e_2+e_3)+\mathrm{conv}(0,e_1-e_2)$	$\mathrm{conv}(0,e_1-e_2,e_1-e_2+e_3)+\mathrm{conv}(0,e_2)+\mathrm{conv}(0,e_1-e_2)$
25.	$\mathrm{conv}(0,e_2)+\mathrm{conv}(0,e_1-e_2)$	$\mathrm{conv}(0,e_1-e_2,e_1-e_2+e_3)+\mathrm{conv}(0,e_1-e_2)$	$\mathrm{conv}(0,e_1-e_2,e_1-e_2+e_3)+2\,\mathrm{conv}(0,e_2)$
26.	$\mathrm{conv}(0,e_2)+\mathrm{conv}(0,e_1-e_2)$	$\mathrm{conv}(0,e_2)+\mathrm{conv}(0,e_2+e_3)$	$\mathrm{conv}(0,e_2+e_3)+\mathrm{conv}(0,e_2)+2\,\mathrm{conv}(0,e_1-e_2)$
27.	$\mathrm{conv}(0,e_2)+\mathrm{conv}(0,e_1-e_2)$	$\mathrm{conv}(0,e_2)+\mathrm{conv}(0,e_2+e_3)$	$2\,\mathrm{conv}(0,e_2)+\mathrm{conv}(0,e_2+e_3)+\mathrm{conv}(0,e_1-e_2)$
28.	$\mathrm{conv}(0,e_2)+\mathrm{conv}(0,e_1-e_2)$	$\mathrm{conv}(0,e_2)+\mathrm{conv}(0,e_2+e_3)$	$\mathrm{conv}(0,e_2+e_3)+3\,\mathrm{conv}(0,e_1-e_2)$
29.	$\mathrm{conv}(0,e_2)+\mathrm{conv}(0,e_1-e_2)$	$\mathrm{conv}(0,e_2)+\mathrm{conv}(0,e_2+e_3)$	$2\,\mathrm{conv}(0,e_2+e_3)+2\,\mathrm{conv}(0,e_1-e_2)$
30.	$\mathrm{conv}(0,e_1,2e_1-e_2)+\mathrm{conv}(-2e_1+e_2)+\mathrm{conv}(0,e_1)$	$\mathrm{conv}(0,e_1,e_1+e_3)+\mathrm{conv}(0,e_1)$	$\mathrm{conv}(0,e_1,e_1+e_3)+\mathrm{conv}(0,e_1,2e_1-e_2)$

Mixed Volume 4, lower-dimensional, $\mathbb{Z}$-maximal but not $\mathbb{R}$-maximal

