MODERN ASPECTS OF CLASSICAL CONVEX GEOMETRY

Habilitation

zur Erlangung des akademischen Grades

doctor rerum naturalium habilitatus
(Dr. rer nat habil.)

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geb. am 20. Februar 1981 in Abarán (Spanien)

genehmigt durch die Fakultät für Mathematik

der Otto-von-Guericke-Universität Magdeburg

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eingereicht am: 31. August 2017
Verteidigung am: 05. April 2018

Señor Pereira deje ya de frecuentar el pasado, frecuente el futuro.
"Sostiene Pereira", A. Tabucchi.

To my children Pablo Basilio and Rodrigo

Il faut exiger de chacun ce que chacun peut donner, reprit le roi. L'autorité repose d'abord sur la raison.
"Le petit Prince", A. Saint-Éxupery

ACKNOWLEDGMENTS

I could not have accomplished this work without the support and encouragement of my family, my colleagues and friends, and all those who in one or another way have accompanied me through.

To all of you, thank you for understanding, for listening, for discussing, and for sharing.

I am grateful for the collaborations that contributed to this work, for many interesting discussions and for the valuable contributions to our joint works.

I thank especially Prof. Henk, who has guided, supported, encouraged and inspired me.

I am much obliged to the Institute of Algebra and Geometry and the Faculty of Mathematics of the Otto-von-Guericke University of Magdeburg.

AUTHORSHIP DECLARATION

The following synopsis describes the co-authorship of the contents of the present work.

Chapter 1 The contents of Chapter 1 are extracted from the paper [38], entitled *Steiner polynomials via ultra-logconcave sequences* and authored by **Henk, Martin; Hernández Cifre, María A.; Saorín Gómez, Eugenia**. This is published in Communications in Contemporary Mathematics, Vol. 14, (2012).

Chapter 2 The contents of Chapter 2 are extracted from the paper [65], entitled *Linearity of the volume. Looking for a characterization of sausages* and authored by **Saorín Gómez, Eugenia; Yepes Nicolás, Jesús**. This is published in Journal of Mathematical Analysis and Applications, Vol. 421, (2015).

Chapter 3 The contents of Chapter 3 are extracted from the papers [45], entitled *Differentiability of quermassintegrals: a classification of convex bodies* and authored by **Hernández Cifre, María A.; Saorín Gómez, Eugenia**, and [49], entitled *Decomposition of polytopes using inner parallel bodies* and authored by **Linke, Eva; Saorín Gómez, Eugenia**. These are published in Transactions of the American Mathematical Society, Vol. 366, (2014) and Monatshefte für Mathematik, Vol. 176, (2015), respectively.

Chapter 4 The contents of Chapter 4 are extracted from the papers [64], entitled *The role of the kernel in Bonnesen-style inradius inequalities* and authored by **Saorín Gómez, Eugenia**, and [57], entitled *On the kernel center of a convex body* and authored by **Moszyńska, Maria; Saorín Gómez, Eugenia**. These are published in Monatshefte für Mathematik, Vol. 171, (2013) and Beiträge zur Algebra und Geometrie. Contributions to Algebra and Geometry, Vol. 55, (2014), respectively.

Chapter 5 The contents of Chapter 5 are extracted from the paper [53], entitled p*-difference: a counterpart of Minkowski difference in the framework of the* L_p*-Brunn-Minkowski theory* and authored by **Martínez Fernández, Antonio. R.; Saorín Gómez, Eugenia; Yepes Nicolás, Jesús**. This paper is published in Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matematicas. RACSAM, Vol. 110, (2016).

Chapter 6 The contents of Chapter 6 are extracted from the paper [39], entitled *Differentiability properties of the family of* p*-parallel bodies* and authored by **Hernández Cifre, María A.; Martínez Fernández, Antonio. R.; Saorín Gómez, Eugenia**. This paper is published in Applicable Analysis and Discrete Mathematics, Vol. 10, (2016).

The contents of Chapter 5 and Chapter 6 are part of the Ph.D. Thesis of A. R. Martínez Fernández [52], defended in January 2016 at the University of Murcia, under the supervision of Prof. M. A. Hernández Cifre and E. Saorín Gómez.

CONTENTS

NOTATION AND BASICS

We start fixing notation and introducing basic definitions. Our reference book for the most of this part is [68].

Batirnos contra la estupidez, la maldad, la superstición, la envidia y la ignorancia [...] Que es como decir contra España, y contra todo.
"El capitán Alatriste", A. Pérez Reverte

BASIC CONVEXITY

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space endowed with the standard scalar product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. We denote by e_i, $1 \leqslant i \leqslant n$ the i-th canonical unit vector in $\mathbb{R}^n$. For a set $A \subseteq \mathbb{R}^n$ we denote by $\mathrm{lin}\, A$ and $\mathrm{aff}\, A$ the linear and affine hulls of the set A, i.e., the smallest linear subspace and the smallest affine subspace, respectively, which contains the set A. Affine subspaces of $\mathbb{R}^n$ will be also called *flats*. By $\mathrm{cl}A$, $\mathrm{bd}\, A$, and $\mathrm{int}\, A$ we denote, respectively, the closure, the boundary, and the interior of A. The set $\mathrm{relint}\, A$ is the relative interior of A, that is, the interior of A relative to its affine hull. For non-empty sets $A, B \subseteq \mathbb{R}^n$, we write $A + B$ to denote the usual vectorial sum of the sets and λA to denote the dilatation of the set A by $\lambda \in \mathbb{R}_{\geqslant 0}$, namely:

$$A + B = \{a + b \,:\, a \in A,\, b \in B\}, \quad \lambda A = \{\lambda a :\, a \in A\}.$$

We denote by B_n the Euclidean unit ball, and by $\mathbb{S}^{n-1}$ its boundary, the unit sphere of $\mathbb{R}^n$. For a Borel subset $A \subseteq \mathbb{R}^n$, we will write $\mathrm{vol}(A)$ to denote its volume, i.e., its n-dimensional Lebesgue measure. Let H_k be a k-dimensional affine subspace of $\mathbb{R}^n$. If $A \subseteq H_k$ we will use $\mathrm{vol}_k(A)$ to denote the k-dimensional volume (or k-dimensional Lebesgue measure) of A, computed in H_k. In particular, we write $\kappa_n = \mathrm{vol}(B_n)$.

The Grassmannian of k-dimensional linear spaces, $1 \leqslant k \leqslant n-1$, of $\mathbb{R}^n$ is denoted by $\mathcal{G}(n,k)$. For $0 \neq u \in \mathbb{R}^n$ (resp. $u \in \mathbb{S}^{n-1}$) and $\alpha \in \mathbb{R}$, we say that u is a normal vector (resp. unit normal vector) of the affine hyperplane $H = \{x \in \mathbb{R}^n \,:\, \langle x, u\rangle = \alpha\}$. For a hyperplane H we denote by H^- and H^+ the closed half-paces $H^- = \{x \in \mathbb{R}^n \,:\, \langle x, u\rangle \leqslant \alpha\}$ and $H^+ = \{x \in \mathbb{R}^n \,:\, \langle x, u\rangle \geqslant \alpha\}$ determined by H.

For $t \in \mathbb{R}^n$, $\lambda \in \mathbb{R}_{>0}$ and $A \subseteq \mathbb{R}^n$ we say that $A + t$ is a translate of A and λA is a dilatate of A. Further, $\lambda A + t$ is homothetic to A for any $t \in \mathbb{R}^n$ and $\lambda \geqslant 0$. For a subspace $H_k \in \mathcal{G}(n,k)$ we denote by $H_k^{\perp}$ the orthogonal complement of H_k. If S is an affine or linear plane of $\mathbb{R}^n$ and $A \subseteq \mathbb{R}^n$, we denote by $A|S$ the orthogonal projection of A onto the space S.

The non-empty set $A \subseteq \mathbb{R}^n$ is convex, if for any two points x, y in A the segment $\{\lambda x + (1-\lambda)y \,:\, \lambda \in [0,1]\}$, denoted by $[x,y]$, is entirely contained in A. The convex hull of A, i.e., the smallest convex set containing A, is denoted by $\mathrm{conv}\, A$.

A convex cone $C \subseteq \mathbb{R}^n$ is a non-empty convex set satisfying that for any $x \in C$ and $\lambda \geqslant 0$, $\lambda x \in C$. For $A \subseteq \mathbb{R}^n$, we denote by $\operatorname{pos} A$ the positive hull of A, that is, the smallest convex cone containing the set A.

For a set $A \subseteq \mathbb{R}^n$, the hyperplane $H \subseteq \mathbb{R}^n$ is a supporting (or support) hyperplane (or plane) of A at x, if $x \in A \cap H$ and A is contained in H^- or H^+. If H is a supporting hyperplane of A at $x \in A$, with normal vector u, and $A \subseteq H^-$, then H^- is called a supporting half-space of A at x. A face F of the convex set $A \subseteq \mathbb{R}^n$ is a convex subset of A such that, if the pair of points $x, y \in A$ is so that $\frac{x+y}{2} \in F$, then $x, y \in F$.

A non-empty convex and compact set will be called a convex body. Convex bodies will be usually denoted with the capital letters K, L, M, and E. The set of all convex bodies in $\mathbb{R}^n$ is denoted by $\mathcal{K}^n$. Further, we will denote by $\mathcal{K}^n_0$, $\mathcal{K}^n_{(0)}$ and $\mathcal{K}^n_n$, respectively, the sub-family of convex bodies which have the origin in them, which have the origin as an interior point, and which have interior points. The dimension of $K \in \mathcal{K}^n$, denoted by $\dim K$, is the dimension of the affine hull of K, i.e., $\dim K = \dim \operatorname{aff} K$. We say that a convex body $K \in \mathcal{K}^n$ is full-dimensional if $\dim K = n$. Otherwise, we say that K is lower-dimensional.

Any convex body K is the intersection of its supporting half-spaces (see e.g. [68, Corollary 1.3.5]). This fact can be described using the support function. The support function of a convex set K, denoted by $h(K, \cdot)$ is defined by

$$h(K, u) := \sup\{\langle x, u\rangle : x \in K\} \text{ for } u \in \mathbb{R}^n. \tag{N.1}$$

Then $x \in K$ if and only if $\langle x, u\rangle \leqslant h(K, u)$ for every $u \in \mathbb{R}^n$. The support function of a convex set K is sublinear and positively homogeneous. Indeed, these properties on a function characterize support functions, namely:

Theorem A ([68, Theorem 1.7.1])**.** *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a sublinear function. Then there is a unique $K \in \mathcal{K}^n$ whose support function is f.*

In particular, $h(K, \cdot)$ is convex. In Appendix B we collect several properties of convex functions, which will be needed in this work.

The support (or supporting) hyperplane H with normal vector u is, thus, $H(K, u) = \{x \in \mathbb{R}^n : \langle x, u\rangle = h(K, u)\}$. If $K \in \mathcal{K}^n$, then, for every $x \in \operatorname{bd} K$, there is a normal vector u and a supporting hyperplane $H(K, u)$, such that $x \in H(K, u) \cap K$. The normal cone of K at x is

$$N_K(x) = \{u \in \mathbb{R}^n \setminus \{0\} : x \in H(K, u)\} \cup \{0\}, \tag{N.2}$$

that is, the set of all outer normal vectors of K at x (with the zero vector). We say that u is an outer normal vector of K at x, if $u \in N_K(x)$. For $K \in \mathcal{K}^n$, $u \in \mathbb{S}^{n-1}$ and $H(K, u) = \{x \in \mathbb{R}^n : \langle x, u\rangle = h(K, u)\}$ the supporting hyperplane to K with outer normal vector u, we denote by $F(K, u) = K \cap H(K, u)$ the corresponding face of K cut off by $H(K, u)$.

A convex body K is said to be regular if every boundary point of K has a unique supporting hyperplane; a point $x \in \operatorname{bd} K$ with unique supporting hyperplane is also called regular. Regular points of a convex body K allow determining the convex body in the following sense:

Theorem B ([68, Theorem 2.2.6])**.** *A convex body with interior points is the intersection of the supporting half-spaces at its regular points.*

A convex body is said to be strictly convex if its boundary does not contain segments. A polytope is the convex hull of finitely many points. Polytopes are examples of convex bodies containing many segments on their boundaries. Every polytope is the intersection of finitely many closed half-spaces and any bounded intersection of finitely many closed half-spaces is a polytope (see e.g. [68, Theorems 2.4.3 and 2.4.6]).

Whenever we refer to a topological property in $\mathcal{K}^n$, we will be considering $\mathcal{K}^n$ as the metric space obtained when it is endowed with the Hausdorff distance δ_H:

$$\delta_H(K, L) = \min\{\lambda \geqslant 0 \, : \, K \subseteq L + \lambda B_n, \; L \subseteq K + \lambda B_n\}. \tag{N.3}$$

For completeness, we state the following two results concerning the metric space $(\mathcal{K}^n, \delta_H)$, which will be used in the work.

Theorem C ([68, Theorem 1.8.7])**.** *(Blaschke selection theorem) Every bounded sequence of convex bodies has a subsequence that converges to a convex body.*

A sequence $(K_i)_{i \in \mathbb{N}}$ of convex bodies is bounded if there exists $M \geqslant 0$, such that $\delta_H(K_i, K_j) \leqslant M$ for any $i, j \in \mathbb{N}$.

Theorem D ([68, Theorem 1.8.8])**.** *Let $(K_i)_{i \in \mathbb{N}}$ be a sequence of convex bodies. Then K_i converges to K if and only if each point in K is the limit of a sequence of points $(k_i)_{i \in \mathbb{N}}$, $k_i \in K_i$ for $i \in \mathbb{N}$ and any convergent sequence of points $(k_{i_j})_{j \in \mathbb{N}}$, with $k_{i_j} \in K_{i_j}$ for $j \in \mathbb{N}$, has its limit in K.*

minkowski's sum and difference

Let $K, L \in \mathcal{K}^n$. The vectorial addition $K + L$ is usually called Minkowski addition. For a convex body $M \in \mathcal{K}^n$, K is a (Minkowski) summand of M if there exists $L \in \mathcal{K}^n$ such that $K + L = M$. The Minkowski sum of two convex bodies is again a convex body and for every $u \in \mathbb{R}^n$, $h(K + L, u) = h(K, u) + h(L, u)$ and $h(\lambda K, u) = \lambda h(K, u)$ for all $\lambda \geqslant 0$.

The so-called Minkowski difference of convex bodies can be seen as the subtraction counterpart of the Minkowski addition. If $K, L \in \mathcal{K}^n$, then the Minkowski difference of K and L is defined as

$$K \sim L := \{x \in \mathbb{R}^n \, : \, x + L \subseteq K\}, \tag{N.4}$$

i.e., the largest set to which we can (Minkowski) sum L and the sum remains contained in K.

Rewriting the definition of Minkowski sum we have that

$$K + L = \bigcup_{k \in K} (k + L) = \bigcup_{l \in L} (l + K).$$

Analogously, for the Minkowski difference, we obtain

$$K \sim L = \bigcap_{l \in L} (K - l).$$

The following lemma states some rules connecting Minkowski addition and subtraction of sets in $\mathbb{R}^n$.

Lemma E ([68, Section 3.1])**.** *Let* $A, B, C \subseteq \mathbb{R}^n$ *be non-empty sets. Then*

(i) $(A + B) \sim B \supseteq A$. *If* $A, B \in \mathcal{K}^n$*, then there is equality.*

(ii) $(A \sim B) + B \subseteq A$. *If* $A, B \in \mathcal{K}^n$*, equality holds if and only if* B *is a summand of* A*.*

(iii) $(A \sim B) + C \subseteq (A + C) \sim B$.

(iv) $(A \sim B) \sim C = A \sim (B + C)$.

(v) $A + B \subseteq C$ *if and only if* $A \subseteq C \sim B$.

For convex bodies $K_1, \dots, K_m \in \mathcal{K}^n$ and reals $\lambda_1, \dots, \lambda_m \geqslant 0$, the convex body $\sum_{i=1}^m \lambda_i K_i$ is called the Minkowski (linear) combination of the convex bodies K_i, $1 \leqslant i \leqslant m$, with scalars λ_i, $1 \leqslant i \leqslant m$. The Minkowski combination of two convex bodies $K, E \in \mathcal{K}^n$ and a non-negative real number λ, $K + \lambda E$, is called the outer parallel body of K (relative to E) at distance λ.

The (relative) inradius $r(K; E)$ of K with respect to E is defined as

$$r(K; E) = \sup\{r : \exists\, x \in \mathbb{R}^n \text{ with } x + r\,E \subseteq K\}. \tag{N.5}$$

Then, for $0 \leqslant \lambda \leqslant r(K; E)$, the inner parallel body of K (relative to E) at distance λ is the Minkowski difference of K and λE:

$$K \sim \lambda E = \{x \in \mathbb{R}^n : \lambda E + x \subseteq K\}. \tag{N.6}$$

From now on we will write K_λ to denote the (relative) inner and outer parallel bodies of K, i.e., the so-called full system of (relative) parallel bodies of K:

$$K_\lambda := \begin{cases} K \sim |\lambda|\, E & \text{for } -r(K; E) \leqslant \lambda \leqslant 0, \\ K + \lambda E & \text{for } 0 \leqslant \lambda < \infty. \end{cases} \tag{N.7}$$

When $\lambda = 0$, it clearly coincides with K, whereas for $\lambda = -r(K; E)$, the convex body $K_{-r(K;E)}$ is called the kernel of K, relative to E, and it will be denoted by $\ker(K;E)$. It is known (see e.g. [8, p. 59]), that for $K, E \in \mathcal{K}^n$, the kernel is lower-dimensional, i.e.,

$$\dim \ker(K;E) \leqslant n-1. \tag{N.8}$$

Next lemma states two useful properties of the full system of relative parallel bodies.

Lemma F ([68, Lemma 3.1.13]). *Let $K, L, E \in \mathcal{K}^n$ be convex bodies, and let $\{K_\lambda\}_{-r(K;E)\leqslant\lambda<\infty}$ and $\{L_\lambda\}_{-r(L;E)\leqslant\lambda<\infty}$ denote the full system of parallel bodies of K and L, respectively, relative to E. Then, for $0 \leqslant \lambda \leqslant 1$, and μ, σ non-smaller than either $-r(K;E)$ or $-r(L;E)$, as applicable:*

(i)

$$K_\mu + L_\sigma \subseteq (K+L)_{\mu+\sigma}. \tag{N.9}$$

(ii)

$$(1-\lambda)K_\mu + \lambda K_\sigma \subseteq K_{(1-\lambda)\mu+\lambda\sigma}. \tag{N.10}$$

Notice that (ii) states that the full system of parallel bodies of $K \in \mathcal{K}^n$, relative to $E \in \mathcal{K}^n$, is concave with respect to inclusion and Minkowski addition.

mixed volumes and related concepts

The Steiner (or Minkowski-Steiner) formula (also known as Steiner polynomial) says that the volume of the outer parallel body $K + \lambda E$, $\lambda \geqslant 0$, is a polynomial of degree at most n in the parameter λ,

$$\mathrm{vol}(K+\lambda E) = \sum_{i=0}^{n} \binom{n}{i} W_i(K;E)\lambda^i. \tag{N.11}$$

The coefficients $W_i(K;E)$ are the *relative quermassintegrals* of K, which are magnitudes associated to the convex bodies K and E carrying important information about it.

The Steiner formula is a particular case of the more general fact, that the volume of a positive linear combination of convex bodies K_i with scalars $\lambda_i \geqslant 0$, $1 \leqslant i \leqslant m$, is a polynomial in λ_i, $1 \leqslant i \leqslant m$.

Theorem G ([68, Theorem 5.1.7]). *There is a nonnegative symmetric function $V : (\mathcal{K}^n)^n \longrightarrow \mathbb{R}$, the mixed volume, such that, for $m \in \mathbb{N}$,*

$$\mathrm{vol}\big(\lambda_1 K_1 + \cdots + \lambda_m K_m\big) = \sum_{i_1,\ldots,i_n=1}^{m} \lambda_{i_1} \ldots \lambda_{i_n} V(K_{i_1}, \ldots, K_{i_n}) \tag{N.12}$$

for convex bodies $K_1, \ldots, K_m \in \mathcal{K}^n$ and numbers $\lambda_1, \ldots, \lambda_m \geqslant 0$.

The coefficients $V(K_{i_1}, \ldots, K_{i_n})$ are the mixed volumes of $K_1, \ldots, K_m$.

For convex bodies $K_1, \dots, K_{n-1} \in \mathcal{K}^n$, the mixed area measure $S(K_1, \dots, K_{n-1}; \cdot)$ is the finite Borel measure on $\mathbb{S}^{n-1}$ such that for all $K \in \mathcal{K}^n$,

$$V(K, K_1, \dots, K_{n-1}) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h(K, u)\, dS(K_1, \dots, K_{n-1}; u). \quad \text{(N.13)}$$

We refer to [68, Chapter 5] for an extensive study of mixed volumes and mixed area measures. For the sake of brevity, we will use the following notation when some convex body appears more than once in the mixed volume or area measure:

$$\big(K_1[r_1], \dots, K_m[r_m]\big) = \big(K_1, \overset{(r_1)}{\dots}, K_1, \dots, K_m, \overset{(r_m)}{\dots}, K_m\big).$$

The mixed area measure $S_m(K; \cdot) = S(K[m], B_n[n-m-1]; \cdot)$ is known as *m-th area measure* of $K \in \mathcal{K}^n$.

The mixed volume and the mixed area measure do not change under arbitrary translation of any of their arguments. Further, for $K, L, K_2, \dots, K_n \in \mathcal{K}^n$,

$$\text{if } K \subseteq L, \text{ then } V(K, K_2, \cdots, K_n) \leqslant V(L, K_2, \cdots, K_n); \quad \text{(N.14)}$$

and by the symmetry, the mixed volume is monotonic in each of its arguments.

