Tightening the Chvátal and split operator via low-codimensional lineality spaces

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Zusammenfassung

Bemerkung: Für eine informellere Einführung in die zentralen Ideen der vorliegenden Arbeit, insbesondere der in Kapitel 4 und Kapitel 5 eingeführten, verweisen wir auf Kapitel 1. Außerdem möchten wir anmerken, dass Kapitel 16 eine deutlich ausführlichere englischsprachige Zusammenfassung dieser Arbeit bildet.

Teil I

Das zentrale Ziel von Teil I, welcher aus Kapitel 2 und Kapitel 3 besteht, ist es, grundlegende Definitionen einzuführen.

Kapitel 2 kann man als das „Mathematische-Grundlagen-Kapitel“ betrachten, in dem wir zahlreiche Definitionen und Resultate aus diversen mathematischen Gebieten, die für die vorliegende Arbeit erforderlich sind, einführen.

Kapitel 3 bildet eine Einführung in verbreitete Klassen von Schnittebenen, welche in der Literatur untersucht wurden. Für diese Zusammenfassung merken wir lediglich an (vgl. Definition 120), dass eine Schnittebene für $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$; hier stehe m für die Anzahl diskreter (ganzzahliger) Variablen und n für die Anzahl kontinuierlicher Variablen) eine lineare Ungleichung $c(\cdot) \leq c_0$ ($c \in (\mathbb{R}^m \times \mathbb{R}^n)^T$ und $c_0 \in \mathbb{R}$) für $P \cap (\mathbb{Z}^m \times \mathbb{R}^n)$ ist.

Teil II

Teil II besteht aus Kapitel 4, 5 und 6. Das Leitthema dieses Teils sind L_k-Schnitte und $L_{k-\frac{1}{2}}$-Schnitte. Diese beiden Klassen von Schnittebenen bilden das zentrale Thema der vorliegenden Arbeit.

Kapitel 4 bildet das zentrale Grundlagenkapitel über diese beiden Klassen von Schnittebenen und ihr Zusammenspiel. In ihm bauen wir das Theoriegebäude der L_k-Schnitte und $L_{k-\frac{1}{2}}$-Schnitte von Grunde auf. Die zentrale Idee hinter L_k-Schnitten und $L_{k-\frac{1}{2}}$-Schnitten ist es, das Problem des Findens von Schnittebenen zu relaxieren, indem wir statt linearen Ungleichungen für $P \cap (\mathbb{Z}^m \times \mathbb{R}^n)$ lineare Ungleichungen für

$$P \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + V)$$

bzw.

$$(P + V) \cap (\mathbb{Z}^m \times \mathbb{R}^n),$$

betrachten, wobei $V \leq \mathbb{R}^m \times \mathbb{R}^n$ ein Vektorraum der Kodimension $k \in \{0, \ldots, m+n\}$ sei (der Fall $k = 0$ ist aus formellen Gründen zugelassen). Die erste Konstruktion bezeichnen wir als L_k**-Schnitte** und die zweite als $L_{k-\frac{1}{2}}$**-Schnitte**. In Abhängigkeit von der Art der Erzeuger von V (Rationalitätsbedingungen) und Existenz kontinuierlicher Variablen führt dies auf diverse Klassen von L_k-Schnitten und $L_{k-\frac{1}{2}}$-Schnitten, namentlich $L_{k,\mathbb{Q}}$-Schnitte, $L_{k,\mathbb{R}}$-Schnitte, $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$-Schnitte, $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$-Schnitte, $L_{k-\frac{1}{2},\mathbb{R}\times\mathbb{R}}$-Schnitte, $L_{k-\frac{1}{2},\mathbb{Q}}$-Schnitte und $L_{k-\frac{1}{2},\mathbb{R}}$-Schnitte.

Betrachten wir die Gliederung von Kapitel 4:

In Abschnitt 4.2 sind L_k-Schnitte das zentrale Thema: In Abschnitt 4.2.1 definieren wir L_k-Schnitte (Definition 161) und ihre Abschlüsse $\mathrm{cl}_{L_{k,\mathbb{Q}}}(\cdot)$ und $\mathrm{cl}_{L_{k,\mathbb{R}}}(\cdot)$ (Definition 165). Der Abschluss bezüglich einer Klasse von Schnittebenen für ein vorgegebenes P is schlicht der Schnitt von P mit sämtlichen von den Schnittebenen des entsprechenden Typs induzierten Halbräumen. In Abschnitt 4.2.2 geht es um die Frage, wie wir L_k-Schnitte auch anders charakterisieren können. In Theorem 168 in Abschnitt 4.2.2.1 zeigen wir, dass wir uns sowohl für $L_{k,\mathbb{Q}}$-Schnitte als auch $L_{k,\mathbb{R}}$-Schnitte auf Vektorräume der Form $V = V' \times \mathbb{R}^n$ beschränken können. In Abschnitt 4.2.2.2 betrachten wir, wie wir $L_{k,\mathbb{Q}}$-Schnitte „auf duale Weise“ charakterisieren können. Dies ermöglicht es uns, die Theorie von $L_{k,\mathbb{Q}}$-Schnitten mit den in [DGMR17] betrachteten „k-dimensional lattice cuts“ in Verbindung zu bringen (siehe Theorem 176).

In Abschnitt 4.3 definieren wir $L_{k-\frac{1}{2}}$-Schnitte (Definition 179) und $L_{k-\frac{1}{2}}$-Abschlüsse (Definition 182), z.B. $\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(\,\cdot\,)$ und $\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(\,\cdot\,)$, analog wie wir es in Abschnitt 4.2.1 für L_k-Schnitte durchführen.

In Remark 156 formulieren wir zentrale Leitfragen, die wir in den restlichen Abschnitten von Kapitel 4 analysieren. Ein zentrales Ziel ist es, zu zeigen, dass wir für rationale Polyeder die beiden Hierarchien von Cutting-Plane-Operatoren in einer gemeinsamen Hierarchie zusammenfassen können, d.h. für rationale Polyeder $P \subseteq \mathbb{R}^m$ ($m \in \mathbb{Z}_{\geq 0}$, obgleich nur der Fall $m \in \mathbb{Z}_{\geq 1}$ von mathematischer Bedeutung ist) gilt

$$P = \mathrm{cl}_{L_{0,\mathbb{Q}}}(P) \supseteq \mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{Q}}}(P) \supseteq \mathrm{cl}_{L_{1,\mathbb{Q}}}(P) \supseteq \ldots \supseteq \mathrm{cl}_{L_{m-1,\mathbb{Q}}}(P) \supseteq \mathrm{cl}_{L_{m-\frac{1}{2},\mathbb{Q}}}(P) = \mathrm{cl}_{L_{m,\mathbb{Q}}}(P) = \mathrm{cl}_I(P) \quad (0.1)$$

und für rationale Polyeder $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m \in \mathbb{Z}_{\geq 0}$ und $n \in \mathbb{Z}_{\geq 1}$, obgleich nur der Fall $m, n \in \mathbb{Z}_{\geq 1}$ von mathematischer Bedeutung ist) gilt

$$\begin{aligned} P &= \mathrm{cl}_{L_{0,\mathbb{Q}}}(P) \supseteq \mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P) \supseteq \mathrm{cl}_{L_{1,\mathbb{Q}}}(P) \supseteq \ldots \supseteq \mathrm{cl}_{L_{m-1,\mathbb{Q}}}(P) \\ &\supseteq \mathrm{cl}_{L_{m-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{m-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P) \supseteq \mathrm{cl}_{L_{m,\mathbb{Q}}}(P) = \mathrm{cl}_{L_{(m+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{(m+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P) = \mathrm{cl}_I(P). \end{aligned} \quad (0.2)$$

$\mathrm{cl}_I(P)$ steht hier für die gemischt-ganzzahlige Hülle von P (siehe Definition 73).

In Abschnitt 4.5 analysieren wir die Unterschiede zwischen den unterschiedlichen Typen von L_k-Schnitten/Abschlüssen und $L_{k-\frac{1}{2}}$-Schnitten/Abschlüssen.

In Abschnitt 4.6 beginnen wir mit dem Projekt, die Inklusionen in (0.1) und (0.2) zu zeigen. Ein wichtiges Resultat hierzu ist Theorem 197, in welchen wir zeigen, dass für ein beliebiges $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) jeder $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$-Schnitt für P ($k \in \{0, \ldots, m+n\}$) auch ein $L_{k,\mathbb{Q}}$-Schnitt für P bezügliches des selben Vektorraums V ist. Dies impliziert insbesondere

$$\mathrm{cl}_{L_{k,\mathbb{Q}}}(P) \subseteq \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P).$$

Man beachte jedoch (vgl. Remark 198), dass die ähnlich aussehende Inklusion

$$\mathrm{cl}_{L_{k,\mathbb{R}}}(P) \subseteq \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{R}\times\mathbb{R}}}(P)$$

im Allgemeinen *nicht* gilt. Ein weiteres wichtiges Resultat ist Theorem 199, in dem wir zeigen, dass für beliebige $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) jeder $L_{k,\mathbb{R}}$-Schnitt für P ($k \in \{0, \ldots, m+n-1\}$) ein $L_{(k+1)-\frac{1}{2},\mathbb{R}\times\mathbb{R}}$-Schnitt für P ist. Dies impliziert

$$\mathrm{cl}_{L_{(k+1)-\frac{1}{2},\mathbb{R}\times\mathbb{R}}}(P) \subseteq \mathrm{cl}_{L_{k,\mathbb{R}}}(P).$$

Falls P ein rationales Polyeder ist (wie für (0.1) und (0.2) gefordert), gilt auch

$$\mathrm{cl}_{L_{(k+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) \subseteq \mathrm{cl}_{L_{k,\mathbb{Q}}}(P).$$

In Abschnitt 4.7 zeigen wir in Theorem 202, dass für beliebige $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m \in \mathbb{Z}_{\geq 0}$ und $n \in \mathbb{Z}_{\geq 1}$)

$$\mathrm{cl}_{L_{m,\mathbb{Q}}}(P) = \mathrm{cl}_{L_{(m+1)-\frac{1}{2},\mathbb{R}\times\mathbb{R}}}(P) = P \cap \mathrm{cl}_{\overline{I}}(P)$$

gilt ($\mathrm{cl}_{\overline{I}}(P)$ ist die abgeschlossene gemischt-ganzzahlige Hülle von P; vgl. Definition 73) und, falls P ein rationales Polyeder ist, sogar

$$\mathrm{cl}_{L_{(m+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_I(P)$$

erfüllt ist. Mit anderen Worten: Die $\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P)$-Hierarchie endet in diesem Fall bereits bei $k = m+1$ und nicht erst bei $k = m+n$ (siehe (0.2)).

Zu Abschnitt 4.8: Wir erinnern uns, dass wir in Theorem 168 in Abschnitt 4.2.2.1 gezeigt haben, dass wir uns für L_k-Schnitte auf Vektorräume der Form $V = V' \times \mathbb{R}^n$ beschränken können. Eine solche Beschränkung ist trivialerweise für $L_{k-\frac{1}{2}}$-Schnitte im Allgemeinen nicht möglich. Nichtsdestotrotz sind $L_{k-\frac{1}{2},\mathbb{Q}}$-Schnitte bezüglich solcher Vektorräume von mathematischer Bedeutung. In Definition 203 definieren wir „essentielle $L_{k-\frac{1}{2},\mathbb{Q}}$-Schnitte“, welche $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$-Schnitte/$L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$-Schnitte bezüglich Vektorräumen V von derartiger Struktur sind. In Theorem 208 werden wir die Bedeutung von essentiellen $L_{k-\frac{1}{2},\mathbb{Q}}$-Schnitten sehen: Dort zeigen wir, dass die einzigen $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$-Schnitte (dies inkludiert $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$-Schnitte), welche nicht bereits $L_{k-1,\mathbb{Q}}$-Schnitte sind, essentielle $L_{k-\frac{1}{2},\mathbb{Q}}$-Schnitte sind. In diesem Sinne kann man salopp sagen, dass essentielle $L_{k-\frac{1}{2},\mathbb{Q}}$-Schnitte „die interessanten“ $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$-Schnitte sind, da diese die einzigen $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$-Schnitte sind, welche möglicherweise mehr Ausdruckskraft besitzen als $L_{k-1,\mathbb{Q}}$-Schnitte. In Theorem 211 nutzen wir dieses Strukturresultat (Theorem 208), um unter anderem die Äquivalenz des $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$-Abschlusses und des $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$-Abschlusses von rationalen Polyedern zu zeigen.

In Abschnitt 4.9 untersuchen wir, was man über L_k-Abschlüsse und $L_{k-\frac{1}{2}}$-Abschlüsse von P aussagen kann, wenn P entweder einen nichttrivialen Linealitätsraum besitzt oder in einem nichttrivialen rationalen affinen Unterraum enthalten ist.

Bis hier ist es keineswegs offensichtlich, wie wir L_k-Schnitte und $L_{k-\frac{1}{2}}$-Schnitte überhaupt konkret berechnen können. In der Tat werden wir in Abschnitt 5.1 sehen, dass ein naiver Ansatz zum Finden von $L_{k-\frac{1}{2},\mathbb{Q}}$-Schnitten in der Praxis leicht zu Komplexitätsproblemen führen kann. Daher ist man daran interessiert, alternative Charakterisierungen für $L_{k,\mathbb{Q}}$-Schnitte/Abschluss, $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$-Schnitte/Abschluss und essentielle(n) $L_{k-\frac{1}{2},\mathbb{Q}}$-Schnitte/Abschluss zu finden. Es gibt zwei natürliche Ansätze, um dieses Problem anzugehen:

- Zeigen von alternativen Charakterisierungen, welche für allgemeine k funktionieren. Dies ist Inhalt von Kapitel 5.
- Zeigen von alternativen Charakterisierungen, welche für ein bestimmtes k (welches typischerweise klein ist) spezifisch sind. Dies ist das Vorgehen in Teil III für $L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$-Schnitte (siehe Kapitel 8, insbesondere Abschnitt 8.1) und $L_{1,\mathbb{Q}}$-Schnitte (siehe Kapitel 9, insbesondere Abschnitt 9.1.1) und in Teil IV (insbesondere Kapitel 11) für $L_{2,\mathbb{Q}}$-Schnitte (Abschnitt 11.1) und essentielle $L_{2-\frac{1}{2},\mathbb{Q}}$-Schnitte (Abschnitt 11.2).

Wir haben es soeben erwähnt: In Kapitel 5 zeigen wir alternative Charakterisierungen für $L_{k,\mathbb{Q}}$-Schnitte/ Abschluss und essentielle(n) $L_{k-\frac{1}{2},\mathbb{Q}}$-Schnitte/Abschluss für ein allgemeines k. Hierzu betrachten wir zwei Klassen von Charakterisierungen:

- Zum einen betrachten wir Charakterisierungen mittels gitterpunktfreien Körpern:
  - In Abschnitt 5.2 charakterisieren wir $L_{k,\mathbb{Q}}$-Schnitte mittels gitterpunktfreien Körpern. Das finale Resultat hierzu befindet sich in Theorem 240.
  - In Abschnitt 5.3 führen wir eine Charakterisierung von essentiellen $L_{k-\frac{1}{2},\mathbb{Q}}$-Schnitten für rationale Polyeder mittels gitterpunktfreien Körpern durch. Das finale Resultat befindet sich in Theorem 246.
- Zum andereren analysieren wir Charakterisierungen mittels t-Branch-Split-Cuts (siehe Definition 143). Dies ist das zentrale Thema von Abschnitt 5.4:
  - In Theorem 259 zeigen wir die Äquivalenz des $L_{k,\mathbb{Q}}$-Abschlusses und des $k, h(k)$-Branch-Split-Closures (siehe Definition 252). Die Bedeutung von $h(k)$ wird in Remark/Definition 248 erklärt.
  - In Theorem 261 zeigen wir die Äquivalenz des essentiellen $L_{k-\frac{1}{2},\mathbb{Q}}$-Abschlusses und des essentiellen $k, h(k)$-Branch-Split-Closures (siehe Definition 253).
  - In Theorem 263 fügen wir diese beiden Charakterisierungen zusammen und charakterisieren den $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$-Abschluss bzw. $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$-Abschluss eines rationalen Polyeders durch den Schnitt seines essentiellen $k, h(k)$-Branch-Split-Closures und seines $k-1, h(k-1)$-Branch-Split-Closures.

Die Charakterisierungvon $L_{k,\mathbb{Q}}$-Schnitten mittels k, t-Branch-Split-Cuts nutzen wir in Abschnitt 5.4.5, um zu zeigen, dass der $L_{k,\mathbb{Q}}$-Abschluss eines rationalen Polyeders wieder ein rationales Polyeder bildet (ein Resultat, welches unabhängig in [DGMR17, Theorem 2] gezeigt wurde).

In Kapitel 6 geben wir eine Übersicht über diverse Resultate bezüglich der Ausdruckskraft unterschiedlicher Klassen von Schnittebenen und deren Operatoren. Diese Resultate sind in erster Linie der Literatur entnommen. Kapitel 6 ist in drei Abschnitte aufgeteilt:

- In Abschnitt 6.1 analysieren wir die Frage, welche Klassen von Schnittebenen bzw. Schnittebenen-Operatoren andere dominieren oder nicht dominieren.
- Abschnitt 6.2 beschäftigt sich mit der Frage, ob es für eine vorgegebene Klasse von Schnittebenen immer möglich ist, den zugeordneten Operator $\mathrm{cl}_{(\cdot)}(\cdot)$ ggf. iterativ auf ein P (typischerweise ein rationales Polyeder) anzuwenden und nach endlich vielen Schritten $\mathrm{cl}_I(P)$ zu erreichen. Falls dies nicht möglich ist: Konvergiert dann wenigstens die Folge

$$\left\{\mathrm{cl}^{(k)}_{(\cdot)}(P)\right\}_{k\in\mathbb{Z}_{\geq 0}}$$

(„Konvergenz“ sei hierbei im Sinne von Definition 308 verstanden) gegen $\mathrm{cl}_I(P)$?

Um diese Frage systematisch zu untersuchen, formulieren wir am Anfang von Abschnitt 6.2 vier Leitfragen zu diesem Thema, welche wir für diverse Klassen von Schnittebenen-Operatoren betrachten.

- In Abschnitt 6.3 betrachten wir für einige Klassen von Schnittebenen die Frage, ob für ein vorgegebenes P und einen vorgegebenen Schnittebenen-Operator $\mathrm{cl}_{(\cdot)}(\cdot)$ die Menge $\mathrm{cl}_{(\cdot)}(P)$ ein (rationales) Polyeder bildet. Hierbei steht natürlich der Fall, dass bereits P ein rationales Polyeder ist, im Zentrum.

Teil III

In Teil III, welcher aus den Kapiteln 7, 8 und 9 besteht, sind die zentralen Themen ganzzahlige Polyeder, $L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$-Schnitte, $L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$-Schnitte und $L_{1,\mathbb{Q}}$-Schnitte (also $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$-Schnitte, $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$-Schnitte und $L_{k,\mathbb{Q}}$-Schnitte im Fall $k = 1$):

In Kapitel 7 ist das zentrale Thema „ganzzahlige Polyeder“. Wir beginnen dieses Kapitel mit Abschnitt 7.1, in welchem wir Ganzzahligkeit (Theorem 345) und Gemischt-Ganzzahligkeit (Theorem 347) von Polyedern mittels Optimierungsproblemen charakterisieren. Wir wollen hierzu anmerken, dass eine solche Charakterisierung von Gemischt-Ganzzahligkeit nach unserem Wissen in der Literatur bislang unbekannt war.

Eine wichtige Rolle zur Beschreibung von ganzzahligen Polyedern besitzen Ungleichungssysteme $Ax \leq b$ (A rational) mit der Eigenschaft, dass, wenn b ganzzahlig ist, das Polyeder $P^{\leq}(A,b) \subseteq \mathbb{R}^m$ ($m \in \mathbb{Z}_{\geq 0}$) ebenfalls ganzzahlig ist (wir definieren für $A \in \mathbb{R}^{l\times d}$ und $b \in \mathbb{R}^l$ ($l,d \in \mathbb{Z}_{\geq 0}$): $P^{\leq}(A,b) := \{x \in \mathbb{R}^d : Ax \leq b\}$). TDI-Systeme sind eine aus der Literatur wohlbekannte Klasse von Systemen mit dieser Eigenschaft, doch in diesem Kapitel führen wir drei weitere Klassen von Systemen mit dieser Eigenschaft ein: $\mathrm{TD}\mathbb{Z} + \{0,1\}$-Systeme, $\mathrm{TD}(I \cap \mathbb{Z}) + \{0,1\}$-Systeme und $\mathrm{TD}\mathbb{Z} + I$-Systeme.

Kapitel 7 beschäftigt sich zentral mit der Untersuchung dieser Klassen von Systemen. So ist beispielsweise in der Literatur wohlbekannt, dass es einen engen Zusammenhang zwischen TDI-Systemen und Hilbertbasen gibt. Um diesen Zusammenhang auf die anderen Klassen von Systeme zu verallgemeinern, führen wir von uns so genannte icone-Systeme, $\mathbb{Z}+$icone-Systeme, $\mathbb{Z}+\{0,1\}$-Systeme und $(\text{icone}\cap\mathbb{Z})+\{0,1\}$-Systeme ein, welche in engem Zusammenhang zu TDI-Systemen, $\mathrm{TD}\mathbb{Z} + I$-Systemen, $\mathrm{TD}\mathbb{Z} + \{0,1\}$-Systemen und $\mathrm{TD}(I \cap \mathbb{Z}) + \{0,1\}$-Systemen stehen.

In Abschnitt 7.6 analysieren wir, wie sich diese verschiedenen Klassen von Systemen in der Größe unterscheiden können.

Kapitel 8 besteht aus zwei recht unabhängigen Teilen, deren Hauptgemeinsamkeit darin besteht, dass es in beiden Teilen um den $L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$-Abschluss bzw. $L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$-Abschluss geht:

- In Abschnitt 8.1 untersuchen wir den Zusammenhang zwischen (projizierten) Chvátal-Gomory-Schnitten (vgl. Definition 122), dualen (projizierten) Chvátal-Gomory-Schnitten (vgl. Definition 382), starken (projizierten) Chvátal-Gomory-Schnitten (vgl. Definition 384), $L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$-Schnitten und $L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$-Schnitten. Eine Übersicht über die Resultate wird in Abschnitt 8.1.2.4 gegeben. Bezüglich der Situation für Polyeder betrachte man insbesondere Theorem 391.
- In Abschnitt 8.2 beschäftigen wir uns mit der Frage, wie man den Chvátal-Gomory-Abschluss eines Polyeders mit rationalen Seitennormalen berechnen kann. Hierzu nutzen wir die Frameworks der $\mathrm{TD}\mathbb{Z}$+I-Systeme und $\mathbb{Z}+$ icone-Systeme aus Kapitel 7:
  - In Theorem 398 in Abschnitt 8.2.3 zeigen wir, wie wir $\mathrm{TD}\mathbb{Z} + I$-Systeme mit ganzzahliger linker Seite nutzen können, um den Chvátal-Gomory-Abschluss eines Polyeders $P \subseteq \mathbb{R}^m$ ($m \in \mathbb{Z}_{\geq 0}$) mit rationalen Seitennormalen darzustellen. Wir merken an, dass ein solches Resultat für TDI-Systeme in der Literatur wohlbekannt ist. Somit beschäftigen wir uns im Rest von Abschnitt 8.2.3 mit der Frage, ob $\mathrm{TD}\mathbb{Z} + I$-Systeme zu diesem Zweck kleiner als TDI-Systeme sein können und ob hier weiteres Potential für Verbesserungen existiert.
  - In Abschnitt 8.2.4 reduzieren wir das Problem, den Chvátal-Gomory-Abschluss eines Polyeders $P = P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right) \subseteq \mathbb{R}^m \times \mathbb{R}^m$ ($m,n \in \mathbb{Z}_{\geq 0}$) mit rationalen Seitennormalen zu berechnen, auf den Spezialfall, dass die Zeilen von $\begin{pmatrix} A & G & -b \end{pmatrix}$ einen LP-Face-Cone (vgl. Definition 356) bilden. Für solche Polyeder können wir den Chvátal-Gomory-Abschluss mittels $\mathbb{Z}+$icone-Systemen charakterisieren. Dieser Ansatz hat gegenüber dem mittels $\mathrm{TD}\mathbb{Z}+I$-Systemen den Vorteil, dass er auch im gemischt-ganzzahligen Fall funktioniert und auf den Split-Closure/MIR-Closure verallgemeinert werden kann; letzteres bildet das Thema von Abschnitt 9.2 in Kapitel 9.

Auch Kapitel 9 besteht aus zwei voneinander nahezu unabhängigen Teilen, deren Hauptgemeinsamkeit darin liegt, dass in beiden der $L_{1,\mathbb{Q}}$-Abschluss betrachtet wird:

- In Abschnitt 9.1 analysieren wir den Zusammenhang zwischen $L_{1,\mathbb{Q}}$-Schnitten, Split-Cuts (vgl. Definition 126) und MIR-Cuts (vgl. Definition 410).

- In Abschnitt 9.2 zeigen wir, dass der Split-Closure eines Polyeders $P \subseteq \mathbb{R}^m \times \mathbb{R}^m$ ($m, n \in \mathbb{Z}_{\geq 0}$) mit rationalen Seitennormalen wieder ein Polyeder ist (ein rationales Polyeder, falls P ein rationales Polyeder ist). Wir merken an, dass für das schwächere Resultat, dass der Split-Closure eines rationalen Polyeders wieder ein rationales Polyeder bildet, zahlreiche Beweise in der Literatur bekannt sind (vgl. Abschnitt 6.3). Unser Beweis benutzt das von uns entwickelte Framework von $\mathbb{Z}+$ icone-Systemen als zentralen Bestandteil. Die Beweisführung spiegelt hierbei sehr eng die Beweisführung unseres zweiten Beweises der Polyedrizität des Chvátal-Gomory-Abschlusses wieder. Bezüglich weiterer Vorteile unseres Ansatzes verweisen wir auf den Anfang von Kapitel 9.

Teil IV

In Teil IV ist das Ziel, eine alternative Charakterisierung von $L_{2,\mathbb{Q}}$-Schnitten/Abschluss und $L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$/ $L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$-Schnitten/Abschluss herzuleiten (bei letzteren liegt der Fokus auf essentiellen $L_{2-\frac{1}{2},\mathbb{Q}}$-Schnitten/ Abschluss).

Um diese Resultate herzuleiten, beweisen wir in Kapitel 10 zwei Theoreme (Theorem 431 und Theorem 434) über die Einbettung volldimensionaler gitterpunktfreier Körper im $\mathbb{R}^2$ in Disjunktionen. Wir merken an, dass eine schwächere Version von Theorem 431 (Theorem 432) bereits in [DDG12] gezeigt wurde.

In Kapitel 11 nutzen wir diese Einbettungsresultate, um $L_{2,\mathbb{Q}}$-Schnitte/Abschluss (in Abschnitt 11.1) und essentielle $L_{2-\frac{1}{2},\mathbb{Q}}$-Schnitte/Abschluss (in Abschnitt 11.2) mittels Disjunktionen zu charakterisieren. Die finalen Resultate sind in Theorem 462 ($L_{2,\mathbb{Q}}$-Abschluss), Theorem 474 (essentieller $L_{2-\frac{1}{2},\mathbb{Q}}$-Abschluss) und Theorem 475 ($L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$-Abschluss und $L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$-Abschluss) zu finden.

Teil V

In Teil V (Kapitel 12 und Kapitel 13) zeigen wir weitergehende Resultate für L_k-Schnitte und $L_{k-\frac{1}{2}}$-Schnitte:

In Kapitel 12 betrachten wir die Frage (vgl. Problem/Definition 476), wie viele Ungleichungen in der Ungleichungsbeschreibung eines Polyeders P wir gleichzeitig betrachten müssen, um alle $L_{k,\mathbb{Q}}$-Schnitte oder $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$-Schnitte, die erforderlich sind, um den $L_{k,\mathbb{Q}}$-Abschluss oder $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$-Abschluss von P zu beschreiben, als $L_{k,\mathbb{Q}}$-Schnitt oder $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$-Schnitt eines solchen Teilsystems herleiten zu können.
Eine Zusammenfassung der wichtigsten oberen und unteren Schranken, die wir in diesem Kapitel beweisen, befindet sich in Abschnitt 12.6. Abschnitt 12.4.3 liefert einen Ausblick darauf, wie diese Resultate auf den essentiellen $L_{k-\frac{1}{2},\mathbb{Q}}$-Abschluss (Theorem 496) und $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$-Abschluss (Theorem 497) erweitert werden können.

In Kapitel 13 betrachten wir Schranken für den $L_{k,\mathbb{Q}}$-Rang und $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$-Rang (insbesondere ersteren) eines Polyeders P, wenn die ganzzahligen Variablen 0/1-wertig sind. Der $L_{k,\mathbb{Q}}$-Rang bzw. $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$-Rang sagt aus, wie oft wir den entsprechenden Cutting-Plane-Operator iterativ auf P anwenden müssen, um die gemischt-ganzzahlige Hülle von P ($\mathrm{cl}_I(P)$) zu erhalten.
In Abschnitt 13.1 betrachten wir den $L_{k,\mathbb{Q}}$-Rang. Das erste zentrale Resultat ist Theorem 515, in dem wir die Schranke

$$\mathrm{rank}_{L_{k,\mathbb{Q}}}(P) \leq \left\lceil \frac{m}{k} \right\rceil$$

zeigen ($m, n \in \mathbb{Z}_{\geq 0}$, $k \in \{1, \ldots, m\}$ und $P \subseteq [0,1]^m \times \mathbb{R}^n$). Ist diese Schranke bestmöglich? Diese Frage beantworten wir in Theorem 526 (das zweite zentrale Resultat von Abschnitt 13.1) positiv, indem wir zeigen, dass für jedes $m \in \mathbb{Z}_{\geq 1}$ ein rationales Polytop $P \subseteq [0,1]^m$ existiert, so dass für alle $k \in \{1, \ldots, m\}$

$$\mathrm{rank}_{L_{k,\mathbb{Q}}}(P) \geq \left\lceil \frac{m}{k} \right\rceil$$

erfüllt ist.

In Abschnitt 13.2 leiten wir hieraus Schranken für den $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$-Rang eines rationalen Polyeders ab, was allerdings keine Herleitung einer oberen Schranke für den $L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$-Rang erlaubt. Glücklicherweise ist das eng verwandte Problem des Findens von Schranken für den Chvátal-Gomory-Rangs eines Polyeders $P \subseteq [0,1]^m$ ($m \in \mathbb{Z}_{\geq 0}$) ein in der Literatur gut untersuchtes Problem. Daher geben wir in Abschnitt 13.2.1 eine Übersicht über wichtige Literaturresultate bezüglich oberer Schranken und in Abschnitt 13.2.2 bezüglich unterer Schranken für den Chvátal-Gomory-Rang.

Teil VI

In Teil VI, welcher aus Kapitel 14 und Kapitel 15 besteht, geht es um folgende Thematik (vgl. Abschnitt 14.1): Können die Inklusionen in (0.1) und (0.2) auch strikt sein? Hierzu formulieren wir folgende Fragen:

- Existiert für jedes $m \in \mathbb{Z}_{\geq 1}$, $n \in \mathbb{Z}_{\geq 0}$ und $k \in \{1, \dots, m\}$ ein rationales Polyeder $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$, für das gilt:
$$\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) \subsetneq \mathrm{cl}_{L_{k-1,\mathbb{Q}}}(P)?$$
- Existiert für jedes $m \in \mathbb{Z}_{\geq 1}$, $n \in \mathbb{Z}_{\geq 0}$ und $k \in \begin{cases} \{1, \dots, m-1\} & \text{falls } n = 0, \\ \{1, \dots, m\} & \text{falls } n \geq 1 \end{cases}$ ein rationales Polyeder $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$, für das gilt:
$$\mathrm{cl}_{L_{k,\mathbb{Q}}}(P) \subsetneq \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P)?$$

Wir betrachten folgende noch stärkere Fragen:

- Existiert für jedes $m \in \mathbb{Z}_{\geq 1}$, $n \in \mathbb{Z}_{\geq 0}$ und $k \in \{1, \dots, m\}$ ein rationales Polyeder $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$, für das gilt:
$$\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) \subsetneq \mathrm{cl}_{L_{k-1,\mathbb{R}}}(P)?$$
- Existiert für jedes $m \in \mathbb{Z}_{\geq 1}, n \in \mathbb{Z}_{\geq 0}$ und $k \in \begin{cases} \{1, \dots, m-1\} & \text{falls } n = 0, \\ \{1, \dots, m\} & \text{falls } n \geq 1 \end{cases}$ ein rationales Polyeder $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$, für das gilt:
$$\mathrm{cl}_{L_{k,\mathbb{Q}}}(P) \subsetneq \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{R}\times\mathbb{R}}}(P)? \tag{0.3}$$

In Kapitel 14 beantworten wir alle diese Fragen positiv mit Ausnahme des am Ende von Kapitel 14 noch offen bleibenden Falles $n \geq 1$ und $k = m$ in (0.3). Die zentrale Übersicht hierzu befindet sich in Abschnitt 14.9. Die zentralen Theoreme zu diesem Zweck sind Theorem 543 und Theorem 544. Man beachte, dass wir in Theorem 543 allgemeiner den $(m-1)$-Branch-Split-Closure und den $L_{m-1,\mathbb{Q}}$-Abschluss gegen den $(m-2)$-Branch-Split-Closure, den $L_{m-2,\mathbb{R}}$-Abschluss und den $L_{(m-1)-\frac{1}{2},\mathbb{R}}$-Abschluss abschätzen.

Die am Ende von Kapitel 14 offene gebliebene Frage nach einer strikten Inklusion in (0.3) im Fall $n \geq 1$ und $k = m$ ist zentrales Thema von Kapitel 15. Hier zeigen wir (Theorem 578) die noch stärkere Aussage, dass für alle $m, n \in \mathbb{Z}_{\geq 1}$ und $k \in \{1, \dots, m\}$ ein a rationales Polyeder $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ existiert, so dass für alle $\ell \in \mathbb{Z}_{\geq 0}$ gilt:
$$\mathrm{cl}_I(P) = \mathrm{cl}_{L_{k,\mathbb{Q}}}(P) \subsetneq \mathrm{cl}^{(\ell)}_{L_{k-\frac{1}{2},\mathbb{R}\times\mathbb{R}}}(P).$$
Dies impliziert, dass wir niemals die gemischt-ganzzahlige Hülle $\mathrm{cl}_I(P)$ dadurch erreichen können, indem wir den $L_{k-\frac{1}{2},\mathbb{R}\times\mathbb{R}}$-Abschluss iterativ auf P anwenden. Das zentrale Theorem für diesen Beweis ist Theorem 563. Ein weiteres ähnliches Resultat für ein Polyeder, welche in der für k row cuts erforderlichen Gleichungsform gegeben ist (vgl. Definition 154), ist Theorem 564.

In Abschnitt 15.5.1 nutzen wir diese Resultate, um strikte Inklusionen für weitere Klassen von Cutting-Plane-Operatoren zu zeigen. Diese Resultate haben wir in Theorem 576 und Theorem 577 aufgeschrieben.

Teil VII

In Teil VII schließen wir die Arbeit mit einer Zusammenfassung (Kapitel 16) und einem Ausblick (Kapitel 17) ab.

Abstract

Remark: For a more casual introducation into the central ideas of this thesis, in particular those that are introduced in chapter 4 and chapter 5, we refer to chapter 1. Additionally, we want to remark that chapter 16 provides a much more detailed summary of this thesis.

Part I

The central goal of part I, which consists of chapter 2 and chapter 3, is to introduce basic definitions that are necessary for this thesis.

Chapter 2 can be considered as the "mathematical basics chapter", in which we introduce numerous definitions and results from various mathematical areas that are used in this thesis.

Chapter 3 is an introduction to common classes of cutting planes that were investigated in the literature. For this abstract, we just remark (cf. Definition 120) that a cutting plane for $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$; here, m denotes the number of discrete (integral) variables and n the number of continuous variables) is a linear inequality $c(\cdot) \leq c_0$ ($c \in (\mathbb{R}^m \times \mathbb{R}^n)^T$ and $c_0 \in \mathbb{R}$) for $P \cap (\mathbb{Z}^m \times \mathbb{R}^n)$.

Part II

Part II consists of chapter 4, 5 and 6. The guiding theme of this part are L_k cuts and $L_{k-\frac{1}{2}}$ cuts. These two classes of cutting planes form the central topic of this thesis.

Chapter 4 forms the central foundations chapter for these two classes of cutting planes and their interplay. In this chapter, we build the whole theory building of L_k cuts and $L_{k-\frac{1}{2}}$ cuts up from its foundations. The central idea behind L_k cuts and $L_{k-\frac{1}{2}}$ cuts is to relax the problem of finding cutting planes by replacing the problem of finding linear inequalities for $P \cap (\mathbb{Z}^m \times \mathbb{R}^n)$ by the problem of finding linear inequalities for

$$P \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + V)$$

or

$$(P + V) \cap (\mathbb{Z}^m \times \mathbb{R}^n),$$

where $V \leq \mathbb{R}^m \times \mathbb{R}^n$ is a vector space of codimension $k \in \{0, \ldots, m+n\}$ (the case $k = 0$ is admitted for formal reasons). We name the first construction L_k **cuts** and the second one $L_{k-\frac{1}{2}}$ **cuts**. Depending on the properties of the generators of V (rationality conditions) and the existence of continuous variables, this leads to various classes of L_k cuts and $L_{k-\frac{1}{2}}$ cuts, which we call $L_{k,\mathbb{Q}}$ cuts, $L_{k,\mathbb{R}}$ cuts, $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts, $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cuts, $L_{k-\frac{1}{2},\mathbb{R}\times\mathbb{R}}$ cuts, $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts and $L_{k-\frac{1}{2},\mathbb{R}}$ cuts.

For the structure of chapter 4:

In section 4.2, L_k cuts are the central topic: in section 4.2.1, we define L_k cuts (Definition 161) and their closures $\mathrm{cl}_{L_{k,\mathbb{Q}}}(\cdot)$ and $\mathrm{cl}_{L_{k,\mathbb{R}}}(\cdot)$ (Definition 165). The closure with respect to some class of cutting planes for a given P is just the intersection of P with all of the half-spaces that are induced by cutting planes of the respective type. In section 4.2.2, we consider the question how we can characterize L_k cuts differently. In Theorem 168 in section 4.2.2.1, we show that we can restrict ourselves to vector spaces of the form $V = V' \times \mathbb{R}^n$ for both $L_{k,\mathbb{Q}}$ cuts and $L_{k,\mathbb{R}}$ cuts. In section 4.2.2.2, we consider how we can characterize $L_{k,\mathbb{Q}}$ cuts "dually". This enables us to connect the theory of $L_{k,\mathbb{Q}}$ cuts to the theory of "k-dimensional lattice cuts" that is studied in [DGMR17] (see Theorem 176).

In section 4.3, we define $L_{k-\frac{1}{2}}$ cuts (Definition 179) and $L_{k-\frac{1}{2}}$ closures (Definition 182), e.g. $\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(\cdot)$ and $\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(\cdot)$, in an analogue way to what we did in section 4.2.1 for L_k cuts.

In Remark 156, we formulate central guiding questions, which we analyze in the remaining sections of chapter 4. A central goal is to show that for rational polyhedra, the two hierarchies of cutting plane operators

can be merged into a unified hierarchy, i.e. for rational polyhedra $P \subseteq \mathbb{R}^m$ ($m \in \mathbb{Z}_{\geq 0}$, even though only the case $m \in \mathbb{Z}_{\geq 1}$ is of mathematical importance), we have

$$P = \mathrm{cl}_{L_{0,\mathbb{Q}}}(P) \supseteq \mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{Q}}}(P) \supseteq \mathrm{cl}_{L_{1,\mathbb{Q}}}(P) \supseteq \ldots \supseteq \mathrm{cl}_{L_{m-1,\mathbb{Q}}}(P) \supseteq \mathrm{cl}_{L_{m-\frac{1}{2},\mathbb{Q}}}(P) = \mathrm{cl}_{L_{m,\mathbb{Q}}}(P) = \mathrm{cl}_I(P) \quad (0.4)$$

and for rational polyhedra $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}_{\geq 1}$, even though only the case $m, n \in \mathbb{Z}_{\geq 1}$ is of mathematical importance), we have

$$\begin{aligned} P = \mathrm{cl}_{L_{0,\mathbb{Q}}}(P) &\supseteq \mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P) \supseteq \mathrm{cl}_{L_{1,\mathbb{Q}}}(P) \supseteq \ldots \supseteq \mathrm{cl}_{L_{m-1,\mathbb{Q}}}(P) \\ &\supseteq \mathrm{cl}_{L_{m-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{m-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P) \supseteq \mathrm{cl}_{L_{m,\mathbb{Q}}}(P) = \mathrm{cl}_{L_{(m+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{(m+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P) = \mathrm{cl}_I(P). \end{aligned} \quad (0.5)$$

$\mathrm{cl}_I(P)$ is the mixed-integer hull of P (see Definition 73).

In section 4.5, we analyze the differences between the different kinds of L_k and $L_{k-\frac{1}{2}}$ cuts/closures.

In section 4.6, we start with the project of showing the inclusions in (0.4) and (0.5). An important result for this is Theorem 197, in which we show that for an arbitrary $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$), every $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cut for P ($k \in \{0, \ldots, m+n\}$) is also an $L_{k,\mathbb{Q}}$ cut for P with respect to the same vector space V. This, in particular, implies

$$\mathrm{cl}_{L_{k,\mathbb{Q}}}(P) \subseteq \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P).$$

Pay attention (cf. Remark 198) that the similar looking inclusion

$$\mathrm{cl}_{L_{k,\mathbb{R}}}(P) \subseteq \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{R}\times\mathbb{R}}}(P)$$

does *not* hold in general. Another important result is Theorem 199, in which we show that for arbitrary $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$), every $L_{k,\mathbb{R}}$ cut for P ($k \in \{0, \ldots, m+n-1\}$) is an $L_{(k+1)-\frac{1}{2},\mathbb{R}\times\mathbb{R}}$ cut for P. This implies

$$\mathrm{cl}_{L_{(k+1)-\frac{1}{2},\mathbb{R}\times\mathbb{R}}}(P) \subseteq \mathrm{cl}_{L_{k,\mathbb{R}}}(P).$$

If P is a rational polyhedron (as we demand for (0.4) and (0.5)), we also have

$$\mathrm{cl}_{L_{(k+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) \subseteq \mathrm{cl}_{L_{k,\mathbb{Q}}}(P).$$

In section 4.7, we show in Theorem 202 that for arbitrary $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}_{\geq 1}$), we have

$$\mathrm{cl}_{L_{m,\mathbb{Q}}}(P) = \mathrm{cl}_{L_{(m+1)-\frac{1}{2},\mathbb{R}\times\mathbb{R}}}(P) = P \cap \mathrm{cl}_{\overline{I}}(P)$$

($\mathrm{cl}_{\overline{I}}(P)$ is the closed mixed-integer hull of P; cf. Definition 73) and if P is a rational polyhedron, even

$$\mathrm{cl}_{L_{(m+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_I(P)$$

holds. In other words: in this situation, the $\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P)$ hierarchy already ends at $k = m+1$ and not only at $k = m+n$ (see (0.5)).

For section 4.8: let us recall that we showed in Theorem 168 in section 4.2.2.1 that for L_k cuts, we can restrict ourselves to vector spaces of the form $V = V' \times \mathbb{R}^n$. Such a restriction is trivially in general not possible for $L_{k-\frac{1}{2}}$ cuts. Nevertheless, $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts with respect to such vector spaces are of mathematical importance. In Definition 203, we define "essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts", which are $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts/$L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cuts with respect to vector spaces V of such a structure. In Theorem 208, we attest the importance of essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts: there, we show that the only $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cuts (this includes $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts) which are not already $L_{k-1,\mathbb{Q}}$ cuts, are essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts. In this sense, one can casually say that essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts are "the interesting" $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cuts, since these are the only $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cuts that have possibly more expressivity than $L_{k-1,\mathbb{Q}}$ cuts. In Theorem 211, we use this structural result (Theorem 208) for proving among other things the equivalence of the $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ closure and the $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ closure of rational polyhedra.

In section 4.9, we investigate what can be said about the L_k closures and $L_{k-\frac{1}{2}}$ closures of P if P either has a nontrivial lineality space or is contained in a nontrivial rational affine subspace.

Until here, it is anything but obvious how to even compute L_k cuts and $L_{k-\frac{1}{2}}$ cuts concretely. Indeed, we see in section 5.1 that a naive approach for finding $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts easily leads to complexity issues in practice. That is why one is interested in finding alternative characterizations of $L_{k,\mathbb{Q}}$ cuts/closure, $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts/closure and essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts/closure. There exist two natural approaches to tackle this problem:

- Show alternative characterizations which work for general k. This is the topic of chapter 5.
- Show alternative characterizations which work for a specific k (that is typically small). This becomes the approach in part III for $L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts (see chapter 8, in particular section 8.1) and $L_{1,\mathbb{Q}}$ cuts (see chapter 9, in particular section 9.1.1) and in part IV (in particular chapter 11) for $L_{2,\mathbb{Q}}$ cuts (section 11.1) and essential $L_{2-\frac{1}{2},\mathbb{Q}}$ cuts (section 11.2).

We just mentioned it: in chapter 5, we show alternative characterizations of $L_{k,\mathbb{Q}}$ cuts/closure and essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts/closure for a general k. For this, we consider two classes of characterizations:

- On one hand, we consider characterizations via lattice-free bodies:
  - In section 5.2, we characterize $L_{k,\mathbb{Q}}$ cuts via lattice-free bodies. The final result can be found in Theorem 240.
  - In section 5.3, we characterize essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts for rational polyhedra via lattice-free bodies. The final result is stated in Theorem 246.
- On the other hand, we examine characterizations via t-branch split cuts (see Definition 143). This is the central topic of section 5.4:
  - In Theorem 259, we show the equivalence of the $L_{k,\mathbb{Q}}$ closure and the $k, h(k)$-branch split closure (see Definition 252). The meaning of $h(k)$ is explained in Remark/Definition 248.
  - In Theorem 261, we show the equivalence of the essential $L_{k-\frac{1}{2},\mathbb{Q}}$ closure and the essential $k, h(k)$ branch-split closure (see Definition 253).
  - In Theorem 263, we put these two characterizations together and characterize the $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ closure or $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ closure, respectively, of a rational polyhedron by the intersection of its $k, h(k)$-branch split closure and its $k-1, h(k-1)$-branch split closure.

In section 5.4.5, we use the characterizations of $L_{k,\mathbb{Q}}$ cuts via k, t-branch split cuts to show that the $L_{k,\mathbb{Q}}$ closure of a rational polyhedron is again a rational polyhedron (a result that has independently been shown in [DGMR17, Theorem 2]).

In chapter 6, we give an overview on various results concerning the expressivity of various classes of cutting planes and their operators. These are predominantly gathered from the literature. Chapter 6 is divided into three sections:

- In section 6.1, we analyze the question which classes of cutting planes or cutting plane operators, respectively, dominate others or not.
- Section 6.2 deals with the question whether for a given class of cutting planes, it is always possible to apply the corresponding operator $\mathrm{cl}_{(\,\cdot\,)}(\,\cdot\,)$ iteratively (if necessary) on P (typically a rational polyhedron) and reach $\mathrm{cl}_I(P)$ in a finite number of steps. If this is not possible: does at least the sequence

  $$\left\{\mathrm{cl}_{(\,\cdot\,)}^{(k)}(P)\right\}_{k\in\mathbb{Z}_{\geq 0}}$$

  converge ("convergence" is to be understood in the sense of Definition 308) against $\mathrm{cl}_I(P)$?

  For analyzing this question systematically, we formulate four guiding questions at the beginning of section 6.2. We consider these guiding questions for several classes of cutting plane operators.
- In section 6.3, for various classes of cutting planes, we consider the question whether for a given P and a given cutting plane operator $\mathrm{cl}_{(\,\cdot\,)}(\,\cdot\,)$, the set $\mathrm{cl}_{(\,\cdot\,)}(P)$ is a (rational) polyhedron. Here, of course, the case that P is already a rational polyhedron is at the center of our considerations.

Part III

In part III, which consists of chapter 7, 8 and 9, the central topics are integral polyhedra $L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts, $L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cuts and $L_{1,\mathbb{Q}}$ cuts (i.e. $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts, $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cuts and $L_{k,\mathbb{Q}}$ cuts in the case $k=1$):

In chapter 7, the central topic is "integral polyhedra". We start this chapter with section 7.1, in which we characterize integrality (Theorem 345) and mixed-integrality (Theorem 347) of polyehdra via optimization problems. We want to remark that such a characterization of mixed-integrality was to our knowledge previously unknown in the literature.

Systems of linear inequalities $Ax \leq b$ (A rational) with the property that if b is integral, so is the polyhedron $P^{\leq}(A, b) \subseteq \mathbb{R}^m$ ($m \in \mathbb{Z}_{\geq 0}$; in general for $A \in \mathbb{R}^{l \times d}$ and $b \in \mathbb{R}^l$ ($l, d \in \mathbb{Z}_{\geq 0}$), we define $P^{\leq}(A, b) := \{x \in \mathbb{R}^d : Ax \leq b\}$), play an important role for describing integral polyhedra. TDI systems are a class of system with this property that is well-known from the literature, but in this chapter we introduce three additional types of systems with this property: TD$\mathbb{Z}+\{0,1\}$ systems, TD$(I \cap \mathbb{Z})+\{0,1\}$ systems and TD$\mathbb{Z}+I$ systems.

Chapter 7 centrally deals with the investigation of these types of systems. So, for example, it is well-known in the literature that there exists a close relationship between TDI systems and Hilbert bases. To generalize this relationship to the other types of systems, we introduce (by us) so-called icone systems, $\mathbb{Z}+$icone systems, $\mathbb{Z}+\{0,1\}$ systems and (icone$\cap\mathbb{Z}$)$+\{0,1\}$ systems which are closely related to TDI systems, TD$\mathbb{Z}+I$ systems, TD$\mathbb{Z}+\{0,1\}$ systems and TD$(I \cap \mathbb{Z}) + \{0,1\}$ systems.

In section 7.6, we analyze how these different types of systems can differ in their sizes.

Chapter 8 consists of two nearly independent parts whose main similarity consists therein that in both, the $L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ closure or $L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ closure, respectively, is considered:

- In section 8.1, we investigate the relationship between (projected) Chvátal-Gomory cuts (cf. Definition 122), dual (projected) Chvátal-Gomory cuts (cf. Definition 382), strong (projected) Chvátal-Gomory cuts (cf. Definition 384), $L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts and $L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cuts. An overview about the results is given in section 8.1.2.4. Concerning the situation for polyhedra, the reader should in particular take a look at Theorem 391.

- In section 8.2, we consider how to compute the Chvátal-Gomory closure of a polyhedron with rational face normals. For this, we use the frameworks of TD$\mathbb{Z}$+I systems and $\mathbb{Z}+$ icone systems from chapter 7:

  - In Theorem 398 in section 8.2.3, we show how we can use TD$\mathbb{Z}+I$ systems with an integral left-hand side to represent the Chvátal-Gomory closure of a polyhedron $P \subseteq \mathbb{R}^m$ ($m \in \mathbb{Z}_{\geq 0}$) with rational face normals. We remark that such a result is well-known in the literature on TDI systems. Thus, in the remainder of section 8.2.3, we focus on the question whether TD$\mathbb{Z}+I$ systems for this purpose can be smaller than TDI systems and whether there is additional potential for improvements.

  - In section 8.2.4, we reduce the problem of computing the Chvátal-Gomory closure of a polyhedron $P = P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right) \subseteq \mathbb{R}^m \times \mathbb{R}^m$ ($m, n \in \mathbb{Z}_{\geq 0}$) with rational face normals to the special case that the rows of $\begin{pmatrix} A & G & -b \end{pmatrix}$ form an LP face cone (cf. Definition 356). For such polyhedra, we can characterize the Chvátal-Gomory closure via $\mathbb{Z}+$ icone systems. This approach has the advantage over the one using TD$\mathbb{Z}+I$ systems that it also works in the mixed-integer case and can be generalized to the split closure; the latter topic is considered in section 9.2 of chapter 9.

Also chapter 9 consist of two nearly independent parts whose main similarity consists therein that in both, the $L_{1,\mathbb{Q}}$ closure is considered:

- In section 9.1, we analyze the relationship between $L_{1,\mathbb{Q}}$ cuts, split cuts (cf. Definition 126) and MIR cuts (cf. Definition 410).

- In section 9.2, we show that the split closure of a polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^m$ ($m, n \in \mathbb{Z}_{\geq 0}$) with rational face normals is again a polyhedron (a rational polyhedron if P is one). We remark that for the weaker result that the split closure of a rational polyhedron is again a rational polyhedron, there exist numerous proofs in the literature (cf. section 6.3). Our proof uses the framework of $\mathbb{Z}+$ icone systems that was developed by us as a central component. Our argumentation closely mirrors the argumentation of our second proof of the polyhecricity of the Chvátal-Gomory closure.

  Regarding further advantages of our approach, we refer to the beginning of chapter 9.

Part IV

In part IV, our goal is to derive an alternative characterization of $L_{2,\mathbb{Q}}$ cuts/closure and $L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}/L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cuts/closure (for the latter one, we focus on essential $L_{2-\frac{1}{2},\mathbb{Q}}$ cuts/closure).

To derive these results, in chapter 10 we show two theorems about embedding full-dimensional lattice-free bodies in $\mathbb{R}^2$ into disjunctions. We remark that a weaker version of Theorem 431 (Theorem 432) was shown in [DDG12].

In chapter 11, we use these embedding results to characterize $L_{2,\mathbb{Q}}$ cuts/closure (in section 11.1) and essential $L_{2-\frac{1}{2},\mathbb{Q}}$ cuts/closure (in section 11.2) via disjunctions. The final results can be found in Theorem 462 ($L_{2,\mathbb{Q}}$ closure), Theorem 474 (essential $L_{2-\frac{1}{2},\mathbb{Q}}$ closure) and Theorem 475 ($L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ closure and $L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ closure).

Part V

In part V (chapter 12 and chapter 13), we show further results about L_k cuts and $L_{k-\frac{1}{2}}$ cuts:

In chapter 12, we consider the question (cf. Problem/Definition 476) how many inequalities in the inequality description of a polyhedron we have to consider simultaneously to be able to derive all $L_{k,\mathbb{Q}}$ cuts or $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts, respectively, that are necessary to describe the $L_{k,\mathbb{Q}}$ closure or $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ closure, respectively, as an $L_{k,\mathbb{Q}}$ cut or $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cut, respectively, of such a subsystem.
A summary of the most important upper and lower bounds for this can be found in section 12.6. Section 12.4.3 delivers an outlook on how these results can be extended to the essential $L_{k-\frac{1}{2},\mathbb{Q}}$ closure (Theorem 496) and $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ closure (Theorem 497).

In chapter 13, we consider bounds for the $L_{k,\mathbb{Q}}$ rank and $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ rank (in particular the former) of a polyhedron P if the integral variables are 0/1-valued. The $L_{k,\mathbb{Q}}$ rank or $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ rank, respectively, tells us how often we have to apply the respective cutting plane operator iteratively on P to obtain the mixed-integer hull of P ($\mathrm{cl}_I(P)$).
In section 13.1, we consider the $L_{k,\mathbb{Q}}$ rank. The first central result is Theorem 515, in which we show the bound

$$\mathrm{rank}_{L_{k,\mathbb{Q}}}(P) \leq \left\lceil \frac{m}{k} \right\rceil$$

($m, n \in \mathbb{Z}_{\geq 0}$, $k \in \{1, \ldots, m\}$ and $P \subseteq [0,1]^m \times \mathbb{R}^n$). Is this bound the best possible? We answer this question positively in Theorem 526 (the second central result of section 13.1) by showing that for all $m \in \mathbb{Z}_{\geq 1}$, there exists a rational polytope $P \subseteq [0,1]^m$ such that for all $k \in \{1, \ldots, m\}$,

$$\mathrm{rank}_{L_{k,\mathbb{Q}}}(P) \geq \left\lceil \frac{m}{k} \right\rceil$$

is satisfied.
In section 13.2, we derive herefrom bounds for the $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ rank of a rational polyhedron, which, however, does not permit to derive an upper bound for the $L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ rank. Luckily, the closely related problem of finding bounds for the Chvátal-Gomory rank of a polyhedron $P \subseteq [0,1]^m$ ($m \in \mathbb{Z}_{\geq 0}$) is a well-investigated problem in the literature. Therefore, in section 13.2.1, we give an overview of important results from the literature about upper bounds and in section 13.2.2 about lower bounds for the Chvátal-Gomory rank.

Part VI

Part VI, which consists of chapter 14 and chapter 15, concerns the following topic (cf. section 14.1): can the inclusions in (0.4) and (0.5) also be strict? On this, we formulate the following questions:

- Does for all $m \in \mathbb{Z}_{\geq 1}$, $n \in \mathbb{Z}_{\geq 0}$ and $k \in \{1, \ldots, m\}$ exist a rational polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ for which we have:

$$\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) \subsetneq \mathrm{cl}_{L_{k-1,\mathbb{Q}}}(P)?$$

- Does for all $m \in \mathbb{Z}_{\geq 1}$, $n \in \mathbb{Z}_{\geq 0}$ and $k \in \begin{cases} \{1, \ldots, m-1\} & \text{if } n = 0, \\ \{1, \ldots, m\} & \text{if } n \geq 1 \end{cases}$ exist a rational polyhedron $P \subseteq$

$\mathbb{R}^m \times \mathbb{R}^n$ for which we have:

$$\mathrm{cl}_{L_{k,\mathbb{Q}}}(P) \subsetneq \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P)?$$

We consider the following even stronger questions:

- Does for all $m \in \mathbb{Z}_{\geq 1}$, $n \in \mathbb{Z}_{\geq 0}$ and $k \in \{1, \ldots, m\}$ exist a rational polyhedron $P \subseteq \mathbb{R}^m$ for which we have:

$$\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) \subsetneq \mathrm{cl}_{L_{k-1,\mathbb{R}}}(P)?$$

- Does for all $m \in \mathbb{Z}_{\geq 1}$, $n \in \mathbb{Z}_{\geq 0}$ and $k \in \begin{cases} \{1, \ldots, m-1\} & \text{if } n = 0, \\ \{1, \ldots, m\} & \text{if } n \geq 1 \end{cases}$ exist a rational polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ for which we have:

$$\mathrm{cl}_{L_{k,\mathbb{Q}}}(P) \subsetneq \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{R}\times\mathbb{R}}}(P)? \tag{0.6}$$

In chapter 14, we answer all of these questions positively with the exception of the case $n \geq 1$ and $k = m$ in (0.6), which (temporarily) remains open at the end of chapter 14. The central overview concerning this can be found in section 14.9. The central theorems on this are Theorem 543 and Theorem 544. Pay attention that in Theorem 543, we more generally estimate the $(m-1)$-branch split closure and the $L_{m-1,\mathbb{Q}}$ closure against the $(m-2)$-branch split closure, the $L_{m-2,\mathbb{R}}$ closure and the $L_{(m-1)-\frac{1}{2},\mathbb{R}}$ closure.

The question about a strict inclusion in (0.6) in the case $n \geq 1$ and $k = m$, which stayed open at the end of chapter 14, is a central topic of chapter 15. Here, we show (Theorem 578) the even stronger statement that for all $m, n \in \mathbb{Z}_{\geq 1}$ and $k \in \{1, \ldots, m\}$, there exists a rational polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ such that for all $\ell \in \mathbb{Z}_{\geq 0}$, we have:

$$\mathrm{cl}_I(P) = \mathrm{cl}_{L_{k,\mathbb{Q}}}(P) \subsetneq \mathrm{cl}^{(\ell)}_{L_{k-\frac{1}{2},\mathbb{R}\times\mathbb{R}}}(P).$$

This implies that we never attain the mixed-integer hull $\mathrm{cl}_I(P)$ by applying the $L_{k-\frac{1}{2},\mathbb{R}\times\mathbb{R}}$ closure iteratively on P. The central theorem for this proof is Theorem 563. Another similar result for a polyhedron which is given in the equation form that is necessary for k row cuts (cf. Definition 154) is Theorem 564.

In section 15.5.1, we use these results to derive strict inclusion for further classes of cutting plane operators. We have written down these results in Theorem 576 and Theorem 577.

Part VII

In part VII, we conclude the thesis with a summary (chapter 16) and an outlook (chapter 17).

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1. Introduction

In this chapter, we want to formulate a gentle introduction into the central topics of this thesis.

Consider some $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m \in \mathbb{Z}_{\geq 1}$ and $n \in \mathbb{Z}_{\geq 0}$, where m denotes the number of integer variables and n the number of continuous variables). We want to find cutting planes for P, i.e. valid linear inequalities $c(\,\cdot\,) \leq c_0$ for $P \cap (\mathbb{Z}^m \times \mathbb{R}^n)$. For the definition of a cutting plane cf. Definition 120. For a short introduction concerning the importance of cutting planes, we refer to the introduction of chapter 3.

The question that forms the center of this thesis is how one can derive cutting planes for P by considering valid inequalities for either

$$P \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + V)$$

or

$$(P + V) \cap (\mathbb{Z}^m \times \mathbb{R}^n),$$

where $V \leq \mathbb{R}^m \times \mathbb{R}^n$ is a subspace of codimension $k \in \{0, \ldots, m+n\}$ with suitable conditions on its generators. The case $k = 0$ is admitted for formal reasons. In this text, we call the first construction L_k **cuts** (cf. Definition 161) and the second construction $L_{k-\frac{1}{2}}$ **cuts** (cf. Definition 179).

To make it a little bit easier for the reader to become acquainted with the central ideas of this thesis, we summarize some essential properties of these constructions, which are represented in a much more formal and thorough way in chapter 4.

For the remainder of this introduction, we assume that P is a rational polyhedron.

Let us start with the case that we have $n = 0$ (i.e. we have no continuous variables), and V is generated by vectors from $\mathbb{Q}^m$ (i.e. V is a rational subspace; cf. Definition 17). Later on in this text, if $n = 0$ and V is generated by rational vectors (V is a rational subspace), we call these cutting planes $L_{k,\mathbb{Q}}$ cuts or $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts, respectively. We define the $L_{k,\mathbb{Q}}$ **closure of** P or $L_{k-\frac{1}{2},\mathbb{Q}}$ **closure of** P, respectively, as the intersection of all $L_{k,\mathbb{Q}}$ cuts or $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts, respectively, for P, i.e.

$$\mathrm{cl}_{L_{k,\mathbb{Q}}}(P) := \bigcap_{\substack{c \in (\mathbb{R}^m)^T, c_0 \in \mathbb{R}: \\ c(\,\cdot\,) \leq c_0 \ L_{k,\mathbb{Q}} \text{ cut for } P}} P^{\leq}(c, c_0),$$

$$\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}}}(P) := \bigcap_{\substack{c \in (\mathbb{R}^m)^T, c_0 \in \mathbb{R}: \\ c(\,\cdot\,) \leq c_0 \ L_{k-\frac{1}{2},\mathbb{Q}} \text{ cut for } P}} P^{\leq}(c, c_0),$$

where $P^{\leq}(c, c_0) := \{x \in \mathbb{R}^m : cx \leq c_0\}$ (cf. Definition 45).

Remark 1. *In general (in particular, if P is not convex or not closed), we additionally add P to the intersection by which the $L_{k,\mathbb{Q}}$ closure or $L_{k-\frac{1}{2},\mathbb{Q}}$ closure, respectively, is defined (cf. Definition 165 and Definition 182).*

This yields two hierarchies of cutting plane operators/closures indexed by k, whose tightness increases with k increasing:

$$P = \mathrm{cl}_{L_{0,\mathbb{Q}}}(P) \supseteq \mathrm{cl}_{L_{1,\mathbb{Q}}}(P) \supseteq \cdots \supseteq \mathrm{cl}_{L_{m-1,\mathbb{Q}}}(P) \supseteq \mathrm{cl}_{L_{m,\mathbb{Q}}}(P) = \mathrm{cl}_I(P)$$

and

$$P \supseteq \mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{Q}}}(P) \supseteq \mathrm{cl}_{L_{2-\frac{1}{2},\mathbb{Q}}}(P) \supseteq \cdots \supseteq \mathrm{cl}_{L_{(m-1)-\frac{1}{2},\mathbb{Q}}}(P) \supseteq \mathrm{cl}_{L_{m-\frac{1}{2},\mathbb{Q}}}(P) = \mathrm{cl}_I(P),$$

where $\mathrm{cl}_I(P)$ is the integer hull of P (cf. Definition 73).

The surprising property is that these two hierarchies can be merged together into *one unified* hierarchy:

$$P = \mathrm{cl}_{L_{0,\mathbb{Q}}}(P) \supseteq \mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{Q}}}(P) \supseteq \mathrm{cl}_{L_{1,\mathbb{Q}}}(P) \supseteq \ldots \supseteq \mathrm{cl}_{L_{m-1,\mathbb{Q}}}(P) \supseteq \mathrm{cl}_{L_{m-\frac{1}{2},\mathbb{Q}}}(P) = \mathrm{cl}_{L_{m,\mathbb{Q}}}(P) = \mathrm{cl}_I(P). \quad (1.1)$$

This unified hierarchy is of high importance for this thesis: indeed, a central goal of chapter 4 is to establish the inclusions between these two hierarchies (cf. Remark 156; in particular guiding question 2 and equation (4.1)).

We now consider the situation that we allow/assume that we have some continuous variables (in other words: we have $n \geq 1$). Let us introduce three types of L_k cuts/$L_{k-\frac{1}{2}}$ cuts:

- $L_{k,\mathbb{Q}}$ **cuts**, which are L_k cuts where we demand that V is generated by vectors from $\mathbb{Q}^m \times \mathbb{R}^n$ (no typo!),
- $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ **cuts**, which are $L_{k-\frac{1}{2}}$ cuts where we demand that V is generated by vectors from $\mathbb{Q}^m \times \mathbb{Q}^n$,
- $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ **cuts**, which are $L_{k-\frac{1}{2}}$ cuts where we demand that V is generated by vectors from $\mathbb{Q}^m \times \mathbb{R}^n$.

Two remarks concerning this list:

- It might seem surprising that for $L_{k,\mathbb{Q}}$ cuts, we demand V to be generated by vectors from $\mathbb{Q}^m \times \mathbb{R}^n$ instead of using the "more natural looking generator condition" that for $L_{k,\mathbb{Q}}$ cuts (in the mixed-integer case), we demand V to be a rational subspace (i.e. V has generators from $\mathbb{Q}^m \times \mathbb{Q}^n$). We soon come back to this objection.
- One might be surprised what the motivation is, also to consider linear inequalities for $(P + V) \cap (\mathbb{Z}^m \times \mathbb{R}^n)$, where V is generated by vectors from $\mathbb{Q}^m \times \mathbb{R}^n$, but not $\mathbb{Q}^m \times \mathbb{Q}^n$. The reason is that in this situation, one can still show that $\operatorname{conv}((P + V) \cap (\mathbb{Z}^m \times \mathbb{R}^n))$ is a polyhedron (a consequence of Theorem 75).

In a completely similar way as in the pure integer case, one defines the associated cutting plane operators and again this yields (this time three) hierarchies of cutting plane operators indexed by k:

$$P = \mathrm{cl}_{L_{0,\mathbb{Q}}}(P) \supseteq \mathrm{cl}_{L_{1,\mathbb{Q}}}(P) \supseteq \mathrm{cl}_{L_{2,\mathbb{Q}}}(P) \supseteq \cdots \supseteq \mathrm{cl}_{L_{m+n,\mathbb{Q}}}(P) = \mathrm{cl}_I(P), \tag{1.2}$$

$$P \supseteq \mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) \supseteq \mathrm{cl}_{L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) \supseteq \cdots \supseteq \mathrm{cl}_{L_{m+n-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_I(P), \tag{1.3}$$

$$P \supseteq \mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P) \supseteq \mathrm{cl}_{L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P) \supseteq \cdots \supseteq \mathrm{cl}_{L_{m+n-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P) = \mathrm{cl}_I(P). \tag{1.4}$$

Concerning these formulas, we first remark that it is not hard to show (cf. Theorem 202) that for $k \in \{m, \ldots, m+n\}$, we have $\mathrm{cl}_{L_{k,\mathbb{Q}}}(P) = \mathrm{cl}_I(P)$. So, (1.2) can be simplified to

$$P = \mathrm{cl}_{L_{0,\mathbb{Q}}}(P) \supseteq \mathrm{cl}_{L_{1,\mathbb{Q}}}(P) \supseteq \cdots \supseteq \mathrm{cl}_{L_{m-1,\mathbb{Q}}}(P) \supseteq \mathrm{cl}_{L_{m,\mathbb{Q}}}(P) = \mathrm{cl}_I(P).$$

In all likelihood, the reader has already conceived that in the mixed-integer setting (recall that we assumed P to be a rational polyhedron), we can even merge these *three* hierarchies of cutting plane operators (equations (1.2), (1.3) and (1.4)) into a unified hierarchy, i.e. we have

$$\begin{aligned} P = \mathrm{cl}_{L_{0,\mathbb{Q}}}(P) &\supseteq \mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P) \supseteq \mathrm{cl}_{L_{1,\mathbb{Q}}}(P) \supseteq \ldots \supseteq \mathrm{cl}_{L_{m-1,\mathbb{Q}}}(P) \\ &\supseteq \mathrm{cl}_{L_{m-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{m-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P) \supseteq \mathrm{cl}_{L_{m,\mathbb{Q}}}(P) = \mathrm{cl}_{L_{(m+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{(m+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P) = \mathrm{cl}_I(P). \end{aligned} \tag{1.5}$$

As a small interlude, let us consider how some elements in the hierarchies of (1.1) and (1.5) can be interpreted in terms of "traditional" cutting plane operators (all under the still existent premise that P is a rational polyhedron):

- $\mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{Q}}}(P)$ in (1.1) is equal to the **Chvátal-Gomory closure of** P and every Chvátal-Gomory cut for P is an $L_{1-\frac{1}{2},\mathbb{Q}}$ cut for P. Similarly, one can show that $\mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P)$ in (1.5) is equivalent to the **projected Chvátal-Gomory closure of** P and every projected Chvátal-Gomory cut for P is an $L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cut for P. For details cf.
  - Definition 122 for the definition of Chvátal-Gomory cuts/projected Chvátal-Gomory cuts,
  - Definition 123 for the definition of the Chvátal-Gomory closure/projected Chvátal-Gomory closure and
  - Theorem 391 (also cf. section 8.1 in general) for the result that every (projected) Chvátal-Gomory cut for P is an $L_{1-\frac{1}{2},\mathbb{Q}}$ cut/ $L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cut for P and the equivalence results

    $$\mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{Q}}}(P) = \mathrm{cl}_{CG}(P)$$

    and

    $$\mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{pCG}(P).$$

- $\mathrm{cl}_{L_{1,\mathbb{Q}}}(P)$ (both in (1.1) and (1.5)) is equal to the **split closure of** P. Even more: split cuts for P are equivalent to $L_{1,\mathbb{Q}}$ cuts for P. For details cf.
  - Definition 126 for the definition of split cuts,
  - Definition 127 for the definition of the split closure and
  - Theorem 409 for the equivalence result between split cuts and $L_{1,\mathbb{Q}}$ cuts.
- $\mathrm{cl}_{L_{2,\mathbb{Q}}}(P)$ (both in (1.1) and (1.5)) is equal to the **crooked cross closure of** P and every crooked cross cut for P is an $L_{2,\mathbb{Q}}$ cut for P. For details cf.
  - Definition 147 for the definition of crooked cross cuts,
  - Definition 148 for the definition of the crooked cross closure,
  - Corollary 279 for the result that every crooked cross cut for P is an $L_{2,\mathbb{Q}}$ cut for P and
  - Theorem 462 for the equivalence result $\mathrm{cl}_{L_{2,\mathbb{Q}}}(P) = \mathrm{cl}_{CC}(P)$.

So, in the hierarchies (1.1) and (1.5), a lot of cutting plane operators occur that have been well-studied in the literature for their independent importance.

Back to the main line of this introduction: with the hierarchy of (1.5) in the back of the mind, we come back to the question of why we defined $L_{k,\mathbb{Q}}$ with respect to a vector space V that is generated by vectors from $\mathbb{Q}^m \times \mathbb{R}^m$ instead of restricting ourselves to vector spaces with generators from $\mathbb{Q}^m \times \mathbb{Q}^m$ for this definition. The central reason for this is that one can show (cf. Theorem 197) that every $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cut with respect to some vector space V that is generated by vectors from $\mathbb{Q}^m \times \mathbb{R}^m$, is an $L_{k,\mathbb{Q}}$ cut with respect to exactly the same vector space. It is clear that this relationship between $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cuts and $L_{k,\mathbb{Q}}$ cuts could not even be formulated if one only allowed to use vector spaces with generators from $\mathbb{Q}^m \times \mathbb{Q}^m$ instead of $\mathbb{Q}^m \times \mathbb{R}^m$ for defining $L_{k,\mathbb{Q}}$ cuts.

Nevertheless there exist good practical reasons why one would (in the mixed-integer setting) prefer to restrict oneself to rational vector spaces for $L_{k,\mathbb{Q}}$ cuts. The good news is that we are not only able to restrict ourselves to rational vector spaces, but even to vector spaces of the form $V = V' \times \mathbb{R}^n$, where $V' \leq \mathbb{R}^m$ is a rational subspace of codimension k (cf. Theorem 168).

It is clear that such a restriction (to vector spaces $V = V' \times \mathbb{R}^n$, where $V' \leq \mathbb{R}^m$ is a rational subspace of codimension k) is not possible for $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts or $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cuts. So, one can legitimately ask: do $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts and $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cuts with respect to such a vector space (which we name **essential** $L_{k-\frac{1}{2},\mathbb{Q}}$ **cuts**; cf. Definition 203) nevertheless have some surprising property? Indeed they have: one can show that if an $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cut or $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cut is not already an $L_{k-1,\mathbb{Q}}$ cut, it has to be an essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cut (cf. Theorem 208; also cf. Theorem 211). In other words: the only cutting planes which in (1.5) make $\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P)$ potentially stronger than $\mathrm{cl}_{k-1,\mathbb{Q}}(P)$ ($k \in \{1, \ldots, m\}$) are essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts.

Now for a second important consideration: how can we compute $L_{k,\mathbb{Q}}$ cuts and $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts with respect to some given rational subspace $V \leq \mathbb{R}^m \times \mathbb{R}^n$ (of codimension k) explicitly? For this discussion, let us restrict ourselves to the pure integer case ($n = 0$). A naive approach for finding $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts for a given rational polyhedron $P \subseteq \mathbb{R}^m$ ($m \in \mathbb{Z}_{\geq 0}$) with respect to a given (rational) subspace $V \leq \mathbb{R}^m$ is:

1. Compute an explicit representation of $P' := \mathrm{proj}^{\perp}_{V^\perp} P$.
2. Compute a lattice basis of $\Lambda := \mathrm{proj}^{\perp}_{V^\perp} \mathbb{Z}^m$.
3. Compute (ideally facet-defining) inequalities $c(\,\cdot\,) \leq c_0$ ($c \in (\mathbb{R}^m)^T$ and $c_0 \in \mathbb{R}$) for $\mathrm{conv}(P' \cap \Lambda)$ such that $c^T \perp V$ (the latter condition can be assumed because of Lemma 159).

The problem with this approach is that for $k \geq 2$, the number of facets of P' can be enormous: for every $d \in \mathbb{Z}_{\geq 2}$, there exists a full-dimensional rational polyhedron $P \subseteq \mathbb{R}^d$ and a rational subspace V of codimension 2 such $\mathrm{proj}^{\perp}_{V^\perp} P$ has 2^d vertices and thus (equivalently in $\mathbb{R}^2$ for full-dimensional polytopes) 2^d facets. See section 5.1 for details regarding this.

Thus, the outlined approach is arguably not feasible in practice and therefore we want to look for different characterizations of $L_{k,\mathbb{Q}}$ cuts, $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts and essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts. For this, one can easily picture two natural approaches:

- Show alternative characterizations that hold for general k. This is the topic of chapter 5.

- Show alternative characterizations for specific (typically small) values of k. This is done in part III for $L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts (see chapter 8, in particular section 8.1) and $L_{1,\mathbb{Q}}$ cuts (see chapter 9, in particular section 9.1.1) and in part IV (in particular chapter 11) for $L_{2,\mathbb{Q}}$ cuts (see section 11.1) and essential $L_{2-\frac{1}{2},\mathbb{Q}}$ cuts (see section 11.2).

To give this introducion a well-rounded conclusion, let us mention the following problem: in (1.1) and (1.5), we saw that the different kinds of closure operators applied to rational polyehdra form a hierarchy with respect to inclusions. Can these inclusions also be *strict*?

Proving that all of these inclusions can indeed be strict is the central topic of part VI (chapter 14 and chapter 15). In chapter 15, we even show that for every $m, n \in \mathbb{Z}_{\geq 1}$ and $k \in \{1, \ldots, m\}$, there exists a rational polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ such that for every $\ell \in \mathbb{Z}_{\geq 0}$, we have

$$\mathrm{cl}_I(P) = \mathrm{cl}_{L_{k,\mathbb{Q}}}(P) \subsetneq \mathrm{cl}^{(\ell)}_{L_{k-\frac{1}{2},\mathbb{R}\times\mathbb{R}}}(P),$$

which implies that we can never attain the mixed-integer hull $\mathrm{cl}_I(P)$ by applying the $L_{k-\frac{1}{2},\mathbb{R}\times\mathbb{R}}$ closure (which is even potentially stronger than the $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ closure and $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ closure) iteratively.

Part I.

Basics

2. Basics

This introductory chapter establishes basic definitions and results. In the subsequent chapter of this part (chapter 3), we then begin to introduce cutting planes, which are the central topic of this dissertation.

2.1. Basic definitions

Definition 2. *For $n \in \mathbb{Z}_{\geq 0}$, we define $[n] := \{1, \ldots, n\}$. For $n \in \mathbb{Z}_{\geq 1}$ and $a, b \in [n]$, we set $a +_n b$ to be the addition modulo n, where the remainder lies in $\{1, \ldots, n\}$ instead of $\{0, \ldots, n\}$. Similarly, we define $a -_n b$ as subtraction modolo n with modulus in $[n]$.*

Definition 3. *For some aritrary set S, let $\mathcal{P}(S)$ denote its power set (set of all subsets of S).*

Definition 4. *Let $f : A \to B$, $g : B' \to C$ be two maps, where $B \subseteq B'$. Then we define*

$$\begin{aligned} g \circ f : A \to C : \\ a \mapsto g(f(a)) \end{aligned}$$

as their composition.

2.2. Algebra

We now define some basic algebraic objects.

2.2.1. Groups

Definition 5. *A **group** is a tuple $(G, *, 1_G)$, where G is a set, $*$ is a map $G \times G \to G$ and $1_G \in G$ such that*

1. *$\forall g_1, g_2, g_3 \in G : g_1 * (g_2 * g_3) = (g_1 * g_2) * g_3$ (associativity),*
2. *$\forall g \in G : 1_G * g = g$ (1_G is left identity element),*
3. *$\forall g \in G \exists g^{-1} \in G : g^{-1} * g = 1_G$ (existence of left inverse element).*

Remark 6. *It is well-known (cf. for example [Fis14, paragraph 1.2.3], [Bos14, chapter 1, Bemerkung 2] and [Bos13, chapter 1, Bemerkung 2]) that in a group $(G, *, 1_G)$:*

- *1_G is also a right identity element in G.*
- *1_G is the* only *left/right identity element in G.*
- *Any left inverse element is also a right inverse element.*
- *The inverse element is uniquely determined.*

Definition 7. *A group $(G, *, 1_G)$ is called **commutative** or **abelian** if $\forall g_1, g_2 \in G : g_1 * g_2 = g_2 * g_1$.*

Since we use group homomorphisms to define fields (cf. Definition 9) and vector spaces (cf. Definition 12), we next define this concept:

Definition 8. *Let $(G, *, 1_G)$ and $(H, \bullet, 1_H)$ be groups. A **group homomorphism** is a map $\varphi : G \to H$ such that*

$$\forall g_1, g_2 \in G : \varphi(g_1 * g_2) = \varphi(g_1) \bullet \varphi(g_2).$$

*If $(G, *, 1_G) = (H, \bullet, 1_H)$, we call φ a **group endomorphism**.*

2.2.2. Fields

Definition 9. *A tuple $(F, +, \cdot, 0_F, 1_F)$ is called a **field** if*

1. *$(F, +, 0_F)$ is an abelian group,*

2. *$(F \setminus \{0_F\}, \cdot, 1_F)$ is an abelian group,*

3. *for all $f \in F$, the map $x \mapsto f \cdot x$ is an endomorphism of the group $(F, +, 0_F)$ (this property is usually called **left distributivity**).*

Definition 10. *Let p be a prime number. Then $(\{0, \ldots, p-1\}, +, \cdot)$, where $+$ and $\cdot$ are addition and multiplication modulo p, forms a field, which we denote by $\mathbb{F}_p$.*

For this thesis, mostly the fields $\mathbb{Q}$, $\mathbb{R}$ and (in particular in section 13.1.2) $\mathbb{F}_2$ are important. Because Theorem 83 and Theorem 87, which we took from the literature, are formulated over skew fields, we next define skew fields. They only differ from fields in the property that multiplication is not necessarily commutative.

Definition 11. *A tuple $(F, +, \cdot, 0_F, 1_F)$ is called a **skew field** if*

1. *$(F, +, 0_F)$ is an abelian group,*

2. *$(F \setminus \{0_F\}, \cdot, 1_F)$ is a group,*

3. *for all $f \in F$, the maps $x \mapsto f \cdot x$ and $x \mapsto x \cdot f$ are endomorphisms of the group $(F, +, 0)$.*

2.2.3. Vector spaces

Definition 12. *Let F be a (skew) field. A **vector space over** F is a tuple $(V, +, \cdot, 0_V)$ $(\cdot : F \times V \to V)$ such that*

1. *$(V, +, 0_V)$ is an abelian group,*

2. *for all $f \in F$, the map $v \mapsto f \cdot v$ is an endomorphism of the group $(V, +, 0_V)$,*

3. *for all $v \in V$, the map $a \mapsto a \cdot v$ is a homomorphism of the group $(F, +, 0_F)$ to the group $(V, +, 0_V)$,*

4. *$\forall a, b \in F, v \in V : a \cdot (b \cdot v) = (a \cdot b) \cdot v$,*

5. *for all $v \in V$, we have $1_F \cdot v = v$.*

We remark that in this text, all vector spaces are over a non-skew field (with the exception of Theorem 83 and Theorem 87). The most important vector spaces for this text are $\mathbb{Q}^d$, $\mathbb{R}^d$ and (in particular in section 13.1.2) $\mathbb{F}_2^d$, where $d \in \mathbb{Z}_{\geq 0}$. We now define some important vectors in $\mathbb{R}^d$:

Definition 13. *Let $d \in \mathbb{Z}_{\geq 0}$. We set*

- *0^d as the zero vector of $\mathbb{R}^d$,*
- *$e^{d,k}$ $(k \in [d])$ as the dth unit vector of $\mathbb{R}^n$. Additionally, we formally define: $e^{d,0} := 0^d$,*
- *$1^d := \sum_{i=1}^{n} e^{n,k}$ as the vector of $\mathbb{R}^d$ which only has ones as components.*

Definition 14. *Let V be a vector space over some field F and let $S \subseteq V$. Then we say S is **linearly independent** if*

$$\forall S' \subseteq S, S' \text{ finite} : \left(\sum_{s' \in S'} \lambda_{s'} s' = 0_V \Rightarrow \forall s' \in S' : \lambda_{s'} = 0_F \right).$$

Definition 15. *Let V be a vector space over $\mathbb{R}$ and let $V' \subseteq V$. S is called a **(linear) subspace** if*

1. *$\mathbb{R} \cdot V' \subseteq V'$,*

2. *$V' + V' \subseteq S$.*

We write $V' \leq V$ in this case.

Definition 16. *Let V be a vector space over $\mathbb{R}$ and let $S \subseteq V$. We set*

$$\operatorname{lin} S := \bigcup_{k \in \mathbb{Z}_{\geq 0}} \bigcup_{\substack{\lambda \in \mathbb{R}^k, \\ s^1, \ldots, s^k \in S}} \left\{ \sum_{i=1}^{k} \lambda_i s^i \right\} = \bigcap_{\substack{S \subseteq S' \subseteq V: \\ S' \text{ subspace of } V}} S'$$

as the ***linear hull of*** *S. If $\operatorname{lin} S = V$, we say S* ***spans*** *V or S* ***generates*** *V.*

As one can see, in this text, we consider many concepts concerning vector spaces (linear hull, span, basis, ...) only for vector spaces over $\mathbb{R}$. We nevertheless remark that it is common in the literature to define such concepts for vector spaces over arbitrary fields, but we do not need this generality for this text.

Definition 17. *Let $V \leq \mathbb{R}^d$ ($d \in \mathbb{Z}_{\geq 0}$). We call V a* ***rational subspace*** *if it is generated by vectors from $\mathbb{Q}^d$.*

Definition 18. *Let V be a vector space over $\mathbb{R}$. A linearly independent set B having $\operatorname{lin} B = V$ is called a* ***basis of*** *V.*

Definition 19. *Let V be a vector space over $\mathbb{R}$ with a basis B having $|B| \in \mathbb{Z}_{\geq 0}$. In this case, we call V* ***finite-dimensional*** *and we define $\dim V := |B|$.*

Definition 20. *Let V and W be vector spaces over some field F.*

- *A* ***linear map*** *(also called* ***homomorphism of vector spaces****) is a map $\varphi : V \to W$ such that*
  1. *$\forall v, v' \in V : \varphi(v + v') = \varphi(v) + \varphi(v')$,*
  2. *$\forall \lambda \in F, v \in V : \varphi(\lambda v) = \lambda \varphi(v)$.*
- *A map $\varphi : V \to W$ of the form $\varphi(x) = \varphi'(x) + w'$, where $\varphi' : V \to W$ is a linear map and $w' \in W$, is called an* ***affine-linear map****.*

Remark/Definition 21. *Let V and W be vector spaces over some field F. Then*

$$\{\varphi : V \to W : \varphi \text{ homomorphism of vector spaces over } F\}$$

is again a vector space with the natural operations. Of particular interest for this text is

$$V^* := \{\varphi : V \to F : \varphi \text{ homomorphism of vector spaces (over } F)\},$$

which we name the ***dual space of*** *V.*

If $\varphi : \mathbb{R}^q \to \mathbb{R}^p$ is a linear map, it it well-known that φ is of the form $x \mapsto Ax$, where $A \in \mathbb{R}^{p \times q}$ and if $\varphi : \mathbb{R}^q \to \mathbb{R}^p$ is an affine-linear map, it is of the form $x \mapsto Ax + b$, where $A \in \mathbb{R}^{p \times q}$ and $b \in \mathbb{R}^p$.

We now define some properties of matrices:

Definition 22. *For $A \in \mathbb{R}^{p \times q}$ ($p, q \in \mathbb{Z}_{\geq 0}$), we define*

$$\operatorname{rowspan} A := \operatorname{lin} \{A_{1,*}, \ldots, A_{p,*}\} \leq (\mathbb{R}^q)^T.$$

Definition 23. *We define for $d \in \mathbb{Z}_{\geq 0}$:*

- *$I^d \in \mathbb{R}^{d \times d}$ as the identity matrix.*
- *A (square) matrix $A \in \mathbb{R}^{d \times d}$ is called* ***regular*** *if $\det A \neq 0$.*
- *A (square) matrix $A \in \mathbb{Z}^{d \times d}$ is called* ***unimodular*** *if $|\det A| = 1$.*
- *A (square) matrix $A \in \mathbb{R}^{d \times d}$ is called* ***symmetric*** *if $A = A^T$.*
- *Let $A \in \mathbb{R}^{d \times d}$ be a regular matrix. Then we define $A^{-T} := \left(A^{-1}\right)^T = \left(A^T\right)^{-1}$.*
- *A symmetric matrix $A \in \mathbb{R}^{d \times d}$ is called* ***positive semidefinite*** *if for all $x \in \mathbb{R}^d \setminus \left\{0^d\right\}$, we have $x^T A x \geq 0$.*
- *A symmetric matrix $A \in \mathbb{R}^{d \times d}$ is called* ***positive definite*** *if for all $x \in \mathbb{R}^d \setminus \left\{0^d\right\}$, we have $x^T A x > 0$.*
- *A map $f : \mathbb{R}^d \to \mathbb{R}^d$ is called* ***affine-unimodular*** *if f is of the form $f : x \mapsto Ux + v$, where U is a unimodular matrix and $v \in \mathbb{Z}^d$.*

Definition 24. *For $A \in \mathbb{R}^{m \times n}$ ($m, n \in \mathbb{Z}_{\geq 0}$), we define $\operatorname{im} A := \operatorname{lin} \{A_{*,1}, \ldots, A_{*,n}\}$. Clearly, $\operatorname{im} A \leq \mathbb{R}^m$.*

2.3. Concepts related to vector spaces

2.3.1. Orthogonal complements and projections

We now define two types of projections for vectors.

Definition 25. *Let $v := \binom{v'}{v''} \in \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$. We define*

$$\operatorname{proj}_{\mathbb{R}^m} v := v',$$
$$\operatorname{proj}_{\mathbb{R}^n} v := v''.$$

Clearly, Definition 25 leads to problems if $m = n$, since in this situation, it is not clear whether for $v = \binom{v'}{v''} \in \mathbb{R}^m \times \mathbb{R}^m$, $\operatorname{proj}_{\mathbb{R}^m} v$ is a projection to the first or second m components. To avoid having to make Definition 25 more complicated, we trust the reader to be able to infer this information from context.

The second kind of projection that we want to introduce is the **orthogonal projection**. For this, we start with the definition of the orthogonal complement:

Definition 26. *Let $V \leq \mathbb{R}^d$ $(d \in \mathbb{Z}_{\geq 0})$. Then we define*

$$V^{\perp} := \left\{x \in \mathbb{R}^d : x^T v = 0 \ \forall v \in V\right\}$$

as the ***orthogonal complement of*** *V. Clearly, $V^{\perp}$ is rational if V is.*

Let $W \leq V$. Then we set

$$W^{\perp_V} := \left\{x \in V : x^T v = 0 \ \forall v \in W\right\}$$

as the ***orthogonal complement of*** *W* ***with respect to*** *V. Clearly, $W^{\perp_V}$ is rational if V and W are.*

Now for the orthogonal projection:

Definition 27. *Let $V \leq \mathbb{R}^d$ $(d \in \mathbb{Z}_{\geq 0})$. We define*

$$\operatorname{proj}_V^{\perp} : \mathbb{R}^d \to V$$

by the property

$$\forall v \in V : \operatorname{proj}_V^{\perp} v = v,$$
$$\forall v \in V^{\perp} : \operatorname{proj}_V^{\perp} v = 0_V = 0^d.$$

$\operatorname{proj}_V^{\perp}$ is called the ***orthogonal projection to*** *V.*

It is clear that such a map is uniquely defined if it exists. We now show that such a map really exists and how it looks like. For this, we start with a small lemma:

Lemma 28. *Let $B \in \mathbb{R}^{d \times k}$ $(d \in \mathbb{Z}_{\geq 0}$ and $k \in \{0, \ldots, d\})$ have linearly independent columns. Then $B^T B$ is positive definite and in particular invertible.*

Proof. Clearly, $B^T B$ is positive semidefinite. For the invertibility of $B^T B$, it suffices to show that the map

$$\mathbb{R}^k \to \mathbb{R}^k : x \mapsto B^T B x$$

is injective. For this, let $B^T B x = 0^k$. This implies

$$0 = x^T B^T B x = (Bx)^T (Bx) = \|Bx\|^2 .$$

Thus, $Bx = 0^d$. Since the columns of B are linearly independent, we have $x = 0^k$. So, $B^T B$ does not have 0 as eigenvalue and is thus positive definite. ☐

Lemma 29. *Let $B \in \mathbb{R}^{d \times k}$ $(d \in \mathbb{Z}_{\geq 0}$, $k \in \{0, \ldots, d\})$ be a matrix with linearly independent columns and let $V := \operatorname{im} B$. Then we have for all $x \in \mathbb{R}^d$:*

$$\operatorname{proj}_V^{\perp} x = B \left(B^T B\right)^{-1} B^T x.$$

Proof. By Lemma 28, $\left(B^T B\right)^{-1}$ exists. Let $x \in V$. Then $x = B\lambda$, where $\lambda \in \mathbb{R}^k$, and we have:

$$\left(B\left(B^T B\right)^{-1} B^T\right) x = B\left(B^T B\right)^{-1} B^T B\lambda = B\lambda = x.$$

On the other hand, let $x \in V^\perp$. Then $B^T x = 0^k$ and thus $\left(B\left(B^T B\right)^{-1} B^T\right) x = 0^m$. □

As an outlook, we want to mention that in Definition 428, we formulate a third type of projection for vectors, which we need to formulate Theorem 429.

2.3.2. Sums

The following lemma is well-known:

Lemma 30. *Let W_1, W_2 be finite-dimensional subspaces of a vector space V. Then*

$$\dim\left(W_1 + W_2\right) = \dim W_1 + \dim W_2 - \dim\left(W_1 \cap W_2\right).$$

Lemma 31. *Let $V_1, V_2 \leq \mathbb{R}^d$ ($d \in \mathbb{Z}_{\geq 0}$) be subspaces. Then*

$$\left(V_1 + V_2\right)^\perp = V_1^\perp \cap V_2^\perp, \tag{2.1}$$

Proof. Clearly, if $w \in \left(V_1 + V_2\right)^\perp$, we have $w \in V_1^\perp \cap V_2^\perp$ (since $0^d \in V_1, V_2$). On the other hand, let $w \in V_1^\perp \cap V_2^\perp$. This means that for all $v^1 \in V^1, v^2 \in V^2$, we have $w^T v^1 = 0$ and $w^T v^2 = 0$. Thus, $w^T\left(v^1 + v^2\right) = 0$. □

Remark 32. *Beside the identity* (2.1) *that we showed in Lemma 31, other important identities concerning orthogonal complements and sums of subspaces are*

$$\left(V_1^\perp\right)^\perp = V_1,$$
$$\left(V_1 \cap V_2\right)^\perp = V_1^\perp + V_2^\perp,$$

where, as in Lemma 31, $V_1, V_2 \leq \mathbb{R}^d$ ($d \in \mathbb{Z}_{\geq 0}$).

Definition 33. *Let V be a vector space and let $W_1, \ldots, W_k$ ($k \in \mathbb{Z}_{\geq 0}$) be subspaces of V. We write*

$$V = W_1 \oplus W_2 \oplus \ldots \oplus W_k$$

if for all $v \in V$, there exist uniquely defined $w^1 \in W_1, \ldots, w^k \in W_k$ such that $v = \sum_{i=1}^k w^i$. If, additionally, $W_1, \ldots, W_k$ are pairwise orthogonal to each other, we write

$$V = W_1 \overset{\perp}{\oplus} W_2 \overset{\perp}{\oplus} \ldots \overset{\perp}{\oplus} W_k.$$

Lemma 34. *Let $V_1, V_2 \leq \mathbb{R}^d$ ($d \in \mathbb{Z}_{\geq 0}$) be subspaces such that*

$$\mathbb{R}^d = V_1 \oplus V_2.$$

Then also

$$\mathbb{R}^d = V_1^\perp \oplus V_2^\perp.$$

Proof. We have to show that for all $x \in \mathbb{R}^d$, there exist unique $w^1 \in V_1^\perp$ and $w^2 \in V_2^\perp$ such that $x = w^1 + w^2$.

For uniqueness: Assume that there exist $x \in \mathbb{R}^d$, $w'^1, w''^1 \in V_1^\perp$ and $w'^2, w''^2 \in V_2^\perp$ having

$$x = w'^1 + w'^2 = w''^1 + w''^2.$$

This implies that there exist $w^1 := w'^1 - w''^1 \in V_1^\perp$ and $w^2 := w'^2 - w''^2 \in V_2^\perp$ such that $0^d = w^1 + w^2$. But this implies

$$\forall v^1 \in V_1 : \left(w^1\right)^T v^1 = 0,$$
$$\forall v^2 \in V_2 : \left(w^2\right)^T v^2 = 0.$$

On the other hand, since $w^1 = -w^2$, we also have

$$\forall v^1 \in V_1 : \left(w^2\right)^T v^1 = 0,$$
$$\forall v^2 \in V_2 : \left(w^1\right)^T v^2 = 0.$$

Thus, $w^1, w^2 \in V_1^\perp \cap V_2^\perp$. But, by Lemma 31, this is equivalent to $w^1, w^2 \in (V_1 + V_2)^\perp$. We have $V_1 + V_2 = \mathbb{R}^d$; thus,

$$(V_1 + V_2)^\perp = \left\{0^d\right\}, \tag{2.2}$$

which implies $w^1 = w^2 = 0^d$, so $w'^1 = w''^1$ and $w'^2 = w''^2$.

For existence: We have

$$\begin{aligned}
\dim\left(V_1^\perp + V_2^\perp\right) &= \dim V_1^\perp + \dim V_2^\perp - \dim\left(V_1^\perp \cap V_2^\perp\right) && \text{(by Lemma 30)}\\
&= \dim V_1^\perp + \dim V_2^\perp - (V_1 + V_2)^\perp && \text{(by Lemma 31)}\\
&= \dim V_1^\perp + \dim V_2^\perp && \text{(by (2.2))}\\
&= d - \dim V_1 + d - \dim V_2 &&\\
&= d. && (\mathbb{R}^d = V_1 \oplus V_2)
\end{aligned}$$

□

2.3.3. Norms and topological properties

Definition 35. *Let V be a vector space over $\mathbb{R}$. A **norm on** V is a map $\|\cdot\| : V \to \mathbb{R}$ such that*

1. $\forall v \in V : \|v\| = 0 \Leftrightarrow v = 0_V$,
2. $\forall \lambda \in \mathbb{R}, v \in V : \|\lambda \cdot v\| = |\lambda| \cdot \|v\|$,
3. $\forall v, v' \in V : \|v + v'\| \leq \|v\| + \|v'\|$

holds.

Important norms on $\mathbb{R}^d$ are $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$. For $x \in \mathbb{R}^d$, these are defined as

$$\|x\|_1 := \sum_{i\in[d]} |x_i|, \qquad \|x\|_2 := \sqrt{\sum_{i\in[d]} (x_i)^2}, \qquad \|x\|_\infty := \max\{|x_i| : i \in [d]\}.$$

If we use no index for a norm on $\mathbb{R}^d$ (i.e. $\|\cdot\|$), we always mean $\|\cdot\|_2$.
Next, we define open balls, closed balls and spheres:

Definition 36. *For $d \in \mathbb{Z}_{\geq 0}$, $p \in \mathbb{R}^d$ and $\epsilon \in \mathbb{R}_{>0}$ we define*

- $B_\epsilon(p) := \left\{x \in \mathbb{R}^d : \|x - p\| < \epsilon\right\}$ *as the **open ball around** p **with radius** ϵ,*
- $\overline{B}_\epsilon(p) := \left\{x \in \mathbb{R}^d : \|x - p\| \leq \epsilon\right\}$ *as the **closed ball around** p **with radius** ϵ,*
- $\mathbb{S}^d := \left\{x \in \mathbb{R}^{d+1} : \sum_{i=1}^{d+1} x_i^2 = 1\right\}$ *as the d-**sphere**.*

Having defined open balls enables us to to define open and closed sets and the (topological) closure:

Definition 37. *Let $S \subseteq \mathbb{R}^d$ ($d \in \mathbb{Z}_{\geq 0}$). S is called **open** if for every $s \in S$, there exists an $\epsilon \in \mathbb{R}_{>0}$ such that $B_\epsilon(s) \subseteq S$.*

Definition 38. *Let $S \subseteq \mathbb{R}^d$ ($d \in \mathbb{Z}_{\geq 0}$). S is called **closed** if $\mathbb{R}^d \backslash S$ is open.*

Definition 39. *Let $S \subseteq \mathbb{R}^d$ ($d \in \mathbb{Z}_{\geq 0}$). Then we define*

$$\overline{S} := \bigcap_{\substack{S \subseteq S' \subseteq \mathbb{R}^d: \\ S' \text{ closed}}} S'$$

*as the **(topological) closure of** S.*

2.4. Cones and convexity

For the remainder of this section, let V be a vector space over $\mathbb{R}$.

2.4.1. Cones and polyhedral cones

Definition 40. *Let $S \subseteq V$. S is called a* ***(convex) cone*** *if for all $\lambda \in \mathbb{R}^2_{\geq 0}$, we have $\lambda_1 S + \lambda_2 S \subseteq S$.*

Definition 41. *Let $S \subseteq V$. We define*

$$\operatorname{cone} S := \bigcup_{k \in \mathbb{Z}_{\geq 0}} \bigcup_{\substack{\lambda \in \mathbb{R}^k_{\geq 0}, \\ s^1,\ldots,s^k \in S}} \left\{ \sum_{i=1}^{k} \lambda_i s^i \right\} = \bigcap_{\substack{S \subseteq S' \subseteq V: \\ S' \text{ cone}}} S'$$

as the ***conic hull of*** *S.*

We now ask: what are the "simplest" cones? One natural criterion for "simplest" is that $C = \operatorname{cone} S$, where $S \subseteq \mathbb{R}^d$ ($d \in \mathbb{Z}_{\geq 0}$) is a *finite* set. Such cones are called **polyhedral cones**:

Definition 42. *Let $C \subseteq \mathbb{R}^d$ ($d \in \mathbb{Z}_{\geq 0}$). We call C a* ***polyhedral cone*** *if there exist $r^1, \ldots, r^k \in \mathbb{R}^d$ ($k \in \mathbb{Z}_{\geq 0}$) such that $C = \operatorname{cone}\left\{r^1, \ldots, r^k\right\}$.*

Often the sitation occurs that one has a matrix be given and wants to consider the cone that is generated by the rows of the matrix. For this, we define similarly to Definition 22:

Definition 43. *For $A \in \mathbb{R}^{p \times q}$ ($p, q \in \mathbb{Z}_{\geq 0}$), we define*

$$\operatorname{rowcone} A := \operatorname{cone}\left\{A_{1,*}, \ldots, A_{p,*}\right\} \subseteq \left(\mathbb{R}^q\right)^T.$$

Definition 44. *We define:*

- *Let $C \subseteq \mathbb{R}^d$ ($d \in \mathbb{Z}_{\geq 0}$) be a (polyhedral) cone. C is called a* ***rational cone*** *if $C = \operatorname{cone}\left\{v^1, \ldots, v^k\right\}$, where $v^i \in \mathbb{Q}^d$ for all $i \in [k]$ ($k \in \mathbb{Z}_{\geq 0}$).*
- *Let $C \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be a (polyhedral) cone. C is called a* ***partially rational cone*** *if $C = \operatorname{cone}\left\{v^1, \ldots, v^k\right\}$, where $v^i \in \mathbb{Q}^m \times \mathbb{R}^n$ for all $i \in [k]$ ($k \in \mathbb{Z}_{\geq 0}$).*

"Surprisingly", it turns out that $C \subseteq \mathbb{R}^d$ being a (rational) polyhedral cone is equivalent to C being an intersection of a finite number of (rational) half-spaces containing 0^d. This is the statement of the Minkowski-Weyl theorem for cones (Theorem 46), for which one can, for example, find a proof in [CCZ10, Theorem 11.9] (also cf. [CCZ10, Remark 11.2]). To formulate this theorem, we define:

Definition 45. *For $A \in \mathbb{R}^{p \times q}$ and $b \in \mathbb{R}^p$ ($p, q \in \mathbb{Z}_{\geq 0}$), we define*

$$\begin{aligned} P^{=}(A, b) &:= \left\{x \in \mathbb{R}^q : Ax = b\right\}, \\ P^{\leq}(A, b) &:= \left\{x \in \mathbb{R}^q : Ax \leq b\right\}, \\ P^{<}(A, b) &:= \left\{x \in \mathbb{R}^q : Ax < b\right\}. \end{aligned}$$

Now for the Minkowski-Weyl theorem for cones:

Theorem 46. *(Minkowski-Weyl theorem for cones; [CCZ10, Theorem 11.9], also cf. [CCZ10, Remark 11.2]) Let $C \subseteq \mathbb{R}^d$ ($d \in \mathbb{Z}_{\geq 0}$). Then the following two statements are equivalent:*

1. *C is a (rational) polyhedral cone (i.e. there exist $R \in \mathbb{Q}^{d \times k}$ or $R \in \mathbb{R}^{d \times k}$ ($k \in \mathbb{Z}_{\geq 0}$), respectively, such that $C = R\mathbb{R}^k_{\geq 0}$).*
2. *There exists an $A \in \mathbb{Q}^{l \times d}$ or $A \in \mathbb{R}^{l \times d}$ ($l \in \mathbb{Z}_{\geq 0}$), respectively, such that $C = P^{\leq}\left(A, 0^l\right)$.*

2.4.2. Convexity and polyhedra

2.4.2.1. Definitions

We now want to define an "affine analogue of cones/the conic hull". For this, we start by defining d-simplices:

Definition 47. *For $d \in \mathbb{Z}_{\geq 0}$ we define $\Delta^d := \left\{x \in \mathbb{R}^{d+1}_{\geq 0} : \left(1^{n+1}\right)^T x = 1\right\}$ as the d-**simplex**.*

We now can define

- convex sets as "the affine analogue of cones" by simply replacing $\mathbb{R}^2_{\geq 0}$ by Δ^1 in Definition 40. This is done in Definition 48.
- the convex hull as "the affine analogue of cones/the conic hull" by simply replacing $\mathbb{R}^k_{\geq 0}$ by Δ^k in Definition 41. This is done in Definition 49.

Definition 48. *Let $S \subseteq V$. S is called **convex** if for all $\lambda \in \Delta^1$, we have $\lambda_1 S + \lambda_2 S \subseteq S$.*

Definition 49. *Let $S \subseteq V$. We define*

$$\operatorname{conv} S := \bigcup_{k \in \mathbb{Z}_{\geq 0}} \bigcup_{\substack{\lambda \in \Delta^k, \\ s^1, \ldots, s^{k+1} \in S}} \left\{\sum_{i=1}^{k+1} \lambda_i s^i\right\} = \bigcap_{\substack{S \subseteq S' \subseteq V: \\ S' \text{ convex}}} S'$$

*as the **convex hull of** S.*

Again we want to describe the "simplest" convex sets – in this case sets $P \subseteq \mathbb{R}^d$ such that $P = \operatorname{conv} S$, where $S \subseteq \mathbb{R}^d$ ($d \in \mathbb{Z}_{\geq 0}$) is a *finite* set. Such convex sets are called **polytopes**:

Definition 50. *Let $P \subseteq \mathbb{R}^d$ ($d \in \mathbb{Z}_{\geq 0}$). We call P a **polytope** if there exist $v^1, \ldots, v^k \in \mathbb{R}^d$ ($k \in \mathbb{Z}_{\geq 0}$) such that $P = \operatorname{conv}\left\{v^1, \ldots, v^k\right\}$.*

Unluckily, Theorem 46 cannot one-to-one be generalized from convex cones to polytopes: while every polytope is an intersection of a finite number of affine halfspces, not every finite intersection of affine half-spaces is a polytope. The arguably simplest counterexample is

$$P = P^{\leq}\left(\begin{pmatrix} 1 \end{pmatrix}, \begin{pmatrix} 0 \end{pmatrix}\right) \subsetneq \mathbb{R}^1,$$

which is surely not bounded, thus not a polytope. One thus introduces another term (polyhedra) for intersections of a finite number of half-spaces:

Definition 51. *A set $P := P^{\leq}(A, b) \subseteq \mathbb{R}^d$ ($d \in \mathbb{Z}_{\geq 0}$), where $A \in \mathbb{R}^{k \times d}$, $b \in \mathbb{R}^k$ ($k \in \mathbb{Z}_{\geq 0}$), is called a **polyhedron**.*

Let us define some specific classes of polyhedra that are important for this thesis:

Definition 52. *We define for $d \in \mathbb{Z}_{\geq 0}$:*

- *A set $P \subseteq \mathbb{R}^d$ is called a **half-space** if it can be represented in the form $P = P^{\leq}(A, b)$, where $A \in \mathbb{R}^{1 \times d} \setminus \left\{0^{1 \times d}\right\}$ and $b \in \mathbb{R}^1$.*
- *A set $P \subseteq \mathbb{R}^d$ is called a **hyperplane** if it can be represented in the form $P = P^{=}(A, b)$, where $A \in \mathbb{R}^{1 \times d} \setminus \left\{0^{1 \times d}\right\}$ and $b \in \mathbb{R}^1$.*
- *A set $P \subseteq \mathbb{R}^d$ is called a **polyhedron with rational face normals** if it can be represented in the form $P = P^{\leq}(A, b)$, where $A \in \mathbb{Q}^{k \times d}$ and $b \in \mathbb{R}^k$ ($k \in \mathbb{Z}_{\geq 0}$).*
- *A set $P \subseteq \mathbb{R}^d$ is called a **rational polyhedron** if it can be represented in the form $P = P^{\leq}(A, b)$, where $A \in \mathbb{Q}^{k \times d}$ and $b \in \mathbb{Q}^k$ ($k \in \mathbb{Z}_{\geq 0}$).*

While we have seen that there is no one-to-one correspondence between polyhedra and convex hulls of a finite number of points (polytopes), there is such a correspondence between (rational) polyhedra and Minkowski sums of (rational) polytopes and (rational) polyhedral cones. So, we can formulate a Minkowski-Weyl theorem for polyhedra, which is, for example, shown in [CCZ10, Theorem 11.10].

Theorem 53. *(Minkowski-Weyl theorem for polyhedra) Let $P \subseteq \mathbb{R}^d$ ($d \in \mathbb{Z}_{\geq 0}$). Then the following two statements are equivalent:*

1. *There exist $v^1, \ldots, v^{k_1} \in \mathbb{Q}^d$ or $\mathbb{R}^d$, respectively, and $r^1, \ldots, r^{k_2} \in \mathbb{Q}^d$ or $\mathbb{R}^d$ ($k_1, k_2 \in \mathbb{Z}_{\geq 0}$), respectively, such that $P = \operatorname{conv}\left\{v^1, \ldots, v^{k_1}\right\} + \operatorname{cone}\left\{r^1, \ldots, r^{k_2}\right\}$.*

2. *P is a (rational) polyhedron (i.e. there exist $A \in \mathbb{Q}^{l\times d}$ or $\mathbb{R}^{l\times d}$, respectively, and $b \in \mathbb{Q}^l$ or $\mathbb{R}^l$ ($l \in \mathbb{Z}_{\geq 0}$), respectively, such that $P = P^{\leq}(A, b)$).*

The next concepts that we define are lineality spaces and recession cones:

Definition 54. *Let $\emptyset \neq S \subseteq \mathbb{R}^d$ ($d \in \mathbb{Z}_{\geq 0}$). Then we define*

$$\begin{aligned} \operatorname{lineal} S &:= \left\{x \in \mathbb{R}^d : \forall s \in S : s + \operatorname{lin}\{x\} \subseteq S\right\}, \\ \operatorname{rec} S &:= \left\{x \in \mathbb{R}^d : \forall s \in S : s + \operatorname{cone}\{x\} \subseteq S\right\} \end{aligned}$$

*as the **lineality space** or **recession cone**, respectively, of S.*

The definition of lineality space and recession cone can be simplified if S is convex. The following lemma is easy to check:

Lemma 55. *Let $\emptyset \neq S \subseteq \mathbb{R}^d$ ($d \in \mathbb{Z}_{\geq 0}$) be convex. Then*

$$\begin{aligned} \operatorname{lineal} S &= \left\{x \in \mathbb{R}^d : \exists s \in S : s + \operatorname{lin}\{x\} \subseteq S\right\}, \\ \operatorname{rec} S &= \left\{x \in \mathbb{R}^d : \exists s \in S : s + \operatorname{cone}\{x\} \subseteq S\right\}. \end{aligned}$$

Definition 56. *Let $\emptyset \neq P \subseteq \mathbb{R}^d$ ($d \in \mathbb{Z}_{\geq 0}$) be a polyhedron. We call P **pointed** if* $\operatorname{lineal} P = \left\{0^d\right\}$.

2.4.2.2. Properties of unions and Minkowski sums of convex sets

We now present some properties about unions and Minkowski sums of convex sets that are used at various places in this text:

Lemma 57. *Let $S_1, S_2 \subseteq V$, where V is a vector space over $\mathbb{R}$. Then*

$$\operatorname{conv}(S_1) + \operatorname{conv}(S_2) \subseteq \operatorname{conv}(S_1 + S_2). \tag{2.3}$$

Proof. We first show that for all $S_1', S_2' \subseteq V$, we have

$$\operatorname{conv}(S_1') + S_2' \subseteq \operatorname{conv}(S_1' + S_2').$$

For this, let $\lambda \in \Delta^k$, $s'^{1,1}, \ldots, s'^{1,k+1} \in S_1'$ ($k \in \mathbb{Z}_{\geq 0}$) and $s'^2 \in S_2'$. Then

$$\sum_{i=1}^{k+1} \lambda_i s'^{1,i} + s'^2 = \sum_{i=1}^{k+1} \lambda_i \left(s'^{1,i} + s'^2\right) \in \operatorname{conv}(S_1' + S_2').$$

Now for (2.3):

$$\operatorname{conv}(S_1) + \operatorname{conv}(S_2) \subseteq \operatorname{conv}(S_1 + \operatorname{conv}(S_2)) \subseteq \operatorname{conv}(\operatorname{conv}(S_1 + S_2)) = \operatorname{conv}(S_1 + S_2).$$

□

Now for three theorems which are shown in [Roc70, Corollary 3.1; p. 16], [Roc70, Corollary 9.8.2; p. 81] and [Roc70, Corollary 9.1.2; p. 75], respectively:

Theorem 58. *([Roc70, Corollary 3.1; p. 16]) Let $S_1, S_2 \subseteq \mathbb{R}^d$ ($d \in \mathbb{Z}_{\geq 0}$) be convex. Then $S_1 + S_2$ is convex.*

Theorem 59. *([Roc70, Corollary 9.8.2; p. 81]) Let $S_1, \ldots, S_k \subseteq \mathbb{R}^d$ ($k, d \in \mathbb{Z}_{\geq 0}$) be compact. Then* $\operatorname{conv} \bigcup_{i=1}^k S_i$ *is compact.*

Theorem 60. *([Roc70, Corollary 9.1.2; p. 75]) Let $\emptyset \neq S_1, S_2 \subseteq \mathbb{R}^d$ ($d \in \mathbb{Z}_{\geq 0}$) be convex and closed. Assume*

$$\nexists r \in \operatorname{rec} S_1 \backslash \left\{0^d\right\} : -r \in \operatorname{rec} S_2$$

(this in particular holds if either S_1 or S_2 is bounded). Then $S_1 + S_2$ is closed and (by Theorem 58) convex and we have

$$\operatorname{rec}(S_1 + S_2) = \operatorname{rec} S_1 + \operatorname{rec} S_2.$$

An important consequence of Theorem 60 is:

Corollary 61. *Let $S \subseteq \mathbb{R}^d$ ($d \in \mathbb{Z}_{\geq 0}$) be convex and compact and let $C \subseteq \mathbb{R}^d$ be a closed convex cone (e.g. a polyhedral cone). Then $S + C$ is convex and closed.*

Proof. W.l.o.g. we can assume $S \neq \emptyset$. Then the statement is an immediate consequence of Theorem 60. □

2.4.2.3. Transformations

We now write down two elementary lemmas about polyhedra (Lemma 62 and Lemma 63). Lemma 62 is easy to check.

Lemma 62. *Let*

$$P := \operatorname{conv}\left\{p^1, \ldots, p^k\right\} + \operatorname{cone}\left\{q^1, \ldots, q^l\right\} \subseteq \mathbb{R}^m \times \mathbb{R}^n$$

($m, n, k, l \in \mathbb{Z}_{\geq 0}$, $p^1, \ldots, p^k, q^1, \ldots, q^l \in \mathbb{R}^m \times \mathbb{R}^n$). Then

$$\operatorname{proj}_{\mathbb{R}^m} P = \operatorname{conv}\left\{\operatorname{proj}_{\mathbb{R}^m} p^1, \ldots, \operatorname{proj}_{\mathbb{R}^m} p^k\right\} + \operatorname{cone}\left\{\operatorname{proj}_{\mathbb{R}^m} q^1, \ldots, \operatorname{proj}_{\mathbb{R}^m} q^l\right\}.$$

Lemma 63. *Let $P := P^{\leq}(A, b) \subseteq \mathbb{R}^d$ ($d \in \mathbb{Z}_{\geq 0}$, $A \in \mathbb{R}^{l \times d}$, $b \in \mathbb{R}^l$, where $l \in \mathbb{Z}_{\geq 0}$) be a polyhedron and let*

$$\begin{aligned} f : \mathbb{R}^d &\to \mathbb{R}^d : \\ x &\mapsto Cx + c \end{aligned}$$

($C \in \mathbb{R}^{d \times d}$ regular, $c \in \mathbb{R}^d$) be an affine map. Then

$$f(P) = P^{\leq}\left(AC^{-1}, b + AC^{-1}c\right), \tag{2.4}$$

$$f^{-1}(P) = P^{\leq}(AC, b - Ac). \tag{2.5}$$

Proof. For (2.4): We have

$$\begin{aligned} f(P) &= CP^{\leq}(A, b) + c \\ &= C\left\{x \in \mathbb{R}^d : Ax \leq b\right\} + c \\ &= \left\{x \in \mathbb{R}^d : AC^{-1}(x - c) \leq b\right\} \\ &= P^{\leq}\left(AC^{-1}, b + AC^{-1}c\right). \end{aligned}$$

For (2.5), consider that $f^{-1}(x) = C^{-1}x - C^{-1}c$. □

We now consider projections of sets that are described by linear inequalities and strict linear inequalities. The following theorem is shown in [DDG11, Appendix A]:

Theorem 64. *Let*

$$P := \left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R}^n : Ax + By \leq e, Cx + Dy < f\right\},$$

where $A \in \mathbb{R}^{s \times m}$, $B \in \mathbb{R}^{s \times n}$, $C \in \mathbb{R}^{t \times m}$, $D \in \mathbb{R}^{t \times n}$, $e \in \mathbb{R}^s$ and $f \in \mathbb{R}^t$ ($m, n, s, t \in \mathbb{Z}_{\geq 0}$). Then

$$\operatorname{proj}_{\mathbb{R}^m} P = S,$$

where

$$\begin{aligned} S := \{ x \in \mathbb{R}^m : & \\ & \lambda Ax \leq \lambda e \ \forall \lambda \in \left(\mathbb{R}^s_{\geq 0}\right)^T : \lambda B = (0^n)^T, & (2.6) \\ & (\lambda A + \mu C)x < \lambda e + \mu f \ \forall \begin{pmatrix} \lambda & \mu \end{pmatrix} \in \left(\mathbb{R}^s_{\geq 0} \times \mathbb{R}^t_{\geq 0}\right)^T : \lambda B + \mu D = (0^n)^T, \exists i \in [t] : \mu_i = 1\Big\}. & (2.7) \end{aligned}$$

In particular, since for (2.6) and (2.7), there exists a finite set of generators, there exist matrices G^1, G^2 and vectors g^1, g^2 that satisfy

$$S = \left\{x \in \mathbb{R}^m : G^1 x \leq g^1, G^2 x < g^2\right\},$$

where G^1 and G^2 can be assumed to be rational if A, B, C and D are rational and g^1 and g^2 can be assumed to be rational if additionally e and f are rational.

One obtains an important special case of Theorem 64 if P is a polyhedron. We formulate it as a corollary:

Corollary 65. *Let*
$$P := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R}^n : Ax + By \leq e \right\},$$
where $A \in \mathbb{R}^{s \times m}$, $B \in \mathbb{R}^{s \times n}$ *and* $e \in \mathbb{R}^s$. *Then*
$$\operatorname{proj}_{\mathbb{R}^m} P = S,$$
where
$$S := \left\{ x \in \mathbb{R}^m : \lambda A x \leq \lambda e \; \forall \lambda \in \left(\mathbb{R}^s_{\geq 0}\right)^T : \lambda B = (0^n)^T \right\}.$$
In particular, since
$$\left\{ \lambda \in \left(\mathbb{R}^s_{\geq 0}\right)^T : \lambda B = (0^n)^T \right\}$$
is finitely generated, there exist a matrix G^1 *and a vector* g^1 *having*
$$S = \left\{ x \in \mathbb{R}^m : G^1 x \leq g^1 \right\},$$
where G^1 *can be assumed to be rational if* A, B *are rational and* g^1 *can be assumed to be rational if additionally* e *is rational.*

At this place a remark concerning extended formulations: let $P \subseteq \mathbb{R}^n$ and $Q \subseteq \mathbb{R}^d$ be polyhedra $(n, d \in \mathbb{Z}_{\geq 0})$ such that $P = \pi(Q)$, where π is a linear or affine-linear map. Then we call Q an **extended formulation of** P.

In [Kai09], the following theorem is shown, which considers the case that Q that is given via linear inequalities and π is a linear map. In this situation, Theorem 66 gives an explicit characterization of $\pi(Q)$ via linear inequalities.

Theorem 66. *[Kai09, Theorem 2] Let* $Q := P^{\leq}(D, g) \subseteq \mathbb{R}^d$ *(*$D \in \mathbb{R}^{q \times d}$ *and* $g \in \mathbb{R}^q$*, where* $d, q \in \mathbb{Z}_{\geq 0}$*) be a polyhedron and let* $\pi : \mathbb{R}^d \to \mathbb{R}^n : x \mapsto Tx$ *be a linear map (*$T \in \mathbb{R}^{n \times d}$*, where* $n \in \mathbb{Z}_{\geq 0}$*). Let* $\overline{T} \in \mathbb{R}^{d \times t}$ *be an arbitrary matrix whose columns form a basis of* $\ker T$. *If* $L \in \mathbb{R}^{m \times q}_{\geq 0}$ *(*$m \in \mathbb{Z}_{\geq 0}$*) is a matrix whose rows generate the* ***projection cone***
$$\left\{ \lambda \in \left(\mathbb{R}^q_{\geq 0}\right)^T : \lambda D \overline{T} = \left(0^t\right)^T \right\} = \operatorname{rowcone} L,$$
then every $A \in \mathbb{R}^{m \times n}$ *with* $AT = LD$ *satisfies*
$$\pi(Q) = P^{\leq}(A, Lg) \cap \pi\left(\mathbb{R}^d\right).$$

Nevertheless, for understanding this thesis, it is not necessary to be familiar with any further results about extended formulations than Theorem 64 and Corollary 65.

2.5. Closed convex sets

2.5.1. Definitions

Definition 67. *Let* $S \subseteq \mathbb{R}^d$ *(*$d \in \mathbb{Z}_{\geq 0}$*). We define*
$$\overline{\operatorname{conv}}\, S := \overline{\operatorname{conv} S} = \bigcap_{\substack{S \subseteq S' \subseteq V: \\ S' \text{ convex and closed}}} S'$$
as the ***closed convex hull of*** S.

Clearly, the closed convex hull of some set S is the smallest closed convex set containing S. We soon consider why the closed convex hull is very important for optimization purposes, but before, we consider separation theorems for convex sets. In [Roc70, section 11; p. 95], the author defines:

Definition 68. *Let* $H := P^{=}(c, c_0) \subseteq \mathbb{R}^d$ *(*$d \in \mathbb{Z}_{\geq 0}$*) be a hyperplane and let* $C_1, C_2 \subseteq \mathbb{R}^d$ *be convex. We say that* H ***separates*** C_1 ***and*** C_2 ***strongly*** *if there exists some* $\epsilon \in \mathbb{R}_{>0}$ *such that*

- $C_1 + \overline{B}_\epsilon\left(0^d\right) \subseteq P^{\leq}(c, c_0)$,

- $C_2 + \overline{B}_\epsilon\left(0^d\right) \subseteq P^{\geq}\left(c, c_0\right)$

or vice versa.

In [Roc70, Theorem 11.4; p. 98], it is shown:

Theorem 69. *Let $\emptyset \neq C_1, C_2 \subseteq \mathbb{R}^d$ $(d \in \mathbb{Z}_{\geq 0})$ be convex. Then there exist a hyperplane that separates C_1 and C_2 strongly if and only if*

$$\inf\left\{\|x - x'\| : x \in C_1, x' \in C_2\right\} > 0,$$

which is equivalent to $0^d \notin \overline{C_1 - C_2}$.

From Theorem 69, we conclude:

Corollary 70. *Let $C \subseteq \mathbb{R}^d$ $(d \in \mathbb{Z}_{\geq 0})$ be convex and closed. Let $x^* \in \mathbb{R}^d \backslash C$. Then there exist $\hat{\varphi} \in \left(\mathbb{R}^d\right)^*$ and $c_0 \in \mathbb{R}$ such that*

- *$\hat{\varphi}(c) < c_0$ for every $c \in C$,*
- *$\hat{\varphi}(x^*) > c_0$.*

Now that we have the tools available, let us consider the importance of the closed convex hull for optimization purposes. We consider the problem

$$\sup\left\{\hat{\varphi}(x) : x \in S\right\},$$

where $S \subseteq \mathbb{R}^d$ $(d \in \mathbb{Z}_{\geq 0})$ is arbitrary and $\hat{\varphi} \in \left(\mathbb{R}^d\right)^*$. Since S might be difficult to describe/enumerate, we consider sets $S' \subseteq \mathbb{R}^d$ that satisfy

$$\forall \varphi \in \left(\mathbb{R}^d\right)^* : \sup\left\{\varphi(x) : x \in S\right\} = \sup\left\{\varphi(x) : x \in S'\right\}. \tag{2.8}$$

Trivially, there can in general exist lots of such sets S'. So one wants to find a specific S' which "stands out" among the others. One very natural such property to look at is the maximum (with respect to inclusion) among all sets S' that satisfy (2.8). We next show that $\overline{\operatorname{conv}} S$ indeed satisfies this property, i.e. $\overline{\operatorname{conv}} S$ is the maximum (with respect to inclusion) among all sets S' that satisfy (2.8):

Theorem 71. *Let $S \subseteq \mathbb{R}^d$ $(d \in \mathbb{Z}_{\geq 0})$. Then:*

1. *Let $\varphi \in \left(\mathbb{R}^d\right)^*$. Then*
$$\sup\left\{\varphi(x) : x \in S\right\} = \sup\left\{\varphi(x) : x \in \overline{\operatorname{conv}} S\right\}.$$

2. *Let $S' \subseteq \mathbb{R}^d$ be such that*
$$\forall \varphi \in \left(\mathbb{R}^d\right)^* : \sup\left\{\varphi(x) : x \in S\right\} = \sup\left\{\varphi(x) : x \in S'\right\}.$$
Then $S' \subseteq \overline{\operatorname{conv}} S$.

Proof.

For 1: W.l.o.g. we can assume $S \neq \emptyset$ and $\sup\left\{\varphi(x) : x \in S\right\} < \infty$. Since $S \subseteq \overline{\operatorname{conv}} S$, we trivially have

$$\sup\left\{\varphi(x) : x \in S\right\} \leq \sup\left\{\varphi(x) : x \in \overline{\operatorname{conv}} S\right\}.$$

For the other direction

$$\sup\left\{\varphi(x) : x \in \overline{\operatorname{conv}} S\right\} \leq \sup\left\{\varphi(x) : x \in S\right\},$$

let $x \in \operatorname{conv} S$. Then there exist $s^1, \ldots, s^{l+1} \in P$ and $\lambda \in \Delta^l$ $(l \in \mathbb{Z}_{\geq 0})$ such that $x = \sum_{i=1}^{l+1} \lambda_i s^i$. So

$$\varphi(x) = \varphi\left(\sum_{i=1}^{l+1} \lambda_i s^i\right) = \sum_{i=1}^{l+1} \lambda_i \varphi\left(s^i\right) \leq \sum_{i=1}^{l+1} \lambda_i \max_{j \in [l+1]} \varphi\left(s^j\right) = \max_{j \in [l+1]} \varphi\left(s^j\right) \leq \sup_{s \in S} \varphi(s).$$

Let $\left\{x^i\right\}_{i \in \mathbb{Z}_{\geq 1}}$ be a convergent (in $\mathbb{R}^d$) sequence of points in $\operatorname{conv} S$ and let $x^* := \lim_{i \to \infty} x^i$. We just saw that for all $i \in \mathbb{Z}_{\geq 1}$, we have $\varphi\left(x^i\right) \leq \sup_{s \in S} \varphi(s)$. Thus, since φ is a continuous function, we conclude $\varphi(x^*) \leq \sup_{s \in S} \varphi(s)$. Thus,

$$\forall s \in \overline{\operatorname{conv}} S : \varphi(s) \leq \sup_{s \in S} \varphi(s).$$

For 2: By 1, we can assume S' to be convex and closed. Assume that there exists an $x^* \in S' \setminus \overline{\operatorname{conv}} S$. Then, by Corollary 70, there exists a $\hat{\varphi} \in \left(\mathbb{R}^d\right)^*$ such that

$$\hat{\varphi}(x^*) > \sup\{\hat{\varphi}(x) : x \in \overline{\operatorname{conv}} S\} = \sup\{\hat{\varphi}(x) : x \in S\}. \lightning$$

□

A very important consequence of Theorem 71 for this text is the following theorem, which is also shown in [Roc70, Corollary 11.5.1; p. 99]:

Theorem 72. *Let $S \subseteq \mathbb{R}^d$ $(d \in \mathbb{Z}_{\geq 0})$. Then*

$$\overline{\operatorname{conv}} S = \bigcap_{H \text{ half-space: } S \subseteq H} H$$

(where the empty intersection is defined as $\mathbb{R}^d$).

Despite the importance of the closed convex hull for optimzation purposes that we saw in Theorem 71, it is often much harder to understand than the convex hull. So, it is very useful to understand under what conditions the closed convex hull of a set is equal to its convex hull.

2.5.2. The closed mixed-integer hull of polyhedra

Central to this text (cf. chapter 3) is optimizing a linear function over $P \cap (\mathbb{Z}^m \times \mathbb{R}^n)$ $(m, n \in \mathbb{Z}_{\geq 0})$, where P is typically a rational polyhedron. So, in the remainder of this section, we consider how the (closed) mixed-integer hull (which we define in Definition 73) of P looks like. In section 2.5.2.1, we consider the "good" cases, where the mixed-integer hull equals the closed mixed-integer hull and is again a polyhedron or at least "well-behaved". In section 2.5.2.2, we consider what can happen if we consider the mixed-integer hull of non-rational polyhedra (more precisely: polyhedra that do not have a partially rational recession cone).

2.5.2.1. The mixed-integer hull in case of a partially rational recession cone

Definition 73. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ be arbitrary. We define*

$$P_I := P \cap (\mathbb{Z}^m \times \mathbb{R}^n)$$

and set

$$\operatorname{cl}_I(P) := \operatorname{conv} P_I,$$
$$\operatorname{cl}_{\overline{I}}(P) := \overline{\operatorname{conv}} P_I$$

*as the **mixed-integer hull of** P or **closed mixed-integer hull of** P, respectively. If $n = 0$, we also use the terms **integer hull** or **closed integer hull**, respectively.*

Remark 74. *From now on, $m \in \mathbb{Z}_{\geq 0}$ mostly stands for the number of integral variables and $n \in \mathbb{Z}_{\geq 0}$ for the number of continuous variables. If it does not matter whether the variables are integral or continuous, we often use the letter d $(d \in \mathbb{Z}_{\geq 0})$ for the dimension.*

The following theorem is proved in [Sch86, Theorem 16.1; p. 231] for the special case that $P \subseteq \mathbb{R}^m$ (pure integer case) is a rational polyhedron. The original proof for $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ (mixed-integer case) being a rational polyhedron is due to Meyer ([Mey74]).

Theorem 75. *Let $P = Q + C \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$, where*

- *$Q \subseteq \mathbb{R}^m \times \mathbb{R}^n$ is arbitrary,*
- *$C := \operatorname{cone}\left\{c^1, \ldots, c^s\right\}$, where for all $i \in [s]$ $(s \in \mathbb{Z}_{\geq 0})$, we have $c^i \in \mathbb{Z}^m \times \mathbb{R}^n$.*

Let

$$B := \left\{\sum_{i=1}^{s} \mu_i c^i : 0 \leq \mu_i \leq 1 \ \forall i \in [s]\right\}.$$

Then

$$\operatorname{cl}_I(P) = \operatorname{cl}_I(Q + B) + C.$$

If $P_I \neq \emptyset$*, we have* $\operatorname{rec}(\operatorname{cl}_I(P)) = \operatorname{rec} P$*. Now let* Q *be bounded. Then*

$$\operatorname{cl}_I(P) = \operatorname{cl}_{\overline{I}}(P)$$

if either

- $n = 0$ *or*
- Q *is closed (which together with the condition "Q is bounded" implies "Q is compact").*

In this case, $\operatorname{cl}_I(Q+B)$ *is closed. In particular:*

- $\operatorname{cl}_I(P)$ *is a polyhedron if either*
  - Q *is a polytope,*
  - $n = 0$ *or*
  - $n = 1$ *and* Q *is closed.*
- $\operatorname{cl}_I(P)$ *is a rational polyhedron if* P *is.*

Proof.

For $\operatorname{cl}_I(P) \subseteq \operatorname{cl}_I(Q+B)+C$**:** We start by showing $P_I \subseteq \operatorname{cl}_I(Q+B)+C$. Let $p \in P_I$. Then $p = q + c$, where $q \in Q$ and $c \in C$. From this, we conclude that there exist $b \in B$ and $c' \in C_I$ having $c = b + c'$. For the reason: let $c = \sum_{i=1}^{s} \mu_i c^i$, where $\mu \in \mathbb{R}^s_{\geq 0}$. Set

$$\begin{aligned} c' &:= \sum_{i=1}^{s} \lfloor \mu_i \rfloor c^i, \\ b &:= \sum_{i=1}^{s} (\mu_i - \lfloor \mu_i \rfloor) c^i = c - c'. \end{aligned}$$

So, we have $p = (q+b)+c'$, where $q+b \in (Q+B)_I$ ($q+b \in Q+B$ is obvious; for $q+b \in \mathbb{Z}^m \times \mathbb{R}^n$, consider that

$$q + b = \underbrace{p}_{\in \mathbb{Z}^m \times \mathbb{R}^n} - \underbrace{c'}_{\in \mathbb{Z}^m \times \mathbb{R}^n} \in \mathbb{Z}^m \times \mathbb{R}^n.$$

Thus, we have $p \in (Q+B)_I + C_I \subseteq \operatorname{cl}_I(Q+B)+C$.

Now for $\operatorname{cl}_I(P) \subseteq \operatorname{cl}_I(Q+B)+C$:

$$\operatorname{cl}_I(P) = \operatorname{conv}(P_I) \subseteq \operatorname{conv}(\operatorname{cl}_I(Q+B)+C) = \operatorname{cl}_I(Q+B)+C.$$

For $\operatorname{cl}_I(Q+B)+C \subseteq \operatorname{cl}_I(P)$**:**

$$\begin{aligned} \operatorname{cl}_I(Q+B)+C &\subseteq \operatorname{cl}_I(P)+C \\ &= \operatorname{cl}_I(P)+\operatorname{cl}_I(C) \\ &= \operatorname{conv}(P_I)+\operatorname{conv}(C_I) \\ &\subseteq \operatorname{conv}(P_I+C_I) && \text{(by Lemma 57)} \\ &= \operatorname{conv}((P\cap(\mathbb{Z}^m\times\mathbb{R}^n))+(C\cap(\mathbb{Z}^m\times\mathbb{R}^n))) \\ &= \operatorname{conv}((P+C)\cap(\mathbb{Z}^m\times\mathbb{R}^n)) && (\mathbb{Z}^m\times\mathbb{R}^n \text{ is closed under addition}) \\ &= \operatorname{cl}_I(P+C) \\ &= \operatorname{cl}_I(P). \end{aligned}$$

For the remaining statements: If either $n = 0$ or Q is closed, the set $(Q+B)_I$ is a finite union of compact sets. Thus, the closure property is immediately implied by Theorem 59. One concludes from Corollary 61 that $\operatorname{cl}_I(Q+B)+C$ is convex and closed with recession cone C if $(Q+B)_I \neq \emptyset$.

If either Q is a polytope, $n = 0$, or $n = 1$ and Q is closed, $\operatorname{cl}_I(Q+B)$ is a polyhedron (a rational polyhedron if P is one), which shows the final statement. □

With Definition 73 and Theorem 75 in mind, we define:

Definition 76. *A rational polyhedron* $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ *(*$m, n \in \mathbb{Z}_{\geq 0}$*) is called* ***mixed-integral*** *if* $P = \mathrm{cl}_I(P)$*. If* $n = 0$*, we also use the term* ***integral****.*

We remark that for future research one might consider relaxing the condition in Definition 76 of P to be a rational polyhedron in particular for the mixed-integral case.

2.5.2.2. Situation for non-rational polyhedra

In Theorem 75, we saw that the mixed-integer hull of a polyhedron with a partially rational recession cone is again a polyhedron. Now, we consider what can happen to $\mathrm{cl}_{\overline{I}}(P)$ and $\mathrm{cl}_I(P)$ if P does not have a partially rational recession cone.

The first thing that can happen is that the situation $\mathrm{cl}_I(P) \subsetneq \mathrm{cl}_{\overline{I}}(P)$ might occur. Since every polyhedron is closed, this implies that $\mathrm{cl}_I(P)$ is not a polyhedron (but $\mathrm{cl}_{\overline{I}}(P)$ might be one). For this, we consider the following example:

Example 77. *(See Figure 2.1) Let*

$$P^{77} := \left\{ x \in \mathbb{R}^2 : x_1 + \sqrt{2}x_2 \leq \frac{1}{2} \right\}.$$

Then

- $\mathrm{cl}_I\left(P^{77}\right)$ *is not a polyhedron. For this, consider that for every* $x \in \left(P^{77}\right)_I$*, we have* $x_1 + \sqrt{2}x_2 < \frac{1}{2}$*, while on the other hand, one can show*

$$\sup\left\{ x_1 + \sqrt{2}x_2 : x \in \left(P^{77}\right)_I \right\} = \frac{1}{2}.$$

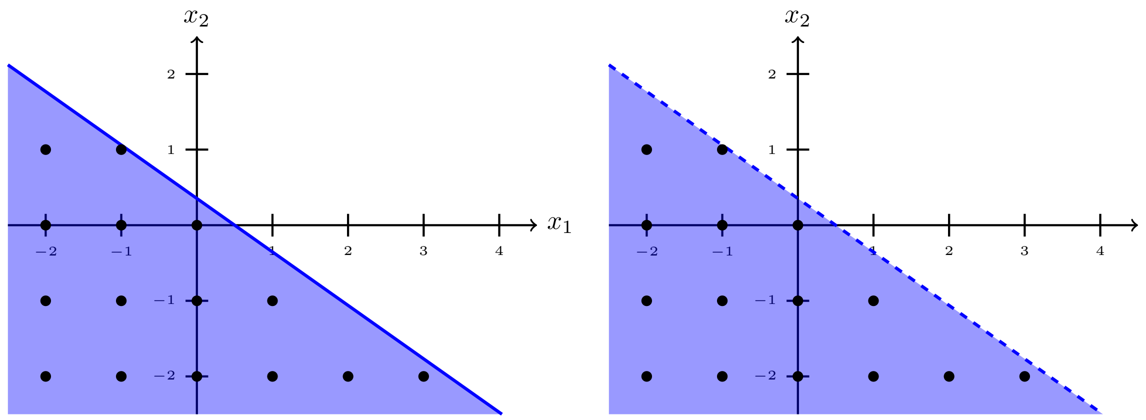
  Thus,

$$\mathrm{cl}_I\left(P^{77}\right) \subseteq \left\{ x \in \mathbb{R}^2 : x_1 + \sqrt{2}x_2 < \frac{1}{2} \right\}.$$

  Indeed, equality holds, i.e. we have

$$\mathrm{cl}_I\left(P^{77}\right) = \left\{ x \in \mathbb{R}^2 : x_1 + \sqrt{2}x_2 < \frac{1}{2} \right\}. \tag{2.9}$$

- *One can conclude from* (2.9) *that* $\mathrm{cl}_{\overline{I}}\left(P^{77}\right) = P^{77}$*.*



(a) $P^{77} = \mathrm{cl}_{\overline{I}}\left(P^{77}\right)$ and $P^{77} \cap \mathbb{Z}^2$ (b) $\mathrm{cl}_I\left(P^{77}\right)$ and $P^{77} \cap \mathbb{Z}^2$

Figure 2.1.: Visualisation of $P^{77} = \mathrm{cl}_{\overline{I}}\left(P^{77}\right)$ and $\mathrm{cl}_I\left(P^{77}\right)$

One might conjecture that, in general, for any polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ at least $\mathrm{cl}_{\overline{I}}(P)$ is again a polyhedron. We next consider an example (Example 80; also cf. Lemma 79) where we have $\mathrm{cl}_I(P) = \mathrm{cl}_{\overline{I}}(P)$, but

- we need a countably infinite number of (rational) linear inequalities to describe $\mathrm{cl}_{\overline{I}}(P)$ using linear inequalities and

- $\operatorname{cl}_{\overline{I}}(P)$ has a countably infinite number of vertices.

Consider the following lemma, for which a proof is sketched in [Rub70]:

Lemma 78. *Let $w \in \mathbb{R}_{>0} \backslash \mathbb{Q}$. Consider the infinite continued fraction*

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\dots + \cfrac{1}{a_k + \frac{1}{\dots}}}}}$$

for w. Let

$$\frac{P_k}{Q_k} := a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\dots + \frac{1}{a_k}}}} \quad \forall k \in \mathbb{Z}_{\geq 1},$$

where $P_k, Q_k \in \mathbb{Z}_{\geq 1}$ and $\gcd(P_k, Q_k) = 1$. Let

$$L^{78,w} := \operatorname{conv}\left\{x \in \mathbb{Z}^2_{\geq 0} : x_2 < w x_1\right\}$$

and let

$$v^i := \begin{cases} \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \text{if } i = 0, \\ \begin{pmatrix} Q_{2i} \\ P_{2i} \end{pmatrix} & \text{if } w < 1 \wedge i \geq 1, \\ \begin{pmatrix} 1 \\ a_0 \end{pmatrix} & \text{if } w > 1 \wedge i = 1, \\ \begin{pmatrix} Q_{2(i-1)} \\ P_{2(i-1)} \end{pmatrix} & \text{if } w > 1 \wedge i \geq 2 \end{cases} \quad \forall i \in \mathbb{Z}_{\geq 0}.$$

Then

$$\begin{aligned} L^{78,w} &= P^{\geq}\left(\begin{pmatrix} 0 & 1 \end{pmatrix}, 0\right) \cap \bigcap_{i \in \mathbb{Z}_{\geq 1}} P^{\leq}\left(\begin{pmatrix} v_2^{k-1} - v_2^k & v_1^k - v_1^{k-1} \end{pmatrix}, v_1^k v_2^{k-1} - v_1^{k-1} v_2^k\right) \\ &= \left(\operatorname{conv} \bigcup_{i \in \mathbb{Z}_{\geq 0}} \{v^i\}\right) + \operatorname{cone}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}, \end{aligned}$$

where all the inequalities are facet-defining for L and the v^i are the vertices of L.

We use Lemma 78 to characterize the facets of $\operatorname{cl}_I(P)$ for a specific class of polyhedra $P \subseteq \mathbb{R}^2$:

Lemma 79. *For $w \in \mathbb{R}_{>0} \backslash \mathbb{Q}$, let*

$$P^{79,w} := \left\{x \in \mathbb{R}^2 : x_1 \geq 1, x_2 \geq 0, x_2 \leq w x_1\right\}.$$

Then

$$\operatorname{cl}_I\left(P^{79,w}\right) = L^{78,w}.$$

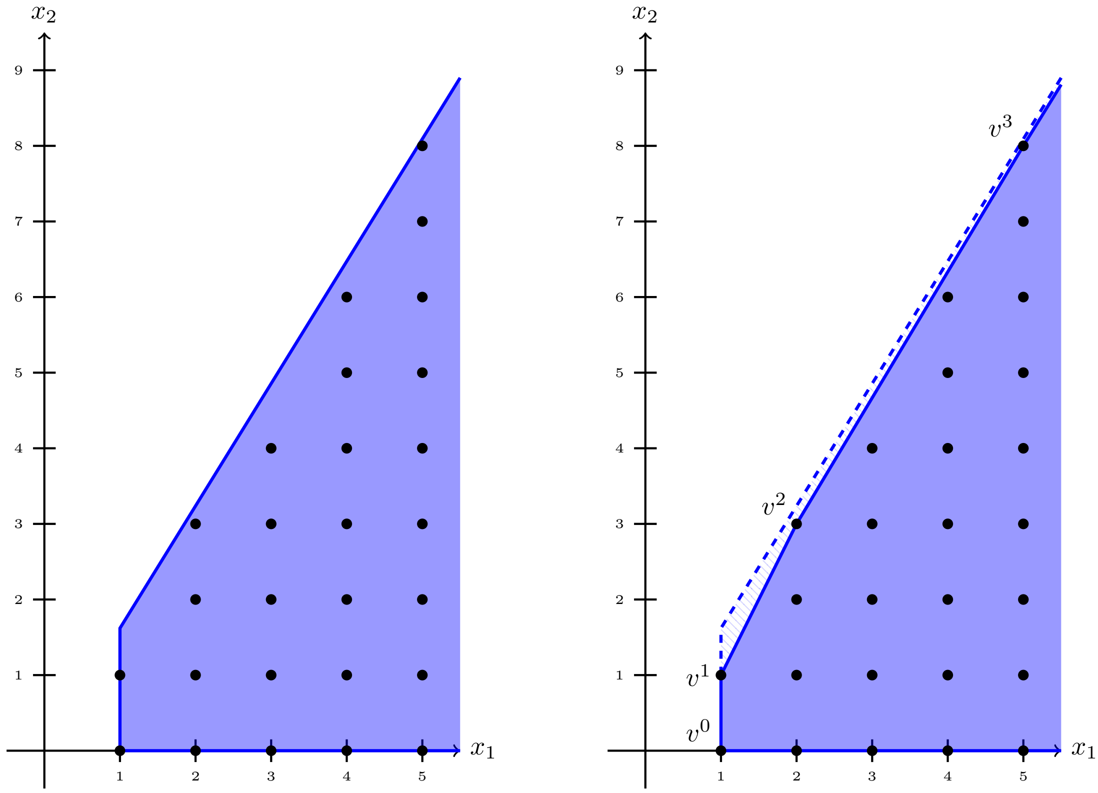
Proof. Since w is irrational, we have

$$\left(L^{78,w}\right)_I = \left\{x \in \mathbb{Z}^2_{\geq 0} : x_2 < w x_1\right\} = \left\{x \in \mathbb{Z}^2 : x_1 \geq 1, x_2 \geq 0, x_2 \leq w x_1\right\} = \left(P^{79,w}\right)_I.$$

Thus, we have $L^{78,w} = \operatorname{cl}_I\left(P^{79,w}\right)$. □

Example 80. *(See Figure 2.2) Let $w := \frac{1+\sqrt{5}}{2}$. It is well-known that the infinite continued fraction representation of w is*

$$w = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\dots + \cfrac{1}{1 + \frac{1}{\dots}}}}},$$



(a) $P^{79,\frac{1+\sqrt{5}}{2}}$ and $P^{79,\frac{1+\sqrt{5}}{2}} \cap \mathbb{Z}^2$ (b) $\mathrm{cl}_I\left(P^{79,\frac{1+\sqrt{5}}{2}}\right)$ and $P^{79,\frac{1+\sqrt{5}}{2}} \cap \mathbb{Z}^2$

Figure 2.2.: Visualisation of $P^{79,\frac{1+\sqrt{5}}{2}}$ and $\mathrm{cl}_I\left(P^{79,\frac{1+\sqrt{5}}{2}}\right)$

i.e. in the notation of Lemma 78, we have: $\forall i \in \mathbb{Z}_{\geq 0} : a_i = 1$*. It is also well-known that for all* $i \in \mathbb{Z}_{\geq 1}$*, we have (again using the notation of Lemma 78)*

$$P_i := F_{i+2}, \qquad\qquad Q_i := F_{i+1},$$

where

$$F_i := \begin{cases} 0 & \textit{if } i = 0, \\ 1 & \textit{if } i = 1, \\ F_{i-2} + F_{i-1} & i \geq 2 \end{cases}$$

(Fibonacci sequence). So, by Lemma 78 and Lemma 79, we have

$$\begin{aligned} \mathrm{cl}_I\left(P^{79,\frac{1+\sqrt{5}}{2}}\right) &= L^{78,\frac{1+\sqrt{5}}{2}} \\ &= P^{\geq}\left(\begin{pmatrix} 0 & 1 \end{pmatrix}, 0\right) \cap \bigcap_{i \in \mathbb{Z}_{\geq 1}} P^{\leq}\left(\begin{pmatrix} v_2^{k-1} - v_2^k & v_1^k - v_1^{k-1} \end{pmatrix}, v_1^k v_2^{k-1} - v_1^{k-1} v_2^k\right) \qquad (2.10) \\ &= \left(\mathrm{conv} \bigcup_{i \in \mathbb{Z}_{\geq 0}} \{v^i\}\right) + \mathrm{cone}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}, \end{aligned}$$

where

$$v^0 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad v^1 := \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \forall i \geq 2 : v^i := \begin{pmatrix} Q_{2(i-1)} \\ P_{2(i-1)} \end{pmatrix} = \begin{pmatrix} F_{2i-1} \\ F_{2i} \end{pmatrix}.$$

If one defines (as it is often done) $F_{-1} := 1$*, we have*

$$\forall i \in \mathbb{Z}_{\geq 0} : v^i = \begin{pmatrix} F_{2i-1} \\ F_{2i} \end{pmatrix}.$$

By Lemma 78, all v^i *(*$i \in \mathbb{Z}_{\geq 0}$*) are vertices of* $\mathrm{cl}_I\left(P^{79,\frac{1+\sqrt{5}}{2}}\right)$ *and all inequalities in* (2.10) *are facet-defining.*

2.6. Theorems of the alternative

The central idea of theorems of the alternative is the following: consider a system of, for example, linear equations or linear inequalities. It is often not too hard to find criteria that are necessary to be satisfied such that this system has a solution. On the other hand, it is typically much harder to find criteria that are additionally sufficient for a solution to exist.

If one is able to find such a criterion, one likes to formulate it in a form where two systems are given, but exactly one of them has a solution. So, if one shows a solution to one of the systems, this automatically gives a certificate that the other one is unsolvable. Examples for such criteria are given in Corollary 84, Lemma 88, Lemma 89 and Theorem 92.

For the outline of this section:

- In section 2.6.1, we consider the situation for systems of linear equations.
- In section 2.6.2, we consider the situation for systems of linear inequalities.
- In section 2.6.3, we consider the situation for systems of linear equations with rational coefficients where we additionally demand integrality conditions on some variables.

2.6.1. Systems of linear equations

Definition 81. *Let* V *be a vector space over an arbitrary (possibly skew) field* F*. Let* $u \in V$*. We define*

$$\begin{aligned} \iota u : F &\to V : \\ \lambda &\mapsto \lambda u. \end{aligned}$$

Remark 82. *Let* u*,* F *and* V *be as in Definition 81 and let* $\alpha : W \to F$ *be a linear form, where* W *is a vector space over* F*. Then*

$$\forall x \in W : ((\iota u) \circ \alpha)(x) = (\alpha(x))\, u.$$

The following formulation of Theorem 83 and its proof are based on ideas that were originally developed by Bartl in [Bar12b] for giving a short algebraic proof of the Farkas lemma in the formulation of Theorem 87 (also cf. [Bar07, Lemma 3.1]):

Theorem 83. *Let* V*,* W *be vector spaces over some (possibly skew) field* F*. Let* $\alpha_1, \ldots, \alpha_m : W \to F$ *(*$m \in \mathbb{Z}_{\geq 0}$*) be linear forms and let* $\gamma : W \to V$ *be a linear mapping. Then*

$$\begin{aligned} &\forall x \in W : (\alpha_1(x) = 0 \wedge \cdots \wedge \alpha_m(x) = 0) \Rightarrow \gamma(x) = 0_V \\ \Leftrightarrow &\exists u_1, \ldots, u_m \in V : \gamma = \sum_{i=1}^{m} (\iota u_i) \circ \alpha_i. \end{aligned}$$

Proof. The direction "⇐" is obvious. So, we only prove "⇒". For $m = 0$, the statement is obvious. Let the statement be proved for some m and assume

$$\forall x \in W : \alpha_1(x) = 0 \wedge \cdots \wedge \alpha_m(x) = 0 \wedge \alpha_{m+1}(x) = 0 \Rightarrow \gamma(x) = 0_V. \tag{2.11}$$

If we also have

$$\forall x \in W : \alpha_1(x) = 0 \wedge \cdots \wedge \alpha_m(x) = 0 \Rightarrow \gamma(x) = 0_V, \tag{2.12}$$

then, by induction hypothesis, there exist $u_1, \ldots, u_m \in V$ having

$$\gamma = \sum_{i=1}^{m} (\iota u_i) \circ \alpha_i,$$

and we can simply set $u_{m+1} := 0_V$.

So, we can assume that (2.12) does not hold. In other words:

$$\exists x^* \in W : \alpha_1(x^*) = 0 \wedge \cdots \wedge \alpha_m(x^*) = 0 \wedge \gamma(x^*) \neq 0_V.$$

By (2.11), we can assume $\alpha_{m+1}(x^*) \neq 0$. By scaling, we can additionally assume that $\alpha_{m+1}(x^*) = 1$. Thus,

$$\forall x \in W : \alpha_{m+1}(x - (\alpha_{m+1}(x))\, x^*) = 0.$$

Thus, by setting $x - (\alpha_{m+1}(x))\, x^*$ for x in (2.11), we obtain

$$\forall x \in W : \left\{ \begin{array}{c} \alpha_1 (x - (\alpha_{m+1}(x))\, x^*) = 0 \\ \vdots \\ \alpha_m (x - (\alpha_{m+1}(x))\, x^*) = 0 \end{array} \right\} \Rightarrow \gamma (x - (\alpha_{m+1}(x))\, x^*) = 0_V,$$

which is equivalent to (cf. Remark 82)

$$\forall x \in W : \left\{ \begin{array}{c} (\alpha_1 - \alpha_1(x^*)\, \alpha_{m+1})(x) = 0 \\ \vdots \\ (\alpha_m - \alpha_m(x^*)\, \alpha_{m+1})(x) = 0 \end{array} \right\} \Rightarrow (\gamma - (\iota(\gamma(x^*))) \circ \alpha_{m+1})(x) = 0_V.$$

Hence, by induction hypothesis, there exist $u_1, \ldots, u_m \in V$ having

$$\gamma - (\iota(\gamma(x^*))) \circ \alpha_{m+1} = \sum_{i=1}^{m} (\iota u_i) \circ (\alpha_i - \alpha_i(x^*)\, \alpha_{m+1}),$$

which is equivalent to

$$\gamma = \sum_{i=1}^{m} (\iota u_i) \circ \alpha_i + \left(\iota\left(\gamma(x^*) - \sum_{i=1}^{m} \alpha_i(x^*)\, u_i\right)\right) \circ \alpha_{m+1}.$$

So simply set

$$u_{m+1} := \gamma(x^*) - \sum_{i=1}^{m} \alpha_i(x^*)\, u_i.$$

□

From Theorem 83, one immediately concludes:

Corollary 84. *Let $A \in F^{m\times n}$ and $b \in F^n$ $(m, n \in \mathbb{Z}_{\geq 0})$, where F is a field. Then*

$$\exists x \in F^n : Ax = b \Leftrightarrow \nexists y \in (F^m)^T : yA = (0^n)^T \wedge yb \neq 0.$$

2.6.2. Systems of linear inequalities (Farkas lemma)

Definition 85. *An* ***ordered (skew) field*** *(also called* ***linearly ordered (skew) field****) is a tuple $(F, \leq)$, where F is a (skew) field and $\leq$ is a total order on F such that*

1. $\forall f_1, f_2, g \in F : f_1 \leq f_2 \Rightarrow f_1 + g \leq f_2 + g$,
2. $\forall f_1, f_2 \in F : 0 \leq f_1, 0 \leq f_2 \Rightarrow 0 \leq f_1 \cdot f_2$.

Definition 86. *Let $(F, \leq)$ be a linearly ordered (skew) field and let V be a vector space over F. Let $\preceq$ be a total order on V. Then $(V, \preceq)$ is called a* ***linearly ordered (skew) vector space (over F)*** *if we have*

1. $\forall v_1, v_2, w \in F : v_1 \preceq v_2 \Rightarrow v_1 + w \preceq v_2 + w$,
2. $\forall \lambda \in F, v \in V : 0 \leq \lambda, 0_V \preceq v \Rightarrow 0_V \preceq \lambda \cdot v$.

For the following theorem, one can find a short proof in [Bar12b] (also cf. [Bar07], [Bar08] and [Bar12a]).

Theorem 87. *Let V, W be vector spaces over some ordered (possibly skew) field F, where $(V, \preceq)$ is a linearly ordered vector space. Let $\alpha_1, \ldots, \alpha_m : W \to F$ $(m \in \mathbb{Z}_{\geq 0})$ be linear forms and let $\gamma : W \to V$ be a linear mapping. Then*

$$\forall x \in W : (\alpha_1(x) \leq 0 \wedge \cdots \wedge \alpha_m(x) \leq 0) \Rightarrow \gamma(x) \preceq 0_V$$
$$\Leftrightarrow \exists u_1, \ldots, u_m \in V : \gamma = \sum_{i=1}^{m} (\iota u_i) \circ \alpha_i \wedge u_1, \ldots, u_m \succeq 0_V.$$

Lemma 88 is an immediate consequence of Theorem 87. In [Sch86, Corollary 7.1d; p. 89], one can find a direct proof for the case $F = \mathbb{R}$:

Lemma 88. *Let F be an ordered field, let $A \in F^{m\times n}$ and let $b \in F^m$ $(m,n \in \mathbb{Z}_{\geq 0})$. Then*

$$\exists x \in F^n_{\geq 0} : Ax = b \Leftrightarrow \nexists y \in (F^m)^T : yA \geq (0^n)^T \wedge yb < 0.$$

Another consequence of Theorem 87 is:

Lemma 89. *Let F be an ordered field, let $A \in F^{m\times n}$ and let $b \in F^m$ $(m,n \in \mathbb{Z}_{\geq 0})$. Then*

$$\exists x \in F^n : Ax \leq b \Leftrightarrow \nexists y \in \left(F^m_{\geq 0}\right)^T : yA = (0^n)^T \wedge yb < 0.$$

For Lemma 89, we give two proofs. The first one is based on the idea of the proof of [Sch86, Corollary 7.1e; p. 89], where Lemma 89 is derived from Lemma 88. The other one is taken from the proof of [Bar07, Lemma 4.2]. In the latter one, Lemma 89 is derived directly from Theorem 87.

Proof. (Lemma 89)

First proof:

$$\begin{aligned} &\exists x \in F^n : Ax \leq b \\ \Leftrightarrow &\exists x' \in F^{2n+m}_{\geq 0} : \begin{pmatrix} A & -A & I^m \end{pmatrix} x' = b \\ \Leftrightarrow &\nexists y \in (F^m)^T : y \begin{pmatrix} A & -A & I^m \end{pmatrix} \geq \left(0^{2n+m}\right)^T \wedge yb < 0 && \text{(by Lemma 88)} \\ \Leftrightarrow &\nexists y \in \left(F^m_{\geq 0}\right)^T : yA = (0^n)^T \wedge yb < 0. \end{aligned}$$

Second proof: There exists no $x \in F^n$ having $Ax \leq b$ if and only if

$$\forall x \in F^n, t \in F : \begin{pmatrix} A & -b \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} \leq 0^m \Rightarrow \begin{pmatrix} (0^n)^T & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} \leq 0.$$

By Theorem 87, this is equivalent to

$$\exists y \in F^m_{\geq 0} : y^T \begin{pmatrix} A & -b \end{pmatrix} = \begin{pmatrix} (0^n)^T & 1 \end{pmatrix},$$

which is again equivalent to

$$\exists y \in \left(F^m_{\geq 0}\right)^T : yA = (0^n)^T \wedge yb < 0.$$

□

An important consequence of the Farkas lemma for this text is the following "affine form", which is a reformulation of [Sch86, Corollary 7.1h; p. 93]. It characterizes all valid inequalities for a non-empty polyhedron $P \subseteq \mathbb{R}^d$ $(d \in \mathbb{Z}_{\geq 0})$.

Lemma 90. *Let $\emptyset \neq P := P^{\leq}(A,b) \subseteq \mathbb{R}^d$ ($A \in \mathbb{R}^{k\times d}$ and $b \in \mathbb{R}^k$, where $d,k \in \mathbb{Z}_{\geq 0}$) be a non-empty polyhedron and let $c(\,\cdot\,) \leq c_0$ ($c \in \left(\mathbb{R}^d\right)^T$ and $c_0 \in \mathbb{R}$) be a linear inequality that is valid for P (i.e. $P \subseteq P^{\leq}(c, c_0)$). Then there exists a $y \in \left(\mathbb{R}^{k+1}_{\geq 0}\right)^T$ having*

$$\begin{pmatrix} c & c_0 \end{pmatrix} = y \begin{pmatrix} A & b \\ (0^d)^T & 1 \end{pmatrix}.$$

On the other hand, every inequality that can be represented in this form, is obviously valid for P.

2.6.3. Systems of mixed-integer linear equations

In [KW04] (also cf. [BW05, Theorem 13.3; p. 491]), it is shown:

Theorem 91. *Let $A \in \mathbb{Z}^{l\times m}$, $G \in \mathbb{Z}^{l\times n}$ and $b \in \mathbb{Z}^l$ $(l,m,n \in \mathbb{Z}_{\geq 0})$ satisfy* $\operatorname{rank}\begin{pmatrix} A & G \end{pmatrix} = l$ *(i.e. $\begin{pmatrix} A & G \end{pmatrix}$ has full row rank). Then*

$$\exists x \in \mathbb{Z}^m \times \mathbb{R}^n : \begin{pmatrix} A & G \end{pmatrix} x = b \Leftrightarrow \nexists y \in \left(\mathbb{R}^l\right)^T : y \begin{pmatrix} A & G \end{pmatrix} \in (\mathbb{Z}^m \times 0^n)^T \wedge yb \notin \mathbb{Z}.$$

We show the following tightening of Theorem 91. Its essential difference to Theorem 91 is that we get rid of the condition that $\begin{pmatrix} A & G \end{pmatrix}$ must have full row rank. Also, for convenience, we write down our formulation in a such a way that A, G and b are not required to be integral, but only rational.

Theorem 92. *Let $A \in \mathbb{Q}^{l\times m}$, $G \in \mathbb{Q}^{l\times n}$ and $b \in \mathbb{Q}^l$ $(l, m, n \in \mathbb{Z}_{\geq 0})$. Then*

$$\exists x \in \mathbb{Z}^m \times \mathbb{R}^n : \begin{pmatrix} A & G \end{pmatrix} x = b \Leftrightarrow \nexists y \in \left(\mathbb{R}^l\right)^T : y \begin{pmatrix} A & G \end{pmatrix} \in (\mathbb{Z}^m \times 0^n)^T \wedge yb \notin \mathbb{Z}.$$

Proof.

For "⇒": Assume that there exist $x^* \in \mathbb{Z}^m \times \mathbb{R}^n$ and $y^* \in \left(\mathbb{R}^l\right)^T$ such that $y^* \begin{pmatrix} A & G \end{pmatrix} \in (\mathbb{Z}^m \times 0^n)^T$, $\begin{pmatrix} A & G \end{pmatrix} x^* = b$ and $y^* b \notin \mathbb{Z}$. Then

$$\mathbb{Z} \not\ni y^* b = y^* \begin{pmatrix} A & G \end{pmatrix} x^* \in (\mathbb{Z}^m \times 0^n)^T x^* \in \mathbb{Z}. \lightning$$

For "⇐": W.l.o.g., by rescaling, we can assume that A, G and b are integral. By assumption, we have

$$\nexists y \in \left(\mathbb{R}^l\right)^T : y \begin{pmatrix} A & G \end{pmatrix} \in (\mathbb{Z}^m \times 0^n)^T \wedge yb \notin \mathbb{Z}. \tag{2.13}$$

We first show

$$\exists x^* \in \mathbb{R}^m \times \mathbb{R}^n : \begin{pmatrix} A & G \end{pmatrix} x^* = b.$$

Assume otherwise. Then, by Corollary 84, there exists a $y^* \in \left(\mathbb{R}^l\right)^T$ having

$$y^* \begin{pmatrix} A & G \end{pmatrix} = (0^m \times 0^n)^T \wedge y^* b = \frac{1}{2}.$$

This is clearly a contradiction to (2.13). Now, let $\operatorname{rank} \begin{pmatrix} A & G \end{pmatrix} =: r$ and assume w.l.o.g. that the first r rows of $\begin{pmatrix} A & G \end{pmatrix}$ are linearly independent. Then there exists some $M \in \mathbb{R}^{(l-r)\times r}$ such that

$$\begin{pmatrix} A & G \end{pmatrix} = \begin{pmatrix} I^r \\ M \end{pmatrix} \begin{pmatrix} A & G \end{pmatrix}_{[r],*}.$$

We claim that

$$b = \begin{pmatrix} I^r \\ M \end{pmatrix} b_{[r]}. \tag{2.14}$$

Assume otherwise. Then there exists an $i^* \in \{r+1, \ldots, l\}$ such that $b_i \neq M_{i^*-r,*} b_{[r]}$. Let

$$y^* := \frac{1}{2\left(M_{i^*-r,*} b_{[r]} - b_i\right)} \begin{pmatrix} M_{i^*-r,*} & -\left(e^{l-r,i^*-r}\right)^T \end{pmatrix} \in \left(\mathbb{R}^l\right)^T.$$

Then

$$\begin{aligned} y^* \begin{pmatrix} A & G \end{pmatrix} &= \frac{1}{2\left(M_{i^*-r,*} b_{[r]} - b_{i^*}\right)} \left(M_{i^*-r,*} \begin{pmatrix} A & G \end{pmatrix}_{[r],*} - \begin{pmatrix} A & G \end{pmatrix}_{i^*,*}\right) \\ &= (0^m \times 0^n)^T, \\ y^* b &= \frac{1}{2\left(M_{i^*-r,*} b_{[r]} - b_{i^*}\right)} \left(M_{i^*-r,*} b_{[r]} - b_{i^*}\right) \\ &= \frac{1}{2}, \end{aligned}$$

which contradicts (2.13).

From (2.13), we obtain

$$\nexists y \in (\mathbb{R}^r)^T : y \begin{pmatrix} A & G \end{pmatrix}_{[r],*} \in (\mathbb{Z}^m \times 0^n)^T \wedge y b_{[r]} \notin \mathbb{Z}.$$

So, by Theorem 91, there exists an $x^* \in \mathbb{Z}^m \times \mathbb{R}^n$ such that $\begin{pmatrix} A & G \end{pmatrix}_{[r],*} x^* = b_{[r]}$; thus, using (2.14), we conclude

$$\begin{pmatrix} A & G \end{pmatrix} x^* = \begin{pmatrix} I^r \\ M \end{pmatrix} \begin{pmatrix} A & G \end{pmatrix}_{[r],*} x^* = \begin{pmatrix} I^r \\ M \end{pmatrix} b_{[r]} = b.$$

□

2.7. Lattices

2.7.1. Definitions and properties

Definition 93. *Let $\{v^1, \ldots, v^m, w^1, \ldots, w^n\} \subseteq \mathbb{R}^d$ ($d, m, n \in \mathbb{Z}_{\geq 0}$) be linearly independent. Then we call the subgroup*

$$\Lambda := \begin{pmatrix} v^1 & \cdots & v^m & w^1 & \cdots & w^n \end{pmatrix} \begin{pmatrix} \mathbb{Z}^m \\ \mathbb{R}^n \end{pmatrix}$$

a ***mixed lattice*** *and denote (m, n) its* ***signature****:*

$$\operatorname{sig} \Lambda := (m, n) \, .$$

If $n = 0$, we call Λ a ***lattice****.*

Definition 94. *A (mixed) lattice Λ is called* ***rational*** *if $v^1, \ldots, v^m, w^1, \ldots, w^n$ in Definition 93 can be assumed to be rational.*

We remark that we found no established definition of a "mixed lattice" in the literature; so, this term and the term "signature of a mixed lattice" is a definition of ours. Nevertheless, for the non-mixed lattice case ($n = 0$), the definition of a lattice in Definition 93 coincides with the one used in [Sch86, p. 47].

In [Bar02, Chapter VII, section 1; p. 279], one can find another definition of a lattice: here, a lattice is defined as an additive subgroup Λ of $\left(\mathbb{R}^d, +\right)$ which spans $\mathbb{R}^d$ and is discrete (i.e. $\exists \epsilon \in \mathbb{R}_{>0} : B_\epsilon\left(0^d\right) \cap \Lambda = \left\{0^d\right\}$). It is clear that every lattice (as in Definition 93) is a discrete subgroup of $\mathbb{R}^d$. In the next theorem (Theorem 95), we show that als the reverse holds and we consider how to find a basis of a mixed lattice. It is a restatement of [Bar02, (1.4) Theorem; p. 284]:

Theorem 95. *Let $\Lambda \subsetneq \mathbb{R}^d$ ($d \in \mathbb{Z}_{\geq 0}$) be a discrete subgroup of $\mathbb{R}^d$. Let $\dim(\operatorname{lin} \Lambda) =: d'$ and let $\{b_1, \ldots, b_{d'}\} \subseteq \Lambda$ be linearly independent. For $k \in \{0, \ldots, d'\}$, define*

$$L_k := \operatorname{lin}\{b_1, \ldots, b_k\} \, .$$

For $k \in \{1, \ldots, d'\}$, let u_k be a lattice point in $L_k \backslash L_{k-1}$ that is closest to L_{k-1}. Then $\Lambda = \begin{pmatrix} v_1 & \cdots & v_{d'} \end{pmatrix} \mathbb{Z}^{d'}$ (i.e. $\{v_1, \ldots, v_{d'}\}$ is a lattice basis of Λ). In particular, a lattice basis always exists.

An immediate consequence of Theorem 95 is:

Corollary 96. *Let $V \leq \mathbb{R}^m$ ($m \in \mathbb{Z}_{\geq 0}$) be a linear subspace. Then $\mathbb{Z}^m \cap V$ is a rational lattice. If V is a rational subspace, then $\mathbb{Z}^m \cap V$ has signature $(\dim V, 0)$.*

We now formulate an immediate consequence of Theorem 95 for mixed lattices, which is easy to prove:

Theorem 97. *Let $\Lambda \subsetneq \mathbb{R}^d$ ($d \in \mathbb{Z}_{\geq 0}$) be a subgroup of $\mathbb{R}^d$. Define*

$$V := \operatorname{lineal} \Lambda, \qquad \Lambda' := \operatorname{proj}^{\perp}_{V^\perp} \Lambda.$$

Let there exists an $\epsilon \in \mathbb{R}_{>0}$ such that $B_\epsilon\left(0^d\right) \cap \Lambda' = \left\{0^d\right\}$. Then Λ' is a lattice. Let

- *$m := \dim(\operatorname{lin} \Lambda') = \dim(\operatorname{lin} \Lambda) - \dim V$,*
- *$n := \dim V$,*
- *$\{v_1, \ldots, v_m\}$ be a lattice basis of Λ' (Theorem 95 tells us how to construct one) and*
- *$\{w_1, \ldots, w_n\} \in V$ be a (vector space) basis of V.*

Then

$$\Lambda = \begin{pmatrix} v^1 & \cdots & v^m & w^1 & \cdots & w^n \end{pmatrix} \cdot \begin{pmatrix} \mathbb{Z}^m \\ \mathbb{R}^n \end{pmatrix}.$$

In particular, Λ has signature (m, n).

2.7.2. Dual representations

We now write down a lemma that gives us a way to represent rational mixed lattices in a dual way:

Lemma 98. *Let $V \leq \mathbb{R}^m$ ($m \in \mathbb{Z}^m$) be a rational subspace of codimension $k \in \{0, \dots, m\}$ and let $\{w^1, \dots, w^k\} \subseteq \mathbb{Z}^m$ be a lattice basis of $\mathbb{Z}^m \cap V^\perp$ (it exists by Corollary 96). Then*

$$\mathbb{Z}^m + V = \left\{ y \in \mathbb{R}^m : \forall i \in [k] : \left(w^i\right)^T y \in \mathbb{Z} \right\}.$$

Proof. Let $\{v^1, \dots, v^{m-k}\} \subseteq \mathbb{Q}^m$ be basis of V that consists of rational vectors. Then

$$\mathbb{Z}^m + V = \left\{ x \in \mathbb{R}^m : \exists \lambda \in \mathbb{Z}^m \times \mathbb{R}^{m-k} : \left(\begin{array}{cccc} I^m & v^1 & \cdots & v^{m-k} \end{array} \right) \lambda = x \right\} =: (2.15).$$

By Theorem 92, $x \in (2.15)$ if and only if there exists no $z \in (\mathbb{R}^m)^T$ having

- $z \in (\mathbb{Z}^m)^T$,
- $zv^i = 0$ for all $i \in [m-k]$ and
- $zx \notin \mathbb{Z}$.

Thus,

$$x \in (2.15) \Leftrightarrow \forall z \in \left(\mathbb{Z}^m \cap V^\perp\right)^T : zx \in \mathbb{Z}.$$

Since $\left\{w^1, \dots, w^k\right\}$ is a lattice basis of $\mathbb{Z}^m \cap V^\perp$, this means

$$x \in (2.15) \Leftrightarrow \forall i \in [k] : \left(w^i\right)^T x \in \mathbb{Z}.$$

□

We now show a simple consequence of Lemma 98, which becomes important in section 4.2.2.2:

Lemma 99. *Let m, V and $w^1, \dots, w^k$ be as in Lemma 98 and let $w'^1, \dots, w'^{k'}$ be arbitrary vectors from $\mathbb{Z}^m \cap V^\perp$ (in particular not necessarily linearly independent or even a lattice basis of this lattice). Then*

$$\left\{ y \in \mathbb{R}^m : \forall i \in [k'] : \left(w'^i\right)^T y \in \mathbb{Z} \right\} \supseteq \left\{ y \in \mathbb{R}^m : \forall i \in [k] : \left(w^i\right)^T y \in \mathbb{Z} \right\} = \mathbb{Z}^m + V.$$

Proof. Since $\left\{w^1, \dots, w^k\right\}$ is a lattice basis, there exist $\lambda^1, \dots, \lambda^{k'} \in \mathbb{Z}^k$ such that for all $i \in [k']$, we have $w'^i = \left(\begin{array}{ccc} w^1 & \cdots & w^k \end{array} \right) \lambda^i$. Let $y \in \mathbb{R}^m$ satisfy $\forall i \in [k] : \left(w^i\right)^T y_{(1,\dots,m)} \in \mathbb{Z}$. Then

$$\left(w'^i\right)^T y_{(1,\dots,m)} = \left(\left(\begin{array}{ccc} w^1 & \cdots & w^k \end{array} \right) \lambda^i\right)^T y_{(1,\dots,m)} = \underbrace{\left(\lambda^i\right)^T}_{\in \mathbb{Z}^k} \underbrace{\left(\begin{array}{ccc} w^1 & \cdots & w^k \end{array} \right)^T y_{(1,\dots,m)}}_{\in \mathbb{Z}^k} \in \mathbb{Z}.$$

□

2.7.3. Projections of lattices

Theorem 100. *Let $V \leq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be a rational subspace of dimension $k \in \{0, \dots, m+n\}$. Let $\left\{b^1, \dots, b^k\right\} \subseteq \mathbb{Q}^m \times \mathbb{Q}^n$ be a basis of V that consists of rational vectors. Define $B := \left(\begin{array}{ccc} b^1 & \dots & b^k \end{array} \right)$. Then $\operatorname{proj}_V^\perp (\mathbb{Z}^m \times \mathbb{R}^n)$ is a rational mixed lattice of signature $(k-s, s)$, where*

$$s = \dim \left(B \left(B^T B\right)^{-1} B^T \left(\begin{array}{c} 0^m \\ \mathbb{R}^n \end{array} \right) \right).$$

In particular

- $\max(0, k-n) \leq k - s \leq \min(m, k)$ *and*
- $\max(0, k-m) \leq s \leq \min(n, k)$.

Proof. Consider

$$\begin{aligned} V_2 &:= \operatorname{proj}_V^\perp (0^m \times \mathbb{R}^n) = B\left(B^T B\right)^{-1} B^T \begin{pmatrix} 0^m \\ \mathbb{R}^n \end{pmatrix}, && \text{(by Lemma 29)} \\ V_1 &:= (V_2)^{\perp_V}. \end{aligned}$$

By definition, $s = \dim V_2$. We surely have

$$0 \le s \le \min\left(\dim\left(B\left(B^T B\right)^{-1} B^T \begin{pmatrix} 0^m \\ \mathbb{R}^n \end{pmatrix}\right), \dim V\right) \le \min(n,k).$$

Additionally, V_1 and V_2 are rational vector spaces. Let

$$\Lambda := \operatorname{proj}_{V_1}^\perp (\mathbb{Z}^m \times 0^n).$$

By Lemma 29,

$$\bigcup_{i=1}^m \left\{\operatorname{proj}_{V_1}^\perp e^{m+n,i}\right\}$$

is a set of rational vectors. So, there exists an $N \in \mathbb{Z}_{\ge 1}$ such that

$$\bigcup_{i=1}^m \left\{\operatorname{proj}_{V_1}^\perp e^{m+n,i}\right\} \subseteq \frac{1}{N}(\mathbb{Z}^m \times \mathbb{Z}^n)$$

and thus

$$\Lambda = \operatorname{proj}_{V_1}^\perp (\mathbb{Z}^m \times 0^n) \subseteq \frac{1}{N}(\mathbb{Z}^m \times \mathbb{Z}^n),$$

which shows that Λ is discrete and thus a lattice.

We now claim that $\operatorname{proj}_V^\perp (\mathbb{Z}^m \times \mathbb{R}^n) = \Lambda + V_2$:

$$\begin{aligned} \operatorname{proj}_V^\perp (\mathbb{Z}^m \times \mathbb{R}^n) &= \operatorname{proj}_V^\perp (\mathbb{Z}^m \times 0^n) + \operatorname{proj}_V^\perp (0^m \times \mathbb{R}^n) \\ &= \operatorname{proj}_{V_1}^\perp (\mathbb{Z}^m \times 0^n) + \operatorname{proj}_{V_2}^\perp (\mathbb{Z}^m \times 0^n) + \operatorname{proj}_V^\perp (0^m \times \mathbb{R}^n) && (V = V_1 \overset{\perp}{\oplus} V_2) \\ &= \operatorname{proj}_{V_1}^\perp (\mathbb{Z}^m \times 0^n) + \operatorname{proj}_{V_2}^\perp (\mathbb{Z}^m \times 0^n) + V_2 \\ &= \operatorname{proj}_{V_1}^\perp (\mathbb{Z}^m \times 0^n) + V_2 \\ &= \Lambda + V_2. \end{aligned}$$

□

In Corollary 96, we already considered that the intersection $\mathbb{Z}^m \cap V$, where V is a (e.g. rational) subspace of $\mathbb{R}^m$, is again a lattice. We now consider

$$(\mathbb{Z}^m \times \mathbb{R}^n) + V, \tag{2.16}$$

where $V \le \mathbb{Z}^m \times \mathbb{R}^n$ is a vector space such that $\operatorname{proj}_{\mathbb{R}^m} V$ is a rational vector space of dimension k. We show that (2.16) is a mixed lattice of signature $(m-k, n+k)$:

Theorem 101. *Let $V \le \mathbb{Z}^m \times \mathbb{R}^n$ be a vector space such that $V' := \operatorname{proj}_{\mathbb{R}^m} V$ is a rational vector space of dimension k. Then $(\mathbb{Z}^m \times \mathbb{R}^n) + V$ is a mixed lattice of signature $(m-k, n+k)$.*

Proof. Clearly,

$$(\mathbb{Z}^m \times \mathbb{R}^n) + V = (\mathbb{Z}^m + V') \times \mathbb{R}^n = \left(\left(\operatorname{proj}_{V'^\perp}^\perp \mathbb{Z}^m\right) + V'\right) \times \mathbb{R}^n.$$

By Theorem 100, $\operatorname{proj}_{V'^\perp}^\perp \mathbb{Z}^m$ is a lattice of signature $(m-k, 0)$. So, by Theorem 97, $\left(\left(\operatorname{proj}_{V'^\perp}^\perp \mathbb{Z}^m\right) + V'\right) \times \mathbb{R}^n$ is a mixed lattice of signature $(m-k, n+k)$. □

2.8. Lattice-free bodies

2.8.1. Basics

Definition 102. *A convex set $S \subseteq \mathbb{R}^m$ ($m \in \mathbb{Z}_{\geq 0}$) is called* ***lattice-free*** *if $(\operatorname{int} S)_I = \emptyset$. If even $S_I = \emptyset$ is satisfied, we call it* ***strictly lattice-free****. A lattice-free set S is called* ***maximal lattice-free*** *if there exists no lattice-free set $S' \subseteq \mathbb{R}^m$ having $S' \supsetneq S$.*

We next show the "obvious" statement that every lattice-free set can be extended to a maximal lattice-free set.

Definition 103. *Let $(P, \leq)$ be a partially ordered set (poset). An element $p \in P$ is called* ***maximal with respect to*** *$\leq$ if for all $p' \in P$, we have $p' \geq p \Rightarrow p' = p$.*

Remark 104. *If one does not one does not demand $\leq$ to be antrisymmetric in Definition 103 (only reflexive and transitive), one defines instead: an element $p \in P$ is called* ***maximal with respect to*** *$\leq$ if for all $p' \in P$, we have: $p' \geq p \Rightarrow p' \leq p$.*

Definition 105. *Let $(P, \leq)$ be a partially ordered set. A totally ordered (with respect to $\leq$) subset of P is called a* ***chain in*** *P.*

Lemma 106. *(Zorn's lemma) Let $(P, \leq)$ ($P \neq \emptyset$) be a partially ordered set such that each chain in P has an upper bound in P. Then P contains at least one maximal element (with respect to $\leq$).*

The following lemma, which shows that every lattice-free body can be embedded into a *maximal* lattice-free body, seems in principle to be well-known in the literature. But at least in none of the papers where one can find a proof of the structure theorem (Theorem 108) for maximal lattice-free bodies or a sketch thereof ([Lov89], [BCCZ10] and [Ave13]; see below), we could find a proof for it. Thus, we came up with an own short proof.

Lemma 107. *Let $P \subsetneq \mathbb{R}^m$ ($m \in \mathbb{Z}_{\geq 0}$) be lattice-free. Then there exists a maximal lattice-free set Q having $P \subseteq Q$.*

Proof. Via "$\subseteq$", one defines a partial order on the set of all lattice-free (convex) subsets of $\mathbb{R}^m$ containing P (which we want to denote $\mathcal{L}_P$ in this proof). Let $\mathcal{C}$ be a totally ordered subset of $\mathcal{L}_P$ (chain in $\mathcal{L}_P$). We show that it has an upper bound in $\mathcal{L}_P$ to be able to apply Zorn's lemma. For this, we consider

$$\operatorname{conv}\left(\bigcup_{C \in \mathcal{C}} C\right) =: \hat{C}.$$

By definition, $\hat{C}$ is convex. Assume $\hat{C}$ is not lattice-free. This is equivalent to the existence of an $x^* \in \left(\operatorname{int} \hat{C}\right) \cap \hat{C}_I$. So, there exists an $\epsilon \in \mathbb{R}_{>0}$ such that $B_\epsilon(x^*) \subseteq \hat{C}$.

