Set–Based Methods for Interconnected Control Systems

Dissertation

zur Erlangung des akademischen Grades

Doktoringenieur (Dr.-Ing.)

von

Benjamin Kern

geboren am 07. März 1983

in Sindelfingen, Deutschland

genehmigt durch die Fakultät für Elektrotechnik und Informationstechnik

der Otto-von-Guericke-Universität Magdeburg

Gutachter:

Prof. Dr.-Ing. Rolf Findeisen

Prof. Dr.-Ing. Christian Ebenbauer

eingereicht am 16. Mai 2017

Promotionskolloquium am 1. Juli 2019

Acknowledgments

This thesis is the result of my PhD studies in the Systems and Control Group at the Faculty of Electrical Engineering and Information Technology at Otto-von-Guericke University of Magdeburg, Germany. I would like to express my gratitude to the people who have made this thesis possible.

I would like to thank Saša V. Raković, Rolf Findeisen and my former colleagues. This thesis would not have been possible without the support of my family and my wife Patricia to whom I am forever grateful for the support, patience, and faith they had in me.

Contents

Abstract

Interconnected systems, such as electrical grids, chemical process plants, CAN-bus systems in modern cars or social networks are becoming even more ubiquitous in our everyday lives. Due to their complexity and size it is often difficult to directly analyze, influence or control their behaviour. For that reason, the complete system with all its interconnections is often split into several interconnected subsystems, which can then be treated in a decentralized or distributed way. A major challenge for the analysis and control of decentralized and distributed systems is the consideration of possible destabilization effects due to the interconnections. In addtion, potential control methods should always handle both disturbance effects as well as physical or logical contraints.

Set-based control methods are modern control schemes, in which time invariant sets are characterized in the state space of a system. Those invariant regions are especially well suited for designing robust controllers subject to input or state contraints. So far the existing invariance concepts are not tailored to interconnected systems. Due to the previously mentioned advantages of set–based methods, it is therefore of great interest to extend them to interconnected systems. In order to achieve this goal, we extend the concept of an invariant set to the notion of an invariant family of sets. With this extension, we can link the interconnections effects of several systems to the dynamics of a collection of sets. Using set–dynamics we can indirectly characterize the system theoretic behaviour of the interconnected system in a reduced state space. In particular, we present a scheme for directly constructing this set–dynamics for linear systems, that are connected either via linear or positively, homogeneous functions.

In addition, in this work a distributed and decentralized control approach exploiting the notion of invariant family of sets is presented. For the distributed controller design, we show how an iterative process based on the solutions of generalized Matrix-Riccati equations leads to a flexible characterization of a family of invariant sets. This design procedure is especially well suited for increasing the closed loop performance, whenever information about the interconnections may be exchanged via the different subsystems. For the decentralized controller, a procedure based on the solutions of LMI's is presented. During this decentralized operation no information is exchanged via the subystems and hence a more generally conservative behaviour is achieved. In conclusion, the flexibility and system theoretic properties of the proposed methods based on the notion of invariant families of sets are presented via several simulation examples.

Deutsche Kurzfassung

Vernetzte Systeme, wie z.B. Energienetzwerke, chemische Prozessanlagen, PKW CAN-Bus Systeme aber auch soziale Netzwerke sind in unserer heutigen Welt nicht mehr wegzudenken. Durch die Komplexität und Größe dieser Prozesse ist es oft schwierig ihr Verhalten direkt zu analysieren und zu beinflussen. Aus diesem Grund wird meist nicht das gesamte System inklusive der Vernetzungskopplungen zentral betrachtet, sondern deren Teilsysteme in einer dezentralen bzw. verteilten Art und Weise. Eine generelle Herausforderung bei der dezentralen bzw. verteilten systemtheoretischen Analyse und Regelung von verkoppelten Teilsystemen ist es etwaige Destabilisierungseffekte aufgrund der Verkopplungen zu berücksichtigen. Zusätzlich gibt es bei diesen Systemen in der Praxis meist Störeinflüße und physikalische Beschränkungen die kompensiert bzw. eingehalten werden müssen.

Mengenbasierte Regelungsmethoden sind moderne Regelungsverfahren, in denen zeitinvariante Mengen im Zustandsraum eines Systems beschrieben werden. Diese invarianten Bereiche eignen sich besonders gut für den Entwurf von robusten Reglern unter Beschränkungen der Systemvariablen verwendet werden. Allerdings sind bisher geläufige Ansätze der Invarianz nicht auf verkoppelte Systeme ausgelegt. Aufgrund der Vorteile der mengebasierten Regelung ist es von Interesse diese Methoden auf verkoppelte Systeme zu erweitern. Hierzu wird in dieser Arbeit der Begriff der invarianten Menge erweitert und das Konzept der invarianten Familie von Mengen eingeführt. Diese Erweiterung erlaubt es die Verkopplungseigenschaften mehrer Teilsysteme auf eine Dynamik von Mengen abzubilden. Durch diese Abbildung ist es möglich die anfänglich hohe Systemordnung der vernetzten Teilsysteme zu reduzieren und deren systemtheoretisches Verhalten anhand dieser Dynamik zu charakterisieren. Desweiteren wird gezeigt wie diese Dynamik von Mengen für lineare und positiv, homogen verkoppelte Systeme konstruiert werden kann.

Weiterhin wird in in dieser Arbeit aufgezeigt wie die invarianten Familien von Mengen für den verteilten und dezentralen Reglerentwurf verwendet werden können. Für den Entwurf der verteilten Reglerstrategie wird ein iterativer Prozess basierend auf der entkoppelten Lösung von erweiterten Matrix-Riccati Gleichungen vorgestellt. Dieser Entwurf bietet sich dann an, wenn es möglich ist Informationen über die Verkopplungen zwischen den Systemen auszutauschen, um ein besseres Systemverhalten zu erzielen. Die dezentrale Regelungsstrategie basiert auf der Lösung von Linearen Matrixungleichungen und bietet sich dann an, wenn es nicht möglich ist Informationen über die Verkopplungen

auszutauschen. Daher führt dieses Verfahren zu einem eher konservativerem Systemverhalten.

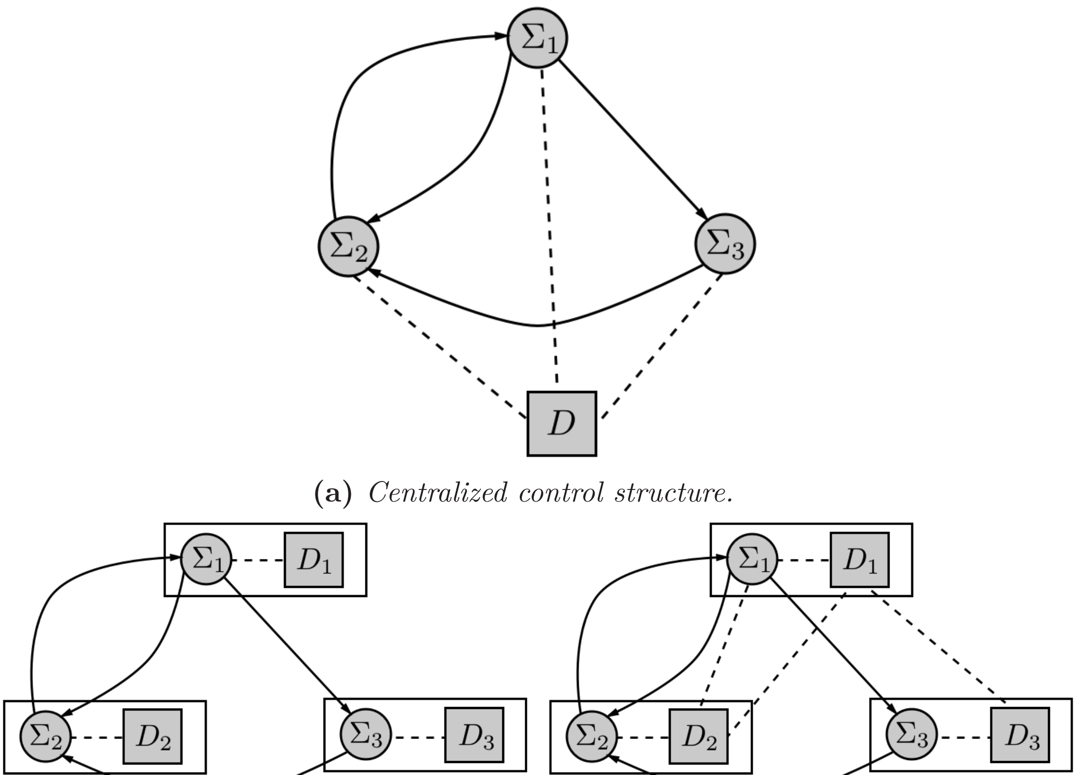
Anhand einer Simulationsstudie werden die vorgestellten Methoden abschließend präsentiert und die Flexibilität der invarianten Familie von Mengen zur systemtheoretischen Beschreibung von verkoppelten Systemen aufgezeigt.

1 Introduction

The design of controllers for large–scale processes is of great theoretical and practical interest. It is driven by many applications, spanning from power networks, chemical plants, battery stacks up to autonomously driven car fleets and social networks. A standard procedure for handling these large networks of connected systems is to divide them into manageable subproblems that are handled independently. However, most often these subproblems are not independent, for instance physical couplings or additional network infrastructure for the transmission of information and data need to be taken into account. Physical interconnection are dominant in many applications, for instance in chemical plants, where a specific amount of a product from one process is continuously needed as an reactant in other processes. Also interconnections due to communication are also relevant in many applications, for instance in the emerging field of autonomously driven car fleets, in which different cars need to negotiate their position and speed in order to avoid collisions.

It is clear that these interconnections can have a huge impact on important system–theoretic properties. Hence, for the design of control strategies, they need to be taken into account. Firstly, one needs to consider how to treat interconnections in general and, secondly, how to distinguish the type of informations available during the design and the operation of the process. A centralized structure, i.e. in which one decision maker has process information of all subsystems and designs one global controller, see Figure 1.1 a), is often not feasible due to the complexity of the global problem and limitations in the network communication infrastructure. In order to circumvent these issues, two important design concepts are usually considered for the control of interconnected systems; decentralized and distributed structures. In decentralized structures the overall system is decomposed into subsystems. Each subsystem has a decision maker and can only utilize local information about the subsystem for the choice of the local controller, cf. Figure 1.1 b). For this type of control structure, there is no additional negotiation of information between the subsystems and hence the network communication load is low.

However, depending on the type of interconnections between the subsystems, the decentralized operation of a plant often leads to a badly performing closed loop behaviour. In distributed structures, there is also a decomposition of the overall system into different subsystems. However, in contrast to the decentralized structure, the local decision makers are allowed to utilize both local information of the subsystem as well as process information of some or all other subsystems. In addition, they can also communicate with the other decision makers, cf. Figure 1.1 c). Due to the additional exchange of information, it is often possible to achieve superior performance in comparison to the decentralized structure, however with an increased network load and often with a more complex control structure.



(a) *Centralized control structure.*

(b) *Decentralized control structure.*

(c) *Distributed control structure.*

Figure 1.1: *Different control structures for interconnected subsystems; D_i denotes the decision maker, Σ_i the subsystem, solid lines the interconnections and dashed lines the information exchange.*

There are many approaches available for the synthesis of distributed and decentralized controllers. However, many of the existing approaches focus on stabilization of set points/trajectories. While this is an appropriate design aspect in many applications, it is not necessarily adequate for other specific tasks. For instance, it is often sufficient to ensure that a chemical stirred reactor is operated in a certain pressure and temperature *region* as opposed to specific temperatures and/or pressures. Set-based methods provide powerful tools, that allow to characterize such regions in a non conservative way. In particular, these methods excel whenever issues such as handling hard constraints on the process variables or guaranteed disturbance rejection, prevalent in all technical applications, need to be addressed. Nevertheless, existing set–based approaches are so far tailored to centralized structures and hence are suitable for systems that are not interconnected with each other. It is therefore of great interest to extend the standard, well established set–based methods to interconnected systems in order to enable new design procedures for distributed and decentralized control.

1.1 Analysis and Control of Large–Scale Processes

The analysis and control of large–scale processes has been under active research since the beginning of modern control. Very early results [Witsenhausen 1968a] already indicate, that distributed algorithm need to be carefully evaluated, since seemingly simple problems can lead to surprising and challenges and results. An important feature for the control and analysis of large–scale systems is the fact that the applied methodologies need to take the distributed or decentralized structure explicitly into account. For instance, global and complex solution can be avoided if the nature of the interconnection and the available information are properly understood: by applying distributed algorithms [Bertsekas 1983] for the determination of fixed points; exploiting separable structures [Bertsekas 2007] with distributed algorithm; properly understanding the network structure for the state estimation and observer based control of distributed and decentralized structures [Rantzer 2006; Necoara, Nedelcu, and Dumitrache 2011; Farina et al. 2011], to name just a few of the relevant works. One important tool for the stability analysis with respect to these considerations is the concept of Vector Lyapunov Functions [Bellman 1962] and generalizations hereof, e.g. [Martynyuk 1998], which can be explicitly exploited for the control synthesis [Nersesov and Haddad 2006; Lakshmikantham, Matrosov, and Sivasundaram 1991]. For strongly coupled systems, it is necessary to properly take destabilizing interconnection effects into account. A standard design procedure for decentralized control is to find appropriate bounds for the interconnections in order to solve them as convex LMI's, [Stipanović and Šiljak 2001; Zečević, Nešković, and Šiljak 2004; Stanković, Stipanović, and Šiljak 2007; Zečević and Šiljak 2004], which in turn can be efficiently solved using standard semidefinite programming approaches, [Boyd and Vandenberghe 2004]. Another alternative approach is to treat the interconnection as disturbances and use modified, robust control approaches, cf. [Shamma 2001]. Issues with time-delays and interconnections were investigated in [Mahmoud and Bingulac 1998; Thanh and Phat 2012] and results for the decentralized an distributed controller synthesis for overlapping decompositions, i.e. when common subsystems have shared states in their dynamics, were provided in [Stanković and Šiljak 2000; Stanković, Stanojević, and Šiljak 2000]. Decentralized controller with adapted controller topology were discussed in [Schuler, Münz, and Allgöwer 2012] and basic properties of positive systems were exploited for designing distributed control algorithms in [Rantzer 2011]. Characterization of invariant subspaces and stability regions for interconnected systems were discussed in [Hayakawa and Šiljak 1988] and [Chiang and Fekih-Ahmed 1990]. In addition, lumped approximations of partial differential equations, can be interpreted as spatially interconnected systems, for which there also exist powerful control strategies, cf. [Andrea and Dullerud 2003]. In order to even start evaluating distributed and decentralized structures for solving interconnected control problems, it is crucial to understand which type of structures lead to tractable problems. This important question was investigated for example in [Rotkowitz and Lall 2006], providing several useful design and analysis tools. Other general design considerations for the decentralized and distributed control of minimum phase system were highlighted in [Johansson and Rantzer 1999].

The discussed methods largely focus on stability issues and can only be used in special cases to determine conditions that lead to explicit constraint satisfaction. Apart from set–based methods, model predictive control schemes provide an attractive approach for the distributed control of interacting and interconnected systems. Since this scheme is based on the repeated solution of online optimal control problems, it can be used whenever constraint satisfactions as well as performance considerations need to be met, cf. [Mayne, Rawlings, et al. 2000; Rawlings and Mayne 2009] for an introduction and general overview of model predictive control. In [Borrelli and Balas 2004; Richards and J. How 2004; Keviczky, Borrelli, and Balas 2006; Richards and J. P. How 2007] several decentralized model predictive control schemes were presented for decoupled systems, which are interconnected through constraints and the cost function. Distributed model predictive control schemes for decoupled systems, in which the controller cooperate [Franco et al. 2008] and distribute informations during the online optimization [Camponogara et al. 2002; Dunbar and Murray 2006] are also available. For coupled systems, there exists distributed model predictive control schemes that exchange state trajectories at each sampling step, cf. [Dunbar 2007], and provide plug and play operation with certain invariance properties, cf. [Zeilinger et al. 2013] and utilizing game–theoretic considerations in order to achieve stability and repeated online feasibility [Venkat, Rawlings, and Wright 2005; Venkat, Hiskens, et al. 2008]. In contrast to the before mentioned methods, it is also possible to achieve stability with a decentralized model predictive control through the use of additional contractive constraints, cf. [Magni and Scattolini 2006].

General issues with model predictive control schemes are either complicated initialization (most of the schemes assume that the problem is feasible during the initialization) as well as difficulties, when facing disturbances. Utilizing basic set–based concepts we can alleviate some of the before mentioned problems, cf. [S. V. Raković 2009] for general set–theoretic consideration in model predictive control. For instance, positively or controlled positively invariant sets can be used to approximate the region of attraction for the initialization of standard model predictive control schemes. Fortunately, there exists several algorithms for testing and determining such sets, cf. [Bitsoris 1988b; Vidal et al. 2000; Gilbert and Tan 1991; S. V. Raković, Kerrigan, et al. 2005; Kolmanovsky and Gilbert 1998; Bitsoris and Athanasopoulos 2011]. Recent advancements were also made in order to overcome the conservatism of several results [S. V. Raković 2007; S. V. Raković and Fiacchini 2008; S. V. Raković and Barić 2009]. In general, the analysis of set–iterates allows a better understanding of invariance properties in general and is therefore a crucial tool in set–theoretic considerations, cf. [Artstein and S. V. Raković 2008; Artstein and S. V. Raković 2011]. Based on the design of of min-max feedback controllers, the determination of target tubes and analysis of reachable regions, cf. [Bertsekas and Rhodes 1971; Bertsekas 1972; Bertsekas and Rhodes 1973; Witsenhausen 1968b], it was possible to circumvent one of the main shortcomings of standard model predictive control schemes; namely the ability to handle disturbances while guaranteeing explicit input and state constraint satisfaction and feasibility during the online optimization. By using tube–based model predictive control schemes these concerns can be circumvented for quite general use scenarios, cf. [S. V. Raković, Kouvaritakis, Cannon, et al. 2012; S. V. Raković, Kouvaritakis, Findeisen, et al. 2012; S. V Raković, Kouvaritakis, and Cannon 2013; Mayne et al. 2006; Mayne et al. 2009; Cannon et al. 2011].

Some results exists with a focus on determining invariant region for decentralized control scenarios [Bitsoris 1988a], they nevertheless focus on the computation of global invariant sets and hence neglect the decentralized or distributed structure of the problem. However, most set–based methods are tailored to the centralized use cases; hence adaptation is necessary in order to use them for distributed and decentralized structures.

For additional information of classic feedback approaches for distributed and decentralized control systems, we refer to [Bakule 2008; Bernussou and Titli 1982; Lunze 1992; Šiljak 1978; Singh and Titli 1978; Lunze 2014]. More considerations about model predictive control and their use cases in decentralized and distributed structures are discussed in [Rawlings and Stewart 2007; Scattolini 2009]. For an in–depth discussion of general set–based methods, we refer to [Aubin 1991; Blanchini 1999; Blanchini and Miani 2008].

1.2 Challenges

An important challenge for the analysis and control of dynamic systems in general is the ability to specify conditions, that lead to explicit constraint satisfaction. For large–scale systems, it is often necessary to divide the overall problem into interconnected subproblems and avoid a centralized solution approach, due to the complexity or nature of the problem. With respect to issues related to guaranteed constraint satisfaction, most of the well established analysis methods are usually tailored to the centralized case or are not explicitly taking the structure of the interconnection into account. For these reasons, it is an important challenge to provide feasible and non conservative notions for the constraint analysis of interconnected systems. Often, the operation of large scale–systems benefits from the use of either decentralized or distributed controllers. For this reason, the provided notions need to be well suited for designing controllers in these use cases and need to fit naturally to the many already available distributed and decentralized control schemes. Eventually, a prevalent challenge in all real applications is the existence of disturbances, for instance due to unmodeled dynamics or measurement uncertainties. Therefore the notions need to be flexible enough to handle these issues as well.

With respect to these challenges we achieve the following contributions which are summarized in this thesis.

1.3 Contributions

We elaborate a mathematical, rigorous set–based framework for the analysis and control of interconnected control systems subject to hard constraints. The methods are well suited for the analysis of positively, homogeneous interconnected systems. Algorithms are provided that allow to specify parametrized regions in which one can safely initialize linear interconnected systems such that these constraints conditions are met. In addition, these methods are flexible enough to handle additive, bounded disturbances. Although a major focus was to provide a framework, which can be used to properly analyze, various forms of invariance properties of interconnected, autonomous systems, it is easily possible to include control synthesis considerations. In particular, we outline two control design methods, which are suitable for linear systems that are interconnected by linear and positively homogeneous functions, respectively. The first method is well suited for distributed control problems. By applying and modifying established results from $\mathcal{H}_\infty$ control, it is possible to obtain positively invariant family of sets based on the iterative solutions of generalized Riccati equations. The second approach is tailored to the synthesis of decentralized controller for positively homogeneous, interconnected linear systems. It is based on the independent, loosely coupled solution of LMI feasibility problems. It can be implemented efficiently using semidefinite programming algorithms. We show how the different proposed methods can be applied to an extended benchmark example in order to discuss various inherent system–theoretical features, such as disturbance rejection and invariance properties.

1.4 Outline

The remainder of this work is structured as follows:

Chapter 2 provides a background for general set–based methods in control theory, introduces the concept of invariance and its advantages and outlines methods and algorithms for the determination of invariant sets for discrete time systems.

Chapter 3 highlights the issues for the analysis of constraint satisfaction for interconnected systems. In particular, we show that a naive extension of the given methodology of invariant sets is not suitable for this specific task, due to the nature of the induced, interconnected set–dynamics. We show, that a parametrized family of sets can be used to properly analyze this problem setting. In addition, methods and algorithms are provided, which can be used to check and construct basic properties of positively invariant family of sets. The results are mainly based on the works [S. V. Raković, Kern, and Findeisen 2010; S. V. Raković, Kern, and Findeisen 2011].

Chapter 4 discusses extensions to the framework provided in Chapter 3. In particular, we highlight how the concept of positively invariant family of sets can be used for the controller synthesis. To this end, a design procedure for a distributed controller, which is based on a max–min/min–max $\mathcal{H}_\infty$ forumlation, and a decentralized controller, that is based on the independent solution of LMI's, is presented. The results are based on the works [S. V. Raković, Kern, and Findeisen 2010; Kern and Findeisen 2013].

Chapter 5 provides details for the synthesis of a distributed and decentralized controller for a specific control application. We provide a model of an extended four tank system and discuss several simulation results. We focus on specific system-theoretic traits of the positively invariant family of sets, such as disturbance rejection properties and regions of initial conditions in which the process will have guaranteed constraint satisfaction.

Chapter 6 closes this work with a summary and outlines interesting, future research directions.

2 Set–Based Concepts in Control

The focus of this thesis is to provide an analysis and design framework for interconnected control systems, subject to constraints and disturbances. Fortunately, set–based methods usually fit these types of problems well. They inherently include questions regarding constraint satisfaction, while allowing to address uncertainties as well as design specifications. The purpose of this chapter is to introduce some well known standard concepts of set–based methods for the analysis and control of discrete time systems, which will be used and extended throughout the remainder of this thesis.

2.1 Nomenclature and Basic Definitions

Positive and non–negative integers are denoted by $\mathbb{N}_+$ and $\mathbb{N}$, respectively, and non-negative reals by $\mathbb{R}_+$. The index set of numbers consisting from 1 to N is defined as $\mathcal{N} := \{1, 2, \ldots, N\}$. A set $\mathcal{X}$ is non–trivial, if it is not a singleton and a non–empty subset of $\mathbb{R}^n$. A set $\mathcal{X} \subseteq \mathbb{R}^n$ is convex, if $(1 - \lambda)x + \lambda y \in \mathcal{X}$, whenever $x \in \mathcal{X}$, $y \in \mathcal{X}$ and $0 < \lambda < 1$. The closed Euclidean unit ball in $\mathbb{R}^n$ is given by $\mathcal{B}^n := \{x \in \mathbb{R}^n : x^{\mathrm{T}}x \leq 1\}$. Given two sets $\mathcal{X} \subseteq \mathbb{R}^n$ and $\mathcal{Y} \subseteq \mathbb{R}^n$, the Minkowski set addition is defined by $\mathcal{X} \oplus \mathcal{Y} := \{x + y \ : \ x \in \mathcal{X}, y \in \mathcal{Y}\}$. Moreover, for a collection of sets $(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_N)$ let $\bigoplus_{i\in\mathcal{N}} \mathcal{S}_i := S_1 \oplus \cdots \oplus S_N$. A *polyhedron* is the set of solutions to a finite system of linear inequalities and a *polytope* is a bounded *polyhedron*. Given a set $\mathcal{X} \subseteq \mathbb{R}^n$ and a function $f : \mathbb{R}^n \to \mathbb{R}^m$ we define the image and preimage of $\mathcal{X}$ under f by $f(\mathcal{X}) := \{f(x) \ : \ x \in \mathcal{X}\}$ and $f^{-1}(\mathcal{X}) := \{x \ : \ f(x) \in \mathcal{X}\}$. Similarly given a matrix or scalar M and a set $\mathcal{X} \subseteq \mathbb{R}^n$, we define the image and preimage of $\mathcal{X}$ under M by $M\mathcal{X} := \{Mx \ : \ x \in \mathcal{X}\}$ and $M^{-1}\mathcal{X} := \{x \ : \ Mx \in \mathcal{X}\}$, respectively. A set $\mathcal{X}$ is symmetric if $-\mathcal{X} = \mathcal{X}$. Given a set $\mathcal{X}$, a function $f : \mathcal{X} \to \mathbb{R}$ is said to be positive definite, if $f(0) = 0$ and $f(x) > 0$ for all $x \in \mathcal{X}$, positive semidefinite, if $f(0) = 0$ and $f(x) \geq 0$ for all $x \in \mathcal{X}$ and negative (semi)definite, if $-f(\cdot)$ is positive (semi)definite. A symmetric matrix $M \in \mathbb{R}^{n\times n}$ is positive or negative (semi)definite, if

$f(x) = x^{\mathrm{T}} M x$ is positive or negative (semi)definite for all $x \in \mathbb{R}^n$. Positive (negative) definite and semi-definite matrices P, Q are denoted by $P \succ (\prec) 0$ and $Q \succeq (\preceq) 0$, respectively. A function $f(\cdot)$ whose domain is a subset $\mathcal{X}$ of $\mathbb{R}^n$ is a convex function, if the set $\{(x, \mu)^{\mathrm{T}} : x \in \mathcal{X}, \mu \in \mathbb{R}, \mu \geq f(x)\}$ is convex. The Hausdorff semi–distance and the Hausdorff distance for two nonempty sets $\mathcal{X} \subseteq \mathbb{R}^n$ and $\mathcal{Y} \subseteq \mathbb{R}^n$ are defined by:

$$\begin{aligned} h(L, \mathcal{X}, \mathcal{Y}) &:= \min_{\alpha} \{\alpha \ : \ \mathcal{X} \subseteq \mathcal{Y} \oplus \alpha L, \ \alpha \geq 0\} \text{ and} \\ H(L, \mathcal{X}, \mathcal{Y}) &:= \max\{h(L, \mathcal{X}, \mathcal{Y}), h(L, \mathcal{Y}, \mathcal{X})\}, \end{aligned}$$

where L is a given, symmetric, compact and convex set in $\mathbb{R}^n$ that contains the origin in its interior. In addition, more used concepts are provided in the Appendix at the end of this thesis.

2.2 Invariance

Informally, we speak of set–based methods/problems, whenever we deal with relations to sets. In basic control problems, the objective is most of the time either to stabilize specific set points or specific trajectories. However, sometimes this control objective is unnatural. For instance, it is more natural to demand, that an autonomously driven car should stay on a specific lane/street rather than on a predefined, specific trajectory. In general, we can consider this as a problem of finding a region that allows us to safely operate a process in certain bounds. In many cases this a more natural and adequate way of specifying a control objective, compared to controlling specific operating set points or trajectories. In an abstract way we can characterize this problem by stating that a process, which is described by a dynamical system, needs to satisfy process constraints, which are given by a set of allowable/admissible process *conditions*. In other words, we can think of a system $x(k+1) = f(x(k))$ and specify the constraints by a set $\mathcal{X}$. This analysis problem can then be interpreted as a set–based problem, namely in characterizing a *set* $\mathcal{X}_0$ in $\mathcal{X}$ that satisfies the *set*–inclusion $x(k) \in \mathcal{X}$ for all $k \geq 0$, where $x(k+1) = f(x(k))$, whenever $x(0) \in \mathcal{X}_0$. This classic set–based problem in the analysis of dynamic processes can be easily extended by including control inputs and/or disturbances, performance specification and so on. For more details, generalization and in depth discussion of set–based considerations we refer to [Blanchini and Miani 2008;

Aubin 1991].

Motivated by the previous discussion, we consider in the remainder the following formalized problem setting:

Problem 1. *Find a set $\mathcal{S} \subset \mathbb{R}^n$, such that $x(0) \in \mathcal{S}$ implies $x(k) \in \mathcal{X}$ for all $k \in \mathbb{N}_+$, for the discrete dynamical system $x(k+1) = f(x(k))$.*

Quite naturally, this leads to the concept of positive invariance, because invariant sets describe basically the regions in which a process in the presence of constraints and disturbances can be operated safely for all time instances $k \geq 0$:

Definition 1. *A set $\mathcal{S} \subseteq \mathcal{X} \subseteq \mathbb{R}^n$ is said to be* positively invariant *for a system $x(k+1) = f(x(k))$ if, for all $x(k) \in \mathcal{S}$, the condition $f(x(k)) \in \mathcal{S}$ holds.*

In other words, the evolution of the state $x(k)$ is contained within a set $\mathcal{S}$ for future time instants larger then k, once $x(k) \in \mathcal{S}$. Note that if $f(x)$ has a fixed point $\bar{x}$, such that $\bar{x} = f(\bar{x})$, then a trivial positively invariant set is $\mathcal{S} = \{\bar{x}\}$. In addition, if $\mathcal{X}$ is the whole state space $\mathbb{R}^n$ and $f(\cdot)$ is properly defined over all $\mathbb{R}^n$ then the whole state space is a positively invariant set. From a practical point of view it makes more sense to characterize non–trivial subsets of $\mathbb{R}^n$ in order to gain a flexible and useful characterization of our *safe* operation region subject to constraints and not just distinctive points.

Obviously the notion of positive invariance is closely related to the stability of a system, and hence it makes sense to use the general concepts of Lyapunov's stability theory for the analysis. Depending on the structure of $\mathcal{X}$ and the properties of $f(\cdot)$, proper candidate sets can be constructed with the help of Lyapunov functions. Note that for simplicity we provide only a special version for such a characterization here, which is sufficiently general for our purposes. For more general results we refer to [Blanchini and Miani 2008].

Theorem 1. *Given the system $x(k+1) = f(x(k))$, where $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function. Assume there exists a continuous positive definite function $V(\cdot) : \mathbb{R}^n \to \mathbb{R}$, such that the difference $V(f(x(k))) - V(x(k))$ is negative semi–definite for all $x(k)$ in a set $\Omega \subseteq \mathbb{R}^n$, then the set $\mathcal{S} := \{x : V(x) \leq \mu\}$ is positively invariant for any $\mu > 0$ as long as $\mathcal{S} \subseteq \Omega$.*

Proof. Pick any $k \geq 0$. For all $x(k) \in \Omega$ we have $V(x(k)) \geq V(f(x(k))) = V(x(k+1))$. By definition of $\mathcal{S}$, it follows that $V(x(k)) \leq \mu$ for all $x(k) \in \mathcal{S}$. Since $V(\cdot)$ is a non–increasing function in Ω and $\mathcal{S} \subseteq \Omega$ for any $\mu > 0$, we have $V(x(k+1)) \leq V(x(k)) \leq \mu$ for all $x(k) \in \mathcal{S}$ and hence $x(k+1) \in \mathcal{S}$ for all $x(k) \in \mathcal{S}$. $\square$

Note, that in Theorem 1 it is not strictly necessary to use positive definite functions $V(\cdot)$ as long as these functions are non–increasing over the set Ω. The main advantage of using positive definite functions is to guarantee compactness of the level sets of $V(\cdot)$ and to highlight the connection between positive invariance and the stability analysis using Lyapunov's second method. Hence, whenever it is possible to construct Lyapunov functions for a certain systems class, we can also quite easily construct positively invariant sets. Fortunately, we can construct in a straightforward way quadratic Lyapunov functions for linear control systems.

Theorem 2. *Given the system $x(k+1) = Ax(k)$, where $A \in \mathbb{R}^{n \times n}$. If there exists a $P \succ 0$, such that $A^T PA - P \preceq 0$, then the set $\mathcal{S} := \{x : x^T Px \leq \mu\}$ is positively invariant for any $\mu > 0$.*

Proof. Let $V(x(k)) := x^T(k)Px(k)$. Note that $V(\cdot)$ is positive definite since $P \succ 0$ and in addition $V(x(k+1)) - V(x(k)) = x^T(k)A^T PAx(k) - x^T(k)Px(k) = x^T(k)(A^T PA - P)x(k)$ is negative semidefinite since $A^T PA - P \preceq 0$ for all $x(k) \in \mathbb{R}^n$. Thus $\mathcal{S}$ is a positively invariant set for all $\mu > 0$ according to Theorem 1. $\square$

It is well known, that the existence of a positive definite matrix P that satisfies the discrete Lyapunov inequality $A^{\mathrm{T}} PA - P \preceq 0$, directly correlates to the stability of the matrix A; i.e. whenever the spectral radius of the matrix A is less than one, then there exists a $P \succ 0$, that satisfies the former conditions. Although quadratic Lyapunov functions provide a lot of simplicity for the analysis of linear systems they still lack flexibility, i.e. invariant sets are restricted to ellipsoidal shaped sets. In addition, if we consider polytopic constraint set, we can increase the domain of attraction by using polyhedral Lyapunov, cf. [Blanchini and Miani 2008]. Nevertheless, we restrict ourselves solely to quadratic Lyapunov functions, since we will focus on properties of sets and not on functions for the analysis of invariance properties.

2.2.1 Robust Invariance

The basic concepts related to positive invariance can be generalized easily to uncertain systems. In this case we speak of robust positive invariance, cf. [Blanchini and Miani 2008].

Definition 2. *A set $\mathcal{S} \subseteq \mathcal{X}$ is said to be* robust positively invariant *for the system $x(k+1) = f(x(k), w)$ if, for all $x(k) \in \mathcal{S}$ and disturbances $w \in \mathcal{W}$, the condition $f(x(k), w) \in \mathcal{S}$ holds.*

In comparison to the disturbance free case, this simply means that the state $x(k)$ is contained in $\mathcal{S}$ for future time instants for all possible disturbance realization $w \in \mathcal{W}$. The computation and characterization of robust positively invariant sets is more challenging compared to determining positively invariant regions, however many results and algorithms are available for such specific sets, see for instance [S. V. Raković, Kerrigan, et al. 2005; S. V. Raković 2007; Artstein and S. V. Raković 2008; S. V. Raković and Fiacchini 2008; Blanchini and Miani 2008].

2.3 Characterization of Set Inclusions with Support Functions

So far we considered the unconstrained problem, i.e. $\mathcal{X} = \mathbb{R}^n$. Obviously, if $\mathcal{X}$ is a subset of $\mathbb{R}^n$, it is easily possible to characterize a positively invariant set within the constraint set $\mathcal{X}$ using Theorem 1. As long as $\mathcal{X}$ is non–empty and/or not a singleton and contained in the set Ω, we *just* need to choose $\mu > 0$ a way that $\mathcal{S} \subseteq \mathcal{X}$. For these reasons, we need a convenient way to ensure algorithmically that a set is contained within another set. Note that for general sets this is a difficult task. Fortunately, we will mainly deal with ellipsoidal and polyhedral convex sets and hence we are able to utilize basic properties of the support function. We only present some basic results, for more details we refer to [Rockafellar 1970; Kolmanovsky and Gilbert 1998].

Definition 3. *The support function for a closed convex set $\mathcal{X} \subseteq \mathbb{R}^n$ and $y \in \mathbb{R}^n$ is defined as*

$$s(\mathcal{X}, y) := \sup\{y^T x : x \in \mathcal{X}\}, \tag{2.1}$$

where $\sup(\cdot)$ *denotes the supremum, i.e. the least upper bound.*

The support function can be evaluated in a straightforward way for polyhedra and ellipsoids. For instance, given an ellipsoidal set $\mathcal{E} := \{x : x^T P x \leq 1\}$ for some positive definite matrix $P \in \mathbb{R}^{n \times n}$, then its support function is given by $s(\mathcal{E}, y) = \sqrt{y^T P^{-1} y}$. Similarly, for a polyhedral set $\mathcal{P} = \{x : c_i^T x \leq r_i,\, i \in \mathcal{N}\}$, for some $x \in \mathbb{R}^n$ the support function can be determined by the optimal value of the linear program $s(\mathcal{P}, y) = \max\{y^T x : c_i^T x \leq r_i,\, i \in \mathcal{N}\}$. With the help of the support function we can conveniently express any closed convex set by a system of inequalities and their support function.

Theorem 3. *Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a closed convex set. Then $x \in \mathcal{X}$ if and only if $y^T x \leq s(\mathcal{X}, y)$ for every $y \in \mathbb{R}^n$.*

Proof. The proof follows directly from the fact that any closed convex set $\mathcal{X}$ is the intersection of the closed half–spaces which contain it, for more details cf. Theorem 11.5 and Theorem 13.1 in [Rockafellar 1970]. □

As a direct consequence we can describe in a dual way if a convex set is included in another convex set using properties of the support function.

