Selected Initial and Boundary Value Problems for Hyperbolic Systems and Kinetic Equations

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Zusammenfassung

In dieser Arbeit sollen mathematische Studien hyperbolischer Systeme von Erhaltungsgleichungen mit ausgewählten physikalischen Anwendungen verbunden werden. Der Schwerpunkt liegt dabei in der mathematischen Analyse von Lorentz-invarianten Systemen der speziellen Relativitätstheorie wie den Maxwell-Gleichungen und den relativistischen Euler Gleichungen. Aber wir studieren auch die sogenannte Boltzmann-Peierls Gleichung (BPG), eine kinetische Evolutionsgleichung, die den Wärmetransport in einem dielektrischen Kristall bei sehr tiefer Temperatur mit Hilfe eines Phonon-Bose Gases beschreibt, sowie ein aus der BPG abgeleitetes hyperbolisches Momentensystem.

Die kinetische Behandlung der relativistischen Euler Gleichungen und die kinetische Boltzmann-Peierls Theorie eines Phonon-Bose Gases beschreiben zwar völlig verschiedene physikalische Vorgänge, zeigen aber trotzdem weitreichende mathematische Analogien.

Für die Untersuchung der relativistischen Euler Gleichungen entwickeln wir die mathematischen Grundlagen der kinetischen Theorie im Rahmen der speziellen Relativitätstheorie. Die von Jüttner angegebene Gleichgewichts-Phasendichte wird mit Hilfe des Maximum Entropie Prinzips begründet. Sie verallgemeinert die klassische Maxwellsche Phasendichte. Dies ermöglicht uns die Entwicklung kinetischer Schemata zur Lösung der relativistischen Euler Gleichungen. Dieser kinetische Zugang erweist sich vor allem bei den ultra-relativistischen Euler Gleichungen als besonders nützlich. In diesem Falle reduzieren sich die aus dem kinetischen Schema gewonnenen Momentenintegrale über die Phasendichte auf einfache Oberflächenintegrale bezüglich der Einheitssphäre, und es läßt sich eine besonders einfache reduzierte Entropiedichte angeben. Wir lösen das Riemannsche Anfangswertproblem für eindimensionale Gasströmungen und vergleichen es mit numerischen Verfahren, die auf dem kinetischen Ansatz beruhen.

Es gibt wichtige Unterschiede zwischen den kinetischen Verfahren für klassische Euler Gleichungen auf der einen Seite und für relativistische Euler Gleichungen bzw. für das Phonon-Bose Gas auf der anderen Seite, die ein detailiertes Studium dieser beiden nichtklassischen Anwendungen rechtfertigt. Der wichtigste Unterschied ist die Möglichkeit, reduzierte Phasendichten, reduzierte Momentenintegrale und reduzierte Entropien für die beiden zuletzt genannten Anwendungen zu finden. Diese sind von wesentlich einfacherer Gestalt als die ursprüglichen Grössen, enthalten aber dieselbe thermodynamische Information und lassen sich ohne Verwendung von Approximationen rigoros herleiten. Die reduzierten Größen haben in beiden nichtklassischen Anwendungen dieselbe Bauart und lassen sich insbesondere für eindimensionale Strömungen weiter sehr stark in einer Weise vereinfachen, die kein Analogon in der klassischen Theorie kennt.

Ein weiterer Unterschied zur klassischen Theorie ist dadurch gegeben, daß sowohl in der relativistischen kinetischen Theorie als auch in der kinetischen Theorie des Phonon-Bose Gases jede Signalgeschwindigkeit global begrenzt ist, nämlich durch die Lichtgeschwindigkeit bzw. durch die sogenannte Debye Geschwindigkeit. Dies hat verschiedene Vorteile für die Analysis und Numerik der kinetischen Schemata, die in dieser Arbeit ebenfalls genutzt werden.

SOLI DEO GLORIA

I dedicate this book to my family

MARIE-PAULETTE & JULIA KUNIK

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"Ich kann es nun einmal nicht lassen, in diesem Drama von Mathematik und Physik - die sich im Dunkeln befruchten, aber von Angesicht zu Angesicht so gerne einander verkennen und verleugnen - die Rolle des (wie ich genugsam erfuhr, oft unerwünschten) Boten zu spielen."

(Hermann Weyl, Gruppentheorie und Quantenmechanik, 1928)

Chapter 1

Preface and Introduction

We aim to combine a mathematical study of hyperbolic systems and conservation laws with specific applications in physics. The main part of this work will consider applications to Lorentz-invariant systems, namely for the Maxwell equations and the relativistic Euler equations. But we will also study the so called Boltzmann-Peierls equation, a kinetic equation for a phonon-Bose gas describing heat conduction in a dielectric solid at very low temperature, and a hyperbolic system resulting from this kinetic equation as a special limiting case. We will see that the latter system shows a very surprising mathematical relationship to the so called ultra-relativistic Euler equations, though the physical applications are totally different in both cases.

Hyperbolic systems describe the propagation of waves with finite velocities, which in special relativity are naturally bounded by the speed of light. This fact is reflected in the beautiful mathematical structure of the equations under consideration. Namely Maxwell's equations and the relativistic Euler equations are very typical representatives for those systems. Though the relativistic Euler equations considered here seem to look complicated, a detailed study shows a simpler mathematical behaviour than the corresponding classical Euler equations. For example, even the solution of the standard shock tube or Riemann problem for the classical Euler equations of gas dynamics may lead to a vacuum region within the shock tube that complicates a rigorous mathematical analysis for the general initial value problem very much. However, we will see that at least for the so called ultra-relativistic Euler equations this behaviour will not occur.

We hope that there are also interested non specialists in relativity which enjoy a short and self consistent elaboration of the electromagnetic theory in Chapters 2 and 3. It is ranging from the mathematical study of the linear wave equation, via the formulation of Maxwell's theory, examples of nontrivial solutions to Maxwell's equations, via the derivation of the electromagnetic balance laws to their formulation with the tensor calculus of special relativity,

which is also presented here.

In Chapter 4 we turn our attention to the relativistic Euler equations. The Euler equations (relativistic or classic) deal with a *perfect gas*, in which mean free paths and collision free times are so short that perfect isotropy is maintained about any point moving with the gas. In this case the *local equilibrium assumption* gives a kinetic distribution function for the microscopic velocities of the representative gas atoms, the so called (relativistic or classic) Maxwellian phase density which depends on the five independent macroscopic variables occuring in the Euler equations, namely the mass density, the velocity and the pressure. These quantities describe uniquely the thermodynamical equilibrium state of the gas in any space-time point. The equilibrium phase density for the relativistic Euler equations was first published by Jüttner in [28] and can be derived like the classical Maxwellian from the so called Maximum Entropy Principle.

Some of these basic ideas can be explained better for the well known classical Boltzmann gas. This will then serve as a useful guideline for the more complicate applications to the relativistic Euler equations and the theory of the phonon-Bose gas presented in this work.

For simplicity we use dimensionless quantitites. A kinetic phase density $f = f(t, \mathbf{x}, \mathbf{q})$ depends on time t, position $\mathbf{x} \in \mathbb{R}^3$ and on the *microscopic velocities* $\mathbf{q}$ of the representative gas atoms. It describes the distribution of the microscopic velocities in such a way that

$$f(t, \mathbf{x}, \mathbf{q})\, d^3x\, d^3q \tag{1.0.1}$$

gives the number of gas particles at time t and position $\mathbf{x}$ in the infinitesimal phase space volume $[\mathbf{x} + \mathbf{dx}, \mathbf{q} + \mathbf{dq}]$. From any phase density f we can recover all macroscopic quantitites which are of interest in thermodynamics by forming integral moments from f with respect to the microscopic velocities. For example, the first five moments denote the particle number density ρ, the momentum density $\rho\mathbf{u} \in \mathbb{R}^3$ and the energy density $\frac{1}{2}\rho\mathbf{u}^2 + \frac{3}{2}\,p$ given by

$$\begin{aligned} \rho(t, \mathbf{x}) &= \int\limits_{\mathbb{R}^3} f(t, \mathbf{x}, \mathbf{q})\, d^3q\,, \\ (\rho\mathbf{u})(t, \mathbf{x}) &= \int\limits_{\mathbb{R}^3} \mathbf{q}\, f(t, \mathbf{x}, \mathbf{q})\, d^3q\,, \\ (\frac{1}{2}\rho\mathbf{u}^2 + \frac{3}{2}\,p)(t, \mathbf{x}) &= \int\limits_{\mathbb{R}^3} \frac{1}{2}\mathbf{q}^2\, f(t, \mathbf{x}, \mathbf{q})\, d^3q\,. \end{aligned} \tag{1.0.2}$$

Here $\mathbf{u}$, T and $p = \rho T$ denote the macroscopic velocity, the absolute temperature and the pressure of the gas, respectively. These formulas hold for the classical monatomic Boltzmann gas, not necessary in local equilibrium.

If the gas is in local equilibrium, then it is described by the classical Maxwellian phase density

$$f_M(\rho, T, \mathbf{u}, \mathbf{q}) = \frac{\rho}{(2\pi T)^{\frac{3}{2}}} \exp\left(-\frac{(\mathbf{q}-\mathbf{u})^2}{2T}\right) . \tag{1.0.3}$$

We conclude that in local equilibrium the microscopic velocities $\mathbf{q}$ of the representative gas atoms are distributed according to a Gaussian density. If we calculate the moments for mass- momentum and energy density with the Maxwellian f_M in (1.0.3), then we get indeed back the equations in (1.0.2).

Next we consider a simplified version of the classical Boltzmann equation, the so called BGK-equation according to Bhatnagar, Gross and Krook, see [1], which describes the evolution of the Boltzmann gas according to the following kinetic equation

$$\frac{\partial f}{\partial t} + \sum_{k=1}^{3} q_k \frac{\partial f}{\partial x_k} = \frac{1}{\tau_R}\left(P_M[f] - f\right) . \tag{1.0.4}$$

The left hand side is the so called transport part of the kinetic equation and the right hand side describes the collision of the particles with the relaxation to the classical Maxwellian $P_M[f]$ and a relaxation time τ_R. Exactly speaking, the "projection" $P_M[f]$ is the Maxwellian phase density which has the same mass- momentum- and energy density as f according to (1.0.2).

A special case is given in the limit $\tau_R \to \infty$, where the collision term on the right hand side in (1.0.4) is zero. Then the kinetic equation is called a collision-free kinetic transport equation, and the solution can be given explicitly in terms of the initial phase density by

$$f_{free}(t, \mathbf{x}, \mathbf{q}) = f(0, \mathbf{x} - t\mathbf{q}, \mathbf{q}) . \tag{1.0.5}$$

Solutions to collision-free kinetic transport equations constitute a basic building block for the formulation of kinetic schemes.

The opposite case is given by the limit $\tau_R \to 0$, where the gas is everywhere described by the Maxwellian phase density in local equilibrium. Then the gas is governed by the hyperbolic system of Euler equations, see the textbook of Cercignani [3]. In this case the Euler equations result formally as a moment system from the BGK-equation as follows, where we omit the arguments t, $\mathbf{x}$ and $\mathbf{q}$ for simplicity.

Put $q_0 = 1$, $q_4 = \frac{1}{2}\mathbf{q}^2$ and define the moment weights W_α as well as the corresponding fluxes $F_{\alpha,k}$ for $\alpha = 0, 1, 2, 3, 4$ and $k = 1, 2, 3$ due to

$$W_\alpha = \int\limits_{\mathbb{R}^3} q_\alpha \, f \, d^3q \, , \qquad F_{\alpha,k} = \int\limits_{\mathbb{R}^3} q_\alpha \, q_k \, f \, d^3q \, . \tag{1.0.6}$$

If we multiply the BGK-equation (1.0.4) by q_α and integrate with respect to $\mathbf{q} \in \mathbb{R}^3$, then we obtain the "five-moment system"

$$\frac{\partial W_\alpha}{\partial t} + \sum_{k=1}^{3} \frac{\partial F_{\alpha,k}}{\partial x_k} = 0\,. \tag{1.0.7}$$

The right hands side is always zero for the "five-moment system", not only in the limit $\tau_R \to 0$, because the first five moments of $P_M[f]$ and f are the same by construction of $P_M[f]$ and by regarding the generally valid equations (1.0.2). Physically speaking, (1.0.7) guarantees the conservation laws for mass, momentum and energy also for $\tau_R > 0$. In order to pass from (1.0.7) to the Euler equations, we just have to calculate the moments W_α and fluxes $F_{\alpha,k}$ with the Maxwellian $P_M[f]$ instead of f and obtain from (1.0.7) in the limit $\tau_R \to 0$ the classical Euler equations

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \sum_{k=1}^{3} \frac{\partial(\rho u_k)}{\partial x_k} &= 0\,,\\
\frac{\partial(\rho u_i)}{\partial t} + \sum_{k=1}^{3} \frac{\partial}{\partial x_k}\Big(\rho\, u_i u_k + p\,\delta_{ik}\Big) &= 0\,,\\
\frac{\partial}{\partial t}\Big(\frac{1}{2}\rho\,\mathbf{u}^2 + \frac{3}{2}p\Big) + \sum_{k=1}^{3} \frac{\partial}{\partial x_k}\Big(\frac{\rho}{2}\,\mathbf{u}^2 u_k + \frac{5}{2}p\,u_k\Big) &= 0\,.
\end{aligned} \tag{1.0.8}$$

The basic kinetic elements presented above serve as building blocks in order to develop kinetic schemes for the classical Euler equations, but also as a guideline for the treatment of the relativistic Euler equations and for the theory of the phonon-Bose gas.

However, there are also important differences between the kinetic approaches for the classical and relativistic Euler equations and for the phonon-Bose gas which justify a detailed study of the non-classical applications! The most important difference is the possibility to define reduced phase densities, reduced kinetic equations and reduced entropies for the latter two applications. Even in three space dimensions the reduced quantitites have a much simpler mathematical structure than the original kinetic quantities. Especially for one-dimensional flow fields they enable a further simplification which has no counterpart in the classical theory. Another difference comes from the fact that in the relativistic kinetic theory as well as in the kinetic theory of the phonon-Bose gas every signal speed is globally bounded by the velocity of light and by the so called Debye velocity, respectively. This has several advantages for the analysis and numerics of the kinetic schemes.

Kinetic approaches in order to solve the classical Euler equations of gas dynamics were applied to several initial- boundary value problems, see for example Dreyer, Kunik and Herrmann [7], [8], [11] and Perthame [41], [42],

[43]. In Dreyer & Kunik [9] the reader will find the kinetic scheme solving the initial-boundary value problem for a phonon Bose gas, which is also presented here.

In [31, 32, 33, 35, 44] Kunik, Qamar and Warnecke have developed a new, purely kinetic approach for the analysis and numerical computation of flows described by the relativistic Euler equations.

There are three basic ingredients of the relativistic kinetic schemes. The first one is the relativistic phase density developed by Jüttner which describes the local equilibrium of the gas. The second one is the solution of a collision free kinetic transport equation, which can be given explicitly in terms of a known initial phase density. For the formulation of the kinetic scheme we prescribe a time step $\tau_M > 0$, define the equidistant times $t_n = n\,\tau_M$ $(n = 0, 1, 2, ...)$, called maximization times, and solve a collision free kinetic transport equation for each time interval $t_n < t < t_{n+1}$, starting with a relativistic Maxwellian as the initial phase density at each maximization time t_n. The third component consists of the continuity conditions, which guarantee that the conservation laws are also satisfied across the maximization times.

The resulting kinetic schemes including all physical important integral moments are formulated in a manifest Lorentz-invariant form. Kunik, Qamar and Warnecke have especially formulated two types of kinetic schemes in order to solve the *ultra-relativistic Euler equations* and a *general form of special relativistic Euler equations* based on Jüttners constitutive relations presented in [28]. The basic ingredients of the kinetic schemes are the phase density in equilibrium and the free flight. The phase density generalizes the non-relativistic Maxwellian for a gas in local equilibrium to Jüttner's relativistic phase density originally presented in [28]. Jüttner's "relativistic Maxwellian" covers the whole range from the classical to the ultra-relativistic limit. The free flight is given by explicit solutions of a collision free kinetic transport equation.

We are also concerned with the kinetic solutions for initial value problems of the Boltzmann-Peierls equation (BPE), as well as with initial-boundary value problems (IBVP) of the derived hyperbolic 4-field moment system. BPE is a kinetic equation which describes the evolution of heat in crystalline solids at very low temperatures. This equation determines the evolution of the phase density of a phonon Bose-gas. The corresponding entropy density is given by the entropy density of a Bose-gas.

We derive an equivalent reduced three-dimensional kinetic equation which has a much simpler structure than the original BPE but the same thermodynamical content. Using a special integration technique for the one-dimensional case, a further important simplification of the reduced kinetic equation can be obtained, as for the first reduction without using any approximation. We develop and study kinetic schemes for the reduced BPE as

well as for the hyperbolic 4-field moment system, based on the theory developed by Dreyer, Herrman, Kunik, Qamar and Warnecke in [9, 10, 12, 34]. Both topics, the relativistic Euler equations and the kinetic approach for the reduced BPE, provide a very similar mathematical structure. We will show a correspondence between the ultra-relativistic Euler equations, the reduced Boltzmann-Peierls equation and between the kinetic schemes used to solve these equations.

Concerning the kinetic solution of hyperbolic initial-boundary value problems, we mention a very successful new method first developed by Dreyer and Kunik in [9, 10] for the four-field system of hyperbolic heat conduction. It uses auxiliary fields and continuity conditions for the boundaries in the free-transport phase of the kinetic scheme. The evaluation of the continuity- and boundary conditions leads to interesting algebraic equations which determine the auxiliary fields needed for the kinetic scheme in an adequate way. Though this method was first applied to a special system with a special boundary condition, it could be extended by Dreyer, Herrmann and Kunik [11] to the classical Euler equations for moving boundary conditions representing an accelerated piston. Boundary conditions for nonlinear hyperbolic systems pose difficult mathematical questions, and up to now the development of appropriate numerical methods is only possible in special cases.

The simple models considered here for the relativistic gas dynamic are mainly studied due to their interesting mathematical structure, though it is clear to us that they must be modified for real physical applications. These modifications should take care for electromagnetic radiation, which will lead to a more complicated description. The Jüttner distribution for a relativistic Boltzmann gas is not realized in nature. On the other hand, Jüttner also derived the relativistic phase densities for a Fermi and Bose gas in [29], and in the ultra relativistic limit the Euler equations for energy and momentum in terms of the pressure and the velocity resulting from these phase densities are the same for the Fermi, Bose and Boltzmann gas. Finally, the models used in our study are fully consistent with the physical basic principles of thermodynamics and the kinetic theory of gases. We namely mentione the mathematical formulation of physical conservation laws and the concept of entropy in these models and in their numerical solution based on the kinetic schemes. The theory presented here may give useful impacts for the analytical as well as the numerical study of more realistic gasdynamical models in relativity.

Chapter 2

Linear Wave Equation

In this chapter we will provide the mathematical tools in order to solve the wave equation for given initial- and boundary data. We consider only the case of one and three space dimensions, which is sufficient for our purpose in order to study the Maxwell equations in Chapter 3. We will mainly follow here the elegant approach of Evans in his textbook [22].

2.1 Representation formula of d'Alembert in one space dimension

We consider the following initial value problem for the linear wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0\,, \qquad u(0,x) = g(x)\,, \qquad \frac{\partial u}{\partial t}(0,x) = h(x)\,. \tag{2.1.1}$$

Here g and h are given initial data, where $g : \mathbb{R} \to \mathbb{R}$ is assumed to be in C^2 and $h : \mathbb{R} \to \mathbb{R}$ is assumed to be in C^1. The unknown solution $u : \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ is assumed to be in C^2. Due to the factoring

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u(t,x) = 0 \tag{2.1.2}$$

we apply the coordinate transformation

$$\xi = x - t\,, \qquad \eta = x + t\,, \qquad \hat{u}(\xi,\eta) = u(\frac{\eta-\xi}{2}, \frac{\eta+\xi}{2}) = u(t,x) \tag{2.1.3}$$

with $\eta - \xi > 0$ in order to obtain

$$\frac{\partial \hat{u}}{\partial \xi} = -\frac{1}{2}\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u}{\partial x}\,, \qquad \frac{\partial \hat{u}}{\partial \eta} = \frac{1}{2}\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u}{\partial x}\,. \tag{2.1.4}$$

Equation (2.1.4) implies for the C^2-function $\hat{u}$:

$$\frac{\partial^2 \hat{u}}{\partial \eta \partial \xi} = 0\,. \tag{2.1.5}$$

We obtain with an appropriate function $a : \mathbb{R} \to \mathbb{R}$ depending only on ξ:

$$\frac{\partial \hat{u}}{\partial \xi} = a(\xi)\,. \tag{2.1.6}$$

Integration of (2.1.6) with respect to ξ yields for a function $A : \mathbb{R} \to \mathbb{R}$ with $A'(\xi) = a(\xi)$ for all ξ and with an appropriate function $B : \mathbb{R} \to \mathbb{R}$ depending only on η:

$$\hat{u}(\xi, \eta) = A(\xi) + B(\eta)\,. \tag{2.1.7}$$

For u itself we obtain the general solution

$$u(t, x) = A(x - t) + B(x + t)\,. \tag{2.1.8}$$

The functions A and B may be determined easily from the given initial data in (2.1.1). It follows from (2.1.8) at $t = 0$:

$$A(x) + B(x) = g(x)\,, \qquad -A'(x) + B'(x) = h(x)\,. \tag{2.1.9}$$

We differentiate the first equation in (2.1.9), add the second equation for h in order to solve for B' and integrate in order to obtain for B

$$B'(x) = \frac{1}{2}\left(h(x) + g'(x)\right)\,, \qquad B(x) = B(0) + \frac{1}{2}\left(g(x) - g(0)\right) + \frac{1}{2}\int_0^x h(z)\,dz\,. \tag{2.1.10}$$

Equation $(2.1.10)_2$ and $(2.1.9)_1$ may be used in order to represent A in the form

$$A(x) = -B(0) + \frac{1}{2}\left(g(x) + g(0)\right) - \frac{1}{2}\int_0^x h(z)\,dz\,. \tag{2.1.11}$$

We replace A and B in the general solution formula (2.1.8) by their expressions derived here in order to obtain the solution of the initial value problem in terms of the initial data g and h:

$$u(t, x) = \frac{1}{2}\left(g(x - t) + g(x + t)\right) + \frac{1}{2}\int_{x-t}^{x+t} h(z)\,dz\,. \tag{2.1.12}$$

This is the **representation formula of d'Alembert** for the solution of the initial value problem (2.1.1).

2.2 Solution of an initial- and boundary value problem by the reflection method

We consider the following initial- and boundary value problem for $t \geq 0$ and $x \geq 0$:

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, && t > 0,\ x > 0, \\ &u(0,x) = g(x), && \frac{\partial u}{\partial t}(0,x) = h(x),\ x > 0, \\ &u(t,0) = 0, && t > 0. \end{aligned} \tag{2.2.1}$$

The given initial data g and h are restricted by the condition

$$g(0) = h(0) = 0\,. \tag{2.2.2}$$

We reduce the initial- and boundary value problem to a pure initial value problem by the so called reflection method, which results if we extend the functions u, g and h to the whole real axis by the definitions

$$\hat{u}(t,x) = u(t,x),\ x \geq 0, \qquad \hat{u}(t,x) = -u(t,-x),\ x < 0\,. \tag{2.2.3}$$

$$\hat{g}(x) = g(x),\ x \geq 0, \qquad \hat{g}(x) = -g(-x),\ x < 0. \tag{2.2.4}$$

$$\hat{h}(x) = h(x),\ x \geq 0, \qquad \hat{h}(x) = -h(-x),\ x < 0. \tag{2.2.5}$$

Equations (2.2.1) may then be rewritten in the form

$$\frac{\partial^2 \hat{u}}{\partial t^2} - \frac{\partial^2 \hat{u}}{\partial x^2} = 0\,, \qquad \hat{u}(0,x) = \hat{g}(x)\,, \qquad \frac{\partial \hat{u}}{\partial t}(0,x) = \hat{h}(x)\,. \tag{2.2.6}$$

We obtain from d'Alemberts formula after replacing the quantities $\hat{u}$, $\hat{g}$ and $\hat{h}$ by u, g and h:

$$u\,(t,x) = \begin{cases} \frac{1}{2}\left[g(x+t) + g(x-t)\right] + \frac{1}{2}\int\limits_{x-t}^{x+t} h(y)\,dy\,,\ 0 \leq t \leq x \\ \\ \frac{1}{2}\left[g(x+t) - g(-x+t)\right] + \frac{1}{2}\int\limits_{-x+t}^{x+t} h(y)\,dy\,,\ 0 \leq x \leq t \end{cases}. \tag{2.2.7}$$

The solution of this initial- and boundary value problem in one space dimension is needed for the solution of the initial value problem of the homogeneous wave equation in three space dimensions, which will be solved next.

2.3 Solution of the homogeneous wave equation in three space dimensions

Let $u \in C^2(\mathbb{R}_0^+ \times \mathbb{R}^3, \mathbb{R})$ be a solution of the initial value problem

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} - \Delta u = 0, \ t > 0, \ \mathbf{x} \in \mathbb{R}^3 \\ &u(0, \mathbf{x}) = g(\mathbf{x}), \qquad \frac{\partial u}{\partial t}(0, \mathbf{x}) = h(\mathbf{x}), \ \mathbf{x} \in \mathbb{R}^3 . \end{aligned} \tag{2.3.1}$$

We solve this initial value problem in the three-dimensional space by taking spherical means and reducing it to the initial-boundary value problem in one space dimension solved in the last section.

In the sequel we assume for the initial data of (2.3.1) that $g \in C^3(\mathbb{R}^3, \mathbb{R})$ and $h \in C^2(\mathbb{R}^3, \mathbb{R})$. $B(r, \mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^3 | \ |\mathbf{x} - \mathbf{y}| \leq r\}$ denotes the compact ball with center $\mathbf{x}$ and radius r. Its boundary is the sphere $\partial B(r, \mathbf{x})$.

We start with an important

Lemma 2.1. *For $\psi \in C^2(\mathbb{R}^3, \mathbb{R})$, $\mathbf{x} \in \mathbb{R}^3$ and $r > 0$ we define the spherical mean*

$$\Psi(r, \mathbf{x}) = \frac{1}{4\pi r^2} \oint_{\partial B(r, \mathbf{x})} \psi(\mathbf{y}) \, dS(\mathbf{y}) .$$

Then $\Psi \in C^2(\mathbb{R}^+ \times \mathbb{R}^3, \mathbb{R})$ has the following two radial derivatives:

(i)
$$\frac{\partial \Psi}{\partial r}(r, \mathbf{x}) = \frac{1}{4\pi r^2} \int_{B(r, \mathbf{x})} (\Delta \psi)(\mathbf{y}) \, d^3 y .$$

(ii)
$$\frac{\partial^2 \Psi}{\partial r^2}(r, \mathbf{x}) + \frac{2}{r} \frac{\partial \Psi}{\partial r}(r, \mathbf{x}) = \frac{1}{4\pi r^2} \oint_{\partial B(r, \mathbf{x})} (\Delta \psi)(\mathbf{y}) \, dS(\mathbf{y}) .$$

Proof: We first make use of the integral substitution $\mathbf{y} = \mathbf{x} + r\mathbf{y}' \to \mathbf{y}'$ in order to conclude that

$$\Psi(r, \mathbf{x}) = \frac{1}{4\pi} \oint_{\partial B(1, \mathbf{0})} \psi(\mathbf{x} + r\mathbf{y}') \, dS(\mathbf{y}') . \tag{2.3.2}$$

For the calculation of $\partial \Psi / \partial r$ we apply the Gaussian integral formula with the outer normal vectors $\mathbf{y}'$ to the unit sphere $\partial B(1, \mathbf{0})$ and regard the chain

rule $\partial/\partial y'_k = r\,\partial/\partial y_k$,

$$\begin{aligned}\frac{\partial \Psi}{\partial r}(r,\mathbf{x}) &= \frac{1}{4\pi}\oint_{\partial B(1,\mathbf{0})} (\nabla\psi)(\mathbf{x}+r\mathbf{y}')\cdot\mathbf{y}'\,dS(\mathbf{y}') \\ &= \frac{r}{4\pi}\int_{B(1,\mathbf{0})} (\Delta\psi)(\mathbf{x}+r\mathbf{y}')\,d^3y' = \frac{1}{4\pi r^2}\int_{B(r,\mathbf{x})} (\Delta\psi)(\mathbf{y})\,d^3y\,.\end{aligned}$$

This is the first part of the Lemma.
If we change to spatial polar coordinates in the last integral we can rewrite $\partial\Psi/\partial r$ in the following way:

$$\frac{\partial \Psi}{\partial r}(r,\mathbf{x}) = \frac{1}{4\pi r^2}\int_0^r \oint_{\partial B(\vartheta,\mathbf{x})} (\Delta\psi)(\mathbf{y})\,dS(\mathbf{y})\,d\vartheta. \tag{2.3.3}$$

Using this representation, we immediately obtain the second derivative

$$\frac{\partial^2 \Psi}{\partial r^2}(r,\mathbf{x}) = \frac{1}{4\pi r^2}\oint_{\partial B(r,\mathbf{x})} (\Delta\psi)(\mathbf{y})\,dS(\mathbf{y}) - \frac{1}{2\pi r^3}\int_{B(r,\mathbf{x})} (\Delta\psi)(\mathbf{y})\,d^3y\,. \tag{2.3.4}$$

Thus we have shown the Lemma. ∎

We first assume that $u \in C^2(\mathbb{R}_0^+ \times \mathbb{R}^3, \mathbb{R})$.

In order to derive an integral representation formula for the solution u, we define the following spherical means, where $\mathbf{x} \in \mathbb{R}^3$, $t > 0$, $r > 0$.

(a)
$$U(t,r,\mathbf{x}) = \frac{1}{4\pi r^2}\oint_{\partial B(r,\mathbf{x})} u(t,\mathbf{y})\,dS(\mathbf{y})$$
is the mean value of the solution (2.3.1) with respect to the sphere $\partial B(r,\mathbf{x})$.

(b) The corresponding spherical means of the initial data are
$$G(r,\mathbf{x}) = \frac{1}{4\pi r^2}\oint_{\partial B(r,\mathbf{x})} g(\mathbf{y})\,dS(\mathbf{y})\,,$$
$$H(r,\mathbf{x}) = \frac{1}{4\pi r^2}\oint_{\partial B(r,\mathbf{x})} h(\mathbf{y})\,dS(\mathbf{y})\,.$$

Let $u = u(t,\mathbf{x})$ be a solution of the wave equation (2.3.1). Then for fixed $\mathbf{x} \in \mathbb{R}^3$ the mean value U is for $t > 0$ and $r > 0$ a solution of the so called

Euler-Poisson-Darboux equation:

$$\frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial r^2} - \frac{2}{r}\,\frac{\partial U}{\partial r} = 0\,, \tag{2.3.5}$$
$$U(0,r,\mathbf{x}) = G(r,\mathbf{x}), \qquad \frac{\partial U}{\partial t}(0,r,\mathbf{x}) = H(r,\mathbf{x})\,.$$

This is an immediate consequence of the second part (ii) of the Lemma, because $u = u(t,\mathbf{x})$ solves the homogeneous three-dimensional wave equation:

$$\begin{aligned}\frac{\partial^2 U}{\partial r^2}(t,r,\mathbf{x}) + \frac{2}{r}\,\frac{\partial U}{\partial r}(t,r,\mathbf{x}) &= \frac{1}{4\pi r^2}\oint_{\partial B(r,\mathbf{x})} (\Delta u)(t,\mathbf{y})\,dS(\mathbf{y}) \\ &= \frac{1}{4\pi r^2}\oint_{\partial B(r,\mathbf{x})} \frac{\partial^2 u}{\partial t^2}(t,\mathbf{y})\,dS(\mathbf{y}) = \frac{\partial^2}{\partial t^2}U(t,r,\mathbf{x})\,.\end{aligned}$$

Moreover, the initial data in (2.3.5) result from the corresponding initial data in (2.3.1) by passing to the mean values. The Euler-Poisson-Darboux equation has a remarkable consequence: For abbreviation we define

$$\tilde{U} = r\,U\,, \qquad \tilde{G} = r\,G\,, \qquad \tilde{H} = r\,H\,, \tag{2.3.6}$$

and conclude that $\tilde{U}$ satisfies the one-dimensional homogeneous wave equation:

$$\frac{\partial^2 \tilde{U}}{\partial r^2} = \frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r} + U\right) = r\frac{\partial^2 U}{\partial r^2} + 2\frac{\partial U}{\partial r} = r\frac{\partial^2 U}{\partial t^2} = \frac{\partial^2 \tilde{U}}{\partial t^2}\,. \tag{2.3.7}$$

Therefore $\tilde{U}$ is a solution of the following initial- and boundary value problem for $t, r \geq 0$, where $\mathbf{x}$ may be regarded as a fixed parameter:

$$\begin{aligned}&\frac{\partial^2 \tilde{U}}{\partial t^2} - \frac{\partial^2 \tilde{U}}{\partial r^2} = 0, && t>0,\ r>0,\\ &\tilde{U}(0,r,\mathbf{x}) = \tilde{G}(r,\mathbf{x}), \qquad \frac{\partial \tilde{U}}{\partial t}(0,r,\mathbf{x}) = \tilde{H}(r,\mathbf{x}), && r>0,\\ &\tilde{U}(t,0,\mathbf{x}) = 0, && t>0.\end{aligned} \tag{2.3.8}$$

The initial data $\tilde{G}$ and $\tilde{H}$ are restricted by the condition

$$\tilde{G}(0,\mathbf{x}) = \tilde{H}(0,\mathbf{x}) = 0 \tag{2.3.9}$$

for every fixed parameter $\mathbf{x} \in \mathbb{R}^3$. The solution results from equation (2.2.7), where we replace $x \geq 0$ by r and regard the condition $0 \leq r \leq t$:

$$\tilde{U}(t,r,\mathbf{x}) = \frac{1}{2}\,\Big[\tilde{G}(t+r,\mathbf{x}) - \tilde{G}(t-r,\mathbf{x})\Big] + \frac{1}{2}\int_{t-r}^{t+r} \tilde{H}(y,\mathbf{x})\,dy\,. \tag{2.3.10}$$

The solution $u = u(t, \mathbf{x})$ of (2.3.1) is given by the limit

$$u(t, \mathbf{x}) = \lim_{r \to 0,\ r>0} U(t, r, \mathbf{x}) = \lim_{r \to 0,\ r>0} \frac{\tilde{U}(t, r, \mathbf{x})}{r}\,. \tag{2.3.11}$$

Now we use (2.3.11), (2.3.10) and (2.3.6) in order to conclude that

$$u(t, \mathbf{x}) = \lim_{r \to 0,\ r>0} \left[\frac{\tilde{G}(t+r, \mathbf{x}) - \tilde{G}(t-r, \mathbf{x})}{2r} + \frac{1}{2r} \int_{t-r}^{t+r} \tilde{H}(y, \mathbf{x})\, dy \right]$$

$$= \frac{\partial \tilde{G}(t, x)}{\partial t} + \tilde{H}(t, \mathbf{x}) = \frac{\partial}{\partial t} \left(\frac{t}{4\pi t^2} \oint_{\partial B(t, \mathbf{x})} g(\mathbf{y})\, dS(\mathbf{y}) \right) + \frac{t}{4\pi t^2} \oint_{\partial B(t, \mathbf{x})} h(\mathbf{y})\, dS(\mathbf{y})$$

$$= \frac{1}{4\pi t^2} \oint_{\partial B(t, \mathbf{x})} (g(\mathbf{y}) + t\, h(\mathbf{y}))\ dS(\mathbf{y}) + t\, \frac{\partial}{\partial t} \left(\frac{1}{4\pi t^2} \oint_{\partial B(t, \mathbf{x})} g(\mathbf{y})\, dS(\mathbf{y}) \right).$$

Finally we apply the integral substitution $\mathbf{y} = \mathbf{x} + t\mathbf{y}'$ and obtain

$$u(t, \mathbf{x}) = \frac{1}{4\pi t^2} \oint_{\partial B(t, \mathbf{x})} (g(\mathbf{y}) + t\, h(\mathbf{y}))\ dS(\mathbf{y}) + t\, \frac{\partial}{\partial t} \left(\frac{1}{4\pi} \oint_{\partial B(1, \mathbf{0})} g(\mathbf{x} + t\, \mathbf{y}')\, dS(\mathbf{y}') \right)$$

$$= \frac{1}{4\pi t^2} \oint_{\partial B(t, \mathbf{x})} (g(\mathbf{y}) + t\, h(\mathbf{y}))\,, dS(\mathbf{y}) + \frac{t}{4\pi} \oint_{\partial B(1, \mathbf{0})} (\nabla g)(\mathbf{x} + t\, \mathbf{y}') \cdot \mathbf{y}'\, dS(\mathbf{y}')$$

$$= \frac{1}{4\pi t^2} \oint_{\partial B(t, \mathbf{x})} [\, g(\mathbf{y}) + t\, h(\mathbf{y}) + (\nabla g)(\mathbf{y}) \cdot (\mathbf{y} - \mathbf{x})\,]\ dS(\mathbf{y})\,.$$

We may summarize our results in the famous

Representation formula of Kirchhoff:

$$u(t, \mathbf{x}) = \frac{1}{4\pi t^2} \oint_{\partial B(t, \mathbf{x})} [\, g(\mathbf{y}) + t\, h(\mathbf{y}) + (\nabla g)(\mathbf{y}) \cdot (\mathbf{y} - \mathbf{x})\,]\ dS(\mathbf{y})$$

$$= \frac{1}{4\pi t^2} \oint_{\partial B(t, \mathbf{x})} (g(\mathbf{y}) + t\, h(\mathbf{y}))\ dS(\mathbf{y}) + t\, \frac{\partial}{\partial t} \left(\frac{1}{4\pi t^2} \oint_{\partial B(t, \mathbf{x})} g(\mathbf{y})\, dS(\mathbf{y}) \right). \tag{2.3.12}$$

This formula was derived under the assumption that u is a given solution of the initial value problem (2.3.1). To complete the study of the initial value problem we **define** for given h the function $v \in C^2(\mathbb{R}^+ \times \mathbb{R}^3, \mathbb{R})$ by

$$v(t, \mathbf{x}) = \frac{1}{4\pi t^2} \oint_{\partial B(t, \mathbf{x})} t\, h(\mathbf{y})\, dS(\mathbf{y}) = tH(t, \mathbf{x})\,. \tag{2.3.13}$$

We obtain the initial data

$$\lim_{t \to 0} v(t, \mathbf{x}) = 0\,, \quad \lim_{t \to 0} \frac{\partial v}{\partial t}(t, \mathbf{x}) = h(\mathbf{x})\,. \tag{2.3.14}$$

In order to show that v satisfies the wave equation, we first conclude that

$$\frac{\partial^2 v}{\partial t^2}(t,\mathbf{x}) = \frac{1}{t}\,\frac{\partial}{\partial t}\{t^2\frac{\partial H}{\partial t}(t,\mathbf{x})\} \tag{2.3.15}$$

from the definition of v. On the other hand

$$\frac{\partial H}{\partial t}(t,\mathbf{x}) = \frac{1}{4\pi t^2}\int_{B(t,\mathbf{x})}(\Delta h)(\mathbf{y})\,d^3y = \frac{1}{4\pi t^2}\int_0^t \oint_{\partial B(\vartheta,\mathbf{x})}(\Delta h)(\mathbf{y})\,dS(\mathbf{y})\,d\vartheta \tag{2.3.16}$$

holds due to the first part of the Lemma, and therefore

$$\frac{\partial^2 v}{\partial t^2}(t,\mathbf{x}) = \frac{1}{4\pi t}\oint_{\partial B(t,\mathbf{x})}(\Delta h)(\mathbf{y})\,dS(\mathbf{y}) = (\Delta v)(t,\mathbf{x})\,. \tag{2.3.17}$$

We obtain that $v(t,\mathbf{x}) = tH(t,\mathbf{x})$ as well as $w(t,\mathbf{x}) = tG(t,\mathbf{x})$ are both solutions of the wave equation and that $u(t,\mathbf{x}) = v(t,\mathbf{x}) + \frac{\partial w}{\partial t}(t,\mathbf{x})$ is the desired Kirchhoff-solution of the initial value problem (2.3.1).

2.4 Solution of the inhomogeneous wave equation in three space dimensions, retarded potentials

In this section we recall the so called principle of Duhamel in order to reduce an inhomogeneous initial value problem for the wave equation to the homogeneous case that we have solved by Kirchhoff's formula in the previous section.

Here we consider the initial value problem

$$\begin{gathered}\frac{\partial^2 u}{\partial t^2}(t,\mathbf{x}) - (\Delta u)(t,\mathbf{x}) = f(t,\mathbf{x}),\ t>0,\ \mathbf{x}\in\mathbb{R}^3 \\ u(0,x) = 0, \qquad \frac{\partial u}{\partial t}(0,\mathbf{x}) = 0,\ \mathbf{x}\in\mathbb{R}^3\end{gathered} \tag{2.4.1}$$

for a given C^1-function $f:\mathbb{R}_0^+\times\mathbb{R}^3\to\mathbb{R}$.

In order to describe the effects of retardation in time for the wave propagation we choose a new initial time $s\geq 0$ as a fixed parameter and solve the following homogeneous initial value problem for $t\geq s$ instead of (2.4.1):

$$\begin{gathered}\frac{\partial^2 v}{\partial t^2}(s,t,\mathbf{x}) - (\Delta v)(s,t,\mathbf{x}) = 0,\ t>s,\ \mathbf{x}\in\mathbb{R}^3 \\ v(s,s,\mathbf{x}) = 0, \qquad \frac{\partial v}{\partial t}(s,s,\mathbf{x}) = f(s,\mathbf{x}),\ \mathbf{x}\in\mathbb{R}^3\,.\end{gathered} \tag{2.4.2}$$

The solution $v = v(s,t,\mathbf{x})$, which is defined for $t \geq s$ and $\mathbf{x} \in \mathbb{R}^3$, is given by Kirchhoff's formula in the case $t > s$:

$$v(s,t,\mathbf{x}) = \frac{1}{4\pi} \oint_{\partial B(t-s,\mathbf{x})} \frac{f(s,\mathbf{y})}{t-s}\, dS(\mathbf{y})\,. \tag{2.4.3}$$

Then Duhamel's principle says that the solution $u = u(t,\mathbf{x})$ of the inhomogeneous problem is given by the integral

$$u(t,\mathbf{x}) = \int_0^t v(s,t,\mathbf{x})\, ds\,. \tag{2.4.4}$$

Here we check this statement, which has several generalizations to other applications, by a simple straightforward calculation: Due to the first initial condition $v(t,t,\mathbf{x}) = 0$ in (2.4.2) we obtain from (2.4.4)

$$\frac{\partial u}{\partial t}(t,\mathbf{x}) = \int_0^t \frac{\partial v}{\partial t}(s,t,\mathbf{x})\, ds \tag{2.4.5}$$

and due to the second initial condition $\frac{\partial v}{\partial t}(t,t,\mathbf{x}) = f(t,\mathbf{x})$ in (2.4.2) there results

$$\frac{\partial^2 u}{\partial t^2}(t,\mathbf{x}) = f(t,\mathbf{x}) + \int_0^t \frac{\partial^2 v}{\partial t^2}(s,t,\mathbf{x})\, ds\,. \tag{2.4.6}$$

But v is a solution of the homogeneous wave equation, and therefore

$$\frac{\partial^2 u}{\partial t^2}(t,\mathbf{x}) = f(t,\mathbf{x}) + \Delta\left(\int_0^t v(s,t,\mathbf{x})\, ds\right) = f(t,\mathbf{x}) + (\Delta u)(t,\mathbf{x}) \tag{2.4.7}$$

i.e. u solves the inhomogeneous wave equation in (2.4.1). From (2.4.4) and (2.4.5) we may read off that also the initial conditions for u in (2.4.1) are satisfied.

We summarize our results, apply the substitution $\vartheta = t - s$ in (2.4.3) and obtain the following representation for the solution $u = u(t,\mathbf{x})$ of the inhomogeneous problem (2.4.1):

$$\begin{aligned} u(t,\mathbf{x}) &= \int_0^t \vartheta \left\{ \frac{1}{4\pi\vartheta^2} \oint_{\partial B(\vartheta,\mathbf{x})} f(t-\vartheta,\mathbf{y})\, dS(\mathbf{y}) \right\} d\vartheta \\ &= \frac{1}{4\pi} \int_{B(t,\mathbf{x})} \frac{f(t-|\mathbf{y}-\mathbf{x}|,\mathbf{y})}{|\mathbf{y}-\mathbf{x}|}\, d^3y\,. \end{aligned} \tag{2.4.8}$$

The last integral on the right-hand side is called a retarded potential. It will be used in Section 3.6 for the calculation of the electromagnetic potentials of a single charged particle.

Chapter 3

Maxwell's Equations

James Clark Maxwell (1831-1879) presented his theory of electromagnetism in 1864. His equations are invariant with respect to Lorentz coordinate transformations, which are linear coordinate transformations of time and space that also leave the speed of light invariant. Maxwell's theory finally gave raise to the replacement of Newton's classical mechanics by the Lorentz invariant mechanics of special relativity mainly published in Einstein's famous papers [20], [21] from 1905.

After we have introduced Maxwell's equations in a simple form in Section 3.2 by using the physical units presented in Section 3.1, we rewrite them as linear wave equations which can be solved using the methods of Chapter 2. This is done in Section 3.3 in terms of the so called potential functions. In the next three sections we present important examples, and in Section 3.7 we derive the balance laws of electrodynamics directly from Maxwell's equations. Section 3.8 is reserved for a detailed study of Lorentz transformations. This study is used in Section 3.9 in order to develop the tensor calculus of special relativity and to write electromagnetism in a manifest Lorentz invariant form.

This chaper is of interest on its own, but it also provides the necessary tools for the relativistic kinetic theory and the relativistic Euler equations presented in Chapter 4.

Important input for this chapter came from the textbooks of Weinberg [48], Landau-Lifschitz [36] and Streater-Wightman [46], but also from Weyl's book [49]. The nice general introduction to Hermann Weyl's scientific work in [45] originates from a DMV seminar (Deutsche Mathematiker-Vereinigung) at Schloss Reisensburg near Günzburg. I also mention useful and inspiring lecture manuscripts of Wolfgang Dreyer (WIAS, Berlin) on special physical basic topics held at the TU Berlin.

3.1 Physical units and fundamental constants

We prescribe the following four standard basic units:

1) 1 m is the unit meter for the length,

2) 1 kg is the unit kilogram for the mass,

3) 1 s is the unit second for the time,

4) 1 A is the unit Ampere for the current strength.

The other units are formed by these basic units, for example

5) 1 N $= kg\,\frac{m}{s^2}$ is the unit Newton for the force,

6) 1 J $= kg\,\frac{m^2}{s^2} = N\,m$ is the unit Joule for the energy,

7) 1 C $= A\,s$ is the unit Coulomb for the charge,

8) 1 V $= \frac{kg\,m^2}{A\,s^3} = \frac{J}{C}$ is the unit Volt for the voltage.

In the following we shall only use these units.

In order to formulate Maxwell's equations and give an interpretation for these equations we need the following physical constants:

$$c = 299792458\,\frac{m}{s} \tag{3.1.1}$$

is the speed of light,

$$e = 1,6021917 \cdot 10^{-19}\,C \tag{3.1.2}$$

the electronic charge.

$$\varepsilon_0 = 8.854188 \cdot 10^{-12}\frac{As}{Vm}\,, \qquad \mu_0 = 4\pi \cdot 10^{-7}\frac{Vs}{Am} \tag{3.1.3}$$

are called electric permittivity and magnetic permeability, respectively, where the index 0 indicates that both quantities are related to the vacuum. These quantities are related by the following important equation:

$$\varepsilon_0 \cdot \mu_0 = \frac{1}{c^2}\,. \tag{3.1.4}$$

3.2 Formulation of Maxwell's equations, electromagnetic forces

Maxwell's equations for the electric and magnetic fields $\mathbf{E}$, $\mathbf{B}$ produced by a given charge density ρ and a given current density $\mathbf{j}$ may be written down in the following form:

$$\nabla \cdot \mathbf{E} = \rho\,, \tag{3.2.1}$$

$$-\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \nabla \times \mathbf{B} = \frac{1}{c}\mathbf{j}\,, \tag{3.2.2}$$

$$\nabla \cdot \mathbf{B} = 0\,, \tag{3.2.3}$$

$$+\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = \mathbf{0}\,. \tag{3.2.4}$$

Here the charge density $\rho = \rho(t, \mathbf{x})$ and the current density $\mathbf{j} = \mathbf{j}(t, \mathbf{x})$ are defined due to the usual technical conventions with the units $\frac{C}{m^3}$ for ρ and the units $\frac{A}{m^2}$ for $\mathbf{j}$.

However, we have redefined the electric field $\mathbf{E} = \mathbf{E}(t, \mathbf{x})$ and the magnetic field $\mathbf{B} = \mathbf{B}(t, \mathbf{x})$ in such a way that the electromagnetic force $\mathbf{F}$ acting on a particle with charge q and velocity $\mathbf{v}$ is given by

$$\mathbf{F} = \frac{q}{\varepsilon_0}\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right)\,. \tag{3.2.5}$$

Then both fields are measured in the new units $\frac{C}{m^2}$. The advantage of this notation is that now Maxwell's equations are free from undesirable numbers 4π and constants ε_0, μ_0. For the conventional electrical field $\mathbf{E}'$ and the conventional magnetic field $\mathbf{B}'$ the electromagnetic force $\mathbf{F}$ on the left hand side in (3.2.5) takes the usual form

$$\mathbf{F} = q\left(\mathbf{E}' + \mathbf{v} \times \mathbf{B}'\right)\,. \tag{3.2.6}$$

We compare (3.2.5) with (3.2.6) and use (3.1.4) in order to conclude that

$$\mathbf{E}' = \frac{1}{\varepsilon_0}\,\mathbf{E}\,, \qquad \mathbf{B}' = \frac{1}{c\varepsilon_0}\,\mathbf{B} = c\mu_0\,\mathbf{B}\,. \tag{3.2.7}$$

We note that ρ and$\frac{1}{c}\mathbf{j}$ on the right hand side of the first two inhomogeneous Maxwell equations are measured in the same units. If we multiply these quatities with the velocity c of light, we obtain the electromagnetic four-vector $(\rho c\,, \mathbf{j})$, which is important in the theory of relativity.

3.3 Potentials for Maxwell's equations

In this section we rewrite Maxwell's equations (3.2.1)-(3.2.4) as wave equations for the so called electromagnetic potential functions. First we take the divergence of (3.2.2) in order to conclude with (3.2.3) that

$$-\frac{1}{c}\frac{\partial(\nabla\cdot\mathbf{E})}{\partial t}=\frac{1}{c}\nabla\cdot\mathbf{j}\,. \tag{3.3.1}$$

Taking into account (3.2.1) we obtain the **continuity equation** for the charge- and current density:

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\mathbf{j}=0\,. \tag{3.3.2}$$

This conservation law is a consequence of (3.2.1)-(3.2.3), i.e. (3.3.2) is a necessary condition in order to solve Maxwell's equations. This means that ρ and $\mathbf{j}$ may not be chosen independently.

Due to the third Maxwellian equation $\nabla\cdot\mathbf{B}=0$ we make the following ansatz:

$$\mathbf{B}=\nabla\times\mathbf{A}\,. \tag{3.3.3}$$

The so called **vector potential** $\mathbf{A}=\mathbf{A}(t,\mathbf{x})$ is in general a function depending on time and space. Using this ansatz for $\mathbf{B}$ we rewrite (3.2.4) in the following form:

$$\nabla\times(\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t})=\mathbf{0}\,. \tag{3.3.4}$$

We conclude that $\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}$ must be a gradient field. There results the so called **scalar potential** $\varphi=\varphi(t,\mathbf{x})$ which satisfies the following equation:

$$\mathbf{E}=-\nabla\varphi-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\,. \tag{3.3.5}$$

Now we replace the fields $\mathbf{E}$ and $\mathbf{B}$ in Maxwell's equations by the derivatives of the potentials (3.3.3), (3.3.5) and obtain by a straight forward calculation:

$$\Box\,\varphi-\frac{1}{c}\frac{\partial}{\partial t}(\frac{1}{c}\frac{\partial\varphi}{\partial t}+\nabla\cdot\mathbf{A})=\rho\,, \tag{3.3.6}$$

$$\Box\,\mathbf{A}+\nabla(\frac{1}{c}\frac{\partial\varphi}{\partial t}+\nabla\cdot\mathbf{A})=\frac{1}{c}\mathbf{j}\,. \tag{3.3.7}$$

Here $\Box=\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\Delta$ is an abbreviation for d'Alembert's wave operator.

Next we use the following freedom to redefine the potentials φ and $\mathbf{A}$: Let $\lambda=\lambda(t,\mathbf{x})$ be a smooth function of time and space and replace φ and $\mathbf{A}$ by the new potentials via the "gauge transformation"

$$\tilde{\varphi}=\varphi-\frac{1}{c}\frac{\partial\lambda}{\partial t}\,, \tag{3.3.8}$$

$$\tilde{\mathbf{A}}=\mathbf{A}+\nabla\lambda\,. \tag{3.3.9}$$

Then the electromagnetic field given by (3.3.3) and (3.3.5) is independent of λ. We solve the following inhomogeneous wave equation for λ:

$$\Box\,\lambda = \frac{1}{c}\frac{\partial\varphi}{\partial t} + \nabla\cdot\mathbf{A} \tag{3.3.10}$$

with the initial conditions

$$\lambda(0,\mathbf{x}) = \lambda_0(\mathbf{x})\ ,\ \frac{\partial\lambda}{\partial t}(0,\mathbf{x}) = 0\ , \tag{3.3.11}$$

where $\lambda_0 = \lambda_0(\mathbf{x})$ is a pure space depending function. The new potentials satisfy the constraint $\frac{1}{c}\frac{\partial\tilde{\varphi}}{\partial t}+\nabla\cdot\tilde{\mathbf{A}} = 0$ and the initial condition $\tilde{\mathbf{A}}_0 = \mathbf{A}_0+\nabla\lambda_0$. Therefore we may require the so called constraint of Lorentz without loss of generality

$$\frac{1}{c}\frac{\partial\varphi}{\partial t} + \nabla\cdot\mathbf{A} = 0 \tag{3.3.12}$$

as well as the following restriction of the initial condition:

$$\frac{\partial\varphi}{\partial t}(0,\mathbf{x}) \;=\; \nabla\cdot\mathbf{A}_0(\mathbf{x}) \;=\; 0\,. \tag{3.3.13}$$

This restriction may also be imposed without loss of generality due to the theorem of Helmholtz, using the freedom to choose the function λ_0 in (3.3.11) in such a way that

$$\mathbf{A}_0 = \nabla\times\mathbf{F}_0\,. \tag{3.3.14}$$

with a vector field $\mathbf{F}_0 = \mathbf{F}_0(\mathbf{x})$ depending only on $\mathbf{x}$, so that $\nabla\cdot\mathbf{A}_0 = 0$ and from the constraint of Lorentz $\frac{\partial\varphi}{\partial t}(0,\mathbf{x}) = 0$. From (3.3.6), (3.3.7) and (3.3.12) there finally result decoupled wave equations for φ and $\mathbf{A}$:

$$\Box\,\varphi = \rho\ ,\qquad \Box\,\mathbf{A} = \frac{1}{c}\mathbf{j}\ . \tag{3.3.15}$$

3.4 Planar light waves in the vacuum

In this section we consider planar wave solutions of Maxwell's equations for $\rho = 0$, $\mathbf{j} = \mathbf{0}$, which describe the propagation of light in the vacuum. Maxwell's equations reduce to

$$\nabla\cdot\mathbf{E} \;=\; 0\,,\qquad \nabla\cdot\mathbf{B} \;=\; 0\,, \tag{3.4.1}$$

$$-\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} + \nabla\times\mathbf{B} \;=\; \mathbf{0}\,,\qquad +\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} + \nabla\times\mathbf{E} \;=\; \mathbf{0}\,. \tag{3.4.2}$$

For the scalar potential φ and the vector potential $\mathbf{A}$ we make the ansatz

$$\varphi(t,\mathbf{x}) = 0\,,\qquad \mathbf{A}(t,\mathbf{x}) = \mathbf{A_0}\,\cos(\omega t - \mathbf{k}\cdot\mathbf{x})\,, \tag{3.4.3}$$

where the constant quantities $\omega \geq 0$, $\mathbf{A_0}, \mathbf{k} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$ must be restricted by simple geometric conditions in order to satisfy Maxwell's equations (3.4.1) and (3.4.2). First we determine these restrictions: Using equations (3.3.5) and (3.3.3) in order to determine the electromagnetic field from the potentials, Maxwell's equations reduce to the constraint of Lorentz and the homogeneous wave equations for the potentials:

$$\frac{1}{c}\frac{\partial \varphi}{\partial t} + \nabla \cdot \mathbf{A} = 0 \tag{3.4.4}$$

$$\Box\, \varphi = 0 \ , \qquad \Box\, \mathbf{A} = \mathbf{0}\,. \tag{3.4.5}$$

Since the scalar potential vanishs identically, equations (3.4.4) and (3.4.5) may be summarized in the form

$$\nabla \cdot \mathbf{A} = 0 \ , \qquad \Box\, \mathbf{A} = \mathbf{0}\,. \tag{3.4.6}$$

We use the ansatz in order to evaluate

$$\nabla \cdot \mathbf{A} = (\mathbf{A_0} \cdot \mathbf{k})\, \sin(\omega t - \mathbf{k} \cdot \mathbf{x}) \ , \ \Box\, \mathbf{A} = -\left(\frac{\omega^2}{c^2} - \mathbf{k} \cdot \mathbf{k}\right)\, \mathbf{A}\,. \tag{3.4.7}$$

We only look for solutions of the electromagnetic field which do not vanish everywhere in time and space, and therefore we compare (3.4.6) with (3.4.7) in order to obtain

$$\mathbf{A_0} \cdot \mathbf{k} = 0\,, \qquad |\mathbf{A_0}| > 0\,, \qquad \omega = c\,|\mathbf{k}| > 0\,. \tag{3.4.8}$$

The corresponding electromagnetic field is given by

$$\mathbf{E}(t, \mathbf{x}) \ = \ |\mathbf{k}|\, \mathbf{A_0}\, \sin(\omega t - \mathbf{k} \cdot \mathbf{x}) \ , \tag{3.4.9}$$

$$\mathbf{B}(t, \mathbf{x}) \ = \ \mathbf{k} \times \mathbf{A_0}\, \sin(\omega t - \mathbf{k} \cdot \mathbf{x}) \ , \tag{3.4.10}$$

where

$$|\mathbf{E}| = |\mathbf{B}|\,, \qquad \mathbf{E} \cdot \mathbf{B} = 0\,. \tag{3.4.11}$$

3.5 Electromagnetic resonance in a box and the Helmholtz-equations

Since Maxwell's equations are linear, we can make the following ansatz for a complex electromagnetic potential describing a field which is changing periodically in time with a constant frequency ω

$$\varphi(t, \mathbf{x}) = 0\,, \qquad \mathbf{A}(t, \mathbf{x}) = e^{i\omega t}\mathbf{A_0}(\mathbf{x})\,. \tag{3.5.1}$$

In order to get back the physical quantities one can split the resulting complex solutions in its real- and imaginary part and then form linear combinations with real coefficients. Then Maxwell's equations reduce to the following conditions between the frequency ω and the vector potential $\mathbf{A_0}$

$$\nabla \cdot \mathbf{A_0} = 0\,, \qquad -\Delta \mathbf{A_0} = \omega^2 \mathbf{A_0}\,. \tag{3.5.2}$$

These are the well known Helmholtz-equations for the monochromatic time-periodic solutions of Maxwell's equations in the vacuum. From the potential $\mathbf{A}$ we obtain the complex electromagnetic field $\mathbf{E}$ and $\mathbf{B}$ according to (3.3.3), (3.3.5), and for these quantities Maxwell's equations reduce to

$$\nabla \times \mathbf{E} = -i\,\frac{\omega}{c}\,\mathbf{B}\,, \qquad \nabla \times \mathbf{B} = i\,\frac{\omega}{c}\,\mathbf{E}\,. \tag{3.5.3}$$

It is important to note that for any solution with frequency ω there corresponds a second solution with frequency $-\omega$. We solve these equations with real numbers $a_1, a_2, a_3 > 0$ for a box $\Omega = [0, a_1] \times [0, a_2] \times [0, a_3]$, where the boundary $\partial\Omega$ is formed by ideal conducting walls, i.e. we prescribe the boundary conditions

$$\nu \times \mathbf{E}|_{\partial\Omega} = \mathbf{0}\,, \qquad \nu \cdot \mathbf{B}|_{\partial\Omega} = 0\,. \tag{3.5.4}$$

Here ν is the outer normal vector field on $\partial\Omega$. Let n_1, n_2, n_3 be non-negative integer numbers such that at most one of them is zero and define the wave-vector $\mathbf{k} = (k_x, k_y, k_z)^T \in \mathbb{R}^3$ and the frequency ω by

$$k_x = \frac{\pi n_1}{a_1}\,,\ k_y = \frac{\pi n_2}{a_2}\,,\ k_z = \frac{\pi n_3}{a_3}\,,\ \omega = \pm c\sqrt{k_x^2 + k_y^2 + k_z^2}\,. \tag{3.5.5}$$

Then any complex vector $\mathbf{C} = (C_x, C_y, C_z)^T \in \mathbb{C}^3 \setminus \{\mathbf{0}\}$ orthogonal on $\mathbf{k}$, i.e. $\mathbf{k} \cdot \mathbf{C} = 0$, gives a solution for the vector-potential $\mathbf{A}$ according to

$$\begin{aligned} A_x(t, x, y, z) &= C_x \cos(k_x x)\sin(k_y y)\sin(k_z z)\, e^{i\omega t}\,, \\ A_y(t, x, y, z) &= C_y \sin(k_x x)\cos(k_y y)\sin(k_z z)\, e^{i\omega t}\,, \\ A_z(t, x, y, z) &= C_z \sin(k_x x)\sin(k_y y)\cos(k_z z)\, e^{i\omega t}\,. \end{aligned} \tag{3.5.6}$$

If we define the vector $\mathbf{C}' = \mathbf{k} \times \mathbf{C}$, then the resulting complex electromagnetic field is

$$\begin{aligned}
\mathbf{E}(t,x,y,z) =& -i\,\frac{\omega}{c}\,\mathbf{A}(t,x,y,z)\,,\\
B_x(t,x,y,z) =& C'_x \sin(k_x x)\cos(k_y y)\cos(k_z z)\,e^{i\omega t}\,,\\
B_y(t,x,y,z) =& C'_y \cos(k_x x)\sin(k_y y)\cos(k_z z)\,e^{i\omega t}\,,\\
B_z(t,x,y,z) =& C'_z \cos(k_x x)\cos(k_y y)\sin(k_z z)\,e^{i\omega t}\,.
\end{aligned} \tag{3.5.7}$$

It satisfies Maxwell's equations in the vacuum as well as the boundary conditions (3.5.4). In the case that all three wave-numbers n_1, n_2, n_3 are positive we obtain for a given frequency ω two linear independent complex solutions, and thus the solution is twofold degenerate. If one of the wave numbers is zero, then the solution is not degenerate and thus uniquely determined up to a given nonzero complex factor. If more than one wave number is zero, then the electromagnetic field vanishes.