This implies the existence of affinely independent points $p^1, \ldots, p^{m+1} \in \mathbb{R}^m$ having

- $\forall i \in [m+1] : p^i \in B_\epsilon(x^*) \subseteq \hat{C}$ and
- $x^* \in \operatorname{int}\left(\operatorname{conv}\left\{p^1, \ldots, p^{m+1}\right\}\right)$.

Since the set $\mathcal{C}$ is totally ordered, there exists a $C^* \in \mathcal{C}$ having $p^1, \ldots, p^{m+1} \in C^*$. But this means that C^* is not lattice-free and thus $\hat{C} \supseteq C^*$ is also not.

So, $\hat{C}$ is indeed lattice-free and we have shown that for all totally ordered subsets of $\mathcal{L}_P$, there exists an upper bound in $\mathcal{L}_P$. So, we conclude from Zorn's lemma that there exists a a maximal lattice-free set Q containing P. $\square$

In [Ave13, Theorem 1 and Theorem 2], it is shown:

Theorem 108. *Let $K \subseteq \mathbb{R}^m$ ($m \in \mathbb{Z}_{\geq 1}$) be a maximal lattice-free body.*

- *If $\dim K = m$, let $k := \dim(\operatorname{rec} K) \in \{0, \ldots, m-1\}$. Then the following conditions hold:*
  - *K is a polyhedron with at most 2^{m-k} facets,*
  - *$\operatorname{rec} K = \operatorname{lineal} K$ forms a linear space and*
  - *the relative interior of every facet of K contains at least one point from $\mathbb{Z}^d$.*

- *If* $\dim K < m$*, then* K *is a translate of an* $(m-1)$*-dimensional linear space* L *having* $L \neq \operatorname{lin}(L \cap \mathbb{Z}^m)$*.*

Remark 109. *The bound* 2^{m-k} *on the number of facets of a full-dimensional lattice-free body* $K \subseteq \mathbb{R}^m$ *(*$m \in \mathbb{Z}_{\geq 1}$*) having* $\dim(\operatorname{lineal} K) = k$ *(*$k \in \{0, \ldots, m-1\}$*) is tight. For this, consider the maximal lattice-free polyhedron*

$$K := \left\{ x \in \mathbb{R}^m : \forall I \in \mathcal{P}([m-k]) : \sum_{i \in I} x_i - \sum_{i \in [m-k] \setminus I} x_i \leq |I| \right\}.$$

We remark that the first complete proof of Theorem 108 can be found in [BCCZ10]. [Lov89, section 3] contains a sketch of a proof for the statements of Theorem 108 in the case that K is a full-dimensional.

We now show a theorem which is in a sense a converse of the characterization of full-dimensional, maximal lattice-free bodies in Theorem 108. This converse is to be understood in the meaning that if we can show that every facet of a lattice-free polyhedron contains an integral points in its relative interior (which was a condition for a full-dimensional maximal lattice-free body in Theorem 108), we have a certificate that the given polyhedron is indeed *maximal* lattice-free (and full-dimensional).

Theorem 110. *Let* $P = P^{\leq}(A, b) \subseteq \mathbb{R}^m$*, where* $A \in \mathbb{R}^{t \times m}$ *and* $b \in \mathbb{R}^t$ *(*$m, t \in \mathbb{Z}_{\geq 1}$*), be a lattice-free polyhedron. Let for every* $i \in [t]$ *exist a* $z^i \in \mathbb{Z}^m$ *such that* $A_{i,*} z^i = b_i \wedge A_{[t] \setminus \{i\},*} z^i < b_{[t] \setminus \{i\}}$*. Then* P *is a full-dimensional maximal lattice-free body.*

Proof.

For P being full-dimensional: It suffices to show that $P^{<}(A, b) \neq \emptyset$. It is easy to check that for P to be lattice-free, we need $t \geq 2$. Let $x^* := \frac{1}{t} \sum_{j \in [t]} z^j \in P$. Then for every $i \in [t]$, we have

$$A_{i,*} x^* = \frac{1}{t} A_{i,*} z^i + \frac{1}{t} \sum_{j \in [t] \setminus \{i\}} A_{i,*} z^j = \frac{1}{t} b_i + \frac{1}{t} \sum_{j \in [t] \setminus \{i\}} A_{i,*} z^j < \frac{1}{t} b_i + \frac{t-1}{t} b_i = b_i.$$

For P being maximal lattice-free: Assume that there exists a lattice-free set $P' \subseteq \mathbb{R}^m$ such that $P' \supsetneq P$. Let $x^* \in P' \setminus P$. Then there exists an $i^* \in [t]$ such that $A_{i^*,*} x^* > b_{i^*}$. For disproving that P' is lattice-free, we show that $z^{i^*} \in \operatorname{int} P'$. Since $z^{i^*} \in \operatorname{relint} P \cap P^{=}(A_{i^*,*}, b_{i^*})$ and P is full-dimensional, there exists an $\epsilon > 0$ such that

$$\overline{B}_\epsilon \left(z^{i^*} \right) \cap P^{\leq}(A_{i^*,*}, b_{i^*}) \subseteq P \subseteq P'. \tag{2.17}$$

Thus, there exist affinely independent points $s^1, \ldots, s^m \in \mathbb{R}^m$ such that for all $i \in [m]$, we have

$$\begin{aligned} \left\| s^i - z^{i^*} \right\| &= \epsilon, \\ s^i - z^{i^*} &\perp (A_{i^*,*})^T, \\ z^{i^*} &\in \operatorname{int}\left(\operatorname{conv}\left\{ s^1, \ldots, s^m \right\} \right). \end{aligned}$$

Surely, $\operatorname{conv}\left\{ s^1, \ldots, s^m, x^* \right\} \subseteq P'$. Let

$$\operatorname{conv}\left\{ s^1, \ldots, s^m, x^* \right\} =: P^{\leq}(C, c) \cap P^{\geq}(A_{i^*,*}, b_{i^*}) \subseteq P',$$

where $C \in \mathbb{R}^{m \times m}$, $c \in \mathbb{R}^m$ and

$$\forall i \in [m] : \|C_{i,*}\| > 0. \tag{2.18}$$

We clearly have $C z^{i^*} < c$. We claim that

$$B_{\epsilon'} \left(z^{i^*} \right) \subseteq P', \tag{2.19}$$

where

$$\epsilon' := \min \left(\epsilon, \frac{c_1 - C_{1,*} z^{i^*}}{\|C_{1,*}\|}, \ldots, \frac{c_m - C_{m,*} z^{i^*}}{\|C_{m,*}\|} \right). \tag{2.20}$$

For (2.19): Let

$$x \in B_{\epsilon'} \left(z^{i^*} \right). \tag{2.21}$$

If $A_{i^*,*}x \leq b_{i^*}$, equation (2.19) is implied by (2.17) and $\epsilon' \leq \epsilon$. On the other hand, if $A_{i^*,*}x > b_{i^*}$, we have, by (2.21), that $x = z^{i^*} + u$, where $A_{i^*,*}u > 0$ and $\|u\| < \epsilon'$. Then for every $i \in [m]$, we have

$$\begin{aligned}
C_{i,*}x &= C_{i,*}z^{i^*} + C_{i,*}u \\
&\leq C_{i,*}z^{i^*} + \|C_{i,*}\| \cdot \|u\| && \text{(Cauchy-Schwarz inequality)} \\
&< C_{i,*}z^{i^*} + \|C_{i,*}\| \cdot \epsilon' && \text{(by (2.18) and } 0 < \|u\| < \epsilon'\text{)} \\
&\leq C_{i,*}z^{i^*} + \|C_{i,*}\| \cdot \frac{c_i - C_{i,*}z^{i^*}}{\|C_{i,*}\|} && \text{(by (2.20))} \\
&= c_i.
\end{aligned}$$

So, we finally conclude from (2.19) that P' cannot be lattice-free. □

2.8.2. Extension of lattice-free bodies

We next show that if we have some strictly lattice-free body $P^<(A,b)$ (A integral) be given such that no inequality in this description is redundant with respect to this property, we can find some integral $b^* \geq b$ such that $P^\leq(A,b^*)$ is full-dimensional and maximal lattice-free.

Theorem 111. *Let $P = P^\leq(A,b) \subseteq \mathbb{R}^m$, where $A \in \mathbb{Z}^{t\times m}$, $b \in \mathbb{R}^t$ and $m,t \in \mathbb{Z}_{\geq 1}$, be a polyhedron with rational face normals that satisfies $P^<(A,b)_I = \emptyset$ (i.e. $P^<(A,b) = \operatorname{int} P$ is strictly lattice-free) such that for every $i \in [t]$, we have: $P^<\left(A_{[t]\setminus\{i\}}, b_{[t]\setminus\{i\}}\right)_I \neq \emptyset$ (i.e. no inequality in the description $A(\cdot) < b$ is redundant for $P^<(A,b)$ to be strictly lattice-free). Then there exists a $b^* \in \mathbb{Z}^t$, where $b^* \geq b$, such that $P^\leq(A,b^*)$ is full-dimensional and maximal lattice-free.*

Proof. For every $i \in [t]$, there exists a $z^i \in \mathbb{Z}^m$ such that $A_{[t]\setminus\{i\}}z^i < b_{[t]\setminus\{i\}}$ and $A_{i,*}z^i =: \overline{b}_i \geq b_i$. We consider the set

$$B := \left\{b' \in \mathbb{Z}^t : b \leq b' \leq \overline{b} \wedge P^\leq(A,b') \text{ is lattice-free}\right\},$$

which is ordered by the natural order $\leq$ on $\mathbb{Z}^t$. Since $B \neq \emptyset$ and B is finite, there exists a maximal element $b^* \in B$. We claim that $P^\leq(A,b^*)$ is maximal lattice-free. For this, we show

$$\forall i \in [t]\; \exists z^i \in \mathbb{Z}^m : A_{[t]\setminus\{i\},*}z^i < b^*_{[t]\setminus\{i\}} \wedge A_{i,*}z^i = b^*_i. \tag{2.22}$$

We then can conclude from Theorem 110 that $P^\leq(A,b^*)$ is full-dimensional and maximal lattice-free.

Assume that there exists an $i^* \in [t]$ for which (2.22) does not hold, i.e.

$$\nexists z \in \mathbb{Z}^m : A_{[t]\setminus\{i^*\},*}z < b^*_{[t]\setminus\{i^*\}} \wedge A_{i^*,*}z = b^*_{i^*}. \tag{2.23}$$

We show that under this assumption, also $P' := P^\leq\left(A, b^* + e^{t,i^*}\right)$ is lattice-free. This yields a contradiction to the maximality of b^* if we can additionally show $b^* + e^{t,i^*} \in B$.

Let $z \in \mathbb{Z}^m$. For proving that P' is lattice-free, we show that $z \notin P^<\left(A, b^* + e^{t,i^*}\right)$. Since $P^\leq(A,b^*)$ is lattice-free, there exists an $i^z \in [t]$ such that $A_{i^z,*}z \geq b^*_{i^z}$. If $i^z \neq i^*$ or $i^z = i^* \wedge A_{i^z,*}z \geq b^*_{i^z} + 1$, there is nothing to show. Thus, we only need to consider the case $i^z = i^* \wedge A_{i^z,*}z = b^*_{i^z}$, i.e. we can assume $A_{i^*,*}z = b^*_{i^*}$. Because of (2.23), we conclude

$$\neg\left(A_{[t]\setminus\{i^*\},*}z < b^*_{[t]\setminus\{i^*\}}\right).$$

Thus, there exists an $i^{**} \in [t] \setminus \{i^*\}$ such that $A_{i^{**},*}z \geq b^*_{i^{**}}$, which shows that $z \notin P^<\left(A, b^* + e^{t,i^*}\right)$.

To derive a final contradiction to the maximality of b^*, we need to show that $b^* + e^{t,i^*} \in B$. Clearly,

$$b_{[t]\setminus\{i^*\}} \leq \left(b^* + e^{t,i^*}\right)_{[t]\setminus\{i^*\}} \leq \overline{b}_{[t]\setminus\{i^*\}}.$$

So, we just need to ensure that $b_{i^*} \leq b^*_{i^*} + 1 \leq \overline{b}_{i^*}$. The inequality $b_{i^*} \leq b^*_{i^*} + 1$ is a consequence of $b_{i^*} \leq b^*_{i^*}$. So for $b^*_{i^*} + 1 \leq \overline{b}_{i^*}$: since we just showed that $P' = P^\leq\left(A, b^* + e^{t,i^*}\right)$ is lattice-free, we have $A_{i^*,*}z^{i^*} \geq b^*_{i^*} + 1$. On the other hand, by construction, we have $A_{i^*,*}z^{i^*} = \overline{b}_{i^*}$. Thus, $b^*_{i^*} + 1 \leq \overline{b}_{i^*}$. □

2.9. Diverse topics

In this last section of chapter 2, we introduce two topics that do not really fit anywhere else in this chapter:

- In section 2.9.1, Lemma 113, we prove some inequalities involving fractional parts. These play an important role in part III, in particular chapters 8 and 9.
- In section 2.9.2, we introduce some polyhedra that are important examples in various subsequent chapters of this text.

2.9.1. Inequalities involving fractional parts

Definition 112. *For $x \in \mathbb{R}$, define*

$$\operatorname{frac} x := x - \lfloor x \rfloor$$

(fractional part of x).

Lemma 113. *Let $x, y \in \mathbb{R}$. Then*

$$\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor, \tag{2.24}$$

$$\operatorname{frac}(x + y) \leq \operatorname{frac} x + \operatorname{frac} y, \tag{2.25}$$

$$\operatorname{frac} x + \operatorname{frac} y \leq 1 + \operatorname{frac}(x + y), \tag{2.26}$$

$$\operatorname{frac}(x + y)(\operatorname{frac} x + \operatorname{frac} y) \leq (\operatorname{frac}(x + y))^2 + \operatorname{frac} y. \tag{2.27}$$

Proof.

For (2.24):

$$\lfloor x \rfloor + \lfloor y \rfloor = \lfloor \lfloor x \rfloor + \lfloor y \rfloor \rfloor \leq \lfloor x + y \rfloor.$$

For (2.25): Using (2.24), we obtain

$$\operatorname{frac}(x + y) = x + y - \lfloor x + y \rfloor \leq x + y - \lfloor x \rfloor - \lfloor y \rfloor = \operatorname{frac} x + \operatorname{frac} y.$$

For (2.26):

$$\begin{aligned} \operatorname{frac} x + \operatorname{frac} y &= \begin{cases} \operatorname{frac}(x + y) & \text{if } \operatorname{frac} x + \operatorname{frac} y < 1, \\ 1 + \operatorname{frac}(x + y) & \text{if } \operatorname{frac} x + \operatorname{frac} y \geq 1 \end{cases} \\ &\leq 1 + \operatorname{frac}(x + y). \end{aligned}$$

For (2.27): We distinguish two cases:

1. $\operatorname{frac}(x + y) = \operatorname{frac} x + \operatorname{frac} y$,
2. $\operatorname{frac}(x + y) = \operatorname{frac} x + \operatorname{frac} y - 1$.

For case 1:

$$\operatorname{frac}(x + y)(\operatorname{frac} x + \operatorname{frac} y) = (\operatorname{frac} x + \operatorname{frac} y)^2 \leq (\operatorname{frac}(x + y))^2 + \operatorname{frac} y.$$

For case 2:

$$\begin{aligned} \operatorname{frac}(x + y)(\operatorname{frac} x + \operatorname{frac} y) &= \operatorname{frac}(x + y)(\operatorname{frac}(x + y) + 1) \\ &= (\operatorname{frac}(x + y))^2 + \operatorname{frac}(x + y) \\ &= (\operatorname{frac}(x + y))^2 + \operatorname{frac} y + \underbrace{\operatorname{frac} x - 1}_{<0} \\ &< (\operatorname{frac}(x + y))^2 + \operatorname{frac} y. \end{aligned}$$

□

2.9.2. Important polyhedra

Definition 114. *Define*

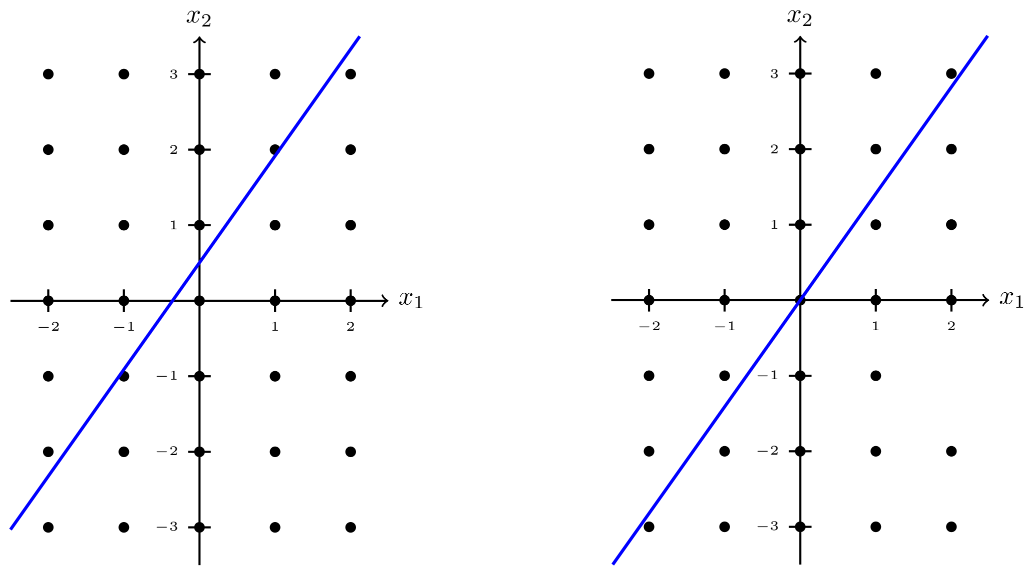
$$P^{114} := \left\{ \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \right\} + \operatorname{lin} \left\{ \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \right\} \subsetneq \mathbb{R}^2.$$

Obviously, $\left(P^{114}\right)_I = \emptyset$ *and* P^{114} *is a maximal lattice-free body (cf. Theorem 108).*

Definition 115. *A variant of* P^{114}*, which contains an integral point in its relative interior, is*

$$P^{115} := \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} + \operatorname{lin} \left\{ \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \right\} \subsetneq \mathbb{R}^2.$$

Obviously, $\left(P^{115}\right)_I = \left\{0^2\right\}$ *and* P^{115} *is a maximal lattice-free body (cf. Theorem 108).*



(a) P^{114} and $\mathbb{Z}^2$ (b) P^{115} and $\mathbb{Z}^2$

Figure 2.3.: Visualization of P^{114} and P^{115}

In Figure 2.3, one can see a visualization of P^{114} and P^{115}.

Definition 116. *Let* $1, h_1, \dots, h_{m-1} \in \mathbb{R}$ $(m \in \mathbb{Z}_{\geq 2})$ *be linearly independent over* $\mathbb{Q}$ $(1, h_1, \dots, h_{m-1}$ *clearly exist). Define*

$$P^{116,m} := \left\{ \frac{1}{2} e^{m,1} \right\} + \operatorname{lin} \dot{\bigcup_{i=2}^{m}} \left\{ h_{i-1} \cdot e^{m,1} + e^{m,i} \right\} \subsetneq \mathbb{R}^m.$$

Obviously, $\left(P^{116,m}\right)_I = \emptyset$ *and* $P^{116,m}$ *is a maximal lattice-free body (cf. Theorem 108).*

Definition 117. *Let* $h_1, \dots, h_{m-1}$ *be as in Definition 116. Define*

$$P^{117,m} := \{0^m\} + \operatorname{lin} \dot{\bigcup_{i=2}^{m}} \left\{ h_{i-1} \cdot e^{m,1} + e^{m,i} \right\} \subsetneq \mathbb{R}^m.$$

Obviously, $\left(P^{117,m}\right)_I = \{0^m\}$ *and* $P^{117,m}$ *is a maximal lattice-free body (cf. Theorem 108).*

Definition 118. *Let*

$$P^{118} := \operatorname{conv} \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{3}{4} \\ 0 \end{pmatrix} \right\} + \operatorname{lin} \left\{ \begin{pmatrix} 0 \\ 1 \\ \sqrt{2} \end{pmatrix} \right\} \subsetneq \mathbb{R}^1 \times \mathbb{R}^2.$$

P^{118} *has a partially rational recession code, which is at the same time its lineality space. We have*

$$\mathrm{cl}_I\left(P^{118}\right) = \mathrm{cl}_{\overline{I}}\left(P^{118}\right) = \mathrm{conv}\left\{\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}\right\} + \mathrm{lin}\left\{\begin{pmatrix}0\\1\\\sqrt{2}\end{pmatrix}\right\}.$$

A variant of Definition 118 is:

Definition 119. *Let*

$$P^{119} := \mathrm{conv}\left\{\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}\frac{1}{2}\\\frac{3}{4}\\0\end{pmatrix}\right\} + \mathrm{cone}\left\{\begin{pmatrix}0\\1\\\sqrt{2}\end{pmatrix}\right\} \subsetneq \mathbb{R}^1 \times \mathbb{R}^2.$$

P^{119} *has a partially rational recsession code, but in contrast to* P^{118}*, its lineality space only consists of* $\left\{\begin{pmatrix}0^1\\0^2\end{pmatrix}\right\}$*. We have*

$$\mathrm{cl}_I\left(P^{119}\right) = \mathrm{cl}_{\overline{I}}\left(P^{119}\right) = \mathrm{conv}\left\{\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}\right\} + \mathrm{cone}\left\{\begin{pmatrix}0\\1\\\sqrt{2}\end{pmatrix}\right\}.$$

3. Cutting planes

In this chapter, we define classes of cutting planes that have been studied in the literature and are important for this thesis. What is the idea behind cutting planes? Consider the problem

$$\sup \{cx : x \in P_I\}, \tag{3.1}$$

where $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) is arbitrary (though typically a rational polyhedron), $c \in (\mathbb{R}^m \times \mathbb{R}^n)^T$ and (recall Definition 73) $P_I := P \cap (\mathbb{Z}^m \times \mathbb{R}^n)$ (i.e. the vector components indexed by $\{1, \ldots, m\}$ are assumed to be integral). By Theorem 71, (3.1) attains the same optimum as

$$\sup \{cx : x \in \operatorname{cl}_{\overline{I}}(P)\}. \tag{3.2}$$

We know from Theorem 75 that under some conditions, $\operatorname{cl}_{\overline{I}}(P) = \operatorname{cl}_I(P)$ is a polyhedron. This means that "we just have to find this finite number of inequalities describing $\operatorname{cl}_I(P)$ and add them to the inequality description of P". Unluckily, finding them directly is in general a very hard problem, since there is no known fast algorithm for this purpose. So, we try a more indirect approach: we consider inequalities $c(\cdot) \leq c_0$ that are valid for P_I, but not for P, and add them iteratively to P, until we obtain $\operatorname{cl}_I(P)$ (or, if P was not convex at the beginning, but $\operatorname{cl}_I(P)$ is a polyhedron, a set *containing* $\operatorname{cl}_I(P) = \operatorname{cl}_{\overline{I}}(P)$).

Another perspective from which one can consider cutting planes is to consider the relaxation

$$\sup \{cx : x \in P\}$$

of (3.1) and (3.2). Assume that we have

$$M_{relax} := \sup \{cx : x \in P\} > \sup \{cx : x \in P_I\} =: M_{opt}. \tag{3.3}$$

Then we are intested in adding linear inequalities to P to decrease the **integrality gap** $M_{relax} - M_{opt}$. This is another application that one can use cutting planes for. Of course, if we find a cutting plane that closes the integrality gap to 0, we have solved (3.1). So, again, one typically rather adds cutting planes iteratively to decrease the gap further and further (hopefully reaching an integrality gap of 0 in a finite number of steps).

At this place, we already want to remark a slightly subtile difference between these two applications of cutting planes: While finding an inequality description for $\operatorname{cl}_{\overline{I}}(P)$ via a finite number of cutting planes (of some arbitrary class) is only possible if $\operatorname{cl}_{\overline{I}}(P)$ is a polyhedron, finding a cutting plane that closes the integrality gap in (3.3) to 0 is *always* possible (just add the cutting plane $c(\cdot) \leq M_{opt}$). This does, of course, *not* imply that closing the integrality gap via a *concrete given class of cutting planes* is always possible.

3.1. Definition of a cutting plane

After we already explained cutting planes informally in the introduction of this chapter, we now introduce them formally:

Definition 120. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be arbitrary. A* ***cutting plane for*** *P is a linear inequality $c(\cdot) \leq c_0$ ($c \in (\mathbb{R}^m \times \mathbb{R}^n)^T$ and $c_0 \in \mathbb{R}$) that is valid for P_I.*

We remark that in the literature, it is often additionally demanded that $c(\cdot) \leq c_0$ is *not* valid for P. Why don't we demand such a condition in this text?

In general, for classes of cutting planes that we consider in this text (for example split cuts; cf. Definition 126), it is easy to find instances of cutting planes of this specific class that are already valid for P. This would mean that only a subset of, for example, split cuts are really cutting planes. In this sense split cuts would *not* be cutting planes; rather only a subset of split cuts would be.

A second reason not to demand such a a condition is that in chapter 4, we introduce the class of what we named $L_{k,\mathbb{Q}}$ cuts ($k \in \{0, \ldots, m+n\}$); cf. Definition 161. For formal reasons, it makes sense also to include $L_{0,\mathbb{Q}}$ cuts as a degenerate case. $L_{0,\mathbb{Q}}$ cuts can be shown to be exactly the inequalities that are valid for P (cf.

Remark 162). Demanding such a condition would imply that $L_{0,\mathbb{Q}}$ cuts are not cutting planes, even though $L_{k,\mathbb{Q}}$ cuts typically are for $k \geq 1$.

This together gives reasonable grounds why we don't demand an inequality $c(\cdot) \leq c_0$ not to be valid for P to be a cutting plane (of course, in practice, one is mostly intested in cutting planes *with* this property).

Since we want to compare cutting planes with respect to their expressivity, we define what it means for an inequality to be dominated:

Definition 121. *Let $c, c^1, \ldots, c^k \in (\mathbb{R}^m \times \mathbb{R}^n)^T$ $(m, n \in \mathbb{Z}_{\geq 0})$ and let $c_0, c_0^1, \ldots, c_0^k \in \mathbb{R}$.*

- *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$. The inequalities $c^1(\cdot) \leq c_0^1$, ..., $c^k(\cdot) \leq c_0^k$* ***dominate the inequality*** *$c(\cdot) \leq v_0$* ***relatively to*** *P* ***if***

$$P \cap P^{\leq}\left(\begin{pmatrix} c^1 \\ \vdots \\ c^k \end{pmatrix}, \begin{pmatrix} c_0^1 \\ \vdots \\ c_0^k \end{pmatrix}\right) \subseteq P \cap P^{\leq}(c, c_0).$$

- *The inequalities $c^1(\cdot) \leq c_0^1$, ..., $c^k(\cdot) \leq c_0^k$* ***dominate the inequality*** *$c(\cdot) \leq v_0$* ***absolutely*** *if $c^1(\cdot) \leq c_0^1$, ..., $c^k(\cdot) \leq c_0^k$ dominate $c(\cdot) \leq c_0$ relatively to $\mathbb{R}^m \times \mathbb{R}^n$, i.e.*

$$P^{\leq}\left(\begin{pmatrix} c^1 \\ \vdots \\ c^k \end{pmatrix}, \begin{pmatrix} c_0^1 \\ \vdots \\ c_0^k \end{pmatrix}\right) \subseteq P^{\leq}(c, c_0).$$

Why do we distinguish between absolute dominance versus relative dominance, which is often not done in the literature? We give three examples:

- In Theorem 385, we show that for $\emptyset \neq P := P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)$, every dual projected Chvátal-Gomory cut with respect to A, G and b is a projected Chvátal-Gomory cut for P. In Remark 386, we state that the reverse only holds up to absolute dominance (i.e. not every projected Chvátal-Gomory cut is a dual projected Chvátal-Gomory cut).
- In Theorem 389, we show that for a convex $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$, every strong projected Chvátal-Gomory cut for P is an $L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cut for P and the reverse holds up to relative dominance, but, as we show in Remark 390, not absolute dominance.
- In Definition 126 (in this chapter), we introduce split cuts and in Definition 410, we introduce MIR cuts. In Theorem 412, we show that every MIR cut is a split cut (thus, split cuts dominate MIR cuts absolutely) and in Theorem 415, we show that every split cut for a polyhedron is dominated relatively to it by a MIR cut. On the other hand, not every split cut is a MIR cut or dominated *absolutely* by a MIR cut (cf. Example 416).

All of these examples have a subtile asymmetry in the dominance behaviour of two different classes of cutting planes, which only becomes apparent if one is very precise about what kind of dominance behaviour is present.

3.2. (Projected) Chvátal-Gomory cuts

The arguably most elemantary class of cutting planes that we consider in this text are the so-called **(projected) Chvátal-Gomory cuts**.

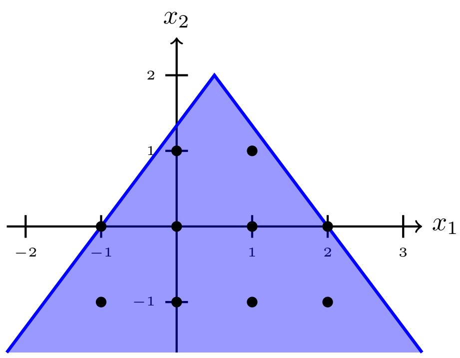
Definition 122. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ be arbitrary. Let $c \in (\mathbb{Z}^m)^T$ and let $c_0 \in \mathbb{R}$ (w.l.o.g. we can assume $c_0 \in \mathbb{R}\backslash\mathbb{Z}$) be such that*

$$P \subseteq P^{\leq}\left(\begin{pmatrix} c & (0^n)^T \end{pmatrix}, c_0\right).$$

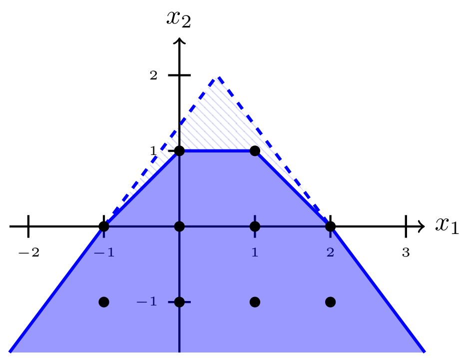
Then the inequality

$$\begin{pmatrix} c & (0^n)^T \end{pmatrix}(\cdot) \leq \lfloor c_0 \rfloor$$

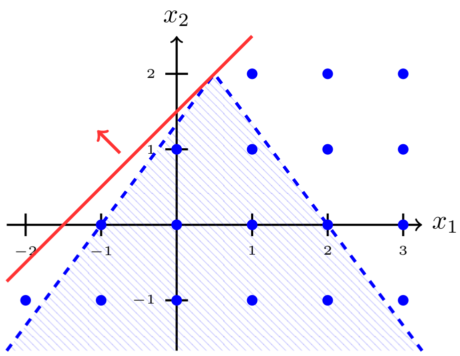
is called a ***projected Chvátal-Gomory cut for*** *P. If $n = 0$, we simply use the term* ***Chvátal-Gomory cut for*** *P.*



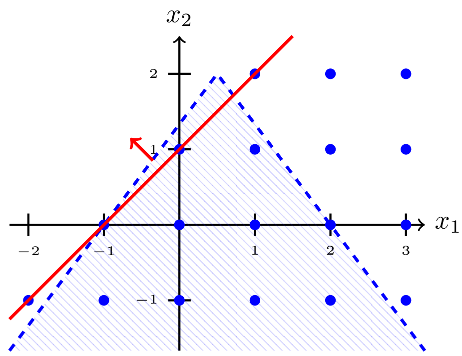
(a) $P := \left\{x \in \mathbb{R}^2 : \begin{pmatrix} -4 & 3 \\ 4 & 3 \end{pmatrix} x \leq \begin{pmatrix} 4 \\ 8 \end{pmatrix}\right\}$ and $P \cap \mathbb{Z}^2$



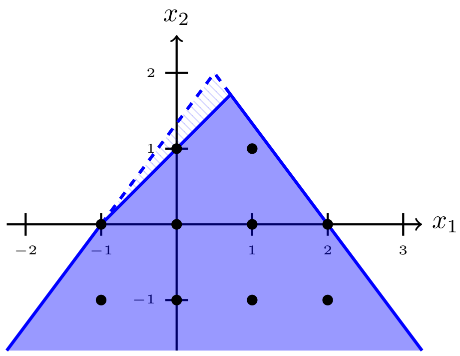
(b) $\operatorname{cl}_I(P)$ (for reference)



(c) $-x_1 + x_2 \leq \frac{3}{2}$ is valid for $x \in P$;
$\left\{x \in \mathbb{Z}^2 : -x_1 + x_2 \leq \frac{3}{2}\right\}$



(d) $-x_1 + x_2 \leq \left\lfloor \frac{3}{2} \right\rfloor$ is a Chvátal-Gomory cut for $x \in P$;
$\left\{x \in \mathbb{Z}^2 : -x_1 + x_2 \leq \frac{3}{2}\right\}$



(e) $P \cap \left\{x \in \mathbb{R}^2 : -x_1 + x_2 \leq \left\lfloor \frac{3}{2} \right\rfloor\right\}$

Figure 3.1.: Illustration of Chvátal-Gomory cuts

We remark that the term "*projected* Chvátal-Gomory cut" was to our knowledge first introduced in [BCD+08], where, as far as we know, for the first time the question was considered how Chvátal-Gomory cuts (in the form of projected Chvátal-Gomory cuts) can also be used for *mixed*-integer linear programs (MILPs).

In Figure 3.1, one can see an illustration of the construction behind Chvátal-Gomory cuts.

We next define the (projected) Chvátal-Gomory closure:

Definition 123. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ be arbitrary. Then we define*

$$\operatorname{cl}_{pCG}(P) := P \cap \bigcap_{\substack{c \in (\mathbb{Z}^m)^T, c_0 \in \mathbb{R}: \\ P \subseteq P^{\leq}\left(\left(\begin{smallmatrix} c & (0^n)^T \end{smallmatrix}\right), c_0\right)}} P^{\leq}\left(\left(\begin{matrix} c & (0^n)^T \end{matrix}\right), \lfloor c_0 \rfloor\right)$$

as the ***projected Chvátal-Gomory closure of*** *P. If $n = 0$, we also use the term* ***Chvátal-Gomory closure of*** *P $(\operatorname{cl}_{CG}(P))$.*

Remark 124. *Why do we, in contrast to the definition that one can often find in the literature, e.g. in [Sch86, p. 339], additionally intersect over P for defining the (projected) Chvátal-Gomory closure of P? The* *reason is that we would otherwise, for example, have* $\mathrm{cl}_{CG}\left(P^{114}\right) = \mathbb{R}^2$. *By adding* P *to the intersection, we can ensure that always* $\mathrm{cl}_{CG}(P) \subseteq P$ *holds; in other words: we can ensure that the Chvátal-Gomory closure of* P *is a relaxation of* P_I *that is "at least as strong as* P*".*

At this place, we state the process of assigning a closure operator with respect a particular class of cutting planes to some $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ explicitely, since this is a construction that arguably pervades the whole text: to define a closure of some P with respect a class of cutting planes, we intersect P with all half-spaces that correspond to a cutting planes of the respective type.

3.3. Split cuts

Definition 125. *Let* $\pi \in \mathbb{Z}^m$ *and* $\gamma \in \mathbb{Z}$ *(*$m \in \mathbb{Z}_{\geq 0}$*). Then the set*

$$D(\pi, \gamma) := \left\{x \in \mathbb{R}^m : \pi^T x \leq \gamma \vee \pi^T x \geq \gamma + 1\right\}$$

is called a ***split disjunction*** *and*

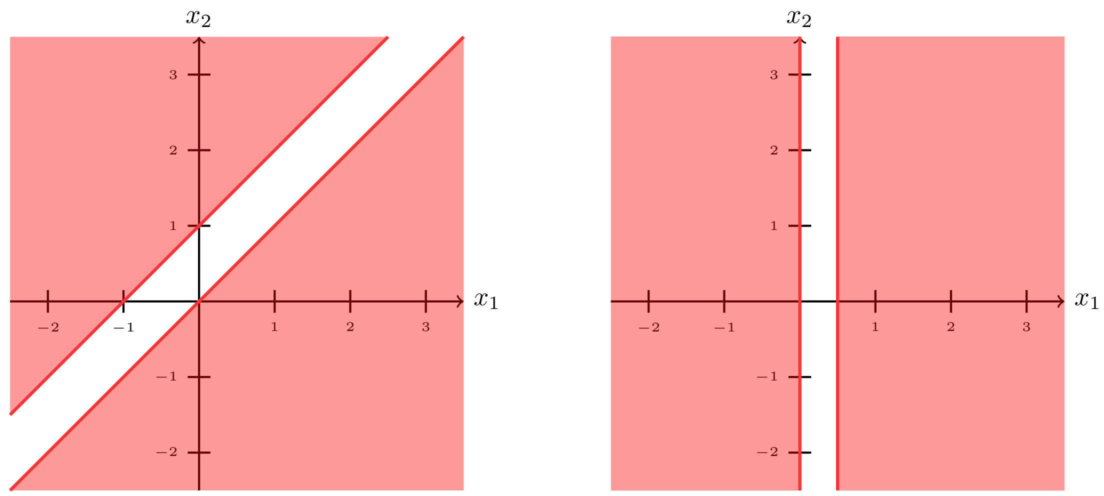
$$S(\pi, \gamma) := \left\{x \in \mathbb{R}^m : \gamma < \pi^T x < \gamma + 1\right\}$$

is called a ***split set****.*

Two split disjunctions are visualized in Figure 3.2.

We next define split cuts:

Definition 126. *Let* $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ *(*$m, n \in \mathbb{Z}_{\geq 0}$*) be arbitrary. A* ***split cut for*** P *is a valid inequality for* $P \cap (D(\pi, \gamma) \times \mathbb{R}^n)$ *for some split disjunction* $D(\pi, \gamma)$*.*



(a) The split disjunction $D\left(\begin{pmatrix}-1\\1\end{pmatrix}, 0\right)$ (b) The split disjunction $D\left(\begin{pmatrix}2\\0\end{pmatrix}, 0\right)$

Figure 3.2.: Two split disjunctions

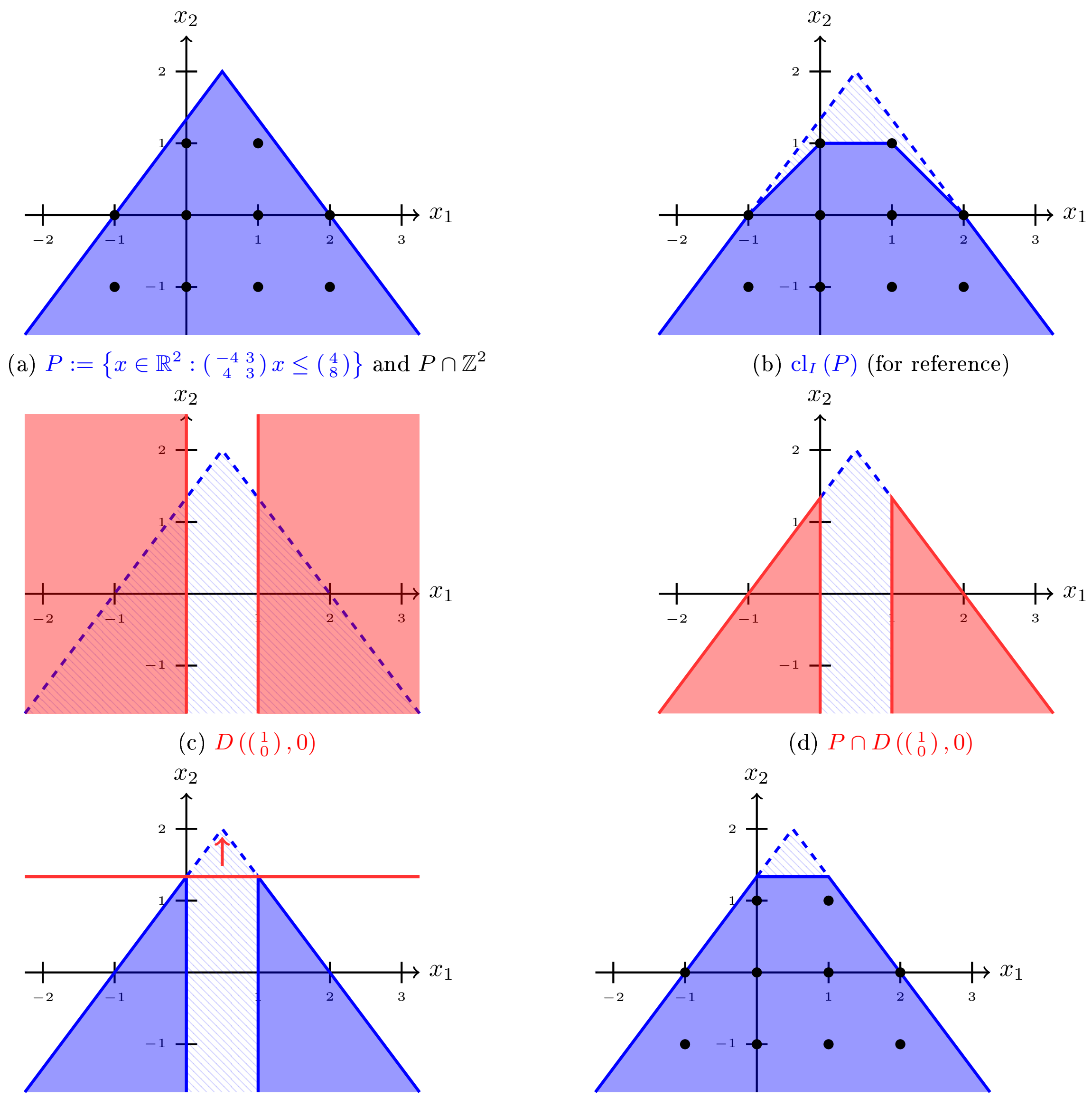
In Figure 3.3, one can see an illustration of the construction behind split cuts.

We now define the split closure in the natural way to assign an operator to a class of cutting planes that we stated at the end of section 3.2:

Definition 127. *Let* $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ *(*$m, n \in \mathbb{Z}_{\geq 0}$*) be arbitrary. Then we define*

$$\begin{aligned}\mathrm{cl}_{split}(P) &:= P \cap \bigcap_{\substack{c \in (\mathbb{R}^m \times \mathbb{R}^n)^T, c_0 \in \mathbb{R}: \\ c(\cdot) \leq c_0 \text{ split cut for } P}} P^{\leq}(c, c_0) \\ &= P \cap \bigcap_{\pi \in \mathbb{Z}^m, \gamma \in \mathbb{Z}} \overline{\mathrm{conv}}\left(P \cap (D(\pi, \gamma) \times \mathbb{R}^n)\right)\end{aligned}$$

as the ***(1-branch) split closure of*** P*.*



(a) $P := \left\{x \in \mathbb{R}^2 : \left(\begin{smallmatrix}-4 & 3\\ 4 & 3\end{smallmatrix}\right) x \leq \left(\begin{smallmatrix}4\\ 8\end{smallmatrix}\right)\right\}$ and $P \cap \mathbb{Z}^2$

(b) $\mathrm{cl}_I(P)$ (for reference)

(c) $D\left(\left(\begin{smallmatrix}1\\ 0\end{smallmatrix}\right), 0\right)$

(d) $P \cap D\left(\left(\begin{smallmatrix}1\\ 0\end{smallmatrix}\right), 0\right)$

(e) $x_2 \leq \frac{4}{3}$ is valid for $x \in P \cap D\left(\left(\begin{smallmatrix}1\\ 0\end{smallmatrix}\right), 0\right)$

(f) $P \cap \left\{x \in \mathbb{R}^2 : x_2 \leq \frac{4}{3}\right\}$

Figure 3.3.: Illustration of split cuts

In [DGMR16a, Lemma 8], the following lemma is shown (by reading its proof, it is easy to check that the rationality conditions hold, even though they are not written down in the statement of [DGMR16a, Lemma 8]):

Lemma 128. *Let $P \subseteq \mathbb{R}^d$ ($d \in \mathbb{Z}_{\geq 0}$) be a polyhedron and let $P_i \subseteq \mathbb{R}^d$ ($i \in [t]$, where $t \in \mathbb{Z}_{\geq 0}$) be polyhedra such that $\operatorname{rec} P_i = \operatorname{lineal} P_i$ for all $i \in [t]$. Then*

$$P' := \operatorname{conv}\left(P \setminus \bigcup_{i=1}^{t} \operatorname{int} P_i\right)$$

is a polyhedron. If $P' \neq \emptyset$, we have $\operatorname{rec} P' = \operatorname{rec} P$ and if $P, P_1, \ldots, P_t$ are rational polyhedra, so is P'.

We remark that in [ALW10], Lemma 128 is shown for the special case $t = 1$. In this reference (for $t = 1$), the authors give an explicit characterization of the P' from Lemma 128 in terms of an extended formulation.

Using Lemma 128, we immediately conclude:

Lemma 129. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be a polyhedron and let $D(\pi, \gamma) \subseteq \mathbb{R}^m \times$ be a split disjunction. Then $\operatorname{conv}(P \cap (D(\pi, \gamma) \times \mathbb{R}^n))$ is a polyhedron (a rational polyhedron if P is rational). In particular, we* *have*

$$\mathrm{cl}_{split}(P) = \bigcap_{\pi\in\mathbb{Z}^m,\gamma\in\mathbb{Z}} \mathrm{conv}\left(P \cap \left(D(\pi,\gamma)\times\mathbb{R}^n\right)\right). \tag{3.4}$$

We state the following conjecture that is related to the statement of Lemma 128:

Conjecture 130. *Let $K \subseteq \mathbb{R}^d$ ($d \in \mathbb{Z}_{\geq 0}$) be a closed convex set and let $P_i \subseteq \mathbb{R}^d$ ($i \in [t]$, where $t \in \mathbb{Z}_{\geq 0}$) be polyhedra such that* $\mathrm{rec}\, P_i = \mathrm{lineal}\, P_i$ *for all $i \in [t]$. Then*

$$K' := \mathrm{conv}\left(K \backslash \bigcup_{i=1}^{t} \mathrm{int}\, P_i\right)$$

is a closed convex set.

In [DDV11a, Lemma 2.3], Conjecture 130 is shown for the case $t = 1$ and P_1 being a split set. Thus, we get the following statement that is related to the statement of Lemma 129:

Lemma 131. *Let $K \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be a closed convex set and let $D(\pi,\gamma) \subseteq \mathbb{R}^m$ be a split disjunction. Then* $\mathrm{conv}(K \cap (D(\pi,\gamma)\times\mathbb{R}^n))$ *is a closed convex set. In particular, we have*

$$\mathrm{cl}_{split}(K) = \bigcap_{\pi\in\mathbb{Z}^m,\gamma\in\mathbb{Z}} \mathrm{conv}\left(K \cap \left(D(\pi,\gamma)\times\mathbb{R}^n\right)\right). \tag{3.5}$$

Indeed, often in the literature (for example cf. [ACL05], [Vie05], [Vie07] and [DGM15]), the split closure is defined as in (3.4) or (3.5). So, Lemma 129 (or Lemma 131) is the connecting link between the "natural" definition of the split closure of P (intersection of P with all split cuts for P, as we did in Definition 127) and the common definition in the literature.

3.4. Generalizations of split cuts

Before we continue with generalizing split cuts, we first want to consider a practical motivation why one is interested in such generalizations. For this, consider Theorem 430, which we show in section 9.2.5:

Theorem 430. *For $\epsilon \in \mathbb{R}_{>0}$, let*

$$\begin{aligned}
P^{430,\epsilon} &:= \mathrm{conv}\left\{\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}2\\0\\0\end{pmatrix},\begin{pmatrix}0\\2\\0\end{pmatrix},\begin{pmatrix}\frac{2}{3}\\\frac{2}{3}\\\epsilon\end{pmatrix}\right\}\\
&= P^{\leq}\left(\left(\begin{array}{cc|c}-1&0&\frac{2}{3\epsilon}\\0&-1&\frac{2}{3\epsilon}\\1&1&\frac{2}{3\epsilon}\\0&0&-1\end{array}\right),\begin{pmatrix}0\\0\\2\\0\end{pmatrix}\right)\\
&=: P^{\leq}\left(\begin{pmatrix}A & G^{\epsilon}\end{pmatrix}, b\right)\\
&\subseteq \mathbb{R}^2\times\mathbb{R}^1.
\end{aligned}$$

Then $\mathrm{cl}_{split}\left(P^{430,\epsilon}\right) = P^{430,\frac{\epsilon}{2}}$. *In particular, for every $t \in \mathbb{Z}_{\geq 0}$, we have*

$$\mathrm{cl}_{split}^{(t)}\left(P^{430,\epsilon}\right) = P^{430,\frac{\epsilon}{2^t}}.$$

Here, $\mathrm{cl}^{(t)}(\,\cdot\,)$ means applying the split closure operator t times iteratively. On the other hand, we clearly have

$$\mathrm{cl}_I\left(P^{430,\epsilon}\right) = \mathrm{conv}\left\{\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}2\\0\\0\end{pmatrix},\begin{pmatrix}0\\2\\0\end{pmatrix}\right\} \subseteq \mathbb{R}^2\times\mathbb{R}^1.$$

In other words: by applying the split closure any finite number of times on $P^{430,\epsilon}$, we never obtain $\mathrm{cl}_I\left(P^{430,\epsilon}\right)$ (though in Theorem 310, we give a result that the iterated split closure $\mathrm{cl}_{split}^{(t)}(P)$ at least *converges* (in a sense that is defined in Definition 308) to $\mathrm{cl}_I(P)$ for a rational polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$)). This is an important reason why one is interested in classes of cutting planes that are more expressive than split cuts.

A rather general method for finding cutting planes is the following: consider some arbitrary set $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) and some set $S \subseteq \mathbb{R}^m$ such that $S \cap \mathbb{Z}^m = \emptyset$. Then every inequality $c(\cdot) \leq c_0$ for $P \cap ((\mathbb{R}^m \backslash S) \times \mathbb{R}^n)$ is a cutting plane for P. On the other hand, for every cutting plane $c(\cdot) \leq c_0$ for P, there trivially exists an $S \subseteq \mathbb{R}^m$ such that $c(\cdot) \leq c_0$ is valid for $P \cap ((\mathbb{R}^m \backslash S) \times \mathbb{R}^n))$ (simply set $S := \operatorname{proj}_{\mathbb{R}^m}(P \cap P^>(c, c_0)))$. Obviously, every split set $S(\pi, \gamma)$ satisfies these conditions on S.

So, in the following two subsections, we consider classes of cutting planes that have been investigated in the literature and are based on specific classes of such sets S. More precisely:

- In section 3.4.1, we consider the situation that S is the interior of a (w.l.o.g. full-dimensional) lattice-free polyhedron (recall that, by Theorem 108, maximal lattice-free bodies are polyhedra).
- In section 3.4.2, we consider the case that $\mathbb{R}^m \backslash S$ forms a (in general non-convex) disjunction.

3.4.1. Generalizing split cuts via lattice-free polyhedra

The first oberservation is that if $L = P^{\leq}(A, b) \subseteq \mathbb{R}^m$ is a lattice-free polyhedron, its interior $\operatorname{int} L = P^{<}(A, b)$ is strictly lattice-free, thus satisfies the conditions imposed on S. We remark that if L is not a *full-dimensional* lattice-free body, we have $P^{<}(A, b) = \emptyset$; thus, only full-dimensional lattice-free bodies are of interest for this construction. In other words: for a given $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ and a given lattice-free body $L := P^{\leq}(A, b) \subseteq \mathbb{R}^m$ ($A \in \mathbb{R}^{l \times m}$ and $b \in \mathbb{R}^l$, where $l \in \mathbb{Z}_{\geq 0}$), we consider valid inequalities for

$$P \cap \bigcup_{i=1}^{l} \left(P^{\geq}(A_{l,*}, b_l) \times \mathbb{R}^n\right) = P \cap ((\mathbb{R}^m \backslash \operatorname{int} P) \times \mathbb{R}^n).$$

Definition 132. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be arbitrary and let $L := P^{\leq}(A, b) \subseteq \mathbb{R}^m$ ($A \in \mathbb{R}^{l \times m}$ and $b \in \mathbb{R}^l$, where $l \in \mathbb{Z}_{\geq 0}$) be a (full-dimensional) lattice-free polyhedron. An inequality $c(\cdot) \leq c_0$ ($c \in (\mathbb{R}^m \times \mathbb{R}^n)^T$ and $c_0 \in \mathbb{R}$) is called a* ***lattice-free cut (with respect to** L**)*** *if it is valid for*

$$P \cap \bigcup_{i=1}^{l} \left(P^{\geq}(A_{l,*}, b_l) \times \mathbb{R}^n\right) = P \cap ((\mathbb{R}^m \backslash \operatorname{int} L) \times \mathbb{R}^n).$$

There still exists a large class of full-dimensional lattice-free bodies; so, one typically wants to restrict the class of lattice-free bodies that one considers even further. This is done in section 3.4.1.2. We remark already here that later on (cf. Theorem 240 in section 5.2.5), $L_{k,\mathbb{Q}}$ cuts (which play a central role in this text; cf. section 4.2 for details) turn out (under some assumptions) to be lattice-free cuts with respect to a special class of lattice-free bodies, too.

3.4.1.1. Lattice-free polyhedra

Lemma 133. *Let $\emptyset \neq P := P^{\leq}(A, b) \subseteq \mathbb{R}^d$, where $A \in \mathbb{R}^{l \times d}$ and $b \in \mathbb{R}^l$ ($l, d \in \mathbb{Z}_{\geq 0}$), be a polyhedron. Define*

$$L(P) := P - \operatorname{rec} P.$$

Then:

1. $\operatorname{lineal} L(P) = (\operatorname{rec} P) + (-\operatorname{rec} P) = \operatorname{rec} L(P)$.
2. *Let $L := \{i \in [l] : A_{i,*}c = 0 \ \forall c \in \operatorname{rec} P\}$. Then $L(P) = P^{\leq}(A_{L,*}, b_L)$. In particular, $L(P)$ has at most the same number of facets as P.*
3. *Let $\operatorname{rec} P$ be generated by rational vectors and let P be a lattice-free polyhedron. Then $L(P)$ is also a lattice-free polyhedron.*
4. *If P is an integral polyhedron, so is $L(P)$.*

Problem 134. *Can in item 3 of Lemma 133 the rationality conditions for the generators of* $\operatorname{rec} P$ *be dropped?*

Proof. (Lemma 133) 1 holds obviously.

For 2: $L(P) \subseteq P^{\leq}(A_{L,*}, b_L)$ holds obviously. So for $L(P) \supseteq P^{\leq}(A_{L,*}, b_L)$: let $\operatorname{rec} P = \operatorname{cone}\{c^1, \ldots, c^k\}$, where $k \in \mathbb{Z}_{\geq 0}$ and $c^1, \ldots, c^k \in \mathbb{R}^d$. We claim that

$$\forall i \in [l] \setminus L \, \exists j(i) \in [k] : A_{i,*} c^{j(i)} < 0. \tag{3.6}$$

We first note that we have (because $\forall j \in [k] : c^j \in \operatorname{rec} P$):

$$\forall i \in [l], j \in [k] : A_{i,*} c^j \leq 0.$$

Assume that

$$\exists i^* \in [l] \setminus L \, \forall j \in [k] : A_{i,*} c^j = 0.$$

This means that $A_{i^*,*} c = 0$ for all $c \in \operatorname{rec} P$. But this implies $i^* \in L$ – a contradiction. So, (3.6) holds.

Now for the main statement: let $x \in P^{\leq}(A_{L,*}, b_L)$. Define $L' := \{i \in [l] \setminus L : A_{i,*} x > b_i\}$ and set

$$x' := x + \underbrace{\sum_{i \in L'} \underbrace{\frac{b_i - A_{i,*} x}{A_{i,*} c^{j(i)}}}_{>0} c^{j(i)}}_{\in \operatorname{rec} P}.$$

We claim that $x' \in P$. For this, let $i' \in [l]$. Then

$$\begin{aligned} A_{i',*} x' &= A_{i',*} \left(x + \sum_{i \in L'} \frac{b_i - A_{i,*} x}{A_{i,*} c^{j(i)}} c^{j(i)} \right) \\ &\leq \begin{cases} A_{i',*} x & \text{if } i' \in [l] \setminus L' = L \,\dot{\cup}\, \{i \in [l] \setminus L : A_{i,*} x \leq b_i\}, \\ A_{i',*} x + \frac{b_{i'} - A_{i',*} x}{A_{i',*} c^{j(i')}} A_{i',*} c^{j(i')} & \text{if } i' \in L' \end{cases} \\ &\leq b_{i'}. \end{aligned} \tag{3.7}$$

So, since $x' \in P$, we immediately conclude $x \in P - \operatorname{rec} P = L(P)$.

We remark that (3.7) holds because for all $i' \in [l]$ and $j \in [k]$, we have $A_{i',*} c^j \leq 0$. On the other hand, we have seen that for $i \in L'$, we have $\frac{b_i - A_{i,*} x}{A_{i,*} c^{j(i)}} > 0$.

For 3: W.l.o.g. we assume $A_{i,*} \neq (0^d)^T$ for all $i \in [l]$. Let again $\operatorname{rec} P = \operatorname{cone}\{c^1, \ldots, c^k\}$ ($k \in \mathbb{Z}_{\geq 0}$). This time, we impose the additional requirement $c^1, \ldots, c^k \in \mathbb{Z}^d$. Assume that there exists an $\epsilon \in \mathbb{R}_{>0}$ and a $z_0 \in \mathbb{Z}^d$ such that $B_\epsilon(z_0) \subseteq L(P)$. Then, by 2, we have

$$\forall x \in B_\epsilon(z_0) : A_{L,*} x \leq b_L,$$

where L is as in the statement of 2. By (3.6), we know that

$$\forall i \in [l] \setminus L \, \exists j(i) \in [k] : A_{i,*} c^{j(i)} < 0.$$

Let

$$L' := \{i \in [l] \setminus L : A_{i,*} z_0 + \epsilon \|A_{i,*}\| > b_i\}.$$

We thus have

$$\forall i \in [l] \setminus (L \,\dot{\cup}\, L') : A_{i,*} z_0 + \epsilon \|A_{i,*}\| \leq b_i. \tag{3.8}$$

Define

$$z_0' := z_0 + \sum_{i \in L'} \underbrace{\left\lceil \underbrace{\frac{b_i - A_{i,*} z_0 - \epsilon \|A_{i,*}\|}{A_{i,*} c^{j(i)}}}_{>0} \right\rceil}_{\geq 1} c^{j(i)} \in \mathbb{Z}^m.$$

We claim that $B_\epsilon(z_0') \subseteq P$. For this, let $e \in \mathbb{S}^{d-1}$, $\epsilon' \in [0, \epsilon)$ and $i' \in [l]$. Then:

- If $i' \in L$, we have

$$\begin{aligned} A_{i',*}\left(z_0' + \epsilon' \cdot e\right) &= A_{i',*}\left(z_0 + \epsilon' \cdot e + \sum_{i \in I'} \left\lceil \frac{b_i - A_{i,*} z_0 - \epsilon \left\| A_{i,*} \right\|}{A_{i,*} c^{j(i)}} \right\rceil c^{j(i)}\right) \\ &\leq A_{i',*}\left(z_0 + \epsilon' \cdot e\right) \\ &\leq b_{i'}. && \left(B_\epsilon\left(z_0\right) \subseteq L\left(P\right)\right) \end{aligned}$$

- If $i' \in [l] \setminus (L \,\dot{\cup}\, L')$, we have

$$\begin{aligned} A_{i',*}\left(z_0' + \epsilon' e\right) &= A_{i',*}\left(z_0 + \epsilon' \cdot e + \sum_{i \in L'} \left\lceil \frac{b_i - A_{i,*} z_0 - \epsilon \left\| A_{i,*} \right\|}{A_{i,*} c^{j(i)}} \right\rceil c^{j(i)}\right) \\ &\leq A_{i',*}\left(z_0 + \epsilon' \cdot e\right) \\ &\leq A_{i',*} z_0 + \epsilon \cdot \left\| A_{i',*} \right\| \cdot \left\| e \right\| && \text{(Cauchy-Schwarz inequality)} \\ &= A_{i',*} z_0 + \epsilon \cdot \left\| A_{i',*} \right\| \\ &\leq b_{i'}. && \text{(by (3.8))} \end{aligned}$$

- If $i' \in L'$, we have

$$\begin{aligned} & A_{i',*}\left(z_0' + \epsilon' \cdot e\right) \\ =& A_{i',*}\left(z_0 + \epsilon' \cdot e + \sum_{i \in L'} \left\lceil \frac{b_i - A_{i,*} z_0 - \epsilon \left\| A_{i,*} \right\|}{A_{i,*} c^{j(i)}} \right\rceil c^{j(i)}\right) \\ \leq& A_{i',*}\left(z_0 + \epsilon' \cdot e + \left\lceil \frac{b_{i'} - A_{i',*} z_0 - \epsilon \left\| A_{i',*} \right\|}{A_{i',*} c^{j(i')}} \right\rceil c^{j(i')}\right) \\ \leq& A_{i',*} z_0 + \epsilon \cdot \left\| A_{i',*} \right\| \cdot \left\| e \right\| + \left\lceil \frac{b_{i'} - A_{i',*} z_0 - \epsilon \left\| A_{i',*} \right\|}{A_{i',*} c^{j(i)}} \right\rceil A_{i',*} c^{j(i')} && \text{(Cauchy-Schwarz inequality)} \\ =& A_{i',*} z_0 + \epsilon \cdot \left\| A_{i',*} \right\| \\ &+ \underbrace{\left(\left\lceil \frac{b_{i'} - A_{i',*} z_0 - \epsilon \left\| A_{i',*} \right\|}{A_{i',*} c^{j(i')}} \right\rceil - \frac{b_{i'} - A_{i',*} z_0 - \epsilon \cdot \left\| A_{i',*} \right\|}{A_{i',*} c^{j(i')}}\right)}_{\geq 0} \underbrace{A_{i',*} c^{j(i')}}_{<0} \\ &+ \frac{b_{i'} - A_{i',*} z_0 - \epsilon \left\| A_{i',*} \right\|}{A_{i',*} c^{j(i')}} A_{i',*} c^{j(i')} \\ \leq& A_{i',*} z_0 + \epsilon \cdot \left\| A_{i',*} \right\| + \frac{b_{i'} - A_{i',*} z_0 - \epsilon \cdot \left\| A_{i',*} \right\|}{A_{i',*} c^{j(i')}} A_{i',*} c^{j(i')} \\ =& b_{i'}. \end{aligned}$$

For 4: By Definition 76, P is an integral polyhedron if and only if

$$P = \operatorname{conv}\left\{q^1, \ldots, q^{k_1}\right\} + \operatorname{cone}\left\{c^1, \ldots, c^{k_2}\right\},$$

where $q^1, \ldots, q^{k_1}, c^1, \ldots, c^{k_2} \in \mathbb{Z}^d$ $(k_1, k_2 \in \mathbb{Z}_{\geq 0})$. Clearly,

$$L\left(P\right) = \operatorname{conv}\left\{q^1, \ldots, q^{k_1}\right\} + \operatorname{cone}\left\{c^1, \ldots, c^{k_2}, -c^1, \ldots, -c^{k_2}\right\},$$

thus also $L\left(P\right)$ can be represented in this form. □

3.4.1.2. k-disjunctive cuts and integral lattice-free cuts

The first class of class of lattice-free cuts that we consider in this section are the so-called **k-disjunctive cuts**. Here, we demand L (as in Definition 132) to be a rational polyhedron and fix the number of inequalities defining L to be bounded by a constant $k \in \mathbb{Z}_{\geq 2}$. This class of cutting planes is considered in [Jör07].

Definition 135. *For $k \in \mathbb{Z}_{\geq 2}$, let $L := P^{\leq}\left(A, b\right) \subseteq \mathbb{R}^m$ $(m \in \mathbb{Z}_{\geq 0})$ be a rational lattice-free polyhedron (i.e. A and b are rational) such that A and b have at most k rows. Then a lattice-free cut with respect to L is called a **k-disjunctive cut**.*

The reason for the naming "k-disjunctive cut" is that in [Jör07], a k-disjunction is defined as a set of integral inequalities $c^1(\cdot) \leq c_0^1, \ldots, c^k(\cdot) \leq c_0^k$ such that for every $z \in \mathbb{Z}^m$, there exists an $i \in [k]$ such that $c^i z \leq c_0^i$. W.l.o.g. we can assume A and b in Definition 135 to be integral. Then we can set

$$A := \begin{pmatrix} -c^1 \\ \vdots \\ -c^k \end{pmatrix}, \qquad b := \begin{pmatrix} -c_0^1 \\ \vdots \\ -c_0^k \end{pmatrix}$$

to derive a lattice-free body L as in Definition 135. On the other hand, we can derive a k-disjunction from A and b in Definition 135 by setting

$$c^i := -A_{i,*}, \qquad c_0^i := -b_i$$

for all $i \in [k]$.

Clearly, every split cut is a 2-disjunctive cut (thus k-disjunctive cut, where $k \in \mathbb{Z}_{\geq 2}$). So, k-disjunctive cuts really generalize split cuts.

We now define the k-disjunctive closure in the natural way:

Definition 136. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be arbitrary and let $k \in \mathbb{Z}_{\geq 2}$. Then we define*

$$\begin{aligned}\mathrm{cl}_{kD}(P) &:= P \cap \bigcap_{\substack{c \in (\mathbb{R}^m \times \mathbb{R}^n)^T, c_0 \in \mathbb{R}: \\ c(\cdot) \leq c_0 \text{ } k\text{-disjunctive cut for } P}} P^{\leq}(c, c_0) \\ &= P \cap \bigcap_{\substack{L \subseteq \mathbb{R}^m \text{ rational lattice-free body} \\ \text{with at most } k \text{ facets}}} \overline{\mathrm{conv}}\left(P \cap ((\mathbb{R}^m \setminus \mathrm{int}\, L) \times \mathbb{R}^n)\right)\end{aligned}$$

*as the k-**disjunctive closure of** P.*

One obtains another important class of lattice-free cuts if one demands L in Definition 132 to be an *integral* lattice-free body. This class is considered in [DPW12].

Definition 137. *Let P be arbitrary. An **integral lattice-free cut** is a lattice-free cut with respect to an* integral *lattice-free polyhedron.*

Definition 138. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be arbitrary. Then we define*

$$\begin{aligned}\mathrm{cl}_{ILF}(P) &:= P \cap \bigcap_{\substack{c \in (\mathbb{R}^m \times \mathbb{R}^n)^T, c_0 \in \mathbb{R}: \\ c(\cdot) \leq c_0 \text{ integral lattice-free cut for } P}} P^{\leq}(c, c_0) \\ &= P \cap \bigcap_{L \subseteq \mathbb{R}^m \text{ integral lattice-free body}} \overline{\mathrm{conv}}\left(P \cap ((\mathbb{R}^m \setminus \mathrm{int}\, L) \times \mathbb{R}^n)\right)\end{aligned}$$

*as the **integral lattice-free closure of** P.*

The integral lattice-free closure has the interesting property that for a rational polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$, it suffices to apply it iteratively a finite number of times on P to obtain the mixed-integer hull $\mathrm{cl}_I(P)$ (cf. Theorem 313).

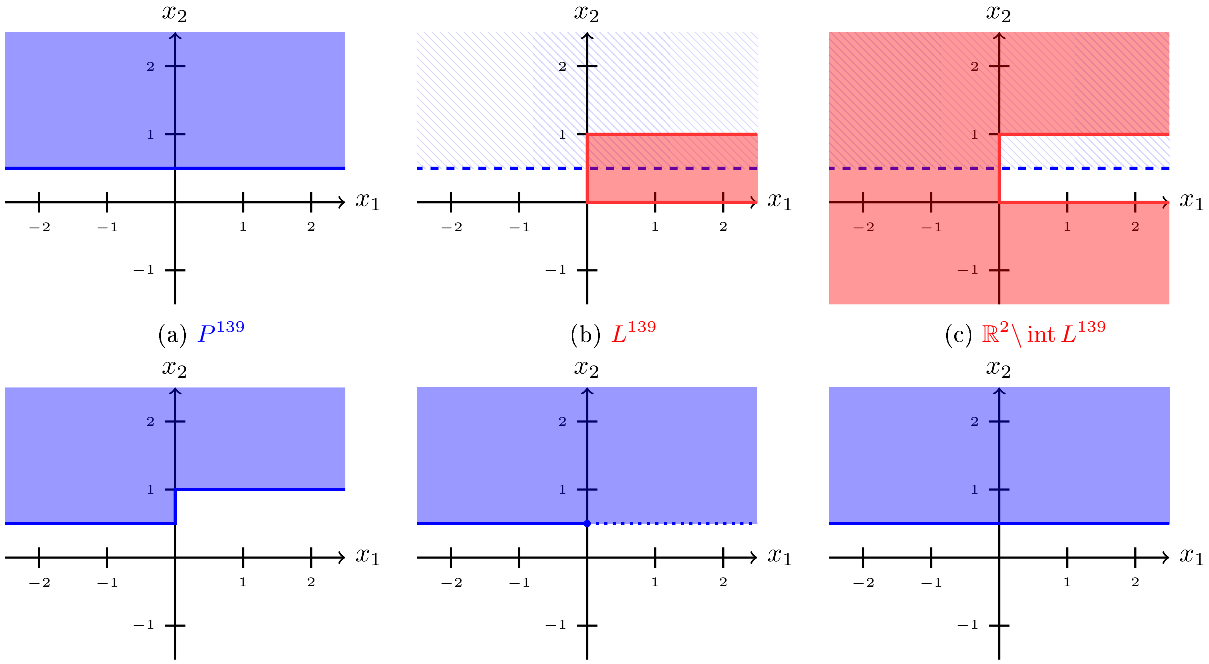
We now consider whether (like in Lemma 129) we can substitute $\overline{\mathrm{conv}}$ by conv in the definition of the k-disjunctive closure or integral lattice-free closure if P is a polyhedron. For *one concretely* given polyhedron P and an L satisfying the conditions of Definition 135 or Definition 137, it can indeed happen that we have

$$\overline{\mathrm{conv}}\left(P \cap ((\mathrm{int}\, L) \times \mathbb{R}^n)\right) \supsetneq \mathrm{conv}\left(P \cap ((\mathrm{int}\, L) \times \mathbb{R}^n)\right).$$

For this, we consider the following example:

Example 139. *(See Figure 3.4) Let*

$$\begin{aligned}P^{139} &:= P^{\leq}\left(\begin{pmatrix} 0 & -1 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \end{pmatrix}\right), \\ L^{139} &:= P^{\leq}\left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right).\end{aligned}$$



(a) P^{139} (b) L^{139} (c) $\mathbb{R}^2 \setminus \operatorname{int} L^{139}$

(d) $P^{139} \cap \left(\mathbb{R}^2 \setminus \operatorname{int} L^{139}\right)$ (e) $\operatorname{conv}\left(P^{139} \cap \left(\mathbb{R}^2 \setminus \operatorname{int} L^{139}\right)\right)$ (f) $\overline{\operatorname{conv}}\left(P^{139} \cap \left(\mathbb{R}^2 \setminus \operatorname{int} L^{139}\right)\right) = P^{139}$

Figure 3.4.: Illustration of Example 139

Clearly, L^{139} is integral, lattice-free and described by 3 rational inequalities. It is easy to check that

$$\operatorname{conv}\left(P^{139} \cap \left(\mathbb{R}^2 \setminus \operatorname{int} L^{139}\right)\right) = P^{139} \setminus \left(\left(\begin{array}{c} 0 \\ \frac{1}{2} \end{array}\right) + \mathbb{R}_{>0} \cdot \left\{\left(\begin{array}{c} 1 \\ 0 \end{array}\right)\right\}\right),$$

which is clearly not closed. On the other hand

$$\overline{\operatorname{conv}}\left(P^{139} \cap \left(\mathbb{R}^2 \setminus \operatorname{int} L^{139}\right)\right) = P^{139}.$$

We can nevertheless substitute $\overline{\operatorname{conv}}$ by conv in Definition 136 and Definition 138 if P is a polyhedron, as the following theorem shows, which holds obviously.

Theorem 140. *Let $L \subseteq \mathbb{R}^m$ be*

- *a rational lattice-free polyhedron that is descibed by at most $k \in \mathbb{Z}_{\geq 2}$ inequalities or*
- *an integral lattic-free polyhedron,*

respectively. Let $L(L)$ be as in Lemma 133. Then, by Lemma 133, also $L(L)$ is

- *a rational lattice-free polyhedron that is descibed by at most $k \in \mathbb{Z}_{\geq 2}$ inequalities or*
- *an integral lattic-free polyhedron,*

respectively, having $L(L) \supseteq L$. So, for every $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$, we have

$$P \cap ((\operatorname{int} L(L)) \times \mathbb{R}^n) \subseteq P \cap ((\operatorname{int} L) \times \mathbb{R}^n).$$

Thus,

$$\operatorname{cl}_{kD} = P \cap \bigcap_{\substack{L \subseteq \mathbb{R}^m \text{ rational lattice-free body} \\ \text{with at most } k \text{ facets and} \\ \operatorname{rec} L = \operatorname{lineal} L}} \overline{\operatorname{conv}}\left(P \cap \left(\left(\mathbb{R}^m \setminus \operatorname{int} L\right) \times \mathbb{R}^n\right)\right),$$

$$\operatorname{cl}_{ILF} = P \cap \bigcap_{\substack{L \subseteq \mathbb{R}^m \text{ integral lattice-free body} \\ \text{having } \operatorname{rec} L = \operatorname{lineal} L}} \overline{\operatorname{conv}}\left(P \cap \left(\left(\mathbb{R}^m \setminus \operatorname{int} L\right) \times \mathbb{R}^n\right)\right).$$

If P is a polyhedron, we thus have using Lemma 128:

$$\begin{aligned}
\mathrm{cl}_{kD} &= \bigcap_{\substack{L\subseteq\mathbb{R}^m \text{ rational lattice-free body}\\ \text{with at most } k \text{ facets and}\\ \mathrm{rec}\, L=\mathrm{lineal}\, L}} \mathrm{conv}\left(P\cap\left(\left(\mathbb{R}^m\setminus \mathrm{int}\, L\right)\times\mathbb{R}^n\right)\right)\\
&= \bigcap_{\substack{L\subseteq\mathbb{R}^m \text{ rational lattice-free body}\\ \text{with at most } k \text{ facets}}} \mathrm{conv}\left(P\cap\left(\left(\mathbb{R}^m\setminus \mathrm{int}\, L\right)\times\mathbb{R}^n\right)\right),\\
\mathrm{cl}_{ILF} &= \bigcap_{\substack{L\subseteq\mathbb{R}^m \text{ integral lattice-free body}\\ \text{having } \mathrm{rec}\, L=\mathrm{lineal}\, L}} \mathrm{conv}\left(P\cap\left(\left(\mathbb{R}^m\setminus \mathrm{int}\, L\right)\times\mathbb{R}^n\right)\right)\\
&= \bigcap_{L\subseteq\mathbb{R}^m \text{ integral lattice-free body}} \mathrm{conv}\left(P\cap\left(\left(\mathbb{R}^m\setminus \mathrm{int}\, L\right)\times\mathbb{R}^n\right)\right).
\end{aligned}$$

3.4.2. Generalizing split cuts via other types of disjunctions

We now consider sets S as in the introduction of section 3.4, which are this time not necessarily convex anymore. A rather general framework for this purpose is what we introduce as "multi-branch disjunctive cuts" in the following definition (we know of no term for this concept in the literature):

Definition 141. *Let $P\subseteq\mathbb{R}^m\times\mathbb{R}^n$ ($m,n\in\mathbb{Z}_{\geq 0}$) be arbitrary and let $D^i := \left\{x\in\mathbb{R}^m : A^i x\leq b^i\right\}$ ($i\in[t]$, $A^i\in\mathbb{R}^{l^i\times m}$ and $b^i\in\mathbb{R}^{l^i}$, where $l^i, t\in\mathbb{Z}_{\geq 0}$) be polyhedra such that*

$$D := \bigcup_{i=1}^{t} D^i \supseteq \mathbb{Z}^m.$$

*An inequality $c(\,\cdot\,)\leq c_0$ ($c\in\left(\mathbb{R}^m\times\mathbb{R}^n\right)^T$ and $c_0\in\mathbb{R}$) is called a **multi-branch disjunctive cut (with respect to D)** if it is valid for $P\cap\left(D\times\mathbb{R}^n\right)$.*

The central place in this dissertation where we use the concept of "multi-branch disjunctive cuts" in its generality is Theorem 277.

3.4.2.1. t-branch split cuts

Definition 142. *Let $\pi^1,\dots,\pi^t\in\mathbb{Z}^m$ ($m\in\mathbb{Z}_{\geq 0}$) and $\gamma_1,\dots,\gamma_t\in\mathbb{Z}$, where $t\in\mathbb{Z}_{\geq 0}$. Then we denote the set*

$$D\left(\pi^1,\dots,\pi^t,\gamma_1,\dots,\gamma_t\right) := \bigcap_{i=1}^{t} D\left(\pi^i,\gamma_i\right)\subseteq\mathbb{R}^m$$

*as t-**branch split disjunction** and the set*

$$S\left(\pi^1,\dots,\pi^k,\gamma_1,\dots,\gamma_k\right) := \bigcup_{i=1}^{k} S\left(\pi^i,\gamma_i\right)\subseteq\mathbb{R}^m$$

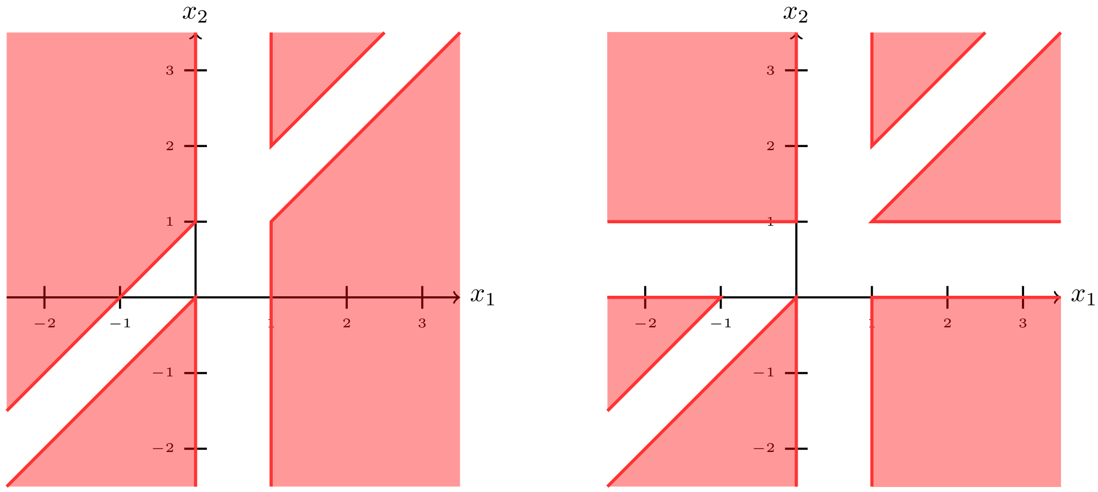
*as t-**branch split set**. 2-branch split disjunctions are also called **cross disjunctions**.*

An example of a 2-branch split disjunction and a 3-branch split disjunction is shown in Figure 3.5.

Definition 143. *A multi-branch disjunctive cut with respect to some t-branch split disjunction*

$$D\left(\pi^1,\dots,\pi^t,\gamma_1,\dots,\gamma_t\right)$$

*is called a t-**branch split cut**. 2-branch split cuts are also called **cross cuts**.*



(a) The 2-branch split disjunction $D\left(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),\left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right),0,0\right)$

(b) The 3-branch split disjunction $D\left(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right),\left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right),0,0,0\right)$

Figure 3.5.: A 2-branch split disjunction and a 3-branch split disjunction

Definition 144. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ be arbitrary and let $t \in \mathbb{Z}_{\geq 1}$. Then we define*

$$\begin{aligned}\operatorname{cl}_{tBS}(P) &:= P \cap \bigcap_{\substack{c \in (\mathbb{R}^m \times \mathbb{R}^n)^T, c_0 \in \mathbb{R}: \\ c(\,\cdot\,) \leq c_0 \text{ } t\text{-branch split cut for } P}} P^{\leq}(c, c_0) \\ &= P \cap \bigcap_{\substack{\pi^1, \ldots, \pi^t \in \mathbb{Z}^m, \\ \gamma_1, \ldots, \gamma_t \in \mathbb{Z}}} \overline{\operatorname{conv}}\left(P \cap \left(D\left(\pi^1, \ldots, \pi^t, \gamma_1, \ldots, \gamma_t\right) \times \mathbb{R}^n\right)\right)\end{aligned}$$

*as the **t-branch split closure of** P.*

Of course, every 1-branch split cut is a split cut and vice versa and thus for every $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$, we have:

$$\operatorname{cl}_{split}(P) = \operatorname{cl}_{1BS}(P).$$

The following lemma is an immediate consequence of Lemma 128 and generalizes Lemma 129:

Lemma 145. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ be a polyhedron and let $D\left(\pi^1, \ldots, \pi^t, \gamma_1, \ldots, \gamma_t\right) \subseteq \mathbb{R}^m \times \mathbb{R}^n$ be a t-branch split disjunction $(t \in \mathbb{Z}_{\geq 1})$. Then $\operatorname{conv}\left(P \cap D\left(\pi^1, \ldots, \pi^t, \gamma_1, \ldots, \gamma_t\right)\right)$ is a polyhedron (a rational polyhedron if P is a rational polyhedron). In particular, we have*

$$\operatorname{cl}_{tBS}(P) = \bigcap_{\substack{\pi^1, \ldots, \pi^t \in \mathbb{Z}^m, \\ \gamma_1, \ldots, \gamma_t \in \mathbb{Z}}} \operatorname{conv}\left(P \cap \left(D\left(\pi^1, \ldots, \pi^t, \gamma_1, \ldots, \gamma_t\right) \times \mathbb{R}^n\right)\right).$$

3.4.2.2. Crooked cross cuts

For the following definition cf. [DGM15]:

Definition 146. *For $\pi^1, \pi^2 \in \mathbb{Z}^m$ $(m \in \mathbb{Z}_{\geq 0})$ and $\gamma_1, \gamma_2 \in \mathbb{Z}$, define*

$$\begin{aligned}D_1^c\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right) &:= \left\{x \in \mathbb{R}^m : {\pi^1}^T x \leq \gamma_1 \wedge \left(\pi^2 - \pi^1\right)^T x \leq \gamma_2 - \gamma_1\right\}, \\ D_2^c\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right) &:= \left\{x \in \mathbb{R}^m : {\pi^1}^T x \leq \gamma_1 \wedge \left(\pi^2 - \pi^1\right)^T x \geq \gamma_2 - \gamma_1 + 1\right\}, \\ D_3^c\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right) &:= \left\{x \in \mathbb{R}^m : {\pi^1}^T x \geq \gamma_1 + 1 \wedge {\pi^2}^T x \leq \gamma_2\right\}, \\ D_4^c\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right) &:= \left\{x \in \mathbb{R}^m : {\pi^1}^T x \geq \gamma_1 + 1 \wedge {\pi^2}^T x \geq \gamma_2 + 1\right\}, \\ D^c\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right) &:= \bigcup_{i=1}^{4} D_i^c\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right).\end{aligned}$$

Then we denote the set $D^c(\pi^1, \pi^2, \gamma_1, \gamma_2)$ *as a* ***crooked cross disjunction*** *and the sets* $D_i^c(\pi^1, \pi^2, \gamma_1, \gamma_2)$ *(*$i \in [4]$*) as* ***atoms of the crooked cross disjunction****.*

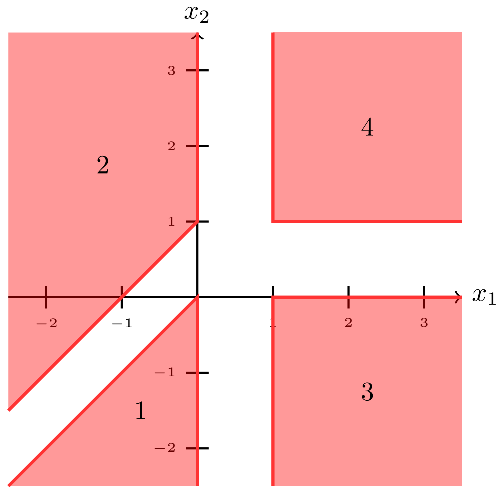


Figure 3.6.: The crooked cross disjunction $D^c\left(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right), 0, 0\right)$

In Figure 3.6, one can see an example of a crooked cross disjunction.

Definition 147. *A multi-branch disjunctive cut with respect to a crooked cross disjunction* $D^c(\pi^1, \pi^2, \gamma_1, \gamma_2)$ *is called a* ***crooked cross cut****.*

Definition 148. *Let* $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ *(*$m, n \in \mathbb{Z}_{\geq 0}$*) be arbitrary. Then we define*

$$\begin{aligned} \operatorname{cl}_{CC}(P) &:= P \cap \bigcap_{\substack{c \in (\mathbb{R}^m \times \mathbb{R}^n)^T, c_0 \in \mathbb{R}: \\ c(\,\cdot\,) \leq c_0 \text{ crooked cross cut for } P}} P^{\leq}(c, c_0) \\ &= P \cap \bigcap_{\substack{\pi^1, \pi^2 \in \mathbb{Z}^m, \\ \gamma_1, \gamma_2 \in \mathbb{Z}}} \overline{\operatorname{conv}}\left(P \cap \left(D^c\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right) \times \mathbb{R}^n\right)\right) \end{aligned}$$

as the ***crooked cross closure of*** P*.*

One may now ask whether we can also simplify the definition of the crooked cross closure to

$$\operatorname{cl}_{CC}(P) = \bigcap_{\substack{\pi^1, \pi^2 \in \mathbb{Z}^m, \\ \gamma_1, \gamma_2 \in \mathbb{Z}}} \operatorname{conv}\left(P \cap \left(D^c\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right) \times \mathbb{R}^n\right)\right). \tag{3.9}$$

Indeed, this is how for example in [DGM15], the crooked cross closure is defined. One obstacle that one encounters if one wants to show (3.9) is that for a rational polyhedron P and a crooked cross disjunction $D^c(\pi^1, \pi^2, \gamma_1, \gamma_2)$, it can happen that $\operatorname{conv}\left(P \cap \left(D^c\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right) \times \mathbb{R}^n\right)\right)$ is not closed, i.e. we have

$$\operatorname{conv}\left(P \cap \left(D^c\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right) \times \mathbb{R}^n\right)\right) \subsetneq \overline{\operatorname{conv}}\left(P \cap \left(D^c\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right) \times \mathbb{R}^n\right)\right)$$

(also recall Example 139 and Theorem 140). For this, we consider the following example, which is inspired by [DGM15, section 2.4 and Figure 3]:

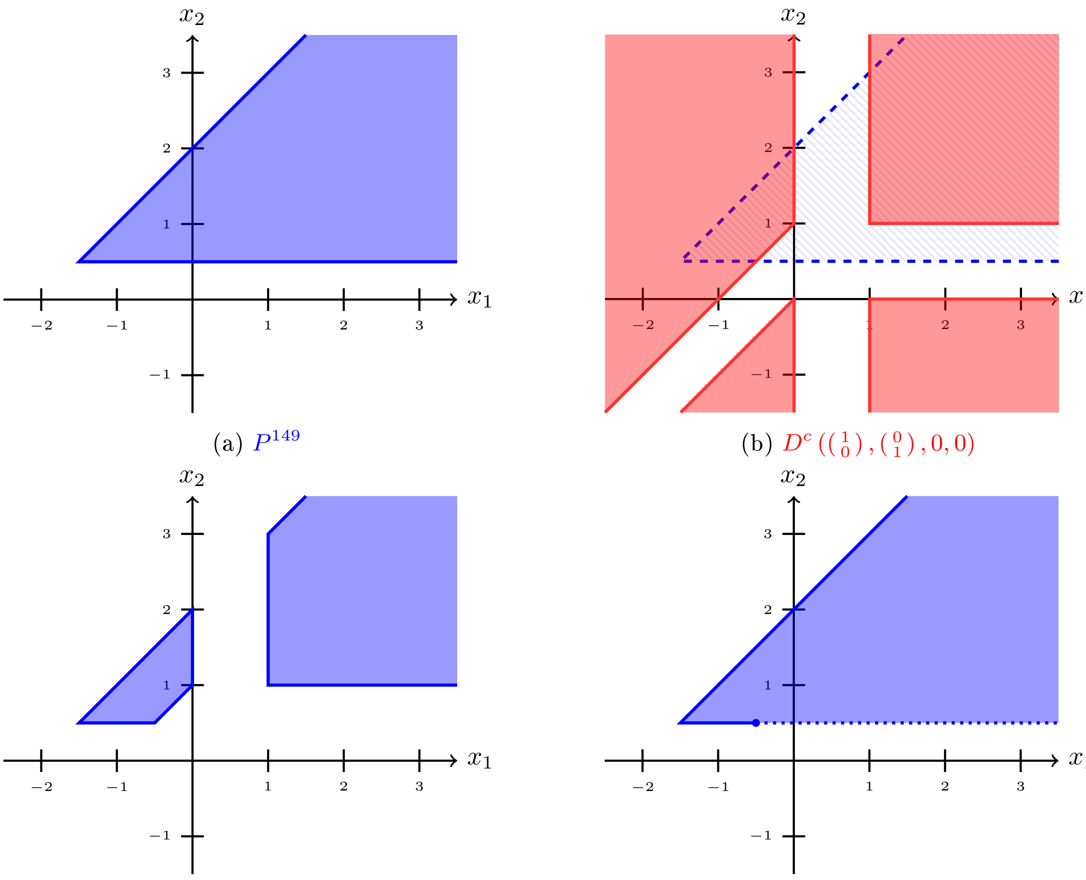
Example 149. *(See Figure 3.7) Let*

$$P^{149} := \left\{\left(\begin{array}{c} -\frac{3}{2} \\ \frac{1}{2} \end{array}\right)\right\} + \operatorname{cone}\left\{\left(\begin{array}{c} 1 \\ 0 \end{array}\right), \left(\begin{array}{c} 1 \\ 1 \end{array}\right)\right\} \subseteq \mathbb{R}^2.$$

Then it is easy to check that

$$\operatorname{conv}\left(P^{149} \cap D^c\left(\left(\begin{array}{c} 1 \\ 0 \end{array}\right), \left(\begin{array}{c} 0 \\ 1 \end{array}\right), 0, 0\right)\right) = P^{149} \backslash \left(\left(\begin{array}{c} -\frac{1}{2} \\ \frac{1}{2} \end{array}\right) + \mathbb{R}_{>0} \cdot \left\{\left(\begin{array}{c} 1 \\ 0 \end{array}\right)\right\}\right),$$

which is clearly not closed.



(a) P^{149} (b) $D^c\left(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right),0,0\right)$

(c) $P^{149} \cap D^c\left(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right),0,0\right)$ (d) $\operatorname{conv}\left(P^{149} \cap D^c\left(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right),0,0\right)\right)$

Figure 3.7.: Illustration of Example 149

In Theorem 464 in section 11.1.4, we show that

$$\mathrm{cl}_{CC}(P) = \bigcap_{\substack{\pi^1,\pi^2\in\mathbb{Z}^m,\\ \gamma_1,\gamma_2\in\mathbb{Z}}} \operatorname{conv}\left(P \cap \left(D^c\left(\pi^1,\pi^2,\gamma_1,\gamma_2\right) \times \mathbb{R}^n\right)\right)$$

nevertheless holds if $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ is either a rational polyhedron or convex and compact.

3.4.2.3. Generalizations of crooked cross cuts

In [DDG12, section 5.2] (also cf. [DDG11, section 3]), the so-called **parametric cross cuts** are introduced. The idea behind them is to provide a unified framework for dealing with 2-branch split cuts (cross cuts) and crooked cross cuts.

Definition 150. *Let $\pi^1, \pi^2 \in \mathbb{Z}^m$ ($m \in \mathbb{Z}_{\geq 0}$), $\gamma_1, \gamma_2 \in \mathbb{Z}$ and $t \in \mathbb{Z}$. Let*

$$\begin{aligned}
D_1^t\left(\pi^1,\pi^2,\gamma_1,\gamma_2\right) &:= \left\{x \in \mathbb{R}^m \times \mathbb{R}^n : {\pi^1}^T x \leq \gamma_1 \wedge \left(\pi^2 - t\pi^1\right)^T x \leq \gamma_2 - t\gamma_1\right\},\\
D_2^t\left(\pi^1,\pi^2,\gamma_1,\gamma_2\right) &:= \left\{x \in \mathbb{R}^m \times \mathbb{R}^n : {\pi^1}^T x \leq \gamma_1 \wedge \left(\pi^2 - t\pi^1\right)^T x \geq \gamma_2 - t\gamma_1 + 1\right\},\\
D_3^t\left(\pi^1,\pi^2,\gamma_1,\gamma_2\right) &:= \left\{x \in \mathbb{R}^m \times \mathbb{R}^n : {\pi^1}^T x \geq \gamma_1 + 1 \wedge {\pi^2}^T x \leq \gamma_2\right\},\\
D_4^t\left(\pi^1,\pi^2,\gamma_1,\gamma_2\right) &:= \left\{x \in \mathbb{R}^m \times \mathbb{R}^n : {\pi^1}^T x \geq \gamma_1 + 1 \wedge {\pi^2}^T x \geq \gamma_2 + 1\right\},\\
D^t\left(\pi^1,\pi^2,\gamma_1,\gamma_2\right) &:= \bigcup_{i=1}^{4} D_i^t\left(\pi^1,\pi^2,\gamma_1,\gamma_2\right).
\end{aligned}$$

Then we denote the set $D^t\left(\pi^1,\pi^2,\gamma_1,\gamma_2\right)$ as ***parametric cross disjunction (with respect to*** t***)*** *and the* *sets* $D_i^t(\pi^1, \pi^2, \gamma_1, \gamma_2)$ $(i \in [4])$ *as* ***atoms of the parametric cross disjunction (with respect to*** t***).***

Definition 151. *A multi-branch disjunctive cut for* P *with respect to some parametric cross disjunction* $D^t(\pi^1, \pi^2, \gamma_1, \gamma_2)$ *is called a* ***parametric cross cut****.*

In Figure 3.8, one can see an example of a parametric cross disjunction (with respect to 2). Obviously, parametric cross disjunctions with respect to 0 are 2-branch split disjunctions (cross disjunctions) and parametric cross disjunctions with respect to 1 are crooked cross disjunctions. Similarly, cuts with respect to a parametric cross disjunction $D^t(\pi^1, \pi^2, \gamma_1, \gamma_2)$, where $t = 0$, are 2-branch split cuts (cross cuts) and cuts with respect to a parametric cross disjunction $D^t(\pi^1, \pi^2, \gamma_1, \gamma_2)$, where $t = 1$, are crooked cross cuts.

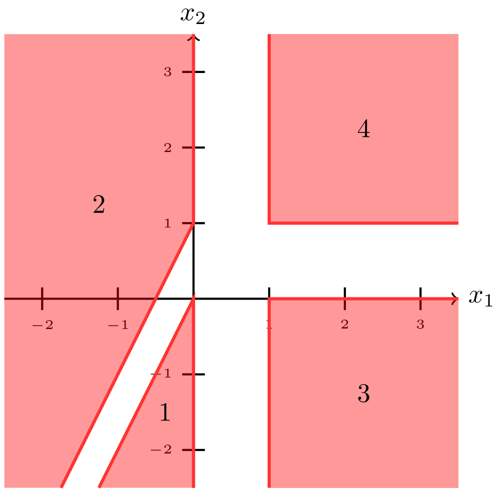


Figure 3.8.: The parametric cross disjunction $D^2\left(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right), 0, 0\right)$

3.5. Cutting planes based on combination of inequalities

We now have a brief look at two classes of cutting planes that use how the polyhedron P is defined via equations/inequalities:

- In section 3.5.1, we consider cutting planes based on a basic relaxation.
- In section 3.5.2, we consider cutting planes based on a k row relaxation.

3.5.1. Cutting planes based on a basic relaxation

For the following definition cf. [DGM15, section 2.5.2] (also cf. [ACL05] and [DGR11]). The only difference to this literature reference is that we do not demand A, G and b to be rational.

Definition 152. *Let*

$$P := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R}^n : Ax + Gy \leq b \right\},$$

where $A \in \mathbb{R}^{r \times m}$, $G \in \mathbb{R}^{r \times n}$ *and* $b \in \mathbb{R}^r$ $(m, n, r \in \mathbb{Z}_{\geq 0}$*, where* $r \geq m + n)$*. Define*

$$P_{[J]} := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R}^n : A_{J,*}x + G_{J,*}y \leq b_J \right\},$$

where $J \subseteq [r]$ *is chosen such that* $\begin{pmatrix} A & G \end{pmatrix}_{J,*}$ *has full row rank (i.e. the rows are linearly independent). Then we call* $P_{[J]}$ *a* ***basic relaxation****. A linear inequality* $c(\cdot) \leq c_0$ $(c \in (\mathbb{R}^m \times \mathbb{R}^n)^T$ *and* $c_0 \in \mathbb{R})$ *for* $\left(P_{[J]}\right)_I$ *is called an* ***inequality of a basic relaxation****.*

We remark that $P_{[J]}$ is an abuse of notation, since $P_{[J]}$ not only depends on P, but on the concrete inequalities that are used to define P. But we stick to this notation that is common in the literature instead of introducing a new one, since in this text, the reader should always be able to infer from context what concrete system of linear equations was used to describe P.

Definition 153. *Let* $P = P^{\leq}\left(\left(\begin{array}{cc} A & G \end{array}\right), b\right)$ *be as in Definition 152. We set*

$$\mathrm{cl}_{BR}(A, G, b) := P \cap \bigcap_{\substack{J \subseteq [r] : \left(\begin{array}{cc} A & G \end{array}\right)_{J,*} \\ \textit{has full row rank}}} \mathrm{cl}_{\overline{I}}\left(P_{[J]}\right).$$

3.5.2. Cutting planes based on a k row relaxation

For the following definition cf. [DGM15, section 2.5.2]. The only difference to this literature reference is that we do not demand A, G and b to be rational.

Definition 154. *Let*

$$P = \left\{\left(\begin{array}{c} x \\ y \end{array}\right) \in \mathbb{R}^m \times \mathbb{R}^n_{\geq 0} : Ax + Gy = b\right\},$$

where $A \in \mathbb{R}^{r \times m}$, $G \in \mathbb{R}^{r \times n}$ *and* $b \in \mathbb{R}^r$ *(*$m, n, r \in \mathbb{Z}_{\geq 0}$*). Let* $k \in \mathbb{Z}_{\geq 0}$ *(even though only the case* $k \in \{0, \ldots, r\}$ *is of mathematical interest) and let* $M \in \mathbb{R}^{k \times r}$*. Then we set*

$$P(M) := \left\{\left(\begin{array}{c} x \\ y \end{array}\right) \in \mathbb{R}^m \times \mathbb{R}^n_{\geq 0} : MAx + MGy = Mb\right\}.$$

We denote $(P(M))_I$ *as a* k ***row relaxation of*** P*. A linear inequality* $c(\cdot) \leq c_0$ *(*$c \in (\mathbb{R}^m \times \mathbb{R}^n)^T$ *and* $c_0 \in \mathbb{R}$*) for* $(P(M))_I$ *is called a* ***cut from a*** k ***row relaxation*** *or shorter a* k ***row cut****.*

Similar to the remark in section 3.5.1, $P(M)$ is, of course, again an abuse of notation.

Definition 155. *Let* P*,* A*,* G*,* b*,* r *and* k *be as in Definition 154. Then we set*

$$\mathrm{cl}_{kR}(A, G, b) := P \cap \bigcap_{M \in \mathbb{R}^{k \times r}} \mathrm{cl}_{\overline{I}}(P(M))$$

as the k ***row closure with respect to*** A***,*** G ***and*** b*.*

Part II.

Cutting planes, L_k cuts and $L_{k-\frac{1}{2}}$ cuts

4. L_k cuts and $L_{k-\frac{1}{2}}$ cuts

In this chapter, we introduce the concepts of L_k cuts and $L_{k-\frac{1}{2}}$ cuts, which are central to this text. The high-level idea for these concepts is the following: let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be arbitrary. We want to find cutting planes for P, i.e. inequalities for $P \cap (\mathbb{Z}^m \times \mathbb{R}^n)$. Let $k \in \{0, \ldots, m\}$ (where $k = 0$ is a degenerate case) and consider a vector space $V \leq \mathbb{R}^m \times \mathbb{R}^n$ of codimension k. We study two ways of using V to relax the problem of finding valid inequalities for $P \cap (\mathbb{Z}^m \times \mathbb{R}^n)$ (in particular cf. Remark 181):

- Add V to the lattice $\mathbb{Z}^m \times \mathbb{R}^n$, i.e. find inequalities for $P \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + V)$; this is the central idea behind L_k **cuts**.
- Add V to P, i.e. find inequalities for $(P + V) \cap (\mathbb{Z}^m \times \mathbb{R}^n)$; this is the central idea behind $L_{k-\frac{1}{2}}$ **cuts**.

By demanding further conditions on the generators of V (in particular with respect to rationality), this leads to a multitude of different classes of L_k cuts and $L_{k-\frac{1}{2}}$ cuts, such as $L_{k,\mathbb{Q}}$ cuts, $L_{k,\mathbb{R}}$ cuts (cf. Definition 161), $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts, $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cuts and $L_{k-\frac{1}{2},\mathbb{R}\times\mathbb{R}}$ cuts (cf. Definition 179). Each of these families yields a hierarchy (indexed by k) of cutting planes or cutting plane operators/closures (intersection of P with all cutting planes of the respective type), such as $\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(\,\cdot\,)$, which become increasingly tight as k increases.

Chapter 4 is structured as follows:

- In section 4.1, we prove some auxiliary results about adding a vector space to a lattice or a polyhedron. These results are later on central for proving results about L_k cuts and $L_{k-\frac{1}{2}}$ cuts. Since the usefulness of these results is not obvious to the readers until they are more familiar with the framework of L_k and $L_{k-\frac{1}{2}}$ cuts, it is perhaps not a bad idea to skim over this section when one reads this chapter for the first time and come back to it later on.
- For section 4.2: in section 4.2.1, we introduce the framework of L_k cuts. In section 4.2.2, we consider how L_k cuts can be represented in an alternative way:
  - In section 4.2.2.1, Theorem 168, we show that for both $L_{k,\mathbb{Q}}$ and $L_{k,\mathbb{R}}$ cuts we can restrict ourselves to vector spaces of the form $V = V' \times \mathbb{R}^n$.
  - In section 4.2.2.2, we consider how $L_{k,\mathbb{Q}}$ cuts can be represented "in a dual way": instead of considering inequalities for
  $$P \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + (V' \times \mathbb{R}^n))$$
  (where $V' \leq \mathbb{R}^m$ is a rational subspace of codimension k), we consider inequalities for
  $$P \cap \left(\left\{ x \in \mathbb{R}^m : \left(w^i\right)^T x \in \mathbb{Z} \ \forall i \in [k] \right\} \times \mathbb{R}^n \right),$$
  where $w^1, \ldots, w^k \in \mathbb{Z}^m$. This perspective is taken in [DGMR17], a paper where a framework called k**-dimensional lattice cuts**, that is very related to $L_{k,\mathbb{Q}}$ cuts, is developed. This relationship is the topic of Definition 175 and Theorem 176.
- In section 4.3, we introduce the framework of $L_{k-\frac{1}{2}}$ cuts.

Remark 156. *Before we continue outlining the structure of chapter 4, we want to characterize the central questions that we want to analyze for L_k cuts/closures and $L_{k-\frac{1}{2}}$ cuts/closures:*

1. *Analyze under what conditions one type of L_k cuts/closure or $L_{k-\frac{1}{2}}$ cuts/closure is more expressive than another one or not.*
2. *Show that the out of themselves unrelated looking hierarchies of operators for $L_{k,\mathbb{Q}}$ cuts, $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts and $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cuts can be combined into a "unified hierarchy" for rational polyehdra, i.e. for a rational polyhedron $P \subseteq \mathbb{R}^m$ ($m \in \mathbb{Z}_{\geq 0}$), we have*

$$P = \mathrm{cl}_{L_{0,\mathbb{Q}}}(P) \supseteq \mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{Q}}}(P) \supseteq \mathrm{cl}_{L_{1,\mathbb{Q}}}(P) \supseteq \ldots \supseteq \mathrm{cl}_{L_{m-1,\mathbb{Q}}}(P) \supseteq \mathrm{cl}_{L_{m-\frac{1}{2},\mathbb{Q}}}(P) = \mathrm{cl}_{L_{m,\mathbb{Q}}}(P) = \mathrm{cl}_I(P) \tag{4.1}$$

and for a rational polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}_{\geq 1}$), the chain of inclusions

$$\begin{aligned} P = \mathrm{cl}_{L_{0,\mathbb{Q}}}(P) &\supseteq \mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P) \supseteq \mathrm{cl}_{L_{1,\mathbb{Q}}}(P) \supseteq \ldots \supseteq \mathrm{cl}_{L_{m-1,\mathbb{Q}}}(P) \\ &\supseteq \mathrm{cl}_{L_{m-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{m-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P) \supseteq \mathrm{cl}_{L_{m,\mathbb{Q}}}(P) = \mathrm{cl}_{L_{(m+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{(m+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P) = \mathrm{cl}_I(P) \end{aligned} \tag{4.2}$$

holds. We remark that in part VI (chapter 14 and chapter 15), we reconsider this hierarchy and analyze whether these inclusions can also be strict.

3. *In section 4.2.2.1, Theorem 168, we show that for both $L_{k,\mathbb{Q}}$ cuts and $L_{k,\mathbb{R}}$ cuts, we can restrict ourselves to vector spaces of the form $V = V' \times \mathbb{R}^n$. It is easy to see that such a restriction is not possible for $L_{k-\frac{1}{2}}$ cuts. Nevertheless, one can ask the question whether $L_{k-\frac{1}{2}}$ cuts, specifically $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts, with respect to such a vector space fill an interesting role or have interesting properties.*

Now on with the outline of this chapter:

- In section 4.4, we prove some technical results about the L_k and $L_{k-\frac{1}{2}}$ closures of some irrational hyperplanes, which are used in the subsequent section (section 4.5).
- By now, we have (even for a fixed k) defined lots of different classes of L_k and $L_{k-\frac{1}{2}}$ cuts. One can legitimately ask whether these really have differences in expressive power. This is what we formulated as guiding question 1 in Remark 156 and is the topic of section 4.5. At the beginning of this section, one can find a much more comprehensive summary of our results on this topic. For this introduction, we only want to mention Theorem 193, which we formulate in section 4.5.4 (though it is proved further back in section 4.8.4). This theorem states that, given a rational polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) and $k \in \{0, \ldots, m\}$, we have
$$\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P).$$
This implies the
$$\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P)$$
equalities in (4.2) of guiding question 2 in Remark 156.
- Up to here, we only considered L_k and $L_{k-\frac{1}{2}}$ cuts independently from each other. The naming already suggests that these two hierarchies can be unified into one hierarchy. Considering this question is the central topic of section 4.6:
  - In Theorem 197, we show that for arbitrary $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$), every $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cut for P ($k \in \{0, \ldots, m+n\}$) is also an $L_{k,\mathbb{Q}}$ cut for P with respect to the same subspace, which in particular implies
  $$\mathrm{cl}_{L_{k,\mathbb{Q}}}(P) \subseteq \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P).$$
  Note however (cf. Remark 198) that the similar looking inclusion
  $$\mathrm{cl}_{L_{k,\mathbb{R}}}(P) \subseteq \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{R}\times\mathbb{R}}}(P)$$
  does *not* hold in general.
  - In Theorem 199, we show that for arbitrary $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$), every $L_{k,\mathbb{R}}$ cut for P ($k \in \{0, \ldots, m+n-1\}$) is an $L_{(k+1)-\frac{1}{2},\mathbb{R}\times\mathbb{R}}$ cut for P. In particular, we have
  $$\mathrm{cl}_{L_{(k+1)-\frac{1}{2},\mathbb{R}\times\mathbb{R}}}(P) \subseteq \mathrm{cl}_{L_{k,\mathbb{R}}}(P)$$
  and if P is a rational polyhedron, we also have
  $$\mathrm{cl}_{L_{(k+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) \subseteq \mathrm{cl}_{L_{k,\mathbb{Q}}}(P).$$
- It is easy to see that we have for every type of L_{m+n} closure and $L_{m+n-\frac{1}{2}}$ closure and any arbitrary $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$):
$$\mathrm{cl}_{L_{m+n,(\cdot)}}(P) = \mathrm{cl}_{L_{m+n-\frac{1}{2},(\cdot)}}(P) = P \cap \mathrm{cl}_{\overline{I}}(P)$$

(for the simple reason that for the only vector space V of codimension $m+n$ that we can add is $\{0^m \times 0^n\}$). In Theorem 202 in section 4.7, we briefly remark that we already always have

$$\operatorname{cl}_{L_{m,\mathbb{Q}}}(P) = \operatorname{cl}_{L_{(m+1)-\frac{1}{2},\mathbb{R}\times\mathbb{R}}}(P) = P \cap \operatorname{cl}_{\overline{I}}(P)$$

and for rational polyhedra P, the identity

$$\operatorname{cl}_{L_{(m+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \operatorname{cl}_I(P)$$

holds. In other words: under these conditions, the $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ and $L_{k-\frac{1}{2},\mathbb{R}\times\mathbb{R}}$ hierarchies "already end at $k = m+1$ or $k = m$, respectively".

- The central topic of section 4.8 is guiding question 3 in Remark 156, i.e. the role of $L_{k-\frac{1}{2}}$ cuts with respect to vector spaces of the form $V = V' \times \mathbb{R}^n$:
  - In Definition 203, we define "essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts". These are $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts with respect to a vector space $V = V' \times \mathbb{R}^n$, where $V' \leq \mathbb{R}^m$ is – as the naming already suggests – generated by rational vectors and has codimension k.
  - In Theorem 208, we see the importance of this concept: we show that the only $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cuts (these include $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts) that are not already $L_{k-1,\mathbb{Q}}$ cuts, are essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts. In this sense, one can very casually say that "essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts are 'the interesting' $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cuts, since these are the only $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cuts that might have potentially more expressive power than $L_{k-1,\mathbb{Q}}$ cuts".
  - In Theorem 211, we use this structure result (Theorem 208) to show that the $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ closure and the $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ closure of a rational polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$ and $k \in \{0, \ldots, m+n\}$) are equal. We already mentioned in the introduction that this implies the

$$\operatorname{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \operatorname{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P)$$

    equalities in (4.2) of guiding question 2 in Remark 156.
  - Since we have just circumscribed how $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}/L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cuts that are not already $L_{k-1,\mathbb{Q}}$ cuts have to look like (they are essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts), we also skim the question of how an $L_{k,\mathbb{Q}}$ cut $c(\cdot) \leq c_0$ with respect to some V has to look like if it is not already an $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ or $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cut with respect to the same V. In Theorem 213, we show that for this to be the case, we need $c^T \not\perp V$.
  - Recall that in section 4.2.2.2 (in particular Theorem 174), we give a dual characterization of $L_{k,\mathbb{Q}}$ cuts. In Theorem 215, we consider how a similar dual characterization for essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts looks like.

- In section 4.9, we consider the following problem: let some arbitrary $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be given. Trivially, we always have

$$\operatorname{cl}_{(m+n)-\frac{1}{2},(\cdot)}(P) = P \cap \operatorname{cl}_{\overline{I}}(P)$$

  and

$$\operatorname{cl}_{m,(\cdot)}(P) = P \cap \operatorname{cl}_{\overline{I}}(P).$$

  But now assume that we can impose some "vector space condition" on P, such as
  - P or $\operatorname{proj}_{\mathbb{R}^m} P$ has a (w.l.o.g. non-trivial) lineality space (section 4.9.1) or
  - P or $\operatorname{proj}_{\mathbb{R}^m} P$ is contained in an (again w.l.o.g. non-trivial) affine subspace (section 4.9.3).

  Can we then show

$$\operatorname{cl}_{(m+n-l)-\frac{1}{2},(\cdot)}(P) = P \cap \operatorname{cl}_{\overline{I}}(P)$$

  or

$$\operatorname{cl}_{m-l,(\cdot)}(P) = P \cap \operatorname{cl}_{\overline{I}}(P),$$

  respectively, for some $l \in \mathbb{Z}_{\geq 1}$?

- Finally for section 4.10. In Definition 224, we consider the following construction: for a given polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$), we construct a polyhedron $P' \supseteq P$ that is defined by k inequalities and consider inequalities for P_I', which we call **k-half-space cuts**. In Theorem 225, we show that every k-half-space cut is an $L_{\min(k,m+n)-\frac{1}{2}}$ cut. On the other hand, in Theorem 226, we prove that for

$$P := P^{=}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right) \cap \left(\mathbb{R}^m \times \mathbb{R}^n_{\geq 0}\right)$$

as in Definition 154 (this is the form that is required for k row cuts), every k-half-space cut for P is a k row cut with respect to A, G and b.

4.1. Preparations for $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ and $L_{k,\mathbb{Q}}$ cuts

As already mentioned in the introduction, we start this chapter with some technical results:

- In section 4.1.1, we introduce technical results (Lemma 157 and Corollary 158) about linear inequalities for $P\cap((\mathbb{Z}^m \times \mathbb{R}^n) + V)$, where $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) is arbitrary. These results become important for $L_{k,\mathbb{Q}}$ cuts, which are introduced in section 4.2.
- In section 4.1.2, we consider results about linear inequalities for $(P + V) \cap (\mathbb{Z}^m \times \mathbb{R}^n)$, where $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) is arbitrary and V is either generated by vectors from $\mathbb{Q}^m \times \mathbb{Q}^n$ or $\mathbb{Q}^m \times \mathbb{R}^n$. This becomes important for $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts and $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cuts, which are both introduced in section 4.3.

4.1.1. Preparations for $L_{k,\mathbb{Q}}$ cuts

The following lemma is about inequalities for $P \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + V)$, where $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ is arbitrary and $V \leq \mathbb{R}^m \times \mathbb{R}^n$ is a rational subspace:

Lemma 157. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be arbitrary and let $V \leq \mathbb{R}^m \times \mathbb{R}^n$ be a rational subspace of codimension $k \in \{0, \dots, m+n\}$. Let $\{v^1, \dots, v^{m+n-k}\} \subseteq \mathbb{Q}^m \times \mathbb{Q}^n$ be a basis of V. Define*

$$C := \begin{pmatrix} w^1 & \cdots & w^{k-s} & w^{k-s+1} & \cdots & w^k & v^1 & \cdots & v^{m+n-k} \end{pmatrix}$$

such that

$$\operatorname{proj}^{\perp}_{V^\perp}(\mathbb{Z}^m \times \mathbb{R}^n) = C_{*,(1,\dots,k)} \begin{pmatrix} \mathbb{Z}^{k-s} \\ \mathbb{R}^s \end{pmatrix}$$

(in other words: $\{w^1, \dots, w^k\} \subseteq \mathbb{Q}^m$ forms a lattice basis of the mixed lattice $\operatorname{proj}^{\perp}_{V^\perp}(\mathbb{Z}^m \times \mathbb{R}^n)$ of signature $(k-s, s)$). Then $c(\cdot) \leq c_0$ is a valid inequality for $P \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + V)$ if and only if $cC(\cdot) \leq c_0$ is a valid inequality for $\left(C^{-1}P\right) \cap \left(\mathbb{Z}^{k-s} \times \mathbb{R}^{m+n-(k-s)}\right)$.

Proof.

For “only if”: Let $c(\cdot) \leq c_0$ be valid for $P \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + V)$ and let

$$x' \in \left(C^{-1}P\right) \cap \left(\mathbb{Z}^{k-s} \times \mathbb{R}^{m+n-(k-s)}\right).$$

Then $x' = \begin{pmatrix} \mu' \\ \lambda' \end{pmatrix}$, where

- $\mu' \in \mathbb{Z}^{k-s} \times \mathbb{R}^s$ and
- $\lambda' \in \mathbb{R}^{m+n-k}$.

Thus,

$$\begin{aligned} Cx' &= \begin{pmatrix} w^1 & \cdots & w^k & v^1 & \cdots & v^{m+n-k} \end{pmatrix} \begin{pmatrix} \mu' \\ \lambda' \end{pmatrix} \\ &= \begin{pmatrix} w^1 & \cdots & w^k \end{pmatrix} \mu' + \begin{pmatrix} v^1 & \cdots & v^{m+n-k} \end{pmatrix} \lambda' \\ &\in \left(\operatorname{proj}^{\perp}_{V^\perp}(\mathbb{Z}^m \times \mathbb{R}^n)\right) + V \\ &= (\mathbb{Z}^m \times \mathbb{R}^n) + V. \end{aligned}$$

On the other hand, $Cx' \in CC^{-1}P = P$. So, $cC(\cdot) \leq c_0$ is a valid inequality for

$$\left(C^{-1}P\right) \cap \left(\mathbb{Z}^{k-s} \times \mathbb{R}^{m+n-(k-s)}\right).$$

For "if": Let $cC(\cdot) \leq c_0$ be a valid inequality for $\left(C^{-1}P\right) \cap \left(\mathbb{Z}^{k-s} \times \mathbb{R}^{m+n-(k-s)}\right)$ and let

$$x \in P \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + V).$$

Since $x \in (\mathbb{Z}^m \times \mathbb{R}^n) + V$, we have $x = w + v$, where

- $w \in \operatorname{proj}^{\perp}_{V^\perp}(\mathbb{Z}^m \times \mathbb{R}^n)$ and
- $v \in V$.

Then $w = C_{*,(1,\ldots,k)}\mu$, where $\mu \in \mathbb{Z}^{k-s} \times \mathbb{R}^s$. Thus,

$$C^{-1}x = \begin{pmatrix} C_{*,(1,\ldots,k)} & v^1 & \cdots & v^{m+n-k} \end{pmatrix}^{-1} \left(C_{*,(1,\ldots,k)}\mu + v\right) \in \begin{pmatrix} \mu \\ \mathbb{R}^{m-k} \end{pmatrix} \subseteq \mathbb{Z}^{k-s} \times \mathbb{R}^{m+n-(k-s)}.$$

On the other hand, $C^{-1}x \in C^{-1}P = P$. So,

$$cx = cC \cdot \underbrace{C^{-1}x}_{\in (C^{-1}P)\cap\left(\mathbb{Z}^{k-s}\times\mathbb{R}^{m+n-(k-s)}\right)} \leq c_0.$$

□

Now let $V \leq \mathbb{R}^m$ be a rational subspace. Note that

$$(\mathbb{Z}^m \times \mathbb{R}^n) + (V \times \{0^n\}) = (\mathbb{Z}^m \times \mathbb{R}^n) + (V \times \mathbb{R}^n).$$

Thus, the following corollary is an immediate consequence of Lemma 157:

Corollary 158. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be arbitrary and let $V \leq \mathbb{R}^m$ be a rational subspace of codimension $k \in \{0, \ldots, m\}$. Let $\left\{v^1, \ldots, v^{m-k}\right\} \subseteq \mathbb{Q}^m$ be a basis of V and let $w^1, \ldots, w^k \in \mathbb{Q}^m$ be a lattice basis of $\operatorname{proj}^{\perp}_{V^\perp} \mathbb{Z}^m$. Define*

$$C := \left(\begin{array}{ccccccc|c} w^1 & \cdots & w^k & v^1 & \cdots & v^{m-k} & & 0^{m\times n} \\ \hline & & & 0^{n\times m} & & & & I^n \end{array}\right).$$

Then $c(\cdot) \leq c_0$ is a valid inequality for

$$P \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + (V \times 0^n)) = P \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + (V \times \mathbb{R}^n))$$

if and only if $cC(\cdot) \leq c_0$ is a valid inequality for

$$\left(C^{-1}P\right) \cap \left(\mathbb{Z}^k \times \mathbb{R}^{m+n-k}\right).$$

4.1.2. Preparations for $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts

In this section, we prove two technical results (Lemma 159 and Lemma 160) about linear inequalities for $(P+V)_I$, where $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ is arbitrary and V is a vector space that is either generated by vectors from $\mathbb{Q}^m \times \mathbb{Q}^n$ or $\mathbb{Q}^m \times \mathbb{R}^n$.

Lemma 159. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be arbitrary, let $V \leq \mathbb{R}^m \times \mathbb{R}^n$ be a subspace with generators from $\mathbb{Q}^m \times \mathbb{R}^n$ and let $c(\cdot) \leq c_0$ be a valid inequality for $(P+V)_I$. Then:*

1. *If $(P+V)_I \neq \emptyset$, we have $c^T \perp V$.*
2. *Let $(P+V)_I = \emptyset$. Then $c(\cdot) \leq c_0$ is dominated absolutely by $(0^m \times 0^n)^T(\cdot) \leq -1$, which is valid for $(P+V)_I$, and we have $0^m \times 0^n \perp V$.*

Proof. 2 holds obviously; so, we only prove 1. Let $\left\{v^1, \ldots, v^l\right\} \subseteq \mathbb{Z}^m \times \mathbb{R}^n$ ($l \in \{0, \ldots, m+n\}$) be a basis of V and let $z \in (P+V)_I$. Assume that there exists some $i \in [l]$ such that $cv^i \neq 0$. W.l.o.g. we can assume $cv^i > 0$. Define

$$M := \left\lceil \frac{c_0 - cz}{cv^i} \right\rceil + 1 \in \mathbb{Z}.$$

Then $z + M \cdot v^i \in (P+V)_I$ and we have

$$\begin{aligned} c\left(z + M \cdot v^i\right) &= cz + cv^i \left(\left\lceil \frac{c_0 - cz}{cv^i} \right\rceil + 1\right) \\ &\geq cz + (c_0 - cz) + cv^i \\ &= c_0 + cv^i \\ &> c_0. \lightning \end{aligned}$$

□

Lemma 160. *Let $V \leq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ be a rational subspace of codimension $k \in \{0, \ldots, m+n\}$ with basis $\{v^1, \ldots, v^{m+n-k}\} \subseteq \mathbb{Q}^m \times \mathbb{Q}^n$. Let*

$$\mathrm{proj}^{\perp}_{V^\perp}(\mathbb{Z}^m \times \mathbb{R}^n) =: \begin{pmatrix} w^1 & \cdots & w^k \end{pmatrix} \begin{pmatrix} \mathbb{Z}^{k-s} \\ \mathbb{R}^s \end{pmatrix}$$

be a mixed lattice of signature $(k-s, s)$. Define

$$C := \begin{pmatrix} w^1 & \cdots & w^k & v^1 & \cdots & v^{m+n-k} \end{pmatrix}.$$

Then:

1. *Let $c(\cdot) \leq c_0$ be a valid inequality for $(P+V) \cap (\mathbb{Z}^m \times \mathbb{R}^n)$. Then $cC(\cdot) \leq c_0$ is a valid inequality for $\left((C^{-1}P) + \left(0^k \times \mathbb{R}^{m+n-k}\right)\right) \cap \left(\mathbb{Z}^{k-s} \times \mathbb{R}^s \times \mathbb{R}^{m+n-k}\right)$. If $c^T \in V^\perp$ (this condition can be assumed w.l.o.g. because of Lemma 159), we have $(cC)^T \in \mathbb{R}^k \times 0^{m+n-k}$.*
2. *Let $c'(\cdot) \leq c_0$ be a valid inequality for $\left((C^{-1}P) + \left(0^k \times \mathbb{R}^{m+n-k}\right)\right) \cap \left(\mathbb{Z}^{k-s} \times \mathbb{R}^s \times \mathbb{R}^{m+n-k}\right)$. Then $c'C^{-1}(\cdot) \leq c_0$ is a valid inequality for $(P+V) \cap (\mathbb{Z}^m \times \mathbb{R}^n)$. If $c' \in \left(\mathbb{R}^k \times 0^{m+n-k}\right)^T$ (this condition can be assumed w.l.o.g. because of Lemma 159), we have $\left(c'C^{-1}\right)^T \in V^\perp$.*

Proof. We first show

$$(P+V) \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + V) = \emptyset \Leftrightarrow (P+V) \cap (\mathbb{Z}^m \times \mathbb{R}^n) = \emptyset. \tag{4.3}$$

The "⇒" implication clearly holds because $0^m \times 0^n \in V$. For "⇐", let $x \in (P+V) \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + V)$. Then $x = p + v = z + v'$, where $p \in P$, $v, v' \in V$ and $z \in \mathbb{Z}^m \times \mathbb{R}^n$. So $z = p + v - v' \in (P+V) \cap (\mathbb{Z}^m \times \mathbb{R}^n)$.

Next, we show

$$(P+V) \cap (\mathbb{Z}^m \times \mathbb{R}^n) = \emptyset \Leftrightarrow \left(C^{-1}P + \left(0^k \times \mathbb{R}^{m+n-k}\right)\right) \cap \left(\mathbb{Z}^{k-s} \times \mathbb{R}^{m+n-(k-s)}\right) = \emptyset. \tag{4.4}$$

For (4.4):

$$\begin{aligned} & \left((C^{-1}P) + \left(0^k \times \mathbb{R}^{m+n-k}\right)\right) \cap \left(\mathbb{Z}^{k-s} \times \mathbb{R}^s \times \mathbb{R}^{m+n-k}\right) = \emptyset \\ \Leftrightarrow\quad & C\left(\left((C^{-1}P) + \left(0^k \times \mathbb{R}^{m+n-k}\right)\right) \cap \left(\mathbb{Z}^{k-s} \times \mathbb{R}^s \times \mathbb{R}^{m+n-k}\right)\right) = \emptyset \\ \Leftrightarrow\quad & (P+V) \cap \left(C\left(\mathbb{Z}^{k-s} \times \mathbb{R}^s \times \mathbb{R}^{m+n-k}\right)\right) = \emptyset \\ \Leftrightarrow\quad & (P+V) \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + V) = \emptyset \\ \Leftrightarrow\quad & (P+V) \cap (\mathbb{Z}^m \times \mathbb{R}^n) = \emptyset. \qquad \text{(by (4.3))} \end{aligned}$$

For 1: If $c \notin \left(V^\perp\right)^T$, we have $(P+V) \cap (\mathbb{Z}^m \times \mathbb{R}^n) = \emptyset$ (by Lemma 159). By (4.4), this is equivalent to $\left((C^{-1}P) + \left(0^k \times \mathbb{R}^{m+n-k}\right)\right) \cap \left(\mathbb{Z}^{k-s} \times \mathbb{R}^s \times \mathbb{R}^{m+n-k}\right) = \emptyset$. So, we can assume $c \in \left(V^\perp\right)^T$. Let

$$x \in \left((C^{-1}P) + \left(0^k \times \mathbb{R}^{m+n-k}\right)\right) \cap \left(\mathbb{Z}^{k-s} \times \mathbb{R}^s \times \mathbb{R}^{m+n-k}\right).$$

Then $x = C^{-1}p + r$, where $p \in P$ and $r \in 0^k \times \mathbb{R}^{m+n-k}$. Since $\{w^1, \ldots, w^k\}$ forms a basis of $\mathrm{proj}^{\perp}_{V^\perp}(\mathbb{Z}^m \times \mathbb{R}^n)$ of signature $(k-s, s)$, for every $i \in \{1, \ldots, k-s\}$, there exists a $v^{\Delta,i} \in V$ such that $w^i + v^{\Delta,i} \in \mathbb{Z}^m \times \mathbb{R}^n$, and for every $i \in \{k-s+1, \ldots, k\}$, there exists a $v^{\Delta,i} \in V$ such that $w^i + v^{\Delta,i} \in 0^m \times \mathbb{R}^n$. Let

$$v^\Delta := \begin{pmatrix} v^{\Delta,1} & \cdots & v^{\Delta,k} \end{pmatrix} x_{(1,\ldots,k)} - \begin{pmatrix} v^1 & \cdots & v^{m+n-k} \end{pmatrix} x_{(k+1,\ldots,m)}.$$

We next show

$$Cx + v^{\Delta} \in (P+V) \cap (\mathbb{Z}^m \times \mathbb{R}^n).$$

For $Cx + v^{\Delta} \in P + V$:

$$Cx + v^{\Delta} = C\left(C^{-1}p + r\right) + v^{\Delta} = p + \underbrace{Cr}_{\in V} + \underbrace{v^{\Delta}}_{\in V} \in P + V.$$

For $Cx + v^{\Delta} \in \mathbb{Z}^m \times \mathbb{R}^n$:

$$\begin{aligned} Cx + v^{\Delta} =& \begin{pmatrix} w^1 + v^{\Delta,1} & \cdots & w^{k-s} + v^{\Delta,k-s} \end{pmatrix} x_{(1,\dots,k-s)} \\ &+ \begin{pmatrix} w^{k-s+1} + v^{\Delta,k-s+1} & \cdots & w^k + v^{\Delta,k} \end{pmatrix} x_{(k-s+1,\dots,k)} \\ &+ \begin{pmatrix} v^1 & \cdots & v^{m+n-k} \end{pmatrix} x_{(k+1,\dots,m)} - \begin{pmatrix} v^1 & \cdots & v^{m+n-k} \end{pmatrix} x_{(k+1,\dots,m)} \\ &\subseteq (\mathbb{Z}^m \times \mathbb{R}^n) + (0^m \times \mathbb{R}^n) \\ &= \mathbb{Z}^m \times \mathbb{R}^n. \end{aligned}$$

Thus, using $v^{\Delta} \in V \perp c^T$ and $Cx + v^{\Delta} \in (P+V) \cap (\mathbb{Z}^m \times \mathbb{R}^n)$, we conclude

$$cCx = c\left(Cx + v^{\Delta}\right) \leq c_0.$$

For the second statement: Using $c^T \perp V$, we obtain

$$(cC)^T = \left(c \begin{pmatrix} w^1 & \cdots & w^k & v^1 & \cdots & v^{m+n-k} \end{pmatrix}\right)^T \in \mathbb{R}^k \times 0^{m+n-k}.$$

For 2: Let

$$x' \in (P+V) \cap (\mathbb{Z}^m \times \mathbb{R}^n).$$

Then $x' = p' + v'$, where $p' \in P$ and $v' \in V$. We want to show that $c'C^{-1}x' \leq c_0$. For this, we show

$$C^{-1}x' \in \left(\left(C^{-1}P\right) + \left(0^k \times \mathbb{R}^{m+n-k}\right)\right) \cap \left(\mathbb{Z}^{k-s} \times \mathbb{R}^s \times \mathbb{R}^{m+n-k}\right).$$

For $C^{-1}x' \in \left(C^{-1}P\right) + \left(0^k \times \mathbb{R}^{m+n-k}\right)$:

$$\begin{aligned} C^{-1}x' &= C^{-1}p' + \begin{pmatrix} w^1 & \cdots & w^k & v^1 & \cdots & v^{m+n-k} \end{pmatrix}^{-1} v' \\ &\in C^{-1}P + \left(0^k \times \mathbb{R}^{m+n-k}\right). \qquad \text{(since } v' \perp w^1, \dots, w^k\text{)} \end{aligned}$$

For $C^{-1}x' \in \mathbb{Z}^{k-s} \times \mathbb{R}^s \times \mathbb{R}^{m+n-k}$:

$$\begin{aligned} C^{-1}x' &\subseteq C^{-1}\left(\mathbb{Z}^m \times \mathbb{R}^n\right) \\ &\subseteq C^{-1}\left(\operatorname{proj}^{\perp}_{V^{\perp}}\left(\mathbb{Z}^m \times \mathbb{R}^n\right) + V\right) \\ &= C^{-1}\left(C_{*,(1,\dots,k)} \begin{pmatrix} \mathbb{Z}^{k-s} \\ \mathbb{R}^s \end{pmatrix} + \underbrace{V}_{=C_{*,(k+1,\dots,m+n)}\mathbb{R}^{m+n-k}}\right) \\ &= \mathbb{Z}^{k-s} \times \mathbb{R}^{m+n-(k-s)}. \end{aligned}$$

Thus, since $c'(\cdot) \leq c_0$ is a valid inequality for $\left(\left(C^{-1}P\right) + \left(0^k \times \mathbb{R}^{m+n-k}\right)\right) \cap \left(\mathbb{Z}^{k-s} \times \mathbb{R}^s \times \mathbb{R}^{m+n-k}\right)$, and $C^{-1}x' \in \left(\left(C^{-1}P\right) + \left(0^k \times \mathbb{R}^{m+n-k}\right)\right) \cap \left(\mathbb{Z}^{k-s} \times \mathbb{R}^s \times \mathbb{R}^{m+n-k}\right)$, we conclude $c'C^{-1}x' \leq c_0$.

For the second statement: Let $v \in V$. Then

$$\left(c'C^{-1}\right)v = c' \begin{pmatrix} w^1 & \cdots & w^k & v^1 & \cdots & v^{m+n-k} \end{pmatrix}^{-1} v \in \begin{pmatrix} \mathbb{R}^k \\ 0^{m+n-k} \end{pmatrix}^T \begin{pmatrix} 0^k \\ \mathbb{R}^{m+n-k} \end{pmatrix} = 0.$$

□

4.2. L_k cuts and closures

4.2.1. Definitions

Definition 161. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ be arbitrary, let $k \in \{0, \ldots, m+n\}$ and let $V \leq \mathbb{R}^m \times \mathbb{R}^n$ be a linear subspace of codimension k that is generated by vectors from*

- $\mathbb{Q}^m \times \mathbb{R}^n$ *or*
- $\mathbb{R}^m \times \mathbb{R}^n$,

respectively. An

- $L_{k,\mathbb{Q}}$ ***cut for*** P *or*
- $L_{k,\mathbb{R}}$ ***cut for*** P,

respectively, is a valid linear inequality $c(\cdot) \leq c_0$ $(c \in (\mathbb{R}^m \times \mathbb{R}^n)^T$ and $c_0 \in \mathbb{R})$ for

$$P \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + V).$$

*If it is of no importance what kind of L_k cuts is used or this is clear from the context, we also use the general term L_k **cuts**.*

In Figure 4.1, one can see an illustration of the construction behind $L_{k,\mathbb{Q}}$ cuts.

Remark 162. *The "degenerated" L_0 cuts for P are simply the valid inequalities for P.*

Remark 163. *We remark that $L_{k,\mathbb{Q}}$ cuts have independently been discovered by Dash, Günlük and Morán (cf. [DGMR17]), even though their definition is "dual" to ours. We defer this discussion to Definition 175 and Theorem 176.*

Remark 164. *If we have a vector space $V \leq \mathbb{R}^m \times \mathbb{R}^n$ of codimension k that is generated by vectors from, for example, $\mathbb{Q}^m \times \mathbb{R}^n$, we can always assume that it is generated by $m+n-k$* linearly independent *vectors from $\mathbb{Q}^m \times \mathbb{R}^n$. From now on, we take the freedom to always implicitely make use of this assumption, if necessary.*

We next define the $L_{k,\mathbb{Q}}$ closure and $L_{k,\mathbb{R}}$ closure in the canonical way:

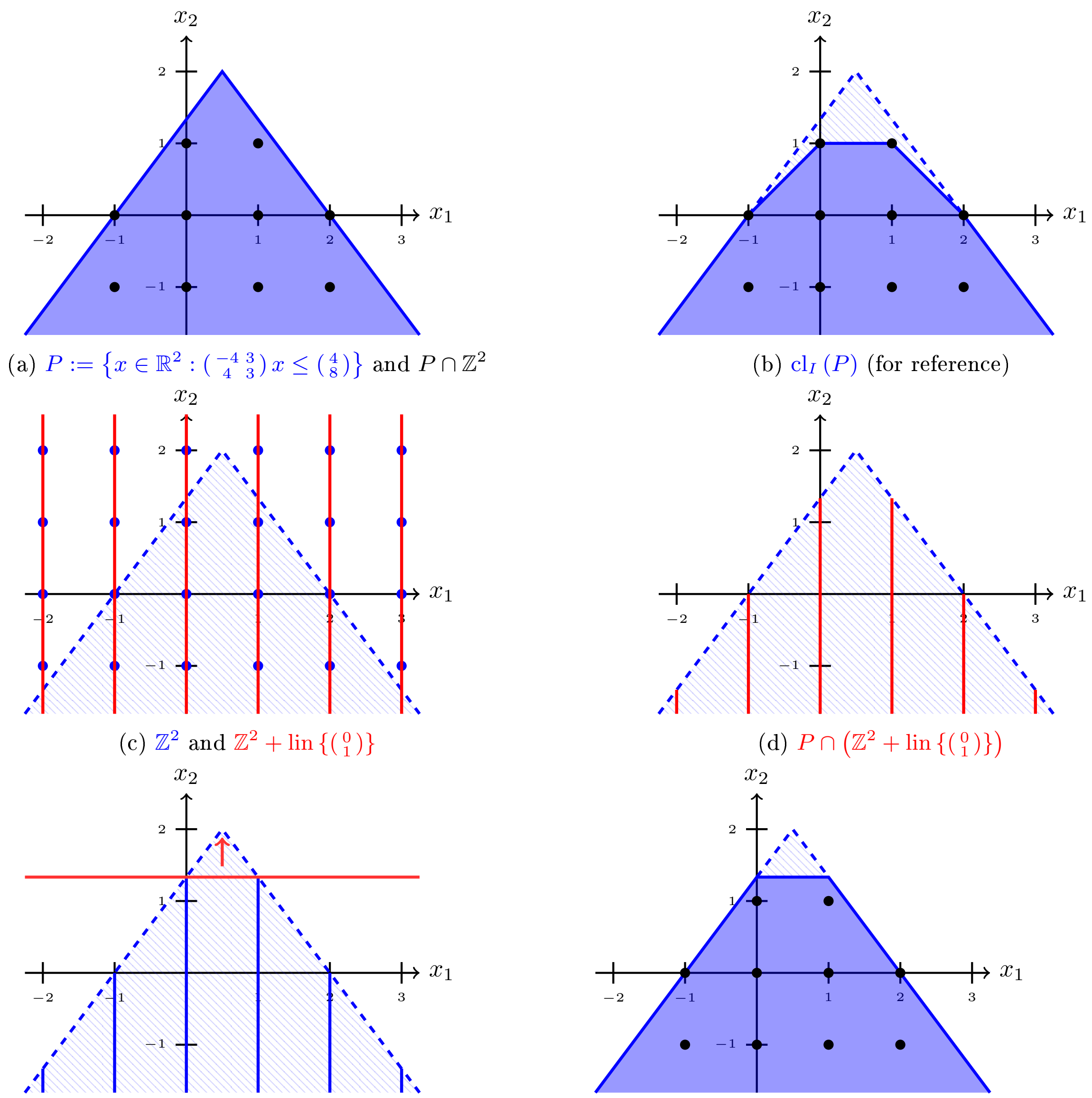
Definition 165. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ be arbitrary and let $k \in \{0, \ldots, m+n\}$. Then we define the $L_{k,\mathbb{Q}}$ **closure of** P $(\mathrm{cl}_{L_{k,\mathbb{Q}}}(P))$ and the $L_{k,\mathbb{R}}$ **closure of** P $(\mathrm{cl}_{L_{k,\mathbb{R}}}(P))$, respectively, via*

$$\mathrm{cl}_{L_{k,\substack{\mathbb{Q}\\ \mathbb{R}}}}(P) := P \cap \bigcap_{\substack{c \in (\mathbb{R}^m \times \mathbb{R}^n)^T, c_0 \in \mathbb{R}: \\ c(\cdot) \leq c_0 \ L_{k,\substack{\mathbb{Q}\\ \mathbb{R}}} \text{ cut for } P}} P^{\leq}(c, c_0).$$

The following lemma is an immediate consequence of Definition 165 and Theorem 72:

Lemma 166. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ be arbitrary and let $k \in \{0, \ldots, m+n\}$. Then*

$$\mathrm{cl}_{L_{k,\substack{\mathbb{Q}\\ \mathbb{R}}}}(P) = P \cap \bigcap_{\substack{V \leq \mathbb{R}^m \times \mathbb{R}^n : \mathrm{codim}\, V = k, \\ V \text{ generated by vectors} \\ \text{from } \substack{\mathbb{Q}^m \times \mathbb{R}^n \\ \mathbb{R}^m \times \mathbb{R}^n}}} \overline{\mathrm{conv}}\,(P \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + V)).$$



(a) $P := \left\{x \in \mathbb{R}^2 : \left(\begin{smallmatrix}-4 & 3\\ 4 & 3\end{smallmatrix}\right) x \leq \left(\begin{smallmatrix}4\\ 8\end{smallmatrix}\right)\right\}$ and $P \cap \mathbb{Z}^2$

(b) $\mathrm{cl}_I(P)$ (for reference)

(c) $\mathbb{Z}^2$ and $\mathbb{Z}^2 + \mathrm{lin}\left\{\left(\begin{smallmatrix}0\\ 1\end{smallmatrix}\right)\right\}$

(d) $P \cap \left(\mathbb{Z}^2 + \mathrm{lin}\left\{\left(\begin{smallmatrix}0\\ 1\end{smallmatrix}\right)\right\}\right)$

(e) $x_2 \leq \frac{4}{3}$ is valid for $x \in P \cap \left(\mathbb{Z}^2 + \mathrm{lin}\left\{\left(\begin{smallmatrix}0\\ 1\end{smallmatrix}\right)\right\}\right)$

(f) $P \cap \left\{x \in \mathbb{R}^2 : x_2 \leq \frac{4}{3}\right\}$

Figure 4.1.: Illustration of $L_{k,\mathbb{Q}}$ cuts

4.2.2. Representation of L_k cuts

4.2.2.1. Restriction to $V = V' \times \mathbb{R}^m$

We now show that in the definition of L_k cuts, we can assume that V is a vector space of the form $V = V' \times \mathbb{R}^m$, where V' is generated by vectors from $\mathbb{Q}^m$ ($L_{k,\mathbb{Q}}$ cuts) or $\mathbb{R}^m$ ($L_{k,\mathbb{R}}$ cuts). For this, we start with a small lemma:

Lemma 167. *Let $V \leq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be a subspace. Then*

$$(\mathbb{Z}^m \times \mathbb{R}^n) + V = (\mathbb{Z}^m \times \mathbb{R}^n) + ((\mathrm{proj}_{\mathbb{R}^m} V) \times \mathbb{R}^n).$$

Proof. Clearly, $V \subseteq (\mathrm{proj}_{\mathbb{R}^m} V) \times \mathbb{R}^n$; thus,

$$(\mathbb{Z}^m \times \mathbb{R}^n) + V \subseteq (\mathbb{Z}^m \times \mathbb{R}^n) + ((\mathrm{proj}_{\mathbb{R}^m} V) \times \mathbb{R}^n).$$

For

$$(\mathbb{Z}^m \times \mathbb{R}^n) + ((\mathrm{proj}_{\mathbb{R}^m} V) \times \mathbb{R}^n) \subseteq (\mathbb{Z}^m \times \mathbb{R}^n) + V,$$

consider some $z + v \in (\mathbb{Z}^m \times \mathbb{R}^n) + ((\mathrm{proj}_{\mathbb{R}^m} V) \times \mathbb{R}^n)$, where $z \in \mathbb{Z}^m \times \mathbb{R}^n$ and $v \in (\mathrm{proj}_{\mathbb{R}^m} V) \times \mathbb{R}^n$. The definition of v implies that there exists a $w \in \mathbb{R}^n$ such that $\left(\begin{smallmatrix} v_{(1,\dots,m)} \\ w \end{smallmatrix}\right) \in V$. So,

$$z + v = \underbrace{z + \begin{pmatrix} 0^n \\ v_{(m+1,\dots,m+n)} - w \end{pmatrix}}_{\in \mathbb{Z}^m \times \mathbb{R}^n} + \underbrace{\begin{pmatrix} v_{(1,\dots,m)} \\ w \end{pmatrix}}_{\in V} \in (\mathbb{Z}^m \times \mathbb{R}^n) + V.$$

□

The following theorem is an immediate consequence of Lemma 167:

Theorem 168. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be arbitrary, let $V \leq \mathbb{R}^m \times \mathbb{R}^n$ be a vector space of codimension $k \in \{0, \dots, m+n\}$ that is generated by vectors from $\mathbb{Q}^m \times \mathbb{R}^n$ or $\mathbb{R}^m \times \mathbb{R}^n$ and let $V' := \operatorname{proj}_{\mathbb{R}^m} V$. Then*

$$P \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + (V' \times \mathbb{R}^n)) = P \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + V). \tag{4.5}$$

In particular, every $L_{k,\mathbb{Q}}$ or $L_{k,\mathbb{R}}$ cut for P with respect to V is an $L_{k,\mathbb{Q}}$ or $L_{k,\mathbb{R}}$ cut for P with respect to $V' \times \mathbb{R}^n$ and for L_k cuts, it thus suffices to consider vector spaces of the form $V := V' \times \mathbb{R}^n$, where V' is generated by vectors from $\mathbb{Q}^m$ or $\mathbb{R}^m$.

Remark 169. *In Theorem 168, let V be generated by linearly independent vectors*

$$\begin{pmatrix} v'^1 \\ v''^1 \end{pmatrix}, \dots, \begin{pmatrix} v'^{m+n-k} \\ v''^{m+n-k} \end{pmatrix} \in \mathbb{Q}^m \times \mathbb{R}^m.$$

Then $V' := \operatorname{proj}_{\mathbb{R}^m} V$ is generated by $\left\{v'^1, \dots, v'^{m+n-k}\right\}$ and there exists a linear independent subset of $\left\{v'^1, \dots, v'^{m+n-k}\right\}$ that forms a basis of V' which consists of vectors from $\mathbb{Q}^m$.

Remark 170. *In Theorem 168, we have:*

$$\operatorname{codim} V' = m - \dim V' = m + n - \dim (V' \times \mathbb{R}^n) \leq m + n - \dim V = m + n - (m + n - k) = k.$$

An immediate consequence of Lemma 166 and Theorem 168 is:

Corollary 171. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be arbitrary and let $k \in \{0, \dots, m\}$. Then*

$$\operatorname{cl}_{L_{k,{\mathbb{Q} \atop \mathbb{R}}}}(P) = P \cap \bigcap_{\substack{V' \leq \mathbb{R}^m : \operatorname{codim} V' = k, \\ V' \text{ generated by vectors} \\ \text{from } {\mathbb{Q}^m \atop \mathbb{R}^m}}} \overline{\operatorname{conv}}(P \cap \underbrace{((\mathbb{Z}^m \times \mathbb{R}^n) + (V' \times \mathbb{R}^n))}_{=(\mathbb{Z}^m \times \mathbb{R}^n) + (V' \times 0^n)}).$$

4.2.2.2. A dual representation of $L_{k,\mathbb{Q}}$ cuts

We next want to show some statements about how one can represent $L_{k,\mathbb{Q}}$ cuts in an alternative, "dual" way.

We immediately conclude from Lemma 98:

Theorem 172. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be arbitrary, let $V \leq \mathbb{R}^m$ be a rational subspace of codimension $k \in \{0, \dots, m\}$ and let $w^1, \dots, w^k$ be a lattice basis of $\mathbb{Z}^m \cap V^\perp$ (as in Lemma 98). Then*

$$P \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + (V \times \mathbb{R}^n)) = P \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + (V \times 0^n)) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in P : \forall i \in [k] : \left(w^i\right)^T x \in \mathbb{Z} \right\}.$$

Theorem 173. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be arbitrary, let $w'^1, \dots, w'^{k'} \in \mathbb{Z}^m$ ($k' \in \mathbb{Z}_{\geq 0}$) and let*

$$W := \operatorname{lin}\left\{w'^1, \dots, w'^{k'}\right\},$$
$$k := \dim W.$$

Then every inequality for

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in P : \forall i \in [k'] : \left(w'^i\right)^T x \in \mathbb{Z} \right\} \tag{4.6}$$

is an $L_{k,\mathbb{Q}}$ cut for P with respect to $W^\perp$.

Proof. Let $w^1, \dots, w^k \in \mathbb{Z}^m$ be a lattice basis of $\mathbb{Z}^m \cap \operatorname{lin}\left\{w'^1, \dots, w'^{k'}\right\}$. By Lemma 99, we have

$$\left\{\begin{pmatrix} x \\ y \end{pmatrix} \in P : \forall i \in [k] : \left(w^i\right)^T x \in \mathbb{Z}\right\} \supseteq \left\{\begin{pmatrix} x \\ y \end{pmatrix} \in P : \forall i \in [k'] : \left(w'^i\right)^T x \in \mathbb{Z}\right\}.$$

So, by Theorem 172, every inequality for (4.6) is an $L_{k,\mathbb{Q}}$ cut for P with respect to $W^\perp$. □

From Theorem 172 and Theorem 173, we immediately conclude:

Theorem 174. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be arbitrary. An inequality $c(\,\cdot\,) \leq c_0$ ($c \in (\mathbb{R}^m \times \mathbb{R}^n)^T$ and $c_0 \in \mathbb{R}$) is an $L_{k,\mathbb{Q}}$ cut for P ($k \in \{0, \dots, m\}$) if and only if there exist vectors $w^1, \dots, w^k \in \mathbb{Z}^m$ (which can w.l.o.g. be assumed to be linearly independent; but this assumption is not necessary) such that $c(\,\cdot\,) \leq c_0$ is a valid inequality for*

$$\left\{\begin{pmatrix} x \\ y \end{pmatrix} \in P : \forall i \in [k] : \left(w^i\right)^T x \in \mathbb{Z}\right\}.$$

We now come back to Remark 163. In [DGMR17], the authors consider the following class of cutting planes:

Definition 175. *([DGMR17, section 1]) Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be arbitrary (in [DGMR17], P is assumed to be a polyhedron) and let $k \in \mathbb{Z}_{\geq 0}$. A k-**dimensional lattice cut for** P is a valid linear inequality $c(\,\cdot\,) \leq c_0$ ($c \in (\mathbb{R}^m \times \mathbb{R}^n)^T$ and $c_0 \in \mathbb{R}$) for*

$$\left\{\begin{pmatrix} x \\ y \end{pmatrix} \in P : \left(\pi^1\right)^T x \in \mathbb{Z}, \dots, \left(\pi^k\right)^T x \in \mathbb{Z}\right\},$$

where $\pi^1, \dots, \pi^k \in \mathbb{Z}^m$.

So, we obtain:

Theorem 176. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be arbitrary.*

- *Let $c(\,\cdot\,) \leq c_0$ ($c \in (\mathbb{R}^m \times \mathbb{R}^n)^T$ and $c_0 \in \mathbb{R}$) be a k-dimensional lattice cut for P with respect to $\pi^1, \dots, \pi^k$, let $W := \operatorname{lin}\left\{\pi^1, \dots, \pi^k\right\}$ and let $d := \dim W$. Then $c(\,\cdot\,) \leq c_0$ is an $L_{d,\mathbb{Q}}$ cut (and, since $d \leq k$ and $d \leq m$, an $L_{\min(k,m),\mathbb{Q}}$ cut) for P with respect to $W^\perp \times \mathbb{R}^n$.*

- *Let $c'(\,\cdot\,) \leq c'_0$ ($c' \in (\mathbb{R}^m \times \mathbb{R}^n)^T$, $c'_0 \in \mathbb{R}$) be an $L_{k,\mathbb{Q}}$ cut for P with respect to $V' \times \mathbb{R}^n$ (by Theorem 168, it suffices to consider this case) and let $w^1, \dots, w^k$ be a lattice basis of $\mathbb{Z}^m \cap V'^\perp$. Then $c'(\,\cdot\,) \leq c'_0$ is a k-dimensional lattice cut for P with respect to $w^1, \dots, w^k$.*

Proof. The first statement is an immediate consequence of Theorem 173. The second statement holds by Theorem 172. □

We finish this section with two consequences of the dual representation of $L_{k,\mathbb{Q}}$ cuts:

Theorem 177. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be*

- *a rational polyhedron,*
- *a polyhedron with a rational recession cone,*
- *a polyhedron with a partially rational recession cone or*
- *convex and compact,*

respectively. Let $w^1, \dots, w^k \in \mathbb{Z}^m$ ($k \in \mathbb{Z}_{\geq 0}$). Then

$$\operatorname{conv}\left(P \cap \left\{\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R}^n : \left(w^i\right)^T x^1 \in \mathbb{Z}\ \forall i \in [k]\right\}\right) =: (4.7)$$

is

- *a rational polyhedron,*
- *a polyhedron with a rational recession cone,*
- *a polyhedron with a partially rational recession cone or*

- *convex and compact,*

respectively.

Proof. Let

$$\Lambda := \left\{x \in \mathbb{R}^m : \left(w^i\right)^T x \in \mathbb{Z}\ \forall i \in [k]\right\} = \begin{pmatrix} C & D \end{pmatrix} \begin{pmatrix} \mathbb{Z}^{k'} \\ \mathbb{R}^{m-k'} \end{pmatrix},$$

where C, D are rational and $0 \leq k' \leq k$ ($k' = k$ holds if and only if the w^i are linearly independent). Then

$$\Lambda \times \mathbb{R}^n = \underbrace{\begin{pmatrix} C & D & 0^{m\times n} \\ 0^{n\times k} & 0^{n\times(m-k)} & I_n \end{pmatrix}}_{=:M} \begin{pmatrix} \mathbb{Z}^{k'} \\ \mathbb{R}^{m-k'} \\ \mathbb{R}^n \end{pmatrix}.$$

We claim that

$$\operatorname{conv}\left(P \cap (\Lambda \times \mathbb{R}^n)\right) = M \cdot \operatorname{conv}\left(\left(M^{-1}P\right) \cap \left(\mathbb{Z}^{k'} \times \mathbb{R}^{m-k'} \times \mathbb{R}^n\right)\right). \tag{4.8}$$

For this, we show

$$P \cap (\Lambda \times \mathbb{R}^n) = M\left(\left(M^{-1}P\right) \cap \left(\mathbb{Z}^{k'} \times \mathbb{R}^{m-k'} \times \mathbb{R}^n\right)\right). \tag{4.9}$$

For (4.9):

$$\begin{aligned} P \cap (\Lambda \times \mathbb{R}^n) &= P \cap \left(\begin{pmatrix} C & D & 0^{m\times n} \\ 0^{n\times k'} & 0^{n\times(m-k')} & I_n \end{pmatrix} \begin{pmatrix} \mathbb{Z}^{k'} \\ \mathbb{R}^{m-k'} \\ \mathbb{R}^n \end{pmatrix}\right) \\ &= M\left(\left(M^{-1}P\right) \cap \left(\mathbb{Z}^{k'} \times \mathbb{R}^{m-k'} \times \mathbb{R}^n\right)\right). \end{aligned}$$

So, considering (4.8), we get from Theorem 75 that (4.7) has the stated properties. □

From Theorem 174 and Theorem 177, we immediately obtain:

Theorem 178. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be*

- *a rational polyhedron,*
- *a polyhedron with a rational recession cone,*
- *a polyhedron with a partially rational recession cone or*
- *convex and compact,*

respectively. Let $V \leq \mathbb{R}^m$ be a rational subspace of codimension k. Then

$$\operatorname{conv}\left(P \cap ((\mathbb{Z}^m + V) \times \mathbb{R}^n)\right)$$

is

- *a rational polyhedron,*
- *a polyhedron with a rational recession cone,*
- *a polyhedron with a partially rational recession cone or*
- *convex and compact,*

respectively.

4.3. $L_{k-\frac{1}{2}}$ cuts and closures

Definition 179. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be arbitrary, let $k \in \{0, \ldots, m+n\}$ and let $V \leq \mathbb{R}^m \times \mathbb{R}^n$ be a linear subspace of codimension k that is generated by vectors from*

- $\mathbb{Q}^m \times \mathbb{Q}^n$,

- $\mathbb{Q}^m \times \mathbb{R}^n$ *or*
- $\mathbb{R}^m \times \mathbb{R}^n$,

respectively. An

- $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ ***cut for*** P,
- $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ ***cut for*** P *or*
- $L_{k-\frac{1}{2},\mathbb{R}\times\mathbb{R}}$ ***cut for*** P,

respectively, is a valid linear inequality $c(\cdot) \leq c_0$ *(*$c \in (\mathbb{R}^m \times \mathbb{R}^n)^T$ *and* $c_0 \in \mathbb{R}$*) for*

$$(P+V)_I = (P+V) \cap (\mathbb{Z}^m \times \mathbb{R}^n).$$

If $n = 0$ *(pure integer case), we also use the terms*

- $L_{k-\frac{1}{2},\mathbb{Q}}$ ***cuts for*** P *or*
- $L_{k-\frac{1}{2},\mathbb{R}}$ ***cut for*** P,

respectively, if V *is generated by vectors from*

- $\mathbb{Q}^m$ *or*
- $\mathbb{R}^m$,

respectively.

If it is of no importance what kind of $L_{k-\frac{1}{2}}$ *cuts is used or this is clear from the context, we also use the general term* $L_{k-\frac{1}{2}}$ ***cuts****.*

In Figure 4.2, one can see an illustration of the construction behind $L_{k-\frac{1}{2}}$ cuts.

Remark 180. *The "degenerated"* $L_{0-\frac{1}{2}}$ *cuts for* P *are*

- *any arbitrary inequality if* $P = \emptyset$,
- *an inequality* $(0^m \times 0^n)^T(\cdot) \leq c_0$, *where* $c_0 \geq 0$, *if* $P \neq \emptyset$.

Remark 181. *Consider the similarity of the definitions of* L_k *cuts and* $L_{k-\frac{1}{2}}$ *cuts. Let* $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ *be arbitrary and let* $V \leq \mathbb{R}^m \times \mathbb{R}^n$ *be a vector space of codimension* k *with suitable restrictions on its generators, for example from* $\mathbb{Q}^m \times \mathbb{R}^n$*. Then*

- *an* L_k *cut is a valid inequality for*
$$P \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + V),$$
- *an* $L_{k-\frac{1}{2}}$ *cut is a valid inequality for*
$$(P+V) \cap (\mathbb{Z}^m \times \mathbb{R}^n).$$

Now, we define the $L_{k-\frac{1}{2}}$ closures:

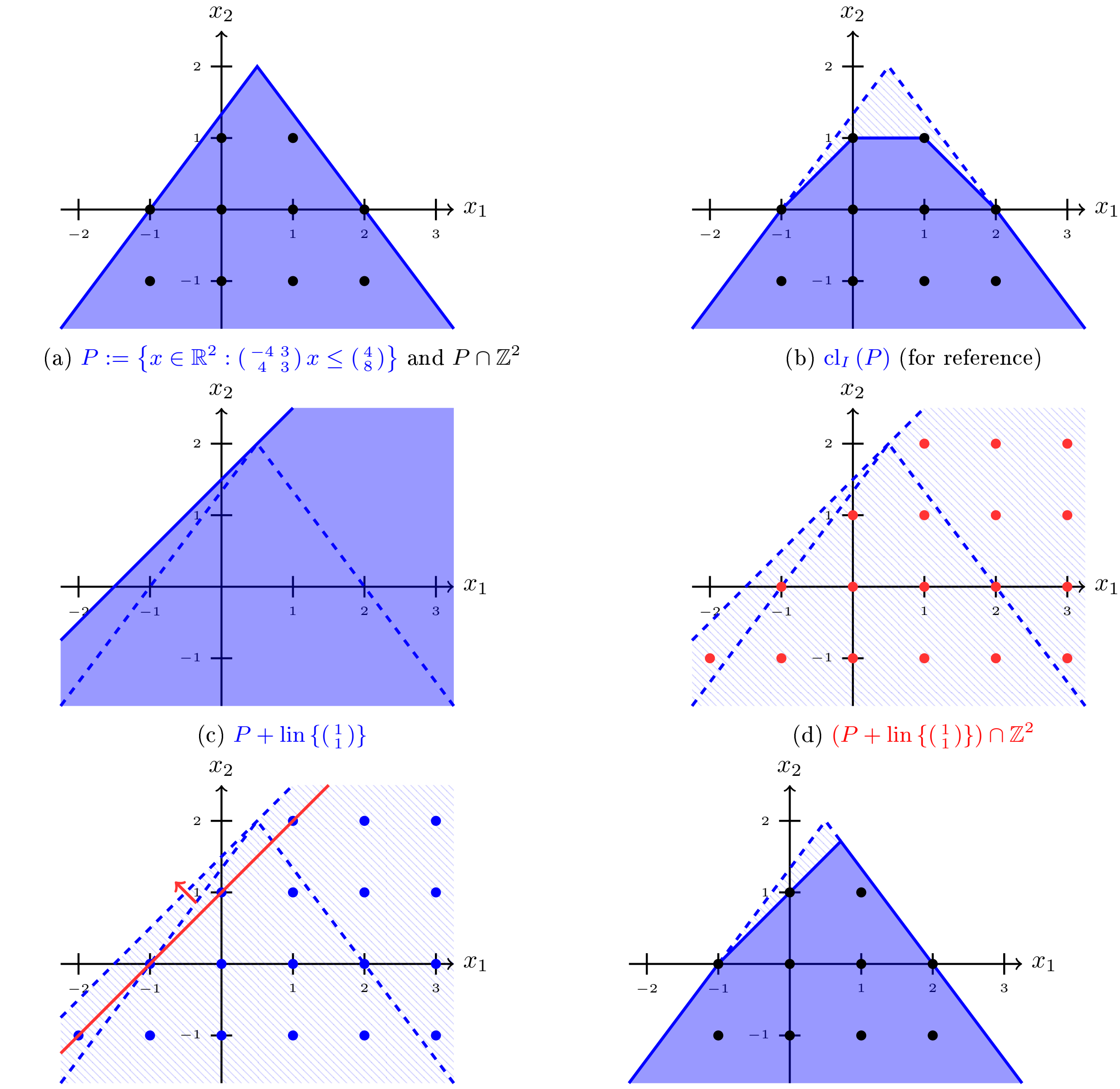
Definition 182. *Let* $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ *(*$m, n \in \mathbb{Z}_{\geq 0}$*) be arbitrary and let* $k \in \{0, \ldots, m+n\}$*. We define*

$$\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) := P \cap \bigcap_{\substack{c \in (\mathbb{R}^m \times \mathbb{R}^n)^T, c_0 \in \mathbb{R}: \\ c(\cdot) \leq c_0 \ L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}} \text{ cut for } P}} P^{\leq}(c, c_0)$$

as the $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ ***closure of*** P*. In a similar way, we define*

- *the* $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ *closure of* P *(*$\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P)$*) and*
- *the* $L_{k-\frac{1}{2},\mathbb{R}\times\mathbb{R}}$ *closure of* P *(*$\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{R}\times\mathbb{R}}}(P)$*).*

For $P \subseteq \mathbb{R}^m$ *(no continuous variables), we additionally define*



(a) $P := \left\{x \in \mathbb{R}^2 : \begin{pmatrix} -4 & 3 \\ 4 & 3 \end{pmatrix} x \leq \begin{pmatrix} 4 \\ 8 \end{pmatrix}\right\}$ and $P \cap \mathbb{Z}^2$

(b) $\mathrm{cl}_I(P)$ (for reference)

(c) $P + \mathrm{lin}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$

(d) $\left(P + \mathrm{lin}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}\right) \cap \mathbb{Z}^2$

(e) $-x_1 + x_2 \leq 1$ is valid for $x \in \left(P + \mathrm{lin}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}\right) \cap \mathbb{Z}^2$

(f) $P \cap \left\{x \in \mathbb{R}^2 : -x_1 + x_2 \leq 1\right\}$

Figure 4.2.: Illustration of $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts

- *the $L_{k-\frac{1}{2},\mathbb{Q}}$ closure of P ($\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}}}(P)$) and*
- *the $L_{k-\frac{1}{2},\mathbb{R}}$ closure of P ($\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{R}}}(P)$).*

The following lemma is an immediate consequence of Definition 182 and Theorem 72:

Lemma 183. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be arbitrary and let $k \in \{0, \ldots, m+n\}$. Then*

$$\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = P \cap \bigcap_{\substack{V \leq \mathbb{R}^m \times \mathbb{R}^n: \\ V \text{ rational subspace} \\ \text{of codimension } k}} \mathrm{cl}_{\overline{I}}(P + V).$$

Similar characterizations also hold for $\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P)$ and $\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{R}\times\mathbb{R}}}(P)$ and, if $P \subseteq \mathbb{R}^m$, for $\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}}}(P)$ and $\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{R}}}(P)$.

Remark 184. *In Lemma 183, if P satisfies the prerequisites of Theorem 75, we can replace $\mathrm{cl}_{\overline{I}}(P+V)$ by $\mathrm{cl}_I(P+V)$ for $\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P)$ and $\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P)$.*

4.4. L_k cuts and $L_{k-\frac{1}{2}}$ cuts for some irrational hyperplanes

As already outlined at the beginning of this chapter, in this section we give proofs how some $L_{k,\mathbb{Q}}$ closures, $L_{k,\mathbb{R}}$ closures, $L_{k-\frac{1}{2},\mathbb{Q}}$ closures and $L_{k-\frac{1}{2},\mathbb{R}}$ closures of some irrational hyperplaness look like. These results are, for example, applied in section 4.5.1 and section 4.5.2 to show differences in the expressiveness of such cutting plane operators. For an outline of this section:

- The purpose of section 4.4.1 is to show an auxiliary lemma (Lemma 185) that is a central tool for the proofs of Theorem 187 and Theorem 189.
- In section 4.4.2, we analyze the irrational hyperplanes $P^{114}, P^{115} \subseteq \mathbb{R}^2$. Specifically:
  - In Theorem 187, we compute $\mathrm{cl}_{L_{1,\mathbb{R}}}\left(P^{114}\right)$, $\mathrm{cl}_{L_{1,\mathbb{Q}}}\left(P^{114}\right)$ and $\mathrm{cl}_{L_{1,\mathbb{R}}}\left(P^{115}\right)$.
  - In Theorem 188, we compute $\mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{Q}}}\left(P^{114}\right)$, $\mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{R}}}\left(P^{114}\right)$, $\mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{Q}}}\left(P^{115}\right)$ and $\mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{R}}}\left(P^{115}\right)$.
- In section 4.4.3, we analyze the irrational hyperplanes $P^{116,m}, P^{117,m} \subseteq \mathbb{R}^m$ ($m \in \mathbb{Z}_{\geq 2}$). Specifically:
  - In Theorem 189, we compute $\mathrm{cl}_{L_{1,\mathbb{R}}}\left(P^{116,m}\right)$ and $\mathrm{cl}_{L_{m-1,\mathbb{Q}}}\left(P^{116,m}\right)$.
  - In Theorem 190, we compute $\mathrm{cl}_{L_{(m-1)-\frac{1}{2},\mathbb{Q}}}\left(P^{116,m}\right)$, $\mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{R}}}\left(P^{116,m}\right)$, $\mathrm{cl}_{L_{(m-1)-\frac{1}{2},\mathbb{Q}}}\left(P^{117,m}\right)$ and $\mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{R}}}\left(P^{117,m}\right)$.

4.4.1. An auxiliary lemma for L_k cuts

We start with a small auxiliary lemma, which allows one to compute the L_{m-1} closures of a hyperplane in $\mathbb{R}^m$ ($m \in \mathbb{Z}_{\geq 1}$; even though only the case $m \in \mathbb{Z}_{\geq 2}$ is of importance for this text) with respect to *one* specifically chosen vector space that satisfies some specific weak property. Lemma 185 is a central tool for the proofs of Theorem 187 and Theorem 189.

Lemma 185. *Let $P := P^{=}(c, c_0) \subseteq \mathbb{R}^m$ ($m \in \mathbb{Z}_{\geq 1}$) be a hyperplane, where $c \in (\mathbb{R}^m)^T \setminus \left\{(0^m)^T\right\}$ and $c_0 \in \mathbb{R}$. Let $v \in \mathbb{R}^m$ be such that $cv \neq 0$. Then*

$$\operatorname{conv}\left(P \cap \left(\mathbb{Z}^m + \operatorname{lin}\{v\}\right)\right) = P.$$

Remark 186. *In Lemma 185, the condition $c \neq (0^m)^T$ is only left in for more clarity. It is redundant, since it is already implied by $cv \neq 0$.*

Proof. (Lemma 185) Since P is convex, clearly

$$\operatorname{conv}\left(P \cap \left(\mathbb{Z}^m + \operatorname{lin}\{v\}\right)\right) \subseteq P$$

holds and we only have to prove

$$\operatorname{conv}\left(P \cap \left(\mathbb{Z}^m + \operatorname{lin}\{v\}\right)\right) \supseteq P.$$

Let $p \in P$. Consider the points

$$\underline{p} := \lfloor p \rfloor \in \mathbb{Z}^m,$$
$$\overline{p} := \lceil p \rceil \in \mathbb{Z}^m,$$

which one obtains by rounding each coordinate of p down or up. Clearly, $p \in \operatorname{conv}\left(\left[\underline{p}, \overline{p}\right]_I\right)$, where $\left[\underline{p}, \overline{p}\right]$ is an axially parallel and not necessarily full-dimensional cuboid, where for each vertex w, we have $w_i \in \left\{\underline{p}_i, \overline{p}_i\right\}$ for $i \in [m]$. Thus, there exist $k \in \mathbb{Z}_{\geq 1}$, $\lambda \in \Delta^{k-1}$ and $p^1, \ldots, p^k \in \left[\underline{p}, \overline{p}\right]_I$ such that

$$p = \sum_{i=1}^{k} \lambda_i p^i. \tag{4.10}$$

For $i \in [k]$, consider

$$p'^i := p^i + \frac{c_0 - cp^i}{cv} v.$$

Clearly, $p'^i \in \mathbb{Z}^m + \operatorname{lin}\{v\}$ for all $i \in [k]$. On the other hand, we have

$$cp'^i = c\left(p^i + \frac{c_0 - cp^i}{cv}v\right) = c_0.$$

So, $p'^i \in P$ for all $i \in [k]$. Finally, using $cp = c_0$, $\lambda \in \mathbb{R}^k$, $\sum_{i=1}^k \lambda_i = 1$ and (4.10), we obtain:

$$\sum_{i=1}^k \lambda_i p'^i = \sum_{i=1}^k \lambda_i \left(p^i + \frac{c_0 - cp^i}{cv}v\right) = p + \frac{1}{cv}\left(c_0 - c\sum_{i=1}^k \lambda_i p^i\right) = p + \frac{1}{cv}(c_0 - cp) = p.$$

Thus, $p \in \operatorname{conv}\{p'^1, \ldots, p'^k\} \subseteq \operatorname{conv}(\mathbb{Z}^m + \operatorname{lin}\{v\})$. □

4.4.2. P^{114} and P^{115}

Theorem 187. *For P^{114} and P^{115}, respectively, we have*

$$\operatorname{cl}_{L_{1,\mathbb{R}}}(P^{114}) = \emptyset = \operatorname{cl}_I(P^{114}) = \operatorname{cl}_{\overline{I}}(P^{114}), \tag{4.11}$$
$$\operatorname{cl}_{L_{1,\mathbb{Q}}}(P^{114}) = P^{114}, \tag{4.12}$$
$$\operatorname{cl}_{L_{1,\mathbb{R}}}(P^{115}) = P^{115} \supsetneq \{0^2\} = \operatorname{cl}_I(P^{115}) = \operatorname{cl}_{\overline{I}}(P^{115}). \tag{4.13}$$

Proof.

For (4.11)**:** Consider $P^{114} \cap \left(\mathbb{Z}^2 + \operatorname{lin}\left\{\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}\right\}\right)$. Obviously, $P^{114} \cap \left(\mathbb{Z}^2 + \operatorname{lin}\left\{\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}\right\}\right) = \emptyset$, from which we conclude the statement.

For (4.12)**:** Let $v \in \mathbb{Q}^2 \setminus \{0^2\}$. Clearly, $\begin{pmatrix} \sqrt{2} & 1 \end{pmatrix} v \neq 0$. By considering

$$P^{114} = P^{=}\left(\begin{pmatrix} -\sqrt{2} & 1 \end{pmatrix}, \begin{pmatrix} 0 \end{pmatrix}\right),$$

the statement is an immediate consequence of Lemma 185.

For (4.13)**:** Let $v \in \mathbb{R}^2 \setminus \{0^2\}$. If $v \in \operatorname{lin}\left\{\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}\right\}$, we have

$$P \cap \left(\mathbb{Z}^2 + \operatorname{lin}\left\{\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}\right\}\right) = P.$$

If, on the other hand, $v \notin \operatorname{lin}\left\{\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}\right\}$, we have $\begin{pmatrix} -\sqrt{2} & 1 \end{pmatrix} v \neq 0$. By considering that

$$P^{115} = P^{=}\left(\begin{pmatrix} -\sqrt{2} & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \end{pmatrix}\right),$$

the statement is an immediate consequence of Lemma 185. □

Theorem 188. *For P^{114} and P^{115}, respectively, we have*

$$\operatorname{cl}_{L_{1-\frac{1}{2},\mathbb{Q}}}(P^{114}) = P^{114}, \tag{4.14}$$
$$\operatorname{cl}_{L_{1-\frac{1}{2},\mathbb{R}}}(P^{114}) = \emptyset = \operatorname{cl}_I(P^{114}) = \operatorname{cl}_{\overline{I}}(P^{114}), \tag{4.15}$$
$$\operatorname{cl}_{L_{1-\frac{1}{2},\mathbb{Q}}}(P^{115}) = P^{115}, \tag{4.16}$$
$$\operatorname{cl}_{L_{1-\frac{1}{2},\mathbb{R}}}(P^{115}) = \{0^2\} = \operatorname{cl}_I(P^{115}) = \operatorname{cl}_{\overline{I}}(P^{115}). \tag{4.17}$$

Proof. Let $v \in \mathbb{Q}^2 \setminus \{0^2\}$. Then v is surely linearly independent to $\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$. Thus, we have

$$P^{114} + \operatorname{lin}\{v\} = P^{115} + \operatorname{lin}\{v\} = \mathbb{R}^2,$$

from which we conclude (4.14) and (4.16). On the other hand, we have

$$P^{114} + \operatorname{lin}\left\{\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}\right\} = P^{114},$$
$$P^{115} + \operatorname{lin}\left\{\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}\right\} = P^{115},$$

which is why any valid inequality for $\left(P^{114}\right)_I$ or $\left(P^{115}\right)_I$ is an $L_{1-\frac{1}{2},\mathbb{Q}}$ cut for this polyhedron. Thus, (4.15) and (4.17) hold. □

4.4.3. $P^{116,m}$ and $P^{117,m}$

Theorem 189. *For $m \in \mathbb{Z}_{\geq 2}$, we have:*

$$\operatorname{cl}_{L_{m-1,\mathbb{Q}}}\left(P^{116,m}\right) = P^{116,m}, \tag{4.18}$$
$$\operatorname{cl}_{L_{1,\mathbb{R}}}\left(P^{116,m}\right) = \emptyset = \operatorname{cl}_I\left(P^{116,m}\right) = \operatorname{cl}_{\overline{I}}\left(P^{116,m}\right). \tag{4.19}$$

Proof. It is easy to check that

$$P^{116,m} = P^{=}\left(\begin{pmatrix} 1 & -h_1 & -h_2 & \cdots & -h_{m-1} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \end{pmatrix}\right),$$

where, of course, $h_1, \ldots, h_{m-1}$ are as in Definition 116.

For (4.18)**:** Let $v \in \mathbb{Q}^m \backslash \{0^m\}$. Clearly, $\begin{pmatrix} 1 & -h_1 & -h_2 & \cdots & -h_{m-1} \end{pmatrix} v \neq 0$. Thus, the statement is an immediate consequence of Lemma 185.

For (4.19)**:** We have

$$P^{116,m} \cap \left(\mathbb{Z}^m + \left(\operatorname{lin}\left\{\begin{pmatrix} 1 & -h_1 & -h_2 & \cdots & -h_{m-1} \end{pmatrix}^T\right\}\right)^{\perp}\right) = \emptyset = \operatorname{cl}_I\left(P^{116,m}\right) = \operatorname{cl}_{\overline{I}}\left(P^{116,m}\right).$$

□

Theorem 190. *For $m \in \mathbb{Z}_{\geq 2}$, we have:*

$$\operatorname{cl}_{L_{(m-1)-\frac{1}{2},\mathbb{Q}}}\left(P^{116,m}\right) = P^{116,m}, \tag{4.20}$$
$$\operatorname{cl}_{L_{1-\frac{1}{2},\mathbb{R}}}\left(P^{116,m}\right) = \emptyset = \operatorname{cl}_I\left(P^{116,m}\right) = \operatorname{cl}_{\overline{I}}\left(P^{116,m}\right), \tag{4.21}$$
$$\operatorname{cl}_{L_{(m-1)-\frac{1}{2},\mathbb{Q}}}\left(P^{117,m}\right) = P^{117,m}, \tag{4.22}$$
$$\operatorname{cl}_{L_{1-\frac{1}{2},\mathbb{R}}}\left(P^{117,m}\right) = \left\{0^2\right\} = \operatorname{cl}_I\left(P^{117,m}\right) = \operatorname{cl}_{\overline{I}}\left(P^{117,m}\right). \tag{4.23}$$

Proof. Let $v \in \mathbb{Q}^m \backslash \{0^m\}$. We first show that

$$\dot{\bigcup_{i=2}^{m}}\left\{h_{i-1} \cdot e^{m,1} + e^{m,i}\right\} \dot{\cup} \{v\}$$

is a linearly independent set. For this, we consider the system of linear equations

$$\sum_{i=1}^{m-1} \lambda_i \left(h_i \cdot e^{m,1} + e^{m,i+1}\right) + \lambda_m v = 0^m, \tag{4.24}$$

which can also be written as

$$\left(\begin{array}{cccc|c} h_1 & h_2 & \cdots & h_{m-1} & \\ 1 & 0 & \cdots & 0 & \\ 0 & 1 & \cdots & 0 & v \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & 1 & \end{array}\right) \lambda = 0^m.$$

We immediately conclude

$$\lambda_i = -v_{i+1} \cdot \lambda_m \ \forall i \in [m-1] \,. \tag{4.25}$$

By plugging (4.25) into the first row of (4.24), we obtain

$$\lambda_m \cdot \left(v_1 \cdot 1 + \sum_{i=1}^{m-1} \left(-v_{i+1} \cdot h_i \right) \right) = 0.$$

If $\lambda_m = 0$, we immediately conclude from (4.25) that $\lambda = 0^m$. Thus, let $\lambda_m \neq 0$. Since by Definition 116 or 117, respectively, $\{1, h_1, \ldots, h_{m-1}\}$ are linearly independent over $\mathbb{Q}$, we conclude $v = 0^m$: a contradiction to the assumption $v \in \mathbb{Q}^m \setminus \{0^m\}$. Because of

$$\operatorname{lineal} P^{116,m} = \operatorname{lineal} P^{117,m} = \operatorname{lin} \dot{\bigcup_{i=2}^{m}} \left\{ h_{i-1} e^{m,1} + e^{m,i} \right\},$$

we conclude

$$P^{116,m} + \operatorname{lin} \{v\} = P^{117,m} + \operatorname{lin} \{v\} = \mathbb{R}^m.$$

Thus, (4.20) and (4.22) hold. On the other hand, we have

$$P^{116,m} + \operatorname{lin} \dot{\bigcup_{i=2}^{m}} \left\{ h_{i-1} \cdot e^{m,1} + e^{m,i} \right\} = P^{116,m},$$
$$P^{117,m} + \operatorname{lin} \dot{\bigcup_{i=2}^{m}} \left\{ h_{i-1} \cdot e^{m,1} + e^{m,i} \right\} = P^{117,m},$$

from which we conclude (4.21) and (4.23). □

4.5. Differences between the types of L_k cuts and $L_{k-\frac{1}{2}}$ cuts

In the previous sections (section 4.2 and section 4.3), we introduced different types of L_k and $L_{k-\frac{1}{2}}$ cuts and formulated in Remark 156 as guiding question 1 whether there exist any differences between these different types.

For the structure of this section:

- In section 4.5.1 and section 4.5.2, respectively, we compare the $L_{k,\mathbb{Q}}$ closure with the $L_{k',\mathbb{R}}$ closure and the $L_{k-\frac{1}{2},\mathbb{Q}}$ closure with the $L_{k'-\frac{1}{2},\mathbb{R}}$ closure, respectively. For this, we summarize results that we showed in section 4.4. We restate that for every $m \in \mathbb{Z}_{\geq 2}$, there exists an irrational hyperplane $P \subseteq \mathbb{R}^m$ such that we have

$$\operatorname{cl}_{L_{1,\mathbb{R}}}(P) \subsetneq \operatorname{cl}_{L_{m-1,\mathbb{Q}}}(P),$$
$$\operatorname{cl}_{L_{1-\frac{1}{2},\mathbb{R}}}(P) \subsetneq \operatorname{cl}_{L_{(m-1)-\frac{1}{2},\mathbb{Q}}}(P).$$

- In section 4.5.3 and section 4.5.4, we compare the $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ closure with the $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ closure. From a bird's eye perspective, section 4.5.3 is about when these two are different and section 4.5.4 is about when these are equal.

  The central result of section 4.5.3 is Theorem 191. Here, we show that for $P^{118} \subseteq \mathbb{R}^1 \times \mathbb{R}^2$, we have

$$P^{118} = \operatorname{cl}_{L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}\left(P^{118}\right) \supsetneq \operatorname{cl}_{L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}\left(P^{118}\right) = \operatorname{cl}_{I}\left(P^{118}\right) = \operatorname{cl}_{\overline{I}}\left(P^{118}\right).$$

  The essential property of P^{118} that enables this proof is that P^{118} has a lineality space that is generated by vectors from $\mathbb{Q}^m \times \mathbb{R}^n$, but is not a rational subspace.

  So, in section 4.5.4, we formulate Conjecture 192, which claims that as long as $P = Q+C+L \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$, where
  - Q is convex and compact,
  - C is a pointed polyhdral cone generated by vectors from $\mathbb{Q}^m \times \mathbb{R}^n$ and

- L is a linear vector space generated by rational vectors (from $\mathbb{Q}^m \times \mathbb{Q}^n$),

we have
$$\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P)$$
for $k \in \{0, \dots, m\}$. While in the general case Conjecture 192 is open, we analyze the important special case that P is a rational polyhedron:

- In Theorem 193, which we prove further back in section 4.8.4, we prove that Conjecture 192 holds if $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) is a rational polyhedron. Moreover: every $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cut for P ($k \in \{0, \dots, m\}$) is dominated absolutely by a *finite* number of rational $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts for P. This, of course, implies the
$$\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P)$$
equalities in (4.1) and (4.2) of guiding question 2 in Remark 156.
- Why do we put an emphasis on this stronger statement (than the statement of Conjecture 192) that every $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ is dominated by a *finite* number of $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts for P if P is a rational polyhedron? The reason is that in the more general setting of Conjecture 192 (even if P is a polyhedron), it can happen that
$$\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P) = \mathrm{cl}_I(P)$$
and we need to apply just one $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cut to P to obtain $\mathrm{cl}_I(P)$, but the intersection of P with an arbitray finite number of half-spaces induced by $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts is always a strict superset of $\mathrm{cl}_I(P)$. This is a central statement of Theorem 194.