1.	$\mathrm{conv}(0, e_1, e_2)$	$\mathrm{conv}(0, 4e_1, 4e_2, 2e_1 - e_3)$	$\mathrm{conv}(0, 4e_1, 4e_2, 2e_1 - e_3)$
2.	$\mathrm{conv}(0, e_1, e_2)$	$\mathrm{conv}(0, 2e_1, 2e_2, e_2 - 2e_3)$	$\mathrm{conv}(0, 2e_1, 2e_2, e_2 - 2e_3)$
3.	$\mathrm{conv}(0, e_1, e_2)$	$\mathrm{conv}(0, e_1, e_2, e_1 - 2e_3, 2e_1 - e_2 - e_3, e_1 - e_2 - e_3)$	$\mathrm{conv}(0, e_1, e_2, e_1 - 2e_3, 2e_1 - e_2 - e_3, e_1 - e_2 - e_3)$
4.	$\mathrm{conv}(0, e_1, e_2)$	$\mathrm{conv}(0, 2e_3, 2e_1 + e_3, 2e_2 + e_3)$	$\mathrm{conv}(0, 2e_3, 2e_1 + e_3, 2e_2 + e_3)$
5.	$\mathrm{conv}(0, e_1, e_2)$	$\mathrm{conv}(0, 2e_1, 2e_2, e_2 + e_3, e_2 - e_3)$	$\mathrm{conv}(0, 2e_1, 2e_2, e_2 + e_3, e_2 - e_3)$
6.	$\mathrm{conv}(0, e_1, e_2)$	$\mathrm{conv}(0, e_1, e_2, e_3, e_1 + e_3, e_2 + e_3, e_1 + e_2 + 2e_3)$	$\mathrm{conv}(0, e_1, e_2, e_3, e_1 + e_3, e_2 + e_3, e_1 + e_2 + 2e_3)$
7.	$\mathrm{conv}(0, e_1, e_2)$	$\mathrm{conv}(0, 2e_1, 2e_2, e_2 - e_3, e_1 + e_2 + e_3)$	$\mathrm{conv}(0, 2e_1, 2e_2, e_2 - e_3, e_1 + e_2 + e_3)$
8.	$\mathrm{conv}(0, e_1, e_2)$	$\mathrm{conv}(0, 2e_1, 2e_2, 2e_1 + e_3, e_1 - e_3)$	$\mathrm{conv}(0, 2e_1, 2e_2, 2e_1 + e_3, e_1 - e_3)$
9.	$\mathrm{conv}(0, e_2) + \mathrm{conv}(0, e_1 - e_2)$	$\mathrm{conv}(0, 2e_1 - 2e_2, 2e_1 - e_2 + e_3) + \mathrm{conv}(0, e_2)$	$\mathrm{conv}(0, 2e_1 - 2e_2, 2e_1 - e_2 + e_3) + \mathrm{conv}(0, e_2)$
10.	$\mathrm{conv}(0, e_2) + \mathrm{conv}(0, e_1 - e_2)$	$\mathrm{conv}(0, 2e_1, 2e_2, 2e_1 - 2e_2, e_1 + e_2 - e_3)$	$\mathrm{conv}(0, 2e_1, 2e_2, 2e_1 - 2e_2, e_1 + e_2 - e_3)$
11.	$\mathrm{conv}(0, e_2) + \mathrm{conv}(0, e_1 - e_2)$	$\mathrm{conv}(0, 2e_1 - 2e_2, e_1 + e_2 + e_3, e_1 - e_2 + e_3)$	$\mathrm{conv}(0, 2e_1 - 2e_2, e_1 + e_2 + e_3, e_1 - e_2 + e_3)$
12.	$\mathrm{conv}(0, e_2) + \mathrm{conv}(0, e_1 - e_2)$	$\mathrm{conv}(0, e_2, 3e_1 - 2e_2, 3e_1 - 3e_2, e_1 - e_3)$	$\mathrm{conv}(0, e_2, 3e_1 - 2e_2, 3e_1 - 3e_2, e_1 - e_3)$
13.	$\mathrm{conv}(0, e_1, 2e_2)$	$\mathrm{conv}(0, 2e_1, 4e_2, 4e_2 - e_3)$	$\mathrm{conv}(0, 2e_1, 4e_2, 4e_2 - e_3)$
14.	$\mathrm{conv}(0, e_1, 2e_1 - e_2) + \mathrm{conv}(-2e_1 + e_2) + \mathrm{conv}(0, e_1)$	$\mathrm{conv}(0, 3e_1, 4e_1 - e_2, 2e_1 - e_2, e_1 - e_3)$	$\mathrm{conv}(0, 3e_1, 4e_1 - e_2, 2e_1 - e_2, e_1 - e_3)$
15.	$\mathrm{conv}(0, 3e_1, 2e_1 + e_2) + \mathrm{conv}(-2e_1)$	$\mathrm{conv}(0, 4e_1, 3e_1 + e_2, 2e_1 + e_2, e_1 + e_3)$	$\mathrm{conv}(0, 4e_1, 3e_1 + e_2, 2e_1 + e_2, e_1 + e_3)$
16.	$\mathrm{conv}(-e_1) + \mathrm{conv}(0, e_1 - e_2) + \mathrm{conv}(0, e_1 + e_2)$	$\mathrm{conv}(0, e_1, e_1 + e_3, e_2 + e_3)$	$\mathrm{conv}(0, e_1 + e_2, e_2 + e_3, e_1 + e_2 + e_3)$
17.	$\mathrm{conv}(-e_1) + \mathrm{conv}(0, e_1 - e_2) + \mathrm{conv}(0, e_1 + e_2)$	$\mathrm{conv}(0, e_1, e_3, e_2 + e_3)$	$\mathrm{conv}(0, e_1, e_1 - e_3, e_1 - e_2 - e_3)$
18.	$\mathrm{conv}(0, 2e_1, e_1 + 2e_2) + \mathrm{conv}(-e_1)$	$\mathrm{conv}(0, e_1, e_3, e_2 + e_3)$	$\mathrm{conv}(0, e_1, e_1 - e_3, e_1 + e_2 - e_3)$
19.	$\mathrm{conv}(-e_1) + \mathrm{conv}(0, 2e_1, 2e_1 - e_2, e_1 + e_2)$	$\mathrm{conv}(0, e_1 - e_2, 2e_1 - e_2 - e_3, e_1 - e_2 - e_3)$	$\mathrm{conv}(0, e_1, 2e_1 - e_3, 2e_1 - e_2 - e_3)$

