Undominated Complexes of Cut Polytopes

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Abstract

Polytopes are basic geometric objects of fundamental importance for Linear Programming (i.e., optimization of linear functions under linear constraints). One of the most important research topic of polytope theory is its combinatorial aspect: the combinatorial structure of polytopes. This thesis is primarily concerned with the concept of the undominated set of a polytope, i.e., the set of all those points of a polytope such that there is no other different point in the polytope which is component-wise less than or equal to it.

The interest in undominated sets originates in the fact that two polytopes (in the first orthant) have the same dominant if and only if their undominated sets coincide. The dominant of a polyhedron is the geometric object that is of interest when minimizing linear objective functions with nonnegative coefficients only. For such minimization problems over a polyhedron, the initial polyhedron can be replaced by another polyhedron with the same dominant but which is easier to describe, e.g. with fewer inequalities.

It turns out that the undominated set of a polytope is a polyhedral complex (the undominated complex), formed by those faces that are also faces of the dominant, or, equivalently, the bounded faces of the dominant. One result we establish about the undominated complexes of general polytopes is that they are contractible. While their topological structure thus is rather simple, we provide some three-dimensional examples showing that their geometry can nevertheless look surprisingly complicated.

The main part of the thesis is devoted to investigating the undominated complexes of the polytopes associated with the cuts in the complete undirected graph on nodes $\{1, 2, \ldots, n\}$ that separate at least one of the nodes $1, 2, \ldots, \xi$ from node n. We provide characterizations of the combinatorial structures of the undominated complexes of those cut polytopes for $\xi = 1, 2, 3$, showing that those complexes are pure simplicial complexes of dimension $n - 2$, $n - 1$, n in these cases, respectively. We also provide a conjecture for the combinatorial structures of those complexes for general ξ, for which a partial prove is given by the main contribution.

Zusammenfassung

Polytope sind geometrische Grundobjekte von fundamentaler Bedeutung für die Lineare Programmierung (d.h. die Optimierung linearer Funktionen unter linearen Nebenbedingungen). Eines der wichtigsten Forschungsthemen der Polytoptheorie ist der kombinatorischer Aspekt: die kombinatorische Struktur von Polytopen. In der vorliegenden Arbeit beschäftigen wir uns hauptsächlich mit dem Konzept der nichtdominierten Menge eines Polytops, d.h. die Menge aller Punkte eines Polytops, für die kein anderer Punkt im Polytop existiert, der komponenten weise kleiner oder gleich ist.

Das Interesse an nichtdominierten Mengen rührt daher, dass zwei Polytope (im ersten Orthanten) genau dann die gleiche Dominante haben, wenn ihre nichtdominierten Mengen gleich sind. Die Dominante eines Polyeders ist das geometrische Objekt, das von Interesse ist, wenn lineare Zielfunktionen nur mit nichtnegativen Koeffizienten minimiert werden. Für ein solches Minimierungsproblem über einem Polyeder kann das ursprüngliche Polyeder durch ein anderes Polyeder mit der gleichen Dominante ersetzt werden, das aber einfacher zu beschreiben ist, z.B. mit weniger Ungleichungen.

Es stellt sich heraus, dass die nichtdominierte Menge eines Polytops ein polyedrischer Komplex (der nichtdominierte Komplex) ist, der aus den Seiten besteht, die auch Seiten der Dominante sind, oder, äquivalent, die beschränkten Seiten der Dominante. Ein Resultat, das wir über die undominierten Komplexe allgemeiner Polytope feststellen, ist, dass sie zusammenziehbar sind. Während ihre topologische Struktur ziemlich einfach ist, geben wir einige dreidimensionale Beispiele, die zeigen, dass ihre Geometrie dennoch überraschend kompliziert aussehen kann.

Der Hauptteil der Arbeit untersucht nichtdominierte Komplexe der Polytope, die mit den Schnitten des vollständigen ungerichteten Graphen mit den Knoten $\{1, 2, \ldots, n\}$ verbunden sind, die mindestens einen von den Knoten 1, 2, ..., ξ vom Knoten n trennen. Wir geben Charakterisierungen der kombinatorischen Strukturen der undominierten Komplexe von diesen Cut Polytopen für $\xi = 1$, 2, 3, welche zeigen, dass diese Komplexe in diesen Fällen reine Simplizialkomplexe der Dimension $n-2$, $n-1$, n sind. Wir formulieren auch eine Vermutung für die kombinatorischen Strukturen dieser Komplexe für allgemeine ξ, für die ein Teilbeweis durch den Hauptbeitrag gegeben ist.

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Contents

Chapter 1

Introduction

Linear programming concerns the problem stated as minimization or maximization of a linear function over a polyhedron. The description of polyhedra by a linear system of equations and inequalities is one of the most important topics in Combinatorial Optimization. This method, known as Polyhedral Combinatorics, makes the linear programming approaches applicable to the problems of the combinatorial optimization. Although the Weyl-Minkowski Theorem (see Weyl [34], Minkowski [22]) guarantees the existence of such a description for every polyhedron, it could be a very challenging problem to find it.

The minimization problem with a nonnegative linear objective function over a polyhedron can be simplified by optimizing over the dominant or by replacing the initial polyhedron by another polyhedron with the same dominant but which is easier to describe, e.g. with fewer inequalities. This follows from the fact that minimizing a nonnegative linear objective function over a polyhedron is equivalent to minimizing the nonnegative linear objective function over its dominant (Observation 2.1.4). For more on dominants of polyhedra see Chapter 2, Section 2.1.

In the search for compact linear descriptions of polytopes, one can try to represent them as affine projections of higher-dimensional polytopes (an *extension*), i.e., they may allow a compact *extended formulation*. In practice this leads to the use of additional variables, which may have a negative impact on the running time of algorithms for solving linear programs. However, higher-dimensional polyhedra may have fewer facets than their images. For example, Figure 1.1 illustrates a regular octagon with 8 facets as an extension of a cube with 6 facets. For more extensive familiarity with extended formulations of combinatorial polytopes we refer to Conforti, Cornuéjols, and Zambelli [4], Pashkovich [25].

Finding compact extended formulations is of great importance for Linear Programming since they allow to formulate corresponding optimization problems as linear programming problems of small sizes, i.e., with small number of inequalities. It is even more important to know how to get small extension formulations with a few additional variables. A nice example of these compact extended formulations is the linear formulation of Carr, Kon-

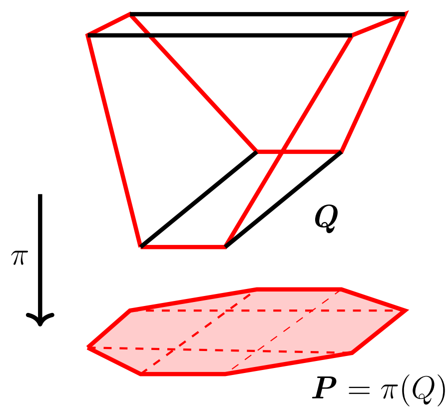

Figure 1.1: A regular octagon P as a projection of a cube Q.
(Fiorini, Kaibel, and Pashkovich [12])

jevod, Little, Natarajan and Parekh [3] for the minimum cut problem for the complete undirected graph on n nodes (see Chapter 1, Section 1.2). Their formulation uses only $O(n^2)$ variables and $O(n^3)$ constraints, which means the reduction of the number of variables by the factor n on the complete graph on n nodes, when compared with the smallest known linear formulation for the minimum cut problem before their work (see Conforti, Rinaldi and Wolsey [7]; Tamir [33]).

Carr, Konjevod, Little, Natarajan and Parekh specify a polyhedron P for which they have a small extended formulation and which has the same dominant as the cut polytope of the complete undirected graph. One of the key approaches in their proof is to consider those vectors x from a polyhedron P such that there is no y in P different from x with $y \leq x$. We define the subset of such vectors of P as the *undominated set* of P and introduce this concept in Chapter 2, Section 2.2. Carr, Konjevod, Little, Natarajan and Parekh did not work out this concept, they argued their result in a different way by giving two proofs. In the first proof they use the graph-theoretical notion of splitting off (see Frank [15], Lovász [21]). The second proof shows how an integral solution can be reconstructed from any optimal fractional solution what makes possible the recovery of a convex combination of cuts dominated by any given feasible fractional solution to the linear program.

In Chapter 2 we examine the structure and general geometrical properties of the undominated set. One of the obtained results is that the undominated set of a polytope is contractible (Corollary 2.3.7). Despite the simple topological structure, however, their geometry can look quite complicated, as one can see from the three-dimensional examples in Section 2.4. Additionally, we obtain that the undominated set of a polytope is a polyhedral complex (the undominated complex), formed by those faces that are also faces of the dominant, or, equivalently, the bounded faces of the dominant. Two polytopes in the nonnegative orthant have the same dominant if and only if their undominated sets are equal. In order to show how this characterization can be applied we rewrite in Chapter 1,

Section 1.2 the second version of Carr's, Konjevod's et al. proof in terms of the undominated set.
Minimizing a nonnegative linear objective function over the dominant of the cut polytope of a graph is equivalent to finding a cut of minimum capacity in the graph. Nevertheless, a description by a finite system of facet-defining inequalities and a structure of the faces for the dominant of the cut polytope are not known. According to blocking polarity (see Schrijver [27], Section 5.8) the vertices of the *Subtour Elimination Relaxation* of the Graphical Traveling Salesman Polyhedron yield an irredundant description of the dominant of the cut polytope. Hence, if we know a facet description of the dominant of the cut polytope then we know a description of the vertices of the subtour elimination relaxation (for detailed information we refer to Conforti, Fiorini, and Pashkovich [6]). This is one reason why the dominant of the cut polytope is of interest.

In this thesis we examine the combinatorial structure of the bounded faces of the dominant, i.e., the undominated complex, of certain cut polytopes. We study the cut polytopes which are the convex hulls of those cuts in the complete undirected graph on nodes $\{1, 2, \ldots, n\}$ that separate at least one of the nodes $1, 2, \ldots, \xi$ from node n. We define these cut polytopes as the $[\xi]$-n-cut polytopes.

The main part of this thesis, Chapter 3, is devoted to the undominated complexes of the $[\xi]$-n-cut polytopes for the complete graph on nodes $\{1, 2, \ldots, n\}$. The main contribution of this thesis is the elaboration of the vertex set description of the facets of the undominated complexes of those cut polytopes for $\xi = 1,\ 2,\ 3$. It turns out that those complexes are pure simplicial complexes of dimension $n-2$, $n-1$, n, respectively. In Section 3.7 we consider the undominated complexes of the $[\xi]$-n-cut polytopes for the complete graph on n nodes for general ξ. We find some faces of those complexes and provide Conjecture 3.7.3 for the complete description of their combinatorial structure, which was proved by the main contribution for the cases $\xi = 1,\ 2,\ 3$.

All images of undominated complexes (undominated sets) provided in Section 2.4 were made using *Polymake* in a collaboration with Kiryl Kukharenka. In Appendix A we present the program code of finding and visualization of the undominated complex of a three-dimensional polytope. Appendices B and C include additional lemmas which give a detailed proof for the results used in Lemma 3.7.2 and Proposition 3.7.1, respectively. Derivation of auxiliary results we collect in appendices for the sake of readability.

1.1 Preliminaries

This thesis borrows a lot of definitions, notation, concepts and statements from two areas of mathematics: convex geometry and combinatorial optimization. To make this work as self-contained as possible we introduce some basic facts in this section.

1.1.1 Basic facts of Convex Geometry

A *polytope* $P \subseteq \mathbb{R}^n$ is the convex hull of a finite set $X = \{x_1, x_2, \dots, x_m\}$, $m \in \mathbb{N}$ of points in $\mathbb{R}^n$, i.e.

$$P := \text{conv}(X) = \left\{ \sum_{i=1}^{m} \lambda_i x_i : \sum_{i=1}^{m} \lambda_i = 1,\ \lambda_1, \dots, \lambda_m \geq 0 \right\}. \tag{1.1}$$

In turn, a polyhedron $P \subseteq \mathbb{R}^n$ is an intersection of finitely many closed halfspaces in $\mathbb{R}^n$ and can be expressed as the Minkowski sum of the convex hull of a finite point set $X \subseteq \mathbb{R}^n$ and the cone generated by a finite set of vectors $Y \subseteq \mathbb{R}^n$, i.e.

$$P = \text{conv}(X) + \text{cone}(Y) \tag{1.2}$$

where $\text{cone}(Y) := \left\{ \sum_{i=1}^{q} \lambda_i y_i :\ y_1, \dots, y_q \in Y,\ \lambda_1, \dots, \lambda_q \geq 0 \right\}$ (see Conforti, Cornuéjols and Zambelli [5], Theorem 3.13).

For a polyhedron $P \subseteq \mathbb{R}^n$ we define any intersection of P with the boundary hyperplane of some affine halfspace containing P as a *face* of P. We consider the empty set and the polyhedron P itself to be faces of P as well. A *vertex* of P is a face of dimension zero; and a *facet* of P is a face of dimension $\dim(P) - 1$, i.e., an inclusion-wise maximal face different from P. A hyperplane that contains a facet of P (but not P itself) we call a *facet hyperplane* of P.

Example 1.1.1. *Let $P_X \subseteq \mathbb{R}^3$ be the polytope and $P_Y \subseteq \mathbb{R}^3$ be the polyhedron as in Figure 1.2 and Figure 1.3, respectively. Then, P_X and P_Y can be represented as follows:*

$$\begin{aligned} P_X &= \text{conv}(X), \\ P_Y &= \text{conv}(X) + \text{cone}(Y), \end{aligned} \tag{1.3}$$

where $X = \{x_1, x_2, \dots, x_{10}\} \subseteq \mathbb{R}^3$, $Y = \{y_1, y_2, \dots, y_5\} \subseteq \mathbb{R}^3$. As we can see, the polytope P_X has 10 vertices and 7 facets, the polyhedron P_Y has 5 vertices and 6 facets.

Every polytope $P \subseteq \mathbb{R}^n$ can be described by the Weyl-Minkowski Theorem (see Weyl [34], Minkowski [22]) as the solution set for a system of linear equalities and inequalities where the number of inequalities is equal to the number of facets of P. Thus polytopes are special kind of polyhedra, but not all polyhedra are polytopes, namely the Weyl-Minkowski Theorem for Polytopes (see Conforti, Cornuéjols and Zambelli [5], Corollary 3.14) states that every polytope is a polyhedron, and only every bounded polyhedron is a polytope.

The Farkas' Lemma gives a characterization if a system of linear inequalities is feasible (it has at least one solution). There are several equivalent variants of this lemma. Lemma 1.1.2 below is an affine version of Farkas' Lemma.

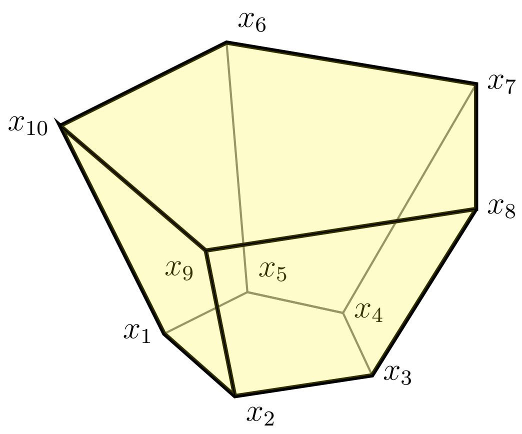

Figure 1.2: A polytope P_X.

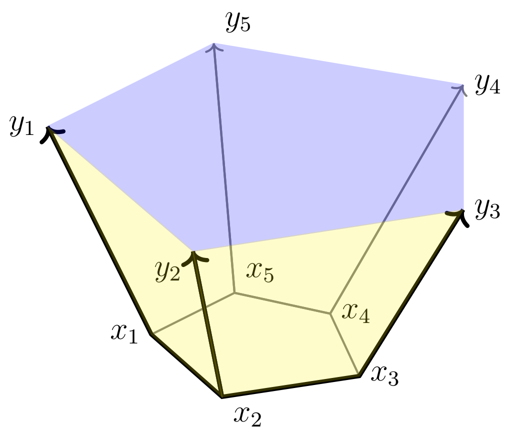

Figure 1.3: An unbounded polyhedron P_Y.

Lemma 1.1.2 (Farkas' Lemma; Schrijver [26], Corollary 7.1h).
Let $Ax \leq b$ be a feasible system of linear inequalities and let $\langle \alpha, x \rangle \leq \beta$ be an inequality that holds for each x with $Ax \leq b$. Then for some $\beta' \leq \beta$ the inequality $\langle \alpha, x \rangle \leq \beta'$ is a nonnegative linear combination of the inequalities in the system $Ax \leq b$.

The *recession cone* of the polyhedron P is defined as

$$\text{rec}(P) := \{y \in \mathbb{R}^n : \ x + y \in P \text{ for all } x \in P\} \tag{1.4}$$

and the *normal cone* of a polyhedron P at $\tilde{x} \in P$ is defined as

$$N_{\tilde{x}}(P) := \{y \in \mathbb{R}^n : \ \langle y, \, x - \tilde{x} \rangle \leq 0 \text{ for all } x \in P\}. \tag{1.5}$$

The normal cone to a polyhedron P at a face F we define as

$$N_F(P) := \bigcap_{\tilde{x} \in F} N_{\tilde{x}}(P). \tag{1.6}$$

The convex hull of any $k+1$ affinely independent points in $\mathbb{R}^n$ with $k \leq n$ is said to be a *simplex* of dimension k. A *simplicial complex* (see Chapter 2, Section 2.3.3) is a finite non-empty collection of simplicies $\mathcal{S}$ in $\mathbb{R}^n$ such that for all $S \in \mathcal{S}$ each face of S is also in $\mathcal{S}$, and for any two simplicies S_1, $S_2 \in \mathcal{S}$ we have that $S_1 \cap S_2$ is a face of each of S_1 and S_2.

For more detailed information from polyhedral theory we refer to the books of Grünbaum [16], Schrijver [26] and Ziegler [35]. Moreover, additional important information can be found in the mentioned references throughout all this thesis.

1.1.2 Basic facts of Combinatorial Optimization

An *undirected graph* is a pair $G = (V, E)$ consisting of a finite set V and a set $E \subseteq \binom{V}{2}$ of unordered pairs from V. The elements of V we call the *vertices*, or the *nodes*, of G, the elements of E are called the *edges* of G. The *complete undirected graph* $K_n = ([n], E_n)$ on n nodes $[n] := \{1, 2, ..., n\}$ is an undirected graph where any two vertices v, $w \in [n]$, $v \neq w$ are adjacent, i.e., for all v, $w \in [n]$ with $v \neq w$ we have $\{v, w\} \in E_n$.

In an undirected graph $G = (V, E)$ a sequence $\{\{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{k-1}, v_k\}\}$, $k \geq 1$ of edges of G is said to be a *v_0-v_k-path* if all v_0, v_1,. . .,$v_k \in V$ are pairwise distinct. If for each pair s, $t \in V$ there is a s-t-path in G then a graph G is called *connected.*

A *directed graph* is a pair $D = (V, A)$ consisting of a finite set V and a set

$$A \subseteq V \times V \setminus \{(v, v) : \; v \in V\} \tag{1.7}$$

of ordered pairs from V. The elements of V we call the *vertices*, or the *nodes*, of D, the elements of A we call the *arcs* of D.

In a directed graph $D = (V, A)$ a sequence $\{(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)\}$, $k \geq 1$ of arcs of D is said to be a (*directed*) *v_0-v_k-path* if all v_0, v_1,. . .,$v_k \in V$ are pairwise distinct.

In an undirected graph $G = (V, E)$ a (*proper*) *cut* is an edge set $\delta(S)$ with $\emptyset \neq S \subsetneq V$ where

$$\delta(S) = \{\{i, j\} \in E : |\{i, j\} \cap S| = 1\}. \tag{1.8}$$

If $s \in S$ and $t \notin S$ then $\delta(S)$ is an s-t-cut of G for s, $t \in V$.

In a directed graph $D = (V, A)$ a (proper) directed cut is a set

$$\delta^{out}(S) = \{(v, w) \in A : \; v \in S, w \notin S\}. \tag{1.9}$$

For s, $t \in V$ an *s-t-cut* in D is a cut $\delta^{out}(S)$ for some $S \subseteq A$ with $s \in S$ and $t \notin S$.

Let $G = (V, E)$ be an undirected graph and let $\mathcal{V}$ be a set of subsets of V. Then the cut-incidence matrix of $\mathcal{V}$ is defined as the matrix $\Theta_{\mathcal{V}} = (\theta_{X,e})_{X \in \mathcal{V}, e \in E}$ where

$$\theta_{X,e} = \begin{cases} 1, & \text{if } e \in \delta(X), \\ 0, & \text{if } e \notin \delta(X). \end{cases} \tag{1.10}$$

A *network* is a pair (D, c) consisting of a directed graph $D = (V, A)$ and nonnegative arcs capacities $c : A \to \mathbb{R}_+$. A *flow* in (D, c) is a function $f : A \to \mathbb{R}_+$ (or briefly $f \in \mathbb{R}^A_+$) with $f(a) \leq c(a)$ for all $a \in A$. For a vertex $v \in V$ we define the *excess* of a flow f at $v \in V$ via

$$\mathrm{ex}_f(v) := \sum_{a \in \delta^{in}(v)} f(a) - \sum_{a \in \delta^{out}(v)} f(a) \tag{1.11}$$

where $\delta^{out}(v) := \{(v, w) : \; w \in V, \; (v, w) \in A\}$ and $\delta^{in}(v) := \{(w, v) : \; w \in V, \; (w, v) \in A\}$. For s, $t \in V$ an *s-t-flow* is a flow $f \in \mathbb{R}^A_+$ satisfying $\mathrm{ex}_f(s) \leq 0$ and $\mathrm{ex}_f(v) = 0$ for all

$v \in V \setminus \{s,t\}$. The *value* of an s-t-flow f is defined as $\text{value}(f) := -\text{ex}_f(s)$, i.e.

$$\text{value}(f) = \sum_{a \in \delta^{out}(s)} f(a) - \sum_{a \in \delta^{in}(s)} f(a). \tag{1.12}$$

Theorem 1.1.3 below is the central result of network flow theory.

Theorem 1.1.3 (Max-Flow Min-Cut Theorem; Dantzig and Fulkerson [8]). *In a network the maximum value of an s-t-flow is equal to the minimum capacity of an s-t-cut.*

An alternative proof of the Max-Flow Min-Cut Theorem was given by Elias, Feinstein, and Shannon [11].

For more detailed familiarity with facts from combinatorial optimization we refer to Korte & Vygen [20], Schrijver [27], Schrijver [28] and Schrijver [29]. Furthermore, additional important information can be found in the mentioned references throughout all this work.

1.2 Small Linear Formulation for the Minimum Cut Problem

Let $G = (V, E)$ be an undirected graph and $c \in \mathbb{R}_+^E$ be a capacity vector with nonnegative edge capacities c_e for each $e \in E$. The problem of finding a cut of minimum total capacity (sum of the capacities of the edges in a cut) in a connected graph G is called the *minimum cut problem*, i.e.

$$\min\{\langle c, \chi(\delta(S))\rangle : \emptyset \neq S \subsetneq V\}. \tag{1.13}$$

The minimum cut problem is polynomial-time solvable (see Korte and Vygen [20], Chapter 8.7). Before Carr's, Konjevod's et al. work [3], the smallest known linear formulation for this problem required $\Theta(|V||E|)$ constraints and variables (see Conforti, Rinaldi and Wolsey [7]; Tamir [33]). Their formulation beats this bound, in a way that it has a variable for each edge and uses only $|V| - 1$ additional variables and $O(|V|^3)$ constrains. An immediate consequence of their result is the following compact linear formulation with $O(|V|^3)$ constrains and $O(|V|^2)$ variables:

$$\begin{array}{llll}
\text{minimize} & \sum_{ij \in E} c_{i,j} x_{i,j} & & \\
\text{subject to:} & \sum_{2 \le i \le n} z_i = 1 & & (1.14.1) \\
& x_{i,k} + x_{j,k} \geq x_{i,j} + 2z_k & \forall ij \in E, k \in V : i,j < k & (1.14.2) \\
& x_{i,k} \geq z_k & \forall i \in V, k \in V : i < k & (1.14.3) \\
& x_{i,j} \geq 0 & \forall ij \in E & (1.14.4) \\
& z_i \geq 0 & \forall i \in V \backslash \{1\} & (1.14.5)
\end{array} \tag{1.14}$$

Carr, Konjevod, Little, Natarajan and Parekh specify a polyhedron for which they have a small extended formulation and which has the same dominant as the cut polytope of the complete graph on n nodes. The key approach in their proof is to consider those vectors x from the polyhedron such that there is no y in P different from x with $y \leq x$. We define the subset of such vectors as the *undominated set*. We introduce this concept in Chapter 2, Section 2.2 and prove that in the nonnegative orthant two sets have the same dominant if and only if their undominated sets are equal. To show how to apply this characterization we rewrite below one of two proofs given in Carr, Konjevod et al. [3] using our notion.

We denote the set of points satisfying system (1.14.1)-(1.14.5) by P_{LF}, i.e.

$$P_{LF} := \{(x, z) \in [0,1]^{E_n} \times [0,1]^{\{2,\dots,n\}} : \\ \sum_{i=2}^{n} z_i = 1, \\ x_{ik} + x_{jk} \geq x_{ij} + 2z_k \quad \text{for all } i, j < k, \quad (1.15.1) \\ x_{ik} \geq z_k \quad \text{for all } i < k, \quad (1.15.2) \\ z \geq \mathbf{0} \quad \}. \tag{1.15}$$

The *cut polytope P_{cut}* for the complete undirected graph K_n on n nodes is the convex hull of the incidence vectors of all edge sets of nonempty cuts in K_n (see Chapter 3, Section 3.1 for more on the cut polytope). We further define also

$$P_{zcut} := \text{conv}\{\chi^{x,z}(S) : \emptyset \neq S \subseteq [n] \setminus \{1\}\} \tag{1.16}$$

with $\chi^{x,z}(S) := \big(\chi(\delta(S)), \mathbf{e}_{\min(S)}\big) \in \mathbb{R}^{E_n} \times \mathbb{R}^{\{2,\dots,n\}}$ for $\emptyset \neq S \subseteq [n] \setminus \{1\}$.

Theorem 1.2.1. *The dominant of the polytope P_{zcut} is equal to the dominant of the polytope which is defined by P_{LF}.*

Proof. Due to $P_{zcut} \subseteq P_{LF}$ and Remark 2.2.6 (ii) the dominant of the cut polytope is equal to the dominant of the polyhedron which is defined by P_{LF} if and only if

$$U(P_{LF}) \subseteq P_{zcut} \subseteq D(P_{LF}) \tag{1.17}$$

where $U(P_{LF})$ (the undominated set of P_{LF}) and $D(P_{LF})$ (the dominant of P_{LF}) are defined in Chapter 2, Sections 2.1 and 2.2, respectively. Thus it suffices to show that

$$U(P_{LF}) \subseteq P_{zcut}. \tag{1.18}$$

Lemma 1.2.2. *If $(x, z) \in U(P_{LF})$ then $x_{ij} \leq x_{ik} + x_{jk}$ for all $i, j, k \in [n]$.*

Proof. Let $(x, z) \in U(P_{LF})$. Choose j such that j is minimal with $x_{ij} > x_{ik} + x_{jk}$ and $i < j$. As $x_{ij} > x_{ik} + x_{jk}$ implies $k < j$, we attain $z_j \leq x_{kj} \leq x_{kj} + x_{ik} < x_{ij}$. Hence due to $(x, z) \in U(P_{LF})$ there is some $p < j$ with

$$x_{ip} = x_{ij} + x_{pj} - 2z_j > x_{ik} + x_{jk} + x_{pj} - 2z_j \geq x_{ik} + x_{kp}$$

what contradicts the minimality of j. □

Lemma 1.2.3. *Let* $(x,z) \in U(P_{LF})$, $(x',z') \in P_{LF}$, *and*

$$(\tilde{x},\tilde{z}) := \frac{1}{1-\lambda}\left((x,z) - \lambda\,(x',z')\right) \in P_{LF} \quad \textit{with} \quad 0 < \lambda < 1. \tag{1.19}$$

Then $(\tilde{x},\tilde{z})$ *is a point of* $U(P_{LF})$ *and* $(x',z') \in U(P_{LF})$.

Proof. The statement follows from Lemma 2.3.1, as if a convex combination of two points of a polyhedron is in a certain face then both those points are as well. □

We continue the proof of Theorem 1.2.1 by proving expression (1.18).

Let $(x^*,z^*) \in U(P_{LF})$ with $\mu := \max\{i : z_i^* > 0\}$ and $S^* := \{\mu\} \cup \{i > \mu : x_{\mu i}^* = 0\}$. Considering in (1.15.1) all triangle inequalities with μ, Lemma 1.2.2 implies that $x_{ij}^* = 0$ $\forall\ \{i,j\} \in E(S^*)$, $x_{ij}^* > 0\ \forall\ \{i,j\} \in \delta(S^*)$ and $\forall i \notin S^*\ \exists \delta_i > 0\ \forall j \in S^*$ such that $\delta_i = x_{ij}^*$.

Let $\lambda := \min\left(\{z_\mu^*\} \cup \{\delta_i : i \notin S^*\}\right) > 0$. Then $\lambda \leq 1$ due to the definition of P_{LF}. If $\lambda = 1$ then $(x^*,z^*) \in \chi^{x,z}(S^*) \in P_{zcut}$. Thus it remains to consider $0 < \lambda < 1$.

It is enough to show that $(\overline{x},\overline{z}) := (x^*,z^*) - \lambda \cdot \chi^{x,z}(S^*)$ satisfies the inequalities (1.15.1) and (1.15.2). Because then $\frac{1}{1-\lambda}(\overline{x},\overline{z}) \in P_{LF}$, hence in $U(P_{LF})$ due to Lemma 1.2.3, with support strictly contained in the support of (x^*,z^*), thus by induction in P_{zcut}, which by $(x^*,z^*) \in \operatorname{conv}\left\{\frac{1}{1-\lambda}(\overline{x},\overline{z}), \chi^{x,z}(S^*)\right\}$ yields $(x^*,z^*) \in P_{zcut}$.

First, it is easy to see that (1.15.2) holds. Indeed, for $i < j$ with $\{i,j\} \in \delta(S^*)$ if $j = \mu$ then $\overline{x}_{ij} - x_{ij}^* = \lambda = \overline{z}_{ij} - z_{ij}^*$, otherwise $j > \mu$ since $\{i,j\} \in \delta(S^*)$ and $\overline{z}_{ij} = z_{ij}^* = 0 \leq \overline{x}_{ij}$.

To show (1.15.1) we choose j such that j is minimal and there exist i, k with $i < j < k$ and $\overline{x}_{ij} > \overline{x}_{ik} + \overline{x}_{jk} - 2\overline{z}_k$. Clearly $k \geq \mu$ (otherwise $\overline{x}_{ij} = x_{ij}^*$, $\overline{x}_{ik} = x_{ik}^*$, $\overline{x}_{jk} = x_{jk}^*$, $\overline{z}_k = z_k^*$). Furthermore, $k \neq \mu$ (otherwise $\overline{x}_{ij} = x_{ij}^*$, $\overline{x}_{ik} + \overline{x}_{jk} - 2z_k = x_{ik}^* + x_{jk}^* - 2z_k^*$). Hence $k > \mu$ what implies $\overline{z}_k = 0$. Thus $\overline{x}_{ij} > \overline{x}_{ik} + \overline{x}_{jk}$ what implies that $\{i,k\}$, $\{j,k\} \in \delta(S^*)$, i.e. $\{i,j\} \notin \delta(S^*)$.

Therefore $\overline{x}_{ik} = x_{ij}^*$, $\overline{x}_{ik} = x_{ik}^* - \lambda$ and $\overline{x}_{jk} = x_{jk}^* - \lambda$, thus we have $x_{ij}^* > x_{ik}^* + x_{jk}^* - 2\lambda \geq 0$. In particular, due to $\{i,j\} \notin \delta(S^*)$ and $x_{ij}^* > 0$ we have $i,j \notin S^*$ and $k \in S^* \setminus \{\mu\}$ thus $x_{ik}^* = \delta_i = x_{i\mu}^*$, $x_{jk}^* = \delta_j = x_{j\mu}^*$, from which we further deduce $x_{ij}^* > x_{i\mu}^* + x_{j\mu}^* - 2\lambda \geq x_{i\mu}^* + x_{j\mu}^* - 2z_\mu^*$ what implies $j > \mu$ and thus $x_{ij}^* > 0 = z_j^*$.

By the minimality of (x^*,z^*) these hence must be some $p < j$ $(p \neq i)$ with

$$\begin{aligned} x_{ip}^* &= x_{ij}^* + x_{pj}^* - z_j^* = x_{ij}^* + x_{pj}^* \\ &> x_{ik}^* + x_{jk}^* - 2\lambda + x_{pj}^* \overset{\text{Lemma 1.2.2}}{\geq} x_{ik}^* + x_{pk}^* - 2\lambda = \overline{x}_{ik} + \overline{x}_{pk} - \lambda. \end{aligned} \tag{1.20}$$

If $p \notin S^*$ then the inequality yields $\overline{x}_{ip} = x_{ip}^* > \overline{x}_{ik} + x_{pk}^* - \lambda = \overline{x}_{ik} + \overline{x}_{pk} = \overline{x}_{ik} + \overline{x}_{pk} - 2\overline{z}_k$ $(\overline{z}_k = 0)$ what contradicts the minimality of j. If $p \in S^*$ we have $x_{ij}^* > \delta_i + \delta_j - 2\lambda$ and $\delta_i = x_{ij}^* + \delta_j$ implying $\delta_i - \delta_j > \delta_i + \delta_j - 2\lambda$, thus $\lambda > \delta_j$ contradicting the choice of λ. □

Chapter 2

Undominated Sets and some Geometrical Properties

This chapter is devoted to the dominants and the undominated sets of polyhedra. We start with a definition and some properties of the dominant of a polyhedron. Then, in Section 2.2, we define the undominated set and elaborate on its relation to the dominant. Furthermore, we present some geometrical properties of undominated sets of polytopes.

2.1 Dominants of Polyhedra

In this section and further below $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : \; x \geq \mathbf{0}\}$.

For any subset X of $\mathbb{R}^n$ we define

$$D(X) := X + \mathbb{R}^n_+. \tag{2.1}$$

This is the *dominant* of X. Thus the dominant $D(X)$ of X is the set of all vectors $y \in \mathbb{R}^n$ such that $y \geq x$ for some $x \in X$, i.e.

$$D(X) = \{y \in \mathbb{R}^n : \; \exists x \in X \text{ with } y \geq x\} \tag{2.2}$$

and it is always full dimensional (if $X \neq \emptyset$). For example, Figure 2.1 illustrates the dominant of some set $P \subseteq \mathbb{R}^2$.

Observation 2.1.1. *For X, $Y \in \mathbb{R}^n$ we have:*

(a) $D(X \cup Y) = D(X) \cup D(Y)$,

(b) $D(X \cap Y) \subseteq D(X) \cap D(Y)$,

(c) $D(X \times Y) = D(X) \times D(Y)$,

(d) $D(\text{conv}(X)) = \text{conv}(D(X))$.

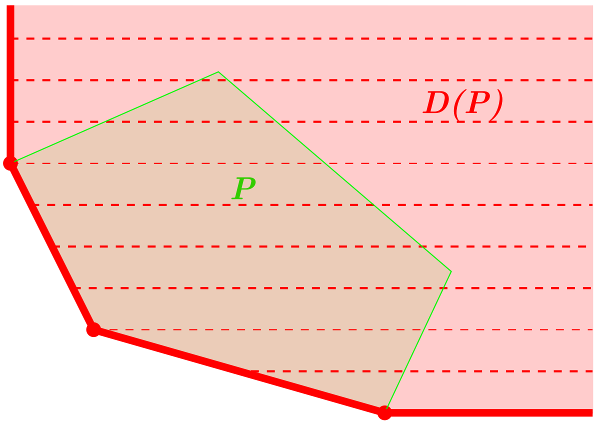

Figure 2.1: The dominant of P.
The set P is in green, and the dominant of P is in red.

Proof. Properties (a)-(c) follow immediately from (2.1) and (2.2).

Let us prove property (d). From (2.1) we have

$$D(\text{conv}(X)) = \text{conv}(X) + \mathbb{R}^n_+, \quad \text{conv}(D(X)) = \text{conv}(X + \mathbb{R}^n_+). \tag{2.3}$$

Let $\sum\limits_i \lambda_i x^i + v \in D(\text{conv}(X))$ where $x^i \in X$, $\lambda_i \geq 0$ with $\sum\limits_i \lambda_i = 1$ and $v \in \mathbb{R}^n_+$. Then we obtain

$$\sum_i \lambda_i x^i + v = \sum_i \lambda_i (x^i + v) \in \text{conv}(X + \mathbb{R}^n_+) = \text{conv}(D(X)). \tag{2.4}$$

Thus, the inclusion $D(\text{conv}(X)) \subseteq \text{conv}(D(X))$ holds.

Now let $\sum\limits_i \lambda_i (x^i + v^i) \in \text{conv}(D(X))$ where $x^i \in X$, $\lambda_i \geq 0$ with $\sum\limits_i \lambda_i = 1$ and $v^i \in \mathbb{R}^n_+$. Then we have

$$\sum_i \lambda_i (x^i + v^i) = \sum_i \lambda_i x^i + \sum_i \lambda_i v^i \ \in \ \text{conv}(X) + \mathbb{R}^n_+ = D(\text{conv}(X)) \tag{2.5}$$

what proves the inclusion $\text{conv}(D(X)) \subseteq D(\text{conv}(X))$.

□

Remark 2.1.2. *Note that, in (b) the reverse inclusion is not true in general. For example, if X, $Y \subseteq \mathbb{R}^n$ with X, $Y \neq \emptyset$ such that $X \cap Y = \emptyset$ then $D(X \cap Y) = \emptyset$ in contrast to $D(X) \cap D(Y) \neq \emptyset$.*

Observation 2.1.3. *Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^n$. Then*

$$\text{conv}(D(X) \cup D(Y)) = D(\text{conv}(X \cup Y)). \tag{2.6}$$

Proof. The proof follows from properties (a) and (d) of Observation 2.1.1. □

The next observation shows that minimization problems over a polyhedron can be solved over its dominant for linear objective functions with nonnegative coefficients only.

Observation 2.1.4. *Let $P \subseteq \mathbb{R}^n$ be a polyhedron. Then*

$$D(P) = \{y \in \mathbb{R}^n : \langle a, y\rangle \geq \inf_{x\in P} \langle a, x\rangle \text{ for all } a \in \mathbb{R}^n_+\}. \tag{2.7}$$

Proof. First, we observe that

$$D(P) \subseteq \{y \in \mathbb{R}^n : \langle a, y\rangle \geq \inf_{x\in P} \langle a, x\rangle \text{ for all } a \in \mathbb{R}^n_+\} \tag{2.8}$$

since for each $\tilde{y} \in D(P)$ there exists some $x \in P$ such that $\tilde{y} \geq x$ and then for all $a \in \mathbb{R}^n_+$ we have

$$\langle a, \tilde{y}\rangle \geq \langle a, x\rangle \geq \inf_{x\in P} \langle a, x\rangle. \tag{2.9}$$

To show the reverse inclusion, we suppose that there exists $y^* \in D(P)$ such that y^* is not contained in the right-hand side of (2.7). Then by a separation theorem (see Stoer and Witzgall [32], Corollary (3.3.8)), since $D(P)$ is closed and convex (in fact: a polyhedron), there is some $\langle a, y\rangle \geq \beta$ with $a \in \mathbb{R}^n_+$, $\beta \in \mathbb{R}$ valid for $D(P)$ with $\langle a, y^*\rangle < \beta$, what leads to a contradiction. Hence the right-hand side of equation (2.7) is contained in $D(P)$.

□

2.2 Undominated Set

Let X be a subset of $\mathbb{R}^n$. We say that $x \in X$ is an *undominated point* of X if $x' \leq x$ and $x' \in X$ imply $x' = x$. The set of all undominated points of X we define as $U(X)$ and call it the *undominated set* of X. For example, Figure 2.2 illustrates the undominated set of some set $P \subseteq \mathbb{R}^2$.

A face of X containing only undominated points of X is said to be an *undominated face* of X. In addition, the *undominated complex* of X is the set of all faces of the undominated set of X and denoted by $U_c(X)$. Note that the undominated set is the support of the undominated complex, i.e., the union of all the faces in the complex.

Observation 2.2.1. *For $X, Y \in \mathbb{R}^n$ we have:*

(a) $U(X \cup Y) \subseteq U(X) \cup U(Y)$,

(b) $U(X \cap Y) \supseteq U(X) \cap U(Y)$,

(c) $U(X \times Y) = U(X) \times U(Y)$,

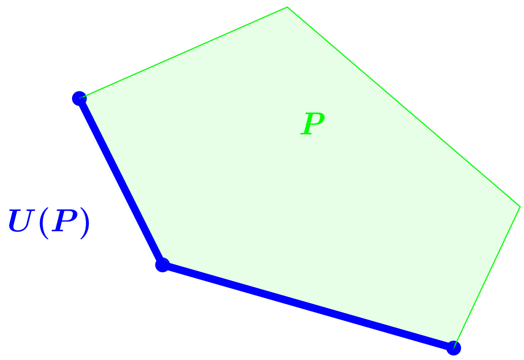

Figure 2.2: The undominated set of P.
The set P is in green, and the undominated set of P is in blue.

(d) if X is a subset of Y then $U(Y) \cap X \subseteq U(X)$, in particular

$$U(\mathrm{conv}(X)) \cap X \subseteq U(X).$$

Remark 2.2.2. *In general the undominated set of $X \in \mathbb{R}^n$ is not contained in the undominated set of the convex hull of X, i.e.*

$$U(X) \nsubseteq U(conv(X)).$$

Proof. Let $X \subseteq \mathbb{R}^2$ be the set as in Figure 2.3 a). Then we can see that the undominated set of X (see Figure 2.3 b)) is not a subset of the undominated set of the convex hull of X (see Figure 2.3 d)). □

Below we provide some properties of the dominant and the undominated set which show relations of the undominated set to the dominant.

Corollary 2.2.3. *Let $X \subseteq \mathbb{R}^n$ be closed and such that $X \cap (\mathbb{R}^n \setminus \mathbb{R}^n_+)$ bounded. Then for all $x \in X$ there is $\tilde{x} \in U(X)$ such that $\tilde{x} \leq x$, in particular*

$$D(X) = D(U(X)). \tag{2.10}$$

Proof. Since $X \cap (\mathbb{R}^n \setminus \mathbb{R}^n_+)$ is bounded there is $z \in \mathbb{R}^n$ such that $X \cap (\mathbb{R}^n \setminus \mathbb{R}^n_+) \subseteq z + \mathbb{R}^n_+$. Let $X^{\leq x} = \{y \in X : \ y \leq x\}$ for $x \in X$. Then the set $X^{\leq x}$ is bounded because for all $y \in X^{\leq x}$, $x \in X$ we have $z \leq y \leq x$ and closed as the intersection of the closed set X with the halfspaces of all $(y_1, y_2, ..., y_n)$ defined by $y_1 \leq x_1, \ldots, y_n \leq x_n$. Therefore $X^{\leq x}$ is compact and choosing $\tilde{x}$ as an optimal solution to $\min\{\langle \mathbf{1}, y\rangle : y \in X^{\leq x}\}$ completes the proof.

□

Remark 2.2.4. *It is important that $X \cap (\mathbb{R}^n \setminus \mathbb{R}^n_+)$ bounded because for $X = \mathbb{R}^n_-$ we have*

$$U(X) = \emptyset, \ D(X) = \mathbb{R}^n \quad \textit{however} \quad D(X) \neq D(U(X)) = \emptyset.$$

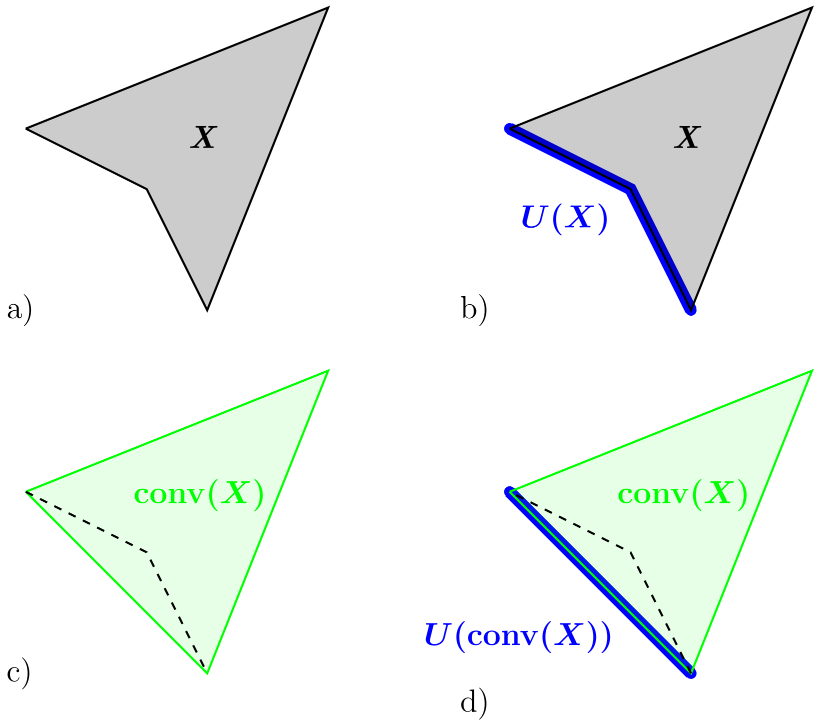

Figure 2.3: Noninclusion of the undominated set of X in the undominated set of the convex hull of X.

Corollary 2.2.5. *If two sets $X_1, X_2 \subseteq \mathbb{R}^n$ have the same dominant then their undominated sets are equal as well.*

Proof. Let $D(X_1) = D(X_2)$ thus

$$\forall x_1 \in X_1 \;\; \exists \phi_2(x_1) \in X_2 \quad \text{with} \;\; \phi_2(x_1) \leq x_1 \tag{2.11}$$

and

$$\forall x_2 \in X_2 \;\; \exists \phi_1(x_2) \in X_1 \quad \text{with} \;\; \phi_1(x_2) \leq x_2. \tag{2.12}$$

Now let $x_1 \in U(X_1)$. In $\phi_1(\phi_2(x_1)) \leq \phi_2(x_1) \leq x_1$ both inequalities are tight due to $x_1 \in U(X_1)$, in particular $x_1 = \phi_2(x_1) \in X_2$. For $y_2 \in X_2$ with $y_2 \leq x_1$ we have that both inequalities in $\phi_1(y_2) \leq y_2 \leq x_1$ are also tight due to $x_1 \in U(X_1)$, thus we conclude $x_1 \in U(X_2)$. Similarly we find $U(X_2) \subseteq U(X_1)$.

□

Remark 2.2.6. *Thus for $X_1, X_2 \subseteq \mathbb{R}^n$ closed such that $X_1 \cap (\mathbb{R}^n \setminus \mathbb{R}^n_+)$ and $X_2 \cap (\mathbb{R}^n \setminus \mathbb{R}^n_+)$ are bounded we have:*

(i) $D(X_1) = D(X_2)$ *if and only if* $U(X_1) = U(X_2)$;

(ii) $D(X_1) = D(X_2)$ *if and only if*

$$U(X_1) \subseteq D(X_2) \quad \text{and} \quad U(X_2) \subseteq D(X_1).$$

Example 2.2.7. *If* $X_1 = \emptyset$ *and* $X_2 = \mathbb{R}^n_-$ *we have*

$$U(X_1) = U(X_2) = \emptyset \quad \textit{however} \quad D(X_1) \neq D(X_2) \tag{2.13}$$

since $D(X_1) = \emptyset,\ D(X_2) = \mathbb{R}^n$.

2.3 General Geometrical Properties of Undominated Set

Let $\mathcal{C}_M(P)$ denote the set of all faces F of a polyhedron $P \subseteq \mathbb{R}^n$ such that the normal cone $N_F(P)$ intersects $\mathbb{R}^n_{<0} := \{x \in \mathbb{R}^n : x < \mathbf{0}\}$, i.e.,

$$\mathcal{C}_M(P) := \{F \in \text{faces}(P) :\ N_F(P) \cap \mathbb{R}^n_{<0} \neq \emptyset\}. \tag{2.14}$$

For example, Figure 2.4 shows the undominated set $U(P)$ of some polytope $P \subseteq \mathbb{R}^2$ with the normal cones of P which have a non-empty intersection with the negative octant $\mathbb{R}^2_{<0}$.

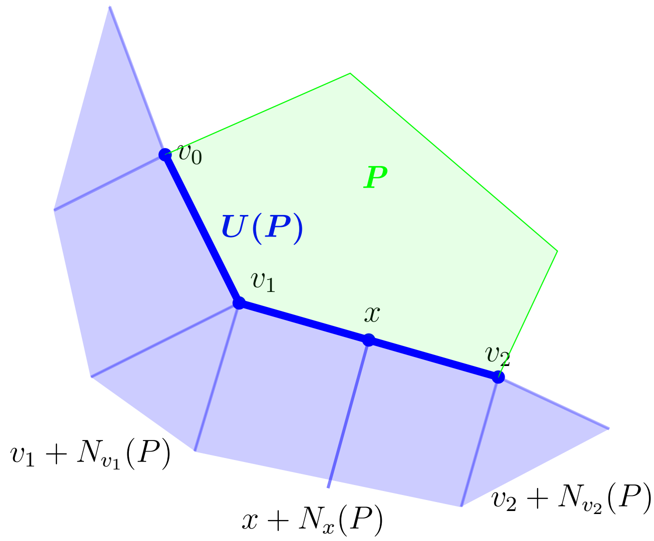

Figure 2.4: Normal cones of $P \subseteq \mathbb{R}^2$ which intersect $\mathbb{R}^2_{<0}$.

Lemma 2.3.1. *For a polyhedron* $P \subseteq \mathbb{R}^n$ *the undominated set* $U(P)$ *is the union of all faces* F *of* P *such that the normal cone* $N_F(P)$ *has a non-empty intersection with the negative orthant* $\mathbb{R}^n_{<0}$.

Proof. Let first F be a face with some $c \in N_F(P)$ with $c_i < 0$ for all i. For every $x^* \in F$ we have $\langle c, x^* \rangle = \max\{\langle c, x \rangle : x \in P\}$, thus for each $x \in P$ with $x \leq x^*$ we have $x = x^*$ (as otherwise $\langle c, x \rangle > \langle c, x^* \rangle$).

Conversely, for $x^* \in U(P)$ let F be the smallest face of P containing x^*. It is clear that $N_{x^*}(P) = N_F(P)$, hence it suffices to exhibit some vector $c \in N_{x^*}(P) \cap \mathbb{R}^n_{<0}$.

Due to $x^* \in U(P)$ for any $x \in P$ with $x \leq x^*$ we have $-x \leq -x^*$. Then by Farkas' Lemma 1.1.2 for all i there exists $(a^i, \beta_i) \in \mathbb{R}^{n+1}$ and $\gamma^i \in \mathbb{R}^n_+$ such that $\langle a^i, x\rangle \leq \beta_i$ is valid for P and the inequality $-x_i \leq -x_i^*$ is the sum of $\langle a^i, x\rangle \leq \beta_i$ and $\langle \gamma^i, x\rangle \leq \langle \gamma^i, x^*\rangle$. Then we have

$$-\mathbf{e}_i = a^i + \gamma^i, \qquad -x_i^* = \beta_i + \langle \gamma^i, x^*\rangle \tag{2.15}$$

what implies

$$a^i = -\mathbf{e}_i - \gamma^i, \quad \beta_i = \langle -\mathbf{e}_i - \gamma^i, x^*\rangle. \tag{2.16}$$

Thus we can choose $c := \sum_{i=1}^{n} a^i$ for our purposes. Indeed, $c_i = -\mathbf{e}_i - \gamma^i < 0$ for all i and

$$\langle c, x\rangle = \sum_{i=1}^{n} \langle -\mathbf{e}_i - \gamma^i, x\rangle = \sum_{i=1}^{n} \langle a^i, x\rangle \leq \sum_{i=1}^{n} \beta_i \tag{2.17}$$

is valid for P with

$$\langle c, x^*\rangle = \sum_{i=1}^{n} \langle -\mathbf{e}_i - \gamma^i, x^*\rangle = \sum_{i=1}^{n} \langle a^i, x^*\rangle = \sum \beta_i. \tag{2.18}$$

□

Lemma 2.3.2. *For each polytope $P \subseteq \mathbb{R}^n$ we have*

$$\mathcal{C}_M(P) = \text{faces}(D(P)) \cap \text{faces}(P) = \{F \in \text{faces}(D(P)) : F \textit{ bounded}\}. \tag{2.19}$$

Proof. First we show that $\mathcal{C}_M(P)$ is contained in the set defined by the second term of (2.19). Let $F \in \mathcal{C}_M(P)$ such that $\langle a, x\rangle \geq \beta$ defines F as a face of P with $a \in \mathbb{R}^n_{>0}$, $\beta \in \mathbb{R}$. Now we want to show that the face $\widetilde{F}$ of $D(P)$ which is defined by $\langle a, x\rangle \geq \beta$ is equal to F.

Clearly, $F \subseteq \widetilde{F}$. Now let $y \in P$, $z \in \mathbb{R}^n_+$ such that $y + z \in \widetilde{F}$ then

$$\beta = \langle a, y + z\rangle = \langle a, y\rangle + \langle a, z\rangle \text{ where } \langle a, y\rangle \geq \beta \text{ and } \langle a, z\rangle \geq 0. \tag{2.20}$$

Hence $\langle a, y\rangle = \beta$ and since $a \in \mathbb{R}^n_{>0}$ and $z \geq \mathbf{0}$ we have $z = \mathbf{0}$. Therefore $y + z = y \in F$ what implies $\widetilde{F} \subseteq F$.

It is clear that the set defined by the second term of (2.19) is contained in the set defined by the third term of (2.19) since each polytope is a bounded polyhedron.

It remains to prove that the set defined by the third term of (2.19) is contained in $\mathcal{C}_M(P)$. It follows from Lemma 2.3.1 since all faces of $D(P)$ are defined by inequalities $\langle a, x\rangle \geq \beta$ with $a \geq \mathbf{0}$ and if such a face is bounded, no a_i can be 0 (as $\mathbb{R}^n_+ \subseteq \text{rec}(D(P))$).

□

From Corollary 2.2.3, Observation 2.1.1 and Lemma 2.3.1, we obtain the following observation.

Observation 2.3.3. *For a polyhedron $P \subseteq \mathbb{R}^n$*

$$D(P) = \bigcup_{F \in \mathcal{C}_M(P)} D(F). \tag{2.21}$$

Remark 2.3.4. *Each $F \in \mathcal{C}_M(P)$ is a face of its dominant $D(F)$.*

2.3.1 Contractibility

Let X be a non-empty set in $\mathbb{R}^n$. If each closed curve in X can within X be contracted to a single point within X then X is said to be *contractible*.

Lemma 2.3.5 (Miller and Sturmfels [23])**.** *Let F be a face of a polytope Q. If K is the subcomplex of* bd(Q) *consisting of all faces of Q that are disjoint from F, then K is contractible.*

Theorem 2.3.6. *The union of all bounded faces of a convex pointed polyhedron P in $\mathbb{R}^n$ is a contractible topological space.*

Proof. Since the polyhedron P is pointed rec(P) is also pointed what implies that the polar recession cone of P is full-dimensional.

Let $H^=$ be a hyperplane such that $H^\leq$ is a halfspace with normal vector from the interior of the polar recession cone of P that contains all vertices of P. Then $P \cap H^\leq$ is bounded and the convex hull of the vertices of P lies strictly on the one side of this hyperplane $H^=$. Now we cut the unbounded polyhedron P with the hyperplane $H^=$. Then the unbounded faces of the polyhedron P are exactly the faces of P that intersect this hyperplane $H^=$. From Lemma 2.3.5 follows that if we take the intersection of P with $H^=$ as the face F then K is exactly the union of all bounded faces of P which build the subcomplex of bd(P) consisting of all faces disjoint from F and K is contractible. □

Corollary 2.3.7. *The undominated set of a polytope $P \subseteq \mathbb{R}^n$ is contractible, and thereby connected.*

Proof. From Lemma 2.3.2 we have that all faces of the undominated set of a polytope P are the bounded faces of its dominant which are also faces of the P. Thus Theorem 2.3.6 implies that the undominated set is contractible, and thereby connected. □

In the next subsection we provide another proof that the undominated set of a polytope $P \subseteq \mathbb{R}^n$ is (path-)connected (for all p_1, $p_2 \in P$ there is a path in P from p_1 to p_2) and for $n \leq 3$ also contractible. Our proof works specifically for the undominated set and not directly for any bounded subcomplex of an unbounded polyhedron.

2.3.2 Connectedness

Let $X \subseteq \mathbb{R}^n$. We say that a point x is a *relative interior point* of X if there exists some $\varepsilon > 0$ such that $B_\varepsilon(x) \cap \text{aff}(X) \subseteq X$, where $\text{aff}(X)$ is the affine hull of X and $B_\varepsilon(x)$ is the ball of radius ε centered at x. The set of all relative interior points of X is called the *relative interior* of X and denoted by $\text{relint}(X)$, i.e.

$$\text{relint}(X) := \{x \in X : \ \exists \varepsilon > 0 \ \text{ with } \ B_\varepsilon(x) \cap \text{aff}(X) \subseteq X\} \tag{2.22}$$

where

$$\begin{aligned} B_\varepsilon(x) &:= \{y \in \mathbb{R}^n : \ \|y - x\| < \varepsilon\}, \\ \text{aff}(X) &:= \left\{ \sum_{i=1}^{m} \lambda_i x_i : \ m \in \mathbb{N}, \ x_i \in X, \ \sum_{i=1}^{m} \lambda_i = 1 \right\}. \end{aligned} \tag{2.23}$$

From Ziegler [35] we attain that every polytope $P \subseteq \mathbb{R}^n$ decomposes into the disjoint union of the relative interiors of its faces

$$P = \biguplus_{F \text{ face of } P} \text{relint}(F). \tag{2.24}$$

Lemma 2.3.8. *If $Q \subseteq \mathbb{R}^n$ is a polyhedron, and H, K are faces of polyhedron Q such that* $\text{relint}(H) \cap K \neq \emptyset$ *then $H \subseteq K$.*

Proof. Let $v \in \text{relint}(H) \cap K$ and

$$\begin{aligned} \mathcal{F}_H &:= \{F \in \text{faces}(Q) : H \subseteq F\}, \\ \mathcal{F}_K &:= \{F \in \text{faces}(Q) : K \subseteq F\}, \\ \mathcal{F}_v &:= \{F \in \text{faces}(Q) : v \in F\}. \end{aligned} \tag{2.25}$$

From $v \in K$ it follows that every face that contains K also contains v, hence $\mathcal{F}_K \subseteq \mathcal{F}_v$.

Due to $v \in H$ every face that contains H also contains v hence $\mathcal{F}_H \subseteq \mathcal{F}_v$.

Let L be a face of Q that contains v, i.e. $L \in \mathcal{F}_v$. Then the intersection of L and H is not empty since it contains v. Each face of Q intersecting with a face of H gives a face of H and the intersection of two faces of a polyhedron is a face of both. Thus $L \cap H = H$ or the intersection of L and H is an edge of H. If $L \cap H = H$ then $L \in \mathcal{F}_H$ what implies $\mathcal{F}_v \subseteq \mathcal{F}_H$. If $L \cap H$ is an edge of H then v is contained in this edge since $v \in L$, but $v \in \text{relint}(H)$. Thus we have $\mathcal{F}_v \subseteq \mathcal{F}_H$ and consequently $\mathcal{F}_H = \mathcal{F}_v$.

It is known that every face of a polyhedron Q is the intersection of all faces of Q that contain it. Thus we have

$$H = \bigcap_{F \in \mathcal{F}_H} F = \bigcap_{F \in \mathcal{F}_v} F \overset{\mathcal{F}_K \subseteq \mathcal{F}_v}{\subseteq} \bigcap_{F \in \mathcal{F}_K} F = K \tag{2.26}$$

what complete the proof.

□

Theorem 2.3.9. *For any polytope $P \subseteq \mathbb{R}^n$ the undominated set $U(P)$ is (path-)connected.*

Proof. Suppose that x, $\tilde{x} \in U(P)$ with $x \in F$, $\tilde{x} \in \tilde{F}$ such that $F, \tilde{F} \in \mathcal{C}_M(P)$. From the definition of $\mathcal{C}_M(P)$, see (2.14), we have

$$\mathcal{C}_M(P) := \{G \in \text{faces}(P) : N_G(P) \cap \mathbb{R}^n_{<0} \neq \emptyset\}. \tag{2.27}$$

It has to be shown that there is a path between x and $\tilde{x}$ which only goes through faces of $U(P)$. For this it should be find a sequence of faces $F = G_1$, G_2, ..., $G_k = \tilde{F} \in \mathcal{C}_M(P)$ with $x \in G_1$, $\tilde{x} \in G_k$ and $G_i \cap G_{i+1} \neq \emptyset$ for $i = 1, \ldots, k-1$.

Let $y \in N_F \cap \mathbb{R}^n_{<0}$ and $\tilde{y} \in N_{\tilde{F}} \cap \mathbb{R}^n_{<0}$. Then y is from the normal cone to P at the face F which contains x and $\tilde{y}$ is from the normal cone to P at the face $\tilde{F}$ which contains $\tilde{x}$. Let us consider $s = \text{conv}\{y, \tilde{y}\} \subseteq \mathbb{R}^n_{<0}$.

Since P is a polytope then the normal cones at faces of P cover all $\mathbb{R}^n$

$$\mathbb{R}^n = \bigcup_{\substack{F \in \text{faces}(P) \\ F \neq \emptyset, P}} N_F \tag{2.28}$$

and, furthermore, the space $\mathbb{R}^n \setminus \{\mathbf{0}\}$ is decomposed into a disjoint union of the relative interiors of normal cones at faces of P

$$\mathbb{R}^n \setminus \{\mathbf{0}\} = \biguplus_{\substack{F \in \text{faces}(P) \\ F \neq \emptyset, P}} \text{relint}(N_F). \tag{2.29}$$

Then for each point on the line segment s we can uniquely define a normal cone in which relative interior it is. On this way we obtain s separated into convex parts since a relative interior of normal cone is convex. This mean that the intersection this line segment with every relative interior of the normal cone will be a (one-point or longer) interval.

Let the relative interior of normal cones of faces $G_i \in \mathcal{C}_M(P)$, $i = 1, ..., k$, separate the line segment s into parts in the following way

$$\begin{aligned}
\text{relint}(N_{G_1}) \cap s &= [y, y_1), \\
\text{relint}(N_{G_2}) \cap s &= y_1, \\
\text{relint}(N_{G_3}) \cap s &= (y_1, y_2), \\
\text{relint}(N_{G_4}) \cap s &= y_2, \\
\text{relint}(N_{G_5}) \cap s &= (y_2, y_3), \\
&\ldots \\
\text{relint}(N_{G_{k-1}}) \cap s &= y_l, \\
\text{relint}(N_{\tilde{F}}) \cap s = \text{relint}(N_{G_k}) \cap s &= (y_l, y]
\end{aligned}$$

what implies by Lemma 2.3.8 that

$$\begin{aligned} N_{G_2} \subseteq N_{G_1}, &\quad N_{G_2} \subseteq N_{G_3} \\ N_{G_4} \subseteq N_{G_3}, &\quad N_{G_4} \subseteq N_{G_5}, \\ \dots & \\ N_{G_{k-1}} \subseteq N_{G_{k-2}}, &\quad N_{G_{k-1}} \subseteq N_{G_k}. \end{aligned}$$

From $N_{G_2} \subseteq N_{G_1}$ and $N_{G_2} \subseteq N_{G_3}$ follows $G_1 \subseteq G_2$ and $G_3 \subseteq G_2$, respectively. Similarly, since $N_{G_i} \subseteq N_{G_{i-1}}$ and $N_{G_i} \subseteq N_{G_{i+1}}$ we have $G_{i-1} \subseteq G_i$, $G_{i+1} \subseteq G_i$ for $i = 3, \dots, k-1$. Thus we obtain the sequence $F = G_1$, G_2, ..., $G_k = \tilde{F}$ of faces of $U(P)$ with $x \in G_1$, $\tilde{x} \in G_k$ and $G_i \cap G_{i+1} \neq \emptyset$ for $i = 1, \dots, k-1$ what complete the proof.

□

Theorem 2.3.10. *For a polytope $P \subseteq \mathbb{R}^n$ the complement of the undominated set in the boundary of P is (path-)connected.*

Proof. Suppose that the complement of the undominated set in the boundary of polytope P $\mathrm{bd}(P) \setminus U(P)$ is not (path-) connected. Then there are two points p and $\tilde{p}$ from the complement such that there is no path connecting p to $\tilde{p}$ in $\mathrm{bd}(P) \setminus U(P)$.

Since p, $\tilde{p}$ do not belong to the undominated set $U(P)$ then there are two points q, $\tilde{q} \in P$ with $q \leq p$, $\tilde{q} \leq \tilde{p}$ and $q \neq p$, $\tilde{q} \neq \tilde{p}$. We can assume that $\tilde{q} \notin U(P)$. If $\tilde{q}$ would be from the undominated set $U(P)$ then we can take as $\tilde{q}$ the point from the half way between $\tilde{q}$ and $\tilde{p}$ which is not $\tilde{p}$ and less than or equal to the point $\tilde{p}$, also from the polytope P and not from the undominated set $U(P)$.

For a point $x \in P$ and a nonzero vector $w \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ we define

$$\Gamma(x, w) := x + \overline{\lambda} w \tag{2.30}$$

with $\overline{\lambda} := \max\{\lambda \in \mathbb{R} : x + \lambda w \in P\}$ the point at which we leave the polytope P when we go from x in the direction w. Then for the vectors $v := p - q \geq \mathbf{0}$ and $\tilde{v} := \tilde{p} - \tilde{q} \geq \mathbf{0}$ we can assume that $\Gamma(q, v) = p$ and $\Gamma(\tilde{q}, \tilde{v}) = \tilde{p}$. This means that if we go from the point q in the direction v we leave the polytope at the point p. If $q \notin \mathrm{relint}(P)$ and $\Gamma(q, v) = p' \neq p$ then we can consider p' instead of p since there is a path from p' to p in $\mathrm{bd}(P) \setminus U(P)$ and the point $p' \notin U(P)$ because of $q \leq p'$. The similar argumentation we can make for the case $\Gamma(\tilde{q}, \tilde{v}) = \tilde{p}$.

Choose a continuous curve parameterized from 0 to 1

$$\alpha : [0, 1] \to P, \quad \alpha(\,]0, 1[\,) \subseteq \mathrm{relint}(P) \tag{2.31}$$

with $\alpha(0) = q$ and $\alpha(1) = \tilde{q}$. Consider $\Gamma(\alpha(t), v)$, $t \in [0, 1]$ which defines a continuous curve in $\mathrm{bd}(P) \setminus U(P)$ connecting p to $\Gamma(\tilde{q}, v)$. This curve lies on the boundary by the definition of the function Γ and outside of the undominated set since all points of $\alpha(t)$ are less than or equal to the points of the curve $\Gamma(\alpha(t), v)$ and

$$\begin{array}{ll} \text{for} \quad t = 0 & \Gamma(\alpha(0), v) = \Gamma(q, v) = p \notin U(P), \\ \text{for} \quad t = 1 & \Gamma(\alpha(1), v) = \Gamma(\tilde{q}, v) \notin U(P). \end{array}$$

Now consider the curve $\Gamma(\tilde{q}, (1-t)v + t\tilde{v})$, $t \in [0, 1]$ defines a continuous curve in $\text{bd}(P) \setminus U(P)$ connecting $\Gamma(\tilde{q}, v)$ to $\tilde{p}$. That is we stay in the point $\tilde{q}$ and change the direction v to $\tilde{v}$. The curve $\Gamma(\tilde{q}, (1-t)v + t\tilde{v})$, $t \in [0, 1]$ goes again through the boundary but not through the undominated set since $\tilde{p} \in \text{bd}(P)$ and $\tilde{q} \notin \text{bd}(P)$, see Figure 2.5.

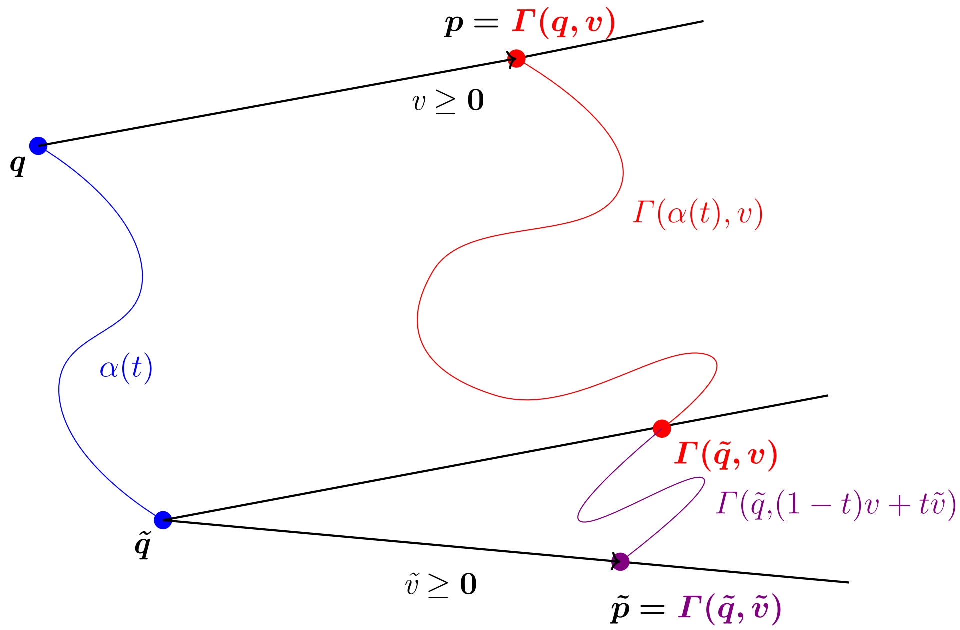

Figure 2.5: The sketch for the proof of Theorem 2.3.10.

Thus we have a connection between p and $\tilde{p}$ on the boundary of the polytope P which is outside the undominated set $U(P)$ what contradicts our assumption. □

Remark 2.3.11. *For a polytope $P \subseteq \mathbb{R}^n$, $n \leq 3$ the undominated set $U(P)$ is contractible (simply connected).*

2.3.3 Pure Complex

A finite non-empty collection of polyhedra $\mathcal{P}$ in $\mathbb{R}^n$ is said to be a *polyhedral complex* if

(1) for all $P \in \mathcal{P}$ each face of P is also in $\mathcal{P}$, and

(2) for any two polyhedra P_1, $P_2 \in \mathcal{P}$ we have that $P_1 \cap P_2$ is a face of each of P_1 and P_2.

If all polyhedra in $\mathcal{P}$ are polytopes then $\mathcal{P}$ is a polytopal complex. If all faces of $\mathcal{P}$ are simplices then $\mathcal{P}$ is a simplicial complex. A complex $\mathcal{P}$ is called *pure* if all facets of $\mathcal{P}$ are of the same dimension (see Figure 2.6 and Figure 2.7).

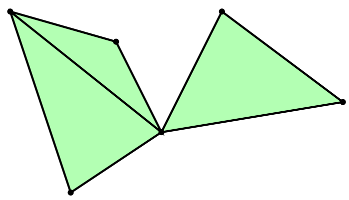

Figure 2.6: A pure complex.

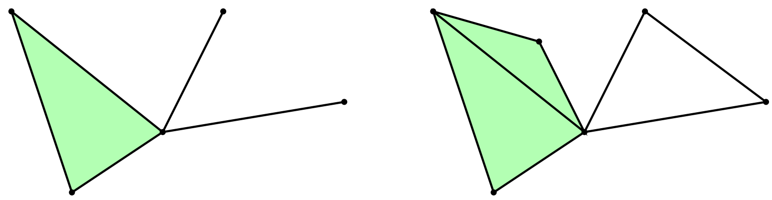

Figure 2.7: A complexes that are not pure.

Observation 2.3.12. *The undominated set is a pure complex for every polytope $P \subseteq \mathbb{R}^2$.*

Proof. The proof follows from the fact that the undominated set is connected set (see Theorem 2.3.9). □

Remark 2.3.13. *Note that this statement is not true in general. For example, in $\mathbb{R}^3$ we have a polytope with the undominated set whose inclusion wise maximal faces are a triangle and an edge (see Figure 2.8).*

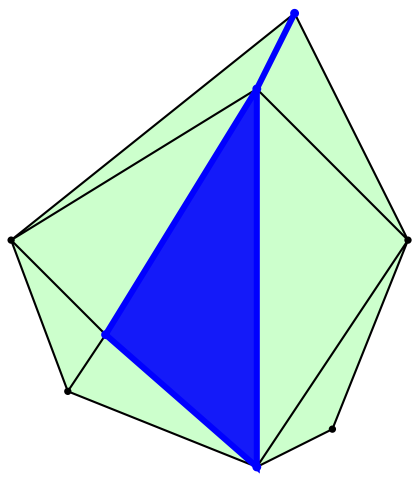

Figure 2.8: Example of the undominated set
of a three-dimensional polytope that is not a pure complex.

2.4 Three-Dimensional Examples of Undominated Complexes

The structure of the undominated complex of a polytope is not known in the general case. However, from Corollary 2.3.7 we have that the undominated complexes are contractible. While their topological structure is rather simple we provide in this section some three-dimensional examples which show that their geometry can nevertheless look quite complicated. In order to obtain more information about the structure of the undominated complex we consider also the intersection of the normal cones at the undominated faces of a polytope $P \subseteq \mathbb{R}^3$ with the hyperplane

$$H^{=}(\mathbf{1}, -1) = \{\mathrm{x} \in \mathbb{R}^3 : \langle \mathbf{1}, \mathrm{x} \rangle = -1\}. \tag{2.32}$$

For each face F of a polytope $P \subseteq \mathbb{R}^n$ we say that $N_F(P) \cap \mathbb{R}^n_{<0}$ is the negative part of the normal cone to P at face F. By Lemma 2.3.1, the undominated complex of a polytope $P \subseteq \mathbb{R}^n$ contains those faces F of P such that the normal cone $N_F(P)$ has a non-empty intersection with the negative orthant $\mathbb{R}^n_{<0}$. Thus, we consider only faces of a polytope P which belong to the undominated complex of P.

Let us consider the *regular dodecahedron* placed into $\mathbb{R}^3$ such that its undominated complex consists of three edges $\{1,2\}$, $\{2,3\}$, $\{4,2\}$ and four vertices $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, see Figure 2.9. For this polytope, the intersection of the negative parts of the normal cones at the undominated faces with the hyperplane $H^{=}(\mathbf{1}, -1)$ is four triangles corresponding to each vertex and three segments corresponding to each edge of the undominated complex, see Figure 2.10. We use the following notation for the unit vectors

$$\boldsymbol{e}_1 = (1,0,0),\ \boldsymbol{e}_2 = (0,1,0),\ \boldsymbol{e}_3 = (0,0,1). \tag{2.33}$$

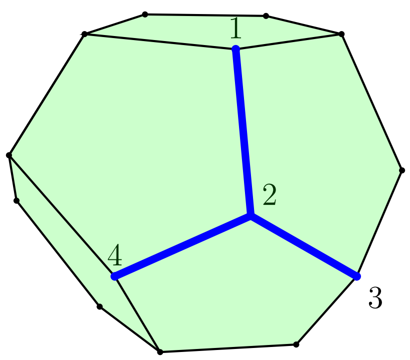

Figure 2.9: Regular dodecahedron and its undominated complex.

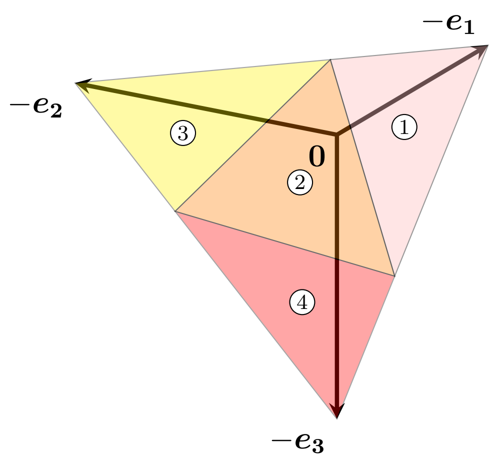

Figure 2.10: Intersection of $H^{=}(\mathbf{1}, -1)$ with the negative parts of the normal cones at the undominated faces of the dodecahedron.

Example 2.4.1 (Truncated cuboctahedron).
Consider the truncated cuboctahedron with its undominated complex, see Figure 2.11. The intersection of the hyperplane $H^{=}(\mathbf{1}, -1)$ with the negative parts of the normal cones at the undominated faces of this polytope is shown in Figure 2.12.

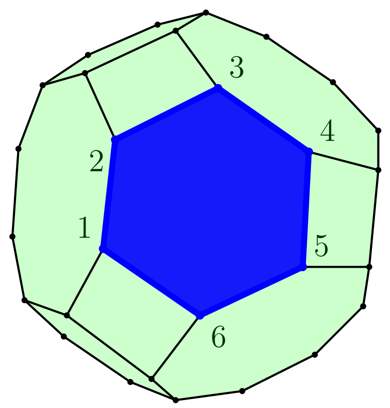

Figure 2.11: Truncated cuboctahedron and its undominated complex.

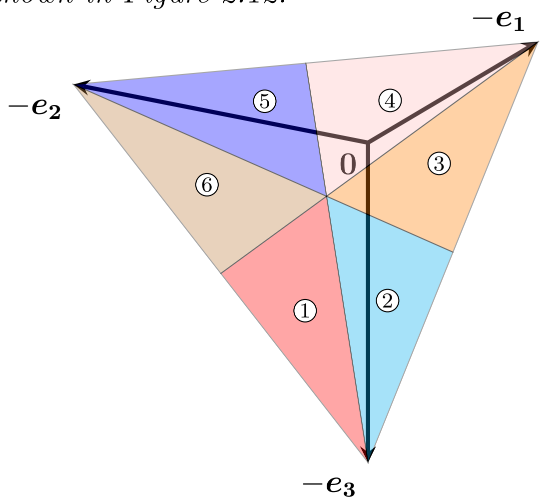

Figure 2.12: Intersection of the negative parts of the normal cones at the undominated faces of the truncated cuboctahedron with $H^{=}(\mathbf{1}, -1)$.

Example 2.4.2 (Elongated Pentagonal Cupola, or Johnson Solid 20).
Consider the elongated pentagonal cupola (Johnson solid 20) with its undominated complex, see Figure 2.13. The intersection of the hyperplane $H^{=}(\mathbf{1}, -1)$ with the negative parts of the normal cones at the undominated faces of this polytope is shown in Figure 2.14.

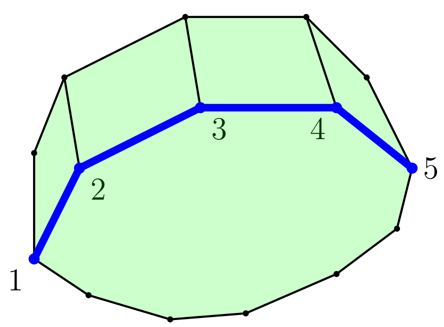

Figure 2.13: Johnson solid 20 and its undominated complex.