From (N.13), it follows that the mixed volume is *Minkowski linear* in each argument, i.e., positively homogeneous and (Minkowski) additive: for $K, L, K_2, \dots, K_n \in \mathcal{K}^n$ and $\lambda, \mu \geqslant 0$,

$$V(\lambda K + \mu L, K_2, \dots, K_n) = \lambda V(K, K_2, \dots, K_n) + \mu V(L, K_2, \dots, K_n). \quad \text{(N.15)}$$

By symmetry, V is Minkowski linear in each of its arguments. The same holds for the mixed area measure

$$\begin{aligned} &S(\lambda K + \mu L, K_2, \dots, K_{n-1}; \cdot) \\ &\qquad = \lambda S(K, K_2, \dots, K_{n-1}; \cdot) + \mu S(L, K_2, \dots, K_{n-1}; \cdot). \end{aligned} \quad \text{(N.16)}$$

The quermassintegrals are thus, special cases of the mixed volumes. In particular, we have $W_0(K; E) = vol(K)$, $W_n(K; E) = vol(E)$, and $W_i(K; E) = W_{n-i}(E; K)$. If $E = B_n$, $nW_1(K; B_n) = S(K)$ is the surface area of K. In analogy, we will denote by $S(K; E) := nW_1(K; E)$ the relative surface area of K, with respect to E.

As a consequence of this polynomial behaviour of mixed volumes with Minkowski combinations and the definition of quermassintegrals, it follows that every quermassintegral of $K + \lambda E$, for $\lambda \geqslant 0$ is a polynomial in λ, namely,

$$W_i(K + \lambda E; E) = \sum_{j=0}^{n-i} \binom{n-i}{j} W_{i+j}(K; E) \lambda^j. \quad \text{(N.17)}$$

Unlike what happens for the volume of outer parallel bodies, where the Steiner formula provides us with an explicit expression of it, the volume of the inner parallel bodies of K relative to E is, in general, not easy to write explicitly in terms of magnitudes associated to K and E.

If E is a summand of K, then

$$W_i(K \sim \lambda E; E) = \sum_{k=0}^{n-i} \binom{n-i}{k} W_{i+k}(K; E)(-\lambda)^k, \qquad \text{(N.18)}$$

for $0 \leqslant \lambda \leqslant 1$ and $i = 0, \dots, n-1$. Matheron [54] proved that the validity of (N.18) for $0 < \lambda < 1$ and $i = 0, \dots, n$ implies that E is a summand of K. He conjectured that it was enough to assume (N.18) just for $i = 0$ and proved the following conjecture for $n = 2$.

Conjecture H (Matheron [54]). *Let $K, E \in \mathcal{K}^n$ be convex bodies with interior points. Then*

$$\mathrm{vol}(K \sim \lambda E) = \sum_{i=0}^{n} \binom{n}{i} W_i(K; E)(-\lambda)^i \qquad \text{(N.19)}$$

for all $0 < \lambda < 1$ if and only if E is a summand of K.

The right-hand side in (N.19) is usually called the *alternating Steiner polynomial* of K with respect to E.

INEQUALITIES FOR MIXED VOLUMES

Inequalities for mixed volumes and, in particular, for the volume, constitute a fundamental part of the Brunn-Minkowski theory. The Brunn-Minkowski inequality, almost as ubiquitous as convexity (see [24] and [10]), can be seen as the *inequality counterpart* of mixed volumes when studying the behaviour of (usual) volume under Minkowski addition. It states that, for convex bodies $K, L \in \mathcal{K}^n$, the n-th root of the volume is a concave function on $\mathcal{K}^n$:

Theorem I ([68, Theorem 7.1.1]). *(Brunn-Minkowski) For convex bodies $K, L \in \mathcal{K}^n$, and $0 \leqslant \lambda \leqslant 1$,*

$$\mathrm{vol}\big(\lambda K + (1-\lambda)L\big)^{1/n} \geqslant \lambda \mathrm{vol}(K)^{1/n} + (1-\lambda)\mathrm{vol}(L)^{1/n}, \qquad \text{(N.20)}$$

with equality for some $\lambda \in (0,1)$ if and only if K and L either lie in parallel hyperplanes or are homothetic.

We will call $\lambda K + (1-\lambda)L$, for $K, L \in \mathcal{K}^n$ and $\lambda \in [0,1]$, a convex combination of the convex bodies K and L.

We notice that the validity of the Brunn-Minkowski inequality is known to be true for sets as general as just measurable. For a complete picture of this inequality, its extensions and further impact of it within Mathematics we refer to the survey article by Gardner [24].

Both, the polynomiality of the volume of a Minkowski combination of convex bodies and the concavity of the n-th root of the volume on

$\mathcal{K}^n$ lead to several inequalities (and other important facts) for mixed volumes. Next, we state some of these inequalities, which will be used throughout the work.

Theorem J ([68, Theorem 7.2.1])**.** *(Minkowski's inequalities) For convex bodies* $K, E \in \mathcal{K}^n$*,*

$$V(K[n-1], E)^n \geqslant \mathrm{vol}(K)^{n-1}\mathrm{vol}(E). \tag{N.21}$$

If K *and* E *have interior points, equality holds if and only if* K *is homothetic to* E*. Further,*

$$V(K[n-1], E)^2 \geqslant \mathrm{vol}(K)V(K[n-2], E[2]). \tag{N.22}$$

If $\dim K < n-1$ *there is equality.*

The complete classification of the equality case in the latter inequality needs the more precise notion of tangential body and will be described in Theorem A.3 of Appendix A.

The first inequality above, (N.21), yields the isoperimetric inequality (classical, if $E = B_n$), which states that the volume and the surface area of an n-dimensional convex body K satisfy

$$\left(\frac{S(K;E)}{S(E;E)}\right)^n \geqslant \left(\frac{\mathrm{vol}(K)}{\mathrm{vol}(E)}\right)^{n-1}. \tag{N.23}$$

If K and E have interior points, equality holds if and only if K is homothetic to E.

The so-called Bonnesen-Blaschke inequality, which strengthens the isoperimetric inequality in the plane, establishes that

$$W_1(K;E)^2 - \mathrm{vol}(K)\mathrm{vol}(E) \geqslant \frac{\mathrm{vol}(E)^2}{4}\big(R(K;E) - r(K;E)\big)^2, \tag{N.24}$$

where $R(K;E) = 1/r(E;K)$ is called the circumradius of K with respect to E. For $E = B_n$, $R(K;B_n) = R(K)$ and $r(K;B_n) = r(K)$ are the classical circumradius and inradius of K. This inequality was first proven by Bonnesen [7], when $E = B_2$, obtaining the classical

$$P(K)^2 - 4\pi\mathrm{vol}(K) \geqslant \pi^2\big(R(K) - r(K)\big)^2, \tag{N.25}$$

where $P(K)$ denotes the perimeter of K. Blaschke [4] generalized it to an arbitrary gauge body $E \in \mathcal{K}^2_2$, i.e., (N.24).

Inequality (N.24) is a consequence of the more general inequality

$$\mathrm{vol}(K) - 2W_1(K;E)x + \mathrm{vol}(E)x^2 \leqslant 0 \tag{N.26}$$

for $r(K;E) \leqslant x \leqslant R(K;E)$.

In the literature, the above inequality for $x = r(K;E)$ is sometimes called Bonnesen's inradius inequality:

$$W_0(K;E) - 2W_1(K;E)r(K;E) + W_2(K;E)r(K;E)^2 \leqslant 0. \tag{N.27}$$

Here equality holds if and only if $K = L + r(K;E)E$ for $\dim L \leqslant 1$ (see e.g. [4, pp.33-36], [7]). We refer to [68, Section 7.2, Note 4] for a detailed study of Bonnesen-type inequalities. Convex bodies which are the Minkowski sum of a (possibly degenerate) segment and a dilatation of E are sometimes called *sausage bodies*. We will say that the pair (of convex bodies) K, E is a sausage if K is the sum of (a dilation of) E and $L \in \mathcal{K}^n$, with $\dim L \leqslant 1$, or E is the sum of (a dilation of) K and $L \in \mathcal{K}^n$, with $\dim L \leqslant 1$.

The quadratic inequality (N.22) is a particular case of a very general system of quadratic inequalities for mixed volumes.

Theorem K ([68, Theorem 7.3.1]). *(Aleksandrov-Fenchel inequality) Let* $K_1, K_2, K_3 \dots, K_n \in \mathcal{K}^n$ *be convex bodies. Then*

$$V(K_1, K_2, K_3, \dots, K_n)^2 \geqslant V(K_1, K_1, K_3, \dots, K_n) V(K_2, K_2, K_3, \dots, K_n). \tag{N.28}$$

If K_1 and K_2 are homothetic, then equality holds. However, a complete classification of equality cases is still open.

An important consequence of the Aleksandrov-Fenchel inequality concerns a generalization of the Brunn-Minkowski theorem.

Theorem L ([68, Theorem 7.4.5]). *(General Brunn-Minkowski theorem) Let a number* $m \in \{1, \dots, n\}$ *and* $K, L, K_{m+1}, \dots, K_n \in \mathcal{K}^n$ *convex bodies be given; define* $K(\lambda) := (1-\lambda)K + \lambda L$ *and*

$$f(\lambda) := \mathrm{vol}\big(K(\lambda)[m], K_{m+1}, \dots, K_n\big)^{1/m} \tag{N.29}$$

for $0 \leqslant \lambda \leqslant 1$. *Then* f *is a concave function on* $[0,1]$.

We remark the following particular case of this, namely, the concavity of the appropriate power of the (relative) quermassintegrals of a convex body.

$$W_i\big((1-\lambda)K + \lambda L; E\big)^{\frac{1}{n-i}} \geqslant (1-\lambda) W_i(K;E)^{\frac{1}{n-i}} + \lambda W_i(L;E)^{\frac{1}{n-i}}, \tag{N.30}$$

$0 \leqslant i \leqslant n-1$, for convex bodies K, L, E and $\lambda \in [0,1]$.

The following inequalities for relative quermassintegrals are also a consequence of the Aleksandrov-Fenchel inequality.

$$W_i(K;E)^2 \geqslant W_{i-1}(K;E) W_{i+1}(K;E), \quad 1 \leqslant i \leqslant n-1. \tag{N.31}$$

A complete classification of the equality cases for this particular case of the Aleksandrov-Fenchel inequality is, to the best of the knowledge of the author, still unsolved. We refer to the book of Schneider [68, Chapter 7] for details, proofs, consequences, and improvements of the Aleksandrov-Fenchel inequality, as well as further inequalities for mixed volumes.

In the literature, there are several improvements and strengthenings of inequalities for mixed volumes. In [68, Section 7.7] several of these results are generalized and unified. There exist also linearization results of the Brunn-Minkowski inequality.

Theorem M ([8, 27, 58]). *Let* $K, E \in \mathcal{K}^n$ *be convex bodies such that there exists a hyperplane* H *with* $\mathrm{vol}_{n-1}(K|H) = \mathrm{vol}_{n-1}(E|H)$. *Then for all* $\lambda \in [0, 1]$,

$$\mathrm{vol}\big(\lambda K + (1-\lambda)E\big) \geqslant \lambda \mathrm{vol}(K) + (1-\lambda)\mathrm{vol}(E).$$

However, the above result does not provide the concavity of the function $\lambda \mapsto \mathrm{vol}\big(\lambda K + (1-\lambda)E\big)$. For further details, we refer to [68, Notes for Section 7.7] and the references therein.

The L_p Brunn-Minkowski theory, Wulff shapes and symmetrizations

As mentioned before, the polynomiality of the volume of a Minkowski combination of convex bodies is a fundamental pillar of the Brunn-Minkowski theory. Indeed, under standard assumptions (see [26], [55], [68, Notes for Section 5.1, Note 11] and the references therein), this polynomiality of the volume is characteristic of the Minkowski addition. We may change the Minkowski addition by other combination of convex bodies, obtaining variations of the Brunn-Minkowski theory. In the next, we will replace the Minkowski sum by the so-called p-*sum* of convex bodies, $p \geqslant 1$. Before defining it, we point out that the position of the convex bodies will be important in this context: the Minkowski sum of convex bodies behaves "well" under translations of the involved convex bodies, which is no longer true for the p-sum. Indeed, for $p > 1$, the p-sum of $K, L \in \mathcal{K}^n$ will be defined only if $0 \in K \cap L$.

Let $p \geqslant 1$ and let the convex bodies K, L contain the origin. The p-*addition* or p-*sum* of K and L (also L_p-*sum*), denoted as $K +_p L$, is the convex body whose support function is given by

$$h(K +_p L, u) = \big(h(K, u)^p + h(L, u)^p\big)^{1/p} \qquad \text{(N.32)}$$

for any $u \in \mathbb{R}^n$.

Using Minkowski's inequality ([33, p. 30], [28, Corollary 1.6]), it can be proven that the right-hand side of the above expression is sublinear and thus, (cf. Theorem A) the set $K +_p L$ is a convex body. A so-called p-scalar multiplication is defined along with the p-sum by

$$\lambda \cdot K = \lambda^{1/p} K \qquad \text{(N.33)}$$

for $\lambda \geqslant 0$, and thus, a p-combination of $K, L \in \mathcal{K}^n_0$ is given by

$$h\big(\lambda \cdot K +_p \mu \cdot L, u\big)^p = \lambda h(K, u)^p + \mu h(L, u)^p \qquad \text{(N.34)}$$

for non-negative reals λ, μ. If $p = \infty$ we set

$$h\big(K +_\infty L, u\big) := \max\big\{h(K, u), h(L, u)\big\} \qquad \text{(N.35)}$$

and so, $K +_\infty L = conv(K \cup L)$.

Firey [22] introduced and studied this p-combination of convex bodies which contain the origin. Several years later, Lutwak [50] started a systematic investigation of the p-sum of convex bodies containing the origin. Both, *Brunn-Minkowski-Firey theory* and L_p-*Brunn-Minkowski theory* can be found in the literature, as names for this extension of the Brunn-Minkowski theory, as well as the name L_p-addition for $+_p$.

We point out that the combination of $+_p$ and the volume does not share the polynomiality, mentioned in (N.11), which the Minkowski sum does (see [26], [55], [68, Notes for Section 5.1, Note 11]).

The first variation of the volume and quermassintegrals of an appropriate p-combination of convex bodies yields new functionals within the L_p-Brunn-Minkowski theory, which also satisfy inequalities in the spirit of those of mixed volumes.

Theorem N ([50], see also [68, Theorems 9.1.1, 9.1.2, and 9.1.3, and Corollary 9.1.5]). *Let* $K, L \in \mathcal{K}^n_{(0)}$ *and* $E \in \mathcal{K}^n_n$. *Let* $1 \leqslant p < \infty$ *and* $0 \leqslant i \leqslant n-1$. *Then*

$$\frac{n-i}{p} W_{p,i}(K, L; E) := \lim_{\varepsilon \to 0^+} \frac{W_i(K +_p \varepsilon \cdot L; E) - W_i(K; E)}{\varepsilon}$$
$$= \frac{n-i}{p}\frac{1}{n}\int_{\mathbb{S}^{n-1}} h(L,u)^p h(K,u)^{1-p}\, dS\big(K[n-i-1], E[i], u\big). \tag{N.36}$$

Moreover, the following inequalities hold:

$$vol(K +_p L)^{p/n} \geqslant vol(K)^{p/n} + vol(L)^{p/n}, \tag{N.37}$$

and

$$W_{p,i}(K, L; E)^{n-i} \geqslant W_i(K; E)^{n-i-p} W_i(L; E)^p. \tag{N.38}$$

Further,

$$W_i\,(K +_p L; B_n)^{\frac{p}{n-i}} \geqslant W_i\,(K; B_n)^{\frac{p}{n-i}} + W_i\,(L; B_n)^{\frac{p}{n-i}}, \tag{N.39}$$

with equality if and only if K *and* L *are dilatates.*

Next, we introduce the notion of Wulff shape or Aleksandrov body. We refer to [68, Section 7.5] for an extensive study of Wulff shapes in the context of the Brunn-Minkowski theory. For a closed subset of the sphere $\Omega \subseteq \mathbb{S}^{n-1}$ which does not lie in a hemisphere, and a non-negative continuous function $f : \mathbb{S}^{n-1} \longrightarrow \mathbb{R}$, the convex body

$$K = \bigcap_{u \in \Omega} \big\{x \in \mathbb{R}^n : \langle x, u\rangle \leqslant f(u)\big\}, \tag{N.40}$$

is called the Wulff shape associated with (Ω, f). We will denote by $[\Omega, f]$ the Wulff shape associated to (Ω, f). If $\Omega = \mathbb{S}^{n-1}$ we will omit it and we will write $[f]$.

We observe that the Minkowski difference $K \sim E$ of the convex bodies $K, E \in \mathcal{K}^n$, $E \subseteq K$, and in consequence, the relative inner parallel bodies, can be described as the Wulff shape $[f]$, associated to $(\mathbb{S}^{n-1}, f)$, for $f(u) = h(K,u) - h(E,u) \geqslant 0$, i.e.,

$$K \sim E = \bigcap_{u \in \mathbb{S}^{n-1}} \left\{ x \in \mathbb{R}^n : \langle x, u \rangle \leqslant h(K,u) - h(E,u) \right\}, \tag{N.41}$$

where a suitable translation might have been done to ensure the positivity of f .

Next theorem collects several aspects of Wulff shapes.

Theorem O ([68, Lemmata 7.5.1 and 7.5.2]). *Let* $K_i = [\Omega, f_i]$, $i \in \mathbb{N}$, *and* $K = [\Omega, f]$, *be the Wulff shapes associated with* (Ω, f_i), *and* (Ω, f), *respectively, where all the functions* f *and* f_i, $i \in \mathbb{N}$, *are positive . Then:*

(i) $S(K[n-1]; \mathbb{S}^{n-1} \setminus \Omega) = 0$.

(ii) $\mathrm{vol}(K) = \frac{1}{n} \int_\Omega f(u) dS(K[n-1]; u)$.

(iii) If $(f_i)_{i \in \mathbb{N}}$ *converges uniformly to* f, *then* $(K_i)_{i \in \mathbb{N}}$ *converges to* K.

Remark P ([68, p. 386]). *The smallest closed set* $\Omega \subset \mathbb{S}^{n-1}$ *that can be taken for a given convex body* K, *so that*

$$K = \bigcap_{u \in \Omega} H^-(K,u), \tag{N.42}$$

is the closure of the set of outer unit normal vectors at regular boundary points of K.

We will finish this short chapter on Notation and Basics recalling the Steiner and Schwarz symmetrization procedures and some of their properties (see e.g. [25, 28], [68, Section 6.3]). In [28] this process is referred to as *Schwarz rounding*.

The so-called Steiner symmetrization is a powerful symmetrization procedure which, among others, within the geometry of convex bodies, allows accomplishing the identification of extreme values of certain functionals.

Let $H \subseteq \mathbb{R}^n$ be a hyperplane. Let $K \in \mathcal{K}^n$. We will denote by $\sigma_H(K)$ the Steiner symmetral of K with respect to H. This is the set having the property that, for each line ℓ orthogonal to H and meeting K, the set $\sigma_H K \cap \ell$ is a (possibly degenerate) closed segment with midpoint in H and length equal to that of the set $\ell \cap K$. The mapping $K \mapsto \sigma_H(K)$ is the Steiner symmetrization with respect to H.

As a generalization of the latter, the so-called Schwarz symmetrization (see e.g. [25], [28], [68]) “uses” flats of arbitrary dimension. For $i = n-1$ the procedure is exactly the Steiner symmetrization. For $1 \leqslant i < n-1$, now $(n-i)$-dimensional balls, orthogonal to a given i-dimensional flat, are constructed. More precisely, for $H_i \in \mathcal{G}(n,i)$ and $K \in \mathcal{K}^n$, the Schwarz symmetral of K with respect to H_i is the set

$\sigma_{H_i}(K)$ such that for each $(n-i)$-dimensional plane G orthogonal to H_i and meeting K, the set $G \cap \sigma_{H_i}(K)$ is a (possibly degenerate) $(n-i)$-dimensional closed ball with center in H_i and $(n-i)$-dimensional volume equal to that of $G \cap K$.

In the following lemma, we collect some properties of it, which are used in the present work.

Lemma Q. *Let* $K, E \in \mathcal{K}^n$ *and* $H_i \in \mathcal{G}(n,i)$, $1 \leqslant i \leqslant n-1$. *Then:*

(i) $\sigma_{H_i}(K) \in \mathcal{K}^n$.

(ii) $\mathrm{vol}(K) = \mathrm{vol}(\sigma_{H_i}(K))$.

(iii) $\sigma_{H_i}\big(\lambda K + (1-\lambda)E\big) \supseteq \lambda\sigma_{H_i}(K) + (1-\lambda)\sigma_{H_i}(E)$.

(iv) $K|H_i = \sigma_{H_i}(K)|H_i = \sigma_{H_i}(K) \cap H_i$.

We refer the reader to [3] and the references therein for recent results on symmetrization in Geometry.

INTRODUCTION

Llegó con tres heridas:
la del amor,
la de la muerte,
la de la vida.

Miguel Hernández

The Brunn-Minkowski inequality and the Steiner formula for convex bodies are some of the most classical results framed on Convex Geometry and, probably, pillars of the so-called Brunn-Minkowski Theory. Both results happen to deal with the behaviour of the combination of the Minkowski sum and the volume. The origins of the Brunn-Minkowski Theory are usually traced to this joint of Minkowski addition and volume, where the powerful notion of mixed volume arises naturally.