Theorem 4. *Given closed and convex sets $\mathcal{X} \subseteq \mathbb{R}^n$ and $\mathcal{Y} \subseteq \mathbb{R}^n$. $\mathcal{X} \subseteq \mathcal{Y}$ if and only if $s(\mathcal{X}, y) \leq s(\mathcal{Y}, y)$ for all $y \in \mathbb{R}^n$.*

Proof. Is a consequence of Theorem 3, cf. Chapter 13 on support functions in [Rockafellar 1970]. □

As an example, given ellipsoids $\mathcal{E}_1 := \{x : x^T P_1 x \leq 1\}$ and $\mathcal{E}_2 := \{x : x^T P_2 x \leq 1\}$ for some $P_1 \succ 0$ and $P_2 \succ 0$, then according to Theorem 4, $\mathcal{E}_1 \subseteq \mathcal{E}_2$ if and only if $\sqrt{y^T P_1^{-1} y} \leq \sqrt{y^T P_2^{-1} y}$ for all $y \in \mathbb{R}^n$, which is the same as the condition $P_2 \preceq P_1$. In a similar way, given a closed convex set $\mathcal{X} \subseteq \mathbb{R}^n$ and a polyhedron $\mathcal{P} = \{x : c_i^T x \leq r_i,\, i \in \mathcal{N}\}$, then $\mathcal{X} \subseteq \mathcal{P}$ if and only if $s(\mathcal{X}, c_i) \leq r_i$ for all $i \in \mathcal{N}$. Evidently, it is very convenient to use polyhedra and ellipsoids when dealing with questions related to set–inclusions.

2.4 Determination of Invariant Sets within Constraints

Utilizing the properties of the support function and the result from Theorem 2 and Theorem 4 we can compute positively invariant sets subject to a polyhedral constraint set:

Theorem 5. *Given the discrete–time linear system* $x(k+1) = Ax(k)$ *and the constraint set* $\mathcal{X} := \{x : c_i^T x \leq 1,\, i \in \{1, 2, \ldots, N\}\}$. *If the following feasibility problem*

$$\begin{aligned} & \textit{find} \quad P && (2.2a)\\ & \begin{pmatrix} P & PA \\ A^T P & P \end{pmatrix} \succeq 0, && (2.2b)\\ & P^T = P \succ 0, && (2.2c)\\ \forall i \in \{1, 2, \ldots, N\} \quad & \begin{pmatrix} P & c_i \\ c_i^T & 1 \end{pmatrix} \succeq 0 && (2.2d) \end{aligned}$$

is feasible, then the ellipsoidal set $\mathcal{E} = \{x : x^T P x \leq 1\}$ *is positively invariant and* $\mathcal{E} \subseteq \mathcal{X}$.

Proof. Assume that (2.2) is feasible, then by using the Schur Complement (see the Appendix for the definition) we know that (2.2b)–(2.2c) is equivalent to $A^T PA - P \preceq 0$, and by Theorem 2, $\mathcal{E}$ is positively invariant. Furthermore, by using the Schur Complement in (2.2d) we can see that for all $i \in \{1, 2, \ldots, N\}$, $c_i^T P^{-1} c_i \leq 1$. Eventually, subsequently taking the square root and using Theorem 4 we can conclude that $\mathcal{E} \subseteq \mathcal{X}$. □

Theorem 5 is the basis which we will use later for the determination of appropriate invariant sets, for the synthesis of decentralized controllers. However, we need to still adapt the LMI appropriately, in order to include the design of a stabilizing feedback controller, that respects possible input constraints. Utilizing the basic properties of the support functions, which was presented in the previous section, we can fortunately, extend the approach from Theorem 5 to synthesize stabilizing feedback controller for general, linear control systems, subject to input and state constraint sets.

Theorem 6. *Given the discrete–time linear control system* $x(k+1) = Ax(k) + Bu(k)$, *with* $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, *the state constraint sets* $\mathcal{X} := \{x \in \mathbb{R}^n : c_i^T x \leq 1,\, i \in \{1, 2, \ldots, N\}\}$, *the input constraint set* $\mathcal{U} := \{u \in \mathbb{R}^m : d_i^T u \leq 1,\, i \in \{1, 2, \ldots, M\}\}$.

Let $0 < \mu \leq 1$. If the following feasibility problem

$$\begin{aligned}
&\textit{find} \quad Q, R && (2.3a)\\
&\begin{pmatrix} Q & AQ+BR \\ (AQ+BR)^T & \mu^2 Q \end{pmatrix} \succeq 0, && (2.3b)\\
&Q^T = Q \succ 0, && (2.3c)\\
\forall i \in \{1,2,\ldots,N\} \quad &\begin{pmatrix} Q & Qc_i \\ c_i^T Q & 1 \end{pmatrix} \succeq 0 && (2.3d)\\
\forall j \in \{1,2,\ldots,M\} \quad &\begin{pmatrix} Q & R^T d_j \\ d_j^T R & 1 \end{pmatrix} \succeq 0. && (2.3e)
\end{aligned}$$

is feasible, then the ellipsoidal set $\mathcal{E} = \{x : x^T Q^{-1} x \leq 1\}$:

i) $\mathcal{E}$ is positively invariant for the closed loop system $x(k+1) = (A + BRQ^{-1})x(k)$,

ii) $(A + BRQ^{-1})\mathcal{E} \subseteq \mu\mathcal{E}$,

iii) $\mathcal{E} \subseteq \mathcal{X}$,

iv) $\forall x \in \mathcal{E}$, $RQ^{-1}x \in \mathcal{U}$.

Proof. i) Applying the Schur Complement we know that (2.3b)–(2.3c) is equivalent to $(AQ+BR)^T Q^{-1}(AQ+BR) - \mu^2 Q \preceq 0$. Multiplying the former relation with Q^{-1} from left and right we obtain

$$(A + BRQ^{-1})^T Q^{-1}(A + BRQ^{-1}) - \mu^2 Q^{-1} \preceq 0, \tag{2.4}$$

which implies $(A + BRQ^{-1})^T Q^{-1}(A + BRQ^{-1}) - Q^{-1} \preceq 0$ since $0 < \mu \leq 1$ and hence by Theorem 2, $\mathcal{E}$ is positively invariant for the closed loop system $x(k+1) = (A + BRQ^{-1})x(k)$. ii) Follows immediately from (2.4). iii) Similarly to Theorem 5 we can deduce from the constraint (2.3d), that $\mathcal{E} \subseteq \mathcal{X}$. iv) We know from (2.3e) that for all $j \in \{1,2,\ldots,M\}$, $d_j^T RQ^{-1}R^T d_j \leq 1$ or equivalently $\sqrt{d_j^T RQ^{-1}QQ^{-1}R^T d_j} \leq 1$. We have $s(\mathcal{E}, (RQ^{-1})^{\mathrm{T}} d_j) \leq 1$ for all $j \in \{1,2,\ldots,M\}$. Using basic properties of the support function we can thus conclude that $\forall x \in \mathcal{E}$, $RQ^{-1}x \in \mathcal{U}$. □

The analysis and design approaches outlined in Theorem 5–6 are appealing since (2.2)–(2.3) are simple LMI's which can be efficiently solved via semidefinite programming algorithms. However, the resulting feasible sets are not necessarily optimal. In order to improve the size of the resulting positively invariant set, we can change the feasibility problems into an optimization problems, for instance maximizing the volume of the ellipsoid subject to constraints. The downside of such a formulation is that we use the level sets of a quadratic Lyapunov functions, i.e. ellipsoidal shaped sets, and try to fit them into polyhedral sets. In general, such a fit will be conservative and the computed set might be of comparatively small size, i.e. there might exist positively invariant sets that are larger then the ellipsoidal sets and still included in $\mathcal{X}$.

In order to improve the size of positively invariant sets, we can employ a different approach similar to the computation of reachable sets via dynamic programming [Blanchini and Miani 2008]. The basic idea is to recursively compute the pre–image set of a target set and their successive intersections. More precisely, given the discrete system $x(k+1) = f(x(k))$ and a set $\mathcal{X} \subseteq \mathbb{R}^n$ with $f : \mathbb{R}^n \to \mathbb{R}^n$ and $x(k) \in \mathbb{R}^n$ for all $k \in \mathbb{N}_+$. The definition of the pre–image is given in the Appendix and basically implies that $x(k) \in f^{-1}(\mathcal{X})$ implies $x(k+1) \in \mathcal{X}$ for any $k \geq 0$. Furthermore, it can be easily seen that it is necessary and sufficient that a set Ω, with the property $x(k) \in \Omega$ implies $x(k+1) \in \mathcal{X}$ exists, if and only if $f^{-1}(\mathcal{X}) \neq \emptyset$. If we intersect the target set $\mathcal{X}$ with the preimage set $f^{-1}(\mathcal{X})$, we can specify a set that guarantees state constraint satisfaction for one time step, i.e. if $x(k) \in f^{-1}(\mathcal{X}) \cap \mathcal{X}$, then $x(k) \in \mathcal{X}$ and $x(k+1) \in \mathcal{X}$, respectively. In order to characterize a set that guarantees state constraint satisfaction for two time steps, we can take an additional recursion and see that whenever $x(k) \in f^{-1}(f^{-1}(\mathcal{X}) \cap \mathcal{X}) \cap \mathcal{X}$ then $x(k) \in \mathcal{X}$, $x(k+1) \in \mathcal{X}$ and $x(k+2) \in \mathcal{X}$, see Figure 2.1. In general, a positively invariant set can be thus obtained by performing the former recursion infinitely, for more general results and the relation to reachable sets, see also [Witsenhausen 1968b; Bertsekas 1972].

Theorem 7. *Given a compact set $\mathcal{X} \subset \mathbb{R}^n$ and the discrete time system $x(k+1) = f(x(k))$, where $f(\cdot)$ is a continuous function. Let*

$$\Omega_{i+1} := f^{-1}(\Omega_i) \cap \Omega_0 \ \textit{ with } \ \Omega_0 = \mathcal{X}, \tag{2.5}$$

and assume the fixed point $\bar{x} = f(\bar{x})$ is in $\mathcal{X}$. Then the following holds:

i) Ω_i is non–empty for all $i \in \mathbb{N}_+$,

ii) $\Omega_\infty = \bigcap_{j=0}^{\infty} \Omega_j$ is the limit of the sequence of sets (2.5),

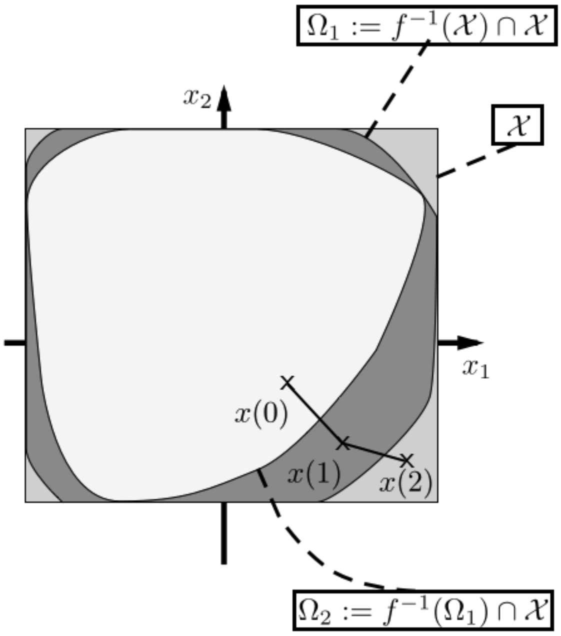


Figure 2.1: *Recursion for the construction of a positively invariant set.*

iii) $\Omega_\infty \subseteq \mathcal{X}$ *and all positively invariant sets included in the constraint set* $\mathcal{X}$ *for the discrete time system* $x(k+1) = f(x(k))$ *are subsets of* Ω_∞.

Proof. i) Follows by the fact that $\bar{x} = f(\bar{x}) \in \Omega_0$.

ii) By construction, Ω_j is a sequence of compact, non–empty nested sets, i.e. $\Omega_{j+1} \subseteq \Omega_j$. A basic fact states, that such a sequence has the limit $\Omega_\infty = \bigcap_{j=0}^{\infty} \Omega_j$, see for instance [Schneider 1993; Kelley 1955].

iii) $\Omega_\infty \subseteq \mathcal{X}$ follows trivially from (2.5) and is positively invariant by construction, since it defines the set to which the states are confined to infinitely. Let $\mathcal{S}$ be a arbitrary positively invariant set which is contained in $\mathcal{X}$. By definition, Ω_1 is the largest set such that $\Omega_1 \subseteq \mathcal{X}$ and $f(\Omega_1) \subseteq \mathcal{X}$, i.e. there exists no $x \in \mathcal{X} \setminus \Omega_1$ such that $f(x) \in \mathcal{X}$. Since $\mathcal{S}$ is a positively invariant subset of $\mathcal{X}$, we have $f(\mathcal{S}) \subseteq \mathcal{S} \subseteq \mathcal{X}$ and hence $\mathcal{S} \subseteq \Omega_1$. In addition, it also follows by positive invariance of $\mathcal{S}$ that $f(\mathcal{S}) \subseteq \Omega_1$, and since Ω_2 is by definition the largest set such that $f(\Omega_2) \subseteq \Omega_1$ and $\Omega_2 \subseteq \mathcal{X}$, we can deduce similarly that $\mathcal{S} \subseteq \Omega_2$ as well as $f(\mathcal{S}) \subseteq \Omega_2$. Eventually, we can conclude by induction that $\mathcal{S} \subseteq \Omega_\infty$. □

Note, that Ω_∞ is the *largest* positively invariant set inside the constraint set $\mathcal{X}$, i.e. all positively invariant sets inside $\mathcal{X}$ are subsets of Ω_∞. As a consequence, we can immediately see that whenever $\Omega_\infty = \emptyset$ then there exists no nontrivial positively invariant set contained in the constraint set $\mathcal{X}$.

In order to perform the set recursion (2.5), we need an efficient way to compute set recursions and pre–images of sets. In addition, it is also necessary to determine an appropriate stopping criteria, since it is obviously impossible to indefinitely perform the recursion. Fortunately, if the sets are equal in two consecutive recursion steps for some $j > 0$, i.e. $\Omega_j = \Omega_{j+1}$, then it is very easy to see that $\Omega_j = \Omega_\infty$. A simple prototype procedure using this fact is given by Algorithm 1: This algorithm can be implemented

Input: $\mathcal{X}$, $f(\cdot)$
Output: Ω_∞
$\Omega_0 \leftarrow \mathcal{X}$;
$\Omega_t \leftarrow \Omega_0$;
while $\Omega_t \neq f^{-1}(\Omega_t) \cap \Omega_0$ **do**
 $\Omega_t \leftarrow f^{-1}(\Omega_t) \cap \Omega_0$;
end
$\Omega_\infty \leftarrow \Omega_t$;

Algorithm 1: Computation of a maximal positively invariant set.

for linear, affine systems subject to polytopic constraint sets $\mathcal{X}$ in a straightforward way. More precisely, for the affine system $x(k+1) = f_{aff}(x(k)) = Ax(k) + g$ and a non empty polytopic set $\mathcal{X} := \{x : c_i^T x \leq 1, i \in \{1, 2, \ldots N\}\}$, the preimage set $f_{aff}^{-1}(\mathcal{X})$ is the polyhedron

$$f_{aff}^{-1}(\mathcal{X}) = \{x : c_i^T Ax \leq 1 - c_i^T g,\ i \in \{1, 2, \ldots, N\}\},$$

while the intersection of $f_{aff}^{-1}(\mathcal{X})$ with the set $\mathcal{X}$ is specified by

$$f_{aff}^{-1}(\mathcal{X}) \cap \mathcal{X} = \{x : c_i^T Ax \leq 1 - c_i^T g,\ c_i^T x \leq 1,\ i \in \{1, 2, \ldots, N\}\}.$$

Note that $f_{aff}^{-1}(\mathcal{X}) \cap \mathcal{X}$ is characterized by the intersection of $2N$ half-spaces as opposed to the intersection of N half-spaces in Ω. By applying Algorithm 1 directly to an affine system, Ω_t would consist in the worst case of the intersection of tN half-spaces at recursion step t and thus might be too complex to handle conveniently. Fortunately, the complexity can be reduced in a straightforward way by detecting and removing redundant half-spaces from the polytope at each recursion step by using the following corollary.

Corollary 1. *Let* $\mathcal{P}_1 = \{x : Ax \leq b, c^T x \leq d\}$ *and* $\mathcal{P}_2 = \{x : Ax \leq b\}$*, where* $A \in \mathbb{R}^{m\times n}$*,* $b \in \mathbb{R}^m$*,* $c \in \mathbb{R}^n$*,* $d \in \mathbb{R}$*. If* $s(\mathcal{P}_2, c) \leq d$ *then* $\mathcal{P}_1 = \mathcal{P}_2$*.*

Proof. Let $\mathcal{H} = \{x : c^T x \leq d\}$ and assume without loss of generality that $\mathbb{R}^n \neq \mathcal{P}_2 \neq \emptyset$. If $s(\mathcal{P}_2, c) \leq d$, then $\mathcal{P}_2 \subseteq \mathcal{H}$ and thus $\mathcal{P}_2 \cap \mathcal{H} = \mathcal{P}_2$. However, since $\mathcal{P}_1 = \mathcal{P}_2 \cap \mathcal{H}$, it follows that $\mathcal{P}_1 = \mathcal{P}_2$. □

Evidently, Corollary 1 shows that detecting and removing redundant constraints in polytopes can be accomplished by a series of linear programs. Furthermore, by a simple modification of Algorithm 1, we are able to compute the maximal positively invariant set for linear systems subject to a polytopic constraint set. To that end, let

$$x(k+1) = Ax(k), \quad \mathcal{X} = \{x : c_i^T x \leq 1,\, i \in \{1, 2, \ldots, N\}\}, \tag{2.6}$$

where $A \in \mathbb{R}^{n\times n}$ and $c_i \in \mathbb{R}^n$ for all $i \in \{1, 2, \ldots, N\}$. A modified algorithm for the determination of the maximal positively invariant set for this use case is given by Algorithm 2. Conceptually Algorithm 1 and Algorithm 2 are similar. The only difference

Input: A, $c_1, c_2, \ldots, c_N$
Output: Ω_∞
$t \leftarrow 0$;
$\Omega_t \leftarrow \{x : c_i^T x \leq 1,\, i \in \{1, 2, \ldots, N\}\}$;
finished ← false;
while *finished* = *false* **do**
  $t \leftarrow t + 1$;
  finished ← true;
  for $i \leftarrow 1$ **to** N **do**
    if $\max_{x\in\Omega_t} c_i^T A^t x \leq 1$ **then**
      $\Omega_t \leftarrow \Omega_t \bigcap \{x : c_i^T A^t x \leq 1\}$;
      finished ← false;
    end
  end
end
$\Omega_\infty \leftarrow \Omega_t$;

Algorithm 2: Computation of the maximal positively invariant set for a linear system inside a polytopic constraint set.

is that we gradually check at each iteration $t > 1$ whether the preimage of the ith–half-space $\mathcal{H}_{i,t} = \{x : c_i^T A^{t-1} x \leq 1\}$ is redundant and thus not need to be added to Ω_t. The algorithm terminates at t if the preimage of the half-spaces $\mathcal{H}_{i,t}$ are redundant for all $i = \{1, 2, \ldots, N\}$ since then $\Omega_t = \Omega_{t+1}$. Furthermore, considering a linear asymptotically stable system and a convex and compact constraint set $\mathcal{X}$, that contains the origin in its interior, we can ensure using a direct modification of standard results [Gilbert and Tan 1991; Blanchini and Miani 2008], that Algorithm 1 terminates in a finite number of steps. In other words, there exists a finite integer t^*, such that $\Omega_{t^*} = \Omega_{t^*+1}$. In addition, as indicated by Algorithm 2, if the set $\mathcal{X}$ is a non–trivial polytope, then the maximal positively invariant set Ω_∞ is also a non-trivial polytope, cf. [Blanchini and Miani 2008; S. V. Raković and Fiacchini 2008].

2.5 Summary

We presented the concept of invariant sets and their usefulness for describing safe operation regions for dynamics processes, that need to respect hard constraints. Ellipsoidal, shaped invariant sets have a strong relation to level sets of Lyapunov functions, and for this reason we could exploit, standard semidefinite algorithms to construct them. In order to guarantee constraint satisfaction, we had to modify these standard algorithms utilizing the essential tool of the support functions. Eventually, since ellipsoidal shaped positively invariant sets are often of very limited size, we presented an algorithm that can be used determine maximal positively invariant sets, that are included in given constraint sets.

3 Invariance for Interconnected Systems

The main focus of this chapter is it to analyze interconnected systems utilizing the concepts presented in the previous chapter. We will focus in this chapter on systems that are autonomous and physically interconnnected, see Figure 3.1. More precisely,

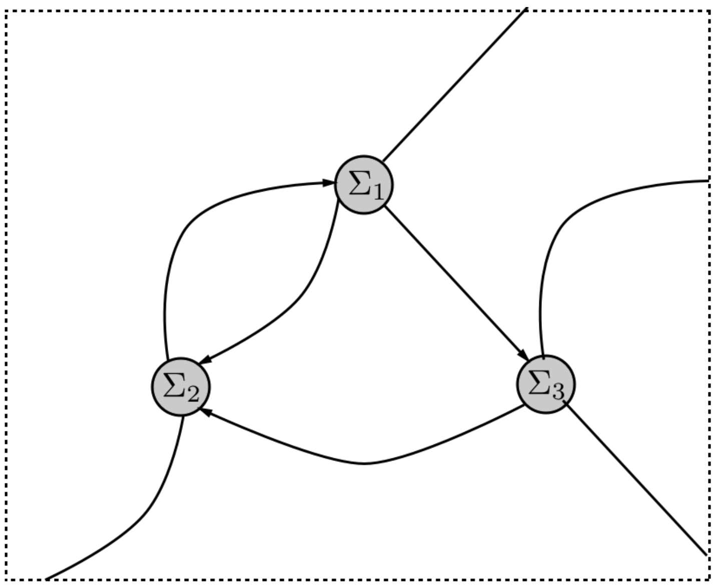


Figure 3.1: *A part of an interconnected, autonomous system, where the solid lines denote the physical interconnections.*

consider for all $i \in \mathcal{N}$ the interconnected, autonomous systems

$$\Sigma_i : \quad x_i(k+1) = f_i(x_i(k)) + g_i(x_j(k) : j \in \mathcal{N} \setminus \{i\}), \tag{3.1}$$

and constraint sets $\mathcal{X}_i \subset \mathbb{R}^{n_i}$, where $x_i(\cdot) \in \mathbb{R}^{n_i}$, $f_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$ and $g_i : \mathbb{R}^{n-n_i} \to \mathbb{R}^{n_i}$, with $n = \sum_{i \in \mathcal{N}} n_i$. Furthermore, we denote Σ_i as the i-th subsystem and the collection of all Σ_i as the overall system Σ. A generalized problem statement, complementary to Problem 1 can be formulated in the following way.

Problem 2 (Interconnected constraint satisfaction)**.** *Find a collection of sets $\mathcal{S} = \{\mathcal{S}_i \subset \mathbb{R}^{n_i},\ i \in \mathcal{N}\}$, such that for all $i \in \mathcal{N}$, $x_i(0) \in \mathcal{S}_i$ implies $x_i(k) \in \mathcal{X}_i$ for all $k \in \mathbb{N}_+$, for the system specified in* (3.1).

One basic difference to the previous problem statement is that we consider several systems Σ_i, which we could either analyze as a whole, a centralized approach, or separately, as an decentralized approach. A fitting framework that works nicely for interconnected systems both from an analytical and computational point of view, should be able to handle both aspects. In addition, the number of subsystems N and n_i might be large numbers, i.e. *many* possible large dynamical systems need to be considered.

A possible solution to Problem 2 is a centralized approach, in which the subsystems Σ_i are treated as one combined big system and not as separate entities. Concatenating all systems Σ_i into one large system Σ, a positively invariant set contained in the constrained set $\mathcal{X} := \mathcal{X}_1 \times \mathcal{X}_2 \cdots \times \mathcal{X}_N$ with the particular structure $\mathcal{S}_1 \times \mathcal{S}_2 \cdots \times \mathcal{S}_N$ can be characterized.

For example, consider that $f_i(\cdot)$ and $g_i(\cdot)$ in (3.1) are linear and the constraints $\mathcal{X}_i$ are non empty polytopes. In this case it is possible to describe the overall dynamics by a linear system $\Sigma : x(k+1) = Ax(k)$, where A is the matrix composed of all $f_i(\cdot)$ and $g_i(\cdot)$, respectively, $x = (x_1(k)^{\mathrm{T}}, x_2(k)^{\mathrm{T}}, \dots, x_N(k)^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^n$ and the concatenated polytope $\mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_N$ (the Cartesian product of polytopes is again a polytope). Hence, we can directly apply the concepts presented in the previous chapter. However, several issues arise treating all subsystems as one large system. For instance, applying this centralized approach can be difficult, even in the linear case, since the dimension n of the overall process might be large. Applying Algorithm 1–2 can thus be challenging. Upon closer inspection, it can be seen, that it is necessary to solve a linear program with n variables and roughly tN constraints m times at recursion step t in the worst case. Although we can slightly increase the efficiency for higher dimension if we use methods based on Farkas Lemma, e.g. [Kerrigan 2000], in general the basic algorithms scale not nicely with the dimension n of the system. In general, the number of half-spaces in the polytope $\mathcal{X}$ is a basic factor in determining whether this particular approach is computationally feasible for many interconnected and/or high dimensional subsystems Σ_i.

A more serious issue, for a combined analysis, is the fact that it is often not trivial to decompose the set $\mathcal{S}$ into the collection of subsets $\{\mathcal{S}_i : i \in \mathcal{N}\}$ without a central acknowledgment for the decision of the initial conditions $x_i(0)$.

As an example, consider a system Σ which consists of two linear, interconnected systems Σ_1 and Σ_2, given by

$$\begin{aligned} \Sigma_1 : \quad & x_1(k+1) = A_1 x_1(k) + G_1 x_2(k) \\ \Sigma_2 : \quad & x_2(k+1) = A_2 x_2(k) + G_2 x_1(k), \end{aligned} \tag{3.2}$$

with convex, polytopic constraint sets $\mathcal{X}_1$ and $\mathcal{X}_2$, respectively. Assume we use Algorithm 2 to compute the maximal, positively invariant set $\mathcal{S}$ which is contained in $\mathcal{X}_1 \times \mathcal{X}_2$ for the concatenated system Σ, with $x_1(k) \in \mathbb{R}^{n_1}$ and $x_2(k) \in \mathbb{R}^{n_2}$. The set can be represented as a convex polytope

$$S := \{x : Cx \leq d\} = \left\{ (x_1^{\mathrm{T}}, x_2^{\mathrm{T}})^{\mathrm{T}} : \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \right\}.$$

In order to decide on initial conditions $x_1(0)$ and $x_2(0)$ that will lead to constraint satisfaction, it is necessary to guarantee that both $C_1x_1(0)+C_2x_2(0) \leq d_1$ and $C_3x_1(0)+C_4x_2(0) \leq d_2$ are simultaneously satisfied. However, the choice of the initial condition for Σ_1 depends on the choice of the initial condition for Σ_2, i.e. either $C_2x_2(0)$ and $C_4x_2(0)$ or $C_1x_1(0)$ and $C_3x_1(0)$ are fixed. This requires communication between Σ_1 and Σ_2, in which admissible initial conditions $x_1(0)$ and $x_2(0)$ are negotiated, thus either a centralized or a distributed structure is necessary for the initialization. This negotiation of course needs to be repeated whenever initial conditions are changed, making the whole process inflexible. From a practical point of view it is often very challenging to distribute the necessary information of all state informations to one entity, especially if a lot of interconnected systems are considered.

In addition, if the structure of the interconnection $G_1x_2(k)$ changes slightly for the subsystem Σ_1, then a precomputed invariant set for the whole concatenated system Σ needs to be recomputed as Σ_1 and Σ_2 mutually depend on each other. Utilizing a decentralized or distributed analysis this problem can be alleviated, since in the worst case only a small part of the overall system needs to be adjusted. In addition, the problem setting for specific applications might be inherently distributed or decentralized, i.e. we are not always interested in finding one *global* solution for the whole system Σ, but instead we would like to find several local ones.

For these reasons, coming back to Problem 2, the sets $\mathcal{S}_i$ should be characterized in such a way that we can easily and in a modular way choose initial conditions $x_i(0) \in \mathcal{S}_i$ such that state constraint satisfaction for all k can be guaranteed. The focus of this chapter is to provide such modularity and in particular to properly extend the basic notion of positive invariance to interconnected systems. For further in depth informations and various other challenges about the analysis and decentralized control of large scale and/or interconnected systems, see for instance [Šiljak 1978; Lunze 1992; Lunze 2014].

3.1 Analysis of Interconnected Systems

A possible solution approach for Problem 2 is to regard the interconnections as additive disturbances and utilize the concept of robust positively invariant sets. More precisely consider the following

Definition 4 (Invariant collection of sets)**.** *A collection of sets* $\Omega := \{\Omega_i : i \in \mathcal{N}\}$*, where* $\Omega_i \subseteq \mathcal{X}_i \subset \mathbb{R}^{n_i}$ *is an invariant collection of sets for the system* (3.1) *if, for all* $i \in \mathcal{N}$ *and all* $x_i \in \Omega_i$*, the condition* $f_i(x_i) + g_i(x_j : j \in \mathcal{N} \setminus \{i\}) \in \Omega_i$ *hold.*

Evidently, this property has similarities to the detection of robust positively sets, e.g. by using $\mathcal{W}_i := g_i(\Omega_j : j \in \mathcal{N} \setminus \{i\})$. Considering the connections, this is an attractive approach, since the sets $\mathcal{S}_i$ can be characterized locally for each system Σ_i. However, the sets Ω_i need to be simultaneously detected and not sequentially as $\mathcal{W}_i$ depend on each other non trivially. For instance, consider again the linear, interconnected systems specified in (3.2). Given the collections of sets (Ω_1, Ω_2), if $\mathcal{W}_1 := G_1\Omega_2$ and $\mathcal{W}_2 := G_2\Omega_1$, then it can be easily seen, that every variation of the set Ω_2 has a direct effect onto the set $\mathcal{W}_1$. Hence a possible set Ω_1 needs to be adjusted, which in turn modifies the set $\mathcal{W}_2$, and so on. In general, although it appears the sets Ω_i can be computed locally, it is very hard to guarantee that such a collection of sets exists or can be detected.

To better understand the properties of the interconnected systems, especially their effects on the computation of invariant sets, it makes sense to more clearly analyze the set–iterates induced by the dynamics (3.1). As indicated by Problem 2, we are interested in the behavior of not only *one* specific initial condition $x_i(0)$ and their resulting trajectory $x_i(k)$, but instead we ultimately want to characterize *all* initial conditions in a specific set $\mathcal{S}_i$. This can be done by considering the set–iterates $\mathcal{X}_{i,k}$, which are induced by the independent set–dynamics for (3.1), given by

$$\mathcal{X}_{i,k+1} = f_i(\mathcal{X}_{i,k}) \oplus g_i(\mathcal{X}_{j,k} : j \in \mathcal{N} \setminus \{i\}), \tag{3.3}$$

where for all $i \in \mathcal{N}$ and $k \in \mathbb{N}_+$, $\mathcal{X}_{i,k} \subseteq \mathbb{R}^{n_i}$. For general results on set–dynamics and various extensions we refer to [Artstein and S. V. Raković 2008; Artstein and S. V. Raković 2011] and focus in the following on the interconnected dynamics.

Using set–dynamics, it is possible to present basic facts in a compact and a general way. For instance, the state trajectories from (3.1) can be directly linked to the set–dynamics (3.3), by considering singleton sets for the initial conditions , i.e. $\mathcal{X}_{i,0} = \{x_i(0)\}$. In addition, a collection of sets $(\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_N)$, where all $\mathcal{S}_i$ are some subsets of $\mathbb{R}^{n_i}$, is a solution to Problem 2, if $\mathcal{X}_{i,0} = \mathcal{S}_i$ implies $\mathcal{X}_{i,k} \subseteq \mathcal{X}_i$ for all $k \in \mathbb{N}_+$ and $i \in \mathcal{N}$.

In order to specify a collection of sets that solves Problem 2, we need to properly characterize the set–iterates $\mathcal{X}_{i,k}$. Unfortunately, this analysis is in general challenging, since it is necessary to determine the exact *orientation*, *size* and *shape* of the sets $\mathcal{X}_{i,k}$. In general, there is little to no hope to obtain meaningful yet computationally tractable results. For these reasons, we assume in the remainder that all $\mathcal{S}_i$ are convex, non–empty subsets of $\mathbb{R}^{n_i}$ and are not singletons. Note that in practice this is often sufficiently general.

One basic problem of the previous approach is that the interconnections should and can *not* be treated as independent *static* disturbances, as the *disturbance size* depends on the the neighbors. A proper way to address invariance for interconnected systems needs to accommodate for this fact. Furthermore, it is also very important to provide notions that are on one hand simple, on the other hand flexible enough. The main challenges in exactly describing the set iterates induced by the set–dynamics (3.3) stem from the fact, that even for convex and compact sets $\mathcal{X}_{i,k}$, the successor sets might be non–connected and non–convex, see Figure 3.2. However, convexity is important to easily perform basic necessary set operations, such as answering questions regarding set inclusions or analyzing properties of Minkowski sums of sets. In addition, it is also very difficult to decide what type of effect the interconnection have on the set–iterates $\mathcal{X}_{i,k}$. Although, we can use convex hulls of the set–iterates for the analysis, from a practical point of view, computing the convex hull for arbitrary sets is a very difficult task and may even be impossible in higher dimensions.

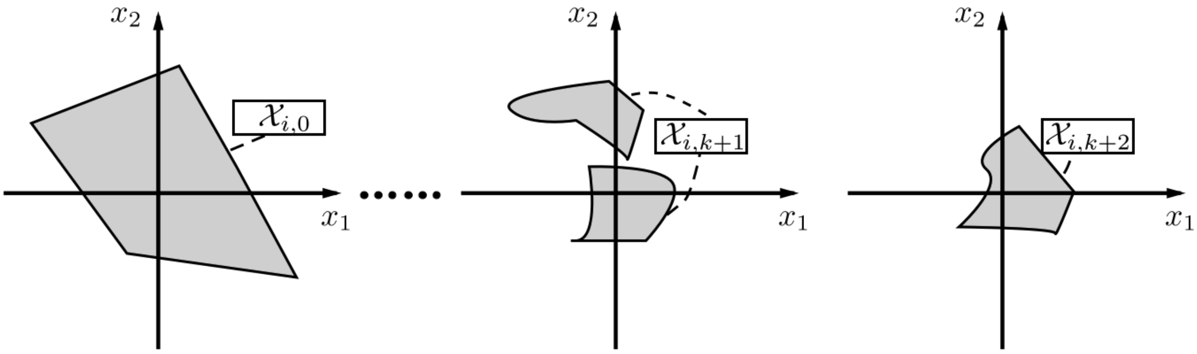


Figure 3.2: *General set–iterates $\mathcal{X}_{i,k}$.*

3.2 Positively Invariant Family of Sets

An alternative way to analyze the general set–iterates $\mathcal{X}_{i,k}$ is to find a flexible and dynamic approximation, which are easier to analyze. More precisely, consider sets $\mathcal{S}_{i,k}$ such that $\mathcal{X}_{i,k} \subseteq \mathcal{S}_{i,k}$, whenever the set of initial conditions is $\mathcal{X}_{i,0} = \mathcal{S}_{i,0}$, cf. Figure 3.3. If such sets have the property $\mathcal{S}_{i,k} \subseteq \mathcal{X}_i$ for all $k \in \mathbb{N}_+$, then the initial sets $\mathcal{S}_{i,0}$ provide a solution to Problem 2.

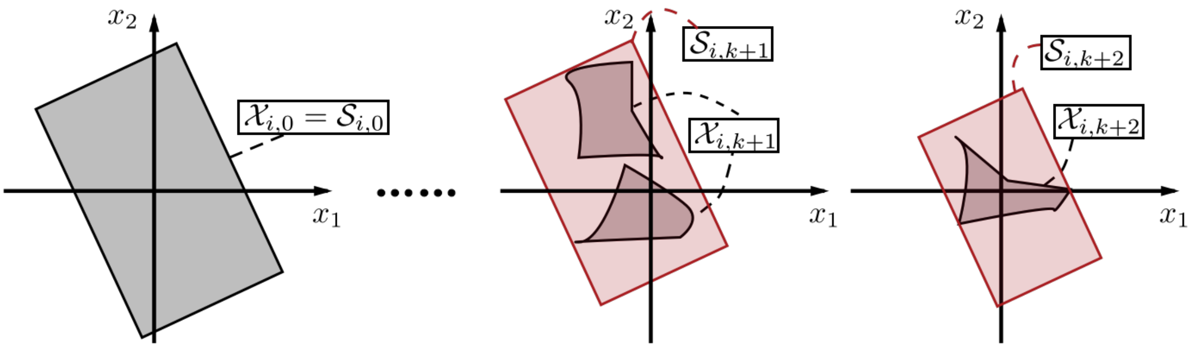


Figure 3.3: *Approximated set–iterates $\mathcal{X}_{i,k}$ via $\mathcal{S}_{i,k}$.*

Obviously, it is desirable that the inclusion $\mathcal{X}_{i,k} \subseteq \mathcal{S}_{i,k}$ is tight, since otherwise one might introduce a lot of conservatism. Another requirement is the ability to easily adjust and change the "size" of the sets $\mathcal{S}_{i,k}$, such that it can be directly linked to the dynamics of $\mathcal{X}_{i,k}$. In particular, if a set $\mathcal{X}_{i,k}$ gets smaller, then $\mathcal{S}_{i,k}$ should shrink similarly.

Parametrized families of sets allow us to flexibly handle these type of requirements. In particular, we focus on the following type of parametrization.