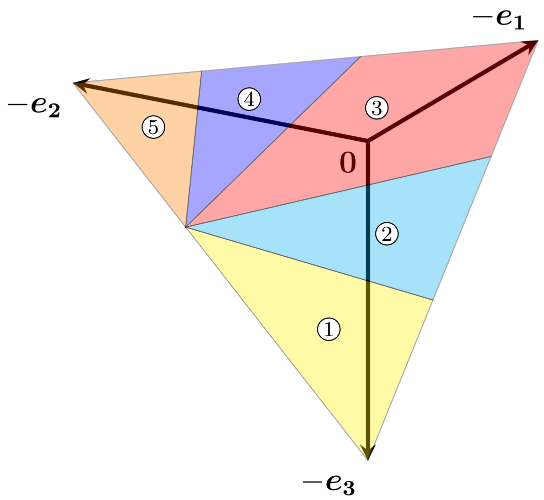

Figure 2.14: Intersection of the negative parts of the normal cones at the undominated faces of the Johnson solid 20 with $H^{=}(\mathbf{1}, -1)$.

Example 2.4.3 (Gyroelongated Square Cupola, or Johnson Solid 23)**.**
Consider the gyroelongated square cupola (Johnson solid 23) with its undominated complex, see Figure 2.15. The intersection of the hyperplane $H^{=}(\mathbf{1}, -1)$ *with the negative parts of the normal cones at the undominated faces of this polytope is shown in Figure 2.16.*

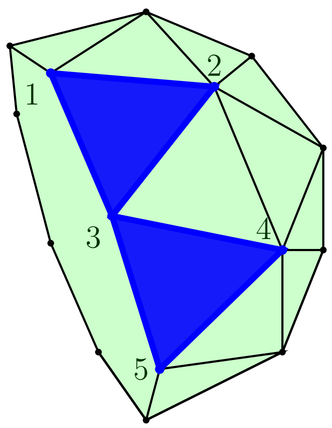

Figure 2.15: Johnson solid 23 and its undominated complex.

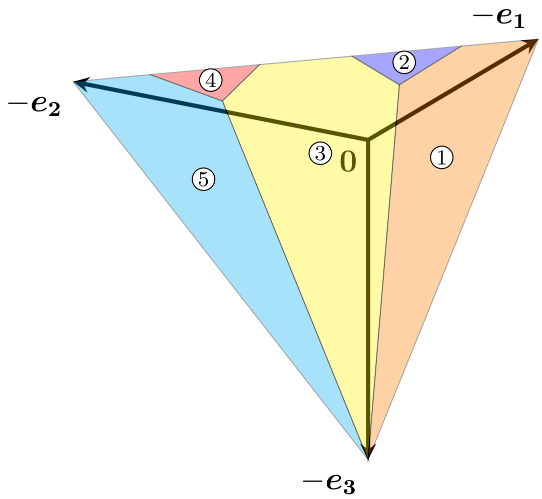

Figure 2.16: Intersection of the negative parts of the normal cones at the undominated faces of the Johnson solid 23 with $H^{=}(\mathbf{1}, -1)$.

Example 2.4.4. *Consider a polytope* $P_1 \subseteq \mathbb{R}^3_+$ *with its undominated complex as in Figure 2.17. The intersection of the hyperplane* $H^{=}(\mathbf{1}, -1)$ *with the negative parts of the normal cones at the undominated faces of* P_1 *is shown in Figure 2.18.*

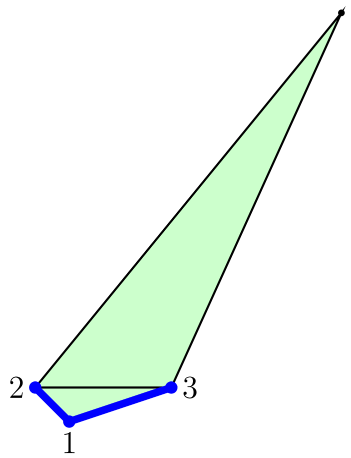

Figure 2.17: Polytope $P_1 \subseteq \mathbb{R}^3_+$ and its undominated complex.

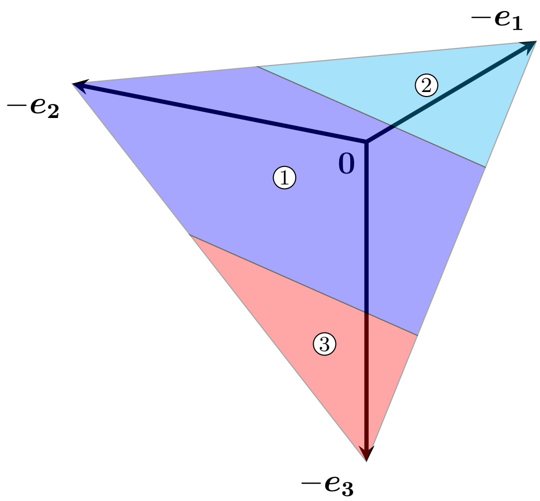

Figure 2.18: Intersection of the negative parts of the normal cones at the undominated faces of P_1 with the hyperplane $H^{=}(\mathbf{1}, -1)$.

Example 2.4.5. *Consider a polytope $P_2 \subseteq \mathbb{R}^3_+$ with its undominated complex as in Figure 2.19. The intersection of the hyperplane $H^=(\mathbf{1}, -1)$ with the negative parts of the normal cones at the undominated faces of P_2 is shown in Figure 2.20.*

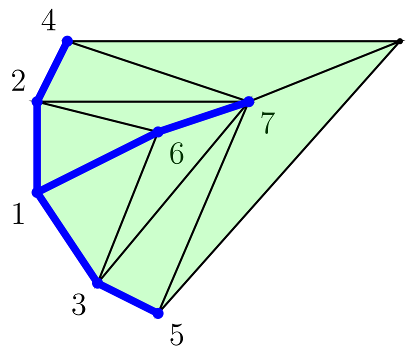

Figure 2.19: Polytope $P_2 \subseteq \mathbb{R}^3_+$ and its undominated complex.

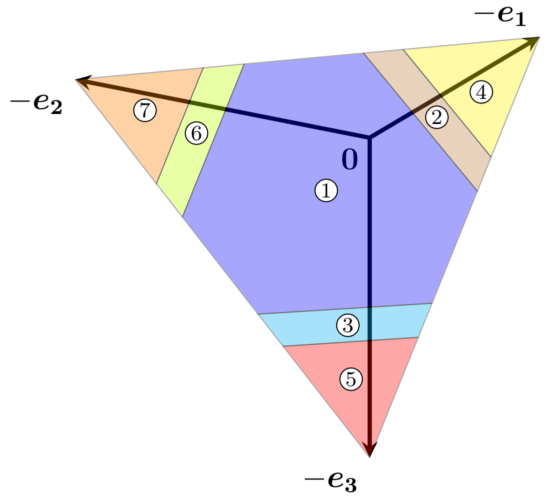

Figure 2.20: Intersection of the negative parts of the normal cones at the undominated faces of P_2 with the hyperplane $H^=(\mathbf{1}, -1)$.

Chapter 3

Undominated Complexes of Cut Polytopes

In this chapter, we present the main results of our work. We start by a definition of the cut polytope and by presenting the description of the dominant for the s-t-cut polytope. Then, in Section 3.3, we define the S-n-cut polytope for the complete undirected graph K_n and provide characterizations of the combinatorial structures of the undominated complexes for the cases $S = \{1\}$, $S = \{1, 2\}$ and $S = \{1, 2, 3\}$ in Sections 3.4, 3.5 and 3.6, respectively. We finish this chapter by considering the general case. We find some faces of the undominated complex of S-n-cut polytope with $S = [\xi]$, $\xi = 4, \ldots, n-1$. We also propose a conjecture for the combinatorial structures of the undominated complexes of the $[\xi]$-n-cut polytope for general ξ, for which a partial proof is given by the main contribution.

3.1 Cut Polytope

In this section and further below we consider only the case of an undirected graph unless otherwise specified.

For a graph $G = (V, E)$ the *cut* defined by $S \subseteq V$ is

$$\delta(S) = \{\{i, j\} \in E : |\{i, j\} \cap S| = 1\}. \tag{3.1}$$

A cut $\delta(S)$ in G is called a *proper* cut of G if $\emptyset \neq S \subsetneq V$. Then the *cut polytope* $P_{cut}(G) \subseteq \mathbb{R}^E$ is the convex hull of the set of characteristic vectors of all edge sets of proper cuts of G, i.e.

$$P_{cut}(G) = \text{conv}\{\chi(\delta(S)) \in \{0, 1\}^E : \emptyset \neq S \subsetneq V\} \tag{3.2}$$

where the vector $\chi(\delta(S)) \in \{0, 1\}^E$ for some $\emptyset \neq S \subsetneq V$ is defined by $\chi(\delta(S))_e = 1$ for $e \in \delta(S)$ and $\chi(\delta(S))_e = 0$ for $e \in E \setminus \delta(S)$. Here, $G = K_n = ([n], E_n)$ with $[n] := \{1, 2, \ldots, n\}$, will always be the complete undirected graph on n nodes. The cut polytope for the complete graph K_n has $2^{n-1} - 1$ vertices and its dimension is $|E| = \binom{n}{2}$.

For example, consider the complete graph $K_3 = ([3], E_3)$ on three nodes $\{1, 2, 3\}$. For this graph we have four different proper cuts. Let (x_{12}, x_{13}, x_{23}) be the order of the coordinates where x_{ij} corresponds to the edge between nodes i and j. Then the set of characteristic vectors of all proper cuts of K_3 is

$$\{\,(0,1,1),\ (1,0,1),\ (1,1,0)\,\}\,. \tag{3.3}$$

Figure 3.1 shows that the cut polytope $P_{cut}(K_3)$ for the complete graph K_3 is a triangle, the two-dimensional polyhedron with three facets.

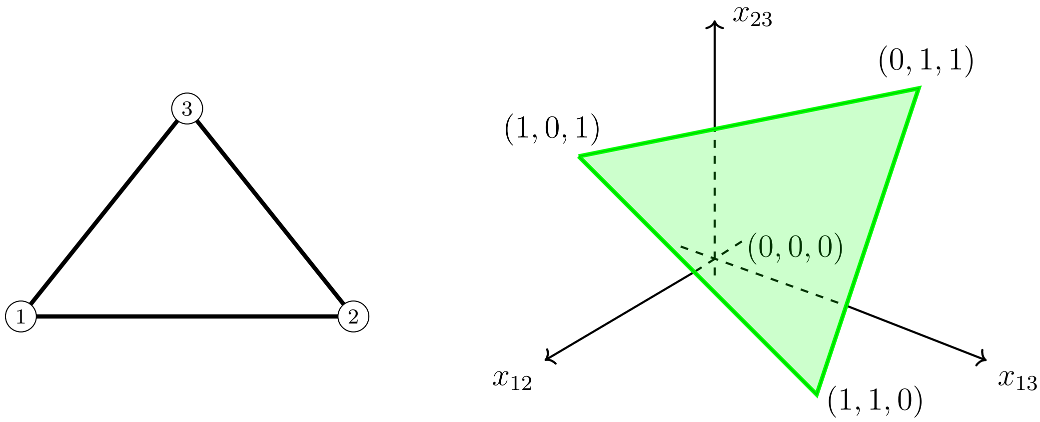

Figure 3.1: Complete graph K_3 and $P_{cut}(K_3)$.

Fiorini et al. [13] have shown that there is no polynomial-size extended formulation for the cut polytope. The currently best known lower-bound is 1.5^n (Kaibel and Weltge [17]). Note that the correlation polytope, called also Boolean quadric polytope, and the cut polytope are linearly isomorphic according to de Simone [10].

3.2 Dominant of the s-t-Cut Polytope

Let $G = (V, E)$ be an undirected graph and $s,\ t \in V,\ s \neq t$. An s-t-cut of G is a cut $\delta(S)$ of G with $s \in S$ and $t \notin S$. Then the convex hull of the set of characteristic vectors of edge sets of all s-t-cuts in G is the *s-t-cut polytope* $P_{s-t\ cut}(G)$ of the graph G. This polytope is hard to describe on the contrary to its dominant since finding a maximum-size s-t-cut in G is NP-hard (see Schrijver [27], Theorem 75.1).

For the dominant of the s-t-cut polytope $P_{s-t\ cut}(G)$ of a graph G we have

$$D(P_{s-t\ cut}(G)) = P_{s-t\ cut}(G) + \mathbb{R}^E_+. \tag{3.4}$$

Let G be the complete undirected graph on n nodes, i.e. $G = K_n = ([n], E_n)$. Let $s,\ t \in [n]$ and $c : E_n \to \mathbb{R}_+^{E_n}$. Consider the following linear formulation of the minimum s-t-cut problem which involves a variable $x_{i,j}$ for each edge $\{i,j\} \in E_n$:

$$
\begin{array}{llll}
\text{minimize} & \sum\limits_{\{i,j\}\in E_n} c_{i,j}x_{i,j} & & \\
\text{subject to:} & x_{s,t} = 1 & & (3.5.1) \\
& x_{i,k} + x_{j,k} \geq x_{i,j} & \forall i,\ j,\ k \in V_n, & (3.5.2) \\
& & i \neq j,\ i \neq k,\ j \neq k & \\
& x \geq \mathbf{0} & & (3.5.3)
\end{array}
\tag{3.5}
$$

Theorem 3.2.1. *Let x^* be an optimal solution for the problem* (3.5) *with $c \in \mathbb{R}_+^{E_n}$. Then, the optimal value of the linear program* (3.5) *is equal to the minimum capacity of an s-t cut, i.e.,*

$$\langle c, x^* \rangle = \min\{\langle c, x\rangle : x = \chi(\delta(S)), \{s,t\} \in \delta(S),\ S \subseteq V_n\}. \tag{3.6}$$

Proof. We clearly have

$$\langle c, x^* \rangle \leq \min\{\langle c, x\rangle : x = \chi(\delta(S)), \{s,t\} \in \delta(S),\ S \subseteq V_n\} \tag{3.7}$$

since all $x \in \{x : x = \chi(\delta(S)),\ \{s,t\} \in \delta(S),\ S \subseteq V_n\}$ satisfy the system (3.5.1)-(3.5.3).

To prove the feasibility of the inverse inequality we construct from the undirected graph K_n the directed graph $\overleftrightarrow{K}_n = (V_n, \overleftrightarrow{E}_n)$ by replacing each edge $e \in E_n$ by two antiparallel arcs both with the same capacity as e. The capacity of a cut in the directed graph is the same as the capacity of the corresponding cut in the undirected graph. In the new graph $\overleftrightarrow{K}_n$ we consider a s-t flow f with the maximum value and decompose it into m s-t-paths $p_1, \ldots, p_m$, i.e., $f = \sum\limits_{\mu=1}^{m} \lambda_\mu \chi(p_\mu)$ where $\lambda_\mu \geq 0$, $\mu \in [m]$. We can assume that there are no cycles in the decomposition.

Let $(s, \mu(1))$, $(\mu(1), \mu(2))$, $\ldots$, $(\mu(k), t)$ be the sequence of arcs forming p_μ. We may assume that p_μ is a simple path, i.e., $\mu(l) \neq s,\ t$ for $l \in [k]$ and $\mu(\alpha) \neq \mu(\beta)$ for all α, $\beta \in [k]$, $\alpha \neq \beta$. Using the metric constrains (3.5.2) we have for each s-t-path p_μ, $\mu \in [m]$

$$x^*_{s,\mu(1)} + x^*_{\mu(1),\mu(2)} + \ldots + x^*_{\mu(k),t} \geq x^*_{s,t} \overset{(3.5.1)}{=} 1. \tag{3.8}$$

Thus, $\lambda_{p_\mu} \sum\limits_{(i,j)\in p_\mu} x^*_{i,j} \geq \lambda_{p_\mu}$ for each $\mu \in [m]$ what implies that $\sum\limits_{\{i,j\}\in E_n} f_{i,j}x^*_{i,j} \geq \sum\limits_{\mu=1}^{m} \lambda_{p_\mu}$ due to

$$
\begin{aligned}
\sum_{\{i,j\}\in E_n} f_{i,j}x^*_{i,j} &= \sum_{\{i,j\}\in E_n} \left(\sum_{\mu:\,(i,j)\in p_\mu} \lambda_{p_\mu} \right) x^*_{i,j} = \sum_{\mu=1}^{m} \sum_{(i,j)\in p_\mu} \lambda_{p_\mu} x^*_{i,j} \\
&= \sum_{\mu=1}^{m} \lambda_{p_\mu} \sum_{(i,j)\in p_\mu} x^*_{i,j} \geq \sum_{\mu=1}^{m} \lambda_{p_\mu}.
\end{aligned}
\tag{3.9}
$$

Therefore,

$$\langle c, x^* \rangle = \sum_{\{i,j\} \in E_n} c_{i,j} x^*_{i,j} \overset{f_{i,j} \leq c_{i,j}}{\geq} \sum_{\{i,j\} \in E_n} f_{i,j} x^*_{i,j} \geq \sum_{\mu=1}^{m} \lambda_{p_\mu}. \tag{3.10}$$

By the Max-Flow-Min-Cut Theorem 1.1.3 the value of the maximum s-t flow f is equal to the minimum capacity over all s-t-cuts, i.e.

$$\text{value}(f) = \sum_{\mu=1}^{m} \lambda_{p_\mu} = \min\{\langle c, x \rangle : x = \chi(\delta(S)),\ \{s,t\} \in \delta(S),\ S \subseteq V_n\} \tag{3.11}$$

what together with (3.9) implies

$$\langle c, x^* \rangle \geq \min\{\langle c, x \rangle : x = \chi(\delta(S)),\ \{s,t\} \in \delta(S),\ S \subseteq V_n\}. \tag{3.12}$$

□

Remark 3.2.2. *From Theorem 3.2.1 and Observation 2.1.4 it follows that the dominant of the s-t-cut polytope $P_{s-t\ cut}(K_n)$ for the complete graph K_n is equal to the dominant of the polytope described by (3.5.1)-(3.5.3), i.e.,*

$$D(P_{s-t\ cut}(K_n)) = D(Q_{s,t}(K_n)) \tag{3.13}$$

where

$$Q_{s,t}(K_n) := \{x \in \mathbb{R}^{E_n} : x \text{ satisfies } (3.5.1) - (3.5.3)\} \tag{3.14}$$

Moreover, by Corollary 2.2.5 their undominated sets are equal as well.

Example 3.2.3. *Consider the complete graph $K_3 = ([3], E_3)$. The 1-3-cut polytope of the graph K_3 is the convex hull of two vectors*

$$P_{1-3\ cut}(K_3) = \text{conv}\{(1,0,1), (0,1,1)\}. \tag{3.15}$$

From (3.14) and (3.5.1)-(3.5.3) it follows that $Q_{1,3}(K_3)$ is described by (3.16.1)-(3.16.4) and by the nonnegativity conditions, i.e.

$$\begin{aligned} Q_{1,3}(K_3) := \{(x_{12}, x_{13}, x_{23}) \geq \boldsymbol{0} : \quad & x_{13} = 1, && (3.16.1) \\ & x_{12} + x_{13} \geq x_{23}, && (3.16.2) \\ & x_{12} + x_{23} \geq x_{13}, && (3.16.3) \\ & x_{13} + x_{23} \geq x_{12} \quad \} && (3.16.4) \end{aligned} \tag{3.16}$$

The dominants of $P_{1-3\ cut}(K_3)$ and $Q_{1,3}(K_3)$ are shown in Figure 3.2 and Figure 3.3, respectively. As we can see $P_{1-3\ cut}(K_3)$ and $Q_{1,3}(K_3)$ have the same dominant and hence the same undominated set which in this case is single one-dimensional polytope described by the following system

$$\begin{cases} x_{13} = 1 \\ x_{12} + x_{23} = 1 \\ x_{12} \leq 1 \\ x_{12} \geq 0 \end{cases} \tag{3.17}$$

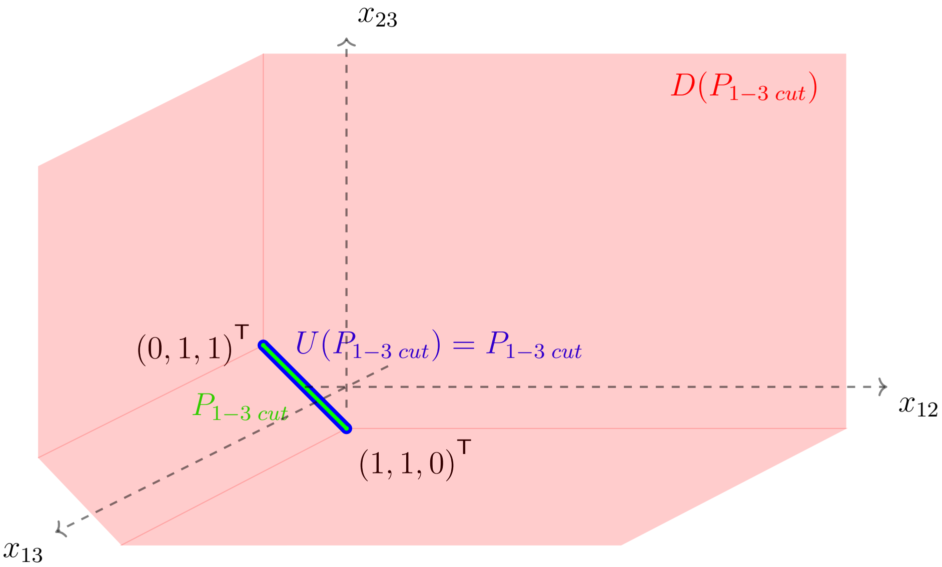

Figure 3.2: The dominant of $P_{1-3\ cut}(K_3)$ (in red).

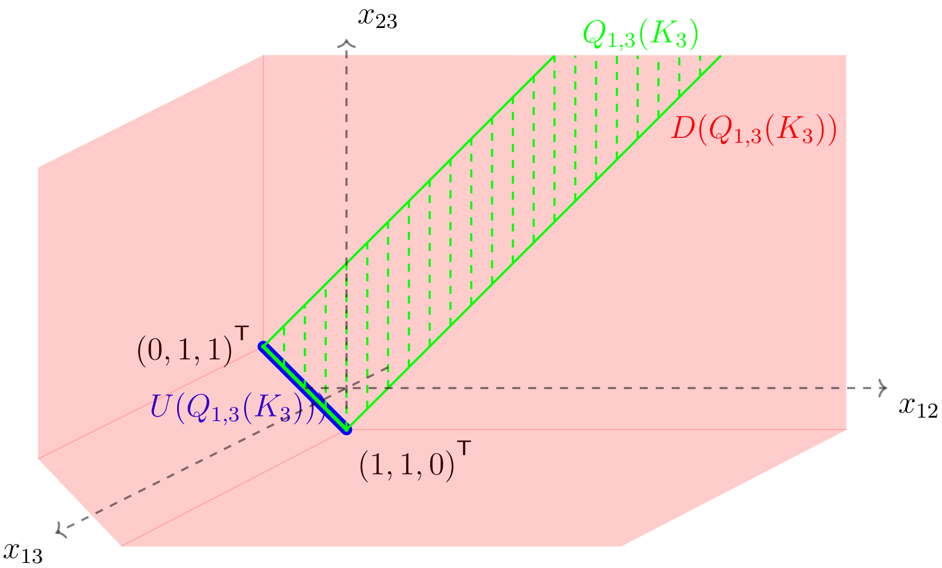

Figure 3.3: The dominant of $Q_{1,3}(K_3)$ (in red).

3.3 S-n-Cut Polytope for the complete graph K_n

Let $K_n = ([n], E_n)$ be the complete undirected graph on n nodes. An S-n-cut of G with $\emptyset \neq S \subseteq [n-1]$ is a cut $\delta(W)$ with $W \in \mathcal{V}(S)$ where

$$\mathcal{V}(S) := \{W \subseteq [n-1] : W \cap S \neq \emptyset\}. \tag{3.18}$$

We define the S-n-cut polytope $P_S(K_n)$ as

$$P_S(K_n) := \operatorname{conv}\{\chi(\delta(W)) : W \in \mathcal{V}(S)\}. \tag{3.19}$$

Example 3.3.1.

- $P_{\{1\}}(K_n)$ *is the* 1*-*n*-cut polytope, the polytope of all cuts that separate* 1 *from* n*;*
- $P_{\{1,2\}}(K_n)$ *is the polytope of all cuts that separate* 1 *from* n *or* 2 *from* n*;*
- $P_{[n-1]}(K_n)$ *is the cut polytope for the undirected complete graph on* n *nodes.*

For $s \in [n-1]$ let Γ_s be the set of all s-n-paths in the complete graph K_n

$$\Gamma_s := \{\gamma \subseteq E_n : \gamma \text{ is a } s\text{-}n\text{-path}\} \tag{3.20}$$

and let F_s be the set of all s-n-flows in K_n

$$F_s := \text{ccone}\{\chi(\gamma) : \gamma \in \Gamma_s\}. \tag{3.21}$$

Then for $f \in F_s$ the value of a flow f we define as value$(f) := f(\delta(n))$ with $f = \sum_{\gamma \in \Gamma_s} \lambda_\gamma \chi(\gamma)$ with $\lambda_\gamma \geq 0$ what implies that value$(f) = \sum_{\gamma \in \Gamma_s} \lambda_\gamma$.

The following theorem was proved by Ford and Fulkerson [14], but nevertheless we provide our proof for the sake of completeness.

Theorem 3.3.2 (Undirected version of Max-Flow Min-Cut Theorem)**.**
Let $K_n = ([n], E_n)$ *be the complete undirected graph on* n *nodes,* $s \in [n-1]$ *and* $c \in \mathbb{R}_+^{E_n}$*. Then*

$$\max\{\text{value}(f) : f \in F_s, f \leq c\} = \min\{c(\delta(W)) : s \in W \subseteq [n-1]\}. \tag{3.22}$$

Proof. Let $\delta(\tilde{W})$, $s \in \tilde{W} \subseteq [n-1]$ be an s-n-cut of K_n with minimum capacity, i.e.

$$c(\delta(\tilde{W})) = \min\{c(\delta(W)) : s \in W \subseteq [n-1]\}. \tag{3.23}$$

Let $f = \sum_{i=1}^{m} \lambda_i \chi(\gamma_i) \in F_s$ be an s-n-flow with $f \leq c$ and m s-n-paths $\gamma_1, \gamma_2, \ldots, \gamma_m \in \Gamma_s$ and $\lambda_i \geq 0$, $i \in [m]$. For each path γ_i, $i \in [m]$ we choose one edge e_{γ_i} from $\delta(\tilde{W})$. Then we have

$$\text{value}(f) = \sum_{i=1}^{m} \lambda_i \leq \sum_{e \in \delta(\tilde{W})} \left(\sum_{\substack{\gamma \in \{\gamma_i : i \in [m]\}: \\ e_\gamma = e}} \lambda_\gamma \right) \leq \sum_{e \in \delta(\tilde{W})} c_e = c(\delta(\tilde{W})). \tag{3.24}$$

Hence, the maximum value of an s-n-flow subject to c is at most the minimum capacity of an s-n-cut.

Consider a directed graph $\overleftrightarrow{K}_n = ([n], \overleftrightarrow{E}_n)$ constructed from the undirected graph $K_n = ([n], E_n)$ by replacing each edge $e \in E_n$ by two antiparallel arcs $\overleftarrow{e}$, $\overrightarrow{e}$ both with the same capacity as e. The capacity of a cut in the directed graph $\overleftrightarrow{K}_n$ is the same as the capacity of the corresponding cut in the undirected graph K_n. By the Max-Flow-Min-Cut Theorem 1.1.3 there is an s-n flow f of value $c(\delta(\tilde{W}))$ in $\overleftrightarrow{K}_n$. It could be that this flow of value $c(\delta(\tilde{W}))$ uses an edge in both ways, for example some edge $e \in E_n$. In this case we change the flow function f by setting the smaller value $\min\{f_{\overleftarrow{e}}, f_{\overrightarrow{e}}\}$ to zero and subtracting that amount from the larger value $\max\{f_{\overleftarrow{e}}, f_{\overrightarrow{e}}\}$, i.e.

$$\begin{aligned} f_{\overrightarrow{e}} &:= f_{\overrightarrow{e}} - \min\{f_{\overleftarrow{e}}, f_{\overrightarrow{e}}\}, \\ f_{\overleftarrow{e}} &:= f_{\overleftarrow{e}} - \min\{f_{\overleftarrow{e}}, f_{\overrightarrow{e}}\} \end{aligned} \tag{3.25}$$

what does not change the flow value, see Figure 3.4.

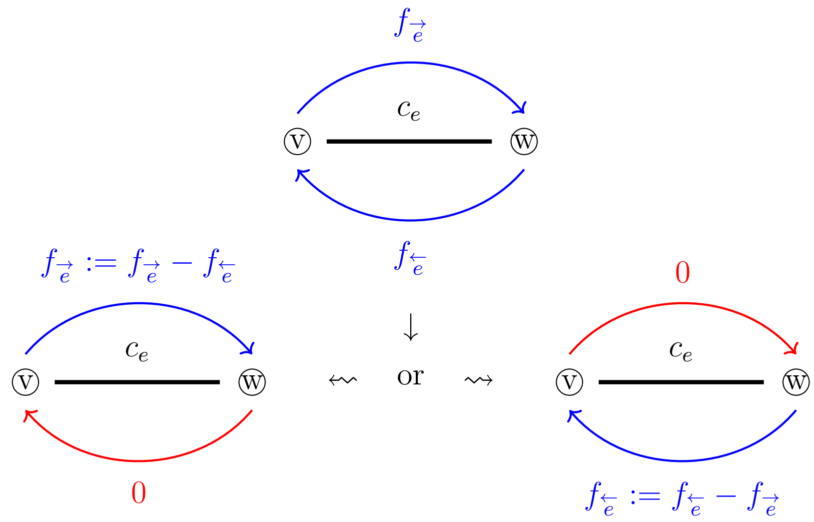

Figure 3.4: Transformation of the flow f in $\overleftrightarrow{K}_n$ in the case of using an edge in both ways.

Thus we have a flow of value $c(\delta(\tilde{W}))$ which does not use antiparallel arcs. This flow can be decomposed into s-n-paths and cycles. We can assume that there are no cycles in the decomposition. Those path can be projected back on undirected edges of the graph K_n. Since every edge is used only in one way then the condition on capacities was respected on each edge. □

Lemma 3.3.3. *For $\tilde{W} \in \mathcal{V}(S)$ holds*

$$\min\{c(\delta(W)) : W \in \mathcal{V}(S)\} = c(\delta(\tilde{W})) \tag{3.26}$$

if and only if for all $s \in S$ there exists $f_s \in F_s$ with $f_s \leq c$ and $\text{value}(f_s) \geq c(\delta(\tilde{W}))$.

Proof. The proof follows from Theorem 3.3.2. □

Consequently, Lemma 3.3.3 implies

Corollary 3.3.4. *For all $W \in \mathcal{V}(S)$ we have $c(\delta(W)) \geq 1$ if and only if for all $s \in S$ there exists $f_s \in F_s$ such that $f_s \leq c$ and* $\text{value}(f_s) \geq 1$.

Observation 3.3.5. *Let $S \subseteq [n-1]$, $S \neq \emptyset$. A vector $x \in \mathbb{R}^{E_n}$ is a vertex of the dominant of $P_S(K_n)$ if and only if $x = \chi(\delta(W))$ for some S-n-cut $\delta(W)$, $W \in \mathcal{V}(S)$ of K_n.*

Proof. Let $G = (V, E)$ be an undirected graph with s, $n \in V$. For the dominant of $P_{s-n-cut}(G)$ Skutella and Weber showed that a vector x is a vertex of $D(P_{s-n-cut}(G))$ if and only if $x = \chi(\delta(W))$ for some s-n-cut $\delta(W)$ (see Skutella and Weber [30], Observation 1). The polytope $P_S(K_n)$ is the convex hull of all s-n-cuts with $s \in S$. Thus characteristic vectors of s-n-cuts, $s \in S$ are vertices of $D(P_S(K_n))$. □

Theorem 3.3.6. *For $\emptyset \neq S \subseteq [n-1]$ the dominant of the S-n-cut polytope $P_S(K_n)$ is equal to the dominant of a polytope $Q(S) \subseteq \mathbb{R}_+^{E_n}$ if and only if*

(1) $\chi(\delta(W)) \in Q(S)$ for all $W \in \mathcal{V}(S)$ and

(2) for all $x \in Q(S)$ holds:

$$\begin{array}{rl} \textit{if} & f_s \in F_s \textit{ with } \text{value}(f_s) \geq 1 \textit{ for all } s \in S \\ \textit{then} & \langle \overline{f}, x \rangle \geq 1 \textit{ with } \overline{f}(e) = \max\{f_s(e) : s \in S\},\ e \in E_n. \end{array} \tag{3.27}$$

Proof. Let $Q(S) \subseteq \mathbb{R}_+^{E_n}$ be a polytope such that both conditions (1) and (2) are satisfied. The inclusion $D(P_S(K_n)) \subseteq D(Q(S))$ holds since $P_S(K_n) \subseteq Q(S)$ by (1).

To prove the reverse inclusion it is enough to show that for every $c \in \mathbb{R}_+^{E_n}$ with

$$c(\delta(W)) \geq 1 \quad \text{for all } W \in \mathcal{V}(S) \tag{3.28}$$

we have that $\langle c, x \rangle \geq 1$ for all $x \in Q(S)$.

From (3.28) by Corollary 3.3.4 it follows that

$$\text{for all } s \in S \quad \exists f_s \in F_s \text{ such that } \ f_s \leq c \text{ and } \ \text{value}(f_s) \geq 1. \tag{3.29}$$

Then by the condition (2) we have

$$\langle \overline{f}, x \rangle \geq 1 \quad \text{for all } x \in Q(S). \tag{3.30}$$

Since $\overline{f} \leq c$ and $Q(S) \subseteq \mathbb{R}_+^{E_n}$ then

$$\langle c, x \rangle \geq 1 \text{ for all } x \in Q(S). \tag{3.31}$$

Now let us have $Q(S) \subseteq \mathbb{R}_+^{E_n}$ such that $D(Q(S)) = D(P_S(K_n))$. By Observation 3.3.5 $\chi(\delta(W))$, $W \in \mathcal{V}(S)$ is a vertex of $D(P_S(K_n))$. Then $\chi(\delta(W))$, $W \in \mathcal{V}(S)$ is also a vertex of $D(Q(S))$ what implies that $\chi(\delta(W)) \in Q(S)$ for all $W \in \mathcal{V}(S)$.

Let $x \in Q(S)$ then $x \in D(Q(S))$ and thereby $x \in D(P_S(K_n))$ since the dominants are equal. Thus there is a convex combination of S-n-cuts $y \in P_S(K_n)$ such that $x \geq y$. By Corollary 3.3.4 and Theorem 3.3.2 we have that (3.27) holds for all S-n-cuts from the convex combination and thereby for y. This implies that (3.27) also holds for x since $\langle \overline{f}, x \rangle \geq \langle \overline{f}, y \rangle$ what completes the proof. □

3.4 Undominated Complex of the 1-n-Cut Polytope

Proposition 3.4.1. *For the complete graph on n nodes $K_n = ([n], E_n)$*

$$\operatorname{conv}\{\chi(\delta([k])) : k \in [n-1]\} \tag{3.32}$$

is a face of the undominated complex of the 1-n-cut polytope $P_{\{1\}}(K_n)$.

Proof. To prove this statement we find some $c \in \mathbb{R}^{E_n}_{>0}$ such that for all $1 \in S^* \subseteq [n-1]$ we have

$$\min\{c(\delta(S)) :\ 1 \in S \subseteq [n-1]\} = c(\delta(S^*)) \tag{3.33}$$

if and only if $S^* = [k]$ for some $k \in [n-1]$ (see the proof of Lemma 2.3.1).

Let $c \in \mathbb{R}^{E_n}_{>0}$ be defined as follows:

$$\begin{aligned}
c_{1,j} &= \frac{1}{n-1} \quad \text{for } \ j = 2, \dots, n;\\
c_{2,j} &= \frac{1}{(n-1)(n-2)} \quad \text{for } \ j = 3, \dots, n;\\
c_{3,j} &= \frac{1}{(n-2)(n-3)} \quad \text{for } \ j = 4, \dots, n;\\
&\dots\\
c_{n-2,j} &= \frac{1}{(n-(n-3))(n-(n-2))} = \frac{1}{3\cdot 2} = \frac{1}{6} \quad \text{for } \ j = n-1, n;\\
c_{n-1,n} &= \frac{1}{(n-(n-2))(n-(n-1))} = \frac{1}{2\cdot 1} = \frac{1}{2}.
\end{aligned} \tag{3.34}$$

Remark 3.4.2. *For all $k \in [n-1]$ and $j = k+1, \dots, n$ we have $\sum\limits_{i=1}^{k} c_{i,j} = \dfrac{1}{n-k}$.*

Proof. Let $k \in [n-1]$. From the definition of $c \in \mathbb{R}^{E_n}_{>0}$, see (3.34), we infer

$$c_{k,j} = \frac{1}{(n-(k-1))(n-k)} \quad \text{for } \ j = k+1, \dots, n \ \text{ with } \ k > 1. \tag{3.35}$$

Hence, from (3.34) and (3.35) we deduce

$$\begin{aligned}
\sum_{i=1}^{k} c_{i,j} &= \frac{1}{(n-1)} + \frac{1}{(n-1)(n-2)} + ... + \frac{1}{(n-(k-1))(n-k)}\\
&= \frac{1}{n-1} - \frac{1}{n-1} + \frac{1}{n-2} - \frac{1}{n-2} + \frac{1}{n-3} + \dots\\
&\qquad - \frac{1}{(n-(k-2))} + \frac{1}{(n-(k-1))} - \frac{1}{(n-(k-1))} + \frac{1}{(n-k)}\\
&= \frac{1}{n-k}.
\end{aligned} \tag{3.36}$$

□

Lemma 3.4.3. *Let $c \in \mathbb{R}^{E_n}_{>0}$ be defined by* (3.34). *Then $c(\delta([k])) = 1$ holds for all $k \in [n-1]$.*

Proof. Let $k \in [n-1]$, then

$$c(\delta([k])) = \sum_{j=k+1}^{n} \sum_{i=1}^{k} c_{i,j} = (n-k) \sum_{i=1}^{k} c_{i,k+1} \overset{\text{Remark 3.4.2}}{=} 1. \tag{3.37}$$

□

Lemma 3.4.4. *Let $w \in \mathbb{R}^{E_n}_{>0}$, $\mu \in [n-1]$ and W_1, W_2 be subsets of $[n-1]$ such that $\mu \in W_1 \cap W_2$ and*

$$1 = w(\delta(W_1)) = w(\delta(W_2)) = \min\{w(\delta(W)) : \mu \in W \subseteq [n-1]\}. \tag{3.38}$$

Then we have $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Proof. Let us define six subsets of the set E_n as follows:

$$\begin{aligned}
A &:= \{\{i,j\} \in E_n : \ i \in W_1 \cap W_2, \ j \in W_1 \setminus (W_1 \cap W_2)\}, \\
B &:= \{\{i,j\} \in E_n : \ i \in W_1 \setminus (W_1 \cap W_2), \ j \in [n] \setminus (W_1 \cup W_2)\}, \\
C &:= \{\{i,j\} \in E_n : \ i \in W_2 \setminus (W_1 \cap W_2), \ j \in [n] \setminus (W_1 \cup W_2)\}, \\
D &:= \{\{i,j\} \in E_n : \ i \in W_1 \cap W_2, \ j \in W_2 \setminus (W_1 \cap W_2)\}, \\
E &:= \{\{i,j\} \in E_n : \ i \in W_1 \cap W_2, \ j \in [n] \setminus (W_1 \cup W_2)\}, \\
F &:= \{\{i,j\} \in E_n : \ i \in W_1 \setminus (W_1 \cap W_2), \ j \in W_2 \setminus (W_1 \cap W_2)\}.
\end{aligned} \tag{3.39}$$

see Figure 3.5.

The cut function is submodular since $w > 0$ (see Schrijver [28], Chapter 44), that is

$$w(\delta(W_1 \cap W_2)) + w(\delta(W_1 \cup W_2)) \leq w(\delta(W_1)) + w(\delta(W_2)). \tag{3.40}$$

Then for a, b, c, d, e and f defined as follows

$$\begin{aligned}
a &:= w(A), \quad d := w(D), \\
b &:= w(B), \quad e := w(E), \\
c &:= w(C), \quad f := w(F),
\end{aligned} \tag{3.41}$$

$w \geq 0$ implies

$$(a+e+d) + (b+e+c) \leq (d+e+f+b) + (a+e+f+c) \tag{3.42}$$

and by (3.38) it follows that

$$\begin{aligned}
w(\delta(W_1 \cap W_2)) = (a+e+d) \geq 1, \quad w(\delta(W_1)) = (d+e+f+b) = 1, \\
w(\delta(W_1 \cup W_2)) = (b+e+c) \geq 1, \quad w(\delta(W_2)) = (a+e+f+c) = 1.
\end{aligned} \tag{3.43}$$

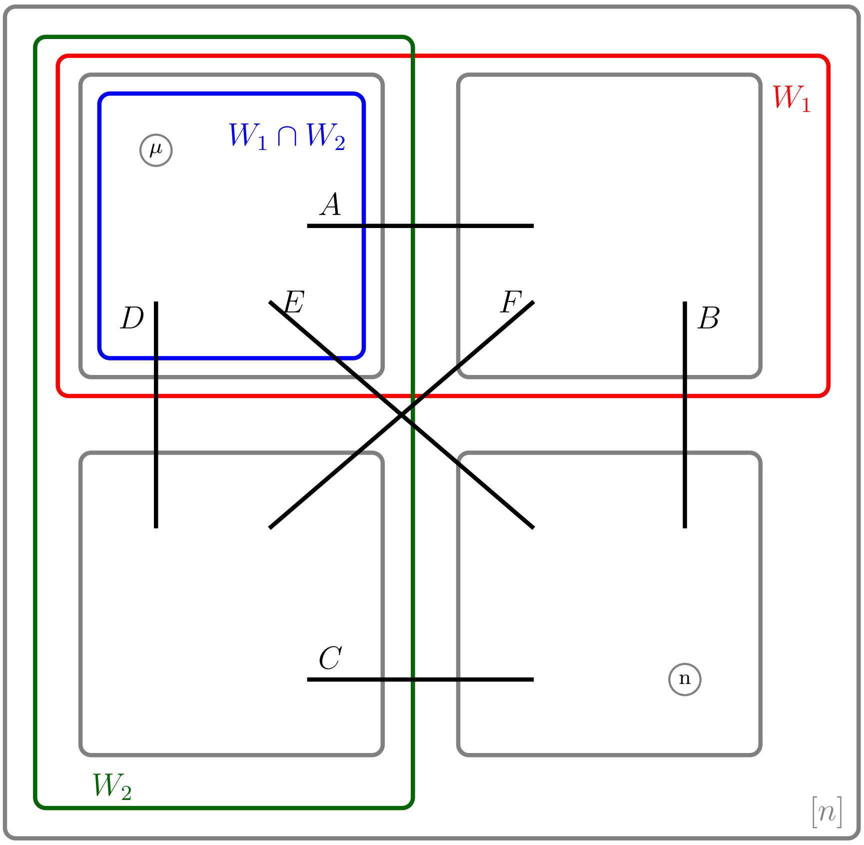

Figure 3.5: Representation of the vertex set $[n]$
in the proof of Lemma 3.4.4.

Thus from (3.42) and (3.43) we have

$$\begin{cases} a+e+d=1, \\ b+e+c=1, \\ d+e+f+b=1, \\ a+e+f+c=1, \end{cases} \Rightarrow \begin{cases} a=f+b, \\ b=a+f, \end{cases} \Rightarrow f=0. \tag{3.44}$$

Due to $w > 0$, however, f can be 0 if and only if F is the empty set. Thus, we have $W_1 \setminus (W_1 \cap W_2) = \emptyset$ or $W_2 \setminus (W_1 \cap W_2) = \emptyset$ what implies $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$. □

Continuing the proof of Proposition 3.4.1 we construct from the undirected graph K_n the directed graph $\overleftrightarrow{K}_n = ([n], \overleftrightarrow{E}_n)$ by replacing each edge $\{i,j\} \in E_n$ by two antiparallel arcs (i,j), (j,i). In the new graph $\overleftrightarrow{K}_n$ we construct a 1-n flow $f : \overleftrightarrow{E}_n \to \mathbb{R}$ with

$$\begin{array}{ll} f_{i,j} := c_{i,j} & \text{for } i < j, \\ f_{i,j} := 0 & \text{for } i > j. \end{array} \tag{3.45}$$

In order to see that f indeed is a 1-n-flow observe that for each $k \in [n] \setminus \{1, n\}$ we have

$$\begin{aligned} f(\delta^{out}(k)) &= \sum_{j=k+1}^{n} c_{k,j} \overset{(3.35)}{=} (n-k) \cdot \frac{1}{(n-(k-1))(n-k)} = \frac{1}{n-k+1}, \\ f(\delta^{in}(k)) &= \sum_{i=1}^{k-1} c_{i,k} \overset{\text{Remark 3.4.2}}{=} \frac{1}{n-k+1}. \end{aligned} \tag{3.46}$$

Hence both conditions of the definition of a flow are satisfied, i.e.

$$\begin{aligned} f_{i,j} &\geq 0 && \text{for each } (i,j) \in \overset{\leftrightarrow}{E}_n, \\ f(\delta^{out}(k)) &= f(\delta^{in}(k)) && \text{for each } k \in [n] \setminus \{1, n\}. \end{aligned} \tag{3.47}$$

The value of this flow f is equal to 1 due to

$$\text{value}(f) = f(\delta^{out}(1)) = c(\delta([1])) \overset{\text{Lemma 3.4.3}}{=} 1. \tag{3.48}$$

For each $S \subseteq [n-1]$ with $1 \in S$ we thus have (due to $c \geq \mathbf{0}$)

$$1 \leq f(\delta^{out}(S)) \leq c(\delta(S)), \tag{3.49}$$

hence Lemma 3.4.3 shows that $\delta([k])$ is c-minimal among the 1-n-cuts for each $k \in [n-1]$.

Now suppose that for some $S^* \subseteq [n-1]$ with $1 \in S^*$ the cut $\delta(S^*)$ is c-minimal. By Lemma 3.4.4 we thus have $S^* \subseteq [|S^*|]$ or $[|S^*|] \subseteq S^*$ hence $S^* = [|S^*|]$.

□

Proposition 3.4.5. *The face in Proposition 3.4.1 is a $(n-2)$-dimensional simplex.*

Proof. Let $\mathcal{K} := \{[k] : k \in [n-1]\}$. The face (3.32) in Proposition 3.4.1 has $n-1$ vertices due to

$$|\mathcal{K}| = |\{[k] : k \in [n-1]\}| = n-1. \tag{3.50}$$

Now we show that these $n-1$ vertices are affinely independent and hence the face is $(n-2)$-dimensional.

Let $\Theta'_{\mathcal{K}}$ be a submatrix of the cut-incidence matrix of $\mathcal{K}$ (see (1.10)) formed by columns corresponding to the edges $e_{1,2}$, $e_{2,3}$, $\dots$, $e_{n-1,n}$ with an additional all ones column, i.e.

$$\Theta'_{\mathcal{K}} := \begin{array}{c} \\ [1] \\ [2] \\ \dots \\ [n-1] \end{array} \begin{array}{c} \begin{array}{ccccc} e_{1,2} & e_{2,3} & \dots & e_{n-1,n} & \end{array} \\ \begin{bmatrix} \mathbf{1} & 0 & 0 & 0 & \mathbf{1} \\ 0 & \mathbf{1} & 0 & 0 & \mathbf{1} \\ 0 & 0 & \ddots & 0 & \mathbf{1} \\ 0 & 0 & 0 & \mathbf{1} & \mathbf{1} \end{bmatrix} \end{array} \tag{3.51}$$

As we have $\operatorname{rank}(\Theta'_{\mathcal{K}}) = n-1$ linear independence of rows of $\Theta'_{\mathcal{K}}$ implies affine independence of rows of the cut-incidence matrix of $\mathcal{K}$. Thus, vertices of the face in Proposition 3.4.1 are affinely independent. □

Remark 3.4.6. *Because of the symmetry, for each permutation $\sigma : [n] \to [n]$ with $\sigma(1) = 1$, $\sigma(n) = n$*

$$\operatorname{conv}\{\chi(\delta(\sigma([k]))) : k \in [n-1]\} \tag{3.52}$$

is also a $(n-2)$-dimensional face of the undominated complex of the 1-n-cut polytope.

Since all faces in Remark 3.4.6 have $n-1$ vertices none of them is contained in any other. Now we show that any other face is contained in one of these faces.

Let $\mathcal{W} \subseteq 2^{[n-1]}$ such that $\{\chi(\delta(W)) : W \in \mathcal{W}\}$ is the set of vertices of some face of the undominated complex of $P_{\{1\}}(K_n)$. Then there is $c \in \mathbb{R}^E_{>0}$ with minimum 1-n-cut value 1 such that $c(\delta(W)) = 1$ if and only if $W \in \mathcal{W}$.

By Lemma 3.4.4 all $W \in \mathcal{W}$ are nested. Let A be the largest set of them, i.e. $A := W$ such that $W \in \mathcal{W}$ and $|W| = \max\{|W| : W \in \mathcal{W}\}$. We number the elements in A, $A = \{a_1 = 1, a_2, \ldots, a_{|A|}\}$ and define $A_I := \{a_1 = 1, a_2, \ldots, a_I\}$, $I \in [|A|]$ such that $\mathcal{W} \subseteq \{A_1, A_2, \ldots, A_{|A|} = A\}$. As $A \subseteq [n-1] = \sigma([n-1])$ we have that $\mathcal{W}$ is a subset of some $\{\sigma([k]) : k \in [n-1]\}$ what complete the proof of the main result of this section:

Theorem 3.4.7. *The undominated complex for the 1-n-cut polytope $P_{\{1\}}(K_n) \subseteq \mathbb{R}^{\binom{n}{2}}$ for the complete graph on n nodes is a pure simplicial complex of dimension $n-2$ whose facets are described in Remark 3.4.6.*

Remark 3.4.8. *Thus the undominated complex for the 1-n-cut polytope $P_{\{1\}}(K_n) \subseteq \mathbb{R}^{\binom{n}{2}}$ for the complete graph on n nodes consists $(n-2)!$ facets which are $(n-2)$-dimensional.*

Proof. We have $(n-2)!$ possibilities to order elements in the set $[n-1]$ such that the first element remains to be 1. Thus we have $(n-2)!$ different sets of 1-n-cuts what implies that the undominated complex of $P_{\{1\}}(K_n)$ has $(n-2)!$ different facets. □

3.5 Undominated Complex of the [2]-n-Cut Polytope

Making similar reasoning as in the previous section for the 1-n-cut polytope for the complete graph on n nodes $K_n = ([n], E_n)$ we obtain the result for the [2]-n-cut polytope for K_n which will be presented in this section.

Let us introduce some notations used in the current section.

Let the set $[n-1] := \{1, 2, \ldots, n-1\}$ be partitioned into three parts A, B and V such that they are all pairwise disjoint

$$\begin{aligned} A &:= \{a_1 = 1, a_2, \ldots, a_{|A|}\}, \\ B &:= \{b_1 = 2, b_2, \ldots, b_{|B|}\}, \\ V &:= \{v_1, v_2, \ldots, v_{|V|}\}, \end{aligned} \tag{3.53}$$

see Figure 3.6. We define the sets A_I, B_I and V_I as follows:

$$\begin{aligned} A_I &:= \{a_1 = 1, a_2, \ldots, a_I\}, & I \in [|A|] \\ B_I &:= \{b_1 = 2, b_2, \ldots, b_I\}, & I \in [|B|] \\ V_I &:= \{v_1, v_2, \ldots, v_I\}, & I \in [|V|] \end{aligned} \tag{3.54}$$

Let $\mathcal{M}$ be the set of the following sets:

$$\begin{aligned} \mathcal{M} = \{A_I :\ I \in [|A|]\ \} &\cup \{B_I :\ I \in [|B|]\ \} \\ &\cup \{A \cup B\} \cup \{A \cup B \cup V_I :\ I \in [|V|]\ \} \end{aligned} \tag{3.55}$$

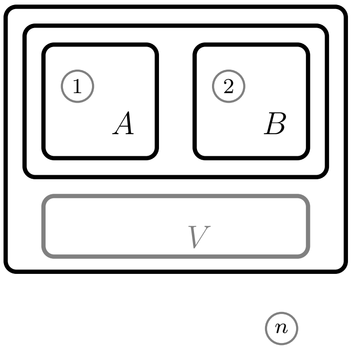

Figure 3.6: Representation of the vertex set $[n]$ in the proof of Proposition 3.5.1.

Proposition 3.5.1. *For each partition of the vertex set* $[n-1]$ *into three parts* A, B *and* V *such that they are all pairwise disjoint and defined as in* (3.53)*, see Figure 3.6,*

$$\mathrm{conv}\{\chi(\delta(M)) :\ M \in \mathcal{M}\} \tag{3.56}$$

where $\mathcal{M}$ *is defined by* (3.55)*, is a face of the undominated complex of the* $[2]$*-*n*-cut polytope* $P_{[2]}(K_n)$ *for the complete graph on* n *nodes* $K_n = ([n], E_n)$*.*

Proof. We first define some $c \in \mathbb{R}^{E_n}_{>0}$ such that for all $M^* \subseteq [n-1]$ with $M^* \cap \{1, 2\} \neq \emptyset$

$$\min\{c(\delta(M)) : M \subseteq [n-1],\ M \cap \{1, 2\} \neq \emptyset\} = c(\delta(M^*)) \tag{3.57}$$

holds if and only if $M^* \in \mathcal{M}$ (see the proof of Lemma 2.3.1).

Let $c \in \mathbb{R}^{E_n}_{>0}$ be defined as follows:

$$
\begin{aligned}
&\text{for } a_i,\ a_j \in A: \quad c_{a_i a_j} = \begin{cases} \dfrac{1}{2\cdot 2^{j-i}}, & 1<i<j,\\ c_{a_2 a_j}, & i=1, \end{cases}\\[2ex]
&\text{for } b_i,\ b_j \in B: \quad c_{b_i b_j} = \begin{cases} \dfrac{1}{2\cdot 2^{j-i}}, & 1<i<j,\\ c_{b_2 b_j}, & i=1, \end{cases}\\[2ex]
&\text{for } v_i,\ v_j \in V: \quad c_{v_i v_j} = \frac{1}{2\cdot 2^{j-i}}, \quad i<j,\\[2ex]
&\text{for } a_i \in A, b_j \in B: \quad c_{a_i b_j} = \begin{cases} \dfrac{1}{2^3\cdot 2^{|A|-i}\cdot 2^{|B|-j}}, & i,\ j>1,\\ c_{a_2 b_j}, & i=1,\\ c_{a_i b_2}, & j=1, \end{cases}\\[2ex]
&\text{for } a_i \in A,\ v_\gamma \in V: \quad c_{a_i v_\gamma} = \begin{cases} \dfrac{1}{2^{\gamma+1}\cdot 2\cdot 2^{|A|-i}}, & i>1,\\ c_{a_2 v_\gamma}, & i=1. \end{cases}\\[2ex]
&\text{for } b_j \in B,\ v_\gamma \in V: \quad c_{b_j v_\gamma} = \begin{cases} \dfrac{1}{2^{\gamma+1}\cdot 2\cdot 2^{|B|-j}}, & j>1,\\ c_{b_2 v_\gamma}, & j=1. \end{cases}\\[2ex]
&\text{for } a_i \in A: \quad c_{a_i n} = \begin{cases} \dfrac{1}{2^{|V|+1}\cdot 2\cdot 2^{|A|-i}}, & i>1,\\ c_{a_2 n}, & i=1. \end{cases}\\[2ex]
&\text{for } b_j \in B: \quad c_{b_j n} = \begin{cases} \dfrac{1}{2^{|V|+1}\cdot 2\cdot 2^{|B|-j}}, & j>1,\\ c_{b_2 n}, & j=1. \end{cases}\\[2ex]
&\text{for } v_\gamma \in V: \quad c_{v_\gamma n} = \frac{1}{2\cdot 2^{|V|-\gamma}}.
\end{aligned}
\tag{3.58}
$$

Lemma 3.5.2. *Let $c \in \mathbb{R}^{E_n}_{>0}$ be defined by* (3.58). *Then the value $c(\delta(M^*))$ of the cut $\delta(M^*)$ is equal to one for all $M^* \in \mathcal{M}$ where $\mathcal{M}$ is defined by* (3.55).

Proof. Consider all four possible cases for a set $M^* \in \mathcal{M}$.

Case 1: $M^* = A_I$ for $I \in [|A|]$.

$$c(\delta(A_I)) = \underbrace{\sum_{i=1}^{I}\sum_{j=1}^{|B|} c_{a_i b_j}}_{\Sigma_1} + \underbrace{\sum_{i=1}^{I}\sum_{\gamma=1}^{|V|} c_{a_i v_\gamma}}_{\Sigma_2} + \underbrace{\sum_{i=1}^{I} c_{a_i n}}_{\Sigma_3} + \underbrace{\sum_{i=1}^{I}\sum_{j=I+1}^{|A|} c_{a_i a_j}}_{\Sigma_4}. \tag{3.59}$$

In view of the definition of $c \in \mathbb{R}^{E_n}_{>0}$, see (3.58), we calculate each term on the right-hand side of equation (3.59).

$$\begin{aligned}
\Sigma_1 &= \sum_{i=1}^{I}\sum_{j=1}^{|B|} c_{a_i b_j} = c_{a_1 b_1} + \sum_{i=2}^{I} c_{a_i b_1} + \sum_{j=2}^{|B|} c_{a_1 b_j} + \sum_{i=2}^{I}\sum_{j=2}^{|B|} c_{a_i b_j} \\
&= c_{a_2 b_2} + \sum_{i=2}^{I} c_{a_i b_2} + \sum_{j=2}^{|B|} c_{a_2 b_j} + \sum_{i=2}^{I}\sum_{j=2}^{|B|} c_{a_i b_j} \\
&= \frac{1}{2^3 \cdot 2^{|A|-2} \cdot 2^{|B|-2}} + \sum_{i=2}^{I} \frac{1}{2^3 \cdot 2^{|A|-i} \cdot 2^{|B|-2}} \\
&\quad + \sum_{j=2}^{|B|} \frac{1}{2^3 \cdot 2^{|A|-2} \cdot 2^{|B|-j}} + \sum_{i=2}^{I}\sum_{j=2}^{|B|} \frac{1}{2^3 \cdot 2^{|A|-i} \cdot 2^{|B|-j}} \\
&= \frac{1}{2^{|A|+|B|-1}} \left(1 + \frac{1}{2^2}\sum_{i=2}^{I} 2^i + \frac{1}{2^2}\sum_{j=2}^{|B|} 2^j + \frac{1}{2^2}\sum_{i=2}^{I} 2^i \frac{1}{2^2}\sum_{j=2}^{|B|} 2^j \right) \\
&= \frac{1}{2^{|A|+|B|-1}} \left(1 + \frac{1}{2^2}\sum_{i=2}^{I} 2^i \right) \left(1 + \frac{1}{2^2}\sum_{j=2}^{|B|} 2^j \right) \\
&= \frac{1}{2^{|A|+|B|-1}} \left(1 + \frac{2^2(1-2^{I-1})}{2^2(1-2)} \right) \left(1 + \frac{2^2(1-2^{|B|-1})}{2^2(1-2)} \right) \\
&= \frac{2^{I-1} \cdot 2^{|B|-1}}{2^{|A|+|B|-1}} = \frac{2^{I-1}}{2^{|A|}}.
\end{aligned} \tag{3.60}$$

$$\begin{aligned}
\Sigma_2 &= \sum_{i=1}^{I}\sum_{\gamma=1}^{|V|} c_{a_i v_\gamma} = \sum_{\gamma=1}^{|V|} c_{a_1 v_\gamma} + \sum_{i=2}^{I}\sum_{\gamma=1}^{|V|} c_{a_i v_\gamma} = \sum_{\gamma=1}^{|V|} c_{a_2 v_\gamma} + \sum_{i=2}^{I}\sum_{\gamma=1}^{|V|} c_{a_i v_\gamma} \\
&= \sum_{\gamma=1}^{|V|} \frac{1}{2^{\gamma+1} \cdot 2 \cdot 2^{|A|-2}} + \sum_{i=2}^{I}\sum_{\gamma=1}^{|V|} \frac{1}{2^{\gamma+1} \cdot 2 \cdot 2^{|A|-i}}
\end{aligned} \tag{3.61}$$

$$
\begin{aligned}
&= \frac{1}{2^{|A|}} \sum_{\gamma=1}^{|V|} \frac{1}{2^{\gamma}} \left(1 + \frac{1}{2^2} \sum_{i=2}^{I} 2^i \right) = \frac{1}{2^{|A|}} \cdot \frac{1}{2} \cdot \frac{1 - \frac{1}{2^{|V|}}}{1 - \frac{1}{2}} \cdot \left(1 + \frac{2^2}{2^2} \cdot \frac{1 - 2^{I-1}}{1-2}\right) \\
&= \frac{2^{I-1} \cdot \left(2^{|V|} - 1\right)}{2^{|A|+|V|}} = \frac{2^{I-1}}{2^{|A|}} - \frac{2^{I-1}}{2^{|A|+|V|}}.
\end{aligned}
\tag{3.62}
$$

$$
\begin{aligned}
\Sigma_3 &= \sum_{i=1}^{I} c_{a_i n} = c_{a_1 n} + \sum_{i=2}^{I} c_{a_i n} = c_{a_2 n} + \sum_{i=2}^{I} c_{a_i n} \\
&= \frac{1}{2^{|V|+1} \cdot 2 \cdot 2^{|A|-2}} + \sum_{i=2}^{I} \frac{1}{2^{|V|+1} \cdot 2 \cdot 2^{|A|-i}} \\
&= \frac{1}{2^{|V|} \cdot 2^{|A|}} \left(1 + \frac{1}{2^2} \sum_{i=2}^{I} 2^i \right) = \frac{1}{2^{|V|} \cdot 2^{|A|}} \left(1 + \frac{2^2}{2^2} \cdot \frac{1 - 2^{I-1}}{1-2}\right) = \frac{2^{I-1}}{2^{|V|+|A|}}.
\end{aligned}
\tag{3.63}
$$

$$
\begin{aligned}
\Sigma_4 &= \sum_{i=1}^{I} \sum_{j=I+1}^{|A|} c_{a_i a_j} = \sum_{j=I+1}^{|A|} c_{a_1 a_j} + \sum_{i=2}^{I} \sum_{j=I+1}^{|A|} c_{a_i a_j} = \sum_{j=I+1}^{|A|} c_{a_2 a_j} + \sum_{i=2}^{I} \sum_{j=I+1}^{|A|} c_{a_i a_j} \\
&= \sum_{j=I+1}^{|A|} \frac{1}{2 \cdot 2^{j-2}} + \sum_{i=2}^{I} \sum_{j=I+1}^{|A|} \frac{1}{2 \cdot 2^{j-i}} = 2 \cdot \sum_{j=I+1}^{|A|} \frac{1}{2^j} \left(1 + \frac{1}{2^2} \cdot \sum_{i=2}^{I} 2^i \right) \\
&= 2 \cdot \frac{1}{2^{I+1}} \cdot \frac{1 - \frac{1}{2^{|A|-I}}}{1 - \frac{1}{2}} \cdot \left(1 + \frac{2^2}{2^2} \cdot \frac{1 - 2^{I-1}}{1-2}\right) = \frac{(2^{|A|-I} - 1) \cdot 2^{I-1}}{2^{|A|-I} \cdot 2^{I-1}} = 1 - \frac{2^I}{2^{|A|}}.
\end{aligned}
\tag{3.64}
$$

Thus for equation (3.59) with $I \in [|A|]$ we have

$$
c(\delta(A_I)) = \frac{2^{I-1}}{2^{|A|}} + \frac{2^{I-1}}{2^{|A|}} - \frac{2^{I-1}}{2^{|A|+|V|}} + \frac{2^{I-1}}{2^{|V|+|A|}} + 1 - \frac{2^I}{2^{|A|}} = 1. \tag{3.65}
$$

Case 2: $M^* = B_I$ for $I \in [|B|]$. Due to the symmetry of c the equation $c(\delta(B_I)) = 1$ follows from Case 1 by exchanging A and B (and nodes 1 and 2).

Case 3: $M^* = A \cup B$.

$$
c(\delta(A \cup B)) = c(\delta(A_{|A|})) + c(\delta(B_{|B|})) - 2 \sum_{i=1}^{|A|} \sum_{j=1}^{|B|} c_{a_i b_j}. \tag{3.66}
$$

The first two terms both evaluate to one (Cases 1 and 2). The third term on the right-hand side of equation (3.66) is a particular case of Σ_1 with $I = |A|$ from Case 1. Thus we have

$$
c(\delta(A \cup B)) = 1 + 1 - 2 \frac{2^{|A|-1}}{2^{|A|}} = 1. \tag{3.67}
$$

Case 4: $M^* = A \cup B \cup V_I$ for $I \in [|V|]$.

$$c(\delta(A \cup B \cup V_I)) = c(\delta(A \cup B)) \\ - \underbrace{\sum_{i=1}^{|A|}\sum_{\gamma=1}^{I} c_{a_i v_\gamma}}_{\Sigma_5} - \underbrace{\sum_{j=1}^{|B|}\sum_{\gamma=1}^{I} c_{b_j v_\gamma}}_{\Sigma_6} + \underbrace{\sum_{\gamma=1}^{I} c_{v_\gamma n}}_{\Sigma_7} + \underbrace{\sum_{\gamma=1}^{I}\sum_{\beta=I+1}^{|V|} c_{v_\gamma v_\beta}}_{\Sigma_8}. \tag{3.68}$$

Calculating each term on the right-hand side of equation (3.68) we obtain

$$\begin{aligned}
\Sigma_5 &= \sum_{i=1}^{|A|}\sum_{\gamma=1}^{I} c_{a_i v_\gamma} = \sum_{\gamma=1}^{I} c_{a_1 v_\gamma} + \sum_{i=2}^{|A|}\sum_{\gamma=1}^{I} c_{a_i v_\gamma} = \sum_{\gamma=1}^{I} c_{a_2 v_\gamma} + \sum_{i=2}^{|A|}\sum_{\gamma=1}^{I} c_{a_i v_\gamma} \\
&= \sum_{\gamma=1}^{I} \frac{1}{2^{\gamma+1}\cdot 2\cdot 2^{|A|-2}} + \sum_{i=2}^{|A|}\sum_{\gamma=1}^{I} \frac{1}{2^{\gamma+1}\cdot 2\cdot 2^{|A|-i}} \\
&= \frac{1}{2^{|A|}}\sum_{\gamma=1}^{I}\frac{1}{2^\gamma}\left(1+\frac{1}{2^2}\sum_{i=2}^{|A|}2^i\right) = \frac{1}{2^{|A|}}\cdot\frac{1}{2}\cdot\frac{1-\frac{1}{2^I}}{1-\frac{1}{2}}\cdot\left(1+\frac{2^2}{2^2}\cdot\frac{1-2^{|A|-1}}{1-2}\right) \\
&= \frac{2^{|A|-1}\cdot\left(2^I-1\right)}{2^{|A|+I}} = \frac{1}{2}-\frac{1}{2^{I+1}}.
\end{aligned} \tag{3.69}$$

$$\begin{aligned}
\Sigma_6 &= \sum_{j=1}^{|B|}\sum_{\gamma=1}^{I} c_{b_j v_\gamma} = \sum_{\gamma=1}^{I} c_{b_1 v_\gamma} + \sum_{j=2}^{|B|}\sum_{\gamma=1}^{I} c_{b_j v_\gamma} = \sum_{\gamma=1}^{I} c_{b_2 v_\gamma} + \sum_{j=2}^{|B|}\sum_{\gamma=1}^{I} c_{b_j v_\gamma} \\
&= \sum_{\gamma=1}^{I} \frac{1}{2^{\gamma+1}\cdot 2\cdot 2^{|B|-2}} + \sum_{j=2}^{|B|}\sum_{\gamma=1}^{I} \frac{1}{2^{\gamma+1}\cdot 2\cdot 2^{|B|-j}} \\
&= \frac{1}{2^{|B|}}\sum_{\gamma=1}^{I}\frac{1}{2^\gamma}\left(1+\frac{1}{2^2}\sum_{j=2}^{|B|}2^j\right) = \frac{1}{2^{|B|}}\cdot\frac{1}{2}\cdot\frac{1-\frac{1}{2^I}}{1-\frac{1}{2}}\cdot\left(1+\frac{2^2}{2^2}\cdot\frac{1-2^{|B|-1}}{1-2}\right) \\
&= \frac{2^{|B|-1}\cdot\left(2^I-1\right)}{2^{|B|+I}} = \frac{1}{2}-\frac{1}{2^{I+1}}.
\end{aligned} \tag{3.70}$$

$$\Sigma_7 = \sum_{\gamma=1}^{I} c_{v_\gamma n} = \sum_{\gamma=1}^{I}\frac{1}{2\cdot 2^{|V|-\gamma}} = \frac{1}{2^{|V|+1}}\sum_{\gamma=1}^{I}2^\gamma = \frac{1}{2^{|V|+1}}\cdot\frac{2\cdot(1-2^I)}{1-2} = \frac{2^I-1}{2^{|V|}}. \tag{3.71}$$

$$
\begin{aligned}
\Sigma_8 &= \sum_{\gamma=1}^{I} \sum_{\beta=I+1}^{|V|} c_{v_\gamma v_\beta} = \sum_{\gamma=1}^{I} \sum_{\beta=I+1}^{|V|} \frac{1}{2 \cdot 2^{\beta-\gamma}} = \frac{1}{2} \sum_{\beta=I+1}^{|V|} \frac{1}{2^\beta} \sum_{\gamma=1}^{I} 2^\gamma \\
&= \frac{1}{2} \cdot \frac{1}{2^{I+1}} \cdot \frac{1-\frac{1}{2^{|V|-I}}}{1-\frac{1}{2}} \cdot \frac{2(1-2^I)}{1-2} = \frac{2^I-1}{2^I} - \frac{2^I-1}{2^{|V|}} = 1 - \frac{1}{2^I} - \frac{2^I-1}{2^{|V|}}.
\end{aligned} \tag{3.72}
$$

Then in view of (3.67) and (3.69)-(3.72) expression (3.68) can be written as

$$
\begin{aligned}
c(\delta(A \cup B \cup V_I)) = 1 - \left(\frac{1}{2} - \frac{1}{2^{I+1}}\right) - \left(\frac{1}{2} - \frac{1}{2^{I+1}}\right) + \cancel{\frac{2^I-1}{2^{|V|}}} \\
+ 1 - \frac{1}{2^I} - \cancel{\frac{2^I-1}{2^{|V|}}} = 1 - \frac{2}{2} + \frac{2}{2^{I+1}} + 1 - \frac{1}{2^I} = 1.
\end{aligned} \tag{3.73}
$$

□

Continuing the proof of Proposition 3.5.1 we construct from the undirected graph K_n the directed graph $\overleftrightarrow{K}_n = ([n], \overleftrightarrow{E}_n)$ by replacing each edge $e \in E_n$ by two antiparallel arcs. In the new graph $\overleftrightarrow{K}_n$ we construct two flows: a 1-n flow $f' : \overleftrightarrow{E}_n \to \mathbb{R}$, see Figure 3.7, with

$$
\begin{array}{ll}
f'_{a_i a_j} := c_{a_i a_j} & \text{for } a_i, a_j \in A \text{ with } i < j, \\
f'_{a_i a_j} := 0 & \text{for } a_i, a_j \in A \text{ with } i > j, \\
\\
f'_{b_i b_j} := 0 & \text{for all } b_i, b_j \in B, \\
\\
f'_{a_i b_j} := c_{a_i b_j} & \text{for all } a_i \in A,\ b_j \in B, \\
f'_{b_j a_i} := 0 & \text{for all } a_i \in A,\ b_j \in B, \\
\\
f'_{v_i v_j} := c_{v_i v_j} & \text{for } v_i, v_j \in V \text{ with } i < j, \\
f'_{v_i v_j} := 0 & \text{for } v_i, v_j \in V \text{ with } i > j, \\
\\
f'_{a_i v_\gamma} := c_{a_i v_\gamma} & \text{for all } a_i \in A,\ v_\gamma \in V, \\
f'_{v_\gamma a_i} := 0 & \text{for all } a_i \in A,\ v_\gamma \in V, \\
\\
f'_{b_j v_\gamma} := c_{b_j v_\gamma} & \text{for all } b_j \in B,\ v_\gamma \in V, \\
f'_{v_\gamma b_j} := 0 & \text{for all } b_j \in B,\ v_\gamma \in V, \\
\\
f'_{a_i n} := c_{a_i n} & \text{for all } a_i \in A, \\
f'_{n a_i} := 0 & \text{for all } a_i \in A, \\
\\
f'_{b_j n} := c_{b_j n} & \text{for all } b_j \in B, \\
f'_{n b_j} := 0 & \text{for all } b_j \in B, \\
\\
f'_{v_\gamma n} := c_{v_\gamma n} & \text{for all } v_\gamma \in V, \\
f'_{n v_\gamma} := 0 & \text{for all } v_\gamma \in V,
\end{array} \tag{3.74}
$$

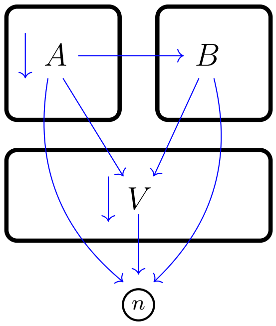

Figure 3.7: Sketch of arcs $(u,w) \in \overset{\leftrightarrow}{E}_n$ with $f'_{u,w} > 0$.

and a 2-n flow $f'' : \overset{\leftrightarrow}{E}_n \to \mathbb{R}$, see Figure 3.8, where

$$\begin{aligned}
f''_{a_i a_j} &:= 0 && \text{for all } a_i, a_j \in A,\\
f''_{b_i b_j} &:= c_{b_i b_j} && \text{for } b_i, b_j \in B \text{ with } i < j,\\
f''_{b_i b_j} &:= 0 && \text{for } b_i, b_j \in B \text{ with } i > j,\\
f''_{a_i b_j} &:= 0 && \text{for all } a_i \in A,\ b_j \in B,\\
f''_{b_j a_i} &:= c_{a_i b_j} && \text{for all } a_i \in A,\ b_j \in B,\\
f''_{v_i v_j} &:= c_{v_i v_j} && \text{for } v_i, v_j \in V \text{ with } i < j,\\
f''_{v_i v_j} &:= 0 && \text{for } v_i, v_j \in V \text{ with } i > j,\\
f''_{a_i v_\gamma} &:= c_{a_i v_\gamma} && \text{for all } a_i \in A,\ v_\gamma \in V,\\
f''_{v_\gamma a_i} &:= 0 && \text{for all } a_i \in A,\ v_\gamma \in V, \qquad (3.75)\\
f''_{b_j v_\gamma} &:= c_{b_j v_\gamma} && \text{for all } b_j \in B,\ v_\gamma \in V,\\
f''_{v_\gamma b_j} &:= 0 && \text{for all } b_j \in B,\ v_\gamma \in V,\\
f''_{a_i n} &:= c_{a_i n} && \text{for all } a_i \in A,\\
f''_{n a_i} &:= 0 && \text{for all } a_i \in A,\\
f''_{b_j n} &:= c_{b_j n} && \text{for all } b_j \in B,\\
f''_{n b_j} &:= 0 && \text{for all } b_j \in B,\\
f''_{v_\gamma n} &:= c_{v_\gamma n} && \text{for all } v_\gamma \in V,\\
f''_{n v_\gamma} &:= 0 && \text{for all } v_\gamma \in V.
\end{aligned}$$

First we consider the 1-n flow $f' : \overset{\leftrightarrow}{E}_n \to \mathbb{R}$ and check the *flow conservation law*: the amount of flow entering a vertex $a_i \in A \setminus \{a_1 = 1\}$, $b_j \in B$ and $v_\gamma \in V$ should be equal to the amount of flow leaving $a_i \in A \setminus \{a_1 = 1\}$, $b_j \in B$ and $v_\gamma \in V$, respectively.

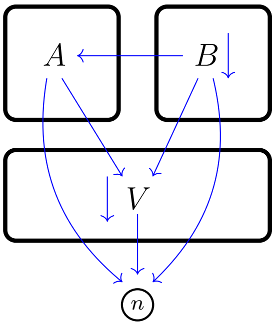

Figure 3.8: Sketch of arcs $(u,w) \in \overset{\leftrightarrow}{E}_n$ with $f''_{u,w} > 0$.

In view of (3.74) for f' we have for each $a_i \in A \setminus \{a_1 = 1\}$

$$\begin{aligned}
f'(\delta^{out}(a_i)) &= \sum_{j=1}^{|B|} c_{a_i b_j} + \sum_{\gamma=1}^{|V|} c_{a_i v_\gamma} + c_{a_i n} + \sum_{j=i+1}^{|A|} c_{a_i a_j} \\
&= c_{a_i b_1} + \sum_{j=2}^{|B|} c_{a_i b_j} + \sum_{\gamma=1}^{|V|} c_{a_i v_\gamma} + c_{a_i n} + \sum_{j=i+1}^{|A|} c_{a_i a_j} \\
&= c_{a_i b_2} + \sum_{j=2}^{|B|} c_{a_i b_j} + \sum_{\gamma=1}^{|V|} c_{a_i v_\gamma} + c_{a_i n} + \sum_{j=i+1}^{|A|} c_{a_i a_j} \\
&= \frac{1}{2^3 \cdot 2^{|A|-i} \cdot 2^{|B|-2}} + \sum_{j=2}^{|B|} \frac{1}{2^3 \cdot 2^{|A|-i} \cdot 2^{|B|-j}} \\
&\quad + \sum_{\gamma=1}^{|V|} \frac{1}{2^{\gamma+1} \cdot 2 \cdot 2^{|A|-i}} + \frac{1}{2^{|V|+1} \cdot 2 \cdot 2^{|A|-i}} + \sum_{j=i+1}^{|A|} \frac{1}{2 \cdot 2^{j-i}} \\
&= \frac{1}{2^{|A|-i} \cdot 2^{|B|+1}} \left(1 + \frac{1}{2^2} \sum_{j=2}^{|B|} 2^j\right) + \frac{1}{2^{|A|-i+2}} \sum_{\gamma=1}^{|V|} \frac{1}{2^\gamma} \\
&\quad + \frac{1}{2^{|V|+1} \cdot 2 \cdot 2^{|A|-i}} + \frac{1}{2^{1-i}} \sum_{j=i+1}^{|A|} \frac{1}{2^j} \\
&= \frac{1}{2^{|A|-i+|B|+1}} \left(1 + \frac{1}{2^2} \cdot \frac{2^2 \cdot (1 - 2^{|B|-1})}{1-2}\right) + \frac{1}{2^{|A|-i+2}} \cdot \frac{1}{2} \cdot \frac{1 - \frac{1}{2^{|V|}}}{1 - \frac{1}{2}} \\
&\quad + \frac{1}{2^{|V|+1} \cdot 2 \cdot 2^{|A|-i}} + \frac{1}{2^{1-i}} \cdot \frac{1}{2^{i+1}} \cdot \frac{1 - \frac{1}{2^{|A|-i}}}{1 - \frac{1}{2}} \\
&= \frac{2^{|B|-1}}{2^{|A|-i} \cdot 2^{|B|+1}} + \frac{1}{2^{|A|-i+2}} - \frac{1}{2^{|A|-i+2+|V|}} + \frac{1}{2^{|V|+1} \cdot 2 \cdot 2^{|A|-i}} \\
&\quad + \frac{1}{2} - \frac{1}{2^{|A|-i+1}} = \frac{1}{2} + \frac{2}{2^{|A|-i+2}} - \frac{1}{2^{|A|-i+1}} = \frac{1}{2},
\end{aligned} \tag{3.76}$$

and for each $a_i \in A \setminus \{a_1 = 1\}$

$$\begin{aligned}
f'(\delta^{in}(a_i)) &= \sum_{j=1}^{i-1} c_{a_i a_j} = c_{a_1 a_i} + \sum_{j=2}^{i-1} c_{a_i a_j} = c_{a_2 a_i} + \sum_{j=2}^{i-1} c_{a_i a_j} \\
&= \frac{1}{2 \cdot 2^{i-2}} + \sum_{j=2}^{i-1} \frac{1}{2 \cdot 2^{i-j}} = \frac{1}{2^{i-1}} \left(1 + \frac{1}{2^2} \sum_{j=2}^{i-1} 2^j \right) \\
&= \frac{1}{2^{i-1}} \left(1 + \frac{1}{2^2} \cdot \frac{2^2 \cdot (1 - 2^{i-2})}{1-2} \right) = \frac{1}{2}.
\end{aligned} \tag{3.77}$$

Note that, from the definition of $c \in \mathbb{R}^{E_n}_{>0}$, see (3.58), we have $c_{b_1,*} = c_{b_2,*}$. Thus, it suffices to calculate $f'(\delta^{out}(b_j))$ and $f'(\delta^{in}(b_j))$ for $j \geq 2$ due to $f'(\delta^{out}(b_1)) = f'(\delta^{out}(b_2))$ and $f'(\delta^{in}(b_1)) = f'(\delta^{in}(b_2))$.