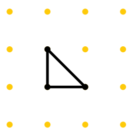
20.	$\mathrm{conv}(0, 2e_1, e_1 + e_2, e_1 + e_3)$	$\mathrm{conv}(0, e_2 - e_3) + \mathrm{conv}(0, e_1)$	$\mathrm{conv}(0, 2e_1, e_1 + e_2, e_1 + e_3) + \mathrm{conv}(0, e_2 - e_3)$
21.	$\mathrm{conv}(0, 2e_1, 3e_1 - e_3, 2e_1 - e_3, 2e_1 + e_2 - e_3)$	$\mathrm{conv}(0, 2e_1, 2e_1 + e_2 - e_3)$	$\mathrm{conv}(0, e_1, 2e_1 - e_3) + \mathrm{conv}(0, 2e_1, 2e_1 + e_2 - e_3)$
22.	$\mathrm{conv}(0, 2e_1, 2e_2, e_1 - e_3)$	$\mathrm{conv}(0, e_1, e_2)$	$2\,\mathrm{conv}(0, 2e_1, 2e_2, e_1 - e_3)$
23.	$\mathrm{conv}(0, e_2, 2e_1 - e_2, 2e_1 - 2e_2, e_1 - e_2 + e_3)$	$\mathrm{conv}(0, e_2) + \mathrm{conv}(0, e_1 - e_2)$	$\mathrm{conv}(0, e_2, 2e_1 - e_2, 2e_1 - 2e_2, e_1 - e_2 + e_3) + \mathrm{conv}(0, e_1 - e_2)$
24.	$\mathrm{conv}(0, e_2, 2e_1 - e_2, 2e_1 - 2e_2, e_1 - e_2 + e_3)$	$\mathrm{conv}(0, e_2) + \mathrm{conv}(0, e_1 - e_2)$	$\mathrm{conv}(0, e_2, 2e_1 - e_2, 2e_1 - 2e_2, e_1 - e_2 + e_3) + \mathrm{conv}(0, e_2)$
25.	$\mathrm{conv}(0, e_1, 3e_2, e_1 + e_2, e_1 - e_3)$	$\mathrm{conv}(0, e_1, 2e_2)$	$\mathrm{conv}(0, 2e_1, 4e_2, e_1 - e_3, e_1 + e_2 - e_3)$
26.	$\mathrm{conv}(0, 3e_1, 3e_2, 3e_2 - e_3)$	$\mathrm{conv}(0, e_1, e_2)$	$\mathrm{conv}(0, 3e_1, 3e_2, 3e_2 - e_3) + \mathrm{conv}(0, e_1, e_2)$
27.	$\mathrm{conv}(0, e_1, e_2)$	$\mathrm{conv}(0, e_1 + e_3, e_1 - e_2 - e_3)$	$\mathrm{conv}(0, 2e_2, 2e_1 + e_2 + 2e_3, 2e_1 - e_2 - 2e_3)$
28.	$\mathrm{conv}(0, e_1, e_2)$	$\mathrm{conv}(0, e_1 + e_3, e_1 - e_2 - e_3)$	$\mathrm{conv}(0, e_1, e_2, e_1 - e_3, 2e_2 + e_3, e_2 + e_3, e_1 + e_2 + 2e_3, e_1 + e_2 + e_3)$
29.	$\mathrm{conv}(0, e_1, e_2)$	$\mathrm{conv}(0, e_1 + e_3, e_1 - e_2 - e_3)$	$\mathrm{conv}(0, e_1, e_2) + \mathrm{conv}(0, e_1 + e_3, e_1 - e_2 - e_3)$
30.	$\mathrm{conv}(0, e_2) + \mathrm{conv}(0, e_1 - e_2)$	$\mathrm{conv}(0, 2e_2, 2e_2 + e_3)$	$\mathrm{conv}(0, 2e_2, 2e_2 + e_3) + 2\,\mathrm{conv}(0, e_1 - e_2)$
31.	$\mathrm{conv}(0, e_2) + \mathrm{conv}(0, e_1 - e_2)$	$\mathrm{conv}(0, 2e_2, 2e_2 + e_3)$	$\mathrm{conv}(0, 2e_2, 2e_2 + e_3) + \mathrm{conv}(0, e_1 - e_2) + \mathrm{conv}(0, e_2)$
32.	$\mathrm{conv}(0, e_2) + \mathrm{conv}(0, e_1 - e_2)$	$\mathrm{conv}(0, 3e_2, e_2 - e_3)$	$\mathrm{conv}(0, 3e_2, e_2 - e_3) + \mathrm{conv}(0, e_1 - e_2)$