Definition 5 (Parametrized family of sets)**.** *A parametrized family of sets over* $\Theta \subseteq \mathbb{R}^N$ *for the collection* $\mathcal{S} = \{\mathcal{S}_i \subset \mathbb{R}^{n_i} : i \in \mathcal{N}\}$ *is given by*

$$\mathbb{S}(\mathcal{S}, \Theta) := ((\theta_1 \mathcal{S}_1, \theta_2 \mathcal{S}_2, \ldots, \theta_N \mathcal{S}_N) : (\theta_1, \theta_2, \ldots, \theta_N) \in \Theta). \tag{3.4}$$

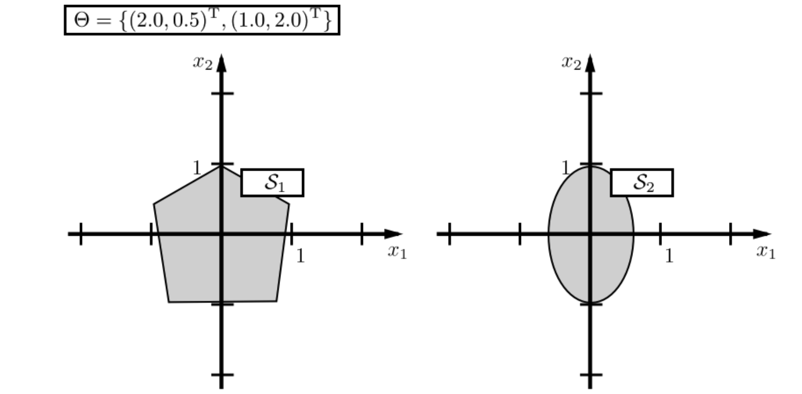
A parametrized family of sets over Θ is basically a family of set, which is constructed by scaling the collection $\{\mathcal{S}_i : i \in \mathcal{N}\}$ over scalars $(\theta_1, \theta_2, \ldots, \theta_N)$, that are picked from the set Θ. If the cardinality of the set Θ and N is finite, we obtain a finite number of scaled sets $\theta_i \mathcal{S}_i$, as illustrated in Figure 3.4. If Θ has infinite members, we obtain infinite numbers of scaled sets $\theta_i \mathcal{S}_i$. The basic idea of this approach is to find shapes $\mathcal{S}_i$, that have "nice" properties for the maps $f_i(\cdot)$ and $g_i(\cdot)$, ideally such that the set $f_i(\theta_i \mathcal{S}_i)$ can be easily characterized for different values of θ_i.

Note that different kind of parametrization for the sets $\mathcal{S}_i$ are plausible, for instance homothetic parametrization or general nonlinear parametrization. Depending on the system structure different kind of parametrization might be more beneficial then others, for instance see also [Gielen, Lazar, and Teel 2012; S. V. Raković, Gielen, and Lazar 2012] for parametrization of a family of sets for time–delay systems.

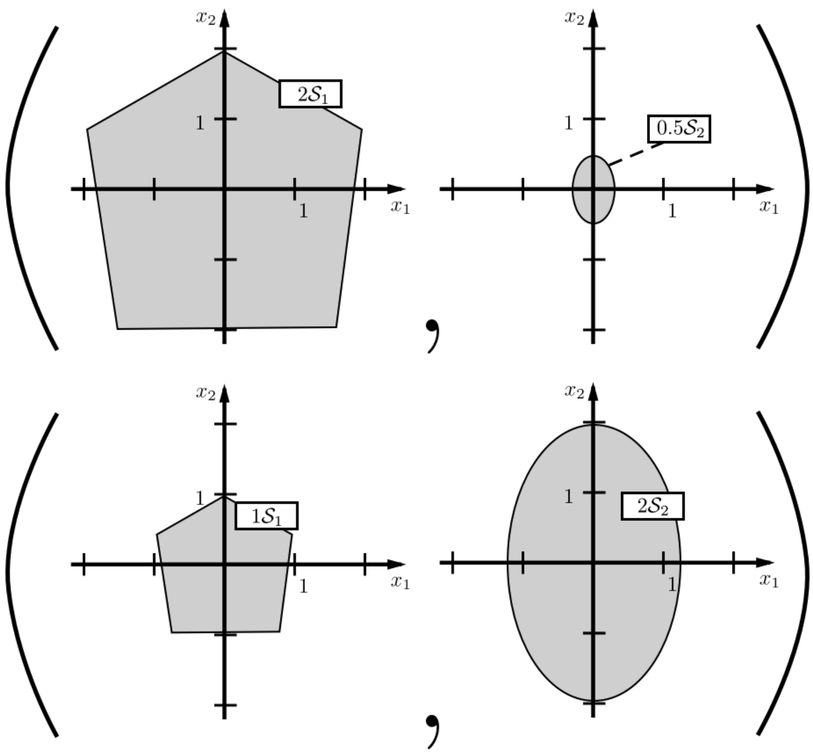
A main goal for using a specific parametrization is the intention to make the problem more manageable. However, choosing general types of parametrization for Problem 2, might further increase the degree of complexity for the analysis. For these reasons we decided to restrict ourselves to the simple form of parametrized sets given in Definition 5.

Motivated by the concept of invariant collection of sets in Definition 4, we present now an adapted notion that describes invariance for a collection of sets scaled over a set Θ. Instead of keeping the collection $\{\mathcal{S}_i : i \in \mathcal{N}\}$ fixed, we allow the sets to vary over time. We achieve these variations through a set of admissible scaling factors Θ. The basic idea is to show that this admissible set of scaling factors has a property that relates to positive invariance and can be used to upper approximate the exact set–dynamics (3.3).

Definition 6 (Invariance for a parametrized family of sets over Θ)**.** *Given a collection of sets* $\mathcal{S} = \{\mathcal{S}_i \subset \mathbb{R}^{n_i} : i \in \mathcal{N}\}$ *and a set* $\Theta \subset \mathbb{R}^N_+$*. The parametrized family of sets* $\mathbb{S}(\mathcal{S}, \Theta)$*, specified in* (3.4) *is a* positively invariant family of sets *for the system* (3.1) *if for all* $\theta = (\theta_1, \theta_2, \ldots, \theta_N)^T \in \Theta$ *and all* $i \in \mathcal{N}$ *there exists* $\theta^+ = (\theta_1^+, \theta_2^+, \ldots, \theta_N^+)^T \in \Theta$*,*



(a) Θ *and sets* $\mathcal{S}_1$ *and* $\mathcal{S}_2$*.*



(b) *Two realizations of parametrized sets* $\theta_i \mathcal{S}_i$*.*

Figure 3.4: *Parametrized family of sets.*

such that $x_i(k) \in \theta_i \mathcal{S}_i \subseteq \mathcal{X}_i$ *implies* $x_i(k+1) \in \theta_i^+ \mathcal{S}_i \subseteq \mathcal{X}_i$*.*

Note, that Definition 6 is a generalization to the concept of positive invariance towards a collection of sets. For instance, given a positively invariant collection of sets $\mathcal{S}$, we can easily construct a positively invariant family of sets $\mathbb{S}(\mathcal{S}, \Theta)$, by choosing $\Theta =$

$\{(1,1,\ldots,1)^{\mathrm{T}}\} \subset \mathbb{R}^N$. Obviously, in that case $\mathbb{S}(\mathcal{S},\{(1,1,\ldots,1)^{\mathrm{T}}\}) = \mathcal{S}$, and hence this particular set Θ induces positive invariance for the family of sets. The basic difference to the concept of positive invariance for a collection of sets can be easily exemplified if we consider a set Θ, that has only a finite amount of members. As an example consider two interconnected systems,

$$x_1(k+1) = f_1(x_1(k)) + g_1(x_2(k)) \tag{3.5}$$

$$x_2(k+1) = f_2(x_2(k)) + g_2(x_1(k)), \tag{3.6}$$

the collection of sets $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2)$ and a set with two members $\Theta = \{(\bar{\theta}_1, \bar{\theta}_2)^{\mathrm{T}}, (\hat{\theta}_1, \hat{\theta}_2)^{\mathrm{T}}\}$. In order to show, that the collection of sets $\mathcal{S}$ is invariant, according to Definitions 4 we need to ensure that $f_1(\mathcal{S}_1) \oplus g_1(\mathcal{S}_2) \subseteq \mathcal{S}_1$ and $f_2(\mathcal{S}_2) \oplus g_2(\mathcal{S}_1) \subseteq \mathcal{S}_2$. As explained before, this leads to very restrictive conditions, i.e. basically the maps $f_i(\cdot)$ need to strongly contract the sets $\mathcal{S}_i$, i.e. $f_i(\mathcal{S}_i) \subseteq \mu_i \mathcal{S}_i$, such that roughly $\mu_1 \mathcal{S}_1 \oplus g_1(\mathcal{S}_2) \subseteq \mathcal{S}_1$ and $\mu_2 \mathcal{S}_2 \oplus g_2(\mathcal{S}_1) \subseteq \mathcal{S}_1$, see Figure 3.5. Hence, the contraction factors $0 < \mu_i < 1$, need to be generally small which restrict the applicable system classes. Also note that μ_i needs to be smaller than one, otherwise the sets would not contract.

By using parametrized family of sets, we can gain a more flexible notion of invariance, since we are not enforcing the successor sets to be static. As an example, consider the parametrized set $\mathbb{S}(\mathcal{S},\Theta) = \{(\bar{\mathcal{S}}_1, \bar{\mathcal{S}}_2), (\hat{\mathcal{S}}_1, \hat{\mathcal{S}}_2)\}$, where $\bar{\theta}_i \mathcal{S}_i = \bar{\mathcal{S}}_i$ and $\hat{\theta}_i \mathcal{S}_i = \hat{\mathcal{S}}_i$ for $i \in \{1,2\}$. In order to show that $\mathbb{S}(\mathcal{S},\Theta)$ is a positively invariant family of sets, we need to first ensure, according to Definition 6 that either

$$f_1(\bar{\mathcal{S}}_1) \oplus g_1(\bar{\mathcal{S}}_2) \subseteq \bar{\mathcal{S}}_1, \quad f_2(\bar{\mathcal{S}}_2) \oplus g_2(\bar{\mathcal{S}}_1) \subseteq \bar{\mathcal{S}}_2 \tag{3.7}$$

or

$$f_1(\bar{\mathcal{S}}_1) \oplus g_1(\bar{\mathcal{S}}_2) \subseteq \hat{\mathcal{S}}_1, \quad f_2(\bar{\mathcal{S}}_2) \oplus g_2(\bar{\mathcal{S}}_1) \subseteq \hat{\mathcal{S}}_2 \tag{3.8}$$

are satisfied and additionally either

$$f_1(\hat{\mathcal{S}}_1) \oplus g_1(\hat{\mathcal{S}}_2) \subseteq \hat{\mathcal{S}}_1, \quad f_2(\hat{\mathcal{S}}_2) \oplus g_2(\hat{\mathcal{S}}_1) \subseteq \hat{\mathcal{S}}_2 \tag{3.9}$$

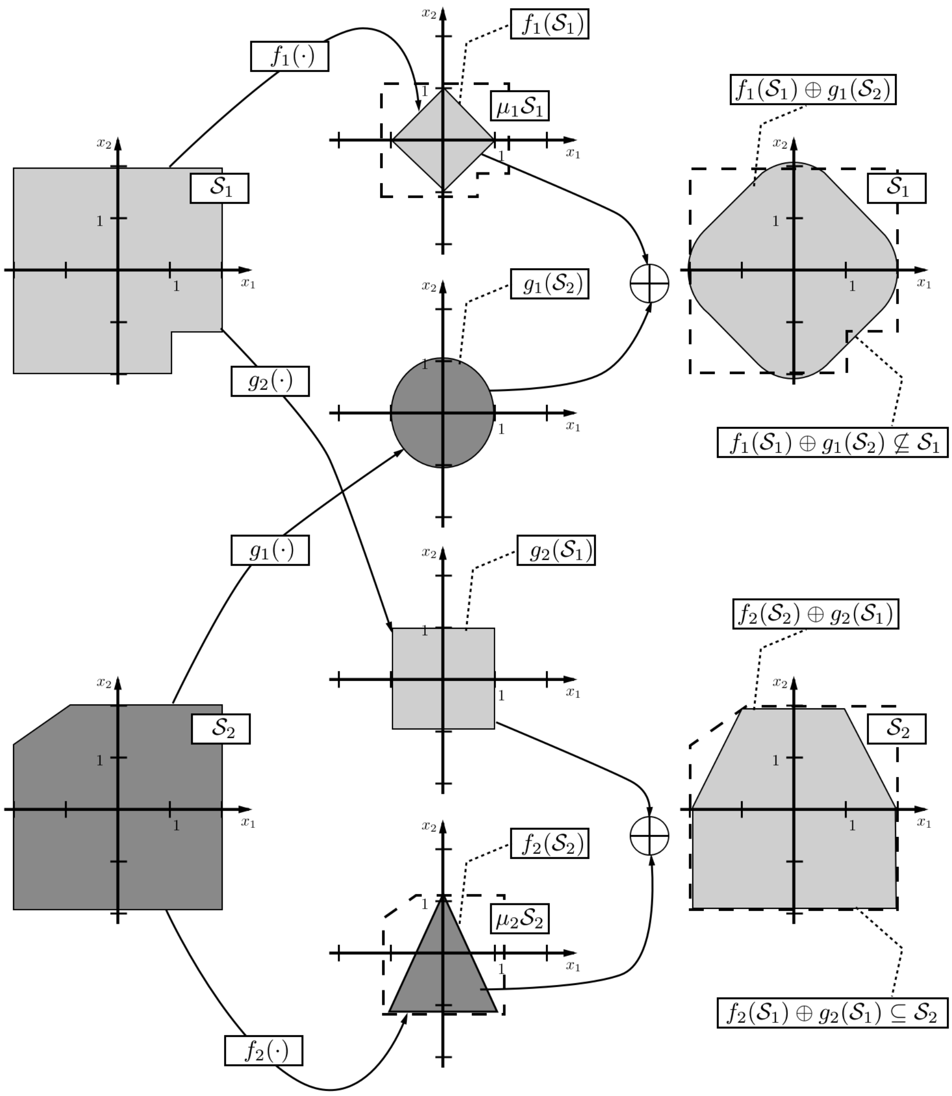


Figure 3.5: *Using a collection of sets $(\mathcal{S}_1, \mathcal{S}_2)$ for invariance. The images of the sets $(\mathcal{S}_1, \mathcal{S}_2)$ (on the left side) are taken over the functions $f_1(\cdot)$, $f_2(\cdot)$, $g_1(\cdot)$ and $g_2(\cdot)$ (depicted in the middle). On the right site the Minkowski sum of those images are compared to the sets $(\mathcal{S}_1, \mathcal{S}_2)$.*

or

$$f_1(\hat{\mathcal{S}}_1) \oplus g_1(\hat{\mathcal{S}}_2) \subseteq \bar{\mathcal{S}}_1, \quad f_2(\hat{\mathcal{S}}_2) \oplus g_2(\hat{\mathcal{S}}_1) \subseteq \bar{\mathcal{S}}_2 \tag{3.10}$$

are satisfied. What makes the analysis for the invariance of parametrized family of sets appealing, is the fact that we conceptually allow successor sets $\theta_i^+\mathcal{S}_i$ to be larger then the sets $\theta_i\mathcal{S}_i$, or in other words for some values $(\theta_1, \theta_2, \ldots, \theta_N)^{\mathrm{T}} \in \Theta$, we have

$\theta_i \mathcal{S}_i \subseteq \theta_i^+ \mathcal{S}_i$, where $(\theta_1^+, \theta_2^+, \dots, \theta_N^+)^{\mathrm{T}} \in \Theta$. As a result we have more degrees of freedom, which allows us to easier capture the interplay of beneficial and non–beneficial effects onto the subsystems.

3.2.1 Indirect Construction for Positively Homogeneous Systems

We intend to approximate the set–iterates $\mathcal{X}_{i,k}$ with approximated sets $\theta_i \mathcal{S}_i$, cf. Figure 3.3. The basic idea is to capture dynamic properties of the induced set–dynamics by changing the scalar values θ_i, i.e. $\mathcal{X}_{i,k} \subseteq \theta_i(k)\mathcal{S}_i$. In order to get a good approximation, it is important to understand the mathematical properties involved in the construction of the sets $\mathcal{X}_{i,k+1}$ based on the knowledge of the precursor sets $\mathcal{X}_{i,k}$, such that we can properly find at every time step k a good tight as possible scaling factor $\theta_i(k)$. Nevertheless, we need to link the dynamic behavior of the interconnected system, appropriately to those scaling factors $\theta_i(k)$. Similar to using comparison functions for Vector Lyapunov functions, cf. [Bellman 1962; Šiljak 1978; Lakshmikantham, Matrosov, and Sivasundaram 1991], we can describe the transition from $\theta(k)$ to $\theta(k+1)$ by a function $\mu : \mathbb{R}_+^N \to \mathbb{R}_+^N$, such that $\theta(k+1) = \mu(\theta(k))$. Motivated by Definition 6, a possible choice for the function $\mu(\cdot)$ is defined as follows:

$$\mu(\mathcal{S}, \theta) = \begin{pmatrix} \mu_1(\mathcal{S}, \theta) \\ \mu_2(\mathcal{S}, \theta) \\ \vdots \\ \mu_N(\mathcal{S}, \theta) \end{pmatrix} := \begin{pmatrix} \min_{\mu_1 \geq 0} \{ f_1(\theta_1 \mathcal{S}_1) \oplus g_1(\theta_j \mathcal{S}_j : j \in \mathcal{N} \setminus \{1\}) \subseteq \mu_1 \mathcal{S}_1 \} \\ \min_{\mu_2 \geq 0} \{ f_2(\theta_2 \mathcal{S}_2) \oplus g_2(\theta_j \mathcal{S}_j : j \in \mathcal{N} \setminus \{2\}) \subseteq \mu_2 \mathcal{S}_2 \} \\ \vdots \\ \min_{\mu_N \geq 0} \{ f_N(\theta_N \mathcal{S}_N) \oplus g_N(\theta_j \mathcal{S}_j : j \in \mathcal{N} \setminus \{N\}) \subseteq \mu_N \mathcal{S}_N \} \end{pmatrix} \tag{3.11}$$

where $\theta = (\theta_1, \theta_2, \dots, \theta_N)^{\mathrm{T}} \in \mathbb{R}_+^N$ and $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_N)$ with $\mathcal{S}_i \subseteq \mathbb{R}^{n_i}$ for all $i \in \mathcal{N}$. Essentially, we capture in equation (3.11) for every i–th subsystem the variation of sets by a minimal *dynamic* scaling factor μ_i.

As we can see from Definition 6 is important to find the set of scaling factors Θ for a given collection of sets $\mathcal{S}$, that leads to a positively invariant family of sets. Using the dynamics (3.11) we can specify such a set in a compact way. Furthermore, in order to guarantee that the function $\mu(\mathcal{S}, \cdot)$ is well defined, we demand the following:

Assumption 1. *For all* $i \in \mathcal{N}$

- $f_i(\cdot)$ *and* $g_i(\cdot)$ *are continuous,*

- $f_i(0) = 0$ *and* $g_i(0) = 0$,
- $\mathcal{X}_i$ *is compact and contains the origin in its interior,*
- $\mathcal{S}_i$ *is convex, compact and contains the origin in its interior.* [1]

We need the former assumption to guarantee that the functions defined in (3.11) are continuous on its domain and can be easily utilized for the analysis.

Lemma 1. *Suppose Assumption 1 is true, then* $\mu_i(\mathcal{S}, \theta)$ *as defined in* (3.11) *are continuous functions for all* $\theta \in \mathbb{R}^N_+$ *and all* $i \in \mathcal{N}$.

Proof. In order to prove continuity, we use its Definition in terms of the limits of sequences. In particular, a function $f(\cdot)$ is continuous at $\bar{x}$ if every converging sequence $x(n)$ implies that the sequence of functions $f(x(n))$ converges to $f(\bar{x})$, i.e. $\lim_{n\to\infty} x(n) = \bar{x} \Rightarrow \lim_{n\to\infty} f(x(n)) = f(\bar{x})$. Let $\theta(n) = (\theta_1(n), \theta_2(n), \ldots, \theta_N(n))^{\mathrm{T}}$, $\bar{\theta} = (\bar{\theta}_1, \bar{\theta}_2, \ldots, \bar{\theta}_N)^{\mathrm{T}}$, and $\lim_{n\to\infty} \theta(n) = \bar{\theta}$. Consider the following sets:

$$\begin{aligned} \Xi_{n,i} &= \{f_i(\theta_i(n)x_i) + g_i(\theta_j(n)x_j \,:\, j \in \mathcal{N} \setminus \{i\}) : x_i \in \mathcal{S}_i, \forall i \in \mathcal{N}\}, \\ \bar{\Xi}_i &= \{f_i(\bar{\theta}_i x_i) + g_i((\bar{\theta}_j x_j \,:\, j \in \mathcal{N} \setminus \{i\}) : x_i \in \mathcal{S}_i, \forall i \in \mathcal{N}\}. \end{aligned}$$

Continuity of $f_i(\cdot)$ and $g_i(\cdot)$ implies that for all fixed $(x_1, x_2, \ldots, x_N)^{\mathrm{T}} \in \mathcal{S}_1 \times \mathcal{S}_2 \times \ldots \times \mathcal{S}_N$

$$\lim_{n\to\infty} f_i(\theta_i(n)x_i) + g_i(\theta_j(n)x_j \,:\, j \in \mathcal{N} \setminus \{i\}) = f_i(\bar{\theta}_i x_i) + g_i(\bar{\theta}_j x_j \,:\, j \in \mathcal{N} \setminus \{i\}) \in \bar{\Xi}_i,$$

or in other words $\lim_{n\to\infty} \Xi_{n,i} \subseteq \bar{\Xi}_i$. On the other hand, pick any $\xi_i \in \bar{\Xi}_i$, then $\xi_i = f_i(\bar{\theta}_i x_i) + g_i((\bar{\theta}_j x_j \,:\, j \in \mathcal{N} \setminus \{i\}) : x_i \in \mathcal{S}_i, \forall i \in \mathcal{N})$. Let $(y_1, y_2, \ldots, y_N)^{\mathrm{T}} \in \mathcal{S}_1 \times \mathcal{S}_2 \times \ldots \times \mathcal{S}_N$, then for all $n \in \mathbb{N}$, we have $f_i(\theta_i(n)y_i) + g_i(\theta_j(n)y_j \,:\, j \in \mathcal{N} \setminus \{i\}) \in \Xi_{n,i}$, and consequently

$$\lim_{n\to\infty} f_i(\theta_i(n)y_i) + g_i(\theta_j(n)y_j \,:\, j \in \mathcal{N} \setminus \{i\}) = f_i(\bar{\theta}_i y_i) + g_i(\bar{\theta}_j y_j \,:\, j \in \mathcal{N} \setminus \{i\}) = \xi_i.$$

Thus, $\bar{\Xi}_i \subseteq \lim_{n\to\infty} \Xi_{n,i}$ and as a result $\lim_{n\to\infty} \Xi_{n,i} = \bar{\Xi}_i$. In other words, for all $\epsilon > 0$, there exists n_0, such that $H(\mathcal{B}^{n_i}, \Xi_{n,i}, \bar{\Xi}_n) \leq \epsilon$. In turn, the definition of the Hausdorff distance implies, that for all $n \geq n_0$, $\Xi_{n,i} \subseteq \bar{\Xi}_i \oplus \epsilon\mathcal{B}^{n_i}$ and $\bar{\Xi}_i \subseteq \Xi_{n,i} \oplus \epsilon\mathcal{B}^{n_i}$, respectively,

[1] Convexity and compactness are crucial properties; for instance if $\mathcal{S}_i$ are compact, star–shaped sets it is impossible to guarantee continuity for the functions defined in (3.11)

and hence

$$\Xi_{n,i} \subseteq \bar{\Xi}_i \oplus \epsilon\mathcal{B}^{n_i} \subseteq \mu_i(\mathcal{S}, \bar{\theta})\mathcal{S}_i \oplus \epsilon\mathcal{B}^{n_i},$$

$$\bar{\Xi}_i \subseteq \Xi_{n,i} \oplus \epsilon\mathcal{B}^{n_i} \subseteq \mu_i(\mathcal{S}, \theta(n))\mathcal{S}_i \oplus \epsilon\mathcal{B}^{n_i}$$

In addition, utilizing the properties of the set $\mathcal{S}_i$ we can always find $\delta > 0$ such that $\epsilon\mathcal{B}^{n_i} \subseteq \delta\mathcal{S}_i$, which implies $\Xi_{n,i} \subseteq (\mu_i(\mathcal{S}, \bar{\theta}) + \delta)\mathcal{S}_i$, and $\bar{\Xi}_i \subseteq (\mu_i(\mathcal{S}, \theta(n)) + \delta)\mathcal{S}_i$. However, optimality of $\mu(\mathcal{S}, \cdot)$ and compactness of $\bar{\Xi}_i$ and $\Xi_{n,i}$ implies $\lim_{n\to\infty} \mu_i(\mathcal{S}, \theta(n)) \leq \mu_i(\mathcal{S}, \bar{\theta})$ and $\lim_{n\to\infty} \mu_i(\mathcal{S}, \theta(n)) \geq \mu_i(\mathcal{S}, \bar{\theta})$, respectively. Eventually we can conclude, that $\lim_{n\to\infty} \mu_i(\mathcal{S}, \theta(n)) = \mu_i(\mathcal{S}, \bar{\theta})$, proving continuity of the function $\mu(\mathcal{S}, \theta)$ for every $\theta \in \mathbb{R}^N_+$. □

Using continuity of the functions $\mu(\mathcal{S}, \cdot)$ we can now check whether a set Θ induces a positively invariant family of sets $\mathbb{S}(\mathcal{S}, \Theta)$.

Proposition 1. *Suppose Assumption 1 is true and let* $\Theta_0 := \{\theta : \forall i \in \mathcal{N}, \theta_i\mathcal{S}_i \subseteq \mathcal{X}_i\}$*. If there is a non empty set* $\Theta \subseteq \Theta_0$ *such that* $\mu(\mathcal{S}, \Theta) \subseteq \Theta$*, then the parametrized family of sets* $\mathbb{S}(\mathcal{S}, \Theta)$ *defined in* (3.4) *is positively invariant.*

Proof. By construction, since $\Theta \subseteq \Theta_0$, we know that Θ_0 is non empty and for every $(\theta_1, \theta_2, \ldots, \theta_N)^{\mathrm{T}} \in \Theta$, we have $\theta_i\mathcal{S}_i \subseteq \mathcal{X}_i$ for all $i \in \mathcal{N}$. Take any $\theta = (\theta_1, \theta_2, \ldots, \theta_N)^{\mathrm{T}} \in \Theta$, then it follows by definition of $\mu(\mathcal{S}, \cdot)$ in (3.11), that for all $i \in \mathcal{N}$

$$f_i(\theta_i\mathcal{S}_i) \oplus g_i(\theta_j\mathcal{S}_j : j \in \mathcal{N} \setminus \{i\}) \subseteq \mu_1(\mathcal{S}, \theta)\mathcal{S}_i. \tag{3.12}$$

In addition, since $\mu(\mathcal{S}, \Theta) \subseteq \Theta$, we know that $\mu_i(\mathcal{S}, \theta)\mathcal{S}_i \subseteq \mathcal{X}_i$ for all $i \in \mathcal{N}$. To summarize, let $\theta^+ = (\theta_1^+, \theta_1^+, \ldots, \theta_N^+)^{\mathrm{T}} = \mu(\mathcal{S}, \theta)$. With equation (3.12) we can see that for all $\theta \in \Theta$, there exists a $\theta^+ \in \Theta$ such that for all $i \in \mathcal{N}$, $x_i(k) \in \theta_i\mathcal{S}_i$ implies $x_i(k+1) \in \theta_i^+\mathcal{S}_i$. Eventually, since $\Theta \subseteq \Theta_0$, whenever $\theta \in \Theta$, we have $\theta_i\mathcal{S}_i \subseteq \mathcal{X}_i$ and $\theta_i^+\mathcal{S}_i \subseteq \mathcal{X}_i$ for all $i \in \mathcal{N}$ and thus $\mathbb{S}(\mathcal{S}, \Theta)$ is a positively invariant family of sets. □

The basic idea to achieve positive invariance for a family of sets is conceptually simple. With the set Θ_0, we define the set of admissible scaling factor, i.e. all scaled sets $\theta_i\mathcal{S}_i$ that lie within the constraint set $\mathcal{X}_i$. Using the function $\mu(\mathcal{S}, \cdot)$ we can approximate the dynamics of the transition from one scaling factor $\theta(k)$ to the successor $\theta(k+1)$. If there is however a set Θ inside the set of admissible scaling factors Θ_0, which is positively invariant for the dynamical systems $\theta(k+1) = \mu(\mathcal{S}, \theta(k))$, we can use this set to from a positively invariant family of sets $\mathbb{S}(\mathcal{S}, \Theta)$.

Although Proposition 1 is easier to handle from an analytical point of view, in order to characterize positively invariant family of sets more constructive methods are necessary. As outlined before, the basic idea is to find a positively invariant set Θ for the dynamics of the scaling factors $\theta(k+1) = \mu(\mathcal{S}, \theta(k))$. In the last chapter, we have discussed possible approaches for finding and analyzing positively invariant sets for general type of systems. However, there are several challenges for directly utilizing these approaches. Directly analyzing stability and invariance properties for the function $\mu(\mathcal{S}, \theta)$ is challenging, due to its nonlinear behavior. This is one reason, why it is challenging to specify a set Θ that satisfies the conditions in Proposition 1. Second of all it is still unclear, how to properly choose the collection of set $\mathcal{S}$. Thus, it is necessary to better understand how the underlying set dynamics (3.3) behave, i.e. how the sets $\mathcal{X}_{i,k+1}$ depend on their predecessor sets $\mathcal{X}_{i,k}$. They are formed by an initial transformation into the sets $f_i(\mathcal{X}_{i,k})$ and $g_i(\mathcal{X}_{j,k} : j \in \mathcal{N} \setminus \{i\})$, which are then linked together through the Minkowski sum. If we consider the set $f_i(\mathcal{X}_{i,k})$ as the *dominant* set, i.e. $g_i(\mathcal{X}_{j,k} : j \in \mathcal{N} \setminus \{i\}) \subseteq \theta f_i(\mathcal{X}_{i,k})$ where θ is a positive very *small* value, then it roughly means that $\mathcal{X}_{i,k+1}$ can be approximated by a slightly enlarged set $f_i(\mathcal{X}_{i,k})$. This type of dominance means, that the effect of the interconnection $g(\cdot)$ is small, which can be justified by the fact that often interconnections should not have a large destabilizing effect on the overall system. Basically, we would like to exploit this type of dominant behaviour for the construction of positively invariant family of sets. We note that this limits the applications but significantly simplifies the calculation.

Since the family of sets are parametrized through scaling factors, we need to understand how the maps $f_i(\cdot)$ and $g_i(\cdot)$ change the sets $\theta_i\mathcal{S}_i$ for different scaling factors θ_i. Obviously, the shape and structure of the sets $f_i(\mathcal{X}_{i,k})$, $g_i(\mathcal{X}_{j,k} : j \in \mathcal{N} \setminus \{i\})$ highly depend on the nature of the maps $f_i(\cdot)$ and $g_i(\cdot)$. For instance, for convex and compact sets $\mathcal{X}_{i,k}$, the set $f_i(\mathcal{X}_{i,k})$ can have arbitrary properties and shapes. Even worse, if we scale the sets by arbitrary scalars θ, it is not possible to predict the resulting image of the scaled sets. For instance, given some sets Ω and assume there is a set $\mathcal{Y}$, such that $f_i(\Omega) \subseteq \mathcal{Y}$, then it is not general true, that $f_i(\theta\Omega) \subseteq \theta\mathcal{Y}$ for any $\theta > 0$. Since we consider parametrization through scalars in the definition of invariance for parametrized family of sets, we need to be able to predict the properties of the $f_i(\theta_i\mathcal{S}_i)$ and $g_i(\theta_i\mathcal{S}_i : j \in \mathcal{N} \setminus \{i\})$, for different values $(\theta_1, \theta_2, \ldots, \theta_N)^{\mathrm{T}} \in \Theta$. In particular, whenever we scale the collection of sets $\mathcal{S}$ by scalar values, we expect that the image of those scaled sets by the maps over $f(\cdot)$ and $g(\cdot)$ should scale similarly. For linear systems, this property is easily verified through homogeneity. However, for nonlinear systems this is general not the case. Fortunately, we can generalize this property to positively homogeneous functions.

Definition 7. *A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be positively homogeneous of degree k, if $f(\lambda x) = \lambda^k f(x)$ for all $x \in \mathbb{R}^n$ and $\lambda > 0$.*

Monomials form homogeneous functions of degree k, for example. Some nontrivial positively homogeneous functions of degree $k = 1$, apart from linear functions, are for instance $f(x) = |x|$ and

$$f(x_1, x_2) = \begin{pmatrix} \sqrt{ax_1^2 + bx_2^2} \\ \sqrt{cx_1^2 + dx_2^2} \end{pmatrix}, \quad a, b, c, d \in \mathbb{R}_+ \text{ and } (x_1, x_2)^{\mathrm{T}} \in \mathbb{R}^2.$$

As a direct result we know that $f(\theta\Omega) = \theta f(\Omega)$ for any positively homogeneous function $f : \mathbb{R}^n \to \mathbb{R}^n$ of degree one, compact sets $\Omega \subset \mathbb{R}^n$ and scalar values $\theta > 0$. Utilizing this property we are now able to state sufficient conditions, to check whether a collection of sets $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_N)$ can be used to construct a nontrivial, positively invariant family of sets, i.e. there exists a parametrization (3.4) with a non–empty and non–trivial set Θ. The basic idea is to find a simpler to analyze approximation of the dynamical systems $\theta(k+1) = \mu(\mathcal{S}, \theta(k))$ induced by (3.11). If we restrict the class of applicable systems and constraint sets to the following

$$\Sigma_i : \quad x_i(k+1) = f_i(x_i(k)) + \sum_{j \in \mathcal{N} \setminus \{i\}} g_{i,j}(x_j(k)),\ i \in \mathcal{N}, \tag{3.13}$$

we obtain the following type of dynamics that govern the transition of the scaling factors from $\theta(k)$ to $\theta(k+1)$:

$$\mu(\mathcal{S}, \theta) = \begin{pmatrix} \mu_1(\mathcal{S}, \theta) \\ \mu_2(\mathcal{S}, \theta) \\ \vdots \\ \mu_N(\mathcal{S}, \theta) \end{pmatrix} := \begin{pmatrix} \min_{\mu_1 \geq 0} \left\{ f_1(\theta_1 \mathcal{S}_1) \bigoplus_{j \in \mathcal{N} \setminus \{1\}} g_{1,j}(\theta_j \mathcal{S}_j) \subseteq \mu_1 \mathcal{S}_1 \right\} \\ \min_{\mu_2 \geq 0} \left\{ f_2(\theta_2 \mathcal{S}_2) \bigoplus_{j \in \mathcal{N} \setminus \{2\}} g_{2,j}(\theta_j \mathcal{S}_j) \subseteq \mu_2 \mathcal{S}_2 \right\} \\ \vdots \\ \min_{\mu_N \geq 0} \left\{ f_N(\theta_N \mathcal{S}_N) \bigoplus_{j \in \mathcal{N} \setminus \{N\}} g_{N,j}(\theta_j \mathcal{S}_j) \subseteq \mu_N \mathcal{S}_N \right\} \end{pmatrix} \tag{3.14}$$

where $\theta = (\theta_1, \theta_2, \ldots, \theta_N)^{\mathrm{T}} \in \mathbb{R}_+^N$ and $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_N)$ with $\mathcal{S}_i \subseteq \mathbb{R}^{n_i}$ for all $i \in \mathcal{N}$, which are easier to handle in the remainder, given the following additional assumption for the collection of sets $(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_N)$ and constraints $(\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_N)$:

Assumption 2. *For all $i \in \mathcal{N}$*

- *$\mathcal{X}_i$ and $\mathcal{S}_i$ are compact, convex sets that contain the origin in their interior.*

Assumption 3. *For all $i \in \mathcal{N}$*

- *$f_i(\cdot)$ and $g_{i,j}(\cdot)$ in* (3.13) *are positively homogeneous and continuous functions of degree 1, for all $j \in \mathcal{N} \setminus \{i\}$,*
- *$f_i(0) = 0$ and $g_{i,j}(0) = 0$, for all $j \in \mathcal{N} \setminus \{i\}$.*

Note, that convexity restrict the type of problems we might be able to analyze, however for most applications it is still general enough, see e.g. Chapter 5 for an example. Using the preceding discussion, it is possible to impose the following properties for the function $\mu(\mathcal{S}, \cdot)$ if we consider the structure (3.13) and Assumption 2.

Proposition 2. *Suppose Assumptions 2–3 are satisfied. Then all $\mu_i(\mathcal{S}, \cdot)$ defined in* (3.14) *are positively homogeneous functions of degree* 1 *for all $i \in \mathcal{N}$.*

Proof. First note that under Assumption 2, $f_i(\theta_i \mathcal{S}_i)$ and $g_{i,j}(\theta_j \mathcal{S}_j)$ are compact sets that contain the origin in their interior, for all $\theta_i > 0, \theta_j > 0$, $i \in \mathcal{N}$ and $j \in \mathcal{N} \setminus \{i\}$. Since the origin is an interior point of $\mathcal{X}_i$ and $\mathcal{S}_i$ we know that $\mu_i(\mathcal{S}, \theta)$ is well defined for all $\theta \in \mathbb{R}^N_+$, $\mu_i(\mathcal{S}, \cdot) : \mathbb{R}^N_+ \to \mathbb{R}_+$ and $\mu_i(\mathcal{S}, 0) = 0$. For any $\lambda > 0$ and any $\theta = (\theta_1, \theta_2, \ldots, \theta_N)^{\mathrm{T}} \in \mathbb{R}^N_+$ we have

$$f_i(\lambda\theta_i \mathcal{S}_i) \bigoplus_{j \in \mathcal{N} \setminus \{i\}} g_{i,j}(\lambda\theta_j \mathcal{S}_j) = \lambda f_i(\theta_i \mathcal{S}_i) \bigoplus_{j \in \mathcal{N} \setminus \{i\}} \lambda g_{i,j}(\theta_j \mathcal{S}_j) = \lambda \left(f_i(\theta_i \mathcal{S}_i) \bigoplus_{j \in \mathcal{N} \setminus \{i\}} g_{i,j}(\theta_j \mathcal{S}_j) \right)$$

due to positive homogeneity and basic rules of Minkowski set–addition. By definition of (3.14) we have

$$f_i(\theta_i \mathcal{S}_i) \bigoplus_{j \in \mathcal{N} \setminus \{i\}} g_{i,j}(\theta_j \mathcal{S}_j) \subseteq \mu_i(\mathcal{S}, \theta) \mathcal{S}_i, \tag{3.15}$$

and

$$\lambda \left(f_i(\theta_i \mathcal{S}_i) \bigoplus_{j \in \mathcal{N} \setminus \{i\}} g_{i,j}(\theta_j \mathcal{S}_j) \right) \subseteq \mu_i(\mathcal{S}, \lambda\theta) \mathcal{S}_i. \tag{3.16}$$

Multiplying (3.15) on both side with $\lambda > 0$ and comparing the right hand sides with (3.16), we can conclude by optimality that $\mu(\mathcal{S}, \lambda\theta) = \lambda \mu(\mathcal{S}, \theta)$. $\square$

We can see that positive homogeneity of the functions $f_i(\cdot)$ and $g_{i,j}(\cdot)$ is directly transferred to $\mu_i(\mathcal{S},\cdot)$. Furthermore, since the dynamic system $\theta(k+1) = \mu(\mathcal{S}, \theta(k))$ is defined on the positive orthant, a possible approach to analyze its system theoretical properties is to find a positive linear system, i.e. a system in which the states are non–negative for all times, that can be used as an upper approximation. A justification for using linear systems can be reasoned by the fact that they are also homogeneous, easier to analyze and if chosen properly they might exhibit a similar qualitative behaviour compared to the transition from $\theta(k)$ to $\theta(k+1)$ under the function $\mu(\mathcal{S},\cdot)$ defined in (3.14).