We can also write down the real solutions for the electromagnetic field in a completely symmetric way. In order to do this we first consider the degenerate case $n_1, n_2, n_3 \in \mathbb{N}$ and choose instead of $\mathbf{C}$ two real polarization vectors $\varepsilon, \tilde{\varepsilon} \in \mathbb{R}^3$ satisfying the orthonormality conditions

$$|\varepsilon| = |\tilde{\varepsilon}| = 1\,, \quad \varepsilon \cdot \tilde{\varepsilon} = 0\,, \quad \varepsilon \times \tilde{\varepsilon} = \frac{1}{|\mathbf{k}|}\mathbf{k}\,. \tag{3.5.8}$$

Here $\mathbf{k}$ and ω are still fixed according to (3.5.5). Now we choose two real parameters a, b with $a^2 + b^2 > 0$ which are multiplied with the physical unit of the electromagnetic field, two angles $\varphi, \psi \in [0, 2\pi)$ and define the two three-vectors $\mathbf{a}(t) = (a_x(t), a_y(t), a_z(t))^T$, $\mathbf{b}(t) = (b_x(t), b_y(t), b_z(t))^T$ with

$$\begin{aligned}
\mathbf{a}(t) =& +a\,(\cos(\varphi)\;\varepsilon - \sin(\varphi)\;\tilde{\varepsilon})\cos(\omega t) + b\,(\cos(\psi)\;\varepsilon - \sin(\psi)\;\tilde{\varepsilon})\sin(\omega t)\,,\\
\mathbf{b}(t) =& -a\,(\sin(\varphi)\;\varepsilon + \cos(\varphi)\;\tilde{\varepsilon})\sin(\omega t) + b\,(\sin(\psi)\;\varepsilon + \cos(\psi)\;\tilde{\varepsilon})\cos(\omega t)\,,\\
\frac{1}{c}\,\dot{\mathbf{a}}(t) =& -\,\mathbf{k}\times\mathbf{b}(t)\,,\;\; \frac{1}{c}\,\dot{\mathbf{b}}(t) = -\,\mathbf{k}\times\mathbf{a}(t)\,,\;\; \mathbf{k}\cdot\mathbf{a}(t) = \mathbf{k}\cdot\mathbf{b}(t) = 0\,.
\end{aligned} \tag{3.5.9}$$

We obtain the following degenerate solution, where we have omitted the time- and spatial arguments of the functions $\mathbf{E}$, $\mathbf{B}$, $\mathbf{a}$, $\mathbf{b}$,

$$\begin{aligned}
E_x =& a_x \cos(k_x x)\sin(k_y y)\sin(k_z z)\,, & B_x =& b_x \sin(k_x x)\cos(k_y y)\cos(k_z z)\,,\\
E_y =& a_y \sin(k_x x)\cos(k_y y)\sin(k_z z)\,, & B_y =& b_y \cos(k_x x)\sin(k_y y)\cos(k_z z)\,,\\
E_z =& a_z \sin(k_x x)\sin(k_y y)\cos(k_z z)\,, & B_z =& b_z \cos(k_x x)\cos(k_y y)\sin(k_z z)\,.
\end{aligned} \tag{3.5.10}$$

It is interesting to note that $\mathbf{a} \cdot \mathbf{b} = 0$ and therefore $\mathbf{E} \cdot \mathbf{B} = 0$, as for the planar light waves studied in the last section.

For the nondegenerate case we consider only two real parameters a, b and restrict ourselves to the case $n_3 = 0$. Then we obtain a solution depending only on t, x and y, namely

$$E_x = E_y = 0\,,\ E_z = [a\cos(\omega t) - b\sin(\omega t)]\sin(k_x x)\sin(k_y y)\,, \tag{3.5.11}$$

$$\begin{aligned} B_x &= -\frac{k_y}{|\mathbf{k}|}[a\sin(\omega t) + b\cos(\omega t)]\sin(k_x x)\cos(k_y y)\,,\\ B_y &= +\frac{k_x}{|\mathbf{k}|}[a\sin(\omega t) + b\cos(\omega t)]\cos(k_x x)\sin(k_y y)\,,\\ B_z &= 0\,. \end{aligned} \tag{3.5.12}$$

For all real parametrizations it is sufficient to consider exclusively positive frequencies $\omega > 0$.

It is important for technical applications, for example for micro-waves, to recover not only the explicit resonance solutions for a box, but also for some other simple geometries. This is indeed possible for further geometries like the sphere and the cylinder.

3.6 The electromagnetic field of a single charged particle

Next we solve Maxwell's equations for a single point-charge q with position

$$\mathbf{x} = \gamma(t) \tag{3.6.1}$$

at time t, where $\gamma : \mathbb{R} \to \mathbb{R}^3$ is a given, three times differentiable curve which satisfies

$$\sup_{t\in\mathbb{R}} \frac{|\dot\gamma(t)|}{c} \le \eta < 1\,. \tag{3.6.2}$$

Using the Dirac-Delta point measure, the charge- and current density are often written in the distributional form

$$\rho(t,\mathbf{x}) = q\,\delta^3(\mathbf{x} - \gamma(t))\,,\quad \mathbf{j}(t,\mathbf{x}) = q\,\dot\gamma(t)\,\delta^3(\mathbf{x} - \gamma(t))\,. \tag{3.6.3}$$

In order to solve Maxwell's equations for this example, we need the so called **retardation function** $\hat\vartheta : \mathbb{R}^4 \to \mathbb{R}$, which is defined as follows: We first note that for any fixed $(t,\mathbf{x}) \in \mathbb{R}^4$ the expression $F(t,\mathbf{x},\vartheta) := t - \frac{1}{c}\,|\mathbf{x} - \gamma(\vartheta)|$ is a contractive function with respect to ϑ, which is a simple consequence of (3.6.2). Using Banach's Fix Point Theorem, we conclude that there is a unique solution $\vartheta = \hat\vartheta(t,\mathbf{x})$ of the equation

$$\vartheta = t - \frac{1}{c}\,|\mathbf{x} - \gamma(\vartheta)|\,, \tag{3.6.4}$$

which just gives the function $\hat{\vartheta}$. It is called retardation function because $\hat{\vartheta}(t, \mathbf{x}) < t$ denotes the time at which a light signal starts on the world-line of the charged particle such that it reaches the space-point $\mathbf{x}$ at the later time t. We present two important properties of the retardation function.

A) The parameter representation of the function $\hat{\vartheta}$:
Let be $\vartheta, d \in \mathbb{R}$ with $d \geq 0$ and $\mathbf{n} \in \mathbb{R}^3$ with $|\mathbf{n}| = 1$. Then

$$\hat{\vartheta}(\vartheta + d,\, \gamma(\vartheta) \,+\, c\, d\mathbf{n}) = \vartheta\,. \tag{3.6.5}$$

B) Iterative scheme for the function $\hat{\vartheta}$:
Using the fix point equation (3.6.4) and the contractivity of $F(t, \mathbf{x}, \vartheta) = t - \frac{1}{c}\,|\mathbf{x} - \gamma(\vartheta)|$ with respect to ϑ, we obtain for any fixed $(t, \mathbf{x}) \in \mathbb{R}^4$:

$$\hat{\vartheta}(t, \mathbf{x}) = \lim_{n\to\infty} \vartheta_n\,, \tag{3.6.6}$$

where $\vartheta_0 := t$ and $\vartheta_{n+1} := t - \frac{1}{c}\,|\gamma(\vartheta_n) - \mathbf{x}|$.

Next we use the Dirac-Delta expressions (3.6.3) for ρ and $\mathbf{j}$ and the representation formula (2.4.8) in order to make the following ansatz for the electromagnetic potentials of the radiating charge:

$$\varphi(t, \mathbf{x}) = \frac{q}{4\pi} \int\limits_{\mathbb{R}^3} \frac{\delta^3(\mathbf{y} - \gamma(t - \frac{1}{c}|\mathbf{y} - \mathbf{x}|))}{|\mathbf{y} - \mathbf{x}|}\, d^3y\,, \tag{3.6.7}$$

$$\mathbf{A}(t, \mathbf{x}) = \frac{q}{4\pi c} \int\limits_{\mathbb{R}^3} \dot{\gamma}(t - \frac{1}{c}|\mathbf{y} - \mathbf{x}|) \frac{\delta^3(\mathbf{y} - \gamma(t - \frac{1}{c}|\mathbf{y} - \mathbf{x}|))}{|\mathbf{y} - \mathbf{x}|}\, d^3y\,. \tag{3.6.8}$$

The integration domain in (2.4.8) has to be replaced by the condition $|\mathbf{y} - \mathbf{x}| \leq c(t - t_0)$ if the initial time is t_0. In our case we have $t_0 = -\infty$, and therefore the integration domain is $\mathbb{R}^3$.

In order to show that the integrals (3.6.7), (3.6.8) are well defined and explicitly solvable, we need the following two propositions:

Proposition 3.1. *For every given $t \in \mathbb{R}$, $\mathbf{x}, \mathbf{z} \in \mathbb{R}^3$ there is exactly one solution $\mathbf{y} \in \mathbb{R}^3$ of the equation*

$$\mathbf{z} = \mathbf{y} - \gamma(t - \frac{1}{c}|\mathbf{y} - \mathbf{x}|)\,,$$

which is given by

$$\mathbf{y} = \mathbf{z} + \gamma(\hat{\vartheta}(t, \mathbf{x} - \mathbf{z}))\,,$$

where $\hat{\vartheta}$ is given by (3.6.6).

Proof: For fixed $t \in \mathbb{R}$, $\mathbf{x}, \mathbf{z} \in \mathbb{R}^3$ we define the function $\mathbf{\Psi} : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\mathbf{\Psi}(\mathbf{y}) := \mathbf{z} + \gamma(t - \frac{1}{c}|\mathbf{y} - \mathbf{x}|) \tag{3.6.9}$$

and show the contractivity of $\mathbf{\Psi}$:

$$\begin{aligned}
|\mathbf{\Psi}(\mathbf{y_1}) - \mathbf{\Psi}(\mathbf{y_2})| = &\left|\gamma(t - \frac{1}{c}|\mathbf{y_1} - \mathbf{x}|) - \gamma(t - \frac{1}{c}|\mathbf{y_2} - \mathbf{x}|)\right| \\
\leq &\sup_{t\in\mathbb{R}} |\dot\gamma(t)| \left|(t - \frac{1}{c}|\mathbf{y_1} - \mathbf{x}|) - (t - \frac{1}{c}|\mathbf{y_2} - \mathbf{x}|)\right| \\
\leq &\sup_{t\in\mathbb{R}} \frac{|\dot\gamma(t)|}{c} \left||\mathbf{y_1} - \mathbf{x}| - |\mathbf{y_2} - \mathbf{x}|\right| \\
\leq &\eta \, |\mathbf{y_1} - \mathbf{y_2}| \,,
\end{aligned}$$

where $0 \leq \eta < 1$ is given by (3.6.2). Banach's Fix Point Theorem shows that $\mathbf{y} = \mathbf{z} + \gamma(t - \frac{1}{c}|\mathbf{y} - \mathbf{x}|)$ has exactly one solution $\mathbf{y} \in \mathbb{R}^3$. In order to prove the representation of this solution by the retardation function we now *define* $\mathbf{y} := \mathbf{z} + \gamma(\hat\vartheta(t, \mathbf{x} - \mathbf{z}))$ and conclude that

$$\begin{aligned}
\hat\vartheta(t, \mathbf{x} - \mathbf{z}) = &t - \frac{1}{c}|\gamma(\hat\vartheta(t, \mathbf{x} - \mathbf{z})) - (\mathbf{x} - \mathbf{z})| \\
= &t - \frac{1}{c}|(\mathbf{y} - \mathbf{z}) - (\mathbf{x} - \mathbf{z})| \\
= &t - \frac{1}{c}|\mathbf{y} - \mathbf{x}| \,,
\end{aligned}$$

and therefore

$$\mathbf{y} - \mathbf{z} = \gamma(\hat\vartheta(t, \mathbf{x} - \mathbf{z})) = \gamma(t - \frac{1}{c}|\mathbf{y} - \mathbf{x}|) \,,$$

which proves the Proposition 3.1. ∎

Proposition 3.2. *(a) For $t \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^3$ we define*

$$N(t, \mathbf{x}) := c(t - \hat\vartheta(t, \mathbf{x})) - \frac{\dot\gamma(\hat\vartheta(t, \mathbf{x}))}{c} \cdot (\mathbf{x} - \gamma(\hat\vartheta(t, \mathbf{x}))) \,.$$

Then $N(t, \mathbf{x}) \geq 0$ and $N(t, \mathbf{x}) = 0 \Leftrightarrow \mathbf{x} = \gamma(t)$.

(b) The partial derivatives of $\hat\vartheta$ are given by

$$\frac{\partial\hat\vartheta}{\partial t}(t, \mathbf{x}) = c\,\frac{t - \hat\vartheta(t, \mathbf{x})}{N(t, \mathbf{x})} \,, \quad \nabla\hat\vartheta(t, \mathbf{x}) = -\frac{1}{c}\,\frac{\mathbf{x} - \gamma(\hat\vartheta(t, \mathbf{x}))}{N(t, \mathbf{x})} \,.$$

(c) There the following equation holds

$$\frac{\partial\hat\vartheta}{\partial t} + \nabla \cdot \gamma(\hat\vartheta) = 1 \,.$$

(d) The transformation given for fixed $t \in \mathbb{R}$, $\mathbf{x} \in \mathbb{R}^3$ by

$$\mathbf{y}(\mathbf{z}) := \mathbf{z} + \gamma(\hat{\vartheta}(t, \mathbf{x} - \mathbf{z}))$$

has the Jacobian determinant

$$\left| \frac{\partial \mathbf{y}}{\partial \mathbf{z}} \right| (\mathbf{z}) = \frac{\partial \hat{\vartheta}}{\partial t}(t, \mathbf{x} - \mathbf{z}) \,.$$

Proof: We first recall that

$$t - \hat{\vartheta}(t, \mathbf{x}) = \frac{1}{c} \left| \gamma(\hat{\vartheta}(t, \mathbf{x})) - \mathbf{x} \right| , \tag{3.6.10}$$

which implies the inequality

$$\hat{\vartheta}(t, \mathbf{x}) \leq t \,. \tag{3.6.11}$$

In order to show (a) we first consider the case $\hat{\vartheta}(t, \mathbf{x}) = t$ and conclude from (3.6.10) that

$$\mathbf{x} = \gamma(\hat{\vartheta}(t, \mathbf{x})) = \gamma(t) \,, \quad N(t, \mathbf{x}) = 0 \,.$$

Due to (3.6.11) there remains to study the case $t - \hat{\vartheta}(t, \mathbf{x}) > 0$. Then we obtain from the definition of N, (3.6.10), the Cauchy-Schwarz inequality and (3.6.2):

$$\frac{N(t, \mathbf{x})}{c(t - \hat{\vartheta}(t, \mathbf{x}))} = 1 + \frac{\dot{\gamma}(\hat{\vartheta}(t, \mathbf{x}))}{c} \cdot \frac{\gamma(\hat{\vartheta}(t, \mathbf{x})) - \mathbf{x}}{|\gamma(\hat{\vartheta}(t, \mathbf{x})) - \mathbf{x}|} \geq 1 - \frac{|\dot{\gamma}(\hat{\vartheta}(t, \mathbf{x})|}{c} \geq 1 - \eta > 0 \,.$$

We have thus confirmed (a).

(b) We differentiate (3.6.10) with respect to t and obtain

$$\begin{aligned} 1 - \frac{\partial \hat{\vartheta}}{\partial t}(t, \mathbf{x}) =& \frac{\gamma(\hat{\vartheta}(t, \mathbf{x})) - \mathbf{x}}{|\gamma(\hat{\vartheta}(t, \mathbf{x})) - \mathbf{x}|} \cdot \frac{\dot{\gamma}(\hat{\vartheta}(t, \mathbf{x}))}{c} \frac{\partial \hat{\vartheta}}{\partial t}(t, \mathbf{x}) \\ =& \frac{\gamma(\hat{\vartheta}(t, \mathbf{x})) - \mathbf{x}}{t - \hat{\vartheta}(t, \mathbf{x})} \cdot \frac{\dot{\gamma}(\hat{\vartheta}(t, \mathbf{x}))}{c^2} \frac{\partial \hat{\vartheta}}{\partial t}(t, \mathbf{x}) \,. \end{aligned}$$

We solve this equation with respect to $\frac{\partial \hat{\vartheta}}{\partial t}$ and obtain the first formula in Proposition 3.2(b). The spatial derivatives of $\hat{\vartheta}$ are obtained by a quite similar calculation.

(c) is a simple consequence of (b):

$$\frac{\partial \hat{\vartheta}}{\partial t} + \nabla \cdot \gamma(\hat{\vartheta}) = c \, \frac{t - \hat{\vartheta}}{N} - \dot{\gamma}(\hat{\vartheta}) \cdot \frac{\mathbf{x} - \gamma(\hat{\vartheta})}{c \, N} = 1 \,.$$

(d) In order to calculate the Jacobian matrix of the transformation

$$\mathbf{z} \to \mathbf{y}(\mathbf{z}) := \mathbf{z} + \gamma(\hat{\vartheta}(t, \mathbf{x} - \mathbf{z}))\,,$$

we introduce the following notations for the components of the quantities

$$\mathbf{x} = (x^1, x^2, x^3)^T\,,\ \mathbf{y} = (y^1, y^2, y^3)^T\,,\ \mathbf{z} = (z^1, z^2, z^3)^T\,,\ \dot{\gamma} = (\dot{\gamma}^1,\ \dot{\gamma}^2,\ \dot{\gamma}^3)^T\,.$$

Moreover we introduce the abbreviations

$$a^k := \dot{\gamma}^k(\hat{\vartheta}(t, \mathbf{x} - \mathbf{z}))\,,\ b_k := -\frac{\partial \hat{\vartheta}}{\partial x^k}(t, \mathbf{x} - \mathbf{z})\,,\ k = 1, 2, 3\,.$$

Here the quantities a^k, b_k are not the spatial components of covariant and contravariant Lorentz invariant four-vectors, which will be defined later. We obtain for any fixed $t \in \mathbb{R}$, $\mathbf{x} \in \mathbb{R}^3$ due to the third part of Proposition 3.2:

$$\begin{aligned}\left|\frac{\partial \mathbf{y}}{\partial \mathbf{z}}\right|(\mathbf{z}) =& \begin{vmatrix} 1 + a^1 b_1 & a^1 b_2 & a^1 b_3 \\ a^2 b_1 & 1 + a^2 b_2 & a^2 b_3 \\ a^3 b_1 & a^3 b_2 & 1 + a^3 b_3 \end{vmatrix} \\ =& 1 + \sum_{k=1}^{3} a^k b_k \ = 1 - \sum_{k=1}^{3} \dot{\gamma}^k(\hat{\vartheta}(t, \mathbf{x} - \mathbf{z}))\, \frac{\partial \hat{\vartheta}}{\partial x^k}(t, \mathbf{x} - \mathbf{z}) \\ =& \frac{\partial \hat{\vartheta}}{\partial t}(t, \mathbf{x} - \mathbf{z})\,.\end{aligned}$$

Thus we have proved Proposition 3.2. ∎

Now we are able to evaluate the integral expressions (3.6.7) and (3.6.8) for φ and $\mathbf{A}$ by applying the integral substitution

$$\mathbf{y} \to \mathbf{z} = \mathbf{y} - \gamma(t - \frac{1}{c}|\mathbf{y} - \mathbf{x}|)\,.$$

In order to evaluate the Dirac-Delta expressions we have to put $\mathbf{z} = \mathbf{0}$, where Proposition 3.1 leads to

$$\mathbf{y} = \gamma(\hat{\vartheta}(t, \mathbf{x}))\,.$$

We also obtain due to (3.6.4) that

$$|\mathbf{x} - \gamma(\hat{\vartheta}(t, \mathbf{x}))| = c\,(t - \hat{\vartheta}(t, \mathbf{x}))\,,$$

and therefore φ, $\mathbf{A}$ may be rewritten due to Proposition 3.2 in the form

$$\varphi(t, \mathbf{x}) = \frac{q}{4\pi}\frac{1}{N(t, \mathbf{x})}\,, \quad \mathbf{A}(t, \mathbf{x}) = \frac{q}{4\pi c}\frac{\dot{\gamma}(\hat{\vartheta}(t, \mathbf{x}))}{N(t, \mathbf{x})}\,. \tag{3.6.12}$$

These are the famous **Liénard-Wiechert** potentials. We have obtained them from the retarded potential formula (2.4.8) in Section 2.4 in a heuristic

way, although we have not justified the validity of (2.4.8) for a Dirac-Delta point measure under the integral. Thus it remains to show that φ and $\mathbf{A}$ are the electromagnetic potentials for the field produced by the single point charge. This is stated in the following

Proposition 3.3. *We abbreviate d'Alembert's wave operator by* $\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta$. *The quantities* φ *and* $\mathbf{A}$ *defined for* $\mathbf{x} \neq \gamma(t)$ *in* (3.6.12) *satisfy the following differential equations:*

$$\Box\,\varphi = 0\,, \qquad \Box\,\mathbf{A} = 0\,, \qquad \frac{1}{c}\frac{\partial\varphi}{\partial t} + \nabla\cdot\mathbf{A} = 0\,.$$

They are electromagnetic potentials for the fields

$$\begin{aligned}
\mathbf{E}(t,\mathbf{x}) \;&=\; \frac{q}{4\pi c^2}\,\frac{1}{|\mathbf{x}-\gamma|}\,\left[\frac{\ddot\gamma\cdot\mathbf{n}}{(1-\frac{\mathbf{n}\cdot\dot\gamma}{c})^3}\,\left(\mathbf{n}-\frac{\dot\gamma}{c}\right) - \frac{\ddot\gamma}{(1-\frac{\mathbf{n}\cdot\dot\gamma}{c})^2}\right] \;+\\
&\quad+\;\frac{q}{4\pi}\,\frac{1}{|\mathbf{x}-\gamma|^2}\,\frac{1-\frac{\dot\gamma^2}{c^2}}{(1-\frac{\mathbf{n}\cdot\dot\gamma}{c})^3}\,\left(\mathbf{n}-\frac{\dot\gamma}{c}\right)\,,\\
\mathbf{B}(t,\mathbf{x}) \;&=\; \frac{q}{4\pi c^2}\,\frac{1}{|\mathbf{x}-\gamma|}\,\left[\frac{\ddot\gamma\cdot\mathbf{n}}{(1-\frac{\mathbf{n}\cdot\dot\gamma}{c})^3}\,\frac{\dot\gamma}{c}\times\mathbf{n} + \frac{\ddot\gamma\times\mathbf{n}}{(1-\frac{\mathbf{n}\cdot\dot\gamma}{c})^2}\right] \;+\\
&\quad+\;\frac{q}{4\pi}\,\frac{1}{|\mathbf{x}-\gamma|^2}\,\frac{1-\frac{\dot\gamma^2}{c^2}}{(1-\frac{\mathbf{n}\cdot\dot\gamma}{c})^3}\,\frac{\dot\gamma}{c}\times\mathbf{n}\,,
\end{aligned}$$

where the symbols $\mathbf{n}$, γ, $\dot\gamma$, $\ddot\gamma$ *are abbreviations for*

$$\frac{\mathbf{x}-\gamma(\hat\vartheta(t,\mathbf{x}))}{|\mathbf{x}-\gamma(\hat\vartheta(t,\mathbf{x}))|}\,,\;\gamma(\hat\vartheta(t,\mathbf{x}))\,,\;\dot\gamma(\hat\vartheta(t,\mathbf{x}))\,,\;\ddot\gamma(\hat\vartheta(t,\mathbf{x}))\;\in\mathbb{R}^3\,,$$

respectively.

Remarks:

(i) If we consider a single point charge at rest with $\gamma(t) = \mathbf{0}$ for all $t \in \mathbb{R}$, then we immediately obtain the special electrostatic case

$$\mathbf{E}(t,\mathbf{x}) \;=\; \frac{q}{4\pi}\,\frac{\mathbf{x}}{|\mathbf{x}|^3}\,, \quad \mathbf{B}(t,\mathbf{x}) \;=\mathbf{0}\,. \tag{3.6.13}$$

(ii) We can simply write the magnetic field in the form

$$\mathbf{B}(t,\mathbf{x}) \;=\mathbf{n}\times\mathbf{E}(t,\mathbf{x}) \tag{3.6.14}$$

with the abbreviation $\mathbf{n} = \mathbf{n}(t,\mathbf{x})$ given above.

(iii) The first term for $\mathbf{E}$ and $\mathbf{B}$ involving the second derivative of γ is due to the so called far-field, since it only decays with the factor $1/|\mathbf{x}-\gamma|$, in contrast to the second term, which decays faster for increasing distance due to the factor $1/|\mathbf{x}-\gamma|^2$. We call them $\mathbf{E}_{far}$, $\mathbf{B}_{far}$, respectively:

$$\mathbf{E}_{far}(t,\mathbf{x}) = \frac{q}{4\pi c^2}\,\frac{1}{|\mathbf{x}-\gamma|}\left[\frac{\ddot{\gamma}\cdot\mathbf{n}}{(1-\frac{\mathbf{n}\cdot\dot{\gamma}}{c})^3}\left(\mathbf{n}-\frac{\dot{\gamma}}{c}\right)-\frac{\ddot{\gamma}}{(1-\frac{\mathbf{n}\cdot\dot{\gamma}}{c})^2}\right], \tag{3.6.15}$$

$$\mathbf{B}_{far}(t,\mathbf{x}) = \frac{q}{4\pi c^2}\,\frac{1}{|\mathbf{x}-\gamma|}\left[\frac{\ddot{\gamma}\cdot\mathbf{n}}{(1-\frac{\mathbf{n}\cdot\dot{\gamma}}{c})^3}\,\frac{\dot{\gamma}}{c}\times\mathbf{n}+\frac{\ddot{\gamma}\times\mathbf{n}}{(1-\frac{\mathbf{n}\cdot\dot{\gamma}}{c})^2}\right]. \tag{3.6.16}$$

Between $\mathbf{E}_{far}$, $\mathbf{B}_{far}$ and $\mathbf{n}$ there hold the nice relations

$$\mathbf{B}_{far} = +\mathbf{n}\times\mathbf{E}_{far}\,, \qquad \mathbf{E}_{far} = -\mathbf{n}\times\mathbf{B}_{far}\,, \tag{3.6.17}$$

and $\mathbf{E}_{far}$, $\mathbf{B}_{far}$ are perpendicular to $\mathbf{n}$:

$$\mathbf{E}_{far}\cdot\mathbf{n} = 0\,, \qquad \mathbf{B}\cdot\mathbf{n} = \mathbf{B}_{far}\cdot\mathbf{n} = 0\,. \tag{3.6.18}$$

But it is important to note that $\mathbf{E}_{far}$, $\mathbf{B}_{far}$ itself will **not** satisfy Maxwell's equations. Nevertheless, $\mathbf{E}_{far}$, $\mathbf{B}_{far}$ form an antisymmetric tensor field like $\mathbf{E}$ and $\mathbf{B}$. The tensor calculus will be considered later.

Proof of Proposition 3.3: We first calculate the partial derivatives of N, where we make use of Proposition 3.2 and of the abbreviations defined above for γ, $\dot{\gamma}$, $\ddot{\gamma}$, and so on. The spatial components for the position are denoted by $\mathbf{x} = (x^1, x^2, x^3)^T$, which will be an appropriate notation for the tensor calculus introduced later:

$$\frac{1}{c}\,\frac{\partial N}{\partial t} = 1-\frac{|\mathbf{x}-\gamma|}{N}\left[1-\frac{\dot{\gamma}^2-\ddot{\gamma}\cdot(\mathbf{x}-\gamma)}{c^2}\right], \tag{3.6.19}$$

$$\frac{\partial N}{\partial x^k} = -\frac{1}{c}\,\dot{\gamma}^k+\frac{x^k-\gamma^k}{N}\left[1-\frac{\dot{\gamma}^2-\ddot{\gamma}\cdot(\mathbf{x}-\gamma)}{c^2}\right],\ k=1,2,3\,. \tag{3.6.20}$$

In order to avoid the often occuring factor $\frac{q}{4\pi}$ in front of the electromagnetic potentials we introduce the new quantities

$$\tilde{\varphi} := \frac{1}{N}\,,\ \tilde{A}^i := \frac{\dot{\gamma}^i}{c\,N} \tag{3.6.21}$$

with the partial derivatives

$$\frac{\partial\tilde{\varphi}}{\partial t} = -\frac{1}{N^2}\,\frac{\partial N}{\partial t}\,,\ \frac{\partial\tilde{\varphi}}{\partial x^k} = -\frac{1}{N^2}\,\frac{\partial N}{\partial x^k}\,, \tag{3.6.22}$$

$$\frac{\partial \tilde{A}^i}{\partial t} = \frac{\ddot{\gamma}^i |\mathbf{x}-\gamma| - \dot{\gamma}^i \frac{\partial N}{\partial t}}{c\, N^2}\,, \quad \frac{\partial \tilde{A}^i}{\partial x^k} = -\frac{\ddot{\gamma}^i (x^k - \gamma^k) + c\,\dot{\gamma}^i \frac{\partial N}{\partial x^k}}{c^2\, N^2}\,, \tag{3.6.23}$$

where we made use of Proposition 3.2 and $c\,(t-\hat{\vartheta}) = |\mathbf{x}-\gamma|$. The Lorentz-condition $\frac{1}{c}\frac{\partial \varphi}{\partial t} + \nabla \cdot \mathbf{A} = 0$ can now be checked by a straight forward calculation, using the formulas (3.6.19)-(3.6.23). In the same way we can also establish $\Box\, \varphi = 0$, $\Box\, \mathbf{A} = 0$ without making any use of the retardation formulas (3.6.7) and (3.6.8), but this requires more effort. Therefore φ, $\mathbf{A}$ are electromagnetic potentials, and the corresponding electromagnetic fields result from (3.6.12), (3.3.3), (3.3.5). ∎

3.7 Balance laws

The energy density W of the electromagnetic field is given by

$$W = \frac{\mathbf{E}^2 + \mathbf{B}^2}{2\varepsilon_0}\,. \tag{3.7.1}$$

We obtain from the second and the fourth Maxwellian equation:

$$\begin{aligned}
\frac{\partial W}{\partial t} &= \frac{c}{\varepsilon_0}\left(\mathbf{E}\cdot\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \mathbf{B}\cdot\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}\right)\\
&= \frac{c}{\varepsilon_0}\left(\mathbf{E}\cdot(\nabla\times\mathbf{B}) - \frac{1}{c}\mathbf{E}\cdot\mathbf{j} - \mathbf{B}\cdot(\nabla\times\mathbf{E})\right)\\
&= \frac{c}{\varepsilon_0}\left(-\nabla\cdot(\mathbf{E}\times\mathbf{B}) - \frac{1}{c}\mathbf{E}\cdot\mathbf{j}\right)\\
&= -c^2\nabla\cdot\left(\frac{\mathbf{E}\times\mathbf{B}}{c\varepsilon_0}\right) - \frac{1}{\varepsilon_0}\mathbf{E}\cdot\mathbf{j}\,.
\end{aligned}$$

There results the energy balance

$$\frac{\partial W}{\partial t} + c^2\nabla\cdot\mathbf{P} + \frac{1}{\varepsilon_0}\mathbf{E}\cdot\mathbf{j} = 0\,. \tag{3.7.2}$$

with the electromagnetic momentum density $\mathbf{P}$ given by

$$\mathbf{P} = \frac{\mathbf{E}\times\mathbf{B}}{c\varepsilon_0}\,. \tag{3.7.3}$$

The quantity $\frac{1}{\varepsilon_0}\mathbf{E}\cdot\mathbf{j}$ is the energy density per time unit of the electromagnetic field. Here only the electric field acts on the charged particles, because the magnetic part of the force is perpendicular to $\mathbf{j}$. In order to derive the

momentum balance we need the following calculation, which does not make use of Maxwell's equations:

$$
\begin{aligned}
(\nabla \times \mathbf{B}) \times \mathbf{B} &= \begin{pmatrix} \partial_y B_z - \partial_z B_y \\ \partial_z B_x - \partial_x B_z \\ \partial_x B_y - \partial_y B_x \end{pmatrix} \times \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \\
&= \begin{pmatrix} (\partial_z B_x) B_z - (\partial_x B_z) B_z - (\partial_x B_y) B_y + (\partial_y B_x) B_y \\ (\partial_x B_y) B_x - (\partial_y B_x) B_x - (\partial_y B_z) B_z + (\partial_z B_y) B_z \\ (\partial_y B_z) B_y - (\partial_z B_y) B_y - (\partial_z B_x) B_x + (\partial_x B_z) B_x \end{pmatrix} = \\
&= \begin{pmatrix} \partial_x (\frac{B_x^2 - B_y^2 - B_z^2}{2}) + \partial_y (B_x B_y) + \partial_z (B_x B_z) \\ \partial_x (B_y B_x) + \partial_y (\frac{-B_x^2 + B_y^2 - B_z^2}{2}) + \partial_z (B_y B_z) \\ \partial_x (B_z B_x) + \partial_y (B_z B_y) + \partial_z (\frac{-B_x^2 - B_y^2 + B_z^2}{2}) \end{pmatrix} - \begin{pmatrix} (\partial_x B_x) B_x + (\partial_y B_y) B_x + (\partial_z B_z) B_x \\ (\partial_x B_x) B_y + (\partial_y B_y) B_y + (\partial_z B_z) B_y \\ (\partial_x B_x) B_z + (\partial_y B_y) B_z + (\partial_z B_z) B_z \end{pmatrix} .
\end{aligned}
$$

An analogous result holds for $(\nabla \times \mathbf{E}) \times \mathbf{E}$. Now we obtain from Maxwell's equations:

$$
\begin{aligned}
\frac{\partial}{\partial t} \Big[\frac{\mathbf{E} \times \mathbf{B}}{c \varepsilon_0} \Big] &= \frac{1}{\varepsilon_0} \left[\frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B} + \mathbf{E} \times \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \right] \\
&= \frac{1}{\varepsilon_0} \left[\Big(\nabla \times \mathbf{B} - \frac{1}{c} \mathbf{j} \Big) \times \mathbf{B} + \mathbf{E} \times (-\nabla \times \mathbf{E}) \right] \\
&= \frac{1}{\varepsilon_0} \Big[(\nabla \times \mathbf{B}) \times \mathbf{B} + (\nabla \times \mathbf{E}) \times \mathbf{E} \Big] - \frac{1}{\varepsilon_0} \frac{\mathbf{j}}{c} \times \mathbf{B} \\
&= - \begin{pmatrix} \partial_x T^{11} + \partial_y T^{12} + \partial_z T^{13} \\ \partial_x T^{21} + \partial_y T^{22} + \partial_z T^{23} \\ \partial_x T^{31} + \partial_y T^{32} + \partial_z T^{33} \end{pmatrix} - \frac{1}{\varepsilon_0} (\nabla \cdot \mathbf{B}) \, \mathbf{B} - \frac{1}{\varepsilon_0} (\nabla \cdot \mathbf{E}) \, \mathbf{E} - \frac{1}{\varepsilon_0} \frac{\mathbf{j}}{c} \times \mathbf{B} \\
&= - \begin{pmatrix} \partial_x T^{11} + \partial_y T^{12} + \partial_z T^{13} \\ \partial_x T^{21} + \partial_y T^{22} + \partial_z T^{23} \\ \partial_x T^{31} + \partial_y T^{32} + \partial_z T^{33} \end{pmatrix} - \frac{1}{\varepsilon_0} \Big(\rho \mathbf{E} + \frac{\mathbf{j}}{c} \times \mathbf{B} \Big) ,
\end{aligned}
$$

where T^{mn} denote the spatial components of the opposite Maxwellian stress tensor T and $\mathbf{E} = (E^1, E^2, E^3) = (E_x, E_y, E_z)$, $\mathbf{B} = (B^1, B^2, B^3) = (B_x, B_y, B_z)$,

$$
T^{mn} = \frac{1}{\varepsilon_0} \left[\frac{\mathbf{E}^2 + \mathbf{B}^2}{2} \, \delta^{mn} - E^m E^n - B^m B^n \right] . \tag{3.7.4}
$$

By $m, n = 1, 2, 3$ we denote exclusively spatial indices. The balance laws for energy and momentum take the form

$$
\frac{1}{c} \frac{\partial W}{\partial t} + \nabla \cdot (c \mathbf{P}) + \frac{1}{\varepsilon_0} \mathbf{E} \cdot \frac{\mathbf{j}}{c} = 0 , \tag{3.7.5}
$$

$$
\frac{1}{c} \frac{\partial}{\partial t} (c \mathbf{P}) + \nabla \cdot T + \frac{1}{\varepsilon_0} \Big(\rho \mathbf{E} + \frac{\mathbf{j}}{c} \times \mathbf{B} \Big) = \mathbf{0} . \tag{3.7.6}
$$

The momentum balance contains the force density $\frac{1}{\varepsilon_0}\Big(\rho\mathbf{E}+\frac{\mathbf{j}}{c}\times\mathbf{B}\Big)$, which is in perfect agreement with the formula (3.2.5) for the electromagnetic force. The formula (3.2.5) was originally used in order to determine the electromagnetic field. Actually, here the corresponding force density is mathematically derived from Maxwell's equations in the momentum balance! Recall that we have redefined the electromagnetic field in such a way that it is connected to the conventional electromagnetic field $\mathbf{E}'$, $\mathbf{B}'$ by equations (3.2.7). However, the technical conventions for all the other physical quantities are untouched.

3.8 Lorentz transformations

One of the main feature of Maxwell's equations is that they reduce to linear wave equations for the electromagnetic potentials. In this section we determine the linear coordinate transformations of time and space which leave the wave equation invariant and show that these are exactly the Lorentz transformations which leave the Einstein-Minkowski pseudo-metric invariant. Moreover, we study the Lorentz-transformations in detail and describe how we can construct them in a very simple way.

The time-space coordinates may be rewritten in terms of a four-quantity $\tilde{x}=(x^0,x^1,x^2,x^3)^T$ according to

$$x^0=ct,\qquad x^1=x,\qquad x^2=y,\qquad x^3=z\,. \tag{3.8.1}$$

$\tilde{x}$ describes an event in time and space. We may identify $\tilde{x}$ with this event. We need the components of the four-matrix

$$G=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\,, \tag{3.8.2}$$

which will be denoted by

$$g_{\mu\nu}=g^{\mu\nu}=\begin{cases}+1 & , \quad \mu=\nu=0\,,\\ -1 & , \quad \mu=\nu=1,2,3\,,\\ 0 & , \quad \mu\neq\nu\,.\end{cases} \tag{3.8.3}$$

We consider the d'Alembertian wave operator

$$\Box=\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\Delta=g^{\mu\nu}\frac{\partial^2}{\partial x^\mu\partial x^\nu}\,, \tag{3.8.4}$$

acting on a time and space depending field $\psi(\tilde{x})$.

Here a Greek index like μ, ν will always run over the four numbers 0,1,2,3, in contrast to the spatial indices, which will be denoted by a Latin letter like m, n in (3.7.4). In this chapter we make use of Einstein's summation convention, i.e. any index, like μ, ν in (3.8.4), that appears twice, once as a subscript and once as a superscript, is understood to be summed over, if not otherwise noted.

We are looking for a linear time-space transformation, given by a four-matrix $\Lambda = (\Lambda^{\mu}_{\ \nu})_{\mu,\nu=0,\dots,3}$ with

$$\tilde{x}' = \tilde{x}'(\tilde{x}) = \Lambda\tilde{x}\,, \qquad x'^{\mu} = \Lambda^{\mu}_{\ \nu}\, x^{\nu}\,, \tag{3.8.5}$$

which leaves the wave operator invariant:

$$g^{\mu\nu}\frac{\partial^2\psi}{\partial x^{\mu}x^{\nu}}(\tilde{x}) = g^{\kappa\lambda}\frac{\partial^2\hat{\psi}}{\partial x'^{\kappa}x'^{\lambda}}(\tilde{x}')\,, \tag{3.8.6}$$

where

$$\hat{\psi}(\tilde{x}'(\tilde{x})) = \psi(\tilde{x})\,. \tag{3.8.7}$$

We apply the chain rule on (3.8.7) and obtain due to (3.8.5):

$$\begin{aligned}\frac{\partial\psi}{\partial x^{\nu}}(\tilde{x}) &= \Lambda^{\lambda}_{\ \nu}\,\frac{\partial\hat{\psi}}{\partial x'^{\lambda}}(\tilde{x}'(\tilde{x}))\,,\\ \frac{\partial^2\psi}{\partial x^{\mu}\partial x^{\nu}}(\tilde{x}) &= \Lambda^{\kappa}_{\ \mu}\,\Lambda^{\lambda}_{\ \nu}\,\frac{\partial^2\hat{\psi}}{\partial x'^{\kappa}\partial x'^{\lambda}}(\tilde{x}'(\tilde{x}))\,.\end{aligned}$$

Now (3.8.6) may be rewritten in the form

$$g^{\kappa\lambda}\,\frac{\partial^2\hat{\psi}}{\partial x'^{\kappa}\partial x'^{\lambda}}(\tilde{x}'(\tilde{x})) = g^{\mu\nu}\,\Lambda^{\kappa}_{\ \mu}\,\Lambda^{\lambda}_{\ \nu}\,\frac{\partial^2\hat{\psi}}{\partial x'^{\kappa}\partial x'^{\lambda}}(\tilde{x}'(\tilde{x}))\,.$$

For a general function ψ this condition is equivalent to the fundamental matrix equation

$$G = \Lambda\, G\, \Lambda^T\,. \tag{3.8.8}$$

Thus Λ leaves the wave operator invariant if and only if (3.8.8) is satisfied. Next we have to show that this invariance is equivalent to the invariance of the Einstein-Minkowski pseudo metric $Q(\tilde{x})$, which is given by the following quadratic form in $\tilde{x} = (x^0, x^1, x^2, x^3)$:

$$Q(\tilde{x}) = \tilde{x}^T\, G\, \tilde{x} = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2\,. \tag{3.8.9}$$

Here we replace $\tilde{x}$ by $\tilde{x}' = \Lambda\,\tilde{x}$ in (3.8.9) in order to obtain in the new reference frame

$$Q'(\tilde{x}') = Q(\Lambda\,\tilde{x}) = \tilde{x}^T\,(\Lambda^T\, G\,\Lambda)\,\tilde{x}\,. \tag{3.8.10}$$

The pseudo metric is invariant with respect to the transformation Λ, i.e. $Q(\Lambda\,\tilde{x}) = Q(\tilde{x})$ for all $\tilde{x}$, if and only if

$$G = \Lambda^T\, G\, \Lambda\,. \tag{3.8.11}$$

Now we prove the following proposition, which states that the conditions (3.8.8) and (3.8.11) are indeed equivalent for any 4×4-matrix Λ.

Proposition 3.4. *Define the Minkowskian matrix G as in* (3.8.2)*. Then the following statements are equivalent for any matrix $\Lambda \in \mathbb{R}^{4\times 4}$:*

(a) $G = \Lambda\, G\, \Lambda^T$, *i.e. the matrix Λ leaves the wave operator $\Box$ invariant.*

(b) Λ *is regular and has the inverse matrix* $\Lambda^{-1} = G\,\Lambda^T\, G$.

(c) $G = \Lambda^T\, G\, \Lambda$, *i.e. the matrix Λ leaves the Einstein-Minkowski pseudo metric invariant.*

Proof: Let $\mathbf{1}$ be the unit matrix in $\mathbb{R}^{4\times 4}$. We obtain

$$\begin{aligned}
G &= \Lambda\, G\, \Lambda^T && \Leftrightarrow \\
\mathbf{1} = G^2 &= \Lambda\,(G\,\Lambda^T\, G) && \Leftrightarrow \\
\Lambda^{-1} &= G\,\Lambda^T\, G && \Leftrightarrow \\
\mathbf{1} = \Lambda^{-1}\,\Lambda &= G\,\Lambda^T\, G\,\Lambda && \Leftrightarrow \\
G &= G^2\,\Lambda^T\, G\,\Lambda = \Lambda^T\, G\,\Lambda\,.
\end{aligned}$$

■

Definition: A constant matrix $\Lambda \in \mathbb{R}^{4\times 4}$ which satisfies the equivalent conditions (a), (b), (c) in Proposition 3.4 is called a Lorentz-matrix.
A Lorentz-matrix Λ and a constant four-quantity $\tilde{a} \in \mathbb{R}^4$ describe a Lorentz-transformation

$$\tilde{x} \to \tilde{x}' = \Lambda\,\tilde{x} + \tilde{a}\,, \qquad x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + a^{\mu} \tag{3.8.12}$$

of the four time-space coordinates. The Lorentz-transformation is called homogeneous if $\tilde{a} = 0$.

A famous example is the homogeneous Lorentz-transformation

$$t' = \frac{t - \frac{v\,x}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}\,, \qquad x' = \frac{x - v\,t}{\sqrt{1 - \frac{v^2}{c^2}}}\,, \qquad y' = y\,, \qquad z' = z \tag{3.8.13}$$

with velocity $v = v_x$ along the x-axis, which reduces for $|v| << c$ to the Galilean transformation

$$t' = t\,, \qquad x' = x - v\,t\,. \tag{3.8.14}$$

Later on we will discuss a generalization of this transformation in detail.

Now let us consider two events A and B, represented by

$$\tilde{x} = (x^0, x^1, x^2, x^3)\,, \qquad \tilde{x} + d\tilde{x} = (x^0 + dx^0, x^1 + dx^1, x^2 + dx^2, x^3 + dx^3)\,,$$

respectively. Note that $d\tilde{x}$ transforms linearly and homogeneously even in the case of the inhomogeneous Lorentz-transformation (3.8.12) according to

$$dx'^{\mu} = \Lambda^{\mu}{}_{\nu}\, dx^{\nu}\,, \tag{3.8.15}$$

whereas $\tilde{x}$ does **not**.

The events A and B can only be connected by a light signal if

$$Q(d\tilde{x}) = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 = 0\,. \tag{3.8.16}$$

In this case $d\tilde{x}$ is called a *light vector*. If $Q(d\tilde{x}) > 0$ then $d\tilde{x}$ is called a *time-vector*. Then A and B can be connected by a signal with speed less then speed of light. If $Q(d\tilde{x}) < 0$ then $d\tilde{x}$ is called a *space-vector*. Then A and B cannot be connected by any signal.

Now we may formulate the following

Proposition 3.5. *Let $\Lambda \in \mathbb{R}^{4\times 4}$ be regular. For any light vector $d\tilde{x}$ let*

$$dx'^{\mu} = \Lambda^{\mu}{}_{\nu}\, dx^{\nu}$$

also be a light vector. Then there exists a constant $\gamma \in \mathbb{R}$ such that $\gamma\,\Lambda$ is a Lorentz-matrix.

Proof: Define the symmetric matrix $L = \Lambda^T\, G\, \Lambda$ with components

$$L_{\mu\nu} = \Lambda^{\kappa}{}_{\mu}\, \Lambda^{\lambda}{}_{\nu}\, g_{\kappa\lambda}\,. \tag{3.8.17}$$

By $d\tilde{x}$ we denote any light vector. Then we obtain due to the assumption of the proposition:

$$d\tilde{x}^T\, L\, d\tilde{x} = 0\,. \tag{3.8.18}$$

In the following we will **not** make use of Einstein's summation convention. Then we may write, since $d\tilde{x}$ is a light vector:

$$(dx^0)^2 = -\sum_{i,k=1}^{3} g_{ik}\, dx^i\, dx^k\,. \tag{3.8.19}$$

From (3.8.18) and (3.8.19) we obtain, keeping in mind that L is a symmetric matrix:

$$d\tilde{x}^T\, L\, d\tilde{x} = \sum_{i,k=1}^{3} (L_{ik} - L_{00}\, g_{ik})\; dx^i\, dx^k + 2\, dx^0 \sum_{i=1}^{3} L_{i0}\, dx^i = 0\,. \qquad (3.8.20)$$

First we choose the six light vectors

$$(\pm 1, 1, 0, 0)^T\,, (\pm 1, 0, 1, 0)^T\,, (\pm 1, 0, 0, 1)^T$$

in order to conclude from (3.8.20) that

$$L_{11} = L_{22} = L_{33} = -L_{00}\,, \qquad L_{10} = L_{01} = L_{20} = L_{02} = L_{30} = L_{03} = 0\,.$$

Then we insert these values in (3.8.20) and choose the three light vectors

$$(\sqrt{2}, 1, 1, 0)^T\,, (\sqrt{2}, 1, 0, 1)^T\,, (\sqrt{2}, 0, 1, 1)^T$$

in order to conclude that

$$L_{12} = L_{21} = L_{13} = L_{31} = L_{23} = L_{32} = 0\,.$$

We have thus proved

$$L = \Lambda^T\, G\, \Lambda = L_{00}\, G\,, \qquad (3.8.21)$$

and it remains to prove that $L_{00} > 0$. This is an application of the well known **inertia law of Sylvester**, which may be shown in this special case as follows: From (3.8.21) we obtain due to the regularity of Λ

$$-Det\,(L) = Det\,(\Lambda)^2 = L_{00}^4 > 0\,. \qquad (3.8.22)$$

Assume that $L_{00} < 0$ and define the regular matrix $\bar{\Lambda} = \frac{1}{\sqrt{|L_{00}|}}\,\Lambda$. Then

$$\bar{\Lambda}^T\, G\, \bar{\Lambda} = -G\,, \qquad (3.8.23)$$

and it is possible to choose four real numbers $\tilde{x} = (x^0, x^1, x^2, x^3)^T$ according to

$$x^0 = 0\,,\; \bar{\Lambda}^0{}_1\, x^1 + \bar{\Lambda}^0{}_2\, x^2 + \bar{\Lambda}^0{}_3\, x^3 = 0\,,\; (x^1)^2 + (x^2)^2 + (x^3)^2 > 0\,. \qquad (3.8.24)$$

Let be $\tilde{y} = \bar{\Lambda}\,\tilde{x} = (y^0, y^1, y^2, y^3)^T$. Due to (3.8.23) and $x^0 = 0$ we obtain

$$\tilde{y}^T\, G\, \tilde{y} = \tilde{x}^T\, (\bar{\Lambda}^T\, G\, \bar{\Lambda})\, \tilde{x}^T = -\tilde{x}^T\, G\, \tilde{x}^T = (x^1)^2 + (x^2)^2 + (x^3)^2 > 0\,. \qquad (3.8.25)$$

On the other hand

$$y^0 = 0\,, \qquad \tilde{y} \neq 0$$

due to (3.8.24) and the regularity of $\bar{\Lambda}$, and therefore

$$\tilde{y}^T G\, \tilde{y} = (y^0)^2 - (y^1)^2 - (y^2)^2 - (y^3)^2 = \\ - \left((y^1)^2 + (y^2)^2 + (y^3)^2\right) < 0\,, \qquad (3.8.26)$$

which contradicts (3.8.25).

We have proved that $\Lambda^T G\, \Lambda = L_{00}\, G$ with $L_{00} > 0$, so that $\frac{1}{\sqrt{L_{00}}}\, \Lambda$ is the desired Lorentz-matrix.

Thus we have proved Propositon 3.5. ∎

Proposition 3.6. *(a) The Lorentz-matrices form a group, the so called homogeneous Lorentz-group L_{hom}.*

(b) If Λ is a Lorentz-matrix, then $Det(\Lambda) = \pm 1$ and $|\Lambda^0{}_0| \geq 1$. The set of the so called **proper** *Lorentz-matrices with $Det(\Lambda) = +1$ and $\Lambda^0{}_0 \geq 1$ form a subgroup L^+_{hom} of the homogeneous Lorentz-group, the* **proper homogeneous Lorentz-group**.

(c) $\mathbf{1}$, G, $-G$, $-\mathbf{1} \in \mathbb{R}^{4\times 4}$ are Lorentz-matrices, which are called identity, space-inversion, time-inversion and space-time-inversion, respectively. For any $\Lambda \in L_{hom}$ there is a matrix $A \in L^+_{hom}$ such that Λ has the following representations

$$\begin{aligned}
&\Lambda = \mathbf{1}\, A = A\, \mathbf{1}\,, && \text{if} \quad \Lambda^0{}_0 \geq +1 \;\&\; Det(\Lambda) = +1\,,\\
&\Lambda = G\, A = (G\, A\, G)\, G\,, && \text{if} \quad \Lambda^0{}_0 \geq +1 \;\&\; Det(\Lambda) = -1\,,\\
&\Lambda = (-G)\, A = (G\, A\, G)\, (-G)\,, && \text{if} \quad \Lambda^0{}_0 \leq -1 \;\&\; Det(\Lambda) = -1\,,\\
&\Lambda = (-\mathbf{1})\, A = A\, (-\mathbf{1})\,, && \text{if} \quad \Lambda^0{}_0 \leq -1 \;\&\; Det(\Lambda) = +1\,.
\end{aligned}$$

Finally, if $A \in L^+_{hom}$, then also $G\, A\, G \in L^+_{hom}$.