4.5.1. $\mathrm{cl}_{L_{k,\mathbb{Q}}}(\cdot)$ vs $\mathrm{cl}_{L_{k,\mathbb{R}}}(\cdot)$

By Theorem 187, we have for $P^{114} \subseteq \mathbb{R}^2$:
$$\begin{aligned}\mathrm{cl}_{L_{1,\mathbb{R}}}\left(P^{114}\right) &= \emptyset = \mathrm{cl}_I\left(P^{114}\right) = \mathrm{cl}_{\overline{I}}\left(P^{114}\right),\\ \mathrm{cl}_{L_{1,\mathbb{Q}}}\left(P^{114}\right) &= P^{114}.\end{aligned}$$

By Theorem 189, we have for $P^{116,m} \subseteq \mathbb{R}^m$ ($m \in \mathbb{Z}_{\geq 2}$):
$$\begin{aligned}\mathrm{cl}_{L_{m-1,\mathbb{Q}}}\left(P^{116,m}\right) &= P^{116,m},\\ \mathrm{cl}_{L_{1,\mathbb{R}}}\left(P^{116,m}\right) &= \emptyset = \mathrm{cl}_I\left(P^{116,m}\right) = \mathrm{cl}_{\overline{I}}\left(P^{116,m}\right).\end{aligned}$$

4.5.2. $\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}}}(\cdot)$ vs $\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{R}}}(\cdot)$

By Theorem 188, we have for $P^{114}, P^{115} \subseteq \mathbb{R}^2$:
$$\begin{aligned}\mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{Q}}}\left(P^{114}\right) &= P^{114},\\ \mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{R}}}\left(P^{114}\right) &= \emptyset = \mathrm{cl}_I\left(P^{114}\right) = \mathrm{cl}_{\overline{I}}\left(P^{114}\right),\\ \mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{Q}}}\left(P^{115}\right) &= P^{115},\\ \mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{R}}}\left(P^{115}\right) &= \left\{0^2\right\} = \mathrm{cl}_I\left(P^{115}\right) = \mathrm{cl}_{\overline{I}}\left(P^{115}\right).\end{aligned}$$

By Theorem 190, we have for $P^{116,m}, P^{117,m} \subseteq \mathbb{R}^m$ ($m \in \mathbb{Z}_{\geq 2}$):
$$\begin{aligned}\mathrm{cl}_{L_{(m-1)-\frac{1}{2},\mathbb{Q}}}\left(P^{116,m}\right) &= P^{116,m},\\ \mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{R}}}\left(P^{116,m}\right) &= \emptyset = \mathrm{cl}_I\left(P^{116,m}\right) = \mathrm{cl}_{\overline{I}}\left(P^{116,m}\right),\\ \mathrm{cl}_{L_{(m-1)-\frac{1}{2},\mathbb{Q}}}\left(P^{117,m}\right) &= P^{117,m},\\ \mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{R}}}\left(P^{117,m}\right) &= \{0^m\} = \mathrm{cl}_I\left(P^{117,m}\right) = \mathrm{cl}_{\overline{I}}\left(P^{117,m}\right).\end{aligned}$$

4.5.3. An example where $\mathrm{cl}_{L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(\,\cdot\,) \subsetneq \mathrm{cl}_{L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(\,\cdot\,)$

We next show that for polyhedra that have a lineality space that is not rational (but is generated by vectors from $\mathbb{Q}^m \times \mathbb{R}^n$), it can happen that the $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ closure is strictly more expressive than the $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ closure:

Theorem 191. *We have:*

1. $\sqrt{2}(\,\cdot\,)_2 - (\,\cdot\,)_3 \leq 0$ *is an* $L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ *cut for* $P^{118} \subseteq \mathbb{R}^1 \times \mathbb{R}^2$ *and we have*

$$\mathrm{cl}_I\left(P^{118} + \mathrm{lin}\left\{\left(\begin{smallmatrix}0\\1\\\sqrt{2}\end{smallmatrix}\right)\right\}\right) = \mathrm{cl}_{L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}\left(P^{118}\right) = \mathrm{cl}_I\left(P^{118}\right) = \mathrm{cl}_{\overline{I}}\left(P^{118}\right).$$

2. *We have*

$$\mathrm{cl}_{L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}\left(P^{118}\right) = P^{118} \supsetneq \mathrm{cl}_I\left(P^{118}\right) = \mathrm{cl}_{\overline{I}}\left(P^{118}\right).$$

Proof.

For 1: Consider $P' := P^{118} + \mathrm{lin}\left\{\left(\begin{smallmatrix}0\\1\\\sqrt{2}\end{smallmatrix}\right)\right\}$. Obviously, $\sqrt{2}x_2 - x_3 \leq 0$ is a valid inequality for P'_I. On the other hand, we have

$$P^{118} \cap P^{\leq}\left(\begin{pmatrix}0 & \sqrt{2} & -1\end{pmatrix}, \begin{pmatrix}0\end{pmatrix}\right) = \mathrm{conv}\left\{\left(\begin{smallmatrix}0\\0\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)\right\} + \mathrm{lin}\left\{\left(\begin{smallmatrix}0\\1\\\sqrt{2}\end{smallmatrix}\right)\right\} = \mathrm{cl}_I\left(P^{118}\right) = \mathrm{cl}_{\overline{I}}\left(P^{118}\right).$$

For 2: We consider valid inequalities for $\left(P^{118} + \mathrm{lin}\{v\}\right)_I$, where $v \in \left(\mathbb{Q}^1 \times \mathbb{Q}^2\right) \setminus \left\{0^1 \times 0^2\right\}$. For this, we analyze the point $\left(\begin{smallmatrix}\frac{1}{2}\\\delta\\0\end{smallmatrix}\right)$, where $\delta \in \left(0, \frac{3}{4}\right]$ is arbitrary. We show that for every $v \in \mathbb{Q}^1 \times \mathbb{Q}^2$, this point lies in $\mathrm{conv}\left(P^{118} + \mathrm{lin}\{v\}\right)_I$. If $v_1 = 0$, we have

$$\begin{aligned}\left(\begin{smallmatrix}\frac{1}{2}\\\delta\\0\end{smallmatrix}\right) =& \frac{1}{2} \cdot \underbrace{\left(\underbrace{\left(\begin{smallmatrix}0\\0\\0\end{smallmatrix}\right) - \frac{v_3 \cdot \delta}{\sqrt{2} \cdot v_2 - v_3} \cdot \left(\begin{smallmatrix}0\\1\\\sqrt{2}\end{smallmatrix}\right)}_{\in (P^{118})_I} + \frac{\sqrt{2} \cdot \delta}{\sqrt{2} \cdot v_2 - v_3} \cdot \underbrace{\left(\begin{smallmatrix}0\\v_2\\v_3\end{smallmatrix}\right)}_{=v}\right)}_{\in (P^{118} + \mathrm{lin}\{v\})_I} \\ &+ \frac{1}{2} \cdot \underbrace{\left(\underbrace{\left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right) - \frac{v_3 \cdot \delta}{\sqrt{2} \cdot v_2 - v_3} \cdot \left(\begin{smallmatrix}0\\1\\\sqrt{2}\end{smallmatrix}\right)}_{\in (P^{118})_I} + \frac{\sqrt{2} \cdot \delta}{\sqrt{2} \cdot v_2 - v_3} \cdot \underbrace{\left(\begin{smallmatrix}0\\v_2\\v_3\end{smallmatrix}\right)}_{=v}\right)}_{\in (P^{118} + \mathrm{lin}\{v\})_I} \\ &\in \mathrm{conv}\left(\left(P^{118} + \mathrm{lin}\{v\}\right)_I\right).\end{aligned}$$

Now for the case $v_1 \neq 0$. We have

$$\left(\begin{smallmatrix}\frac{1}{2}\\\delta\\0\end{smallmatrix}\right) = \frac{1}{2}\underbrace{\left(\underbrace{\left(\begin{smallmatrix}\frac{1}{2}\\\delta\\0\end{smallmatrix}\right)}_{\in P^{118}} + \frac{1}{2v_1} \cdot \left(\begin{smallmatrix}v_1\\v_2\\v_3\end{smallmatrix}\right)\right)}_{\in (P^{118} + \mathrm{lin}\{v\})_I} + \frac{1}{2}\underbrace{\left(\underbrace{\left(\begin{smallmatrix}\frac{1}{2}\\\delta\\0\end{smallmatrix}\right)}_{\in P^{118}} - \frac{1}{2v_1} \cdot \left(\begin{smallmatrix}v_1\\v_2\\v_3\end{smallmatrix}\right)\right)}_{\in (P^{118} + \mathrm{lin}\{v\})_I} \in \mathrm{conv}\left(\left(P^{118} + \mathrm{lin}\{v\}\right)_I\right).$$

□

4.5.4. $\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(\,\cdot\,)$ vs $\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(\,\cdot\,)$: cases of equality

We conjecture that the reason why in Theorem 191, the $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ closure is strictly weaker than the $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ closure lies in the fact that P^{118} has a lineality space that cannot be generated by rational vectors. To formalize this, we state the following conjecture:

Conjecture 192. *Let* $P = Q + C + L \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$*, where*

- Q *is convex and compact,*

- *C is a pointed polyhdral cone generated by vectors from $\mathbb{Q}^m \times \mathbb{R}^n$ and*
- *L is a linear vector space generated by rational vectors (from $\mathbb{Q}^m \times \mathbb{Q}^n$).*

Let $k \in \{0, \ldots, m\}$. Then

$$\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P).$$

Additionally, we have

$$\mathrm{cl}_{L_{(m+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_I(P).$$

At least if P is a rational polyhedron, an even stronger statement holds than what we hypothesize in Conjecture 192. In this case, we do not just have $\mathrm{cl}_{L_{(m+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{(m+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P)$, but also that every $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cut for P is dominated absolutely by a *finite* set of rational $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts:

Theorem 193. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be a rational polyhedron. Then every $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cut for P ($k \in \{0, \ldots, m\}$) is dominated absolutely by a finite set of either*

- *rational essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts for P (essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts are defined in Definition 203) or*
- *rational $L_{k-1,\mathbb{Q}}$ cuts for P (which, by Theorem 199, are $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts for P) if $k \geq 1$.*

More concisely: every $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cut for P is dominated by a finite set of rational $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts for P. Thus, we conclude

$$\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P).$$

Theorem 193 is proved in section 4.8.4. Why do we point out the dominance by a *finite* number of $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts in Theorem 193 so explicitly? The reason is that if we have a polyhedron P for which the finite set of generators of its recession cone consists of vectors from $\mathbb{Q}^m \times \mathbb{R}^n$ (instead of $\mathbb{Q}^m \times \mathbb{Q}^n$, as it can be assumed for rational polyhedra), it can happen that $\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P)$, but there exist $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cuts for P that are not dominated (not even relatively to P) by a finite set of $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts for P. This is a consequence of the following theorem (observe the similarity to Theorem 191):

Theorem 194. *We have:*

1. *$\sqrt{2}\,(\cdot)_2 - (\cdot)_3 \leq 0$ is an $L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cut for $P^{119} \subseteq \mathbb{R}^1 \times \mathbb{R}^2$ and*
$$P^{119} \cap \mathrm{cl}_I\left(P^{119} + \mathrm{lin}\left\{\begin{pmatrix} 0 \\ 1 \\ \sqrt{2} \end{pmatrix}\right\}\right) = \mathrm{cl}_{L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}\left(P^{119}\right) = \mathrm{cl}_I\left(P^{119}\right) = \mathrm{cl}_{\overline{I}}\left(P^{119}\right).$$
2. *Let $l \in \mathbb{Z}_{\geq 1}$ and let $v^1, \ldots, v^l \in \mathbb{Q}^1 \times \mathbb{Q}^2 \setminus \{0^1 \times 0^2\}$. Then*
$$\bigcap_{i=1}^{l} \mathrm{cl}_I\left(P^{119} + \mathrm{lin}\left\{v^i\right\}\right) \supsetneq \mathrm{cl}_I\left(P^{119}\right) = \mathrm{cl}_{\overline{I}}\left(P^{119}\right).$$
3. *For all $p \in P^{119} \setminus \mathrm{cl}_I\left(P^{119}\right)$, there exists a $v \in \mathbb{Q}^1 \times \mathbb{Q}^2$ such that $p \notin \mathrm{cl}_I(P + \mathrm{lin}\{v\})$. This, in particular, implies*
$$\mathrm{cl}_{L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}\left(P^{119}\right) = \mathrm{cl}_I\left(P^{119}\right) = \mathrm{cl}_{\overline{I}}\left(P^{119}\right),$$
which shows that Conjecture 192 holds for $P = P^{119}$ and $k = 2$.

Remark 195. *Taking 2 and 3 of Theorem 194 together shows that we have $\mathrm{cl}_{L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}\left(P^{119}\right) = \mathrm{cl}_I\left(P^{119}\right)$, but in contrast to the situation of Theorem 193, no* finite *number of $L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts suffices to describe $\mathrm{cl}_{L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}\left(P^{119}\right)$.*

Proof. (Theorem 194)

For 1: Because of

- $P^{119} \subsetneq P^{118}$ and
- $\begin{pmatrix} 0 & \sqrt{2} & -1 \end{pmatrix}(\cdot) \leq 0$ is a valid $L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cut for P^{118} (cf. Theorem 191),

we conclude that $\begin{pmatrix} 0 & \sqrt{2} & -1 \end{pmatrix}(\cdot) \leq 0$ is also an $L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cut for $P^{119} \subseteq \mathbb{R}^1 \times \mathbb{R}^2$.

The second statement is a consequence of

$$\mathrm{cl}_I\left(P^{119}\right) = P^{119} \cap P^{\leq}\left(\left(\begin{array}{c|cc} 0 & \sqrt{2} & -1 \end{array}\right), \left(\begin{array}{c} 0 \end{array}\right)\right).$$

For 2: We show that for every $i \in \{1, \ldots, l\}$ and $\delta \in \left(0, \frac{3}{4}\right]$, there exists a $\lambda^*(i, \delta) \in \mathbb{R}_{\geq 0}$ such that for all $\lambda \geq \lambda^*(i, \delta)$, we have

$$\begin{pmatrix} \frac{1}{2} \\ \delta \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ \sqrt{2} \end{pmatrix} \in \mathrm{cl}_I\left(P^{119} + \mathrm{lin}\left\{v^i\right\}\right).$$

By setting $\lambda^*(\delta) := \max\{\lambda(1, \delta), \ldots, \lambda(l, \delta)\}$, we thus obtain

$$\forall \lambda \geq \lambda^*(\delta) : \begin{pmatrix} \frac{1}{2} \\ \delta \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ \sqrt{2} \end{pmatrix} \in \left(\bigcap_{i=1}^{l} \mathrm{cl}_I\left(P^{119} + \mathrm{lin}\left\{v^i\right\}\right)\right) \setminus P^{119}.$$

Let $i \in \{1, \ldots, l\}$ and let $\delta \in \left(0, \frac{3}{4}\right]$ be fixed. We distinguish three cases:

a. $v_1^i = 0$ and $\frac{v_3^i}{v_3^i - \sqrt{2} v_2^i} > 0$,

b. $v_1^i = 0$ and $\frac{v_3^i}{v_3^i - \sqrt{2} v_2^i} < 0$,

c. $v_1^i \neq 0$.

For case a: Set $\lambda^*(i, \delta) := 0$ and let $\lambda \in \mathbb{R}_{\geq 0}$. Then

$$\begin{aligned}
\begin{pmatrix} \frac{1}{2} \\ \delta \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ \sqrt{2} \end{pmatrix} =& \frac{1}{2} \cdot \underbrace{\Bigg(\underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \left(\frac{v_3^i \cdot \delta}{v_3^i - \sqrt{2} v_2^i} + \lambda \right) \cdot \begin{pmatrix} 0 \\ 1 \\ \sqrt{2} \end{pmatrix}}_{\in (P^{119})_I} - \frac{\sqrt{2} \cdot \delta}{v_3^i - \sqrt{2} v_2^i} \cdot \underbrace{\begin{pmatrix} 0 \\ v_2^i \\ v_3^i \end{pmatrix}}_{= v^i} \Bigg)}_{\in (P^{119} + \mathrm{lin}\{v^i\})_I} \\
&+ \frac{1}{2} \cdot \underbrace{\Bigg(\underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \left(\frac{v_3^i \cdot \delta}{v_3^i - \sqrt{2} v_2^i} + \lambda \right) \cdot \begin{pmatrix} 0 \\ 1 \\ \sqrt{2} \end{pmatrix}}_{\in (P^{119})_I} - \frac{\sqrt{2} \cdot \delta}{v_3^i - \sqrt{2} v_2^i} \cdot \underbrace{\begin{pmatrix} 0 \\ v_2^i \\ v_3^i \end{pmatrix}}_{= v^i} \Bigg)}_{\in (P^{119} + \mathrm{lin}\{v^i\})_I} \\
&\in \mathrm{conv}\left(\left(P^{119} + \mathrm{lin}\left\{v^i\right\}\right)_I\right).
\end{aligned}$$

For case b: Set

$$\lambda^*(i, \delta) := -\frac{v_3^i \cdot \delta}{v_3^i - \sqrt{2} v_2^i} > 0$$

and let $\lambda \in \mathbb{R}$ satisfy $\lambda \geq \lambda^*(i, \delta) > 0$. Then

$$\begin{aligned}
\begin{pmatrix} \frac{1}{2} \\ \delta \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ \sqrt{2} \end{pmatrix} &= \begin{pmatrix} \frac{1}{2} \\ \delta \\ 0 \end{pmatrix} + \lambda^* \begin{pmatrix} 0 \\ 1 \\ \sqrt{2} \end{pmatrix} + (\lambda - \lambda^*) \begin{pmatrix} 0 \\ 1 \\ \sqrt{2} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2} \\ \delta \\ 0 \end{pmatrix} - \frac{v_3^i \cdot \delta}{v_3^i - \sqrt{2} v_2^i} \begin{pmatrix} 0 \\ 1 \\ \sqrt{2} \end{pmatrix} + (\lambda - \lambda^*) \begin{pmatrix} 0 \\ 1 \\ \sqrt{2} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \delta - \frac{v_3^i \cdot \delta}{v_3^i - \sqrt{2} v_2^i} \\ -\frac{v_3^i \cdot \delta}{v_3^i - \sqrt{2} v_2^i} \sqrt{2} \end{pmatrix} + (\lambda - \lambda^*) \begin{pmatrix} 0 \\ 1 \\ \sqrt{2} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{-\sqrt{2} v_2^i \cdot \delta}{v_3^i - \sqrt{2} v_2^i} \\ -\frac{v_3^i \cdot \delta}{v_3^i - \sqrt{2} v_2^i} \sqrt{2} \end{pmatrix} + (\lambda - \lambda^*) \begin{pmatrix} 0 \\ 1 \\ \sqrt{2} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \delta \\ 0 \end{pmatrix} - \frac{1}{\sqrt{2}} \cdot \frac{2\delta}{v_3^i - \sqrt{2} v_2^i} \begin{pmatrix} 0 \\ v_2^i \\ v_3^i \end{pmatrix} + (\lambda - \lambda^*) \begin{pmatrix} 0 \\ 1 \\ \sqrt{2} \end{pmatrix} \\
&= \frac{1}{2} \cdot \underbrace{\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + (\lambda - \lambda^*) \begin{pmatrix} 0 \\ 1 \\ \sqrt{2} \end{pmatrix} \right)}_{\in (P^{119} + \mathrm{lin}\{v^i\})_I} + \frac{1}{2} \cdot \underbrace{\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{\sqrt{2}} \cdot \frac{4\delta}{v_3^i - \sqrt{2} v_2^i} \begin{pmatrix} 0 \\ v_2^i \\ v_3^i \end{pmatrix} + (\lambda - \lambda^*) \begin{pmatrix} 0 \\ 1 \\ \sqrt{2} \end{pmatrix} \right)}_{\in (P^{119} + \mathrm{lin}\{v^i\})_I} \\
&\in \mathrm{cl}_I\left(P^{119} + \mathrm{lin}\left\{v^i\right\}\right).
\end{aligned}$$

For case c: Set $\lambda^*(i,\delta) := 0$ and let $\lambda \in \mathbb{R}_{\geq 0}$. Then

$$\begin{pmatrix} \frac{1}{2} \\ \delta \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ \sqrt{2} \end{pmatrix} = \frac{1}{2} \underbrace{\Bigg(\underbrace{\begin{pmatrix} \frac{1}{2} \\ \delta \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ \sqrt{2} \end{pmatrix}}_{\in P^{119}} + \frac{1}{2v_1^i} \cdot \begin{pmatrix} v_1^i \\ v_2^i \\ v_3^i \end{pmatrix} \Bigg)}_{\in (P^{119}+\mathrm{lin}\{v^i\})_I} + \frac{1}{2} \underbrace{\Bigg(\underbrace{\begin{pmatrix} \frac{1}{2} \\ \delta \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ \sqrt{2} \end{pmatrix}}_{\in P^{119}} - \frac{1}{2v_1^i} \cdot \begin{pmatrix} v_1^i \\ v_2^i \\ v_3^i \end{pmatrix} \Bigg)}_{\in (P^{119}+\mathrm{lin}\{v^i\})_I}$$
$$\in \mathrm{conv}\left(\left(P^{119} + \mathrm{lin}\left\{v^i\right\}\right)_I\right).$$

For 3: Since $p \in P^{119} \backslash \mathrm{cl}_I\left(P^{119}\right)$, we have $-\sqrt{2}p_2 + p_3 > 0$. Let $c \in \mathbb{Z}^2_{\geq 1}$ be such that

$$-c_2 p_2 + c_1 p_3 =: C > 0 \tag{4.26}$$

and

$$c_2 > \sqrt{2} c_1 \tag{4.27}$$

(one can easily show that such a c exists). Additionally, define $v := \begin{pmatrix} 0 \\ c_1 \\ c_2 \end{pmatrix}$. It is easy to check that

$$\mathrm{cl}_I\left(P^{119} + \mathrm{lin}\{v\}\right) = \mathrm{conv}\left(\{0,1\} \times \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \mathrm{cone}\left\{\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}\right\} + \mathrm{lin}\left\{\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}\right\}\right)\right). \tag{4.28}$$

We now show

$$\begin{pmatrix} 0 & -c_2 & c_1 \end{pmatrix} p > 0, \tag{4.29}$$
$$\forall x \in \mathrm{cl}_I\left(P^{119} + \mathrm{lin}\{v\}\right) : \begin{pmatrix} 0 & -c_2 & c_1 \end{pmatrix} x \leq 0. \tag{4.30}$$

For (4.29)**:** (4.29) is an immediate consequence of (4.26).

For (4.30)**:** By (4.28), there exist $\lambda \in \mathbb{R}_{\geq 0}$ and $\mu \in \mathbb{R}$ such that

$$x_2 = \lambda + \mu c_1,$$
$$x_3 = \lambda\sqrt{2} + \mu c_2.$$

Thus, using (4.27), we obtain

$$\begin{pmatrix} 0 & -c_2 & c_1 \end{pmatrix} x = -c_2(\lambda + \mu c_1) + c_1\left(\lambda\sqrt{2} + \mu c_2\right) = \lambda\left(-c_2 + \sqrt{2}c_1\right) \leq 0.$$

□

4.6. Inclusions between $\mathrm{cl}_{L_{k-\frac{1}{2}}}(\cdot)$ vs $\mathrm{cl}_{L_k}(\cdot)$ vs $\mathrm{cl}_{L_{k+\frac{1}{2}}}(\cdot)$

In Remark 156, we formulated as question 2 whether if $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) is a rational polyhedron, the out of themselves unrelated looking hierarchies

$$\begin{array}{llllll} P = \mathrm{cl}_{L_{0,\mathbb{Q}}}(P) \supseteq & \mathrm{cl}_{L_{1,\mathbb{Q}}}(P) \supseteq & \mathrm{cl}_{L_{2,\mathbb{Q}}}(P) \supseteq \cdots \supseteq & \mathrm{cl}_{L_{m,\mathbb{Q}}}(P) = \mathrm{cl}_I(P), \\ P & \supseteq \mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{Q}}}(P) \supseteq & \mathrm{cl}_{L_{2-\frac{1}{2},\mathbb{Q}}}(P) \supseteq \cdots \supseteq & \mathrm{cl}_{L_{m-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P) = \mathrm{cl}_I(P) \end{array}$$

(if $n = 0$) or

$$\begin{array}{llllll} P = \mathrm{cl}_{L_{0,\mathbb{Q}}}(P) \supseteq & \mathrm{cl}_{L_{1,\mathbb{Q}}}(P) \supseteq & \mathrm{cl}_{L_{2,\mathbb{Q}}}(P) \supseteq \cdots \supseteq & \mathrm{cl}_{L_{m+n,\mathbb{Q}}}(P) = \mathrm{cl}_I(P), \\ P & \supseteq \mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) \supseteq & \mathrm{cl}_{L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) \supseteq \cdots \supseteq & \mathrm{cl}_{L_{m+n-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_I(P), \\ P & \supseteq \mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P) \supseteq & \mathrm{cl}_{L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P) \supseteq \cdots \supseteq & \mathrm{cl}_{L_{m+n-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P) = \mathrm{cl}_I(P) \end{array}$$

(if $n \geq 1$), respectively, can be merged together into a "unified" hierarchy, i.e. whether we have

$$P = \mathrm{cl}_{L_{0,\mathbb{Q}}}(P) \supseteq \mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{Q}}}(P) \supseteq \mathrm{cl}_{L_{1,\mathbb{Q}}}(P) \supseteq \ldots \supseteq \mathrm{cl}_{L_{m-1,\mathbb{Q}}}(P) \supseteq \mathrm{cl}_{L_{m-\frac{1}{2},\mathbb{Q}}}(P) = \mathrm{cl}_{L_{m,\mathbb{Q}}}(P) = \mathrm{cl}_I(P) \quad (4.31)$$

for $n = 0$ and

$$\begin{aligned} P = \mathrm{cl}_{L_{0,\mathbb{Q}}}(P) &\supseteq \mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P) \supseteq \mathrm{cl}_{L_{1,\mathbb{Q}}}(P) \supseteq \ldots \supseteq \mathrm{cl}_{L_{m-1,\mathbb{Q}}}(P) \\ &\supseteq \mathrm{cl}_{L_{m-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{m-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P) \supseteq \mathrm{cl}_{L_{m,\mathbb{Q}}}(P) = \mathrm{cl}_{L_{(m+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{(m+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P) = \mathrm{cl}_I(P) \end{aligned} \quad (4.32)$$

for $n \geq 1$.

We have already seen the equivalence $\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P)$ ($k \in \{0, \ldots, m\}$) from (4.32) in section 4.5.4, Theorem 193 (though it is proved later on in section 4.8.4).

The topic of this section are the inclusions in (4.31) and (4.32):

- In section 4.6.1, we consider the inclusion

  $$\mathrm{cl}_{L_{k,\mathbb{Q}}}(P) \subseteq \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P), \quad (4.33)$$

  where $m, n \in \mathbb{Z}_{\geq 0}$ and $k \in \{0, \ldots, m+n\}$. This inclusion does hold for arbitrary $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$. In other words: we don't demand P to be a rational polyhedron. The final result is formulated in Theorem 197. In Theorem 196, we prove a property that is essential for this proof, but is in our opinion also by itself of mathematical interest.

  We remark that the inclusion (4.33) holds "in an even stronger sense", i.e. we show that every $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cut for P with respect to some vector space V is an $L_{k,\mathbb{Q}}$ cut with respect to the same vector space V.

  Note that the "obvious" analogue of (4.33) for the $L_{k,\mathbb{R}}$ closure vs the $L_{k-\frac{1}{2},\mathbb{R}\times\mathbb{R}}$ closure

  $$\mathrm{cl}_{L_{k,\mathbb{R}}}(P) \subseteq \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{R}\times\mathbb{R}}}(P) \quad (4.34)$$

  does *not* hold in general: in Remark 198, we consider a counterexample $P \subseteq \mathbb{R}^2$ for (4.34), where P is a (non-rational) polyhedron.

- In section 4.6.2, we consider the inclusion

  $$\mathrm{cl}_{L_{(k+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) \subseteq \mathrm{cl}_{L_{k,\mathbb{Q}}}(P) \quad (4.35)$$

  ($m, n \in \mathbb{Z}_{\geq 0}$ and $k \in \{0, \ldots, m+n-1\}$), which holds for every rational polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$. As we see in Example 201, the condition that P is a rational polyhedron is essential.

  (4.35) has an analogue for the $L_{(k+1)-\frac{1}{2},\mathbb{R}\times\mathbb{R}}$ closure vs $L_{k,\mathbb{R}}$ closure:

  $$\mathrm{cl}_{L_{(k+1)-\frac{1}{2},\mathbb{R}\times\mathbb{R}}}(P) \subseteq \mathrm{cl}_{L_{k,\mathbb{R}}}(P) \quad (4.36)$$

  ($m, n \in \mathbb{Z}_{\geq 0}$ and $k \in \{0, \ldots, m+n-1\}$), which holds for *arbitrary* $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$. Both (4.35) and (4.36) are shown in Theorem 199.

The final property of (4.32), that the hierarchy ends at $\mathrm{cl}_{L_{m,\mathbb{Q}}}(P) = \mathrm{cl}_{L_{(m+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{(m+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P)$ if P is a rational polyhedron, is the topic of section 4.7, even though this section rather brings together results that are shown in the preceding sections.

4.6.1. $\mathrm{cl}_{L_{k,(\cdot)}}(\cdot) \subseteq \mathrm{cl}_{L_{k-\frac{1}{2},(\cdot)}}(\cdot)$?

Theorem 196. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be an arbitrary set and let $V \leq \mathbb{R}^m \times \mathbb{R}^n$ be a subspace with generators from $\mathbb{Q}^m \times \mathbb{R}^n$. Then*

$$P \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + V) \subseteq \mathrm{cl}_I(P + V).$$

Proof. Clearly, V is finitely generated. Let $\{v^1, \ldots, v^l\} \subseteq \mathbb{Z}^m \times \mathbb{R}^n$ ($l \in \mathbb{Z}_{\geq 0}$) be a generating system for V and let $p \in P \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + V)$. Then

$$p = z + \sum_{i=1}^{l} \lambda_i \cdot v^i,$$

where $z \in \mathbb{Z}^m \times \mathbb{R}^n$ and $\lambda \in \mathbb{R}^l$. For $I \in \{0,1\}^l$, consider

$$q^I := \underbrace{z + \sum_{i=1}^{l} (\lfloor \lambda_i \rfloor + I_i)\, v^i}_{\in \mathbb{Z}^m \times \mathbb{R}^n} = \underbrace{p}_{\in P} \underbrace{- \sum_{i=1}^{l} \lambda_i \cdot v^i + \sum_{i=1}^{l} (\lfloor \lambda_i \rfloor + I_i)\, v^i}_{\in V} \in (P+V)_I. \tag{4.37}$$

For $i \in [l]$, let

$$\lambda'_i := \lambda_i - \lfloor \lambda_i \rfloor \in [0,1).$$

Define $\mu \in [0,1]^{(\{0,1\}^l)}$ via

$$\mu_I := \prod_{i=1}^{l} \begin{cases} 1 - \lambda'_i & \text{if } I_i = 0, \\ \lambda'_i & \text{if } I_i = 1. \end{cases}$$

We have

$$\sum_{I \in \{0,1\}^l} \mu_I = \sum_{I \in \{0,1\}^l} \prod_{j=1}^{l} \begin{cases} 1 - \lambda'_j & \text{if } I_j = 0, \\ \lambda'_j & \text{if } I_j = 1 \end{cases} = \prod_{j=1}^{l} \left((1 - \lambda'_j) + \lambda'_j \right) = 1, \tag{4.38}$$

$$\forall i \in [l] : \sum_{\substack{I \in \{0,1\}^l : \\ I_i = 1}} \mu_I = \sum_{\substack{I \in \{0,1\}^l : \\ I_i = 1}} \lambda'_i \prod_{j=1, j \neq i}^{l} \begin{cases} 1 - \lambda'_j & \text{if } I_i = 0, \\ \lambda'_j & \text{if } I_i = 1 \end{cases} = \lambda'_i \prod_{j=1, j \neq i}^{l} \left((1 - \lambda'_j) + \lambda'_j \right) = \lambda'_i. \tag{4.39}$$

Using (4.37) and (4.38), we infer

$$\sum_{I \in \{0,1\}^l} \mu_I q^I \in \mathrm{cl}_I (P + V).$$

On the other hand,

$$\begin{aligned}
\sum_{I \in \{0,1\}^l} \mu_I q^I &= z + \sum_{I \in \{0,1\}^l} \mu_I \sum_{i=1}^{l} (\lfloor \lambda_i \rfloor + I_i)\, v^i && \text{(by (4.37) and (4.38))} \\
&= z + \sum_{i=1}^{l} \left(\lfloor \lambda_i \rfloor + \sum_{I \in \{0,1\}^l} \mu_I I_i \right) v^i && \text{(by (4.38))} \\
&= z + \sum_{i=1}^{l} \left(\lfloor \lambda_i \rfloor + \sum_{\substack{I \in \{0,1\}^l : \\ I_i = 1}} \mu_I \right) v^i \\
&= z + \sum_{i=1}^{l} (\lfloor \lambda_i \rfloor + \lambda'_i)\, v^i && \text{(by (4.39))} \\
&= z + \sum_{i=1}^{l} \lambda_i v^i \\
&= p.
\end{aligned}$$

Thus, $p \in \mathrm{cl}_I (P + V)$. □

An immediate consequence of Theorem 196 is:

Theorem 197. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be arbitrary. Then every $L_{k-\frac{1}{2}, \mathbb{Q} \times \mathbb{R}}$ cut for P ($k \in \{0, \ldots, m+n\}$) with respect to some V is also an $L_{k,\mathbb{Q}}$ cut for P with respect to the same V. In particular,*

we have

$$\mathrm{cl}_{L_{k,\mathbb{Q}}}(P) \subseteq \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P).$$

Remark 198. *Note that a similar statement to Theorem 197 for $L_{k,\mathbb{R}}$ cuts vs $L_{k-\frac{1}{2},\mathbb{R}\times\mathbb{R}}$ cuts, i.e. that for arbitrary $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ and $k \in \{0, \ldots, m+n\}$, we have*

$$\mathrm{cl}_{L_{k,\mathbb{R}}}(P) \subseteq \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{R}\times\mathbb{R}}}(P),$$

does not *hold in general, even if we restrict P to be a polyhedron and only consider the pure integer case. For this, consider that, by Theorem 188, we have*

$$\mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{R}}}\left(P^{115}\right) = \left\{0^2\right\},$$

but, on the other hand, by equation (4.13) *in Theorem 187, we have*

$$\mathrm{cl}_{L_{1,\mathbb{R}}}\left(P^{115}\right) = P^{115} = P \supsetneq \left\{0^2\right\}.$$

4.6.2. $\mathrm{cl}_{L_{(k+1)-\frac{1}{2},(\,\cdot\,)}}(\,\cdot\,) \subseteq \mathrm{cl}_{L_{k,(\,\cdot\,)}}(\,\cdot\,)$?

Theorem 199. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ be arbitary, let $k \in \{0, \ldots, m+n-1\}$, let $V^{pre} \leq \mathbb{R}^m \times \mathbb{R}^n$ be a subspace of codimension k and let $c(\,\cdot\,) \leq c_0$ be an L_k cut for P with respect to V^{pre} (i.e. valid for $P \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + V^{pre})$). Define*

$$W^{pre} := \left(\mathrm{lin}\left\{c^T\right\}\right)^{\perp},$$
$$V := V^{pre} \cap W^{pre}.$$

Then $c(\,\cdot\,) \leq c_0$ is valid for $(P+V) \cap (\mathbb{Z}^m \times \mathbb{R}^n)$ (i.e. an $L_{k-\frac{1}{2}}$ cut or $L_{k+1-\frac{1}{2}}$ cut (depending on the codimension of V) for P with respect to V). Thus:

1. *Every $L_{k,\mathbb{R}}$ cut for P is an $L_{(k+1)-\frac{1}{2},\mathbb{R}\times\mathbb{R}}$ cut for P. In particular, we have*

$$\mathrm{cl}_{L_{(k+1)-\frac{1}{2},\mathbb{R}\times\mathbb{R}}}(P) \subseteq \mathrm{cl}_{L_{k,\mathbb{R}}}(P).$$

2. *Let $c(\,\cdot\,) \leq c_0$ be an $L_{k,\mathbb{Q}}$ cut for P having $c \in (\mathbb{Q}^m \times \mathbb{Q}^n)^T$. Then $c(\,\cdot\,) \leq c_0$ is also an $L_{(k+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cut for P. In particular, if P is a rational polyhedron, we have*

$$\mathrm{cl}_{L_{(k+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) \subseteq \mathrm{cl}_{L_{k,\mathbb{Q}}}(P).$$

Proof. We start with a proof that 2 can be concluded from the main statement (for 1, this is obvious).

Since $\mathrm{cl}_{L_{m,\mathbb{Q}}}(P) = \mathrm{cl}_I(P)$ (just consider that every inequality for P_I is an $L_{m,\mathbb{Q}}$ cut with respect to $0^m \times \mathbb{R}^n$), we can assume $k \leq m$. By Theorem 168, we can additionally assume that $V^{pre} = V' \times \mathbb{R}^n$, where V' is a rational subspace of codimension k. So, V is a rational subspace if $c \in (\mathbb{Q}^m \times \mathbb{Q}^n)^T$. Additionally, note that $\mathrm{cl}_I(P+V)$ is a rational polyhedron; thus, any facet-defining inequality for it can be assumed to have rational coefficients. Thus, we can restrict ourselves to the case $c \in (\mathbb{Q}^m \times \mathbb{Q}^n)^T$.

Now for a proof of the main statement: we have

$$m+n-k-1 \leq \dim V \leq m+n-k,$$

since

$$\begin{aligned}
m+n-k &= \dim V^{pre} \\
&\geq \dim V \\
&= \dim\left(V^{pre} \cap W^{pre}\right) \\
&= \dim V^{pre} + \dim W^{pre} - \dim\left(V^{pre} + W^{pre}\right) && \text{(by Lemma 30)} \\
&\geq (m+n-k) + (m+n-1) - \underbrace{\dim\left(V^{pre} + W^{pre}\right)}_{\leq \dim(\mathbb{R}^m \times \mathbb{R}^n) = m+n} \\
&\geq (m+n-k) + (m+n-1) - (m+n) \\
&= m+n-k-1.
\end{aligned}$$

Remark 200. *One can even tell exactly:*

$$\dim V = \begin{cases} m+n-k & \text{if } c^T \in \left(V^{pre}\right)^{\perp}, \\ m+n-k-1 & \text{otherwise.} \end{cases}$$

Proof. (Remark 200) If $c^T = 0^m \times 0^n$, the statement is clear. So, from now on, we assume $c^T \neq 0^m \times 0^n$, which is equivalent to $\dim W^{pre} = m+n-1$. Looking at the chain of inequalities, we just have to prove

$$\dim\left(V^{pre} + W^{pre}\right) = \begin{cases} m+n-1 & \text{if } c^T \in \left(V^{pre}\right)^{\perp}, \\ m+n & \text{if } c^T \notin \left(V^{pre}\right)^{\perp}. \end{cases}$$

We first show

$$c^T \in \left(V^{pre}\right)^{\perp} \Leftrightarrow V^{pre} \subseteq W^{pre}. \tag{4.40}$$

For $\Rightarrow$: Let $w \in \left(V^{pre}\right)^{\perp}$. Then $cw = 0$, thus $w \in W^{pre}$.

For $\Leftarrow$: Let $v \in V^{pre}$. Then, since by assumption $v \in W^{pre} = \left(\operatorname{lin}\left\{c^T\right\}\right)^{\perp}$, we have $cv = 0$.

Now, if $c^T \in \left(V^{pre}\right)^{\perp}$, by (4.40), we have $V^{pre} \subseteq W^{pre}$. Thus,

$$\dim\left(V^{pre} + W^{pre}\right) = \dim W^{pre} = m+n-1.$$

If, on the other hand, $c \notin \left(V^{pre}\right)^{\perp}$, we show that $V^{pre} + W^{pre} = \mathbb{R}^m \times \mathbb{R}^n$: since $\dim W^{pre} = m+n-1$, the situation $\dim\left(V^{pre} + W^{pre}\right) = m+n-1$ can only happen if $V^{pre} \subseteq W^{pre}$. But, by (4.40), this is a contradiction to the case assumption. ☐

Now on with the proof of Theorem 199: we claim that $c(\,\cdot\,) \leq c_0$ is a valid inequality for $(P+V)_I$. For this, let $p \in P$ and $v \in V$ be such that $p + v \in \mathbb{Z}^m \times \mathbb{R}^n$. Then, clearly, $p \in P \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + V)$. Thus,

$$cp \leq c_0. \tag{4.41}$$

On the other hand, we have using $c^T \perp V$ and (4.41):

$$c(p+v) = cp \leq c_0.$$

☐

We next want to demonstrate that the condition that P is a rational polyhedron is essential for the

$$\mathrm{cl}_{L_{(k+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) \subseteq \mathrm{cl}_{L_{k,\mathbb{Q}}}(P)$$

inclusion in Theorem 199 to hold. For this, we consider the following example:

Example 201. *We have:*

$$\mathrm{cl}_{L_{1,\mathbb{Q}}}\left(P^{118}\right) = \mathrm{cl}_{\overline{I}}\left(P^{118}\right) = \mathrm{cl}_{I}\left(P^{118}\right) \subsetneq P^{118} = \mathrm{cl}_{L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}\left(P^{118}\right).$$

Proof. Set $V := 0^1 \times \mathbb{R}^2$ and consider

$$\overline{\operatorname{conv}}\big(P^{118} \cap \underbrace{\big((\mathbb{Z}^1 \times \mathbb{R}^2) + V\big)}_{=\mathbb{Z}^1 \times \mathbb{R}^2}\big) = \operatorname{cl}_{\overline{I}}\left(P^{118}\right) = \operatorname{cl}_I\left(P^{118}\right).$$

The rest is a consequence of Theorem 191. □

4.7. Termination of the $L_k/L_{k-\frac{1}{2}}$ hierarchy

Of the questions that we formulated at the beginning of section 4.6, it is still open that for rational polyhedra, the hierarchy (4.32) ends at $\operatorname{cl}_{L_{m,\mathbb{Q}}}(P) = \operatorname{cl}_{L_{(m+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \operatorname{cl}_{L_{(m+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P)$, i.e. that we have

$$\operatorname{cl}_{L_{m,\mathbb{Q}}}(P) = \operatorname{cl}_{L_{(m+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \operatorname{cl}_{L_{(m+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P) = \operatorname{cl}_I(P).$$

We also write down that for arbitrary $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m \in \mathbb{Z}_{\geq 0}$, $n \in \mathbb{Z}_{\geq 1}$), we have

$$\operatorname{cl}_{L_{m,\mathbb{Q}}}(P) = \operatorname{cl}_{L_{(m+1)-\frac{1}{2},\mathbb{R}\times\mathbb{R}}}(P) = P \cap \operatorname{cl}_{\overline{I}}(P).$$

The only argument that is necessary for the proof of the following theorem, is to consider that for P, m and n as in Theorem 202, we have

$$P_I = P \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + \underbrace{(0^m \times \mathbb{R}^n)}_{=:V}).$$

Theorem 202. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be arbitrary. Then every linear inequality for P_I is an $L_{m,\mathbb{Q}}$ cut for P. Thus, we have*

$$\operatorname{cl}_{L_{m,\mathbb{Q}}}(P) = P \cap \operatorname{cl}_{\overline{I}}(P).$$

This has the following consequences if $n \geq 1$:

- *If P is a rational polyhedron, every cutting plane $c(\,\cdot\,) \leq c_0$ for P such that $c \in (\mathbb{Q}^m \times \mathbb{Q}^n)^T$ is, by Theorem 199, an $L_{(m+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cut for P, and every arbitrary cutting plane for P is dominated absolutely by some set of rational $L_{(m+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts for P. Thus, the identity*

  $$\operatorname{cl}_{L_{(m+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \operatorname{cl}_I(P).$$

  holds.

- *In the general case (P arbitrary), every cutting plane for P is an $L_{(m+1)-\frac{1}{2},\mathbb{R}\times\mathbb{R}}$ cut for P (again a consequence of Theorem 199). We thus have*

  $$\operatorname{cl}_{L_{(m+1)-\frac{1}{2},\mathbb{R}\times\mathbb{R}}}(P) = P \cap \operatorname{cl}_{\overline{I}}(P).$$

So, summarizingly, for $n \geq 1$, we have:

- *If P is a rational polyehdron, we have*

  $$\operatorname{cl}_{L_{m,\mathbb{Q}}}(P) = \operatorname{cl}_{L_{(m+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \operatorname{cl}_I(P).$$

- *In the general case (P arbitrary), we have*

  $$\operatorname{cl}_{L_{m,\mathbb{Q}}}(P) = \operatorname{cl}_{L_{(m+1)-\frac{1}{2},\mathbb{R}\times\mathbb{R}}}(P) = P \cap \operatorname{cl}_{\overline{I}}(P).$$

We remark that in Theorem 217, Theorem 218, Theorem 222 and Theorem 223, we generalize Theorem 202.

4.8. Essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts

The central topic of this section is guiding question 3 of Remark 156, i.e. the role of $L_{k-\frac{1}{2}}$ cuts with respect to vector spaces of the form $V = V' \times \mathbb{R}^n$. The motivation for this is that in section 4.2.2.1, Theorem 168, we showed that a restriction to this subset of vector spaces is possible for L_k cuts. While it is easy to see that this restriction, in general, does not suffice to describe *all possible* $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts, it should at this point at least be plausible to the reader that this restricted subset of $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts might have interesting properties.

4.8.1. Definitions

We start by giving a formal definition of "essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts" (Definition 203). These are $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts with respect to a vector space $V = V' \times \mathbb{R}^n$, where V' is – as the naming already suggests – generated by rational vectors and has codimension k. We then define their respective closure ("essential $L_{k-\frac{1}{2},\mathbb{Q}}$ closure") in Definition 205.

Definition 203. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ be arbitrary, let $k \in \{0, \ldots, m\}$ and let $V' \leq \mathbb{R}^m$ be a linear subspace of codimension k that is generated by rational vectors (i.e. a rational subspace of codimension k). An **essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cut for** P is a valid linear inequality for $(P + (V' \times \mathbb{R}^n))_I$.*

Remark 204. *Let $P \subseteq \mathbb{R}^m$ $(m \in \mathbb{Z}_{\geq 0})$ be arbitrary. Then every $L_{k-\frac{1}{2},\mathbb{Q}}$ cut for P $(k \in \{0, \ldots, m\})$ is also an essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cut for P. In other words: in the pure integer case, it makes no sense to distinguish between $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts and essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts.*

Definition 205. *We define for an arbitrary $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$*

$$\operatorname{cl}_{\text{ess}\, L_{k-\frac{1}{2},\mathbb{Q}}}(P) := P \cap \bigcap_{\substack{c \in (\mathbb{R}^m \times \mathbb{R}^n)^T, c_0 \in \mathbb{R}: \\ c(\,\cdot\,) \leq c_0 \text{ essential } L_{k-\frac{1}{2},\mathbb{Q}} \\ \text{cut for } P}} P^{\leq}(c, c_0)$$

*as the **essential $L_{k-\frac{1}{2},\mathbb{Q}}$ closure of** P.*

Remark 206. *In an analogue way as in Lemma 183, one can show that for arbitrary $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$, where $m, n \in \mathbb{Z}_{\geq 0}$ and $k \in \{0, \ldots, m\}$, we have*

$$\operatorname{cl}_{\text{ess}\, L_{k-\frac{1}{2},\mathbb{Q}}}(P) = P \cap \bigcap_{\substack{V' \leq \mathbb{R}^m: \\ V' \text{ rational subspace} \\ \text{of codimension } k}} \operatorname{cl}_{\overline{I}}(P + (V' \times \mathbb{R}^n)).$$

4.8.2. Characterizing the essential $L_{k-\frac{1}{2},\mathbb{Q}}$ closure

Our first theorem about the essential $L_{k-\frac{1}{2},\mathbb{Q}}$ closure states that to describe the essential $L_{k-\frac{1}{2},\mathbb{Q}}$ closure of some arbitrary $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$, we just need to consider the $L_{k-\frac{1}{2},\mathbb{Q}}$ closure of the projection of P into the integer variables:

Theorem 207. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ be arbitrary and let $k \in \{0, \ldots, m\}$. Then*

$$\operatorname{cl}_{\text{ess}\, L_{k-\frac{1}{2},\mathbb{Q}}}(P) = P \cap \left(\left(\operatorname{cl}_{L_{k-\frac{1}{2},\mathbb{Q}}}(\operatorname{proj}_{\mathbb{R}^m} P) \right) \times \mathbb{R}^n \right).$$

Proof.

$$\begin{aligned}\operatorname{cl}_{\text{ess}\, L_{k-\frac{1}{2},\mathbb{Q}}}(P) &= P \cap \bigcap_{\substack{V' \leq \mathbb{R}^m: \\ V' \text{ rational subspace} \\ \text{of codimension } k}} \operatorname{cl}_{\overline{I}}(P + (V' \times \mathbb{R}^n)) \\ &= P \cap \bigcap_{\substack{V' \leq \mathbb{R}^m: \\ V' \text{ rational subspace} \\ \text{of codimension } k}} (\operatorname{cl}_{\overline{I}}((\operatorname{proj}_{\mathbb{R}^m} P) + V') \times \mathbb{R}^n)\end{aligned}$$

$$= P \cap \left(\left((\mathrm{proj}_{\mathbb{R}^m} P) \cap \bigcap_{\substack{V' \leq \mathbb{R}^m: \\ V' \text{ rational subspace} \\ \text{of codimension } k}} (\mathrm{cl}_{\overline{I}}((\mathrm{proj}_{\mathbb{R}^m} P) + V')) \right) \times \mathbb{R}^n \right)$$
$$= P \cap \left(\left(\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}}} (\mathrm{proj}_{\mathbb{R}^m} P) \right) \times \mathbb{R}^n \right).$$

□

4.8.3. $L_{(k+1)-\frac{1}{2},\mathbb{Q}\times(\,\cdot\,)}$ cuts vs $L_{k,\mathbb{Q}}$ cuts

We now come to a result that can be considered as an analogue of Theorem 168 and Corollary 171 for $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts and $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cuts. We recapitulate that in these theorems, we showed that for $L_{k,\mathbb{Q}}$ cuts ($k \in \{0, \ldots, m\}$), we can restrict ourselves to vector spaces of the form $V := V' \times \mathbb{R}^n$ (cf. Theorem 168).

It is easy to verify that such a restriction (i.e. restrict oneself to essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts) is not possible for $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts and $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cuts. In the next theorem, we show that such a restriction is possible if we only consider $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cuts that are not already $L_{k-1,\mathbb{Q}}$ cuts.

Theorem 208. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be arbitrary and let $k \in \{1, \ldots, m\}$. Let $c(\,\cdot\,) \leq c_0$ ($c \in (\mathbb{R}^m \times \mathbb{R}^n)^T$ and $c_0 \in \mathbb{R}$) be an $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cut for P that is not an $L_{k-1,\mathbb{Q}}$ cut for P. Then $c(\,\cdot\,) \leq c_0$ is an essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cut for P.*

Remark 209. *The reason why in Theorem 208 the condition $k \in \{1, \ldots, m\}$ (instead of $k \in \{1, \ldots, m+n\}$) is no restriction (beside the fact that we only defined essential $L_{k,\mathbb{Q}}$ cuts for $k \in \{0, \ldots, m\}$) is that*

$$\mathrm{cl}_{L_{m,\mathbb{Q}}}(P) = P \cap \mathrm{cl}_{\overline{I}}(P).$$

In other words: for $k \geq m+1$, there exist no $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cuts for P which are not already $L_{k-1,\mathbb{Q}}$ cuts for P.

Remark 210. *Since every $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cut is an $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cut, Theorem 208 also holds for $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts.*

Proof. (Theorem 208) Let V be a subspace of codimension k that is generated by vectors from $\mathbb{Q}^m \times \mathbb{R}^n$ and let $c(\,\cdot\,) \leq c_0$ be valid for $(P+V)_I$ (i.e. $c(\,\cdot\,) \leq c_0$ is an $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cut for P with respect to V). If $0^m \times \mathbb{R}^n \leq V$, we are done. Otherwise, let $V' := \mathrm{proj}_{\mathbb{R}^m} V$.

By Theorem 196, $c(\,\cdot\,) \leq c_0$ is valid for

$$P \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + V).$$

On the other hand, by Theorem 168, $c(\,\cdot\,) \leq c_0$ is also valid for

$$P \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + (V' \times \mathbb{R}^n)).$$

But we have $V < V' \times \mathbb{R}^n$; thus,

$$\mathrm{codim}(V' \times \mathbb{R}^n) < \mathrm{codim}\, V = k.$$

Thus, $c(\,\cdot\,) \leq c_0$ is an $L_{k-1,\mathbb{Q}}$ cut for P. □

Theorem 211. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be arbitrary, let $k \in \{1, \ldots, m\}$ and let $c(\,\cdot\,) \leq c_0$ ($c \in (\mathbb{R}^m \times \mathbb{R}^n)^T$ and $c_0 \in \mathbb{R}$) be an $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cut for P. Then $c(\,\cdot\,) \leq c_0$ is either an $L_{k-1,\mathbb{Q}}$ cut for P or an essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cut for P. This has the following consequences:*

- *If P is a rational polyhedron, every $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cut for P is dominated (absolutely) by a finite set of either*
  - *rational essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts for P or*
  - *rational $L_{k-1,\mathbb{Q}}$ cuts for P.*

  In particular,

  $$\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P) = \mathrm{cl}_{L_{k-1,\mathbb{Q}}}(P) \cap \mathrm{cl}_{\mathrm{ess}\, L_{k-\frac{1}{2},\mathbb{Q}}}(P). \tag{4.42}$$

  Furthermore, we have for $k \in \{0, \ldots, m+n\}$:

  $$\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P). \tag{4.43}$$

- *In general, the weaker identity*

$$\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P) \supseteq \mathrm{cl}_{L_{k-1,\mathbb{Q}}}(P) \cap \mathrm{cl}_{\mathrm{ess}\, L_{k-\frac{1}{2},\mathbb{Q}}}(P) \tag{4.44}$$

holds.

Proof. From Theorem 208, one concludes that if $c(\cdot) \leq c_0$ is an $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cut, but no $L_{k-1,\mathbb{Q}}$ cut, for P, the inequality $c(\cdot) \leq c_0$ is an essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cut for P. This shows the first statement and equation (4.44).

Now for the situation if P is a rational polyhedron: since for every rational subspace $V \leq \mathbb{R}^m$,

$$\mathrm{cl}_I(P + (V \times \mathbb{R}^n))$$

is a rational polyhedron, every essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cut for P is dominated by a finite set of rational essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts for P. Let $c(\cdot) \leq c_0$ be an $L_{k-1,\mathbb{Q}}$ cut for P with respect to some appropriate vector space V. By Theorem 168, we can assume $V = V' \times \mathbb{R}^n$, where V' is a rational subspace. Then, by Theorem 172 and Theorem 177, $P \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + V)$ is a rational polyhedron; thus, every $L_{k-1,\mathbb{Q}}$ cut for P with respect to V is dominated by some finite set of rational $L_{k-1,\mathbb{Q}}$ cuts for P (with respect to V). Thus, we can assume $c \in (\mathbb{Q}^m \times \mathbb{Q}^n)^T$. So, by Theorem 199, $c(\cdot) \leq c_0$ is an $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cut for P.

Finally, for (4.43): for $k \in \{1, \ldots, m\}$, (4.43) is a consequence of (4.42). For $k = 0$, (4.43) is a consequence of Remark 180. Finally, for $k \in \{m+1, \ldots, m+n\}$, we conclude using Theorem 202:

$$\mathrm{cl}_I(P) \subseteq \mathrm{cl}_{L_{(k+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P) \subseteq \mathrm{cl}_{L_{(k+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) \subseteq \mathrm{cl}_{L_{(m+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_I(P).$$

□

From Theorem 211, we immediately conclude Theorem 193, which we already stated in section 4.5.4:

Theorem 193. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be a rational polyhedron. Then every $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cut for P ($k \in \{0, \ldots, m\}$) is dominated absolutely by a finite set of either*

- *rational essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts for P (essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts are defined in Definition 203) or*
- *rational $L_{k-1,\mathbb{Q}}$ cuts for P (which, by Theorem 199, are $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts for P) if $k \geq 1$.*

More concisely: every $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cut for P is dominated by a finite set of rational $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts for P. Thus, we conclude

$$\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P).$$

An immediate consequence of Theorem 193 is:

Theorem 212. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be a rational polyhedron and let $k \in \{0, \ldots, m+n\}$. Then*

$$\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P).$$

Proof. If $k \in \{0, \ldots, m\}$, the statement holds by Theorem 193. On the other hand, if $k \geq m$, we have using Theorem 75 and Theorem 202:

$$\mathrm{cl}_I(P) \subseteq \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P) \subseteq \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_I(P).$$

Thus, equality holds. □

4.8.4. $L_{k,\mathbb{Q}}$ cuts vs $L_{k-\frac{1}{2},\mathbb{Q}\times(\cdot)}$ cuts

Recall that in Theorem 208, we showed how an $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cut that is not already an $L_{k-1,\mathbb{Q}}$ cut has to look like (it is an essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cut). Additionally, recall that by Theorem 197, every $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cut or $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cut with respect to some V is also an $L_{k,\mathbb{Q}}$ cut with respect to the same V.

In this section, in Theorem 213, we consider some kind of reverse to Theorem 208: let an $L_{k,\mathbb{Q}}$ cut $c(\cdot,) \leq c_0$ with respect to some V be given. We show that if $c^T \perp V$, then it already is an $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cut or $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cut, respectively, with respect to the same V. Thus, every $L_{k,\mathbb{Q}}$ cut $c(\cdot,) \leq c_0$ with respect to some V that is not already an $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cut or $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cut, respectively, has to satisfy $c^T \not\perp V$.

Theorem 213. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be arbitrary, let $V \leq \mathbb{R}^m \times \mathbb{R}^n$ be a subspace of codimension k that is generated by vectors from $\mathbb{Q}^m \times \mathbb{Q}^n$ or $\mathbb{Q}^m \times \mathbb{R}^n$, respectively, and let $c(\cdot) \leq c_0$ ($c \in (\mathbb{R}^m \times \mathbb{R}^n)^T$ and $c_0 \in \mathbb{R}$) be a valid inequality for $P \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + V)$ (in other words: an $L_{k,\mathbb{Q}}$ cut for P with respect to V), where $c^T \perp V$. Then $c(\cdot) \leq c_0$ is already an $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cut or $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cut, respectively, for P with respect to V.*

Proof. Let $z \in (P+V)_I$. Then $x = p + v$, where $p \in P$ and $v \in V$. Since $p + v \in \mathbb{Z}^m \times \mathbb{R}^n$, we have

$$p \in P \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + V). \tag{4.45}$$

Since we have $cp' \leq c_0$ for all $p' \in P \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + V)$, we thus get from (4.45):

$$cp \leq c_0. \tag{4.46}$$

Thus,

$$cz = c(p+v) = \underbrace{cp}_{\leq c_0 \text{ (by (4.46))}} + \underbrace{cv}_{=0 \text{ (since } v \in V \perp c^T)} \leq c_0.$$

□

4.8.5. A dual representation of essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts

In the previous subsections of section 4.8, we saw the importance of the concept of essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts. In this last subsection of section 4.8, we want to study an alternative characterization of essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts. Recall that in section 4.2.2.2 (in particular Theorem 174), we gave a “dual” characterization of $L_{k,\mathbb{Q}}$ cuts. In this section, in Theorem 215, we now derive a “dual” characterization of essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts. Before we state and prove Theorem 215, we give a lemma that is used for its proof:

Lemma 214. *Let $P \subseteq \mathbb{R}^m$ ($m \in \mathbb{Z}_{\geq 0}$) be arbitrary, let $V' \leq \mathbb{R}^m$ be a rational subspace of codimension $k \in \{0, \ldots, m\}$ and let $\{w^1, \ldots, w^k\}$ be a lattice basis of $\mathbb{Z}^m \cap V'^{\perp}$. Then the following statements are equivalent:*

1. $(P + V')_I = \emptyset$,
2. $P \cap (\mathbb{Z}^m + V') = \emptyset$,
3. $\left\{x \in P : \forall i \in [k] : \left(w^i\right)^T x \in \mathbb{Z}\right\} = \emptyset$.

Proof.

For $1 \Rightarrow 2$**:** Since $(P + V')_I = \emptyset$, we also have $\mathrm{cl}_I(P + V') = \emptyset$. Using Theorem 196, we conclude

$$\emptyset = \mathrm{cl}_I(P + V') \supseteq P \cap (\mathbb{Z}^m + V').$$

For $2 \Leftrightarrow 3$**:** By Theorem 172, we have

$$P \cap (\mathbb{Z}^m + V') = \emptyset \Leftrightarrow \left\{x \in P : \forall i \in [k] : \left(w^i\right)^T x \in \mathbb{Z}\right\} = \emptyset.$$

For $3 \Rightarrow 1$**:** Let $\left\{x \in \mathrm{proj}_{\mathbb{R}^m} P : \forall i \in [k] : \left(w^i\right)^T x \in \mathbb{Z}\right\} = \emptyset$. By “$2 \Leftrightarrow 3$”, we have

$$\emptyset = P \cap (\mathbb{Z}^m + V'). \tag{4.47}$$

Now let $z = p + v \in (P + V')_I$, where $z \in \mathbb{Z}^m$, $p \in P$ and $v \in V'$. Then $z - v \in P \cap (\mathbb{Z}^m + V')$. But this is a contradiction to (4.47). □

Theorem 215. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be arbitrary, let $V' \leq \mathbb{R}^m$ be a rational subspace of codimension $k \in \{0, \ldots, m\}$, let $c \in \left(V'^{\perp}\right)^T$ and let $c_0 \in \mathbb{R}$. Then:*

1. *Let $\{w^1, \ldots, w^k\}$ be a lattice basis of $\mathbb{Z}^m \cap V'^{\perp}$. Then $\begin{pmatrix} c & (0^n)^T \end{pmatrix} (\cdot) \leq c_0$ is valid for $(P + (V' \times \mathbb{R}^n))_I$ (in other words: $\begin{pmatrix} c & (0^n)^T \end{pmatrix} (\cdot) \leq c_0$ is an essential $L_{k,\mathbb{Q}}$ cut for P with respect to V') if and only if $\begin{pmatrix} c & (0^n)^T \end{pmatrix} (\cdot) \leq c_0$ is valid for*
$$\left\{x \in \operatorname{proj}_{\mathbb{R}^m} P : \forall i \in [k] : \left(w^i\right)^T x \in \mathbb{Z}\right\} \times \mathbb{R}^n.$$

2. *Let $w'^1, \ldots, w'^k$ be arbitrary vectors of $\mathbb{Z}^m \cap V'^{\perp}$ and let $\begin{pmatrix} c & (0^n)^T \end{pmatrix} (\cdot) \leq c_0$ be valid for*
$$\left\{x \in \operatorname{proj}_{\mathbb{R}^m} P : \forall i \in [k] : \left(w'^i\right)^T x \in \mathbb{Z}\right\} \times \mathbb{R}^n.$$
Then $\begin{pmatrix} c & (0^n)^T \end{pmatrix} (\cdot) \leq c_0$ is valid for $(P + (V' \times \mathbb{R}^n))_I$ (in other words: it is an essential $L_{k,\mathbb{Q}}$ cut for P with respect to V).

Remark 216. *The condition $c^T \perp V'$ and the restricition to linear inequalities of the form*
$$\begin{pmatrix} c & (0^n)^T \end{pmatrix} (\cdot) \leq c_0$$
in Theorem 215 can be assumed without loss of generality (cf. Lemma 159).

Proof. (Theorem 215) We first show
$$\left((\operatorname{proj}_{\mathbb{R}^m} P) \cap P^{>}(c, c_0)\right) + V' = ((\operatorname{proj}_{\mathbb{R}^m} P) + V') \cap P^{>}(c, c_0). \tag{4.48}$$

For $\subseteq$: We have, using $c^T \perp V'$:
$$\left((\operatorname{proj}_{\mathbb{R}^m} P) \cap P^{>}(c, c_0)\right) + V' \subseteq ((\operatorname{proj}_{\mathbb{R}^m} P) + V') \cap \left(P^{>}(c, c_0) + V'\right) = ((\operatorname{proj}_{\mathbb{R}^m} P) + V') \cap P^{>}(c, c_0).$$

For $\supseteq$: Let $p \in \operatorname{proj}_{\mathbb{R}^m} P$ and $v \in V'$ be such that $c(p + v) > c_0$. Then, using $c^T \perp v \in V'$, we obtain
$$cp = c(p + v) > c_0.$$
So, $p \in P^{>}(c, c_0)$ and thus $p \in (\operatorname{proj}_{\mathbb{R}^m} P) \cap P^{>}(c, c_0)$. This shows
$$p + v \in \left((\operatorname{proj}_{\mathbb{R}^m} P) \cap P^{>}(c, c_0)\right) + V'.$$

Now for the main statements:

For 1:
$$\begin{aligned}
&\left\{x \in \operatorname{proj}_{\mathbb{R}^m} P : \forall i \in [k] : \left(w^i\right)^T x \in \mathbb{Z}\right\} \times \mathbb{R}^n \subseteq P^{\leq}\left(\begin{pmatrix} c & (0^n)^T \end{pmatrix}, c_0\right) \\
\Leftrightarrow &\left\{x \in (\operatorname{proj}_{\mathbb{R}^m} P) \cap P^{>}(c, c_0) : \forall i \in [k] : \left(w^i\right)^T x \in \mathbb{Z}\right\} = \emptyset \\
\Leftrightarrow &\left(\left((\operatorname{proj}_{\mathbb{R}^m} P) \cap P^{>}(c, c_0)\right) + V'\right)_I = \emptyset && \text{(by Lemma 214)} \\
\Leftrightarrow &\left(((\operatorname{proj}_{\mathbb{R}^m} P) + V') \cap P^{>}(c, c_0)\right)_I = \emptyset && \text{(by (4.48))} \\
\Leftrightarrow &((\operatorname{proj}_{\mathbb{R}^m} P) + V') \cap \mathbb{Z}^m \subseteq P^{\leq}(c, c_0) \\
\Leftrightarrow &(P + (V' \times \mathbb{R}^n))_I \subseteq P^{\leq}\left(\begin{pmatrix} c & (0^n)^T \end{pmatrix}, c_0\right).
\end{aligned}$$

For 2: 2 is a consequence of 1 and
$$\begin{aligned}
&\left\{x \in \operatorname{proj}_{\mathbb{R}^m} P : \forall i \in [k] : \left(w'^i\right)^T x \in \mathbb{Z}\right\} \times \mathbb{R}^n \subseteq P^{\leq}\left(\begin{pmatrix} c & (0^n)^T \end{pmatrix}, c_0\right) \\
\Rightarrow &\left\{x \in \operatorname{proj}_{\mathbb{R}^m} P : \forall i \in [k] : \left(w^i\right)^T x \in \mathbb{Z}\right\} \times \mathbb{R}^n \subseteq P^{\leq}\left(\begin{pmatrix} c & (0^n)^T \end{pmatrix}, c_0\right).
\end{aligned}$$

□

4.9. Lineality spaces and affine subspaces

In the introduction of this chapter, we already gave an idea of the problem that we consider in this section. Let us repeat it here: let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ be arbitrary. Trivially, we always have

$$\mathrm{cl}_{(m+n)-\frac{1}{2},(\,\cdot\,)}(P) = P \cap \mathrm{cl}_{\overline{I}}(P),$$

and by section 4.7, we have

$$\mathrm{cl}_{m,(\,\cdot\,)}(P) = P \cap \mathrm{cl}_{\overline{I}}(P).$$

But now assume that we can impose some "vector space condition" on P:

- P or $\mathrm{proj}_{\mathbb{R}^m} P$ has a (w.l.o.g. non-trivial) lineality space,
- P or $\mathrm{proj}_{\mathbb{R}^m} P$ is contained in an (again w.l.o.g. non-trivial) affine subspace.

Can we then show

$$\mathrm{cl}_{(m+n-l)-\frac{1}{2},(\,\cdot\,)}(P) = P \cap \mathrm{cl}_{\overline{I}}(P)$$

or

$$\mathrm{cl}_{m-l,(\,\cdot\,)}(P) = P \cap \mathrm{cl}_{\overline{I}}(P),$$

respectively, where $l \in \mathbb{Z}_{\geq 1}$?

For the structure and results of this section: in section 4.9.1, we consider the situation that P or $\mathrm{proj}_{\mathbb{R}^m} P$ has a (w.l.o.g. non-trivial) lineality space:

- Let $L \leq \mathrm{lineal}\, P$ satisfy some appropriate rationality conditions (that depend on the type of $L_{k-\frac{1}{2}}$ cut that we consider). In Theorem 217, we show that under these circumstances, for $L_{k-\frac{1}{2}}$ cuts, we can restrict ourselves to the ones with respect to vector spaces that contain L. In particular, we have

  $$\mathrm{cl}_{L_{m+n-l-\frac{1}{2},(\,\cdot\,)}}(P) = P \cap \mathrm{cl}_{\overline{I}}(P),$$

  where $l := \dim L$.

- Let $L \leq \mathrm{lineal}\,(\mathrm{proj}_{\mathbb{R}^m} P)$ be a rational subspace. In Theorem 218, we show that under these circumstances, for L_k cuts, we can restrict ourselves to the ones with respect to vector spaces that contain $L \times \mathbb{R}^n$. In particular, we have

  $$\mathrm{cl}_{L_{m-l,\mathbb{Q}}}(P) = P \cap \mathrm{cl}_{\overline{I}}(P),$$

  where $l := \dim L$.

In section 4.9.2, Theorem 220, we consider the situation that some $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ is given, for which we "understand" how some of its $L_{k-\frac{1}{2}}$ closure or L_k closure looks like. This theorem tells us how we can find similar examples $\hat{P} \subseteq \mathbb{R}^{\hat{m}} \times \mathbb{R}^{\hat{n}}$, where $\hat{m} \geq m$ and $\hat{n} \geq n$, for which we also "understand" its $L_{k-\frac{1}{2}}$ closure or L_k closure. This becomes important in part VI, but since it is very related to the results in section 4.9.1, we prove it in a subsection of section 4.9.

In section 4.9.3, we consider the situation that P or $\mathrm{proj}_{\mathbb{R}^m} P$ is contained in an (w.l.o.g. non-trivial) affine subspace:

- Let P be contained in an affine translate of a rationally generated subspace of codimension l. In Theorem 222, we show that under these circumstances, we have

  $$\mathrm{cl}_{L_{\max(l,1)-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = P \cap \mathrm{cl}_{\overline{I}}(P).$$

- Let $\mathrm{proj}_{\mathbb{R}^m} P$ be contained in an affine translate of a rationally generated subspace of dimension l. In Theorem 223, we show that under these circumstances, we have

  $$\mathrm{cl}_{L_{\max(l,1),\mathbb{Q}}}(P) = P \cap \mathrm{cl}_{\overline{I}}(P).$$

4.9.1. Lineality spaces

4.9.1.1. $L_{k-\frac{1}{2}}$ cuts

Theorem 217. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be arbitrary, let $L \leq \mathbb{R}^m \times \mathbb{R}^n$ be a subspace that satisfies $L \leq \operatorname{lineal} P$ and let $V \leq \mathbb{R}^m \times \mathbb{R}^n$ be another subspace. Then*

$$(P + V)_I = (P + (V + L))_I \,. \tag{4.49}$$

So, if $L \leq \mathbb{R}^m \times \mathbb{R}^n$ is generated by vectors from

- $\mathbb{Q}^m \times \mathbb{Q}^n$,
- $\mathbb{Q}^m \times \mathbb{R}^n$ *or*
- $\mathbb{R}^m \times \mathbb{R}^n$,

respectively, for

- $L_{k-\frac{1}{2}, \mathbb{Q}^m \times \mathbb{Q}^n}$ *cuts,*
- $L_{k-\frac{1}{2}, \mathbb{Q}^m \times \mathbb{R}^n}$ *cuts or*
- $L_{k-\frac{1}{2}, \mathbb{R}^m \times \mathbb{R}^n}$ *cuts,*

respectively ($k \in \{0, \ldots, m+n\}$), we can restrict ourselves to those with respect to subspaces containing L.
Additionally, under this "generator condition for L", we have, using $l := \dim L$:

$$\begin{aligned} \operatorname{cl}_{L_{m+n-l-\frac{1}{2}, \mathbb{Q}\times\mathbb{Q}}}(P) &= P \cap \operatorname{cl}_{\overline{I}}(P)\,, \\ \operatorname{cl}_{L_{m+n-l-\frac{1}{2}, \mathbb{Q}\times\mathbb{R}}}(P) &= P \cap \operatorname{cl}_{\overline{I}}(P) \quad \textit{or} \\ \operatorname{cl}_{L_{m+n-l-\frac{1}{2}, \mathbb{R}\times\mathbb{R}}}(P) &= P \cap \operatorname{cl}_{\overline{I}}(P)\,, \end{aligned} \tag{4.50}$$

respectively.

Proof.

For (4.49): Since $P + L = P$, we have

$$(P + V)_I = ((P + L) + V)_I = (P + (V + L))_I \,.$$

For (4.50): By definition, we have $P + L = P$. Thus,

$$P \cap \operatorname{cl}_{\overline{I}}(P + L) = P \cap \operatorname{cl}_{\overline{I}}(P)\,.$$

Depending on the generators of L, we thus obtain

$$\begin{aligned} P \cap \operatorname{cl}_{\overline{I}}(P) \subseteq \operatorname{cl}_{L_{m+n-l-\frac{1}{2}, \mathbb{Q}\times\mathbb{Q}}}(P) &\subseteq P \cap \operatorname{cl}_{\overline{I}}(P + L) = P \cap \operatorname{cl}_{\overline{I}}(P)\,, \\ P \cap \operatorname{cl}_{\overline{I}}(P) \subseteq \operatorname{cl}_{L_{m+n-l-\frac{1}{2}, \mathbb{Q}\times\mathbb{R}}}(P) &\subseteq P \cap \operatorname{cl}_{\overline{I}}(P + L) = P \cap \operatorname{cl}_{\overline{I}}(P) \text{ or} \\ P \cap \operatorname{cl}_{\overline{I}}(P) \subseteq \operatorname{cl}_{L_{m+n-l-\frac{1}{2}, \mathbb{R}\times\mathbb{R}}}(P) &\subseteq P \cap \operatorname{cl}_{\overline{I}}(P + L) = P \cap \operatorname{cl}_{\overline{I}}(P)\,, \end{aligned}$$

respectively. □

4.9.1.2. L_k cuts

Theorem 218. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be arbitrary, let $L \leq \mathbb{R}^m$ be a rational subspace that satisfies $L \leq \operatorname{proj}_{\mathbb{R}^m}(\operatorname{lineal} P)$ and let $V \leq \mathbb{R}^m$ be another subspace. Then*

$$\operatorname{conv}(P \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + (V \times \mathbb{R}^n))) = \operatorname{conv}(P \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + ((V + L) \times \mathbb{R}^n)))\,, \tag{4.51}$$

i.e. for L_k cuts for P, we can restrict ourselves to subspaces containing $L \times \mathbb{R}^n$.
Additionally, we have

$$P \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + (L \times \mathbb{R}^n)) \subseteq \operatorname{cl}_I(P)\,, \tag{4.52}$$

which implies

$$\mathrm{cl}_{L_{m-l,\mathbb{Q}}}(P) = P \cap \mathrm{cl}_{\overline{I}}(P), \tag{4.53}$$

where $l := \dim L$.

Proof.

For (4.51)**:**

$$\mathrm{conv}\left(P \cap \left((\mathbb{Z}^m \times \mathbb{R}^n) + (V \times \mathbb{R}^n)\right)\right) \subseteq \mathrm{conv}\left(P \cap \left((\mathbb{Z}^m \times \mathbb{R}^n) + ((V+L) \times \mathbb{R}^n)\right)\right)$$

is obvious. So for

$$\mathrm{conv}\left(P \cap \left((\mathbb{Z}^m \times \mathbb{R}^n) + ((V+L) \times \mathbb{R}^n)\right)\right) \subseteq \mathrm{conv}\left(P \cap \left((\mathbb{Z}^m \times \mathbb{R}^n) + (V \times \mathbb{R}^n)\right)\right):$$

We show

$$(\mathrm{proj}_{\mathbb{R}^m} P) \cap (\mathbb{Z}^m + (V+L)) \subseteq \mathrm{conv}\left((\mathrm{proj}_{\mathbb{R}^m} P) \cap (\mathbb{Z}^m + V)\right)$$

by a construction that is similar to the one that we used in the proof of Theorem 196.

Let

$$p = z + v + w,$$

where $p \in \mathrm{proj}_{\mathbb{R}^m} P$, $z \in \mathbb{Z}^m$, $v \in V$ and $w \in L$. Let $\{w^1, \dots, w^l\} \subseteq \mathbb{Z}^m$ be a basis of L, where, of course, $l := \dim L$. So there exists a $\lambda \in \mathbb{R}^l$ such that $w = \sum_{i=1}^{l} \lambda_i w^i$.

For $I \in \{0,1\}^l$, consider

$$q^I := \underbrace{v + \underbrace{z + \sum_{i=1}^{l} (\lfloor \lambda_i \rfloor + I_i) w^i}_{\in \mathbb{Z}^m}}_{\in \mathbb{Z}^m + V} = \underbrace{p - \sum_{i=1}^{l} \lambda_i \cdot w^i + \sum_{i=1}^{l} (\lfloor \lambda_i \rfloor + I_i) w^i}_{\in \mathrm{proj}_{\mathbb{R}^m} P \text{ (since } L \leq \mathrm{lineal}\, P)} \in (\mathrm{proj}_{\mathbb{R}^m} P) \cap (\mathbb{Z}^m + V). \tag{4.54}$$

For $i \in [l]$, let

$$\lambda'_i := \lambda_i - \lfloor \lambda_i \rfloor \in [0,1).$$

Define $\mu \in [0,1]^{\left(\{0,1\}^l\right)}$ via

$$\mu_I := \prod_{i=1}^{l} \begin{cases} 1 - \lambda'_i & \text{if } I_i = 0, \\ \lambda'_i & \text{if } I_i = 1. \end{cases}$$

Like in the proof of Theorem 196, we have

$$\sum_{I \in \{0,1\}^l} \mu_I = 1, \tag{4.55}$$

$$\forall i \in [l]: \sum_{\substack{I \in \{0,1\}^l: \\ I_i = 1}} \mu_I = \lambda'_i. \tag{4.56}$$

So,

$$\begin{aligned}
\sum_{I \in \{0,1\}^l} \mu_I q^I &= z + v + \sum_{I \in \{0,1\}^l} \mu_I \sum_{i=1}^{l} (\lfloor \lambda_i \rfloor + I_i) w^i && \text{(by (4.54) and (4.55))} \\
&= z + v + \sum_{i=1}^{l} \left(\lfloor \lambda_i \rfloor + \sum_{I \in \{0,1\}^l} \mu_I I_i \right) w^i && \text{(by (4.55))} \\
&= z + v + \sum_{i=1}^{l} \left(\lfloor \lambda_i \rfloor + \sum_{\substack{I \in \{0,1\}^l: \\ I_i = 1}} \mu_I \right) w^i
\end{aligned}$$

$$= z + v + \sum_{i=1}^{l} \left(\lfloor \lambda_i \rfloor + \lambda'_i\right) w^i \qquad \text{(by (4.56))}$$
$$= z + v + \sum_{i=1}^{l} \lambda_i w^i$$
$$= p.$$

Thus, $p \in \operatorname{conv}\left(\left(\operatorname{proj}_{\mathbb{R}^m} P\right) \cap \left(\mathbb{Z}^m + V\right)\right)$.

For (4.52) **and** (4.53)**:** Since L is a rational subspace and satisfies $L \leq \operatorname{proj}_{\mathbb{R}^m}(\operatorname{lineal} P)$, there exists a subspace $\hat{L} \leq \mathbb{R}^m \times \mathbb{R}^n$ having

- $\hat{L} \leq \operatorname{lineal} P$,
- $\operatorname{proj}_{\mathbb{R}^m} \hat{L} = L$ and
- $\hat{L}$ has generators from $\mathbb{Q}^m \times \mathbb{R}^n$.

We have

$$\begin{aligned} P \cap \left(\left(\mathbb{Z}^m \times \mathbb{R}^n\right) + \left(L \times \mathbb{R}^n\right)\right) &= P \cap \left(\left(\mathbb{Z}^m \times \mathbb{R}^n\right) + \hat{L}\right) \\ &\subseteq \operatorname{cl}_I\left(P + \hat{L}\right) \qquad \text{(by Theorem 196)} \\ &= \operatorname{cl}_I(P). \end{aligned}$$

Thus,

$$\begin{aligned} P \cap \operatorname{cl}_{\overline{I}}(P) &\subseteq \operatorname{cl}_{L_{m-l,\mathbb{Q}}}(P) \\ &\subseteq P \cap \overline{\operatorname{conv}}\left(P \cap \left(\left(\mathbb{Z}^m \times \mathbb{R}^n\right) + \left(L \times \mathbb{R}^n\right)\right)\right) \\ &= P \cap \operatorname{cl}_{\overline{I}}(P). \end{aligned}$$

□

Remark/Problem 219. *In contrast to* (4.50) *in Theorem 217, equation* (4.53) *in Theorem 218 does not hold if L is not a rational subspace. For this, consider* $P^{115} \subseteq \mathbb{R}^2$. *Clearly,*

$$\dim\left(\operatorname{lineal} P^{115}\right) = 1,$$

but, by equation (4.13) *in Theorem 187, we have* $\operatorname{cl}_{L_{1,\mathbb{R}}}\left(P^{115}\right) \supsetneq \operatorname{cl}_I\left(P^{115}\right)$. *We want to state the problem to decide whether* (4.51) *also holds if V is not a rational subspace as a research question.*

4.9.2. Increasing the dimension

As explained in the introduction of this section, we now prove a theorem (Theorem 220) about how we can "lift" some $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$, for which we "understand" how some of its $L_{k-\frac{1}{2}}$ closure or L_k closure looks like, into higher dimensions, i.e. find a $\hat{P} \subseteq \mathbb{R}^{\hat{m}} \times \mathbb{R}^{\hat{n}}$, where $\hat{m} \geq m$ and $\hat{n} \geq n$, for which we also "understand" its $L_{k-\frac{1}{2}}$ closure or L_k closure. This result becomes important in part VI.

Theorem 220. *Let* $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ *be arbitrary, let* $k \in \{0, \ldots, m+n\}$ *and let* $m', n' \in \mathbb{Z}_{\geq 0}$. *Then*

$$\operatorname{cl}_{L_{k-\frac{1}{2},\substack{\mathbb{Q}\times\mathbb{Q}\\ \mathbb{Q}\times\mathbb{R}\\ \mathbb{R}\times\mathbb{R}}}}\left(P \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}\right) = \operatorname{cl}_{L_{k-\frac{1}{2},\substack{\mathbb{Q}\times\mathbb{Q}\\ \mathbb{Q}\times\mathbb{R}\\ \mathbb{R}\times\mathbb{R}}}}(P) \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}, \tag{4.57}$$

$$\operatorname{cl}_{L_{k,\substack{\mathbb{Q}\\ \mathbb{R}}}}\left(P \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}\right) = \operatorname{cl}_{L_{k,\substack{\mathbb{Q}\\ \mathbb{R}}}}(P) \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}. \tag{4.58}$$

All of the closures are with respect to the lattice $\mathbb{Z}^m \times \mathbb{R}^n \times \mathbb{Z}^{m'} \times \mathbb{R}^{n'}$.

Remark 221. *One could also prove the "⊇" inclusions in Theorem 220 as consequences of Theorem 217 and Theorem 218. While this would be very much in the spirit of the preceding section 4.9.1 and perhaps more* *elegant, it probably would not make the proof much shorter. So, we decided to keep the self-contained proof of Theorem 220.*

Proof. (Theorem 220) In the following proof, we replace some required conditions for V in the intersections by "..." for better readability. For the meaning of this "..." cf. Lemma 166 and Lemma 183.

For $\subseteq$: Let $V \leq \mathbb{R}^m \times \mathbb{R}^n$ be a vector space of codimension k for which there exists a generating system that satisfies the imposed rationality conditions. We have

$$\begin{aligned}
&\operatorname{conv}\left(\left(\left(P \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}\right) + \left(V \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}\right)\right) \cap \left(\mathbb{Z}^m \times \mathbb{R}^n \times \mathbb{Z}^{m'} \times \mathbb{R}^{n'}\right)\right) \\
&= \operatorname{conv}\left(\left(\left(P \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}\right) + \left(V \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}\right)\right) \cap \left(\mathbb{Z}^m \times \mathbb{R}^n \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}\right)\right) \\
&= \operatorname{conv}\left((P + V) \cap (\mathbb{Z}^m \times \mathbb{R}^n)\right) \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}. \qquad (4.59)
\end{aligned}$$

For (4.57):

$$\begin{aligned}
&\operatorname{cl}_{L_{k-\frac{1}{2}},(\cdot)}\left(P \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}\right) \\
&= \left(P \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}\right) \\
&\quad \cap \bigcap_{\substack{V \leq \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}: \\ \operatorname{codim} V = k, \ldots}} \overline{\operatorname{conv}}\left(\left(\left(P \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}\right) + V\right) \cap \left(\mathbb{Z}^m \times \mathbb{R}^n \times \mathbb{Z}^{m'} \times \mathbb{R}^{n'}\right)\right) \\
&\subseteq \left(P \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}\right) \\
&\quad \cap \bigcap_{\substack{V \leq \mathbb{R}^m \times \mathbb{R}^n: \\ \operatorname{codim} V = k, \ldots}} \overline{\operatorname{conv}}\left(\left(\left(P \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}\right) + \left(V \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}\right)\right) \cap \left(\mathbb{Z}^m \times \mathbb{R}^n \times \mathbb{Z}^{m'} \times \mathbb{R}^{n'}\right)\right) \\
&= \left(P \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}\right) \cap \bigcap_{\substack{V \leq \mathbb{R}^m \times \mathbb{R}^n: \\ \operatorname{codim} V = k, \ldots}} \left(\overline{\operatorname{conv}}\left((P + V) \cap (\mathbb{Z}^m \times \mathbb{R}^n)\right) \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}\right) \qquad \text{(by (4.59))} \\
&= \left(P \cap \bigcap_{\substack{V \leq \mathbb{R}^m \times \mathbb{R}^n: \\ \operatorname{codim} V = k, \ldots}} \overline{\operatorname{conv}}\left(((P + V)) \cap \left(\mathbb{Z}^m \times \mathbb{R}^n \times \mathbb{Z}^{m'} \times \mathbb{R}^{n'}\right)\right)\right) \times \mathbb{R}^{m'} \times \mathbb{R}^{n'} \\
&= \operatorname{cl}_{L_{k-\frac{1}{2}},(\cdot)}(P) \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}.
\end{aligned}$$

For (4.58):

$$\begin{aligned}
&\operatorname{cl}_{L_{k},(\cdot)}\left(P \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}\right) \\
&= \left(P \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}\right) \cap \bigcap_{\substack{V \leq \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}: \\ \operatorname{codim} V = k, \ldots}} \overline{\operatorname{conv}}\left(\left(P \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}\right) \cap \left(\left(\mathbb{Z}^m \times \mathbb{R}^n \times \mathbb{Z}^{m'} \times \mathbb{R}^{n'}\right) + V\right)\right) \\
&\subseteq \left(P \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}\right) \\
&\quad \cap \bigcap_{\substack{V \leq \mathbb{R}^m \times \mathbb{R}^n: \\ \operatorname{codim} V = k, \ldots}} \overline{\operatorname{conv}}\left(\left(P \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}\right) \cap \left(\left(\mathbb{Z}^m \times \mathbb{R}^n \times \mathbb{Z}^{m'} \times \mathbb{R}^{n'}\right) + \left(V \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}\right)\right)\right) \\
&= \left(P \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}\right) \cap \bigcap_{\substack{V \leq \mathbb{R}^m \times \mathbb{R}^n: \\ \operatorname{codim} V = k, \ldots}} \left(\overline{\operatorname{conv}}\left(P \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + V)\right) \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \left(P \cap \bigcap_{\substack{V \leq \mathbb{R}^m \times \mathbb{R}^n: \\ \operatorname{codim} V = k, \ldots}} \overline{\operatorname{conv}} \left(P \cap \left(\left(\mathbb{Z}^m \times \mathbb{R}^n \right) + V \right) \right) \right) \times \mathbb{R}^{m'} \times \mathbb{R}^{n'} \\
&= \operatorname{cl}_{L_{k,(\,\cdot\,)}} (P) \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}.
\end{aligned}
$$

For $\supseteq$: Let $V \leq \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}$ be a vector space of codimension k that is generated by vectors $v^1, \ldots, v^{m+n+m'+n'-k}$ that satisfy the rationality conditions that are imposed on the generating system of V. Since $\dim V = m + n + m' + n' - k$, by the Steinitz exchange lemma, there exist $i_1, \ldots, i_{m+n-k}$ such that

$$
\dim \left(\left(0^m \times 0^n \times \mathbb{R}^{m'} \times \mathbb{R}^{n'} \right) + \operatorname{lin} \left\{ v^{i_1}, \ldots, v^{i_{m+n-k}} \right\} \right) = m + n + m' + n' - k.
$$

Define

$$
V'(V) := \operatorname{proj}_{\mathbb{R}^m \times \mathbb{R}^n} \left(\operatorname{lin} \left\{ v^{i_1}, \ldots, v^{i_{m+n-k}} \right\} \right) \leq \mathbb{R}^m \times \mathbb{R}^n.
$$

We clearly have

$$
\dim \left(V'(V) \times \mathbb{R}^{m'} \times \mathbb{R}^{n'} \right) = m + n + m' + n' - k \tag{4.60}
$$

and

$$
\begin{aligned}
V + \left(0^m \times 0^n \times \mathbb{R}^{m'} \times \mathbb{R}^{n'} \right) &\supseteq \left(V'(V) \times 0^{m'} \times 0^{n'} \right) + \left(0^m \times 0^n \times \mathbb{R}^{m'} \times \mathbb{R}^{n'} \right) \\
&= V'(V) \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}.
\end{aligned} \tag{4.61}
$$

For (4.57):

$$
\begin{aligned}
&\operatorname{cl}_{L_{k-\frac{1}{2},(\,\cdot\,)}} (P) \times \mathbb{R}^{m'} \times \mathbb{R}^{n'} \\
&= \left(P \cap \bigcap_{\substack{V \leq \mathbb{R}^m \times \mathbb{R}^n: \\ \operatorname{codim} V = k, \ldots}} \overline{\operatorname{conv}} \left(\left(P + V \right) \cap \left(\mathbb{Z}^m \times \mathbb{R}^n \right) \right) \right) \times \mathbb{R}^{m'} \times \mathbb{R}^{n'} \\
&= \left(P \times \mathbb{R}^{m'} \times \mathbb{R}^{n'} \right) \cap \bigcap_{\substack{V \leq \mathbb{R}^m \times \mathbb{R}^n: \\ \operatorname{codim} V = k, \ldots}} \overline{\operatorname{conv}} \left(\left(\left(P \times \mathbb{R}^{m'} \times \mathbb{R}^{n'} \right) \right.\right.\right. \\
&\quad \left.\left. + \left(V \times \mathbb{R}^{m'} \times \mathbb{R}^{n'} \right) \right) \cap \left(\mathbb{Z}^m \times \mathbb{R}^n \times \mathbb{Z}^{m'} \times \mathbb{R}^{n'} \right) \right) \\
&\subseteq \left(P \times \mathbb{R}^{m'} \times \mathbb{R}^{n'} \right) \cap \bigcap_{\substack{V \leq \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}: \\ \operatorname{codim} V = k, \ldots}} \overline{\operatorname{conv}} \left(\left(\left(P \times \mathbb{R}^{m'} \times \mathbb{R}^{n'} \right) \right.\right.\right. \\
&\quad \left.\left. + \left(V'(V) \times \mathbb{R}^{m'} \times \mathbb{R}^{n'} \right) \right) \cap \left(\mathbb{Z}^m \times \mathbb{R}^n \times \mathbb{Z}^{m'} \times \mathbb{R}^{n'} \right) \right) && \text{(by (4.60))} \\
&\subseteq \left(P \times \mathbb{R}^{m'} \times \mathbb{R}^{n'} \right) \cap \bigcap_{\substack{V \leq \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}: \\ \operatorname{codim} V = k, \ldots}} \overline{\operatorname{conv}} \left(\left(\left(P \times \mathbb{R}^{m'} \times \mathbb{R}^{n'} \right) \right.\right.\right. \\
&\quad \left.\left. + \left(V + \left(0^m \times 0^n \times \mathbb{R}^{m'} \times \mathbb{R}^{n'} \right) \right) \right) \cap \left(\mathbb{Z}^m \times \mathbb{R}^n \times \mathbb{Z}^{m'} \times \mathbb{R}^{n'} \right) \right) && \text{(by (4.61))} \\
&= \left(P \times \mathbb{R}^{m'} \times \mathbb{R}^{n'} \right) \cap \bigcap_{\substack{V \leq \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}: \\ \operatorname{codim} V = k, \ldots}} \overline{\operatorname{conv}} \left(\left(\left(P \times \mathbb{R}^{m'} \times \mathbb{R}^{n'} \right) + V \right) \right. \\
&\quad \left. \cap \left(\mathbb{Z}^m \times \mathbb{R}^n \times \mathbb{Z}^{m'} \times \mathbb{R}^{n'} \right) \right) \\
&= \operatorname{cl}_{L_{k-\frac{1}{2},(\,\cdot\,)}} \left(P \times \mathbb{R}^{m'} \times \mathbb{R}^{n'} \right).
\end{aligned}
$$

For (4.58)**:**

$$
\begin{aligned}
&\mathrm{cl}_{L_{k,(\cdot)}}(P) \times \mathbb{R}^{m'} \times \mathbb{R}^{n'} \\
=&\left(P \cap \bigcap_{\substack{V \leq \mathbb{R}^m \times \mathbb{R}^n: \\ \operatorname{codim} V = k,\dots}} \overline{\operatorname{conv}}\left(P \cap \left(\left(\mathbb{Z}^m \times \mathbb{R}^n\right) + V\right)\right)\right) \times \mathbb{R}^{m'} \times \mathbb{R}^{n'} \\
=&\left(P \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}\right) \cap \bigcap_{\substack{V \leq \mathbb{R}^m \times \mathbb{R}^n: \\ \operatorname{codim} V = k,\dots}} \overline{\operatorname{conv}}\left(\left(P \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}\right)\right. \\
&\left.\cap\left(\left(\mathbb{Z}^m \times \mathbb{R}^n \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}\right) + \left(V \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}\right)\right)\right) \\
\subseteq&\left(P \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}\right) \cap \bigcap_{\substack{V \leq \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}: \\ \operatorname{codim} V = k,\dots}} \overline{\operatorname{conv}}\left(\left(P \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}\right)\right. \\
&\left.\cap\left(\left(\mathbb{Z}^m \times \mathbb{R}^n \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}\right) + \left(V'(V) \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}\right)\right)\right) && \text{(by (4.60))} \\
\subseteq&\left(P \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}\right) \cap \bigcap_{\substack{V \leq \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}: \\ \operatorname{codim} V = k,\dots}} \overline{\operatorname{conv}}\left(\left(P \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}\right)\right. \\
&\left.\cap\left(\left(\mathbb{Z}^m \times \mathbb{R}^n \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}\right) + V\right)\right) && \text{(by (4.61))} \\
=&\left(P \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}\right) \cap \bigcap_{\substack{V \leq \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}: \\ \operatorname{codim} V = k,\dots}} \overline{\operatorname{conv}}\left(\left(P \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}\right)\right. \\
&\left.\cap\left(\left(\mathbb{Z}^m \times \mathbb{R}^n \times \mathbb{Z}^{m'} \times \mathbb{R}^{n'}\right) + V\right)\right) \\
=&\,\mathrm{cl}_{L_{k,(\cdot)}}\left(P \times \mathbb{R}^{m'} \times \mathbb{R}^{n'}\right).
\end{aligned}
$$

□

4.9.3. Affine subspaces

4.9.3.1. $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts

Theorem 222. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ be arbitrary. Let there exist an affine subspace*

$$q + L \subseteq \mathbb{R}^m \times \mathbb{R}^n,$$

where $L \leq \mathbb{R}^m \times \mathbb{R}^n$ is a rational subspace, $\dim L =: l$ and $q \in \mathbb{R}^m$, such that

$$P \subseteq q + L.$$

Let $(\mathbb{Z}^m \times \mathbb{R}^n) \cap L$ be a lattice of signature $(l-s, s)$, let $b^1, \dots, b^{l-s}, c^1, \dots, c^s \in \mathbb{Z}^m \times \mathbb{Q}^n$ be vectors that form a mixed lattice basis of $(\mathbb{Z}^m \times \mathbb{R}^n) \cap L$, i.e.

$$(\mathbb{Z}^m \times \mathbb{R}^n) \cap L = \begin{pmatrix} b^1 & \cdots & b^{l-s} & c^1 & \cdots & b^s \end{pmatrix} \begin{pmatrix} \mathbb{Z}^{l-s} \\ \mathbb{R}^s \end{pmatrix},$$

let $b^{l-s+1}, \dots, b^m, c^{s+1}, \dots, c^n \in \mathbb{Z}^m \times \mathbb{R}^n$ be vectors that complete $b^1, \dots, b^{l-s}, c^1, \dots, c^s \in \mathbb{Z}^m \times \mathbb{Q}^n$ to a mixed lattice basis of $\mathbb{Z}^m \times \mathbb{R}^n$, i.e.

$$\mathbb{Z}^m \times \mathbb{R}^n = \begin{pmatrix} b^1 & \cdots & b^m & c^1 & \cdots & c^n \end{pmatrix} \begin{pmatrix} \mathbb{Z}^m \\ \mathbb{R}^n \end{pmatrix}$$

and let

$$q = \sum_{i=1}^m \eta_i^q b^i + \sum_{i=1}^n \vartheta_i^q c^i,$$

where $\eta^q \in \mathbb{R}^m$ and $\vartheta^q \in \mathbb{R}^n$ (they exist because $\left\{b^1, \ldots, b^m, c^1, \ldots, c^n\right\}$ is a basis of $\mathbb{R}^m \times \mathbb{R}^n$). Then:

- ***Case 1:*** $\exists i^* \in \{l-s+1, \ldots, m\} : \eta^q_{i^*} \notin \mathbb{Z}$.
  Define
  $$\hat{L} := \operatorname{lin}\left\{b^1, \ldots, b^{i^*-1}, b^{i^*+1}, \ldots, b^m, c^1, \ldots, c^n\right\}.$$
  Then
  $$\left(P + \hat{L}\right)_I = \emptyset, \tag{4.62}$$
  which immediately implies
  $$\operatorname{cl}_{L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \emptyset.$$

- ***Case 2:*** $\forall i \in \{l-s+1, \ldots, m\} : \eta^q_i \in \mathbb{Z}$.
  Define
  $$\tilde{L} := \operatorname{lin}\left\{b^{l-s+1}, \ldots, b^m, c^{s+1}, \ldots, c^n\right\}.$$
  Then
  $$P \cap \operatorname{cl}_{\overline{I}}\left(P + \tilde{L}\right) = P \cap \operatorname{cl}_{\overline{I}}(P); \tag{4.63}$$
  thus,
  $$\operatorname{cl}_{L_{l-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = P \cap \operatorname{cl}_{\overline{I}}(P).$$

In any case, we have
$$\operatorname{cl}_{L_{\max(l,1)-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = P \cap \operatorname{cl}_{\overline{I}}(P).$$

Proof.

For (4.62)**:** We want to show
$$\left(P + \hat{L}\right)_I = \emptyset.$$

Let $z \in \left(P + \hat{L}\right)_I$. Then, using $p \in P \subseteq q + L$, we obtain

$$z = \underbrace{\sum_{i=1}^m \eta^q_i b^i + \sum_{i=1}^n \vartheta^q_i c^i}_{=q} + \underbrace{\sum_{i=1}^{l-s} \eta^{w^L}_i b^i + \sum_{i=1}^{s} \vartheta^{w^L}_i c^i}_{\in L} + \underbrace{\sum_{i\in[m]\setminus\{i^*\}} \eta^w_i b^i + \sum_{i=1}^n \vartheta^w_i c^i}_{\in \hat{L}},$$

where

- $\eta^{w^L} \in \mathbb{R}^{l-s}$,
- $\vartheta^{w^L} \in \mathbb{R}^s$,
- $\eta^w \in \mathbb{R}^{[m]\setminus\{i^*\}}$ and
- $\vartheta^w \in \mathbb{R}^n$.

Thus, there exist $\eta^z \in \mathbb{R}^{[m]\setminus\{i^*\}}$ and $\vartheta^z \in \mathbb{R}^n$ such that

$$z = \eta^q_{i^*} b^{i^*} + \sum_{i\in[m]\setminus\{i^*\}} \eta^z_i b^i + \sum_{i=1}^n \vartheta^z c^i.$$

By definition, $\left(b^1, \ldots, b^m, c^1, \ldots, c^n\right)$ is a mixed lattice basis of $\mathbb{Z}^m \times \mathbb{R}^n$. On the other hand, $\eta^q_{i^*} \notin \mathbb{Z}$. Thus, $z \notin \mathbb{Z}^m \times \mathbb{R}^n$, which is a contradiction.

For (4.63)**:** Clearly
$$P \cap \operatorname{cl}_{\overline{I}}\left(P + \tilde{L}\right) \supseteq P \cap \operatorname{cl}_{\overline{I}}(P);$$
so, we only have to show
$$P \cap \operatorname{cl}_{\overline{I}}\left(P + \tilde{L}\right) \subseteq P \cap \operatorname{cl}_{\overline{I}}(P).$$

Let $c(\cdot) \le c_0$ ($c \in (\mathbb{R}^m \times \mathbb{R}^n)^T$ and $c_0 \in \mathbb{R}$) be a valid inequality for P_I and let $p^* \in P$ be such that $cp^* > c_0$. We show that there exists a valid inequality $c'(\cdot) \le c_0'$ for $\left(P + \tilde{L}\right)_I$ such that $c'p^* > c_0'$.

Clearly, $\mathbb{R}^m \times \mathbb{R}^n = L \oplus \tilde{L}$, i.e. every vector in $\mathbb{R}^m \times \mathbb{R}^n$ can uniquely be decomposed into vectors of these subspaces. Thus, by Lemma 34, there exists a decomposition $\mathbb{R}^m \times \mathbb{R}^n = L^\perp \oplus \tilde{L}^\perp$. Let

$$c = c^{L^\perp} + c^{\tilde{L}^\perp},$$

where $\left(c^{L^\perp}\right)^T \in L^\perp$ and $\left(c^{\tilde{L}^\perp}\right)^T \in \tilde{L}^\perp$. Define

$$c' := c^{\tilde{L}^\perp}, \qquad\qquad c_0' := c_0 - c^{L^\perp} q.$$

We claim that

1. $c'(\cdot) \le c_0'$ is valid for $\left(P + \tilde{L}\right)_I$ and
2. $c'p^* > c_0'$.

For 1: Let $z = p + w^{\tilde{L}} \in \left(P + \tilde{L}\right)_I$, where $p \in P$ and $w^{\tilde{L}} \in \tilde{L}$. We claim that this implies $p \in P_I$. Since $p \in P \subseteq q + L$, there exists a $w^L \in L$ such that $p = q + w^L$. Let

$$\begin{aligned} p &= q + \sum_{i=1}^{l-s} \eta_i^{w^L} b^i + \sum_{i=1}^{s} \vartheta_i^{w^L} c^i \\ &= \sum_{i=1}^{m} \eta_i^q b^i + \sum_{i=1}^{n} \vartheta_i^q c^i + \sum_{i=1}^{l-s} \eta_i^{w^L} b^i + \sum_{i=1}^{s} \vartheta_i^{w^L} c^i \\ &= \sum_{i=1}^{l-s} \left(\eta_i^q + \eta_i^{w^L}\right) b^i + \sum_{i=l-s+1}^{m} \eta_i^q b^i + \sum_{i=1}^{s} \left(\vartheta_i^q + \vartheta_i^{w^L}\right) c^i + \sum_{i=s+1}^{n} \vartheta_i^q c^i, \end{aligned} \tag{4.64}$$

where

- $\eta^{w^L} \in \mathbb{R}^{l-s}$ and
- $\vartheta^{w^L} \in \mathbb{R}^s$.

Additionally, there exist $\eta^{w^{\tilde{L}}} \in \mathbb{R}^{(s+1,\ldots,l)}$ and $\vartheta^{w^{\tilde{L}}} \in \mathbb{R}^{(s+1,\ldots,l)}$ such that

$$w^{\tilde{L}} = \sum_{i=l-s+1}^{m} \eta_i^{w^{\tilde{L}}} b^i + \sum_{i=s+1}^{n} \vartheta_i^{w^{\tilde{L}}} c^i.$$

Thus,

$$\begin{aligned} z &= p + w^{\tilde{L}} \\ &= \sum_{i=1}^{l-s} \left(\eta_i^q + \eta_i^{w^L}\right) b^i + \sum_{i=l-s+1}^{m} \left(\eta_i^q + \eta_i^{w^{\tilde{L}}}\right) b^i + \sum_{i=1}^{s} \left(\vartheta_i^q + \vartheta_i^{w^L}\right) c^i + \sum_{i=s+1}^{n} \left(\vartheta_i^q + \vartheta_i^{w^{\tilde{L}}}\right) c^i. \end{aligned}$$

Considering that $z \in \mathbb{Z}^m \times \mathbb{R}^n$ and $\left(b^1, \ldots, b^m, c^1, \ldots, c^n\right)$ is a mixed lattice basis of $\mathbb{Z}^m \times \mathbb{R}^n$, we obtain

$$\forall i \in \{1, \ldots, l-s\} : \eta_i^q + \eta_i^{w^L} \in \mathbb{Z}.$$

Thus, (4.64) and

$$\forall i \in \{l-s+1, \ldots, m\} : \underbrace{\eta_i^q}_{\in \mathbb{Z}} + \eta_i^{w^{\tilde{L}}} \in \mathbb{Z}$$

imply $p \in \mathbb{Z}^m \times \mathbb{R}^n$.

Now for the proof of statement 1:

$$\begin{aligned}
c'z &= c^{\tilde{L}^\perp}\left(p + w^{\tilde{L}}\right) \\
&= cp - c^{L^\perp}p + c^{\tilde{L}^\perp}w^{\tilde{L}} \\
&= cp - c^{L^\perp}p && \left(c^{\tilde{L}^\perp} \in \left(\tilde{L}^\perp\right)^T,\ w^{\tilde{L}} \in \tilde{L}\right) \\
&= cp - c^{L^\perp}(q + w) \\
&= cp - c^{L^\perp}q && \left(c^{L^\perp} \in \left(L^\perp\right)^T,\ w \in L\right) \\
&\le c_0 - c^{L^\perp}q && (p \in P_I) \\
&= c_0'.
\end{aligned}$$

For 2: Since $P \subseteq q + L$, there exists a $w^* \in L$ such that $p^* = q + w^*$. So, we have

$$\begin{aligned}
c'p^* &= cp^* - c^{L^\perp}p^* \\
&= cp^* - c^{L^\perp}(q + w^*) \\
&= cp^* - c^{L^\perp}q && \left(c^{L^\perp} \in \left(L^\perp\right)^T,\ w^* \in L\right) \\
&> c_0 - c^{L^\perp}q \\
&= c_0'.
\end{aligned}$$

□

4.9.3.2. $L_{k,\mathbb{Q}}$ cuts

Theorem 223. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\ge 0})$ be arbitrary and let there exist an affine subspace*

$$q + L \subseteq \mathbb{R}^m,$$

where $L \le \mathbb{R}^m$ is a rational subspace, $\dim L =: l$ and $q \in \mathbb{R}^m$, such that

$$\operatorname{proj}_{\mathbb{R}^m} P \subseteq q + L.$$

Let $\{b^1, \ldots, b^l\} \subseteq \mathbb{Z}^m$ be a lattice basis of $\mathbb{Z}^m \cap L$. Let $b^{l+1}, \ldots, b^m \in \mathbb{Z}^m$ be such that $\{b^1, \ldots, b^m\}$ is a lattice basis of $\mathbb{Z}^m$ (these exist because of Theorem 95) and let

$$q = \sum_{i=1}^m \eta_i^q b^i,$$

where $\eta^q \in \mathbb{R}^m$ (it exists because $\{b^1, \ldots, b^m\}$ is a basis of $\mathbb{R}^m$). Then:

- ***Case 1:** $\exists i^* \in \{l+1, \ldots, m\} : \eta_{i^*}^q \notin \mathbb{Z}$.*
  Define
  $$\hat{L} := \operatorname{lin}\left\{b^1, \ldots, b^{i^*-1}, b^{i^*+1}, \ldots, b^m\right\}.$$
  Then
  $$\left(\operatorname{proj}_{\mathbb{R}^m} P\right) \cap \left(\mathbb{Z}^m + \hat{L}\right) = \emptyset, \tag{4.65}$$
  which immediately implies
  $$P \cap \left(\left(\mathbb{Z}^m \times \mathbb{R}^n\right) + \left(\hat{L} \times \mathbb{R}^n\right)\right) = \emptyset;$$
  thus,
  $$\operatorname{cl}_{L_{1,\mathbb{Q}}}(P) = \emptyset.$$
- ***Case 2:** $\forall i \in \{l+1, \ldots, m\} : \eta_i^q \in \mathbb{Z}$.*

Define

$$\tilde{L} := \operatorname{lin}\left\{b^{l+1}, \ldots, b^m\right\}.$$

Then

$$(\operatorname{proj}_{\mathbb{R}^m} P) \cap \left(\mathbb{Z}^m + \tilde{L}\right) = (\operatorname{proj}_{\mathbb{R}^m} P) \cap \mathbb{Z}^m; \tag{4.66}$$

thus,

$$P \cap \left((\mathbb{Z}^m \times \mathbb{R}^n) + \left(\tilde{L} \times \mathbb{R}^n\right)\right) = P \cap (\mathbb{Z}^m \times \mathbb{R}^n), \tag{4.67}$$

which implies

$$\operatorname{cl}_{L_{l,\mathbb{Q}}}(P) = P \cap \operatorname{cl}_{\overline{I}}(P).$$

In any case, we have

$$\operatorname{cl}_{L_{\max(l,1),\mathbb{Q}}}(P) = P \cap \operatorname{cl}_{\overline{I}}(P).$$

Proof.

For (4.65)**:** We want to show

$$(\operatorname{proj}_{\mathbb{R}^m} P) \cap \left(\mathbb{Z}^m + \hat{L}\right) = \emptyset.$$

Let $p \in (\operatorname{proj}_{\mathbb{R}^m} P) \cap \left(\mathbb{Z}^m + \hat{L}\right)$. Then, using $p \in \operatorname{proj}_{\mathbb{R}^m} P \subseteq q + L$, we obtain

$$p = \underbrace{\sum_{i=1}^m \eta_i^q b^i}_{=q} + \underbrace{\sum_{i=1}^l \eta_i^{w^L} b^i}_{\in L} = \underbrace{\sum_{i=1}^m \eta_i^z b^i}_{=\mathbb{Z}^m} + \underbrace{\sum_{i \in [m] \setminus \{i^*\}} \eta_i^{w^{\hat{L}}} b^i}_{\in \hat{L}},$$

where

- $\eta^{w^L} \in \mathbb{R}^l$,
- $\eta^z \in \mathbb{Z}^m$ (since $\{b^1, \ldots, b^m\}$ is a lattice basis of $\mathbb{Z}^m$) and
- $\eta^{w^{\hat{L}}} \in \mathbb{R}^{[m] \setminus \{i^*\}}$.

Thus, by comparing coefficients and using that $\{b^1, \ldots, b^m\}$ is linearly independent, we obtain

$$\mathbb{Z} \ni \eta_{i^*}^z = \eta_{i^*}^q \notin \mathbb{Z}. \;\lightning$$

For (4.66) **and** (4.67)**:** Clearly,

$$(\operatorname{proj}_{\mathbb{R}^m} P) \cap \left(\mathbb{Z}^m + \tilde{L}\right) \supseteq (\operatorname{proj}_{\mathbb{R}^m} P) \cap \mathbb{Z}^m;$$

so, we only have to show

$$(\operatorname{proj}_{\mathbb{R}^m} P) \cap \left(\mathbb{Z}^m + \tilde{L}\right) \subseteq (\operatorname{proj}_{\mathbb{R}^m} P) \cap \mathbb{Z}^m.$$

Let

$$p = q + w^L = z + w^{\hat{L}},$$

where $p \in \operatorname{proj}_{\mathbb{R}^m} P$, $w^L \in L$, $z \in \mathbb{Z}^m$ and $w^{\tilde{L}} \in \tilde{L}$. Thus, there exist η^{w^L}, η^z and $\eta^{w^{\tilde{L}}}$ having

$$w^L = \sum_{i=1}^l \eta_i^{w^L} b^i, \qquad z = \sum_{i=1}^m \eta_i^z b^i, \qquad w^{\tilde{L}} = \sum_{i=l+1}^m \eta_i^{w^{\tilde{L}}} b^i,$$

where

- $\eta^{w^L} \in \mathbb{R}^l$,
- $\eta^z \in \mathbb{Z}^m$ and
- $\eta^{w^{\tilde{L}}} \in \mathbb{R}^{\{l+1,\ldots,m\}}$.

So, we obtain

$$p=\sum_{i=1}^{l}\left(\eta_i^q+\eta_i^{w^L}\right)b^i+\sum_{i=l+1}^{m}\underbrace{\eta_i^q}_{\in\mathbb{Z}}b^i=\sum_{i=1}^{l}\underbrace{\eta_i^z}_{\in\mathbb{Z}}b^i+\sum_{i=l+1}^{m}\big(\underbrace{\eta_i^z}_{\in\mathbb{Z}}+\eta_i^{w^L}\big)b^i, \tag{4.68}$$

which, by comparing coefficients and using that $\{b^1,\ldots,b^m\}$ is linearly independent, implies

$$\forall i\in\{l+1,\ldots,m\}:\underbrace{\eta_i^q}_{\in\mathbb{Z}}=\underbrace{\eta_i^z}_{\in\mathbb{Z}}+\eta_i^{w^L}.$$

Thus,

$$\forall i\in\{l+1,\ldots,m\}:\eta_i^{w^L}\in\mathbb{Z}.$$

So, from (4.68), one concludes, using the fact that $\{b^1,\ldots,b^m\}$ is a lattice basis of $\mathbb{Z}^m$, that $p\in\mathbb{Z}^m$.

Now for (4.67):

$$\begin{aligned}\begin{pmatrix}p^1\\p^2\end{pmatrix}\in P\cap\left((\mathbb{Z}^m\times\mathbb{R}^n)+\left(\tilde{L}\times\mathbb{R}^n\right)\right)&\Leftrightarrow\begin{pmatrix}p^1\\p^2\end{pmatrix}\in P\wedge p^1\in(\operatorname{proj}_{\mathbb{R}^m}P)\cap\left(\mathbb{Z}^m+\tilde{L}\right)\\&\Leftrightarrow\begin{pmatrix}p^1\\p^2\end{pmatrix}\in P\wedge p^1\in(\operatorname{proj}_{\mathbb{R}^m}P)\cap\mathbb{Z}^m\\&\Leftrightarrow\begin{pmatrix}p^1\\p^2\end{pmatrix}\in P\cap(\mathbb{Z}^m\times\mathbb{R}^n).\end{aligned}$$

□

4.10. k-half-space cuts

Definition 224. *Let $P\subseteq\mathbb{R}^m\times\mathbb{R}^n$ $(m,n\in\mathbb{Z}_{\geq0})$ be arbitrary and let $P':=P^{\leq}(A',b')\supseteq P$, where $A'\in\mathbb{R}^{k\times(m+n)}$ and $b'\in\mathbb{R}^k$ $(k\in\mathbb{Z}_{\geq0})$. Then we denote an inequality $c(\,\cdot\,)\leq c_0$ $(c\in(\mathbb{R}^m\times\mathbb{R}^n)^T$ and $c_0\in\mathbb{R})$ for P'_I as $k,\mathbb{R}\times\mathbb{R}$-**half-space cut for** P. Additionally, we define:*

- *If* lineal P' *is generated by vectors from $\mathbb{Q}^m\times\mathbb{R}^n$, we call $c(\,\cdot\,)\leq c_0$ a $k,\mathbb{Q}\times\mathbb{R}$-**half-space cut for** P.*
- *If* lineal P' *is generated by vectors from $\mathbb{Q}^m\times\mathbb{Q}^n$, we call $c(\,\cdot\,)\leq c_0$ a $k,\mathbb{Q}\times\mathbb{Q}$-**half-space cut for** P.*

Theorem 225. *Let $P\subseteq\mathbb{R}^m\times\mathbb{R}^n$ $(m,n\in\mathbb{Z}_{\geq0})$ be arbitrary and let $c(\,\cdot\,)\leq c_0$ $(c\in(\mathbb{R}^m\times\mathbb{R}^n)^T$ and $c_0\in\mathbb{R})$ be a*

- *$k,\mathbb{R}\times\mathbb{R}$-half-space cut,*
- *$k,\mathbb{Q}\times\mathbb{R}$-half-space cut or*
- *$k,\mathbb{Q}\times\mathbb{Q}$-half-space cut,*

respectively, for P with respect to $P':=P^{\leq}(A',b')$. Let $V:=\operatorname{lineal}P'$ and let $k':=m+n-\dim V$. Then $k'\leq k$ and $c(\,\cdot\,)\leq c_0$ is an

- *$L_{k'-\frac{1}{2},\mathbb{R}\times\mathbb{R}}$ cut,*
- *$L_{k'-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cut or*
- *$L_{k'-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cut,*

respectively, for P with respect to V. In particular, it is an $L_{\min(k,m+n)-\frac{1}{2}}$ cut for P.

Proof. W.l.o.g. we assume $P\neq\emptyset$; thus, $P'\neq\emptyset$.

For $k'\leq k$: It is easy to conclude from $P'\neq\emptyset$ that

$$\operatorname{lineal}P'=P^{=}\left(A',0^k\right).$$

Thus,

$$k'=m+n-\dim V=m+n-\dim(\ker A')=m+n-(m+n-\dim(\operatorname{im}A'))\leq k.$$

For $c(\cdot) \leq c_0$ being an $L_{k'-\frac{1}{2},\cdot\times\cdot}$ cut: We have $P'_I = (P' + V)_I \supseteq (P + V)_I$. Thus, every inequality that is valid for P'_I is also valid for $(P + V)_I$. □

Theorem 226. *Let $A \in \mathbb{R}^{l \times m}$, $G \in \mathbb{R}^{l \times n}$ and $b \in \mathbb{R}^l$ $(l, m, n \in \mathbb{Z}_{\geq 0})$. Let*

$$P := P^{=}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right) \cap \left(\mathbb{R}^m \times \mathbb{R}^n_{\geq 0}\right)$$

(as in Definition 154) and let $c(\cdot) \leq c_0$ $(c \in (\mathbb{R}^m \times \mathbb{R}^n)^T$ and $c_0 \in \mathbb{R})$ be a $k, \mathbb{R} \times \mathbb{R}$-half-space cut for P $(k \in \mathbb{Z}_{\geq 0})$. Then $c(\cdot) \leq c_0$ is a k row cut for P with respect to A, G and b.

Proof. W.l.o.g. let $P \neq \emptyset$. Let $P' := P^{\leq}(A', b') \supseteq P$ $(A' \in \mathbb{R}^{k \times (m+n)}$ and $b' \in \mathbb{R}^k)$. Then, by Lemma 90, we have:

$$\exists M \in \mathbb{R}^{k \times l}, S \in \mathbb{R}^{k \times n}_{\geq 0}, s \in \mathbb{R}^k_{\geq 0} : A' = \begin{pmatrix} MA & MG - S \end{pmatrix} \wedge b' = Mb + s.$$

We claim that

$$P^{=}\left(M \begin{pmatrix} A & G \end{pmatrix}, Mb\right) \cap \left(\mathbb{R}^m \times \mathbb{R}^n_{\geq 0}\right) \subseteq P'. \tag{4.69}$$

If (4.69) is shown, we immediately conclude the statement because of

$$P^{\leq}(c, c_0)_I \supseteq P'_I \supseteq \left(P^{=}\left(M \begin{pmatrix} A & G \end{pmatrix}, Mb\right) \cap \left(\mathbb{R}^m \times \mathbb{R}^n_{\geq 0}\right)\right)_I.$$

So for (4.69): let $x \in P^{=}\left(M \begin{pmatrix} A & G \end{pmatrix}, Mb\right) \cap \left(\mathbb{R}^m \times \mathbb{R}^n_{\geq 0}\right)$. Then

$$A'x = \begin{pmatrix} MA & MG \end{pmatrix} x - \begin{pmatrix} 0^{k \times m} & S \end{pmatrix} x = Mb - \begin{pmatrix} 0^{k \times m} & S \end{pmatrix} x \leq Mb \leq Mb + s = b'.$$

□

5. Alternative characterizations of $L_{k,\mathbb{Q}}$ cuts and essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts

5.1. Motivation and outline

A naive approach for finding $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts for a given rational polyhedron $P \subseteq \mathbb{R}^m$ ($m \in \mathbb{Z}_{\geq 0}$) with respect to a given (rational) subspace $V \leq \mathbb{R}^m$ is:

1. Compute an explicit representation of $P' := \operatorname{proj}^{\perp}_{V^\perp} P$.
2. Compute a lattice basis of $\Lambda := \operatorname{proj}^{\perp}_{V^\perp} \mathbb{Z}^m$.
3. Compute (ideally facet-defining) inequalities $c(\cdot) \leq c_0$ ($c \in (\mathbb{R}^m)^T$ and $c_0 \in \mathbb{R}$) for $\operatorname{conv}(P' \cap \Lambda)$ such that $c^T \perp V$ (the latter condition can be assumed because of Lemma 159).

We now explain why this approach causes problems in practice. The short answer is that the number of facets of P' can be very large if $k \geq 2$.

In [GHOT13], it is shown that if the matrix A of $P = P^{\leq}(A, b) \subseteq \mathbb{R}^d$ ($d \in \mathbb{Z}_{\geq 2}$) contains merely two entries in each row that are different from 0, there can already exist a subspace $V \leq \mathbb{R}^d$ of codimension 2 such that $\operatorname{proj}^{\perp}_{V^\perp} P$ has 2^d vertices and thus (equivalently in $\mathbb{R}^2$ for full-dimensional polytopes) has 2^d facets. Let us sketch the proof idea. For this, let us for $\epsilon \in \left(0, \frac{1}{2}\right)$ consider the d-dimensional Kleene-Minty cube (cf. [GHOT13, Definition 5]):

$$\begin{aligned} P^{KM,d,\epsilon} := \{x \in \mathbb{R}^d : {} & 0 \leq x_1 \leq 1, \\ & \epsilon x_{j-1} \leq x_j \leq 1 - \epsilon x_{j-1} \ \forall j \in \{2, \dots, d\}\}. \end{aligned}$$

For $u \in \{0,1\}^d$, we define $x(u) \in \mathbb{R}^d$ recursively via

$$x_j(u) := (1 - u_j)\, \epsilon x_{j-1}(u)$$

for $j \in \{1, \dots, d\}$, where we formally set $x_0(u) := 0$ (cf. [GHOT13, Definition 6]). The following lemma is easy to check:

Lemma 227. *For all $u \in \{0,1\}^d$ ($d \in \mathbb{Z}_{\geq 2}$), the point $x(u)$ is a vertex of $P^{KM,d,\epsilon}$.*

For given $r, s \in \mathbb{R}^d$, we define:

$$\begin{aligned} \pi_{r,s} : \mathbb{R}^d & \to \mathbb{R}^2, \\ x & \mapsto \begin{pmatrix} r^T x \\ s^T x \end{pmatrix}. \end{aligned}$$

Then (cf. [GHOT13, Definition 9, Definition 11 and Lemma 12]):

Lemma 228. *Let*

$$r := \begin{pmatrix} \epsilon^{3\cdot(d-1)} \\ \epsilon^{3\cdot(d-2)} \\ \vdots \\ \epsilon^{3\cdot 1} \\ 0 \end{pmatrix},$$

$$s := e^{d,d},$$

where $d \in \mathbb{Z}_{\geq 2}$, and let $u \in \{0,1\}^d$. Then the maximum of the function

$$\left(r - \sum_{j=0}^{d-1} p_{j+1}^{d}(u)\,\epsilon^{2(d-j)} s \right)^T (\cdot)$$

over $P^{KM,d,\epsilon}$ is uniquely attained at the vertex $x(u)$. Here,

$$p_i^j(u) := \prod_{k=i}^{j} (1 - 2u_k) \in \{-1, 1\}$$

is defined as the $\{-1,1\}$ coding of the parity of the bit vector $(u_i, u_{i+1}, \ldots, u_j)$.

This, of course, has the consequence that $\pi_{r,s}\left(P^{KM,d,\epsilon}\right)$ has 2^d vertices and facets.

So, we want to look for different characterizations of $L_{k,\mathbb{Q}}$ cuts, $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts and essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts. To do this, one can easily picture two natural approaches:

- Show alternative characterizations for specific (typically small) values of k. This is done in part III for $L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts and $L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cuts (chapter 8; in particular section 8.1) and $L_{1,\mathbb{Q}}$ cuts (chapter 9; in particular section 9.1.1), and in part IV (in particular chapter 11) for $L_{2,\mathbb{Q}}$ cuts (section 11.1) and essential $L_{2-\frac{1}{2},\mathbb{Q}}$ cuts (section 11.2). For the latter, recall that by Theorem 208 and Theorem 211, "the most interesting" $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts/$L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cuts are the essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts, since if an $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ is not already an $L_{k-1,\mathbb{Q}}$ cut, it has to be an essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cut.
- Show alternative characterizations that hold for general $k \in \{1, \ldots, m\}$. This is what this chapter is about.

So, for the outline of this chapter:

- In section 5.2, we consider the relationship between $L_{k,\mathbb{Q}}$ cuts and lattice-free bodies:
  - In section 5.2.1, we prove some auxiliary results about full-dimensional projections that are used in section 5.2.2.
  - In section 5.2.2, we show that for
    * a rational polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ (Theorem 231),
    * $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ convex (Theorem 233) and
    * $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ convex and compact (Theorem 234)

    ($m, n \in \mathbb{Z}_{\geq 0}$) and a valid inequality for P_I satisfying some conditions (rationality in Theorem 231; being a strict inequality in Theorem 234), the projection of the points that are cut away into the $\mathbb{R}^m$ can be embedded into a lattice-free body in a specific way.
  - In section 5.2.3, Theorem 235, Theorem 236 and Theorem 237, we extend this result to $L_{k,\mathbb{Q}}$ cuts, where $k \in \{0, \ldots, m\}$.
  - In section 5.2.4, Theorem 239, we show the reverse: if we have a lattice-free polyhedron $L \subseteq \mathbb{R}^m$ ($m \in \mathbb{Z}_{\geq 0}$) with a rational lineality space such that $\dim(\operatorname{lineal} L) \geq k$ be given, it can be used to derive an $L_{k,\mathbb{Q}}$ cut for P.
  - In section 5.2.5, Theorem 240, we put all the parts together and characterize $L_{k,\mathbb{Q}}$ cuts using lattice-free bodies.
- In section 5.3, we consider the relationship between essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts and lattice-free bodies:
  - In section 5.3.1, Theorem 241, we show how rational essential $L_{m-\frac{1}{2},\mathbb{Q}}$ cuts for rational polyhedra $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) can be represented via lattice-free bodies.
  - In section 5.3.2, Theorem 244, we extend this result to rational essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts for rational polyhedra $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$), where $k \in \{0, \ldots, m\}$.
  - In section 5.3.3, Theorem 245, we show the reverse: if we have a cutting plane be given that can be represented via a full-dimensional lattice-free body as in the previous section, it is an essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cut.

   - In section 5.3.4, Theorem 246, we put all these parts together to describe essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts for rational polyhedra using lattice-free bodies.

- In section 5.4, we consider how $L_{k,\mathbb{Q}}$ cuts and essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts can be characterized via t-branch split disjunctions:
   - In section 5.4.1, we present some results from the literature about covering lattice-free bodies with t-branch split disjunctions. Additionally, we define k,t-branch split cuts (Definition 252) and essential k,t-branch split cuts (Definition 253) and their respective closures. These are restrictions of t-branch split cuts (cf. Definition 143) that are used later on for characterizing $L_{k,\mathbb{Q}}$ cuts and essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts via t-branch split cuts.
   - In section 5.4.2, we show in Theorem 254 that if $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ is either a rational polyhedron or convex and compact, we have

$$\mathrm{cl}_{m,h(m)BS}(P) = \mathrm{cl}_{L_{m,\mathbb{Q}}}(P) = \mathrm{cl}_{\overline{I}}(P)$$

   ($h(\cdot)$ is defined in Remark/Definition 248 and $\mathrm{cl}_{m,h(m)BS}(\cdot)$ is defined in Definition 252). In Theorem 255, we show that a similar statement does not hold if $P \subseteq \mathbb{R}^2$ is instead an irrational hyperplane.
   - In section 5.4.3, Theorem 256, we extend this to the situation where we have $k \in \{1, \dots, m\}$, but $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ is still either a rational polyhedron or convex and compact. Here, we show that

$$\mathrm{cl}_{k,h(k)BS}(P) \subseteq \mathrm{cl}_{L_{k,\mathbb{Q}}}(P).$$

   In Theorem 258, we extend this result to essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts and show that

$$\mathrm{cl}_{\mathrm{ess}\,k,h(k)BS}(P) \subseteq \mathrm{cl}_{\mathrm{ess}\,L_{k-\frac{1}{2},\mathbb{Q}}}(P)$$

   ($\mathrm{cl}_{\mathrm{ess}\,k,h(m)BS}(\cdot)$ is defined in Definition 253).
   - In section 5.4.4, we put all the pieces together and show:
      * If $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ is either a rational polyhedron or convex and compact, we have for $k \in \{0, \dots, m\}$ (Theorem 259):

$$\mathrm{cl}_{L_{k,\mathbb{Q}}}(P) = \mathrm{cl}_{L_{k,h(k)BS}}(P).$$

      * If $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ is either a rational polyhedron or convex and compact, we have for $k \in \{0, \dots, m\}$ (Theorem 261):

$$\mathrm{cl}_{\mathrm{ess}\,L_{k-\frac{1}{2},\mathbb{Q}}}(P) = \mathrm{cl}_{\mathrm{ess}\,k,h(k)BS}(P).$$

      .
      * If $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ is a rational polyhedron, we have for $k \in \{1, \dots, m\}$ (Theorem 263):

$$\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P) = \mathrm{cl}_{\mathrm{ess}\,k,h(k)BS}(P) \cap \mathrm{cl}_{k-1,h(k-1)BS}(P).$$

   - In section 5.4.5, we use these structural results to show in Theorem 264 that the $L_{k,\mathbb{Q}}$ closure of a rational polyhedron is again a rational polyhedron.

5.2. Characterizing $L_{k,\mathbb{Q}}$ cuts via lattice-free bodies

5.2.1. Full-dimensional projections

Lemma 229. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ be convex, let $\mathrm{proj}_{\mathbb{R}^m} P$ be full-dimensional and let $c(\cdot) \geq c_0$ ($c \in (\mathbb{R}^m \times \mathbb{R}^n)^T$ and $c_0 \in \mathbb{R}$) be an inequality such that $P \cap P^<(c, c_0) \neq \emptyset$. Then $\mathrm{proj}_{\mathbb{R}^m}(P \cap P^<(c, c_0))$ is full-dimensional.*

Proof. If $c = (0^m \times 0^n)^T$ (which implies $c_0 > 0$ because of the assumption), the statement holds trivially. Thus, let $c \neq (0^m \times 0^n)^T$. Let $\hat{x} \in P \cap P^<(c, c_0)$. Since $\mathrm{proj}_{\mathbb{R}^m} P$ is full-dimensional, there exist linearly independent $r^1, \ldots, r^m \in \mathbb{R}^m$ for which there exist $s^1, \ldots, s^m \in \mathbb{R}^n$ having $\hat{x} + \binom{r^i}{s^i} \in P$ for all $i \in [m]$. Let

$$\mu_i := \begin{cases} \min\left(\frac{c_0 - c\hat{x}}{2c\binom{r^i}{s^i}}, 1\right) & \text{if } c\binom{r^i}{s^i} > 0, \\ 1 & \text{if } c\binom{r^i}{s^i} \leq 0 \end{cases} \quad \forall i \in [m].$$

Using $0^m < \mu \leq 1^m$ (the strict inequality holds because $\hat{x} \in P \cap P^{<}(c, c_0)$) and that P is convex, we obtain

$$\forall i \in [m] : \hat{x} + \mu_i \begin{pmatrix} r^i \\ s^i \end{pmatrix} \in P.$$

Additionally, we claim

$$\forall i \in [m] : c\left(\hat{x} + \mu_i \begin{pmatrix} r^i \\ s^i \end{pmatrix}\right) < c_0.$$

For this:

$$c\left(\hat{x} + \mu_i \begin{pmatrix} r^i \\ s^i \end{pmatrix}\right) = (c\hat{x} - c_0) + c_0 + \mu_i c \begin{pmatrix} r^i \\ s^i \end{pmatrix} =: (5.1).$$

If $c\binom{r^i}{s^i} \leq 0$, we have

$$(5.1) = \underbrace{(c\hat{x} - c_0)}_{<0} + c_0 + \underbrace{1 \cdot c \begin{pmatrix} r^i \\ s^i \end{pmatrix}}_{\leq 0} < c_0.$$

On the other hand, if $c\binom{r^i}{s^i} > 0$, we conclude

$$(5.1) \leq (c\hat{x} - c_0) + c_0 + \frac{c_0 - c\hat{x}}{2c\begin{pmatrix} r^i \\ s^i \end{pmatrix}} c \begin{pmatrix} r^i \\ s^i \end{pmatrix} = \frac{1}{2}\underbrace{(c\hat{x} - c_0)}_{<0} + c_0 < c_0.$$

□

For the role of Lemma 229 in the later context of this text: by using cutting planes, one cuts away points of P to approach $\mathrm{cl}_{\overline{I}}(P)$. The set of the points that are cut off obviously forms a lattice-free body, which, by Theorem 111, can be embedded into a maximal lattice-free body. By Theorem 108, a maximal lattice-free body in $\mathbb{R}^m$ can also be an irrational hyperplane of codimension $m-1$ (which is not full-dimensional), which is often an inconvenient situation. So one wishes to avoid this degenerate constellation. Lemma 229 guarantees that if $\mathrm{proj}_{\mathbb{R}^m} P$ is full-dimensional, so is the set of points that are cut off and the maximal lattice-free body into which these points can be embedded.

We next consider a lemma (Lemma 230) that guarantees that a projection is full-dimensional, so that we can apply Lemma 229.

Lemma 230. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ be convex and let $\mathrm{proj}_{\mathbb{R}^m} P$ be full-dimensional. For linearly independent vectors $\pi^1, \ldots, \pi^k \in \mathbb{Z}^m$ $(k \in \{0, \ldots, m\})$, define*

$$S\left(P, \pi^1, \ldots, \pi^k\right) := \mathrm{conv}\left\{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in P \times \mathbb{R}^k : \left(\pi^i\right)^T x = z_i \ \forall i \in [k]\right\}.$$

Then $\mathrm{proj}_{\mathbb{R}^k} S\left(P, \pi^1, \ldots, \pi^k\right)$ is convex and full-dimensional.

Proof. Convexity is obvious; so, we only prove full-dimensionality. Let $\binom{x}{y} \in P$ be such that $x \in \mathrm{int}\left(\mathrm{proj}_{\mathbb{R}^m} P\right)$ (such a point clearly exists). Thus, there exists an $\epsilon > 0$ such that

$$x + \epsilon\pi^1, \ldots, x^* + \epsilon\pi^k \in \mathrm{proj}_{\mathbb{R}^m} P.$$

So, we can find a $y \in \mathbb{R}^n$ and $\mu^1, \ldots, \mu^k \in \mathbb{R}^n$ such that

$$\begin{pmatrix} x \\ y \end{pmatrix} + \epsilon \begin{pmatrix} \pi^1 \\ \mu^1 \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} + \cdots + \epsilon \begin{pmatrix} \pi^k \\ \mu^k \end{pmatrix} \in P.$$

This means

$$\begin{pmatrix} x \\ y \\ \left(\pi^1\right)^T x \\ \vdots \\ \left(\pi^k\right)^T x \end{pmatrix} \in S\left(P, \pi^1, \ldots, \pi^k\right) \wedge \forall i \in [k] : \begin{pmatrix} x \\ y \\ \left(\pi^1\right)^T x \\ \vdots \\ \left(\pi^k\right)^T x \end{pmatrix} + \epsilon \begin{pmatrix} \pi^i \\ \mu^i \\ \left(\pi^1\right)^T \pi^i \\ \vdots \\ \left(\pi^k\right)^T \pi^i \end{pmatrix} \in S\left(P, \pi^1, \ldots, \pi^k\right).$$

So, it suffices to show that

$$\begin{pmatrix} \left(\pi^1\right)^T \pi^1 & \cdots & \left(\pi^1\right)^T \pi^k \\ \vdots & \ddots & \vdots \\ \left(\pi^k\right)^T \pi^1 & \cdots & \left(\pi^k\right)^T \pi^k \end{pmatrix}$$

has full rank. But this holds by Lemma 28. □

5.2.2. $k = m$

The following theorem generalizes statements that are proved implicitly in the proof of [DDG11, Lemma 2.1] to arbitrary $m \in \mathbb{Z}_{\geq 0}$.

Theorem 231. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be a rational polyhedron and let $c(\,\cdot\,) \geq c_0$, where $c \in (\mathbb{Q}^m \times \mathbb{Q}^n)^T$ and $c_0 \in \mathbb{R}$ (even though only the case $c_0 \in \mathbb{Q}$ is of interest for this text), be a valid inequality for P_I (in other words: an $L_{m,\mathbb{Q}}$ cut for P). Let $R := P \cap P^{<}(c, c_0)$. Then there exists a full-dimensional, maximal lattice-free, rational polyhedron $L \subseteq \mathbb{R}^m$ such that*

$$\operatorname{proj}_{\mathbb{R}^m} R \subseteq \operatorname{int} L.$$

In particular, $c(\,\cdot\,) \geq c_0$ is valid for $P \setminus ((\operatorname{int} L) \times \mathbb{R}^n)$.

Proof. W.l.o.g. we can assume $R \neq \emptyset$ and $P = P^{\leq}(A, b)$, where A is rational. Obviously, we have $R_I = \emptyset$ and thus also $(\operatorname{proj}_{\mathbb{R}^m} R)_I = \emptyset$. By Theorem 64, there exist $G^1 \in \mathbb{Q}^{t_1 \times m}$, $G^2 \in \mathbb{Q}^{t_2 \times m}$ and vectors $g^1 \in \mathbb{Q}^{t_1}$, $g^2 \in \mathbb{Q}^{t_2}$ ($t_1, t_2 \in \mathbb{Z}_{\geq 0}$) such that

$$\operatorname{proj}_{\mathbb{R}^m} R = \left\{x \in \mathbb{R}^m : G^1 x \leq g^1, G^2 x < g^2\right\}.$$

W.l.o.g. let G^1, G^2 be integral. Define $g'^1 \in \mathbb{Z}^{t_1}$ via

$$g_i'^1 := \begin{cases} \lceil g_i \rceil & \text{if } g_i^1 \notin \mathbb{Z}, \\ g_i^1 + 1 & \text{if } g_i^1 \in \mathbb{Z} \end{cases} \quad \forall i \in [t_1].$$

Consider (since G^1 is integral) that for $x \in \mathbb{Z}^m$, we have $\forall i \in [t_1]$: $G^1_{i,*} x \leq g^1_i \Leftrightarrow G^1_{i,*} x < g_i'^1$. Let

$$S := P^{<}\left(\begin{pmatrix} G^1 \\ G^2 \end{pmatrix}, \begin{pmatrix} g'^1 \\ g^2 \end{pmatrix}\right).$$

Clearly, $S_I = \emptyset$. Observe that S is full-dimensional, since

- $S \neq \emptyset$ (since $\emptyset \neq \operatorname{proj}_{\mathbb{R}^m} R \subseteq S$) and
- S is generated by strict inequalities.

Thus, $\overline{S} = P^{\leq}\left(\begin{pmatrix} G^1 \\ G^2 \end{pmatrix}, \begin{pmatrix} g'^1 \\ g^2 \end{pmatrix}\right)$ (the topological closure of S) is surely lattice-free. By Theorem 111, there exists a rational, maximal lattice-free polyhedron $L \subseteq \mathbb{R}^m$ having $L \supseteq \overline{S}$. So, we have

$$\operatorname{proj}_{\mathbb{R}^m} R \subseteq S \subseteq \operatorname{int} \overline{S} \subseteq \operatorname{int} L.$$

□

Before we formulate the next theorem (Theorem 233), we show a small lemma:

Lemma 232. *Let $L = P^{\leq}\left(A^L, b^L\right) \subseteq \mathbb{R}^m$ be a lattice-free body, where $A^L \in \mathbb{R}^{l\times m}$ and $b^L \in \mathbb{R}^l$ ($l, m \in \mathbb{Z}_{\geq 0}$). Then*

$$\bigcap_{i=1}^{l} \left(P^{<}\left(A^L_{i,*}, b^L_i\right) \dot\cup \left(P^{=}\left(A^L_{i,*}, b^L_i\right) \setminus \left(P^{=}\left(A^L_{i,*}, b^L_i\right)_I\right)\right)\right) = (\operatorname{int} L) \dot\cup ((\operatorname{bd} L) \setminus (\operatorname{bd} L)_I).$$

Proof.

For $\subseteq$: Let

$$x \in \bigcap_{i=1}^{l} \left(P^{<}\left(A^L_{i,*}, b^L_i\right) \dot\cup \left(P^{=}\left(A^L_{i,*}, b^L_i\right) \setminus \left(P^{=}\left(A^L_{i,*}, b^L_i\right)_I\right)\right)\right).$$

If $A^L x < b^L$, we have $x \in \operatorname{int} L$. On the other hand, if $\exists i\,[l] : A^L_{i,*}x = b^L_i$, we conclude (using $x \in L$): $x \in \operatorname{bd} L$. By construction, x is not integral. So, $x \notin (\operatorname{bd} L)_I$.

For $\supseteq$: Let $x \in (\operatorname{int} L) \dot\cup ((\operatorname{bd} L) \setminus (\operatorname{bd} L)_I)$. If $x \in \operatorname{int} L$, we have $A^L x < b^L$. On the other hand, if $x \in (\operatorname{bd} L) \setminus (\operatorname{bd} L)_I$, we have $A^L x \leq b^L$ and $\exists i \in [l] : A^L_{i,*}x = b_i$. So, since x is not integral, we conclude

$$x \in \bigcap_{i=1}^{l} \left(P^{<}\left(A^L_{i,*}, b^L_i\right) \dot\cup \left(P^{=}\left(A^L_{i,*}, b^L_i\right) \setminus \left(P^{=}\left(A^L_{i,*}, b^L_i\right)_I\right)\right)\right).$$

□

Theorem 233. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be convex and let $c(\cdot) \geq c_0$, where $c \in (\mathbb{R}^m \times \mathbb{R}^n)^T$ and $c_0 \in \mathbb{R}$, be a valid inequality for P_I (in other words: an $L_{m,\mathbb{Q}}$ cut for P). Set $R := P \cap P^{<}(c, c_0)$. Then there exists a maximal lattice-free polyhedron $L \subseteq \mathbb{R}^m$ such that*

$$\operatorname{proj}_{\mathbb{R}^m} R \subseteq (\operatorname{int} L) \dot\cup ((\operatorname{bd} L) \setminus (\operatorname{bd} L)_I).$$

In particular, $c(\cdot) \geq c_0$ is valid for

$$P \setminus (((\operatorname{int} L) \dot\cup ((\operatorname{bd} L) \setminus (\operatorname{bd} L)_I)) \times \mathbb{R}^n).$$

If $\operatorname{proj}_{\mathbb{R}^m} P$ is full-dimensional, so is L.

Proof. W.l.o.g. we can assume $R \neq \emptyset$. Since $R_I = \emptyset$, $\operatorname{proj}_{\mathbb{R}^m} R$ is surely lattice-free. Thus, by Lemma 107, there exists a maximal lattice-free body $L \supseteq \operatorname{proj}_{\mathbb{R}^m} R$. By Theorem 108, L is a polyhedron; so, there exist $A^L \in \mathbb{R}^{l\times m}$ and $b^L \in \mathbb{R}^l$ ($l \in \mathbb{Z}_{\geq 0}$) such that $L = P^{\leq}\left(A^L, b^L\right)$. Since $R \subseteq L$, we surely have for all $i \in [l]$:

$$\operatorname{proj}_{\mathbb{R}^m} R \subseteq P^{\leq}\left(A^L_{i,*}, b^L_i\right). \tag{5.2}$$

Since $R_I = \emptyset$, also clearly $P^{=}\left(A^L, b^L\right)_I \cap \operatorname{proj}_{\mathbb{R}^m} R = \emptyset$ holds. Thus, we can tighten (5.2) to

$$\begin{aligned}\operatorname{proj}_{\mathbb{R}^m} R &\subseteq \bigcap_{i=1}^{l} \left(P^{<}\left(A^L_{i,*}, b^L_i\right) \dot\cup \left(P^{=}\left(A^L_{i,*}, b^L_i\right) \setminus \left(P^{=}\left(A^L_{i,*}, b^L_i\right)_I\right)\right)\right) \\ &= (\operatorname{int} L) \dot\cup ((\operatorname{bd} L) \setminus (\operatorname{bd} L)_I). \qquad \text{(by Lemma 232)}\end{aligned}$$

If $\operatorname{proj}_{\mathbb{R}^m} P$ is full-dimensional, we get from Lemma 229 that $\operatorname{proj}_{\mathbb{R}^m} R$ is full-dimensional, too. Since $\operatorname{proj}_{\mathbb{R}^m} R \subseteq L$, we then obtain the full-dimensionality of L. □

The following theorem generalizes statements that are proved implicitly in the proof of [DDG11, Theorem 5.1] to arbitrary $m \in \mathbb{Z}_{\geq 0}$:

Theorem 234. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be convex and compact and let $c \in (\mathbb{R}^m \times \mathbb{R}^n)^T$ and $c_0 \in \mathbb{R}$ be given such that*

$$\forall x \in P_I : cx > c_0. \tag{5.3}$$

Set $R := P \cap P^{<}(c, c_0)$. Then there exists a full-dimensional, maximal lattice-free polyhedron $L \subseteq \mathbb{R}^m$ such that $\operatorname{proj}_{\mathbb{R}^m} R \subseteq \operatorname{int} L$. In particular, $c(\cdot) \geq c_0$ is valid for $P \setminus ((\operatorname{int} L) \times \mathbb{R}^n)$.

Proof. W.l.o.g. we can assume $R \neq \emptyset$. Set $S := P \cap P^{\leq}(c, c_0)$. Since S is convex, closed and bounded, also $\operatorname{proj}_{\mathbb{R}^m} S$ is. We have $(\operatorname{proj}_{\mathbb{R}^m} S)_I = \emptyset$. To realize this, assume otherwise, i.e. assume $\exists x \in (\operatorname{proj}_{\mathbb{R}^m} S)_I$. Then there exists a $y \in \mathbb{R}^n$ such that $\binom{x}{y} \in S_I = \left(P \cap P^{\leq}(c, c_0)\right)_I$, which is a contradiction to (5.3).

Since $\operatorname{proj}_{\mathbb{R}^m} S$ is bounded, there exists a full-dimensional polytope $U =: P^{\leq}\left(A^U, b^U\right) \subseteq \mathbb{R}^m$ such that $\operatorname{proj}_{\mathbb{R}^m} S \subseteq \operatorname{int} U$. Let $\left\{v^1, \dots, v^k\right\} := U_I$ (since U is bounded, the number of integral points in U is finite). Then there exist $c^1, \dots, c^k \in (\mathbb{R}^m)^T$ and $\epsilon_1, \dots, \epsilon_k \in \mathbb{R}_{>0}$ such that

$$\forall i \in [k], x \in \operatorname{proj}_{\mathbb{R}^m} S : c^i v^i - c^i x > \epsilon_i. \tag{5.4}$$

Let $V := U \cap \bigcap_{i=1}^k P^{\leq}\left(c^i, c^i v^i\right)$. We claim that

1. V is lattice-free,

2. $\operatorname{proj}_{\mathbb{R}^m} S \subseteq \operatorname{int} V$ (thus, in particular, V is full-dimensional, since $\operatorname{proj}_{\mathbb{R}^m} S \neq \emptyset$).

For 1: Let $x \in V_I$. Then $x = v^i$ for some $i \in [k]$, since $V \subseteq U$. On the other hand, we have $c^i v^i \leq c^i v^i$. Since $c^i(\,\cdot\,) \leq c^i v^i$ is one of the inequalities defining V, we thus conclude $x \in \operatorname{bd} V$ and obtain $x \notin \operatorname{int} V$.

For 2: Let $x \in \operatorname{proj}_{\mathbb{R}^m} S$. Since $\operatorname{proj}_{\mathbb{R}^m} S \subseteq \operatorname{int} U$, we have $A^U x < b^U$. On the other hand, by (5.4), we have $c^i x < c^i v^i - \epsilon_i < c^i v^i$.

So, by Lemma 107, there exists a full-dimensional, maximal lattice-free polyhedron L having $V \subseteq L \subseteq \mathbb{R}^m$. By Theorem 108, L is a polyhedron. We thus conclude

$$\operatorname{proj}_{\mathbb{R}^m} R = \operatorname{proj}_{\mathbb{R}^m}\left(P \cap P^{<}(c, c_0)\right) \subseteq \operatorname{proj}_{\mathbb{R}^m}\left(P \cap P^{\leq}(c, c_0)\right) = \operatorname{proj}_{\mathbb{R}^m} S \subseteq \operatorname{int} L.$$

□

5.2.3. $k \leq m$

Theorem 235. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be a rational polyhedron and let $V' \leq \mathbb{R}^m$ be a rational subspace of codimension $k \in \{0, \dots, m\}$. Define $V := V' \times \mathbb{R}^n$. Let $c(\,\cdot\,) \geq c_0$, where $c \in (\mathbb{Q}^m \times \mathbb{Q}^n)^T$ and $c_0 \in \mathbb{R}$ (even though only the case $c_0 \in \mathbb{Q}$ is of interest for this text), be a valid inequality for $P \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + V)$ (in other words: an $L_{k,\mathbb{Q}}$ cut for P with respect to V; recall that, by Theorem 168, it suffices to consider vector spaces of this form for $L_{k,\mathbb{Q}}$ cuts). Set $R := P \cap P^{<}(c, c_0)$.*

Then there exists a full-dimensional, maximal lattice-free, rational polyhedron $L \subseteq \mathbb{R}^m$, where $\operatorname{proj}_{\mathbb{R}^m} R \subseteq \operatorname{int} L$ *and* $\operatorname{lineal} L \geq V'$. *In particular, $c(\,\cdot\,) \geq c_0$ is valid for $P \setminus ((\operatorname{int} L) \times \mathbb{R}^n)$.*

Theorem 236. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be convex and let $V' \leq \mathbb{R}^m$ be a rational subspace of codimension $k \in \{0, \dots, m\}$. Define $V := V' \times \mathbb{R}^n$. Let $c(\,\cdot\,) \geq c_0$, where $c \in (\mathbb{R}^m \times \mathbb{R}^n)^T$ and $c_0 \in \mathbb{R}$, be a valid inequality for $P \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + V)$ (in other words: an $L_{k,\mathbb{Q}}$ cut for P). Set $R := P \cap P^{<}(c, c_0)$.*

Then there exists a maximal lattice-free polyhedron $L \subseteq \mathbb{R}^m$ such that we have

$$\operatorname{proj}_{\mathbb{R}^m} R \subseteq (\operatorname{int} L) \,\dot{\cup}\, ((\operatorname{bd} L) \setminus ((\operatorname{bd} L)_I + V'))$$

and $\operatorname{lineal} L \geq V'$. *In particular, $c(\,\cdot\,) \geq c_0$ is valid for*

$$P \setminus (((\operatorname{int} L) \,\dot{\cup}\, ((\operatorname{bd} L) \setminus ((\operatorname{bd} L)_I + V'))) \times \mathbb{R}^n).$$

If $\operatorname{proj}_{\mathbb{R}^m} P$ *is full-dimensional, so is L.*

Theorem 237. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be convex, closed and bounded. Let $V' \leq \mathbb{R}^m$ be a rational subspace of codimension $k \in \{0, \dots, m\}$. Define $V := V' \times \mathbb{R}^n$. Let $c(\,\cdot\,) > c_0$, where $c \in (\mathbb{R}^m \times \mathbb{R}^n)^T$ and $c_0 \in \mathbb{R}$, be a valid inequality for $P \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + V)$ (in other words: an $L_{k,\mathbb{Q}}$ cut for P, though in this case* a strict *linear inequality). Set $R := P \cap P^{<}(c, c_0)$.*

Then there exists a full-dimensional maximal lattice-free polyhedron L, where $\operatorname{proj}_{\mathbb{R}^m} R \subseteq \operatorname{int} L$ *and* $\operatorname{lineal} L \geq V'$. *In particular, $c(\,\cdot\,) \geq c_0$ is valid for $P \setminus ((\operatorname{int} L) \times \mathbb{R}^n)$.*

Before we prove Theorem 235, Theorem 236 and Theorem 237, we show a small helper statement:

Lemma 238. *Let $Q^{pre} \subseteq \mathbb{R}^k$ ($k \in \mathbb{Z}_{\geq 0}$) be arbitrary and let $x^{*,pre} \in Q^{pre}$ and $\epsilon^{pre} \in \mathbb{R}_{>0}$ be such that $B_{\epsilon^{pre}}(x^{*,pre}) \subseteq Q^{pre}$. Define $Q := \{x \in \mathbb{R}^m : W^T x \in Q^{pre}\}$, where $m \in \mathbb{Z}_{\geq 0}$, and $W \in \mathbb{R}^{m \times k}$ is a matrix with linearly independent columns. Set $x^* := W (W^T W)^{-1} x^{*,pre}$. Then there exists an $\epsilon \in \mathbb{R}_{>0}$ such that $B_\epsilon(x^*) \subseteq Q$.*

Proof. Let

$$\epsilon := \frac{1}{\max\{\|W^T x\| : x \in \mathbb{S}^{m-1}\}}.$$

By the Weierstrass extreme value theorem, the maximum in the denominator clearly exists. We claim:

1. $\forall x \in Q, v \in (\operatorname{im} W)^\perp : x + v \in Q$,
2. $\forall w \in \operatorname{im} W, \|w\| < 1 : x^* + \epsilon \cdot w \in Q$.

For 1: Let $x \in Q$. Then $W^T x \in Q^{pre}$. Using $v \perp \operatorname{im} W$, we obtain

$$x + v \in Q \Leftrightarrow W^T(x+v) \in Q^{pre} \Leftrightarrow W^T x \in Q^{pre}.$$

For 2: If $w = 0^m$ (which is equivalent to $\|w\| = 0$), we trivially have $\|\epsilon W^T w\| = 0 < 1$. On the other hand, if $\|w\| > 0$, we obtain

$$\|\epsilon \cdot W^T w\| = \epsilon \cdot \|w\| \cdot \left\| W^T \frac{w}{\|w\|} \right\| \leq \epsilon \cdot \|w\| \cdot \max\{\|W^T x\| : x \in \mathbb{S}^{m-1}\} = \|w\| < 1.$$

So, we conclude:

$$\begin{aligned} B_{\epsilon^{pre}}(x^{*,pre}) \subseteq Q^{pre} &\Rightarrow x^{*,pre} + \epsilon \cdot W^T w \in Q^{pre} \\ &\Leftrightarrow W^T \left(W (W^T W)^{-1} x^{*,pre} + \epsilon \cdot w \right) \in Q^{pre} \qquad ((W^T W)^{-1} \text{ exists by Lemma 28}) \\ &\Leftrightarrow x^* + \epsilon \cdot w \in Q. \end{aligned}$$

Now let $r \in B_1(0^m)$ (i.e. $\|r\| < 1$). r can be partitioned into $r = v + w$, where $v \in (\operatorname{im} W)^\perp$ and $w \in \operatorname{im} W$, where $\|v\|, \|w\| < 1$. Thus,

$$x^* + \epsilon \cdot r = x^* + \epsilon \cdot (v + w) = \underbrace{\underbrace{x^* + \epsilon \cdot w}_{\in Q \text{ (by 2)}} + \underbrace{\epsilon \cdot v}_{\in (\operatorname{im} W)^\perp}}_{\in Q \text{ (by 1)}} \in Q.$$

□

We now prove Theorem 235, Theorem 236 and Theorem 237:

Proof. (Theorem 235, Theorem 236 and Theorem 237) W.l.o.g. we can assume $R \neq \emptyset$. By Theorem 172, there exist linearly independent vectors $w^1, \ldots, w^k \in \mathbb{Z}^m$ such that

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in P : (w^1)^T x \in \mathbb{Z}, \ldots, (w^k)^T x \in \mathbb{Z} \right\} = P \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + V). \tag{5.5}$$

Let $W := \begin{pmatrix} w^1 & \cdots & w^k \end{pmatrix}$ and recall that, by Lemma 98 and Theorem 172, we have

$$\mathbb{Z}^m + V' = \{x \in \mathbb{R}^k : W^T x \in \mathbb{Z}^k\}. \tag{5.6}$$

Define

$$S^{LP} := \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in P \times \mathbb{R}^k : z = W^T x \right\},$$
$$S := S^{LP} \cap (\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{Z}^k).$$

We claim that

$$\left(c_{(1,\ldots,m)}\left(I^m - \left(W\left(W^TW\right)^{-1}\right)W^T\right)\right)x + c_{(m+1,\ldots,m+n)}y + \left(c_{(1,\ldots,m)}W\left(W^TW\right)^{-1}\right)z \geq c_0 \tag{5.7}$$

is a valid linear inequality for $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in S$ (the inverse $\left(W^TW\right)^{-1}$ exists by Lemma 28). For this, let $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in S$. Then

$$\begin{aligned}
&\left(c_{(1,\ldots,m)}\left(I^m - \left(W\left(W^TW\right)^{-1}\right)W^T\right)\right)x + c_{(m+1,\ldots,m+n)}y + \left(c_{(1,\ldots,m)}W\left(W^TW\right)^{-1}\right)z \\
=&\left(c_{(1,\ldots,m)}\left(I^m - W\left(W^TW\right)^{-1}W^T\right)\right)x + c_{(m+1,\ldots,m+n)}y + c_{(1,\ldots,m)}W\left(W^TW\right)^{-1}W^Tx \\
=&c_{(1,\ldots,m)}x + c_{(m+1,\ldots,m+n)}y \\
\geq&c_0 \text{ (since } \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in P \text{ and } W^Tx \in \mathbb{Z}^k).
\end{aligned}$$

We claim that there exists a maximal lattice-free body $L^{pre} := P^{\leq}\left(A^{pre}, b^{pre}\right) \subseteq \mathbb{R}^k$ such that (5.7) is a valid linear inequality for

$$\begin{aligned}
&S^{LP} \setminus \left(\mathbb{R}^m \times \mathbb{R}^n \times \operatorname{int} L^{pre}\right) && \text{(for Theorem 235 and Theorem 237)}, \\
&S^{LP} \cap \left(\mathbb{R}^m \times \mathbb{R}^n \times \left(\left(\operatorname{int} L^{pre}\right) \dot\cup \left(\left(\operatorname{bd} L^{pre}\right) \setminus \left(\operatorname{bd} L^{pre}\right)_I\right)\right)\right) && \text{(for Theorem 236)}
\end{aligned}$$

and we additionally have

$$\begin{aligned}
&A^{pre} \in \mathbb{Q}^{l \times k} \wedge b^{pre} \in \mathbb{Q}^l && \text{(for Theorem 235)}, \\
&A^{pre} \in \mathbb{R}^{l \times k} \wedge b^{pre} \in \mathbb{R}^l && \text{(for Theorem 236 and Theorem 237)}
\end{aligned}$$

($l \in \mathbb{Z}_{\geq 0}$) and L^{pre} is full-dimensional if necessary (for Theorem 235 and Theorem 237; for Theorem 236 if $\operatorname{proj}_{\mathbb{R}^m} P$ is full-dimensional).

- In the case of Theorem 235, this is a consequence of Theorem 231: because P is a rational polyhedron, so is S^{LP}.
- In the case of Theorem 236, this is a consequence of Theorem 233. Note that if $\operatorname{proj}_{\mathbb{R}^m} P$ is full-dimensional, by Lemma 230, $\operatorname{proj}_{\mathbb{R}^k} S^{LP}$ is full-dimensional, too.
- In case of Theorem 237, note that, since P is convex and compact, so is also S and we can apply Theorem 234.

We claim that

- $L := P^{\leq}\left(A^{pre}W^T, b^{pre}\right)$ is a rational (in case of Theorem 235), full-dimensional (except for Theorem 236 if $\operatorname{proj}_{\mathbb{R}^m} P$ is not full-dimensional), maximal lattice-free body with $\operatorname{lineal} L \geq V'$.
- $c(\cdot) \geq c_0$ is valid for
  - $P \setminus \left(\left(\operatorname{int} L\right) \times \mathbb{R}^n\right)$ (for Theorem 235 and Theorem 237),
  - $P \setminus \left(\left(\left(\operatorname{int} L\right) \dot\cup \left(\left(\operatorname{bd} L\right) \setminus \left(\left(\operatorname{bd} L\right)_I + V'\right)\right)\right) \times \mathbb{R}^n\right)$ (for Theorem 236).

Since $V' \perp w^1, \ldots, w^k$, we clearly have $A^{pre}W^Tv = A^{pre}0^m = 0^l$ for all $v \in V'$. Thus, $\operatorname{lineal} L \geq V'$. To see that L is lattice-free, let $z \in \mathbb{Z}^m$. Then, clearly,

$$W^Tz \in \mathbb{Z}^k \tag{5.8}$$

and, since L^{pre} is lattice-free, there exists an $i \in [l]$ such that $A_{i,*}W^Tz \geq b_i$.

For L being full-dimensional (if applicable): by construction, $L^{pre} \subseteq \mathbb{R}^k$ is full-dimensional in the cases where we want to prove that L is full-dimensional. So,

$$\exists x^{*,pre} \in L^{pre}, \epsilon^{pre} \in \mathbb{R}_{>0} : B_{\epsilon^{pre}}\left(x^{*,pre}\right) \subseteq L^{pre}.$$

Now consider that $L = \left\{x \in \mathbb{R}^m : W^Tx \in L^{pre}\right\}$. By Lemma 238, there exists an $\epsilon \in \mathbb{R}_{>0}$ such that $B_\epsilon\left(x^*\right) \subseteq L$, where $x^* := W\left(W^TW\right)^{-1}x^{*,pre}$.

For L being *maximal* lattice-free: by construction, L^{pre} is maximal lattice-free. Thus, by Theorem 108, L^{pre} can only be of two forms:

1. L^{pre} is a full-dimensional polyhedron with an integral point in the relative interior of each facet,

2. L^{pre} is a translate of an irrational hyperplane.

For case 1: We can w.l.o.g. assume that every inequality of $P^{\leq}(A^{pre}, b^{pre})$ is facet-defining. Thus, by Theorem 108,

$$\forall i \in [l]\, \exists z^{*,pre,i} \in \mathbb{Z}^k : A^{pre}_{i,*} z^{*,pre,i} = b^{pre}_i \wedge A^{pre}_{[l]\backslash\{i\},*} z^{*,pre,i} < b^{pre}_{[l]\backslash\{i\}}.$$

Because of (5.6), we have

$$\forall i \in [l]\, \exists z^{*,i} \in \mathbb{Z}^m, v'^{*,i} \in V' : W^T \left(z^{*,i} + v'^{*,i}\right) = z^{*,pre,i}.$$

This implies that for all $i \in [l]$, we have

$$\begin{aligned} A^{pre}_{i,*} W^T z^{*,i} &= A^{pre}_{i,*} W^T \left(z^{*,i} + v'^{*,i}\right) &&= A^{pre}_{i,*} z^{*,pre,i} &&= b_i, \\ A^{pre}_{[l]\backslash\{i\},*} W^T z^{*,i} &= A^{pre}_{[l]\backslash\{i\},*} W^T \left(z^{*,i} + v'^{*,i}\right) &&= A^{pre}_{[l]\backslash\{i\},*} z^{*,pre,i} &&< b^{pre}_{[l]\backslash\{i\}}. \end{aligned}$$

By Theorem 110, this implies that $L = P^{\leq}\left(A^{pre} W^T, b^{pre}\right)$ is a full-dimensional, maximal lattice-free body.

For case 2: We can assume that $l = 1$ and $A^{pre}_{1,*} \neq (0^m)^T$. Denote $d := A^{pre}_{1,*}$ and $d_0 := b^{pre}_1$. Assume that there exists a lattice-free body $\hat{L} \supsetneq L$ and let $\hat{x}^* \in \hat{L}\backslash L$. By definition, $dW^T \hat{x}^* \neq d_0$. W.l.o.g. let $dW^T \hat{x}^* > d_0$. Consider

$$\begin{aligned} \hat{L}^- &:= \left\{x \in \mathbb{R}^m : d_0 \leq dW^T x < dW^T \hat{x}^*\right\}, \\ \hat{L}^{pre,-} &:= \left\{x \in \mathbb{R}^k : d_0 \leq dx < d\hat{x}^*\right\}. \end{aligned}$$

Clearly,

$$\hat{L}^- = \left\{x \in \mathbb{R}^m : W^T x \in \hat{L}^{pre,-}\right\}. \tag{5.9}$$

Additionally,

$$\hat{L}^- \subseteq \hat{L} \tag{5.10}$$

holds (we show this later on). Let $z^{*,pre} \in \mathbb{Z}^k$ and $\epsilon^{*,pre} \in \mathbb{R}_{>0}$ be such that

$$B_{\epsilon^{*,pre}}\left(z^{*,pre}\right) \subseteq \operatorname{int}\left(\operatorname{conv}\left(L^{pre} \dot\cup \left\{W^T \hat{x}^*\right\}\right)\right) \subseteq \left\{x \in \mathbb{R}^m : d_0 \leq dW^T x \leq dW^T \hat{x}^*\right\}.$$

Since $B_{\epsilon^{*,pre}}\left(z^{*,pre}\right)$ is an open set, we even have

$$\begin{aligned} B_{\epsilon^{*,pre}}\left(z^{*,pre}\right) &\subseteq \operatorname{int}\left\{x \in \mathbb{R}^m : d_0 \leq dW^T x \leq dW^T \hat{x}^*\right\} \\ &= \left\{x \in \mathbb{R}^m : d_0 < dW^T x < dW^T \hat{x}^*\right\} \\ &\subseteq \left\{x \in \mathbb{R}^m : d_0 \leq dW^T x < dW^T \hat{x}^*\right\} \\ &= \hat{L}^{pre,-}. \end{aligned}$$

So, by Lemma 238 and (5.9), there exists an $\epsilon^* \in \mathbb{R}_{>0}$ such that $B_\epsilon\left(W\left(W^T W\right)^{-1} z^{*,pre}\right) \subseteq \hat{L}^-$. Now consider that

$$W^T\left(W\left(W^T W\right)^{-1} z^{*,pre}\right) = z^{*,pre} \in \mathbb{Z}^k.$$

Thus, by (5.6), we have $W\left(W^T W\right)^{-1} z^{*,pre} =: z^* + v^*$, where $z^* \in \mathbb{Z}^m$ and $v^* \in V'$. Since $V' \leq \operatorname{lineal} \hat{L}^-$, we thus conclude using (5.10):

$$B_\epsilon\left(z^*\right) \subseteq \hat{L}^- \subseteq \hat{L}.$$

So, clearly, $\hat{L}$ is not lattice-free.

Now for (5.10): let $x \in \hat{L}^-$, i.e. $d_0 \leq dW^T x < dW^T \hat{x}^*$ and let $x = x^{Wd^T} + x^{\left(Wd^T\right)^\perp}$, where

$$x^{Wd^T} \in \operatorname{lin}\left\{Wd^T\right\}, \qquad x^{\left(Wd^T\right)^\perp} \perp Wd^T.$$

Clearly,

$$x = \frac{dW^T(\hat{x}^* - x)}{dW^T\hat{x}^* - d_0} \cdot \underbrace{\left(\frac{\left(dW^T\hat{x}^*\right)\cdot x - \left(dW^Tx\right)\cdot \hat{x}^* + d_0 \cdot (\hat{x}^* - x)}{dW^T(\hat{x}^* - x)}\right)}_{=:y^*} + \frac{dW^Tx - d_0}{dW^T\hat{x}^* - d_0}\cdot \hat{x}^*$$
$$\in \operatorname{conv}\{y^*, \hat{x}^*\}.$$

What remains to be shown is $y^* \in \hat{L}$. We show that even $y^* \in L$ holds:

$$dW^Ty^* = dW^T\frac{\left(dW^T\hat{x}^*\right)\cdot x - \left(dW^Tx\right)\cdot \hat{x}^* + d_0 \cdot (\hat{x}^* - x)}{dW^T(\hat{x}^* - x)} = d_0.$$

We now show that $c(\cdot) \geq c_0$ is valid for

1. $P\backslash((\operatorname{int} L)\times\mathbb{R}^n)$ (for Theorem 235 and Theorem 237),
2. $P\backslash(((\operatorname{int} L)\,\dot\cup\,((\operatorname{bd} L)\setminus((\operatorname{bd} L)_I + V')))\times\mathbb{R}^n)$ (for Theorem 236).

For 1: Let $\binom{x}{y} \in P\backslash((\operatorname{int} L)\times\mathbb{R}^n)$. So, there exists an $i^* \in [l]$ such that $A^{pre}_{i^*,*}W^Tx \geq b^{pre}_{i^*}$. This means $W^Tx \in \mathbb{R}^k\backslash(\operatorname{int} L^{pre})$ and thus

$$\begin{pmatrix} x \\ y \\ W^Tx \end{pmatrix} \in S^{LP}\backslash(\mathbb{R}^m \times \mathbb{R}^n \times (\operatorname{int} L^{pre})).$$

So, (5.7) is a valid inequality for $\begin{pmatrix} x \\ y \\ W^Tx \end{pmatrix}$ and we conclude using (5.7):

$$\begin{aligned} &c_{(1,\dots,m)}x + c_{(m+1,\dots,m+n)}y \\ =&c_{(1,\dots,m)}\left(I^m - W\left(W^TW\right)^{-1}W^T\right)x + c_{(m+1,\dots,m+n)}y + \left(c_{(1,\dots,m)}W\left(W^TW\right)^{-1}W^T\right)x \\ \geq& c_0. \end{aligned}$$

For 2: Let

$$\begin{pmatrix} x \\ y \end{pmatrix} \in P\backslash(((\operatorname{int} L)\,\dot\cup\,((\operatorname{bd} L)\setminus((\operatorname{bd} L)_I + V')))\times\mathbb{R}^n)$$

This implies that there exists some $i^* \in [l]$ such that

1. $A^{pre}_{i^*,*}W^Tx > b_{i^*}$ or
2. $A^{pre}_{i^*,*}W^Tx = b_{i^*} \wedge x \notin \mathbb{Z}^m + V'$.

We claim that in both cases

$$W^Tx \in \mathbb{R}^k\backslash((\operatorname{int} L^{pre})\,\dot\cup\,((\operatorname{bd} L^{pre})\setminus(\operatorname{bd} L^{pre})_I)) \tag{5.11}$$

holds.

In case 1, we clearly have

$$W^Tx \in \mathbb{R}^k\backslash L^{pre} \subseteq \mathbb{R}^k\backslash((\operatorname{int} L^{pre})\,\dot\cup\,((\operatorname{bd} L^{pre})\setminus(\operatorname{bd} L^{pre})_I)),$$

which implies (5.11).

In case 2, if $W^Tx \notin L^{pre}$, equation (5.11) clearly holds. So, let $W^Tx \in L^{pre}$. Then, by case assumption, clearly $W^Tx \in \operatorname{bd} L^{pre}$. Assuming $W^Tx \in \mathbb{Z}^k$ leads by (5.6) to the contradiction $x \in \mathbb{Z}^m + V'$. So, $W^Tx \in \operatorname{bd} L^{pre}\backslash(\operatorname{bd} L^{pre})_I$, again showing (5.11).

Using (5.11), we obtain

$$\begin{pmatrix} x \\ y \\ W^Tx \end{pmatrix} \in S^{LP}\backslash(\mathbb{R}^m \times \mathbb{R}^n \times ((\operatorname{int} L^{pre})\,\dot\cup\,((\operatorname{bd} L^{pre})\setminus(\operatorname{bd} L^{pre})_I))).$$

So, (5.7) is a valid inequality for $\begin{pmatrix} x \\ y \\ W^T x \end{pmatrix}$ and we conclude using (5.7):

$$
\begin{aligned}
&c_{(1,\dots,m)}x + c_{(m+1,\dots,m+n)}y \\
=&c_{(1,\dots,m)}\left(I^m - W\left(W^TW\right)^{-1}W\right)x + c_{(m+1,\dots,m+n)}y + c_{(1,\dots,m)}W\left(W^TW\right)^{-1}\left(W^Tx\right) \\
\geq&c_0.
\end{aligned}
$$

□

5.2.4. Reverse inclusions

Theorem 239. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ be arbitrary and let $V' \leq \mathbb{R}^m$ be a rational subspace of codimension $k \in \{0, \dots, m\}$. Let $\emptyset \neq L = P^{\leq}\left(A^L, b^L\right) \subseteq \mathbb{R}^m$ $(A^L \in \mathbb{R}^{l\times m}$ and $b^L \in \mathbb{R}^l$, where $l \in \mathbb{Z}_{\geq 0})$ be a lattice-free polyhedron such that* $\operatorname{lineal} L \geq V'$. *Then every linear inequality for*

$$P \backslash \left(\left(\left(\operatorname{int} L\right) \dot\cup \left(\left(\operatorname{bd} L\right) \backslash \left(\left(\operatorname{bd} L\right)_I + V'\right)\right)\right) \times \mathbb{R}^n\right) \tag{5.12}$$

(this in particular includes the special case $P \backslash \left(\left(\operatorname{int} L\right) \times \mathbb{R}^n\right)$) is an $L_{k,\mathbb{Q}}$ cut for P with respect to $V := V' \times \mathbb{R}^n$.

Proof. Let $c(\cdot) \geq c_0$ be valid for (5.12) and let $\binom{x}{y} := \underbrace{z}_{\in \mathbb{Z}^m \times \mathbb{R}^n} + \underbrace{v}_{\in V} \in P \cap \left(\left(\mathbb{Z}^m \times \mathbb{R}^n\right) + V\right)$. If we can show $\binom{x}{y} \in (5.12)$, we are done. Clearly, $\binom{x}{y} \in P$; so, only

$$x \in \mathbb{R}^m \backslash \left(\left(\operatorname{int} L\right) \dot\cup \left(\left(\operatorname{bd} L\right) \backslash \left(\left(\operatorname{bd} L\right)_I + V'\right)\right)\right)$$

remains to be shown. Because of $\operatorname{lineal} L \geq V'$ and $L \neq \emptyset$, we have $A^L v_{(1,\dots,m)} = 0^l$. Thus,

$$A^L x = A^L\left(z_{(1,\dots,m)} + v_{(1,\dots,m)}\right) = A^L z_{(1,\dots,m)}. \tag{5.13}$$

Since $z_{(1,\dots,m)} \in \mathbb{Z}^m$ and L is lattice-free, we have

$$\exists i \in [l] : A_{i,*} z_{(1,\dots,m)} \geq b_i. \tag{5.14}$$

We consider two cases:

1. $\exists i \in [l] : A_{i,*} z_{(1,\dots,m)} > b_i$,
2. $\forall i \in [l] : A_{i,*} z_{(1,\dots,m)} \leq b_i$.

For case 1: Let $i^* \in [l]$ be such that $A_{i^*,*} z_{(1,\dots,m)} > b_{i^*}$. Using (5.13), we obtain $A^L_{i^*,*} x > b^L_{i^*}$, which implies $x \in \mathbb{R}^m \backslash L$. Finally, since

$$L \supseteq \left(\operatorname{int} L\right) \dot\cup \left(\left(\operatorname{bd} L\right) \backslash \left(\left(\operatorname{bd} L\right)_I + V'\right)\right),$$

we conclude $x \in \mathbb{R}^m \backslash \left(\left(\operatorname{int} L\right) \dot\cup \left(\left(\operatorname{bd} L\right) \backslash \left(\left(\operatorname{bd} L\right)_I + V'\right)\right)\right)$.

For case 2: Let $i^* \in [l]$ be such that $A_{i^*,*} z_{(1,\dots,m)} = b_{i^*}$ (i^* exists by case assumption and (5.14)). Combined with the case assumption, we thus obtain $z_{(1,\dots,m)} \in \operatorname{bd} L$, from which we conclude $z_{(1,\dots,m)} \in \left(\operatorname{bd} L\right)_I$. So, $x \in \left(\operatorname{bd} L\right)_I + V'$. Because of

$$\left(\operatorname{bd} L\right)_I + V' \subseteq \mathbb{R}^m \backslash \left(\left(\operatorname{int} L\right) \dot\cup \left(\left(\operatorname{bd} L\right) \backslash \left(\left(\operatorname{bd} L\right)_I + V'\right)\right)\right),$$

we conclude $x \in \mathbb{R}^m \backslash \left(\left(\operatorname{int} L\right) \dot\cup \left(\left(\operatorname{bd} L\right) \backslash \left(\left(\operatorname{bd} L\right)_I + V'\right)\right)\right)$. □

5.2.5. Conclusion

We now summarize the central results that we showed in section 5.2. Recall that in section 3.4.1, we looked ahead and mentioned that rational $L_{k,\mathbb{Q}}$ cuts for a rational polyhedron are often equivalent to lattice-free cuts with respect to (maximal) lattice-free bodies $L \subseteq \mathbb{R}^m$ having

$$\operatorname{codim}\left(\operatorname{lineal} L\right) \geq k.$$

Formulating this relationship more precisely is one of the statements of the following theorem (Theorem 240).

Theorem 240. *Let $m,n \in \mathbb{Z}_{\geq 0}$, let $k \in \{0,\dots,m\}$ and let $V' \leq \mathbb{R}^m$ be a rational subspace of codimension k. Then:*

1. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ be a rational polyhedron, let $c \in (\mathbb{Q}^m \times \mathbb{Q}^n)^T$ and let $c_0 \in \mathbb{R}$ (recall that, by Theorem 178, for rational polyhedra, every $L_{k,\mathbb{Q}}$ cut is dominated absolutely by a set of* rational *$L_{k,\mathbb{Q}}$ cuts). Let $c(\cdot) \geq c_0$ be an $L_{k,\mathbb{Q}}$ cut for P with respect to $V' \times \mathbb{R}^n$. Then there exists a rational maximal lattice-free polyhedron $L \subseteq \mathbb{R}^m$ having* $\operatorname{lineal} L \geq V'$ *such that $c(\cdot) \geq c_0$ is valid for*
$$P \backslash ((\operatorname{int} L) \times \mathbb{R}^n),$$
which in particular implies that $c(\cdot) \geq c_0$ is valid for
$$P \backslash (((\operatorname{int} L) \,\dot{\cup}\, ((\operatorname{bd} L) \setminus (\operatorname{bd} L)_I)) \times \mathbb{R}^n).$$
Together with 3 of this enumeration, we thus get: an inequality $c(\cdot) \geq c_0$, where $c \in (\mathbb{Q}^m \times \mathbb{Q}^n)^T$ and $c_0 \in \mathbb{R}$, is an $L_{k,\mathbb{Q}}$ cut for P with respect to $V' \times \mathbb{R}^n$ if and only if there exists a rational lattice-free polyhedron $L \subseteq \mathbb{R}^m$ such that $\operatorname{lineal} L \geq V'$ *and $c(\cdot) \geq c_0$ is valid for*
$$P \backslash ((\operatorname{int} L) \times \mathbb{R}^n).$$

2. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ be convex, let $c \in (\mathbb{R}^m \times \mathbb{R}^n)^T$ and let $c_0 \in \mathbb{R}$. Let $c(\cdot) \geq c_0$ be an $L_{k,\mathbb{Q}}$ cut for P with respect to $V' \times \mathbb{R}^n$. Then there exists a maximal lattice-free polyhedron $L \subseteq \mathbb{R}^m$ having* $\operatorname{lineal} L \geq V'$ *such that $c(\cdot) \geq c_0$ is valid for*
$$P \backslash (((\operatorname{int} L) \,\dot{\cup}\, ((\operatorname{bd} L) \setminus (\operatorname{bd} L)_I)) \times \mathbb{R}^n).$$
If $\operatorname{proj}_{\mathbb{R}^m} P$ *is full-dimensional, we can assume L to be full-dimensional, too.*

   Together with 3 of this enumeration, we thus get: $c(\cdot) \geq c_0$ is an $L_{k,\mathbb{Q}}$ cut for P with respect to $V' \times \mathbb{R}^n$ if and only if there exists a lattice-free polyhedron $L \subseteq \mathbb{R}^m$ such that $\operatorname{lineal} L \geq V'$ *and $c(\cdot) \geq c_0$ is valid for*
$$P \backslash (((\operatorname{int} L) \,\dot{\cup}\, ((\operatorname{bd} L) \setminus (\operatorname{bd} L)_I)) \times \mathbb{R}^n).$$

3. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ be arbitrary, let $L \subseteq \mathbb{R}^m$ be a lattice-free polyhedron having* $\operatorname{lineal} L \geq V'$, *let $c \in (\mathbb{R}^m \times \mathbb{R}^n)^T$ and let $c_0 \in \mathbb{R}$ be such that $c(\cdot) \geq c_0$ is valid for*
$$P \backslash (((\operatorname{int} L) \,\dot{\cup}\, ((\operatorname{bd} L) \setminus (\operatorname{bd} L)_I)) \times \mathbb{R}^n)$$
(this is, in particular, satisfied if $c(\cdot) \geq c_0$ is valid for $P \backslash ((\operatorname{int} L) \times \mathbb{R}^n)$). Then $c(\cdot) \geq c_0$ is an $L_{k,\mathbb{Q}}$ cut for P with respect to $V' \times \mathbb{R}^n$.

Proof. 1, 2 and 3, respectively, hold by Theorem 235, Theorem 236 and Theorem 239, respectively. □

5.3. Characterizing essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts via lattice-free bodies

In this section, we want to characterize essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts for rational polyhedra via lattice-free bodies. Let us first recapitulate (cf. Theorem 208) that for an arbitrary $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m,n \in \mathbb{Z}_{\geq 0}$) and $k \in \{1,\dots,m\}$, every $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cut (or $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cut) for P that is not already an $L_{k-1,\mathbb{Q}}$ cut for P, is an *essential* $L_{k-\frac{1}{2},\mathbb{Q}}$ cut for P. We also want to recapitulate (cf. Lemma 159) that for an essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cut $c(\cdot) \leq c_0$ ($k \in \{0,\dots,m\}$) with respect to $V' \times \mathbb{R}^n$, we can assume $c = \begin{pmatrix} c' & (0^n)^T \end{pmatrix}$, where $c'^T \perp V'$.