This idea is used in the following Theorem to obtain a posteriori check to verify that a given collection of sets $\mathcal{S}$ can be used to form a positively invariant family of sets. But first we need to assert, that the set of admissible scaling factors behaves nicely.

Lemma 2. *Suppose Assumptions 2–3 are satisfied, then the set* $\Theta_0 = \{(\theta_1, \theta_2, \ldots, \theta_N)^T \in \mathbb{R}^N_+ : \forall i \in \mathcal{N},\, \theta_i\mathcal{S}_i \subseteq \mathcal{X}_i\}$ *is a convex, compact and full-dimensional subset of* $\mathbb{R}^N_+$ *that contains the origin.*

Proof. First note, $0 \in \Theta_0$ follows trivially. Pick any $\hat{\theta} \in \Theta_0$ and $\bar{\theta} \in \Theta_0$. We have $(1-\lambda)\hat{\theta} + \lambda\bar{\theta} \in \mathbb{R}^N_+$ for all $0 \leq \lambda \leq 1$. By Assumption 2 it follows that

$$((1-\lambda)\hat{\theta} + \lambda\bar{\theta})\mathcal{S}_i = (1-\lambda)\hat{\theta}\mathcal{S}_i \oplus \lambda\bar{\theta}\mathcal{S}_i \subseteq (1-\lambda)\mathcal{X}_i \oplus \lambda\mathcal{X}_i = ((1-\lambda)+\lambda)\mathcal{X}_i = \mathcal{X}_i,$$

for all $0 \leq \lambda \leq 1$, which makes Θ_0 a convex subset of $\mathbb{R}^N_+$. Furthermore, the set Θ_0 is clearly closed and due to Assumption 2 bounded for all $i \in \mathcal{N}$. Let $\Omega = \{\eta_i e_i : i \in \mathcal{N}\} \cup \{0\}$, where $e_i \in \mathbb{R}^N$ are the unit vectors of the N-dimensional canonical basis and $\eta_i := \max_{\eta \geq 0}\{\eta \ :\ \eta S_i \subseteq \mathcal{X}_i\}$. Note, by convexity and compactness of the sets $\mathcal{X}_i$ and $\mathcal{S}_i$, we have $0 < \eta_i < \infty$. By construction, we know that $\operatorname{convh}(\Omega)$ is a full-dimensional subset of $\mathbb{R}^N_+$ and since for every $i \in \mathcal{N}$, $\omega_i \in \Theta_0$, we have $\operatorname{convh}(\Omega) \subseteq \Theta_0$ and can conclude that Θ_0 is a full-dimensional subset of $\mathbb{R}^N_+$ as well. $\square$

Theorem 8. *Suppose Assumption 2–3 are satisfied. Let* $\Theta_0 = \{(\theta_1, \theta_2, \ldots, \theta_N)^T \in \mathbb{R}^N_+ : \forall i \in \mathcal{N},\, \theta_i\mathcal{S}_i \subseteq \mathcal{X}_i\}$ *and*

$$M = \begin{pmatrix} \mu_{1,1} & \cdots & \mu_{1,N} \\ \vdots & \ddots & \vdots \\ \mu_{N,1} & \cdots & \mu_{N,N} \end{pmatrix},$$

where $f_i(\mathcal{S}_i) \subseteq \mu_{i,i}\mathcal{S}_i$, $g_{i,j}(\mathcal{S}_j) \subseteq \mu_{i,j}\mathcal{S}_i$, $\mu_{i,j} \geq 0$ and $\mu_{i,i} \geq 0$ for all $i \in \mathcal{N}$ and $j \in \mathcal{N}\setminus\{i\}$. If the origin is in the interior of Θ_0 and the matrix M is strictly stable, i.e. the spectral radius $\rho(M)$ is less than 1, then there exists a parametrized positively invariant family of sets for the collection $\mathcal{S}$.

Proof. The idea of the proof is to construct a non-trivial set $\Theta \subseteq \Theta_0$ in such a way, that for all $\theta \in \Theta$, $\mu(\mathcal{S}, \theta) \in \Theta$, similar to Proposition 1. We are not directly utilizing the system induced by the function (3.14), but instead approximate the dynamics by a linear, positive system.

Let

$$\theta(k+1) = \begin{pmatrix} \theta_1(k+1) \\ \vdots \\ \theta_N(k+1) \end{pmatrix} = \begin{pmatrix} \sum_{i\in\mathcal{N}} \mu_{1,i}\theta_i(k) \\ \vdots \\ \sum_{i\in\mathcal{N}} \mu_{N,i}\theta_i(k) \end{pmatrix} = M\theta(k). \tag{3.17}$$

From Lemma 1 we know that Θ_0 is convex, compact and a full-dimensional subset of $\mathbb{R}^N$ and whenever $\theta(k) \in \Theta_0$ then $\theta_i(k)\mathcal{S}_i \subseteq \mathcal{X}_i$. Thus, due to the fact that $\rho(M) < 1$, we can infer that there exists some non–trivial positively invariant region $\Theta \subseteq \Theta_0$, such that $\theta(0) \in \Theta$ implies $\theta(k) \in \Theta \subseteq \Theta_0$ for all $k > 0$. By construction, we know that all entries of M are non–negative, and hence $\theta(0) \in \mathbb{R}^N_+$ implies $\theta(k) \in \mathbb{R}^N_+$ for all $k > 0$. Hence, by positive homogeneity and the fact that $\theta(k) \in \mathbb{R}^N_+$, we know for all $i \in \mathcal{N}$ and $j \in \mathcal{N} \setminus \{i\}$ and every $\theta_i(k) \geq 0$, $\theta_j(k) \geq 0$, that $\theta_i(k)f_i(\mathcal{S}_i) = f_i(\theta_i(k)\mathcal{S}_i) \subseteq \mu_{i,i}\theta_i(k)\mathcal{S}_i$ and $\theta_j(k)g_{i,j}(\mathcal{S}_j) = g_{i,j}(\theta_j(k)\mathcal{S}_j) \subseteq \mu_{i,j}\theta_j(k)\mathcal{S}_i$. Also note, that the former relations are true if $\theta_i(k) = 0$ and $\theta_j(k) = 0$, since the origin is an interior point of $\mathcal{S}_i$. Taking the sum over all $j \in \mathcal{N} \setminus \{i\}$ and $i \in \mathcal{N}$, we have

$$\begin{aligned} &f_i(\theta_i(k)\mathcal{S}_i) \bigoplus_{j\in\mathcal{N}\setminus\{i\}} g_{i,j}(\theta_j(k)\mathcal{S}_j) \subseteq \\ &\quad \mu_{i,i}\theta_i(k)\mathcal{S}_i \bigoplus_{j\in\mathcal{N}\setminus\{i\}} \mu_{i,j}\theta_j(k)\mathcal{S}_i = \sum_{j\in\mathcal{N}} \mu_{i,j}\theta_j(k)\mathcal{S}_i = \theta_i(k+1)\mathcal{S}_i. \end{aligned} \tag{3.18}$$

It follows from Equation 3.18, that whenever $x_i(k) \in \theta_i(k)\mathcal{S}_i \subseteq \mathcal{X}_i$, we have $x_i(k+1) \in \theta_i(k+1)\mathcal{S}_i \subseteq \mathcal{X}_i$, for all $i \in \mathcal{N}$ and all $\theta(k) \in \mathbb{R}^N_+$. Take any $\theta(0) \in \Theta \subseteq \Theta_0$, where Θ is positively invariant set for (3.17), then $\theta(k) \in \Theta$ for all $k > 0$. Hence by construction $x_i(k) \in \theta_i(k)\mathcal{S}_i \subseteq \mathcal{X}_i$ for all $i \in \mathcal{N}$ and $k > 0$ and as a result $\mathbb{S}(\mathcal{S}, \Theta)$ is a positively invariant family of sets. □

Note, that we need to find a relation for all the sets $f_i(\mathcal{S}_i)$ and $g_{i,j}(\mathcal{S}_j)$ with respect to a scaled set $\mathcal{S}_i$. The idea is that all the functions need to exhibit a contracting behaviour in order to guarantee the existence for a positively invariant family of sets. Furthermore, the conditions in Theorem 8 are only sufficient for the existence of a positively invariant family of sets for a given collection of sets $\mathcal{S}$.

As also pointed out in the proof, we utilize a linear upper approximation of the positively homogeneous function $\mu(\mathcal{S}, \cdot)$ defined (3.14). As a result, the approach might be conservative. Note, that linear systems are by default positive homogeneous and can therefore be used as well to establish invariance for a parametrized family of sets. Furthermore, in that case we can also see that the function $\mu(\mathcal{S}, \cdot)$ defined in (3.14) behaves like a sublinear function, and therefore the justification to use a linear system to approximate the dynamical system $\theta(k+1) = \mu(\mathcal{S}, \theta(k))$ is even more apparent. Thus we assume:

Assumption 4. *For all $i \in \mathcal{N}$ and all $j \in \mathcal{N} \setminus \{i\}$*

- *$f_i(\cdot)$ and $g_{i,j}(\cdot)$ are linear functions of compatible dimension.*

Proposition 3. *Suppose Assumptions 2 and 4 are satisfied. Then all $\mu_i(\mathcal{S}, \cdot)$ defined in* (3.14) *are sublinear for all $i \in \mathcal{N}$.*

Proof. Due to Assumption 4, we know that $f_i(\cdot)$ and $g_{i,j}(\cdot)$ are homogeneous and hence using Proposition 2, we have $\mu_i(\mathcal{S}, \lambda\theta) = \lambda\mu_i(\mathcal{S}, \theta)$ for all $\lambda > 0$ and $\theta \in \mathbb{R}^N_+$. In addition take any $\bar{\theta} \in \mathbb{R}^N_+$ and $\hat{\theta} \in \mathbb{R}^N_+$, then we have

$$
\begin{aligned}
&f_i((\bar{\theta}_i + \hat{\theta}_i)\mathcal{S}_i) \bigoplus_{j \in \mathcal{N}\setminus\{i\}} g_{i,j}((\bar{\theta}_j + \hat{\theta}_j)\theta_j \mathcal{S}_j) = \\
&\qquad f_i(\bar{\theta}_i\mathcal{S}_i) \oplus f_i(\hat{\theta}_i\mathcal{S}_i) \oplus \bigoplus_{j \in \mathcal{N}\setminus\{i\}} g_{i,j}(\bar{\theta}_j\mathcal{S}_j) \oplus \bigoplus_{j \in \mathcal{N}\setminus\{i\}} g_{i,j}(\hat{\theta}_j\mathcal{S}_j) \subseteq \mu_i(\mathcal{S}, \bar{\theta})\mathcal{S}_i \oplus \mu_i(\mathcal{S}, \hat{\theta})\mathcal{S}_i,
\end{aligned}
$$

and hence $\mu_i(\mathcal{S}, \bar{\theta} + \hat{\theta}) \leq \mu_i(\mathcal{S}, \bar{\theta}) + \mu_i(\mathcal{S}, \hat{\theta})$. □

We can check for interconnected linear systems, if a positively invariant family of sets for a collection of sets $\mathcal{S}$ exists, using Theorem 8.

3.2.2 Stability and Convergence

Although our focus is on guaranteed constraint satisfaction, i.e. issues related to invariance, it is also interesting to see that whether it is possible to obtain conclusions related to stability. In particular we consider the following interconnected systems with given structure and their induced set–dynamics

$$\mathcal{X}_{i,k+1} = f_i(\mathcal{X}_{i,k}) \bigoplus_{j\in\mathcal{N}\setminus\{i\}} g_{i,j}(\mathcal{X}_{j,k}). \tag{3.19}$$

Fortunately, for linear interconnected systems, it is possible to establish stability of the family of sets with respect to the Hausdorff distance, which measures the distance between sets.

Regarding the set–sequence induced by (3.19), it is possible to derive the following result.

Theorem 9. *Suppose Theorem 8 holds under Assumptions 2 and 4. Consider the parametrized family of sets* $\mathbb{S}(\mathcal{S},\Theta)$ *given by* (3.4) *and any set–sequence* $\mathcal{X}_{i,k}$ *generated by* (3.19) *with* $(\mathcal{X}_{1,0},\mathcal{X}_{2,0},\dots,\mathcal{X}_{N,0}) \in \mathbb{S}(\mathcal{S},\Theta)$ *for all* $i\in\mathcal{N}$ *and some collection of non-empty, convex, symmetric sets that contain the origin in their interior,* $(L_1,L_2,\dots,L_N)$*. Then, for all* $k\in\mathbb{N}_+$*,*

(i) $(\mathcal{X}_{1,k},\mathcal{X}_{2,k},\dots,\mathcal{X}_{N,k}) \in \mathbb{S}(\mathcal{S},\Theta)$*,*

(ii) $\sum_{i\in\mathcal{N}} H(L_i,\mathcal{X}_{i,k},\{0\}) \le a^k b \sum_{i\in\mathcal{N}} H(L_i,\mathcal{X}_{i,0},\{0\})$ *for some scalars* $a\in[0,1)$ *and* $b\in(0,\infty)$*,*

(iii) $\forall i\in\mathcal{N}$*,* $H(L_i,\mathcal{X}_{i,k},\{0\}) \to 0$ *as* $k\to\infty$*.*

Proof. (i) First note, that under the given Assumptions, whenever $\mathcal{X}_{i,k} \subseteq \theta_i(k)\mathcal{S}_i$, we know that

$$\mathcal{X}_{i,k+1} = f_i(\mathcal{X}_{i,k}) \bigoplus_{j\in\mathcal{N}\setminus\{i\}} g_{i,j}(\mathcal{X}_{j,k}) \subseteq f_i(\theta_i(k)\mathcal{S}_i) \bigoplus_{j\in\mathcal{N}\setminus\{i\}} g_{i,j}(\theta_j(k)\mathcal{S}_j) \subseteq \theta_i(k+1)\mathcal{S}_i,\ \forall i\in\mathcal{N}. \tag{3.20}$$

Since $\mathbb{S}(\mathcal{S},\Theta)$ is a positively invariant family of sets it follows by construction, that $(\mathcal{X}_{1,k},\mathcal{X}_{2,k},\dots,\mathcal{X}_{N,k}) \in \mathbb{S}(\mathcal{S},\Theta)$ implies $(\mathcal{X}_{1,k+1},\mathcal{X}_{2,k+1},\dots,\mathcal{X}_{N,k+1}) \in \mathbb{S}(\mathcal{S},\Theta)$. Hence, by induction we can conclude $(\mathcal{X}_{1,k},\mathcal{X}_{2,k},\dots,\mathcal{X}_{N,k}) \in \mathbb{S}(\mathcal{S},\Theta)$ for all $k\in\mathbb{N}$, since $(\mathcal{X}_{1,0},\mathcal{X}_{2,0},\dots,\mathcal{X}_{N,0}) \in \mathbb{S}(\mathcal{S},\Theta)$.

(ii) Due to Assumption 2 there exists a pair of positive, real scalars η_1 and η_2 such that, for all $i\in\mathcal{N}$, $\eta_1 L_i \subseteq \mathcal{S}_i \subseteq \eta_2 L_i$, where L_i are some non-empty, convex, symmetric sets that contain the origin in their interior. It follows, that for any $\theta = (\theta_1, \theta_2, \ldots, \theta_N)^{\mathrm{T}} \in \mathbb{R}^N_+$, we have

$$\eta_1 \theta_i L_i \subseteq \theta_i \mathcal{S}_i \subseteq \eta_2 \theta_i L_i$$

and hence

$$\eta_1 \sum_{i \in \mathcal{N}} \theta_i \leq \sum_{i \in \mathcal{N}} H(L_i, \theta_i \mathcal{S}_i, \{0\}) \leq \eta_2 \sum_{i \in \mathcal{N}} \theta_i. \tag{3.21}$$

Similar to Theorem 8, we can form the dynamics of the scaling factors as defined in (3.17). Since $\rho(M) < 1$, we know that there exists $\tilde{a} \in [0, 1)$ and $\tilde{b} \in (0, \infty)$ such that, for all $k \in \mathbb{N}$,

$$|\theta(k)|_L \leq \tilde{a}^k \tilde{b} |\theta(0)|_L, \tag{3.22}$$

where $|x|_L := \min_{\mu \geq 0}\{x \in \mu L\}$ is a induced vector norm and L is some non–empty, convex, symmetric set that contains the origin in its interior.

In addition, (3.21) implies, that there exists an additional pair of positive scalars η_3 and η_4 such that $\eta_3 |\theta|_L \leq \sum_{i \in \mathcal{N}} H(L_i, \theta_i \mathcal{S}_i, \{0\}) \leq \eta_4 |\theta|_L$. Using (3.20) and (3.22), we can conclude that there exists a pair of scalars $a \in [0, 1)$ and $b \in (0, \infty)$ such that, for all $k \in \mathbb{N}$, $\sum_{i \in \mathcal{N}} H(L_i, \mathcal{X}_{i,k}, \{0\}) \leq a^k b \sum_{i \in \mathcal{N}} H(L_i, \mathcal{X}_{i,0}, \{0\})$.

(*iii*) By (*ii*), we have $\sum_{i \in \mathcal{N}} H(L_i, \mathcal{X}_{i,k}, \{0\}) \to 0$ as $k \to \infty$ so that, $\forall i \in \mathcal{N}$, $H(L_i, \mathcal{X}_{i,k}, \{0\}) \to 0$ as $k \to \infty$. □

Using the previous Theorem, we can state now actual convergence and stability properties of the interconnected subsystems. In particular, given a positively invariant family of sets $\mathbb{S}(\mathcal{S}, \Theta)$, that satisfies the conditions in Theorem 8-9, we have for all $k \in \mathbb{N}_+$ and all $i \in \mathcal{N}$, that $x_i(0) \in \theta_i(0)\mathcal{S}_i \subseteq \mathcal{X}_i$ implies $x_i(k) \in \theta_i(k)\mathcal{S}_i \subseteq \mathcal{X}_i$ for any $(\theta_1(0), \theta_2(0), \ldots, \theta_N(0))^{\mathrm{T}} \in \Theta$, where $\theta_i(k)$ is generated by (3.17). Furthermore, the global state trajectory $x(k) = (x_1(k)^{\mathrm{T}}, x_2(k)^{\mathrm{T}}, \ldots, x_N(k)^{\mathrm{T}})^{\mathrm{T}}$ converges exponentially fast, in a stable manner. Hence, Theorem 9 implies that the origin is an exponentially stable attractor for the dynamics (3.13) subject to the state constraints $\mathcal{X}_i$ with the basin of attraction induced by and depending on the set Θ. More importantly, the individual subsystems do not require the exact knowledge of the initial conditions of the other subsystems but merely that they belong to appropriate sets; in other words the only requirement for the safe and independent operation of the dynamics (3.13) is the condition that for all $i \in \mathcal{N}$, $x_i(0) \in \theta_i(0) S_i$ for some $\theta_0 = (\theta_1(0), \theta_2(0), \ldots, \theta_N(0))^{\mathrm{T}} \in \Theta$.

3.2.3 Direct Construction for Linear Systems

As shown in the previous section, we use an outer linear approximation for the dynamics (3.11) in order to obtain a positively invariant family of sets for the positively homogeneous interconnected systems (3.13). For linear interconnected systems we could show that the accompanying $\mu(\mathcal{S},\cdot)$ dynamics are linear subhomogeneous, cf. Proposition 3. Thus, the conditions specified in Theorem 8 might lead to conservative conditions, i.e. the there might exist a larger set Θ. In order to obtain stronger results, we will utilize an approach similar to the recursive set–iteration approach exemplified in the previous chapter. As a motivation consider, the following Corollary, where we utilize the comparison function $\mu(\mathcal{S},\cdot)$ defined in (3.14) in an analogous way to the approach outlined in Theorem 7, to obtain a positively invariant family of sets $\mathbb{S}(\mathcal{S},\Theta)$.

Corollary 2. *Suppose Assumption 2 and 4 are satisfied. Let*

$$\Theta_{i+1} := \mu^{-1}(\mathcal{S},\Theta_i)\bigcap\Theta_0, \tag{3.23}$$

where $\mu_i(\mathcal{S},\cdot)$ are defined in (3.14)*, $\mu^{-1}(\cdot)$ denotes the pre–image set and the initial set $\Theta_0 = \{(\theta_1,\theta_2,\ldots,\theta_N)^T \in \mathbb{R}^N_+ : \forall i \in \mathcal{N},\, \theta_i\mathcal{S}_i \subseteq \mathcal{X}_i\}$. Assume Θ_∞ exists and is non–empty, then $\mathcal{S}$ and Θ_∞ form a positively invariant family of sets $\mathbb{S}(\mathcal{S},\Theta_\infty)$.*

Proof. According to Theorem 7, we know that $\mu(\mathcal{S},\Theta_\infty) \subseteq \Theta_\infty \subseteq \Theta_0$. Using Proposition 1 we can conclude that $\mathbb{S}(\mathcal{S},\Theta_\infty)$ is a positively invariant family of sets. □

In order to use the recursion outlined in Corollary 2, we need to be able to compute the pre–image set $\mu^{-1}(\mathcal{S},\Theta)$ for a given set Θ. However, this is even in the linear case with polytopic sets Θ nontrivial. In the remainder of this section we want to construct a similar approach to Algorithm 1 motivated by the use of the recursion (3.23). But instead of using the comparison function $\mu(\mathcal{S},\cdot)$, we intend to directly utilize the definition of positively invariant family of sets as given by Definition 6. As a motivation, let us first define the set of admissible scaling factors Θ,

$$\Theta_0 = \{(\theta_1,\theta_2,\ldots,\theta_N)^{\mathrm{T}} \in \mathbb{R}^N_+ : \forall i \in \mathcal{N},\, \theta_i\mathcal{S}_i \subseteq \mathcal{X}_i\}, \tag{3.24}$$

and assume that Assumptions 2–3 hold, which in turn induces nice properties for Θ_0 according to Lemma 2. For simplicity, let us consider the interconnected, positively homogeneous systems specified in (3.13) for a collection of sets $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_N)$ and let for all $i \in \mathcal{N}$,

$$F_i(\mathcal{S}, \theta) := f_i(\theta_i \mathcal{S}_i) \bigoplus_{j \in \mathcal{N} \setminus \{i\}} g_{i,j}(\theta_j \mathcal{S}_j), \tag{3.25}$$

$$p(\mathcal{S}, \mathcal{Y}) := \{\theta \in \mathbb{R}^N_+ : \exists (\theta_1^+, \theta_2^+, \ldots, \theta_N^+)^{\mathrm{T}} \in \mathcal{Y}, \text{ s.t.} \forall i \in \mathcal{N},\ F_i(\mathcal{S}, \theta) \subseteq \theta_i^+ \mathcal{S}_i\}. \tag{3.26}$$

The set $p(\mathcal{S}, \mathcal{Y})$ given by (3.26) has similar properties as the preimage set $\mu^{-1}(\mathcal{S}, \mathcal{Y})$. In particular, note that for all $\bar{\theta} \in p(\mathcal{S}, \Omega_0)$, there exists a $\bar{\theta}^+ \in \Omega_0$ such that $F_i(\mathcal{S}, \bar{\theta}) \subseteq \bar{\theta}_i^+ \mathcal{S}_i \subseteq \mathcal{X}_i$ for all $i \in \mathcal{N}$. The basic idea we would like to exploit is to find some set $\Theta \subseteq \Theta_0$, such that $p(\mathcal{S}, \Theta) \subseteq \Theta$. This set would induce a positively invariant family of sets $\mathbb{S}(\mathcal{S}, \Theta)$, since for every $\theta \in p(\mathcal{S}, \Theta) \subseteq \Theta$ there would exist another $\theta^+ \in \Theta$, such that $F_i(\mathcal{S}, \theta) \subseteq \theta_i^+ \mathcal{S}_i \subseteq \mathcal{X}_i$ for all $i \in \mathcal{N}$. In order to construct such a set, we can adapt the iteration given in (3.23), but instead of using the preimage set $\mu^{-1}(\mathcal{S}, \cdot)$, we exploit the some basic properties of $p(\mathcal{S}, \cdot)$. In particular, consider the following recursion

$$\Omega_{i+1} := p(\mathcal{S}, \Omega_i) \bigcap \Omega_0,\ \Omega_0 := \Theta_0. \tag{3.27}$$

Note, that all the sets Ω_i are either empty or included in Θ_0 for all $i \in \mathbb{N}$. Furthermore, if we can find an index t^* such that $\Omega_{t^*} = \Omega_{t^*+1}$, we can stop the recursion given by (3.26) and use the set Ω_{t^*} to form the positively invariant family of sets $\mathbb{S}(\mathcal{S}, \Omega_{t^*})$. For the linear case and considering a collections of non–empty, convex polytopic sets $\mathcal{S}$ and constraint sets $\mathcal{X}_i$, we can easily construct an algorithm to compute the set $p(\mathcal{S}, \Theta)$ for any non–empty, convex and polytopic set Θ. In fact, the set $p(\mathcal{S}, \Theta)$ is in that case also a polytope. In order to highlight the method, we assume in the remainder of this section the following:

Assumption 5. *For all $i \in \mathcal{N}$,*

i) $\mathcal{X}_i$ is non–empty, compact and a convex polytope given by

$$\mathcal{X}_i := \{x \in \mathbb{R}^{n_i} : \phi_{i,j}^T x \le 1,\ j = \{1, 2, \ldots, s_i\}\},$$

ii) $\mathcal{S}_i$ is non–empty, compact and a convex polytope given by

$$\mathcal{S}_i := \{x \in \mathbb{R}^{n_i} : \rho_{i,j}^T x \le 1,\ j = \{1, 2, \ldots, t_i\}.$$

First of we can show, that the set Θ_0 is in that case a polytope.

Corollary 3. *Suppose Assumption 5 holds. Then* Θ_0 *given by* (3.24) *is a convex polytope.*

Proof. Using Theorem 4, we know that $\theta_i \mathcal{S}_i \subseteq \mathcal{X}_i$ implies $s(\theta_i \mathcal{S}_i, y) \leq s(\mathcal{X}, y)$ for all $y \in \mathbb{R}^{n_i}$ and all $i \in \mathcal{N}$. Since, $\mathcal{S}_i$ are $\mathcal{X}_i$ are polytopes, this is equivalent to $s(\theta_i \mathcal{S}_i, \phi_{i,j}) \leq 1$, $\forall j \in \{1, 2, \ldots, s_i\}$. Exploiting basic properties of the support function $s(\mathcal{X}, \cdot)$, we know that this is the same as $\theta_i s(\mathcal{S}_i, \phi_{i,j}) \leq 1$, $\forall j \in \{1, 2, \ldots, s_i\}$. Compactness and the fact that $\mathcal{S}_i$ is non–empty implies the existence and boundedness of $s(\mathcal{S}_i, \phi_{i,j})$ for every $\phi_{i,j} \in \mathbb{R}^{n_i}$, which implies that

$$\Theta_0 = \left\{ \theta \in \mathbb{R}^N_+ : C\theta \leq \mathbf{1} \right\},$$

where

$$C = \begin{pmatrix} s(\mathcal{S}_1, \phi_{1,1}) & 0 & 0 & \ldots & 0 \\ s(\mathcal{S}_1, \phi_{1,2}) & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ s(\mathcal{S}_1, \phi_{1,s_1}) & 0 & 0 & \ldots & 0 \\ 0 & s(\mathcal{S}_2, \phi_{2,1}) & 0 & \ldots & 0 \\ 0 & s(\mathcal{S}_2, \phi_{2,2}) & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & s(\mathcal{S}_2, \phi_{2,s_2}) & 0 & \ldots & 0 \\ 0 & 0 & \vdots & \ldots & \vdots \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & s(\mathcal{S}_N, \phi_{N,1}) \\ 0 & 0 & 0 & \ldots & s(\mathcal{S}_N, \phi_{N,2}) \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & s(\mathcal{S}_N, \phi_{N,s_N}) \end{pmatrix} \tag{3.28}$$

Furthermore, compactness of $\mathcal{X}_i$ implies that the set Θ_0 is bounded. In conclusion, we know that the set Θ_0 is given by the intersection of a finite number of half-spaces and hence a convex polytope. □

As mentioned before, if we consider linear interconnected systems, we can show that all Ω_i given by the recursion (3.27) are convex polytopes in case of polytopic constraint sets $\mathcal{X}_i$ and sets $\mathcal{S}_i$

Corollary 4. *Suppose Assumption 4–5 are satisfied. Then* Ω_i *given by* (3.27) *is a convex polytope for any* $i \in \mathbb{N}$.

Proof. First note, that the intersection of a polyhedron and a polytope is again a polytope, since the intersection of a bounded and another arbitrary set in $\mathbb{R}^n$ is again bounded. According to Corollary 3, we know that $\Omega_0 = \Theta_0$ is a polytope. Hence, if $p(\mathcal{S}, \Omega_0)$ is a polyhedron, then it follows by induction, that Ω_i is a polytope for all $i \in \mathbb{N}$. Without loss of generality, let $\Omega_0 := \{\theta \in \mathbb{R}^N : C\theta \leq \bar{\theta}\}$, where $C \in \mathbb{R}^{m \times N}$ and $\bar{\theta} \in \mathbb{R}^m$. In addition, for notational convenience (implied by linearity) we set

$$A_{i,i}x_i := f_i(x_i) \text{ and } A_{i,j}x_j := g_{i,j}(x_j),$$

where $A_{i,i} \in \mathbb{R}^{n_i \times n_i}$ and $A_{i,j} \in \mathbb{R}^{n_i \times n_j}$ for all $i \in \mathcal{N}$ and $j \in \mathcal{N} \setminus \{i\}$. Pick any $\theta = (\theta_1, \theta_2, \ldots, \theta_N)^{\mathrm{T}} \in p(\mathcal{S}, \Omega_0)$, then using (3.26), we know that for all $i \in \mathcal{N}$ there exists $\theta^+ = (\theta_1^+, \theta_2^+, \ldots, \theta_N^+)^{\mathrm{T}} \in \Omega_0$, such that $\bigoplus_{j \in \mathcal{N}} \theta_j A_{i,j} \mathcal{S}_j \subseteq \theta_i^+ \mathcal{S}_i$. Using Theorem 4 and the fact that all $\mathcal{S}_i$ are polytopes, we can exploit the properties of the support function $s(\mathcal{S}_j, \cdot)$ to show that there exists $\theta^+ \in \Omega_0$, such that

$$\sum_{j \in \mathcal{N}} \theta_j s(\mathcal{S}_j, A_{i,j}^{\mathrm{T}} \rho_{i,k}) \leq \theta_i^+, \, \forall k \in \{1, 2, \ldots, t_i\}. \tag{3.29}$$

for all $i \in \mathcal{N}$. By Assumption, all $\mathcal{S}_j$ are compact and non–empty, hence $s(\mathcal{S}_j, A_{i,j}^{\mathrm{T}} \rho_{i,k})$ exists and is bounded. This in turn implies, that for all $\theta \in p(\mathcal{S}, \Omega_0)$ there exists $\theta^+ \in \Omega_0$, such that

$$\underbrace{\begin{pmatrix} s(\mathcal{S}_1, A_{1,1}^{\mathrm{T}}\rho_{1,1}) & s(\mathcal{S}_2, A_{1,2}^{\mathrm{T}}\rho_{1,1}) & \dots & s(\mathcal{S}_N, A_{1,N}^{\mathrm{T}}\rho_{1,1}) \\ s(\mathcal{S}_1, A_{1,1}^{\mathrm{T}}\rho_{1,2}) & s(\mathcal{S}_2, A_{1,2}^{\mathrm{T}}\rho_{1,2}) & \dots & s(\mathcal{S}_N, A_{1,N}^{\mathrm{T}}\rho_{1,2}) \\ \vdots & \vdots & \dots & \vdots \\ s(\mathcal{S}_1, A_{1,1}^{\mathrm{T}}\rho_{1,t_1}) & s(\mathcal{S}_2, A_{1,2}^{\mathrm{T}}\rho_{1,t_1}) & \dots & s(\mathcal{S}_N, A_{1,N}^{\mathrm{T}}\rho_{1,t_1}) \\ s(\mathcal{S}_1, A_{2,1}^{\mathrm{T}}\rho_{2,1}) & s(\mathcal{S}_2, A_{2,2}^{\mathrm{T}}\rho_{2,1}) & \dots & s(\mathcal{S}_N, A_{2,N}^{\mathrm{T}}\rho_{2,1}) \\ s(\mathcal{S}_1, A_{2,1}^{\mathrm{T}}\rho_{2,2}) & s(\mathcal{S}_2, A_{2,2}^{\mathrm{T}}\rho_{2,2}) & \dots & s(\mathcal{S}_N, A_{2,N}^{\mathrm{T}}\rho_{2,2}) \\ \vdots & \vdots & \dots & \vdots \\ s(\mathcal{S}_1, A_{2,1}^{\mathrm{T}}\rho_{2,t_2}) & s(\mathcal{S}_2, A_{2,2}^{\mathrm{T}}\rho_{2,t_2}) & \dots & s(\mathcal{S}_N, A_{2,N}^{\mathrm{T}}\rho_{2,t_2}) \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ s(\mathcal{S}_1, A_{N,1}^{\mathrm{T}}\rho_{N,1}) & s(\mathcal{S}_2, A_{N,2}^{\mathrm{T}}\rho_{N,1}) & \dots & s(\mathcal{S}_N, A_{N,N}^{\mathrm{T}}\rho_{N,1}) \\ s(\mathcal{S}_1, A_{N,1}^{\mathrm{T}}\rho_{N,2}) & s(\mathcal{S}_2, A_{N,2}^{\mathrm{T}}\rho_{N,2}) & \dots & s(\mathcal{S}_N, A_{N,N}^{\mathrm{T}}\rho_{N,2}) \\ \vdots & \vdots & \vdots & \vdots \\ s(\mathcal{S}_1, A_{N,1}^{\mathrm{T}}\rho_{N,t_N}) & s(\mathcal{S}_2, A_{N,2}^{\mathrm{T}}\rho_{N,t_N}) & \dots & s(\mathcal{S}_N, A_{N,N}^{\mathrm{T}}\rho_{N,t_N}) \end{pmatrix}}_{:=P} \theta \leq \underbrace{\begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & \dots & 0 \\ 0 & 0 & \vdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}}_{:=L} \theta^+. \tag{3.30}$$

Eventually, we can conclude that

$$p(\mathcal{S}, \Omega_0) := \left\{ \theta \in \mathbb{R}^N : \exists \theta^+ \in \mathbb{R}^N, \text{s.t.} \begin{pmatrix} P \\ \mathbf{0} \end{pmatrix} \theta + \begin{pmatrix} -L \\ C \end{pmatrix} \theta^+ \leq \begin{pmatrix} \mathbf{0} \\ \bar{\theta} \end{pmatrix} \right\}, \tag{3.31}$$

is a convex polyhedron, since the projection of a polyhedron onto a subspace is again a polyhedron. □

Note, that under the given Assumptions, we can easily compute the set $p(\mathcal{S}, \Omega_0)$ for some given polytopic set Ω_0 and collection of polytopic sets $\mathcal{S}$. In fact, we can see from (3.31), that it only involves solving a series of linear programs and performing a orthogonal projection onto a subspace of N coordinates, which in turn can be performed by applying the well known Fourier-Motzkin elimination algorithm, cf. [Keerthi and Gilbert 1987]. As explained before, we need an appropriate stopping criteria to stop the recursion defined in (3.27). This can be performed similarly to Algorithm 2 in which we check if there is an index t^* in which the generated polytopes Ω_i are not changing. An example of this method can be found in Chapter 5.

3.3 Robust Positively Invariant Family of Sets for Linear Systems

Conceptually, we can analyze the robust, interconnected systems defined in (3.32) in the same way as outlined in the previous section. As in most of the cases, this leads to simple and basic extensions. Only the most practical and applicable results are provided in this section, namely by considering linear systems and linear interconnections. In particular, consider the set of N interconnected, autonomous systems given for all $i \in \mathcal{N}$ as

$$\Sigma_i : \quad x_i(k+1) = A_{i,i} x_i(k) + \sum_{j \in \mathcal{N} \setminus \{i\}} A_{i,j} x_j(k) + w_i, \tag{3.32}$$

with the disturbance sets $w_i \in \mathcal{W}_i \subset \mathbb{R}^{n_i}$ and the constraint sets $\mathcal{X}_i \subset \mathbb{R}^{n_i}$. Furthermore, we have $x_i(\cdot) \in \mathbb{R}^{n_i}$, $A_{i,i} \in \mathbb{R}^{n_i \times n_i}$ and $A_{i,j} \in \mathbb{R}^{n_i \times n_j}$.

We can extend Definition 6 to handle the system class given by (3.32).

Definition 8. *Given a collection of sets* $\mathcal{S} = \{\mathcal{S}_i \subset \mathbb{R}^{n_i} : i \in \mathcal{N}\}$ *and a set* $\Theta \subset \mathbb{R}^N_+$. *The parametrized family of sets* $\mathbb{S}(\mathcal{S}, \Theta)$*, specified in* (3.4) *is a* robust positively invariant family of sets *for the system* (3.32) *if for all* $\theta = (\theta_1, \theta_2, \ldots, \theta_N)^T \in \Theta$ *and all* $i \in \mathcal{N}$ *there exists* $\theta^+ = (\theta_1^+, \theta_2^+, \ldots, \theta_N^+)^T \in \Theta$*, such that* $x_i(k) \in \theta_i \mathcal{S}_i \subseteq \mathcal{X}_i$ *implies* $x_i(k+1) \in \theta_i^+ \mathcal{S}_i \subseteq \mathcal{X}_i$ *for all* $w_i \in \mathcal{W}_i$.