B. Enumeration data for dimension 2

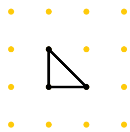
For the reader's convenience we present a complete list of maximal pairs of lattice polygons of mixed volume up to 4. Note that these have already been classified in [EG16].

Mixed Volume 1

1.



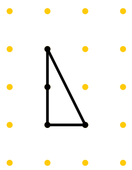
$\mathrm{conv}(0, e_1, e_2)$



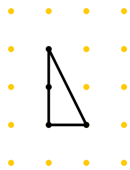
$\mathrm{conv}(0, e_1, e_2)$

Mixed Volume 2

1.

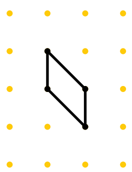


$\mathrm{conv}(0, e_1, 2e_2)$

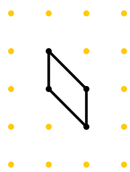


$\mathrm{conv}(0, e_1, 2e_2)$

2.

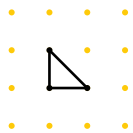


$\mathrm{conv}(0, e_1 - e_2) + \mathrm{conv}(0, e_2)$

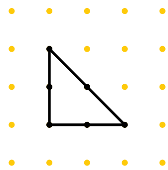


$\mathrm{conv}(0, e_1 - e_2) + \mathrm{conv}(0, e_2)$

3.



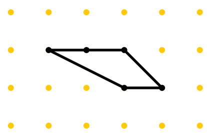
$\mathrm{conv}(0, e_1, e_2)$



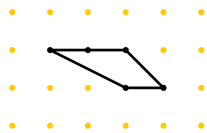
$2\,\mathrm{conv}(0, e_1, e_2)$

Mixed Volume 3

1.