Regarding Steiner formula, it is natural to ask whether, when considering more "algebraically" this formal polynomial(s), the roots carry some information of geometrical significance on the pair of convex bodies K, E. For given $K, E \in \mathcal{K}^n$ and regarding the polynomial expression providing the volume of $K + zE$, $z \geqslant 0$, as a formal polynomial in the now complex variable $z \in \mathbb{C}$, we are interested in the location of the roots and the structure of the sets of roots (if all convex bodies K and/or all gauge bodies E are considered) of these polynomials. A study of the roots of Steiner polynomials for convex bodies K, E, aiming to understand their structure and the information they may provide, has been carried out in [36] and [37], where great progress was made and some of the still main open goals in this direction were posed. Before that, the three-dimensional case had been already addressed in [40] and [41]. Regarding the right-hand side of (N.11) as a formal polynomial in a complex variable $z \in \mathbb{C}$, we will write it as follows,

$$f_{K;E}(z) := \sum_{i=0}^{n} \binom{n}{i} W_i(K;E) z^i.$$

We will focus on the location of the roots of $f_{K;E}(z)$, for which we introduce, for $n \geqslant 2$,

$$\mathcal{R}(n) = \left\{z \in \mathbb{C}^+ : f_{K;E}(z) = 0 \text{ for } K, E \in \mathcal{K}^n, \dim(K+E) = n\right\},$$

the set of all roots of all non-trivial Steiner polynomials in the upper half-plane $\mathbb{C}^+ = \{z \in \mathbb{C} : \mathrm{Im}(z) \geqslant 0\}$. By the isoperimetric inequality (N.23) for arbitrary gauge bodies E, it is easy to see that $\mathcal{R}(2) = \mathbb{R}_{\leqslant 0}$ is exactly the non-positive real axis and, in particular, it is a convex cone. For arbitrary dimensions, it was proven in [37] that $\mathcal{R}(n)$ is a convex cone containing $\mathbb{R}_{\leqslant 0}$.

In the first chapter of this work, the structure of the set of roots of Steiner polynomials, namely, $\mathcal{R}(n)$, is investigated and properties, such as closeness, or monotonicity depending on the dimension of the underlying space, are understood. We also establish some connections of the roots of Steiner polynomials with fundamental geometric inequalities, as some particular cases of the Aleksandrov-Fenchel inequality (cf. (N.31)), based on the *ultra logconcavity property* (see Chapter 1 for the precise definition) of the coefficients of Steiner polynomials. We provide an exact description of the cone $\mathcal{R}(4)$, similarly to the characterization of $\mathcal{R}(3)$ carried out in [37]. We call a pair of convex bodies $(K, E) \in \mathcal{K}^n \times \mathcal{K}^n$ a *boundary-pair* if the Steiner polynomial $f_{K;E}(z)$ has a root on the boundary part $\big(\mathrm{bd}\,\mathcal{R}(n)\big)\backslash\mathbb{R}_{\leqslant 0}$ of the cone of roots. Unlike the case $n = 3$, we are not aware of a completely geometric description of the boundary-pairs (K, E) in dimension 4. From the results in [37], it easily follows that one ray of the boundary of $\mathcal{R}(n)$ consists of the non-positive real axis $\mathbb{R}_{\leqslant 0}$, and that any odd-degree Steiner polynomial has necessarily a root on this (ray of the) boundary. The remaining part of the boundary of $\mathcal{R}(n)$, i.e., the "other ray" seems to have a more intricated geometric structure; some aspects of which will be treated in the first chapter. Since the cones $\mathcal{R}(2), \mathcal{R}(3), \mathcal{R}(4)$ happen to be closed, the question whether this is a general fact arises naturally, i.e., whether $\mathcal{R}(n)$ is closed for any other $n > 4$. We will prove that this is also the case in any dimension. Analogously, since $\mathcal{R}(2) \subsetneq \mathcal{R}(3) \subsetneq \mathcal{R}(4)$, we address the question whether the cones $\mathcal{R}(n)$ are nested, and prove that they are strictly nested for increasing dimension. The latter leads to the question whether, for sufficiently large n, the cones $\mathcal{R}(n)$ can cover $\mathbb{C}^+ \setminus \mathbb{R}_{\geqslant 0}$ to which we provide an affirmative answer.

If instead of looking at the precise expression for the volume of the combination of convex bodies $K + \lambda E$ for $\lambda \geqslant 0$, we rather focus on the volume of the convex combinations of two convex bodies, one of the most powerful inequalities in the framework of Convex Geometry (and beyond), the Brunn-Minkowski inequality (N.20) arises naturally:

$$\mathrm{vol}\big(\lambda K + (1-\lambda)L\big)^{1/n} \geqslant \lambda \mathrm{vol}(K)^{1/n} + (1-\lambda)\mathrm{vol}(L)^{1/n}.$$

There are several improvements and strengthenings of this inequality, in very different directions. We refer to [24] for an excellent survey on the Brunn-Minkowski inequality and [68, Chapters 7 and 9] for further references and other aspects of this inequality and related ones. The question whether this inequality, i.e., (N.20), providing us with the concavity of the n-th root of the volume, could also provide us with concavity of the volume itself under particular circumstances has already been addressed. In other words, we are asking whether in the Brunn-Minkowski inequality (N.20) the powers $1/n$ can be removed. Several results in this direction can be found in the literature.

Under special assumptions on the convex bodies K, E ([8, s. 50], [58], [27, ss. 1.2.4], [46]) the classical Brunn-Minkowski inequality can be refined obtaining that

$$\mathrm{vol}\big(\lambda K + (1-\lambda)E\big) \geqslant \lambda \mathrm{vol}(K) + (1-\lambda)\mathrm{vol}(E).$$

The main goal along Chapter 2 is to understand the (pairs of) convex bodies K, E for which there is equality in this inequality, i.e., for which the polynomial expression $\mathrm{vol}(\lambda K + (1-\lambda)E)$ has degree one. In this case, we would have

$$\mathrm{vol}\big(\lambda K + (1-\lambda)E\big) = \lambda \mathrm{vol}(K) + (1-\lambda)\mathrm{vol}(E),$$

which can be seen as a certain linearity behaviour of the volume. In general, the latter inequality will not provide us with the concavity of the function $\mathrm{vol}_{K;E}(\lambda) = \mathrm{vol}\big(\lambda K + (1-\lambda)E\big)$. For further details, we refer to [68, Notes for Section 7.7] and the references therein. For K, E, L convex bodies, $\dim L \leqslant 1$, and $K = L + E$ or $E = L + K$, a sausage, we know that the above equality holds true. We will investigate whether there are further pairs of convex bodies for which the equality also holds true, with the objective of characterizing sausages in this context. We prove that under the sole assumption that K and E have an equal volume projection (or a common maximal volume section), if the above equality holds for just one value in $(0, 1)$, then $K = L + E$, or $E = L + K$, with $\dim L \leqslant 1$, i.e., the pair K, E is a sausage. However, even having equality for all $\lambda \in [0, 1]$, if no extra assumption on K, E is done, such a characterization is not possible. This problem happens to be connected with a conjectured result relating the roots of the Steiner polynomial of a pair of convex bodies and the relative inradius of them. We provide counterexamples for a general version of this conjecture. In the same spirit, a counterexample to Conjecture H on inner parallel bodies is also explicitly given.

Intrinsically connected to the Minkowski sum, the Minkowski difference gives rise to the notion of inner parallel bodies. Matheron [54] (see also [68, p. 225]) pointed out that if $r(K; E)E \in \mathcal{K}^n$ is a summand of K, then the volume (and any quermassintegral relative to E) of $K \sim \lambda E$, for $0 \leqslant \lambda \leqslant r(K; E)$, can be expressed as a polynomial in λ:

$$W_i\big(K \sim \lambda E; E\big) = \sum_{k=0}^{n-i} \binom{n-i}{k} W_{i+k}(K; E)(-\lambda)^k,$$

for $i \in \{0, 1, \ldots, n-1\}$. However, in general, for the volume of the Minkowski difference $K \sim \lambda E$, $0 \leqslant \lambda \leqslant r(K; E)$, the *natural* analog of the Steiner formula (see [68, Section 4.2, Note 6] and the references therein) does not hold true. Of particular interest when considering a Minkowski difference and the possible explicit expression of its volume is the family of tangential bodies, about which Appendix A provides some further details. A convex body K is a tangential body of

$E \in \mathcal{K}^n_n$ if and only if through each boundary point of K there exists a support plane to K that also supports E. The very close existing connection between tangential bodies and the Minkowski difference is given by the fact that K is (homothetic to) a tangential body of E if and only if $K \sim \lambda E$ is homothetic to K for some $\lambda \in (0, r(K, E))$ (see Theorem 3.1.1). If K is a tangential body of E, the latter yields that $K \sim \lambda E$ is a summand of K. Furthermore, $r(K; E) = 1$ and $K_{-\lambda} = (1 - \lambda)K$ for all $\lambda \in (0, 1)$. The (positive) homogeneity of the quermassintegrals (or the above-mentioned fact noticed by Matheron) provides us with a polynomial expression in $\lambda \in \big(0, r(K; E)\big)$ for the quermassintegrals of the inner parallel bodies of tangential bodies, namely

$$W_i(K_{-\lambda}) = (1 - \lambda)^{n-i} W_i(K)$$

for $i \in \{0, 1, \ldots, n-1\}$. This special situation leads naturally to ask whether there exist other convex bodies which have (some of) their inner parallel bodies as summands of themselves and whether there are conditions on the convex bodies for this to hold true. Some answers to this question were given by Sangwine-Yager in [62], where several connections of the so-called form body of a convex body (see Chapter 3 for precise definitions) with inner parallel bodies are proven. The form body of a convex body K, with respect to $E \in \mathcal{K}^n$, is a particular example of a Wulff-shape (cf. (N.40)). The form body of K, with respect to E, may be geometrically understood to be constructed *based on* E keeping some essential boundary properties of K. Indeed, the form body of K, with respect to E, is a tangential body of E, which, by the mentioned connection between tangential bodies and Minkowski difference provides us already with a link between form bodies and inner parallel bodies. This connection of inner parallel bodies and form bodies appeared earlier in the literature, e.g., in the works of Bol [6] and Dinghas [15]. In Chapter 3 we first focus on the boundary structure aspects of the intertwining among inner parallel bodies, summands and the mentioned form bodies. This is used to characterize certain special decompositions of convex bodies via their inner parallel bodies and form bodies, as well as decompositions of (the special class of) polytopes via their inner parallel bodies, relative to the Euclidean unit ball. As a consequence of (some of) these results, a complete answer to an open question posed by G. Bol [6] in 1943 is provided.

Inequalities for convex bodies, as it happens with the classical isoperimetric inequality and the Bonnesen-Blaschke inequality *may profit* if we let the inradius of K, relative to E, enter into play.

A possible extension of the Bonnesen-Blaschke inequality to higher dimensions was conjectured by Wills [72] and proven simultaneously by Bokowski [5] and Diskant [20] for $E = B_n$, and later by Sangwine-Yager [63] for a general gauge body E with interior points:

$$\mathrm{vol}(K) - n r(K; E) W_1(K; E) + (n-1) r(K; E)^n \mathrm{vol}(E) \leqslant 0. \qquad \text{(I.1)}$$

In Chapter 4, we provide new inequalities for the volume of a convex body in terms of its quermassintegrals, using the technique of inner parallel bodies. We also prove that equality conditions rely on the decomposition of the convex body through its kernel. These results will strengthen the Wills conjecture inequality (I.1). Sangwine-Yager [63] proved a more general inequality than (I.1) bounding the volume of every inner parallel body of K in terms of $\mathrm{vol}(K)$, $W_1(K;E)$, $W_2(K;E)$, and some mixed volumes involving inner parallel bodies, from which (I.1) follows as a consequence. She also provided sufficient conditions for equality. Later, in [11], Brannen proved a strengthening of the Wills conjecture inequality (I.1) by introducing in the inequality the quermassintegrals of the form body of the involved convex body. This last result was improved in [43, Theorem 2.3], where also equality conditions were provided. In the proofs of these results a crucial use of results about inner parallel bodies is made.

As already mentioned in the Notation and Basics part, the extension or modification of the Brunn-Minkowski Theory based on the replacement of the Minkowski sum by the p-addition (or L_p-addition), where the position of the origin is of essential importance, is known as L_p-Brunn-Minkowski Theory. Considering Minkowski's difference as the subtraction counterpart of the Minkowski sum, the question arises whether an L_p-analog of the Minkowski difference can be defined. In Chapter 5, we introduce a notion of p-difference, which is an extension of the Minkowski difference to the setting of the L_p-Brunn-Minkowski theory.

For $1 \leqslant p < \infty$ and $K, E \in \mathcal{K}^n_0$ with $E \subseteq K$, the p-difference of K and E is given by

$$K \sim_p E = \bigcap_{u \in S^{n-1}} \left\{ x \in \mathbb{R}^n : \langle x, u \rangle \leqslant \left(h(K,u)^p - h(E,u)^p \right)^{1/p} \right\}.$$

For $p = 1$ it coincides with the Minkowski difference. Along Chapter 5, we prove several properties of this new operation. We introduce, for any $1 \leqslant p < \infty$, the notion of p-parallel bodies of a convex body K (relative to $E \subseteq K$),

$$K^p_\lambda = \begin{cases} K \sim_p |\lambda| E & \text{if } \; -r(K;E) \leqslant \lambda \leqslant 0, \\ K +_p \lambda E & \text{if } \qquad\quad 0 \leqslant \lambda < \infty. \end{cases}$$

For that, we will restrict to an appropriate position of the origin in the convex body, namely the origin lies in $K \sim r(K;E)E$.

We prove an analog of the concavity of the family of classical parallel bodies for the p-parallel ones, as well as the continuity of this new family, in its definition parameter.

Further results on (classical) inner parallel bodies are extended to p-inner ones. In particular, we prove that tangential bodies are characterized as the only convex bodies whose p-inner parallel bodies are homothetic copies of them, as it occurs for classical inner parallel bodies. Indeed, we prove that for any $1 < p < \infty$, the p-inner parallel bodies of tangential bodies are dilatations of the (classical) inner parallel bodies.

In the last chapter, we aim to understand the behaviour of some functionals defined on p-inner and outer parallel bodies, according to the parameter of definition of the whole system of p-parallel bodies, namely, relative quermassintegrals and the support function. Although this study is essentially the analog of the study of the differentiability of quermassintegrals with respect to the parameter of definition of the (classical) inner parallel bodies, already the case $\lambda > 0$, which is straightforward for the classical case ($p = 1$), turns out to be challenging if $p > 1$, where even the existence of right and left derivatives is not clear and should be first proven. We recall that for $p > 1$ there is no polynomial behaviour of the quermassintegrals of $K +_p \lambda E$ (see [26], [55], [68, Notes for Section 5.1, Note 11]).

We will also approach the differentiability of the quermassintegrals $W_i(\lambda) := W_i(K^p_\lambda; E)$, as functions of the parameter $\lambda \in \big(-r(K;E), \infty\big)$, proving that in $(0, \infty)$ they are always differentiable and providing an explicit expression for the derivative while, in general, we only have differentiability a.e. $\lambda \in \big(-r(K;E), 0\big)$. As a consequence of results by Lutwak [50] relating the volume and the p-sum of convex bodies, we establish that $\mathrm{vol}(K^p_\lambda)$ is differentiable on $\big(-r(K;E), \infty\big)$ with an explicit expression for its derivative.

Finally, we deal with the differentiability of the support function $h(\lambda, u) := h(K^p_\lambda, u)$ in terms of λ. We prove that a.e. $\lambda \in (-r(K;E), 0)$ and for all $u \in \mathbb{S}^{n-1}$

$$\frac{d}{d\lambda} h(\lambda, u) \geqslant \frac{|\lambda|^{p-1} h(E, u)^p}{h(\lambda, u)^{p-1}}.$$

Equality holds for all $u \in \mathbb{S}^{n-1}$ and a.e. $\lambda \in [-r(K;E), 0]$, if and only if $K = K^p_{-r(K;E)} +_p r(K;E)E$. The latter result allows us to finish the chapter establishing a connection, as in the classical case, of decomposition of convex bodies, now using the p-sum and p-inner parallel bodies, and differentiability of quermassintegrals, as in Chapter 3.

Part I

ON THE VOLUME OF MINKOWSKI COMBINATIONS

PROLOGUE TO PART I

In this first part of the work we will focus on two different aspects of the expression (N.11) providing the volume of the parallel set $K + \lambda E$, for $\lambda \geqslant 0$, which from now on, will be denoted by $f_{K;E}(\lambda)$:

$$\mathrm{vol}(K + \lambda E) = \sum_{i=0}^{n} \binom{n}{i} W_i(K; E)\lambda^i =: f_{K;E}(\lambda).$$

First, we will deal with it as a formal polynomial, being its roots our focus.

More specifically, we investigate structural properties of the cone of roots of relative Steiner polynomials of convex bodies. We prove that they are closed, monotonic in the dimension (with respect to inclusion), and that, when the dimension tends to infinity, they cover the whole upper half-plane, up to the positive real axis. We prove that when a pair of convex bodies K, E, whose relative Steiner polynomial $f_{K;E}(z)$ has a complex root on the boundary of the cone of roots $\mathcal{R}(n)$, i.e., if the pair (K, E) is a boundary-pair, then the convex bodies K, E have to satisfy some Aleksandrov-Fenchel inequality in (N.31) with equality.

In the second chapter we will devote to investigating the possibility of *linearizing* $f_{K;E}(\lambda)$, in the sense of looking for conditions and/or families of bodies for which

$$\mathrm{vol}\,(\lambda K + (1 - \lambda)E) = \lambda \mathrm{vol}\,(K) + (1 - \lambda)\,\mathrm{vol}\,(E)\,,$$

i.e., the special case in which the volume of the convex combination $\lambda K + (1 - \lambda)E$, is a polynomial of degree one, namely, linear.

In the two-dimensional case, if we let the inradius of a planar convex body enter into play, we obtain the answer by considering, essentially, (N.24) and its equality case. In order to consider the higher dimensional case, let $K, E, L \in \mathcal{K}^n$ be convex bodies, with $\dim L \leqslant 1$ and $K = L + E$ (a sausage). Then it is easy to verify the identity $\mathrm{vol}(\lambda K + (1 - \lambda)E) = \lambda\mathrm{vol}(K) + (1 - \lambda)\mathrm{vol}(E)$. We prove that under the assumption that K and E have an equal volume projection, if the latter equality holds for just one value in $(0, 1)$, then $K = L + E$ with $\dim L \leqslant 1$.

We further obtain, that, if no extra assumption is made on the convex bodies K, E, having $\mathrm{vol}(\lambda K + (1 - \lambda)E) = \lambda\mathrm{vol}(K) + (1 - \lambda)\mathrm{vol}(E)$ for all $\lambda \in [0, 1]$, does not yield that $K = L + E$ with $\dim L \leqslant 1$. This "linearity" of the volume is linked to a conjecture relating the roots of the Steiner polynomial of a pair of convex bodies and the relative inradius of one with respect to the other. We construct explicit counterexamples for the general case of this conjecture.

1 STEINER POLYNOMIAL(S)

Los cobardes tienen miedo de sí mismos.
"El lector de Julio Verne",
A. Grandes.

1.1 THE SET OF THE ROOTS OF STEINER POLYNOMIALS

In this chapter, we consider the polynomial expression appearing in (N.11) as a formal polynomial in a complex variable $z \in \mathbb{C}$:

$$f_{K;E}(z) := \sum_{i=0}^{n} \binom{n}{i} W_i(K;E) z^i. \tag{1.1}$$

By [68, Theorem 5.1.8], we know that $W_i(K;E) \geqslant 0$, with equality if and only if $\dim K < n - i$ or $\dim E < i$. Thus, $f_{K;E}(z) = \binom{n}{k} W_k(K;E) z^k$ if and only if $\dim K = n - k$, $\dim E = k$, and $\dim(K + E) = n$. Thus, we can write

$$f_{K;E}(z) = \sum_{i=n-\dim K}^{\dim E} \binom{n}{i} W_i(K;E) z^i.$$

Moreover, from (N.15), it follows that $W_i(\mu_1 K; \mu_2 E) = \mu_1^{n-i} \mu_2^i W_i(K;E)$ for $\mu_1, \mu_2 \geqslant 0$. Further, $f_{K;E}(z) = z^n f_{E;K}(1/z)$, since for quermassintegrals we have $W_i(K;E) = W_{n-i}(E;K)$, and thus, up to multiplication by real constants,

$$f_{K;E}(z) \text{ and } f_{E;K}(z) \text{ have the same non-trivial roots.} \tag{1.2}$$

We are interested in the location of the roots of $f_{K;E}(z)$. To this end, let $\mathbb{C}^+ = \{z \in \mathbb{C} : \operatorname{Im}(z) \geqslant 0\}$ be the set of complex numbers with non-negative imaginary part. For any dimension $n \geqslant 2$, let

$$\mathcal{R}(n) = \left\{ z \in \mathbb{C}^+ : f_{K;E}(z) = 0 \text{ for } K, E \in \mathcal{K}^n, \dim(K + E) = n \right\} \tag{1.3}$$

be the set of all roots of all non-trivial Steiner polynomials in the upper half-plane. We note that if $\dim(K + E) < n$ then all relative quermassintegrals vanish and so $f_{K;E}(z) \equiv 0$.

The isoperimetric inequality (N.23) yields that $\mathcal{R}(2) = \mathbb{R}_{\leqslant 0}$, i.e., it is exactly the non-positive real axis and, in particular, it is a convex cone. For arbitrary dimensions, this was verified in [37]. More precisely, the following result was proven.