Note, that the notion of robust positively invariant family of sets, is similar to the extension of robust positively invariant sets to positively invariant sets. As pointed out previously, it is possible to gain additional advantages, depending on the choice of parametrization of the family of sets as well as the choice of the collection of sets $\mathcal{S}$. Fortunately, with the simple parametrization given by (3.4), we can perform an adequate and similar analysis of the problem.

In a similar fashion, as in the non–robust case, we can use an extended type of comparison function defined in (3.11) to establish the existence of a robust positively invariant family of sets. Obviously, it is necessary to slightly adjust the functions:

$$\mu^e(\mathcal{S},\theta) = \begin{pmatrix} \mu_1^e(\mathcal{S},\theta) \\ \mu_2^e(\mathcal{S},\theta) \\ \vdots \\ \mu_N^e(\mathcal{S},\theta) \end{pmatrix} := \begin{pmatrix} \min\limits_{\mu_1 \geq 0} \left\{ A_{1,1}\theta_1\mathcal{S}_1 \oplus \bigoplus\limits_{j\in\mathcal{N}\setminus\{1\}} A_{1,j}\theta_j\mathcal{S}_j \oplus \mathcal{W}_1 \subseteq \mu_1\mathcal{S}_1 \right\} \\ \min\limits_{\mu_2 \geq 0} \left\{ A_{2,2}\theta_2\mathcal{S}_2 \oplus \bigoplus\limits_{j\in\mathcal{N}\setminus\{2\}} A_{2,j}\theta_j\mathcal{S}_j \oplus \mathcal{W}_2 \subseteq \mu_2\mathcal{S}_2 \right\} \\ \vdots \\ \min\limits_{\mu_N \geq 0} \left\{ A_{N,N}\theta_N\mathcal{S}_N \oplus \bigoplus\limits_{j\in\mathcal{N}\setminus\{N\}} A_{N,j}\theta_j\mathcal{S}_j \oplus \mathcal{W}_N \subseteq \mu_N\mathcal{S}_N \right\} \end{pmatrix} \tag{3.33}$$

In order to obtain, results that are easily verifiable and applicable, we need to assume certain conditions on the disturbance sets $\mathcal{W}_i$.

Assumption 6. *The disturbance sets $\mathcal{W}_i$ are compact, convex sets that contain the origin for all $i \in \mathcal{N}$.*

Fortunately, the properties of this extended type of comparison function defined in (3.33) is again favorably, as summarized in the following Proposition.

Proposition 4. *Suppose Assumptions 2 and 6 are true. Then all $\mu_i^e(\mathcal{S},\cdot)$ defined in* (3.33) *are convex and continuous functions for all $i \in \mathcal{N}$.*

Proof. Pick any λ such that $0 \leq \lambda \leq 1$ and any $\bar{\theta} \in \mathbb{R}_+^N$, $\hat{\theta} \in \mathbb{R}_+^N$. By properties of Minkowski set addition and definition of $\mu_i^e(\mathcal{S},\cdot)$ the relations

$$\begin{aligned} &\bigoplus_{j\in\mathcal{N}} (\lambda\bar{\theta}_j + (1-\lambda)\hat{\theta}_j)A_{i,j}\mathcal{S}_j \oplus \mathcal{W}_i = \\ &\qquad \bigoplus_{j\in\mathcal{N}} \lambda\bar{\theta}_j A_{i,j}\mathcal{S}_j \oplus \bigoplus_{j\in\mathcal{N}} (1-\lambda)\hat{\theta}_j A_{i,j}\mathcal{S}_j \oplus \mathcal{W}_i \subseteq \mu_i^e(\mathcal{S}, \lambda\bar{\theta} + (1-\lambda)\hat{\theta})\mathcal{S}_i \end{aligned} \tag{3.34}$$

hold true and, similarly, we also have:

$$\bigoplus_{j\in\mathcal{N}} \lambda\bar{\theta}_j A_{i,j}\mathcal{S}_j \oplus \lambda\mathcal{W}_i = \lambda(\bigoplus_{j\in\mathcal{N}} \bar{\theta}_j A_{i,j}\mathcal{S}_j \oplus \mathcal{W}_i) \subseteq \lambda\mu_i^e(\mathcal{S},\bar{\theta})\mathcal{S}_i, \tag{3.35}$$

and

$$\bigoplus_{j\in\mathcal{N}} (1-\lambda)\hat{\theta}_j A_{i,j}\mathcal{S}_j \oplus (1-\lambda)\mathcal{W}_i \subseteq (1-\lambda)\mu_i^e(\mathcal{S},\hat{\theta})\mathcal{S}_i. \tag{3.36}$$

By assumption the sets $\mathcal{W}_i$ are convex and compact and, hence, $\mathcal{W}_i = (1-\lambda)\mathcal{W}_i \oplus \lambda\mathcal{W}_i$

so that by utilizing (3.35) and (3.36) we obtain:

$$\begin{aligned}\bigoplus_{j\in\mathcal{N}}\lambda\bar{\theta}_j A_{i,j}\mathcal{S}_j \oplus \bigoplus_{j\in\mathcal{N}}(1-\lambda)\hat{\theta}_j A_{i,j}\mathcal{S}_j \oplus \mathcal{W}_i \subseteq \\ \lambda\mu_i^e(\bar{\theta})\mathcal{S}_i \oplus (1-\lambda)\mu_i^e(\hat{\theta})\mathcal{S}_i = ((1-\lambda)\mu_i^e(\mathcal{S},\hat{\theta}) + \lambda\mu_i^e(\mathcal{S},\bar{\theta}))\mathcal{S}_i.\end{aligned} \tag{3.37}$$

Hence, in view of (3.34) and (3.37), optimality of $\mu_i^e(\mathcal{S},\lambda\bar{\theta}+(1-\lambda)\hat{\theta})$ yields that $\mu_i^e(\mathcal{S},\lambda\bar{\theta}+(1-\lambda)\hat{\theta}) \leq \lambda\mu_i^e(\mathcal{S},\bar{\theta}) + (1-\lambda)\mu_i^e(\mathcal{S},\hat{\theta})$ verifying convexity of $\mu_i^e(\mathcal{S},\cdot)$. Continuity follows similar to the proof of continuity in Lemma 1. □

Although convexity and continuity are certainly nice properties for the collection of functions $\mu_i^e(\mathcal{S},\cdot)$, they are still bothersome to analyze from a system-theoretic point of view. Henceforth, in order to establish nicely verifiable conditions, it makes sense to use a collection of simpler functions to properly approximate the exact functions $\mu_i^e(\mathcal{S},\cdot)$ defined in (3.33). In particular, a collection of affine functions is used to find an upper approximation for the exact functions. Similar to Theorem 8, we can find conditions that guarantee the existence of a robust positively invariant family of sets.

Theorem 10. *Suppose Assumption 2 and 6 hold. Let* $\Theta_0 = \{(\theta_1,\theta_2,\ldots,\theta_N)^T \in \mathbb{R}_+^N : \forall i \in \mathcal{N},\ \theta_i\mathcal{S}_i \subseteq \mathcal{X}_i\}$,

$$M = \begin{pmatrix}\mu_{1,1} & \cdots & \mu_{1,N}\\ \vdots & \ddots & \vdots \\ \mu_{N,1} & \cdots & \mu_{N,N}\end{pmatrix} \text{ and } \alpha = \begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_N\end{pmatrix},$$

where $A_{i,j}\mathcal{S}_j \subseteq \mu_{i,j}\mathcal{S}_i$, $\mathcal{W}_i \subseteq \alpha_i\mathcal{S}_i$, $\alpha_i \geq 0$, *and* $\mu_{i,j} \geq 0$ *for all* $i \in \mathcal{N}$ *and* $j \in \mathcal{N}$. *If the matrix* M *is strictly stable, i.e.* $\rho(M) < 1$ *and* $\bar{\theta} = (I-M)^{-1}\alpha$ *is an interior point of* Θ_0, *then there exists a parametrized robust, positively invariant family of sets for the collection* $\mathcal{S}$.

Proof. Conceptually, the proof is very similar to the proof of Theorem 8, i.e. we want to construct a nontrivial set $\Theta \subseteq \Theta_0$, such that $\theta \in \Theta$ implies $\mu^e(\mathcal{S},\theta) \in \Theta$. In fact, consider the affine system

$$\theta(k+1) = \begin{pmatrix}\theta_1(k+1)\\ \vdots \\ \theta_N(k+1)\end{pmatrix} = \begin{pmatrix}\sum_{i\in\mathcal{N}}\mu_{1,i}\theta_i(k)\\ \vdots \\ \sum_{i\in\mathcal{N}}\mu_{N,i}\theta_i(k)\end{pmatrix} + \begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_N\end{pmatrix} = M\theta(k) + \alpha, \tag{3.38}$$

then clearly by construction $\theta(k) \geq 0$ for all $\theta(0) \in \mathbb{R}_+^N$ and $\theta_i(k+1) \geq \mu_i^e(\mathcal{S}, \theta(k))$ for every $\theta(k) \in \mathbb{R}_+^N$. Hence, if we can find a set $\Theta \subseteq \Theta_0$ such that $\theta(k) \in \Theta$ implies $M\theta(k) + \alpha \in \Theta$, then we know that $\mu^e(\mathcal{S}, \theta(k)) \in \Theta$ as well. According to Lemma 2, Θ_0 is a convex, compact and a full-dimensional subset of $\mathbb{R}_+^N$. Since the fixed point $\bar{\theta}$ is an interior point of Θ_0, we know that the set $\Theta_0 \oplus \{-\bar{\theta}\}$ has the origin is an interior point, and is also convex, compact and a full-dimensional subset of $\mathbb{R}_+^N$. Furthermore, since M is strictly stable we can use the change of coordinates $\theta^*(k) = \theta(k) - \bar{\theta}$ to show similarly to the proof of Theorem 8, that there exists a nontrivial, positively invariant set Θ^*, such that $\theta^*(k) \in \Theta^*$ implies $M\theta^*(k) \in \Theta^* \subseteq \Theta_0 \oplus \{-\bar{\theta}\}$. It follows, that

$$M\theta^*(k) + \alpha = M\theta(k) - M\bar{\theta} + \alpha \in \Theta^* \oplus \{\alpha\} \subseteq \Theta_0 \oplus \{-\bar{\theta} + \alpha\},$$

and therefore

$$M\theta(k) + \alpha \in \Theta^* \oplus \{M\bar{\theta} + \alpha\} = \Theta^* \oplus \{\bar{\theta}\} \subseteq \Theta_0\{-\bar{\theta} + M\bar{\theta} + \alpha\} = \Theta_0.$$

Let, $\Theta := \{\bar{\theta}\} \oplus \Theta^*$, then we know that Θ is nontrivial, $\Theta \subseteq \Theta_0$ and $\theta(k) \in \Theta$ implies $M\theta(k) + \alpha \in \Theta$ and as a result $\mu^e(\mathcal{S}, \theta(k)) \in \Theta$ for every $\theta(k) \in \Theta$.

Using the fact, that $A_{i,j}\mathcal{S}_j \subseteq \mu_{i,j}\mathcal{S}_j$, we can show that $A_{i,j}\theta_j(k)\mathcal{S}_j \subseteq \mu_{i,j}\theta_j(k)\mathcal{S}_j$ for every $\theta(k) \in \mathbb{R}_+^N$ and all $i \in \mathcal{N}$ and $j \in \mathcal{N}$. Hence, by taking the sum over all $j \in \mathcal{N}$ we have

$$\bigoplus_{i \in \mathcal{N}} A_{i,j}\theta_j(k)\mathcal{S}_j \oplus \mathcal{W}_i \subseteq \bigoplus_{j \in \mathcal{N}} \mu_{i,j}\theta_j(k)\mathcal{S}_i \oplus \alpha_i\mathcal{S}_i = (\sum_{j \in \mathcal{N}} \mu_{i,j}\theta_j(k) + \alpha_i)\mathcal{S}_i = \theta_i(k+1)\mathcal{S}_i.$$

Eventually, we can argue similar as in the proof of Theorem 8 and conclude that there exists a robust positively invariant family of sets $\mathbb{S}(\mathcal{S}, \Theta)$. □

Convergence and Stability

Similar to the non–robust case, we can characterize the qualitative behavior of the global trajectories induced by the N interconnected systems defined in (3.32) subject to the collection of state constraints $(\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_N)$. We will exploit the θ–dynamics induced by the dynamics specified (3.33) and employ the Hausdorff distance as a measure. In particular, consider the set–dynamics for all $i \in \mathcal{N}$

$$\mathcal{X}_{i,k+1} = A_{i,i}\mathcal{X}_{i,k} \bigoplus_{j\in\mathcal{N}} A_{i,j}\mathcal{X}_{j,k} \oplus \mathcal{W}_i. \tag{3.39}$$

induced by (3.32). We proceed to demonstrate how the stability properties of the θ–dynamics in (3.33) can be utilized to obtain guaranteed robust stability properties of the exact induced, independent set–dynamics in (3.39) as well as the original set of N systems specified in (3.32). In particular, we have the following:

Theorem 11. *Suppose Theorem 10 holds under Assumptions 2 and 6. Consider the parametrized family of sets $\mathbb{S}(\mathcal{S},\Theta)$ given by (3.4), the fixed point $\bar{\theta} = (I-M)^{-1}\alpha$ and any set–sequence $\mathcal{X}_{i,k}$ generated by (3.39) with $(\mathcal{X}_{1,0},\mathcal{X}_{2,0},\ldots,\mathcal{X}_{N,0}) \in \mathbb{S}(\mathcal{S},\Theta)$ for all $i\in\mathcal{N}$. Then, for all $k\in\mathbb{N}$,*

- *(i)* $(\mathcal{X}_{1,k},\mathcal{X}_{2,k},\ldots,\mathcal{X}_{N,k}) \in \mathbb{S}(\mathcal{S},\Theta)$,
- *(ii)* $\sum_{i\in\mathcal{N}} H(L_i,\theta_i(k)\mathcal{S}_i,\bar{\theta}_i\mathcal{S}_i) \leq a^k b \sum_{i\in\mathcal{N}} H(L_i,\theta_i(0)\mathcal{S}_i,\bar{\theta}_i\mathcal{S}_i)$ *for some scalars* $a\in[0,1)$ *and* $b\in(0,\infty)$,
- *(iii)* $\forall i\in\mathcal{N}$, $H(L_i,\theta_i(k)\mathcal{S}_i,\bar{\theta}_i\mathcal{S}_i) \to 0$ *as* $k\to\infty$.
- *(iv)* $\forall i\in\mathcal{N}$, $h(L_i,\mathcal{X}_{i,k},\bar{\theta}_i\mathcal{S}_i) \to 0$ *as* $k\to\infty$.

Proof. (i) Similarly, to the the non–robust case and under the given assumptions, $\mathcal{X}_{i,k} \subseteq \theta_i(k)\mathcal{S}_i$ implies for all $i\in\mathcal{N}$,

$$\mathcal{X}_{i,k+1} = A_{i,i}\mathcal{X}_{i,k} \bigoplus_{j\in\mathcal{N}\setminus\{i\}} A_{i,j}\mathcal{X}_{j,k} \oplus \mathcal{W}_i \subseteq A_{i,i}\theta_i(k)\mathcal{S}_i \bigoplus_{j\in\mathcal{N}\setminus\{i\}} A_{i,j}\theta_j(k)\mathcal{S}_j \oplus \mathcal{W}_i \subseteq \theta_i(k+1)\mathcal{S}_i. \tag{3.40}$$

Furthermore, $\mathbb{S}(\mathcal{S},\Theta)$ is a robust positively invariant family of sets and therefore we have that $(\mathcal{X}_{1,k},\mathcal{X}_{2,k},\ldots,\mathcal{X}_{N,k}) \in \mathbb{S}(\mathcal{S},\Theta)$ implies $(\mathcal{X}_{1,k+1},\mathcal{X}_{2,k+1},\ldots,\mathcal{X}_{N,k+1}) \in \mathbb{S}(\mathcal{S},\Theta)$. But, $(\mathcal{X}_{1,0},\mathcal{X}_{2,0},\ldots,\mathcal{X}_{N,0}) \in \mathbb{S}(\mathcal{S},\Theta)$ and hence the claim follows by induction.

(ii) By assumption, $\mathcal{S}_i$ are compact and convex sets that contain the origin in their interior for all $i\in\mathcal{N}$. Thus, there exist scalars $\eta_1\in(0,\infty)$, $\eta_2\in(0,\infty)$, such that $\eta_1 L_i \subseteq S_i \subseteq \eta_2 L_i$, where L_i are some compact, symmetric and convex sets that contain the origin in their interior. If $\bar{\theta}_i \leq \theta_i$, then trivially $\bar{\theta}_i\mathcal{S}_i \subseteq \theta_i\mathcal{S}_i$ and

$$\bar{\theta}_i\mathcal{S}_i \oplus \eta_1(\theta_i-\bar{\theta}_i)L_i \subseteq \bar{\theta}_i\mathcal{S}_i \oplus (\theta_i-\bar{\theta}_i)\mathcal{S}_i = \theta_i\mathcal{S}_i \subseteq \bar{\theta}_i\mathcal{S}_i \oplus \eta_2(\theta_i-\bar{\theta}_i)L_i.$$

Similarly, if $\theta_i \leq \bar{\theta}_i$ then $\theta_i\mathcal{S}_i \subseteq \bar{\theta}_i\mathcal{S}_i$ and

$$\begin{aligned}\theta_i\mathcal{S}_i \oplus \eta_1(\theta_i - \bar{\theta}_i)L_i = \theta_i\mathcal{S}_i \oplus \eta_1(\bar{\theta}_i - \theta_i)L_i \subseteq \theta_i\mathcal{S}_i \oplus (\bar{\theta}_i - \theta_i)\mathcal{S}_i = \bar{\theta}_i\mathcal{S}_i \subseteq \\ \theta_i\mathcal{S}_i \oplus \eta_2(\bar{\theta}_i - \theta_i)L_i = \theta_i\mathcal{S}_i \oplus \eta_2(\theta_i - \bar{\theta}_i)L_i.\end{aligned}$$

By the definition of the Hausdorff distance it follows that:

$$\eta_1|\theta_i - \bar{\theta}_i| \leq H(L_i, \theta_i\mathcal{S}_i, \bar{\theta}_i\mathcal{S}_i) \leq \eta_2|\theta_i - \bar{\theta}_i|.$$

Summing over $i \in \mathcal{N}$ we obtain:

$$\eta_1 \sum_{i\in\mathcal{N}} |\theta_i - \bar{\theta}_i| \leq \sum_{i\in\mathcal{N}} H(L_i, \theta_i\mathcal{S}_i, \bar{\theta}_i\mathcal{S}_i) \leq \eta_2 \sum_{i\in\mathcal{N}} |\theta_i - \bar{\theta}_i|.$$

Since $\rho(M) < 1$, there exist scalars $\tilde{a} \in [0, 1)$ and $\tilde{b} \in (0, \infty)$ such that:

$$|\theta(k) - \bar{\theta}|_L \leq \tilde{a}^k\tilde{b}|\theta(0) - \bar{\theta}|_L,$$

where $|x|_L := \min_{\mu\geq 0}\{x \in \mu L\}$ is a induced vector norm and L is some non–empty, convex, symmetric set that contains the origin in its interior. Furthermore, there exists scalars $\eta_3 \in (0, \infty)$ and $\eta_4 \in (0, \infty)$ such that:

$$\eta_3|\theta - \bar{\theta}|_L \leq \sum_{i\in\mathcal{N}} H(L_i, \theta_i S_i, \bar{\theta}_i S_i) \leq \eta_4|\theta - \bar{\theta}|_L.$$

Eventually, by using the preceding relations, we can always find scalars $a \in [0, 1)$ and $b \in (0, \infty)$ such that

$$\sum_{i\in\mathcal{N}} H(L_i, \theta_i(k)\mathcal{S}_i, \bar{\theta}_i\mathcal{S}_i) \leq a^k b \sum_{i\in\mathcal{N}} H(L_i, \theta_i(0)\mathcal{S}_i, \bar{\theta}_i\mathcal{S}_i).$$

(iii) By (ii), we have $\sum_{i\in\mathcal{N}} H(L_i, \theta_{(k,i)}S_i, \bar{\theta}_i S_i) \to 0$ as $k \to \infty$ and therefore it follows $H(L_i, \theta_i(k)\mathcal{S}_i, \bar{\theta}_i\mathcal{S}_i) \to 0$ as $k \to \infty$ for all $i \in \mathcal{N}$.

(iv) Convergence with respect to the Hausdorff upper semi-distance $h(L, \cdot, \cdot)$ follows immediately from (i), (iii) and (3.40). □

The knowledge of a robust positively invariant family of sets $\mathbb{S}(\mathcal{S}, \Theta)$, satisfying the conditions in Theorem 11, allow us to analyze the convergence properties of the state trajectories $x_i(k)$ given by (3.32) for all $k \in \mathbb{N}$ and $i \in \mathcal{N}$. In fact, whenever $x_i(0) \in \theta_i(0)\mathcal{S}_i$, for some $(\theta_1(0), \theta_2(0), \ldots, \theta_N(0))^{\mathrm{T}} \in \Theta$, we have $x_i(k) \in \theta_i(k)\mathcal{S}_i$, for all $k \in \mathbb{N}$ and all $w_i \in \mathcal{W}_i$, where $\theta_i(k)$ is generated by (3.38). Since, the definition of a robust positively invariant family of sets guarantees that $\theta_i(k) \in \mathcal{X}_i$ for all $k \in \mathbb{N}$, i.e. the state constraints are respected for all times $k > 0$. In addition, Theorem 11 (iii)-(iv) implies that the states $x_i(k)$ will converge to the set $\bar{\theta}_i\mathcal{S}_i$ exponentially fast. An example application of this method can be found in Chapter 5.

3.4 Summary

We could show, that the concept of invariant regions benefits from the use of parametrized family of sets, when the analysis of interconnected systems is concerned. By introducing the concept of invariant family of sets, we provided a strong analytical tool, allowing us to specify flexible regions in the state space, that lead to guaranteed constraint satisfaction, even with respect to additive, bounded disturbances. In addition, we presented easily verifiable conditions, guaranteeing existence of nontrivial, positively invariant family of sets for nonlinear interconnected systems, that are positively homogeneous of degree one. For linear interconnected systems, we also presented algorithms for the determination of these family of sets. Furthermore we showed, that under specific circumstances strong convergence properties can be guaranteed.

4 Control Synthesis using Positively Invariant Family of Sets

The main focus of this chapter is to extend the methodology we provided in the previous chapter, by incorporating contrained input controls and employing different control strategies. We have outlined how to analyze autonomous and interconnected systems, with the objective to achieve state constraint satisfaction and robust stability. Obviously, if we consider control systems with additional input constraints, this leads to a more complicated setup. For instance, consider the set of N linear interconnected discrete–time, time–invariant, systems given by:

$$\Sigma_i : \qquad x_i(k+1) = A_i x_i(k) + B_i u_i(k) + \sum_{j \in \mathcal{N} \setminus \{i\}} g_{i,j}(x_j(k)), \tag{4.1}$$

where $\forall i \in \mathcal{N},\ x_i(\cdot) \in \mathbb{R}^{n_i}$ is the current state of the i^{th} subsystem, $u_i(\cdot) \in \mathbb{R}^{m_i}$ is the current control of the i^{th} subsystem, $x(k) = (x(k)_1^{\mathrm{T}}, x(k)_2^{\mathrm{T}}, \ldots, x(k)_N^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^n$ with $n = \sum_{i \in \mathcal{N}} n_i$ is the current state of the overall system, $u(k) = (u(k)_1^{\mathrm{T}}, u(k)_2^{\mathrm{T}}, \ldots, u(k)_N^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^m$ with $m = \sum_{i \in \mathcal{N}} m_i$ is the current control of the overall system. In addition, we have for each $i \in \mathcal{N},\ A_i \in \mathbb{R}^{n_i \times n_i}$, $B_i \in \mathbb{R}^{n_i \times m_i}$ and the interconnections $g_{i,j}(\cdot) : \mathbb{R}^{n_j} \to \mathbb{R}^{n_i}$.

Furthermore, the subsystem variables $x_i(\cdot) \in \mathbb{R}^{n_i}$ and $u_i(\cdot) \in \mathbb{R}^{m_i}$ are subject to hard constraints, namely:

$$\forall i \in \mathcal{N},\ x_i(\cdot) \in \mathcal{X}_i \text{ and } u_i(\cdot) \in \mathcal{U}_i, \tag{4.2}$$

where $\forall i \in \mathcal{N},\ \mathcal{X}_i \subseteq \mathbb{R}^{n_i}$ and $\mathcal{U}_i \subseteq \mathbb{R}^{m_i}$ are the state and control constraint sets for the i^{th} subsystem. Similarly, to the autonomous case we could augment the subsystems into one big global control system Σ and try to design a global controller and one positively invariant set. For instance in the linear case, e.g. when $g_{i,j}(\cdot)$ are linear functions, we could use the approach summarized in Theorem 6 to analyze one linear $u(k) = Kx(k)$ to find one positively invariant set $\mathcal{S}$ subject to the dynamics of one *large* process $\Sigma : x(k+1) = Ax(k)+Bu(k)$. Despite the complexity issues outlined, we have an additional problem by applying this centralized approach. In fact, we obtain one global control law, i.e. the matrix K is possible dense, and hence global state information needs to be transmitted to a central entity for the stabilization of the complete process. This is, however, often not desirable. For instance a decentralized operation might be more robust against changes in the interconnection structure or losses in the other subsystems.

Enforcing a specific structure in the K matrix, i.e. exploiting that only certain informations can be used in the local controllers, is, at least for the constrained case challenging and can be obtained only in special cases, see for instance [Rotkowitz and Lall 2006] for an in depth discussion. As outlined we would like to obtain a *collection* of feedback controllers instead and a related modular design procedure, not requiring to consider a overall description of the complete interconnected systems. Even though we might not achieve the same performance as the global solution, we can gain more flexibility from this modular design.

One of the main objectives in this chapter is to exemplify, how positively invariant family of sets can provide a design tool for the given setup. In particular, as argued in the autonomous case, besides state constraint satisfaction, we want to be able to easily handle input constraint satisfaction in a decentralized or distributed way, i.e. with only partial knowledge of the overall interconnected plant, avoiding a centralized solution. This is done by introducing a trade–off between the local and global design aspects for decentralized or distributed controllers of interconnected systems. As a result we gain flexibility without introducing too much conservatism. To achieve this goal, we assume for the constraint sets and the system dynamics of the non–interconnected systems the following:

Assumption 7. *For each $i \in \mathcal{N}$,*

(i) the matrix pairs (A_i, B_i) are controllable, and,

(ii) $\mathcal{X}_i$ and $\mathcal{U}_i$ are polytopes given by

$$\mathcal{X}_i := \{x \in \mathbb{R}^{n_i} : \phi_{i,j}^T x \leq 1, \forall j = \{1, 2, \ldots, s_i\}\},$$
$$\mathcal{U}_i := \{u \in \mathbb{R}^{m_i} : \psi_{i,j}^T u \leq 1, \forall j = \{1, 2, \ldots, t_i\}\}.$$

Assuming controllability allows us to present the remaining results in a more straightforward way. The advantage of using polytopic constraint sets $\mathcal{X}_i$ and $\mathcal{U}_i$ allows us to flexibly approximate general compact and convex sets sufficiently, cf. [Schneider 1993]. For the subsequent analysis we will mainly consider the cases, where the interconnections are either linear or positively homogeneous functions.

4.1 Distributed Linear Control Systems subject to Linear Interconnections

For the distributed case, we restrict our attention to linear interconnected systems. We assume that (4.1) can be written in the following way:

$$\Sigma_i : x_i(k+1) = A_i x(k) + B_i u_i(k) + \sum_{j \in \mathcal{N} \setminus \{i\}} G_{i,j} x_j(k), \tag{4.3}$$

where for all $i \in \mathcal{N}$ and $j \in \mathcal{N} \setminus \{i\}$, $G_{i,j} \in \mathbb{R}^{n_i \times n_j}$.

The main objective of this section, is to 1) examine practical set invariance and stability notions for the set of discrete–time, time–invariant, linear control systems of (4.3), and 2) obtain a local controller design, that renders the overall system stable. In particular we want to show how the concept of positively invariant family of sets can be properly extended and applied. Mainly, we want to analyze two types of structures for the employed controllers. In the first case we assume that each of the subsystems Σ_i has only partial knowledge of the overall plant for the synthesis of the local controllers. More precisely, for any $i \in \mathcal{N}$ and at any time instance $k \in \mathbb{N}$, the current state $x_i(k)$ of the subsystem Σ_i and the value of the total sum $\sum_{j \in \mathcal{N} \setminus \{i\}} G_{i,j} x_j(k)$ is know to the i^{th} decision maker for the synthesis of the control action $u_i(\cdot)$. In particular, the decision maker of subsystem Σ_i has no knowledge of the exact state information $x_j(k)$ of the other subsystems Σ_j or the individual summands $G_{i,j} x_j(k)$. In the second case we want to assume, that for any $i \in \mathcal{N}$ and at any time instance k, the individual summands $G_{i,j} x_j(k)$ are known to the decision maker of the subsystem Σ_i, when deciding on the control action $u_i(\cdot)$.

Note, that this is a reasonable scenario, since complete knowledge of individual state information for the i^{th} subsystem would imply, that all the state information need to be transmitted at every instance k. Often this is not feasible or desirable, as explained in the previous chapters, and therefore we focus on a decentralized or distributed operations. Also note, that the decision maker for each of the subsystem Σ_i has no knowledge of the exact state information of the other subsystems Σ_j. This also implies, that it is in general not possible to reconstruct $x_j(k)$ from either $G_{i,j}x_j(k)$ or $\sum_{j\in\mathcal{N}\setminus\{i\}} G_{i,j}x_j(k)$.

Due to linearity of the overall interconnected system, we restrict ourselves to linear feedbacks. Furthermore, as mentioned previously, the decision maker of the subsystem Σ_i has either knowledge of the individual interconnection effects or of the cumulative interconnection effects. Under these consideration it makes sense to employ linear feedbacks that counteracts these effects by either employing one or several additional gains. In particular, we consider two types of control structures. The first one is given by,

$$u_i(k) = K_i x_i(k) + L_i \sum_{j\in\mathcal{N}\setminus\{i\}} G_{i,j}x_j(k),\ \forall i \in \mathcal{N} \tag{4.4}$$

with $K_i \in \mathbb{R}^{n_i\times n_i}$, $L_i \in \mathbb{R}^{n_i\times n_i}$ and the second one is given by

$$u_i(k) = K_i x_i(k) + \sum_{j\in\mathcal{N}\setminus\{i\}} L_{i,j}G_{i,j}x_j(k),\ \forall i \in \mathcal{N} \tag{4.5}$$

with $K_i \in \mathbb{R}^{n_i\times n_i}$ and $L_{i,j} \in \mathbb{R}^{n_i\times n_i}$ for all $j \in \mathcal{N}\setminus\{i\}$. Note, that in (4.4) we are looking for a pair of gains (K_i, L_i) as compared to several gains $(K_i, (L_{i,j} : j \in \mathcal{N}\setminus\{i\}))$ in (4.5).

As mentioned, we want to investigate how the concept of positively invariant family of sets can be properly applied to the interconnected control system specified in (4.3) with the control structures given by (4.4) and (4.5), respectively. In particular, we consider the closed loop form given by

$$x_i(k+1) = A_{i,i}x_i(k) + \sum_{j\in\mathcal{N}\setminus\{i\}} A_{i,j}x_j(k), \tag{4.6}$$

where for all $i \in \mathcal{N}$, $A_{i,i} := A_i + B_iK_i$, $\forall j \in \mathcal{N}\setminus\{i\}$, in the case of control structure (4.4), $A_{i,j} := (I + B_iL_i)G_{i,j}$, while in case of control structure (4.5), $A_{i,j} := (I + B_iL_{i,j})G_{i,j}$. With these definitions, we utilize the form (4.6) for the analysis throughout the remainder of this section.

We considered so far only autonomous systems, that are subject to state constraints only. Recall from Proposition 1, that we used Θ_0 to form a set of admissible θ that obey the state constraints. In other words, whenever we pick a θ from this set, we know that the scaled set $\theta_i\mathcal{S}_i$ is within the state constraint $\mathcal{X}_i$. By employing the $\mu(\mathcal{S},\cdot)$–dynamics, see Lemma 1, and determining a positively invariant set Θ within this set Θ_0, i.e $\mu(\mathcal{S},\Theta) \subseteq \Theta \subseteq \Theta_0$, we can simply guarantee that Θ together with the collection of sets $\mathcal{S}$ form a positively invariant family of sets. Clearly, in order to enforce additional input constraint satisfaction, we simply need to adjust the set Θ_0 in such a way, that it defines the set of all valid θ that obey the state and input constraints. In that case, we can use the same argumentation to provide a parametrized collection of sets $\mathbb{S}(\mathcal{S},\Theta)$ that satisfies both input and state constraint satisfaction. This set can be characterized, if we consider the control input structures induced by (4.4)–(4.5):

$$\Theta_0 = \{(\theta_1, \theta_2, \ldots, \theta_N)^{\mathrm{T}} \in \mathbb{R}_+^N :, \forall i \in \mathcal{N}, \theta_i \mathcal{S}_i \subseteq \mathcal{X}_i,\ K_i \theta_i \mathcal{S}_i \oplus \bigoplus_{j \in \mathcal{N} \setminus \{i\}} K_{i,j} \theta_j \mathcal{S}_j \subseteq \mathcal{U}_i\}, \tag{4.7}$$

where $K_{i,j} := L_i G_{i,j}$ in case of control structure (4.5) and $K_{i,j} := L_{i,j} G_{i,j}$ in case of control structure (4.6), respectively. Similarly, to the previous chapter, convexity and compactness of the set Θ_0 simplifies the subsequent characterization of positively invariant family of sets.

Lemma 3. *Suppose Assumption 2 and 7 are satisfied, then the set* Θ_0 *as defined in* (4.7) *is a convex, compact and full-dimensional, proper subset of* $\mathbb{R}_+^N$ *that contains the origin.*

Proof. The proof follows the lines of the proof of Lemma 2 and therefore only a sketch is given here. Trivially, we have $0 \in \Theta_0$. Pick any $\hat{\theta} \in \Theta_0$ and $\bar{\theta} \in \Theta_0$. We have $(1-\lambda)\hat{\theta} + \lambda\bar{\theta} \in \mathbb{R}_+^N$ for all $0 \leq \lambda \leq 1$. By Assumption 2 it follows that for all $i \in \mathcal{N}$,

$$((1-\lambda)\hat{\theta}_i + \lambda\bar{\theta}_i)\mathcal{S}_i = (1-\lambda)\hat{\theta}_i\mathcal{S}_i \oplus \lambda\bar{\theta}_i\mathcal{S}_i \subseteq (1-\lambda)\mathcal{X}_i \oplus \lambda\mathcal{X}_i = ((1-\lambda)+\lambda)\mathcal{X}_i = \mathcal{X}_i,$$

and

$$\begin{aligned} &K_i((1-\lambda)\hat{\theta}_i + \lambda\bar{\theta}_i)\mathcal{S}_i \oplus \bigoplus_{j \in \mathcal{N} \setminus \{i\}} K_{i,j}((1-\lambda)\hat{\theta}_j + \lambda\bar{\theta}_j)\mathcal{S}_j = \\ &(1-\lambda)(K_i\hat{\theta}_i\mathcal{S}_i \oplus \bigoplus_{j \in \mathcal{N} \setminus \{i\}} K_{i,j}\hat{\theta}_j\mathcal{S}_j) \oplus \lambda(K_i\bar{\theta}_i\mathcal{S}_i \oplus \bigoplus_{j \in \mathcal{N} \setminus \{i\}} K_{i,j}\bar{\theta}_j\mathcal{S}_j) \subseteq (1-\lambda)\mathcal{U}_i \oplus \lambda\mathcal{U}_i = \mathcal{U}_i \end{aligned}$$

for all $0 \leq \lambda \leq 1$, which makes Θ_0 a convex subset of $\mathbb{R}_+^N$. The fact that Θ_0 is compact and a full-dimensional subset of $\mathbb{R}_+^N$ follows analogously to the proof of Lemma 2. □

For the linear case, we can now easily extend the results presented in Theorem 8.

Corollary 5. *Suppose Assumption 2 and 7 are satisfied. Consider the set* Θ_0 *defined in* (4.7) *and let*

$$M = \begin{pmatrix} \mu_{1,1} & \cdots & \mu_{1,N} \\ \vdots & \ddots & \vdots \\ \mu_{N,1} & \cdots & \mu_{N,N} \end{pmatrix}, \tag{4.8}$$

where $A_{i,i}\mathcal{S}_i \subseteq \mu_{i,i}\mathcal{S}_i$, $A_{i,j}\mathcal{S}_j \subseteq \mu_{i,j}\mathcal{S}_i$, $\mu_{i,j} \geq 0$ *and* $\mu_{i,i} \geq 0$ *for all* $i \in \mathcal{N}$ *and* $j \in \mathcal{N} \setminus \{i\}$. *If the origin is in the interior of* Θ_0 *and the matrix* M *is strictly stable, then there exists a parametrized positively invariant family of sets* $\mathbb{S}(\mathcal{S}, \Theta)$. *In particular,* $(\theta_1, \theta_2, \ldots, \theta_N)^T \in \Theta$ *implies that whenever* $x_i(0) \in \theta_i\mathcal{S}_i$, *we have* $x_i(k) \in \mathcal{X}_i$ *and* $K_i x_i(k) + \sum_{j\in\mathcal{N}\setminus\{i\}} K_{i,j}x_j(k) \in \mathcal{U}_i$ *for all* $i \in \mathcal{N}$ *and all* $k \in \mathbb{N}$.