Thus for each $b_j \in B$ we obtain

$$\begin{aligned}
f'(\delta^{out}(b_j)) &= \sum_{\gamma=1}^{|V|} c_{b_j v_\gamma} + c_{b_j n} = \sum_{\gamma=1}^{|V|} \frac{1}{2^{\gamma+1} \cdot 2 \cdot 2^{|B|-j}} + \frac{1}{2^{|V|+1} \cdot 2 \cdot 2^{|B|-j}} \\
&= \frac{1}{2^{|B|-j+2}} \sum_{\gamma=1}^{|V|} \frac{1}{2^\gamma} + \frac{1}{2^{|V|+1} \cdot 2 \cdot 2^{|B|-j}} \\
&= \frac{1}{2^{|B|-j+2}} \cdot \frac{1}{2} \cdot \frac{1 - \frac{1}{2^{|V|}}}{1 - \frac{1}{2}} + \frac{1}{2^{|V|+|B|-j+2}} \\
&= \frac{1}{2^{|B|-j+2}} - \cancel{\frac{1}{2^{|V|+|B|-j+2}}} + \cancel{\frac{1}{2^{|V|+|B|-j+2}}} = \frac{1}{2^{|B|-j+2}},
\end{aligned} \tag{3.78}$$

$$\begin{aligned}
f'(\delta^{in}(b_j)) &= \sum_{i=1}^{|A|} c_{a_i b_j} = c_{a_1 b_j} + \sum_{i=2}^{|A|} c_{a_i b_j} = c_{a_2 b_j} + \sum_{i=2}^{|A|} c_{a_i b_j} \\
&= \frac{1}{2^3 \cdot 2^{|A|-2} \cdot 2^{|B|-j}} + \sum_{i=2}^{|A|} \frac{1}{2^3 \cdot 2^{|A|-i} \cdot 2^{|B|-j}} \\
&= \frac{1}{2^3 \cdot 2^{|A|-2} \cdot 2^{|B|-j}} \left(1 + \frac{1}{2^2} \sum_{i=2}^{|A|} 2^i \right) \\
&= \frac{1}{2^{|A|+1} \cdot 2^{|B|-j}} \cdot \left(1 + \frac{1}{2^2} \cdot \frac{2^2(1 - 2^{|A|-1})}{1-2} \right) = \frac{2^{|A|-1}}{2^{|A|+1+|B|-j}} = \frac{1}{2^{|B|-j+2}}.
\end{aligned} \tag{3.79}$$

For each $v_\gamma \in V$ we have

$$\begin{aligned} f'(\delta^{out}(v_\gamma)) &= \sum_{i=\gamma+1}^{|V|} c_{v_\gamma v_i} + c_{v_\gamma n} = \sum_{i=\gamma+1}^{|V|} \frac{1}{2\cdot 2^{i-\gamma}} + \frac{1}{2\cdot 2^{|V|-\gamma}} \\ &= \frac{1}{2^{1-\gamma}} \sum_{i=\gamma+1}^{|V|} \frac{1}{2^i} + \frac{1}{2\cdot 2^{|V|-\gamma}} = \frac{1}{2^{1-\gamma}} \cdot \frac{1}{2^{\gamma+1}} \cdot \frac{1-\frac{1}{2^{|V|-\gamma}}}{1-\frac{1}{2}} + \frac{1}{2\cdot 2^{|V|-\gamma}} \\ &= \frac{1}{2} - \cancel{\frac{1}{2\cdot 2^{|V|-\gamma}}} + \cancel{\frac{1}{2\cdot 2^{|V|-\gamma}}} = \frac{1}{2}, \end{aligned} \tag{3.80}$$

$$\begin{aligned} f'(\delta^{in}(v_\gamma)) &= \sum_{i=1}^{|A|} c_{a_i v_\gamma} + \sum_{j=1}^{|B|} c_{b_j v_\gamma} + \sum_{i=1}^{\gamma-1} c_{v_i v_\gamma} \\ &= c_{a_1 v_\gamma} + \sum_{i=2}^{|A|} c_{a_i v_\gamma} + c_{b_1 v_\gamma} + \sum_{j=2}^{|B|} c_{b_j v_\gamma} + \sum_{i=1}^{\gamma-1} c_{v_i v_\gamma} \\ &= c_{a_2 v_\gamma} + \sum_{i=2}^{|A|} c_{a_i v_\gamma} + c_{b_2 v_\gamma} + \sum_{j=2}^{|B|} c_{b_j v_\gamma} + \sum_{i=1}^{\gamma-1} c_{v_i v_\gamma} \\ &= \frac{1}{2^{\gamma+1}\cdot 2\cdot 2^{|A|-2}} + \sum_{i=2}^{|A|} \frac{1}{2^{\gamma+1}\cdot 2\cdot 2^{|A|-i}} \\ &\quad + \frac{1}{2^{\gamma+1}\cdot 2\cdot 2^{|B|-2}} + \sum_{j=2}^{|B|} \frac{1}{2^{\gamma+1}\cdot 2\cdot 2^{|B|-j}} + \sum_{i=1}^{\gamma-1} \frac{1}{2\cdot 2^{\gamma-i}} \\ &= \frac{1}{2^{\gamma+|A|}} \left(1 + \frac{1}{2^2} \sum_{i=2}^{|A|} 2^i\right) + \frac{1}{2^{\gamma+|B|}} \left(1 + \frac{1}{2^2} \sum_{j=2}^{|B|} 2^j\right) + \frac{1}{2^{\gamma+1}} \sum_{i=1}^{\gamma-1} 2^i \\ &= \frac{1}{2^{\gamma+|A|}} \left(1 + \frac{1}{2^2} \cdot \frac{2^2\cdot(1-2^{|A|-1})}{1-2}\right) \\ &\quad + \frac{1}{2^{\gamma+|B|}} \left(1 + \frac{1}{2^2} \cdot \frac{2^2\cdot(1-2^{|B|-1})}{1-2}\right) + \frac{1}{2^{\gamma+1}} \cdot \frac{2\cdot(1-2^{\gamma-1})}{1-2} \\ &= \frac{2^{|A|-1}}{2^{\gamma+|A|}} + \frac{2^{|B|-1}}{2^{\gamma+|B|}} + \frac{2^{\gamma-1}}{2^\gamma} - \frac{1}{2^\gamma} = \frac{1}{2^{\gamma+1}} + \frac{1}{2^{\gamma+1}} + \frac{1}{2} - \frac{1}{2^\gamma} = \frac{1}{2}. \end{aligned} \tag{3.81}$$

Hence, both conditions of the definition of a flow are satisfied for our 1-n flow f', i.e.

$$\begin{aligned} f'_e &\geq 0 && \text{for each } e \in \overset{\leftrightarrow}{E}_n, \\ f'(\delta^{out}(v)) &= f'(\delta^{in}(v)) && \text{for each } v \in [n] \setminus \{1, n\}. \end{aligned} \tag{3.82}$$

The value of this flow f' is equal to 1 due to:

$$\text{value}(f') = f'(\delta^{out}(1)) = f(\delta^{out}(a_1)) = c(\delta(A_1)) \overset{\text{Lemma 3.5.2}}{=} 1. \tag{3.83}$$

Now consider the 2-n flow $f'' : \overleftrightarrow{E}_n \to \mathbb{R}$ and check again the *flow conservation law*: the amount of flow entering a vertex $a_i \in A$, $b_j \in B \setminus \{b_1 = 2\}$ and $v_\gamma \in V$ should be equal to the amount of flow leaving $a_i \in A$, $b_j \in B \setminus \{b_1 = 2\}$ and $v_\gamma \in V$, respectively.

Note that, from the definition of $c \in \mathbb{R}^{E_n}_{>0}$, see (3.58), we have $c_{a_1,*} = c_{a_2,*}$. Thus, it suffices to calculate $f''(\delta^{out}(a_i))$ and $f''(\delta^{in}(a_i))$ for $i \geq 2$ due to $f''(\delta^{out}(a_1)) = f''(\delta^{out}(a_2))$ and $f''(\delta^{in}(a_1)) = f''(\delta^{in}(a_2))$.

Thus, in view of (3.75) for f'' we have for each $a_i \in A$, $i \geq 2$

$$\begin{aligned} f''(\delta^{out}(a_i)) &= \sum_{\gamma=1}^{|V|} c_{a_i v_\gamma} + c_{a_i n} = \sum_{\gamma=1}^{|V|} \frac{1}{2^{\gamma+1} \cdot 2 \cdot 2^{|A|-i}} + \frac{1}{2^{|V|+1} \cdot 2 \cdot 2^{|A|-i}} \\ &= \frac{1}{2^{|A|-i+2}} \sum_{\gamma=1}^{|V|} \frac{1}{2^\gamma} + \frac{1}{2^{|V|+1} \cdot 2 \cdot 2^{|A|-i}} \\ &= \frac{1}{2^{|A|-i+2}} \cdot \frac{\frac{1}{2} \cdot (1 - \frac{1}{2^{|V|}})}{1 - \frac{1}{2}} + \frac{1}{2^{|V|+2+|A|-i}} \\ &= \frac{1}{2^{|A|-i+2}} - \cancel{\frac{1}{2^{|A|-i+2+|V|}}} + \cancel{\frac{1}{2^{|V|+2+|A|-i}}} = \frac{1}{2^{|A|-i+2}}, \end{aligned} \tag{3.84}$$

$$\begin{aligned} f''(\delta^{in}(a_i)) &= \sum_{j=1}^{|B|} c_{a_i b_j} = c_{a_i b_1} + \sum_{j=2}^{|B|} c_{a_i b_j} c_{a_i b_2} + \sum_{j=2}^{|B|} c_{a_i b_j} \\ &= \frac{1}{2^3 \cdot 2^{|A|-i} \cdot 2^{|B|-2}} + \sum_{j=2}^{|B|} \frac{1}{2^3 \cdot 2^{|A|-i} \cdot 2^{|B|-j}} \\ &= \frac{1}{2^3 \cdot 2^{|A|-i} \cdot 2^{|B|-2}} \left(1 + \frac{1}{2^2} \sum_{j=2}^{|B|} 2^j\right) \\ &= \frac{1}{2^{|A|-i} 2^{|B|+1}} \cdot \left(1 + \frac{1}{2^2} \cdot \frac{2^2(1 - 2^{|B|-1})}{1-2}\right) = \frac{2^{|B|-1}}{2^{|A|-i+|B|+1}} = \frac{1}{2^{|A|-i+2}}. \end{aligned} \tag{3.85}$$

For each $b_j \in B \setminus \{b_1 = 2\}$ we obtain

$$\begin{aligned} f''(\delta^{out}(b_j)) &= \sum_{i=1}^{|A|} c_{a_i b_j} + \sum_{\gamma=1}^{|V|} c_{b_j v_\gamma} + c_{b_j n} + \sum_{i=j+1}^{|B|} c_{b_j b_i} \\ &= c_{a_1 b_j} + \sum_{i=2}^{|A|} c_{a_i b_j} + \sum_{\gamma=1}^{|V|} c_{b_j v_\gamma} + c_{b_j n} + \sum_{i=j+1}^{|B|} c_{b_j b_i} \end{aligned} \tag{3.86}$$

$$
\begin{aligned}
&= c_{a_2b_j} + \sum_{i=2}^{|A|} c_{a_ib_j} + \sum_{\gamma=1}^{|V|} c_{b_jv_\gamma} + c_{b_jn} + \sum_{i=j+1}^{|B|} c_{b_jb_i} \\
&= \frac{1}{2^3\cdot 2^{|A|-2}\cdot 2^{|B|-j}} + \sum_{i=2}^{|A|} \frac{1}{2^3\cdot 2^{|A|-i}\cdot 2^{|B|-j}} \\
&\quad + \sum_{\gamma=1}^{|V|} \frac{1}{2^{\gamma+1}\cdot 2\cdot 2^{|B|-j}} + \frac{1}{2^{|V|+1}\cdot 2\cdot 2^{|B|-j}} + \sum_{i=j+1}^{|B|} \frac{1}{2\cdot 2^{i-j}} \\
&= \frac{1}{2^{|A|+1}\cdot 2^{|B|-j}} \left(1 + \frac{1}{2^2}\sum_{i=2}^{|A|} 2^i\right) + \frac{1}{2^{|B|-j+2}} \sum_{\gamma=1}^{|V|} \frac{1}{2^\gamma} \\
&\quad + \frac{1}{2^{|V|+1}\cdot 2\cdot 2^{|B|-j}} + \frac{1}{2^{1-j}} \sum_{i=j+1}^{|B|} \frac{1}{2^i} \\
&= \frac{1}{2^{|A|+1+|B|-j}} \left(1 + \frac{1}{2^2}\cdot \frac{2^2\cdot(1-2^{|A|-1})}{1-2}\right) + \frac{1}{2^{|B|-j+2}}\cdot\frac{1}{2}\cdot\frac{1-\frac{1}{2^{|V|}}}{1-\frac{1}{2}} \\
&\quad + \frac{1}{2^{|V|+1}\cdot 2\cdot 2^{|B|-j}} + \frac{1}{2^{1-j}}\cdot\frac{1}{2^{j+1}}\cdot\frac{1-\frac{1}{2^{|B|-j}}}{1-\frac{1}{2}} \\
&= \frac{2^{|A|-1}}{2^{|B|-j}\cdot 2^{|A|+1}} + \frac{1}{2^{|B|-j+2}} - \frac{1}{2^{|B|-j+2+|V|}} + \frac{1}{2^{|V|+1}\cdot 2\cdot 2^{|B|-j}} \\
&\quad + \frac{1}{2} - \frac{1}{2^{|B|-j+1}} = \frac{1}{2} + \frac{2}{2^{|B|-j+2}} - \frac{1}{2^{|B|-j+1}} = \frac{1}{2},
\end{aligned} \tag{3.87}
$$

and for each $b_j \in B \setminus \{b_1 = 2\}$

$$
\begin{aligned}
f''(\delta^{in}(b_j)) &= \sum_{i=1}^{j-1} c_{b_ib_j} = c_{b_1b_j} + \sum_{i=2}^{j-1} c_{b_ib_j} = c_{b_2b_j} + \sum_{i=2}^{j-1} c_{b_ib_j} \\
&= \frac{1}{2\cdot 2^{j-2}} + \sum_{i=2}^{j-1} \frac{1}{2\cdot 2^{j-i}} = \frac{1}{2^{j-1}}\left(1 + \frac{1}{2^2}\sum_{i=2}^{j-1} 2^i\right) \\
&= \frac{1}{2^{j-1}}\left(1 + \frac{1}{2^2}\cdot\frac{2^2\cdot(1-2^{j-2})}{1-2}\right) = \frac{1}{2}.
\end{aligned} \tag{3.88}
$$

For each $v_\gamma \in V$ we have

$$
f''(\delta^{out}(v_\gamma)) = \sum_{i=\gamma+1}^{|V|} c_{v_\gamma v_i} + c_{v_\gamma n} = \sum_{i=\gamma+1}^{|V|} \frac{1}{2\cdot 2^{i-\gamma}} + \frac{1}{2\cdot 2^{|V|-\gamma}} \tag{3.89}
$$

$$
\begin{aligned}
&= \frac{1}{2^{1-\gamma}} \sum_{i=\gamma+1}^{|V|} \frac{1}{2^i} + \frac{1}{2 \cdot 2^{|V|-\gamma}} = \frac{1}{2^{1-\gamma}} \cdot \frac{1}{2^{\gamma+1}} \cdot \frac{1 - \frac{1}{2^{|V|-\gamma}}}{1 - \frac{1}{2}} + \frac{1}{2 \cdot 2^{|V|-\gamma}} \\
&= \frac{1}{2} - \cancel{\frac{1}{2 \cdot 2^{|V|-\gamma}}} + \cancel{\frac{1}{2 \cdot 2^{|V|-\gamma}}} = \frac{1}{2},
\end{aligned} \tag{3.90}
$$

$$
\begin{aligned}
f''(\delta^{in}(v_\gamma)) &= \sum_{i=1}^{|A|} c_{a_i v_\gamma} + \sum_{j=1}^{|B|} c_{b_j v_\gamma} + \sum_{i=1}^{\gamma-1} c_{v_i v_\gamma} \\
&= c_{a_1 v_\gamma} + \sum_{i=2}^{|A|} c_{a_i v_\gamma} + c_{b_1 v_\gamma} + \sum_{j=2}^{|B|} c_{b_j v_\gamma} + \sum_{i=1}^{\gamma-1} c_{v_i v_\gamma} \\
&= c_{a_2 v_\gamma} + \sum_{i=2}^{|A|} c_{a_i v_\gamma} + c_{b_2 v_\gamma} + \sum_{j=2}^{|B|} c_{b_j v_\gamma} + \sum_{i=1}^{\gamma-1} c_{v_i v_\gamma} \\
&= \frac{1}{2^{\gamma+1} \cdot 2 \cdot 2^{|A|-2}} + \sum_{i=2}^{|A|} \frac{1}{2^{\gamma+1} \cdot 2 \cdot 2^{|A|-i}} \\
&\quad + \frac{1}{2^{\gamma+1} \cdot 2 \cdot 2^{|B|-2}} + \sum_{j=2}^{|B|} \frac{1}{2^{\gamma+1} \cdot 2 \cdot 2^{|B|-j}} + \sum_{i=1}^{\gamma-1} \frac{1}{2 \cdot 2^{\gamma-i}} \\
&= \frac{1}{2^{\gamma+|A|}} \left(1 + \frac{1}{2^2} \sum_{i=2}^{|A|} 2^i \right) + \frac{1}{2^{\gamma+|B|}} \left(1 + \frac{1}{2^2} \sum_{j=2}^{|B|} 2^j \right) + \frac{1}{2^{\gamma+1}} \sum_{i=1}^{\gamma-1} 2^i \\
&= \frac{1}{2^{\gamma+|A|}} \left(1 + \frac{1}{2^2} \cdot \frac{2^2 \cdot (1 - 2^{|A|-1})}{1-2} \right) \\
&\quad + \frac{1}{2^{\gamma+|B|}} \left(1 + \frac{1}{2^2} \cdot \frac{2^2 \cdot (1 - 2^{|B|-1})}{1-2} \right) + \frac{1}{2^{\gamma+1}} \cdot \frac{2 \cdot (1 - 2^{\gamma-1})}{1-2} \\
&= \frac{2^{|A|-1}}{2^{\gamma+|A|}} + \frac{2^{|B|-1}}{2^{\gamma+|B|}} + \frac{2^{\gamma-1}}{2^\gamma} - \frac{1}{2^\gamma} = \frac{1}{2^{\gamma+1}} + \frac{1}{2^{\gamma+1}} + \frac{1}{2} - \frac{1}{2^\gamma} = \frac{1}{2}.
\end{aligned} \tag{3.91}
$$

Hence, both conditions of the definition of a flow are satisfied for f'', i.e.

$$
\begin{aligned}
f''_e &\geq 0 && \text{for each } e \in \overleftrightarrow{E}_n, \\
f''(\delta^{out}(v)) &= f''(\delta^{in}(v)) && \text{for each } v \in [n] \setminus \{2, n\}.
\end{aligned} \tag{3.92}
$$

The value of this flow f'' is equal to 1 due to:

$$
\text{value}(f'') = f''(\delta^{out}(2)) = f(\delta^{out}(b_1)) = c(\delta(B_1)) \overset{\text{Lemma 3.5.2}}{=} 1. \tag{3.93}
$$

For each $M \subseteq [n-1]$ with $1 \in M$ or $2 \in M$ we thus have (due to $c \geq \mathbf{0}$)

$$1 \leq f'(\delta^{out}(M)) \leq c(\delta(M)), \tag{3.94}$$

or

$$1 \leq f''(\delta^{out}(M)) \leq c(\delta(M)), \tag{3.95}$$

respectively.

Hence Lemma 3.5.2 shows that $\delta(M^*)$ is c-minimal among the [2]-n-cuts for each $M^* \in \mathcal{M}$.

Now we show that all other [2]-n-cuts are not c-minimal. Before we continue the proof, we prove two auxiliary Lemmas 3.5.3 and 3.5.4 which will be used later.

Lemma 3.5.3. *Let $w \in \mathbb{R}^{E_n}_{>0}$ and W_1, W_2 be subsets of $[n-1]$ such that $1 \in W_1 \setminus W_2$, $2 \in W_2 \setminus W_1$ and*

$$1 = w(\delta(W_1)) = w(\delta(W_2)) = \min\{w(\delta(W)) : \ W \subseteq [n-1], \ W \cap \{1,2\} \neq \emptyset\}. \tag{3.96}$$

Then $w(\delta(W_1 \cap W_2)) = 0$, i.e., $W_1 \cap W_2 = \emptyset$.

Proof. Let us define six subsets of the set E_n as follows:

$$\begin{aligned}
A &:= \{\{i,j\} \in E_n : \ i \in W_1 \cap W_2, \ j \in W_1 \setminus (W_1 \cap W_2)\},\\
B &:= \{\{i,j\} \in E_n : \ i \in W_1 \setminus (W_1 \cap W_2), \ j \in V \setminus (W_1 \cup W_2)\},\\
C &:= \{\{i,j\} \in E_n : \ i \in W_2 \setminus (W_1 \cap W_2), \ j \in V \setminus (W_1 \cup W_2)\},\\
D &:= \{\{i,j\} \in E_n : \ i \in W_1 \cap W_2, \ j \in W_2 \setminus (W_1 \cap W_2)\},\\
E &:= \{\{i,j\} \in E_n : \ i \in W_1 \cap W_2, \ j \in V \setminus (W_1 \cup W_2)\},\\
F &:= \{\{i,j\} \in E_n : \ i \in W_1 \setminus (W_1 \cap W_2), \ j \in W_2 \setminus (W_1 \cap W_2)\}.
\end{aligned} \tag{3.97}$$

see Figure 3.9. For simplicity we introduce notations for the w-value of the cuts defined by A, B, C, D, E and F as follows:

$$\begin{aligned}
a &:= w(A), \quad d := w(D),\\
b &:= w(B), \quad e := w(E),\\
c &:= w(C), \quad f := w(F).
\end{aligned} \tag{3.98}$$

Then, in view of (3.96) it holds that

$$\begin{aligned}
w(\delta(W_1 \setminus W_2)) &= (a+f+b) \geq 1, \quad w(\delta(W_1)) = (d+e+f+b) = 1,\\
w(\delta(W_2 \setminus W_1)) &= (d+f+c) \geq 1, \quad w(\delta(W_2)) = (a+e+f+c) = 1.
\end{aligned} \tag{3.99}$$

Therefore

$$a+b+2f+c+d \geq 2 = a+b+c+d+2f+2e \tag{3.100}$$

what implies that $e \leq 0$ and thereby $e = 0$. Due to $w > 0$ e can be 0 only if E is the empty set, i.e. $W_1 \cap W_2 = \emptyset$.

□

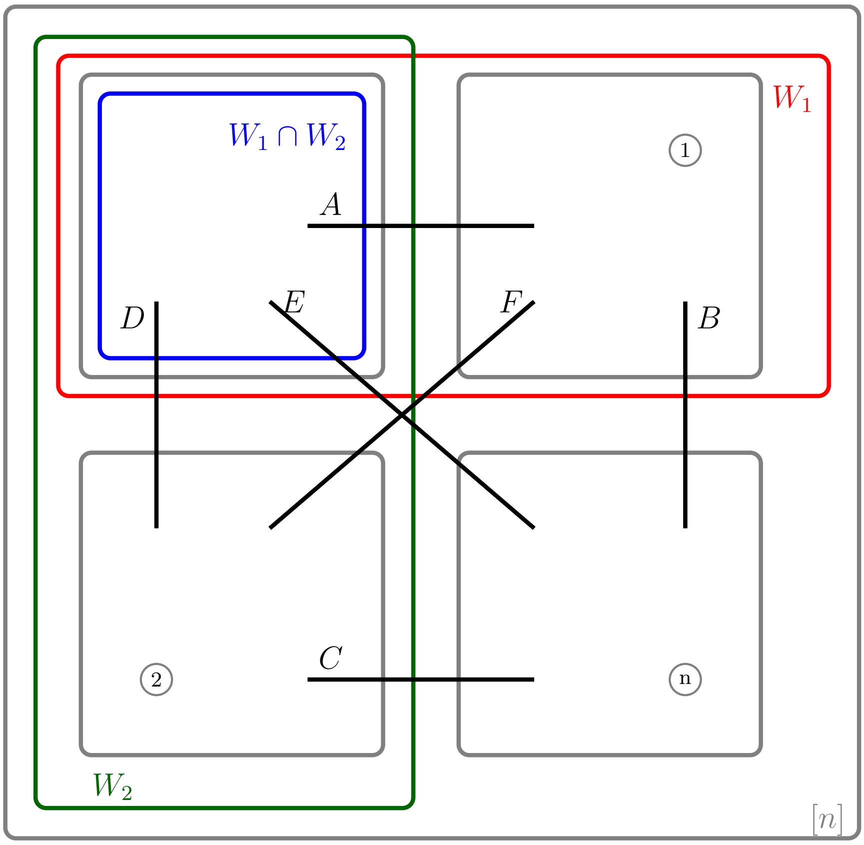

Figure 3.9: Representation of the vertex set $[n]$ in the proof of Lemma 3.5.3.

Lemma 3.5.4. *Let* $X_1 \subsetneq X_2 \subsetneq \ldots \subsetneq X_k \subseteq [n-1]$ *with* $|X_{i+1} \setminus X_i| = 1$, $i \in [k-1]$ *and* $X_1 \subseteq Y \subseteq X_k$ *such that for all* $i \in [k]$ *either* $X_i \subseteq Y$ *or* $Y \subseteq X_i$. *Then there exists* $i \in [k]$ *such that* $Y = X_i$.

Proof. Let $i^* := \max\{i : X_i \subseteq Y\}$. If $i^* = k$ then $Y = X_k$.

If $i^* < k$ then $X_{i^*+1} \nsubseteq Y$ since i^* is the maximum index such that $X_{i^*} \subseteq Y$. Hence $Y \subseteq X_{i^*+1}$ must be satisfied. Thus we have $X_i^* \subseteq Y \subseteq X_{i^*+1}$. Because of $|X_{i+1} \setminus X_i| = 1$ either $Y = X_{i^*}$ or $Y \subseteq X_{i^*+1}$.

□

We continue the proof of Proposition 3.5.1.

Let $W \subseteq [n-1]$ such that $W \cap \{1,2\} \neq \emptyset$ and $W \notin \mathcal{M}$ with $c(\delta(W)) = 1$.

Case 1: $1 \in W$, $2 \notin W$. On the one hand by Lemma 3.4.4 $W \subseteq A \cup B$ or $A \cup B \subseteq W$ (where the latter is impossible due to $2 \in B \setminus W$) since $c(\delta(W)) = c(\delta(A \cup B)) = 1$

with $1 \in W$ and $1 \in A \cup B$. On the other hand by Lemma 3.5.3 $W \cap B = \emptyset$ since $c(\delta(W)) = c(\delta(B)) = 1$ with $1 \in W \setminus B$ and $2 \in B \setminus W$. Thus $W \subseteq A$.

As we have $\{1\} = A_1 \subsetneq A_2 \subsetneq \ldots \subsetneq A_{|A|} = A$ with $|A_{i+1} \setminus A_i| = 1$ for all $i \in [|A| - 1]$ $\{1\} \subseteq W \subseteq A$ by Lemma 3.5.4 implies $W \in \{A_1, \ldots, A_{|A|}\}$ (note that for each i we have $A_i \subseteq W$ or $W \subseteq A_i$ by Lemma 3.4.4).

Case 2: $2 \in W$, $1 \notin W$. Similarly to case 1 we obtain $W \in \{B_1, \ldots, B_{|B|}\}$.

Case 3: $1, 2 \in W$. On the one hand by Lemma 3.4.4 $W \subseteq A$ (impossible due to $2 \in W \setminus A$) or $A \subseteq W$ holds due to $c(\delta(W)) = c(\delta(A)) = 1$ with $1 \in W$, $1 \in A$ and $W \subseteq B$ (impossible due to $1 \in W \setminus B$) or $B \subseteq W$ holds due to $c(\delta(W)) = c(\delta(B)) = 1$ with $2 \in W$, $2 \in B$. Thus we have $A \cup B \subseteq W$.

By the definition of A, B, V and A_I, B_I, V_I, $I \in [|V|]$, see (3.53)-(3.54), we have

$$A \cup B \subsetneq A \cup B \cup V_1 \subsetneq A \cup B \cup V_2 \subsetneq \ldots \subsetneq A \cup B \cup V_{|V|} = A \cup B \cup V \tag{3.101}$$

with $|(A \cup B \cup V_{i+1}) \setminus (A \cup B \cup V_i)| = 1$ for all $i \in [|V| - 1]$. Then, as we have

$$A \cup B \subseteq W \subseteq A \cup B \cup V = [n-1]$$

Lemma 3.5.4 implies $W \in \{A \cup B, A \cup B \cup V_1, \ldots, A \cup B \cup V_{|V|}\}$ (note that for each i we have $A \cup B \cup V_i \subseteq W$ or $W \subseteq A \cup B \cup V_i$ by Lemma 3.4.4).

Thereby all cuts $\delta(W)$ such that $W \subseteq [n-1]$ with $W \cap \{1, 2\} \neq \emptyset$ and $W \notin \mathcal{M}$ are not c-minimal. □

Proposition 3.5.5. *The faces in Proposition 3.5.1 are $(n-1)$-dimensional simplices.*

Proof. Each face in Proposition 3.5.1 has n vertices since for $\mathcal{M}$ defined by (3.55) we have

$$|\mathcal{M}| = |A| + |B| + 1 + |V| = n - 1 + 1 = n. \tag{3.102}$$

Now we show that these n vertices are affinely independent and hence all those faces are $(n-1)$-dimensional simplices.

Let $\Theta'_{\mathcal{M}}$ be a submatrix of the cut-incidence matrix of $\mathcal{M}$ (see (1.10)) formed by columns corresponding to the edges e_{a_1,a_2}, $\ldots$, $e_{a_{|A|-1},a_{|A|}}$, $e_{a_{|A|},n}$, e_{b_1,b_2}, $\ldots$, $e_{b_{|B|-1},b_{|B|}}$, $e_{b_{|B|},n}$, e_{v_1,v_2}, $\ldots$, $e_{v_{|v|-1},v_{|V|}}$, $e_{v_{|V|},n}$ (see Figure 3.10) with an additional unit column, i.e.,

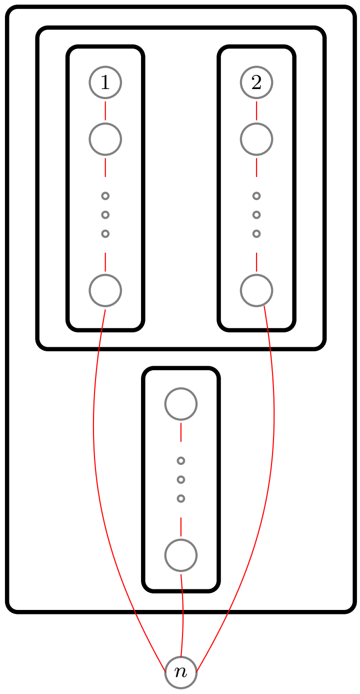

Figure 3.10: Sketch of the $n-1$ chosen edges used to create the matrix $\Theta'_{\mathcal{M}}$ in the proof of Proposition 3.5.5.

$$
\Theta'_{\mathcal{M}} := \begin{array}{c} \\ A_1 \\ \dots \\ A_{|A|} \\ B_1 \\ \dots \\ B_{|B|} \\ A\cup B \\ A\cup B\cup V_1 \\ \dots \\ A\cup B\cup V_{|V|} \end{array}
\begin{array}{c}
\begin{array}{cccccccccc} e_{a_1,a_2} & \dots & e_{a_{|A|},n} & e_{b_1,b_2} & \dots & e_{b_{|B|},n} & e_{v_1,v_2} & \dots & e_{v_{|V|},n} & \end{array} \\
\left[\begin{array}{cccccccccc}
\mathbf{1} & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \mathbf{1} \\
0 & \ddots & 0 & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \mathbf{1} & 0 & \dots & 0 & 0 & \dots & 0 & \mathbf{1} \\
0 & \dots & 0 & \mathbf{1} & 0 & 0 & 0 & \dots & 0 & \mathbf{1} \\
\vdots & \ddots & \vdots & 0 & \ddots & 0 & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & 0 & 0 & 0 & \mathbf{1} & 0 & \dots & 0 & \mathbf{1} \\
0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & 0 & \dots & 0 & \mathbf{1} \\
0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & \mathbf{1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & \ddots & 0 & \vdots \\
0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & \mathbf{1}
\end{array}\right]
\end{array}
\begin{array}{c} \\ \\ \text{I.} \\ \\ \\ \text{II.} \\ \\ \text{III.} \\ \\ \text{IV.} \\ \\ \end{array}
$$

Performing following elementary row operations:

1. multiply the row III by (-1) and add that result to each row of IV to get new rows IV′;
2. multiply the sum of the last rows of I and II by (-1) and add that result to the row III to get a new row III′;

3. add the row III′ to each row of I and II

the $n \times n$-matrix $\Theta'_{\mathcal{M}}$ takes the following form:

$$\begin{bmatrix}
\mathbf{1} & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\
0 & \ddots & 0 & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \mathbf{1} & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\
0 & \dots & 0 & \mathbf{1} & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & 0 & \ddots & 0 & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & 0 & 0 & 0 & \mathbf{1} & 0 & \dots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \mathbf{-1} \\
0 & \dots & 0 & 0 & \dots & 0 & \mathbf{1} & 0 & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & 0 & \ddots & 0 & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & \mathbf{1} & 0
\end{bmatrix}$$

As we have $\operatorname{rank}(\Theta'_{\mathcal{M}}) = n$ linear independence of rows of $\Theta'_{\mathcal{M}}$ implies affine independence of rows of the cut-incidence matrix of $\mathcal{M}$. Thus, we have for each face in Proposition 3.5.1 that its vertices are affinely independent.

□

Since all faces in Proposition 3.5.1 have n vertices none of them is contained in any other. Now we show that any further face contains in one of these faces.

Let $\mathcal{W} \subseteq 2^{[n-1]}$ such that $\{\chi(\delta(W)) : W \in \mathcal{W}\}$ is the set of vertices of some face of the undominated complex of $P_{[2]}(K_n)$. Then there is $c \in \mathbb{R}^E_{>0}$ with minimum [2]-n-cut value 1 such that $c(\delta(W)) = 1$ if and only if $W \in \mathcal{W}$. Let

$$\begin{aligned}
\mathcal{W}_1 &:= \{W \in \mathcal{W} :\ 1 \in W,\ 2 \notin W\}, \\
\mathcal{W}_2 &:= \{W \in \mathcal{W} :\ 2 \in W,\ 1 \notin W\}, \\
\mathcal{W}_{12} &:= \{W \in \mathcal{W} :\ 1, 2 \in W\}.
\end{aligned} \tag{3.103}$$

By Lemma 3.4.4 all $W \in \mathcal{W}_1$ are nested. Let A be the largest set of them, i.e. $A := W$ such that $W \in \mathcal{W}_1$ and $|W| = \max\{|W| : W \in \mathcal{W}_1\}$. We number the elements in A, $A = \{a_1 = 1, a_2, \dots, a_{|A|}\}$ and define $A_I := \{a_1 = 1, a_2, \dots, a_I\}$, $I \in [|A|]$ such that $\mathcal{W}_1 \subseteq \{A_1, A_2, \dots, A_{|A|}\}$. Similarly, by Lemma 3.4.4 all $W \in \mathcal{W}_2$ are nested. Let B be the largest set of them, i.e. $B := W$ such that $W \in \mathcal{W}_2$ and $|W| = \max\{|W| : W \in \mathcal{W}_2\}$. We number the elements in B, $B = \{b_1 = 2, b_2, \dots, b_{|B|}\}$ and define $B_I := \{b_1 = 2, b_2, \dots, b_I\}$, $I \in [|B|]$ such that $\mathcal{W}_2 \subseteq \{B_1, B_2, \dots, B_{|B|}\}$.

By Lemma 3.4.4 $A \subseteq W$ for all $W \in \mathcal{W}_{12}$ and $B \subseteq W$ for all $W \in \mathcal{W}_{12}$ what implies that $A \cup B \subseteq W$ for all $W \in \mathcal{W}_{12}$. We choose $V = [n-1] \setminus (A \cup B)$. By Lemma 3.4.4 all $W \in \mathcal{W}_{12}$ are nested. Thus, we can number the elements in V such that for each $W \in \mathcal{W}_{12}$ there is some $I \in [|V|]$ with $W = A \cup B \cup V_I$ where $V_I = \{v_1, \ldots, v_I\}$.

Thereby it is shown that $\mathcal{W}$ is a subset of some $\mathcal{M}$, as in (3.55), what complete the proof of the main result of this section:

Theorem 3.5.6. *The undominated complex for the* $[2]$*-*n*-cut polytope* $P_{[2]}(K_n) \subseteq \mathbb{R}^{\binom{n}{2}}$ *for the complete graph on* n *nodes is a pure simplicial complex of dimension* $n-1$ *whose facets are described in Proposition 3.5.1.*

Remark 3.5.7. *Thus the undominated complex for the* $[2]$*-*n*-cut polytope* $P_{[2]}(K_n)$ *consists of* $\frac{(n-1)!}{2}$ *facets which are* $(n-1)$*-dimensional.*

Proof. We have $\frac{(n-1)!}{2}$ different sets of $[2]$-n cuts what implies that the undominated complex of $P_{[2]}(K_n)$ has $\frac{(n-1)!}{2}$ different facets. It follows from the fact that there are $(n-3)! \sum_{k=1}^{2} \binom{n-2}{k}$ possibilities to partition the set $[n-1]$ of $n-1$ elements into three parts as in (3.55): $(n-3)!$ possibilities to order the elements in $[n-1] \setminus \{1,2\}$ and in each of these cases we have $\binom{n-2}{2} + \binom{n-2}{1}$ possibilities to partition the elements into three parts. Thus, as a result we have $\frac{(n-1)!}{2}$ different sets of $[2]$-n cuts due to:

$$
\begin{aligned}
(n-3)! \sum_{k=1}^{2} \binom{n-2}{k} &= (n-3)! \left(\frac{(n-2)!}{2!(n-4)!} + \frac{(n-2)!}{1!(n-3)!} \right) \\
&= (n-3)! \left(\frac{(n-2)(n-3)}{2} + (n-2) \right) = \frac{(n-1)!}{2}.
\end{aligned}
\tag{3.104}
$$

□

3.6 Undominated Complex of the $[3]$-n-Cut Polytope

Making similar reasoning as in the two previous sections 3.4 and 3.5 for the 1-n-cut polytope and for the $[2]$-n-cut polytope, respectively, we obtain the result for the $[3]$-n-cut polytope for the complete graph on n nodes which will be presented in this section.

Let us introduce some notations used in the current section.

Let the set $[n-1] := \{1, 2, \ldots, n-1\}$ be partitioned into five parts A^γ, $\gamma \in [3]$ and V^λ, $\lambda \in [2]$ such that they are all pairwise disjoint

$$
\begin{aligned}
A^\gamma \; &:= \{a_1^\gamma = \gamma,\, a_2^\gamma,\, \ldots,\, a_{|A_\gamma|}^\gamma\}, \quad \gamma \in [3], \\
V^\lambda \; &:= \{v_1^\lambda,\, v_2^\lambda,\, \ldots,\, v_{|V_\lambda|}^\lambda\}, \qquad \lambda \in [2],
\end{aligned}
\tag{3.105}
$$

see Figure 3.11. We define the sets A_γ^I, $\gamma \in [3]$ and V_λ^I, $\lambda \in [2]$ as follows:

$$\begin{aligned} A_I^\gamma &:= \{a_1^\gamma = \gamma,\, a_2^\gamma,\, \ldots,\, a_I^\gamma\}, \quad I \in [|A^\gamma|] \\ V_I^\lambda &:= \{v_1^\lambda,\, v_2^\lambda,\, \ldots,\, v_I^\lambda\}, \qquad I \in [|V^\lambda|] \end{aligned} \tag{3.106}$$

Note that, $A^\gamma_{|A_\gamma|} = A^\gamma$ for all $\gamma \in [3]$ and $V^\lambda_{|V^\lambda|} = V^\lambda$ for all $\lambda \in [2]$.

Let $\mathcal{M}$ be the set of the following sets:

$$\begin{aligned} \mathcal{M} = \{A_I^1 :\ I \in [|A^1|]\,\} &\cup \{A_I^2 :\ I \in [|A^2|]\,\} \cup \{A_I^3 :\ I \in [|A^3|]\,\} \\ &\cup \{A^1 \cup A^2\} \cup \{A^1 \cup A^2 \cup V_I^1 :\ I \in [|V^1|]\,\} \cup \{A^1 \cup A^2 \cup V^1 \cup A^3\} \\ &\cup \{A^1 \cup A^2 \cup V^1 \cup A^3 \cup V_I^2 :\ I \in [|V^2|]\,\} \end{aligned} \tag{3.107}$$

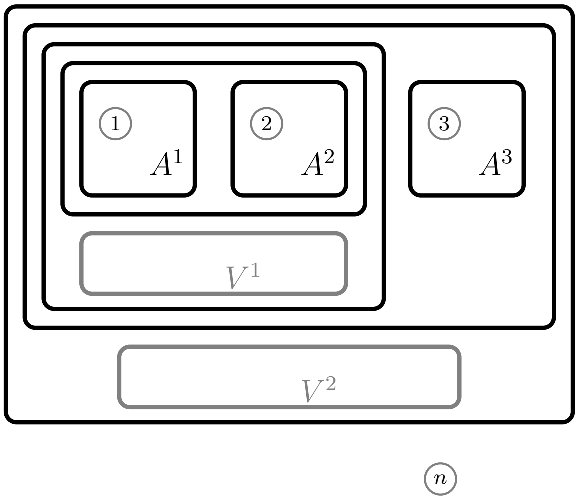

Figure 3.11: Representation of the vertex set $[n]$ in the proof of Proposition 3.6.1.

Proposition 3.6.1. *For each partition of the vertex set* $[n-1]$ *into five parts* A_γ, $\gamma \in [3]$ *and* V_λ, $\lambda \in [2]$ *such that they are all pairwise disjoint and defined as in* (3.105)*, see Figure 3.11,*

$$\operatorname{conv}\{\chi(\delta(M)) :\ M \in \mathcal{M}\} \tag{3.108}$$

where $\mathcal{M}$ *is defined by* (3.107)*, is a face of the undominated complex of the* [3]*-n-cut polytope* $P_{[3]}(K_n)$ *for the complete graph on n nodes* $K_n = ([n], E_n)$*.*

Proof. We first define some $c \in \mathbb{R}^{E_n}_{>0}$ such that for all $M^* \subseteq [n-1]$ with $M^* \cap [3] \neq \emptyset$

$$\min\{c(\delta(M)) : M \subseteq [n-1], M \cap [3] \neq \emptyset\} = c(\delta(M^*)) \tag{3.109}$$

holds if and only if $M^* \in \mathcal{M}$ (see the proof of Lemma 2.3.1).

Let $c \in \mathbb{R}^{E_n}_{>0}$ be defined as follows:

$$
\begin{aligned}
&\text{for } \ a_i^\gamma,\ a_j^\gamma \in A^\gamma, \gamma \in [3]:\ c_{a_i^\gamma a_j^\gamma} = \begin{cases} \dfrac{1}{2 \cdot 2^{j-i}}, & 1 < i < j, \\ c_{a_2^\gamma a_j^\gamma}, & i = 1; \end{cases} \\
&\text{for } \ v_i^\lambda,\ v_j^\lambda \in V^\lambda, \lambda \in [2]:\ c_{v_i^\lambda v_j^\lambda} = \ \frac{1}{2 \cdot 2^{j-i}}, \quad i < j; \\
&\text{for } \ a_i^1 \in A^1, a_j^2 \in A^2:\ c_{a_i^1 a_j^2} = \begin{cases} \dfrac{1}{2^3 \cdot 2^{|A^1|-i} \cdot 2^{|A^2|-j}}, & i,\ j > 1, \\ c_{a_2^1 a_j^2}, & i = 1, \\ c_{a_i^1 a_2^2}, & j = 1; \end{cases} \\
&\text{for } a_i^\gamma \in A^\gamma, \gamma \in [2], a_j^3 \in A^3:\ c_{a_i^\gamma a_j^3} = \begin{cases} \dfrac{1}{2^3 \cdot 2^{|A^\gamma|-i} \cdot 2^{|V^1|}\, 2 \cdot 2^{|A^3|-j}}, & i,\ j > 1, \\ c_{a_2^\gamma a_j^3}, & i = 1, \\ c_{a_i^\gamma a_j^3}, & j = 1; \end{cases} \\
&\text{for } \ a_i^\gamma \in A^\gamma, \gamma \in [2],\ v_\gamma^1 \in V^1:\ c_{a_i^\gamma v_\gamma^1} = \begin{cases} \dfrac{1}{2^{\gamma+1} \cdot 2 \cdot 2^{|A^\gamma|-i}}, & i > 1, \\ c_{a_2^\gamma v_\gamma^1}, & i = 1; \end{cases} \\
&\text{for } \ a_i^\gamma \in A^\gamma, \gamma \in [2],\ v_\beta^2 \in V^2:\ c_{a_i^\gamma v_\gamma^2} = \begin{cases} \dfrac{1}{2^{\beta+1} \cdot 2^{|V^1|+1} \cdot 2 \cdot 2^{|A^\gamma|-i}}, & i > 1, \\ c_{a_2^\gamma v_\beta^2}, & i = 1; \end{cases} \\
&\text{for } \ a_\alpha^3 \in A^3,\ v_\gamma^1 \in V^1:\ c_{a_\alpha^3 v_\gamma^1} = \begin{cases} \dfrac{1}{2^3 \cdot 2^{|V^1|-\gamma} \cdot 2^{|A^3|-\alpha}}, & \alpha > 1, \\ c_{a_2^3 v_\gamma^1}, & \alpha = 1; \end{cases} \\
&\text{for } \ a_\alpha^3 \in A^3,\ v_\beta^2 \in V^2:\ c_{a_\alpha^3 v_\beta^2} = \begin{cases} \dfrac{1}{2^{\beta+1} \cdot 2 \cdot 2^{|A^3|-\alpha}}, & \alpha > 1, \\ c_{a_2^3 v_\beta^2}, & \alpha = 1; \end{cases} \\
&\text{for } \ v_\gamma^1 \in V^1,\ v_\beta^2 \in V^2:\ c_{v_\gamma^1 v_\beta^2} = \frac{1}{2^{\beta+1} \cdot 2 \cdot 2^{|V^1|-\gamma}};
\end{aligned}
\tag{3.110}
$$

$$\text{for } a_i^\gamma \in A^\gamma, \gamma \in [2] : \; c_{a_i^\gamma n} = \begin{cases} \dfrac{1}{2^{|V^2|+1} \cdot 2^{|V^1|+1} \cdot 2 \cdot 2^{|A^\gamma|-i}}, & i > 1, \\ c_{a_2^\gamma n}, & i = 1; \end{cases}$$

$$\text{for } a_\alpha^3 \in A^3 : \; c_{a_\alpha^3 n} = \begin{cases} \dfrac{1}{2^{|V^2|+1} \cdot 2 \cdot 2^{|A^3|-\alpha}}, & \alpha > 1, \\ c_{a_2^3 n}, & \alpha = 1; \end{cases} \tag{3.111}$$

$$\text{for } v_\gamma^1 \in V^1 : \; c_{v_\gamma^1 n} = \frac{1}{2^{|V^2|+1} \cdot 2 \cdot 2^{|V^1|-\gamma}};$$

$$\text{for } v_\beta^2 \in V^2 : \; c_{v_\beta^2 n} = \frac{1}{2 \cdot 2^{|V^2|-\beta}}.$$

Lemma 3.6.2. *Let $c \in \mathbb{R}_{>0}^{E_n}$ be defined by* (3.110)-(3.111)*. Then the cut $\delta(M^*)$ has value one for all $M^* \in \mathcal{M}$ where $\mathcal{M}$ is defined by* (3.107)*.*

Proof. Consider all seven possible cases for a set $M^* \in \mathcal{M}$.

Case 1: $M^* = A_I^1$ for $I \in [|A^1|]$.

$$\begin{aligned} c(\delta(A_I^1)) = & \underbrace{\sum_{i=1}^{I} \sum_{j=1}^{|A^2|} c_{a_i^1 a_j^2}}_{\Sigma_1} + \underbrace{\sum_{i=1}^{I} \sum_{\gamma=1}^{|V^1|} c_{a_i^1 v_\gamma^1}}_{\Sigma_2} + \underbrace{\sum_{i=1}^{I} \sum_{\alpha=1}^{|A^3|} c_{a_i^1 a_\alpha^3}}_{\Sigma_3} \\ & + \underbrace{\sum_{i=1}^{I} \sum_{\gamma=1}^{|V^2|} c_{a_i^1 v_\gamma^2}}_{\Sigma_4} + \underbrace{\sum_{i=1}^{I} c_{a_i^1 n}}_{\Sigma_5} + \underbrace{\sum_{i=1}^{I} \sum_{j=I+1}^{|A^1|} c_{a_i^1 a_j^1}}_{\Sigma_6}. \end{aligned} \tag{3.112}$$

In view of the definition of $c \in \mathbb{R}_{>0}^{E_n}$, see (3.110)-(3.111), we calculate each term of the right-hand side of equation (3.112).

$$\begin{aligned} \Sigma_1 &= \sum_{i=1}^{I} \sum_{j=1}^{|A^2|} c_{a_i^1 a_j^2} = c_{a_1^1 a_1^2} + \sum_{i=2}^{I} c_{a_i^1 a_1^2} + \sum_{j=2}^{|A^2|} c_{a_1^1 a_j^2} + \sum_{i=2}^{I} \sum_{j=2}^{|A^2|} c_{a_i^1 a_j^2} \\ &= c_{a_2^1 a_2^2} + \sum_{i=2}^{I} c_{a_i^1 a_2^2} + \sum_{j=2}^{|A^2|} c_{a_2^1 a_j^2} + \sum_{i=2}^{I} \sum_{j=2}^{|A^2|} c_{a_i^1 a_j^2} \\ &= \frac{1}{2^3 \cdot 2^{|A^1|-2} \cdot 2^{|A^2|-2}} + \sum_{i=2}^{I} \frac{1}{2^3 \cdot 2^{|A^1|-i} \cdot 2^{|A^2|-2}} \end{aligned} \tag{3.113}$$

$$+\sum_{j=2}^{|A^2|}\frac{1}{2^3\cdot 2^{|A^1|-2}\cdot 2^{|A^2|-j}}+\sum_{i=2}^{I}\sum_{j=2}^{|A^2|}\frac{1}{2^3\cdot 2^{|A^1|-i}\cdot 2^{|A^2|-j}}$$

$$=\frac{1}{2^{|A^1|+|A^2|-1}}\left(1+\frac{1}{2^2}\sum_{i=2}^{I}2^i+\frac{1}{2^2}\sum_{j=2}^{|A^2|}2^j+\frac{1}{2^2}\sum_{i=2}^{I}2^i\frac{1}{2^2}\sum_{j=2}^{|A^2|}2^j\right)$$

$$=\frac{1}{2^{|A^1|+|A^2|-1}}\left(1+\frac{1}{2^2}\sum_{i=2}^{I}2^i\right)\left(1+\frac{1}{2^2}\sum_{j=2}^{|A^2|}2^j\right) \tag{3.114}$$

$$=\frac{1}{2^{|A^1|+|A^2|-1}}\left(1+\frac{2^2}{2^2}\cdot\frac{1-2^{I-1}}{1-2}\right)\left(1+\frac{2^2}{2^2}\cdot\frac{1-2^{|A^2|-1}}{1-2}\right)$$

$$=\frac{2^{I-1}\cdot 2^{|A^2|-1}}{2^{|A^1|+|A^2|-1}}=\frac{2^{I-1}}{2^{|A^1|}}.$$

$$\Sigma_2=\sum_{i=1}^{I}\sum_{\gamma=1}^{|V^1|}c_{a_i^1v_\gamma^1}=\sum_{\gamma=1}^{|V^1|}c_{a_1^1v_\gamma^1}+\sum_{i=2}^{I}\sum_{\gamma=1}^{|V^1|}c_{a_i^1v_\gamma^1}=\sum_{\gamma=1}^{|V^1|}c_{a_2^1v_\gamma^1}+\sum_{i=2}^{I}\sum_{\gamma=1}^{|V^1|}c_{a_i^1v_\gamma^1}$$

$$=\sum_{\gamma=1}^{|V^1|}\frac{1}{2^{\gamma+1}\cdot 2\cdot 2^{|A^1|-2}}+\sum_{i=2}^{I}\sum_{\gamma=1}^{|V^1|}\frac{1}{2^{\gamma+1}\cdot 2\cdot 2^{|A^1|-i}}$$

$$=\frac{1}{2^{|A^1|}}\sum_{\gamma=1}^{|V^1|}\frac{1}{2^\gamma}\left(1+\frac{1}{2^2}\sum_{i=2}^{I}2^i\right)=\frac{1}{2^{|A^1|}}\cdot\frac{1}{2}\cdot\frac{1-\frac{1}{2^{|V^1|}}}{1-\frac{1}{2}}\cdot\left(1+\frac{2^2}{2^2}\cdot\frac{1-2^{I-1}}{1-2}\right) \tag{3.115}$$

$$=\frac{2^{I-1}\cdot\left(2^{|V^1|}-1\right)}{2^{|A^1|+|V^1|}}=\frac{2^{I-1}}{2^{|A^1|}}-\frac{2^{I-1}}{2^{|A^1|+|V^1|}}.$$

$$\Sigma_3=\sum_{i=1}^{I}\sum_{\alpha=1}^{|A^3|}c_{a_i^1a_\alpha^3}=c_{a_1^1a_1^3}+\sum_{i=2}^{I}c_{a_i^1a_1^3}+\sum_{\alpha=2}^{|A^3|}c_{a_1^1a_\alpha^3}+\sum_{i=2}^{I}\sum_{\alpha=2}^{|A^3|}c_{a_i^1a_\alpha^3}$$

$$=c_{a_2^1a_2^3}+\sum_{i=2}^{I}c_{a_i^1a_2^3}+\sum_{\alpha=2}^{|A^3|}c_{a_2^1a_\alpha^3}+\sum_{i=2}^{I}\sum_{\alpha=2}^{|A^3|}c_{a_i^1a_\alpha^3}$$

$$=\frac{1}{2^3\cdot 2^{|A^1|-2}\cdot 2^{|V^1|}\cdot 2\cdot 2^{|A^3|-2}}+\sum_{i=2}^{I}\frac{1}{2^3\cdot 2^{|A^1|-i}\cdot 2^{|V^1|}\cdot 2\cdot 2^{|A^3|-2}} \tag{3.116}$$

$$+\sum_{\alpha=2}^{|A^3|}\frac{1}{2^3\cdot 2^{|A^1|-2}\cdot 2^{|V^1|}\cdot 2\cdot 2^{|A^3|-\alpha}}+\sum_{i=2}^{I}\sum_{\alpha=2}^{|A^3|}\frac{1}{2^3\cdot 2^{|A^1|-i}\cdot 2^{|V^1|}\cdot 2\cdot 2^{|A^3|-\alpha}}$$

$$\begin{aligned}
&= \frac{1}{2^{|A^1|+|V^1|+|A^3|}} \left(1 + \frac{1}{2^2} \sum_{i=2}^{I} 2^i + \frac{1}{2^2} \sum_{\alpha=2}^{|A^3|} 2^\alpha + \frac{1}{2^2} \sum_{i=2}^{I} 2^i \frac{1}{2^2} \sum_{\alpha=2}^{|A^3|} 2^\alpha \right) \\
&= \frac{1}{2^{|A^1|+|V^1|+|A^3|}} \left(1 + \frac{1}{2^2} \sum_{i=2}^{I} 2^i \right) \left(1 + \frac{1}{2^2} \sum_{\alpha=2}^{|A^3|} 2^\alpha \right) \\
&= \frac{1}{2^{|A^1|+|V^1|+|A^3|}} \left(1 + \frac{2^2}{2^2} \cdot \frac{1 - 2^{I-1}}{1-2} \right) \left(1 + \frac{2^2}{2^2} \cdot \frac{1 - 2^{|A^3|-1}}{1-2} \right) \\
&= \frac{2^{I-1} \cdot 2^{|A^3|-1}}{2^{|A^1|+|V^1|+|A^3|}} = \frac{2^{I-1}}{2^{|A^1|+|V^1|+1}}.
\end{aligned} \tag{3.117}$$

$$\begin{aligned}
\Sigma_4 &= \sum_{i=1}^{I} \sum_{\gamma=1}^{|V^2|} c_{a_i^1 v_\gamma^2} = \sum_{\gamma=1}^{|V^2|} c_{a_1^1 v_\gamma^2} + \sum_{i=2}^{I} \sum_{\gamma=1}^{|V^2|} c_{a_i^1 v_\gamma^2} = \sum_{\gamma=1}^{|V^2|} c_{a_2^1 v_\gamma^2} + \sum_{i=2}^{I} \sum_{\gamma=1}^{|V^2|} c_{a_i^1 v_\gamma^2} \\
&= \sum_{\gamma=1}^{|V^2|} \frac{1}{2^{\gamma+1} \cdot 2^{|V^1|+1} \cdot 2 \cdot 2^{|A^1|-2}} + \sum_{i=2}^{I} \sum_{\gamma=1}^{|V^2|} \frac{1}{2^{\gamma+1} \cdot 2^{|V^1|+1} \cdot 2 \cdot 2^{|A^1|-i}} \\
&= \frac{1}{2^{|A^1|+|V^1|+1}} \sum_{\gamma=1}^{|V^2|} \frac{1}{2^\gamma} \left(1 + \frac{1}{2^2} \sum_{i=2}^{I} 2^i \right) \\
&= \frac{1}{2^{|A^1|+|V^1|+1}} \cdot \frac{\frac{1}{2} \cdot (1 - \frac{1}{2^{|V^2|}})}{1 - \frac{1}{2}} \cdot \left(1 + \frac{2^2}{2^2} \cdot \frac{1 - 2^{I-1}}{1-2} \right) \\
&= \frac{2^{I-1} \cdot \left(2^{|V^2|} - 1 \right)}{2^{|A^1|+|V^1|+|V^2|+1}} = \frac{2^{I-1}}{2^{|A^1|+|V^1|+1}} - \frac{2^{I-1}}{2^{|A^1|+|V^1|+|V^2|+1}}.
\end{aligned} \tag{3.118}$$

$$\begin{aligned}
\Sigma_5 &= \sum_{i=1}^{I} c_{a_i^1 n} = c_{a_1^1 n} + \sum_{i=2}^{I} c_{a_i^1 n} = c_{a_2^1 n} + \sum_{i=2}^{I} c_{a_i^1 n} \\
&= \frac{1}{2^{|V^2|+1} \cdot 2^{|V^1|+1} \cdot 2 \cdot 2^{|A^1|-2}} + \sum_{i=2}^{I} \frac{1}{2^{|V^2|+1} \cdot 2^{|V^1|+1} \cdot 2 \cdot 2^{|A^1|-i}} \\
&= \frac{1}{2^{|A^1|+|V^1|+|V^2|+1}} \left(1 + \frac{1}{2^2} \cdot \sum_{i=2}^{I} 2^i \right) \\
&= \frac{1}{2^{|A^1|+|V^1|+|V^2|+1}} \left(1 + \frac{2^2}{2^2} \cdot \frac{1 - 2^{I-1}}{1-2} \right) = \frac{2^{I-1}}{2^{|A^1|+|V^1|+|V^2|+1}}.
\end{aligned} \tag{3.119}$$

$$
\begin{aligned}
\Sigma_6 &= \sum_{i=1}^{I} \sum_{j=I+1}^{|A^1|} c_{a_i^1 a_j^1} = \sum_{j=I+1}^{|A^1|} c_{a_1^1 a_j^1} + \sum_{i=2}^{I} \sum_{j=I+1}^{|A^1|} c_{a_i^1 a_j^1} \\
&= \sum_{j=I+1}^{|A^1|} c_{a_2^1 a_j^1} + \sum_{i=2}^{I} \sum_{j=I+1}^{|A^1|} c_{a_i^1 a_j^1} = \sum_{j=I+1}^{|A^1|} \frac{1}{2 \cdot 2^{j-2}} + \sum_{i=2}^{I} \sum_{j=I+1}^{|A^1|} \frac{1}{2 \cdot 2^{j-i}} \\
&= 2 \sum_{j=I+1}^{|A^1|} \frac{1}{2^j} \left(1 + \frac{1}{2^2} \sum_{i=2}^{I} 2^i \right) = 2 \cdot \frac{1}{2^{I+1}} \cdot \frac{1 - \frac{1}{2^{|A^1|-I}}}{1 - \frac{1}{2}} \cdot \left(1 + \frac{2^2}{2^2} \cdot \frac{1 - 2^{I-1}}{1-2} \right) \\
&= \frac{(2^{|A^1|-I} - 1) \cdot 2^{I-1}}{2^{|A^1|-I} \cdot 2^{I-1}} = 1 - \frac{2^I}{2^{|A^1|}}.
\end{aligned}
\tag{3.120}
$$

Thus for expression (3.112) with $I \in [|A^1|]$ we have

$$
\begin{aligned}
c(\delta(A_I^1)) = {} & \cancel{\frac{2^{I-1}}{2^{|A^1|}}} + \cancel{\frac{2^{I-1}}{2^{|A^1|}}} - \cancel{\frac{2^{I-1}}{2^{|A^1|+|V^1|}}} + \cancel{\frac{2^{I-1}}{2^{|A^1|+|V^1|+1}}} + \cancel{\frac{2^{I-1}}{2^{|A^1|+|V^1|+1}}} \\
& - \cancel{\frac{2^{I-1}}{2^{|A^1|+|V^1|+|V^2|+1}}} + \cancel{\frac{2^{I-1}}{2^{|V^2|+|V^1|+|A^1|+1}}} + 1 - \cancel{\frac{2^{I}}{2^{|A^1|}}} = 1.
\end{aligned}
\tag{3.121}
$$

Case 2: $M^* = A_I^2$ for $I \in [|A^2|]$. Due to the symmetry of c the equation $c(\delta(A_I^2)) = 1$ follows from Case 1 by exchanging A^1 and A^2 (and nodes 1 and 2).

Case 3: $M^* = A_I^3$ for $I \in [|A^3|]$.

$$
\begin{aligned}
c(\delta(A_I^3)) = {} & \underbrace{\sum_{\alpha=1}^{I} \sum_{i=1}^{|A^1|} c_{a_\alpha^3 a_i^1}}_{\Sigma_7} + \underbrace{\sum_{\alpha=1}^{I} \sum_{j=1}^{|A^2|} c_{a_\alpha^3 a_j^2}}_{\Sigma_8} + \underbrace{\sum_{i=1}^{I} \sum_{\gamma=1}^{|V^1|} c_{a_i^3 v_\gamma^1}}_{\Sigma_9} \\
& + \underbrace{\sum_{i=1}^{I} \sum_{\gamma=1}^{|V^2|} c_{a_i^3 v_\gamma^2}}_{\Sigma_{10}} + \underbrace{\sum_{\alpha=1}^{I} c_{a_\alpha^3 n}}_{\Sigma_{11}} + \underbrace{\sum_{i=1}^{I} \sum_{j=I+1}^{|A^3|} c_{a_i^3 a_j^3}}_{\Sigma_{12}} .
\end{aligned}
\tag{3.122}
$$

Calculating each term on the right-hand side of equation (3.122) we obtain

$$
\begin{aligned}
\Sigma_7 &= \sum_{\alpha=1}^{I} \sum_{i=1}^{|A^1|} c_{a_\alpha^3 a_i^1} = c_{a_1^3 a_1^1} + \sum_{\alpha=2}^{I} c_{a_\alpha^3 a_1^1} + \sum_{i=2}^{|A^1|} c_{a_1^3 a_i^1} + \sum_{\alpha=2}^{I} \sum_{i=2}^{|A^1|} c_{a_\alpha^3 a_i^1} \\
&= c_{a_2^3 a_2^1} + \sum_{\alpha=2}^{I} c_{a_\alpha^3 a_2^1} + \sum_{i=2}^{|A^1|} c_{a_2^3 a_i^1} + \sum_{\alpha=2}^{I} \sum_{i=2}^{|A^1|} c_{a_\alpha^3 a_i^1}
\end{aligned}
\tag{3.123}
$$

$$
\begin{aligned}
&= \frac{1}{2^3 \cdot 2^{|A^1|-2} \cdot 2^{|V^1|} \cdot 2 \cdot 2^{|A^3|-2}} + \sum_{\alpha=2}^{I} \frac{1}{2^3 \cdot 2^{|A^1|-2} \cdot 2^{|V^1|} \cdot 2 \cdot 2^{|A^3|-\alpha}} \\
&\quad + \sum_{i=2}^{|A^1|} \frac{1}{2^3 \cdot 2^{|A^1|-i} \cdot 2^{|V^1|} \cdot 2 \cdot 2^{|A^3|-2}} + \sum_{\alpha=2}^{I}\sum_{i=2}^{|A^1|} \frac{1}{2^3 \cdot 2^{|A^1|-i} \cdot 2^{|V^1|} \cdot 2 \cdot 2^{|A^3|-\alpha}} \\
&= \frac{1}{2^{|A^1|+|V^1|+|A^3|}} \left(1 + \frac{1}{2^2}\sum_{\alpha=2}^{I} 2^\alpha + \frac{1}{2^2}\sum_{i=2}^{|A^1|} 2^i + \frac{1}{2^2}\sum_{\alpha=2}^{I} 2^\alpha \frac{1}{2^2}\sum_{i=2}^{|A^1|} 2^i \right) \\
&= \frac{1}{2^{|A^1|+|V^1|+|A^3|}} \left(1 + \frac{1}{2^2}\sum_{\alpha=2}^{I} 2^\alpha \right)\left(1 + \frac{1}{2^2}\sum_{i=2}^{|A^1|} 2^i \right) \\
&= \frac{1}{2^{|A^1|+|V^1|+|A^3|}} \left(1 + \frac{2^2}{2^2} \cdot \frac{1-2^{I-1}}{1-2}\right)\left(1 + \frac{2^2}{2^2} \cdot \frac{1-2^{|A^1|-1}}{1-2}\right) \\
&= \frac{2^{I-1} \cdot 2^{|A^1|-1}}{2^{|A^1|+|V^1|+|A^3|}} = \frac{2^{I-2}}{2^{|V^1|+|A^3|}}.
\end{aligned}
\tag{3.124}
$$

$$
\begin{aligned}
\Sigma_8 &= \sum_{\alpha=1}^{I}\sum_{j=1}^{|A^2|} c_{a^3_\alpha a^2_j} = c_{a^3_1 a^2_1} + \sum_{\alpha=2}^{I} c_{a^3_\alpha a^2_1} + \sum_{j=2}^{|A^2|} c_{a^3_1 a^2_j} + \sum_{\alpha=2}^{I}\sum_{j=2}^{|A^2|} c_{a^3_\alpha a^2_j} \\
&= c_{a^3_2 a^2_2} + \sum_{\alpha=2}^{I} c_{a^3_\alpha a^2_2} + \sum_{j=2}^{|A^2|} c_{a^3_2 a^2_j} + \sum_{\alpha=2}^{I}\sum_{j=2}^{|A^2|} c_{a^3_\alpha a^2_j} \\
&= \frac{1}{2^3 \cdot 2^{|A^2|-2} \cdot 2^{|V^1|} \cdot 2 \cdot 2^{|A^3|-2}} + \sum_{\alpha=2}^{I} \frac{1}{2^3 \cdot 2^{|A^2|-2} \cdot 2^{|V^1|} \cdot 2 \cdot 2^{|A^3|-\alpha}} \\
&\quad + \sum_{j=2}^{|A^2|} \frac{1}{2^3 \cdot 2^{|A^2|-j} \cdot 2^{|V^1|} \cdot 2 \cdot 2^{|A^3|-2}} + \sum_{\alpha=2}^{I}\sum_{j=2}^{|A^2|} \frac{1}{2^3 \cdot 2^{|A^2|-j} \cdot 2^{|V^1|} \cdot 2 \cdot 2^{|A^3|-\alpha}} \\
&= \frac{1}{2^{|A^2|+|V^1|+|A^3|}} \left(1 + \frac{1}{2^2}\sum_{\alpha=2}^{I} 2^\alpha + \frac{1}{2^2}\sum_{j=2}^{|A^2|} 2^j + \frac{1}{2^2}\sum_{\alpha=2}^{I} 2^\alpha \frac{1}{2^2}\sum_{j=2}^{|A^2|} 2^j \right) \\
&= \frac{1}{2^{|A^2|+|V^1|+|A^3|}} \left(1 + \frac{1}{2^2}\sum_{\alpha=2}^{I} 2^\alpha \right)\left(1 + \frac{1}{2^2}\sum_{j=2}^{|A^2|} 2^j \right) \\
&= \frac{1}{2^{|A^2|+|V^1|+|A^3|}} \left(1 + \frac{2^2}{2^2} \cdot \frac{1-2^{I-1}}{1-2}\right)\left(1 + \frac{2^2}{2^2} \cdot \frac{1-2^{|A^2|-1}}{1-2}\right) \\
&= \frac{2^{I-1} \cdot 2^{|A^2|-1}}{2^{|A^2|+|V^1|+|A^3|}} = \frac{2^{I-2}}{2^{|V^1|+|A^3|}}.
\end{aligned}
\tag{3.125}
$$

$$\begin{aligned}
\Sigma_9 &= \sum_{i=1}^{I}\sum_{\gamma=1}^{|V^1|} c_{a_i^3 v_\gamma^1} = \sum_{\gamma=1}^{|V^1|} c_{a_1^3 v_\gamma^1} + \sum_{i=2}^{I}\sum_{\gamma=1}^{|V^1|} c_{a_i^3 v_\gamma^1} = \sum_{\gamma=1}^{|V^1|} c_{a_2^3 v_\gamma^1} + \sum_{i=2}^{I}\sum_{\gamma=1}^{|V^1|} c_{a_i^3 v_\gamma^1} \\
&= \sum_{\gamma=1}^{|V^1|} \frac{1}{2^3 \cdot 2^{|V^1|-\gamma} \cdot 2^{|A^3|-2}} + \sum_{i=2}^{I}\sum_{\gamma=1}^{|V^1|} \frac{1}{2^3 \cdot 2^{|V^1|-\gamma} \cdot 2^{|A^3|-i}} \\
&= \frac{1}{2^{|V^1|+|A^3|+1}} \sum_{\gamma=1}^{|V^1|} 2^\gamma \left(1 + \frac{1}{2^2}\sum_{i=2}^{I} 2^i\right) \qquad (3.126) \\
&= \frac{1}{2^{|V^1|+|A^3|+1}} \cdot 2 \cdot \frac{1-2^{|V^1|}}{1-2} \cdot \left(1 + \frac{2^2}{2^2} \cdot \frac{1-2^{I-1}}{1-2}\right) \\
&= \frac{2^{I-1} \cdot \left(2^{|V^1|} - 1\right)}{2^{|V^1|+|A^3|}} = \frac{2^{I-1}}{2^{|A^3|}} - \frac{2^{I-1}}{2^{|V^1|+|A^3|}}.
\end{aligned}$$

$$\begin{aligned}
\Sigma_{10} &= \sum_{i=1}^{I}\sum_{\gamma=1}^{|V^2|} c_{a_i^3 v_\gamma^2} = \sum_{\gamma=1}^{|V^2|} c_{a_1^3 v_\gamma^2} + \sum_{i=2}^{I}\sum_{\gamma=1}^{|V^2|} c_{a_i^3 v_\gamma^2} = \sum_{\gamma=1}^{|V^2|} c_{a_2^3 v_\gamma^2} + \sum_{i=2}^{I}\sum_{\gamma=1}^{|V^2|} c_{a_i^3 v_\gamma^2} \\
&= \sum_{\gamma=1}^{|V^2|} \frac{1}{2^{\gamma+1} \cdot 2 \cdot 2^{|A^3|-2}} + \sum_{i=2}^{I}\sum_{\gamma=1}^{|V^2|} \frac{1}{2^{\gamma+1} \cdot 2 \cdot 2^{|A^3|-i}} \\
&= \frac{1}{2^{|A^3|}} \sum_{\gamma=1}^{|V^2|} \frac{1}{2^\gamma} \left(1 + \frac{1}{2^2}\sum_{i=2}^{I} 2^i\right) = \frac{1}{2^{|A^3|}} \cdot \frac{1}{2} \cdot \frac{1 - \frac{1}{2^{|V^2|}}}{1 - \frac{1}{2}} \cdot \left(1 + \frac{2^2}{2^2} \cdot \frac{1-2^{I-1}}{1-2}\right) \qquad (3.127) \\
&= \frac{2^{I-1} \cdot \left(2^{|V^2|} - 1\right)}{2^{|A^3|} \cdot 2^{|V^2|}} = \frac{2^{I-1}}{2^{|A^3|}} - \frac{2^{I-1}}{2^{|V^2|+|A^3|}}.
\end{aligned}$$

$$\begin{aligned}
\Sigma_{11} &= \sum_{\alpha=1}^{I} c_{a_\alpha^3 n} = c_{a_1^3 n} + \sum_{\alpha=2}^{I} c_{a_\alpha^3 n} = c_{a_2^3 n} + \sum_{\alpha=2}^{I} c_{a_\alpha^3 n} \\
&= \frac{1}{2^{|V^2|+1} \cdot 2 \cdot 2^{|A^3|-2}} + \sum_{\alpha=2}^{I} \frac{1}{2^{|V^2|+1} \cdot 2 \cdot 2^{|A^3|-\alpha}} \qquad (3.128) \\
&= \frac{1}{2^{|V^2|+|A^3|}} \left(1 + \frac{1}{2^2}\sum_{\alpha=2}^{I} 2^\alpha\right) = \frac{1}{2^{|V^2|+|A^3|}} \left(1 + \frac{2^2}{2^2} \cdot \frac{1-2^{I-1}}{1-2}\right) = \frac{2^{I-1}}{2^{|V^2|+|A^3|}}.
\end{aligned}$$

$$\Sigma_{12} = \sum_{i=1}^{I}\sum_{j=I+1}^{|A^3|} c_{a_i^3 a_j^3} = \sum_{j=I+1}^{|A^3|} c_{a_1^3 a_j^3} + \sum_{i=2}^{I}\sum_{j=I+1}^{|A^3|} c_{a_i^3 a_j^3} = \sum_{j=I+1}^{|A^3|} c_{a_2^3 a_j^3} + \sum_{i=2}^{I}\sum_{j=I+1}^{|A^3|} c_{a_i^3 a_j^3} \qquad (3.129)$$

$$\begin{aligned}
&= \sum_{j=I+1}^{|A^3|} \frac{1}{2\cdot 2^{j-2}} + \sum_{i=2}^{I}\sum_{j=I+1}^{|A^3|} \frac{1}{2\cdot 2^{j-i}} = 2\cdot \sum_{j=I+1}^{|A^3|} \frac{1}{2^j}\left(1+\frac{1}{2^2}\cdot\sum_{i=2}^{I} 2^i\right) \\
&= 2\cdot\frac{1}{2^{I+1}}\cdot\frac{1-\frac{1}{2^{|A^3|-I}}}{1-\frac{1}{2}}\cdot\left(1+\frac{2^2}{2^2}\cdot\frac{1-2^{I-1}}{1-2}\right) = \frac{(2^{|A^3|-I}-1)\cdot 2^{I-1}}{2^{|A^3|-I}\cdot 2^{I-1}} = 1-\frac{2^I}{2^{|A^3|}}.
\end{aligned} \tag{3.130}$$

Thus for expression (3.122) with $I\in[|A^3|]$ we have

$$\begin{aligned}
c(\delta(A^3_I)) = &\frac{2^{I-2}}{2^{|V^1|+|A^3|}} + \frac{2^{I-2}}{2^{|V^1|+|A^3|}} + \frac{2^{I-1}}{2^{|A^3|}} - \frac{2^{I-1}}{2^{|V^1|+|A^3|}} \\
&+ \frac{2^{I-1}}{2^{|A^3|}} - -\frac{2^{I-1}}{2^{|V^2|+|A^3|}} + \frac{2^{I-1}}{2^{|V^2|+|A^3|}} + 1 - \frac{2^I}{2^{|A^3|}} = 1.
\end{aligned} \tag{3.131}$$

Case 4: $M^* = A^1\cup A^2$.

$$c(\delta(A^1\cup A^2)) = c(\delta(A^1_{|A^1|})) + c(\delta(A^2_{|A^2|})) - 2\sum_{i=1}^{|A^1|}\sum_{j=1}^{|A^2|} c_{a^1_i a^2_j}. \tag{3.132}$$

The first two terms both evaluate to one (Cases 1 and 2). The third term on the right-hand side of equation (3.132) is a particular case of Σ_1 with $I=|A^1|$ from Case 1. Thus we have

$$c(\delta(A^1\cup A^2)) = 1+1-2\frac{2^{|A^1|-1}}{2^{|A^1|}} = 1. \tag{3.133}$$

Case 5: $M^* = A^1\cup A^2\cup V^1_I$ for $I\in[|V^1|]$.