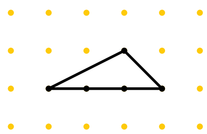


$\mathrm{conv}(0, e_1, 2e_1 - e_2) + \mathrm{conv}(0, e_1)$

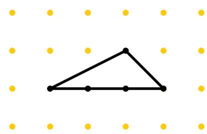


$\mathrm{conv}(0, e_1, 2e_1 - e_2) + \mathrm{conv}(0, e_1)$

2.

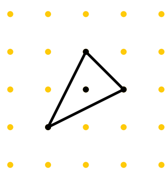


$\mathrm{conv}(0, 3e_1, 2e_1 + e_2)$

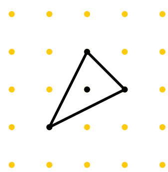


$\mathrm{conv}(0, 3e_1, 2e_1 + e_2)$

3.

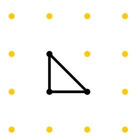


$\mathrm{conv}(0, 2e_1 + e_2, e_1 + 2e_2)$

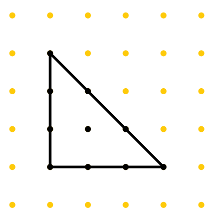


$\mathrm{conv}(0, 2e_1 + e_2, e_1 + 2e_2)$

4.

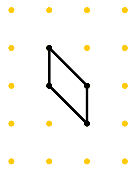


$\mathrm{conv}(0, e_1, e_2)$

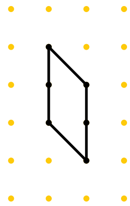


$3\,\mathrm{conv}(0, e_1, e_2)$

5.

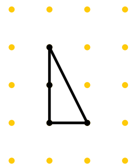


$\mathrm{conv}(0, e_1 - e_2) + \mathrm{conv}(0, e_2)$

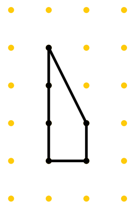


$\mathrm{conv}(0, e_1 - e_2) + 2\,\mathrm{conv}(0, e_2)$

6.



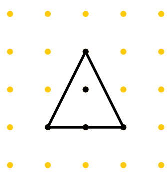
$\mathrm{conv}(0, e_1, 2e_2)$



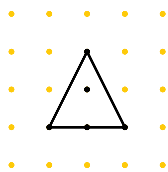
$\mathrm{conv}(0, e_1, 2e_2) + \mathrm{conv}(0, e_2)$

Mixed Volume 4

1.

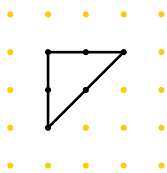


$\mathrm{conv}(0, 2e_1, e_1 + 2e_2)$

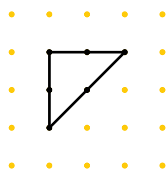


$\mathrm{conv}(0, 2e_1, e_1 + 2e_2)$

2.

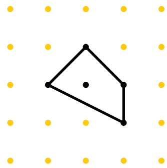


$2\,\mathrm{conv}(0, e_2, e_1 + e_2)$

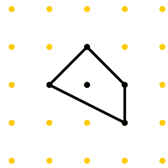


$2\,\mathrm{conv}(0, e_2, e_1 + e_2)$

3.

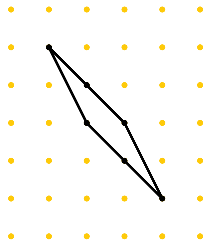


$\mathrm{conv}(0, 2e_1, 2e_1 - e_2, e_1 + e_2)$

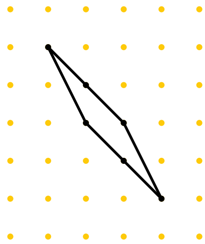


$\mathrm{conv}(0, 2e_1, 2e_1 - e_2, e_1 + e_2)$

4.

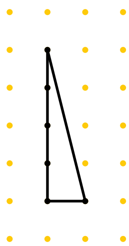


$2\,\mathrm{conv}(0, e_1 - e_2) + \mathrm{conv}(0, e_1 - 2e_2)$

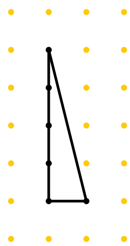


$2\,\mathrm{conv}(0, e_1 - e_2) + \mathrm{conv}(0, e_1 - 2e_2)$

5.