Proof. Follows directly from Theorem 9, the definition of positively invariant family of sets and construction of the set Θ_0 defined in (4.7). □

Corollary 5 provides a guideline to design distributed controller that induce a positively invariant family of sets. In particular, the basic idea is to design the gains K_i and $K_{i,j}$ in such a way that the matrix M given by (4.8) is asymptotically stable. In addition, upon closer inspection, we can see that the diagonal values of the matrix M can be directly linked to the stability properties of the corresponding local subsystems, while the off–diagonal elements characterize the physical interconnection effects. Hence, an approach for the design of local controllers for the subsystem Σ_i needs to incorporate these effects directly. For instance, if the off–diagonal values of the matrix M in the i^{th} row are relatively large, then the local controller for the subsystem Σ_i needs to better accommodate for these destabilizing effect.

A possible approach for the determination of the different gains is to treat the interconnections as disturbances and design controllers with certain robustness properties. Furthermore, the controller structure specified in (4.4) and (4.5) implies that we can measure this disturbance at each instance k. For these reasons and due to the fact that static and linear feedbacks should be used, it is possible to apply a well established approach from $\mathcal{H}_\infty$ optimal control and use a *Minimax* design procedure. In the remainder, we will briefly outline only the basic and to our approach relevant ideas. For more details, generalizations and an in depth discussion of the concepts, we refer to [Başar and Bernhard 1995]. As a motivation, consider the following system

$$x(k+1) = Ax(k) + Bu(k) + Dw(k), \tag{4.9}$$

where $x(\cdot) \in \mathbb{R}^n$ is the state, $u(\cdot) \in \mathbb{R}^m$ is the control input, $w(\cdot) \in \mathcal{W} \subset \mathbb{R}^n$ is a measurable disturbance within a compact set $\mathcal{W}$ and A, B and D are matrices of compatible dimension. With respect to stabilization of the system, we can interpret the whole setting as a two player game in which the objective of the input $u(\cdot)$ is to stabilize the plant, while the objective of the disturbance $w(\cdot)$ is to destabilize the plant. Since the disturbances can be measured, it makes sense for the controller to leverage from this information, i.e. the control strategy should depend on both the state as well as the disturbance. A well known approach to find a solution to this problem is to consider the following *max–min/min–max* control problem:

$$V(x) = \max_w \min_u \{x^{\mathrm{T}}Qx + u^{\mathrm{T}}u - \gamma^2 w^{\mathrm{T}}w + V(Ax + Bu + w)\}, \forall x \in \mathbb{R}^n, \tag{4.10}$$

where $Q \succ 0$ and $\gamma > 0$ and $V(\cdot)$ is unknown. It is well known, cf. [Başar and Bernhard 1995], that if (A, B) is controllable and (A, D) is observable we can find a solution $V(x) = x^{\mathrm{T}}Px$, with a positive definite matrix P and a stabilizing feedback $u = u(x, w)$. To be more precise, let

$$\tilde{V}(x, u, w) = x^{\mathrm{T}}Qx + u^{\mathrm{T}}u - \gamma^2 w^{\mathrm{T}}w + (Ax + Bu + Dw)^{\mathrm{T}}P(Ax + Bu + Dw).$$

Upon closer inspection, we can see that if $\gamma^2 I - D^{\mathrm{T}}PD \succ 0$, then $\tilde{V}(x, u, w)$ is convex in x and u for all $w \in \mathbb{R}^n$ and concave in w for all $u \in \mathbb{R}^m$, since $Q + A^{\mathrm{T}}PA \succ 0$ and $I + B^{\mathrm{T}}PB \succ 0$ due to $P \succ 0$. Using these properties we can solve (4.10) by first computing

$$\bar{V}(x, w) = \min_u \tilde{V}(x, u, w), \quad \bar{u}(x, w) = \operatorname*{argmin}_u \tilde{V}(x, u, w)$$

and

$$V^*(x) = \max_w \bar{V}(x, w), \quad \bar{w}^*(x) = \operatorname*{argmax}_w \bar{V}(x, w).$$

By simply setting $x^{\mathrm{T}}Px = V^*(x)$ and some algebraic reformulations using the Matrix Inversion Lemma, we obtain as a solution to (4.10) the generalized algebraic Riccati equation

$$P = Q + A^{\mathrm{T}}P(I + (BB^{\mathrm{T}} - \frac{1}{\gamma^2}DD^{\mathrm{T}})P)^{-1}A. \tag{4.11}$$

We have as a feedback $\bar{u}(x, w)$

$$\bar{u}(x, w) = Kx + Lw = -(I + B^{\mathrm{T}}PB)^{-1}B^{\mathrm{T}}P(Ax + Dw) \tag{4.12}$$

and the $\mathcal{H}_\infty$ optimal feedback, after some algebraic reformulations, becomes

$$u^*(x) = \bar{u}(x, w^*(x)) = K^* x = -B^\mathrm{T} P(I + (BB^\mathrm{T} - \frac{1}{\gamma^2} DD^\mathrm{T})P)^{-1} Ax. \tag{4.13}$$

Equation (4.11) can be solved by investigating the limit of the following sequence of matrices

$$P_{k+1} = Q + A^\mathrm{T} P_k (I + (BB^\mathrm{T} - \frac{1}{\gamma^2} DD^\mathrm{T})P_k)^{-1} A,\ P_0 = Q. \tag{4.14}$$

Note that under the given conditions, we can ensure that $P_k \succ 0$ for all k, cf. [Başar and Bernhard 1995] where more general properties for (4.11) and (4.14) can be found. Furthermore, we can also assert by construction that for a given $\gamma > 0$ and a solution P that satisfies (4.11) and $\gamma^2 I - D^\mathrm{T} P D \succ 0$, we have

$$\tilde{V}(x, \bar{u}(x, w), w) \leq x^\mathrm{T} P x, \text{ and } \tilde{V}(x, \bar{u}^*(x), w) \leq x^\mathrm{T} P x,\ \forall w \in W,$$

which means that we can use $V(x) = x^\mathrm{T} P x$ as a Lyapunov function and both controllers $\bar{u}(x, w)$ and $u^*(x)$ for stabilization purposes. In addition, we can also guarantee that the following performance index is satisfied if we apply the controller $u^*(x)$ to the system (4.9)

$$\begin{aligned}\min_u \max_w (x^\mathrm{T} Q x + u^\mathrm{T} u - \gamma^2 w^\mathrm{T} w + V(Ax + Bu + Dw)) = \\ \max_w \min_u (x^\mathrm{T} Q x + u^\mathrm{T} u - \gamma^2 w^\mathrm{T} w + V(Ax + Bu + Dw)),\end{aligned}$$

which implies that by utilizing $\bar{u}(x, w)$ we cannot achieve a performance that is worse than with $u^*(x)$. For more details, see [Başar and Bernhard 1995].

Returning to our setting in (4.3), the main advantage of using the *max–min* framework, in particular for designing distributed controllers to obtain positively invariant family of sets, is the ability to tune the disturbance attenuation. In particular we can modify the disturbance rejection conveniently by changing the scalar value γ. As shown in Corollary 5, we need to be able to design the controllers in such a way that the matrix M in (4.8) is asymptotically stable. As also mentioned before, the off diagonal elements correspond to the interconnection effects of the other systems, hence they can be understood as a disturbance trying to destabilize the plant. Hence, by tuning the value γ it is easier to obtain a stable matrix M and thus also a positively invariant family of sets.

As motivated, it is reasonable for the i^{th} controller to consider the uncertain system:

$$x_i(k+1) = A_i x_i(k) + B_i u_i(k) + D_i w_i(k), \tag{4.15}$$

where the disturbance $w_i(k)$ and matrix D_i are specified accordingly to our different control structures (4.4) and (4.5). In the first case we would have $D_i = I$ and $w_i(k) = \sum_{j\in\mathcal{N}\setminus\{i\}} G_{i,j}x_j(k)$, while in the second case $D_i = (I\ I\ \ldots\ I)$ and $w_i(k) = (G_{i,1}x_1(k), \ldots, G_{i,j-1}x_{j-1}(k), G_{i,j+1}x_{j+1}(k), \ldots G_{i,N}x_N(k))$. Within this framework, the i^{th} decision maker can construct the linear control rules specified in (4.4) or (4.5) by solving the local version of the *max–min* infinite–horizon control problem, specified in (4.10), where the matrices A, B, D, Q and the scalar γ are replaced by A_i, B_i, D_i, Q_i, γ_i, respectively. Under the before mentioned assumptions, for the solution of the local *max–min* infinite–horizon control problem, we would obtain for all $i \in \mathcal{N}$ the collection of value functions

$$V_i(x_i) = x_i^{\mathrm{T}} P_i x_i, \tag{4.16}$$

where P_i satisfied the local generalized Riccati equation specified in (4.11). In addition, the gains K_i, L_i and $L_{i,j}$ for the different control structures specified in (4.4) and (4.5) are then computed according to (4.12).

Now in order to properly apply the results from Corollary 5, we can use the collection of value functions specified in (4.16) to form the collection of ellipsoidal sets $\mathcal{S}$, given by

$$\mathcal{S}_i := \{x_i\ :\ x_i^{\mathrm{T}} P_i x_i \leq 1\},\ \forall i \in \mathcal{N}. \tag{4.17}$$

Furthermore, in this case, the matrix M specified in (4.8) can be easily constructed for all $i \in \mathcal{N}$ exploiting the properties of the support function, cf. Theorem 4, by evaluating the smallest non–negative scalars $\mu_{i,j}$ satisfying

$$A_{i,i}^{\mathrm{T}} P_i A_{i,i} \preceq \mu_{i,i}^2 P_i, \tag{4.18}$$

where $A_{i,i} := (A_i + B_i K_i)$, and

$$A_{(i,j)}^{\mathrm{T}} P_j A_{(i,j)} \preceq \mu_{(i,j)}^2 P_i,\ j \in \mathcal{N}\setminus\{i\} \tag{4.19}$$

where either $A_{i,j} := (I + B_i L_i)G_{i,j}$ in case of (4.4) or $A_{(i,j)} := (I + B_i L_{i,j})G_{i,j}$ in case of

(4.5).

Now in order to properly apply the results from Corollary 5, we only need to construct the set (4.7). Under Assumption 7, this set is given by

$$\Theta_0 := \{\theta \in \mathbb{R}_+^N : C\theta \leq \mathbf{1}\}, \tag{4.20}$$

where

$$C = \begin{pmatrix} s(\mathcal{S}_1, \phi_{1,1}) & 0 & \dots & 0 \\ s(\mathcal{S}_1, \phi_{1,2}) & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ s(\mathcal{S}_1, \phi_{1,s_1}) & 0 & \dots & 0 \\ 0 & s(\mathcal{S}_2, \phi_{2,1}) & \dots & 0 \\ 0 & s(\mathcal{S}_2, \phi_{2,2}) & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & s(\mathcal{S}_2, \phi_{2,s_2}) & \dots & 0 \\ 0 & 0 & \dots & \vdots \\ \vdots & \vdots & \dots & s(\mathcal{S}_N, \phi_{N-1,s_{N-1}}) \\ 0 & 0 & \dots & s(\mathcal{S}_N, \phi_{N,1}) \\ 0 & 0 & \dots & s(\mathcal{S}_N, \phi_{N,2}) \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & s(\mathcal{S}_N, \phi_{N,s_N}) \\ s(\mathcal{S}_1, K_1^{\mathrm{T}}\psi_{1,1}) & s(\mathcal{S}_2, K_{1,2}^{\mathrm{T}}\psi_{1,1}) & \dots & s(\mathcal{S}_N, K_{1,N}^{\mathrm{T}}\psi_{1,1}) \\ s(\mathcal{S}_1, K_1^{\mathrm{T}}\psi_{1,2}) & s(\mathcal{S}_2, K_{1,2}^{\mathrm{T}}\psi_{1,2}) & \dots & s(\mathcal{S}_N, K_{1,N}^{\mathrm{T}}\psi_{1,2}) \\ \vdots & \vdots & \dots & \vdots \\ s(\mathcal{S}_1, K_1^{\mathrm{T}}\psi_{1,t_1}) & s(\mathcal{S}_2, K_{1,2}^{\mathrm{T}}\psi_{1,t_1}) & \dots & s(\mathcal{S}_N, K_{1,N}^{\mathrm{T}}\psi_{1,t_1}) \\ s(\mathcal{S}_1, K_{2,1}^{\mathrm{T}}\psi_{2,1}) & s(\mathcal{S}_2, K_2^{\mathrm{T}}\psi_{2,1}) & \dots & s(\mathcal{S}_N, K_{2,N}^{\mathrm{T}}\psi_{2,1}) \\ s(\mathcal{S}_1, K_{2,2}^{\mathrm{T}}\psi_{2,2}) & s(\mathcal{S}_2, K_2^{\mathrm{T}}\psi_{2,2}) & \dots & s(\mathcal{S}_N, K_{2,N}^{\mathrm{T}}\psi_{2,2}) \\ \vdots & \vdots & \dots & \vdots \\ s(\mathcal{S}_1, K_{2,t_2}^{\mathrm{T}}\psi_{2,2}) & s(\mathcal{S}_2, K_2^{\mathrm{T}}\psi_{2,t_2}) & \dots & s(\mathcal{S}_N, K_{2,N}^{\mathrm{T}}\psi_{2,t_2}) \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ s(\mathcal{S}_1, K_{N,1}^{\mathrm{T}}\psi_{N,1}) & s(\mathcal{S}_2, K_{N,2}^{\mathrm{T}}\psi_{N,1}) & \dots & s(\mathcal{S}_N, K_N^{\mathrm{T}}\psi_{N,1}) \\ s(\mathcal{S}_1, K_{N,1}^{\mathrm{T}}\psi_{N,2}) & s(\mathcal{S}_2, K_{N,2}^{\mathrm{T}}\psi_{N,2}) & \dots & s(\mathcal{S}_N, K_N^{\mathrm{T}}\psi_{N,2}) \\ \vdots & \vdots & \vdots & \vdots \\ s(\mathcal{S}_1, K_{N,1}^{\mathrm{T}}\psi_{N,t_N}) & s(\mathcal{S}_2, K_{N,2}^{\mathrm{T}}\psi_{N,t_N}) & \dots & s(\mathcal{S}_N, K_N^{\mathrm{T}}\psi_{N,t_N}) \end{pmatrix}. \tag{4.21}$$

Note that in this case, we can analytically evaluate the support function in the following way

$$s(\mathcal{S}_i, K^{\mathrm{T}}x) := \sqrt{x^{\mathrm{T}} K P_i^{-1} K^{\mathrm{T}} x}.$$

In conclusion, we can apply directly the approach from Corollary 5 using the set Θ_0 defined in (4.20), using the scalar values from (4.18) and (4.19) to form the matrix M to design the distributed controllers given by (4.4) and (4.5), that induce a positively invariant family of sets and eventually stabilize the system, while respecting state and input constraints. Once we have found candidate feedback gains K_i and matrices P_i determining a set Θ_0 according to (4.20) involves only algebraic manipulations in a reduced space. Thus the complexity of the overall distrubuted control problem is noticably reduced. An example application of this method can be found in Chapter 5.

4.2 Decentralized Linear Control Systems with Nonlinear Interconnections

A disadvantage of the before mentioned approach is the fact that by tuning the controller parameters, we change the structure of the set Θ_0 as well as the matrix M specified in Corollary 5. This leads to nontrivial couplings, that need to be satisfied during the controller synthesis. Furthermore, in a fully distributed case it is also difficult to enforce conditions that lead to an asymptotically stable matrix M. Fortunately, in the decentralized setting, i.e. in which the controllers only use the local information for the state feedback, we can alleviate these problems. In particular, it is possible to derive feedback controllers $u_i(k) = K_i x_i(k)$ inducing a positively invariant family of sets by utilizing *LMI's*. In this section, we consider the interconnected control system specified in (4.1) and assume the following:

Assumption 8. *For all $i \in \mathcal{N}$ and $j \in \mathcal{N} \setminus \{i\}$*

(i) $g_{i,j}(\cdot)$ in (4.1) are continuous, positive homogeneous functions of degree one.

As a way to synthesize the collection of controllers in a decentralized way, we need to be able to find an worst case approximation of the possible, destabilizing interconnection effects. As explained in the previous section, we can again think of it as a type of disturbance. One way to accomplish this is to look at the map $g_{i,j}(\mathcal{X}_j)$, where $\mathcal{X}_j$ is the constraint set of the j^{th} subsystem. Since, the objective is to enforce state constraint satisfaction for each of the subsystems Σ_i, we know that each state trajectory should be ultimately part of $\mathcal{X}_j$. Hence, if we know how to bound the set $g_{i,j}(\mathcal{X}_j)$, we get an conservative but reasonable bound on the possible destabilizing interconnection effects. In particular we assume the following:

Assumption 9. *For all $i \in \mathcal{N}$ there exists $\eta_{i,j}$ such that $g_{i,j}(\mathcal{X}_j) \subseteq \eta_{i,j}\mathcal{B}_i$ for all $j \in \mathcal{N} \setminus \{i\}$, where $\mathcal{B}^{n_i} := \{x \in \mathbb{R}^{n_i \times n_i} : \sqrt{x^T x} \leq 1\}$.*

This means, we can bound the set $g_{i,j}(\mathcal{X}_j)$ by a scaled Euclidean ball $\eta_{i,j}\mathcal{B}_i$. Note, that compactness of $\mathcal{X}_i$ and continuity of $g_{i,j}(\cdot)$ essentially implies that we can always find a finite $\eta_{i,j}$. Small values of $\eta_{i,j}$ imply, that the interconnection effects from the j^{th} subsystem to the i^{th} subsystem are small as well. Hence, the off–diagonal elements in the matrix M given in Corollary 5 will also have smaller values. Furthermore, to obtain valid $\eta_{i,j}$, we can use outer approximation approaches, for instance [Borchers et al. 2009]. The restriction given by Assumption 9, are similiar to the conditions assumed in [Šiljak and Zečević 2005] that allow us to design the controllers in a modular way.

For the synthesis of the decentralized controllers and the positively invariant family of sets we will utilize ideas similiar to Corollary 5 and Theorem 8. We choose the controller for the i-th subsystem in such a way that an ellipsoidal set $\mathcal{S}_i$ exhibits a contraction with the rate of μ_i for the closed loop local system $x_i(k+1) = (A_i + B_i K_i)x_i(k)$. This can be done by a simple adaptation of the controller design discussed in Theorem 6. As a result, we obtain sets $\mathcal{S}_i$ which upper bound separately the interconnection effects with a rate of μ_i. In order to compensate for the effects appropriately the rate of contraction needs to dominate the effects of interconnections. Once we can assert that the contraction of the control system is large enough, we can use the contraction factors μ_i and μ_j to approximate the evolution of sets $\mathcal{S}_i$ with respect to the whole interconnected system and then easily describe a positively invariant family of sets. These values μ_i and μ_j correspond to the entries of the matrix M specified in Corollary 5. Enforcing stability of the matrix M is achieved by adding an additional constraint. This can be achieved by a simple test of feasibility for a optimization problem. However, we first need to provide some necessary intermediate results, allowing us to provide an approach that is computational tractable.

Lemma 4. *$f(x) := \sum_{i \in \mathcal{N}} x_i^{-0.5}$ is convex function for positive $x = (x_1, x_2, \ldots, x_N)^T \in \mathbb{R}^N$.*

Proof. The Hessian of $f(x)$ is obviously positive semidefinite whenever $x_i > 0$ and hence $f(\cdot)$ is convex on the positive orthant. □

Theorem 12. *The feasible region of the following problem*

$$\text{find} \quad Q, R, \alpha, \xi_2, \xi_3, \ldots, \xi_N \tag{4.22a}$$

$$\begin{pmatrix} Q & (AQ+BR) \\ (AQ+BR)^T & \xi Q \end{pmatrix} \succeq 0 \tag{4.22b}$$

$$Q^T = Q \succ 0 \tag{4.22c}$$

$$\forall j = \{2, 3, \ldots, N\}, \quad Q \succeq \eta_j^2 \xi_j I, \quad \xi_j > 1 \tag{4.22d}$$

$$\forall k = \{1, 2, \ldots, s\}, \quad \begin{pmatrix} Q & Q\phi_k \\ (Q\phi_k)^T & \alpha \end{pmatrix} \succeq 0 \tag{4.22e}$$

$$\forall l = \{1, 2, \ldots, t\}, \quad \begin{pmatrix} Q & R^T\psi_l \\ (R^T\psi_l)^T & \alpha \end{pmatrix} \succeq 0 \tag{4.22f}$$

$$1 \geq \alpha > 0 \tag{4.22g}$$

$$\sum_{j \in \mathcal{N} \setminus \{1\}} \xi_j^{-0.5} < 1 - \sqrt{\xi}, \tag{4.22h}$$

is convex for given $0 < \xi < 1$, $\eta_j \geq 0$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, ϕ_k *and* ψ_l *of compatible dimension.*

Proof. Convexity of the feasible region is implied by Lemma 4 and the fact that the constraints are composed of either LMI's, simple algebraic expressions or convex functions. □

Note, that (4.22) has a very similar structure to the controller determined in Theorem 6. As a consequence, we obtain a ellipsoidal set with the following properties:

Corollary 6. *Assume* (4.22) *is feasible for given* $0 < \xi < 1$, $\eta_j \geq 0$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, g_k *and* h_l *of compatible dimension. Let,* $\mathcal{S} := \{x : x^T Q^{-1} x \leq 1\}$, $\mathcal{X} := \{x : \phi_k^T x \leq 1,\ k = \{1, 2, \ldots, s\}\}$ *and* $\mathcal{U} := \{u : \psi_l^T u \leq 1,\ l = \{1, 2, \ldots, t\}\}$, *then*

i) $(A + BRQ^{-1})\mathcal{S} \subseteq \sqrt{\xi}\mathcal{S}$,

ii) $\alpha^{-0.5}\mathcal{S} \subseteq \mathcal{X}$,

iii) $RQ^{-1}\alpha^{-0.5}\mathcal{S} \subseteq \mathcal{U}$, *iv)* $\eta_j \mathcal{B}^n \subseteq \xi_j^{-0.5} \mathcal{S}$.

Proof. i)–iii) follow immediately from Theorem 6. iv) follows from (4.22d). □

From i), we can see that the controller $K = RQ^{-1}$ contract the set $\mathcal{S}$ with a ratio of $\sqrt{\xi}$. Furthermore, the scaled set $\xi_j^{-0.5}\mathcal{S}$ is a upper bound for the scaled ball $\eta_j \mathcal{B}^n$. In addition, ii) and iii) imply that $\theta\mathcal{S} \subseteq \mathcal{X}$ and $K\theta\mathcal{S} \subseteq \mathcal{U}$ for all $0 \leq \theta \leq \alpha^{-0.5}$. Combining the results, we can use (4.22) as a basic tool to construct LMI's that induce a positively invariant family of sets.

Theorem 13. *Suppose Assumption 7–9 are satisfied. Let $0 < \xi_i < 1$ for all $i \in \mathcal{N}$ and assume that the problem*

$$\textit{find} \quad Q_i, R_i, \alpha_i, (\xi_{i,j} : j \in \mathcal{N} \setminus \{i\}) \tag{4.23a}$$

$$\begin{pmatrix} Q_i & (A_iQ_i + B_iR_i) \\ (A_iQ_i + B_iR_i)^T & \xi_i Q_i \end{pmatrix} \succeq 0 \tag{4.23b}$$

$$Q_i^T = Q_i \succ 0 \tag{4.23c}$$

$$\forall j \in \mathcal{N} \setminus \{i\}, \quad Q_i \succeq \eta_{i,j}^2 \xi_{i,j} I, \quad \xi_{i,j} > 1 \tag{4.23d}$$

$$\forall k = \{1, 2, \ldots, s_i\}, \quad \begin{pmatrix} Q_i & Q_i\phi_{i,k} \\ Q_i\phi_{i,k}^T & \alpha_i \end{pmatrix} \succeq 0 \tag{4.23e}$$

$$\forall l = \{1, 2, \ldots, t_i\}, \quad \begin{pmatrix} Q_i & R_i^T\psi_{i,l} \\ (R_i^T\psi_{i,l})^T & \alpha_i \end{pmatrix} \succeq 0 \tag{4.23f}$$

$$1 \geq \alpha_i > 0 \tag{4.23g}$$

$$\sum_{j \in \mathcal{N}\setminus\{i\}} \xi_{i,j}^{-0.5} < 1 - \sqrt{\xi_i} \tag{4.23h}$$

has a solution for all $i \in \mathcal{N}$, then there exists a non–trivial set Θ and a collection of ellipsoids $\mathcal{S}_i := \{x : x^T Q_i^{-1} x \leq 1\}$ forming a positively invariant family of sets $\mathbb{S}(\mathcal{S}, \Theta)$. In particular, $(\theta_1, \theta_2, \ldots, \theta_N)^T \in \Theta$ implies that whenever $x_i(0) \in \theta_i \mathcal{S}_i$ we have $x_i(k) \in \mathcal{X}_i$ and $R_i Q_i^{-1} x_i(k) \in \mathcal{U}_i$ for all $i \in \mathcal{N}$ and $k \in \mathbb{N}$.

Proof. Note, that the proof is similar to the proof of Theorem 8. Recall from the definition of positively invariant families of sets, that we need to find a non–trivial set $\hat{\Theta}$, such that that for all $\theta(k) \in \hat{\Theta}$, we have I) $\theta_i(k)\mathcal{S}_i \in \mathcal{X}_i$, II) $K_i\theta_i(k)\mathcal{S}_i \in \mathcal{U}_i$ and III) $\forall x_i(k) \in \theta_i(k)\mathcal{S}_i \Rightarrow \exists \theta(k+1) \in \hat{\Theta}$, such that $x_i(k+1) \in \theta_i(k+1)S_i$. This relationship between $\theta(k)$, $\theta(k+1)$ will be again constructed by an auxiliary scaling set $\bar{\Theta}$ and an auxiliary dynamical system $\theta(k+1) = M\theta(k)$.

For all $i \in \mathcal{N}$ and $j \in \mathcal{N} \setminus \{i\}$, $K_i = R_i Q_i^{-1}$, $\mu_{i,j} = \xi_{i,j}^{-0.5}$, $\mu_{i,i} = \sqrt{\xi_i}$ and $\bar{\theta}_i = \alpha^{-0.5}$. Given the set of admissible $\bar{\Theta} := \{\theta \in \mathbb{R}_+^N : \theta_i \leq \bar{\theta}_i,\, i \in \mathcal{N}\}$, we can clearly see, that for every $\theta \in \bar{\Theta}$ condition I) and II) are satisfied as a result of ii) and iii) from Corollary 6. Given the auxiliary system $\theta(k+1) = M\theta(k)$, where $M = (\mu_{i,j})_{i\in\mathcal{N},j\in\mathcal{N}}$, we know that M is non–negative matrix and hence $\theta(0) \in \mathbb{R}_+^N$ implies $\theta(k) \in \mathbb{R}_+^N$ for all $k \in \mathbb{N}$.

Relation (4.23h) implies that for every $i \in \mathcal{N}$, $\sum_{j\in\mathcal{N}} \mu_{i,j} < 1$. Using the Gershgorin Circle Theorem we can infer that auxiliary system is asymptotically stable, since the maximum row–sum of M is less then one. As result there exists a non–trivial region $\hat{\Theta}$, that includes the origin and is a subset of $\bar{\Theta}$, which is positively invariant for the auxiliary system. We can infer from the structure of the feasibility problem (4.23) using Corollary 6 that for every $i \in \mathcal{N}$, $(A_i + B_i K_i)\mathcal{S}_i \subseteq \mu_{i,i}\mathcal{S}_i$ $\eta_{i,j}\mathcal{B}^{n_j} \subseteq \mu_{i,j}\mathcal{S}_i$, $\mathcal{S}_i \subseteq \mathcal{X}_i$. With Assumption 9 we get $g_{i,j}(\mathcal{X}_j) \subseteq \eta_{i,j}\mathcal{B}^{n_j} \subseteq \mu_{i,j}\mathcal{S}_i$ for all $j \in \mathcal{N} \setminus \{i\}$ and by continuity of $g_{i,j}(\cdot)$, we have $g_{i,j}(\mathcal{S}_j) \subseteq g_{i,j}(\mathcal{X}_j)$ since $\mathcal{S}_j \subseteq \mathcal{X}_j$. Due to positive homogeneity of $g_{i,j}(\cdot)$, we have for every $\theta_j > 0$, $\theta_j g_{i,j}(\mathcal{S}_j) = g_{i,j}(\theta_j\mathcal{S}_j) \subseteq \mu_{i,j}\theta_j\mathcal{S}_i$, and additionally for every $\theta_i(k) > 0, (A_i + B_i K_i)\theta_i(k)\mathcal{S}_i \subseteq \mu_{i,i}\theta_i(k)\mathcal{S}_i$. Taking the sum over all $j \in \mathcal{N} \setminus \{i\}$ and using the two former relations we obtain

$$\begin{aligned} &(A_i + B_i K_i)\theta_i(k)\mathcal{S}_i \oplus \bigoplus_{j\in\mathcal{N}\setminus\{i\}} g_{i,j}(\theta_j\mathcal{S}_j) \subseteq \\ &\mu_{i,i}\theta_i(k)\mathcal{S}_i \oplus \bigoplus_{j\in\mathcal{N}\setminus\{i\}} \mu_{i,j}\theta_j(k)\mathcal{S}_i = \left(\sum_{j\in\mathcal{N}} \mu_{i,j}\theta_j(k)\right)\mathcal{S}_i = \theta_i(k+1)\mathcal{S}_i. \end{aligned} \tag{4.24}$$

To summarize, it follows from Equation (4.24) that whenever $x_i(k) \in \theta_i(k)\mathcal{S}_i$ we have $x_i(k+1) \in \theta_i(k+1)\mathcal{S}_i$, for all $i \in \mathcal{N}$ and all $\theta \in \mathbb{R}_+^N$. Since $\theta(0) \in \hat{\Theta} \subset \bar{\Theta}$ implies by construction $\theta(k) \in \hat{\Theta}$, $\theta_i(k)\mathcal{S}_i \subseteq \mathcal{X}_i$ and $K_i\theta_i(k)\mathcal{S}_i \subseteq \mathcal{U}_i$ for all $k \in \mathbb{N}$ we can eventually assert that $\hat{\Theta}$ satisfies the conditions I)–III). $\square$

Using Theorem 13 we can design a collection of decentralized controllers $u_i(k) = K_i x(k)$ that keep the interconnected systems (4.1) within the state constraints $\mathcal{X}_i$ for all times $k \in \mathbb{N}$, while obeying the input constraints $\mathcal{U}_i$. The strict inequalities given in (4.23d) and (4.23h) are unfortunately problematic from an algorithmic point of view. However by introducing artificial, positive, sufficiently small slack variables, we can relax this conditions. Note, that the constraint (4.23h) can be further approximated in order to transform (4.23) to simple LMI's.

Lemma 5. *Given* $f(x) := \sum_{i \in \mathcal{N}} x_i^{-0.5}$, *where* $x = (x_1, x_2, \ldots, x_N)^T \in \mathbb{R}^N$, $x_i > 1$ *and some* $0 < k < 1$. *Let* $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_N)^T \in \mathbb{R}^N$, *where* $\bar{x}_i > \left(\frac{N}{k}\right)^2$ *for all* $i \in \mathcal{N}$, *then* $f(\bar{x}) < k$.

Proof. Let $\bar{x}_i = N^2 k^{-2} + \alpha_i$ with $\alpha_i \geq 0$. Hence $f(\bar{x}) := \sum_{i \in \mathcal{N}} \bar{x}_i^{-0.5} = \sum_{i \in \mathcal{N}} (N^2 k^{-2} + \alpha_i)^{-0.5} \leq \sum_{i \in \mathcal{N}} (N^2 k^{-2})^{-0.5} = k$ □

Hence, one way to transform (4.23) into simple LMI's is to replace the constraint (4.23h) by

$$\xi_{i,j} > \left(\frac{N-1}{1-\sqrt{\xi_i}} \right)^2$$

for all $i \in \mathcal{N}$ and $j \in \mathcal{N} \setminus \{i\}$. However, note that this approximation used is not optimal, i.e. the resulting positively invariant family of sets might have a smaller set of admissible scaling factors Θ. In that case the decentralized controller can only guarantee in a smaller region, that the input and state constraints for the closed loop systems are satisfied. In general, the complexity of the design process is reduced since in that case, only LMI's and not a nonlinear problem need be solved.

4.3 Summary

We could show, that the concept of positively invariant family of sets can be extended to a class of interconnected control systems. Based on this fact, we exemplified how to design decentralized and distributed controllers for positively homogeneous and linear interconnected systems, respectively. For the distributed controller, we presented an iterative design procedure exploiting generalized Riccati equation, which can be tuned to determine positively invariant family of sets. Eventually, we presented a decentralized control synthesis, based on feasibility conditions of LMI's, which can be solved independently for each of the subsystems and hence leading to a modular design procedure.

5 Control Applications

We begin by presenting the considered problem and its basic features. We utilize a distributed and decentralized control methodology in order to compute the maximal tolerable disturbances this system can exhibit with respect to state and input constraints.

5.1 Multiple Tank System

The four tank system is a classical benchmark example for evaluating decentralized and distributed control schemes. The basic scheme is depicted in Figure 5.1. The system

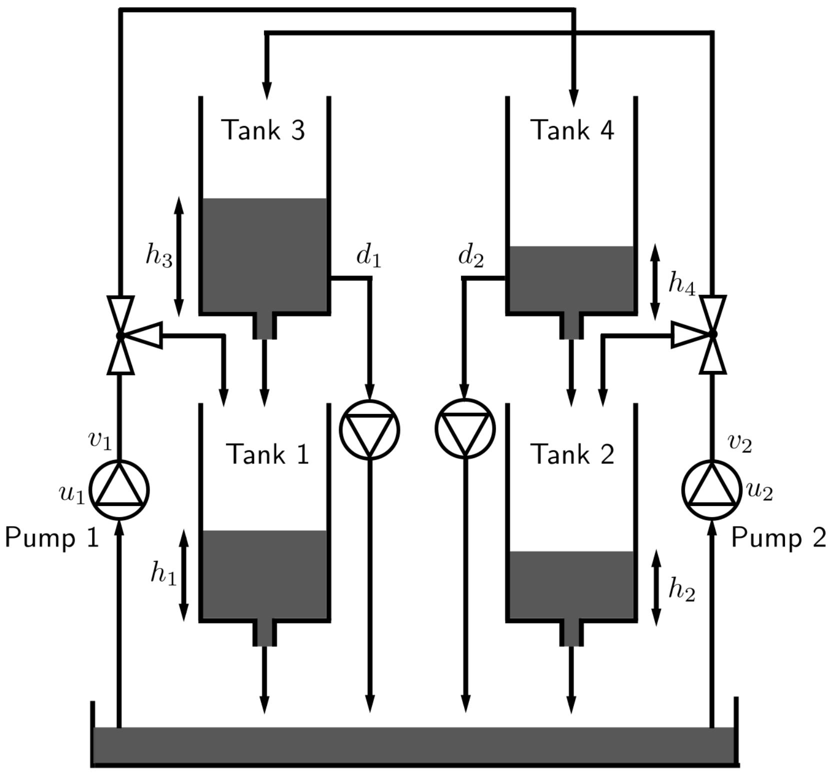


Figure 5.1: *Quadruple tank system.*

consists of four tanks and the objective is to control the water level for each of the tanks. The control variables are in this case two pumps, that transfer water from a common basin to the four overhead tanks. Through a special piping system, we can adjust the influence and the ratio of water that is pumped from one pump to these two tanks and hence modify the interaction between the pumps and water levels, cf. [Johansson 2000] for more details. In order to achieve a more challenging control task we use a modified model of the quadruple tank system, adapted from [Mercangöz and Doyle 2007]. In contrast to the standard system described in [Johansson 2000], there are additional pumps that can withdraw water from two of the tanks but are not directly controlled and the pump dynamics are modelled via a first order lag. Using Bernoulli's law we obtain as a model for the system depicted in Figure (5.1) the following equations:

$$
\begin{aligned}
\frac{\mathrm{d}h_1}{\mathrm{d}t} &= -\frac{a_1}{S}\sqrt{2gh_1} + \frac{a_3}{S}\sqrt{2gh_3} + \frac{\gamma_1 k_1}{S}v_1,\\
\frac{\mathrm{d}h_2}{\mathrm{d}t} &= -\frac{a_2}{S}\sqrt{2gh_2} + \frac{a_4}{S}\sqrt{2gh_4} + \frac{\gamma_2 k_2}{S}v_2,\\
\frac{\mathrm{d}h_3}{\mathrm{d}t} &= -\frac{a_3}{S}\sqrt{2gh_3} + \frac{(1-\gamma_2)k_2}{S}v_2 - \frac{d_1}{S},\\
\frac{\mathrm{d}h_4}{\mathrm{d}t} &= -\frac{a_4}{S}\sqrt{2gh_4} + \frac{(1-\gamma_1)k_1}{S}v_1 - \frac{d_2}{S},\\
\frac{\mathrm{d}v_1}{\mathrm{d}t} &= -\frac{v_1}{\tau_1} - \frac{1}{\tau_1}u_1,\\
\frac{\mathrm{d}v_2}{\mathrm{d}t} &= -\frac{v_2}{\tau_2} - \frac{1}{\tau_2}u_2,
\end{aligned}
\tag{5.1}
$$

where h_i and a_i refer to the water level and the cross section of the outlet hole of tank i, S is cross section for all the tanks and g is the gravitational constant. The control signal and flow speed of tank i is denoted by u_i and v_i, respectively, τ_i is the time constant for pump i and k_i is the corresponding gain. γ_i is the proportion of flow that goes to the upper tank from pump i and we have non–minimum phase behavior whenever $0 < \gamma_1 + \gamma_2 < 1$ with respect to the water level of the first and the second tank. d_i are possible disturbances that additionally modify the liquid level of tank 3 and 4, respectively. The overall goal of our simulation study is to synthesize distributed and decentralized control schemes and characterize a set of tolerable disturbance rates subject to hard state and input constraints.