Theorem 1.1.1 ([37, Theorem 1.1]). *$\mathcal{R}(n)$ is a convex cone containing $\mathbb{R}_{\leqslant 0}$.*

Considering the boundary of $\mathcal{R}(n)$, one ray consists of the non-positive real axis $\mathbb{R}_{\leqslant 0}$, and, of course, any odd-degree Steiner polynomial has a root on this boundary. The "other ray" of the boundary of $\mathcal{R}(n)$ seems to have more geometric structure. We recall that a pair of convex bodies $(K, E) \in \mathcal{K}^n \times \mathcal{K}^n$ is a boundary-pair if the Steiner polynomial $f_{K;E}(z)$ has a root on the boundary $\operatorname{bd}\mathcal{R}(n)\backslash\mathbb{R}_{\leqslant 0}$, and in view of (1.2) we may additionally assume $\dim K \leqslant \dim E$.

Regarding the 3-dimensional case, in [37] the following characterization was given.

Proposition 1.1.2 ([37, Theorem 1.2]).

$$\mathcal{R}(3) = \left\{x + yi \in \mathbb{C}^+ : x + \sqrt{3}\,y \leqslant 0\right\}.$$

A pair (K, E) *is a boundary-pair if and only if* $\dim K = 2$, $\dim E = 3$ *and* $W_2(K; E)^2 = W_1(K; E)W_3(K; E)$.

We notice that two convex bodies $K, E \in \mathcal{K}^3$ satisfy the above conditions if and only if $E \in \mathcal{K}^3_0$ is a cap-body of (a homothetic copy of) a planar convex body K (see [6]). A convex body $L \in \mathcal{K}^n$ is called a cap-body of $M \in \mathcal{K}^n$ if L is the convex hull of M and countably many points such, that the line segment joining any pair of these points intersects M. Cap-bodies are special cases of tangential bodies, more precisely, they are 1-tangential bodies (see Appendix A).

In this chapter, we will analyse several aspects of the roots of Steiner polynomials. First, we deal with topological properties of the set $\mathcal{R}(n)$, such as closure and boundary. We also deal with the monotonicity of $\mathcal{R}(n)$ depending on $n \in \mathbb{N}$ and its behaviour when n grows. For $n = 4$ we prove an analogue result to Proposition 1.1.2. Along the way to these results, we obtain a characterization of Steiner polynomials in terms of inequalities for their coefficients.

1.2 ULTRA-LOGCONCAVE SEQUENCES

A sequence of non-negative real numbers $a_0, \dots, a_n$ is said to be *ultra-logconcave* if

$$c_{i,n}\, a_i^2 \geqslant a_{i-1}\, a_{i+1} \quad \text{with} \quad c_{i,n} = \frac{\binom{n}{i-1}\binom{n}{i+1}}{\binom{n}{i}^2} = \frac{i}{i+1}\frac{n-i}{n-i+1}, \tag{1.4}$$

$1 \leqslant i \leqslant n-1$. For further information on ultra-logconcave sequences we refer to [30, 48] and the references inside. This property for real numbers allows us to characterize Steiner polynomials.

Lemma 1.2.1 ([38, Lemma 2.1]). *A real polynomial* $\sum_{i=0}^{n} a_i z^i$, $a_i \geqslant 0$, *is a Steiner polynomial* $f_{K;E}(z)$ *for a pair of convex bodies* $K, E \in \mathcal{K}^n$, *with* $\dim E = r$, $\dim K = s$, $\dim(K + E) = n$, *if and only if*

i) $a_i > 0$ *for all* $n - s \leqslant i \leqslant r$, *and* $a_i = 0$ *otherwise, and*

ii) the sequence $a_0, \dots, a_n$ *is ultra-logconcave, i.e.,*

$$c_{i,n}\, a_i^2 \geqslant a_{i-1}\, a_{i+1} \ \text{ for } 1 \leqslant i \leqslant n-1.$$

This follows essentially from a theorem of Shephard ([69, Theorem 4], see also [68, Section 7.4], and for the 2-dimensional case see [35]), which states that any given set of $n+1$ non-negative real numbers $W_0, \dots, W_n \geqslant 0$ satisfying the inequalities $W_i W_j \geqslant W_{i-1} W_{j+1}$, $1 \leqslant i \leqslant j \leqslant n-1$, arises as the set of relative quermassintegrals of two convex bodies. There, an explicit construction of the two convex bodies is given in the case when all $W_i > 0$, whereas the general case is obtained by a rather non-constructive topological argument. Here we reduce the number of involved inequalities and extend the construction of the two convex bodies to $W_i \geqslant 0$.

Proof of Lemma 1.2.1. If $\sum_{i=0}^n a_i z^i$ is the Steiner polynomial of two convex bodies $K, E \in \mathcal{K}^n$, then $a_i = \binom{n}{i} W_i(K; E)$. Now (i), (ii) are well-known properties of quermassintegrals. For (i) see [68, Theorem 5.1.8] and (ii) is (N.31).

Now we assume (i) and (ii). If $s = 0$ or $r = 0$ then both $a_n z^n$ and a_0 are obviously Steiner polynomials, and so we may assume $r, s \geqslant 1$. Setting $W_i = a_i / \binom{n}{i}$, the assumptions yield then that

$$\begin{aligned} &W_i > 0 \text{ for all } n-s \leqslant i \leqslant r \text{ and } W_i = 0 \text{ otherwise, and} \\ &W_i^2 \geqslant W_{i-1} W_{i+1}, \; 1 \leqslant i \leqslant n-1. \end{aligned} \tag{1.5}$$

The rest of the proof is devoted to constructing two convex bodies K, E, with $\dim K = s$, $\dim E = r$, $\dim(K+E) = n$ and $W_i = W_i(K; E)$, and so that $\sum_{i=0}^n a_i z^i = f_{K;E}(z)$. To this end, we extend the construction in [69] to handle lower-dimensional bodies as well and the sets K, E will be simplices, as in [69].

Let $q_i = \alpha_i e_i$, where $\alpha_i > 0$ for the values $i = 1, \dots, r$, $\alpha_i = 0$ for $i = r+1, \dots, n$, and $\alpha_i \geqslant \alpha_{i+1}$ if $i = n-s+1, \dots, r-1$. These numbers α_i's will be fixed at the end of the proof. Let K, E be the, respectively, s- and r-dimensional simplices

$$K = \operatorname{conv}\{0, e_{n-s+1}, \dots, e_n\}, \quad E = \operatorname{conv}\{0, q_1, \dots, q_r\}. \tag{1.6}$$

Then $K + E = \operatorname{conv}\{0, e_i, q_j, e_i + q_j : n-s+1 \leqslant i \leqslant n, 1 \leqslant j \leqslant r\}$, but since $\alpha_j \geqslant \alpha_{j+1}$ for $j = n-s+1, \dots, r-1$, if $i < j$, the points $e_i + q_j \in \operatorname{conv}\{0, e_i + q_i, e_j + q_j\}$, and thus

$$\begin{aligned} K + E = \operatorname{conv}\{0, e_i + q_j : j \leqslant i, n-s+1 \leqslant i \leqslant n, 1 \leqslant j \leqslant r, \\ e_i, q_j : r+1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n-s\}. \end{aligned} \tag{1.7}$$

Now, for $n-s+1 \leqslant m \leqslant r+1$, let

$$K_m = \operatorname{conv}\{0, e_m, \dots, e_n\}, \quad E_m = \operatorname{conv}\{q_1, \dots, q_m\},$$

with $K_{n+1} = \{0\}$, $E_{n+1} = \operatorname{conv}\{0, q_1, \dots, q_n\}$. Notice that $q_{r+1} = 0$. In the following, we will prove, by induction on the dimension, that $K + E$ is the interior-disjoint union of the sets $K_m + E_m$, i.e.,

$$K + E = \dot{\bigcup_{m=n-s+1}^{r+1}} (K_m + E_m), \tag{1.8}$$

where $\dot\cup$ denotes interior-disjoint union.

For $n = 1$ the assertion is trivial. So let $n \geqslant 2$, and let

$$\overline{K} = \mathrm{conv}\{0, e_{n-s+1}, \dots, e_{n-1}\},$$
$$\overline{E} = \begin{cases} \mathrm{conv}\{0, q_1, \dots, q_r\} & \text{if } r < n, \\ \mathrm{conv}\{0, q_1, \dots, q_{n-1}\} & \text{if } r = n, \end{cases}$$

with $\overline{K} = \{0\}$ if $s = 1$. In both cases, $\dim \overline{K} + \dim \overline{E} = s - 1 + r \geqslant n - 1$ holds. Similarly as before, we consider, for $n - s + 1 \leqslant m \leqslant r + 1$ (if $r < n$), or $n - s + 1 \leqslant m < n$ (if $r = n$),

$$\overline{K}_m = \mathrm{conv}\{0, e_m, \dots, e_{n-1}\}, \quad \overline{E}_m = \mathrm{conv}\{q_1, \dots, q_m\},$$

where $\overline{K}_n = \{0\}$ and $\overline{E}_n = \mathrm{conv}\{0, q_1, \dots, q_{n-1}\}$ (also for $m = r = n$). By induction hypothesis,

$$\overline{K} + \overline{E} = \begin{cases} \dot\bigcup_{m=n-s+1}^{r+1} (\overline{K}_m + \overline{E}_m) & \text{if } r < n \\ \dot\bigcup_{m=n-s+1}^{n} (\overline{K}_m + \overline{E}_m) & \text{if } r = n \end{cases}$$
$$=: \dot\bigcup_{m=n-s+1}^{r+1,n} (\overline{K}_m + \overline{E}_m),$$

and taking the orthogonal projection π_n onto the coordinate hyperplane $e_n = 0$ and the restriction $\pi := (\pi_n)|_{K+E}$, we get

$$K + E = \pi^{-1}(\overline{K} + \overline{E}) = \dot\bigcup_{m=n-s+1}^{r+1,n} \pi^{-1}(\overline{K}_m + \overline{E}_m).$$

Notice that $\pi^{-1}(\overline{K}_m + \overline{E}_m) = K_m + E_m$ for $m = n - s + 1, \dots, r + 1$ when $r < n$ and $m = n - s + 1, \dots, n - 1$ when $r = n$. So we get the required union for $K + E$ in $r + s - n + 1$ interior-disjoint parts (cf. (1.8)) when $r < n$. Finally, if $r = n$,

$$\pi^{-1}(\overline{K}_n + \overline{E}_n) = \mathrm{conv}\{0, q_j, q_j + e_n : 1 \leqslant j \leqslant n\}$$
$$= (K_{n+1} + E_{n+1}) \dot\cup (K_n + E_n),$$

providing the $s + 1$ interior-disjoint parts in (1.8) when $r = n$.

Based on relation (1.8), we can compute the volume of the polytope $K + E$. Since $(\mathrm{aff}\, K_m) \cap (\mathrm{aff}\, E_m) = \{q_m\}$, for all $n - s + 1 \leqslant m \leqslant r + 1$ $(m \neq n + 1)$, we get that

$$\mathrm{vol}(K_m + E_m) = \mathrm{vol}\left(K_m + \left(E_m | (\mathrm{lin}\ K_m)^{\perp}\right)\right)$$
$$= \mathrm{vol}_{n-m+1}(K_m)\mathrm{vol}_{m-1}\left(\mathrm{conv}\{0, q_1, \dots, q_{m-1}\}\right)$$
$$= \frac{1}{(n-m+1)!} \frac{\alpha_1 \dots \alpha_{m-1}}{(m-1)!} = \frac{1}{n!} \binom{n}{m-1} \alpha_1 \dots \alpha_{m-1}.$$

Observe that $\mathrm{vol}(K_{n+1}+E_{n+1}) = \mathrm{vol}(E_{n+1}) = (1/n!)\,\alpha_1\ldots\alpha_n$, if $r = n$. Thus, by (1.8),

$$\mathrm{vol}(K+E) = \sum_{m=n-s+1}^{r+1} \mathrm{vol}(K_m+E_m) = \sum_{i=n-s}^{r} \binom{n}{i}\frac{1}{n!}\,\alpha_1\ldots\alpha_i,$$

where, if $s = n$, the first summand ($i = 0$) is just $1/n!$. This says that $W_i(K;E) = (1/n!)\,\alpha_1\ldots\alpha_i$ for $n-s \leqslant i \leqslant r$, and $W_i(K;E) = 0$ otherwise.

Let $W_0,\ldots,W_n \geqslant 0$ be our given sequence of real numbers satisfying (1.5). Let

$$\alpha_i = \begin{cases} (n!W_{n-s})^{1/(n-s)} & \text{for } i = 1,\ldots,n-s,\\ W_i/W_{i-1} & \text{for } i = n-s+1,\ldots,r,\\ 0 & \text{for } i = r+1,\ldots,n. \end{cases}$$

Since $W_i^2 \geqslant W_{i-1}W_{i+1}$ we have $\alpha_i \geqslant \alpha_{i+1}$ for $n-s+1 \leqslant i \leqslant r$, and, taking K, E as defined in (1.6), we get, for all $i = n-s,\ldots,r$,

$$W_i(K;E) = \frac{1}{n!}\,\alpha_1\ldots\alpha_{n-s}\alpha_{n-s+1}\ldots\alpha_i = \frac{1}{n!}(n!W_{n-s})\frac{W_i}{W_{n-s}} = W_i,$$

and $W_i = 0$ otherwise. □

The following remark contains essentially the proof of the "if" part of the previous result. We state it again for future references.

Remark 1.2.2. *Let $\sum_{i=0}^{n} a_i z^i$ be the Steiner polynomial of the two convex bodies $K, E \in \mathcal{K}^n$. Then the sequence $\{a_i = \binom{n}{i} W_i(K;E)\}$ is ultra-logconcave by means of* (N.31)*.*

For complex numbers $z_1,\ldots,z_r \in \mathbb{C}$ let

$$\sigma_i\left(z_1,\ldots,z_r\right) = \sum_{\substack{J\subseteq\{1,\ldots,r\}\\ \#J=i}} \prod_{j\in J} z_j$$

denote the i-th elementary symmetric function of $z_1,\ldots,z_r$, $1 \leqslant i \leqslant r$. In addition, we set $\sigma_0\left(z_1,\ldots,z_r\right) = 1$. Using this notation the following corollary is an immediate consequence of Lemma 1.2.1.

Corollary 1.2.3 ([38, Corollary 2.1])**.** *The complex numbers $\gamma_1,\ldots,\gamma_r \in \mathbb{C}$ are the roots of a Steiner polynomial $f_{K;E}(z)$ of degree $r \leqslant n$, for convex bodies $K, E \in \mathcal{K}^n$, $\dim E = r$, $\dim K = s$, $\dim(K+E) = n$, if and only if*

$$\begin{aligned} &\text{i)}\quad (-1)^i\sigma_i\left(\gamma_1,\ldots,\gamma_r\right) > 0, \quad 0 \leqslant i \leqslant r+s-n,\\ &\qquad\quad \sigma_i\left(\gamma_1,\ldots,\gamma_r\right) = 0, \quad r+s-n+1 \leqslant i \leqslant r,\\ &\text{ii)}\quad c_{r-i,n}\,\sigma_i\left(\gamma_1,\ldots,\gamma_r\right)^2 \geqslant \sigma_{i-1}\left(\gamma_1,\ldots,\gamma_r\right)\sigma_{i+1}\left(\gamma_1,\ldots,\gamma_r\right),\\ &\qquad 1 \leqslant i \leqslant r-1. \end{aligned} \tag{1.9}$$

In the next, we provide three direct applications of Lemma 1.2.1. First, for $0 \leqslant j < k \leqslant n$, we define

$$P^n_{j,k}(z) := \sum_{i=j}^{k} \binom{n}{i} z^i,$$

the truncation of the binomial polynomial $(z+1)^n$ with indices $j < k$.

Proposition 1.2.4 ([38, Proposition 2.1]). *All truncated binomial polynomials* $P^n_{j,k}(z) = \sum_{i=j}^{k} \binom{n}{i} z^i$, $0 \leqslant j < k \leqslant n$, *are Steiner polynomials of convex bodies* $K, E \in \mathcal{K}^n$ *with* $\dim K = n - j$, $\dim E = k$ *and* $\dim(K + E) = n$.

Hence, in the following, we consider $P^n_{j,k}(z)$ as Steiner polynomials. In fact, by the proof of Lemma 1.2.1, $P^n_{j,k}(z)$ can be realized as the Steiner polynomial $f_{K;E}(z)$ of the bodies $K = \mathrm{conv}\{0, e_{j+1}, \dots, e_n\}$ and $E = \mathrm{conv}\{0, c\,e_1, \dots, c\,e_j, e_{j+1}, \dots, e_k\}$ with $c = (n!)^{1/j}$.

The second application deals with the derivative and antiderivative of Steiner polynomials.

Proposition 1.2.5 ([38, Proposition 2.2]). *Let* $f_{K;E}(z) = \sum_{i=0}^{n} a_i z^i$ *be the Steiner polynomial of two convex bodies* $K, E \in \mathcal{K}^n$, $\dim(K + E) = n$. *Then both, its derivative as well as its antiderivative,*

$$f'_{K;E}(z) = \sum_{i=0}^{n-1} (i+1)\, a_{i+1}\, z^i \quad \textit{and} \quad \int f_{K;E}(z)\, dz = \sum_{i=1}^{n+1} \frac{a_{i-1}}{i}\, z^i,$$

are Steiner polynomials of appropriate convex bodies in $\mathcal{K}^{n-1}$ *and* $\mathcal{K}^{n+1}$, *respectively.*

If $\dim K = n$, we may also add any constant term c to the antiderivative as long as $c \leqslant n a_0^2 / \big((n+1) a_1\big)$.

The last consequence regards Steiner polynomials with only real roots.

Proposition 1.2.6 ([38, Proposition 2.3]). *For any given* n *real numbers* $\gamma_i \leqslant 0$, $i = 1, \dots, n$, *there exist* $K, E \in \mathcal{K}^n$ *such that* $f_{K;E}(\gamma_i) = 0$ *for all* $i = 1, \dots, n$.

This is, for instance, due to the fact that the elementary symmetric functions form an ultra-logconcave sequence (Newton inequalities, see e.g. [33]),

$$\left(\frac{\sigma_i\,(\gamma_1, \dots, \gamma_n)}{\binom{n}{i}} \right)^2 \geqslant \frac{\sigma_{i-1}\,(\gamma_1, \dots, \gamma_n)}{\binom{n}{i-1}} \, \frac{\sigma_{i+1}\,(\gamma_1, \dots, \gamma_n)}{\binom{n}{i+1}},$$

and so Lemma 1.2.1 gives the result. In the case $n = 2$ this means that given any pair $\gamma, \gamma' \in \mathcal{R}(2)$, we can find a Steiner polynomial having these two roots. This property is, however, not true in higher dimension if we also allow complex (non-real) numbers to be involved. Indeed, in [37, pp. 160-161] it is shown that if $-a + bi \in \mathcal{R}(3)$, then $-a + bi, -a - bi, -c$ are the roots of a Steiner polynomial if and only if either $c \leqslant a - \sqrt{3}\, b$ or $c \geqslant (a^2 + b^2)/(a - \sqrt{3}\, b)$.

The 4-dimensional cone

Next, we describe completely the cone of roots of Steiner polynomials of convex bodies in the 4-dimensional Euclidean space. For, it suffices to determine its boundary, according to Theorem 1.1.1.

Proposition 1.2.7 ([38, Proposition 1.2]).

$$\mathcal{R}(4) = \left\{x + yi \in \mathbb{C}^+ : x + y \leqslant 0\right\}.$$

Moreover, a pair of convex bodies (K, E) *is a boundary-pair if and only if* $\dim K = 3$, $\dim E = 4$, *and,* $W_i(K; E)^2 = W_{i-1}(K; E)W_{i+1}(K; E)$, *for* $i = 2, 3$.

Proof. First, we notice that $\{x + yi \in \mathbb{C}^+ : x + y \leqslant 0\} \subseteq \mathcal{R}(4)$. Indeed, since $\mathcal{R}(4)$ is a convex cone containing $\mathbb{R}_{\leqslant 0}$ (Theorem 1.1.1), it suffices to prove that $-1 + i \in \mathcal{R}(4)$, which follows from $P^4_{1,4}(-1 + i) = 0$ and Proposition 1.2.4.

Next, we determine conditions satisfied by a pair of convex bodies whose Steiner polynomial has $-1 + i$ as a root. We have to distinguish two cases. If $E \in \mathcal{K}^4_4$, then such a polynomial has to take the form

$$f_{K;E}(z) = \sum_{i=0}^{4} \binom{4}{i} W_i(K; E)z^i = W_4(K; E)(z^2 + 2z + 2)(z^2 + cz + d),$$

for certain $c, d \geqslant 0$, because it is weakly stable ([37, Proposition 1.1], cf. Proposition 1.3.4). Then we have the identities

$$\begin{aligned} 2 + c &= 4\frac{W_3(K; E)}{W_4(K; E)}, \quad 2(c + 1) + d = 6\frac{W_2(K; E)}{W_4(K; E)}, \\ 2(c + d) &= 4\frac{W_1(K; E)}{W_4(K; E)}, \quad 2d = \frac{W_0(K; E)}{W_4(K; E)}. \end{aligned} \tag{1.10}$$

Inequalities (N.31) for $i = 3$, $i = 2$ and $i = 1$ yield, in terms of c, d, respectively,

$$\begin{aligned} 3c^2 - 4c - 8d - 4 &\geqslant 0, \\ c^2 + (d + 2)c - 2(d^2 - 5d + 4) &\leqslant 0, \\ 3c^2 - 2dc - d^2 - 8d &\geqslant 0, \end{aligned}$$

which, since $c, d \geqslant 0$, are equivalent to

$$\begin{aligned} &c \geqslant \frac{2}{3}\left(1 + \sqrt{2}\sqrt{2 + 3d}\right), \\ &c \leqslant d - 4 \ \text{ if } \ d \geqslant 2 \quad \text{ and } \quad c \leqslant 2(1 - d) \ \text{ if } \ d \leqslant 2, \\ &c \geqslant \frac{1}{3}\left(d + 2\sqrt{d(d + 6)}\right), \end{aligned}$$

respectively. A straightforward computation allows us to conclude that the three above inequalities hold simultaneously if and only if $d = 0$ and $c = 2$. Then

$$\begin{aligned} f_{K;E}(z) &= W_4(K; E)(z^4 + 4z^3 + 6z^2 + 4z) \\ &= W_4(K; E)P^4_{1,4}(z). \end{aligned}$$

In particular, $W_0(K;E) = 0$ (cf. (1.10)) and, taking $W_1(K;E) > 0$ into consideration, this shows that $\dim K = 3$ and moreover, we obtain that $W_1(K;E) = W_2(K;E) = W_3(K;E) = W_4(K;E)$.