Proof: (a) The unit matrix $\mathbf{1}$ is a Lorentz-matrix. In the following we make use of Proposition 3.4: Let Λ be a Lorentz-matrix. Then Λ is regular due to $\Lambda^T G\, \Lambda = G$, which implies $Det(\Lambda) = \pm 1$. Moreover, $Q(\Lambda \tilde{x}) = Q(\tilde{x})$ for every $\tilde{x} \in \mathbb{R}^4$ implies $Q(\tilde{y}) = Q(\Lambda^{-1}\tilde{y})$ for every $\tilde{y} = \Lambda \tilde{x}$ in $\mathbb{R}^4$, i.e. Λ^{-1} is a Lorentz-matrix. Finally, if Λ_1, Λ_2 are Lorentz-matrices, then also $\Lambda_3 = \Lambda_1\, \Lambda_2$ due to

$$Q(\Lambda_3 \tilde{x}) = Q(\Lambda_1(\Lambda_2 \tilde{x})) = Q(\Lambda_2 \tilde{x}) = Q(\tilde{x})$$

for every $\tilde{x} \in \mathbb{R}^4$.
(b) Let Λ be a Lorentz-matrix. We have already seen that $Det(\Lambda) = \pm 1$. We evaluate the 00-component of the matrix equation $\Lambda^T G \Lambda = G$ in order to conclude that

$$(\Lambda^0{}_0)^2 = 1 + (\Lambda^1{}_0)^2 + (\Lambda^2{}_0)^2 + (\Lambda^3{}_0)^2 \geq 1\,,$$

which shows that $|\Lambda^0{}_0| \geq 1$. Next we have to prove that L^+_{hom} forms a subgroup:
First we note that $\mathbf{1} \in L^+_{hom}$.
Assume that $\Lambda^0{}_0 \geq 1$ and $Det(\Lambda) = +1$ for a Lorentz-matrix Λ. We first note that $Det(\Lambda^{-1}) = +1$. Due to Proposition 3.4 we have $\Lambda^{-1} = G\,\Lambda^T\,G$, and evaluating the 00-component of this matrix equation we obtain

$$(\Lambda^{-1})^0{}_0 = \Lambda^0{}_0 \geq 1\,,$$

i.e. $\Lambda^{-1} \in L^+_{hom}$. Let A, B be Lorentz-matrices with $Det(A) = +1$, $A^0{}_0 \geq 1$, $Det(B) = +1$, $B^0{}_0 \geq 1$. Then $Det(A\,B) = Det(A)\,Det(B) = +1$, and we obtain from the Cauchy-Schwarz inequality

$$\begin{aligned}(A\,B)^0{}_0 &= A^0{}_0\,B^0{}_0 + A^0{}_1\,B^1{}_0 + A^0{}_2\,B^2{}_0 + A^0{}_3\,B^3{}_0 \\ &\geq A^0{}_0\,B^0{}_0 - \sqrt{\sum_{k=1}^{3}(A^0{}_k)^2}\,\sqrt{\sum_{k=1}^{3}(B^k{}_0)^2}\,.\end{aligned}$$

But the last expression is ≥ 0, which can be seen from the evaluation

$$A^0{}_0 \geq \sqrt{\sum_{k=1}^{3}(A^0{}_k)^2}$$

of the 00-component of $A\,G\,A^T = G$ and from the evaluation

$$B^0{}_0 \geq \sqrt{\sum_{k=1}^{3}(B^k{}_0)^2}$$

of the 00-component of $B^T\,G\,B = G$. Thus $A\,B \in L^+_{hom}$.
(c) may be proved by a very simple straight forward calculation. ∎

The conditions $\Lambda^0{}_0 \geq 1$ and $\det\Lambda = +1$ for the proper Lorentz-matrices are necessary in order to exclude inversions in time and space. The (not necessarily homogeneous) Lorentz-transformation (3.8.12) is called **proper** if the Lorentz-matrix Λ is proper. Due to Proposition 3.6 it is sufficient to construct the proper Lorentz-matrices $\Lambda \in L^+_{hom}$ in order to obtain any other Lorentz-matrix. This will be done next:

We restrict to proper, homogeneous Lorentz-transformations and rewrite any event $\tilde{x} \in \mathbb{R}^4$ as a Hermitean 2×2 matrix $\tilde{X}$, see the textbook of Streater and Wightman [46], according to

$$\tilde{x} = (x^0, x^1, x^2, x^3)^T \;\rightarrow\; \tilde{X} = \begin{pmatrix} x^0 + x^3 & x^1 - i\,x^2 \\ x^1 + i\,x^2 & x^0 - x^3 \end{pmatrix}\,. \tag{3.8.27}$$

Then we choose any complex matrix M and their adjoint M^+

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad M^+ = \begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{pmatrix}$$

with $Det(M) = +1$, which maps the Hermitean 2×2 matrix $\tilde{X}$ into another Hermitean 2×2 matrix $\tilde{X}'$ according to

$$\tilde{X}' = M\, \tilde{X}\, M^+ . \tag{3.8.28}$$

Afterwards we construct from the transformed Hermitean matrix $\tilde{X}'$ the corresponding event $\tilde{x}'$:

$$\tilde{X}' = \begin{pmatrix} x'^0 + x'^3 & x'^1 - i\, x'^2 \\ x'^1 + i\, x'^2 & x'^0 - x'^3 \end{pmatrix} \;\rightarrow\; \tilde{x}' = (x'^0, x'^1, x'^2, x'^3)^T . \tag{3.8.29}$$

From (3.8.27), (3.8.28) and (3.8.29) we obtain that the transformation $\tilde{x} \rightarrow \tilde{x}'$ of the events is linear and homogeneous and leaves the Einstein-Minkowski metric invariant:

$$Det(\tilde{X}') = Q(\tilde{x}') = Det(\tilde{X}) = Q(\tilde{x}) . \tag{3.8.30}$$

By a straight forward calculation we identify the following components of the Lorentz-matrix Λ, which maps $\tilde{x}$ on $\tilde{x}' = \Lambda\, \tilde{x}$:

$$\begin{aligned}
\Lambda^0{}_0 &= \frac{1}{2}(\alpha\bar{\alpha} + \beta\bar{\beta} + \gamma\bar{\gamma} + \delta\bar{\delta}) , & \Lambda^0{}_1 &= \frac{1}{2}(\alpha\bar{\beta} + \beta\bar{\alpha} + \gamma\bar{\delta} + \delta\bar{\gamma}) , \\
\Lambda^0{}_2 &= \frac{i}{2}(\beta\bar{\alpha} - \alpha\bar{\beta} + \delta\bar{\gamma} - \gamma\bar{\delta}) , & \Lambda^0{}_3 &= \frac{1}{2}(\alpha\bar{\alpha} - \beta\bar{\beta} + \gamma\bar{\gamma} - \delta\bar{\delta}) , \\
\\
\Lambda^1{}_0 &= \frac{1}{2}(\gamma\bar{\alpha} + \alpha\bar{\gamma} + \delta\bar{\beta} + \beta\bar{\delta}) , & \Lambda^1{}_1 &= \frac{1}{2}(\alpha\bar{\delta} + \delta\bar{\alpha} + \beta\bar{\gamma} + \gamma\bar{\beta}) , \\
\Lambda^1{}_2 &= \frac{i}{2}(\delta\bar{\alpha} - \alpha\bar{\delta} + \beta\bar{\gamma} - \gamma\bar{\beta}) , & \Lambda^1{}_3 &= \frac{1}{2}(\gamma\bar{\alpha} + \alpha\bar{\gamma} - \delta\bar{\beta} - \beta\bar{\delta}) , \\
\\
\Lambda^2{}_0 &= \frac{i}{2}(\beta\bar{\delta} - \delta\bar{\beta} - \gamma\bar{\alpha} + \alpha\bar{\gamma}) , & \Lambda^2{}_1 &= \frac{i}{2}(\beta\bar{\gamma} - \gamma\bar{\beta} + \alpha\bar{\delta} - \delta\bar{\alpha}) , \\
\Lambda^2{}_2 &= \frac{1}{2}(\alpha\bar{\delta} + \delta\bar{\alpha} - \beta\bar{\gamma} - \gamma\bar{\beta}) , & \Lambda^2{}_3 &= \frac{i}{2}(\alpha\bar{\gamma} - \gamma\bar{\alpha} + \delta\bar{\beta} - \beta\bar{\delta}) , \\
\\
\Lambda^3{}_0 &= \frac{1}{2}(\alpha\bar{\alpha} + \beta\bar{\beta} - \gamma\bar{\gamma} - \delta\bar{\delta}) , & \Lambda^3{}_1 &= \frac{1}{2}(\beta\bar{\alpha} + \alpha\bar{\beta} - \delta\bar{\gamma} - \gamma\bar{\delta}) , \\
\Lambda^3{}_2 &= \frac{i}{2}(\beta\bar{\alpha} - \alpha\bar{\beta} - \delta\bar{\gamma} + \gamma\bar{\delta}) , & \Lambda^3{}_3 &= \frac{1}{2}(\alpha\bar{\alpha} - \beta\bar{\beta} - \gamma\bar{\gamma} + \delta\bar{\delta}) .
\end{aligned}$$

Since ${\Lambda^0}_0 \geq 0$ immediately implies ${\Lambda^0}_0 \geq 1$ for the Lorentz-matrix Λ, we obtain from these representations

$$ {\Lambda^0}_0 = \frac{1}{2}(\alpha\bar{\alpha} + \beta\bar{\beta} + \gamma\bar{\gamma} + \delta\bar{\delta}) \geq 1\,, \tag{3.8.31} $$

which is also an immediate consequence of $Det(M) = \alpha\delta - \beta\gamma = 1$ and the inequality $(\alpha - \bar{\delta})(\bar{\alpha} - \delta) + (\beta + \bar{\gamma})(\bar{\beta} + \gamma) \geq 0$. In order to show that in addition $Det(\Lambda) = +1$, i.e. $\Lambda = \Lambda(M) \in L^+_{hom}$, we apply the following continuity argument: Starting with the complex 2×2 unit matrix $\mathbf{1}$ in order to generate $\Lambda = \mathbf{1} \in \mathbb{R}^{4\times 4}$ with determinant 1, we may continuously change the components of M in order to reach any other complex 2×2 matrix with determinant 1. Then the corresponding Lorentz-matrix $\Lambda(M)$ cannot jump from $Det(\Lambda(M)) = +1$ to $Det(\Lambda(M)) = -1$.

The next proposition characterizes the spatial rotations, which form a subgroup of L^+_{hom}.

Proposition 3.7. *(a) Let $R \in \mathbb{R}^{3\times 3}$ be a spatial rotation, i.e.*

$$ R\,R^T = \mathbf{1} \in \mathbb{R}^{3\times 3}\,, \qquad Det(R) = +1\,. $$

Let $(\varphi_1, \varphi_2, \varphi_3)^T$ be the vector in direction of the rotational axis of R whose absolute value

$$ \varphi = \sqrt{\varphi_1^2 + \varphi_2^2 + \varphi_3^2} > 0 $$

is the positive oriented rotational angle of R, i.e. positive oriented with respect to $(\varphi_1, \varphi_2, \varphi_3)^T$. For the antisymmetric 3×3 matrix

$$ A = \begin{pmatrix} 0 & -\varphi_3 & +\varphi_2 \\ +\varphi_3 & 0 & -\varphi_1 \\ -\varphi_2 & +\varphi_1 & 0 \end{pmatrix}\,, $$

there results

$$ A^2 + \varphi^2\,\mathbf{1} = \begin{pmatrix} \varphi_1\,\varphi_1 & \varphi_1\,\varphi_2 & \varphi_1\,\varphi_3 \\ \varphi_2\,\varphi_1 & \varphi_2\,\varphi_2 & \varphi_2\,\varphi_3 \\ \varphi_3\,\varphi_1 & \varphi_3\,\varphi_2 & \varphi_3\,\varphi_3 \end{pmatrix}\,, \quad A^3 + \varphi^2\,A = \mathbf{0}\,, $$

and the rotational matrix R has the following representation:

$$ R = \exp(A) = \mathbf{1} + \frac{\sin\varphi}{\varphi}\,A + \frac{1 - \cos\varphi}{\varphi^2}\,A^2\,. $$

For $\mathbf{R} = \mathbf{1}$ we set $A = \mathbf{0}$ with $\mathbf{R} = \exp(A)$.

(b) Let be $\Lambda \in L^+_{hom}$. Then $\Lambda^T \Lambda = \mathbf{1} \Leftrightarrow \Lambda^0{}_0 = 1$. $\Lambda^T \Lambda = \mathbf{1}$ is a necessary and sufficient condition for Λ to describe a spatial rotation. Let us assume that Λ satisfies this condition. Then there exists a purely spatial rotation $R = R^m{}_n \in \mathbb{R}^{3\times 3}$ such that $\tilde{x}' = \Lambda\, \tilde{x}$ is given by

$$x'^0 = x^0\,, \qquad x'^k = \sum_{j=1}^{3} R^k{}_j\, x^j\,,$$

and the Lorentz-matrix Λ itself is called a rotation. Let $(\varphi_1, \varphi_2, \varphi_3)^T$ be the angle vector of R as described above. Then Λ is generated by the following unitary 2×2 matrix with determinante 1:

$$M = \cos(\varphi/2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\, \frac{\sin(\varphi/2)}{\varphi} \begin{pmatrix} \varphi_3 & \varphi_1 - i\varphi_2 \\ \varphi_1 + i\varphi_2 & -\varphi_3 \end{pmatrix}. \quad (3.8.32)$$

Remark: The unitary matrix M in Proposition 3.7 b which generates the rotation Λ may also be written in the exponential form

$$M = \exp\left(-i/2 \begin{pmatrix} \varphi_3 & \varphi_1 - i\varphi_2 \\ \varphi_1 + i\varphi_2 & -\varphi_3 \end{pmatrix} \right). \quad (3.8.33)$$

Proof of Proposition 3.7: By straight forward calculations. ∎

The next proposition gives a representation for the generalization of the Lorentz-transformation in (3.8.13).

Proposition 3.8. *Let us consider $\mathbf{v} = (v_1, v_2, v_3)^T \in \mathbb{R}^3$ with absolute value less then speed of light and put*

$$\lambda = \lambda(\mathbf{v}) = \frac{1}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}}\,, \quad \mathbf{u} = \frac{\lambda(\mathbf{v})}{c}\, \mathbf{v} = (u^1, u^2, u^3)^T\,.$$

Then we define the components $\Lambda^\mu{}_\nu$ of the Lorentz-boost $\Lambda = \Lambda(\mathbf{u})$ by

$$\Lambda^0{}_0 = \sqrt{1 + \mathbf{u}^2} = \lambda(\mathbf{v})\,, \;\; \Lambda^0{}_j = \Lambda^j{}_0 = -u^j\,,$$
$$\Lambda^j{}_k = \delta^{j\,k} + \frac{u^j u^k}{1 + \sqrt{1 + \mathbf{u}^2}}\,, \quad j,k = 1,2,3\,.$$

The matrix $\Lambda = \Lambda(\mathbf{u})$ is generated by the Hermitean 2×2-matrix M

$$M = \begin{pmatrix} \frac{1+(1-v_3/c)\,\lambda}{\sqrt{2+2\lambda}} & \frac{\lambda}{\sqrt{2+2\lambda}}\, \frac{-v_1 + i\, v_2}{c} \\ \frac{\lambda}{\sqrt{2+2\lambda}}\, \frac{-v_1 - i\, v_2}{c} & \frac{1+(1+v_3/c)\,\lambda}{\sqrt{2+2\lambda}} \end{pmatrix}$$

with determinant 1 and has the inverse matrix $\Lambda(\mathbf{u})^{-1} = \Lambda(-\mathbf{u})$.

Remarks:

(i) For $v = v_1$ and $v_2 = v_3 = 0$ we obtain the transformation (3.8.13).

(ii) The Hermitean 2×2 matrix M which generates the Lorentz-matrix $\Lambda(\mathbf{u})$ may also be written in the exponential form

$$M = \exp\left(-\kappa \begin{pmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{pmatrix}\right) \tag{3.8.34}$$

for the constant $\kappa = \dfrac{1}{c} \dfrac{\lambda}{\sqrt{\lambda^2 - 1}} \ln\left(\sqrt{\dfrac{\lambda+1}{2}} + \sqrt{\dfrac{\lambda-1}{2}}\right)$.

It is also important to note that the set of all Lorentz-boosts $\Lambda(\mathbf{u})$ with $\mathbf{u} \in \mathbb{R}^3$ does not form a subgroup of L^+_{hom} like the rotations.

(iii) If $|\mathbf{v}|$ is very small compared to c, then $\Lambda(\mathbf{u})$ reduces to the special Galilean transformation

$$t' = t\,,\ \mathbf{x}' = \mathbf{x} - t\,\mathbf{v}\,. \tag{3.8.35}$$

We also omit the tedious proof of the straight forward calculations. ∎

Proposition 3.9. *Any Lorentz-matrix $\Lambda \in L^+_{hom}$ may be written in the form*

$$\Lambda = \Lambda_S\,\Lambda_R\,, \tag{3.8.36}$$

where $\Lambda_S \in L^+_{hom}$ is the Lorentz-matrix $\Lambda_S = \Lambda(\mathbf{u})$ with the velocity

$$\mathbf{v} = c\,\frac{\mathbf{u}}{\sqrt{1+\mathbf{u}^2}} = c\,\left(-\frac{\Lambda^1{}_0}{\Lambda^0{}_0}, -\frac{\Lambda^2{}_0}{\Lambda^0{}_0}, -\frac{\Lambda^3{}_0}{\Lambda^0{}_0}\right)^T \tag{3.8.37}$$

less then speed of light and $\Lambda_R \in L^+_{hom}$ is a rotation. If M_S and M_R are the corresponding complex 2×2 matrices which generate Λ_S and Λ_R, respectively, then

$$M = M_S\,M_R$$

generates the Lorentz-matrix Λ. M has determinant 1.

Proof: The matrix Λ_S with the velocity $\mathbf{v}$ less then speed of light given in (3.8.37) is a member of L^+_{hom} due to Proposition 3.8. Using Proposition 3.6(b) we conclude that the matrix $\Lambda_R := \Lambda_S^{-1}\Lambda$ is also a member of L^+_{hom}. Due to Proposition 3.7(b) it is sufficient to show that the 00-component of Λ_R is equal to 1, which is an easy task. The corresponding matrices M_S and M_R may be constructed due to Propositions 3.7 and 3.8. ∎

Remark: If we apply this representation theorem to Λ^{-1} instead of Λ and take the inverse matrix, we conclude that Λ may also be written in the form

$$\Lambda = \Lambda'_R\,\Lambda(\mathbf{u}') \tag{3.8.38}$$

with a new rotation Λ'_R and a new $\mathbf{u}'$.

"Es ist vielfach versucht worden, in unserem Gebiet eine solche invariante, mit den Tensoren selbst und nicht mit ihren Komponenten arbeitende Bezeichnungsweise auszubilden, wie sie in der Vektorrechnung besteht. Was aber dort am Platze ist, erweist sich für den viel weiter gespannten Rahmen des Tensorkalküls als äußerst unzweckmäßig. Es werden eine solche Fülle von Namen, Bezeichnungen und ein solcher Apparat von Rechenregeln nötig (wenn man nicht doch immer wieder auf die Komponenten zurückgreifen will), daß damit ein Gewinn von sehr erheblichem negativem Betrag erreicht wird. Man muß gegen diese Orgien des Formalismus, mit dem man heute sogar die Techniker zu belästigen beginnt, nachdrücklich protestieren."

(Hermann Weyl, Raum-Zeit-Materie, 1918)

3.9 Relativistic formulation of electrodynamics

In the last section we have studied the Lorentz-transformations which leave the wave equation invariant. Now we develope the corresponding Lorentz-invariant tensor calculus which enables us to recognize at a glance that an equation is Lorentz-invariant. Then we will recover the four-vectors and four-tensors that describe the electromagnetic phenomena. Using these quantities the electromagnetic laws, namely Maxwell's equations and the conservation laws, may be written in a compact tensor notation.

The tensor calculus: In order to formulate the Lorentz-invariant tensor calculus, we make use of the notations used in the textbook of Weinberg [48], with only slight modifications.

A *contravariant* four-vector V^μ is a quantity with a single upper index which undergoes like dx^μ in (3.8.15) the following linear and homogeneous transformation with respect to a proper Lorentz-transformation (3.8.12):

$$V^\mu(\tilde{x}) \to V'^\mu(\tilde{x}') = \Lambda^\mu{}_\nu \, V^\nu(\tilde{x}) \,. \tag{3.9.1}$$

Then we define the matrix $\Lambda_\mu{}^\nu$ by

$$\Lambda_\mu{}^\nu = g_{\mu\kappa} \, g^{\nu\lambda} \, \Lambda^\kappa{}_\lambda \tag{3.9.2}$$

and state that $\Lambda_\mu{}^\nu$ is just the inverse of the matrix $\Lambda^\nu{}_\mu$:

$$\Lambda_\mu{}^\alpha \, \Lambda^\mu{}_\beta = g_{\mu\gamma} \, g^{\alpha\delta} \, \Lambda^\gamma{}_\delta \, \Lambda^\mu{}_\beta = g^{\alpha\delta} \, g_{\delta\beta} = \delta^\alpha{}_\beta = \begin{cases} +1 & , \quad \alpha = \beta = 0, 1, 2, 3 \,, \\ 0 & , \quad \alpha \neq \beta \,. \end{cases} \tag{3.9.3}$$

A *covariant* four-vector U_μ is a quantity with a single lower index which undergoes the following linear and homogeneous transformation with respect to a proper Lorentz transformation (3.8.12):

$$U_\mu(\tilde{x}) \to U'_\mu(\tilde{x}') = \Lambda_\mu{}^\kappa \, U_\kappa(\tilde{x}) \,. \tag{3.9.4}$$

The scalar product of a contravariant vector V^μ with a covariant vector U_μ is *invariant* with respect to (3.8.12):

$$V^\mu \, U_\mu \to V'^\mu \, U'_\mu = \Lambda^\mu{}_\nu \, \Lambda_\mu{}^\kappa \, V^\nu \, U_\kappa = \delta^\kappa{}_\nu \, V^\nu \, U_\kappa = V^\kappa \, U_\kappa \,. \tag{3.9.5}$$

A co- or contravariant vector is a *tensor with one index*, and a *scalar* (invariant expression) is a *tensor without indices*.

In general a *tensor* has several upper contravariant and/or lower covariant indices and transforms linear and homogeneous with respect to a proper Lorentz-transformation (3.8.12), for example

$$T^{\alpha\ \gamma\delta}_{\ \beta\ \ \varepsilon}(\tilde{x}) \to T'^{\alpha\ \gamma\delta}_{\ \beta\ \ \varepsilon}(\tilde{x}') = \Lambda^\alpha{}_\kappa \, \Lambda_\beta{}^\lambda \, \Lambda^\gamma{}_\mu \, \Lambda^\delta{}_\nu \, \Lambda_\varepsilon{}^\xi \; T^{\kappa\ \mu\nu}_{\ \lambda\ \ \xi}(\tilde{x}) \,. \tag{3.9.6}$$

Important tensors are the

- *zero tensor*, whose components are zero in any reference frame for an arbitrary but fixed combination of upper and lower indices,
- *metric tensor*, which transforms according to (3.8.11)

$$g_{\mu\nu} \to g'_{\mu\nu} = \Lambda_\mu{}^\kappa \, \Lambda_\nu{}^\lambda \, g_{\kappa\lambda} = g_{\mu\nu} \,, \tag{3.9.7}$$

- *Kronecker tensor* introduced in (3.9.3)

$$\delta'^\alpha_{\ \beta} = \Lambda^\alpha{}_\kappa \, \Lambda_\beta{}^\lambda \, \delta^\kappa{}_\lambda = \Lambda^\alpha{}_\kappa \, \Lambda_\beta{}^\kappa = \delta^\alpha{}_\beta \,, \tag{3.9.8}$$

- *Levi-Civita tensor*

$$\epsilon_{\alpha\beta\gamma\delta} = \begin{cases} +1 \;\; , \;\; \alpha\beta\gamma\delta \text{ even permutation of } 0123, \\ -1 \;\; , \;\; \alpha\beta\gamma\delta \text{ odd permutation of } 0123, \\ \;\; 0 \;\; , \;\; \text{otherwise.} \end{cases}$$

  For the Levi-Civita tensor the restriction to proper Lorentz-transformations is necessary. Note that in the textbook of Weinberg [48] the Levi-Civita tensor as well as the metric tensor both take the sign opposite to the notation used here.

The components of these tensors are the same in every reference frame.

The **tensor calculus** itself is very nice, since it consists of only **four rules** in order to form new tensors from old ones. These rules may be combined with each others under certain constraints to obtain every possible tensor.

Rule A: Linear combinations
A linear combination of two tensors R and S with the same upper and lower indices is a new tensor T with these indices, for example let be

$$T_{\alpha}^{\ \beta} = a\, R_{\alpha}^{\ \beta} + b\, S_{\alpha}^{\ \beta}$$

with given tensors $R_{\alpha}^{\ \beta}$, $S_{\alpha}^{\ \beta}$. Then $T_{\alpha}^{\ \beta}$ is also a tensor according to

$$\begin{aligned} T'_{\ \alpha}^{\ \beta} &= a\, R'_{\ \alpha}^{\ \beta} + b\, S'_{\ \alpha}^{\ \beta} \\ &= a\, \Lambda_{\alpha}^{\ \kappa}\, \Lambda^{\beta}_{\ \lambda}\, R_{\kappa}^{\ \lambda} + b\, \Lambda_{\alpha}^{\ \kappa}\, \Lambda^{\beta}_{\ \lambda}\, S_{\kappa}^{\ \lambda} \\ &= \Lambda_{\alpha}^{\ \kappa}\, \Lambda^{\beta}_{\ \lambda}\, T_{\kappa}^{\ \lambda}\,. \end{aligned}$$

Rule B: Direct products
The product of two tensors R and S is a new tensor T whose upper and lower indices consists of all the upper and lower indices of the original tensors, for example let be

$$T^{\alpha}_{\ \beta}{}^{\gamma} = R^{\alpha}\, S_{\beta}^{\ \gamma}$$

with given tensors R^{α}, $S_{\beta}^{\ \gamma}$. Then $T^{\alpha}_{\ \beta}{}^{\gamma}$ is also a tensor according to

$$\begin{aligned} T'^{\alpha}_{\ \ \beta}{}^{\gamma} &= R'^{\alpha}\, S'_{\ \beta}^{\ \gamma} \\ &= \Lambda^{\alpha}_{\ \kappa}\, R^{\kappa}\ \Lambda_{\beta}^{\ \lambda}\, \Lambda^{\gamma}_{\ \mu}\, S_{\lambda}^{\ \mu} \\ &= \Lambda^{\alpha}_{\ \kappa}\, \Lambda_{\beta}^{\ \lambda}\, \Lambda^{\gamma}_{\ \mu}\, T^{\kappa}_{\ \lambda}{}^{\mu}. \end{aligned}$$

Rule C: Contractions
Setting an upper and lower index of a tensor equal and summing it over its values 0,1,2,3 we obtain a new tensor without these two indices. For example, let $T^{\alpha}_{\ \beta\gamma\delta}$ be a tensor. Then

$$T_{\beta\delta} = T^{\alpha}_{\ \beta\alpha\delta}$$

is also a tensor according to

$$\begin{aligned} T'_{\beta\delta} &= T'^{\alpha}_{\ \ \beta\alpha\delta} \\ &= \Lambda^{\alpha}_{\ \kappa}\, \Lambda_{\beta}^{\ \lambda}\, \Lambda_{\alpha}^{\ \mu}\, \Lambda_{\delta}^{\ \nu}\, T^{\kappa}_{\ \lambda\mu\nu} \\ &= \delta^{\mu}_{\ \kappa}\, \Lambda_{\beta}^{\ \lambda}\, \Lambda_{\delta}^{\ \nu}\, T^{\kappa}_{\ \lambda\mu\nu} \\ &= \Lambda_{\beta}^{\ \lambda}\, \Lambda_{\delta}^{\ \nu}\, T^{\mu}_{\ \lambda\mu\nu} \\ &= \Lambda_{\beta}^{\ \lambda}\, \Lambda_{\delta}^{\ \nu}\, T_{\ \lambda\nu}\,. \end{aligned}$$

Rule D: Differentiation
Differentiation of a tensor with respect to x^α yields a tensor with an additional covariant index α. For example, let $T^\beta_{\ \gamma}$ be a tensor and define

$$T_{\alpha\ \gamma}^{\ \beta} = \frac{\partial T^\beta_{\ \gamma}}{\partial x^\alpha}\,.$$

Then $T_\alpha^{\ \beta}{}_\gamma$ is also a tensor according to

$$\begin{aligned} T'^{\ \beta}_{\alpha\ \gamma} &= \frac{\partial T'^\beta_{\ \gamma}}{\partial x'^\alpha} \\ &= \Lambda_\alpha^{\ \kappa}\,\frac{\partial}{\partial x^\kappa}\left(\Lambda^\beta_{\ \lambda}\,\Lambda_\gamma^{\ \mu}\,T^\lambda_{\ \mu}\right) \\ &= \Lambda_\alpha^{\ \kappa}\,\Lambda^\beta_{\ \lambda}\,\Lambda_\gamma^{\ \mu}\,\frac{\partial T^\lambda_{\ \mu}}{\partial x^\kappa}\,. \end{aligned}$$

Finally we mention some important combinations of these rules:

(a) *Lowering of contravariant and raising of covariant indices*
(contractions of direct products with the metric tensor),
for example the first index of a tensor $T^\alpha_{\ \beta}$ may be lowered according to

$$T_{\alpha\beta} = g_{\alpha\kappa}\,T^\kappa_{\ \beta}\,.$$

Raising the same index afterwards according to

$$T^\alpha_{\ \beta} = g^{\alpha\kappa}\,T_{\kappa\beta}$$

gives back the original tensor. This also explaines the two original notations for $g^{\mu\nu} = g_{\mu\nu}$.

(b) *The scalar product* $V^\alpha\,U_\alpha$ results from the contraction of the direct product $V^\alpha\,U_\beta$, which shows once again its Lorentz-invariance.

(c) *The covariant divergence*

$$\frac{\partial V^\alpha}{\partial x^\alpha}$$

of a contravariant vector V^α (Rules C and D) is Lorentz- invariant.

(d) *The wave operator* (3.8.4), acting on an arbitrary tensor T, is Lorentz-invariant (Rules B, C, D), in agreement with Proposition 3.4.

If we replace the time-space arguments $(t, \mathbf{x})$ of the fields by the four-quantity $\tilde{x} = (x^0, x^1, x^2, x^3)^T$ according to (3.8.1), we can define the so called (electromagnetic) four-potential $A^\mu(\tilde{x})$ according to

$$\begin{aligned} A^0(\tilde{x}) = \varphi(\frac{x^0}{c}, \mathbf{x})\,,\ A^1(\tilde{x}) = A_x(\frac{x^0}{c}, \mathbf{x})\,, \\ A^2(\tilde{x}) = A_y(\frac{x^0}{c}, \mathbf{x})\,,\ A^3(\tilde{x}) = A_z(\frac{x^0}{c}, \mathbf{x})\,, \end{aligned} \tag{3.9.10}$$

where $\mathbf{A}(t,\mathbf{x}) = (A_x(t,\mathbf{x}), A_y(t,\mathbf{x}), A_z(t,\mathbf{x}))^T$ is the vector potential and $\varphi(t,\mathbf{x})$ the scalar potential, respectively, which are related to the electromagnetic field by the conditions (3.3.3) and (3.3.5). If we change from one Lorentz-frame to another according to the Lorentz-transformation (3.8.12) for a proper Lorentz-matrix Λ, then we require the following

First Postulate: The four-potential A^μ transforms like a contravariant four-vector, i.e. we require for each proper Lorentz-transformation (3.8.12):

$$A'^\mu(\tilde{x}') = \Lambda^\mu{}_\nu \, A^\nu(\tilde{x}) \,. \tag{3.9.11}$$

This postulate is compatible with the "gauge transformations" (3.3.8) and (3.3.9), because we can rewrite (3.3.8) and (3.3.9) in the following Lorentz-invariant form for a Lorentz-invariant scalar function $\lambda = \lambda(\tilde{x})$:

$$\tilde{A}^\mu(\tilde{x}) = A^\mu(\tilde{x}) - g^{\mu\nu} \frac{\partial \lambda}{\partial x^\nu}(\tilde{x}) \,.$$

The advantadge of definition (3.9.10) and the First Postulate can be seen when we calculate the antisymmetric tensor $F_{\mu\nu}$ defined by

$$F_{\mu\nu}(\tilde{x}) = \frac{\partial A_\nu}{\partial x^\mu}(\tilde{x}) - \frac{\partial A_\mu}{\partial x^\nu}(\tilde{x}) \,. \tag{3.9.12}$$

Recall that the covariant four-vector $A_\mu = g_{\mu\nu} A^\nu$ differs from A^μ only in its spatial components, which have the opposite sign.

This tensor is called the **electromagnetic field tensor** because it comes out that

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} . \tag{3.9.13}$$

For the contravariant tensor $F^{\mu\nu} = g^{\mu\kappa} g^{\nu\lambda} F_{\kappa\lambda}$ we obtain the representation

$$(F^{\mu\nu}) = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} . \tag{3.9.14}$$

If we change to another Lorentz frame according to (3.8.12) then the transformation rules for these tensors are

$$F'_{\mu\nu}(\tilde{x}') = \Lambda_\mu{}^\kappa \Lambda_\nu{}^\lambda F_{\kappa\lambda}(\tilde{x}) \,, \qquad F'^{\mu\nu}(\tilde{x}') = \Lambda^\mu{}_\kappa \Lambda^\nu{}_\lambda F^{\kappa\lambda}(\tilde{x}) \,. \tag{3.9.15}$$

Now we recognize that definitions (3.9.10), (3.9.12) and equation (3.9.13) have exactly the same meaning as the conditions (3.3.3) and (3.3.5) for the vector potential and the scalar potential in Section 3.3. Therefore we know that Maxwell's equations for $\mathbf{E}$ and $\mathbf{B}$ are equivalent to the conditions (3.3.6), (3.3.7) which now take the form

$$\Box A^\mu - g^{\mu\nu} \frac{\partial}{\partial x^\nu} \left(\frac{\partial A^\kappa}{\partial x^\kappa} \right) = \frac{1}{c} j^\mu \,, \tag{3.9.16}$$

where the so called electromagnetic four-density j^μ is defined by

$$j^0 = c\,\rho, \qquad (j^1, j^2, j^3)^T = \mathbf{j} \,. \tag{3.9.17}$$

The divergence of (3.9.16) again gives the conservation of charge,

$$\frac{\partial j^\mu}{\partial x^\mu} = 0 \,. \tag{3.9.18}$$

Since the left hand side in (3.9.16) is already a four-vector due to the First Postulate, we see that Maxwell's equations are Lorentz-invariant if and only if we require the

Second Postulate: The electromagnetic four-density j^μ transforms like a four-vector with respect to proper Lorentz-transformations (3.8.12).

Recall that we have the freedom to impose additionally the Lorentz-condition on the electromagnetic potentials in order to obtain decoupled wave equations for the electromagnetic potential, i.e. we can replace (3.9.16) by

$$\frac{\partial A^\kappa}{\partial x^\kappa} = 0 \,, \qquad \Box A^\mu = \frac{1}{c} j^\mu \,. \tag{3.9.19}$$

Here we should stop for a moment and develop an alternative way to obtain the Lorentz invariance of Maxwell's equations without making any use of the electromagnetic four-potential. This is important, since the electromagnetic potential itself is not measurable and may change according to the transformations (3.3.8) and (3.3.9).

Assume now that an electromagnetic field is given in terms of $\mathbf{E} = \mathbf{E}(t, \mathbf{x})$ and $\mathbf{B} = \mathbf{B}(t, \mathbf{x})$, which satisfy Maxwell's equations. Then we **define** the matrix $(F^{\mu\nu})$ by (3.9.14) and replace the First Postulate by the following

Alternative formulation of the First Postulate: The matrix $(F^{\mu\nu})$ defined in (3.9.14) (without electromagnetic potentials) is a tensor, i.e. there holds the transformation law $(3.9.15)_2$ for any proper Lorentz-transformation (3.8.12).

Of course, the requirement that $F^{\mu\nu}$ is a tensor is equivalent the requirement that $F_{\mu\nu}$ is a tensor. Then we will find that the inhomogeneous Maxwell equations take the Lorentz-invariant form

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} + \frac{1}{c} j^\mu = 0\,. \tag{3.9.20}$$

Also the homogeneous Maxwell equations may be written in a Lorentz-invariant form by

$$\frac{\partial F_{\beta\gamma}}{\partial x^\alpha} + \frac{\partial F_{\gamma\alpha}}{\partial x^\beta} + \frac{\partial F_{\alpha\beta}}{\partial x^\gamma} = 0\,. \tag{3.9.21}$$

The new formulation (3.9.21) of the homogeneous Maxwell equations needs some explanations: If at least two of the three indices α, β, γ are equal, then both sides of this equation are identically zero since $F_{\mu\nu}$ is an antisymmetric tensor. On the other hand an odd permutation of the indices α, β, γ will only change the sign on the left hand side of the equation, which is again a consequence of the antisymmetry of $F_{\mu\nu}$. Finally there only result three independent equations from (3.9.21) for the homogeneous Maxwell equations which may also be written in the compact form

$$\varepsilon^{\alpha\beta\gamma\mu}\,\frac{\partial F_{\alpha\beta}}{\partial x^\gamma} = 0\,. \tag{3.9.22}$$

Therefore Maxwell's equations (3.9.20) and (3.9.21) have again Lorentz-invariant tensor form when we replace the original First Postulate by its alternative version.

In the following we also rewrite the electromagnetic balance laws for energy (3.7.5) and momentum (3.7.6) in Lorentz-invariant form by defining the tensor quantities

$$T^{\mu\nu} = \frac{1}{\varepsilon_0}\left[F^{\mu\kappa}F_{\kappa\lambda}g^{\lambda\nu} + \frac{1}{4}\,F^{\lambda\kappa}F_{\lambda\kappa}g^{\mu\nu}\right], \qquad f^\mu = \frac{1}{c\varepsilon_0}\,F^{\mu\nu}j_\nu\,. \tag{3.9.23}$$

By a straight forward calculation it comes out that

$$(T^{\mu\nu}) = \left(\begin{array}{c|c} W & c\mathbf{P} \\ \hline c\mathbf{P} & T \end{array}\right), \qquad (f^\mu) = \left(\frac{\mathbf{j}}{\varepsilon_0 c}\cdot\mathbf{E}\;,\; \frac{1}{\varepsilon_0}(\rho\mathbf{E} + \frac{\mathbf{j}}{c}\times\mathbf{B})\right)^T, \tag{3.9.24}$$

and therefore we can also rewrite the balance laws of Section 3.7 in the Lorentz-invariant form

$$\frac{\partial T^{\mu\nu}}{\partial x^\nu} + f^\mu = 0\,. \tag{3.9.25}$$

In view of (3.9.25) the quantities $T^{\mu\nu}$ and f^μ are called the **energy-momentum tensor** of the electromagnetic field and the **electromagnetic force density**, respectively. Recall that $T^{\mu\nu}$ and f^μ only take the special form (3.9.23)

because we have redefined the units of the electromagnetic field according to (3.2.7) in order to keep Maxwell's equations free from the quantities ε_0, μ_0 and 4π.

Since the invariant balance laws (3.9.25) are only derived from Maxwell's equations, we conclude that the postulated Lorentz-invariance of the four-potential A^μ and of the electromagnetic current four-density j^μ are sufficient in order to rewrite the electromagnetic theory in terms of the tensor calculus developed in this section.

Up to here we have developed the mathematical structure of the classical electromagnetic field theory. It is an important remaining task to derive from these Lorentz-tensors all important physical quantities of electrodynamics which can be measured in appropriate gedanken experiments with respect to an arbitrary Lorentz frame. Then the two postulates formulated above become physical statements about the validity of electrodynamical equations in all Lorentz frames. This was done by Einstein in his famous article [20].

Chapter 4

Relativistic Euler Equations

4.1 Introduction

In the kinetic theory of gases the **phase density** $f = f(t, \mathbf{x}, \mathbf{q})$ is a fundamental quantity. It is usually a function of time t, position $\mathbf{x}$ and of the velocity or momentum $\mathbf{q}$ related to the single gas atoms. It usually results from a kinetic equation like the *Boltzmann-* or *BGK-equation*, see Cercignani [3] and Bhatnagar, Gross, Krook [1]. Then the macroscopic, thermodynamic quantities like particle- and energy density, velocity and pressure are tensor-algebraic combinations of some basic integral moments of the phase density f, where the integration is performed with respect to the momentum $\mathbf{q}$ of the gas atoms. Therefore the macroscopic quantities depend only on time and space. They are all completely determined by the phase density f.

Kinetic approaches in order to solve the classical Euler equations of gas dynamics were applied to several initial- and boundary value problems, see for example Dreyer, Kunik and Herrmann [7], [8], [11] or Perthame [41], [42], [43]. In Dreyer and Kunik [9] the reader will also find a kinetic scheme in order to solve an initial and boundary value problem for a phonon Bose gas. In [31, 32, 33, 35, 44] Kunik, Qamar and Warnecke have formulated two types of kinetic schemes in order to solve the initial and boundary value problems for the *ultra-relativistic Euler equations* and a *general form of special relativistic Euler equations* based on Jüttners constitutive relations presented in [28].

The hyperbolic systems that can be treated by the kinetic method are those which may be generated from kinetic transfer equations and from the *Maximum Entropy Principle*. Since these systems lead to a convex entropy function, they enable several rigorous mathematical results, see for example Friedrichs and Lax [23], Godlewski & Raviart [24] as well as Dafermos [6]. In the case of *thermodynamical equilibrium* the Maximum Entropy Principle constitutes a successful method in order to obtain the Maxwellian phase

density for the Boltzmann gas as well as the corresponding phase densities for the Fermi- and Bose gas in equilibrium from the corresponding kinetic entropy definitions. But the application of the Maximum Entropy Principle may lead to serious problems in *non-equilibrium*, for example for the moment-systems of the Fokker-Planck equation and of the Boltzmann equation. These difficulties arise due to a singular behaviour of some integral moments of the nonequilibrium phase density. This was shown explicitly for Levermore's five-momemt system in one space dimension by Junk in [30] and for the Fokker-Planck equation by Dreyer, Junk and Kunik [14]. An exception is the phonon Bose gas considered in [9], where the singular behaviour of the integral moments cannot occure.

A few years after Einstein's famous paper [20] "Zur Elektrodynamik bewegter Körper", Jüttner [28] extended the kinetic theory of gases which was developed by D. Bernoulli, Clausius, Maxwell and Boltzmann, to the domain of relativity. He succeeded in deriving the relativistic generalization of the Maxwellian equilibrium phase density. Later on this phase density and the whole relativistic kinetic theory was structured in a well organized Lorentz invariant form, see Chernikov [4], [5], Israel [26], Müller [38] and the textbook of deGroot, van Leeuven and van Weert [25]. Jüttner [29] also established the relativistic form of equilibrium phase densities and the corresponding equations of state for the systems of bosons and fermions.

Euler's equations (relativistic or classic) deal with a *perfect gas*, in which mean free paths and collision free times are so short that perfect isotropy is maintained about any point moving with the gas. In this case the *local equilibrium assumption* is satisfied and the corresponding phase-densities are obtained from the Maximum Entropy Principle in equilibrium. In the textbook of Weinberg [48] one can find a short introduction to special relativity and relativistic hydrodynamics with further literature also for the imperfect fluid (gas), see for example the papers of Eckart [17], [18], [19] for the classical and relativistic thermodynamics.

There are three basic ingredients of the relativistic kinetic schemes. The first one is the relativistic phase density developed by Jüttner. The second one is the solution of a collision free kinetic transport equation, which can be given explicitly in terms of a known initial phase density. For the formulation of the kinetic scheme we prescribe a time step $\tau_M > 0$, define the equidistant times $t_n = n\,\tau_M$ ($n = 0, 1, 2, ...$), called maximization times, and solve a collision free kinetic transport equation for each time interval $t_n < t < t_{n+1}$, starting with a relativistic Maxwellian as the initial phase density at each maximization time t_n. The third component consists of the continuity conditions, which guarantee that the conservation laws are also satisfied across the maximization times. They also determine the new initial data for the next free-flight period. Finally, it is also possible to incorporate adiabatic boundary conditions into the kinetic scheme in a quite natural

way. By taking moments of the corresponding phase densities we obtain every macroscopic quantity like particle density, energy density, pressure and velocity four-vector. These macroscopic quantities will solve the relativistic Euler equations in the limit $\tau_M \to 0$.

Now we give a short overview of this chapter:

In Section 4.2 we will present the basic definitions of the relativistic kinetic theory, namely the macroscopic quantities considered in thermodynamics which are obtained from a kinetic phase density. Moreover the relativistic Maxwellian studied by Jüttner in [28] is introduced and two limiting cases are considered, namely the classical Maxwellian for a cool, non-relativistic gas and the ultra-relativistic Jüttner phase density.

In Section 4.3 we calculate the energy density, pressure and entropy density from Jüttner's phase density. Then we determine the macroscopic moments of this relativistic Maxwellian, which gives the so called constitutive relations. The conservation laws for the particle number, the momentum and energy and these constitutive relations are representing the relativistic Euler equations. The Euler equations are written in differential form as well as in a weak integral form, which takes care of the evolution of shock waves. There holds an entropy inequality in terms of a specific entropy function which satisfies Gibbs equation. In order to prove that the relativistic Maxwellian satisfies the Maximum Entropy Principle we have first proved four lemmas which are needed for this purpose. After this we formulate and prove the Maximum Entropy Principle. In order to get the general formulation of the relativistic Euler equations and the Maximum Entropy Principle, we use some relations concerning the Bessel functions, which can be found in the hand book of Jeffrey [27]. We have also evaluated the limiting cases for the energy density, pressure and entropy density for the non-relativistic limit as well as the ultra-relativic limit. The eigenvalues from the differential form of the Euler equations are also presented in this section. Moreover, the Rankine-Hugoniot shock conditions and the entropy inequality are used in order to derive a simple parameter representation for the admissible single shock fronts.

In Section 4.4 we consider the ultra-relativistic limit and the corresponding ultra-relativistic Euler equations. A very characteristic feature of these equations is that the energy- and momentum balance law decouples from the continuity equation and thus form a hyperbolic subsystem for the pressure and velocity four-vector, the so called $(p, \mathbf{u})$-subsystem. In one space dimension this subsystem admits an extensive study and especially a complete solution of the Riemannian initial value problem, which will be studied in this section.

In Section 4.5 we first formulate the kinetic scheme in order to solve the three-dimensional ultra-relativistic Euler equations. In contrast to the classical three-dimensional Euler equations for a non-relativistic gas we will show

that the three-fold momentum integrals for the particle-density four-vector and for the energy-momentum tensor reduce simply to surface integrals where the integration is performed with respect to the unit sphere. A similar idea was used by Dreyer, Herrmann and Kunik [12], [9] in order to solve the Boltzmann-Peierls equation and their moment systems for a phonon Bose-gas. Later on we will see by an appropriate transformation in phase space that the kinetic formulation for both problems are so closely related that the scheme for the phonon Bose-gas contains the kinetic scheme for the ultra-relativistic Euler equations as a limiting case. We prove that the conservation laws and the entropy inequality are satisfied for the ultra-relativistic scheme. For this purpose the continuity conditions for the zero components of the moments play a crucial role, more precisely they constitute necessary conditions in order to guarantee that the conservation laws and the entropy inequality are also satisfied across the maximization times. The continuity conditions are also required in order to initialize the kinetic scheme for the next time step.

In Section 4.6 we are looking at the special case of spatially one-dimensional solutions which are nevertheless solutions to the three-dimensional ultra-relativistic Euler equations. In this case the surface integrals of the three-dimensional kinetic scheme reduce again to single integrals which range from -1 to $+1$. They indicate the finite domain of dependence on the preceeding initial data, which is covered by the backward light cones. This property does not hold for classical kinetic schemes.

In Section 4.7 we discuss the Eulerian limit $\tau_M \to 0$ of the kinetic scheme where weak solutions are obtained from the initial value problems including arbitrary complicated shock interactions.

In Section 4.8 we present an analysis of the kinetic scheme for the $(p, \mathbf{u})$-subsystem. We derive a reduced kinetic phase density and an own reduced kinetic entropy for this subsystem that are much easier than the original ones and prove the corresponding Maximum Entropy Principle.

We close in Section 4.9 with the derivation of two completely discretized numerical schemes for the ultra-relativistic Euler equations in one space dimension and test them for a known solution of the Riemann problem.

4.2 Moments of the relativistic kinetic phase density

In this section we describe a relativistic gas consisting of many microscopic structureless particles in terms of the relativistic kinetic phase density. From this fundamental phase density we calculate tensorial moments which give the

local macroscopic physical quantities of the gas such as the particle density, the velocity, the pressure, the temperature and so on.

Recall the basic definitions and conventions from Sections 3.8 and 3.9 which are necessary in order to formulate the tensor calculus.

First we take a microscopic look at the gas and start with the kinematics of a representative gas atom with particle trajectory $\mathbf{x} = \mathbf{x}(t)$, where the time coordinate t and the space coordinate $\mathbf{x}$ are related to an arbitrary Lorentz-frame. The invariant mass of all structureless particles is assumed to be the same and is denoted by m_0. The microscopic velocity of the gas atom is $\frac{d\,\mathbf{x}(t)}{dt}$, and its microscopic velocity four-vector is given by $c\,q^\mu$, where the *dimensionless microscopic velocity four-vector* q^μ is defined by

$$\big(q^0, \mathbf{q}\big)^T\,, \quad q^0 = q_0 = \sqrt{1+\mathbf{q}^2}\,, \quad \mathbf{q} = \frac{\frac{1}{c}\frac{d\,\mathbf{x}}{dt}}{\sqrt{1-\big(\frac{1}{c}\frac{d\,\mathbf{x}}{dt}\big)^2}}\,. \tag{4.2.1}$$

The *relativistic phase density* $f(t,\mathbf{x},\mathbf{q}) > 0$ is the basic quantity of the kinetic theory. In the sequel we assume that $f(t,x,\cdot) \in V$, where

$$V = \{\, f \in L^\infty(\mathbb{R}^3) \mid 0 < f(\mathbf{q}) \le \frac{C}{1+|\mathbf{q}|^{4+\epsilon}} \text{ for some } \epsilon, C > 0 \,\}\,. \tag{4.2.2}$$

The following definitions of the macroscopic moments and the entropy four-vector make use of the fact that the so called *proper volume element* d^3q/q_0 is invariant with respect to Lorentz-transformations. Note that $f \in V$ guarantees that all the three-fold integrals below can be formed with respect to $\frac{d^3q}{q^0}$ due to $q^0 \ge |\mathbf{q}|$. This can easily be seen by using spherical coordinates.

Macroscopic moments and entropy four-vector:

(i) *Particle-density four-vector*

$$N^\mu = N^\mu(t,\mathbf{x}) = \int_{\mathbb{R}^3} q^\mu\, f(t,\mathbf{x},\mathbf{q})\,\frac{d^3q}{q^0}\,, \tag{4.2.3}$$

(ii) *energy-momentum tensor*

$$T^{\mu\nu} = T^{\mu\nu}(t,\mathbf{x}) = m_0c^2 \int_{\mathbb{R}^3} q^\mu q^\nu\, f(t,\mathbf{x},\mathbf{q})\,\frac{d^3q}{q^0}\,, \tag{4.2.4}$$

(iii) *entropy four-vector*

$$S^\mu = S^\mu(t,\mathbf{x}) = -k_B \int_{\mathbb{R}^3} q^\mu\, f(t,\mathbf{x},\mathbf{q})\,\ln\left(\frac{f(t,\mathbf{x},\mathbf{q})}{\chi}\right)\frac{d^3q}{q^0}\,. \tag{4.2.5}$$

Here $k_B = 1.38062 \times 10^{-23} J/K$ is Boltzmann's constant and $\chi = (\frac{m_0 c}{\hbar})^3$ with Planck's constant $\hbar = 1.05459 \times 10^{-34} Jsec$. Note that χ has the same dimension as f, namely 1/volume. We also state here that the entropy formula (4.2.5) can be generalized easily in such a way, that the well known case of a Fermi- or Bose gas is also included in this kinetic framework. Then formula (4.2.5) reads in the general case

$$S^\mu = -k_B \int_{\mathbb{R}^3} q^\mu \left[f \ln \frac{f}{\chi} - \eta\chi \left(1 + \eta \frac{f}{\chi}\right) \ln \left(1 + \eta \frac{f}{\chi}\right) \right] \frac{d^3q}{q^0} \,. \qquad (4.2.6)$$

Here $\eta = 0$ reduces to (4.2.5), which is valid for the relativistic generalization of Boltzmann's statistic, whereas $\eta > 0$ is required for the Bose-Einstein statistic and $\eta < 0$ for the Fermi statistic. Note that η is dimensionless, but η may depend on the spin of the particles.

The spatial part $\mathbf{q} \in \mathbb{R}^3$ of the dimensionless microscopic velocity four-vector is used as an integration variable in the relativistic kinetic theory.
Now we may use the macroscopic moments N^μ, $T^{\mu\nu}$ and S^μ of the relativistic phase density f in order to calculate the other macroscopic quantities of the gas, which are

Tensor algebraic combinations of these moments:

(i) *The proper particle density*

$$n = \sqrt{N^\mu N_\mu} \,, \qquad (4.2.7)$$

(ii) *the dimensionless velocity four-vector*

$$u^\mu = \frac{1}{n} N^\mu \,, \qquad (4.2.8)$$

(iii) *the proper energy density*

$$e = u_\mu u_\nu \, T^{\mu\nu} \,, \qquad (4.2.9)$$

(iv) *the proper pressure and temperature*

$$p = \frac{1}{3} \left(u_\mu u_\nu - g_{\mu\nu}\right) T^{\mu\nu} = k_B \, nT \,, \qquad (4.2.10)$$

(v) *the proper entropy density*

$$\sigma = S^\mu u_\mu \,. \qquad (4.2.11)$$

Remarks:

(i) Since $f > 0$, it can be seen by a simple Cauchy-Schwarz argumentation that N^μ is a time-like vector, i.e. $N^\mu N_\mu > 0$. It follows that the particle density n is well defined and positive. In order to see that the energy density is always positive we write it in the form

$$e = m_0 c^2 \int_{\mathbb{R}^3} (u_\mu q^\mu)^2 \, f(t, \mathbf{x}, \mathbf{q}) \, \frac{d^3 q}{q^0} \,. \tag{4.2.12}$$

(ii) The *macroscopic velocity* $\mathbf{v}$ of the gas can be obtained easily from the spatial part $\mathbf{u} = (u^1, u^2, u^3)^T$ of the dimensionless velocity four-vector by

$$\mathbf{v} = c \, \frac{\mathbf{u}}{\sqrt{1 + \mathbf{u}^2}} \,. \tag{4.2.13}$$

From this formula we can immediately read off that $|\mathbf{v}| < c$, i.e. the absolute value of the velocity is bounded by the speed of light. Note also that $u^0 = \sqrt{1 + \mathbf{u}^2}$.

Let $\mathbf{u} = (u^1, u^2, u^3)^T \in \mathbb{R}^3$ be fixed. In order to present a short derivation of the relativistic Euler equations, we will later use the relations for the Lorentz boost $\Lambda^\alpha{}_\beta = \Lambda^\alpha{}_\beta(\mathbf{u})$ presented in Section 3.8,

$$\begin{aligned} &\Lambda^0{}_0 = \sqrt{1 + \mathbf{u}^2} \,, \;\; \Lambda^0{}_j = \Lambda^j{}_0 = -u^j \,, \\ &\Lambda^j{}_k = \delta_{j\,k} + \frac{u^j u^k}{1 + \sqrt{1 + \mathbf{u}^2}} \,. \end{aligned} \tag{4.2.14}$$

where $j, \, k \in \{1, 2, 3\}$ are spatial indices. Let G be the Minkowskian matrix defined in (3.8.2). Recall that the above relations imply

(a) $G = \Lambda(\mathbf{u}) \, G \, \Lambda(\mathbf{u})^T$, $\Lambda^0{}_0(\mathbf{u}) \geq 1$ and $Det(\Lambda(\mathbf{u})) = 1$, i.e. $\Lambda(\mathbf{u})$ is a proper Lorentz-matrix.

(b) $\Lambda^{-1}(\mathbf{u}) = \Lambda(-\mathbf{u})$.

The attribute "proper" for n, e, p, T and σ denotes quantities, which are invariant with respect to proper Lorentz-transformations. They take their simplest form in the Lorentz rest frame. Since all quantities under consideration are written down in Lorentz-invariant form, we may omit the word "proper" in the following.
These definitions are valid for any relativistic phase-density $f = f(t, \mathbf{x}, \mathbf{q})$, which has to be determined from a *kinetic equation* of the following form

$$q^\mu \frac{\partial f}{\partial x^\mu} = Q(f) \,. \tag{4.2.15}$$

As in the non-relativistic kinetic theory we have a corresponding transport part on the left-hand side and a collision part $Q(f)$ on the right-hand side. In the simplest case $Q(f)$ is determined in such a way that the following five conservation laws hold for the particle number, the energy, and the momentum

$$\frac{\partial N^\mu}{\partial x^\mu} = 0\,, \qquad \frac{\partial T^{\mu\nu}}{\partial x^\nu} = 0\,. \tag{4.2.16}$$

This simple case holds if the particles interact only during elastic collisions without other forces and radiation. The case $Q(f) = 0$ leads to pure free flight, without any collision of the gas particles. A detailed study of the relativistic Boltzmann-equation can be found in the textbook of de Groot, van Leeuwen and van Weert [25].

Relativistic Jüttner Phase density:

Jüttner extended the classical velocity distribution of Maxwell for a gas in equilibrium to the relativistic case. The resulting *Jüttner distribution* $f_J(n, T, \mathbf{u}, \mathbf{q})$ depends on five constant parameters, which describe the state of the gas in equilibrium, namely the particle density n, the absolute temperature T and the spatial part $\mathbf{u} \in \mathbb{R}^3$ of the dimensionless four-velocity. It is given by

$$\begin{aligned} f_J(n, T, \mathbf{u}, \mathbf{q}) &= \frac{n}{M(\beta)}\, \exp\left(-\beta\, u_\mu q^\mu\right) \\ &= \frac{n}{M(\beta)}\, \exp\left(-\beta\, \left(\sqrt{(1+\mathbf{u}^2)(1+\mathbf{q}^2)} - \mathbf{u}\cdot\mathbf{q}\right)\right), \end{aligned} \tag{4.2.17}$$

where $\beta = \frac{m_0 c^2}{k_B T}$ and

$$\begin{aligned} M(\beta) &= \int_{\mathbb{R}^3} \exp(-\beta\sqrt{1+\mathbf{q}^2}) d^3q \\ &= 4\pi \int_0^\infty \vartheta^2\, \exp(-\beta\,\sqrt{1+\vartheta^2})\, d\vartheta\,. \end{aligned} \tag{4.2.18}$$

The function $M(\beta)$ is chosen in such a way that

$$n\, u^\mu = \int\limits_{\mathbb{R}^3} q^\mu\, f_J(n, T, \mathbf{u}, \mathbf{q})\, \frac{d^3q}{q^0} \tag{4.2.19}$$

holds for the spatial part $\mathbf{u} = (u^1, u^2, u^3)^T$ of the dimensionless macroscopic velocity four-vector. This is equation (4.2.8), where $\mathbf{u}$ and n are in addition parameters of Jüttner's relativistic phase density. Using the Bessel functions for the non-negative integer numbers j

$$K_j(\beta) = \int\limits_0^\infty \cosh(js)\, \exp(-\beta\, \cosh(s))\, ds\,, \tag{4.2.20}$$

and applying the integral substitution $\vartheta = \sinh(s)$ we may also write $M(\beta)$ in the form

$$M(\beta) = \frac{4\pi}{\beta} K_2(\beta)\,. \tag{4.2.21}$$

We have in addition recursion relations for the modified Bessel functions, which can be found in the hand book of Jeffrey [27],

$$K_{j+1}(\beta) = \frac{2j}{\beta} K_j(\beta) + K_{j-1}(\beta)\,, \tag{4.2.22}$$

where j is the integer order of the modified Bessel functions. Using (4.2.20), (4.2.21) and (4.2.22) we can write

$$M'(\beta) = -4\pi \left(\frac{K_1(\beta)}{\beta} + \frac{3K_2(\beta)}{\beta^2} \right)\,, \tag{4.2.23}$$

$$\eta(\beta) = 4\pi \int_0^\infty \frac{\vartheta^2}{\sqrt{1+\vartheta^2}} \exp(-\beta\sqrt{1+\theta^2}) d\vartheta = \frac{4\pi}{\beta} K_1(\beta)\,. \tag{4.2.24}$$

Note that

$$\eta'(\beta) = M(\beta)\,. \tag{4.2.25}$$

The equations (4.2.21) and (4.2.23) will be used in Section 4.3 in order to find the constitutive relations for e, p, σ.

In order to formulate the Euler equations and other relations coming in next sections in a nice form, we introduce the function

$$\Psi(\beta) = \frac{3}{\beta} + \frac{K_1(\beta)}{K_2(\beta)}\,. \tag{4.2.26}$$

In Section 4.3 it turns out that $\Psi(\beta)$ is just the specific energy $\frac{e}{n}$ for the gas in equilibrium.

Limiting cases of the relativistic Jüttner Phase density:

Here we discuss two important special cases for this phase density, namely the non-relativistic limit for a cool gas and the ultra-relativistic limit $m_0 \to 0$.

Case 1: The non-relativistic limit (small temperatures, small velocities)

For the first case we rewrite (4.2.17) in the form

$$f_J(n, T, \mathbf{u}, \mathbf{q}) = \frac{n}{M_1(\beta)} \exp\left(-\beta \frac{(\mathbf{q}-\mathbf{u})^2 + \mathbf{u}^2\mathbf{q}^2 - (\mathbf{u}\cdot\mathbf{q})^2}{1 + \sqrt{(1+\mathbf{u}^2)(1+\mathbf{q}^2)} + \mathbf{u}\cdot\mathbf{q}} \right)\,, \tag{4.2.27}$$

where

$$M_1(\beta) = M(\beta)\, \exp(\beta)\,. \tag{4.2.28}$$

If we apply for $\vartheta > 0$ the integral substitution

$$\xi = \sqrt{2\beta(\sqrt{1+\vartheta^2} - 1)}\,, \tag{4.2.29}$$

then we can rewrite $M_1(\beta)$ in the form

$$M_1(\beta) = \left(\frac{2\pi}{\beta}\right)^{\frac{3}{2}} \cdot 2 \int_0^\infty \left(1 + \frac{\xi^2}{2\beta}\right) \sqrt{1 + \frac{\xi^2}{4\beta}}\; \frac{\exp(-\frac{\xi^2}{2})}{\sqrt{2\pi}}\, d\xi\,. \tag{4.2.30}$$

For β very large compared to 1, i.e. for small temperature, we can conclude from (4.2.30) that

$$M_1(\beta) = (\frac{2\pi}{\beta})^{\frac{3}{2}} + O(\beta^{-\frac{5}{2}}) = \left(\frac{2\pi K_B T}{m_0 c^2}\right)^{\frac{3}{2}} + O(T^{\frac{5}{2}})\,,$$

and the representation (4.2.27) shows that the Jüttner phase density reduces to the non-relativistic Maxwellian for $|\mathbf{u}|$, $|\mathbf{q}|$ very small, namely

$$f_c(n, T, \mathbf{u}, \mathbf{q}) = n \left(\frac{m_0 c^2}{2\pi K_B T}\right)^{\frac{3}{2}} \exp\left(-\frac{m_0 c^2 (\mathbf{q} - \mathbf{u})^2}{2 K_B T}\right)\,. \tag{4.2.31}$$

Case 2: The ultra-relativistic limit (zero rest mass of the particles)

For the ultra-relativistic limit $m_0 \to 0$ with fixed temperature we apply the substitution $\mathbf{q}' = m_0.\mathbf{q}$ in order to write (4.2.17) in the form

$$f_J(n, T, \mathbf{u}, \mathbf{q}) = m_0^3 \frac{n}{M_2(\tilde{\beta})}\, \exp\left(-\tilde{\beta}\, \left(\sqrt{(1+\mathbf{u}^2)(m_0^2 + \mathbf{q}'^2)} - \mathbf{u}\cdot\mathbf{q}'\right)\right)\,, \tag{4.2.32}$$

where

$$\tilde{\beta} = \frac{\beta}{m_0} = \frac{c^2}{k_B T}\,, \quad M_2(\tilde{\beta}) = \frac{8\pi}{\tilde{\beta}^3} \int_0^\infty \frac{\xi^2}{2}\, \exp\left(-\sqrt{m_0^2 \tilde{\beta}^2 + \xi^2}\right)\, d\xi\,. \tag{4.2.33}$$

In the following we do not use primes for the new integration variable $\mathbf{q}$.