For our analysis, we linearize the model (5.1) around the nominal steady states u_i^0, v_i^0 and h_i^0 with $d_i = 0$ and use a Euler-forward discretization with sampling time $\tau = 1\frac{1}{s}$

Table 5.1: Parameters.

h_1^0	[cm]	3.48		h_2^0	[cm]	3.09
h_3^0	[cm]	4.4		h_4^0	[cm]	3.1
v_1^0	[V]	19.44		v_2^0	[V]	27.41
u_1^0	[V]	19.44		u_2^0	[V]	27.41
k_1	[cm^3 / Vs]	7.45		k_2	[cm^3 / Vs]	7.3
a_1	[cm^2]	2.1		a_2	[cm^2]	2.2
a_3	[cm^2]	1.4		a_4	[cm^2]	1.3
τ_1	[s]	2.0		τ_2	[s]	2.1
S	[cm^2]	730		g	[cm/s^2]	981
γ_1		0.3		γ_2		0.35

using the parameters given in Table 5.1. In this case we obtain the following, linear discrete time model:

$$\Sigma: \quad x(k+1) = \begin{pmatrix} 0.966 & 0 & 0.003 & 0 & 0.02 & 0 \\ 0 & 0.978 & 0.007 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0.022 & 0 & 0.962 & 0 & 0.004 \\ 0 & 0 & 0 & 0 & 0.98 & 0.007 \\ 0 & 0 & 0 & 0 & 0 & 0.5 \end{pmatrix} x(k) + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0.5 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0.476 \end{pmatrix} u(k) \tag{5.2}$$

where $x = (h_1 - h_1^0, h_4 - h_4^0, v_1 - v_1^0, h_2 - h_2^0, h_3 - h_3^0, v_2 - v_2^0)^{\mathrm{T}} \in \mathbb{R}^6$, $u = (u_1 - u_1^0, u_2 - u_2^0)^{\mathrm{T}} \in \mathbb{R}^2$.

Due to the symmetry of the given structure, we decided to partition the overall system given by (5.2) into two interconnected systems of the following form:

$$\Sigma_1: \; x_1(k+1) = \underbrace{\begin{pmatrix} 0.966 & 0 & 0.003 \\ 0 & 0.978 & 0.007 \\ 0 & 0 & 0.5 \end{pmatrix}}_{:=A_1} x_1(k) + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0.5 \end{pmatrix}}_{:=B_1} u_1(k) + \underbrace{\begin{pmatrix} 0 & 0.02 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{:=G_1} x_2(k) + w_1, \tag{5.3}$$

where $x_1 = (x_{1,1}, x_{1,2}, x_{1,3})^{\mathrm{T}} = (h_1 - h_1^0, h_4 - h_4^0, v_1 - v_1^0)^{\mathrm{T}}$, $u_1 = (u_1 - u_1^0)^{\mathrm{T}}$, $w_1 \in \mathcal{W}_1 \subset \mathbb{R}^3$

and

$$\Sigma_2: \ x_2(k+1) = \underbrace{\begin{pmatrix} 0.962 & 0 & 0.004 \\ 0 & 0.98 & 0.007 \\ 0 & 0 & 0.5 \end{pmatrix}}_{:=A_2} x_2(k) + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0.476 \end{pmatrix}}_{:=B_2} u_2(k) + \underbrace{\begin{pmatrix} 0 & 0.022 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{:=G_2} x_1(k) + w_2, \tag{5.4}$$

where $x_2 = (x_{2,1}, x_{2,2}, x_{2,3})^{\mathrm{T}} = (h_2 - h_2^0, h_3 - h_3^0, v_2 - v_2^0)^{\mathrm{T}}$, $u_1 = (u_2 - u_2^0)^{\mathrm{T}}$, $w_2 \in \mathcal{W}_2 \subset \mathbb{R}^3$. In order to compute tolerable flow disturbances d_1 and d_2, we introduced additional disturbance terms w_i in (5.3) and (5.4). Hence, specifying the sets $\mathcal{W}_i$ allows us to properly characterize a set of tolerable disturbances d_i.

We furthermore impose the following constraints for the states

$$\mathcal{X}_1 = \left\{ x_1 : \begin{pmatrix} -1 \\ -1 \\ -5 \end{pmatrix} \leq x_1 \leq \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} \right\}, \quad \mathcal{X}_2 = \left\{ x_2 : \begin{pmatrix} -1 \\ -1 \\ -5 \end{pmatrix} \leq x_2 \leq \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} \right\},$$

and the following constraints for the control variables

$$\mathcal{U}_1 = \{u_1 : -2 \leq u_1 \leq 2\}, \quad \mathcal{U}_2 = \{u_2 : -2 \leq u_2 \leq 2\}.$$

In the following sections, we want to design different collection of controllers and positively invariant family of sets, by utilizing the concepts that have been introduced in the previous chapters. In addition, we want to show how these family of sets can be used to easily specify the sets $\mathcal{W}_i$, which in turn can be directly related to the region of tolerable leakage flows d_i. This is done by synthesizing the controllers for the disturbance free case and later on inspecting their robustness properties.

5.2 Decentralized Control of the Tank System

As mentioned before, we consider the disturbance free case for the synthesis, i.e. $\mathcal{W}_i = \{0\}$ for all $i \in \{1, 2\}$. Upon closer inspection, we can immediately see, that the approach presented in Theorem 13 is directly applicable, since (A_1, B_1), (A_2, B_2) are both controllable, $G_1 x_2(k)$ and $G_2 x_1(k)$ are linear functions and the input and state constraints are simple polytopes. Due to the linear structure of the interconnection terms, it is fortunately very easy to find in this special case η_1 and η_2, such that $G_1\mathcal{X}_2 \subseteq \eta_1\mathcal{B}^3$ and $G_2\mathcal{X}_1 \subseteq \eta_1\mathcal{B}^3$, respectively. In particular, $\eta_{1,2} = 0.0202$ and $\eta_{2,1} = 0.0224$ are suitable and fulfill Assumption 9. Since we only have two interconnected systems, we can directly solve the LMI's in (4.23) using semidefinite programming algorithms, since the constraint (4.23h) can be transformed for all $i \in \{1,2\}$ into the simple algebraic expression $\xi_{1,2} > (1-\sqrt{\xi_1})^{-2}$ and $\xi_{2,1} > (1-\sqrt{\xi_2})^{-2}$, respectively. We used SEDUMI and YALMIP, see [Sturm 1999] and [Löfberg 2004], to solve the LMI's given by (4.23) for the interconnected systems (5.3) and (5.4). As a feasible solution we obtain

$$
\begin{aligned}
Q_1 &= \begin{pmatrix} 0.993 & 0.609 & -0.549 \\ 0.609 & 0.965 & -1.914 \\ -0.549 & -1.914 & 24.403 \end{pmatrix}, \quad R_1 = \begin{pmatrix} -0.518 & -1.765 & 3.498 \end{pmatrix}, \\
\alpha_1 &= 0.996, \quad \xi_{1,2} = 789.02
\end{aligned}
$$

for $\xi_1 = 0.93$ and

$$
\begin{aligned}
Q_2 &= \begin{pmatrix} 0.888 & 0.068 & 0.399 \\ 0.068 & 0.53 & -1.475 \\ 0.399 & -1.475 & 21.776 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0.421 & -1.31 & 2.958 \end{pmatrix}, \\
\alpha_2 &= 0.957, \quad \xi_{2,1} = 798.14
\end{aligned}
$$

for $\xi_2 = 0.93$. Thus, according to Theorem 13, we can use the collection of ellipsoids $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2)$, where

$$
\mathcal{S}_1 = \{x : x^{\mathrm{T}} Q_1^{-1} x \leq 1\} = \left\{ x : x^{\mathrm{T}} \begin{pmatrix} 1.703 & -1.183 & -0.055 \\ -1.183 & 2.049 & 0.134 \\ -0.055 & 0.134 & 0.05 \end{pmatrix} x^{\mathrm{T}} \leq 1 \right\},
$$

and

$$
\mathcal{S}_2 = \{x : x^{\mathrm{T}} Q_2^{-1} x \leq 1\} = \left\{ x : x^{\mathrm{T}} \begin{pmatrix} 1.1629 & -0.256 & -0.039 \\ -0.256 & 2.379 & 0.166 \\ -0.039 & 0.166 & 0.058 \end{pmatrix} x^{\mathrm{T}} \leq 1 \right\},
$$

to form a positively invariant family of sets $\mathbb{S}(\mathcal{S}, \Theta)$ for the interconnected systems (5.3) and (5.4). Existence for a non–trivial set of scaling factors Θ is also implied by

Theorem 13. In order to compute the actual set, we utilize the dynamics of the scaling factors (see proof of Theorem 13)

$$\theta(k+1) = M\theta(k) = \begin{pmatrix} 0.964 & 0.021 \\ 0.021 & 0.964 \end{pmatrix} \theta(k), \tag{5.5}$$

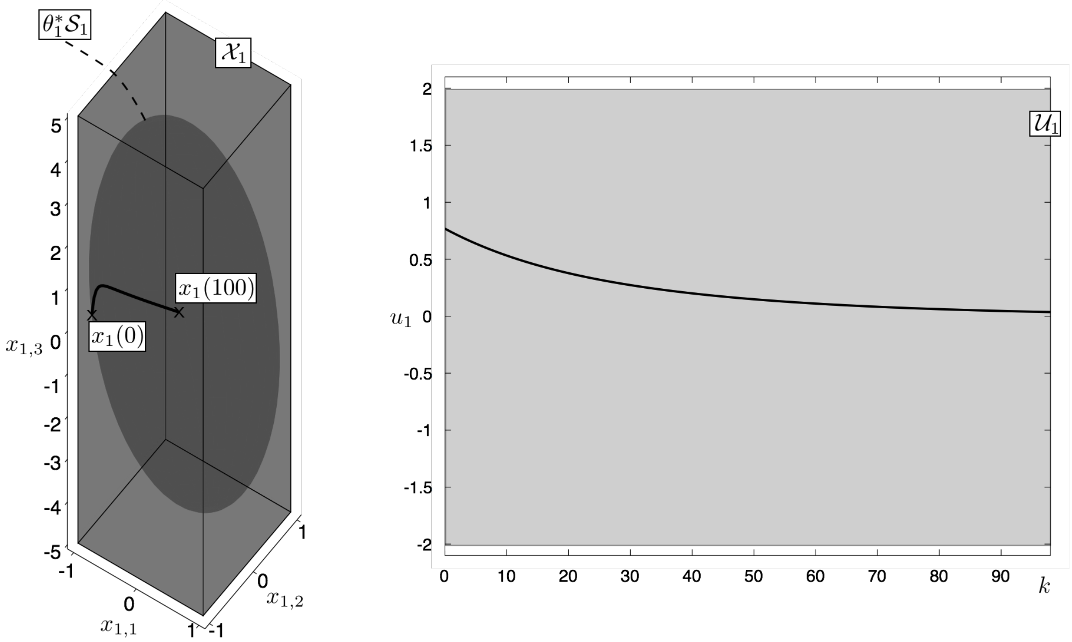
where the elements of the matrix M can be obtained similarly as presented in (4.17) and (4.18) and the set of admissible scaling factors

$$\Theta_0 = \{(\theta_1, \theta_2)^{\mathrm{T}} : 0 \leq \theta_1 \leq 1.003,\, 0 \leq \theta_2 \leq 1.061\} \tag{5.6}$$

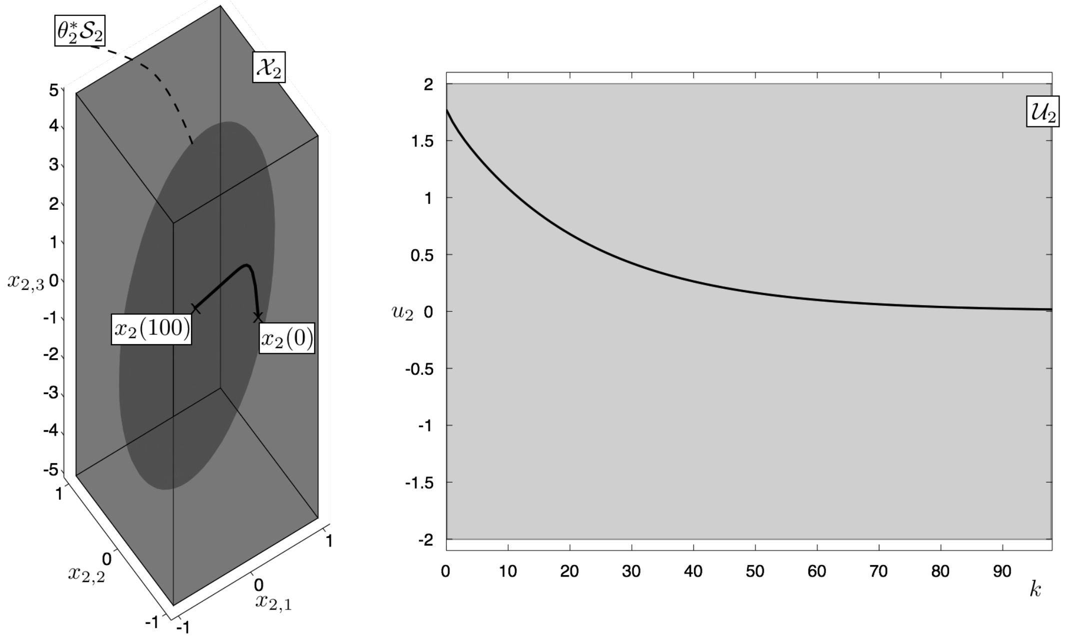
which is computed according to (4.20) using the gains $K_1 = R_1 Q_1^{-1}$, $K_2 = R_2 Q_2^{-1}$ and $K_{1,2} = K_{2,1} = 0$ and after removing all redundant half-spaces using the approach outlined in Corollary 1. In order to form the positively invariant family of sets, we need to determine a set $\Theta \subseteq \Theta_0$, such that $M\Theta \subseteq \Theta \subseteq \Theta_0$. We have several options to compute such a set, for instance by solving the LMI from Theorem 5 or by applying Algorithm 2. However, to determine the maximal possible set Θ of scaling factors we decided to use Algorithm 2. As result, we can verify that the whole set Θ_0 is in fact positively invariant for the dynamical system $\theta(k+1) = M\theta(k)$, i.e. $M\Theta_0 \subseteq \Theta_0$. Hence this set induces a positively invariant family of sets $\mathbb{S}(\mathcal{S}, \Theta_0)$. As a consequence we know from Theorem 13, that we can pick any $(\theta_1^*, \theta_2^*)^{\mathrm{T}} \in \Theta_0$ to guarantee that $\theta_1^* \mathcal{S}_1 \subseteq \mathcal{X}_1$ and $\theta_2^* \mathcal{S}_2 \subseteq \mathcal{X}_2$, respectively. In addition, whenever $x_1(0) \in \theta_1^* \mathcal{S}_1$ and $x_2(0) \in \theta_2^* \mathcal{S}_2$, we have $x_1(k) \in \mathcal{X}_1$, $x_2(k) \in \mathcal{X}_2$, $K_1 x_1(k) \in \mathcal{U}_1$ and $K_2 x_2(k) \in \mathcal{U}_2$ for all $k \in \mathbb{N}_+$, cf. Figure 5.2. For an illustration we picked the largest possible scaling factors $\theta_1^* = 1.003$ and $\theta_2^* = 1.006$.

5.2.1 Robustness Properties of the Decentralized controller

Note, that one of our initial goals was to investigate the robustness properties of the collection of decentralized controllers, which we computed for the disturbance free case, i.e. $\mathcal{W}_1 = \{0\}$ and $\mathcal{W}_2 = \{0\}$, respectively. Using the results from Theorem 10, we can quite easily determine sets $\mathcal{W}_1$ and $\mathcal{W}_2$ that lead to a robust positively invariant family of sets. Note, that we considered in Theorem 10 the autonomous case without input constraints. However, in an analogous way to Corollary 5, the results from Theorem 10 can be extended to include input constraints by a simple and direct adjustment of the set



(a) *State and input trajectories for subsystem Σ_1.*



(b) *State and input trajectories for subsystem Σ_2.*

Figure 5.2: *Simulation of input and state trajectories for the decentralized controller $u_1(k) = K_1x_1(k)$ and $u_2(k) = K_2x_2(k)$ with $x_1(0) = (-0.99, -0.633, 0)^T \in \theta_1^*\mathcal{S}_1$ and $x_2(0) = (0.5, -0.541, 0)^T \in \theta_2^*\mathcal{S}_2$.*

of scaling factors Θ_0, which directly corresponds to the set Θ_0 given by (5.4) in this case. The simplest way to determine a collection of valid sets $(\mathcal{W}_1, \mathcal{W}_2)$ is to directly relate them to the collection of sets $(\mathcal{S}_1, \mathcal{S}_2)$ that form a robust positively invariant family of sets. First note, that the matrix M from (5.5) directly corresponds to the matrix M from Theorem 10 and is by construction asymptotically stable. In addition to the former theorem, whenever $\alpha = (\alpha_1, \alpha_2)^{\mathrm{T}} \in \mathbb{R}^2_+$ is chosen such that $\bar{\theta} = (I - M)^{-1}\alpha$ is in the interior of Θ_0 then the collection of sets $(\mathcal{S}_1, \mathcal{S}_2)$ can be used to form a robust positively invariant family of sets if the collection of disturbance sets $(\mathcal{W}_1, \mathcal{W}_2)$ satisfy $\mathcal{W}_1 \subseteq \alpha_1\mathcal{S}_1$ and $\mathcal{W}_2 \subseteq \alpha_2\mathcal{S}_2$. Hence, if we consider disturbance sets $\mathcal{W}_1 = \alpha_1\mathcal{S}_1$ and $\mathcal{W}_2 = \alpha_2\mathcal{S}_2$, we simply need to adjust α_1 and α_2, accordingly. In fact, the set Ω which defines valid α can be easily constructed if Θ_0 has the structure $\Theta_0 = \{\theta : C\theta \leq f\}$. In that case

$$\Omega := \{\alpha : (I - M)^{-1}\alpha \in \mathrm{int}\Theta_0\} = \{\alpha : C(I - M)^{-1}\alpha < f\}. \tag{5.7}$$

For our particular example, using the set Θ_0 given by (5.4), we have

$$\Omega := \left\{ \alpha \in \mathbb{R}^2 : \begin{pmatrix} 30.832 & 22.173 \\ 26.155 & 36.322 \\ -1.0 & 0 \\ 0 & -1.0 \end{pmatrix} \alpha \leq \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\},$$

which is also depicted in Figure 5.3. Note, that a larger α_i corresponds to a larger tolerable disturbance set $\mathcal{W}_i$. By inspection of Figure 5.3, we can also see the level of interplay between α_1 and α_2, i.e. we can increase the size of the set $\mathcal{W}_1 = \alpha_1\mathcal{S}_1$ as long as we decrease the set of the other disturbance set $\mathcal{W}_2 = \alpha_2\mathcal{S}_2$ accordingly. As a result, the set Ω gives us more flexibility in the analysis for the tolerable disturbance flows d_1 and d_2, respectively. For instance, if we can assert that $d_2 \approx 0$ in our example, then the set of tolerable disturbance flows $\mathcal{W}_2$ for the flow d_1 has a larger size.

In order to characterize a robust positively invariant family of sets for the collection of sets $(\mathcal{S}_1, \mathcal{S}_2)$, it is still necessary to adjust the set of admissible scaling factors Θ. As explained in the proof of Theorem 10, this set needs to have the property that $M\Theta \oplus \{\alpha\} \subseteq \Theta \subseteq \Theta_0$. As pointed out we can compute a set Θ^*, such that $M\Theta^* \subseteq \Theta^* \subseteq \Theta_0 \oplus \{-\bar{\theta}\}$, then we can ensure that $\Theta = \Theta^* \oplus \{\bar{\theta}\}$, where $\bar{\theta} = (I - M)^{-1}\alpha$. Hence, we can again apply the procedures from Theorem 5 or Algorithm 2, however with the

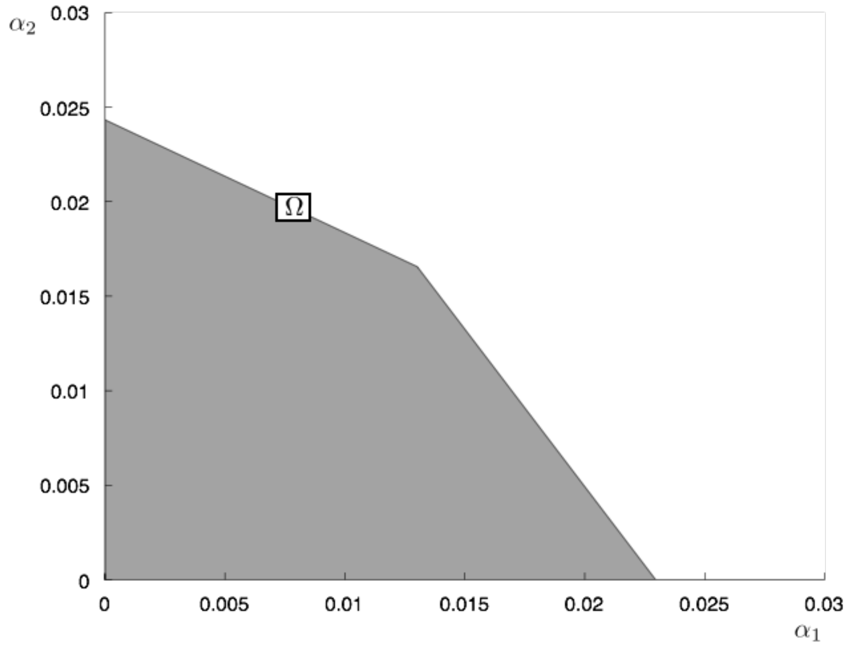


Figure 5.3: *Set of valid α_1 and α_2 for the decentralized controller.*

modified constraint set $\Theta_0 \oplus \{-\bar{\theta}\}$. As a specific example, we assume that the dominant disturbance set is $\mathcal{W}_2$ and that $\mathcal{W}_1$ is very small. As a first step we need to pick an appropriate $\alpha = (\alpha_1, \alpha_2)^{\mathrm{T}}$ out of the set Ω, say $\alpha = (0.001, 0.023)^{\mathrm{T}}$, which corresponds to a large tolerable set $\mathcal{W}_2$ and a smaller tolerable set $\mathcal{W}_1$. For this specific α, we have $\bar{\theta} = (0.647, 1.029)^{\mathrm{T}}$. In the next step, we again choose Algorithm 2 to compute the maximal positively invariant set Θ^*, such that $M\Theta^* \subseteq \Theta^* \subseteq \Theta_0 \oplus \{-\bar{\theta}\}$ to eventually obtain the set $\Theta = \Theta^* \oplus \{\bar{\theta}\}$ that induces a robust positively invariant family of sets for the collection of sets $(\mathcal{S}_1, \mathcal{S}_2)$. This set is depicted in Figure 5.4. Also note, that in contrast to the disturbance free setting, we cannot guarantee that the whole set Θ_0 induces a robust positively invariant family of sets and hence we have only a smaller set of admissible scaling factors Θ. Using this set Θ, whenever $(\theta_1, \theta_2)^{\mathrm{T}} \in \Theta$, we know that $x_1(0) \in \theta_1 \mathcal{S}_1$ and $x_2(0) \in \theta_2 \mathcal{S}_2$ implies $x_1(k) \in \mathcal{X}_1$, $x_2(k) \in \mathcal{X}_2$, $K_1 x_1(k) \in \mathcal{U}_1$ and $K_2 x_2(k) \in \mathcal{U}_2$ for all $k \in \mathbb{N}_+$ and all $w_1 \in \alpha_1 \mathcal{S}_1$ and $w_1 \in \alpha_2 \mathcal{S}_2$, respectively.

A depiction of this fact can be seen in Figure 5.5, where we simulated the state and input trajectories using the previously synthesized decentralized controller and initial conditions $x_1(0) \in \theta_1^* \mathcal{S}_1$ and $x_2(0) \in \theta_2^* \mathcal{S}_2$ with $(\theta_1^*, \theta_2^*)^{\mathrm{T}} = (0.704, 1.061)^{\mathrm{T}} \in \Theta$. For this simulation we set $w_1 = w_2 = \mathbf{0}$ for the first 80 time steps. At time step 81 we set $w_1 = \mathbf{0}$ and $w_2 = (0, -0.0137, 0)^{\mathrm{T}} \in 0.023\mathcal{S}_2$, which roughly translates to $d_1 = 10\frac{cm^3}{s}$.

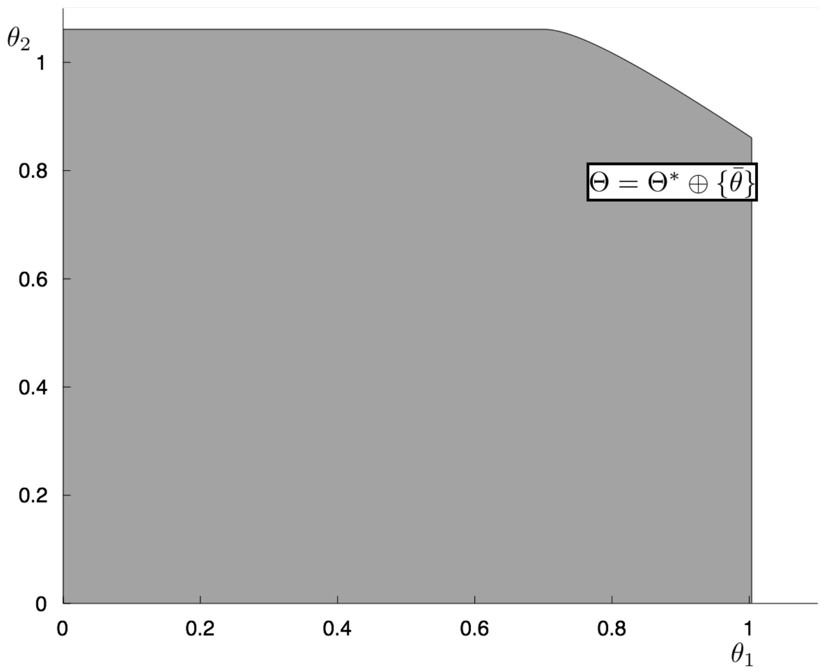
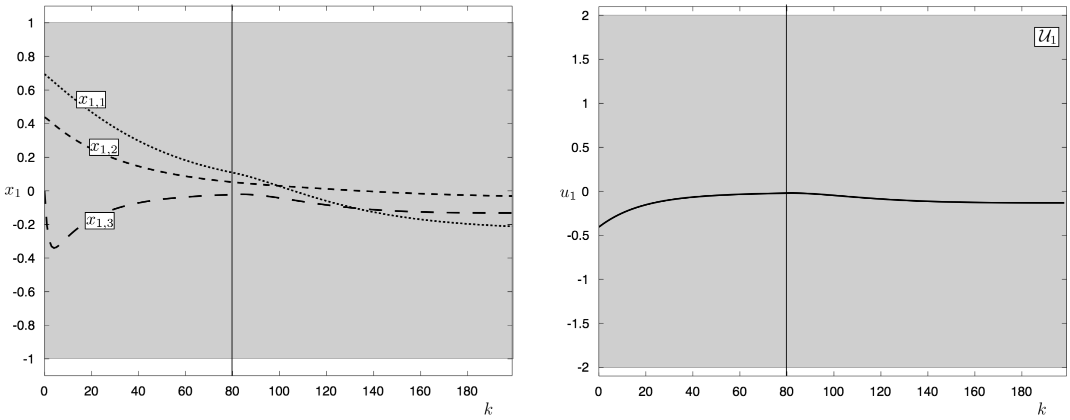


Figure 5.4: *Set of admissible scaling factors Θ that induces a robust positively invariant family of sets for the decentralized controller with disturbance sets $\mathcal{W}_1 := 0.001\mathcal{S}_1$ and $\mathcal{W}_2 := 0.023\mathcal{S}_2$.*

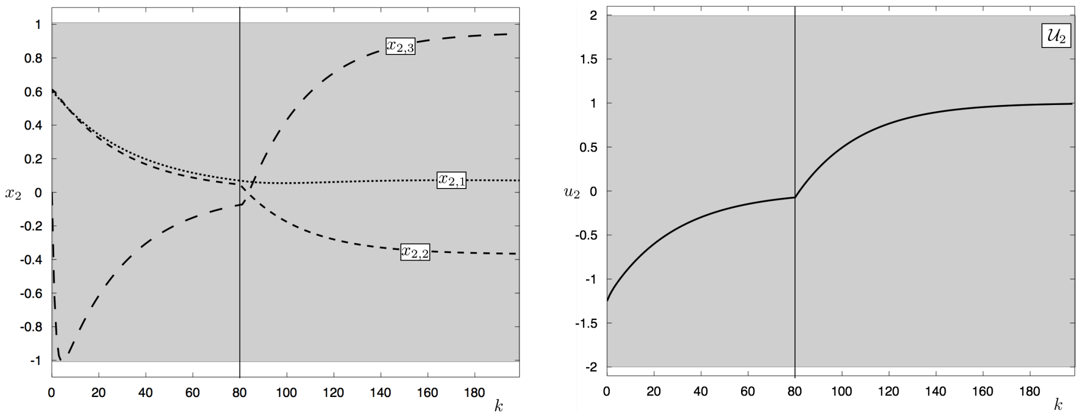
Note, that we highlighted only the first and second coordinate of the state constraints in Figure 5.5 to improve the visibility of the trajectories. We can conclude from Figure 5.5, that the decentralized controller keeps the states and the inputs within the constraints even for this type of disturbances. The results are quite conservative, since the tolerable disturbance sets are comparatively small and the region of initial conditions within the robust positively family of sets is also much smaller compared to the disturbance free case. However, note that during our design process our focus was purely on the guaranteed constraint satisfaction. In order to improve the performance, it is therefore still necessary to adjust the procedure for the synthesis of the controllers.

5.3 Distributed Control of the Tank System

Although, we could find a pair of decentralized controller for the interconnected tank system, subject to the state constraints $\mathcal{X}_1$, $\mathcal{X}_2$ and input constraints $\mathcal{U}_1$, $\mathcal{U}_2$, the performance from the closed loop system was not optimal, since we derived rather large feedback gains (K_1, K_2) for the open loop stable interconnected systems Σ_1 and Σ_2.



(a) *State and input trajectories for subsystem* Σ_1.



(b) *State and input trajectories for subsystem* Σ_2.

Figure 5.5: *Simulation of input and state trajectories for the decentralized controller* $u_1(k) = K_1 x_1(k)$ *and* $u_2(k) = K_2 x_2(k)$ *with* $x_1(0) = (0.695, 0.439, 0)^T \in \theta_1^* \mathcal{S}_1$, $x_2(0) = (0.601, 0.613, 0)^T \in \theta_2^* \mathcal{S}_2$ *and constant* $w_2 = (0, -0.0137, 0)^T$ *at time step 80.*

The main focus of this section is to derive a pair of controllers that achieves better closed loop performance.

In this section we assume that each subsystem Σ_i can utilize local state information $x_i(\cdot)$ as well as the cumulative value $G_i x_j(\cdot)$ of the other subsystem Σ_j. In particular, we want to derive the pair of distributed controller

$$u_1(k) = K_1 x_1(k) + L_1 G_1 x_2(k)$$

$$u_2(k) = K_2 x_2(k) + L_2 G_2 x_1(k)$$

for the interconnected linear model of the multiple tank system given by the equation

(5.3) and (5.4) with the same input $(\mathcal{U}_1, \mathcal{U}_2)$ and state constraint sets $(\mathcal{X}_1, \mathcal{X}_2)$ as in the decentralized case.

In order to obtain these controllers, we utilize the min–max/ max–min approach from Section 4.1. Similarly, to the synthesis of the decentralized controller in the previous section, we consider again for the design of the controller the disturbance free case in which the disturbance sets $\mathcal{W}_i$ consist only of the element $\{0\}$. In addition, due to controllability of the matrix pairs (A_i, B_i) and the fact that the constraint sets are polytopes, we can directly apply the approach from Section 4.1. Note, that in contrast to the LMI approach from Section 4.2, we cannot easily guarantee that there exists a distributed controller that fulfills all the conditions of Corollary 5. In particular, we need to iteratively adjust the parameters (Q_1, γ_1) and (Q_2, γ_2) for each generalized Riccati equation (4.11) and check after each step if the corresponding matrix M from (4.8) is strictly stable. Fortunately, we could show in the previous section that there exists a decentralized controller for this specific task and for this reason there should also exist a pair of parameters (Q_1, γ_1) and (Q_2, γ_2) that leads to a distributed controller that satisfies the conditions from Corollary 5.

We can see from the prototype infinite horizon control problem (4.10), that the optimal value $V^*(\cdot)$ depends on both terms $x^{\mathrm{T}}Qx$ and $u^{\mathrm{T}}u$. With the intention of obtaining less aggressive controllers, we therefore decided to decrease the penalty for state deviations and chose the following weighting matrices for the states of the subsystem Σ_1 and Σ_2, respectively:

$$Q_1 = Q_2 = \begin{pmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{pmatrix}.$$

By following the approach presented in Section 4.1, we are able to solve the generalized Riccati equation utilizing the sequence of matrices given by (4.14) with using $\gamma_1 = \gamma_2 = 24$. We have as a solution to the generalized Riccati equation the following matrices for the subsystems Σ_1 and Σ_2, respectively:

$$P_1 = \begin{pmatrix} 8.692 & -0.028 & 0.045 \\ -0.028 & 14.343 & 0.176 \\ 0.045 & 0.176 & 0.64 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 7.667 & -0.256 & 0.045 \\ -0.256 & 16.202 & 0.183 \\ 0.045 & 0.183 & 0.642 \end{pmatrix}.$$

Using (4.12) and the solution of the generalized Riccati equation we can easily compute the corresponding gain pairs (K_1, L_1) and (K_2, L_2) of the distributed controllers:

$$K_1 = \begin{pmatrix}-0.019 & -0.074 & -0.139\end{pmatrix}, \; L_1 = \begin{pmatrix}-0.019 & -0.076 & -0.276\end{pmatrix}$$
$$K_2 = \begin{pmatrix}-0.018 & -0.075 & -0.134\end{pmatrix}, \; L_2 = \begin{pmatrix}-0.019 & -0.076 & -0.267\end{pmatrix}.$$

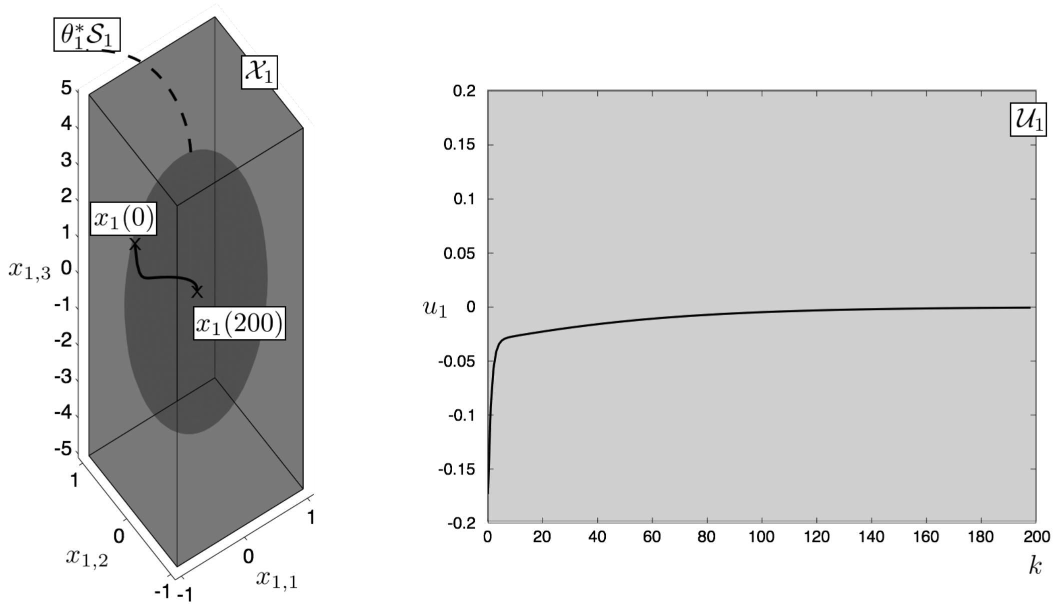
In the next step we need to compute the transition matrix M from (4.8), which described the dynamics of the scaling factors $\theta(k+1) = M\theta(k)$. According to (4.18) and (4.19), we can use the collection of sets $(\mathcal{S}_1, \mathcal{S}_2)$, where $\mathcal{S}_1 = \{x : x^{\mathrm{T}} P_1 x \leq 1\}$ and $\mathcal{S}_2 = \{x : x^{\mathrm{T}} P_2 x \leq 1\}$ to directly compute the entries of the matrix M:

$$\theta(k+1) = M\theta(k) = \begin{pmatrix}0.977 & 0.015 \\ 0.016 & 0.979\end{pmatrix} \theta(k). \tag{5.8}$$

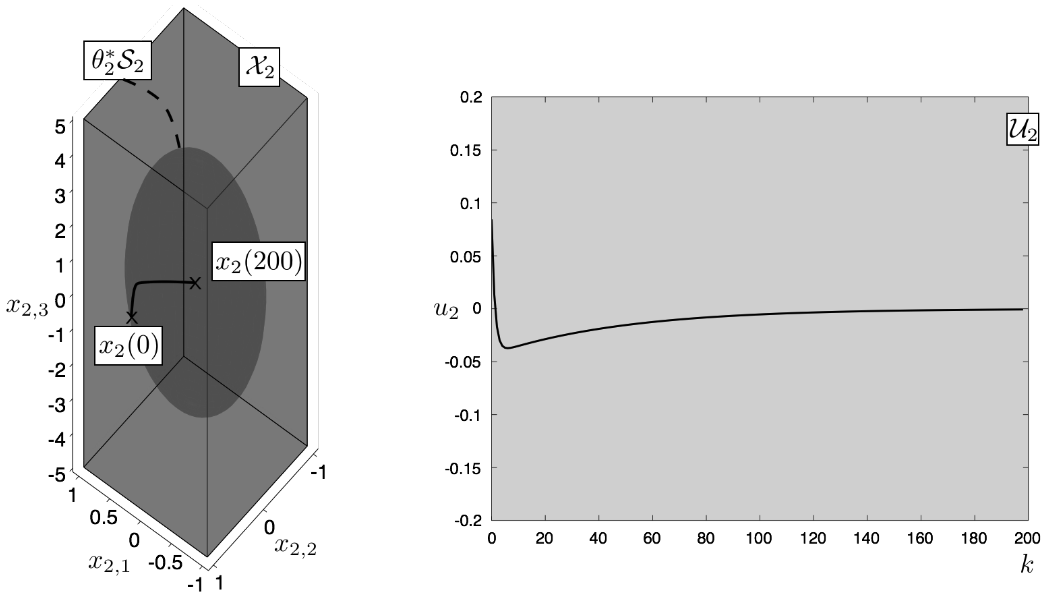
Since the matrix M is strictly stable, i.e. $\rho(M) < 1$, we can directly apply the results from Corollary 5 and use the collection of sets $(\mathcal{S}_1, \mathcal{S}_2)$ and the previously computed distributed controller to specify a positively invariant family of sets $\mathbb{S}((\mathcal{S}_1, \mathcal{S}_2), \Theta)$ for the interconnected systems Σ_1 and Σ_2. For the actual computation of a valid set of scaling factors Θ, we need to first specify the set of admissible scaling factors Θ_0 for the distributed controller and the collection of sets $(\mathcal{S}_1, \mathcal{S}_2)$, which is computed according to (4.20). After removing all redundant half-spaces from Θ_0, using the approach outlined in Corollary 1, we have

$$\Theta_0 = \{(\theta_1, \theta_2)^{\mathrm{T}} : 0 \leq \theta_1 \leq 2.948, \; 0 \leq \theta_2 \leq 2.769\}. \tag{5.9}$$

As already explained in the previous section, in order to obtain the actual set of scaling factors Θ, that induces a positively invariant family of sets, we need to first find a positively invariant set Θ for the scaling dynamics, that is included in Θ_0, i.e. $M\Theta \subseteq \Theta \subseteq \Theta_0$. Similarly to the decentralized synthesis, we again decided to apply Algorithm 2 to determine the maximal possible set Θ in order to increase the region of admissible initial conditions for the interconnected closed loop system. Eventually, after applying Algorithm 2 we can again verify that the whole set Θ_0 specified in (5.9) is positively invariant for the dynamics of the scaling factors, i.e. $M\Theta_0 \subseteq \Theta_0$. Thus, $\mathbb{S}(\mathcal{S}, \Theta_0)$ is a positively invariant family of sets, so according to Theorem 13, we can pick any $(\theta_1^*, \theta_2^*)^{\mathrm{T}} \in \Theta_0$ to guarantee that $\theta_1^* \mathcal{S}_1 \subseteq \mathcal{X}_1$ and $\theta_2^* \mathcal{S}_2 \subseteq \mathcal{X}_2$, respectively and whenever $x_1(0) \in \theta_1^* \mathcal{S}_1$ and $x_2(0) \in \theta_2^* \mathcal{S}_2$, we have $x_1(k) \in \mathcal{X}_1$, $x_2(k) \in \mathcal{X}_2$, $K_1 x_1(k) \in \mathcal{U}_1$ and



(a) *State and input trajectories for subsystem* Σ_1.