$$\begin{aligned}
c(\delta(A^1\cup A^2\cup V^1_I)) = c(\delta(A^1\cup A^2)) &- \underbrace{\sum_{i=1}^{|A^1|}\sum_{\gamma=1}^{I} c_{a^1_i v^1_\gamma}}_{\Sigma_{13}} - \underbrace{\sum_{\gamma=1}^{I}\sum_{j=1}^{|A^2|} c_{v^1_\gamma a^2_j}}_{\Sigma_{14}} \\
&+ \underbrace{\sum_{\alpha=1}^{|A^3|}\sum_{\gamma=1}^{I} c_{a^3_\alpha v^1_\gamma}}_{\Sigma_{15}} + \underbrace{\sum_{\gamma=1}^{I}\sum_{\beta=1}^{|V^2|} c_{v^1_\gamma v^2_\beta}}_{\Sigma_{16}} + \underbrace{\sum_{\gamma=1}^{I} c_{v^1_\gamma n}}_{\Sigma_{17}} + \underbrace{\sum_{\gamma=1}^{I}\sum_{\beta=I+1}^{|V^1|} c_{v^1_\gamma v^1_\beta}}_{\Sigma_{18}}.
\end{aligned} \tag{3.134}$$

In view of the definition of $c\in\mathbb{R}^{E_n}_{>0}$, see (3.110)-(3.111), we calculate each term of the right-hand side of equation (3.134).

$$\begin{aligned}
\Sigma_{13} &= \sum_{\gamma=1}^{I}\sum_{i=1}^{|A^1|} c_{a^1_i v^1_\gamma} = \sum_{\gamma=1}^{I} c_{a^1_1 v^1_\gamma} + \sum_{i=2}^{|A^1|}\sum_{\gamma=1}^{I} c_{a^1_i v^1_\gamma} = \sum_{\gamma=1}^{I} c_{a^1_2 v^1_\gamma} + \sum_{i=2}^{|A^1|}\sum_{\gamma=1}^{I} c_{a^1_i v^1_\gamma} \\
&= \sum_{\gamma=1}^{I} \frac{1}{2^{\gamma+1}\cdot 2\cdot 2^{|A^1|-2}} + \sum_{i=2}^{|A^1|}\sum_{\gamma=1}^{I} \frac{1}{2^{\gamma+1}\cdot 2\cdot 2^{|A^1|-i}}
\end{aligned} \tag{3.135}$$

$$= \frac{1}{2^{|A^1|}} \sum_{\gamma=1}^{I} \frac{1}{2^\gamma} \left(1 + \frac{1}{2^2} \sum_{i=2}^{|A^1|} 2^i \right) = \frac{1}{2^{|A^1|}} \cdot \frac{1}{2} \cdot \frac{1 - \frac{1}{2^I}}{1 - \frac{1}{2}} \cdot \left(1 + \frac{2^2}{2^2} \cdot \frac{1 - 2^{|A^1|-1}}{1-2} \right)$$
$$= \frac{2^{|A^1|-1} \cdot \left(2^I - 1\right)}{2^{|A^1|+I}} = \frac{1}{2} - \frac{1}{2^{I+1}}. \tag{3.136}$$

$$\Sigma_{14} = \sum_{\gamma=1}^{I} \sum_{j=1}^{|A^2|} c_{a_j^2 v_\gamma^1} = \sum_{\gamma=1}^{I} c_{a_1^2 v_\gamma^1} + \sum_{j=2}^{|A^2|} \sum_{\gamma=1}^{I} c_{a_j^2 v_\gamma^1} = \sum_{\gamma=1}^{I} c_{a_2^2 v_\gamma^1} + \sum_{j=2}^{|A^2|} \sum_{\gamma=1}^{I} c_{a_j^2 v_\gamma^1}$$
$$= \sum_{\gamma=1}^{I} \frac{1}{2^{\gamma+1} \cdot 2 \cdot 2^{|A^2|-2}} + \sum_{j=2}^{|A^2|} \sum_{\gamma=1}^{I} \frac{1}{2^{\gamma+1} \cdot 2 \cdot 2^{|A^2|-j}}$$
$$= \frac{1}{2^{|A^2|}} \sum_{\gamma=1}^{I} \frac{1}{2^\gamma} \left(1 + \frac{1}{2^2} \sum_{j=2}^{|A^2|} 2^j \right) = \frac{1}{2^{|A^2|}} \cdot \frac{1}{2} \cdot \frac{1 - \frac{1}{2^I}}{1 - \frac{1}{2}} \cdot \left(1 + \frac{2^2}{2^2} \cdot \frac{1 - 2^{|A^2|-1}}{1-2} \right) \tag{3.137}$$
$$= \frac{2^{|A^2|-1} \cdot \left(2^I - 1\right)}{2^{|A^2|+I}} = \frac{1}{2} - \frac{1}{2^{I+1}}.$$

$$\Sigma_{15} = \sum_{\alpha=1}^{|A^3|} \sum_{\gamma=1}^{I} c_{a_\alpha^3 v_\gamma^1} = \sum_{\gamma=1}^{I} c_{a_1^3 v_\gamma^1} + \sum_{\alpha=2}^{|A^3|} \sum_{\gamma=1}^{I} c_{a_\alpha^3 v_\gamma^1} = \sum_{\gamma=1}^{I} c_{a_2^3 v_\gamma^1} + \sum_{\alpha=2}^{|A^3|} \sum_{\gamma=1}^{I} c_{a_\alpha^3 v_\gamma^1}$$
$$= \sum_{\gamma=1}^{I} \frac{1}{2^3 \cdot 2^{|V^1|-\gamma} \cdot 2^{|A^3|-2}} + \sum_{\alpha=2}^{|A^3|} \sum_{\gamma=1}^{I} \frac{1}{2^3 \cdot 2^{|V^1|-\gamma} \cdot 2^{|A^3|-\alpha}}$$
$$= \frac{1}{2^{|V^1|+|A^3|+1}} \sum_{\gamma=1}^{I} 2^\gamma \left(1 + \frac{1}{2^2} \sum_{\alpha=2}^{|A^3|} 2^\alpha \right) \tag{3.138}$$
$$= \frac{1}{2^{|V^1|+|A^3|+1}} \cdot 2 \cdot \frac{1 - 2^I}{1-2} \cdot \left(1 + \frac{2^2}{2^2} \cdot \frac{1 - 2^{|A^3|-1}}{1-2} \right)$$
$$= \frac{2^{|A^3|-1} \cdot \left(2^I - 1\right)}{2^{|V^1|+|A^3|}} = \frac{2^I - 1}{2^{|V^1|+1}}.$$

$$\Sigma_{16} = \sum_{\gamma=1}^{I} \sum_{\beta=1}^{|V^2|} c_{v_\gamma^1 v_\beta^2} = \sum_{\gamma=1}^{I} \sum_{\beta=1}^{|V^2|} \frac{1}{2^{\beta+1} \cdot 2 \cdot 2^{|V^1|-\gamma}} = \frac{1}{2^{|V^1|+2}} \sum_{\gamma=1}^{I} 2^\gamma \sum_{\beta=1}^{|V^2|} \frac{1}{2^\beta}$$
$$= \frac{1}{2^{|V^1|+2}} \cdot \frac{2 \cdot (1 - 2^I)}{1-2} \cdot \frac{\frac{1}{2} \cdot \left(1 - \frac{1}{2^{|V^2|}}\right)}{1 - \frac{1}{2}} = \frac{(2^I - 1)(2^{|V^2|} - 1)}{2^{|V^1|+|V^2|+1}} \tag{3.139}$$
$$= \frac{2^I - 1}{2^{|V^1|+1}} - \frac{2^I - 1}{2^{|V^1|+|V^2|+1}}.$$

$$\begin{aligned}\Sigma_{17} &= \sum_{\gamma=1}^{I} c_{v_\gamma^1 n} = \sum_{\gamma=1}^{I} \frac{1}{2^{|V^2|+1}\cdot 2\cdot 2^{|V^1|-\gamma}} = \frac{1}{2^{|V^1|+|V^2|+2}} \sum_{\gamma=1}^{I} 2^{\gamma} \\ &= \frac{1}{2^{|V^1|+|V^2|+2}} \cdot \frac{2\cdot(1-2^I)}{1-2} = \frac{2^I-1}{2^{|V^1|+|V^2|+1}}\end{aligned} \tag{3.140}$$

$$\begin{aligned}\Sigma_{18} &= \sum_{\gamma=1}^{I} \sum_{\beta=I+1}^{|V^1|} c_{v_\gamma^1 v_\beta^1} = \sum_{\gamma=1}^{I} \sum_{\beta=I+1}^{|V^1|} \frac{1}{2\cdot 2^{\beta-\gamma}} = \frac{1}{2} \sum_{\beta=I+1}^{|V^1|} \frac{1}{2^{\beta}} \sum_{\gamma=1}^{I} 2^{\gamma} \\ &= \frac{1}{2}\cdot \frac{\frac{1}{2^{I+1}}\cdot\left(1-\frac{1}{2^{|V^1|-I}}\right)}{1-\frac{1}{2}} \cdot \frac{2\cdot(1-2^I)}{1-2} = \frac{2^I-1}{2^I} - \frac{2^I-1}{2^{|V^1|}} = 1 - \frac{1}{2^I} - \frac{2^I-1}{2^{|V^1|}}.\end{aligned} \tag{3.141}$$

Then, in view of (3.133) and (3.135)-(3.141) for expression (3.134) we deduce

$$\begin{aligned}c(\delta(A^1\cup A^2\cup V_I^1)) &= 1 - \left(\frac{1}{2} - \frac{1}{2^{I+1}}\right) - \left(\frac{1}{2} - \frac{1}{2^{I+1}}\right) + \cancel{\frac{2^I-1}{2^{|V^1|+1}}} + \cancel{\frac{2^I-1}{2^{|V^1|+1}}} \\ &\quad - \cancel{\frac{2^I-1}{2^{|V^1|+|V^2|+1}}} + \cancel{\frac{2^I-1}{2^{|V^1|+|V^2|+1}}} + 1 - \frac{1}{2^I} - \cancel{\frac{2^I-1}{2^{|V^1|}}} \\ &= 1 - 2\cdot\frac{1}{2} + 2\cdot\frac{1}{2^{I+1}} + 1 - \frac{1}{2^I} = 1.\end{aligned} \tag{3.142}$$

Case 6: $M^* = A^1 \cup A^2 \cup V^1 \cup A^3$.

$$\begin{aligned}c(\delta(A^1\cup A^2\cup V^1\cup A^3)) &= c(\delta(A^1\cup A^2\cup V^1_{|V^1|})) \\ &\quad - \underbrace{\sum_{\alpha=1}^{|A^3|}\sum_{i=1}^{|A^1|} c_{a_\alpha^3 a_i^1}}_{\Sigma_{19}} - \underbrace{\sum_{\alpha=1}^{|A^3|}\sum_{j=1}^{|A^2|} c_{a_\alpha^3 a_j^2}}_{\Sigma_{20}} - \underbrace{\sum_{i=1}^{|A^3|}\sum_{\gamma=1}^{|V^1|} c_{a_i^3 v_\gamma^1}}_{\Sigma_{21}} + \underbrace{\sum_{i=1}^{|A^3|}\sum_{\gamma=1}^{|V^2|} c_{a_i^3 v_\gamma^2}}_{\Sigma_{22}} + \underbrace{\sum_{\alpha=1}^{|A^3|} c_{a_\alpha^3 n}}_{\Sigma_{23}}.\end{aligned} \tag{3.143}$$

In this case Σ_{19}, ..., Σ_{23} are equal to Σ_7, ..., Σ_{11} with $I = |A^3|$ from Case 3, respectively. Using (3.123)-(3.130) and (3.142) with $I = |V^1|$ from Case 3 we obtain

$$\begin{aligned}c(\delta(A^1\cup A^2\cup V^1\cup A^3)) &= 1 - \cancel{\frac{2^{|A^3|-2}}{2^{|V^1|+|A^3|}}} - \cancel{\frac{2^{|A^3|-2}}{2^{|V^1|+|A^3|}}} - \cancel{\frac{2^{|A^3|-1}}{2^{|A^3|}}} + \cancel{\frac{2^{|A^3|-1}}{2^{|V^1|+|A^3|}}} \\ &\quad + \cancel{\frac{2^{|A^3|-1}}{2^{|A^3|}}} - \cancel{\frac{2^{|A^3|-1}}{2^{|V^2|+|A^3|}}} + \cancel{\frac{2^{|A^3|-1}}{2^{|V^2|+|A^3|}}} + 1 - \frac{2^{|A^3|}}{2^{|A^3|}} \\ &= 2 - 1 = 1.\end{aligned} \tag{3.144}$$

Case 7: $M^* = A^1 \cup A^2 \cup V^1 \cup A^3 \cup V_I^2$ for $I \in [|V^2|]$.

$$\begin{aligned} c(\delta(A^1 \cup A^2 \cup V^1 \cup A^3 \cup V_I^2)) &= c(\delta(A^1 \cup A^2 \cup V^1 \cup A^3)) - \underbrace{\sum_{i=1}^{|A^1|}\sum_{\gamma=1}^{I} c_{a_i^1 v_\gamma^2}}_{\Sigma_{24}} \\ &- \underbrace{\sum_{j=1}^{|A^2|}\sum_{\gamma=1}^{I} c_{a_j^2 v_\gamma^2}}_{\Sigma_{25}} - \underbrace{\sum_{\alpha=1}^{|A^3|}\sum_{\gamma=1}^{I} c_{a_\alpha^3 v_\gamma^2}}_{\Sigma_{26}} - \underbrace{\sum_{\gamma=1}^{|V^1|}\sum_{\beta=1}^{I} c_{v_\gamma^1 v_\beta^2}}_{\Sigma_{27}} + \underbrace{\sum_{\gamma=1}^{I} c_{v_\gamma^2 n}}_{\Sigma_{28}} + \underbrace{\sum_{\gamma=1}^{I}\sum_{\beta=I+1}^{|V^2|} c_{v_\gamma^2 v_\beta^2}}_{\Sigma_{29}}. \end{aligned} \tag{3.145}$$

Calculating each term on the right-hand side of equation (3.145) we obtain

$$\begin{aligned} \Sigma_{24} &= \sum_{i=1}^{|A^1|}\sum_{\gamma=1}^{I} c_{a_i^1 v_\gamma^2} = \sum_{\gamma=1}^{I} c_{a_1^1 v_\gamma^2} + \sum_{i=2}^{|A^1|}\sum_{\gamma=1}^{I} c_{a_i^1 v_\gamma^2} = \sum_{\gamma=1}^{I} c_{a_2^1 v_\gamma^2} + \sum_{i=2}^{|A^1|}\sum_{\gamma=1}^{I} c_{a_i^1 v_\gamma^2} \\ &= \sum_{\gamma=1}^{I} \frac{1}{2^{\gamma+1}\cdot 2^{|V^1|+1}\cdot 2\cdot 2^{|A^1|-2}} + \sum_{i=2}^{|A^1|}\sum_{\gamma=1}^{I} \frac{1}{2^{\gamma+1}\cdot 2^{|V^1|+1}\cdot 2\cdot 2^{|A^1|-i}} \\ &= \frac{1}{2^{|A^1|+|V^1|+1}} \sum_{\gamma=1}^{I} \frac{1}{2^\gamma}\left(1 + \frac{1}{2^2}\sum_{i=2}^{|A^1|} 2^i\right) \\ &= \frac{1}{2^{|A^1|+|V^1|+1}} \cdot \frac{1}{2} \cdot \frac{1-\frac{1}{2^I}}{1-\frac{1}{2}} \cdot \left(1 + \frac{2^2}{2^2}\cdot\frac{1-2^{|A^1|-1}}{1-2}\right) \\ &= \frac{2^{|A^1|-1}\cdot\left(2^I - 1\right)}{2^{|A^1|+|V^1|+I+1}} = \frac{1}{2^{|V^1|+2}} - \frac{1}{2^{|V^1|+I+2}}. \end{aligned} \tag{3.146}$$

$$\begin{aligned} \Sigma_{25} &= \sum_{j=1}^{|A^2|}\sum_{\gamma=1}^{I} c_{a_j^2 v_\gamma^2} = \sum_{\gamma=1}^{I} c_{a_1^2 v_\gamma^2} + \sum_{j=2}^{|A^2|}\sum_{\gamma=1}^{I} c_{a_j^2 v_\gamma^2} = \sum_{\gamma=1}^{I} c_{a_2^2 v_\gamma^2} + \sum_{j=2}^{|A^2|}\sum_{\gamma=1}^{I} c_{a_j^2 v_\gamma^2} \\ &= \sum_{\gamma=1}^{I} \frac{1}{2^{\gamma+1}\cdot 2^{|V^1|+1}\cdot 2\cdot 2^{|A^2|-2}} + \sum_{j=2}^{|A^2|}\sum_{\gamma=1}^{I} \frac{1}{2^{\gamma+1}\cdot 2^{|V^1|+1}\cdot 2\cdot 2^{|A^2|-j}} \\ &= \frac{1}{2^{|A^2|+|V^1|+1}} \sum_{\gamma=1}^{I} \frac{1}{2^\gamma}\left(1 + \frac{1}{2^2}\sum_{j=2}^{|A^2|} 2^j\right) \end{aligned} \tag{3.147}$$

$$
\begin{aligned}
&= \frac{1}{2^{|A^2|+|V^1|+1}} \cdot \frac{1}{2} \cdot \frac{1-\frac{1}{2^I}}{1-\frac{1}{2}} \cdot \left(1 + \frac{2^2}{2^2} \cdot \frac{1-2^{|A^2|-1}}{1-2}\right) \\
&= \frac{2^{|A^2|-1} \cdot \left(2^I - 1\right)}{2^{|A^2|+|V^1|+I+1}} = \frac{1}{2^{|V^1|+2}} - \frac{1}{2^{|V^1|+I+2}}.
\end{aligned} \tag{3.148}
$$

$$
\begin{aligned}
\Sigma_{26} &= \sum_{\alpha=1}^{|A^3|} \sum_{\gamma=1}^{I} c_{a^3_\alpha v^2_\gamma} = \sum_{\gamma=1}^{I} c_{a^3_1 v^2_\gamma} + \sum_{\alpha=2}^{|A^3|} \sum_{\gamma=1}^{I} c_{a^3_\alpha v^2_\gamma} = \sum_{\gamma=1}^{I} c_{a^3_2 v^2_\gamma} + \sum_{\alpha=2}^{|A^3|} \sum_{\gamma=1}^{|V^2|} c_{a^3_\alpha v^2_\gamma} \\
&= \sum_{\gamma=1}^{I} \frac{1}{2^{\gamma+1} \cdot 2 \cdot 2^{|A^3|-2}} + \sum_{\alpha=2}^{|A^3|} \sum_{\gamma=1}^{I} \frac{1}{2^{\gamma+1} \cdot 2 \cdot 2^{|A^3|-\alpha}} \\
&= \frac{1}{2^{|A^3|}} \sum_{\gamma=1}^{I} \frac{1}{2^\gamma} \left(1 + \frac{1}{2^2} \sum_{\alpha=2}^{|A^3|} 2^\alpha\right) = \frac{1}{2^{|A^3|}} \cdot \frac{1}{2} \cdot \frac{1-\frac{1}{2^I}}{1-\frac{1}{2}} \cdot \left(1 + \frac{2^2}{2^2} \cdot \frac{1-2^{|A^3|-1}}{1-2}\right) \\
&= \frac{2^{|A^3|-1} \cdot \left(2^I - 1\right)}{2^{|A^3|} \cdot 2^I} = \frac{1}{2} - \frac{1}{2^{I+1}}.
\end{aligned} \tag{3.149}
$$

$$
\begin{aligned}
\Sigma_{27} &= \sum_{\gamma=1}^{|V^1|} \sum_{\beta=1}^{I} c_{v^1_\gamma v^2_\beta} = \sum_{\gamma=1}^{|V^1|} \sum_{\beta=1}^{I} \frac{1}{2^{\beta+1} \cdot 2 \cdot 2^{|V^1|-\gamma}} = \frac{1}{2^{|V^1|+2}} \sum_{\gamma=1}^{|V^1|} 2^\gamma \sum_{\beta=1}^{I} \frac{1}{2^\beta} \\
&= \frac{1}{2^{|V^1|+2}} \cdot \frac{2 \cdot (1-2^{|V^1|})}{1-2} \cdot \frac{\frac{1}{2} \cdot \left(1-\frac{1}{2^I}\right)}{1-\frac{1}{2}} = \frac{(2^{|V^1|}-1)(2^I-1)}{2^{|V^1|+I+1}} \\
&= \frac{2^{|V^1|}-1}{2^{|V^1|+1}} - \frac{2^{|V^1|}-1}{2^{|V^1|+I+1}} = \frac{1}{2} - \frac{1}{2^{|V^1|+1}} - \frac{1}{2^{I+1}} + \frac{1}{2^{|V^1|+I+1}}.
\end{aligned} \tag{3.150}
$$

$$
\Sigma_{28} = \sum_{\beta=1}^{I} c_{v^2_\beta n} = \sum_{\beta=1}^{I} \frac{1}{2 \cdot 2^{|V^2|-\beta}} = \frac{1}{2^{|V^2|+1}} \sum_{\beta=1}^{I} 2^\beta = \frac{1}{2^{|V^2|+1}} \cdot \frac{2 \cdot (1-2^I)}{1-2} = \frac{2^I-1}{2^{|V^2|}}. \tag{3.151}
$$

$$
\begin{aligned}
\Sigma_{29} &= \sum_{\gamma=1}^{I} \sum_{\beta=I+1}^{|V^2|} c_{v^2_\gamma v^2_\beta} = \sum_{\gamma=1}^{I} \sum_{\beta=I+1}^{|V^2|} \frac{1}{2 \cdot 2^{\beta-\gamma}} = \frac{1}{2} \sum_{\beta=I+1}^{|V^2|} \frac{1}{2^\beta} \sum_{\gamma=1}^{I} 2^\gamma \\
&= \frac{1}{2} \cdot \frac{1}{2^{I+1}} \cdot \frac{1-\frac{1}{2^{|V^2|-I}}}{1-\frac{1}{2}} \cdot \frac{2 \cdot (1-2^I)}{1-2} = \frac{2^I-1}{2^I} - \frac{2^I-1}{2^{|V^2|}} \\
&= 1 - \frac{1}{2^I} - \frac{2^I-1}{2^{|V^2|}}.
\end{aligned} \tag{3.152}
$$

Then, in view of (3.144) and (3.146)-(3.152) we deduce for expression (3.145)

$$
\begin{aligned}
c(\delta(A^1 \cup A^2 \cup V^1 \cup A^3 \cup V_I^2)) = 1 - \xcancel{\frac{1}{2^{|V^1|+2}}} + \cancel{\frac{1}{2^{|V^1|+I+2}}} - \xcancel{\frac{1}{2^{|V^1|+2}}} + \cancel{\frac{1}{2^{|V^1|+I+2}}} \\
-\frac{1}{2} + \frac{1}{2^{I+1}} - \frac{1}{2} + \xcancel{\frac{1}{2^{|V^1|+1}}} + \frac{1}{2^{I+1}} - \cancel{\frac{1}{2^{|V^1|+I+1}}} + \cancel{\frac{2^I-1}{2^{|V^2|}}} + 1 - \frac{1}{2^I} - \cancel{\frac{2^I-1}{2^{|V^2|}}} = 1.
\end{aligned}
\tag{3.153}
$$

□

Continuing the proof of Proposition 3.6.1 we construct from the undirected graph K_n the directed graph $\overleftrightarrow{K}_n = ([n], \overleftrightarrow{E}_n)$ by replacing each edge $e \in E_n$ by two antiparallel arcs. In the new graph $\overleftrightarrow{K}_n$ we construct three flows: a 1-n flow $f' : \overleftrightarrow{E}_n \to \mathbb{R}$, see Figure 3.12, with

$$
\begin{array}{lll}
f'_{a_i^1 a_j^1} := c_{a_i^1 a_j^1} & \text{for } a_i^1, a_j^1 \in A^1 \text{ with } i < j, \\
f'_{a_i^1 a_j^1} := 0 & \text{for } a_i^1, a_j^1 \in A^1 \text{ with } i > j, \\
f'_{a_i^2 a_j^2} := 0 & \text{for all } a_i^2, a_j^2 \in A^2, \\
f'_{a_i^1 a_j^2} := c_{a_i^1 a_j^2} & \text{for all } a_i^1 \in A^1,\ a_j^2 \in A^2, \\
f'_{a_j^2 a_i^1} := 0 & \text{for all } a_i^1 \in A^1,\ a_j^2 \in A^2, \\
f'_{v_i^1 v_j^1} := c_{v_i^1 v_j^1} & \text{for } v_i^1, v_j^1 \in V^1 \text{ with } i < j, \\
f'_{v_i^1 v_j^1} := 0 & \text{for } v_i^1, v_j^1 \in V^1 \text{ with } i > j, \\
f'_{a_i^1 v_\gamma^1} := c_{a_i^1 v_\gamma^1} & \text{for all } a_i^1 \in A^1,\ v_\gamma^1 \in V^1, \\
f'_{v_\gamma^1 a_i^1} := 0 & \text{for all } a_i^1 \in A^1,\ v_\gamma^1 \in V^1, \\
f'_{a_j^2 v_\gamma^1} := c_{a_j^2 v_\gamma^1} & \text{for all } a_j^2 \in A^2,\ v_\gamma^1 \in V^1, \\
f'_{v_\gamma^1 a_j^2} := 0 & \text{for all } a_j^2 \in A^2,\ v_\gamma^1 \in V^1, \\
f'_{a_i^3 a_j^3} := 0 & \text{for all } a_i^3, a_j^3 \in A^3, \\
f'_{a_\alpha^3 v_\gamma^1} := 0 & \text{for all } a_\alpha^3 \in A^3,\ v_\gamma^1 \in V^1, \\
f'_{v_\gamma^1 a_\alpha^3} := c_{a_\alpha^3 v_\gamma^1} & \text{for all } a_\alpha^3 \in A^3,\ v_\gamma^1 \in V^1, \\
f'_{a_i^1 a_\alpha^3} := c_{a_i^1 a_\alpha^3} & \text{for all } a_i^1 \in A^1,\ a_\alpha^3 \in A^3, \\
f'_{a_\alpha^3 a_i^1} := 0 & \text{for all } a_i^1 \in A^1,\ a_\alpha^3 \in A^3, \\
f'_{v_i^2 v_j^2} := c_{v_i^2 v_j^2} & \text{for } v_i^2, v_j^2 \in V^2 \text{ with } i < j, \\
f'_{v_i^2 v_j^2} := 0 & \text{for } v_i^2, v_j^2 \in V^2 \text{ with } i > j, \\
f'_{a_j^2 a_\alpha^3} := c_{a_j^2 a_\alpha^3} & \text{for all } a_j^2 \in A^2,\ a_\alpha^3 \in A^3, \\
f'_{a_\alpha^3 a_j^2} := 0 & \text{for all } a_j^2 \in A^2,\ a_\alpha^3 \in A^3,
\end{array}
\tag{3.154}
$$

$$
\begin{aligned}
f'_{a_i^1 v_\beta^2} &:= c_{a_i^1 v_\beta^2} && \text{for all } a_i^1 \in A^1,\ v_\beta^2 \in V^2,\\
f'_{v_\beta^2 a_i^1} &:= 0 && \text{for all } a_i^1 \in A^1,\ v_\beta^2 \in V^2,\\
f'_{a_j^2 v_\beta^2} &:= c_{a_j^2 v_\beta^2} && \text{for all } a_j^2 \in A^2,\ v_\beta^2 \in V^2,\\
f'_{v_\beta^2 a_j^2} &:= 0 && \text{for all } a_j^2 \in A^2,\ v_\beta^2 \in V^2,\\
f'_{a_\alpha^3 v_\beta^2} &:= c_{a_\alpha^3 v_\beta^2} && \text{for all } a_\alpha^3 \in A^3,\ v_\beta^2 \in V^2,\\
f'_{v_\beta^2 a_\alpha^3} &:= 0 && \text{for all } a_\alpha^3 \in A^3,\ v_\beta^2 \in V^2,\\
f'_{v_\gamma^1 v_\beta^2} &:= c_{v_\gamma^1 v_\beta^2} && \text{for all } v_\gamma^1 \in V^1,\ v_\beta^2 \in V^2,\\
f'_{v_\beta^2 v_\gamma^1} &:= 0 && \text{for all } v_\gamma^1 \in V^1,\ v_\beta^2 \in V^2,\\
f'_{a_i^1 n} &:= c_{a_i^1 n} && \text{for all } a_i^1 \in A^1,\\
f'_{n a_i^1} &:= 0 && \text{for all } a_i^1 \in A^1,\\
f'_{a_j^2 n} &:= c_{a_j^2 n} && \text{for all } a_j^2 \in A^2,\\
f'_{n a_j^2} &:= 0 && \text{for all } a_j^2 \in A^2,\\
f'_{v_\gamma^1 n} &:= c_{v_\gamma^1 n} && \text{for all } v_\gamma^1 \in V^1,\\
f'_{n v_\gamma^1} &:= 0 && \text{for all } v_\gamma^1 \in V^1,\\
f'_{a_\alpha^3 n} &:= c_{a_\alpha^3 n} && \text{for all } a_\alpha^3 \in A^3,\\
f'_{n a_\alpha^3} &:= 0 && \text{for all } a_\alpha^3 \in A^3,\\
f'_{v_\beta^2 n} &:= c_{v_\beta^2 n} && \text{for all } v_\beta^2 \in V^2,\\
f'_{n v_\beta^2} &:= 0 && \text{for all } v_\beta^2 \in V^2.
\end{aligned} \tag{3.155}
$$

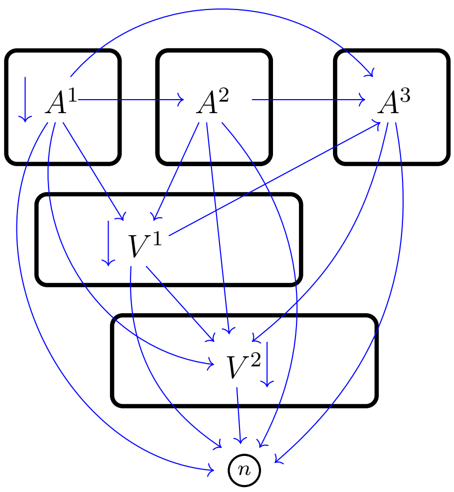

Figure 3.12: Sketch of arcs $(u, w) \in \overleftrightarrow{E}_n$ with $f'_{u,w} > 0$.

a 2-n flow $f'' : \overset{\leftrightarrow}{E}_n \to \mathbb{R}$, see Figure 3.13, with

$$
\begin{aligned}
&f''_{a^1_i a^1_j} := 0 && \text{for all } a^1_i, a^1_j \in A^1,\\
&f''_{a^2_i a^2_j} := c_{a^2_i a^2_j} && \text{for } a^2_i, a^2_j \in A^2 \text{ with } i < j,\\
&f''_{a^2_i a^2_j} := 0 && \text{for } a^2_i, a^2_j \in A^2 \text{ with } i > j,\\
&f''_{a^1_i a^2_j} := 0 && \text{for all } a^1_i \in A^1,\ a^2_j \in A^2,\\
&f''_{a^2_j a^1_i} := c_{a^1_i a^2_j} && \text{for all } a^1_i \in A^1,\ a^2_j \in A^2,\\
&f''_{v^1_i v^1_j} := c_{v^1_i v^1_j} && \text{for } v^1_i, v^1_j \in V^1 \text{ with } i < j,\\
&f''_{v^1_i v^1_j} := 0 && \text{for } v^1_i, v^1_j \in V^1 \text{ with } i > j,\\
&f''_{a^1_i v^1_\gamma} := c_{a^1_i v^1_\gamma} && \text{for all } a^1_i \in A^1,\ v^1_\gamma \in V^1,\\
&f''_{v^1_\gamma a^1_i} := 0 && \text{for all } a^1_i \in A^1,\ v^1_\gamma \in V^1,\\
&f''_{a^2_j v^1_\gamma} := c_{a^2_j v^1_\gamma} && \text{for all } a^2_j \in A^2,\ v^1_\gamma \in V^1,\\
&f''_{v^1_\gamma a^2_j} := 0 && \text{for all } a^2_j \in A^2,\ v^1_\gamma \in V^1,\\
&f''_{a^3_i a^3_j} := 0 && \text{for all } a^3_i, a^3_j \in A^3,\\
&f''_{a^3_\alpha v^1_\gamma} := 0 && \text{for all } a^3_\alpha \in A^3,\ v^1_\gamma \in V^1,\\
&f''_{v^1_\gamma a^3_\alpha} := c_{a^3_\alpha v^1_\gamma} && \text{for all } a^3_\alpha \in A^3,\ v^1_\gamma \in V^1,\\
&f''_{a^1_i a^3_\alpha} := c_{a^1_i a^3_\alpha} && \text{for all } a^1_i \in A^1,\ a^3_\alpha \in A^3,\\
&f''_{a^3_\alpha a^1_i} := 0 && \text{for all } a^1_i \in A^1,\ a^3_\alpha \in A^3,\\
&f''_{a^2_j a^3_\alpha} := c_{a^2_j a^3_\alpha} && \text{for all } a^2_j \in A^2,\ a^3_\alpha \in A^3,\\
&f''_{a^3_\alpha a^2_j} := 0 && \text{for all } a^2_j \in A^2,\ a^3_\alpha \in A^3,\\
&f''_{v^2_i v^2_j} := c_{v^2_i v^2_j} && \text{for } v^2_i, v^2_j \in V^2 \text{ with } i < j,\\
&f''_{v^2_i v^2_j} := 0 && \text{for } v^2_i, v^2_j \in V^2 \text{ with } i > j,\\
&f''_{a^1_i v^2_\beta} := c_{a^1_i v^2_\beta} && \text{for all } a^1_i \in A^1,\ v^2_\beta \in V^2,\\
&f''_{v^2_\beta a^1_i} := 0 && \text{for all } a^1_i \in A^1,\ v^2_\beta \in V^2,\\
&f''_{a^2_j v^2_\beta} := c_{a^2_j v^2_\beta} && \text{for all } a^2_j \in A^2,\ v^2_\beta \in V^2,\\
&f''_{v^2_\beta a^2_j} := 0 && \text{for all } a^2_j \in A^2,\ v^2_\beta \in V^2,\\
&f''_{a^3_\alpha v^2_\beta} := c_{a^3_\alpha v^2_\beta} && \text{for all } a^3_\alpha \in A^3,\ v^2_\beta \in V^2,\\
&f''_{v^2_\beta a^3_\alpha} := 0 && \text{for all } a^3_\alpha \in A^3,\ v^2_\beta \in V^2,\\
&f''_{v^1_\gamma v^2_\beta} := c_{v^1_\gamma v^2_\beta} && \text{for all } v^1_\gamma \in V^1,\ v^2_\beta \in V^2,\\
&f''_{v^2_\beta v^1_\gamma} := 0 && \text{for all } v^1_\gamma \in V^1,\ v^2_\beta \in V^2,
\end{aligned}
\tag{3.156}
$$

$$
\begin{aligned}
f''_{a_i^1 n} &:= c_{a_i^1 n} && \text{for all } a_i^1 \in A^1, \\
f''_{n a_i^1} &:= 0 && \text{for all } a_i^1 \in A^1, \\
f''_{a_j^2 n} &:= c_{a_j^2 n} && \text{for all } a_j^2 \in A^2, \\
f''_{n a_j^2} &:= 0 && \text{for all } a_j^2 \in A^2, \\
f''_{v_\gamma^1 n} &:= c_{v_\gamma^1 n} && \text{for all } v_\gamma^1 \in V^1, \\
f''_{n v_\gamma^1} &:= 0 && \text{for all } v_\gamma^1 \in V^1, \\
f''_{a_\alpha^3 n} &:= c_{a_\alpha^3 n} && \text{for all } a_\alpha^3 \in A^3, \\
f''_{n a_\alpha^3} &:= 0 && \text{for all } a_\alpha^3 \in A^3, \\
f''_{v_\beta^2 n} &:= c_{v_\beta^2 n} && \text{for all } v_\beta^2 \in V^2, \\
f''_{n v_\beta^2} &:= 0 && \text{for all } v_\beta^2 \in V^2.
\end{aligned}
\tag{3.157}
$$

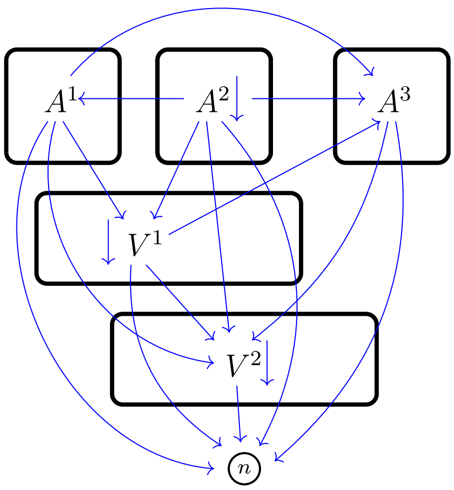

Figure 3.13: Sketch of arcs $(u, w) \in \overset{\leftrightarrow}{E}_n$ with $f''_{u,w} > 0$.

and a 3-n flow $f''' : \overset{\leftrightarrow}{E}_n \to \mathbb{R}$, see Figure 3.14, with

$$
\begin{aligned}
f'''_{a_i^1 a_j^1} &:= 0 && \text{for all } a_i^1, a_j^1 \in A^1, \\
f'''_{a_i^2 a_j^2} &:= 0 && \text{for all } a_i^2, a_j^2 \in A^2, \\
f'''_{a_i^1 a_j^2} &:= 0 && \text{for all } a_i^1 \in A^1,\ a_j^2 \in A^2, \\
f'''_{a_j^2 a_i^1} &:= 0 && \text{for all } a_i^1 \in A^1,\ a_j^2 \in A^2, \\
f'''_{v_i^1 v_j^1} &:= 0 && \text{for all } v_i, v_j \in V^1, \\
f'''_{a_i^1 v_\gamma^1} &:= 0 && \text{for all } a_i^1 \in A^1,\ v_\gamma^1 \in V^1, \\
f'''_{v_\gamma^1 a_i^1} &:= 0 && \text{for all } a_i^1 \in A^1,\ v_\gamma^1 \in V^1,
\end{aligned}
\tag{3.158}
$$

$$\begin{aligned}
f'''_{a_j^2 v_\gamma^1} &:= 0 && \text{for all } a_j^2 \in A^2,\ v_\gamma^1 \in V^1,\\
f'''_{v_\gamma^1 a_j^2} &:= 0 && \text{for all } a_j^2 \in A^2,\ v_\gamma^1 \in V^1,\\
f'''_{a_i^3 a_j^3} &:= c_{a_i^3 a_j^3} && \text{for } a_i^3, a_j^3 \in A^3 \text{ with } i < j,\\
f'''_{a_i^3 a_j^3} &:= 0 && \text{for } a_i^3, a_j^3 \in A^3 \text{ with } i > j,\\
f'''_{a_\alpha^3 v_\gamma^1} &:= c_{a_\alpha^3 v_\gamma^1} && \text{for all } a_\alpha^3 \in A^3,\ v_\gamma^1 \in V^1,\\
f'''_{v_\gamma^1 a_\alpha^3} &:= 0 && \text{for all } a_\alpha^3 \in A^3,\ v_\gamma^1 \in V^1,\\
f'''_{a_i^1 a_\alpha^3} &:= 0 && \text{for all } a_i^1 \in A^1,\ a_\alpha^3 \in A^3,\\
f'''_{a_\alpha^3 a_i^1} &:= c_{a_i^1 a_\alpha^3} && \text{for all } a_i^1 \in A^1,\ a_\alpha^3 \in A^3,\\
f'''_{a_j^2 a_\alpha^3} &:= 0 && \text{for all } a_j^2 \in A^2,\ a_\alpha^3 \in A^3,\\
f'''_{a_\alpha^3 a_j^2} &:= c_{a_j^2 a_\alpha^3} && \text{for all } a_j^2 \in A^2,\ a_\alpha^3 \in A^3,\\
f'''_{v_i^2 v_j^2} &:= c_{v_i^2 v_j^2} && \text{for } v_i^2, v_j^2 \in V^2 \text{ with } i < j,\\
f'''_{v_i^2 v_j^2} &:= 0 && \text{for } v_i^2, v_j^2 \in V^2 \text{ with } i > j,\\
f'''_{a_i^1 v_\beta^2} &:= c_{a_i^1 v_\beta^2} && \text{for all } a_i^1 \in A^1,\ v_\beta^2 \in V^2,\\
f'''_{v_\beta^2 a_i^1} &:= 0 && \text{for all } a_i^1 \in A^1,\ v_\beta^2 \in V^2,\\
f'''_{a_j^2 v_\beta^2} &:= c_{a_j^2 v_\beta^2} && \text{for all } a_j^2 \in A^2,\ v_\beta^2 \in V^2,\\
f'''_{v_\beta^2 a_j^2} &:= 0 && \text{for all } a_j^2 \in A^2,\ v_\beta^2 \in V^2, \qquad (3.159)\\
f'''_{a_\alpha^3 v_\beta^2} &:= c_{a_\alpha^3 v_\beta^2} && \text{for all } a_\alpha^3 \in A^3,\ v_\beta^2 \in V^2,\\
f'''_{v_\beta^2 a_\alpha^3} &:= 0 && \text{for all } a_\alpha^3 \in A^3,\ v_\beta^2 \in V^2,\\
f'''_{v_\gamma^1 v_\beta^2} &:= c_{v_\gamma^1 v_\beta^2} && \text{for all } v_\gamma^1 \in V^1,\ v_\beta^2 \in V^2,\\
f'''_{v_\beta^2 v_\gamma^1} &:= 0 && \text{for all } v_\gamma^1 \in V^1,\ v_\beta^2 \in V^2,\\
f'''_{a_i^1 n} &:= c_{a_i^1 n} && \text{for all } a_i^1 \in A^1,\\
f'''_{n a_i^1} &:= 0 && \text{for all } a_i^1 \in A^1,\\
f'''_{a_j^2 n} &:= c_{a_j^2 n} && \text{for all } a_j^2 \in A^2,\\
f'''_{n a_j^2} &:= 0 && \text{for all } a_j^2 \in A^2,\\
f'''_{v_\gamma^1 n} &:= c_{v_\gamma^1 n} && \text{for all } v_\gamma^1 \in V^1,\\
f'''_{n v_\gamma^1} &:= 0 && \text{for all } v_\gamma^1 \in V^1,\\
f'''_{a_\alpha^3 n} &:= c_{a_\alpha^3 n} && \text{for all } a_\alpha^3 \in A^3,\\
f'''_{n a_\alpha^3} &:= 0 && \text{for all } a_\alpha^3 \in A^3,\\
f'''_{v_\beta^2 n} &:= c_{v_\beta^2 n} && \text{for all } v_\beta^2 \in V^2,\\
f'''_{n v_\beta^2} &:= 0 && \text{for all } v_\beta^2 \in V^2.
\end{aligned}$$

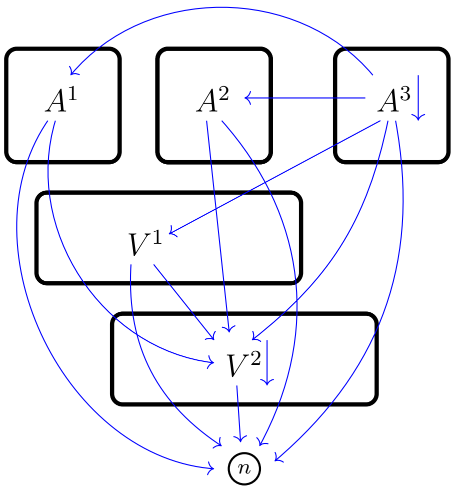

Figure 3.14: Sketch of arcs $(u, w) \in \overset{\leftrightarrow}{E}_n$ with $f'''_{u,w} > 0$.

First we consider the 1-n flow $f' : \overset{\leftrightarrow}{E}_n \to \mathbb{R}$ and check the *flow conservation law*: the amount of flow entering a vertex $a_i^1 \in A^1 \setminus \{a_1^1 = 1\}$, $a_j^2 \in A^2$, $a_\alpha^3 \in A^3$, $v_\gamma^1 \in V^1$ and $v_\beta^2 \in V^2$ should be equal to the amount of flow leaving $a_i^1 \in A^1 \setminus \{a_1^1 = 1\}$, $a_j^2 \in A^2$, $a_\alpha^3 \in A^3$, $v_\gamma^1 \in V^1$ and $v_\beta^2 \in V^2$, respectively.

Note that, from the definition of $c \in \mathbb{R}^{E_n}_{>0}$, see (3.223)-(3.232), we have $c_{a_1^\gamma,*} = c_{a_2^\gamma,*}$ for all $\gamma \in [3]$. Thus, it suffices to calculate $f'(\delta^{out}(a_j^\gamma))$ and $f'(\delta^{in}(a_j^\gamma))$ for $j \geq 2$ due to $f'(\delta^{out}(a_1^\gamma)) = f'(\delta^{out}(a_2^\gamma))$ and $f'(\delta^{in}(a_1^\gamma)) = f'(\delta^{in}(a_2^\gamma))$ for all $\gamma \in [3]$.

In view of (3.154)-(3.155) for f' we have for each $a_i^1 \in A^1 \setminus \{a_1^1 = 1\}$

$$\begin{aligned}
f'(\delta^{out}(a_i^1)) &= \sum_{j=1}^{|A^2|} c_{a_i^1 a_j^2} + \sum_{\gamma=1}^{|V^1|} c_{a_i^1 v_\gamma^1} + \sum_{\alpha=1}^{|A^3|} c_{a_i^1 a_\alpha^3} + \sum_{\beta=1}^{|V^2|} c_{a_i^1 v_\beta^2} + c_{a_i^1 n} + \sum_{j=i+1}^{|A^1|} c_{a_i^1 a_j^1} \\
&= c_{a_i^1 a_1^2} + \sum_{j=2}^{|A^2|} c_{a_i^1 a_j^2} + \sum_{\gamma=1}^{|V^1|} c_{a_i^1 v_\gamma^1} + c_{a_i^1 a_1^3} + \sum_{\alpha=2}^{|A^3|} c_{a_i^1 a_\alpha^3} \\
&\quad + \sum_{\beta=1}^{|V^2|} c_{a_i^1 v_\beta^2} + c_{a_i^1 n} + \sum_{j=i+1}^{|A^1|} c_{a_i^1 a_j^1} \qquad (3.160) \\
&= c_{a_i^1 a_2^2} + \sum_{j=2}^{|A^2|} c_{a_i^1 a_j^2} + \sum_{\gamma=1}^{|V^1|} c_{a_i^1 v_\gamma^1} + c_{a_i^1 a_2^3} + \sum_{\alpha=2}^{|A^3|} c_{a_i^1 a_\alpha^3} \\
&\quad + \sum_{\beta=1}^{|V^2|} c_{a_i^1 v_\beta^2} + c_{a_i^1 n} + \sum_{j=i+1}^{|A^1|} c_{a_i^1 a_j^1}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2^3 \cdot 2^{|A^1|-i} \cdot 2^{|A^2|-2}} + \sum_{j=2}^{|A^2|} \frac{1}{2^3 \cdot 2^{|A^1|-i} \cdot 2^{|A^2|-j}} + \sum_{\gamma=1}^{|V^1|} \frac{1}{2^{\gamma+1} \cdot 2 \cdot 2^{|A^1|-i}} \\
&\quad + \frac{1}{2^3 \cdot 2^{|A^1|-i} \cdot 2^{|V^1|} \cdot 2 \cdot 2^{|A^3|-2}} + \sum_{\alpha=2}^{|A^3|} \frac{1}{2^3 \cdot 2^{|A^1|-i} \cdot 2^{|V^1|} \cdot 2 \cdot 2^{|A^3|-\alpha}} \\
&\quad + \sum_{\beta=1}^{|V^2|} \frac{1}{2^{\beta+1} \cdot 2^{|V^1|+1} \cdot 2 \cdot 2^{|A^1|-i}} + \frac{1}{2^{|V^2|+1} \cdot 2^{|V^1|+1} \cdot 2 \cdot 2^{|A^1|-i}} + \sum_{j=i+1}^{|A^1|} \frac{1}{2 \cdot 2^{j-i}} \\
&= \frac{1}{2^{|A^1|+|A^2|-i+1}} \left(1 + \frac{1}{2^2} \sum_{j=2}^{|A^2|} 2^j \right) + \frac{1}{2^{|A^1|-i+2}} \sum_{\gamma=1}^{|V^1|} \frac{1}{2^\gamma} \\
&\quad + \frac{1}{2^{|A^1|+|V^1|+|A^3|-i+2}} \left(1 + \frac{1}{2^2} \sum_{\alpha=2}^{|A^3|} 2^i \right) + \frac{1}{2^{|A^1|+|V^1|-i+3}} \sum_{\beta=1}^{|V^2|} \frac{1}{2^\beta} \\
&\quad + \frac{1}{2^{|A^1|+|V^1|+|V^2|-i+3}} + \frac{1}{2^{1-i}} \sum_{j=i+1}^{|A^1|} \frac{1}{2^j} \\
&= \frac{1}{2^{|A^1|+|A^2|-i+1}} \left(1 + \frac{2^2(1-2^{|A^2|-1})}{2^2(1-2)} \right) + \frac{1}{2^{|A^1|-i+2}} \cdot \frac{1}{2} \cdot \frac{1-\frac{1}{2^{|V^1|}}}{1-\frac{1}{2}} \\
&\quad + \frac{1}{2^{|A^1|+|V^1|+|A^3|-i+2}} \left(1 + \frac{2^2(1-2^{|A^3|-1})}{2^2(1-2)} \right) + \frac{1}{2^{|A^1|+|V^1|-i+3}} \cdot \frac{1}{2} \cdot \frac{1-\frac{1}{2^{|V^2|}}}{1-\frac{1}{2}} \\
&\quad + \frac{1}{2^{|A^1|+|V^1|+|V^2|-i+3}} + \frac{1}{2^{1-i}} \cdot \frac{1}{2^{i+1}} \cdot \frac{1-\frac{1}{2^{|A^1|-i}}}{1-\frac{1}{2}} \\
&= \frac{2^{|A^2|-1}}{2^{|A^1|+|A^2|-i+1}} + \frac{1}{2^{|A^1|-i+2}} \left(1 - \frac{1}{2^{|V^1|}} \right) + \frac{2^{|A^3|-1}}{2^{|A^1|+|V^1|+|A^3|-i+2}} \\
&\quad + \frac{1}{2^{|A^1|+|V^1|-i+3}} \left(1 - \frac{1}{2^{|V^2|}} \right) + \frac{1}{2^{|A^1|+|V^1|+|V^2|-i+3}} + \frac{1}{2} \left(1 - \frac{1}{2^{|A^1|-i}} \right) \\
&= \cancel{\frac{1}{2^{|A^1|-i+2}}} + \cancel{\frac{1}{2^{|A^1|-i+2}}} - \cancel{\frac{1}{2^{|A^1|+|V^1|-i+2}}} + \cancel{\frac{1}{2^{|A^1|+|V^1|-i+3}}} + \cancel{\frac{1}{2^{|A^1|+|V^1|-i+3}}} \\
&\quad - \xcancel{\frac{1}{2^{|A^1|+|V^1|+|V^2|-i+3}}} + \xcancel{\frac{1}{2^{|A^1|+|V^1|+|V^2|-i+3}}} + \frac{1}{2} - \cancel{\frac{1}{2^{|A^1|-i+1}}} = \frac{1}{2},
\end{aligned} \tag{3.161}
$$

and for each $a_i^1 \in A^1 \setminus \{a_1^1 = 1\}$

$$
f'(\delta^{in}(a_i^1)) = \sum_{j=1}^{i-1} c_{a_i^1 a_j^1} = c_{a_1^1 a_i^1} + \sum_{j=2}^{i-1} c_{a_i^1 a_j^1} = c_{a_2^1 a_i^1} + \sum_{j=2}^{i-1} c_{a_i^1 a_j^1} \tag{3.162}
$$

$$
\begin{aligned}
&= \frac{1}{2 \cdot 2^{i-2}} + \sum_{j=2}^{i-1} \frac{1}{2 \cdot 2^{i-j}} = \frac{1}{2^{i-1}} \left(1 + \frac{1}{2^2} \sum_{j=2}^{i-1} 2^j \right) \\
&= \frac{1}{2^{i-1}} \left(1 + \frac{1}{2^2} \cdot \frac{2^2 \cdot (1 - 2^{i-2})}{1-2} \right) = \frac{1}{2},
\end{aligned} \tag{3.163}
$$

For each $a_j^2 \in A^2$, $j \geq 2$ we obtain

$$
\begin{aligned}
f'(\delta^{out}(a_j^2)) &= \sum_{\alpha=1}^{|A^3|} c_{a_j^2 a_\alpha^3} + \sum_{\gamma=1}^{|V^1|} c_{a_j^2 v_\gamma^1} + \sum_{\beta=1}^{|V^2|} c_{a_j^2 v_\beta^2} + c_{a_j^2 n} \\
&= c_{a_j^2 a_1^3} + \sum_{\alpha=2}^{|A^3|} c_{a_j^2 a_\alpha^3} + \sum_{\gamma=1}^{|V^1|} c_{a_j^2 v_\gamma^1} + \sum_{\beta=1}^{|V^2|} c_{a_j^2 v_\beta^2} + c_{a_j^2 n} \\
&= c_{a_j^2 a_2^3} + \sum_{\alpha=2}^{|A^3|} c_{a_j^2 a_\alpha^3} + \sum_{\gamma=1}^{|V^1|} c_{a_j^2 v_\gamma^1} + \sum_{\beta=1}^{|V^2|} c_{a_j^2 v_\beta^2} + c_{a_j^2 n} \\
&= \frac{1}{2^3 \cdot 2^{|A^2|-j} \cdot 2^{|V^1|} \cdot 2 \cdot 2^{|A^3|-2}} + \sum_{\alpha=2}^{|A^3|} \frac{1}{2^3 \cdot 2^{|A^2|-j} \cdot 2^{|V^1|} \cdot 2 \cdot 2^{|A^3|-\alpha}} \\
&\quad + \sum_{\gamma=1}^{|V^1|} \frac{1}{2^{\gamma+1} \cdot 2 \cdot 2^{|A^2|-j}} + \sum_{\beta=1}^{|V^2|} \frac{1}{2^{\beta+1} \cdot 2^{|V^1|+1} \cdot 2 \cdot 2^{|A^2|-j}} \\
&\quad + \frac{1}{2^{|V^2|+1} \cdot 2^{|V^1|+1} \cdot 2 \cdot 2^{|A^2|-j}} \\
&= \frac{1}{2^{|A^2|+|V^1|+|A^3|-j+2}} \left(1 + \frac{1}{2^2} \cdot \frac{2^2(1 - 2^{|A^3|-1})}{1-2} \right) \\
&\quad + \frac{1}{2^{|A^2|-j+2}} \sum_{\gamma=1}^{|V^1|} \frac{1}{2^\gamma} + \frac{1}{2^{|A^2|+|V^1|-j+3}} \sum_{\beta=1}^{|V^2|} \frac{1}{2^\beta} + \frac{1}{2^{|Av2|+|V^1|+|V^2|-j+3}} \\
&= \frac{2^{|A^3|-1}}{2^{|A^2|+|V^1|+|A^3|-j+2}} + \frac{1}{2^{|A^2|-j+2}} \cdot \frac{\frac{1}{2}(1 - \frac{1}{2^{|V^1|}})}{1 - \frac{1}{2}} \\
&\quad + \frac{1}{2^{|A^2|+|V^1|-j+3}} \cdot \frac{\frac{1}{2}(1 - \frac{1}{2^{|V^2|}})}{1 - \frac{1}{2}} + \frac{1}{2^{|A^2|+|V^1|+|V^2|-j+3}} \\
&= \frac{1}{2^{|A^2|+|V^1|-j+3}} + \frac{1}{2^{|A^2|-j+2}} \left(1 - \frac{1}{2^{|V^1|}} \right) \\
&\quad + \frac{1}{2^{|A^2|+|V^1|-j+3}} \cdot \left(1 - \frac{1}{2^{|V^2|}} \right) + \frac{1}{2^{|A^2|+|V^1|+|V^2|-j+3}}
\end{aligned} \tag{3.164}
$$

$$= \cancel{\frac{1}{2^{|A^2|+|V^1|-j+3}}} + \frac{1}{2^{|A^2|-j+2}} - \cancel{\frac{1}{2^{|A^2|+|V^1|-j+2}}} + \cancel{\frac{1}{2^{|A^2|+|V^1|-j+3}}} - \cancel{\frac{1}{2^{|A^2|+|V^1|+|V^2|-j+3}}} + \cancel{\frac{1}{2^{|A^2|+|V^1|+|V^2|-j+3}}} = \frac{1}{2^{|A^2|-j+2}}, \tag{3.165}$$

$$\begin{aligned} f'(\delta^{in}(a_j^2)) &= \sum_{i=1}^{|A^1|} c_{a_i^1 a_j^2} = c_{a_1^1 a_j^2} + \sum_{i=2}^{|A^1|} c_{a_i^1 a_j^2} = c_{a_2^1 a_j^2} + \sum_{i=2}^{|A^1|} c_{a_i^1 a_j^2} \\ &= \frac{1}{2^3 \cdot 2^{|A^1|-2} \cdot 2^{|A^2|-j}} + \sum_{i=2}^{|A^1|} \frac{1}{2^3 \cdot 2^{|A^1|-i} \cdot 2^{|A^2|-j}} \\ &= \frac{1}{2^{|A^1|+|A^2|-j+1}} \left(1 + \frac{1}{2^2} \sum_{i=2}^{|A^1|} 2^i\right) \\ &= \frac{1}{2^{|A^1|+|A^2|-j+1}} \cdot \left(1 + \frac{2^2(1-2^{|A^1|-1})}{2^2(1-2)}\right) = \frac{2^{|A^1|-1}}{2^{|A^1|+|A^2|-j+1}} = \frac{1}{2^{|A^2|-j+2}}. \end{aligned} \tag{3.166}$$

For each $v_\gamma^1 \in V^1$ we have

$$\begin{aligned} f'(\delta^{out}(v_\gamma^1)) &= \sum_{i=\gamma+1}^{|V^1|} c_{v_\gamma^1 v_i^1} + \sum_{\alpha=1}^{|A^3|} c_{a_\alpha^3 v_\gamma^1} + \sum_{\beta=1}^{|V^2|} c_{v_\gamma^1 v_\beta^2} + c_{v_\gamma^1 n} \\ &= \sum_{i=\gamma+1}^{|V^1|} c_{v_\gamma^1 v_i^1} + c_{a_1^3 v_\gamma^1} + \sum_{\alpha=2}^{|A^3|} c_{a_\alpha^3 v_\gamma^1} + \sum_{\beta=1}^{|V^2|} c_{v_\gamma^1 v_\beta^2} + c_{v_\gamma^1 n} \\ &= \sum_{i=\gamma+1}^{|V^1|} c_{v_\gamma^1 v_i^1} + c_{a_2^3 v_\gamma^1} + \sum_{\alpha=2}^{|A^3|} c_{a_\alpha^3 v_\gamma^1} + \sum_{\beta=1}^{|V^2|} c_{v_\gamma^1 v_\beta^2} + c_{v_\gamma^1 n} \\ &= \sum_{i=\gamma+1}^{|V^1|} \frac{1}{2 \cdot 2^{i-\gamma}} + \frac{1}{2^3 \cdot 2^{|V^1|-\gamma} \cdot 2^{|A^3|-2}} + \sum_{\alpha=2}^{|A^3|} \frac{1}{2^3 \cdot 2^{|V^1|-\gamma} \cdot 2^{|A^3|-\alpha}} \\ &\quad + \sum_{\beta=1}^{|V^2|} \frac{1}{2^{\beta+1} \cdot 2 \cdot 2^{|V^1|-\gamma}} + \frac{1}{2^{|V^2|+1} \cdot 2 \cdot 2^{|V^1|-\gamma}} \\ &= \frac{1}{2^{1-\gamma}} \sum_{i=\gamma+1}^{|V^1|} \frac{1}{2^i} + \frac{1}{2^{|V^1|+|A^3|-\gamma+1}} \left(1 + \frac{1}{2^2} \sum_{\alpha=2}^{|A^3|} 2^\alpha\right) \\ &\quad + \frac{1}{2^{|V^1|-\gamma+2}} \sum_{\beta=1}^{|V^2|} \frac{1}{2^\beta} + \frac{1}{2^{|V^1|+|V^2|-\gamma+2}} \end{aligned} \tag{3.167}$$

$$
\begin{aligned}
&= \frac{1}{2^{1-\gamma}} \cdot \frac{\frac{1}{2^{\gamma+1}}\left(1-\frac{1}{2^{|V^1|-\gamma}}\right)}{1-\frac{1}{2}} + \frac{1}{2^{|V^1|+|A^3|-\gamma+1}}\left(1+\frac{2^2(1-2^{|A^3|-1})}{2^2(1-2)}\right)\\
&\quad + \frac{1}{2^{|V^1|-\gamma+2}} \cdot \frac{\frac{1}{2}\cdot\left(1-\frac{1}{2^{|V^2|}}\right)}{1-\frac{1}{2}} + \frac{1}{2^{|V^1|+|V^2|-\gamma+2}}\\
&= \frac{1}{2} - \frac{1}{2^{|V^1|-\gamma+1}} + \frac{2^{|A^3|-1}}{2^{|V^1|+|A^3|-\gamma+1}}\\
&\quad + \frac{1}{2^{|V^1|-\gamma+2}}\left(1-\frac{1}{2^{|V^2|}}\right) + \frac{1}{2^{|V^1|+|V^2|-\gamma+2}}\\
&= \frac{1}{2} - \cancel{\frac{1}{2^{|V^1|-\gamma+1}}} + \cancel{\frac{1}{2^{|V^1|-\gamma+2}}}\\
&\quad + \cancel{\frac{1}{2^{|V^1|-\gamma+2}}} - \cancel{\frac{1}{2^{|V^1|+|V^2|-\gamma+2}}} + \cancel{\frac{1}{2^{|V^1|+|V^2|-\gamma+2}}} = \frac{1}{2},
\end{aligned}
\tag{3.168}
$$

$$
\begin{aligned}
f'(\delta^{in}(v^1_\gamma)) &= \sum_{i=1}^{|A^1|} c_{a^1_i v^1_\gamma} + \sum_{j=1}^{|A^2|} c_{a^2_j v^1_\gamma} + \sum_{i=1}^{\gamma-1} c_{v^1_i v^1_\gamma}\\
&= c_{a^1_1 v^1_\gamma} + \sum_{i=2}^{|A^1|} c_{a^1_i v^1_\gamma} + c_{a^2_1 v^1_\gamma} + \sum_{j=2}^{|A^2|} c_{a^2_j v^1_\gamma} + \sum_{i=1}^{\gamma-1} c_{v^1_i v^1_\gamma}\\
&= c_{a^1_2 v^1_\gamma} + \sum_{i=2}^{|A^1|} c_{a^1_i v^1_\gamma} + c_{a^2_2 v^1_\gamma} + \sum_{j=2}^{|A^2|} c_{a^2_j v^1_\gamma} + \sum_{i=1}^{\gamma-1} c_{v^1_i v^1_\gamma}\\
&= \frac{1}{2^{\gamma+1}\cdot 2\cdot 2^{|A^1|-2}} + \sum_{i=2}^{|A^1|} \frac{1}{2^{\gamma+1}\cdot 2\cdot 2^{|A^1|-i}}\\
&\quad + \frac{1}{2^{\gamma+1}\cdot 2\cdot 2^{|A^2|-2}} + \sum_{j=2}^{|A^2|} \frac{1}{2^{\gamma+1}\cdot 2\cdot 2^{|A^2|-j}} + \sum_{i=1}^{\gamma-1} \frac{1}{2\cdot 2^{\gamma-i}}\\
&= \frac{1}{2^{|A^1|+\gamma}}\left(1+\frac{1}{2^2}\sum_{i=2}^{|A^1|} 2^i\right) + \frac{1}{2^{|A^2|+\gamma}}\left(1+\frac{1}{2^2}\sum_{j=2}^{|A^2|} 2^j\right) + \frac{1}{2^{\gamma+1}}\sum_{i=1}^{\gamma-1} 2^i\\
&= \frac{1}{2^{|A^1|+\gamma}}\left(1+\frac{1}{2^2}\cdot\frac{2^2(1-2^{|A^1|-1})}{1-2}\right)\\
&\quad + \frac{1}{2^{|A^2|+\gamma}}\left(1+\frac{1}{2^2}\cdot\frac{2^2(1-2^{|A^2|-1})}{1-2}\right) + \frac{1}{2^{\gamma+1}}\cdot\frac{2(1-2^{\gamma-1})}{1-2}\\
&= \frac{2^{|A^1|-1}}{2^{|A^1|+\gamma}} + \frac{2^{|A^2|-1}}{2^{|A^2|+\gamma}} + \frac{2^{\gamma-1}}{2^\gamma} - \frac{1}{2^\gamma} = \cancel{\frac{1}{2^{\gamma+1}}} + \cancel{\frac{1}{2^{\gamma+1}}} + \frac{1}{2} - \cancel{\frac{1}{2^\gamma}} = \frac{1}{2},
\end{aligned}
\tag{3.169}
$$

For each $a^3_\alpha \in A^3$, $\alpha \geq 2$ we have

$$\begin{aligned} f'(\delta^{out}(a^3_\alpha)) &= \sum_{\beta=1}^{|V^2|} c_{a^3_\alpha v^2_\beta} + c_{a^3_\alpha n} = \sum_{\beta=1}^{|V^2|} \frac{1}{2^{\beta+1}\cdot 2\cdot 2^{|A^3|-\alpha}} + \frac{1}{2^{|V^2|+1}\cdot 2\cdot 2^{|A^3|-\alpha}} \\ &= \frac{1}{2^{|A^3|-\alpha+2}} \sum_{\beta=1}^{|V^2|} \frac{1}{2^\beta} + \frac{1}{2^{|V^2|+1}\cdot 2\cdot 2^{|A^3|-\alpha}} \\ &= \frac{1}{2^{|A^3|-\alpha+2}} \cdot \frac{1}{2} \cdot \frac{1-\frac{1}{2^{|V^2|}}}{1-\frac{1}{2}} + \frac{1}{2^{|V^2|+|A^3|-\alpha+2}} \\ &= \frac{1}{2^{|A^3|-\alpha+2}} - \frac{1}{2^{|A^3|+|V^2|-\alpha+2}} + \frac{1}{2^{|A^3|+|V^2|-\alpha+2}} = \frac{1}{2^{|A^3|-\alpha+2}}, \end{aligned} \tag{3.170}$$

$$\begin{aligned} f'(\delta^{in}(a^3_\alpha)) &= \sum_{i=1}^{|A^1|} c_{a^1_i a^3_\alpha} + \sum_{j=1}^{|A^2|} c_{a^2_j a^3_\alpha} + \sum_{\gamma=1}^{|V^1|} c_{a^3_\alpha v^1_\gamma} \\ &= c_{a^1_1 a^3_\alpha} + \sum_{i=2}^{|A^1|} c_{a^1_i a^3_\alpha} + c_{a^2_1 a^3_\alpha} + \sum_{j=2}^{|A^2|} c_{a^2_j a^3_\alpha} + \sum_{\gamma=1}^{|V^1|} c_{a^3_\alpha v^1_\gamma} \\ &= c_{a^1_2 a^3_\alpha} + \sum_{i=2}^{|A^1|} c_{a^1_i a^3_\alpha} + c_{a^2_2 a^3_\alpha} + \sum_{j=2}^{|A^2|} c_{a^2_j a^3_\alpha} + \sum_{\gamma=1}^{|V^1|} c_{a^3_\alpha v^1_\gamma} \\ &= \frac{1}{2^3\cdot 2^{|A^1|-2}\cdot 2^{|V^1|}\cdot 2\cdot 2^{|A^3|-\alpha}} + \sum_{i=2}^{|A^1|} \frac{1}{2^3\cdot 2^{|A^1|-i}\cdot 2^{|V^1|}\cdot 2\cdot 2^{|A^3|-\alpha}} \\ &\quad + \frac{1}{2^3\cdot 2^{|A^2|-2}\cdot 2^{|V^1|}\cdot 2\cdot 2^{|A^3|-\alpha}} + \sum_{j=2}^{|A^2|} \frac{1}{2^3\cdot 2^{|A^2|-j}\cdot 2^{|V^1|}\cdot 2\cdot 2^{|A^3|-\alpha}} \\ &\quad + \sum_{\gamma=1}^{|V^1|} \frac{1}{2^3\cdot 2^{|V^1|-\gamma}\cdot 2^{|A^3|-\alpha}} = \frac{1}{2^{|A^1|+|V^1|+|A^3|-\alpha+2}} \left(1 + \frac{1}{2^2} \sum_{i=2}^{|A^1|} 2^i\right) \\ &\quad + \frac{1}{2^{|A^2|+|V^1|+|A^3|-\alpha+2}} \left(1 + \frac{1}{2^2} \sum_{j=2}^{|A^2|} 2^j\right) + \frac{1}{2^{|V^1|+|A^3|-\alpha+3}} \sum_{\gamma=1}^{|V^1|} 2^\gamma \\ &= \frac{1}{2^{|A^1|+|V^1|+|A^3|-\alpha+2}} \left(1 + \frac{1}{2^2} \cdot \frac{2^2(1-2^{|A^1|-1})}{1-2}\right) \\ &\quad + \frac{1}{2^{|A^2|+|V^1|+|A^3|-\alpha+2}} \left(1 + \frac{1}{2^2} \cdot \frac{2^2(1-2^{|A^2|-1})}{1-2}\right) \\ &\quad + \frac{1}{2^{|V^1|+|A^3|-\alpha+3}} \cdot \frac{2(1-2^{|V^1|})}{1-2} \end{aligned} \tag{3.171}$$

$$\begin{aligned}
&= \frac{2^{|A^1|-1}}{2^{|A^1|+|V^1|+|A^3|-\alpha+2}} + \frac{2^{|A^2|-1}}{2^{|A^2|+|V^1|+|A^3|-\alpha+2}} + \frac{(2^{|V^1|}-1)}{2^{|V^1|+|A^3|-\alpha+2}} \\
&= \cancel{\frac{1}{2^{|V^1|+|A^3|-\alpha+3}}} + \cancel{\frac{1}{2^{|V^1|+|A^3|-\alpha+3}}} + \frac{1}{2^{|A^3|-\alpha+2}} - \cancel{\frac{1}{2^{|V^1|+|A^3|-\alpha+2}}} = \frac{1}{2^{|A^3|-\alpha+2}},
\end{aligned} \tag{3.172}$$

and for each $v^2_\beta \in V^2$ we obtain

$$\begin{aligned}
f'(\delta^{out}(v^2_\beta)) &= \sum_{i=\beta+1}^{|V^2|} c_{v^2_\beta v^2_i} + c_{v^2_\beta n} = \sum_{i=\beta+1}^{|V^2|} \frac{1}{2\cdot 2^{i-\beta}} + \frac{1}{2\cdot 2^{|V^2|-\beta}} \\
&= \frac{1}{2^{1-\beta}} \sum_{i=\beta+1}^{|V^2|} \frac{1}{2^i} + \frac{1}{2\cdot 2^{|V^2|-\beta}} = \frac{1}{2^{1-\beta}} \cdot \frac{1}{2^{\beta+1}} \cdot \frac{1-\frac{1}{2^{|V^2|-\beta}}}{1-\frac{1}{2}} + \frac{1}{2\cdot 2^{|V^2|-\beta}} \\
&= \frac{1}{2}\left(1 - \frac{1}{2^{|V^2|-\beta}}\right) + \frac{1}{2\cdot 2^{|V^2|-\beta}} = \frac{1}{2} - \cancel{\frac{1}{2^{|V^2|-\beta+1}}} + \cancel{\frac{1}{2^{|V^2|-\beta+1}}} = \frac{1}{2},
\end{aligned} \tag{3.173}$$

$$\begin{aligned}
f'(\delta^{in}(v^2_\beta)) &= \sum_{i=1}^{|A^1|} c_{a^1_i v^2_\beta} + \sum_{j=1}^{|A^2|} c_{a^2_j v^2_\beta} + \sum_{\alpha=1}^{|A^3|} c_{a^3_\alpha v^2_\beta} + \sum_{\gamma=1}^{|V^1|} c_{v^1_\gamma v^2_\beta} + \sum_{i=1}^{\beta-1} c_{v^2_i v^2_\beta} \\
&= c_{a^1_1 v^2_\beta} + \sum_{i=2}^{|A^1|} c_{a^1_i v^2_\beta} + c_{a^2_1 v^2_\beta} + \sum_{j=2}^{|A^2|} c_{a^2_j v^2_\beta} \\
&\quad + c_{a^3_1 v^2_\beta} + \sum_{\alpha=2}^{|A^3|} c_{a^3_\alpha v^2_\beta} + \sum_{\gamma=1}^{|V^1|} c_{v^1_\gamma v^2_\beta} + \sum_{i=1}^{\beta-1} c_{v^2_i v^2_\beta} \\
&= c_{a^1_2 v^2_\beta} + \sum_{i=2}^{|A^1|} c_{a^1_i v^2_\beta} + c_{a^2_2 v^2_\beta} + \sum_{j=2}^{|A^2|} c_{a^2_j v^2_\beta} \\
&\quad + c_{a^3_2 v^2_\beta} + \sum_{\alpha=2}^{|A^3|} c_{a^3_\alpha v^2_\beta} + \sum_{\gamma=1}^{|V^1|} c_{v^1_\gamma v^2_\beta} + \sum_{i=1}^{\beta-1} c_{v^2_i v^2_\beta} \\
&= \frac{1}{2^{\beta+1}\cdot 2^{|V^1|+1}\cdot 2\cdot 2^{|A^1|-2}} + \sum_{i=2}^{|A^1|} \frac{1}{2^{\beta+1}\cdot 2^{|V^1|+1}\cdot 2\cdot 2^{|A^1|-i}} \\
&\quad + \frac{1}{2^{\beta+1}\cdot 2^{|V^1|+1}\cdot 2\cdot 2^{|A^2|-2}} + \sum_{j=2}^{|A^2|} \frac{1}{2^{\beta+1}\cdot 2^{|V^1|+1}\cdot 2\cdot 2^{|A^2|-j}} \\
&\quad + \frac{1}{2^{\beta+1}\cdot 2\cdot 2^{|A^3|-2}} + \sum_{\alpha=2}^{|A^3|} \frac{1}{2^{\beta+1}\cdot 2\cdot 2^{|A^3|-\alpha}} \\
&\quad + \sum_{\gamma=1}^{|V^1|} \frac{1}{2^{\beta+1}\cdot 2\cdot 2^{|V^1|-\gamma}} + \sum_{i=1}^{\beta-1} \frac{1}{2\cdot 2^{\beta-i}}
\end{aligned} \tag{3.174}$$

$$
\begin{aligned}
&= \frac{1}{2^{|A^1|+|V^1|+\beta+1}}\left(1+\frac{1}{2^2}\sum_{i=2}^{|A^1|}2^i\right)+\frac{1}{2^{|A^2|+|V^1|+\beta+1}}\left(1+\frac{1}{2^2}\sum_{j=2}^{|A^2|}2^j\right)\\
&\quad+\frac{1}{2^{|A^3|+\beta}}\left(1+\frac{1}{2^2}\sum_{\alpha=2}^{|A^3|}2^\alpha\right)+\frac{1}{2^{|V^1|+\beta+2}}\sum_{\gamma=1}^{|V^1|}2^\gamma+\frac{1}{2^{\beta+1}}\sum_{i=1}^{\beta-1}2^i\\
&= \frac{1}{2^{|A^1|+|V^1|+\beta+1}}\left(1+\frac{1}{2^2}\cdot\frac{2^2(1-2^{|A^1|-1})}{1-2}\right)\\
&\quad+\frac{1}{2^{|A^2|+|V^1|+\beta+1}}\left(1+\frac{1}{2^2}\cdot\frac{2^2(1-2^{|A^2|-1})}{1-2}\right)\\
&\quad+\frac{1}{2^{|A^3|+\beta}}\left(1+\frac{1}{2^2}\cdot\frac{2^2(1-2^{|A^3|-1})}{1-2}\right)\\
&\quad+\frac{1}{2^{|V^1|+\beta+2}}\cdot\frac{2(1-2^{|V^1|})}{1-2}+\frac{1}{2^{\beta+1}}\cdot\frac{2(1-2^{\beta-1})}{1-2}\\
&= \frac{2^{|A^1|-1}}{2^{|A^1|+|V^1|+\beta+1}}+\frac{2^{|A^2|-1}}{2^{|A^2|+|V^1|+\beta+1}}+\frac{2^{|A^3|-1}}{2^{|A^3|+\beta}}+\frac{2^{|V^1|}-1}{2^{|V^1|+\beta+1}}+\frac{2^{\beta-1}-1}{2^\beta}\\
&= \cancel{\frac{1}{2^{|V^1|+\beta+2}}}+\cancel{\frac{1}{2^{|V^1|+\beta+2}}}+\cancel{\frac{1}{2^{\beta+1}}}+\cancel{\frac{1}{2^{\beta+1}}}-\cancel{\frac{1}{2^{|V^1|+\beta+1}}}+\frac{1}{2}-\cancel{\frac{1}{2^\beta}}=\frac{1}{2}.
\end{aligned}
\tag{3.175}
$$

Hence, both conditions of the definition of a flow are satisfied for our 1-n flow f', i.e.

$$
\begin{aligned}
f'_e \geq 0 &\quad \text{for each } e \in \overset{\leftrightarrow}{E}_n,\\
f'(\delta^{out}(v)) = f'(\delta^{in}(v)) &\quad \text{for each } v \in [n]\setminus\{1,n\}.
\end{aligned}
\tag{3.176}
$$

The value of this flow f' is equal to 1 due to

$$
\text{value}(f') = f'(\delta^{out}(1)) = f(\delta^{out}(a_1^1)) = c(\delta(A_1^1)) \overset{\text{Lemma 3.6.2}}{=} 1. \tag{3.177}
$$

Now consider the 2-n flow $f'' : \overset{\leftrightarrow}{E}_n \to \mathbb{R}$ and check again the *flow conservation law*: the amount of flow entering a vertex $a_i^1 \in A^1$, $a_j^2 \in A^2 \setminus \{a_1^2 = 2\}$, $a_\alpha^3 \in A^3$, $v_\gamma^1 \in V^1$ and $v_\beta^2 \in V^2$ should be equal to the amount of flow leaving $a_i^1 \in A^1$, $a_j^2 \in A^2 \setminus \{a_1^2 = 2\}$, $a_\alpha^3 \in A^3$, $v_\gamma^1 \in V^1$ and $v_\beta^2 \in V^2$, respectively.

Note that, from the definition of $c \in \mathbb{R}_{>0}^{E_n}$, see (3.223)-(3.232), we have $c_{a_1^\gamma,*} = c_{a_2^\gamma,*}$ for all $\gamma \in [3]$. Thus, it suffices to calculate $f''(\delta^{out}(a_j^\gamma))$ and $f''(\delta^{in}(a_j^\gamma))$ for $j \geq 2$ due to $f''(\delta^{out}(a_1^\gamma)) = f''(\delta^{out}(a_2^\gamma))$ and $f''(\delta^{in}(a_1^\gamma)) = f''(\delta^{in}(a_2^\gamma))$ for all $\gamma \in [3]$.