Now we suppose $\dim E < 4$. Then the polynomial has to take the form

$$f_{K;E}(z) = (z^2 + 2z + 2)(cz + d) = cz^3 + (d + 2c)z^2 + 2(c + d)z + 2d,$$

for certain $c, d \geqslant 0$. Applying Lemma 1.2.1 it is easy to check that it is a Steiner polynomial if and only if $c = d$. Notice that $c = d = 0$ is not possible. Hence $f_{K;E}(z) = cz^3 + 3cz^2 + 4cz + 2c$, implying that

$$\frac{1}{2}W_0(K;E) = W_1(K;E) = 2W_2(K;E) = 4W_3(K;E) = c \neq 0$$

and, in particular, that $\dim K = 4$. In both cases we get the required equalities $W_i(K;E)^2 = W_{i-1}(K;E)W_{i+1}(K;E)$, for $i = 2, 3$.

Finally we prove that $\mathcal{R}(4) = \{x + yi \in \mathbb{C}^+ : x + y \leqslant 0\}$. Let $\varepsilon > 0$, and $\gamma = -1 + (1 + \varepsilon)i \in \mathcal{R}(4)$ be such that there exist $K, E \in \mathcal{K}^n$ with $f_{K;E}(\gamma) = 0$. Then (see [37, Lemma 2.1]) $\gamma - \varepsilon$ is a root of $f_{K+\varepsilon E;E}(z)$. But since $\gamma - \varepsilon = -(1 + \varepsilon) + (1 + \varepsilon)i$, the previous property implies that either $\dim(K + \varepsilon E) = 3$ with $E \in \mathcal{K}^4_4$, which is clearly not possible, or $\dim E = 3$ and $\mathrm{vol}(K + \varepsilon E) = W_i(K + \varepsilon E; 2E) \neq 0$, $i = 1, 2, 3$, which also leads to a contradiction. Indeed, if

$$W_0(K + \varepsilon E; 2E) = W_1(K + \varepsilon E; 2E) = W_2(K + \varepsilon E; 2E) = W_3(K + \varepsilon E; 2E),$$

we find using (N.17) that

$$W_2(K;E) = 2(1 - \varepsilon)W_3(K;E) \quad \text{and} \quad W_1(K;E) = (4 + 3\varepsilon^2 - 6\varepsilon)W_3(K;E).$$

Notice that this implies $\varepsilon < 1$. However, substitution of the above expressions in inequality (N.31) for $i = 2$ leads to $\varepsilon \geqslant 2$, a contradiction. □

1.3 CLOSURE AND BOUNDARY OF $\mathcal{R}(n)$

Next, we prove that all cones $\mathcal{R}(n)$ are closed.

Theorem 1.3.1 ([38, Theorem 1.2]). *The cone* $\mathcal{R}(n)$ *is closed.*

Proof. Let $\gamma \in \mathrm{bd}\,\mathcal{R}(n)$. Since we already know that the non-positive real axis is always contained in $\mathcal{R}(n)$, we assume that $\gamma \notin \mathbb{R}$. Let $(\gamma_j)_{j\in\mathbb{N}} \subsetneq \mathrm{int}\,\mathcal{R}(n)$ be a sequence of complex numbers converging to γ. For each $j \in \mathbb{N}$, since $\gamma_j \in \mathrm{int}\,\mathcal{R}(n)$, there exists a pair of convex bodies $(K_j, E_j) \in \mathcal{K}^n \times \mathcal{K}^n$, $\dim(K_j + E_j) = n$, such that $f_{K_j;E_j}(\gamma_j) = 0$.

Notice that we can always choose the convex bodies K_j, E_j such that $\mathrm{vol}(K_j + E_j) = 1$. Otherwise, since $\mathrm{vol}(K_j + E_j) > 0$, it suffices to consider the new convex bodies $K_j' = 1/\mathrm{vol}(K_j + E_j)^{1/n} K_j$ and

$E_j' = 1/\mathrm{vol}(K_j+E_j)^{1/n}E_j$, for which the following equality clearly holds $f_{K_j';E_j'}(\gamma_j) = \big(1/\mathrm{vol}(K_j+E_j)\big)f_{K_j;E_j}(\gamma_j) = 0$. Moreover,

$$\mathrm{vol}(K_j'+E_j') = f_{K_j';E_j'}(1) = \frac{1}{\mathrm{vol}(K_j+E_j)}f_{K_j;E_j}(1) = 1.$$

Observe that, since $\mathrm{vol}(K_j+E_j) = \sum_{i=0}^{n}\binom{n}{i}W_i(K_j;E_j) = 1$, all quermassintegrals $W_i(K_j;E_j) \in [0,1]$, $i=0,\dots,n$, and not all of them are zero. Then, denoting by $W_{i,j} = W_i(K_j;E_j)$, we can assure that the bounded sequence of $(n+1)$-tuples of numbers $(W_{0,j},\dots,W_{n,j})_{j\in\mathbb{N}}$ has a convergent subsequence to an $(n+1)$-tuple $(W_0,\dots,W_n)$, and without loss of generality we assume that $(W_{0,j},\dots,W_{n,j})_{j\in\mathbb{N}}$ is the convergent subsequence.

By continuity, (the numbers) $W_0,\dots,W_n$ also satisfy inequalities (N.31), and thus, the sequence $\left\{a_i = \binom{n}{i}W_i : i=0,\dots,n\right\}$ is ultra-logconcave (see Remark 1.2.2). Moreover,

$$\sum_{i=0}^{n}\binom{n}{i}W_i = \lim_{j\to\infty}\sum_{i=0}^{n}\binom{n}{i}W_{i,j} = \lim_{j\to\infty}\mathrm{vol}(K_j+E_j) = 1,$$

i.e., the polynomial $\sum_{i=0}^{n}\binom{n}{i}W_iz^i = \sum_{i=0}^{n}a_iz^i \neq 0$. Therefore, property (i) in Lemma 1.2.1, i.e., $a_i > 0$ for all $n-s \leqslant i \leqslant r$ and $a_i = 0$ otherwise, holds for suitable $r,s \in \{1,\dots,n\}$. And thus, Lemma 1.2.1 ensures that $\sum_{i=0}^{n}\binom{n}{i}W_iz^i$ is a Steiner polynomial of two convex bodies $K,E \in \mathcal{K}^n$ satisfying $\dim K = s$, $\dim E = r$. By continuity, since $f_{K_j;E_j}(\gamma_j) = 0$, for all $j \in \mathbb{N}$, and the sequence of complex numbers $(\gamma_j)_{j\in\mathbb{N}}$ converges to γ, we have $f_{K;E}(\gamma) = 0$, i.e., $\gamma \in \mathcal{R}(n)$. This proves that the cone $\mathcal{R}(n)$ is closed. □

Since $\mathcal{R}(n)$ is closed, it is natural to ask about its boundary, and what is more important to us, which pairs of convex bodies or Steiner polynomials determine the boundary $\mathrm{bd}\,\mathcal{R}(n)\backslash\mathbb{R}_{\leqslant 0}$. To this end, we recall that (cf. Proposition 1.1.2) if $E \in \mathcal{K}_0^3$ is a cap-body of a planar convex body K, then (K,E) is a boundary-pair. We also notice that if $E \in \mathcal{K}_0^4$ is a cap-body of K with $\dim K = 3$, then the condition for the boundary in Proposition 1.2.7 is satisfied, i.e., (K,E) is also a boundary-pair in dimension 4. However this is not true in general, as it is the case for $n \geqslant 5$. Indeed, if $K \in \mathcal{K}^n$ with $\dim K = n-1$ and $E \in \mathcal{K}_n^n$ is a cap-body of K, then using Remark A.1, we can write the equalities $\mathrm{vol}(E) = W_0(E;K) = \cdots = W_{n-1}(E;K) \neq 0$. Since $W_0(K;E) = 0$ we get

$$\begin{aligned} f_{K;E}(z) &= \sum_{i=1}^{n}\binom{n}{i}W_i(K;E)z^i = \sum_{i=1}^{n}\binom{n}{i}W_{n-i}(E;K)z^i \\ &= \mathrm{vol}(E)\sum_{i=1}^{n}\binom{n}{i}z^i = \mathrm{vol}(E)P_{1,n}^n(z). \end{aligned}$$

All roots of the Steiner polynomial $P_{1,5}^5(z)$ lie in the interior of the cone determined by the complex number $-0.5000 + 0.8660i$, which is a root of the Steiner polynomial $P^5_{1,4}(z)$ (cf. Table 1). Analogously for dimensions $n = 6, 7, 8, 9$.

$n = 3$	$j = 1, k = 3$	$\gamma = -1.5000 + 0.8660i$	$\alpha = 2.6179$
$n = 4$	$j = 1, k = 4$	$\gamma = -1.0000 + 1.0000i$	$\alpha = 2.3561$
$n = 5$	$j = 1, k = 4$	$\gamma = -0.5000 + 0.8660i$	$\alpha = 2.0943$
$n = 6$	$j = 1, k = 5$	$\gamma = -0.3856 + 0.9226i$	$\alpha = 1.9667$
$n = 7$	$j = 2, k = 6$	$\gamma = -0.3249 + 1.2279i$	$\alpha = 1.8294$
$n = 8$	$j = 2, k = 6$	$\gamma = -0.1464 + 0.9892i$	$\alpha = 1.7177$
$n = 9$	$j = 2, k = 7$	$\gamma = -0.0698 + 0.9975i$	$\alpha = 1.6406$
$n = 10$	$j = 3, k = 8$	$\gamma = 0.0158 + 1.1903i$	$\alpha = 1.5574$
$n = 11$	$j = 3, k = 8$	$\gamma = 0.0854 + 0.9963i$	$\alpha = 1.4852$
$n = 12$	$j = 4, k = 9$	$\gamma = 0.1533 + 1.1549i$	$\alpha = 1.4388$
$n = 13$	$j = 4, k = 10$	$\gamma = 0.2127 + 1.1256i$	$\alpha = 1.3840$
$n = 14$	$j = 4, k = 10$	$\gamma = 0.2400 + 0.9707i$	$\alpha = 1.3284$
$n = 15$	$j = 5, k = 11$	$\gamma = 0.3139 + 1.0864i$	$\alpha = 1.2895$
$n = 16$	$j = 5, k = 11$	$\gamma = 0.3121 + 0.9500i$	$\alpha = 1.2533$
$n = 17$	$j = 5, k = 12$	$\gamma = 0.3452 + 0.9384i$	$\alpha = 1.2182$
$n = 18$	$j = 6, k = 13$	$\gamma = 0.4186 + 1.0258i$	$\alpha = 1.1833$
$n = 19$	$j = 6, k = 13$	$\gamma = 0.4076 + 0.9131i$	$\alpha = 1.1509$
$n = 20$	$j = 7, k = 14$	$\gamma = 0.4727 + 0.9917i$	$\alpha = 1.1259$

Table 1: Numerical computations for $\operatorname{bd}\mathcal{R}(n)\backslash\mathbb{R}_{\leqslant 0}$, $n \leqslant 20$.

Further, it can be easily seen (cf. also [37, Corollary 3.1]) that all roots of $P^n_{1,n}(z)$ have non-positive real part. Thus, because of the non-stability of the Steiner polynomial for $n \geqslant 10$, which we will, in shortly, prove in Proposition 1.3.4, they cannot determine the boundary.

Remark 1.3.2 ([38, Remark 3.1])**.** *Numerical computations suggest that for each* n *and suitable* $0 < j < k \leqslant n$, *the Steiner polynomials*

$$P^n_{j,k}(z) = \sum_{i=j}^{k} \binom{n}{i} z^i$$

have a root on the boundary $\operatorname{bd}\mathcal{R}(n)\backslash\mathbb{R}_{\leqslant 0}$ *(cf. Conjecture 1.3.3). Table 1 lists, for* $n \leqslant 20$, *the indices* j *and* k *of those Steiner polynomials* $P^n_{j,k}(z)$ *having a root* γ *of minimal angle* α *with the positive real axis.*

Now, instead of further looking at boundary-pairs, we briefly consider the polynomials, of pairs of convex bodies having a root on the boundary of $\mathcal{R}(n)$, themselves.

According to Propositions 1.1.2 and 1.2.7, all Steiner polynomials, for $n = 3, 4$, of boundary-pairs are (up to multiplication by a constant) of the type

$$\sum_{i=1}^{3} \binom{3}{i} \lambda^{3-i} z^i = P^3_{1,3}(\mu z) \quad \text{and} \quad \sum_{i=1}^{4} \binom{4}{i} \lambda^{4-i} z^i = P^4_{1,4}(\nu z),$$

for all $\lambda \geqslant 0$, where μ, ν are appropriately chosen reals. Since the parameter λ implies just a multiplication of the roots and $\mathcal{R}(n)$ is a convex cone, we can say that a representative of the Steiner polynomials of boundary-pairs is given by a truncated binomial polynomial (setting $\lambda = 1$) for $n = 3, 4$. We believe that this is true in general.

Conjecture 1.3.3 ([38, Conjecture 1.1])**.** *Let $n \geqslant 5$ and $\gamma \in \operatorname{bd} \mathcal{R}(n) \backslash \mathbb{R}_{\leqslant 0}$. Then there exist a truncated binomial polynomial $P^n_{j,k}(z)$, $0 < j < k < n$, and $\lambda > 0$, such that $P^n_{j,k}(\lambda\gamma) = 0$.*

Notice that the conjecture would directly imply that if (K, E) is a boundary-pair, then K, E are extremal sets for exactly $n - 3$ cases of the Aleksandrov-Fenchel inequalities (N.31) (cf. Corollary 1.4.2, where we will obtain a necessary condition for two convex bodies to be a boundary-pair).

Stability of Steiner polynomials

The entry for dimension $n = 10$ in Table 1 is particularly interesting. Here we encounter a root γ with positive real part.

A real polynomial is called (Hurwitz) stable if all its zeros have strict negative real part. We say that a real polynomial is weakly stable if all its zeros have non-positive real part.

Thus, the entry for the tenth dimension tells us that $P^{10}_{3,8}(z)$ is a non-weakly stable Steiner polynomial.

The property that all roots of n-dimensional Steiner polynomials lie in the left half-plane was part of a conjecture posed by Sangwine-Yager [63], motivated by a problem of Teissier [71]. There, it was claimed that Steiner polynomials satisfy $\mathcal{R}(n) \subseteq \{z \in \mathbb{C}^+ : \operatorname{Re}(z) \leqslant 0\}$. This inclusion, as mentioned, was known to be true for dimensions n, $n \leqslant 9$. In fact, in [37, Proposition 1.1], it was shown that

$$\mathcal{R}(n) \subseteq \{z \in \mathbb{C}^+ : \operatorname{Re}(z) < 0\} \cup \{0\} \quad \text{for } n \leqslant 9, \tag{1.11}$$

i.e., all nontrivial roots are in the open left half-plane. We have called this property "weak" stability above. In [36], the conjecture was shown to be false in dimensions n, with $n \geqslant 12$, for a special family of bodies (see also [47] for another family of high dimensional convex bodies with this property). Considering the roots of particular truncated polynomials, we get rid of the gap, showing that for $n = 10, 11$ Steiner polynomials are also not weakly stable.

Together with the mentioned results [37, Proposition 1.1] and [36, Remark 3.2], this settles the question when Steiner polynomials are weakly stable.

Proposition 1.3.4 ([38, Proposition 1.3]). *Steiner polynomials are weakly stable polynomials, i.e.,* $\mathcal{R}(n) \subseteq \{z \in \mathbb{C}^+ : \mathrm{Re}(z) < 0\} \cup \{0\}$*, if and only if* $n \leqslant 9$*.*

Proof. The (weak) stability of the Steiner polynomial was shown for all dimensions $n \leqslant 9$ in [37, Proposition 1.1], as well as its non-stability when $n \geqslant 12$, in [36, Remark 3.2]. Thus just the two cases $n = 10, 11$ remain to be considered, but Table 1 provides two non-weakly stable Steiner polynomials in these dimensions. □

1.4 MONOTONICITY OF $\mathcal{R}(n)$

First, we observe that it is easy to see that $\mathcal{R}(n) \subseteq \mathcal{R}(n+1)$. To this end, let $\gamma \in \mathcal{R}(n)$ and $K, E \in \mathcal{K}^n$ such that $f_{K;E}(\gamma) = 0$. Identifying K and E with their canonical embedding in the hyperplane $\{e_{n+1}\}^\perp \subsetneq \mathbb{R}^{n+1}$, let $E' = E \times \mathrm{conv}\{0, e_{n+1}\}$ be the prism over E of height 1 in the direction e_{n+1}. Then we observe that

$$\begin{aligned} \mathrm{vol}(K + \lambda E') &= \mathrm{vol}\big((K + \lambda E) \times \lambda \mathrm{conv}\{0, e_{n+1}\}\big) \\ &= \lambda\, \mathrm{vol}_n(K + \lambda E), \end{aligned}$$

i.e., $f_{K;E'}(z) = z f_{K;E}(z)$ and thus, we have $f_{K;E'}(\gamma) = 0$. Hence, $\gamma \in \mathcal{R}(n+1)$, which shows that $\mathcal{R}(n) \subseteq \mathcal{R}(n+1)$. The next result states that this inclusion is strict.

Theorem 1.4.1 ([38, Theorem 1.3]). $\mathcal{R}(n) \subsetneq \mathcal{R}(n+1)$.

Proof. Let $\gamma_1 \in \mathrm{bd}\,\mathcal{R}(n) \backslash \mathbb{R}_{\leqslant 0}$. By Theorem 1.3.1, $\mathcal{R}(n)$ is closed, and hence, γ_1 is a root of some Steiner polynomial $f_{K;E}(z)$ of degree $r \leqslant n$, with $K, E \in \mathcal{K}^n$, such that $\dim E = r$, $\dim K = s$, and $\dim(K + E) = n$. Let $\gamma_2, \dots, \gamma_r$ be the remaining roots of the polynomial, where $\gamma_2 = \overline{\gamma}_1$ is the complex conjugate of γ_1. We may assume that $\gamma_1, \dots, \gamma_{r+s-n} \neq 0$ and $\gamma_{r+s-n+1} = \cdots = \gamma_r = 0$. So, 0 is (exactly) an $(n-s)$-fold root.

In the next, we will show that γ_1 lies in the interior of $\mathcal{R}(n+1)$. More precisely, we prove that there exists $\varepsilon_0 > 0$, such that for any $z \in \mathbb{C}$ with $|z| = 1$, the following $r+1$ complex numbers $\rho_1 = \gamma_1 + \varepsilon_0 z, \rho_2 = \gamma_2 + \varepsilon_0 \overline{z}, \gamma_3, \dots, \gamma_r, 0$ are the roots of a Steiner polynomial $f_{K';E'}(z)$ of degree $r+1$, with $K', E' \in \mathcal{K}^{n+1}$, $\dim E' = r+1$, $\dim K' = s$ and $\dim(K' + E') = n+1$. According to Corollary 1.2.3, this is equivalent to prove that

I) $(-1)^i \sigma_i(\rho_1, \rho_2, \gamma_3 \dots, \gamma_r, 0) > 0, \quad 0 \leqslant i \leqslant r+s-n,$

$\sigma_i(\rho_1, \rho_2, \gamma_3 \dots, \gamma_r, 0) = 0, \quad r+s-n+1 \leqslant i \leqslant r+1,$

II) $c_{r+1-i,n+1}\, \sigma_i(\rho_1, \rho_2, \gamma_3 \dots, \gamma_r, 0)^2$

$\geqslant \sigma_{i-1}(\rho_1, \rho_2, \gamma_3 \dots, \gamma_r, 0)\, \sigma_{i+1}(\rho_1, \rho_2, \gamma_3 \dots, \gamma_r, 0),$

for $1 \leqslant i \leqslant r$.