(b) *State and input trajectories for subsystem* Σ_2.

Figure 5.6: *Simulation of input and state trajectories for the distributed controller* $u_1(k) = K_1x_1(k) + L_1G_1x_2(k)$ *and* $u_2(k) = K_2x_2(k) + L_2G_2x_1(k)$ *with* $x_1(0) = (-0.55, -0.6, 1.0)^T \in \theta_1^*\mathcal{S}_1$ *and* $x_2(0) = (0.63, -0.51, -1.0)^T \in \theta_2^*\mathcal{S}_2$.

$K_2 x_2(k) \in \mathcal{U}_2$ for all $k \in \mathbb{N}_+$, cf. Figure 5.6 for a illustration where we picked the largest possible scaling factors $\theta_1^* = 2.948$ and $\theta_2^* = 2.769$ for a depiction of this fact. In comparison to the decentralized controller, we can see from Figure 5.6 that the inputs of the distributed controller are much smaller. However, the region of admissible initial conditions is so far larger and the closed loop system converges faster to the origin for the aggressive decentralized controller, cf. Figure 5.2.

5.3.1 Robustness Properties of the Distributed Controller

As shown previously, we utilize the concept of robust positively invariant family of sets to specify guaranteed robustness properties for the distributed controller. The first step is to parametrize the tolerable collection of disturbance sets $(\mathcal{W}_1, \mathcal{W}_2)$ using the collection of sets $(\mathcal{S}_1, \mathcal{S}_2)$ and $\alpha = (\alpha_1, \alpha_2)^{\mathrm{T}} \in \mathbb{R}_+^2$ such that $\mathcal{W}_1 \subseteq \alpha_1 \mathcal{S}_1$ and $\mathcal{W}_2 \subseteq \alpha_2 \mathcal{S}_2$, respectively. As explained in the previous section, we can guarantee in this case, that the collection of sets $(\mathcal{S}_1, \mathcal{S}_2)$ form a robust positively invariant family of sets, if $\bar{\theta} = (I - M)^{-1}\alpha$ is in the interior of Θ_0. The set of α that satisfies the former relation is specified according to (5.7). Using the set Θ_0 from (5.9) and M from (5.8), we have the following set Ω of tolerable scaling factors α for the disturbances:

$$\Omega := \left\{ \alpha \in \mathbb{R}^2 : \begin{pmatrix} 30.832 & 22.173 \\ 26.155 & 36.322 \\ -1.0 & 0 \\ 0 & -1.0 \end{pmatrix} \alpha \leq \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\},$$

which we depicted in Figure 5.7 as well. Thus whenever we pick $\alpha \in \Omega$, we can ensure that there is robust positively invariant family of sets $\mathbb{S}((\mathcal{S}_1, \mathcal{S}_2), \Theta)$ with a nontrivial set of scaling factors Θ.

In comparison to decentralized controller, we can only ensure guaranteed robustness properties for relatively small disturbance sets if we use the previously designed distributed controller, since the basic shape sets (S_1, S_2) for the distributed controller are much smaller than the shape sets for the decentralized controller, while the set of tolerable scaling factors for the disturbances has a roughly similar size. For the determination of an actual robust positively invariant family of sets, we use the same rationale as in decentralized case, where we tried to maximize the level of uncertainty of the disturbance set $\mathcal{W}_2$, which later on implies that the tolerable disturbance flow d_2 is small

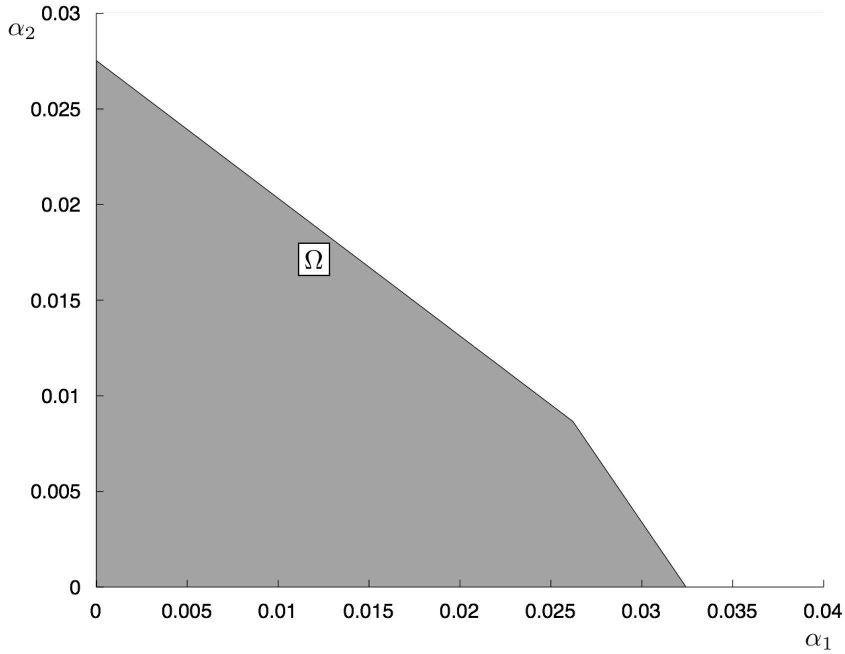


Figure 5.7: *Set of valid α_1 and α_2 for the distributed controller.*

compared to the flow d_1. For this specific example, we picked $\alpha = (0.001, 0.026)^{\mathrm{T}}$. As explained in the previous section, to determine a robust positively invariant family of sets $\mathbb{S}((\mathcal{S}_1, \mathcal{S}_2), \Theta)$, we simply need to find a set $\Theta = \Theta^* \oplus \{\bar{\theta}\}$, with $M\Theta^* \subseteq \Theta^* \subseteq \Theta_0 \oplus \{-\bar{\theta}\}$ and $\bar{\theta} = (I - M)^{-1}\alpha = (1.79, 2.687)^{\mathrm{T}}$, see the previous section for more details. For the determination of this specific set, we utilized again Algorithm 2 to compute the maximal positively invariant set Θ^* to later on determine the set Θ, which is depicted in Figure 5.8. Also note here, that the set of scaling factors Θ that induce the robust positively invariant family of sets is again smaller than the set of admissible scaling factors Θ_0. As a consequence, we can only guarantee in a smaller region, that the input and state constraints will be satisfied whenever we apply our distributed controller to the interconnected system for this specific disturbance scenario. Nevertheless, we have that $x_1(0) \in \theta_1\mathcal{S}_1$ and $x_2(0) \in \theta_2\mathcal{S}_2$ implies $x_1(k) \in \mathcal{X}_1$, $x_2(k) \in \mathcal{X}_2$, $K_1x_1(k) + L_1G_1x_2(k) \in \mathcal{U}_1$ and $K_2x_2(k) + L_2G_2x_1(k) \in \mathcal{U}_2$ for all $k \in \mathbb{N}_+$ and all $w_1 \in \alpha_1\mathcal{S}_1$ and $w_1 \in \alpha_2\mathcal{S}_2$, respectively.

In Figure 5.9, we can see the closed loop behaviour of the interconnected system with the distributed controller. For the simulation, we picked initial conditions $x_1(0) \in \theta_1^*\mathcal{S}_1$ and $x_2(0) \in \theta_2^*\mathcal{S}_2$ with $(\theta_1^*, \theta_2^*)^{\mathrm{T}} = (1.9, 2.75)^{\mathrm{T}} \in \Theta$. We again set $w_1 = w_2 = \mathbf{0}$ for the first 80 time steps, while starting with time step 81 we changed the value of the

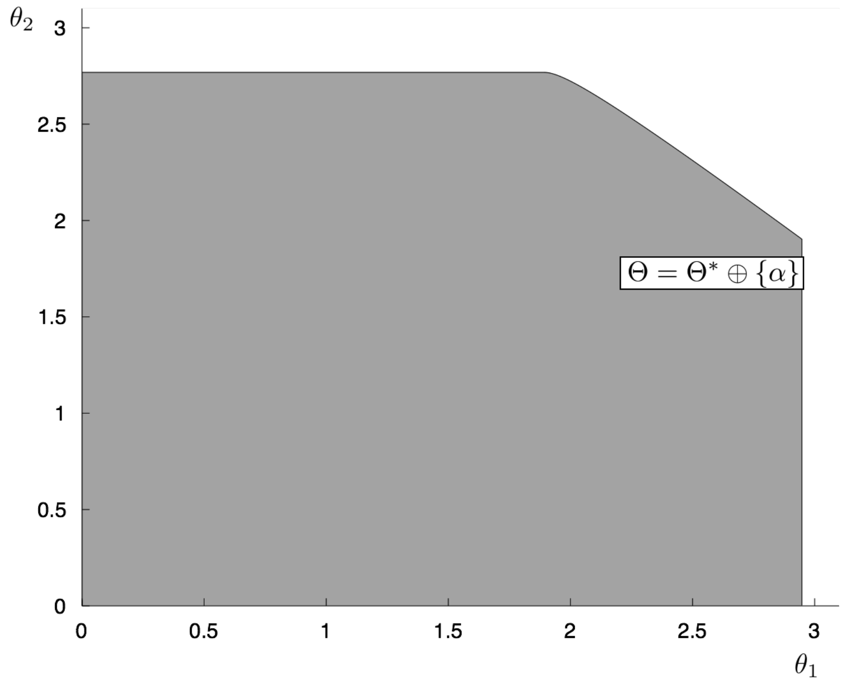
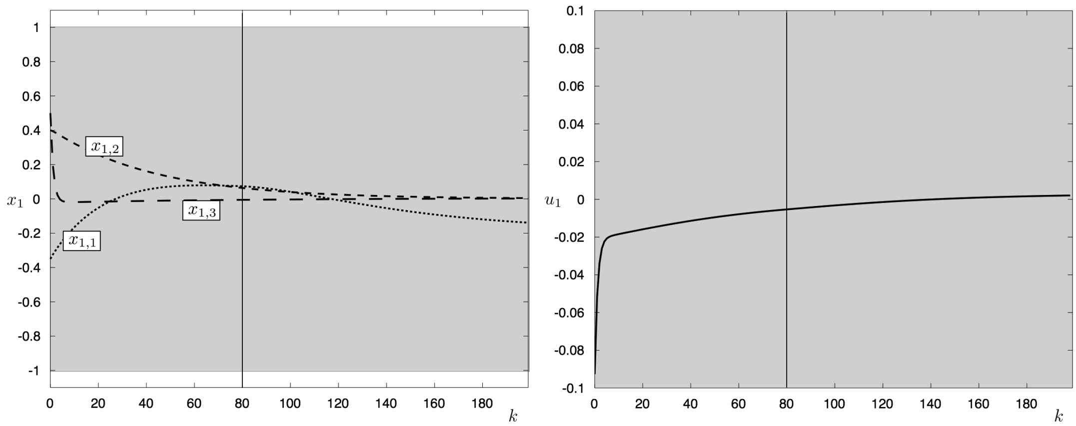


Figure 5.8: *Set of admissible scaling factors Θ that induces a robust positively invariant family of sets for the distributed controller with disturbance sets $\mathcal{W}_1 := 0.001\mathcal{S}_1$ and $\mathcal{W}_2 := 0.026\mathcal{S}_2$.*

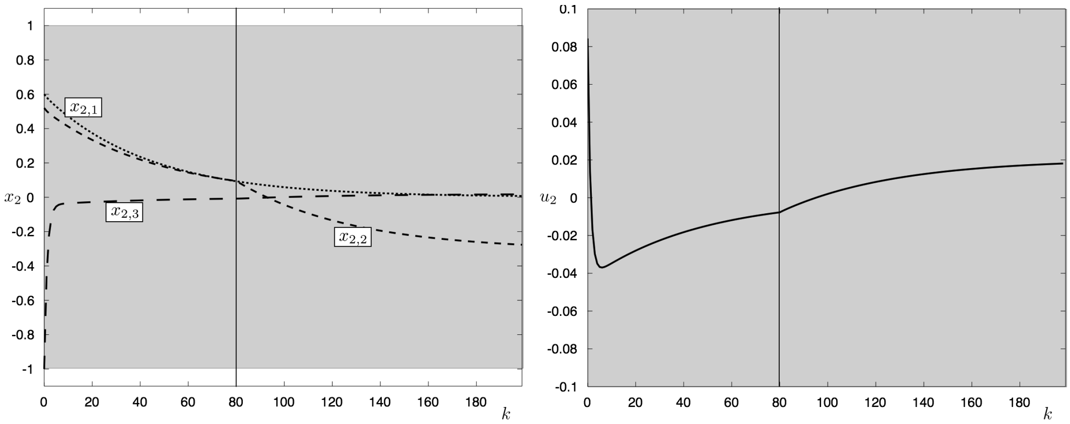
second disturbance to $w_2 = (0, -0.0064, 0)^{\mathrm{T}} \in 0.026\mathcal{S}_2$, which roughly translates to $d_1 = 4.67\frac{cm^3}{s}$. Clearly, the state and input constraints are satisfied for this simulation examples, as we can see from Figure 5.9 and hence this type of disturbance scenario can be easily handled. Note that, we can only guarantee state and input constraint satisfaction for a relatively small disturbance set. Furthermore, the initial conditions in which we can assert this fact is also relatively small compared to the disturbance free case. However, we can argue similarly to the decentralized control case, that the objective of the controller was to characterize a region in which we can guarantee state and input constraint satisfaction. Hence, in order to improve the performance of the controller several disturbance cases, it is still necessary to adapt the synthesis of the distributed controller appropriately.

5.4 Polytopic Sets

We noted so far, that the region of admissible initial conditions for the decentralized controller might be larger compared to the distributed controller. In addition, we used



(a) *State and input trajectories for subsystem* Σ_1.



(b) *State and input trajectories for subsystem* Σ_2.

Figure 5.9: *Simulation of input and state trajectories for the distributed controller* $u_1(k) = K_1x_1(k)+L_1G_1x_2(k)$ *and* $u_2(k) = K_2x_2(k)+L_2G_2x_1(k)$ *with* $x_1(0) = (-0.35, 0.4, 0.5)^T \in \theta_1^*\mathcal{S}_1$, $x_2(0) = (0.6, 0.52, -1.0)^T \in \theta_2^*\mathcal{S}_2$ *and constant* $w_2 = (0, -0.0064, 0)^T$ *at time step 80.*

in both cases, family of ellipsoidal shaped sets to characterize the positively invariant family of sets. Restricting ourselves to this type of sets might produce conservative results, since the constraint sets $\mathcal{X}_1$ and $\mathcal{X}_2$ can only be badly approximated by ellipsoidal sets. In this section, we want to check, if we can enlarge this region for both control cases, by using differently shaped, polytopic family of sets $\mathbb{S}((\mathcal{S}_1^*, \mathcal{S}_2^*), \Theta)$. In particular, we want to see if it is possible to initialize the system on the boundary of the constraint sets $\mathcal{X}_1$, $\mathcal{X}_2$ or both of them, without fear of violating the constraint sets. Furthermore, in order to reduce the complexity for the characterization of the shape sets $(\mathcal{S}_1, \mathcal{S}_2)$ we restrict them to be simple rectangular sets.

The approach highlighted in Section 3.2.3 allows us to properly analyze this specific task, although it was motivated for the autonomous case only. Nevertheless, we can easily extend it to include input constraints, by adjusting the set of admissible scaling factors Θ_0 given by (3.24). This adjustment was highlighted also in Corollary 5 and we computed this set Θ_0 in the distributed control case using ellipsoidal shaped sets via the use of the support function in (4.7). In order to compute this set for our case, i.e. for the case of polytopic shaped sets $(\mathcal{S}_1^*, \mathcal{S}_2^*)$, we simply need to evaluate the support function for polytopic sets in (4.22). However, extending the approach of Section 3.2.3 to include additive disturbances is unfortunately not–trivial and for this reason we assume that $\mathcal{W}_1 = \mathcal{W}_2 = \{0\}$ in the remainder of this section.

As outlined in Section 3.2.3, we need basic shape sets $(\mathcal{S}_1^*, \mathcal{S}_2^*)$ to perform the recursion defined in (3.27). Furthermore, we intend to maximize the size of the sets in order to check if we can initialize the interconnected system properly on the boundaries of the constraint sets $(\mathcal{X}_1, \mathcal{X}_2)$. In the first step, we tried to use the whole constraint sets as basic shape sets to perform the recursion, i.e. $\mathcal{S}_1^* = \mathcal{X}_1$ and $\mathcal{S}_2^* = \mathcal{X}_2$. In that case, considering the discussion from before, we have $\Theta_0 = \{x : |x|_\infty \leq 1\}$. However, for both the decentralized as well as the distributed controller we got as $\Theta_\infty = \{0\}$ after performing several recursion steps utilizing (3.27). This also implies that we cannot initialize the interconnected systems in the corners of the constraint sets $\mathcal{X}_1, \mathcal{X}_2$, which can be easily verified by a simple simulation using appropriate initial conditions, which we decided to not to include here due to its triviality.

In our second step, we decided to analyze if it is possible to maximize the basic shape sets $(\mathcal{S}_1^*, \mathcal{S}_2^*)$ with the intention of achieving a large region of admissible initial conditions for the water levels of the tanks. Note, that the first two states of the interconnected systems Σ_1 and Σ_2 describe the water level of the tanks while the third state describes the flow speed of the first and second pump, cf. (5.3) and (5.4). A reasonable way to choose appropriate shape sets for this task is therefore to consider the following parametrized polytopes:

$$\mathcal{S}_1^* = \mathcal{S}_2^* = \{(x_1, x_2, x_3)^{\mathrm{T}} : -1 \leq x_1 \leq 1,\ -1 \leq x_2 \leq 1, -a \leq x_3 \leq a\}. \tag{5.10}$$

A possible way to check, if we can use the parametrized polytopes given in (5.10) to form a positively invariant family of sets $\mathbb{S}((\mathcal{S}_1^*, \mathcal{S}_2^*, \Theta)$, is to perform the recursion defined in (3.27) for different parameters a until we obtain a non–trivial and non–empty set Θ_∞. Therefore, we can use a simple bisection algorithm for the parameter a in order to check when the result of the recursion (3.27) is non–empty and non–trivial. Fortunately, for the distributed controller we were able to obtain a non–trivial solution of the recursion (3.27) for the value $a = 2.1912$. In that case, we have $\Theta_0 = \{x : |x|_\infty \leq 1\}$ and the set of admissible scaling factors $\Theta = \Theta_\infty$, which is depicted in Figure 5.10. By construction,

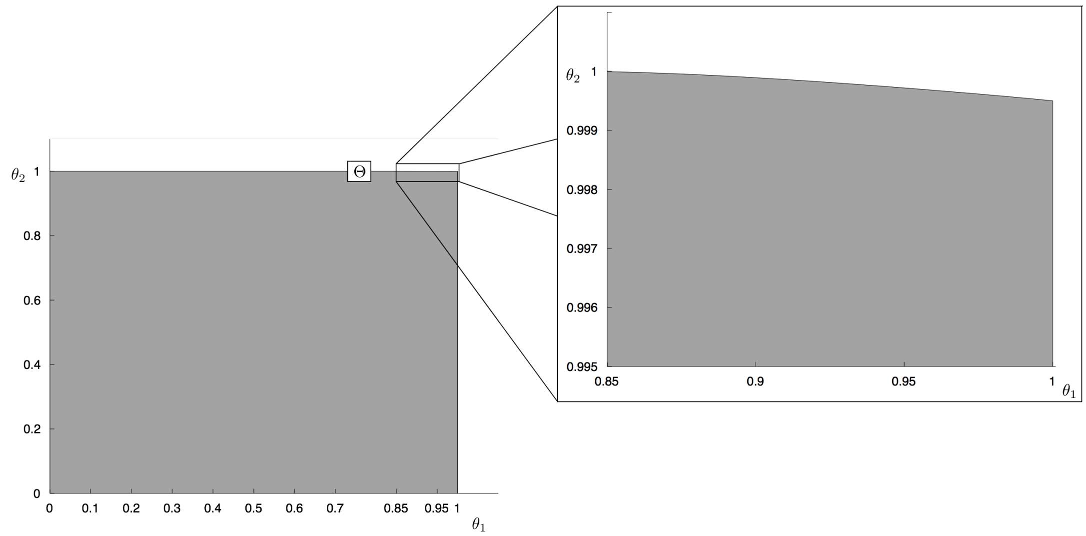
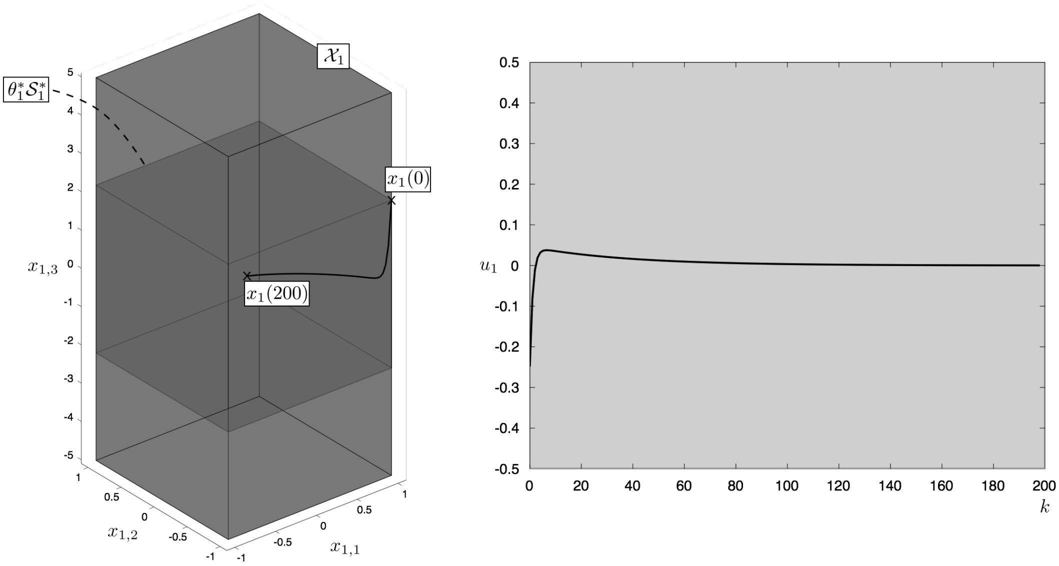


Figure 5.10: *Set of admissible scaling factors Θ that induce a positively invariant family of sets for the distributed controller and the collection of sets $(\mathcal{S}_1^*, \mathcal{S}_2^*)$.*

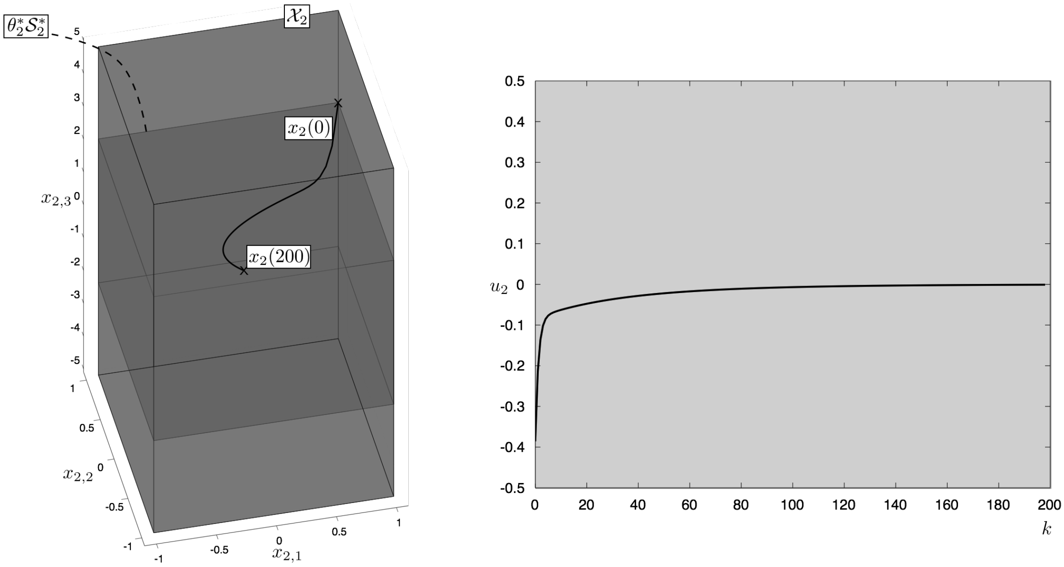
we therefore know that $\mathbb{S}(\mathcal{S}, \Theta)$ is a positively invariant family of sets. Which allows us to easily constructs regions of initial conditions, in which we can guarantee that state and input constraint satisfaction are guaranteed. As an example, we depicted the state and input trajectories in Figure 5.11 and choose the $\theta_1^* = 1$ and $\theta_2^* = 0.9995$ as admissible scaling factors. In comparison to the results from the previous section, we were able to greatly increase the region of admissible initial conditions for the distributed controller. Unfortunately, for the decentralized controller, we were not able to obtain a set of non–trivial scaling factors Θ_∞ for any parameter a.

5.5 Summary

We applied several methodologies highlighting the basic concepts and flexibility of positively invariant family of sets for the controller synthesis and analysis of a multiple tank system. We exemplified the synthesis of the distributed and decentralized controller based on invariant family of sets, which was highlighted in Section 4.1 and 4.2, respec-



(a) *State and input trajectories for subsystem* Σ_1.



(b) *State and input trajectories for subsystem* Σ_2.

Figure 5.11: *Simulation of input and state trajectories for the distributed controller* $u_1(k) = K_1x_1(k) + L_1G_1x_2(k)$ *and* $u_2(k) = K_2x_2(k) + L_2G_2x_1(k)$ *with* $x_1(0) = (1, -1, 2.192)^T \in \theta_1^*\mathcal{S}_1^*$ *and* $x_2(0) = (0.999, 0.999, 2.191)^T \in \theta_2^*\mathcal{S}_2^*$.

tively. For both of the controllers, we could successfully determine, flexible regions of admissible initial conditions that achieve explicit state and input constraint satisfaction. We could also show, that on the one hand the distributed controller has a better performance in the disturbance free case. On the other hand, we highlighted the fact, that the synthesis is non–trivial, due to its iterative procedure. In contrast to this, the decentralized controller is simpler to synthesize, due to its one shot procedure, i.e. it is only necessary to check feasibility of a LMI for each of the subsystem.

6 Conclusions

The main intention of this thesis was to present an appropriate analysis design tool for interconnected systems subject to constraints. We showed that the standard concepts of invariance should not be naively extended, since otherwise the synthesis and analysis for these types of problems might lead to conservative results. By using parametrized family of sets, we could show that interconnected systems can be analyzed in a flexible and appropriate way. We presented the concept of positive invariance for a family of sets and provided several methods, that can be either used to check for existence or used to construct such family of sets. Furthermore, we presented flexible and easily applicable tools for the distributed and decentralized controller synthesis exploiting the concepts of positively invariant family of sets and exemplified in a standard benchmark example for decentralized and distributed control problems, their applicability and effectiveness.

Set–based frameworks are very well–suited for the characterization of safe process regions with respect to constraints. In particular, the concept of invariance is a beneficial tool for the analysis and computation of such regions and is very well established for centralized use cases. Nevertheless, as argued in Chapter 1, due to complexity of the plant and/or informational constraints, it is often not possible to employ a centralized methodology and therefore it was necessary to adapt the concept of invariance in order to make them more useful for distributed and decentralized control tasks. By analyzing the induced set–dynamics (3.3) in Chapter 2, we could reveal the interplay of the specific set–iterates $\mathcal{X}_{i,k}$ and hence gain additional insight for an adapted notion of invariance. The main idea was to capture the dynamic interplay of the set iterates $\mathcal{X}_{i,k}$, with a flexible parametrization in form of family of sets $\mathbb{S}(\mathcal{S}, \Theta)$ given by (3.4). With this family of sets, we could capture possible shrinking and increasing effects of the set–iterates $\mathcal{X}_{i,k}$ and hence arrive to a more suitable notion of invariance for interconnected systems. Using this type of parametrization it was possible to provide strong and easily verifiable results on invariance, convergence and extensions to additive disturbances for positively homogeneous and linear interconnected systems, exemplified in Section 3.2.1

and Section 3.3, respectively. The analysis was made possible by relating the set–iterates $\mathcal{X}_{i,k}$ to a auxiliary system $\theta(k+1) = M\theta(k)$, which describes the dynamics of admissible scaling factors $\theta(k)$. Motivated by the recursion (2.5) for the direct determination of maximal positively invariant sets, we provided an adapted approach, that allowed us to investigate invariance properties for a given collection of polytopic sets $(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_N)$, subject to a collections of polytopic constraint sets $(\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_N)$ in Section 3.2.3. For interconnected linear systems, we could show that this approach is based on the repeated solution of linear programs and can therefore be easily implemented using standard numerical software tools. We exploited for the distributed and decentralized controller synthesis based on positively invariant family of sets the dynamic properties of the helper system $\theta(k+1) = M\theta(k)$. In order to construct an appropriate, positively invariant family of sets, we needed to guarantee asymptotic stability of this helper system by tuning the different controller parameters. Fortunately, the entries of the matrix M have a direct relation to the interconnection effects and stability properties of the different subsystems. In the linear case, we can think of the control setting as a class of systems that are interconnected through artificial outputs. As a result we showed that for the distributed controller synthesis an $\mathcal{H}_\infty$ max–min approach is a suitable design tool, since it allows us to easily tune input–output properties and hence influence in an iterative way the stability properties of the matrix M. An alternative approach, which is suitable for the decentralized controller design of linear systems, that are interconnected through positively homogeneous functions, utilizes the same, basic idea. However instead of employing an iterative procedure, we showed that a modular, LMI approach can be used, whenever it is possible to appropriately bound the interconnections. We highlighted in the benchmark example, considering a four tank systems, the different invariance and performance properties of the previously explained concepts. In particular, although the performance and robustness properties of the synthesized controllers have still potential of improvement, we exemplified the strong invariance properties and the flexibility to adapt to different disturbance scenarios.

6.1 Outlook and Future Directions

As highlighted in Chapter 3, we restricted our attention to specific parametrization of family of sets, since our focus was mainly on positively homogeneous and linear interconnected systems. An interesting future research direction is to investigate more general parametrization and their connection to specific classes of interconnected processes, such as bilinear or polynomial systems. In many cases we have interconnected systems that have shared state variables stemming from an overlapping decomposition of a large, complex processes. Further work should be focused on extending the concept of invariance of family of sets to this type of problem setting. In Section 3.2.3, we highlighted an algorithm for the determination of a possible large or maximal sets of scaling factors Θ for a given collection of sets $\mathcal{S}$. An interesting question is to investigate generalization of this algorithm, for instance by including additive disturbances or inputs. Furthermore, by relating it to the approach highlighted in (3.23), it might be possible rigorously analyze its properties with respect to convergence and maximality of the set Θ. Robust invariant sets are extensively used in modern, tube based model predictive control schemes. Robust positively invariant family of sets might be beneficial, if these predictive control methods are extended to interconnected systems. In Chapter 5, we presented some controller design procedures exploiting positively invariant family of sets. The focus for the synthesis of these controllers was solely on describing regions, that lead to a safe operation of the plants with respect to constraints. Although we could see in Chapter 6 that the methods work as intended, the overall performance was often not satisfactory. Hence, an interesting research questions is to improve the general performance of these methods, for instance by including an appropriate objective function in the LMI's of Theorem 13 or by employing a distributed algorithm in which the γ_i are exchanged and negotiated in the max–min/min–max approach of Section 4.1. A major focus of this work was to provide strong and rigorous theoretical framework, however it is still necessary to evaluate these methods for real plants in a realistic and challenging setting.

7 Appendix

Topological, Algebraic and Convex Concepts

In order to provide a self–contained thesis, we provide some additional useful facts related to convexity and topology, which were used throughout this thesis. For more details on these topics, we refer to standard textbooks such as [Kelley 1955; Schneider 1993; Boyd and Vandenberghe 2004].

Lemma 6. *$f(\mathcal{X})$ is compact, if $\mathcal{X} \subset \mathbb{R}^n$ is compact and $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is continuous.*

Proof. Elementary. $\square$

Lemma 7. *If $\mathcal{X} \subset \mathbb{R}^n$ and $\mathcal{Y} \subset \mathbb{R}^n$ are compact sets, then $\mathcal{X} \oplus \mathcal{Y}$ is a compact set.*

Proof. The Minkowski sum is a continuous operation, and hence the image $\mathcal{X} \oplus \mathcal{Y}$ of the compact set $\mathcal{X} \times \mathcal{Y}$ is compact. $\square$

Lemma 8.

$$f^{-1}\Big(\bigcap_{i\in\mathcal{I}} \mathcal{X}_i\Big) = \bigcap_{i\in\mathcal{I}} f^{-1}(\mathcal{X}_i),$$

where $\mathcal{I}$ is some index set.

Proof. By definition $x \in f^{-1}(\bigcap_{i\in\mathcal{I}} \mathcal{X}_i)$ implies $f(x) \in \bigcap_{i\in\mathcal{I}} \mathcal{X}_i$ and hence $f(x) \in \mathcal{X}_i$ for every $i \in \mathcal{I}$. This is the same as $f^{-1}(x) \in \mathcal{X}_i$ for every $i \in \mathcal{I}$ or $x \in \bigcap_{i\in\mathcal{I}} f^{-1}(\mathcal{X}_i)$. Conversely, if $x \in \bigcap_{i\in\mathcal{I}} f^{-1}(\mathcal{X}_i)$, then $x \in f^{-1}(\bigcap_{i\in\mathcal{I}} \mathcal{X}_i)$ by a similar argument. $\square$

Lemma 9. ***Matrix Inversion Lemma:***

$$(A + UCV)^{-1} = A^{-1} - A^{-1}U(C^{-1} + VA^{-1}U)^{-1}VA^{-1}$$

Proof.

$$
\begin{aligned}
&(A^{-1} - A^{-1}U(C^{-1} + VA^{-1}U)^{-1}VA^{-1})(A + UCV) = \\
&\quad I + A^{-1}UCV - A^{-1}U(C^{-1} + VA^{-1}U)^{-1}V - A^{-1}U(C^{-1} + VA^{-1}U)^{-1}VA^{-1}UCV = \\
&\quad I + A^{-1}U(C^{-1} + VA^{-1}U)^{-1}(I + VA^{-1}UC - I - VA^{-1}UC)V = I
\end{aligned}
$$

□

Lemma 10. *Given a convex set $\mathcal{X}$ and $\mathcal{Y}$, we have*

i) $s(G\mathcal{X}, y) = s(\mathcal{X}, G^T y)$,

ii) $s(\mathcal{X} \oplus \mathcal{Y}, y) = s(\mathcal{X}, y) + s(\mathcal{Y}, y)$,

iii) $s(\alpha\mathcal{X}, y) = \alpha s(\mathcal{X}, y)$,

for any matrix G of compatible dimension and any scalar α.

Proof. i)–iii) follows directly from the definition of the support function $s(\cdot, \cdot)$. □

Lemma 11. ***Schur Complement:*** *Let*

$$
S = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix},
$$

where $A = A^T \succ 0$, $C = C^T$ and $S = S^T$. S is positive semidefinite if and only if $C - B^T A^{-1} B$ is positive semidefinite.

Proof. First note that for any nonsingular matrix M, MSM^T is positive semidefinite iff S is positive semidefinite. We have

$$
\begin{pmatrix} I & 0 \\ -B^{\mathrm{T}}A^{-1} & I \end{pmatrix} \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \begin{pmatrix} I & 0 \\ -B^{\mathrm{T}}A^{-1} & I \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} A & 0 \\ 0 & C - B^T A^{-1} B \end{pmatrix}
$$

and since $A \succ 0$, in order for S to be positive semidefinite, $C - B^T A^{-1} B$ needs to be positive semidefinite. □

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