Now we are able to pass to the ultra-relativistic limit $m_0 \to 0$. In order to do this we first have to replace the four-vector q^μ defined in (4.2.1) by the light vector

$$\left(q^0, \mathbf{q}\right)^T\,, \qquad q^0 = q_0 = |\mathbf{q}|\,. \tag{4.2.34}$$

Next we will introduce dimensionless quantities by setting $c = k_B = \hbar = 1$. Then the ultra-relativistic moments and entropy four-vector take a similar form as given in (4.2.3), (4.2.4) and (4.2.5)

$$N^\mu = N^\mu(t, \mathbf{x}) = \int_{\mathbb{R}^3} q^\mu \, f(t, \mathbf{x}, \mathbf{q}) \, \frac{d^3q}{|\mathbf{q}|} \,, \tag{4.2.35}$$

$$T^{\mu\nu} = T^{\mu\nu}(t, \mathbf{x}) = \int_{\mathbb{R}^3} q^\mu q^\nu \, f(t, \mathbf{x}, \mathbf{q}) \, \frac{d^3q}{|\mathbf{q}|} \,, \tag{4.2.36}$$

and the macroscopic *entropy four-vector*

$$S^\mu = S^\mu(t, \mathbf{x}) = -\int_{\mathbb{R}^3} q^\mu \, f(t, \mathbf{x}, \mathbf{q}) \, \ln\left(\frac{f(t, \mathbf{x}, \mathbf{q})}{\chi}\right) \, \frac{d^3q}{|\mathbf{q}|} \,, \tag{4.2.37}$$

where $\chi = (\frac{m_0 c}{\hbar})^3$.
Here f is first taken as the ultra-relativistic Jüttner phase density (4.2.32) in its dimensionless form as

$$\begin{aligned} f_J^*(n, T, \mathbf{u}, \mathbf{q}) &= \frac{n}{8\pi T^3} \, \exp\left(-\frac{u_\mu q^\mu}{T}\right) \\ &= \frac{n}{8\pi T^3} \, \exp\left(-\frac{|\mathbf{q}|}{T}\left(\sqrt{1+\mathbf{u}^2} - \mathbf{u}\cdot\frac{\mathbf{q}}{|\mathbf{q}|}\right)\right) \,. \end{aligned} \tag{4.2.38}$$

But we will also use equations (4.2.35), (4.2.36) and (4.2.37) in order to define new moments for a general phase density f. In this case we will again call (4.2.35), (4.2.36) and (4.2.37) the moments for the ultra-relativistic limit. For that reason we have used the general symbol f for the new moments instead of f_J^*. These more general definitions will be important for the formulation of kinetic schemes in order to solve the fluid dynamic equations in the ultra-relativistic limit.

Finally it is important to note that all the definitions given for the particle density n, velocity four-vector u^μ, energy density e and for the pressure p, which are tensor invariant algebraic combinations of the basic moments N^μ and $T^{\mu\nu}$, are still valid for an arbitrary phase density f in the ultra-relativistic limit. Neverthless we can still simplify the generally valid formula (4.2.10) for the pressure in the ultra-relativistic limit

$$\begin{aligned} p &= \frac{1}{3} \, u_\mu u_\nu T^{\mu\nu} - \frac{1}{3} \, g_{\mu\nu} T^{\mu\nu} \\ &= \frac{1}{3} \, u_\mu u_\nu T^{\mu\nu} - \int_{\mathbb{R}^3} g_{\mu\nu} q^\mu q^\nu \, f \, \frac{d^3q}{q^0} \,. \end{aligned} \tag{4.2.39}$$

Since $g_{\mu\nu} q^\mu q^\nu = q_\nu q^\nu = 0$ holds due to (4.2.34), we immediately conclude that

$$p = \frac{e}{3} = \frac{1}{3} \, T^{\mu\nu} u_\mu u_\nu = nT \tag{4.2.40}$$

in the ultra-relativistic case.

4.3 Relativistic Euler equations

4.3.1 The constitutive relations, general case

Using the relativistic Jüttner distribution (4.2.17), the relations (4.2.21) and (4.2.23), we calculate the originally given moments (4.2.3), (4.2.4) and (4.2.5) with $q^0 = \sqrt{1+\mathbf{q}^2}$. In the global rest frame, where u^μ has components $(1,0,0,0)^T$, we first obtain

$$\begin{aligned} e &= -n\frac{M'(\beta)}{M(\beta)} = n\Psi(\beta)\,, \\ p &= -\frac{n}{3M(\beta)}(M'(\beta)+\eta(\beta)) = nT = \frac{n}{\beta}\,, \\ \sigma &= -n\ln\left(\frac{n\beta}{K_2(\beta)}\right) + \beta n\Psi(\beta) + \gamma n\,. \end{aligned} \tag{4.3.1}$$

Here the choice of the entropy constant γ is not so important in general. Since e, p and σ are Lorentz invariant, we see that (4.3.1) is already true in any other Lorentz-frame, without the restriction that $(u^\mu) = (1,0,0,0)^T$.

Starting with the rest frame and then applying the inverse of the Lorentz boost given by (4.2.14), we get the following relations for the general frame, where $\mathbf{u}$ may or may not be zero,

$$N^\mu = n\,u^\mu\,, \quad T^{\mu\nu} = -p\,g^{\mu\nu} + (e+p)u^\mu u^\nu\,, \quad S^\mu = \sigma u^\mu\,. \tag{4.3.2}$$

Moreover one can see by simple calculations that σ in $(4.3.1)_3$ obeys the Gibbs equation

$$T\,d\left(\frac{\sigma}{n}\right) = p\,d\left(\frac{1}{n}\right) + d\left(\frac{e}{n}\right). \tag{4.3.3}$$

Since the relativistic moments (4.3.2) are valid in a special Lorentz frame and since these equations are written in tensor invariant form, they are generally valid in every Lorentz frame. This can also be seen directly without making use of the Lorentz-boosts.

4.3.2 The constitutive relations, limiting cases

Case 1: The classical limit

In order to get the classical limit with $\beta \gg 1$, we first regard the asymptotic

relations which hold for $\beta \to \infty$,

$$K_1(\beta) = \sqrt{\frac{\pi}{2\beta}} \exp(-\beta) \left(1 + \frac{3}{8\beta}\right) + O\left(\frac{\exp(-\beta)}{\beta^{\frac{5}{2}}}\right),$$
$$K_2(\beta) = \sqrt{\frac{\pi}{2\beta}} \exp(-\beta) \left(1 + \frac{15}{8\beta}\right) + O\left(\frac{\exp(-\beta)}{\beta^{\frac{5}{2}}}\right), \tag{4.3.4}$$
$$\Psi(\beta) = 1 + \frac{3}{2\beta} + O\left(\frac{\exp(-\beta)}{\beta^{\frac{5}{2}}}\right).$$

To obtain the non-relativistic values of the energy density, pressure and entropy density, we substitute the asymptotic relations (4.3.4) for $K_2(\beta)$ and $\Psi(\beta)$ into the constitutive relations (4.3.1). There result the following expressions in the classical limit

$$e = n + \frac{3}{2} nT\,, \qquad p = nT\,, \qquad \sigma = n \ln \frac{p^{\frac{3}{2}}}{n^{\frac{5}{2}}} + \gamma n\,. \tag{4.3.5}$$

For the detailed study of the classical case the reader is refered to Dreyer and Kunik [7]. In this paper the particle density n is replaced by the mass density ρ and entropy density σ is denoted by h. The term n in $(4.3.5)_1$ is the energy density of the rest mass.

Case 2: The ultra-relativistic limit

In order to obtain the ultra-relativistic limit with $\beta << 1$, we first regard the asymptotic relations which hold for $\beta \to 0$,

$$K_1(\beta) = \frac{1}{\beta} + O\left(\beta \ln \beta\right)\,,$$
$$K_2(\beta) = \frac{2}{\beta^2} + O(1)\,, \quad \Psi(\beta) = \frac{3}{\beta} + O(\beta). \tag{4.3.6}$$

We substitute them into the constitutive relations (4.3.1) and (4.3.2) and get

$$e = 3nT\,, \qquad p = nT\,, \qquad \sigma = n \ln \frac{p^3}{n^4} + \gamma n\,, \tag{4.3.7}$$

$$N^\mu = n\, u^\mu\,,\ T^{\mu\nu} = -p\, g^{\mu\nu} \ + \ 4pu^\mu u^\nu\,,$$
$$S^\mu = N^\mu \ln \frac{p^3}{n^4} + \gamma N^\mu\,. \tag{4.3.8}$$

A detailed study of the ultra-relativistic case including its kinetic solution was first presented by Kunik, Qamar and Warnecke [31]. We extend this study in the following sections.

4.3.3 Formulation of Euler's equations, general case

Now we use the constitutive relations (4.3.1), (4.3.2) and the conservation laws (4.2.16) in order to get the general form of three-dimensional Euler equations at regular points. For this purpose we introduce the abbreviation

$$\chi(\beta) = \Psi(\beta) + \frac{1}{\beta}\,, \tag{4.3.9}$$

and obtain for $p = \frac{n}{\beta}$

$$\frac{\partial}{\partial t}\left(n\sqrt{1+\mathbf{u}^2}\right) + \sum_{k=1}^{3} \frac{\partial (nu^k)}{\partial x^k} = 0\,, \tag{4.3.10}$$

$$\frac{\partial}{\partial t}\left(n\chi(\beta)u^i\sqrt{1+\mathbf{u}^2}\right) + \sum_{k=1}^{3} \frac{\partial}{\partial x^k}\left(p\delta^{ik} + n\chi(\beta)u^i u^k\right) = 0\,, \tag{4.3.11}$$

$$\frac{\partial}{\partial t}\left(-p + n\chi(\beta)(1+\mathbf{u}^2)\right) + \sum_{k=1}^{3} \frac{\partial}{\partial x^k}\left(n\chi(\beta)u^k\sqrt{1+\mathbf{u}^2}\right) = 0\,. \tag{4.3.12}$$

Remarks:

(i) These equations constitute a closed system in terms of the unknown fields n, $\mathbf{u}$, and β. Also note that $\beta = \frac{1}{T}$.

(ii) The classical Euler equations result using (4.3.4) in the following way: From (4.3.10) we obtain the classical continuity equation by neglecting the second order terms in $\mathbf{u}$, whereas the classical momentum equations are obtained from (4.3.11) by setting $\chi(\beta)$ equal to one and neglecting the third order terms in $\mathbf{u}$. Finally, the classical energy equation results if we subtract (4.3.10) from (4.3.12) and then neglect the fourth order terms in $\mathbf{u}$ and terms which contain $p\mathbf{u}^2$ as a factor.

(iii) The ultra-relativistic Euler equations result directly from (4.3.8) and (4.2.16). We will discuss and study them in detail in the sequel.

4.3.4 Maximum Entropy Principle

The Jüttner distribution f_J (4.2.17) has some important properties. First of all it generalizes the classical Maxwellian of a gas in equilibrium to the relativistic case, and secondly f_J satisfies the so called Maximum Entropy Principle in equilibrium, which will be formulated and proved below. For this purpose we need the following lemmas.

Lemma 4.1. *Let be* $\mathbf{u}, \mathbf{q} \in \mathbb{R}^3$. *Also let be* $u^\mu = (\sqrt{1+\mathbf{u}^2}, \mathbf{u})^T$ *and* $q^\mu = (\sqrt{1+\mathbf{q}^2}, \mathbf{q})^T$. *Then the scalar product* $q^\mu u_\mu$ *satisfies the inequality* $q^\mu u_\mu \geq 1$, *where* $q^\mu u_\mu = 1$ *if and only if* $\mathbf{u} = \mathbf{q}$.

Proof: We consider

$$\begin{aligned} q^\mu u_\mu - 1 &= \sqrt{1+\mathbf{q}^2}\sqrt{1+\mathbf{u}^2} - \mathbf{q}\cdot\mathbf{u} - 1 \\ &= \frac{(\mathbf{q}-\mathbf{u})^2 + \mathbf{q}^2\mathbf{u}^2 - (\mathbf{q}\cdot\mathbf{u})^2}{\sqrt{1+\mathbf{q}^2}\sqrt{1+\mathbf{u}^2} + \mathbf{q}\cdot\mathbf{u} + 1}\,. \end{aligned} \tag{4.3.13}$$

Due to the Cauchy-Schwarz inequality we know that

$$\mathbf{q}^2\mathbf{u}^2 - (\mathbf{q}\cdot\mathbf{u})^2 \geq 0. \tag{4.3.14}$$

If $\mathbf{q} \neq \mathbf{u}$ then from (4.3.13) we have

$$(\mathbf{q}-\mathbf{u})^2 + \mathbf{q}^2\mathbf{u}^2 - (\mathbf{q}\cdot\mathbf{u})^2 > 0\,, \tag{4.3.15}$$

and this implies again from (4.3.13) that $q^\mu u_\mu - 1 > 0$ or $q^\mu u_\mu > 1$. ∎

Lemma 4.2. *The derivative* $\Psi' : \mathbb{R}^+ \to \mathbb{R}$ *has the representation*

$$\begin{aligned} \Psi'(\beta) &= \frac{d}{d\beta}\left(\frac{3}{\beta} + \frac{K_1(\beta)}{K_2(\beta)}\right) \\ &= -\frac{3}{\beta^2} + \frac{3}{\beta}\cdot\frac{K_1(\beta)}{K_2(\beta)} + \left(\frac{K_1(\beta)}{K_2(\beta)}\right)^2 - 1\,, \end{aligned} \tag{4.3.16}$$

and is negative for any $\beta > 0$. *Moreover* $\Psi(\beta)$ *satisfies the inequality*

$$\Psi(\beta) > 1\,, \tag{4.3.17}$$

which indicates that the specific energy is larger than the rest mass energy of a single atom.

Proof: We divide the proof of this lemma into two cases as follow.

Case 1 when $0 < \beta < 1$: From the definition of the modified Bessel functions we have $0 < K_1(\beta) < K_2(\beta)$. So we can write due to (4.3.16)

$$\begin{aligned} \Psi'(\beta) &= -\frac{3}{\beta^2} + \frac{3}{\beta}\cdot\frac{K_1(\beta)}{K_2(\beta)} + \left(\frac{K_1(\beta)}{K_2(\beta)}\right)^2 - 1 \\ &< -\frac{3}{\beta^2} + \frac{3}{\beta} + 1 - 1 = \frac{3}{\beta^2}(\beta - 1) < 0. \end{aligned} \tag{4.3.18}$$

Thus we have proved for $0 < \beta < 1$ that $\Psi'(\beta) < 0$.

Case 2 when $\beta \geq 1$: Recall the integral definition (4.2.20) for the modified Bessel functions $K_j(\beta)$. We also know that $\cosh(s) = 1+2\sinh^2(\frac{s}{2})$ and make the substitution $\alpha = \sinh(\frac{s}{2})$ with $ds = \frac{2d\alpha}{\sqrt{1+\alpha^2}}$ in (4.2.20) in order to get

$$\frac{1}{4}\exp\left(\frac{\beta}{4}\right)K_0\left(\frac{\beta}{4}\right) = \frac{1}{2}\int_0^\infty \frac{\exp\left(\frac{-\beta\alpha^2}{2}\right)}{\sqrt{1+\alpha^2}}d\alpha\,, \tag{4.3.19}$$

$$\frac{1}{4}\exp\left(\frac{\beta}{4}\right)\left[K_0\left(\frac{\beta}{4}\right)+K_1\left(\frac{\beta}{4}\right)\right] = \int_0^\infty \sqrt{1+\alpha^2}\exp\left(\frac{-\beta\alpha^2}{2}\right)d\alpha\,. \tag{4.3.20}$$

Also we have the following estimates

$$\frac{1}{\sqrt{1+\alpha^2}} \geq 1-\frac{\alpha^2}{2}\,, \qquad \sqrt{1+\alpha^2} \leq 1+\frac{\alpha^2}{2}\,. \tag{4.3.21}$$

Keeping in view (4.3.19) and (4.3.20), we obtain from (4.3.21)

$$\frac{1}{4}\exp\left(\frac{\beta}{4}\right)K_0\left(\frac{\beta}{4}\right) \geq \frac{1}{2}\sqrt{\frac{\pi}{2\beta}}\left(1-\frac{1}{2\beta}\right)\,, \tag{4.3.22}$$

$$\frac{1}{4}\exp\left(\frac{\beta}{4}\right)\left[K_0\left(\frac{\beta}{4}\right)+K_1\left(\frac{\beta}{4}\right)\right] \leq \sqrt{\frac{\pi}{2\beta}}\left(1+\frac{1}{2\beta}\right)\,. \tag{4.3.23}$$

Note that also the right hand side of (4.3.22) is positive due to $\beta \geq 1$. We take the inverse of (4.3.22) and multiply with (4.3.23) in order to get

$$1+\frac{K_1\left(\frac{\beta}{4}\right)}{K_0\left(\frac{\beta}{4}\right)} \leq 2\left(1+\frac{1}{2\beta}\right)\cdot\left(1-\frac{1}{2\beta}\right)^{-1}\,. \tag{4.3.24}$$

Using the recursion relation (4.2.22) for $j=1$ and replacing β by $\frac{\beta}{4}$ we get

$$\frac{K_2\left(\frac{\beta}{4}\right)}{K_1\left(\frac{\beta}{4}\right)} - \frac{K_0\left(\frac{\beta}{4}\right)}{K_1\left(\frac{\beta}{4}\right)} = \frac{8}{\beta}\,. \tag{4.3.25}$$

Now using (4.3.24) and (4.3.25) we get the following inequality after some manipulations and replacing $\frac{\beta}{4}$ by β

$$\frac{K_1(\beta)}{K_2(\beta)} \leq \frac{1+\frac{3}{8\beta}}{1+\frac{15}{8\beta}+\frac{3}{4\beta^2}}\,. \tag{4.3.26}$$

Substituting (4.3.26) in (4.3.16) for $\frac{K_1(\beta)}{K_2(\beta)}$, we finally get after simplification

$$\Psi'(\beta) \leq -\frac{9}{64\beta^6}\cdot\frac{12+60\beta+105\beta^2+69\beta^3+8\beta^4}{\left(1+\frac{15}{8\beta}+\frac{3}{4\beta^2}\right)^2} < 0\,. \tag{4.3.27}$$

It follows that $\Psi'(\beta) < 0$ for any $\beta > 0$.

Hence we have proved that $\Psi(\beta)$ is a strictly monotonically decreasing function which satisfies due to $(4.3.4)_3$ the asymptotic relation

$$\lim_{\beta \to \infty} \Psi(\beta) = 1\,. \tag{4.3.28}$$

Thus we conclude that $\Psi(\beta)$ is strictly bounded below by one. ∎

Lemma 4.3. *(i) Let $f \in V$ with V defined in (4.2.2) be any phase density with its moments N^μ, $T^{\mu\nu}$. Let n and e be the corresponding particle density and energy density, respectively. Then there hold the inequalities $0 < n < e$.*

(ii) Let $0 < n < e$ and $\mathbf{u} \in \mathbb{R}^3$ be given parameters, corresponding to the particle density, energy density and the spatial part of the macroscopic four-velocity. Then there exists exactly one temperature $T > 0$ such that the Jüttner phase density $f_J(n, T, \mathbf{u}, \mathbf{q})$ gives the prescribed energy density $e > n$.

Proof: In order to prove (i) we use from Lemma 4.1 that $q^\mu u_\mu > 1$ for $\mathbf{q} \neq \mathbf{u}$. Therefore we can write

$$(q^\mu u_\mu)(q^\nu u_\nu) > q^\mu u_\mu\,. \tag{4.3.29}$$

This implies that

$$\int_{\mathbb{R}^3} (q^\mu u_\mu)(q^\nu u_\nu) f\, \frac{d^3q}{q^0} > \int_{\mathbb{R}^3} q^\mu u_\mu f\, \frac{d^3q}{q^0}\,, \tag{4.3.30}$$

Now using the definitions (4.2.35), (4.2.36) for N^μ and $T^{\mu\nu}$, respectively, we can write (4.3.30) in the following form

$$u_\mu u_\nu T^{\mu\nu} > u_\mu N^\mu\,. \tag{4.3.31}$$

Now using the definitions (4.2.8) and (4.2.9) for n and e we finally conclude that $e > n$.

Next we prove part (ii). We know from the part (i) of this lemma that the restriction $e > n > 0$ is necessary. Moreover we know from Lemma 4.2 that $\frac{e}{n} = \Psi(\beta) > 1$ has exactly one solution for $\beta > 0$. Let $T = \frac{1}{\beta} > 0$ be the corresponding temperature. Then we know from $(4.3.1)_1$ that the Jüttner phase density $f_J(n, T, \mathbf{u}, \mathbf{q})$ leads to the prescribed energy density e. ∎

Remark: The restriction $e > n$ is also natural from the physical point of view since it states that the energy density is always larger then the rest-mass energy due to non-zero temperature in the gas.

Lemma 4.4. *For $u, v > 0$ we have*

$$v \ln v - u \ln u = [\ln u + 1](v - u) + R(u, v)\,, \tag{4.3.32}$$

with $R(u, v) \geq 0$.

Proof: Due to the Taylor formula there is a $\xi > 0$ between $u, v > 0$ such that

$$v \ln v = u \ln u + (\ln u + 1)(v - u) + \frac{1}{2\xi}(v - u)^2. \tag{4.3.33}$$

We conclude that $R(u, v) = \frac{1}{2\xi}(v - u)^2 \geq 0$. ∎

Proposition 4.5. *Let $f \in V$ with V defined in (4.2.2) be any phase density with its moments N^μ, $T^{\mu\nu}$. Let n, $\mathbf{u}$, e be the values resulting from f for the particle density, the spatial part of the velocity four-vector and the energy density, respectively. Then there is exactly one temperature $T > 0$ for which the Jüttner phase density $f_J(n, T, \mathbf{u}, \mathbf{q})$ leads to the prescribed energy density e. Let σ and σ_J be the entropy densities corresponding to f and f_J, respectively. Then there holds the Maximum Entropy Inequality $\sigma_J \geq \sigma$. In the case $\sigma_J = \sigma$ we obtain the uniqueness result $f = f_J$ a.e. on $\mathbb{R}^3$.*

Proof: Due to Lemma 4.3 we have exactly one temperature $T > 0$ for which the Jüttner phase density $f_J(n, T, \mathbf{u}, \mathbf{q})$ gives the prescribed energy density $e > n$ coming from the general phase density $f(\mathbf{q})$. In the following proof we will fix this temperature T and the corresponding phase density $f_J(n, T, \mathbf{u}, \mathbf{q})$. Using the definition (4.2.37) and (4.2.11) we have due to the constraint on $\mathbf{u}$

$$\begin{aligned}\sigma_J - \sigma &= S^\mu_J u_\mu - S^\mu u_\mu \\ &= -u_\mu \int_{\mathbb{R}^3} q^\mu \, [(f_J \ln f_J)(n, T, \mathbf{u}, \mathbf{q}) - (f \ln f)(\mathbf{q})] \, \frac{d^3q}{q^0}\,. \end{aligned} \tag{4.3.34}$$

In the following we omit the arguments of f and f_J for the sake of simplicity, which will not lead to confusion here. We use Lemma 4.4 for $u = f_J$, $v = f$ and get

$$f \ln f - f_J \ln f_J = (\ln f_J + 1)(f - f_J) + R(f_J, f), \tag{4.3.35}$$

Using equation (4.3.35) in (4.3.34) we have

$$\begin{aligned}\sigma_J - \sigma =& u_\mu \int_{\mathbb{R}^3} q^\mu \, \ln f_J \, (f - f_J) \, \frac{d^3q}{q^0} + u_\mu \int_{\mathbb{R}^3} q^\mu \, (f - f_J) \frac{d^3q}{q^0} \\ &+ u_\mu \int_{\mathbb{R}^3} q^\mu \, R(f_J, f) \, \frac{d^3q}{q^0}\,. \end{aligned} \tag{4.3.36}$$

The second integral in (4.3.36) is zero due to the constraints on n and $\mathbf{u}$,

$$u_\mu N^\mu - u_\mu N_J^\mu = n - n = 0\,, \tag{4.3.37}$$

so we are left with

$$\begin{aligned}\sigma_J - \sigma =& u_\mu \int_{\mathbb{R}^3} q^\mu\,(f - f_J)\,\ln f_J\,\frac{d^3q}{q^0} \\ &+ u_\mu \int_{\mathbb{R}^3} q^\mu\,R(f_J, f)\,\frac{d^3q}{q^0}\,.\end{aligned} \tag{4.3.38}$$

From Jüttner's phase density we have

$$\begin{aligned}\ln f_J &= \ln\left(\frac{n}{M(\beta)}\exp\left(-\frac{1}{T}u_\nu q^\nu\right)\right) \\ &= \ln\left(\frac{n}{M(\beta)}\right) - \frac{1}{T}u_\nu q^\nu\,.\end{aligned} \tag{4.3.39}$$

Using (4.3.39) in (4.3.38) and the fact that n and T are independent of the integration variable $\mathbf{q}$ we get

$$\begin{aligned}\sigma_J - \sigma =& \ln\left(\frac{n}{M(\beta)}\right)\,u_\mu \int_{\mathbb{R}^3} q^\mu\,(f - f_J)\,\frac{d^3q}{q^0} \\ &- \frac{1}{T}\,u_\mu u_\nu \int_{\mathbb{R}^3} q^\mu q^\nu\,(f - f_J)\frac{d^3q}{q^0} \\ &+ u_\mu \int_{\mathbb{R}^3} q^\mu\,R(f_J, f)\,\frac{d^3q}{q^0}\,.\end{aligned} \tag{4.3.40}$$

In (4.3.40) the first integral is zero due to (4.3.37). Also we know from (4.2.9) and our constraints on the velocity and energy density that

$$e = u_\mu u_\nu T^{\mu\nu} = e_J = u_\mu u_\nu T_J^{\mu\nu}\,, \tag{4.3.41}$$

where $T^{\mu\nu}$ and $T_J^{\mu\nu}$ are the energy momentum tensors for f and f_J, respectively. Thus equation (4.3.40) finally reduces to

$$\sigma_J - \sigma = \int_{\mathbb{R}^3} u_\mu q^\mu\,R(f_J, f)\,\frac{d^3q}{q^0} \geq 0. \tag{4.3.42}$$

The integral in (4.3.42) is non-negative because $u_\mu q^\mu > 1$ due to Lemma 4.1 and because $R(f_J, f)$ is non-negative due to Lemma 4.4. Hence we have proved that Jüttner's phase density satisfies the Maximum Entropy Principle, i.e. $\sigma_J \geq \sigma$. For the uniqueness proof we assume in addition that $\sigma_J = \sigma$ and obtain from (4.3.42) with $R(u, v) = \frac{1}{2\xi}(v - u)^2 \geq 0$ for some ξ between $u, v > 0$ that $f = f_J$ a.e. on $\mathbb{R}^3$. ∎

4.3.5 The one-dimensional relativistic Euler equations, general case

Now we are looking for spatial one-dimensional solutions of the three-dimensional Euler equations, which will not depend on x^2, x^3 but only on $x = x^1$. Moreover we restrict to a one-dimensional flow field $\mathbf{u} = (u(t,x), 0, 0)^T$

$$\begin{aligned} (n\sqrt{1+u^2})_t + (nu)_x &= 0\,, \\ (n\chi(\beta)u\sqrt{1+u^2})_t + (\tfrac{n}{\beta} + n\chi(\beta)u^2)_x &= 0\,, \\ (-\tfrac{n}{\beta} + n\chi(\beta)(1+u^2))_t + (n\chi(\beta)u\sqrt{1+u^2})_x &= 0\,. \end{aligned} \tag{4.3.43}$$

These differential equations constitute a strictly hyperbolic system with the sound speed

$$\lambda_* = \left[\frac{1 - 1/(\beta^2\Psi'(\beta))}{1+\beta\Psi(\beta)}\right]^{\frac{1}{2}}\,, \tag{4.3.44}$$

and the characteristic velocities

$$\begin{aligned} \lambda_1 &= \frac{-\lambda_*\sqrt{1+u^2}+u}{\sqrt{1+u^2}-\lambda_* u}\,,\ \lambda_2 = \frac{u}{\sqrt{1+u^2}}\,, \\ \lambda_3 &= \frac{\lambda_*\sqrt{1+u^2}+u}{\sqrt{1+u^2}+\lambda_* u}\,. \end{aligned} \tag{4.3.45}$$

Note that $\Psi'(\beta) < 0$ was proved in Lemma 4.2.

These eigenvalues may first be obtained in the Lorentz zero rest frame where u=0. Then using the additivity law for the velocities in the general Lorentz frame we can easily obtain (4.3.45).

Note that one can easily get from (4.3.45), the eigenvalues for the classical and ultra-relativistic cases by using the asymptotic relations (4.3.4) and (4.3.6) respectively as given below.

Classical eigenvalues:

$$\lambda_1 = v - \sqrt{\frac{5}{3}T}\,,\ \lambda_2 = v\,,\ \lambda_3 = v + \sqrt{\frac{5}{3}T}\,, \tag{4.3.46}$$

with the velocity $v = \frac{u}{\sqrt{1+u^2}}$, the temperature $T = \frac{1}{\beta}$ and the speed of sound $\lambda_* = \sqrt{\frac{5}{3}T}$.

Ultra-relativistic eigenvalues:

$$\begin{aligned} \lambda_1 &= \frac{2u\sqrt{1+u^2}-\sqrt{3}}{3+2u^2}\,,\ \lambda_2 = \frac{u}{\sqrt{1+u^2}}\,, \\ \lambda_3 &= \frac{2u\sqrt{1+u^2}+\sqrt{3}}{3+2u^2}\,. \end{aligned} \tag{4.3.47}$$

Here we obtain the speed of sound $\lambda_* = \frac{1}{\sqrt{3}}$.

The differential equations (4.3.43) are not sufficient if we take shock discontinuities into account. Therefore we use a weak integral formulation of the one-dimensional hyperbolic system for $n, u, \beta : [0,\infty] \times \mathbb{R} \to \mathbb{R}$ with $n, \beta > 0$, which is given for piecewise C^1-solutions n, u, β according to Oleinik [39] by

$$\begin{aligned} \oint_{\partial\Omega} n\sqrt{1+u^2}dx - nudt = 0\,, \\ \oint_{\partial\Omega} (n\chi(\beta)u\sqrt{1+u^2})dx - (\tfrac{n}{\beta} + n\chi(\beta)u^2)dt = 0\,, \\ \oint_{\partial\Omega} (-\tfrac{n}{\beta} + n\chi(\beta)(1+u^2))dx - (n\chi(\beta)u\sqrt{1+u^2})dt = 0\,. \end{aligned} \tag{4.3.48}$$

Here $\Omega \subset \mathbb{R}_0^+ \times \mathbb{R}$ is a bounded and convex region in space-time with piecewise smooth, positive oriented boundary. Note that this weak formulation takes discontinuities into account, since there are no longer derivatives of the fields. If we apply the Gauss Divergence Theorem in regular time-space regions to the weak formulation (4.3.48) we come back to the differential form of Euler's equation (4.3.43).

Furthermore we require that the weak solution (4.3.48) must also satisfy the one-dimensional *entropy-inequality*

$$\oint_{\partial\Omega} S^0 dx - S^1 dt \geq 0\,, \tag{4.3.49}$$

where

$$\begin{aligned} S^0 &= -n\sqrt{1+u^2}\left(\ln\tfrac{n\beta}{K_2(\beta)} + \beta\Psi(\beta)\right)\,, \\ S^1 &= -nu\left(\ln\tfrac{n\beta}{K_2(\beta)} + \beta\Psi(\beta)\right)\,. \end{aligned} \tag{4.3.50}$$

Now we consider bounded and integrable *initial data* for a positive particle density n, transformed velocity u and absolute temperature T, which may have jumps

$$n(0,x) = n_0(x) > 0, \quad u(0,x) = u_0(x), \quad T(0,x) = T_0(x) > 0. \tag{4.3.51}$$

Parametrizations of single shock solutions

If $x = x(t)$ is a shock-discontinuity of the weak solution (4.3.48) with speed $v_s = \dot{x}(t)$, $W_- = (n_-, u_-, p_-)$ the state left to the shock and $W_+ = (n_+, u_+, p_+)$ the state to the right, then (4.3.48) leads to the RANKINE-HUGONIOT *jump conditions*

$$\begin{aligned} v_s(N_+^0 - N_-^0) &= N_+^1 - N_-^1\,, \\ v_s(T_+^{01} - T_-^{01}) &= T_+^{11} - T_-^{11}\,, \\ v_s(T_+^{00} - T_-^{00}) &= T_+^{01} - T_-^{01}\,, \end{aligned} \tag{4.3.52}$$

where

$$N_\pm^0 = n_\pm\sqrt{1+u_\pm^2}\,, \quad N_\pm^1 = n_\pm u_\pm\,, \quad T_\pm^{01} = n_\pm\chi(\beta_\pm)u_\pm\sqrt{1+u_\pm^2}\,,$$
$$T_\pm^{11} = p_\pm + n_\pm\chi(\beta_\pm)u_\pm^2\,, \quad T_\pm^{00} = -p_\pm + n_\pm\chi(\beta_\pm)(1+u_\pm^2)\,.$$

Also in singular points the local form of (4.3.49) reads

$$-v_s(S_+^0 - S_-^0) + (S_+^1 - S_-^1) \geq 0, \tag{4.3.53}$$

which must be satisfied at each shock curve of (4.3.48). The shock that satisfies (4.3.52) and (4.3.53) is called *entropy shock*.

Now we give parameter representations for the single entropy shocks. For this purpose we choose the initial data as follows:

Let be $(n_*, u_*, \beta_*) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+$ and define $p_* = n_* T_* = \frac{n_*}{\beta_*}$. We use the inverse temperature β as a shock-parameter, impose the restriction $\beta > \beta_*$ and obtain from (4.3.52) and (4.3.53) the following parametrization of the particle density and the pressure

$$\begin{aligned}
\gamma(\beta) &= \Psi(\beta_*)\,\chi(\beta_*) - \Psi(\beta)\,\chi(\beta)\,,\\
n(\beta) &= \frac{n_*\beta}{2\chi(\beta_*)}\left[\sqrt{\gamma(\beta)^2 + 4\frac{\chi(\beta_*)\chi(\beta)}{\beta_*\beta}} - \gamma(\beta)\right],\\
p(\beta) &= \frac{n(\beta)}{\beta}\,.
\end{aligned} \tag{4.3.54}$$

For the parametrization of the velocities we have

$$\begin{aligned}
\hat{u}(\beta) &= \left(\frac{(p_* - p(\beta))\cdot(n_*\Psi(\beta_*) - n(\beta)\,\Psi(\beta))}{n(\beta)\,n_*\chi(\beta)\chi(\beta_*)}\right)^{\frac{1}{2}},\\
u(\beta) &= u_*\sqrt{1+\hat{u}(\beta)^2} \pm \hat{u}(\beta)\sqrt{1+u_*^2}\,,
\end{aligned} \tag{4.3.55}$$

$$\begin{aligned}
\hat{u}_s(\beta) &= \left[\frac{(p_* - p(\beta))\,(p_* + n(\beta)\Psi(\beta))}{n_*\,\chi(\beta_*)\left[n_*(\Psi(\beta_*) - \frac{1}{\beta_*}) - n(\beta)\,(\Psi(\beta) - \frac{1}{\beta})\right]}\right]^{\frac{1}{2}},\\
u_s(\beta) &= u_*\sqrt{1+\hat{u}_s(\beta)^2} \pm \hat{u}_s(\beta)\sqrt{1+{u_*}^2}\,,\\
v_s &= \frac{u_s}{\sqrt{1+u_s^2}}\,,\ v = \frac{u}{\sqrt{1+u^2}}\,,\ v_* = \frac{u_*}{\sqrt{1+u_*^2}}\,.
\end{aligned} \tag{4.3.56}$$

Remarks: All the expressions under the square roots are positive because $n(\beta)$, $\Psi(\beta)$, $\Psi(\beta) - \frac{1}{\beta}$ and $p(\beta)$ are strictly monotonically decreasing and positive functions.

Now there results the following parametrization for the different kind of shock waves:

The "+" sign in (4.3.55), (4.3.56) and $\beta > \beta_*$ gives the so called *3-shocks* with the constant state (n_*, u_*, β_*) on the left

$$(n_-, u_-, \beta_-) = (n_*, u_*, \beta_*), \quad (n_+, u_+, \beta_+) = (n(\beta), u(\beta), \beta).$$

These 3-shocks satisfy both the RANKINE-HUGONIOT conditions (4.3.52) as well as the entropy condition (4.3.53).

The "-" sign in (4.3.55), (4.3.56) and $\beta > \beta_*$ gives the so called *1-shocks* with the constant state (n_*, u_*, β_*) on the right:

$$(n_-, u_-, \beta_-) = (n(\beta), u(\beta), \beta), \quad (n_+, u_+, \beta_+) = (n_*, u_*, \beta_*).$$

These 1-shocks satisfy both the RANKINE-HUGONIOT conditions (4.3.52) as well as the entropy condition (4.3.53).

Now we define the *2-shocks*, that turn out to be *contact-discontinuities without entropy-production*:

Only for these we choose $n > 0$ instead of β as a shock parameter and set

$$(n_-, u_-, \beta_-) = (n_*, u_*, \beta_*), \quad (n_+, u_+, \beta_+) = \left(n, u_*, \frac{n\beta_*}{n_*}\right).$$

These shocks satisfy the RANKINE-HUGONIOT- and entropy conditions. Note that velocity and pressure are constant across a 2-shock. Here the shock-speed is $v_s = v_* = \frac{u_*}{\sqrt{1+u_*^2}}$.

Remark. One can prove that the only shocks satisfying (4.3.52) and (4.3.53) are 1-, 2- and 3-shocks.

4.4 The ultra-relativistic Euler equations

Using the ultra-relativistic Jüttner distribution $f_J^*(n, T, \mathbf{u}, \mathbf{q})$ in (4.2.38), we obtain for the moments (4.2.35), (4.2.36) and (4.2.37)

$$N^\mu = n\, u^\mu, \qquad T^{\mu\nu} = -p\, g^{\mu\nu} + 4pu^\mu u^\nu\,, \tag{4.4.1}$$

$$S^\mu = -N^\mu \ln \frac{n^4}{p^3} + \gamma N^\mu, \qquad \sigma = -n \ln \frac{n^4}{p^3} + \gamma n\,. \tag{4.4.2}$$

Note that due to the mass conservation $(4.2.16)_1$ the divergence of S^μ, which will give rise to the H-theorem formulated later, will not change when we add some multiple of N^μ to S^μ. Moreover σ obeys the Gibbs equation

$$T d\left(\frac{\sigma}{n}\right) = p\, d\left(\frac{1}{n}\right) + d\left(\frac{e}{n}\right). \tag{4.4.3}$$

These formulas can be easily checked for a special Lorentz frame where $u^0 = 1$, $u^1 = u^2 = u^3 = 0$, i.e. where the gas is locally at rest. Since the ultra-relativistic moments (4.4.1) are valid in a special Lorentz frame and since these equations are written in tensor invariant form, they are generally valid in every Lorentz frame. Using the moments (4.4.1) and the conservation laws (4.2.16), we get at regular points the three-dimensional Euler equation in differential form

$$\frac{\partial}{\partial t}\,(n\,\sqrt{1+\mathbf{u}^2}) + \nabla\cdot(n\,\mathbf{u}) = 0\,, \tag{4.4.4}$$

$$\frac{\partial}{\partial t}(4pu^i\sqrt{1+\mathbf{u}^2}) + \sum_{k=1}^{3}\frac{\partial}{\partial x^k}\,(p\,\delta^{ik} + 4pu^iu^k) = 0\,, \tag{4.4.5}$$

$$\frac{\partial}{\partial t}\,(3p + 4p\mathbf{u}^2) + \sum_{k=1}^{3}\frac{\partial}{\partial x^k}\,(4pu^k\sqrt{1+\mathbf{u}^2}) = 0\,. \tag{4.4.6}$$

The equations for the conservation of momentum and energy (4.4.5), (4.4.6) form a closed 4 by 4 subsystem for p and $\mathbf{u}$, the $(p,\mathbf{u})$-subsystem, where the relativistic continuity equation (4.4.4) for n decouples from this subsystem. This is an important feature of the ultra-relativistic Euler equations which will be studied in the sequel.

Now we are looking for special solutions of the three-dimensional ultra-relativistic Euler equations, which will not depend on x^2, x^3 but only on $x = x^1$. Moreover we restrict to a one-dimensional flow field $\mathbf{u} = (u(t,x),0,0)^T$

$$\begin{aligned}(n\sqrt{1+u^2})_t + (nu)_x &= 0\,,\\ (4pu\sqrt{1+u^2})_t + (p(1+4u^2))_x &= 0\,,\\ (p(3+4u^2))_t + (4pu\sqrt{1+u^2})_x &= 0\,.\end{aligned} \tag{4.4.7}$$

These differential equations constitute a strictly hyperbolic system with the characteristic velocities

$$\lambda_1 = \frac{2u\sqrt{1+u^2}-\sqrt{3}}{3+2u^2} < \lambda_2 = \frac{u}{\sqrt{1+u^2}} < \lambda_3 = \frac{2u\sqrt{1+u^2}+\sqrt{3}}{3+2u^2}\,. \tag{4.4.8}$$

These eigenvalues may first be obtained in the Lorentz rest frame where $u = 0$. Then using the relativistic additivity law for the velocities, we can easily obtain (4.4.8) in the general Lorentz frame. In the Lorentz rest frame we obtain the positive speed of sound $\lambda = \frac{1}{\sqrt{3}}$, which is independent of the spatial direction.

But there is another useful derivation for the characteristic speeds. For this purpose we rewrite the 2×2 subsystem for p and u in (4.4.7) in the form

$$\begin{pmatrix} p_t \\ u_t \end{pmatrix} + \begin{pmatrix} \frac{2u\sqrt{1+u^2}}{3+2u^2} & \frac{4p}{\sqrt{1+u^2}(3+2u^2)} \\ \frac{3\sqrt{1+u^2}}{4p(3+2u^2)} & \frac{2u\sqrt{1+u^2}}{3+2u^2} \end{pmatrix} \begin{pmatrix} p_x \\ u_x \end{pmatrix} = 0\,. \tag{4.4.9}$$

Thus we obtain that also the 2×2 subsystem for p and u turns out to be strictly hyperbolic with the characteristic velocities $\lambda_1 < \lambda_3$ in (4.4.8).

The differential equations (4.4.7) are not sufficient if we take shock discontinuities into account. Therefore we use a weak integral formulation with a piecewise C^1-solution $n, u, p : [0, \infty] \times \mathbb{R} \to \mathbb{R}$, $n, p > 0$, which is given according to Oleinik [39] by

$$\begin{aligned} \oint_{\partial\Omega} n\sqrt{1+u^2}dx - nudt = 0\,, \\ \oint_{\partial\Omega} 4pu\sqrt{1+u^2}dx - p(1+4u^2)dt = 0\,, \\ \oint_{\partial\Omega} p(3+4u^2)dx - 4pu\sqrt{1+u^2}dt = 0\,. \end{aligned} \tag{4.4.10}$$

Here $\Omega \subset \mathbb{R}_0^+ \times \mathbb{R}$ is a bounded and convex region in space-time and with a piecewise smooth, positively oriented boundary. If we apply the Gaussian divergence theorem to the weak formulation (4.4.10) in time-space regions where the solution is regular we come back to the differential equation form of the Euler equations (4.4.7).

Furthermore we require that the weak solution (4.4.10) must also satisfy the *entropy-inequality*

$$\oint_{\partial\Omega} S^0 dx - S^1 dt \geq 0 \tag{4.4.11}$$

where

$$S^0 = -n\sqrt{1+u^2}\,\ln\frac{n^4}{p^3}\,, \qquad S^1 = -nu\,\ln\frac{n^4}{p^3}\,. \tag{4.4.12}$$

The subsystem has an own entropy inequality, which reads in one space dimension

$$\oint_{\partial\Omega} \tilde{S}^0 dx - \tilde{S}^1 dt \geq 0\,. \tag{4.4.13}$$

Here the *reduced thermodynamical entropy four-vector* is defined in one space dimension by

$$\tilde{S}^0 = \alpha\, p^{3/4}\sqrt{1+u^2}\,, \quad \tilde{S}^1 = \alpha\, p^{3/4}u\,, \tag{4.4.14}$$

where $\alpha > 0$ is constant. This entropy four-vector is independent of n and satisfies an additional conservation law in the points (t, x) of smoothness, namely

$$\frac{\partial \tilde{S}^0}{\partial t} + \frac{\partial \tilde{S}^1}{\partial x} = 0\,, \tag{4.4.15}$$

which can be obtained with the help of (4.4.9) . Formally this equation can be obtained by putting $n = \alpha p^{3/4}$ in the continuity equation $(4.4.7)_1$ of the larger system. However, when shocks are involved in the weak solutions of the subsystem, then the reduced entropy will increase. We will show that (4.4.11) and (4.4.13) are indeed equivalent for piecewise smooth solutions.

Parametrizations of single shock solutions

If $x = x(t)$ is a shock-discontinuity of the weak solution (4.4.10) with speed $v_s = \dot{x}(t)$, $W_- = (n_-, u_-, p_-)$ the state left to the shock and $W_+ = (n_+, u_+, p_+)$ the state to the right, then (4.4.10) leads to the RANKINE-HUGONIOT *jump conditions*

$$\begin{aligned}
v_s[n_+\sqrt{1+u_+^2} - n_-\sqrt{1+u_-^2}] &= n_+u_+ - n_-u_-\,,\\
v_s[4p_+u_+\sqrt{1+u_+^2} - 4p_-u_-\sqrt{1+u_-^2}] &= (p_+ + 4p_+u_+^2) - (p_- + 4p_-u_-^2)\,,\\
v_s[(3p_+ + 4p_+u_+^2) - (3p_- + 4p_-u_-^2)] &= 4p_+u_+\sqrt{1+u_+^2} - 4p_-u_-\sqrt{1+u_-^2}\,.
\end{aligned} \tag{4.4.16}$$

Also in singular points the local form of (4.4.11) reads

$$-v_s(S_+^0 - S_-^0) + (S_+^1 - S_-^1) \geq 0, \tag{4.4.17}$$

which must be satisfied on each shock curve of (4.4.10). A shock that satifies (4.4.16) and (4.4.17) is called an *entropy shock*.

Now we give parameter representations for single entropy shocks. For this purpose we choose the initial data as follows:

Let be $(n_*, u_*, T_*) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+$ and define $p_* = n_*T_*$.

We use the pressure p as a parameter which determines the strength of an entropy shock. Equations (4.4.16) and (4.4.17) are solved by

$$n(p) = n_* \sqrt{\frac{p}{p_*}\left(\frac{3p+p_*}{p+3p_*}\right)}, \tag{4.4.18}$$

$$u(p) = \frac{u_*\sqrt{p_*+3p}\sqrt{p+3p_*} \pm \sqrt{3}(p-p_*)\sqrt{1+u_*^2}}{4\sqrt{pp_*}}, \tag{4.4.19}$$

$$T(p) = \frac{p}{n(p)}, \tag{4.4.20}$$

$$u_s(p) = \frac{u_*\sqrt{3(p+3p_*)} \pm \sqrt{p_*+3p}\sqrt{1+u_*^2}}{\sqrt{8p_*}}, \tag{4.4.21}$$

$$v_s = \frac{u_s}{\sqrt{1+u_s^2}}, \quad v = \frac{u}{\sqrt{1+u^2}}, \quad v_* = \frac{u_*}{\sqrt{1+u_*^2}} \tag{4.4.22}$$

in the following way:

- The "+" sign in (4.4.19), (4.4.21) and $p < p_*$ gives the so called *3-shocks* with the constant state (n_*, u_*, T_*) on the left

  $$(n_-, u_-, T_-) = (n_*, u_*, T_*), \quad (n_+, u_+, T_+) = (n(p), u(p), T(p)).$$

  These 3-shocks satisfy both the RANKINE-HUGONIOT conditions (4.4.16) as well as the entropy condition (4.4.17).

- The "-" sign in (4.4.19), (4.4.21) and $p < p_*$ gives the so called *1-shocks* with the constant state (n_*, u_*, T_*) on the right:

  $$(n_-, u_-, T_-) = (n(p), u(p), T(p)), \quad (n_+, u_+, T_+) = (n_*, u_*, T_*).$$

  These 1-shocks satisfy both the RANKINE-HUGONIOT conditions (4.4.16) as well as the entropy condition (4.4.17).

Now we define the *2-shocks*, that turn out to be *contact-discontinuities without entropy-production*:

Only for these we choose $n > 0$ instead of p as the parameter and set

$$(n_-, u_-, T_-) = (n_*, u_*, T_*), \quad (n_+, u_+, T_+) = \left(n, u_*, \frac{n_*T_*}{n}\right).$$

These shocks satisfy the RANKINE-HUGONIOT- and entropy conditions.

Note that velocity and pressure are constant across a 2-shock. Here the shock-speed is $v_s = v_* = \frac{u_*}{\sqrt{1+u_*^2}}$.

Remark. One can prove that the only shocks satisfying (4.4.16) and (4.4.17) are 1-, 2- and 3-shocks.

The structure of these shock solutions is quite similar to the classical Euler equations.

Since the subsystem in p and u has its own entropy, we obtain for the subsystem the following local form for the increasing of the reduced entropy across the shock front

$$-v_s(\tilde{S}^0_+ - \tilde{S}^0_-) + (\tilde{S}^1_+ - \tilde{S}^1_-) \geq 0\,. \tag{4.4.23}$$

It is important to note that both entropies, the original one given by (4.4.2) and the reduced entropy for the subsystem given by (4.4.14), lead to the same entropy shocks, i.e. they lead to equivalent shock selection criteria. The equivalence of the local entropy inequalities (4.4.17) and (4.4.23) across a single shock front can be checked without big effort by applying a proper Lorentz-transformation which transforms u_* to 0 in the shock parameter representations above. The inverse Lorentz-transformation will then preserve these inequalities. We will make essential use of this trick when we construct the general Riemann solution for the $(p, \mathbf{u})$-subsystem.

In view of (4.4.15) and in view of the equivalence of the local shock conditions (4.4.17) and (4.4.23) we thus conclude that (4.4.11) and (4.4.13) are indeed equivalent. This equivalence holds at least for piecewise smooth solutions due to standard arguments for the decomposition of curve integrals.

In the case of the classical Euler equations the so called fan-type shaped rarefaction waves play a key role as building blocks for the Riemann solutions beside the shock waves. The same is also true for the relativistic Euler equations. Before we construct the Riemann solutions we need the

Parametrizations of rarefaction waves

A rarefaction wave with center at the origin $t = 0$, $x = 0$ is a smooth solution of Euler's equations which depends only on the characteristic speed $s = \frac{x}{t}$. If we define the 2×2-matrix in (4.4.9) by $A = A(p, u)$ and assume that p and u depend only on $s = \frac{x}{t}$, then we immediately obtain from these equations with the chain rule

$$A(p,u)\begin{pmatrix}\dot{p}\\ \dot{u}\end{pmatrix} = s\begin{pmatrix}\dot{p}\\ \dot{u}\end{pmatrix}\,. \tag{4.4.24}$$

It follows that s is indeed a characteristic speed of the subsystem, i.e.

$$s = \frac{2u\sqrt{1+u^2} \pm \sqrt{3}}{3+2u^2}\,, \quad u = \sqrt{\frac{3}{2}}\,\frac{s \mp \frac{1}{\sqrt{3}}}{\sqrt{1-s^2}}\,. \tag{4.4.25}$$

Inserting $(4.4.25)_1$ in (4.4.24) leads without any effort to the condition

$$\frac{\dot{p}}{p} = \pm\frac{4}{\sqrt{3}}\frac{\dot{u}}{\sqrt{1+u^2}}\,, \tag{4.4.26}$$

and the general primitive with respect to s on both sides of this equation is with a constant C

$$\ln p = C + \frac{4}{\sqrt{3}} \ln(\sqrt{1+u^2} \pm u)\,. \tag{4.4.27}$$

In the same way we obtain from the continuity equation for n with (4.4.25)

$$\ln n = C' + \sqrt{3} \ln(\sqrt{1+u^2} \pm u)\,. \tag{4.4.28}$$

We finally obtain for positive real numbers a, b the parametrization of the rarefaction fans with respect to the characteristic speed s

$$u(s) = \sqrt{\frac{3}{2}}\, \frac{s \mp \frac{1}{\sqrt{3}}}{\sqrt{1-s^2}}, \tag{4.4.29}$$

$$p(s) = a \left(\sqrt{1+u(s)^2} \pm u(s)\right)^{\frac{4}{\sqrt{3}}} = a \left((2-\sqrt{3})\frac{1 \pm s}{1 \mp s}\right)^{\frac{2}{\sqrt{3}}}\,, \tag{4.4.30}$$

$$n(s) = b \left(\sqrt{1+u(s)^2} \pm u(s)\right)^{\sqrt{3}} = b \left((2-\sqrt{3})\frac{1 \pm s}{1 \mp s}\right)^{\frac{\sqrt{3}}{2}}\,. \tag{4.4.31}$$

Sometimes it is also useful to take the pressure p as a parameter for the rarefaction waves and to rewrite (4.4.29)-(4.4.31) in the form

$$u(p) = \pm \frac{\left(\frac{p}{a}\right)^{\frac{\sqrt{3}}{2}} - 1}{2\left(\frac{p}{a}\right)^{\frac{\sqrt{3}}{4}}}, \tag{4.4.32}$$

$$n(p) = b \left(\frac{p}{a}\right)^{\frac{3}{4}}\,, \tag{4.4.33}$$

$$s(p) = \pm \frac{\left(\frac{p}{a}\right)^{\frac{\sqrt{3}}{2}} + \sqrt{3} - 2}{\left(\frac{p}{a}\right)^{\frac{\sqrt{3}}{2}} - \sqrt{3} + 2}\,. \tag{4.4.34}$$

In both parametrizations the upper sign represents the 3-waves and the lower sign the 1-waves.

Solution of the Riemann problem

Now we solve the weak form of the hyperbolic 2×2 subsystem

$$\begin{aligned} (4pu\sqrt{1+u^2})_t + (p(1+4u^2))_x &= 0\,, \\ (p(3+4u^2))_t + (4pu\sqrt{1+u^2})_x &= 0 \end{aligned} \tag{4.4.35}$$

for given Riemannian initial data

$$p_0(x) = \begin{cases} p_-\,, & x \le 0\,, \\ p_+\,, & x > 0\,, \end{cases} \quad , \quad u_0(x) = \begin{cases} u_-\,, & x \le 0\,, \\ u_+\,, & x > 0\,. \end{cases} \tag{4.4.36}$$

The basic ingredients for Riemann solutions are the parametrizations of shocks and rarefaction waves studied before. Moreover, whenever the weak solution for the subsystem is available, it is not difficult to extend it to a weak solution of the one-dimensional 3×3 system (4.4.7). The continuity equation can be solved afterwards by using the parametrizations of n across the shocks and rarefaction waves. In Section 4.9 this is illustrated for an explicit Riemann solution of the 3×3 system which serves as a numerical test case. The correct entropy conditions are always satisfied, because we only make use of the parameter representations for admissible shock fronts.

In order to prepare the construction of the Riemann solutions we start with two important simplifications.

The first simplification is based on the fact that whenever (p, u) is a weak solution of (4.4.35), then also $(\kappa p, u)$ for a positive constant κ, i.e. we can apply a homogeneous scaling of the pressure with the scaling factor κ.

The second simplification is more interesting. It is based on the fact that the weak solutions are invariant with respect to the following homogeneous Lorentz-transformations in dimensionless form

$$t' = at + bx\,, \quad x' = bt + ax\,, \tag{4.4.37}$$

where $a > 1$ and b are real parameters which satisfy the condition $a^2 - b^2 = 1$. These are the Lorentz-boosts in x-directions studied in Section 3.8.

Lemma 4.6. *The expression*

$$\frac{\sqrt{1+u_+^2} - u_+}{\sqrt{1+u_-^2} - u_-}$$

is invariant with respect to the Lorentz-boost (4.4.37)*, if applied to the relativistic velocity vectors from the initial data* (4.4.36)*.*

Proof: The relativistic velocity vectors from the initial data and their Lorentz-transforms are $(\sqrt{1+u_\pm^2}, u_\pm)^T$ and

$$\begin{pmatrix} \sqrt{1+u_\pm'^2} \\ u_\pm' \end{pmatrix} = \begin{pmatrix} a\sqrt{1+u_\pm^2} + b\,u_\pm \\ b\sqrt{1+u_\pm^2} + a\,u_\pm \end{pmatrix}.$$

We obtain that

$$\begin{aligned} \frac{\sqrt{1+u_+'^2} - u_+'}{\sqrt{1+u_-'^2} - u_-'} &= \frac{(a\sqrt{1+u_+^2} + b\,u_+) - (b\sqrt{1+u_+^2} + a\,u_+)}{(a\sqrt{1+u_-^2} + b\,u_-) - (b\sqrt{1+u_-^2} + a\,u_-)} \\ &= \frac{(a-b)\sqrt{1+u_+^2} - (a-b)\,u_+}{(a-b)\sqrt{1+u_-^2} - (a-b)\,u_-} = \frac{\sqrt{1+u_+^2} - u_+}{\sqrt{1+u_-^2} - u_-}\,, \end{aligned}$$

which is the statement of the Lemma. ∎

We define the equivalence of two Riemann solutions if it is possible to map them on each other by an appropriate scaling of the pressure and by applying an appropriate Lorentz-transformation (4.4.37). Then we can state that equivalent Riemann solutions depend only on the two positive parameters

$$\alpha = \frac{p_+}{p_-}\,, \quad \beta = \frac{\sqrt{1+u_+^2}-u_+}{\sqrt{1+u_-^2}-u_-}\,. \tag{4.4.39}$$

This equivalence also allows us to simplify calculations with the single-shock parametrizations (4.4.18)-(4.4.22) considerably by assuming first the special case

$$p_* = 1\,, \quad u_* = 0\,. \tag{4.4.40}$$

Afterwards one can easily remove these restrictions by applying an appropriate Lorentz-transformation (4.4.37) and an appropriate scaling of the pressure.

Our next aim is to rewrite the parametrizations for single shocks and rarefaction waves in terms of the Lorentz-invariant quantities α and β. Then we are also able to handle the Riemann problem in general.

For this purpose we first define two functions which turn out to be very useful in order to perform these parametrizations in a completely unified way. For $\alpha > 0$ we put

$$K_S(\alpha) = \frac{\sqrt{1+3\alpha}\,\sqrt{3+\alpha}+\sqrt{3}\,(\alpha-1)}{4\sqrt{\alpha}}\,, \quad K_R(\alpha) = \alpha^{\sqrt{3}/4}\,. \tag{4.4.41}$$

Then the following Lemma gives a simple characterization for the case that the Riemannian initial data (4.4.36) can be connected by a single shock or rarefaction wave, respectively.