In view of (3.156)-(3.157) for f'' we have for each $a_i^1 \in A^1$

$$
\begin{aligned}
f''(\delta^{out}(a_i^1)) &= \sum_{\alpha=1}^{|A^3|} c_{a_i^1 a_\alpha^3} + \sum_{\gamma=1}^{|V^1|} c_{a_i^1 v_\gamma^1} + \sum_{\beta=1}^{|V^2|} c_{a_i^1 v_\beta^2} + c_{a_i^1 n} \\
&= c_{a_i^1 a_1^3} + \sum_{\alpha=2}^{|A^3|} c_{a_i^1 a_\alpha^3} + \sum_{\gamma=1}^{|V^1|} c_{a_i^1 v_\gamma^1} + \sum_{\beta=1}^{|V^2|} c_{a_i^1 v_\beta^2} + c_{a_i^1 n} \\
&= c_{a_i^1 a_2^3} + \sum_{\alpha=2}^{|A^3|} c_{a_i^1 a_\alpha^3} + \sum_{\gamma=1}^{|V^1|} c_{a_i^1 v_\gamma^1} + \sum_{\beta=1}^{|V^2|} c_{a_i^1 v_\beta^2} + c_{a_i^1 n} \\
&= \frac{1}{2^3 \cdot 2^{|A^1|-i} \cdot 2^{|V^1|} \cdot 2 \cdot 2^{|A^3|-2}} + \sum_{\alpha=2}^{|A^3|} \frac{1}{2^3 \cdot 2^{|A^1|-i} \cdot 2^{|V^1|} \cdot 2 \cdot 2^{|A^3|-\alpha}} \\
&\quad + \sum_{\gamma=1}^{|V^1|} \frac{1}{2^{\gamma+1} \cdot 2 \cdot 2^{|A^1|-i}} + \sum_{\beta=1}^{|V^2|} \frac{1}{2^{\beta+1} \cdot 2^{|V^1|+1} \cdot 2 \cdot 2^{|A^1|-i}} \\
&\quad + \frac{1}{2^{|V^2|+1} \cdot 2^{|V^1|+1} \cdot 2 \cdot 2^{|A^1|-i}} \\
&= \frac{1}{2^{|A^1|+|V^1|+|A^3|-i+2}} \left(1 + \frac{2^2(1-2^{|A^3|-1})}{2^2(1-2)}\right) + \frac{1}{2^{|A^1|-i+2}} \sum_{\gamma=1}^{|V^1|} \frac{1}{2^\gamma} \qquad (3.178) \\
&\quad + \frac{1}{2^{|A^1|+|V^1|-i+3}} \sum_{\beta=1}^{|V^2|} \frac{1}{2^\beta} + \frac{1}{2^{|A^1|+|V^1|+|V^2|-i+3}} \\
&= \frac{2^{|A^3|-1}}{2^{|A^1|+|V^1|+|A^3|-i+2}} + \frac{1}{2^{|A^1|-i+2}} \cdot \frac{\frac{1}{2}(1-\frac{1}{2^{|V^1|}})}{1-\frac{1}{2}} \\
&\quad + \frac{1}{2^{|A^1|+|V^1|-i+3}} \cdot \frac{\frac{1}{2} \cdot (1-\frac{1}{2^{|V^2|}})}{1-\frac{1}{2}} + \frac{1}{2^{|A^1|+|V^1|+|V^2|-i+3}} \\
&= \frac{1}{2^{|A^1|+|V^1|-i+3}} + \frac{1}{2^{|A^1|-i+2}} \left(1 - \frac{1}{2^{|V^1|}}\right) \\
&\quad + \frac{1}{2^{|A^1|+|V^1|-i+3}} \left(1 - \frac{1}{2^{|V^2|}}\right) + \frac{1}{2^{|A^1|+|V^1|+|V^2|-i+3}} \\
&= \cancel{\frac{1}{2^{|A^1|+|V^1|-i+3}}} + \frac{1}{2^{|A^2|-j+2}} - \cancel{\frac{1}{2^{|A^1|+|V^1|-i+2}}} \\
&\quad + \cancel{\frac{1}{2^{|A^1|+|V^1|+-i+3}}} - \cancel{\frac{1}{2^{|A^1|+|V^1|+|V^2|-i+3}}} + \cancel{\frac{1}{2^{|A^1|+|V^1|+|V^2|-i+3}}} = \frac{1}{2^{|A^1|-i+2}},
\end{aligned}
$$

$$\begin{aligned}
f''(\delta^{in}(a_i^1)) &= \sum_{j=1}^{|A^2|} c_{a_i^1 a_j^2} = c_{a_i^1 a_1^2} + \sum_{j=2}^{|A^2|} c_{a_i^1 a_j^2} = c_{a_i^1 a_2^2} + \sum_{j=2}^{|A^2|} c_{a_i^1 a_j^2} \\
&= \frac{1}{2^{|A^1|+|A^2|-i+1}} \left(1 + \frac{1}{2^2} \sum_{j=2}^{|A^2|} 2^j \right) \\
&= \frac{1}{2^{|A^1|+|A^2|-i+1}} \cdot \left(1 + \frac{2^2(1-2^{|A^2|-1})}{2^2(1-2)}\right) = \frac{2^{|A^2|-1}}{2^{|A^1|+|A^2|-i+1}} = \frac{1}{2^{|A^1|-i+2}}.
\end{aligned} \tag{3.179}$$

For each $a_j^2 \in A^2 \setminus \{a_1^2 = 2\}$ we have

$$\begin{aligned}
f''(\delta^{out}(a_j^2)) &= \sum_{i=1}^{|A^1|} c_{a_i^1 a_j^2} + \sum_{\gamma=1}^{|V^1|} c_{a_j^2 v_\gamma^1} + \sum_{\alpha=1}^{|A^3|} c_{a_j^2 a_\alpha^3} + \sum_{\beta=1}^{|V^2|} c_{a_j^2 v_\beta^2} + c_{a_j^2 n} + \sum_{i=j+1}^{|A^2|} c_{a_j^2 a_i^2} \\
&= c_{a_1^1 a_j^2} + \sum_{i=2}^{|A^1|} c_{a_i^1 a_j^2} + \sum_{\gamma=1}^{|V^1|} c_{a_j^2 v_\gamma^1} + c_{a_j^2 a_1^3} + \sum_{\alpha=2}^{|A^3|} c_{a_j^2 a_\alpha^3} + \sum_{\beta=1}^{|V^2|} c_{a_j^2 v_\beta^2} + c_{a_j^2 n} + \sum_{i=j+1}^{|A^2|} c_{a_j^2 a_i^2} \\
&= c_{a_2^1 a_j^2} + \sum_{i=2}^{|A^1|} c_{a_i^1 a_j^2} + \sum_{\gamma=1}^{|V^1|} c_{a_j^2 v_\gamma^1} + c_{a_j^2 a_2^3} + \sum_{\alpha=2}^{|A^3|} c_{a_j^2 a_\alpha^3} + \sum_{\beta=1}^{|V^2|} c_{a_j^2 v_\beta^2} + c_{a_j^2 n} + \sum_{i=j+1}^{|A^2|} c_{a_j^2 a_i^2} \\
&= \frac{1}{2^3 \cdot 2^{|A^1|-2} \cdot 2^{|A^2|-j}} + \sum_{i=2}^{|A^1|} \frac{1}{2^3 \cdot 2^{|A^1|-i} \cdot 2^{|A^2|-j}} + \sum_{\gamma=1}^{|V^1|} \frac{1}{2^{\gamma+1} \cdot 2 \cdot 2^{|A^2|-j}} \\
&\quad + \frac{1}{2^3 \cdot 2^{|A^2|-j} \cdot 2^{|V^1|} \cdot 2 \cdot 2^{|A^3|-2}} + \sum_{\alpha=2}^{|A^3|} \frac{1}{2^3 \cdot 2^{|A^2|-j} \cdot 2^{|V^1|} \cdot 2 \cdot 2^{|A^3|-\alpha}} \\
&\quad + \sum_{\beta=1}^{|V^2|} \frac{1}{2^{\beta+1} \cdot 2^{|V^1|+1} \cdot 2 \cdot 2^{|A^2|-j}} + \frac{1}{2^{|V^2|+1} \cdot 2^{|V^1|+1} \cdot 2 \cdot 2^{|A^2|-j}} + \sum_{i=j+1}^{|A^2|} \frac{1}{2 \cdot 2^{i-j}} \\
&= \frac{1}{2^{|A^1|+|A^2|-j+1}} \left(1 + \frac{1}{2^2} \sum_{i=2}^{|A^1|} 2^i \right) + \frac{1}{2^{|A^2|-j+2}} \sum_{\gamma=1}^{|V^1|} \frac{1}{2^\gamma} \\
&\quad + \frac{1}{2^{|A^2|+|V^1|+|A^3|-j+2}} \left(1 + \frac{1}{2^2} \sum_{\alpha=2}^{|A^3|} 2^\alpha \right) + \frac{1}{2^{|A^2|+|V^1|-j+3}} \sum_{\beta=1}^{|V^2|} \frac{1}{2^\beta} \\
&\quad + \frac{1}{2^{|A^2|+|V^1|+|V^2|-j+3}} + \frac{1}{2^{1-j}} \sum_{i=j+1}^{|A^2|} \frac{1}{2^i}
\end{aligned} \tag{3.180}$$

$$
\begin{aligned}
&= \frac{1}{2^{|A^1|+|A^2|-j-2}}\left(1+\frac{2^2(1-2^{|A^1|-1})}{2^2(1-2)}\right)+\frac{1}{2^{|A^2|-j+2}}\cdot\frac{1}{2}\cdot\frac{1-\frac{1}{2^{|V^1|}}}{1-\frac{1}{2}} \\
&\quad+\frac{1}{2^{|A^2|+|V^1|+|A^3|-j+2}}\left(1+\frac{2^2(1-2^{|A^3|-1})}{2^2(1-2)}\right)+\frac{1}{2^{|A^2|+|V^1|-j+3}}\cdot\frac{1}{2}\cdot\frac{1-\frac{1}{2^{|V^2|}}}{1-\frac{1}{2}} \\
&\quad+\frac{1}{2^{|A^2|+|V^1|+|V^2|-j+3}}+\frac{1}{2^{1-j}}\cdot\frac{1}{2^{j+1}}\cdot\frac{1-\frac{1}{2^{|A^2|-j}}}{1-\frac{1}{2}} \\
&= \frac{2^{|A^1|-1}}{2^{|A^1|+|A^2|-j+1}}+\frac{1}{2^{|A^2|-j+2}}\cdot\left(1-\frac{1}{2^{|V^1|}}\right)+\frac{2^{|A^3|-1}}{2^{|A^2|+|V^1|+|A^3|-j+2}} \\
&\quad+\frac{1}{2^{|A^2|+|V^1|-j+3}}\cdot\left(1-\frac{1}{2^{|V^2|}}\right)+\frac{1}{2^{|A^2|+|V^1|+|V^2|-j+3}}+\frac{1}{2}\left(1-\frac{1}{2^{|A^2|-j}}\right) \\
&= \frac{1}{2^{|A^2|-j+2}}+\frac{1}{2^{|A^2|-j+2}}-\frac{1}{2^{|A^2|+|V^1|-j+2}}+\frac{1}{2^{|A^2|+|V^1|-j+3}}+\frac{1}{2^{|A^2|+|V^1|-j+3}} \\
&\quad-\frac{1}{2^{|A^2|+|V^1|+|V^2|-j+3}}+\frac{1}{2^{|A^2|+|V^1|+|V^2|-j+3}}+\frac{1}{2}-\frac{1}{2^{|A^2|-j+1}}=\frac{1}{2},
\end{aligned}
\tag{3.181}
$$

and for each $a_j^2 \in A^2 \setminus \{a_1^2 = 2\}$

$$
\begin{aligned}
f''(\delta^{in}(a_j^2)) &= \sum_{i=1}^{j-1} c_{a_i^2 a_j^2} = c_{a_1^2 a_j^2} + \sum_{i=2}^{j-1} c_{a_i^2 a_j^2} = c_{a_2^2 a_j^2} + \sum_{i=2}^{j-1} c_{a_i^2 a_j^2} \\
&= \frac{1}{2\cdot 2^{j-2}} + \sum_{i=2}^{j-1}\frac{1}{2\cdot 2^{j-i}} = \frac{1}{2^{j-1}}\left(1+\frac{1}{2^2}\sum_{j=2}^{j-1}2^i\right) \\
&= \frac{1}{2^{j-1}}\left(1+\frac{1}{2^2}\cdot\frac{2^2\cdot(1-2^{j-2})}{1-2}\right) = \frac{2^{j-2}}{2^{j-1}} = \frac{1}{2}.
\end{aligned}
\tag{3.182}
$$

For each $v_\gamma^1 \in V^1$ we have

$$
\begin{aligned}
f''(\delta^{out}(v_\gamma^1)) &= \sum_{i=\gamma+1}^{|V^1|} c_{v_\gamma^1 v_i} + \sum_{\alpha=1}^{|A^3|} c_{a_\alpha^3 v_\gamma^1} + \sum_{\beta=1}^{|V^2|} c_{v_\gamma^1 v_\beta^2} + c_{v_\gamma^1 n} \\
&= \sum_{i=\gamma+1}^{|V^1|} c_{v_\gamma^1 v_i} + c_{a_1^3 v_\gamma^1} + \sum_{\alpha=2}^{|A^3|} c_{a_\alpha^3 v_\gamma^1} + \sum_{\beta=1}^{|V^2|} c_{v_\gamma^1 v_\beta^2} + c_{v_\gamma^1 n} \\
&= \sum_{i=\gamma+1}^{|V^1|} c_{v_\gamma^1 v_i} + c_{a_2^3 v_\gamma^1} + \sum_{\alpha=2}^{|A^3|} c_{a_\alpha^3 v_\gamma^1} + \sum_{\beta=1}^{|V^2|} c_{v_\gamma^1 v_\beta^2} + c_{v_\gamma^1 n}
\end{aligned}
\tag{3.183}
$$

$$
\begin{aligned}
&= \sum_{i=\gamma+1}^{|V^1|} \frac{1}{2 \cdot 2^{i-\gamma}} + \frac{1}{2^3 \cdot 2^{|V^1|-\gamma} \cdot 2^{|A^3|-2}} + \sum_{\alpha=2}^{|A^3|} \frac{1}{2^3 \cdot 2^{|V^1|-\gamma} \cdot 2^{|A^3|-\alpha}} \\
&\quad + \sum_{\beta=1}^{|V^2|} \frac{1}{2^{\beta+1} \cdot 2 \cdot 2^{|V^1|-\gamma}} + \frac{1}{2^{|V^2|+1} \cdot 2 \cdot 2^{|V^1|-\gamma}} \\
&= \frac{1}{2^{1-\gamma}} \sum_{i=\gamma+1}^{|V^1|} \frac{1}{2^i} + \frac{1}{2^{|V^1|+|A^3|-\gamma+1}} \left(1 + \frac{1}{2^2} \sum_{\alpha=2}^{|A^3|} 2^{\alpha}\right) \\
&\quad + \frac{1}{2^{|V^1|-\gamma+2}} \sum_{\beta=1}^{|V^2|} \frac{1}{2^{\beta}} + \frac{1}{2^{|V^1|+|V^2|-\gamma+2}} \\
&= \frac{1}{2^{1-\gamma}} \cdot \frac{\frac{1}{2^{\gamma+1}} \cdot (1 - \frac{1}{2^{|V^1|-\gamma}})}{1-\frac{1}{2}} + \frac{1}{2^{|V^1|+|A^3|-\gamma+1}} \left(1 + \frac{2^2(1-2^{|A^3|-1})}{2^2(1-2)}\right) \\
&\quad + \frac{1}{2^{|V^1|-\gamma+2}} \cdot \frac{\frac{1}{2}(1-\frac{1}{2^{|V^2|}})}{1-\frac{1}{2}} + \frac{1}{2^{|V^1|+|V^2|-\gamma+2}} \\
&= \frac{1}{2} - \frac{1}{2^{|V^1|-\gamma+1}} + \frac{2^{|A^3|-1}}{2^{|V^1|+|A^3|-\gamma+1}} + \frac{1}{2^{|V^1|-\gamma+2}}\left(1 - \frac{1}{2^{|V^2|}}\right) + \frac{1}{2^{|V^1|+|V^2|-\gamma+2}} \\
&= \frac{1}{2} - \cancel{\frac{1}{2^{|V^1|-\gamma+1}}} + \cancel{\frac{1}{2^{|V^1|-\gamma+2}}} + \cancel{\frac{1}{2^{|V^1|-\gamma+2}}} - \cancel{\frac{1}{2^{|V^1|+|V^2|-\gamma+2}}} + \cancel{\frac{1}{2^{|V^1|+|V^2|-\gamma+2}}} = \frac{1}{2},
\end{aligned}
\tag{3.184}
$$

$$
\begin{aligned}
f''(\delta^{in}(v^1_\gamma)) &= \sum_{i=1}^{|A^1|} c_{a^1_i v^1_\gamma} + \sum_{j=1}^{|A^2|} c_{a^2_j v^1_\gamma} + \sum_{i=1}^{\gamma-1} c_{v^1_i v^1_\gamma} \\
&= c_{a^1_1 v^1_\gamma} + \sum_{i=2}^{|A^1|} c_{a^1_i v^1_\gamma} + c_{a^2_1 v^1_\gamma} + \sum_{j=2}^{|A^2|} c_{a^2_j v^1_\gamma} + \sum_{i=1}^{\gamma-1} c_{v^1_i v^1_\gamma} \\
&= c_{a^1_2 v^1_\gamma} + \sum_{i=2}^{|A^1|} c_{a^1_i v^1_\gamma} + c_{a^2_2 v^1_\gamma} + \sum_{j=2}^{|A^2|} c_{a^2_j v^1_\gamma} + \sum_{i=1}^{\gamma-1} c_{v^1_i v^1_\gamma} \\
&= \frac{1}{2^{\gamma+1} \cdot 2 \cdot 2^{|A^1|-2}} + \sum_{i=2}^{|A^1|} \frac{1}{2^{\gamma+1} \cdot 2 \cdot 2^{|A^1|-i}} \\
&\quad + \frac{1}{2^{\gamma+1} \cdot 2 \cdot 2^{|A^2|-2}} + \sum_{j=2}^{|A^2|} \frac{1}{2^{\gamma+1} \cdot 2 \cdot 2^{|A^2|-j}} + \sum_{i=1}^{\gamma-1} \frac{1}{2 \cdot 2^{\gamma-i}} \\
&= \frac{1}{2^{|A^1|+\gamma}} \left(1 + \frac{1}{2^2} \sum_{i=2}^{|A^1|} 2^i\right) + \frac{1}{2^{|A^2|+\gamma}} \left(1 + \frac{1}{2^2} \sum_{j=2}^{|A^2|} 2^j\right) + \frac{1}{2^{\gamma+1}} \sum_{i=1}^{\gamma-1} 2^i
\end{aligned}
\tag{3.185}
$$

$$\begin{aligned}
&= \frac{1}{2^{|A^1|+\gamma}} \left(1 + \frac{1}{2^2} \cdot \frac{2^2(1-2^{|A^1|-1})}{1-2}\right) \\
&\quad + \frac{1}{2^{|A^2|+\gamma}} \left(1 + \frac{1}{2^2} \cdot \frac{2^2(1-2^{|A^2|-1})}{1-2}\right) + \frac{1}{2^{\gamma+1}} \cdot \frac{2(1-2^{\gamma-1})}{1-2} \\
&= \frac{2^{|A^1|-1}}{2^{|A^1|+\gamma}} + \frac{2^{|A^2|-1}}{2^{|A^2|+\gamma}} + \frac{2^{\gamma-1}}{2^\gamma} - \frac{1}{2^\gamma} = \frac{1}{2^{\gamma+1}} + \frac{1}{2^{\gamma+1}} + \frac{1}{2} - \frac{1}{2^\gamma} = \frac{1}{2},
\end{aligned} \tag{3.186}$$

for each $a^3_\alpha \in A^3$, $\alpha \geq 2$ we obtain

$$\begin{aligned}
f''(\delta^{out}(a^3_\alpha)) &= \sum_{\beta=1}^{|V^2|} c_{a^3_\alpha v^2_\beta} + c_{a^3_\alpha n} = \sum_{\beta=1}^{|V^2|} \frac{1}{2^{\beta+1} \cdot 2 \cdot 2^{|A^3|-\alpha}} + \frac{1}{2^{|V^2|+1} \cdot 2 \cdot 2^{|A^3|-\alpha}} \\
&= \frac{1}{2^{|A^3|-\alpha+2}} \sum_{\beta=1}^{|V^2|} \frac{1}{2^\beta} + \frac{1}{2^{|V^2|+|A^3|-\alpha+2}} \\
&= \frac{1}{2^{|A^3|-\alpha+2}} \cdot \frac{1}{2} \cdot \frac{1-\frac{1}{2^{|V^2|}}}{1-\frac{1}{2}} + \frac{1}{2^{|V^2|+|A^3|-\alpha+2}} \\
&= \frac{1}{2^{|A^3|-\alpha+2}} - \frac{1}{2^{|A^3|-\alpha+2+|V^2|}} + \frac{1}{2^{|V^2|+2+|A^3|-\alpha}} = \frac{1}{2^{|A^3|-\alpha+2}},
\end{aligned} \tag{3.187}$$

$$\begin{aligned}
f''(\delta^{in}(a^3_\alpha)) &= \sum_{i=1}^{|A^1|} c_{a^1_i a^3_\alpha} + \sum_{j=1}^{|A^2|} c_{a^2_j a^3_\alpha} + \sum_{\gamma=1}^{|V^1|} c_{a^3_\alpha v^1_\gamma} \\
&= c_{a^1_1 a^3_\alpha} + \sum_{i=2}^{|A^1|} c_{a^1_i a^3_\alpha} + c_{a^2_1 a^3_\alpha} + \sum_{j=2}^{|A^2|} c_{a^2_j a^3_\alpha} + \sum_{\gamma=1}^{|V^1|} c_{a^3_\alpha v^1_\gamma} \\
&= c_{a^1_2 a^3_\alpha} + \sum_{i=2}^{|A^1|} c_{a^1_i a^3_\alpha} + c_{a^2_2 a^3_\alpha} + \sum_{j=2}^{|A^2|} c_{a^2_j a^3_\alpha} + \sum_{\gamma=1}^{|V^1|} c_{a^3_\alpha v^1_\gamma} \\
&= \frac{1}{2^3 \cdot 2^{|A^1|-2} \cdot 2^{|V^1|} \cdot 2 \cdot 2^{|A^3|-\alpha}} + \sum_{i=2}^{|A^1|} \frac{1}{2^3 \cdot 2^{|A^1|-i} \cdot 2^{|V^1|} \cdot 2 \cdot 2^{|A^3|-\alpha}} \\
&\quad + \frac{1}{2^3 \cdot 2^{|A^2|-2} \cdot 2^{|V^1|} \cdot 2 \cdot 2^{|A^3|-\alpha}} + \sum_{j=2}^{|A^2|} \frac{1}{2^3 \cdot 2^{|A^2|-j} \cdot 2^{|V^1|} \cdot 2 \cdot 2^{|A^3|-\alpha}} \\
&\quad + \sum_{\gamma=1}^{|V^1|} \frac{1}{2^3 \cdot 2^{|V^1|-\gamma} \cdot 2^{|A^3|-\alpha}}
\end{aligned} \tag{3.188}$$

$$
\begin{aligned}
&= \frac{1}{2^{|A^1|+|V^1|+|A^3|-\alpha+2}}\left(1+\frac{1}{2^2}\sum_{i=2}^{|A^1|}2^i\right) \\
&\quad + \frac{1}{2^{|A^2|+|V^1|+|A^3|-\alpha+2}}\left(1+\frac{1}{2^2}\sum_{j=2}^{|A^2|}2^j\right) + \frac{1}{2^{|V^1|+|A^3|-\alpha+3}}\sum_{\gamma=1}^{|V^1|}2^\gamma \\
&= \frac{1}{2^{|A^1|+|V^1|+|A^3|-\alpha+2}}\left(1+\frac{1}{2^2}\cdot\frac{2^2(1-2^{|A^1|-1})}{1-2}\right) \\
&\quad + \frac{1}{2^{|A^2|+|V^1|+|A^3|-\alpha+2}}\left(1+\frac{1}{2^2}\cdot\frac{2^2(1-2^{|A^2|-1})}{1-2}\right) \qquad (3.189)\\
&\quad + \frac{1}{2^{|V^1|+|A^3|-\alpha+3}}\cdot\frac{2(1-2^{|V^1|})}{1-2} \\
&= \frac{2^{|A^1|-1}}{2^{|A^1|+|V^1|+|A^3|-\alpha+2}} + \frac{2^{|A^2|-1}}{2^{|A^2|+|V^1|+|A^3|-\alpha+2}} + \frac{2^{|V^1|}-1}{2^{|V^1|+|A^3|-\alpha+2}} \\
&= \cancel{\frac{1}{2^{|V^1|+|A^3|-\alpha+3}}} + \cancel{\frac{1}{2^{|V^1|+|A^3|-\alpha+3}}} + \frac{1}{2^{|A^3|-\alpha+2}} - \cancel{\frac{1}{2^{|V^1|+|A^3|-\alpha+2}}} = \frac{1}{2^{|A^3|-\alpha+2}},
\end{aligned}
$$

for each $v_\beta^2 \in V^2$ we have

$$
\begin{aligned}
f''(\delta^{out}(v_\beta^2)) &= \sum_{i=\beta+1}^{|V^2|} c_{v_\beta^2 v_i^2} + c_{v_\beta^2 n} = \sum_{i=\beta+1}^{|V^2|}\frac{1}{2\cdot 2^{i-\beta}} + \frac{1}{2\cdot 2^{|V^2|-\beta}} \\
&= \frac{1}{2^{1-\beta}}\sum_{i=\beta+1}^{|V^2|}\frac{1}{2^i} + \frac{1}{2\cdot 2^{|V^2|-\beta}} = \frac{1}{2^{1-\beta}}\cdot\frac{1}{2^{\beta+1}}\cdot\frac{1-\frac{1}{2^{|V^2|-\beta}}}{1-\frac{1}{2}} + \frac{1}{2\cdot 2^{|V^2|-\beta}} \qquad (3.190)\\
&= \frac{1}{2}\left(1-\frac{1}{2^{|V^2|-\beta}}\right) + \frac{1}{2\cdot 2^{|V^2|-\beta}} = \frac{1}{2} - \cancel{\frac{1}{2^{|V^2|-\beta+1}}} + \cancel{\frac{1}{2^{|V^2|-\beta+1}}} = \frac{1}{2},
\end{aligned}
$$

$$
\begin{aligned}
f''(\delta^{in}(v_\beta^2)) &= \sum_{i=1}^{|A^1|} c_{a_i^1 v_\beta^2} + \sum_{j=1}^{|A^2|} c_{a_j^2 v_\beta^2} + \sum_{\alpha=1}^{|A^3|} c_{a_\alpha^3 v_\beta^2} + \sum_{\gamma=1}^{|V^1|} c_{v_\gamma^1 v_\beta^2} + \sum_{i=1}^{\beta-1} c_{v_i^2 v_\beta^2} \\
&= c_{a_1^1 v_\beta^2} + \sum_{i=2}^{|A^1|} c_{a_i^1 v_\beta^2} + c_{a_1^2 v_\beta^2} + \sum_{j=2}^{|A^2|} c_{a_j^2 v_\beta^2} \qquad (3.191)\\
&\quad + c_{a_1^3 v_\beta^2} + \sum_{\alpha=2}^{|A^3|} c_{a_\alpha^3 v_\beta^2} + \sum_{\gamma=1}^{|V^1|} c_{v_\gamma^1 v_\beta^2} + \sum_{i=1}^{\beta-1} c_{v_i^2 v_\beta^2}
\end{aligned}
$$

$$
\begin{aligned}
&= c_{a^1_2 v^2_\beta} + \sum_{i=2}^{|A^1|} c_{a^1_i v^2_\beta} + c_{a^2_2 v^2_\beta} + \sum_{j=2}^{|A^2|} c_{a^2_j v^2_\beta} + c_{a^3_2 v^2_\beta} + \sum_{\alpha=2}^{|A^3|} c_{a^3_\alpha v^2_\beta} + \sum_{\gamma=1}^{|V^1|} c_{v^1_\gamma v^2_\beta} + \sum_{i=1}^{\beta-1} c_{v^2_i v^2_\beta} \\
&= \frac{1}{2^{\beta+1}\cdot 2^{|V^1|+1}\cdot 2\cdot 2^{|A^1|-2}} + \sum_{i=2}^{|A^1|} \frac{1}{2^{\beta+1}\cdot 2^{|V^1|+1}\cdot 2\cdot 2^{|A^1|-i}} \\
&\quad + \frac{1}{2^{\beta+1}\cdot 2^{|V^1|+1}\cdot 2\cdot 2^{|A^2|-2}} + \sum_{j=2}^{|A^2|} \frac{1}{2^{\beta+1}\cdot 2^{|V^1|+1}\cdot 2\cdot 2^{|A^2|-j}} \\
&\quad + \frac{1}{2^{\beta+1}\cdot 2\cdot 2^{|A^3|-2}} + \sum_{\alpha=2}^{|A^3|} \frac{1}{2^{\beta+1}\cdot 2\cdot 2^{|A^3|-\alpha}} \\
&\quad + \sum_{\gamma=1}^{|V^1|} \frac{1}{2^{\beta+1}\cdot 2\cdot 2^{|V^1|-\gamma}} + \sum_{i=1}^{\beta-1} \frac{1}{2\cdot 2^{\beta-i}} \\
&= \frac{1}{2^{|A^1|+|V^1|+\beta+1}} \left(1 + \frac{1}{2^2}\sum_{i=2}^{|A^1|} 2^i\right) + \frac{1}{2^{|A^2|+|V^1|+\beta+1}} \left(1 + \frac{1}{2^2}\sum_{j=2}^{|A^2|} 2^j\right) \qquad (3.192) \\
&\quad + \frac{1}{2^{|A^3|+\beta}} \left(1 + \frac{1}{2^2}\sum_{\alpha=2}^{|A^3|} 2^\alpha\right) + \frac{1}{2^{|V^1|+\beta+2}} \sum_{\gamma=1}^{|V^1|} 2^\gamma + \frac{1}{2^{\beta+1}} \sum_{i=1}^{\beta-1} 2^i \\
&= \frac{1}{2^{|A^1|+|V^1|+\beta+1}} \left(1 + \frac{2^2(1-2^{|A^1|-1})}{2^2(1-2)}\right) + \frac{1}{2^{|A^2|+|V^1|+\beta+1}} \left(1 + \frac{2^2(1-2^{|A^2|-1})}{2^2(1-2)}\right) \\
&\quad + \frac{1}{2^{|A^3|+\beta}} \left(1 + \frac{2^2(1-2^{|A^3|-1})}{2^2(1-2)}\right) + \frac{1}{2^{|V^1|+\beta+2}} \cdot \frac{2(1-2^{|V^1|})}{1-2} + \frac{2(1-2^{\beta-1})}{2^{\beta+1}(1-2)} \\
&= \frac{2^{|A^1|-1}}{2^{|A^1|+|V^1|+\beta+1}} + \frac{2^{|A^2|-1}}{2^{|A^2|+|V^1|+\beta+1}} + \frac{2^{|A^3|-1}}{2^{|A^3|+\beta}} + \frac{2^{|V^1|}-1}{2^{|V^1|+\beta+1}} + \frac{2^{\beta-1}-1}{2^\beta} \\
&= \cancel{\frac{1}{2^{\beta+|V^1|+2}}} + \cancel{\frac{1}{2^{\beta+|V^1|+2}}} + \cancel{\frac{1}{2^{\beta+1}}} + \cancel{\frac{1}{2^{\beta+1}}} - \cancel{\frac{1}{2^{\beta+|V^1|+1}}} + \frac{1}{2} - \cancel{\frac{1}{2^\beta}} = \frac{1}{2}.
\end{aligned}
$$

Hence, both conditions of the definition of a flow are satisfied for our 2-n flow f'', i.e.

$$
\begin{aligned}
f''_e &\geq 0 && \text{for each } e \in \overset{\leftrightarrow}{E}_n, \\
f''(\delta^{out}(v)) &= f''(\delta^{in}(v)) && \text{for each } v \in [n] \setminus \{2, n\}.
\end{aligned}
\qquad (3.193)
$$

The value of this flow f'' is equal to 1 due to

$$
\text{value}(f'') = f''(\delta^{out}(2)) = f''(\delta^{out}(a^2_1)) = c(\delta(A^2_1)) \overset{\text{Lemma 3.6.2}}{=} 1. \qquad (3.194)
$$

Now consider the 3-n flow $f''' : \overleftrightarrow{E}_n \to \mathbb{R}$ and check again the *flow conservation law*: the amount of flow entering a vertex $a^1_i \in A^1$, $a^2_j \in A^2$, $a^3_\alpha \in A^3 \setminus \{a^3_1 = 3\}$, $v^1_\gamma \in V^1$ and $v^2_\beta \in V^2$ should be equal to the amount of flow leaving $a^1_i \in A^1$, $a^2_j \in A^2$, $a^3_\alpha \in A^3 \setminus \{a^3_1 = 3\}$, $v^1_\gamma \in V^1$ and $v^2_\beta \in V^2$, respectively.

Note that, from the definition of $c \in \mathbb{R}^{E_n}_{>0}$, see (3.223)-(3.232), we have $c_{a^\gamma_1,*} = c_{a^\gamma_2,*}$ for all $\gamma \in [3]$. Thus, it suffices to calculate $f'''(\delta^{out}(a^\gamma_j))$ and $f'''(\delta^{in}(a^\gamma_j))$ for $j \geq 2$ due to $f'''(\delta^{out}(a^\gamma_1)) = f'''(\delta^{out}(a^\gamma_2))$ and $f'''(\delta^{in}(a^\gamma_1)) = f'''(\delta^{in}(a^\gamma_2))$ for all $\gamma \in [3]$.

In view of (3.158)-(3.159) for f''' we have for each $a^1_i \in A^1$, $i \geq 2$

$$\begin{aligned}
f'''(\delta^{out}(a^1_i)) &= \sum_{\beta=1}^{|V^2|} c_{a^1_i v^2_\beta} + c_{a^1_i n} \\
&= \sum_{\beta=1}^{|V^2|} \frac{1}{2^{\beta+1} \cdot 2^{|V^1|+1} \cdot 2 \cdot 2^{|A^1|-i}} + \frac{1}{2^{|V^2|+1} \cdot 2^{|V^1|+1} \cdot 2 \cdot 2^{|A^1|-i}} \\
&= \frac{1}{2^{|A^1|+|V^1|-i+3}} \sum_{\beta=1}^{|V^2|} \frac{1}{2^\beta} + \frac{1}{2^{|A^1|+|V^1|+|V^2|-i+3}} \\
&= \frac{1}{2^{|A^1|+|V^1|-i+3}} \cdot \frac{\frac{1}{2}(1-\frac{1}{2^{|V^2|}})}{1-\frac{1}{2}} + \frac{1}{2^{|A^1|+|V^1|+|V^2|-i+3}} \\
&= \frac{1}{2^{|A^1|+|V^1|-i+3}} \left(1 - \frac{1}{2^{|V^2|}}\right) + \frac{1}{2^{|A^1|+|V^1|+|V^2|-i+3}} \\
&= \frac{1}{2^{|A^1|+|V^1|-i+3}} - \cancel{\frac{1}{2^{|A^1|+|V^1|+|V^2|-i+3}}} + \cancel{\frac{1}{2^{|A^1|+|V^1|+|V^2|-i+3}}} = \frac{1}{2^{|A^1|+|V^1|-i+3}},
\end{aligned} \tag{3.195}$$

$$\begin{aligned}
f'''(\delta^{in}(a^1_i)) &= \sum_{\alpha=1}^{|A^3|} c_{a^1_i a^3_\alpha} = c_{a^1_i a^3_1} + \sum_{\alpha=2}^{|A^3|} c_{a^1_i a^3_\alpha} = c_{a^1_i a^3_2} + \sum_{\alpha=2}^{|A^3|} c_{a^1_i a^3_\alpha} \\
&= \frac{1}{2^3 \cdot 2^{|A^1|-i} \cdot 2^{|V^1|} \cdot 2 \cdot 2^{|A^3|-2}} + \sum_{\alpha=2}^{|A^3|} \frac{1}{2^3 \cdot 2^{|A^1|-i} \cdot 2^{|V^1|} \cdot 2 \cdot 2^{|A^3|-\alpha}} \\
&= \frac{1}{2^{|A^1|+|V^1|+|A^3|-i+2}} \left(1 + \frac{1}{2^2} \sum_{\alpha=2}^{|A^3|} 2^\alpha\right) \\
&= \frac{1}{2^{|A^1|+|V^1|+|A^3|-i+2}} \left(1 + \frac{2^2(1-2^{|A^3|-1})}{2^2(1-2)}\right) = \frac{2^{|A^3|-1}}{2^{|A^1|+|V^1|+|A^3|-i+2}} = \frac{1}{2^{|A^1|-i+|V^1|+3}},
\end{aligned} \tag{3.196}$$

for each $a^2_j \in A^2$, $j \geq 2$ we obtain

$$
\begin{aligned}
f'''(\delta^{out}(a^2_j)) &= \sum_{\beta=1}^{|V^2|} c_{a^2_j v^2_\beta} + c_{a^2_j n} \\
&= \sum_{\beta=1}^{|V^2|} \frac{1}{2^{\beta+1} \cdot 2^{|V^1|+1} \cdot 2 \cdot 2^{|A^2|-j}} + \frac{1}{2^{|V^2|+1} \cdot 2^{|V^1|+1} \cdot 2 \cdot 2^{|A^2|-j}} \\
&= \frac{1}{2^{|A^2|+|V^1|-j+3}} \sum_{\beta=1}^{|V^2|} \frac{1}{2^\beta} + \frac{1}{2^{|A^2|+|V^1|+|V^2|-j+3}} \\
&= \frac{1}{2^{|A^2|+|V^1|-j+3}} \cdot \frac{\frac{1}{2} \cdot (1 - \frac{1}{2^{|V^2|}})}{1 - \frac{1}{2}} + \frac{1}{2^{|A^2|+|V^1|+|V^2|-j+3}} \\
&= \frac{1}{2^{|A^2|+|V^1|-j+3}} \cdot \left(1 - \frac{1}{2^{|V^2|}}\right) + \frac{1}{2^{|A^2|+|V^1|+|V^2|-j+3}} \\
&= \frac{1}{2^{|A^2|+|V^1|-j+3}} - \cancel{\frac{1}{2^{|A^2|+|V^1|+|V^2|-j+3}}} + \cancel{\frac{1}{2^{|A^2|+|V^1|+|V^2|-j+3}}} = \frac{1}{2^{|A^2|+|V^1|-j+3}},
\end{aligned}
\tag{3.197}
$$

$$
\begin{aligned}
f'''(\delta^{in}(a^2_j)) &= \sum_{\alpha=1}^{|A^3|} c_{a^2_j a^3_\alpha} = c_{a^2_j a^3_1} + \sum_{\alpha=2}^{|A^3|} c_{a^2_j a^3_\alpha} = c_{a^2_j a^3_2} + \sum_{\alpha=2}^{|A^3|} c_{a^2_j a^3_\alpha} \\
&= \frac{1}{2^3 \cdot 2^{|A^2|-j} \cdot 2^{|V^1|} \cdot 2 \cdot 2^{|A^3|-2}} + \sum_{\alpha=2}^{|A^3|} \frac{1}{2^3 \cdot 2^{|A^2|-j} \cdot 2^{|V^1|} \cdot 2 \cdot 2^{|A^3|-\alpha}} \\
&= \frac{1}{2^{|A^2|+|V^1|+|A^3|-j+2}} \left(1 + \frac{1}{2^2} \cdot \sum_{\alpha=2}^{|A^3|} 2^\alpha\right) \\
&= \frac{1}{2^{|A^2|+|V^1|+|A^3|-j+2}} \left(1 + \frac{1}{2^2} \cdot \frac{2^2(1 - 2^{|A^3|-1})}{1-2}\right) \\
&= \frac{2^{|A^3|-1}}{2^{|A^2|+|V^1|+|A^3|-j+2}} = \frac{1}{2^{|A^2|+|V^1|-j+3}},
\end{aligned}
\tag{3.198}
$$

for each $v^1_\gamma \in V^1$ we have

$$
\begin{aligned}
f'''(\delta^{out}(v^1_\gamma)) &= \sum_{\beta=1}^{|V^2|} c_{v^1_\gamma v^2_\beta} + c_{v^1_\gamma n} = \sum_{\beta=1}^{|V^2|} \frac{1}{2^{\beta+1} \cdot 2 \cdot 2^{|V^1|-\gamma}} + \frac{1}{2^{|V^2|+1} \cdot 2 \cdot 2^{|V^1|-\gamma}} \\
&= \frac{1}{2^{|V^1|-\gamma+2}} \sum_{\beta=1}^{|V^2|} \frac{1}{2^\beta} + \frac{1}{2^{|V^1|+|V^2|-\gamma+2}}
\end{aligned}
\tag{3.199}
$$

$$
\begin{aligned}
&= \frac{1}{2^{|V^1|-\gamma+2}} \cdot \frac{\frac{1}{2} \cdot (1 - \frac{1}{2^{|V^2|}})}{1 - \frac{1}{2}} + \frac{1}{2^{|V^1|+|V^2|-\gamma+2}} \\
&= \frac{1}{2^{|V^1|-\gamma+2}} \left(1 - \frac{1}{2^{|V^2|}}\right) + \frac{1}{2^{|V^1|+|V^2|-\gamma+2}} \\
&= \frac{1}{2^{|V^1|-\gamma+2}} - \cancel{\frac{1}{2^{|V^1|+|V^2|-\gamma+2}}} + \cancel{\frac{1}{2^{|V^1|+|V^2|-\gamma+2}}} = \frac{1}{2^{|V^1|-\gamma+2}},
\end{aligned}
\tag{3.200}
$$

$$
\begin{aligned}
f'''(\delta^{in}(v^1_\gamma)) &= \sum_{\alpha=1}^{|A^3|} c_{a^3_\alpha v^1_\gamma} = c_{a^3_1 v^1_\gamma} + \sum_{\alpha=2}^{|A^3|} c_{a^3_\alpha v^1_\gamma} = c_{a^3_2 v^1_\gamma} + \sum_{\alpha=2}^{|A^3|} c_{a^3_\alpha v^1_\gamma} \\
&= \frac{1}{2^3 \cdot 2^{|V^1|-\gamma} \cdot 2^{|A^3|-2}} + \sum_{\alpha=2}^{|A^3|} \frac{1}{2^3 \cdot 2^{|V^1|-\gamma} \cdot 2^{|A^3|-\alpha}} \\
&= \frac{1}{2^{|V^1|+|A^3|-\gamma+1}} \left(1 + \frac{1}{2^2} \sum_{\alpha=2}^{|A^3|} 2^\alpha\right) \\
&= \frac{1}{2^{|V^1|+|A^3|-\gamma+1}} \left(1 + \frac{2^2(1 - 2^{|A^3|-1})}{2^2(1-2)}\right) = \frac{2^{|A^3|-1}}{2^{|V^1|+|A^3|-\gamma+1}} = \frac{1}{2^{|V^1|-\gamma+2}}.
\end{aligned}
\tag{3.201}
$$

For each $a^3_\alpha \in A^3 \setminus \{a^3_1 = 3\}$ we have

$$
\begin{aligned}
f'''(\delta^{out}(a^3_\alpha)) = &\sum_{i=1}^{|A^1|} c_{a^1_i a^3_\alpha} + \sum_{j=1}^{|A^2|} c_{a^2_j a^3_\alpha} + \sum_{\gamma=1}^{|V^1|} c_{a^3_\alpha v^1_\gamma} \\
&+ \sum_{\beta=1}^{|V^2|} c_{a^3_\alpha v^2_\beta} + c_{a^3_\alpha n} + \sum_{i=\alpha+1}^{|A^3|} c_{a^3_\alpha a^3_i} \\
= &\, c_{a^1_1 a^3_\alpha} + \sum_{i=2}^{|A^1|} c_{a^1_i a^3_\alpha} + c_{a^2_1 a^3_\alpha} + \sum_{j=2}^{|A^2|} c_{a^2_j a^3_\alpha} + \sum_{\gamma=1}^{|V^1|} c_{a^3_\alpha v^1_\gamma} \\
&+ \sum_{\beta=1}^{|V^2|} c_{a^3_\alpha v^2_\beta} + c_{a^3_\alpha n} + \sum_{i=\alpha+1}^{|A^3|} c_{a^3_\alpha a^3_i} \\
= &\, c_{a^1_2 a^3_\alpha} + \sum_{i=2}^{|A^1|} c_{a^1_i a^3_\alpha} + c_{a^2_2 a^3_\alpha} + \sum_{j=2}^{|A^2|} c_{a^2_j a^3_\alpha} + \sum_{\gamma=1}^{|V^1|} c_{a^3_\alpha v^1_\gamma} \\
&+ \sum_{\beta=1}^{|V^2|} c_{a^3_\alpha v^2_\beta} + c_{a^3_\alpha n} + \sum_{i=\alpha+1}^{|A^3|} c_{a^3_\alpha a^3_i}
\end{aligned}
\tag{3.202}
$$

$$
\begin{aligned}
&= \frac{1}{2^3\cdot 2^{|A^1|-2}\cdot 2^{|V^1|}\cdot 2\cdot 2^{|A^3|-\alpha}} + \sum_{i=2}^{|A^1|}\frac{1}{2^3\cdot 2^{|A^1|-i}\cdot 2^{|V^1|}\cdot 2\cdot 2^{|A^3|-\alpha}}\\
&\quad + \frac{1}{2^3\cdot 2^{|A^2|-2}\cdot 2^{|V^1|}\cdot 2\cdot 2^{|A^3|-\alpha}} + \sum_{j=2}^{|A^2|}\frac{1}{2^3\cdot 2^{|A^2|-j}\cdot 2^{|V^1|}\cdot 2\cdot 2^{|A^3|-\alpha}}\\
&\quad + \sum_{\gamma=1}^{|V^1|}\frac{1}{2^3\cdot 2^{|V^1|-\gamma}\cdot 2^{|A^3|-\alpha}} + \sum_{\beta=1}^{|V^2|}\frac{1}{2^{\beta+1}\cdot 2\cdot 2^{|A^3|-\alpha}}\\
&\quad + \frac{1}{2^{|V^2|+1}\cdot 2\cdot 2^{|A^3|-\alpha}} + \sum_{i=\alpha+1}^{|A^3|}\frac{1}{2\cdot 2^{i-\alpha}}\\
&= \frac{1}{2^{|A^1|+|V^1|+|A^3|-\alpha+2}}\left(1+\frac{1}{2^2}\sum_{i=2}^{|A^1|}2^i\right) + \frac{1}{2^{|A^2|+|V^1|+|A^3|-\alpha+2}}\left(1+\frac{1}{2^2}\sum_{j=2}^{|A^2|}2^j\right)\\
&\quad + \frac{1}{2^{|V^1|+|A^3|-\alpha+3}}\sum_{\gamma=1}^{|V^1|}2^\gamma + \frac{1}{2^{|A^3|-\alpha+2}}\sum_{\beta=1}^{|V^2|}\frac{1}{2^\beta} + \frac{1}{2^{|V^2|+|A^3|-\alpha+2}} + \frac{1}{2^{1-\alpha}}\sum_{i=\alpha+1}^{|A^3|}\frac{1}{2^i}\\
&= \frac{1}{2^{|A^1|+|V^1|+|A^3|-\alpha+2}}\left(1+\frac{1}{2^2}\cdot\frac{2^2(1-2^{|A^1|-1})}{1-2}\right) \qquad (3.203)\\
&\quad + \frac{1}{2^{|A^2|+|V^1|+|A^3|-\alpha+2}}\left(1+\frac{1}{2^2}\cdot\frac{2^2(1-2^{|A^2|-1})}{1-2}\right)\\
&\quad + \frac{1}{2^{|V^1|+|A^3|-\alpha+3}}\cdot\frac{2(1-2^{|V^1|})}{1-2} + \frac{1}{2^{|A^3|-\alpha+2}}\cdot\frac{1}{2}\cdot\frac{1-\frac{1}{2^{|V^2|}}}{1-\frac{1}{2}}\\
&\quad + \frac{1}{2^{|V^2|+|A^3|-\alpha+2}} + \frac{1}{2^{1-\alpha}}\cdot\frac{1}{2^{\alpha+1}}\cdot\frac{1-\frac{1}{2^{|A^3|-\alpha}}}{1-\frac{1}{2}}\\
&= \frac{2^{|A^1|-1}}{2^{|A^1|+|V^1|+|A^3|-\alpha+2}} + \frac{2^{|A^2|-1}}{2^{|A^2|+|V^1|+|A^3|-\alpha+2}} + \frac{2^{|V^1|}-1}{2^{|V^1|+|A^3|-\alpha+2}}\\
&\quad + \frac{1}{2^{|A^3|-\alpha+2}} - \frac{1}{2^{|V^2|+|A^3|-\alpha+2}} + \frac{1}{2^{|V^2|+|A^3|-\alpha+2}} + \frac{1}{2} - \frac{1}{2^{|A^3|-\alpha+1}}\\
&= \cancel{\frac{1}{2^{|V^1|+|A^3|-\alpha+3}}} + \cancel{\frac{1}{2^{|V^1|+|A^3|-\alpha+3}}} + \cancel{\frac{1}{2^{|A^3|-\alpha+2}}} - \cancel{\frac{1}{2^{|V^1|+|A^3|-\alpha+2}}}\\
&\quad + \cancel{\frac{1}{2^{|A^3|-\alpha+2}}} - \cancel{\frac{1}{2^{|V^2|+|A^3|-\alpha+2}}} + \cancel{\frac{1}{2^{|V^2|+|A^3|-\alpha+2}}} + \frac{1}{2} - \cancel{\frac{1}{2^{|A^3|-\alpha+1}}} = \frac{1}{2},
\end{aligned}
$$

and for each $a^3_\alpha \in A^3 \setminus \{a^3_1 = 3\}$ we have

$$\begin{aligned} f'''(\delta^{in}(a^3_\alpha)) &= \sum_{i=1}^{\alpha-1} c_{a^3_i a^3_\alpha} = c_{a^3_1 a^3_\alpha} + \sum_{i=2}^{\alpha-1} c_{a^3_i a^3_\alpha} = c_{a^3_2 a^3_\alpha} + \sum_{i=2}^{\alpha-1} c_{a^3_i a^3_\alpha} \\ &= \frac{1}{2 \cdot 2^{\alpha-2}} + \sum_{i=2}^{\alpha-1} \frac{1}{2 \cdot 2^{\alpha-i}} = \frac{1}{2 \cdot 2^{\alpha-2}} \left(1 + \frac{1}{2^2} \cdot \sum_{i=2}^{\alpha-1} 2^i\right) \\ &= \frac{1}{2 \cdot 2^{\alpha-2}} \left(1 + \frac{1}{2^2} \cdot \frac{2^2(1-2^{\alpha-2})}{1-2}\right) = \frac{2^{\alpha-2}}{2^{\alpha-1}} = \frac{1}{2}, \end{aligned} \tag{3.204}$$

for each $v^2_\beta \in V^2$ we obtain

$$\begin{aligned} f'''(\delta^{out}(v^2_\beta)) &= \sum_{i=\beta+1}^{|V^2|} c_{v^2_\beta v^2_i} + c_{v^2_\beta n} = \sum_{i=\beta+1}^{|V^2|} \frac{1}{2 \cdot 2^{i-\beta}} + \frac{1}{2 \cdot 2^{|V^2|-\beta}} \\ &= \frac{1}{2^{1-\beta}} \sum_{i=\beta+1}^{|V^2|} \frac{1}{2^i} + \frac{1}{2^{|V^2|-\beta+1}} = \frac{1}{2^{1-\beta}} \cdot \frac{1}{2^{\beta+1}} \cdot \frac{1-\frac{1}{2^{|V^2|-\beta}}}{1-\frac{1}{2}} + \frac{1}{2^{|V^2|-\beta+1}} \\ &= \frac{1}{2}\left(1 - \frac{1}{2^{|V^2|-\beta}}\right) + \frac{1}{2^{|V^2|-\beta+1}} = \frac{1}{2} - \cancel{\frac{1}{2^{|V^2|-\beta+1}}} + \cancel{\frac{1}{2^{|V^2|-\beta+1}}} = \frac{1}{2}, \end{aligned} \tag{3.205}$$

$$\begin{aligned} f'''(\delta^{in}(v^2_\beta)) &= \sum_{i=1}^{|A^1|} c_{a^1_i v^2_\beta} + \sum_{j=1}^{|A^2|} c_{a^2_j v^2_\beta} + \sum_{\alpha=1}^{|A^3|} c_{a^3_\alpha v^2_\beta} + \sum_{\gamma=1}^{|V^1|} c_{v^1_\gamma v^2_\beta} + \sum_{i=1}^{\beta-1} c_{v^2_i v^2_\beta} \\ &= c_{a^1_1 v^2_\beta} + \sum_{i=2}^{|A^1|} c_{a^1_i v^2_\beta} + c_{a^2_1 v^2_\beta} + \sum_{j=2}^{|A^2|} c_{a^2_j v^2_\beta} \\ &\quad + c_{a^3_1 v^2_\beta} + \sum_{\alpha=2}^{|A^3|} c_{a^3_\alpha v^2_\beta} + \sum_{\gamma=1}^{|V^1|} c_{v^1_\gamma v^2_\beta} + \sum_{i=1}^{\beta-1} c_{v^2_i v^2_\beta} \\ &= c_{a^1_2 v^2_\beta} + \sum_{i=2}^{|A^1|} c_{a^1_i v^2_\beta} + c_{a^2_2 v^2_\beta} + \sum_{j=2}^{|A^2|} c_{a^2_j v^2_\beta} \\ &\quad + c_{a^3_2 v^2_\beta} + \sum_{\alpha=2}^{|A^3|} c_{a^3_\alpha v^2_\beta} + \sum_{\gamma=1}^{|V^1|} c_{v^1_\gamma v^2_\beta} + \sum_{i=1}^{\beta-1} c_{v^2_i v^2_\beta} \\ &= \frac{1}{2^{\beta+1} \cdot 2^{|V^1|+1} \cdot 2 \cdot 2^{|A^1|-2}} + \sum_{i=2}^{|A^1|} \frac{1}{2^{\beta+1} \cdot 2^{|V^1|+1} \cdot 2 \cdot 2^{|A^1|-i}} \\ &\quad + \frac{1}{2^{\beta+1} \cdot 2^{|V^1|+1} \cdot 2 \cdot 2^{|A^2|-2}} + \sum_{j=2}^{|A^2|} \frac{1}{2^{\beta+1} \cdot 2^{|V^1|+1} \cdot 2 \cdot 2^{|A^2|-j}} \end{aligned} \tag{3.206}$$

$$
\begin{aligned}
&+\frac{1}{2^{\beta+1}\cdot 2\cdot 2^{|A^3|-2}}+\sum_{\alpha=2}^{|A^3|}\frac{1}{2^{\beta+1}\cdot 2\cdot 2^{|A^3|-\alpha}}+\sum_{\gamma=1}^{|V^1|}\frac{1}{2^{\beta+1}\cdot 2\cdot 2^{|V^1|-\gamma}}+\sum_{i=1}^{\beta-1}\frac{1}{2\cdot 2^{\beta-i}}\\
&=\frac{1}{2^{|A^1|+|V^1|+\beta+1}}\left(1+\frac{1}{2^2}\sum_{i=2}^{|A^1|}2^i\right)+\frac{1}{2^{|A^2|+|V^1|+\beta+1}}\left(1+\frac{1}{2^2}\sum_{j=2}^{|A^2|}2^j\right)\\
&+\frac{1}{2^{|A^3|+\beta}}\left(1+\frac{1}{2^2}\sum_{\alpha=2}^{|A^3|}2^\alpha\right)+\frac{1}{2^{|V^1|+\beta+2}}\sum_{\gamma=1}^{|V^1|}2^\gamma+\frac{1}{2^{\beta+1}}\sum_{i=1}^{\beta-1}2^i\\
&=\frac{1}{2^{|A^1|+|V^1|+\beta+1}}\left(1+\frac{2^2(1-2^{|A^1|-1})}{2^2(1-2)}\right)+\frac{1}{2^{|A^2|+|V^1|+\beta+1}}\left(1+\frac{2^2(1-2^{|A^2|-1})}{2^2(1-2)}\right)\\
&+\frac{1}{2^{|A^3|+\beta}}\left(1+\frac{2^2(1-2^{|A^3|-1})}{2^2(1-2)}\right)+\frac{1}{2^{|V^1|+\beta+2}}\cdot\frac{2(1-2^{|V^1|})}{1-2}+\frac{1}{2^{\beta+1}}\cdot\frac{2(1-2^{\beta-1})}{1-2}\\
&=\frac{2^{|A^1|-1}}{2^{|A^1|+|V^1|+\beta+1}}+\frac{2^{|A^2|-1}}{2^{|A^2|+|V^1|+\beta+1}}+\frac{2^{|A^3|-1}}{2^{|A^3|+\beta}}+\frac{2^{|V^1|}-1}{2^{|V^1|+\beta+1}}+\frac{2^{\beta-1}-1}{2^\beta}\\
&=\frac{1}{2^{|V^1|+\beta+2}}+\frac{1}{2^{|V^1|+\beta+2}}+\frac{1}{2^{\beta+1}}+\frac{1}{2^{\beta+1}}-\frac{1}{2^{|V^1|+\beta+1}}+\frac{1}{2}-\frac{1}{2^\beta}=\frac{1}{2}.
\end{aligned}
\tag{3.207}
$$

Hence both conditions of the definition of a flow are satisfied for our 3-n flow f''', i.e.

$$
\begin{aligned}
f'''_e &\geq 0 && \text{for each } e\in \overset{\leftrightarrow}{E}_n,\\
f'''(\delta^{out}(v)) &= f'''(\delta^{in}(v)) && \text{for each } v\in[n]\setminus\{3,n\}.
\end{aligned}
\tag{3.208}
$$

The value of this flow f''' is equal to 1 due to

$$
\text{value}(f''') = f'''(\delta^{out}(3)) = f'''(\delta^{out}(a_1^3)) = c(\delta(A_1^3)) \overset{\text{Lemma 3.6.2}}{=} 1. \tag{3.209}
$$

For each $M\subseteq[n-1]$ with $1\in M$ or $2\in M$ or $3\in M$ we thus have (due to $c\geq\mathbf{0}$)

$$
1\leq f'(\delta^{out}(M))\leq c(\delta(M)), \tag{3.210}
$$

or

$$
1\leq f''(\delta^{out}(M))\leq c(\delta(M)), \tag{3.211}
$$

or

$$
1\leq f'''(\delta^{out}(M))\leq c(\delta(M)), \tag{3.212}
$$

respectively.

Hence Lemma 3.6.2 shows that $\delta(M^*)$ is c-minimal among the [3]-n-cuts for each $M^*\in\mathcal{M}$.

Now we show that all other [3]-n-cuts are not c-minimal.

Let $W \subseteq [n-1]$ such that $W \cap [3] \neq \emptyset$, $W \notin \mathcal{M}$ and $c(\delta(W)) = 1$.

Case 1: $1 \in W$, $2, 3 \notin W$. On the one hand by Lemma 3.4.4 $W \subseteq A^1 \cup A^2$ or $A^1 \cup A^2 \subseteq W$ (where the latter is impossible due to $2 \in A^2 \setminus W$) since $c(\delta(W)) = c(\delta(A^1 \cup A^2)) = 1$ with $1 \in W$ and $1 \in A^1 \cup A^2$. On the other hand by Lemma 3.5.3 $W \cap A^2 = \emptyset$ since $c(\delta(W)) = c(\delta(A^2)) = 1$ with $1 \in W \setminus A^2$ and $2 \in A^2 \setminus W$. Thus $W \subseteq A^1$.

As we have $\{1\} = A^1_1 \subsetneqq A^1_2 \subsetneqq \ldots \subsetneqq A^1_{|A^1|} = A^1$ with $|A^1_{i+1} \setminus A^1_i| = 1$ for all $i \in [|A^1| - 1]$ and $\{1\} \subseteq W \subseteq A^1$ Lemma 3.5.4 implies $W \in \{A^1_1, \ldots, A^1_{|A^1|}\}$ (note that for each i we have $A^1_i \subseteq W$ or $W \subseteq A^1_i$ by Lemma 3.4.4).

Case 2: $2 \in W$, $1, 3 \notin W$. Similarly to case 1 we obtain $W \in \{A^2_1, \ldots, A^2_{|A^2|}\}$.

Case 3: $3 \in W$, $1, 2 \notin W$. On the one hand by Lemma 3.4.4 $W \subseteq A^1 \cup A^2 \cup V^1 \cup A^3$ or $A^1 \cup A^2 \cup V^1 \cup A^3 \subseteq W$ (where the latter is impossible due to $1 \in A^1 \setminus W$, $2 \in A^2 \setminus W$) since $c(\delta(W)) = c(\delta(A^1 \cup A^2 \cup V^1 \cup A^3)) = 1$ with $3 \in W$, $3 \in A^1 \cup A^2 \cup V^1 \cup A^3$. On the other hand by Lemma 3.5.3 $W \cap (A^1 \cup A^2 \cup V^1) = \emptyset$ since $c(\delta(W)) = c(\delta(A^1 \cup A^2 \cup V^1)) = 1$ with $2 \in (A^1 \cup A^2 \cup V^1) \setminus W$, $3 \in W \setminus (A^1 \cup A^2 \cup V^1)$. Thus $W \subseteq A^3$.

As we have $\{3\} = A^3_1 \subsetneqq A^3_2 \subsetneqq \ldots \subsetneqq A^3_{|A^3|} = A^3$ with $|A^3_{i+1} \setminus A^3_i| = 1$ for all $i \in [|A^3| - 1]$ and $\{3\} \subseteq W \subseteq A^3$ Lemma 3.5.4 implies $W \in \{A^3_1, \ldots, A^3_{|A^3|}\}$ (note that for each i we have $A^3_i \subseteq W$ or $W \subseteq A^3_i$ by Lemma 3.4.4).

Case 4: $1, 2 \in W$, $3 \notin W$. On the one hand by Lemma 3.4.4 $W \subseteq A^1$ (impossible due to $1 \in A^1 \setminus W$) or $A^1 \subseteq W$ since $c(\delta(W)) = c(\delta(A^1)) = 1$ with $1 \in W$, $1 \in A^1$ and $W \subseteq A^2$ (impossible due to $2 \in A^2 \setminus W$) or $A^2 \subseteq W$ since $c(\delta(W)) = c(\delta(A^2)) = 1$ with $2 \in W$, $2 \in A^2$. By the definition of A^1 and A^2, see (3.105), $A^1 \cap A^2 = \emptyset$. Thus we have $A^1 \cup A^2 \subseteq W$.

By Lemma 3.4.4 $W \subseteq A^1 \cup A^2 \cup V^1 \cup A^3$ or $A^1 \cup A^2 \cup V^1 \cup A^3 \subseteq W$ (where the latter is impossible due to $3 \in A^3 \setminus W$) since $c(\delta(W)) = c(\delta(A^1 \cup A^2 \cup V^1 \cup A^3)) = 1$ with $1 \in W$ and $1 \in A^1 \cup A^2 \cup V^1 \cup A^3$. By Lemma 3.5.3 $W \cap A^3 = \emptyset$ since $c(\delta(W)) = c(\delta(A^3)) = 1$ with $1 \in W \setminus A^3$ and $3 \in A^3 \setminus W$. Thus $W \subseteq A^1 \cup A^2 \cup V^1$.

By the definition of A^1, A^2 and V^1, V^1_I, $I \in [|V^1|]$, see (3.105)-(3.106), we have

$$\begin{aligned} A^1 \cup A^2 \subsetneqq A^1 \cup A^2 \cup V^1_1 \subsetneqq A^1 \cup A^2 \cup V^1_2 \subsetneqq \ldots \\ \subsetneqq A^1 \cup A^2 \cup V^1_{|V^1|} = A^1 \cup A^2 \cup V^1 \end{aligned} \tag{3.213}$$

with $|(A^1 \cup A^2 \cup V^1_1) \setminus (A^1 \cup A^2)| = 1$ and $|(A^1 \cup A^2 \cup V^1_{i+1}) \setminus (A^1 \cup A^2 \cup V^1_i)| = 1$ for all $i \in [|V^1| - 1]$. Then, as we have $A^1 \cup A^2 \subseteq W \subseteq A^1 \cup A^2 \cup V^1$ Lemma 3.5.4 implies $W \in \{A^1 \cup A^2, A^1 \cup A^2 \cup V_1^1, \dots, A^1 \cup A^2 \cup V_{|V^1|}^1\}$ (note that for each i we have $A^1 \cup A^2 \cup V_i^1 \subseteq W$ or $W \subseteq A^1 \cup A^2 \cup V_i^1$ by Lemma 3.4.4).

Case 5: $1, 3 \in W$, $2 \notin W$. By Lemma 3.5.3 $W \cap (A^1 \cup A^2 \cup V^1) = \emptyset$ since $c(\delta(W)) = c(\delta(A^1 \cup A^2 \cup V^1)) = 1$ with $2 \in (A^1 \cup A^2 \cup V^1) \setminus W$ and $3 \in W \setminus (A^1 \cup A^2 \cup V^1)$, contradicting $1 \in W \cap A^1$.

Case 6: $2, 3 \in W$, $1 \notin W$. Similarly to case 5 we obtain a contradiction to $2 \in W \cap A^2$.

Case 7: $1, 2, 3 \in W$. On the one hand by Lemma 3.4.4 $W \subseteq A^1$ (impossible due to $2 \in W \setminus A^1$) or $A^1 \subseteq W$ holds due to $c(\delta(W)) = c(\delta(A^1)) = 1$ with $1 \in W$, $1 \in A^1$, $W \subseteq A^2$ (impossible due to $1 \in W \setminus A^2$) or $A^2 \subseteq W$ holds due to $c(\delta(W)) = c(\delta(A^2)) = 1$ with $2 \in W$, $2 \in A^2$ and $W \subseteq A^3$ (impossible due to $1 \in W \setminus A^3$) or $A^3 \subseteq W$ holds due to $c(\delta(W)) = c(\delta(A^3)) = 1$ with $3 \in W$, $3 \in A^3$. Hence we have $A^1 \cup A^2 \cup A^3 \subseteq W$.

Let $W_1 := W$ and $W_2 := A^1 \cup A^2 \cup V^1$. By Lemma 3.4.4 $W_1 \subseteq W_2$ (impossible due to $3 \in W_1 \setminus W_2$) or $W_2 \subseteq W_1$ holds due to $c(\delta(W_1)) = c(\delta(W_2)) = 1$ with $1 \in W_1$, $1 \in W_2$. Thus $A^1 \cup A^2 \cup V^1 \cup A^3 \subseteq W$.

By the definition of A^1 A^2, A^3 and V^1, V^2, V_I^2, $I \in [|V^2|]$, see (3.105)-(3.106), we have

$$\begin{aligned} A^1 \cup A^2 \cup V^1 \cup A^3 &\subsetneq A^1 \cup A^2 \cup V^1 \cup A^3 \cup V_1^2 \subsetneq A^1 \cup A^2 \cup V^1 \cup A^3 \cup V_2^2 \subsetneq \dots \\ &\subsetneq A^1 \cup A^2 \cup V^1 \cup A^3 \cup V_{|V^2|}^2 = A^1 \cup A^2 \cup V^1 \cup A^3 \cup V^2 \end{aligned} \tag{3.214}$$

with $|(A^1 \cup A^2 \cup V^1 \cup A^3 \cup V_1^2) \setminus (A^1 \cup A^2 \cup V^1 \cup A^3)| = 1$ and

$$|(A^1 \cup A^2 \cup V^1 \cup A^3 \cup V_{i+1}^2) \setminus (A^1 \cup A^2 \cup V^1 \cup A^3 \cup V_i^2)| = 1$$

for all $i \in [|V^2| - 1]$. Then, as we have

$$A^1 \cup A^2 \cup V^1 \cup A^3 \subseteq W \subseteq A^1 \cup A^2 \cup V^1 \cup A^3 \cup V^2 = [n-1]$$

Lemma 3.5.4 implies

$$W \in \{A^1 \cup A^2 \cup V^1 \cup A^3,\ A^1 \cup A^2 \cup V^1 \cup A^3 \cup V_1^2,\ \dots,\ A^1 \cup A^2 \cup V^1 \cup A^3 \cup V_{|V^2|}^2\}$$

(note that for each i we have $A^1 \cup A^2 \cup V^1 \cup A^3 \cup V_i^2 \subseteq W$ or $W \subseteq A^1 \cup A^2 \cup V^1 \cup A^3 \cup V_i^2$ by Lemma 3.4.4).

Thereby all cuts $\delta(W)$ with $W \subseteq [n-1]$, $W \cap [3] \neq \emptyset$ and $W \notin \mathcal{M}$ are not c-minimal. □

Proposition 3.6.3. *The faces in Proposition 3.6.1 are n-dimensional simplices.*

Proof. Each face in Proposition 3.6.1 has $n+1$ vertices since for $\mathcal{M}$ defined by (3.107) we have

$$|\mathcal{M}| = |A^1| + |A^2| + 1 + |V^1| + |A^3| + 1 + |V^2| = n - 1 + 2 = n + 1. \tag{3.215}$$

Now we show that these $n+1$ vertices are affinely independent and hence all those faces are n-dimensional simplices.

Let $\Theta'_{\mathcal{M}}$ be a submatrix of the cut-incidence matrix of $\mathcal{M}$ (see (1.10)) formed by columns corresponding to the edges $e_{a_1^1,a_2^1}$, $\ldots$, $e_{a^1_{|A^1|-1},a^1_{|A^1|}}$, $e_{a^1_{|A^1|},n}$, $e_{a_1^2,a_2^2}$, $\ldots$, $e_{a^2_{|A^2|-1},a^2_{|A^2|}}$, $e_{a^2_{|A^2|},n}$, $e_{v_1^1,v_2^1}$, $\ldots$, $e_{v^1_{|V^1|-1},v^1_{|V^1|}}$, $e_{v^1_{|V^1|},n}$, $e_{a_1^3,a_2^3}$, $\ldots$, $e_{a^3_{|A^3|-1},a^3_{|A^3|}}$, $e_{a^3_{|A^3|},n}$, $e_{a^2_{|A^2|},a^3_{|A^3|}}$, $e_{v_1^2,v_2^2}$, $\ldots$, $e_{v^2_{|V^2|-1},v^2_{|V^2|}}$, $e_{v^2_{|V^2|},n}$ (see Figure 3.15) with an additional all ones column.

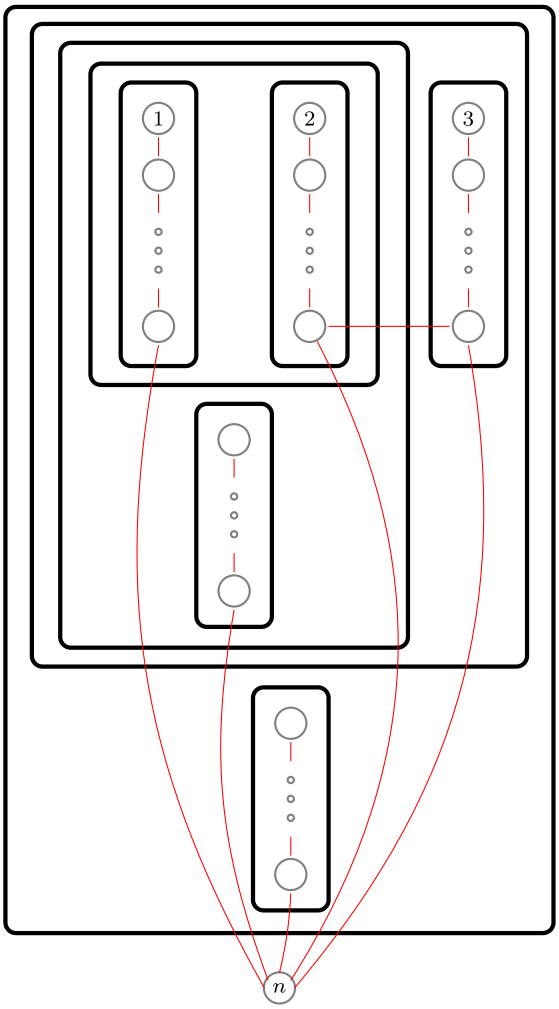

Figure 3.15: Sketch of the n chosen edges used
to create the matrix $\Theta'_{\mathcal{M}}$ in the proof of Proposition 3.6.3.

Hence, $\Theta'_{\mathcal{M}}$ is a $(n+1)\times(n+1)$-matrix whose rows correspond A_1^1, $\ldots$, $A^1_{|A^1|}$, A_1^2, $\ldots$, $A^2_{|A^2|}$, $A^1\cup A^2$, $A^1\cup A^2\cup V_1^1$, $\ldots$, $A^1\cup A^2\cup V^1_{|V^1|}$, A_1^3, $\ldots$, $A^3_{|A^3|}$, $A^1\cup A^2\cup V^1\cup A^3$, $A^1\cup A^2\cup V^1\cup A^3\cup V_1^2$, $\ldots$, $A^1\cup A^2\cup V^1\cup A^3\cup V^2_{|V^2|}$, respectively, i.e. $\Theta'_{\mathcal{M}} :=$

	$e_{a^1_1,a^1_2}$	…	$e_{a^1_{\|A^1\|},n}$	$e_{a^2_1,a^2_2}$	…	$e_{a^2_{\|A^2\|},n}$	$e_{v^1_1,v^1_2}$	…	$e_{v^1_{\|V^1\|},n}$	$e_{a^3_1,a^3_2}$	…	$e_{a^3_{\|A^3\|},n}$	$e_{a^2_{\|A^2\|},a^3_{\|A^3\|}}$	$e_{v^2_1,v^2_2}$	…	$e_{v^2_{\|V^2\|},n}$	
	1	0	0	0	…	0	0	…	0	0	…	0	0	0	…	0	**1**
I.	0	⋱	0	⋮	⋱	⋮	⋮	⋱	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮
	0	0	**1**	0	…	0	0	…	0	0	…	0	0	0	…	0	**1**
	0	…	0	**1**	0	0	0	…	0	0	…	0	0	0	…	0	**1**
II.	⋮	⋱	⋮	0	⋱	0	⋮	⋱	⋮	⋮	⋱	⋮	0	⋮	⋱	⋮	⋮
	0	…	0	0	0	**1**	0	…	0	0	…	0	**1**	0	…	0	**1**
III.	0	0	**1**	0	0	**1**	0	…	0	0	…	0	**1**	0	…	0	**1**
	0	0	**1**	0	0	**1**	**1**	0	0	0	…	0	**1**	0	…	0	**1**
IV.	⋮	⋮	⋮	⋮	⋮	⋮	0	⋱	0	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮
	0	0	**1**	0	0	**1**	0	0	**1**	0	…	0	**1**	0	…	0	**1**
	0	…	0	0	…	0	0	…	0	**1**	0	0	0	0	…	0	**1**
V.	⋮	⋱	⋮	⋮	⋱	⋮	⋮	⋱	⋮	0	⋱	0	0	⋮	⋱	⋮	⋮
	0	…	0	0	…	0	0	…	0	0	0	**1**	**1**	0	…	0	**1**
VI.	0	0	**1**	0	0	**1**	0	0	**1**	0	0	**1**	0	0	…	0	**1**
	0	0	**1**	0	0	**1**	0	0	**1**	0	0	**1**	0	**1**	0	0	**1**
VII.	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	0	⋱	0	⋮	
	0	0	**1**	0	0	**1**	0	0	**1**	0	0	**1**	0	0	0	**1**	**1**

Performing following elementary row operations:

1. multiply the row III by (-1) and add that result to each row of IV to get new rows IV$'$;
2. multiply the row VI by (-1) and add that result to each row of VII to get new rows VII$'$;
3. multiply the sum of the last rows of I and II by (-1) and add that result to the row III to get a new row III$'$;
4. add the row III$'$ to each row of I, II, V and to the row VI to get new rows I$'$, II$'$, V$'$ and VI$'$;
5. multiply the sum of the last rows of I$'$, II$'$, IV$'$ and V$'$ by (-1) and add that result to the row VI$'$ to get a row VI$''$;
6. multiply the row VI$''$ by $1/2$ and add that result to the last rows of II$'$ and V$'$

the $(n+1) \times (n+1)$-matrix $\Theta'_{\mathcal{S}}$ takes the form:

$$\begin{bmatrix}
\mathbf{1} & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & \ddots & 0 & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \mathbf{1} & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & \dots & 0 & \mathbf{1} & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & 0 & \ddots & 0 & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & 0 & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & 0 & 0 & 0 & \mathbf{1} & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \mathbf{-1} \\
0 & \dots & 0 & 0 & \dots & 0 & \mathbf{1} & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & 0 & \ddots & 0 & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & \mathbf{1} & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & 0 & \ddots & 0 & 0 & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & \dots & 0 & 0 \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{-2} & 0 & \dots & 0 & 0 \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & 0 & \ddots & 0 & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \mathbf{1} & 0
\end{bmatrix}$$

As we have $\mathrm{rank}(\Theta'_{\mathcal{M}}) = n+1$ linear independence of rows of $\Theta'_{\mathcal{M}}$ implies affine independence of rows of the cut-incidence matrix of $\mathcal{M}$. Thus, we have for each face in Proposition 3.6.1 that its vertices are affinely independent. □

Since all faces in Proposition 3.6.1 have $n+1$ vertices none of them is contained in any other. Now we show that any further face is contained in one of these faces.

Let $\mathcal{W} \subseteq 2^{[n-1]}$ such that $\{\chi(\delta(W)) : W \in \mathcal{W}\}$ is the set of vertices of some face of the undominated complex of $P_{[3]}(K_n)$. Then there is $c \in \mathbb{R}^E_{>0}$ with minimum [3]-n-cut value 1 such that $c(\delta(W)) = 1$ if and only if $W \in \mathcal{W}$. Let

$$\begin{aligned} \mathcal{W}_1 &:= \{W \in \mathcal{W} : \ 1 \in W, \ 2,3 \notin W\}, \\ \mathcal{W}_2 &:= \{W \in \mathcal{W} : \ 2 \in W, \ 1,3 \notin W\}, \\ \mathcal{W}_3 &:= \{W \in \mathcal{W} : \ 3 \in W, \ 1,2 \notin W\}, \\ \mathcal{W}_{12} &:= \{W \in \mathcal{W} : \ 1,2 \in W, \ 3 \notin W\}, \\ \mathcal{W}_{123} &:= \{W \in \mathcal{W} : \ 1,2,3 \in W\}. \end{aligned} \tag{3.216}$$

By Lemma 3.4.4 all $W \in \mathcal{W}_1$ are nested. Let A^1 be the largest set of them, i.e. $A^1 := W$ such that $W \in \mathcal{W}_1$ and $|W| = \max\{|W| : W \in \mathcal{W}_1\}$. We number the elements in A^1, $A^1 = \{a^1_1 = 1, a^1_2, ..., a^1_{|A^1|}\}$ and define $A^1_I := \{a^1_1 = 1, a^1_2, ..., a^1_I\}$, $I \in [|A^1|]$ such that $\mathcal{W}_1 \subseteq \{A^1_1, A^1_2, \ldots, A^1_{|A^1|}\}$. Similarly, by Lemma 3.4.4 all $W \in \mathcal{W}_2$ are nested. Let A^2 be the largest set of them, i.e. $A^2 := W$ such that $W \in \mathcal{W}_2$ and $|W| = \max\{|W| : W \in \mathcal{W}_2\}$. We number the elements in A^2, $A^2 = \{a^2_1 = 2, a^2_2, ..., a^2_{|A^2|}\}$ and define $A^2_I := \{a^2_1 = 2, a^2_2, ..., a^2_I\}$, $I \in [|A^2|]$ such that $\mathcal{W}_2 \subseteq \{A^2_1, A^2_2, \ldots, A^2_{|A^2|}\}$. Similarly, by Lemma 3.4.4 all $W \in \mathcal{W}_3$ are nested. Let A^3 be the largest set of them, i.e. $A^3 := W$ such that $W \in \mathcal{W}_3$ and $|W| = \max\{|W| : W \in \mathcal{W}_3\}$. We number the elements in A^3, $A^3 = \{a^3_1 = 3, a^3_2, ..., a^3_{|A^3|}\}$ and define $A^3_I := \{a^3_1 = 3, a^3_2, ..., a^3_I\}$, $I \in [|A^3|]$ such that $\mathcal{W}_3 \subseteq \{A^3_1, A^3_2, \ldots, A^3_{|A^3|}\}$.