In order to show this, we note that, for $0 \leqslant i \leqslant r$ and $\varepsilon > 0$,

$$\begin{aligned} &\sigma_i\left(\gamma_1+\varepsilon z, \gamma_2+\varepsilon \bar{z}, \gamma_3, \ldots, \gamma_r, 0\right) \\ &= \sigma_i\left(\gamma_1+\varepsilon z, \gamma_2+\varepsilon \bar{z}, \gamma_3, \ldots, \gamma_r\right) \end{aligned} \tag{1.12}$$

and

$$\sigma_{r+1}\left(\gamma_1+\varepsilon z, \gamma_2+\varepsilon \bar{z}, \gamma_3, \ldots, \gamma_r, 0\right) = 0. \tag{1.13}$$

Since $n+1-s$ of the $r+1$ numbers $\gamma_1+\varepsilon z, \gamma_2+\varepsilon \bar{z}, \gamma_3, \ldots, \gamma_r, 0$ are zero, we also have that, for any $\varepsilon > 0$,

$$\sigma_i\left(\gamma_1+\varepsilon z, \gamma_2+\varepsilon \bar{z}, \gamma_3, \ldots, \gamma_r, 0\right) = 0 \quad \text{for } i \geqslant r+s-n+1. \tag{1.14}$$

The numbers $\gamma_1+\varepsilon z, \gamma_2+\varepsilon \bar{z}, \gamma_3, \ldots, \gamma_r, 0$ are roots of a polynomial with real coefficients. Hence, in view of (1.12), (1.13), (1.9) (i), and the continuity of polynomials, there exists $\varepsilon_1 > 0$ such that for any $0 < \varepsilon \leqslant \varepsilon_1$

$$\begin{aligned} &(-1)^i \sigma_i\left(\gamma_1+\varepsilon z, \gamma_2+\varepsilon \bar{z}, \gamma_3, \ldots, \gamma_r, 0\right) \\ &\quad = (-1)^i \sigma_i\left(\gamma_1+\varepsilon z, \gamma_2+\varepsilon \bar{z}, \gamma_3, \ldots, \gamma_r\right) > 0, \quad 0 \leqslant i \leqslant r+s-n. \end{aligned}$$

So, with (1.14) both conditions in I) are satisfied for $\varepsilon \leqslant \varepsilon_1$.

Relation (1.14) also implies that the inequalities in II) are certainly satisfied for $r+s-n \leqslant i \leqslant r$. Thus, it remains to consider the cases $1 \leqslant i < r+s-n$. By (1.9) (ii), we know that

$$c_{r-i,n}\, \sigma_i\left(\gamma_1, \ldots, \gamma_r\right)^2 \geqslant \sigma_{i-1}\left(\gamma_1, \ldots, \gamma_r\right) \sigma_{i+1}\left(\gamma_1, \ldots, \gamma_r\right).$$

Since $c_{r+1-i,n+1} > c_{r-i,n}$ for $1 \leqslant i \leqslant r$, and $\sigma_i\left(\gamma_1, \ldots, \gamma_r\right)^2 > 0$ for $0 \leqslant i \leqslant r+s-n$ (cf. (1.9) (i)), we get that

$$c_{r+1-i,n+1}\, \sigma_i\left(\gamma_1, \ldots, \gamma_r\right)^2 > \sigma_{i-1}\left(\gamma_1, \ldots, \gamma_r\right) \sigma_{i+1}\left(\gamma_1, \ldots, \gamma_r\right)$$

for all $1 \leqslant i < r+s-n$. Hence, as before, by continuity of polynomials, there exists $\varepsilon_2 > 0$ such that

$$\begin{aligned} &c_{r+1-i,n+1}\, \sigma_i\left(\gamma_1+\varepsilon z, \gamma_2+\varepsilon \bar{z}, \gamma_3, \ldots, \gamma_r\right)^2 \\ &> \sigma_{i-1}\left(\gamma_1+\varepsilon z, \gamma_2+\varepsilon \bar{z}, \gamma_3, \ldots, \gamma_r\right) \sigma_{i+1}\left(\gamma_1+\varepsilon z, \gamma_2+\varepsilon \bar{z}, \gamma_3, \ldots, \gamma_r\right) \end{aligned}$$

for all $0 < \varepsilon \leqslant \varepsilon_2$ and $1 \leqslant i < r+s-n$. Considering (1.12) and (1.13) we obtain II) for $\varepsilon \leqslant \varepsilon_2$, and thus, the assertion follows for $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\}$. □

As a corollary of the above proof we obtain a necessary condition for convex bodies forming a boundary-pair.

Corollary 1.4.2 ([38, Corollary 1.1]). *For $n \geqslant 3$, let (K, E) be a boundary-pair. Then there exists $i \in \{1, \ldots, n-1\}$ such that*

$$W_i(K;E)^2 = W_{i-1}(K;E) W_{i+1}(K;E), \tag{1.15}$$

i.e., K, E are extremal sets for at least one Aleksandrov-Fenchel inequality.

Proof. For $\gamma \in \operatorname{bd}\mathcal{R}(n)\backslash\mathbb{R}_{\leqslant 0}$, $n \geqslant 3$, let $K, E \in \mathcal{K}^n$ be such that $f_{K;E}(\gamma) = 0$, and let $\overline{\gamma}, \gamma_3, \dots, \gamma_n$ be the remaining roots of $f_{K;E}(z)$.

If we assume that K, E are not extremal sets in any Aleksandrov-Fenchel inequality, i.e., if we have strict inequalities in (N.31), then, for all $1 \leqslant i \leqslant n-1$, we get by Corollary 1.2.3

$$c_{r-i,n}\,\sigma_i\left(\gamma, \overline{\gamma}, \gamma_3, \dots, \gamma_n\right)^2 \\ > \sigma_{i-1}\left(\gamma, \overline{\gamma}, \gamma_3, \dots, \gamma_n\right)\sigma_{i+1}\left(\gamma, \overline{\gamma}, \gamma_3, \dots, \gamma_n\right).$$

By the continuity of the elementary symmetric functions, for $\varepsilon > 0$ small enough, the numbers $\gamma + \varepsilon z, \overline{\gamma} + \varepsilon\overline{z}, \gamma_3, \dots, \gamma_n$ are roots of a polynomial with real coefficients, satisfying also conditions (i) and (ii) of Corollary 1.2.3 for any $z \in \mathbb{C}$ with $|z| = 1$. This implies that $\{\gamma + \varepsilon z : |z| = 1\} \subsetneq \mathcal{R}(n)$, contradicting that $\gamma \in \operatorname{bd}\mathcal{R}(n)\backslash\mathbb{R}_{\leqslant 0}$. □

We finish the chapter studying the behaviour of the cones for increasing dimensions. Taking the last result into consideration the following question arises in a natural way: does $\mathcal{R}(n)$ cover the whole upper half-plane $\mathbb{C}^+$, except $\mathbb{R}_{>0}$, when n tends to infinity? Next theorem gives an affirmative answer to it.

Theorem 1.4.3 ([38, Theorem 1.4]). *Let $\gamma \in \mathbb{C}^+ \setminus \mathbb{R}_{>0}$. Then there exists $n_\gamma \in \mathbb{N}$ with $\gamma \in \mathcal{R}(n)$ for all $n \geqslant n_\gamma$.*

Proof. The proof is based on known results on the distribution of the roots of the truncated binomial polynomials $P^n_{0,k}(z) = \sum_{i=0}^k \binom{n}{i} z^i$, $0 < k \leqslant n$, which are also Steiner polynomials (cf. Proposition 1.2.4).

Let $\{k_n : n \in \mathbb{N}\}$ be any sequence of positive integer numbers such that $\alpha = \lim_{n\to\infty} k_n/n \in (0,1)$. By [59, Remark 1] we have that the set of accumulation points of $\bigcup_{n=1}^\infty \left\{z \in \mathbb{C} : P^n_{0,k_n}(z) = 0\right\}$ coincides with the set

$$\left\{z \in \mathbb{C} : |z| = \alpha\,(1-\alpha)^{1/\alpha - 1}\,|1+z|^{1/\alpha} \text{ and } \left|z - \frac{\alpha^2}{1-\alpha^2}\right| \leqslant \frac{\alpha}{1-\alpha^2}\right\}.$$

Hence, taking $k_n = \lfloor n/2 \rfloor$, it can be checked that 1 is contained in the above set of accumulation points. Thus, we know that there exists a sequence $\gamma_n \in \mathbb{C}^+ \setminus \mathbb{R}_{>0}$, $n \in \mathbb{N}$, such that, for each $n \in \mathbb{N}$, there is $m_n \in \mathbb{N}$ with

$$\lim_{n\to\infty} \gamma_n = 1 \quad \text{and} \quad P^{m_n}_{0,\lfloor m_n/2 \rfloor}(\gamma_n) = 0. \tag{1.16}$$

Now let $\gamma \in \mathbb{C}^+ \setminus \mathbb{R}_{>0}$. By the choice of the sequence γ_n (cf. (1.16)), we can find an $n_\gamma \in \mathbb{N}$, such that γ is contained in the interior of the cone generated by the negative x-axis and γ_{n_γ}, which, in particular, implies, by the convexity of the cone $\mathcal{R}(n_\gamma)$ (cf. Theorem 1.1.1), that $\gamma \in \mathcal{R}(n_\gamma)$. By Theorem 1.4.1 we get the desired statement. □

2 LINEARITY OF THE VOLUME AND SAUSAGES

Perhaps one did not want to be loved so much as to be understood.
"1984", G. Orwell

2.1 BONNESEN'S INEQUALITY & THE INRADIUS CONJECTURE

For planar convex bodies K, E, Bonnesen's inradius inequality (N.27) states that

$$W_0(K;E) - 2W_1(K;E)r(K;E) + W_2(K;E)r(K;E)^2 \leqslant 0,$$

where equality holds if and only if $K = L + r(K;E)E$ for $\dim L \leqslant 1$.

Equality in this inequality, in turn, ensures that the inradius $r(K;E)$ is a root of the polynomial appearing on the left hand side of the more general inequality (N.26)

$$\mathrm{vol}(K) - 2W_1(K;E)x + \mathrm{vol}(E)x^2 \leqslant 0,$$

which holds for $r(K;E) \leqslant x \leqslant R(K;E)$. We notice that this coincides, for dimension $n = 2$ and $\lambda = r(K;E) = 1$, with (N.19) in Conjecture H, i.e., $f_{K;E}(-r(K;E)) = 0$.

In [37], the following statement was conjectured

Conjecture 2.1.1 ([37]). *Let $K \in \mathcal{K}^n$ with inradius $r(K;B_n) = 1$. Then -1 is an $(n-1)$-fold root of $f_{K;B_n}(z)$ if and only if K is a sausage with respect to B_n, i.e., $K = L + B_n$ with $L \in \mathcal{K}^n$, and $\dim L \leqslant 1$.*

In this chapter the pair (of convex bodies) K, E is called a sausage if $K = E + L$ and $L \in \mathcal{K}^n$, with $\dim L \leqslant 1$, or $E = K + L$ and $L \in \mathcal{K}^n$, with $\dim L \leqslant 1$, i.e., we do not allow dilatations of K, respectively, E, in the definition of sausage introduced after (N.27).

From the Bonnesen inequality (N.27), it follows that Conjecture 2.1.1 is true in dimension 2 for any *gauge* body E, which might be seen as indicating that the above conjecture could be true for any gauge body E and not only $E = B_n$.

In this chapter, we prove that this -extended- conjecture is not true for all gauge bodies E. More precisely, we prove that for $n \geqslant 3$, there exist convex bodies $K, E \in \mathcal{K}^n$, with $-r(K;E)$ as an $(n-1)$-fold root of $f_{K;E}(z)$, but such that the pair K, E is not a sausage. However, Conjecture 2.1.1, i.e., the case $E = B_n$, to the best of author's knowledge, remains open. Indeed, known results (see Remark 2.3.8) ensure the validity of Conjecture 2.1.1 in some special cases where additional hypotheses, such as a common/equal volume projection onto a hyperplane, are assumed. Some of these known hypotheses, under which Conjecture 2.1.1 holds, happen to ensure *linear refinements* of the classical Brunn-Minkowski inequality (N.20) too. More precisely, let us consider

$$\mathrm{vol}_{K;E}(\lambda) = \mathrm{vol}(\lambda K + (1-\lambda)E),$$

the volume of the convex combination of $K, E \in \mathcal{K}^n$ for $\lambda \in [0,1]$. From (N.12), it follows that $\mathrm{vol}_{K;E}(\lambda)$ is a polynomial of degree at most n, namely,

$$\mathrm{vol}_{K;E}(\lambda) = \sum_{i=0}^{n} \binom{n}{i} W_i(K;E)\lambda^{n-i}(1-\lambda)^i.$$

Brunn-Minkowski's inequality ensures that the function $\mathrm{vol}_{K;E}^{1/n}(\lambda)$ defined on $\lambda \in [0,1]$ is concave. It is known that under special assumptions on the convex bodies K, E ([58], [8], [27], [46]) the classical Brunn-Minkowski inequality can be refined. An example of this behaviour is Theorem M, that we state again for completeness.

Theorem 2.1.2 ([8, 27, 58]). *Let $K, E \in \mathcal{K}^n$ be convex bodies such that there exists a hyperplane $H \in \mathcal{G}(n, n-1)$ with $\mathrm{vol}_{n-1}(K|H) = \mathrm{vol}_{n-1}(E|H)$. Then, for all $\lambda \in [0,1]$,*

$$\mathrm{vol}_{K;E}(\lambda) \geqslant \lambda \mathrm{vol}(K) + (1-\lambda)\mathrm{vol}(E).$$

However, as mentioned in the introduction, the above result does not provide the concavity of the function $\mathrm{vol}_{K;E}(\lambda) = \mathrm{vol}\big(\lambda K + (1-\lambda)E\big)$. For further details we refer to [68, Notes for Section 7.7] and the references therein.

The inequality in Theorem 2.1.2 holds true if the assumption on projections is replaced by the following one:

$$\max_{x \in H^\perp} \mathrm{vol}_{n-1}\big(K \cap (H+x)\big) = \max_{x \in H^\perp} \mathrm{vol}_{n-1}\big(E \cap (H+x)\big),$$

for $K, E \in \mathcal{K}^n$ and some hyperplane $H \in \mathcal{G}(n, n-1)$.

In this chapter we aim to understand the pairs of convex bodies K, E for which there is equality in this inequality, i.e., for which $\mathrm{vol}_{K;E}(\lambda)$ is a polynomial of degree one in $\lambda \in [0,1]$. In this case, we have

$$\mathrm{vol}_{K;E}(\lambda) = \lambda \mathrm{vol}(K) + (1-\lambda)\mathrm{vol}(E), \tag{2.1}$$

and we will say that $\mathrm{vol}_{K;E}(\lambda)$ is *linear* in $\lambda \in [0,1]$. From now on, whenever we refer to linearity of the volume we will be meaning (2.1).

After these considerations it is rather natural to ask whether there is some closer connection between both, the validity of Conjecture 2.1.1 and the linearity of the volume. The answer is indeed positive, as we will prove in Lemma 2.3.1: linearity of the volume for K, E is equivalent to the fact that -1 is an $(n-1)$-fold root of $f_{K;E}(z)$. The latter, together with the validity of Conjecture 2.1.1 in some special cases where hypotheses, such as a common/equal volume projection onto a hyperplane, are assumed (see Remark 2.3.8), and Theorem 2.1.2, suggest that one may get a characterization of the linearity of the volume under the additional assumption of a common projection onto a hyperplane. This motivates a following section, where we will characterize linearity of the volume under such assumptions.

2.2 A COUNTEREXAMPLE TO THE INRADIUS SAUSAGE CONJECTURE

Next, we provide a counterexample to the extended version of Conjecture 2.1.1. More precisely, we prove the following result.

Theorem 2.2.1 ([65, Theorem 1.1])**.** *For* $n \geqslant 3$, *there exist convex bodies* $K, E \in \mathcal{K}^n$, *with* $-r(K;E)$ *as an* $(n-1)$*-fold root of* $f_{K;E}(z)$ *and such that* K, E *is not a sausage.*

Proof. In order to describe such convex bodies, we fix the notation. Let $C_1 = \mathrm{conv}\{(0,0,0)^\top, (1,0,0)^\top\}$, $C_1^\perp = \mathrm{conv}\{(0,0,0)^\top, (0,1,0)^\top\}$ and $C_2 = C_1 + C_1^\perp$ be the 2-dimensional unit cube.

Let $L = \mathrm{conv}\{(0,0,0)^\top, (0,0,1)^\top\}$ be a segment orthogonal to C_2 of length one, and $\overline{C_1} = \mathrm{conv}\{(0,1,1)^\top, (1,1,1)^\top\}$.

For $\tau \in [0,1]$ fixed, we define by $A_\tau = C_1 + \tau\, C_1^\perp \subset C_2$, the orthogonal box of sides length 1 and τ.

Let $L_1 = \mathrm{conv}\{(0,\tau,\tau)^\top, (1,\tau,\tau)^\top\}$ be the segment, parallel to C_1 lying in the diagonal face $\mathrm{conv}\left(C_1 \cup \overline{C_1}\right)$ of the unit cube C_3, whose projection onto C_2 is the edge $\mathrm{conv}\{(0,\tau,0)^\top, (1,\tau,0)^\top\}$ of A_τ.

We consider $K = C_3 = L + C_2$ and $E = \mathrm{conv}\,(A_\tau \cup L_1)$ the triangular prism determined by L_1 and A_τ (see Figure 1).

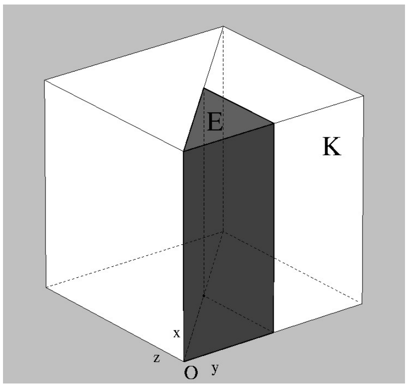

Figure 1: Counterexample proving Theorem 2.2.1.

Then, on the one hand, and taking all the above into consideration, it is clear that $r(K;E) = 1$. On the other hand, for $\lambda \geqslant 0$, and denoting by $M(s)$ the section of $M \in \mathcal{K}^3$ with the plane defined by $\{x_3 = s\}$ we have

$$\begin{aligned} f_{K;E}(\lambda) &= \mathrm{vol}(K+\lambda E) \\ &= \mathrm{vol}(L+C_2+\lambda E) \\ &= \mathrm{vol}_2\big((C_2+\lambda E)|L^{\perp}\big)+\mathrm{vol}(C_2+\lambda E) \\ &= \mathrm{vol}_2(C_2+\lambda A_\tau)+\int_0^{\lambda\tau}\mathrm{vol}_2\big((C_2+\lambda E)(s)\big)\,ds \\ &= (\lambda+1)(\lambda\tau+1)+\int_0^{\lambda\tau}\mathrm{vol}_2\big((C_2+\lambda E)(s)\big)\,ds. \end{aligned}$$

Since

$$\begin{aligned} \int_0^{\lambda\tau} & \mathrm{vol}_2\big((C_2+\lambda E)(s)\big)\,ds \\ &= \int_0^{\lambda\tau}\mathrm{vol}_2\Big((1-\frac{s}{\lambda\tau})\lambda A_\tau+\frac{s}{\lambda\tau}\lambda L_1+C_2\Big)\,ds \\ &= \lambda\tau\int_0^1\mathrm{vol}_2\big((1-t)\lambda A_\tau+t\lambda L_1+C_2\big)\,dt \\ &= \lambda\tau\Bigg(\int_0^1\mathrm{vol}_2\big((1-t)\lambda A_\tau+C_2\big)\,dt \\ &\quad+\int_0^1 t\lambda\mathrm{vol}_1\Big(\big((1-t)\lambda A_\tau+C_2\big)|L_1^{\perp}\Big)\,dt\Bigg) \\ &= \lambda\tau\int_0^1\big((1-t)\lambda\tau+1\big)\big((1-t)\lambda+1+t\lambda\big)\,dt \\ &= \lambda\tau(\lambda+1)\Big(1+\lambda\tau\int_0^1(1-t)\,dt\Big) \\ &= \lambda\tau(\lambda+1)\Big(\frac{\lambda\tau}{2}+1\Big), \end{aligned}$$

we have

$$f_{K;E}(\lambda)=\frac{1}{2}(\lambda+1)(\lambda^2\tau^2+4\lambda\tau+2).$$

Finally, as $\tau^2-4\tau+2=0$ if and only if $\tau=2\pm\sqrt{2}$, if we take $\tau=2-\sqrt{2}\in[0,1]$, then we have that $-1=-r(K;E)$ is a 2-fold root of $f_{K;E}(z)$. However, clearly, K is not a sausage with respect to E, which concludes the proof. □

The (most direct) extension of this construction to higher dimension was not successful. Nevertheless, if degenerated gauge bodies E are considered, a pair of convex bodies $K, E \in \mathcal{K}^n$ providing a counterexample can be obtained as follows:

Remark 2.2.2 ([65, Remark 3.4]). *Following the same notation as in the proof of Theorem 2.2.1, we consider the unit cube* $K=C_n=L+C_{n-1}$, *and* $E=\mathrm{conv}\{C_{n-2},\frac{1}{2}(C_{n-2}+\overline{C_{n-2}})\}$ *the diagonal 'half-face' of the cube* C_n

determined by C_{n-2}. *It is clear that* K *is not a sausage with respect to* E *and* $r(K;E) = 1$. *However we have*

$$\begin{aligned} f_{K;E}(\lambda) &= \mathrm{vol}(K + \lambda E) \\ &= \mathrm{vol}(L + C_{n-1} + \lambda E) \\ &= \mathrm{vol}(C_{n-1} + \lambda E) + \mathrm{vol}_{n-1}\big((C_{n-1} + \lambda E)|L^{\perp}\big) \\ &= \frac{\lambda}{2}(\lambda+1)^{n-2} + \big(\frac{\lambda}{2}+1\big)(\lambda+1)^{n-2} \\ &= (\lambda+1)^{n-1}. \end{aligned}$$

To the best of the author knowledge it is not known whether for some other fixed gauge body E, in particular, for the Euclidean ball B_n, Conjecture 2.1.1 holds true. In fact, the problem of classifying the gauge bodies E, if there are any, for which Conjecture 2.1.1 is true for any K remains open. So far, it is only known that they are not the whole $\mathcal{K}^n$, as the above results show.