5.3.1. $k = m$

Theorem 241. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m,n \in \mathbb{Z}_{\geq 0}$) be a rational polyhedron. Then:*

1. *If $(P + (0^n \times \mathbb{R}^n))_I = \emptyset$ (which is equivalent to $P_I = \emptyset$), there exists a rational, full-dimensional, maximal lattice-free polyhedron $L \subseteq \mathbb{R}^m$ such that $P \subseteq (\operatorname{int} L) \times \mathbb{R}^n$. In particular, we have*
$$P \backslash ((\operatorname{int} L) \times \mathbb{R}^n) = \emptyset.$$

2. If $(P + (0^n \times \mathbb{R}^n))_I \neq \emptyset$ (which is equivalent to $P_I \neq \emptyset$), let

$$\underbrace{\begin{pmatrix} c' & (0^n)^T \end{pmatrix}}_{=:c} (\,\cdot\,) \geq c_0,$$

where $c' \in (\mathbb{Q}^m)^T$ and $c_0 \in \mathbb{R}$, be a linear inequality for $(P + (0^n \times \mathbb{R}^n))_I$ that is not already valid for P (because of the particular structure of c, the statements $c(\,\cdot\,) \geq c_0$ being valid for $(P + (0^n \times \mathbb{R}^n))_I$ and $c(\,\cdot\,) \geq c_0$ being valid for P_I are equivalent). Then there exists a rational, full-dimensional polyhedron $\tilde{L} \subseteq \mathbb{R}^m$ such that

a) $L := \tilde{L} \cap P^{\leq}(c', \tilde{c}_0)$ is a rational, full-dimensional, maximal lattice-free polyhedron, where $\tilde{c}_0 \in \mathbb{Q}$ satisfies $\tilde{c}_0 \geq c_0$,

b) $P \subseteq \left(\operatorname{int} \tilde{L}\right) \times \mathbb{R}^n$.

Remark 242. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be a rational polyhedron (as in Theorem 241). Consider an essential $L_{m-\frac{1}{2},\mathbb{Q}}$ cut for P, i.e. a linear inequality for $(P + (0^m \times \mathbb{R}^n))_I$. Then, if $P_I \neq \emptyset$, by Lemma 159, every linear inequality $c(\,\cdot\,) \geq c_0$ for $P + (0^m \times \mathbb{R}^n)$ has to satisfy $c^T \perp 0^m \times \mathbb{R}^n$; so $c = \begin{pmatrix} c' & (0^n)^T \end{pmatrix}$, where $c' \in (\mathbb{R}^m)^T$.*

On the other hand, $P + (0^m \times \mathbb{R}^n)$ is also a rational polyhedron and so is $\operatorname{cl}_I(P + (0^m \times \mathbb{R}^n))$. Thus, every essential $L_{m-\frac{1}{2},\mathbb{Q}}$ cut for P is dominated by a finite *set of* rational *essential $L_{m-\frac{1}{2},\mathbb{Q}}$ cuts for P and we can assume $c' \in (\mathbb{Q}^m)^T$, i.e. the assumptions in 2 of Theorem 241 can be made without loss of generality.*

Similar considerations also hold for Theorem 244 in section 5.3.2.

Proof. (Theorem 241) Since $(P + (0^n \times \mathbb{R}^n))_I = \emptyset$ if and only if $P_I = \emptyset$, 1 is a direct consequence of Theorem 231 if we set $c := (0^m \times 0^n)^T$. So, we only have to show 2.

Let $P' := \operatorname{proj}_{\mathbb{R}^m} P = P^{\leq}(A', b')$, where $A' \in \mathbb{Z}^{t' \times m}$ and $b' \in \mathbb{Z}^{t'}$ ($t' \in \mathbb{Z}_{\geq 0}$). Such a representation exists by Corollary 65. By assumption, we have $(P' \cap P^{<}(c', c_0))_I = \emptyset$. We also know that $P'_I \neq \emptyset$. Let $\mathcal{T}' \subseteq [t']$ be minimal such that $\left(P^{\leq}\left(A'_{\mathcal{T}',*}, b'_{\mathcal{T}'}\right) \cap P^{<}(c', c_0)\right)_I = \emptyset$. If we set

$$\tilde{A} := A'_{\mathcal{T}',*},$$

$$b'' \in \mathbb{Z}^{\mathcal{T}'}, \forall i \in \mathcal{T}' : b''_i := \begin{cases} \lceil b'_i \rceil & \text{if } b'_i \notin \mathbb{Z}, \\ b'_i + 1 & \text{if } b'_i \in \mathbb{Z}, \end{cases}$$

we have by construction

$$P^{<}\left(\begin{pmatrix} \tilde{A} \\ c' \end{pmatrix}, \begin{pmatrix} b'' \\ c_0 \end{pmatrix}\right)_I = \emptyset.$$

So,

$$P^{\leq}\left(\begin{pmatrix} \tilde{A} \\ c' \end{pmatrix}, \begin{pmatrix} b'' \\ c_0 \end{pmatrix}\right)$$

is lattice-free. To be able to apply Theorem 111, we need to show that $P^{<}\left(\tilde{A}, b''\right)_I \neq \emptyset$ (i.e. we need the inequality $c'(\,\cdot\,) \leq c_0$). For this: clearly,

$$P^{<}\left(\tilde{A}, b''\right) \supseteq P^{\leq}(A', b') = P'.$$

On the other hand (by case assumption), we have $P_I \neq \emptyset$ and thus $P'_I \neq \emptyset$. This shows that we really need $c'(\,\cdot\,) \leq c_0$.

So, we can apply Theorem 111 and get that there exists some $\mathbb{Z}^{\mathcal{T}'} \ni \tilde{b} \geq b''$ and some $\mathbb{Q} \ni \tilde{c}_0 \geq c_0$ such that

$$L = P^{\leq}\left(\begin{pmatrix} \tilde{A} \\ c \end{pmatrix}, \begin{pmatrix} \tilde{b} \\ \tilde{c}_0 \end{pmatrix}\right)$$

is maximal lattice-free.

Finally, for $P \subseteq \left(\operatorname{int} \tilde{L}\right) \times \mathbb{R}^n$:

$$P \subseteq P^{<}\left(\tilde{A}, b''\right) \times \mathbb{R}^n \subseteq P^{<}\left(\tilde{A}, \tilde{b}\right) \times \mathbb{R}^n \subseteq \left(\operatorname{int} \tilde{L}\right) \times \mathbb{R}^n.$$

□

5.3.2. $k \leq m$

Before we state our main statement in Theorem 244, we prove a small lemma:

Lemma 243. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be arbitrary and let $V' \leq \mathbb{R}^m$ be a rational subspace. Then*

$$(P + (V' \times \mathbb{R}^n))_I = \emptyset \Leftrightarrow P \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + (V' \times \mathbb{R}^n)) = \emptyset.$$

Proof. Let $p \in P$, $z \in \mathbb{Z}^m \times \mathbb{R}^n$ and $v \in V' \times \mathbb{R}^n$. Then $p + v = z \Leftrightarrow p = z - v$. From this, we easily conclude

$$(P + (V' \times \mathbb{R}^n))_I = \emptyset \Leftrightarrow P \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + (V' \times \mathbb{R}^n)) = \emptyset.$$

□

Theorem 244. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be a rational polyhedron and let $V' \leq \mathbb{R}^m$ be a rational subspace of codimension $k \in \{0, \ldots, m\}$. Then:*

1. *If $(P + V)_I = \emptyset$, there exists a rational, full-dimensional, maximal lattice-free polyhedron $L \subseteq \mathbb{R}^m$ such that $V' \leq \operatorname{lineal} L$ and $P \subseteq (\operatorname{int} L) \times \mathbb{R}^n$. In particular, we have*

$$P \backslash ((\operatorname{int} L) \times \mathbb{R}^n) = \emptyset.$$

2. *If $(P + V)_I \neq \emptyset$, let $\begin{pmatrix} c' & (0^n)^T \end{pmatrix} (\cdot) \geq c_0$, where $c' \in (\mathbb{Q}^m)^T \cap V'^{\perp}$ and $c_0 \in \mathbb{R}$, be a linear inequality for P_I that is not already valid for P. Then there exists a rational, full-dimensional polyhedron $\tilde{L} \subseteq \mathbb{R}^m$ having*
   - $\operatorname{lineal} \tilde{L} \geq V'$,
   - $P \subseteq \left(\operatorname{int} \tilde{L}\right) \times \mathbb{R}^n$ *and*
   - $L := \tilde{L} \cap P^{\leq}(c', \tilde{c}_0)$ *is a rational, full-dimensional, maximal lattice-free polyhedron, where $\tilde{c}_0 \in \mathbb{Q}$ satisfies $\tilde{c}_0 \geq c_0$.*

Proof. By Lemma 243, we have

$$(P + (V' \times \mathbb{R}^n))_I = \emptyset \Leftrightarrow P \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + (V' \times \mathbb{R}^n)) = \emptyset.$$

With this in mind, 1 is a direct consequence of Theorem 235 if we set $c := (0^m \times 0^n)^T$. So, we only need to show 2.

By Theorem 215 and $c'^T \perp V'$, $\begin{pmatrix} c' & (0^n)^T \end{pmatrix} (\cdot) \geq c_0$ is an essential $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cut for P if and only if there exist linearly independent $w^1, \ldots, w^k \in \mathbb{Z}^k$ such that $\begin{pmatrix} c' & (0^n)^T \end{pmatrix} (\cdot) \geq c_0$ is valid for

$$\left\{x \in \operatorname{proj}_{\mathbb{R}^m} P : \forall i \in [k] : \left(w^i\right)^T x \in \mathbb{Z}\right\} \times \mathbb{R}^n.$$

This is equivalent to $c'(\cdot) \geq c_0$ being valid for

$$\left\{x \in \operatorname{proj}_{\mathbb{R}^m} P : \forall i \in [k] : \left(w^i\right)^T x \in \mathbb{Z}\right\}.$$

Let $W := \begin{pmatrix} w^1 & \cdots & w^k \end{pmatrix}$ and define

$$\begin{aligned} S^{LP} &:= \left\{\begin{pmatrix} x \\ z \end{pmatrix} \in (\operatorname{proj}_{\mathbb{R}^m} P) \times \mathbb{R}^k : z = W^T x\right\}, \\ S &:= S^{LP} \cap \left(\mathbb{R}^m \times \mathbb{Z}^k\right). \end{aligned}$$

We claim that $S \neq \emptyset$: by assumption, there exists an $\left(\begin{smallmatrix} x^* \\ y^* \end{smallmatrix}\right) \in (P + V)_I$. Thus, $x^* = p^* + v^*$, where $p^* \in \operatorname{proj}_{\mathbb{R}^m} P$ and $v^* \in V'$. We have using $\operatorname{im} W \perp V'$:

$$W^T p^* = W^T (x^* - v^*) = W^T z^* \in \mathbb{Z}^k.$$

Thus, $\binom{x^*}{W^T x^*} \in S$.

We next claim that

$$\left(c'W\left(W^TW\right)^{-1}\right)z \geq c_0 \tag{5.15}$$

is a valid inequality for $\binom{x}{z} \in S$ (the inverse $\left(W^TW\right)^{-1}$ exists by Lemma 28). For this, we show

$$c'W\left(W^TW\right)^{-1}W^T = c'. \tag{5.16}$$

Since $c'^T \perp V'$ and $\{w^1, \ldots, w^k\}$ forms a basis of $V'^\perp$, we have by Lemma 29:

$$W\left(W^TW\right)^{-1}W^T(\,\cdot\,) = \operatorname{proj}^\perp_{V'^\perp}(\,\cdot\,).$$

Thus,

$$\begin{aligned} c'W\left(W^TW\right)^{-1}W^T &= \left(\left(W\left(W^TW\right)^{-1}W^T\right)^T c'^T\right)^T \\ &= \left(\operatorname{proj}^\perp_{V^\perp} c'^T\right)^T \\ &= c'. && (c^T \in V^\perp) \end{aligned}$$

To show (5.15), let $\binom{x}{z} \in S$. Then

$$\begin{aligned} \left(c'W\left(W^TW\right)^{-1}\right)z &= c'W\left(W^TW\right)^{-1}W^Tx \\ &= c'x && \text{(by (5.16))} \\ &\geq c_0. && (x \in \operatorname{proj}_{\mathbb{R}^m} P,\ W^Tx \in \mathbb{Z}^k) \end{aligned}$$

Since (5.15) is valid for $\binom{x}{z} \in S$, by Theorem 241, there exists a rational, full-dimensional polyhedron

$$\tilde{L}^{pre} := P^{\leq}\left(\tilde{A}^{pre}, \tilde{b}^{pre}\right) \subseteq \mathbb{R}^k$$

that satisfies $\tilde{A} \in \mathbb{Q}^{\tilde{t}\times k}$, $\tilde{b} \in \mathbb{Q}^{\tilde{t}}$ and

$$S^{LP} \subseteq \operatorname{int} \tilde{L}^{pre}, \tag{5.17}$$

for which there exists a $\tilde{c}_0 \in \mathbb{Q}$ having $\tilde{c}_0 \geq c_0$ such that

$$L^{pre} := P^{\leq}\left(\begin{pmatrix} \tilde{A}^{pre} \\ c'W\left(W^TW\right)^{-1} \end{pmatrix}, \begin{pmatrix} \tilde{b}^{pre} \\ \tilde{c}_0 \end{pmatrix}\right)$$

is a rational, full-dimensional, lattice-free polyhedron. Set

$$\tilde{L} := P^{\leq}\left(\tilde{A}^{pre}W^T, \tilde{b}^{pre}\right).$$

Proving that

$$L := P^{\leq}\left(\begin{pmatrix} \tilde{A}^{pre}W^T \\ c'W\left(W^TW\right)^{-1}W^T \end{pmatrix}, \begin{pmatrix} \tilde{b}^{pre} \\ \tilde{c}_0 \end{pmatrix}\right) \tag{5.18}$$

is a full-dimensional maximal lattice-free body with lineal $L \geq V'$ works completely similar as in the proof of Theorem 235. In (5.16), we already saw that $cW\left(W^TW\right)^{-1}W^T = c$ (thus, the last row in the definition of L (cf. (5.18)) is indeed c); so, the only remaining statement to show is $P \subseteq \left(\operatorname{int} \tilde{L}\right) \times \mathbb{R}^n$:

For this, let $\binom{x}{y} \in P$. This implies $\binom{x}{W^Tx} \in S^{LP}$. Using (5.17), we obtain

$$\begin{pmatrix} x \\ W^Tx \end{pmatrix} \in S^{LP} \subseteq \mathbb{R}^m \times \left(\operatorname{int} \tilde{L}^{pre}\right).$$

Thus, $\left(\tilde{A}^{pre}W^T\right)x = \tilde{A}^{pre}\left(W^Tx\right) < \tilde{b}^{pre}$, which implies $x \in \operatorname{int}\tilde{L}$. Thus, clearly, $\binom{x}{y} \in \left(\operatorname{int}\tilde{L}\right) \times \mathbb{R}^n$. $\square$

5.3.3. Reverse inclusions

Theorem 245. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be arbitrary, let $V' \leq \mathbb{R}^m$ be a rational subspace of codimension $k \in \{0, \ldots, m\}$ and let*

$$L := \tilde{L} \cap P^{\leq}(c', c_0) \subseteq \mathbb{R}^m$$

($c' \in (\mathbb{R}^m)^T$, $c'^T \perp V'$ and $c_0 \in \mathbb{R}$) be a lattice-free body, where

- *$\tilde{L} \subseteq \mathbb{R}^m$ is a polyhedron,*
- *$\operatorname{lineal} \tilde{L} \geq V'$ and*
- *$P \subseteq \operatorname{int} \tilde{L} \times \mathbb{R}^m$.*

Then $\begin{pmatrix} c' & (0^n)^T \end{pmatrix}(\cdot) \geq c_0$ is an essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cut for P with respect to $V := V' \times \mathbb{R}^n$.

Proof. We have to show that $\begin{pmatrix} c' & (0^n)^T \end{pmatrix}(\cdot) \geq c_0$ is valid for $(P+V)_I$. Let

$$z := \underbrace{p}_{\in P} + \underbrace{v}_{\in V} \in (P+V)_I .$$

Since $z_{(1,\ldots,m)} \in \mathbb{Z}^m$ and L is lattice-free, we have $z_{(1,\ldots,m)} \notin \operatorname{int} L$, which implies that we either have $z_{(1,\ldots,m)} \notin \operatorname{int} \tilde{L}$ or $c' z_{(1,\ldots,m)} \geq c_0$ (the latter is what we want to show).

To show that the latter holds, assume that $z_{(1,\ldots,m)} \notin \operatorname{int} \tilde{L}$. Since $\operatorname{lineal} \tilde{L} \geq V'$, we obtain $p_{(1,\ldots,m)} \notin \operatorname{int} \tilde{L}$. But this is a contradiction to $P \subseteq \operatorname{int} \tilde{L} \times \mathbb{R}^m$.

So, $\begin{pmatrix} c' & (0^n)^T \end{pmatrix}(\cdot) \geq c_0$ is indeed valid for $(P+V)_I$. □

5.3.4. Conclusion

Theorem 246. *Let*

- *$P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be a rational polyhedron,*
- *$V' \leq \mathbb{R}^m$ be a rational subspace of codimension $k \in \{0, \ldots, m\}$,*
- *$c' \in (\mathbb{Q}^m)^T$ where $c'^T \perp V'$ and*
- *$c_0 \in \mathbb{R}$*

be such that $\begin{pmatrix} c' & (0^n)^T \end{pmatrix}(\cdot) \geq c_0$ is not valid for P. Then $\begin{pmatrix} c' & (0^n)^T \end{pmatrix}(\cdot) \geq c_0$ is an essential $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cut for $\binom{x}{y} \in P$ with respect to $V \times \mathbb{R}^n$ if and only if there exists a rational polyhedron $\tilde{L} \subseteq \mathbb{R}^m$ having

- *$\operatorname{lineal} \tilde{L} \geq V'$,*
- *$P \subseteq \left(\operatorname{int} \tilde{L}\right) \times \mathbb{R}^n$ and*
- *$L := \tilde{L} \cap P^{\leq}(c', \tilde{c}_0)$ is a rational, full-dimensional, maximal lattice-free polyhedron, where $\tilde{c}_0 \in \mathbb{Q}$ satisfies $\tilde{c}_0 \geq c_0$.*

Proof. "If" holds by Theorem 245 and "only if" by Theorem 244. □

5.4. Characterizing $L_{k,\mathbb{Q}}$/essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts via t-branch split cuts

5.4.1. Definitions and properties

Definition 247. *(Cf. [DDG+13, section 3]) Let $B \subseteq \mathbb{R}^m$ ($m \in \mathbb{Z}_{\geq 0}$) be closed, bounded and convex, and let $c \in \mathbb{Z}^m$. Then we define*

$$w(B, c) := \max\left\{c^T x : x \in B\right\} - \min\left\{c^T x : x \in B\right\}.$$

*as **width of** B **along the direction** c. The **lattice width of** B, which we denote by $w(B)$, is defined as*

$$w(B) := \min_{c \in \mathbb{Z}^m \setminus \{0^m\}} w(B, c).$$

If B is not closed, we set $w(B) := w(\overline{B})$.

Remark/Definition 248. *There exists a function $f : \mathbb{Z}_{\geq 1} \to \mathbb{R}_{\geq 1}$ such that for any strictly lattice-free, bounded, convex set $B \subsetneq \mathbb{R}^m$ ($m \in \mathbb{Z}_{\geq 1}$), we have*

$$w(B) \leq f(m).$$

For an overview of functions with this property, cf. [DDG+13, section 3]. According to this text, the best known asymptotic upper bound for such an f is of order $O\left(n^{\frac{4}{3}} log^c n\right)$ for some constant c. From now on, let f be such a function. Define for $m \in \mathbb{Z}_{\geq 1}$:

$$\overline{f}_m := 1 + \lceil f(m) \rceil$$

and let

$$h : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0} : \quad m \mapsto \sum_{i=1}^{m} \prod_{j=m-i+1}^{m} \overline{f}_j.$$

In [DDG+13, Lemma 3.3], it is shown:

Lemma 249. *Any (convex) bounded, strictly lattice-free set (cf. Definition 102) $B \subseteq \mathbb{R}^m$ ($m \in \mathbb{Z}_{\geq 0}$) is contained in the union of some $h(m)$ split sets.*

In [DDG+13, Lemma 3.4], this is tightened to

Lemma 250. *Let $B \subseteq \mathbb{R}^m$ ($m \in \mathbb{Z}_{\geq 0}$) be a (convex) unbounded strictly lattice-free set. If B is contained in the interior of a maximal lattice-free convex set in $\mathbb{R}^m$, then it is contained in the union of some $h(m)$ split sets.*

From Lemma 249 and Lemma 250, we immediately obtain:

Lemma 251. *Let $B \subseteq \mathbb{R}^m$ ($m \in \mathbb{Z}_{\geq 0}$) be a (convex) full-dimensional strictly lattice-free set. Then B is contained in the union of some $h(m)$ split sets.*

Since a central topic of this section (section 5.4) is to unify the theory of $L_{k,\mathbb{Q}}$ cuts and t-branch split cuts, we define:

Definition 252. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be arbitrary, let $k \in \{0, \ldots, m\}$ (even though the case $k = 0$ is only interesting for formal purposes) and let $t \in \mathbb{Z}_{\geq 0}$. A* ***$k, t$-branch split cut for*** *P is an inequality $c(\cdot) \geq c_0$ that is valid for some*

$$P \cap \left(D\left(\pi^1, \ldots, \pi^t, \gamma_1, \ldots, \gamma_t\right) \times \mathbb{R}^n\right),$$

where $D\left(\pi^1, \ldots, \pi^t, \gamma_1, \ldots, \gamma_t\right)$ is a t-branch split disjunction such that $\dim\left(\operatorname{lin}\left\{\pi^1, \ldots, \pi^t\right\}\right) \leq k$. We define the ***$k, t$-branch split closure of*** *P as*

$$\begin{aligned}\operatorname{cl}_{k,tBS}(P) &:= P \cap \bigcap_{\substack{\pi^1,\ldots,\pi^t \in \mathbb{Z}^m,\\ \gamma_1,\ldots,\gamma_t \in \mathbb{Z}:\\ \dim\left(\operatorname{lin}\left\{\pi^1,\ldots,\pi^t\right\}\right) \leq k}} \bigcap_{\substack{c \in (\mathbb{R}^m \times \mathbb{R}^n)^T, c_0 \in \mathbb{R}:\\ P \cap \left(D\left(\pi^1,\ldots,\pi^t,\gamma_1,\ldots,\gamma_t\right) \times \mathbb{R}^n\right) \subseteq P^{\leq}(c, c_0)}} P^{\leq}(c, c_0) \\ &= P \cap \bigcap_{\substack{\pi^1,\ldots,\pi^t \in \mathbb{Z}^m,\\ \gamma_1,\ldots,\gamma_t \in \mathbb{Z}:\\ \dim\left(\operatorname{lin}\left\{\pi^1,\ldots,\pi^t\right\}\right) \leq k}} \overline{\operatorname{conv}}\left(P \cap \left(D\left(\pi^1, \ldots, \pi^t, \gamma_1, \ldots, \gamma_t\right) \times \mathbb{R}^n\right)\right).\end{aligned}$$

Similarly, we want to unify the theory of essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts with the theory of t-branch split cuts. To prepare this, we define:

Definition 253. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be arbitrary, let $k \in \{0, \ldots, m\}$ (even though the case $k = 0$ is only interesting for formal purposes) and let $t \in \mathbb{Z}_{\geq 0}$. An* ***essential k, t-branch split cut for*** *P is an inequality $\begin{pmatrix} c' & (0^n)^T \end{pmatrix}(\cdot) \geq c_0$ that is valid for some*

$$P \cap \left(D\left(\pi^1, \ldots, \pi^t, \gamma_1, \ldots, \gamma_t\right) \times \mathbb{R}^n\right),$$

where $D\left(\pi^1,\dots,\pi^t,\gamma_1,\dots,\gamma_t\right)$ is a t-branch split disjunction such that $\dim\left(\operatorname{lin}\left\{\pi^1,\dots,\pi^t\right\}\right)\le k$ and we have $c'^T\in\operatorname{lin}\left\{\pi^1,\dots,\pi^t\right\}$ (this is the central property that distinguishes essential k,t-branch split cuts from "ordinary" k,t-branch split cuts!). We define the ***essential** k,t**-branch split closure of*** *P as*

$$\operatorname{cl}_{\mathrm{ess}\,k,tBS}(P):=P\cap\bigcap_{\substack{\pi^1,\dots,\pi^t\in\mathbb{Z}^m,\\ \gamma_1,\dots,\gamma_t\in\mathbb{Z}:\\ \dim\left(\operatorname{lin}\left\{\pi^1,\dots,\pi^t\right\}\right)\le k}}\ \bigcap_{\substack{c'\in(\mathbb{R}^m)^T,c_0\in\mathbb{R}:\\ c'^T\in\operatorname{lin}\left\{\pi^1,\dots,\pi^t\right\},\\ P\cap\left(D\left(\pi^1,\dots,\pi^t,\gamma_1,\dots,\gamma_t\right)\times\mathbb{R}^n\right)\subseteq P^{\le}\left(\left(\begin{array}{cc}c' & (0^n)^T\end{array}\right),c_0\right)}} P^{\le}\left(\left(\begin{array}{cc}c' & (0^n)^T\end{array}\right),c_0\right).$$

We remark that in Definition 253, we demand that the inequality is of the form $\left(\begin{array}{cc}c' & (0^n)^T\end{array}\right)(\,\cdot\,)\ge c_0$, where c' satisfies $c'^T\in\operatorname{lin}\left\{\pi^1,\dots,\pi^t\right\}$. But we allow P to have a non-empty intersections with as many atoms of the t-branch split disjunction as one desires. On the other hand, for the related **essential T cuts** and **essential crooked cross cuts**, which we define in Definition 465 (also cf. Definition 466), we allow a non-empty intersection with at most one of the atoms of the respective disjunction. The reason for this difference in the definitions is the following:

- For the purpose of section 5.4, we want to make it as easy as possible to extend the results that we show on the relationship between $L_{k,\mathbb{Q}}$ cuts vs $k,h\,(k)$-branch split cuts to essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts.

  Indeed, extending Theorem 256 about the relationship between $L_{k,\mathbb{Q}}$ cuts vs $k,h\,(k)$-branch split cuts to essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts vs essential $k,h\,(k)$-branch split cuts, which we do in Theorem 258, turns out to be rather easy (see the proof of Theorem 258). A similar "easy transferability" exists between Theorem 259 (about $L_{k,\mathbb{Q}}$ cuts vs $k,h\,(k)$-branch split cuts) and Theorem 261 (about essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts vs essential $k,h\,(k)$-branch split cuts).

  The results for essential $L_{k-\frac{1}{2},\mathbb{Q}}$ vs essential T cuts/essential crooked cross cuts in chapter 11 less closely mirror the results for $L_{2,\mathbb{Q}}$ cuts vs crooked cross cuts in that chapter.

- On the other hand, for the definition of essential T cuts and essential crooked cross cuts (Definition 465; also cf. Definition 466) in section 11.2.1, let $P\subseteq\mathbb{R}^m\times\mathbb{R}^n$, $\tilde{L}:=P^{\le}(A,b)$ and $L=\tilde{L}\cap P^{\le}(c',\tilde{c}_0)$ be as Theorem 246 ($A\in\mathbb{R}^{l\times m}$, $b\in\mathbb{R}^l$, $c'\in(\mathbb{R}^m)^T$ and $\tilde{c}_0\in\mathbb{R}$, where $l,m,n\in\mathbb{Z}_{\ge0}$). Then the condition $P\subseteq\left(\operatorname{int}\tilde{L}\right)\times\mathbb{R}^n$ in Theorem 246 means

  $$\forall i\in[l]:P\cap P^{\ge}(A_{i,*},b_i)=\emptyset,$$

  i.e. for all inequalities defining L (except for $c'(\,\cdot\,)\le\tilde{c}_0$), we demand that the associated half-spaces have an empty intersection with P and the remaining half-space defines the essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cut.

  Similarly, for essential T cuts and essential crooked cross cuts, we define in Definition 465 (also cf. Definition 466) that all except one of the atoms of the disjunction that we consider have an empty intersection with P and the remaining atom is the central tool for constructing the essential T cut/essential crooked cross cut.

So to summarize:

- The definition of t,k-branch split cuts makes it easier to extend results on $L_{k,\mathbb{Q}}$ cuts/closure vs t,k-branch split cuts/closure to essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts/closure vs essential t,k-branch split cuts/closure.
- The definitions of essential T cuts and essential crooked cross cuts more closely mirror the characterization of essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts via lattice-free bodies that we gave in Theorem 246.

5.4.2. $k=m$

Theorem 254. *Let $P\subseteq\mathbb{R}^m\times\mathbb{R}^n$ ($m,n\in\mathbb{Z}_{\ge0}$) be*

- *a rational polyhedron or*
- *convex and compact,*

respectively, and let

- *$c(\,\cdot\,)\ge c_0$, where $c\in(\mathbb{Q}^m\times\mathbb{Q}^n)^T$ and $c_0\in\mathbb{R}$, be a valid inequality for P_I, or*

- *$c(\cdot) > c_0$, where $c \in (\mathbb{R}^m \times \mathbb{R}^n)^T$ and $c_0 \in \mathbb{R}$, be a valid* strict *inequality for P_I,*

respectively. Set $R := P \cap P^< (c, c_0) \neq \emptyset$ and let $h(m)$ be as in Remark/Definition 248. Then there exists an $h(m)$-branch split disjunction $D(\pi^1, \dots, \pi^{h(m)}, \gamma_1, \dots, \gamma_{h(m)})$ such that $c(\cdot) \geq c_0$ is a valid inequality for

$$P \cap \left(D\left(\pi^1, \dots, \pi^{h(m)}, \gamma_1, \dots, \gamma_{h(m)}\right) \times \mathbb{R}^n\right).$$

In particular, we have

$$\mathrm{cl}_{m,h(m)BS}(P) = \mathrm{cl}_{L_{m,\mathbb{Q}}}(P) = \mathrm{cl}_{\overline{I}}(P) = \mathrm{cl}_I(P). \tag{5.19}$$

Proof. By Theorem 231 or Theorem 234, respectively, there exists a full-dimensional maximal lattice-free body $L \subseteq \mathbb{R}^m$ such that $\mathrm{proj}_{\mathbb{R}^m} R \subseteq \mathrm{int}\, L$. Thus, by Lemma 251, there exists an $h(m)$-branch split disjunction $D(\pi^1, \dots, \pi^{h(m)}, \gamma_1, \dots, \gamma_{h(m)})$ having

$$R \cap \left(D\left(\pi^1, \dots, \pi^{h(m)}, \gamma_1, \dots, \gamma_{h(m)}\right) \times \mathbb{R}^n\right) = \emptyset; \tag{5.20}$$

thus,

$$P \cap P^<(c, c_0) \cap \left(D\left(\pi^1, \dots, \pi^{h(m)}, \gamma_1, \dots, \gamma_{h(m)}\right) \times \mathbb{R}^n\right) = \emptyset.$$

This means that $c(\cdot) \geq c_0$ is a valid inequality for $P \cap D(\pi^1, \dots, \pi^{h(m)}, \gamma_1, \dots, \gamma_{h(m)})$.

Now for (5.19): because of Theorem 75 and Theorem 202, we have

$$\mathrm{cl}_{L_{m,\mathbb{Q}}}(P) = \mathrm{cl}_{\overline{I}}(P) = \mathrm{cl}_I(P).$$

For $\mathrm{cl}_{\overline{I}}(P) \subseteq \mathrm{cl}_{m,h(m)BS}(P)$: Because P is convex and closed, we have $\mathrm{cl}_{\overline{I}}(P) \subseteq P$. On the other hand, every $m, h(m)$-branch split cut for P is a valid inequality for $\mathrm{cl}_{\overline{I}}(P)$.

For $\mathrm{cl}_{\overline{I}}(P) \supseteq \mathrm{cl}_{m,h(m)BS}(P)$: For the case that P is a rational polyhedron, note that, by Theorem 75, $\mathrm{cl}_I(P)$ is a rational polyhedron; thus, any linear inequality for P_I is dominated absolutely by some finite set of rational linear inequalities for P_I. In the case that P is convex and compact, let $c'(\cdot) \geq c'_0$ be a valid inequality for P_I. We have proved that for every $\epsilon \in \mathbb{R}_{>0}$, the inequality $c'(\cdot) \geq c'_0 - \epsilon$ is an $h(m)$-branch split cut for P. □

We now show that a similar statement to Theorem 254 does *not* hold if $P \subseteq \mathbb{R}^2$ is instead an irrational hyperplane:

Theorem 255. *Let $t \in \mathbb{Z}_{\geq 0}$, $\pi^1, \dots, \pi^t \in \mathbb{Z}^2$ and $\gamma_1, \dots, \gamma_t \in \mathbb{Z}$. Then*

$$\mathrm{conv}\left(P^{114} \cap D\left(\pi^1, \dots, \pi^t, \gamma_1, \dots, \gamma_t\right)\right) = P^{114}. \tag{5.21}$$

In particular, for all parametric cross disjunctions $D^{t'}(\pi^1, \pi^2, \gamma_1, \gamma_2)$ (cf. Definition 150) with respect to some $t' \in \mathbb{Z}$ (this includes crooked cross disjunctions), we have:

$$\mathrm{conv}\left(P^{114} \cap D^{t'}\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right)\right) = P^{114}. \tag{5.22}$$

On the other hand, $(0^2)^T(\cdot) \leq -1$ is an $L_{2,\mathbb{Q}}$ cut for P^{114}.

Proof. The statement that $(0^2)^T(\cdot) \leq -1$ is an $L_{2,\mathbb{Q}}$ cut for P^{114} holds by definition, since $P^{114} \subseteq \mathbb{R}^2$ and $(P^{114})_I = \emptyset$.

For (5.21): If $\dim\{\pi^1, \dots, \pi^t\} = 0$, we have

$$P^{114} \cap D\left(\pi^1, \dots, \pi^t, \gamma_1, \dots, \gamma_t\right) = P^{114}.$$

If, on the other hand, $\dim\{\pi^1, \dots, \pi^t\} \geq 1$, we can w.l.o.g. assume $\forall i \in [t] : \pi^i \neq 0^2$. It is easy to check that then

$$\forall i \in [t] : \left(\pi^i\right)^T \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \neq 0.$$

Additionally, we can w.l.o.g. assume that

$$\forall i \in [t] : \left(\pi^i\right)^T \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} > 0$$

(if this is not the case for some $i \in [t]$, replace π^i by $-\pi^i$ and γ_i by $-\gamma_i + 1$). Let

$$\overline{M} := \max \left\{ \frac{\gamma_i + 1 - \left(\pi^i\right)^T \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}}{\left(\pi^i\right)^T \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}} : i \in [t] \right\},$$

$$\underline{M} := \min \left\{ \frac{\gamma_i - \left(\pi^i\right)^T \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}}{\left(\pi^i\right)^T \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}} : i \in [t] \right\}.$$

Let $x \in P$. Then $x = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} + \lambda^* \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$, where $\lambda^* \in \mathbb{R}$. We distinguish three cases:

1. $\lambda^* \geq \overline{M}$,
2. $\lambda^* \leq \underline{M}$,
3. $\underline{M} < \lambda^* < \overline{M}$.

For case 1: For showing $x \in D\left(\pi^1, \dots, \pi^t, \gamma_1, \dots, \gamma_t\right)$, we show that for all $i \in [t]$, we have $\left(\pi^i\right)^T x \geq \gamma_i + 1$.. Let $i \in [t]$. Then

$$\begin{aligned}
\left(\pi^i\right)^T x &= \left(\pi^i\right)^T \left(\begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} + \overline{M} \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \right) + \left(\lambda^* - \overline{M}\right) \left(\pi^i\right)^T \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \\
&\geq \left(\pi^i\right)^T \left(\begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} + \overline{M} \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \right) \\
&\geq \left(\pi^i\right)^T \left(\begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} + \frac{\gamma_i + 1 - \left(\pi^i\right)^T \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}}{\left(\pi^i\right)^T \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}} \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \right) \\
&= \gamma_i + 1.
\end{aligned}$$

For case 2: For showing $x \in D\left(\pi^1, \dots, \pi^t, \gamma_1, \dots, \gamma_t\right)$, we show that for all $i \in [t]$, we have $\left(\pi^i\right)^T x \leq \gamma_i$. Let $i \in [t]$. Then

$$\begin{aligned}
\left(\pi^i\right)^T x &= \left(\pi^i\right)^T \left(\begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} + \underline{M} \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \right) + \left(\lambda^* - \underline{M}\right) \left(\pi^i\right)^T \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \\
&\leq \left(\pi^i\right)^T \left(\begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} + \underline{M} \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \right) \\
&\leq \left(\pi^i\right)^T \left(\begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} + \frac{\gamma_i - \left(\pi^i\right)^T \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}}{\left(\pi^i\right)^T \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}} \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \right) \\
&= \gamma_i.
\end{aligned}$$

For case 3: We have

$$\begin{aligned}
x &= \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} + \lambda^* \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \\
&= \frac{\overline{M} - \lambda^*}{\overline{M} - \underline{M}} \left(\begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} + \underline{M} \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \right) + \frac{\lambda^* - \underline{M}}{\overline{M} - \underline{M}} \left(\begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} + \overline{M} \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \right) \\
&\in \operatorname{conv}\left(P^{114} \cap D\left(\pi^1, \dots, \pi^t, \gamma_1, \dots, \gamma_t\right)\right). && \text{(by case 1 and 2)}
\end{aligned}$$

For (5.22)**:** Notice that (cf. Definition 150) $D^{t'}\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right) \supseteq D\left(\pi^1, \pi^2, \pi^2 - t'\pi^1, \gamma_1, \gamma_2, \gamma_2 - t'\gamma_1\right)$. □

5.4.3. $k \leq m$

5.4.3.1. $L_{k,\mathbb{Q}}$ cuts

Theorem 256. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ be*

- *a rational polyhedron or*
- *convex and compact,*

respectively, let $k \in \{0, \ldots, m\}$, let $V' \leq \mathbb{R}^m$ be a rational subspace of codimension k and let

- *$c(\cdot) \geq c_0$, where $c \in (\mathbb{Q}^m \times \mathbb{Q}^n)^T$ and $c_0 \in \mathbb{R}$, be an $L_{k,\mathbb{Q}}$ cut for P with respect to $V' \times \mathbb{R}^n$, or*
- *$c(\cdot) > c_0$, where $c \in (\mathbb{R}^m \times \mathbb{R}^n)^T$ and $c_0 \in \mathbb{R}$, be a strict $L_{k,\mathbb{Q}}$ cut for P with respect to $V' \times \mathbb{R}^n$,*

respectively. Then there exists an $h(k)$-branch split disjunction $D\left(\pi^1, \ldots, \pi^{h(k)}, \gamma_1, \ldots, \gamma_{h(k)}\right)$ where $V' \perp \pi^1, \ldots, \pi^{h(k)}$ such that $c(\cdot) \geq c_0$ is a valid inequality for

$$P \cap \left(D\left(\pi^1, \ldots, \pi^{h(k)}, \gamma_1, \ldots, \gamma_{h(k)}\right) \times \mathbb{R}^n\right).$$

In particular, we have

$$\mathrm{cl}_{k,h(k)BS}(P) \subseteq \mathrm{cl}_{L_{k,\mathbb{Q}}}(P).$$

Remark 257. *In Definition 161, we defined $L_{k,\mathbb{Q}}$ cuts as linear inequalities $c(\cdot) \leq c_0$ for $P \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + V)$, while in Theorem 256, a strict inequality $c(\cdot) > c_0$ for $P \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + V)$ is called a "strict $L_{k,\mathbb{Q}}$ cut for P" (in both cases, of course, $V \leq \mathbb{R}^m \times \mathbb{R}^n$ is some rational subspace of codimension k). This is obviously some abuse of notation, but we believe it is clear to the reader how Definition 161 can easily be adapted to define "strict $L_{k,\mathbb{Q}}$ cuts" formally.*

Proof. (Theorem 256) Let $S, S^{LP}, w^1, \ldots, w^k$ and W be as in the proof in section 5.2.3. Keep in mind that $W \in \mathbb{Z}^{m \times k}$. Since $c(\cdot) \geq c_0$ is valid for

$$\left\{\begin{pmatrix} x \\ y \end{pmatrix} \in P : \left(w^1\right)^T x, \ldots, \left(w^k\right)^T x \in \mathbb{Z}\right\},$$

we saw in the proof in section 5.2.3 that

$$\left(c_{(1,\ldots,m)}\left(I^m - \left(W\left(W^T W\right)^{-1}\right) W^T\right)\right) x + c_{(m+1,\ldots,m+n)} y + \left(c_{(1,\ldots,m)} W\left(W^T W\right)^{-1}\right) z \geq c_0 \tag{5.23}$$

is valid for $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in S$. Thus, by Theorem 254, there exists an $h(k)$-branch split disjunction

$$D\left(\pi^{1,pre}, \ldots, \pi^{h(k),pre}, \gamma_1, \ldots, \gamma_{h(k)}\right) \subseteq \mathbb{R}^k$$

such that (5.23) is a valid inequality for

$$S^{LP} \cap \left(\mathbb{R}^m \times \mathbb{R}^n \times D\left(\pi^{1,pre}, \ldots, \pi^{h(k),pre}, \gamma_1, \ldots, \gamma_{h(k)}\right)\right).$$

We first note that, because of the dimension of W, we have

$$\dim\left(\mathrm{lin}\left\{W\pi^{1,pre}, \ldots, W\pi^{h(k),pre}\right\}\right) \leq \mathrm{rank}\, W \leq \min(m, k) \leq k.$$

We now claim that $c(\cdot) \geq c_0$ is valid for

$$P \cap \left(D\left(W\pi^{1,pre}, \ldots, W\pi^{h(k),pre}, \gamma_1, \ldots, \gamma_{h(k)}\right) \times \mathbb{R}^n\right).$$

For this, let

$$\begin{pmatrix} x \\ y \end{pmatrix} \in P \cap \left(D\left(W\pi^{1,pre}, \ldots, W\pi^{h(k),pre}, \gamma_1, \ldots, \gamma_{h(k)}\right) \times \mathbb{R}^n\right).$$

Then, using

$$\begin{pmatrix} x \\ y \\ W^T x \end{pmatrix} \in S^{LP} \cap \left(\mathbb{R}^m \times \mathbb{R}^n \times D^c\left(\pi^{1,pre}, \ldots, \pi^{h(k),pre}, \gamma_1, \ldots, \gamma_{h(k)}\right)\right)$$

and (5.23), we get

$$c\begin{pmatrix} x \\ y \end{pmatrix} = c_{(1,\ldots,m)}\left(I^m - W\left(W^TW\right)^{-1}W^T\right)x + c_{(m+1,\ldots,m+n)}y + c_{(1,\ldots,m)}W\left(W^TW\right)^{-1}\left(W^Tx\right) \geq c_0.$$

Finally, observe that, because $\operatorname{im} W = \operatorname{lin}\left\{w^1,\ldots,w^k\right\} \perp V'$, we have $W\pi^{1,pre},\ldots,W\pi^{h(k),pre} \perp V'$. □

5.4.3.2. Essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts

Theorem 258. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m,n \in \mathbb{Z}_{\geq 0})$ be*

- *a rational polyhedron or*
- *convex and compact,*

respectively, let $k \in \{0,\ldots,m\}$, let $V' \leq \mathbb{R}^m$ be a rational subspace of codimension k and let

- *$\begin{pmatrix} c' & (0^n)^T \end{pmatrix}(\,\cdot\,) \geq c_0$, where $c \in (\mathbb{Q}^m)^T$, be a valid essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cut for P with respect to $V' \times \mathbb{R}^n$ or*
- *$\begin{pmatrix} c & (0^n)^T \end{pmatrix}(\,\cdot\,) > c_0$, where $c \in (\mathbb{R}^m)^T$, be a valid strict essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cut for P with respect to $V' \times \mathbb{R}^n$,*

respectively, be such that $c^T \perp V'$. Then there exists a $h(k)$-branch split disjunction

$$D\left(\pi^1,\ldots,\pi^{h(m)},\gamma_1,\ldots,\gamma_{h(m)}\right),$$

where $V' \perp \pi^1,\ldots,\pi^{h(k)}$, such that $\begin{pmatrix} c' & (0^n)^T \end{pmatrix}(\,\cdot\,) \geq c_0$ is a valid inequality for

$$P \cap \left(D\left(\pi^1,\ldots,\pi^{h(m)},\gamma_1,\ldots,\gamma_{h(k)}\right) \times \mathbb{R}^n\right).$$

In particular, we have

$$\operatorname{cl}_{\operatorname{ess} k,h(k)BS}(P) \subseteq \operatorname{cl}_{\operatorname{ess} L_{k-\frac{1}{2},\mathbb{Q}}}(P). \tag{5.24}$$

Proof. By Theorem 196, we have

$$P \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + (V' \times \mathbb{R}^n)) \subseteq \operatorname{conv}((P + (V' \times \mathbb{R}^n)) \cap (\mathbb{Z}^m \times \mathbb{R}^n)).$$

Thus, the first statement is a consequence of Theorem 256. So, only (5.24) remains to be shown.

For this, consider essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts with respect to $V' \times \mathbb{R}^n$, where $V' \leq \mathbb{R}^m$ is a subspace of codimension k. W.l.o.g. we can assume $(P + (V' \times \mathbb{R}^n))_I \neq \emptyset$. Thus, by Lemma 159, every essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cut $c(\,\cdot\,) \geq c_0$ for P with respect to $V' \times \mathbb{R}^n$ can be assumed to satisfy $c^T \perp V' \times \mathbb{R}^n$, i.e. $c = \begin{pmatrix} c' & (0^n)^T \end{pmatrix}$, where $c' \perp V'$.

If P is convex and compact, by the first statement, for all $\epsilon > 0$, the inequality $\begin{pmatrix} c' & (0^n)^T \end{pmatrix}(\,\cdot\,) \geq c_0 + \epsilon$ is an essential $k, h(k)$-branch split cut for P with respect to a $h(k)$-branch split disjunction

$$D\left(\pi^1,\ldots,\pi^{h(m)},\gamma_1,\ldots,\gamma_{h(m)}\right),$$

where $c' \perp V' \perp \pi^1,\ldots,\pi^{h(k)}$. Thus, $\begin{pmatrix} c' & (0^n)^T \end{pmatrix}(\,\cdot\,) \geq c_0$ is also valid for $\operatorname{cl}_{\operatorname{ess} k,h(k)BS}(P)$.

If, on the other hand, P is a rational polyhedron, by Theorem 75, $\operatorname{cl}_I(P + (V' \times \mathbb{R}^n))$ is also a rational polyhedron. Thus, we can w.l.o.g. assume c' to be rational. Again, by the first part of this proof, $\begin{pmatrix} c' & (0^n)^T \end{pmatrix}(\,\cdot\,) \geq c_0$ is valid for $\operatorname{cl}_{\operatorname{ess} k,h(k)BS}(P)$. □

5.4.4. Conclusion

5.4.4.1. $L_{k,\mathbb{Q}}$ closure

Theorem 259. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m,n \in \mathbb{Z}_{\geq 0})$ be*

- *a rational polyhedron or*

- *convex and compact,*

respectively, let $k \in \{0, \ldots, m\}$ and let $h(\cdot)$ be as in Remark/Definition 248. Then

$$\mathrm{cl}_{L_{k,\mathbb{Q}}}(P) = \mathrm{cl}_{k,h(k)BS}(P).$$

Remark 260. *The condition that P is either a rational polyhedron or convex and compact is essential for Theorem 259 to hold. For example if P is an irrational hyperplane, it does not hold. For this, consider the irrational hyperplane $P^{114} \subseteq \mathbb{R}^2$. By Theorem 255, we have*

$$\mathrm{cl}_{L_{2,\mathbb{Q}}}\left(P^{114}\right) = \emptyset \subsetneq P^{114} = \mathrm{cl}_{2,h(2)BS}\left(P^{114}\right),$$

i.e. the $L_{2,\mathbb{Q}}$ closure of P^{114} is stronger than the $2, h(2)$-branch split closure of P^{114}.

Proof. (Theorem 259) The inclusion $\mathrm{cl}_{L_{k,\mathbb{Q}}}(P) \subseteq \mathrm{cl}_{k,h(k)BS}(P)$ is an immediate consequence of Corollary 278, which we show in section 6.1.4. On the other hand, the inclusion $\mathrm{cl}_{k,h(k)BS}(P) \subseteq \mathrm{cl}_{L_{k,\mathbb{Q}}}(P)$ holds by Theorem 256. □

5.4.4.2. Essential $L_{k-\frac{1}{2},\mathbb{Q}}$ closure

Theorem 261. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ be*

- *a rational polyhedron or*
- *convex and compact,*

respectively, let $k \in \{0, \ldots, m\}$ and let $h(\cdot)$ be as in Remark/Definition 248. Then

$$\mathrm{cl}_{\mathrm{ess}\, L_{k-\frac{1}{2},\mathbb{Q}}}(P) = \mathrm{cl}_{\mathrm{ess}\, k,h(k)BS}(P).$$

Remark 262. *Similarly to what we wrote in Remark 260 about Theorem 259, also in Theorem 261, the condition that P is either a rational polyhedron or convex and compact is essential. Again, consider $P^{114} \subseteq \mathbb{R}^2$. Using Theorem 255, we obtain*

$$\mathrm{cl}_{\mathrm{ess}\, L_{2-\frac{1}{2},\mathbb{Q}}}\left(P^{114}\right) = \mathrm{cl}_{L_{2-\frac{1}{2},\mathbb{Q}}}\left(P^{114}\right) = \emptyset \subsetneq P^{114} = \mathrm{cl}_{2,h(2)BS}\left(P^{114}\right) \subseteq \mathrm{cl}_{\mathrm{ess}\, 2,h(2)BS}\left(P^{114}\right).$$

Proof. (Theorem 261) Since "$\supseteq$" is a direct consequence of Theorem 258, we only have to show "$\subseteq$".

For "$\subseteq$": Let $\begin{pmatrix} c' & (0^n)^T \end{pmatrix}(\cdot) \geq c_0$ be an essential $k, h(k)$-branch split cut for P with respect to the $k, h(k)$-branch split disjunction

$$D\left(\pi^1, \ldots, \pi^{h(m)}, \gamma_1, \ldots, \gamma_{h(m)}\right).$$

Since essential $k, h(k)$-branch split cut are by definition also $k, h(k)$-branch split cuts, by Corollary 278, $\begin{pmatrix} c' & (0^n)^T \end{pmatrix}(\cdot) \geq c_0$ is an $L_{k,\mathbb{Q}}$ cut for P with respect to $V' \times \mathbb{R}^n$, where

$$V' \leq \left(\mathrm{lin}\left\{\pi^1, \ldots, \pi^{h(m)}\right\}\right)^{\perp}$$

is a rational subspace. By the definition of essential $k, h(k)$-branch split cuts, we have

$$\begin{pmatrix} c & (0^n)^T \end{pmatrix}^T \perp \left(\mathrm{lin}\left\{\pi^1, \ldots, \pi^{h(m)}\right\}\right) \times \mathbb{R}^n;$$

thus,

$$\begin{pmatrix} c & (0^n)^T \end{pmatrix}^T \perp V' \times \mathbb{R}^n.$$

So, by Theorem 213, $\begin{pmatrix} c & (0^n)^T \end{pmatrix}(\cdot) \geq c_0$ is also a valid inequality for $(P + (V' \times \mathbb{R}^n))_I$; in other words: an essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cut for P. □

5.4.4.3. $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ closure and $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ closure

Theorem 263. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ be a rational polyhedron, let $k \in \{1, \ldots, m\}$ and let $h(\cdot)$ be as in Remark/Definition 248. Then*

$$\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P) = \mathrm{cl}_{\mathrm{ess}\, k,h(k)BS}(P) \cap \mathrm{cl}_{k-1,h(k-1)BS}(P).$$

Proof. The equalitity

$$\mathrm{cl}_{\mathrm{ess}\, L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{\mathrm{ess}\, k,h(k)BS}(P)$$

holds by Theorem 261. By Theorem 211, we know that

$$\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P) = \mathrm{cl}_{L_{k-1,\mathbb{Q}}}(P) \cap \mathrm{cl}_{\mathrm{ess}\, L_{k-\frac{1}{2},\mathbb{Q}}}(P).$$

Finally, by Theorem 259, we have

$$\mathrm{cl}_{L_{k-1,\mathbb{Q}}}(P) = \mathrm{cl}_{L_{k-1,h(k)BS}}(P).$$

□

5.4.5. Polyhedricity

We remark that a theorem that can be shown to be equivalent to the following one was independently proved using a different reasoning in [DGMR17, Theorem 2].

Theorem 264. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ be a rational polyhedron and let $k \in \{0, \ldots, m\}$. Then there exists a finite set $\mathcal{V}$ of rational subspaces of $\mathbb{R}^m$ of codimension k such that*

$$\mathrm{cl}_{L_{k,\mathbb{Q}}}(P) = \bigcap_{V \in \mathcal{V}} \mathrm{conv}\left(P \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + (V \times \mathbb{R}^n))\right).$$

In particular, $\mathrm{cl}_{L_{k,\mathbb{Q}}}(P)$ is a rational polyhedron.

Proof. The inclusion "⊆" is trivial; so, we only have to show "⊇". We first note that it suffices to show a weaker statement where we only demand all elements of $\mathcal{V}$ to have codimension $\leq k$ instead of exactly k, since if there exists some $V \in \mathcal{V}$ having $\mathrm{codim}\, V < k$, we can simply replace V by a rational subspace $V' \leq \mathbb{R}^m$ where $V' \geq V$ has codimension k; this only tightens the statement.

By Theorem 259, we have

$$\mathrm{cl}_{L_{k,\mathbb{Q}}}(P) = \mathrm{cl}_{k,h(k)BS}(P),$$

where $h(\cdot)$ is as in Remark/Definition 248. Let

$$\begin{aligned}\mathcal{T}^{split}_{h(k),m,k} := \Big\{ & S\left(\pi^1, \ldots, \pi^{h(k)}, \gamma_1, \ldots, \gamma_{h(k)}\right) : \pi^1, \ldots, \pi^{h(k)} \in \mathbb{Z}^m, \gamma_1, \ldots, \gamma_{h(k)} \in \mathbb{Z}, \\ & \dim\left(\mathrm{lin}\left\{\pi^1, \ldots, \pi^{h(k)}\right\}\right) \leq k \Big\} \\ \subseteq\ & \mathcal{T}^*_{h(k),1,m}\end{aligned}$$

(for the definition of $S\left(\pi^1, \ldots, \pi^{h(m)}, \gamma_1, \ldots, \gamma_{h(m)}\right)$ cf. Definition 142 and for $\mathcal{T}^*_{h(m),1,m}$ cf. Definition 332). By Theorem 335, there exists a finite set $\mathcal{T}_f \subseteq \mathcal{T}^{split}_{h(m),m,k}$ such that

$$\mathrm{cl}_{k,h(k)BS}(P) = \bigcap_{T \in \mathcal{T}^{split}_{h(m),m,k}} \mathrm{conv}\left(P \setminus (T \times \mathbb{R}^n)\right) = \bigcap_{T \in \mathcal{T}_f} \mathrm{conv}\left(P \setminus (T \times \mathbb{R}^n)\right).$$

By Corollary 278, every inequality for $P \setminus \left(S\left(\pi^1, \ldots, \pi^{h(m)}, \gamma_1, \ldots, \gamma_{h(m)}\right) \times \mathbb{R}^n\right)$ is an $L_{k',\mathbb{Q}}$ cut for P with respect to $V := \left(\mathrm{lin}\left\{\pi^1, \ldots, \pi^{h(m)}\right\}\right)^\perp$, where $k' := \dim\left(\mathrm{lin}\left\{\pi^1, \ldots, \pi^{h(m)}\right\}\right)$. Set

$$\mathcal{V} := \left\{ \left(\mathrm{lin}\left\{\pi^1, \ldots, \pi^{h(m)}\right\}\right)^\perp : S\left(\pi^1, \ldots, \pi^{h(m)}, \gamma_1, \ldots, \gamma_{h(m)}\right) \in \mathcal{T}^{split}_{h(m),m,k} \right\}.$$

So, we have

$$\begin{aligned}\bigcap_{V\in\mathcal{V}} \operatorname{conv}\left(P\cap\left(\left(\mathbb{Z}^m\times\mathbb{R}^n\right)+\left(V\times\mathbb{R}^n\right)\right)\right) &\subseteq \bigcap_{T\in\mathcal{T}_f} \operatorname{conv}\left(P\backslash\left(T\times\mathbb{R}^n\right)\right) && \text{(by Corollary 278)}\\ &= \operatorname{cl}_{k,h(k)BS}\left(P\right)\\ &= \operatorname{cl}_{L_{k,\mathbb{Q}}}\left(P\right).\end{aligned}$$

Finally, by Theorem 178, for all $V\in\mathcal{V}$, $\operatorname{conv}\left(P\cap\left(\left(\mathbb{Z}^m\times\mathbb{R}^n\right)+\left(V\times\mathbb{R}^n\right)\right)\right)$ is a rational polyhedron. □

6. Expressivity of various classes of cutting planes

This chapter consists of three parts:

- In section 6.1, we look at results concerning what cutting planes/cutting plane operators are dominated by others or not.
- In section 6.2, we concern ourselves with the following question: let a rational polyhedron P and a cutting plane operator $\mathrm{cl}_{(\,\cdot\,)}(\cdot)$ be given. Is it always possible to obtain $\mathrm{cl}_I(P)$ in a finite number of steps by applying the cutting plane operator iteratively on P? Or if this is not possible: does the sequence $\left\{\mathrm{cl}^{(k)}_{(\,\cdot\,)}(P)\right\}_{k\in\mathbb{Z}_{\geq 0}}$ at least converge (in a sense that we formalize in Definition 308) to $\mathrm{cl}_I(P)$?
- In section 6.3, we consider whether for a given cutting plane operator $\mathrm{cl}_{(\,\cdot\,)}(\cdot)$ and a given P, also $\mathrm{cl}_{(\,\cdot\,)}(P)$ is a (rational) polyhedron – in particular in the case that P already is a rational polyhedron.

Note however that such topics are also treated in other chapters for specific cases.

6.1. (Non-)Inclusions

6.1.1. Split cuts vs integral lattice-free cuts and k-disjunctive cuts

We already saw at the discussion at the beginning of section 3.4.1.2 that every split cut is a 2-disjunctive cut. On the other hand, is easy to check that for $m \in \mathbb{Z}_{\geq 0}$, every full-dimensional rational lattice-free body in $\mathbb{R}^m$ with (at most) two facets is contained in a split set (though not necessarily forms a split set). Putting these parts together, we obtain:

Theorem 265. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ be arbitrary. Then every split cut for P is also an k-disjunctive cut for P $(k \in \mathbb{Z}_{\geq 2})$. In particular, we have*

$$\mathrm{cl}_{kD}(P) \subseteq \mathrm{cl}_{split}(P).$$

For $k = 2$, also the reverse holds, i.e. every 2-disjunctive cut for P is also a split cut for P and we thus have

$$\mathrm{cl}_{2D}(P) = \mathrm{cl}_{split}(P).$$

Similarly, also by the discussion at the beginning of section 3.4.1.2, every split cut is an integral lattice-free cut. We thus get:

Theorem 266. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ be arbitrary. Then every split cut for P is an integral lattice-free cut for P. In particular, we have*

$$\mathrm{cl}_{ILF}(P) \subseteq \mathrm{cl}_{split}(P).$$

6.1.2. t-branch split cuts vs cuts from basic relaxations

In [ACL05], the following theorem is shown, which states that every split cut is a split cut of a basic relaxation:

Theorem 267. *Let $A \in \mathbb{R}^{l\times(m+n)}$ and $b \in \mathbb{R}^l$ $(l, m, n \in \mathbb{Z}_{\geq 0})$. Additionally, let $\pi \in \mathbb{Z}^m$ and $\gamma \in \mathbb{Z}$ be given. Then*

$$\mathrm{conv}\left(P^{\leq}(A, b) \cap (D(\pi, \gamma) \times \mathbb{R}^n)\right) = \bigcap_{S\in\mathcal{B}^*(A)} \mathrm{conv}\left(P^{\leq}(A_{S,*}, b_S) \cap (D(\pi, \gamma) \times \mathbb{R}^n)\right),$$

where $\mathcal{B}^*(A)$ is as in Definition 392.

We remark (cf. [ACL05]) that in Theorem 267, one also has to consider infeasible bases.

Remark 268. *Let $P := P^{\leq}(A,b)$, π and γ_0 be as in Theorem 267. Then, by Lemma 128,*

$$\operatorname{conv}\left(P \cap \left(D\left(\pi,\gamma\right) \times \mathbb{R}^n\right)\right)$$

is a polyhedron and is thus closed. So, we have

$$\operatorname{conv}\left(P \cap \left(D\left(\pi,\gamma\right) \times \mathbb{R}^n\right)\right) = \overline{\operatorname{conv}}\left(P \cap \left(D\left(\pi,\gamma\right) \times \mathbb{R}^n\right)\right).$$

In Theorem 409, we show the equivalence between $L_{1,\mathbb{Q}}$ cuts and split cuts for a given convex $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$, where $m+n \geq 1$). Using this, we obtain the following result from Theorem 267 and Remark 268.

Theorem 269. *Let $P := P^{\leq}(A,b) \subseteq \mathbb{R}^m \times \mathbb{R}^n$, where $A \in \mathbb{R}^{l\times(m+n)}$ and $b \in \mathbb{R}^l$ ($l, m, n \in \mathbb{Z}_{\geq 0}$). Then*

$$\begin{matrix}\mathrm{cl}_{L_{1,\mathbb{Q}}} \\ \mathrm{cl}_{split}\end{matrix}(P) = \bigcap_{S \in \mathcal{B}^*(A)} \begin{matrix}\mathrm{cl}_{L_{1,\mathbb{Q}}} \\ \mathrm{cl}_{split}\end{matrix}\left(P^{\leq}\left(A_{S,*}, b_S\right)\right).$$

Here, the equation for $\mathrm{cl}_{L_{1,\mathbb{Q}}}(\cdot)$, of course, only holds if $m+n \geq 1$ (otherwise, $\mathrm{cl}_{L_{1,\mathbb{Q}}}(\cdot)$ is not defined).

We remark that the statement of Theorem 269 for split cuts was originally formulated in [ACL05]. In Theorem 427 of section 9.2.2, we present a tightening of Theorem 269.

From Theorem 269, one concludes:

Theorem 270. *Let $A \in \mathbb{R}^{l\times m}$, $G \in \mathbb{R}^{l\times n}$ and $b \in \mathbb{R}^l$ ($l, m, n \in \mathbb{Z}_{\geq 0}$). Then*

$$\mathrm{cl}_{BR}(A, G, b) \subseteq \mathrm{cl}_{split}\left(P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)\right).$$

Next, we ask whether the inclusion that is formulated in Theorem 270 can also be strict. This can be the case, as we see in Theorem 576 (also cf. the related Theorem 577): there, we see that for every $m \in \mathbb{Z}_{\geq 1}$, there exists a rational polytope $P = P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right) \subseteq \mathbb{R}^m \times \mathbb{R}^1$ such that for every $\ell \in \mathbb{Z}_{\geq 0}$, we have

$$\mathrm{cl}_{BR}(A, G, b) = \mathrm{cl}_I(P) \subsetneq \mathrm{cl}^{(\ell)}_{(m-1)BS}(P).$$

We now consider another result that is shown in [DGM15], which states that for every $m \in \mathbb{Z}_{\geq 2}$, there exists a rational polyhedron $P^{\leq}(A,b) \subseteq \mathbb{R}^m$ such that

- its *second* Chvátal-Gomory closure (applying the Chvátal-Gomory closure two times) and thus the second split closure and
- its closure with respect to cuts from a basic relaxation

can be strictly included in its t-branch split closure for $t \in \{0, \ldots, m-2\}$. For this, we define:

Definition 271. *For $m \in \mathbb{Z}_{\geq 2}$ and $\epsilon \in \mathbb{R}_{>0}$, let*

$$\begin{aligned} A^{271,m} &:= \begin{pmatrix} \left(1^{m-1}\right)^T & 2 \\ -I^{m-1} & 0^{m-1} \end{pmatrix} \in \mathbb{Z}^{m\times m}, \\ b^{271,m,\epsilon} &:= \begin{pmatrix} m+1-\epsilon \\ -\epsilon \cdot 1^{m-1} \end{pmatrix} \in \mathbb{R}^m, \\ P^{271,m,\epsilon} &:= P^{\leq}\left(A^{271,m}, b^{271,m,\epsilon}\right) \subseteq \mathbb{R}^m. \end{aligned}$$

We have for $\epsilon \in (0,1)$ (cf. [DGM15, proof of Theorem 1.1 (restated)]):

$$\mathrm{cl}_I\left(P^{271,m,\epsilon}\right) = P^{\leq}\left(\begin{pmatrix} \left(1^{m-1}\right)^T & 2 \\ -I^{m-1} & 0^{m-1} \\ \left(0^{m-1}\right)^T & 1 \end{pmatrix}, \begin{pmatrix} m \\ -1^{m-1} \\ 0 \end{pmatrix}\right).$$

The following theorem is a consequence of [DGM15, section 4; in particular cf. Lemmas 4.1, Lemma 4.4, Theorem 1.1 (Restated) and Corollary 4.5]. We remark that the authors formulate the statement for the second split closure, but actually prove it for the second Chvátal-Gomory closure. The central consideration for $\mathrm{cl}_{BR}\left(A^{271,m}, b^{271,m,\epsilon}\right) = \mathrm{cl}_I\left(P^{271,m,\epsilon}\right)$ is to observe that the rows of $A^{271,m}$ are linearly independent.

Theorem 272. *For every $m \in \mathbb{Z}_{\geq 2}$, there exists an $\epsilon^* \in (0,1)$ such that for all $\epsilon \in (0, \epsilon^*]$ and $t \in \{0, \dots, m-2\}$, we have*

$$\mathrm{cl}_{CG}^{(2)}\left(P^{271,m,\epsilon}\right) = \mathrm{cl}_{split}^{(2)}\left(P^{271,m,\epsilon}\right) = \mathrm{cl}_{BR}\left(A^{271,m}, b^{271,m,\epsilon}\right) = \mathrm{cl}_I\left(P^{271,m,\epsilon}\right) \subsetneq \mathrm{cl}_{tBS}\left(P^{271,m,\epsilon}\right).$$

So, the central statement of Theorem 272 is that both the second Chvátal-Gomory closure (and thus the second split closure) and the closure with respect to cuts from a basic relaxations can be stronger than *any* t-branch split closure.

We next come to two results that state that also the reverse can happen: the 2-branch split closure can be stronger than the closure with respect to cuts from a basic relaxations. This is the statement of Theorem 273 and Theorem 274, which constitute the remainder of this section.

In [DGM15, Lemma 7.1, Lemma 7.2 and Theorem 1.6 (restated)], the following theorem is shown:

Theorem 273. *Let*

$$A := \begin{pmatrix} -1 & -1 \\ 1 & 1 \\ -1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad G := \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \qquad b := \begin{pmatrix} 0 \\ 2 \\ 1 \\ 1 \end{pmatrix}.$$

Define $P := P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right) \subseteq \mathbb{R}^2 \times \mathbb{R}^1$. Then:

1. *The inequality $w_1 \leq 0$ is a valid 2-branch split cut for $\binom{x}{w} \in P$ derived from the 2-branch split disjunction $D\left(e^{2,1}, e^{2,2}, 0, 0\right)$.*
2. $\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \in \bigcap_{i=1}^{4} \mathrm{cl}_I\left(P\left(\begin{pmatrix} A & G \end{pmatrix}_{[4]\setminus\{i\},*}, b_{[4]\setminus\{i\}}\right)\right)$.

In particular, we have

$$\mathrm{cl}_{BR}(A, G, b) \nsubseteq \mathrm{cl}_{2BS}(P).$$

We now prove that an analogous statement of the statement of Theorem 273 already holds in the pure integer case. In other words: we do not have to assume that one of the variables of the polytope P in Theorem 273 is continuous. The proof of the next theorem (Theorem 274) is very similar to the proof in [DGM15, Lemma 7.1, Lemma 7.2 and Theorem 1.6 (restated)] for what we state here as Theorem 273.

Theorem 274. *Let*

$$A := \begin{pmatrix} -1 & -1 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}, \qquad b := \begin{pmatrix} 0 \\ 2 \\ 1 \\ 1 \end{pmatrix}.$$

Define $P := P^{\leq}(A, b) \subseteq \mathbb{R}^3$. Then:

1. *The inequality $x_3 \leq 0$ is a valid 2-branch split cut for $x \in P$ derived from the 2-branch split disjunction $D\left(e^{3,1}, e^{3,2}, 0, 0\right)$.*
2. $\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \in \bigcap_{i=1}^{4} \mathrm{cl}_I\left(P^{\leq}\left(A_{[4]\setminus\{i\},*}, b_{[4]\setminus\{i\}}\right)\right)$.

In particular, we have

$$\mathrm{cl}_{BR}(A, b) \nsubseteq \mathrm{cl}_{2BS}(P).$$

Proof.

For 1: Let P' be the P of Theorem 273. By Theorem 273, $w_1 \leq 0$ is a valid 2-branch split for $\binom{x}{w} \in P'$ derived from the 2-branch split disjunction $D\left(e^{2,1}, e^{2,2}, 0, 0\right)$; in other words: it is a valid inequality for $\binom{x}{w} \in P' \cap \left(D\left(e^{2,1}, e^{2,2}, 0, 0\right) \times \mathbb{R}^1\right)$. Since $\underbrace{D\left(e^{2,1}, e^{2,2}, 0, 0\right) \times \mathbb{R}}_{\subseteq \mathbb{R}^2 \times \mathbb{R}} = \underbrace{D\left(e^{3,1}, e^{3,2}, 0, 0\right)}_{\subseteq \mathbb{R}^3}$ (under identification of $\mathbb{R}^2 \times \mathbb{R}^1$ with $\mathbb{R}^3$), we conclude the statement.

For 2: We prove this statement in a similar way to the proof from [DGM15] for statement 2 of Theorem 273. The points

$$p^1 := \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad p^2 := \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad p^3 := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad p^4 := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

lie in P_I and thus for every $i \in [4]$ in $P^{\leq}\left(A_{[4]\setminus\{i\},*}, b_{[4]\setminus\{i\}}\right)_I$. Also the points

$$q^1 := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad q^2 := \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad q^3 := \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \qquad q^4 := \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

satisfy the property $q^i \in P^{\leq}\left(A_{[4]\setminus\{i\},*}, b_{[4]\setminus\{i\}}\right)_I$ for $i \in [4]$ (since q^i only violates the inequality $A_{i,*}\left(\cdot\right) \leq b_i$). So for all $i \in [4]$, we have

$$\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \frac{p^i + q^i}{2} \in \operatorname{conv}\left(P^{\leq}\left(A_{[4]\setminus\{i\},*}, b_{[4]\setminus\{i\}}\right)_I\right) = \operatorname{cl}_I\left(P^{\leq}\left(A_{[4]\setminus\{i\},*}, b_{[4]\setminus\{i\}}\right)\right).$$

We thus conclude

$$\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \in \bigcap_{i=1}^{4} \operatorname{cl}_I\left(P^{\leq}\left(A_{[4]\setminus\{i\},*}, b_{[4]\setminus\{i\}}\right)\right).$$

□

We remark that later on (Theorem 576; also cf. Theorem 577, Theorem 560, Theorem 561 and Theorem 562), we prove that for every $m \in \mathbb{Z}_{\geq 1}$, there exists a rational polytope $P = P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right) \subseteq \mathbb{R}^m \times \mathbb{R}^1$ such that for every $\ell \in \mathbb{Z}_{\geq 0}$, we have:

$$\operatorname{cl}_{BR}(A, G, b) \subsetneq \operatorname{cl}^{(\ell)}_{(m-1)BS}(P),$$

which is some kind of reverse non-inclusion to Theorem 273 and Theorem 274.

6.1.3. t-branch split cuts vs integral lattice-free cuts

A proof of the following lemma is sketched in [DPW12]:

Lemma 275. *Let*

$$P^{275} := \operatorname{conv}\left\{\begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{3}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}\right\} \subsetneq \mathbb{R}^2 \times \mathbb{R}^1.$$

Then

$$\operatorname{cl}_I\left(P^{275}\right) = \operatorname{conv}\left\{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right\},$$

but

$$\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{4} \end{pmatrix} \in \operatorname{cl}_{ILF}\left(P^{275}\right) \setminus \operatorname{cl}_I(P).$$

On the other hand:

Theorem 276. *We have*

$$\operatorname{cl}_{2BS}\left(P^{275}\right) = \operatorname{cl}_I\left(P^{275}\right).$$

To see this, just consider the 2-branch split disjunction $D\left(e^{2,1}, e^{2,2}, 0, 0\right)$. Thus, in the light of Lemma 275, there exists a rational polytope $P \subseteq \mathbb{R}^2 \times \mathbb{R}^1$ that satisfies

$$\operatorname{cl}_{2BS}(P) \subsetneq \operatorname{cl}_{ILF}(P).$$

We remark that in Theorem 576 (also cf. Theorem 577, Theorem 560, Theorem 561 and Theorem 562), we prove that for every $m \in \mathbb{Z}_{\geq 1}$, there exists a rational polytope $P \subseteq \mathbb{R}^m \times \mathbb{R}^1$ such that for every $\ell \in \mathbb{Z}_{\geq 1}$, we have

$$\operatorname{cl}_{ILF}(P) \subsetneq \operatorname{cl}^{(\ell)}_{(m-1)BS}(P),$$

which is some kind of reverse non-inclusion to Theorem 276.

6.1.4. L_k cuts, crooked cross cuts and t-branch lattice-free cuts

Theorem 277. *Let $c(\cdot) \leq c_0$ be a multi-branch disjunctive cut (cf. Definition 141) for an arbitrary $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ with respect to $D := \bigcup_{i=1}^{t} D^i =: D \supseteq \mathbb{Z}^m$ $(t \in \mathbb{Z}_{\geq 0})$, where the*

$$D^i := \left\{x \in \mathbb{R}^m : A^i x \leq b^i\right\}$$

$(i \in [t])$ are as in Definition 141. Let $V' \leq \mathbb{R}^m$ be a subspace that satisfies

$$\operatorname{lin}\left(\bigcup_{i=1}^{t} \left(\operatorname{rowspan} A^i\right)^T\right) \leq V'.$$

Then $c(\cdot) \leq c_0$ is a valid inequality for $P \cap \left((\mathbb{Z}^m \times \mathbb{R}^n) + \left(V'^{\perp} \times \mathbb{R}^n\right)\right)$. In particular, if we set $k := \dim V$, we have:

- *If V' is a rational subspace, then $c(\cdot) \leq c_0$ is an $L_{k,\mathbb{Q}}$ cut for P.*
- *In general, $c(\cdot) \leq c_0$ is an $L_{k,\mathbb{R}}$ cut for P.*

Proof. Since $c(\cdot) \leq c_0$ is a multi-branch disjunctive cut for P, we have

$$P \cap (D \times \mathbb{R}^n) \subseteq P^{\leq}(c, c_0).$$

Because $\mathbb{Z}^m \subseteq D$ and $V'^{\perp} \leq \operatorname{lineal} D$, we deduce $\mathbb{Z}^m + V'^{\perp} \subseteq D$, from which we immediately conclude the statement:

$$P \cap \left((\mathbb{Z}^m \times \mathbb{R}^n) + \left(V'^{\perp} \times \mathbb{R}^n\right)\right) \subseteq P \cap (D \times \mathbb{R}^n) \subseteq P^{\leq}(c, c_0).$$

□

An immediate consequence of Theorem 277 is:

Corollary 278. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ be arbitrary, let $k \in \{0, \ldots, m\}$ and let $t \in \mathbb{Z}_{\geq 0}$. Every k, t-branch split cut for P (cf. Definition 252) with respect to $D\left(\pi^1, \ldots, \pi^t, \gamma_1, \ldots, \gamma_t\right)$ is an $L_{k,\mathbb{Q}}$ cut for P with respect to $V' \times \mathbb{R}^n$, where $V' \leq \left(\operatorname{lin}\left\{\pi^1, \ldots, \pi^t\right\}\right)^{\perp}$ is a rational subspace of codimension k. In particular, we have*

$$\operatorname{cl}_{L_{k,\mathbb{Q}}}(P) \subseteq \operatorname{cl}_{k,tBS}(P),$$

and more specifically

$$\operatorname{cl}_{L_{k,\mathbb{Q}}}(P) \subseteq \operatorname{cl}_{k,kBS}(P) = \operatorname{cl}_{kBS}(P).$$

Another consequence of Theorem 277 is:

Corollary 279. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ be arbitrary, where $m + n \geq 2$. Then every parametric cross cut for P (cf. Definition 150) and, in particular, every crooked cross cut for P (cf. Definition 146) is an $L_{2,\mathbb{Q}}$ cut for P. In particular, we have*

$$\operatorname{cl}_{L_{2,\mathbb{Q}}}(P) \subseteq \operatorname{cl}_{CC}(P).$$

In Theorem 462, we see that in many cases, the crooked cross closure is equivalent to the $L_{2,\mathbb{Q}}$ closure. On the other hand, by Corollary 278, every 2-branch split cut is an $L_{2,\mathbb{Q}}$ cut. So, for the next step, we analyze the relationship between 2-branch split cuts and crooked cross cuts for rational polyhedra. The following theorem is shown in [DGM15, Theorem 1.2 (Restated)]:

Theorem 280. *There exists a rational polytope $P \subsetneq \mathbb{R}^2 \times \mathbb{R}^1$ such that*

$$\operatorname{cl}_{CC}(P) \subsetneq \operatorname{cl}_{2BS}(P).$$

The central idea for the proof of Theorem 280 is the following result, which is proved in [DDG+13]. We present it in a formulation that is based on the one that is given in [DGM15, Theorem 5.1].

Theorem 281. *There exists a rational triangle $T^* \subseteq \mathbb{R}^2$ satisfying*

1. $(\operatorname{int} T^*) \cap \mathbb{Z}^2 = \emptyset$,

2. *$e^{2,i} \in \operatorname{bd} T^*$ for $i \in \{0,1,2\}$ and*
3. *$\exists \delta \in \mathbb{R}_{>0}$ such that for any pair S_1, S_2 of split sets for $\mathbb{Z}^2$, we have $\operatorname{vol}_2 (T^* (T^*, x^*) \setminus (S_1 \cup S_2)) \geq \delta$, where vol_2 denotes the 2-dimensional volume.*

Now, let $T^* \subsetneq \mathbb{R}^2$ be as Theorem 281 and let $x^* \in (\operatorname{int} T^*) \cap \mathbb{Q}^2$. Set

$$P := \operatorname{conv} ((T^* \times \{0\}) \cup (\{x^*\} \times \{1\})).$$

This P satisfies the conditions that are imposed upon it in Theorem 280.

From Corollary 279 and Theorem 280, we conclude:

Corollary 282. *There exists a rational polytope $P \subsetneq \mathbb{R}^2 \times \mathbb{R}^1$ such that*

$$\operatorname{cl}_{L_{2,\mathbb{Q}}} (P) \subsetneq \operatorname{cl}_{2BS} (P).$$

We now consider how Corollary 282 can be generalized to t-branch split cuts, where $t \geq 3$. In [DDG+13, Lemma 4.9, Theorem 4.10 and Theorem 4.11], it is shown:

Theorem 283. *For all $m \in \mathbb{Z}_{\geq 3}$, there exists a rational, full-dimensional, lattice-free polytope $T^\epsilon \subseteq \mathbb{R}^m$ (the ϵ has a specific meaning in [DDG+13], which does not matter here) such that for any given set of $3 \cdot 2^{m-2} - 1$ split sets $S_1, \ldots, S_{3 \cdot 2^{m-2}-1}$, we have:*

$$\operatorname{int} T^\epsilon \not\subseteq \bigcup_{i=1}^{3 \cdot 2^{m-2}-1} S_i.$$

Additionally, we have ([DDG+13], Theorem 4.13):

Theorem 284. *Let $t \in \mathbb{Z}_{\geq 1}$ and let $P \subseteq \mathbb{R}^m$ ($m \in \mathbb{Z}_{\geq 1}$) be a rational, full-dimensional, lattice-free polytope. Assume that the integer hull of P has dimension m and $\operatorname{int} P$ cannot be covered by t split sets. Let $\overline{x} \in \operatorname{int} P$ (w.l.o.g. $\overline{x} \in \mathbb{Q}^m$) and let*

$$Q := \operatorname{conv} \left((B \times \{0\}) \dot\cup \left\{ \left(\begin{pmatrix} \overline{x} \\ \frac{1}{2} \end{pmatrix} \right) \right\} \right) \subseteq \mathbb{R}^m \times \mathbb{R}^1.$$

Then $y_1 \leq 0$ is a facet-defining inequality for $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in \operatorname{cl}_I (Q)$ which cannot be expressed as a t-branch split cut for Q.

We thus conclude, since the polytope T^ϵ from Theorem 283 satisfies the conditions for P in Theorem 284:

Theorem 285. *For all $m \in \mathbb{Z}_{\geq 3}$, there exists a rational polytope $P \subsetneq \mathbb{R}^m \times \mathbb{R}^1$ such that*

- *$y_1 \leq 0$ is a valid inequality for $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in P_I$ and a valid $L_{m,\mathbb{Q}}$ cut for P,*
- *$y_1 \leq 0$ is not a valid $\left(3 \cdot 2^{m-2} - 1\right)$-branch split cut for $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in P$.*

We remark that one *cannot* conclude from Theorem 285 that there exists a polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^1$ ($m \in \mathbb{Z}_{\geq 3}$) having

$$\operatorname{cl}_{L_{m,\mathbb{Q}}} (P) \subsetneq \operatorname{cl}_{(3 \cdot 2^{m-2}-1)BS} (P).$$

For the question how $t-1$-branch split cuts relate to t-branch split cuts: in Theorem 560 and Theorem 562 (also cf. Theorem 561, Theorem 576 and Theorem 577), we see that for every $m \in \mathbb{Z}_{\geq 1}$, there exists a rational polytope $P \subseteq \mathbb{R}^m \times \mathbb{R}^1$ having

$$\operatorname{cl}_{mBS} (P) \subsetneq \operatorname{cl}_{(m-1)BS} (P).$$

We now consider the question how crooked cross cuts are related to 3-branch split cuts. One can easily show that for every $\pi^1, \pi^2 \in \mathbb{Z}^m$ and $\gamma_1, \gamma_2 \in \mathbb{Z}$, we have:

$$D\left(\pi^1, \pi^2, \pi^2 - \pi^1, \gamma_1, \gamma_2, \gamma_2 - \gamma_1\right) \subseteq D^c\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right).$$

From this, we immediately conclude:

Theorem 286. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be arbitrary. Then every crooked cross cut for P is also a 2,3-branch split cut for P (recall Definition 252), thus a 3-branch split cut for P. In particular, we have*

$$\operatorname{cl}_{3BS} (P) \subseteq \operatorname{cl}_{2,3BS} (P) \subseteq \operatorname{cl}_{CC} (P).$$

We now consider that the inclusion $\mathrm{cl}_{3BS}(P) \subseteq \mathrm{cl}_{CC}(P)$ in Theorem 286 can also be strict. This is the statement of the following theorem, which is shown in [DGM15, Theorem 1.2 (restated)]:

Theorem 287. *Let $P := P^{516,3} \subseteq \mathbb{R}^3$ (cf. Definition 516). Then $\emptyset = \mathrm{cl}_I(P) = \mathrm{cl}_{3BC}(P)$, but $\frac{1}{2} \cdot 1^3 \in \mathrm{cl}_{CC}(P)$, thus $\mathrm{cl}_{3BC}(P) \subsetneq \mathrm{cl}_{CC}(P)$.*

We remark that we analyze the $L_{k,\mathbb{Q}}$ closures of $P^{516,3}$ (or rather of the series $P^{516,m}$ of polytopes) much more deeply in section 13.1 of chapter 13.

6.1.5. k-disjunctive cuts vs $L_{k',\mathbb{Q}}$ cuts

We now show that for a rational polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$, the k-disjunctive closure is at least as strong as the $L_{k',\mathbb{Q}}$ closure as long as k is chosen sufficiently large (only depending on k'):

Theorem 288. *Let $\emptyset \neq P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ be a rational polyehdron. Let $k \in \{0, \dots, m\}$ and let $c(\cdot) \geq c_0$ $(c \in (\mathbb{Q}^m \times \mathbb{Q}^n)^T$ and $c_0 \in \mathbb{Q})$ be an $L_{k,\mathbb{Q}}$ cut for P that is not already valid for P. Then $c(\cdot) \geq c_0$ is a 2^k-disjunctive cut for P. In particular, we have*

$$\mathrm{cl}_{2^k D}(P) \subseteq \mathrm{cl}_{L_{k,\mathbb{Q}}}(P).$$

Remark 289. *The condition of Theorem 288 that c and c_0 are rational is not a restriction, since it is a consequence of Theorem 178 that for a rational polyhedron P, every $L_{k,\mathbb{Q}}$ cut is dominated absolutely by a finite set of* rational *$L_{k,\mathbb{Q}}$ cuts.*

Proof. (Theorem 288) Since $c(\cdot) \geq c_0$ is an $L_{k,\mathbb{Q}}$ cut for P that is not valid for P, and P is rational, by Theorem 240, there exists a rational, full-dimensional, maximal lattice-free body $L := P^{\leq}(A', b') \subseteq \mathbb{R}^m$ with $\dim(\mathrm{lineal}\, L) \geq m - k$ such that $c(\cdot) \geq c_0$ is valid for

$$P \cap \bigcup_{i=1}^{k'} \left(P^{\geq}\left(A'_{i,*}, b'_i\right) \times \mathbb{R}^n\right).$$

Since L is maximal lattice-free, by Theorem 108, L has at most 2^k facets, i.e. we can assume $k' \leq 2^k$. Thus, $c(\cdot) \geq c_0$ is a 2^k-disjunctive cut for P. ☐

6.1.6. k row cuts, split cuts, crooked cross cuts and $L_{k',\mathbb{Q}}$ cuts

For the remainder of this section, we define (cf. [DDG12, section 5]):

Definition 290. *Let $A \in \mathbb{R}^{l \times m}$, $G \in \mathbb{R}^{l \times n}$ and $b \in \mathbb{R}^l$ $(l, m, n \in \mathbb{Z}_{\geq 0})$. An inequality $\begin{pmatrix} c & d \end{pmatrix}(\cdot) \geq f$ $(c \in (\mathbb{R}^m)^T$, $d \in (\mathbb{R}^n)^T$ and $f \in \mathbb{R})$ is **a translation of** $\begin{pmatrix} c' & d' \end{pmatrix}(\cdot) \geq f'$ **with respect to** A, G **and** b $(c' \in (\mathbb{R}^m)^T$, $d' \in (\mathbb{R}^n)^T$ and $f' \in \mathbb{R})$ if there exists a $\mu \in \left(\mathbb{R}^l\right)^T$ and a $\delta \in \mathbb{R}_{>0}$ such that*

$$\begin{pmatrix} c & d & f \end{pmatrix} = \mu \begin{pmatrix} A & G & b \end{pmatrix} + \delta \begin{pmatrix} c' & d' & f' \end{pmatrix}.$$

The motivation for Definition 290 should be clear if one keeps in mind that in [DDG12], the authors consider linear inequalities for $P^{=}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right) \cap \left(\mathbb{R}^m \times \mathbb{R}^n_{\geq 0}\right)$.

6.1.6.1. Results

The following theorem is shown in [DDG12, Lemma 5.1] for the case that A, G and b are rational, even though the argumentation does not make any use of the rationality:

Theorem 291. *Let*

$$P := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R}^n_{\geq 0} : Ax + Gy = b \right\},$$

where $A \in \mathbb{R}^{l \times m}$, $G \in \mathbb{R}^{l \times n}$ and $b \in \mathbb{R}^l$ $(l, m, n \in \mathbb{Z}_{\geq 0})$. Let $\pi \in \mathbb{Z}^m$ and $\gamma \in \mathbb{Z}$ be such that

$$P \cap (D(\pi, \gamma) \times \mathbb{R}^n) \neq \emptyset$$

and let $\begin{pmatrix} c & d \end{pmatrix}(\cdot) \geq f$ $(c \in (\mathbb{R}^m)^T$, $d \in (\mathbb{R}^n)^T$ and $f \in \mathbb{R})$ be a nontrivial split cut for P (i.e. $\begin{pmatrix} c & d \end{pmatrix}(\cdot) \geq f$ is not valid for P) with respect to $D(\pi, \gamma)$. Then there exists a $\lambda \in \left(\mathbb{R}^l\right)^T$ having $\pi = \lambda A$ such that

$\begin{pmatrix} \pi & d' \end{pmatrix}(\cdot) \geq \gamma + 1$ is a translation of $\begin{pmatrix} c & d \end{pmatrix}(\cdot) \geq f$ with respect to A, G and b for some $d' \in (\mathbb{R}^n)^T$. Furthermore, $\begin{pmatrix} \pi & d' \end{pmatrix}(\cdot) \geq \gamma + 1$ can be derived as a split cut for the 1-row relaxation

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R}^n_{\geq 0} : \underbrace{\lambda A}_{=\pi^T} x + \lambda G y = \lambda b \right\}.$$

In [DDG12, section 5], the authors analyze that in Theorem 291, the translation of the split cut is really necessary. In other words: not every split cut for P as in Theorem 291 can be obtained as a split cut of a 1-row relaxation *without* translation. Also, Theorem 291 does not hold in the case that $P \cap (D(\pi, \gamma) \times \mathbb{R}^n) = \emptyset$. For this, we consider the following example:

Example 292. *([DDG12, section 5]) Consider the polyhedron*

$$P := \left\{ x \in \mathbb{R}^2 : \frac{3}{10} \leq x_1 \leq \frac{7}{10} \right\} \subseteq \mathbb{R}^2.$$

By introducing slack variables, we can bring P into the form that is required for Theorem 291:

$$\overline{P} := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \times \mathbb{R}^2_{\geq 0} : \underbrace{\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}}_{=:A} x + \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{=:G} y = \underbrace{\begin{pmatrix} \frac{7}{10} \\ \frac{3}{10} \end{pmatrix}}_{=:b} \right\}.$$

Clearly, $\overline{P} \cap \left(D\left(e^{2,1}, 0\right) \times \mathbb{R}^2\right) = \emptyset$, thus $x_2 \leq 0$ is a split cut for $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in \overline{P}$ with respect to the split disjunction $D\left(e^{2,1}, 0\right)$. On the other hand, every 1-row relaxation of $\overline{P}$ with respect to A, G, b and some $\lambda\left(\mathbb{R}^2\right)^T$ is of the form

$$\overline{P}' := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \times \mathbb{R}^2_{\geq 0} : (\lambda_1 + \lambda_2) x_1 + \lambda_1 y_1 - \lambda_2 y_2 = \frac{7}{10}\lambda_1 + \frac{3}{10}\lambda_2 \right\},$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$. This relaxation is feasible for every $x_2 \in \mathbb{R}$; thus, $x_2 \leq 0$ is not valid for any arbitrary 1-row relaxation. It can also not be obtained by a translation, as x_2 does not appear in the constraints defining $\overline{P}$.

What, according to [DDG12, section 5.1], holds in the case $P \cap D(\pi, \gamma) = \emptyset$, is the following version of Theorem 291:

Theorem 293. *Let*

$$P := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R}^n_{\geq 0} : Ax + Gy = b \right\},$$

where $A \in \mathbb{R}^{l \times m}$, $G \in \mathbb{R}^{l \times n}$ and $b \in \mathbb{R}^l$ ($l, m, n \in \mathbb{Z}_{\geq 0}$). Let $\pi \in \mathbb{Z}^m$ and $\gamma \in \mathbb{Z}$ be such that

$$P \cap (D(\pi, \gamma) \times \mathbb{R}^n) = \emptyset.$$

Then there exists a $\lambda \in \left(\mathbb{R}^l\right)^T$ such that a translation of $\begin{pmatrix} (0^m)^T & (0^n)^T \end{pmatrix}(\cdot) \geq 1$ can be derived as a split cut from the 1-row relaxation

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R}^n_{\geq 0} : \underbrace{\lambda A}_{=\pi^T} x + \lambda G y = \lambda b \right\}.$$

From Theorem 291 and Theorem 293, we obtain:

Theorem 294. *Let*

$$P := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R}^n_{\geq 0} : Ax + Gy = b \right\},$$

where $A \in \mathbb{R}^{l \times m}$, $G \in \mathbb{R}^{l \times n}$ and $b \in \mathbb{R}^l$ ($l, m, n \in \mathbb{Z}_{\geq 0}$). Then

$$\mathrm{cl}_{1R}(A, G, b) \subseteq \mathrm{cl}_{split}(P).$$

We next consider 2-row relaxations. For the following theorem, a proof idea is sketched in [DGM15, section 1.2]:

Theorem 295. *There exist rational A, G and b such that*

$$\mathrm{cl}_{2R}(A, G, b) \subsetneq \mathrm{cl}_{1R}(A, G, b).$$

The idea that the authors sketch is to bring the well-known Cook-Kannan-Schrijver example (cf. [CKS90], but also cf. [LR08]) into the form that Definition 154 requires for k row cuts, just as we do in Definition 557 for the more general Li-Richard example to define $P^{557,m,\epsilon,=}$. One can show that one can derive the missing facet-defining inequality from a 2-row relaxation (cf. [ALWW07] for the original derivation for 2-row cuts). On the other hand, the authors remark that it is possible to show the non-trivial statement that for this polyhedron, all cuts from a 1-row relaxation are split cuts; thus, 1-row cuts are not sufficient to obtain the mixed-integer hull.

For the remainder of this section, we define (cf. [DDG12, section 2]):

Definition 296. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ be arbitrary. A parametric cross cut for P (cf. Definition 151) is called* ***non-trivial*** *if it not valid for the split closure of P.*

In [DDG12, Theorem 5.8], the following analogue of Theorem 291 for parametric cross cuts is shown:

Theorem 297. *Let*

$$P := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R}^n_{\geq 0} : Ax + Gy = b \right\},$$

where $A \in \mathbb{R}^{l\times m}$, $G \in \mathbb{R}^{l\times n}$ and $b \in \mathbb{R}^l$ $(l, m, n \in \mathbb{Z}_{\geq 0})$ and let $\begin{pmatrix} c & d \end{pmatrix}(\cdot) \geq f$ $(c \in (\mathbb{R}^m)^T$, $d \in (\mathbb{R}^n)^T$ and $f \in \mathbb{R})$ be a non-trivial parametric cross cut for P derived from the parametric cross disjunction $D^t(\pi^1, \pi^2, \gamma_1, \gamma_2)$ $(\pi^1, \pi^2 \in \mathbb{Z}^m$ and $\gamma_1, \gamma_2, t \in \mathbb{Z})$. If $P \cap D^t(\pi^1, \pi^2, \gamma_1, \gamma_2) \neq \emptyset$, then there exist $\lambda^1, \lambda^2, \lambda^3 \in (\mathbb{R}^l)^T$ having

$$\lambda^1 A = (\pi^1)^T, \qquad \lambda^2 A = (\pi^2)^T, \qquad \lambda^3 A = (0^m)^T$$

such that a translation of $\begin{pmatrix} c & d \end{pmatrix}(\cdot) \geq f$ is a parametric cross cut with respect to $D^t(\pi^1, \pi^2, \gamma_1, \gamma_2)$ for the 3-row relaxation

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R}^n_{\geq 0} : \begin{pmatrix} \lambda^1 \\ \lambda^2 \\ \lambda^3 \end{pmatrix} Ax + \begin{pmatrix} \lambda^1 \\ \lambda^2 \\ \lambda^3 \end{pmatrix} Gy = \begin{pmatrix} \lambda^1 \\ \lambda^2 \\ \lambda^3 \end{pmatrix} b \right\}.$$

In [DDG12, Lemma 5.5, Corollary 5.6], the following analogue of Theorem 293 for parametric cross cuts is shown:

Theorem 298. *Let*

$$P := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R}^n_{\geq 0} : Ax + Gy = b \right\},$$

where $A \in \mathbb{R}^{l\times m}$, $G \in \mathbb{R}^{l\times n}$ and $b \in \mathbb{R}^l$ $(l, m, n \in \mathbb{Z}_{\geq 0})$. Let a parametric cross disjunction $D^t(\pi^1, \pi^2, \gamma_1, \gamma_2)$ be given such that

- *$P \cap D^t(\pi^1, \pi^2, \gamma_1, \gamma_2) = \emptyset$ and*
- *$\begin{pmatrix} (0^m)^T & (0^n)^T \end{pmatrix}(\cdot) \geq 1$ is a* non-trivial *parametric cross cut for P with respect to $D^t(\pi^1, \pi^2, \gamma_1, \gamma_2)$.*

Then the inequality $\begin{pmatrix} (0^m)^T & (0^n)^T \end{pmatrix}(\cdot) \geq 1$ is a conic combination of two parametric cross cuts with respect to $D^t(\pi^1, \pi^2, \gamma_1, \gamma_2)$ of two 2-row relaxations of P.

Note that for the statements in [DDG12] for what we state here as Theorem 297 and Theorem 298, A, G and b are demanded be rational. However the proofs in this paper do not depend on this fact.

From Theorem 297 and Theorem 298, we obtain:

Corollary 299. *Let*

$$P := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R}^n_{\geq 0} : Ax + Gy = b \right\},$$

where $A \in \mathbb{R}^{l\times m}$, $G \in \mathbb{R}^{l\times n}$ and $b \in \mathbb{R}^l$ $(l, m, n \in \mathbb{Z}_{\geq 0})$. Then

$$\mathrm{cl}_{3R}(A, G, b) \subseteq \mathrm{cl}_{CC}(P).$$

We next get to a result that is shown in [DGM15, Theorem 1.6 (Restated)], which states that the 2-branch split closure can be stronger than the 2-row closure:

Theorem 300. *Let*

$$P := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \times \mathbb{R}^4_{\geq 0} : \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}}_{=:A} x + \underbrace{\begin{pmatrix} -\frac{1}{2} & 0 & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ -1 & -1 & 1 & 1 \end{pmatrix}}_{=:G} y = \underbrace{\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}}_{=:b} \right\}.$$

We have

$$P \cap \left(\bigcap_{M \in \mathbb{R}^{2\times 3}} \operatorname{conv}\left(\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \times \mathbb{R}^4_{\geq 0} : MAx + MGy = Mb \right\}_I \right) \right) \not\subseteq \mathrm{cl}_{2BC}(P).$$

This obviously means

$$\mathrm{cl}_{2R}(A, G, b) \not\subseteq \mathrm{cl}_{2BC}(P).$$

We remark that in Theorem 577, we show an additional non-inclusion result for k rows cuts versus t-branch split cuts and $L_{k'}$ cuts.

We now show that for polyhedra that are neither rational nor full-dimensional, even the 1-row closure can be stronger than any t-branch split closure (thus any parametric cross closure, which in particular includes the crooked cross closure). For this, let us recall Theorem 255 before we state our result in Theorem 301.

Theorem 255. *Let $t \in \mathbb{Z}_{\geq 0}$, $\pi^1, \ldots, \pi^t \in \mathbb{Z}^2$ and $\gamma_1, \ldots, \gamma_t \in \mathbb{Z}$. Then*

$$\operatorname{conv}\left(P^{114} \cap D\left(\pi^1, \ldots, \pi^t, \gamma_1, \ldots, \gamma_t\right)\right) = P^{114}. \tag{5.21}$$

In particular, for all parametric cross disjunctions $D^{t'}\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right)$ (cf. Definition 150) with respect to some $t' \in \mathbb{Z}$ (this includes crooked cross disjunctions), we have:

$$\operatorname{conv}\left(P^{114} \cap D^{t'}\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right)\right) = P^{114}. \tag{5.22}$$

On the other hand, $\left(0^2\right)^T(\,\cdot\,) \leq -1$ is an $L_{2,\mathbb{Q}}$ cut for P^{114}.

We thus obtain:

Theorem 301. *Let*

$$P := P^{=}\big(\underbrace{\begin{pmatrix} \sqrt{2} & -1 \end{pmatrix}}_{=:A}, \underbrace{\begin{pmatrix} -\frac{1}{2} \end{pmatrix}}_{=:b}\big) \subseteq \mathbb{R}^2.$$

Clearly, $P = P^{114}$ and $\left(0^2\right)^T(\,\cdot\,) \leq -1$ is a 1-row cut for P. On the other hand, we have for all $t \in \mathbb{Z}_{\geq 0}$:

$$\mathrm{cl}_{1R}(A, b) = \emptyset \subsetneq P = \mathrm{cl}_{CC}(P) = \mathrm{cl}_{tBS}(P).$$

Proof. By definition, $\left(0^2\right)^T(\,\cdot\,) \leq -1$ is a 1-row cut for P with respect to A and b. $\mathrm{cl}_{CC}\left(P^{114}\right) = \mathrm{cl}_{tBS}\left(P^{114}\right) = P^{114}$ is a direct consequence of Theorem 255. □

We write down another consequence of Theorem 255:

Theorem 302. *We have for all $t \in \mathbb{Z}_{\geq 0}$:*

$$\begin{aligned} &\mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{R}}}\left(P^{114}\right) = \mathrm{cl}_{L_{1,\mathbb{R}}}\left(P^{114}\right) = \mathrm{cl}_{L_{2-\frac{1}{2},\mathbb{Q}}}\left(P^{114}\right) = \mathrm{cl}_{L_{2,\mathbb{Q}}}\left(P^{114}\right) = \mathrm{cl}_I\left(P^{114}\right) = \emptyset \\ \subsetneq & P^{114} = \mathrm{cl}_{tBS}\left(P^{114}\right). \end{aligned}$$

Proof. $\mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{R}}}\left(P^{114}\right) = \mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{R}}}\left(P^{114}\right) = \mathrm{cl}_{L_{2-\frac{1}{2},\mathbb{Q}}}\left(P^{114}\right) = \mathrm{cl}_{L_{2,\mathbb{Q}}}\left(P^{114}\right) = \mathrm{cl}_I\left(P^{114}\right) = \emptyset$ holds by Theorem 187, Theorem 188 and Theorem 202. $\mathrm{cl}_{tBS}\left(P^{114}\right) = P^{114}$ is a direct consequence of Theorem 255. □

6.1.6.2. k row cuts, $L_{k',\mathbb{Q}}$ cuts and lattice-free bodies

The goal of this section is to give an inclusion result for the k row closure versus the $L_{k',\mathbb{Q}}$ closure. This result is formulated in Theorem 304. To show it, we first prove a result about lattice-free bodies and k row cuts, which we believe is also out of itself of mathematical relevance.

Theorem 303. *Let*

$$\emptyset \neq P = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R}^n_{\geq 0} : Ax + Gy = b \right\},$$

where $A \in \mathbb{R}^{r \times m}$, $G \in \mathbb{R}^{r \times n}$ and $b \in \mathbb{R}^r$ ($r, m, n \in \mathbb{Z}_{\geq 0}$). Clearly, P is in the form that is required in Definition 154. Let $P' := P^{\leq}(A', b') \subseteq \mathbb{R}^m$ be a (w.l.o.g full-dimensional) lattice-free polyhedron, where $A' \in \mathbb{R}^{m \times k'}$ and $b' \in \mathbb{R}^{k'}$ ($k' \in \mathbb{Z}_{\geq 2}$).

- *If*

$$P \cap \bigcup_{i=1}^{k'} \left(P^{\geq} \left(A'_{i,*}, b'_i \right) \times \mathbb{R}^n \right) = \emptyset,$$

  define

$$\begin{pmatrix} c^1 & c^2 \end{pmatrix} := \begin{pmatrix} (0^m)^T & (0^n)^T \end{pmatrix}, \qquad c_0 := 1.$$

- *Otherwise, i.e. if*

$$P \cap \bigcup_{i=1}^{k'} \left(P^{\geq} \left(A'_{i,*}, b'_i \right) \times \mathbb{R}^n \right) \neq \emptyset,$$

  let $\begin{pmatrix} c^1 & c^2 \end{pmatrix} (\cdot) \geq c_0$ ($c^1 \in (\mathbb{R}^m)^T$, $c^2 \in (\mathbb{R}^n)^T$ and $c_0 \in \mathbb{R}$) be a cutting plane that is not *valid for P, but for*

$$P \cap \bigcup_{i=1}^{k'} \left(P^{\geq} \left(A'_{i,*}, b'_i \right) \times \mathbb{R}^n \right).$$

Then there exists an $M \in \mathbb{R}^{(k'-1) \times r}$ such that $\begin{pmatrix} c^1 & c^2 \end{pmatrix} (\cdot) \geq c_0$ is a translation (cf. Definition 290) of a valid inequality for

$$\underbrace{\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R}^n_{\geq 0} : MAx + MGy = Mb \right\}}_{=:P(M)} \cap \bigcup_{i=1}^{k'} \left(P^{\geq} \left(A'_{i,*}, b'_i \right) \times \mathbb{R}^n \right).$$

Proof. W.l.o.g. we assume that in the second case, we have $P \cap \left(P^{\geq} \left(A'_{1,*}, b'_1 \right) \times \mathbb{R}^n \right) \neq \emptyset$. Additionally, we assume that the rows of A' and b' are ordered such that for all $2 \leq i \leq k''$, we have $P \cap \left(P^{\geq} \left(A'_{i,*}, b'_i \right) \times \mathbb{R}^n \right) \neq \emptyset$ and for $k'' + 1 \leq i \leq k'$, we have $P \cap \left(P^{\geq} \left(A'_{i,*}, b'_i \right) \times \mathbb{R}^n \right) = \emptyset$ ($k'' \in \{1, \dots, k'\}$). Thus, it is a consequence of Lemma 88, Lemma 89 and Lemma 90 that there exist multipliers $\lambda^1, \dots, \lambda^{k'} \in (\mathbb{R}^r)^T$ and $\mu^1, \dots, \mu^{k'} \in \mathbb{R}_{\geq 0}$ such that

$$c^1 = \lambda^1 A + \mu^1 A'_{1,*}, \tag{6.1}$$

$$c^2 \geq \lambda^1 G, \tag{6.2}$$

$$c_0 \leq \lambda^1 b + \mu^1 b'_1, \tag{6.3}$$

$$c^1 = \lambda^i A + \mu^i A'_{i,*} \qquad \forall i \in \{2, \dots, k''\}, \tag{6.4}$$

$$c^2 \geq \lambda^i G \qquad \forall i \in \{2, \dots, k''\}, \tag{6.5}$$

$$c_0 \leq \lambda^i b + \mu^i b'_i \qquad \forall i \in \{2, \dots, k''\}, \tag{6.6}$$

$$(0^m)^T = \lambda^i A + \mu^i A'_{i,*} \qquad \forall i \in \{k'' + 1, \dots, k'\}, \tag{6.7}$$

$$(0^n)^T \geq \lambda^i G \qquad \forall i \in \{k'' + 1, \dots, k'\}, \tag{6.8}$$

$$1 = \lambda^i b + \mu^i b'_i \qquad \forall i \in \{k'' + 1, \dots, k'\} \tag{6.9}$$

holds. Let

$$M := \begin{pmatrix} \lambda^2 - \lambda^1 \\ \vdots \\ \lambda^{k''} - \lambda^1 \\ \lambda^{k''+1} \\ \vdots \\ \lambda^{k'} \end{pmatrix}$$

and let

$$\begin{pmatrix} x \\ y \end{pmatrix} \in P(M) \cap \bigcup_{i=1}^{k'} \left(P^{\geq}\left(A'_{i,*}, b'_i\right) \times \mathbb{R}^n\right).$$

By definition, there exists an $i \in [k']$ such that $A'_{i,*}x \geq b'_i$. We distinguish three cases:

1. $i = 1$,
2. $i \in \{2, \ldots, k''\}$,
3. $i \in \{k''+1, \ldots, k'\}$.

We show that in all three cases, the identitity

$$c^1 x + c^2 y - \lambda^1 A x - \lambda^1 G y \geq c_0 - \lambda^1 b \tag{6.10}$$

holds.

Case 1 ($i = 1$):

$$\begin{aligned} c^1 x + c^2 y - \lambda^1 A x - \lambda^1 G y &\geq \mu^1 A'_{1,*} x && \text{(by (6.1), (6.2), } y \geq 0) \\ &\geq \mu^1 b'_1 && \text{(case assumption, } \mu^1 \geq 0) \\ &\geq c_0 - \lambda^1 b. && \text{(by (6.3))} \end{aligned}$$

Case 2 ($i \in \{2, \ldots, k''\}$):

$$\begin{aligned} c^1 x + c^2 y - \lambda^1 A x - \lambda^1 G y =& c^1 x + c^2 y - \lambda^1 A x - \lambda^1 G y \\ & - \left(\lambda^i - \lambda^1\right) A x - \left(\lambda^i - \lambda^1\right) G y + \left(\lambda^i - \lambda^1\right) b && \left(\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in P(M)\right) \\ =& c^1 x - \lambda^i A x + c^2 y - \lambda^i G y + \left(\lambda^i - \lambda^1\right) b \\ \geq& \mu^i A'_{i,*} x + \left(\lambda^i - \lambda^1\right) b && \text{(by (6.4), (6.5))} \\ \geq& \mu^i b'_i + \left(\lambda^i - \lambda^1\right) b && \text{(case assumption, } \mu^i \geq 0) \\ \geq& c_0 - \lambda^1 b. && \text{(by (6.6))} \end{aligned}$$

Case 3 ($i \in \{k''+1, \ldots, k'\}$): We show that $P(M) \cap P^{\geq}(A'_i, b'_i) = \emptyset$. Thus, any inequality (in particular (6.10)) is valid for $P(M) \cap P^{\geq}(A'_i, b'_i)$:

$$\begin{aligned} \left(0^m\right)^T x + \left(0^n\right)^T y &= -\lambda^i A x - \lambda^i G y + \lambda^i b && \left(\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in P(M)\right) \\ &\geq \mu^i A'_{i,*} x + \lambda^i b && \text{(by (6.7), (6.8))} \\ &\geq \mu^i b'_i + \lambda^i b && \text{(case assumption, } \mu^i \geq 0) \\ &= 1. && \text{(by (6.9))} \end{aligned}$$

□

Theorem 304. *Let*

$$P := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R}^n_{\geq 0} : Ax + Gy = b \right\},$$

where $A \in \mathbb{Q}^{r \times m}$, $G \in \mathbb{Q}^{r \times n}$ and $b \in \mathbb{Q}^r$ $(r, m, n \in \mathbb{Z}_{\geq 0})$. Let $k \in \{0, \ldots, m\}$ and let $c(\cdot) \geq c_0$ be an $L_{k,\mathbb{Q}}$ cut for P, where $c \in (\mathbb{Q}^m \times \mathbb{Q}^n)^T$ and $c_0 \in \mathbb{Q}$. If $P \neq \emptyset$, then $c(\cdot) \geq c_0$ is a translate of a $2^k - 1$ row cut with respect to A, G and b. In particular, we have

$$\mathrm{cl}_{(2^k-1)R}(A, G, b) \subseteq \mathrm{cl}_{L_{k,\mathbb{Q}}}(P).$$

Remark 305. *The condition of Theorem 304 that c and c_0 are rational is not a restriction, since it is a consequence of Theorem 178 that for a rational polyhedron P, every $L_{k,\mathbb{Q}}$ cut is dominated absolutely by a finite set of* rational *$L_{k,\mathbb{Q}}$ cuts.*

Proof. (Theorem 304) Since $c(\cdot) \geq c_0$ is an $L_{k,\mathbb{Q}}$ cut for P and P is rational, by Theorem 240, there exists a rational, full-dimensional, maximal lattice-free body $L := P^{\leq}(A', b') \subseteq \mathbb{R}^m$ with $\dim(\mathrm{lineal}\, L) \geq m - k$ such that $c(\cdot) \geq c_0$ is valid for

$$P \cap \bigcup_{i=1}^{k'} \left(P^{\geq}\left(A'_{i,*}, b'_i\right) \times \mathbb{R}^n\right).$$

Since L is maximal lattice-free, by Theorem 108, L has at most 2^k facets. Thus, we immediately conclude from Theorem 303 that $c(\cdot) \geq c_0$ is a $2^k - 1$ row cut with respect to A, G and b. □

We remark that in Theorem 316 in section 6.2.5, we state an important consequence of Theorem 304.

6.2. Convergence to the (mixed-)integral closure

Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ be a rational polyhedron and let $\mathrm{cl}_{(\cdot)}(P)$ be the closure of P with respect to the class $(\cdot)$ of cutting planes. For $t \in \mathbb{Z}_{\geq 0}$, we define the iterated closure with respect to the class $(\cdot)$ as

$$\mathrm{cl}^{(t)}_{(\cdot)}(P) := \begin{cases} P & \text{for } t = 0, \\ \mathrm{cl}_{(\cdot)}\left(\mathrm{cl}^{(t-1)}_{(\cdot)}(P)\right) & \text{for } t > 0. \end{cases}$$

In this section, we consider the following questions for various classes of cutting planes (in order of essentially decreasing strength):

1. Does $\mathrm{cl}_{(\cdot)}(P) = \mathrm{cl}_I(P)$ hold?

2. Does there exist a $t \in \mathbb{Z}_{\geq 0}$ such that $\mathrm{cl}^{(t)}_{(\cdot)}(P) = \mathrm{cl}_I(P)$ holds? In general (i.e. $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ arbitrary), we define:
$$\mathrm{rank}_{(\cdot)}(P) := \min\{t \in \mathbb{Z}_{\geq 0} : \mathrm{cl}^{(t)}_{(\cdot)}(P) \subseteq \mathrm{cl}_{\overline{I}}(P)\} \in \mathbb{Z}_{\geq 0} \,\dot{\cup}\, \{\infty\} \tag{6.11}$$
as the $(\cdot)$ **rank of** P (e.g. Chvátal-Gomory rank of P).

3. Does $\lim_{i \to \infty} \mathrm{cl}^{(i)}_{(\cdot)}(P) = \mathrm{cl}_I(P)$ hold? We define the Hausdorff convergence that is used here in Definition 308.

4. Unter what conditions on, for example, $c \in (\mathbb{Q}^m \times \mathbb{Q}^n)^T$ and P does
$$\exists t \in \mathbb{Z}_{\geq 0} : \max\left\{cx : x \in \mathrm{cl}^{(t)}_{(\cdot)}(P)\right\} = \max\{cx : x \in \mathrm{cl}_I(P)\} \tag{6.12}$$
hold?

6.2.1. Chvátal-Gomory closure

If $P \subseteq \mathbb{R}^m$ is a rational polyhedron, where $m \in \mathbb{Z}_{\geq 0}$ (i.e. we consider the pure integral case), there exists a positive answer to question 2: in [CGST86] (also cf. [Sch86, Theorem 23.4; p. 345]), the following theorem is shown:

Theorem 306. *Let $P \subseteq \mathbb{R}^m$ $(m \in \mathbb{Z}_{\geq 0})$ be a rational polyhedron. Then there exists a $t \in \mathbb{Z}_{\geq 0}$ such that $\mathrm{cl}^{(t)}_{CG}(P) = \mathrm{cl}_I(P)$.*

For question 1: already for $P \subseteq \mathbb{R}^2$, one can find examples such that the Chvátal-Gomory rank of the rational polytope P is arbitrarily large:

Lemma 307. *(cf. [Sch86, section 23.3]) For $k \in \mathbb{Z}_{\geq 0}$, define*

$$P^{307,k} := \operatorname{conv}\left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{k}{2} \\ \frac{1}{2} \end{pmatrix}\right\} \subseteq \mathbb{R}^2.$$

Then for all $k \in \mathbb{Z}_{\geq 1}$, we have

$$\operatorname{cl}_{CG}\left(P^{307,k}\right) = P^{307,k-1}.$$

Since for all $k \in \mathbb{Z}_{\geq 0}$, we have $\operatorname{cl}_I\left(P^{307,k}\right) = P^{307,0}$, we thus conclude

$$\forall k \in \mathbb{Z}_{\geq 0} : \operatorname{rank}_{CG}\left(P^{307,k}\right) = k.$$

Proof.

For $k = 1$: Clearly

$$P^{307,1} \subseteq P^{\leq}\left(\begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \end{pmatrix}\right).$$

Thus,

$$\operatorname{cl}_{CG}\left(P^{307,1}\right) \subseteq P^{307,1} \cap P^{\leq}\left(\begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} \lfloor \frac{1}{2} \rfloor \end{pmatrix}\right) = P^{307,0}.$$

On the other hand, we have

$$P^{307,0} = \operatorname{cl}_I\left(P^{307,1}\right) \subseteq \operatorname{cl}_{CG}\left(P^{307,1}\right).$$

So, we conclude $\operatorname{cl}_{CG}\left(P^{307,1}\right) = P^{307,0}$.

For $k \geq 2$: Obviously,

$$\begin{pmatrix} 1 & k-1 \end{pmatrix}(\,\cdot\,) \leq \underbrace{\left\lfloor \frac{k}{2} + \frac{k-1}{2} \right\rfloor}_{=\lfloor k - \frac{1}{2} \rfloor = k-1}$$

and

$$\begin{pmatrix} 1 & -(k-1) \end{pmatrix}(\,\cdot\,) \leq \underbrace{\left\lfloor \frac{k}{2} - \frac{k-1}{2} \right\rfloor}_{=\lfloor \frac{1}{2} \rfloor = 0}$$

are valid Chvátal-Gomory cuts for $P^{307,k}$. So, we have

$$\operatorname{cl}_{CG}\left(P^{307,k}\right) \subseteq P^{307,k} \cap P^{\leq}\left(\begin{pmatrix} 1 & k-1 \\ 1 & -(k-1) \end{pmatrix}, \begin{pmatrix} k-1 \\ 0 \end{pmatrix}\right) = P^{307,k-1}.$$

For $\operatorname{cl}_{CG}\left(P^{307,k}\right) \supseteq P^{307,k-1}$, let $c(\,\cdot\,) \leq c_0$ ($c \in \left(\mathbb{Z}^2\right)^T$ and $c_0 \in \mathbb{R}$, where w.l.o.g. $c_0 \notin \mathbb{Z}$) be a valid inequality for $P^{307,t}$ such that $c(\,\cdot\,) \leq \lfloor c_0 \rfloor$ is not valid for $P^{307,t}$. This clearly implies $c_1 \geq 1$. Since $\begin{pmatrix} \frac{k}{2} \\ \frac{1}{2} \end{pmatrix} \in P^{307,k}$, we obtain

$$c_1 \frac{k}{2} + c_2 \frac{1}{2} \leq c_0. \tag{6.13}$$

Clearly, $c_1 \frac{k}{2} + c_2 \frac{1}{2} \in \frac{1}{2}\mathbb{Z}$. If $c_1 \frac{k}{2} + c_2 \frac{1}{2} \in \mathbb{Z}$, we thus obtain using (6.13) and $c_1 \geq 1$:

$$c\begin{pmatrix} \frac{k-1}{2} \\ \frac{1}{2} \end{pmatrix} = c_1 \frac{k-1}{2} + c_2 \frac{1}{2} = \underbrace{c_1 \frac{k}{2} + c_2 \frac{1}{2}}_{\leq \lfloor c_0 \rfloor} - c_1 \frac{1}{2} \leq \lfloor c_0 \rfloor - c_1 \frac{1}{2} < \lfloor c_0 \rfloor.$$

On the other hand, if $c_1 \frac{k}{2} + c_2 \frac{1}{2} \in \mathbb{Z} + \frac{1}{2}$, we conclude using (6.13) and $c_1 \geq 1$:

$$c\begin{pmatrix} \frac{k-1}{2} \\ \frac{1}{2} \end{pmatrix} = c_1 \frac{k}{2} + c_2 \frac{1}{2} - \frac{1}{2} - (c_1 - 1)\frac{1}{2} = \underbrace{\left\lfloor c_1 \frac{k}{2} + c_2 \frac{1}{2} - \frac{1}{2} \right\rfloor}_{\leq \lfloor c_0 \rfloor} \underbrace{- (c_1 - 1)\frac{1}{2}}_{\leq 0} \leq \lfloor c_0 \rfloor.$$

This shows

$$c\begin{pmatrix} \frac{k-1}{2} \\ \frac{1}{2} \end{pmatrix} \leq \lfloor c_0 \rfloor.$$

□

6.2.2. Split closure

We start with a theorem, which we show in section 9.2.5:

Theorem 430. *For $\epsilon \in \mathbb{R}_{>0}$, let*

$$\begin{aligned} P^{430,\epsilon} &:= \operatorname{conv}\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \epsilon \end{pmatrix} \right\} \\ &= P^{\leq}\left(\left(\begin{array}{cc|c} -1 & 0 & \frac{2}{3\epsilon} \\ 0 & -1 & \frac{2}{3\epsilon} \\ 1 & 1 & \frac{2}{3\epsilon} \\ 0 & 0 & -1 \end{array} \right), \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix} \right) \\ &=: P^{\leq}\left(\left(\begin{array}{cc} A & G^{\epsilon} \end{array} \right), b \right) \\ &\subseteq \mathbb{R}^2 \times \mathbb{R}^1. \end{aligned}$$

Then $\operatorname{cl}_{split}\left(P^{430,\epsilon}\right) = P^{430,\frac{\epsilon}{2}}$. In particular, for every $t \in \mathbb{Z}_{\geq 0}$, we have

$$\operatorname{cl}_{split}^{(t)}\left(P^{430,\epsilon}\right) = P^{430,\frac{\epsilon}{2^t}}.$$

So, there exists a rational polytope $P \subsetneq \mathbb{R}^2 \times \mathbb{R}^1$ such that for all $t \in \mathbb{Z}_{\geq 0}$, we have: $\operatorname{cl}_{split}^{(t)}(P) \supsetneq \operatorname{cl}_I(P)$. In Theorem 306, we saw that in the pure integer case, the iterated Chátal-Gomory closure of a rational polyhedron P is equal to $\operatorname{cl}_I(P)$ after some iterations. Theorem 430 shows that there is no analogue theorem for the iterated split closure in the mixed-integer setting. More concisely: question 2 (and thus trivially also question 1) from the list at the beginning of section 6.2 can be answered with "no" for the split closure in the mixed-integer case.

Nevertheless, a weaker property is satisfied: the iterated split closure of a rational polyedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) at least *converges* to the mixed-integer hull $\operatorname{cl}_I(P)$. To be able to formalize this property, we introduce the concept of Hausdorff covergence of closed sets. This concept is defined in [DPW12, section 2] the following way (also cf. [OM01] and [SW79]):

Definition 308. *Let $\tilde{P}, \left\{P^i\right\}_{i\in\mathbb{N}} \subseteq \mathbb{R}^d$ ($d \in \mathbb{Z}_{\geq 0}$) be closed sets such that $\tilde{P} \subseteq P_{i+1} \subseteq P_i$ for every $i \in \mathbb{N}$ (i.e. $\left\{P^i\right\}_{i\in\mathbb{N}}$ is antitone). We say that $\left\{P^i\right\}_{i\in\mathbb{N}}$ **(Hausdorff) converges to** $\tilde{P}$ ($\lim_{i\to\infty} P^i = \tilde{P}$) if for every $\epsilon > 0$, there exists a $k \in \mathbb{N}$ such that $P^k \subseteq \tilde{P} + \overline{B}_\epsilon$.*

Remark 309. *(Cf. [DPW12, section 2]) If $\left\{P^i\right\}_{i\in\mathbb{N}}$ converges, we have $\lim_{i\to\infty} P^i = \bigcap_{i\in\mathbb{N}} P^i$. This is a consequence of [SW79, Proposition 2 and Theorem 2]). For this to hold, of course, the antitonicity condition for $\left\{P^i\right\}_{i\in\mathbb{N}}$ from Definition 308 has to be satisfied.*

The following theorem is shown in [OM01] for P being polytope and in [DPW12, Theorem 2] for the case that P is a rational polyhedron:

Theorem 310. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be a polytope or rational polyhedron. Then*

$$\lim_{i\to\infty} \operatorname{cl}_{split}^{(i)}(P) = \operatorname{cl}_I(P).$$

Remark 311. *It is a natural question to ask whether Theorem 310 also holds for arbitrary polyhedra. This is not the case. For this, consider that we have shown in Theorem 301 that $\operatorname{cl}_{split}\left(P^{114}\right) = P^{114}$. On the other hand, by Definition 114, we have $\left(P^{114}\right)_I = \emptyset$.*

We now consider question 4 for the split closure, i.e. what conditions on the rational polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) and $c \in (\mathbb{Q}^m \times \mathbb{Q}^n)^T$ need to be satisfied so that we have (cf. (6.12))

$$\exists t \in \mathbb{Z}_{\geq 0} : \max\left\{cx : x \in \operatorname{cl}_{split}^{(t)}(P)\right\} = \max\left\{cx : x \in P_I\right\}.$$

The following theorem, which is shown in [Jör07, Theorem 4], provides a sufficient condition for this:

Theorem 312. *Let $\emptyset \neq P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be a rational polyhedron and let $c \in (\mathbb{Q}^m \times \mathbb{Q}^n)^T$ be a vector such that* $\max\{cx : x \in P\} < \infty$. *Set $\gamma^* := \max\{cx : x \in P_I\}$ and $M := \mathrm{cl}_I(P) \cap P^{=}(c, \gamma^*)$. If*

$$(\mathrm{relint}\,(\mathrm{proj}_{\mathbb{R}^m} M)) \cap \mathbb{Z}^m \neq \emptyset, \tag{6.14}$$

then there exists a $t \in \mathbb{Z}_{\geq 0}$ such that $\max\left\{cx : x \in \mathrm{cl}^{(t)}_{split}(P)\right\} = \gamma^*$.

We now consider what Theorem 312 tells us for the situation occuring in Theorem 430. Let $\epsilon \in \mathbb{Q}_{>0}$, $c \in \left(\mathbb{Q}^2 \times \mathbb{Q}\right)^T$ and consider $P^{430,\epsilon}$ as the P in Theorem 312. Then:

- If $c \notin \mathrm{lin}\left\{\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}\right\}$, it is easy to check that (6.14) holds for M. Thus, by Theorem 312, we have:

$$\exists t \in \mathbb{Z}_{\geq 0} : \max\left\{cx : x \in \mathrm{cl}^{(t)}_{split}\left(P^{430,\epsilon}\right)\right\} = \max\left\{cx : x \in \left(P^{430,\epsilon}\right)_I\right\}.$$

- On the other hand, if $c \in \mathrm{lin}\left\{\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}\right\}$, we have

$$M = \mathrm{conv}\left\{\begin{pmatrix} e^{2,0} \\ 0 \end{pmatrix}, \begin{pmatrix} e^{2,1} \\ 0 \end{pmatrix}, \begin{pmatrix} e^{2,2} \\ 0 \end{pmatrix}\right\}.$$

  Clearly, (6.14) does not hold for M. Indeed, if $c \in \left(\mathrm{cone}\left\{\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}\right\}\right) \setminus \left\{\begin{pmatrix} 0 & 0 & 0 \end{pmatrix}\right\}$, we have by Theorem 430:

$$\forall t \in \mathbb{Z}_{\geq 0} : \max\left\{cx : x \in \mathrm{cl}^{(t)}_{split}\left(P^{430,\epsilon}\right)\right\} = \frac{\epsilon}{2^t} > 0 = \max\left\{cx : x \in \left(P^{430,\epsilon}\right)_I\right\}.$$

  On the other hand, if $c \in \mathrm{cone}\left\{\begin{pmatrix} 0 & 0 & -1 \end{pmatrix}\right\}$, we have

$$\max\left\{cx : x \in \mathrm{cl}^{(0)}_{split}\left(P^{430,\epsilon}\right)\right\} = \max\left\{cx : x \in \left(P^{430,\epsilon}\right)_I\right\},$$

  but (6.14) does not hold. Thus, the condition of Theorem 312 is only sufficient, but not necessary.

6.2.3. Integral lattice-free closure

In [DPW12], it is shown (on the basis of Theorem 310 and [ALW10]):

Theorem 313. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be rational polyhedron. Then there exists a $k \in \mathbb{Z}_{\geq 0}$ such that $\mathrm{cl}^{(k)}_{ILF}(P) = \mathrm{cl}_I(P)$.*

So question 2 can be answered positively with respect to the integral lattice-free closure. We now recall the stronger question 1, i.e. whether for every rational polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$, we even have

$$\mathrm{cl}_{ILF}(P) = \mathrm{cl}_I(P).$$

We have seen in Lemma 275 and Theorem 276 that this is in general not the case.

6.2.4. k-disjunctive closure

For k-disjunctive cuts, we can answer question 1 positively as long as we choose k sufficiently large. The following Theorem is shown in [Jör07, Lemma 2 and Theorem 2]. Another possible proof is by setting $k := m$ in Theorem 288 and considering that for a rational polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$, we have $\mathrm{cl}_{L_{m,\mathbb{Q}}}(P) = \mathrm{cl}_I(P)$. This is similar to how we derive Theorem 316 from Theorem 304 later on in this text.

Theorem 314. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be a rational polyhedron and let $c(\cdot) \leq c_0$ be a valid inequality for P_I ($c \in (\mathbb{Q}^m \times \mathbb{Q}^n)^T$ and $c_0 \in \mathbb{Q}$). Then $c(\cdot) \leq c_0$ is a 2^m-disjunctive cut for P. Since every facet-defining inequality for $\mathrm{cl}_I(P)$ can be assumed to have rational coefficients, we have*

$$\mathrm{cl}_{2^m D}(P) = \mathrm{cl}_I(P).$$

Since the 2^m in Theorem 314 grows exponentially in m, one is interested in answering question 4 positively for a smaller k than the 2^m from Theorem 314, i.e. one wants to find sufficient conditions on $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$,

$c \in (\mathbb{Q}^m \times \mathbb{Q}^n)^T$ and $k \in \mathbb{Z}_{\geq 2}$ such that we have (recall (6.12)):

$$\exists t \in \mathbb{Z}_{\geq 0} : \max \left\{ cx : x \in \mathrm{cl}_{kD}^{(t)}(P) \right\} = \max \{cx : x \in P_I\}.$$

The following theorem, which is shown in [Jör07, Theorem 5], delivers an analogue of Theorem 312 for k-disjunctive cuts:

Theorem 315. *Let $\emptyset \neq P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ be a rational polyhedron and let $c \in (\mathbb{Q}^m \times \mathbb{Q}^n)^T$ be such that $\max\{cx : x \in P\} < \infty$. Set $\gamma^* := \max\{cx : x \in P_I\}$ and $M := \mathrm{cl}_I(P) \cap P^=(c, \gamma^*)$. Let F denote the set of faces of M that satisfy*

$$f \in F \Rightarrow (\mathrm{relint}(\mathrm{proj}_{\mathbb{R}^m}(f))) \cap \mathbb{Z}^m = \emptyset$$

and let $k \in \mathbb{Z}_{\geq 2}$. If

- *there exist $A^M \in \mathbb{Q}^{k \times m}$ and $b^M \in \mathbb{Q}$ such that $P^{\leq}(A^M, b^M)$ is lattice-free and*

$$x \in \mathrm{relint}(\mathrm{proj}_{\mathbb{R}^m}(M)) \Rightarrow x \in P^{<}(A^M, b^M)$$

  and

- *for every $f \in F$, there exist $A^f \in \mathbb{Q}^{k \times m}$ and $b^f \in \mathbb{Q}$ such that $P^{\leq}(A^f, b^f)$ is lattice-free and*

$$x \in \mathrm{relint}(\mathrm{proj}_{\mathbb{R}^m}(f)) \Rightarrow x \in P^{<}(A^f, b^f),$$

then there exists a $t \in \mathbb{Z}_{\geq 0}$ such that $\max\left\{cx : x \in \mathrm{cl}_{kD}^{(t)}(P)\right\} = \gamma^*$.

Let us revisit the example that we analyzed at the end of section 6.2.2 with respect to Theorem 312. Here, the remaining situation was $c \in \left(\mathrm{lin}\{\begin{pmatrix} 0 & 0 & -1 \end{pmatrix}\}\right) \cap (\mathbb{Q}^2 \times \mathbb{Q})^T$. Indeed, for

$$M = \mathrm{conv}\left\{ \begin{pmatrix} e^{2,0} \\ 0 \end{pmatrix}, \begin{pmatrix} e^{2,1} \\ 0 \end{pmatrix}, \begin{pmatrix} e^{2,2} \\ 0 \end{pmatrix} \right\},$$

all faces f of M satisfy $(\mathrm{relint}(\mathrm{proj}_{\mathbb{R}^m}(f))) \cap \mathbb{Z}^m \neq \emptyset$. Thus, $F = \emptyset$. For M, there exist

$$A^M := \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{pmatrix}, \qquad b^M := \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

such that $x \in \mathrm{relint}(\mathrm{proj}_{\mathbb{R}^m}(M)) \Rightarrow x \in P^{<}(A^M, b^M)$. Thus, we have by Theorem 315:

$$\exists t \in \mathbb{Z}_{\geq 0} : \max\left\{cx : x \in \mathrm{cl}_{3D}^{(t)}\left(P^{430,\epsilon}\right)\right\} = \max\left\{cx : x \in \left(P^{430,\epsilon}\right)_I\right\}.$$

We remark that one can easily show that in this case, $t = 1$ satisfies this property by just considering the 3-disjunctive cut $y_1 \leq 0$ for $\binom{x}{y} \in P^{430,\epsilon}$ with respect to the (integral) lattice-free body

$$P^{\leq}\left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \right).$$

6.2.5. k row closure

A similar statement to the situation for k-disjunctive cuts that we considered in section 6.2.4, where we saw in Theorem 315 that as long as "k is large enough", we have $\mathrm{cl}_{kD}(P) = \mathrm{cl}_I(P)$, also holds for k row cuts:

Theorem 316. *Let*

$$P := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R}^n_{\geq 0} : Ax + Gy = b \right\},$$

where $A \in \mathbb{Q}^{r \times m}$, $G \in \mathbb{Q}^{r \times n}$ and $b \in \mathbb{Q}^r$ (i.e. P is a rational polyhedron), where $r, m, n \in \mathbb{Z}_{\geq 0}$. Let $c(\cdot) \geq c_0$ be a valid linear inequality for P_I, where $c \in (\mathbb{Q}^m \times \mathbb{Q}^n)^T$ and $c_0 \in \mathbb{Q}$. If $P \neq \emptyset$, then $c(\cdot) \geq c_0$ is a translate *of a $2^m - 1$ row cut with respect to A, G and b. In particular, we have*

$$\mathrm{cl}_{(2^m-1)R}(A, G, b) = \mathrm{cl}_I(P).$$

Proof. For $P = \emptyset$, the statement is trivial. If $P \neq \emptyset$, the statement is an immediate consequence of Theorem 304 and the considerations that

- if P is a rational polyhedron, so is $\mathrm{cl}_I(P)$ and
- every linear inequality for P_I is an $L_{m,\mathbb{Q}}$ cut for P.

□

6.2.6. t-branch split closure

For t-branch split cuts, we can answer question 1 positively as long as we choose t sufficiently large, since in [DDG+13, Theorem 3.5], it is shown:

Theorem 317. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ be a rational polyhedron. Then every facet-defining inequality for $\mathrm{cl}_I(P)$ is an $h(m)$-branch split cut for P, where $h(\cdot)$ is as in Remark/Definition 248. In particular, we have*

$$\mathrm{cl}_{h(m)BS}(P) = \mathrm{cl}_I(P).$$

6.3. Polyhedricity

There are two natural approaches how one can approach the problem of proving that the closure of some P with respect to a given class of cutting planes is a polyhedron:

- Prove a polyhedricity result that is rather specific to the class of cutting planes that one considers.
- Prove a rather general polyhedricity result and apply it to the concrete class of cutting planes.

As one can imagine, this is rather a rough guide than a precise classification. Nevertheless, we use it to structure the remainder of this section.

6.3.1. Polyhedricity results that are specific to a particular class of cutting planes

6.3.1.1. Chvátal-Gomory closure

For a rational polyhedron $P \subseteq \mathbb{R}^m$, its Chvátal-Gomory closure is a again a rational polyhedron:

Theorem 318. *Let $P \subseteq \mathbb{R}^m$ $(m \in \mathbb{Z}_{\geq 0})$ be a rational polyhedron. Then $\mathrm{cl}_{CG}(P)$ is a again a rational polyhedron.*

This is proved in [Chv73, Corollary 3.3] if P is a rational polytope and in [Sch80, Theorem 1] if P is a rational polyhedron (also cf. [Sch86, Theorem 23.1; p. 340]). In [CCZ10, section 11.6.1], the authors simplify the proof from [Chv73] and generalize it from rational polytopes to arbitrary rational polyhedra.

We remark that in Theorem 399, we show that the Chvátal-Gomory closure of a polyhedron $P \subseteq \mathbb{R}^m$ $(m \in \mathbb{Z}_{\geq 0})$ with rational face normals is a rational polyhedron and in Theorem 405, we prove a similar result for the projected Chvátal-Gomory closure of a polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ (i.e. mixed-integer setting) with rational face normals.

We now consider the situation for $\mathrm{cl}_{CG}(K)$ if $K \subseteq \mathbb{R}^m$ is not a rational polyhedron. For this, we start with a definition (cf. [DV10, section 1]):

Definition 319. *A **rational ellipsoid** is the image of an (Euclidian) unit ball under a rational affine map.*

In [DV10, Theorem 2], it is shown:

Theorem 320. *Let $K \subseteq \mathbb{R}^m$ $(m \in \mathbb{Z}_{\geq 0})$ be a full-dimensional rational ellipsoid. Then $\mathrm{cl}_{CG}(K)$ is a rational polytope that is defined by a finite number of Chvátal-Gomory cuts for K.*

In [DS11], it is shown:

Theorem 321. *Let $K \subseteq \mathbb{R}^m$ $(m \in \mathbb{Z}_{\geq 0})$ be a (not necessarily rational) polytope. Then $\mathrm{cl}_{CG}(K)$ is a rational polytope that is defined by a finite number of Chvátal-Gomory cuts for K.*

We next define (cf. [DDV11b, section 2]):

Definition 322. *Let $K \subseteq \mathbb{R}^d$ ($d \in \mathbb{Z}_{\geq 0}$). We call K* ***strictly convex*** *if for all $k^1, k^2 \in K$ and $\lambda \in (0,1)$, we have $\lambda k^1 + (1-\lambda) k^2 \in \operatorname{relint} K$.*

In [DDV11b, Theorem 2.3], the authors prove the following generalization of Theorem 320:

Theorem 323. *Let $K \subseteq \mathbb{R}^m$ ($m \in \mathbb{Z}_{\geq 0}$) be compact and strictly convex and let $P \subseteq \mathbb{R}^m$ be a rational polyhedron. Then $\operatorname{cl}_{CG}(K \cap P)$ is a rational polytope that is defined by a finite number of Chvátal-Gomory cuts for $K \cap P$.*

In Theorem 323, one can, of course, set $P := \mathbb{R}^m$; in this case, one obtains that the Chvátal-Gomory closure of a strictly convex body is a rational polytope, which we state as corollary and is the statement of [DDV11b, Theorem 2.2]:

Corollary 324. *Let $K \subseteq \mathbb{R}^m$ ($m \in \mathbb{Z}_{\geq 0}$) be compact and strictly convex. Then $\operatorname{cl}_{CG}(K)$ is a rational polytope that is defined by a finite number of Chvátal-Gomory cuts for K.*

In [DDV14, Theorem 1], the statements of Theorem 320, Theorem 321 and Theorem 323 are generalized to the case that $K \subseteq \mathbb{R}^m$ is convex and compact. In [BP14, Theorem 6], one can find a simplified proof of this statement. We write this down as the following theorem:

Theorem 325. *Let $K \subseteq \mathbb{R}^m$ ($m \in \mathbb{Z}_{\geq 0}$) be convex and compact. Then $\operatorname{cl}_{CG}(K)$ is a rational polytope that is defined by a finite number of Chvátal-Gomory cuts for K.*

6.3.1.2. Split closure

For a rational polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$), its split closure is again a rational polyhedron:

Theorem 326. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be a rational polyhedron. Then $\operatorname{cl}_{split}(P)$ is a again a rational polyhedron.*

There exist several proofs of Theorem 326 in the literature. Here, we only consider proofs that are specific to the (1-branch) split closure; for a more general perspective cf. section 6.3.1.3 and section 6.3.2.1:

- The first proof of Theorem 326 can be found in [CKS90].
- A second proof can be found in [ACL05].
- While the previous two proofs are non-constructive, in [Vie07] or the extended version [Vie05], respectively, one can find a constructive proof of Theorem 326.
- In [DGL10, section 4], one can find a proof of Theorem 326 for (rational) polyhedra of the form

  $$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n_{\geq 0} : Ax + Gy = b \right\}$$

  (A, G and b rational) that is based on the framework of MIR cuts, which we introduce in chapter 9. This proof is simplified and generalized in [CCZ10, section 11.6.2] to polyhedra of the form

  $$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n_{\geq 0} : Ax + Gy \leq b \right\}$$

  (A, G and b again rational).

Our proof of Theorem 417 later on in section 9.2.4 constitutes an independent and constructive proof of Theorem 326. In this theoren, we additionally show that if $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ is a polyhedron with rational face normals, then $\operatorname{cl}_{split}(P)$ is a polyhedron.

Similarly to how we structured section 6.3.1.1, we now again consider the situation if $K \subseteq \mathbb{R}^m$ is not a rational polyhedron.

In [DDV11a, Theorem 1.3], it is shown (recall the related result by the same authors concerning the Chvátal-Gomory closure of a compact and strictly convex set that we stated in Theorem 323):

Theorem 327. *Let $K \subseteq \mathbb{R}^m$ ($m \in \mathbb{Z}_{\geq 0}$) be compact and strictly convex and let* aff K *be a rational affine subspace. Then* $\mathrm{cl}_{split}(K)$ *is finitely defined, i.e. there exists a finite set $D \subseteq \mathbb{Z}^m \times \mathbb{Z}$ such that*

$$\mathrm{cl}_{split}(K) = \bigcap_{(\pi,\gamma)\in D} \mathrm{conv}\left(K \cap D(\pi,\gamma)\right).$$

Does Theorem 327 imply that the split closure of a compact and strictly convex body $K \subseteq \mathbb{R}^m$, where aff K is a rational affine subspace, is a polytope? For $m \in \{0,1,2\}$, this is the case, as the following theorem shows:

Theorem 328. *Let $K \subseteq \mathbb{R}^m$ ($m \in \{0,1,2\}$) be compact and strictly convex and let* aff K *be a rational affine subspace. Then* $\mathrm{cl}_{split}(K)$ *is a polytope.*

The statement of Theorem 328 is easy to verify for $m \in \{0,1\}$. The case $m = 2$ of Theorem 328 is shown in [DDV11a, Corollary 3.2]. We now consider the situation for $m = 3$. In this case, it can happen that for $K \subseteq \mathbb{R}^3$ compact and strictly convex, $\mathrm{cl}_{split}(K)$ is not a polyhedron. For this, we consider the following example:

Example 329. *(cf. [DDV11a, Example 3.8]) Let*

$$K := \left\{x \in \mathbb{R}^3 : \left\| x - \frac{1}{2}\cdot 1^3 \right\|_A \leq 1\right\},$$

where

$$A := \frac{64}{33}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{10000} \end{pmatrix},$$

$$\|y\|_A := y^T A y \text{ for } y \in \mathbb{R}^3.$$

Then

$$\left\{\begin{pmatrix} t \\ t \\ \frac{1+25\sqrt{17-64\left(t-\frac{1}{2}\right)^2}}{2} \end{pmatrix} : t \in \left[\frac{1}{2}, \frac{1}{2}+\frac{1}{100}\right]\right\} \subseteq \mathrm{bd}\left(\mathrm{cl}_{split}(K)\right).$$

Thus, clearly, $\mathrm{cl}_{split}(K)$ *is not a polyhedron.*

6.3.1.3. 2-branch split closure

In [DGMR16b, Corollary 3], it is shown (also cf. [DGMR13], an older version of [DGMR16b]):

Theorem 330. *For $m \in \mathbb{Z}_{\geq 0}$, define*

$$\mathcal{C}^* := \{\{S_1, S_2\} : S_1, S_2 \subseteq \mathbb{R}^m \text{ split sets}\}.$$

Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($n \in \mathbb{Z}_{\geq 0}$) be a rational polyhedron and let $\mathcal{C} \subseteq \mathcal{C}^*$. Then there exists a finite set $\hat{\mathcal{C}} \subseteq \mathcal{C}$ such that

$$\bigcap_{\{S_1,S_2\}\in\mathcal{C}} \mathrm{conv}\left(P \backslash \left((S_1 \cup S_2) \times \mathbb{R}^n\right)\right) = \bigcap_{\{S_1,S_2\}\in\hat{\mathcal{C}}} \mathrm{conv}\left(P \backslash \left((S_1 \cup S_2) \times \mathbb{R}^n\right)\right).$$

In particular (using Lemma 128), this implies that $\mathrm{cl}_{2BS}(P)$ *is a again a rational polyhedron.*

We remark that in Theorem 335 and Corollary 336, Theorem 330 is generalized.

6.3.1.4. $L_{k,\mathbb{Q}}$ closure

We recall that in Theorem 264 in section 5.4.5, we showed that the $L_{k,\mathbb{Q}}$ closure of a rational polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) is again a rational polyhedron, which was independently proved in [DGMR17, Theorem 2].

6.3.2. General methods for proofs of polyhedricity

6.3.2.1. Theorems

Definition 331. *(cf. [Ave12]; also cf. [DGMR16a, section 4.1])) Let* $L \subseteq \mathbb{R}^d$ *(*$d \in \mathbb{Z}_{\geq 0}$*) be a d-dimensional rational polyhedron. Then we denote by* $m(L)$ *the minimal* $m \in \mathbb{Z}_{\geq 0}$ *such that*

$$L = \left\{x \in \mathbb{R}^d : b_i - m \leq a^i x \leq b_i \forall i \in [n]\right\},$$

where $n \in \mathbb{Z}_{\geq 1}$, $a^1, \ldots, a^n \in \left(\mathbb{Z}^d\right)^T \setminus \left\{\left(0^d\right)^T\right\}$ *and* $b_1, \ldots, b_n \in \mathbb{Z}$*. If* $L = \mathbb{R}^d$ *or* $(\operatorname{rec} L) \setminus (\operatorname{lineal} L) \neq \emptyset$*, set* $m(L) := +\infty$*.*

Definition 332. *(cf. [DGMR16a, section 4.2]) For* $l \in \mathbb{Z}_{\geq 1}$ *and* $d \in \mathbb{Z}_{\geq 0}$*, set*

$$\mathcal{M}_{l,d} := \left\{\operatorname{int} P \subsetneq \mathbb{R}^d : P \textit{ is a rational polyhedron having } \dim P = d \textit{ and } m(P) \leq l\right\}$$

and for $l \in \mathbb{Z}_{\geq 1}$ *and* $t, d \in \mathbb{Z}_{\geq 0}$*, set*

$$\mathcal{T}^*_{t,l,d} := \left\{T \subsetneq \mathbb{R}^d : T = \bigcup_{j=1}^{t} M_j, \textit{ where } M_j \in \mathcal{M}_{l,d} \textit{ for } j \in [t]\right\}$$

(in [DGMR16a], the notations $\mathcal{M}_l$ *and* $\mathcal{T}^*_{t,l}$ *are used instead of our notations* $\mathcal{M}_{l,d}$ *and* $\mathcal{T}^*_{t,l,d}$*).*

Definition 333. *(cf. [DGMR16a, section 2] Let* $Q \subseteq \mathbb{R}^d$ *(*$d \in \mathbb{Z}_{\geq 0}$*) and let* $\mathcal{V} \subseteq \mathcal{P}\left(\mathbb{R}^d\right)$ *be a family of subsets of* $\mathbb{R}^d$*. Then we say that* $V' \in \mathcal{V}$ ***dominates*** $V \in \mathcal{V}$ ***on*** Q *if* $\operatorname{conv}(Q \setminus V') \subseteq \operatorname{conv}(Q \setminus V)$*. For this, we write* $V' \preceq_Q V$*.*

Clearly, $\preceq_Q$ forms a quasi-order (also called preorder) on $\mathcal{V}$, since it is reflexive and transitive (though not necessarily antisymmetric). Thus, for all Q and $\mathcal{V}$ as in Definition 333, $(\mathcal{V}, \preceq_Q)$ forms a quasi-ordered set (qoset).

Definition 334. *(cf. [DGMR16a, Definition 1]) Given a qoset* $(X, \preceq)$*, we say that* $Y \subseteq X$ *is a* ***dominating subset of*** X *if for all* $x \in X$*, there exists a* $y \in Y$ *such that* $y \succeq x$*. The qoset* $(X, \preceq)$ *is called* ***fairly well-ordered*** *if* X' *has a finite dominating subset for each* $X' \subseteq X$*.*

The following theorem is shown for $t = 1$ in [Ave12] (also cf. [ALW10]) and for general t in [DGMR16a, Theorem 2 and section 5.1]:

Theorem 335. *Let* $l \in \mathbb{Z}_{\geq 1}$*, let* $t, d \in \mathbb{Z}_{\geq 0}$*, let* $P \subseteq \mathbb{R}^d$ *be a rational polyhedron and let* $\mathcal{T} \subseteq \mathcal{T}^*_{t,l,d}$*. Then* $(\mathcal{T}, \preceq_P)$ *is fairly well-ordered, i.e.* $\mathcal{T}$ *has a finite dominating subset* $\mathcal{T}_f$*. By Lemma 128, this implies that*

$$\bigcap_{T \in \mathcal{T}} \operatorname{conv}(P \setminus T) = \bigcap_{T \in \mathcal{T}_f} \operatorname{conv}(P \setminus T)$$

is a rational polyhedron. For mixed-integer programming we are in particular interested in the following case: let $m, n \in \mathbb{Z}_{\geq 0}$*, let* $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ *be a rational polyhedron and let* $\mathcal{T}' \times \mathbb{R}^n := \mathcal{T} \subseteq \mathcal{T}^*_{t,l,m} \times \mathbb{R}^n \subseteq \mathcal{T}^*_{t,l,m+n}$*. Then there exists a finite dominating subset* $\mathcal{T}'_f$ *of* $\mathcal{T}'$*. In particular (using Lemma 128), this implies that*

$$\bigcap_{T \in \mathcal{T}'} \operatorname{conv}(P \setminus (T \times \mathbb{R}^n)) = \bigcap_{T \in \mathcal{T}'_f} \operatorname{conv}(P \setminus (T \times \mathbb{R}^n))$$

is a rational polyhedron.

An immediate consequence of Theorem 335 is that the t-branch split closure of a rational polyhedron is again a rational polyhedron, which is also stated in [DGMR16a, Theorem 1]:

Corollary 336. *Let* $m, n \in \mathbb{Z}_{\geq 0}$*, let* $t \in \mathbb{Z}_{\geq 1}$*, let* $\mathcal{T}^*$ *be a collection of t-branch split sets in* $\mathbb{R}^m$*, let* $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ *be a rational polyhedron and let* $\mathcal{T} \subseteq \mathcal{T}^*$*. Then there exists a finite set* $\hat{\mathcal{T}} \subseteq \mathcal{T}$ *such that*

$$\bigcap_{S \in \mathcal{T}} \operatorname{conv}(P \setminus (S \times \mathbb{R}^n)) = \bigcap_{S \in \hat{\mathcal{T}}} \operatorname{conv}(P \setminus (S \times \mathbb{R}^n)).$$

In particular (using Lemma 128), $\operatorname{cl}_{tBS}(P)$ *is a again a rational polyhedron.*

6.3.2.2. Consequences for the integral lattice-free closure

For $d \in \mathbb{Z}_{\geq 1}$, let (for the notation cf. [AWW11, section 2])

- $\mathscr{P}_{if}^d$ denote the set of of integral lattice-free (with respect to the lattice $\mathbb{Z}^d$) polyhedra in $\mathbb{R}^d$,
- $\mathscr{P}_{ifm}^d$ denote the elements of $\mathscr{P}_{if}^d$ that are maximal in $\mathscr{P}_{if}^d$ with respect to inclusion,
- $\mathscr{C}_{fm}^d$ denote the set of maximal lattice-free (with respect to the lattice $\mathbb{Z}^d$) polyhedra in $\mathbb{R}^d$ and
- $\mathscr{P}_{fmi}^d$ denote the polyhedra from $\mathscr{C}_{fm}^d$ which are integral.

Remark 337. *In [AWW11, section 2], instead the symbols $\mathscr{P}_{if}(1)$, $\mathscr{P}_{ifm}(1)$, $\mathscr{C}_{fm}(1)$ and $\mathscr{P}_{fmi}(1)$ are used for what we denote $\mathscr{P}_{if}^d$, $\mathscr{P}_{ifm}^d$, $\mathscr{C}_{fm}^d$ and $\mathscr{P}_{fmi}^d$.*

Definition 338. *For $d \in \mathbb{Z}_{\geq 1}$, let $P_1, P_2 \in \mathscr{P}_{ifm}^d$. We define*

$$P_1 \equiv P_2 \mod \operatorname{Aff}\left(\mathbb{Z}^d\right) \Leftrightarrow \exists \text{ affine-unimodular map } f : P_2 = f\left(P_1\right).$$

This, of course, defines an equivalence relation on $\mathscr{P}_{ifm}^d$.

We have:

Theorem 339. *Let $d \in \mathbb{Z}_{\geq 1}$. Then there exists an $N(d) \in \mathbb{Z}_{\geq 1}$ and polyhedra $P_1^d, \ldots, P_{N(d)}^d \in \mathscr{P}_{ifm}^d$ such that for every $P \in \mathscr{P}_{ifm}^d$, one has $P \equiv P_j^d \mod \operatorname{Aff}\left(\mathbb{Z}^d\right)$ for some $j \in \{1, \ldots, N(d)\}$.*

Theorem 339 is shown in [AWW11, Theorem 1.1] and independently in [NZ11, Corollary 1.1]. This, of course, implies $\mathscr{P}_{ifm}^d \subseteq \operatorname{int} \mathcal{M}_{l,d}$ (for the definition of $\mathcal{M}_{l,d}$, cf. Definition 332) for some $l \geq 1$, from which we conclude using Theorem 335 (also cf. [DPW12]):

Theorem 340. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be a rational polyhedron. Then $\operatorname{cl}_{ILF}(P)$ is again a rational polyhedron.*

For the end of the section, we consider for specific values of d:

1. what the relationship between $\mathscr{P}_{ifm}^d$ and $\mathscr{P}_{fmi}^d$ is and
2. how polyhedra $P_1^d, \ldots, P_{N(d)}^d$ that satify the conditions of Theorem 339 look like.

For 1: It is easy to check that for all $d \in \mathbb{Z}_{\geq 1}$, the inclusion

$$\mathscr{P}_{fmi}^d \subseteq \mathscr{P}_{ifm}^d \tag{6.15}$$

holds (cf. [AWW11, section 2]). One might ask whether this inclusion can be strict. For $d \in \{1, 2\}$, one can show that equality holds (cf. [AWW11, section 2], [NZ11, section 1.3]). The proof that equality also holds in (6.15) for $d = 3$ was recently done in [AKW15]. On the other hand, in [NZ11, Theorem 1.2], it is shown that for $d \geq 4$, the inclusion (6.15) is strict. So, we obtain:

Theorem 341. *Let $d \in \mathbb{Z}_{\geq 1}$. Then $\mathscr{P}_{fmi}^d = \mathscr{P}_{ifm}^d$ if $1 \leq d \leq 3$ and $\mathscr{P}_{fmi}^d \subsetneq \mathscr{P}_{ifm}^d$ if $d \geq 4$.*

For 2: We would like to know how the polyhedra $P_1^d, \ldots, P_{N(d)}^d$ in Theorem 339 concretely look like. W.l.o.g. we additionally want to assume

$$P_i^d \not\equiv P_j^d \mod \operatorname{Aff}\left(\mathbb{Z}^d\right) \text{ for } i, j \in N(d) \text{ having } i \neq j.$$

We first claim that we can restrict ourselves to the case that the P_i^d are polytopes. This is the statement of Theorem 342, which is shown in [AWW11, Proposition 3.1]:

Theorem 342. *([AWW11, Proposition 3.1]) Let $d \in \mathbb{Z}_{\geq 1}$ and let $P \in \mathscr{P}_{ifm}^d$. Then there exists a $k \in \{1, \ldots, d\}$ and a polytope $P' \in \mathscr{P}_{ifm}^k$ such that $P \equiv P' \times \mathbb{R}^{d-k} \mod \operatorname{Aff}\left(\mathbb{Z}^d\right)$.*

The statement of the following theorem

- is well-known for $\mathscr{P}_{ifm}^1$, $\mathscr{P}_{fmi}^1$, $\mathscr{P}_{ifm}^2$ and $\mathscr{P}_{fmi}^2$ (cf. [AWW11, section 2] and [NZ11, section 1.3]),

- is shown in [AWW11, Theorem 2.2] for $\mathscr{P}^3_{fmi}$ and
- is shown in [AKW15] for $\mathscr{P}^3_{ifm}$.

Theorem 343. *We have:*

- *Let* $P \in \mathscr{P}^1_{ifm} = \mathscr{P}^1_{fmi}$ *be a polytope. Then* $P \equiv [0,1] \mod \operatorname{Aff}\left(\mathbb{Z}^1\right)$.
- *Let* $P \in \mathscr{P}^2_{ifm} = \mathscr{P}^2_{fmi}$ *be a polytope. Then* $P \equiv \operatorname{conv}\left\{\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}2\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\2\end{smallmatrix}\right)\right\} \mod \operatorname{Aff}\left(\mathbb{Z}^2\right)$.
- *Let* $P \in \mathscr{P}^3_{ifm} = \mathscr{P}^3_{fmi}$ *be a polytope. Then* $P \equiv P_i \mod \operatorname{Aff}\left(\mathbb{Z}^3\right)$ *for some* $i \in \{1, \ldots, 12\}$*, where* $P_1, \ldots, P_{12} \in \mathscr{P}^3_{ifm} = \mathscr{P}^3_{fmi}$ *are the 12 polytopes that are listed in [AWW11, Theorem 2.2].*

For $d \geq 4$, we are not aware of a similar classification of the polytopes in either $\mathscr{P}^d_{fmi}$ or $\mathscr{P}^d_{ifm}$.

Part III.

(Mixed-)Integral polyhedra, $L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}/L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cuts and $L_{1,\mathbb{Q}}$ cuts

7. (Mixed-)Integral polyhedra

This chapter is centrally about integral polyhedra with a small digression into mixed-integral polyhedra in Theorem 347.

In section 7.1, we consider how we can characterize integral polyhedra (Theorem 345) and mixed-integral polyhedra (Theorem 347) in terms of optimization problems. We remark that such a characterization of mixed-integrality has to our knowledge not appeared in the literature before.

In section 7.2, we analyze a class of systems of linear inequalities $Ax \leq b$, where A is rational, with the property that if b is integral, the polyhedron $P^{\leq}(A, b) \subseteq \mathbb{R}^m$ is integral. A well-known class of systems of linear inequalities with this property are **TDI systems** (cf. [Sch86, section 22.1]), which we redefine in Definition 348 in a slightly more general way than what is common in the literature (we do not demand rationality of b).

But TDI systems are just one class of systems of linear inequalities that satisfy this property. Another class of systems, which we introduce, that satisfies this property are what we call **TD**$\mathbb{Z} + \{0, 1\}$ **systems** (cf. Definition 349). These can be considered as a formulation of the fact that a polyhedron $P \subseteq \mathbb{R}^m$ with rational face normals is integral if and only

$$\forall c \in (\mathbb{Z}^m)^T : \max\{cx : x \in P\} \in \mathbb{R} \Rightarrow \max\{cx : x \in P\} \in \mathbb{Z}$$

(cf. Theorem 345) in terms of dual programs.

These two classes (TDI systems and TD$\mathbb{Z} + \{0, 1\}$ systems) are our basic building blocks. Since one can show (cf. section 7.6) that there are both polyhedra for which, under weak assumptions, there exists a TDI system that is smaller (with respect to number of inequalities) than every TD$\mathbb{Z} + \{0, 1\}$ system for it and vice versa, we develop **TD**$\mathbb{Z} + I$ **systems** as generalization of both TDI systems and TD$\mathbb{Z} + \{0, 1\}$ systems (cf. Definition 348). On the other hand, we define **TD**$(I \cap \mathbb{Z}) + \{0, 1\}$ **systems** (cf. Definition 349). These form a class of systems which are *both* TDI systems and TD$\mathbb{Z} + \{0, 1\}$ systems.

Now something interesting happens: as we state in Theorem 354, for integral polyhedra, *any* description of them via rational inequalities already forms a TD$\mathbb{Z} + \{0, 1\}$ system. So, one can ask: if we know that the system is additionally TDI: does an even stronger property than TDI hold here? The answer is (cf. Theorem 355): the TD$(I \cap \mathbb{Z}) + \{0, 1\}$ property exactly serves this role.

In section 7.3 and section 7.4, we consider the following problem: it is known that $Ax \leq b$ is TDI if and only if for each minimal face F of $P^{\leq}(A, b)$, the rows of A that are active in F form a Hilbert basis (cf. Theorem 367). So, one wants to find other types of generating systems than Hilbert bases that work for the other types of systems that we just outlined. The central idea for this is to consider not only the left-hand side matrix A, but also the right-hand side vector b to define suitable generalizations, but on the other hand restrict oneself to some very specific class of cones that form what we named **LP face cones** (cf. Definition 356). These types of generating systems are defined in Definition 358 and Definition 359. In this new framework, Hilbert bases are reinterpreted as **"**icone **systems"**. In Lemma 361, we write down how Hilbert bases relate to these icone systems.

The central topic of section 7.4 is to show Theorem 368, which relates the systems of dual integrality from section 7.2 with the generating systems from section 7.3.

In section 7.5, Theorem 369, we show that for any polyhedron $P \subseteq \mathbb{R}^m$ with rational face normals ($m \in \mathbb{Z}_{\geq 0}$), there exists a TDI system $Ax \leq b$ describing it. If P is a rational polyhedron, there even exists a TD$(I \cap \mathbb{Z}) + \{0, 1\}$ system for it. In both cases, b can be assumed to be integral if P is an integral polyhedron.

Finally, in section 7.6, we compare sizes of the different types of systems of dual integrality and generating systems and find examples where the respective types are "small" versus "large" (with respect to the number of inequalities or number of elements).

7.1. Definitions and (mixed-)integral polyhedra

We start by defining "icone" as a variant of "cone" (cf. Definition 41) where we only allow integral coefficients in the conic combinations:

Definition 344. *Let $S \subseteq V$, where V is a vector space over $\mathbb{R}$. We define*

$$\operatorname{icone} S := \bigcup_{k \in \mathbb{Z}_{\geq 0}} \bigcup_{\substack{\lambda \in \mathbb{Z}^k_{\geq 0}, \\ s^1, \ldots, s^k \in S}} \left\{ \sum_{i=1}^{k} \lambda_i s^i \right\}.$$

The following theorem generalizes [Sch86, Corollary 22.1a; p. 310] from rational polyhedra to polyhedra with rational face normals. Its proof is an immediate consequence of the proofs of [Sch86, Theorem 22.1; p. 310] and [Sch86, Corollary 22.1a; p. 310] if one considers the argument that if we have an inequality description $Ax \leq b$ ($A \in \mathbb{Z}^{l \times m}$, $b \in \mathbb{R}^l$, where $l, m \in \mathbb{Z}_{\geq 0}$) of an integral polyhedron with no redundant inequalities, we have $b \in \mathbb{Z}^l$.

Theorem 345. *Let $P \subseteq \mathbb{R}^m$ ($m \in \mathbb{Z}_{\geq 0}$) be a polyhedron with rational face normals. Then P is integral (cf. Definition 76) if and only if*

$$\forall c \in (\mathbb{Z}^m)^T : \max \{cx : x \in P\} \in \mathbb{R} \Rightarrow \max \{cx : x \in P\} \in \mathbb{Z}.$$

We now consider how an analogue of Theorem 345 for the mixed-integer case looks like. This is the statement of Theorem 347, but before, we state a simple observation in the following proposition:

Proposition 346. *Let $P = P^{\leq}(A, b) \subseteq \mathbb{R}^d$, where $A \in \mathbb{R}^{l \times d}$ and $b \in \mathbb{R}^l$ ($l, d \in \mathbb{Z}_{\geq 0}$). Let $\emptyset \neq F$ be a face of P, let $L_F \subseteq [l]$ denote the rows of $A(\,\cdot\,) \leq b$ that are active in F and let $c \in \operatorname{rowcone} A_{L_F,*}$. Then*

$$\max \{cx : x \in P\} = cx^*,$$

where $x^* \in F$ is an arbitrary point in F.

Proof. Since $c \in \operatorname{rowcone} A_{L_F,*}$, there exists a $y^* \in \left(\mathbb{R}^l_{\geq 0}\right)^T$ having $\operatorname{supp} y^* \subseteq L_F$ and $y^* A = c$. Thus,

$$Ax^* \leq b, \qquad cx^* = y^* A x^* = y^* b, \qquad z^* A = c.$$

So, x^* and y^* form a primal-dual pair. ☐

Theorem 347. *Let $\emptyset \neq P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be a rational polyhedron. Then P is mixed-integral (cf. Definition 76) if and only if for all $c^1, c^2 \in (\mathbb{R}^m \times \mathbb{R}^n)^T$ having*

$$\max \left\{c^1 x : x \in P\right\} =: c^1_0 \in \mathbb{R} \wedge \max \left\{c^2 x : x \in P\right\} =: c^2_0 \in \mathbb{R} \wedge c^2 - c^1 \in (\mathbb{Z}^m \times 0^n)^T,$$

we have

$$\max \left\{ \left(f c^1 + (1-f) c^2\right) x : x \in P \right\} \leq f c^1_0 + (1-f) c^2_0 - f(1-f), \tag{7.1}$$

where $f := \operatorname{frac}\left(c^2_0 - c^1_0\right)$.

Proof. Let $\emptyset \neq P = P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)$, where $A \in \mathbb{Q}^{l \times m}$, $G \in \mathbb{Q}^{l \times n}$ and $b \in \mathbb{Q}^l$ ($l \in \mathbb{Z}_{\geq 0}$).

For "if": Assume that P is not mixed-integral. Let F be a minimal face of P such that $F \neq \operatorname{cl}_I(F)$. Let $L_F \subseteq [l]$ denote the rows of $Ax + Gy \leq b$ that are active in F. Since F is a chosen minimally, we have $F = \operatorname{aff} F$ and $(\operatorname{aff} F)_I = \emptyset$. Thus,

$$F = P^{=}\left(\begin{pmatrix} A & G \end{pmatrix}_{L_F,*}, b_{L_F,*}\right),$$
$$\nexists x \in \mathbb{Z}^m \times \mathbb{R}^n : \begin{pmatrix} A & G \end{pmatrix}_{L_F,*} x = b_{L_F}. \tag{7.2}$$

By Theorem 92, (7.2) is equivalent to

$$\exists y \in \left(\mathbb{R}^{L_F}\right)^T : y \begin{pmatrix} A & G \end{pmatrix}_{L_F,*} \in (\mathbb{Z}^m \times 0^n)^T \wedge y b_{L_F,*} \notin \mathbb{Z}. \tag{7.3}$$

Let (Definition 410 gives a hint about what we want to achieve)

$$\begin{aligned}
&\mu \in \left(\mathbb{R}^l\right)^T : \\
&\mu_i = \begin{cases} y_i & \text{if } i \in L_F, \\ 0 & \text{otherwise} \end{cases} \quad \forall i \in [l], \\
&c^1 := \mu^- \begin{pmatrix} A & G \end{pmatrix}, \\
&c^2 := \mu^+ \begin{pmatrix} A & G \end{pmatrix}, \\
&x^* \in F.
\end{aligned}$$

By Proposition 346, we get using

- L_F is the set of rows of $Ax + Gy \leq b$ that are active in F,
- $\operatorname{supp} \mu^-, \operatorname{supp} \mu^+ \subseteq L_F$,
- $c^1 \in \operatorname{rowcone} \begin{pmatrix} A & G \end{pmatrix}_{\operatorname{supp} \mu^-,*} \subseteq \operatorname{rowcone} \begin{pmatrix} A & G \end{pmatrix}_{L_F,*}$ and
- $c^2 \in \operatorname{rowcone} \begin{pmatrix} A & G \end{pmatrix}_{\operatorname{supp} \mu^+,*} \subseteq \operatorname{rowcone} \begin{pmatrix} A & G \end{pmatrix}_{L_F,*}$

that

$$\begin{aligned}
\max \left\{c^1 x : Ax \leq b\right\} = c^1 x^* =: c_0^1, \\
\max \left\{c^2 x : Ax \leq b\right\} = c^2 x^* =: c_0^2.
\end{aligned}$$

So let

$$\begin{aligned}
f &:= \operatorname{frac}\left(c_0^2 - c_0^1\right) \\
&= \operatorname{frac}\left(\left(c^2 - c^1\right) x^*\right) \\
&= \operatorname{frac}\left(\left(\mu^+ - \mu^-\right) \begin{pmatrix} A & G \end{pmatrix} x^*\right) \\
&= \operatorname{frac}\left(\mu \begin{pmatrix} A & G \end{pmatrix} x^*\right) \\
&= \operatorname{frac}(\mu b) \\
&\in (0, 1). && (\mu b = y b_{L_F} \notin \mathbb{Z} \text{ (using (7.3))})
\end{aligned}$$

To use Proposition 346 again, consider that

$$\begin{aligned}
\left(f c^1 + (1-f) c^2\right) &= \left(f \mu^- + (1-f) \mu^+\right) \begin{pmatrix} A & G \end{pmatrix} \\
&\in \operatorname{rowcone} \begin{pmatrix} A & G \end{pmatrix}_{(\operatorname{supp} \mu^-) \dot\cup (\operatorname{supp} \mu^+),*} \\
&\subseteq \operatorname{rowcone} \begin{pmatrix} A & G \end{pmatrix}_{L_F,*}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\max \left\{\left(f c^1 + (1-f) c^2\right) x : x \in P\right\} &= \left(f c^1 + (1-f) c^2\right) x^* \\
&= f c_0^1 + (1-f) c_0^2 \\
&> f c_0^1 + (1-f) c_0^2 - f(1-f).
\end{aligned}$$

For "only if": Consider dual variables $y^{1,*}, y^{2,*} \in \left(\mathbb{R}^l_{\geq 0}\right)^T$ having

$$\begin{aligned}
y^{1,*} \begin{pmatrix} A & G \end{pmatrix} &= c^1, & c_0^1 &= y^{1,*} b, \\
y^{2,*} \begin{pmatrix} A & G \end{pmatrix} &= c^2, & c_0^2 &= y^{2,*} b.
\end{aligned}$$

Additionally, let

$$\begin{aligned}
&\mu := y^{2,*} - y^{1,*}, \\
&f = \operatorname{frac}\left(c_0^2 - c_0^1\right) = \operatorname{frac}\left(y^{2,*} b - y^{1,*} b\right) = \operatorname{frac}(\mu b).
\end{aligned}$$

Then for $x \in P_I$, we have

$$\begin{aligned}\left(fc^1 + (1-f)\,c^2\right) x &= \left(fy^{1,*}\begin{pmatrix} A & G\end{pmatrix} + (1-f)\,y^{2,*}\begin{pmatrix} A & G\end{pmatrix}\right) x - fc_0^1 - (1-f)\,c_0^2 + f\,(1-f)\\ &= \left((1-\operatorname{frac}(\mu b))\,y^{2,*}\begin{pmatrix} A & G\end{pmatrix} + \operatorname{frac}(\mu b)\,y^{1,*}\begin{pmatrix} A & G\end{pmatrix}\right) x\\ &\quad -(1-\operatorname{frac}(\mu b))\,y^{2,*}b - \operatorname{frac}(\mu b)\,y^{1,*}b + \operatorname{frac}(\mu b)\,(1-\operatorname{frac}(\mu b))\\ &= \left((1-\operatorname{frac}(\mu b))\,y^{2,*} + \operatorname{frac}(\mu b)\,y^{1,*}\right)\left(\begin{pmatrix} A & G\end{pmatrix} x - b\right) + \operatorname{frac}(\mu b)\,(1-\operatorname{frac}(\mu b))\\ &\leq \left((1-\operatorname{frac}(\mu b))\,\mu^+ + \operatorname{frac}(\mu b)\,\mu^-\right)\left(\begin{pmatrix} A & G\end{pmatrix} x - b\right) + \operatorname{frac}(\mu b)\,(1-\operatorname{frac}(\mu b)) \quad (7.4)\\ &= f_{A,G,b}^{\leq 0, MIR, \mu}(x) \quad (7.5)\\ &\leq 0. \quad (7.6)\end{aligned}$$

For the explanations why the individual equations hold:

- (7.4) holds since $\begin{pmatrix} A & G\end{pmatrix} x - b \leq 0$, $\left(0^l\right)^T \leq y^{1,*} \leq \mu^-$ and $\left(0^l\right)^T \leq y^{2,*} \leq \mu^+$.
- (7.5) holds by Definition 410.
- (7.6) holds by Theorem 412.

□

7.2. Systems of dual integrality

In section 7.2.1, we define the classes of systems whose properties we already outlined at the introduction of this chapter.

For section 7.2.2: in Lemma 350 and Lemma 351, we state how the different types of systems of dual integrality relate to each other (we also gave an overview in the introduction of this chapter about this). What we want to show is that if $Ax \leq b$ is such a system and b is integral, then the polyhedron $P^{\leq}(A, b)$ is integral. By these two lemmas, it suffices to consider TD$\mathbb{Z} + I$ systems, for which we show the statement in Theorem 352. We then conslude the general case in Corollary 353.

On the other hand, we also mentioned in the introduction that if we have some integral polyhedron $P^{\leq}(A, b)$ in terms of rational inequalities be given, we have:

- the system $Ax \leq b$ is TD$\mathbb{Z} + \{0, 1\}$,
- if $Ax \leq b$ is TDI, it even satisfies the TD$(I \cap \mathbb{Z}) + \{0, 1\}$ property.

These are statements of Theorem 354 and Theorem 355.

7.2.1. Definitions

Definition 348. *Let $A \in \mathbb{Q}^{l\times m}$ and $b \in \mathbb{R}^l$, where $l, m \in \mathbb{Z}_{\geq 0}$. If for every $c \in (\mathbb{Z}^m)^T$ such that*

$$(7.7) := \max\{cx : Ax \leq b, x \in \mathbb{R}^m\} \in \mathbb{R},$$

we have

- $(7.7) = \min\left\{zb : zA = c, z \in \left(\mathbb{Z}^l_{\geq 0}\right)^T\right\}$, *then we call the system $Ax \leq b$ **TDI** (totally dual integral),*
- $(7.7) = \min\left\{\left(z^1 + z^2\right) b : \left(z^1 + z^2\right) A = c, z^1 b \in \mathbb{Z}, z^1 \in \left(\mathbb{R}^l_{\geq 0}\right)^T, z^2 \in \left(\mathbb{Z}^l_{\geq 0}\right)^T\right\}$, *then we call the system $Ax \leq b$ **TD**$\mathbb{Z} + I$.*

Definition 349. *Let $A \in \mathbb{Q}^{l\times m}$ and $b \in \mathbb{Q}^l$, where $l, m \in \mathbb{Z}_{\geq 0}$. If for every $c \in (\mathbb{Z}^m)^T$ such that*

$$(7.8) := \max\{cx : Ax \leq b, x \in \mathbb{R}^m\} \in \mathbb{R},$$

we have

- $(7.8) = \min\left\{\left(z^1 + z^2\right) b : \left(z^1 + z^2\right) A = c, z^1 b \in \mathbb{Z}, z^1 \in \left(\mathbb{R}^l_{\geq 0}\right)^T, z^2 \in \left(\mathbb{Z}^l_{\geq 0}\right)^T, \left\|z^2\right\|_1 \leq 1\right\}$, *then we call the system $Ax \leq b$ **TD**$\mathbb{Z} + \{0, 1\}$,*
- $(7.8) = \min\left\{\left(z^1 + z^2\right) b : \left(z^1 + z^2\right) A = c, z^1 b \in \mathbb{Z}, z^1 \in \left(\mathbb{Z}^l_{\geq 0}\right)^T, z^2 \in \left(\mathbb{Z}^l_{\geq 0}\right)^T, \left\|z^2\right\|_1 \leq 1\right\}$, *then we call the system $Ax \leq b$ **TD**$(I \cap \mathbb{Z}) + \{0, 1\}$.*

7.2.2. Properties

The following two lemmas clearly hold:

Lemma 350. *Let $Ax \leq b$ be TD($I \cap \mathbb{Z}$) $+ \{0,1\}$. Then $Ax \leq b$ is TD$\mathbb{Z} + \{0,1\}$ and TDI.*

Lemma 351. *Let $Ax \leq b$ be TD$\mathbb{Z} + \{0,1\}$ or TDI. Then $Ax \leq b$ is TD$\mathbb{Z} + I$.*

After these preparations, we give a short remark about the importance of the different types of systems:

- The most important type of system for this text are TD$\mathbb{Z} + I$ systems:
  - If $Ax \leq b$ is a TD$\mathbb{Z} + I$ system, where b is integral, then $P^{\leq}(A,b)$ is an integral polyhedron (Theorem 352) and, as we saw in the two preceding lemmas, every other type of dual integrality that we consider, implies the TD$\mathbb{Z} + I$ property.
  - If $Ax \leq b$ is a TD$\mathbb{Z} + I$ system, where A is integral, then this system can be used to represent the Chvátal-Gomory closure of $P^{\leq}(A,b)$ (Theorem 398).
  - One can show that TD$\mathbb{Z}+I$ systems can be significantly smaller than TDI systems or TD$\mathbb{Z}+\{0,1\}$ systems (Theorem 372 and Theorem 376).
- The importance of TD$\mathbb{Z} + \{0,1\}$ lies in the property that one can show that if $P^{\leq}(A,b)$ is integral (where A and b are rational), then $Ax \leq b$ is TD$\mathbb{Z} + \{0,1\}$.
- The importance of TDI and TD$(I \cap \mathbb{Z}) + \{0,1\}$ comes from the fact that for a given $P^{\leq}(A,b)$, where $A \in \mathbb{Q}^{l \times m}$ and $b \in \mathbb{R}^l$ or $b \in \mathbb{Q}^l$, respectively, there exists a system $A'x \leq b'$ that satisfies the TDI property or TD$(I \cap \mathbb{Z}) + \{0,1\}$ property, respectively, such that $P^{\leq}(A,b) = P^{\leq}(A',b')$ (Theorem 369).

Theorem 352. *Let $Ax \leq b$ be TD$\mathbb{Z} + I$ and let b be integral. Then $P^{\leq}(A,b)$ is an integral polyhedron.*

Proof. Let $c \in (\mathbb{Z}^m)^T$ be such that

$$\max\{cx : Ax \leq b, x \in \mathbb{R}^m\} =: c_0 \in \mathbb{R}$$

and let $\begin{pmatrix} z^{1,*} & z^{2,*} \end{pmatrix}$ be a minimizer for the dual program in the TD$\mathbb{Z} + I$ definition. Then

$$c_0 = \underbrace{z^{1,*}b}_{\in\mathbb{Z}} + \underbrace{z^{2,*}}_{\in\mathbb{Z}}\underbrace{b}_{\in\mathbb{Z}} \in \mathbb{Z}.$$

Thus, by Theorem 345, $P^{\leq}(A,b)$ is integral. □

Using Lemma 350 and Lemma 351, we conclude from Theorem 352:

Corollary 353. *Let $Ax \leq b$ be TD$\mathbb{Z} + I$, TD$(I \cap \mathbb{Z}) + \{0,1\}$, TD$\mathbb{Z} + \{0,1\}$ or TDI and let b be integral. Then $P^{\leq}(A,b)$ is integral.*

On the other hand, we have the following two theorems, which are obvious consequences of Theorem 345, Definition 348 and Definition 349.

Theorem 354. *Let $P^{\leq}(A,b)$ be an integral polyhedron, where $A \in \mathbb{Q}^{l \times m}$ and $b \in \mathbb{Q}^l$ ($l, m \in \mathbb{Z}_{\geq 0}$). Then $Ax \leq b$ is TD$\mathbb{Z} + \{0,1\}$.*

Theorem 355. *Let $P^{\leq}(A,b)$ be an integral polyhedron, where $A \in \mathbb{Q}^{l \times m}$ and $b \in \mathbb{Q}^l$ ($l, m \in \mathbb{Z}_{\geq 0}$), such that additionally $Ax \leq b$ is a TDI system. Then $Ax \leq b$ is TD$(I \cap \mathbb{Z}) + \{0,1\}$.*

7.3. Generating systems for integral vectors in cones

Our goal is to generalize Hilbert bases (cf. Definition 360) to more general basis concepts. A well-known theorem that links Hilbert bases to TDI systems is the following one:

Theorem 367. *([Sch86, Theorem 22.5; p. 315]) A rational system $Ax \leq b$ is TDI if and only if for each face F of the polyhedron $P^{\leq}(A,b)$, the rows of A which are active in F form a Hilbert basis.*

We remark that Theorem 368, which we show section 7.4, is a generalization of Theorem 367.

A natural approach to construct more general generating systems is not to just consider cones that are generated by subsets of rows of A, but by subsets of rows of $\begin{pmatrix} A & -b \end{pmatrix}$. On the other hand, we restrict ourself to a specific subset of such cones, which we named **LP face cones**. We define LP face cones in Definition 356 and prove an important characterization of them in Theorem 357.

After this, we define multiple classes of generating systems for LP face cones (icone systems, $\mathbb{Z}+$ icone systems, $\mathbb{Z}+\{0,1\}$ systems and $(\text{icone}\cap\mathbb{Z})+\{0,1\}$ systems) in Definition 358 and Definition 359 of section 7.3.2. In Lemma 361, we write down how icone systems are related to Hilbert bases (cf. Definition 360). Finally, in section 7.3.4, we prove existence results for $(\text{icone}\cap\mathbb{Z})+\{0,1\}$ systems and icone systems.