By Lemma 3.4.4 $A^1 \subseteq W$ for all $W \in \mathcal{W}_{12}$ and $A^2 \subseteq W$ for all $W \in \mathcal{W}_{12}$ what implies that $A^1 \cup A^2 \subseteq W$ for all $W \in \mathcal{W}_{12}$. By Lemma 3.4.4 all $W \in \mathcal{W}_{12}$ are nested. Let V' be the largest set of them, i.e. $V' := W$ such that $W \in \mathcal{W}_{12}$ and $|W| = \max\{|W| : W \in \mathcal{W}_{12}\}$. We choose $V^1 = V' \setminus (A^1 \cup A^2)$ and number the elements in V^1, $V^1 = \{v^1_1, \ldots, v^1_{|V^1|}\}$ and define $V^1_I := \{v^1_1, \ldots, v^1_I\}$, $I \in [|V^1|]$ such that $\mathcal{W}_{12} \subseteq \{A^1 \cup A^2 \cup V^1_1, A^1 \cup A^2 \cup V^1_2, \ldots, A^1 \cup A^2 \cup V^1_{|V^1|}\}$.

By Lemma 3.4.4 $A^1 \cup A^2 \cup V^1 \subseteq W$ for all $W \in \mathcal{W}_{123}$ and $A^3 \subseteq W$ for all $W \in \mathcal{W}_{123}$ what implies that $A^1 \cup A^2 \cup V^1 \cup A^3 \subseteq W$ for all $W \in \mathcal{W}_{123}$. We choose $V^2 = [n-1] \setminus (A^1 \cup A^2 \cup V^1 \cup A^3)$. By Lemma 3.4.4 all $W \in \mathcal{W}_{123}$ are nested. Thus we can number the elements in V^2 such that for each $W \in \mathcal{W}_{123}$ there is some $I \in [|V^2|]$ with $W = A^1 \cup A^2 \cup V^1 \cup A^3 \cup V^2_I$ where $V^2_I = \{v^2_1, \ldots, v^2_I\}$.

Thereby it is shown that $\mathcal{W}$ is a subset of some $\mathcal{M}$, as in (3.107), what complete the proof of the main result of this section:

Theorem 3.6.4. *The undominated complex of the* $[3]$*-*n*-cut polytope* $P_{[3]}(K_n) \subseteq \mathbb{R}^{\binom{n}{2}}$ *for the complete graph on* n *nodes is a pure simplicial complex of dimension* n *whose facets are described in Proposition 3.6.1.*

Remark 3.6.5. *Thus the undominated complex for the* $[3]$*-*n*-cut polytope* $P_{[3]}(K_n) \subseteq \mathbb{R}^{\binom{n}{2}}$ *for the complete graph on* n *nodes consists of* $\frac{(n-2)!}{24}(n^2 - 9n + 32)$ *facets which are* n*-dimensional.*

Proof. We have $(n-4)! \sum_{k=1}^{4} \binom{n-3}{k}$ possibilities to partition the set $[n-1]$ of $n-1$ elements into five parts as in (3.107): $(n-4)!$ possibilities to order the elements in $[n-1] \setminus \{1,2,3\}$ and in each of these cases $\sum_{k=1}^{4} \binom{n-3}{k}$ possibilities to partition the elements into four parts. Thus, as a result we have $(n-4)! \sum_{k=1}^{4} \binom{n-3}{k}$ different sets of $[3]$-n cuts due to:

$$\begin{aligned}
(n-4)! \sum_{k=1}^{4} \binom{n-3}{k} &= (n-4)! \left(\frac{(n-3)!}{4!(n-7)!} + \frac{(n-3)!}{3!(n-6)!} + \frac{(n-3)!}{2(n-5)!} + \frac{(n-3)!}{(n-4)!} \right) \\
&= (n-4)! \frac{(n-3)}{4!} ((n-4)(n-5)(n-6) \\
&\qquad + 4(n-4)(n-5) + 12(n-4) + 24) \qquad (3.217) \\
&= \frac{(n-3)!}{24}(n-2)(n^2 - 9n + 32) \\
&= \frac{(n-2)!}{24}(n^2 - 9n + 32).
\end{aligned}$$

□

3.7 Undominated Complex of the $[\xi]$-n-Cut Polytope

Generalizing the result from Section 3.6 for the $[3]$-n-cut polytope for the complete graph $K_n = ([n], E_n)$ we obtain the result for the $[\xi]$-n-cut polytope, $\xi \in [n-1]$ for the complete graph K_n which will be presented in this section.

Let us introduce some notations used in the current section.

Let the set $[n-1] := \{1, 2, \ldots, n-1\}$ be partitioned into $2\xi - 1$ parts A^γ, $\gamma \in [\xi]$ and V^λ, $\lambda \in [\xi - 1]$ such that they are all pairwise disjoint

$$\begin{aligned}
A^\gamma &:= \{a_1^\gamma = \gamma,\, a_2^\gamma,\, \ldots,\, a_{|A^\gamma|}^\gamma\}, \quad \gamma \in [\xi] \\
V^\lambda &:= \{v_1^\lambda,\, v_2^\lambda,\, \ldots,\, v_{|V^\lambda|}^\lambda\}, \quad \lambda \in [\xi - 1]
\end{aligned} \qquad (3.218)$$

see Figure 3.16. We define the sets A_I^γ, $\gamma \in [\xi]$ and V_I^λ, $\lambda \in [\xi - 1]$ as follows:

$$\begin{aligned}
A_I^\gamma &:= \{a_1^\gamma = \gamma,\, a_2^\gamma,\, \ldots,\, a_I^\gamma\}, \quad I \in [|A^\gamma|] \\
V_I^\lambda &:= \{v_1^\lambda,\, v_2^\lambda,\, \ldots,\, v_I^\lambda\}, \qquad I \in [|V^\lambda|]
\end{aligned} \qquad (3.219)$$

Note that, $A^{\gamma}_{|A^{\gamma}|} = A^{\gamma}$ for all $\gamma \in [\xi]$ and $V^{\lambda}_{|V^{\lambda}|} = V^{\lambda}$ for all $\lambda \in [\xi - 1]$.

Let $\mathcal{M}$ be the set of the following sets:

$$\begin{aligned}\mathcal{M} = \{A^1_I :\ I \in [|A^1|]\ \} &\cup \{A^2_I :\ I \in [|A^2|]\ \} \cup \{A^1 \cup A^2\ \} \\ &\cup \{A^1 \cup A^2 \cup V^1_I :\ I \in [|V^1|]\ \} \cup \{A^3_I :\ I \in [|A^3|]\ \} \\ &\cup \{A^1 \cup A^2 \cup V^1 \cup A^3\} \cup \{A^1 \cup A^2 \cup V^1 \cup A^3 \cup V^2_I :\ I \in [|V^2|]\ \} \\ &\cup \{A^4_I :\ I \in [|A^4|]\ \} \cup \{A^1 \cup A^2 \cup V^1 \cup A^3 \cup V^2 \cup A^4\ \} \cup \ldots \\ &\cup \{A^1 \cup A^2 \cup V^1 \cup A^3 \cup V^2 \cup \ldots \cup A^{\xi} \cup V^{\xi-1}_I :\ I \in [|V^{\xi-1}|]\ \}\end{aligned} \tag{3.220}$$

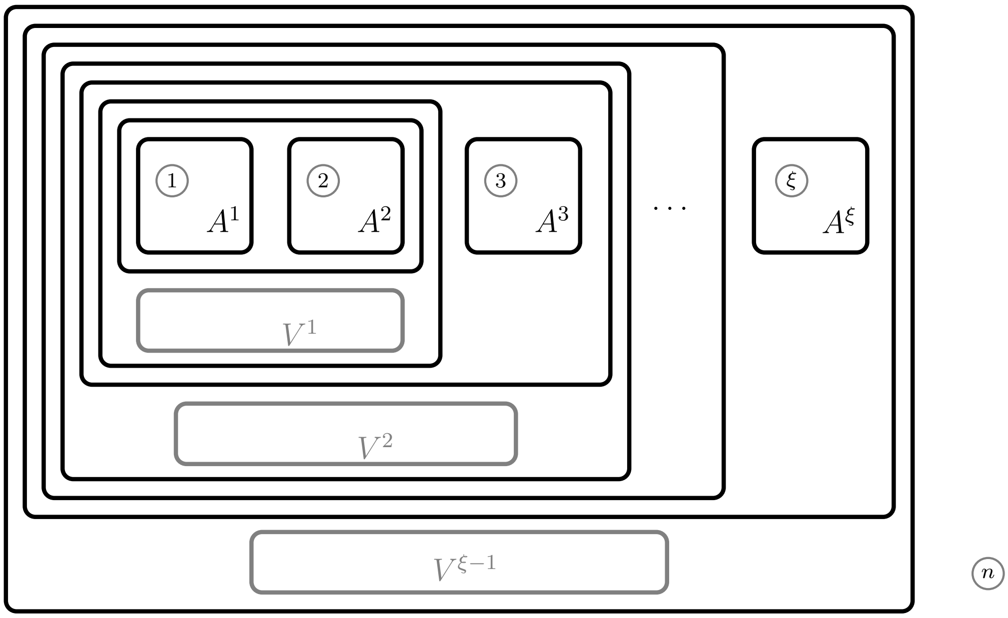

Figure 3.16: Representation of the vertex set $[n]$
in the proof of Proposition 3.7.1.

Proposition 3.7.1. *For each partition of the vertex set $[n-1]$ into $2\xi - 1$ parts A^{γ}, $\gamma \in [\xi]$ and V^{λ}, $\lambda \in [\xi - 1]$ such that they are all pairwise disjoint and defined as in* (3.218)*, see Figure 3.16,*

$$\text{conv}\{\chi(\delta(M)) :\ M \in \mathcal{M}\}, \tag{3.221}$$

where $\mathcal{M}$ is defined by (3.220)*, is a face of the undominated complex of the $[\xi]$-n-cut polytope $P_{[\xi]}(K_n)$ for the complete graph on n nodes $K_n = ([n], E_n)$.*

Proof. We first define some $c \in \mathbb{R}^{E_n}_{>0}$ such that for all $M^* \subseteq [n-1]$ with $M^* \cap [n-1] \neq \emptyset$

$$\min\{c(\delta(M)) : M \subseteq [n-1], M \cap [n-1] \neq \emptyset\} = c(\delta(M^*)) \tag{3.222}$$

holds if and only if $M^* \in \mathcal{M}$.

Let $c \in \mathbb{R}^{E_n}_{>0}$ be defined as follows:

for a_i^γ, $a_j^\gamma \in A^\gamma$, $\gamma \in [\xi]$:

$$c_{a_i^\gamma a_j^\gamma} := \begin{cases} \dfrac{1}{2 \cdot 2^{j-i}}, & 1 < i < j, \\ c_{a_2^\gamma a_j^\gamma}, & i = 1; \end{cases} \tag{3.223}$$

for v_i^λ, $v_j^\lambda \in V^\lambda$, $\lambda \in [\xi - 1]$:

$$c_{v_i^\lambda v_j^\lambda} := \frac{1}{2 \cdot 2^{j-i}}, \quad 1 \leq i < j; \tag{3.224}$$

for $a_i^\lambda \in A^\lambda$, $a_j^\gamma \in A^\gamma$, $\lambda, \gamma \in [\xi]$ with $\lambda < \gamma$:

$$c_{a_i^\lambda a_j^\gamma} := \begin{cases} \dfrac{1}{2^3 \cdot 2^{|A^\lambda|-i} \cdot 2^{|A^\gamma|-j} \prod_{k=\lambda-1}^{\gamma-2} 2^{|V^k|+1}}, & \lambda > 1,\ i, j > 1 \\ \dfrac{1}{2^3 \cdot 2^{|A^\lambda|-i} \cdot 2^{|A^\gamma|-j} \prod_{k=1}^{\gamma-2} 2^{|V^k|+1}}, & \lambda = 1,\ \gamma > 2,\ i, j > 1 \\ \dfrac{1}{2^3 \cdot 2^{|A^\lambda|-i} \cdot 2^{|A^\gamma|-j}}, & \lambda = 1,\ \gamma = 2,\ i, j > 1 \\ c_{a_2^\lambda a_j^\gamma}, & i = 1 \\ c_{a_i^\lambda a_2^\gamma}, & j = 1 \end{cases} \tag{3.225}$$

for $v_i^\lambda \in V^\lambda$, $v_j^\gamma \in V^\gamma$, $\lambda, \gamma \in [\xi - 1]$ with $\lambda < \gamma$:

$$c_{v_i^\lambda v_j^\gamma} := \begin{cases} \dfrac{1}{2^{j-i+1} \prod_{k=\lambda}^{\gamma-1} 2^{|V^k|+1}}, & \gamma > \lambda + 1, \\ \dfrac{1}{2^{j+1} \cdot 2 \cdot 2^{|V^\lambda|-i}}, & \gamma = \lambda + 1; \end{cases} \tag{3.226}$$

for $a_i^\gamma \in A^\gamma$, $\gamma \in [\xi]$, $v_j^\lambda \in V^\lambda$, $\lambda \in [\xi - 1]$ with $\gamma > \lambda + 2$:

$$c_{a_i^\gamma v_j^\lambda} := \begin{cases} \dfrac{1}{2^{2-j} \cdot 2^{|A^\gamma|-i} \prod_{k=\lambda}^{\gamma-2} 2^{|V^k|+1}}, & i > 1, \\ c_{a_2^\gamma v_j^\lambda}, & i = 1; \end{cases} \tag{3.227}$$

for $a_i^\gamma \in A^\gamma$, $\gamma \in [\xi]$, $v_j^\lambda \in V^\lambda$, $\lambda \in [\xi - 1]$ with $\gamma = \lambda + 2$:

$$c_{a_i^\gamma v_j^\lambda} := \begin{cases} \dfrac{1}{2^3 \cdot 2^{|A^\gamma|-i} \cdot 2^{|V^\lambda|-j}}, & i > 1, \\ c_{a_2^\gamma v_j^\lambda}, & i = 1; \end{cases} \tag{3.228}$$

for $a_i^\gamma \in A^\gamma$, $\gamma \in [\xi]$, $v_j^\lambda \in V^\lambda$, $\lambda \in [\xi - 1]$ with $\gamma = \lambda + 1$:

$$c_{a_i^\gamma v_j^\lambda} := \begin{cases} \dfrac{1}{2^{j+1} \cdot 2 \cdot 2^{|A^\gamma|-i}}, & i > 1, \\ c_{a_2^\gamma v_j^\lambda}, & i = 1; \end{cases} \tag{3.229}$$

for $a_i^\gamma \in A^\gamma$, $\gamma \in [\xi]$, $v_j^\lambda \in V^\lambda$, $\lambda \in [\xi - 1]$ with $\gamma < \lambda + 1$:

$$c_{a_i^\gamma v_j^\lambda} := \begin{cases} \dfrac{1}{2^{j+1} \cdot 2 \cdot 2^{|A^\gamma|-i} \prod\limits_{k=\gamma-1}^{\lambda-1} 2^{|V^k|+1}}, & \gamma > 1,\ i > 1, \\ \dfrac{1}{2^{j+1} \cdot 2 \cdot 2^{|A^\gamma|-i} \prod\limits_{k=1}^{\lambda-1} 2^{|V^k|+1}}, & \gamma = 1,\ \lambda > 1,\ i > 1, \\ \dfrac{1}{2^{j+1} \cdot 2 \cdot 2^{|A^\gamma|-i}}, & \gamma = 1,\ \lambda = 1,\ i > 1, \\ c_{a_2^\gamma v_j^\lambda}, & i = 1; \end{cases} \tag{3.230}$$

for $a_i^\gamma \in A^\gamma$, $\gamma \in [\xi]$:

$$c_{a_i^\gamma n} := \begin{cases} \dfrac{1}{2 \cdot 2^{|A^\gamma|-i} \prod\limits_{k=\gamma-1}^{\xi-1} 2^{|V^k|+1}}, & i > 1, \\ c_{a_2^\gamma n}, & i = 1; \end{cases} \tag{3.231}$$

for $v_j^\lambda \in V^\lambda$, $\lambda \in [\xi - 1]$:

$$c_{v_j^\lambda n} := \frac{2^j}{\prod\limits_{k=\lambda}^{\xi-1} 2^{|V^k|+1}}. \tag{3.232}$$

Lemma 3.7.2. *Let $c \in \mathbb{R}^{E_n}_{>0}$ be defined by* (3.223)-(3.232)*. Then the cut $\delta(M^*)$ has value one for all $M^* \in \mathcal{M}$ where $\mathcal{M}$ is defined by* (3.220)*.*

Proof. For brevity, we introduce the following notations:

$$\begin{aligned} C_{A_I^\gamma n} &:= \sum_{i=1}^{I} c_{a_i^\gamma n}, \quad \gamma \in [\xi],\ I \in [|A^\gamma|], \\ C_{V_I^\lambda n} &:= \sum_{j=1}^{I} c_{v_j^\lambda n}, \quad \lambda \in [\xi - 1],\ I \in [|V^\lambda|], \\ C_{A_I^\lambda A^\gamma} &:= \sum_{i=1}^{I} \sum_{j=1}^{|A^\gamma|} c_{a_i^\lambda a_j^\gamma}, \quad \lambda, \gamma \in [\xi],\ \lambda \neq \gamma,\ I \in [|A^\lambda|], \\ C_{A^\lambda A_I^\gamma} &:= \sum_{i=1}^{|A^\lambda|} \sum_{j=1}^{I} c_{a_i^\lambda a_j^\gamma}, \quad \lambda, \gamma \in [\xi],\ \lambda \neq \gamma,\ I \in [|A^\gamma|], \\ C_{V_I^\lambda V^\gamma} &:= \sum_{i=1}^{I} \sum_{j=1}^{|V^\gamma|} c_{v_i^\lambda v_j^\gamma}, \quad \lambda, \gamma \in [\xi - 1],\ \lambda \neq \gamma,\ I \in [|V^\lambda|], \end{aligned} \tag{3.233}$$

$$\begin{aligned}
C_{V^\lambda V_I^\gamma} &:= \sum_{i=1}^{|V^\lambda|} \sum_{j=1}^{I} c_{v_i^\lambda v_j^\gamma}, \quad \lambda, \gamma \in [\xi - 1],\ \lambda \neq \gamma,\ I \in [|V^\gamma|], \\
C_{A_I^\gamma V^\lambda} &:= \sum_{i=1}^{I} \sum_{j=1}^{|V^\lambda|} c_{a_i^\gamma v_j^\lambda}, \quad \gamma \in [\xi],\ \lambda \in [\xi - 1],\ I \in [|A^\gamma|], \\
C_{A^\gamma V_I^\lambda} &:= \sum_{i=1}^{|A^\gamma|} \sum_{j=1}^{I} c_{a_i^\gamma v_j^\lambda}, \quad \gamma \in [\xi],\ \lambda \in [\xi - 1],\ I \in [|V^\lambda|].
\end{aligned} \tag{3.234}$$

Consider all possible cases for a set $M^* \in \mathcal{M}$.

Case 1: $M^* = A_I^\gamma$ for $I \in [|A^\gamma|]$ and $\gamma \in [\xi]$.

$$\begin{aligned}
c(\delta(A_I^\gamma)) = \underbrace{\sum_{i=1}^{I} \sum_{j=I+1}^{|A^\gamma|} c_{a_i^\gamma a_j^\gamma}}_{\Sigma_1} + \underbrace{\sum_{\lambda=1}^{\gamma-1} C_{A^\lambda A_I^\gamma}}_{\Sigma_2} + \underbrace{\sum_{\lambda=\gamma+1}^{\xi} C_{A_I^\gamma A^\lambda}}_{\Sigma_3} \\
+ \underbrace{\sum_{\lambda=1}^{\xi-2} C_{A_I^\gamma V^\lambda}}_{\Sigma_4} + C_{A_I^\gamma V^{\xi-1}} + C_{A_I^\gamma n}.
\end{aligned} \tag{3.235}$$

First, we reproduce the results obtained in Appendix B for the terms on the right-hand side of equation (3.235). From Lemma B.0.2 we have

$$\begin{aligned}
\Sigma_2 = \sum_{\lambda=1}^{\gamma-1} C_{A^\lambda A_I^\gamma} &= \frac{2^I}{2^{|A^\gamma|}} \sum_{\lambda=1}^{\gamma-1} C_{A^\lambda A^\gamma} \\
\Sigma_3 = \sum_{\lambda=\gamma+1}^{\xi} C_{A_I^\gamma A^\lambda} &= \frac{2^I}{2^{|A^\gamma|}} \sum_{\lambda=\gamma+1}^{\xi} C_{A^\lambda A^\gamma} \\
C_{A^\gamma A^{\gamma+1}} &\overset{(B.11)}{=} \frac{1}{2 \cdot 2^{|V^{\gamma-1}|+1}}, \quad 1 < \gamma < \xi \\
C_{A^1 A^2} &= \frac{1}{2}
\end{aligned} \tag{3.236}$$

Lemma B.0.5 implies

$$\Sigma_4 \begin{cases}
\sum_{\lambda=1}^{\gamma-3} C_{A_I^\gamma V^\lambda} &= \frac{2^I}{2^{|A^\gamma|}} \sum_{\lambda=1}^{\gamma-3} (C_{A^{\lambda+2} A^\gamma} - 2C_{A^{\lambda+1} A^\gamma}), \\
C_{A_I^\gamma V^{\gamma-2}} &= \frac{2^{I-1}}{2^{|A^\gamma|}} \left(1 - \frac{1}{2^{|V^{\gamma-2}|}}\right), \quad 2 < \gamma \leq \xi \\
C_{A_I^\gamma V^{\gamma-1}} &= \frac{2^{I-1}}{2^{|A^\gamma|}} \left(1 - \frac{1}{2^{|V^{\gamma-1}|}}\right), \quad 1 < \gamma \leq \xi \\
\sum_{\lambda=\gamma}^{\xi-2} C_{A_I^\gamma V^\lambda} &= \frac{2^I}{2^{|A^\gamma|}} \sum_{\lambda=\gamma}^{\xi-2} (C_{A^\gamma A^{\lambda+1}} - 2C_{A^\gamma A^{\lambda+2}})
\end{cases} \tag{3.237}$$

From (3.64), see the proof of Lemma 3.5.2, we obtain

$$\Sigma_1 = \sum_{i=1}^{I} \sum_{j=I+1}^{|A^\gamma|} c_{a_i^\gamma a_j^\gamma} = 1 - \frac{2^I}{2^{|A^\gamma|}}. \tag{3.238}$$

Thus, using the relations (3.236)-(3.238) equation (3.235) with $\gamma > 2$ takes the form

$$\begin{aligned}
c(\delta(A_I^\gamma)) &= 1 - \frac{2^I}{2^{|A^\gamma|}} + \frac{2^I}{2^{|A^\gamma|}} \sum_{\lambda=1}^{\gamma-1} C_{A^\lambda A^\gamma} + \frac{2^I}{2^{|A^\gamma|}} \sum_{\lambda=\gamma+1}^{\xi} C_{A^\lambda A^\gamma} \\
&\quad + \frac{2^I}{2^{|A^\gamma|}} \sum_{\lambda=1}^{\gamma-3} (C_{A^{\lambda+2} A^\gamma} - 2C_{A^{\lambda+1} A^\gamma}) + \frac{2^{I-1}}{2^{|A^\gamma|}} \left(1 - \frac{1}{2^{|V^{\gamma-2}|}}\right) \\
&\quad + \frac{2^{I-1}}{2^{|A^\gamma|}} \left(1 - \frac{1}{2^{|V^{\gamma-1}|}}\right) + \frac{2^I}{2^{|A^\gamma|}} \sum_{\lambda=\gamma}^{\xi-2} (C_{A^\gamma A^{\lambda+1}} - 2C_{A^\gamma A^{\lambda+2}}) \\
&\quad + C_{A_I^\gamma V^{\xi-1}} + C_{A_I^\gamma n} \\
&= 1 - \cancel{\frac{2^I}{2^{|A^\gamma|}}} + \frac{2^I}{2^{|A^\gamma|}} \sum_{\lambda=1}^{\gamma-1} C_{A^\lambda A^\gamma} + \frac{2^I}{2^{|A^\gamma|}} \sum_{\lambda=\gamma+1}^{\xi} C_{A^\lambda A^\gamma} \\
&\quad + \frac{2^I}{2^{|A^\gamma|}} \sum_{\lambda=3}^{\gamma-1} C_{A^\lambda A^\gamma} - 2\frac{2^I}{2^{|A^\gamma|}} \sum_{\lambda=2}^{\gamma-2} C_{A^\lambda A^\gamma} + \cancel{\frac{2^{I-1}}{2^{|A^\gamma|}}} - \frac{2^{I-1}}{2^{|A^\gamma|}} \cdot \frac{1}{2^{|V^{\gamma-2}|}} \\
&\quad + \cancel{\frac{2^{I-1}}{2^{|A^\gamma|}}} - \frac{2^{I-1}}{2^{|A^\gamma|}} \cdot \frac{1}{2^{|V^{\gamma-1}|}} + \frac{2^I}{2^{|A^\gamma|}} \sum_{\lambda=\gamma+1}^{\xi-1} C_{A^\gamma A^\lambda} - 2\frac{2^I}{2^{|A^\gamma|}} \sum_{\lambda=\gamma+2}^{\xi} C_{A^\gamma A^\lambda} \\
&\quad + C_{A_I^\gamma V^{\xi-1}} + C_{A_I^\gamma n} \\
&= 1 - \frac{2^{I-1}}{2^{|A^\gamma|}} \cdot \frac{1}{2^{|V^{\gamma-2}|}} - \frac{2^{I-1}}{2^{|A^\gamma|}} \cdot \frac{1}{2^{|V^{\gamma-1}|}} \\
&\quad + \frac{2^I}{2^{|A^\gamma|}} (\cancel{C_{A_1 A^\gamma}} - \cancel{C_{A_2 A^\gamma}} + 2C_{A^{\gamma-1} A^\gamma}) + \frac{2^I}{2^{|A^\gamma|}} (2C_{A^\gamma A^{\gamma+1}} - C_{A^\gamma A^\xi}) \\
&\quad + C_{A_I^\gamma V^{\xi-1}} + C_{A_I^\gamma n} \\
&= 1 - \cancel{\frac{2^{I-1}}{2^{|A^\gamma|}} \cdot \frac{1}{2^{|V^{\gamma-2}|}}} - \cancel{\frac{2^{I-1}}{2^{|A^\gamma|}} \cdot \frac{1}{2^{|V^{\gamma-1}|}}} + \cancel{\frac{2^I}{2^{|A^\gamma|}} \cdot 2 \cdot \frac{1}{2 \cdot 2^{|V^{\gamma-2}|+1}}} \\
&\quad + \cancel{\frac{2^I}{2^{|A^\gamma|}} \cdot 2 \cdot \frac{1}{2 \cdot 2^{|V^{\gamma-1}|+1}}} - C_{A_I^\gamma A^\xi} + C_{A_I^\gamma V^{\xi-1}} + C_{A_I^\gamma n} \\
&= 1 - C_{A_I^\gamma A^\xi} + C_{A_I^\gamma V^{\xi-1}} + C_{A_I^\gamma n} \overset{\text{Lemma } B.0.6}{=} 1.
\end{aligned} \tag{3.239}$$

For expression (3.235) with $\gamma = 2$ we deduce

$$\begin{aligned}
c(\delta(A_I^2)) &= 1 - \frac{2^I}{2^{|A^2|}} + \frac{2^I}{2^{|A^2|}} C_{A^1A^2} + \frac{2^I}{2^{|A^2|}} \sum_{\lambda=3}^{\xi} C_{A^2A^\lambda} + \frac{2^{I-1}}{2^{|A^2|}} \left(1 - \frac{1}{2^{|V^1|}}\right) \\
&\quad + \frac{2^I}{2^{|A^2|}} \sum_{\lambda=2}^{\xi-2} (C_{A^2A^{\lambda+1}} - 2C_{A^2A^{\lambda+2}}) + C_{A_I^2V^{\xi-1}} + C_{A_I^2n} \\
&= 1 - \cancel{\frac{2^I}{2^{|A^2|}}} + \cancel{\frac{2^I}{2^{|A^2|}} \cdot \frac{1}{2}} + \frac{2^I}{2^{|A^2|}} \sum_{\lambda=3}^{\xi} C_{A^2A^\lambda} + \cancel{\frac{2^{I-1}}{2^{|A^2|}}} - \frac{2^{I-1}}{2^{|A^2|}} \cdot \frac{1}{2^{|V^1|}} \\
&\quad + \frac{2^I}{2^{|A^2|}} \sum_{\lambda=3}^{\xi-1} C_{A^2A^\lambda} - 2\frac{2^I}{2^{|A^2|}} \sum_{\lambda=4}^{\xi} C_{A^2A^\lambda} + C_{A_I^2V^{\xi-1}} + C_{A_I^2n} \\
&= 1 - \frac{2^{I-1}}{2^{|A^2|}} \cdot \frac{1}{2^{|V^1|}} + 2 \cdot \frac{2^I}{2^{|A^2|}} C_{A^2A^3} - \frac{2^I}{2^{|A^2|}} C_{A^2A^\xi} + C_{A_I^2V^{\xi-1}} + C_{A_I^2n} \\
&= 1 - \cancel{\frac{2^{I-1}}{2^{|A^2|}} \cdot \frac{1}{2^{|V^1|}}} + \cancel{2 \cdot \frac{2^I}{2^{|A^2|}} \cdot \frac{1}{2 \cdot 2^{|V^1|+1}}} - C_{A_I^2A^\xi} + C_{A_I^2V^{\xi-1}} + C_{A_I^2n} \\
&= 1 - C_{A_I^2A^\xi} + C_{A_I^2V^{\xi-1}} + C_{A_I^2n} \overset{\text{Lemma } B.0.6}{=} 1
\end{aligned} \tag{3.240}$$

and for expression (3.235) with $\gamma = 1$ we obtain

$$\begin{aligned}
c(\delta(A_I^1)) &= 1 - \frac{2^I}{2^{|A^1|}} + \frac{2^I}{2^{|A^1|}} \sum_{\lambda=2}^{\xi} C_{A^1A^\lambda} + \frac{2^I}{2^{|A^1|}} \sum_{\lambda=1}^{\xi-2} (C_{A^1A^{\lambda+1}} - 2C_{A^1A^{\lambda+2}}) \\
&\quad + C_{A_I^1V^{\xi-1}} + C_{A_I^1n} \\
&= 1 - \frac{2^I}{2^{|A^1|}} + \frac{2^I}{2^{|A^1|}} \sum_{\lambda=2}^{\xi} C_{A^1A^\lambda} + \frac{2^I}{2^{|A^1|}} \sum_{\lambda=2}^{\xi-1} C_{A^1A^\lambda} \\
&\quad - 2\frac{2^I}{2^{|A^1|}} \sum_{\lambda=3}^{\xi} C_{A^1A^\lambda} + C_{A_I^1V^{\xi-1}} + C_{A_I^1n} \\
&= 1 - \frac{2^I}{2^{|A^1|}} + 2 \cdot \frac{2^I}{2^{|A^1|}} \cdot C_{A^1A^2} - \frac{2^I}{2^{|A^1|}} C_{A^1A^\xi} + C_{A_I^1V^{\xi-1}} + C_{A_I^1n} \\
&= 1 - \cancel{\frac{2^I}{2^{|A^1|}}} + \cancel{2 \cdot \frac{2^I}{2^{|A^1|}} \cdot \frac{1}{2}} - C_{A_I^1A^\xi} + C_{A_I^1V^{\xi-1}} + C_{A_I^1n} \\
&= 1 - C_{A_I^1A^\xi} + C_{A_I^1V^{\xi-1}} + C_{A_I^1n} \overset{\text{Lemma } B.0.6}{=} 1.
\end{aligned} \tag{3.241}$$

Case 2: $M^* = A^1 \cup A^2$.

$$c(\delta\left(A^1 \cup A^2\right)) = c(\delta(A^1)) + c(\delta(A^2)) - 2C_{A^1A^2} \tag{3.242}$$

The first two terms both evaluate to one (Case 1 with $I = |A^1|$ and $I = |A^2|$, respectively). For the third term on the right hand side of equation (3.242) it follows from Lemma B.0.2 that $C_{A^1A^2} = \frac{1}{2}$. Thus, for the cut $\delta(A^1 \cup A^2)$ we have

$$c(\delta\left(A^1 \cup A^2\right)) = 2 - 2 \cdot \frac{1}{2} = 1. \tag{3.243}$$

Case 3: $M^* = \bigcup_{\gamma=1}^{k+1} A^\gamma \cup \bigcup_{\lambda=1}^{k-1} V^\lambda$ for $1 < k \leq \xi - 1$.

$$\begin{aligned} c(\delta(M^*)) = c\left(\delta\left(\bigcup_{\gamma=1}^{k} A^\gamma \cup \bigcup_{\lambda=1}^{k-1} V^\lambda\right)\right) - \sum_{\gamma=1}^{k} C_{A^{k+1}A^\gamma} - \sum_{\lambda=1}^{k-1} C_{A^{k+1}V^\lambda} \\ + \sum_{\gamma=k+2}^{\xi} C_{A^{k+1}A^\gamma} + \sum_{\lambda=k}^{\xi-1} C_{A^{k+1}V^\lambda} + C_{A^{k+1}n} \end{aligned} \tag{3.244}$$

Applying Lemma B.0.7 to (3.244) we have

$$\begin{aligned} c(\delta(M^*)) = 1 - \sum_{\gamma=1}^{k} C_{A^{k+1}A^\gamma} - \underbrace{\sum_{\gamma=1}^{k-1} C_{A^{k+1}V^\gamma}}_{\Sigma_5} + \sum_{\gamma=k+2}^{\xi} C_{A^{k+1}A^\gamma} \\ + \underbrace{\sum_{\gamma=k}^{\xi-1} C_{A^{k+1}V^\gamma}}_{\Sigma_6} + C_{A^{k+1}n} \end{aligned} \tag{3.245}$$

Let us reproduce the results obtained in Appendix B for the terms on the right-hand side of equation (3.245). Lemma B.0.5 implies

$$\begin{aligned} &\Sigma_5 \quad \begin{cases} \sum_{\gamma=1}^{k-2} C_{A^{k+1}V^\gamma} &= \sum_{\gamma=1}^{k-2} (C_{A^{\gamma+2}A^{k+1}} - 2C_{A^{\gamma+1}A^{k+1}}), \\ C_{A^{k+1}V^{k-1}} &= \frac{1}{2}\left(1 - \frac{1}{2^{|V^{k-1}|}}\right), \end{cases} \\ &\Sigma_6 \quad \begin{cases} C_{A^{k+1}V^k} &= \frac{1}{2}\left(1 - \frac{1}{2^{|V^k|}}\right), \\ \sum_{\gamma=k+1}^{\xi-1} C_{A^{k+1}V^\gamma} &= \sum_{\gamma=k+1}^{\xi-2} (C_{A^{k+1}A^{\lambda+1}} - 2C_{A^{k+1}A^{\lambda+2}}) + C_{A^{k+1}V^{\xi-1}} \end{cases} \end{aligned} \tag{3.246}$$

From Lemma B.0.2 we have

$$C_{A^\gamma A^{\gamma+1}} = \frac{1}{2\cdot 2^{|V^{\gamma-1}|+1}}, \quad 1 < \gamma \leq \xi - 1. \tag{3.247}$$

Thus, using relations (3.246)-(3.247) equation (3.245) takes the form

$$\begin{aligned}
c(\delta(M^*)) &= 1 - \sum_{\gamma=1}^{k} C_{A^{k+1}A^\gamma} - \sum_{\gamma=1}^{k-2}(C_{A^{\gamma+2}A^{k+1}} - 2C_{A^{\gamma+1}A^{k+1}}) - C_{A^{k+1}V^{k-1}} \\
&\quad + \sum_{\gamma=k+2}^{\xi} C_{A^{k+1}A^\gamma} + C_{A^{k+1}V^k} + \sum_{\gamma=k+1}^{\xi-2}(C_{A^{k+1}A^{\lambda+1}} - 2C_{A^{k+1}A^{\lambda+2}}) + C_{A^{k+1}V^{\xi-1}} + C_{A^{k+1}n} \\
&= 1 - \sum_{\gamma=1}^{k} C_{A^\gamma A^{k+1}} - \sum_{\gamma=3}^{k} C_{A^\gamma A^{k+1}} + 2\sum_{\gamma=2}^{k-1} C_{A^\gamma A^{k+1}} - C_{A^{k+1}V^{k-1}} \\
&\quad + \sum_{\gamma=k+2}^{\xi} C_{A^{k+1}A^\gamma} + C_{A^{k+1}V^k} + \sum_{\gamma=k+2}^{\xi-1} C_{A^{k+1}A^\gamma} - 2\sum_{\gamma=k+3}^{\xi} C_{A^{k+1}A^\gamma} + C_{A^{k+1}V^{\xi-1}} + C_{A^{k+1}n} \\
&= 1 - C_{A^1A^{k+1}} - 2C_{A^kA^{k+1}} + C_{A^2A^{k+1}} - C_{A^{k+1}V^{k-1}} + 2C_{A^{k+1}A^{k+2}} \\
&\quad \underbrace{-C_{A^{k+1}A^\xi} + C_{A^{k+1}V^k} + C_{A^{k+1}n}}_{=0 \text{ by Lemma } B.0.6} + C_{A^{k+1}V^{\xi-1}} \\
&\overset{(B.11)}{=} 1 - \cancel{C_{A^1A^{k+1}}} - 2C_{A^kA^{k+1}} + \cancel{C_{A^1A^{k+1}}} - C_{A^{k+1}V^{k-1}} + 2C_{A^{k+1}A^{k+2}} + C_{A^{k+1}V^k} \\
&= 1 - 2\cdot\frac{1}{2\cdot 2^{|V^{k-1}|+1}} - \frac{1}{2}\left(1 - \frac{1}{2^{|V^{k-1}|}}\right) + 2\cdot\frac{1}{2\cdot 2^{|V^k|+1}} + \frac{1}{2}\left(1 - \frac{1}{2^{|V^k|}}\right) \\
&= 1 - \cancel{\frac{1}{2\cdot 2^{|V^{k-1}|}}} - \frac{1}{2} + \cancel{\frac{1}{2\cdot 2^{|V^{k-1}|}}} + \cancel{\frac{1}{2\cdot 2^{|V^k|}}} + \frac{1}{2} - \cancel{\frac{1}{2\cdot 2^{|V^k|}}} = 1.
\end{aligned}$$

Case 4: $M^* = \bigcup_{\gamma=1}^{k+1} A^\gamma \cup \bigcup_{\lambda=1}^{k-1} V^\lambda \cup V_I^k$ for $1 \leq k \leq \xi - 1$ and $I \in [|V^k|]$.

$$\begin{aligned}
c(\delta(M^*)) &= c\left(\delta\left(\bigcup_{\gamma=1}^{k+1} A^\gamma \cup \bigcup_{\lambda=1}^{k-1} V^\lambda\right)\right) - \underbrace{\sum_{\gamma=1}^{k+1} C_{A^\gamma V_I^k}}_{\Sigma_7} - \underbrace{\sum_{\lambda=1}^{k-1} C_{V^\lambda V_I^k}}_{\Sigma_8} \\
&\quad + \underbrace{\sum_{\gamma=k+2}^{\xi} C_{A^\gamma V_I^k}}_{\Sigma_9} + \underbrace{\sum_{\lambda=k+1}^{\xi-1} C_{V_I^k V^\lambda}}_{\Sigma_{10}} + C_{V_I^k n} + \underbrace{\sum_{i=1}^{I}\sum_{j=M+1}^{|V^k|} c_{v_i^k v_j^k}}_{\Sigma_{11}}
\end{aligned} \tag{3.248}$$

First, we reproduce the results obtained in Appendix B for the terms on the right-hand side of equation (3.248). Lemma B.0.1 implies

$$C_{V_I^k n} = 2(2^I - 1)C_{A^{k+1}n}, \quad k \in [\xi - 1] \tag{3.249}$$

From Lemma B.0.2 we have

$$\begin{aligned} C_{A^k A^{k+1}} &= \frac{1}{2 \cdot 2^{|V^{k-1}|+1}}, \quad 1 < k < \xi \\ C_{A^{k+1} A^{k+3}} &= \frac{1}{2 \cdot 2^{|V^k|+1} \cdot 2^{|V^{k+1}|+1}}, \quad k \in [\xi - 1] \end{aligned} \tag{3.250}$$

Using Lemma B.0.3 we deduce

$$\begin{aligned} \Sigma_8 &= \sum_{\lambda=1}^{k-1} C_{V^\lambda V_I^k} = \sum_{\lambda=1}^{k-2} C_{V^\lambda V_I^k} + C_{V^{k-1} V_I^k} \\ &= \left(1 - \frac{1}{2^I}\right) \sum_{\lambda=1}^{k-2} (C_{A^{\lambda+2} A^{k+1}} - 2C_{A^{\lambda+1} A^{k+1}}) + \frac{1}{2}\left(1 - \frac{1}{2^I}\right)\left(1 - \frac{1}{2^{|V^{k-1}|}}\right) \end{aligned} \tag{3.251}$$

$$\begin{aligned} \Sigma_{10} &= \sum_{\lambda=k+1}^{\xi-1} C_{V_I^k V^\lambda} = C_{V_I^k V^{k+1}} + \sum_{\lambda=k+2}^{\xi-2} C_{V_I^k V^\lambda} + C_{V_I^k V^{\xi-1}} \\ &= \frac{2^I - 1}{2^{|V^k|+1}}\left(1 - \frac{1}{2^{|V^{k+1}|}}\right) + (2^I - 1) \sum_{\lambda=k+2}^{\xi-2} (2C_{A^{k+1} A^{\lambda+1}} - 4C_{A^{k+1} A^{\lambda+2}}) \\ &\quad + 2(2^I - 1)\left(C_{A^{k+1} A^\xi} - C_{A^{k+1} n}\right) \end{aligned} \tag{3.252}$$

From Lemma B.0.5 we obtain

$$\Sigma_7 = \sum_{\gamma=1}^{k} C_{A^\gamma V_I^k} + C_{A^{k+1} V_I^k} = \left(1 - \frac{1}{2^I}\right) \sum_{\gamma=1}^{k} C_{A^\gamma A^{k+1}} + \frac{1}{2}\left(1 - \frac{1}{2^I}\right), \tag{3.253}$$

$$\Sigma_9 = C_{A^{k+2} V_I^k} + \sum_{\gamma=k+3}^{\xi} C_{A^\gamma V_I^k} = (2^I - 1) \cdot \frac{1}{2^{|V^k|+1}} + 2(2^I - 1) \sum_{\gamma=k+3}^{\xi} C_{A^{k+1} A^\gamma}. \tag{3.254}$$

From (3.64), see the proof of Lemma 3.5.2, we have

$$\Sigma_{11} = \sum_{i=1}^{I} \sum_{j=M+1}^{|V^k|} c_{v_i^k v_j^k} = 1 - \frac{1}{2^I} - \frac{2^I - 1}{2^{|V^k|}}. \tag{3.255}$$

Thus, using relations (3.249)-(3.255) equation (3.248) takes the form

$$
\begin{aligned}
c(\delta(M^*)) &= 1 - \left(1 - \frac{1}{2^I}\right) \sum_{\gamma=1}^{k} C_{A^\gamma A^{k+1}} - \frac{1}{2}\left(1 - \frac{1}{2^I}\right) \\
&\quad - \left(1 - \frac{1}{2^I}\right) \sum_{\lambda=1}^{k-2} (C_{A^{\lambda+2} A^{k+1}} - 2C_{A^{\lambda+1} A^{k+1}}) - \frac{1}{2}\left(1 - \frac{1}{2^I}\right)\left(1 - \frac{1}{2^{|V^{k-1}|}}\right) \\
&\quad + (2^I - 1)\frac{1}{2^{|V^k|+1}} + 2(2^I - 1) \sum_{\gamma=k+3}^{\xi} C_{A^{k+1} A^\gamma} + \frac{2^I - 1}{2^{|V^k|+1}}\left(1 - \frac{1}{2^{|V^{k+1}|}}\right) \\
&\quad + (2^I - 1) \sum_{\lambda=k+2}^{\xi-2} (2C_{A^{k+1} A^{\lambda+1}} - 4C_{A^{k+1} A^{\lambda+2}}) + 2(2^I - 1)(C_{A^{k+1} A^\xi} - C_{A^{k+1} n}) \\
&\quad + 2(2^I - 1) C_{A^{k+1} n} + 1 - \frac{1}{2^I} - \frac{2^I - 1}{2^{|V^k|}} \\
&= 1 - \left(1 - \frac{1}{2^I}\right) \sum_{\gamma=1}^{k} C_{A^\gamma A^{k+1}} - \frac{1}{2}\left(1 - \frac{1}{2^I}\right) - \left(1 - \frac{1}{2^I}\right) \sum_{\lambda=3}^{k} C_{A^\lambda A^{k+1}} \\
&\quad + 2\left(1 - \frac{1}{2^I}\right) \sum_{\lambda=2}^{k-1} C_{A^\lambda A^{k+1}} - \frac{1}{2}\left(1 - \frac{1}{2^I}\right) + \frac{1}{2}\left(1 - \frac{1}{2^I}\right)\frac{1}{2^{|V^{k-1}|}} \\
&\quad + \frac{2^I - 1}{2^{|V^k|+1}} + 2(2^I - 1) \sum_{\gamma=k+3}^{\xi} C_{A^{k+1} A^\gamma} + \frac{2^I - 1}{2^{|V^k|+1}} - \frac{2^I - 1}{2^{|V^k|+1} \cdot 2^{|V^{k+1}|}} \\
&\quad + 2(2^I - 1) \sum_{\lambda=k+3}^{\xi-1} C_{A^{k+1} A^\lambda} - 4(2^I - 1) \sum_{\lambda=k+4}^{\xi} C_{A^{k+1} A^\lambda} + 2(2^I - 1) C_{A^{k+1} A^\xi} \\
&\quad - 2(2^I - 1) C_{A^{k+1} n} + 2(2^I - 1) C_{A^{k+1} n} + 1 - \frac{1}{2^I} - \frac{2^I - 1}{2^{|V^k|}} \\
&= 1 - \left(1 - \frac{1}{2^I}\right)(C_{A_1 A^{k+1}} + 2C_{A^k A^{k+1}} - C_{A_2 A^{k+1}}) + \frac{1}{2}\left(1 - \frac{1}{2^I}\right)\frac{1}{2^{|V^{k-1}|}} \\
&\quad + 2(2^I - 1)(2C_{A^{k+1} A^{k+3}} - C_{A^{k+1} A^\xi}) - \frac{2^I - 1}{2^{|V^k|+1} \cdot 2^{|V^{k+1}|}} + 2(2^I - 1) C_{A^{k+1} A^\xi} \\
&= 1 - 2\left(1 - \frac{1}{2^I}\right)\frac{1}{2 \cdot 2^{|V^{k-1}|+1}} + \frac{1}{2}\left(1 - \frac{1}{2^I}\right)\frac{1}{2^{|V^{k-1}|}} + \frac{4(2^I - 1)}{2 \cdot 2^{|V^k|+1} \cdot 2^{|V^{k+1}|+1}} \\
&\quad - 2(2^I - 1) C_{A^{k+1} A^\xi} - \frac{2^I - 1}{2^{|V^k|+1} \cdot 2^{|V^{k+1}|}} + 2(2^I - 1) C_{A^{k+1} A^\xi} = 1.
\end{aligned}
$$

□

Continuing the proof of Proposition 3.7.1 we construct from the undirected graph K_n the directed graph $\overleftrightarrow{K}_n = ([n], \overleftrightarrow{E}_n)$ by replacing each edge $e \in E_n$ by two antiparallel arcs. In the new graph $\overleftrightarrow{K}_n$ we construct a γ-n flow $f : \overleftrightarrow{E}_n \to \mathbb{R}$ for each $\gamma \in [\xi]$, see Figures 3.17-3.21, with

$$
\begin{aligned}
&f_{a_i^\gamma a_j^\gamma} := c_{a_i^\gamma a_j^\gamma} && \text{for all } a_i^\gamma, a_j^\gamma \in A^\gamma \text{ with } i < j,\\
&f_{a_i^\gamma a_j^\gamma} := 0 && \text{for all } a_i^\gamma, a_j^\gamma \in A^\gamma \text{ with } i > j,\\
&f_{a_i^\lambda a_j^\lambda} := 0 && \text{for all } a_i^\lambda, a_j^\lambda \in A^\lambda \text{ with } \lambda \in [\xi] \setminus \{\gamma\},\\
&f_{a_i^\gamma a_j^\lambda} := c_{a_i^\gamma a_j^\lambda} && \text{for all } a_i^\gamma \in A^\gamma,\ a_j^\lambda \in A^\lambda \text{ with } \lambda \in [\xi] \setminus \{\gamma\},\\
&f_{a_j^\lambda a_i^\gamma} := 0 && \text{for all } a_i^\gamma \in A^\gamma,\ a_j^\lambda \in A^\lambda \text{ with } \lambda \in [\xi] \setminus \{\gamma\},\\
&f_{a_j^\lambda a_i^\beta} := 0 && \text{for all } a_j^\lambda \in A^\lambda,\ a_i^\beta \in A^\beta \text{ with } \lambda < \beta < \gamma,\\
&f_{a_i^\beta a_j^\lambda} := 0 && \text{for all } a_j^\lambda \in A^\lambda,\ a_i^\beta \in A^\beta \text{ with } \lambda < \beta < \gamma,\\
&f_{a_j^\lambda a_i^\beta} := c_{a_i^\lambda a_j^\beta} && \text{for all } a_j^\lambda \in A^\lambda,\ a_i^\beta \in A^\beta \text{ with } \lambda < \gamma < \beta \le \xi,\\
&f_{a_i^\beta a_j^\lambda} := 0 && \text{for all } a_j^\lambda \in A^\lambda,\ a_i^\beta \in A^\beta \text{ with } \lambda < \gamma < \beta \le \xi,\\
&f_{a_j^\lambda a_i^\beta} := c_{a_i^\lambda a_j^\beta} && \text{for all } a_j^\lambda \in A^\lambda,\ a_i^\beta \in A^\beta \text{ with } \gamma < \lambda < \beta \le \xi,\\
&f_{a_i^\beta a_j^\lambda} := 0 && \text{for all } a_j^\lambda \in A^\lambda,\ a_i^\beta \in A^\beta \text{ with } \gamma < \lambda < \beta \le \xi,\\
&f_{v_i^\lambda v_j^\lambda} := 0 && \text{for all } v_i^\lambda, v_j^\lambda \in V^\lambda \text{ with } \lambda < \gamma - 1,\\
&f_{v_i^\lambda v_j^\lambda} := c_{v_i^\lambda v_j^\lambda} && \text{for all } v_i^\lambda, v_j^\lambda \in V^\lambda \text{ with } i < j,\ \gamma - 1 \le \lambda \le \xi - 1\\
&f_{v_i^\lambda v_j^\lambda} := 0 && \text{for all } v_i^\lambda, v_j^\lambda \in V^\lambda \text{ with } i > j,\ \gamma - 1 \le \lambda \le \xi - 1\\
&f_{v_i^\lambda v_j^\beta} := 0 && \text{for all } v_i^\lambda \in V^\lambda, v_j^\beta \in V^\beta \text{ with } \lambda < \beta < \gamma - 1\\
&f_{v_i^\beta v_j^\lambda} := 0 && \text{for all } v_i^\lambda \in V^\lambda, v_j^\beta \in V^\beta \text{ with } \lambda < \beta < \gamma - 1\\
&f_{v_i^\lambda v_j^\beta} := c_{v_i^\beta v_j^\lambda} && \text{for all } v_i^\lambda \in V^\lambda, v_j^\beta \in V^\beta \text{ with } \lambda < \gamma - 1 \le \beta \le \xi - 1\\
&f_{v_i^\beta v_j^\lambda} := 0 && \text{for all } v_i^\lambda \in V^\lambda, v_j^\beta \in V^\beta \text{ with } \lambda < \gamma - 1 \le \beta \le \xi - 1\\
&f_{v_i^\lambda v_j^\beta} := c_{v_i^\lambda v_j^\beta} && \text{for all } v_i^\lambda \in V^\lambda, v_j^\beta \in V^\beta \text{ with } \gamma - 1 \le \lambda < \beta \le \xi - 1\\
&f_{v_i^\beta v_j^\lambda} := 0 && \text{for all } v_i^\lambda \in V^\lambda, v_j^\beta \in V^\beta \text{ with } \gamma - 1 \le \lambda < \beta \le \xi - 1
\end{aligned}
\tag{3.256}
$$

$$
\begin{aligned}
&f_{a_i^\gamma v_j^\lambda} := c_{a_i^\gamma v_j^\lambda} && \text{for all } a_i^\gamma \in A^\gamma,\ v_j^\lambda \in V^\lambda \text{ with } \lambda \in [\xi-1],\\
&f_{v_j^\lambda a_i^\gamma} := 0 && \text{for all } a_i^\gamma \in A^\gamma,\ v_j^\lambda \in V^\lambda \text{ with } \lambda \in [\xi-1],\\
&f_{a_i^\beta v_j^\lambda} := c_{a_i^\beta v_j^\lambda} && \text{for all } a_i^\beta \in A^\beta,\ v_j^\lambda \in V^\lambda \text{ with } \gamma-1 \le \lambda \le \xi-1,\ \beta \le \lambda+1\\
&f_{v_j^\lambda a_i^\beta} := 0 && \text{for all } a_i^\beta \in A^\beta,\ v_j^\lambda \in V^\lambda \text{ with } \gamma-1 \le \lambda \le \xi-1,\ \beta \le \lambda+1\\
&f_{a_i^\beta v_j^\lambda} := 0 && \text{for all } a_i^\beta \in A^\beta,\ v_j^\lambda \in V^\lambda \text{ with } \gamma-1 \le \lambda \le \xi-1,\ \beta > \lambda+1\\
&f_{v_j^\lambda a_i^\beta} := c_{v_j^\lambda a_i^\beta} && \text{for all } a_i^\beta \in A^\beta,\ v_j^\lambda \in V^\lambda \text{ with } \gamma-1 \le \lambda \le \xi-1,\ \beta > \lambda+1\\
&f_{a_i^\beta v_j^\lambda} := 0 && \text{for all } a_i^\beta \in A^\beta,\ v_j^\lambda \in V^\lambda \text{ with } \beta \in [\xi]\setminus\{\gamma\},\ \lambda < \gamma-1,\\
&f_{v_j^\lambda a_i^\beta} := 0 && \text{for all } a_i^\beta \in A^\beta,\ v_j^\lambda \in V^\lambda \text{ with } \lambda < \gamma-1,\ \beta < \gamma\\
&f_{v_j^\lambda a_i^\beta} := c_{v_j^\lambda a_i^\beta} && \text{for all } a_i^\beta \in A^\beta,\ v_j^\lambda \in V^\lambda \text{ with } \lambda < \gamma-1,\ \gamma < \beta \le \xi\\
&f_{a_i^\beta n} := c_{a_i^\beta n} && \text{for all } a_i^\beta \in A^\beta \text{ with } \beta \in [\xi],\\
&f_{n a_i^\beta} := 0 && \text{for all } a_i^\beta \in A^\beta \text{ with } \beta \in [\xi],\\
&f_{v_i^\lambda n} := c_{v_i^\lambda n} && \text{for all } v_i^\lambda \in V^\lambda \text{ with } \lambda \in [\xi-1],\\
&f_{n v_i^\lambda} := 0 && \text{for all } v_i^\lambda \in V^\lambda \text{ with } \lambda \in [\xi-1].
\end{aligned} \tag{3.257}
$$

For each $\gamma \in [\xi]$ we consider the γ-n-flow $f : \overset{\leftrightarrow}{E}_n \to \mathbb{R}$ and check the *flow conservation law*: the amount of flow entering a vertex $a_i^\gamma \in A^\gamma \setminus \{a_1^\gamma = \gamma\}$, $a_i^\lambda \in A^\lambda$, $\lambda \in [\xi] \setminus \{\gamma\}$ and $v_j^\lambda \in V^\lambda$, $\lambda \in [\xi-1]$ should be equal to the amount of flow leaving $a_i^\gamma \in A^\gamma \setminus \{a_1^\gamma = \gamma\}$, $a_i^\lambda \in A^\lambda$, $\lambda \in [\xi] \setminus \{\gamma\}$ and $v_j^\lambda \in V^\lambda$, $\lambda \in [\xi-1]$, respectively.

For brevity, we introduce the following notations

$$
\begin{aligned}
C_{a_i^\lambda A^\beta} &:= \sum_{j=1}^{|A^\beta|} c_{a_i^\lambda a_j^\beta}, \quad \lambda, \beta \in [\xi],\ \lambda \ne \beta,\ i \in [|A^\lambda|]\\
C_{a_i^\beta V^\lambda} &:= \sum_{j=1}^{|V^\lambda|} c_{a_i^\beta v_j^\lambda}, \quad \beta \in [\xi],\ \lambda \in [\xi-1],\ i \in [|A^\beta|]\\
C_{v_i^\lambda A^\beta} &:= \sum_{j=1}^{|A^\beta|} c_{v_i^\lambda a_j^\beta}, \quad \beta \in [\xi],\ \lambda \in [\xi-1],\ i \in [|V^\lambda|]\\
C_{v_i^\lambda V^\beta} &:= \sum_{j=1}^{|V^\beta|} c_{v_i^\lambda v_j^\beta}, \quad \lambda, \beta \in [\xi-1],\ \lambda \ne \beta,\ i \in [|V^\lambda|]
\end{aligned} \tag{3.258}
$$

Note that, from the definition of $c \in \mathbb{R}^{E_n}_{>0}$, see (3.223)-(3.232), we have $c_{a_1^\lambda,*} = c_{a_2^\lambda,*}$ for all $\lambda \in [\xi]$ what implies that $C_{a_1^\lambda,*} = C_{a_2^\lambda,*}$ for all $\lambda \in [\xi]$. Thus, it suffices to calculate $f(\delta^{out}(a_i^\lambda))$ and $f(\delta^{in}(a_i^\lambda))$ for $i \geq 2$ due to $f(\delta^{out}(a_1^\lambda)) = f(\delta^{out}(a_2^\lambda))$ and $f(\delta^{in}(a_1^\lambda)) = f(\delta^{in}(a_2^\lambda))$ for all $\lambda \in [\xi]$.

1. In view of (3.256)-(3.257) we have for each $a_i^\gamma \in A^\gamma \setminus \{a_1^\gamma = \gamma\}$, see Figure 3.17,

$$\begin{aligned} f(\delta^{out}(a_i^\gamma)) = \underbrace{\sum_{\lambda=1}^{\gamma-1} C_{a_i^\gamma A^\lambda}}_{\Sigma_{12}} + \underbrace{\sum_{\lambda=\gamma+1}^{\xi} C_{a_i^\gamma A^\lambda}}_{\Sigma_{13}} + \underbrace{\sum_{\lambda=1}^{\gamma-1} C_{a_i^\gamma V^\lambda}}_{\Sigma_{14}} \\ + \underbrace{\sum_{\lambda=\gamma}^{\xi-2} C_{a_i^\gamma V^\lambda}}_{\Sigma_{15}} + C_{a_i^\gamma V^{\xi-1}} + c_{a_i^\gamma n} + \underbrace{\sum_{j=i+1}^{|A^\gamma|} c_{a_i^\gamma a_j^\gamma}}_{\Sigma_{16}}, \end{aligned} \tag{3.259}$$

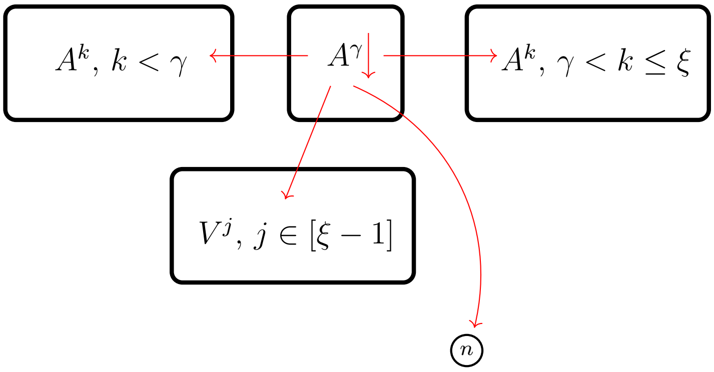

Figure 3.17: Sketch of arcs $(u, w) \in \overset{\leftrightarrow}{E}_n$ with $f_{u,w} > 0$ for the vertices from A^γ.

First, we reproduce the results obtained in Appendix C for the terms on the right-hand side of equation (3.259). Lemma C.0.1 implies

$$\begin{aligned} \Sigma_{12} = \sum_{\lambda=1}^{\gamma-1} C_{a_i^\gamma A^\lambda} = \frac{1}{2^{|A^\gamma|-i+1}} \sum_{\lambda=1}^{\gamma-1} C_{A^\lambda A^\gamma}, \\ \Sigma_{13} = \sum_{\lambda=\gamma+1}^{\xi} C_{a_i^\gamma A^\lambda} = \frac{1}{2^{|A^\gamma|-i+1}} \sum_{\lambda=\gamma+1}^{\xi} C_{A^\gamma A^\lambda}. \end{aligned} \tag{3.260}$$

From Lemma C.0.3 we have

$$\Sigma_{15} = \sum_{\lambda=\gamma}^{\xi-2} C_{a_i^\gamma V^\lambda} = \frac{1}{2^{|A^\gamma|-i+1}} \sum_{\lambda=\gamma}^{\xi-2} (C_{A^\gamma A^{\lambda+1}} - 2C_{A^\gamma A^{\lambda+2}}) \tag{3.261}$$

$$\Sigma_{14}\begin{cases} \sum\limits_{\lambda=1}^{\gamma-3} C_{a_i^\gamma V^\lambda} &= \dfrac{1}{2^{|A^\gamma|-i+1}} \sum\limits_{\lambda=1}^{\gamma-3} (C_{A^{\lambda+2}A^\gamma} - 2C_{A^{\lambda+1}A^\gamma}), \\ C_{a_i^\gamma V^{\gamma-1}} &= \dfrac{1}{2^{|A^\gamma|-i+2}} \left(1 - \frac{1}{2^{|V^{\gamma-1}|}}\right), \quad 1 < \gamma \le \xi \\ C_{a_i^\gamma V^{\gamma-2}} &= \dfrac{1}{2^{|A^\gamma|-i+2}} \left(1 - \frac{1}{2^{|V^{\gamma-2}|}}\right), \quad 2 < \gamma \le \xi \end{cases} \tag{3.262}$$

For the last term on the right-hand side of equation (3.259) we deduce

$$\begin{aligned} \Sigma_{16} = \sum_{j=i+1}^{|A^\gamma|} c_{a_i^\gamma a_j^\gamma} &= \sum_{j=i+1}^{|A^\gamma|} \frac{1}{2 \cdot 2^{j-i}} = \frac{1}{2^{1-i}} \sum_{j=i+1}^{|A^\gamma|} \frac{1}{2^j} \\ &= \frac{1}{2^{1-i}} \cdot \frac{1}{2^{i+1}} \cdot \frac{1 - \frac{1}{2^{|A^\gamma|-i}}}{1 - \frac{1}{2}} = \frac{1}{2}\left(1 - \frac{1}{2^{|A^\gamma|-i}}\right) = \frac{1}{2} - \frac{1}{2^{|A^\gamma|-i+1}}. \end{aligned} \tag{3.263}$$

In view of (3.260)-(3.263) equation (3.259) takes the form

$$\begin{aligned} f(\delta^{out}(a_i^\gamma)) &= \frac{1}{2^{|A^\gamma|-i+1}} \sum_{\lambda=1}^{\gamma-1} C_{A^\lambda A^\gamma} + \frac{1}{2^{|A^\gamma|-i+1}} \sum_{\lambda=\gamma+1}^{\xi} C_{A^\gamma A^\lambda} \\ &+ \frac{1}{2^{|A^\gamma|-i+1}} \sum_{\lambda=1}^{\gamma-3} (C_{A^{\lambda+2}A^\gamma} - 2C_{A^{\lambda+1}A^\gamma}) + \frac{1}{2^{|A^\gamma|-i+2}} \left(1 - \frac{1}{2^{|V^{\gamma-1}|}}\right) \\ &+ \frac{1}{2^{|A^\gamma|-i+2}} \left(1 - \frac{1}{2^{|V^{\gamma-2}|}}\right) \frac{1}{2^{|A^\gamma|-i+1}} \sum_{\lambda=\gamma}^{\xi-2} (C_{A^\gamma A^{\lambda+1}} - 2C_{A^\gamma A^{\lambda+2}}) \\ &+ C_{a_i^\gamma V^{\xi-1}} + c_{a_i^\gamma n} + \frac{1}{2} - \frac{1}{2^{|A^\gamma|-i+1}} \\ &= \frac{1}{2^{|A^\gamma|-i+1}} \left(C_{A_1 A^\gamma} + \sum_{\lambda=2}^{\gamma-1} C_{A^\lambda A^\gamma} + \sum_{\lambda=\gamma+1}^{\xi} C_{A^\gamma A^\lambda} + \sum_{\lambda=3}^{\gamma-1} C_{A^\lambda A^\gamma} \right. \\ &\left. - 2\sum_{\lambda=2}^{\gamma-2} C_{A^\lambda A^\gamma} + \sum_{\lambda=\gamma+1}^{\xi-1} C_{A^\gamma A^\lambda} - 2\sum_{\lambda=\gamma+2}^{\xi} C_{A^\gamma A^\lambda} \right) + \frac{1}{2^{|A^\gamma|-i+1}} \\ &- \frac{1}{2^{|A^\gamma|-i+1}} \left(\frac{1}{2^{|V^{\gamma-1}|}} + \frac{1}{2^{|V^{\gamma-2}|}} \right) + C_{a_i^\gamma V^{\xi-1}} + c_{a_i^\gamma n} + \frac{1}{2} - \frac{1}{2^{|A^\gamma|-i+1}} \\ &= \frac{1}{2^{|A^\gamma|-i+1}} (\cancel{C_{A_2 A^\gamma}} - \cancel{C_{A^2 A^\gamma}} + 2C_{A^{\gamma-1}A^\gamma} + 2C_{A^\gamma A^{\gamma+1}} - C_{A^\gamma A^\xi}) \\ &- \frac{1}{2^{|A^\gamma|-i+1}} \left(\frac{1}{2^{|V^{\gamma-1}|}} + \frac{1}{2^{|V^{\gamma-2}|}} \right) + C_{a_i^\gamma V^{\xi-1}} + c_{a_i^\gamma n} + \frac{1}{2} \end{aligned} \tag{3.264}$$

$$= \frac{1}{2^{|A^\gamma|-i+1}} \left(\frac{2}{2 \cdot 2^{|V^{\gamma-2}|}} + \frac{2}{2 \cdot 2^{|V^{\gamma-1}|}} \right) - \frac{1}{2^{|A^\gamma|-i+1}} C_{A^\gamma A^\xi}$$
$$- \frac{1}{2^{|A^\gamma|-i+1}} \left(\frac{1}{2^{|V^{\gamma-1}|}} + \frac{1}{2^{|V^{\gamma-2}|}} \right) + C_{a_i^\gamma V^{\xi-1}} + c_{a_i^\gamma n} + \frac{1}{2} \tag{3.265}$$
$$= -C_{a_i^\gamma A^\xi} + C_{a_i^\gamma V^{\xi-1}} + c_{a_i^\gamma n} + \frac{1}{2} \overset{\text{Lemma } C.0.4}{=} \frac{1}{2}.$$

Calculating the amount of flow f entering $a_i^\gamma \in A^\gamma \setminus \{a_1^\gamma = \gamma\}$, see Figure 3.17, we deduce

$$f(\delta^{in}(a_i^\gamma)) = \sum_{j=1}^{i-1} c_{a_j^\gamma a_i^\gamma} = c_{a_1^\gamma a_i^\gamma} + \sum_{j=2}^{i-1} c_{a_j^\gamma a_i^\gamma} = c_{a_2^\gamma a_i^\gamma} + \sum_{j=2}^{i-1} c_{a_j^\gamma a_i^\gamma}$$
$$= \frac{1}{2 \cdot 2^{i-2}} + \sum_{j=2}^{i-1} \frac{1}{2 \cdot 2^{i-j}} = \frac{1}{2^{i-1}} \left(1 + \frac{1}{2^2} \sum_{j=2}^{i-1} 2^j \right) \tag{3.266}$$
$$= \frac{1}{2^{i-1}} \left(1 + \frac{1}{2^2} \cdot \frac{2^2 \cdot (1 - 2^{i-2})}{1-2} \right) = \frac{2^{i-2}}{2^{i-1}} = \frac{1}{2}.$$

2. For each $a_i^\lambda \in A^\lambda$, $i \geq 2$ with $\gamma < \lambda \leq \xi$ we have due to (3.256)-(3.257), see Figure 3.18,

$$f(\delta^{out}(a_i^\lambda)) = \underbrace{\sum_{\beta=\lambda+1}^{\xi} C_{a_i^\lambda A^\beta}}_{\Sigma_{17}} + \underbrace{\sum_{\beta=\lambda-1}^{\xi-2} C_{a_i^\lambda V^\beta}}_{\Sigma_{18}} + C_{a_i^\lambda V^{\xi-1}} + c_{a_i^\lambda n}. \tag{3.267}$$

$$f(\delta^{in}(a_i^\lambda)) = \underbrace{\sum_{\beta=1}^{\lambda-1} C_{a_i^\lambda A^\beta}}_{\Sigma_{19}} + \underbrace{\sum_{\beta=1}^{\lambda-2} C_{a_i^\lambda V^\beta}}_{\Sigma_{20}}. \tag{3.268}$$

First, we reproduce the results obtained in Appendix C for the terms on the right-hand side of equations (3.267), (3.268) with $i \in [|A^\lambda|]$, $i \geq 2$. Lemma C.0.1 implies

$$\Sigma_{17} = \sum_{\beta=\lambda+1}^{\xi} C_{a_i^\lambda A^\beta} = \frac{1}{2^{|A^\lambda|-i+1}} \sum_{\beta=\lambda+1}^{\xi} C_{A^\lambda A^\beta},$$
$$\Sigma_{19} = \sum_{\beta=1}^{\lambda-1} C_{a_i^\lambda A^\beta} = \frac{1}{2^{|A^\lambda|-i+1}} \sum_{\beta=1}^{\lambda-1} C_{A^\beta A^\lambda}, \tag{3.269}$$

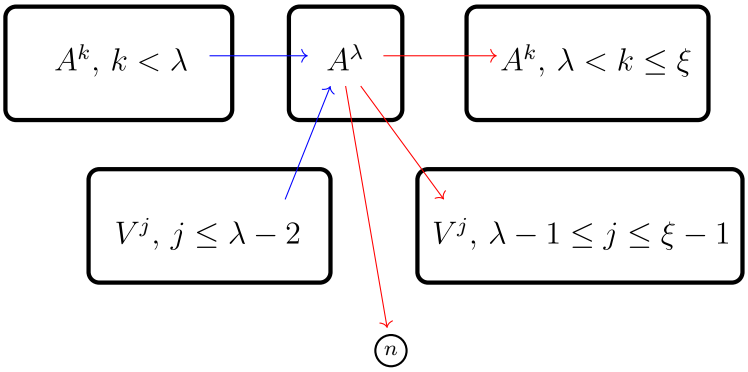

Figure 3.18: Sketch of ingoing and outgoing arcs $(u, w) \in \overset{\leftrightarrow}{E}_n$ with $f_{u,w} > 0$ for the vertices from A^λ, $\gamma < \lambda \le \xi$.

From Lemma C.0.3 we have

$$\begin{aligned}
&\Sigma_{18}\begin{cases} \sum\limits_{\beta=\lambda}^{\xi-2} C_{a_i^\lambda V^\beta} &= \dfrac{1}{2^{|A^\lambda|-i+1}} \sum\limits_{\beta=\lambda}^{\xi-2} (C_{A^\lambda A^{\beta+1}} - 2C_{A^\lambda A^{\beta+2}}), \\ C_{a_i^{\lambda+1} V^\lambda} &= \dfrac{1}{2^{|A^{\lambda+1}|-i+2}} \left(1 - \frac{1}{2^{|V^\lambda|}}\right), \end{cases} \\
&\Sigma_{20}\begin{cases} C_{a_i^{\lambda+2} V^\lambda} &= \dfrac{1}{2^{|A^{\lambda+2}|-i+2}} \left(1 - \frac{1}{2^{|V^\lambda|}}\right), \\ \sum\limits_{\beta=1}^{\lambda-3} C_{a_i^\lambda V^\beta} &= \dfrac{1}{2^{|A^\lambda|-i+1}} \sum\limits_{\beta=1}^{\lambda-3} (C_{A^{\beta+2} A^\lambda} - 2C_{A^{\beta+1} A^\lambda}). \end{cases}
\end{aligned} \tag{3.270}$$

In view of (3.269)-(3.270) we deduce for equations (3.267) and (3.268):

$$\begin{aligned}
f(\delta^{out}(a_i^\lambda)) &= \frac{1}{2^{|A^\lambda|-i+1}} \sum_{\beta=\lambda+1}^{\xi} C_{A^\lambda A^\beta} + \frac{1}{2^{|A^\lambda|-i+2}} \left(1 - \frac{1}{2^{|V^{\lambda-1}|}}\right) \\
&+ \frac{1}{2^{|A^\lambda|-i+1}} \sum_{\beta=\lambda}^{\xi-2} (C_{A^\lambda A^{\beta+1}} - 2C_{A^\lambda A^{\beta+2}}) + C_{a_i^\lambda V^{\xi-1}} + c_{a_i^\lambda n} \\
&= \frac{1}{2^{|A^\lambda|-i+1}} \left(\sum_{\beta=\lambda+1}^{\xi} C_{A^\lambda A^\beta} + \sum_{\beta=\lambda+1}^{\xi-1} C_{A^\lambda A^\beta} - 2 \sum_{\beta=\lambda+2}^{\xi} C_{A^\lambda A^\beta} \right) \\
&+ \frac{1}{2^{|A^\lambda|-i+2}} \left(1 - \frac{1}{2^{|V^{\lambda-1}|}}\right) + C_{a_i^\lambda V^{\xi-1}} + c_{a_i^\lambda n} \\
&= \frac{1}{2^{|A^\lambda|-i+1}} (2C_{A^\lambda A^{\lambda+1}} - C_{A^\lambda A^\xi}) + \frac{1}{2^{|A^\lambda|-i+1}} \left(1 - \frac{1}{2^{|V^{\lambda-1}|}}\right) \\
&+ C_{a_i^\lambda V^{\xi-1}} + c_{a_i^\lambda n} = \frac{1}{2^{|A^\lambda|-i+1}} \cdot \frac{2}{2 \cdot 2^{|V^{\lambda-1}|+1}} - \frac{1}{2^{|A^\lambda|-i+1}} \cdot C_{A^\lambda A^\xi}
\end{aligned} \tag{3.271}$$

$$+\frac{1}{2^{|A^\lambda|-i+2}} - \frac{1}{2^{|A^\lambda|-i+2}} \cdot \frac{1}{2^{|V^{\lambda-1}|}} + C_{a_i^\lambda V^{\xi-1}} + c_{a_i^\lambda n}$$

$$= -C_{a_i^\lambda A^\xi} + C_{a_i^\lambda V^{\xi-1}} + c_{a_i^\lambda n} + \frac{1}{2^{|A^\lambda|-i+1}} \overset{\text{Lemma } C.0.4}{=} \frac{1}{2^{|A^\lambda|-i+2}},\ \lambda < \xi \tag{3.272}$$

$$f(\delta^{out}(a_i^\xi)) = \frac{1}{2^{|A^\xi|-i+2}}\left(1 - \frac{1}{2^{|V^{\xi-1}|}}\right) + \frac{1}{2 \cdot 2^{|A^\xi|-i} \cdot 2^{|V^{\xi-1}|+1}} = \frac{1}{2^{|A^\lambda|-i+2}},\ \lambda = \xi \tag{3.273}$$

$$f(\delta^{in}(a_i^\lambda)) = \frac{1}{2^{|A^\lambda|-i+1}} \sum_{\beta=1}^{\lambda-1} C_{A^\beta A^\lambda} + \frac{1}{2^{|A^\lambda|-i+1}} \sum_{\beta=1}^{\lambda-3} (C_{A^{\beta+2}A^\lambda} - 2C_{A^{\beta+1}A^\lambda})$$

$$+ \frac{1}{2^{|A^\lambda|-i+2}}\left(1 - \frac{1}{2^{|V^{\lambda-2}|}}\right) = \frac{1}{2^{|A^\lambda|-i+1}}\left(C_{A_1 A^\lambda} + \sum_{\beta=2}^{\lambda-1} C_{A^\beta A^\lambda}\right.$$

$$\left. + \sum_{\beta=3}^{\lambda-1} C_{A^\beta A^\lambda} - 2\sum_{\beta=2}^{\lambda-2} C_{A^\beta A^\lambda}\right) + \frac{1}{2^{|A^\lambda|-i+2}}\left(1 - \frac{1}{2^{|V^{\lambda-2}|}}\right) \tag{3.274}$$

$$= \frac{1}{2^{|A^\lambda|-i+1}}\left(C_{A_2 A^\lambda} - C_{A^2 A^\lambda} + 2C_{A^{\lambda-1}A^\lambda}\right) + \frac{1}{2^{|A^\lambda|-i+2}}\left(1 - \frac{1}{2^{|V^{\lambda-2}|}}\right)$$

$$= \frac{1}{2^{|A^\lambda|-i+1}} \cdot \frac{2}{2 \cdot 2^{|V^{\lambda-2}|+1}} + \frac{1}{2^{|A^\lambda|-i+2}} - \frac{1}{2^{|A^\lambda|-i+2}} \cdot \frac{1}{2^{|V^{\lambda-2}|}} = \frac{1}{2^{|A^\lambda|-i+2}}.$$

3. For each $a_i^\lambda \in A^\lambda$, $i \geq 2$ with $\lambda < \gamma$ we obtain due to (3.256)-(3.257), see Figure 3.19,

$$f(\delta^{out}(a_i^\lambda)) = \underbrace{\sum_{\beta=\gamma+1}^{\xi} C_{a_i^\lambda A^\beta}}_{\Sigma_{21}} + \underbrace{\sum_{\beta=\gamma-1}^{\xi-2} C_{a_i^\lambda V^\beta}}_{\Sigma_{22}} + C_{a_i^\lambda V^{\xi-1}} + c_{a_i^\lambda n}, \tag{3.275}$$

$$f(\delta^{in}(a_i^\lambda)) = C_{a_i^\lambda A^\gamma}. \tag{3.276}$$

Let us reproduce the results obtained in Appendix C for the terms on the right-hand side of equations (3.275) and (3.276) with $i \in [|A^\lambda|]$, $i \geq 2$. Lemma C.0.1 implies

$$\Sigma_{21} = \sum_{\beta=\gamma+1}^{\xi} C_{a_i^\lambda A^\beta} = \frac{1}{2^{|A^\lambda|-i+1}} \sum_{\beta=\gamma+1}^{\xi} C_{A^\lambda A^\beta},$$
$$C_{a_i^\lambda A^\gamma} = \frac{1}{2^{|A^\lambda|-i+1}} \cdot C_{A^\lambda A^\gamma}. \tag{3.277}$$

From Lemma C.0.3 we have

$$\Sigma_{22} = \sum_{\beta=\gamma-1}^{\xi-2} C_{a_i^\lambda V^\beta} = \frac{1}{2^{|A^\lambda|-i+1}} \sum_{\beta=\gamma-1}^{\xi-2} (C_{A^\lambda A^{\beta+1}} - 2C_{A^\lambda A^{\beta+2}}). \tag{3.278}$$

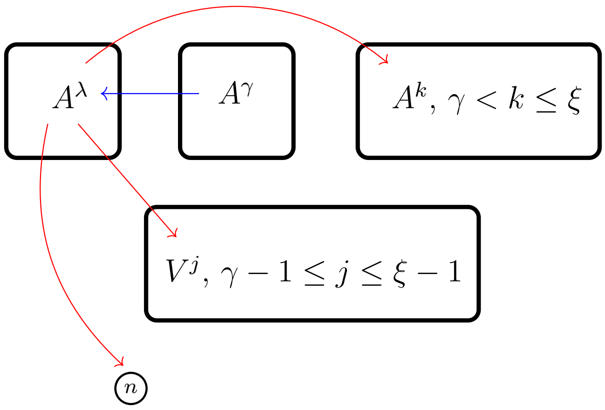

Figure 3.19: Sketch of ingoing and outgoing arcs $(u,w) \in \overset{\leftrightarrow}{E}_n$ with $f_{u,w} > 0$ for the vertices from A^λ, $\lambda < \gamma$.