Lemma 4.7. *The functions $K_S(\alpha)$ and $K_R(\alpha)$ are strictly monotonically increasing for $\alpha > 0$ and they satisfy*

$$K_S(1/\alpha) = 1/K_S(\alpha)\,, \quad K_R(1/\alpha) = 1/K_R(\alpha)\,.$$

Moreover, $\alpha = 1$ is a tangent point for both curves with

$$K_S(1) = K_R(1) = 1\,, \quad K_S'(1) = K_R'(1) = \frac{\sqrt{3}}{4}\,, \quad K_S''(1) = K_R''(1) = \frac{3-4\sqrt{3}}{16}\,.$$

Define α and β according to (4.4.39) with $p_\pm > 0$.

(i) *For $\beta\, K_R(\alpha) = 1$ the two conditions $\alpha > 1$ and $\beta < K_S(\alpha)$ are equivalent. If all these conditions are satisfied, then the initial data* (4.4.36) *can be connected by a single 3-rarefaction wave.*

(ii) *For $\beta\, K_S(\alpha) = 1$ the two conditions $\alpha < 1$ and $\beta > K_R(\alpha)$ are equivalent. If all these conditions are satisfied, then the initial data* (4.4.36) *can be connected by a single 3-shock.*

(iii) *For $\beta = K_R(\alpha)$ the two conditions $\alpha < 1$ and $\beta\, K_S(\alpha) < 1$ are equivalent. If all these conditions are satisfied, then the initial data* (4.4.36) *can be connected by a single 1-rarefaction wave.*

(iv) *For $\beta = K_S(\alpha)$ the two conditions $\alpha > 1$ and $\beta\, K_R(\alpha) > 1$ are equivalent. If all these conditions are satisfied, then the initial data* (4.4.36) *can be connected by a single 1-shock.*

Remark: The trivial case $\beta = \alpha = 1$ means that $p_- = p_+$ and $u_- = u_+$.

Proof: The first part of the Lemma can be obtained by straight forward calculations.

We first consider a single 1-shock. According to (4.4.40) we put $p = p_- < p_* = p_+ = 1$ with $\alpha > 1$ and $u_* = u_+ = 0$ in (4.4.19) with the lower minus sign and obtain that

$$u_- = -\frac{\sqrt{3}}{4}\,\frac{(p_- - p_+)}{\sqrt{p_-\,p_+}} = \frac{\sqrt{3}}{4}\,\left(\sqrt{\alpha} - \sqrt{\frac{1}{\alpha}}\right)\,.$$

From this equation we obtain that

$$\beta = \sqrt{1 + u_-^2} + u_- = K_S(\alpha) > 1\,,$$

which is the proof of case (iv) of the lemma. The case (ii) for a single 3-shock follows in the same way.

Next we consider a single 1-rarefaction wave. In this case we can directly conclude from (4.4.30) with the lower minus sign that

$$\frac{p_+}{p_-} = \alpha = \beta^{\frac{4}{\sqrt{3}}}\,,$$

and this gives the desired statement $\beta = K_R(\alpha)\,.$

The case (i) for a single 3-rarefaction wave follows in the same way. ∎

After we have studied the single shock and single wave Riemann solutions we are now able to construct the general Riemann solutions by studying four cases. These cases are illustrated in Figure 4.1, where the behaviour of the

solution depends on the corresponding four regions depicted in this figure. For each case we will employ monotonicity arguments as well as the Mean Value Theorem in order to construct an intermediate state (p_*, u_*) in the so called "star-region" which can be connected with a single shock or rarefaction wave to the prescribed left and right Riemannian initial data. The boundaries of these four regions characterize the initial data for which the Riemann solution consists only of a single shock or rarefaction wave in the sense of Lemma 4.7.

Case 1: $\beta < K_S(\alpha)$ and $\beta\, K_R(\alpha) > 1$.

Here we can construct a solution which is composed on a lower 1-shock and an upper 3-rarefaction wave. Define the intermediate pressure and velocity by the implicit equations

$$K_S(\frac{p_*}{p_-})\, K_R(\frac{p_*}{p_+}) = \beta\,, \quad \frac{\sqrt{1+u_*^2} - u_*}{\sqrt{1+u_-^2} - u_-} = K_S(\frac{p_*}{p_-})\,. \tag{4.4.42}$$

Since $K_S(\frac{p_*}{p_-})\, K_R(\frac{p_*}{p_+})$ is strictly monotonically increasing with

$$\lim_{p_*\to 0} K_S(\frac{p_*}{p_-})\, K_R(\frac{p_*}{p_+}) = 0\,, \quad \lim_{p_*\to\infty} K_S(\frac{p_*}{p_-})\, K_R(\frac{p_*}{p_+}) = \infty\,,$$

we conclude that the implicit equation for $p_* > 0$ and hence for u_* has a unique solution. From $1 < \beta\, K_R(\alpha) < K_S(\alpha)\, K_R(\alpha)$ we obtain that $\alpha > 1$ and hence $p_- < p_+$. Since

$$K_S(\frac{p_-}{p_-})\, K_R(\frac{p_-}{p_+}) = 1/K_R(\alpha) < \beta\,, \quad K_S(\frac{p_+}{p_-})\, K_R(\frac{p_+}{p_+}) = K_S(\alpha) > \beta\,,$$

we obtain from the mean value theorem that $p_- < p_* < p_+$. This implies $K_S(\frac{p_*}{p_-}) > 1$ and $K_S(\frac{p_*}{p_+}) < 1$. From the last two inequalities and (4.4.42) we obtain $u_* < \min(u_-, u_+)$.

We summarize these inequalities for the first case:

$$\alpha > 1\,, \quad p_- < p_* < p_+ \quad \text{and} \quad u_* < \min(u_-, u_+)\,. \tag{4.4.43}$$

From (4.4.42), (4.4.43) and the preceeding Lemma we conclude that the states (p_-, u_-) and (p_*, u_*) can be connected by a 1-shock and that the states (p_*, u_*) and (p_+, u_+) can be connected by a 3-rarefaction wave.

In order to guarantee that the 1-shock with velocity v_s and the 3-rarefaction wave fit together to a complete Riemann solution of (4.4.35) and (4.4.36), we only have to check that

$$v_s < \lambda_3(u_*)\,.$$

This inequality is valid in the special case $p_* = 1$ and $u_* = 0$ with $v_s = -\frac{1}{\sqrt{3}}\sqrt{\frac{1+3p_-}{3+p_-}} < 0$, $\lambda_3(u_*) = \lambda_3(0) = \frac{1}{\sqrt{3}} > 0$ and hence valid in the general case, because any ordering of propagation velocities is invariant with respect to proper Lorentz-transformations (4.4.37).

Case 2: $\beta > K_S(\alpha)$ and $\beta\, K_S(\alpha) > 1$.

Here we can construct a solution which is composed on a lower 1-shock and an upper 3-shock. Define the intermediate pressure and velocity by the implicit equations

$$K_S(\frac{p_*}{p_-})\, K_S(\frac{p_*}{p_+}) = \beta\,, \quad \frac{\sqrt{1+u_*^2}-u_*}{\sqrt{1+u_-^2}-u_-} = K_S(\frac{p_*}{p_-})\,. \tag{4.4.44}$$

From $\beta > K_S(\alpha)$ and $\beta > 1/K_S(\alpha) = K_S(1/\alpha)$ we conclude that $\beta > 1$, because one of the two numbers α or $1/\alpha$ is larger than 1. Since $K_S(\frac{p_*}{p_-})\, K_S(\frac{p_*}{p_+})$ is strictly monotonically increasing in p_* with

$$\lim_{p_*\to 0} K_S(\frac{p_*}{p_-})\, K_S(\frac{p_*}{p_+}) = 0\,, \quad \lim_{p_*\to\infty} K_S(\frac{p_*}{p_-})\, K_S(\frac{p_*}{p_+}) = \infty\,,$$

we conclude that the implicit equation for $p_* > 0$ and hence for u_* has a unique solution. Since

$$K_S(\frac{p_-}{p_-})\, K_S(\frac{p_-}{p_+}) = 1/K_S(\alpha) < \beta\,, \quad K_S(\frac{p_+}{p_-})\, K_S(\frac{p_+}{p_+}) = K_S(\alpha) < \beta\,,$$

we obtain from the monotonicity of $K_S(\frac{p_*}{p_-})\, K_S(\frac{p_*}{p_+})$ with respect to p_* that $p_* > \max(p_-, p_+)$. This implies $K_S(\frac{p_*}{p_\pm}) > 1$. From the last two inequalities and (4.4.44) we obtain $u_- > u_* > u_+$.

We summarize these inequalities for the second case:

$$\beta > 1\,, \quad p_* > \max(p_-, p_+) \quad \text{and} \quad u_- > u_* > u_+\,. \tag{4.4.45}$$

From (4.4.44), (4.4.45) and the preceeding Lemma we conclude that the states (p_-, u_-) and (p_*, u_*) can be connected by a 1-shock and that the states (p_*, u_*) and (p_+, u_+) can be connected by a 3-shock.

In order to guarantee that the 1-shock with velocity $v_s^{(1)}$ and the 3-shock with velocity $v_s^{(3)}$ fit together to a complete Riemann solution of (4.4.35) and (4.4.36), we only have to check that

$$v_s^{(1)} < v_s^{(3)}\,.$$

This inequality is valid in the special case $p_* = 1$ and $u_* = 0$ with $v_s^{(1)} = -\frac{1}{\sqrt{3}}\sqrt{\frac{1+3p_-}{3+p_-}} < 0\,,\ v_s^{(3)} = \frac{1}{\sqrt{3}}\sqrt{\frac{1+3p_+}{3+p_+}} > 0\,,$ and hence valid in the general case, because any ordering of propagation velocities is invariant with respect to proper Lorentz-transformations (4.4.37).

Case 3: $\beta > K_R(\alpha)$ and $\beta\, K_S(\alpha) < 1$.

Here we can construct a solution which is composed on a lower 1-rarefaction wave and an upper 3-shock. Define the intermediate pressure and velocity

by the implicit equations

$$K_R(\frac{p_*}{p_-})\,K_S(\frac{p_*}{p_+}) = \beta\,, \quad \frac{\sqrt{1+u_*^2}-u_*}{\sqrt{1+u_-^2}-u_-} = K_R(\frac{p_*}{p_-})\,. \tag{4.4.46}$$

Again the implicit equations have a unique solution for p_* and u_*, and similar to the first case we obtain in the third case the inequalities

$$\alpha < 1\,, \quad p_- > p_* > p_+ \quad \text{and} \quad u_* > \max(u_-, u_+)\,. \tag{4.4.47}$$

From (4.4.46), (4.4.47) and the preceeding Lemma we conclude that the states (p_-, u_-) and (p_*, u_*) can be connected by a 1-rarefaction wave and that the states (p_*, u_*) and (p_+, u_+) can be connected by a 3-shock. They form a complete Riemann solution because in the special case $p_* = 1$ and $u_* = 0$ the characteristic slopes of the 1-rarefaction wave are negative, whereas the velocity of the upper 3-shock is positive.

Case 4: $\beta < K_R(\alpha)$ and $\beta\,K_R(\alpha) < 1$.

Here we can construct a solution which is composed on a lower 1-rarefaction and an upper 3-rarefaction wave. Define the intermediate pressure and velocity by the implicit equations

$$K_R(\frac{p_*}{p_-})\,K_R(\frac{p_*}{p_+}) = \beta\,, \quad \frac{\sqrt{1+u_*^2}-u_*}{\sqrt{1+u_-^2}-u_-} = K_R(\frac{p_*}{p_-})\,. \tag{4.4.48}$$

In this case the intermediate fields can be solved explicitly by

$$\frac{p_*}{p_-} = \alpha^{1/2}\,\beta^{2/\sqrt{3}}\,, \quad \frac{\sqrt{1+u_*^2}-u_*}{\sqrt{1+u_-^2}-u_-} = \alpha^{\sqrt{3}/8}\,\beta^{1/2}\,. \tag{4.4.49}$$

From $\beta < K_R(\alpha)$ and $\beta < 1/K_R(\alpha) = K_R(1/\alpha)$ we conclude that $\beta < 1$, because one of the two numbers α or $1/\alpha$ is less than 1. Similar to the second case we obtain in the last case the inequalities

$$\beta < 1\,, \quad p_* < \min(p_-, p_+) \quad \text{and} \quad u_- < u_* < u_+\,. \tag{4.4.50}$$

From (4.4.48), (4.4.50) and the preceeding Lemma we conclude that the states (p_-, u_-) and (p_*, u_*) can be connected by a 1-rarefaction wave and that the states (p_*, u_*) and (p_+, u_+) can be connected by a 3-rarefaction wave.

They form a complete Riemann solution because in the special case $p_* = 1$ and $u_* = 0$ the characteristic slopes of the 1-rarefaction wave are negative, whereas the characteristic slopes of the 3-rarefaction wave are positive. ∎

The following sketch describes the behaviour of the Riemann solutions in each of the four cases.

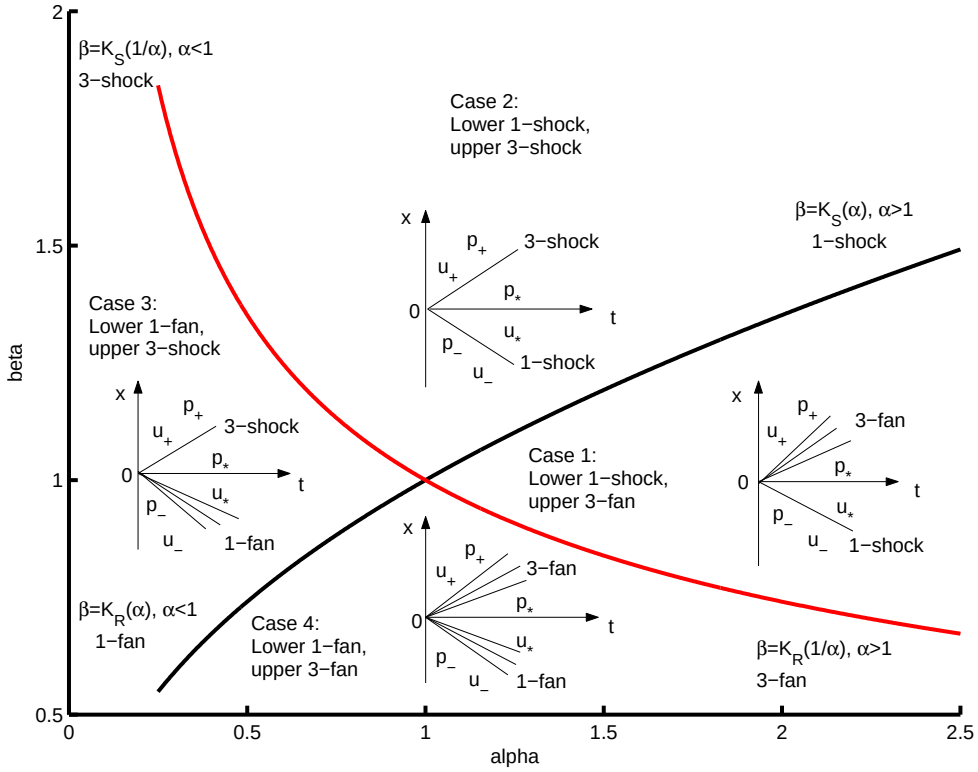

Figure 4.1: Classification of the Riemann solutions for the subsystem.

4.5 A kinetic scheme for the ultra-relativistic Euler equations

We first formulate the scheme for the three-dimensional Euler equations. After that we solve the one-dimensional Euler equations, using a special integration technique. Recalling the ultra-relativistic Jüttner phase density (4.2.38), we start with the given initial data

$$n_I(\mathbf{x}) = n(0,\mathbf{x}) \geq \epsilon_1, \quad T_I(\mathbf{x}) = T(0,\mathbf{x}) \geq \epsilon_2, \quad \mathbf{u}_I(\mathbf{x}) = u(0,\mathbf{x}) \in \mathbb{R}^3$$

for sufficiently small $\epsilon_1, \epsilon_2 > 0$. We assume that at least $n_I, T_I, \mathbf{u}_I \in L^\infty(\mathbb{R}^3)$. We prescribe a time step $\tau_M > 0$ and let $t_n = n\,\tau_M$ for $n = 0, 1, 2, 3...$ be the maximization times. Then we define the moments and the entropy four-vector in the free flight for $0 < \tau < \tau_M$ as

$$\begin{aligned} N^\mu(t_n + \tau, \mathbf{x}) &= \int_{\mathbb{R}^3} q^\mu \, f_n(\mathbf{x} - \tau \frac{\mathbf{q}}{|\mathbf{q}|}, \mathbf{q}) \frac{d^3q}{|\mathbf{q}|}\,, \\ T^{\mu\nu}(t_n + \tau, \mathbf{x}) &= \int_{\mathbb{R}^3} q^\mu q^\nu \, f_n(\mathbf{x} - \tau \frac{\mathbf{q}}{|\mathbf{q}|}, \mathbf{q}) \frac{d^3q}{|\mathbf{q}|}\,, \end{aligned} \tag{4.5.1}$$

$$S^\mu(t_n+\tau,\mathbf{x}) = -\int_{\mathbb{R}^3} q^\mu \,(f_n \ln f_n)(\mathbf{x}-\tau\frac{\mathbf{q}}{|\mathbf{q}|},\mathbf{q})\frac{d^3q}{|\mathbf{q}|} \tag{4.5.2}$$

with the ultra-relativistic initial phase density at the maximization time t_n given as

$$f_n(\mathbf{y},\mathbf{q}) = f_J^*(n(t_n^+,\mathbf{y}),T(t_n^+,\mathbf{y}),\mathbf{u}(t_n^+,\mathbf{y}),\mathbf{q})\,. \tag{4.5.3}$$

Moreover n, T, u^μ are calculated from N^μ and $T^{\mu\nu}$ for the next time step from the following generally valid definitions

$$n = \sqrt{N^\mu N_\mu}, \qquad u^\mu = \frac{1}{n}N^\mu\,, \qquad T = \frac{1}{3n}u_\mu u_\nu\, T^{\mu\nu}\,. \tag{4.5.4}$$

In order to initialize the kinetic scheme for the next time step, we first require the following continuity conditions for the zero-components of the moments across the maximization time t_n, $n \geq 1$

$$\begin{aligned} N^0(t_n^+,\mathbf{x}) &= N^0(t_n^-,\mathbf{x})\,,\\ T^{0k}(t_n^+,\mathbf{x}) &= T^{0k}(t_n^-,\mathbf{x}),\ k=1,2,3\,,\\ T^{00}(t_n^+,\mathbf{x}) &= T^{00}(t_n^-,\mathbf{x})\,. \end{aligned} \tag{4.5.5}$$

Here we have used the following abbreviations for the one-sided limits across the maximization time t_n, $n \geq 1$, where ε is a positive real number

$$\begin{aligned} N^\mu(t_n^\pm,\mathbf{x}) &= \lim_{\varepsilon\to 0} N^\mu(t_n\pm\varepsilon,\mathbf{x})\,,\\ T^{\mu\nu}(t_n^\pm,\mathbf{x}) &= \lim_{\varepsilon\to 0} T^{\mu\nu}(t_n\pm\varepsilon,\mathbf{x})\,. \end{aligned}$$

Later on we will see in Proposition 4.9 that these conditions are necessary in order to guarantee the conservation laws for mass, momentum and energy across the maximization time t_n. Moreover we start again with a ultra-relativistic Jüttner distribution for the next time step. Then we obtain, using the constitutive relations (4.4.1), for the three-dimensional Euler equations which are valid for the t_n^+ side of the maximization time

$$\begin{aligned} N^0(t_n^+,\mathbf{x}) &= n(t_n^+,\mathbf{x})\sqrt{1+\mathbf{u}^2(t_n^+,\mathbf{x})}\,,\\ T^{0k}(t_n^+,\mathbf{x}) &= 4\,p(t_n^+,\mathbf{x})\,u^k(t_n^+,\mathbf{x})\sqrt{1+\mathbf{u}^2(t_n^+,\mathbf{x})}\,,\\ T^{00}(t_n^+,\mathbf{x}) &= p(t_n^+,\mathbf{x})\left[3+4\,\mathbf{u}^2(t_n^+,\mathbf{x})\right]\,. \end{aligned} \tag{4.5.6}$$

Here $k = 1,2,3$ again denote a spatial indices. Since these components of the moments are continuous across the maximization time t_n, we can replace them by the free-flight moments for t_n^- and solve the equations (4.5.6) for $p, \mathbf{u}, n$ in order to initialize the kinetic scheme for the next time step by the following relations

$$
\begin{aligned}
p(t_n^+,\mathbf{x}) &= \frac{1}{3}\left[-T^{00}+\sqrt{4(T^{00})^2-3\sum_{k=1}^{3}(T^{0k})^2}\,\right],\\
u^k(t_n^+,\mathbf{x}) &= \frac{T^{0k}}{\sqrt{4p(t_n^+,\mathbf{x})[p(t_n^+,\mathbf{x})+T^{00}]}},\\
n(t_n^+,\mathbf{x}) &= \frac{N^0}{\sqrt{1+\sum_{k=1}^{3}[u^k(t_n^+,\mathbf{x})]^2}}.
\end{aligned}
\tag{4.5.7}
$$

In these formulas N^0, T^{00} and T^{0k} are abbreviations for the free-flight moments $N^0(t_n^\pm,\mathbf{x})$, $T^{00}(t_n^\pm,\mathbf{x})$ and $T^{0k}(t_n^\pm,\mathbf{x})$ respectively, which are continuous across each maximization time.

Note that the quantities on the left hand side have to be calculated in the prescribed order from the free-flight moments N^0, T^{00} and T^{0k}. Since they initialize the scheme for the next time step they conclude the formulation of the kinetic scheme.

But, we can still apply an important simplification of the volume integrals (4.5.1) and (4.5.2) for the free-flight moments. We can see with (4.5.3) that the fields $n(t,\mathbf{y})$, $T(t,\mathbf{y})$ and $\mathbf{u}(t,\mathbf{y})$ are not depending on $|\mathbf{q}|$ but only on the unit vector $\mathbf{w}=(w^1,w^2,w^3)^T=\frac{\mathbf{q}}{|\mathbf{q}|}$. This fact enables us to reduce the three-fold volume integrals to two-fold surface integrals over the unit sphere by applying polar coordinates. Then the integration with respect to $|\mathbf{q}|$ can be carried out explicitly. We introduce the abbreviations

$$
\begin{aligned}
\Phi(\mathbf{y},\mathbf{w}) &= \frac{1}{4\pi}\frac{n(t_n^+,\mathbf{y})}{(\sqrt{1+\mathbf{u}^2(t_n^+,\mathbf{y})}-\mathbf{w}\cdot\mathbf{u}(t_n^+,\mathbf{y}))^3},\\
\Psi(\mathbf{y},\mathbf{w}) &= \frac{1}{4\pi}\frac{3p(t_n^+,\mathbf{y})}{(\sqrt{1+\mathbf{u}^2(t_n^+,\mathbf{y})}-\mathbf{w}\cdot\mathbf{u}(t_n^+,\mathbf{y}))^4}.
\end{aligned}
\tag{4.5.8}
$$

Carrying out the integration with respect to $|\mathbf{q}|$ explicitly and defining the quantity $w^0=1$ in addition to $w^k=\frac{q^k}{|\mathbf{q}|}$, we obtain the following reduced surface integrals for the moments, where $0<\tau<\tau_M$

$$
N^\mu(t_n+\tau,\mathbf{x}) = \oint_{\partial B(1,\mathbf{0})} w^\mu\,\Phi(\mathbf{x}-\tau\mathbf{w},\mathbf{w})\,dS(\mathbf{w}),
\tag{4.5.9}
$$

$$
T^{\mu\nu}(t_n+\tau,\mathbf{x}) = \oint_{\partial B(1,\mathbf{0})} w^\mu\,w^\nu\,\Psi(\mathbf{x}-\tau\mathbf{w},\mathbf{w})\,dS(\mathbf{w}).
\tag{4.5.10}
$$

Here $B(r, \mathbf{x}_0)$ denotes the ball with radius r and center $\mathbf{x}_0$, and its boundary is the sphere $\partial B(r, \mathbf{x}_0)$. These surface integrals with respect to the unit sphere reflect the fact that in the ultra-relativistic case the particles are moving on the surface of the light cone. Using the Cauchy-Schwarz inequality one can prove that n, p, e resulting from the moment integrals (4.5.9) and (4.5.10) are well defined and positive quantities for all times and positions.

Proposition 4.8. *Let $0 < \tau < \tau_M$ and $n = 0, 1, 2, ...$ We consider the moments in the free flight between the two maximization times t_n and t_{n+1}. Within this free-flight zone the moments $N^\mu(t_n + \tau, \mathbf{x})$, $T^{\mu\nu}(t_n + \tau, \mathbf{x})$ and the entropy four-vector $S^\mu(t_n + \tau, \mathbf{x})$ satisfy the following conservation laws in weak integral form*

$$\oint\limits_{\partial\Omega} N^\nu(t_n + \tau, \mathbf{x}) do_\nu = 0, \ \oint\limits_{\partial\Omega} T^{\mu\nu}(t_n + \tau, \mathbf{x}) do_\nu = 0, \ \oint\limits_{\partial\Omega} S^\nu(t_n + \tau, \mathbf{x}) do_\nu = 0\,.$$

Here the covariant vector do_ν is a positively oriented surface element to the boundary $\partial\Omega$. It can be written in covariant form as

$$do_\kappa = \varepsilon_{\kappa\lambda\mu\nu} \sum_{i,j,m=1}^{3} \frac{\partial x^\lambda}{\partial u^i} \frac{\partial x^\mu}{\partial u^j} \frac{\partial x^\nu}{\partial u^m} du^i\, du^j\, du^m\,,$$

where $x^\alpha = x^\alpha(u^1, u^2, u^3)$ is a positively oriented parameterization of the boundary $\partial\Omega$.

Remark. Note that these weak formulations correspond to the differential equations

$$\frac{\partial N^\nu}{\partial x^\nu}(t_n + \tau, \mathbf{x}) = 0, \quad \frac{\partial T^{\mu\nu}}{\partial x^\nu}(t_n + \tau, \mathbf{x}) = 0, \quad \frac{\partial S^\nu}{\partial x^\nu}(t_n + \tau, \mathbf{x}) = 0. \quad (4.5.11)$$

Proof: For $0 < \tau < \tau_M$ let be $t = t_n + \tau$. If we start with the relativistic Maxwellian (4.5.3) as the initial phase density at the time t_n then we obtain within the time-region $0 < t_n < t < t_n + \tau_M$ the free-flight density $f(t, \mathbf{x}, \mathbf{q}) = f(\mathbf{x} - \tau\frac{\mathbf{q}}{|\mathbf{q}|}, \mathbf{q})$, which satisfies the following weak form of the free-flight equation

$$\oint\limits_{\partial\Omega} q^\nu\, f(t, \mathbf{x}, \mathbf{q}) do_\nu = q^\nu \oint\limits_{\partial\Omega} f(t, \mathbf{x}, \mathbf{q}) do_\nu = 0\,. \quad (4.5.12)$$

The equation (4.5.12) and its multiplication with q^μ leads after integration

with respect to $\mathbf{q}$ to the following equations

$$\int\limits_{\mathbb{R}^3} \left(\oint\limits_{\partial\Omega} q^\nu \, f(t,\mathbf{x},\mathbf{q}) do_\nu \right) \frac{d^3q}{|\mathbf{q}|} = 0\,,$$
$$\int\limits_{\mathbb{R}^3} \left(\oint\limits_{\partial\Omega} q^\mu \, q^\nu \, f(t,\mathbf{x},\mathbf{q}) do_\nu \right) \frac{d^3q}{|\mathbf{q}|} = 0\,. \tag{4.5.13}$$

Since the volume integral with respect to $\mathbf{q}$ and the surface integral with respect to t and $\mathbf{x}$ are interchangeable, we can rewrite equations (4.5.1) in order to get the conservation laws

$$\oint\limits_{\partial\Omega} \left(\int\limits_{\mathbb{R}^3} q^\nu \, f(t,\mathbf{x},\mathbf{q}) \frac{d^3q}{|\mathbf{q}|} \right) do_\nu = \oint\limits_{\partial\Omega} N^\nu(t_n + \tau\mathbf{x},\mathbf{q}) do_\nu = 0\,,$$
$$\oint\limits_{\partial\Omega} \left(\int\limits_{\mathbb{R}^3} q^\mu \, q^\nu \, f(t,\mathbf{x},\mathbf{q}) \frac{d^3q}{|\mathbf{q}|} \right) do_\nu = \oint\limits_{\partial\Omega} T^{\mu\nu}(t_n + \tau\mathbf{x},\mathbf{q}) do_\nu = 0\,. \tag{4.5.14}$$

Next we define

$$\psi(t,\mathbf{x},\mathbf{q}) = -(f_n \ln f_n)(\mathbf{x} - \tau\frac{\mathbf{q}}{|\mathbf{q}|},\mathbf{q})\,, \tag{4.5.15}$$

and we conclude due to the chain rule that ψ satisfies the weak form of the free-flight equation $q^\nu \frac{\partial\psi}{\partial x^\nu} = 0$, namely

$$\oint\limits_{\partial\Omega} q^\nu \, \psi(t,\mathbf{x},\mathbf{q}) do_\nu = 0\,. \tag{4.5.16}$$

This is coming from the Gauss Divergence Theorem.

Integrating this equation with respect to $\mathbf{q}$ and interchanging the volume and surface integrals, we finally get, using equation (4.5.2)

$$\oint\limits_{\partial\Omega} \left(\int\limits_{\mathbb{R}^3} q^\nu \, \psi(t,\mathbf{x},\mathbf{q}) \frac{d^3q}{|\mathbf{q}|} \right) do_\nu = \oint\limits_{\partial\Omega} S^\nu(t_n + \tau\mathbf{x},\mathbf{q}) do_\nu = 0\,. \tag{4.5.17}$$

■

Proposition 4.9. *Let $\Omega \subset \mathbb{R}_0^+ \times \mathbb{R}^3$ be any bounded convex region in time and space. By do_ν we denote the positively oriented surface element of $\partial\Omega$. Let $\tau_M > 0$ be a fixed time step. The moment representations* (4.5.1) *and* (4.5.2) *calculated by the iterated scheme defined above have the following properties:*

(i) The conservation laws for the particle number, the momentum and energy hold, i.e.

$$\oint\limits_{\partial\Omega} N^\nu do_\nu = 0, \quad \oint\limits_{\partial\Omega} T^{\mu\nu} do_\nu = 0. \tag{4.5.18}$$

(ii) The following entropy inequality is satisfied

$$\oint\limits_{\partial\Omega} S^\nu do_\nu \geq 0. \tag{4.5.19}$$

Proof: Let be $\tau_M > 0$. We first prove part (i) of the proposition. The time axis is divided by the maximization times $0 = t_0 < t_1 < t_2 < \cdots$, so that the convex domain Ω can be decomposed into the subdomains

$$\left\{ \begin{array}{ll} \Omega_0 & = \left\{ (\delta, \mathbf{x}) \in \Omega \middle| \, 0 \leq \delta \leq \frac{t_0+t_1}{2} \right\}, \\ \Omega_n & = \left\{ (\delta, \mathbf{x}) \in \Omega \middle| \, \frac{t_{n-1}+t_n}{2} \leq \delta \leq \frac{t_n+t_{n+1}}{2} \right\} \quad n = 1, 2, 3, \ldots. \end{array} \right. \tag{4.5.20}$$

Since $\oint_{\partial\Omega} N^\nu do_\nu = \sum_{n\geq 0} \oint_{\partial\Omega_n} N^\nu do_\nu$ and $\oint_{\partial\Omega} T^{\mu\nu} do_\nu = \sum_{n\geq 0} \oint_{\partial\Omega_n} T^{\mu\nu} do_\nu$, it is sufficient to assume without loss of generality that the time range

$$\Theta_\Omega = \left\{ t \geq 0 \middle| \text{ there exists an } \mathbf{x} \in \mathbb{R}^3 \text{ such that } (t, \mathbf{x}) \in \Omega \right\}$$

of Ω contains at most one maximization time t.

Then for ε in the range $0 < \varepsilon < \frac{1}{2}\tau_M$ we define a further decomposition of each $\Omega_n, n \geq 1$, into three parts

$$\left\{ \begin{array}{ll} \Omega_{n,L}^\varepsilon & = \left\{ (\delta, \mathbf{x}) \in \Omega_n \middle| \, \delta \leq t_n - \varepsilon \right\}, \\ \Omega_{n,M}^\varepsilon & = \left\{ (\delta, \mathbf{x}) \in \Omega_n \middle| \, t_n - \varepsilon \leq \delta \leq t_n + \varepsilon \right\}, \\ \Omega_{n,R}^\varepsilon & = \left\{ (\delta, \mathbf{x}) \in \Omega_n \middle| \, \delta \geq t_n + \varepsilon \right\}. \end{array} \right. \tag{4.5.21}$$

The decompositions which are illustrated in the following figure, were also applied in order to prove the conservation laws and the entropy inequality for the classical Euler equations, see [7].

We obtain

$$\oint\limits_{\partial\Omega_n} N^\nu do_\nu = \oint\limits_{\partial\Omega_{n,L}^\varepsilon} N^\nu do_\nu + \oint\limits_{\partial\Omega_{n,R}^\varepsilon} N^\nu do_\nu + \oint\limits_{\partial\Omega_{n,M}^\varepsilon} N^\nu do_\nu,$$

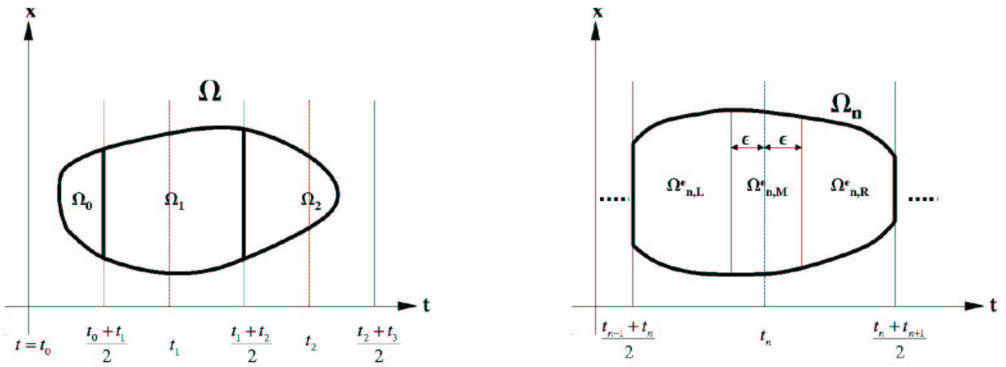

Figure 4.2: The decompositions of Ω and Ω_n.

$$\oint\limits_{\partial\Omega_n} T^{\mu\nu} do_\nu = \oint\limits_{\partial\Omega^\varepsilon_{n,L}} T^{\mu\nu} do_\nu + \oint\limits_{\partial\Omega^\varepsilon_{n,R}} T^{\mu\nu} do_\nu + \oint\limits_{\partial\Omega^\varepsilon_{n,M}} T^{\mu\nu} do_\nu$$

since the first two integrals on the right hand side are in the free-flight zone, so we conclude from equation (4.5.14) that these integrals vanish, i.e.,

$$\oint\limits_{\partial\Omega^\varepsilon_{n,L}} N^\nu do_\nu = \oint\limits_{\partial\Omega^\varepsilon_{n,R}} N^\nu do_\nu = 0, \qquad \oint\limits_{\partial\Omega^\varepsilon_{n,L}} T^{\mu\nu} do_\nu = \oint\limits_{\partial\Omega^\varepsilon_{n,R}} T^{\mu\nu} do_\nu = 0.$$

This implies, using $\Omega^*_n = \{\mathbf{x} \in \mathbb{R}^3 | (t_n, \mathbf{x}) \in \Omega\}$

$$\begin{aligned} \oint\limits_{\partial\Omega_n} N^\nu do_\nu &= \oint\limits_{\partial\Omega^\varepsilon_{n,M}} N^\nu do_\nu = \lim_{\varepsilon\to 0} \oint\limits_{\partial\Omega^\varepsilon_{n,M}} N^\nu do_\nu \\ &= \int_{\Omega^*_n} \left\{ \int q^0 \left[f_n(\mathbf{x},\mathbf{q}) - f_{n-1}(\mathbf{x} - \tau_M \frac{\mathbf{q}}{|\mathbf{q}|}) \right] \frac{d^3q}{|\mathbf{q}|} \right\} d^3\mathbf{x} \end{aligned}$$

and

$$\begin{aligned} \oint\limits_{\partial\Omega_n} T^{\mu\nu} do_\nu &= \oint\limits_{\partial\Omega^\varepsilon_{n,M}} T^{\mu\nu} do_\nu = \lim_{\varepsilon\to 0} \oint\limits_{\partial\Omega^\varepsilon_{n,M}} T^{\mu\nu} do_\nu \\ &= \int_{\Omega^*_n} \left\{ \int q^0 q^\mu \left[f_n(\mathbf{x},\mathbf{q}) - f_{n-1}(\mathbf{x} - \tau_M \frac{\mathbf{q}}{|\mathbf{q}|}) \right] \frac{d^3q}{|\mathbf{q}|} \right\} d^3x, \end{aligned}$$

where t_{n-1} is the maximization time that preceedes the maximization time t_n. The phase density f_n has to be taken to be the ultra-relativistic Jüttner phase density (4.2.38).

The last integral expression in these equations vanishes due to the continuity conditions (4.5.5) across the maximization time t_n, which yields

$$\int_{\mathbb{R}^3} q^0 f_n(\mathbf{x},\mathbf{q})\frac{d^3q}{|\mathbf{q}|} = \int_{\mathbb{R}^3} q^0 f_{n-1}(\mathbf{x}-\tau_M\frac{\mathbf{q}}{|\mathbf{q}|},\mathbf{q})\frac{d^3q}{|\mathbf{q}|}\,,$$
$$\int_{\mathbb{R}^3} q^0 q^\mu f_n(\mathbf{x},\mathbf{q})\frac{d^3q}{|\mathbf{q}|} = \int_{\mathbb{R}^3} q^0 q^\mu f_{n-1}(\mathbf{x}-\tau_M\frac{\mathbf{q}}{|\mathbf{q}|},\mathbf{q})\frac{d^3q}{|\mathbf{q}|}\,. \tag{4.5.22}$$

This expresses the constraints that were used for the maximization procedure. We have thus established that the weak form (4.5.18) for an arbitrary bounded convex domain Ω is implied by the representations (4.5.1).

Regarding the second part (ii) which states the existence of the entropy inequality (4.5.19), we start the proof again with the decompositions (4.5.20) and (4.5.21) of Ω. Since $\int_{\partial\Omega} S^\nu do_\nu = \sum_{n\geq 0}\int_{\partial\Omega_n} S^\nu do_\nu$, it is sufficient to prove $\int_{\partial\Omega_n} S^\nu do_\nu \geq 0$ for each n. We obtain

$$\oint_{\partial\Omega_n} S^\nu do_\nu = \oint_{\partial\Omega^\varepsilon_{n,L}} S^\nu do_\nu + \oint_{\partial\Omega^\varepsilon_{n,R}} S^\nu do_\nu + \oint_{\partial\Omega^\varepsilon_{n,M}} S^\nu do_\nu\,. \tag{4.5.23}$$

Again the first two integrals lie in the free-flight zone. We can see from equation (4.5.17) that these integrals vanishes i.e., $\oint_{\partial\Omega^\varepsilon_{n,R}} S^\nu do_\nu = 0$, and $\oint_{\partial\Omega^\varepsilon_{n,L}} S^\nu do_\nu = 0$.
For every sufficiently small $\varepsilon > 0$ the following holds

$$\begin{aligned}\oint_{\partial\Omega_n} S^\nu do_\nu &= \lim_{\varepsilon\to 0}\oint_{\partial\Omega^\varepsilon_{n,M}} S^\nu do_\nu \qquad (4.5.24)\\ &= \int_{\Omega_n^*}\left\{\int q^0\left[-(f_n\ln f_n)(\mathbf{x},\mathbf{q}) + (f_{n-1}\ln f_{n-1})(\mathbf{x}-\tau_M\frac{\mathbf{q}}{|\mathbf{q}|},\mathbf{q})\right]\frac{d^3q}{|\mathbf{q}|}\right\}d^3x,\end{aligned}$$

where $\Omega_n^* = \{\mathbf{x}\in\mathbb{R}^3 | (t_n,\mathbf{x})\in\Omega\}$, and $t_{n-1} < t_n$ is the maximization time that preceeds t_n. ∎

Next we shall show that the integral (4.5.24) is non-negative. To this end we need the following

Lemma 4.10. *For $u, v > 0$ we have*

$$v\ln v - u\ln u = [\ln u + 1](v-u) + R(u,v) \tag{4.5.25}$$

with a function $R(u,v)\geq 0$.

Proof of the Lemma: Due to TAYLORs formula there is a $\xi > 0$ between $u, v > 0$ such that

$$v\ln v = u\ln u + (\ln u + 1)(v-u) + \frac{1}{2\xi}(v-u)^2. \tag{4.5.26}$$

We conclude that $R(u,v) = \frac{1}{2\xi}(v-u)^2 \geq 0$. ∎

Continuation of proof of Proposition 4.9: Now we apply Lemma 4.10 to $u = f_n(\mathbf{x},\mathbf{q})$ and $v = f_{n-1}(\mathbf{x} - \tau_M \frac{\mathbf{q}}{|\mathbf{q}|}, \mathbf{q})$

$$\begin{aligned}\int_{\mathbb{R}^3} q^0 \left[-(f_n \ln f_n)(\mathbf{x},\mathbf{q}) + (f_{n-1}\ln f_{n-1})(\mathbf{x}-\tau_M\frac{\mathbf{q}}{|\mathbf{q}|},\mathbf{q})\right]\frac{d^3q}{|\mathbf{q}|} \\ = -\int_{\mathbb{R}^3} q^0\left[1+\ln f_n(\mathbf{x},\mathbf{q})\right]\left[f_n(\mathbf{x},\mathbf{q}) - f_{n-1}(\mathbf{x}-\tau_M\frac{\mathbf{q}}{|\mathbf{q}|},\mathbf{q})\right]\frac{d^3q}{|\mathbf{q}|} \\ + \int_{\mathbb{R}^3} R\left(f_n(\mathbf{x},\mathbf{q}), f_{n-1}(\mathbf{x}-\tau_M\frac{\mathbf{q}}{|\mathbf{q}|},\mathbf{q})\right)\frac{d^3q}{|\mathbf{q}|}. \qquad (4.5.27)\end{aligned}$$

The second integral is non-negative and the first one vanishes due to the following reasons. Using Jüttner's phase density for $f_n(\mathbf{x},\mathbf{q})$ we have

$$\ln f_n(\mathbf{x},\mathbf{q}) = \ln\left[\frac{n(\mathbf{x})}{8\pi T^3(\mathbf{x})}\exp\left(\frac{-u_\nu q^\nu}{T(\mathbf{x})}\right)\right] \quad = A(x) - B(x)u_\nu q^\nu\,, \quad (4.5.28)$$

where $A(\mathbf{x}) = \ln\frac{n(\mathbf{x})}{8\pi T^3(\mathbf{x})}$ and $B(\mathbf{x}) = \frac{1}{T(\mathbf{x})}$. We use the value (4.5.28) of $\ln f_n(\mathbf{x},\mathbf{q})$ in (4.5.27). Using the definitions (4.5.1) for N^μ, $T^{\mu\nu}$ and the continuity conditions (4.5.5) for the zero components N^0, $T^{0\nu}$, we can see that the first integral in (4.5.27) is zero.

We have thus established the entropy inequality (4.5.19). ∎

4.6 A kinetic scheme for the one-dimensional ultra-relativistic equations

In the following we are looking for spatially one-dimensional solutions, which are nevertheless solutions to the fully three-dimensional equations. We only consider solutions which depend on t and $x = x^1$ and satisfy $n = n(t,x)$, $\mathbf{u} = (u(t,x),0,0)$, $p = p(t,x)$. We will use the generally valid equation $p = nT$ and go back to the fully three-dimensional scheme.

In order to calculate the surface integrals (4.5.9) and (4.5.10) we introduce instead of the unit vector $\mathbf{w}$ the new variables $-1 \leq \xi \leq 1$ and $0 \leqq \varphi \leqq 2\pi$ by

$$w^1 = \xi\,, \qquad w^2 = \sqrt{1-\xi^2}\sin\varphi\,, \qquad w^3 = \sqrt{1-\xi^2}\cos\varphi \qquad (4.6.1)$$

with the surface element $dS(\mathbf{w}) = d\xi d\varphi$.

Note that the quantities n, p, u in the integrals (4.5.9) and (4.5.10) do not depend on the variable φ. This fact enables us to carry out the integration with respect to φ directly. Thus the two-fold surface integral reduces to a simple $\xi-$integral. For abbreviation we introduce

$$\phi(y,\xi) = \frac{1}{2}\frac{n(t_n^+, y)}{(\sqrt{1+u^2(t_n^+, y)} - \xi u(t_n^+, y))^3}\,,$$
$$\psi(y,\xi) = \frac{3}{2}\frac{p(t_n^+, y)}{(\sqrt{1+u^2(t_n^+, y)} - \xi u(t_n^+, y))^4}\,, \tag{4.6.2}$$

then the reduced integrals for the moments can be written as

$$N^0(t_n+\tau, x) = \int_{-1}^{1} \phi(x-\tau\xi, \xi)\, d\xi\,, \quad N^1(t_n+\tau, x) = \int_{-1}^{1} \xi\phi(x-\tau\xi, \xi)\, d\xi\,, \tag{4.6.3}$$

$$\begin{aligned} T^{00}(t_n+\tau, x) &= \int_{-1}^{1} \psi(x-\tau\xi, \xi)\, d\xi\,, \\ T^{01}(t_n+\tau, x) &= \int_{-1}^{1} \xi\psi(x-\tau\xi, \xi)\, d\xi\,, \\ T^{11}(t_n+\tau, x) &= \int_{-1}^{1} \xi^2\psi(x-\tau\xi, \xi)\, d\xi\,. \end{aligned} \tag{4.6.4}$$

As in (4.5.9) and (4.5.10), these finite integrals reflect the fact that in the ultra-relativistic case the particles are moving on the surface of the light cone. Moreover we obtain

$$N^2(t_n+\tau, x) = N^3(t_n+\tau, x) = 0\,, \quad T^{10}(t_n+\tau, x) = T^{01}(t_n+\tau, x)\,,$$
$$T^{22}(t_n+\tau, x) = T^{33}(t_n+\tau, x) = \frac{1}{2}[T^{00}(t_n+\tau, x) - T^{11}(t_n+\tau, x)]\,,$$

where all the other components of $T^{\mu\nu}$ are zero. So in the one-dimensional case n, u, p and T can be found from the generally valid relations given in Section 4.2 as follows

$$n(t_n+\tau, x) = \sqrt{(N^0(t_n+\tau, x))^2 - (N^1(t_n+\tau, x))^2}\,, \tag{4.6.5}$$
$$u(t_n+\tau, x) = \frac{1}{n}N^1(t_n+\tau, x)\,, \tag{4.6.6}$$
$$\begin{aligned} p(t_n+\tau, x) = \frac{1}{3}[\{1+u^2(t_n+\tau, x)\}T^{00}(t_n+\tau, x) - 2u\sqrt{1+u^2(t_n+\tau, x)} \\ \cdot T^{01}(t_n+\tau, x) + u^2(t_n+\tau, x)T^{11}(t_n+\tau, x)]\,. \end{aligned} \tag{4.6.7}$$

We can now simplify the equations (4.5.7), which are used in order to initialize the general three-dimensional scheme, and obtain for the one-dimensional case

$$p(t_n^+, \mathbf{x}) = \frac{1}{3}\left[-T^{00} + \sqrt{4(T^{00})^2 - 3(T^{01})^2}\,\right], \tag{4.6.8}$$

$$u(t_n^+, \mathbf{x}) = \frac{T^{01}}{\sqrt{4p(t_n^+, \mathbf{x})[p(t_n^+, \mathbf{x}) + T^{00}]}}, \tag{4.6.9}$$

$$n(t_n^+, \mathbf{x}) = \frac{N^0}{\sqrt{1 + u(t_n^+, \mathbf{x})^2}}. \tag{4.6.10}$$

Here again $N^0 = N^0(t_n^\pm, \mathbf{x})$, $T^{00} = T^{00}(t_n^\pm, \mathbf{x})$ and $T^{01} = T^{01}(t_n^\pm, \mathbf{x})$ are given by the free-flight moments which are continuous across the maximization times.

4.7 From the kinetic scheme to the Eulerian limit ($\tau_M \to 0$)

In the previous sections we have shown how to calculate the solution of the kinetic scheme in the three- and one-dimensional case, respectively. This was done for the prescribed initial data of n, u and p for a given free-flight time step $\tau_M > 0$. If we calculate these solutions for $\tau_M \to 0$ then we get the Eulerian limit

$$N^\mu \to n\,u^\mu, \qquad T^{\mu\nu} \to -p\,g^{\mu\nu} + 4pu^\mu u^\nu, \qquad S^\mu \to n\,u^\mu \ln\frac{n^4}{p^3}. \tag{4.7.1}$$

First we pass to the Eulerian limit (4.7.1) at the points of smoothness in the following way using (4.2.15) with $Q(f) = 0$

$$\begin{aligned}
\lim_{\tau\to 0}\frac{\partial}{\partial\tau}N^0(t_n+\tau,\mathbf{x}) &= \lim_{\tau\to 0}\frac{\partial}{\partial\tau}(n(t_n+\tau,\mathbf{x})\sqrt{1+u^2(t_n+\tau,\mathbf{x})}\,) \\
&= \lim_{\tau\to 0}\frac{\partial}{\partial\tau}\int_{\mathbb{R}^3}|\mathbf{q}|f_n(\mathbf{x}-\tau\frac{\mathbf{q}}{|\mathbf{q}|},\mathbf{q})\frac{d^3q}{|\mathbf{q}|} \\
&= -\lim_{\tau\to 0}\int_{\mathbb{R}^3}|\mathbf{q}|\sum_{k=1}^3\frac{q^k}{|\mathbf{q}|}\frac{\partial}{\partial x^k}f_n(\mathbf{x}-\tau\frac{\mathbf{q}}{|\mathbf{q}|},\mathbf{q})\frac{d^3q}{|\mathbf{q}|} \\
&= -\int_{\mathbb{R}^3}\sum_{k=1}^3 q^k\frac{\partial}{\partial x^k}f_n(\mathbf{x},\mathbf{q})\frac{d^3q}{|\mathbf{q}|} \\
&= -\sum_{k=1}^3\frac{\partial}{\partial x^k}(u^k(t_n^+,\mathbf{x})n(t_n^+,\mathbf{x})) \\
&= -\nabla\cdot(\mathbf{u}(t_n^+,\mathbf{x})n(t_n^+,\mathbf{x}))\,.
\end{aligned}$$

This implies

$$\frac{\partial}{\partial t}(n(t_n+\tau,\mathbf{x})\sqrt{1+u^2(t_n^+,\mathbf{x})}\,)+\nabla\cdot(\mathbf{u}(t_n^+,\mathbf{x})n(t_n^+,\mathbf{x}))=0\,, \qquad (4.7.2)$$

which is the first ultra-relativistic Euler equation (4.4.4). Similarly we get the other Euler equations (4.4.5) and (4.4.6) if we differentiate $T^{0k}(t_n+\tau,\mathbf{x})$ and $T^{00}(t_n+\tau,\mathbf{x})$ with respect to τ and pass to the limit $\tau\to 0$.

Secondly, on the left hand sides of (4.7.1) there are the moments N^μ, $T^{\mu\nu}$ and S^μ as calculated by the kinetic scheme see (4.5.1) and (4.5.2). Since we have already established the conservation laws and the entropy inequality for the solution of the kinetic scheme in Proposition 4.9, we conclude from (4.7.1) that this also results for the weak entropy solution in the Eulerian limit $\tau_M\to 0$. The weak entropy solution in the Eulerian limit in one space dimension is given by (4.4.10), (4.4.11) and (4.4.12).

4.8 The kinetic scheme for the subsystem

Here we shall also use the kinetic scheme with the reduced surface integrals (4.5.10) to solve the $(p,\mathbf{u})$-subsystem. In order to calculate p and $\mathbf{u}$ from the previous scheme, the function Φ defined in $(4.5.8)_2$ is not used, the same with the equations involving N^μ or n in $(4.5.5)_1$, $(4.5.6)_1$, $(4.5.7)_3$, (4.5.9). These parts have to be replaced by a kinetic definition of a reduced entropy four-vector exclusively defined for the subsystem.

In order to avoid confusions, we formulate the whole reduced scheme again, even if there are some repetitions, but we will complement them by a kinetic redefinition of the entropy four-vector.

We define the two functions

$$\Psi(\mathbf{y},\mathbf{w})=\frac{1}{4\pi}\frac{3p(t_n^+,\mathbf{y})}{(\sqrt{1+\mathbf{u}^2(t_n^+,\mathbf{y})}-\mathbf{w}\cdot\mathbf{u}(t_n^+,\mathbf{y}))^4}\,,\quad \tilde{\Phi}(\mathbf{y},\mathbf{w})=\Psi(\mathbf{y},\mathbf{w})^{3/4}\,, \qquad (4.8.1)$$

where the function Ψ is given as before, but we will call it now the reduced phase density (for the subsystem). The new quantity $\tilde{\Phi}=\Psi^{3/4}$ will be used to define a much easier entropy equivalent to (4.5.2) for the kinetic scheme of the subsystem.

The reduced surface integrals for the energy-momentum tensor in the subsystem are the same as before, namely for $0\le\tau<\tau_M$

$$T^{\mu\nu}(t_n+\tau,\mathbf{x})=\oint\limits_{\partial B(1,\mathbf{0})} w^\mu w^\nu\,\Psi(\mathbf{x}-\tau\mathbf{w},\mathbf{w})\,dS(\mathbf{w})\,. \qquad (4.8.2)$$

However, the kinetic definition of the following reduced entropy four-vector $\tilde{S}^\mu$ now overtakes the role of N^μ in (4.5.9) and of S^μ in (4.5.2), but with the redefined function $\tilde{\Phi} = \Psi^{3/4}$:

$$\tilde{S}^\mu(t_n + \tau, \mathbf{x}) = \alpha \oint\limits_{\partial B(1,\mathbf{0})} w^\mu \, \Psi^{3/4}(\mathbf{x} - \tau\mathbf{w}, \mathbf{w}) \, dS(\mathbf{w}) \, . \tag{4.8.3}$$

Here $\alpha > 0$ is an arbitrary positive constant.

We initialize the scheme as before, i.e. the **equilibrium state** is used to calculate p and $\mathbf{u}$ in the next time step $t_{n+1} = t_n + \tau_M$:

$$p(t_{n+1}^+, \mathbf{x}) = \frac{1}{3} \left[-T^{00} + \sqrt{4(T^{00})^2 - 3 \sum_{k=1}^{3} (T^{0k})^2} \right] ,$$

$$u^k(t_{n+1}^+, \mathbf{x}) = \frac{T^{0k}}{\sqrt{4p(t_{n+1}^+, \mathbf{x})[p(t_{n+1}^+, \mathbf{x}) + T^{00}]}}$$

with $T^{00} = T^{00}(t_{n+1}, \mathbf{x})$, $T^{0k} = T^{0k}(t_{n+1}, \mathbf{x})$.

Recall that the tensor components $T^{0\mu}$ are continuous across the time steps!

The Maximum Entropy Principle for the reduced entropy of the $(p, \mathbf{u})$ subsystem

The following considerations do not depend on time-space arguments of the fields and phase densities. We thus supress the time-space arguments in the sequel. Here macroscopic fields like p, $\mathbf{u}$, $T^{0\mu}$ are regarded as parameters.

Recall the definition of the zero-component of the reduced entropy four-vector in terms of a general reduced phase density $\Psi = \Psi(\mathbf{w}) \in L^\infty(\partial B(1, \mathbf{0}))$, $\Psi \geq \eta$ for some constant $\eta > 0$, where we have put $\alpha = 1$ without loss of generality:

$$\tilde{S}^0[\Psi] = \oint\limits_{\partial B(1,\mathbf{0})} \Psi^{3/4}(\mathbf{w}) \, dS(\mathbf{w}) \, .$$

We want to maximize $\tilde{S}^0$ under the following four constraints on the reduced phase density $\Psi(\mathbf{w})$:

$$T^{00} = \oint\limits_{\partial B(1,\mathbf{0})} \Psi(\mathbf{w}) \, dS(\mathbf{w}) \, ,$$

$$T^{0k} = \oint\limits_{\partial B(1,\mathbf{0})} w^k \, \Psi(\mathbf{w}) \, dS(\mathbf{w}) \, .$$

Lemma 4.11. *For $\Psi_1, \Psi_2 > 0$ we have*

$$\Psi_1^{3/4} = \Psi_2^{3/4} - \tfrac{3}{4}\Psi_1^{-1/4}(\Psi_2 - \Psi_1) + \tilde{R}(\Psi_1, \Psi_2)$$

with the function $\tilde{R}(\Psi_1, \Psi_2) = \Psi_1^{3/4} - \Psi_2^{3/4} + \frac{3}{4}\Psi_1^{-1/4}(\Psi_2 - \Psi_1) \geq 0$.

Proof of the Lemma: Due to TAYLORs formula there is a $\xi > 0$ between $\Psi_1, \Psi_2 > 0$ such that

$$\Psi_2^{3/4} = \Psi_1^{3/4} + \frac{3}{4}\Psi_1^{-1/4}(\Psi_2 - \Psi_1) - \frac{3}{32}\xi^{-5/4}(\Psi_2 - \Psi_1)^2.$$

We conclude that $\tilde{R}(\Psi_1, \Psi_2) = \frac{3}{32}\xi^{-5/4}(\Psi_2 - \Psi_1)^2 \geq 0$. ∎

Proposition 4.12. *(Maximum Entropy Principle for the reduced entropy)*
Define the reduced phase density in equilibrium $\Psi_{eq} : \partial B(1, \mathbf{0}) \to \mathbb{R}$ depending on the four given parameters $p > 0, \mathbf{u} \in \mathbb{R}^3$ by

$$\Psi_{eq}(p, \mathbf{u}, \mathbf{w}) = \frac{3}{4\pi} \frac{p}{(\sqrt{1 + \mathbf{u}^2} - \mathbf{w} \cdot \mathbf{u})^4}, \qquad \mathbf{w} \in \partial B(1, \mathbf{0}). \tag{4.8.4}$$

Then there result the four constraints

$$T^{00} := \oint_{\partial B(1,\mathbf{0})} \Psi_{eq}(p, \mathbf{u}, \mathbf{w})\, dS(\mathbf{w}) = p(3 + 4\mathbf{u}^2),$$
$$T^{0k} := \oint_{\partial B(1,\mathbf{0})} w^k\, \Psi_{eq}(p, \mathbf{u}, \mathbf{w})\, dS(\mathbf{w}) = 4pu^k\sqrt{1 + \mathbf{u}^2} \quad \textit{with}$$

$$p = \frac{1}{3}\left[-T^{00} + \sqrt{4(T^{00})^2 - 3\sum_{k=1}^{3}(T^{0k})^2}\right], \quad u^k = \frac{T^{0k}}{\sqrt{4p[p + T^{00}]}}.$$

Assume that $\Psi : \partial B(1, \mathbf{0}) \to \mathbb{R}$ with $\Psi \in L^\infty(\partial B(1, \mathbf{0}))$ and $\Psi \geq \eta$ for some positive constant η is a second reduced phase density with the same four constraints T^{00}, T^{0k}. Then Ψ_{eq} maximizes the entropy due to

$$\tilde{S}^0[\Psi_{eq}] = \oint_{\partial B(1,\mathbf{0})} \Psi_{eq}^{3/4}(p, \mathbf{u}, \mathbf{w})\, dS(\mathbf{w}) \geq \tilde{S}^0[\Psi] = \oint_{\partial B(1,\mathbf{0})} \Psi^{3/4}(\mathbf{w})\, dS(\mathbf{w}).$$

Moreover, if the entropy inequality reduces to an equality, then $\Psi = \Psi_{eq}$ a.e. on $\partial B(1, \mathbf{0})$.