7.3.1. LP face cones

Definition 356. *Let $C \subseteq \mathbb{R}^d \times \mathbb{R}$ ($d \in \mathbb{Z}_{\geq 0}$) be a polyhedral cone. C is called an **LP face cone** if for every $c \in \operatorname{proj}_{\mathbb{R}^d} C$, there exists a unique $\overline{c} \in \mathbb{R}$ such that $\begin{pmatrix} c \\ \overline{c} \end{pmatrix} \in C$.*

Theorem 357. *Let*

$$C := \operatorname{cone}\left\{ \begin{pmatrix} r^1 & \overline{r}^1 \end{pmatrix}, \ldots, \begin{pmatrix} r^k & \overline{r}^k \end{pmatrix} \right\} \subseteq \left(\mathbb{R}^d \times \mathbb{R}\right)^T,$$

where $k, d \in \mathbb{Z}_{\geq 0}$, $r^1, \ldots, r^k \in \left(\mathbb{R}^d\right)^T$ and $\overline{r}^1, \ldots, \overline{r}^k \in \mathbb{R}$. Then C is an LP face cone if and only if

$$\exists z^* \in \mathbb{R}^d : \begin{pmatrix} r^1 & \overline{r}^1 \\ \vdots & \vdots \\ r^k & \overline{r}^k \end{pmatrix} \begin{pmatrix} z^* \\ 1 \end{pmatrix} = 0^k.$$

Proof.

$$\begin{aligned}
& \nexists c \in \operatorname{proj}_{\mathbb{R}^m} C, \overline{c}^1, \overline{c}^2 \in \mathbb{R}, \overline{c}^1 \neq \overline{c}^2 : \begin{pmatrix} c & \overline{c}^1 \end{pmatrix}, \begin{pmatrix} c & \overline{c}^2 \end{pmatrix} \in C \\
\Leftrightarrow \quad & \nexists \begin{pmatrix} \lambda^1 & \lambda^2 \end{pmatrix} \in \left(\mathbb{R}^k_{\geq 0} \times \mathbb{R}^k_{\geq 0}\right)^T : \\
& \begin{pmatrix} \lambda^1 & \lambda^2 \end{pmatrix} \begin{pmatrix} r^1 \\ \vdots \\ r^k \\ -r^1 \\ \vdots \\ -r^k \end{pmatrix} = \left(0^d\right)^T \wedge \begin{pmatrix} \lambda^1 & \lambda^2 \end{pmatrix} \begin{pmatrix} \overline{r}^1 \\ \vdots \\ \overline{r}^k \\ -\overline{r}^1 \\ \vdots \\ -\overline{r}^k \end{pmatrix} \neq 0 \\
\Leftrightarrow \quad & \nexists \lambda \in \left(\mathbb{R}^k\right)^T : \lambda \begin{pmatrix} r^1 \\ \vdots \\ r^k \end{pmatrix} = \left(0^d\right)^T \wedge \lambda \begin{pmatrix} \overline{r}^1 \\ \vdots \\ \overline{r}^k \end{pmatrix} \neq 0 \\
\Leftrightarrow \quad & \exists z^* \in \mathbb{R}^d : \begin{pmatrix} r^1 \\ \vdots \\ r^k \end{pmatrix} z^* = \begin{pmatrix} -\overline{r}^1 \\ \vdots \\ -\overline{r}^k \end{pmatrix} \qquad \text{(by Corollary 84)} \\
\Leftrightarrow \quad & \exists z^* \in \left(\mathbb{R}^d\right)^T : \begin{pmatrix} r^1 & \overline{r}^1 \\ \vdots & \vdots \\ r^k & \overline{r}^k \end{pmatrix} \begin{pmatrix} z^* \\ 1 \end{pmatrix} = 0^k.
\end{aligned}$$

□

7.3.2. Definitions

Definition 358. *Let $\mathcal{S} := \left\{ \begin{pmatrix} r^1 \\ \overline{r}^1 \end{pmatrix}, \ldots, \begin{pmatrix} r^k \\ \overline{r}^k \end{pmatrix} \right\} \subseteq \mathbb{Q}^m \times \mathbb{R}$ ($k, m \in \mathbb{Z}_{\geq 0}$) be given such that $C := \operatorname{cone} \mathcal{S}$ forms an LP face cone.*

- *$\mathcal{S}$ is called an* icone ***system*** *if $\forall c \in C \cap (\mathbb{Z}^m \times \mathbb{R}) : c \in \operatorname{icone} \mathcal{S}$.*

- $\mathcal{S}$ *is called a* $\mathbb{Z} + \text{icone}$ ***system*** *if* $\forall c \in C \cap (\mathbb{Z}^m \times \mathbb{R}) : c = c^1 + c^2$, *where*

$$c^1 \in C \cap (\mathbb{Q}^m \times \mathbb{Z}),$$
$$c^2 \in \operatorname{icone} \mathcal{S}.$$

Definition 359. *Let* $\mathcal{S} := \left\{ \begin{pmatrix} r^1 \\ \overline{r}^1 \end{pmatrix}, \ldots, \begin{pmatrix} r^k \\ \overline{r}^k \end{pmatrix} \right\} \subseteq \mathbb{Q}^m \times \mathbb{Q}$ *($k, m \in \mathbb{Z}_{\geq 0}$) be given such that* $C := \operatorname{cone} \mathcal{S}$ *forms an LP face cone.*

- $\mathcal{S}$ *is called a* $\mathbb{Z} + \{0,1\}$ ***system*** *if* $\forall c \in C \cap (\mathbb{Z}^m \times \mathbb{R}) : c = c^1 + c^2$, *where*

$$c^1 \in C \cap (\mathbb{Q}^m \times \mathbb{Z}),$$
$$c^2 \in \mathcal{S} \cup \left\{ \begin{pmatrix} 0^m \\ 0 \end{pmatrix} \right\}.$$

- $\mathcal{S}$ *is called an* $(\text{icone} \cap \mathbb{Z}) + \{0,1\}$ ***system*** *if* $\forall c \in C \cap (\mathbb{Z}^m \times \mathbb{R}) : c = c^1 + c^2$, *where*

$$c^1 \in (\operatorname{icone} \mathcal{S}) \cap (\mathbb{Q}^m \times \mathbb{Z}),$$
$$c^2 \in \mathcal{S} \cup \left\{ \begin{pmatrix} 0^m \\ 0 \end{pmatrix} \right\}.$$

Let us have a look how icone systems are related to Hilbert bases. For this, we, of course, have to define Hilbert bases:

Definition 360. *(cf. [Sch86, section 16.4; p. 232f]) Let* $a^1, \ldots, a^k \in \mathbb{Q}^m$ *($k, m \in \mathbb{Z}_{\geq 0}$). Then the set* $\{a^1, \ldots, a^k\}$ *is called a **Hilbert basis** if* $\left(\operatorname{cone}\{a^1, \ldots, a^k\}\right)_I \subseteq \operatorname{icone}\{a^1, \ldots, a^k\}$.

The following lemma clearly holds:

Lemma 361. *We have:*

1. *Let* C' *be a polyhedral cone generated by* $\binom{r^1}{\overline{r}^1}, \ldots, \binom{r^k}{\overline{r}^k} \in \mathbb{R}^m \times \mathbb{R}$ *($k, m \in \mathbb{Z}_{\geq 0}$) and let* $C := \operatorname{proj}_{\mathbb{R}^m} C'$. *Then*
$$C = \operatorname{cone}\{r^1, \ldots, r^k\}.$$

2. *Let* C, C' *be as in 1, but this time let* $\binom{r^1}{\overline{r}^1}, \ldots, \binom{r^k}{\overline{r}^k} \in \mathbb{Q}^m \times \mathbb{R}$ *($k, m \in \mathbb{Z}_{\geq 0}$). Additionally, let* $\left\{\binom{s^1}{\overline{s}^1}, \ldots, \binom{s^l}{\overline{s}^l}\right\}$ *($l \in \mathbb{Z}_{\geq 0}$) be an* icone *system of* C'. *Then* $\{s^1, \ldots, s^l\}$ *forms a Hilbert basis of* C.

3. *Let* $C \subseteq \mathbb{R}^m$ *be a polyhedral cone generated by* $r^1, \ldots, r^k \in \mathbb{R}^m$ *($k, m \in \mathbb{Z}_{\geq 0}$) and let* $C' \subseteq \mathbb{R}^m \times \mathbb{R}$ *be an LP face cone having* $\operatorname{proj}_{\mathbb{R}^d} C' = C$. *Then there exist unique* $\overline{r}^1, \ldots, \overline{r}^k \in \mathbb{R}$ *such that*
$$C' = \operatorname{cone}\left\{ \begin{pmatrix} r^1 \\ \overline{r}^1 \end{pmatrix}, \ldots, \begin{pmatrix} r^k \\ \overline{r}^k \end{pmatrix} \right\}.$$

4. *Let* C, C' *be as in 3, but this time let* $r^1, \ldots, r^k \in \mathbb{Q}^m$. *Additionally, let* $s^1, \ldots, s^l$ *($l \in \mathbb{Z}_{\geq 0}$) be a Hilbert basis of* C. *Then there exist unique* $\overline{s}^1, \ldots, \overline{s}^l \in \mathbb{R}$ *such that* $\left\{\binom{s^1}{\overline{s}^1}, \ldots, \binom{s^l}{\overline{s}^l}\right\}$ *is an* icone *system of* C.

In the sense of Lemma 361, icone systems can be considered as the analogue of Hilbert bases for LP face cones.

7.3.3. Properties

In section 7.2.2, we saw how the different kind of systems of dual integrality relate to each other. We now do the same for the different kinds of systems of integral vectors in cones that we defined in the previous section.

The following two lemmas clearly hold:

Lemma 362. *Let* $\mathcal{S}$ *be an* $(\text{icone} \cap \mathbb{Z}) + \{0,1\}$ *system. Then* $\mathcal{S}$ *is a* $\mathbb{Z} + \{0,1\}$ *system and an* icone *system.*

Lemma 363. *Let* $\mathcal{S}$ *be a* $\mathbb{Z} + \{0,1\}$ *or an* icone *system. Then* $\mathcal{S}$ *is a* $\mathbb{Z} + \text{icone}$ *system.*

Just as we already remarked in section 7.2.2 about the importance of the different types of systems of dual integrality, we again bring up a few points concerning which of the generating systems are important for which purpose:

- The most important type of system in this text are $\mathbb{Z}+$icone systems:
  - $\mathbb{Z}+$icone systems are deeply related to Chvátal-Gomory cuts (cf. Theorem 404) and MIR cuts (cf. Theorem 429).
  - One can show that $\mathbb{Z}+$icone systems can be significantly smaller than icone systems or $\mathbb{Z}+\{0,1\}$ systems (Theorem 370 and Theorem 375).
- The importance of icone systems lies in the following properties:
  - They bear a strong relationship to Hilbert bases (cf. Lemma 361).
  - For LP face cones generated by vectors from $\mathbb{Q}^m \times \mathbb{R}$, one can show their existence (Theorem 366) and use this existence result to prove the existence of $\mathbb{Z}+$icone systems (a consequence of Lemma 363).
- $(\text{icone}\cap\mathbb{Z})+\{0,1\}$ are important, since for LP face cones generated by vectors from $\mathbb{Q}^m \times \mathbb{Q}$, one can show their existence (Theorem 365) and use this existence result to prove the existence of $\mathbb{Z}+\{0,1\}$ systems, icone systems and $\mathbb{Z}+$icone systems (a consequence of Lemma 362 and Lemma 363).

7.3.4. Existence

Before we state our existence results for $(\text{icone}\cap\mathbb{Z})+\{0,1\}$ systems and icone systems in Theorem 365 and Theorem 366, we prove a small lemma about rational vectors in LP face cones:

Lemma 364. *Let $C \subseteq \mathbb{R}^m \times \mathbb{R}$ $(m \in \mathbb{Z}_{\geq 0})$ be an LP face cone generated by*

$$\begin{pmatrix} r^1 \\ \overline{r}^1 \end{pmatrix}, \dots, \begin{pmatrix} r^k \\ \overline{r}^k \end{pmatrix} \in \mathbb{Q}^m \times \mathbb{Q},$$

where $k, m \in \mathbb{Z}_{\geq 0}$, and let $\begin{pmatrix} c \\ \overline{c} \end{pmatrix} \in C \cap (\mathbb{Z}^m \times \mathbb{R})$. Then $\overline{c} \in \mathbb{Q}$.

Proof. Since, by the LP face cone property, $\overline{c}$ is uniquely determined, it suffices to show that there exists a $\lambda \in \mathbb{Q}^k_{\geq 0}$ having $\begin{pmatrix} r^1 & \cdots & r^k \end{pmatrix} \lambda = c$. So, we have using Lemma 88:

$$\begin{aligned} \exists \lambda \in \mathbb{R}^k_{\geq 0} : \begin{pmatrix} r^1 & \cdots & r^k \end{pmatrix} \lambda = c &\Leftrightarrow \nexists y \in (\mathbb{R}^m)^T : y \begin{pmatrix} r^1 & \cdots & r^k \end{pmatrix} \geq \left(0^k\right)^T \wedge yc < 0 \\ &\Rightarrow \nexists y \in (\mathbb{Q}^m)^T : y \begin{pmatrix} r^1 & \cdots & r^k \end{pmatrix} \geq \left(0^k\right)^T \wedge yc < 0 \\ &\Leftrightarrow \exists \lambda \in \mathbb{Q}^k_{\geq 0} : \begin{pmatrix} r^1 & \cdots & r^k \end{pmatrix} \lambda = c. \end{aligned}$$

The initial condition in this chain of implications holds since $c \in \text{proj}_{\mathbb{R}^m} C$. □

Theorem 365. *Let $C \subseteq \mathbb{R}^m \times \mathbb{R}$ $(m \in \mathbb{Z}_{\geq 0})$ be an LP face cone that is generated by*

$$\begin{pmatrix} r^1 \\ \overline{r}^1 \end{pmatrix}, \dots, \begin{pmatrix} r^k \\ \overline{r}^k \end{pmatrix} \in \mathbb{Z}^m \times \mathbb{Z},$$

where $k \in \mathbb{Z}_{\geq 0}$. Then

$$\mathcal{S} := \underbrace{\left(\left\{ \sum_{i=1}^k \lambda_i \begin{pmatrix} r^i \\ \overline{r}^i \end{pmatrix} : 0^k \lneq \lambda < 1^k \right\} \cap (\mathbb{Z}^m \times \mathbb{R}) \right)}_{=:\mathcal{S}'} \cup \underbrace{\bigcup_{i=1}^k \left\{ \begin{pmatrix} r^i \\ \overline{r}^i \end{pmatrix} \right\}}_{=:\mathcal{S}''}$$

is an $(\text{icone}\cap\mathbb{Z})+\{0,1\}$ system that generates C and consists of vectors from $\mathbb{Z}^m \times \mathbb{Q}$.

Proof. Clearly, $\operatorname{cone}\mathcal{S} = C$. Since $\operatorname{proj}_{\mathbb{R}^m}\mathcal{S}'$ is finite, by the LP face cone property, also $\mathcal{S}'$ and thus $\mathcal{S}$ is finite. Let $\begin{pmatrix} c \\ \overline{c} \end{pmatrix} \in C \cap (\mathbb{Z}^m \times \mathbb{R})$. We claim that

$$\begin{pmatrix} c \\ \overline{c} \end{pmatrix} \in \underbrace{\operatorname{icone}\mathcal{S}''}_{\subseteq(\operatorname{icone}\mathcal{S})\cap(\mathbb{Q}^m\times\mathbb{Z})} + \underbrace{\left(\mathcal{S}' \cup \left\{\begin{pmatrix} 0^m \\ 0 \end{pmatrix}\right\}\right)}_{\subseteq\left(\mathcal{S}\cup\left\{\begin{pmatrix} 0^m \\ 0 \end{pmatrix}\right\}\right)}.$$

For this, let $\lambda \in \mathbb{R}^k_{\geq 0}$ be such that $\begin{pmatrix} c \\ \overline{c} \end{pmatrix} = \sum_{j=1}^{k} \lambda_j \begin{pmatrix} r^j \\ \overline{r}^j \end{pmatrix}$. Then

$$\begin{pmatrix} c \\ \overline{c} \end{pmatrix} = \underbrace{\sum_{j=1}^{k} \underbrace{\lfloor\lambda_j\rfloor}_{\in\mathbb{Z}_{\geq 0}} \begin{pmatrix} r^j \\ \overline{r}^j \end{pmatrix}}_{\in\operatorname{icone}\mathcal{S}''} + \underbrace{\sum_{j=1}^{d} \underbrace{(\lambda_j - \lfloor\lambda_j\rfloor)}_{<1} \begin{pmatrix} r^j \\ \overline{r}^j \end{pmatrix}}_{\in\mathcal{S}'\cup\left\{\begin{pmatrix} 0^m \\ 0 \end{pmatrix}\right\}}.$$

The statement that $\mathcal{S}$ consists of vectors from $\mathbb{Z}^m \times \mathbb{Q}$ is a consequence of Lemma 364. □

Theorem 366. *Let $C \subseteq \mathbb{R}^m \times \mathbb{R}$ ($m \in \mathbb{Z}_{\geq 0}$) be an LP face cone that is generated by*

$$\begin{pmatrix} r^1 \\ \overline{r}^1 \end{pmatrix}, \dots, \begin{pmatrix} r^k \\ \overline{r}^k \end{pmatrix} \in \mathbb{Z}^m \times \mathbb{R},$$

where $k \in \mathbb{Z}_{\geq 0}$. Then

$$\mathcal{S} := \underbrace{\left(\left\{\sum_{i=1}^{k} \lambda_i \begin{pmatrix} r^i \\ \overline{r}^i \end{pmatrix} : 0^k \lneq \lambda < 1^k\right\} \cap (\mathbb{Z}^m \times \mathbb{R})\right)}_{=:\mathcal{S}'} \cup \underbrace{\bigcup_{i=1}^{k} \left\{\begin{pmatrix} r^i \\ \overline{r}^i \end{pmatrix}\right\}}_{=:\mathcal{S}''}$$

is an icone *system that generates C and consists of vectors from $\mathbb{Z}^m \times \mathbb{R}$. If $\overline{r}^1, \dots, \overline{r}^k \in \mathbb{Q}$, then $\mathcal{S}$ consists of vectors from $\mathbb{Z}^m \times \mathbb{Q}$.*

Proof. Clearly, $\operatorname{cone}\mathcal{S} = C$. Since $\operatorname{proj}_{\mathbb{R}^m}\mathcal{S}'$ is finite, by the LP face cone property, also $\mathcal{S}'$ and thus $\mathcal{S}$ is finite. Let $\begin{pmatrix} c \\ \overline{c} \end{pmatrix} \in C \cap (\mathbb{Z}^m \times \mathbb{R})$. We claim that

$$\begin{pmatrix} c \\ \overline{c} \end{pmatrix} \in \underbrace{\operatorname{icone}\mathcal{S}'' + \left(\mathcal{S}' \cup \left\{\begin{pmatrix} 0^m \\ 0 \end{pmatrix}\right\}\right)}_{\subseteq\operatorname{icone}\mathcal{S}}.$$

For this, let $\lambda \in \mathbb{R}^k_{\geq 0}$ be such that $\begin{pmatrix} c \\ \overline{c} \end{pmatrix} = \sum_{j=1}^{k} \lambda_j \begin{pmatrix} r^j \\ \overline{r}^j \end{pmatrix}$. Then

$$\begin{pmatrix} c \\ \overline{c} \end{pmatrix} = \underbrace{\sum_{j=1}^{k} \underbrace{\lfloor\lambda_j\rfloor}_{\in\mathbb{Z}_{\geq 0}} \begin{pmatrix} r^j \\ \overline{r}^j \end{pmatrix}}_{\in\operatorname{icone}\mathcal{S}''} + \underbrace{\sum_{j=1}^{d} \underbrace{(\lambda_j - \lfloor\lambda_j\rfloor)}_{<1} \begin{pmatrix} r^j \\ \overline{r}^j \end{pmatrix}}_{\in\mathcal{S}'\cup\left\{\begin{pmatrix} 0^m \\ 0 \end{pmatrix}\right\}}.$$

The statement that $\mathcal{S}$ consists of vectors from $\mathbb{Z}^m \times \mathbb{Q}$ if $\overline{r}^1, \dots, \overline{r}^k \in \mathbb{Q}$, is a consequence of Lemma 364. □

7.4. Systems of dual integrality and generating systems

We already mentioned the following result from the literature, which relates TDI systems and Hilbert bases, in the introduction of this chapter and in the introduction of section 7.3.

Theorem 367. *([Sch86, Theorem 22.5; p. 315]) A rational system $Ax \leq b$ is TDI if and only if for each face F of the polyhedron $P^{\leq}(A, b)$, the rows of A which are active in F form a Hilbert basis.*

In the following theorem, we consider how the systems of dual integrality that we introduced in section 7.2 relate to the generating systems for cones that we introduced in section 7.3. Its proof generalizes the proof of Theorem 367 in [Sch86, p. 315f].

Theorem 368. *Let $A \in \mathbb{Q}^{l \times m}$ and let $b \in \mathbb{R}^l$ or $b \in \mathbb{Q}^l$, respectively, where $l, m \in \mathbb{Z}_{\geq 0}$. Then:*

1. *Let $Ax \leq b$ be*
   - *TDI,*
   - *TD$\mathbb{Z} + \{0, 1\}$,*
   - *TD$(I \cap \mathbb{Z}) + \{0, 1\}$ or*
   - *TD$\mathbb{Z} + I$,*

   respectively. For every face F of $P^{\leq}(A, b)$, let $S_F \subseteq [l]$ denote the indices of the rows of $Ax \leq b$ that are active in F. Then for each such F, the rows of $\begin{pmatrix} A & -b \end{pmatrix}_{S_F,*}$ form
   - *an* icone *system,*
   - *a $\mathbb{Z} + \{0, 1\}$ system,*
   - *an* $(\text{icone} \cap \mathbb{Z}) + \{0, 1\}$ *system or*
   - *a* $\mathbb{Z} + \text{icone}$ *system,*

   respectively.
2. *For each* minimal *face $\emptyset \neq F$ of $P^{\leq}(A, b)$, let $S_F \subseteq [l]$ denote the indices of rows of $Ax \leq b$ that are active in F. If for each such F, the rows of $\begin{pmatrix} A & -b \end{pmatrix}_{S_F,*}$ form*
   - *an* icone *system,*
   - *a $\mathbb{Z} + \{0, 1\}$ system,*
   - *an* $(\text{icone} \cap \mathbb{Z}) + \{0, 1\}$ *system or*
   - *a* $\mathbb{Z} + \text{icone}$ *system,*

   respectively, then $Ax \leq b$ is
   - *TDI,*
   - *TD$\mathbb{Z} + \{0, 1\}$,*
   - *TD$(I \cap \mathbb{Z}) + \{0, 1\}$ or*
   - *TD$\mathbb{Z} + I$,*

   respectively.

Proof. W.l.o.g. we can assume $P^{\leq}(A, b) \neq \emptyset$.

For 1: Let $\emptyset \neq F$ be a face of $P^{\leq}(A, b)$ and let $c \in \left(\text{cone} \bigcup_{s \in S_F} \{A_{s,*}\}\right)_I$. Then, clearly,

$$\max\{cx : Ax \leq b\} =: c_0 \in \mathbb{R}.$$

By Proposition 346, the maximum is attained for every $x \in F$. So, let $x^* \in \operatorname{relint} F$. On the other hand, the dual minimum (where the dual program is defined as in the definition of the respective system of dual integrality) is attained for a

- $z^* \in \left(\mathbb{Z}^l_{\geq 0}\right)^T$,
- $z^{1,*} \in \left(\mathbb{R}^l_{\geq 0}\right)^T$, $z^{2,*} \in \left(\mathbb{Z}^l_{\geq 0}\right)^T$ having
  - $z^{1,*} b \in \mathbb{Z}$,
  - $\left\|z^{2,*}\right\|_1 \leq 1$,
- $z^{1,*} \in \left(\mathbb{Z}^l_{\geq 0}\right)^T$, $z^{2,*} \in \left(\mathbb{Z}^l_{\geq 0}\right)^T$ having
  - $z^{1,*} b \in \mathbb{Z}$,
  - $\left\|z^{2,*}\right\|_1 \leq 1$ or

- $z^{1,*} \in \left(\mathbb{R}^l_{\geq 0}\right)^T$, $z^{2,*} \in \left(\mathbb{Z}^l_{\geq 0}\right)^T$ having $z^{1,*}b \in \mathbb{Z}$,

respectively. So, x^* and z^* or $\begin{pmatrix} z^{1,*} & z^{2,*} \end{pmatrix}$, respectively, form a primal-dual pair. Since $x^* \in \operatorname{relint} F$, we have $\forall i \in [l] \setminus S_F : A_{i,*}x^* < b_i$. So, by complementary slackness, we have

$$\forall i \in [l] \setminus S_F : z_i^* = 0$$

or

$$\forall i \in [l] \setminus S_F : z_i^{1,*} = 0 \wedge z_i^{2,*} = 0,$$

respectively. Thus,

$$\begin{pmatrix} c & -c_0 \end{pmatrix} \in \left(\mathbb{R}^{S_F}_{\geq 0}\right)^T \begin{pmatrix} A & -b \end{pmatrix}_{S_F,*}.$$

By the additional properties of the dual variables, it is easy to check that $\begin{pmatrix} c & -c_0 \end{pmatrix}$ can be represented by the rows of $\begin{pmatrix} A & -b \end{pmatrix}_{S_F,*}$ as the property (TDI, $\mathrm{TD}\mathbb{Z} + \{0,1\}$, $\mathrm{TD}(I \cap \mathbb{Z}) + \{0,1\}$ or $\mathrm{TD}\mathbb{Z} + I$) requires.

For 2: Let $c \in (\mathbb{Z}^m)^T$ be such that

$$\max\{cx : Ax \leq b\} =: c_0 \in \mathbb{R}.$$

Let F be a minimal face of $P^{\leq}(A,b)$ such that for each point in F the maximum is attained. We claim that

$$c \in \operatorname{cone} \bigcup_{s \in S_F} \{A_{s,*}\}. \tag{7.9}$$

For this, let $x^* \in \operatorname{relint} F$. Clearly (since any point in F is by construction such a maximizer), x^* is a maximizer for

$$\max\{cx : Ax \leq b\} = c_0 = \min\left\{zb : zA = c, z \in \left(\mathbb{R}^l_{\geq 0}\right)^T\right\}.$$

On the other hand, since $x^* \in \operatorname{relint} F$, we have $A_{i,*}x^* < b_i$ for any $i \in [l] \setminus S_F$. Let $z^{pre,*}$ be a minimizer for the dual program. Then, by complementary slack, we have $z_i^{pre,*} = 0$ for any $i \in [l] \setminus S_F$. This shows (7.9).

Since $\bigcup_{s \in S_F} \left\{\begin{pmatrix} A & -b \end{pmatrix}_{s,*}\right\}$ forms

- an icone system,
- a $\mathbb{Z} + \{0,1\}$ system,
- an (icone$\cap\mathbb{Z}$) $+ \{0,1\}$ system or
- a $\mathbb{Z} +$ icone system,

respectively, there exists

- $z^{pre,*} \in \left(\mathbb{Z}^{S_F}_{\geq 0}\right)^T$ having
$$\begin{pmatrix} c & -c_0 \end{pmatrix} = z^{pre,*} \begin{pmatrix} A & -b \end{pmatrix}_{S_F},$$
- $z^{1,pre,*} \in \left(\mathbb{R}^{S_F}_{\geq 0}\right)^T$, $z^{2,pre,*} \in \left(\mathbb{Z}^{S_F}_{\geq 0}\right)^T$, $\left\|z^{2,pre,*}\right\| \leq 1$ having
$$\begin{aligned} &z^{1,pre,*}(-b)_{S_F} \in \mathbb{Z}, \\ &\begin{pmatrix} c & -c_0 \end{pmatrix} = \left(z^{1,pre,*} + z^{2,pre,*}\right) \begin{pmatrix} A & -b \end{pmatrix}_{S_F}, \end{aligned}$$
- $z^{1,pre,*} \in \left(\mathbb{Z}^{S_F}_{\geq 0}\right)^T$, $z^{2,pre,*} \in \left(\mathbb{Z}^{S_F}_{\geq 0}\right)^T$, $\left\|z^{2,pre,*}\right\| \leq 1$ having
$$\begin{aligned} &z^{1,pre,*}(-b)_{S_F} \in \mathbb{Z}, \\ &\begin{pmatrix} c & -c_0 \end{pmatrix} = \left(z^{1,pre,*} + z^{2,pre,*}\right) \begin{pmatrix} A & -b \end{pmatrix}_{S_F} \end{aligned}$$

  or

- $z^{1,pre,*} \in \left(\mathbb{R}^{S_F}_{\geq 0}\right)^T$, $z^{2,pre,*} \in \left(\mathbb{Z}^{S_F}_{\geq 0}\right)^T$ having

$$\begin{aligned} z^{1,pre,*}(-b)_{S_F} &\in \mathbb{Z}, \\ \begin{pmatrix} c & -c_0 \end{pmatrix} &= \left(z^{1,pre,*} + z^{2,pre,*}\right) \begin{pmatrix} A & -b \end{pmatrix}_{S_F}, \end{aligned}$$

respectively. Set

$$\begin{aligned} z^* &\in \mathbb{Z}^l, \\ z_i^* &:= \begin{cases} z_i^{pre,*} & \text{if } i \in S_F, \\ 0 & \text{if } i \in [l] \setminus S_F \end{cases} \quad \forall i \in [l] \end{aligned}$$

or

$$\begin{aligned} z^{1,*}, z^{2,*} &\in \mathbb{R}^l, \\ z_i^{1,*} &:= \begin{cases} z_i^{1,pre,*} & \text{if } i \in S_F, \\ 0 & \text{if } i \in [l] \setminus S_F \end{cases} \quad \forall i \in [l], \\ z_i^{2,*} &:= \begin{cases} z_i^{2,pre,*} & \text{if } i \in S_F, \\ 0 & \text{if } i \in [l] \setminus S_F \end{cases} \quad \forall i \in [l], \end{aligned}$$

respectively. It is easy to check that z^* or $\begin{pmatrix} z^{1,*} & z^{2,*} \end{pmatrix}$ is a minimizer for the dual program

$$\min \left\{ zb : zA = c, z \in \left(\mathbb{R}^l_{\geq 0}\right)^T \right\}$$

or

$$\min \left\{ \begin{pmatrix} z^1 & z^2 \end{pmatrix} \begin{pmatrix} b \\ b \end{pmatrix} : z \begin{pmatrix} A \\ A \end{pmatrix} = c, \begin{pmatrix} z^1 & z^2 \end{pmatrix} \in \left(\mathbb{R}^l_{\geq 0} \times \mathbb{R}^l_{\geq 0}\right)^T \right\},$$

respectively, and satisfies the additional dual minimizer properties that defines a

- TDI system,
- TD$\mathbb{Z} + \{0,1\}$ system,
- TD$(I \cap \mathbb{Z}) + \{0,1\}$ system or
- TD$\mathbb{Z} + I$ system,

respectively. □

7.5. Existence of systems of dual integrality

We now show that

- for every polyhedron with rational face normals, one can find a TDI system with an integral left-hand side describing it,
- for every rational polyhedron, one can even find a TD$(I \cap \mathbb{Z}) + \{0,1\}$ system with an integral left-hand side describing it.

For the case of rational polyhedra and TDI systems, this is well-known in literature (cf. [Sch86, Theorem 22.6; p. 316]). The proof of Theorem 369 generalizes the idea for the proof of this result.

Theorem 369. *Let $P \subseteq \mathbb{R}^m$ ($m \in \mathbb{Z}_{\geq 0}$). Then:*

- *If P is a polyhedron with rational face normals, there exists a TDI system $A'x \leq b'$, where $A' \in \mathbb{Z}^{l' \times m}$ and $b' \in \mathbb{R}^{l'}$ ($l' \in \mathbb{Z}_{\geq 0}$), having $P = P^{\leq}(A', b')$. Here, we can assume that b' is integral if P is an integral polyhedron.*
- *If P is a rational polyhedron, there exists a TD$(I \cap \mathbb{Z}) + \{0,1\}$ system $A'x \leq b'$, where $A' \in \mathbb{Z}^{l' \times m}$ and $b' \in \mathbb{Q}^{l'}$ ($l' \in \mathbb{Z}_{\geq 0}$), having $P = P^{\leq}(A', b')$. Here, we can assume that b' is integral if P is an integral polyhedron.*

Proof. W.l.o.g. we can assume $P \neq \emptyset$. For each minimal face F of P, let

$$C_F := \left\{ \begin{pmatrix} c & -\max\{cx : x \in P\} \end{pmatrix} : c \in (\mathbb{R}^m)^T, \max\{cx : x \in P\} \text{ is attained for all } x \in F \right\}.$$

Clearly, for each minimal face F of P, C_F is an LP face cone that has generators from

- $(\mathbb{Z}^m \times \mathbb{R})^T$ or
- $(\mathbb{Z}^m \times \mathbb{Z})^T$,

respectively. Thus, by Theorem 365 or Theorem 366, respectively, there exists an

- icone system or
- (icone$\cap\mathbb{Z}$) + $\{0, 1\}$ system,

respectively, for C_F that consists of vectors from

- $(\mathbb{Z}^m \times \mathbb{R})^T$ or
- $(\mathbb{Z}^m \times \mathbb{Z})^T$,

respectively. We denote it by $\mathcal{S}_F$. For every $\begin{pmatrix} c & \overline{c} \end{pmatrix} \in \mathcal{S}_F$, the inequality $cx \leq -\overline{c}$ valid for $x \in P$. Let

$$\mathcal{S} := \bigcap_{\substack{F \text{ minimal} \\ \text{face of } P}} \mathcal{S}_F =: \left\{ \begin{pmatrix} c^1 & \overline{c}^1 \end{pmatrix}, \ldots, \begin{pmatrix} c^{|\mathcal{S}|} & \overline{c}^{|\mathcal{S}|} \end{pmatrix} \right\},$$
$$l' := |\mathcal{S}|$$

and set for $i \in [l']$:

$$A'_{i,*} := c^i, \qquad b'_i := -\overline{c}^i.$$

Clearly, $P = P^{\leq}(A', b')$. By Theorem 368, $A'x \leq b'$ is TDI or TD$(I \cap \mathbb{Z}) + \{0, 1\}$, respectively. □

7.6. Sizes of generating systems for integral vectors in cones/systems of dual integrality

In this section, we consider the following questions:

- How large has a minimal system of dual integrality (in the sense of section 7.2) to be for some polyhedra (with respect to the number inequalities).
- How large has a minimal generating system (in the sense of section 7.3) to be for some cones (with respect to the number of elements).

For the outline:

- In section 7.6.1, we construct a series of cones such that any icone system for one of them that consists of vectors from $\left(\mathbb{Z}^2 \times \mathbb{R}\right)^T$ or $\left(\mathbb{Q}^2 \times \mathbb{Z}\right)^T$ has to be "large" (with respect to the number of elements), but there exists a "small" $\mathbb{Z} + \{0, 1\}$ system for them (Theorem 370).

  We use this result to construct integral polyhedra in $\mathbb{R}^2$ such that any TDI system $Ax \leq b$ for them, where either A or b is integral, needs a "large number" of rows (Theorem 372). We remark (Remark 373) that these integrality conditions are essential. On the other hand, for any polyhedron from this series, there exists a "small" TD $\mathbb{Z} + \{0, 1\}$ system $Ax \leq b$ describing it, where A and b are integral.

- In section 7.6.2, we similarly construct a series of cones such that any $\mathbb{Z} + \{0, 1\}$ system that generates one of them contains a "large" number of vectors, but for which there exists a "small" icone system (Theorem 375).

  We use this to construct a series of rational polyhedra with the property that any TD$\mathbb{Z} + \{0, 1\}$ system for them has a "large" number of rows, but for which there exists a "small" TDI system $Ax \leq b$ describing it, where, additionally, A is integral.

- In section 7.6.3, we construct a series of polyhedra P^{377,k_1,k_2} indexed by $(k_1,k_2) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 2}$, where for suitable k_1, k_2, the minimal sizes of a
  - $\mathrm{TD}(I \cap \mathbb{Z}) + \{0,1\}$ system,
  - TDI system,
  - $\mathrm{TD}\mathbb{Z} + \{0,1\}$ system and
  - $\mathrm{TD}\mathbb{Z} + I$ system

  $Ax \leq b$ for P^{377,k_1,k_2}, where A is integral, differ (Theorem 379). The high-level idea for this is to construct a rational polyhedron in $\mathbb{R}^2$ with two vertices such that for one vertex, the cone of rows that are active in it requires a large icone system and for the other vertex, the similarly defined cone warrants a large $\mathbb{Z} + \{0,1\}$ system.

- In section 7.6.4, we construct a series of (rational, non-integral) polyhedra in $\mathbb{R}^2$ with two facets such that any $\mathrm{TD}\mathbb{Z}+I$ system $Ax \leq b$ for it, where A is integral, requires a "large" number of rows (Theorem 381).

  This shows that there exist polyhedra that have a simple facet structure, but still require a "large" $\mathrm{TD}\mathbb{Z} + I$ system $Ax \leq b$ to describe them if one demands A to be integral.

  Another application of this series of polyhedra is later on presented in section 8.2.3.3, where this result turns out to be an important counterexample for a minimality problem (Problem 401) about a rounding procedure for $\mathrm{TD}\mathbb{Z} + I$ systems to represent the Chvátal-Gomory closure of a polyhedron with rational face normals.

7.6.1. TDI systems and icone systems

Theorem 370. *For $k \in \mathbb{Z}_{\geq 1}$, define*

$$C^{370,k} := \operatorname{cone}\left\{\begin{pmatrix} 1 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & k & -1 \end{pmatrix}\right\}.$$

Let $\mathcal{S}$ be an icone *system for $C^{370,k}$, where either*

- $\mathcal{S} \subseteq \left(\mathbb{Z}^2 \times \mathbb{R}\right)^T$ *or*
- $\mathcal{S} \subseteq \left(\mathbb{Q}^2 \times \mathbb{Z}\right)^T$.

Then

$$\underbrace{\dot{\bigcup_{i=0}^{k}} \left\{\begin{pmatrix} 1 & i & -1 \end{pmatrix}\right\}}_{=:\mathcal{S}'} \subseteq \mathcal{S}.$$

In particular, $|\mathcal{S}| \geq k+1$. On the other hand,

- *$\mathcal{S}'$ is an* $(\text{icone} \cap \mathbb{Z}) + \{0,1\}$ *system (and thus an* icone *system) for $C^{370,k}$ that consists of exactly $k+1$ elements from $\left(\mathbb{Z}^2 \times \mathbb{Z}\right)^T$,*
- *$\left\{\begin{pmatrix} 1 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & k & -1 \end{pmatrix}\right\} =: \mathcal{S}''$ is a $\mathbb{Z} + \{0,1\}$ system for $C^{370,k}$ that consists of exactly 2 elements from $\left(\mathbb{Z}^2 \times \mathbb{Z}\right)^T$.*

Before we prove Theorem 370, we first have a look at how we use the statement of Theorem 370 to construct a polyhedron that can only be represented by a "large" TDI system $Ax \leq b$ if one demands that either A or b is integral:

Definition 371. *For $k \in \mathbb{Z}_{\geq 1}$, define*

$$P^{371,k} := \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\} + \operatorname{cone}\left\{\begin{pmatrix} -k \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right\} = P^{\leq}\left(\begin{pmatrix} 1 & 0 \\ 1 & k \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) \subseteq \mathbb{R}^2.$$

Clearly, for every $k \in \mathbb{Z}_{\geq 1}$, the polyhedron $P^{371,k}$ is integral, but on the other hand, we have:

Theorem 372. *Let $k \in \mathbb{Z}_{\geq 1}$ and let $Ax \leq b$ be a TDI system for $P^{371,k}$ such that either A or b is integral. Then $\left(\begin{array}{c|c} A & -b \end{array}\right)$ contains the rows*

$$\dot{\bigcup_{i=0}^{k}} \left\{ \left(\begin{array}{cc|c} 1 & i & -1 \end{array}\right) \right\}.$$

In particular, $\left(\begin{array}{c|c} A & -b \end{array}\right)$ has at least $k+1$ rows. On the other hand,

- $$\underbrace{\begin{pmatrix} 1 & 0 \\ \vdots & \vdots \\ 1 & k \end{pmatrix}}_{=:A^{372,k}} x \leq \underbrace{\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}}_{=:b^{372,k}}$$

  is a TD$(I \cap \mathbb{Z}) + \{0,1\}$ system (and thus TDI system) for $P^{371,k}$ with both an integral left-hand and right-hand side that consists of exactly $k+1$ rows,

- $$\underbrace{\begin{pmatrix} 1 & 0 \\ 1 & k \end{pmatrix}}_{=:A'^{372,k}} x \leq \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{=:b'^{372,k}}$$

  is a TD$\mathbb{Z}+\{0,1\}$ system for $P^{371,k}$ with an integral left-hand and right-hand side that consists of exactly 2 rows.

Remark 373. *The restrictions*

- *$\mathcal{S} \subseteq \mathbb{Z}^2 \times \mathbb{Q}$ or $\mathcal{S} \subseteq \mathbb{Q}^2 \times \mathbb{Z}$ in Theorem 370,*
- *A or b is integral in Theorem 372*

are necessary for the respective statements to hold, since one can easily check that

- $\left\{ \begin{pmatrix} \frac{1}{k} & 0 & -\frac{1}{k} \end{pmatrix}, \begin{pmatrix} \frac{1}{k} & 1 & -\frac{1}{k} \end{pmatrix} \right\}$ *is an* (icone $\cap \mathbb{Z}) + \{0,1\}$ *system (and thus an* icone *system) for $C^{370,k}$,*
- $\begin{pmatrix} \frac{1}{k} & 0 \\ \frac{1}{k} & 1 \end{pmatrix} x \leq \begin{pmatrix} \frac{1}{k} \\ \frac{1}{k} \end{pmatrix}$ *is a TD$(I \cap \mathbb{Z}) + \{0,1\}$ system (and thus a TDI system) for $P^{371,k}$.*

Remark 374. *Let $k \in \mathbb{Z}_{\geq 1}$. The polyhedron*

$$P^{374,k} := \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \operatorname{cone}\left\{ \begin{pmatrix} -k \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\} = P^{\leq}\left(\begin{pmatrix} 1 & 0 \\ 1 & k \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) \subseteq \mathbb{R}^2,$$

which is simply an integral translate of $P^{371,k}$, can be represented by a TDI system with two inequalities and an integral right-hand side:

$$P^{374,k} = P^{\leq}\left(\begin{pmatrix} \frac{1}{k} & 0 \\ \frac{1}{k} & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right).$$

Thus, a "short" integrality proof for $P^{371,k}$ via TDI systems is that $P^{371,k}$ is an integral translate of $P^{374,k}$, which can be represented by a "small" TDI system with an integral right-hand side.

Now for the proofs of Theorem 370 and Theorem 372:

Proof. (Theorem 370)

For $\mathcal{S}' \subseteq \mathcal{S}$: Let $i \in \{0, \ldots, k\}$. Consider the vector $c^i := \begin{pmatrix} 1 & i & -1 \end{pmatrix} \in C^{370,k}$. We have

$$\forall c \in \left(C^{370,k} \cap \left((\mathbb{Z}^2 \times \mathbb{Q})^T \cup (\mathbb{Q}^2 \times \mathbb{Z})^T \right) \right) \setminus \left\{ \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \right\} : c_1 \geq 1 \wedge c_3 \leq -1.$$

From this, we conclude $c^i \in \operatorname{icone} \mathcal{S} \Rightarrow c^i \in \mathcal{S}$.

$\mathcal{S}'$ is an (icone $\cap \mathbb{Z}) + \{0,1\}$ system: Clearly, $\operatorname{cone} \mathcal{S}' = C^{370,k}$. The rest is a consequence of Theorem 365.

$\mathcal{S}''$ is a $\mathbb{Z}+\{0,1\}$ system: Clearly, cone $\mathcal{S}'' = C^{370,k}$. Let $c \in C^{370,k} \cap \left(\mathbb{Z}^2 \times \mathbb{Q}\right)^T$. Then $c_3 = -c_1$ and thus $c \in C^{370,k} \cap \left(\mathbb{Z}^2 \times \mathbb{Z}\right)^T \subseteq C^{370,k} \cap \left(\mathbb{Q}^2 \times \mathbb{Z}\right)^T$. So

$$c = \underbrace{c}_{\in C^{370,k} \cap (\mathbb{Q}^2 \times \mathbb{Z})^T} + \underbrace{\begin{pmatrix} (0^2)^T & 0 \end{pmatrix}}_{\in \mathcal{S}'' \cup \left\{ \begin{pmatrix} (0^2)^T & 0 \end{pmatrix} \right\}} .$$

□

Proof. (Theorem 372)

For the first statement: By Theorem 368, we have to show that the rows of $\begin{pmatrix} A & -b \end{pmatrix}$ that are active in $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (the only minimal face of $P^{371,k}$) form an icone system of

$$\operatorname{cone}\left\{ \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & k & -1 \end{pmatrix} \right\} = C^{370,k}.$$

By Theorem 370, every icone system for $C^{370,k}$ that consists of vectors from $\left(\mathbb{Z}^2 \times \mathbb{Q}\right)^T$ or $\left(\mathbb{Q}^2 \times \mathbb{Z}\right)^T$, has to contain

$$\dot{\bigcup_{i=0}^{k}} \left\{ \begin{pmatrix} 1 & i \mid -1 \end{pmatrix} \right\}.$$

$A^{372,k}x \le b^{372,k}$ is a TD$(I \cap \mathbb{Z})+\{0,1\}$ system for $P^{371,k}$: Clearly, all the rows of $A^{372,k}x \le b^{372,k}$ are active in $\{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\}$, the only minimal face of P^{371}. By Theorem 370, the rows of $\begin{pmatrix} A^{372} & -b^{372} \end{pmatrix}$ form a $\mathbb{Z}+$ icone system. Thus, we conclude the statement using Theorem 368.

$A'^{372,k}x \le b'^{372,k}$ is a TD$\mathbb{Z}+\{0,1\}$ system for $P^{371,k}$: For this statement, we present two different arguments:

- For every $k \in \mathbb{Z}_{\ge 1}$, the polyhedron $P^{371,k}$ is integral. By Theorem 354, we thus know that $A'^{372,k}x \le b'^{372,k}$ is a TD$\mathbb{Z}+\{0,1\}$ system for $P^{371,k}$.
- By Theorem 370, the rows of $\begin{pmatrix} A'^{372} & -b'^{372} \end{pmatrix}$ form a $\mathbb{Z}+\{0,1\}$ system. All rows of $A'^{372,k}\,(\cdot) \le b'^{372,k}$ are active in $\{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\}$, the only minimal face of P^{371}. Thus, we conclude the statement from Theorem 368.

□

7.6.2. TD$\mathbb{Z}+\{0,1\}$ systems and $\mathbb{Z}+\{0,1\}$ systems

Theorem 375. *For $k \in \mathbb{Z}_{\ge 2}$, define*

$$C^{375,k} := \operatorname{cone}\left\{ \begin{pmatrix} 1 & -\frac{1}{k} \end{pmatrix} \right\}.$$

Let $\mathcal{S}$ be an $\mathbb{Z}+\{0,1\}$ system for $C^{375,k}$. Then

$$\underbrace{\dot{\bigcup_{i=1}^{k-1}} \left\{ \begin{pmatrix} i & -\frac{i}{k} \end{pmatrix} \right\}}_{=:\mathcal{S}'} \subseteq \mathcal{S}.$$

In particular, $|\mathcal{S}| \ge k-1$. On the other hand,

- *$\mathcal{S}'$ is an* (icone $\cap\mathbb{Z}$) $+ \{0,1\}$ *system for $C^{375,k}$ that consists of exactly $k-1$ elements from $(\mathbb{Z} \times \mathbb{Q})^T$,*
- *$\left\{ \begin{pmatrix} 1 & -\frac{1}{k} \end{pmatrix} \right\} =: \mathcal{S}''$ is an* icone *system for $C^{375,k}$ that consists of exactly* 1 *element from $(\mathbb{Z} \times \mathbb{Q})^T$.*

Again, before we prove Theorem 375, let us have a look at how the construction behind $C^{375,k}$ can be used to construct a polyhedron that requires a large TD$\mathbb{Z}+\{0,1\}$ system to describe:

Theorem 376. *For $k \in \mathbb{Z}_{\geq 2}$, define*

$$P^{376,k} := P^{\leq}\left(\left(\begin{array}{c} 1 \end{array}\right), \left(\begin{array}{c} \frac{1}{k} \end{array}\right)\right) \subseteq \mathbb{R}^1.$$

Then every TD$\mathbb{Z}$ + $\{0,1\}$ system $Ax \leq b$ for $P^{376,k}$ must contain

$$\dot{\bigcup_{i=1}^{k-1}} \left\{\left(\begin{array}{cc} i & -\frac{i}{k} \end{array}\right)\right\}$$

as rows of $\left(\begin{array}{cc} A & -b \end{array}\right)$. On the other hand,

- $$\underbrace{\left(\begin{array}{c} 1 \\ \vdots \\ k-1 \end{array}\right)}_{=:A^{376,k}} x \leq \underbrace{\left(\begin{array}{c} \frac{1}{k} \\ \vdots \\ \frac{k-1}{k} \end{array}\right)}_{=:b^{376,k}}$$

  is a TD$(I \cap \mathbb{Z}) + \{0,1\}$ system for $P^{376,k}$ that has an integral left-hand side and consists of exactly $k-1$ rows,

- $$\underbrace{\left(\begin{array}{c} 1 \end{array}\right)}_{=:A'^{376,k}} x \leq \underbrace{\left(\begin{array}{c} \frac{1}{k} \end{array}\right)}_{=:b'^{376,k}}$$

  is a TDI system for $P^{376,k}$ that has an integral left-hand side and consists of exactly 1 row.

Now for the proofs of Theorem 375 and Theorem 376:

Proof. (Theorem 375)

For $\mathcal{S}' \subseteq \mathcal{S}$: Let $i \in \{1, \ldots, k-1\}$. Then, clearly, $c := \left(\begin{array}{cc} i & -\frac{i}{k} \end{array}\right) \in C^{375,k} \cap (\mathbb{Z} \times \mathbb{R})^T$. Assume $c = c^1 + c^2$, where

$$c^1 \in C \cap (\mathbb{Q} \times \mathbb{Z})^T,$$
$$c^2 \in \mathcal{S} \cup \left\{\left(\begin{array}{cc} 0 & 0 \end{array}\right)\right\} \subseteq C^{375,k}.$$

Since $x \in C^{375,k} \Rightarrow x_2 \leq 0$, we conclude $c^1_2 = 0$. Thus, $c^1 = \left(\begin{array}{cc} 0 & 0 \end{array}\right)$ and we have $c^2 = c \in \mathcal{S}$.

$\mathcal{S}'$ is an (icone$\cap\mathbb{Z}$) + $\{0,1\}$ **system:** Consider that $C^{375,k} = \operatorname{cone}\left\{\left(\begin{array}{cc} k & -1 \end{array}\right)\right\}$. Then the statement is a consequence of Theorem 365.

$\mathcal{S}''$ is an icone **system:** Let $c \in C^{375,k} \cap (\mathbb{Z} \times \mathbb{R})^T$. Then $c = \underbrace{c_1}_{\in \mathbb{Z}} \underbrace{\left(\begin{array}{cc} 1 & -\frac{1}{k} \end{array}\right)}_{\in \mathcal{S}''}$. □

Proof. (Theorem 376)

For the first statement: By Theorem 368, we have to show that the rows of $\left(\begin{array}{cc} A & -b \end{array}\right)$ that are active in $\left(\begin{array}{c} 1 \end{array}\right)$ (the only minimal face of $P^{376,k}$) form a $\mathbb{Z} + \{0,1\}$ system of $\operatorname{cone}\left\{\left(\begin{array}{cc} 1 & -\frac{1}{k} \end{array}\right)\right\} = C^{375,k}$. By Theorem 375, any such system has to contain the stated vectors.

$A^{376,k}x \leq b^{376,k}$ is a TD$(I \cap \mathbb{Z}) + \{0,1\}$ system for $P^{376,k}$: Clearly, all the rows of $A^{376,k}x \leq b^{376,k}$ are active in $\left(\begin{array}{c} \frac{1}{k} \end{array}\right)$, the only minimal face of P^{376}. By Theorem 375, the rows of $\left(\begin{array}{cc} A^{376} & -b^{376} \end{array}\right)$ form an icone system. Thus, the statement is a consequence of Theorem 368.

$A'^{376,k}x \leq b'^{376,k}$ is a TDI system for $P^{376,k}$: By Theorem 375, the rows of $\left(\begin{array}{cc} A'^{376} & -b'^{376} \end{array}\right)$ form an icone system. All rows of $A'^{376,k}(\cdot) \leq b'^{376,k}$ are active in $\left(\begin{array}{c} \frac{1}{k} \end{array}\right)$, the only minimal face of P^{376}. Thus, the statement is a consequence of Theorem 368. □

7.6.3. Comparing all kinds of systems of dual integrality

We now come up with an example where all of the presented systems of dual integrality can be of different sizes (with respect to the number of ineuqalities):

Definition 377. *For $k_1 \in \mathbb{Z}_{\geq 1}$ and $k_2 \in \mathbb{Z}_{\geq 2}$, define*

$$\begin{aligned} P^{377,k_1,k_2} &:= \operatorname{conv}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -\frac{1}{k_2} \end{pmatrix}\right\} + \operatorname{cone}\left\{\begin{pmatrix} -k_1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right\} \\ &= P^{\leq}\left(\begin{pmatrix} 1 & 0 \\ 1 & k_1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ \frac{1}{k_2} \end{pmatrix}\right) \\ &\subseteq \mathbb{R}^2. \end{aligned}$$

Before we state and show Theorem 379, we show a small proposition:

Proposition 378. *For $k_2 \in \mathbb{Z}_{\geq 2}$, let $C := \operatorname{cone}\left\{\begin{pmatrix} 1 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & -\frac{1}{k_2} \end{pmatrix}\right\}$. Then:*

1. *Any* icone *system $\mathcal{S}$ for C that consists of vectors from $\left(\mathbb{Z}^2 \times \mathbb{R}\right)^T$ contains*

$$\begin{pmatrix} 1 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & -\frac{1}{k_2} \end{pmatrix}.$$

2. *Any $\mathbb{Z} + \{0,1\}$ system $\mathcal{S}$ for C that consists of vectors from $\left(\mathbb{Z}^2 \times \mathbb{R}\right)^T$ contains*

$$\dot{\bigcup_{i=1}^{k_2-1}} \left\{\begin{pmatrix} 0 & -i \mid -\frac{i}{k_2} \end{pmatrix}\right\}.$$

Proof.

For 1: Any vector $c \in C \cap \left(\mathbb{Z}^2 \times \mathbb{R}\right)^T$ satisfies $c_1 \geq 0$ and $c_2 \leq 0$. Thus, we have

$$\begin{pmatrix} 1 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & -\frac{1}{k_2} \end{pmatrix} \in \operatorname{icone} \mathcal{S} \Rightarrow \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & -\frac{1}{k_2} \end{pmatrix} \in \mathcal{S}.$$

For 2:

$$\dot{\bigcup_{i=1}^{k_2-1}} \left\{\begin{pmatrix} 0 & -i \mid -\frac{i}{k_2} \end{pmatrix}\right\} \subseteq \mathcal{S}$$

is shown similarly to Theorem 375. □

Theorem 379. *Let $k_1 \in \mathbb{Z}_{\geq 1}$, $k_2 \in \mathbb{Z}_{\geq 2}$. Then:*

1. $$\begin{pmatrix} 1 & 0 \\ 1 & k_1 \\ 0 & -1 \end{pmatrix} x \leq \begin{pmatrix} 1 \\ 1 \\ \frac{1}{k_2} \end{pmatrix}$$

   is a TD$\mathbb{Z} + I$ system for P^{377,k_1,k_2} with an integral left-hand side that consists of 3 rows.

2. *If $Ax \leq b$ is a TDI system for P^{377,k_1,k_2} where A is integral, $\begin{pmatrix} A \mid -b \end{pmatrix}$ contains the following $k_1 + 2$ rows:*

$$\left(\dot{\bigcup_{i=0}^{k_1}} \left\{\begin{pmatrix} 1 & i \mid -1 \end{pmatrix}\right\}\right) \dot\cup \left\{\begin{pmatrix} 0 & -1 \mid -\frac{1}{k_2} \end{pmatrix}\right\}.$$

   On the other hand,

$$\begin{pmatrix} 1 & 0 \\ \vdots & \vdots \\ 1 & k_1 \\ 0 & -1 \end{pmatrix} x \leq \begin{pmatrix} 1 \\ \vdots \\ 1 \\ \frac{1}{k_2} \end{pmatrix}$$

   is a TDI system for P^{377,k_1,k_2} with an integral left-hand side that consists of $k_1 + 2$ rows.

3. *If $Ax \leq b$ is a TD$\mathbb{Z}+\{0,1\}$ system for P^{377,k_1,k_2} where A is integral, $\begin{pmatrix} A \mid -b \end{pmatrix}$ contains the following $k_2 - 1$ rows:*

$$\dot{\bigcup_{i=1}^{k_2-1}} \left\{ \begin{pmatrix} 0 & -i \mid -\frac{i}{k_2} \end{pmatrix} \right\}.$$

On the other hand,

$$\begin{pmatrix} 1 & 0 \\ 1 & k_1 \\ 0 & -1 \\ \vdots & \vdots \\ 0 & -(k_2-1) \end{pmatrix} x \leq \begin{pmatrix} 1 \\ 1 \\ \frac{1}{k_2} \\ \vdots \\ \frac{k_2-1}{k_2} \end{pmatrix}$$

is a TD$\mathbb{Z}+\{0,1\}$ system for P^{377,k_1,k_2} with an integral left-hand side that consists of k_2+1 rows.

4. *If $Ax \leq b$ is a TD$(I \cap \mathbb{Z}) + \{0,1\}$ system for P^{377,k_1,k_2} where A is integral, $\begin{pmatrix} A \mid -b \end{pmatrix}$ contains the following $k_1 + k_2$ rows:*

$$\left(\dot{\bigcup_{i=0}^{k_1}} \left\{ \begin{pmatrix} 1 & i \mid -1 \end{pmatrix} \right\} \right) \dot\cup \left(\dot{\bigcup_{i=1}^{k_2-1}} \left\{ \begin{pmatrix} 0 & -i \mid -\frac{i}{k_2} \end{pmatrix} \right\} \right).$$

On the other hand,

$$\begin{pmatrix} 1 & 0 \\ \vdots & \vdots \\ 1 & k_1 \\ 0 & -1 \\ \vdots & \vdots \\ 0 & -(k_2-1) \end{pmatrix} x \leq \begin{pmatrix} 1 \\ \vdots \\ 1 \\ \frac{1}{k_2} \\ \vdots \\ \frac{k_2-1}{k_2} \end{pmatrix}$$

is a TD$(I \cap \mathbb{Z})+\{0,1\}$ system for P^{377,k_1,k_2} with an integral left-hand side that consists of k_1+k_2 rows.

Proof.

For 1: We have to show that

$$\left\{ \begin{pmatrix} 1 & 0 \mid -1 \end{pmatrix}, \begin{pmatrix} 1 & k_1 \mid -1 \end{pmatrix} \right\} \tag{7.10}$$

(the subset of the rows of $\begin{pmatrix} A \mid -b \end{pmatrix}$ that are active in $\binom{1}{0}$) and

$$\left\{ \begin{pmatrix} 1 & 0 \mid -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \mid -\frac{1}{k_1} \end{pmatrix} \right\} \tag{7.11}$$

(the subset of the rows of $\begin{pmatrix} A \mid -b \end{pmatrix}$ that are active in $\begin{pmatrix} 1 \\ -\frac{1}{k_2} \end{pmatrix}$), respectively, each form a $\mathbb{Z}+$icone system. By Theorem 370, (7.10) forms a $\mathbb{Z}+\{0,1\}$ system, thus a $\mathbb{Z}+$icone system. By Theorem 366, (7.11) forms an icone system, thus a $\mathbb{Z}+$icone system.

For 2:

For the first statement: Since $Ax \leq b$ is a TDI system for P^{377,k_1,k_2}, the rows of $\begin{pmatrix} A \mid -b \end{pmatrix}$ that are active in $\binom{1}{0}$ form an icone system. By Theorem 370, any icone system for

$$\operatorname{cone}\left\{ \begin{pmatrix} 1 & 0 \mid -1 \end{pmatrix}, \begin{pmatrix} 1 & k_1 \mid -1 \end{pmatrix} \right\}$$

that consists of vectors from $\left(\mathbb{Z}^2 \times \mathbb{R}\right)^T$ has to contain

$$\dot{\bigcup_{i=0}^{k_1}} \left\{ \begin{pmatrix} 1 & i \mid -1 \end{pmatrix} \right\}.$$

On the other hand, we know from Proposition 378 that any icone system for

$$\operatorname{cone}\left\{\begin{pmatrix} 1 & 0 \mid -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \mid -\frac{1}{k_2} \end{pmatrix}\right\}$$

(cone generated by the rows of $\begin{pmatrix} A \mid -b \end{pmatrix}$ that are active in $\begin{pmatrix} 1 \\ -\frac{1}{k_2} \end{pmatrix}$) has to contain

$$\left\{\begin{pmatrix} 1 & 0 \mid -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \mid -\frac{1}{k_2} \end{pmatrix}\right\}.$$

For the second statement: We have to show that

$$\dot{\bigcup_{i=0}^{k_1}} \left\{\begin{pmatrix} 1 & i \mid -1 \end{pmatrix}\right\} \tag{7.12}$$

(the subset of the rows of $\begin{pmatrix} A \mid -b \end{pmatrix}$ that are active in $\binom{1}{0}$) and

$$\left\{\begin{pmatrix} 1 & 0 \mid -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \mid -\frac{1}{k_2} \end{pmatrix}\right\} \tag{7.13}$$

(the subset of the rows of $\begin{pmatrix} A \mid -b \end{pmatrix}$ that are active in $\begin{pmatrix} 1 \\ -\frac{1}{k_2} \end{pmatrix}$), respectively, each form an icone system. By Theorem 370, (7.12) forms an icone system. By Theorem 366, (7.13) forms an icone system.

For 3:

For the first statement: Since $Ax \leq b$ is a TD$\mathbb{Z} + \{0,1\}$ system for P^{377,k_1,k_2}, the rows of $\begin{pmatrix} A \mid -b \end{pmatrix}$ that are active in $\begin{pmatrix} 1 \\ -\frac{1}{k_2} \end{pmatrix}$ form a $\mathbb{Z} + \{0,1\}$ system. By Proposition 378, any $\mathbb{Z} + \{0,1\}$ system for

$$\operatorname{cone}\left\{\begin{pmatrix} 1 & 0 \mid -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \mid -\frac{1}{k_2} \end{pmatrix}\right\}$$

has to contain

$$\left\{\begin{pmatrix} 1 & 0 \mid -1 \end{pmatrix}, \begin{pmatrix} 1 & k_1 \mid -1 \end{pmatrix}\right\}.$$

For the second statement: We have to show that

$$\left\{\begin{pmatrix} 1 & 0 \mid -1 \end{pmatrix}, \begin{pmatrix} 1 & k_1 \mid -1 \end{pmatrix}\right\} \tag{7.14}$$

(the subset of the rows of $\begin{pmatrix} A \mid -b \end{pmatrix}$ that are active in $\binom{1}{0}$) and

$$\left\{\begin{pmatrix} 1 & 0 \mid -1 \end{pmatrix}\right\} \dot\cup \dot{\bigcup_{i=1}^{k_1-1}} \left\{\begin{pmatrix} 0 & -i \mid -\frac{i}{k_1} \end{pmatrix}\right\} \tag{7.15}$$

(the subset of the rows of $\begin{pmatrix} A \mid -b \end{pmatrix}$ that are active in $\begin{pmatrix} 1 \\ -\frac{1}{k_2} \end{pmatrix}$), respectively, each form a $\mathbb{Z} + \{0,1\}$ system. Using Theorem 365, it is easy to check that (7.15) forms an (icone$\cap\mathbb{Z}$)$+\{0,1\}$ system (thus $\mathbb{Z}+\{0,1\}$ system). So, what remains is to show that (7.14) forms a $\mathbb{Z} + \{0,1\}$ system. Let

$$c \in \underbrace{\left(\operatorname{cone}\left\{\begin{pmatrix} 1 & 0 \mid -1 \end{pmatrix}, \begin{pmatrix} 1 & k_1 \mid -1 \end{pmatrix}\right\}\right)}_{=:C} \cap (\mathbb{Z}^m \times \mathbb{R}).$$

Then, clearly, $c = c^1 + c^2$, where

$$\begin{aligned} c^1 &:= c \in C \cap (\mathbb{Q}^m \times \mathbb{Z}), \\ c^2 &:= \begin{pmatrix} 0 & 0 \mid 0 \end{pmatrix} \in \left\{\begin{pmatrix} 0 & 0 \mid 0 \end{pmatrix}\right\} \dot\cup \left\{\begin{pmatrix} 1 & 0 \mid -1 \end{pmatrix}, \begin{pmatrix} 1 & k_1 \mid -1 \end{pmatrix}\right\}. \end{aligned}$$

For 4:

For the first statement: Since $Ax \leq b$ is a TD$(I \cap \mathbb{Z})+\{0,1\}$ system for P^{377,k_1,k_2}, the rows of $\begin{pmatrix} A \mid -b \end{pmatrix}$ that are active in $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -\frac{1}{k_2} \end{pmatrix}$, respectively, each form an (icone$\cap\mathbb{Z}$) $+ \{0,1\}$ system. By Theorem

370, any icone system (thus (icone $\cap \mathbb{Z}$) + $\{0,1\}$ system) for

$$\operatorname{cone}\left\{\begin{pmatrix} 1 & 0 & | & -1 \end{pmatrix}, \begin{pmatrix} 1 & k_1 & | & -1 \end{pmatrix}\right\}$$

that consists of vectors from $\left(\mathbb{Z}^2 \times \mathbb{R}\right)^T$ has to contain

$$\dot{\bigcup_{i=0}^{k_1}} \left\{\begin{pmatrix} 1 & i & | & -1 \end{pmatrix}\right\}.$$

On the other hand, by Proposition 378, any $\mathbb{Z} + \{0,1\}$ system (thus (icone $\cap \mathbb{Z}$) + $\{0,1\}$ system) for

$$\operatorname{cone}\left\{\begin{pmatrix} 1 & 0 & | & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & | & -\frac{1}{k_2} \end{pmatrix}\right\}$$

has to contain

$$\dot{\bigcup_{i=1}^{k_2-1}} \left\{\begin{pmatrix} 0 & -i & | & -\frac{i}{k_2} \end{pmatrix}\right\}.$$

For the second statement: We have to show that

$$\dot{\bigcup_{i=0}^{k_1}} \left\{\begin{pmatrix} 1 & i & | & -1 \end{pmatrix}\right\} \tag{7.16}$$

(the subset of the rows of $\begin{pmatrix} A & | & -b \end{pmatrix}$ that are active in $\binom{1}{0}$) and

$$\left\{\begin{pmatrix} 1 & 0 & | & -1 \end{pmatrix}\right\} \dot\cup \dot{\bigcup_{i=1}^{k_2-1}} \left\{\begin{pmatrix} 0 & -i & | & -\frac{i}{k_2} \end{pmatrix}\right\} \tag{7.17}$$

(the subset of the rows of $\begin{pmatrix} A & | & -b \end{pmatrix}$ that are active in $\begin{pmatrix} 1 \\ -\frac{1}{k_2} \end{pmatrix}$), respectively, each form an (icone $\cap \mathbb{Z}$) + $\{0,1\}$ system. By Theorem 370, (7.16) forms an (icone $\cap \mathbb{Z}$) + $\{0,1\}$ system. Using Theorem 365, it is easy to check that (7.17) forms an (icone $\cap \mathbb{Z}$) + $\{0,1\}$ system. □

7.6.4. Polyhedra that require a large TD$\mathbb{Z} + I$ system

In this section, we prove that there exist polyhedra in $\mathbb{R}^2$ which are defined by just two rational inequalities and require large TD$\mathbb{Z} + I$ systems (the weakest type of system that we considered in section 7.2) $Ax \leq b$ if one demands that A is integral. This result becomes important in section 8.2.3.3, but since the statement and its proof are very related to the other results of section 7.6, we state and prove it here.

Definition 380. *For $k \in \mathbb{Z}_{\geq 2}$, define*

$$P^{380,k} := \left\{\begin{pmatrix} \frac{k}{2} \\ \frac{1}{2} \end{pmatrix}\right\} + \operatorname{cone}\left\{\begin{pmatrix} -\frac{k}{2} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\frac{k}{2} \\ -\frac{1}{2} \end{pmatrix}\right\} = P^{\leq}\left(\begin{pmatrix} 1 & k \\ 1 & -k \end{pmatrix}, \begin{pmatrix} k \\ 0 \end{pmatrix}\right) \subseteq \mathbb{R}^2.$$

Theorem 381. *Let $k \in \mathbb{Z}_{\geq 2}$. Then:*

1. *Let $Ax \leq b$, where $A \in \mathbb{Z}^{l \times 2}$ and $b \in \mathbb{R}^l$ ($l \in \mathbb{Z}_{\geq 0}$), be a TDI system such that $P^{\leq}(A, b) = P^{380,k}$. Then*

$$\left\{\begin{pmatrix} 1 & i & | & -\frac{k+i}{2} \end{pmatrix} : i \in \{-k, \dots, k\}\right\}$$

are rows of $\begin{pmatrix} A & | & -b \end{pmatrix}$*, i.e.* $Ax \leq b$ *consists of at least* $2k+1$ *rows. On the other hand,*

$$\begin{pmatrix} 1 & k \\ 1 & k-1 \\ 1 & k-2 \\ \vdots & \vdots \\ 1 & -(k-2) \\ 1 & -(k-1) \\ 1 & -k \end{pmatrix} x \leq \begin{pmatrix} k \\ k-\frac{1}{2} \\ k-1 \\ \vdots \\ 1 \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

is a $TD(I \cap \mathbb{Z}) + \{0,1\}$ *system (and thus TDI system) for* $P^{380,k}$ *with an integral left-hand side that consists of* $2k+1$ *rows.*

2. *Let* $Ax \leq b$*, where* $A \in \mathbb{Z}^{l \times 2}$ *and* $b \in \mathbb{R}^l$ *(*$l \in \mathbb{Z}_{\geq 0}$*), be a* $TD\mathbb{Z}+I$ *system such that* $P^{\leq}(A,b) = P^{380,k}$*. Then*

$$\left\{ \begin{pmatrix} 1 & i & | & -\frac{k+i}{2} \end{pmatrix} : i \in \{-(k-1), \dots, k-1\}, k+i \text{ odd} \right\}$$

are rows of $\begin{pmatrix} A & | & -b \end{pmatrix}$*, i.e.* $Ax \leq b$ *consists of at least* k *rows. On the other hand,*

$$\begin{pmatrix} 1 & k \\ 1 & -k \\ 1 & k-1 \\ 1 & k-3 \\ \vdots & \vdots \\ 1 & -(k-3) \\ 1 & -(k-1) \end{pmatrix} x \leq \begin{pmatrix} k \\ 0 \\ k-\frac{1}{2} \\ k-\frac{3}{2} \\ \vdots \\ \frac{3}{2} \\ \frac{1}{2} \end{pmatrix}$$

is a $TD\mathbb{Z}+\{0,1\}$ *system (and thus* $TD\mathbb{Z}+I$ *system) for* $P^{380,k}$ *that consists of* $k+2$ *rows.*

Proof. Define $C := \operatorname{cone}\left\{ \begin{pmatrix} 1 & k & | & -k \end{pmatrix}, \begin{pmatrix} 1 & -k & | & 0 \end{pmatrix} \right\}$. It is easy to verify that

$$\forall c \in C \cap \left(\mathbb{Z}^2 \times \mathbb{R}\right)^T : c_1 = 0 \Rightarrow c = \begin{pmatrix} 0 & 0 & | & 0 \end{pmatrix}, \tag{7.18}$$

$$\forall c \in \mathbb{R}^2 \times \mathbb{R} : c \in C \Leftrightarrow \left(c_1 \geq 0 \wedge -kc_1 \leq c_2 \leq kc_1 \wedge c_3 = -\frac{kc_1 + c_2}{2} \right) \tag{7.19}$$

hold.

For 1:

For the first statement: By Theorem 368, the rows of $\begin{pmatrix} A & | & -b \end{pmatrix}$ that are active in $\begin{pmatrix} \frac{k}{2} \\ \frac{1}{2} \end{pmatrix}$ form an icone system of C. We denote these active rows of $\begin{pmatrix} A & | & -b \end{pmatrix}$ by $\mathcal{S}$. Let $i \in \{-k, \dots, k\}$ be fixed and set

$$c := \begin{pmatrix} 1 & i & | & -\frac{k+i}{2} \end{pmatrix} \in C \cap \left(\mathbb{Z}^2 \times \mathbb{R}\right)^T.$$

From (7.18) and (7.19), we conclude that c cannot be represented by an integral conic combination of vectors from $C \cap \left(\mathbb{Z}^2 \times \mathbb{R}\right)^T \setminus \left\{ \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \right\}$. Thus, $c \in \mathcal{S}$.

For the second statement: Clearly, $P^{\leq}(A', b') = P^{380,k}$. We have to show that the rows of $\begin{pmatrix} A' & | & -b' \end{pmatrix}$ that are active in $\begin{pmatrix} \frac{k}{2} \\ \frac{1}{2} \end{pmatrix}$ (all rows) form an (icone$\cap\mathbb{Z}$)$+\{0,1\}$ system of C. From Theorem 365, we conclude that this is indeed the case.

For 2:

For the first statement: By Theorem 368, the rows of $\begin{pmatrix} A & | & -b \end{pmatrix}$ that are active in $\begin{pmatrix} \frac{k}{2} \\ \frac{1}{2} \end{pmatrix}$ form a $\mathbb{Z}+$icone system of C. We denote these active rows of $\begin{pmatrix} A & | & -b \end{pmatrix}$ by $\mathcal{S}$. Let $i \in \{-(k-1), \dots, k-1\}$ be fixed such that $k+i$ is odd and set

$$c := \begin{pmatrix} 1 & i \mid -\frac{k+i}{2} \end{pmatrix} \in C \cap \left(\mathbb{Z}^2 \times \mathbb{R}\right)^T.$$

We want to show that $c \in \mathcal{S}$.

Since $\mathcal{S}$ is a $\mathbb{Z}+$ icone system of C, we can represent c as $c = c^1 + c^2$, where $c^1 \in C \cap \left(\mathbb{Q}^2 \times \mathbb{Z}\right)^T$ and $c^2 \in \operatorname{icone} \mathcal{S}$. Since $\mathcal{S} \subseteq \left(\mathbb{Z}^2 \times \mathbb{R}\right)^T$, we have $c^1 \in C \cap \left(\mathbb{Z}^2 \times \mathbb{Z}\right)^T$. From $c^1 \in C$, we conclude $c^1_1 \geq 0$.

Case 1: $c^1_1 = 0$**:** From $c^1_1 = 0$, we obtain $c^1 = \left(0^2 \times 0\right)^T$; thus, $c = c^2 \in \operatorname{icone} \mathcal{S}$. Using (7.18) and (7.19), we conclude that $c^2 \in \mathcal{S}$.

Case 2: $c^1_1 = 1$**:** From the case assumption, we obtain $c^2_1 = 0$, which implies $c^2 = \begin{pmatrix} 0 & 0 \mid 0 \end{pmatrix}$. So, we have $c = c^1$. But this cannot happen, since

$$k+i \in 2\mathbb{Z}+1 \Rightarrow \frac{k+i}{2} \in \mathbb{Z} + \frac{1}{2} \Rightarrow c_3 = c^1_3 \in \mathbb{Z} + \frac{1}{2},$$

which is a contradiction to $c^1 \in C \cap \left(\mathbb{Z}^2 \times \mathbb{Z}\right)^T$.

Case 3: $c^1_1 \geq 2$**:** Since $c^2 \in C$, by (7.19), we have $c^2_1 \geq 0$. But on the other hand, we have

$$c^2_1 = c_1 - c^1_1 = 1 - c^1_1 \leq -1,$$

which is clearly a contradiction.

For the second statement: Clearly, $P^{\leq}(A'', b'') = P^{380,k}$. We have to show that the rows of $\begin{pmatrix} A \mid -b \end{pmatrix}$ that are active in $\begin{pmatrix} \frac{k}{2} \\ \frac{1}{2} \end{pmatrix}$ (all rows), which we denote by $\mathcal{S}$, form a $\mathbb{Z} + \{0,1\}$ system of C. Let

$$c \in C \cap \left(\mathbb{Z}^2 \times \mathbb{R}\right)^T.$$

Case 1: $c \in \left(\mathbb{Z}^2 \times \mathbb{Z}\right)^T$**:** Let

$$c^1 := c, \qquad c^2 := \begin{pmatrix} 0 & 0 \mid 0 \end{pmatrix}.$$

Indeed $c = c^1 + c^2$, where $c^1 \in C \cap \left(\mathbb{Q}^2 \times \mathbb{Z}\right)^T$ and $c^2 \in \mathcal{S} \cup \left\{\begin{pmatrix} 0 & 0 & 0 \end{pmatrix}\right\}$.

Case 2: $c \in \left(\mathbb{Z}^2 \times \left(\mathbb{Z} + \frac{1}{2}\right)\right)^T$**:** We clearly have $c_1 \geq 1$. We distinguish the following subcases:

a $c_1 = 1$,

b $c_1 \geq 2 \wedge k \in 2\mathbb{Z}+1 \wedge -k\left(c_1 - 1\right) < c_2 < k\left(c_1 - 1\right)$,

c $c_1 \geq 2 \wedge k \in 2\mathbb{Z} \wedge -k\left(c_1 - 1\right) < c_2 < k\left(c_1 - 1\right)$,

d $c_1 \geq 2 \wedge c_2 \geq k\left(c_1 - 1\right)$,

e $c_1 \geq 2 \wedge c_2 \leq -k\left(c_1 - 1\right)$.

For case a: It is easy to see that

$$c \in \dot{\bigcup_{\substack{i=-(k-1), \\ k+i \text{ odd}}}^{k-1}} \left\{\begin{pmatrix} 1 & i \mid -\frac{k+i}{2} \end{pmatrix}\right\} \subseteq \mathcal{S}.$$

Thus, $c = c^1 + c^2$, where

$$\begin{aligned} c^1 &:= \begin{pmatrix} 0 & 0 \mid 0 \end{pmatrix} \in C \cap \left(\mathbb{Q}^2 \times \mathbb{Z}\right)^T, \\ c^2 &:= c \in \mathcal{S} \subseteq \mathcal{S} \cup \left\{\begin{pmatrix} 0 & 0 \mid 0 \end{pmatrix}\right\}. \end{aligned}$$

For case b: Let $c = c^1 + c^2$, where

$$\begin{aligned} c^1 &:= \begin{pmatrix} c_1 - 1 & c_2 & c_3 + \frac{k}{2} \end{pmatrix} \in C \cap \left(\mathbb{Q}^2 \times \mathbb{Z}\right)^T, \\ c^2 &:= \begin{pmatrix} 1 & 0 & -\frac{k}{2} \end{pmatrix} \in \mathcal{S} \subseteq \mathcal{S} \cup \left\{\begin{pmatrix} 0 & 0 \mid 0 \end{pmatrix}\right\}. \end{aligned}$$

The statement $c^1 \in C$ is easy to check via (7.19): $c_1^1 \geq 0$ is clear, $-kc_1^1 \leq c_2^1 \leq kc_1^1$ holds by case assumption (even with strict inequalities). Finally, using $c \in C$ and (7.19), we obtain

$$c_3^1 = c_3 + \frac{k}{2} = -\frac{kc_1 + c_2}{2} + \frac{k}{2} = -\frac{k\,(c_1 - 1) + c_2}{2} = -\frac{kc_1^1 + c_2^1}{2}.$$

For case c: Let $c = c^1 + c^2$, where

$$\begin{aligned} c^1 &:= \begin{pmatrix} c_1 - 1 & c_2 - 1 & c_3 + \frac{k+1}{2} \end{pmatrix} \in C \cap \left(\mathbb{Q}^2 \times \mathbb{Z}\right)^T, \\ c^2 &:= \begin{pmatrix} 1 & 1 & -\frac{k+1}{2} \end{pmatrix} \in \mathcal{S} \subseteq \mathcal{S} \cup \left\{\begin{pmatrix} 0 & 0 \mid 0 \end{pmatrix}\right\}. \end{aligned}$$

The statement $c^1 \in C$ is easy to check via (7.19): $c_1^1 \geq 0$ is clear, $-kc_1^1 \leq c_2^1 \leq kc_1^1$ holds by case assumption (even with strict inequalities). Finally, using $c \in C$ and (7.19), we obtain

$$c_3^1 = c_3 + \frac{k+1}{2} = -\frac{kc_1 + c_2}{2} + \frac{k+1}{2} = -\frac{k\,(c_1 - 1) + (c_2 - 1)}{2} = -\frac{kc_1^1 + c_2^1}{2}.$$

For case d: Let $c = c^1 + c^2$, where

$$\begin{aligned} c^1 &:= \begin{pmatrix} c_1 - 1 & k\,(c_1 - 1) & k\,(c_1 - 1) \end{pmatrix} \in C \cap \left(\mathbb{Q}^2 \times \mathbb{Z}\right)^T, \\ c^2 &:= \begin{pmatrix} 1 & c_2 - k\,(c_1 - 1) & c_3 - k\,(c_1 - 1) \end{pmatrix} \in \mathcal{S} \subseteq \mathcal{S} \cup \left\{\begin{pmatrix} 0 & 0 \mid 0 \end{pmatrix}\right\}. \end{aligned}$$

Note that $k\,(c_1 - 1) \leq c_2 \leq kc_1$; thus, $0 \leq c_2 - k\,(c_1 - 1) \leq k$.

For case e: Let $c = c^1 + c^2$, where

$$\begin{aligned} c^1 &:= \begin{pmatrix} c_1 - 1 & -k\,(c_1 - 1) & 0 \end{pmatrix} \in C \cap \left(\mathbb{Q}^2 \times \mathbb{Z}\right)^T, \\ c^2 &:= \begin{pmatrix} 1 & c_2 + k\,(c_1 - 1) & c_3 \end{pmatrix} \in \mathcal{S} \subseteq \mathcal{S} \cup \left\{\begin{pmatrix} 0 & 0 \mid 0 \end{pmatrix}\right\}. \end{aligned}$$

Note that $-kc_1 \leq c_2 \leq -k\,(c_1 - 1)$; thus, $-k \leq c_2 + k\,(c_1 - 1) \leq 0$. $\square$

8. $L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}/L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cuts, Chvátal-Gomory cuts and strong Chvátal-Gomory cuts

This chapter consists of two rather independent parts:

- In the first part (section 8.1), we consider the following classes of cutting planes:
  - (projected) Chvátal-Gomory cuts (cf. Definition 122),
  - dual (projected) Chvátal-Gomory cuts (which we define in Definition 382 and are sometimes used in the literature interchangeably with (projected) Chvátal-Gomory cuts because they are deeply related to each other for polyhedra (cf. Theorem 385)),
  - strong (projected) Chvátal-Gomory cuts (cf. Definition 384),
  - $L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts and
  - $L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cuts.

  We analyze in section 8.1.2 in what sense these classes are equivalent or not. In section 8.1.2.4, we give a summary of all these results. Concerning this, in particular cf. Theorem 391 for the equivalence results for the associated closure operators for polyhedra.

- In the second part (section 8.2), we consider the question of how one can compute the Chvátal-Gomory closure of a polyhedron with rational face normals:
  - In section 8.2.1, in particular Theorem 394, we consider that, when we want to compute the Chvátal-Gomory closure of a polyhedron $P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right) \subseteq \mathbb{R}^m \times \mathbb{R}^n$, it suffices to consider polyhedra that are described by a subset of the inequalities such that the cone spanned by the normal vectors of these inequalities forms an LP face cone (recall Definition 356). We remark that these results are in principle already known in the literature, though often only formulated for non-projected Chvátal-Gomory cuts (i.e. only pure integer case) and implicitly.
  - In section 8.2.2, we derive some abstract dominance relationships for dual projected Chvátal-Gomory cuts, which turn out to be important for an explicit characterization of the Chvátal-Gomory closure in the subsequent two sections (section 8.2.3 and section 8.2.4).
  - In section 8.2.3, we have a look at how TD$\mathbb{Z}+I$ systems with an integral left-hand side can be used for computing the Chvátal-Gomory closure (Theorem 398). We remark that this result is well-known for the weaker TDI systems (cf. [Sch86, Theorem 23.1; p. 340]). So, in the subsequent subsections, we analyze the size of TD$\mathbb{Z}+I$ systems for this purpose and ask whether the number of rows of such a TD$\mathbb{Z}+I$ system can be smaller than the size of a TDI system (section 8.2.3.2) and whether there is potential for future improvements (section 8.2.3.3).
  - In section 8.2.4, we give an alternative procedure for computing the projected Chvátal-Gomory closure of a polyhedron by reducing this problem to the problem of computing the Chvátal-Gomory closure of a polyhedron $P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)$ where the rows of $\begin{pmatrix} A & G & -b \end{pmatrix}$ form an LP face cone. This approach has the advantage that it also works in the mixed-integer case (projected Chvátal-Gomory closure) and can be generalized to the MIR closure. The latter topic is considered later on in section 9.2.3 of chapter 9 (Theorem 429).

8.1. Equivalences/non-equivalences

8.1.1. Definitions

For convenience, we restate Definition 122 and Definition 123, in which we define (projected) Chvátal-Gomory cuts and their closure:

Definition 122. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be arbitrary. Let $c \in (\mathbb{Z}^m)^T$ and let $c_0 \in \mathbb{R}$ (w.l.o.g. we can assume $c_0 \in \mathbb{R}\backslash\mathbb{Z}$) be such that*

$$P \subseteq P^{\leq}\left(\begin{pmatrix} c & (0^n)^T \end{pmatrix}, c_0\right).$$

Then the inequality

$$\begin{pmatrix} c & (0^n)^T \end{pmatrix}(\cdot) \leq \lfloor c_0 \rfloor$$

is called a ***projected Chvátal-Gomory cut for*** *P. If $n = 0$, we simply use the term* ***Chvátal-Gomory cut for*** *P.*

Definition 123. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be arbitrary. Then we define*

$$\mathrm{cl}_{pCG}(P) := P \cap \bigcap_{\substack{c \in (\mathbb{Z}^m)^T, c_0 \in \mathbb{R}: \\ P \subseteq P^{\leq}\left(\begin{pmatrix} c & (0^n)^T \end{pmatrix}, c_0\right)}} P^{\leq}\left(\begin{pmatrix} c & (0^n)^T \end{pmatrix}, \lfloor c_0 \rfloor\right)$$

as the ***projected Chvátal-Gomory closure of*** *P. If $n = 0$, we also use the term* ***Chvátal-Gomory closure of*** *P ($\mathrm{cl}_{CG}(P)$).*

We now give a kind of dual definition for (projected) Chvátal-Gomory cuts. We clearly state that the term "dual (projected) Chvátal-Gomory cut" is to our knowledge used nowhere else in the literature – we invented it to distinguish these cuts from the (projected) Chvátal-Gomory cuts that are defined in Definition 122.

Definition 382. *Let $A \in \mathbb{R}^{l \times m}$, $G \in \mathbb{R}^{l \times n}$ and $b \in \mathbb{R}^l$, where $l, m, n \in \mathbb{Z}_{\geq 0}$. Define*

$$\mathcal{M}_{CG}(A, G) := \left\{\mu \in \left(\mathbb{R}^l_{\geq 0}\right)^T : \mu \begin{pmatrix} A & G \end{pmatrix} \in (\mathbb{Z}^m \times 0^n)^T\right\}.$$

If $n = 0$, we also use the notation $\mathcal{M}_{CG}(A)$. Let $\mu \in \mathcal{M}_{CG}(A, G)$. Then we call the inequality

$$\underbrace{\mu(Ax + Gy - b)}_{= \mu Ax - \mu b} \leq -\operatorname{frac}(\mu b)$$

a ***dual projected Chvátal-Gomory cut for*** *$\binom{x}{y} \in P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)$* ***with respect to*** *A, G* ***and*** *b. We define*

$$f^{\leq, CG, \mu}_{A,G,b} : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R},$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \mu(Ax - b) + \operatorname{frac}(\mu b).$$

Definition 383. *Let A, G, b, m and n be as in Definition 382. We define*

$$\begin{aligned}\mathrm{cl}_{dpCG}(A, G, b) :=& P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right) \\ &\cap \bigcap_{\mu \in \mathcal{M}_{CG}(A,G)} \left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R}^n : f^{\leq 0, CG, \mu}_{A,G,b}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) \leq 0\right\}\end{aligned}$$

as the ***dual projected Chvátal-Gomory closure with respect to*** *A, G* ***and*** *b.*

Note that the definition of dual projected Chvátal-Gomory cuts in Definition 382 closely mirrors the definition of MIR cuts that we introduce in Definition 410 and the definition of the dual projected Chvátal-Gomory closure in Definition 383 closely mirrors the definition of the MIR closure that we introduce in Definition 411.

Next, we define "strong projected Chvátal-Gomory cuts" and their respective closure. This is also a term that we invented.

Definition 384. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ be arbitrary. Let $c \in (\mathbb{Z}^m)^T$ and $c_0 \in \mathbb{Z}$ be such that*

$$P \cap P^{\geq}\left(\begin{pmatrix} c & (0^n)^T \end{pmatrix}, c_0 + 1\right) = \emptyset.$$

Then we call the inequality $\begin{pmatrix} c & (0^n)^T \end{pmatrix}(\cdot) \leq c_0$ a ***strong projected Chvátal-Gomory cut for*** *P. If*

$n = 0$, we simply use the term ***strong Chvátal-Gomory cut for*** *P. We define*

$$\mathrm{cl}_{spCG}(P) := P \cap \bigcap_{\substack{c \in (\mathbb{Z}^m)^T, c_0 \in \mathbb{Z}: \\ P \cap P^{\geq}\left(\begin{pmatrix} c & (0^n)^T \end{pmatrix}, c_0+1\right) = \emptyset}} P^{\leq}\left(\begin{pmatrix} c & (0^n)^T \end{pmatrix}, c_0\right)$$

as the ***strong (projected) Chvátal-Gomory closure of*** *P.*

8.1.2. Equivalences and inclusions

8.1.2.1. Dual projected Chvátal-Gomory cuts vs projected Chvátal-Gomory cuts

In the following theorem, we show that projected Chvátal-Gomory cuts are "mostly identical" to dual projected Chvátal-Gomory cuts and their closures are identical:

Theorem 385. *Let $P = P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)$ be a polyhedron, where $A \in \mathbb{R}^{l \times m}$, $G \in \mathbb{R}^{l \times n}$ and $b \in \mathbb{R}^l$ ($l, m, n \in \mathbb{Z}_{\geq 0}$). Then:*

1. *Every dual projected Chvátal-Gomory cut (for P) with respect to A, G and b is a projected Chvátal-Gomory cut for P.*
2. *Let $\begin{pmatrix} c & (0^n)^T \end{pmatrix}(\cdot) \leq \lfloor c_0 \rfloor$ be a projected Chvátal-Gomory cut for P.*
   a) *If $P = \emptyset$, there exists a $\mu \in \mathcal{M}_{CG}(A, G)$ such that $f^{\leq, CG, \mu}_{A,G,b}(\cdot) = -1$.*
   b) *If $P \neq \emptyset$, but $\begin{pmatrix} c & (0^n)^T \end{pmatrix}(\cdot) \leq \lfloor c_0 \rfloor$ is already valid for P, there exists a $\mu \in \mathcal{M}_{CG}(A, G)$ such that $f^{\leq, CG, \mu}_{A,G,b}\left(\binom{x}{y}\right) = cx - c_0'$ ($\binom{x}{y} \in \mathbb{R}^m \times \mathbb{R}^n$), where $c_0' \in \mathbb{Z}$ and $c_0' \leq \lfloor c_0 \rfloor$, i.e. $f^{\leq, CG, \mu}_{A,G,b}(\cdot) \leq 0$ dominates $\begin{pmatrix} c & (0^n)^T \end{pmatrix}(\cdot) \leq \lfloor c_0 \rfloor$ absolutely.*
   c) *Otherwise ($P \neq \emptyset$ and $\begin{pmatrix} c & (0^n)^T \end{pmatrix}(\cdot) \leq \lfloor c_0 \rfloor$ is not valid for P), there exists a $\mu \in \mathcal{M}_{CG}(A, G)$ such that $f^{\leq, CG, \mu}_{A,G,b}\left(\binom{x}{y}\right) = cx - \lfloor c_0 \rfloor$ ($x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$).*

   In any case: every projected Chvátal-Gomory cut for P is dominated dominated absolutely by a dual projected Chvátal-Gomory cut with respect to A, G and b.

In particular, we have

$$\mathrm{cl}_{pCG}(P) = \mathrm{cl}_{dpCG}(A, G, b).$$

Remark 386. *We remark that in Theorem 385 in point 2a and point 2b, the cutting plane*

$$\begin{pmatrix} c & (0^n)^T \end{pmatrix}(\cdot) \leq \lfloor c_0 \rfloor$$

is not *necessarily a dual projected Chvátal-Gomory cut for with respect to A, G and b (as we stated in Theorem 385: it is* only dominated absolutely *by a dual projected Chvátal-Gomory cut with respect to A, G and b):*

For 2a: *Consider*

$$\emptyset = P := P^{\leq}\left(\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right) =: P^{\leq}(A, b) \subseteq \mathbb{R}^2.$$

It is easy to check that any dual Chvátal-Gomory cut for with respect to A and b is of the form

$$f^{\leq, CG, \mu}_{A,b}(x) = C_1 x_1 - C_2 \tag{8.1}$$

($x \in \mathbb{R}^2$), where $C_1, C_2 \in \mathbb{Z}$. On the other hand, $x_2 \leq 0$ clearly is a valid Chvátal-Gomory cut for $x \in P$. But, by (8.1)*, there exists no dual Chvátal-Gomory cut for with respect to A, b that induces the same half-space as $f^{\leq, CG, \mu}_{A,b}$.*

For 2b: *Consider*

$$P := P^{\leq}\left(\begin{pmatrix} 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \end{pmatrix}\right) =: P^{\leq}(A, b) \subseteq \mathbb{R}^1.$$

Clearly, $x_1 \leq 1$ is a valid Chvátal-Gomory cut for $x \in P$ (though this inequality is already valid for P: thus, it is a trivial Chvátal-Gomory cut for P). On the other hand, it is easy to check that if one has a dual Chvátal-Gomory cut $f^{\leq, CG, \mu}_{A,b}(x) = C_1 x_1 - C_2$ ($x \in \mathbb{R}^1$) with respect to A and b be given, one can show that

- *$C_1 \geq 0$ and*
- *if $C_1 > 0$, we have $C_2 \leq \frac{1}{2}C_1$.*

Thus, $(\cdot)_1 \leq 1$ cannot be represented as a dual Chvátal-Gomory cut with respect to A and b (but is dominated absolutely by one).

Proof. (Theorem 385)

For 1: Clearly, for $\mu \in \mathcal{M}_{CG}(A, G)$, the inequality $\mu Ax \leq \mu b$ is a valid for $\binom{x}{y} \in P$ (since $\mu \in \left(\mathbb{R}^l_{\geq 0}\right)^T$ and $Gy = (0^n)^T$). On the other hand, μA is integral and we have

$$\mu b - \operatorname{frac}(\mu b) = \lfloor \mu b \rfloor .$$

Thus, $\begin{pmatrix} \mu A & (0^n)^T \end{pmatrix}(\cdot) \leq \lfloor \mu b \rfloor$ is a projected Chvátal-Gomory cut for P and we have

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R}^n : f^{\leq, CG, \mu}_{A,G,b}\left(\begin{pmatrix} x \\ y \end{pmatrix} \right) \leq 0 \right\} = P^{\leq}\left(\begin{pmatrix} \mu A & (0^n)^T \end{pmatrix}, \lfloor \mu b \rfloor \right).$$

For 2a: Since $P = \emptyset$, by the Farkas lemma (Lemma 89), there exists a $\mu \in \left(\mathbb{R}^l_{\geq 0}\right)^T$ having

$$\mu \begin{pmatrix} A & G \end{pmatrix} = \begin{pmatrix} (0^m)^T & (0^n)^T \end{pmatrix}, \qquad \mu b = -1.$$

Clearly, $f^{\leq, CG, \mu}_{A,G,b}(\cdot) \leq 0$ is a projected dual Chvátal-Gomory cut with respect to A, G and b and we have $f^{\leq, CG, \mu}_{A,G,b}(\cdot) = -1$.

For 2b and 2c: Let $\begin{pmatrix} c & (0^n)^T \end{pmatrix}(\cdot) \leq c_0$ be valid for $P \neq \emptyset$ (recall that, by case assumption, we have $c \in (\mathbb{Z}^m)^T$). Let $c_0' := \max\left\{ cx : \begin{pmatrix} x \\ y \end{pmatrix} \in P \right\}$. By LP duality, there exists a $\mu \in \left(\mathbb{R}^l_{\geq 0}\right)^T$ such that

$$\mu \begin{pmatrix} A & G \end{pmatrix} = \begin{pmatrix} c & (0^n)^T \end{pmatrix}, \qquad \mu b = c_0'.$$

Clearly, $\mu \in \mathcal{M}_{CG}(A, G)$. Thus, $f^{\leq, CG, \mu}_{A,G,b}$ is defined. Since $c_0' \leq c_0$, we clearly have $\lfloor c_0' \rfloor \leq \lfloor c_0 \rfloor$. We now distinguish two cases:

1. $\lfloor c_0' \rfloor < \lfloor c_0 \rfloor$,
2. $\lfloor c_0' \rfloor = \lfloor c_0 \rfloor$.

Note that the situation of 2c immediately implies the second case, since by the case assumption and $\lfloor c_0' \rfloor < \lfloor c_0 \rfloor$, using

$$c_0' = \max\left\{ cx : \begin{pmatrix} x \\ y \end{pmatrix} \in P \right\} > \lfloor c_0 \rfloor > \lfloor c_0' \rfloor ,$$

we obtain $\lfloor c_0' \rfloor < \lfloor c_0 \rfloor < c_0'$. But this chain of inequalities clearly cannot be satisfied.

For case 1: We have for $\binom{x}{y} \in \mathbb{R}^m \times \mathbb{R}^n$:

$$f^{\leq, CG, \mu}_{A,G,b}\left(\tbinom{x}{y}\right) = \mu(Ax + Gy - b) + \operatorname{frac}(\mu b) = \mu Ax - \lfloor \mu b \rfloor = cx - \lfloor c_0' \rfloor = cx - \lfloor c_0 \rfloor - \underbrace{\left(\lfloor c_0' \rfloor - \lfloor c_0 \rfloor\right)}_{<0}.$$

Thus, $f^{\leq, CG, \mu}_{A,G,b}\left(\binom{x}{y}\right) \leq 0$ clearly dominates $cx \leq \lfloor c_0 \rfloor$ absolutely.

For case 2: We have for $\binom{x}{y} \in \mathbb{R}^m \times \mathbb{R}^n$:

$$f^{\leq, CG, \mu}_{A,G,b}\left(\tbinom{x}{y}\right) = \mu(Ax + Gy - b) + \operatorname{frac}(\mu b) = \mu Ax - \lfloor \mu b \rfloor = cx - \lfloor c_0' \rfloor = cx - \lfloor c_0 \rfloor .$$

Thus, $f^{\leq, CG, \mu}_{A,G,b}\left(\binom{x}{y}\right) \leq 0$ is equivalent to $cx \leq \lfloor c_0 \rfloor$. □

8.1.2.2. Projected Chvátal-Gomory cuts vs strong projected Chvátal-Gomory cuts

We next compare projected Chvátal-Gomory cuts to strong projected Chvátal-Gomory cuts:

Theorem 387. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ be arbitrary. Then:*

1. *Every projected Chvátal-Gomory cut for P is a strong projected Chvátal-Gomory cut for P.*
2. *If P is either convex and compact or a polyhedron, also the reverse holds.*
3. *If, on the other hand, P is convex, but either unbounded or not closed, there can exist strong (projected) Chvátal-Gomory cuts for P that are not (projected) Chvátal-Gomory cuts for P. Furthermore: there exists a convex and bounded (but, of course, not closed) $P' \subseteq \mathbb{R}^1$ such that*

$$\mathrm{cl}_{spCG}(P) \subsetneq \mathrm{cl}_{CG}(P).$$

Proof.

For 1: Let $\begin{pmatrix} c & (0^n)^T \end{pmatrix} (\,\cdot\,) \leq c_0$ be a projected Chvátal-Gomory cut for P. This implies

$$\sup \left\{ cx : \begin{pmatrix} x \\ y \end{pmatrix} \in P \right\} < c_0 + 1.$$

So, $P \cap P^{\geq} \left(\begin{pmatrix} c & (0^n)^T \end{pmatrix}, c_0 + 1 \right) = \emptyset$ and thus $\begin{pmatrix} c & (0^n)^T \end{pmatrix} (\,\cdot\,) \leq c_0$ is a strong Chvátal-Gomory cut for P.

For 2: Let P either be convex and compact or a polyhedron and let $\begin{pmatrix} c & (0^n)^T \end{pmatrix} (\,\cdot\,) \leq c_0$ be a strong projected Chvátal-Gomory cut for P. Then

$$\max \left\{ cx : \begin{pmatrix} x \\ y \end{pmatrix} \in P \right\} =: c_0' < c_0 + 1$$

is attained and thus $\begin{pmatrix} c & (0^n)^T \end{pmatrix} (\,\cdot\,) \leq c_0$ is a projected Chvátal-Gomory cut for P.

For 3: For the first statement, consider

- $P := [0, 1) \subseteq \mathbb{R}^1$ or
- $P := \left\{ x \in \mathbb{R}^2_{\geq 0} : x_1 \cdot x_2 \geq 1 \right\} \subseteq \mathbb{R}^2$,

respectively, and the respective strong Chvátal-Gomory cuts

- $(\,\cdot\,)_1 \leq 0$ or
- $-(\,\cdot\,)_1 \leq -1$,

respectively.
For the second statement, let $P' := [0, 1) \subseteq \mathbb{R}^1$. Clearly,

$$\left\{ 0^1 \right\} = \mathrm{cl}_I(P') = \mathrm{cl}_{spCG}(P').$$

On the other hand, we claim that for every Chvátal-Gomory cut $c(\,\cdot\,) \leq \lfloor c_0 \rfloor$ for P', we have

$$P' \cap P^{\leq}(c, \lfloor c_0 \rfloor) = P'.$$

For this, we distinguish three cases:

1. $c_1 = 0$,
2. $c_1 < 0$,
3. $c_1 > 0$.

In case 1, we have $c_0 \geq 0$, since $c(\cdot) \leq c_0$ is valid for $P' \neq \emptyset$. Thus,

$$P' \cap P^{\leq}(c, \lfloor c_0 \rfloor) = P' \cap \mathbb{R}^1 = P'.$$

In case 2, we have $c_0 \leq 0$, since $c(\cdot) \leq c_0$ is valid for P' and $0^1 \in P'$. Thus,

$$P' \cap \underbrace{P^{\leq}(c, \lfloor c_0 \rfloor)}_{\supseteq \mathbb{R}^1_{\geq 0}} = P'.$$

In case 3, we have $c_0 \geq c_1$, since $c(\cdot) \leq c_0$ is valid for P'. Thus,

$$P' \cap \underbrace{P^{\leq}(c, \lfloor c_0 \rfloor)}_{\supseteq P^{\leq}((1),(1))} = P'.$$

□

8.1.2.3. Strong projected Chvátal-Gomory cuts vs $L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts and $L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cuts

Now for the relationship between strong projected Chvátal-Gomory cuts and $L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts/$L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cuts. The final result is stated in Theorem 389, but before, we show a small proposition:

Proposition 388. *Let $a \in (\mathbb{Z}^m)^T \setminus \left\{(0^m)^T\right\}$ and $\alpha \in \mathbb{Z}$. Let $k := \gcd\{a_1, \ldots, a_m\} \geq 1$. Then*

$$\mathcal{F} := \operatorname{conv}(\{x \in \mathbb{Z}^m : ax \leq \alpha\}) = \left\{x \in \mathbb{R}^m : ax \leq k \left\lfloor \frac{\alpha}{k} \right\rfloor\right\}, \tag{8.2}$$

$$\mathcal{F}' := \operatorname{conv}(\{x \in \mathbb{Z}^m : ax < \alpha\}) = \left\{x \in \mathbb{R}^m : ax \leq k \left\lceil \frac{\alpha}{k} - 1 \right\rceil\right\}. \tag{8.3}$$

Proof. (8.2) is shown in [BW05, Proposition 6.1, p. 209]. So, we only prove (8.3). If $\alpha \notin k\mathbb{Z}$, we clearly have $\{x \in \mathbb{Z}^m : ax < \alpha\} = \{x \in \mathbb{Z}^m : ax \leq \alpha\}$ and $\left\lfloor \frac{\alpha}{k} \right\rfloor = \left\lceil \frac{\alpha}{k} - 1 \right\rceil$, so that we immediately obtain (8.3) from (8.2). On the other hand, if $\alpha \in k\mathbb{Z}$, we have $\{x \in \mathbb{Z}^m : ax < \alpha\} = \{x \in \mathbb{Z}^m : ax \leq \alpha - k\}$ and we thus obtain $\left\lfloor \frac{\alpha-k}{k} \right\rfloor = \left\lceil \frac{\alpha}{k} - 1 \right\rceil$. Again, we immediately conclude (8.3) from (8.2). □

Theorem 389. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$, where $m + n \geq 1$) be arbitrary. Then:*

1. *Every strong projected Chvátal-Gomory cut for P is an $L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cut for P (and thus an $L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cut for P).*
2. *Let P be convex and let $c(\cdot) \leq c_0$ be an $L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cut for P with respect to some $V \leq \mathbb{R}^m \times \mathbb{R}^n$.*
   a) *Let $\operatorname{proj}_{\mathbb{R}^n} V < \mathbb{R}^n$. Then every inequality for $(P + V)_I$ is already valid for P.*
   b) *Let $\operatorname{proj}_{\mathbb{R}^n} V = \mathbb{R}^n$ (i.e. there exists a V' such that $V = V' \times \mathbb{R}^n$).*
      i. *Let $(P + V)_I = \emptyset$. Then $c(\cdot) \leq c_0$ is dominated relatively to P by a strong projected Chvátal-Gomory cut for P.*
      ii. *Let $(P + V)_I \neq \emptyset$. Then $c(\cdot) \leq c_0$ is dominated absolutely by a strong projected Chvátal-Gomory cut for P.*

   In any case: every $L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cut for P is dominated relatively to P by a strong projected Chvátal-Gomory cut for P.

So, if P is convex, we have

$$\operatorname{cl}_{L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \operatorname{cl}_{L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \operatorname{cl}_{spCG}(P).$$

On the other hand, if P is not convex, the situation

$$\operatorname{cl}_{L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) \subsetneq \operatorname{cl}_{spCG}(P) \tag{8.4}$$

can occur.

Remark 390. *For the tightness of 2(b)i and 2(b)ii in Theorem 389:*

- *2(b)i can only hold up to relative (not absolute) dominance. For this, consider*

$$P := P^{\leq}\left(\left(\begin{array}{cc} 1 & 0 \\ -1 & 0 \end{array}\right), \left(\begin{array}{c} \frac{2}{3} \\ -\frac{1}{3} \end{array}\right)\right) \subsetneq \mathbb{R}^2.$$

Clearly, $(\cdot)_2 \leq 0$ *is an* $L_{1-\frac{1}{2},\mathbb{Q}}$ *cut for* P *with respect to* $0^1 \times \mathbb{R}^1$*. It is easy to check that there exists no Chvátal-Gomory cut that dominates* $(\cdot)_2 \leq 0$ *absolutely. On the other hand,* $(\cdot)_1 \leq 0$ *dominates* $(\cdot)_2 \leq 0$ *relatively to* P*.*

- *In 2(b)ii, not every* $L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ *cut (or* $L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ *cut)* $c(\cdot) \leq c_0$ *for* P *is a strong projected Chvátal-Gomory cut for* P *(there only exists a strong projected Chvátal-Gomory cut for* P *that dominates* $c(\cdot) \leq c_0$ *absolutely), even if we consider only the half-spaces that are induced by the respective type of cutting plane. For this, consider*

$$P := P^{\leq}\left(\left(1 \right), \left(\tfrac{1}{2} \right)\right) \subsetneq \mathbb{R}^1.$$

Clearly, $(\cdot)_1 \leq \frac{1}{2\sqrt{2}}$ *is an* $L_{1-\frac{1}{2},\mathbb{Q}}$ *cut for* P *with respect to* $\mathbb{R}^1$*, but not a strong Chvátal-Gomory cut for* P*. On the other hand, the strong Chvátal-Gomory cut* $(\cdot)_1 \leq 0$ *clearly dominates* $(\cdot)_1 \leq \frac{1}{2\sqrt{2}}$ *absolutely.*

Proof. (Theorem 389)

For 1: Let $\left(\begin{array}{cc} c & (0^n)^T \end{array}\right)(\cdot) \leq c_0$ be a strong projected Chvátal-Gomory cut for P, where w.l.o.g. $c \in (\mathbb{Z}^m)^T$ has coprime components and $c \neq (0^m)^T$. By definition, we have $c_0 \in \mathbb{Z}$ and

$$P \cap P^{\geq}\left(\left(\begin{array}{cc} c & (0^n)^T \end{array}\right), c_0\right) = \emptyset;$$

thus,

$$\forall \left(\begin{array}{c} x \\ y \end{array}\right) \in P : cx < c_0 + 1.$$

Let $V := \left(\operatorname{lin}\left\{\left(\begin{smallmatrix} c^T \\ 0^n \end{smallmatrix}\right)\right\}\right)^{\perp}$. We surely have

$$P + V \subseteq P^{<}\left(\left(\begin{array}{cc} c & (0^n)^T \end{array}\right), c_0 + 1\right).$$

Thus, we conclude using Proposition 388:

$$\operatorname{cl}_{\overline{I}}(P + V) \subseteq P^{\leq}\left(\left(\begin{array}{cc} c & (0^n)^T \end{array}\right), \lceil c_0 + 1 - 1 \rceil\right) = P^{\leq}\left(\left(\begin{array}{cc} c & (0^n)^T \end{array}\right), c_0\right).$$

For 2: W.l.o.g. we assume that $P \neq \emptyset$ and $(\cdot) \leq c_0$ is not be valid for P.

For 2a: We show that if $c(\cdot) \leq c_0$ is not valid for P, we have $\operatorname{proj}_{\mathbb{R}^m} V < \mathbb{R}^m$. We first recapitulate (cf. Remark 162) that $L_{0,\mathbb{Q}}$ cuts for P are simply linear inequalities that are valid for P. So, $c(\cdot) \leq c_0$ is not an $L_{0,\mathbb{Q}}$ cut for P. Since $c(\cdot) \leq c_0$ is not an $L_{0,\mathbb{Q}}$ cut for P, by Theorem 208, it is an essential $L_{1-\frac{1}{2},\mathbb{Q}}$ cut for P with respect to a rational subspace $V := V' \times \mathbb{R}^n \leq \mathbb{R}^m \times \mathbb{R}^n$, where $\operatorname{codim} V' = 1$.

For 2b: Let $c' \in (\mathbb{Z}^m \backslash \{0^m\})^T$ be an integral vector with coprime component that satisfies $c'^T \perp V'$. We first show that for $c_0 \in \mathbb{Z}$, we have

$$\left(\exists \left(\begin{array}{c} x \\ y \end{array}\right) \in P : c'x = c_0\right) \Leftrightarrow \left(\exists \left(\begin{array}{c} x' \\ y' \end{array}\right) \in (P + V)_I : c'x' = c_0\right). \tag{8.5}$$

For "$\Leftarrow$" in (8.5)**:** Let $\left(\begin{smallmatrix} x' \\ y' \end{smallmatrix}\right) = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) + v \in (P + V)_I$, where $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in P$, $v \in V$ and $c'x' = c_0$. Then, by definition, $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in P$ and we have using $c' \perp V'$: $c'x = c'\left(x' - v_{(1,\ldots,m)}\right) = c'x' = c_0$.

For "$\Rightarrow$" in (8.5)**:** Let $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in P$, where $c'x = c_0$. Since

- c' is integral,

- the components of c' are coprime and
- $c' \perp V'$,

$\{c'\}$ is a lattice basis of $\mathbb{Z}^m \cap V'^{\perp}$. By Lemma 98,

$$\mathbb{Z}^m + V' = \{q \in \mathbb{R}^m : c'q \in \mathbb{Z}\}.$$

So, since $c_0 \in \mathbb{Z}$, there exists a $z \in \mathbb{Z}^m \times \mathbb{R}^n$ and a $v \in V$ such that $\binom{x}{y} = z+v$. This implies $\binom{x}{y} - v \in (P+V)_I$.

For 2(b)i: We claim that

$$\nexists c_0 \in \mathbb{Z} : \exists \binom{x}{y} \in P : c'x = c_0.$$

If there existed such a c_0, we would by (8.5) get a contradiction to $(P+V)_I = \emptyset$. Let

$$\underline{c}_0 := \inf\left\{c'x : \begin{pmatrix} x \\ y \end{pmatrix} \in P\right\} = \inf\left\{c'x : \begin{pmatrix} x \\ y \end{pmatrix} \in P+V\right\},$$
$$\overline{c}_0 := \sup\left\{c'x : \begin{pmatrix} x \\ y \end{pmatrix} \in P\right\} = \sup\left\{c'x : \begin{pmatrix} x \\ y \end{pmatrix} \in P+V\right\}.$$

We note that $-\infty < \underline{c}_0 \leq \overline{c}_0 < \infty$ (in particular $\underline{c}_0, \overline{c}_0 \in \mathbb{R}$): since $P \neq \emptyset$, we have $\underline{c}_0 \in \mathbb{R} \,\dot{\cup}\, \{-\infty\}$ and $\overline{c}_0 \in \mathbb{R} \,\dot{\cup}\, \{\infty\}$. On the other hand, if $\pm\infty$ were attained, we would have (since P is convex):

$$\exists c_0 \in \mathbb{Z}, \begin{pmatrix} x \\ y \end{pmatrix} \in P : c'x = c_0,$$

which, by (8.5), is a contradiction to $(P+V)_I = \emptyset$. We next claim that

$$\lceil \overline{c}_0 \rceil - \lfloor \underline{c}_0 \rfloor = 1. \tag{8.6}$$

For $\geq$ in (8.6): $\lceil \overline{c}_0 \rceil = \lfloor \underline{c}_0 \rfloor$ implies $\underline{c}_0 = \overline{c}_0 \in \mathbb{Z}$. Because of (8.5) and $(P+V)_I = \emptyset$, this cannot happen.

For $\leq$ in (8.6): Assume

$$\lceil \overline{c}_0 \rceil - \lfloor \underline{c}_0 \rfloor \geq 2.$$

Then there exists a $c_0 \in \mathbb{Z}$ having

$$\underline{c}_0 < c_0 < \overline{c}_0.$$

Thus, since P is convex, there exists an $\binom{x}{y} \in P$ such that $cx = c_0$. So, using (8.5), we get a contradiction to $(P+V)_I = \emptyset$.

Finally, note that there exists no $\binom{x}{y} \in P$ having either $c'x = \underline{c}_0$ or $c'x = \overline{c}_0$, since otherwise, we again get the contradiction $(P+V)_I \neq \emptyset$ from (8.5). Thus,

$$P \cap P^{\geq}\left(\begin{pmatrix} c' & (0^n)^T \end{pmatrix}, \overline{c}_0\right) = \emptyset.$$

So, $\begin{pmatrix} c' & (0^n)^T \end{pmatrix}(\cdot) \leq \overline{c}_0 - 1$ is a strong projected Chvátal-Gomory cut for P and we have

$$P \cap P^{\leq}\left(\begin{pmatrix} c' & (0^n)^T \end{pmatrix}, \overline{c}_0 - 1\right) = \emptyset.$$

For 2(b)ii: By Lemma 159, we have $c^T \perp V$. On the other hand, since we assumed that $c(\cdot) \leq c_0$ is not valid for P, we have $c^T \neq \binom{0^m}{0^n}$. Thus, there exists a $\lambda \in \mathbb{R}_{\neq 0}$ such that

$$\begin{pmatrix} c' & (0^n)^T \end{pmatrix} = \lambda c.$$

W.l.o.g. we can assume that $\lambda = 1$. In other words: $c = \begin{pmatrix} c' & (0^n)^T \end{pmatrix}$. By definition, we have

$$(P+V)_I \cap P^{>}(c, c_0) = (P+V)_I \cap P^{>}\left(\begin{pmatrix} c' & (0^n)^T \end{pmatrix}, c_0\right) = \emptyset. \tag{8.7}$$

We now show that $\begin{pmatrix} c' & (0^n)^T \end{pmatrix}(\cdot) \leq \lfloor c_0 \rfloor$ is a strong projected Chvátal-Gomory cut for P. For this, we prove

$$P \cap P^{\geq}\left(\begin{pmatrix} c' & (0^n)^T \end{pmatrix}, \lfloor c_0 \rfloor + 1\right) = \emptyset.$$

Assume that there exists an $\binom{x^{1,*}}{y^{1,*}} \in P$ such that $c'x^{1,*} \geq \lfloor c_0 \rfloor + 1$. Since $(P+V)_I \neq \emptyset$, there exists an $\binom{x^{2,*}}{y^{2,*}} \in (P+V)_I$, which, by (8.5), implies $c_0^* := c'x^{2,*} \in \mathbb{Z}$. Because of (8.7) and $x^{2,*} \in \mathbb{Z}^m$, we have $c_0^* \leq \lfloor c_0 \rfloor$. So, by (8.5), there exists a $\binom{x^{3,*}}{y^{3,*}} \in P$ having $c_0^* := c'x^{3,*}$.

Let

$$\begin{pmatrix} x^{4,*} \\ y^{4,*} \end{pmatrix} := \lambda \begin{pmatrix} x^{1,*} \\ y^{1,*} \end{pmatrix} + (1-\lambda) \begin{pmatrix} x^{3,*} \\ y^{3,*} \end{pmatrix},$$

where

$$\lambda := \frac{\lfloor c_0 \rfloor + 1 - c_0^*}{c'x^{1,*} - c_0^*}.$$

By convexity of P, we have $\binom{x^{4,*}}{y^{4,*}} \in P$. On the other hand,

$$\begin{aligned} c'x^{4,*} &= \lambda c'x^{1,*} + (1-\lambda)\, c'x^{3,*} \\ &= \lambda c'x^{1,*} + (1-\lambda)\, c_0^* \\ &= \frac{\lfloor c_0 \rfloor + 1 - c_0^*}{c'x^{1,*} - c_0^*} c'x^{1,*} + \left(1 - \frac{\lfloor c_0 \rfloor + 1 - c_0^*}{c'x^{1,*} - c_0^*}\right) c_0^* \\ &= \frac{\lfloor c_0 \rfloor + 1 - c_0^*}{c'x^{1,*} - c_0^*} c'x^{1,*} + \frac{c'x^{1,*} - \lfloor c_0 \rfloor - 1}{c'x^{1,*} - c_0^*} c_0^* \\ &= \lfloor c_0 \rfloor + 1. \end{aligned}$$

By (8.5), this implies

$$\exists \begin{pmatrix} x \\ y \end{pmatrix} \in (P+V)_I : c'x = \lfloor c_0 \rfloor + 1,$$

which is a contradiction to (8.7).

For (8.4)**:** Consider

$$P := (-2,-1) \,\dot{\cup}\, (-1,0) \,\dot{\cup}\, (0,1) \subseteq \mathbb{R}^1.$$

It is easy to check that $\mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{Q}}}(P) = \emptyset$. On the other hand, every strong Chvátal-Gomory cut for P is dominated absolutely by either $(\cdot)_1 \leq 0$ or $-(\cdot)_1 \leq 1$. Thus, $\mathrm{cl}_{sCG}(P) = (-1,0) \subseteq \mathbb{R}^1$. □

8.1.2.4. Conclusion and summary

Using Theorem 385, Theorem 387 and Theorem 389, we obtain the following result for the types of cutting planes that we presented in section 8.1.1:

Theorem 391. *Let $P = P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)$ be a polyhedron, where $A \in \mathbb{R}^{l \times m}$, $G \in \mathbb{R}^{l \times n}$ and $b \in \mathbb{R}^l$ ($l, m, n \in \mathbb{Z}_{\geq 0}$). Then:*

- *Every dual projected Chvátal-Gomory cut (for P) with respect to A, G and b is a projected Chvátal-Gomory cut for P.*
- *Every projected Chvátal-Gomory cut for P is a strong projected Chvátal-Gomory cut for P.*
- *Every strong projected Chvátal-Gomory cut for P is an $L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cut (and thus an $L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cut) for P.*

Additionally, we have

$$\mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P) = \mathrm{cl}_{spCG}(P) = \mathrm{cl}_{pCG}(P) = \mathrm{cl}_{dpCG}(A, G, b).$$

Now for a more encompassing summary of the results that we have shown in section 8.1.2:

- In Theorem 385, we saw that for a polyhedron $P := P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right) \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($A \in \mathbb{R}^{l\times m}$, $G \in \mathbb{R}^{l\times n}$ and $b \in \mathbb{R}^l$, where $l, m, n \in \mathbb{Z}_{\geq 0}$), every dual projected Chvátal-Gomory cut (for P) with respect to A, G and b is a projected Chvátal-Gomory cut for P and, on the other hand, every projected Chvátal-Gomory cut for P is dominated absolutely by a dual projected Chvátal-Gomory cut (for P) with respect to A, G and b. Thus, in particular, we have

  $$\mathrm{cl}_{pCG}(P) = \mathrm{cl}_{dpCG}(A, G, b).$$

  Note however (cf. Remark 386) that not every projected Chvátal-Gomory cut for P is necessarily a dual projected Chvátal-Gomory cut with respect to A, G and b. In other words: projected Chvátal-Gomory cuts and dual projected Chvátal-Gomory cuts are nevertheless not completely identical concepts.

  Additionally, note that it does not make sense to ask the question about the relationship between projected Chvátal-Gomory cut vs dual projected Chvátal-Gomory cuts for non-polyhedra, since (recall Definition 382 and Definition 383) we only defined dual projected Chvátal-Gomory cuts and their closure for polyhedra.

- In Theorem 387, we showed that every projected Chvátal-Gomory cut for an arbitrary $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) is also a strong projected Chvátal-Gomory cut for P (as the naming implies) and the reverse does hold if P is either convex and compact or a polyhedron.

  On the other hand, if P is convex, but either unbounded or not closed, there can exist strong projected Chvátal-Gomory cuts for P that are not projected Chvátal-Gomory cuts for P. Indeed: if P is convex, but not closed, the situation

  $$\mathrm{cl}_{spCG}(P) \subsetneq \mathrm{cl}_{CG}(P).$$

  can occur.

- Finally, in Theorem 389, we proved that for an arbitrary $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$), every strong projected Chvátal-Gomory cut for P is an $L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cut (and thus an $L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cut) for P. If P is convex (but not in general), also the reverse holds up to dominance relatively to P (cf. Remark 390 for the technical details concerning the reverse inclusion).

8.2. Dominance and representation of the Chvátal-Gomory closure

8.2.1. Restriction to smaller support

Definition 392. *For $A \in \mathbb{R}^{l\times(m+n)}$ and $b \in \mathbb{R}^l$ ($l, m, n \in \mathbb{Z}_{\geq 0}$), let $r := \operatorname{rank} A$. Define*

$$\begin{aligned}
\mathcal{B}^*(A) &:= \{S \subseteq [l] : |S| = r \wedge \operatorname{rank} A_{S,*} = r\},\\
\mathcal{B}^{*,feas}(A, b) &:= \left\{S \subseteq [l] : |S| = r \wedge \operatorname{rank} A_{S,*} = r \wedge \left(\exists x \in \mathbb{R}^d : A_{S,*}x = b_S \wedge Ax \leq b\right)\right\},\\
\mathcal{F}^{*,feas}(A, b) &:= \left\{S \subseteq [l] : \operatorname{rank} A_{S,*} = r \wedge \left(\exists x \in \mathbb{R}^d : A_{S,*}x = b_S \wedge A_{[m+n]\setminus S}x < b_{[m+n]\setminus S}\right)\right\}
\end{aligned}$$

as the row indices of bases, feasible bases and minimal faces, respectively.

Proposition 393. *Let*

$$\emptyset \neq P := P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right) \subseteq \mathbb{R}^m \times \mathbb{R}^n,$$

where A, G b, m and n are as in Definition 382. Then for every $c \in (\mathbb{Z}^m \times 0^n)^T$ that satisfies

$$\max\left\{cx : x \in P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)\right\} < \infty,$$

we have

$$\begin{aligned}
&P^{\leq}\left(c, \left\lfloor \max\left\{cx : x \in P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)\right\}\right\rfloor\right) \\
=& \bigcap_{\substack{S \in \mathcal{F}^{*,feas}((A\ G),b):\\ \max\left\{cx : x \in P^{\leq}\left((A\ G)_{S,*}, b_S\right)\right\} < \infty}} P^{\leq}\left(c, \left\lfloor \max\left\{cx : x \in P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}_{S,*}, b_S\right)\right\}\right\rfloor\right) \\
=& \bigcap_{\substack{S \in \mathcal{B}^{*,feas}((A\ G),b):\\ \max\left\{cx : x \in P^{\leq}\left((A\ G)_{S,*}, b_S\right)\right\} < \infty}} P^{\leq}\left(c, \left\lfloor \max\left\{cx : x \in P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}_{S,*}, b_S\right)\right\}\right\rfloor\right) \\
=& \bigcap_{\substack{S \in \mathcal{B}^{*}((A\ G)):\\ \max\left\{cx : x \in P^{\leq}\left((A\ G)_{S,*}, b_S\right)\right\} < \infty}} P^{\leq}\left(c, \left\lfloor \max\left\{cx : x \in P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}_{S,*}, b_S\right)\right\}\right\rfloor\right).
\end{aligned}$$

Proof. Let $\mathcal{F}$ be a minimal face of $P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)$ in which $\max\left\{cx : x \in P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)\right\}$ is attained. Then there exist r linearly independent rows with row indices S such that

$$\max\left\{cx : x \in P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)\right\} = \max\left\{cx : x \in P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}_{S,*}, b_S\right)\right\}.$$

From this, the statements immediately follow. □

Theorem 394. *Let $P := P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right) \subseteq \mathbb{R}^m \times \mathbb{R}^n$ be given, where A, G b, m and n are as in Definition 382. Then*

$$\begin{aligned}
\mathrm{cl}_{pCG}\left(P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)\right) =& \bigcap_{S \in \mathcal{F}^{*,feas}((A\ G),b)} \mathrm{cl}_{pCG}\left(P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}_{S,*}, b_S\right)\right) \qquad (8.8)\\
=& \bigcap_{S \in \mathcal{B}^{*,feas}((A\ G),b)} \mathrm{cl}_{pCG}\left(P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}_{S,*}, b_S\right)\right) \\
=& \bigcap_{S \in \mathcal{B}^{*}((A\ G))} \mathrm{cl}_{pCG}\left(P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}_{S,*}, b_S\right)\right).
\end{aligned}$$

By Theorem 391, we can replace $\mathrm{cl}_{pCG}(\cdot)$ by $\mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(\cdot)$, $\mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(\cdot)$, $\mathrm{cl}_{spCG}(\cdot)$ or $\mathrm{cl}_{dpCG}(\cdot,\cdot,\cdot)$ in these equations.

Proof. If $P = \emptyset$, there is nothing to show. So, let $P \neq \emptyset$. W.l.o.g. we only show (8.8). Using the definition of the projected Chvátal-Gomory closure (Definition 122) and Proposition 393, we obtain

$$\begin{aligned}
&\mathrm{cl}_{pCG}\left(P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)\right) \\
=& \bigcap_{\substack{c \in (\mathbb{Z}^m)^T, c_0 \in \mathbb{R}:\\ P \subseteq P^{\leq}\left(\left(c\ (0^n)^T\right), c_0\right)}} P^{\leq}\left(\begin{pmatrix} c & (0^n)^T \end{pmatrix}, \lfloor c_0 \rfloor\right) \\
=& \bigcap_{\substack{c \in (\mathbb{Z}^m \times 0^n)^T:\\ \max\left\{cx : x \in P^{\leq}((A\ G),b)\right\} < \infty}} P^{\leq}\left(c, \left\lfloor \max\left\{cx : x \in P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)\right\}\right\rfloor\right) \\
=& \bigcap_{S \in \mathcal{F}^{*,feas}((A\ G),b)} \bigcap_{\substack{c \in (\mathbb{Z}^m \times 0^n)^T:\\ \max\left\{cx : x \in P^{\leq}\left((A\ G)_{S,*}, b_S\right)\right\} < \infty}} P^{\leq}\left(c, \left\lfloor \max\left\{cx : x \in P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}_{S,*}, b_S\right)\right\}\right\rfloor\right) \\
=& \bigcap_{S \in \mathcal{F}^{*,feas}((A\ G),b)} \left(P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}_{S,*}, b_S\right)\right. \\
&\cap \bigcap_{\substack{S \in \mathcal{F}^{*,feas}((A\ G),b):\\ \max\left\{cx : x \in P^{\leq}\left((A\ G)_{S,*}, b_S\right)\right\} < \infty}} \left. P^{\leq}\left(c, \left\lfloor \max\left\{cx : x \in P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}_{S,*}, b_S\right)\right\}\right\rfloor\right)\right) \\
=& \bigcap_{S \in \mathcal{F}^{*,feas}((A\ G),b)} \left(P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}_{S,*}, b_S\right) \cap \bigcap_{\substack{c \in (\mathbb{Z}^m)^T, c_0 \in \mathbb{R}:\\ P^{\leq}\left((A\ G)_{S,*}, b_S\right) \subseteq P^{\leq}\left(\left(c\ (0^n)^T\right), c_0\right)}} P^{\leq}\left(\begin{pmatrix} c & (0^n)^T \end{pmatrix}, \lfloor c_0 \rfloor\right)\right)
\end{aligned}$$

$$= \bigcap_{S\in\mathcal{F}^{*,feas}((\begin{array}{cc} A & G \end{array}),b)} \mathrm{cl}_{pCG}\left(P^{\leq}\left(\left(\begin{array}{cc} A & G \end{array}\right)_{S,*}, b_S\right)\right).$$

□

8.2.2. Dominance

Lemma 395. *Let A, G and b be as in Definition 382 and let $\mu^1, \mu^2 \in \mathcal{M}_{CG}(A,G)$ be such that $\mu^1 b \in \mathbb{Z}$. Then the dual projected Chvátal-Gomory cut $f_{A,G,b}^{\leq 0,CG,\mu^1+\mu^2}(\cdot) \leq 0$ is dominated relatively to*

$$P^{\leq}\left(\left(\begin{array}{cc} A & G \end{array}\right)_{\mathrm{supp}\,\mu,*}, b_{\mathrm{supp}\,\mu}\right) \supseteq P^{\leq}\left(\left(\begin{array}{cc} A & G \end{array}\right), b\right)$$

by the dual projected Chvátal-Gomory cut $f_{A,G,b}^{\leq 0,CG,\mu^2}(\cdot) \leq 0$.

Proof. Let $\binom{x}{y} \in P^{\leq}\left(\left(\begin{array}{cc} A & G \end{array}\right), b\right)$ satisfy $f_{A,G,b}^{\leq,CG,\mu^2}\left(\binom{x}{y}\right) \leq 0$. Then

$$\begin{aligned}
f_{A,G,b}^{\leq,CG,\mu^1+\mu^2}\left(\left(\begin{array}{c} x \\ y \end{array}\right)\right) &= \left(\mu^1+\mu^2\right)(Ax-b) + \mathrm{frac}\left(\left(\mu^1+\mu^2\right)b\right) \\
&= \sum_{i\in\mathrm{supp}\,\mu^1} \mu_i^1 (Ax+Gy-b)_i + \mu^2(Ax-b) + \mathrm{frac}\left(\mu^2 b\right) \\
&\leq \mu^2(Ax-b) + \mathrm{frac}\left(\mu^2 b\right) \\
&= f_{A,G,b}^{\leq,CG,\mu^2}\left(\left(\begin{array}{c} x \\ y \end{array}\right)\right) \\
&\leq 0.
\end{aligned}$$

□

Lemma 396. *Let A, G and b be as in Definition 382 and let $\mu^1, \mu^2 \in \mathcal{M}_{CG}(A,G)$. Then $f_{A,G,b}^{\leq 0,CG,\mu^1+\mu^2}(\cdot) \leq 0$ is dominated absolutely by $f_{A,G,b}^{\leq 0,CG,\mu^1}(\cdot) + f_{A,G,b}^{\leq 0,CG,\mu^2}(\cdot) \leq 0$ (in particular by the dual projected Chvátal-Gomory cuts $f_{A,G,b}^{\leq 0,CG,\mu^1}(\cdot) \leq 0$ and $f_{A,G,b,\mu^2}^{\leq,CG}(\cdot) \leq 0$).*

Proof. Let $\binom{x}{y} \in \mathbb{R}^m \times \mathbb{R}^n$ satisfy $f_{A,G,b}^{\leq 0,CG,\mu^1}\left(\binom{x}{y}\right) + f_{A,G,b}^{\leq 0,CG,\mu^1}\left(\binom{x}{y}\right) \leq 0$. Then

$$\begin{aligned}
f_{A,G,b}^{\leq,CG,\mu^1+\mu^2}\left(\left(\begin{array}{c} x \\ y \end{array}\right)\right) &= \left(\mu^1+\mu^2\right)(Ax-b) + \mathrm{frac}\left(\left(\mu^1+\mu^2\right)b\right) \\
&\leq \left(\mu^1+\mu^2\right)(Ax-b) + \mathrm{frac}\left(\mu^1 b\right) + \mathrm{frac}\left(\mu^2 b\right) && \text{(by (2.25))} \\
&= \mu^1(Ax-b) + \mathrm{frac}\left(\mu^1 b\right) + \mu^2(Ax-b) + \mathrm{frac}\left(\mu^2 b\right) \\
&= f_{A,G,b}^{\leq,CG,\mu^1}\left(\left(\begin{array}{c} x \\ y \end{array}\right)\right) + f_{A,G,b}^{\leq,CG,\mu^2}\left(\left(\begin{array}{c} x \\ y \end{array}\right)\right) \\
&\leq 0.
\end{aligned}$$

□

Lemma 397. *Let A, G and b be as in Definition 382 and let $\mu^1, \mu^2 \in \mathcal{M}_{CG}(A,G)$ satisfy*

$$\mu^1\left(\begin{array}{cc} A & G \end{array}\right) = \mu^2\left(\begin{array}{cc} A & G \end{array}\right), \qquad \mu^1 b \geq \mu^2 b.$$

Then $f_{A,G,b}^{\leq,CG,\mu^1}(\cdot) \leq 0$ is dominated absolutely by $f_{A,G,b}^{\leq,CG,\mu^2}(\cdot) \leq 0$.

Proof. Let $\binom{x}{y} \in \mathbb{R}^m \times \mathbb{R}^n$ satisfy $f_{A,G,b}^{\leq,CG,\mu^2}\left(\binom{x}{y}\right) \leq 0$. Then

$$\begin{aligned}
f_{A,G,b}^{\leq,CG,\mu^1}\left(\left(\begin{array}{c} x \\ y \end{array}\right)\right) &= \mu^1(Ax-b) + \mathrm{frac}\left(\mu^1 b\right) \\
&= \mu^1 Ax - \left\lfloor \mu^1 b \right\rfloor \\
&\leq \mu^1 Ax - \left\lfloor \mu^2 b \right\rfloor
\end{aligned}$$

$$= \mu^2 Ax - \lfloor \mu^2 b \rfloor$$
$$= \mu^2 (Ax - b) + \operatorname{frac}(\mu^2 b)$$
$$= f_{A,G,b}^{\leq, CG, \mu^2}\left(\begin{pmatrix} x \\ y \end{pmatrix} \right)$$
$$\leq 0.$$

□

8.2.3. Representation via TD$\mathbb{Z} + I$ systems with an integral left-hand sides

8.2.3.1. Statement

The following theorem is well-known for TDI systems (instead of the more general TD$\mathbb{Z} + I$ systems); cf. [Sch86, Theorem 23.1; p. 340]:

Theorem 398. *Let $Ax \leq b$ be a TD$\mathbb{Z}$+I system, where $A \in \mathbb{Z}^{l \times m}$ (i.e. A is integral) and $b \in \mathbb{R}^l$ ($l, m \in \mathbb{Z}_{\geq 0}$). Then*

$$\operatorname{cl}_{CG}\left(P^{\leq}(A, b)\right) = P^{\leq}(A, \lfloor b \rfloor).$$

Proof. (Theorem 398) W.l.o.g. let $P^{\leq}(A, b) \neq \emptyset$. The inclusion $\operatorname{cl}_{CG}\left(P^{\leq}(A, b)\right) \subseteq P^{\leq}(A, \lfloor b \rfloor)$ is obvious. So for $\operatorname{cl}_{CG}\left(P^{\leq}(A, b)\right) \supseteq P^{\leq}(A, \lfloor b \rfloor)$: let $\mu \in \mathcal{M}_{CG}(A)$. We, of course, consider the dual projected Chvátal-Gomory cut $f_{A,b}^{\leq, CG, \mu}(\,\cdot\,) \leq 0$.

By the TD$\mathbb{Z} + I$ property, we have

$$\max\left\{ cx : Ax \leq b, \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R}^n \right\}$$
$$= \min\left\{ \left(z^1 + z^2\right) b : \left(z^1 + z^2\right) A = c, z^1 b \in \mathbb{Z}, z^1 \in \left(\mathbb{R}^l_{\geq 0}\right)^T, z^2 \in \left(\mathbb{Z}^l_{\geq 0}\right)^T \right\}.$$

Let $\begin{pmatrix} z^{1,*} & z^{2,*} \end{pmatrix}$ be a minimizer of the dual program. Now we argue:

- Since $\mu A(\,\cdot\,) \leq \mu b$ is a valid inequality for $P^{\leq}(A, b)$, we have $\mu b \geq \left(z^{1,*} + z^{2,*}\right) b$. Thus, by Lemma 397,

  $$f_{A,b}^{\leq, CG, \mu}(\,\cdot\,) \leq 0$$

  is dominated absolutely by

  $$f_{A,b}^{\leq, CG, z^{1,*}+z^{2,*}}(\,\cdot\,) \leq 0.$$

- By Lemma 395 (using $z^1 b \in \mathbb{Z}$),

  $$f_{A,b}^{\leq, CG, z^{1,*}+z^{2,*}}(\,\cdot\,) \leq 0$$

  is dominated relatively to $P^{\leq}(A, b)$ by

  $$f_{A,b}^{\leq, CG, z^{2,*}}(\,\cdot\,) \leq 0.$$

- By Lemma 396 (using that A and $z^{2,*}$ are integral),

  $$f_{A,b}^{\leq, CG, z^{2,*}}(\,\cdot\,) \leq 0,$$

  is dominated absolutely by the inequalities

  $$\left\{ f_{A,b}^{\leq, CG, \left(e^{l,i}\right)^T}(\,\cdot\,) \leq 0 : i \in \underbrace{\operatorname{supp} z^{2,*}}_{\subseteq [l]} \right\}.$$

But for $x \in P^{\leq}(A, \lfloor b \rfloor)$ and $i \in \operatorname{supp} z^{2,*}$, we have

$$f_{A,b}^{\leq, CG, \left(e^{l,i}\right)^T}(x) = A_{i,*}x - b_i x + \operatorname{frac}(b_i) = A_{i,*}x - \lfloor b_i \rfloor \leq 0.$$

□

Since, by Theorem 369, for every polyhedron $P \subseteq \mathbb{R}^m$ with rational face normals, there exists a TDI system (thus TD$\mathbb{Z}+I$ system) $Ax \leq b$ having $P = P^{\leq}(A, b)$, where A is integral, we immediately conclude from Theorem 398:

Theorem 399. *Let $P \subseteq \mathbb{R}^m$ ($m \in \mathbb{Z}_{\geq 0}$) be a polyhedron with rational face normals. Then $\mathrm{cl}_{CG}(P)$ is a rational polyhedron.*

We remark that in Theorem 405, we write down a similar polyhedricity result for the mixed-integer case.

8.2.3.2. Advantages of TD$\mathbb{Z}+I$ systems over TDI systems and TD$\mathbb{Z}+\{0,1\}$ systems

One can use Theorem 398 to compute the Chvátal-Gomory closure of a polyhedron $P \subseteq \mathbb{R}^m$ ($m \in \mathbb{Z}_{\geq 0}$), where all variables are integral. Unluckily, the TD$\mathbb{Z}+I$ system that is necessary to use Theorem 398 might potentially be large (in the sense of the number of necessary inequalities). In this section, we analyze how using a TD$\mathbb{Z}+I$ system instead of a TDI system or TD$\mathbb{Z}+\{0,1\}$ system can reduce the number of inequalities. Let us recapitulate the definition of P^{377,k_1,k_2}:

Definition 377. *For $k_1 \in \mathbb{Z}_{\geq 1}$ and $k_2 \in \mathbb{Z}_{\geq 2}$, define*

$$\begin{aligned} P^{377,k_1,k_2} &:= \mathrm{conv}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -\frac{1}{k_2} \end{pmatrix}\right\} + \mathrm{cone}\left\{\begin{pmatrix} -k_1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right\} \\ &= P^{\leq}\left(\begin{pmatrix} 1 & 0 \\ 1 & k_1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ \frac{1}{k_2} \end{pmatrix}\right) \\ &\subseteq \mathbb{R}^2. \end{aligned}$$

We obviously conclude from Theorem 398, since by Theorem 379, $\begin{pmatrix} 1 & 0 \\ 1 & k_1 \\ 0 & -1 \end{pmatrix} x \leq \begin{pmatrix} 1 \\ 1 \\ \frac{1}{k_2} \end{pmatrix}$ is a TD$\mathbb{Z}+I$ system for P^{377,k_1,k_2}:

Lemma 400. *Let $k_1 \in \mathbb{Z}_{\geq 1}$, $k_2 \in \mathbb{Z}_{\geq 2}$. Then*

$$\mathrm{cl}_{CG}\left(P^{377,k_1,k_2}\right) = P^{\leq}\left(\begin{pmatrix} 1 & 0 \\ 1 & k_1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right).$$

On the other hand, we saw in Theorem 379, which we restate here:

Theorem 379. *Let $k_1 \in \mathbb{Z}_{\geq 1}$, $k_2 \in \mathbb{Z}_{\geq 2}$. Then:*

1.
$$\begin{pmatrix} 1 & 0 \\ 1 & k_1 \\ 0 & -1 \end{pmatrix} x \leq \begin{pmatrix} 1 \\ 1 \\ \frac{1}{k_2} \end{pmatrix}$$
is a TD$\mathbb{Z}+I$ system for P^{377,k_1,k_2} with an integral left-hand side that consists of 3 rows.

2. *If $Ax \leq b$ is a TDI system for P^{377,k_1,k_2} where A is integral, $\begin{pmatrix} A \mid -b \end{pmatrix}$ contains the following $k_1 + 2$ rows:*
$$\left(\dot{\bigcup_{i=0}^{k_1}} \left\{\begin{pmatrix} 1 & i \mid -1 \end{pmatrix}\right\}\right) \dot{\cup} \left\{\begin{pmatrix} 0 & -1 \mid -\frac{1}{k_2} \end{pmatrix}\right\}.$$
On the other hand,
$$\begin{pmatrix} 1 & 0 \\ \vdots & \vdots \\ 1 & k_1 \\ 0 & -1 \end{pmatrix} x \leq \begin{pmatrix} 1 \\ \vdots \\ 1 \\ \frac{1}{k_2} \end{pmatrix}$$
is a TDI system for P^{377,k_1,k_2} with an integral left-hand side that consists of $k_1 + 2$ rows.

3. *If $Ax \leq b$ is a TD$\mathbb{Z}+\{0,1\}$ system for P^{377,k_1,k_2} where A is integral, $\begin{pmatrix} A \mid -b \end{pmatrix}$ contains the following $k_2 - 1$ rows:*
$$\dot{\bigcup_{i=1}^{k_2-1}} \left\{\begin{pmatrix} 0 & -i \mid -\frac{i}{k_2} \end{pmatrix}\right\}.$$

On the other hand,

$$\begin{pmatrix} 1 & 0 \\ 1 & k_1 \\ 0 & -1 \\ \vdots & \vdots \\ 0 & -(k_2-1) \end{pmatrix} x \leq \begin{pmatrix} 1 \\ 1 \\ \frac{1}{k_2} \\ \vdots \\ \frac{k_2-1}{k_2} \end{pmatrix}$$

is a TD$\mathbb{Z} + \{0,1\}$ *system for* P^{377,k_1,k_2} *with an integral left-hand side that consists of* k_2+1 *rows.*

4. *If* $Ax \leq b$ *is a TD*$(I \cap \mathbb{Z}) + \{0,1\}$ *system for* P^{377,k_1,k_2} *where* A *is integral,* $\left(A \mid -b \right)$ *contains the following* k_1+k_2 *rows:*

$$\left(\bigcup_{i=0}^{k_1} \left\{ \left(1 \quad i \mid -1 \right) \right\} \right) \dot\cup \left(\bigcup_{i=1}^{k_2-1} \left\{ \left(0 \quad -i \mid -\tfrac{i}{k_2} \right) \right\} \right).$$

On the other hand,

$$\begin{pmatrix} 1 & 0 \\ \vdots & \vdots \\ 1 & k_1 \\ 0 & -1 \\ \vdots & \vdots \\ 0 & -(k_2-1) \end{pmatrix} x \leq \begin{pmatrix} 1 \\ \vdots \\ 1 \\ \frac{1}{k_2} \\ \vdots \\ \frac{k_2-1}{k_2} \end{pmatrix}$$

is a TD$(I \cap \mathbb{Z}) + \{0,1\}$ *system for* P^{377,k_1,k_2} *with an integral left-hand side that consists of* k_1+k_2 *rows.*

This shows that if we use Theorem 398 to represent the Chvátal-Gomory closure, it can be advantageous to use TD$\mathbb{Z} + I$ systems instead of "only" using TDI systems, TD$\mathbb{Z} + \{0,1\}$ systems or TD$(I \cap \mathbb{Z}) + \{0,1\}$ systems (recall that by Lemma 350 and Lemma 351, these classes of systems are all TD$\mathbb{Z} + I$ systems), since through this, it can happen that we have to consider much less inequalities.

8.2.3.3. Considerations about minimality

Now one might ask whether the following "reverse" of Theorem 398 holds:

Problem 401. *Let* $A \in \mathbb{Z}^{l \times m}$ *and* $b \in \mathbb{R}^l$ *(*$l, m \in \mathbb{Z}_{\geq 0}$*) be such that*

- $P^{\leq}(A,b) \neq \emptyset$,
- $\nexists b' \in \mathbb{R}^l : b' \lneq b \wedge P^{\leq}(A,b) = P^{\leq}(A,b')$ *(i.e.* b *is chosen minimally) and*
- $\mathrm{cl}_{CG}\left(P^{\leq}(A,b)\right) = P^{\leq}(A, \lfloor b \rfloor)$.

Is $Ax \leq b$ *then a TD*$\mathbb{Z} + I$ *system?*

If the answer to Problem 401 was "yes", this would imply that TD$\mathbb{Z} + I$ systems (with an integral left-hand side) are in some sense the "best possible" systems from which we can derive the Chátal-Gomory closure by rounding the right-hand side down. Unluckily, the answer to Problem 401 is "no", even if we restrict ourselves to simple polyhedra $P^{\leq}(A,b)$ that have exactly one vertex.

For convenience, we restate Definition 380:

Definition 380. *For* $k \in \mathbb{Z}_{\geq 2}$*, define*

$$P^{380,k} := \left\{ \begin{pmatrix} \frac{k}{2} \\ \frac{1}{2} \end{pmatrix} \right\} + \operatorname{cone} \left\{ \begin{pmatrix} -\frac{k}{2} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\frac{k}{2} \\ -\frac{1}{2} \end{pmatrix} \right\} = P^{\leq}\left(\begin{pmatrix} 1 & k \\ 1 & -k \end{pmatrix}, \begin{pmatrix} k \\ 0 \end{pmatrix} \right) \subseteq \mathbb{R}^2.$$

The following lemma is easy to show (its proof is very similar to the proof of Lemma 307):

Lemma 402. *Let $k \in \mathbb{Z}_{\geq 1}$. Then*

$$\begin{aligned} P^{402,k} &:= \mathrm{cl}_{CG}\left(P^{380,k}\right) \\ &= \mathrm{conv}\left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{k-1}{2} \\ \frac{1}{2} \end{pmatrix}\right\} + \mathrm{cone}\left\{\begin{pmatrix} -\frac{k}{2} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\frac{k}{2} \\ -\frac{1}{2} \end{pmatrix}\right\} \\ &= P^{\leq}\left(\begin{pmatrix} 1 & k \\ 1 & -k \\ 1 & k-1 \\ 1 & -(k-1) \end{pmatrix}, \begin{pmatrix} k \\ 0 \\ k-1 \\ 0 \end{pmatrix}\right). \end{aligned}$$

Theorem 403 is a simple consequence of Lemma 402:

Theorem 403. *Let $k \in \mathbb{Z}_{\geq 2}$. Define*

$$A^{403,k} := \begin{pmatrix} 1 & k \\ 1 & -k \\ 1 & k-1 \\ 1 & -(k-1) \end{pmatrix}, \qquad b^{403,k} := \begin{pmatrix} k \\ 0 \\ k-\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$$

Then

- $P^{\leq}\left(A^{403,k}, b^{403,k}\right) = P^{380,k}$,
- *there exists no $b' \in \mathbb{R}^4$ such that $b' \lneq b^{403,k}$ and $P^{\leq}\left(A^{403,k}, b'\right) = P^{380,k}$ and*
- $P^{\leq}\left(A^{403,k}, \left\lfloor b^{403,k}\right\rfloor\right) = P^{402,k} = \mathrm{cl}_{CG}\left(P^{\leq}\left(A^{403,k}, b^{403,k}\right)\right)$.

In other words: the conditions of Problem 401 are satisfied for $A^{403,k}$ and $b^{403,k}$.

On the other hand, we saw in Theorem 381, which we restate here:

Theorem 381. *Let $k \in \mathbb{Z}_{\geq 2}$. Then:*

1. *Let $Ax \leq b$, where $A \in \mathbb{Z}^{l\times 2}$ and $b \in \mathbb{R}^l$ ($l \in \mathbb{Z}_{\geq 0}$), be a TDI system such that $P^{\leq}(A,b) = P^{380,k}$. Then*

   $$\left\{\left(\begin{array}{cc|c} 1 & i & -\frac{k+i}{2} \end{array}\right) : i \in \{-k, \ldots, k\}\right\}$$

   are rows of $\left(\begin{array}{c|c} A & -b \end{array}\right)$, i.e. $Ax \leq b$ consists of at least $2k+1$ rows. On the other hand,

   $$\begin{pmatrix} 1 & k \\ 1 & k-1 \\ 1 & k-2 \\ \vdots & \vdots \\ 1 & -(k-2) \\ 1 & -(k-1) \\ 1 & -k \end{pmatrix} x \leq \begin{pmatrix} k \\ k-\frac{1}{2} \\ k-1 \\ \vdots \\ 1 \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

   is a TD($I \cap \mathbb{Z}$) + $\{0,1\}$ system (and thus TDI system) for $P^{380,k}$ with an integral left-hand side that consists of $2k+1$ rows.

2. *Let $Ax \leq b$, where $A \in \mathbb{Z}^{l\times 2}$ and $b \in \mathbb{R}^l$ ($l \in \mathbb{Z}_{\geq 0}$), be a TD$\mathbb{Z}+I$ system such that $P^{\leq}(A,b) = P^{380,k}$. Then*

   $$\left\{\left(\begin{array}{cc|c} 1 & i & -\frac{k+i}{2} \end{array}\right) : i \in \{-(k-1), \ldots, k-1\}, k+i \text{ odd}\right\}$$

   are rows of $\left(\begin{array}{c|c} A & -b \end{array}\right)$, i.e. $Ax \leq b$ consists of at least k rows. On the other hand,

   $$\begin{pmatrix} 1 & k \\ 1 & -k \\ 1 & k-1 \\ 1 & k-3 \\ \vdots & \vdots \\ 1 & -(k-3) \\ 1 & -(k-1) \end{pmatrix} x \leq \begin{pmatrix} k \\ 0 \\ k-\frac{1}{2} \\ k-\frac{3}{2} \\ \vdots \\ \frac{3}{2} \\ \frac{1}{2} \end{pmatrix}$$

is a TD$\mathbb{Z}+\{0,1\}$ system (and thus TD$\mathbb{Z}+I$ system) for $P^{380,k}$ that consists of $k+2$ rows.

So the answer to Problem 401 is "no", which means that it is arguably an open research question to find a type of system (presumably generalizing TD$\mathbb{Z}+I$ systems) which satisfies the property of Problem 401.

8.2.4. Representation via LP face cones

In section 8.2.3, we saw (Theorem 398) how one can represent the Chvátal-Gomory closure via a TD$\mathbb{Z}$+I system. This representation has the disadvantage that it is very specific to the Chvátal-Gomory closure and additionally only works in the situation where all variables are integral (i.e. no variable is continuous, which means that we cannot generalize it easily to the mixed-integer case (projected Chvátal-Gomory closure)). In particular, being very specific to the Chvátal-Gomory closure means that we are not aware of any method how this result can be generalized to other types of closures, in particular the split closure. In this section, we formulate a way of representing the Chvátal-Gomory closure for which such a generalization to the split closure exists. For this, compare Theorem 404, which we show here, to Theorem 429, which one can consider as its analogue for the split closure (MIR closure).

To find an alternative representation, we first remark that it suffices to consider the situation that the rows of $\begin{pmatrix} A & G & -b \end{pmatrix}$ in $P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)$ form an LP face cone. Indeed, by Theorem 394, the computation of $\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}\left(P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)\right)$ can be reduced to the computation of some set of

$$\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}\left(P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}_{S,*}, b_S\right)\right),$$

where either $S \in \mathcal{F}^{*,feas}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)$, $S \in \mathcal{B}^{*,feas}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)$ or $S \in \mathcal{B}^{*}\left(\begin{pmatrix} A & G \end{pmatrix}\right)$. By Theorem 357, rowcone $\begin{pmatrix} A & G & -b \end{pmatrix}_{S,*}$ is indeed an LP face cone for such an S.

The next theorem (Theorem 404) provides a method for computing the Chvátal-Gomory closure of such a polyhedron if A and G are rational matrices. We recall that Theorem 366 provides a method to compute an icone system of a cone with generators from $\mathbb{Z}^d \times \mathbb{R}$ ($d \in \mathbb{Z}_{\geq 0}$). Recall that, by Lemma 363, the property of being an icone system implies being a $\mathbb{Z}$+icone system (the latter is what is necessary for the characterization in Theorem 404). These components can be used together to formulate an algorithm for computing the projected Chvátal-Gomory closure of either a rational polyhedron or a polyhedron with rational face normals.

Theorem 404. *Let $P := P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)$, where $A \in \mathbb{R}^{l\times m}$, $G \in \mathbb{R}^{l\times n}$ and $b \in \mathbb{R}^l$ ($l, m, n \in \mathbb{Z}_{\geq 0}$), be such that*

$$C := \mathrm{rowcone}\begin{pmatrix} A & G & -b \end{pmatrix}$$

is an LP face cone. Additionally, let

$$C' := C \cap \left(\mathbb{R}^m \times 0^n \times \mathbb{R}\right)^T$$

be finitely generated by vectors from $\left(\mathbb{Q}^m \times 0^n \times \mathbb{R}\right)^T$ (this is surely the case if A and G are rational). Let $\mathcal{S}$ be a $\mathbb{Z}+$icone system generating C' that consists of vectors from $\left(\mathbb{Z}^m \times 0^n \times \mathbb{R}\right)^T$. Then

$$\mathrm{cl}_{pCG}(P) = P \cap \bigcap_{\substack{\begin{pmatrix} \hat{a} & (0^n)^T & -\hat{b} \end{pmatrix} \in \mathcal{S}: \\ \mathrm{frac}\,\hat{b} \neq 0}} P^{\leq}\left(\begin{pmatrix} \hat{a} & (0^n)^T \end{pmatrix}, \hat{b}\right).$$

Proof. W.l.o.g. let $P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right) \neq \emptyset$. Let $\mu \in \mathcal{M}_{CG}(A, G)$. Then, clearly,

$$\mu \begin{pmatrix} A & G & -b \end{pmatrix} \in \underbrace{C' \cap \left(\mathbb{Z}^m \times 0^n \times \mathbb{R}\right)^T}_{=C' \cap \left(\mathbb{Z}^m \times \mathbb{Z}^n \times \mathbb{R}\right)^T}.$$

On the other hand, since $\mathcal{S}$ is an $\mathbb{Z}+$icone system for C', there exist

$$\begin{pmatrix} \hat{a}^1 & (0^n)^T & -\hat{b}^1 \end{pmatrix} \in C' \cap \left(\mathbb{Z}^m \times \mathbb{Z}^n \times \mathbb{Z}\right)^T = C' \cap \left(\mathbb{Z}^m \times 0^n \times \mathbb{Z}\right)^T$$

and

$$\begin{pmatrix} \hat{a}^{2,1} & (0^n)^T & -\hat{b}^{2,1} \end{pmatrix}, \ldots, \begin{pmatrix} \hat{a}^{2,k} & (0^n)^T & -\hat{b}^{2,k} \end{pmatrix} \in \mathcal{S}$$

($k \in \mathbb{Z}_{\geq 0}$) such that

$$\mu \begin{pmatrix} A & G & -b \end{pmatrix} = \begin{pmatrix} \hat{a}^1 & (0^n)^T & -\hat{b}^1 \end{pmatrix} + \sum_{i=1}^{k} \begin{pmatrix} \hat{a}^{2,i} & (0^n)^T & -\hat{b}^{2,i} \end{pmatrix}.$$

W.l.o.g. we can assume that

$$\forall i \in [k] : \begin{pmatrix} \hat{a}^{2,i} & (0^n)^T & -\hat{b}^{2,i} \end{pmatrix} \notin (\mathbb{Z}^m \times 0^n \times \mathbb{Z})^T.$$

There exist (not necessarily uniquely defined) $\mu^1, \mu^{2,1}, \ldots, \mu^{2,k} \in \left(\mathbb{R}^l_{\geq 0}\right)^T$ having

$$\begin{aligned} \begin{pmatrix} \hat{a}^1 & (0^n)^T & -\hat{b}^1 \end{pmatrix} &= \mu^1 \begin{pmatrix} A & G & -b \end{pmatrix}, \\ \forall i \in [k] : \begin{pmatrix} \hat{a}^{2,i} & (0^n)^T & -\hat{b}^{2,i} \end{pmatrix} &= \mu^{2,i} \begin{pmatrix} A & G & -b \end{pmatrix}. \end{aligned}$$

Let

$$\mu' := \mu^1 + \sum_{i=1}^{k} \mu^{2,i}.$$

Since $\mu \begin{pmatrix} A & G & -b \end{pmatrix} = \mu' \begin{pmatrix} A & G & -b \end{pmatrix}$, clearly $f^{\leq,CG,\mu}_{A,G,b} = f^{\leq,CG,\mu'}_{A,G,b}$. On the other hand, by Lemma 395 (since $\mu^1 b \in \mathbb{Z}$) and Lemma 396, the inequality $f^{\leq,CG,\mu'}_{A,G,b}(\,\cdot\,) \leq 0$ is dominated relatively to P by

$$\left\{ f^{\leq,CG,\mu^{2,1}}_{A,G,b}(\,\cdot\,) \leq 0, \ldots, f^{\leq,CG,\mu^{2,k}}_{A,G,b}(\,\cdot\,) \leq 0 \right\}.$$

We finally consider that for all $i \in [k]$, we have

$$f^{\leq,CG,\mu^{2,i}}_{A,G,b}(\,\cdot\,) = \begin{pmatrix} \hat{a}^{2,i} & (0^n)^T \end{pmatrix}(\,\cdot\,) - \hat{b}^{2,i}.$$

□

Now for a second polyhedricity proof for the (projected) Chvátal-Gomory closure of a polyhedron. In contrast to Theorem 399, this time, we do not only consider the pure integer case, but also the mixed-integer case.

Theorem 405. *We have:*

1. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be a rational polyhedron. Then $\mathrm{cl}_{pCG}(P)$ is again a rational polyhedron.*
2. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) be a polyhedron with rational face normals. Then $\mathrm{cl}_{pCG}(P)$ is again a polyhedron with rational face normals such that $\mathrm{proj}_{\mathbb{R}^m}(\mathrm{cl}_{pCG}(P))$ is a rational polyhedron.*

Proof. Let $P := P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right) \subseteq \mathbb{R}^m \times \mathbb{R}^n$ be given, where $A \in \mathbb{Q}^{l \times m}$, $G \in \mathbb{Q}^{l \times n}$ and $b \in \mathbb{R}^l$ or $b \in \mathbb{Q}^l$, respectively ($l \in \mathbb{Z}_{\geq 0}$). Then, by Theorem 394, we have

$$\mathrm{cl}_{pCG}(P) = \bigcap_{S \in \mathcal{F}^{*,feas}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)} \mathrm{cl}_{pCG}\left(P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}_{S,*}, b_S\right)\right).$$

By Theorem 357, for every $S \in \mathcal{F}^{*,feas}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)$, the cone

$$\mathrm{rowcone}\begin{pmatrix} A & G & -b \end{pmatrix}_{S,*} =: C_S$$

is an LP face cone. For $S \in \mathcal{F}^{*,feas}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)$, let $\mathcal{S}(S)$ be a $\mathbb{Z}+$icone system that generates C_S and consists of vectors from $(\mathbb{Z}^m \times 0^n \times \mathbb{R})^T$ or $(\mathbb{Z}^m \times 0^n \times \mathbb{Q})^T$, respectively (such a $\mathbb{Z}+$icone system exists by Theorem 366). Using Theorem 404, we thus conclude:

$$\mathrm{cl}_{pCG}(P) = P \cap \bigcap_{S \in \mathcal{F}^{*,feas}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)} \bigcap_{\substack{\begin{pmatrix} \hat{a} & (0^n)^T & -\hat{b} \end{pmatrix} \in \mathcal{S}(S): \\ \mathrm{frac}\,\hat{b} \neq 0}} P^{\leq}\left(\begin{pmatrix} \hat{a} & (0^n)^T \end{pmatrix}, \hat{b}\right). \tag{8.9}$$

The statement that $\mathrm{cl}_{pCG}(P)$ is a rational polyhedron/a polyhedron with rational face normals if P is, is then an immediate consequence of (8.9).

For $\mathrm{proj}_{\mathbb{R}^m}(\mathrm{cl}_{CG}(P))$ being a rational polyhedron in 2: if $n = 0$, this is an immediate consequence of (8.9) by considering that for each $i \in [l]$, the inequality $A_{i,*}(\cdot) \leq b_i$ is dominated by the Chvátal-Gomory cut $A_{i,*}(\cdot) \leq \lfloor b_i \rfloor$. In the general case, it is easy to check that

$$\mathrm{cl}_{pCG}(P) = P \cap (\mathrm{cl}_{CG}(\mathrm{proj}_{\mathbb{R}^m} P) \times \mathbb{R}^n)$$

holds. Thus,

$$\mathrm{proj}_{\mathbb{R}^m}(\mathrm{cl}_{pCG}(P)) = (\mathrm{proj}_{\mathbb{R}^m} P) \cap \mathrm{cl}_{CG}(\mathrm{proj}_{\mathbb{R}^m} P) = \mathrm{cl}_{CG}(\mathrm{proj}_{\mathbb{R}^m} P).$$

The final statement is then a consequence of the fact that, by Corollary 65, $\mathrm{proj}_{\mathbb{R}^m} P$ is a polyhedron with rational face normals. □

9. $L_{1,\mathbb{Q}}$ cuts, split cuts and MIR cuts

This chapter consists of two rather independent parts:

- In section 9.1, we analyze the relationship between $L_{1,\mathbb{Q}}$ cuts (cf. Definition 161), split cuts (cf. Definition 126) and MIR cuts (which we define in Definition 410):
  - In section 9.1.1, Theorem 409, we prove the the equivalence of $L_{1,\mathbb{Q}}$ cuts and split cuts for a given $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) if P is convex (or more precisely: prove the missing part (Lemma 406) for a complete proof of this statement) and see (Remark 407) that convexity is indeed essential for this equivalence to hold.
  - In section 9.1.2, we define MIR cuts and prove the equivalence of the MIR closure and the split closure. This equivalence is in principle well-known in the literature. What is the reason why we reprove these results from ground up?
    * There exist multiple definitions of MIR cuts in the literature. For example, a classic definition of MIR cuts can be found in [NW90]. This construction is simplified in [DGL10] by Dash, Günlük and Lodi. Another different definition can be found in [CCZ14, section 5.1.5].

      Because our goal is to generalize Theorem 404 (which stated how one can compute the Chvátal-Gomory closure of a polyhedron $P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)$ with rational face normals, where the rows of $\begin{pmatrix} A & G & -b \end{pmatrix}$ form an LP face cone) from the (dual projected) Chvátal-Gomory closure to the MIR closure (the final result of this journey is Theorem 429 in section 9.2), we define MIR cuts in Definition 410 in a way that is as similar as possible to the definition of dual (projected) Chvátal-Gomory cuts that we gave in Definition 382 .

      Our presentation of MIR cuts is guided by the presentation that is given in [DGL10] (though not completely equivalent).
    * The "equivalence" of MIR cuts and split cuts is not symmetric. While every MIR cut is a split cut (Theorem 412), not every split cut is a MIR cut. One can only show (Theorem 415) that every split cut for some $P := P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)$ is dominated *relatively to* P by a MIR cut with respect to A, G and b (not even dominated absolutely; cf. Example 416). This subtle asymmetry is typically only touched on in the literature.
- In section 9.2, we show that the split closure of a polyhedron $P = P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right) \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) with rational face normals is a polyhedron (a rational polyhedron if P was a rational polyhedron). This is the statement of Theorem 417, which we show in section 9.2.4. While there exist multiple proofs of the (rational) polyhedricity of the split closure of a rational polyhedron (in section 6.3, we gave an overview), our method has multiple advantages:
  - It also works for polyhedra with rational face normals and sometimes even for non-rational polyhedra (in section 9.2.5, we consider such an example), while in the literature typically only rational polyhedra are considered.
  - Theorem 429 is a natural generalization of the method that we used in Theorem 398 to compute the Chvátal-Gomory closure of a polyhedron $P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)$, where the rows of $\begin{pmatrix} A & G & -b \end{pmatrix}$ form an LP face cone:
    * The dominance result that we formulate in Lemma 423 is the analogue of Lemma 395.
    * The dominance result that we formulate in Lemma 424 is the analogue of Lemma 396.
    * Theorem 427, which tells us that we can restrict ourselves to "MIR closures of basic relaxations", is the analogue of Theorem 394.
  - Our method can easily be turned into an algorithm. The only necessary algorithmic ingredients for this are:
    * a method to compute $\mathcal{B}^*\left(\begin{pmatrix} A & G \end{pmatrix}\right)$ (cf. Definition 392) so that we can apply Theorem 267,
    * a method to compute generating rays from $\mathbb{Z}^m \times 0^n \times \mathbb{R}$ for $C \cap (\mathbb{R}^m \times 0^n \times \mathbb{R})$, where C is a simplicial cone with rows from $\begin{pmatrix} A & G & -b \end{pmatrix}$,

* a method to compute a $\mathbb{Z}$+icone system of a polyhedral cone that is given in terms of generating rays from $\mathbb{Z}^m \times 0^n \times \mathbb{R}$. Theorem 366 gives a method to compute an icone system, which, by Lemma 363, is an even stronger property.

In Theorem 430, we execute the steps by hand that a computer would perform to find an explicit representation of the split/MIR closure of the Cook-Kannan-Schrijver example.

9.1. Equivalences/non-equivalences

9.1.1. Equivalence of $L_{1,\mathbb{Q}}$ cuts and split cuts

We already proved in Corollary 278 that for arbitrary sets $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$, where $m + n \geq 1$), every split cut for P is an $L_{1,\mathbb{Q}}$ cut for P. We now show that if P is convex, also the converse holds:

Lemma 406. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$, where $m + n \geq 1$) be convex. Then every $L_{1,\mathbb{Q}}$ cut for P is a split cut for P.*

Remark 407. *The condition that P is convex is essential for Theorem 406 to hold. For this, consider*

$$P := \left(\mathbb{R}^1 \times \mathbb{R}^1\right) \setminus \left(\mathbb{Z}^1 \times \mathbb{R}^1\right).$$

Obviously, $0(\,\cdot\,) \leq -1$ is a valid $L_{1,\mathbb{Q}}$ cut for P (cf. Theorem 202), but for all split disjunctions $D(\pi, \gamma) \subseteq \mathbb{R}^1$, we have

$$\operatorname{conv}\left(P \cap \left(D(\pi,\gamma) \times \mathbb{R}^1\right)\right) = \mathbb{R}^1 \times \mathbb{R}^1.$$

Proof. (Lemma 406) W.l.o.g. we can assume $m \geq 1$. Let $c(\,\cdot\,) \leq c_0$ be an $L_{1,\mathbb{Q}}$ cut for P. By Lemma 98, there thus exists a $w \in \mathbb{Z}^m \setminus \{0^m\}$ such that $c(\,\cdot\,) \leq c_0$ is a valid inequality for

$$P \cap \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R}^n : w^T x \in \mathbb{Z} \right\} =: (9.1).$$

W.l.o.g. let $c(\,\cdot\,) \leq c_0$ not be a valid inequality for P. We claim that there exists at most one $z^* \in \mathbb{Z}$ such that

$$\exists p \in P \cap \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R}^n : z^* < w^T x < z^* + 1 \right\} : cp > c_0. \tag{9.2}$$

If we can show this, we are done, since this demonstrates that $c(\,\cdot\,) \leq c_0$ is valid for $P \cap D\left(w^T, z^*\right)$.

Remark 408. *One can show that a z^* such that* (9.2) *is satisfied does indeed exist, since if none existed, $c(\,\cdot\,) \leq c_0$ would already be valid for P, which we have excluded.*

So assume that the statement does not hold, i.e. there exist $z^{*,1}, z^{*,2} \in \mathbb{Z}$, where $z^{*,1} < z^{*,2}$, and $\begin{pmatrix} p^{*,1} \\ q^{*,1} \end{pmatrix}, \begin{pmatrix} p^{*,2} \\ q^{*,2} \end{pmatrix} \in P$ having

$$\begin{aligned} \forall i \in [2] : c \begin{pmatrix} p^{*,i} \\ q^{*,i} \end{pmatrix} &> c_0, \\ \forall i \in [2] : \qquad w^T p^{*,i} &\in \left(z^{*,i}, z^{*,i} + 1\right). \end{aligned}$$

Let

$$\begin{aligned} \lambda_1 &:= \frac{w^T p^{*,2} - z^{*,2}}{w^T \left(p^{*,2} - p^{*,1}\right)}, \\ \lambda_2 &:= \frac{z^{*,2} - w^T p^{*,1}}{w^T \left(p^{*,2} - p^{*,1}\right)}, \\ \begin{pmatrix} p^* \\ q^* \end{pmatrix} &:= \lambda_1 \begin{pmatrix} p^{*,1} \\ q^{*,1} \end{pmatrix} + \lambda_2 \begin{pmatrix} p^{*,2} \\ q^{*,2} \end{pmatrix}. \end{aligned}$$

Obviously, we have $\lambda_1, \lambda_2 > 0$ and $\lambda_1 + \lambda_2 = 1$. Thus, $\begin{pmatrix} p^* \\ q^* \end{pmatrix} \in \operatorname{conv} P = P$. On the other hand, we have

$$w^T p^* = \frac{w^T p^{*,2} - z^{*,2}}{w^T \left(p^{*,2} - p^{*,1}\right)} w^T p^{*,1} + \frac{z^{*,2} - w^T p^{*,1}}{w^T \left(p^{*,2} - p^{*,1}\right)} w^T p^{*,2} = z^{*,2} \in \mathbb{Z}.$$

So, $\binom{p^*}{q^*} \in (9.1)$. Finally, we have

$$c\begin{pmatrix} p^* \\ q^* \end{pmatrix} = c\left(\lambda_1 \begin{pmatrix} p^{*,1} \\ q^{*,1} \end{pmatrix} + \lambda_2 \begin{pmatrix} p^{*,2} \\ q^{*,2} \end{pmatrix}\right) = \lambda_1 \cdot c\begin{pmatrix} p^{*,1} \\ q^{*,1} \end{pmatrix} + \lambda_2 \cdot c\begin{pmatrix} p^{*,2} \\ q^{*,2} \end{pmatrix} > \lambda_1 \cdot c_0 + \lambda_2 \cdot c_0 = c_0,$$

which is a contradiction, since we have seen that $\binom{p^*}{q^*} \in (9.1)$, but $c(\cdot) \le c_0$ is valid for (9.1). □

So, we get the following theorem:

Theorem 409. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\ge 0}$, where $m + n \ge 1$) be convex. Then every $L_{1,\mathbb{Q}}$ cut for P is a split cut for P and vice versa. We thus have*

$$\mathrm{cl}_{L_{1,\mathbb{Q}}}(P) = \mathrm{cl}_{split}(P).$$

9.1.2. Equivalence of the MIR closure and the split closure

9.1.2.1. Definitions for MIR cuts

Definition 410. *Let $A \in \mathbb{R}^{l \times m}$, $G \in \mathbb{R}^{l \times n}$ and $b \in \mathbb{R}^l$, where $l, m, n \in \mathbb{Z}_{\ge 0}$. Define*

$$\mathcal{M}_{MIR}(A, G) := \left\{\mu \in \left(\mathbb{R}^l\right)^T : \mu \begin{pmatrix} A & G \end{pmatrix} \in (\mathbb{Z}^m \times 0^n)^T\right\}.$$

Let $\mu \in \mathcal{M}_{MIR}(A, G)$. Then we call the inequality

$$\left((1 - \mathrm{frac}(\mu b))\mu^+ + \mathrm{frac}(\mu b)\mu^-\right)(Ax + Gy - b) \le -\mathrm{frac}(\mu b)(1 - \mathrm{frac}(\mu b))$$

a ***MIR cut for*** $\binom{x}{y} \in P^{\le}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)$ ***with respect to*** A, G ***and*** b*. We set*

$$\begin{aligned} f_{A,G,b}^{\le 0,MIR,\mu} : \mathbb{R}^m \times \mathbb{R}^n \to& \mathbb{R}, \\ \begin{pmatrix} x \\ y \end{pmatrix} \mapsto& \left((1 - \mathrm{frac}(\mu b))\mu^+ + \mathrm{frac}(\mu b)\mu^-\right)(Ax + Gy - b) \\ &+ \mathrm{frac}(\mu b)(1 - \mathrm{frac}(\mu b)) \\ =& \left(\mu^+ - \mathrm{frac}(\mu b)\mu\right)(Ax + Gy - b) + \mathrm{frac}(\mu b)(1 - \mathrm{frac}(\mu b)) \\ =& \left(\mu^- + (1 - \mathrm{frac}(\mu b))\mu\right)(Ax + Gy - b) + \mathrm{frac}(\mu b)(1 - \mathrm{frac}(\mu b)) \end{aligned}$$

(see also [DGL10]). If $n = 0$, we also use $f_{A,b}^{\le 0,MIR,\mu}$ instead of $f_{A,G,b}^{\le 0,MIR,\mu}$

Now one might ask what the idea behind the definition of a MIR cut is. For an alternative presentation of the explanation cf. [Wol98, section 8.7] and [DGL10, section 2].

We start with the mixed-integral set

$$X_b := \left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{Z}^1 \times \mathbb{R}^1 : x + y \ge b, y \ge 0\right\},$$

where $b \in \mathbb{R}$. It is easy to check that

$$\mathrm{cl}_I(X_b) = \left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^1 \times \mathbb{R}^1 : x + y \ge b, y \ge 0, (\mathrm{frac}\, b)\, x + y \ge (\mathrm{frac}\, b)(b + 1 - \mathrm{frac}\, b)\right\}$$

(if $b \in \mathbb{Z}$, the inequality $(\mathrm{frac}\, b)\, x + y \ge (\mathrm{frac}\, b)(b + 1 - \mathrm{frac}\, b)$ is redundant); in particular,

$$(\mathrm{frac}\, b)\, x + y \ge (\mathrm{frac}\, b)(b + 1 - \mathrm{frac}\, b) \tag{9.3}$$

is valid for X_b. Now we consider the polyhedron $P := P^{\le}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right) \subseteq \mathbb{R}^m \times \mathbb{R}^n$, where A, G and b are as in Definition 410. Let $\mu \in \mathcal{M}_{MIR}(A, G)$ and let $\binom{x}{y} \in P_I$. Then

$$\begin{pmatrix} \mu A x \\ \mu^+ (b - Ax - Gy) \end{pmatrix} \in X_{\mu b}. \tag{9.4}$$

For $\mu Ax + \mu^{+}(b - Ax - Gy) \geq \mu b$: using $\mu G = (0^n)^T$, we get

$$\mu Ax + \mu^{+}(b - Ax - Gy) = \underbrace{(\mu^{+} - \mu)}_{\geq (0^l)^T} \underbrace{(b - Ax - Gy)}_{\geq 0^l} + \mu b \geq \mu b.$$

Thus, by combining (9.3) and (9.4), we obtain the inequality

$$(\mathrm{frac}(\mu b))(\mu Ax) + \mu^{+}(b - Ax - Gy) \geq (\mathrm{frac}(\mu b))(\mu b + 1 - \mathrm{frac}(\mu b)).$$

for P_I. Finally, note that

$$\begin{aligned}
&(\mathrm{frac}(\mu b))(\mu b + 1 - \mathrm{frac}(\mu b)) - \mathrm{frac}(\mu b)(\mu Ax) - \mu^{+}(b - Ax - Gy) \\
=&\left(\mu^{+} - \mathrm{frac}(\mu b)\mu\right)(Ax + Gy - b) + \mathrm{frac}(\mu b)(1 - \mathrm{frac}(\mu b)) && (\mu G = (0^n)^T) \\
=& f^{\leq 0, MIR, \mu}_{A,G,b}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right).
\end{aligned}$$

Definition 411. *Let A, G, b, m and n be as in Definition 410. We define*

$$\mathrm{cl}_{MIR}(A, G, b) := P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right) \cap \bigcap_{\mu \in \mathcal{M}_{MIR}(A,G)} \left\{x \in \mathbb{R}^m \times \mathbb{R}^n : f^{\leq 0, MIR, \mu}_{A,G,b}(x) \leq 0\right\}$$

as the ***MIR closure with respect to*** A***,*** G ***and*** b*.*

In section 8.1.1, we already remarked that the definition of MIR cuts (Definition 410) closely mirrors the definition of dual projected Chvátal-Gomory cuts (Definition 382) and, similarly, the definition of the MIR closure (Definition 411) closely mirrors the definition of the dual (projected) Chvátal-Gomory closure (Definition 383).

9.1.2.2. MIR cuts are split cuts

Theorem 412. *Let A, G and b be as in Definition 410 and let $\mu \in \mathcal{M}_{MIR}(A, G)$. Then*

$$f^{\leq 0, MIR, \mu}_{A,G,b}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) \leq 0$$

is a valid inequality for

$$\begin{pmatrix} x \\ y \end{pmatrix} \in P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}_{\mathrm{supp}\,\mu, *}, b_{\mathrm{supp}\,\mu}\right) \cap \left(D\left((\mu A)^T, \lfloor \mu b \rfloor\right) \times \mathbb{R}^n\right).$$

In other words: every MIR cut is a split cut.

Proof. Let

$$\begin{pmatrix} x \\ y \end{pmatrix} \in P^{\leq}((\begin{matrix} A & G \end{matrix}))_{\mathrm{supp}\,\mu, *}, b)_{\mathrm{supp}\,\mu}) \cap \left(D\left((\mu A)^T, \lfloor \mu b \rfloor\right) \times \mathbb{R}^n\right)$$

and let $\mu \in \mathcal{M}_{MIR}(A, G)$. We distinguish two cases:

1. $\mu Ax \leq \lfloor \mu b \rfloor$,
2. $\mu Ax \geq \lfloor \mu b \rfloor + 1$.

For case 1:

$$\begin{aligned}
f^{\leq 0, MIR, \mu}_{A,G,b}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) &= \left((1 - \mathrm{frac}(\mu b))\mu^{+} + \mathrm{frac}(\mu b)\mu^{-}\right)(Ax + Gy - b) + \mathrm{frac}(\mu b)(1 - \mathrm{frac}(\mu b)) \\
&= (\mu^{-} + \underbrace{(1 - \mathrm{frac}(\mu b))\mu^{+} - (1 - \mathrm{frac}(\mu b))\mu^{-}}_{=(1 - \mathrm{frac}(\mu b))\mu})(Ax + Gy - b) \\
&\quad + \mathrm{frac}(\mu b)(1 - \mathrm{frac}(\mu b)) \\
&= \mu^{-}(Ax + Gy - b) + (1 - \mathrm{frac}(\mu b))\underbrace{\mu(Ax + Gy - b)}_{=\mu Ax - \mu b} + \mathrm{frac}(\mu b)(1 - \mathrm{frac}(\mu b))
\end{aligned}$$

$$
\begin{aligned}
&= \underbrace{\mu^-_{\operatorname{supp}\mu} (Ax + Gy - b)_{\operatorname{supp}\mu}}_{\leq 0} + (1 - \operatorname{frac}(\mu b)) (\mu A x - \mu b) + \operatorname{frac}(\mu b) (1 - \operatorname{frac}(\mu b)) \\
&\leq (1 - \operatorname{frac}(\mu b)) (\mu A x - \mu b) + \operatorname{frac}(\mu b) (1 - \operatorname{frac}(\mu b)) \\
&\leq (1 - \operatorname{frac}(\mu b)) (\lfloor \mu b \rfloor - \mu b) + \operatorname{frac}(\mu b) (1 - \operatorname{frac}(\mu b)) \\
&= (1 - \operatorname{frac}(\mu b)) \operatorname{frac}(\mu b) + \operatorname{frac}(\mu b) (1 - \operatorname{frac}(\mu b)) \\
&= 0.
\end{aligned}
$$

For case 2:

$$
\begin{aligned}
f^{\leq 0, MIR, \mu}_{A,G,b}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) &= \left((1 - \operatorname{frac}(\mu b)) \mu^+ + \operatorname{frac}(\mu b) \mu^-\right) (Ax + Gy - b) + \operatorname{frac}(\mu b) (1 - \operatorname{frac}(\mu b)) \\
&= \mu^+ (Ax + Gy - b) - \operatorname{frac}(\mu b) \underbrace{\mu (Ax + Gy - b)}_{= \mu A x - \mu b} + \operatorname{frac}(\mu b) (1 - \operatorname{frac}(\mu b)) \\
&= \underbrace{\mu^+_{\operatorname{supp}\mu} (Ax + Gy - b)_{\operatorname{supp}\mu}}_{\leq 0} - \operatorname{frac}(\mu b) (\mu A x - \mu b) + \operatorname{frac}(\mu b) (1 - \operatorname{frac}(\mu b)) \\
&\leq \underbrace{- \operatorname{frac}(\mu b)}_{\leq 0} (\underbrace{\mu A x}_{\geq \lfloor \mu b \rfloor + 1} - \mu b) + \operatorname{frac}(\mu b) (1 - \operatorname{frac}(\mu b)) \\
&\leq - \operatorname{frac}(\mu b) (1 - (\mu b - \lfloor \mu b \rfloor)) + \operatorname{frac}(\mu b) (1 - \operatorname{frac}(\mu b)) \\
&= - \operatorname{frac}(\mu b) (1 - \operatorname{frac}(\mu b)) + \operatorname{frac}(\mu b) (1 - \operatorname{frac}(\mu b)) \\
&= 0.
\end{aligned}
$$

□

9.1.2.3. Split cuts are dominated by MIR cuts

We start this section with two propositions (Proposition 413 and Proposition 414), before we write down our final result in Theorem 415.