In view of (3.277)-(3.278) equations (3.275) and (3.276) take the form

$$\begin{aligned} f(\delta^{out}(a_i^\lambda)) &= \frac{1}{2^{|A^\lambda|-i+1}} \sum_{\beta=\gamma+1}^{\xi} C_{A^\lambda A^\beta} + \frac{1}{2^{|A^\lambda|-i+1}} \sum_{\beta=\gamma-1}^{\xi-2} (C_{A^\lambda A^{\beta+1}} - 2C_{A^\lambda A^{\beta+2}}) \\ &\quad + C_{a_i^\lambda V^{\xi-1}} + c_{a_i^\lambda n} \\ &= \frac{1}{2^{|A^\lambda|-i+1}} \left(\sum_{\beta=\gamma+1}^{\xi} C_{A^\lambda A^\beta} + \sum_{\beta=\gamma}^{\xi-1} C_{A^\lambda A^\beta} - 2 \sum_{\beta=\gamma+1}^{\xi} C_{A^\lambda A^\beta} \right) \\ &\quad + C_{a_i^\lambda V^{\xi-1}} + c_{a_i^\lambda n} \\ &= \frac{1}{2^{|A^\lambda|-i+1}} (C_{A^\lambda A^\gamma} - C_{A^\lambda A^\xi}) + C_{a_i^\lambda V^{\xi-1}} + c_{a_i^\lambda n} \\ &= C_{a_i^\lambda V^{\xi-1}} + c_{a_i^\lambda n} - \frac{1}{2^{|A^\lambda|-i+1}} \cdot C_{A^\lambda A^\xi} + \frac{1}{2^{|A^\lambda|-i+1}} \cdot C_{A^\lambda A^\gamma} \\ &= C_{a_i^\lambda V^{\xi-1}} + c_{a_i^\lambda n} - C_{a_i^\lambda A^\xi} + \frac{1}{2^{|A^\lambda|-i+1}} \cdot C_{A^\lambda A^\gamma} \\ &\overset{\text{Lemma } C.0.4}{=} \frac{1}{2^{|A^\lambda|-i+1}} \cdot C_{A^\lambda A^\gamma}, \end{aligned} \tag{3.279}$$

$$f(\delta^{in}(a_i^\lambda)) = C_{a_i^\lambda A^\gamma} = \frac{1}{2^{|A^\lambda|-i+1}} \cdot C_{A^\lambda A^\gamma}. \tag{3.280}$$

4. For each $v_j^\lambda \in V^\lambda$, $\lambda < \gamma - 1$ we have due to (3.256)-(3.257), see Figure 3.20,

$$f(\delta^{out}(v_i^\lambda)) = \underbrace{\sum_{\beta=\gamma+1}^{\xi} C_{v_i^\lambda A^\beta}}_{\Sigma_{23}} + \underbrace{\sum_{\beta=\gamma-1}^{\xi-2} C_{v_i^\lambda V^\beta}}_{\Sigma_{24}} + C_{v_i^\lambda V^{\xi-1}} + c_{v_i^\lambda n}, \tag{3.281}$$

$$f(\delta^{in}(v_i^\lambda)) = C_{v_i^\lambda A^\gamma}. \tag{3.282}$$

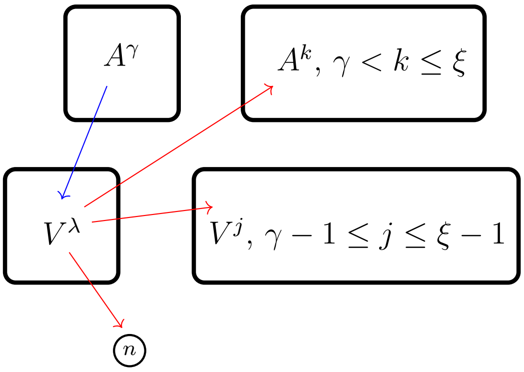

Figure 3.20: Sketch of ingoing and outgoing arcs $(u, w) \in \overleftrightarrow{E}_n$ with $f_{u,w} > 0$ for the vertices from V^λ, $\lambda < \gamma - 1$.

First, we reproduce the results obtained in Appendix C for the terms on the right-hand side of equations (3.281) and (3.282). It follows from Lemma C.0.2 that

$$\begin{aligned} \Sigma_{24} = \sum_{\beta=\gamma-1}^{\xi-2} C_{v_i^\lambda V^\beta} &= 2^i \sum_{\beta=\gamma-1}^{\xi-2} (C_{A^{\lambda+1}A^{\beta+1}} - 2C_{A^{\lambda+1}A^{\beta+2}}),\ \lambda < \gamma - 2 \\ C_{v_i^\lambda V^{\xi-1}} &= 2^i (C_{A^{\lambda+1}A^\xi} - C_{A^{\lambda+1}n}), \\ C_{v_i^\lambda V^{\lambda+1}} &= \frac{1}{2^{|V^\lambda|-i+2}} \left(1 - \frac{1}{2^{|V^{\lambda+1}|}}\right). \end{aligned} \tag{3.283}$$

From Lemma C.0.3 we have

$$\begin{aligned} \Sigma_{23} = \sum_{\beta=\gamma+1}^{\xi} C_{v_i^\lambda A^\beta} &= 2^i \sum_{\beta=\gamma+1}^{\xi} C_{A^{\lambda+1}A^\beta}, \\ C_{v_i^\lambda A^\gamma} &= 2^i \cdot C_{A^{\lambda+1}A^\gamma}. \end{aligned} \tag{3.284}$$

Using Lemma B.0.1 we obtain for (3.232)

$$\begin{aligned} c_{v_i^\lambda n} &= \frac{1}{2 \cdot 2^{|V^\lambda|-i} \prod\limits_{k=\lambda+1}^{\xi-1} 2^{|V^k|+1}} = \frac{1}{2 \cdot 2^{|V^\lambda|-i} \cdot 2^{\sum_{k=\lambda+1}^{\xi-1}(|V^k|+1)}} \\ &= \frac{2^i}{2^{\sum_{k=\lambda}^{\xi-1}(|V^k|+1)}} \overset{(B.3)}{=} 2^i C_{A^{\lambda+1}n}. \end{aligned} \tag{3.285}$$

If $\lambda < \gamma - 2$ then in view of (3.283)-(3.285) equations (3.281) and (3.282) take the form

$$\begin{aligned} f(\delta^{out}(v_i^\lambda)) = 2^i \sum_{\beta=\gamma+1}^{\xi} C_{A^{\lambda+1}A^\beta} &+ 2^i \sum_{\beta=\gamma-1}^{\xi-2} (C_{A^{\lambda+1}A^{\beta+1}} - 2C_{A^{\lambda+1}A^{\beta+2}}) \\ &+ 2^i (C_{A^{\lambda+1}A^\xi} - C_{A^{\lambda+1}n}) + 2^i C_{A^{\lambda+1}n} \end{aligned} \tag{3.286}$$

$$= 2^i \left(\sum_{\beta=\gamma+1}^{\xi} C_{A^{\lambda+1}A^\beta} + \sum_{\beta=\gamma}^{\xi-1} C_{A^{\lambda+1}A^\beta} - 2 \sum_{\beta=\gamma+1}^{\xi} C_{A^{\lambda+1}A^\beta} \right) + 2^i C_{A^{\lambda+1}A^\xi}$$
$$= 2^i C_{A^{\lambda+1}A^\gamma} - 2^i C_{A^{\lambda+1}A^\xi} + 2^i C_{A^{\lambda+1}A^\xi} = 2^i C_{A^{\lambda+1}A^\gamma}, \tag{3.287}$$

$$f(\delta^{in}(v_i^\lambda)) = C_{v_i^\lambda A^\gamma} = 2^i \cdot C_{A^{\lambda+1}A^\gamma}, \tag{3.288}$$

respectively.

If $\lambda = \gamma - 2$ equations (3.281) and (3.282) take the form

$$\begin{aligned} f(\delta^{out}(v_i^\lambda)) &= 2^i \sum_{\beta=\gamma+1}^{\xi} C_{A^{\gamma-1}A^\beta} + C_{v_i^{\gamma-2}V^{\gamma-1}} + 2^i \sum_{\beta=\gamma}^{\xi-2} (C_{A^{\gamma-1}A^{\beta+1}} - 2C_{A^{\gamma-1}A^{\beta+2}}) \\ &\quad + 2^i (C_{A^{\gamma-1}A^\xi} - C_{A^{\gamma-1}n}) + 2^i C_{A^{\gamma-1}n} \\ &= 2^i \left(\sum_{\beta=\gamma+1}^{\xi} C_{A^{\gamma-1}A^\beta} + \sum_{\beta=\gamma+1}^{\xi-1} C_{A^{\gamma-1}A^\beta} - 2 \sum_{\beta=\gamma+2}^{\xi} C_{A^{\gamma-1}A^\beta} \right) \\ &\quad + \frac{1}{2^{|V^{\gamma-2}|-i+2}} \left(1 - \frac{1}{2^{|V^{\gamma-1}|}} \right) + 2^i C_{A^{\gamma-1}A^\xi} \\ &= 2 \cdot 2^i C_{A^{\gamma-1}A^{\gamma+1}} - 2^i C_{A^{\gamma-1}A^\xi} + \frac{1}{2^{|V^{\gamma-2}|-i+2}} \\ &\quad - \frac{1}{2^{|V^{\gamma-2}|-i+2} \cdot 2^{|V^{\gamma-1}|}} + 2^i C_{A^{\gamma-1}A^\xi} \\ &= \frac{2 \cdot 2^i}{2 \cdot 2^{|V^{\gamma-2}|+1} \cdot 2^{|V^{\gamma-1}|+1}} + \frac{1}{2^{|V^{\gamma-2}|-i+2}} - \frac{1}{2^{|V^{\gamma-2}|-i+2} \cdot 2^{|V^{\gamma-1}|}} \\ &= \frac{2^i}{2 \cdot 2^{|V^{\gamma-2}|+1}} = 2^i C_{A^{\gamma-1}A^\gamma}, \end{aligned} \tag{3.289}$$

$$f(\delta^{in}(v_i^\lambda)) = C_{v_i^{\gamma-2}A^\gamma} = 2^i \cdot C_{A^{\gamma-1}A^\gamma}, \tag{3.290}$$

respectively.

5. For each $v_i^\lambda \in V^\lambda$ with $\lambda \geq \gamma - 1$, $\lambda \in [\xi - 1]$ we have due to (3.256)-(3.257), see Figure 3.21,

$$f(\delta^{out}(v_i^\lambda)) = \underbrace{\sum_{\beta=\lambda+2}^{\xi} C_{v_i^\lambda A^\beta}}_{\Sigma_{25}} + \underbrace{\sum_{\beta=\lambda+1}^{\xi-1} C_{v_i^\lambda V^\beta}}_{\Sigma_{26}} + c_{v_i^\lambda n} + \underbrace{\sum_{j=i+1}^{|V^\lambda|} c_{v_i^\lambda v_j^\lambda}}_{\Sigma_{27}}, \tag{3.291}$$

$$f(\delta^{in}(v_i^\lambda)) = \underbrace{\sum_{\beta=1}^{\lambda} C_{v_i^\lambda A^\beta}}_{\Sigma_{28}} + C_{v_i^\lambda A^{\lambda+1}} + \underbrace{\sum_{\beta=1}^{\lambda-1} C_{v_i^\lambda V^\beta}}_{\Sigma_{29}} + \underbrace{\sum_{j=1}^{i-1} c_{v_j^\lambda v_i^\lambda}}_{\Sigma_{30}}. \tag{3.292}$$

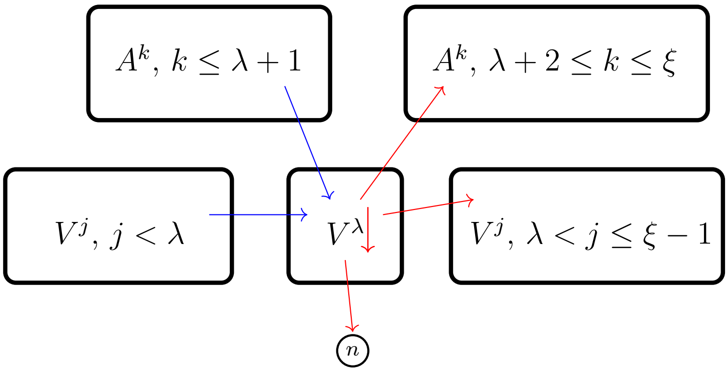

Figure 3.21: Sketch of ingoing and outgoing arcs $(u, w) \in \overleftrightarrow{E}_n$ with $f_{u,w} > 0$ for the vertices from V^λ, $\lambda \geq \gamma - 1$, $\lambda \in [\xi - 1]$.

Let us reproduce the results obtained in Appendix C for the terms on the right-hand side of equations (3.291) and (3.292). It follows from Lemma C.0.2 that

$$
\begin{aligned}
&\Sigma_{26}\left\{\begin{aligned}
C_{v_i^\lambda V^{\lambda+1}} &= \frac{1}{2^{|V^\lambda|-i+2}}\left(1 - \frac{1}{2^{|V^{\lambda+1}|}}\right),\\
\sum_{\beta=\lambda+2}^{\xi-2} C_{v_i^\lambda V^\beta} &= 2^i \sum_{\beta=\lambda+2}^{\xi-2} (C_{A^{\lambda+1}A^{\beta+1}} - 2C_{A^{\lambda+1}A^{\beta+2}}),\\
C_{v_i^\lambda V^{\xi-1}} &= 2^i (C_{A^{\lambda+1}A^\xi} - C_{A^{\lambda+1}n}),
\end{aligned}\right.\\
&\Sigma_{29}\left\{\begin{aligned}
C_{v_i^\lambda V^{\lambda-1}} &= \frac{1}{2^{i+1}}\left(1 - \frac{1}{2^{|V^{\lambda-1}|}}\right), \quad 1 < \lambda \leq \xi - 1\\
\sum_{\beta=1}^{\lambda-2} C_{v_i^\lambda V^\beta} &= \frac{1}{2^i}\sum_{\beta=1}^{\lambda-2} (C_{A^{\beta+2}A^{\lambda+1}} - 2C_{A^{\beta+1}A^{\lambda+1}}),
\end{aligned}\right.
\end{aligned}
\tag{3.293}
$$

From Lemma C.0.3 we have

$$
\begin{aligned}
\Sigma_{25} = \sum_{\beta=\lambda+2}^{\xi} C_{v_i^\lambda A^\beta} &= 2^i \sum_{\beta=\lambda+2}^{\xi} C_{A^{\lambda+1}A^\beta},\\
C_{v_i^\lambda A^{\lambda+1}} &= \frac{1}{2^{i+1}},\\
\Sigma_{28} = \sum_{\beta=1}^{\lambda} C_{v_i^\lambda A^\beta} &= \frac{1}{2^i}\sum_{\beta=1}^{\lambda} C_{A^\beta A^{\lambda+1}}.
\end{aligned}
\tag{3.294}
$$

From the definition of $c \in \mathbb{R}^{E_n}_{>0}$, see (3.224), we obtain

$$\begin{aligned}\Sigma_{27} &= \sum_{j=i+1}^{|V^\lambda|} c_{v_i^\lambda v_j^\lambda} = \sum_{j=i+1}^{|V^\lambda|} \frac{1}{2\cdot 2^{j-i}} = \frac{1}{2^{1-i}} \sum_{j=i+1}^{|V^\lambda|} \frac{1}{2^j} \\ &= \frac{1}{2^{1-i}} \cdot \frac{1}{2^{i+1}} \cdot \frac{(1-\frac{1}{2^{|V^\lambda|-i}})}{1-\frac{1}{2}} = \frac{1}{2} - \frac{1}{2^{|V^\lambda|-i+1}},\end{aligned} \tag{3.295}$$

$$\begin{aligned}\Sigma_{30} &= \sum_{j=1}^{i-1} c_{v_j^\lambda v_i^\lambda} = \sum_{j=1}^{i-1} \frac{1}{2\cdot 2^{i-j}} = \frac{1}{2^{i+1}} \sum_{j=1}^{i-1} 2^j \\ &= \frac{1}{2^{i+1}} \cdot \frac{2\cdot(1-2^{i-1})}{1-2} = \frac{2^{i-1}-1}{2^i} = \frac{1}{2} - \frac{1}{2^i}.\end{aligned} \tag{3.296}$$

In view of (3.285) and (3.293)-(3.295) equations (3.291) and (3.292) take the form

$$\begin{aligned}f(\delta^{out}(v_i^\lambda)) &= 2^i \sum_{\beta=\lambda+2}^{\xi} C_{A^{\lambda+1}A^\beta} + \frac{1}{2^{|V^\lambda|-i+2}}\left(1-\frac{1}{2^{|V^{\lambda+1}|}}\right) \\ &\quad + 2^i \sum_{\beta=\lambda+2}^{\xi-2} (C_{A^{\lambda+1}A^{\beta+1}} - 2C_{A^{\lambda+1}A^{\beta+2}}) + 2^i(C_{A^{\lambda+1}A^\xi} - C_{A^{\lambda+1}n}) \\ &\quad + 2^i C_{A^{\lambda+1}n} + \frac{1}{2} - \frac{1}{2^{|V^\lambda|-i+1}} \\ &= 2^i \left(\sum_{\beta=\lambda+2}^{\xi} C_{A^{\lambda+1}A^\beta} + \sum_{\beta=\lambda+3}^{\xi-1} C_{A^{\lambda+1}A^\beta} - 2\sum_{\beta=\lambda+4}^{\xi} C_{A^{\lambda+1}A^\beta}\right) \\ &\quad + \frac{1}{2^{|V^\lambda|-i+2}}\left(1-\frac{1}{2^{|V^{\lambda+1}|}}\right) + 2^i C_{A^{\lambda+1}A^\xi} - \cancel{2^i C_{A^{\lambda+1}n}} + \cancel{2^i C_{A^{\lambda+1}n}} \\ &\quad + \frac{1}{2} - \frac{1}{2^{|V^\lambda|-i+1}}\end{aligned} \tag{3.297}$$

$$\begin{aligned}&= 2^i \cdot C_{A^{\lambda+1}A^{\lambda+2}} + 2\cdot 2^i C_{A^{\lambda+1}A^{\lambda+3}} - \cancel{2^i C_{A^{\lambda+1}A^\xi}} + \cancel{2^i C_{A^{\lambda+1}A^\xi}} \\ &\quad + \frac{1}{2} - \frac{1}{2^{|V^\lambda|-i+1}} + \frac{1}{2^{|V^\lambda|-i+2}} - \frac{1}{2^{|V^\lambda|-i+2}\cdot 2^{|V^{\lambda+1}|}} \\ &\overset{(B.11)}{=} \cancel{\frac{2^i}{2\cdot 2^{|V^\lambda|+1}}} + \cancel{\frac{2\cdot 2^i}{2\cdot 2^{|V^\lambda|+1}\cdot 2^{|V^{\lambda+1}|+1}}} + \frac{1}{2} - \cancel{\frac{1}{2^{|V^\lambda|-i+1}}} \\ &\quad + \cancel{\frac{1}{2^{|V^\lambda|-i+2}}} - \cancel{\frac{1}{2^{|V^\lambda|-i+2}\cdot 2^{|V^{\lambda+1}|}}} = \frac{1}{2}, \quad \lambda < \xi - 1\end{aligned} \tag{3.298}$$

$$f(\delta^{out}(v_i^{\xi-1})) = \frac{2^i}{2^{|V^{\xi-1}|+1}} + \frac{1}{2} - \frac{1}{2^{|V^{\xi-1}|-i+1}} = \frac{1}{2}, \quad \lambda = \xi - 1 \tag{3.299}$$

$$
\begin{aligned}
f(\delta^{in}(v_i^\lambda)) &= \frac{1}{2^i}\sum_{\beta=1}^{\lambda} C_{A^\beta A^{\lambda+1}} + \frac{1}{2^{i+1}} + \frac{1}{2^i}\sum_{\beta=1}^{\lambda-2}(C_{A^{\beta+2}A^{\lambda+1}} - 2C_{A^{\beta+1}A^{\lambda+1}}) \\
&\quad + \frac{1}{2^{i+1}}\left(1-\frac{1}{2^{|V^{\lambda-1}|}}\right) + \frac{1}{2} - \frac{1}{2^i} \\
&= \frac{1}{2^i}\left(\sum_{\beta=1}^{\lambda} C_{A^\beta A^{\lambda+1}} + \sum_{\beta=3}^{\lambda} C_{A^\beta A^{\lambda+1}} - 2\sum_{\beta=2}^{\lambda-1} C_{A^\beta A^{\lambda+1}}\right) \\
&\quad + \cancel{\frac{1}{2^{i+1}}} + \cancel{\frac{1}{2^{i+1}}} - \frac{1}{2^{|V^{\lambda-1}|+i+1}} + \frac{1}{2} - \cancel{\frac{1}{2^i}} \\
&= \frac{1}{2^i}(\cancel{C_{A^1A^{\lambda+1}}} + 2C_{A^\lambda A^{\lambda+1}} - \cancel{C_{A^2A^{\lambda+1}}}) - \frac{1}{2^{|V^{\lambda-1}|+i+1}} + \frac{1}{2} \\
&= \frac{1}{2^i}\cdot\frac{1}{2^{|V^{\lambda-1}|+1}} - \frac{1}{2^{|V^{\lambda-1}|+i+1}} + \frac{1}{2} = \frac{1}{2}.
\end{aligned}
\tag{3.300}
$$

Hence, both conditions of the definition of a flow are satisfied for f, i.e.

$$
\begin{aligned}
&f_e \geq 0 && \text{for each } e \in \overset{\leftrightarrow}{E}_n, \\
&f(\delta^{out}(v)) = f(\delta^{in}(v)) && \text{for each } v \in [n] \setminus \{\gamma, n\}.
\end{aligned}
\tag{3.301}
$$

The value of this flow f is equal to 1 due to

$$
\text{value}(f) = f(\delta^{out}(\gamma)) = f(\delta^{out}(a_1^\gamma)) = c(\delta(A_1^\gamma)) \overset{\text{Lemma 3.7.2}}{=} 1. \tag{3.302}
$$

For each $M \subseteq [n-1]$ with $[\xi] \subseteq M$ we thus have (due to $c \geq \mathbf{0}$)

$$
1 \leq f(\delta^{out}(M)) \leq c(\delta(M)), \tag{3.303}
$$

hence Lemma 3.5.2 shows that $\delta(M^*)$ is c-minimal among the $[\xi]$-n-cuts for each $M^* \in \mathcal{M}$.

Now we show that all others $[\xi]$-n-cuts are not c-minimal.

Let $W \subseteq [n-1]$ such that $W \cap [\xi] \neq \emptyset$, $W \notin \mathcal{M}$ and $c(\delta(W)) = 1$.

Case 1: $W \cap [\xi] = \lambda$, $\mu \notin W$ for all $\mu \in [\xi] \setminus \{\lambda\}$. Let $W_1 := \bigcup_{i=1}^{\lambda} A^i \cup \bigcup_{j=1}^{\lambda-2} V^j$ and $W_2 := \bigcup_{i=1}^{\lambda-1} A^i \cup \bigcup_{j=1}^{\lambda-2} V^j$. On the one hand by Lemma 3.4.4 $W \subseteq W_1$ or $W_1 \subseteq W$ since $c(\delta(W)) = c(\delta(W_1)) = 1$ with $\lambda \in W$, $\lambda \in W_1$. On the other hand by Lemma 3.5.3 $W \cap W_2 = \emptyset$ since $c(\delta(W)) = c(\delta(W_2)) = 1$ with $\mu \in W_2 \setminus W$, $\lambda \in W \setminus W_2$. Thus $W \subseteq A^\lambda$.

As we have $\{\lambda\} = A_1^\lambda \subsetneq A_2^\lambda \subsetneq \ldots \subsetneq A_{|A^\lambda|}^\lambda = A^\lambda$ with $|A_{i+1}^\lambda \setminus A_i^\lambda| = 1$ for all $i \in [|A^\lambda|-1]$ and $\{\lambda\} \subseteq W \subseteq A^\lambda$ Lemma 3.5.4 implies $W \in \{A_1^\lambda, \ldots, A_{|A^\lambda|}^\lambda\}$.

Case 2: $W = [k] \subseteq [\xi]$. Let $W_1 := A^\mu$, $\mu \in [k]$ and $W_2 := \bigcup\limits_{i=1}^{k} A^i \cup \bigcup\limits_{j=1}^{k-2} V^j$. On the one hand by Lemma 3.4.4 $W \subseteq W_1$ or $W_1 \subseteq W$ since $c(\delta(W)) = c(\delta(W_1)) = 1$ with $\mu \in W$ and $\mu \in W_1$ for all $\mu \in [k]$. On the other hand by Lemma 3.4.4 $W \subseteq W_2$ or $W_2 \subseteq W$ since $c(\delta(W)) = c(\delta(W_2)) = 1$ with $1 \in W$ and $1 \in W_2$. Thus $\left(\bigcup\limits_{i=1}^{k} A^i \cup \bigcup\limits_{j=1}^{k-2} V^j\right) \subseteq W$.

Let $W' := \bigcup\limits_{i=1}^{k+1} A^i \cup \bigcup\limits_{j=1}^{k-1} V^j$. On the one hand by Lemma 3.4.4 $W \subseteq W'$ or $W' \subseteq W$ since $c(\delta(W)) = c(\delta(W')) = 1$ with $1 \in W$ and $1 \in W'$. On the other hand by Lemma 3.5.3 $W \cap A^{k+1} = \emptyset$ since $c(\delta(W)) = c(\delta(A^{k+1})) = 1$ with $1 \in W \setminus A^{k+1}$ and $k+1 \in A^{k+1} \setminus W$. Thus $W \subseteq \left(\bigcup\limits_{i=1}^{k} A^i \cup \bigcup\limits_{j=1}^{k-1} V^j\right)$.

By the definition of A^i, A^i_I, $I \in [|A^i|]$, $i \in [k]$ and V^j, V^j_I, $I \in [|V^j|]$, $j \in [k-1]$, see. (3.218)-(3.219), we have

$$\begin{aligned}\bigcup_{i=1}^{k} A^i \cup \bigcup_{j=1}^{k-2} V^j \subseteq \bigcup_{i=1}^{k} A^i \cup \bigcup_{j=1}^{k-2} V^j \cup V^{k-1}_1 \subseteq \bigcup_{i=1}^{k} A^i \cup \bigcup_{j=1}^{k-2} V^j \cup V^{k-1}_2 \subseteq \dots \\ \subseteq \bigcup_{i=1}^{k} A^i \cup \bigcup_{j=1}^{\lambda-2} V^j \cup V^{k-1}_{|V_{k-1}|} = \bigcup_{i=1}^{k} A^i \cup \bigcup_{j=1}^{k-1} V^j\end{aligned} \tag{3.304}$$

with $\left|\left(\bigcup\limits_{i=1}^{k} A^i \cup \bigcup\limits_{j=1}^{k-2} V^j \cup V^{k-1}_1\right) \setminus \left(\bigcup\limits_{i=1}^{k} A^i \cup \bigcup\limits_{j=1}^{k-2} V^j\right)\right| = 1$ and

$$\left|\left(\bigcup_{i=1}^{k} A^i \cup \bigcup_{j=1}^{k-2} V^j \cup V^{k-1}_{i+1}\right) \setminus \left(\bigcup_{i=1}^{k} A^i \cup \bigcup_{j=1}^{k-2} V^j \cup V^{k-1}_{i}\right)\right| = 1 \tag{3.305}$$

for all $i \in [|V^{k-1}| - 1]$. Then, as we have $\bigcup\limits_{i=1}^{k} A^i \cup \bigcup\limits_{j=1}^{k-2} V^j \subseteq W \subseteq \bigcup\limits_{i=1}^{k} A^i \cup \bigcup\limits_{j=1}^{k-1} V^j$ Lemma 3.5.4 implies

$$W \in \left\{\bigcup_{i=1}^{k} A^i \cup \bigcup_{j=1}^{k-2} V^j,\quad \bigcup_{i=1}^{k} A^i \cup \bigcup_{j=1}^{k-2} V^j \cup V^{k-1}_1,\quad \dots,\quad \bigcup_{i=1}^{k} A^i \cup \bigcup_{j=1}^{k-2} V^j \cup V^{k-1}_{|V^{k-1}|}\right\}.$$

Case 3: $|W \cap [\xi]| > 1$, $W \neq [k]$ for all $k \in [\xi]$. Let α, $\beta \in W$ with $\beta < \alpha$ and $\tilde{\alpha} \in [\xi] \setminus W$ with $\tilde{\alpha} < \alpha$ and $\mu := \max\{\beta, \tilde{\alpha}\}$. Let us define $W' := \bigcup\limits_{i=1}^{\mu} A^i \cup \bigcup\limits_{j=1}^{\mu-2} V^j$ then by Lemma 3.5.3 $W \cap W' = \emptyset$ since $c(\delta(W)) = c(\delta(W')) = 1$ with $\alpha \in W \setminus W'$ and $\tilde{\alpha} \in W' \setminus W$. On the other hand by Lemma 3.4.4 $W \subseteq W'$ or $W' \subseteq W$ since $\beta \in W$ and $\beta \in W'$. Thus $W = \emptyset$.

Thereby all cuts $\delta(W)$ with $W \subseteq [n-1]$, $W \cap [\xi] \neq \emptyset$ and $W \notin \mathcal{M}$ are not minimal. $\square$

The above proposition 3.7.1 provide the combinatorial structure for a part of the faces of the undominated complex for the $[\xi]$-n-cut polytope $P_{[\xi]}(K_n)$ for the complete graph on n nodes. Based on the results obtained in the previous sections 3.4–3.7, we propose a conjecture for the complete description of the combinatorial structure of the undominated complex of the $[\xi]$-n-cut polytope for the complete graph on n nodes.

Conjecture 3.7.3. *The undominated complex of the $[\xi]$-n-cut polytope $P_{[\xi]}(K_n) \subseteq \mathbb{R}^{\binom{n}{2}}$ for the complete graph on n nodes is a pure simplicial complex of dimension $n + \xi - 3$ whose facets are described by*

$$\operatorname{conv}\{\chi(\delta(M)) : \ M \in \mathcal{M}\} \tag{3.306}$$

for all $\mathcal{M} \subseteq 2^{[n-1]}$ generated by Algorithm 3.7.4.

Algorithm 3.7.4.
Finding the vertex set of a facet of the undominated complex for $P_{[\xi]}(K_n)$.

Input: Finite sets $[n-1] = \{1, 2, \ldots, n-1\}$, $n \geq 2$ and $[\xi] = \{1, 2, \ldots, \xi\}$, $\xi < n$.

Output: A set $\mathcal{M} \subseteq 2^{[n-1]}$ such that $\operatorname{conv}\{\chi(\delta(M)) : M \in \mathcal{M}\}$ is a facet of $P_{[\xi]}(K_n)$.

Set $\overline{V} := [n-1] \setminus [\xi]$.
Set $\mathcal{M}^{max} := \{\{1\}, \{2\}, \ldots, \{\xi\}\}$.
Set $\mathcal{M} := \mathcal{M}^{max}$.
while $\overline{V} \neq \emptyset$ **do**
  Choose M^1, $M^2 \in \mathcal{M}^{max}$.
  Choose V^1, $V^2 \subseteq \overline{V}$ with $V^1 \cap V^2 = \emptyset$
    orderings $V^1 = \{v_1^1, v_2^1, \ldots, v_{|V^1|}^1\}$ with $V_I^1 := \{v_1^1, v_2^1, \ldots, v_I^1\}$,
    $V^2 = \{v_1^2, v_2^2, \ldots, v_{|V^2|}^2\}$ with $V_I^2 := \{v_1^2, v_2^2, \ldots, v_I^2\}$.
  Set $\overline{V} := \overline{V} \setminus (V^1 \cup V^2)$.
  Set $\mathcal{M}^{max} := \mathcal{M}^{max} \setminus (M^1 \cup M^2) \cup \{M^1 \cup V^1 \cup M^2 \cup V^2\}$.
  Set $\mathcal{M} := \mathcal{M} \cup \{M^1 \cup V_I^1 : I \in [|V^1|]\} \cup \{M^2 \cup V_I^2 : I \in [|V^2|]\} \cup \{M^1 \cup V^1 \cup M^2 \cup V^2\}$.
end

Remark 3.7.5. *For the case $\xi = 1$, $\xi = 2$ and $\xi = 3$ we proved Conjecture 3.7.3 in Theorem 3.4.7, Theorem 3.5.6 and Theorem 3.6.4, respectively. To prove Conjecture 3.7.3 for general ξ it should be found for each $\mathcal{M}$ generated by Algorithm 3.7.4 some $c \in \mathbb{R}_{>0}^{E_n}$ such that for all $M^* \subseteq [n-1]$ with $M^* \cap [\xi] \neq \emptyset$*

$$\min\{c(\delta(M)) : M \subseteq [n-1], \ M \cap \{1, 2\} \neq \emptyset\} = c(\delta(M^*)) \tag{3.307}$$

holds if and only if $M^* \in \mathcal{M}$*. In the proof of Proposition 3.7.1 we presented the function* $c \in \mathbb{R}^{E_n}_{>0}$ *for* $\mathcal{M}$ *with the structure as in* (3.220) *(see Figure 3.16). For the other cases of* $\mathcal{M}$ *it is still an open problem to construct such a function* $c \in \mathbb{R}^{E_n}_{>0}$*.*

Remark 3.7.6. *In the particular case considered in Proposition 3.7.1, the notation used for the structure of* $\mathcal{M}$ *defined by* (3.220) *differs from the notation used for a set* $\mathcal{M}$ *computed by Algorithm 3.7.4. In* (3.220)*, there are* A^γ_I*,* $I \in [|A^\gamma|]$ *and* A^γ *that would have been written similar to Algorithm 3.7.4 as* $A^\gamma_I = \{\gamma\} \cup V_I$ *with* $V_I = \{a^\gamma_2, a^\gamma_3, \ldots, a^\gamma_I\}$ *and* $A^\gamma = \{\gamma\} \cup V$ *with* $V = \{a^\gamma_2, a^\gamma_3, \ldots, a^\gamma_{|A^\gamma|}\}$ *for all* $\gamma \in [\xi]$*, respectively.*

Example 3.7.7. *Consider the* $[\xi]$*-*n*-cut polytope for the complete graph on* n *nodes with* $\xi = 6$*. Figure 3.23 illustrates one of the possible variants of the set* $\mathcal{M}$ *generated by Algorithm 3.7.4 for* $P_{[6]}(K_n)$*.*

A set $\mathcal{M} \subseteq 2^{[n-1]}$ generated by Algorithm 3.7.4 is a maximal laminar family of non-empty subsets of $[n-1]$ with singleton sets $\{1\}$, $\{2\}$, ..., $\{\xi\}$. A family of sets $\mathcal{F}$ is said to be *laminar* if $F_1 \cap F_2 = \emptyset$ or $F_1 \subseteq F_2$ or $F_2 \subseteq F_1$ for all F_1, $F_2 \in \mathcal{F}$, i.e., a laminar family $\mathcal{F}$ is a family of sets with no pair of intersecting sets, see Figure 3.22.

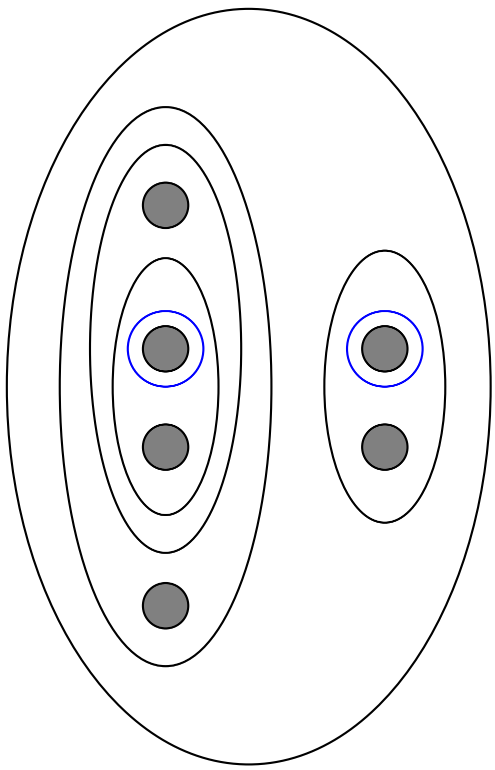

Figure 3.22: A sample laminar family with two singleton sets.

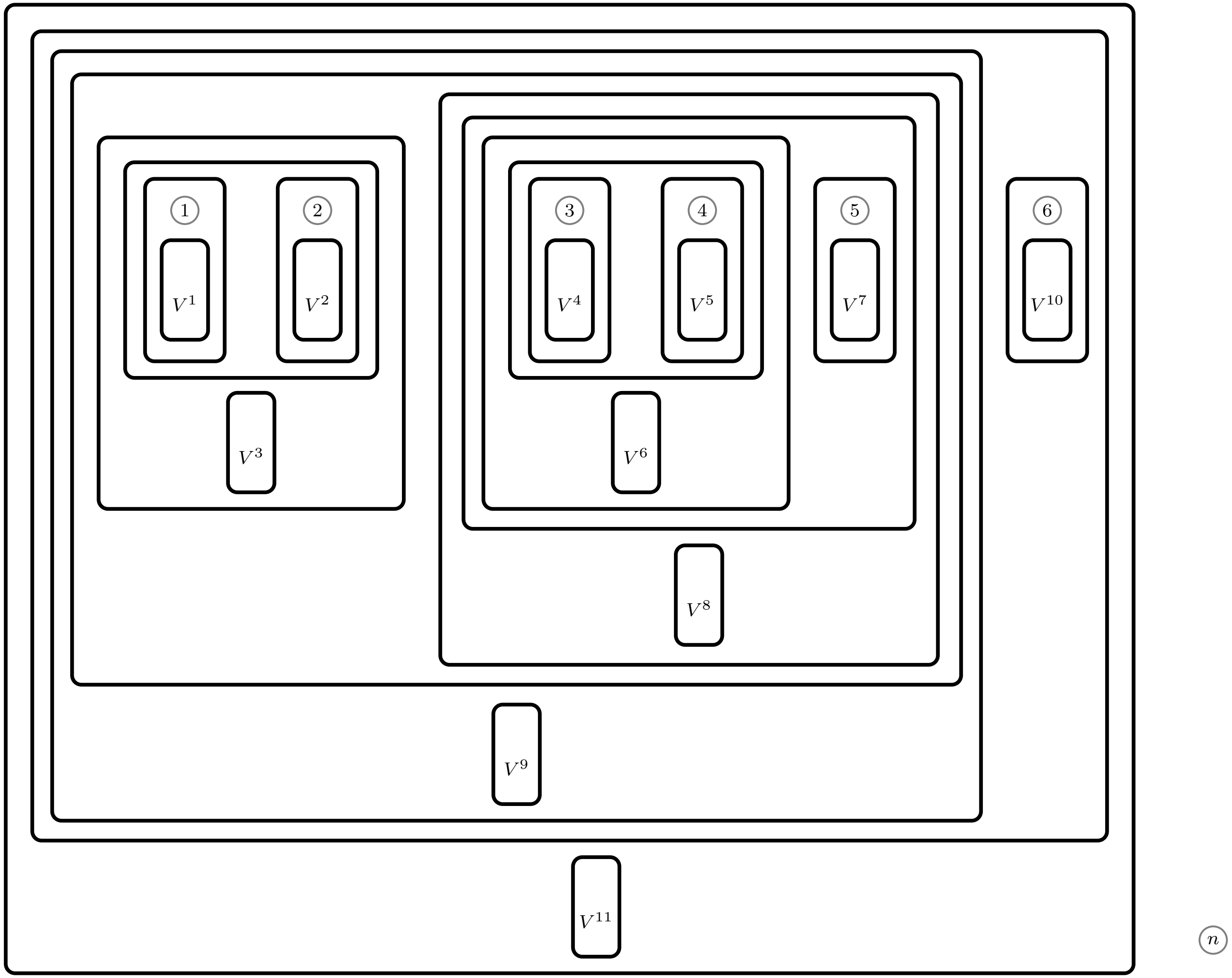

Figure 3.23: Representation of a set $\mathcal{M}$ generated by Algorithm 3.7.4 for $P_{[6]}(K_n)$.

Appendix A

A.1 Undominated Complex of a polytope in $\mathbb{R}^3$ (Using polymake)

In this section will be presented the program code of finding the undominated complex of a polytope $P \subseteq \mathbb{R}^3$ which is given as a convex hull of a finite number of vertices, i.e.

$$P := \text{conv}(V) = \text{conv}\{v_1, v_2, ..., v_n\} \subseteq \mathbb{R}^3.$$

The function **UndomComplex 3D(P)** is written in *Polymake*/Perl programming language. It computes undominated complex $U_c(P)$ of the given polytope $P \subseteq \mathbb{R}^3$ and visualizes it. For more information about this language, we refer by the following link

`https://polymake.org/doku.php/tutorial/start.`

A.1.1 Structure of the function UndomComplex 3D

The function **UndomComplex 3D** has the following structure.

Input: a polytope $P \subseteq \mathbb{R}^3$ as a convex hull of a finite number of vertices v_1,v_2,. . .,$v_n \in \mathbb{R}^3$.

Output: an array of the size three, where

- the first element is an array of the vertices belonging to $U_c(P)$;
- the second element is an array of the edges belonging to $U_c(P)$;
- the third element is an array of the facets belonging to $U_c(P)$.

Note that, each edge and each facet is represented as a set of vertices.

A.1.2 Description of the function UndomComplex 3D

The function **UndomComplex 3D(P)** works as follows:

First, the function computes all vertices of the undominated complex of P. By Lemma 2.3.1 a vertex v_i of P belongs to the undominated complex of P if and only if the normal cone of the polytope P at the vertex v_i has a non-empty intersection with the negative octant $\mathbb{R}^3_{<0}$. The function produces a normal cone $N_{v_i}(P)$ of P at each vertex v_i, $i \in [n]$ and chooses v_i with $N_{v_i}(P) \cap \mathbb{R}^3_{<0} \neq \emptyset$.

The second step is to compute all edges of the undominated complex of P. By Lemma 2.3.1 an edge e of P belongs to the undominated complex of P if and only if the normal cone of the polytope P at the edge e has a non-empty intersection with the negative octant $\mathbb{R}^3_{<0}$. The function computes a normal cone of the polytope P at each edge $e = \{v_i, v_j\}$, i, $j \in [n]$, $i \neq j$ as an intersection of normal cones at endpoints v_i, v_j of the edge e and chooses e with $N_e(P) \cap \mathbb{R}^3_{<0} \neq \emptyset$.

Finally, the function computes all facets of the undominated complex of P. By Lemma 2.3.1 a facet of P belongs to the undominated complex of P if and only if the normal cone of the polytope P at the facet intersects the negative octant $\mathbb{R}^3_{<0}$, i.e., all coordinates of the outer normal vector to the facet are negative. The function chooses facets with negative outer normal vector.

A.1.3 Manual of the function UndomComplex 3D

Step 1: Launch polymake and enter

```
prefer("threejs");
```

It leads to visualization of a polytope P in a browser tab (a browser must support javascript, e.g. Google Chrome, Internet Explorer, Opera, Mozilla Firefox etc.)

Step 2: Define a polytope P as the convex hull of a finite point set, for example

```
$my_polytope = new Polytope(POINTS=>[
                [1,0,0,1],[1,-0.2,1,1],[1,0,1,0],[1,0.8,0,0],
                     [1,1,0,1],[1,1,1,1],[1,1,1,0]]);
```

Note that, Polymake uses homogeneous coordinates what implies the additional coordinate $x_0 = 1$. For more on the construction of a polytope, we refer by the following link

https://polymake.org/doku.php/tutorial/apps_polytope.

Step 3: Call the script "UndomComplex 3D" with the code provided in section A.1.4 and pass the polytope P as an argument, for example

```
@array_of_undominated_faces=
        script("path_to_the_script/UndomComplex 3D", $my_polytope);
```

It prints out all faces of the undominated complex of P and visualizes it. Vertices, edges and facets of the undominated complex are stored separately in

```
@array_of_undominated_faces.
```

Thus, the function **UndomComplex 3D** returns for the polytope P defined above in Step 2 the following result

```
UNDOMINATED COMPLEX:

Vertices: {0 1 2 3}

Edges:
{0 1}
{0 2}
{1 2}
{0 3}
{2 3}

Facets:
{0 1 2}
{0 2 3}
```

with the visualization in a browser as in Figure A.1.

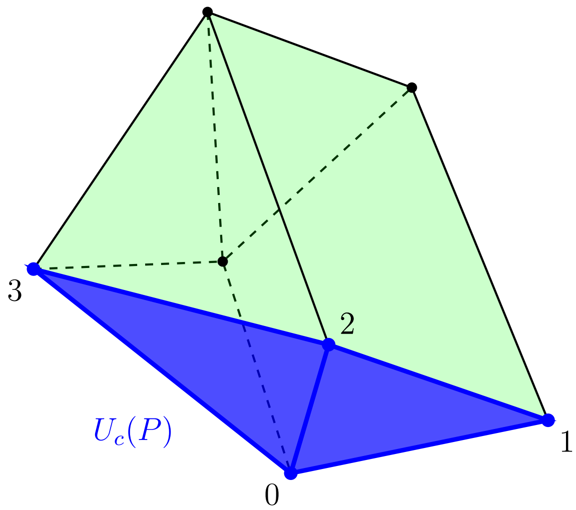

Figure A.1: Visualization of the polytope P and its undominated complex $U_c(P)$ computed by the function UndomComplex 3D.

A.1.4 Program code of the function UndomComplex 3D

```
use application 'polytope';
package threejs;
package ppl;
package Polymake::User;

#polytope:
my $f_pol = $_[0];
my $set_vert = new Set();

my @facets = $f_pol -> FACETS;
my @vert_in_f =  $f_pol -> VERTICES_IN_FACETS;
my @edges = $f_pol -> GRAPH -> EDGES;

#arrays of vertices, edges aund facets of the undominated complex:
my @undom_vertex_arr = new Array < Set >(1);
my @undom_edges_arr = new Array < Set >($f_pol -> N_EDGES);
my @undom_facets_arr = new Array < Set >($f_pol -> N_FACETS);

my $i = 0; my $j = 0; my $k = 0;
my $arr_i = 0;


#computes vertices of the undominated complex:
my $R_ = new Cone ( INEQUALITIES => [ [-1, 0, 0], [0, -1, 0], [0, 0, -1] ]);
for ($i = 0; $i < $f_pol -> N_VERTICES; $i++)
{
  my $cone_of_v = new Cone ( normal_cone($f_pol, $i, 1) );
  my $inter = intersection($R_, $cone_of_v);
  if ($inter -> DIM == 3)
 {
   my $t_set = new Set($i);
   $set_vert += $t_set;
  }
}
print "\nUNDOMINATED COMPLEX:\n";

print "Vertices: ", $set_vert, "\n";
$undom_vertex_arr[0][0] = $set_vert;

print "Edges:\n";
```

```
#computes edges of the undominated complex:
for ($i = 0; $i < $f_pol -> N_EDGES; $i++)
{
  my $t_edge = $set_vert * $edges[0][$i];
  if ($t_edge -> size() == 2)
 {
   my $v = new Cone (normal_cone ($f_pol, $t_edge -> front(), 1) );
   my $u = new Cone (normal_cone ($f_pol, $t_edge -> back(), 1) );
   my $v_u = intersection($v, $u);
   my $inter = intersection($v_u, $R_);
   if ($inter -> DIM == 2)
  {
    print $t_edge, "\n";
    $undom_edges_arr[0][$arr_i] = $t_edge;
    $arr_i++;
   }
  }
 }


#makes array of edges of the undominated complex:
my @undom_edges_arr1 = new Array < Set >($arr_i);
for ($i = 0; $i < $arr_i; $i++)
{
  $undom_edges_arr1[0][$i] = $undom_edges_arr[0][$i];
 }

print "Facets:\n";

#computes facets of the undominated complex:
$f_pol ->VISUAL( VertexColor => sub
{
  $i = shift;
  if ($set_vert -> contains($i) == 1)
  {
    new RGB(255, 255, 0);
   }
  else
 {
   new RGB(255, 0, 0);
  }
 },
  VertexThickness => 3,
```

```
  FacetColor => sub
 {
   $j = shift;
  if ($facets[0][$j][1] > 0 && $facets[0][$j][2] > 0 && $facets[0][$j][3] > 0)
    {
      print $vert_in_f[0][$j], "\n";
      $undom_facets_arr[0][$arr_i] = $vert_in_f[0][$j];
      $arr_i++;
      return new RGB(255, 255, 0);
    }
  else
   {
     new RGB(0, 255, 0);
    }
});


#makes array of facets of the undominated complex:
my @undom_facets_arr1 = new Array < Set >($arr_i);
for ($i = 0; $i < $arr_i; $i++)
{
  $undom_facets_arr1[0][$i] = $undom_facets_arr[0][$i];
 }


#creates array of undominated faces as a list of vertices, edges and facets:
my @undom_faces_arr = new Array < Array < Set > >(3);
$undom_faces_arr[0][0] = $undom_vertex_arr[0];
$undom_faces_arr[0][1] = $undom_edges_arr1[0];
$undom_faces_arr[0][2] = $undom_facets_arr1[0];

return @undom_faces_arr;
```

Appendix B

In this chapter, we provide the proofs of the auxiliary results which were used in the proof of Lemma 3.7.2.

Lemma B.0.1. *Let* $c \in \mathbb{R}^{E_n}_{>0}$ *be defined by* (3.223)-(3.232) *and*

$$C_{A^\gamma_I n} := \sum_{i=1}^{I} c_{a^\gamma_i n},\ I \in [|A^\gamma|] \quad \textit{and} \quad C_{V^\lambda_I n} := \sum_{j=1}^{I} c_{v^\lambda_j n},\ I \in [|V^\lambda|]. \tag{B.1}$$

Then, for all $\gamma \in [\xi]$ *and* $\lambda \in [\xi - 1]$, *the equations*

$$\begin{aligned} C_{A^\gamma_I n} &= \frac{2^I}{2^{|A^\gamma|}} \cdot C_{A^\gamma n}, \quad I \in [|A^\gamma|] \\ C_{V^\lambda_I n} &= 2 \cdot (2^I - 1) \cdot C_{A^{\lambda+1} n}, \quad I \in [|V^\lambda|] \end{aligned} \tag{B.2}$$

hold, where

$$C_{A^\gamma n} = C_{A^\gamma_{|A^\gamma|} n} = \frac{1}{2^{\sum\limits_{k=\gamma-1}^{\xi-1}(|V^k|+1)}}. \tag{B.3}$$

Proof. Let $\gamma \in [\xi]$. For the first term of (B.1), we have

$$\begin{aligned} C_{A^\gamma_I n} &= \sum_{i=1}^{I} c_{a^\gamma_i n} = c_{a^\gamma_1 n} + \sum_{i=2}^{I} c_{a^\gamma_i n} = c_{a^\gamma_2 n} + \sum_{i=2}^{I} c_{a^\gamma_i n} \\ &= \frac{1}{2 \cdot 2^{|A^\gamma|-2} \cdot \prod\limits_{k=\gamma-1}^{\xi-1} 2^{|V^k|+1}} + \sum_{i=2}^{I} \frac{1}{2 \cdot 2^{|A^\gamma|-i} \cdot \prod\limits_{k=\gamma-1}^{\xi-1} 2^{|V^k|+1}} \\ &= \frac{1}{2 \cdot 2^{|A^\gamma|-2} \cdot 2^{\sum\limits_{k=\gamma-1}^{\xi-1}(|V^k|+1)}} + \sum_{i=2}^{I} \frac{1}{2 \cdot 2^{|A^\gamma|-i} \cdot 2^{\sum\limits_{k=\gamma-1}^{\xi-1}(|V^k|+1)}} \\ &= \frac{1}{2 \cdot 2^{|A^\gamma|-2} \cdot 2^{\sum\limits_{k=\gamma-1}^{\xi-1}(|V^k|+1)}} \left(1 + \frac{1}{2^2} \sum_{i=2}^{I} 2^i \right) \end{aligned} \tag{B.4}$$

$$
\begin{aligned}
&= \frac{1}{2 \cdot 2^{|A^\gamma|-2} \cdot 2^{\sum_{k=\gamma-1}^{\xi-1}(|V^k|+1)}} \left(1 + \frac{1}{2^2} \cdot \frac{2^2 \cdot (1-2^{I-1})}{1-2}\right) \\
&= \frac{2^{I-1}}{2 \cdot 2^{|A^\gamma|-2} \cdot 2^{\sum_{k=\gamma-1}^{\xi-1}(|V^k|+1)}} = \frac{2^I}{2^{|A^\gamma|}} \cdot \frac{1}{2^{\sum_{k=\gamma-1}^{\xi-1}(|V^k|+1)}}
\end{aligned} \tag{B.5}
$$

From the definition of A^γ and A_I^γ, $I \in [|A^\gamma|]$, see (3.218)-(3.219), we have $A^\gamma = A_{|A^\gamma|}^\gamma$ for all $\gamma \in [\xi]$. Thus, we deduce

$$
C_{A^\gamma n} = C_{A_{|A^\gamma|}^\gamma n} = \frac{2^{|A^\gamma|}}{2^{|A^\gamma|} \cdot 2^{\sum_{k=\gamma-1}^{\xi-1}(|V^k|+1)}} = \frac{1}{2^{\sum_{k=\gamma-1}^{\xi-1}(|V^k|+1)}} \tag{B.6}
$$

what implies that $C_{A_I^\gamma n}$ can be written as

$$
C_{A_I^\gamma n} = \frac{2^I}{2^{|A^\gamma|}} \cdot C_{A^\gamma n}, \quad I \in [|A^\gamma|]. \tag{B.7}
$$

Let $\lambda \in [\xi - 1]$. Calculating the second term of (B.1) we obtain

$$
\begin{aligned}
C_{V_I^\lambda n} &= \sum_{j=1}^{I} c_{v_j^\lambda n} = \sum_{j=1}^{I} \frac{2^j}{\prod_{k=\lambda}^{\xi-1} 2^{|V^k|+1}} = \sum_{j=1}^{I} \frac{2^j}{2^{\sum_{k=\lambda}^{\xi-1}(|V^k|+1)}} = \frac{1}{2^{\sum_{k=\lambda}^{\xi-1}(|V^k|+1)}} \sum_{j=1}^{I} 2^j \\
&= \frac{1}{2^{\sum_{k=\lambda}^{\xi-1}(|V^k|+1)}} \cdot \frac{2 \cdot (1-2^I)}{1-2} = \frac{2 \cdot (2^I - 1)}{2^{\sum_{k=\lambda}^{\xi-1}(|V^k|+1)}} \overset{(B.6)}{=} 2 \cdot (2^I - 1) \cdot C_{A^{\lambda+1} n}
\end{aligned} \tag{B.8}
$$

□

Lemma B.0.2. *Let $c \in \mathbb{R}_{>0}^{E_n}$ be defined by (3.223)-(3.232) and*

$$
C_{A_I^\lambda A^\gamma} := \sum_{i=1}^{I} \sum_{j=1}^{|A^\gamma|} c_{a_i^\lambda a_j^\gamma},\ I \in [|A^\lambda|] \quad \text{and} \quad C_{A^\lambda A_I^\gamma} := \sum_{i=1}^{|A^\lambda|} \sum_{j=1}^{I} c_{a_i^\lambda a_j^\gamma},\ I \in [|A^\gamma|]. \tag{B.9}
$$

Then, for all $1 \leq \lambda < \gamma \leq \xi$, the equations

$$
C_{A_I^\lambda A^\gamma} = \frac{2^I}{2^{|A^\lambda|}} C_{A^\lambda A^\gamma},\ I \in [|A^\lambda|] \quad \text{and} \quad C_{A^\lambda A_I^\gamma} = \frac{2^I}{2^{|A^\gamma|}} C_{A^\lambda A^\gamma},\ I \in [|A^\gamma|] \tag{B.10}
$$

hold, where

$$
C_{A^\lambda A^\gamma} = C_{A_{|A^\lambda|}^\lambda A^\gamma} = \begin{cases} \dfrac{1}{2 \cdot 2^{\sum_{k=\lambda-1}^{\gamma-2}(|V^k|+1)}}, & \lambda > 1,\ \gamma > 2 \\ C_{A^2 A^\gamma}, & \lambda = 1,\ \gamma > 2 \\ \dfrac{1}{2}, & \lambda = 1,\ \gamma = 2 \end{cases} \tag{B.11}
$$

Proof. Let $1 < \lambda < \gamma \leq \xi$. For the first term of (B.9) we obtain

$$
\begin{aligned}
C_{A_I^\lambda A^\gamma} &= \sum_{i=1}^{I} \sum_{j=1}^{|A^\gamma|} c_{a_i^\lambda a_j^\gamma} = c_{a_1^\lambda a_1^\gamma} + \sum_{i=2}^{I} c_{a_i^\lambda a_1^\gamma} + \sum_{j=2}^{|A^\gamma|} c_{a_1^\lambda a_j^\gamma} + \sum_{i=2}^{I} \sum_{j=2}^{|A^\gamma|} c_{a_i^\lambda a_j^\gamma} \\
&= c_{a_2^\lambda a_2^\gamma} + \sum_{i=2}^{I} c_{a_i^\lambda a_2^\gamma} + \sum_{j=2}^{|A^\gamma|} c_{a_2^\lambda a_j^\gamma} + \sum_{i=2}^{I} \sum_{j=2}^{|A^\gamma|} c_{a_i^\lambda a_j^\gamma} \\
&= \frac{1}{2^3 \cdot 2^{|A^\lambda|-2} \cdot 2^{|A^\gamma|-2} \prod\limits_{k=\lambda-1}^{\gamma-2} 2^{|V^k|+1}} + \sum_{i=2}^{I} \frac{1}{2^3 \cdot 2^{|A^\lambda|-i} \cdot 2^{|A^\gamma|-2} \prod\limits_{k=\lambda-1}^{\gamma-2} 2^{|V^k|+1}} \\
&\quad + \sum_{j=2}^{|A^\gamma|} \frac{1}{2^3 \cdot 2^{|A^\lambda|-2} \cdot 2^{|A^\gamma|-j} \prod\limits_{k=\lambda-1}^{\gamma-2} 2^{|V^k|+1}} + \sum_{i=2}^{I} \sum_{j=2}^{|A^\gamma|} \frac{1}{2^3 \cdot 2^{|A^\lambda|-i} \cdot 2^{|A^\gamma|-j} \prod\limits_{k=\lambda-1}^{\gamma-2} 2^{|V^k|+1}} \\
&= \frac{1}{2^3 \cdot 2^{|A^\lambda|-2} \cdot 2^{|A^\gamma|-2} \cdot 2^{\sum_{k=\lambda-1}^{\gamma-2} (|V^k|+1)}} \\
&\qquad \times \left(1 + \frac{1}{2^2} \sum_{i=2}^{I} 2^i + \frac{1}{2^2} \sum_{j=2}^{|A^\gamma|} 2^j + \frac{1}{2^2} \sum_{i=2}^{I} 2^i + \frac{1}{2^2} \sum_{j=2}^{|A^\gamma|} 2^j \right) \qquad \text{(B.12)} \\
&= \frac{1}{2^3 \cdot 2^{|A^\lambda|-2} \cdot 2^{|A^\gamma|-2} \cdot 2^{\sum_{k=\lambda-1}^{\gamma-2} (|V^k|+1)}} \left(1 + \frac{1}{2^2} \sum_{i=2}^{I} 2^i \right) \left(1 + \frac{1}{2^2} \sum_{j=2}^{|A^\gamma|} 2^j \right) \\
&= \frac{1}{2^3 \cdot 2^{|A^\lambda|-2} \cdot 2^{|A^\gamma|-2} \cdot 2^{\sum_{k=\lambda-1}^{\gamma-2} (|V^k|+1)}} \\
&\qquad \times \left(1 + \frac{1}{2^2} \cdot \frac{2^2 \cdot (1 - 2^{I-1})}{1-2} \right) \left(1 + \frac{1}{2^2} \cdot \frac{2^2 (1 - 2^{|A^\gamma|-1})}{1-2} \right) \\
&= \frac{2^{I-1} \cdot 2^{|A^\gamma|-1}}{2^3 \cdot 2^{|A^\lambda|-2} \cdot 2^{|A^\gamma|-2} \cdot 2^{\sum_{k=\lambda-1}^{\gamma-2} (|V^k|+1)}} = \frac{2^I}{2^{|A^\lambda|} \cdot 2 \cdot 2^{\sum_{k=\lambda-1}^{\gamma-2} (|V^k|+1)}}.
\end{aligned}
$$

Due to the symmetry of c it follows from (B.12) by exchanging A^γ and A^λ that the second term of (B.9) takes the form

$$
C_{A^\lambda A_I^\gamma} = \frac{2^I}{2^{|A^\gamma|} \cdot 2 \cdot 2^{\sum_{k=\lambda-1}^{\gamma-2} (|V^k|+1)}}. \qquad \text{(B.13)}
$$

From the definition of A^γ and A^γ_I, $I \in [|A^\gamma|]$, see (3.218)-(3.219), we have $A^\gamma = A^\gamma_{|A^\gamma|}$ for all $\gamma \in [\xi]$. Thus, we deduce

$$C_{A^\lambda A^\gamma} = C_{A^\lambda_{|A^\lambda|} A^\gamma} = \frac{2^{|A^\lambda|}}{2^{|A^\lambda|} \cdot 2 \cdot 2^{\sum_{k=\lambda-1}^{\gamma-2}(|V^k|+1)}} = \frac{1}{2 \cdot 2^{\sum_{k=\lambda-1}^{\gamma-2}(|V^k|+1)}} \tag{B.14}$$

what implies that the terms of (B.9) can be written as

$$C_{A^\lambda_I A^\gamma} = \frac{2^I}{2^{|A^\lambda|}} \cdot C_{A^\lambda A^\gamma},\ I \in [|A^\lambda|] \quad \text{and} \quad C_{A^\lambda A^\gamma_I} = \frac{2^I}{2^{|A^\gamma|}} \cdot C_{A^\lambda A^\gamma},\ I \in [|A^\gamma|]. \tag{B.15}$$

Similarly to the case $\lambda > 1$ we obtain due to the definition of c, see (3.225),

$$C_{A^1 A^\gamma} = \frac{1}{2 \cdot 2^{\sum_{k=1}^{\gamma-2}(|V^k|+1)}},\ \gamma > 2 \quad \text{and} \quad C_{A^1 A^2} = \frac{1}{2}. \tag{B.16}$$

what implies

$$C_{A^1 A^\gamma} = C_{A^2 A^\gamma} \quad \text{for all} \quad 2 < \gamma \leq \xi. \tag{B.17}$$

□

Lemma B.0.3. *Let* $c \in \mathbb{R}^{E_n}_{>0}$ *be defined by* (3.223)-(3.232), λ, $\gamma \in [\xi - 1]$ *and*

$$C_{V^\lambda V^\gamma_I} := \sum_{i=1}^{|V^\lambda|} \sum_{j=1}^{I} c_{v^\lambda_i v^\gamma_j},\ I \in [|V^\gamma|] \quad \textit{and} \quad C_{V^\lambda_I V^\gamma} := \sum_{i=1}^{I} \sum_{j=1}^{|V^\gamma|} c_{v^\lambda_i v^\gamma_j},\ I \in [|V^\lambda|]. \tag{B.18}$$

1. *For* $\gamma > \lambda + 1$ *we have*

$$\begin{aligned} C_{V^\lambda V^\gamma_I} &= \left(1 - \frac{1}{2^I}\right) \cdot (C_{A^{\lambda+2} A^{\gamma+1}} - 2 \cdot C_{A^{\lambda+1} A^{\gamma+1}}), \quad \gamma \leq \xi - 1 \\ C_{V^\lambda_I V^\gamma} &= (2^I - 1) \cdot (2 \cdot C_{A^{\lambda+1} A^{\gamma+1}} - 4 \cdot C_{A^{\lambda+1} A^{\gamma+2}}), \quad \gamma < \xi - 1 \\ C_{V^\lambda_I V^{\xi-1}} &= 2 \cdot (2^I - 1) \cdot (C_{A^{\lambda+1} A^\xi} - C_{A^{\lambda+1} n}), \quad \gamma = \xi - 1 \end{aligned} \tag{B.19}$$

2. *For* $\gamma = \lambda + 1$ *we have*

$$\begin{aligned} C_{V^\lambda V^\gamma_I} &= \frac{1}{2}\left(1 - \frac{1}{2^I}\right)\left(1 - \frac{1}{2^{|V^\lambda|}}\right), \\ C_{V^\lambda_I V^\gamma} &= (2^I - 1)\frac{1}{2^{|V^\lambda|+1}}\left(1 - \frac{1}{2^{|V^\gamma|}}\right). \end{aligned} \tag{B.20}$$

Proof. 1. Let $\xi - 1 \geq \gamma > \lambda + 1$. Then, for the first term of (B.18) we deduce

$$\begin{aligned} C_{V^\lambda V^\gamma_I} &= \sum_{i=1}^{|V^\lambda|} \sum_{j=1}^{I} c_{v^\lambda_i v^\gamma_j} = \sum_{i=1}^{|V^\lambda|} \sum_{j=1}^{I} \frac{1}{2^{j-i+1} \prod_{k=\lambda}^{\gamma-1} 2^{|V^k|+1}} = \sum_{i=1}^{|V^\lambda|} \sum_{j=1}^{I} \frac{1}{2^{j-i+1} \cdot 2^{\sum_{k=\lambda}^{\gamma-1}(|V^k|+1)}} \\ &= \frac{1}{2 \cdot 2^{\sum_{k=\lambda}^{\gamma-1}(|V^k|+1)}} \sum_{i=1}^{|V^\lambda|} 2^i \sum_{j=1}^{I} \frac{1}{2^j} = \frac{1}{2 \cdot 2^{\sum_{k=\lambda}^{\gamma-1}(|V^k|+1)}} \cdot \frac{2 \cdot (1 - 2^{|V^\lambda|})}{1 - 2} \cdot \frac{\frac{1}{2}(1 - \frac{1}{2^I})}{1 - \frac{1}{2}} \end{aligned} \tag{B.21}$$

$$= \left(1 - \frac{1}{2^I}\right) \cdot \frac{(2^{|V^\lambda|} - 1)}{2^{\sum_{k=\lambda}^{\gamma-1}(|V^k|+1)}} = \left(1 - \frac{1}{2^I}\right) \cdot \left(\frac{2^{|V^\lambda|}}{2^{|V^\lambda|+1} \cdot 2^{\sum_{k=\lambda+1}^{\gamma-1}(|V^k|+1)}} - \frac{1}{2^{\sum_{k=\lambda}^{\gamma-1}(|V^k|+1)}} \right)$$

$$= \left(1 - \frac{1}{2^I}\right) \cdot \left(\frac{1}{2 \cdot 2^{\sum_{k=\lambda+1}^{\gamma-1}(|V^k|+1)}} - \frac{2}{2 \cdot 2^{\sum_{k=\lambda}^{\gamma-1}(|V^k|+1)}} \right) \tag{B.22}$$

$$\overset{(B.11)}{=} \left(1 - \frac{1}{2^I}\right) \cdot (C_{A^{\lambda+2}A^{\gamma+1}} - 2 \cdot C_{A^{\lambda+1}A^{\gamma+1}}).$$

For the second term of (B.18) we obtain

$$C_{V_I^\lambda V^\gamma} = \sum_{i=1}^{I} \sum_{j=1}^{|V^\gamma|} c_{v_i^\lambda v_j^\gamma} = \sum_{i=1}^{I} \sum_{j=1}^{|V^\gamma|} \frac{1}{2^{j-i+1} \prod_{k=\lambda}^{\gamma-1} 2^{|V^k|+1}} = \sum_{i=1}^{I} \sum_{j=1}^{|V^\gamma|} \frac{1}{2^{j-i+1} \cdot 2^{\sum_{k=\lambda}^{\gamma-1}(|V^k|+1)}}$$

$$= \frac{1}{2 \cdot 2^{\sum_{k=\lambda}^{\gamma-1}(|V^k|+1)}} \sum_{i=1}^{I} 2^i \sum_{j=1}^{|V^\gamma|} \frac{1}{2^j} = \frac{1}{2 \cdot 2^{\sum_{k=\lambda}^{\gamma-1}(|V^k|+1)}} \cdot \frac{2 \cdot (1 - 2^I)}{1 - 2} \cdot \frac{\frac{1}{2}(1 - \frac{1}{2^{|V^\gamma|}})}{1 - \frac{1}{2}}$$

$$= (2^I - 1) \left(\frac{1}{2^{\sum_{k=\lambda}^{\gamma-1}(|V^k|+1)}} - \frac{1}{2^{|V^\gamma|} \cdot 2^{\sum_{k=\lambda}^{\gamma-1}(|V^k|+1)}} \right) \tag{B.23}$$

$$= (2^I - 1) \left(\frac{2}{2 \cdot 2^{\sum_{k=\lambda}^{\gamma-1}(|V^k|+1)}} - \frac{4}{2 \cdot 2^{\sum_{k=\lambda}^{\gamma}(|V^k|+1)}} \right).$$

Applying (B.3) and (B.11) to (B.23) we infer for the second term of (B.18)

$$\begin{aligned} C_{V_I^\lambda V^\gamma} &= (2^I - 1) \cdot (2C_{A^{\lambda+1}A^{\gamma+1}} - 4C_{A^{\lambda+1}A^{\gamma+2}}), \quad \gamma < \xi - 1 \\ C_{V_I^\lambda V^{\xi-1}} &= 2 \cdot (2^I - 1) \cdot (C_{A^{\lambda+1}A^\xi} - C_{A^{\lambda+1}n}), \quad \gamma = \xi - 1 \end{aligned} \tag{B.24}$$

2. Let $\xi - 1 \geq \gamma = \lambda + 1$. Then, for the first term of (B.18) we have

$$C_{V^\lambda V_I^\gamma} = \sum_{i=1}^{|V^\lambda|} \sum_{j=1}^{I} c_{v_i^\lambda v_j^\gamma} = \sum_{i=1}^{|V^\lambda|} \sum_{j=1}^{I} \frac{1}{2^{j+1} \cdot 2 \cdot 2^{|V^\lambda|-i}} = \frac{1}{2^{|V^\lambda|+2}} \sum_{i=1}^{|V^\lambda|} 2^i \sum_{j=1}^{I} \frac{1}{2^j} \tag{B.25}$$

$$
\begin{aligned}
&= \frac{1}{2^{|V^\lambda|+2}} \cdot \frac{2\cdot(1-2^{|V^\lambda|})}{1-2} \cdot \frac{\frac{1}{2}(1-\frac{1}{2^I})}{1-\frac{1}{2}} = \left(1-\frac{1}{2^I}\right)\cdot\frac{(2^{|V^\lambda|}-1)}{2^{|V^\lambda|+1}} \\
&= \frac{1}{2}\left(1-\frac{1}{2^I}\right)\cdot\left(1-\frac{1}{2^{|V^\lambda|}}\right).
\end{aligned} \tag{B.26}
$$

For the second term of (B.18) we deduce

$$
\begin{aligned}
C_{V_I^\lambda V^\gamma} &= \sum_{i=1}^{I}\sum_{j=1}^{|V^\gamma|} c_{v_i^\lambda v_j^\gamma} = \sum_{i=1}^{I}\sum_{j=1}^{|V^\gamma|}\frac{1}{2^{j+1}\cdot 2\cdot 2^{|V^\lambda|-i}} = \frac{1}{2^{|V^\lambda|+2}}\sum_{i=1}^{I}2^i\sum_{j=1}^{|V^\gamma|}\frac{1}{2^j} \\
&= \frac{1}{2^{|V^\lambda|+2}}\cdot\frac{2\cdot(1-2^I)}{1-2}\cdot\frac{\frac{1}{2}(1-\frac{1}{2^{|V^\gamma|}})}{1-\frac{1}{2}} = (2^I-1)\frac{1}{2^{|V^\lambda|+1}}\left(1-\frac{1}{2^{|V^\gamma|}}\right).
\end{aligned} \tag{B.27}
$$

□

Remark B.0.4. *Setting $I = |V^\lambda|$ in the case $\gamma > \lambda + 1$ of Lemma B.0.3, we obtain*

$$C_{V^\lambda V^\gamma} = C_{A^{\lambda+2}A^{\gamma+1}} - 2C_{A^{\lambda+2}A^{\gamma+2}} - 2C_{A^{\lambda+1}A^{\gamma+1}} + 4C_{A^{\lambda+1}A^{\gamma+2}}, \quad \gamma < \xi - 1 \tag{B.28}$$

$$C_{V^\lambda V^{\xi-1}} = C_{A^{\lambda+2}A^{\xi}} - 2C_{A^{\lambda+1}A^{\xi}} - C_{A^{\lambda+2}n} + 2C_{A^{\lambda+1}n}, \qquad \gamma = \xi - 1 \tag{B.29}$$

Proof. From the definition of V^λ and V_I^λ, $I \in [|V^\lambda|]$, see (3.218)-(3.219), we have $V^\lambda = V^\lambda_{|V^\lambda|}$ for all $\lambda \in [\xi-1]$. Thus, setting $I = |V^\lambda|$ in (B.23) we deduce

$$
\begin{aligned}
C_{V^\lambda V^\gamma} &= (2^{|V^\lambda|}-1)\cdot\left(\frac{2}{2\cdot 2^{\sum_{k=\lambda}^{\gamma-1}(|V^k|+1)}} - \frac{4}{2\cdot 2^{\sum_{k=\lambda}^{\gamma}(|V^k|+1)}}\right) \\
&= \frac{2\cdot 2^{|V^\lambda|}}{2\cdot 2^{|V^\lambda|+1}\cdot 2^{\sum_{k=\lambda+1}^{\gamma-1}(|V^k|+1)}} - \frac{2\cdot 2^{|V^\lambda|}}{2\cdot 2^{|V^\lambda|+1}\cdot 2^{\sum_{k=\lambda+1}^{\gamma}(|V^k|+1)}} \\
&\quad - \frac{2}{2\cdot 2^{\sum_{k=\lambda}^{\gamma-1}(|V^k|+1)}} + \frac{4}{2\cdot 2^{\sum_{k=\lambda}^{\gamma}(|V^k|+1)}} \\
&= \frac{1}{2\cdot 2^{\sum_{k=\lambda+1}^{\gamma-1}(|V^k|+1)}} - \frac{2}{2\cdot 2^{\sum_{k=\lambda+1}^{\gamma}(|V^k|+1)}} - \frac{2}{2\cdot 2^{\sum_{k=\lambda}^{\gamma-1}(|V^k|+1)}} + \frac{4}{2\cdot 2^{\sum_{k=\lambda}^{\gamma}(|V^k|+1)}}.
\end{aligned} \tag{B.30}
$$

In view of (B.3), (B.11) and (B.30) the term $C_{V^\lambda V^\gamma}$ can be written as

$$C_{V^\lambda V^\gamma} = C_{A^{\lambda+2}A^{\gamma+1}} - 2C_{A^{\lambda+2}A^{\gamma+2}} - 2C_{A^{\lambda+1}A^{\gamma+1}} + 4C_{A^{\lambda+1}A^{\gamma+2}}, \quad \gamma < \xi - 1 \tag{B.31}$$

$$C_{V^\lambda V^{\xi-1}} = C_{A^{\lambda+2}A^{\xi}} - 2C_{A^{\lambda+1}A^{\xi}} - C_{A^{\lambda+2}n} + 2C_{A^{\lambda+1}n}, \qquad \gamma = \xi - 1 \tag{B.32}$$

□

Lemma B.0.5. *Let $c \in \mathbb{R}^{E_n}_{>0}$ be defined by* (3.223)-(3.232)*, $\gamma \in [\xi]$, $\lambda \in [\xi - 1]$ and*

$$C_{A^\gamma_I V^\lambda} := \sum_{i=1}^{I} \sum_{j=1}^{|V^\lambda|} c_{a^\gamma_i v^\lambda_j},\ I \in [|A^\gamma|] \quad \textit{and} \quad C_{A^\gamma V^\lambda_I} := \sum_{i=1}^{|A^\gamma|} \sum_{j=1}^{I} c_{a^\gamma_i v^\lambda_j},\ I \in [|V^\lambda|]. \tag{B.33}$$

1. *For $\gamma < \lambda + 1$ we have*

$$\begin{aligned} C_{A^\gamma_I V^\lambda} &= \frac{2^I}{2^{|A^\gamma|}} \cdot (C_{A^\gamma A^{\lambda+1}} - 2 \cdot C_{A^\gamma A^{\lambda+2}}),\ \lambda < \xi - 1 \\ C_{A^\gamma_I V^{\xi-1}} &= \frac{2^I}{2^{|A^\gamma|}} \cdot (C_{A^\gamma A^\xi} - C_{A^\gamma n}), \quad \lambda = \xi - 1 \\ C_{A^\gamma V^\lambda_I} &= \left(1 - \frac{1}{2^I}\right) \cdot C_{A^\gamma A^{\lambda+1}}, \quad \lambda \le \xi - 1 \end{aligned} \tag{B.34}$$

2. *For $\gamma = \lambda + 1$ we have*

$$\begin{aligned} C_{A^\gamma_I V^\lambda} &= \frac{2^{I-1}}{2^{|A^\gamma|}} \cdot \left(1 - \frac{1}{2^{|V^\lambda|}}\right), \\ C_{A^\gamma V^\lambda_I} &= \frac{1}{2} \cdot \left(1 - \frac{1}{2^I}\right) \end{aligned} \tag{B.35}$$

3. *For $\gamma = \lambda + 2$ we have*

$$\begin{aligned} C_{A^\gamma_I V^\lambda} &= \frac{2^{I-1}}{2^{|A^\gamma|}} \cdot \left(1 - \frac{1}{2^{|V^\lambda|}}\right), \\ C_{A^\gamma V^\lambda_I} &= (2^I - 1) \cdot \frac{1}{2^{|V^\lambda|+1}} \end{aligned} \tag{B.36}$$

4. *For $\gamma > \lambda + 2$ we have*

$$\begin{aligned} C_{A^\gamma_I V^\lambda} &= \frac{2^I}{2^{|A^\gamma|}} \cdot (C_{A^{\lambda+2} A^\gamma} - 2 \cdot C_{A^{\lambda+1} A^\gamma}), \\ C_{A^\gamma V^\lambda_I} &= 2(2^I - 1) \cdot C_{A^{\lambda+1} A^\gamma}. \end{aligned} \tag{B.37}$$

Proof. 1. Let $1 < \gamma < \lambda + 1$. Then, for the first term of (B.33) we obtain

$$\begin{aligned} C_{A^\gamma_I V^\lambda} &= \sum_{i=1}^{I} \sum_{j=1}^{|V^\lambda|} c_{a^\gamma_i v^\lambda_j} = \sum_{j=1}^{|V^\lambda|} c_{a^\gamma_1 v^\lambda_j} + \sum_{i=2}^{I} \sum_{j=1}^{|V^\lambda|} c_{a^\gamma_i v^\lambda_j} = \sum_{j=1}^{|V^\lambda|} c_{a^\gamma_2 v^\lambda_j} + \sum_{i=2}^{I} \sum_{j=1}^{|V^\lambda|} c_{a^\gamma_i v^\lambda_j} \\ &= \sum_{j=1}^{|V^\lambda|} \frac{1}{2^{j+1} \cdot 2 \cdot 2^{|A^\gamma|-2} \prod\limits_{k=\gamma-1}^{\lambda-1} 2^{|V^k|+1}} + \sum_{i=2}^{I} \sum_{j=1}^{|V^\lambda|} \frac{1}{2^{j+1} \cdot 2 \cdot 2^{|A^\gamma|-i} \prod\limits_{k=\gamma-1}^{\lambda-1} 2^{|V^k|+1}} \end{aligned} \tag{B.38}$$

$$
\begin{aligned}
&= \sum_{j=1}^{|V^\lambda|} \frac{1}{2^j \cdot 2^{|A^\gamma|} \cdot 2^{\sum_{k=\gamma-1}^{\lambda-1}(|V^k|+1)}} + \sum_{i=2}^{I}\sum_{j=1}^{|V^\lambda|} \frac{1}{2^j \cdot 2^{|A^\gamma|-i+2} \cdot 2^{\sum_{k=\gamma-1}^{\lambda-1}(|V^k|+1)}} \\
&= \frac{1}{2^{|A^\gamma|} \cdot 2^{\sum_{k=\gamma-1}^{\lambda-1}(|V^k|+1)}} \cdot \sum_{j=1}^{|V^\lambda|} \frac{1}{2^j} \left(1 + \frac{1}{2^2} \cdot \sum_{i=2}^{I} 2^i\right) \\
&= \frac{1}{2^{|A^\gamma|} \cdot 2^{\sum_{k=\gamma-1}^{\lambda-1}(|V^k|+1)}} \cdot \frac{\frac{1}{2}(1-\frac{1}{2^{|V^\lambda|}})}{1-\frac{1}{2}} \left(1 + \frac{2^2(1-2^{I-1})}{2^2(1-2)}\right) \\
&= \frac{2^{I-1}}{2^{|A^\gamma|} \cdot 2^{\sum_{k=\gamma-1}^{\lambda-1}(|V^k|+1)}} \cdot \left(1 - \frac{1}{2^{|V^\lambda|}}\right) \qquad \text{(B.39)} \\
&= \frac{2^I}{2^{|A^\gamma|}} \cdot \left(\frac{1}{2 \cdot 2^{\sum_{k=\gamma-1}^{\lambda-1}(|V^k|+1)}} - \frac{1}{2^{|V^\lambda|+1} \cdot 2^{\sum_{k=\gamma-1}^{\lambda-1}(|V^k|+1)}} \right) \\
&= \frac{2^I}{2^{|A^\gamma|}} \cdot \left(\frac{1}{2 \cdot 2^{\sum_{k=\gamma-1}^{\lambda-1}(|V^k|+1)}} - \frac{2}{2 \cdot 2^{\sum_{k=\gamma-1}^{\lambda}(|V^k|+1)}} \right)
\end{aligned}
$$

Applying (B.3) and (B.11) to (B.38)-(B.39) we infer for the first term of (B.33)

$$
C_{A_I^\gamma V^\lambda} = \frac{2^I}{2^{|A^\gamma|}} \cdot (C_{A^\gamma A^{\lambda+1}} - 2 \cdot C_{A^\gamma A^{\lambda+2}}), \quad \lambda < \xi - 1 \qquad \text{(B.40)}
$$

$$
C_{A_I^\gamma V^{\xi-1}} = \frac{2^I}{2^{|A^\gamma|}} \cdot (C_{A^\gamma A^\xi} - C_{A^\gamma n}), \quad \lambda = \xi - 1 \qquad \text{(B.41)}
$$

For the second term of (B.33) we have

$$
\begin{aligned}
C_{A^\gamma V_I^\lambda} &= \sum_{i=1}^{|A^\gamma|}\sum_{j=1}^{I} c_{a_i^\gamma v_j^\lambda} = \sum_{j=1}^{I} c_{a_1^\gamma v_j^\lambda} + \sum_{i=2}^{|A^\gamma|}\sum_{j=1}^{I} c_{a_i^\gamma v_j^\lambda} = \sum_{j=1}^{I} c_{a_2^\gamma v_j^\lambda} + \sum_{i=2}^{|A^\gamma|}\sum_{j=1}^{I} c_{a_i^\gamma v_j^\lambda} \\
&= \sum_{j=1}^{I} \frac{1}{2^{j+1} \cdot 2 \cdot 2^{|A^\gamma|-2} \prod_{k=\gamma-1}^{\lambda-1} 2^{|V^k|+1}} + \sum_{i=2}^{|A^\gamma|}\sum_{j=1}^{I} \frac{1}{2^{j+1} \cdot 2 \cdot 2^{|A^\gamma|-i} \prod_{k=\gamma-1}^{\lambda-1} 2^{|V^k|+1}} \qquad \text{(B.42)} \\
&= \sum_{j=1}^{I} \frac{1}{2^j \cdot 2^{|A^\gamma|} \cdot 2^{\sum_{k=\gamma-1}^{\lambda-1}(|V^k|+1)}} + \sum_{i=2}^{|A^\gamma|}\sum_{j=1}^{I} \frac{1}{2^j \cdot 2^{|A^\gamma|-i+2} \cdot 2^{\sum_{k=\gamma-1}^{\lambda-1}(|V^k|+1)}}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2^{|A^\gamma|}\cdot 2^{\sum\limits_{k=\gamma-1}^{\lambda-1}(|V^k|+1)}} \cdot \sum_{j=1}^{I} \frac{1}{2^j}\left(1+\frac{1}{2^2}\cdot\sum_{i=2}^{|A^\gamma|}2^i\right)\\
&= \frac{1}{2^{|A^\gamma|}\cdot 2^{\sum\limits_{k=\gamma-1}^{\lambda-1}(|V^k|+1)}} \cdot \frac{\frac{1}{2}(1-\frac{1}{2^I})}{1-\frac{1}{2}}\left(1+\frac{2^2(1-2^{|A^\gamma|-1})}{2^2(1-2)}\right)\\
&= \frac{2^{|A^\gamma|-1}}{2^{|A^\gamma|}\cdot 2^{\sum\limits_{k=\gamma-1}^{\lambda-1}(|V^k|+1)}} \cdot \left(1-\frac{1}{2^I}\right)\\
&= \left(1-\frac{1}{2^I}\right)\cdot \frac{1}{2\cdot 2^{\sum\limits_{k=\gamma-1}^{\lambda-1}(|V^k|+1)}} \overset{(B.11)}{=} \left(1-\frac{1}{2^I}\right) C_{A^\gamma A^{\lambda+1}}.
\end{aligned}
\tag{B.43}
$$

Similarly to the case $\gamma > 1$ we obtain due to the definition of c, see (3.230), that equations in (B.34) hold for $\gamma = 1$.