We will now prove that a slight modification of the convex bodies given in the proof of Theorem 2.2.1 provide us with a counterexample for (the equality case in) Conjecture H.

Theorem 2.2.3 ([65, Theorem 3.1]). *If* $n \geqslant 3$, *there exist convex bodies* $K, E \in \mathcal{K}^n$ *with interior points satisfying*

$$\mathrm{vol}(K \sim \lambda E) = f_{K;E}(-\lambda)$$

for all $0 < \lambda < r(K;E)$ *and such that* E *is not a summand of* K.

Proof. Following the same notation as in the proof of Theorem 2.2.1, we define by $A = \frac{1}{4}C_1 + \frac{3}{4}\,C_1^{\perp} \subset C_2$, the orthogonal box of sides length $1/4$ and $3/4$ and let $L_1 = \mathrm{conv}\{(0,3/4,3/4)^{\top},(1/4,3/4,3/4)^{\top}\}$ be the segment (of length $1/4$) parallel to C_1 lying in the diagonal face $\mathrm{conv}\left(C_1 \cup \overline{C_1}\right)$ of the unit cube C_3.

Thus, if we consider $K = C_3 = L + C_2$ and $E = \mathrm{conv}\,(A \cup L_1)$ the triangular prism determined by L_1 and A (cf. Figure 1), it is easy to check that

$$\mathrm{vol}(K \sim \lambda E) = \left(1 - \frac{3}{4}\lambda\right)^2 \left(1 - \frac{1}{4}\lambda\right), \quad \text{for all } 0 \leqslant \lambda \leqslant 4/3 = r(K;E).$$

On the other hand, a similar computation as in the proof of Theorem 2.2.1 shows that, for $\lambda \geqslant 0$,

$$f_{K;E}(\lambda) = \left(1 + \frac{3}{4}\lambda\right)^2 \left(1 + \frac{1}{4}\lambda\right),$$

which concludes the proof. □

To finish this section we prove a sufficient condition, relying on the Schwarz symmetrization (see paragraph before Lemma Q for the definition), for a pair K, E to be a sausage.

Lemma 2.2.4 ([65, Lemma 2.2]). *Let $n \geqslant 3$, and let $K, E \in \mathcal{K}^n$ be convex bodies. Let K have interior points, and let $H \in \mathcal{G}(n, n-1)$ be a hyperplane. If*

$$\sigma_{H^\perp}(\lambda_0 K + (1-\lambda_0)E) = \lambda_0 \sigma_{H^\perp}(K) + (1-\lambda_0)\sigma_{H^\perp}(E) \tag{2.2}$$

for some $\lambda_0 \in (0,1)$ and

$$\sigma_{H^\perp}(K) = L + \sigma_{H^\perp}(E), \textit{ with } \dim L \leqslant 1, \tag{2.3}$$

then K is a sausage with respect to E.

Proof. By an appropriate choice of coordinates we may assume that $H = \{x \in \mathbb{R}^n : x_1 = 0\}$. Further, we may assume that the origin is an interior point of K. By definition of the Schwarz symmetrization, $L \subset H^\perp$ and then $L = [\tilde{a}, \tilde{b}]$ with $\tilde{a} = (a, 0, \dots, 0)$ and $\tilde{b} = (b, 0, \dots, 0)$, for some $a \leqslant b$.

We use the following notation. We write $H_t = \{x \in \mathbb{R}^n : x_1 = t\}$ and $H_t^+ = \{x \in \mathbb{R}^n : x_1 \geqslant t\}$ (respectively $H_t^- = \{x \in \mathbb{R}^n : x_1 \leqslant t\}$) and, for any convex body M, we will also write $M_t = M \cap H_t$ and $M_t^+ = M \cap H_t^+$ (respectively $M_t^- = M \cap H_t^-$).

Without loss of generality, we may also assume that (one of) the maximum volume section(s) of E through hyperplanes parallel to H contains the origin. So, condition (2.3) implies that

$$\max_{x \in H^\perp} \mathrm{vol}_{n-1}\big(K \cap (x + H)\big) = \max_{x \in H^\perp} \mathrm{vol}_{n-1}\big(E \cap (x + H)\big) = m > 0$$

(since 0 is an interior point of K) and also that $\mathrm{vol}_{n-1}(K_t) = m$ for all $t \in [a, b]$.

Moreover, from the inclusion $K_{\lambda a + (1-\lambda)b} \supset \lambda K_a + (1-\lambda)K_b$, for $\lambda \in [0,1]$, it follows

$$\begin{aligned} m &= \mathrm{vol}_{n-1}(K_{\lambda a+(1-\lambda)b}) \geqslant \mathrm{vol}_{n-1}(\lambda K_a + (1-\lambda)K_b) \\ &\geqslant \Big(\lambda \mathrm{vol}_{n-1}(K_a)^{1/(n-1)} + (1-\lambda)\mathrm{vol}_{n-1}(K_b)^{1/(n-1)}\Big)^{n-1} = m, \end{aligned}$$

and hence, the equality case in Brunn-Minkowski's inequality allows us to conclude that (up to translation)

$$K_{\lambda a + (1-\lambda)b} = K_a \quad (\text{for all } \lambda \in [0,1]). \tag{2.4}$$

Finally, we study what happens on the 'leftmost and rightmost parts' of K. To this aim, using Lemma Q (i) and the inclusion

$$\lambda_0 K_b^+ + (1-\lambda_0)E_0^+ \subset (\lambda_0 K + (1-\lambda_0)E)^+_{\lambda_0 b},$$

we obtain, on the one hand,

$$\begin{aligned}
\mathrm{vol}(\lambda_0 K_b^+ + (1-\lambda_0)E_0^+) &= \mathrm{vol}(\sigma_{H^\perp}(\lambda_0 K_b^+ + (1-\lambda_0)E_0^+)) \\
&\leqslant \mathrm{vol}\Big(\sigma_{H^\perp}\big((\lambda_0 K + (1-\lambda_0)E)^+_{\lambda_0 b}\big)\Big) \\
&= \mathrm{vol}\Big(\sigma_{H^\perp}(\lambda_0 K + (1-\lambda_0)E)^+_{\lambda_0 b}\Big) \\
&= \mathrm{vol}\Big(\big(\lambda_0\sigma_{H^\perp}(K) + (1-\lambda_0)\sigma_{H^\perp}(E)\big)^+_{\lambda_0 b}\Big) \\
&= \mathrm{vol}\Big(\big(\lambda_0 L + \sigma_{H^\perp}(E)\big)^+_{\lambda_0 b}\Big) \\
&= \mathrm{vol}(\sigma_{H^\perp}(E)^+_0) \\
&= \mathrm{vol}(E_0^+).
\end{aligned} \tag{2.5}$$

On the other hand,

$$\begin{aligned}
\mathrm{vol}(\lambda_0 K_b^+ + (1-\lambda_0)E_0^+) &\geqslant \Big(\lambda_0 \mathrm{vol}(K_b^+)^{1/n} + (1-\lambda_0)\mathrm{vol}(E_0^+)^{1/n}\Big)^n \\
&= \Big(\lambda_0 \mathrm{vol}(\sigma_{H^\perp}(K)_b^+)^{1/n} + (1-\lambda_0)\mathrm{vol}(E_0^+)^{1/n}\Big)^n \\
&= \Big(\lambda_0 \mathrm{vol}(\sigma_{H^\perp}(E)_0^+)^{1/n} + (1-\lambda_0)\mathrm{vol}(E_0^+)^{1/n}\Big)^n \\
&= \mathrm{vol}(E_0^+),
\end{aligned} \tag{2.6}$$

and hence, from (2.5) and (2.6), we have equality in Brunn-Minkowski's inequality (for K_b^+ and E_0^+). Therefore, there are two possibilities depending on the dimension of E_0^+ and K_b^+:

(i) if $\mathrm{vol}(E_0^+) = \mathrm{vol}(K_b^+) = 0$, then we have

$$\begin{aligned}
&\mathrm{vol}_{n-1}(\lambda_0 K_b + (1-\lambda_0)E_0) \\
&\qquad = \mathrm{vol}_{n-1}\big((\lambda_0 K + (1-\lambda_0)E)_{\lambda_0 b}\big) \\
&\qquad = \mathrm{vol}_{n-1}\big(\sigma_{H^\perp}(\lambda_0 K + (1-\lambda_0)E)_{\lambda_0 b}\big) \\
&\qquad = \mathrm{vol}_{n-1}\big((\lambda_0 L + \sigma_{H^\perp}(E))_{\lambda_0 b}\big) \\
&\qquad = \mathrm{vol}_{n-1}(E_0) = m > 0,
\end{aligned}$$

and thus (again by the equality case in Brunn-Minkowski's inequality), $K_b = y_0 + E_0$, for some $y_0 \in \mathbb{R}^n$.

Hence $K_b^+ = K_b = y_0 + E_0 = y_0 + E_0^+$.

(ii) If $\mathrm{vol}(E_0^+), \mathrm{vol}(K_b^+) > 0$, then, as they are homothetic with the same volume, $K_b^+ = y_0 + E_0^+$ for some $y_0 \in \mathbb{R}^n$.

In any case we have that $K_b^+ = y_0 + E_0^+$ for some $y_0 \in \mathbb{R}^n$ and, arguing in the same way as before, we may assert that $K_a^- = x_0 + E_0^-$ for some $x_0 \in \mathbb{R}^n$. These facts, together with (2.4) and the convexity of all involved bodies, imply that $K = [x_0, y_0] + E$, i.e., K is a sausage with respect to E. □

Remark 2.2.5 ([65, Remark 2.1]). *If we ask whether (only) one of the conditions (2.2), (2.3) is enough in order to characterize sausages, we obtain a negative answer in both cases.*

(i) for (2.2), it is enough to consider $E = B_n$ the n-dimensional unit ball, and $K = L + B_{n-1}$ a cylinder, with $\dim L = 1$ and $L \perp \operatorname{aff} B_{n-1}$. As both bodies (and the convex combination of them) are rotationally symmetric about the axis determined by L, it is clear that condition (2.2) holds (for all $\lambda \in [0,1]$) but K is not a sausage with respect to E.

(ii) for (2.3), we may consider $E = C_n$ the n-dimensional unit cube, and $K = L + C_{n-1}$ a parallelepiped, where L is a segment of appropriate length and so, that $L \not\perp \operatorname{aff} C_{n-1}$ and K is not parallel to $\operatorname{aff} C_{n-1}$. These bodies satisfy (2.3) for $H = \operatorname{aff} C_{n-1}$ and also K is not a sausage with respect to E.

Notice that in both cases it is also fulfilled that

$$\max_{x \in H^\perp} \operatorname{vol}_{n-1}\big(K \cap (x + H)\big) = \max_{x \in H^\perp} \operatorname{vol}_{n-1}\big(E \cap (x + H)\big), \qquad (2.7)$$

for some hyperplane H. Thus, even under this additional assumption, none of conditions (2.2), (2.3) is enough to determine sausages. However, as we shall see in Theorems 2.3.9 and 2.3.11, if we further assume linearity of the volume, then we can characterize sausages.

2.3 LINEARITY OF THE VOLUME, SAUSAGES AND THEIR INTERPLAY

Our aim in this section is to understand and characterize the (pairs of) convex bodies K, E for which $\operatorname{vol}_{K;E}(\lambda)$ is a linear function, i.e., those bodies for which (2.1) holds. From now on, along this chapter, we will write $K(\lambda) = \lambda K + (1-\lambda)E$, for $\lambda \in [0,1]$.

To start with, we will prove the mentioned relation between the linearity of the volume (2.1) and Conjecture 2.1.1.

Lemma 2.3.1 ([65, Lemma 3.1]). *Let $K, E \in \mathcal{K}^n$ be convex bodies. Then, for $i = 0, \dots, n$,*

$$W_i(\lambda K + (1-\lambda)E; E) = \left(\sum_{j=i}^{n} \binom{j}{i} \frac{f^{(j)}_{K;E}(-1)}{j!} \lambda^{n-j} \right) \frac{1}{\binom{n}{i}}.$$

Here, $f^{(j)}$ denotes the j-th derivative of the real valued function f.

Proof of Lemma 2.3.1. Using the linearity of mixed volumes (N.15), we can write the quermassintegrals $W_i\,(K(\lambda); E)$, for $i = 0, \dots, n-1$, as polynomials in λ:

$$\begin{aligned} W_i\,(K(\lambda); E) &= V\,(K(\lambda)[n-i], E[i]) \\ &= \sum_{k=0}^{n-i} \binom{n-i}{k} W_{i+k}(K; E) \lambda^{n-(i+k)} (1-\lambda)^k. \end{aligned}$$

By rearranging the terms we obtain that

$$\begin{aligned} W_i\,(K(\lambda);E) &= \sum_{k=0}^{n-i}\binom{n-i}{k}W_{i+k}(K;E)\lambda^{n-(i+k)}(1-\lambda)^k \\ &= \sum_{k=0}^{n-i}\binom{n-i}{k}W_{i+k}(K;E)\sum_{j=0}^{k}\binom{k}{j}(-1)^j\lambda^{n-i-(k-j)} \\ &= \sum_{l=0}^{n-i}\left(\sum_{k=l}^{n-i}\binom{n-i}{k}W_{i+k}(K;E)\binom{k}{l}(-1)^{k-l}\right)\lambda^{n-i-l} \\ &= \sum_{l=0}^{n-i}\left(\sum_{k=l}^{n-i}\frac{\binom{n}{i+k}\binom{i+k}{i}}{\binom{n}{i}}W_{i+k}(K;E)\binom{k}{l}(-1)^{k-l}\right)\lambda^{n-i-l}. \end{aligned}$$

Thus,

$$\begin{aligned} W_i\,(K(\lambda);E) &= \frac{1}{\binom{n}{i}}\sum_{j=i}^{n}\left(\sum_{r=j}^{n}\binom{n}{r}W_r(K;E)\binom{r-i}{j-i}\binom{r}{i}(-1)^{r-j}\right)\lambda^{n-j} \\ &= \frac{1}{\binom{n}{i}}\sum_{j=i}^{n}\left(\binom{j}{i}\sum_{r=j}^{n}\binom{n}{r}W_r(K;E)\binom{r}{j}(-1)^{r-j}\right)\lambda^{n-j} \\ &= \frac{1}{\binom{n}{i}}\left(\sum_{j=i}^{n}\binom{j}{i}\frac{f_{K;E}^{(j)}(-1)}{j!}\lambda^{n-j}\right). \end{aligned}$$

□

The particular case $i = 0$ yields

$$\mathrm{vol}(\lambda K + (1-\lambda)E) = \sum_{j=0}^{n}\frac{f_{K;E}^{(n-j)}(-1)}{(n-j)!}\lambda^j, \tag{2.8}$$

and hence, from the above result, we immediately get the following corollary.

Corollary 2.3.2 ([65, Corollary 3.1]). *Let $K, E \in \mathcal{K}^n$ be convex bodies. Then $\mathrm{vol}_{K;E}(\lambda)$ is linear if and only if -1 is an $(n-1)$-fold root of $f_{K;E}(z)$. In that case, we also have linearity for every quermassintegral $W_i\,(K(\lambda);E)$, $i = 0,\dots,n$.*

Next, we state the following equivalent formulation of Lemma 1.2.1, which will be useful in the following (cf. Remark 1.2.2).

Corollary 2.3.3 ([65, Proposition 2.1]). *Let $a_0,\dots,a_n \geqslant 0$ be a sequence of real numbers. If $a_0,\dots,a_n \geqslant 0$ satisfy inequalities (N.31), then there exist simplices $K, E \in \mathcal{K}^n$, such that $W_i(K;E) = a_i$.*

Using this, we observe the following fact.

Remark 2.3.4 ([65, Remark 3.1]). *Using Lemma 2.3.1, we know that, if for some* $i_0 \in \{0, \dots, n-2\}$, $W_{i_0}(K(\lambda); E)$ *is linear, then* $W_i(K(\lambda); E)$ *is also linear for all* $i > i_0$. *The converse is not true.*

For $n = 2$ *the situation is clear, since* W_1 *is always linear. For* $n = 3$, *consider the numbers* $W_0 = 9, W_1 = 7, W_2 = 4$ *and* $W_3 = 1$. *They satisfy inequalities (N.31). Hence, Corollary 2.3.3 ensures the existence convex bodies* K, E *such that* $W_i(K; E) = W_i$, *which yields* $f_{K;E}(z) = 9 + 21z + 12z^2 + z^3$. *Thus* $f_{K;E}(-1) = -1$, $f'_{K;E}(-1) = 0$ *and we have that* $W_i(K(\lambda); E)$ *is linear for* $i = 1, 2, 3$, *but* $W_0(K(\lambda); E)$ *is not. In higher dimension similar examples can be constructed.*

Good candidates for (pairs of) convex bodies characterizing the linearity of the volume are sausages: fix a convex body E and consider $K = L + E$, with $\dim L \leqslant 1$. Indeed, for these bodies, we have

$$\begin{aligned} \mathrm{vol}\big(\lambda K + (1-\lambda)E\big) &= \mathrm{vol}\big(\lambda L + E\big) = nW_{n-1}(L; E)\lambda + \mathrm{vol}(E) \\ &= \lambda \mathrm{vol}(K) + (1-\lambda)\mathrm{vol}(E), \end{aligned} \tag{2.9}$$

where we have used again the linearity of mixed volumes (N.15). One might think that this family may allow us to characterize the linearity of the volume. In fact, considering full-dimensional convex bodies K, E having equal volume, the following remark ensures that, in this case, only 'degenerated' sausages, i.e., $K = L + E$ with $\dim L = 0$, can turn up.

Remark 2.3.5 ([65, Remark 3.2]). *Let* $K, E \in \mathcal{K}^n$. *The following facts hold:*

(i) if $\mathrm{vol}(K) = \mathrm{vol}(E)$ *and, for some* $\lambda \in (0, 1)$,

$$\mathrm{vol}_{K;E}(\lambda) = \mathrm{vol}\big(\lambda K + (1-\lambda)E\big) = \lambda \mathrm{vol}(K) + (1-\lambda)\mathrm{vol}(E),$$

$$\text{then } \mathrm{vol}_{K;E}(\lambda) = \mathrm{vol}(K) = \left(\lambda \mathrm{vol}(K)^{1/n} + (1-\lambda)\mathrm{vol}(E)^{1/n}\right)^n.$$

Equality case in Brunn-Minkowski's inequality allows us to assert that either K *and* E *lie in parallel hyperplanes (if* $\dim K, \dim E < n$*), or* $K = E$ *(since* $\mathrm{vol}(K) = \mathrm{vol}(E)$*).*

(ii) if, for some $\lambda \in (0, 1)$,

$$\begin{aligned} \mathrm{vol}_{K;E}(\lambda) &= \lambda \mathrm{vol}(K) + (1-\lambda)\mathrm{vol}(E) \\ &= \left(\lambda \mathrm{vol}(K)^{1/n} + (1-\lambda)\mathrm{vol}(E)^{1/n}\right)^n, \end{aligned}$$

then, from the strict concavity of $x \mapsto x^{1/n}$, *it follows that either* K *and* E *lie in parallel hyperplanes or* $K = E$.

In the following, we will suppose, without loss of generality, that $\mathrm{vol}(K) \neq \mathrm{vol}(E)$. Despite all the signs, sausages are not the only (pairs of) convex bodies satisfying linearity of the volume as we shall see in the following proposition. They are, in turn, not so far from being the ones, as it follows from Theorems 2.3.9 and 2.3.11. There, the sole additional assumption that the bodies have a common volume projection or a common maximum volume section provides a characterization of sausages.

Proposition 2.3.6 ([65, Proposition 3.1]). *There exist* $K, E \in \mathcal{K}^n$, $n \geqslant 2$, *such that the pair* K, E *is not a sausage, and so, that the equality*

$$\mathrm{vol}\big(\lambda K + (1-\lambda)E\big) = \lambda \mathrm{vol}(K) + (1-\lambda)\mathrm{vol}(E), \quad \text{for all } \lambda \in [0,1],$$

is satisfied.

This proposition is a straightforward consequence of Corollary 2.3.2 and Theorem 2.2.1. For completeness, we include the proof contained in the mentioned reference, where the result was originally proven.

Proof of Proposition 2.3.6. Let $E_0 = B_n$ and $K_0 = L + B_n$, with $L \in \mathcal{K}^n$ and $\dim L = 1$, and denote by $W_0, \dots, W_n$ the quermassintegrals of K_0, with respect to E_0. Since, for $1 \leqslant i \leqslant n-1$, W_i satisfy inequalities (N.31), by Proposition 2.3.3, there exist simplices K and E, such that $W_i(K;E) = W_i$. Thus, $f_{K;E}(z) = f_{K_0;E_0}(z)$, together with (2.9) yield the linearity of $\mathrm{vol}_{K;E}(\lambda)$.

Finally, notice that a simplex K is a sausage with respect to another simplex E if and only if they coincide (up to a translation), which, as $\mathrm{vol}(K) = W_0(K_0;E_0) = \mathrm{vol}(K_0) > \mathrm{vol}(E_0) = W_n(K_0;E_0) = \mathrm{vol}(E)$, cannot be the case. ☐

The (pairs of) convex bodies for which $\mathrm{vol}_{K;E}(\lambda)$ is linear enjoy also other useful properties, as the following result indicates.

Lemma 2.3.7 ([65, Lemma 3.2]). *Let* $K, E \in \mathcal{K}^n$ *be convex bodies. If* -1 *is an* $(n-1)$*-fold root of* $f_{K;E}(z)$ *(cf. Corollary 2.3.2), then*

(i) $W_0(K;E) - W_1(K;E) = W_i(K;E) - W_{i+1}(K;E)$, $i = 0, \dots, n-1$.