Proposition 413. *Let A, G, b, l, m and n be as in Definition 410. Let $\pi \in \mathbb{Z}^m$, $\gamma \in \mathbb{Z}$, $c \in (\mathbb{R}^m \times \mathbb{R}^n)^T$, $c_0 \in \mathbb{R}$ and $z^1, z^2 \in \left(\mathbb{R}^{l+1}_{\geq 0}\right)^T$ be such that*

$$
z^1 \begin{pmatrix} A & G \\ \pi^T & (0^n)^T \end{pmatrix} = c, \qquad z^1 \begin{pmatrix} b \\ \gamma \end{pmatrix} = c_0,
$$

$$
z^2 \begin{pmatrix} A & G \\ -\pi^T & (0^n)^T \end{pmatrix} = c, \qquad z^2 \begin{pmatrix} b \\ -(\gamma+1) \end{pmatrix} = c_0,
$$

where $z^1_{l+1}, z^2_{l+1} > 0$. Define

$$
\begin{aligned}
\mu &:= \frac{z^2_{[l]} - z^1_{[l]}}{z^1_{l+1} + z^2_{l+1}}, \\
S &:= \left(\operatorname{supp} z^1\right) \cup \left(\operatorname{supp} z^2\right).
\end{aligned}
$$

Then

$$
\mu \begin{pmatrix} A & G \end{pmatrix} = \begin{pmatrix} \pi^T & (0^n)^T \end{pmatrix}, \tag{9.5}
$$

$$
\mu b = \gamma + \frac{z^2_{l+1}}{z^1_{l+1} + z^2_{l+1}} \in (\gamma, \gamma + 1), \tag{9.6}
$$

and $c(\cdot) \leq c_0$ is dominated by $\begin{pmatrix} A & G \end{pmatrix}_{S,*}(\cdot) \leq b_S$ and $f^{\leq 0, MIR, \mu}_{A,G,b}(\cdot) \leq 0$.

Proof.

For (9.5):

$$\begin{aligned}\mu \begin{pmatrix} A & G \end{pmatrix} =& \frac{z^2}{z^1_{l+1}+z^2_{l+1}} \begin{pmatrix} A & G \\ -\pi^T & (0^n)^T \end{pmatrix} - \frac{z^1}{z^1_{l+1}+z^2_{l+1}} \begin{pmatrix} A & G \\ \pi^T & (0^n)^T \end{pmatrix} + \begin{pmatrix} \pi^T & (0^n)^T \end{pmatrix} \\ =& \frac{1}{z^1_{l+1}+z^2_{l+1}} c - \frac{1}{z^1_{l+1}+z^2_{l+1}} c + \begin{pmatrix} \pi^T & (0^n)^T \end{pmatrix} \\ =& \begin{pmatrix} \pi^T & (0^n)^T \end{pmatrix}.\end{aligned}$$

For (9.6):

$$\begin{aligned}\mu b =& \frac{z^2}{z^1_{l+1}+z^2_{l+1}} \begin{pmatrix} b \\ -(\gamma+1) \end{pmatrix} - \frac{z^1}{z^1_{l+1}+z^2_{l+1}} \begin{pmatrix} b \\ \gamma \end{pmatrix} + \gamma + \frac{z^2_{l+1}}{z^1_{l+1}+z^2_{l+1}} \\ =& c_0 - c_0 + \gamma + \frac{z^2_{l+1}}{z^1_{l+1}+z^2_{l+1}} \\ =& \gamma + \frac{z^2_{l+1}}{z^1_{l+1}+z^2_{l+1}}.\end{aligned}$$

For the dominance: We have

$$\frac{z^2_{[l]}}{z^1_{l+1}+z^2_{l+1}} - \mu^+ = \frac{z^2_{[l]}}{z^1_{l+1}+z^2_{l+1}} - \left(\frac{z^2_{[l]} - z^1_{[l]}}{z^1_{l+1}+z^2_{l+1}}\right)^+ \geq \left(0^l\right)^T. \tag{9.7}$$

Let $\binom{x}{y} \in P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)$ satisfy $f^{\leq 0, MIR, \mu}_{A,G,b}\left(\binom{x}{y}\right) \leq 0$. Then

$$\begin{aligned}c\begin{pmatrix} x \\ y \end{pmatrix} &= \left(z^1_{l+1}+z^2_{l+1}\right) \frac{z^2}{z^1_{l+1}+z^2_{l+1}} \begin{pmatrix} A & G \\ -\pi^T & (0^n)^T \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \left(z^1_{l+1}+z^2_{l+1}\right) \left(\frac{z^2_{[l]}}{z^1_{l+1}+z^2_{l+1}} - \frac{z^2_{l+1}}{z^1_{l+1}+z^2_{l+1}} \mu \right) \begin{pmatrix} A & G \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \underbrace{\left(z^1_{l+1}+z^2_{l+1}\right)}_{\geq 0} \left(\mu^+ - \underbrace{\frac{z^2_{l+1}}{z^1_{l+1}+z^2_{l+1}}}_{=\operatorname{frac}(\mu b)} \mu \right) \begin{pmatrix} A & G \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &\quad + \left(z^1_{l+1}+z^2_{l+1}\right) \left(\frac{z^2_{[l]}}{z^1_{l+1}+z^2_{l+1}} - \mu^+ \right) \begin{pmatrix} A & G \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &\leq \left(z^1_{l+1}+z^2_{l+1}\right) \left(\mu^+ - \frac{z^2_{l+1}}{z^1_{l+1}+z^2_{l+1}} \mu \right) b - \frac{z^1_{l+1} z^2_{l+1}}{z^1_{l+1}+z^2_{l+1}} \\ &\quad + \underbrace{\left(z^1_{l+1}+z^2_{l+1}\right)}_{\geq 0} \underbrace{\left(\frac{z^2_{[l]}}{z^1_{l+1}+z^2_{l+1}} - \mu^+ \right)}_{\geq (0^l)^T \text{ (by (9.7))}} \begin{pmatrix} A & G \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} && \left(f^{\leq 0, MIR, \mu}_{A,G,b}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) \leq 0\right) \\ &\leq \left(z^1_{l+1}+z^2_{l+1}\right) \left(\mu^+ - \frac{z^2_{l+1}}{z^1_{l+1}+z^2_{l+1}} \mu \right) b - \frac{z^1_{l+1} z^2_{l+1}}{z^1_{l+1}+z^2_{l+1}} \\ &\quad + \left(z^1_{l+1}+z^2_{l+1}\right) \left(\frac{z^2_{[l]}}{z^1_{l+1}+z^2_{l+1}} - \mu^+ \right) b && \left(\begin{pmatrix} A & G \end{pmatrix}_{S,*} \begin{pmatrix} x \\ y \end{pmatrix} \leq b_S\right) \\ &= -z^2_{l+1} \mu b + z^2_{[l]} b - \frac{z^1_{l+1} z^2_{l+1}}{z^1_{l+1}+z^2_{l+1}} \\ &= z^2_{[l]} b - z^2_{l+1} - z^2_{l+1} \left(\mu b - \frac{z^2_{l+1}}{z^1_{l+1}+z^2_{l+1}} \right) && \left(-\frac{z^1_{l+1} z^2_{l+1}}{z^1_{l+1}+z^2_{l+1}} = -z^2_{l+1} + \frac{\left(z^2_{l+1}\right)^2}{z^1_{l+1}+z^2_{l+1}}\right)\end{aligned}$$

$$=z_{[l]}^2 b - z_{l+1}^2 (\gamma + 1) \qquad \text{(by (9.6))}$$
$$=c_0.$$

□

Proposition 414. *Let A, G, b, l, m and n be as in Definition 410, let*

$$A' := \begin{pmatrix} A \\ (0^m)^T \end{pmatrix}, \qquad G' := \begin{pmatrix} G \\ (0^n)^T \end{pmatrix}, \qquad b' := \begin{pmatrix} b \\ 1 \end{pmatrix}.$$

and let $\mu' \in \mathcal{M}_{MIR}(A', G')$ satisfy $\mu'_{l+1} > 0$ ($\mu'_{l+1} \geq 0$ can be assumed by Lemma 421 and if $\mu'_{l+1} = 0$, the statement is trivial). Define $\mu := \mu'_{[l]}$. Then the MIR cut $f_{A',G',b'}^{\leq 0,MIR,\mu'}(\cdot) \leq 0$ is dominated relatively to $P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}_{\operatorname{supp}\mu,*}, b_{\operatorname{supp}\mu}\right)$ by $f_{A,G,b}^{\leq 0,MIR,\mu}(\cdot) \leq 0$.

Proof. By Lemma 424, which we state and prove later on, the MIR cut $f_{A',G',b'}^{\leq 0,MIR,\mu'}(\cdot) \leq 0$ is dominated relatively to

$$P^{\leq}\left(\begin{pmatrix} A' & G' \end{pmatrix}_{\operatorname{supp}\mu',*}, b'_{\operatorname{supp}\mu'}\right) = P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}_{\operatorname{supp}\mu,*}, b_{\operatorname{supp}\mu}\right)$$

by the MIR cuts $f_{A',G',b'}^{\leq 0,MIR,(\mu\ 0)}(\cdot) \leq 0$ and $f_{A',G',b'}^{\leq 0,MIR,\left((0^l)^T\ \mu'_{l+1}\right)}(\cdot) \leq 0$, where $f_{A',G',b'}^{\leq 0,MIR,(\mu\ 0)}(\cdot) \leq 0$ is equivalent to $f_{A,G,b}^{\leq 0,MIR,\mu}(\cdot) \leq 0$. On the other hand, for $\binom{x}{y} \in \mathbb{R}^m \times \mathbb{R}^n$, we have

$$\begin{aligned} f_{A',G',b'}^{\leq 0,MIR,\left((0^l)^T\ \mu'_{l+1}\right)}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) &= \left(1 - \operatorname{frac}\mu'_{l+1}\right)\begin{pmatrix} (0^l)^T & \mu'_{l+1} \end{pmatrix}(A'x + G'y - b') \\ &\quad + \operatorname{frac}\mu'_{l+1}\left(1 - \operatorname{frac}\mu'_{l+1}\right) \\ &= \left(1 - \operatorname{frac}\mu'_{l+1}\right)\left(-\mu'_{l+1} + \operatorname{frac}\mu'_{l+1}\right) \\ &\leq 0. \end{aligned}$$

Thus, $f_{A',G',b'}^{\leq 0,MIR,\left((0^l)^T\ \mu'_{l+1}\right)}(\cdot) \leq 0$ is valid for $P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}_{\operatorname{supp}\mu,*}, b_{\operatorname{supp}\mu}\right)$ and we conclude that the MIR cut $f_{A',G',b'}^{\leq 0,MIR,\mu'}(\cdot) \leq 0$ is dominated relatively to $P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}_{\operatorname{supp}\mu,*}, b_{\operatorname{supp}\mu}\right)$ by $f_{A,G,b}^{\leq 0,MIR,\mu}(\cdot) \leq 0$. □

Theorem 415. *Let A, G, b, l and m be as in Definition 410. Let $D(\pi,\gamma) \subseteq \mathbb{R}^m$ be a split disjunction and let $c(\cdot) \leq c_0$ be a split cut for $P := P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)$ with respect to $D(\pi,\gamma)$ that is not already valid for P. Then there exists a MIR cut $f_{A,G,b}^{\leq 0,MIR,\mu}(\cdot) \leq 0$ that dominates $c(\cdot) \leq c_0$ relatively to P such that*

$$\mu A = \pi^T, \qquad \lfloor \mu b \rfloor = \gamma.$$

Proof. W.l.o.g. we can assume that $P \neq \emptyset$ and $c(\cdot) \leq c_0$ is not already valid for P. We distinguish three cases:

1. $P \cap (D(\pi,\gamma) \times \mathbb{R}^n) = \emptyset$,
2. $P \cap P^{\leq}\left(\begin{pmatrix} \pi^T & (0^n)^T \end{pmatrix}, \gamma\right) = \emptyset \wedge P \cap P^{\geq}\left(\begin{pmatrix} \pi^T & (0^n)^T \end{pmatrix}, \gamma + 1\right) \neq \emptyset$ (or the other way round) and
3. $P \cap P^{\leq}\left(\begin{pmatrix} \pi^T & (0^n)^T \end{pmatrix}, \gamma\right) \neq \emptyset \wedge P \cap P^{\geq}\left(\begin{pmatrix} \pi^T & (0^n)^T \end{pmatrix}, \gamma + 1\right) \neq \emptyset$.

For case 1: By the Farkas lemma (Lemma 89), there exist $z^1, z^2 \in \left(\mathbb{R}_{\geq 0}^{l+1}\right)^T$ such that

$$z^1 \begin{pmatrix} A & G \\ \pi^T & (0^n)^T \end{pmatrix} = \begin{pmatrix} (0^m)^T & (0^n)^T \end{pmatrix}, \qquad z^1 \begin{pmatrix} b \\ \gamma \end{pmatrix} = -1,$$

$$z^2 \begin{pmatrix} A & G \\ -\pi^T & (0^n)^T \end{pmatrix} = \begin{pmatrix} (0^m)^T & (0^n)^T \end{pmatrix}, \qquad z^2 \begin{pmatrix} b \\ -(\gamma + 1) \end{pmatrix} = -1.$$

Clearly, $z^1_{l+1}, z^2_{l+1} > 0$ (since otherwise $P^{\leq}\left(\left(\begin{array}{cc} A & G \end{array}\right), b\right) = \emptyset$). Let

$$\mu := \frac{z^2_{[l]} - z^1_{[l]}}{z^1_{l+1} + z^2_{l+1}}.$$

The rest is a consequence of Proposition 413 (let $c := \left(\begin{array}{cc} (0^m)^T & (0^n)^T \end{array}\right)$ and $c_0 := -1$).

For case 2: W.l.o.g. let

$$P \cap P^{\geq}\left(\left(\begin{array}{cc} \pi^T & (0^n)^T \end{array}\right), \gamma + 1\right) = \emptyset$$

(otherwise replace π by $-\pi$ and γ by $-(\gamma+1)$). Clearly, every inequality for $P \cap P^{\leq}\left(\left(\begin{array}{cc} \pi^T & (0^n)^T \end{array}\right), \gamma\right)$ is dominated relatively to P by

$$\left(\begin{array}{cc} \pi^T & (0^n)^T \end{array}\right)(\,\cdot\,) \leq \gamma.$$

Consider the primal-dual pair

$$\begin{aligned} \gamma + 1 > \gamma^* :=& \max\left\{\pi^T x : \left(\begin{array}{cc} A & G \end{array}\right)\left(\begin{array}{c} x \\ y \end{array}\right) \leq b, \left(\begin{array}{c} x \\ y \end{array}\right) \in \mathbb{R}^m \times \mathbb{R}^n\right\} \\ =& \min\left\{zb : z\left(\begin{array}{cc} A & G \end{array}\right) = \left(\begin{array}{cc} \pi^T & (0^n)^T \end{array}\right), z \in \left(\mathbb{R}^l_{\geq 0}\right)^T\right\} \end{aligned}$$

and let z^* be a minimizer of the dual program. Then, clearly, $z^* \in \mathcal{M}_{MIR}(A,G)$ (even $z^* \in \mathcal{M}_{MIR,[l]}(A,G)$ (cf. Definition 418)). The only remaining statement to show is that the MIR cut $f^{\leq 0, MIR, z^*}_{A,G,b}(\,\cdot\,) \leq 0$ dominates $\left(\begin{array}{cc} \pi^T & (0^n)^T \end{array}\right)(\,\cdot\,) \leq \gamma$ relatively to P (we show that it even dominates this inequality *absolutely*). For this, let $\binom{x}{y} \in \mathbb{R}^m \times \mathbb{R}^n$ be such that $f^{\leq 0, MIR, z^*}_{A,G,b}\left(\binom{x}{y}\right) \leq 0$. Then

$$\begin{aligned} \pi^T x &= z^*(Ax + Gy - b) + z^* b \\ &= \frac{1}{1 - \operatorname{frac}(z^* b)}(1 - \operatorname{frac}(z^* b)) z^*(Ax + Gy - b) + z^* b \\ &= \frac{1}{1 - \operatorname{frac}(z^* b)}\left((1 - \operatorname{frac}(z^* b)) z^{*+} + \operatorname{frac}(z^* b) z^{*-}\right)(Ax + Gy - b) + z^* b \quad \left(z^* \geq \left(0^l\right)^T\right) \\ &\leq -\frac{1}{1 - \operatorname{frac}(z^* b)} \operatorname{frac}(z^* b)(1 - \operatorname{frac}(z^* b)) + z^* b \qquad \left(f^{\leq 0, MIR, z^*}_{A,G,b}\left(\binom{x}{y}\right) \leq 0\right) \\ &= -\operatorname{frac}(z^* b) + z^* b \\ &= \gamma^* - \operatorname{frac}\gamma^* \\ &\leq \gamma. \end{aligned}$$

For case 3: We have

$$\begin{aligned} c_0 \geq c_0^1 =& \max\left\{c\left(\begin{array}{c} x \\ y \end{array}\right) : \left(\begin{array}{cc} A & G \\ \pi^T & (0^n)^T \end{array}\right)\left(\begin{array}{c} x \\ y \end{array}\right) \leq \left(\begin{array}{c} b \\ \gamma \end{array}\right), \left(\begin{array}{c} x \\ y \end{array}\right) \in \mathbb{R}^m \times \mathbb{R}^n\right\} \\ =& \min\left\{z\left(\begin{array}{c} b \\ \gamma \end{array}\right) : z\left(\begin{array}{cc} A & G \\ \pi^T & (0^n)^T \end{array}\right) = c, z \in \left(\mathbb{R}^l_{\geq 0}\right)^T\right\}, \\ c_0 \geq c_0^2 =& \max\left\{c\left(\begin{array}{c} x \\ y \end{array}\right) : \left(\begin{array}{cc} A & G \\ -\pi^T & (0^n)^T \end{array}\right)\left(\begin{array}{c} x \\ y \end{array}\right) \leq \left(\begin{array}{c} b \\ -(\gamma+1) \end{array}\right), \left(\begin{array}{c} x \\ y \end{array}\right) \in \mathbb{R}^m \times \mathbb{R}^n\right\} \\ =& \min\left\{z\left(\begin{array}{c} b \\ -(\gamma+1) \end{array}\right) : z\left(\begin{array}{cc} A & G \\ -\pi^T & (0^n)^T \end{array}\right) = c, z \in \left(\mathbb{R}^l_{\geq 0}\right)^T\right\}. \end{aligned}$$

Let $z^{*,1}$ and $z^{*,2}$ be minimizers for the dual programs and let

$$\begin{aligned} A' &:= \left(\begin{array}{c} A \\ (0^m)^T \end{array}\right), & G' &:= \left(\begin{array}{c} G \\ (0^n)^T \end{array}\right), & b' &:= \left(\begin{array}{c} b \\ 1 \end{array}\right), \\ z'^{*,1} &:= \left(\begin{array}{ccc} z^{*,1}_{[l]} & c_0^2 - c_0^1 & z^{*,1}_{l+1} \end{array}\right), & z'^{*,2} &:= \left(\begin{array}{ccc} z^{*,2} & 0 & z^{*,2}_{l+1} \end{array}\right). \end{aligned}$$

Then

$$c_0 = z'^{*,1}\begin{pmatrix} b \\ 1 \\ \gamma \end{pmatrix} = z'^{*,2}\begin{pmatrix} b \\ 1 \\ -(\gamma+1) \end{pmatrix},$$

$$c = z'^{*,1}\begin{pmatrix} A & G \\ (0^m)^T & (0^n)^T \\ \pi^T & (0^n)^T \end{pmatrix} = z'^{*,2}\begin{pmatrix} A & G \\ (0^m)^T & (0^n)^T \\ -\pi^T & (0^n)^T \end{pmatrix}.$$

By Proposition 413, $c(\cdot) \leq c_0$ is dominated relatively to $P^{\leq}\left(\begin{pmatrix} A' & G' \end{pmatrix}, b'\right)$, and thus $P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)$, by $f^{\leq 0,\mu'}_{A',G',b'}(\cdot) \leq 0$, where

$$\mu' := \frac{z'^{*,2}_{[l+1]} - z'^{*,1}_{[l+1]}}{z'^{*,1}_{l+2} + z'^{*,2}_{l+2}}.$$

Finally, by Proposition 414, $f^{\leq 0,\mu'}_{A',G',b'}(\cdot) \leq 0$ is dominated relatively to $P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)$ by $f^{\leq 0,\mu}_{A,G,b}(\cdot) \leq 0$, where

$$\mu := \frac{z'^{*,2}_{[l]} - z'^{*,1}_{[l]}}{z'^{*,1}_{l+2} + z'^{*,2}_{l+2}} = \frac{z^{*,2}_{[l]} - z^{*,1}_{[l]}}{z'^{*,1}_{l+1} + z'^{*,2}_{l+1}}.$$

□

We remark that not every split cut is a MIR cut and neither every split cut is dominated *absolutely* by a MIR cut. For this, we consider the following example:

Example 416. *(See Figure 9.1) Let*

$$P^{416} := P^{\leq}\left(\begin{pmatrix} A \mid G \end{pmatrix}, b\right) := P^{\leq}\left(\left(\begin{array}{c|c} 1 & 1 \\ 1 & -1 \end{array}\right), \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \subseteq \mathbb{R}^1 \times \mathbb{R}^1$$

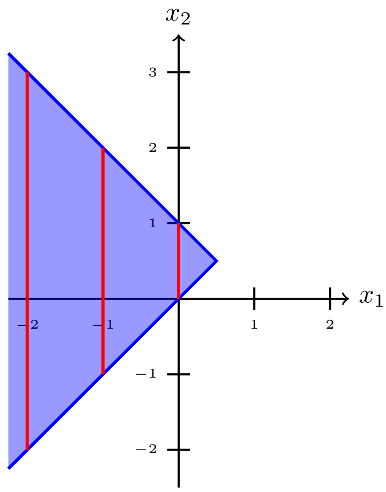
(see Figure 9.1a). Then:

1. $2x_1 + x_2 \leq 1$ *is a split cut for* $x \in P^{416}$ *(see Figure 9.1e) with respect to the split disjunction* $D\left(\begin{pmatrix} 1 \end{pmatrix}, 0\right)$ *(see Figure 9.1c and Figure 9.1d).*
2. *There exists no MIR cut with respect to A, G and b that dominates* $\begin{pmatrix} 2 & 1 \end{pmatrix}(\cdot) \leq 1$ *absolutely.*
3. *Let* $\mu := \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix}$*. Then* $f^{\leq 0,MIR,\mu}_{A,b}(x) \leq 0$ *(*$\frac{1}{2}x_1 \leq 0$*, which is equivalent to* $x_1 \leq 0$*) is a MIR cut for* $x \in P^{416}$ *that dominates* $2x_1 + x_2 \leq 1$ *relatively to* P^{416}*, but not absolutely. By Theorem 412,* $\frac{1}{2}x_1 \leq 0$ *is also a split cut for* P^{416} *with respect to the split disjunction* $D\left(\begin{pmatrix} 1 \end{pmatrix}, 0\right)$ *(see Figure 9.1f).*

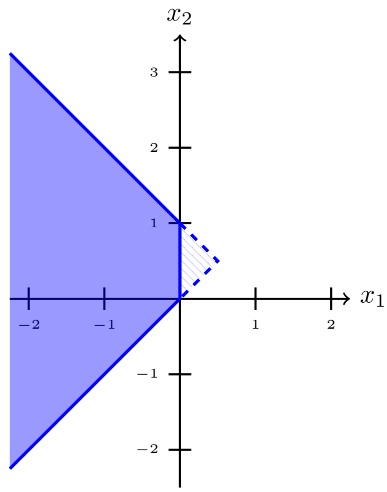
Proof. We only show 2. The validity of the other statements can be seen from Figure 9.1. Assume that there exists a $\mu \in \mathcal{M}_{MIR}(A,G)$ such that $f^{\leq 0,MIR,\mu}_{A,b}(\cdot) \leq 0$ dominates $\begin{pmatrix} 2 & 1 \end{pmatrix}(\cdot) \leq -1$ absolutely. Since $\mu \in \mathcal{M}_{MIR}(A,G)$, we have $\mu_1 = \mu_2$. On the other hand, it is easy to check (we write this down formally in Lemma 420) that if $\mu b \in \mathbb{Z}$, then $f^{\leq 0,MIR,\mu}_{A,b}(\cdot) \leq 0$ is already valid for P. So, we can assume $\mu \neq \left(0^2\right)^T$. Additionally, by Lemma 421, we can assume $\mu_1 > 0$ (thus, $\mu > \left(0^2\right)^T$). But then

$$\begin{aligned} f^{\leq 0,MIR,\mu}_{A,G,b}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) &= (1 - \operatorname{frac}(\mu b))\,\mu\,(Ax + Gy - b) + \operatorname{frac}(\mu b)\,(1 - \operatorname{frac}(\mu b)) && (\mu > \left(0^2\right)^T) \\ &= \underbrace{(1 - \operatorname{frac}\mu_1)\begin{pmatrix} 2\mu_1 & 0 \end{pmatrix}}_{\notin \operatorname{cone}\left\{\begin{pmatrix} 2 & 1 \end{pmatrix}\right\}}\begin{pmatrix} x \\ y \end{pmatrix} - (1 - \operatorname{frac}\mu_1)(\mu_1 - \operatorname{frac}\mu_1). \end{aligned}$$

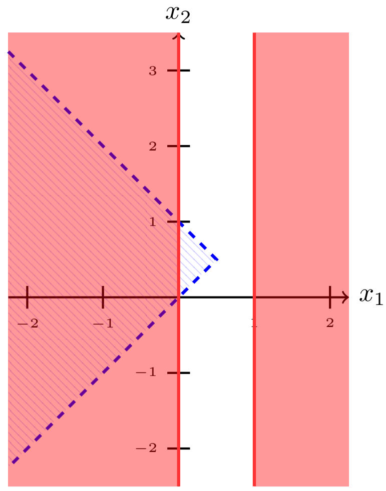
□



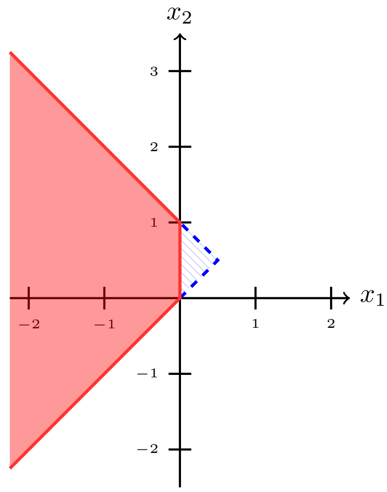
(a) P^{416} and $P^{416} \cap \left(\mathbb{Z}^1 \times \mathbb{R}^1\right)$



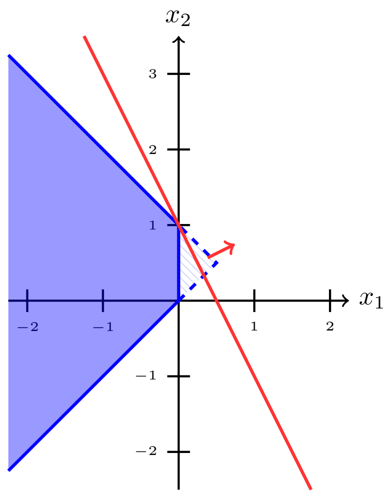
(b) $\mathrm{cl}_I\left(P^{416}\right)$ (for reference)



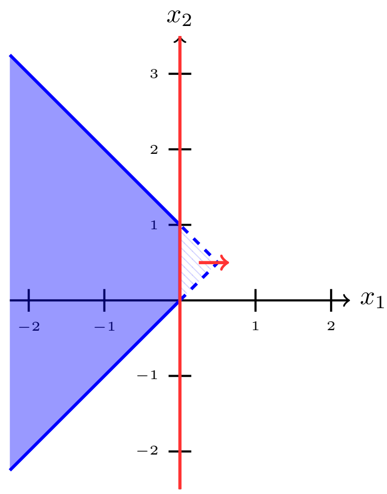
(c) $D\left(\left(\begin{array}{c} 1 \end{array}\right), 0\right) \times \mathbb{R}^1$



(d) $P^{416} \cap \left(D\left(\left(\begin{array}{c} 1 \end{array}\right), 0\right) \times \mathbb{R}^1\right)$



(e) $2x_1 + x_2 \leq 2$ is a valid inequality for $x \in P^{416} \cap \left(D\left(\left(\begin{array}{c} 1 \end{array}\right), 0\right) \times \mathbb{R}^1\right)$



(f) $\frac{1}{2}x_1 \leq 0$ is a valid inequality for $x \in P^{416} \cap \left(D\left(\left(\begin{array}{c} 1 \end{array}\right), 0\right) \times \mathbb{R}^1\right)$

Figure 9.1.: Visualisation of Example 416

9.2. Polyhedricity of the MIR closure

The goal of this section is to prove the following theorem:

Theorem 417. *Let $P = P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)$ be a polyhedron, where $A \in \mathbb{Q}^{l\times m}$, $G \in \mathbb{Q}^{l\times n}$ and $b \in \mathbb{R}^l$ (i.e. P is a polyhedron with rational face normals). Then $\mathrm{cl}_{split}(P) = \mathrm{cl}_{MIR}(A, G, b)$ is a polyhedron and this polyhedron is rational if b is.*

Theorem 417 is proved in section 9.2.4. But we believe that here the journey is the destination: for example in section 9.2.5, Theorem 430, we compute the split closure of the so-called Cook-Kannan-Schrijver example (i.e. the polyhedron $P^{430,\epsilon}$, which we define in Theorem 430). For $\epsilon \in \mathbb{R}\backslash\mathbb{Q}$, this is a polyhedron that not even has rational face normals. Nevertheless, using the methods that we developed on the journey, we can still compute its split closure.

We begin by generalizing some definitions that we introduced in section 9.1.2.1 (concretely, $\mathcal{M}_{MIR}(A, G)$ in Definition 410 and the MIR closure $\mathrm{cl}_{MIR}(A, G, b)$ in Definition 411):

Definition 418. *Let A, G, b, l, m and n be as in Definition 410. For $L \in \mathcal{P}([l])$, define*

$$\begin{aligned}
\mathcal{M}_{MIR,L}(A, G) :=& \mathcal{M}_{MIR}(A, G) \cap \left\{\mu \in \left(\mathbb{R}^l\right)^T : \mu_i \geq 0\ \forall i \in L, \mu_i \leq 0\ \forall i \in [l]\backslash L\right\},\\
\mathrm{cl}_{MIR,L}(A, G, b) :=& P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right) \cap \bigcap_{\mu\in\mathcal{M}_{MIR,L}(A,G)} \left\{x \in \mathbb{R}^m \times \mathbb{R}^n : f^{\leq 0,MIR,\mu}_{A,G,b}(x) \leq 0\right\}.
\end{aligned}$$

We clearly have:

Lemma 419. *Let A, G, b and l be as in Definition 410. Then*

$$\mathrm{cl}_{MIR}(A, G, b) = \bigcap_{L\in\mathcal{P}([l])} \mathrm{cl}_{MIR,L}(A, G, b).$$

We remark that in Theorem 422, we tighten Lemma 419.

9.2.1. Dominance of MIR cuts

Lemma 420. *Let A, G, b and l be as in Definition 410, and let $\mu \in \mathcal{M}_{MIR}(A, G)$ be such that $\mu b \in \mathbb{Z}$. Then $f^{\leq 0,MIR,\mu}_{A,G,b}(\cdot) \leq 0$ is already valid for*

$$P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}_{\mathrm{supp}\,\mu^+,*}, b_{\mathrm{supp}\,\mu^+}\right) \supseteq P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right).$$

Proof. Let $\binom{x}{y} \in P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}_{\mathrm{supp}\,\mu^+,*}, b_{\mathrm{supp}\,\mu^+}\right)$. Then

$$\begin{aligned}
f^{\leq 0,MIR,\mu}_{A,G,b}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) &= \left((1 - \mathrm{frac}(\mu b))\,\mu^+ + \mathrm{frac}(\mu b)\,\mu^-\right)(Ax + Gy - b) + \mathrm{frac}(\mu b)(1 - \mathrm{frac}(\mu b))\\
&= \mu^+(Ax + Gy - b)\\
&= \sum_{i\in\mathrm{supp}\,\mu^+} \mu^+_i (Ax + Gy - b)_i\\
&\leq 0.
\end{aligned}$$

□

Lemma 421. *Let A, G, b and l be as in Definition 410 and let $\mu \in \mathcal{M}_{MIR}$. Then also $-\mu \in \mathcal{M}_{MIR}$ and if $\mu b \notin \mathbb{Z}$, we have*

$$f^{\leq 0,MIR,\mu}_{A,G,b} = f^{\leq 0,MIR,-\mu}_{A,G,b}.$$

Proof. The statement that $-\mu \in \mathcal{M}_{MIR}$ is obvious. For the second statement: let $\binom{x}{y} \in \mathbb{R}^m \times \mathbb{R}^n$. Then

$$\begin{aligned}
f^{\leq 0,MIR,\mu}_{A,G,b}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) &= \left((1 - \mathrm{frac}(\mu b))\,\mu^+ + \mathrm{frac}(\mu b)\,\mu^-\right)(Ax + Gy - b)\\
&\quad + \mathrm{frac}(\mu b)(1 - \mathrm{frac}(\mu b))\\
&= \left((1 - (1 - \mathrm{frac}((-\mu)\,b)))\,\mu^+ + (1 - \mathrm{frac}((-\mu)\,b))\,\mu^-\right)(Ax + Gy - b)\\
&\quad + (1 - \mathrm{frac}((-\mu)\,b))(1 - (1 - \mathrm{frac}((-\mu)\,b))) && (\mu b \notin \mathbb{Z})\\
&= \left(\mathrm{frac}((-\mu)\,b)\,\mu^+ + (1 - \mathrm{frac}((-\mu)\,b))\,\mu^-\right)(Ax + Gy - b)
\end{aligned}$$

$$\begin{aligned}
&+ (1 - \operatorname{frac}((-\mu)\, b)) \operatorname{frac}((-\mu)\, b) \\
=& \left((1 - \operatorname{frac}((-\mu)\, b))\, (-\mu)^{+} + \operatorname{frac}((-\mu)\, b)\, (-\mu)^{-}\right) (Ax + Gy - b) \\
&+ \operatorname{frac}((-\mu)\, b)\, (1 - \operatorname{frac}((-\mu)\, b)) \\
=& f_{A,G,b}^{\leq 0, MIR, -\mu} \left(\begin{pmatrix} x \\ y \end{pmatrix} \right).
\end{aligned}$$

□

From Lemma 419 and Lemma 421, we immediately conclude:

Theorem 422. *Let A, G, b and l be as in Definition 410 and let $\mathcal{L} \subseteq \mathcal{P}([l])$ be such that*

$$\forall L \in \mathcal{P}([l]) : L \in \mathcal{L} \vee [l] \setminus L \in \mathcal{L}.$$

Then

$$\operatorname{cl}_{MIR}(A, G, b) = \bigcap_{L \in \mathcal{L}} \operatorname{cl}_{MIR,L}(A, G, b).$$

Lemma 423. *Let A, G, b and l be as in Definition 410, let $L \in \mathcal{P}([l])$ and let $\mu^1, \mu^2 \in \mathcal{M}_{MIR,L}(A, G)$, where $\mu^1 b \in \mathbb{Z}$. Then $f_{A,G,b}^{\leq 0, MIR, \mu^1+\mu^2}(\,\cdot\,) \leq 0$ is dominated relatively to*

$$P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}_{\operatorname{supp}\mu^1, *}, b_{\operatorname{supp}\mu^1}\right) \supseteq P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)$$

by the MIR cut $f_{A,G,b}^{\leq 0, MIR, \mu^2}(\,\cdot\,) \leq 0$.

Proof. Let $\begin{pmatrix} x \\ y \end{pmatrix} \in P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}_{\operatorname{supp}\mu, *}, b_{\operatorname{supp}\mu}\right)$ satisfy $f_{A,G,b}^{\leq 0, MIR, \mu^2}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) \leq 0$. Then

$$\begin{aligned}
& f_{A,G,b}^{\leq 0, MIR, \mu^1+\mu^2}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) \\
=& \left(\left(1 - \operatorname{frac}\left(\left(\mu^1 + \mu^2\right) b\right)\right) \left(\mu^1 + \mu^2\right)^{+} + \operatorname{frac}\left(\left(\mu^1 + \mu^2\right) b\right) \left(\mu^1 + \mu^2\right)^{-}\right) (Ax + Gy - b) \\
&+ \operatorname{frac}\left(\left(\mu^1 + \mu^2\right) b\right) \left(1 - \operatorname{frac}\left(\left(\mu^1 + \mu^2\right) b\right)\right) \\
=& \left(\left(1 - \operatorname{frac}\left(\mu^2 b\right)\right) \left(\mu^1 + \mu^2\right)^{+} + \operatorname{frac}\left(\mu^2 b\right) \left(\mu^1 + \mu^2\right)^{-}\right) (Ax + Gy - b) \\
&+ \operatorname{frac}\left(\mu^2 b\right) \left(1 - \operatorname{frac}\left(\mu^2 b\right)\right) && (\mu^1 b \in \mathbb{Z}) \\
=& \left(\left(1 - \operatorname{frac}\left(\mu^2 b\right)\right) \left(\mu^1\right)^{+} + \operatorname{frac}\left(\mu^2 b\right) \left(\mu^1\right)^{-}\right) (Ax + Gy - b) \\
&+ \left(\left(1 - \operatorname{frac}\left(\mu^2 b\right)\right) \left(\mu^2\right)^{+} + \operatorname{frac}\left(\mu^2 b\right) \left(\mu^2\right)^{-}\right) (Ax + Gy - b) \\
&+ \operatorname{frac}\left(\mu^2 b\right) \left(1 - \operatorname{frac}\left(\mu^2 b\right)\right) \\
=& \left(\left(1 - \operatorname{frac}\left(\mu^2 b\right)\right) \left(\mu^1\right)^{+} + \operatorname{frac}\left(\mu^2 b\right) \left(\mu^1\right)^{-}\right)_{\operatorname{supp}\mu^1} (Ax + Gy - b)_{\operatorname{supp}\mu^1} + f_{A,G,b}^{\leq 0, MIR, \mu^2}\left(\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right) \\
\leq & 0.
\end{aligned}$$

□

Lemma 424. *Let A, G, b and l be as in Definition 410, let $L \in \mathcal{P}([l])$ and let $\mu^1, \mu^2 \in \mathcal{M}_{MIR,L}(A, G)$. Then $f_{A,G,b}^{\leq 0, MIR, \mu^1+\mu^2}(\,\cdot\,) \leq 0$ is dominated relatively to*

$$P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}_{(\operatorname{supp}\mu^1) \cup (\operatorname{supp}\mu^2), *}, b_{(\operatorname{supp}\mu^1) \cup (\operatorname{supp}\mu^2)}\right) \supseteq P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)$$

by the MIR cuts $f_{A,G,b}^{\leq 0, MIR, \mu^1}(\,\cdot\,) \leq 0$ and $f_{A,G,b}^{\leq 0, MIR, \mu^2}(\,\cdot\,) \leq 0$.

Proof. We first note that the case

$$\operatorname{frac}\left(\mu^1 b\right) \in \mathbb{Z} \vee \operatorname{frac}\left(\mu^2 b\right) \in \mathbb{Z}$$

has already been analyzed in Lemma 423. So, we can assume

$$\operatorname{frac}\left(\mu^1 b\right), \operatorname{frac}\left(\mu^2 b\right) \notin \mathbb{Z}.$$

For $i \in \{1,2\}$, let

$$\begin{aligned} C^i :=& \left(\left(1-\operatorname{frac}\left(\left(\mu^1+\mu^2\right)b\right)\right)\left(\mu^i\right)^+ + \operatorname{frac}\left(\left(\mu^1+\mu^2\right)b\right)\left(\mu^i\right)^-\right) \\ &- \min\left(\frac{1-\operatorname{frac}\left(\left(\mu^1+\mu^2\right)b\right)}{1-\operatorname{frac}\left(\mu^i b\right)}, \frac{\operatorname{frac}\left(\left(\mu^1+\mu^2\right)b\right)}{\operatorname{frac}\left(\mu^i b\right)}\right)\left(\left(1-\operatorname{frac}\left(\mu^i b\right)\right)\left(\mu^i\right)^+ + \operatorname{frac}\left(\mu^i b\right)\left(\mu^i\right)^-\right). \end{aligned}$$

Clearly, for $i \in \{1,2\}$, we have $\operatorname{supp} C^i \subseteq \operatorname{supp} \mu^i$. We now show that for $i \in \{1,2\}$, we have $C^i \geq 0$:

$$\begin{aligned} C^i =& \left(\left(1-\operatorname{frac}\left(\left(\mu^1+\mu^2\right)b\right)\right) - \min\left(\frac{1-\operatorname{frac}\left(\left(\mu^1+\mu^2\right)b\right)}{1-\operatorname{frac}\left(\mu^i b\right)}, \frac{\operatorname{frac}\left(\left(\mu^1+\mu^2\right)b\right)}{\operatorname{frac}\left(\mu^i b\right)}\right)\left(1-\operatorname{frac}\left(\mu^i b\right)\right)\right)\left(\mu^i\right)^+ \\ &+ \left(\operatorname{frac}\left(\left(\mu^1+\mu^2\right)b\right) - \min\left(\frac{1-\operatorname{frac}\left(\left(\mu^1+\mu^2\right)b\right)}{1-\operatorname{frac}\left(\mu^i b\right)}, \frac{\operatorname{frac}\left(\left(\mu^1+\mu^2\right)b\right)}{\operatorname{frac}\left(\mu^i b\right)}\right)\operatorname{frac}\left(\mu^i b\right)\right)\left(\mu^i\right)^- \\ \geq& \left(\left(1-\operatorname{frac}\left(\left(\mu^1+\mu^2\right)b\right)\right) - \frac{1-\operatorname{frac}\left(\left(\mu^1+\mu^2\right)b\right)}{1-\operatorname{frac}\left(\mu^i b\right)}\left(1-\operatorname{frac}\left(\mu^i b\right)\right)\right)\left(\mu^i\right)^+ \\ &+ \left(\operatorname{frac}\left(\left(\mu^1+\mu^2\right)b\right) - \frac{\operatorname{frac}\left(\left(\mu^1+\mu^2\right)b\right)}{\operatorname{frac}\left(\mu^i b\right)}\operatorname{frac}\left(\mu^i b\right)\right)\left(\mu^i\right)^- \\ =& \left(0^l\right)^T. \end{aligned}$$

Now for the main statement: let

$$\begin{pmatrix} x \\ y \end{pmatrix} \in P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}_{(\operatorname{supp}\mu^1)\cup(\operatorname{supp}\mu^2),*}, b_{(\operatorname{supp}\mu^1)\cup(\operatorname{supp}\mu^2)}\right)$$

satisfy $f^{\leq 0,MIR,\mu^1}_{A,G,b}\left(\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right) \leq 0$ and $f^{\leq 0,MIR,\mu^2}_{A,G,b}\left(\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right) \leq 0$. Then

$$\begin{aligned} & f^{\leq 0,MIR,\mu^1+\mu^2}_{A,G,b}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) \\ =& \left(\left(1-\operatorname{frac}\left(\left(\mu^1+\mu^2\right)b\right)\right)\left(\mu^1+\mu^2\right)^+ + \operatorname{frac}\left(\left(\mu^1+\mu^2\right)b\right)\left(\mu^1+\mu^2\right)^-\right)(Ax+Gy-b) \\ &+ \operatorname{frac}\left(\left(\mu^1+\mu^2\right)b\right)\left(1-\operatorname{frac}\left(\left(\mu^1+\mu^2\right)b\right)\right) \\ =& \min\left(\frac{1-\operatorname{frac}\left(\left(\mu^1+\mu^2\right)b\right)}{1-\operatorname{frac}\left(\mu^1 b\right)}, \frac{\operatorname{frac}\left(\left(\mu^1+\mu^2\right)b\right)}{\operatorname{frac}\left(\mu^1 b\right)}\right)\underbrace{\left(\left(1-\operatorname{frac}\left(\mu^1 b\right)\right)\left(\mu^1\right)^+ + \operatorname{frac}\left(\mu^1 b\right)\left(\mu^1\right)^-\right)\cdot(Ax+Gy-b)}_{\leq -\operatorname{frac}(\mu^1 b)(1-\operatorname{frac}(\mu^1 b))\ (\text{since } f^{\leq 0,MIR,\mu^1}_{A,G,b}((\begin{smallmatrix} x \\ y \end{smallmatrix}))\leq 0)} \\ &+ \min\left(\frac{1-\operatorname{frac}\left(\left(\mu^1+\mu^2\right)b\right)}{1-\operatorname{frac}\left(\mu^2 b\right)}, \frac{\operatorname{frac}\left(\left(\mu^1+\mu^2\right)b\right)}{\operatorname{frac}\left(\mu^2 b\right)}\right)\underbrace{\left(\left(1-\operatorname{frac}\left(\mu^2 b\right)\right)\left(\mu^2\right)^+ + \operatorname{frac}\left(\mu^2 b\right)\left(\mu^2\right)^-\right)\cdot(Ax+Gy-b)}_{\leq -\operatorname{frac}(\mu^2 b)(1-\operatorname{frac}(\mu^2 b))\ (\text{since } f^{\leq 0,MIR,\mu^2}_{A,G,b}((\begin{smallmatrix} x \\ y \end{smallmatrix}))\leq 0)} \\ &+ \operatorname{frac}\left(\left(\mu^1+\mu^2\right)b\right)\left(1-\operatorname{frac}\left(\left(\mu^1+\mu^2\right)b\right)\right) + \underbrace{\underbrace{\left(C^1+C^2\right)}_{\operatorname{supp}(C^1+C^2)\subseteq(\operatorname{supp}\mu^1)\cup(\operatorname{supp}\mu^2)}(Ax+Gy-b)}_{\leq 0} \\ \leq& -\min\left(\frac{1-\operatorname{frac}\left(\left(\mu^1+\mu^2\right)b\right)}{1-\operatorname{frac}\left(\mu^1 b\right)}, \frac{\operatorname{frac}\left(\left(\mu^1+\mu^2\right)b\right)}{\operatorname{frac}\left(\mu^1 b\right)}\right)\operatorname{frac}\left(\mu^1 b\right)\left(1-\operatorname{frac}\left(\mu^1 b\right)\right) \\ &- \min\left(\frac{1-\operatorname{frac}\left(\left(\mu^1+\mu^2\right)b\right)}{1-\operatorname{frac}\left(\mu^2 b\right)}, \frac{\operatorname{frac}\left(\left(\mu^1+\mu^2\right)b\right)}{\operatorname{frac}\left(\mu^2 b\right)}\right)\operatorname{frac}\left(\mu^2 b\right)\left(1-\operatorname{frac}\left(\mu^2 b\right)\right) \\ &+ \operatorname{frac}\left(\left(\mu^1+\mu^2\right)b\right)\left(1-\operatorname{frac}\left(\left(\mu^1+\mu^2\right)b\right)\right) \\ =:& (9.8) \end{aligned}$$

We distinguish four cases:

1. $\frac{1-\text{frac}\left(\left(\mu^1+\mu^2\right)b\right)}{1-\text{frac}\left(\mu^1 b\right)} \leq \frac{\text{frac}\left(\left(\mu^1+\mu^2\right)b\right)}{\text{frac}\left(\mu^1 b\right)} \wedge \frac{1-\text{frac}\left(\left(\mu^1+\mu^2\right)b\right)}{1-\text{frac}\left(\mu^2 b\right)} \leq \frac{\text{frac}\left(\left(\mu^1+\mu^2\right)b\right)}{\text{frac}\left(\mu^2 b\right)}$,
2. $\frac{1-\text{frac}\left(\left(\mu^1+\mu^2\right)b\right)}{1-\text{frac}\left(\mu^1 b\right)} \geq \frac{\text{frac}\left(\left(\mu^1+\mu^2\right)b\right)}{\text{frac}\left(\mu^1 b\right)} \wedge \frac{1-\text{frac}\left(\left(\mu^1+\mu^2\right)b\right)}{1-\text{frac}\left(\mu^2 b\right)} \geq \frac{\text{frac}\left(\left(\mu^1+\mu^2\right)b\right)}{\text{frac}\left(\mu^2 b\right)}$,
3. $\frac{1-\text{frac}\left(\left(\mu^1+\mu^2\right)b\right)}{1-\text{frac}\left(\mu^1 b\right)} \leq \frac{\text{frac}\left(\left(\mu^1+\mu^2\right)b\right)}{\text{frac}\left(\mu^1 b\right)} \wedge \frac{1-\text{frac}\left(\left(\mu^1+\mu^2\right)b\right)}{1-\text{frac}\left(\mu^2 b\right)} \geq \frac{\text{frac}\left(\left(\mu^1+\mu^2\right)b\right)}{\text{frac}\left(\mu^2 b\right)}$,
4. $\frac{1-\text{frac}\left(\left(\mu^1+\mu^2\right)b\right)}{1-\text{frac}\left(\mu^1 b\right)} \geq \frac{\text{frac}\left(\left(\mu^1+\mu^2\right)b\right)}{\text{frac}\left(\mu^1 b\right)} \wedge \frac{1-\text{frac}\left(\left(\mu^1+\mu^2\right)b\right)}{1-\text{frac}\left(\mu^2 b\right)} \leq \frac{\text{frac}\left(\left(\mu^1+\mu^2\right)b\right)}{\text{frac}\left(\mu^2 b\right)}$.

For case 1:

$$\begin{aligned}
(9.8) =& -\frac{1-\text{frac}\left(\left(\mu^1+\mu^2\right)b\right)}{1-\text{frac}\left(\mu^1 b\right)}\text{frac}\left(\mu^1 b\right)\left(1-\text{frac}\left(\mu^1 b\right)\right) - \frac{1-\text{frac}\left(\left(\mu^1+\mu^2\right)b\right)}{1-\text{frac}\left(\mu^2 b\right)}\text{frac}\left(\mu^2 b\right)\left(1-\text{frac}\left(\mu^2 b\right)\right)\\
&+\text{frac}\left(\left(\mu^1+\mu^2\right)b\right)\left(1-\text{frac}\left(\left(\mu^1+\mu^2\right)b\right)\right)\\
=& -\left(1-\text{frac}\left(\left(\mu^1+\mu^2\right)b\right)\right)\underbrace{\left(\text{frac}\left(\mu^1 b\right)+\text{frac}\left(\mu^2 b\right)-\text{frac}\left(\left(\mu^1+\mu^2\right)b\right)\right)}_{\geq 0 \text{ (by (2.25))}}\\
\leq& 0.
\end{aligned}$$

For case 2:

$$\begin{aligned}
(9.8) =& -\frac{\text{frac}\left(\left(\mu^1+\mu^2\right)b\right)}{\text{frac}\left(\mu^1 b\right)}\text{frac}\left(\mu^1 b\right)\left(1-\text{frac}\left(\mu^1 b\right)\right) - \frac{\text{frac}\left(\left(\mu^1+\mu^2\right)b\right)}{\text{frac}\left(\mu^2 b\right)}\text{frac}\left(\mu^2 b\right)\left(1-\text{frac}\left(\mu^2 b\right)\right)\\
&+\text{frac}\left(\left(\mu^1+\mu^2\right)b\right)\left(1-\text{frac}\left(\left(\mu^1+\mu^2\right)b\right)\right)\\
=& -\text{frac}\left(\left(\mu^1+\mu^2\right)b\right)\underbrace{\left(1+\text{frac}\left(\left(\mu^1+\mu^2\right)b\right)-\text{frac}\left(\mu^1 b\right)-\text{frac}\left(\mu^2 b\right)\right)}_{\geq 0 \text{ (by (2.26))}}\\
\leq& 0.
\end{aligned}$$

For case 3 and 4: By exchanging μ^1 and μ^2 if necessary, we can assume that case 3 holds.

$$\begin{aligned}
(9.8) =& -\frac{1-\text{frac}\left(\left(\mu^1+\mu^2\right)b\right)}{1-\text{frac}\left(\mu^1 b\right)}\text{frac}\left(\mu^1 b\right)\left(1-\text{frac}\left(\mu^1 b\right)\right) - \frac{\text{frac}\left(\left(\mu^1+\mu^2\right)b\right)}{\text{frac}\left(\mu^2 b\right)}\text{frac}\left(\mu^2 b\right)\left(1-\text{frac}\left(\mu^2 b\right)\right)\\
&+\text{frac}\left(\left(\mu^1+\mu^2\right)b\right)\left(1-\text{frac}\left(\left(\mu^1+\mu^2\right)b\right)\right)\\
=& \underbrace{\text{frac}\left(\left(\mu^1+\mu^2\right)b\right)\left(\text{frac}\left(\mu^1 b\right)+\text{frac}\left(\mu^2 b\right)\right)-\left(\text{frac}\left(\left(\mu^1+\mu^2\right)b\right)\right)^2-\text{frac}\left(\mu^1 b\right)}_{\leq 0 \text{ (by (2.27): set } x := \mu^1 b \text{ and } y := \mu^2 b)}\\
\leq& 0.
\end{aligned}$$

□

9.2.2. Restriction to linearly independent support

Let us recall Theorem 267 and Theorem 269:

Theorem 267. *Let $A \in \mathbb{R}^{l\times(m+n)}$ and $b \in \mathbb{R}^l$ ($l, m, n \in \mathbb{Z}_{\geq 0}$). Additionally, let $\pi \in \mathbb{Z}^m$ and $\gamma \in \mathbb{Z}$ be given. Then*

$$\text{conv}\left(P^{\leq}(A,b)\cap\left(D(\pi,\gamma)\times\mathbb{R}^n\right)\right) = \bigcap_{S\in\mathcal{B}^*(A)}\text{conv}\left(P^{\leq}\left(A_{S,*},b_S\right)\cap\left(D(\pi,\gamma)\times\mathbb{R}^n\right)\right),$$

where $\mathcal{B}^*(A)$ is as in Definition 392.

Theorem 269. *Let* $P := P^{\leq}(A,b) \subseteq \mathbb{R}^m \times \mathbb{R}^n$, *where* $A \in \mathbb{R}^{l\times(m+n)}$ *and* $b \in \mathbb{R}^l$ $(l,m,n \in \mathbb{Z}_{\geq 0})$. *Then*

$$\begin{smallmatrix}\mathrm{cl}_{L_{1,\mathbb{Q}}}\\ \mathrm{cl}_{split}\end{smallmatrix}(P) = \bigcap_{S\in\mathcal{B}^*(A)} \begin{smallmatrix}\mathrm{cl}_{L_{1,\mathbb{Q}}}\\ \mathrm{cl}_{split}\end{smallmatrix}\left(P^{\leq}(A_{S,*}, b_S)\right).$$

Here, the equation for $\mathrm{cl}_{L_{1,\mathbb{Q}}}(\cdot)$, *of course, only holds if* $m+n \geq 1$ *(otherwise,* $\mathrm{cl}_{L_{1,\mathbb{Q}}}(\cdot)$ *is not defined).*

We already mentioned in section 6.1.2 that in this section, we want to introduce Theorem 427 as a tightening of Theorem 269. For this, we present a theorem (Theorem 425) which is shown in [DGR11] and is very related to Theorem 267 and Theorem 269. The original formulation that is given in [DGR11] is weaker, but it is easy to check that the proof given in that paper also goes through for this stronger formulation.

Theorem 425. *Let* A, G, b, l, m *and* n *be as in Definition 410. Let* $x^* \in P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)$ *and let* $\mu \in \mathcal{M}_{MIR}(A,G)$ *be such that* $f^{\leq 0,MIR,\mu}_{A,G,b}(x^*) > 0$ *(by Theorem 420, this implies* $\mu b \notin \mathbb{Z}$*). Then there exists a* $\mu' \in \mathcal{M}_{MIR}(A,G)$ *having*

$$\begin{aligned}
\mu'\begin{pmatrix} A & G \end{pmatrix} &= \mu\begin{pmatrix} A & G \end{pmatrix},\\
\lfloor \mu' b\rfloor &= \lfloor \mu b\rfloor,\\
\{i\in[l] : \mu'_i > 0\} &\subseteq \{i \in [l] : \mu_i > 0\},\\
\{i\in[l] : \mu'_i < 0\} &\subseteq \{i \in [l] : \mu_i < 0\},\\
\exists B \in \mathcal{B}^*\left(\begin{pmatrix} A & G \end{pmatrix}\right) &: \operatorname{supp}\mu' \subseteq B,\\
f^{\leq 0,MIR,\mu'}_{A,G,b}(x^*) &> 0
\end{aligned} \tag{9.9}$$

(by Theorem 420, (9.9) *implies* $\mu' b \notin \mathbb{Z}$*).*

Remark 426. *It is easy to check that Theorem 267 can immediately be concluded from Theorem 415, Remark 268 and Theorem 425.*

From Theorem 425, we finally obtain the following tightening of Theorem 269:

Theorem 427. *Let* A, G, b *and* l *be as in Definition 410. Let* $L \in \mathcal{P}([l])$. *Then*

$$\mathrm{cl}_{MIR,L}(A,G,b) = \bigcap_{S\in\mathcal{B}^*\left(\begin{pmatrix} A & G\end{pmatrix}\right)} \mathrm{cl}_{MIR,L}(A_{S,*}, G_{S,*}, b_S).$$

9.2.3. Representation

Definition 428. *Let* $V^1, V^2 \leq V$ *be subspaces of a vector space* V *that satisfy* $V_1 \cap V_2 = \{0_V\}$. *Let* $v \in V_1 + V_2$. *Then there exist unique* $v^1 \in V^1$, $v^2 \in V^2$ *such that* $v = v^1 + v^2$. *We define*

$$\begin{aligned}
\mathrm{proj}^1_{V^1,V^2} v &:= v^1,\\
\mathrm{proj}^2_{V^1,V^2} v &:= v^2.
\end{aligned}$$

Theorem 429. *Let* A, G, b *and* l *be as in Definition 410 such that additionally the rows of* $\begin{pmatrix} A & G \end{pmatrix}$ *are linearly independent, and let* $L \in \mathcal{P}([l])$. *Let*

$$C' := \left(\mathrm{cone}\left(\left(\dot\bigcup_{i\in L}\left\{\begin{pmatrix} A & G & -b\end{pmatrix}_{i,*}\right\}\right) \dot\cup \left(\dot\bigcup_{i\in[l]\setminus L}\left\{-\begin{pmatrix} A & G & -b\end{pmatrix}_{i,*}\right\}\right)\right)\right) \cap (\mathbb{R}^m \times 0^n \times \mathbb{R})$$

be a cone that is generated by vectors from $(\mathbb{Q}^m \times 0^n \times \mathbb{R})^T$ *(this is surely the case if* A *and* G *are rational). Let* $\mathcal{S}$ *be a* $\mathbb{Z} + \mathrm{icone}$ *system for* C' *that consists of vectors from* $\mathbb{Z}^m \times 0^n \times \mathbb{R}$ *(the existence is assured by Theorem 366). Set*

$$\begin{aligned}
V^1 &:= \mathrm{lin}\dot\bigcup_{i\in L}\left\{\begin{pmatrix} A & G & -b\end{pmatrix}_{i,*}\right\},\\
V^2 &:= \mathrm{lin}\dot\bigcup_{i\in[l]\setminus L}\left\{\begin{pmatrix} A & G & -b\end{pmatrix}_{i,*}\right\}.
\end{aligned}$$

Then

$$\begin{aligned}&\mathrm{cl}_{MIR,L}(A,G,b)\\ =&P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right) \cap \bigcap_{\substack{\begin{pmatrix} \hat{a} & \hat{g} & -\hat{b}\end{pmatrix}\in\mathcal{S}:\,\mathrm{frac}\,\hat{b}\neq 0,\\ \begin{pmatrix} \hat{a}^1 & \hat{g}^1 & -\hat{b}^1\end{pmatrix}:=\mathrm{proj}^1_{V^1,V^2}\begin{pmatrix} \hat{a} & \hat{g} & -\hat{b}\end{pmatrix},\\ \begin{pmatrix} \hat{a}^2 & \hat{g}^2 & -\hat{b}^2\end{pmatrix}:=\mathrm{proj}^2_{V^1,V^2}\begin{pmatrix} \hat{a} & \hat{g} & -\hat{b}\end{pmatrix}}}\\ &P^{\leq}\left(\left(1-\mathrm{frac}\,\hat{b}\right)\begin{pmatrix} \hat{a}^1 & \hat{g}^1 \end{pmatrix} - \left(\mathrm{frac}\,\hat{b}\right)\begin{pmatrix} \hat{a}^2 & \hat{g}^2 \end{pmatrix}, \left(1-\mathrm{frac}\,\hat{b}\right)\hat{b}^1 - \left(\mathrm{frac}\,\hat{b}\right)\hat{b}^2 - \left(1-\mathrm{frac}\,\hat{b}\right)\left(\mathrm{frac}\,\hat{b}\right)\right).\end{aligned} \tag{9.10}$$

Proof. Let $\begin{pmatrix} \hat{a} & \hat{g} & -\hat{b} \end{pmatrix} \in C' \cap (\mathbb{Z}^m \times 0^n \times \mathbb{R})^T$. Because of the linear independence of the rows of $\begin{pmatrix} A & G \end{pmatrix}$, there exists a unique $\mu \in \left(\mathbb{R}^l\right)^T$ such that

$$\mu \begin{pmatrix} A & G & -b \end{pmatrix} = \begin{pmatrix} \hat{a} & \hat{g} & -\hat{b} \end{pmatrix}. \tag{9.11}$$

Let

$$\begin{aligned}\begin{pmatrix} \hat{a}^1 & \hat{g}^1 & -\hat{b}^1 \end{pmatrix} &:= \mathrm{proj}^1_{V^1,V^2}\begin{pmatrix} \hat{a} & \hat{g} & -\hat{b} \end{pmatrix},\\ \begin{pmatrix} \hat{a}^2 & \hat{g}^2 & -\hat{b}^2 \end{pmatrix} &:= \mathrm{proj}^2_{V^1,V^2}\begin{pmatrix} \hat{a} & \hat{g} & -\hat{b} \end{pmatrix}.\end{aligned}$$

Since $\begin{pmatrix} \hat{a} & \hat{g} & -\hat{b} \end{pmatrix} \in C'$ and because of the construction of C', we have

$$\begin{aligned}\begin{pmatrix} \hat{a}^1 & \hat{g}^1 & -\hat{b}^1 \end{pmatrix} &= \mu^+\begin{pmatrix} A & G & -b \end{pmatrix},\\ \begin{pmatrix} \hat{a}^2 & \hat{g}^2 & -\hat{b}^2 \end{pmatrix} &= \mu^-\begin{pmatrix} A & G & -b \end{pmatrix}.\end{aligned}$$

We have for $\binom{x}{y} \in \mathbb{R}^m \times \mathbb{R}^n$:

$$\begin{aligned}f^{\leq 0,MIR,\mu}_{A,G,b}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) &= \left(\left(1-\mathrm{frac}\,(\mu b)\right)\mu^+ + \mathrm{frac}\,(\mu b)\,\mu^-\right)(Ax+Gy-b) + \mathrm{frac}\,(\mu b)\left(1-\mathrm{frac}\,(\mu b)\right)\\ &= \left(\left(1-\mathrm{frac}\,(\mu b)\right)\mu^+ + \mathrm{frac}\,(\mu b)\,\mu^-\right)(Ax+Gy)\\ &\quad - \left(\left(1-\mathrm{frac}\,(\mu b)\right)\mu^+ + \mathrm{frac}\,(\mu b)\,\mu^-\right)b + \mathrm{frac}\,(\mu b)\left(1-\mathrm{frac}\,(\mu b)\right)\\ &= \left(\left(1-\mathrm{frac}\,\hat{b}\right)\mu^+ + \left(\mathrm{frac}\,\hat{b}\right)\mu^-\right)(Ax+Gy)\\ &\quad - \left(\left(1-\mathrm{frac}\,\hat{b}\right)\mu^+ + \left(\mathrm{frac}\,\hat{b}\right)\mu^-\right)b + \left(\mathrm{frac}\,\hat{b}\right)\left(1-\mathrm{frac}\,\hat{b}\right)\\ &= \left(1-\mathrm{frac}\,\hat{b}\right)\left(\hat{a}^1x+\hat{g}^1y\right) - \left(\mathrm{frac}\,\hat{b}\right)\left(\hat{a}^2x+\hat{g}^2y\right)\\ &\quad - \left(1-\mathrm{frac}\,\hat{b}\right)\hat{b}^1 + \left(\mathrm{frac}\,\hat{b}\right)\hat{b}^2 + \left(\mathrm{frac}\,\hat{b}\right)\left(1-\mathrm{frac}\,\hat{b}\right).\end{aligned} \tag{9.12}$$

For $\subseteq$ in (9.10)**:** Let $\begin{pmatrix} \hat{a} & \hat{g} & -\hat{b} \end{pmatrix} \in \mathcal{S} \subseteq C' \cap (\mathbb{Z}^m \times 0^n \times \mathbb{R})^T$ and let μ satisfy (9.11) (we saw that such a μ always exists). Then $\mu \in \mathcal{M}_{MIR,L}(A,G)$. Using (9.12), we obtain

$$\begin{aligned}f^{\leq 0,MIR,\mu}_{A,G,b}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) &= \left(1-\mathrm{frac}\,\hat{b}\right)\left(\hat{a}^1x+\hat{g}^1y\right) - \left(\mathrm{frac}\,\hat{b}\right)\left(\hat{a}^2x+\hat{g}^2y\right)\\ &\quad - \left(1-\mathrm{frac}\,\hat{b}\right)\hat{b}^1 + \left(\mathrm{frac}\,\hat{b}\right)\hat{b}^2 + \left(\mathrm{frac}\,\hat{b}\right)\left(1-\mathrm{frac}\,\hat{b}\right),\end{aligned}$$

which shows the statement.

For $\supseteq$ in (9.10)**:** Let $\mu \in \mathcal{M}_{MIR}(A,G)$. If $\mathrm{frac}\,(\mu b) = 0$, then $f^{\leq 0,MIR,\mu}_{A,G,b} \leq 0$ is already valid for $P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)$ (by Lemma 420); so, we can assume $\mathrm{frac}\,(\mu b) \neq 0$. Let

$$\begin{pmatrix} \hat{a} & \hat{g} & -\hat{b} \end{pmatrix} := \mu\begin{pmatrix} A & G & -b \end{pmatrix}.$$

Since $\mathcal{S}$ is a $\mathbb{Z}+$ icone system that consists of vectors from $\left(\mathbb{Z}^m \times 0^n \times \mathbb{R}\right)^T$, there exist

- $\left(\begin{array}{ccc} \overline{a}^1 & \overline{g}^1 & -\overline{b}^1 \end{array} \right) \in C' \cap \left(\mathbb{Z}^m \times 0^n \times \mathbb{Z}\right)^T$ and
- $\left(\begin{array}{ccc} \overline{a}^{2,1} & \overline{g}^{2,1} & -\overline{b}^{2,1} \end{array} \right), \dots, \left(\begin{array}{ccc} \overline{a}^{2,k} & \overline{g}^{2,k} & -\overline{b}^{2,k} \end{array} \right) \in \mathcal{S} \subseteq C' \cap \left(\mathbb{Z}^m \times 0^n \times (\mathbb{R} \backslash \mathbb{Z})\right)^T$ $(k \in \mathbb{Z}_{\geq 0})$

such that

$$\left(\begin{array}{ccc} \hat{a} & \hat{g} & -\hat{b} \end{array} \right) = \left(\begin{array}{ccc} \overline{a}^1 & \overline{g}^1 & -\overline{b}^1 \end{array} \right) + \sum_{i=1}^{k} \left(\begin{array}{ccc} \overline{a}^{2,i} & \overline{g}^{2,i} & -\overline{b}^{2,i} \end{array} \right).$$

There exist uniquely defined $\overline{\mu}^1, \overline{\mu}^{2,1}, \dots, \overline{\mu}^{2,k} \in \left(\mathbb{R}^l\right)^T$ such that

$$\begin{aligned} \overline{\mu}^1 \left(\begin{array}{ccc} A & G & -b \end{array} \right) &= \left(\begin{array}{ccc} \overline{a}^1 & \overline{g}^1 & -\overline{b}^1 \end{array} \right), \\ \forall j \in [k] : \overline{\mu}^{2,j} \left(\begin{array}{ccc} A & G & -b \end{array} \right) &= \left(\begin{array}{ccc} \overline{a}^{2,j} & \overline{g}^{2,j} & -\overline{b}^{2,j} \end{array} \right). \end{aligned}$$

These satisfay

$$\begin{aligned} \left\{ i \in [l], \overline{\mu}^1_i > 0 \right\} &\subseteq L, \\ \forall j \in [k] : \left\{ i \in [l], \overline{\mu}^{2,j}_i > 0 \right\} &\subseteq L, \\ \left\{ i \in [l], \overline{\mu}^1_i < 0 \right\} &\subseteq [l] \backslash L, \\ \forall j \in [k] : \left\{ i \in [l], \overline{\mu}^{2,j}_i < 0 \right\} &\subseteq [l] \backslash L. \end{aligned}$$

By Lemma 423 and Lemma 424, $f^{\leq 0, MIR, \mu}_{A,G,b}(\cdot) \leq 0$ is dominated relatively to $P^{\leq}\left(\left(\begin{array}{cc} A & G \end{array} \right), b\right)$ by the MIR cuts $f^{\leq 0, MIR, \overline{\mu}^{2,1}}_{A,G,b}(\cdot) \leq 0$, …, $f^{\leq 0, MIR, \overline{\mu}^{2,k}}_{A,G,b}(\cdot) \leq 0$. Thus, we can assume $\left(\begin{array}{ccc} \overline{a} & \overline{g} & -\overline{b} \end{array} \right) \in \mathcal{S}$. Let

$$\begin{aligned} \left(\begin{array}{ccc} \hat{a}^1 & \hat{g}^1 & -\hat{b}^1 \end{array} \right) &:= \operatorname{proj}^1_{V^1, V^2} \left(\begin{array}{ccc} \hat{a} & \hat{g} & -\hat{b} \end{array} \right), \\ \left(\begin{array}{ccc} \hat{a}^2 & \hat{g}^2 & -\hat{b}^2 \end{array} \right) &:= \operatorname{proj}^2_{V^1, V^2} \left(\begin{array}{ccc} \hat{a} & \hat{g} & -\hat{b} \end{array} \right). \end{aligned}$$

Then, by (9.12), we have

$$\begin{aligned} f^{\leq 0, MIR, \mu}_{A,G,b}\left(\left(\begin{array}{c} x \\ y \end{array} \right)\right) &= \left(1 - \operatorname{frac} \hat{b}\right) \left(\hat{a}^1 x + \hat{g}^1 y\right) - \left(\operatorname{frac} \hat{b}\right) \left(\hat{a}^2 x + \hat{g}^2 y\right) \\ &\quad - \left(1 - \operatorname{frac} \hat{b}\right) \hat{b}^1 + \left(\operatorname{frac} \hat{b}\right) \hat{b}^2 + \left(\operatorname{frac} \hat{b}\right) \left(1 - \operatorname{frac} \hat{b}\right), \end{aligned}$$

which shows the statement. □

9.2.4. Proof of Theorem 417

We now prove Theorem 417. For convenience, let us recapitulate it here:

Theorem 417. *Let $P = P^{\leq}\left(\left(\begin{array}{cc} A & G \end{array} \right), b\right)$ be a polyhedron, where $A \in \mathbb{Q}^{l \times m}$, $G \in \mathbb{Q}^{l \times n}$ and $b \in \mathbb{R}^l$ (i.e. P is a polyhedron with rational face normals). Then $\operatorname{cl}_{split}(P) = \operatorname{cl}_{MIR}(A, G, b)$ is a polyhedron and this polyhedron is rational if b is.*

Proof. $\operatorname{cl}_{split}(P) = \operatorname{cl}_{MIR}(A, G, b)$ holds by Theorem 412 and Theorem 415. By Lemma 419 and Theorem 427, we have:

$$\operatorname{cl}_{MIR}(A, G, b) = \bigcap_{L \in \mathcal{P}([l])} \bigcap_{S \in \mathcal{B}^*\left(\left(\begin{array}{cc} A & G \end{array} \right)\right)} \operatorname{cl}_{MIR, L}\left(A_{S,*}, G_{S,*}, b_S\right).$$

By Theorem 429, for each $L \in \mathcal{P}([l])$ and $S \in \mathcal{B}^*\left(\left(\begin{array}{cc} A & G \end{array} \right)\right)$, the set $\operatorname{cl}_{MIR,L}\left(A_{S,*}, G_{S,*}, b_S\right)$ is a polyhedron (a rational polyhedron if b is rational). □

9.2.5. Cook-Kannan-Schrijver example

Theorem 430. *For $\epsilon \in \mathbb{R}_{>0}$, let*

$$\begin{aligned} P^{430,\epsilon} &:= \operatorname{conv}\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \epsilon \end{pmatrix} \right\} \\ &= P^{\leq}\left(\left(\begin{array}{cc|c} -1 & 0 & \frac{2}{3\epsilon} \\ 0 & -1 & \frac{2}{3\epsilon} \\ 1 & 1 & \frac{2}{3\epsilon} \\ 0 & 0 & -1 \end{array} \right), \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix} \right) \\ &=: P^{\leq}\left(\begin{pmatrix} A & G^{\epsilon} \end{pmatrix}, b \right) \\ &\subseteq \mathbb{R}^2 \times \mathbb{R}^1. \end{aligned}$$

Then $\operatorname{cl}_{split}\left(P^{430,\epsilon}\right) = P^{430,\frac{\epsilon}{2}}$. *In particular, for every $t \in \mathbb{Z}_{\geq 0}$, we have*

$$\operatorname{cl}_{split}^{(t)}\left(P^{430,\epsilon}\right) = P^{430,\frac{\epsilon}{2^t}}.$$

Proof. We first note that

$$\operatorname{cl}_{MIR}\left(P^{430,\epsilon}\right) = P^{430,\epsilon} \cap \operatorname{cl}_{MIR}\left(P^{\leq}\left(\begin{pmatrix} A & G^{\epsilon} \end{pmatrix}_{(1,2,3),*}, b_{(1,2,3)}\right)\right). \tag{9.13}$$

For (9.13): since $\operatorname{rank}\begin{pmatrix} A & G^{\epsilon} \end{pmatrix} = 3$, by Lemma 419 and Theorem 427, we just have to consider subsets of 3 rows. Note that for $J \in \binom{[4]}{3}$, where $J \neq \{1,2,3\}$, we have

$$P^{\leq}\left(\begin{pmatrix} A & G^{\epsilon} \end{pmatrix}_{J,*}, b^{\epsilon}_{J,*}\right) = \operatorname{cl}_I\left(P^{\leq}\left(\begin{pmatrix} A & G^{\epsilon} \end{pmatrix}_{J,*}, b_{J,*}\right)\right);$$

so, we only have to consider $\operatorname{cl}_{MIR}\left(P^{\leq}\left(\begin{pmatrix} A & G^{\epsilon} \end{pmatrix}_{(1,2,3),*}, b_{(1,2,3)}\right)\right)$.

By Theorem 422, it suffices to compute $\operatorname{cl}_{MIR,L}\left(P^{430,\epsilon}\right)$ where

$$L \in \{\{1,2,3\}, \{2,3\}, \{1,3\}, \{1,2\}\}.$$

So, we just have to apply Theorem 429 to the cones

1. $C' := \left(\operatorname{cone}\left\{\begin{pmatrix} -1 & 0 & \frac{2}{3\epsilon} & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & \frac{2}{3\epsilon} & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & \frac{2}{3\epsilon} & -2 \end{pmatrix}\right\}\right) \cap \left(\mathbb{R}^2 \times 0^1 \times \mathbb{R}\right)^T,$
2. $C' := \left(\operatorname{cone}\left\{-\begin{pmatrix} -1 & 0 & \frac{2}{3\epsilon} & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & \frac{2}{3\epsilon} & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & \frac{2}{3\epsilon} & -2 \end{pmatrix}\right\}\right) \cap \left(\mathbb{R}^2 \times 0^1 \times \mathbb{R}\right)^T,$
3. $C' := \left(\operatorname{cone}\left\{\begin{pmatrix} -1 & 0 & \frac{2}{3\epsilon} & 0 \end{pmatrix}, -\begin{pmatrix} 0 & -1 & \frac{2}{3\epsilon} & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & \frac{2}{3\epsilon} & -2 \end{pmatrix}\right\}\right) \cap \left(\mathbb{R}^2 \times 0^1 \times \mathbb{R}\right)^T,$
4. $C' := \left(\operatorname{cone}\left\{\begin{pmatrix} -1 & 0 & \frac{2}{3\epsilon} & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & \frac{2}{3\epsilon} & 0 \end{pmatrix}, -\begin{pmatrix} 1 & 1 & \frac{2}{3\epsilon} & -2 \end{pmatrix}\right\}\right) \cap \left(\mathbb{R}^2 \times 0^1 \times \mathbb{R}\right)^T.$

We clearly have in case 1:

$$\left(\operatorname{cone}\left\{\begin{pmatrix} -1 & 0 & \frac{2}{3\epsilon} & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & \frac{2}{3\epsilon} & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & \frac{2}{3\epsilon} & -2 \end{pmatrix}\right\}\right) \cap \left(\mathbb{R}^2 \times 0^1 \times \mathbb{R}\right)^T = \left(0^2 \times 0^1 \times 0\right)^T.$$

So for the remaining three cases:

For case 2: Let

$$x \in C' := \left(\operatorname{cone}\left\{-\begin{pmatrix} -1 \\ 0 \\ \frac{2}{3\epsilon} \\ 0 \end{pmatrix}^T, \begin{pmatrix} 0 \\ -1 \\ \frac{2}{3\epsilon} \\ 0 \end{pmatrix}^T, \begin{pmatrix} 1 \\ 1 \\ \frac{2}{3\epsilon} \\ -2 \end{pmatrix}^T\right\}\right) \cap \left(\mathbb{R}^2 \times 0^1 \times \mathbb{R}\right)^T.$$

Then

$$x = -(\lambda_2+\lambda_3)\begin{pmatrix}-1\\0\\\frac{2}{3\epsilon}\\0\end{pmatrix}^T + \lambda_2\begin{pmatrix}0\\-1\\\frac{2}{3\epsilon}\\0\end{pmatrix}^T + \lambda_3\begin{pmatrix}1\\1\\\frac{2}{3\epsilon}\\-2\end{pmatrix}^T = \lambda_2\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}^T + \lambda_3\begin{pmatrix}2\\1\\0\\-2\end{pmatrix}^T,$$

where $\lambda_2, \lambda_3 \in \mathbb{R}_{\geq 0}$. Indeed, any such vector lies in C'. Thus,

$$C' = \operatorname{cone}\left\{\begin{pmatrix}1 & -1 & 0 & 0\end{pmatrix}, \begin{pmatrix}2 & 1 & 0 & -2\end{pmatrix}\right\}.$$

We have

$$\left(\left([0,1)^2 \setminus \{0^2\}\right)^T \begin{pmatrix}1 & -1 & 0 & 0\\ 2 & 1 & 0 & -2\end{pmatrix}\right) \cap \left(\mathbb{Z}^2 \times 0^1 \times \mathbb{R}\right)^T = \left\{\begin{pmatrix}1 & 0 & 0 & -\frac{2}{3}\end{pmatrix}, \begin{pmatrix}2 & 0 & 0 & -\frac{4}{3}\end{pmatrix}\right\}.$$

So, we can apply either Theorem 365 or Theorem 366 to compute a $\mathbb{Z}+$icone system that generates C'. Note that the vector $\begin{pmatrix}2 & 0 & 0 & -\frac{4}{3}\end{pmatrix}$ is redundant for such a $\mathbb{Z}+$icone system.

For $\begin{pmatrix}1 & 0 & 0 & -\frac{2}{3}\end{pmatrix}$:

$$\begin{aligned}
\underbrace{\begin{pmatrix}1 & 0 & 0 & -\frac{2}{3}\end{pmatrix}}_{=:\begin{pmatrix}\hat{a} & \hat{g} & -\hat{b}\end{pmatrix}} &= \frac{1}{3}\begin{pmatrix}1 & -1 & 0 & 0\end{pmatrix} + \frac{1}{3}\begin{pmatrix}2 & 1 & 0 & -2\end{pmatrix}\\
&= \frac{1}{3}\begin{pmatrix}0 & -1 & \frac{2}{3\epsilon} & 0\end{pmatrix} + \frac{1}{3}\begin{pmatrix}1 & 1 & \frac{2}{3\epsilon} & -2\end{pmatrix} - \frac{2}{3}\begin{pmatrix}-1 & 0 & \frac{2}{3\epsilon} & 0\end{pmatrix}\\
&= \underbrace{\begin{pmatrix}\frac{1}{3} & 0 & \frac{4}{9\epsilon} & -\frac{2}{3}\end{pmatrix}}_{=:\begin{pmatrix}\hat{a}^1 & \hat{g}^1 & -\hat{b}^1\end{pmatrix}} - \underbrace{\begin{pmatrix}-\frac{2}{3} & 0 & \frac{4}{9\epsilon} & 0\end{pmatrix}}_{=:-\begin{pmatrix}\hat{a}^2 & \hat{g}^2 & -\hat{b}^2\end{pmatrix}}.
\end{aligned}$$

By Theorem 429, this yields the following half-space:

$$\begin{aligned}
&P^{\leq}\left(\left(1-\operatorname{frac}\hat{b}\right)\begin{pmatrix}\hat{a}^1 & \hat{g}^1\end{pmatrix} - \left(\operatorname{frac}\hat{b}\right)\begin{pmatrix}\hat{a}^2 & \hat{g}^2\end{pmatrix}, \left(1-\operatorname{frac}\hat{b}\right)\hat{b}^1 - \left(\operatorname{frac}\hat{b}\right)\hat{b}^2 - \left(1-\operatorname{frac}\hat{b}\right)\left(\operatorname{frac}\hat{b}\right)\right)\\
=&P^{\leq}\left(\frac{1}{3}\begin{pmatrix}\frac{1}{3} & 0 & \frac{4}{9\epsilon}\end{pmatrix} - \frac{2}{3}\begin{pmatrix}\frac{2}{3} & 0 & -\frac{4}{9\epsilon}\end{pmatrix}, \frac{1}{3}\cdot\frac{2}{3} - \frac{2}{3}\cdot 0 - \frac{1}{3}\cdot\frac{2}{3}\right)\\
=&P^{\leq}\left(\begin{pmatrix}-1 & 0 & 2\cdot\frac{2}{3\epsilon}\end{pmatrix}, 0\right)\\
=&P^{\leq}\left(\begin{pmatrix}A & G^{\frac{\epsilon}{2}}\end{pmatrix}_{1,*}, b_1\right).
\end{aligned}$$

For case 3: Note that

$$C' := \left(\operatorname{cone}\left\{\begin{pmatrix}-1\\0\\\frac{2}{3\epsilon}\\0\end{pmatrix}^T, -\begin{pmatrix}0\\-1\\\frac{2}{3\epsilon}\\0\end{pmatrix}^T, \begin{pmatrix}1\\1\\\frac{2}{3\epsilon}\\-2\end{pmatrix}^T\right\}\right) \cap \left(\mathbb{R}^2 \times 0^1 \times \mathbb{R}\right)^T$$

is the same C' as in case 2, just with the first two coordinates flipped. So, if we append $\begin{pmatrix}0 & 1 & 0 & -\frac{2}{3}\end{pmatrix}$ to the generators

$$\left\{\begin{pmatrix}-1 & 1 & 0 & 0\end{pmatrix}, \begin{pmatrix}1 & 2 & 0 & -2\end{pmatrix}\right\}$$

of C', we obtain a $\mathbb{Z}+$icone system that generates C' and consists of vectors from $\mathbb{Z}^2 \times 0^1 \times \mathbb{R}$. Putting the vector $\begin{pmatrix}\hat{a} & \hat{g} & -\hat{b}\end{pmatrix} := \begin{pmatrix}0 & 1 & 0 & -\frac{2}{3}\end{pmatrix}$ into Theorem 429, one obtains (similarly to case 2) the half-space $P^{\leq}\left(\begin{pmatrix}A & G^{\frac{\epsilon}{2}}\end{pmatrix}_{2,*}, b_2\right)$.

For case 4: Let

$$x \in C' := \left(\operatorname{cone}\left\{\begin{pmatrix} -1 \\ 0 \\ \frac{2}{3\epsilon} \\ 0 \end{pmatrix}^T, \begin{pmatrix} 0 \\ -1 \\ \frac{2}{3\epsilon} \\ 0 \end{pmatrix}^T, -\begin{pmatrix} 1 \\ 1 \\ \frac{2}{3\epsilon} \\ -2 \end{pmatrix}^T\right\}\right) \cap \left(\mathbb{R}^2 \times 0^1 \times \mathbb{R}\right)^T.$$

Then

$$x = \lambda_1 \begin{pmatrix} -1 \\ 0 \\ \frac{2}{3\epsilon} \\ 0 \end{pmatrix}^T + \lambda_2 \begin{pmatrix} 0 \\ -1 \\ \frac{2}{3\epsilon} \\ 0 \end{pmatrix}^T - (\lambda_1 + \lambda_2) \begin{pmatrix} 1 \\ 1 \\ \frac{2}{3\epsilon} \\ -2 \end{pmatrix}^T = \lambda_1 \begin{pmatrix} -2 \\ -1 \\ 0 \\ 2 \end{pmatrix}^T + \lambda_2 \begin{pmatrix} -1 \\ -2 \\ 0 \\ 2 \end{pmatrix}^T,$$

where $\lambda_1, \lambda_2 \in \mathbb{R}_{\geq 0}$. Indeed, any such vector lies in C'. Thus,

$$C' = \operatorname{cone}\left\{\begin{pmatrix} -2 & -1 & 0 & -2 \end{pmatrix}, \begin{pmatrix} -1 & -2 & 0 & -2 \end{pmatrix}\right\}.$$

We have

$$\left(\left([0,1)^2 \setminus \{0^2\}\right)^T \begin{pmatrix} -2 & -1 & 0 & 2 \\ -1 & -2 & 0 & 2 \end{pmatrix}\right) \cap \left(\mathbb{Z}^2 \times 0^1 \times \mathbb{R}\right)^T = \left\{\begin{pmatrix} -1 & -1 & 0 & \frac{4}{3} \end{pmatrix}, \begin{pmatrix} -2 & -2 & 0 & \frac{8}{3} \end{pmatrix}\right\}.$$

So, we can apply either Theorem 365 or Theorem 366 to compute a $\mathbb{Z}+$icone system that generates C'. Note that the vector $\begin{pmatrix} -2 & -2 & 0 & \frac{8}{3} \end{pmatrix}$ is redundant for such a $\mathbb{Z}+$icone system.

For $\begin{pmatrix} -1 & -1 & 0 & \frac{4}{3} \end{pmatrix}$:

$$\begin{aligned} \underbrace{\begin{pmatrix} -1 & -1 & 0 & \frac{4}{3} \end{pmatrix}}_{=:\begin{pmatrix} \hat{a} & \hat{g} & -\hat{b} \end{pmatrix}} &= \frac{1}{3}\begin{pmatrix} -2 & -1 & 0 & 2 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} -1 & -2 & 0 & 2 \end{pmatrix} \\ &= \frac{1}{3}\begin{pmatrix} -1 & 0 & \frac{2}{3\epsilon} & 0 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 0 & -1 & \frac{2}{3\epsilon} & 0 \end{pmatrix} - \frac{2}{3}\begin{pmatrix} 1 & 1 & \frac{2}{3\epsilon} & -2 \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} -\frac{1}{3} & -\frac{1}{3} & \frac{4}{9\epsilon} & 0 \end{pmatrix}}_{=:\begin{pmatrix} \hat{a}^1 & \hat{g}^1 & -\hat{b}^1 \end{pmatrix}} - \underbrace{\begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \frac{4}{9\epsilon} & -\frac{4}{3} \end{pmatrix}}_{=:-\begin{pmatrix} \hat{a}^2 & \hat{g}^2 & -\hat{b}^2 \end{pmatrix}}. \end{aligned}$$

By Theorem 429, this yields the following half-space:

$$\begin{aligned} &P^{\leq}\left(\left(1 - \operatorname{frac}\hat{b}\right)\begin{pmatrix} \hat{a}^1 & \hat{g}^1 \end{pmatrix} - \left(\operatorname{frac}\hat{b}\right)\begin{pmatrix} \hat{a}^2 & \hat{g}^2 \end{pmatrix}, \left(1 - \operatorname{frac}\hat{b}\right)\hat{b}^1 - \left(\operatorname{frac}\hat{b}\right)\hat{b}^2 - \left(1 - \operatorname{frac}\hat{b}\right)\left(\operatorname{frac}\hat{b}\right)\right) \\ =&P^{\leq}\left(\frac{1}{3}\begin{pmatrix} -\frac{1}{3} & -\frac{1}{3} & \frac{4}{9\epsilon} \end{pmatrix} - \frac{2}{3}\begin{pmatrix} -\frac{2}{3} & -\frac{2}{3} & -\frac{4}{9\epsilon} \end{pmatrix}, \frac{1}{3}\cdot 0 - \frac{2}{3}\cdot\left(-\frac{4}{3}\right) - \frac{2}{3}\cdot\frac{1}{3}\right) \\ =&P^{\leq}\left(\begin{pmatrix} 1 & 1 & 2\cdot\frac{2}{3\epsilon} \end{pmatrix}, 2\right) \\ =&P^{\leq}\left(\begin{pmatrix} A & G^{\frac{\epsilon}{2}} \end{pmatrix}_{3,*}, b_3\right). \end{aligned}$$

Thus,

$$\operatorname{cl}_{MIR}\left(P^{430,\epsilon}\right) = P^{\leq}\left(\begin{pmatrix} A & G^{\epsilon} \\ A_{[3],*} & G^{\frac{\epsilon}{2}}_{[3],*} \end{pmatrix}, \begin{pmatrix} b \\ b_{[3]} \end{pmatrix}\right) = P^{\leq}\left(\begin{pmatrix} A & G^{\frac{\epsilon}{2}} \end{pmatrix}, b\right) = P^{\frac{\epsilon}{2}}.$$

□

Part IV.

Disjunctions, $L_{2,\mathbb{Q}}$ cuts and essential $L_{2-\frac{1}{2},\mathbb{Q}}$ cuts

10. Embedding two-dimensional lattice-free bodies into disjunctions

10.1. Central statements

The purpose of this chapter is to prove Theorem 431 and Theorem 434. These two theorems are used in chapter 11 to derive characterizations of the $L_{2,\mathbb{Q}}$ and essential $L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ closure via disjunctions. Specifically, Theorem 431 is used in section 11.1 for the proof of Theorem 459 and Theorem 434 is used in section 11.2.2 for the proof of Theorem 470.

Theorem 431. *Let $S \subseteq \mathbb{R}^2$ be convex and let $L = P^{\leq}(A, b)$ ($A \in \mathbb{R}^{l\times 2}$ and $b \in \mathbb{R}^l$, where $l \in \mathbb{Z}_{\geq 0}$) be a full-dimensional maximal lattice-free polyhedron such that we have*

$$S \subseteq (\operatorname{int} L) \,\dot{\cup}\, ((\operatorname{bd} L) \setminus (\operatorname{bd} L)_I).$$

Then there exists a crooked cross disjunction (cf. Definition 146) $D^c(\pi^1, \pi^2, \gamma_1, \gamma_2) \subseteq \mathbb{R}^2$ such that

$$S \cap D^c(\pi^1, \pi^2, \gamma_1, \gamma_2) = \emptyset.$$

We remark that in [DDG12], the following weaker version of Theorem 431 is shown:

Theorem 432. *Let $L = P^{\leq}(A, b)$ ($A \in \mathbb{R}^{l\times 2}$ and $b \in \mathbb{R}^l$, where $l \in \mathbb{Z}_{\geq 0}$) be a full-dimensional maximal lattice-free polyhedron. Then there exists a crooked cross disjunction $D^c(\pi^1, \pi^2, \gamma_1, \gamma_2) \subseteq \mathbb{R}^2$ such that*

$$D^c(\pi^1, \pi^2, \gamma_1, \gamma_2) \cap \operatorname{int} L = \emptyset.$$

We now define **T disjunctions**, which we need to formulate Theorem 434:

Definition 433. *Let $\pi^1, \pi^2 \in \mathbb{Z}^m$ and $\gamma_1, \gamma_2 \in \mathbb{Z}$, where $m \in \mathbb{Z}_{\geq 0}$. Let*

$$\begin{aligned}
D_1^T(\pi^1, \pi^2, \gamma_1, \gamma_2) &:= \left\{x \in \mathbb{R}^m : {\pi^1}^T x \leq \gamma_1\right\},\\
D_2^T(\pi^1, \pi^2, \gamma_1, \gamma_2) &:= \left\{x \in \mathbb{R}^m : {\pi^1}^T x \geq \gamma_1 + 1 \wedge {\pi^2}^T x \leq \gamma_2\right\},\\
D_3^T(\pi^1, \pi^2, \gamma_1, \gamma_2) &:= \left\{x \in \mathbb{R}^m : {\pi^1}^T x \geq \gamma_1 + 1 \wedge {\pi^2}^T x \geq \gamma_2 + 1\right\},\\
D^T(\pi^1, \pi^2, \gamma_1, \gamma_2) &:= \bigcup_{i=1}^{3} D_i^T(\pi^1, \pi^2, \gamma_1, \gamma_2).
\end{aligned}$$

*Then we denote the set $D^T(\pi^1, \pi^2, \gamma_1, \gamma_2)$ as **T disjunction** and the sets $D_i^T(\pi^1, \pi^2, \gamma_1, \gamma_2)$ ($i \in [3]$) as the **atoms of the T disjunction**.*

Figure 10.1 shows an example of a T disjunction.

Theorem 434. *Let $L = P^{\leq}(A, b)$ ($A \in \mathbb{R}^{l\times 2}$ and $b \in \mathbb{R}^l$, where $l \in \mathbb{Z}_{\geq 0}$) be a full-dimensional maximal lattice-free polyhedron such that every inequality is facet-defining. Let $j^* \in [l]$ be arbitrary. Then there either exists*

- *a T disjunction (cf. Definition 433) $D^T(\pi^1, \pi^2, \gamma_1, \gamma_2) \subseteq \mathbb{R}^2$ and an $i^* \in [3]$ such that*

$$\begin{aligned}
P^{\geq}(A_{j^*,*}, b_{j^*}) &\supseteq D_{i^*}^T(\pi^1, \pi^2, \gamma_1, \gamma_2),\\
\bigcup_{j\in[l]\setminus\{j^*\}} P^{\geq}(A_{j,*}, b_j) &\supseteq \bigcup_{i\in[3]\setminus\{i^*\}} D_i^T(\pi^1, \pi^2, \gamma_1, \gamma_2)
\end{aligned}$$

  or

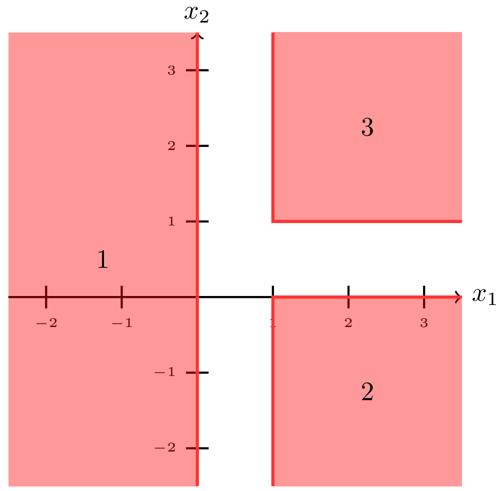


Figure 10.1.: The T disjunction $D^T\left(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right), 0, 0\right)$

- *a crooked cross disjunction (cf. Definition 146)* $D^c\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right) \subseteq \mathbb{R}^2$ *and an* $i^* \in [4]$ *such that*

$$P^{\geq}\left(A_{j^*,*}, b_{j^*}\right) \supseteq D^c_{i^*}\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right),$$
$$\bigcup_{j \in [l] \setminus \{j^*\}} P^{\geq}\left(A_{j,*}, b_j\right) \supseteq \bigcup_{i \in [4] \setminus \{i^*\}} D^c_i\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right),$$

where π^1, π^2 *are linearly independent (i.e. the T disjunction or crooked cross disjunction, respectively, is not degenerate).*

Theorem 432 is sufficient to show the cases

- P polyhedron with rational face normals and
- P convex and compact,

but *not* the situation if

- P is convex and $\operatorname{proj}_{\mathbb{R}^2} P$ is full-dimensional

in Theorem 459, Theorem 461 and Theorem 462 (for this, we indeed need the stronger Theorem 431).

The high level strategy for the proof of Theorem 431 and Theorem 434 is the following:

- Every lattice-free body in $\mathbb{R}^2$ can by an affine-unimodular map be transformed into a standard form with a specific standard description (this is defined in Theorem 437).
- In section 10.3, Lemma 441, we show it suffices to prove the respective theorems for lattice-free bodies that are in standard description.
- In section 10.5, we show Theorem 431 for L in standard description.
- In section 10.6, we show Theorem 434 for L in standard description.

10.2. Latticefree bodies in $\mathbb{R}^2$

10.2.1. Classification

The following theorem is proved in [DW10] (also cf. [DDG12], from where we obtained the following version):

Theorem 435. *a full-dimensional maximal lattice-free convex set in* $\mathbb{R}^2$ *is one of the following sets:*

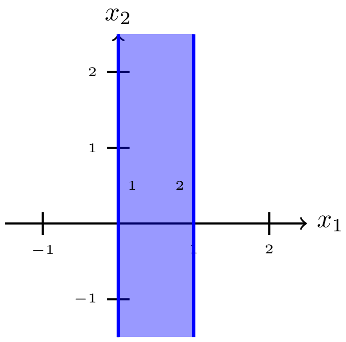
1. *a split set* $\left\{\left(\begin{smallmatrix}x_1\\x_2\end{smallmatrix}\right) \in \mathbb{R}^2 : b \leq a_1 x_1 + a_2 x_2 \leq b+1\right\}$*, where* a_1, a_2 *are coprime integers and* $b \in \mathbb{Z}$ *(see Figure 10.2a);*

2. *a triangle with at least one integral point in the relative interior of each of its sides, which in turn is either:*
   a) *a type 1 triangle, i.e. a triangle with integral vertices and exactly one integral point in the relative interior of each side (see Figure 10.2b),*
   b) *a type 2 triangle, i.e. a triangle with at least one fractional vertex v, exactly one integral point in the relative interior of the two sides incedent to v and at least two integral points in the relative interior of the third side (see Figure 10.2c),*
   c) *a type 3 triangle, i.e. a triangle with exactly three integral points on the boundary, one in the relative interior of each side (see Figure 10.2d);*
3. *a quadrilateral containing exactly one integral point in the relative interior of each of its sides (see Figure 10.2e).*

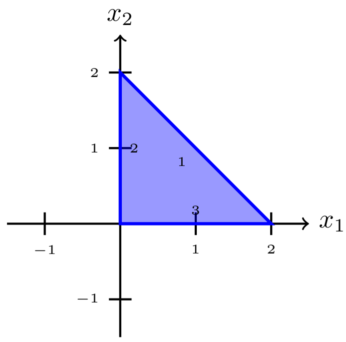
Remark 436. *Concerning Theorem 435, we remark:*

- *If we don't assume the maximal lattice-free body in Theorem 435 to be full-dimensional, it can also be an irrational, one-dimensional hyperplane (cf. Theorem 108). For this type, P^{114} and P^{115} are concrete examples.*
- *If we demand the maximal lattice-free body in Theorem 435 to be a rational polyhedron, it is full-dimensional (cf. Theorem 111).*

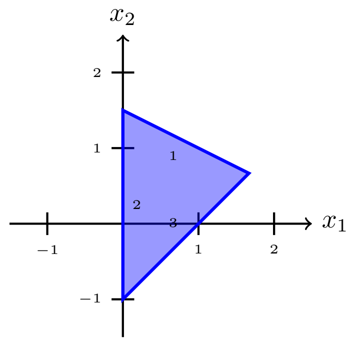
10.2.2. Standard forms



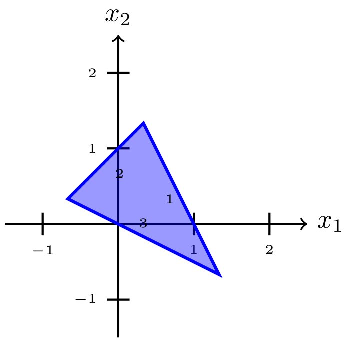
(a) Split set



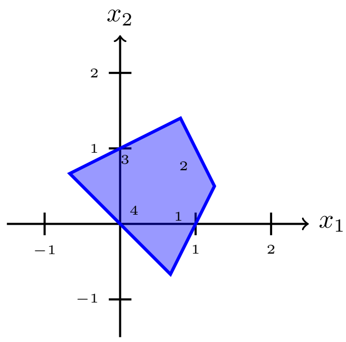
(b) Type 1 triangle



(c) Type 2 triangle



(d) Type 3 triangle



(e) Quadrilateral

Figure 10.2.: The five types of two-dimensional lattice-free bodies in $\mathbb{R}^2$ in standard form

The following theorem can be concluded from results that are proved in [DW10] and [DDG12]:

Theorem 437. *Let $L \subseteq \mathbb{R}^2$ be maximal lattice-free and either a rational polyhedron or full-dimensional. We first remark that then for every affine-unimodular map $f : \mathbb{R}^2 \to \mathbb{R}^2$, we have by construction that $f(L)$ is also maximal lattice-free.*

There exists an affine-unimodular map $f : \mathbb{R}^2 \to \mathbb{R}^2$ (cf. Definition 23) such that for $L' := f(L)$, we have:

- *If L is a split disjunction, we have (see Figure 10.2a)*

$$L' = P^{\leq}\left(\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) =: P^{\leq}(A, b).$$

- *If L is a type 1 triangle, we have (see Figure 10.2b)*

$$L' = P^{\leq}\left(\begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}\right) =: P^{\leq}(A, b).$$

- *If L is a type 2 triangle, we have $L' = P^{\leq}(A, b)$, where $A \in \mathbb{R}^{3\times 2}$ and $b \in \mathbb{R}^3$, such that (see Figure 10.2c)*

$$\left(\operatorname{relint}\left(L' \cap P^{=}(A_{1,*}, b_1)\right)\right)_I = \left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\},$$

$$A_{2,*} = \begin{pmatrix} -1 & 0 \end{pmatrix}, \tag{10.1}$$

$$b_2 = 0, \tag{10.2}$$

$$\left(\operatorname{relint}\left(L' \cap P^{=}(A_{2,*}, b_2)\right)\right)_I \supseteq \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\},$$

$$\left(\operatorname{relint}\left(L' \cap P^{=}(A_{3,*}, b_3)\right)\right)_I = \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}.$$

On the other hand, the polyhedron

$$P^{\leq}\left(\begin{pmatrix} A_{1,1} & A_{1,2} \\ -1 & 0 \\ A_{3,1} & A_{3,2} \end{pmatrix}, \begin{pmatrix} A_{1,1} + A_{1,2} \\ 0 \\ A_{3,1} \end{pmatrix}\right),$$

where $A_{1,1}, A_{1,2}, A_{3,1} > 0$ and $A_{3,2} < 0$, is a maximal lattice-free type 2 triangle.

- *If L is a type 3 triangle, we have $L' = P^{\leq}(A, b)$, where $A \in \mathbb{R}^{3\times 2}$ and $b \in \mathbb{R}^3$, such that (see Figure 10.2d)*

$$\left(L' \cap P^{=}(A_{1,*}, b_1)\right)_I = \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\},$$

$$\left(L' \cap P^{=}(A_{2,*}, b_2)\right)_I = \left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\},$$

$$\left(L' \cap P^{=}(A_{3,*}, b_3)\right)_I = \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}. \tag{10.3}$$

- *If L is a quadrilateral, we have $L' = P^{\leq}(A, b)$, where $A \in \mathbb{R}^{4\times 2}$ and $b \in \mathbb{R}^4$, such that (see Figure 10.2e)*

$$\left(L' \cap P^{=}(A_{1,*}, b_1)\right)_I = \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\},$$

$$\left(L' \cap P^{=}(A_{2,*}, b_2)\right)_I = \left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\},$$

$$\left(L' \cap P^{=}(A_{3,*}, b_3)\right)_I = \left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\},$$

$$\left(L' \cap P^{=}(A_{4,*}, b_4)\right)_I = \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}.$$

On the other hand, the polyhedron

$$P^{\leq}\left(\begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \\ A_{3,1} & A_{3,2} \\ A_{4,1} & A_{4,2} \end{pmatrix}, \begin{pmatrix} A_{1,1} \\ A_{2,1} + A_{2,2} \\ A_{3,2} \\ 0 \end{pmatrix}\right),$$

where

$$\begin{aligned} A_{1,1} &> 0, & A_{1,2} &< 0, \\ A_{2,1} &> 0, & A_{2,2} &> 0, \\ A_{3,1} &< 0, & A_{3,2} &> 0, \\ A_{4,1} &< 0, & A_{4,2} &< 0 \end{aligned}$$

is a maximal lattice-free quadrilateral.

Additionally, every inequality of $A(\cdot) \leq b$ (which describes L') is facet-defining. We then say that L' ***is in standard form*** *and we call the desciption $A(\cdot) \leq b$ a* ***standard description of*** *L'.*

We finish this section with some properties that one can show for the maximal lattice-free bodies in standard description as in Theorem 437:

Remark 438. *Let $L = P^{\leq}(A, b)$ be a maximal lattice-free type 2 triangle in standard description. Then*

$$\begin{aligned} A_{1,1} &> 0, & A_{1,2} &> 0, \\ A_{3,1} &> 0, & A_{3,2} &< 0. \end{aligned}$$

Remark 439. *Let $L = P^{\leq}(A, b)$ be a maximal lattice-free type 3 triangle in standard description. Then we have $A_{3,1} \neq A_{3,2}$ and if we assume $A_{3,1} > A_{3,2}$ (which, using (10.3), is equivalent to $\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right) \notin P^{\leq}(A_{3,*}, b_3)$), we have*

$$\begin{aligned} A_{1,1} > A_{1,2} &> 0, \\ A_{2,1} &< 0, \\ A_{2,2} &> 0, \\ A_{3,2}, A_{3,1} &< 0 \end{aligned}$$

(some of these identities also hold if $A_{3,2} > A_{3,1}$).

Remark 440. *Let $L = P^{\leq}(A, b)$ be a maximal lattice-free quadrilateral in standard description. Then*

$$\begin{aligned} A_{1,1} &> 0, & A_{1,2} &< 0, \\ A_{2,1} &> 0, & A_{2,2} &> 0, \\ A_{3,1} &< 0, & A_{3,2} &> 0, \\ A_{4,1} &< 0, & A_{4,2} &< 0. \end{aligned}$$

10.3. Reducing Theorem 431 and Theorem 434 to lattice-free bodies in standard form

The following Lemma is easy to check using Lemma 63:

Lemma 441. *Let*

$$\begin{aligned} f : \mathbb{R}^2 \to \mathbb{R}^2 : \\ x \mapsto Ux + v \end{aligned}$$

be an affine-unimodular map (cf. Definition 23). Then:

- *Let $D^T(\pi^1, \pi^2, \gamma_1, \gamma_2)$ or $D^c(\pi^1, \pi^2, \gamma_1, \gamma_2)$ be a T disjunction or crooked cross disjunction, respectively.*

Then for any $i \in [3]$ or $i \in [4]$, respectively, we have

$$\begin{aligned}
f\left(D_i^T\left(\pi^1,\pi^2,\gamma_1,\gamma_2\right)\right) &= D_i^T\left(U^{-T}\pi^1, U^{-T}\pi^2, \gamma_1 + \left(U^{-T}\pi^1\right)^T v, \gamma_2 + \left(U^{-T}\pi^2\right)^T v\right),\\
f\left(D_i^c\left(\pi^1,\pi^2,\gamma_1,\gamma_2\right)\right) &= D_i^c\left(U^{-T}\pi^1, U^{-T}\pi^2, \gamma_1 + \left(U^{-T}\pi^1\right)^T v, \gamma_2 + \left(U^{-T}\pi^2\right)^T v\right),\\
f^{-1}\left(D_i^T\left(\pi^1,\pi^2,\gamma_1,\gamma_2\right)\right) &= D_i^T\left(U^T\pi^1, U^T\pi^2, \gamma_1 - \left(\pi^1\right)^T v, \gamma_2 - \left(\pi^2\right)^T v\right),\\
f^{-1}\left(D_i^c\left(\pi^1,\pi^2,\gamma_1,\gamma_2\right)\right) &= D_i^c\left(U^T\pi^1, U^T\pi^2, \gamma_1 - \left(\pi^1\right)^T v, \gamma_2 - \left(\pi^2\right)^T v\right).
\end{aligned}$$

- *Let the prerequisites of Theorem 431 hold for S and L. Then they also hold for*
  - $f(L)$ *and* $f(S)$,
  - $f^{-1}(L)$ *and* $f^{-1}(S)$.

- *Let the prerequisites of Theorem 434 hold for $P^{\leq}(A,b)$ and j^*. Then they also hold for*
  - $f\left(P^{\leq}(A,b)\right) = P^{\leq}\left(AU^{-1}, b + AU^{-1}v\right)$ *and* j^*,
  - $f^{-1}\left(P^{\leq}(A,b)\right) = P^{\leq}(AU, b - Av)$ *and* j^*.

- *Let Theorem 431 hold for S and L with respect to $D^c\left(\pi^1,\pi^2,\gamma_1,\gamma_2\right)$. Then Theorem 431 also holds for*
  - $f(L)$ *with respect to*
  $$f\left(D^c\left(\pi^1,\pi^2,\gamma_1,\gamma_2\right)\right) = D^c\left(U^{-T}\pi^1, U^{-T}\pi^2, \gamma_1 + \left(U^{-T}\pi^1\right)^T u, \gamma_2 + \left(U^{-T}\pi^2\right)^T u\right),$$
  - $f^{-1}(L)$ *with respect to*
  $$f^{-1}\left(D^c\left(\pi^1,\pi^2,\gamma_1,\gamma_2\right)\right) = D^c\left(U^T\pi^1, U^T\pi^2, \gamma_1 - \left(\pi^1\right)^T u, \gamma_2 - \left(\pi^2\right)^T u\right).$$

- *Let Theorem 434 hold for $P^{\leq}(A,b)$ and j^* with respect to $D^T\left(\pi^1,\pi^2,\gamma_1,\gamma_2\right)$ or $D^c\left(\pi^1,\pi^2,\gamma_1,\gamma_2\right)$, respectively, and i^*. Then Theorem 431 also holds for*
  - $f\left(P^{\leq}(A,b)\right) = P^{\leq}\left(AU^{-1}, b + AU^{-1}u\right)$ *and j^* with respect to*
  $$\begin{aligned}
  f\left(D^T\left(\pi^1,\pi^2,\gamma_1,\gamma_2\right)\right) &= D^T\left(U^{-T}\pi^1, U^{-T}\pi^2, \gamma_1 + \left(U^{-T}\pi^1\right)^T u, \gamma_2 + \left(U^{-T}\pi^2\right)^T u\right) \quad \text{or}\\
  f\left(D^c\left(\pi^1,\pi^2,\gamma_1,\gamma_2\right)\right) &= D^c\left(U^{-T}\pi^1, U^{-T}\pi^2, \gamma_1 + \left(U^{-T}\pi^1\right)^T u, \gamma_2 + \left(U^{-T}\pi^2\right)^T u\right),
  \end{aligned}$$
  respectively, and i^*,
  - $f^{-1}\left(P^{\leq}(A,b)\right) = P^{\leq}(AU, b - Au)$ *and j^* with respect to*
  $$\begin{aligned}
  f^{-1}\left(D^T\left(\pi^1,\pi^2,\gamma_1,\gamma_2\right)\right) &= D^T\left(U^T\pi^1, U^T\pi^2, \gamma_1 - \left(\pi^1\right)^T u, \gamma_2 - \left(\pi^2\right)^T u\right) \quad \text{or}\\
  f^{-1}\left(D^c\left(\pi^1,\pi^2,\gamma_1,\gamma_2\right)\right) &= D^c\left(U^T\pi^1, U^T\pi^2, \gamma_1 - \left(\pi^1\right)^T u, \gamma_2 - \left(\pi^2\right)^T u\right),
  \end{aligned}$$
  respectively, and i^*.

By Theorem 437 and Lemma 441, it thus suffices to prove Theorem 431 and Theorem 434, respectively, in the case that L is in standard form. These proofs are done in section 10.5 and 10.6, respectively.

But before that, in section 10.4, we write down some statements that are needed in both sections. Note that to save space, we typically only state the individual lemmas and sometimes give high-level ideas for their proofs. The reason why we do this is that the proofs are in principle typically rather simple, but have lots of tedious technical details to care about.

10.4. Embedding lattice-free bodies in standard form into disjunctions

10.4.1. Type 2 triangles

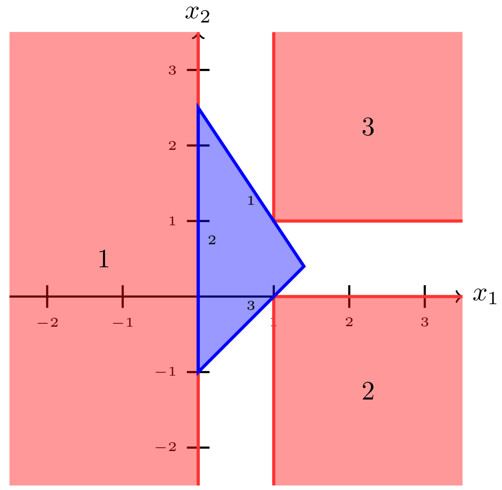


Figure 10.3.: Situation for type 2 triangles

Remark 438 is a central tool for proving the following lemma:

Lemma 442. *Let $L = P^{\leq}(A, b)$ be a lattice-free type 2 triangle in standard description. Then (see Figure 10.3)*

$$P^{\leq}\left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}\right) \subseteq P^{\geq}(A_{1,*}, b_1), \tag{10.4}$$

$$P^{\leq}\left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}\right) \cap P^{=}(A_{1,*}, b_1) = \left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}, \tag{10.5}$$

$$P^{\leq}\left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right) \subseteq P^{\geq}(A_{3,*}, b_3), \tag{10.6}$$

$$P^{\leq}\left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right) \cap P^{=}(A_{3,*}, b_3) = \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}. \tag{10.7}$$

10.4.2. Type 3 triangles

Lemma 443. *Let $L = P^{\leq}(A, b)$ be a lattice-free type 3 triangle in standard description. Let*

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \notin P^{\leq}(A_{3,*}, b_3) \tag{10.8}$$

(which, as we have noted in Remark 439, is equivalent to $A_{3,1} > A_{3,2}$). Then (see Figure 10.4)

$$\underbrace{P^{\leq}\left(\begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right)}_{=D_1^c(e^{2,2}, -e^{2,1}, 0, -1)} \subseteq P^{\leq}(A_{1,*}, b_1),$$

$$\underbrace{P^{\leq}\left(\begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right)}_{=D_1^c(e^{2,2}, -e^{2,1}, 0, -1)} \cap P^{\leq}(A_{1,*}, b_1) = \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\},$$

$$\underbrace{P^{\leq}\left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}\right)}_{=D_3^c(e^{2,2}, -e^{2,1}, 0, -1)} \cap P^{\leq}(A_{1,*}, b_1) = \emptyset,$$

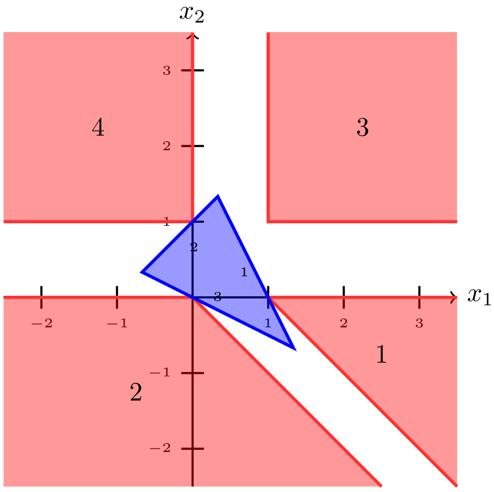


Figure 10.4.: Situation for type 3 triangles

$$\underbrace{P^{\leq}\left(\left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right), \left(\begin{array}{c} 0 \\ -1 \end{array}\right)\right)}_{=D_4^c(e^{2,2},-e^{2,1},0,-1)} \subseteq P^{\leq}\left(A_{2,*}, b_2\right),$$

$$\underbrace{P^{\leq}\left(\left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right), \left(\begin{array}{c} 0 \\ -1 \end{array}\right)\right)}_{=D_4^c(e^{2,2},-e^{2,1},0,-1)} \cap P^{\leq}\left(A_{2,*}, b_2\right) = \left\{\left(\begin{array}{c} 0 \\ 1 \end{array}\right)\right\},$$

$$\underbrace{P^{\leq}\left(\left(\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}\right), \left(\begin{array}{c} 0 \\ 0 \end{array}\right)\right)}_{=D_2^c(e^{2,2},-e^{2,1},0,-1)} \subseteq P^{\leq}\left(A_{3,*}, b_3\right),$$

$$\underbrace{P^{\leq}\left(\left(\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}\right), \left(\begin{array}{c} 0 \\ 0 \end{array}\right)\right)}_{=D_2^c(e^{2,2},-e^{2,1},0,-1)} \cap P^{\leq}\left(A_{3,*}, b_3\right) = \left\{\left(\begin{array}{c} 0 \\ 0 \end{array}\right)\right\}.$$

Lemma 444. *Let $L = P^{\leq}(A, b)$ be a type 3 triangle in standard description such that $A_{3,1} < A_{3,2}$ (by Remark 439, $A_{3,2} = A_{3,1}$ cannot happen). Let*

$$\begin{aligned} f : \mathbb{R}^2 &\to \mathbb{R}^2 : \\ x &\mapsto \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right) x. \end{aligned}$$

Then

$$f(L) = P^{\leq}\left(A\left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right), b\right) =: P^{\leq}\left(A', b\right)$$

is a type 3 triangle in standard description such that $A'_{3,1} > A'_{3,2}$ (as required for the second part of Remark 439 and for Lemma 443).

10.4.3. Quadrilaterals

Remark 440 is a central tool for proving the following lemma:

Lemma 445. *Let $L = P^{\leq}(A, b)$ be a maximal lattice-free quadrilateral in standard description. Let $L = \operatorname{conv}\left\{v^1, \ldots, v^4\right\}$ be such that for $i \in [4]$, we have $L \cap P^{=}\left(A_{i,*}, b_i\right) = \operatorname{conv}\left\{v^i, v^{i+_4 1}\right\}$, where the points $v^1, \ldots, v^4$ are in counter-clockwise order. For $i \in [4]$, let $u^i \in L \cap P^{=}\left(A_{i,*}, b_i\right)_I$ (i.e.*

$$u^1 = \left(\begin{array}{c} 1 \\ 0 \end{array}\right), \qquad u^2 = \left(\begin{array}{c} 1 \\ 1 \end{array}\right), \qquad u^3 = \left(\begin{array}{c} 0 \\ 1 \end{array}\right), \qquad u^4 = \left(\begin{array}{c} 0 \\ 0 \end{array}\right))$$

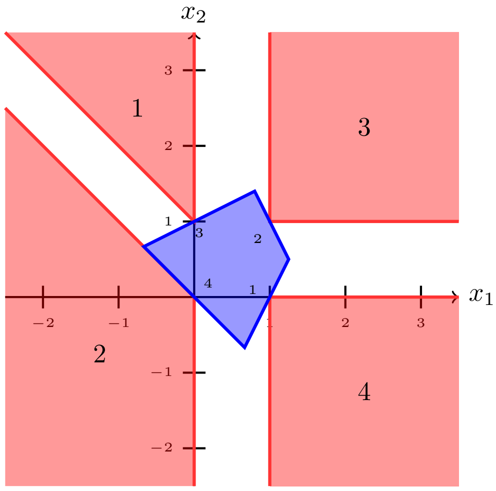


Figure 10.5.: Situation for quadrilaterals

and let

$$A_{4,1} \le A_{4,2}. \tag{10.9}$$

Then (see Figure 10.5)

$$\underbrace{P^{\le}\left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right)}_{=D_4^c(e^{2,1},-e^{2,2},0,-1)} \subseteq P^{\ge}\left(A_{1,*}, b_1\right),$$

$$\underbrace{P^{\le}\left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right)}_{=D_4^c(e^{2,1},-e^{2,2},0,-1)} \cap P^{=}\left(A_{1,*}, b_1\right) = \left\{u^1\right\},$$

$$\underbrace{P^{\le}\left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}\right)}_{=D_3^c(e^{2,1},-e^{2,2},0,-1)} \subseteq P^{\ge}\left(A_{2,*}, b_2\right),$$

$$\underbrace{P^{\le}\left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}\right)}_{=D_3^c(e^{2,1},-e^{2,2},0,-1)} \cap P^{=}\left(A_{2,*}, b_2\right) = \left\{u^2\right\},$$

$$\underbrace{P^{\le}\left(\begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right)}_{=D_1^c(e^{2,1},-e^{2,2},0,-1)} \subseteq P^{\ge}\left(A_{3,*}, b_3\right),$$

$$\underbrace{P^{\le}\left(\begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right)}_{=D_1^c(e^{2,1},-e^{2,2},0,-1)} \cap P^{=}\left(A_{3,*}, b_3\right) = \left\{u^3\right\},$$

$$\underbrace{P^{\le}\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)}_{=D_2^c(e^{2,1},-e^{2,2},0,-1)} \subseteq P^{\ge}\left(A_{4,*}, b_4\right),$$

$$\underbrace{P^{\le}\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)}_{=D_2^c(e^{2,1},-e^{2,2},0,-1)} \cap L = \operatorname{conv}\left\{v^4, u^4\right\} \text{ if } A_{4,1} = A_{4,2},$$

$$\underbrace{P^{\le}\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)}_{=D_2^c(e^{2,1},-e^{2,2},0,-1)} \cap P^{=}\left(A_{4,*}, b_4\right) = \left\{u^4\right\} \text{ if } A_{4,1} < A_{4,2}.$$

10.5. Proof of Theorem 431 for L in standard form

In this section, we consider the five different cases for a full-dimensional lattice-free body L in standard form (cf. Theorem 437):

- Split set: section 10.5.1, Theorem 446,
- Type 1 triangle: section 10.5.2, Theorem 448,
- Type 2 triangle: section 10.5.3, Theorem 451,
- Type 3 triangle: section 10.5.4, Theorem 452,
- Quadrilateral: section 10.5.5, Theorem 453

and prove that in each of them, Theorem 431 holds. Recall that we saw at the end of section 10.3 that this suffices to prove Theorem 431.

10.5.1. Split sets

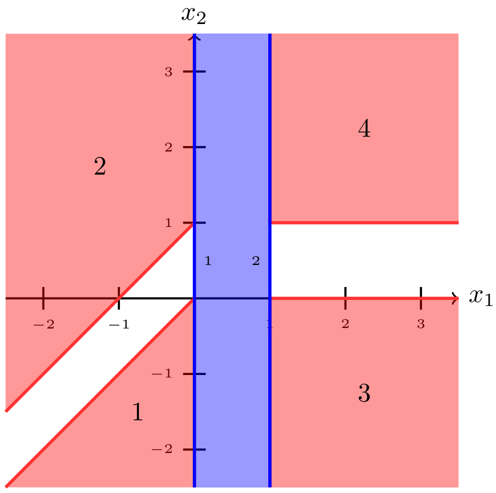


Figure 10.6.: Situation for split sets after applying f in the proof of Theorem 446

Theorem 446. *Theorem 431 holds if L is a split set in standard form.*

Proof. There exists at most one of each $\gamma_1, \gamma_2 \in \mathbb{Z}$ such that

$$\begin{aligned} (\{0\} \times (\gamma_1, \gamma_1 + 1)) \cap S &\neq \emptyset, \\ (\{1\} \times (\gamma_2, \gamma_2 + 1)) \cap S &\neq \emptyset. \end{aligned}$$

For the reason: assume that there exist two $\delta, \delta' \in \mathbb{R}\backslash\mathbb{Z}$ such that $\lfloor \delta \rfloor < \lfloor \delta' \rfloor$, but (w.l.o.g.) $\left(\begin{smallmatrix} 0 \\ \delta \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ \delta' \end{smallmatrix}\right) \in S$. Then, since S is convex:

$$\frac{\delta' - \lfloor \delta' \rfloor}{\delta' - \delta} \begin{pmatrix} 0 \\ \delta \end{pmatrix} + \frac{\lfloor \delta' \rfloor - \delta}{\delta' - \delta} \begin{pmatrix} 0 \\ \delta' \end{pmatrix} = \begin{pmatrix} 0 \\ \lfloor \delta' \rfloor \end{pmatrix} \in S \cap \mathbb{Z}. \lightning$$

If no such γ_1 or γ_2 exists, set the respective variable to an arbitrary values from $\mathbb{Z}$. Apply the affine-unimdular shearing

$$f : z \mapsto \begin{pmatrix} 1 & 0 \\ \gamma_1 - \gamma_2 & 1 \end{pmatrix} z + \begin{pmatrix} 0 \\ -\gamma_1 \end{pmatrix}$$

on L. Then $f(L) = L$, but

$$f(S) \cap (\{0\} \times \mathbb{R}_{\geq 1}), f(S) \cap (\{0\} \times \mathbb{R}_{\leq 0}), f(S) \cap (\{1\} \times \mathbb{R}_{\geq 1}), f(S) \cap (\{1\} \times \mathbb{R}_{\leq 0}) = \emptyset. \tag{10.10}$$

Additionally, we have

$$f(S) \cap (\mathbb{R}_{<0} \times \mathbb{R}), f(S) \cap (\mathbb{R}_{>1} \times \mathbb{R}) = \emptyset. \tag{10.11}$$

From now on, w.l.o.g. (cf. Lemma 441) assume that S is transformed by f, if necessary, such that (10.10) and (10.11) are satisfied. Consider the crooked cross disjunction $D^c\left(e^{2,1}, e^{2,2}, 0, 0\right)$. We clearly have (see Figure 10.6)

$$D_1^c\left(e^{2,1}, e^{2,2}, 0, 0\right), D_2^c\left(e^{2,1}, e^{2,2}, 0, 0\right) \subseteq \left(\mathbb{R}_{<0} \times \mathbb{R}\right) \dot{\cup} \left(\{0\} \times \mathbb{R}_{\leq 0}\right) \dot{\cup} \left(\{0\} \times \mathbb{R}_{\geq 1}\right),$$

since

$$\begin{aligned} D_1^c\left(e^{2,1}, e^{2,2}, 0, 0\right) &= P^{\leq}\left(\left(\begin{array}{cc} 1 & 0 \\ -1 & 1 \end{array}\right), \left(\begin{array}{c} 0 \\ 0 \end{array}\right)\right), \\ D_2^c\left(e^{2,1}, e^{2,2}, 0, 0\right) &= P^{\leq}\left(\left(\begin{array}{cc} 1 & 0 \\ 1 & -1 \end{array}\right), \left(\begin{array}{c} 0 \\ -1 \end{array}\right)\right). \end{aligned}$$

Finally, we have

$$D_3^c\left(e^{2,1}, e^{2,2}, 0, 0\right), D_4^c\left(e^{2,1}, e^{2,2}, 0, 0\right) \subseteq \left(\mathbb{R}_{>1} \times \mathbb{R}\right) \dot{\cup} \left(\{1\} \times \mathbb{R}_{\leq 0}\right) \dot{\cup} \left(\{1\} \times \mathbb{R}_{\geq 1}\right),$$

since

$$\begin{aligned} D_3^c\left(e^{2,1}, e^{2,2}, 0, 0\right) &= P^{\leq}\left(\left(\begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array}\right), \left(\begin{array}{c} -1 \\ 0 \end{array}\right)\right), \\ D_4^c\left(e^{2,1}, e^{2,2}, 0, 0\right) &= P^{\leq}\left(\left(\begin{array}{cc} -1 & 0 \\ 0 & -1 \end{array}\right), \left(\begin{array}{c} -1 \\ -1 \end{array}\right)\right). \end{aligned}$$

□

10.5.2. Type 1 triangles

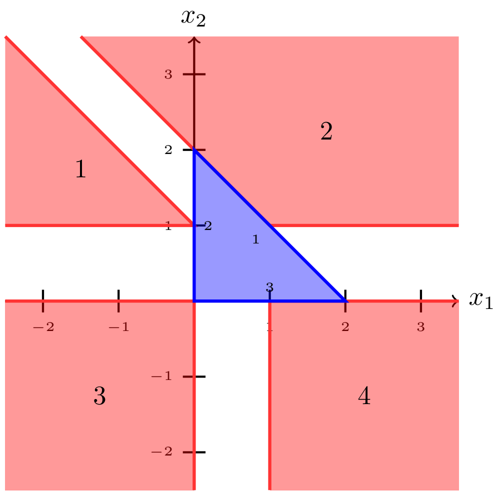


Figure 10.7.: Situation for type 1 triangles

The following lemma is easy to check:

Lemma 447. *Let $L = P^{\leq}(A, b)$ be a type 1 triangle in standard description. Then (see Figure 10.7)*

$$\underbrace{P^{\leq}\left(\left(\begin{array}{cc} 1 & 1 \\ 0 & -1 \end{array}\right), \left(\begin{array}{c} 1 \\ -1 \end{array}\right)\right)}_{=D_1^c\left(-e^{2,2}, e^{2,1}, 0, 0\right)} \subseteq P^{\geq}\left(A_{2,*}, b_2\right),$$

$$\underbrace{P^{\leq}\left(\left(\begin{array}{cc} 1 & 1 \\ 0 & -1 \end{array}\right), \left(\begin{array}{c} 1 \\ -1 \end{array}\right)\right)}_{=D_1^c\left(-e^{2,2}, e^{2,1}, 0, 0\right)} \cap L = \left\{\left(\begin{array}{c} 0 \\ 1 \end{array}\right)\right\},$$

$$\underbrace{P^{\leq}\left(\left(\begin{array}{cc} -1 & -1 \\ 0 & -1 \end{array}\right), \left(\begin{array}{c} -2 \\ -1 \end{array}\right)\right)}_{=D_2^c\left(-e^{2,2}, e^{2,1}, 0, 0\right)} \subseteq P^{\geq}\left(A_{1,*}, b_1\right),$$

$$\underbrace{P^{\leq}\left(\left(\begin{array}{cc} -1 & -1 \\ 0 & -1 \end{array}\right), \left(\begin{array}{c} -2 \\ -1 \end{array}\right)\right)}_{=D_2^c(-e^{2,2},e^{2,1},0,0)} \cap L = \operatorname{conv}\left\{\left(\begin{array}{c} 0 \\ 2 \end{array}\right), \left(\begin{array}{c} 1 \\ 1 \end{array}\right)\right\},$$

$$\underbrace{P^{\leq}\left(\left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right), \left(\begin{array}{c} 0 \\ 0 \end{array}\right)\right)}_{D_3^c(-e^{2,2},e^{2,1},0,0)} \subseteq P^{\geq}\left(A_{2,*}, b_2\right), P^{\geq}\left(A_{3,*}, b_3\right),$$

$$\underbrace{P^{\leq}\left(\left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right), \left(\begin{array}{c} 0 \\ 0 \end{array}\right)\right)}_{=D_3^c(-e^{2,2},e^{2,1},0,0)} \cap L = \left\{\left(\begin{array}{c} 0 \\ 0 \end{array}\right)\right\},$$

$$\underbrace{P^{\leq}\left(\left(\begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array}\right), \left(\begin{array}{c} -1 \\ 0 \end{array}\right)\right)}_{=D_4^c(-e^{2,2},e^{2,1},0,0)} \subseteq P^{\geq}\left(A_{3,*}, b_3\right),$$

$$\underbrace{P^{\leq}\left(\left(\begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array}\right), \left(\begin{array}{c} -1 \\ 0 \end{array}\right)\right)}_{=D_4^c(-e^{2,2},e^{2,1},0,0)} \cap L = \operatorname{conv}\left\{\left(\begin{array}{c} 1 \\ 0 \end{array}\right), \left(\begin{array}{c} 2 \\ 0 \end{array}\right)\right\}.$$

Theorem 448. *Theorem 431 holds if L is a type 1 triangle.*

Proof. Let

$$u^1 := \left(\begin{array}{c} 2 \\ 0 \end{array}\right), \qquad u^2 := \left(\begin{array}{c} 0 \\ 2 \end{array}\right), \qquad u^3 := \left(\begin{array}{c} 0 \\ 0 \end{array}\right),$$
$$v^1 := \left(\begin{array}{c} 1 \\ 1 \end{array}\right), \qquad v^2 := \left(\begin{array}{c} 0 \\ 1 \end{array}\right), \qquad v^3 := \left(\begin{array}{c} 1 \\ 0 \end{array}\right).$$

Since (by assumption of Theorem 431) we have

$$\forall i \in [l] : S \cap P^{=}\left(A_{i,*}, b_i\right)_I = \emptyset,$$

we conclude by convexity of L that for every $i \in [3]$, at most one of

- $\left(\operatorname{relint}\left(\operatorname{conv}\left\{u^i, v^i\right\}\right)\right) \cap S \neq \emptyset$,
- $\left(\operatorname{relint}\left(\operatorname{conv}\left\{v^i, u^{i+_3 1}\right\}\right)\right) \cap S \neq \emptyset$

holds. In other words: for every $i \in [3]$, we have at least one of

- $\left(\operatorname{relint}\left(\operatorname{conv}\left\{u^i, v^i\right\}\right)\right) \cap S = \emptyset$,
- $\left(\operatorname{relint}\left(\operatorname{conv}\left\{v^i, u^{i+_3 1}\right\}\right)\right) \cap S = \emptyset$.

This in particular implies (by case distinction) that at least one of

$$\exists i \in [3] : \left(\operatorname{relint}\left(\operatorname{conv}\left\{v^{i-_3 1}, u^i\right\}\right)\right) \cap S = \emptyset \wedge \left(\operatorname{relint}\left(\operatorname{conv}\left\{v^i, u^{i+_3 1}\right\}\right)\right) \cap S = \emptyset, \tag{10.12}$$
$$\exists i \in [3] : \left(\operatorname{relint}\left(\operatorname{conv}\left\{u^{i-_3 1}, v^{i-_3 1}\right\}\right)\right) \cap S = \emptyset \wedge \left(\operatorname{relint}\left(\operatorname{conv}\left\{u^i, v^i\right\}\right)\right) \cap S = \emptyset \tag{10.13}$$

holds. By applying

$$f_1 : \mathbb{R}^2 \to \mathbb{R}^2 : \quad x \mapsto \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right) x$$

to L and S in the case of (10.13) (by Lemma 441, it suffices to prove the statement for $f_1(L) = L$ and $f_1(S)$), we can assume that (10.12) holds. For the sake of exactness, let

$$S' := \begin{cases} S & \text{in the first case,} \\ f_1(S) & \text{in the second case.} \end{cases}$$

In Lemma 441, we saw that it suffices to prove the statement for L and S'. Let $i^* \in [3]$ be an i for which the inner condition of (10.12) holds with respect to S'. Define

$$f_2 : \mathbb{R}^2 \to \mathbb{R}^2 :$$
$$x \mapsto \begin{cases} x & \text{if } i^* = 1, \\ \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 2 \end{pmatrix} & \text{if } i^* = 2, \\ \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 2 \\ 0 \end{pmatrix} & \text{if } i^* = 3. \end{cases}$$

Again, it suffices to prove the statement for $f_2(L) = L$ and $S'' := f_2(S')$. By the transformation f_2, (10.12) holds for $i = 1$ with respect to S''; thus,

$$\begin{aligned} & \left(\operatorname{relint}\left(\operatorname{conv}\left\{v^{1-_3 1}, u^1\right\}\right)\right) \cap S'' = \emptyset \wedge \left(\operatorname{relint}\left(\operatorname{conv}\left\{v^1, u^{1+_3 1}\right\}\right)\right) \cap S'' = \emptyset \\ \Leftrightarrow & \left(\operatorname{relint}\left(\operatorname{conv}\left\{v^3, u^1\right\}\right)\right) \cap S'' = \emptyset \wedge \left(\operatorname{relint}\left(\operatorname{conv}\left\{v^1, u^2\right\}\right)\right) \cap S'' = \emptyset \\ \Leftrightarrow & \left(\operatorname{relint}\left(\operatorname{conv}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}\right\}\right)\right) \cap S'' = \emptyset \wedge \left(\operatorname{relint}\left(\operatorname{conv}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}\right\}\right)\right) \cap S'' = \emptyset. \end{aligned}$$

Thus, by Lemma 447, the statement holds for $L = (f_2 \circ f_1)(L)$ and S''. So, the statement can be concluded from Lemma 441. □

10.5.3. Type 2 triangles

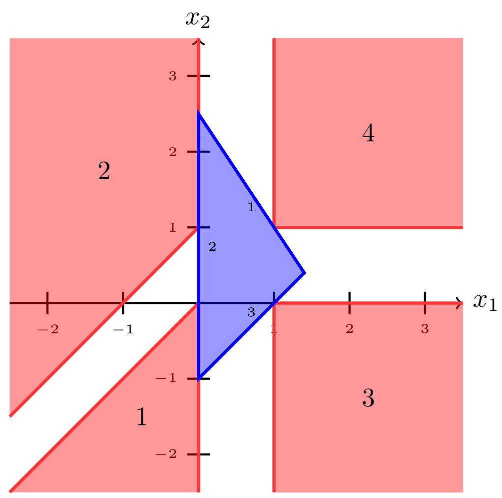


Figure 10.8.: Situation for type 2 triangles: case 1

In the following lemma (Lemma 449)

- (10.14) and (10.15) hold by (10.1) and (10.2),
- (10.16), (10.17), (10.18) and (10.19) hold by Lemma 442 (equations (10.6), (10.7), (10.4) and (10.5)).

Lemma 449. *Let $L = P^{\leq}(A, b)$ be a type 2 triangle in standard description. Let $L = \operatorname{conv}\left\{v^1, \dots, v^3\right\}$ be such that for all $i \in [3]$, we have $L \cap P^{=}(A_{i,*}, b_i) = \operatorname{conv}\left\{v^i, v^{i+_3 1}\right\}$, where the points $v^1, \dots, v^3$ are in counter-clockwise order. Then (see Figure 10.8)*

$$\underbrace{P^{\leq}\left(\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)}_{=D_1^c(e^{2,1}, e^{2,1}, 0, 0)} \subseteq P^{\geq}(A_{2,*}, b_2), \tag{10.14}$$

$$\underbrace{P^{\leq}\left(\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)}_{=D_1^c(e^{2,1}, e^{2,1}, 0, 0)} \cap L = \operatorname{conv}\left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, v^3\right\},$$

$$\underbrace{P^{\leq}\left(\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right)}_{=D_2^c(e^{2,1},e^{2,1},0,0)} \subseteq P^{\geq}(A_{2,*}, b_2), \tag{10.15}$$

$$\underbrace{P^{\leq}\left(\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right)}_{=D_2^c(e^{2,1},e^{2,1},0,0)} \cap L = \operatorname{conv}\left\{v^2, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\},$$

$$\underbrace{P^{\leq}\left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right)}_{=D_3^c(e^{2,1},e^{2,1},0,0)} \subseteq P^{\geq}(A_{3,*}, b_3), \tag{10.16}$$

$$\underbrace{P^{\leq}\left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right)}_{=D_3^c(e^{2,1},e^{2,1},0,0)} \cap P^{=}(A_{3,*}, b_1) = \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}, \tag{10.17}$$

$$\underbrace{P^{\leq}\left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}\right)}_{=D_4^c(e^{2,1},e^{2,1},0,0)} \subseteq P^{\geq}(A_{1,*}, b_1), \tag{10.18}$$

$$\underbrace{P^{\leq}\left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}\right)}_{=D_4^c(e^{2,1},e^{2,1},0,0)} \cap P^{=}(A_{1,*}, b_1) = \left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}. \tag{10.19}$$

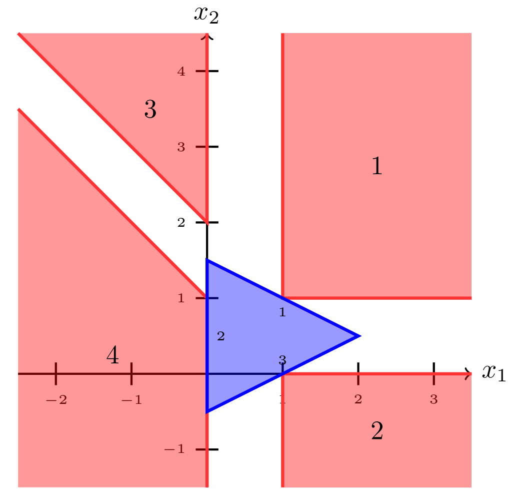


Figure 10.9.: Situation for type 2 triangles: case 2

In the following lemma (Lemma 450)

- (10.25) and the inclusion $D_4^c\left(\binom{-1}{0}, \binom{-1}{-1}, -1, -2\right) \subseteq P^{\geq}(A_{2,*}, b_2)$ in (10.24) hold by (10.1) and (10.2),
- (10.20), (10.21), (10.22) and (10.23) hold by Lemma 442 (equations (10.4), (10.5), (10.6) and (10.7)).

Lemma 450. *Let $L = P^{\leq}(A, b)$ be a type 2 triangle in standard description. Let $L = \operatorname{conv}\{v^1, \ldots, v^3\}$ be such that for $i \in [3]$, we have $L \cap P^{=}(A_{i,*}, b_i) = \operatorname{conv}\{v^i, v^{i+_3 1}\}$, where the points $v^1, \ldots, v^3$ are in counter-clockwise order. Additionally, let*

$$A_{1,1} \leq A_{1,2}.$$

Then (see Figure 10.9)

$$\underbrace{P^{\leq}\left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}\right)}_{=D_1^c\left(\binom{-1}{0},\binom{-1}{-1},-1,-2\right)} \subseteq P^{\geq}(A_{1,*}, b_1), \tag{10.20}$$

$$\underbrace{P^{\leq}\left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}\right)}_{=D_1^c\left(\binom{-1}{0},\binom{-1}{-1},-1,-2\right)} \cap L = \left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}, \tag{10.21}$$

$$\underbrace{P^{\leq}\left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right)}_{=D_2^c\left(\binom{-1}{0},\binom{-1}{-1},-1,-2\right)} \subseteq P^{\geq}\left(A_{3,*}, b_3\right), \tag{10.22}$$

$$\underbrace{P^{\leq}\left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right)}_{=D_2^c\left(\binom{-1}{0},\binom{-1}{-1},-1,-2\right)} \cap L = \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}, \tag{10.23}$$

$$\underbrace{P^{\leq}\left(\begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \end{pmatrix}\right)}_{=D_3^c\left(\binom{-1}{0},\binom{-1}{-1},-1,-2\right)} \subseteq P^{\geq}\left(A_{1,*}, b_1\right), P^{\geq}\left(A_{2,*}, b_2\right), \tag{10.24}$$

$$\underbrace{P^{\leq}\left(\begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \end{pmatrix}\right)}_{=D_3^c\left(\binom{-1}{0},\binom{-1}{-1},-1,-2\right)} \cap L = \begin{cases} \emptyset & \textit{if } A_{1,1} < A_{1,2}, \\ \begin{pmatrix} 0 \\ 2 \end{pmatrix} & \textit{if } A_{1,1} = A_{1,2}, \end{cases}$$

$$\underbrace{P^{\leq}\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)}_{=D_4^c\left(\binom{-1}{0},\binom{-1}{-1},-1,-2\right)} \subseteq P^{\geq}\left(A_{2,*}, b_2\right), \tag{10.25}$$

$$\underbrace{P^{\leq}\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)}_{=D_4^c\left(\binom{-1}{0},\binom{-1}{-1},-1,-2\right)} \cap P^{=}\left(A_{2,*}, b_2\right) = \operatorname{conv}\left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, v^3\right\}.$$

Theorem 451. *Theorem 431 holds if $L = P^{\leq}(A,b)$ is a type 2 triangle in standard description.*

Proof. It is easy to check that

$$v^2 := \begin{pmatrix} 0 \\ 1 + \frac{A_{1,1}}{A_{1,2}} \end{pmatrix}, \qquad v^3 := \begin{pmatrix} 0 \\ \frac{A_{3,1}}{A_{3,2}} \end{pmatrix} \tag{10.26}$$

are vertices of L and we have

$$\begin{aligned} \left\{v^2\right\} &= P^{=}\left(A_{1,*}, b_1\right) \cap P^{=}\left(A_{2,*}, b_2\right), \\ \left\{v^3\right\} &= P^{=}\left(A_{2,*}, b_2\right) \cap P^{=}\left(A_{3,*}, b_3\right). \end{aligned}$$

By convexity of S and $S \cap P^{=}\left(A_{2,*}, b_2\right)_I = \emptyset$, there can exist at most one $z^* \in \mathbb{Z}$ such that

$$S \cap L \cap \left\{\begin{pmatrix} 0 \\ z^* + \lambda \end{pmatrix} : \lambda \in (0,1)\right\} \neq \emptyset. \tag{10.27}$$

If no such z^* exists, we set w.l.o.g. $z^* := 0$. From (10.26), (10.27) and Remark 438, we immediately obtain

$$\left\lfloor \frac{A_{3,1}}{A_{3,2}} \right\rfloor \leq z^* \leq \left\lceil \frac{A_{1,1}}{A_{1,2}} \right\rceil.$$

We first note that, by Lemma 441, it suffices to prove Theorem 431 for $L' = f(L)$, where f is an affine-unimodular map. We distinguish three cases:

1. $\left\lfloor \frac{A_{3,1}}{A_{3,2}} \right\rfloor + 1 \leq z^* \leq \left\lceil \frac{A_{1,1}}{A_{1,2}} \right\rceil - 1$,
2. $z^* = \left\lceil \frac{A_{1,1}}{A_{1,2}} \right\rceil$,
3. $z^* = \left\lfloor \frac{A_{3,1}}{A_{3,2}} \right\rfloor$.

In the third case, we mirror L and S at the line $\left\{x \in \mathbb{R}^2 : x_2 = \frac{1}{2}\right\}$, i.e. we apply the affine-unimodular map

$$f : \mathbb{R}^2 \to \mathbb{R}^2,$$
$$x \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then, using Lemma 63, we have:

$$\begin{aligned}
f(L) &= P^{\leq}\left(\underbrace{\begin{pmatrix} A_{1,1} & A_{1,2} \\ -1 & 0 \\ A_{3,1} & A_{3,2} \end{pmatrix}}_{=A}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{-1}, \underbrace{\begin{pmatrix} A_{1,1} + A_{1,2} \\ 0 \\ A_{3,1} \end{pmatrix}}_{=b} + \underbrace{\begin{pmatrix} A_{1,1} & A_{1,2} \\ -1 & 0 \\ A_{3,1} & A_{3,2} \end{pmatrix}}_{=A}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \\
&= P^{\leq}\left(\begin{pmatrix} A_{3,1} & -A_{3,2} \\ -1 & 0 \\ A_{1,1} & -A_{1,2} \end{pmatrix}, \begin{pmatrix} A_{3,1} - A_{3,2} \\ 0 \\ A_{1,1} \end{pmatrix}\right) \\
&=: P^{\leq}(A', b').
\end{aligned}$$

It is easy to check, using Lemma 441, that $P^{\leq}(A', b')$ is in standard description. Additionally, we have for $z \in \mathbb{Z}\setminus\{0\}$:

$$\begin{aligned}
& f(S) \cap f(L) \cap \left\{\begin{pmatrix} 0 \\ \left\lceil \frac{A'_{1,1}}{A'_{1,2}} \right\rceil + z + \lambda \end{pmatrix} : \lambda \in (0,1)\right\} \\
=& f(S) \cap f(L) \cap f\left(\left\{\begin{pmatrix} 0 \\ -\left(\left\lceil \frac{A'_{1,1}}{A'_{1,2}} \right\rceil + z\right) + (1-\lambda) \end{pmatrix} : \lambda \in (0,1)\right\}\right) \\
=& f(S) \cap f(L) \cap f\left(\left\{\begin{pmatrix} 0 \\ -\left(\left\lceil \frac{A'_{1,1}}{A'_{1,2}} \right\rceil + z\right) + \lambda \end{pmatrix} : \lambda \in (0,1)\right\}\right) \\
=& f\left(S \cap L \cap \left\{\begin{pmatrix} 0 \\ -\left(\left\lceil \frac{A'_{1,1}}{A'_{1,2}} \right\rceil + z\right) + \lambda \end{pmatrix} : \lambda \in (0,1)\right\}\right) && (f \text{ bijective}) \\
=& f\left(S \cap L \cap \left\{\begin{pmatrix} 0 \\ -\left(\left\lceil -\frac{A_{3,1}}{A_{3,2}} \right\rceil + z\right) + \lambda \end{pmatrix} : \lambda \in (0,1)\right\}\right) \\
=& f\left(S \cap L \cap \left\{\begin{pmatrix} 0 \\ \left\lfloor \frac{A_{3,1}}{A_{3,2}} \right\rfloor - z + \lambda \end{pmatrix} : \lambda \in (0,1)\right\}\right) \\
=& \emptyset. && \text{(by (10.27) and case assumption)}
\end{aligned}$$

So, we have reduced the statement for $f(S)$ and $f(L)$ to case 2 (since we showed that if $z' \in \mathbb{Z}\setminus\left\{\frac{A'_{1,1}}{A'_{1,2}}\right\}$, we have

$$f(S) \cap f(L) \cap \left\{\begin{pmatrix} 0 \\ z' + \lambda \end{pmatrix} : \lambda \in (0,1)\right\} = \emptyset)$$

and we thus only have to consider case 1 and 2.

For case 1 and 2: Let

$$z^{**} := \begin{cases} z^* & \text{in case 1,} \\ \left\lceil \frac{A_{1,1}}{A_{1,2}} \right\rceil - 1 & \text{in case 2} \end{cases}$$

and consider the map

$$f : \mathbb{R}^2 \to \mathbb{R}^2 :$$
$$x \mapsto \begin{pmatrix} 1 & 0 \\ z^{**} & 1 \end{pmatrix} x + \begin{pmatrix} 0 \\ -z^{**} \end{pmatrix}.$$

We first note that in case 1, we have

$$\forall z \in \mathbb{Z}\setminus\{0\} : f(S) \cap f(L) \cap \left\{ \begin{pmatrix} 0 \\ z+\lambda \end{pmatrix} : \lambda \in (0,1) \right\} = \emptyset \tag{10.28}$$

and in case 2, we have

$$\forall z \in \mathbb{Z}\setminus\{1\} : f(S) \cap f(L) \cap \left\{ \begin{pmatrix} 0 \\ z+\lambda \end{pmatrix} : \lambda \in (0,1) \right\} = \emptyset. \tag{10.29}$$

W.l.o.g. we only show (10.28): Let $z \in \mathbb{Z}\setminus\{0\}$. Then

$$\begin{aligned}
& f(S) \cap f(L) \cap \left\{ \begin{pmatrix} 0 \\ z+\lambda \end{pmatrix} : \lambda \in (0,1) \right\} \\
=& f(S) \cap f(L) \cap f\left(\left\{ \begin{pmatrix} 0 \\ z+z^{**}+\lambda \end{pmatrix} : \lambda \in (0,1) \right\} \right) \\
=& f\left(S \cap L \cap \left\{ \begin{pmatrix} 0 \\ z+z^{**}+\lambda \end{pmatrix} : \lambda \in (0,1) \right\} \right) && (f \text{ bijective}) \\
=& \emptyset. && (\text{by } (10.27))
\end{aligned}$$

By Lemma 441, it suffices to prove the statement for $f(L)$ and $f(S)$. We note that also $f(S)$ and $f(L)$ satisfy the prerequisites of Theorem 431 and we have

$$\begin{aligned}
f(L) =& P^{\leq}\left(\underbrace{\begin{pmatrix} A_{1,1} & A_{1,2} \\ -1 & 0 \\ A_{3,1} & A_{3,2} \end{pmatrix}}_{=A} \begin{pmatrix} 1 & 0 \\ z^{**} & 1 \end{pmatrix}^{-1}, \right. \\
& \left. \underbrace{\begin{pmatrix} A_{1,1}+A_{1,2} \\ 0 \\ A_{3,1} \end{pmatrix}}_{=b} + \underbrace{\begin{pmatrix} A_{1,1} & A_{1,2} \\ -1 & 0 \\ A_{3,1} & A_{3,2} \end{pmatrix}}_{=A} \begin{pmatrix} 1 & 0 \\ z^{**} & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ -z^{**} \end{pmatrix} \right) \\
=& P^{\leq}\left(\begin{pmatrix} A_{1,1} & A_{1,2} \\ -1 & 0 \\ A_{3,1} & A_{3,2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -z^{**} & 1 \end{pmatrix}, \right. \\
& \left. \begin{pmatrix} A_{1,1}+A_{1,2} \\ 0 \\ A_{3,1} \end{pmatrix} + \begin{pmatrix} A_{1,1} & A_{1,2} \\ -1 & 0 \\ A_{3,1} & A_{3,2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -z^{**} & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -z^{**} \end{pmatrix} \right) \\
=& P^{\leq}\left(\begin{pmatrix} A_{1,1}-z^{**}A_{1,2} & A_{1,2} \\ -1 & 0 \\ A_{3,1}-z^{**}A_{3,2} & A_{3,2} \end{pmatrix}, \begin{pmatrix} A_{1,1}+(1-z^{**})A_{1,2} \\ 0 \\ A_{3,1}-z^{**}A_{3,2} \end{pmatrix} \right) \\
=:& P^{\leq}(A', b').
\end{aligned}$$

By definition, we have $A'_{1,2} > 0$ and $A'_{3,2} < 0$. What remains to be shown is $A'_{1,1}, A'_{3,1} > 0$:

For $A'_{1,1} > 0$:

$$\begin{aligned}
A'_{1,1} &= A_{1,1} - z^{**}A_{1,2} \\
&= \underbrace{\left(\frac{A_{1,1}}{A_{1,2}} - \left(\left\lceil \frac{A_{1,1}}{A_{1,2}} \right\rceil - 1 \right) \right)}_{>0} \underbrace{A_{1,2}}_{>0} + \left(\left(\left\lceil \frac{A_{1,1}}{A_{1,2}} \right\rceil - 1 \right) - z^{**} \right) A_{1,2} \\
&> \underbrace{\left(\left(\left\lceil \frac{A_{1,1}}{A_{1,2}} \right\rceil - 1 \right) - z^{**} \right)}_{\geq 0} \underbrace{A_{1,2}}_{>0}
\end{aligned}$$

$$\geq 0.$$

For $A'_{3,1} > 0$:

$$\begin{aligned}
A'_{3,1} &= A_{3,1} - z^{**} A_{3,2} \\
&= \underbrace{\left(\frac{A_{3,1}}{A_{3,2}} - \left(\left\lfloor\frac{A_{3,1}}{A_{3,2}}\right\rfloor + 1\right)\right)}_{<0} \underbrace{A_{3,2}}_{<0} + \left(\left(\left\lfloor\frac{A_{3,1}}{A_{3,2}}\right\rfloor + 1\right) - z^{**}\right) A_{3,2} \\
&> \underbrace{\left(\left(\left\lfloor\frac{A_{3,1}}{A_{3,2}}\right\rfloor + 1\right) - z^{**}\right)}_{\leq 0} \underbrace{A_{3,2}}_{<0} \\
&\geq 0.
\end{aligned}$$

Thus, by Theorem 437, $P^{\leq}(A', b')$ is again a type 2 triangle in standard description. Because of (10.28) or (10.29), respectively, the statement holds by Lemma 449 (case 1) or Lemma 450 (case 2), respectively. □

10.5.4. Type 3 triangles

Theorem 452. *Theorem 431 holds if $L = P^{\leq}(A, b)$ is a type 3 triangle in standard description.*

Proof. By Remark 439, the condition (10.8) is equivalent to $A_{3,1} > A_{3,2}$. If this is the case, the statement is an immediate consequence of Lemma 443. On the other hand, if $A_{3,1} < A_{3,2}$, we can use Lemma 444 and Lemma 441 to reduce the statement to Lemma 443 again. □

10.5.5. Quadrilaterals

Theorem 453. *Theorem 431 holds if $L = P^{\leq}(A, b)$ is a quadrilateral in standard description.*

Proof. Let $u^1, \ldots, u^4, v^1, \ldots, v^4$ be as in Lemma 445 (with respect to L). If we have

$$A_{4,1} < A_{4,2} \vee \left(A_{4,1} = A_{4,2} \wedge \left(\operatorname{conv}\left\{v^4, u^4\right\}\right) \cap S = \emptyset\right),$$

the statement is an immediate consequence of Lemma 445. So, we assume that either

- $A_{4,1} > A_{4,2}$ or
- $A_{4,1} = A_{4,2} \wedge \left(\operatorname{conv}\left\{v^4, u^4\right\}\right) \cap S \neq \emptyset$

holds. Consider

$$\begin{aligned}
f : \mathbb{R}^2 &\to \mathbb{R}^2 : \\
x &\mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x.
\end{aligned}$$

Then, by Lemma 441, we have

$$L' := f(L) = P^{\leq}\left(\begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \\ A_{3,1} & A_{3,2} \\ A_{4,1} & A_{4,2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, b\right) = P^{\leq}\left(\begin{pmatrix} A_{3,2} & A_{3,1} \\ A_{2,2} & A_{2,1} \\ A_{1,2} & A_{1,1} \\ A_{4,2} & A_{4,1} \end{pmatrix}, \begin{pmatrix} b_3 \\ b_2 \\ b_1 \\ b_4 \end{pmatrix}\right) =: P^{\leq}(A', b').$$

By Theorem 437, $P^{\leq}(A', b)$ is a lattice-free quadrilateral in standard description. If we have $A_{4,1} > A_{4,2}$, we conclude $A'_{4,1} < A'_{4,2}$; thus, the statement holds by Lemma 445 for $f(S)$ and L'.

On the other hand, if $A_{4,1} = A_{4,2}$ (from which we conclude $A'_{4,1} = A'_{4,2}$), we have $\left(\operatorname{conv}\left\{u^4, v^1\right\}\right) \cap S = \emptyset$, since if there exist $s^1 \in \left(\operatorname{conv}\left\{v^4, u^4\right\}\right) \cap S$ (by case assumption) and $s^2 \in \left(\operatorname{conv}\left\{u^4, v^1\right\}\right) \cap S$, we immediately conclude $u^4 \in S$. But we also have $u^4 \in (\operatorname{bd} L)_I$ – a contradiction to the conditions on S and L in Theorem 431.

Let $u'^1, \ldots, u'^4, v'^1, \ldots, v'^4$ be the points $u^1, \ldots, u^4, v^1, \ldots, v^4$ from Lemma 445 for L'. It is easy to check that

- $f(v^4) = v'^1$,
- $f(v^1) = v'^4$,
- $f(u^4) = u'^4$.

We thus get:

$$A'_{4,1} = A'_{4,2} \wedge \left(\operatorname{conv}\left\{v'^4, u'^4\right\}\right) \cap f(S) = \emptyset.$$

So, we again conclude from Lemma 445 that the statement holds for $f(S)$ and L'. Finally, we conclude from Lemma 441 that if the statement holds for $f(S)$ and L' (which we just proved), it also holds for S and L. □

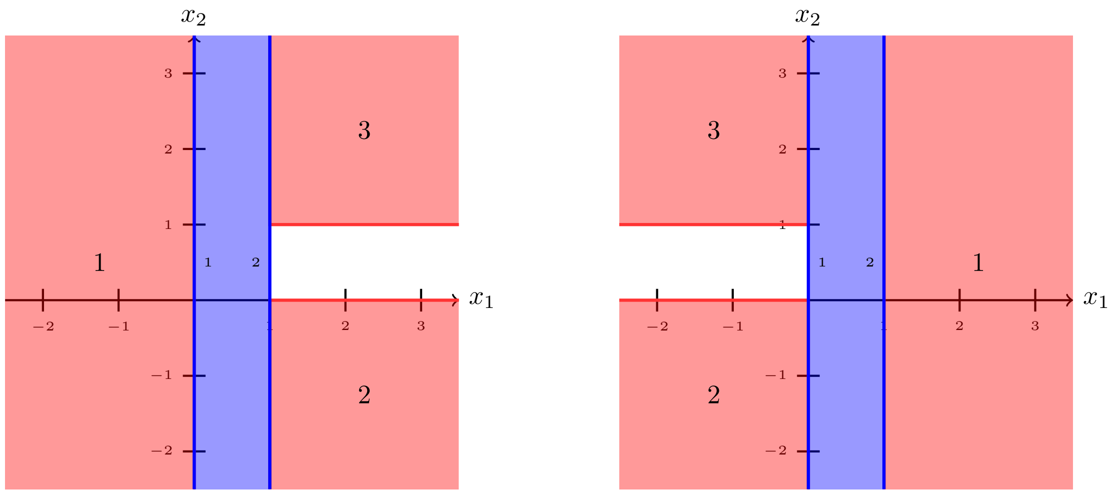
10.6. Proof of Theorem 434 for L in standard form

In this section we consider the five different cases for a lattice-free L in standard form (cf. Theorem 437):

- Split set: section 10.6.1, Theorem 454,
- Type 1 triangle: section 10.6.2, Theorem 455,
- Type 2 triangle: section 10.6.3, Theorem 456,
- Type 3 triangle: section 10.6.4, Theorem 457,
- Quadrilateral: section 10.6.5, Theorem 458

and prove that in each of them, Theorem 434 holds. Recall that we saw at the end of section 10.3 that this suffices to prove Theorem 434.

10.6.1. Split sets



(a) Situation for split sets: case 1 (b) Situation for split sets: case 2

Figure 10.10.: Situation for split sets

Theorem 454. *Theorem 434 holds if L is a split set in standard form.*

Proof. If $j^* = 1$, we have (see Figure 10.10a)

$$\begin{aligned}
P^{\geq}\left(A_{1,*}, b_1\right) &= P^{\leq}\left(\left(e^{2,1}\right)^T, 0\right) \\
&= D_1^T\left(e^{2,1}, e^{2,2}, 0, 0\right), \\
\bigcup_{j \in [2] \setminus \{1\}} P^{\geq}\left(A_{j,*}, b_j\right) &= P^{\geq}\left(A_{2,*}, b_2\right) \\
&= P^{\geq}\left(\left(e^{2,1}\right)^T, 1\right) \\
&\supseteq P^{\geq}\left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right) \cup P^{\geq}\left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}\right) \\
&= D_2^T\left(e^{2,1}, e^{2,2}, 0, 0\right) \cup D_3^T\left(e^{2,1}, e^{2,2}, 0, 0\right) \\
&= \bigcup_{i \in [3] \setminus \{1\}} D_i^T\left(e^{2,1}, e^{2,2}, 0, 0\right).
\end{aligned}$$

On the other hand, if $j^* = 2$, we have (see Figure 10.10b)

$$\begin{aligned}
P^{\geq}\left(A_{2,*}, b_2\right) &= P^{\leq}\left(\left(-e^{2,1}\right)^T, -1\right) \\
&= D_1^T\left(-e^{2,1}, e^{2,2}, -1, 0\right), \\
\bigcup_{j \in [2] \setminus \{2\}} P^{\geq}\left(A_{j,*}, b_j\right) &= P^{\geq}\left(A_{1,*}, b_1\right) \\
&= P^{\geq}\left(\left(e^{2,1}\right)^T, 1\right) \\
&\supseteq P^{\geq}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) \cup P^{\geq}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right) \\
&= D_2^T\left(-e^{2,1}, e^{2,2}, -1, 0\right), \cup D_3^T\left(-e^{2,1}, e^{2,2}, -1, 0\right) \\
&= \bigcup_{i \in [3] \setminus \{1\}} D_i^T\left(-e^{2,1}, e^{2,2}, -1, 0\right).
\end{aligned}$$

□

10.6.2. Type 1 triangles

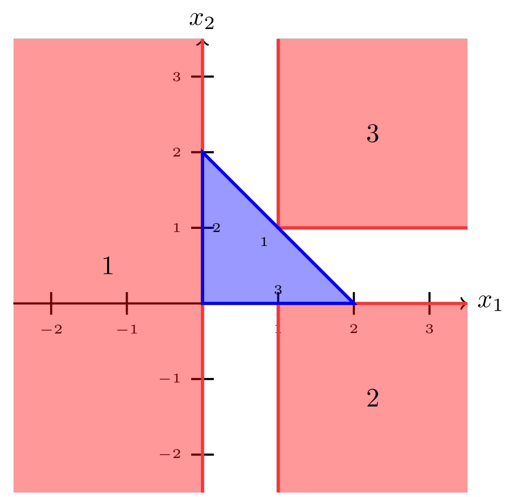


Figure 10.11.: Situation for type 1 triangles

Theorem 455. *Theorem 434 holds if $L = P^{\leq}(A, b)$ is a type 1 triangle in standard description.*

Proof. We have (see Figure 10.11)

$$D_1^T\left(e^{2,1}, e^{2,2}, 0, 0\right) = \left\{x \in \mathbb{R}^2 : x_1 \leq 0\right\} = P^{\geq}\left(A_{2,*}, b_2\right), \tag{10.30}$$
$$D_2^T\left(e^{2,1}, e^{2,2}, 0, 0\right) = \left\{x \in \mathbb{R}^2 : x_1 \geq 1 \wedge x_2 \leq 0\right\} \subseteq \left\{x \in \mathbb{R}^2 : x_2 \leq 0\right\} = P^{\geq}\left(A_{3,*}, b_3\right), \tag{10.31}$$
$$D_3^T\left(e^{2,1}, e^{2,2}, 0, 0\right) \subseteq P^{\geq}\left(A_{1,*}, b_1\right). \tag{10.32}$$

(10.30) and (10.31) are obvious. For (10.32): let $x \in D_3^T\left(e^{2,1}, e^{2,2}, 0, 0\right)$. Then $x_1 \geq 1$ and $x_2 \geq 1$. So $x_1 + x_2 \geq 1 + 1 = 2$. Thus, $x \in P^{\geq}\left(\begin{pmatrix} 1 & 1 \end{pmatrix}, 2\right) = P^{\geq}\left(A_{1,*}, b_1\right)$.

So from (10.30)-(10.32), we conclude

$$P^{\geq}\left(A_{2,*}, b_2\right) = D_1^T\left(e^{2,1}, e^{2,2}, 0, 0\right),$$
$$\bigcup_{j \in [3] \setminus \{2\}} P^{\geq}\left(A_{j,*}, b_j\right) \supseteq \bigcup_{i \in [3] \setminus \{1\}} D_i^T\left(e^{2,1}, e^{2,2}, 0, 0\right).$$

□

10.6.3. Type 2 triangles

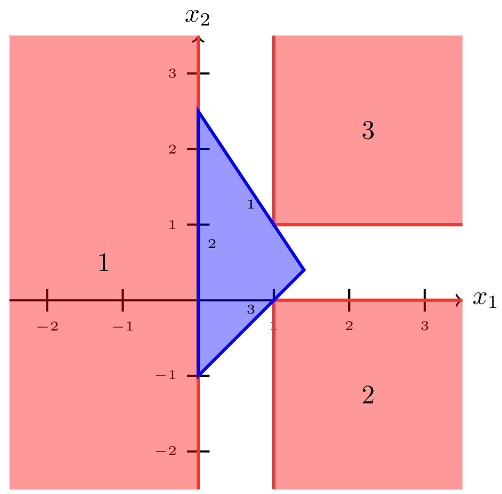


Figure 10.12.: Situation for type 2 triangles

Theorem 456. *Theorem 434 holds if L is a type 2 triangle in standard form.*

Proof. (See Figure 10.12) Using Lemma 442, we get

$$D_1^T\left(e^{2,1}, e^{2,2}, 0, 0\right) = \left\{x \in \mathbb{R}^2 : x_1 \leq 0\right\} = P^{\geq}\left(A_{2,*}, b_2\right),$$
$$D_2^T\left(e^{2,1}, e^{2,2}, 0, 0\right) = \left\{x \in \mathbb{R}^2 : x_1 \geq 1 \wedge x_2 \leq 0\right\} \supseteq P^{\geq}\left(A_{3,*}, b_3\right),$$
$$D_3^T\left(e^{2,1}, e^{2,2}, 0, 0\right) = \left\{x \in \mathbb{R}^2 : x_1 \geq 1 \wedge x_2 \geq 1\right\} \supseteq P^{\geq}\left(A_{1,*}, b_1\right).$$

Thus,

$$P^{\geq}\left(A_{2,*}, b_2\right) = D_1^T\left(e^{2,1}, e^{2,2}, 0, 0\right),$$
$$\bigcup_{j \in [3] \setminus \{2\}} P^{\geq}\left(A_{j,*}, b_j\right) \supseteq \bigcup_{i \in [3] \setminus \{1\}} D_i^T\left(e^{2,1}, e^{2,2}, 0, 0\right).$$

□

10.6.4. Type 3 triangles

Theorem 457. *Theorem 434 holds if $L = P^{\leq}(A, b)$ is a type 3 triangle in standard description.*

Proof. By Remark 439, the condition (10.8) is equivalent to $A_{3,1} > A_{3,2}$. If this is the case, the statement is an immediate consequence of Lemma 443. On the other hand, if $A_{3,1} < A_{3,2}$ (recall that by Remark 439, the situation $A_{3,1} = A_{3,2}$ cannot happen), we can use Lemma 444 and Lemma 441 to reduce the statement to Lemma 443 again. □

10.6.5. Quadrilaterals

Theorem 458. *Theorem 434 holds if $L = P^{\leq}(A, b)$ is a quadrilateral in standard description.*

Proof. If $A_{4,1} \leq A_{4,2}$ (equation (10.9)), the statement holds by Lemma 445. On the other hand, if $A_{4,1} > A_{4,2}$, let

$$\begin{aligned} f : \mathbb{R}^2 &\to \mathbb{R}^2 : \\ x &\mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x. \end{aligned}$$

Then, by Lemma 441, we have

$$\begin{aligned} L' &:= f(L) \\ &= P^{\leq}\left(\begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \\ A_{3,1} & A_{3,2} \\ A_{4,1} & A_{4,2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} \right) \\ &= P^{\leq}\left(\begin{pmatrix} A_{3,2} & A_{3,1} \\ A_{2,2} & A_{2,1} \\ A_{1,2} & A_{1,1} \\ A_{4,2} & A_{4,1} \end{pmatrix}, \begin{pmatrix} b_3 \\ b_2 \\ b_1 \\ b_4 \end{pmatrix} \right) \\ &=: P^{\leq}(A', b'). \end{aligned}$$

Thus, by Theorem 437, $P^{\leq}(A', b)$ is a lattice-free quadrilateral in standard description and we have $A'_{4,1} < A'_{4,2}$; thus, the statement holds by Lemma 445 for L'. So, we conclude from Lemma 441 that the statement also holds for $L = f^{-1}(f(L)) = f^{-1}(L')$. □

11. Characterizing $L_{2,\mathbb{Q}}$ cuts/closure and essential $L_{2-\frac{1}{2},\mathbb{Q}}$ cuts/closure via disjunctions

The central goal of this section is to give an alternative characterization of the $L_{2,\mathbb{Q}}$ closure (in section 11.1) and the essential $L_{2-\frac{1}{2},\mathbb{Q}}$ closure (in section 11.2).

- The goal of section 11.1 is to establish the equivalence between the $L_{2,\mathbb{Q}}$ closure $\mathrm{cl}_{L_{2,\mathbb{Q}}}(P)$ and the crooked cross closure $\mathrm{cl}_{CC}(P)$ for $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ in the situation that P is either
  - a rational polyhedron,
  - convex and $\mathrm{proj}_{\mathbb{R}^2} P$ be full-dimensional or
  - convex and compact.

  Note that parts of these statements have already been shown in [DDG11].

  For the structure of section 11.1:
  - In section 11.1.1, Theorem 459, we show the statement for $m = 2$.
  - In section 11.1.2, Theorem 461, we generalize this to arbitrary $m \in \mathbb{Z}_{\geq 2}$ by showing that in this case, we have $\mathrm{cl}_{CC}(P) \subseteq \mathrm{cl}_{L_{2,\mathbb{Q}}}(P)$.
  - In section 11.1.3, Theorem 462, we show the equivalence of the $L_{2,\mathbb{Q}}$ closure $\mathrm{cl}_{L_{2,\mathbb{Q}}}(P)$ and the crooked cross closure $\mathrm{cl}_{CC}(P)$ under the stated conditions on P.
  - In section 11.1.4, Theorem 464, we show that we can replace "$\overline{\mathrm{conv}}$" by "conv" in the definition of the crooked cross closure $\mathrm{cl}_{CC}(P)$ (cf. Definition 148) if P is either a rational polyhedron or convex and compact.

- The goal of section 11.2 is to show that for rational polyhedra, the essential $L_{2-\frac{1}{2},\mathbb{Q}}$ closure is equivalent the intersection of the essential T closure and the essential crooked cross closure. We use this statement to give an alternative characterization of the $L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ closure of a rational polyhedron (which, by Theorem 193, is equivalent to its $L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ closure).

  The structure of section 11.2 is the following:
  - In section 11.2.1, we define essential T cuts and essential crooked cross cuts (Definition 465) and their respective closures (Definition 466).
  - In section 11.2.2, Theorem 470, we show that for a rational polyhedron $P \subseteq \mathbb{R}^2 \times \mathbb{R}^n$ $(n \in \mathbb{Z}_{\geq 0})$, we have
  $$\mathrm{cl}_{\mathrm{ess}\,CC}(P) \cap \mathrm{cl}_{\mathrm{ess}\,T}(P) \subseteq \mathrm{cl}_{\mathrm{ess}\,L_{2-\frac{1}{2},\mathbb{Q}}}(P).$$
  - In section 11.2.3, Theorem 471, we extend this result to arbitrary rational polyhedra $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m \in \mathbb{Z}_{\geq 2}$ and $n \in \mathbb{Z}_{\geq 0})$.
  - In section 11.2.4, we consider the other direction of inclusions: in Theorem 472, we show that for an arbitrary $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m \in \mathbb{Z}_{\geq 2}$ and $n \in \mathbb{Z}_{\geq 0})$, we have
  $$\mathrm{cl}_{\mathrm{ess}\,L_{2-\frac{1}{2},\mathbb{Q}}}(P) \subseteq \mathrm{cl}_{\mathrm{ess}\,CC}(P) \cap \mathrm{cl}_{\mathrm{ess}\,T}(P).$$

    In Theorem 473, we use this to show that if P is a rational polyhedron, we have
  $$\mathrm{cl}_{L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) \subseteq \mathrm{cl}_{split}(P) \cap \mathrm{cl}_{\mathrm{ess}\,T}(P) \cap \mathrm{cl}_{\mathrm{ess}\,CC}(P).$$

– In section 11.2.5, we finally conclude that for rational polyhedra $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ (with obvious conditions on m and n), we have

$$\mathrm{cl}_{\mathrm{ess}\,L_{2-\frac{1}{2},\mathbb{Q}}}(P) = \mathrm{cl}_{\mathrm{ess}\,T}(P) \cap \mathrm{cl}_{\mathrm{ess}\,CC}(P), \quad \text{(Theorem 474)}$$
$$\mathrm{cl}_{L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P) = \mathrm{cl}_{split}(P) \cap \mathrm{cl}_{\mathrm{ess}\,T}(P) \cap \mathrm{cl}_{\mathrm{ess}\,CC}(P). \quad \text{(Theorem 475)}$$

11.1. $L_{2,\mathbb{Q}}$ closure

11.1.1. $m = 2$

We want to mention that the following cases that occur in the next theorem (Theorem 459) are well-known in the literature:

- The case that P is a rational polyhedron of the form $\left(\mathbb{R}^2 \times \mathbb{R}^n_{\geq 0}\right) \cap P^{=}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)$ is shown in [DDG11, Lemma 2.1]. Despite this restriction, the proof given there is easy to extend to arbitrary rational polyhedra.
- The case that P is convex and compact is shown in [DDG11, Theorem 5.1].

So, only the case "P is convex and $\mathrm{proj}_{\mathbb{R}^2} P$ is full-dimensional" has to our knowledge not yet been considered in the literature. We remark that both of these proofs from [DDG11] depend on results that are shown in [DDG12].

Theorem 459. *Let $P \subseteq \mathbb{R}^2 \times \mathbb{R}^n$ ($n \in \mathbb{Z}_{\geq 0}$) be*

1. *a rational polyhedron,*
2. *convex and $\mathrm{proj}_{\mathbb{R}^2} P$ be full-dimensional or*
3. *convex and compact,*

respectively, and let

1. *$c(\cdot) \geq c_0$, where $c \in \left(\mathbb{Q}^2 \times \mathbb{Q}^n\right)^T$, be a valid inequality for P_I,*
2. *$c(\cdot) \geq c_0$, where $c \in \left(\mathbb{R}^2 \times \mathbb{R}^n\right)^T$, be a valid inequality for P_I or*
3. *$c(\cdot) > c_0$, where $c \in \left(\mathbb{R}^2 \times \mathbb{R}^n\right)^T$, be a valid* strict *inequality for P_I,*

respectively. Let $P \cap P^{<}(c, c_0) \neq \emptyset$. Then there exists a crooked cross disjunction $D^c\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right)$ such that $c(\cdot) \geq c_0$ is a valid inequality for $P \cap D^c\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right)$. In particular, we have

$$\mathrm{cl}_{CC}(P) = \mathrm{cl}_{\overline{I}}(P) \tag{11.1}$$

in case 1 and 3 and, if P is closed, also in case 2. In all three cases, the more general identities

$$\mathrm{cl}_{CC}(P) \subseteq \mathrm{cl}_{\overline{I}}(P), \tag{11.2}$$
$$\mathrm{cl}_{CC}(P) = P \cap \mathrm{cl}_{\overline{I}}(P) \tag{11.3}$$

hold.

Remark 460. *In case 2 of Theorem 459, in general only identity* (11.2)*, but not the stronger identity* (11.1)*, holds if P is not closed. For this, consider*

$$P := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \times \mathbb{R}^1 : y_1 < 1 \right\}.$$

It is easy to check that

$$\mathrm{cl}_{\overline{I}}(P) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \times \mathbb{R}^1 : y_1 \leq 1 \right\}.$$

So, we have using (11.3)*:*

$$\mathrm{cl}_{CC}(P) = P \cap \mathrm{cl}_{\overline{I}}(P) = P \subsetneq \mathrm{cl}_{\overline{I}}(P).$$

Proof. (Theorem 459)

For the first part: Define $R := P \cap P^{<}(c, c_0) \neq \emptyset$. By Theorem 231, Theorem 233 or Theorem 234, respectively, there exists a full-dimensional maximal lattice-free body $L \subseteq \mathbb{R}^m$ which satisfies

$$\operatorname{proj}_{\mathbb{R}^m} R \subseteq (\operatorname{int} L) \,\dot{\cup}\, ((\operatorname{bd} L) \setminus (\operatorname{bd} L)_I).$$

Thus, by Theorem 431, there exists a crooked cross disjunction $D^c(\pi^1, \pi^2, \gamma_1, \gamma_2)$ having

$$(\operatorname{proj}_{\mathbb{R}^m} R) \cap D^c(\pi^1, \pi^2, \gamma_1, \gamma_2) = \emptyset. \tag{11.4}$$

From (11.4), we obtain

$$\operatorname{proj}_{\mathbb{R}^2}(P \cap P^{<}(c, c_0)) \cap D^c(\pi^1, \pi^2, \gamma_1, \gamma_2) = \emptyset$$

and thus

$$P \cap P^{<}(c, c_0) \cap \left(D^c(\pi^1, \pi^2, \gamma_1, \gamma_2) \times \mathbb{R}^n\right) = \emptyset.$$

This means that $c(\,\cdot\,) \geq c_0$ is a valid inequality for $P \cap D^c(\pi^1, \pi^2, \gamma_1, \gamma_2)$.

For (11.1)**:** In case 1, note that (cf. Theorem 75) $\operatorname{cl}_I(P)$ is a rational polyhedron and thus closed. So, every inequality for P_I is dominated absolutely by a finite set of linear inequalities with rational coefficients. In case 3, note that, because P is convex and compact, we have $\operatorname{cl}_{\overline{I}}(P) \subseteq P$. Let $c'(\,\cdot\,) \geq c'_0$ be a valid inequality for P_I. In the first part of this proof, we proved that for every $\epsilon > 0$, $c'(\,\cdot\,) \geq c'_0 - \epsilon$ is a crooked cross cut for P. Thus, $c'(\,\cdot\,) \geq c'_0$ is valid for $\operatorname{cl}_{CC}(P)$. In case 2, (11.1) holds by definition.

For (11.2)**:** For $\operatorname{cl}_{\overline{I}}(P) \subseteq \operatorname{cl}_{CC}(P)$, consider that, since P is closed and convex, we have $\operatorname{cl}_{\overline{I}}(P) \subseteq P$. Also note that for every crooked cross cut $c(\,\cdot\,) \geq c_0$ for P ($c \in (\mathbb{R}^m \times \mathbb{R}^n)^T$ and $c_0 \in \mathbb{R}$), we have

$$\operatorname{cl}_{\overline{I}}(P) \subseteq P^{\geq}(c, c_0).$$

Thus,

$$\operatorname{cl}_{\overline{I}}(P) \subseteq \operatorname{cl}_{CC}(P).$$

For (11.3)**:** By (11.2), we have $\operatorname{cl}_{CC}(P) \subseteq \operatorname{cl}_{\overline{I}}(P)$. Additionally, by definition, we have $\operatorname{cl}_{\overline{I}}(P) \subseteq P$. Thus, $\operatorname{cl}_{CC}(P) \subseteq P \cap \operatorname{cl}_{\overline{I}}(P)$. For the inclusion $\operatorname{cl}_{CC}(P) \supseteq P \cap \operatorname{cl}_{\overline{I}}(P)$, it suffices to consider case 2, since by (11.1), the statement holds in case 1 and 3. We saw that in this case, every valid inequality for P_I is a crooked cross cut for P. Thus,

$$\begin{aligned} P \cap \operatorname{cl}_{\overline{I}}(P) &= P \cap \bigcap_{\substack{c \in (\mathbb{R}^m \times \mathbb{R}^n)^T, c_0 \in \mathbb{R}: \\ P_I \subseteq P^{\leq}(c, c_0)}} P^{\leq}(c, c_0) \\ &= P \cap \bigcap_{\substack{c \in (\mathbb{R}^m \times \mathbb{R}^n)^T, c_0 \in \mathbb{R}: \\ c(\,\cdot\,) \leq c_0 \text{ crooked cross cut for } P}} P^{\leq}(c, c_0) \\ &= \operatorname{cl}_{CC}(P). \end{aligned}$$

□

Now we consider that the conditions of Theorem 459 that are imposed on P are necessary by presenting an example $P \subsetneq \mathbb{R}^2$ that is convex and closed, but not a rational polyhedron, full-dimensional or bounded, where we have $\operatorname{cl}_I(P) = \operatorname{cl}_{\overline{I}}(P) = \operatorname{cl}_{L_{2,\mathbb{Q}}}(P) \subsetneq \operatorname{cl}_{CC}(P)$. For this, let us recall Theorem 255:

Theorem 255. *Let $t \in \mathbb{Z}_{\geq 0}$, $\pi^1, \ldots, \pi^t \in \mathbb{Z}^2$ and $\gamma_1, \ldots, \gamma_t \in \mathbb{Z}$. Then*

$$\operatorname{conv}\left(P^{114} \cap D(\pi^1, \ldots, \pi^t, \gamma_1, \ldots, \gamma_t)\right) = P^{114}. \tag{5.21}$$

In particular, for all parametric cross disjunctions $D^{t'}(\pi^1, \pi^2, \gamma_1, \gamma_2)$ (cf. Definition 150) with respect to some $t' \in \mathbb{Z}$ (this includes crooked cross disjunctions), we have:

$$\operatorname{conv}\left(P^{114} \cap D^{t'}(\pi^1, \pi^2, \gamma_1, \gamma_2)\right) = P^{114}. \tag{5.22}$$

On the other hand, $(0^2)^T(\,\cdot\,) \leq -1$ is an $L_{2,\mathbb{Q}}$ cut for P^{114}.