Proof: The first part is a straight forward calculation of the integrals and the tensor-algebraic relations. In order to show that Ψ_{eq} uniquely maximizes the entropy we use the Lemma 4.11 and conclude that

$$\tilde{S}^0[\Psi_{eq}] = \tilde{S}^0[\Psi] - \frac{3}{4} \oint\limits_{\partial B(1,\mathbf{0})} \Psi_{eq}^{-1/4}(\Psi - \Psi_{eq}) dS(\mathbf{w}) + \oint\limits_{\partial B(1,\mathbf{0})} \tilde{R}(\Psi_{eq}, \Psi) dS(\mathbf{w}) \,.$$

The second integral is non-negative due to the Lemma and the first integral vanishes because Ψ_{eq}, Ψ have the same constraints and

$$\Psi_{eq}^{-1/4} = \left(\frac{4\pi}{3p}\right)^{1/4} \left(\sqrt{1+\mathbf{u}^2} - \mathbf{w}\cdot\mathbf{u}\right)$$

is a linear combination of the moment-weights 1 and w^k, $k = 1, 2, 3$. Now we assume that $\tilde{S}^0[\Psi_{eq}] = \tilde{S}^0[\Psi]$ and conclude in this case that

$$\oint\limits_{\partial B(1,\mathbf{0})} \tilde{R}(\Psi_{eq}, \Psi) dS(\mathbf{w}) = 0 \,.$$

But for all $\Psi_1, \Psi_2 > 0$ we have for an appropriate ξ between Ψ_1, Ψ_2

$$\tilde{R}(\Psi_1, \Psi_2) = \frac{3}{32} \xi^{-5/4} (\Psi_2 - \Psi_1)^2 \,,$$

and therefore $\Psi = \Psi_{eq}$ a.e. on $\partial B(1, \mathbf{0})$. ∎

Remark: In the preceeding Proposition we have solved the Maximum Entropy problem in a simple way without using Langrange-multipliers, and the result presented here is also important for the theory of the phonon-Bose gas developed in the last two Chapters. This will be explained in Section 6.5.

A closer look at the proof of Proposition 4.9(ii) shows that only the increase of the entropy zero-component S^0 across the maximization time is needed in order to derive the entropy inequality for the kinetic scheme. Since the same holds for the kinetic definition $\tilde{S}^0$ for the reduced scheme and since the moment definitions of $T^{0\mu}$ have not been changed, we conclude that also the reduced entropy increases for the kinetic scheme of the subsystem. This is stated in

Proposition 4.13. *Let $\Omega \subset \mathbb{R}_0^+ \times \mathbb{R}^3$ be any bounded convex region in time and space. By do_ν we denote the positively oriented surface element of $\partial\Omega$. Let $\tau_M > 0$ be a fixed time step. The moment representations for $\tilde{S}^\mu$ in* (4.8.3) *calculated by the reduced iterated scheme for the $(p, \mathbf{u})$-subsystem satisfies the entropy inequality*

$$\oint\limits_{\partial\Omega} \tilde{S}^\nu do_\nu \geq 0. \tag{4.8.5}$$

Remark: If we calculate the reduced entropy defined in (4.8.3) for $\tau = 0$ with the reduced phase density (4.8.4) in equilibrium, then we get back the reduced entropy for the subsystem of the ultra-relativistic Euler equations defined in (4.4.14). We have thus used the tilde-symbol for expressions involved in the reduced entropies.

4.9 First order numerical schemes for the ultra-relativistic Euler equations

For numerical purposes the one-dimensional ultra-relativistic Euler equations may be written down in the dimensionless "vector form"

$$\frac{\partial W}{\partial t} + \frac{\partial F_{Eul}(W)}{\partial x} = 0\,, \tag{4.9.1}$$

where

$$W = \begin{pmatrix} N^0 \\ T^{01} \\ T^{00} \end{pmatrix} = \begin{pmatrix} n\sqrt{1+u^2} \\ 4pu\sqrt{1+u^2} \\ 3p + 4pu^2 \end{pmatrix}\,,$$
$$F_{Eul}(W) = \begin{pmatrix} N^1 \\ T^{11} \\ T^{01} \end{pmatrix} = \begin{pmatrix} n\,u \\ p + 4pu^2 \\ 4pu\sqrt{1+u^2} \end{pmatrix}\,. \tag{4.9.2}$$

Note that from each quantity $W = \begin{pmatrix} n\sqrt{1+u^2} \\ 4pu\sqrt{1+u^2} \\ 3p + 4pu^2 \end{pmatrix}$ we can easily get back the values of p, u and n by using the formulas

$$p = \frac{1}{3}\left[\sqrt{4(T^{00})^2 - 3(T^{01})^2} - T^{00}\right]\,, \quad u = \frac{T^{01}}{\sqrt{4p(p+T^{00})}}\,, \quad n = \frac{N^0}{\sqrt{1+u^2}}\,, \tag{4.9.3}$$

which also justifies the notation $F_{Eul} = F_{Eul}(W)$. This will be used in the sequel.

We also prescribe initial data for (n, u, p) at $t = 0$ which are only restricted by the positivity-conditions $n > 0$, $p > 0$. It is sufficient for our purpose to assume that this initial data is piecewise constant on some equidistant spatial grid which will be fixed later.

In this section we develop two fully discretized numerical schemes for this initial value problem. They are rigorously based on the integral conservation laws in terms of curve integrals adapted to the choice of the numerical grid for the discretization of time and space. Both schemes are of first order with

respect to time and space, but they can be extended to second order with limiters for accuracy by using a general standard method called the MUSCL-type approach, which is important for very time consuming computations in several space dimensions. For details we refer to the textbook of Toro [47] and the papers of Kunik, Qamar and Warnecke [33, 35].

The first scheme is the so called central scheme. It is very simple and can be obtained from the conservation laws on a light cone grid without making use of the kinetic theory. The advantage of this scheme is that it can be generalized very easily to some other systems of conservation laws.

The second scheme is the so called kinetic flux vector splitting (KFVS) scheme and can be obtained from the conservation laws on a special rectangular grid. It may also be regarded as a fully discretized version of the semidiscrete scheme given by (4.5.7)-(4.5.10) in Section 4.5. Beside the integral conservation laws, the projection on the Maximum Entropy phase density at the update times and a consistent discretization of the reduced moment integrals in the free-flight phases play a key role for this method.

We conclude this section with an example where we compare an explicit solution of a Riemann problem with the numerical results.

For both schemes we use some common notations in order to discretize time and space. We prescribe a fixed time step $\Delta t > 0$ and calculate the spatial mesh size Δx in terms of the natural Courant-Friedrichs-Lewy (CFL) condition according to

$$\Delta x = 2\Delta t\,. \tag{4.9.4}$$

Later on this condition will guarantee that neighbouring light cones will not interact within the time step of the numerical scheme. Note that in the theory of the classical Euler equations one has to assume a bound for the characteristic speeds which depend on the choice of the initial data in order to obtain a CFL-condition. This is not necessary in our case, since in the relativistic theory every signal speed is bounded by the velocity of light, which is set to $c = 1$ in dimensionless form.

For each integer number $i \in \mathbb{Z}$ we define the equidistant spatial values

$$x_i = i\Delta x\,, \quad x_{i+\frac{1}{2}} = (i + \frac{1}{2})\Delta x\,. \tag{4.9.5}$$

We also define the equidistant time steps $t_n = n\Delta t$, $n \geq 0$, which are also called update times.

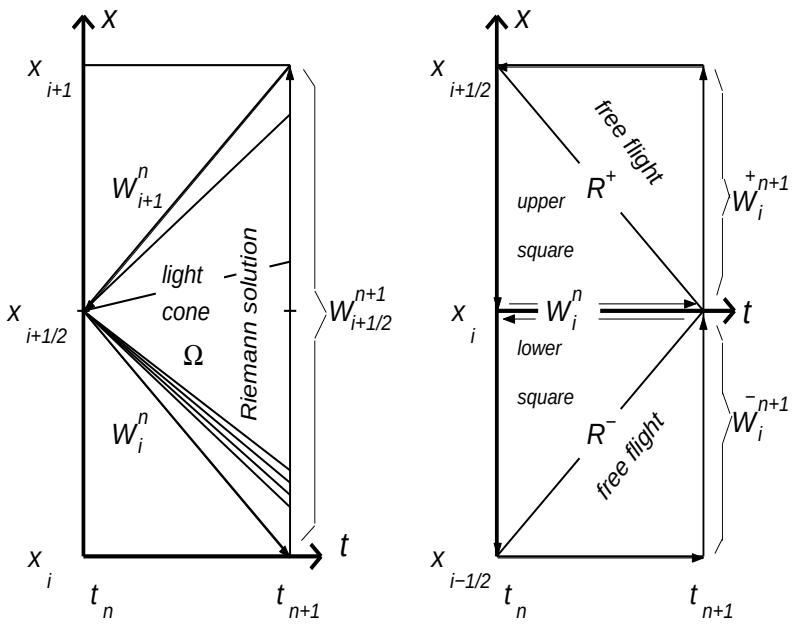

Figure 4.3: Balance regions for the central scheme (light cone Ω on the left) and for the KFVS-scheme (squares $R^{\pm}$ on the right)

The central scheme

For the central scheme we consider the Riemann solution of the ultra-relativistic Euler equations inside the light cone Ω depicted in Figure 4.3. At the next update time t_{n+1} this Riemann solution has the integral mean value $W_{i+1/2}^{n+1}$ with respect to the spatial interval $x_i \le x \le x_{i+1}$. For the formulation of the central scheme we calculate $W_{i+1/2}^{n+1}$ by using the conservation laws with respect to the domain Ω depicted in the left part of Figure 4.3, namely

$$\oint_{\partial\Omega} W(t,x)\,dx - F_{Eul}(W(t,x))\,dt = 0\,. \tag{4.9.6}$$

There results immediately

$$W_{i+1/2}^{n+1} = \frac{1}{2}\left(W_i^n + W_{i+1}^n\right) - \frac{1}{2}\left(F_{Eul}(W_{i+1}^n) - F_{Eul}(W_i^n)\right)\,. \tag{4.9.7}$$

Note that the update value $W_{i+1/2}^{n+1}$ shows a half shift in the spatial index because the central scheme is formulated on a light cone grid.

Recall that we can recover the values of (p,u,n) from $W_{i+1/2}^{n+1}$ by using (4.9.3). However, this requires that there hold the inequalities $|T^{01}| < T^{00}$ and $N^0 > 0$ between the components N^0, T^{01} and T^{00} of $W_{i+1/2}^{n+1}$ in order to obtain positive values for p and n. In order to prove these positivity properties we assume

that W^n_{i+1} and W^n_i are already well defined and put

$$W^n_{i+1} = \begin{pmatrix} n_+\sqrt{1+u_+^2} \\ 4p_+u_+\sqrt{1+u_+^2} \\ p_+(3+4u_+^2) \end{pmatrix}, \quad W^n_i = \begin{pmatrix} n_-\sqrt{1+u_-^2} \\ 4p_-u_-\sqrt{1+u_-^2} \\ p_-(3+4u_-^2) \end{pmatrix},$$

$$F_{Eul}(W^n_{i+1}) = \begin{pmatrix} n_+\,u_+ \\ p_+(1+4u_+^2) \\ 4p_+u_+\sqrt{1+u_+^2} \end{pmatrix}, \quad F_{Eul}(W^n_i) = \begin{pmatrix} n_-\,u_- \\ p_-(1+4u_-^2) \\ 4p_-u_-\sqrt{1+u_-^2} \end{pmatrix}.$$

It follows from (4.9.7) that

$$W^{n+1}_{i+1/2} = \begin{pmatrix} \frac{n_+}{2}(\sqrt{1+u_+^2}-u_+) + \frac{n_-}{2}(\sqrt{1+u_-^2}+u_-) \\ -\frac{p_+}{2}(1+4u_+^2-4u_+\sqrt{1+u_+^2}) + \frac{p_-}{2}(1+4u_-^2+4u_-\sqrt{1+u_-^2}) \\ \frac{p_+}{2}(3+4u_+^2-4u_+\sqrt{1+u_+^2}) + \frac{p_-}{2}(3+4u_-^2+4u_-\sqrt{1+u_-^2}) \end{pmatrix}.$$

We can use the conservation laws satisfied by the Riemann solution $(N^0(t,x), T^{01}(t,x), T^{00}(t,x))$ of the ultra-relativistic Euler equations in the compact domain Ω in order to rewrite the update value $W^{n+1}_{i+1/2}$ in (4.9.7) as the integral mean values

$$W^{n+1}_{i+1/2} = \frac{1}{\Delta x}\int\limits_{x_i}^{x_{i+1}} \begin{pmatrix} N^0(t_{n+1},x) \\ T^{01}(t_{n+1},x) \\ T^{00}(t_{n+1},x) \end{pmatrix} dx = \begin{pmatrix} \bar{N}^0_{i+1/2}(t_{n+1}) \\ \bar{T}^{01}_{i+1/2}(t_{n+1}) \\ \bar{T}^{00}_{i+1/2}(t_{n+1}) \end{pmatrix}. \qquad (4.9.8)$$

We obtain that $\bar{N}^0_{i+1/2}(t_{n+1}) > 0$ as well as

$$\bar{T}^{00}_{i+1/2}(t_{n+1}) - \bar{T}^{01}_{i+1/2}(t_{n+1}) = p_- + 2p_+(\sqrt{1+u_+^2}-u_+)^2 > 0\,,$$
$$\bar{T}^{00}_{i+1/2}(t_{n+1}) + \bar{T}^{01}_{i+1/2}(t_{n+1}) = p_+ + 2p_-(\sqrt{1+u_-^2}+u_-)^2 > 0\,.$$

Thus we conclude that the central scheme guarantees a positive pressure and particle density for all later times, provided that these positivity properties are satisfied for the initial data.

Now we finish the study of the central scheme with a proof of the discrete entropy inequality for the (p,u)-subsystem with respect to the domain Ω in Figure 4.3.

Let $(p(t,x), u(t,x))$ be the Riemann solution of the (p,u)-subsystem in the compact domain Ω due to Section 4.4. Then the reduced one-dimensional entropy four vector $(\tilde{S}^0, \tilde{S}^1)^T$ in (4.4.14) reads with $\alpha = 1$:

$$\tilde{S}^0(t,x) = p(t,x)^{3/4}\sqrt{1+u(t,x)^2}\,,\ \tilde{S}^1(t,x) = p(t,x)^{3/4}u(t,x)\,. \qquad (4.9.9)$$

Note that (t,x) must be chosen in (4.9.9) in the compact domain Ω. We know that the Riemann solution in Ω satisfies the entropy inequality with

the curve integral along $\partial\Omega$, which can be rewritten in the form

$$\frac{1}{\Delta x}\int\limits_{x_i}^{x_{i+1}} \tilde{S}^0(t_{n+1},x)dx \geq \frac{p_+^{3/4}}{2}\left(\sqrt{1+u_+^2}-u_+\right)+\frac{p_-^{3/4}}{2}\left(\sqrt{1+u_-^2}+u_-\right). \tag{4.9.10}$$

The right hand side of this entropy inequality is already written in the desired discrete form. In order to estimate the integral mean value on the left hand side we rewrite $\tilde{S}^0 = p^{3/4}\sqrt{1+u^2}$ in terms of the so called "conservative variables" $T^{01} = 4pu\sqrt{1+u^2}$ and $T^{00} = p(3+4u^2)$ under the time derivative of the conservation laws for the (p,u)-subsystem and obtain from (4.9.3) that

$$\begin{aligned}\tilde{S}^0 &= p^{3/4}\sqrt{1+u^2} = \frac{1}{2}p^{1/4}\left(T^{00}+p\right)^{1/2}\\ &= \frac{1}{2}\left(\frac{1}{3}\right)^{\frac{3}{4}}\left[\sqrt{4(T^{00})^2-3(T^{01})^2}-T^{00}\right]^{\frac{1}{4}}\left[\sqrt{4(T^{00})^2-3(T^{01})^2}+2T^{00}\right]^{\frac{1}{2}}.\end{aligned} \tag{4.9.11}$$

Now we put $\eta = \frac{T^{01}}{T^{00}}$ and consider for a moment $\tilde{S}^0$ as a function $\tilde{S}^0_*(T^{00},T^{01})$ depending on T^{00}, T^{01}, restricted by the condition $|\eta|<1$. We obtain that

$$\mathrm{Det}\begin{pmatrix} \frac{\partial^2\tilde{S}^0_*}{\partial T^{00}\partial T^{00}} & \frac{\partial^2\tilde{S}^0_*}{\partial T^{00}\partial T^{01}} \\ \frac{\partial^2\tilde{S}^0_*}{\partial T^{00}\partial T^{01}} & \frac{\partial^2\tilde{S}^0_*}{\partial T^{01}\partial T^{01}}\end{pmatrix} = \frac{1}{48p^{\frac{5}{2}}}\frac{\left(1-\frac{3}{4}\eta^2\right)^{\frac{3}{2}}+1-\frac{9}{8}\eta^2}{(4-3\eta^2)^{\frac{1}{2}}\left(2+\sqrt{4-3\eta^2}\right)^2}. \tag{4.9.12}$$

The function $f:[-1,1]\to\mathbb{R}$ with $f(\eta) = \left(1-\frac{3}{4}\eta^2\right)^{\frac{3}{2}}+1-\frac{9}{8}\eta^2$ is even with $f(0)=2$ and $f(1)=0$. Its derivative is negative on $(0,1)$ and f is strictly monotonically decreasing on $[0,1]$. It follows that $f(\eta)>0$ for $|\eta|<1$, and the determinant in (4.9.12) is also positive for $|\eta|<1$. However, the second derivative

$$\frac{\partial^2\tilde{S}^0_*}{\partial T^{00}\partial T^{00}} = -\frac{1}{96p^{\frac{5}{4}}}\frac{8+21\eta^2+\frac{16+36\eta^2}{\sqrt{4-3\eta^2}}}{\left(\sqrt{4-3\eta^2}-1\right)^{\frac{1}{2}}\left(2+\sqrt{4-3\eta^2}\right)^{\frac{3}{2}}} \tag{4.9.13}$$

is negative for $|\eta|<1$. We conclude that $\tilde{S}^0_*(T^{00},T^{01})$ is a stricly concave function of T^{00}, T^{01} whenever $|\eta|<1$.

We obtain from Jensens inequality with the concavity of $\tilde{S}^0_*(T^{00},T^{01})$ that

$$\tilde{S}^0_*(\bar{T}^{00}_{i+1/2}(t_{n+1}),\bar{T}^{01}_{i+1/2}(t_{n+1})) \geq \frac{1}{\Delta x}\int\limits_{x_i}^{x_{i+1}}\tilde{S}^0(t_{n+1},x)dx. \tag{4.9.14}$$

We finally get with (4.9.10) the desired discrete entropy inequality

$$\tilde{S}^0_*(\bar{T}^{00}_{i+1/2}(t_{n+1}), \bar{T}^{01}_{i+1/2}(t_{n+1})) \geq \frac{p_+^{3/4}}{2}\left(\sqrt{1+u_+^2} - u_+\right) + \frac{p_-^{3/4}}{2}\left(\sqrt{1+u_-^2} + u_-\right). \tag{4.9.15}$$

The KFVS scheme

At each time step t_n we consider a discrete solution of the initial value problem which has the constant value W_i^n in each cell $I_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ with midpoint x_i. Thus we require that the piecewise constant solution of the KFVS scheme has only jumps in the nodal points $x_{i\pm\frac{1}{2}}$ at each time step t_n. Inside each forward light cone starting in $x_{i+1/2}$ at time t_n we solve the kinetic free-flight transport equation for $t_n \leq t \leq t_{n+1}$ and the corresponding moments of the fluxes.

We start at $t_n = 0$, i.e. $n = 0$, with the given piecewise constant initial data $W(0,x)$ over the numerical cells I_i which is represented by the numbers W_i^0 with $i \in \mathbb{Z}$.

Next we assume that the values W_i^n at time step t_n are already known. Our aim is to calculate the new quantitites $\overset{\pm}{W}{}_i^{n+1}$ at the next time step t_{n+1}, which are the integral mean values resulting from the conservation laws with respect to the quadratic domains $R^{\pm}$ depicted in Figure 4.3.

In the time interval $t_n \leq t \leq t_{n+1}$ the one-dimensional free-flight moments are given by

$$W(t,x) = \begin{pmatrix} N^0(t,x) \\ T^{01}(t,x) \\ T^{00}(t,x) \end{pmatrix} = \int_{-1}^{1} \begin{pmatrix} \Phi(x-t\xi,\xi) \\ \xi\,\Psi(x-t\xi,\xi) \\ \Psi(x-t\xi,\xi) \end{pmatrix} d\xi, \tag{4.9.16}$$

$$F(t,x) = \begin{pmatrix} N^1(t,x) \\ T^{11}(t,x) \\ T^{01}(t,x) \end{pmatrix} = \int_{-1}^{1} \begin{pmatrix} \xi\,\Phi(x-t\xi,\xi) \\ \xi^2\,\Psi(x-t\xi,\xi) \\ \xi\ \ \Psi(x-t\xi,\xi) \end{pmatrix} d\xi\,, \tag{4.9.17}$$

$$\Phi(y,\xi) = \frac{1}{2}\frac{n(y)}{(\sqrt{1+u^2(y)} - \xi u(y))^3}\,, \quad \Psi(y,\xi) = \frac{3}{2}\frac{p(y)}{(\sqrt{1+u^2(y)} - \xi u(y))^4}\,.$$

Note that the free-flight moments and fluxes defined in (4.9.16) and (4.9.17) must not be mixed up with the corresponding Eulerian moments from the solution of the Riemann problems, except in the backward light-cone region with the constant value W_i^n in the right part of Figure 4.3, where the Eulerian quantities and the free-flight moments are the same.

Due to Figure 4.3 we define the integral mean values

$$\overset{+}{W}{}_i^{n+1} = \frac{2}{\Delta x} \int_{x_i}^{x_{i+1/2}} W(t,x)\,dx\,, \tag{4.9.18}$$

$$\overset{-}{W}{}_i^{n+1} = \frac{2}{\Delta x} \int_{x_{i-1/2}}^{x_i} W(t,x)\,dx\,. \tag{4.9.19}$$

Then the evaluation of the free-flight conservation laws with respect to the rectangles $R^{\pm}$ in Figure 4.3 leads to

$$\overset{+}{W}{}_i^{n+1} = W_i^n - \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} F(t, x_{i+1/2})\,dt + F_{Eul}(W_i^n)\,, \tag{4.9.20}$$

$$\overset{-}{W}{}_i^{n+1} = W_i^n + \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} F(t, x_{i-1/2})\,dt - F_{Eul}(W_i^n)\,. \tag{4.9.21}$$

Then we apply the conservation law with respect to the rectangle R consisting of the lower and upper parts $R^{\pm}$ in order to obtain the following integral mean value with respect to the intervall I_i as "update values" at t_n:

$$W_i^{n+1} = \frac{1}{2}\left(\overset{+}{W}{}_i^{n+1} + \overset{-}{W}{}_i^{n+1}\right) = W_i^n - \frac{1}{2}F_{i+1/2}^n + \frac{1}{2}F_{i-1/2}^n\,, \tag{4.9.22}$$

$$F_{i+1/2}^n = \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} F(t, x_{i+1/2})\,dt\,, \quad F_{i-1/2}^n = \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} F(t, x_{i-1/2})\,dt\,. \tag{4.9.23}$$

In contrast to the central scheme there is no half-shift in the spatial index of the new value W_i^{n+1}, and we say that the KFVS scheme has upwind form.

In order to complete the formulation of the KFVS scheme the remains the calculation of the fluxes $F_{i\pm1/2}^n$ in (4.9.23). This can be done explicitly thanks to the fact that the free-flight solutions in neighbouring light-cone regions corresponding to the numerical cells are independent from each other. This is not the case for the classical Euler equations, where the free-flight moments have an infinite domain of dependence!

A straight forward calculation shows that $F(t, x_{i\pm1/2}) = F_{i\pm1/2}^n$ are independent of t within the integration domain $t_n \le t \le t_{n+1}$. There results the "upwind flux" from (4.9.17), where we have omitted for simplicity the time

index n with respect to the piecewise constant initial data at $t = t_n$:

$$F^n_{i+1/2} = \begin{pmatrix} \frac{n_i}{4\sqrt{1+u_i^2}} \left(\sqrt{1+u_i^2}+u_i\right)^2 - \frac{n_{i+1}}{4\sqrt{1+u_{i+1}^2}} \left(\sqrt{1+u_{i+1}^2}-u_{i+1}\right)^2 \\ \frac{p_i}{2\sqrt{1+u_i^2}} \left(\sqrt{1+u_i^2}+u_i\right)^3 + \frac{p_{i+1}}{2\sqrt{1+u_{i+1}^2}} \left(\sqrt{1+u_{i+1}^2}-u_{i+1}\right)^3 \\ \frac{p_i}{4(1+u_i^2)} \frac{(3\sqrt{1+u_i^2}-u_i)}{(\sqrt{1+u_i^2}-u_i)^3} - \frac{p_{i+1}}{4(1+u_{i+1}^2)} \frac{(3\sqrt{1+u_{i+1}^2}+u_{i+1})}{(\sqrt{1+u_{i+1}^2}+u_{i+1})^3} \end{pmatrix}, \tag{4.9.24}$$

$$F^n_{i-1/2} = \begin{pmatrix} \frac{n_{i-1}}{4\sqrt{1+u_{i-1}^2}} \left(\sqrt{1+u_{i-1}^2}+u_{i-1}\right)^2 - \frac{n_i}{4\sqrt{1+u_i^2}} \left(\sqrt{1+u_i^2}-u_i\right)^2 \\ \frac{p_{i-1}}{2\sqrt{1+u_{i-1}^2}} \left(\sqrt{1+u_{i-1}^2}+u_{i-1}\right)^3 + \frac{p_i}{2\sqrt{1+u_i^2}} \left(\sqrt{1+u_i^2}-u_i\right)^3 \\ \frac{p_{i-1}}{4(1+u_{i-1}^2)} \frac{(3\sqrt{1+u_{i-1}^2}-u_{i-1})}{(\sqrt{1+u_{i-1}^2}-u_{i-1})^3} - \frac{p_i}{4(1+u_i^2)} \frac{(3\sqrt{1+u_i^2}+u_i)}{(\sqrt{1+u_i^2}+u_i)^3} \end{pmatrix}. \tag{4.9.25}$$

This is exactly the kinetic flux vector splitting scheme for the initial value problem of the ultra-relativistic Euler equations. It has a similar structure as the first order Godunov scheme, which also results from (4.9.22), (4.9.23) if we replace there the free-flight moment fluxes $F(t, x_{i\pm1/2})$ by the Eulerian fluxes $F_{Eul}(W(t, x_{i\pm1/2}))$ resulting from the solution of the Riemann problems in the forward light cones starting at $t = t_n$ and $x = x_{i\pm1/2}$.

It is easy to prove the positivity of particle density and pressure for the central, the KFVS- and the Godunov scheme in a unified way. To see this we first define the state space

$$\Omega = \{W = \begin{pmatrix} N^0 \\ T^{01} \\ T^{00} \end{pmatrix} \in \mathbb{R}^3 \mid N^0 > 0 \text{ and } |T^{01}| < T^{00} \}. \tag{4.9.26}$$

If $W_1, W_2 \in \Omega$ and $a > 0$, then it is clear that $aW_1 \in \Omega$ and $W_1 + W_2 \in \Omega$. It follows that $a_1W_1 + ... + a_mW_m \in \Omega$ for $W_1, ..., W_m \in \Omega$ and $a_1, ..., a_m > 0$. Moreover, in the limiting case any integral mean value of "W-state vectors" is also a "W-state vector". But we can write each "W-state vector" of the conservative schemes at the new update time t_{n+1} in terms of such an integral mean value due to the conservation laws for particle density, momentum- and energy density. Thus the central scheme, the KFVS- and the Godunov scheme preserve the positivity of particle density and pressure.

The derivation of the discrete entropy inequality for the Godunov scheme follows exactly the same line as for the central scheme, whereas the derivation of a discrete entropy inequality for the KFVS scheme uses the entropy inequalities derived for the semidiscrete scheme. It is technically more difficult but does not contain new ideas and will thus be omitted here.

A numerical example for a Riemann solution

We solve the weak form of the one-dimensional system of Euler equations

$$\begin{aligned} (n\sqrt{1+u^2})_t + (nu)_x &= 0\,, \\ (4pu\sqrt{1+u^2})_t + (p(1+4u^2))_x &= 0\,, \\ (p(3+4u^2))_t + (4pu\sqrt{1+u^2})_x &= 0 \end{aligned} \tag{4.9.27}$$

for given Riemannian initial data at $t = 0$, namely

$$p_0(x) = \begin{cases} 4\,, & x \le 0\,, \\ 1\,, & x > 0\,, \end{cases} \quad , \quad u_0(x) \equiv 0\,, \quad n_0(x) = \begin{cases} 1\,, & x \le 0\,, \\ 3\,, & x > 0\,. \end{cases} \tag{4.9.28}$$

We use the notation of Section 4.4 and obtain first that

$$\alpha = \frac{p_+}{p_-} = 1/4\,, \quad \beta = \frac{\sqrt{1+u_+^2} - u_+}{\sqrt{1+u_-^2} - u_-} = 1\,. \tag{4.9.29}$$

There holds $\beta > K_R(\alpha)$ and $\beta K_S(\alpha) < 1$ due to Case 3 for the classification of the Riemann solution, i.e. we have a lower 1-fan and an upper 3-shock. For the intermediate "star region" we obtain approximately the values

$$p_* = 1.996674\,, \qquad u_* = 0.305422\,.$$

We define the slopes

$$s_1 = -0.577350\,,\ s_2 = -0.343113\,,\ s_3 = 0.292102\,,\ s_4 = 0.682870$$

and the constants

$$\delta_1 = 0.874242\,,\ \delta_2 = 0.319653\,,\ n_1 = 0.593862\,,\ n_2 = 5.013875\,.$$

The slopes s_1 and s_2 are the bounds for the lower 1-rarefaction fan, s_3 is the slope of the contact discontinuity and s_4 the slope of the upper 3-shock.

Then the Riemann solution at time $t = 1$ is given by

$$p(1,x) = \begin{cases} 4\,, & x \le s_1\,, \\ \delta_1 \left(\frac{1-x}{1+x}\right)^{2/\sqrt{3}}\,, & s_1 < x \le s_2\,, \\ p_*\,, & s_2 < x \le s_4\,, \\ 1\,, & x > s_4\,, \end{cases}$$

$$u(1,x) = \begin{cases} 0\,, & x \le s_1\,, \\ \sqrt{\frac{3}{2}}\,\frac{x+\frac{1}{\sqrt{3}}}{\sqrt{1-x^2}}\,, & s_1 < x \le s_2\,, \\ u_*\,, & s_2 < x \le s_4\,, \\ 0\,, & x > s_4\,, \end{cases}$$

$$n(1,x) = \begin{cases} 1\,, & x \le s_1\,, \\ \delta_2 \left(\frac{1-x}{1+x}\right)^{\sqrt{3}/2}\,, & s_1 < x \le s_2\,, \\ n_1\,, & s_2 < x \le s_3\,, \\ n_2\,, & s_3 < x \le s_4\,, \\ 3\,, & x > s_4\,. \end{cases}$$

In Figures 4.4-4.6 the exact Riemann solution is plotted at $t = 1$ together with the numerical solution in the spatial range $-0.75 \le x \le 0.75$ of the semidiscrete scheme in one space dimension described in Section 4.6 with $n = 1$, $n = 20$ and $n = 400$ times steps, respectively.

In all these figures the contact discontinuity of the particle density n is not resolved very well, because the free-flight intervals cause a large numerical diffusion at the 2-shocks. This phenomenon is also very well known for the classical Euler equations. However, the upper 3-shock in Figure 4.6 at position $s_4 = 0.682870$ is resolved much better.

For the ξ-integration from -1 to 1 we have used the trapezoidal rule with at least 1000 nodal points. For the spatial meshsize of the interval $[-0.75, 0.75]$ we use at most $\Delta x \le 10^{-4}$, but under these restrictions the same pictures could be created for different kinds of spatial meshsizes and different numbers of integration nodal points. Recall that in contrast to the fully discretized schemes there is no CFL condition for the semidiscrete scheme. But of course also in this kinetic scheme each propagation velocity is bounded by the speed of light, whereas the velocity domain for the corresponding semidiscrete scheme of the classical Euler equations is unbounded even for usual Riemannian initial data, see Dreyer and Kunik [7].

In Figure 4.7 the exact Riemann solution is plotted at $t = 1$ together with the numerical solutions of the central- and KFVS scheme in the spatial range $-0.75 \le x \le 0.75$. The meshsize is $\Delta x = 0.005$ with 300 subdivisions of the interval $-0.75 \le x \le 0.75$, and there are 400 time steps with the time step size $\Delta t = 0.0025 = \frac{1}{400}$, which is comparable with the semidiscrete solution for $n = 400$ time steps in Figure 4.6. However, the resolutions of the central scheme (dashed and dotted curve) and of the KFVS scheme (dashed curve) are lower, which can be seen clearly near the position of the upper 3-shock. In our example the results of the central- and KFVS scheme are very similar, where the resolution of the central scheme is only slightly better than the resolution of the KFVS scheme.

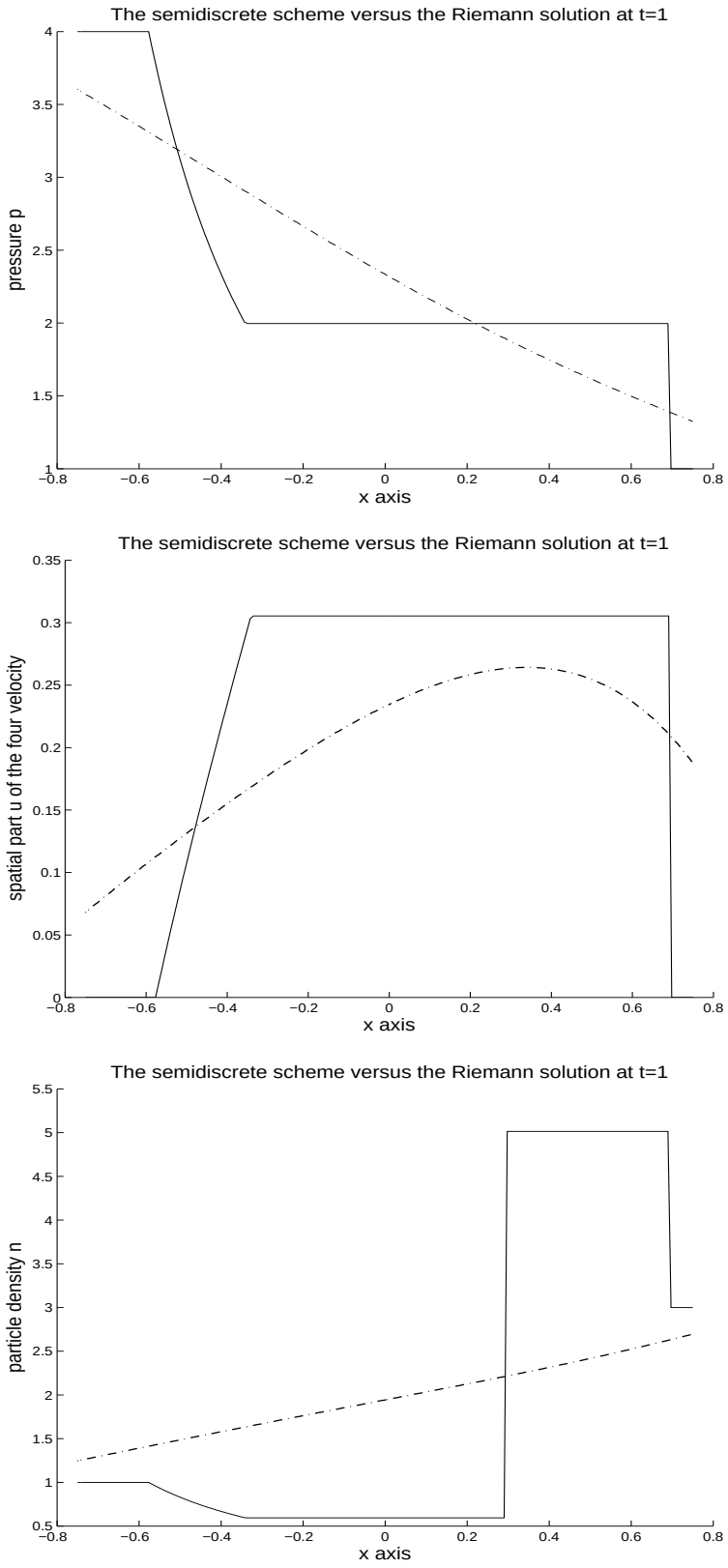

Figure 4.4: The semidiscrete scheme, 1 time step.

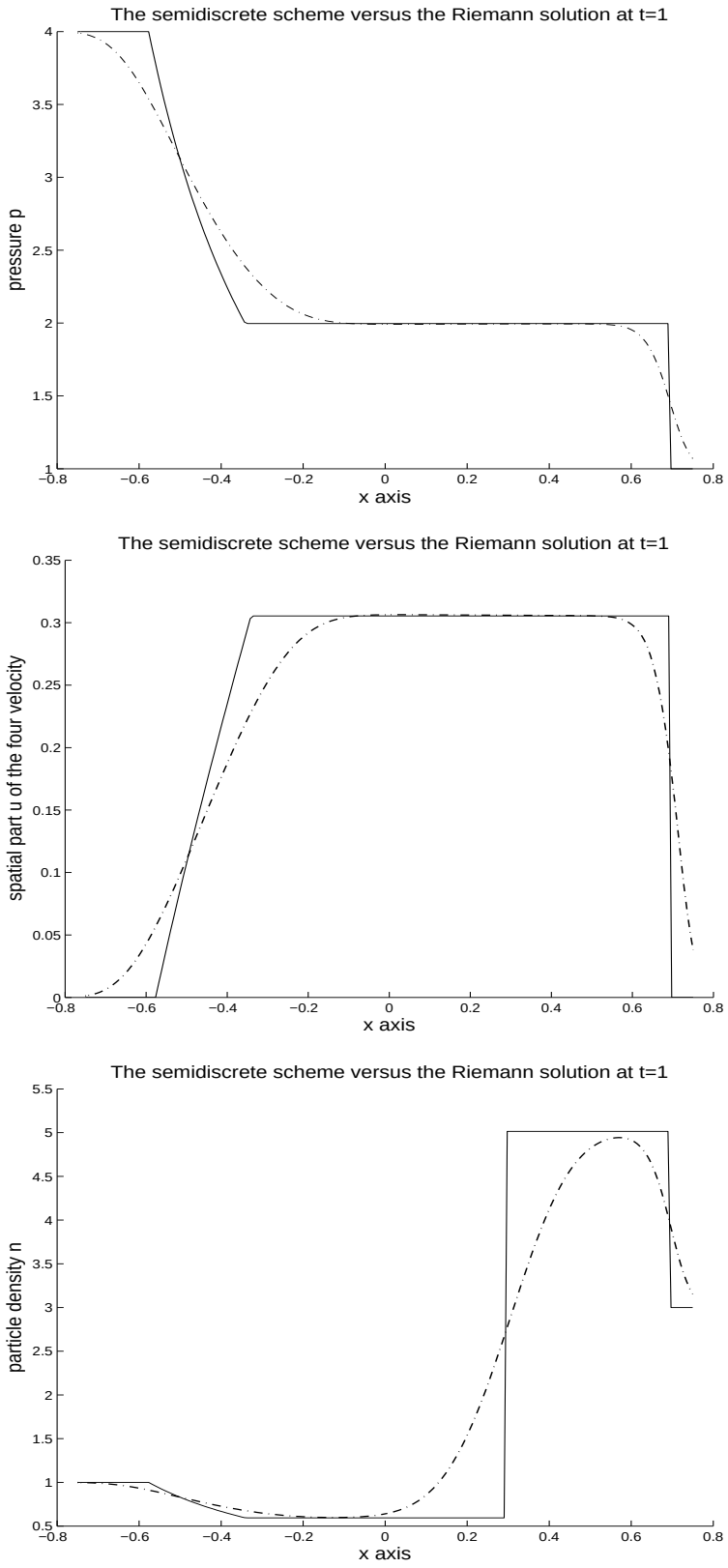

Figure 4.5: The semidiscrete scheme, 20 time steps.

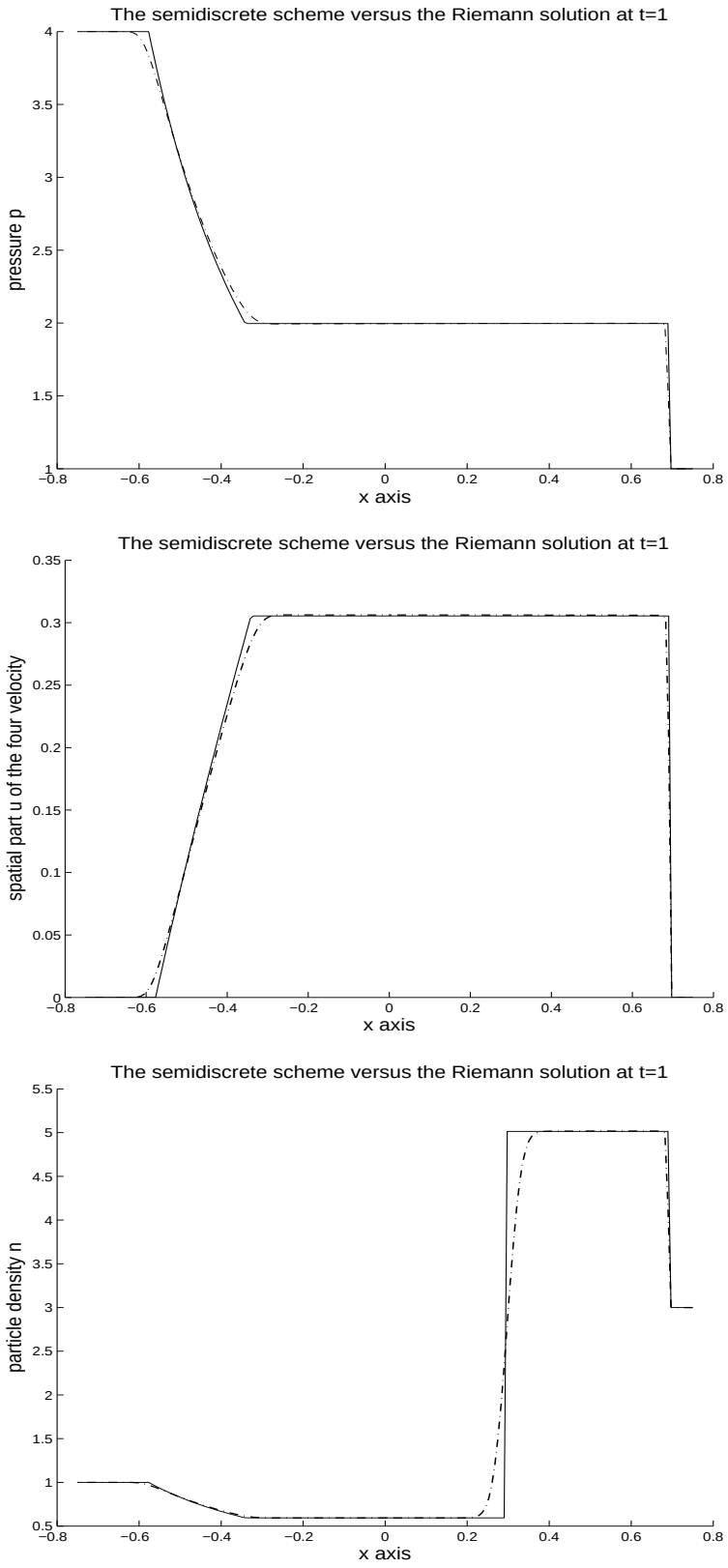

Figure 4.6: The semidiscrete scheme, 400 time steps.

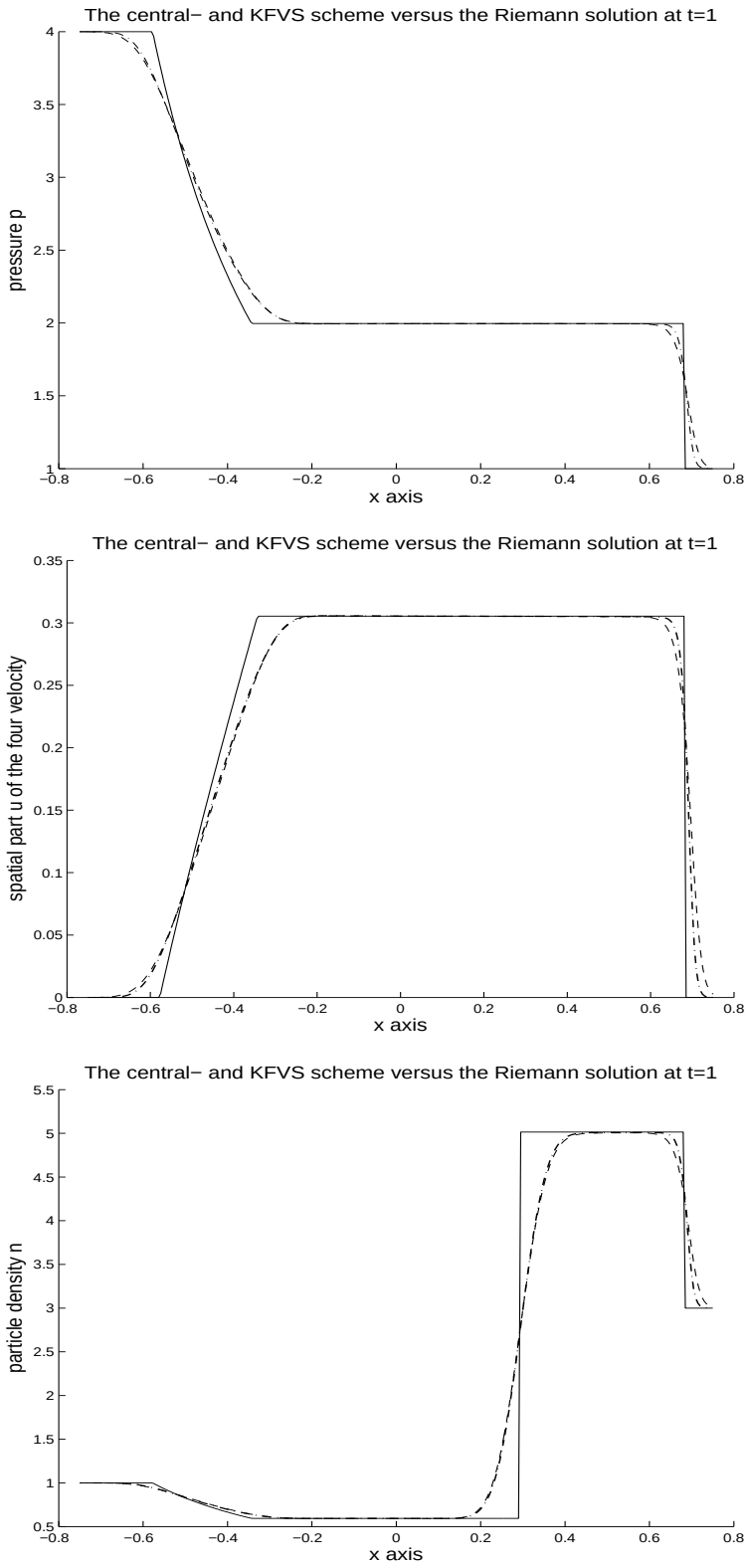

Figure 4.7: The central scheme (dashed and dotted curve) and the KFVS scheme (dashed curve), both with 400 time steps.

Chapter 5

The Boltzmann-Peierls Equation for a Phonon-Bose Gas

5.1 Overview and introduction to the Boltzmann-Peierls equation

In 1929, Peierls [40] proposed his celebrated theoretical model based on the Boltzmann-Peierls equation. According to him the lattice vibrations responsible for heat transport can be described as an interacting gas of phonons. The Boltzmann-Peierls approach is one of the milestones of the theory of thermal transport in crystalline solids at very low temperatures.

The Boltzmann-Peierls equation describes the evolution of the phase density $f(t, \mathbf{x}, \mathbf{k})$, where $f(t, \mathbf{x}, \mathbf{k})\, d^3x\, d^3k$ is interpreted as the number of phonons which are at time t in a small spatial volume element d^3x at location $\mathbf{x}$ and which have a momentum $\hbar\,\mathbf{k}$ in the range d^3k. Here $\mathbf{k}$ denotes the wave vector and $\hbar$ is Planck's constant. It is important to note that f as well as $d^3x\, d^3k$ are dimensionless quantities. The Boltzmann-Peierls equation reads

$$\frac{\partial f}{\partial t} + \sum_{j=1}^{3} \frac{\partial \omega(\mathbf{k})}{\partial k_j} \frac{\partial f}{\partial x_j} \;=\; \zeta(f)\,, \quad \text{where } \omega(\mathbf{k}) = c|\mathbf{k}|\,. \tag{5.1.1}$$

Even a simplified version of the collision term $\zeta(f)$ on the right-hand side given by Callaway [2] needs several explanations and will therefore be given later. Like the classical Boltzmann equation, the Boltzmann-Peierls equation is also a kinetic evolution equation. However, in contrast to the classical case there is a physical constant, the so called Debye velocity c in the dispersion relation $(5.1.1)_2$, which is an upper bound for the sound speed of the actual

crystal. There is no danger to confuse the Debye velocity with the speed of light, which is also denoted by c but not considered in the sequel.

It is important to mention that Fourier theory of heat flow fails to describe heat conduction processes in a crystal at low temperatures, see for example the review article of Dreyer and Struchtrup [16] as well as the literature therein. Heat conduction processes are usually described by a parabolic system. It results from a diffusion law, where the heat flux is proportional to the temperature gradient.

However, Dreyer and Struchtrup report in [16] on special circumstances that are met in quite pure crystals at not too low temperature, where the state of a crystal is sufficiently described by only four thermodynamic fields as the basic variables. These are the energy density e, or the temperature T, and the heat flux $\mathbf{Q} = (Q_i)_{i=1,2,3}$. In this case the heat conduction is described by the four fields which obey a nonlinear hyperbolic system of field equations.

Independent of the necessary restrictions for the physical applicability of the hyperbolic four-field system, these equations have been studied in their own right, because they exhibit several interesting and important mathematical aspects of the initial-boundary value problem.

The four-field system of hyperbolic heat conduction was studied and solved by Dreyer and Kunik in [9, 10]. They consider this system in one space dimension and solved its pure initial value problem as well as the initial-boundary value problem by using kinetic representations for the unknown fields. The system consists of a conservation law for the energy density e and of a balance law for the heat flux $\mathbf{Q}$, and it is derived as a moment system from the Boltzmann-Peierls equation by averaging with respect to the wave vector $\mathbf{k}$. The closure problem is solved by the Maximum Entropy Principle, see [15].

Dreyer, Herrmann and Kunik [12] have introduced a reduced kinetic equation which has a much simpler structure than the original Boltzmann-Peierls equation, but leads to the same physical moments for energy density, heat flux and so on. They performed the reduction in the microscopically two-dimensional case and thus could construct a kinetic scheme in order to solve the initial value problem for the reduced Boltzmann-Peierls equation.

Kunik, Qamar and Warnecke [34] have extended this approach to the usual microscopically three-dimensional Boltzmann-Peierls equation (BPE) given by (5.1.1), which is very important from the physical point of view. The schemes presented in [34] can be used for the solution of the reduced BPE as well as for the four-field system of hyperbolic heat conduction. Regarding the solution of the reduced BPE they have also developed a kinetic scheme which is fully discretized in the whole phase space and thus more efficient and faster than the scheme used in [12], because the kinetic scheme used

there is discrete in time but continuous in space, so that an interpolation polynomial was needed in order to calculate the free-flight phase density. The fully discretized scheme for the reduced BPE in [34] uses the idea of the kinetic flux vector splitting. Kinetic flux vector splitting was already successfully applied to the ultra-relativistic Euler equations in [33]. In both cases the schemes use a natural CFL condition, in the relativistic application given by the speed of light and in the theory of the phonon-Bose gas by the Debye-velocity, the upper bounds for the velocity of propagating waves. For a short overview see also the report [13].

The moments of the phase density f reflect the kinetic processes on the scale of continuum physics. The most important moments are

$$e(t,\mathbf{x}) = \hbar c \int\limits_{\mathbb{R}^3} |\mathbf{k}|\, f(t,\mathbf{x},\mathbf{k})\, d^3k\,, \tag{5.1.2}$$

$$Q_i(t,\mathbf{x}) = \hbar c^2 \int\limits_{\mathbb{R}^3} k_i f(t,\mathbf{x},\mathbf{k})\, d^3k\,, \tag{5.1.3}$$

$$N_{ij}(t,\mathbf{x}) = \hbar c \int\limits_{\mathbb{R}^3} \frac{k_i k_j}{|\mathbf{k}|} f(t,\mathbf{x},\mathbf{k})\, d^3k\,, \quad i,j = 1,2,3\,. \tag{5.1.4}$$

The fields e, $\mathbf{Q} = (Q_1, Q_2, Q_3)$ and the Matrix $\mathbf{N} = (N_{ij})$ are the energy density, heat flux and momentum flux, respectively.

Phonons are classified as Bose particles due to their statistic distribution, see [40, 15]. The entropy density h and the entropy flux Φ_k of the phonon-Bose gas are given according to the kinetic theory [16] as

$$\begin{aligned} h &= -k_B \int_{\mathbb{R}^3} \left[f \ln(\frac{f}{y}) - y(1+\frac{f}{y}) \ln(1+\frac{f}{y}) \right]\, d^3k\,, \\ \Phi_k &= -k_B \int_{\mathbb{R}^3} c\frac{\mathbf{k}_k}{|\mathbf{k}|} \left[f \ln(\frac{f}{y}) - y(1+\frac{f}{y}) \ln(1+\frac{f}{y}) \right]\, d^3k\,. \end{aligned} \tag{5.1.5}$$

Here $k_B = 1,38062 \cdot 10^{-23} J/K$ is Boltzmann's constant and $y = 3/8\pi^3$. Recall that f and hence y are dimensionless.

In contrast to the ordinary gas atoms, the phonons may interact by two different collision processes, called R- and N-processes. R-processes include interactions of phonons with lattice impurities which destroy the periodicity of the crystal, while N-processes can be interpreted as phonon-phonon interactions which are due to the deviations from harmonicity of the crystal forces. N-processes conserve both, energy and momentum, while R-processes only conserve energy. The Callaway approximation of the collision operator

[2, 15] is a suitable simplification of the actual interaction processes. The Callaway collision operator is written as the sum of two relaxation operators modelling the R- and N-processes seperately. We write

$$\zeta(f) = \zeta_R(f) + \zeta_N(f)\,, \qquad \zeta_\alpha(f) = \frac{1}{\tau_\alpha}\left(P_\alpha f - f\right)\,, \quad \alpha \in \{R, N\}\,. \tag{5.1.6}$$

The positive constants τ_R and τ_N are the relaxation times, while P_R and P_N are two nonlinear projectors. Here $P_R f$ and $P_N f$ represent the phase densities in the limiting case when the relaxation time tends to zero. Explicitly, we define $P_R f$ and $P_N f$ as the solutions of two optimization problems, namely

$$h(P_R f) = \max_{f'}\left\{h(f) \;:\; e(f') = e(f)\right\}\,, \tag{5.1.7}$$

$$h(P_N f) = \max_{f'}\left\{h(f) \;:\; e(f') = e(f),\, \mathbf{Q}(f') = \mathbf{Q}(f)\right\}\,, \tag{5.1.8}$$

where $e(f)$, $\mathbf{Q}(f)$ are given by (5.1.2),(5.1.3).

The maximization problems can be solved by means of Lagrange multipliers Λ_R^0 and Λ_N^0, Λ_N^1, Λ_N^2, Λ_N^3. We get

$$P_R f(\mathbf{k}) = \frac{y}{\exp(\Sigma_R) - 1}\,, \qquad P_N f(\mathbf{k}) = \frac{y}{\exp(\Sigma_N) - 1}\,, \tag{5.1.9}$$

where

$$\Sigma_R(t, \mathbf{x}, \mathbf{k}) = \hbar c\,|\mathbf{k}|\,\Lambda_R^0\,, \tag{5.1.10}$$

$$\Sigma_N(t, \mathbf{x}, \mathbf{k}) = \hbar c\,|\mathbf{k}|\,\Lambda_N^0(t, \mathbf{x}) + \sum_{i=1}^{3} \hbar\, k_i\, \Lambda_N^i(t, \mathbf{x})\,. \tag{5.1.11}$$

From (5.1.7) and (5.1.8) the Lagrange multipliers can be calculated explicitly. They are given by, see [9], [12],

$$\Lambda_R^0 = \gamma\left(\frac{3}{e}\right)^{\frac{1}{4}}\,, \quad \Lambda_N^0 = \gamma\frac{\left(\frac{F}{e}\right)^{\frac{1}{4}}}{(4-F)^{\frac{3}{4}}}\,, \quad \Lambda_N^i = -\frac{\gamma}{4}\frac{\left(\frac{F}{e}\right)^{\frac{5}{4}}}{(4-F)^{\frac{3}{4}}}Q_i\,, \tag{5.1.12}$$

$$F = \frac{6}{1 + \sqrt{1 - \frac{3}{4}\left(\frac{|\mathbf{Q}|}{ce}\right)^2}}\,, \quad \gamma = \left(\frac{\pi^2}{30\hbar^3 c^3}\right)^{\frac{1}{4}}\,. \tag{5.1.13}$$

For experimental purposes it is sometimes useful to use the absolute temperature T instead of the energy density e. Both quantities are related to each other so that the Stefan-Boltzmann law for phonons is established, viz.

$$T = \frac{1}{k_B}\left(\frac{10\hbar^3 c^3}{\pi^2}e\right)^{\frac{1}{4}} = \frac{1}{\gamma k_B}\left(\frac{e}{3}\right)^{\frac{1}{4}}\,. \tag{5.1.14}$$

Now we have explained the collision term of the BPE (5.1.1) in detail. The special form of the collision term enables the derivation of an entropy inequality for the entropy-density entropy-flux pair given in (5.1.5), i.e. any solution f of (5.1.1) satisfies

$$\frac{\partial h(f)}{\partial t} + \sum_{k=1}^{3} \frac{\partial \Phi_k(f)}{\partial x_k} \geq 0\,. \tag{5.1.15}$$

This inequality has a counterpart for the classical Boltzmann-equation and is well known as the so called H-theorem. In Section 5.3 we will give a short proof of the H-Theorem for the reduced phase density which is sufficient for our purposes.

Now we give a short overview of this chapter.

In Section 5.2 we obtain the four-field system of hyperbolic heat conduction from the moments for e and Q_i of the BPE (5.1.1) in the limit $\tau_N \to 0$.

In Section 5.3 we derive a kinetic equation for a reduced kinetic phase density. Moreover, we present the corresponding kinetic definition of the equivalent reduced entropy density-entropy flux pair. All physical moments of thermodynamics with respect to the reduced kinetic phase density are simple surface integrals with respect to the unit sphere in $\mathbb{R}^3$. Like in the theory of the ultra-relativistic Euler equations, these simplified integrals result from the exact evaluation of the corresponding three-fold integral moments with respect to the phase density f, where f satisfies the original BPE.

The time integral form of the reduced kinetic equations is obtained by using Duhamel's principle. We replace the time integral by finite Riemann sums in order to obtain a time discrete kinetic scheme which solves the initial value problem for the reduced kinetic phase density. This scheme is still continuous in space and was already used by Dreyer, Herrmann and Kunik in [12].

In Section 5.4 we further simplify the already reduced kinetic equation by using a special integral coordinate transformation adapted to the spatially one-dimensional flow-field. In three lemmas we show that this reduction is valid for all times. Like in the first reduction, this additional simplification of the surface integrals is exact, without any approximation.