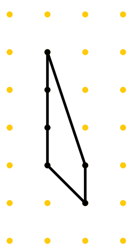


$\mathrm{conv}(0, e_1, 4e_2)$

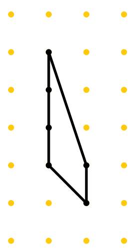


$\mathrm{conv}(0, e_1, 4e_2)$

6.

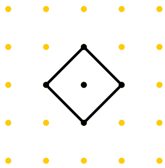


$\mathrm{conv}(0, 2e_2, e_1 - e_2) + \mathrm{conv}(0, e_2)$

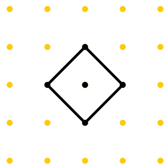


$\mathrm{conv}(0, 2e_2, e_1 - e_2) + \mathrm{conv}(0, e_2)$

7.

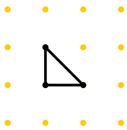


$\mathrm{conv}(0, e_1 + e_2) + \mathrm{conv}(0, e_1 - e_2)$

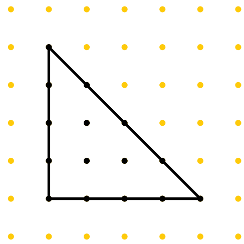


$\mathrm{conv}(0, e_1 + e_2) + \mathrm{conv}(0, e_1 - e_2)$

8.

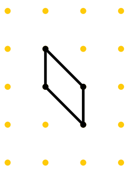


$\mathrm{conv}(0, e_1, e_2)$

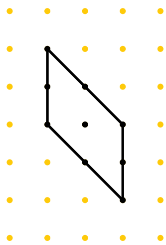


$4\,\mathrm{conv}(0, e_1, e_2)$

9.

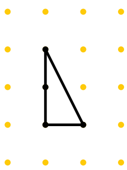


$\mathrm{conv}(0, e_1 - e_2) + \mathrm{conv}(0, e_2)$

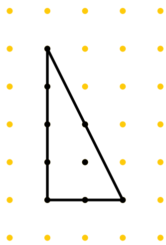


$2\,\mathrm{conv}(0, e_1 - e_2) + 2\,\mathrm{conv}(0, e_2)$

10.

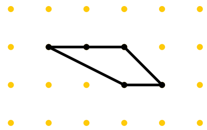


$\mathrm{conv}(0, e_1, 2e_2)$

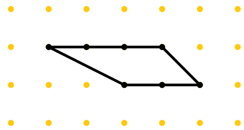


$2\,\mathrm{conv}(0, e_1, 2e_2)$

11.

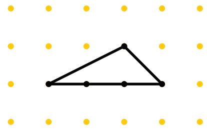


$\mathrm{conv}(0, e_1, 2e_1 - e_2) + \mathrm{conv}(0, e_1)$

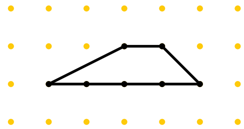


$\mathrm{conv}(0, e_1, 2e_1 - e_2) + 2\,\mathrm{conv}(0, e_1)$

12.

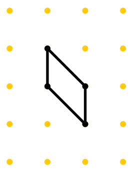


$\mathrm{conv}(0, 3e_1, 2e_1 + e_2)$

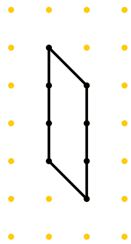


$\mathrm{conv}(0, 3e_1, 2e_1 + e_2) + \mathrm{conv}(0, e_1)$

13.



$\mathrm{conv}(0, e_1 - e_2) + \mathrm{conv}(0, e_2)$



$\mathrm{conv}(0, e_1 - e_2) + 3\,\mathrm{conv}(0, e_2)$