11.1.2. $m \geq 2$

We already stated that some of the cases in Theorem 459 are well-known in the literature. A similar statement also holds for following theorem (Theorem 461):

- The case that P is rational polyhedron of the form $\left(\mathbb{R}^m \times \mathbb{R}^n_{\geq 0}\right) \cap P^= \left(\left(\begin{array}{cc} A & G \end{array} \right), b\right)$ is shown in [DDG11, Theorem 3.1]. Despite this restriction, the proof given there is easy to extend to arbitrary rational polyhedra.
- The case that P convex and compact is shown in [DDG11, Proposition 5.2].

So, only the case that P is convex and $\operatorname{proj}_{\mathbb{R}^m} P$ is full-dimensional has to our knowledge not yet been considered in the literature. We remark that the proofs in [DDG11] are not for $L_{2,\mathbb{Q}}$ cuts, but for an implicit version of 2-dimensional lattice cuts (cf. Definition 175 and Theorem 176), even though the framework of k-dimensional lattice cuts was only later on introduced in the literature ([DGMR17]; see section 4.2.2.2 for details).

Theorem 461. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m \in \mathbb{Z}_{\geq 2}$ and $n \in \mathbb{Z}_{\geq 0}$) be*

1. *a rational polyhedron,*
2. *convex and $\operatorname{proj}_{\mathbb{R}^m} P$ be full-dimensional or*
3. *convex and compact,*

respectively, and let

1. *$c(\cdot) \geq c_0$, where $c \in (\mathbb{Q}^m \times \mathbb{Q}^n)^T$, be a valid $L_{2,\mathbb{Q}}$ cut for P,*
2. *$c(\cdot) \geq c_0$, where $c \in (\mathbb{R}^m \times \mathbb{R}^n)^T$, be a valid $L_{2,\mathbb{Q}}$ cut for P or*
3. *$c(\cdot) > c_0$, where $c \in (\mathbb{R}^m \times \mathbb{R}^n)^T$, be a valid strict $L_{2,\mathbb{Q}}$ cut for P,*

respectively. Then there exists a crooked cross disjunction $D^c\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right)$ such that $c(\cdot) \geq c_0$ is a valid inequality for $P \cap D^c\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right)$. In particular, we have

$$\operatorname{cl}_{CC}(P) \subseteq \operatorname{cl}_{L_{2,\mathbb{Q}}}(P). \tag{11.5}$$

Proof. Let $W \in \mathbb{Z}^{m \times 2}$ be as in the proof in section 5.2.3 and let

$$S^{LP} := \left\{ \left(\begin{array}{c} x \\ y \\ z \end{array} \right) \in P \times \mathbb{R}^2 : z = W^T x \right\},$$
$$S := S^{LP} \cap \left(\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{Z}^2\right)$$

(also as in the proof in section 5.2.3). Since $c(\cdot) \geq c_0$ is valid for

$$\left\{ \left(\begin{array}{c} x \\ y \end{array} \right) \in P : \left(w^1\right)^T x, \left(w^2\right)^T x \in \mathbb{Z} \right\},$$

we have seen in the proof in section 5.2.3 that

$$\left(c_{(1,\ldots,m)} \left(I^m - \left(W \left(W^T W\right)^{-1}\right) W^T\right)\right) x + c_{(m+1,\ldots,m+n)} y + \left(c_{(1,\ldots,m)} W \left(W^T W\right)^{-1}\right) z \geq c_0 \tag{11.6}$$

is valid for $\left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right) \in S \subseteq \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{Z}^2$. Thus, by Theorem 459, there exists a crooked cross disjunction $D^c\left(\pi^{1,pre}, \pi^{2,pre}, \gamma_1, \gamma_2\right)$ ($\pi^{1,pre}, \pi^{2,pre} \in \mathbb{Z}^2$ and $\gamma_1, \gamma_2 \in \mathbb{Z}$) such that (11.6) is a valid inequality for

$$S^{LP} \cap \left(\mathbb{R}^m \times \mathbb{R}^n \times D^c\left(\pi^{1,pre}, \pi^{2,pre}, \gamma_1, \gamma_2\right)\right).$$

We claim that $c(\cdot) \geq c_0$ is valid for $P \cap \left(D^c\left(W\pi^{1,pre}, W\pi^{2,pre}, \gamma_1, \gamma_2\right) \times \mathbb{R}^n\right)$. For this, let

$$\left(\begin{array}{c} x \\ y \end{array} \right) \in P \cap \left(D^c\left(W\pi^{1,pre}, W\pi^{2,pre}, \gamma_1, \gamma_2\right) \times \mathbb{R}^n\right).$$

Then, using

$$\begin{pmatrix} x \\ y \\ W^T x \end{pmatrix} \in S^{LP} \cap \left(\mathbb{R}^m \times \mathbb{R}^n \times D^c\left(\pi^{1,pre}, \pi^{2,pre}, \gamma_1, \gamma_2\right)\right),$$

we obtain

$$c\begin{pmatrix} x \\ y \end{pmatrix} = c_{(1,\dots,m)}\left(I^m - W\left(W^T W\right)^{-1} W^T\right)x + c_{(m+1,\dots,m+n)}y + c_{(1,\dots,m)}W\left(W^T W\right)^{-1}\left(W^T x\right) \geq c_0.$$

Finally for (11.5): for case 1, consider that, by Theorem 177,

$$\left\{\begin{pmatrix} x \\ y \end{pmatrix} \in P : \left(w^1\right)^T x, \left(w^2\right)^T x \in \mathbb{Z}\right\}$$

is a rational polyhedron if P is. So, it suffices to consider linear inequalities $c(\,\cdot\,) \geq c_0$ with rational coefficients. For case 3, one argues similarly to how we showed in the proof of Theorem 459 that (11.1) holds in case 3. □

11.1.3. Conclusion

Again (see the remarks before Theorem 459 and Theorem 461), the following result (Theorem 462) is known to be true

- in the case that P rational polyhedron of the form $\left(\mathbb{R}^m \times \mathbb{R}^n_{\geq 0}\right) \cap P^{=}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)$ (cf. [DDG11, Theorem 3.1]) and
- in the case that P is convex and compact (cf. [DDG11, Proposition 5.2]).

So, only the case that P is convex and $\operatorname{proj}_{\mathbb{R}^m} P$ is full-dimensional has to our knowledge not yet been considered in the literature. Again, we remark that in the given literature reference, the results are shown for 2-dimensional lattice cuts.

Theorem 462. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$, where $m + n \geq 2$) be either*

- *a rational polyhedron,*
- *convex and $\operatorname{proj}_{\mathbb{R}^m} P$ be full-dimensional or*
- *convex and compact.*

Then

$$\operatorname{cl}_{L_{2,\mathbb{Q}}}(P) = \operatorname{cl}_{CC}(P).$$

Proof. The inclusion $\operatorname{cl}_{L_{2,\mathbb{Q}}}(P) \subseteq \operatorname{cl}_{CC}(P)$ is an immediate consequence of Corollary 279. On the other hand, the inclusion $\operatorname{cl}_{L_{2,\mathbb{Q}}}(P) \supseteq \operatorname{cl}_{CC}(P)$ clearly holds for $m = 0$ and is for $m \geq 2$ a direct consequence of Theorem 461. So, we only have to show

$$\operatorname{cl}_{L_{2,\mathbb{Q}}}(P) \supseteq \operatorname{cl}_{CC}(P) \tag{11.7}$$

for $m = 1$. In this case, we have

$$\operatorname{cl}_{L_{2,\mathbb{Q}}}(P) = \operatorname{cl}_{L_{1,\mathbb{Q}}}(P) = P \cap \operatorname{cl}_{\overline{I}}(P).$$

By Theorem 409, if P is convex (which is the case here), every cutting plane for P is a split cut for P. Let $D(\pi, \gamma)$ be the corresponding split disjunction. Then, clearly, $D(\pi, \gamma) = D^c\left(\pi, 0^1, \gamma, 0\right)$, from which we conclude that every cutting plane for P is a crooked cross cut for P, which implies (11.7). □

11.1.4. Replacing $\overline{\operatorname{conv}}$ by conv in the definition of the crooked cross closure

Recall that in Definition 148, we defined the crooked cross closure as

$$\operatorname{cl}_{CC}(P) = P \cap \bigcap_{\substack{\pi^1, \pi^2 \in \mathbb{Z}^m, \\ \gamma_1, \gamma_2 \in \mathbb{Z}}} \overline{\operatorname{conv}}\left(P \cap \left(D^c\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right) \times \mathbb{R}^n\right)\right) \tag{11.8}$$

and we saw in Example 149 that for a concrete rational polyhedron $P \subseteq \mathbb{R}^2$ and a concrete crooked cross disjunction $D^c\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right)$, the situation

$$\overline{\operatorname{conv}}\left(P \cap D^c\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right)\right) \supsetneq \operatorname{conv}\left(P \cap D^c\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right)\right)$$

can occur. Thus, in comparison to the situation for the t-branch split closure and the integral lattice-free closure (recall Theorem 140), it is much less obvious that we can simply replace "$\overline{\text{conv}}$" by "conv" in (11.8).

Nevertheless, in the literature (see for example [DGM15]), the crooked cross closure is often defined as

$$\operatorname{cl}_{CC}(P) = \bigcap_{\substack{\pi^1, \pi^2 \in \mathbb{Z}^m, \\ \gamma_1, \gamma_2 \in \mathbb{Z}}} \operatorname{conv}\left(P \cap \left(D^c\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right) \times \mathbb{R}^n\right)\right).$$

So, at the end of section 3.4.2.2, we gave an outlook that if $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ is either a rational polyhedron or convex and compact, these definitions are equivalent. This is the statement of Theorem 464, which forms the center of this section. To show it, we buid on a related result that is shown in [DDG11]:

Theorem 463. *([DDG11, Theorem 3.1 and Proposition 5.2]) Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ be either*

- *a rational polyhedron or*
- *convex and compact.*

Then

$$\bigcap_{\substack{\pi^1, \pi^2 \in \mathbb{Z}^m, \\ \gamma_1, \gamma_2 \in \mathbb{Z}}} \operatorname{conv}\left(P \cap \left(D^c\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right) \times \mathbb{R}^n\right)\right) = \bigcap_{\pi^1, \pi^2 \in \mathbb{Z}^m} \operatorname{conv}\left\{\begin{pmatrix} x \\ y \end{pmatrix} \in P : \left(\pi^1\right)^T x \in \mathbb{Z}, \left(\pi^2\right)^T x \in \mathbb{Z}\right\}.$$

Theorem 464. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ be either*

- *a rational polyhedron or*
- *convex and compact.*

Then

$$\operatorname{cl}_{CC}(P) = \bigcap_{\substack{\pi^1, \pi^2 \in \mathbb{Z}^m, \\ \gamma_1, \gamma_2 \in \mathbb{Z}}} \operatorname{conv}\left(P \cap \left(D^c\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right) \times \mathbb{R}^n\right)\right),$$

i.e. we can replace $\overline{\text{conv}}$ by conv *in the definition of the crooked cross closure of P (recall Definition 148).*

Proof.

$$\begin{aligned}
\operatorname{cl}_{CC}(P) &= P \cap \bigcap_{\substack{\pi^1, \pi^2 \in \mathbb{Z}^m, \\ \gamma_1, \gamma_2 \in \mathbb{Z}}} \overline{\operatorname{conv}}\left(P \cap \left(D^c\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right) \times \mathbb{R}^n\right)\right) \\
&= P \cap \bigcap_{\pi^1, \pi^2 \in \mathbb{Z}^m} \overline{\operatorname{conv}}\left\{\begin{pmatrix} x \\ y \end{pmatrix} \in P : \left(\pi^1\right)^T x \in \mathbb{Z}, \left(\pi^2\right)^T x \in \mathbb{Z}\right\} && \text{(by Theorem 462)} \\
&= \bigcap_{\pi^1, \pi^2 \in \mathbb{Z}^m} \overline{\operatorname{conv}}\left\{\begin{pmatrix} x \\ y \end{pmatrix} \in P : \left(\pi^1\right)^T x \in \mathbb{Z}, \left(\pi^2\right)^T x \in \mathbb{Z}\right\} && (P \text{ convex and closed}) \\
&= \bigcap_{\pi^1, \pi^2 \in \mathbb{Z}^m} \operatorname{conv}\left\{\begin{pmatrix} x \\ y \end{pmatrix} \in P : \left(\pi^1\right)^T x \in \mathbb{Z}, \left(\pi^2\right)^T x \in \mathbb{Z}\right\} && \text{(by Theorem 177)} \\
&= \bigcap_{\substack{\pi^1, \pi^2 \in \mathbb{Z}^m, \\ \gamma_1, \gamma_2 \in \mathbb{Z}}} \operatorname{conv}\left(P \cap \left(D^c\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right) \times \mathbb{R}^n\right)\right). && \text{(by Theorem 463)}
\end{aligned}$$

□

11.2. Essential $L_{2-\frac{1}{2},\mathbb{Q}}$ closure

11.2.1. Essential crooked cross closure and essential T closure

Definition 465. *Let $D^T(\pi^1,\pi^2,\gamma_1,\gamma_2) \subseteq \mathbb{R}^m$ ($m \in \mathbb{Z}_{\geq 0}$) be a T disjunction (cf. Definition 433) and let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ be arbitrary. Then we call $D^T(\pi^1,\pi^2,\gamma_1,\gamma_2)$ **essentially valid for** P if there exists an $i^* \in [3]$ such that*

$$\forall i \in [3] \setminus \{i^*\} : P \cap \left(D_i^T\left(\pi^1,\pi^2,\gamma_1,\gamma_2\right) \times \mathbb{R}^n\right) = \emptyset.$$

*In a similar way, we call the crooked cross disjunction (cf. Definition 146) $D^c(\pi^1,\pi^2,\gamma_1,\gamma_2) \subseteq \mathbb{R}^m \times \mathbb{R}^n$ **essentially valid for** P if there exists a $i^* \in [4]$ such that*

$$\forall i \in [4] \setminus \{i^*\} : P \cap \left(D_i^c\left(\pi^1,\pi^2,\gamma_1,\gamma_2\right) \times \mathbb{R}^n\right) = \emptyset.$$

*A valid linear inequality $c(\cdot) \leq c_0$ ($c \in (\mathbb{R}^m \times \mathbb{R}^n)^T$ and $c_0 \in \mathbb{R}$) for $P \cap D_{i^*}^T(\pi^1,\pi^2,\gamma_1,\gamma_2)$ is called an **essential T cut** and, similarly, a linear inequality for $P \cap D_{i^*}^c(\pi^1,\pi^2,\gamma_1,\gamma_2)$ is called an **essential crooked cross cut**.*

Definition 466. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m,n \in \mathbb{Z}_{\geq 0}$) be arbitrary. We define*

$$\mathrm{cl}_{\mathrm{ess}\,T}(P) := P \cap \bigcap_{\substack{\pi^1,\pi^2 \in \mathbb{Z}^m,\\ \gamma_1,\gamma_2 \in \mathbb{Z}}} \bigcap_{\substack{i^* \in [3]: \forall i \in [3] \setminus \{i^*\}:\\ P \cap \left(D_i^T\left(\pi^1,\pi^2,\gamma_1,\gamma_2\right) \times \mathbb{R}^n\right) = \emptyset}} \bigcap_{\substack{c \in (\mathbb{R}^m \times \mathbb{R}^n)^T, c_0 \in \mathbb{R}:\\ P^{\leq}(c,c_0) \subseteq P \cap \left(D_{i^*}^T\left(\pi^1,\pi^2,\gamma_1,\gamma_2\right) \times \mathbb{R}^n\right)}} P^{\leq}(c,c_0),$$

$$\mathrm{cl}_{\mathrm{ess}\,CC}(P) := P \cap \bigcap_{\substack{\pi^1,\pi^2 \in \mathbb{Z}^m,\\ \gamma_1,\gamma_2 \in \mathbb{Z}}} \bigcap_{\substack{i^* \in [4]: \forall i \in [4] \setminus \{i^*\}:\\ P \cap \left(D_i^c\left(\pi^1,\pi^2,\gamma_1,\gamma_2\right) \times \mathbb{R}^n\right) = \emptyset}} \bigcap_{\substack{c \in (\mathbb{R}^m \times \mathbb{R}^n)^T, c_0 \in \mathbb{R}:\\ P^{\leq}(c,c_0) \subseteq P \cap \left(D_{i^*}^c\left(\pi^1,\pi^2,\gamma_1,\gamma_2\right) \times \mathbb{R}^n\right)}} P^{\leq}(c,c_0)$$

*as the **essential T closure of** P and the **essential crooked cross closure of** P, respectively.*

At the end of section 5.4.1, we already discussed the differences in the definitions of essential t,k-branch split cuts vs essential T cuts/essential crooked cross cuts (Definition 465). To recall the central points:

- The definition of t,k-branch split cuts makes it easier to prove analogues of results on $L_{k,\mathbb{Q}}$ cuts/closure vs k,t-branch split cuts/closure for essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts/closure vs essential k,t-branch split cuts/closure.
- The definitions of essential T cuts and essential crooked cross cuts more closely mirror the characterization of essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts via lattice-free bodies that we gave in Theorem 246.

The following lemma is a very simple, but important observation:

Lemma 467. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m,n \in \mathbb{Z}_{\geq 0}$) and let $D^T(\pi^1,\pi^2,\gamma_1,\gamma_2) \subseteq \mathbb{R}^m$ or $D^c(\pi^1,\pi^2,\gamma_1,\gamma_2) \subseteq \mathbb{R}^m$, respectively, be given ($\pi^1,\pi^2 \in \mathbb{Z}^m$ and $\gamma_1,\gamma_2 \in \mathbb{Z}$). Let there exist an $i^* \in [3]$ or $i^* \in [4]$, respectively, such that*

$$\forall i \in [3] \setminus \{i^*\} : P \cap \left(D_i^T\left(\pi^1,\pi^2,\gamma_1,\gamma_2\right) \times \mathbb{R}^n\right) = \emptyset$$

or

$$\forall i \in [4] \setminus \{i^*\} : P \cap \left(D_i^c\left(\pi^1,\pi^2,\gamma_1,\gamma_2\right) \times \mathbb{R}^n\right) = \emptyset,$$

respectively. Denote $D := D_{i^*}^T(\pi^1,\pi^2,\gamma_1,\gamma_2)$ or $D := D_{i^*}^c(\pi^1,\pi^2,\gamma_1,\gamma_2)$, respectively, and let $c(\cdot) \leq c_0$ be a valid linear inequality for $P \cap (D \times \mathbb{R}^n)$. Then

$$P \cap (D \times \mathbb{R}^n) \subseteq P \cap P^{\leq}(c,c_0).$$

Proof. Let $x \in P \cap (D \times \mathbb{R}^n)$. We clearly have $x \in P$. On the other hand, $c(\cdot) \leq c_0$ is a valid linear inequality for $P \cap (D \times \mathbb{R}^n)$. Thus, $cx \leq c_0$. □

Remark 468. *By Lemma 467, it suffices to consider linear inequalities for $D_{i^*}^T(\pi^1,\pi^2,\gamma_1,\gamma_2) \times \mathbb{R}^n$ instead of $P \cap \left(D_{i^*}^T(\pi^1,\pi^2,\gamma_1,\gamma_2) \times \mathbb{R}^n\right)$ and $D_{i^*}^c(\pi^1,\pi^2,\gamma_1,\gamma_2) \times \mathbb{R}^n$ instead of $P \cap \left(D_{i^*}^c(\pi^1,\pi^2,\gamma_1,\gamma_2) \times \mathbb{R}^n\right)$ as in Definition 465.*

From Definition 466, Lemma 467 and Remark 468, we immediately conclude:

Lemma 469. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ be arbitrary. Then*

$$\begin{aligned}
\mathrm{cl}_{\mathrm{ess}\,T}(P) &:= P \cap \bigcap_{\substack{\pi^1,\pi^2\in\mathbb{Z}^m,\\ \gamma_1,\gamma_2\in\mathbb{Z}}} \bigcap_{\substack{i^*\in[3]:\forall i\in[3]\setminus\{i^*\}:\\ P\cap\left(D_i^T(\pi^1,\pi^2,\gamma_1,\gamma_2)\times\mathbb{R}^n\right)=\emptyset}} \bigcap_{\substack{c\in(\mathbb{R}^m\times\mathbb{R}^n)^T,c_0\in\mathbb{R}:\\ P^{\leq}(c,c_0)\subseteq P\cap\left(D_{i^*}^T(\pi^1,\pi^2,\gamma_1,\gamma_2)\times\mathbb{R}^n\right)}} P^{\leq}(c,c_0) \\
&= P \cap \bigcap_{\substack{\pi^1,\pi^2\in\mathbb{Z}^m,\\ \gamma_1,\gamma_2\in\mathbb{Z}}} \bigcap_{\substack{i^*\in[3]:\forall i\in[3]\setminus\{i^*\}:\\ P\cap\left(D_i^T(\pi^1,\pi^2,\gamma_1,\gamma_2)\times\mathbb{R}^n\right)=\emptyset}} \left(D_{i^*}^T\left(\pi^1,\pi^2,\gamma_1,\gamma_2\right)\times\mathbb{R}^n\right), \\
\mathrm{cl}_{\mathrm{ess}\,CC}(P) &:= P \cap \bigcap_{\substack{\pi^1,\pi^2\in\mathbb{Z}^m,\\ \gamma_1,\gamma_2\in\mathbb{Z}}} \bigcap_{\substack{i^*\in[4]:\forall i\in[4]\setminus\{i^*\}:\\ P\cap\left(D_i^c(\pi^1,\pi^2,\gamma_1,\gamma_2)\times\mathbb{R}^n\right)=\emptyset}} \bigcap_{\substack{c\in(\mathbb{R}^m\times\mathbb{R}^n)^T,c_0\in\mathbb{R}:\\ P^{\leq}(c,c_0)\subseteq P\cap\left(D_{i^*}^c(\pi^1,\pi^2,\gamma_1,\gamma_2)\times\mathbb{R}^n\right)}} P^{\leq}(c,c_0) \\
&= P \cap \bigcap_{\substack{\pi^1,\pi^2\in\mathbb{Z}^m,\\ \gamma_1,\gamma_2\in\mathbb{Z}}} \bigcap_{\substack{i^*\in[4]:\forall i\in[4]\setminus\{i^*\}:\\ P\cap\left(D_i^c(\pi^1,\pi^2,\gamma_1,\gamma_2)\times\mathbb{R}^n\right)=\emptyset}} \left(D_{i^*}^c\left(\pi^1,\pi^2,\gamma_1,\gamma_2\right)\times\mathbb{R}^n\right).
\end{aligned}$$

11.2.2. $m = 2$

Theorem 470. *Let $\emptyset \neq P \subseteq \mathbb{R}^2 \times \mathbb{R}^n$ $(n \in \mathbb{Z}_{\geq 0})$ be a rational polyhedron. Then:*

1. *If $P_I = \emptyset$, there exist linearly independent $\pi^1, \pi^2 \in \mathbb{Z}^m$ and $\gamma_1, \gamma_2 \in \mathbb{Z}$ such that*

$$P \cap \left(D^T\left(\pi^1,\pi^2,\gamma_1,\gamma_2\right)\times\mathbb{R}^n\right) = \emptyset \vee P \cap \left(D^c\left(\pi^1,\pi^2,\gamma_1,\gamma_2\right)\times\mathbb{R}^n\right) = \emptyset.$$

2. *If $P_I \neq \emptyset$, let $\begin{pmatrix} c' & (0^n)^T \end{pmatrix}(\cdot) \geq c_0$, where $c' \in \left(\mathbb{Q}^2\right)^T$ and $c_0 \in \mathbb{R}$, be a valid inequality for*

$$\left(P + \left(0^2 \times \mathbb{R}^n\right)\right)_I = \left(\mathrm{proj}_{\mathbb{R}^2} P\right)_I \times \mathbb{R}^n,$$

which is not already valid for P (the fact that we only consider inequalities of the type $\begin{pmatrix} c' & (0^n)^T \end{pmatrix}(\cdot) \geq c_0$ is, by Lemma 159, no loss of generality). Then there exist $\pi^1, \pi^2 \in \mathbb{Z}^2$ (π^1, π^2 linearly indepdendent), $\gamma_1, \gamma_2 \in \mathbb{Z}$ and an $i^* \in [3]$ or $i^* \in [4]$, respectively, such that

$$\begin{aligned}\forall i \in [3]\setminus\{i^*\} : P \cap \left(D_i^T\left(\pi^1,\pi^2,\gamma_1,\gamma_2\right)\times\mathbb{R}^n\right) &= \emptyset, \\ D_{i^*}^T\left(\pi^1,\pi^2,\gamma_1,\gamma_2\right) &\subseteq P^{\leq}\left(c',c_0\right)\end{aligned}$$

or

$$\begin{aligned}\forall i \in [4]\setminus\{i^*\} : P \cap \left(D_i^c\left(\pi^1,\pi^2,\gamma_1,\gamma_2\right)\times\mathbb{R}^n\right) &= \emptyset, \\ D_{i^*}^c\left(\pi^1,\pi^2,\gamma_1,\gamma_2\right) &\subseteq P^{\leq}\left(c',c_0\right),\end{aligned}$$

respectively.

We thus have

$$\mathrm{cl}_{essT}(P) \cap \mathrm{cl}_{\mathrm{ess}\,CC}(P) \subseteq \mathrm{cl}_{\mathrm{ess}\,L_{2-\frac{1}{2},\mathbb{Q}}}(P). \tag{11.9}$$

Proof.

For 1: Since $P_I = \emptyset$, by Theorem 241, there exists a full-dimensional rational maximal lattice-free polyhedron $L := P^{\leq}\left(A^L, b^L\right) \subseteq \mathbb{R}^2$ ($A^L \in \mathbb{Z}^{l\times 2}$ and $b^L \in \mathbb{Z}^l$, where $l \in \mathbb{Z}_{\geq 0}$) such that

$$\forall j \in [l] : P \cap \left(P^{\geq}\left(A_{j,*}^L, b_j^L\right)\times\mathbb{R}^n\right) = \emptyset.$$

Thus, by Theorem 434, there exist linearly independent $\pi^1, \pi^2 \in \mathbb{Z}^2$ and $\gamma_1, \gamma_2 \in \mathbb{Z}$ such that either

$$\bigcup_{j\in[l]} P^{\geq}\left(A_{j,*}^L, b_j^L\right) \supseteq \bigcup_{i\in[3]} D_i^T\left(\pi^1,\pi^2,\gamma_1,\gamma_2\right) = D^T\left(\pi^1,\pi^2,\gamma_1,\gamma_2\right)$$

or

$$\bigcup_{j\in[l]} P^{\geq}\left(A_{j,*}^L, b_j^L\right) \supseteq \bigcup_{i\in[4]} D_i^c\left(\pi^1,\pi^2,\gamma_1,\gamma_2\right) = D^c\left(\pi^1,\pi^2,\gamma_1,\gamma_2\right)$$

holds. So, we either have

$$P \cap \left(D^T\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right) \times \mathbb{R}^n\right) \subseteq P \cap \left(\bigcup_{j\in[l]} P^{\geq}\left(A^L_{j,*}, b^L_j\right) \times \mathbb{R}^n\right) = \emptyset$$

or

$$P \cap \left(D^c\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right) \times \mathbb{R}^n\right) \subseteq P \cap \left(\bigcup_{j\in[l]} P^{\geq}\left(A^L_{j,*}, b^L_j\right) \times \mathbb{R}^n\right) = \emptyset.$$

For 2: Since $P_I \neq \emptyset$, consider that, by Theorem 241, there exists a full-dimensional, rational, maximal lattice-free polyhedron

$$L = P^{\leq}\left(\left(\begin{array}{c} A^L \\ c' \end{array}\right), \left(\begin{array}{c} b^L \\ \tilde{c}_0 \end{array}\right)\right) \subseteq \mathbb{R}^m$$

($A^L \in \mathbb{Z}^{l\times 2}$ and $b^L \in \mathbb{Z}^l$, where $l \in \mathbb{Z}_{\geq 0}$) having

- $P \cap P^{<}\left(\left(\begin{array}{cc} c' & (0^n)^T \end{array}\right), c_0\right) \subseteq (\operatorname{int} L) \times \mathbb{R}^n$,
- $\forall j \in [l] : \left(P^{\geq}\left(A^L_{j,*}, b^L_j\right) \times \mathbb{R}^n\right) \cap P = \emptyset$ and
- $\tilde{c}_0 \geq c_0$.

Thus, by Theorem 434, there exist linearly independent $\pi^1, \pi^2 \in \mathbb{Z}^2$, $\gamma_1, \gamma_2 \in \mathbb{Z}$ and an $i^* \in [3]$ or $i^* \in [4]$, respectively, such that either

$$\bigcup_{j\in[l]} P^{\geq}\left(A^L_{j,*}, b^L_j\right) \supseteq \bigcup_{i\in[3]\setminus\{i^*\}} D^T_i\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right),$$
$$P^{\geq}\left(c', \tilde{c}_0\right) \supseteq D^T_{i^*}\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right)$$

or

$$\bigcup_{j\in[l]} P^{\geq}\left(A_{j,*}, b_j\right) \supseteq \bigcup_{i\in[4]\setminus\{i^*\}} D^c_i\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right),$$
$$P^{\geq}\left(c', \tilde{c}_0\right) \supseteq D^c_{i^*}\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right),$$

respectively, holds. Thus, in the first case (T disjunction), we obtain

$$\forall i \in [3] \setminus \{i^*\} : P \cap D^T_i\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right) \subseteq P \cap \bigcup_{j\in[l]} P^{\geq}\left(A^L_{j,*}, b^L_j\right) = \emptyset$$

and

$$D^T_{i^*}\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right) \subseteq P^{\geq}\left(\left(\begin{array}{cc} c' & (0^n)^T \end{array}\right), \tilde{c}_0\right) \subseteq P^{\geq}\left(\left(\begin{array}{cc} c' & (0^n)^T \end{array}\right), c_0\right).$$

The proof for the second case (crooked cross disjunction) is completely similar.

For (11.9)**:** If $P_I = \emptyset$, by 1, $\left(0^2 \times 0^n\right)^T(\,\cdot\,) \leq -1$ is either an essential T cut or an essential crooked cross cut for P.

If $P_I \neq \emptyset$, by Lemma 159, every essential $L_{2-\frac{1}{2},\mathbb{Q}}$ cut $c(\,\cdot\,) \geq c_0$ satisfies $c^T \perp 0^2 \times \mathbb{R}^n$, thus $c \in \left(\mathbb{R}^2 \times 0^n\right)^T$. On the other hand, $\operatorname{cl}_I\left(P + \left(0^2 \times \mathbb{R}^n\right)\right)$ is a rational polyhedron. So, every essential $L_{2-\frac{1}{2},\mathbb{Q}}$ cut for P is dominated absolutely by a finite set of essential $L_{2-\frac{1}{2},\mathbb{Q}}$ cuts for P with rational coefficients. So, we can assume $c \in \left(\mathbb{Q}^2 \times 0^n\right)^T$ and conclude the statement from 2. □

11.2.3. $m \geq 2$

Theorem 471. *Let $\emptyset \neq P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m \in \mathbb{Z}_{\geq 0}$, $n \in \mathbb{Z}_{\geq 0}$) be a rational polyhedron. Let $V' \leq \mathbb{R}^m$ be a rational subspace of codimension 2. Define $V := V' \times \mathbb{R}^n$. Then:*

1. *If $(P + V)_I = \emptyset$, there exist linearly independent $\pi^1, \pi^2 \in \mathbb{Z}^m$ and $\gamma_1, \gamma_2 \in \mathbb{Z}$ such that $\pi^1, \pi^2 \perp V'$ and*

$$P \cap \left(D^T\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right) \times \mathbb{R}^n\right) = \emptyset \vee P \cap \left(D^c\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right) \times \mathbb{R}^n\right) = \emptyset.$$

2. *If $(P+V)_I \neq \emptyset$, let $\begin{pmatrix} c' & (0^n)^T \end{pmatrix}(\cdot) \geq c_0$, where $c' \in (\mathbb{Q}^m)^T \cap V'^{\perp}$ and $c_0 \in \mathbb{R}$, be a valid inequality for P_I which is not already valid for P (the fact that we only consider inequalities of the type $\begin{pmatrix} c' & (0^n)^T \end{pmatrix}(\cdot) \geq c_0$, where $c' \perp V'$ is, by Lemma 159, no loss of generality). Then there exist $\pi^1, \pi^2 \in \mathbb{Z}^2$ (π^1, π^2 linearly indepdendent) and $\gamma_1, \gamma_2 \in \mathbb{Z}$ such that $\pi^1, \pi^2 \perp V'$ and an $i^* \in [3]$ or $i^* \in [4]$, respectively, such that*
$$\forall i \in [3] \setminus \{i^*\} : P \cap \left(D_i^T\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right) \times \mathbb{R}^n\right) = \emptyset,$$
$$D_{i^*}^T\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right) \subseteq P^{\leq}\left(\begin{pmatrix} c' & (0^n)^T \end{pmatrix}, c_0\right)$$
or
$$\forall i \in [4] \setminus \{i^*\} : P \cap \left(D_j^c\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right) \times \mathbb{R}^n\right) = \emptyset,$$
$$D_{i^*}^c\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right) \subseteq P^{\leq}\left(\begin{pmatrix} c' & (0^n)^T \end{pmatrix}, c_0\right),$$
respectively.

We thus have
$$\mathrm{cl}_{essT}(P) \cap \mathrm{cl}_{\mathrm{ess}\,CC}(P) \subseteq \mathrm{cl}_{\mathrm{ess}\,L_{2-\frac{1}{2},\mathbb{Q}}}(P). \tag{11.10}$$

Proof. Let $W \in \mathbb{Z}^{m \times 2}$ be as in the proof of Theorem 244 and let
$$S^{LP} := \left\{\begin{pmatrix} x \\ z \end{pmatrix} \in (\mathrm{proj}_{\mathbb{R}^m} P) \times \mathbb{R}^2 : z = W^T x\right\},$$
$$S := \left(S^{LP}\right) \cap \left(\mathbb{R}^m \times \mathbb{Z}^2\right)$$
(also as in the proof of Theorem 244).

For 1: By construction, $S = \emptyset$. Thus, by Theorem 470, there exist linearly independent $\pi^{1,pre}, \pi^{2,pre} \in \mathbb{Z}^2$ and $\gamma_1, \gamma_2 \in \mathbb{Z}$ such that
$$\left(\mathbb{R}^m \times D^T\left(\pi^{1,pre}, \pi^{2,pre}, \gamma_1, \gamma_2\right)\right) \cap S^{LP} = \emptyset \vee \left(\mathbb{R}^m \times D^c\left(\pi^{1,pre}, \pi^{2,pre}, \gamma_1, \gamma_2\right)\right) \cap S^{LP} = \emptyset$$
holds. We claim that
$$P \cap \left(D^T\left(W\pi^{1,pre}, W\pi^{2,pre}, \gamma_1, \gamma_2\right) \times \mathbb{R}^n\right) = \emptyset \vee P \cap \left(D^c\left(W\pi^{1,pre}, W\pi^{2,pre}, \gamma_1, \gamma_2\right) \times \mathbb{R}^n\right) = \emptyset.$$
W.l.o.g. let $\left(\mathbb{R}^m \times D^T\left(\pi^{1,pre}, \pi^{2,pre}, \gamma_1, \gamma_2\right)\right) \cap S^{LP} = \emptyset$ (the proof for $\left(\mathbb{R}^m \times D^c\left(\pi^{1,pre}, \pi^{2,pre}, \gamma_1, \gamma_2\right)\right) \cap S^{LP} = \emptyset$ works completely similarly). Assume that
$$\exists \begin{pmatrix} x^* \\ y^* \end{pmatrix} \in P \cap \left(D^T\left(W\pi^{1,pre}, W\pi^{2,pre}, \gamma_1, \gamma_2\right) \times \mathbb{R}^n\right).$$
Then it is easy to check that $W^T x^* \in D^T\left(\pi^{1,pre}, \pi^{2,pre}, \gamma_1, \gamma_2\right)$. Additionally, by definition, $x^* \in \mathrm{proj}_{\mathbb{R}^m} P$. Thus,
$$\begin{pmatrix} x^* \\ W^T x^* \end{pmatrix} \in \left(\mathbb{R}^m \times D^T\left(\pi^{1,pre}, \pi^{2,pre}, \gamma_1, \gamma_2\right)\right) \cap S^{LP}.$$
But this set is, as we have seen above, empty. So, $P \cap \left(D^T\left(W\pi^{1,pre}, W\pi^{2,pre}, \gamma_1, \gamma_2\right) \times \mathbb{R}^n\right) = \emptyset$.

Next, consider that $W\pi^{1,pre}, W\pi^{2,pre}$ are linearly independent as $\pi^{1,pre}, \pi^{2,pre}$ are linearly independent and W has linearly independent columns.

Finally, since by construction $\operatorname{im} W \perp V'$, we have $W\pi^{1,pre}, W\pi^{2,pre} \perp V'$.

For 2: We have seen in the proof of Theorem 244 that
$$\left(cW\left(W^T W\right)^{-1}\right) z \geq c_0 \tag{11.11}$$
is a valid inequality for $\binom{x}{z} \in S$. Thus, by Theorem 470, there exist linearly independent $\pi^{1,pre}, \pi^{2,pre} \in \mathbb{Z}^2$, $\gamma_1, \gamma_2 \in \mathbb{Z}$ and an $i^* \in [3]$ or $i^* \in [4]$, respectively, such that
$$\forall i \in [3] \setminus \{i^*\} : S^{LP} \cap \left(\mathbb{R}^m \times D_i^T\left(\pi^{1,pre}, \pi^{2,pre}, \gamma_1, \gamma_2\right)\right) = \emptyset$$

or

$$\forall i \in [4] \setminus \{i^*\} : S^{LP} \cap \left(\mathbb{R}^m \times D_i^c\left(\pi^{1,pre}, \pi^{2,pre}, \gamma_1, \gamma_2\right)\right) = \emptyset,$$

respectively, and

$$D_{i^*}^T\left(\pi^{1,pre}, \pi^{2,pre}, \gamma_1, \gamma_2\right) \subseteq P^{\geq}\left(cW\left(W^T W\right)^{-1}, c_0\right) \tag{11.12}$$

or

$$D_{i^*}^c\left(\pi^{1,pre}, \pi^{2,pre}, \gamma_1, \gamma_2\right) \subseteq P^{\geq}\left(cW\left(W^T W\right)^{-1}, c_0\right), \tag{11.13}$$

respectively, hold.

Consider the disjunction $D^T\left(W\pi^{1,pre}, W\pi^{2,pre}, \gamma_1, \gamma_2\right)$ or $D^c\left(W\pi^{1,pre}, W\pi^{2,pre}, \gamma_1, \gamma_2\right)$, respectively. The proof for

$$\forall i \in [3] \setminus \{i^*\} : P \cap D_i^T\left((W\pi^{1,pre}, W\pi^{2,pre}, \gamma_1, \gamma_2\right) = \emptyset$$

or

$$\forall i \in [4] \setminus \{i^*\} : P \cap D_i^c\left((W\pi^{1,pre}, W\pi^{2,pre}, \gamma_1, \gamma_2\right) = \emptyset$$

respectively, works completely similar to the proof of statement 1. So, we only have to show

$$D_{i^*}^T\left(W\pi^{1,pre}, W\pi^{2,pre}, \gamma_1, \gamma_2\right) \times \mathbb{R}^n \subseteq P^{\leq}\left(\begin{pmatrix} c' & (0^n)^T \end{pmatrix}, c_0\right) \tag{11.14}$$

or

$$D_{i^*}^c\left(W\pi^{1,pre}, W\pi^{2,pre}, \gamma_1, \gamma_2\right) \times \mathbb{R}^n \subseteq P^{\leq}\left(\begin{pmatrix} c' & (0^n)^T \end{pmatrix}, c_0\right),$$

respectively. W.l.o.g. let (11.12) hold (the argumentation for (11.13) is similar) and we show (11.14). Let

$$\begin{pmatrix} x^* \\ y^* \end{pmatrix} \in D_{i^*}^T\left(W\pi^{1,pre}, W\pi^{2,pre}, \gamma_1, \gamma_2\right) \times \mathbb{R}^n.$$

This means

$$\begin{pmatrix} x^* \\ W^T x^* \end{pmatrix} \in \mathbb{R}^m \times D_{i^*}^T\left(\pi^{1,pre}, \pi^{2,pre}, \gamma_1, \gamma_2\right).$$

Using (11.12), we obtain

$$cW\left(W^T W\right)^{-1} W^T x^* \geq c_0.$$

By equation (5.16), which we showed in the proof of Theorem 244, we have $c'W\left(W^T W\right)^{-1} W^T = c'$. Thus, using (11.11), we conclude

$$c'x^* = c'W\left(W^T W\right)^{-1} W^T x^* \geq c_0.$$

The properties that $W\pi^{1,pre}$, $W\pi^{2,pre}$ are linearly independent and $W\pi^{1,pre}, W\pi^{2,pre} \perp V'$ are shown completely similar as as in the proof of statement 1.

For (11.10)**:** Let $V = V' \times \mathbb{R}^n \leq \mathbb{R}^m \times \mathbb{R}^n$ be a fixed rational subspace of codimension 2.

If $(P + V)_I = \emptyset$, by statement 1, $\left(0^2 \times 0^n\right)^T(\,\cdot\,) \leq -1$ is either an essential T cut or an essential crooked cross cut for P.

If $(P + V)_I \neq \emptyset$, by Lemma 159, every essential $L_{2-\frac{1}{2},\mathbb{Q}}$ cut $c(\,\cdot\,) \geq c_0$ for P with respect to V satisfies $c^T \perp V' \times \mathbb{R}^n$, thus $c \in \left(\mathbb{R}^m \times 0^n\right)^T$ and $\left(c_{(1,\dots,m)}\right)^T \perp V'$. On the other hand, $\mathrm{cl}_I(P + V)$ is a rational polyhedron. So, every essential $L_{2-\frac{1}{2},\mathbb{Q}}$ cut for P with respect to V is dominated by a finite set of essential $L_{2-\frac{1}{2},\mathbb{Q}}$ cuts for P with respect to V that have rational coefficients. So, we can assume that $c \in \left(\mathbb{Q}^m \times 0^n\right)^T$ and thus conclude (11.10) from statement 2. □

11.2.4. Reverse inclusions

Theorem 472. *Let $\emptyset \neq P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m \in \mathbb{Z}_{\geq 2}$ and $n \in \mathbb{Z}_{\geq 0}$) be arbitrary. Let there exist a*

$$D^T\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right) \subseteq \mathbb{R}^m \times \mathbb{R}^n \text{ or } D^c\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right) \subseteq \mathbb{R}^m \times \mathbb{R}^n,$$

respectively ($\pi^1, \pi^2 \in \mathbb{Z}^m$ and $\gamma_1, \gamma_2 \in \mathbb{Z}$), and an $i^* \in [3]$ or $i^* \in [4]$, respectively, such that

$$\forall i \in [3] \setminus \{i^*\} : P \cap \left(D_i^T\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right) \times \mathbb{R}^n\right) = \emptyset \tag{11.15}$$

or

$$\forall i \in [4] \setminus \{i^*\} : P \cap \left(D_i^c\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right) \times \mathbb{R}^n\right) = \emptyset, \tag{11.16}$$

respectively, holds. Define $D := D_{i^*}^T\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right)$ or $D := D_{i^*}^c\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right)$, respectively. Additionally, let $V' := \left(\operatorname{lin}\left\{\pi^1, \pi^2\right\}\right)^\perp$. Then every linear inequality $c(\cdot) \leq c_0$ ($c \in (\mathbb{R}^m \times \mathbb{R}^n)^T$ and $c_0 \in \mathbb{R}$) for D (recall that by Remark 468, it suffices to consider linear inequalities for D instead of $P \cap D$) is an essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cut for P with respect to $V := V' \times \mathbb{R}^n$, where $k := \operatorname{codim} V' \leq 2$. In particular, we have

$$\operatorname{cl}_{\operatorname{ess} L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) \subseteq \operatorname{cl}_{\operatorname{ess} CC}(P) \cap \operatorname{cl}_{\operatorname{ess} T}(P).$$

Proof. W.l.o.g. we assume $D \neq \emptyset$. Note that for every non-empty atom D' of our disjunction D, we have

$$V' \leq \operatorname{lineal} D'. \tag{11.17}$$

Thus, since $D \neq \emptyset$ and $c(\cdot) \leq c_0$ is valid for D, we have

$$c^T \perp V. \tag{11.18}$$

Additionally, we have

$$\bigcup_{j=1}^{3} D_j^T\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right) \supseteq \mathbb{Z}^m \times \mathbb{R}^n \text{ or } \bigcup_{j=1}^{4} D_j^c\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right) \supseteq \mathbb{Z}^m \times \mathbb{R}^n, \tag{11.19}$$

respectively. Combining (11.17) and (11.19), we obtain

$$\bigcup_{j=1}^{3} D_j^T\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right) \supseteq (\mathbb{Z}^m \times \mathbb{R}^n) + V \text{ or } \bigcup_{j=1}^{4} D_j^c\left(\pi^1, \pi^2, \gamma_1, \gamma_2\right) \supseteq (\mathbb{Z}^m \times \mathbb{R}^n) + V, \tag{11.20}$$

respectively.

Let $z = p + v \in (P + V)_I$, where, of course, $z \in \mathbb{Z}^m \times \mathbb{R}^n$, $p \in P$ and $v \in V$. We have to show $cz \leq c_0$. Using (11.15) or (11.16), respectively, and (11.20), we obtain $p \in D$. Thus, using (11.18) and the fact that $c(\cdot) \leq c_0$ is valid for D, we conclude

$$cz = cp + cv = cp \leq c_0.$$

□

Theorem 473. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$, where $m + n \geq 2$) be a rational polyhedron. Then*

$$\operatorname{cl}_{L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) \subseteq \operatorname{cl}_{split}(P) \cap \operatorname{cl}_{\operatorname{ess} T}(P) \cap \operatorname{cl}_{\operatorname{ess} CC}(P).$$

Proof. For $m = 0$, the statement is obvious. We distinguish between the two remaning cases:

1. $m = 1$,
2. $m \geq 2$.

In the first case, we conclude from Theorem 202:

$$\operatorname{cl}_{L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \operatorname{cl}_I(P) \subseteq \operatorname{cl}_{split}(P) \cap \operatorname{cl}_{\operatorname{ess} T}(P) \cap \operatorname{cl}_{\operatorname{ess} CC}(P).$$

In the second case, we have

$$\begin{aligned}
\operatorname{cl}_{L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) &= \operatorname{cl}_{L_{1,\mathbb{Q}}}(P) \cap \operatorname{cl}_{\operatorname{ess} L_{2-\frac{1}{2},\mathbb{Q}}}(P) && \text{(by Theorem 211)}\\
&= \operatorname{cl}_{split}(P) \cap \operatorname{cl}_{\operatorname{ess} L_{2-\frac{1}{2},\mathbb{Q}}}(P) && \text{(by Theorem 409)}\\
&\subseteq \operatorname{cl}_{split}(P) \cap \operatorname{cl}_{\operatorname{ess} T}(P) \cap \operatorname{cl}_{\operatorname{ess} CC}(P). && \text{(by Theorem 472)}
\end{aligned}$$

□

11.2.5. Conclusions

Theorem 474. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m \in \mathbb{Z}_{\geq 2}$ and $n \in \mathbb{Z}_{\geq 0}$) be a rational polyhedron. Then*

$$\mathrm{cl}_{\mathrm{ess}\, L_{2-\frac{1}{2},\mathbb{Q}}}(P) = \mathrm{cl}_{\mathrm{ess}\, T}(P) \cap \mathrm{cl}_{\mathrm{ess}\, CC}(P).$$

Proof. The inclusion "$\subseteq$" holds by Theorem 472. "$\supseteq$" is a consequence of Theorem 471. □

Theorem 475. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$, where $m + n \geq 2$) be a rational polyhedron. Then*

$$\mathrm{cl}_{L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P) = \mathrm{cl}_{split}(P) \cap \mathrm{cl}_{\mathrm{ess}\, T}(P) \cap \mathrm{cl}_{\mathrm{ess}\, CC}(P).$$

Proof. The equivalence $\mathrm{cl}_{L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P)$ is shown in Theorem 212. So, it suffices to show

$$\mathrm{cl}_{L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{split}(P) \cap \mathrm{cl}_{\mathrm{ess}\, T}(P) \cap \mathrm{cl}_{\mathrm{ess}\, CC}(P). \tag{11.21}$$

The inclusion "$\subseteq$" in (11.21) holds by Theorem 473. For "$\supseteq$": for $m = 0$, the statement is obvious. So, assume $m \geq 1$. We distinguish between the two remaning cases:

1. $m = 1$,
2. $m \geq 2$.

In the first case, we have

$$\begin{aligned}
\mathrm{cl}_{L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) &= \mathrm{cl}_I(P) && \text{(by Theorem 202)}\\
&= \mathrm{cl}_{L_{1,\mathbb{Q}}}(P) && \text{(by Theorem 202)}\\
&= \mathrm{cl}_{split}(P) && \text{(by Theorem 409)}\\
&\supseteq \mathrm{cl}_{split}(P) \cap \mathrm{cl}_{\mathrm{ess}\, T}(P) \cap \mathrm{cl}_{\mathrm{ess}\, CC}(P).
\end{aligned}$$

In the second case, we have

$$\begin{aligned}
\mathrm{cl}_{L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) &= \mathrm{cl}_{L_{1,\mathbb{Q}}}(P) \cap \mathrm{cl}_{\mathrm{ess}\, L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) && \text{(by Theorem 211)}\\
&= \mathrm{cl}_{split}(P) \cap \mathrm{cl}_{\mathrm{ess}\, L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) && \text{(by Theorem 409)}\\
&\supseteq \mathrm{cl}_{split}(P) \cap \mathrm{cl}_{\mathrm{ess}\, T}(P) \cap \mathrm{cl}_{\mathrm{ess}\, CC}(P). && \text{(by Theorem 471)}
\end{aligned}$$

□

Part V.

Further results on L_k cuts/$L_{k-\frac{1}{2}}$ cuts

12. Sizes of subsets of inequalities to consider

12.1. Problem statement and outline

For the remainder of this chapter, let always $A \in \mathbb{R}^{l \times m}$, $G \in \mathbb{R}^{l \times n}$ and $b \in \mathbb{R}^l$, where $l, m, n \in \mathbb{Z}_{\geq 0}$. We saw in Theorem 394 and Theorem 269 that we have:

$$\operatorname{cl}_{L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}\left(P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)\right) = \bigcap_{S \in \mathcal{B}^{*,feas}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)} \operatorname{cl}_{L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}\left(P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}_{S,*}, b_S\right)\right),$$
$$\operatorname{cl}_{L_{1,\mathbb{Q}}}\left(P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)\right) = \bigcap_{S \in \mathcal{B}^{*}\left(\begin{pmatrix} A & G \end{pmatrix}\right)} \operatorname{cl}_{L_{1,\mathbb{Q}}}\left(P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}_{S,*}, b_S\right)\right),$$

where of course

$$\mathcal{B}^{*,feas}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right), \mathcal{B}^{*}\left(\begin{pmatrix} A & G \end{pmatrix}\right) \subseteq \binom{[l]}{\operatorname{rank}\begin{pmatrix} A & G \end{pmatrix}}.$$

Thus, for all

$$S \in \mathcal{B}^{*,feas}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right), \mathcal{B}^{*}\left(\begin{pmatrix} A & G \end{pmatrix}\right),$$

we have $|S| = \operatorname{rank}\begin{pmatrix} A & G \end{pmatrix} \leq \min(l, m+n)$. So, in particular, we have

$$\operatorname{cl}_{L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}\left(P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)\right) = \bigcap_{S \in \binom{[l]}{\min(l,m+n)}} \operatorname{cl}_{L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}\left(P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}_{S,*}, b_S\right)\right),$$
$$\operatorname{cl}_{L_{1,\mathbb{Q}}}\left(P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)\right) = \bigcap_{S \in \binom{[l]}{\min(l,m+n)}} \operatorname{cl}_{L_{1,\mathbb{Q}}}\left(P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}_{S,*}, b_S\right)\right),$$

In contrast, for general k, the identities

$$\operatorname{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}\left(P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)\right) = \bigcap_{S \in \binom{[l]}{\min(l,m+n)}} \operatorname{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}\left(P^{\leq}\left(A_{S,*}, b_S\right)\right),$$
$$\operatorname{cl}_{L_{k,\mathbb{Q}}}\left(P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)\right) = \bigcap_{S \in \binom{[l]}{\min(l,m+n)}} \operatorname{cl}_{L_{k,\mathbb{Q}}}\left(P^{\leq}\left(A_{S,*}, b_S\right)\right)$$

do *not* hold. For this, let $A \in \mathbb{Q}^{4 \times 3}$ and $b \in \mathbb{Q}^4$ be as in Theorem 274. Then

$$\begin{aligned}
\operatorname{cl}_{L_{3-\frac{1}{2},\mathbb{Q}}}\left(P^{\leq}(A,b)\right) &\subseteq \operatorname{cl}_{L_{2,\mathbb{Q}}}\left(P^{\leq}(A,b)\right) && \text{(by Theorem 199)} \\
&\subseteq \operatorname{cl}_{2BS}\left(P^{\leq}(A,b)\right) && \text{(by Corollary 278)} \\
&\subsetneq \operatorname{cl}_{BR}(A,b) && \text{(by Theorem 274)} \\
&= \bigcap_{S \in \binom{[4]}{3}} \operatorname{cl}_{I}\left(P^{\leq}\left(A_{S,*}, b_S\right)\right) \\
&\subseteq \bigcap_{S \in \binom{[4]}{3}} \operatorname{cl}_{L_{3-\frac{1}{2},\mathbb{Q}}}\left(P^{\leq}\left(A_{S,*}, b_S\right)\right) \\
&\subseteq \bigcap_{S \in \binom{[4]}{3}} \operatorname{cl}_{L_{2,\mathbb{Q}}}\left(P^{\leq}\left(A_{S,*}, b_S\right)\right). && \text{(by Theorem 199)}
\end{aligned}$$

So, only considering subsets of $m+n$ rows at the same time does in general not suffice to characterize the $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ closure or $L_{k,\mathbb{Q}}$ closure of $P^{\leq}(A,b)$. Thus, we define the following problem:

Problem/Definition 476. *Let* $P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right) \subseteq \mathbb{R}^m \times \mathbb{R}^n$ *be given and let* $k \in \{0, \ldots, m+n\}$. *What is* *the smallest* $h \in \{0, \ldots, l\}$ *such that*

$$\mathrm{cl}_{L_{k,\mathbb{Q}}} \left(P^{\leq} \left(\begin{pmatrix} A & G \end{pmatrix}, b\right)\right) = \bigcap_{S \in \binom{[l]}{h}} \mathrm{cl}_{L_{k,\mathbb{Q}}} \left(P^{\leq} \left(\begin{pmatrix} A & G \end{pmatrix}_{S,*}, b_S\right)\right) \quad or$$

$$\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}} \left(P^{\leq} \left(\begin{pmatrix} A & G \end{pmatrix}, b\right)\right) = \bigcap_{S \in \binom{[l]}{h}} \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}} \left(P^{\leq} \left(\begin{pmatrix} A & G \end{pmatrix}_{S,*}, b_S\right)\right),$$

respectively, holds? This smallest h *is referred to as* $h^*_{L_{k,\mathbb{Q}}}(A, G, b)$ *or* $h^*_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(A, G, b)$*, respectively. If* $n = 0$*, we also use the notations* $h^*_{L_{k,\mathbb{Q}}}(A, b)$ *or* $h^*_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(A, b)$*, respectively.*

In particular, we are, of course, interested whether there exists an upper bound for $h^*_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(A, G, b)$ or $h^*_{L_{k,\mathbb{Q}}}(A, G, b)$, respectively, that only depends on m and n. This is the topic of this chapter. For its outline:

- In section 12.2, we give an overview about results on Helly numbers that follows the exposition in [AW12].
- Section 12.3 is about upper bounds for $h^*_{L_{k,\mathbb{Q}}}(A, G, b)$. The central result of this section is Theorem 487. The central helper statement for its proof is Theorem 485.
- Section 12.4 is about upper bounds for $h^*_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(A, G, b)$. The central result of this section is Theorem 495. For its proof, we show two central helper statements: Theorem 490 (its proof is centrally what section 12.4.1 is about) and Theorem 493. In section 12.4.3, we generalize these results to other types of $L_{k-\frac{1}{2}}$ closures: the essential $L_{k-\frac{1}{2},\mathbb{Q}}$ closure is the topic of Theorem 496 and the $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ closure is considered in Theorem 497.
- In section 12.5, we show lower bounds for $h^*_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(A, G, b)$ and $h^*_{L_{k,\mathbb{Q}}}(A, G, b)$.
- In section 12.6, we summarize our results on the bounds for $h^*_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(A, G, b)$ and $h^*_{L_{k,\mathbb{Q}}}(A, G, b)$: Theorem 504 summarizes the upper bounds and Theorem 505 summarizes the lower bounds.

12.2. Helly numbers

Most of the material that is presented in this section (except for Corollary 483) is taken from [AW12].

Definition 477. *Let* $M \neq \emptyset$ *be a closed subset of* $\mathbb{R}^d$ ($d \in \mathbb{Z}_{\geq 0}$)*. A subset* $C \subseteq M$ *is called* M***-convex*** *if* $C = C' \cap M$*, where* $C' \subseteq \mathbb{R}^d$ *is convex.*

Definition 478. *Let* M *be as in Definition 477.* $h(M) \in \mathbb{Z}_{\geq 0} \dot{\cup} \{\infty\}$ *is called the* ***Helly number of*** M *if* $h(M)$ *is the minimal possible* h *satisfying the following condition:*

(12.1) *Every finite collection* $C_1, \ldots, C_m$ ($m \geq h$) *of* M*-convex sets, for which every subcollection of* h *elements has a non-empty intersection, necessarily satisfies* $\bigcap_{i=1}^m C_i \neq \emptyset$*.*

If h *as above does not exist, we set* $h(M) := \infty$*.*

How is this definition of the Helly number (Definition 478) related to mixed-integer linear optimization? There exists the following theorem, of which the first part tells us that for Helly numbers, one only has to consider linear half-spaces in (12.1) and the second part relates Helly numbers to optimization over M (which is mixed-integer linear optimization if one sets $M := \mathbb{Z}^m \times \mathbb{R}^n$).

Theorem 479. *([AW12, Proposition 1.2]) Let* M *be as in Definition 477 and let* $h \in \mathbb{Z}_{\geq 0}$*. Then* (12.1) *is equivalent to each of the following two conditions:*

(12.2) *For every collection of affine-linear functions* $a_1, \ldots, a_m : \mathbb{R}^d \to \mathbb{R}$ ($m \geq h$)

- *either* $\exists x \in M \,\forall j \in [m] : a_j(x) \geq 0$
- *or* $\exists i_1, \ldots, i_h \in [m] \,\nexists x \in M \,\forall j \in [h] : a_{i_j}(x) \geq 0$.

(12.3) *For every collection of affine-linear functions* $b_1, \ldots, b_m, c : \mathbb{R}^d \to \mathbb{R}$ $(m \geq h-1)$ *such that*

$$\mu := \sup \{c(x) : x \in M, b_j(x) \geq 0 \, \forall j \in [m]\} \in \mathbb{R}$$

(in other words: $\mu \neq \pm\infty$*), there exist* $i_1, \ldots, i_{h-1} \in [m]$ *such that*

$$\mu = \sup \left\{c(x) : x \in M, b_{i_j}(x) \geq 0 \, \forall j \in [h-1]\right\}.$$

For $M = \mathbb{Z}^m \times \mathbb{R}^n$, the following bounds are important:

Theorem 480. *([AW12, Theorem 1.1]) Let* $M \subseteq \mathbb{R}^d$ $(d \in \mathbb{Z}_{\geq 0})$ *be as in Definition 477 and let* $m, n \in \mathbb{Z}_{\geq 0}$*. Then*

$$h(M \times \mathbb{R}^n) \leq (n+1)\, h(M), \tag{12.4}$$
$$h(\mathbb{Z}^m \times M) \geq 2^m h(M). \tag{12.5}$$

Additionally, we have:

Theorem 481. *Let* $m, n \in \mathbb{Z}_{\geq 0}$*. Then*

$$h(\mathbb{Z}^m) = 2^m, \tag{12.6}$$
$$h(\mathbb{R}^n) = n+1. \tag{12.7}$$

A proof for (12.6) can be found in [Sch86, Theorem 16.5; p. 234]. Equation (12.7) is implied by the Farkas lemma and Carathéodory's theorem (cf. [Sch86, section 7.7]).

From (12.4)–(12.7), one obtains:

Corollary 482. *([AW12, Theorem 1.1]) Let* $m, n \in \mathbb{Z}_{\geq 0}$*. Then*

$$h(\mathbb{Z}^m \times \mathbb{R}^n) = 2^m (n+1). \tag{12.8}$$

We thus obtain the following corollary:

Corollary 483. *Let* $c(\cdot) \leq c_0$ $(c \in (\mathbb{R}^m \times \mathbb{R}^n)^T$ *and* $c_0 \in \mathbb{R})$ *be a valid inequality for* $P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)_I \subseteq \mathbb{Z}^m \times \mathbb{R}^n$*. Then there exists an* $S \subseteq [l]$ *having*

- $|S| \leq 2^m (n+1) - 1$ *if* $P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)_I \neq \emptyset$*,*
- $|S| \leq 2^m (n+1)$ *in general,*

respectively, such that $c(\cdot) \leq c_0$ *is a valid inequality for* $P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}_{S,*}, b_S\right)_I$*.*

Proof. The first case is an immediate consequence of (12.3) and (12.8). In the second case, we only have to consider the case $P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)_I = \emptyset$. In this case, one concludes from (12.2) and (12.8) that there exists an $S \in \mathcal{P}([l])$ having $|S| \leq 2^m (n+1)$ such that $P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}_{S,*}, b_S\right)_I = \emptyset$. Thus, any inequality (in particular $c(\cdot) \leq c_0$) is valid for $P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}_{S,*}, b_S\right)_I$. □

We note that in general, (12.5) cannot be improved to equality (and thus the case $M = \mathbb{R}^n$, for which equality holds, is quite likely an exception). For this, consider the following example:

Example 484. *(cf. [AW12]) Let* $M := \left\{0, 1, 2, \frac{5}{2}\right\} \subsetneq \mathbb{R}^1$*. Obviously,* $h(M) = 2$*. We show that*

$$h(M \times \mathbb{Z}) \geq 5 > 4 = 2h(M).$$

For this, consider the set

$$A := \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{5}{2} \\ 2 \end{pmatrix}\right\} =: \{a_1, \ldots, a_5\}$$

(also see Figure 12.1). Let $A_i := A \backslash \{a_i\}$ *for* $i \in [5]$*. Then every* A_i *is* $M \times \mathbb{Z}$*-convex and* $\{A_1, \ldots, A_5\}$ *does not satisfy* (12.1) *for* $h = 4$*, since for every* $i \in [5]$*, we have* $a_i \in \bigcap_{j \in [5] \backslash \{i\}} A_j$*. On the other hand,* $\bigcap_{j \in [5]} A_j = \emptyset$ *holds.*

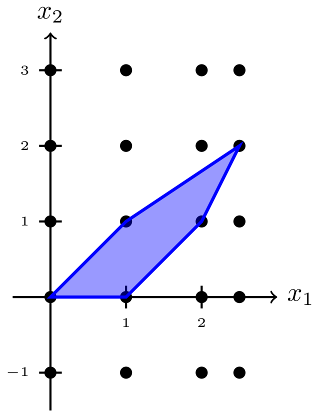


Figure 12.1.: Example of $h(M \times \mathbb{Z}^m) > 2^m h(M)$ for $m = 1$ (also cf. [AW12, Figure 1])

12.3. Upper bounds for $h^*_{L_{k,\mathbb{Q}}}(A, G, b)$

In this section, we derive a general upper bound for $h^*_{L_{k,\mathbb{Q}}}(A, G, b)$. The final result of this section can be found in Theorem 487.

Theorem 485. *Let $P := P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)$, let $k \in \{0, \ldots, m+n\}$ and let $V \leq \mathbb{R}^m \times \mathbb{R}^n$ be a rational subspace of codimension k. Let $(k-s, s)$ be the signature of the lattice $\operatorname{proj}^{\perp}_{V^{\perp}}(\mathbb{Z}^m \times \mathbb{R}^n)$. Then there exists some $p \in \{0, \ldots, l\}$ such that*

$$\begin{aligned}&\overline{\operatorname{conv}}\left(P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right) \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + V)\right)\\ =&\bigcap_{S \in \binom{[l]}{p}} \overline{\operatorname{conv}}\left(P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}_{S,*}, b_S\right) \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + V)\right)\end{aligned}$$

(which implies

$$\begin{aligned}&P \cap \overline{\operatorname{conv}}\left(P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right) \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + V)\right)\\ =&P \cap \bigcap_{S \in \binom{[l]}{p}} \overline{\operatorname{conv}}\left(P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}_{S,*}, b_S\right) \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + V)\right))\end{aligned}$$

having:

- *If $P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right) \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + V) \neq \emptyset$ (this is, in particular, satisfied if $P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)_I \neq \emptyset$), we have*
$$p \leq 2^{k-s}(m + n - (k-s) + 1) - 1.$$
- *In general, we have*
$$p \leq 2^{k-s}(m + n - (k-s) + 1).$$

Proof. Let C be as in Lemma 157 and let $c(\cdot) \leq c_0$ ($c \in (\mathbb{R}^m \times \mathbb{R}^n)^T$ and $c_0 \in \mathbb{R}$) be a valid inequality for $P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right) \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + V)$. Then, by Lemma 157, $cC(\cdot) \leq c_0$ is a valid inequality for

$$\begin{aligned}&\left(C^{-1} P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)\right) \cap \left(\mathbb{Z}^{k-s} \times \mathbb{R}^{m+n-(k-s)}\right)\\ =&P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix} C, b\right) \cap \left(\mathbb{Z}^{k-s} \times \mathbb{R}^{m+n-(k-s)}\right). \qquad \text{(by Lemma 63)}\end{aligned}$$

By Corollary 483, there exists some $S \subseteq [l]$ having

$$|S| \leq \begin{cases} 2^{k-s}(m+n-(k-s)+1) - 1 & \text{if } P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right) \cap \left(\mathbb{Z}^{k-s} \times \mathbb{R}^{m+n-(k-s)}\right) \neq \emptyset, \\ 2^{k-s}(m+n-(k-s)+1) & \text{in general,} \end{cases}$$

respectively, such that $cC(\cdot) \le c_0$ is a valid inequality for

$$P^{\le}\left(\left(\begin{pmatrix} A & G \end{pmatrix} C\right)_{S,*}, b_S\right) \cap \left(\mathbb{Z}^{k-s} \times \mathbb{R}^{m+n-(k-s)}\right) = P^{\le}\left(\begin{pmatrix} A & G \end{pmatrix}_{S,*} C, b_S\right) \cap \left(\mathbb{Z}^{k-s} \times \mathbb{R}^{m+n-(k-s)}\right).$$

This means (again because of Lemma 157) that $c(\cdot) \le c_0$ is a valid inequality for

$$P^{\le}\left(\begin{pmatrix} A & G \end{pmatrix}_{S,*}, b_S\right) \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + V).$$

□

Proposition 486. *Let $d \in \mathbb{Z}_{\ge 0}$, $k \in \{0, \dots, d\}$ and $k' \in \{0, \dots, k\}$. Then*

$$2^{k'}(d - k' + 1) \le 2^k (d - k + 1).$$

Proof. We have

$$\begin{aligned} 2^k(d-k+1) - 2^{k'}(d-k'+1) &= \sum_{i=0}^{k-k'-1} \left(2^{k-i}\left(d-(k-i)+1\right) - 2^{k-(i+1)} \underbrace{(d-(k-(i+1))+1)}_{=(d-(k-i))+2}\right) \\ &= \sum_{i=0}^{k-k'-1} \underbrace{2^{k-(i+1)}}_{\ge 0} \underbrace{(d - \underbrace{(k-i)}_{\le k})}_{\ge 0} \\ &\ge 0. \end{aligned}$$

□

Theorem 487. *We have for $k \in \{0, \dots, m+n\}$:*

$$h^*_{L_{k,\mathbb{Q}}}(A,G,b) \le \begin{cases} 2^{\min(k,m)}(m+n-\min(k,m)+1) - 1 & \text{if } P^{\le}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)_I \ne \emptyset, \\ 2^{\min(k,m)}(m+n-\min(k,m)+1) & \text{in general.} \end{cases}$$

Remark 488. *Because of Theorem 269, for all $l, m, n \in \mathbb{Z}_{\ge 0}$, the statement*

$$h^*_{L_{1,\mathbb{Q}}}(A,G,b) \le m+n$$

holds for arbitrary $A \in \mathbb{R}^{l \times m}$, $G \in \mathbb{R}^{l \times n}$ and $b \in \mathbb{R}^l$. From Theorem 487, we only get the weaker bound

$$h^*_{L_{1,\mathbb{Q}}}(A,G,b) \le \begin{cases} 2(m+n) - 1 & \text{if } P^{\le}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right) \ne \emptyset, \\ 2(m+n) & \text{in general.} \end{cases}$$

Proof. (Theorem 487) Let V be as in Theorem 485. Consider the signature $(k-s, s)$ of $\operatorname{proj}^{\perp}_{V^{\perp}}(\mathbb{Z}^m \times \mathbb{R}^n)$. We know from Theorem 100 that $0 \le k - s \le \min(k,m)$. Thus,

$$\begin{aligned} & h^*_{L_{k,\mathbb{Q}}}(A,G,b) \\ \le & \max_{i \in \{0,\dots,\min(k,m)\}} \left\{ \begin{cases} 2^i(m+n-i+1) - 1 & \text{if } P^{\le}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)_I \ne \emptyset, \\ 2^i(m+n-i+1) & \text{in general} \end{cases} \right\} && \text{(by Theorem 485)} \\ \le & \begin{cases} 2^{\min(k,m)}(m+n-\min(k,m)+1) - 1 & \text{if } P^{\le}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)_I \ne \emptyset, \\ 2^{\min(k,m)}(m+n-\min(k,m)+1) & \text{in general,} \end{cases} && \text{(by Proposition 486)} \end{aligned}$$

where in Proposition 486, we set

- $d := m + n$,
- $k := \min(k,m)$,
- $k' := i$.

□

12.4. Upper bounds for $h^*_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(A, G, b)$

12.4.1. Considerations for the case $P_I \neq \emptyset$

For the remainder of this section, we introduce the following notations:

Definition 489. *Let $f : \mathbb{R}^d \to \mathbb{R}$ be an affine-linear function, where $d \in \mathbb{Z}_{\geq 0}$. We set*

$$P^{\geq 0}(f) := \left\{x \in \mathbb{R}^d : f(x) \geq 0\right\},$$
$$P^{=0}(f) := \left\{x \in \mathbb{R}^d : f(x) = 0\right\}.$$

The following theorem generalizes ideas that are used in [AW12] to prove (12.4) in Theorem 480:

Theorem 490. *Let $P := \left\{x \in \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} : f_1(x) \geq 0, \ldots, f_l(x) \geq 0\right\} \neq \emptyset$ ($k_1, k_2, l \in \mathbb{Z}_{\geq 0}$), where the f_i ($i \in [l]$) are affine-linear functions. Let $M \subseteq \mathbb{R}^{k_2}$ be as in Definition 477, let $P \cap \left(M \times \mathbb{R}^{k_2}\right) \neq \emptyset$, but let there exist some affine-linear function $f_{l+1} : \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \to \mathbb{R}$ which is constant on $0^{k_1} \times \mathbb{R}^{k_2}$ such that*

$$\left(P \cap P^{\geq 0}(f_{l+1})\right) \cap \left(M \times \mathbb{R}^{k_2}\right) = \emptyset.$$

Then there exists some $S \subseteq [l]$ having $|S| \leq (h(M) - 1)(k_2 + 1)$ such that

$$\left\{x \in \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} : \forall i \in S : f_i(x) \geq 0 \wedge f_{l+1}(x) \geq 0\right\} \cap \left(M \times \mathbb{R}^{k_2}\right) = \emptyset.$$

W.l.o.g. we assume that $h(M) < \infty$ for the following proofs, since otherwise Theorem 490 holds trivially. The proof of Theorem 490 is done in multiple steps, similarly to the approach that is used in [AW12] to show (12.4) in Theorem 480:

- In Proposition 491, we show Theorem 480 in the case where $P \cap P^{\geq 0}(f_{l+1})$ is full-dimensional and bounded.
- In Proposition 492, we generalize this to the case where $P \cap P^{\geq 0}(f_{l+1})$ is bounded, but not necessarily full-dimensional.
- After these preparations, we finally show Theorem 490 in general.

Proposition 491. *Theorem 490 holds if $P \cap P^{\geq 0}(f_{l+1})$ is full-dimensional and bounded.*

Proof. Let $P' := \operatorname{proj}_{\mathbb{R}^{k_1}} P$. Then

$$P' = \left\{x' \in \mathbb{R}^{k_1} : f'_1(x') \geq 0, \ldots, f'_{l'}(x') \geq 0\right\},$$

where all $f'_{i'}$ ($i' \in [l']$) are facet-defining. In other words: for all $i' \in [l']$,

$$F'_{i'} := P^{=0}(f'_{i'}) \cap P'$$

is a facet of P'. Let

$$f'_{l'+1} : \mathbb{R}^{k_1} \to \mathbb{R} :$$
$$x' \mapsto f_{l+1}\left(\begin{pmatrix} x' \\ 0^{k_2} \end{pmatrix}\right).$$

One can easily check (since f_{l+1} is constant on $0^{k_1} \times \mathbb{R}^{k_2}$) that

$$\operatorname{proj}_{\mathbb{R}^{k_1}}\left(P \cap P^{\geq 0}(f_{l+1})\right) = P' \cap P^{\geq 0}\left(f'_{l'+1}\right).$$

Since $P \cap P^{\geq 0}(f_{l+1})$ is full-dimensional, so is $P' \cap P^{\geq 0}\left(f'_{l'+1}\right)$.

So, for $i' \in [l']$, we have $\dim F'_{i'} = k_1 - 1$ and thus $G_{i'} := \left((\operatorname{proj}_{\mathbb{R}^{k_1}})^{-1} F'_{i'}\right) \cap P$ is a face of P having

$$\dim G_{i'} \geq k_1 - 1. \tag{12.9}$$

Also $G_{l'+1} := P \cap \left(\left(\mathrm{proj}_{\mathbb{R}^{k_1}}\right)^{-1} F'_{l'+1}\right)$ is a face of P, but (because f_{l+1} is constant on $\left\{0^{k_1}\right\} \times \mathbb{R}^{k_2}$) with

$$G_{l'+1} = P^{=0}\left(f_{l+1}\right) \cap P. \tag{12.10}$$

By assumption, $P \cap P^{\geq 0}\left(f_{l+1}\right)$ is full-dimensional and we have

- $P \cap \left(M \times \mathbb{R}^{k_2}\right) \neq \emptyset$ and
- $P \cap P^{\geq 0}\left(f_{l+1}\right) \cap \left(M \times \mathbb{R}^{k_2}\right) = \emptyset$.

Thus, f_{l+1} is facet-defining for $P \cap P^{\geq 0}\left(f_{l+1}\right)$. So, one concludes from (12.10) together with the full-dimensionality of $P \cap P^{\geq 0}\left(f_{l+1}\right)$:

$$\dim G_{l'+1} = k_1 + k_2 - 1. \tag{12.11}$$

For $i' \in [l'+1]$, consider the cone $N_{i'}$ of affine-linear functions $\mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \to \mathbb{R}$ vanishing on $G_{i'}$ that are non-negative on P. We have for all $i' \in [l'+1]$:

$$N_{i'} = \mathrm{cone}\left\{f_i : i \in [l], f_i \text{ vanishes on some facet of } P \cap P^{\geq 0}\left(f_{l+1}\right) \text{ containing } G_{i'}\right\}$$

(we remark that $N_{l'+1} = \mathrm{cone}\left\{f_{l+1}\right\}$). Thus, using (12.9) and (12.11), respectively, we obtain

$$\begin{aligned} \forall i' \in [l'] : \dim N_{i'} &= (k_1 + k_2) - \dim G_{i'} \leq (k_1 + k_2) - (k_1 - 1) = k_2 + 1, \\ \dim N_{l'+1} &= (k_1 + k_2) - \dim G_{l'+1} = (k_1 + k_2) - (k_1 + k_2 - 1) = 1. \end{aligned} \tag{12.12}$$

For $i' \in [l'+1]$, let

$$\begin{aligned} \hat{f}'_{i'} : \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} &\to \mathbb{R} : \\ \begin{pmatrix} x' \\ x'' \end{pmatrix} &\mapsto f'_{i'}\left(x'\right). \end{aligned}$$

Since for all $i' \in [l']$, we have $\hat{f}'_{i'} \in N_{i'}$, we obtain

$$\hat{f}'_{i'} = \sum_{i \in I_{i'}} \lambda_{i,i'} f_i \ \forall i' \in [l'+1],$$

where

$$\begin{aligned} \forall i' \in [l'] : \quad & I_{i'} \subseteq [l], \\ \forall i' \in [l'], i \in I_{i'} : & \lambda_{i,i'} \geq 0. \end{aligned}$$

Using (12.12), we thus get from Carathéodory's theorem for convex cones (for example cf. [Sch86, section 7.7]) that we can assume

$$\forall i' \in [l'] : \left|I_{i'}\right| \leq \dim N_{i'} \leq k_2 + 1. \tag{12.13}$$

Additionally, we surely have $\hat{f}'_{l'+1} \in \mathrm{cone}\left\{f_{l+1}\right\} \setminus \{0\}$. Since $P' \cap P^{\geq 0}\left(f'_{l'+1}\right) \cap M = \emptyset$, there exists an $S'_{pre} \subseteq [l'+1]$ such that

$$\left|S'_{pre}\right| \leq h(M) \tag{12.14}$$

and

$$\left\{x' \in \mathbb{R}^{k_1} : f'_{s'}\left(x'\right) \geq 0 \ \forall s' \in S'_{pre}\right\} \cap M = \emptyset. \tag{12.15}$$

Since $P' \cap M \neq \emptyset$, we have $l'+1 \in S'_{pre}$. Thus, by (12.14) or (12.15), respectively, there exists an $S' := S'_{pre} \setminus \{l'+1\}$ having

$$\left|S'\right| \leq h(M) - 1 \tag{12.16}$$

and

$$\left\{x' \in \mathbb{R}^{k_1} : f'_{s'}\left(x'\right) \geq 0 \ \forall s' \in S'\right\} \cap P^{\geq 0}\left(f'_{l'+1}\right) \cap M = \emptyset. \tag{12.17}$$

Because of (12.17), we have

$$\left\{x \in \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} : f_i(x) \geq 0 \ \forall i \in \bigcup_{i' \in S'} I_{i'}\right\} \cap P^{\geq 0}\left(f_{l+1}\right) \cap M = \emptyset.$$

Thus, using (12.13) and (12.16), we conclude

$$\left|\bigcup_{i'\in S'} I_{i'}\right| \leq \sum_{i'\in S'} |I_{i'}| \leq (h(M)-1)(k_2+1).$$

□

Proposition 492. *Theorem 490 holds if $P \cap P^{\geq 0}(f_{l+1})$ is bounded.*

Proof. Since M is closed and P is compact and (by assumption) non-empty, there exists an $\epsilon > 0$ such that

$$P_\epsilon := \left\{x \in \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} : f_1(x) + \epsilon \geq 0, \ldots, f_l(x) + \epsilon \geq 0\right\} \cap P^{\geq 0}(f_{l+1} + \epsilon)$$

is full-dimensional and satisfies $P_\epsilon \cap M = \emptyset$. Because of Proposition 491, there exists some $S \subseteq [l]$ such that $|S| \leq (h(M)-1)(k_2+1)$ and

$$\left\{x \in M \times \mathbb{R}^{k_2} : f_s(x) + \epsilon \geq 0 \ \forall s \in S, f_{l+1} + \epsilon \geq 0\right\} = \emptyset.$$

From this, we obtain the statement of Theorem 490, since for all $i \in [l+1]$, we have

$$f_i(x) + \epsilon \geq 0 \Rightarrow f_i(x) \geq 0.$$

□

Proof. (Theorem 490) For $t \in \mathbb{Z}_{\geq 1}$, let

$$Q_t := \{x \in P, x_1 + t \geq 0, -x_1 + t \geq 0, \ldots, x_{k_1+k_2} + t \geq 0, -x_{k_1+k_2} + t \geq 0\}.$$

Because of Proposition 492, there exist I_t, J_t^+, J_t^- such that

$$|I_t| + \left|J_t^+\right| + \left|J_t^-\right| \leq (h(M)-1)(k_2+1) \tag{12.18}$$

and

$$\begin{aligned} \{x \in M \times \mathbb{R}^{k_2} : & \\ x_j + t \geq 0 \qquad & \forall j \in J_t^+, \\ -x_j + t \geq 0 \qquad & \forall j \in J_t^-, \\ f_i(x) \geq 0 \qquad & \forall i \in I_t, \\ f_{l+1}(x) \geq 0\} = \emptyset & \end{aligned} \tag{12.19}$$

holds. Since $P \cap \left(M \times \mathbb{R}^{k_2}\right) \neq \emptyset$ and $\left(P \cap P^{\geq 0}(f_{l+1})\right) \cap \left(M \times \mathbb{R}^{k_2}\right) = \emptyset$, there exists a $t^* \in \mathbb{Z}_{\geq 1}$ such that $l+1 \in I_t$ for all $t \geq t^*$. Next, note that, because there is only a finite amount of possibilities of sets for I_t, J_t^+ and J_t^-, there exist I_*, J_*^+ and J_*^- which satisfy (12.18) and (12.19) for an infinite number of $t \in \mathbb{Z}_{\geq 1}$ as I_t, J_t^+ and J_t^-. This means that we can assume $J_*^+, J_*^- = \emptyset$. Thus, we have

$$|I_*| \leq (h(M)-1)(k_2+1) \qquad \text{(by (12.18))}$$

and

$$\left\{x \in M \times \mathbb{R}^{k_2} : f_i(x) \geq 0 \ \forall i \in I_*, f_{l+1}(x) \geq 0\right\} = \emptyset. \qquad \text{(by (12.19) and } J^+, J^- = \emptyset)$$

□

12.4.2. Proof of the bounds

We now have all the tools available to prove an upper bound for $h^*_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(A, G, b)$. The final result is formulated in Theorem 495.

Theorem 493. *Let $P := P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)$, let $V \leq \mathbb{R}^m \times \mathbb{R}^n$ be a rational subspace of codimension $k \in \{0, \ldots, m+n\}$ and let $(k-s, s)$ be the signature of the mixed lattice $\operatorname{proj}^{\perp}_{V^\perp}(\mathbb{Z}^m \times \mathbb{R}^n)$. Then there* *exists some $p \in \{0,\dots,l\}$ having*

$$\overline{\operatorname{conv}}\left(\left(P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right) + V\right)_I\right) = \bigcap_{S \in \binom{[l]}{p}} \overline{\operatorname{conv}}\left(\left(P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}_{S,*}, b_S\right) + V\right)_I\right)$$

(which implies

$$P \cap \overline{\operatorname{conv}}\left(\left(P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right) + V\right)_I\right) = P \cap \bigcap_{S \in \binom{[l]}{p}} \overline{\operatorname{conv}}\left(\left(P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}_{S,*}, b_S\right) + V\right)_I\right))$$

that satisfies the following bound:

- *If $(P+V)_I \neq \emptyset$ (this is in particular satisfied if $P_I \neq \emptyset$), then*

$$p \leq \begin{cases} \left(2^k - 1\right)(m+n-k+1) & \text{if } s = 0 \text{ and } k \leq m, \\ 2^{k-s}(m+n-(k-s)+1) - 1 & \text{in general.} \end{cases}$$

- *In general*

$$p \leq 2^{k-s}(m+n-(k-s)+1).$$

Proof. Let C be as in Lemma 157 and let $c(\cdot) \leq c_0$ ($c \in (\mathbb{R}^m \times \mathbb{R}^n)^T$ and $c_0 \in \mathbb{R}$) be a valid inequality for $(P+V)_I$. W.l.o.g. (see Lemma 159) we can assume $c^T \perp V$. We first prove that there exists an $S \in \mathcal{P}([l])$ having

$$|S| \leq \begin{cases} 2^{k-s}(m+n-(k-s)+1) - 1 & \text{if } (P+V)_I \neq \emptyset, \\ 2^{k-s}(m+n-(k-s)+1) & \text{in general} \end{cases} \tag{12.20}$$

such that $c(\cdot) \leq c_0$ is a valid inequality for $\left(P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}_{S,*}, b_S\right) + V\right)_I$.

It is a consequence of Theorem 485 that there exists some $S \subseteq [l]$ having

$$|S| \leq \begin{cases} 2^{k-s}(m+n-(k-s)+1) - 1 & \text{if } P \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + V) \neq \emptyset, \\ 2^{k-s}(m+n-(k-s)+1) & \text{in general} \end{cases}$$

such that $c(\cdot) \leq c_0$ is a valid inequality for

$$P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}_{S,*}, b_S\right) \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + V).$$

Next, note that, by Theorem 213 (because $c^T \perp V$), $c(\cdot) \leq c_0$ is a valid inequality for

$$\left(P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}_{S,*}, b_S\right) + V\right)_I.$$

Finally, by additionally observing that

$$P \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + V) \neq \emptyset \Leftrightarrow (P+V)_I \neq \emptyset,$$

we conclude (12.20).

Now for the bound if $s = 0$ and $k \leq m$. Let $c(\cdot) \leq c_0$ be an $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cut for P with respect to V. This is equivalent to

$$\forall \epsilon \in \mathbb{R}_{>0} : \left(P^{\leq}(A,b) + V\right) \cap P^{\geq}(c, c_0 + \epsilon) \cap (\mathbb{Z}^m \times \mathbb{R}^n) = \emptyset.$$

Let C be as in Lemma 160. Then, by this lemma, $c(\cdot) \leq c_0$ is valid for $(P+V)_I$ if and only if $cC(\cdot) \leq c_0$ is a valid inequality for

$$\begin{aligned} &\left(\left(C^{-1}P\right) + \left(0^k \times \mathbb{R}^{m+n-k}\right)\right) \cap \left(\mathbb{Z}^k \times \mathbb{R}^{m+n-k}\right) \\ =& \left(P^{\leq}(AC, b) + \left(0^k \times \mathbb{R}^{m+n-k}\right)\right) \cap \left(\mathbb{Z}^k \times \mathbb{R}^{m+n-k}\right) \qquad \text{(by Lemma 63)} \end{aligned}$$

and we then have $cC \in \left(\mathbb{R}^k \times 0^{m+n-k}\right)^T$. But this is the case if and only if for every $\epsilon > 0$, we have

$$P^{\leq}(AC, b) \cap P^{\geq}(cC, c_0 + \epsilon) = \emptyset.$$

Let us check the conditions of Theorem 490:

1. $P^{\leq}(AC, b) \cap (\mathbb{Z}^k \times \mathbb{R}^{m+n-k}) \neq \emptyset$,
2. $cC(\cdot)$ is constant on $0^k \times \mathbb{R}^{m+n-k}$.

Concerning 1:

$$\begin{aligned} & P^{\leq}(AC, b) \cap (\mathbb{Z}^k \times \mathbb{R}^{m+n-k}) \neq \emptyset \\ \Leftrightarrow & (0^m \times 0^n)^T (\cdot) \leq -1 \text{ is not valid for } P^{\leq}(AC, b) \cap (\mathbb{Z}^k \times \mathbb{R}^{m+n-k}) \\ \Leftrightarrow & (0^m \times 0^n)^T (\cdot) = (0^m \times 0^n)^T C^{-1} (\cdot) \leq c_0 \text{ is not valid for } (P+V)_I \qquad \text{(by Lemma 160)} \\ \Leftrightarrow & (P+V)_I \neq \emptyset. \end{aligned}$$

Concerning 2: In Lemma 160, we defined

$$C := \begin{pmatrix} w^1 & \cdots & w^k & v^1 & \cdots & v^{m+n-k} \end{pmatrix},$$

where $\{v^1, \ldots, v^{m+n-k}\}$ is a basis of V. Because of Lemma 159, we have $c^T \perp V$. From this, the statement immediately follows.

So, by Theorem 490, there exists some $S \subseteq [l]$ having

$$|S| \leq (\underbrace{h(\mathbb{Z}^k)}_{=2^k \text{ (by (12.6))}} - 1)((m+n-k)+1)$$

such that

$$\begin{aligned} \emptyset &= P^{\leq}\left((AC)_{S,*}, b_S\right) \cap P^{\geq}(cC, c_0 + \epsilon) \cap (\mathbb{Z}^k \times \mathbb{R}^{m+n-k}) \\ &= P^{\leq}(A_{S,*}C, b_S) \cap P^{\geq}(cC, c_0 + \epsilon) \cap (\mathbb{Z}^k \times \mathbb{R}^{m+n-k}). \end{aligned}$$

Again, by Lemma 160, this is equivalent to

$$\left(P^{\leq}(A_{S,*}, b_S) + V\right) \cap P^{\geq}(c, c_0 + \epsilon) \cap (\mathbb{Z}^m \times \mathbb{R}^n) = \emptyset.$$

Finally, we conclude from

$$\forall \epsilon \in \mathbb{R}_{>0} \exists S \subseteq [l] : |S| \leq (2^k - 1)(m+n-k+1) \wedge \left(P^{\leq}(A_{S,*}, b_S) + V\right) \cap P^{\geq}(c, c_0 + \epsilon) \cap (\mathbb{Z}^m \times \mathbb{R}^n) = \emptyset$$

that

$$\exists S \subseteq [l] : |S| \leq (2^k - 1)(m+n-k+1) \wedge \left(P^{\leq}(A_{S,*}, b_S) + V\right) \cap P^{\geq}(c, c_0) \cap (\mathbb{Z}^m \times \mathbb{R}^n) = \emptyset.$$

□

Proposition 494. *Let $m, n \in \mathbb{Z}_{\geq 0}$ and let $k \in \{0, \ldots, m+n\}$. Then*

$$(2^k - 1)(m+n-k+1) \leq 2^k (m+n-k+1) - 1, \qquad (12.21)$$

$$(2^k - 1)(m+n-k+1) \geq 2^{k-1}(m+n-(k-1)+1) - 1. \qquad (12.22)$$

Proof.

For (12.21):

$$\left(2^k(m+n-k+1) - 1\right) - (2^k - 1)(m+n-k+1) = m+n-k \geq 0.$$

For (12.22):

$$(2^k - 1)(m+n-k+1) - \left(2^{k-1}(m+n-(k-1)+1) - 1\right) = (2^{k-1} - 1)(m+n-k) \geq 0.$$

□

Theorem 495. *We have for $k \in \{0,\dots,m+n\}$:*

$$h^*_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(A,G,b) \le \begin{cases} \left(2^k-1\right)(m+n-k+1) & \text{if } P^{\le}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)_I \neq \emptyset \wedge k \in \{0,\dots,m\}, \\ 2^{\min(k,m)}(m+n-\min(k,m)+1)-1 & \text{if } P^{\le}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)_I \neq \emptyset, \\ 2^{\min(k,m)}(m+n-\min(k,m)+1) & \text{in general.} \end{cases}$$

Proof. Consider the signature $(k-s,s)$ in Theorem 493. From Theorem 100, we obtain $0 \le k-s \le \min(k,m)$ and $0 \le s \le \min(k,n)$.

For the bound if $P_I \neq \emptyset$ and $k \in \{0,\dots,m\}$: By case distinction between $s=0$ and $1 \le s \le k$, we obtain from Theorem 493:

$$\begin{aligned} h^*_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(A,G,b) \le \max\Big(&\underbrace{\min\left(\left(2^k-1\right)(m+n-k+1), 2^k(m+n-k+1)-1\right)}_{=(2^k-1)(m+n-k+1) \text{ (by (12.21))}}, \\ &\underbrace{\max_{s\in\{1,\dots,k\}}\left\{2^{k-s}(m+n-(k-s)+1)-1\right\}\Big)}_{=2^{k-1}(m+n-(k-1)+1)-1 \text{ (by Proposition 486)}} && \text{(by Theorem 493)} \\ = \max&\left(\left(2^k-1\right)(m+n-k+1), 2^{k-1}(m+n-(k-1)+1)-1\right) \\ = &\left(2^k-1\right)(m+n-k+1). && \text{(by (12.22))} \end{aligned}$$

For the two other bounds: By computing the maximum for $i := k-s \in \{0,\dots,\min(k,m)\}$, we obtain:

$$\begin{aligned} &h^*_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(A,G,b) \\ &\le \max_{i\in\{0,\dots,\min(k,m)\}}\left\{\begin{cases} 2^i(m+n-i+1)-1 & \text{if } P^{\le}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)_I \neq \emptyset, \\ 2^i(m+n-i+1) & \text{in general} \end{cases}\right\} && \text{(by Theorem 493)} \\ &= \begin{cases} 2^{\min(k,m)}(m+n-\min(k,m)+1)-1 & \text{if } P^{\le}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)_I \neq \emptyset, \\ 2^{\min(k,m)}(m+n-\min(k,m)+1) & \text{in general.} \end{cases} && \text{(by Proposition 486)} \end{aligned}$$

□

12.4.3. Generalizations of the upper bounds for $h^*_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(A,G,b)$

In a completely similar way as in Problem/Definition 476, we can also define $h^*_{(\cdot)}(A,G,b)$ with respect to other types of cutting planes, in particular

- $h^*_{\text{ess}\,L_{k-\frac{1}{2},\mathbb{Q}}}(A,G,b)$: with respect to essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts and
- $h^*_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(A,G,b)$: with respect to $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cuts.

In this section, we show two small results for these two generalizations of $h^*_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(A,G,b)$.

Theorem 496. *Let $k \in \{0,\dots,m\}$ (otherwise essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts are not defined). Then*

$$h^*_{\text{ess}\,L_{k-\frac{1}{2},\mathbb{Q}}}(A,G,b) \le \begin{cases} \left(2^k-1\right)(m+n-k+1) & \text{if } P^{\le}(A,b)_I \neq \emptyset, \\ 2^k(m+n-k+1) & \text{in general.} \end{cases}$$

Proof. The statement is an immediate consequence of Theorem 493 by considering that for essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts, we only consider rational subspaces of the form $V' \times \mathbb{R}^n$. Thus, we have $s=0$ in the signature $(k-s,s)$ of $\operatorname{proj}^{\perp}_{V^{\perp}}(\mathbb{Z}^m \times \mathbb{R}^n)$. □

Theorem 497. *Let A, G and b be rational and let $k \in \{0,\dots,m+n\}$. Then*

$$h^*_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(A,G,b) = h^*_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(A,G,b).$$

Proof. For brevity, let $h := h^*_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(A,G,b) < \infty$ (the "$< \infty$" is a consequence of Theorem 495). Using Theorem 193 and the definition of $h^*_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(A,G,b)$ (Problem/Definition 476), we obtain

$$\begin{aligned}\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}\left(P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)\right) &= \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}\left(P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)\right) \\ &= \bigcap_{S\in\binom{[l]}{h}} \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}\left(P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}_{S,*}, b_S\right)\right) \\ &= \bigcap_{S\in\binom{[l]}{h}} \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}\left(P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}_{S,*}, b_S\right)\right),\end{aligned}$$

and if $h \geq 1$, we have

$$\begin{aligned}\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}\left(P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)\right) &= \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}\left(P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)\right) \\ &\subsetneq \bigcap_{S\in\binom{[l]}{h-1}} \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}\left(P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}_{S,*}, b_S\right)\right) \\ &= \bigcap_{S\in\binom{[l]}{h-1}} \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}\left(P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}_{S,*}, b_S\right)\right).\end{aligned}$$

Thus,

$$h^*_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(A,G,b) = h^*_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(A,G,b) = h.$$

□

12.5. Lower bounds for $h^*_{L_{k,\mathbb{Q}}}(A,G,b)$ and $h^*_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(A,G,b)$

After we have considered upper bounds for $h^*_{L_{k,\mathbb{Q}}}(A,G,b)$ and $h^*_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(A,G,b)$ in section 12.3 and section 12.4, we now consider lower bounds for them.

12.5.1. Remarks concerning lower bounds for $h^*_{L_{m+n,\mathbb{Q}}}(A,G,b)$ and $h^*_{L_{m+n-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(A,G,b)$

In this section, we have a look at the case $k = m+n$ for lower bounds for $h^*_{L_{k,\mathbb{Q}}}(A,G,b)$ and $h^*_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(A,G,b)$. At first sight, it seems to be reasonable for finding lower bounds for $h_{L_{k,\mathbb{Q}}}(A,G,b)$ and $h_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(A,G,b)$ to just consider an instance of (12.3) where we need $h(M)-1$ inequalities such that

$$\mu := \sup\{c(x) : x \in M, b_j(x) \geq 0 \,\forall j \in [h(M)-1]\} \in \mathbb{R},$$

but for all $S \subsetneq [h(M)-1]$, we have

$$\mu < \sup\{c(x) : x \in M, b_j(x) \geq 0 \,\forall j \in S\} \in \mathbb{R}.$$

In this section, we write down why this strategy does in general not suffice to find a best possible lower bound for $h_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(A,G,b)$ and $h_{L_{k,\mathbb{Q}}}(A,G,b)$. For this, we consider the following example:

Example 498. *(See Figure 12.2) Let*

$$P^{498} := P^{\leq}\left(A^{498}, b^{498}\right) = P^{\leq}\left(\begin{pmatrix} -1 & 1 \\ 1 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{3}{2} \end{pmatrix}\right)$$

(see Figure 12.2a). Then $\max\left\{x_1+x_2 : x \in P^{498}_I\right\} = 0$, *but for every* $S \subsetneq [3] := \left[h\left(\mathbb{Z}^2\right)-1\right]$, *we have*

$$\max\left\{x_1+x_2 : x \in P^{\leq}\left(A^{498}_{S,*}, b^{498}_S\right)_I\right\} \geq 1.$$

On the other hand, $h_{L_{2-\frac{1}{2},\mathbb{Q}}}\left(A^{498}, b^{498}\right) = h_{L_{2,\mathbb{Q}}}\left(A^{498}, b^{498}\right) = 2$. *For this, consider that*

$$\begin{aligned}\mathrm{cl}_{L_{2-\frac{1}{2},\mathbb{Q}}}\left(P^{498}\right) &= \mathrm{cl}_{L_{2,\mathbb{Q}}}\left(P^{498}\right) = \mathrm{cl}_I\left(P^{498}\right)\\ &= \left\{x \in \mathbb{R}^2 : x_1 = x_2, x_1 + x_2 \leq 0\right\} = \left\{x \in \mathbb{R}^2 : x_1 = x_2, x_1 \leq 0, x_2 \leq 0\right\}\end{aligned}$$

(see Figure 12.2b). On the other hand:

- $$\begin{aligned}\mathrm{cl}_{L_{2-\frac{1}{2},\mathbb{Q}}}\left(P^{\leq}\left(A^{498}_{\{1,2\},*}, b^{498}_{\{1,2\}}\right)\right) &= \mathrm{cl}_{L_{2,\mathbb{Q}}}\left(P^{\leq}\left(A^{498}_{\{1,2\},*}, b^{498}_{\{1,2\}}\right)\right)\\ &= \mathrm{cl}_I\left(P^{\leq}\left(A^{498}_{\{1,2\},*}, b^{498}_{\{1,2\}}\right)\right)\\ &= \left\{x \in \mathbb{R}^2 : x_1 = x_2\right\}\end{aligned}$$

  (see Figure 12.2c and Figure 12.2d),

- $$\begin{aligned}\mathrm{cl}_{L_{2-\frac{1}{2},\mathbb{Q}}}\left(P^{\leq}\left(A^{498}_{\{1,3\},*}, b^{498}_{\{1,3\}}\right)\right) &= \mathrm{cl}_{L_{2,\mathbb{Q}}}\left(P^{\leq}\left(A^{498}_{\{1,3\},*}, b^{498}_{\{1,3\}}\right)\right)\\ &= \mathrm{cl}_I\left(P^{\leq}\left(A^{498}_{\{1,3\},*}, b^{498}_{\{1,3\}}\right)\right)\\ &\subseteq P^{\leq}\left(\begin{pmatrix} 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \end{pmatrix}\right)\end{aligned}$$

  (see Figure 12.2e and Figure 12.2f) and

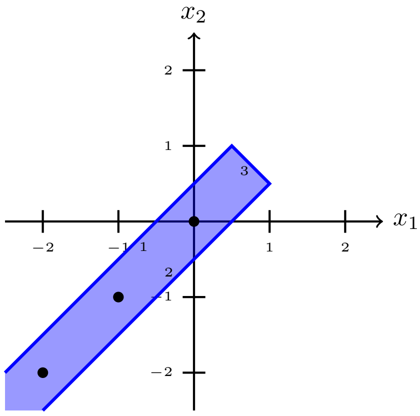
- $$\begin{aligned}\mathrm{cl}_{L_{2-\frac{1}{2},\mathbb{Q}}}\left(P^{\leq}\left(A^{498}_{\{2,3\},*}, b^{498}_{\{2,3\}}\right)\right) &= \mathrm{cl}_{L_{2,\mathbb{Q}}}\left(P^{\leq}\left(A^{498}_{\{2,3\},*}, b^{498}_{\{2,3\}}\right)\right)\\ &= \mathrm{cl}_I\left(P^{\leq}\left(A^{498}_{\{2,3\},*}, b^{498}_{\{2,3\}}\right)\right)\\ &\subseteq P^{\leq}\left(\begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 \end{pmatrix}\right)\end{aligned}$$

  (see Figure 12.2g and Figure 12.2h).

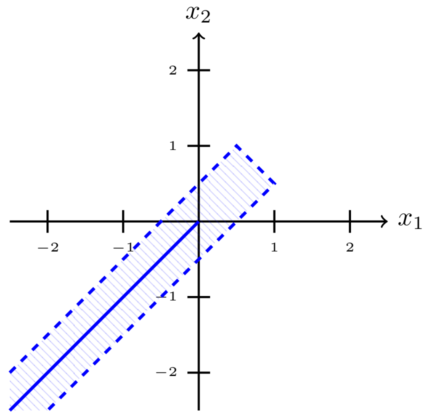
Thus,

$$\begin{matrix}\mathrm{cl}_{L_{2-\frac{1}{2},\mathbb{Q}}}\left(P^{498}\right)\\ \mathrm{cl}_{L_{2,\mathbb{Q}}}\left(P^{498}\right)\\ \mathrm{cl}_I\left(P^{498}\right)\end{matrix} = \bigcap_{S \in \binom{[3]}{2}} \begin{matrix}\mathrm{cl}_{L_{2-\frac{1}{2},\mathbb{Q}}}\left(P^{\leq}\left(A^{498}_{S,*}, b^{498}_S\right)\right)\\ \mathrm{cl}_{L_{2,\mathbb{Q}}}\left(P^{\leq}\left(A^{498}_{S,*}, b^{498}_S\right)\right)\\ \mathrm{cl}_I\left(P^{\leq}\left(A^{498}_{S,*}, b^{498}_S\right)\right).\end{matrix}$$

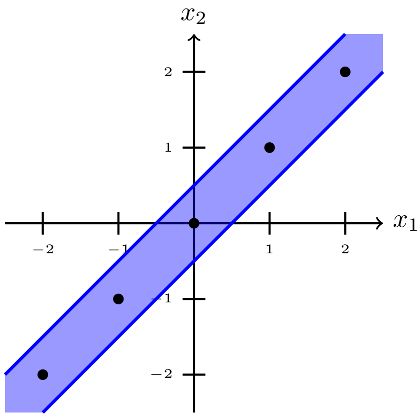
By Theorem 479, there always exists a polyhedron $P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)$ ($A \in \mathbb{R}^{l\times m}$, $G \in \mathbb{R}^{l\times n}$ and $b \in \mathbb{R}^l$, where $m, n \in \mathbb{Z}_{\geq 0}$ and $l := h\left(\mathbb{Z}^m \times \mathbb{R}^n\right) - 1$) and an inequality $c(\cdot) \leq c_0$ such that there exists no subset $S \subsetneq [l]$ such that $c(\cdot) \leq c_0$ is valid for $P := P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}_{S,*}, b_S\right)_I$. Example 498 gives a concrete instance for $m = 2$ and $n = 0$. But, as we saw in Example 498, this does not necessarily imply that for the facet-defining inequalities of $\mathrm{cl}_{\overline{I}}(P)$ (or more general: of $\mathrm{cl}_{L_{k,\mathbb{Q}}}(P)$ or $\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P)$), this has to be true.



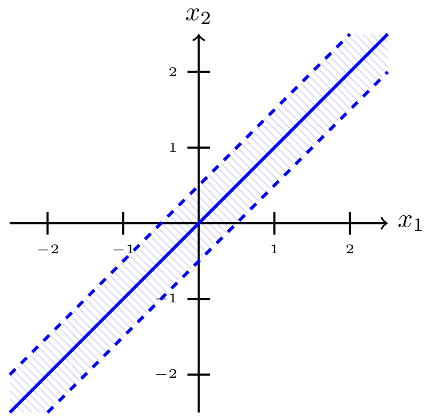
(a) $P^{498} = P^{\leq}\left(A^{498}, b^{498}\right)$ and $P^{498} \cap \mathbb{Z}^2$



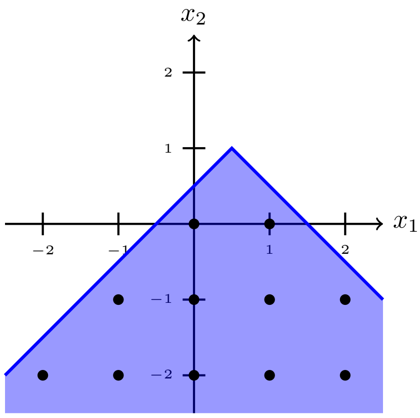
(b) $\mathrm{cl}_I\left(P^{\leq}\left(A^{498}, b^{498}\right)\right)$



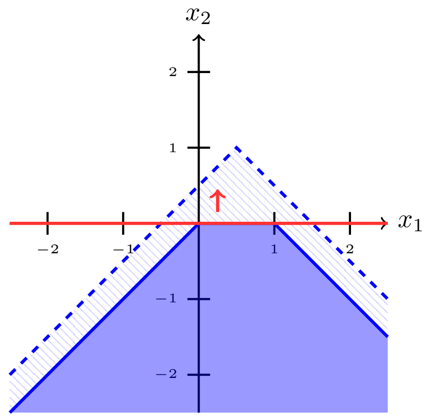
(c) $P^{\leq}\left(A^{498}_{\{1,2\},*}, b^{498}_{\{1,2\}}\right)$ and $P^{\leq}\left(A^{498}_{\{1,2\},*}, b^{498}_{\{1,2\}}\right) \cap \mathbb{Z}^2$



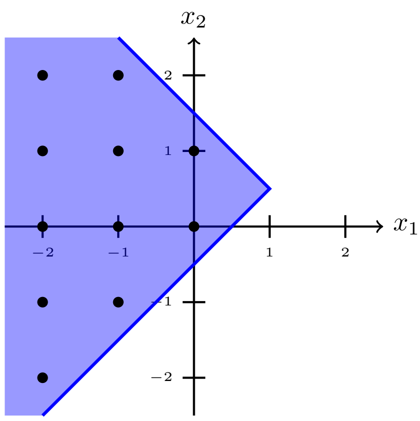
(d) $\mathrm{cl}_I\left(P^{\leq}\left(A^{498}_{\{1,2\},*}, b^{498}_{\{1,2\}}\right)\right)$



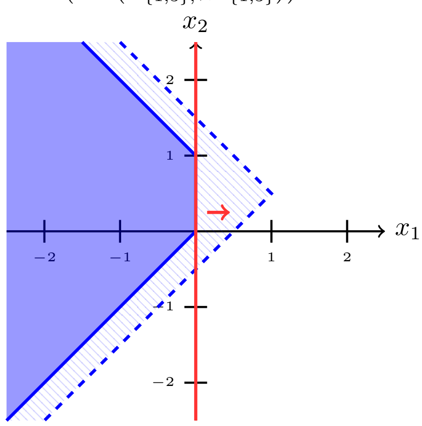
(e) $P^{\leq}\left(A^{498}_{\{1,3\},*}, b^{498}_{\{1,3\}}\right)$ and $P^{\leq}\left(A^{498}_{\{1,3\},*}, b^{498}_{\{1,3\}}\right) \cap \mathbb{Z}^2$



(f) $\mathrm{cl}_I\left(P^{\leq}\left(A^{498}_{\{1,3\},*}, b^{498}_{\{1,3\}}\right)\right)$; $(\cdot)_2 \leq 0$ is valid for $\mathrm{cl}_I\left(P^{\leq}\left(A^{498}_{\{1,3\},*}, b^{498}_{\{1,3\}}\right)\right)$



(g) $P^{\leq}\left(A^{498}_{\{2,3\},*}, b^{498}_{\{2,3\}}\right)$ and $P^{\leq}\left(A^{498}_{\{2,3\},*}, b^{498}_{\{2,3\}}\right) \cap \mathbb{Z}^2$



(h) $\mathrm{cl}_I\left(P^{\leq}\left(A^{498}_{\{2,3\},*}, b^{498}_{\{2,3\}}\right)\right)$; $(\cdot)_1 \leq 0$ is valid for $\mathrm{cl}_I\left(P^{\leq}\left(A^{498}_{\{2,3\},*}, b^{498}_{\{2,3\}}\right)\right)$

Figure 12.2.: Illustration of Example 498

12.5.2. Proofs of lower bounds

12.5.2.1. $k = 1$

The following theorem, which considers the case $m = 1 \wedge n = 0$, clearly holds.

Theorem 499. *Let $m = 1$ and $n = 0$. Let $A := \begin{pmatrix} 1 \end{pmatrix} \in \mathbb{Q}^{1\times 1} = \mathbb{Q}^{1\times m}$ and $b := \begin{pmatrix} \frac{1}{2} \end{pmatrix} \in \mathbb{Q}^1$. Then*

$$h^*_{L_{1,\mathbb{Q}}}(A,b) = h^*_{L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(A,b) = 1 = m+n.$$

Now for the case $m \geq 1 \wedge m+n \geq 2$:

Theorem 500. *Let*

$$A := \begin{pmatrix} 1^{m-1} & -I^{m-1} \\ 1^n & 0^{n\times(m-1)} \\ 1 & \left(1^{m-1}\right)^T \end{pmatrix}, \qquad G := \begin{pmatrix} 0^{(m-1)\times n} \\ -I^n \\ (1^n)^T \end{pmatrix}, \qquad b := \begin{pmatrix} 0^{m-1} \\ 0^n \\ 1 \end{pmatrix},$$

where $m \in \mathbb{Z}_{\geq 1}$ and $n \in \mathbb{Z}_{\geq 0}$ are such that $m+n \geq 2$. Let $P := P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)$. Then:

- *$(\cdot)_1 \leq 0$ is a valid $L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cut (and thus also $L_{1,\mathbb{Q}}$ cut) for P with respect to $\left(0^1 \times \mathbb{R}^{m-1}\right) \times \mathbb{R}^n$.*

- *For every $i \in [m+n]$, we have*

$$\frac{1}{m+n}\begin{pmatrix} 1 \\ 1^{m+n-1} \end{pmatrix} \in \mathrm{cl}_I\left(P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}_{[m+n]\setminus\{i\},*}, b_{[m+n]\setminus\{i\}}\right)\right).$$

Thus, in particular, we have

$$h^*_{L_{1,\mathbb{Q}}}(A,G,b) = h^*_{L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(A,G,b) = m+n.$$

Proof.

For the first statement: For $x \in P$, we have

$$\begin{aligned} x_1 &= \frac{1}{m+n}\left(\left(x_1 + \sum_{i=2}^{m+n} x_i\right) + \sum_{i=2}^{m+n}(x_1 - x_i)\right) \\ &\leq \frac{1}{m+n}\left(1 + \sum_{i=2}^{m+n} 0\right) \\ &= \frac{1}{m+n} < 1. \end{aligned}$$

Thus, $x_1 \leq \frac{1}{m+n}$ is valid for $x \in P + \left(\left(0^1 \times \mathbb{R}^{m-1}\right) \times \mathbb{R}^n\right)$. Since $\frac{1}{m+n} < 1$, this implies that $x_1 \leq 0$ is valid for $x \in \left(P + \left(\left(0^1 \times \mathbb{R}^{m-1}\right) \times \mathbb{R}^n\right)\right)_I$.

For the second statement: Consider that for all $j \in \{0, \ldots, m+n-1\}$, we have $\begin{pmatrix} 0 \\ e^{m+n-1,j} \end{pmatrix} \in P_I$. Let $i' := i \mod m+n \in \{0, \ldots, m+n-1\}$. We claim that

$$\delta := \begin{pmatrix} 1 \\ 1^{m+n-1} \end{pmatrix} - \begin{pmatrix} 0 \\ (m+n-1)\, e^{m+n-1,i'} \end{pmatrix} \in P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}_{[m+n]\setminus\{i\},*}, b_{[m+n]\setminus\{i\}}\right) \cap (\mathbb{Z}^m \times \mathbb{R}^n). \tag{12.23}$$

If (12.23) holds, we have

$$\begin{aligned}&\frac{1}{m+n}\begin{pmatrix}1\\ 1^{m+n-1}\end{pmatrix}\\ =&\frac{m+n-1}{m+n}\underbrace{\begin{pmatrix}0\\ e^{m+n-1,i'}\end{pmatrix}}_{\in P_I\subseteq P^{\leq}\left(\left(\begin{array}{cc}A & G\end{array}\right),b\right)_I}\\ &+\frac{1}{m+n}\left(\begin{pmatrix}1\\ 1^{m+n-1}\end{pmatrix}-\begin{pmatrix}0\\ (m+n-1)\,e^{m+n-1,i'}\end{pmatrix}\right)\\ \in&\operatorname{cl}_I\left(P^{\leq}\left(\left(\begin{array}{cc}A & G\end{array}\right)_{[m+n]\setminus\{i\},*},b_{[m+n]\setminus\{i\}}\right)\right)\end{aligned}$$

and we are thus done.

So for (12.23): we show that for $j\in[m+n]\setminus\{i\}$, we have $\left(\begin{array}{cc}A & G\end{array}\right)_{j,*}\delta=b_j$.

Case 1: $j\in[m+n-1]$**:**

$$A_{j,*}\delta=\delta_1-\delta_{j+1}=1-1=0=b_j.$$

Case 2: $j=m+n$**:**

$$A_{m+n,*}\delta=\delta_1+\sum_{j'=2}^{m+n}\delta_{j'}=1+(m+n-1)-(m+n-1)=1=b_{m+n}.$$

□

12.5.2.2. k,m,n arbitrary

Before we state our result in Theorem 503, we show two results (Theorem 501 and Theorem 502), which build on one another.

Theorem 501. *Let $m\in\mathbb{Z}_{\geq1}$ and $n\in\mathbb{Z}_{\geq0}$ be such that $m+n\geq2$. Let*

$$m':=\begin{cases}m-1 & \text{if } n=0,\\ m & \text{if } n\geq1\end{cases}$$

and let $k\in\{1,\dots,m'\}$. Let $A^{501,m,n,k}\in\mathbb{Z}^{\mathcal{P}([k])\times(m+n)}$ and $b^{501,m,n,k}\in\mathbb{Z}^{\mathcal{P}([k])}$, where

$$\begin{aligned}A^{501,m,n,k}_{I,j}&:=\begin{cases}1 & \text{if } j\in I,\\ -1 & \text{if } j\in[k]\setminus I,\\ 0 & \text{if } j\in\{k+1,\dots,m'\},\\ 1 & \text{if } j=m'+1,\\ 0 & \text{if } k\in\{m'+2,\dots,m+n\},\end{cases}\\ b^{501,m,n,k}_I&:=|I|.\end{aligned}$$

Then

$$\emptyset\neq P\left(A^{501,m,n,k},b^{501,m,n,k}\right)_I\subseteq P^{\leq}\left(\left(e^{m+n,m'+1}\right)^T,0\right)\tag{12.24}$$

and for every $J\in\mathcal{P}([k])$, $y\in\mathbb{Z}^{m'-k}$ and $y'\in\mathbb{R}^{n-1}$ (if $n\geq1$), we have

$$\begin{pmatrix}\frac{1}{2}\cdot 1^k\\ y\\ \frac{1}{2}\\ y'\end{pmatrix}\in\operatorname{cl}_I\left(P^{\leq}\left(A^{501,m,n,k}_{\mathcal{P}([k])\setminus\{J\},*},b^{501,m,n,k}_{\mathcal{P}([k])\setminus\{J\}}\right)\right),\tag{12.25}$$

where the $(\,\cdot\,)_I$ and $\operatorname{cl}_I(\,\cdot\,)$ in (12.24) and (12.25) are with respect to the lattice $\mathbb{Z}^m\times\mathbb{R}^n$.

Proof. (12.24) is obvious.

For (12.25)**:** We surely have

$$x^1 := \begin{pmatrix} 1^k - \chi_J^{[k]} \\ y \\ 0 \\ y' \end{pmatrix} \in P^{\leq}\left(A^{501,m,n,k}, b^{501,m,n,k}\right)_I \subseteq P^{\leq}\left(A^{501,m,n,k}_{\mathcal{P}([k])\setminus J,*}, b^{501,m,n,k}_{\mathcal{P}([k])\setminus J}\right)_I .$$

Additionally, we claim that

$$x^2 := \begin{pmatrix} \chi_J^{[k]} \\ y \\ 1 \\ y' \end{pmatrix} \in P^{\leq}\left(A^{501,m,n,k}_{\mathcal{P}([k])\setminus J,*}, b^{501,m,n,k}_{\mathcal{P}([k])\setminus J}\right)_I .$$

For the reason: let $I \in \mathcal{P}([k]) \setminus \{J\}$. Then

$$\begin{aligned} A^{501,m,n,k}_{I,*} x^2 &= \sum_{i\in I} \left(x^2\right)_i - \sum_{i\in[k]\setminus I} \left(x^2\right)_i + \left(x^2\right)_{m'+1} \\ &= \sum_{i\in I\cap J} \underbrace{\left(x^2\right)_i}_{=1} + \sum_{i\in I\setminus J} \underbrace{\left(x^2\right)_i}_{=0} - \sum_{i\in J\setminus I} \underbrace{\left(x^2\right)_i}_{=1} - \sum_{i\in[k]\setminus(I\cup J)} \underbrace{\left(x^2\right)_i}_{=0} + 1 \\ &= |I\cap J| - |J\backslash I| + 1 \\ &=: (12.26). \end{aligned}$$

We distinguish two cases (since $I \neq J$, at least one of them has to occur):

1. $J\backslash I \neq \emptyset$,
2. $I\backslash J \neq \emptyset$.

In case 1, we have

$$(12.26) = \underbrace{|I\cap J|}_{\leq |I|} - \underbrace{|J\backslash I|}_{\geq 1} + 1 \leq |I| .$$

In case 2, we have

$$(12.26) = |I\cap J| - |J\backslash I| + 1 \leq |I| - \underbrace{|I\backslash J|}_{\geq 1} - \underbrace{|J\backslash I|}_{\geq 0} + 1 \leq |I| .$$

In any case, the inequality

$$\sum_{i\in I} \left(x^2\right)_i - \sum_{i\in[k]\setminus I} \left(x^2\right)_i + \left(x^2\right)_{m'+1} \leq |I|$$

is satisfied, from which $x^2 \in P^{\leq}\left(A^{501,m,n,k}_{\mathcal{P}([k])\setminus J,*}, b^{501,m,n,k}_{\mathcal{P}([k])\setminus J}\right)$ and, thus, $x^2 \in P^{\leq}\left(A^{501,m,n,k}_{\mathcal{P}([k])\setminus J,*}, b^{501,m,n,k}_{\mathcal{P}([k])\setminus J}\right)_I$ follows. Thus, we obviously obtain

$$\frac{1}{2}\left(x^1 + x^2\right) \in \mathrm{cl}_I\left(P^{\leq}\left(A^{501,m,n,k}_{\mathcal{P}([k])\setminus J,*}, b^{501,m,n,k}_{\mathcal{P}([k])\setminus J}\right)\right).$$

□

Theorem 502. *Let $m \in \mathbb{Z}_{\geq 1}$ and $n \in \mathbb{Z}_{\geq 0}$ be such that $m + n \geq 2$. Let*

$$m' := \begin{cases} m-1 & \text{if } n = 0, \\ m & \text{if } n \geq 1 \end{cases}$$

and let $k \in \{1, \ldots, m'\}$. Then there exist $A^{502,m,n,k} \in \mathbb{Z}^{2^k\times m}$, $G^{502,m,n,k} \in \mathbb{Z}^{2^k\times n}$ and $b^{502,m,n,k} \in \mathbb{Z}^{2^k}$ such that

$$P := P^{\leq}\left(\begin{pmatrix} A^{502,m,n,k} & G^{502,m,n,k} \end{pmatrix}, b^{502,m,n,k}\right)_I \neq \emptyset \tag{12.27}$$

and

$$P^{\leq}\left(\left(e^{m+n,m'+1}\right)^T,0\right) \supseteq P \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + \underbrace{((0^k \times \mathbb{R}^{m-k}) \times \mathbb{R}^n)}_{\operatorname{codim}(\,\cdot\,)=k}), \tag{12.28}$$

$$P^{\leq}\left(\left(e^{m+n,m'+1}\right)^T,0\right) \supseteq \begin{cases} \left(P + \underbrace{\left(0^k \times \mathbb{R}^{m'-k} \times 0\right)}_{\operatorname{codim}(\,\cdot\,)=k+1}\right)_I & \text{if } n = 0, \\ \left(P + \underbrace{\left(0^k \times \mathbb{R}^{m'-k} \times 0 \times \mathbb{R}^{n-1}\right)}_{\operatorname{codim}(\,\cdot\,)=k+1}\right)_I & \text{if } n \geq 1. \end{cases} \tag{12.29}$$

On the other hand, for every $i \in [2^k]$, $y \in \mathbb{Z}^{m'-k}$ and $y' \in \mathbb{R}^{n-1}$ (if $n \geq 1$), we have

$$\begin{pmatrix} \frac{1}{2} \cdot 1^k \\ y \\ \frac{1}{2} \\ y' \end{pmatrix} \in \operatorname{cl}_I\left(P^{\leq}\left(\begin{pmatrix} A^{502,m,n,k} & G^{502,m,n,k} \end{pmatrix}_{[2^k]\setminus\{i\},*}, b^{502,m,n,k}_{[2^k]\setminus\{i\}}\right)\right). \tag{12.30}$$

Proof. Let

$$A^{502,m,n,k} := A^{501,m,n,k}_{*,(1,\dots,m)}, \qquad G^{502,m,n,k} := A^{501,m,n,k}_{*,(m+1,\dots,m+n)}, \qquad b^{502,m,n,k} := b^{501,m,n,k}.$$

Then:

- (12.27) holds by Theorem 501, equation (12.24).
- For (12.28), consider that for $x \in P$, where $x_{(1,\dots,k)} \in \mathbb{Z}^k$, we have $x_{m'+1} \leq 0$ (this is easy to verify). From this, (12.28) is an immediate consequence.
- For (12.29), consider that

$$\operatorname{lineal} P \supseteq \begin{cases} 0^k \times \mathbb{R}^{m-(k+1)} \times 0 & \text{if } n = 0, \\ 0^k \times \mathbb{R}^{m-k} \times 0 \times \mathbb{R}^{n-1} & \text{if } n \geq 1. \end{cases}$$

  Thus, we conclude (12.29) from Theorem 217 and Theorem 501, equation (12.24).
- (12.30) holds by Theorem 501, equation (12.25).

□

Theorem 503. *Let $m \in \mathbb{Z}_{\geq 1}$, $n \in \mathbb{Z}_{\geq 0}$ and $k \in \{1, \dots, m+n\}$.*

- *Let*

$$h := \begin{cases} 2^{\min(k,m-1)} & \text{if } n = 0, \\ 2^{\min(k,m)} & \text{if } n \geq 1. \end{cases}$$

  Then there exist $A \in \mathbb{Q}^{h \times m}$, $G \in \mathbb{Q}^{h \times n}$ and $b \in \mathbb{Q}^h$ such that

$$P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)_I \neq \emptyset \tag{12.31}$$

  and

$$h^*_{L_{k,\mathbb{Q}}}(A, G, b) = h. \tag{12.32}$$

- *Let*

$$h := \begin{cases} 2^{k-1} & \text{if } n = 0, \\ 2^{\min(k-1,m)} & \text{if } n \geq 1. \end{cases}$$

  Then there exist $A \in \mathbb{Q}^{h \times m}$, $G \in \mathbb{Q}^{h \times n}$ and $b \in \mathbb{Q}^h$ such that

$$P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)_I \neq \emptyset \tag{12.33}$$

  and

$$h^*_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(A, G, b) = h. \tag{12.34}$$

Proof.

For (12.31) **and** (12.32)**:** If $m = 1$ and $n = 0$, the statement holds by Theorem 499. For $m + n \geq 2$, let m' be as in Theorem 502 and consider

$$A := A^{502,m,n,\min\left(k,m'\right)}, \qquad G := G^{502,m,n,\min\left(k,m'\right)}, \qquad b := b^{502,m,n,\min\left(k,m'\right)}.$$

Then (12.31) holds by (12.27). If $k \in \{1, \ldots, m'\}$, (12.32) is an immediate consequence of (12.28) and (12.30). On the other hand, if $k \geq m' + 1$, we immediately conclude (12.32) from (12.28), (12.30) and

$$\begin{aligned} &P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right) \cap \left((\mathbb{Z}^m \times \mathbb{R}^n) + \left(0^m \times \left(\mathbb{R}^{m+n-k} \times 0^{k-m}\right)\right)\right) \\ \subseteq &P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right) \cap \left((\mathbb{Z}^m \times \mathbb{R}^n) + \left(\left(0^{m'} \times \mathbb{R}^{m-m'}\right) \times \mathbb{R}^n\right)\right). \end{aligned}$$

For (12.33) **and** (12.34)**:** If $k = 1$, the statement holds by Theorem 499. For $k \geq 2$ (this implies $m + n \geq 2$), let m' be as in Theorem 502 and consider

$$A := A^{502,m,n,\min\left(k-1,m'\right)}, \qquad G := G^{502,m,n,\min\left(k-1,m'\right)}, \qquad b := b^{502,m,n,\min\left(k-1,m'\right)}.$$

Then (12.31) holds by (12.27). If $k - 1 \in \{1, \ldots, m'\}$ (thus, $k \in \{2, \ldots, m' + 1\}$), (12.34) is an immediate consequence of (12.29) and (12.30). On the other hand, if $k \geq m' + 2$ (which implies $n \geq 2$), we immediately conclude (12.34) from (12.29), (12.30) and

$$\begin{aligned} &\Big(P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right) + \underbrace{\left(0^m \times \left(0^{k-m} \times \mathbb{R}^{m+n-k}\right)\right)}_{\operatorname{codim}(\,\cdot\,)=k}\Big)_I \\ \subseteq &\Big(P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right) + \underbrace{\left(\left(0^m \times \mathbb{R}^{m-m}\right) \times \left(0 \times \mathbb{R}^{n-1}\right)\right)}_{\operatorname{codim}(\,\cdot\,)=m+1=m'}\Big)_I. \end{aligned}$$

□

12.6. Summary of bounds for $h^*_{L_{m+n-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(A,G,b)$ and $h^*_{L_{m+n,\mathbb{Q}}}(A,G,b)$

12.6.1. Upper bounds

Theorem 504. *Let $k \in \{0, \ldots, m+n\}$. Then*

$$h^*_{L_{k,\mathbb{Q}}}(A,G,b) \leq \begin{cases} \operatorname{rank}\begin{pmatrix} A & G \end{pmatrix} \ (\leq m+n) & \text{if } k = 1, \\ 2^{\min(k,m)}\left(m+n-\min(k,m)+1\right) - 1 & \text{if } P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)_I \neq \emptyset, \\ 2^{\min(k,m)}\left(m+n-\min(k,m)+1\right) & \text{in general,} \end{cases}$$

$$h^*_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(A,G,b) \leq \begin{cases} \operatorname{rank}\begin{pmatrix} A & G \end{pmatrix} \ (\leq m+n) & \text{if } k = 1, \\ \left(2^k - 1\right)(m+n-k+1) & \text{if } P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)_I \neq \emptyset \wedge k \in \{0, \ldots, m\}, \\ 2^{\min(k,m)}\left(m+n-\min(k,m)+1\right) - 1 & \text{if } P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)_I \neq \emptyset, \\ 2^{\min(k,m)}\left(m+n-\min(k,m)+1\right) & \text{in general.} \end{cases}$$

Proof. The first bound for $h^*_{L_{k,\mathbb{Q}}}(A,G,b)$ is a consequence of Theorem 269. The second and third bound for $h^*_{L_{k,\mathbb{Q}}}(A,G,b)$ are shown in Theorem 487. The first bound for $h^*_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(A,G,b)$ holds by Theorem 394. The other three bounds for $h^*_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(A,G,b)$ are shown in Theorem 495. □

12.6.2. Lower bounds

Theorem 505. *Let $m \in \mathbb{Z}_{\geq 1}$, $n \in \mathbb{Z}_{\geq 0}$ and $k \in \{1, \ldots, m+n\}$. Then there exist rational A, G and b such that $P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right) \neq \emptyset$ and*

$$h^*_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(A, G, b) = \begin{cases} m+n, \\ 2^{k-1} & \text{if } n = 0, \\ 2^{\min(k-1,m)} & \text{if } n \geq 1, \end{cases}$$

$$h^*_{L_{k,\mathbb{Q}}}(A, G, b) = \begin{cases} m+n, \\ 2^{\min(k,m-1)} & \text{if } n = 0, \\ 2^{\min(k,m)} & \text{if } n \geq 1. \end{cases}$$

Proof. The $m+n$ bounds for $h^*_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(A, G, b)$ and $h^*_{L_{k,\mathbb{Q}}}(A, G, b)$ are immediately implied

- by Theorem 499 if $m = 1$ and $n = 0$ and
- by Theorem 500 if $m + n \geq 2$.

The remaining bounds are immediate consequences of Theorem 503. □

13. Bounds on the $L_{k,\mathbb{Q}}$ rank and $L_{k-\frac{1}{2},\mathbb{Q}}$ rank of polyhedra with 0/1 integer variables

In this chapter, we consider bounds on the $L_{k,\mathbb{Q}}$ and $L_{k-\frac{1}{2},\mathbb{Q}}$ rank for polyhedra where the integer variables are 0/1 variables (recall (6.11) for the definition of the $L_{k,\mathbb{Q}}$ and $L_{k-\frac{1}{2},\mathbb{Q}}$ rank ($\operatorname{rank}_{L_{k,\mathbb{Q}}}(\cdot)$ and $\operatorname{rank}_{L_{k-\frac{1}{2},\mathbb{Q}}}(\cdot)$)).

- In section 13.1.1, Theorem 515, we show the bound

$$\operatorname{rank}_{L_{k,\mathbb{Q}}}(P) \leq \left\lceil \frac{m}{k} \right\rceil \tag{13.1}$$

  for the $L_{k,\mathbb{Q}}$ rank of a polyhedron $P \subseteq [0,1]^m \times \mathbb{R}^n$, where $m, n \in \mathbb{Z}_{\geq 0}$ and $k \in \{1, \ldots, m\}$ (we actually prove that this bound holds for a somewhat larger class of convex sets; cf. Definition 506).

- Now one can ask whether this bound is tight. This is the topic of section 13.1.2. In Theorem 526, we prove that for every $m \in \mathbb{Z}_{\geq 1}$, there exists a rational polytope $P \subseteq [0,1]^m$ such that for all $k \in \{1, \ldots, m\}$, we have

$$\operatorname{rank}_{L_{k,\mathbb{Q}}}(P) \geq \left\lceil \frac{m}{k} \right\rceil.$$

  Thus, the bound (13.1) is indeed tight.

- At the beginning of section 13.2, in Theorem 528, we use these results that we showed on the $L_{k,\mathbb{Q}}$ rank to estimate the $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ rank of a rational polyhedron. In the case $k = 1$, this gives us only a lower bound. Luckily, the problem of finding bounds for the $L_{1-\frac{1}{2},\mathbb{Q}}$ rank (Chvátal-Gomory rank) of, for example, a polytope $P \subseteq [0,1]^m$ ($m \in \mathbb{Z}_{\geq 0}$) is a well-studied problem in the literature. So, in section 13.2.1, we give an overview about some results from the literature concerning upper bounds for the Chvátal-Gomory rank and in section 13.2.2, we do the same for lower bounds.

13.1. $L_{k,\mathbb{Q}}$ rank

13.1.1. Upper bounds for the $L_{k,\mathbb{Q}}$ rank

Definition 506. *Let $P \subseteq [0,1]^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) satisfy*

$$P = Q + C,$$

where $Q \subseteq [0,1]^m \times \mathbb{R}^n$ is convex and compact and $C \subseteq 0^m \times \mathbb{R}^n$ is a polyhedral cone generated by vectors from $0^m \times \mathbb{R}^n$. For $\mathcal{K} \subseteq [m]$, we define

$$P_{\mathcal{K},I} := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in P : x_{\mathcal{K}} \in \{0,1\}^{\mathcal{K}} \right\},$$
$$P_{\mathcal{K}} := \operatorname{conv} P_{\mathcal{K},I}.$$

Remark 507. *The conditions on P in Definition 506 include the situation that $P \subseteq [0,1]^m \times \mathbb{R}^n$ is an arbitrary polyhedron (i.e. not "only" a rational polyhedron) as special case.*

Lemma 508. *Let $P = Q + C$, where P, Q and C are as in Definition 506. Additionally, let also $\mathcal{K}$ be as in Definition 506. Then $P_{\mathcal{K}} = Q_{\mathcal{K}} + C$, where the sets $P_{\mathcal{K}}$ and $Q_{\mathcal{K}}$ are closed.*

Proof. W.l.o.g. let $\mathcal{K} := \{1, \ldots, k\}$, where $k \in \{0, \ldots, m\}$. Obviously,

$$P_{\mathcal{K}} = \operatorname{conv}\left(P \cap \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in P : x_{\mathcal{K}} \in \mathbb{Z}^{\mathcal{K}} \right\} \right).$$

In other words:

$$P_{\mathcal{K}} = \operatorname{conv}\left(P \cap \left(\mathbb{Z}^k \times \mathbb{R}^{m+n-k}\right)\right).$$

The cone C is finitely generated. Each generator $\begin{pmatrix} c^1 \\ c^2 \end{pmatrix} \in 0^m \times \mathbb{R}^n$ satisfies $c^1_i = 0 \in \mathbb{Z}$ for $i \in [k]$. Thus, the conditions of Theorem 75 are satisfied for P with respect to a mixed lattice that is isomorphic to the mixed lattice $\mathbb{Z}^k \times \mathbb{R}^{m+n-k}$. So, $P_{\mathcal{K}} = Q_{\mathcal{K}} + C$, and the sets $P_{\mathcal{K}}$ and $Q_{\mathcal{K}}$ are closed. □

Lemma 509. *Let m, n and P be as in Definition 506. Let $\mathcal{K} \in \binom{[m]}{k}$, where $k \in \{0, \ldots, m\}$. Then*

$$\operatorname{cl}_{L_{k,\mathbb{Q}}}(P) \subseteq \operatorname{cl}_{kBS}(P) \subseteq P_{\mathcal{K}}.$$

Proof. Let $\mathcal{K} := \{i_1, \ldots, i_k\}$. Obviously,

$$\operatorname{cl}_{kBS}(P) \subseteq \overline{P_{\mathcal{K}}} \tag{13.2}$$

holds, since any valid inequality for $P_{\mathcal{K}}$ is a valid inequality for $P \cap D\left(e^{m,i_1}, \ldots, e^{m,i_k}, 0, \ldots, 0\right)$. Because of Lemma 508, we have $P_{\mathcal{K}} = \overline{P_{\mathcal{K}}}$. Combined with (13.2), we get the inclusion $\operatorname{cl}_{kBS}(P) \subseteq P_{\mathcal{K}}$. The inclusion $\operatorname{cl}_{L_{k,\mathbb{Q}}}(P) \subseteq \operatorname{cl}_{kBS}(P)$ is an immediate consequence of Corollary 278. □

The proof of the following Lemma is taken from [CCZ10, Lemma 11.10].

Lemma 510. *Let $H := P^{=}(c, c_0) \subseteq \mathbb{R}^d$ ($c \in \left(\mathbb{R}^d\right)^T \setminus \left\{\left(0^d\right)^T\right\}$ and $c_0 \in \mathbb{R}$, where $d \in \mathbb{Z}_{\geq 0}$) be a hyperplane and let $S \subseteq P^{\leq}(c, c_0)$. Then*

$$(\operatorname{conv} S) \cap H = \operatorname{conv}(S \cap H).$$

Remark 511. *(Also cf. [CCZ10, section 11.7.4]) The condition $S \subseteq P^{\leq}(c, c_0)$ in Lemma 510 is essential for the inclusion $(\operatorname{conv} S) \cap H \subseteq \operatorname{conv}(S \cap H)$ to hold. To that end, let a hyperplane $H = P^{=}(c, c_0) \subseteq \mathbb{R}^d$ ($c \in \left(\mathbb{R}^d\right)^T \setminus \left\{\left(0^d\right)^T\right\}$ and $c_0 \in \mathbb{R}$) be given. Let $S := \left\{x^1, x^2\right\}$, where $cx^1 > c_0$ and $cx^2 < c_0$. Then*

$$\operatorname{conv}(S \cap H) = \emptyset,$$

but

$$\frac{c_0 - cx^2}{c\left(x^1 - x^2\right)} x^1 + \frac{cx^1 - c_0}{c\left(x^1 - x^2\right)} x^2 \in (\operatorname{conv} S) \cap H.$$

Proof. (Lemma 510) Because of $\operatorname{conv}(S \cap H) \subseteq \operatorname{conv} S$ and $\operatorname{conv}(S \cap H) \subseteq H$, we immediately obtain

$$\operatorname{conv}(S \cap H) \subseteq (\operatorname{conv} S) \cap H.$$

Thus, we only have to show $(\operatorname{conv} S) \cap H \subseteq \operatorname{conv}(S \cap H)$. Let $x \in (\operatorname{conv} S) \cap H$. This means $cx = c_0$ and $x = \begin{pmatrix} x^1 & \cdots & x^k \end{pmatrix} \lambda$, where $k \in \mathbb{Z}_{\geq 1}$, $\lambda \in \Delta^{k-1}$ and for $i \in [k]$, $x^i \in S$. We have

$$c_0 = cx = \sum_{i=1}^{k} \lambda_i c x^i \leq \sum_{i=1}^{k} \lambda_i c_0 = c_0.$$

Thus, equality holds. In other words, we have $cx^i = c_0$ for all $i \in [k]$. Therefore $x^i \in S \cap H$. This implies $x \in \operatorname{conv}(S \cap H)$. □

Lemma 512. *Let $k \in \mathbb{Z}_{\geq 1}$ and let $\{H_i\}_{i=1}^{k}$ be a collection of hyperplanes, where $H^i := P^{=}\left(c^i, c_{0,i}\right) \subseteq \mathbb{R}^d$ ($c^i \in \left(\mathbb{R}^d\right)^T \setminus \left\{\left(0^d\right)^T\right\}$, $c_{0,i} \in \mathbb{R}$ for all $i \in [k]$, where $d \in \mathbb{Z}_{\geq 0}$) such that for all $i \in [k]$, we have $S \subseteq P^{\leq}\left(\left(c^i\right)^T, c_{0,i}\right)$. Then*

$$(\operatorname{conv} S) \cap \bigcap_{i=1}^{k} H_i = \operatorname{conv}\left(S \cap \bigcap_{i=1}^{k} H_i\right).$$

Proof. We do a proof by induction. For $k = 1$, this is the statement of Lemma 510. For the induction step, assume that the statement holds for k^*. Then

$$(\operatorname{conv} S) \cap \bigcap_{i=1}^{k^*+1} H_i = (\operatorname{conv} S) \cap \bigcap_{i=1}^{k^*} H_i \cap H_{k^*+1}$$

$$
\begin{aligned}
&= \left(\operatorname{conv}\left(S \cap \bigcap_{i=1}^{k^*} H_i\right)\right) \cap H_{k^*+1} && \text{(induction hypothesis)} \\
&= \operatorname{conv}\left(S \cap \bigcap_{i=1}^{k^*+1} H_i\right). && \text{(Lemma 510)}
\end{aligned}
$$

□

Lemma 513. *Let m and P be as in Definition 506 and let $\mathcal{K}_1, \mathcal{K}_2 \subseteq [m]$. Then*

$$(P_{\mathcal{K}_1})_{\mathcal{K}_2} = P_{\mathcal{K}_1 \cup \mathcal{K}_2}.$$

Proof.

$$
\begin{aligned}
(P_{\mathcal{K}_1})_{\mathcal{K}_2} &= (\operatorname{conv} P_{\mathcal{K}_1,I})_{\mathcal{K}_2} \\
&= \operatorname{conv} \bigcup_{J \in \{0,1\}^{\mathcal{K}_2}} \left((\operatorname{conv} P_{\mathcal{K}_1,I}) \cap \left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R}^n : x_{\mathcal{K}_2} = J\right\}\right) \\
&= \operatorname{conv} \bigcup_{J \in \{0,1\}^{\mathcal{K}_2}} \operatorname{conv}\left(P_{\mathcal{K}_1,I} \cap \left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R}^n : x_{\mathcal{K}_2} = J\right\}\right) && \text{(Lemma 512)} \\
&= \operatorname{conv} \bigcup_{J \in \{0,1\}^{\mathcal{K}_2}} \left(P_{\mathcal{K}_1,I} \cap \left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R}^n : x_{\mathcal{K}_2} = J\right\}\right) \\
&= \operatorname{conv} P_{\mathcal{K}_1 \cup \mathcal{K}_2, I} \\
&= P_{\mathcal{K}_1 \cup \mathcal{K}_2}.
\end{aligned}
$$

□

We note that, if $\mathcal{K} \subseteq [m]$ and $P = Q + C$ are as in Definition 506, we have (cf. Lemma 508) $P_{\mathcal{K}} = Q_{\mathcal{K}} + C$ and $Q_{\mathcal{K}}$ satisfies the conditions for Q in Definition 506. Thus, we may apply Lemma 513 inductively and get:

Corollary 514. *Let m and P be as in Definition 506 and let let $\mathcal{K}_1, \ldots, \mathcal{K}_t \subseteq [m]$, where $t \in \mathbb{Z}_{\geq 1}$. Then*

$$\left(\ldots\left((P_{\mathcal{K}_1})_{\mathcal{K}_2}\right)_{\ldots}\right)_{\mathcal{K}_t} = P_{\bigcup_{i=1}^{t} \mathcal{K}_i}.$$

In particular, if $\bigcup_{i=1}^{t} \mathcal{K}_i = [m]$, we have

$$\left(\ldots\left((P_{\mathcal{K}_1})_{\mathcal{K}_2}\right)_{\ldots}\right)_{\mathcal{K}_t} = \operatorname{cl}_I(P).$$

Now we come to the central theorem of this section:

Theorem 515. *Let $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m \in \mathbb{Z}_{\geq 1}$, $n \in \mathbb{Z}_{\geq 0}$) be as in Definition 506 and let $k \in [m]$. Then*

$$\operatorname{rank}_{L_{k,\mathbb{Q}}}(P) \leq \left\lceil \frac{m}{k} \right\rceil$$

(recall (6.11) for the definition of the $L_{k,\mathbb{Q}}$ rank).

Proof. By setting

$$
\begin{aligned}
\mathcal{K}_1 &:= \{1, \ldots, k\}, \\
\mathcal{K}_2 &:= \{k+1, \ldots, 2k\}, \\
&\vdots \\
\mathcal{K}_{\lceil \frac{m}{k} \rceil - 1} &:= \left\{\left(\left\lceil \frac{m}{k} \right\rceil - 2\right) k + 1, \ldots, \left(\left\lceil \frac{m}{k} \right\rceil - 1\right) k\right\}, \\
\mathcal{K}_{\lceil \frac{m}{k} \rceil} &:= \{m - k + 1, \ldots, m\}
\end{aligned}
$$

in Corollary 514, one obtains:

$$\left(\dots\left(\left(P_{\mathcal{K}_1}\right)_{\mathcal{K}_2}\right)_{\dots}\right)_{\mathcal{K}_{\lceil \frac{m}{k} \rceil}} = \mathrm{cl}_I\left(P\right).$$

By observing that for all $i \in \lceil \frac{m}{k} \rceil$, we have $|\mathcal{K}_i| = k$, we obtain $\mathrm{rank}_{L_{k,\mathbb{Q}}}(P) \leq \lceil \frac{m}{k} \rceil$ from Lemma 509. □

13.1.2. Lower bounds for the $L_{k,\mathbb{Q}}$ rank

Definition 516. *For $m \in \mathbb{Z}_{\geq 1}$, let*

$$P^{516,m} := \left\{ x \in [0,1]^m : \sum_{j \in J} x_j + \sum_{j \in [m] \setminus J} (1 - x_j) \geq \frac{1}{2} \ \forall J \in \mathcal{P}([m]) \right\}.$$

Remark 517. *We have $\left(P^{516,m}\right)_I = \emptyset$.*

Proof. Let $x \in \{0,1\}^m$ and let $J := \{j \in [m] : x_j = 0\}$. Then

$$\sum_{j \in J} x_j + \sum_{j \in [m] \setminus J} (1 - x_j) = \sum_{j \in J} 0 + \sum_{j \in [m] \setminus J} (1 - 1) = 0 \ngeq \frac{1}{2}.$$

□

Remark 518. *The statement of Theorem 515 particularly holds for $P = P^{516,m}$, i.e. for $m \in \mathbb{Z}_{\geq 1}$ and $k \in \{1, \dots, m\}$, we have*

$$\mathrm{rank}_{L_{k,\mathbb{Q}}}\left(P^{516,m}\right) \leq \left\lceil \frac{m}{k} \right\rceil.$$

Definition 519. *For $m \in \mathbb{Z}_{\geq 1}$ and $j \in \{1, \dots, m\}$, define the set*

$$F_{j,m} := \left\{ x \in \left\{0, \frac{1}{2}, 1\right\} : \left| \left\{ j' \in [m] : x_{j'} = \frac{1}{2} \right\} \right| = j \right\}.$$

Remark 520. *We have (cf. Example 11.2 in [CCZ10]) for $m \in \mathbb{Z}_{\geq 1}$:*

$$\mathrm{conv}\, F_{1,m} = P^{516,m}.$$

Lemma 521. *For $m \in \mathbb{Z}_{\geq 1}$ and $k_1, k_2 \in \{1, \dots, m\}$, where $k_1 \leq k_2$, the identity $F_{k_2,m} \subseteq \mathrm{conv}\, F_{k_1,m}$ holds.*

Proof. For $k_1 = k_2$ the statement is trivial. This shows the induction basis. Concerning the induction step: let the statement hold for $k_2 - k_1 =: l^*$. Let $k_2 - k_1 = l^* + 1$ and let $v \in F_{k_2,m}$. Since $k_2 \geq 1$, there exists at least one $i \in [m]$ such that $v_i = \frac{1}{2}$. Let

$$v^1 := v + \frac{1}{2} e^{m,i},$$
$$v^2 := v - \frac{1}{2} e^{m,i}.$$

Then $v^1, v^2 \in F_{k_2-1,m}$ and $v = \frac{1}{2} v^1 + \frac{1}{2} v^2 \in \mathrm{conv}\, F_{k_2-1,m}$. Thus,

$$v = \frac{1}{2} v^1 + \frac{1}{2} v^2 \in \underbrace{\mathrm{conv}\, F_{k_2-1,m}}_{\substack{\subseteq \mathrm{conv}\left(\mathrm{conv}\, F_{k_1,m}\right) \\ \text{(induction hypothesis)}}} \subseteq \mathrm{conv}\left(\mathrm{conv}\, F_{k_1,m}\right) = \mathrm{conv}\, F_{k_1,m}.$$

□

Lemma 522. *Let $A \in F^{l \times d}$ ($l, d \in \mathbb{Z}_{\geq 0}$) be a matrix with linearly independent rows over an arbitrary field F and let $v^* \in F^d$. Then there exists a $v \in F^d$ such that*

$$v^* - v \in \ker A$$

and v has at most l non-zero components.

Proof. Clearly, $v := v^*$ satisfies $v^* - v \in \ker A$. So, there (trivially) exists a $v \in F^d$ such that $v^* - v \in \ker A$. Now let $v \in F^d$ be such that $v^* - v \in \ker A$ and $|\operatorname{supp} v|$ is minimal. Assume $|\operatorname{supp} v| \geq l+1$. Since $\operatorname{rank} A = l$, there exist linearly dependent vectors among $\bigcup_{i\in\operatorname{supp} v} \{A_{*,i}\}$. This means that there exist a $\lambda \in F^l \setminus \{0^l\}$ such that $A\lambda = 0^l$ and $\operatorname{supp}\lambda \subseteq \operatorname{supp} v$. Now let $i^* \in \operatorname{supp}\lambda \subseteq \operatorname{supp} v$ and let

$$v' := v - \frac{v_{i^*}}{\lambda_{i^*}}\lambda.$$

Clearly, $\operatorname{supp} v' \subsetneq \operatorname{supp} v$, since $\operatorname{supp} v' \subseteq \operatorname{supp} v$ and $v'_{i^*} = v_{i^*} - \frac{v_{i^*}}{\lambda_{i^*}}\lambda_{i^*} = 0$. On the other hand

$$A(v^* - v') = Av^* - A\left(v - \frac{v_{i^*}}{\lambda_{i^*}}\lambda\right) = Av^* - Av = 0^l,$$

which is a contradiction to the fact that $|\operatorname{supp} v|$ is chosen minimally. □

By setting $F := \mathbb{F}_2$, $v^* := 1^l$ in Lemma 522 and observing that in $\mathbb{F}_2$

- 1 is the only element different from 0 and
- for all $x \in \mathbb{F}_2$, we have $-x = x$,

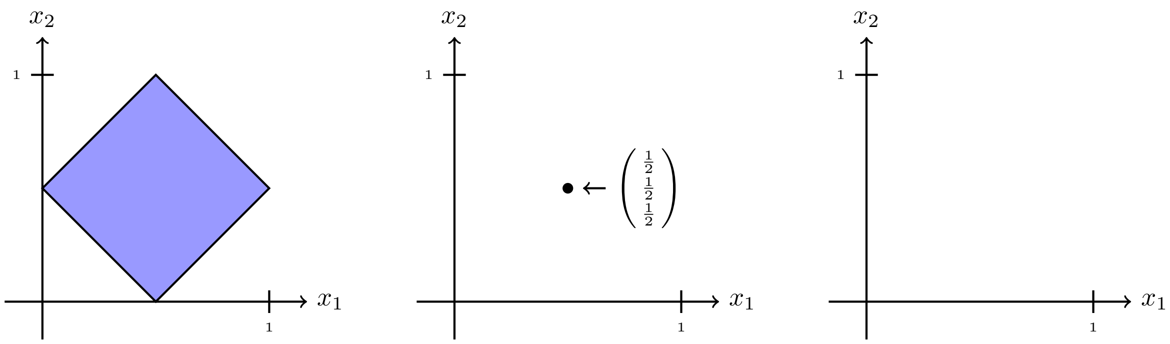
we obtain:

Corollary 523. *Let $A \in \mathbb{F}_2^{l\times d}$ $(l, d \in \mathbb{Z}_{\geq 0})$ be a matrix with linearly independent rows. Then there exist distinct $i_1, \ldots, i_{l'} \in [d]$, where $0 \leq l' \leq l$, such that*

$$1^d + \sum_{j=1}^{l'} e^{d,i_j} \in \ker A.$$

The following theorem generalizes the statement of [CCZ10, Lemma 11.14]:

Theorem 524. *Let $m \in \mathbb{Z}_{\geq 1}$, $k \in [m-1]$ and $j \in [m-k]$. Then $F_{j+k,m} \subseteq \operatorname{cl}_{L_{k,\mathbb{Q}}}(\operatorname{conv} F_{j,m})$.*



(a) $\operatorname{conv} F_{1,2} = \operatorname{cl}^{(0)}_{L_{1,\mathbb{Q}}}(P^{516,2})$ (b) $\left\{\begin{pmatrix}\frac12\\\frac12\end{pmatrix}\right\} = \operatorname{conv} F_{2,2} = \operatorname{cl}^{(1)}_{L_{1,\mathbb{Q}}}(P^{516,2})$ (c) $\emptyset = \operatorname{cl}^{(2)}_{L_{1,\mathbb{Q}}}(P^{516,2}) = \operatorname{cl}^{(1)}_{L_{2,\mathbb{Q}}}(P^{516,2})$

Figure 13.1.: Iterative $L_{k,\mathbb{Q}}$ closure of $P^{516,2} = \operatorname{conv} F_{1,2}$

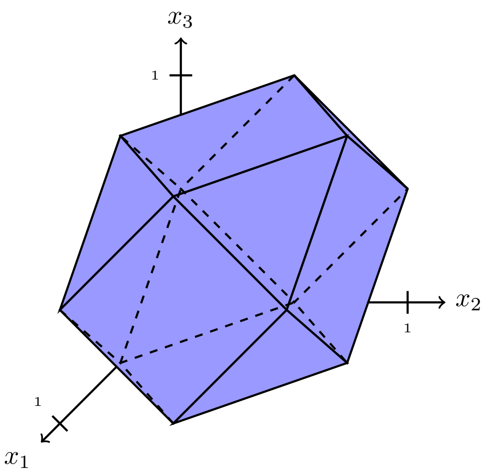
Remark 525. *By combining Theorem 524 with Corollary 514, one can even show that for m, k and j as in Theorem 524, we have*

$$\operatorname{cl}_{L_{k,\mathbb{Q}}}(\operatorname{conv} F_{j,m}) = \operatorname{conv} F_{j+k,m}.$$

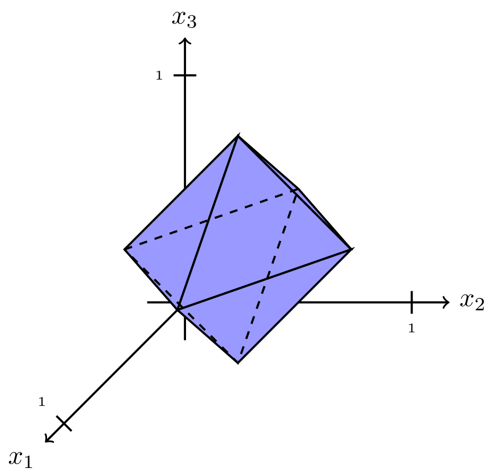
This enables us to compute all iterated $L_{k,\mathbb{Q}}$ closures of $P^{516,m}$. For $m = 2$ and $m = 3$, we visualized these in Figure 13.1 and Figure 13.2.

Proof. (Theorem 524) Let $w^1, \ldots, w^k \in \mathbb{Z}^m$ be as in Theorem 172. We show that

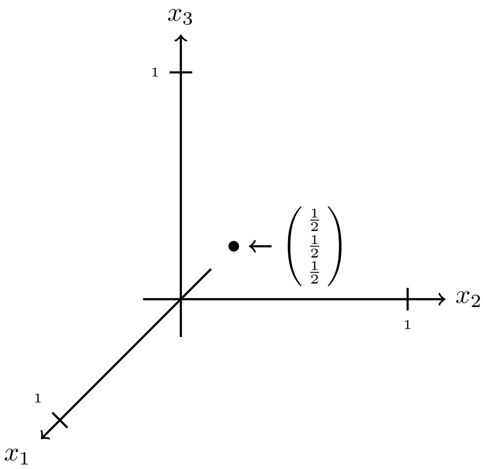
$$F_{j+k,m} \subseteq \operatorname{conv}\left((\operatorname{conv} F_{j,m}) \cap \left\{x \in \mathbb{R}^m : \left(w^i\right)^T x \in \mathbb{Z}\ \forall i \in [k]\right\}\right).$$



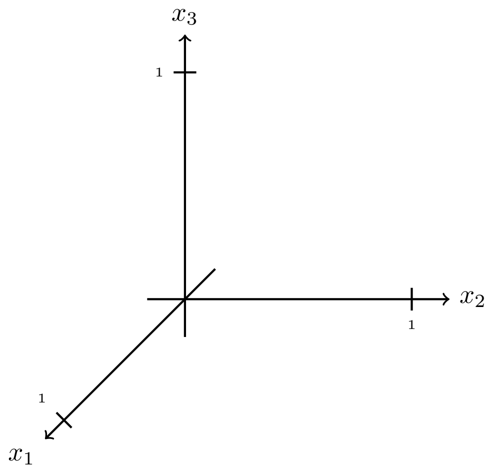
(a) $\operatorname{conv} F_{1,3} = P^{516,3} = \mathrm{cl}^{(0)}_{L_{1,\mathbb{Q}}}\left(P^{516,3}\right) = \mathrm{cl}^{(0)}_{L_{2,\mathbb{Q}}}\left(P^{516,3}\right)$



(b) $\operatorname{conv} F_{2,3} = \mathrm{cl}^{(1)}_{L_{1,\mathbb{Q}}}\left(P^{516,3}\right)$



(c) $\left\{\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}\right\} = \operatorname{conv} F_{3,3} = \mathrm{cl}^{(2)}_{L_{1,\mathbb{Q}}}\left(P^{516,3}\right)$
$= \mathrm{cl}^{(1)}_{L_{2,\mathbb{Q}}}\left(P^{516,3}\right)$



(d) $\emptyset = \mathrm{cl}^{(3)}_{L_{1,\mathbb{Q}}}\left(P^{516,3}\right) = \mathrm{cl}^{(2)}_{L_{2,\mathbb{Q}}}\left(P^{516,3}\right)$
$= \mathrm{cl}^{(1)}_{L_{3,\mathbb{Q}}}\left(P^{516,3}\right)$

Figure 13.2.: Iterative $L_{k,\mathbb{Q}}$ closure of $P^{516,3} = \operatorname{conv} F_{1,3}$

Let $v \in F_{j+k,m}$. W.l.o.g. let

$$v \in \left\{\frac{1}{2}\right\}^{j+k} \times \{0,1\}^{m-(j+k)} \tag{13.3}$$

(if this is not the case, do a permutation of the coordinates). For a vector $x \in \mathbb{Z}^d$ (where d may be chosen freely and is apparent from context), let

$$\Psi_{\mathbb{F}_2}(x) := x \mod 2 \cdot \mathbf{1}^d \in \mathbb{F}_2^d.$$

Conversely, we define for $x \in \mathbb{F}_2^d$

$$\Psi_{\mathbb{Z}}(x) := x \in \mathbb{Z}^d$$

as the embedding of x into $\mathbb{Z}^d$. Let

$$w'^i := \Psi_{\mathbb{F}_2}\left(w^i_{(1,\ldots,j+k)}\right) \in \mathbb{F}_2^{j+k}$$

for all $i \in [k]$. W.l.o.g. let $w'^1, \ldots, w'^r$ ($r \in \{0, \ldots, k\}$) be linearly independent over $\mathbb{F}_2$ and let $w'^{r+1}, \ldots, w'^k$

be representable as linear combination of $w'^1, \ldots, w'^r$ (over $\mathbb{F}_2$). Define

$$W' := \begin{pmatrix} \left(w'^1\right)^T \\ \vdots \\ \left(w'^r\right)^T \end{pmatrix} \in \mathbb{F}_2^{r \times (j+k)}.$$

By Corollary 523, there exist distinct $i_1, \ldots, i_{r^*} \in [r]$ ($r^* \in \{0, \ldots, r\}$) having

$$1^{j+k} + \sum_{j'=1}^{r^*} e^{j+k, i_{j'}} \in \ker W' \tag{13.4}$$

(over $\mathbb{F}_2$). For $I \in \mathcal{P}([r^*])$, define

$$v^I := v + \frac{1}{2} \left(\sum_{j' \in I} e^{m, i_{j'}} - \sum_{j' \in [r^*] \setminus I} e^{m, i_{j'}} \right). \tag{13.5}$$

Using these definitions, we obtain the following two statements:

- $$v = \frac{1}{2^{r^*}} \sum_{I \in \mathcal{P}([r^*])} v^I,$$

- $$v_i^I \begin{cases} \in \{0,1\} & \text{if } i \in \{i_1, \ldots, i_{r^*}\} \\ = \frac{1}{2} & \text{if } i \in [j+k] \setminus \{i_1, \ldots, i_{r^*}\} \\ \in \{0,1\} & \text{if } i \in \{j+k+1, \ldots, m\} \end{cases} \forall I \in \mathcal{P}([r^*]). \tag{13.6}$$

  For (13.6):

  - If $i \in \{i_1, \ldots, i_{r^*}\}$, we have using $i \in \{i_1, \ldots, i_{r^*}\} \subseteq [r] \subseteq [k] \subseteq [j+k]$:

    $$v_i^I = v_i + \frac{1}{2} \left(\sum_{j' \in I} e_i^{m, i_{j'}} - \sum_{j' \in [r^*] \setminus I} e_i^{m, i_{j'}} \right) = \frac{1}{2} \pm \frac{1}{2} \in \{0,1\}.$$

  - If $i \in [j+k] \setminus \{i_1, \ldots, i_{r^*}\}$, we have:

    $$v_i^I = v_i + \frac{1}{2} \left(\sum_{j' \in I} e_i^{m, i_{j'}} - \sum_{j' \in [r^*] \setminus I} e_i^{m, i_{j'}} \right) = \frac{1}{2} + 0 = \frac{1}{2}.$$

  - If $i \in \{j+k+1, \ldots, m\}$, we have using $\{i_1, \ldots, i_{r^*}\} \subseteq [r] \subseteq [k] \subseteq [j+k]$:

    $$v_i^I = v_i + \frac{1}{2} \left(\sum_{j' \in I} e_i^{m, i_{j'}} - \sum_{j' \in [r^*] \setminus I} e_i^{m, i_{j'}} \right) \in \{0,1\} + 0 = \{0,1\}.$$

Because $v^I \in F_{j+k-r^*, m}$ for all $I \in \mathcal{P}([r^*])$, we surely have using Lemma 521: $v^I \in \operatorname{conv} F_{j,m}$. What remains to be shown is

$$\forall i \in [k], I \in \mathcal{P}([r^*]) : \left(w^i\right)^T v^I \in \mathbb{Z}. \tag{13.7}$$

For this purpose, we show the following auxiliary statement: for all $I \in \mathcal{P}([r^*])$ and $\mathfrak{r} \in [r]$, we have

$$\left(w'^{\mathfrak{r}}\right)^T \Psi_{\mathbb{F}_2} \left(2 v^I_{(1, \ldots, j+k)}\right) = 0. \tag{13.8}$$

For (13.8):

$$\left(w'^{\mathfrak{r}}\right)^T \Psi_{\mathbb{F}_2} \left(2 v^I_{(1, \ldots, j+k)}\right)$$

$$\begin{aligned}
&= \left(w'^{\mathfrak{r}}\right)^T \Psi_{\mathbb{F}_2}\left(2\left(v_{(1,\dots,j+k)} + \frac{1}{2}\left(\sum_{j' \in I} e^{j+k,i_{j'}} - \sum_{j' \in [r^*]\setminus I} e^{j+k,i_{j'}}\right)\right)\right) && \text{(by (13.5))}\\
&= \left(w'^{\mathfrak{r}}\right)^T \Psi_{\mathbb{F}_2}\left(\underbrace{2v_{(1,\dots,j+k)}}_{=1^{j+k} \text{ (by (13.3))}} + \sum_{j' \in I} e^{j+k,i_{j'}} + \sum_{j' \in [r^*]\setminus I} e^{j+k,i_{j'}}\right)\\
&= \left(w'^{\mathfrak{r}}\right)^T \Psi_{\mathbb{F}_2}\left(1^{j+k} + \sum_{j' \in [r^*]} e^{j+k,i_{j'}}\right)\\
&= 0. && \text{(by (13.4))}
\end{aligned}$$

Now for the proof of (13.7): because of $\forall i \in [k] : w'^i = \Psi_{\mathbb{F}_2}\left(w^i_{(1,\dots,j+k)}\right)$, there exists a $\delta^i \in (2\mathbb{Z})^{j+k}$ having

$$w^i_{(1,\dots,j+k)} = \Psi_{\mathbb{Z}}\left(w'^i\right) + \delta^i. \tag{13.9}$$

Additionally, note that for all $i \in [k]$, there exists a $\lambda \in \mathbb{F}_2^r$ having

$$w'^i = \sum_{i'=1}^{r} \lambda_{i'} w'^{i'}. \tag{13.10}$$

For $i \in \{1, \dots, r\}$, this is obvious (just let $\lambda := e^{r,i}$). For $i \in \{r+1, \dots, k\}$, this is implied by the fact that $w'^{r+1}, \dots, w'^k$ can be represented as a linear combination of $w'^1, \dots, w'^r$ (over $\mathbb{F}_2$). Thus, for $i \in [k]$ and $I \in \mathcal{P}([r^*])$, we have

$$\begin{aligned}
\left(w^i\right)^T v^I &= \sum_{i'=1}^{j+k} w^i_{i'} v^I_{i'} + \sum_{i'=j+k+1}^{m} \underbrace{w^i_{i'}}_{\in\mathbb{Z}} \underbrace{v^I_{i'}}_{\in\mathbb{Z} \text{ (by (13.6))}}\\
&\in \sum_{i'=1}^{j+k} \left(\Psi_{\mathbb{Z}}\left(w'^i\right) + \delta^i\right)_{i'} v^I_{i'} + \mathbb{Z} && \text{(by (13.9))}\\
&= \sum_{i'=1}^{j+k} \left(\Psi_{\mathbb{Z}}\left(w'^i\right)\right)_{i'} v^I_{i'} + \sum_{i'=1}^{j+k} \underbrace{\delta^i_{i'}}_{\in 2\mathbb{Z}} \underbrace{v^I_{i'}}_{\substack{\in\left\{0,\frac{1}{2},1\right\}\\ \text{(by (13.6))}}} + \mathbb{Z}\\
&= \frac{1}{2}\sum_{i'=1}^{j+k} \left(\Psi_{\mathbb{Z}}\left(w'^i\right)\right)_{i'} \left(2v^I_{i'}\right) + \mathbb{Z}\\
&= \frac{1}{2}\sum_{i'=1}^{j+k} \left(\Psi_{\mathbb{Z}}\left(\sum_{j'=1}^{r} \lambda_{j'} w'^{j'}\right)\right)_{i'} \left(2v^I_{i'}\right) + \mathbb{Z} && \text{(by (13.10))}\\
&\subseteq \frac{1}{2}\left(\Psi_{\mathbb{Z}}\left(\sum_{i'=1}^{j+k}\left(\sum_{j'=1}^{r} \lambda_{j'} w'^{j'}\right)_{i'} \Psi_{\mathbb{F}_2}\left(2v^I_{i'}\right)\right) + 2\mathbb{Z}\right) + \mathbb{Z} && (13.11)\\
&= \underbrace{\frac{1}{2}\Psi_{\mathbb{Z}}\left(\sum_{j'=1}^{r} \lambda_{j'} \underbrace{\sum_{i'=1}^{j+k} w'^{j'}_{i'} \left(\Psi_{\mathbb{F}_2}\left(2v^I\right)\right)_{i'}}_{=0 \text{ (by (13.8))}}\right)}_{=0} + \mathbb{Z}\\
&= \mathbb{Z}.
\end{aligned}$$

We remark that (13.11) holds because for all $z \in \mathbb{Z}$, we have $z \in \Psi_{\mathbb{Z}}\left(\Psi_{\mathbb{F}_2}(z)\right) + 2\mathbb{Z}$. □

Theorem 526. *Let $m \in \mathbb{Z}_{\geq 1}$ and $k \in \{1, \ldots, m\}$. Then*

$$\operatorname{rank}_{L_{k,\mathbb{Q}}}\left(P^{516,m}\right) \geq \left\lceil \frac{m}{k} \right\rceil$$

(recall (6.11) *for the definition of the $L_{k,\mathbb{Q}}$ rank).*

Proof. For $k \in \{1, \ldots, m-1\}$, the statement is an immediate consequence of Theorem 524. For $k = m$, we know by Remark 517 that $\left(P^{516,m}\right)_I = \emptyset$. On the other hand, clearly $P^{516,m} \neq \emptyset$. Thus,

$$\operatorname{rank}_{L_{k,\mathbb{Q}}}\left(P^{516,m}\right) \geq 1 = \left\lceil \frac{m}{k} \right\rceil.$$

□

We immediately obtain as corollary of Theorem 515 and Theorem 526:

Corollary 527. *Let P and k be as in Definition 506. Then*

$$\operatorname{rank}_{L_{k,\mathbb{Q}}}(P) \leq \left\lceil \frac{m}{k} \right\rceil \tag{13.12}$$

and there exists a rational polytope $P \subseteq [0,1]^m$ having

$$\operatorname{rank}_{L_{k,\mathbb{Q}}}(P) = \left\lceil \frac{m}{k} \right\rceil$$

(in particular $P^{516,m}$). So, the bound (13.12) *is tight.*

13.2. $L_{k-\frac{1}{2},\mathbb{Q}\times(\,\cdot\,)}$ rank

We now consider the situation for the $L_{k-\frac{1}{2},\mathbb{Q}\times(\,\cdot\,)}$ rank. From Theorem 197, Theorem 199, Theorem 515 and Theorem 526, we conclude:

Theorem 528. *Let $P \subseteq [0,1]^m \times \mathbb{R}^n$ ($m \in \mathbb{Z}_{\geq 1}$ and $n \in \mathbb{Z}_{\geq 0}$) be a rational polyhedron. Then:*

- *If $n = 0$, we have for every $k \in \{1, \ldots, m-1\}$:*

$$\operatorname{rank}_{L_{k+1-\frac{1}{2},\mathbb{Q}}}(P) \leq \left\lceil \frac{m}{k} \right\rceil.$$

- *If $n \geq 1$, we have for every $k \in \{1, \ldots, m\}$:*

$$\operatorname{rank}_{L_{k+1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \operatorname{rank}_{L_{k+1-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P) \leq \left\lceil \frac{m}{k} \right\rceil.$$

On the other hand, for every $k \in \{1, \ldots, m-1\}$, we have

$$\operatorname{rank}_{L_{k-\frac{1}{2},\mathbb{Q}}}\left(P^{516,m}\right) \geq \left\lceil \frac{m}{k} \right\rceil.$$

Theorem 528 allows to bound the $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ rank/$L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ rank for $k \geq 2$ both from above and below, so that the upper and lower bound differ only by a constant factor. The situation is different for $k = 1$: Theorem 528 gives no upper bound for the $L_{k-\frac{1}{2},\mathbb{Q}}$ rank of a (w.l.o.g. rational) polyhedron $P \subseteq [0,1]^m$ (only a lower bound of m). Thus, it gives no idea whether there exists a polyhedron $P \subseteq [0,1]^m$ such that

$$\operatorname{rank}_{L_{1-\frac{1}{2},\mathbb{Q}}}\left(P^{516,m}\right) > \left\lceil \frac{m}{1} \right\rceil = m$$

holds. In other words: we don't know whether the lower bound in Theorem 528 is tight or not for $k = 1$.

Luckily, the problem of estimating the Chvátal-Gomory rank of a polyhedron $P \subseteq [0,1]^m$ (which, by Theorem 391, is equivalent to the $L_{1-\frac{1}{2},\mathbb{Q}}$ rank of P) has been studied extensively in the literature. In the remainder of this section, we give a short overview about some results concerning this topic.

13.2.1. Upper bounds for the Chvátal-Gomory rank

We now summarize some results from the literature about the Chvátal-Gomory rank of a polytope contained in the 0/1 cube. In [BEHS99, Lemma 3 and Corollary 4], it is shown:

Theorem 529. *Let $\emptyset \neq P \subseteq [0,1]^m$ $(m \in \mathbb{Z}_{\geq 1})$ be a rational polytope of dimension d with $P_I = \emptyset$. Then*

$$\operatorname{rank}_{CG}(P) \leq \max(1, d).$$

This implies that for every (not necessarily rational) polytope $P \subseteq [0,1]^m$, we have

$$\operatorname{rank}_{CG}(P) \leq m.$$

We now can ask: “How do polytopes $P \subseteq [0,1]^m$ $(m \in \mathbb{Z}_{\geq 1})$ having $\operatorname{rank}_{CG}(P) = m$ look like?”. In [ES03, Proposition 2.4], it is shown (we remark that some of the statements in Theorem 530 are not stated explicitly in [ES03, Proposition 2.4], but are shown in the proof):

Theorem 530. *Let $P \subseteq [0,1]^m$ $(m \in \mathbb{Z}_{\geq 1})$ be a polytope with $P_I = \emptyset$ and $\operatorname{rank}_{CG}(P) = m$. Then any inequality description $A(\cdot) \leq b$ $(A \in \mathbb{R}^{l \times m}, b \in \mathbb{R}^l$, where $l \in \mathbb{Z}_{\geq 1})$ of P satisfies the following two properties:*

- *For each point $x \in \{0,1\}^m$, there exists an $i \in [l]$ such that $A_{i,*}x > b_i$.*
- *For every $i \in [l]$, there exists at most one $x \in \{0,1\}^m$ having $A_{i,*}x > b_i$.*

Thus, in particular, we have $l \geq 2^m$, i.e. P has at least 2^m facets.

This characterization is made more precise in [PS11a]:

Theorem 531. *[PS11a, Theorem 3.12] For $m \in \mathbb{Z}_{\geq 1}$, let*

$$P^{531,m} := \left\{ x \in [0,1]^m : \sum_{j \in J} x_j + \sum_{j \in [m] \setminus J} (1 - x_j) \geq 1 \ \forall J \in \mathcal{P}([m]) \right\}$$

(note how the definitions of $P^{516,m}$ and $P^{531,m}$ are very related: the only difference is the coefficient at the right-hand side of the inequalities ($\frac{1}{2}$ vs 1)). Let $P \subseteq [0,1]^m$ $(m \in \mathbb{Z}_{\geq 1})$ be a polytope with $P_I = \emptyset$. Then the following statements are equivalent:

1. $\operatorname{rank}_{CG}(P) = m$,
2. $\operatorname{cl}_{CG}(P) = P^{531,m}$,
3. $P \cap F \neq \emptyset$ *for all 1-dimensional faces F of $[0,1]^m$,*
4. $\operatorname{rank}_{CG}(P \cap F) = k$ *for all k-dimensional faces F of $[0,1]^m$ $(k \in \{1, \ldots, m\})$.*

Now what if $P_I \neq \emptyset$? In [BEHS99, Theorem 10 and Theorem 11], it is shown:

Theorem 532. *Let $P \subseteq [0,1]^m$ $(m \in \mathbb{Z}_{\geq 1})$ be a rational polytope of dimension d. Then*

$$\operatorname{rank}_{CG}(P) \leq md \left(1 + \left\lfloor \frac{m}{2} \log_2 m \right\rfloor \right).$$

This implies that for every (not necessarily rational) polytope $P \subseteq [0,1]^m$, we have

$$\operatorname{rank}_{CG}(P) \leq m^2 \left(1 + \left\lfloor \frac{m}{2} \log_2 m \right\rfloor \right).$$

This bound is improved in [ES03, Theorem 3.3]:

Theorem 533. *Let $P \subseteq [0,1]^m$ $(m \in \mathbb{Z}_{\geq 1})$ be a polytope. Then*

$$\operatorname{rank}_{CG}(P) \leq m^2 (1 + \log m),$$

where

$$\log : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0} :$$
$$n \mapsto \begin{cases} 1 & \text{if } n = 0, \\ 1 + \lfloor \log_2 m \rfloor & \text{if } n \geq 1. \end{cases}$$

A different kind of upper bound is shown in [ES03, Theorem 4.6]. For this, we need the following definition:

Definition 534. *(cf. [ES03, section 2]) Let $P \subseteq \mathbb{R}^m$ ($m \in \mathbb{Z}_{\geq 1}$) be a polyhedron and let $c(\,\cdot\,) \leq c_0$ ($c \in (\mathbb{R}^m)^T$ and $c_0 \in \mathbb{R}$) be a linear inequality. The* ***depth of*** *$c(\,\cdot\,) \leq c_0$* ***with respect to*** *P is the smallest $k \in \mathbb{Z}_{\geq 0} \dot{\cup} \{\infty\}$ such that $c(\,\cdot\,) \leq c_0$ is valid for $\mathrm{cl}_{CG}^{(k)}(P)$.*

Theorem 535. *([ES03, Theorem 4.6]) Let $\emptyset \neq P \subseteq [0,1]^m$ ($m \in \mathbb{Z}_{\geq 1}$) be a polytope and let $c(\,\cdot\,) \leq c_0$ be a linear inequality for P_I with $c \in (\mathbb{Z}^m)^T$. Then $c(\,\cdot\,) \leq c_0$ has depth at most $n + \|c\|_1$ with respect to P.*

13.2.2. Lower bounds for the Chvátal-Gomory rank

From Theorem 528, we immediately obtain that $\mathrm{rank}_{CG}(P) = m$ can be attained for a full-dimensional rational polyhedron $P \subseteq [0,1]^m$ having $P_I = \emptyset$. So, the upper bound from Theorem 529 is tight in this situation and we only have to consider the case $P_I \neq \emptyset$. In [ES03, section 5], it is shown:

Theorem 536. *There exists an $\epsilon \in \mathbb{R}_{>0}$ and an infinite number of $m \in \mathbb{Z}_{\geq 1}$ for which there exists a rational polytope $P \subseteq [0,1]^m$ having $\mathrm{rank}_{CG}(P) > (1+\epsilon)\, m$.*

We remark that according to [ES03], this is the first time that someone gave an example of a polytope $P \subseteq [0,1]^m$ for some $m \in \mathbb{Z}_{\geq 1}$ such that $\mathrm{rank}_{CG}(P) > m$. What are the polyhedra P that the authors consider for Theorem 536? They define

$$P := \mathrm{conv}\left(P^{516,m} \cup Q_G\right) \subseteq [0,1]^m ,$$

where Q_G is a fractional stable set polytope associated with a suitable graph $G := ([m], E)$ (i.e.

$$Q_G := \left\{x \in \mathbb{R}^m_{\geq 0}, \forall C \in \mathcal{C} : x(C) \leq 1\right\} \subseteq [0,1]^m ,$$

where $\mathcal{C}$ is the family of all cliques in G).

Nevertheless, in [PS11b], it is remarked that for the bound in Theorem 536, the linear factor is very small. They also admit that the proof only yields a bound $\mathrm{rank}_{CG}(P^m) \geq (1+\epsilon)\, m - 1$, where $\epsilon \leq 3.12 \cdot 10^{-6}$. In [PS11b], the authors construct a family of polytopes, where the linear factor is larger:

Theorem 537. *[PS11b, Theorem 3.5] For any $\epsilon > 0$ and any $m_0 \in \mathbb{Z}_{\geq 1}$, there exists an $m \in \mathbb{Z}_{\geq 1}$ having $m \geq m_0$ and a polytope $P \subseteq [0,1]^m$ with $\mathrm{rank}_{CG}(P) \geq \left(1 + \frac{1}{e}\right) n - 1 - \epsilon$.*

The polytope that the authors consider in Theorem 537 is

$$P := \mathrm{conv}\left(P^{516,m} \cup \left\{x \in [0,1]^m : (1^m)^T x \leq d\right\} \cup \left\{\frac{1}{2} 1^m\right\}\right) \subseteq [0,1]^m ,$$

where $d \in [m]$ is suitably chosen.

This leaves the question open whether there exists a series of polytopes $\left\{P^i\right\}_{i \in \mathbb{Z}_{\geq 1}}$, where $P^i \subseteq [0,1]^i$, such that

$$\mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1} : t \mapsto \mathrm{rank}_{CG}\left(P^t\right)$$

grows superlinearly. This question is answered positively in [RS17]:

Theorem 538. *([RS17, Theorem 1, section 7 and Theorem 3]) For every $m \in \mathbb{Z}_{\geq 1}$, there exists a vector $c \in \left\{0, \ldots, 2^{\frac{m}{16}}\right\}^m$ such that the polytope*

$$\mathrm{conv}\left(\left\{x \in \{0,1\}^m : c^T x \leq \frac{\|c\|_1}{2}\right\} \cup \left\{\frac{3}{4} \cdot 1^m\right\}\right) \subseteq [0,1]^m$$

has Chvátal-Gomory rank $\Theta\left(m^2\right)$. Its linear relaxation

$$\mathrm{conv}\left(\left\{x \in [0,1]^m : c^T x \leq \frac{\|c\|_1}{2}\right\} \cup \left\{\frac{3}{4} \cdot 1^m\right\}\right) \subseteq [0,1]^m$$

has Chvátal-Gomory rank $\Theta\left(m^2\right)$, too, and can be decribed with $O(m)$ inequalities with integral coefficients of size (at most) $2^{O(m)}$.

Even though the authors admit in [RS17, section 7] that they have no insight whether the worst case Chvátal-Gomory rank of a polytope $P \subseteq [0,1]^m$ should be $\Theta\left(m^2\right)$ or $\Theta\left(m^2 \log m\right)$, they nevertheless give a rough sketch of an observation that might be used to tighten the upper bound of Theorem 533 even further.

Part VI.