We also apply a complete discretization in phase space in order to solve the reduced kinetic equation in terms of a KFVS scheme. The initial data for the scheme are the average values of the phase density on phase-space cells, and we obtain the average values of the phase density at the next time step for a spatial one-dimensional flow field.

In Section 5.5 we use the same KFVS-scheme for the reduced BPE in order to compute several initial value problems in one space dimension for both, the reduced BPE as well as for the hyperbolic four-field system. Some of these numerical results will directly be compared with explicit solutions.

This chapter has three main contributions to the theory of kinetic solutions of the BPE and its four-field moment system. The first one is the kinetic solution of the reduced microscopic three-dimensional BPE. The second contribution is the use of a special integration technique in order to calculate the moments of a macroscopic one-dimensional flow field. We show by three lemmas that this reduction is valid for all times. This second reduction for a one-dimensional flow field enables the third contribution, the use of a kinetic flux-vector splitting scheme as a numerical approximation for the reduced phase density which is fully discretized in the phase space. This is a new aspect in our work which has no counterpart in the classical theory.

5.2 The hyperbolic four-field system as a limiting case

Taking the integral-moments of the BPE, one can generate an infinite number of further balance equations, because there follows for any vector $\mathbf{m}(\mathbf{n})$ of moment weights

$$\frac{\partial \mathbf{u}}{\partial t} + \sum_{i=1}^{3} \frac{\partial \mathbf{F}_i}{\partial x_i} = \Gamma \,. \tag{5.2.1}$$

Here, $\mathbf{u}$ and $\mathbf{F}_i$ are the vectors of densities and fluxes, respectively, and Γ is the production. They are defined as

$$\mathbf{u} = \int\limits_{\mathbb{R}^3} \mathbf{m}(\mathbf{k}) f(\mathbf{k})\, d^3k \,,\ \mathbf{F}_i = \int\limits_{\mathbb{R}^3} \frac{ck_i}{|\mathbf{k}|}\, \mathbf{m}(\mathbf{k}) f(\mathbf{k})\, d^3k \,,\ \Gamma = \int\limits_{\mathbb{R}^3} \mathbf{m}(\mathbf{k}) \zeta(f)(\mathbf{k})\, d^3k \,. \tag{5.2.2}$$

When the thermodynamic state is described by four fields e and Q_i only, then we can derive the following balance equations from the Boltzmann-Peierls equation obtained by multiplication of (5.1.1) with $\hbar c|\mathbf{k}|$ and $\hbar c^2\, k_i$ and integration over $\mathbf{k}$. There results the equations of balance for the energy density e and the heat flux Q_i

$$\begin{aligned} \frac{\partial e}{\partial t} + \sum_{k=1}^{3} \frac{\partial Q_k}{\partial x_k} &= \hbar\, c \int_{\mathbb{R}^3} |\mathbf{k}|\, \zeta\, d^3k \,, \\ \frac{\partial Q_i}{\partial t} + \sum_{k=1}^{3} \frac{\partial (c^2\, N_{ik})}{\partial x_k} &= \hbar\, c^2 \int_{\mathbb{R}^3} k_i\, \zeta\, d^3k \,. \end{aligned} \tag{5.2.3}$$

When we consider e and Q_i as the (macroscopic) basic variables for which initial-boundary value problems have to be solved, we must close the system

(5.2.3) so that the flux N_{ik} and the production terms on the right hand sides are related to the variables. This objective is achieved by applying the Maximum Entropy Principle. We first define the phase density
$w^{(4)} = P_N w^{(4)}$, see $(5.1.9)_2$, with the quantities y and $\Sigma_N = \Sigma_N(e, \mathbf{Q}, \mathbf{k})$ defined in (5.1.11)-(5.1.13) according to

$$w^{(4)}(e, \mathbf{Q}, \mathbf{k}) = \frac{y}{\exp(\Sigma_N) - 1}\,. \tag{5.2.4}$$

It satisfies the Maximum Entropy Principle for the given constraints on e and $\mathbf{Q}$. These are the four parameters of $w^{(4)}$, like the parameters n, $\mathbf{u}$ and p of Jüttner's relativistic Maxwellian. Next we replace these parameters by dynamical fields $e = e(t, \mathbf{x})$ and $\mathbf{Q} = \mathbf{Q}(t, \mathbf{x})$ and the Boltzmann-Peierls phase density f in the moment system (5.2.3) by the new "Maximum Entropy phase-density" $f^{(4)}$ given by

$$f^{(4)}(t, \mathbf{x}, \mathbf{k}) = w^{(4)}(e(t, \mathbf{x}), \mathbf{Q}(t, \mathbf{x}), \mathbf{k})\,. \tag{5.2.5}$$

By forming the integral moments for e, Q_i and N_{ij} with respect to $f^{(4)}$ we finally obtain the following hyperbolic four-field system, which will be studied in detail in Chapter 6

$$\frac{\partial e}{\partial t} + \sum_{i=1}^{3} \frac{\partial Q_i}{\partial x_i} = 0\,, \tag{5.2.6}$$

$$\frac{\partial Q_i}{\partial t} + \sum_{j=1}^{3} \frac{\partial \left(c^2 N_{ij}\right)}{\partial x_j} = -\frac{1}{\tau_R} Q_i\,, \quad i, j = 1, 2, 3, \tag{5.2.7}$$

$$N_{ij} = \frac{1}{3} e\, \delta_{ij} + \frac{1}{2} e(3\chi - 1) \left(\frac{Q_i Q_j}{|\mathbf{Q}|^2} - \frac{1}{3}\delta_{ij} \right)\,. \tag{5.2.8}$$

Here χ is the so called Eddington-factor

$$\chi = \frac{5}{3} - \frac{4}{3}\sqrt{1 - \frac{3}{4}\left(\frac{|\mathbf{Q}|}{ce}\right)^2}\,. \tag{5.2.9}$$

Note that in above equations the τ_N term does not appear on the right hand side. Indeed, the solutions of the Boltzmann-Peierls equation will only lead to a solution of the four-field system in the limit $\tau_N \to 0$.

By use of $f^{(4)}$ we may also calculate the entropy density h, the entropy flux ϕ_k and the entropy production σ as local functions of e and $\mathbf{Q}$, which turn

out to be, see [15] and [37],

$$\begin{aligned} h &= \left(\frac{2a}{3}\right)^{\frac{1}{4}} e^{\frac{3}{4}}(3-\chi)^{\frac{1}{2}}(1-\chi)^{\frac{1}{4}}, \\ \phi_k &= 2\left(\frac{2a}{3}\right)^{\frac{1}{4}} e^{-\frac{1}{4}}(3-\chi)^{-\frac{1}{2}}(1-\chi)^{\frac{1}{4}}\, Q_k, \\ \sigma &= -\frac{k_B}{\tau_R}\sum_{i=1}^{3}\Lambda_i Q_i \ , \quad a := \frac{\pi^2}{10}\, k_B^4/(\hbar^3 c^3) \ . \end{aligned} \tag{5.2.10}$$

Entropy density, entropy flux and entropy production are related to each other by an additional balance law, viz.

$$\frac{\partial h}{\partial t} + \sum_{k=1}^{3}\frac{\partial \phi_k}{\partial x_k} = \sigma \geq 0 \ . \tag{5.2.11}$$

Furthermore, it follows that the entropy production is zero in equilibrium and otherwise positive.

5.3 Reduced Boltzmann-Peierls equation

In the following two sections we extend the study of Dreyer, Herrmann and Kunik in [12], where a reduced kinetic equation for a reduced phase density was derived. We will present the solution for the microscopic three-dimensional equation. Also in this case the reduced kinetic equation simplifies considerably for a macroscopic one-dimensional flow field, which will be shown in Section 5.4 by three lemmata. In [12] we have mainly considered the microscopic two-dimensional case.

The procedure relies on the fact that for any solution f of (5.1.1) there exists a corresponding solution of a reduced equation that determines exactly the same moments as the original phase density. It uses the specific form of the transport term and the collision-operator in (5.1.1) and thus has no counterpart for the classical Boltzmann-equation.

The moments of the reduced phase density have a much simpler structure, because they are surface integrals over the unit ball with respect to the reduced phase density. We will also present the expression for the reduced kinetic entropy density-entropy flux pair.

The phase density f in (5.1.1) depends on the wave vector $\mathbf{k} \in \mathbb{R}^3$. We define the reduced phase density Ψ according to the radial integration in

polar-coordinates by

$$\Psi(\mathbf{n}) = \hbar c \int_0^\infty |\mathbf{k}|^3 \, f(|\mathbf{k}|\,\mathbf{n}) \, d|\mathbf{k}| \,. \tag{5.3.1}$$

Note that Ψ only depends on the unit vector $\mathbf{n} = (n_1, n_2, n_2) = \dfrac{\mathbf{k}}{|\mathbf{k}|}$.

Let m be a homogeneous moment weight of degree one, i.e. $m(\lambda \mathbf{k}) = \lambda m(\mathbf{k})$ for all $\lambda > 0$, and recall the definition of the unit sphere in $\mathbb{R}^3$, namely $\partial B(1, \mathbf{0}) = \{\mathbf{n} \in \mathbb{R}^3 \mid \ |\mathbf{n}| = 1\}$ in $\mathbb{R}^3$. Also let $\mathbf{u}$ be the corresponding moment function

$$\begin{aligned} \mathbf{u} &= \hbar c \int_{\mathbb{R}^3} m(\mathbf{k}) f(\mathbf{k}) \, d^3k \\ &= \hbar c \oint_{\partial B(1,\mathbf{0})} \int_0^\infty |\mathbf{k}|^3 \, m(\mathbf{n}) f(|\mathbf{k}|\,\mathbf{n}) \, d|\mathbf{k}| \, dS(\mathbf{n}) \\ &= \oint_{\partial B(1,\mathbf{0})} m(\mathbf{n}) \Psi(\mathbf{n}) \, dS(\mathbf{n}) \,. \end{aligned} \tag{5.3.2}$$

The moment $\mathbf{u}$ of f is thus given by the corresponding moments of Ψ. All the moments with physical interpretation are formed by homogeneous moment weights of degree one. In particular, we conclude for the moments defined in (5.1.2), (5.1.3), (5.1.4)

$$e = e(\Psi) \,, \quad Q_i = Q_i(\Psi) \,, \quad N_{ij} = N_{ij}(\Psi) \,, \tag{5.3.3}$$

where

$$e(\Psi) = \oint_{\partial B(1,\mathbf{0})} \Psi(\mathbf{n}) \, dS(\mathbf{n}) \,, \quad Q_i(\Psi) = c \oint_{\partial B(1,\mathbf{0})} n_i \Psi(\mathbf{n}) \, dS(\mathbf{n}) \,, \tag{5.3.4}$$

$$N_{ij}(\Psi) = \oint_{\partial B(1,\mathbf{0})} n_i n_j \Psi(\mathbf{n}) \, dS(\mathbf{n}) \,. \tag{5.3.5}$$

Thanks to the fact that only these moments with a homogeneous weight of degree one enter the collision operators defined in (5.1.6) and (5.1.9)-(5.1.13), we can also obtain the following reduced collision operators Φ, Φ_R and Φ_N

$$\Phi_\alpha = \frac{1}{\tau_\alpha} \left(\Theta_\alpha \Psi - \Psi \right) \,, \quad \alpha \in \{R, N\} \,, \quad \Phi = \Phi_R + \Phi_N \,, \tag{5.3.6}$$

where

$$\Theta_R \Psi = \frac{e}{4\pi} \,, \quad \Theta_N \Psi = \frac{3}{4\pi} \frac{e(4-F)^3}{F\left(1 - F\,\frac{\mathbf{n}\cdot\mathbf{Q}}{4ce}\right)^4} \,, \quad F = \frac{6}{1 + \sqrt{1 - \frac{3}{4}\left(\frac{|\mathbf{Q}|}{ce}\right)^2}} \tag{5.3.7}$$

and

$$\Phi_\alpha(\mathbf{n}) = \hbar c \int_0^\infty |\mathbf{k}|^3 \, P_\alpha f(|\mathbf{k}|\,\mathbf{n})\, d|\mathbf{k}|, \quad \alpha \in \{R, N\} \; . \tag{5.3.8}$$

We finally conclude that any solution $f(t, \mathbf{x}, \mathbf{k})$ of (5.1.1) induces a solution $\Psi(t, x, \mathbf{n})$ of the following reduced Boltzmann-Peierls equation

$$\frac{\partial \Psi}{\partial t}(t, \mathbf{x}, \mathbf{n}) + \sum_{k=1}^{3} c\, n_k \, \frac{\partial \Psi}{\partial x_k}(t, \mathbf{x}, \mathbf{n}) = \Phi(t, \mathbf{x}, \mathbf{n}) \, . \tag{5.3.9}$$

This reduced kinetic equation is again an evolution equation for Ψ. It can be rewritten as a time integral by using Duhamel's principle

$$\Psi(t+\tau, \mathbf{x}, \mathbf{n}) = \Psi(t, \mathbf{x} - c\tau\mathbf{n}, \mathbf{n}) + \sum_{\alpha \in \{R,N\}} \int_0^\tau \Phi_\alpha(t+s, \mathbf{x} - c(\tau - s)\mathbf{n}, \mathbf{n})\, ds \, , \tag{5.3.10}$$

where $\Psi(t, \mathbf{x} - c\tau\mathbf{n}, \mathbf{n})$ is the solution at time $t + \tau$ of the collisionless kinetic equation

$$\frac{\partial \Psi}{\partial t}(t, \mathbf{x}, \mathbf{n}) + \sum_{k=1}^{3} c\, n_k \, \frac{\partial \Psi}{\partial x_k}(t, \mathbf{x}, \mathbf{n}) = 0 \, . \tag{5.3.11}$$

In particular, $\Psi_0(t, \mathbf{x} - c\tau\mathbf{n}, \mathbf{n})$ is the free-flight solution of (5.3.11) for initial data Ψ_0.

Finally we introduce an entropy density-entropy flux pair for the reduced equation (5.3.9). The definition is not so straight forward as before, because in general the entropy density h formed with respect to the original phase density f cannot be determined from the reduced phase density Ψ. The following definition is proposed in [12]

$$\begin{aligned} \tilde{h}(\Psi) &= \mu \oint_{\partial B(1,\mathbf{0})} \Psi^{\frac{3}{4}}(\mathbf{n})\, dS(\mathbf{n}) \, , \\ \tilde{\Phi}_k(\Psi) &= \mu\, c \oint_{\partial B(1,\mathbf{0})} n_k \, \Psi^{\frac{3}{4}}(\mathbf{n})\, dS(\mathbf{n}) \, , \quad \mu = \frac{4\pi}{3} \left(\frac{y}{15}\right)^{\frac{1}{4}} \, , \quad y = \frac{3}{8\pi^3} \, . \end{aligned} \tag{5.3.12}$$

It was shown in [12] that in view of the hyperbolic moment systems this new kinetic entropy definition for the reduced BPE leads to the same results as the original kinetic entropy definition (5.1.5) for the BPE (5.1.1).

In order to establish this result, we present here a new and short proof of the

Theorem 5.1. *H-Theorem*

Assume that $\Psi : \mathbb{R}_0^+ \times \mathbb{R}^3 \times \partial B(1, \mathbf{0}) \to \mathbb{R}^+$ *is a solution of the reduced BPE* (5.3.9) *and that* $(\tilde{h}, \tilde{\Phi}_1, \tilde{\Phi}_2, \tilde{\Phi}_3,)$ *is the corresponding reduced entropy density-entropy flux pair given in* (5.3.12). *Then there holds the entropy inequality*

$$\frac{\partial \tilde{h}}{\partial t} + \sum_{k=1}^{3} \frac{\partial \tilde{\Phi}_k}{\partial x_k} \geq 0 \,.$$

Proof:

Multiplication of the reduced BPE with $\frac{3}{4}\Psi^{-1/4}$ gives

$$\begin{aligned}
&\frac{\partial \Psi^{3/4}}{\partial t} + c \sum_{k=1}^{3} n_k \frac{\partial \Psi^{3/4}}{\partial x_k} = \frac{3\Psi^{-1/4}}{4\tau_N}(\Theta_N \Psi - \Psi) + \frac{3\Psi^{-1/4}}{4\tau_R}(\Theta_R \Psi - \Psi) \\
=&\frac{3}{4\tau_N}\Big[(\Psi^{-1/4} - (\Theta_N \Psi)^{-1/4})(\Theta_N \Psi - \Psi)\Big] + \\
&\frac{3}{4\tau_R}\Big[(\Psi^{-1/4} - (\Theta_R \Psi)^{-1/4})(\Theta_R \Psi - \Psi)\Big] + \\
&\frac{3}{4\tau_N}\Big[(\Theta_N \Psi)^{-1/4}(\Theta_N \Psi - \Psi)\Big] + \frac{3}{4\tau_R}\Big[(\Theta_R \Psi)^{-1/4}(\Theta_R \Psi - \Psi)\Big] \,.
\end{aligned}$$

We perform from this the surface integral with respect to $\mathbf{n}$. The first two brackets $\Big[...\Big]$ are non-negative and the surface integrals with respect to $\mathbf{n}$ over the last two brackets vanish because Ψ, $\Theta_N \Psi$ and $\Theta_R \Psi$ have the same moment for the energy density e and Ψ, $\Theta_N \Psi$ have the same moments for the heat flux $\mathbf{Q}$. ∎

Time discretization of the reduced kinetic equation

Now we replace the time integrals in (5.3.10) by finite Riemann sums at the update times $t_n = n\tau$, where $\tau > 0$ is a fixed time step. We will choose τ in such a way that there holds the inequality

$$0 < \frac{\tau}{\tau_R} + \frac{\tau}{\tau_N} \leq 1 \tag{5.3.13}$$

with the fixed relaxation times $\tau_R, \tau_N > 0$ and will also include the important limiting case that $\tau_R = \infty$. Then we obtain the first kinetic scheme which is discrete in time but continuous in the phase space

$$\begin{aligned}
\Psi(t_{n+1}, \mathbf{x}, \mathbf{n}) = &\sum_{\alpha \in \{R,N\}} \frac{\tau}{\tau_\alpha}(\Theta_\alpha \Psi)(t_n, \mathbf{x} - c\tau\mathbf{n}, \mathbf{n}) \\
&+ \left(1 - \frac{\tau}{\tau_R} - \frac{\tau}{\tau_N}\right) \Psi(t_n, \mathbf{x} - c\tau\mathbf{n}, \mathbf{n}) \,.
\end{aligned} \tag{5.3.14}$$

In order to get the first term on the right hand side of (5.3.14), we first calculate the moments of energy and heat flux at the update-time t_n, namely

$$e(t_n, \mathbf{y}) = \oint\limits_{\partial B(1,\mathbf{0})} \Psi(t_n, \mathbf{y}, \mathbf{n})\, dS(\mathbf{n})\,,$$
$$Q_i(t_n, \mathbf{y}) = c \oint\limits_{\partial B(1,\mathbf{0})} n_i \Psi(t_n, \mathbf{y}, \mathbf{n})\, dS(\mathbf{n})\,. \tag{5.3.15}$$

Next we perform the projections $\Theta_R\Psi$ and $\Theta_N\Psi$ from these moments on the reduced Maximum Entropy phase densities given in (5.3.7) at time t_n. Finally we use Ψ and these projections as initial phase densities at time t in order to solve the collisionless reduced kinetic transport equation in the time interval $[t, t+\tau]$. There results the convex-combination of free-flight solutions given in (5.3.14) for the phase-density at the next update-time t_{n+1}.

Since the scheme (5.3.14) is only first order in time and space, we can simplify it in such a way that we just replace $\frac{\tau}{\tau_\alpha}(\Theta_\alpha\Psi)(t_n, \mathbf{x} - c\tau\mathbf{n}, \mathbf{n})$ by $\frac{\tau}{\tau_\alpha}(\Theta_\alpha\Psi)(t_n, \mathbf{x}, \mathbf{n})$. The modified scheme may now be written as

$$\Psi(t_{n+1}, \mathbf{x}, \mathbf{n}) = \Psi(t_n, \mathbf{x} - c\tau\mathbf{n}, \mathbf{n}) + \sum_{\alpha\in\{R,N\}} \frac{\tau}{\tau_\alpha}\Big((\Theta_\alpha\Psi)(t_n, \mathbf{x}, \mathbf{n}) - \Psi(t_n, \mathbf{x}, \mathbf{n})\Big)\,. \tag{5.3.16}$$

5.4 Reduced kinetic equation in one space dimension

Here we are interested in reducing the already reduced Boltzmann-Peierls equation (5.3.9) further. Later on we show by three lemmas that this reduced phase density describes a spatially one-dimensional flow.

In the one-dimensional case we put $\mathbf{x} = (x, 0, 0)$, $\mathbf{Q} = (Q(t, x), 0, 0)$ and $e = e(t, x)$. We introduce the new variables $-1 \le \xi \le 1$, $0 \le \vartheta \le 2\pi$ by

$$n_1 = \xi\,, \qquad n_2 = \sqrt{1-\xi^2}\sin\vartheta\,, \qquad n_3 = \sqrt{1-\xi^2}\cos\vartheta\,, \tag{5.4.1}$$

and the surface element is $dS(\mathbf{n}) = d\xi d\vartheta$. Since in the one-dimensional case the macroscopic fields inside the phase density $\Psi(t, x, \mathbf{n})$ will not depend on the angle ϑ, we can further reduce Ψ to

$$\psi(t, x, \xi) = \int_0^{2\pi} \Psi(t, x, 0, 0, \mathbf{n})\, d\vartheta = 2\pi\Psi(t, x, \xi). \tag{5.4.2}$$

The reduced Boltzmann-Peierls equation(5.3.9) then further simplifies to

$$\frac{\partial\psi}{\partial t}(t,x,\xi)+c\,\xi\,\frac{\partial\psi}{\partial x}(t,x,\xi)=\sum_{\alpha\in\{R,N\}}\frac{1}{\tau_\alpha}\left(\Theta_\alpha\psi-\psi\right)(t,x,\xi)\,, \tag{5.4.3}$$

where

$$\Theta_R\psi=\frac{e}{2}\,,\quad \Theta_N\psi=\frac{3}{2}\frac{e(4-F)^3}{F\left(1-\xi\,\frac{FQ}{4ce}\right)^4}\,,\quad F=\frac{6}{1+\sqrt{1-\frac{3}{4}\left(\frac{Q}{ce}\right)^2}}\,. \tag{5.4.4}$$

We have used the same notation for the projection as in (5.3.7), but this will not lead to confusion within the context. Also the reduced moments integrals are given by

$$e(t,x)=\int_{-1}^{1}\psi(t,x,\xi)\,d\xi,\quad Q(t,x)=c\int_{-1}^{1}\xi\psi(t,x,\xi)\,d\xi\,, \tag{5.4.5}$$

$$N(t,x)=N_{11}(t,x)=\int_{-1}^{1}\xi^2\psi(t,x,\xi)\,d\xi. \tag{5.4.6}$$

In the following three lemmas we are going to show that the above reduction is valid for all later times.

Definition: Any function $g:\mathbb{R}^3\times\partial B(1,\mathbf{0})\to\mathbb{R}$ has **property 1D** if and only if $g(\mathbf{x},\mathbf{n})$ depends only on $x=x_1$ and $\xi=n_1$, $\mathbf{x}=(x_1,x_2,x_3)\in\mathbb{R}^3$ and $\mathbf{n}=(n_1,n_2,n_3)\in\partial B(1,\mathbf{0})$.

Assumption
For the next three lemmas we assume that the initial C^1-phase density $\Psi(t_0,\cdot,\cdot):\mathbb{R}^3\times\partial B(1,\mathbf{0})\to\mathbb{R}^+$ at fixed initial time t_0 has property 1D.

Lemma 1: Under the above assumption $\Theta_N\left(\Psi(t_0,\cdot,\cdot)\right)$ and $\Theta_R\left(\Psi(t_0,\cdot,\cdot)\right)$ have also property 1D, furthermore the energy density and heat flux moment vector of $\Psi(t_0,\cdot,\cdot)$ are of the form

$$e(t_0,\mathbf{x})=e(t_0,x)\,,\quad \mathbf{Q}(t_0,\mathbf{x})=(Q_1(t_0,x),0,0)\,,\quad \text{where } x=x_1\,.$$

Proof: Due to the property 1D we can write $\Psi(t_0,x,\xi)$ instead of $\Psi(t_0,\mathbf{x},\mathbf{n})$. We rewrite the surface integrals in (5.3.4) in terms of the new integration

variables ξ, ϑ. Using (5.4.2) we have

$$e(t_0,x) = \int_{-1}^{1}\int_{0}^{2\pi} \Psi(t_0,x,\xi)d\vartheta d\xi$$
$$= \int_{-1}^{1} 2\pi\, \Psi(t_0,x,\xi)d\xi = \int_{-1}^{1} \psi(t_0,x,\xi)d\xi\,, \qquad (5.4.7)$$

$$Q(t_0,x) = Q_1(t_0,x) = c\int_{-1}^{1}\int_{0}^{2\pi} \xi\Psi(t_0,x,\xi)d\vartheta d\xi$$
$$= c\int_{-1}^{1} \xi\, 2\pi\, \Psi(t_0,x,\xi)d\xi = c\int_{-1}^{1} \xi\psi(t_0,x,\xi)d\xi\,, \qquad (5.4.8)$$

$$Q_2(t_0,x) = c\int_{-1}^{1}\int_{0}^{2\pi} \sin\vartheta\sqrt{1-\xi^2}\,\Psi(t_0,x,\xi)d\vartheta d\xi = 0\,, \qquad (5.4.9)$$

similarly $Q_3(t_0,x) = 0$.

Keeping in view (5.4.7),(5.4.8) and (5.4.9) we can see from the definitions (5.3.7) of $\Theta_R(\Psi)$ and $\Theta_N(\Psi)$ that they are only depending on $e(t_0,x)$, $Q_1(t_0,x)$ and $\xi = n_1$. Therefore they also satisfy property 1D. ∎

Lemma 2: The time derivative $\dfrac{\partial\Psi}{\partial t}(t_0,\cdot,\cdot)$ has property 1D, if Ψ satisfies the reduced Boltzmann-Peierls equation (BPE) (5.3.9).

Proof: The reduced BPE (5.3.9) can be rewritten as

$$\frac{\partial\Psi}{\partial t}(t,\mathbf{x},\mathbf{n}) = \Phi(t,\mathbf{x},\mathbf{n}) - c\sum_{k=1}^{3} n_k \frac{\partial\Psi}{\partial x_k}(t,\mathbf{x},\mathbf{n}). \qquad (5.4.10)$$

The first term $\Phi(t,\mathbf{x},\mathbf{n})$ on the right hand side of (5.4.10) has property 1D for $t = t_0$ due to the assumption and Lemma 1. Also since $\dfrac{\partial\Psi}{\partial x_2} = \dfrac{\partial\Psi}{\partial x_3} = 0$, the second term on the right hand side of (5.4.10) also satisfies property 1D. This implies that the left hand side of (5.4.10) satisfies the property 1D. ∎

Lemma 3: The free-flight phase density $\Psi_{free}(t_0+\tau,\mathbf{x},\mathbf{n}) = \Psi(t_0,\mathbf{x}-\tau\mathbf{n},\mathbf{n})$ also satisfies the property 1D for all later times $t_0+\tau$, $\tau > 0$.

Proof: The proof of this lemma is obvious from $\Psi_{free}(t_0+\tau, \mathbf{x}, \mathbf{n})$, since due to the assumption that Ψ is only depending on $x = x_1$, therefore Ψ_{free} will maintain this property for all later times. ∎

5.4.1 Discrete kinetic solution in the one-dimensional case

We present a kinetic solution of the one-dimensional reduced kinetic equation (5.3.10) for the fully discretized phase space. In the one-dimensional case the semidiscrete scheme (5.3.16) with the time steps $t_n = n\tau$ can be written in terms of the fully reduced phase density ψ as

$$\psi(t_{n+1}, x, \xi) = \psi(t_n, \mathbf{x} - c\tau\xi, \xi) + \sum_{\alpha\in\{R,N\}} \frac{\tau}{\tau_\alpha}\Big((\Theta_\alpha\psi)(t_n, x, \xi) - \psi(t_n, x, \xi)\Big)\,. \tag{5.4.11}$$

This equation is discrete in time but continuous in the reduced x, ξ phase space. In order to get a fully discretized piecewise constant solution, we first define a grid in the reduced phase space consisting of the cells

$$C_{i,j} = \left\{(x,\xi)\in\mathbb{R}^2 \,|\; |x - x_i| < \frac{1}{2}\,\Delta x\,, \quad \xi_j \le \xi < \xi_{j+1}\right\}\,,$$

where $x_i = i\Delta x$, $x_{i\pm\frac{1}{2}} = i \pm \frac{1}{2}\Delta x$ and $\xi_j = j/N_\xi$ with the integers i, j, N_ξ and $|j| \le N_\xi$, $N_\xi > 0$. The cell-average of ψ at time $t = t_n$ is given by the following integral mean value, where $\Delta\xi = 1/(2N_\xi)$

$$\psi_{i,j}(t) = \frac{1}{\Delta x\Delta\xi}\int\limits_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int\limits_{\xi_j}^{\xi_{j+1}} \psi(t,x,\xi)\,d\xi dx\,. \tag{5.4.12}$$

With the characteristic function $\chi_{i,j}(x,\xi)$ of the cell $C_{i,j}$ we can write the fully discretized piecewise constant phase density in the form $\sum \psi_{i,j}(t_n)\chi_{i,j}(x,\xi)$. Due to the upwind approach we will choose the time step according to the natural CFL-condition

$$\tau = \Delta t = \frac{\Delta x}{2c}\,. \tag{5.4.13}$$

Note that for the semidiscrete case there is no such CFL-condition, but we will assume in the sequel that $\tau = \Delta t$ still satisfies the inequality (5.3.13).

The heart-piece of the scheme for the fully discretized phase space is the integral mean value $\psi_{i,j}^{free}(t_{n+1})$ of the free-flight phase density over the cell $C_{i,j}$ at the next time step t_{n+1}, provided the initial data of the phase density

at the actual time step t_n is assumed to be piecewise constant and thus determined by the coefficients $\psi_{i,j}(t_n)$. Then the integration can be performed explicitly and gives the fully discretized free-flight solution in upwind form

$$\psi_{i,j}^{free}(t_{n+1}) = \begin{cases} -\frac{1}{2}\xi_{j+\frac{1}{2}}\psi_{i+1,j}(t_n) + (1+\frac{1}{2}\xi_{j+\frac{1}{2}})\,\psi_{i,j}(t_n)\,, & \xi_j < 0\,, \\ \\ \frac{1}{2}\xi_{j+\frac{1}{2}}\psi_{i-1,j}(t_n) + (1-\frac{1}{2}\xi_{j+\frac{1}{2}})\,\psi_{i,j}(t_n)\,, & \xi_j \geq 0\,, \end{cases} \tag{5.4.14}$$

where $\xi_{j+\frac{1}{2}} = \xi_j + \frac{1}{2}\Delta\xi$. Integrating (5.4.11) over $\left[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right] \times [\xi_j, \xi_{j+1}]$ and dividing by $\Delta x \Delta\xi$, we get with (5.4.14) the fully discretized phase density at the next time step t_{n+1} in terms of its coefficients

$$\psi_{i,j}(t_{n+1}) = \psi_{i,j}^{free}(t_{n+1}) + \sum_{\alpha\in\{R,N\}} \frac{\tau}{\tau_\alpha}\Big((\Theta_\alpha\psi)_{i,j}(t_n) - \psi_{i,j}(t_n)\Big)\,. \tag{5.4.15}$$

The right-hand side in (5.4.15) needs some explanations. We first note that the discretized moments are given at each time step t_n in the spatial interval $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ by the constant values

$$e_i(t_n) = \Delta\xi \sum_{j=-N_\xi}^{N_\xi-1} \psi_{i,j}(t_n)\,, \quad Q_i(t_n) = \Delta\xi \sum_{j=-N_\xi}^{N_\xi-1} \xi_{j+\frac{1}{2}}\psi_{i,j}(t_n)\,. \tag{5.4.16}$$

Then we need the integral mean values of the Maximum Entropy phase densities $\Theta_\alpha\psi$ formed by the moments $e_i(t_n)$ and $Q_i(t_n)$ over the small intervals $[\xi_j, \xi_{j+1}]$, namely

$$(\Theta_\alpha\psi)_{i,j}(t_n) = \frac{1}{\Delta\xi}\int_{\xi_j}^{\xi_{j+1}} (\Theta_\alpha\psi)(e_i(t_n), Q_i(t_n), \xi)\,d\xi\,, \tag{5.4.17}$$

which may be approximated by the trapezoidal rule with only the two nodal points ξ_j and ξ_{j+1} or by the midpoint rule with the single nodal point $\xi_{j+\frac{1}{2}}$.

5.5 Numerical examples

Now we present some numerical test cases, where we use the kinetic scheme for the reduced BPE in order to compute several initial value problems in one space dimension for both, the reduced BPE as well as for the hyperbolic four-field system. Some of these numerical results will directly be compared with explicit solutions. The numerical implementations were carried out by Shamsul Qamar.

5.5.1 Energy pulse

In this example we assume that $\tau_R = \infty$, so that from the physical point of view, we study a phonon gas in a pure crystal at low temperature. Since there are no lattice impurities, diffusion cannot appear. In particular, the propagation of heat behaves like a wave phenomenon. Further we assume that the phase density only depends on $x = x_1$. In order to simulate the problem, we consider the following macroscopic initial data for energy density e and the momentum density Q:

$$e(0,x) = \begin{cases} 1.5 & \text{if} \quad 0.99 \le x \le 1.01 \\ 1.0 & \text{otherwise} \end{cases}, \qquad Q(0,x) = 0.0\,. \tag{5.5.1}$$

Furthermore, the Debye speed c is set to 0.5.

Figures 5.1 shows the spatial dependence of the energy density, heat flux and momentum flux at different times for the relaxation time $\tau_N = 0.5$. While Figures 5.2 gives the distribution functions ψ and $\Theta_N\psi$ at time t=1.2.

5.5.2 Two interacting heat pulses

This test problem demonstrates the interaction of two heat pulses, which leads to a large increase of the energy density at the collision point during a short time interval. The initial data are

$$e(0,x) = \begin{cases} 1\,, & x \le 0.3, \\ 2\,, & 0.3 \le x \le 0.4 \\ 1\,, & 0.4 \le x \le 0.6 \\ 2\,, & 0.6 \le x \le 0.7 \\ 1\,, & x \le 1.0 \end{cases}, \quad Q(0,x) = \begin{cases} 0\,, & x \le 0.3, \\ 1\,, & 0.3 \le x \le 0.4 \\ 0\,, & 0.4 \le x \le 0.6 \\ -1\,, & 0.6 \le x \le 0.7 \\ 0\,, & x \le 1.0\,. \end{cases} \tag{5.5.2}$$

We solve the above problem at time $t = 0.2$ for two values of τ_N, i.e., $\tau_N = 1$ and $\tau_N = 0.1$, while $\tau_R = 1.0$. Figure 5.3 shows the results. From the comparison of the initial and final curves of energy density, we observe a large increase of the energy density e at the collision point $x = 0.5$.

5.5.3 Initial phase density independent of space

Here we consider an initial phase density which is independent of the x-coordinate, i.e., $\psi_0 = \psi_0(\xi)$. Furthermore we assume that $\tau_R = \infty$, this implies that the Boltzmann-Peierls equation (5.4.3) will only contain τ_N and Θ_N. Therefore we have to solve the following simplified Boltzmann-Peierls equation with unit Debye constant, i.e, $c = 1$.

$$\frac{\partial\psi}{\partial t}(t,\xi) + \xi\,\frac{\partial\psi}{\partial x}(t,\xi) = \frac{1}{\tau_N}\left(\Theta_N\psi - \psi\right)(t,\xi)\,. \tag{5.5.3}$$

The energy density e and the heat flux Q are constant with respect to the time, which is clear from the following conservation laws

$$\frac{\partial e}{\partial t}(t) = \int_{-1}^{1} \frac{\partial \psi}{\partial t} d\xi = \int_{-1}^{1} \frac{\Theta_N \psi - \psi}{\tau_N} d\xi = \frac{1}{\tau_N} (e_N - e) = 0 \,,$$

$$\frac{\partial Q}{\partial t}(t) = c \int_{-1}^{1} \xi \frac{\partial \psi}{\partial t} d\xi = c \int_{-1}^{1} \xi \frac{\Theta_N \psi - \psi}{\tau_N} d\xi = \frac{c}{\tau_N} (Q_N - Q) = 0 \,.$$

Both of the above integrals are zero because of the constraints $e_N = e$ and $Q_N = Q$. Therefore equation (5.5.3) is simply an ordinary differential equation with initial data
$\psi_0 = \psi(\xi)$. The explicit solution of this equation is given by

$$\psi_{exact}(t, \xi) = \psi_0(\xi) \, \exp\left(-\frac{t}{\tau_N}\right) + \left(1 - \exp\left(-\frac{t}{\tau_N}\right)\right) \Theta_N \psi_0(\xi) \,. \quad (5.5.4)$$

Now we consider two examples in which the inital phase density ψ_0 only depend on ξ.

Example 1: Here we consider the initial phase density as a linear function, $\psi_0 = \dfrac{1+\xi}{2}$. Figure $5.4_{1,2}$ gives the numerical value of phase density ψ at $t = 0.0$ and $t = 1.0$, while Figure 5.4_3 gives the difference between the exact and numerical solutions.

Example 2: The initial phase density is a shifted hat function

$$\psi_0(\xi) = \begin{cases} \frac{4}{7}(\xi + 1) \,, & -1 \le \xi \le \frac{3}{4} \,, \\ 4(-\xi + 1) \,, & \frac{3}{4} \le \xi \le 1 \,, \end{cases} \quad . \quad (5.5.5)$$

Figure $5.5_{1,2}$ gives the numerical value of phase density ψ at $t = 0.0$ and $t = 1.0$, while Figure 5.5_3 gives the difference between the exact and numerical solutions.

5.5.4 A single shock solution

We consider now a single shock solution of the hyperbolic four-field system. The initial data are

$$(e, Q)(0, x) = \begin{cases} \left(2, \frac{1}{\sqrt{3}} \sqrt{\frac{3\sqrt{2}-1}{\sqrt{2}+1}}\right) , & x \le 0.5, \\ (1, 0) \,, & x \ge 0.5. \end{cases} \quad (5.5.6)$$

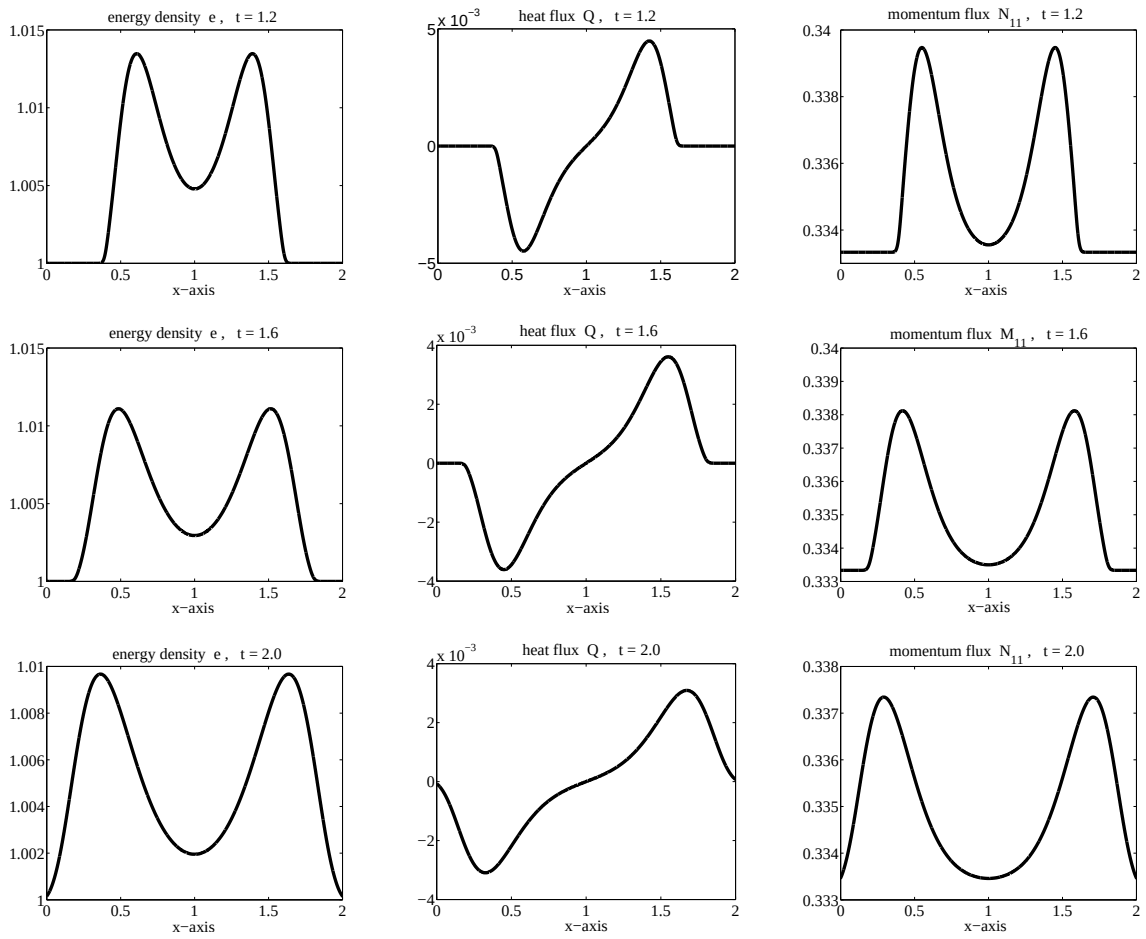

Figure 5.1: Evolution of energy, heat flux and momentum flux pulses at $\tau_N = 0.5$.

This problem was considered by [9] using Rankine-Hugoniot conditions which were derived in [15] for the hyperbolic four-field moment system. We will consider it in detail in Section 6.3, however the explanations which are given here are sufficient to perform this numerical test.

The computational domain is $0 \leq x \leq 1$. In this example $\tau_R = \infty$, while $\frac{\tau}{\tau_N} = 1$, where τ is a small numerical time step. Then the Boltzmann-Peierls equation (5.4.3) approximates the solution of the hyperbolic four-field moments system with right-hand side zero. The x, ξ reduced phase-space is divided by 1000×1000 mesh points. The results are shown in Figure 5.6.

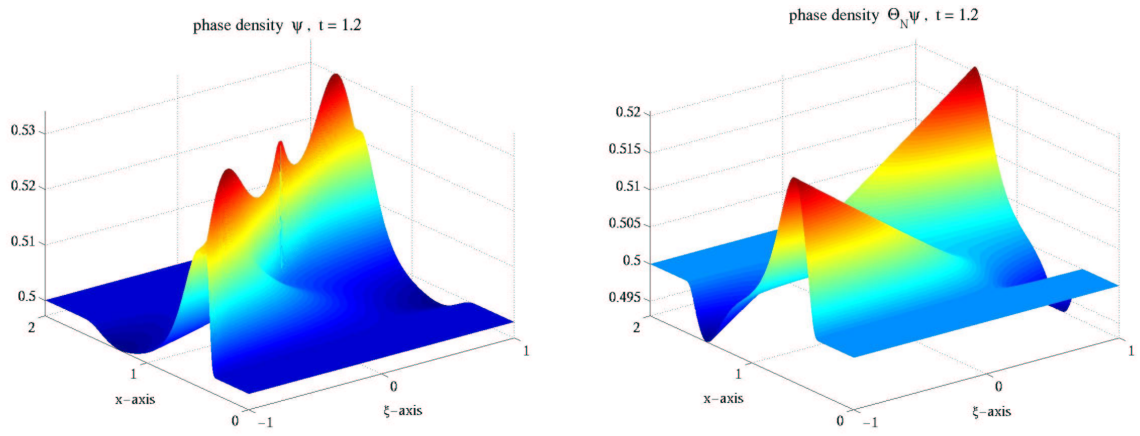

Figure 5.2: Representation of the phase densities at $\tau_N = 0.5$ and t=1.2.

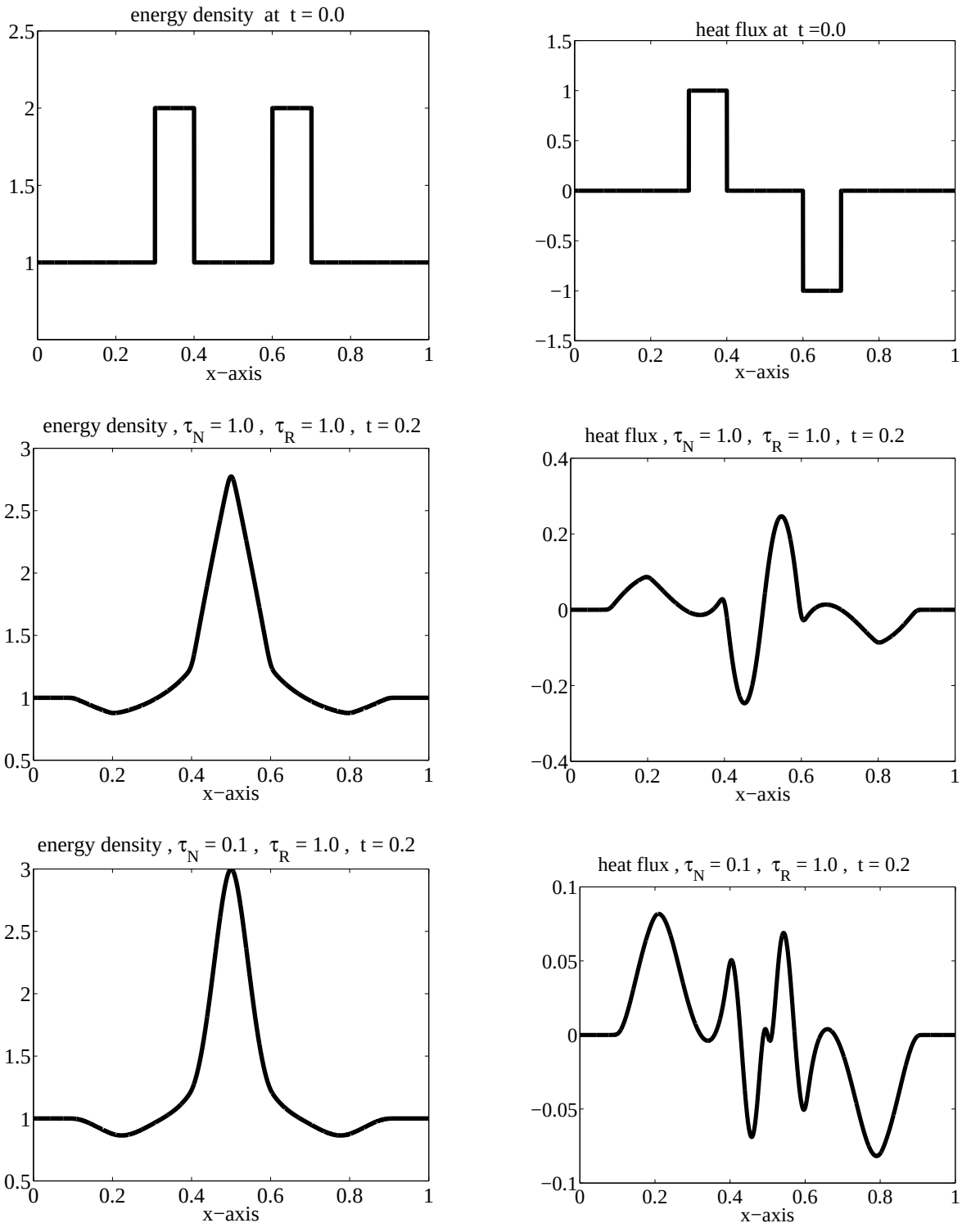

Figure 5.3: Evolution of energy and heat flux.

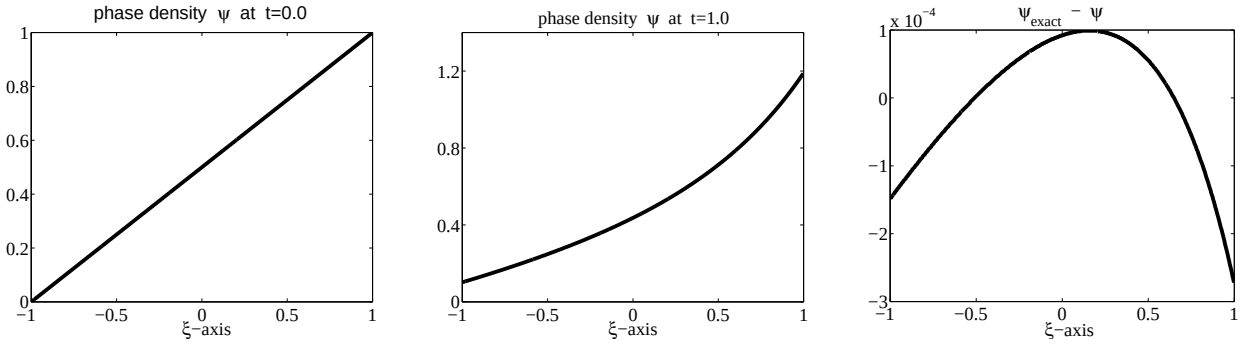

Figure 5.4: Phase densities and the error difference between exact the numerical values.

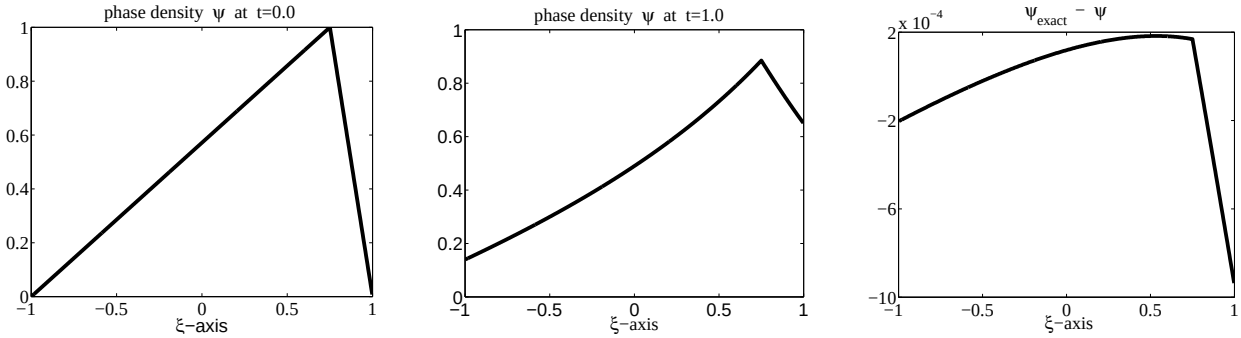

Figure 5.5: Phase densities and the error difference between exact the numerical values.

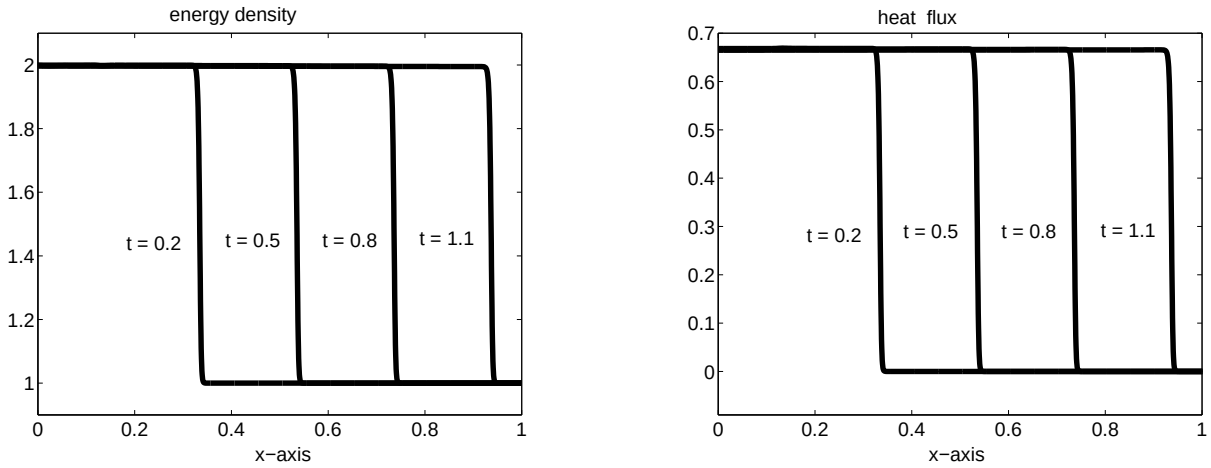

Figure 5.6: Evolution of a single shock at different times.

Chapter 6

Kinetic Solution of the Hyperbolic Four-Field System

6.1 Motivation

One may wonder why we present an additional chapter for the numerical solution of the four-field system, because the kinetic approach for the reduced Boltzmann-Peierls equation developed in Chapter 5 has already captured this hyperbolic four-field system as a limiting case.

However, for the initial value problem of the four-field system we will present an alternative scheme which is discrete in time and continuous in space. It uses a modified transport part which reproduces the special production term on the right-hand side of the momentum-balance law of the four-field system. Moreover, we will extend the modified scheme in order to solve an interesting initial-boundary value problem for this system which is not covered by the schemes presented in Chapter 5.

Finally, we will also show by an appropriate transformation in the state-space that the four-field system and its kinetic solution contains the ultra-relativistic Euler equations and their kinetic solution as a special case. The work reported in this chapter is joint work with Wolfgang Dreyer (WIAS, Berlin), see [9, 10].

The four-field system was introduced in Section 5.2. Especially in the one-dimensional case we can write down its weak formulation with a bounded convex region Ω in space and time as

$$\begin{aligned} &\int\limits_{\partial\Omega} (e\,dx \;-\; Q\,dt) = 0 \;, \\ &\int\limits_{\partial\Omega} (Q\,dx \;-\; c^2\,e\chi\,dt) = -\tfrac{1}{\tau_R} \iint\limits_{\Omega} Q\;dt\,dx \;, \\ &\chi \;=\; \tfrac{5}{3} - \tfrac{4}{3}\sqrt{1 - \tfrac{3}{4}\left(\tfrac{Q}{ce}\right)^2} \;. \end{aligned} \tag{6.1.1}$$

We will also include the important limiting case that the relaxation time $\tau_R > 0$ is infinite. Here $e : \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}_0^+$ and $Q : \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ is a piecewise C^1-solution of the hyperbolic system which may contain a finite number of C^1-shock curves. We will prescribe appropriate initial- and boundary data for e and Q.

6.2 A modified kinetic transport equation for the four-field system

We have already seen that solutions of collisionless kinetic transport equations constitutes a basic building block for the development of kinetic schemes which are solving special hyperbolic systems and special kinetic equations. Now we go one step further into this direction and develop a modified kinetic transport equation which is able to reproduce the specific relaxation terms on the right-hand side of the four-field system (5.2.6)-(5.2.9). This also includes a useful modification of the moments which will again be given in terms of surface integrals with respect to the reduced phase density $\Psi = \Psi(t, \mathbf{x}, \mathbf{n})$.

This section is a motivation for the kinetic schemes which will be considered next and introduces a modified collisionless kinetic transport equation as a basic ingredient for these schemes.

We start with a reduced initial phase density $\Psi_0(\mathbf{x}, \mathbf{n}) = \Psi(0, \mathbf{x}, \mathbf{n})$ at some initial time t_0. For $t > t_0$ we define the modification of the reduced free-flight phase density Ψ according to

$$\Psi(t_0 + \tau, \mathbf{x}, \mathbf{n}) = \Psi(t_0, \mathbf{x} - c\,\gamma(\tau)\mathbf{n})\,, \tag{6.2.1}$$

where the function $\gamma(\tau)$ is defined for $\tau \geq 0$ as

$$\gamma(\tau) = \tau_R\,(1 - \exp(-\frac{\tau}{\tau_R})) < \tau_R\,. \tag{6.2.2}$$

We conclude that the phase density (6.2.1) satisfies the new kinetic transport equation

$$\frac{\partial \Psi}{\partial \tau} + \sum_{i=1}^{3} \dot{\gamma}(\tau)\, c\, n_i \frac{\partial \Psi}{\partial x_i} = 0\ . \tag{6.2.3}$$

According to this transport equation we also redefine the moments (5.3.4). While the energy density e will be defined as before, but with the phase density (6.2.1), the heat flux Q_i and its flux N_{ik} involve in addition the functions $\dot{\gamma}(\tau)$ and $\dot{\gamma}^2(\tau)$ as new time-dependent factors:

$$
\begin{aligned}
e(t_0+\tau,\mathbf{x}) &= \oint\limits_{\partial B(1,\mathbf{0})} \Psi(t_0+\tau,\mathbf{x},\mathbf{n})\; dS(\mathbf{n})\,, \\
Q_i(t_0+\tau,\mathbf{x}) &= c\dot{\gamma}(\tau) \oint\limits_{\partial B(1,\mathbf{0})} n_i \Psi(t_0+\tau,\mathbf{x},\mathbf{n})\; dS(\mathbf{n})\,, \qquad (6.2.4)\\
N_{ik}(t_0+\tau,\mathbf{x}) &= \dot{\gamma}^2(\tau) \oint\limits_{\partial B(1,\mathbf{0})} n_i n_k \Psi(t_0+\tau,\mathbf{x},\mathbf{n})\; dS(\mathbf{n}).
\end{aligned}
$$

These definitions imply equations of balance that have the same structure as the corresponding hyperbolic four-field system (5.2.6)-(5.2.9), namely

$$
\begin{aligned}
\frac{\partial e}{\partial \tau} + \sum_{k=1}^{3} \frac{\partial Q_k}{\partial x_k} &= 0, \\
\frac{\partial Q_i}{\partial \tau} + \sum_{k=1}^{3} \frac{\partial (c^2\, N_{ik})}{\partial x_k} &= -\frac{1}{\tau_R}\, Q_i.
\end{aligned} \qquad (6.2.5)
$$

It is important to note that in contrast to the local hyperbolic system (5.2.6)-(5.2.9) these moment system is non-local in time and space. In the next sections we will show how the modified kinetic representations can be used in order to solve the initial-boundary value problem for the hyperbolic system.

6.3 The pure initial value problem (IVP)

6.3.1 Kinetic solution of the IVP

In order to solve the initial value problem for the nonlinear four- field system (5.2.6)-(5.2.9), we start at a fixed time t_0 with given initial data e_0 for the energy density and $\mathbf{Q}_0$ for the heat flux. With e_0, Q_0 we form the four-field Maximum Entropy phase density Ψ_{eq} at time t_0 as the initial phase density in order to solve the modified kinetic transport equation (6.2.3) in the finite time interval $t_0 \le t \le t_0 + \tau_M$ for a given time step τ_M. Then we put $t_0 + \tau_M$ as a new initial time and proceed as before, and so on.

To initialize the scheme we start with

- Bounded and integrable initial data for $\mathbf{x} \in I\!R^3$ at time $t = 0$, $e(0,\mathbf{x}) = e_0(\mathbf{x}) \ge \epsilon > 0$, $\mathbf{Q}(0,\mathbf{x}) = \mathbf{Q}_0(\mathbf{x})$ with $|\mathbf{Q}| < c\, e$.

- A fixed time step $\tau_M > 0$, so that the maximization of entropy is carried out at the equidistant times $t_n = n\tau_M$, $n = 0, 1, 2, ...$ The four-field Maximum Entropy phase density is then used as an initial phase density in order to solve the modified kinetic transport equation within each subsequent time interval $[t_n, t_{n+1}]$.