2. Let $\gamma = \lambda + 1$. Then, for the first term of (B.33) we deduce

$$
\begin{aligned}
C_{A_I^\gamma V^\lambda} &= \sum_{i=1}^{I}\sum_{j=1}^{|V^\lambda|} c_{a_i^\gamma v_j^\lambda} = \sum_{j=1}^{|V^\lambda|} c_{a_1^\gamma v_j^\lambda} + \sum_{i=2}^{I}\sum_{j=1}^{|V^\lambda|} c_{a_i^\gamma v_j^\lambda} = \sum_{j=1}^{|V^\lambda|} c_{a_2^\gamma v_j^\lambda} + \sum_{i=2}^{I}\sum_{j=1}^{|V^\lambda|} c_{a_i^\gamma v_j^\lambda}\\
&= \sum_{j=1}^{|V^\lambda|} \frac{1}{2^{j+1}\cdot 2\cdot 2^{|A^\gamma|-2}} + \sum_{i=2}^{I}\sum_{j=1}^{|V^\lambda|} \frac{1}{2^{j+1}\cdot 2\cdot 2^{|A^\gamma|-i}}\\
&= \frac{1}{2^{|A^\gamma|}}\sum_{j=1}^{|V^\lambda|}\frac{1}{2^j}\left(1+\frac{1}{2^2}\sum_{i=2}^{I}2^i\right) = \frac{1}{2^{|A^\gamma|}}\cdot\frac{\frac{1}{2}(1-\frac{1}{2^{|V^\lambda|}})}{1-\frac{1}{2}}\left(1+\frac{2^2(1-2^{I-1})}{2^2(1-2)}\right)\\
&= \frac{2^{I-1}}{2^{|A^\gamma|}}\cdot\left(1-\frac{1}{2^{|V^\lambda|}}\right)
\end{aligned}
\tag{B.44}
$$

For the second term of (B.33) we obtain

$$
\begin{aligned}
C_{A^\gamma V_I^\lambda} &= \sum_{i=1}^{|A^\gamma|}\sum_{j=1}^{I} c_{a_i^\gamma v_j^\lambda} = \sum_{j=1}^{I} c_{a_1^\gamma v_j^\lambda} + \sum_{i=2}^{|A^\gamma|}\sum_{j=1}^{I} c_{a_i^\gamma v_j^\lambda} = \sum_{j=1}^{I} c_{a_2^\gamma v_j^\lambda} + \sum_{i=2}^{|A^\gamma|}\sum_{j=1}^{I} c_{a_i^\gamma v_j^\lambda}\\
&= \sum_{j=1}^{I} \frac{1}{2^{j+1}\cdot 2\cdot 2^{|A^\gamma|-2}} + \sum_{i=2}^{|A^\gamma|}\sum_{j=1}^{I} \frac{1}{2^{j+1}\cdot 2\cdot 2^{|A^\gamma|-i}}
\end{aligned}
\tag{B.45}
$$

$$
\begin{aligned}
&= \frac{1}{2^{|A^\gamma|}} \sum_{j=1}^{I} \frac{1}{2^j} \left(1 + \frac{1}{2^2} \sum_{i=2}^{|A^\gamma|} 2^i \right) = \frac{\frac{1}{2}(1-\frac{1}{2^I})}{2^{|A^\gamma|}(1-\frac{1}{2})} \left(1 + \frac{2^2(1-2^{|A^\gamma|-1})}{2^2(1-2)} \right) \\
&= \frac{2^{|A^\gamma|-1}}{2^{|A^\gamma|}} \cdot \left(1 - \frac{1}{2^I} \right) = \frac{1}{2} \left(1 - \frac{1}{2^I} \right)
\end{aligned}
\tag{B.46}
$$

3. Let $\gamma = \lambda + 2$. Then, for the first term of (B.33) we have

$$
\begin{aligned}
C_{A_I^\gamma V^\lambda} &= \sum_{i=1}^{I} \sum_{j=1}^{|V^\lambda|} c_{a_i^\gamma v_j^\lambda} = \sum_{j=1}^{|V^\lambda|} c_{a_1^\gamma v_j^\lambda} + \sum_{i=2}^{I} \sum_{j=1}^{|V^\lambda|} c_{a_i^\gamma v_j^\lambda} = \sum_{j=1}^{|V^\lambda|} c_{a_2^\gamma v_j^\lambda} + \sum_{i=2}^{I} \sum_{j=1}^{|V^\lambda|} c_{a_i^\gamma v_j^\lambda} \\
&= \sum_{j=1}^{|V^\lambda|} \frac{1}{2^3 \cdot 2^{|A^\gamma|-2} \cdot 2^{|V^\lambda|-j}} + \sum_{i=2}^{I} \sum_{j=1}^{|V^\lambda|} \frac{1}{2^3 \cdot 2^{|A^\gamma|-i} \cdot 2^{|V^\lambda|-j}} \\
&= \frac{1}{2^{|A^\gamma|+1} \cdot 2^{|V^\lambda|}} \cdot \sum_{j=1}^{|V^\lambda|} 2^j \left(1 + \frac{1}{2^2} \cdot \sum_{i=2}^{I} 2^i \right) \\
&= \frac{1}{2^{|A^\gamma|+1} \cdot 2^{|V^\lambda|}} \cdot \frac{2(1-2^{|V^\lambda|})}{1-2} \left(1 + \frac{1}{2^2} \cdot \frac{2^2 \cdot (1-2^{I-1})}{1-2} \right) \\
&= \frac{2^{I-1}}{2^{|A^\gamma|}} \cdot \frac{(2^{|V^\lambda|}-1)}{2^{|V^\lambda|}} = \frac{2^{I-1}}{2^{|A^\gamma|}} \cdot \left(1 - \frac{1}{2^{|V^\lambda|}} \right).
\end{aligned}
\tag{B.47}
$$

For the second term of (B.33) we obtain

$$
\begin{aligned}
C_{A^\gamma V_I^\lambda} &= \sum_{i=1}^{|A^\gamma|} \sum_{j=1}^{I} c_{a_i^\gamma v_j^\lambda} = \sum_{j=1}^{I} c_{a_1^\gamma v_j^\lambda} + \sum_{i=2}^{|A^\gamma|} \sum_{j=1}^{I} c_{a_i^\gamma v_j^\lambda} = \sum_{j=1}^{I} c_{a_2^\gamma v_j^\lambda} + \sum_{i=2}^{|A^\gamma|} \sum_{j=1}^{I} c_{a_i^\gamma v_j^\lambda} \\
&= \sum_{j=1}^{I} \frac{1}{2^3 \cdot 2^{|A^\gamma|-2} \cdot 2^{|V^\lambda|-j}} + \sum_{i=2}^{|A^\gamma|} \sum_{j=1}^{I} \frac{1}{2^3 \cdot 2^{|A^\gamma|-i} \cdot 2^{|V^\lambda|-j}} \\
&= \frac{1}{2^{|A^\gamma|+1} \cdot 2^{|V^\lambda|}} \cdot \sum_{j=1}^{I} 2^j \left(1 + \frac{1}{2^2} \cdot \sum_{i=2}^{|A^\gamma|} 2^i \right) \\
&= \frac{1}{2^{|A^\gamma|+1} \cdot 2^{|V^\lambda|}} \cdot \frac{2(1-2^I)}{1-2} \left(1 + \frac{1}{2^2} \cdot \frac{2^2 \cdot (1-2^{|A^\gamma|-1})}{1-2} \right) \\
&= \frac{2^{|A^\gamma|-1}}{2^{|A^\gamma|}} \cdot \frac{(2^I-1)}{2^{|V^\lambda|}} = (2^I-1) \cdot \frac{1}{2^{|V^\lambda|+1}}.
\end{aligned}
\tag{B.48}
$$

4. Let $\gamma > \lambda + 2$. Calculating the first term of (B.33) we arrive at

$$\begin{aligned}
C_{A_I^\gamma V^\lambda} &= \sum_{i=1}^{I}\sum_{j=1}^{|V^\lambda|} c_{a_i^\gamma v_j^\lambda} = \sum_{j=1}^{|V^\lambda|} c_{a_1^\gamma v_j^\lambda} + \sum_{i=2}^{I}\sum_{j=1}^{|V^\lambda|} c_{a_i^\gamma v_j^\lambda} = \sum_{j=1}^{|V^\lambda|} c_{a_2^\gamma v_j^\lambda} + \sum_{i=2}^{I}\sum_{j=1}^{|V^\lambda|} c_{a_i^\gamma v_j^\lambda} \\
&= \sum_{j=1}^{|V^\lambda|} \frac{1}{2^{2-j}\cdot 2^{|A^\gamma|-2}\prod_{k=\lambda}^{\gamma-2} 2^{|V^k|+1}} + \sum_{i=2}^{I}\sum_{j=1}^{|V^\lambda|} \frac{1}{2^{2-j}\cdot 2^{|A^\gamma|-i}\prod_{k=\lambda}^{\gamma-2} 2^{|V^k|+1}} \\
&= \sum_{j=1}^{|V^\lambda|} \frac{1}{2^{2-j}\cdot 2^{|A^\gamma|-2}\cdot 2^{\sum_{k=\lambda}^{\gamma-2}(|V^k|+1)}} + \sum_{i=2}^{I}\sum_{j=1}^{|V^\lambda|} \frac{1}{2^{2-j}\cdot 2^{|A^\gamma|-i}\cdot 2^{\sum_{k=\lambda}^{\gamma-2}(|V^k|+1)}} \\
&= \frac{1}{2^{|A^\gamma|}\cdot 2^{\sum_{k=\lambda}^{\gamma-2}(|V^k|+1)}} \cdot \sum_{j=1}^{|V^\lambda|} 2^j \left(1 + \frac{1}{2^2}\cdot\sum_{i=2}^{I} 2^i\right) \\
&= \frac{1}{2^{|A^\gamma|}\cdot 2^{\sum_{k=\lambda}^{\gamma-2}(|V^k|+1)}} \cdot \frac{2(1-2^{|V^\lambda|})}{1-2}\cdot\left(1 + \frac{1}{2^2}\cdot\frac{2^2(1-2^{I-1})}{1-2}\right) \\
&= \frac{2^{I-1}\cdot 2(2^{|V^\lambda|}-1)}{2^{|A^\gamma|}\cdot 2^{\sum_{k=\lambda}^{\gamma-2}(|V^k|+1)}} = \frac{2^I}{2^{|A^\gamma|}}\left(\frac{2^{|V^\lambda|}}{2^{\sum_{k=\lambda}^{\gamma-2}(|V^k|+1)}} - \frac{1}{2^{\sum_{k=\lambda}^{\gamma-2}(|V^k|+1)}}\right) \\
&= \frac{2^I}{2^{|A^\gamma|}}\left(\frac{1}{2\cdot 2^{\sum_{k=\lambda+1}^{\gamma-2}(|V^k|+1)}} - \frac{2}{2\cdot 2^{\sum_{k=\lambda}^{\gamma-2}(|V^k|+1)}}\right) \\
&\overset{(B.11)}{=} \frac{2^I}{2^{|A^\gamma|}}\cdot\left(C_{A^{\lambda+2}A^\gamma} - 2\cdot C_{A^{\lambda+1}A^\gamma}\right).
\end{aligned} \tag{B.49}$$

For the second term of (B.33) we obtain

$$\begin{aligned}
C_{A^\gamma V_I^\lambda} &= \sum_{i=1}^{|A^\gamma|}\sum_{j=1}^{I} c_{a_i^\gamma v_j^\lambda} = \sum_{j=1}^{I} c_{a_1^\gamma v_j^\lambda} + \sum_{i=2}^{|A^\gamma|}\sum_{j=1}^{I} c_{a_i^\gamma v_j^\lambda} = \sum_{j=1}^{I} c_{a_2^\gamma v_j^\lambda} + \sum_{i=2}^{|A^\gamma|}\sum_{j=1}^{I} c_{a_i^\gamma v_j^\lambda} \\
&= \sum_{j=1}^{I} \frac{1}{2^{2-j}\cdot 2^{|A^\gamma|-2}\prod_{k=\lambda}^{\gamma-2} 2^{|V^k|+1}} + \sum_{i=2}^{|A^\gamma|}\sum_{j=1}^{I} \frac{1}{2^{2-j}\cdot 2^{|A^\gamma|-i}\prod_{k=\lambda}^{\gamma-2} 2^{|V^k|+1}} \\
&= \sum_{j=1}^{I} \frac{1}{2^{2-j}\cdot 2^{|A^\gamma|-2}\cdot 2^{\sum_{k=\lambda}^{\gamma-2}(|V^k|+1)}} + \sum_{i=2}^{|A^\gamma|}\sum_{j=1}^{I} \frac{1}{2^{2-j}\cdot 2^{|A^\gamma|-i}\cdot 2^{\sum_{k=\lambda}^{\gamma-2}(|V^k|+1)}}
\end{aligned} \tag{B.50}$$

$$
\begin{aligned}
&= \frac{1}{2^{|A^\gamma|}\cdot 2^{\sum_{k=\lambda}^{\gamma-2}(|V^k|+1)}} \cdot \sum_{j=1}^{I} 2^j \cdot \left(1 + \frac{1}{2^2}\cdot \sum_{i=2}^{|A^\gamma|} 2^i\right) \\
&= \frac{1}{2^{|A^\gamma|}\cdot 2^{\sum_{k=\lambda}^{\gamma-2}(|V^k|+1)}} \cdot \frac{2(1-2^I)}{1-2}\left(1 + \frac{1}{2^2}\cdot \frac{2^2(1-2^{|A^\gamma|-1})}{1-2}\right) \\
&= \frac{2^{|A^\gamma|-1}\cdot 2(2^I-1)}{2^{|A^\gamma|}\cdot 2^{\sum_{k=\lambda}^{\gamma-2}(|V^k|+1)}} = (2^I-1)\cdot \frac{1}{2^{\sum_{k=\lambda}^{\gamma-2}(|V^k|+1)}} \\
&= (2^I-1)\cdot \frac{2}{2\cdot 2^{\sum_{k=\lambda}^{\gamma-2}(|V^k|+1)}} \overset{(B.11)}{=} 2\cdot(2^I-1)\cdot C_{A^{\lambda+1}A^\gamma}.
\end{aligned}
\tag{B.51}
$$

□

Lemma B.0.6. *Let $c \in \mathbb{R}^{E_n}_{>0}$ be defined by* (3.223)-(3.232). *Then, for all $\gamma \in [\xi-1]$ and $I \in [|A^\gamma|]$ we have*

$$C_{A_I^\gamma n} - C_{A_I^\gamma A^\xi} + C_{A_I^\gamma V^{\xi-1}} = 0, \tag{B.52}$$

where $C_{A_I^\gamma n}$, $C_{A_I^\gamma A^\xi}$ and $C_{A_I^\gamma V^{\xi-1}}$ are defined in (B.1), (B.9) *and* (B.33), *respectively.*

Proof. Let $\gamma \in [\xi-1]$ and $I \in [|A^\gamma|]$. From Lemma B.0.1 we have

$$C_{A_I^\gamma n} = \frac{2^I}{2^{|A^\gamma|}} \cdot C_{A^\gamma n}. \tag{B.53}$$

It follows from Lemma B.0.2 that

$$C_{A_I^\gamma A^\xi} = \frac{2^I}{2^{|A^\gamma|}} \cdot C_{A^\gamma A^\xi}. \tag{B.54}$$

Thus, in view of (B.53), (B.54) and (B.41) we obtain

$$
\begin{aligned}
&C_{A_I^\gamma n} - C_{A_I^\gamma A^\xi} + C_{A_I^\gamma V^{\xi-1}} \\
&\quad = \frac{2^I}{2^{|A^\gamma|}} \cdot C_{A^\gamma n} - \frac{2^I}{2^{|A^\gamma|}} \cdot C_{A^\gamma A^\xi} + \frac{2^I}{2^{|A^\gamma|}} \cdot C_{A^\gamma A^\xi} - \frac{2^I}{2^{|A^\gamma|}} \cdot C_{A^\gamma n} = 0.
\end{aligned}
\tag{B.55}
$$

□

Lemma B.0.7. *Let $c \in \mathbb{R}^{E_n}_{>0}$ be defined by* (3.223)-(3.232). *Then, for $1 < k \leq \xi$ we have*

$$c\left(\delta\left(\bigcup_{\gamma=1}^{k} A^\gamma \cup \bigcup_{\lambda=1}^{k-1} V^\lambda\right)\right) = 1, \tag{B.56}$$

where A^γ, $\gamma \in [\xi]$ and V^λ, $\lambda \in [\xi-1]$ are defined by (3.218).

Proof. For brevity, we introduce the following notations:

$$\begin{aligned}
&C_{A^\gamma n} := \sum_{i=1}^{|A^\gamma|} c_{a_i^\gamma n},\ \gamma \in [\xi] && C_{A^\gamma A^\lambda} := \sum_{i=1}^{|A^\gamma|}\sum_{j=1}^{|A^\lambda|} c_{a_i^\gamma a_j^\lambda},\ \gamma,\lambda \in [\xi],\ \gamma \neq \lambda \\
&C_{V^\lambda n} := \sum_{j=1}^{|V^\lambda|} c_{v_j^\lambda n},\ \lambda \in [\xi-1] && C_{V^\lambda V^\gamma} := \sum_{i=1}^{|V^\lambda|}\sum_{j=1}^{|V^\gamma|} c_{v_i^\lambda v_j^\gamma},\ \lambda,\gamma \in [\xi-1],\ \lambda \neq \gamma \\
&C_{A^\gamma V^\lambda} := \sum_{i=1}^{|V^\gamma|}\sum_{j=1}^{|V^\lambda|} c_{a_i^\gamma v_j^\lambda},\ \gamma \in [\xi],\ \lambda \in [\xi-1].
\end{aligned} \tag{B.57}$$

Then, for the left-hand side of equation (B.56) we have

$$\begin{aligned}
c\left(\delta\left(\bigcup_{\gamma=1}^{k} A^\gamma \cup \bigcup_{\lambda=1}^{k-1} V^\lambda\right)\right) = &\underbrace{\sum_{\gamma=1}^{k} C_{A^\gamma n}}_{\Sigma_1} + \underbrace{\sum_{\gamma=1}^{k}\sum_{\lambda=k}^{\xi-1} C_{A^\gamma V^\lambda}}_{\Sigma_2} + \underbrace{\sum_{\gamma=1}^{k}\sum_{\lambda=k+1}^{\xi} C_{A^\gamma A^\lambda}}_{\Sigma_3} \\
&+ \underbrace{\sum_{\lambda=1}^{k-1} C_{V^\lambda n}}_{\Sigma_4} + \underbrace{\sum_{\lambda=1}^{k-1}\sum_{\gamma=k}^{\xi-1} C_{V^\lambda V^\gamma}}_{\Sigma_5} + \underbrace{\sum_{\lambda=1}^{k-1}\sum_{\gamma=k+1}^{\xi} C_{A^\gamma V^\lambda}}_{\Sigma_6}
\end{aligned} \tag{B.58}$$

Using equations (B.2) from Lemma B.0.1 we obtain

$$\begin{aligned}
\Sigma_1 + \Sigma_4 &= \sum_{\gamma=1}^{k} C_{A^\gamma n} + \sum_{\gamma=1}^{k-1}(C_{A^{\gamma+2} n} - 2C_{A^{\gamma+1} n}) \\
&= \sum_{\gamma=1}^{k} C_{A^\gamma n} + \sum_{\gamma=3}^{k+1} C_{A^\gamma n} - 2\sum_{\gamma=2}^{k} C_{A^\gamma n} = C_{A^1 n} - C_{A^2 n} + C_{A^{k+1} n} \\
&\overset{(3.231)}{=} C_{A^1 n} - C_{A^1 n} + C_{A^{k+1} n} = C_{A^{k+1} n}.
\end{aligned} \tag{B.59}$$

From Lemma B.0.5 we have for $\gamma \in [\xi]$, $\lambda \in [\xi-1]$ that

$$\begin{aligned}
C_{A^\gamma V^\lambda} &= C_{A^\gamma_{|A^\gamma|} V^\lambda} = C_{A^\gamma A^{\lambda+1}} - 2 \cdot C_{A^\gamma A^{\lambda+2}} \quad \text{if} \quad \gamma < \lambda + 1, \\
C_{A^\gamma V^\lambda} &= C_{A^\gamma_{|A^\gamma|} V^\lambda} = C_{A^{\lambda+2} A^\gamma} - 2 \cdot C_{A^{\lambda+1} A^\gamma} \quad \text{if} \quad \gamma > \lambda + 2.
\end{aligned} \tag{B.60}$$

Thus, for the second term of the right-hand side of (B.58) we deduce

$$\begin{aligned}
\Sigma_2 &= \sum_{\gamma=1}^{k}\sum_{\lambda=k}^{\xi-2}(C_{A^\gamma A^{\lambda+1}} - 2 \cdot C_{A^\gamma A^{\lambda+2}}) + \sum_{\gamma=1}^{k} C_{A^\gamma V^{\xi-1}} \\
&= \sum_{\gamma=1}^{k}\sum_{\lambda=k+1}^{\xi-1} C_{A^\gamma A^\lambda} - 2\sum_{\gamma=1}^{k}\sum_{\lambda=k+2}^{\xi} C_{A^\gamma A^\lambda} + \sum_{\gamma=1}^{k} C_{A^\gamma V^{\xi-1}} \\
&= \sum_{\gamma=1}^{k} C_{A^\gamma A^{k+1}} - \sum_{\gamma=1}^{k} C_{A^\gamma A^\xi} - \sum_{\gamma=1}^{k}\sum_{\lambda=k+2}^{\xi} C_{A^\gamma A^\lambda} + \sum_{\gamma=1}^{k} C_{A^\gamma V^{\xi-1}}
\end{aligned} \tag{B.61}$$

Calculating the sixth term of the right-hand side of (B.58) we obtain

$$
\begin{aligned}
\Sigma_6 &= \sum_{\lambda=1}^{k-1} \sum_{\gamma=k+1}^{\xi} C_{A^\gamma V^\lambda} = \sum_{\lambda=1}^{k-2} \sum_{\gamma=k+1}^{\xi} C_{A^\gamma V^\lambda} + \sum_{\gamma=k+1}^{\xi} C_{A^\gamma V^{k-1}} \\
&= \sum_{\lambda=1}^{k-2} \sum_{\gamma=k+1}^{\xi} (C_{A^{\lambda+2} A^\gamma} - 2C_{A^{\lambda+1} A^\gamma}) + \sum_{\gamma=k+2}^{\xi} C_{A^\gamma V^{k-1}} + C_{A^{k+1} V^{k-1}} \\
&= \sum_{\lambda=3}^{k} \sum_{\gamma=k+1}^{\xi} C_{A^\lambda A^\gamma} - 2 \sum_{\lambda=2}^{k-1} \sum_{\gamma=k+1}^{\xi} C_{A^\lambda A^\gamma} + \sum_{\gamma=k+2}^{\xi} (C_{A^{k+1} A^\gamma} - 2C_{A^k A^\gamma}) + C_{A^{k+1} V^{k-1}} \\
&= \sum_{\gamma=k+1}^{\xi} C_{A^k A^\gamma} - \sum_{\gamma=k+1}^{\xi} C_{A^2 A^\gamma} - \sum_{\lambda=2}^{k-1} \sum_{\gamma=k+1}^{\xi} C_{A^\lambda A^\gamma} \\
&\quad + \sum_{\gamma=k+2}^{\xi} C_{A^{k+1} A^\gamma} - 2 \sum_{\gamma=k+2}^{\xi} C_{A^k A^\gamma} + C_{A^{k+1} V^{k-1}}.
\end{aligned}
\tag{B.62}
$$

For the fifth term of (B.58) we deduce

$$
\begin{aligned}
\Sigma_5 &= \sum_{\lambda=1}^{k-1} \sum_{\gamma=k}^{\xi-1} C_{V^\lambda V^\gamma} = \sum_{\lambda=1}^{k-2} \sum_{\gamma=k}^{\xi-2} C_{V^\lambda V^\gamma} + \sum_{\gamma=k}^{\xi-2} C_{V^{k-1} V^\gamma} + C_{V^{k-1} V^{\xi-1}} + \sum_{\lambda=1}^{k-2} C_{V^\lambda V^{\xi-1}} \\
&= \sum_{\lambda=1}^{k-2} \sum_{\gamma=k}^{\xi-2} C_{V^\lambda V^\gamma} + C_{V^{k-1} V^k} + \sum_{\gamma=k+1}^{\xi-2} C_{V^{k-1} V^\gamma} + C_{V^{k-1} V^{\xi-1}} + \sum_{\lambda=1}^{k-2} C_{V^\lambda V^{\xi-1}} \\
&\overset{\text{Remark } B.0.4}{=} \sum_{\lambda=1}^{k-2} \sum_{\gamma=k}^{\xi-2} (C_{A^{\lambda+2} A^{\gamma+1}} - 2 \cdot C_{A^{\lambda+2} A^{\gamma+2}} - 2 \cdot C_{A^{\lambda+1} A^{\gamma+1}} + 4 \cdot C_{A^{\lambda+1} A^{\gamma+2}}) \\
&\quad + C_{V^{k-1} V^k} + \sum_{\gamma=k+1}^{\xi-2} (C_{A^{k+1} A^{\gamma+1}} - 2 \cdot C_{A^{k+1} A^{\gamma+2}} - 2 \cdot C_{A^k A^{\gamma+1}} + 4 \cdot C_{A^k A^{\gamma+2}}) \\
&\quad + C_{A^{k+1} A^\xi} - 2 \cdot C_{A^k A^\xi} - C_{A^{k+1} n} + 2 \cdot C_{A^k n} \\
&\quad + \sum_{\lambda=1}^{k-2} (C_{A^{\lambda+2} A^\xi} - 2 \cdot C_{A^{\lambda+1} A^\xi} - C_{A^{\lambda+2} n} + 2 \cdot C_{A^{\lambda+1} n}) \\
&= \sum_{\lambda=3}^{k} \sum_{\gamma=k+1}^{\xi-1} C_{A^\lambda A^\gamma} - 2 \sum_{\lambda=3}^{k} \sum_{\gamma=k+2}^{\xi} C_{A^\lambda A^\gamma} - 2 \sum_{\lambda=2}^{k-1} \sum_{\gamma=k+1}^{\xi-1} C_{A^\lambda A^\gamma} \\
&\quad + 4 \sum_{\lambda=2}^{k-1} \sum_{\gamma=k+2}^{\xi} C_{A^\lambda A^\gamma} + C_{V^{k-1} V^k} + \sum_{\gamma=k+2}^{\xi-1} C_{A^{k+1} A^\gamma} - 2 \sum_{\gamma=k+3}^{\xi} C_{A^{k+1} A^\gamma}
\end{aligned}
\tag{B.63}
$$

$$
\begin{aligned}
&- 2\sum_{\gamma=k+2}^{\xi-1} C_{A^kA^\gamma} + 4\sum_{\gamma=k+3}^{\xi} C_{A^kA^\gamma} + C_{A^{k+1}A^\xi} - 2\cdot C_{A^kA^\xi} - C_{A^{k+1}n} \\
&+ 2\cdot C_{A^kn} + \sum_{\lambda=3}^{k} C_{A^\lambda A^\xi} - 2\sum_{\lambda=2}^{k-1} C_{A^\lambda A^\xi} - \sum_{\lambda=3}^{k} C_{A^\lambda n} + 2\sum_{\lambda=2}^{k-1} C_{A^\lambda n} \\
&= \sum_{\lambda=3}^{k}\sum_{\gamma=k+1}^{\xi-1} C_{A^\lambda A^\gamma} + 2\sum_{\gamma=k+2}^{\xi} C_{A^2A^\gamma} - 2\sum_{\gamma=k+2}^{\xi} C_{A^kA^\gamma} + 2\sum_{\lambda=2}^{k-1} C_{A^\lambda A^\xi} \\
&- 2\sum_{\lambda=2}^{k-1} C_{A^\lambda A^{k+1}} + \sum_{\gamma=k+2}^{\xi-1} C_{A^{k+1}A^\gamma} - 2\sum_{\gamma=k+3}^{\xi} C_{A^{k+1}A^\gamma} - 2\sum_{\gamma=k+2}^{\xi-1} C_{A^kA^\gamma} \\
&+ 4\sum_{\gamma=k+3}^{\xi} C_{A^kA^\gamma} + C_{V^{k-1}V^k} + C_{A^{k+1}A^\xi} - 2\cdot C_{A^kA^\xi} - C_{A^{k+1}n} + 2\cdot C_{A^kn} \\
&+ \sum_{\lambda=3}^{k} C_{A^\lambda A^\xi} - 2\sum_{\lambda=2}^{k-1} C_{A^\lambda A^\xi} - \sum_{\lambda=3}^{k} C_{A^\lambda n} + 2\sum_{\lambda=2}^{k-1} C_{A^\lambda n}
\end{aligned} \tag{B.64}
$$

In view of (B.60)-(B.64) equation (B.58) takes the form

$$
\begin{aligned}
c\left(\delta\left(\bigcup_{\gamma=1}^{k} A^\gamma \cup \bigcup_{\lambda=1}^{k-1} V^\lambda\right)\right) &= \cancel{C_{A^{k+1}n}} + \sum_{\gamma=1}^{k} C_{A^\gamma A^{k+1}} - \sum_{\gamma=1}^{k} C_{A^\gamma A^\xi} - \sum_{\gamma=1}^{k}\sum_{\lambda=k+2}^{\xi} C_{A^\gamma A^\lambda} \\
&+ \sum_{\gamma=1}^{k} C_{A^\gamma V^{\xi-1}} + \sum_{\gamma=1}^{k}\sum_{\lambda=k+1}^{\xi} C_{A^\gamma A^\lambda} + \sum_{\lambda=3}^{k}\sum_{\gamma=k+1}^{\xi-1} C_{A^\lambda A^\gamma} + 2\sum_{\gamma=k+2}^{\xi} C_{A^2A^\gamma} \\
&- 2\sum_{\gamma=k+2}^{\xi} C_{A^kA^\gamma} + 2\sum_{\lambda=2}^{k-1} C_{A^\lambda A^\xi} - 2\sum_{\lambda=2}^{k-1} C_{A^\lambda A^{k+1}} + C_{V^{k-1}V^k} + \sum_{\gamma=k+2}^{\xi-1} C_{A^{k+1}A^\gamma} \\
&- 2\sum_{\gamma=k+3}^{\xi} C_{A^{k+1}A^\gamma} - 2\sum_{\gamma=k+2}^{\xi-1} C_{A^kA^\gamma} + 4\sum_{\gamma=k+3}^{\xi} C_{A^kA^\gamma} + C_{A^{k+1}A^\xi} - 2C_{A^kA^\xi} \\
&- \cancel{C_{A^{k+1}n}} + 2\cdot C_{A^kn} + \sum_{\lambda=3}^{k} C_{A^\lambda A^\xi} - 2\sum_{\lambda=2}^{k-1} C_{A^\lambda A^\xi} - \sum_{\lambda=3}^{k} C_{A^\lambda n} + 2\sum_{\lambda=2}^{k-1} C_{A^\lambda n} \\
&+ \sum_{\gamma=k+1}^{\xi} C_{A^kA^\gamma} - \sum_{\gamma=k+1}^{\xi} C_{A^2A^\gamma} - \sum_{\lambda=2}^{k-1}\sum_{\gamma=k+1}^{\xi} C_{A^\lambda A^\gamma} + \sum_{\gamma=k+2}^{\xi} C_{A^{k+1}A^\gamma} \\
&- 2\sum_{\gamma=k+2}^{\xi} C_{A^kA^\gamma} + C_{A^{k+1}V^{k-1}}
\end{aligned} \tag{B.65}
$$

$$
\begin{aligned}
&= \sum_{\gamma=1}^{k} C_{A^\gamma A^{k+1}} - \sum_{\gamma=k+1}^{\xi-1} C_{A^2 A^\gamma} + \sum_{\gamma=k+1}^{\xi-1} C_{A^k A^\gamma} - \sum_{\gamma=2}^{k-1} C_{A^\gamma A^\xi} - \sum_{\gamma=2}^{k-1} C_{A^\gamma A^{k+1}} \\
&\quad + \cancel{C_{A^1 A^{k+1}}} + C_{A^k A^{k+1}} + C_{A^{k+1} A^{k+2}} - \cancel{C_{A^{k+1} A^\xi}} + C_{A^{k+1} A^{k+2}} + \cancel{C_{A^{k+1} A^\xi}} \\
&\quad - \cancel{2C_{A^k A^\xi}} - C_{A^1 A^\xi} - C_{A^2 A^\xi} + \sum_{\gamma=1}^{k} C_{A^\gamma V^{\xi-1}} - \sum_{\gamma=k+2}^{\xi} C_{A^k A^\gamma} + C_{A^k A^{k+1}} \\
&\quad - 2C_{A^k A^{k+2}} - 2C_{A^k A^{k+2}} + \cancel{2C_{A^k A^\xi}} + \sum_{\gamma=k+2}^{\xi} C_{A^2 A^\gamma} + C_{A^{k+1} V^{k-1}} - \cancel{C_{A^2 A^{k+1}}} \\
&\quad + C_{V^{k-1} V^k} + 2C_{A^k n} + \sum_{\lambda=2}^{k-1} C_{A^\lambda n} + C_{A^2 n} - C_{A^k n} \\
&= C_{A^k A^{k+1}} + C_{A^1 A^{k+1}} + C_{A^2 A^\xi} - C_{A^2 A^{k+1}} + C_{A^k A^{k+1}} - C_{A^k A^\xi} - \sum_{\gamma=1}^{k-1} C_{A^\gamma A^\xi} \\
&\quad + \sum_{\gamma=1}^{k} C_{A^\gamma V^{\xi-1}} + C_{A^k A^{k+1}} + C_{A^{k+1} A^{k+2}} - C_{A^2 A^\xi} + \sum_{\gamma=2}^{k-1} C_{A^\gamma A^\xi} + C_{A^k A^{k+1}} \\
&\quad - 4C_{A^k A^{k+2}} + 2C_{A^k A^\xi} + C_{A^{k+1} V^{k-1}} + C_{V^k V^{k-1}} + \sum_{\gamma=1}^{k} C_{A^\gamma n} \qquad \text{(B.66)} \\
&= 4C_{A^k A^{k+1}} - 4C_{A^k A^{k+2}} + 2C_{A^{k+1} A^{k+2}} + C_{A^{k+1} V^{k-1}} + C_{V^k V^{k-1}} \\
&\qquad + \underbrace{\sum_{\gamma=1}^{k} C_{A^\gamma n} - \sum_{\gamma=1}^{k} C_{A^\gamma A^\xi} + \sum_{\gamma=1}^{k} C_{A^\gamma V^{\xi-1}}}_{=0 \text{ by Lemma } B.0.6} \\
&= \frac{4}{2 \cdot 2^{|V^{k-1}|+1}} - \frac{4}{2 \cdot 2^{|V^{k-1}|+|V^k|+2}} \\
&\qquad + \frac{2}{2 \cdot 2^{|V^k|+1}} + \frac{(2^{|V^k|}-1)(2^{|V^{k-1}|}-1)}{2 \cdot 2^{|V^k|} \cdot 2^{|V^{k-1}|}} + \frac{1}{2}\left(1 - \frac{1}{2^{|V^{k-1}|}}\right) \\
&= \frac{2 \cdot 2^{|V^k|} - 1 + 2^{|V^{k-1}|} + 2^{|V^k|}2^{|V^{k-1}|} - 2^{|V^k|} - 2^{|V^{k-1}|} + 1 + 2^{|V^{k-1}|}2^{|V^k|} - 2^{|V^k|}}{2 \cdot 2^{|V^k|} \cdot 2^{|V^{k-1}|}} \\
&= \frac{2 \cdot 2^{|V^k|} \cdot 2^{|V^{k-1}|}}{2 \cdot 2^{|V^k|} \cdot 2^{|V^{k-1}|}} = 1.
\end{aligned}
$$

□

Appendix C

In this chapter, we present four auxiliary lemmas, which were used in the proof of Proposition 3.7.1. Note that, from the definition of $c \in \mathbb{R}^{E_n}_{>0}$, see (3.223)-(3.232), we have $c_{a_1^\lambda,*} = c_{a_2^\lambda,*}$ for all $\lambda \in [\xi]$. Thus, in Lemmas C.0.1 and C.0.3 it suffices to calculate $C_{a_i^\lambda,*}$ for $i \geq 2$.

Lemma C.0.1. *Let $c \in \mathbb{R}^{E_n}_{>0}$ be defined by* (3.223)-(3.232) *and*

$$C_{a_i^\lambda A^\beta} := \sum_{j=1}^{|A^\beta|} c_{a_i^\lambda a_j^\beta}. \tag{C.1}$$

Then, for $i \in [|A^\lambda|]$, $i \geq 2$ and $\lambda, \beta \in [\xi]$, $\lambda \neq \beta$ we have

$$\begin{aligned} C_{a_i^\lambda A^\beta} &= \frac{C_{A^\lambda A^\beta}}{2^{|A^\lambda|-i+1}}, \qquad \lambda < \beta \\ C_{a_i^\lambda A^\beta} &= \frac{C_{A^\beta A^\lambda}}{2^{|A^\lambda|-i+1}}, \qquad \lambda > \beta \end{aligned} \tag{C.2}$$

Proof. Let $\lambda, \beta \in [\xi]$ and $i \in [|A^\lambda|]$, $i \geq 2$.

If $\lambda < \beta$ then calculating expression (C.1) we obtain

$$\begin{aligned} C_{a_i^\lambda A^\beta} &= \sum_{j=1}^{|A^\beta|} c_{a_i^\lambda a_j^\beta} = c_{a_i^\lambda a_1^\beta} + \sum_{j=2}^{|A^\beta|} c_{a_i^\lambda a_j^\beta} = c_{a_i^\lambda a_2^\beta} + \sum_{j=2}^{|A^\beta|} c_{a_i^\lambda a_j^\beta} \\ &= \frac{1}{2^3 \cdot 2^{|A^\lambda|-i} \cdot 2^{|A^\beta|-2} \prod\limits_{k=\lambda-1}^{\beta-2} 2^{|V^k|+1}} + \sum_{j=2}^{|A^\beta|} \frac{1}{2^3 \cdot 2^{|A^\lambda|-i} \cdot 2^{|A^\beta|-j} \prod\limits_{k=\lambda-1}^{\beta-2} 2^{|V^k|+1}} \\ &= \frac{1}{2^3 \cdot 2^{|A^\lambda|-i} \cdot 2^{|A^\beta|-2} \cdot 2^{\sum_{k=\lambda-1}^{\beta-2}(|V^k|+1)}} \left(1 + \frac{1}{2^2} \cdot \sum_{j=2}^{|A^\beta|} 2^j\right) \end{aligned} \tag{C.3}$$

$$
\begin{aligned}
&= \frac{1}{2^3 \cdot 2^{|A^\lambda|-i} \cdot 2^{|A^\beta|-2} \cdot 2^{\sum_{k=\lambda-1}^{\beta-2}(|V^k|+1)}} \cdot \left(1 + \frac{1}{2^2} \cdot \frac{2^2 \cdot (1-2^{|A^\beta|-1})}{1-2}\right) \\
&= \frac{2^{|A^\beta|-1}}{2^3 \cdot 2^{|A^\lambda|-i} \cdot 2^{|A^\beta|-2} \cdot 2^{\sum_{k=\lambda-1}^{\beta-2}(|V^k|+1)}} = \frac{1}{2^{|A^\lambda|-i+1}} \cdot \frac{1}{2 \cdot 2^{\sum_{k=\lambda-1}^{\beta-2}(|V^k|+1)}}.
\end{aligned}
\tag{C.4}
$$

Now, let $\lambda > \beta$. Then, we deduce for expression (C.1)

$$
\begin{aligned}
C_{a_i^\lambda A^\beta} &= \sum_{j=1}^{|A^\beta|} c_{a_j^\beta a_i^\lambda} = c_{a_1^\beta a_i^\lambda} + \sum_{j=2}^{|A^\beta|} c_{a_j^\beta a_i^\lambda} = c_{a_2^\beta a_i^\lambda} + \sum_{j=2}^{|A^\beta|} c_{a_j^\beta a_i^\lambda} \\
&= \frac{1}{2^3 \cdot 2^{|A^\beta|-2} \cdot 2^{|A^\lambda|-i} \prod_{k=\beta-1}^{\lambda-2} 2^{|V^k|+1}} + \sum_{j=2}^{|A^\beta|} \frac{1}{2^3 \cdot 2^{|A^\beta|-j} \cdot 2^{|A^\lambda|-i} \prod_{k=\beta-1}^{\lambda-2} 2^{|V^k|+1}} \\
&= \frac{1}{2^3 \cdot 2^{|A^\beta|-2} \cdot 2^{|A^\lambda|-i} \cdot 2^{\sum_{k=\beta-1}^{\lambda-2}(|V^k|+1)}} \left(1 + \frac{1}{2^2} \cdot \sum_{j=2}^{|A^\beta|} 2^j\right) \\
&= \frac{1}{2^3 \cdot 2^{|A^\beta|-2} \cdot 2^{|A^\lambda|-i} \cdot 2^{\sum_{k=\beta-1}^{\lambda-2}(|V^k|+1)}} \left(1 + \frac{1}{2^2} \cdot \frac{2^2 \cdot (1-2^{|A^\beta|-1})}{1-2}\right) \\
&= \frac{2^{|A^\beta|-1}}{2^3 \cdot 2^{|A^\beta|-2} \cdot 2^{|A^\lambda|-i} \cdot 2^{\sum_{k=\beta-1}^{\lambda-2}(|V^k|+1)}} = \frac{1}{2^{|A^\lambda|-i+1}} \cdot \frac{1}{2 \cdot 2^{\sum_{k=\beta-1}^{\lambda-2}(|V^k|+1)}}.
\end{aligned}
\tag{C.5}
$$

Equation (B.11) from Lemma B.0.2 implies that (C.1) can be written as

$$
\begin{aligned}
C_{a_i^\lambda A^\beta} &= \frac{C_{A^\lambda A^\beta}}{2^{|A^\lambda|-i+1}}, \qquad \lambda < \beta \\
C_{a_i^\lambda A^\beta} &= \frac{C_{A^\beta A^\lambda}}{2^{|A^\lambda|-i+1}}, \qquad \lambda > \beta
\end{aligned}
\tag{C.6}
$$

□

Lemma C.0.2. *Let* $c \in \mathbb{R}^{E_n}_{>0}$ *be defined by* (3.223)-(3.232), λ, $\beta \in [\xi - 1]$ *and*

$$
C_{v_i^\lambda V^\beta} := \sum_{j=1}^{|V^\beta|} c_{v_i^\lambda v_j^\beta}, \quad i \in [|V^\lambda|]. \tag{C.7}
$$

1. *For* $\beta > \lambda + 1$ *we have*

$$
\begin{aligned}
C_{v_i^\lambda V^\beta} &= 2^i (C_{A^{\lambda+1} A^{\beta+1}} - 2C_{A^{\lambda+1} A^{\beta+2}}), \quad \beta < \xi - 1 \\
C_{v_i^\lambda V^{\xi-1}} &= 2^i (C_{A^{\lambda+1} A^{\xi}} - C_{A^{\lambda+1} n}), \quad \beta = \xi - 1
\end{aligned}
\tag{C.8}
$$

2. For $\beta = \lambda + 1$ *we have*

$$C_{v_i^\lambda V^\beta} = C_{v_i^\lambda V^{\lambda+1}} = \frac{1}{2^{|V^\lambda|-i+2}} \left(1 - \frac{1}{2^{|V^{\lambda+1}|}}\right). \tag{C.9}$$

3. For $\beta = \lambda - 1$ *we have*

$$C_{v_i^\lambda V^\beta} = C_{v_i^\lambda V^{\lambda-1}} = \frac{1}{2^{i+1}} \left(1 - \frac{1}{2^{|V^{\lambda-1}|}}\right). \tag{C.10}$$

4. For $\beta < \lambda - 1$ *we have*

$$C_{v_i^\lambda V^\beta} = \frac{1}{2^i}(C_{A^{\beta+2}A^{\lambda+1}} - 2C_{A^{\beta+1}A^{\lambda+1}}). \tag{C.11}$$

Proof. Let λ, $\beta \in [\xi - 1]$.

1. Let $\beta > \lambda + 1$. Then, we deduce for expression (C.7)

$$\begin{aligned}
C_{v_i^\lambda V^\beta} &= \sum_{j=1}^{|V^\beta|} c_{v_i^\lambda v_j^\beta} = \sum_{j=1}^{|V^\beta|} \frac{1}{2^{j-i+1} \prod_{k=\lambda}^{\beta-1} 2^{(|V^k|+1)}} = \sum_{j=1}^{|V^\beta|} \frac{1}{2^{j-i+1} \cdot 2^{\sum_{k=\lambda}^{\beta-1}(|V^k|+1)}} \\
&= \frac{2^i}{2 \cdot 2^{\sum_{k=\lambda}^{\beta-1}(|V^k|+1)}} \sum_{j=1}^{|V^\beta|} \frac{1}{2^j} = \frac{2^i}{2 \cdot 2^{\sum_{k=\lambda}^{\beta-1}(|V^k|+1)}} \cdot \frac{\frac{1}{2}(1 - \frac{1}{2^{|V^\beta|}})}{1 - \frac{1}{2}} \\
&= \frac{2^i}{2 \cdot 2^{\sum_{k=\lambda}^{\beta-1}(|V^k|+1)}} - \frac{2^i}{2 \cdot 2^{|V^\beta|} \cdot 2^{\sum_{k=\lambda}^{\beta-1}(|V^k|+1)}} = \frac{2^i}{2 \cdot 2^{\sum_{k=\lambda}^{\beta-1}(|V^k|+1)}} - \frac{2 \cdot 2^i}{2 \cdot 2^{\sum_{k=\lambda}^{\beta}(|V^k|+1)}}.
\end{aligned} \tag{C.12}$$

Applying (B.3) and (B.11) to (C.12) we infer

$$\begin{aligned}
C_{v_i^\lambda V^\beta} &= 2^i(C_{A^{\lambda+1}A^{\beta+1}} - 2C_{A^{\lambda+1}A^{\beta+2}}), \quad \beta < \xi - 1 \\
C_{v_i^\lambda V^{\xi-1}} &= 2^i(C_{A^{\lambda+1}A^\xi} - C_{A^{\lambda+1}n}), \quad \beta = \xi - 1
\end{aligned} \tag{C.13}$$

2. If $\beta = \lambda + 1$ then we obtain

$$\begin{aligned}
C_{v_i^\lambda V^{\lambda+1}} &= \sum_{j=1}^{|V^{\lambda+1}|} c_{v_i^\lambda v_j^{\lambda+1}} = \sum_{j=1}^{|V^{\lambda+1}|} \frac{1}{2^{j+1} \cdot 2 \cdot 2^{|V^\lambda|-i}} = \frac{1}{2^{|V^\lambda|-i+2}} \sum_{j=1}^{|V^{\lambda+1}|} \frac{1}{2^j} \\
&= \frac{1}{2^{|V^\lambda|-i+2}} \cdot \frac{\frac{1}{2}(1 - \frac{1}{2^{|V^{\lambda+1}|}})}{1 - \frac{1}{2}} = \frac{1}{2^{|V^\lambda|-i+2}} \left(1 - \frac{1}{2^{|V^{\lambda+1}|}}\right).
\end{aligned} \tag{C.14}$$

3. If $\beta = \lambda - 1$ then we have

$$
\begin{aligned}
C_{v_i^\lambda V^{\lambda-1}} &= \sum_{j=1}^{|V^{\lambda-1}|} c_{v_j^{\lambda-1} v_i^\lambda} = \sum_{j=1}^{|V^{\lambda-1}|} \frac{1}{2^{i+1}\cdot 2 \cdot 2^{|V^{\lambda-1}|-j}} = \frac{1}{2^i \cdot 2^{|V^{\lambda-1}|+2}} \sum_{j=1}^{|V^{\lambda-1}|} 2^j \\
&= \frac{1}{2^i \cdot 2^{|V^{\lambda-1}|+2}} \frac{2(1-2^{|V^{\lambda-1}|})}{1-2} = \frac{1}{2^{i+1}}\left(1 - \frac{1}{2^{|V^{\lambda-1}|}}\right).
\end{aligned}
\tag{C.15}
$$

4. Let $\beta < \lambda - 1$. Then, expression (C.7) takes the form

$$
\begin{aligned}
C_{v_i^\lambda V^\beta} &= \sum_{j=1}^{|V^\beta|} c_{v_j^\beta v_i^\lambda} = \sum_{j=1}^{|V^\beta|} \frac{1}{2^{i-j+1} \prod_{k=\beta}^{\lambda-1} 2^{(|V^k|+1)}} = \sum_{j=1}^{|V^\beta|} \frac{1}{2^{i-j+1}\cdot 2^{\sum_{k=\beta}^{\lambda-1}(|V^k|+1)}} \\
&= \frac{1}{2^{i+1}\cdot 2^{\sum_{k=\beta}^{\lambda-1}(|V^k|+1)}} \sum_{j=1}^{|V^\beta|} 2^j = \frac{1}{2^{i+1}\cdot 2^{\sum_{k=\beta}^{\lambda-1}(|V^k|+1)}} \cdot \frac{2(1-2^{|V^\beta|})}{1-2} \\
&= \frac{2^{|V^\beta|}-1}{2^i \cdot 2^{\sum_{k=\beta}^{\lambda-1}(|V^k|+1)}} = \frac{2^{|V^\beta|}}{2^i \cdot 2^{\sum_{k=\beta}^{\lambda-1}(|V^k|+1)}} - \frac{1}{2^i \cdot 2^{\sum_{k=\beta}^{\lambda-1}(|V^k|+1)}} \\
&= \frac{1}{2^{i+1}\cdot 2^{\sum_{k=\beta+1}^{\lambda-1}(|V^k|+1)}} - \frac{1}{2^i \cdot 2^{\sum_{k=\beta}^{\lambda-1}(|V^k|+1)}} \overset{(B.11)}{=} \frac{1}{2^i}(C_{A^{\beta+2}A^{\lambda+1}} - 2C_{A^{\beta+1}A^{\lambda+1}}).
\end{aligned}
\tag{C.16}
$$

□

Lemma C.0.3. *Let $c \in \mathbb{R}^{E_n}_{>0}$ be defined by* (3.223)-(3.232)*, $\beta \in [\xi]$, $\lambda \in [\xi - 1]$ and*

$$
C_{a_i^\beta V^\lambda} := \sum_{j=1}^{|V^\lambda|} c_{a_i^\beta v_j^\lambda},\ i \in [|A^\beta|], i \geq 2 \quad \textit{and} \quad C_{v_j^\lambda A^\beta} := \sum_{i=1}^{|A^\beta|} c_{v_j^\lambda a_i^\beta},\ j \in [|V^\lambda|]. \tag{C.17}
$$

1. For $\beta \leq \lambda$ we have

$$
\begin{aligned}
C_{a_i^\beta V^\lambda} &= \frac{1}{2^{|A^\beta|-i+1}} \cdot (C_{A^\beta A^{\lambda+1}} - 2C_{A^\beta A^{\lambda+2}}), \quad \lambda < \xi - 1 \\
C_{a_i^\beta V^{\xi-1}} &= \frac{1}{2^{|A^\beta|-i+1}} \cdot (C_{A^\beta A^\xi} - C_{A^\beta n}), \quad \lambda = \xi - 1 \\
C_{v_j^\lambda A^\beta} &= \frac{1}{2^j} \cdot C_{A^\beta A^{\lambda+1}}
\end{aligned}
\tag{C.18}
$$

2. *For $\beta = \lambda + 1$ we have*

$$C_{a_i^\beta V^\lambda} = C_{a_i^{\lambda+1} V^\lambda} = \frac{1}{2^{|A^{\lambda+1}|-i+2}} \cdot \left(1 - \frac{1}{2^{|V^\lambda|}}\right),$$
$$C_{v_j^\lambda A^\beta} = C_{v_j^\lambda A^{\lambda+1}} = \frac{1}{2^{j+1}}. \tag{C.19}$$

3. *For $\beta = \lambda + 2$ we have*

$$C_{a_i^\beta V^\lambda} = C_{a_i^{\lambda+2} V^\lambda} = \frac{1}{2^{|A^{\lambda+2}|-i+2}} \cdot \left(1 - \frac{1}{2^{|V^\lambda|}}\right),$$
$$C_{v_j^\lambda A^\beta} = C_{v_j^\lambda A^{\lambda+2}} = \frac{1}{2^{|V^\lambda|-j+2}}. \tag{C.20}$$

4. *For $\beta > \lambda + 2$ we have*

$$C_{a_i^\beta V^\lambda} = \frac{1}{2^{|A^\beta|-i+1}} \cdot (C_{A^{\lambda+2} A^\beta} - 2 \cdot C_{A^{\lambda+1} A^\beta}),$$
$$C_{v_j^\lambda A^\beta} = 2^j C_{A^{\lambda+1} A^\beta}. \tag{C.21}$$

Proof. Let $\beta \in [\xi]$ and $\lambda \in [\xi - 1]$.

1. Let $\beta \leq \lambda$. Then, we have for the first term of (C.17)

$$\begin{aligned}
C_{a_i^\beta V^\lambda} &= \sum_{j=1}^{|V^\lambda|} c_{a_i^\beta v_j^\lambda} = \sum_{j=1}^{|V^\lambda|} \frac{1}{2^{j+1} \cdot 2 \cdot 2^{|A^\beta|-i} \prod\limits_{k=\beta-1}^{\lambda-1} 2^{|V^k|+1}} \\
&= \sum_{j=1}^{|V^\lambda|} \frac{1}{2^{j+1} \cdot 2 \cdot 2^{|A^\beta|-i} \cdot 2^{\sum_{k=\beta-1}^{\lambda-1} (|V^k|+1)}} = \frac{1}{2^{|A^\beta|-i+2} \cdot 2^{\sum_{k=\beta-1}^{\lambda-1} (|V^k|+1)}} \sum_{j=1}^{|V^\lambda|} \frac{1}{2^j} \\
&= \frac{1}{2^{|A^\beta|-i+2} \cdot 2^{\sum_{k=\beta-1}^{\lambda-1} (|V^k|+1)}} \cdot \frac{\frac{1}{2}(1 - \frac{1}{2^{|V^\lambda|}})}{1 - \frac{1}{2}} \\
&= \frac{1}{2^{|A^\beta|-i+2} \cdot 2^{\sum_{k=\beta-1}^{\lambda-1} (|V^k|+1)}} \cdot \left(1 - \frac{1}{2^{|V^\lambda|}}\right) \\
&= \frac{1}{2^{|A^\beta|-i+1}} \left(\frac{1}{2 \cdot 2^{\sum_{k=\beta-1}^{\lambda-1} (|V^k|+1)}} - \frac{1}{2^{|V^\lambda|+1} \cdot 2^{\sum_{k=\beta-1}^{\lambda-1} (|V^k|+1)}} \right) \\
&= \frac{1}{2^{|A^\beta|-i+1}} \left(\frac{1}{2 \cdot 2^{\sum_{k=\beta-1}^{\lambda-1} (|V^k|+1)}} - \frac{2}{2 \cdot 2^{\sum_{k=\beta-1}^{\lambda} (|V^k|+1)}} \right).
\end{aligned} \tag{C.22}$$

Applying (B.3) and (B.11) to (C.22) we infer

$$C_{a_i^\beta V^\lambda} = \frac{1}{2^{|A^\beta|-i+1}} \cdot (C_{A^\beta A^{\lambda+1}} - 2 \cdot C_{A^\beta A^{\lambda+2}}), \quad \lambda < \xi - 1 \tag{C.23}$$

$$C_{a_i^\beta V^{\xi-1}} = \frac{1}{2^{|A^\beta|-i+1}} \cdot (C_{A^\beta A^\xi} - C_{A^\beta n}), \quad \lambda = \xi - 1 \tag{C.24}$$

Calculating the second term of (C.17) we arrive at

$$\begin{aligned}
C_{v_j^\lambda A^\beta} &= \sum_{i=1}^{|A^\beta|} c_{a_i^\beta v_j^\lambda} = c_{a_1^\beta v_j^\lambda} + \sum_{i=2}^{|A^\beta|} c_{a_i^\beta v_j^\lambda} = c_{a_2^\beta v_j^\lambda} + \sum_{i=2}^{|A^\beta|} c_{a_i^\beta v_j^\lambda} \\
&= \frac{1}{2^{j+1} \cdot 2 \cdot 2^{|A^\beta|-2} \prod\limits_{k=\beta-1}^{\lambda-1} 2^{|V^k|+1}} + \sum_{i=2}^{|A^\beta|} \frac{1}{2^{j+1} \cdot 2 \cdot 2^{|A^\beta|-i} \prod\limits_{k=\beta-1}^{\lambda-1} 2^{|V^k|+1}} \\
&= \frac{1}{2^{j+1} \cdot 2 \cdot 2^{|A^\beta|-2} \cdot 2^{\sum_{k=\beta-1}^{\lambda-1}(|V^k|+1)}} + \sum_{i=2}^{|A^\beta|} \frac{1}{2^{j+1} \cdot 2 \cdot 2^{|A^\beta|-i} \cdot 2^{\sum_{k=\beta-1}^{\lambda-1}(|V^k|+1)}} \\
&= \frac{1}{2^{|A^\beta|+j} \cdot 2^{\sum_{k=\beta-1}^{\lambda-1}(|V^k|+1)}} \left(1 + \frac{1}{2^2} \cdot \sum_{i=2}^{|A^\beta|} 2^i \right) \\
&= \frac{1}{2^{|A^\beta|+j} \cdot 2^{\sum_{k=\beta-1}^{\lambda-1}(|V^k|+1)}} \left(1 + \frac{1}{2^2} \cdot \frac{2^2 \cdot (1 - 2^{|A^\beta|-1})}{1-2} \right) \\
&= \frac{2^{|A^\beta|-1}}{2^{|A^\beta|+j} \cdot 2^{\sum_{k=\beta-1}^{\lambda-1}(|V^k|+1)}} = \frac{1}{2^{j+1} \cdot 2^{\sum_{k=\beta-1}^{\lambda-1}(|V^k|+1)}} = \frac{1}{2^j} C_{A^\beta A^{\lambda+1}}
\end{aligned} \tag{C.25}$$

2. If $\beta = \lambda + 1$ then we have for the first term of (C.17)

$$\begin{aligned}
C_{a_i^{\lambda+1} V^\lambda} &= \sum_{j=1}^{|V^\lambda|} c_{a_i^{\lambda+1} v_j^\lambda} = \sum_{j=1}^{|V^\lambda|} \frac{1}{2^{j+1} \cdot 2 \cdot 2^{|A^{\lambda+1}|-i}} = \frac{1}{2^{|A^{\lambda+1}|-i+2}} \sum_{j=1}^{|V^\lambda|} \frac{1}{2^j} \\
&= \frac{1}{2^{|A^{\lambda+1}|-i+2}} \cdot \frac{\frac{1}{2}(1 - \frac{1}{2^{|V^\lambda|}})}{1 - \frac{1}{2}} = \frac{1}{2^{|A^{\lambda+1}|-i+2}} \left(1 - \frac{1}{2^{|V^\lambda|}} \right).
\end{aligned} \tag{C.26}$$

For the second term of (C.17) with $\beta = \lambda + 1$ we deduce

$$\begin{aligned}
C_{v_j^\lambda A^{\lambda+1}} &= \sum_{i=1}^{|A^{\lambda+1}|} c_{a_i^{\lambda+1} v_j^\lambda} = c_{a_1^{\lambda+1} v_j^\lambda} + \sum_{i=2}^{|A^{\lambda+1}|} c_{a_i^{\lambda+1} v_j^\lambda} = c_{a_2^{\lambda+1} v_j^\lambda} + \sum_{i=2}^{|A^{\lambda+1}|} c_{a_i^{\lambda+1} v_j^\lambda} \\
&= \frac{1}{2^{j+1} \cdot 2 \cdot 2^{|A^{\lambda+1}|-2}} + \sum_{i=2}^{|A^{\lambda+1}|} \frac{1}{2^{j+1} \cdot 2 \cdot 2^{|A^{\lambda+1}|-i}} \\
&= \frac{1}{2^{|A^{\lambda+1}|+j}} \left(1 + \frac{1}{2^2} \cdot \sum_{i=2}^{|A^{\lambda+1}|} 2^i \right) = \frac{1}{2^{|A^{\lambda+1}|+j}} \left(1 + \frac{1}{2^2} \cdot \frac{2^2(1 - 2^{|A^{\lambda+1}|-1})}{1-2} \right) \\
&= \frac{2^{|A^{\lambda+1}|-1}}{2^{|A^{\lambda+1}|+j}} = \frac{1}{2^{j+1}}.
\end{aligned} \tag{C.27}$$

3. If $\beta = \lambda + 2$ then we obtain for the first term of (C.17)

$$\begin{aligned}
C_{a_i^{\lambda+2} V^\lambda} &= \sum_{j=1}^{|V^\lambda|} c_{a_i^{\lambda+2} v_j^\lambda} = \sum_{j=1}^{|V^\lambda|} \frac{1}{2^3 \cdot 2^{|A^{\lambda+2}|-i} \cdot 2^{|V^\lambda|-j}} = \frac{1}{2^3 \cdot 2^{|A^{\lambda+2}|-i} \cdot 2^{|V^\lambda|}} \sum_{j=1}^{|V^\lambda|} 2^j \\
&= \frac{1}{2^3 \cdot 2^{|A^{\lambda+2}|-i} \cdot 2^{|V^\lambda|}} \cdot \frac{2(1 - 2^{|V^\lambda|})}{1-2} = \frac{1}{2^2 \cdot 2^{|A^{\lambda+2}|-i}} \cdot \frac{(2^{|V^\lambda|} - 1)}{2^{|V^\lambda|}} \\
&= \frac{1}{2^{|A^{\lambda+2}|-i+2}} \left(1 - \frac{1}{2^{|V^\lambda|}} \right).
\end{aligned} \tag{C.28}$$

For the second term of (C.17) with $\beta = \lambda + 2$ we have

$$\begin{aligned}
C_{v_j^\lambda A^{\lambda+2}} &= \sum_{i=1}^{|A^{\lambda+2}|} c_{a_i^\beta v_j^\lambda} = c_{a_1^{\lambda+2} v_j^\lambda} + \sum_{i=2}^{|A^{\lambda+2}|} c_{a_i^{\lambda+2} v_j^\lambda} = c_{a_2^{\lambda+2} v_j^\lambda} + \sum_{i=2}^{|A^\beta|} c_{a_i^{\lambda+2} v_j^\lambda} \\
&= \frac{1}{2^3 \cdot 2^{|A^{\lambda+2}|-2} \cdot 2^{|V^\lambda|-j}} + \sum_{i=2}^{|A^\beta|} \frac{1}{2^3 \cdot 2^{|A^{\lambda+2}|-i} \cdot 2^{|V^\lambda|-j}} \\
&= \frac{1}{2^{|A^{\lambda+2}|+1} \cdot 2^{|V^\lambda|-j}} \cdot \left(1 + \frac{1}{2^2} \cdot \sum_{i=2}^{|A^{\lambda+2}|} 2^i \right) \\
&= \frac{1}{2^{|A^{\lambda+2}|+1} \cdot 2^{|V^\lambda|-j}} \cdot \left(1 + \frac{1}{2^2} \cdot \frac{2^2 \cdot (1 - 2^{|A^{\lambda+2}|-1})}{1-2} \right) \\
&= \frac{2^{|A^{\lambda+2}|-1}}{2^{|A^{\lambda+2}|+1} \cdot 2^{|V^\lambda|-j}} = \frac{1}{2^{|V^\lambda|-j+2}}.
\end{aligned} \tag{C.29}$$

4. Let $\beta > \lambda + 2$. Then, the first term of (C.17) takes the form

$$C_{a_i^\beta V^\lambda} = \sum_{j=1}^{|V^\lambda|} c_{a_i^\beta v_j^\lambda} = \sum_{j=1}^{|V^\lambda|} \frac{1}{2^{2-j} \cdot 2^{|A^\beta|-i} \prod_{k=\lambda}^{\beta-2} 2^{|V^k|+1}} = \sum_{j=1}^{|V^\lambda|} \frac{1}{2^{2-j} \cdot 2^{|A^\beta|-i} \cdot 2^{\sum_{k=\lambda}^{\beta-2}(|V^k|+1)}}$$

$$= \frac{1}{2^{|A^\beta|-i+2} \cdot 2^{\sum_{k=\lambda}^{\beta-2}(|V^k|+1)}} \sum_{j=1}^{|V^\lambda|} 2^j = \frac{1}{2^{|A^\beta|-i+2} \cdot 2^{\sum_{k=\lambda}^{\beta-2}(|V^k|+1)}} \cdot \frac{2(1-2^{|V^\lambda|})}{1-2}$$

$$= \frac{(2^{|V^\lambda|}-1)}{2^{|A^\beta|-i+1} \cdot 2^{\sum_{k=\lambda}^{\beta-2}(|V^k|+1)}} = \frac{1}{2^{|A^\beta|-i+1}} \left(\frac{2^{|V^\lambda|}}{2^{\sum_{k=\lambda}^{\beta-2}(|V^k|+1)}} - \frac{1}{2^{\sum_{k=\lambda}^{\beta-2}(|V^k|+1)}} \right) \tag{C.30}$$

$$= \frac{1}{2^{|A^\beta|-i+1}} \left(\frac{1}{2 \cdot 2^{\sum_{k=\lambda+1}^{\beta-2}(|V^k|+1)}} - \frac{2}{2 \cdot 2^{\sum_{k=\lambda}^{\beta-2}(|V^k|+1)}} \right)$$

$$= \frac{1}{2^{|A^\beta|-i+1}} (C_{A^{\lambda+2}A^\beta} - 2C_{A^{\lambda+1}A^\beta}).$$

For the second term of (C.17) we deduce

$$C_{v_j^\lambda A^\beta} = \sum_{i=1}^{|A^\beta|} c_{a_i^\beta v_j^\lambda} = c_{a_1^\beta v_j^\lambda} + \sum_{i=2}^{|A^\beta|} c_{a_i^\beta v_j^\lambda} = c_{a_2^\beta v_j^\lambda} + \sum_{i=2}^{|A^\beta|} c_{a_i^\beta v_j^\lambda}$$

$$= \frac{1}{2^{2-j} \cdot 2^{|A^\beta|-2} \prod_{k=\lambda}^{\beta-2} 2^{|V^k|+1}} + \sum_{i=2}^{|A^\beta|} \frac{1}{2^{2-j} \cdot 2^{|A^\beta|-i} \prod_{k=\lambda}^{\beta-2} 2^{|V^k|+1}}$$

$$= \frac{1}{2^{2-j} \cdot 2^{|A^\beta|-2} \cdot 2^{\sum_{k=\lambda}^{\beta-2}(|V^k|+1)}} + \sum_{i=2}^{|A^\beta|} \frac{1}{2^{2-j} \cdot 2^{|A^\beta|-i} \cdot 2^{\sum_{k=\lambda}^{\beta-2}(|V^k|+1)}} \tag{C.31}$$

$$= \frac{1}{2^{|A^\beta|-j} \cdot 2^{\sum_{k=\lambda}^{\beta-2}(|V^k|+1)}} \left(1 + \frac{1}{2^2} \sum_{i=2}^{|A^\beta|} 2^i \right)$$

$$= \frac{1}{2^{|A^\beta|-j} \cdot 2^{\sum_{k=\lambda}^{\beta-2}(|V^k|+1)}} \left(1 + \frac{1}{2^2} \cdot \frac{2^2(1-2^{|A^\beta|-1})}{1-2} \right)$$

$$= \frac{1}{2^{|A^\beta|-j} \cdot 2^{\sum_{k=\lambda}^{\beta-2}(|V^k|+1)}} \left(1 + \frac{1}{2^2} \cdot \frac{2^2(1-2^{|A^\beta|-1})}{1-2}\right)$$

$$= \frac{2^{|A^\beta|-1}}{2^{|A^\beta|-j} \cdot 2^{\sum_{k=\lambda}^{\beta-2}(|V^k|+1)}} = \frac{1}{2^{1-j} \cdot 2^{\sum_{k=\lambda}^{\beta-2}(|V^k|+1)}} = 2^j \cdot C_{A^{\lambda+1}A^\beta} \tag{C.32}$$

□

Lemma C.0.4. *Let $c \in \mathbb{R}^{E_n}_{>0}$ be defined by (3.223)-(3.232). Then, for all $\beta \in [\xi - 1]$ and $i \in [|A^\beta|]$, $i \geq 2$ we have*

$$c_{a_i^\beta n} - C_{a_i^\beta A^\xi} + C_{a_i^\beta V^{\xi-1}} = 0 \tag{C.33}$$

where $C_{a_i^\beta A^\xi}$ and $C_{a_i^\beta V^{\xi-1}}$ are defined in (C.1) and (C.17), respectively.

Proof. Let $\beta \in [\xi - 1]$ and $i \in [|A^\beta|]$, $i \geq 2$. Applying (B.3) to (3.231) we deduce

$$c_{a_i^\beta n} = \frac{1}{2 \cdot 2^{|A^\beta|-i} \prod_{k=\beta-1}^{\xi-1} 2^{|V^k|+1}} = \frac{1}{2^{|A^\beta|-i+1} \cdot 2^{\sum_{k=\beta-1}^{\xi-1}(|V^k|+1)}} = \frac{C_{A^\beta n}}{2^{|A^\beta|-i+1}}. \tag{C.34}$$

From Lemma C.0.1 we have

$$C_{a_i^\beta A^\xi} = \frac{C_{A^\beta A^\xi}}{2^{|A^\beta|-i+1}}. \tag{C.35}$$

Thus, in view of (C.34), (C.35) and (C.24) we obtain

$$c_{a_i^\beta n} - C_{a_i^\beta A^\xi} + C_{a_i^\beta V^{\xi-1}} = \frac{C_{A^\beta n}}{2^{|A^\beta|-i+1}} - \frac{C_{A^\beta A^\xi}}{2^{|A^\beta|-i+1}} + \frac{C_{A^\beta A^\xi}}{2^{|A^\beta|-i+1}} - \frac{C_{A^\beta n}}{2^{|A^\beta|-i+1}} = 0. \tag{C.36}$$

□

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