Strictness of the $L_{k-\frac{1}{2}}/L_k$ hierarchy

14. Pyramids over cross polytopes

14.1. Motivation

We saw in Theorem 197, Theorem 199 and Theorem 202 that for every rational polyhedron $P \subseteq \mathbb{R}^m$ ($m \in \mathbb{Z}_{\geq 0}$), we have

$$P = \mathrm{cl}_{L_{0,\mathbb{Q}}}(P) \supseteq \mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{Q}}}(P) \supseteq \mathrm{cl}_{L_{1,\mathbb{Q}}}(P) \supseteq \ldots \supseteq \mathrm{cl}_{L_{m-1,\mathbb{Q}}}(P) \supseteq \mathrm{cl}_{L_{m-\frac{1}{2},\mathbb{Q}}}(P) = \mathrm{cl}_I(P)$$

and for all rational polyhedra $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m \in \mathbb{Z}_{\geq 0}$, $n \in \mathbb{Z}_{\geq 1}$), the chain of inclusions

$$\begin{aligned} P = \mathrm{cl}_{L_{0,\mathbb{Q}}}(P) &\supseteq \mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P) \supseteq \mathrm{cl}_{L_{1,\mathbb{Q}}}(P) \supseteq \ldots \supseteq \mathrm{cl}_{L_{m-1,\mathbb{Q}}}(P) \\ &\supseteq \mathrm{cl}_{L_{m-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{m-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P) \supseteq \mathrm{cl}_{L_{m,\mathbb{Q}}}(P) = \mathrm{cl}_{L_{(m+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{(m+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P) = \mathrm{cl}_I(P) \end{aligned}$$

holds (recall that the equality of the $L_{(\,\cdot\,)-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ and $L_{(\,\cdot\,)-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ closure in this chain of inclusions holds by Theorem 193). What is still open up to here is whether for fixed m and n, there can be a *strict* inclusion for any arbitrarily chosen inclusion of this chain. In other words:

- Does for every $m \in \mathbb{Z}_{\geq 1}$, $n \in \mathbb{Z}_{\geq 0}$ and $k \in \{1, \ldots, m\}$ exist a rational polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ such that
$$\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) \subsetneq \mathrm{cl}_{L_{k-1,\mathbb{Q}}}(P)?$$
- Does for every $m \in \mathbb{Z}_{\geq 1}$, $n \in \mathbb{Z}_{\geq 0}$ and $k \in \begin{cases} \{1, \ldots, m-1\} & \text{if } n = 0, \\ \{1, \ldots, m\} & \text{if } n \geq 1 \end{cases}$ exist a rational polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ such that
$$\mathrm{cl}_{L_{k,\mathbb{Q}}}(P) \subsetneq \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P)?$$

For this, we consider the following even stronger questions:

- Does for every $m \in \mathbb{Z}_{\geq 1}$, $n \in \mathbb{Z}_{\geq 0}$ and $k \in \{1, \ldots, m\}$ exist a rational polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ such that
$$\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) \subsetneq \mathrm{cl}_{L_{k-1,\mathbb{R}}}(P)? \tag{14.1}$$
- Does for every $m \in \mathbb{Z}_{\geq 1}, n \in \mathbb{Z}_{\geq 0}$ and $k \in \begin{cases} \{1, \ldots, m-1\} & \text{if } n = 0, \\ \{1, \ldots, m\} & \text{if } n \geq 1 \end{cases}$ exist a rational polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ such that
$$\mathrm{cl}_{L_{k,\mathbb{Q}}}(P) \subsetneq \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{R}\times\mathbb{R}}}(P)? \tag{14.2}$$

We resolve these questions in the remainder of this chapter, except for the situation in (14.2) if $n \geq 1$ and $k = m$. In section 14.9.3, we give an outlook for this remaining inclusion, which we finally resolve at the end of chapter 15 (section 15.5.2).

14.2. Definition of the polytopes

Definition 539. *For $m \in \mathbb{Z}_{\geq 1}$, let*

$$C^{539,m} := \mathrm{conv} \dot{\bigcup_{i=1}^{m}} \left\{ \frac{1}{2} \cdot 1^m \pm \frac{m}{2} e^{m,i} \right\}.$$

We have

$$C^{539,m} = \left\{ x \in \mathbb{R}^m : \sum_{i \in I} x_i - \sum_{i \in [m] \setminus I} x_i \leq |I| \;\; \forall I \in \mathcal{P}([m]) \right\}. \tag{14.3}$$

(14.3) holds because of:

Lemma 540. *Let $d \in \mathbb{Z}_{\geq 1}$ and let $C \in \mathbb{R}_{\geq 0}$. Then*

$$\left\{ x \in \mathbb{R}^d : \left\| x - \frac{1}{2} \cdot 1^d \right\|_1 \leq C \right\} = \left\{ x \in \mathbb{R}^d : \sum_{i \in I} x_i - \sum_{i \in [d] \setminus I} x_i \leq |I| + C - \frac{d}{2} \;\forall I \in \mathcal{P}([d]) \right\}.$$

Proof.

$$\begin{aligned} \left\{ x \in \mathbb{R}^d : \left\| x - \frac{1}{2} \cdot 1^d \right\|_1 \leq C \right\} &= \left\{ x \in \mathbb{R}^d : \sum_{i \in I} \left(x_i - \frac{1}{2} \right) - \sum_{i \in [d] \setminus I} \left(x_i - \frac{1}{2} \right) \leq C \;\forall I \in \mathcal{P}([d]) \right\} \\ &= \left\{ x \in \mathbb{R}^d : \sum_{i \in I} x_i - \frac{|I|}{2} - \sum_{i \in [d] \setminus I} x_i + \frac{d - |I|}{2} + \leq C \;\forall I \in \mathcal{P}([d]) \right\} \\ &= \left\{ x \in \mathbb{R}^d : \sum_{i \in I} x_i - \sum_{i \in [d] \setminus I} x_i \leq |I| + C - \frac{d}{2} \;\forall I \in \mathcal{P}([d]) \right\}. \end{aligned}$$

□

Definition 541. *Let $m \in \mathbb{Z}_{\geq 2}$ and let $h \in \mathbb{R}_{>0}$ (h – "height"). Define*

$$P^{541,m,h} := \operatorname{conv}\left(\left(C^{539,m-1} \times \{0\} \right) \dot\cup \left\{ \begin{pmatrix} \frac{1}{2} \cdot 1^{m-1} \\ h \end{pmatrix} \right\} \right) \subseteq \mathbb{R}^m.$$

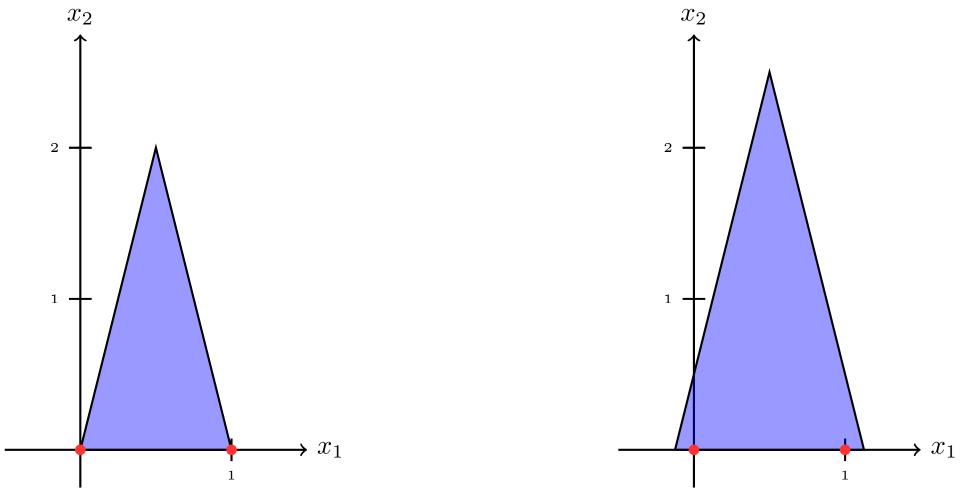
We have

$$P^{541,m,h} = \left\{ x \in \mathbb{R}^m : x_m \geq 0 \wedge \sum_{i \in I} x_i - \sum_{i \in [m-1] \setminus I} x_i + \frac{m-1}{2h} x_m \leq |I| \;\; \forall I \in \mathcal{P}([m-1]) \right\}.$$

Definition 542. *Let $m \in \mathbb{Z}_{\geq 2}$ and let $h \in \mathbb{R}_{>0}$. Define*

$$P^{542,m,h} = \left\{ x \in \mathbb{R}^m : x_m \geq 0 \wedge \sum_{i \in I} x_i - \sum_{i \in [m-1] \setminus I} x_i + \frac{m-1}{2h} x_m \leq |I| + \frac{m-1}{4h} \;\forall I \in \mathcal{P}([m-1]) \right\}.$$

In Figure 14.1 and Figure 14.2, we visualized instances of $P^{541,m,h}$ and $P^{542,m,h}$ for $m = 2$ and $m = 3$.



(a) $P^{541,2,2}$ and $P^{541,2,2} \cap \mathbb{Z}^2$ (b) $P^{542,2,2}$ and $P^{542,2,2} \cap \mathbb{Z}^2$

Figure 14.1.: $P^{541,2,2}$ and $P^{542,2,2}$

14.3. Statements

Theorem 543. *For $m \in \mathbb{Z}_{\geq 2}$, we have*

$$\begin{pmatrix} \frac{1}{2} \cdot \mathbb{1}^{m-1} \\ \frac{1}{2} \end{pmatrix} \in \operatorname{cl}_{L_{m-2,\mathbb{R}}} \left(P^{541,m,2(m-1)} \right) \tag{14.4}$$

$$\subseteq \operatorname{cl}_{(m-2)BC} \left(P^{541,m,2(m-1)} \right), \tag{14.5}$$

$$\begin{pmatrix} \frac{1}{2} \cdot \mathbb{1}^{m-1} \\ 1 \end{pmatrix} \in \operatorname{cl}_{L_{m-1-\frac{1}{2},\mathbb{R}}} \left(P^{541,m,2(m-1)} \right), \tag{14.6}$$

$$[0,1]^{m-1} \times \{0\} = \operatorname{cl}_I \left(P^{541,m,2(m-1)} \right)$$

$$= \operatorname{cl}_{(m-1)BS} \left(P^{541,m,2(m-1)} \right) \tag{14.7}$$

$$= \operatorname{cl}_{L_{m-1,\mathbb{Q}}} \left(P^{541,m,2(m-1)} \right). \tag{14.8}$$

In particular, $x_m \leq 0$ is not valid for $x \in \operatorname{cl}_{L_{m-2,\mathbb{R}}} \left(P^{541,m,2(m-1)} \right)$ and $x \in \operatorname{cl}_{L_{m-1-\frac{1}{2},\mathbb{R}}} \left(P^{541,m,2(m-1)} \right)$.

Theorem 544. *For $m \in \mathbb{Z}_{\geq 2}$, we have*

$$\begin{pmatrix} \frac{1}{2} \cdot \mathbb{1}^{m-1} \\ 1 \end{pmatrix} \in \operatorname{cl}_{L_{m-1-\frac{1}{2},\mathbb{R}}} \left(P^{542,m,2(m-1)} \right), \tag{14.9}$$

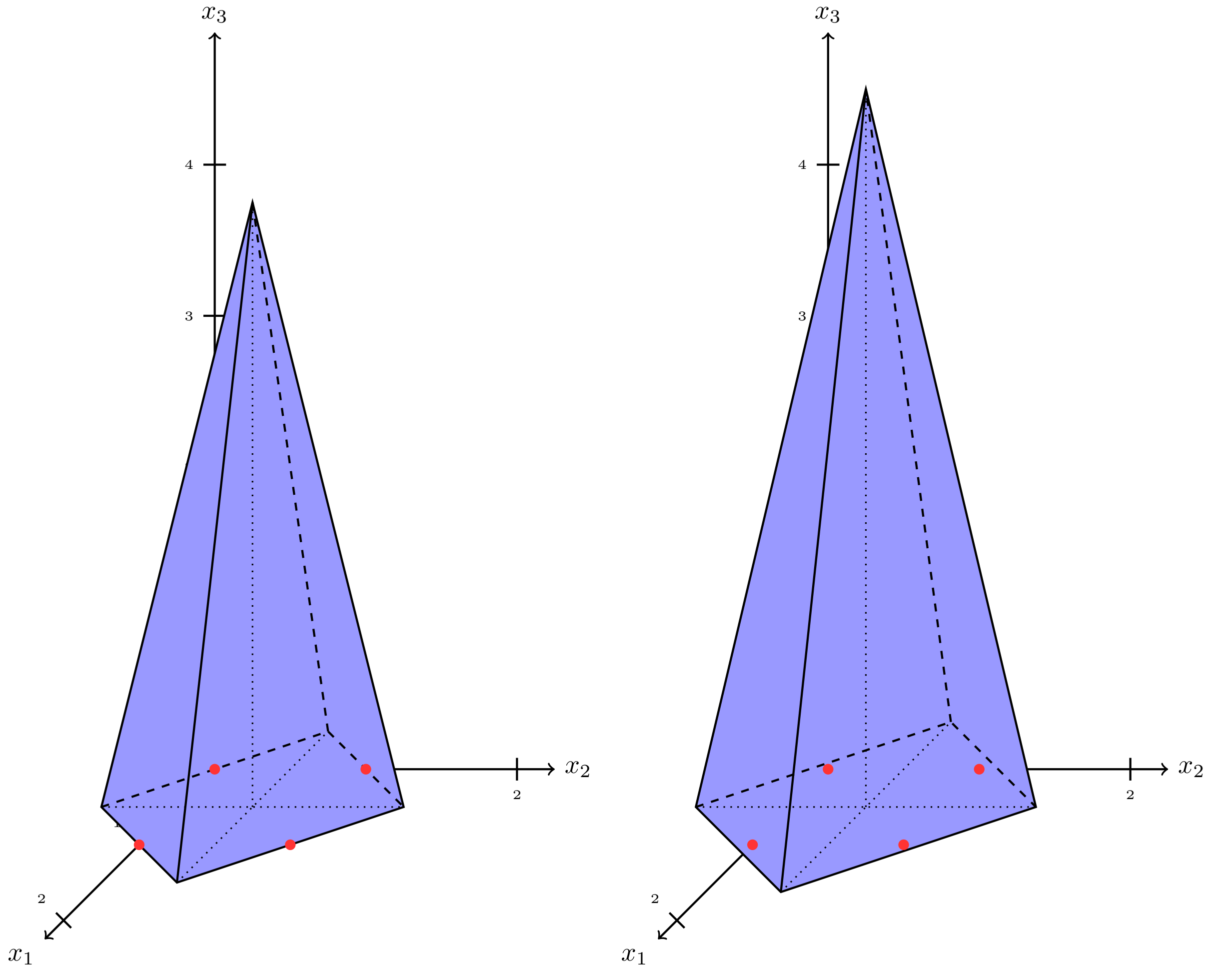
$$\begin{pmatrix} \frac{1}{2} \cdot \mathbb{1}^{m-1} \\ \frac{1}{2} \end{pmatrix} \in \operatorname{cl}_{L_{m-1,\mathbb{R}}} \left(P^{542,m,2(m-1)} \right), \tag{14.10}$$

$$[0,1]^{m-1} \times \{0\} = \operatorname{cl}_I \left(P^{542,m,2(m-1)} \right)$$

$$= \operatorname{cl}_{L_{m-\frac{1}{2},\mathbb{Q}}} \left(P^{542,m,2(m-1)} \right). \tag{14.11}$$

In particular, $x_m \leq 0$ is not valid for $x \in \operatorname{cl}_{L_{m-1-\frac{1}{2},\mathbb{R}}} \left(P^{542,m,2(m-1)} \right)$ and $x \in \operatorname{cl}_{L_{m-1,\mathbb{R}}} \left(P^{542,m,2(m-1)} \right)$.

Theorem 543 and Theorem 544, respectively, are shown in section 14.7 and section 14.8, respectively. In Figure 14.1, one can see a visualization of the polytopes of Theorem 543 and Theorem 544 for $m = 2$ and in Figure 14.2 for $m = 3$.



(a) $P^{542,3,4}$ and $P^{542,3,4} \cap \mathbb{Z}^3$

(b) $P^{541,3,4}$ and $P^{541,3,4} \cap \mathbb{Z}^3$

Figure 14.2.: $P^{541,3,4}$ and $P^{542,3,4}$

14.4. General preparations

Lemma 545. *We have for $m \in \mathbb{Z}_{\geq 2}$:*

$$\left\{x \in \mathbb{R}^{m-1} : \left\| x - \frac{1}{2} \cdot 1^{m-1} \right\|_1 \leq \frac{m-2}{2}\right\} \times [0,2] \subseteq P^{541,m,2(m-1)}.$$

Proof. Using Lemma 540, we conclude:

$$\begin{aligned} &\left\{x \in \mathbb{R}^{m-1} : \left\| x_{[m-1]} - \frac{1}{2} \cdot 1^{m-1} \right\|_1 \leq \frac{m-2}{2}\right\} \\ =& \left\{x \in \mathbb{R}^{m-1} : \sum_{i \in I} x_i - \sum_{i \in [m-1] \setminus I} x_i \leq |I| + \frac{m-2}{2} - \frac{m-1}{2} \ \forall I \in \mathcal{P}([m-1])\right\} \\ =& \left\{x \in \mathbb{R}^{m-1} : \sum_{i \in I} x_i - \sum_{i \in [m-1] \setminus I} x_i \leq |I| - \frac{1}{2} \ \forall I \in \mathcal{P}([m-1])\right\}. \end{aligned}$$

Let $x \in \left\{x' \in \mathbb{R}^{m-1} : \left\|x' - \frac{1}{2} \cdot 1^{m-1}\right\|_1 \leq \frac{m-2}{2}\right\} \times [0,2]$. Then surely $x_m \geq 0$. Let $I \in \mathcal{P}([m-1])$. Then

$$\sum_{i \in I} x_i - \sum_{i \in [m-1] \setminus I} x_i + \frac{m-1}{2 \cdot 2(m-1)} x_m \leq \sum_{i \in I} x_i - \sum_{i \in [m-1] \setminus I} x_i + \frac{1}{2} \qquad (x_m \leq 2)$$

$$\leq |I| - \frac{1}{2} + \frac{1}{2}$$
$$= |I| .$$

Thus, $x \in P^{541,m,m-1}$. □

Lemma 546. *Let $v \in \mathbb{R}^d$ ($d \in \mathbb{Z}_{\geq 1}$) satisfy $\|v\|_1 = \frac{1}{2}$. Let $J(v) \in \{0,1\}^d$ be such that*

$$J(v)_i = \begin{cases} 1 & \text{if } v_i \geq 0, \\ 0 & \text{if } v_i < 0 \end{cases} \quad \forall i \in [d] .$$

Then

$$\left\| J(v) - v - \frac{1}{2} \cdot 1^d \right\|_1 = \frac{d-1}{2}.$$

Proof.

$$\begin{aligned}
\left\| J(v) - v - \frac{1}{2} \cdot 1^d \right\|_1 &= \sum_{i=1}^{d} \left| J(v)_i - v_i - \frac{1}{2} \right| \\
&= \sum_{i=1}^{d} \begin{cases} \left|-v_i + \frac{1}{2}\right| & \text{if } v_i \geq 0, \\ \left|-v_i - \frac{1}{2}\right| & \text{if } v_i < 0 \end{cases} \\
&= \sum_{i=1}^{d} \begin{cases} \left|\frac{1}{2} - v_i\right| & \text{if } v_i \geq 0, \\ \left|\frac{1}{2} + v_i\right| & \text{if } v_i < 0 \end{cases} \\
&= \sum_{i=1}^{d} \left| \frac{1}{2} - |v_i| \right| \\
&= \sum_{i=1}^{d} \left(\frac{1}{2} - |v_i| \right) && (\|v\|_1 = \frac{1}{2}) \\
&= \frac{d}{2} - \|v\|_1 \\
&= \frac{d-1}{2}. && (\|v\|_1 = \frac{1}{2})
\end{aligned}$$

□

14.5. Preparations for $L_{k-\frac{1}{2},\mathbb{R}}$ cuts

Lemma 547. *Let $v \in \mathbb{R}^m \setminus \{0^m\}$ ($m \in \mathbb{Z}_{\geq 2}$). Then there exists an $x(v) \in \{0,1\}^{m-1} \times \mathbb{Z}_{\geq 2}$ such that*

$$x(v) \in P^{541,m,2(m-1)} + \operatorname{lin}\{v\} .$$

Proof. If $v_{[m-1]} = 0^{m-1}$, the statement is obvious, since using Lemma 545, we have

$$\{0,1\}^{m-1} \times \{2\} \subseteq P^{541,m,2(m-1)} \subseteq P^{541,m,2(m-1)} + \operatorname{lin}\{v\} .$$

So assume $v_{[m-1]} \neq 0^{m-1}$. W.l.o.g. let

$$\left\| v_{[m-1]} \right\|_1 = \frac{1}{2},$$
$$v_m \geq 0.$$

Let $J\left(v_{[m-1]}\right) \in \{0,1\}^{m-1}$ be as in Lemma 546. Set

$$x(v) := \begin{pmatrix} J\left(v_{[m-1]}\right) \\ 2 + \lfloor v_m \rfloor \end{pmatrix},$$
$$p(v) := x(v) - v.$$

Clearly, $x(v) \in \{0,1\}^{m-1} \times \mathbb{Z}_{\geq 2}$ and $x(v)_m \geq 2$. We now show

$$p(v) \in P^{541,m,2(m-1)}. \tag{14.12}$$

From Lemma 546, we obtain

$$\left\| p(v)_{[m-1]} - \frac{1}{2} \cdot 1^{m-1} \right\|_1 = \left\| J(v)_{[m-1]} - v_{[m-1]} - \frac{1}{2} \cdot 1^{m-1} \right\|_1 \leq \frac{m-2}{2}.$$

Thus, (14.12) is implied by Lemma 545, since $1 < p(v)_m \leq 2$. □

Lemma 548. *Let $v \in \mathbb{R}^m \setminus \{0^m\}$ $(m \in \mathbb{Z}_{\geq 2})$. Then*

$$\begin{pmatrix} \frac{1}{2} \cdot 1^{m-1} \\ 1 \end{pmatrix} \in \mathrm{cl}_I \left(P^{541,m,m-1} + \mathrm{lin}\{v\} \right).$$

Proof. By Lemma 547, there exists an $x(v)$ having

$$x(v) \in \left(\{0,1\}^{m-1} \times \mathbb{Z}_{\geq 2} \right) \cap \left(P^{541,m,m-1} + \mathrm{lin}\{v\} \right).$$

Surely

$$\begin{pmatrix} 1^{m-1} - x(v)_{[m-1]} \\ 0 \end{pmatrix} \in \left(P^{541,m,m-1} \right)_I \subseteq \mathrm{cl}_I \left(P^{541,m,m-1} + \mathrm{lin}\{v\} \right).$$

Additionally, since for every $J \in \{0,1\}^{m-1}$, we have $\begin{pmatrix} J \\ 0 \end{pmatrix} \in \left(P^{541,m,m-1} \right)_I$, we obtain

$$\begin{pmatrix} \frac{1}{2} \cdot 1^{m-1} \\ 0 \end{pmatrix} \in \mathrm{cl}_I \left(P^{541,m,m-1} \right) \subseteq \mathrm{cl}_I \left(P^{541,m,m-1} + \mathrm{lin}\{v\} \right).$$

Thus,

$$\begin{aligned} \begin{pmatrix} \frac{1}{2} \cdot 1^{m-1} \\ 1 \end{pmatrix} &= \left(1 - \frac{2}{x(v)_m} \right) \begin{pmatrix} \frac{1}{2} \cdot 1^{m-1} \\ 0 \end{pmatrix} + \frac{1}{x(v)_m} x(v) + \frac{1}{x(v)_m} \begin{pmatrix} 1^{m-1} - x(v)_{[m-1]} \\ 0 \end{pmatrix} \\ &\in \mathrm{cl}_I \left(P^{541,m,m-1} + \mathrm{lin}\{v\} \right) \end{aligned}$$

(since $x(v)_m \geq 2$, we have $0 < \frac{2}{x(v)_m} \leq 1$). □

14.6. Preparations for $L_{k,\mathbb{R}}$ cuts

Lemma 549. *Let $v \in \mathbb{R}^m$ $(m \in \mathbb{Z}_{\geq 2})$ be given such that $v_{[m-1]} \neq 0^{m-1}$. Then there exists a $p(v) \in \{0,1\}^{m-1} \times (1,2]$ such that*

$$p(v) \in P^{541,m,2(m-1)} \cap (\mathbb{Z}^m + \mathrm{lin}\{v\}).$$

Proof. W.l.o.g. let

$$\left\| v_{[m-1]} \right\|_1 = \frac{1}{2}.$$

Let $J\left(v_{[m-1]}\right)$ be as in Lemma 546 and let

$$\begin{aligned} p(v) &:= \begin{pmatrix} J\left(v_{[m-1]}\right) \\ 2 + \lfloor v_m \rfloor \end{pmatrix} - v, \\ x(v) &:= p(v) + v. \end{aligned}$$

Clearly, $x(v) \in \mathbb{Z}^m$ and $p(v)_m \in (1,2]$. So, $p(v) = x(v) - v \in \mathbb{Z}^m + \mathrm{lin}\{v\}$. We claim that

$$p(v) \in P^{541,m,2(m-1)}. \tag{14.13}$$

Using Lemma 546, we obtain

$$\left\|p(v)_{[m-1]} - \frac{1}{2}1^{m-1}\right\|_1 = \left\|J(v)_{[m-1]} - v_{[m-1]} - \frac{1}{2}1^{m-1}\right\|_1 \leq \frac{m-1-1}{2} = \frac{m-2}{2}.$$

Thus, (14.13) is implied by Lemma 545, since $1 < p(v)_m \leq 2$. □

Lemma 550. *Let $v \in \mathbb{R}^m$ ($m \in \mathbb{Z}_{\geq 2}$) be such that $v_{[m-1]} \neq 0^{m-1}$. Then*

$$\begin{pmatrix} \frac{1}{2} \cdot 1^{m-1} \\ \frac{1}{2} \end{pmatrix} \in \operatorname{conv}\left(P^{541,m,2(m-1)} \cap (\mathbb{Z}^m + \operatorname{lin}\{v\})\right).$$

Proof. By Lemma 549, there exists some $p(v) \in \{0,1\}^{m-1} \times (1,2]$ such that

$$p(v) \in P^{541,m,2(m-1)} \cap (\mathbb{Z}^m + \operatorname{lin}\{v\}).$$

So, clearly,

$$\begin{pmatrix} \frac{1}{2} \cdot 1^{m-1} \\ 0 \end{pmatrix}, p(v), \begin{pmatrix} 1^{m-1} - p(v)_{[m-1]} \\ 0 \end{pmatrix} \in \operatorname{conv}\left(P^{541,m,2(m-1)} \cap (\mathbb{Z}^m + \operatorname{lin}\{v\})\right).$$

Thus, we just have to show that $\begin{pmatrix} \frac{1}{2} \cdot 1^{m-1} \\ \frac{1}{2} \end{pmatrix}$ is a convex combination of these three points:

$$\begin{pmatrix} \frac{1}{2} \cdot 1^{m-1} \\ \frac{1}{2} \end{pmatrix} = \left(1 - \frac{1}{p(v)_m}\right)\begin{pmatrix} \frac{1}{2} \cdot 1^{m-1} \\ 0 \end{pmatrix} + \frac{1}{2p(v)_m} p(v) + \frac{1}{2p(v)_m}\begin{pmatrix} 1^{m-1} - p(v)_{[m-1]} \\ 0 \end{pmatrix}.$$

□

14.7. Proof of Theorem 543

We restate Theorem 543:

Theorem 543. *For $m \in \mathbb{Z}_{\geq 2}$, we have*

$$\begin{pmatrix} \frac{1}{2} \cdot 1^{m-1} \\ \frac{1}{2} \end{pmatrix} \in \operatorname{cl}_{L_{m-2,\mathbb{R}}}\left(P^{541,m,2(m-1)}\right) \tag{14.4}$$

$$\subseteq \operatorname{cl}_{(m-2)BC}\left(P^{541,m,2(m-1)}\right), \tag{14.5}$$

$$\begin{pmatrix} \frac{1}{2} \cdot 1^{m-1} \\ 1 \end{pmatrix} \in \operatorname{cl}_{L_{m-1-\frac{1}{2},\mathbb{R}}}\left(P^{541,m,2(m-1)}\right), \tag{14.6}$$

$$[0,1]^{m-1} \times \{0\} = \operatorname{cl}_I\left(P^{541,m,2(m-1)}\right)$$

$$= \operatorname{cl}_{(m-1)BS}\left(P^{541,m,2(m-1)}\right) \tag{14.7}$$

$$= \operatorname{cl}_{L_{m-1,\mathbb{Q}}}\left(P^{541,m,2(m-1)}\right). \tag{14.8}$$

In particular, $x_m \leq 0$ is not valid for $x \in \operatorname{cl}_{L_{m-2,\mathbb{R}}}\left(P^{541,m,2(m-1)}\right)$ and $x \in \operatorname{cl}_{L_{m-1-\frac{1}{2},\mathbb{R}}}\left(P^{541,m,2(m-1)}\right)$.

Proof. For (14.4): Let $V = \operatorname{lin}\{v^1, v^2\} \leq \mathbb{R}^m$ be a subspace of codimension 2, where $v^1, v^2 \in \mathbb{R}^m$ are linearly independent. Then at least one of these two vectors does not lie in $\operatorname{lin}\{e^{m,m}\}$ (w.l.o.g. let $v^1 \notin \operatorname{lin}\{e^{m,m}\}$). So, by Lemma 550, we have

$$\begin{pmatrix} \frac{1}{2} \cdot 1^{m-1} \\ \frac{1}{2} \end{pmatrix} \in \operatorname{conv}\left(P^{541,m,2(m-1)} \cap \left(\mathbb{Z}^m + \operatorname{lin}\{v^1\}\right)\right).$$

(14.5) holds because of

$$\operatorname{cl}_{L_{m-2,\mathbb{R}}}\left(P^{541,m,2(m-1)}\right) \subseteq \operatorname{cl}_{L_{m-2,\mathbb{Q}}}\left(P^{541,m,2(m-1)}\right) \subseteq \operatorname{cl}_{(m-2)BC}\left(P^{541,m,2(m-1)}\right),$$

where the second inclusion is a consequence of Corollary 278.

(14.6) holds by Lemma 548.

For (14.8), consider that

$$\operatorname{conv}\left(P^{541,m,2(m-1)} \cap \left(\mathbb{Z}^m + \operatorname{lin}\left\{e^{m,m}\right\}\right)\right) = [0,1]^{m-1} \times \{0\} = \operatorname{cl}_I\left(P^{541,m,2(m-1)}\right).$$

For (14.7), we show that

$$x_m \leq 0, \tag{14.14}$$
$$x_i \geq 0 \ \forall j \in [m-1], \tag{14.15}$$
$$x_i \leq 1 \ \forall j \in [m-1] \tag{14.16}$$

are valid inequalities for $x \in P^{541,m,2(m-1)} \cap D\left(e^{m,1}, \ldots, e^{m,m-1}, 0, \ldots, 0\right)$.

For (14.14): Let $I \in \mathcal{P}([m-1])$. We show that (14.14) is valid for

$$x \in P^{541,m,2(m-1)} \cap \bigcap_{i\in[m-1]\setminus I} P^{\leq}\left(\left(e^{m,i}\right)^T, 0\right) \cap \bigcap_{i\in I} P^{\geq}\left(\left(e^{m,i}\right)^T, 1\right).$$

For this, let such an x be given. Then

$$\begin{aligned}
x_m &= 4\left(\sum_{i\in I} x_i - \sum_{i\in[m-1]\setminus I} x_i + \frac{m-1}{2\cdot 2(m-1)}x_m\right) - 4\left(\sum_{i\in I} x_i - \sum_{i\in[m-1]\setminus I} x_i\right) \\
&\leq 4|I| - 4\left(\sum_{i\in I} \underbrace{x_i}_{\geq 1} - \sum_{i\in[m-1]\setminus I} \underbrace{x_i}_{\leq 0}\right) && \left(x \in P^{541,m,2(m-1)}\right) \\
&\leq 4|I| - 4|I| \\
&= 0.
\end{aligned}$$

For (14.15) **and** (14.16): W.l.o.g. we only prove (14.15). Let $j \in [m-1]$ and let $I \in \mathcal{P}([m-1])$. We show that (14.15) is valid for

$$x \in P^{541,m,2(m-1)} \cap \bigcap_{i\in[m-1]\setminus I} P^{\leq}\left(\left(e^{m,i}\right)^T, 0\right) \cap \bigcap_{i\in I} P^{\geq}\left(\left(e^{m,i}\right)^T, 1\right).$$

If $j \in I$, there is nothing to prove; so w.l.o.g. let $j \in [m-1] \setminus I$. Then

$$\begin{aligned}
x_j =& x_j + \left(\sum_{i\in I} x_i - \sum_{i\in[m-1]\setminus I} x_i + \frac{m-1}{2\cdot 2(m-1)}x_m\right) \\
&- \left(\sum_{i\in I} x_i - \sum_{i\in[m-1]\setminus I} x_i + \frac{m-1}{2\cdot 2(m-1)}x_m\right) \\
\geq& x_j + \left(\sum_{i\in I} x_i - \sum_{i\in[m-1]\setminus I} x_i + \frac{m-1}{2\cdot 2(m-1)}x_m\right) - |I| && \left(x \in P^{541,m,2(m-1)}\right) \\
=& \left(\sum_{i\in I} \underbrace{x_i}_{\geq 1} - \sum_{i\in[m-1]\setminus(I\cup\{j\})} \underbrace{x_i}_{\leq 0} + \underbrace{\frac{m-1}{2\cdot 2(m-1)}x_m}_{\geq 0}\right) - |I| \\
\geq& |I| - |I| \\
=& 0.
\end{aligned}$$

Finally, for (14.8), consider that

$$\begin{aligned}\mathrm{cl}_I\left(P^{541,m,2(m-1)}\right) &\subseteq \mathrm{cl}_{L_{m-1,\mathbb{Q}}}\left(P^{541,m,2(m-1)}\right)\\ &\subseteq \mathrm{cl}_{(m-1)BS}\left(P^{541,m,2(m-1)}\right) && \text{(by Corollary 278)}\\ &= \mathrm{cl}_I\left(P^{541,m,2(m-1)}\right). && \text{(by (14.7))}\end{aligned}$$

□

14.8. Proof of Theorem 544

We restate Theorem 544:

Theorem 544. *For $m \in \mathbb{Z}_{\geq 2}$, we have*

$$\begin{pmatrix} \frac{1}{2}\cdot \mathbb{1}^{m-1} \\ 1 \end{pmatrix} \in \mathrm{cl}_{L_{m-1-\frac{1}{2},\mathbb{R}}}\left(P^{542,m,2(m-1)}\right), \tag{14.9}$$

$$\begin{pmatrix} \frac{1}{2}\cdot \mathbb{1}^{m-1} \\ \frac{1}{2} \end{pmatrix} \in \mathrm{cl}_{L_{m-1,\mathbb{R}}}\left(P^{542,m,2(m-1)}\right), \tag{14.10}$$

$$\begin{aligned}[0,1]^{m-1}\times\{0\} &= \mathrm{cl}_I\left(P^{542,m,2(m-1)}\right)\\ &= \mathrm{cl}_{L_{m-\frac{1}{2},\mathbb{Q}}}\left(P^{542,m,2(m-1)}\right).\end{aligned} \tag{14.11}$$

In particular, $x_m \leq 0$ is not valid for $x \in \mathrm{cl}_{L_{m-1-\frac{1}{2},\mathbb{R}}}\left(P^{542,m,2(m-1)}\right)$ and $x \in \mathrm{cl}_{L_{m-1,\mathbb{R}}}\left(P^{542,m,2(m-1)}\right)$.

Proof. (14.9) is a consequence of Lemma 548 by considering that $P^{542,m,2(m-1)} \supseteq P^{541,m,2(m-1)}$. Thus, we obtain from (14.6) in Theorem 545:

$$\begin{pmatrix} \frac{1}{2}\cdot \mathbb{1}^{m-1} \\ 1 \end{pmatrix} \in \mathrm{cl}_{L_{m-1-\frac{1}{2},\mathbb{R}}}\left(P^{541,m,2(m-1)}\right) \subseteq \mathrm{cl}_{L_{m-1-\frac{1}{2},\mathbb{R}}}\left(P^{542,m,2(m-1)}\right).$$

(14.11) holds by Theorem 202 using

$$\left(P^{541,m,2(m-1)}\right)_I = \left(P^{542,m,2(m-1)}\right)_I = \{0,1\}^{m-1}\times\{0\}.$$

So for (14.10): let $v \in \mathbb{R}^m \setminus \{0^m\}$. We claim that

$$\begin{pmatrix} \frac{1}{2}\cdot \mathbb{1}^{m-1} \\ \frac{1}{2} \end{pmatrix} \in \mathrm{conv}\left(P^{542,m,2(m-1)} \cap (\mathbb{Z}^m + \mathrm{lin}\{v\})\right). \tag{14.17}$$

For this, we distinguish two cases:

1. $v_{[m-1]} = 0^{m-1}$,
2. $v_{[m-1]} \neq 0^{m-1}$.

For case 1: If $v_{[m-1]} = 0^{m-1}$, we have $\mathrm{lin}\{v\} = 0^{m-1}\times\mathbb{R}^1$. We claim that for every $J \in \mathcal{P}([m-1])$, we have

$$x^J := \begin{pmatrix} \chi(J) \\ \frac{1}{2} \end{pmatrix} \in P^{542,m,2(m-1)} \cap (\mathbb{Z}^m + \mathrm{lin}\{v\}). \tag{14.18}$$

For (14.18): $x^J \in \mathbb{Z}^m + \mathrm{lin}\{v\}$ is obvious; so, we only have to show $x^J \in P^{542,m,2(m-1)}$. Clearly, $x^J_m \geq 0$. Let $I \in \mathcal{P}([m-1])$. We have to show

$$\sum_{i\in I} x^J_i - \sum_{i\in[m-1]\setminus I} x^J_i + \frac{m-1}{2h}x^J_m \leq |I| + \frac{m-1}{4h}.$$

If $I = J$, we have

$$\sum_{i\in I} x_i^J - \sum_{i\in[m-1]\setminus I} x_i^J + \frac{m-1}{2h}x_m^J = \sum_{i\in I} 1 - \sum_{i\in[m-1]\setminus I} 0 + \frac{m-1}{2h}\cdot\frac{1}{2} = |I| + \frac{m-1}{4h} \left(\leq |I| + \frac{m-1}{4h}\right).$$

Otherwise ($I \neq J$), we have

$$\begin{aligned}\sum_{i\in I} x_i^J - \sum_{i\in[m-1]\setminus I} x_i^J + \frac{m-1}{2h}x_m^J &= \sum_{i\in I\cap J} x_i^J + \sum_{i\in I\setminus J} x_i^J - \sum_{i\in[m-1]\setminus(I\cup J)} x_i^J - \sum_{i\in J\setminus I} x_i^J + \frac{m-1}{2h}x_m^J \\ &= \sum_{i\in I\cap J} 1 + \sum_{i\in I\setminus J} 0 - \sum_{i\in[m-1]\setminus(I\cup J)} 0 - \sum_{i\in J\setminus I} 1 + \frac{m-1}{2h}\cdot\frac{1}{2} \\ &= |I\cap J| - |J\setminus I| + \frac{m-1}{4h} \\ &\leq |I| + \frac{m-1}{4h}.\end{aligned}$$

From (14.18), we conclude (14.17).

For case 2: (14.17) is a consequence of Lemma 550, since $P^{542,m,2(m-1)} \supseteq P^{541,m,2(m-1)}$. □

14.9. Back to the initial question

Now we go back to the initial question that we introduced in section 14.1.

14.9.1. $m = 1$

For the situation of (14.2) in the case $k = m = 1$ (and $n \geq 1$, as required in the statement of (14.2)), consider the outlook in section 14.9.3. Thus, we only consider the situation for (14.1) here.

The only inclusion that we have to consider for $m = 1$ in (14.1) is

$$\mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) \subsetneq \mathrm{cl}_{L_{0,\mathbb{R}}}(P).$$

The following lemma holds obviously:

Lemma 551. *Let* $P := \left\{\left(\ \tfrac{1}{2}\ \right)\right\} \subseteq \mathbb{R}^1$. *Then*

$$\mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_I(P) = \emptyset \subsetneq P \subsetneq \mathrm{cl}_{L_{0,\mathbb{R}}}(P).$$

Thus, by combining Lemma 551 and Theorem 220, we obtain:

Theorem 552. *For every* $n \in \mathbb{Z}_{\geq 0}$, *there exists a rational polyhedron* $P \subseteq \mathbb{R}^1 \times \mathbb{R}^n$ *such that*

$$\mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_I(P) = \emptyset \subsetneq P = \mathrm{cl}_{L_{0,\mathbb{R}}}(P).$$

14.9.2. $m \geq 2$

Theorem 553. *For every* $m \in \mathbb{Z}_{\geq 2}$, $n \in \mathbb{Z}_{\geq 0}$ *and* $k \in \{1, \ldots, m-1\}$, *there exists a rational polyhedron* $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ *such that*

$$\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) \subsetneq \mathrm{cl}_{L_{k-1,\mathbb{R}}}(P).$$

Proof. By Theorem 544, we have

$$\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}}}\Big(\underbrace{P^{542,k,2(k-1)}}_{\subseteq\mathbb{R}^k}\Big) \subsetneq \mathrm{cl}_{L_{k-1,\mathbb{R}}}\Big(\underbrace{P^{542,k,2(k-1)}}_{\subseteq\mathbb{R}^k}\Big).$$

Thus, by Theorem 220, there exists a rational polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ such that

$$\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) \subsetneq \mathrm{cl}_{L_{k-1,\mathbb{R}}}(P).$$

□

Theorem 554. *For every $m \in \mathbb{Z}_{\geq 2}$, $n \in \mathbb{Z}_{\geq 0}$ and $k \in \{1, \ldots, m-1\}$, there exists a rational polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ such that*

$$\mathrm{cl}_{L_{k,\mathbb{Q}}}(P) \subsetneq \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{R}\times\mathbb{R}}}(P).$$

Proof. By Theorem 543, we have

$$\mathrm{cl}_{L_{k,\mathbb{Q}}}\Big(\underbrace{P^{541,k+1,2k}}_{\subseteq \mathbb{R}^{k+1}}\Big) \subsetneq \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{R}}}\Big(\underbrace{P^{541,k+1,2k}}_{\subseteq \mathbb{R}^{k+1}}\Big).$$

Thus, by Theorem 220, there exists a rational polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ such that

$$\mathrm{cl}_{L_{k,\mathbb{Q}}}(P) \subsetneq \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{R}\times\mathbb{R}}}(P).$$

□

14.9.3. Summary and outlook

From Theorem 552, Theorem 553 and Theorem 554, we conclude:

Theorem 555. *For every $m \in \mathbb{Z}_{\geq 1}$, $n \in \mathbb{Z}_{\geq 0}$ and $k \in \{1, \ldots, m-1\}$, there exists a rational polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ such that*

$$\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) \subsetneq \mathrm{cl}_{L_{k-1,\mathbb{R}}}(P)$$

and for every $m \in \mathbb{Z}_{\geq 2}$, $n \in \mathbb{Z}_{\geq 0}$ and $k \in \{1, \ldots, m-1\}$, there exists a rational polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ such that

$$\mathrm{cl}_{L_{k,\mathbb{Q}}}(P) \subsetneq \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{R}\times\mathbb{R}}}(P).$$

Considering the inclusion problem in (14.2), i.e. whether there exists a rational polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m \in \mathbb{Z}_{\geq 1}$ and $n \in \mathbb{Z}_{\geq 0}$) such that for a suitable k, we have

$$\mathrm{cl}_{L_{k,\mathbb{Q}}}(P) \subsetneq \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{R}\times\mathbb{R}}}(P),$$

we thus only have not yet considered the case $k = m$ and $n \geq 1$. This is done in section 15.5.2 (more precisely: Theorem 578), where we prove the even tighter inclusion that for every $m, n \in \mathbb{Z}_{\geq 1}$ and $k \in \{1, \ldots, m\}$, there exists a rational polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ such that for every $\ell \in \mathbb{Z}_{\geq 0}$, we have

$$\mathrm{cl}_{L_{k,\mathbb{Q}}}(P) \subsetneq \mathrm{cl}^{(\ell)}_{L_{k-\frac{1}{2},\mathbb{R}\times\mathbb{R}}}(P).$$

15. Li-Richard example

15.1. Definition of the polytopes

In this chapter, we want to consider the polytopes of the Li-Richard example (cf. [LR08]). Since different kinds of cutting planes expect different representations of this polyhedron, we define two variants in this section. The central motivation to analyze this polytope is to prove some separation results for various classes of cutting planes. The central results for this are stated in section 15.2 and the inclusions which these imply are stated in section 15.5.1. Finally, in section 15.5.2, we answer the question that was still open at the end of section 14.9.3.

Definition 556. *For $m \in \mathbb{Z}_{\geq 1}$ and $\epsilon \in \mathbb{R}_{>0}$, let*

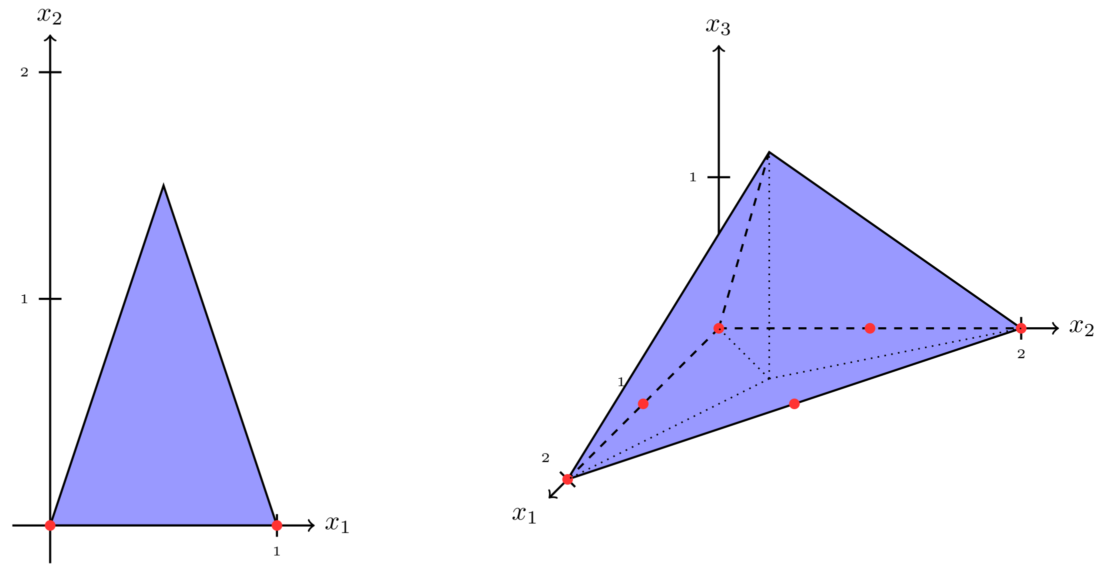
$$\begin{aligned} X_m^m &:= \operatorname{conv} \dot{\bigcup_{i=0}^{m}} \left\{ m \cdot e^{m,i} \right\} \subsetneq \mathbb{R}^m, \\ P^{556,m,\epsilon,\leq} &:= \operatorname{conv} \left(\left(X_m^m \times \{0\} \right) \dot{\cup} \left\{ \begin{pmatrix} \frac{m}{m+1} \cdot \mathbb{1}^m \\ \epsilon \end{pmatrix} \right\} \right) \subsetneq \mathbb{R}^m \times \mathbb{R}^1. \end{aligned} \tag{15.1}$$

We have

$$P^{556,m,\epsilon,\leq} = P^{\leq}\left(A^{556,m,\epsilon,\leq}, b^{556,m,\epsilon,\leq}\right),$$

where

$$A^{556,m,\epsilon,\leq} := \begin{pmatrix} -I_m & \frac{m}{\epsilon(m+1)} \mathbb{1}^m \\ \left(\mathbb{1}^m\right)^T & \frac{m}{\epsilon(m+1)} \\ \left(0^m\right)^T & -1 \end{pmatrix}, \qquad b^{556,m,\epsilon,\leq} := \begin{pmatrix} 0^m \\ m \\ 0 \end{pmatrix}.$$



(a) $P^{556,2,\frac{3}{2},\leq}$ and $P^{556,2,\frac{3}{2},\leq} \cap \left(\mathbb{Z}^1 \times \mathbb{R}\right)$ (b) $P^{556,3,\frac{3}{2},\leq}$ and $P^{556,3,\frac{3}{2},\leq} \cap \left(\mathbb{Z}^2 \times \mathbb{R}\right)$

Figure 15.1.: $P^{556,2,\frac{3}{2},\leq}$ and $P^{556,3,\frac{3}{2},\leq}$

A visualization of $P^{556,2,\frac{3}{2},\leq}$ and $P^{556,3,\frac{3}{2},\leq}$ can be found in Figure 15.1. The series of polyhedra $P^{556,m,\epsilon,\leq}$ was originally conceived by Li and Richard ([LR08]) as a generalization of the famous Cook-Kannan-Schrijver example (cf. [CKS90]). Li and Richard considered the following question: on one hand (Theorem 562), it is easy to show that applying a single m-branch split cut to $P^{556,m,\epsilon,\leq}$ suffices to obtain $\mathrm{cl}_I\left(P^{556,m,\epsilon,\leq}\right)$. On the other hand, they conjectured (Theorem 560) that even applying the $m-1$-branch split closure iteratively an arbitrary amount of times does not suffice to obtain $\mathrm{cl}_I\left(P^{556,m,\epsilon,\leq}\right)$. This conjecture was finally resolved by Dash and Günlük (cf. [DG12]).

Since k row cuts expect the polyhedron in equation form (recall Definition 154), we define:

Definition 557. *Let $m \in \mathbb{Z}_{\geq 1}$ and $\epsilon \in \mathbb{R}_{>0}$. Then*

$$\begin{aligned} P^{557,m,\epsilon,=} := \Bigg\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R}_{\geq 0}^{1+(m+1)} : \\ x_i - \frac{m}{\epsilon(m+1)} y_1 - y_{i+1} = 0 \ \forall i \in [m], \\ -\sum_{i=1}^{m} x_i - \frac{m}{\epsilon(m+1)} y_1 - y_{m+2} = -m \Bigg\}. \end{aligned} \tag{15.2}$$

One obtains this polyhedron by multiplying all facet-defining inequalities of $P^{556,m,\epsilon,\leq}$ by -1 and introducing slack variables afterwards. Let

$$A^{557,m,\epsilon,=} := \begin{pmatrix} I^m \\ -\left(1^m\right)^T \end{pmatrix}, \quad G^{557,m,\epsilon,=} := \begin{pmatrix} -\frac{m}{\epsilon(m+1)} 1^{m+1} & -I^{m+1} \end{pmatrix}, \quad b^{557,m,\epsilon,=} := \begin{pmatrix} 0^m \\ -m \end{pmatrix}.$$

Then

$$P^{557,m,\epsilon,=} = P^{=}\left(\left(A^{557,m,\epsilon,=} \mid G^{557,m,\epsilon,=} \right), b^{557,m,\epsilon,=}\right) \cap \left(\mathbb{R}^m \times \mathbb{R}_{\geq 0}^{1+m+1}\right).$$

For a given $M \in \mathbb{R}^{k \times (m+1)}$, where $k \in \mathbb{Z}_{\geq 0}$, let (cf. Definition 154)

$$P^{557,m,\epsilon,=}(M) := P^{=}\left(\left(MA^{557,m,\epsilon,=} \mid MG^{557,m,\epsilon,=} \right), Mb^{557,m,\epsilon,=}\right) \cap \left(\mathbb{R}^m \times \mathbb{R}_{\geq 0}^{1+m+1}\right).$$

15.2. Statements

15.2.1. Statements for cuts of basic relaxations

In the following theorem, we consider the strength of cuts of a basic relaxation.

Theorem 558. *Let $m \in \mathbb{Z}_{\geq 1}$ and $\epsilon \in \mathbb{R}_{>0}$. Then $y_1 \leq 0$ is a valid inequality for*

$$\begin{pmatrix} x \\ y \end{pmatrix} \in P^{\leq}\left(\left(A^{556,m,\epsilon,\leq}\right)_{[m+1],*}, \left(b^{556,m,\epsilon,\leq}\right)_{[m+1]}\right)_I,$$

i.e. it is a cut of a basic relaxation with respect to $A^{556,m,\epsilon,\leq}$ and $b^{556,m,\epsilon,\leq}$.

Proof. We consider

$$P^{\leq}\left(\left(A^{556,m,\epsilon,\leq}\right)_{[m+1],*}, \left(b^{556,m,\epsilon,\leq}\right)_{[m+1]}\right) = P^{\leq}\left(\begin{pmatrix} -I_m & \frac{m}{\epsilon(m+1)} \cdot 1^m \\ \left(1^m\right)^T & \frac{m}{\epsilon(m+1)} \end{pmatrix}, \begin{pmatrix} 0^m \\ m \end{pmatrix}\right).$$

Clearly, the rows at the left-hand side are linearly independent. To see that $y_1 \leq 0$ is a valid inequality for

$$\begin{pmatrix} x \\ y \end{pmatrix} \in P^{\leq}\left(\left(A^{556,m,\epsilon,\leq}\right)_{[m+1],*}, \left(b^{556,m,\epsilon,\leq}\right)_{[m+1]}\right)_I,$$

consider that it is a valid inequality for

$$P^{\leq}\left(\left(A^{556,m,\epsilon,\leq}\right)_{[m+1],*}, \left(b^{556,m,\epsilon,\leq}\right)_{[m+1]}\right) \cap \left(\left(\mathbb{R}^m \setminus \operatorname{int} X_m^m\right) \times \mathbb{R}\right),$$

where X_m^m is clearly a lattice-free polyhedron. □

15.2.2. Statements for integral lattice-free cuts

Theorem 559. *Let $m \in \mathbb{Z}_{\geq 1}$ and $\epsilon \in \mathbb{R}_{>0}$. Then $y_1 \leq 0$ is a valid integral lattice-free cut with respect to X_m^m for*

$$\begin{pmatrix} x \\ y \end{pmatrix} \in P^{556,m,\epsilon,\leq} \subseteq \mathbb{R}^m \times \mathbb{R}$$

and

$$\begin{pmatrix} x \\ y \end{pmatrix} \in P^{557,m,\epsilon,=} \subseteq \mathbb{R}^m \times \mathbb{R}^{1+(m+1)}.$$

Proof. It is easy to check that $y_1 \leq 0$ is valid for

$$\begin{pmatrix} x \\ y \end{pmatrix} \in P^{556,m,\epsilon,\leq} \cap \left((\mathbb{R}^m \setminus \operatorname{int} X_m^m) \times \mathbb{R}^1 \right)$$

or

$$\begin{pmatrix} x \\ y \end{pmatrix} \in P^{557,m,\epsilon,=} \cap \left((\mathbb{R}^m \setminus \operatorname{int} X_m^m) \times \mathbb{R}^{1+(m+1)} \right),$$

respectively. □

15.2.3. Statements for t-branch split cuts

In [DG12, Theorem 2.7 and following remark], the following theorem is shown:

Theorem 560. *Let $m \in \mathbb{Z}_{\geq 1}$ and $\epsilon \in \mathbb{R}_{>0}$. Then there exists an $\epsilon^* \in \mathbb{R}_{>0}$ (having $\epsilon^* \leq \epsilon$) such that*

$$\begin{pmatrix} \frac{m}{m+1} \cdot 1^m \\ \epsilon^* \end{pmatrix} \in \operatorname{cl}_{(m-1)BS} \left(P^{556,m,\epsilon,\leq} \right).$$

A similar result to Theorem 560 holds for $P^{557,m,\epsilon,=}$:

Theorem 561. *Let $m \in \mathbb{Z}_{\geq 1}$ and $\epsilon \in \mathbb{R}_{>0}$. Then there exists an $\epsilon^* \in \mathbb{R}_{>0}$ (having $\epsilon^* \leq \epsilon$) such that*

$$\begin{pmatrix} \frac{m}{m+1} \cdot 1^m \\ \epsilon^* \\ \frac{m}{m+1} \left(1 - \frac{\epsilon^*}{\epsilon}\right) \cdot 1^{m+1} \end{pmatrix} \in \operatorname{cl}_{(m-1)BS} \left(P^{557,m,\epsilon,=} \right).$$

Proof. We have by Corollary 278:

$$\operatorname{cl}_{(m-1)BS} \left(P^{557,m,\epsilon,=} \right) \supseteq \operatorname{cl}_{L_{m-1,\mathbb{Q}}} \left(P^{557,m,\epsilon,=} \right).$$

Thus, by Theorem 565 (which we prove further below), there exists an $\epsilon^* > 0$ such that

$$\begin{pmatrix} \frac{m}{m+1} \cdot 1^m \\ \epsilon^* \\ \frac{m}{m+1} \left(1 - \frac{\epsilon^*}{\epsilon}\right) \cdot 1^{m+1} \end{pmatrix} \in \operatorname{cl}_{(m-1)BS} \left(P^{557,m,\epsilon,=} \right).$$

□

On the other hand, we have:

Theorem 562. *For $m \in \mathbb{Z}_{\geq 1}$ and $\epsilon \in \mathbb{R}_{>0}$, the inequality $y_1 \leq 0$ is an m-branch split cut for $\binom{x}{y} \in P^{556,m,\epsilon,\leq}$ and $\binom{x}{y} \in P^{557,m,\epsilon,=}$ with respect to the split disjunction $D\left(e^{m,1}, \ldots, e^{m,m}, 0, \ldots, 0\right)$. In particular*

$$\operatorname{cl}_{mBS} \left(P^{556,m,\epsilon,\leq} \right) = \operatorname{cl}_I \left(P^{556,m,\epsilon,\leq} \right),$$
$$\operatorname{cl}_{mBS} \left(P^{557,m,\epsilon,=} \right) = \operatorname{cl}_I \left(P^{557,m,\epsilon,=} \right).$$

Proof. Clearly $X_m^m \subseteq D\left(e^{m,1}, \ldots, e^{m,m}, 0, \ldots, 0\right)$. Thus, the statement is a consequence of Theorem 559. □

15.2.4. Statements for $L_{k-\frac{1}{2}}/L_k$ cuts

Theorem 563. *Let $m \in \mathbb{Z}_{\geq 1}$ and $\epsilon \in \mathbb{R}_{>0}$. Then there exists an $\epsilon^* \in \mathbb{R}_{>0}$ (having $\epsilon^* \leq \epsilon$) such that*

$$\begin{pmatrix} \frac{m}{m+1} \cdot 1^m \\ \epsilon^* \end{pmatrix} \in \mathrm{cl}_{L_{m-\frac{1}{2},\mathbb{R}\times\mathbb{R}}}\left(P^{556,m,\epsilon,\leq}\right).$$

In particular, for every $\ell \in \mathbb{Z}_{\geq 1}$, the inequality $y_1 \leq 0$ is not valid for

$$\begin{pmatrix} x \\ y \end{pmatrix} \in \mathrm{cl}^{(\ell)}_{L_{m-\frac{1}{2},\mathbb{R}\times\mathbb{R}}}\left(P^{556,m,\epsilon,\leq}\right).$$

Theorem 564. *Let $m \in \mathbb{Z}_{\geq 1}$ and $\epsilon \in \mathbb{R}_{>0}$. Then there exists an $\epsilon^* \in \mathbb{R}_{>0}$ (having $\epsilon^* \leq \epsilon$) such that*

$$\begin{pmatrix} \frac{m}{m+1} \cdot 1^m \\ \epsilon^* \\ \frac{m}{m+1}\left(1 - \frac{\epsilon^*}{\epsilon}\right) \cdot 1^{m+1} \end{pmatrix} \in \mathrm{cl}_{L_{m-\frac{1}{2},\mathbb{R}\times\mathbb{R}}}\left(P^{557,m,\epsilon,=}\right).$$

Theorem 565. *Let $m \in \mathbb{Z}_{\geq 1}$ and $\epsilon \in \mathbb{R}_{>0}$. Then there exists an $\epsilon^* \in \mathbb{R}_{>0}$ (having $\epsilon^* \leq \epsilon$) such that*

$$\begin{pmatrix} \frac{m}{m+1} \cdot 1^m \\ \epsilon^* \end{pmatrix} \in \mathrm{cl}_{L_{m-1,\mathbb{R}}}\left(P^{556,m,\epsilon,\leq}\right),$$

$$\begin{pmatrix} \frac{m}{m+1} \cdot 1^m \\ \epsilon^* \\ \frac{m}{m+1}\left(1 - \frac{\epsilon^*}{\epsilon}\right) \cdot 1^{m+1} \end{pmatrix} \in \mathrm{cl}_{L_{m-1,\mathbb{R}}}\left(P^{557,m,\epsilon,=}\right).$$

Proof. The statement is a consequence of Theorem 563, Theorem 564 and Theorem 199. ☐

Theorem 566. *We have for all $m \in \mathbb{Z}_{\geq 1}$ and $\epsilon \in \mathbb{R}_{>0}$:*

$$\mathrm{cl}_{L_{(m+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}\left(P^{556,m,\epsilon,\leq}\right) = \mathrm{cl}_{L_{m,\mathbb{Q}}}\left(P^{556,m,\epsilon,\leq}\right) = \mathrm{cl}_I\left(P^{556,m,\epsilon,\leq}\right) = X_m^m \times \{0\}.$$

Proof. Clearly, $y_1 \leq 0$ is valid for $\binom{x}{y} \in P_I^{556,m,\epsilon,\leq}$. Thus, by Theorem 202, it is an $L_{m,\mathbb{Q}}$ cut for $P^{556,m,\epsilon,\leq}$ with respect to $V := 0^m \times \mathbb{R}^1$. From Theorem 199, we conclude that it is also an $L_{(m+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cut for $P^{556,m,\epsilon,\leq}$. ☐

We conclude this section with a remark about a result from the literature. Let us recapitulate (cf. Definition 175 and Theorem 176) that Dash, Günlük and Morán developed $L_{k,\mathbb{Q}}$ cuts independently from us under the name "k-dimensional lattice cuts" in [DGMR17]. In this paper, the authors prove in section 6 (in particular cf. [DGMR17, Theorem 3]):

Definition/Theorem 567. *For $m \in \mathbb{Z}_{\geq 1}$ and $\epsilon \in \mathbb{R}_{>0}$, let*

$$P^{567,m,\epsilon} := \mathrm{conv}\left(\left(X_m^m \times \{0\}\right) \dot\cup \left\{\begin{pmatrix} \frac{1}{2} \cdot 1^m \\ \epsilon \end{pmatrix}\right\}\right) \subsetneq \mathbb{R}^m \times \mathbb{R}^1.$$

Then for all $\ell \in \mathbb{Z}_{\geq 0}$, we have

$$\mathrm{cl}^{(\ell)}_{L_{m-1,\mathbb{R}}}\left(P^{567,m,\epsilon}\right) \supsetneq \mathrm{cl}_I\left(P^{567,m,\epsilon}\right).$$

We remark that the central difference between $P^{556,m,\epsilon,\leq}$ and $P^{567,m,\epsilon}$ lies in the position of the apex of $P^{556,m,\epsilon,\leq}$ vs $P^{567,m,\epsilon}$.

One additional remark concerning Definition/Theorem 567: Dash, Günlük and Morán only proved Definition/Theorem 567 for the closure with respect to $m-1$-dimensional lattices, which, as we saw in Theorem 176 is equivalent to the $L_{m-1,\mathbb{Q}}$ closure. But if one uses the concept of $L_{m-1,\mathbb{R}}$ cuts instead of $m-1$-dimensional lattice cuts (cf. Definition 175), their proof goes through for the $L_{m-1,\mathbb{R}}$ closure.

15.2.5. Statements for k row cuts

Theorem 568. *There exists a matrix $M \in \mathbb{Q}^{m\times(m+1)}$ such that $y_1 \leq 0$ is a translate of (cf. Definition 290) a valid inequality for*

$$\begin{pmatrix} x \\ y \end{pmatrix} \in P^{557,m,\epsilon,=}(M)_I .$$

Proof. By Theorem 559, $y_1 \leq 0$ is a valid inequality for

$$\begin{pmatrix} x \\ y \end{pmatrix} \in P^{557,m,\epsilon,=} \cap \left(X_m^m \times \mathbb{R}^{1+(m+1)}\right).$$

Clearly, X_m^m has $m+1$ facets. Thus, by Theorem 303, $y_1 \leq 0$ is a translate of a valid inequality of an m row relaxation of

$$P^{=}\left(\left(\begin{array}{cc} A^{557,m,\epsilon,=} & G^{557,m,\epsilon,=} \end{array}\right), b^{557,m,\epsilon,=}\right) \cap \left(\mathbb{R}^m \times \mathbb{R}^{1+(m+1)}\right).$$

□

15.3. Auxiliary statements

15.3.1. Height Lemma

In [DGM15, Lemma 3.1 and following remark], the following lemma is shown:

Lemma 569. *(Height Lemma) Consider some $a \in \left(\mathbb{R}^d \setminus \left\{0^d\right\}\right)^T$ and $b \in \mathbb{R}$, where $d \in \mathbb{Z}_{\geq 1}$. Let $s^1, \dots, s^d$ be affinely independent points in the hyperplane $H := P^{=}(a,b)$, let $b' \in \mathbb{R}$ be such that $b' > b$ and let $U \in \mathbb{R}_{\geq 0}$. Finally, let*

$$Q := \left\{x \in \mathbb{R}^d : ax \geq b', \left\|\operatorname{proj}_H^{\perp} x\right\| \leq U\right\} \neq \emptyset.$$

Then there exists a point

$$x^* \in \bigcap_{q \in Q} \operatorname{conv}\left\{s^1, \dots, s^d, q\right\}$$

that satisfies $ax^* > b$.

Remark 570. *Let a, b', d, $s^1, \dots, s^d$, U and Q be as as in Lemma 569. Define*

$$Q' := \left\{x \in \mathbb{R}^d : ax \geq b', \|x\| \leq U\right\},$$

Then, obviously, $Q' \subseteq Q$ and thus, by Lemma 569, there exists a point

$$x^* \in \bigcap_{q \in Q} \operatorname{conv}\left\{s^1, \dots, s^d, q\right\} \subseteq \bigcap_{q \in Q'} \operatorname{conv}\left\{s^1, \dots, s^d, q\right\}.$$

So in Lemma 569 we can replace Q by Q'.

Lemma 571. *Let a, d, $s^1, \dots, s^d$ and Q be as in Lemma 569 and let $s^* \in \operatorname{relint}\left(\operatorname{conv}\left\{s^1, \dots, s^d\right\}\right)$. Then there exists an $\epsilon \in \mathbb{R}_{>0}$ such that*

$$t^* := s^* + \epsilon a \in \bigcap_{q \in Q} \operatorname{conv}\left\{s^1, \dots, s^d, q\right\}.$$

Proof. Let x^* be as in Lemma 569 and let

$$s^* = \sum_{i=1}^{d} \lambda_i s^i,$$

where $\lambda \in \operatorname{relint} \Delta^{d-1}$ (i.e. $\lambda \in \mathbb{R}^d_{>0}$ and $\sum_{i=1}^d \lambda_i = 1$). Additionally, let

$$x^* = \sum_{i=1}^{d} \lambda'_i s^i + \delta a,$$

where $\sum_{i=1}^{d} \lambda'_i = 1$ and $\delta > 0$. Set

$$M^* := \max \left\{ \frac{\lambda'_i}{\lambda_i} : i \in [d] \right\}.$$

Since $\lambda > 0^d$ and since there exists an $i^* \in [d]$ such that $\lambda'_{i^*} > 0$, we clearly have $M^* > 0$. Set

$$\begin{aligned} t^* &:= \underbrace{\sum_{i=1}^{d} \lambda_i s^i}_{=s^*} + \frac{1}{M^*} \underbrace{\delta\, a}_{=:\epsilon} \\ &= \frac{1}{M^*} \sum_{i=1}^{d} \left(\lambda'_i + (M^* \lambda_i - \lambda'_i)\right) s^i + \frac{1}{M^*} \delta a \\ &= \frac{1}{M^*} \left(\sum_{i=1}^{d} \lambda'_i s^i + \delta a \right) + \sum_{i=1}^{d} \left(\lambda_i - \frac{1}{M^*} \lambda'_i \right) s^i \\ &= \frac{1}{M^*} x^* + \sum_{i=1}^{d} \left(\lambda_i - \frac{1}{M^*} \lambda'_i \right) s^i. \end{aligned}$$

We claim that $t^* \in \operatorname{conv} \left\{ x^*, s^1, \ldots, s^d \right\}$: clearly, for all $i \in [d]$, we have

$$\lambda_i - \frac{1}{M^*} \lambda'_i = \frac{1}{M^*} \left(M^* \lambda_i - \lambda'_i \right) \geq \frac{1}{M^*} \left(\frac{\lambda'_i}{\lambda_i} \lambda_i - \lambda'_i \right) = 0.$$

Additionally, we have

$$\frac{1}{M^*} + \sum_{i=1}^{d} \left(\lambda_i - \frac{1}{M^*} \lambda'_i \right) = \frac{1}{M^*} + \sum_{i=1}^{d} \lambda_i - \frac{1}{M^*} \sum_{i=1}^{d} \lambda'_i = \frac{1}{M^*} + 1 - \frac{1}{M^*} = 1.$$

□

15.3.2. Monotonicity

Remark 572. *Let $P \subseteq P' \subseteq \mathbb{R}^m \times \mathbb{R}^n$ be arbitrary and let $k \in \mathbb{Z}_{\geq 0}$ ($k \leq m + n$ in the case of L_k cuts and $L_{k-\frac{1}{2}}$ cuts). Let $\operatorname{cl}_{(\,\cdot\,)}(\cdot)$ be the closure operator for either some L_k, $L_{k-\frac{1}{2}}$ or k-branch split closure. Then*

$$\operatorname{cl}_{(\,\cdot\,)}(P) \subseteq \operatorname{cl}_{(\,\cdot\,)}(P').$$

15.3.3. Auxiliar statements for investigating the $L_{m-\frac{1}{2}, \mathbb{R} \times \mathbb{R}}$ closure of $P^{556,m,\epsilon,\leq}$

Lemma 573. *Let*

$$v \in \mathbb{S}^{m-1,\infty} := \{ x \in \mathbb{R}^m : \|x\|_\infty = 1 \},$$

where $m \in \mathbb{Z}_{\geq 1}$. Let $\epsilon \in \mathbb{R}$ be such that $-1 \leq \epsilon < 1$ and $v \leq \epsilon \cdot 1^m$. Then

$$-\sum_{i=1}^{m} v_i \geq 1 - (m-1)\,\epsilon.$$

Proof. Since $v \in \mathbb{S}^{m-1,\infty}$ and $v \leq \epsilon \cdot 1^m < 1^m$, there must exist a coordinate $i^* \in [m]$ such that $v_{i^*} = -1$. So

$$-\sum_{i=1}^{m} v_i = 1 - \sum_{i=1, i \neq i^*}^{m} v_i \geq 1 - (m-1)\,\epsilon.$$

□

Lemma 574. *Let $m \in \mathbb{Z}_{\geq 1}$ and $\epsilon > 0$. Then for every $z \in (X_m^m)_I$, there exists a polyhedral cone $C_z \subseteq \mathbb{R}^m$ such that*

- $\bigcup_{z \in (X_m^m)_I} C_z = \mathbb{R}^m$ *and*

- $$\forall z \in (X^m_m)_I, v \in C_z \setminus \{0^m\} : \max\left\{y : \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in P^{556,m,\epsilon,\leq} \wedge x \in z + \operatorname{cone}\{v\}\right\} \geq \frac{\epsilon}{m^2}. \tag{15.3}$$

Proof. For $m = 1$, the statement is obvious: set $C_{0^1} := \mathbb{R}^1_{\geq 0}$ and $C_{1^1} := \mathbb{R}^1_{\leq 0}$. So, we can assume $m \geq 2$. By Lemma 573 (set $\epsilon := \frac{1}{m}$), we have for all $v \in \mathbb{S}^{m-1,\infty}$:

$$\left(e^{m,1}\right)^T v \geq \frac{1}{m} \vee \ldots \vee \left(e^{m,m}\right)^T v \geq \frac{1}{m} \vee \left(-1^m\right)^T v \geq \frac{1}{m}. \tag{15.4}$$

So for a given

$$z^i \in \dot{\bigcup_{i=1}^{m}}\{\underbrace{1^m - e^{m,i}}_{=:z^i}\} \,\dot{\cup}\, \{\underbrace{1^m}_{=:z^{m+1}}\},$$

consider the cones C_{z^i}, which we define for $i \in \{1, \ldots, m+1\}$ the following way:

- For $i \in \{1, \ldots, m\}$:

$$\begin{aligned} C_{z^i} &:= \left\{v \in \mathbb{R}^m : v_i \geq \frac{1}{m}\|v\|_\infty\right\} \\ &= \operatorname{proj}_{\mathbb{R}^m}\left\{\begin{pmatrix} v \\ \delta \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R} : v_i \geq \frac{1}{m}\delta, \forall j \in [m] : \delta \geq v_j, \delta \geq -v_j\right\}. \end{aligned}$$

- For $i = m+1$:

$$\begin{aligned} C_{z^{m+1}} &:= \left\{v \in \mathbb{R}^m : -\sum_{k=1}^{m} v_k \geq \frac{1}{m}\|v\|_\infty\right\} \\ &= \operatorname{proj}_{\mathbb{R}^m}\left\{\begin{pmatrix} v \\ \delta \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R} : -\sum_{k=1}^{m} v_k \geq \frac{1}{m}\delta, \forall j \in [m] : \delta \geq v_j, \delta \geq -v_j\right\}. \end{aligned}$$

If $z \in (X^m_m)_I \setminus \bigcup_{i\in[m+1]}\{z^i\}$, simply set $C_z := \{0^m\}$. By (15.4), for all $v \in \mathbb{R}^m$ (w.l.o.g. $v \neq 0^m$), there exists an $i \in [m+1]$ such that $v \in C_{z^i}$. W.l.o.g. let $\|v\|_\infty = 1$. Let

$$\begin{aligned} \hat{x} &:= z^i + \frac{1}{m+1}v, \\ \hat{y} &:= \begin{cases} \frac{1}{m+1} \cdot \frac{\epsilon(m+1)}{m} v_i & \text{if } i \in [m], \\ -\frac{1}{m+1} \cdot \frac{\epsilon(m+1)}{m} \sum_{j=1}^{m} v_j & \text{if } i = m+1. \end{cases} \end{aligned}$$

Clearly (using the definitions of the cones C_{z^i}), $\hat{y} \geq \frac{\epsilon}{m^2} > 0$ and we have $A^{556,m,\epsilon,\leq}_{i,*}\left(\begin{smallmatrix} \hat{x} \\ \hat{y} \end{smallmatrix}\right) = b^{556,m,\epsilon,\leq}_i$. What is still to be shown is that for $j \in \{1, \ldots, m+1\} \setminus \{i\}$, we have $A^{556,m,\epsilon,\leq}_{j,*}\left(\begin{smallmatrix} \hat{x} \\ \hat{y} \end{smallmatrix}\right) \leq b^{556,m,\epsilon,\leq}_j$

Case 1: $i, j \in \{1, \ldots, m\}$**:** We have

$$\begin{aligned} A^{556,m,\epsilon,\leq}_{j,*}\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} &= -\left(1 + \frac{1}{m+1}v_j\right) + \frac{1}{m+1}v_i \\ &\leq -1 + \frac{2}{m+1} && (\|v\|_\infty = 1) \\ &\leq 0 && (m \geq 1) \\ &= b^{556,m,\epsilon,\leq}_j. \end{aligned}$$

Case 2: $i \in \{1, \ldots, m\}, j = m+1$**:** We have

$$A^{556,m,\epsilon,\leq}_{j,*}\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} = (m-1) + \frac{1}{m+1}\sum_{k=1}^{m} v_k + \frac{1}{m+1}v_i$$

$$\begin{aligned}
&\leq m-1+m\cdot\frac{1}{m+1}+\frac{1}{m+1} && (\|v\|_\infty=1)\\
&=m\\
&=b_j^{556,m,\epsilon,\leq}.
\end{aligned}$$

Case 3: $i=m+1$**:** We have

$$\begin{aligned}
A_{j,*}^{556,m,\epsilon,\leq}\begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix} &= -\left(1+\frac{1}{m+1}v_j\right)-\frac{1}{m+1}\sum_{k=1}^{m}v_k\\
&\leq -1+\frac{1}{m}+m\cdot\frac{1}{m+1} && (\|v\|_\infty=1)\\
&=0\\
&=b_j^{556,m,\epsilon,\leq}.
\end{aligned}$$

□

Lemma 575. *Let* $m\in\mathbb{Z}_{\geq 1}$ *and* $\epsilon\in\mathbb{R}_{>0}$. *Then*

$$\forall v\in\left(\mathbb{R}^m\times\mathbb{R}^1\right)\setminus\left(0^m\times 0^1\right)\exists x^*\in\left(X_m^m\right)_I:\begin{pmatrix}x^*\\ \frac{\epsilon}{m^2}\end{pmatrix}\in P^{556,m,\epsilon,\leq}+\operatorname{lin}\{v\}.$$

Proof. W.l.o.g. we can assume $v_{m+1}\leq 0$. Additionally, we can assume $v_{[m]}\neq 0^m$. By Lemma 574, there exists a (not necessarily uniquely defined) $z(v)\in\left(X_m^m\right)_I$ such that $v_{[m]}\in C_{z(v)}$. So, by this lemma, there exists a $\lambda\in\mathbb{R}_{>0}$ and an $y^*\in\mathbb{R}$ having $y^*\geq\frac{\epsilon}{m^2}$ such that

$$\begin{pmatrix}z(v)+\lambda v_{[m]}\\ y^*\end{pmatrix}\in P^{556,m,\epsilon,\leq}.$$

So

$$\begin{pmatrix}z(v)\\ y^{**}\end{pmatrix}:=\begin{pmatrix}z(v)+\lambda v_{[m]}\\ y^*\end{pmatrix}-\lambda v\in P^{556,m,\epsilon,\leq}+\operatorname{lin}\{v\}$$

and we have $y^{**}=y^*-\lambda v_{m+1}\geq y^*\geq\frac{\epsilon}{m^2}$. On the other hand

$$\begin{pmatrix}z(v)\\ 0\end{pmatrix}\in X_m^m\times\{0\}\subseteq P^{556,m,\epsilon,\leq}\subseteq P^{556,m,\epsilon,\leq}+\operatorname{lin}\{v\}.$$

So, since $P^{556,m,\epsilon,\leq}+\operatorname{lin}\{v\}$ is convex, we conclude

$$\begin{pmatrix}z(v)\\ \frac{\epsilon}{m^2}\end{pmatrix}\in P^{556,m,\epsilon,\leq}+\operatorname{lin}\{v\}.$$

□

15.4. Proofs of the statements from section 15.2

15.4.1. Proof of Theorem 563

Proof. (Theorem 563) The statement is an immediate consequence of Lemma 571 and Lemma 575. □

15.4.2. Proof of Theorem 564

Proof. (Theorem 564) Consider a vector space $V\leq\mathbb{R}^m\times\mathbb{R}^{1+(m+1)}$ of codimension m, thus dimension $m+2$. Let $W:=\ker\begin{pmatrix}A^{557,m,\epsilon,=} & G^{557,m,\epsilon,=}\end{pmatrix}$. We clearly have

$$\dim\left(\operatorname{im}\begin{pmatrix}A^{557,m,\epsilon,=} & G^{557,m,\epsilon,=}\end{pmatrix}\right)=m+1;$$

thus,

$$\begin{aligned}\dim W &= \dim\left(\ker\left(\begin{array}{cc} A^{557,m,\epsilon,=} & G^{557,m,\epsilon,=}\end{array}\right)\right)\\ &= (m+(1+(m+1))) - \dim\left(\operatorname{im}\left(\begin{array}{cc} A^{557,m,\epsilon,=} & G^{557,m,\epsilon,=}\end{array}\right)\right)\\ &= 2(m+1)-(m+1)\\ &= m+1.\end{aligned}$$

So, by Lemma 30, we have

$$\dim(V\cap W) = \dim V + \dim W - \dim(V+W) \geq (m+2)+(m+1)-(2m+2) = 1.$$

Let $\left(\begin{smallmatrix} v \\ v'\end{smallmatrix}\right) \in (V\cap W)\setminus\left\{\left(\begin{smallmatrix} 0^m \\ 0^{1+(m+1)}\end{smallmatrix}\right)\right\}$. It is easy to check that $\left(\begin{smallmatrix} v \\ v'_1\end{smallmatrix}\right) \neq \left(\begin{smallmatrix} 0^m \\ 0\end{smallmatrix}\right)$, since

$$\left(\begin{array}{c} v \\ v'\end{array}\right) \neq \left(\begin{array}{c} 0^m \\ 0^{1+(m+1)}\end{array}\right)$$

and by the structure of $\left(\begin{array}{cc} A^{557,m,\epsilon,=} & G^{557,m,\epsilon,=}\end{array}\right)$, if

$$\left(\begin{array}{c} v \\ v'\end{array}\right) \in \ker\left(\begin{array}{cc} A^{557,m,\epsilon,=} & G^{557,m,\epsilon,=}\end{array}\right)$$

such that $v'_{(2,\ldots,m+2)} \neq 0^{m+1}$, we also have $\left(\begin{smallmatrix} v \\ v'_1\end{smallmatrix}\right) \neq \left(\begin{smallmatrix} 0^m \\ 0\end{smallmatrix}\right)$. Since

$$\left(\begin{array}{c} v \\ v'\end{array}\right) \in \ker\left(\begin{array}{cc} A^{557,m,\epsilon,=} & G^{557,m,\epsilon,=}\end{array}\right),$$

we conclude

$$v - \frac{m}{\epsilon(m+1)}v'_1 - v'_{(2,\ldots,m+1)} = 0^m, \tag{15.5}$$

$$-\sum_{i=1}^{m} v_i - \frac{m}{\epsilon(m+1)}v'_1 - v'_{m+2} = 0. \tag{15.6}$$

By Lemma 575, there exists an $\left(\begin{smallmatrix} x^1 \\ y^1\end{smallmatrix}\right) \in P^{556,m,\epsilon,\leq}$ and $\mu \in \mathbb{R}$ such that

$$\begin{aligned}\left(\begin{array}{c} x^2 \\ \frac{\epsilon}{m^2}\end{array}\right) &:= \left(\begin{array}{c} x^1 \\ y^1\end{array}\right) + \mu\left(\begin{array}{c} v \\ v'_1\end{array}\right)\\ &\in \left(P^{556,m,\epsilon,\leq} + \operatorname{lin}\left\{\left(\begin{array}{c} v \\ v'_1\end{array}\right)\right\}\right) \cap \left(\left(X^m_m\right)_I \times \mathbb{R}\right)\\ &\subseteq \left(P^{556,m,\epsilon,\leq} + \operatorname{lin}\left\{\left(\begin{array}{c} v \\ v'_1\end{array}\right)\right\}\right)_I.\end{aligned} \tag{15.7}$$

By Lemma 571, there thus exists an $\epsilon^* \in \mathbb{R}_{>0}$ (only depending on ϵ) having $\epsilon^* \leq \epsilon$ such that

$$\left(\begin{array}{c} \frac{m}{m+1}\cdot 1^m \\ \epsilon^*\end{array}\right) \in \operatorname{conv}\left(\dot{\bigcup_{i=0}^{m}}\left\{\left(\begin{array}{c} m\cdot e^{m,i} \\ 0\end{array}\right)\right\} \dot{\cup} \left\{\left(\begin{array}{c} x^2 \\ \frac{\epsilon}{m^2}\end{array}\right)\right\}\right).$$

This means that there exists a $\lambda \in \Delta^{m+1}$ having

$$\left(\begin{array}{c} \frac{m}{m+1}\cdot 1^m \\ \epsilon^*\end{array}\right) = \sum_{i=1}^{m}\lambda_i\left(\begin{array}{c} m\cdot e^{m,i} \\ 0\end{array}\right) + \lambda_{m+1}\left(\begin{array}{c} 0^m \\ 0\end{array}\right) + \lambda_{m+2}\left(\begin{array}{c} x^2 \\ \frac{\epsilon}{m^2}\end{array}\right). \tag{15.8}$$

Clearly,

$$\begin{pmatrix} m\cdot e^{m,1} \\ 0 \\ m\cdot e^{m,1} \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} m\cdot e^{m,m} \\ 0 \\ m\cdot e^{m,m} \\ 0 \end{pmatrix}, \begin{pmatrix} 0^m \\ 0 \\ 0^m \\ m \end{pmatrix}, \begin{pmatrix} x^1 \\ y^1 \\ x^1 - \frac{m}{\epsilon(m+1)} y^1 \\ m - \sum_{i=1}^m x_i^1 - \frac{m}{\epsilon(m+1)} y^1 \end{pmatrix} + \mu \begin{pmatrix} v \\ v' \end{pmatrix}$$
$$\in \left(P^{557,m,\epsilon,=} + \operatorname{lin}\left\{ \begin{pmatrix} v \\ v' \end{pmatrix} \right\} \right)_I . \tag{15.9}$$

We now finally claim that

$$\begin{pmatrix} \frac{m}{m+1}\cdot 1^m \\ \epsilon^* \\ \frac{m}{m+1}\left(1-\frac{\epsilon^*}{\epsilon}\right)1^{m+1} \end{pmatrix} = \sum_{i=1}^m \lambda_i \begin{pmatrix} m\cdot e^{m,i} \\ 0 \\ m\cdot e^{m,i} \\ 0 \end{pmatrix} + \lambda_{m+1} \begin{pmatrix} 0^m \\ 0 \\ 0^m \\ m \end{pmatrix}$$
$$+ \lambda_{m+2} \left(\begin{pmatrix} x^1 \\ y^1 \\ x^1 - \frac{m}{\epsilon(m+1)} y^1 1^m \\ m - \sum_{i=1}^m x_i^1 - \frac{m}{\epsilon(m+1)} y^1 \end{pmatrix} + \mu \begin{pmatrix} v \\ v' \end{pmatrix} \right). \tag{15.10}$$

(15.10) and (15.9) together with the easily verifiable fact

$$\begin{pmatrix} \frac{m}{m+1}\cdot 1^m \\ \epsilon^* \\ \frac{m}{m+1}\left(1-\frac{\epsilon^*}{\epsilon}\right)1^{m+1} \end{pmatrix} \in P^{557,m,\epsilon,=}$$

show

$$\begin{pmatrix} \frac{m}{m+1}\cdot 1^m \\ \epsilon^* \\ \frac{m}{m+1}\left(1-\frac{\epsilon^*}{\epsilon}\right)1^{m+1} \end{pmatrix} \in \operatorname{cl}_{m-\frac{1}{2},\mathbb{R}\times\mathbb{R}} \left(P^{557,m,\epsilon,=}\right).$$

For (15.10)**:** The equality in the first $m+1$ components holds by (15.7) and (15.8). So, we only have to check the last $m+1$ components:

$$\begin{aligned}
&\frac{m}{m+1}\left(1-\frac{\epsilon^*}{\epsilon}\right)1^m \\
=&\frac{m}{m+1}\cdot 1^m - \frac{\epsilon^*}{\epsilon} m^2 \frac{1}{(m+1)}\cdot\frac{1}{m}1^m \\
=&\frac{m}{m+1}\cdot 1^m - \lambda_{m+2}\frac{1}{(m+1)}\cdot\frac{1}{m}1^m && \text{(by (15.8))} \\
=&\sum_{i=1}^m \lambda_i m\cdot e^{m,i} + \lambda_{m+1}0^m + \lambda_{m+2}\left(x^2 - \frac{1}{(m+1)}\cdot\frac{1}{m}1^m\right) && \text{(by (15.8))} \\
=&\sum_{i=1}^m \lambda_i m\cdot e^{m,i} + \lambda_{m+1}0^m + \lambda_{m+2}\left(x^2 - \frac{m}{\epsilon\,(m+1)}\cdot\frac{\epsilon}{m^2}1^m\right) \\
=&\sum_{i=1}^m \lambda_i m\cdot e^{m,i} + \lambda_{m+1}0^m + \lambda_{m+2}\left(\left(x^1+\mu v\right) - \frac{m}{\epsilon\,(m+1)}\left(y^1+\mu v_1'\right)1^m\right) && \text{(by (15.7))} \\
=&\sum_{i=1}^m \lambda_i m\cdot e^{m,i} + \lambda_{m+1}0^m + \lambda_{m+2}\left(x^1 - \frac{m}{\epsilon\,(m+1)}y^1 1^m + \mu\left(v - \frac{m}{\epsilon\,(m+1)}v_1'\right)\right) \\
=&\sum_{i=1}^m \lambda_i m\cdot e^{m,i} + \lambda_{m+1}0^m + \lambda_{m+2}\left(x^1 - \frac{m}{\epsilon\,(m+1)}y^1 1^m + \mu v'_{(2,\dots,m+1)}\right) && \text{(by (15.5))}.
\end{aligned}$$

and

$$\begin{aligned}
\frac{m}{m+1}\left(1-\frac{\epsilon^*}{\epsilon}\right) &= \frac{m}{m+1} - \frac{\epsilon^*}{\epsilon}m^2 \cdot \frac{1}{m+1}\cdot\frac{1}{m} \\
&= \frac{m}{m+1} - \lambda_{m+2}\cdot\frac{1}{m+1}\cdot\frac{1}{m} && \text{(by (15.8))} \\
&= \frac{m}{m+1} - \lambda_{m+2}\frac{m}{\epsilon(m+1)}\frac{\epsilon}{m^2} \\
&= \frac{m}{m+1} - \lambda_{m+2}\frac{m}{\epsilon(m+1)}\left(\mu v'_1 + y^1\right) && \text{(by (15.7))} \\
&= \frac{m}{m+1} + \lambda_{m+2}\left(-\mu\frac{m}{\epsilon(m+1)}v'_1 - \frac{m}{\epsilon(m+1)}y^1\right) \\
&= \frac{m}{m+1} + \lambda_{m+2}\left(\mu\sum_{i=1}^{m} v_i - \frac{m}{\epsilon(m+1)}y^1 + \mu v'_{m+2}\right) && \text{(by (15.6))} \\
&= \frac{m}{m+1} + \lambda_{m+2}\left(\sum_{i=1}^{m} x_i^2 - \sum_{i=1}^{m} x_i^1 - \frac{m}{\epsilon(m+1)}y^1 + \mu v'_{m+2}\right) && \text{(by (15.7))} \\
&= m - \left(\frac{m^2}{m+1} - \lambda_{m+2}\sum_{i=1}^{m} x_i^2\right) + \lambda_{m+2}\left(-\sum_{i=1}^{m} x_i^1 - \frac{m}{\epsilon(m+1)}y^1 + \mu v'_{m+2}\right) \\
&= m - \sum_{i=1}^{m}\left(\frac{m}{m+1} - \lambda_{m+2}x_i^2\right) + \lambda_{m+2}\left(-\sum_{i=1}^{m} x_i^1 - \frac{m}{\epsilon(m+1)}y^1 + \mu v'_{m+2}\right) \\
&= m - m\sum_{i=1}^{m}\lambda_i + \lambda_{m+2}\left(-\sum_{i=1}^{m} x_i^1 - \frac{m}{\epsilon(m+1)}y^1 + \mu v'_{m+2}\right) && \text{(by (15.8))} \\
&= m\left(1 - \sum_{i=1}^{m}\lambda_i - \lambda_{m+2}\right) + \lambda_{m+2}\left(m - \sum_{i=1}^{m} x_i^1 - \frac{m}{\epsilon(m+1)}y^1 + \mu v'_{m+2}\right) \\
&= \lambda_{m+1}m + \lambda_{m+2}\left(m - \sum_{i=1}^{m} x_i^1 - \frac{m}{\epsilon(m+1)}y^1 + \mu v'_{m+2}\right). && (\lambda \in \Delta^{m+1})
\end{aligned}$$

□

15.5. Implications

15.5.1. Differences between various classes of cutting planes

We now write down some implications of the results that we stated in section 15.2.

Theorem 576. *For every $m \in \mathbb{Z}_{\geq 1}$, there exists a rational polytope*

$$P := P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right) \subseteq \mathbb{R}^m \times \mathbb{R}^1$$

(concretely, one can choose $P := P^{556,m,\epsilon,\leq} = P^{\leq}\left(\begin{pmatrix} A^{556,m,\epsilon,\leq} & G \end{pmatrix}, b^{556,m,\epsilon,\leq}\right)$ for some $\epsilon \in \mathbb{Q}_{>0}$) such that for every $\ell \in \mathbb{Z}_{\geq 0}$, we have

$$\begin{aligned}
&\mathrm{cl}_I(P) = \mathrm{cl}_{mBS}(P) = \mathrm{cl}_{L_{m,\mathbb{Q}}}(P) = \mathrm{cl}_{L_{(m+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{ILF}(P) = \mathrm{cl}_{BR}(A,G,b) \\
&\subsetneq \mathrm{cl}^{(\ell)}_{(m-1)BS}(P), \mathrm{cl}^{(\ell)}_{L_{m-\frac{1}{2},\mathbb{R}\times\mathbb{R}}}(P).
\end{aligned}$$

Proof. The statements are consequences of Theorem 558, Theorem 559, Theorem 560, Theorem 562, Theorem 563, Theorem 565 and Theorem 566 together with Remark 572. □

Theorem 577. *For every $m \in \mathbb{Z}_{\geq 1}$, there exist*

$$A \in \mathbb{Q}^{(m+1)\times m}, \qquad G \in \mathbb{Q}^{(m+1)\times(m+2)}, \qquad b \in \mathbb{Q}^{m+1}$$

(concretely, one can choose $A^{557,m,\epsilon,=}$, $G^{557,m,\epsilon,=}$ and $b^{557,m,\epsilon,=}$ for some $\epsilon \in \mathbb{Q}_{>0}$) such that for

$$P := P^{=}\left(\left(\begin{array}{cc} A & G \end{array}\right), b\right) \cap \left(\mathbb{R}^m \times \mathbb{R}_{\geq 0}^{1+(m+1)}\right)$$

and $\ell \in \mathbb{Z}_{\geq 0}$, we have

$$\begin{aligned} &\mathrm{cl}_I(P) = \mathrm{cl}_{mBS}(P) = \mathrm{cl}_{L_{m,\mathbb{Q}}}(P) = \mathrm{cl}_{L_{(m+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{ILF}(P) = \mathrm{cl}_{mR}(A, G, b) \\ \subsetneq\; &\mathrm{cl}^{(\ell)}_{(m-1)BS}(P), \mathrm{cl}^{(\ell)}_{L_{m-\frac{1}{2},\mathbb{R}\times\mathbb{R}}}(P). \end{aligned}$$

Proof. The statements are consequences of Theorem 559, Theorem 561, Theorem 562, Theorem 564, Theorem 565, Theorem 566 and Theorem 568 together with Remark 572. ☐

15.5.2. The final inclusion in the $L_{k-\frac{1}{2}}/L_k$ hierarchy

We finally return to the question that was still open in section 14.9.3 and show:

Theorem 578. *For every $m, n \in \mathbb{Z}_{\geq 1}$ and $k \in \{1, \ldots, m\}$, there exists a rational polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ such that for every $\ell \in \mathbb{Z}_{\geq 0}$, we have*

$$\mathrm{cl}_I(P) = \mathrm{cl}_{L_{k,\mathbb{Q}}}(P) \subsetneq \mathrm{cl}^{(\ell)}_{L_{k-\frac{1}{2},\mathbb{R}\times\mathbb{R}}}(P).$$

Proof. For $m = k$ and $n = 1$, let $P := P^{556,m,\epsilon,\leq}$, where $\epsilon \in \mathbb{Q}_{>0}$. In this case, the statement holds by Theorem 576. The general statement is then a consequence of Theorem 220. ☐

Part VII.

Summary and outlook

16. Summary

16.1. Basic definitions and cutting planes

The purpose of part I, which consists of chapters 2 and 3, was mainly to set up definitions. In chapter 2, we established some basic definitions and results that were used throughout this text. In chapter 3, we defined common cutting planes and their associated operators that have been studied in the literature.

16.2. L_k cuts and $L_{k-\frac{1}{2}}$ cuts

In chapter 4, we started with the central topic of this dissertation: the frameworks of L_k cuts and $L_{k-\frac{1}{2}}$ cuts. The idea behind both is that one can derive cutting planes for some $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$ (even though the case $m = 0$ is of hardly any mathematical interest), where m denotes the number of integer variables and n denotes the number of continuous variables) by considering valid inequalities for either

$$P \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + V)$$

(L_k cuts) or

$$(P + V) \cap (\mathbb{Z}^m \times \mathbb{R}^n)$$

($L_{k-\frac{1}{2}}$ cuts), where $V \leq \mathbb{R}^m \times \mathbb{R}^n$ is a subspace of codimension $k \in \{0, \ldots, m+n\}$ (the case $k = 0$ is admitted for formal reasons). By demanding different rationality conditions on the generators of V, this defines different types of L_k cuts and $L_{k-\frac{1}{2}}$ cuts such as

- $L_{k,\mathbb{Q}}$ cuts: V has generators from $\mathbb{Q}^m \times \mathbb{R}^n$,
- $L_{k,\mathbb{R}}$ cuts: V has generators from $\mathbb{R}^m \times \mathbb{R}^n$,
- $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts: V has generators from $\mathbb{Q}^m \times \mathbb{Q}^n$,
- $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cuts: V has generators from $\mathbb{Q}^m \times \mathbb{R}^n$,
- $L_{k-\frac{1}{2},\mathbb{R}\times\mathbb{R}}$ cuts: V has generators from $\mathbb{R}^m \times \mathbb{R}^n$.

Each of these families yields a hierarchy (indexed by k) of cutting planes or cutting plane operators/closures (intersection of P with all cutting planes of the respective type), such as $\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(\,\cdot\,)$, which is increasingly tight as k increases.

For the structure of chapter 4:

- In section 4.2, we started with the framework of L_k cuts (section 4.1 was used for proving some auxiliary results). Here, in section 4.2.1, we defined L_k cuts (Definition 161) and their closures (Definition 165). In section 4.2.2, we considered how L_k cuts can be represented in an alternative way:
  - In section 4.2.2.1, Theorem 168, we showed that for both $L_{k,\mathbb{Q}}$ and $L_{k,\mathbb{R}}$ cuts, we can restrict ourselves to vector spaces of the form $V = V' \times \mathbb{R}^n$.
  - In section 4.2.2.2, we considered how $L_{k,\mathbb{Q}}$ cuts can be represented "in a dual way": instead of considering inequalities for

    $$P \cap ((\mathbb{Z}^m \times \mathbb{R}^n) + (V' \times \mathbb{R}^n)),$$

    we considered inequalities for

    $$P \cap \left(\left\{ x \in \mathbb{R}^m : \left(w^i\right)^T x \in \mathbb{Z}\ \forall i \in [k] \right\} \times \mathbb{R}^n \right),$$

    where $w^1, \ldots, w^k \in \mathbb{Z}^m$ for $k \in \mathbb{Z}_{\geq 0}$. This perspective is taken in [DGMR17], a paper where a class of cutting planes called **k-dimensional lattice cuts** (cf. Definition 175) is considered. Indeed, k-dimensional lattice cuts turn out to be closely related to $L_{k,\mathbb{Q}}$ cuts. We formalized this relationship in Theorem 176.

- In section 4.3, we defined $L_{k-\frac{1}{2}}$ cuts (Definition 179) and their closures (Definition 182) similarly to how we did for L_k cuts in section 4.2.1.

In Remark 156, we gave some central guiding questions for chapter 4. We thus restate it here:

Remark 156. *Before we continue outlining the structure of chapter 4, we want to characterize the central questions that we want to analyze for L_k cuts/closures and $L_{k-\frac{1}{2}}$ cuts/closures:*

1. *Analyze under what conditions one type of L_k cuts/closure or $L_{k-\frac{1}{2}}$ cuts/closure is more expressive than another one or not.*

2. *Show that the out of themselves unrelated looking hierarchies of operators for $L_{k,\mathbb{Q}}$ cuts, $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts and $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cuts can be combined into a "unified hierarchy" for rational polyehdra, i.e. for a rational polyhedron $P \subseteq \mathbb{R}^m$ ($m \in \mathbb{Z}_{\geq 0}$), we have*

$$P = \mathrm{cl}_{L_{0,\mathbb{Q}}}(P) \supseteq \mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{Q}}}(P) \supseteq \mathrm{cl}_{L_{1,\mathbb{Q}}}(P) \supseteq \ldots \supseteq \mathrm{cl}_{L_{m-1,\mathbb{Q}}}(P) \supseteq \mathrm{cl}_{L_{m-\frac{1}{2},\mathbb{Q}}}(P) = \mathrm{cl}_{L_{m,\mathbb{Q}}}(P) = \mathrm{cl}_I(P) \tag{4.1}$$

and for a rational polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}_{\geq 1}$), the chain of inclusions

$$\begin{aligned} P = \mathrm{cl}_{L_{0,\mathbb{Q}}}(P) \supseteq \mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P) \supseteq \mathrm{cl}_{L_{1,\mathbb{Q}}}(P) \supseteq \ldots \supseteq \mathrm{cl}_{L_{m-1,\mathbb{Q}}}(P) \\ \supseteq \mathrm{cl}_{L_{m-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{m-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P) \supseteq \mathrm{cl}_{L_{m,\mathbb{Q}}}(P) = \mathrm{cl}_{L_{(m+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{(m+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P) = \mathrm{cl}_I(P) \end{aligned} \tag{4.2}$$

holds. We remark that in part VI (chapter 14 and chapter 15), we reconsider this hierarchy and analyze whether these inclusions can also be strict.

3. *In section 4.2.2.1, Theorem 168, we show that for both $L_{k,\mathbb{Q}}$ cuts and $L_{k,\mathbb{R}}$ cuts, we can restrict ourselves to vector spaces of the form $V = V' \times \mathbb{R}^n$. It is easy to see that such a restriction is not possible for $L_{k-\frac{1}{2}}$ cuts. Nevertheless, one can ask the question whether $L_{k-\frac{1}{2}}$ cuts, specifically $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts, with respect to such a vector space fill an interesting role or have interesting properties.*

For guiding question 1: In section 4.5, we started with the journey that was outlined in Remark 156 and analyzed the different types of L_k and $L_{k-\frac{1}{2}}$ cuts/closures for differences:

- In section 4.5.1, we summarized results for L_k cuts that we showed in section 4.4: by Theorem 189, for every $m \in \mathbb{Z}_{\geq 2}$, we have for $P^{116,m} \subseteq \mathbb{R}^m$ (an irrational hyperplane that contains no integral point):

$$\begin{aligned} \mathrm{cl}_{L_{m-1,\mathbb{Q}}}\left(P^{116,m}\right) &= P^{116,m}, \\ \mathrm{cl}_{L_{1,\mathbb{R}}}\left(P^{116,m}\right) &= \emptyset = \mathrm{cl}_I\left(P^{116,m}\right) = \mathrm{cl}_{\overline{I}}\left(P^{116,m}\right). \end{aligned}$$

  In this sense, $L_{1,\mathbb{R}}$ cuts/the $L_{1,\mathbb{R}}$ closure can be stronger than $L_{m-1,\mathbb{Q}}$ cuts/the $L_{m-1,\mathbb{Q}}$ closure.

- In section 4.5.2, we summarized results for $L_{k-\frac{1}{2}}$ cuts that we showed in section 4.4: by Theorem 190, for every $m \in \mathbb{Z}_{\geq 2}$, we have for $P^{116} \subseteq \mathbb{R}^m$ (an irrational hyperplane that contains no integral point) and $P^{117} \subseteq \mathbb{Z}^m$ (an irrational hyperplane that contains 0^m as the only integral point):

$$\begin{aligned} \mathrm{cl}_{L_{(m-1)-\frac{1}{2},\mathbb{Q}}}\left(P^{116,m}\right) &= P^{116,m}, \\ \mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{R}}}\left(P^{116,m}\right) &= \emptyset = \mathrm{cl}_I\left(P^{116,m}\right) = \mathrm{cl}_{\overline{I}}\left(P^{116,m}\right), \\ \mathrm{cl}_{L_{(m-1)-\frac{1}{2},\mathbb{Q}}}\left(P^{117,m}\right) &= P^{117,m}, \\ \mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{R}}}\left(P^{117,m}\right) &= \left\{0^2\right\} = \mathrm{cl}_I\left(P^{117,m}\right) = \mathrm{cl}_{\overline{I}}\left(P^{117,m}\right). \end{aligned}$$

So, until here, we have cleared up that there can exist a difference in expressivity between

- $L_{(\,\cdot\,),\mathbb{Q}}$ cuts/closure versus $L_{(\,\cdot\,),\mathbb{R}}$ cuts/closure and
- $L_{(\,\cdot\,)-\frac{1}{2},\mathbb{Q}}$ cuts/closure versus $L_{(\,\cdot\,)-\frac{1}{2},\mathbb{R}}$ cuts/closure.

We next compared $L_{k,\mathbb{Q}\times\mathbb{Q}}$ cuts/closure to $L_{k,\mathbb{Q}\times\mathbb{R}}$ cuts/closure. Section 4.5.3 was about the situation where we have $\mathrm{cl}_{L_{k,\mathbb{Q}\times\mathbb{Q}}}(\cdot) \supsetneq \mathrm{cl}_{L_{k,\mathbb{Q}\times\mathbb{R}}}(\cdot)$, while section 4.5.4 was about the situation where equality between these closures holds.

The central result of section 4.5.3 is Theorem 191. Here, we showed that for $P^{118} \subseteq \mathbb{R}^1 \times \mathbb{R}^2$, we have:

$$P^{118} = \mathrm{cl}_{L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}\left(P^{118}\right) \supsetneq \mathrm{cl}_{L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}\left(P^{118}\right) = \mathrm{cl}_I\left(P^{118}\right) = \mathrm{cl}_{\overline{I}}\left(P^{118}\right).$$

The essential property of P^{118} that enabled this proof was that P^{118} has a lineality space that is generated by vectors from $\mathbb{Q}^m \times \mathbb{R}^n$, but is not a rational subspace.

So, we formulated Conjecture 192, which claims that as long as $P = Q + C + L \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$, where

- Q is convex and compact,
- C is a pointed polyhdral cone generated by vectors from $\mathbb{Q}^m \times \mathbb{R}^n$ and
- L is a linear vector space generated by rational vectors (from $\mathbb{Q}^m \times \mathbb{Q}^n$),

we have

$$\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P)$$

for $k \in \{0, \ldots, m\}$. While Conjecture 192 is open, we considered important special cases in section 4.5.4:

- In Theorem 193, which we proved further back in section 4.8.4, we saw that Conjecture 192 holds if $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ is a rational polyhedron. Moreover: every $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cut for P $(k \in \{0, \ldots, m\})$ is dominated absolutely by a *finite* number of rational $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts for P. This, of course, implies the
$$\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P)$$
equalities in (4.1) and (4.2) of guiding question 2 in Remark 156.
- Why do we put an emphasis on this stronger statement (than the statement of Conjecture 192) that every $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ is dominated by a *finite* number of $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts for P if P is a rational polyhedron? The reason is that in the more general setting of Conjecture 192 (even if P is a polyhedron), it can happen that we have
$$\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P) = \mathrm{cl}_I(P)$$
and we need to apply just one $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cut to P to obtain $\mathrm{cl}_I(P)$, but the intersection of P with an arbitray finite number of half-spaces induced by $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts is always a strict superset of $\mathrm{cl}_I(P)$. In Theorem 194, we gave an example of this phenomenon.

For guiding question 2:

Inclusions/non-inclusions: In section 4.6, we considered the inclusions *between* the two hierarchies of $L_{k-\frac{1}{2}}$ cuts and L_k cuts (i.e. the inclusions in (4.1) and (4.2) with the particular goal to put these on a solid foundation, but also related inclusions/non-inclusions). In Theorem 197, we showed that for an arbitrary $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$, every $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cut for it $(k \in \{0, \ldots, m+n\})$ with respect to some vector space V is also an $L_{k,\mathbb{Q}}$ cut with respect to the same vector space, which, of course, implies the

$$\mathrm{cl}_{L_{k,\mathbb{Q}}}(P) \subseteq \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P)$$

inclusions in (4.1) and (4.2). Note that (cf. Remark 198) a similar result for $L_{k,\mathbb{R}}$ cuts/closure versus $L_{k,\mathbb{R}}$ cuts/closure of an arbitrary set $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ does *not* hold in general, i.e. we don't have

$$\mathrm{cl}_{L_{k,\mathbb{R}}}(P) \subseteq \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{R}\times\mathbb{R}}}(P)$$

in general (even if we assume P to be a polyhedron). For this, recall from Remark 198 that

$$cl_{L_{1,\mathbb{R}}}\left(P^{115}\right) = P^{115} \supsetneq \left\{0^2\right\} = \mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{R}}}\left(P^{115}\right).$$

On the other hand, for the inclusion type $\mathrm{cl}_{L_{(k+1)-\frac{1}{2},(\,\cdot\,)}}(\,\cdot\,) \subseteq \mathrm{cl}_{L_{k,(\,\cdot\,)}}(\,\cdot\,)$, we know by Theorem 199 that for an arbitrary $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ and $k \in \{0, \ldots, m+n-1\}$, every $L_{k,\mathbb{R}}$ cut for P is an $L_{(k+1)-\frac{1}{2},\mathbb{R}\times\mathbb{R}}$ cut for P, from which we conclude

$$\mathrm{cl}_{L_{(k+1)-\frac{1}{2},\mathbb{R}\times\mathbb{R}}}(P) \subseteq \mathrm{cl}_{L_{k,\mathbb{R}}}(P).$$

We also proved there that if P is a rational polyhedron, we have

$$\mathrm{cl}_{L_{(k+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) \subseteq \mathrm{cl}_{L_{k,\mathbb{Q}}}(P),$$

which showed the other type of inclusion in (4.1) and (4.2). Recall (cf. Example 201) that P being a rational polyhedron is essential for this inclusion to hold.

Termination: We claimed in (4.2) that if $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}_{\geq 1}$) is a rational polyhedron, the chain of inclusions in the hierarchy already ends at

$$\mathrm{cl}_{L_{m,\mathbb{Q}}}(P) = \mathrm{cl}_{L_{(m+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{(m+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P) = \mathrm{cl}_I(P)$$

instead of

$$\mathrm{cl}_{L_{m+n,\mathbb{Q}}}(P) = \mathrm{cl}_{L_{m+n-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{m+n-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P) = \mathrm{cl}_I(P),$$

as one might expect intuitively. In Theorem 202 in section 4.7, we showed that the hierarchy indeed terminates at this place under the stated rationality condition for P. Additionally, we showed in this theorem that for general $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$, we have

$$\mathrm{cl}_{L_{m,\mathbb{R}}}(P) = \mathrm{cl}_{L_{(m+1)-\frac{1}{2},\mathbb{R}\times\mathbb{R}}}(P) = P \cap \mathrm{cl}_{\overline{I}}(P).$$

For guiding question 3/essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts: In section 4.8, we considered what we denoted "guiding question 3" in Remark 156. For this, in Definition 203, we introduced **essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts**, which are $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts/$L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cuts with respect to a vector space $V' \times \mathbb{R}^n$, where $V' \leq \mathbb{R}^m$ $(m, n \in \mathbb{Z}_{\geq 0})$ is a rational subspace of codimension $k \in \{0, \ldots, m\}$. In Definition 205, we defined their respective closure, for which we gave an alternative characterization in Theorem 207.

In Theorem 208, we proved a result that shows the importance of the concept of essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts: we showed that if for some arbitrary $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$, there exists an $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cut ($k \in \{1, \ldots, m\}$) which is not already an $L_{k-1,\mathbb{Q}}$ cut for P, then it is an essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cut for P. This exhibits that the only "interesting" $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cuts (since these are not "already" $L_{k-1,\mathbb{Q}}$ cuts) are essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts.

In Theorem 211, we elaborated on this statement in particular for the case that P is a rational polyhedron. In this case, one can even show that every $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cut ($k \in \{1, \ldots, m\}$) for P is dominated (absolutely) by a finite set of either rational essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts for P or rational $L_{k-1,\mathbb{Q}}$ cuts for P.

As a next step, we showed a reverse to Theorem 208. We just mentioned that in this theorem, we characterized how $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cuts that are not already $L_{k-1,\mathbb{Q}}$ cuts look like (they are essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts). So, the reverse question to ask is how $L_{k,\mathbb{Q}}$ cuts that are not already $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts or $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cuts look like. In Theorem 213, we showed that if we have an $L_{k,\mathbb{Q}}$ cut $c(\,\cdot\,) \leq c_0$ for some $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ with respect to some vector space V be given, it is already an $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ or $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cut for P with respect to the same vector space V if $c^T \perp V$. Thus, every $L_{k,\mathbb{Q}}$ cut $c(\,\cdot\,) \leq c_0$ with respect to some V that is not already an $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cut or $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cut, respectively, must satisfy $c^T \not\perp V$.

Finally, in Theorem 215, we considered how essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts can be characterized in a "dual way" similar to the dual characterization of $L_{k,\mathbb{Q}}$ cuts in Theorem 174.

Lineality spaces and affine subspaces: The central topic of section 4.9 was the following: let some arbitrary $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ be given. Trivially, we always have

$$\mathrm{cl}_{(m+n)-\frac{1}{2},(\,\cdot\,)}(P) = P \cap \mathrm{cl}_{\overline{I}}(P)$$

and

$$\mathrm{cl}_{m,(\,\cdot\,)}(P) = P \cap \mathrm{cl}_{\overline{I}}(P).$$

But now assume that we can impose some "vector space condition" on P:

- P or $\operatorname{proj}_{\mathbb{R}^m} P$ has a (w.l.o.g. non-trivial) lineality space or
- P or $\operatorname{proj}_{\mathbb{R}^m} P$ is contained in an (again w.l.o.g. non-trivial) affine subspace.

Can we then show

$$\operatorname{cl}_{(m+n-l)-\frac{1}{2},(\,\cdot\,)}(P) = P \cap \operatorname{cl}_{\overline{I}}(P)$$

or

$$\operatorname{cl}_{m-l,(\,\cdot\,)}(P) = P \cap \operatorname{cl}_{\overline{I}}(P),$$

respectively, where $l \in \mathbb{Z}_{\geq 1}$? For the results:

- In Theorem 217, we showed that given some vector space $L \leq \operatorname{lineal} P$ that satisfies some rationality conditions if relevant, we can restrict ourselves to $L_{k-\frac{1}{2}}$ cuts with respect to subspaces that contain L. If we set $l := \dim L$, we have (depending on the rationality conditions that hold for the system of generators of L)

$$\operatorname{cl}_{L_{m+n-l-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = P \cap \operatorname{cl}_{\overline{I}}(P),$$
$$\operatorname{cl}_{L_{m+n-l-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P) = P \cap \operatorname{cl}_{\overline{I}}(P) \text{ or}$$
$$\operatorname{cl}_{L_{m+n-l-\frac{1}{2},\mathbb{R}\times\mathbb{R}}}(P) = P \cap \operatorname{cl}_{\overline{I}}(P),$$

  respectively.

- In Theorem 218, we showed that given some rational vector space $L \leq \operatorname{proj}_{\mathbb{R}^m}(\operatorname{lineal} P)$, we can restrict ourselves to L_k cuts with respect to subspaces that contain $L \times \mathbb{R}^n$. See Remark/Problem 219 for an explanation why the rationality condition for L is important here. Additionally, if we set $l := \dim L$, we have

$$\operatorname{cl}_{L_{m-l,\mathbb{Q}}}(P) = P \cap \operatorname{cl}_{\overline{I}}(P).$$

- In Theorem 222, we showed that if $P \subseteq \mathbb{R}^m$ is contained in an affine translate of a rational subspace of dimension l, we have

$$\operatorname{cl}_{L_{\max(l,1)-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = P \cap \operatorname{cl}_{\overline{I}}(P).$$

- In Theorem 223, we showed that if for $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$, the projection $\operatorname{proj}_{\mathbb{R}^m} P$ is contained in an affine translate of a rational subspace of dimension l, we have

$$\operatorname{cl}_{L_{\max(l,1),\mathbb{Q}}}(P) = P \cap \operatorname{cl}_{\overline{I}}(P).$$

k-half-space cuts: In section 4.10, we briefly considered the concept of **k-half-space cuts** (cf. Definition 224), which, by Theorem 225, indeed form a special class of $L_{k-\frac{1}{2}}$ cuts. On the other hand, we showed (cf. Theorem 226) that k-half-space cuts are k row cuts.

16.3. Alternative characterizations of $L_{k,\mathbb{Q}}$ and essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts

As we have outlined in section 5.1, there exist two natural ways to derive alternative characterizations of L_k cuts and $L_{k-\frac{1}{2}}$ cuts: one can either show alternative characterizations for specific (typically small) values of k or prove alternative characterizations that hold for a large general range of values for k.

16.3.1. Alternative characterizations that hold for a general range of values for k

The second approach was considered in chapter 5. Here, we gave two approaches to characterize $L_{k,\mathbb{Q}}$ cuts and essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts in a different way: via lattice-free bodies and via t-branch split cuts.

16.3.1.1. Characterizing $L_{k,\mathbb{Q}}$ cuts/essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts cuts using lattice-free bodies

In section 5.2, we characterized $L_{k,\mathbb{Q}}$ cuts for a given $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) using lattice-free bodies. We stated the final result in Theorem 240.

In section 5.3, we characterized essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts cuts using lattice-free bodies. The final result is stated in Theorem 246.

16.3.1.2. Characterizing $L_{k,\mathbb{Q}}$ cuts/essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts via t-branch split cuts

In section 5.4, we characterized $L_{k,\mathbb{Q}}$ and essential $L_{k-\frac{1}{2},\mathbb{Q}}$ cuts via t-branch split cuts. For this, in Definition 252, we defined the concept of k, t**-branch split cuts** and the k, t**-branch split closure**. In Definition 253, we defined **essential** k, t**-branch split cuts** and the **essential** k, t**-branch split closure**.

In Theorem 259, we subsequently proved that if $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ is either a rational polyhedron or convex and compact, then for $k \in \{1, \ldots, m\}$, we have

$$\mathrm{cl}_{L_{k,\mathbb{Q}}}(P) = \mathrm{cl}_{k,h(k)BS}(P),$$

where $h(\cdot)$ is as in Remark/Definition 248. Note that (cf. Theorem 255) the condition that P is either a rational polyhedron or convex and compact is essential for Theorem 259 to hold, i.e. if P is an irrational hyperplane (as in Theorem 255), the $L_{k,\mathbb{Q}}$ closure can be stronger than the $k, h(k)$-branch split closure.

In Theorem 261, we showed that if $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$ is again either a rational polyhedron or convex and compact, then for $k \in \{1, \ldots, m\}$, we have

$$\mathrm{cl}_{\mathrm{ess}\, L_{k-\frac{1}{2},\mathbb{Q}}}(P) = \mathrm{cl}_{\mathrm{ess}\, k,h(k)BS}(P),$$

where $h(\cdot)$ is as in Remark/Definition 248.

In section 5.4.5, Theorem 264, we used this characterization of the $L_{k,\mathbb{Q}}$ closure of a rational polyhedron to show that the $L_{k,\mathbb{Q}}$ closure of a rational polyhedron (even with respect to an arbitrarily chosen set of rationally generated subspaces of codimension k) is again a rational polyhedron – a statement which has also independently from us been proved in [DGMR17]. Our proof reduced the statement to Corollary 336, i.e. the statement that the t-branch split closure of a rational polyehdron with respect to an arbitrary set of t-branch split disjunctions is again a rational polyhedron.

16.3.2. Alternative characterizations that hold for specific values of k

We now summarize the results with respect to the first approach for giving alternative characterizations of L_k cuts and $L_{k-\frac{1}{2}}$ cuts, i.e. giving alternative characterizations that are specific to some particular (small) value of k.

For $L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts and $L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cuts: We considered the case $k = 1$ for $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts and $L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cuts in section 8.1. There, we analyzed the relationship between

- (projected) Chvátal-Gomory cuts (cf. Definition 122),
- dual (projected) Chvátal-Gomory cuts (cf. Definition 382),
- strong (projected) Chvátal-Gomory cuts (cf. Definition 384) and
- $L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts and $L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cuts (cf. Definition 179).

In section 8.1.2.4, we gave an overview about the relationbship between these classes of cutting planes/cutting plane operators; so, at this place, we only recall that for *polyhedra*, all these types of cutting planes yield identical cutting plane *operators*.

For $L_{1,\mathbb{Q}}$ cuts: The case $k = 1$ for $L_{k,\mathbb{Q}}$ cuts was considered in section 9.1 of chapter 9. There, we analyzed the relationship between

- $L_{1,\mathbb{Q}}$ cuts (cf. Definition 161),
- split cuts (cf. Definition 126) and
- MIR cuts (cf. Definition 410).

In section 9.1.1, we showed the missing part for a proof of the equivalence between $L_{1,\mathbb{Q}}$ cuts and split cuts for convex sets $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$; a result that we wrote down as Theorem 409. Note that convexity is essential for this equivalence to hold; otherwise, $L_{1,\mathbb{Q}}$ cuts can be more expressive than split cuts (cf. Remark 407).

In section 9.1.2, we gave an equivalence proof of the split closure and the MIR closure. This equivalence is in principle well-known in the literature. Nevertheless, we think we had good reasons to reprove these results: on one hand, there exist multiple definitions of MIR cuts in the literature. This makes it hard to reuse existing results from the literature since these proofs often rely on slightly different definitions of MIR cuts. On the other hand, many authors tend to skim over the subtile asymmetry in the equivalence between split cuts and MIR cuts: while every MIR cut is a split cut (cf. Theorem 412) – thus, split cuts dominate MIR cuts *absolutely* – not every split cut is a MIR cut (cf. Example 416), even though one can always find a MIR cut that dominates it *relatively to the polyhedron* (cf. Theorem 415).

For $L_{2,\mathbb{Q}}$ cuts and essential $L_{2-\frac{1}{2},\mathbb{Q}}$ cuts: To analyze these types of cuts, we first considered lattice-free bodies in $\mathbb{R}^2$. For this, in chapter 10, we proved two theorems (Theorem 431 and Theorem 434) about how to embed two-dimensional full-dimensional lattice-free bodies into disjunctions. We want to recall that a weaker version of Theorem 431 (Theorem 432) has already been proved in [DDG12].

After this, in chapter 11, we analyzed $L_{2,\mathbb{Q}}$ cuts (section 11.1) and essential $L_{2-\frac{1}{2},\mathbb{Q}}$ cuts (section 11.2):

- In Theorem 462, we showed that if $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m \in \mathbb{Z}_{\geq 1}$, $n \in \mathbb{Z}_{\geq 0}$, $m + n \geq 2$) is either
  - a rational polyhedron,
  - convex and $\operatorname{proj}_{\mathbb{R}^2} P$ is full-dimensional or
  - convex and compact,

  we have $\operatorname{cl}_{L_{2,\mathbb{Q}}}(P) = \operatorname{cl}_{CC}(P)$. We remark that results for
  - rational polyhedra of the form $\left(\mathbb{R}^2 \times \mathbb{R}^n_{\geq 0}\right) \cap P^{=}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)$ (the proof is easy to extend to arbitrary rational polyehdra) and
  - P convex and compact

  were already proved in [DDG11] (for details cf. section 11.1). So, mostly the case "P is convex and $\operatorname{proj}_{\mathbb{R}^2} P$ is full-dimensional" is fundamentally new.

  Building on an existing result from the literature (Theorem 463), we proved that if $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) is either
  - a rational polyhedron or
  - convex and compact,

  we can indeed replace $\overline{\operatorname{conv}}$ by conv in the definition of the crooked cross closure (Definition 148), as it is commonly done in the literature.

- In section 11.1.4, we followed a similar route as we did in section 11.1 for the $L_{2,\mathbb{Q}}$ closure to show a characterization of the
  - essential $L_{2-\frac{1}{2},\mathbb{Q}}$ closure,
  - $L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ closure and
  - $L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ closure

  of a rational polyhedron. In Theorem 474, we saw that if $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m \in \mathbb{Z}_{\geq 2}$, $n \in \mathbb{Z}_{\geq 0}$) is a rational polyhedron, we have

  $$\operatorname{cl}_{\operatorname{ess} L_{2-\frac{1}{2},\mathbb{Q}}}(P) = \operatorname{cl}_{\operatorname{ess} T}(P) \cap \operatorname{cl}_{\operatorname{ess} CC}(P). \tag{16.1}$$

  In Theorem 475, we extended this result to show that for a rational polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$, where $m + n \geq 2$), we have

  $$\operatorname{cl}_{L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \operatorname{cl}_{split}(P) \cap \operatorname{cl}_{\operatorname{ess} T}(P) \cap \operatorname{cl}_{\operatorname{ess} CC}(P). \tag{16.2}$$

  In both (16.1) and (16.2), $\operatorname{cl}_{\operatorname{ess} T}(\cdot)$ denotes the **essential T closure** and $\operatorname{cl}_{\operatorname{ess} CC}(\cdot)$ denotes the **essential crooked cross closure** (cf. Definition 466).

16.4. Strictness of the $L_{k-\frac{1}{2}}/L_k$ hierarchy

The central question guiding part VI (chapter 14 and chapter 15) was the following (cf. section 14.1 for a much more detailed outline): we know (recall (4.1) and (4.2) from Remark 156) that for every rational polyhedron $P \subseteq \mathbb{R}^m$ ($m \in \mathbb{Z}_{\geq 0}$), we have

$$P = \mathrm{cl}_{L_{0,\mathbb{Q}}}(P) \supseteq \mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{Q}}}(P) \supseteq \mathrm{cl}_{L_{1,\mathbb{Q}}}(P) \supseteq \ldots \supseteq \mathrm{cl}_{L_{m-1,\mathbb{Q}}}(P) \supseteq \mathrm{cl}_{L_{m-\frac{1}{2},\mathbb{Q}}}(P) = \mathrm{cl}_{L_{m,\mathbb{Q}}}(P) = \mathrm{cl}_I(P)$$

and for every rational polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m \in \mathbb{Z}_{\geq 0}$, $n \in \mathbb{Z}_{\geq 1}$), the chain of inclusions

$$\begin{aligned} P = \mathrm{cl}_{L_{0,\mathbb{Q}}}(P) &\supseteq \mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P) \supseteq \mathrm{cl}_{L_{1,\mathbb{Q}}}(P) \supseteq \ldots \supseteq \mathrm{cl}_{L_{m-1,\mathbb{Q}}}(P) \\ &\supseteq \mathrm{cl}_{L_{m-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{m-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P) \supseteq \mathrm{cl}_{L_{m,\mathbb{Q}}}(P) = \mathrm{cl}_{L_{(m+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{(m+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P) = \mathrm{cl}_I(P) \end{aligned}$$

holds. We asked: can each of the inclusions in this chain also be strict? This was the initial motivation for chapter 14 and chapter 15.

In chapter 14, we resolved this question positively (cf. section 14.9.3) in "almost all cases" (the remaining case was considered in chapter 15) and proved the following even tighter inclusions (cf. Theorem 555):

- For every $m \in \mathbb{Z}_{\geq 1}$, $n \in \mathbb{Z}_{\geq 0}$ and $k \in \{1, \ldots, m-1\}$, there exists a rational polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ such that $\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) \subsetneq \mathrm{cl}_{L_{k-1,\mathbb{R}}}(P)$ holds.

- For every $m \in \mathbb{Z}_{\geq 2}$, $n \in \mathbb{Z}_{\geq 0}$ and $k \in \{1, \ldots, m-1\}$, there exists a rational polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ such that $\mathrm{cl}_{L_{k,\mathbb{Q}}}(P) \subsetneq \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{R}\times\mathbb{R}}}(P)$ holds.

The only case that is absent here is $k = m$ and $n \geq 1$.

This missing case was a central topic in chapter 15. In section 15.5.2, Theorem 578, we considered it and showed an even stricter result: for every $m, n \in \mathbb{Z}_{\geq 1}$ and $k \in \{1, \ldots, m\}$, there exists a rational polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}$ such that for every $\ell \in \mathbb{Z}_{\geq 0}$, we have

$$\mathrm{cl}_I(P) = \mathrm{cl}_{L_{k,\mathbb{Q}}}(P) \subsetneq \mathrm{cl}^{(\ell)}_{L_{k-\frac{1}{2},\mathbb{R}\times\mathbb{R}}}(P) \subseteq \mathrm{cl}^{(\ell)}_{L_{k-1,\mathbb{R}}}(P),$$

i.e. "we never attain $\mathrm{cl}_I(P)$ via $L_{k-\frac{1}{2},\mathbb{R}\times\mathbb{R}}$ cuts or $L_{k-1,\mathbb{R}}$ cuts, even if we apply them iteratively".

16.5. Further results on L_k cuts/$L_{k-\frac{1}{2}}$ cuts

16.5.1. Sizes of subsets of inequalities to consider

In chapter 12, we analyzed how many inequalities in the inequality description of the respective polyhedron we have to consider at the same time to derive a specific $L_{k,\mathbb{Q}}$ cut or $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cut for it. We defined this in Problem/Definition 476, which we restate here. Recall that we defined in section 12.1 that $A \in \mathbb{R}^{l\times m}$, $G \in \mathbb{R}^{l\times n}$ and $b \in \mathbb{R}^l$, where $l, m, n \in \mathbb{Z}_{\geq 0}$:

Problem/Definition 476. *Let $P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right) \subseteq \mathbb{R}^m \times \mathbb{R}^n$ be given and let $k \in \{0, \ldots, m+n\}$. What is the smallest $h \in \{0, \ldots, l\}$ such that*

$$\mathrm{cl}_{L_{k,\mathbb{Q}}}\left(P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)\right) = \bigcap_{S \in \binom{[l]}{h}} \mathrm{cl}_{L_{k,\mathbb{Q}}}\left(P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}_{S,*}, b_S\right)\right) \text{ or}$$

$$\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}\left(P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)\right) = \bigcap_{S \in \binom{[l]}{h}} \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}\left(P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}_{S,*}, b_S\right)\right),$$

respectively, holds? This smallest h is referred to as $h^*_{L_{k,\mathbb{Q}}}(A, G, b)$ or $h^*_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(A, G, b)$, respectively. If $n = 0$, we also use the notations $h^*_{L_{k,\mathbb{Q}}}(A, b)$ or $h^*_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(A, b)$, respectively.

Theorem 504 and Theorem 505, which we restate here, summarize the upper and lower bounds that we showed:

Theorem 504. *Let $k \in \{0, \ldots, m+n\}$. Then*

$$h^*_{L_{k,\mathbb{Q}}}(A,G,b) \leq \begin{cases} \operatorname{rank}\begin{pmatrix} A & G \end{pmatrix} \; (\leq m+n) & \text{if } k=1, \\ 2^{\min(k,m)}(m+n-\min(k,m)+1)-1 & \text{if } P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)_I \neq \emptyset, \\ 2^{\min(k,m)}(m+n-\min(k,m)+1) & \text{in general,} \end{cases}$$

$$h^*_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(A,G,b) \leq \begin{cases} \operatorname{rank}\begin{pmatrix} A & G \end{pmatrix} \; (\leq m+n) & \text{if } k=1, \\ (2^k-1)(m+n-k+1) & \text{if } P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)_I \neq \emptyset \wedge k \in \{0,\ldots,m\}, \\ 2^{\min(k,m)}(m+n-\min(k,m)+1)-1 & \text{if } P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)_I \neq \emptyset, \\ 2^{\min(k,m)}(m+n-\min(k,m)+1) & \text{in general.} \end{cases}$$

Theorem 505. *Let $m \in \mathbb{Z}_{\geq 1}$, $n \in \mathbb{Z}_{\geq 0}$ and $k \in \{1,\ldots,m+n\}$. Then there exist rational A, G and b such that $P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right) \neq \emptyset$ and*

$$h^*_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(A,G,b) = \begin{cases} m+n, \\ 2^{k-1} & \text{if } n=0, \\ 2^{\min(k-1,m)} & \text{if } n \geq 1, \end{cases}$$

$$h^*_{L_{k,\mathbb{Q}}}(A,G,b) = \begin{cases} m+n, \\ 2^{\min(k,m-1)} & \text{if } n=0, \\ 2^{\min(k,m)} & \text{if } n \geq 1. \end{cases}$$

In section 12.4.3, we gave a short glimpse into how the results on $h^*_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(A,G,b)$ can be generalized to other types of cutting planes that are related to $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts:

- In Theorem 496, we showed an upper bound for $h^*_{\text{ess}\,L_{k-\frac{1}{2},\mathbb{Q}}}(A,G,b)$.
- In Theorem 497, we showed that $h^*_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(A,G,b) = h^*_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(A,G,b)$ if A, G and b are rational.

16.5.2. Bounds on the $L_{k,\mathbb{Q}}$ rank and $L_{k-\frac{1}{2},\mathbb{Q}}$ rank of polyhedra with 0/1 integer variables

In section 13.1 of chapter 13, we considered bounds on the $L_{k,\mathbb{Q}}$ rank (see (6.11)) of polyhedra $P \subseteq [0,1]^m \times \mathbb{R}^n$ ($m \in \mathbb{Z}_{\geq 1}$ and $n \in \mathbb{Z}_{\geq 0}$), i.e. the integer variables are 0/1-valued. In Theorem 515, we showed the bound

$$\operatorname{rank}_{L_{k,\mathbb{Q}}}(P) \leq \left\lceil \frac{m}{k} \right\rceil \tag{16.3}$$

(this bound even holds if P is a more general closed convex set; cf. Definition 506). Now we asked whether this bound is tight. This was the topic of section 13.1.2: in Theorem 526, we proved that for every $m \in \mathbb{Z}_{\geq 1}$, there exists a rational polytope $P \subseteq [0,1]^m$ such that for all $k \in \{1,\ldots,m\}$, we have

$$\operatorname{rank}_{L_{k,\mathbb{Q}}}(P) \geq \left\lceil \frac{m}{k} \right\rceil.$$

Thus, the bound in (16.3) is indeed tight.

After this result on the $L_{k,\mathbb{Q}}$ rank, we turned our focus to the $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ rank of rational polyhedra. At the beginning of section 13.2, in Theorem 528, we used the results on the $L_{k,\mathbb{Q}}$ rank to estimate the $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ rank of rational polyhedra $P \subseteq [0,1]^m \times \mathbb{R}^n$ ($m \in \mathbb{Z}_{\geq 1}$ and $n \in \mathbb{Z}_{\geq 0}$). In the case $k=1$, this only allowed a rather disappointing estimate. Luckily, the problem of finding bounds for the $L_{1-\frac{1}{2},\mathbb{Q}}$ rank (Chvátal-Gomory rank) of a polytope $P \subseteq [0,1]^m$ is a well-studied problem in the literature. So, in section 13.2.1, we gave an overview on upper bounds for the Chvátal-Gomory rank from the literature and in section 13.2.2, we did the same for lower bounds.

16.6. Expressivity of various classes of cutting planes

16.6.1. Inclusions and non-inclusions

The question of inclusions and non-inclusions between the various cutting plane operators was a central topic in section 6.1, but also in chapter 14 and chapter 15 (here, in particular, cf. section 15.5.1). Many of the results that we list in the remainder of this section are taken from the literature.

Split cuts vs integral lattice-free cuts and k-disjunctive cuts: By Theorem 265 and Theorem 266, for an arbitrary $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$), every split cut for P is also an k-disjunctive cut ($k \in \mathbb{Z}_{\geq 2}$) and an integral lattice-free cut for P. So, we have

$$\mathrm{cl}_{kD}(P), \mathrm{cl}_{ILF}(P) \subseteq \mathrm{cl}_{split}(P).$$

Moreover, for $k = 2$, we saw in Theorem 265 that also the reverse holds, i.e. every 2-disjunctive cut for P is also a split cut for P and we thus have

$$\mathrm{cl}_{2D}(P) = \mathrm{cl}_{split}(P).$$

t-branch split cuts vs cuts from basic relaxations: In Theorem 267, which was first shown in [ACL05], we saw that for an arbitrary polyhedron $P := P^{\leq}\left(\left(\begin{array}{cc} A & G \end{array}\right), b\right)$ ($A \in \mathbb{R}^{l \times m}$, $G \in \mathbb{R}^{l \times n}$ and $b \in \mathbb{R}^l$, where $l, m, n \in \mathbb{Z}_{\geq 0}$), every split cut for P is a split cut of a basic relaxation of $P^{\leq}\left(\left(\begin{array}{cc} A & G \end{array}\right), b\right)$. Thus, one obtains (cf. Theorem 270):

$$\mathrm{cl}_{BR}(A, G, b) \subseteq \mathrm{cl}_{split}\left(P^{\leq}\left(\left(\begin{array}{cc} A & G \end{array}\right), b\right)\right).$$

This inclusion can also be strict, as we saw in Theorem 576 (which we summarize further below).

We next considered how the t-branch split closure relates to the closure with respect to cuts from basic relaxations. In Theorem 272, which was originally shown in [DGM15], we saw that for every $m \in \mathbb{Z}_{\geq 2}$, there exists a rational polyhedron $P := P^{\leq}(A, b)$ ($A \in \mathbb{Q}^{m \times m}$ and $b \in \mathbb{Q}^m$) such that for every $t \in \{0, \dots, m-2\}$, we have

$$\mathrm{cl}_{CG}^{(2)}(P) = \mathrm{cl}_{split}^{(2)}(P) = \mathrm{cl}_{BR}(A, b) = \mathrm{cl}_I(P) \subsetneq \mathrm{cl}_{tBS}(P).$$

So both the second Chvátal-Gomory closure (and thus the second split closure) and the closure with respect to cuts from a basic relaxations can be stronger than the t-branch split closure where $t \in \{0, \dots, m-2\}$.

We saw that also the reverse can happen: in Theorem 273, which was first shown in [DGM15], we saw that there exist $A \in \mathbb{Q}^{4 \times 2}$, $G \in \mathbb{Q}^{4 \times 1}$ and $b \in \mathbb{Q}^4$ such that

$$\mathrm{cl}_{BR}(A, G, b) \not\subseteq \mathrm{cl}_{2BS}\left(P^{\leq}\left(\left(\begin{array}{cc} A & G \end{array}\right), b\right)\right).$$

In Theorem 274, we showed that a similar statement also holds in the pure integer case, i.e. there exist $A \in \mathbb{Q}^{4 \times 3}$ and $b \in \mathbb{Q}^4$ such that

$$\mathrm{cl}_{BR}(A, b) \not\subseteq \mathrm{cl}_{2BS}\left(P^{\leq}(A, b)\right).$$

t-branch split cuts vs integral lattice-free cuts: In Theorem 276, we saw that there exists a rational polytope $P \subseteq \mathbb{R}^2 \times \mathbb{R}^1$ having

$$\mathrm{cl}_{2BS}(P) \subsetneq \mathrm{cl}_{ILF}(P).$$

A proof of Lemma 275, on which this theorem is essentially based, is sketched in [DPW12].

L_k cuts, crooked cross cuts and t-branch lattice-free cuts: These were the central topic of section 6.1.4. In Theorem 277, we proved a rather general statement that if we have a cutting plane with respect to some disjunction, it is an L_k cut, where

- k is the dimension of the lineality space of the disjunction and
- the type of L_k cut ($L_{k,\mathbb{Q}}$ cut or $L_{k,\mathbb{R}}$ cut) depends of the existence of rational generators for this lineality space.

From this, we consluded that for arbitrary $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$), we have:

- In Corollary 278, we showed that for $k \in \{0, \dots, m\}$ and $t \in \mathbb{Z}_{\geq 1}$, every k,t-branch split cut (cf. Definition 252) for P is an $L_{k,\mathbb{Q}}$ cut for P. In particular, this implies
$$\mathrm{cl}_{L_{k,\mathbb{Q}}}(P) \subseteq \mathrm{cl}_{k,tBS}(P)$$
and more specifically
$$\mathrm{cl}_{L_{k,\mathbb{Q}}}(P) \subseteq \mathrm{cl}_{k,kBS}(P) = \mathrm{cl}_{kBS}(P).$$
- In Corollary 279, we showed that if $m + n \geq 2$, every parametric cross cut for P (cf. Definition 150) and especially every crooked cross cut (cf. Definition 146) for P is an $L_{2,\mathbb{Q}}$ cut for P. In particular, this implies
$$\mathrm{cl}_{L_{2,\mathbb{Q}}}(P) \subseteq \mathrm{cl}_{CC}(P).$$

We next considered a result from the literature (Theorem 280, which was originally shown in [DGM15]), which states that there exists a rational polytope $P \subsetneq \mathbb{R}^2 \times \mathbb{R}^1$ having
$$\mathrm{cl}_{CC}(P) \subsetneq \mathrm{cl}_{2BS}(P).$$
From this, together with Corollary 279, we could immediately conclude that there exists a rational polytope $P \subsetneq \mathbb{R}^2 \times \mathbb{R}^1$ having
$$\mathrm{cl}_{L_{2,\mathbb{Q}}}(P) \subsetneq \mathrm{cl}_{2BS}(P).$$
We wrote this down explicitly in Corollary 282.

As a next step, we considered how this result can be generalized to arbitrary t-branch split cuts versus $L_{k,\mathbb{Q}}$ cuts. In Theorem 285, we saw that for all $m \in \mathbb{Z}_{\geq 3}$, there exists a rational polytope $P \subsetneq \mathbb{R}^m \times \mathbb{R}^1$ such that $y_1 \leq 0$ is a valid inequality for $\binom{x}{y} \in P_I$ and thus a valid $L_{m,\mathbb{Q}}$ cut for P, but not a valid $\left(3 \cdot 2^{m-2} - 1\right)$-branch split cut for $\binom{x}{y} \in P$. Note that this does *not* imply
$$\mathrm{cl}_{L_{m,\mathbb{Q}}}(P) \subsetneq \mathrm{cl}_{(3 \cdot 2^{m-2}-1)BS}(P)$$
(even though it is plausible that this strict inclusion does indeed hold).

After considering the relationship between 2-branch split cuts and crooked cross cuts, we next considered the relationship between crooked cross cuts and 3-branch split cuts. In Theorem 286, we showed that for an arbitrary $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$), every crooked cross cut for P is also a 2,3-branch split cut for P, thus, in particular, a 3-branch split cut for P. This, in particular, implies
$$\mathrm{cl}_{3BS}(P) \subseteq \mathrm{cl}_{2,3BS}(P) \subseteq \mathrm{cl}_{CC}(P).$$
Can the inclusion between $\mathrm{cl}_{3BS}(P)$ and $\mathrm{cl}_{CC}(P)$ also be strict? In Theorem 287, which was originally shown in [DGM15], we saw that there exists a rational polytope $P \subseteq \mathbb{R}^3$ having
$$\emptyset = \mathrm{cl}_{3BS}(P) \subsetneq \mathrm{cl}_{CC}(P).$$

k-disjunctive cuts vs $L_{k',\mathbb{Q}}$ cuts: In Theorem 288, we showed that for a rational polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) and $k \in \{0, \dots, m\}$, we have
$$\mathrm{cl}_{2^k D}(P) \subseteq \mathrm{cl}_{L_{k,\mathbb{Q}}}(P).$$

k row cuts, split cuts, crooked cross cuts and $L_{k',\mathbb{Q}}$ cuts: In Theorem 294, which is a consequence of results that are shown in [DDG12], we saw that if we have a polyhedron
$$P := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R}^n_{\geq 0} : Ax + Gy = b \right\}$$
be given, where $A \in \mathbb{R}^{l \times m}$, $G \in \mathbb{R}^{l \times n}$ and $b \in \mathbb{R}^l$ ($l, m, n \in \mathbb{Z}_{\geq 0}$), we have
$$\mathrm{cl}_{1R}(A, G, b) \subseteq \mathrm{cl}_{split}(P)$$
(this inclusion can also be strict if A, G and b are not rational; see further below).

Next, one can ask the question whether 2-row cuts can be stronger than 1-row cuts. In Theorem 295, for which a proof was sketched in [DGM15], we saw that there exist rational A, G and b such that

$$\mathrm{cl}_{2R}(A,G,b) \subsetneq \mathrm{cl}_{1R}(A,G,b).$$

Corollary 299, which is again a consequence of results shown in [DDG12], served as an analogue of Theorem 294 for 3-row closure versus crooked cross closure (we just recalled that Theorem 294 was about 1-row closure versus split closure). Here, we saw that given a polyhedron

$$P := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R}^n_{\geq 0} : Ax + Gy = b \right\},$$

where $A \in \mathbb{R}^{l\times m}$, $G \in \mathbb{R}^{l\times n}$ and $b \in \mathbb{R}^l$ $(l, m, n \in \mathbb{Z}_{\geq 0})$, we have

$$\mathrm{cl}_{3R}(A,G,b) \subseteq \mathrm{cl}_{CC}(P).$$

In Theorem 300, which was shown in [DGM15], we saw that there exists a rational polytope

$$P := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \times \mathbb{R}^4_{\geq 0} : Ax + Gy = b \right\}$$

having

$$\mathrm{cl}_{2R}(A,G,b) \not\subseteq \mathrm{cl}_{2BC}(P).$$

On the other hand, for polyhedra that are neither rational nor full-dimensional, even the 1-row closure can be stronger than the crooked cross closure or t-branch split closure. We showed this in Theorem 301, where we gave an example of a polyhedron

$$P := P^{=}(A,b) \subseteq \mathbb{R}^2$$

that forms a non-rational hyperplane and satisfies

$$\mathrm{cl}_{1R}(A,b) = \emptyset \subsetneq P = \mathrm{cl}_{CC}(P) = \mathrm{cl}_{tBS}(P)$$

for all $t \in \mathbb{Z}_{\geq 0}$. In Theorem 302, we wrote down a related result: for all $t \in \mathbb{Z}_{\geq 0}$, we have

$$\begin{aligned}&\mathrm{cl}_{L_{1-\frac{1}{2},\mathbb{R}}}\left(P^{114}\right) = \mathrm{cl}_{L_{1,\mathbb{R}}}\left(P^{114}\right) = \mathrm{cl}_{L_{2-\frac{1}{2},\mathbb{Q}}}\left(P^{114}\right) = \mathrm{cl}_{L_{2,\mathbb{Q}}}\left(P^{114}\right) = \mathrm{cl}_{I}\left(P^{114}\right) = \emptyset \\ \subsetneq & P^{114} = \mathrm{cl}_{tBS}\left(P^{114}\right).\end{aligned}$$

Now for the relationship between k row cuts and $L_{k',\mathbb{Q}}$ cuts: in Theorem 304, we showed that if a rational polyhedron

$$P := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R}^n_{\geq 0} : Ax + Gy = b \right\}$$

is given, where A, G and b are rational, we have

$$\mathrm{cl}_{(2^k-1)R}(A,G,b) \subseteq \mathrm{cl}_{L_{k,\mathbb{Q}}}(P).$$

Properties from pyramids over cross polytopes: In Theorem 543, which we proved in chapter 14, we showed that for every $m \in \mathbb{Z}_{\geq 2}$, there exists a rational polytope $P \subseteq \mathbb{R}^m$ for which

$$\mathrm{cl}_{(m-1)BS}(P) = \mathrm{cl}_{L_{m-1,\mathbb{Q}}}(P) = \mathrm{cl}_I(P) \subsetneq \mathrm{cl}_{L_{m-1-\frac{1}{2},\mathbb{R}}}(P) \subseteq \mathrm{cl}_{L_{m-2,\mathbb{R}}}(P)$$

holds.

Properties from the Li-Richard example: In Theorem 576, which we proved in chapter 15, we showed that for every $m \in \mathbb{Z}_{\geq 1}$, there exists a rational polytope $P \subseteq \mathbb{R}^m \times \mathbb{R}^1$ such that for every $\ell \in \mathbb{Z}_{\geq 0}$, we have

$$\begin{aligned}&\mathrm{cl}_{mBS}(P) = \mathrm{cl}_{L_{m,\mathbb{Q}}}(P) = \mathrm{cl}_{L_{(m+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{ILF}(P) = \mathrm{cl}_{BR}(A,G,b) = \mathrm{cl}_I(P) \\ \subsetneq & \mathrm{cl}^{(\ell)}_{(m-1)BS}(P), \mathrm{cl}^{(\ell)}_{L_{m-\frac{1}{2},\mathbb{R}\times\mathbb{R}}}(P).\end{aligned}$$

Relatedly, we showed in Theorem 577 that for every $m \in \mathbb{Z}_{\geq 1}$, there exist

$$A \in \mathbb{Q}^{(m+1)\times m}, \qquad G \in \mathbb{Q}^{(m+1)\times(m+2)}, \qquad b \in \mathbb{Q}^{m+1}$$

such that for $P := P^{=}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right) \cap \left(\mathbb{R}^m \times \mathbb{R}^{1+(m+1)}_{\geq 0}\right)$, we have for every $\ell \in \mathbb{Z}_{\geq 0}$:

$$\begin{aligned}&\mathrm{cl}_{mBS}(P) = \mathrm{cl}_{L_{m,\mathbb{Q}}}(P) = \mathrm{cl}_{L_{(m+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{ILF}(P) = \mathrm{cl}_{mR}(A,G,b) = \mathrm{cl}_I(P)\\ &\subsetneq \mathrm{cl}^{(\ell)}_{(m-1)BS}(P), \mathrm{cl}^{(\ell)}_{L_{m-\frac{1}{2},\mathbb{R}\times\mathbb{R}}}(P).\end{aligned}$$

16.6.2. Convergence to the (mixed-)integral closure

In section 6.2 of chapter 6, we mostly gave an overview of results mostly taken from the literature concerning the following four questions (in order of essentially decreasing strength) for a cutting plane operator $\mathrm{cl}_{(\,\cdot\,)}(\,\cdot\,)$ with respect to some rational polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ (where $m, n \in \mathbb{Z}_{\geq 0}$):

1. Does $\mathrm{cl}_{(\,\cdot\,)}(P) = \mathrm{cl}_I(P)$ hold?

2. Does there exist a $t \in \mathbb{Z}_{\geq 0}$ such that $\mathrm{cl}^{(t)}_{(\,\cdot\,)}(P) = \mathrm{cl}_I(P)$ holds?

3. Does $\lim_{i\to\infty} \mathrm{cl}^{(i)}_{(\,\cdot\,)}(P) = \mathrm{cl}_I(P)$ hold ($\lim_{i\to\infty} \ldots$ is the Hausdorff convergence of closed convex sets; cf. Definition 308)?

4. Unter what conditions on, for example, $c \in (\mathbb{Q}^m \times \mathbb{Q}^n)^T$ and P does

   $$\exists t \in \mathbb{Z}_{\geq 0} : \max\left\{cx : x \in \mathrm{cl}^{(t)}_{(\,\cdot\,)}(P)\right\} = \max\left\{cx : x \in \mathrm{cl}_I(P)\right\}$$

   hold?

Chvátal-Gomory closure: In Theorem 306, we saw that for a rational polyhedron $P \subseteq \mathbb{R}^m$, if we apply the Chvátal-Gomory closure iteratively a sufficient number of times, we obtain the integer hull $\mathrm{cl}_I(P)$. Thus, question 2 could be answered positively for the Chvátal-Gomory closure. On the other hand, there exist examples of rational polytopes $P \subseteq \mathbb{R}^2$ with an arbitrarily large Chvátal-Gomory rank (cf. Lemma 307). Thus, question 1 could be answered negatively for $\mathrm{cl}_{CG}(\,\cdot\,)$ and question 2 positively.

Split closure: In Theorem 430, we gave an independent proof of the statement that there exists a rational polytope $P \subsetneq \mathbb{R}^2 \times \mathbb{R}^1$ such that for all $t \in \mathbb{Z}_{\geq 0}$, we have: $\mathrm{cl}^{(t)}_{split}(P) \supsetneq \mathrm{cl}_I(P)$. Thus, in contrast to the iterated Chvátal-Gomory closure in the pure-integer setting, the iterated split closure of a rational polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ does not always converge to $\mathrm{cl}_I(P)$ in a finite number of steps if $m \in \mathbb{Z}_{\geq 2}$ and $n \in \mathbb{Z}_{\geq 1}$. So, question 2 could be answered negatively in the mixed-integer setting for the split closure.

On the other hand, we saw in Theorem 310 that if $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) is a polytope or rational polyhedron, we have $\lim_{i\to\infty} \mathrm{cl}^{(i)}_{split}(P) = \mathrm{cl}_I(P)$. Thus, question 3 could be answered positively for the split closure. We showed in Remark 311 that the condition that P is either a polytope or a rational polyehdron is necessary for Theorem 310 to hold. In other words: Theorem 310 does not hold in the general case that P is an *arbitrary* polyehdron.

Finally, for question 4, a sufficient condition was given in Theorem 312.

Integral lattice-free closure: In Theorem 313, we saw that for a rational polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$, there always exists a $k \in \mathbb{Z}_{\geq 0}$ such that $\mathrm{cl}^{(k)}_{ILF}(P) = \mathrm{cl}_I(P)$. So, question 2 could be answered positively for the integral lattice-free closure. On the other hand, we saw in Lemma 275 and Theorem 276 that there exists a polytope $P \subsetneq \mathbb{R}^2 \times \mathbb{R}^1$ having $\mathrm{cl}_I(P) \subsetneq \mathrm{cl}_{ILF}(P)$. Thus, question 1 has in general a negative answer for the integral lattice-free closure.

k-disjunctive closure: In Theorem 314, we saw that for a rational polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$, we have $\mathrm{cl}_I(P) = \mathrm{cl}_{2^m D}(P)$. This answers question 1 positively for the k-disjunctive closure as long as "k is large enough" (where "large enough" only depends on m). Since 2^m is exponential in m, one is interested in answering question 4 positively for a smaller value of k than 2^m. In Theorem 315, such a sufficient condition was formulated. This theorem can be considered as the analogue of Theorem 312 (where a sufficient condition for split cuts was formulated) for k-disjunctive cuts.

k **row closure** In Theorem 316, we showed that for

$$P := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R}^n_{\geq 0} : Ax + Gy = b \right\},$$

where A, G and b are rational, we have $\mathrm{cl}_{(2^m-1)R}(A, G, b) = \mathrm{cl}_I(P)$.

t**-branch split closure:** For t-branch split cuts, question 1 can be answered positively if t is "sufficiently large": in Theorem 317, we saw that for a rational polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$, every facet-defining inequality for $\mathrm{cl}_I(P)$ is a t-branch split cut for P for a sufficiently large t (only depending on m).

16.6.3. Polyhedricity

In particular in section 6.3, we considered various polyhedricity results that were mostly taken from the literature.

Chvátal-Gomory closure: Further below, in section 16.7.2, we summarize that the Chvátal-Gomory closure of a polyhedron $P \subseteq \mathbb{R}^m$ ($m \in \mathbb{Z}_{\geq 0}$) with rational face normals is a rational polyhedron (which we showed in Theorem 399 in section 8.2.3.1). In section 6.3.1.1, we mentioned other proofs from the literature for the result that the Chvátal-Gomory closure of a rational polyhedron $P \subseteq \mathbb{R}^m$ ($m \in \mathbb{Z}_{\geq 0}$) is again a rational polyhedron.

Additionally, we summarized results from the literature which state that the Chvátal-Gomory closure of the following objects is a rational polytope that is defined by a finite number of Chvátal-Gomory cuts:

- rational ellipsoids (cf. Theorem 320 and [DV10]),
- arbitrary polytopes (cf. Theorem 321 and [DS11]),
- intersections of a rational polyhedron and a strictly convex body (Theorem 323). From this, of course, a result for the Chvátal-Gomory of a strictly convex body follows (Corollary 324) (cf. [DDV11b] for both results) and
- convex and compact sets (cf. Theorem 325 and [DDV14]). This result subsumes all results of this list.

Split closure: In section 6.3.1.2, we listed different proofs from the literature for the statement that the split closure of a rational polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) is again a rational polyehdron. Further below, in section 16.7.3, we summarize our independent proof for this, in which we additionally showed that the split closure of a polyhedron with rational face normals is again a polyhedron.

We next looked at results from the literature concerning the split closure of convex bodies that are not already rational polyhedra. In Theorem 327 (cf. [DDV11a]), we saw that if $K \subseteq \mathbb{R}^m$ ($m \in \mathbb{Z}_{\geq 0}$) is a strictly convex and compact body such that $\operatorname{aff} K$ is a rational affine subspace, then $\mathrm{cl}_{split}(K)$ is finitely defined. Does this imply that $\mathrm{cl}_{split}(K)$ is a polyhedron (polytope)? For $m \in \{0, 1, 2\}$, this is the case (Theorem 328), but for $m = 3$, there exists a rational ellipsoid $K \subseteq \mathbb{R}^3$ such that $\mathrm{cl}_{split}(K)$ is not a polyhedron/polytope (cf. Example 329, which is taken from [DDV11a]).

A very general result: In Theorem 335, we stated a very general polyhedricity result from the literature ([DGMR16a]).

t**-branch split closure:** A consequence of Theorem 335 (Corollary 336) is that the t-branch split closure of a rational polyehdron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) with respect to an arbitrary set of t-branch split disjunctions is again a rational polyhedron.

Integral lattice-free closure: Using Theorem 339, another consequence of Theorem 335 is that the integral lattice-free closure of a rational polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) is again a rational polyhedron. In section 6.3.2.2, we gave a short overview about further results from the literature about integral lattice-free polyhedra.

$L_{k,\mathbb{Q}}$ closure: In Theorem 264, we showed that the $L_{k,\mathbb{Q}}$ closure of a rational polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) with respect to an arbitrary subset of suitable subspaces is again a rational polyhedron. This was independently proved in [DGMR17]. The central idea of our proof is to reduce the problem of showing the polyhedricity of the $L_{k,\mathbb{Q}}$ closure to the stated polyhedricity result for the t-branch split closure.

16.7. Integral polyhedra, Chvátal-Gomory cuts and split cuts

16.7.1. Integral polyhedra

In chapter 7, we considered systems $Ax \leq b$ (A rational) with the property that if b is integral, then $P^{\leq}(A, b)$ is an integral polyhedron. TDI systems are an example of such systems that is known in the literature, but, as we saw in this chapter, there exist other types of systems with this property: we additionally introduced $\text{TD}\mathbb{Z} + I$ systems, $\text{TD}\mathbb{Z} + \{0, 1\}$ systems and $\text{TD}(I \cap \mathbb{Z}) + \{0, 1\}$ systems. A very interesting property to keep in mind is that every description of an integral polyhedron using rational inequalities is already a $\text{TD}\mathbb{Z} + \{0, 1\}$ system (cf. Theorem 354).

It is also well-known in the literature that TDI systems are related to Hilbert bases. The central idea to find the analogues of Hilbert bases for the other kinds of system is not to just consider the matrix A, but also the right-hand side vector b of $Ax \leq b$. This lead to the framework of **LP face cones** (cf. Definition 356). In this framework, the analogue of Hilbert bases are icone systems (cf. Lemma 361), which correspond to TDI systems. The analogues of the other three kinds of systems ($\text{TD}\mathbb{Z} + I$ systems, $\text{TD}\mathbb{Z} + \{0, 1\}$ systems and $\text{TD}(I \cap \mathbb{Z}) + \{0, 1\}$ systems) are what we named $\mathbb{Z}+$icone systems, $\mathbb{Z} + \{0, 1\}$ systems and $(\text{icone} \cap \mathbb{Z}) + \{0, 1\}$ systems.

In chapter 7, we also answered questions about how these concepts are related to each other (section 7.2.2 and section 7.3.3) and how for some polyhedra/LP face cones, the size of a minimal system of the different types can differ (cf. section 7.6).

Another important result that we proved in chapter 7 was to characterize mixed-integrality in terms of an optimization problem for the first time: we stated our characterization in Theorem 347.

16.7.2. Projected Chvátal-Gomory closure

We gave two proofs for the statement that the (projected) Chvátal-Gomory closure of a polyhedron with rational face normals is again a polyhedron.

First proof: The first proof of the statement that the Chvátal-Gomory closure of a polyhedron $P \subseteq \mathbb{R}^m$ ($m \in \mathbb{Z}_{\geq 0}$) with rational face normals is a rational polyhedron (Theorem 399) is based on the fact that the Chvátal-Gomory closure of a polyehdron that is described by a $\text{TD}\mathbb{Z} + I$ system with an integral left-hand side can be obtained by a simple rounding procedure (Theorem 398). This was the topic of section 8.2.3.1 (for the polyhedricity result) and, more generally, section 8.2.3 (mathematical properties of such a rounding procedure).

While this rounding procedure is well-known in the literature for TDI systems with an integral left-hand side, the fact that the weaker $\text{TD}\mathbb{Z} + I$ property suffices is new. So, in section 8.2.3.2, we considered a polyhedron where the sizes of a minimal

- $\text{TD}\mathbb{Z} + I$ system,
- TDI system,
- $\text{TD}\mathbb{Z} + \{0, 1\}$ system or
- $\text{TD}(I \cap \mathbb{Z}) + \{0, 1\}$ system,

respectively, with an integral left-hand size are all different. Thus, we know, in particular, that $\text{TD}\mathbb{Z} + I$ systems can be stronger than TDI systems in the sense that a smaller system (in the sense of the number of rows) suffices to represent the Chvátal-Gomory closure via the stated rounding procedure.

In Problem 401, we formulated the reverse question to the rounding procedure for $\text{TD}\mathbb{Z} + I$ systems: if we have a system $Ax \leq b$ such that b is chosen minimally and $\text{cl}_{CG}\left(P^{\leq}(A, b)\right) = P^{\leq}(A, \lfloor b \rfloor)$: does this imply that $Ax \leq b$ is $\text{TD}\mathbb{Z} + I$? In section 8.2.3.3, we saw that this is in general not the case.

Second proof: The second proof, for which we found an analogue for the split/MIR closure in section 9.2, was given in section 8.2.4. We wrote down the final polyhedricity result in Theorem 405. Here, we stated that for a given polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ $(m, n \in \mathbb{Z}_{\geq 0})$, we have:

- If P is a rational polyehdron, so is $\mathrm{cl}_{pCG}(P)$.
- If P is a polyhedron with rational face normals, so is $\mathrm{cl}_{pCG}(P)$. Additionally, $\mathrm{proj}_{\mathbb{R}^m}(\mathrm{cl}_{pCG}(P))$ is a rational polyhedron.

This proof depends on the fact that, by Theorem 394 and Theorem 357, we can restrict ourselvelves to polyhedra $P^{\leq}\left(\left(\begin{array}{cc} A & G \end{array}\right), b\right)$ where the rows of $\left(\begin{array}{ccc} A & G & -b \end{array}\right)$ form an LP face cone. Theorem 404 gives a description of the projected Chvátal-Gomory closure in such a situation.

16.7.3. Split closure

In section 9.2, we showed (Theorem 417) that the split/MIR closure of a polyhedron P with rational face normals is again a polyhedron and a rational polyhedron if P is a rational polyhedron. For this, in Definition 410, we gave a definition of MIR cuts that is as similar as possible to how we defined dual projected Chvátal-Gomory cuts in Definition 382. This way, we could write down “MIR cut analogues” of results that we showed for dual projected Chvátal-Gomory cuts:

- The dominance result that we formulated in Lemma 423 is the analogue of Lemma 395.
- The dominance result that we formulated in Lemma 424 is the analogue of Lemma 396.
- Theorem 427, which states that we can restrict ourselves to bases, is the analogue of Theorem 394.

Using these results, we could prove Theorem 429, in which we gave an explicit representation of the split/MIR closure (or more precisely: $\mathrm{cl}_{MIR,L}(\cdot)$; cf. Definition 418) of a basic relaxation of a polyhedron with rational face normals via a $\mathbb{Z}+$ icone system. This theorem is the analogue of Theorem 404, in which we proved a similar result for the projected Chvátal-Gomory closure. From there, it was only a small step to prove Theorem 417, which, as we mentioned, states that the split/MIR closure of a polyhedron P with rational face normals is again a polyhedron and a rational polyhedron if P is a rational polyhedron.

For the scientific importance of these results: while the fact that the split closure of a rational polyhedron is again a rational polyhedron is well-known in the literature (cf. in particular section 6.3.1.2, but also section 6.3.2.1), our approach has some innovations to offer:

- The fact that the split closure of a polyhedron with rational face normals is again a polyhedron has to our knowledge not been proved before.
- Our approach can easily be turned into an algorithm for computing the split closure explicitly (for details, recall the summary at the beginning of chapter 9).
- The relationship between MIR cuts and specific generating systems for cones (in this case $\mathbb{Z}+$ icone systems, which, by Lemma 363, also include the special case of icone systems (these are, as we saw in Lemma 361, deeply related to Hilbert bases), $\mathbb{Z}+\{0,1\}$ systems and $(\mathrm{icone}\cap\mathbb{Z})+\{0,1\}$ systems) has not been noticed in the literature before.

17. Outlook

We now present some open research questions that one might find interesting to do further research on to build on this thesis.

17.1. Systems that ensure mixed-integrality

In chapter 7, we considered systems $Ax \leq b$ that ensure that $P^{\leq}(A, b)$ is integral if b is (for example TD$\mathbb{Z}+I$ systems; cf. Theorem 352). We ask: does there exist a property for systems $Ax + Gy \leq b$ that ensures that

$$P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right) \subseteq \mathbb{R}^m \times \mathbb{R}^n$$

is a mixed-integral polyhedron (with respect to the mixed lattice $\mathbb{Z}^m \times \mathbb{R}^n$)? A reason why one could be interested to do further research into this direction is the following: such a class of system might be suitable to find an analogue of Theorem 398 for characterizing the split closure via such a system (Theorem 398 characterized the Chvátal-Gomory closure of a polyhedron using a TD$\mathbb{Z}+I$ system). For this, recall that we could generalize a result for representing the projected Chvátal-Gomory closure of a polyhedron $P^{\leq}\left(C := \begin{pmatrix} A & G \end{pmatrix}, b\right)$, where rowcone $\begin{pmatrix} A & G & -b \end{pmatrix}$ forms an LP face cone and

$$C' := C \cap (\mathbb{R}^m \times 0^n \times \mathbb{R})^T$$

is finitely generated by vectors from $(\mathbb{Q}^m \times 0^n \times \mathbb{R})^T$ (Theorem 404), to the split/MIR closure (Theorem 429).

How might a roadmap to achieve this goal of finding mixed-integer analogues of properties like TDI systems, TD$\mathbb{Z}+I$ systems, TD$\mathbb{Z}+\{0,1\}$ systems and TD$(I \cap \mathbb{Z})+\{0,1\}$ systems (cf. Definition 348 and Definition 349) look like? For this, consider that these types of systems are all relaxations of the characterization of integrality of polyhedra in Theorem 345.

Recall that in Theorem 347, we gave a similar characterization of mixed-integrality of polyhedra. It should not be too hard to formulate properties for systems $Ax+Gy \leq b$, which under specific conditions (similar to the condition that b is integral in the case of, for example, TD$\mathbb{Z}+I$ systems $Ax \leq b$) ensure that $P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)$ is mixed-integral. But the central and interesting property is that one can show that for

- a rational polyhedron P or
- a polyhedron P with rational face normals,

respectively, one can *always* find a

- TD$(I \cap \mathbb{Z})+\{0,1\}$ or
- TDI system,

respectively, that describes it. This was the statement of Theorem 369. Recall that its proof with all the required helper statements was far from trivial and needed considerations about generating systems for cones (cf. section 7.3); in particular statements about their existence (cf. section 7.3.4) and how one can describe the systems of dual integrality using generating systems for cones (cf. Theorem 368).

So, if one wants to show that one got "the correct" generalization of, for example, TDI systems for the mixed-integer case, one has to provide a proof that e.g. for rational polyhedra such a system *always* exists. This is in our opinion the harder part of this roadmap.

17.2. More general systems for representing the Chvátal-Gomory closure

In Theorem 398, we saw how TD$\mathbb{Z}+I$ systems with an integral left-hand side can be used to compute the Chvátal-Gomory closure of a polyhedron with rational face normals. On the other hand, in Problem 401, we considered the question that if we have a a system $Ax \leq b$ with integral A be given such that

- $P^{\leq}(A,b) \neq \emptyset$,
- b is chosen minimally with respect to $\leq$, i.e. there exists no $b' \leq b$ such that $P^{\leq}(A,b) = P^{\leq}(A,b')$ and
- $\mathrm{cl}_{CG}\left(P^{\leq}(A,b)\right) = P^{\leq}(A,\lfloor b \rfloor)$:

does this imply that $Ax \leq b$ is TD$\mathbb{Z}+I$? In section 8.2.3.3, we saw that this is not the case even if $P^{\leq}(A,b)$ has exactly one vertex. So, it is an open research question to find a property X (presumably generalizing TD$\mathbb{Z}+I$ systems) such that

- for every rational polyhedron/polyhedron with rational face normals, there exists an inequality description $Ax \leq b$ with integral A for it that satisfies property X (recall Theorem 369),
- if $Ax \leq b$ is a system with integral A that satisfies property X, then $\mathrm{cl}_{CG}\left(P^{\leq}(A,b)\right) = P^{\leq}(A,\lfloor b \rfloor)$ and
- if we have A and b be given, where
  - A is integral,
  - $P^{\leq}(A,b) \neq \emptyset$,
  - there exists no $b' \lneq b$ such that $P^{\leq}(A,b) = P^{\leq}(A,b')$ and
  - $\mathrm{cl}_{CG}\left(P^{\leq}(A,b)\right) = P^{\leq}(A,\lfloor b \rfloor)$,

  then $Ax \leq b$ satisfies property X.

17.3. Polyhedricity and finitely defined closures

In Theorem 264, we showed that the $L_{k,\mathbb{Q}}$ closure of a rational polyhedron P ($\mathrm{cl}_{L_{k,\mathbb{Q}}}(P)$) is again a rational polyhedron as it was done independently in [DGMR17]. A next step would be to show that the $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ closure of a rational polyhedron P ($\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P)$) is again a rational polyhedron. How might such a proof work? In Theorem 211, we saw that for a rational polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m,n \in \mathbb{Z}_{\geq 0}$), we have for $k \in \{1,\ldots,m\}$:

$$\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{\mathrm{ess}\,L_{k-\frac{1}{2},\mathbb{Q}}}(P) \cap \mathrm{cl}_{L_{k-1,\mathbb{Q}}}(P).$$

For $\mathrm{cl}_{L_{k-1,\mathbb{Q}}}(P)$, we already know that it is a rational polyhedron; so, it suffices to show that $\mathrm{cl}_{\mathrm{ess}\,L_{k-\frac{1}{2},\mathbb{Q}}}(P)$ is also a rational polyhedron. We consider it as plausible that the ideas of at least one of the proofs of Theorem 264 (either the one from [DGMR17, Theorem 2] or ours) can be generalized to show that for a rational polyhedron P, also $\mathrm{cl}_{\mathrm{ess}\,L_{k-\frac{1}{2},\mathbb{Q}}}(P)$ is a rational polyhedron.

Let us continue with some questions related to polyhedricity. Recall that in Theorem 325, we saw that if $K \subseteq \mathbb{R}^m$ ($m \in \mathbb{Z}_{\geq 0}$) is convex and compact, then $\mathrm{cl}_{CG}(K)$ is a rational polytope that is defined by a finite number of Chvátal-Gomory cuts for K. This, in particular, includes Theorem 320 as special case, where the situation that $K \subseteq \mathbb{R}^m$ ($m \in \mathbb{Z}_{\geq 0}$) is a full-dimensional rational ellipsoid was considered.

In [DDV11a], the authors attempted to generalize such results to the split closure and could show (cf. Theorem 327) that if $K \subseteq \mathbb{R}^m$ ($m \in \mathbb{Z}_{\geq 0}$) is compact and strictly convex and $\mathrm{aff}\,K$ is a rational affine subspace, then $\mathrm{cl}_{split}(K)$ is finitely defined, i.e. one only needs to consider a finite number of split disjunctions to describe $\mathrm{cl}_{split}(K)$. On the other hand, the authors could show (cf. Example 329) that in this situation, the fact that the split closure is finitely defined does *not* imply that the split closure is a rational polytope. Concretely, they could find a rational ellipsoid $K \subseteq \mathbb{R}^3$ such that $\mathrm{cl}_{split}(K)$ is not a polytope. This fact suggests the following problem:

Problem 579. *Let $K \subseteq \mathbb{R}^m$ ($m \in \mathbb{Z}_{\geq 1}$) be*

- *a full-dimensional rational ellipsoid,*
- *a (not necessarily rational) polytope,*
- *strictly convex and compact or*
- *convex and compact,*

respectively (these situations are inspired by Theorem 320, Theorem 321, Corollary 324 and Theorem 325). Let $k \in \{1, \ldots, m\}$. Is then $\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}}}(K)$ or $\mathrm{cl}_{L_{k,\mathbb{Q}}}(K)$ a polytope or even a rational polytope?

Example 329 lets us conjecture that this is in general not the case for $\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}}}(K)$ and $\mathrm{cl}_{L_{k,\mathbb{Q}}}(K)$ if $k \geq 2$ (but we only know the answer for $k = 1$). So, we also ask whether in one of these situations, $\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}}}(K)$ or $\mathrm{cl}_{L_{k,\mathbb{Q}}}(K)$ is finitely defined, i.e. whether there exist a finite set $\mathcal{V}$ of rational subspaces of $\mathbb{R}^m$ of codimension k such that we have

$$\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}}}(K) = \bigcap_{V \in \mathcal{V}} \mathrm{conv}\left((K + V) \cap \mathbb{Z}^m\right) \tag{17.1}$$

or

$$\mathrm{cl}_{L_{k,\mathbb{Q}}}(K) = \bigcap_{V \in \mathcal{V}} \mathrm{conv}\left(K \cap (\mathbb{Z}^m + V)\right), \tag{17.2}$$

respectively. For $k = 1$, we know that (17.1) and (17.2) hold (the latter is a consequence of Theorem 327).

If one generalizes Problem 579 to the mixed-inter case, one, of course, wants to replace the $L_{k-\frac{1}{2},\mathbb{Q}}$ closure in Problem 579 by either

- the $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ closure,
- the $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ closure or
- the essential $L_{k-\frac{1}{2},\mathbb{Q}}$ closure.

17.4. L_k cuts vs k' row cuts

In Theorem 304, we saw that for all $A \in \mathbb{Q}^{r\times m}$, $G \in \mathbb{Q}^{r\times n}$ and $b \in \mathbb{Q}^r$ $(r, m, n \in \mathbb{Z}_{\geq 0})$, where

$$\emptyset \neq P = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R}^n_{\geq 0} : Ax + Gy = b \right\}, \tag{17.3}$$

we have

$$\mathrm{cl}_{(2^k-1)R}(A, G, b) \subseteq \mathrm{cl}_{L_{k,\mathbb{Q}}}(P), \tag{17.4}$$

where $k \in \{0, \ldots, m\}$.

Now one can ask: is the bound $2^k - 1$ in (17.4) the best possible? In other words: do there exist rational A, G and b such that we have

$$\mathrm{cl}_{(2^k-2)R}(A, G, b) \nsubseteq \mathrm{cl}_{L_{k,\mathbb{Q}}}(P), \tag{17.5}$$

where P is as in (17.3) and $k \in \{1, \ldots, m\}$? For $k = 1$, this is obviously the case. For $k = 2$, we conclude from Theorem 300 that there exist rational A, G and b having

$$\mathrm{cl}_{2R}(A, G, b) \nsubseteq \mathrm{cl}_{2BC}(P) \supseteq \mathrm{cl}_{L_{2,\mathbb{Q}}}(P);$$

thus,

$$\mathrm{cl}_{2R}(A, G, b) \nsubseteq \mathrm{cl}_{L_{2,\mathbb{Q}}}(P),$$

where P is again as in (17.3). So, the question that we formulated in (17.5) is open for $k \geq 3$.

Now for some other inclusion: in Theorem 577, we saw that for $m \in \mathbb{Z}_{\geq 1}$ and $\epsilon \in \mathbb{Q}_{>0}$, we have for every $\ell \in \mathbb{Z}_{\geq 0}$:

$$\mathrm{cl}_{mR}\left(A^{557,m,\epsilon,=}, G^{557,m,\epsilon,=}, b^{557,m,\epsilon,=}\right) = \mathrm{cl}_I(P) \subsetneq \mathrm{cl}^{(\ell)}_{(m-1)BS}(P), \mathrm{cl}^{(\ell)}_{L_{m-\frac{1}{2},\mathbb{R}\times\mathbb{R}}}(P),$$

where

$$P := P^{=}\left(\begin{pmatrix} A^{557,m,\epsilon,=} & G^{557,m,\epsilon,=} \end{pmatrix}, b^{557,m,\epsilon,=}\right) \cap \left(\mathbb{R}^m \times \mathbb{R}^{1+(m+1)}_{\geq 0}\right).$$

So one can ask the reverse question, i.e. whether we also have

$$\mathrm{cl}_{L_{(m+1)-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{m,\mathbb{Q}}}(P) = \mathrm{cl}_{mBS}(P) = \mathrm{cl}_I(P) \subsetneq \mathrm{cl}_{(m-1)R}\left(A^{557,m,\epsilon,=}, G^{557,m,\epsilon,=}, b^{557,m,\epsilon,=}\right).$$

For $m = 1$, this statement is trivial and for $m = 2$, it can be concluded from the proof sketch for Theorem 295 in [DGM15]. So, this question is open for $m \in \mathbb{Z}_{\geq 3}$.

17.5. $\mathrm{cl}_{L_{m,\mathbb{Q}}}(P) \subsetneq \mathrm{cl}_{(3\cdot 2^{m-2}-1)BS}(P)$?

Recall that in Theorem 285 (also cf. section 16.6.1, where we summarized this result), we saw that for all $m \in \mathbb{Z}_{\geq 3}$, there exists a rational polytope $P \subsetneq \mathbb{R}^m \times \mathbb{R}^1$ such that $y_1 \leq 0$ is a valid inequality for $\binom{x}{y} \in P_I$ and thus a valid $L_{m,\mathbb{Q}}$ cut for P, but not a valid $\left(3 \cdot 2^{m-2} - 1\right)$-branch split cut for $\binom{x}{y} \in P$. But this result does *not* imply

$$\mathrm{cl}_{L_{m,\mathbb{Q}}}(P) \subsetneq \mathrm{cl}_{(3\cdot 2^{m-2}-1)BS}(P). \tag{17.6}$$

This leads to two natural research questions:

1. Does for the polytope $P \subsetneq \mathbb{R}^m \times \mathbb{R}^1$ ($m \in \mathbb{Z}_{\geq 3}$) from Theorem 285 also the strict inclusion (17.6) hold?
2. If not: does for every $m \in \mathbb{Z}_{\geq 3}$ exist a rational polytope $P \subsetneq \mathbb{R}^m \times \mathbb{R}^1$ for which (17.6) is satisfied?

Note that it does not make sense to replace "rational polytope" by "arbitrary polyhedron" in the second question, since we saw in Theorem 302 that already for the $L_{2,\mathbb{Q}}$ closure of the irrational hyperplane $P^{114} \subseteq \mathbb{R}^2$, we have for every $t \in \mathbb{Z}_{\geq 1}$:

$$\mathrm{cl}_{L_{2,\mathbb{Q}}}\left(P^{114}\right) = \emptyset \subsetneq P^{114} = \mathrm{cl}_{tBS}\left(P^{114}\right).$$

17.6. Role of $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts vs $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cuts

We saw in Theorem 193 that for a rational polyhedron $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$), the $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ closure ($k \in \{0, \ldots, m+n\}$) equals the $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ closure and in Theorem 194, we showed that there exists a polyhedron $P \subseteq \mathbb{R}^2 \times \mathbb{R}^1$ with a partially rational recession cone having

$$\mathrm{cl}_{L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P) = \mathrm{cl}_{L_{2-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P),$$

but for which no finite amount of $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ cuts suffice to describe $\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(P)$ (both of these results were summarized in section 16.2). On the other hand, the much more general statement of Conjecture 192 is still open. If one could prove it, this would imply that for a very large class of closed convex sets, equality between their $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ closure and their $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ closure indeed holds.

17.7. Role of $L_{k,\mathbb{R}}$ cuts and $L_{k-\frac{1}{2},\mathbb{R}\times\mathbb{R}}$ cuts

We saw examples where $L_{k,\mathbb{R}}$ cuts and $L_{k-\frac{1}{2},\mathbb{R}\times\mathbb{R}}$ cuts are more expressive than $L_{k,\mathbb{Q}}$ cuts and $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ cuts (cf. section 4.5.1 and section 4.5.2; specifically Theorem 189, Theorem 188 and Theorem 190). But all of these examples are irrational hyperplanes (i.e. neither full-dimensional nor compact or rational polyhedra). So, we ask: do there exist examples $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ ($m, n \in \mathbb{Z}_{\geq 0}$) such that

$$\mathrm{cl}_{L_{k,\mathbb{R}}}(P) \subsetneq \mathrm{cl}_{L_{k,\mathbb{Q}}}(P)$$

or

$$\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{R}\times\mathbb{R}}}(P) \subsetneq \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P),$$

respectively ($k \in \{1, \ldots, m-1\}$ for L_k cuts and $k \in \{1, \ldots, m\}$ for $L_{k-\frac{1}{2}}$ cuts), where $P \subseteq \mathbb{R}^m \times \mathbb{R}^n$ satisfies properties such as

- being a rational polytope,
- being a rational polyhedron,
- being convex, full-dimensional and closed or
- being convex and compact,

respectively, or does in some of these situations always

$$\mathrm{cl}_{L_{k,\mathbb{R}}}(P) = \mathrm{cl}_{L_{k,\mathbb{Q}}}(P)$$

or

$$\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{R}\times\mathbb{R}}}(P) = \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(P),$$

respectively, hold?

17.8. Bounds for sizes of subsets of inequalities to consider

In chapter 12 (also recall the summary in section 16.5.1), we considered how many inequalities of the inequality description $Ax+Gy \leq b$ of $P := P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)$ we have to consider at the same time to derive all necessary inequalities to describe the $L_{k,\mathbb{Q}}$ closure or $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}$ closure of P. We formulated this in Problem/Definition 476 (recall that in section 12.1, we defined that $A \in \mathbb{R}^{l\times m}$, $G \in \mathbb{R}^{l\times n}$ and $b \in \mathbb{R}^l$, where $l, m, n \in \mathbb{Z}_{\geq 0}$):

Problem/Definition 476. *Let $P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right) \subseteq \mathbb{R}^m \times \mathbb{R}^n$ be given and let $k \in \{0, \dots, m+n\}$. What is the smallest $h \in \{0, \dots, l\}$ such that*

$$\mathrm{cl}_{L_{k,\mathbb{Q}}}\left(P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)\right) = \bigcap_{S \in \binom{[l]}{h}} \mathrm{cl}_{L_{k,\mathbb{Q}}}\left(P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}_{S,*}, b_S\right)\right) \quad \textit{or}$$

$$\mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}\left(P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}, b\right)\right) = \bigcap_{S \in \binom{[l]}{h}} \mathrm{cl}_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}\left(P^{\leq}\left(\begin{pmatrix} A & G \end{pmatrix}_{S,*}, b_S\right)\right),$$

respectively, holds? This smallest h is referred to as $h^*_{L_{k,\mathbb{Q}}}(A, G, b)$ or $h^*_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(A, G, b)$, respectively. If $n = 0$, we also use the notations $h^*_{L_{k,\mathbb{Q}}}(A, b)$ or $h^*_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(A, b)$, respectively.

Obviously, this problem can easily be generalized to other types of cutting plane closures such as

- essential $L_{k-\frac{1}{2},\mathbb{Q}}$ closure,
- $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ closure,
- $L_{k-\frac{1}{2},\mathbb{R}\times\mathbb{R}}$ closure,
- $L_{k,\mathbb{R}}$ closure,
- t-branch split closure etc.

The generalizations for the $L_{k-\frac{1}{2},\mathbb{R}\times\mathbb{R}}$ closure, $L_{k,\mathbb{R}}$ closure and t-branch split closure were not considered in this text. For the other two closures in this list (essential $L_{k-\frac{1}{2},\mathbb{Q}}$ closure and $L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}$ closure), we gave a glimpse in section 12.4.3:

- In Theorem 496, we showed an upper bound for
$$h^*_{\mathrm{ess}\, L_{k-\frac{1}{2},\mathbb{Q}}}(A, G, b).$$
- In Theorem 497, we showed that
$$h^*_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(A, G, b) = h^*_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{Q}}}(A, G, b)$$
if A, G and b are rational. Whether this equality also holds for arbitrary A, G and b is an open research question.

But we believe that a lot more results can be shown about $h^*_{\mathrm{ess}\, L_{k-\frac{1}{2},\mathbb{Q}}}(A, G, b)$ and $h^*_{L_{k-\frac{1}{2},\mathbb{Q}\times\mathbb{R}}}(A, G, b)$.

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