Due to this description there hold the following moment representations for each index $n \geq 0$ and for all $\tau \in [0, \tau_M]$

$$\begin{aligned} e(t_n + \tau, \mathbf{x}) &= c \int\limits_{\partial B(1,\mathbf{0})} \Psi_n(\mathbf{x} - c\gamma(\tau)\mathbf{n}, \mathbf{n})\, dS(\mathbf{n}) \\ Q_i(t_n + \tau, \mathbf{x}) &= c\,\dot{\gamma}(\tau) \int\limits_{\partial B(1,\mathbf{0})} n_i\, \Psi_n(\mathbf{x} - c\gamma(\tau)\mathbf{n}, \mathbf{n})\, dS(\mathbf{n})\,. \end{aligned} \tag{6.3.1}$$

Here the phase-density Ψ_n at the maximization time t_n reads

$$\Psi_n(\mathbf{y}, \mathbf{n}) = \frac{3}{4\pi} \frac{e(4-F)^3}{F\left(1 - F\,\frac{\mathbf{n}\cdot\mathbf{Q}}{4ce}\right)^4}\,, \quad F = \frac{6}{1 + \sqrt{1 - \frac{3}{4}\left(\frac{|\mathbf{Q}|}{ce}\right)^2}}\,, \tag{6.3.2}$$

where we have set for abbreviation $e = e(t_n, \mathbf{y})$, $Q_i = Q_i(t_n, \mathbf{y})$ and where $\gamma(\tau)$ is defined in (6.2.2).

In the following we are especially interested in one-dimensional solutions, which do not depend on x_2 and x_3. In this case we can also use the integration variables given in (4.6.1) for the one-dimensional evaluation of the surface integrals for the ultra-relativistic Euler equations. Then the moments reduce to one-dimensional integrals over the interval $[-1, 1]$ with

$$\begin{aligned} e(t_n + \tau, x) &= \frac{3}{2} \int_{-1}^{+1} \frac{e\,(4-F)^3}{F\,(1 - \frac{F}{4}\frac{Q\xi}{ce})^4}(t_n, x - c\,\gamma(\tau)\,\xi)\; d\,\xi\;, \\ Q(t_n + \tau, x) &= \frac{3}{2}\, c\,\dot{\gamma}(\tau) \int_{-1}^{+1} \frac{e\,(4-F)^3\,\xi}{F\,(1 - \frac{F}{4}\frac{Q\xi}{ce})^4}(t_n, x - c\,\gamma(\tau)\,\xi)\; d\,\xi\,. \end{aligned} \tag{6.3.3}$$

6.3.2 Riemannian initial data and shock condition

We consider now a single shock front which propagates into a region of thermal equilibrium. The Rankine-Hugoniot conditions for this case were already solved by Dreyer & Seelecke in [15] and will be sufficient for our purpose. However, we will prove a correspondence between the four-field system and the ultra-relativistic Euler equations which allows us to solve the complete Riemann problem of the four-field system in the limit $\tau_R \to \infty$.

In this section we will compare our numerical results with their analytical predictions from the Rankine-Hugoniot conditions for a large relaxation time τ_R. But we will also study the influence of a small relaxation time τ_R, which leads to a fast relaxation of the heat flux, and especially the smoothing of a single shock front under the influence of this fast relaxation.

We restrict ourselves to the one-dimensional case and recall the weak formulation of the four-field system in (6.1.1) with a convex region Ω in space and time:

$$\begin{aligned} &\int\limits_{\partial\Omega} (e\,dx \;-\; Q\,dt) = 0 \;, \\ &\int\limits_{\partial\Omega} (Q\,dx \;-\; c^2\,e\chi\,dt) = -\tfrac{1}{\tau_R} \iint\limits_{\Omega} Q \;dt\,dx \;, \\ &\chi \;=\; \tfrac{5}{3} - \tfrac{4}{3}\sqrt{1 - \tfrac{3}{4}\left(\tfrac{Q}{ce}\right)^2} \;. \end{aligned} \tag{6.3.4}$$

We prescribe Riemannian initial data

$$e_0(x) = \begin{cases} e_- & , \quad x \le 0 \\ e_+ & , \quad x > 0 \end{cases} \;, \qquad Q_0(x) = \begin{cases} Q_- & , \quad x \le 0 \\ Q_+ & , \quad x > 0 \end{cases} \;. \tag{6.3.5}$$

In order to guarantee that only a single shock solution occurs, we introduce the shock-parameter $X = e_-/e_+ > 1$, which determines the strength of the shock. Then we choose the equilibrium state $e_+ > 0$, $Q_+ = 0$ to the right of the shock and calculate the state e_-, Q_- to the left of the shock according to the Dreyer-Seelecke condition

$$e_- = X e_+ \;, \quad Q_- = (X-1)e_+ \,\frac{c}{\sqrt{3}}\sqrt{\frac{3\sqrt{X}-1}{\sqrt{X}+1}} \;. \tag{6.3.6}$$

The shock speed $V_s > 0$ is also taken from [15] and reads

$$V_s = \frac{c}{\sqrt{3}}\sqrt{\frac{3\sqrt{X}-1}{\sqrt{X}+1}} \;. \tag{6.3.7}$$

The condition $X > 1$ and the representations (6.3.6) select a single shock solution which corresponds to a 3-shock in the parametrizations given for the ultra-relativistic Euler equations. These conditions also imply that always $V_s > c/\sqrt{3}$.

Sometimes it is useful to define another shock parameter via the shock speed. The new shock parameter gives the deviation from the Deby velocity c and is defined as

$$\alpha = \frac{V_s}{c} = \frac{1}{\sqrt{3}}\sqrt{\frac{3\sqrt{X}-1}{\sqrt{X}+1}} \;. \tag{6.3.8}$$

Note that V_s is restricted to the range $c/\sqrt{3} < V_s < c$. Using α, with $1/\sqrt{3} < \alpha < 1$, instead of X, the Dreyer-Seelecke condition reads

$$e_- = \frac{1}{9}\, e_+ \, \frac{(3\alpha^2+1)^2}{(1-\alpha)^2} \;, \quad Q_- = \frac{8}{9}\, c e_+ \alpha \, \frac{3\alpha^2-1}{1-\alpha^2} \;. \tag{6.3.9}$$

In the following we put $c = 1$ for simplicity.

Figure 6.1 shows for Riemannian initial data of type (6.3.5) three initial value problems for the two fields energy density e and heat flux Q. The space region is $-0.5 \leq x \leq 0.5$ and the time is restricted to $0 \leq t \leq 0.5$. The first row displays a single shock solution resulting from the initial data $e_+ = 1$, $Q_+ = 0$, $X = 2$ for large relaxation time $\tau_R = 8$. The light and dark colours correspond to small and large values of the fields, respectively, ranging from $e_{min} = 1$, $Q_{min} = 0$ (light colour) to $e_{max} = 2$, $Q_{max} = 0.67$ (dark colour).

The second row displays the same initial value problem but for small relaxation time $\tau_R = 0.2$. This value corresponds to a dominant right-hand side and causes a strong diffusion of the original shock front. The extreme values of e and Q are the same as before.

The third row displays the development of initial conditions that violate the Dreyer-Seelecke shock condition $X > 1$. Here the initial data result from $e_+ = 1$, $Q_+ = 0$, $X = 0.3$ for a large relaxation time $\tau_R = 8$. The extreme values of the fields range from $e_{min} = 0.3$, $Q_{min} = -0.26$ (light colour) to $e_{max} = 1$, $Q_{max} = 0$ (dark colour).

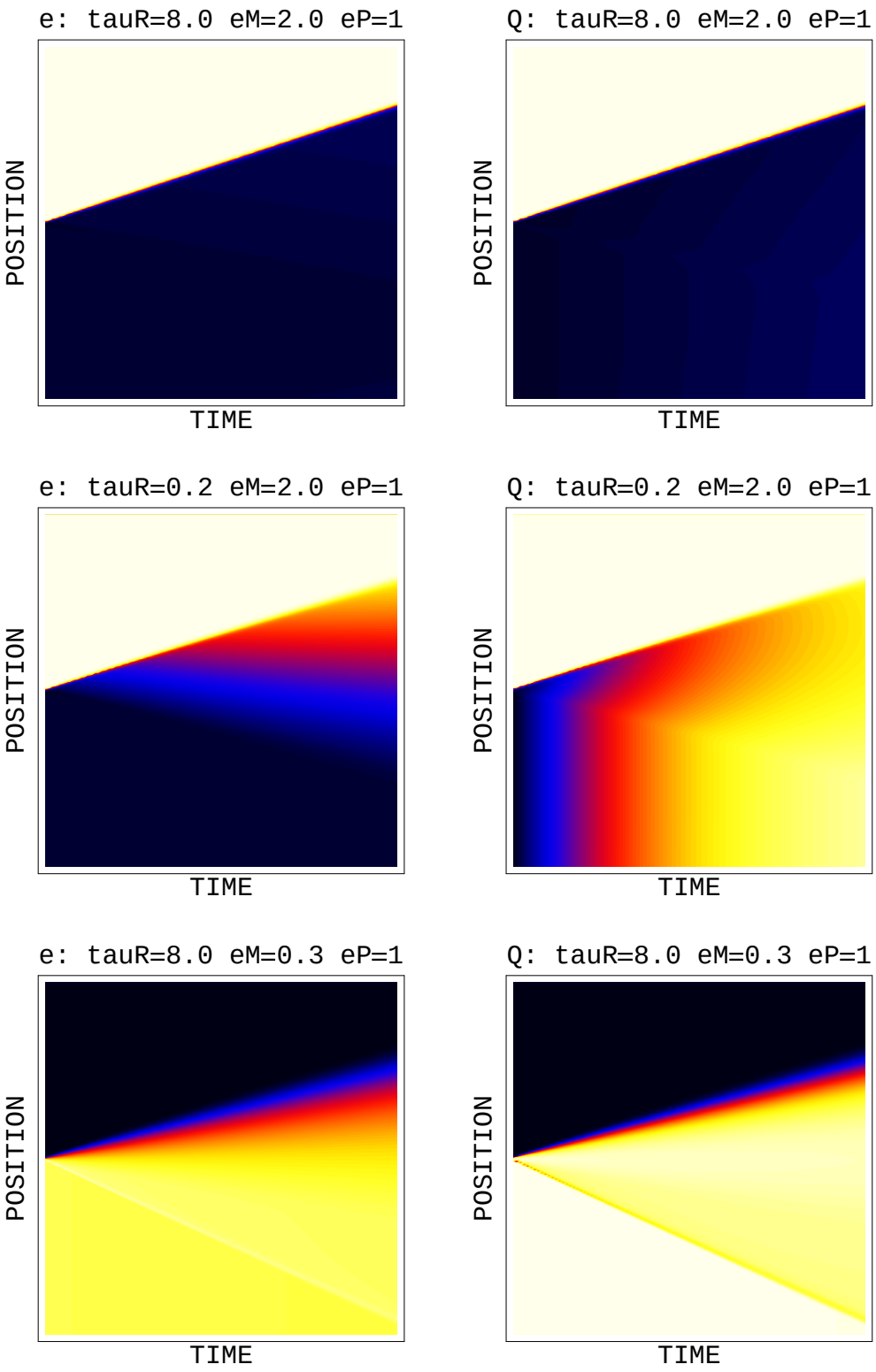

Figure 6.1: Fields of energy density and heat flux
for various inital conditions and relaxation times

6.4 The initial-boundary value problem (IBVP)

6.4.1 Kinetic solution of the IBVP

Boundary value problems that are solved by integral representations of an underlying kinetic model confront us with a serious problem. For a discussion we consider a half space problem of a one-dimensional crystal with a boundary at $x = 0$. We will solve the inital-boundary value problem with prescribed boundary data for the energy density e by a suitable extension of the kinetic scheme developed for the initial value problem in Section 6.3.

Our objective is the calculation of the fields of energy density and heat flux at location $\bar{x}$ and at time $0 < \bar{t} \leq \tau_M$. To this end we rely on an extension of the representations (6.3.3). For every value of the integration variable $\xi \in [-1, 1]$ there is a micro trajectory

$$x(t) = \bar{x} - c\xi \left(\gamma(\bar{t}) - \gamma(t)\right) \tag{6.4.1}$$

through the point $(\bar{t}, \bar{x})$ which starts for $\xi < \xi_0$ from the initial line $t = 0$ and for $\xi > \xi_0$ from the boundary $x = 0$. The critical value

$$\xi_0 = \frac{\bar{x}}{c\gamma(\bar{t})} > 0 \tag{6.4.2}$$

corresponds to the micro trajectory that originates at the point $(0, 0)$.

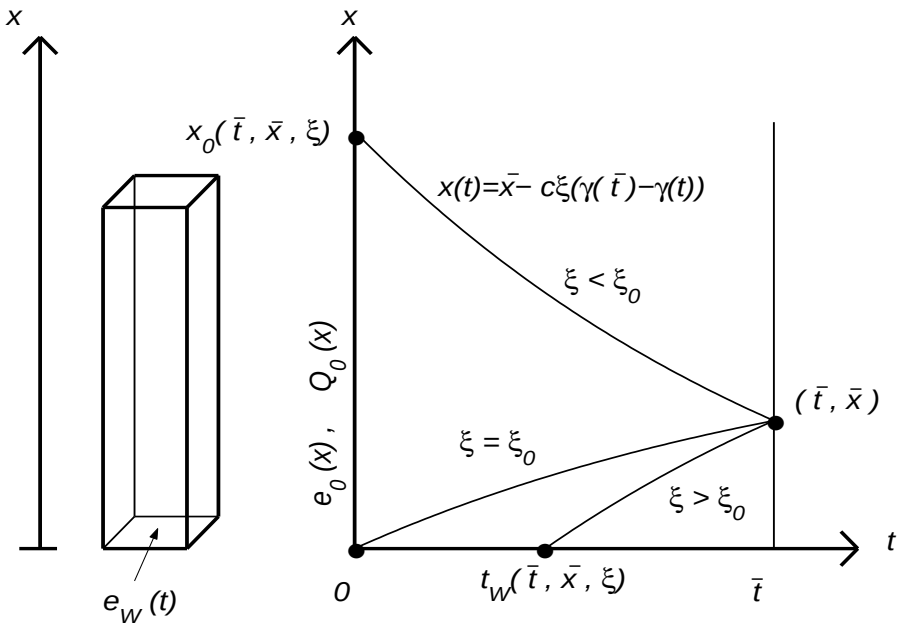

Figure 6.2: Micro trajectories relating ($\bar{t}$,$\bar{x}$) to the initial and boundary line

Figure 6.2 illustrates three selected micro trajectories and the functions

$$x_0(\bar{t},\bar{x},\xi) = \bar{x} - c\gamma(\bar{t})\xi, \qquad t_W(\bar{t},\bar{x},\xi) = -\tau_R \ln\left[\exp\left(-\frac{\bar{t}}{\tau_R}\right) + \frac{\bar{x}}{c\,\tau_R\xi}\right], \tag{6.4.3}$$

which denote the intersections of the micro trajectories with the initial axes $t = 0$ and with the boundary $x = 0$, respectively. These play an important role for the following representation formulas. Note that micro trajectories are given by a purely formal definition and that there are neither physical particles nor phonons corresponding to them.

We introduce the abbreviation

$$U(e,Q,\xi) = \frac{3}{2}\frac{e\,(4-F)^3}{F\left(1-\frac{F}{4}\frac{Q}{ce}\xi\right)^4} \qquad \text{with} \qquad F = \frac{6}{1+\sqrt{1-\frac{3}{4}\left(\frac{Q}{ce}\right)^2}}, \tag{6.4.4}$$

and form the representation formula of the initial-boundary value problem (in the following abbreviated as IBVP) by means of two auxiliary functions $e_H(t)$ and $Q_H(t)$ which will be determined later on. To calculate the fields $e(\bar{t},\bar{x})$ and $Q(\bar{t},\bar{x})$ we write

$$\begin{aligned} e(\bar{t},\bar{x}) \;&=\; \int_{-1}^{\xi_0} U(e_0(x_0(\bar{t},\bar{x},\xi)),Q_0(x_0(\bar{t},\bar{x},\xi)),\xi)\,d\xi \\ &+\; \int_{\xi_0}^{1} U(e_H(t_W(\bar{t},\bar{x},\xi)),Q_H(t_W(\bar{t},\bar{x},\xi)),\xi)\,d\xi, \end{aligned} \tag{6.4.5}$$

$$\begin{aligned} Q(\bar{t},\bar{x}) = \;& c\,\exp\left(-\tfrac{\bar{t}}{\tau_R}\right)\left[\int_{-1}^{\xi_0} U(e_0(x_0(\bar{t},\bar{x},\xi)),Q_0(x_0(\bar{t},\bar{x},\xi)),\xi)\,\xi\,d\xi\right. \\ &\left. +\int_{\xi_0}^{1} U(e_H(t_W(\bar{t},\bar{x},\xi)),Q_H(t_W(\bar{t},\bar{x},\xi)),\xi)\,\xi\,d\xi\right]. \end{aligned} \tag{6.4.6}$$

The initial and boundary data are denoted by

$$e(0,x) = e_0(x), \qquad Q(0,x) = Q_0(x), \qquad e(t,0) = e_W(t), \qquad Q(t,0) = Q_W(t). \tag{6.4.7}$$

Note that either $(6.7)_3$ or $(6.7)_4$ can be prescribed as independent boundary data, but not both. This restriction is also very well known from the reflection problem in the linear limit, which will be studied in detail in Section 6.4.3. Here we will discuss the dependencies of the boundary values for e and Q on the corresponding kinetic level.

The solution of the IBVP, i.e. the representations (6.4.5) and (6.4.6), must satisfy the following two continuity conditions at the boundary

$$\lim_{\bar{x}\to 0} e(\bar{t},\bar{x}) = e_W(\bar{t}), \qquad \lim_{\bar{x}\to 0} Q(\bar{t},\bar{x}) = Q_W(\bar{t}). \tag{6.4.8}$$

These read explicitly at any time t

$$e_W(t) = \int_{-1}^{0} U(e_0(x_0(t,0,\xi)), Q_0(x_0(t,0,\xi)), \xi)\, d\xi + \int_{0}^{1} U(e_H(t), Q_H(t), \xi)\, d\xi, \tag{6.4.9}$$

$$\begin{aligned} Q_W(t) = \;\; c\, \exp\left(-\tfrac{t}{\tau_R}\right) \;\; \Bigg[& \int_{-1}^{0} U(e_0(x_0(t,0,\xi)), Q_0(x_0(t,0,\xi)), \xi)\, \xi\, d\xi \\ & + \int_{0}^{1} U(e_H(t), Q_H(t), \xi)\, \xi\, d\xi \Bigg] \; . \end{aligned} \tag{6.4.10}$$

Here the auxiliary functions e_H and Q_H do no longer depend on the integration variable ξ. Thus the integrals that contain e_H and Q_H can be carried out and the continuity conditions (6.4.9) and (6.4.10) turn out to be algebraic equations for e_H and Q_H.

We introduce the abbreviations

$$a = \frac{F_H}{4}\frac{Q_H}{c\, e_H}, \qquad f(a) = \frac{1}{2}\,\frac{a^2-3a+3}{a^2+3}(1+a)^3, \qquad g(a) = \frac{1}{4}\,\frac{3-a}{a^2+3}(1+a)^3, \tag{6.4.11}$$

and obtain from (6.4.9) and (6.4.10)

$$e_W(t) \;=\; e_H(t) f(a(t)) \;+\; \int_{-1}^{0} U(e_0(x_0(t,0,\xi)), Q_0(x_0(t,0,\xi)), \xi)\, d\xi, \tag{6.4.12}$$

$$Q_W(t)\frac{\exp\left(\frac{t}{\tau_R}\right)}{c} \;=\; e_H(t) g(a(t)) + \int_{-1}^{0} U(e_0(x_0(t,0,\xi)), Q_0(x_0(t,0,\xi)), \xi)\, \xi\, d\xi. \tag{6.4.13}$$

We conclude that the auxiliary functions $e_H(t)$ and $Q_H(t)$ must satisfy two non-linear algebraic equations at the boundary!

Now we **assume** for the moment that $e_W(t)$ and $Q_W(t)$ are prescribed independently of each other. Under this assumption we consider two cases in order to demonstrate that they lead to a contradiction. The reason for this

failure comes from the fact that a consistent kinetic approximation to the solution of the hyperbolic system in the limit $\tau_M \to 0$ must also take care of the fact that the boundary data for e and Q cannot be prescribed independently.

In the first case we additionally assume that we could choose the auxiliary functions such that they coincide with the boundary data, i.e. $e_H = e_W$ and $Q_H = Q_W$. Then it follows that the algebraic equations (6.4.12) and (6.4.13) are not satisfied in general, i.e.

$$\lim_{\bar{x}\to 0} e(\bar{t},\bar{x}) \neq e_W(\bar{t}), \qquad \lim_{\bar{x}\to 0} Q(\bar{t},\bar{x}) \neq Q_W(\bar{t}). \tag{6.4.14}$$

In the second case we assume that e_H and Q_H follow from the algebraic system (6.4.12) and (6.4.13) for given e_W and Q_W. Here another contradiction appears because the quantity a is restricted to the range $[-1,+1]$ according to its definition (6.4.11). However, the corresponding solution of the necessary continuity conditions (6.4.12) and (6.4.13) leads even for very simple examples with constant initial- and boundary data to values of a out of that range. In these cases we cannot prescribe $e_W(t)$ and $Q_W(t)$ independently.

To proceed the discussion we consider now exclusively the case that $e_W(t)$ but not $Q_W(t)$ is prescribed. Consequently, the function $Q_W(t)$ must also be calculated. We need one further condition that allows the determination of the auxiliary functions $e_H(t)$ and $Q_H(t)$ and of $Q_W(t)$. In addition to the two algebraic conditions (6.4.12) and (6.4.13) we found that it is necessary to require a third continuity condition for $0 < t \leq \tau_M$, namely

$$e_H(t) = e_W(t) \ , \tag{6.4.15}$$

which guarantees that the continuity conditions (6.4.8) are also satisfied in the limit $\tau_M \to 0$. For the evaluation we use Newtons method in order to solve the resulting equation for $a(t)$, which is a combination of (6.4.12) and the third continuity condition (6.4.15)

$$1 - f(a(t)) = \frac{1}{e_W(t)} \int_{-1}^{0} U(e_0(x_0(t,0,\xi)), Q_0(x_0(t,0,\xi)), \xi)\, d\xi \ . \tag{6.4.16}$$

Since $f(a)$ is monotonically increasing from $f(-1) = 0$ to $f(1) = 1$, a unique solution $a = a(t)$ of (6.4.16) exists whenever the right-hand side of (6.4.16) is in the range $[0,1]$, which is the case for all interesting examples of practical importance considered here.

We may now determine the auxiliary field $Q_H(t)$ according to the definition of $a = a(t)$ by

$$Q_H(t) = \frac{4c\, a(t)}{a(t)^2 + 3}\, e_H(t) \ . \tag{6.4.17}$$

Finally we obtain the dependent boundary data $Q_W(t)$ from (6.4.13).

The next two examples exhibit a surprising consequence of condition (6.4.15): Immediately after the initial time and after a sufficient number of maximizations of entropy were carried out, the boundary values e_W and Q_W are related to the initial data according to the Rankine-Hugoniot conditions!

6.4.2 Two explicit examples for IBVPs

The following numerical results serve to illustrate this observation and additionally record three nonlinear phenomena: a) the formation and steepening of shock fronts,
b) the speed of shock fronts is apparently larger then $c/\sqrt{3}$,
c) the broadening of initial heat pulses at later times.

Figures 6.3 and 6.4 display the propagation of the heat pulse

$$e_W(t) = \begin{cases} 1 & , \quad t \leq 0 \\ 3 & , \quad 0 < t \leq 0.5 \\ 1 & , \quad t > 0.5 \end{cases} , \tag{6.4.18}$$

created at the lower boundary. The initial data are $e_0 = 1$ and $Q_0 = 0$.

In Figure 6.3 we consider the undamped case. The first row of Figure 6.3 shows the boundary data. Note that only $e_W(t)$ is prescribed but $Q_W(t)$ is calculated according to (6.4.15)-(6.4.17). The second and third row show the solution at times $t = 0.5$ and $t = 1.5$, respectively, for $0 \leq x \leq 1.5$.

We observe that the pulse front remains a shock moving with the speed $0.72\,c$, which is confirmed by the Dreyer-Seelecke condition (6.3.6). The rear side of the pulse changes into a rarefaction wave. Thereby it comes to a broadening, even if there is no damping.

Figure 6.4 illustrates the effect of large damping due to the relaxation time $\tau_R = 0.5$. In contrast to the undamped case, the heat flux may become negative here. The last row of Figure 6.4 shows the fields e and Q at time $t = 1.5$ exhibiting a large broadening of the rear side of the initial pulse. Note that this phenomenon cannot be observed in the undamped case, although a rarefaction wave appears at the rear side of the pulse here. Furthermore the solution decays rapidly to an equilibrium state.

In the last example which is represented in Figure 6.5 we create the periodic heat signal

$$e_W(t) = 2 - \cos(8\pi\, t) \tag{6.4.19}$$

at the lower boundary. The initial data are again $e_0 = 1$ and $Q_0 = 0$. The left and right columns show the effect of zero damping ($\tau_R \to \infty$) and high damping ($\tau_R = 0.5$), respectively. The first two rows of Figure 6.5 depict the boundary data. Note again that only $e_W(t)$ is prescribed but

$Q_W(t)$ is calculated according to (6.4.15)-(6.4.17). Surprisingly even in this example $Q_W(t)$ meets the value Q that we obtain by the Dreyer-Seelecke condition with $e_+(t) = e_0 = 1$, $Q_+(t) = Q_0 = 0$ and $e_-(t) = e_W(t)$, at least in the undamped case. The damped case requires a more detailed study. The last two rows illustrate the solution at time $t = 1.5$ for $0 \leq x \leq 1.5$. The formation and steepening of shock fronts is clearly visible. As before, we observe regions in space with a negative heat flux which is due to the damping.

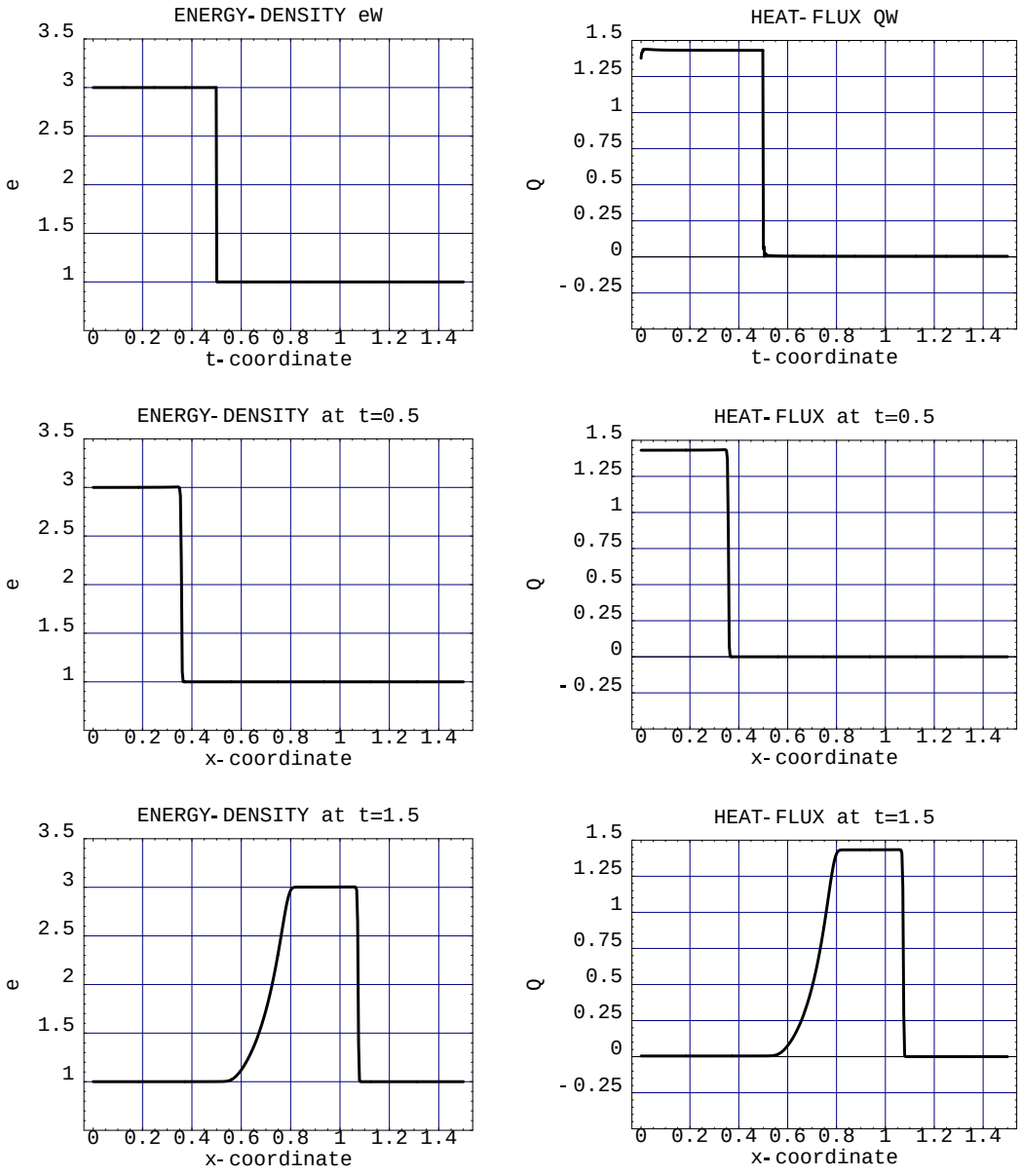

Figure 6.3: Creation of a heat pulse for $\tau_R \to \infty$. First row: boundary data for the energy density and the resulting heat flux, second and third row: energy density and heat flux at time $t = 0.5$ and at $t = 1.5$, respectively.

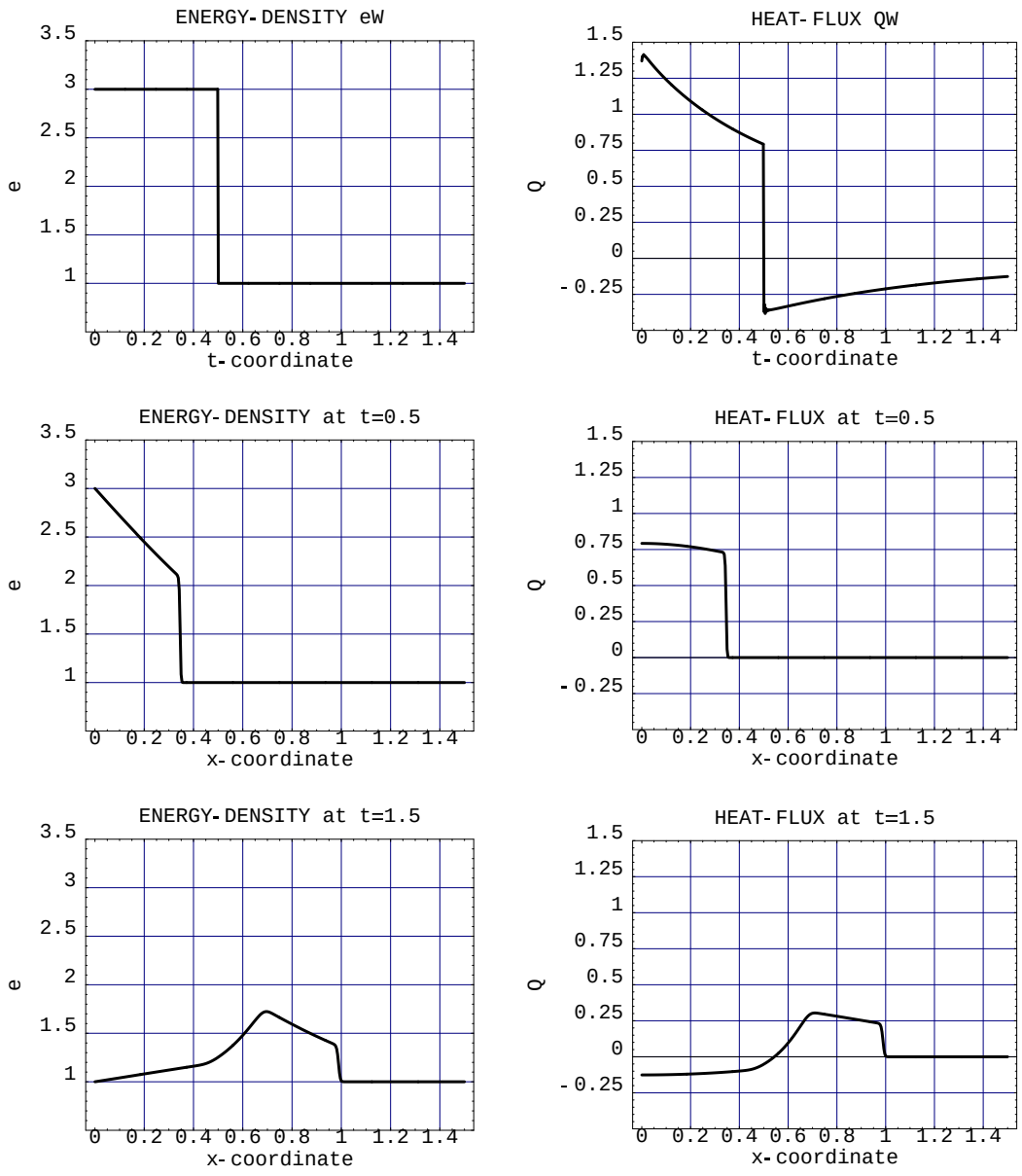

Figure 6.4: Creation of a heat pulse for $\tau_R = 0.5$. First row: boundary data for the energy density and the resulting heat flux, second and third row: energy density and heat flux at time $t = 0.5$ and at $t = 1.5$, respectively.

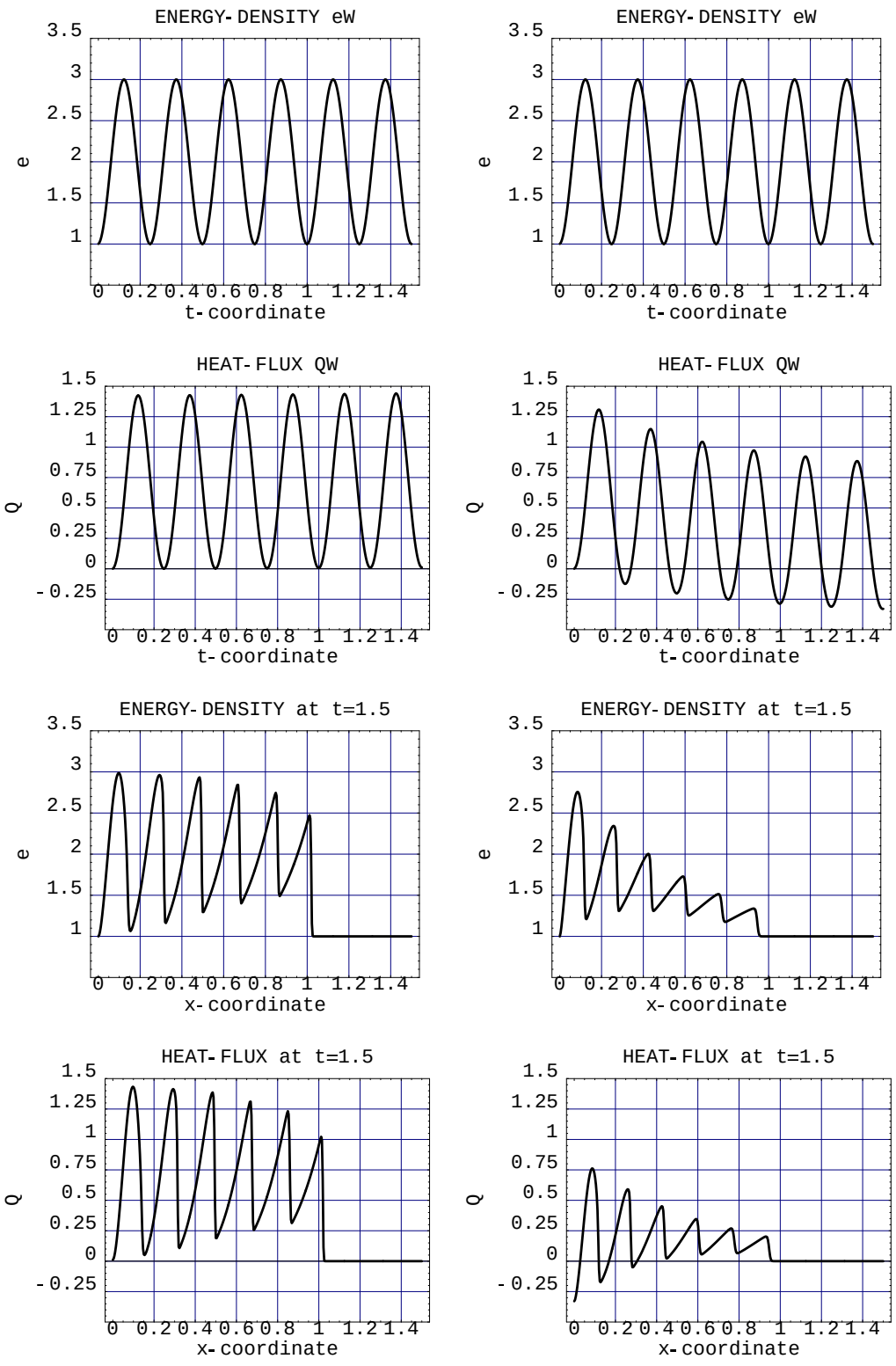

Figure 6.5: A periodic boundary condition. Left and right column: $\tau_R \to +\infty$ and $\tau_R = 0.5$, respectively.

6.4.3 The linear limit

In this section we choose initial- and boundary data so that the solution of the non-linear system (6.1.1) agrees approximately with the solution of the linear limit. The latter is obtained from the full system by neclegting terms of the order Q^2. For simplicity we set $c = \sqrt{3}$ and consider the case $\tau_R \to \infty$.

The initial data are

$$e_0(x) = \begin{cases} 1 & , \quad 0.0 < x \leq 0.4 \\ 1+\epsilon & , \quad 0.4 < x \leq 0.8 \\ 1 & , \quad x > 0.8 \end{cases} , \qquad Q_0(x) = \begin{cases} 0 & , \quad 0.0 < x \leq 0.4 \\ -\epsilon & , \quad 0.4 < x \leq 0.8 \\ 0 & , \quad x > 0.8 \end{cases} . \tag{6.4.20}$$

Here $\epsilon > 0$ is a fixed positive constant; in particular we choose $\epsilon = 0.01$. At the boundary we prescribe the energy density to be

$$e_W(t) = 1 \ . \tag{6.4.21}$$

The solution of this problem according to the representation formulas (6.4.5) and (6.4.6) can be read off from Figure 6.6 for $0 \leq t \leq 1$ and $0 \leq x \leq 1$.

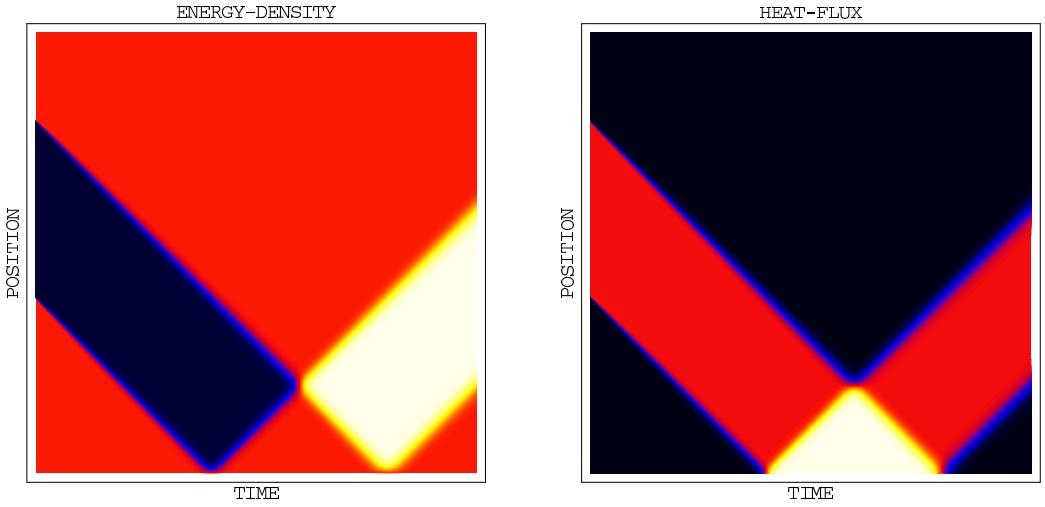

Figure 6.6: The nonlinear solution of the IBVP for $\varepsilon = 0.01$

The Figure reveals the well known behaviour of a linear wave equation because ϵ was chosen so that terms of order Q^2 show no influence. Furthermore we observe that the prescribed constant boundary data $e_W = 1$ causes a reflection of the incoming wave from the initial line.

In order to establish agreement with the linear theory we will compare the solution of the non-linear system (6.1.1) in Figure 6.6 with the solution of its

linearized form that we study now. The linearized version of (6.1.1) reads

$$\int_{\partial\Omega} (e\,dx \;-\; Q\,dt) = 0\ , \qquad \int_{\partial\Omega} (Q\,dx \;-\; e\,dt) = 0\ . \tag{6.4.22}$$

(6.4.22) leads to the following system of wave equations:

$$\frac{\partial e}{\partial t} \;+\; \frac{\partial Q}{\partial x} = 0\ , \qquad \frac{\partial Q}{\partial t} \;+\; \frac{\partial e}{\partial x} = 0\ . \tag{6.4.23}$$

Across a shock with velocity V_s we obtain from (6.4.22) the jump conditions

$$V_s\,(e_+ - e_-) = Q_+ - Q_-\ , \qquad V_s\,(Q_+ - Q_-) = e_+ - e_-\ , \tag{6.4.24}$$

These equations immediately imply $V_s = +1$ or $V_s = -1$, which is a well known result. In the following we use the jump conditions (6.4.24) in order to construct the analytical solution of the IBVP from above.

Figure 6.7 shows the piecewise constant analytical solution of the linear problem. The various regions with constant states (e, Q) are bounded by jumps with slopes $+1$ and -1 or by the t- and x-axis, respectively.

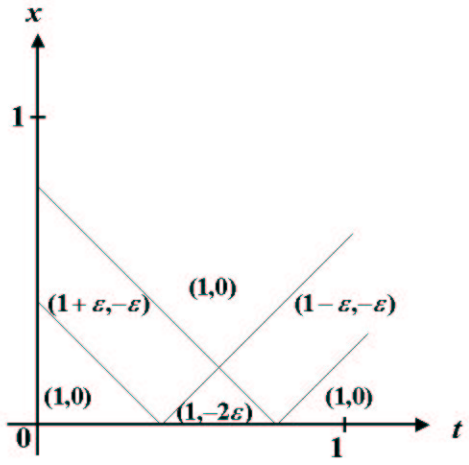

Figure 6.7: The analytical linear solution of the IBVP for $\varepsilon = 0.01$

Due to the small variation of the initial data there is good agreement between the nonlinear numerical solution of Figure 6.6 and the linear analytical solution of Figure 6.7. Note that the light colors in Figure 6.6 correspond to small values of the fields while large values are indicated by dark colors.

6.4.4 The stationary boundary value problem

Finally we study the analytical solution of the stationary boundary value problem. As before we consider the one-dimensional case. We reduce the local three-dimensional system (5.2.6)-(5.2.9) to the stationary one-dimensional case and obtain

$$\frac{d}{dx}Q = 0 \ , \qquad \frac{d}{dx}(c^2\,N) = -\frac{1}{\tau_R}\,Q \ , \qquad N = \frac{5}{3}e - \frac{4\,e}{3}\sqrt{1-\frac{3}{4}\left(\frac{Q}{c\,e}\right)^2} \ , \tag{6.4.25}$$

where $e = e(x)$, $Q = Q(x)$ for $0 \le x \le L$. We prescribe values for e at the upper and lower boundary, viz.

$$e_- = e(0) \ , \quad e_+ = e(L). \tag{6.4.26}$$

The constant Debye velocity is again $c = 1$. $(6.4.25)_1$ implies that Q =constant, whereas $(6.4.25)_2$ leads to the algebraic equation

$$\frac{5}{3}\,e - \frac{4}{3}\sqrt{e^2 - \frac{3}{4}Q^2} = \gamma - \frac{x}{\tau_R}\,Q. \tag{6.4.27}$$

The boundary conditions at $x = 0$ and at $x = L$ are used for the determination of the integration constants γ and Q

$$\frac{5}{3}\,e_- - \frac{4}{3}\sqrt{e_-^2 - \frac{3}{4}Q^2} = \gamma, \qquad \frac{5}{3}\,e_+ - \frac{4}{3}\sqrt{e_+^2 - \frac{3}{4}Q^2} = \gamma - \frac{L}{\tau_R}\,Q. \tag{6.4.28}$$

Subtracting these equations leads to a single equation for Q, namely

$$\frac{5}{3}\,(e_- - e_+) - \frac{4}{3}\left(\sqrt{e_-^2 - \frac{3}{4}Q^2} - \sqrt{e_+^2 - \frac{3}{4}Q^2}\right) - \frac{LQ}{\tau_R} \ = \ 0 \ , \tag{6.4.29}$$

which may be solved by Newtons method.

The other constant γ results then immediately from $(6.4.28)_1$. With known values for Q and γ we may solve the equation for the energy density e in (6.4.27). It turns out that the solution only admits the "+" sign and reads

$$e(x) \ = \ -\frac{5}{3}\left(\frac{Qx}{\tau_R} - \gamma\right) \ + \ \sqrt{\frac{16}{9}\left(\frac{Qx}{\tau_R} - \gamma\right)^2 - \frac{4}{3}Q^2}. \tag{6.4.30}$$

The same representation can be used in order to solve the mixed boundary value problem for given e_- and $Q = Q_+$. In this case γ can be read off from the equation $(6.4.28)_1$.

6.5 The correspondence with the ultra-relativistic Euler equations

Now we are able to explain a correspondence with the $(p, \mathbf{u})$-subsystem of the ultra-relativistic Euler equations for both, for the hyperbolic systems as well as for the kinetic schemes in order to solve them.

We first compare the conservation laws of momentum- and energy for the ultra-relativistic Euler equations given by (4.4.5),(4.4.6) with the hyperbolic four-field system in (5.2.6)-(5.2.9). It is sufficient to assume that the field equations are written down in dimensionless form with $c = 1$ for the velocity of light as well as for the Debye velocity. The relaxation term in the momentum balance (5.2.7) of the four-field system has no counterpart in the $(p, \mathbf{u})$-subsystem of the ultra-relativistic Euler equations. Thus we will only consider the limit $\tau_R \to \infty$ without production term on the right-hand side. The natural domains Ω_{rel} and Ω_{phon} for the $(p, \mathbf{u})$- and the $(e, \mathbf{Q})$ state space are given by

$$\begin{aligned} \Omega_{rel} &= \{\, (p, \mathbf{u}) \in \mathbb{R} \times \mathbb{R}^3 \mid p > 0 \} \,, \\ \Omega_{phon} &= \{\, (e, \mathbf{Q}) \in \mathbb{R} \times \mathbb{R}^3 \mid |\mathbf{Q}| < \mathbf{e} \} \,, \end{aligned} \tag{6.5.1}$$

respectively. Then we compare the $(p, \mathbf{u})$-subsystem with the four-field system and make the following ansatz for a transformation between the $(p, \mathbf{u})$- and the $(e, \mathbf{Q})$-state space:

$$e = p(3 + 4\mathbf{u}^2) \,, \quad \mathbf{Q} = 4p\mathbf{u}\sqrt{1 + \mathbf{u}^2} \,. \tag{6.5.2}$$

The inverse transformation is given by

$$p = \frac{1}{3}\left[\sqrt{4e^2 - 3\mathbf{Q}^2} - e\right] \,, \quad \mathbf{u} = \frac{\mathbf{Q}}{\sqrt{4p(p+e)}} \,. \tag{6.5.3}$$

These transformations were already used in (4.5.6) and (4.5.7) in order to initialize the kinetic scheme of the ultra-relativistic Euler equations for the next time step. We conclude that the mapping $\Gamma : \Omega_{rel} \to \Omega_{phon}$ with

$$\Gamma(p, \mathbf{u}) = \begin{pmatrix} p(3 + 4\mathbf{u}^2) \\ 4p\mathbf{u}\sqrt{1 + \mathbf{u}^2} \end{pmatrix} \tag{6.5.4}$$

is one-to-one. We can also use (6.5.3) and replace e and $\mathbf{Q}$ by p and $\mathbf{u}$ in the flux N_{ij} of the four-field system in (5.2.8). There results with $c = 1$ for all $i, j = 1, 2, 3$ the momentum-flux of the ultra-relativistic Euler equations

$$N_{ij} = p\delta_{ij} + 4pu_iu_j \,. \tag{6.5.5}$$

Note that $\mathbf{u} = (u^1, u^2, u^3)^T = -(u_1, u_2, u_3)^T$, and therefore $N_{ij} = N^{ij}$.

However, there is a deeper reasoning for this analogy. This will be explained next and will also give a relationship between the kinetic schemes for the relativistic Euler equations and for the phonon-Bose gas.

First we note that the kinetic schemes in both theories have the same collisionless kinetic transport equations in the free-flight phases. Next we consider the entropies defined for the reduced phase densities $\Psi_{rel}(\mathbf{w})$ and $\Psi_{phon}(\mathbf{n})$, which are also of the same mathematical form in both theories, namely the zero-component

$$\tilde{S}^0[\Psi_{rel}] = \oint\limits_{\partial B(1,\mathbf{0})} \Psi_{rel}^{3/4}(\mathbf{w})\, dS(\mathbf{w})$$

of the reduced entropy four-vector of the relativistic $(p, \mathbf{u})$-subsystem and

$$\tilde{h}(\Psi_{phon}) = \mu \oint\limits_{\partial B(1,\mathbf{0})} \Psi_{phon}^{3/4}(\mathbf{n})\, dS(\mathbf{n})\,, \quad \mu = \frac{4\pi}{3}\left(\frac{1}{40\pi^3}\right)^{\frac{1}{4}}$$

for the reduced entropy density of the phonon-Bose gas, where $\mathbf{w}, \mathbf{n} \in \partial B(1, \mathbf{0})$ denotes the unit vector field in both cases. In order to maximize the entropy with four constraints, we also prescribe integral moments of the same mathematical form in both theories, namely the integral moments $T^{0\mu}$ given by (4.2.36) in the ultra-relativistic theory and the integral moments for e and $\mathbf{Q}$ given by (5.3.4) in the theory of the phonon-Bose gas, both with respect to the unit sphere $\partial B(1, \mathbf{0})$. But due to Proposition 4.12 the Maximum Entropy problem for the given four constraints leads to a unique Maximum Entropy phase density. We conclude that the reduced four-field phase densities which maximize the entropy are the same in both theories up to a transformation of the state spaces. This transformation is given by (6.5.2). For the kinetic schemes we obtain

- If $\tau_R \to \infty$ and $\tau_N = \tau_M$, then the kinetic scheme for the Boltzmann-Peierls equation given by (5.3.14), (5.3.15) in Section 5.3 reduces to the scheme for the $(p, \mathbf{u})$-subsystem of the ultra-relativistic Euler equations given in Section 4.8 in view of the transformation (6.5.2).
- If $\tau_R \to \infty$ and $\tau_N = \tau_M$, then the kinetic scheme for the hyperbolic four-field system given in Section 6.3.1 reduces to the scheme for the $(p, \mathbf{u})$-subsystem of the ultra-relativistic Euler equations given in Section 4.8 in view of the transformation (6.5.2).
- The analogous results follows for the spatial one-dimensional versions of the kinetic schemes described above, since in both theories the same integral substitution (4.6.1) could be used in the one-dimensional case.

However, these are only purely mathematical analogies between two totally different physical applications.

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Thesen zur Habilitationsschrift

Ausgewählte Anfangs-Randwertprobleme für hyperbolische Systeme und kinetische Gleichungen

1. Die vorliegende Arbeit schlägt in den Kapiteln 2 und 3 eine Brücke von den mathematischen Grundlagen zur Behandlung der Wellengleichungen zu den Maxwellschen Gleichungen inklusive wichtiger Anwendungsbeispiele. Die Maxwellschen Gleichungen werden sowohl klassisch als auch im Rahmen der speziellen Relativitätstheorie mathematisch beleuchtet, wobei der mitentwickelte Tensorkalkül auch die Grundlage für die Behandlung der relativistischen Eulergleichungen im folgenden Kapitel 4 bildet. Obwohl die wesentlichen Einzelresultate der Kapitel 2 und 3 schon in der mathematischen bzw. physikalischen Literatur behandelt worden sind, hoffen wir doch, mit dieser bündigen und selbsterklärenden Einführung der Elektrodynamik von der mathematischen Analysis bis hin zu konkreten physikalischen Anwendungen der Maxwellschen Gleichungen eine wichtige Lücke geschlossen zu haben.

2. Während der kinetische Zugang zu den klassischen Eulergleichungen vom analytischen und numerischen Standpunkt aus in der bestehenden Literatur umfangreich untersucht worden ist, stellt die kinetische Behandlung der relativistischen Eulergleichungen noch weitgehendes Neuland dar. Im Kapitel 4 dieser Arbeit wird hierzu ein wesentlicher Beitrag geliefert. Die hier dargestellten Eulergleichungen erlauben zwei Grenzfälle, den Übergang zu den klassischen Eulergleichungen wie auch den Übergang zu den sogenannten ultra-relativistischen Eulergleichungen. Die letzteren nehmen eine besonders einfache mathematische Gestalt an, da die Gleichungen für Impuls und Energie ein eigenes Subsystem bilden, was sich von der Kontinuitätsgleichung abkoppelt. In einer Raumdimension lösen wir für die ultra-relativistischen Eulergleichungen das Riemannsche Anfangswertproblem komplett und vergleichen es mit verschiedenen numerischen Verfahren. Die hier entwickelten Verfahren erfüllen die diskreten Formulierungen der physikalischen Erhaltungssätze und liefern insbesondere immer einen positiven Druck und eine positive Dichte, was auch in der Arbeit gezeigt wird.

3. In den Kapiteln 3 und 4 haben wir Lorentz-invariante Systeme der speziellen Relativitätstheorie mathematisch untersucht. In den Kapiteln 5 und 6 wird die sogenannte Boltzmann-Peierls Gleichung (BPG) studiert, eine kinetische Evolutionsgleichung, die den Wärmetransport in einem dielektrischen Kristall bei sehr tiefer Temperatur mit Hilfe eines Phonon-Bose Gases beschreibt, sowie ein aus der BPG abgeleitetes hyperbolisches Momentensystem.

Die kinetische Theorie der relativistischen Euler Gleichungen und die kinetische Boltzmann-Peierls Theorie eines Phonon-Bose Gases beschreiben zwar völlig verschiedene physikalische Vorgänge, zeigen aber trotzdem weitreichende mathematische Analogien. Diese Analogien werden in der vorliegenden Schrift schrittweise herausgearbeitet.

4. Sowohl das kinetische Verfahren für die ultra-relativistischen Euler Gleichungen als auch für das Phonon-Bose Gas erlauben die Möglichkeit, reduzierte Phasendichten, reduzierte Momentenintegrale und reduzierte Entropien zu finden. Die reduzierten Momentenintegrale sind Integrale über die kompakte Einheitssphäre und damit von wesentlich einfacherer mathematischer Gestalt als die ursprünglichen 3-fach Momentenintegrale, enthalten aber dieselbe thermodynamische Information und lassen sich ohne Verwendung von Approximationen rigoros herleiten. Hierzu werden die 3-fach Momentenintegrale der kinetischen Phasendichte in Polarkoordinaten umgeschrieben. Die mathematische Struktur dieser Integrale erlaubt sodann die explizite Integration des radialen Anteils. Dabei haben die reduzierten Größen für beide Anwendungen dieselbe Bauart und lassen sich insbesondere für eindimensionale Strömungen weiter sehr stark vereinfachen. Dieser Zugang kennt kein Analogon in der klassischen Theorie, d.h. weder die klassische Boltzmann-Gleichung noch die sogenannte kinetische BGK-Gleichung, eine vereinfachte Variante der Boltzmann-Gleichung, lassen sich in dieser oder ähnlicher Form vereinfachen, aber auch nicht die kinetischen Schemata der klassischen Euler Gleichungen.

5. In der relativistischen kinetischen Theorie sowie in der kinetischen Theorie der Boltzmann-Peierls Gleichung hat man es nur mit Signalgeschwindigkeiten zu tun, die jeweils durch die Geschwindigkeit des Lichtes bzw. die sogenannte Debye-Geschwindigkeit global begrenzt sind. Dies hat verschiedene Vorteile für die Analysis und Numerik der entsprechenden kinetischen Schemata, die in der vorliegenden Arbeit ebenfalls genutzt werden. Die unter den Thesen 4 und 5 genannten Punkte nennen signifikante Unterschiede zur klassischen Theorie, die ein detailiertes Studium der hier behandelten nichtklassischen Anwendungen rechtfertigen.

6. Die von Jüttner aus physikalischen Überlegungen gewonnene relativistische Verallgemeinerung der klassischen Maxwellverteilung für ein ideales Gas wird mathematisch mit Hilfe des Maximum Entropie Prinzips begründet. Der von uns gewählte Zugang zum Maximum Entropie Prinzip kommt ohne Lagrange-Multiplikatoren aus. Er zeigt nicht nur das Erfülltsein der Entropieungleichung, sondern auch die Eindeutigkeit der Phasendichte, die dem Maximum Entropie Prinzip mit den vorgegebenen Nebenbedingungen genügt.

Mit diesem Prinzip begründen wir auch die reduzierten Phasendichten für die ultra-relativistischen Eulergleichungen bzw. für das Phonon-Bose Gas. Wir zeigen, daß bis auf eine Transformation im Zustandsraum die reduzierten Phasendichten für das Subsystem der ultra-relativistischen Eulergleichungen bzw. für das hyperbolische 4-Feld Momentensystem der Boltzmann-Peierls Gleichung übereinstimmen. Die Grundlage hierfür ist eine vereinfachte Entropie der reduzierten Größen, die sich in beiden Anwendungen als äquivalent zu den ursprünglich gegebenen Entropien herausstellt.

7. In den Kapiteln 5 und 6 werden kinetische Verfahren für die reduzierte Boltzmann-Peierls Gleichung und das hyperbolische 4-Feld Momentensystem der Boltzmann-Peierls Gleichung entwickelt und mit verschiedenen analytisch bekannten Testfällen numerisch verglichen. Dabei zeigt sich eine sehr gute Übereinstimmung auch für die nichttrivialen Testfälle, die teilweise selbst einer umfangreichen Analysis der entsprechenden Erhaltungsgleichungen entspringen. Die Verfahren werden zunächst semidiskret bzgl. der Zeit in drei Raumdimensionen formuliert. Deren weitere Reduktion auf eindimensionale Strömungen führt auf einfache Momentenintegrale über das Intervall $[-1, 1]$ sowie auf ein im gesamten Phasenraum konsistent diskretisiertes kinetisches Verfahren.

8. Das oben erwähnte hyperbolische 4-Feld Momentensystem der Boltzmann-Peierls Gleichung weist einen Relaxationsterm in der Impulsbilanz auf. Im Kapitel 6 entwicklen wir in einer Raumdimension ein kinetisches Schema, das mit diesem Produktionsterm ein vorgegebenes Anfangs- Randwertproblem erfolgreich löst. Die Produktionsterme lassen sich durch eine geschickte Modifikation der reduzierten Freiflug-Phasendichte im kinetische Schema berücksichtigen. Die Behandlung der weniger einfachen Randbedingung erfolgt durch Einführung geeigneter Hilfsfelder in der modifizierten Freiflugphase des kinetischen Schemas. Die Auswertung von Stetigkeits- und Randbedingungen führt sodann auf interessante algebraische Gleichungen zur Bestimmung der Hilfsfelder. Randbedingungen für nichtlineare hyperbolische Erhaltungsgleichungen stellen uns vor schwierige mathematische Aufgaben, und bis jetzt ist die mathematische Analysis sowie die Entwicklung geeigneter numerischer Verfahren für Anfangs-Randwertprobleme nur in Spezialfällen möglich. Die vom Autor dieser Arbeit entwickelte kinetische Hilfsfeldmethode ist nun nicht nur für die oben genannte Anwendung gut geeignet, sondern konnte inzwischen auch von Dreyer, Herrmann et al. zur Lösung der klassischen Eulergleichungen mit den Randbedingungen für einen beschleunigten Kolben erfolgreich erweitert werden.

(Dr. rer. nat. Matthias Kunik)