(ii) $f^{(i)}_{K(\lambda);E}(-1) = 0$ *for* $i = 0, \dots, n-2$, *and any* $\lambda \in [0,1]$.

Proof. First, if -1 is an $(n-1)$-fold root of $f_{K;E}(z)$, then it is also a root of the $(n-2)$-th derivative, namely,

$$0 = W_{n-2}(K;E) - 2W_{n-1}(K;E) + W_n(K;E).$$

We notice that the above equality is exactly

$$W_{n-2}(K;E) - W_{n-1}(K;E) = W_{n-1}(K;E) - W_n(K;E).$$

Now, if we suppose, by reverse induction on $s \leqslant n-1$, that

$$W_s(K;E) - W_{s+1}(K;E) = W_{n-1}(K;E) - W_n(K;E)$$

holds for all $s > j$, and we substitute this in the j-th derivative of $f_{K;E}(z)$, then, by arranging the terms, we obtain

$$
\begin{aligned}
0 &= \frac{f^{(j)}_{K;E}(-1)}{\binom{n}{j} j!} = \sum_{i=0}^{n-j} \binom{n-j}{i} W_{j+i}(K;E)(-1)^i \\
&= W_j(K;E) + \sum_{i=1}^{n-j-1} \left(\binom{n-j-1}{i-1} + \binom{n-j-1}{i} \right) W_{j+i}(K;E)(-1)^i \\
&+ W_n(K;E)(-1)^{(n-j)} \\
&= W_j(K;E) - W_{j+1}(K;E) \\
&+ \sum_{i=1}^{n-j-1} \binom{n-j-1}{i} \big(W_{j+i}(K;E) - W_{j+i+1}(K;E)\big)(-1)^i \\
&= W_j(K;E) - W_{j+1}(K;E) \\
&+ \big(W_{n-1}(K;E) - W_n(K;E)\big) \sum_{i=1}^{n-j-1} \binom{n-j-1}{i} (-1)^i \\
&= W_j(K;E) - W_{j+1}(K;E) - \big(W_{n-1}(K;E) - W_n(K;E)\big),
\end{aligned}
$$

which concludes the proof of (i).

In order to prove the second assertion, notice that, since -1 is an $(n-1)$-fold root of $f_{K;E}(z)$, we have, by using Corollary 2.3.2, the equality $W_i\,(K(\lambda);E) = \lambda W_i(K;E) + (1-\lambda)W_i(E;E)$. Thus, also the equality $W_i\,(K(\lambda);E) - W_{i+1}\,(K(\lambda);E) = \lambda(W_i(K;E) - W_{i+1}(K;E))$ holds true for $i \in \{0, 1, \dots, n-1\}$. Hence, we obtain, for $i = 0, \dots, n-1$, that $W_0\,(K(\lambda);E) - W_1\,(K(\lambda);E) = W_i\,(K(\lambda);E) - W_{i+1}\,(K(\lambda);E)$. Finally, by substituting on successive derivatives of $f_{K(\lambda);E}(z)$, we obtain that, $f^{(j)}_{K(\lambda);E}(-1) = 0$, $j = 0, 1, \dots, n-2$, as it also happens for K. □

We observe that under the assumption of a common projection of K and E, it is known (see [68, Theorem 7.7.2]) that (i) implies that the pair K, E is a sausage.

Indeed, this is a consequence of some results which support Conjecture 2.1.1. The validity of it is known in some special cases where additional hypotheses, such as a common/equal volume projection onto a hyperplane, are assumed. For a convex body M satisfying that $\dim M = j \leqslant n-1$, we denote by $W_i^{(j)}$, $i = 0, \dots, j$, the i-th quermassintegral of M in $\operatorname{aff} M (\equiv \mathbb{R}^j)$.

For completeness, we finish the section collecting, in the following remark, (some of) the known cases dealing with the validity of Conjecture 2.1.1.

Remark 2.3.8 ([65, Remark 3.3])**.**

(i) If there exists a hyperplane $H \in \mathcal{G}(n, n-1)$ *for which we have that* $W^{(n-1)}_{n-2}(K|H; B_{n-1}) = \kappa_{n-1}$, *that is, the mean width of* $K|H$, *in the ambient space* H, *coincides with the mean width of the unit ball in* H, *i.e., 2, then equality in the main result in [13] yields that* K *is the sum of a segment and the unit ball.*

In other words, from this result, it follows that if K *is a convex body having a common projection with the unit ball,* $K|H = B_{n-1} = B_n|H$*, then* -1 *is an* $(n-1)$*-fold root of* $f_{K;B_n}(z)$ *if and only if* K *is a sausage with respect to* B_n*.*

(ii) If there exists a hyperplane $H \in \mathcal{G}(n, n-1)$ *so, that* $K|H = E|H$*, with* $\dim(E|H) = n-1$*, then the validity of the conjecture follows from [68, Theorem 7.7.3].*

(iii) These above two cases are closely related to [37, Theorem 3.3], since this latter one can be obtained from them when the set of incenters of K *is not a unique point. Indeed, let* K *have inradius equal to one. If all the two-dimensional projections of* K *have inradius (considered now in* $\mathbb{R}^2$*) equal to the inradius of* K*, the set of incenters of* K *is at most one dimensional; otherwise, some of the projections would have larger inradius. Since the set of incenters is not a singleton, there is at least a one-dimensional (convex and compact) set of incenters, which we may call* ℓ*. Furthermore, if there exists a point* $p \in K$*,* $p \notin \ell + B_n$*, then* $\mathrm{conv}\,(p \cup (\ell + B_n))\,|\mathrm{aff\ conv}\,(\ell \cup \{p\})$ *has inradius larger than* 1*, a contradiction. Thus,* K *has an* $(n-1)$*-dimensional projection being an* $(n-1)$*-unit ball.*

2.3.1 *Characterizing sausages and linearity of the volume at one point*

In this section we provide several characterizations of sausages which rely on the linearity of the volume (cf. Proposition 2.3.6) and some additional assumption on common/equal volume projection or maximal volume section through parallel hyperplanes to a given one.

We will prove that the sole assumption of linearity at one point, together with the equal 'size' of a projection or a section, in the already mentioned sense, allows us to characterize sausages.

In general, linearity of the volume at some point $\lambda_0 \in (0,1)$ does not imply linearity of the volume. Indeed, defining the numbers $W_0 = 5$, $W_1 = 4, W_2 = 2$ and $W_3 = 1$, these numbers satisfy inequalities (N.31) and hence, by Corollary 2.3.3, there exist convex bodies $K, E \in \mathcal{K}^3$, such that $W_i(K;E) = W_i$, which yields $f_{K;E}(z) = 5 + 12z + 6z^2 + z^3$. Hence,

$$f_{K;E}(-1) = -2,\ f'_{K;E}(-1) = 3,\ f''_{K;E}(-1) = 6,\ f'''_{K;E}(-1) = 6,$$

and thus, by Lemma 2.3.1, $\mathrm{vol}(\lambda K + (1-\lambda)E) = 1 + 3\lambda + 3\lambda^2 - 2\lambda^3$. Therefore, the volume of $K(\lambda)$ is not linear but satisfies

$$\mathrm{vol}\left(\frac{K+E}{2}\right) = 3 = \frac{1}{2}W_0 + \frac{1}{2}W_3 = \frac{1}{2}\mathrm{vol}(K) + \frac{1}{2}\mathrm{vol}(E),$$

i.e., there is linearity at $\lambda_0 = 1/2$.

Next, we will see that, under the assumptions of common/equal volume projection or maximum volume section, linearity of the volume at some point $\lambda_0 \in (0,1)$ implies linearity of the volume.

Theorem 2.3.9 ([65, Theorem 4.1]). *Let $K, E \in \mathcal{K}^n$ be convex bodies, such that there exists a hyperplane $H \in \mathcal{G}(n, n-1)$ for which it is satisfied that $\mathrm{vol}_{n-1}(K|H) = \mathrm{vol}_{n-1}(E|H)$. Then we have*

$$\mathrm{vol}\big(\lambda K + (1-\lambda)E\big) = \lambda \mathrm{vol}(K) + (1-\lambda)\mathrm{vol}(E) \quad \text{for all } \lambda \in [0,1],$$

if and only if either K and E lie in parallel hyperplanes, or the pair K, E is a sausage.

Proof. On account of Remark 2.3.5, we may assume, without loss of generality, that $\mathrm{vol}(K) > \mathrm{vol}(E)$ and also, that $\mathrm{vol}_{n-1}(K|H) > 0$ (otherwise we would have $\mathrm{vol}(K) = \mathrm{vol}(E) = 0$).

Because of the linearity of the volume and by means of (2.8), we have that $f_{K;E}^{(n-1)}(-1)/(n-1)! = \mathrm{vol}(K) - \mathrm{vol}(E)$ and $f_{K;E}^{(n-j)}(-1) = 0$ for all $j \geqslant 2$, and thus,

$$f_{K;E}(z) = \mathrm{vol}(E)(z+1)^n + (\mathrm{vol}(K) - \mathrm{vol}(E))(z+1)^{n-1}.$$

We define $\ell = (\mathrm{vol}(K) - \mathrm{vol}(E))\,/\mathrm{vol}_{n-1}(K|H) > 0$ and $L = \ell\,[0, u]$, where $u \in \mathbb{S}^{n-1}$ is a normal vector of H. Therefore,

$$\mathrm{vol}(K) = f_{K;E}(0) = \mathrm{vol}(E) + \ell\,\mathrm{vol}_{n-1}(K|H)$$

and

$$\mathrm{vol}(L+E) = nW_{n-1}(L;E) + \mathrm{vol}(E) = \ell\,\mathrm{vol}_{n-1}(E|H) + \mathrm{vol}(E).$$

As a consequence we have

$$\begin{aligned}
\mathrm{vol}(K,\dots,K,L+E)^n &= (\mathrm{vol}(K,\dots,K,L) + W_1(K;E))^n \\
&= \left(\frac{\ell\mathrm{vol}_{n-1}(K|H)}{n} + \frac{n\mathrm{vol}(E) + (n-1)\ell\,\mathrm{vol}_{n-1}(K|H)}{n}\right)^n \\
&= \mathrm{vol}(K)^{n-1}\,\mathrm{vol}(L+E),
\end{aligned}$$

and hence, by the equality case in Minkowski's first inequality (N.21), together with the common volume projection hypothesis, K and $L+E$ are equal (up to translation).

The converse is immediately verified (cf. (2.9)). □

Notice that if $K = L + E$, with $L \in \mathcal{K}^n$ and $\dim L \leqslant 1$, then we have that $K|H = E|H$, where $H = L^{\perp}$. Besides, if K and E lie in parallel hyperplanes H_1 and H_2 then for any $H = u^{\perp}$, where u is a line parallel to H_i, $i = 1, 2$, we have $\mathrm{vol}_{n-1}(K|H) = \mathrm{vol}_{n-1}(E|H) = 0$. Hence, we can assert that the following result holds.

Theorem 2.3.10 ([65, Theorem 4.2]). *Let $K, E \in \mathcal{K}^n$ be convex bodies. Then we have*

$$\begin{aligned}
&\mathrm{vol}\big(\lambda K + (1-\lambda)E\big) = \lambda \mathrm{vol}(K) + (1-\lambda)\mathrm{vol}(E) \quad \text{for all } \lambda \in [0,1], \text{ and} \\
&\mathrm{vol}_{n-1}(K|H) = \mathrm{vol}_{n-1}(E|H) \quad \text{for some hyperplane } H \in \mathcal{G}(n, n-1),
\end{aligned}$$

if and only if either K and E lie in parallel hyperplanes, or the pair K, E is a sausage.

Replacing a common volume projection by a common maximal volume section we obtain the same characterization.

Theorem 2.3.11 ([65, Theorem 4.3]). *Let* $K, E \in \mathcal{K}^n$ *be convex bodies such that there exists a hyperplane* $H \in \mathcal{G}(n, n-1)$ *with*

$$\max_{x\in H^\perp} \mathrm{vol}_{n-1}\big(K \cap (x+H)\big) = \max_{x\in H^\perp} \mathrm{vol}_{n-1}\big(E \cap (x+H)\big).$$

Then we have

$$\mathrm{vol}\big(\lambda K + (1-\lambda)E\big) = \lambda \mathrm{vol}(K) + (1-\lambda)\mathrm{vol}(E) \quad \textit{for all } \lambda \in [0,1],$$

if and only if either K *and* E *lie in parallel hyperplanes, or the pair* K, E *is a sausage.*

Proof. On account of Remark 2.3.5, we may assume, without loss of generality, that $\mathrm{vol}(K) > \mathrm{vol}(E)$.

If we denote by $\nu = \max_{x\in H^\perp} \mathrm{vol}_{n-1}\big(K \cap (x+H)\big)$, then we have that the orthogonal projections onto H of the Schwarz symmetrals of K and E with respect to $H^\perp$, namely, $\sigma_{H^\perp}(K)$, $\sigma_{H^\perp}(E)$, are equal. More precisely, $\big(\sigma_{H^\perp}(K)\big)|H = (\nu/\kappa_{n-1})^{1/(n-1)}B_{n-1} = \big(\sigma_{H^\perp}(E)\big)|H$.

Thus, we can apply Theorem 2.1.2 with the convex bodies $\sigma_{H^\perp}(K)$, $\sigma_{H^\perp}(E)$ which, together with Lemma Q (ii), (iii), yields

$$\begin{aligned}
\mathrm{vol}\big(\lambda K + (1-\lambda)E\big) &= \mathrm{vol}\big(\sigma_{H^\perp}(\lambda K + (1-\lambda)E)\big) \\
&\geqslant \mathrm{vol}\big(\lambda\sigma_{H^\perp}(K) + (1-\lambda)\sigma_{H^\perp}(E)\big) \\
&\geqslant \lambda \mathrm{vol}\big(\sigma_{H^\perp}(K)\big) + (1-\lambda)\mathrm{vol}\big(\sigma_{H^\perp}(E)\big) \\
&= \lambda \mathrm{vol}(K) + (1-\lambda)\mathrm{vol}(E).
\end{aligned}$$

Thus, linearity of the volume for the bodies K, E implies, on the one hand, that

$$\sigma_{H^\perp}\big(\lambda K + (1-\lambda)E\big) = \lambda\sigma_{H^\perp}(K) + (1-\lambda)\sigma_{H^\perp}(E).$$

On the other hand, linearity for the volume of the bodies $\sigma_{H^\perp}(K), \sigma_{H^\perp}(E)$ is also obtained, which, by Theorem 2.3.9, yields

$$\sigma_{H^\perp}(K) = L + \sigma_{H^\perp}(E), \text{ with } \dim L \leqslant 1.$$

Now, the result follows directly from Lemma 2.2.4. □

In order to reduce the assumption on the linearity of the volume for the range $[0, 1]$ to a single point in $(0, 1)$ we need first the following result, where not just equal volume projections are needed, but common projections of K and E.

Lemma 2.3.12 ([65, Lemma 4.1]). *Let $K,E\in\mathcal{K}^n$ be convex bodies such that there exists a hyperplane $H\in\mathcal{G}(n,n-1)$ with $K|H=E|H$. Then we have*

$$\mathrm{vol}\big(\lambda_0 K+(1-\lambda_0)E\big)=\lambda_0\mathrm{vol}(K)+(1-\lambda_0)\mathrm{vol}(E)\quad \textit{for some } \lambda_0\in(0,1),$$

if and only if either K and E lie in parallel hyperplanes, or the pair K,E is a sausage.

Proof. Since $K|H=E|H$, the function $f(t)=\mathrm{vol}(tK+(1-t)E)$ is concave (cf. [46, proof of Theorem 1.1]), which, together with linearity at λ_0, implies that f is an affine function on $[0,1]$ (see Remark B.2 and the lines before). Now, the result follows from Theorem 2.3.9. □

Theorem 2.3.13 ([65, Theorem 1.2]). *Let $K,E\in\mathcal{K}^n$ be such that there exists a hyperplane $H\in\mathcal{G}(n,n-1)$ with $\mathrm{vol}_{n-1}(K|H)=\mathrm{vol}_{n-1}(E|H)$. Then we have*

$$\mathrm{vol}_{K;E}(\lambda_0)=\lambda_0\mathrm{vol}(K)+(1-\lambda_0)\mathrm{vol}(E)\quad \textit{for some } \lambda_0\in(0,1),$$

if and only if either K and E lie in parallel hyperplanes, or the pair K,E is a sausage.

Proof. Without loss of generality (see Remark 2.3.5), we may assume that $\mathrm{vol}(K)>\mathrm{vol}(E)$. Denoting by $\sigma_H,\sigma_{H^\perp}$ the Schwarz symmetrizations with respect to H and $H^\perp$, respectively, and using Lemma Q (iii), we have that

$$\big(\sigma_{H^\perp}(\sigma_H(K))\big)|H=\big(\sigma_{H^\perp}(\sigma_H(E))\big)|H.$$

Therefore, we can apply Theorem 2.1.2 with the convex bodies $\sigma_{H^\perp}(K)$ and $\sigma_{H^\perp}(E)$, which, together with Lemma Q (ii), (iii), yields

$$\begin{aligned}\mathrm{vol}\big(\lambda_0K+(1-\lambda_0)E\big)&=\mathrm{vol}\Big(\sigma_{H^\perp}\big(\sigma_H(\lambda_0K+(1-\lambda_0)E)\big)\Big)\\&\geqslant\mathrm{vol}\Big(\lambda_0\sigma_{H^\perp}\big(\sigma_H(K)\big)+(1-\lambda_0)\sigma_{H^\perp}\big(\sigma_H(E)\big)\Big)\\&\geqslant\lambda_0\mathrm{vol}\Big(\sigma_{H^\perp}\big(\sigma_H(K)\big)\Big)+(1-\lambda_0)\mathrm{vol}\Big(\sigma_{H^\perp}\big(\sigma_H(E)\big)\Big)\\&=\lambda_0\mathrm{vol}(K)+(1-\lambda_0)\mathrm{vol}(E).\end{aligned}\tag{2.10}$$

Thus, linearity of the volume at λ_0 for the bodies K,E is equivalent to the same property for $\sigma_{H^\perp}\big(\sigma_H(K)\big),\sigma_{H^\perp}\big(\sigma_H(E)\big)$ and hence, by Lemma 2.3.12, we obtain

$$\sigma_{H^\perp}\big(\sigma_H(K)\big)=L+\sigma_{H^\perp}\big(\sigma_H(E)\big),\ \text{with } L\subset H^\perp,\quad \dim L=1,\quad \text{and}\tag{2.11}$$

$$\sigma_{H^\perp}\big(\lambda_0\sigma_H(K)+(1-\lambda_0)\sigma_H(E)\big)=\lambda_0\sigma_{H^\perp}\big(\sigma_H(K)\big)+(1-\lambda_0)\sigma_{H^\perp}\big(\sigma_H(E)\big).\tag{2.12}$$

Now, conditions (2.11), (2.12) yield, by Lemma 2.2.4,

$$\sigma_H(K)=L_1+\sigma_H(E),\quad \dim L_1=1,$$

where, from the common/equal volume projection hypothesis, it follows that $L_1 \perp H$.

Therefore, (up to translations) we have

$$K|H = \sigma_H(K)|H = \big(L_1 + \sigma_H(E)\big)|H = \sigma_H(E)|H = E|H,$$

and hence, Lemma 2.3.12 allows us to assert that $K = L_0 + E$ with $\dim L_0 = 1$. □

Theorem 2.3.14 ([65, Theorem 1.3]). *Let $K, E \in \mathcal{K}^n$ be convex bodies such that there exists a hyperplane $H \in \mathcal{G}(n, n-1)$ with*

$$\max_{x \in H^\perp} \mathrm{vol}_{n-1}\big(K \cap (x + H)\big) = \max_{x \in H^\perp} \mathrm{vol}_{n-1}\big(E \cap (x + H)\big).$$

Then we have

$$\mathrm{vol}_{K;E}(\lambda_0) = \lambda_0 \mathrm{vol}(K) + (1 - \lambda_0)\mathrm{vol}(E) \quad \text{for some } \lambda_0 \in (0, 1),$$

if and only if either K and E lie in parallel hyperplanes, or the pair K, E is a sausage.

Proof. On account of Remark 2.3.5, we may assume, without loss of generality, that $\mathrm{vol}(K) > \mathrm{vol}(E)$.

By reasoning like in (2.10) with the convex bodies $\sigma_{H^\perp}(K), \sigma_{H^\perp}(E)$, and by Lemma 2.3.12, we get that $\sigma_{H^\perp}(K)$ is a sausage with respect to $\sigma_{H^\perp}(E)$, and that $\sigma_{H^\perp}(\lambda_0 K + (1 - \lambda_0)E) = \lambda_0 \sigma_{H^\perp}(K) + (1 - \lambda_0)\sigma_{H^\perp}(E)$. Hence, by Lemma 2.2.4, we may conclude that K is also a sausage with respect to E. □

To end this section, we show that if we assume linearity at some point $\lambda_0 \in (0, 1)$ for all quermassintegrals, then all of them are linear, in the same sense we have understood linearity, here, for $\mathrm{vol}_{K;E}(\lambda)$.

Proposition 2.3.15 ([65, Proposition 4.1]). *If there exists $\lambda_0 \in (0, 1)$ such that*

$$W_i(K(\lambda_0); E) = \lambda_0 W_i(K; E) + (1 - \lambda_0) W_i(E; E), \ \text{for all } i = 0, \ldots, n-2,$$

then $W_i\,(K(\lambda); E)$ is linear for all $i = 0, \ldots, n$.

Proof. We will prove the result by induction on $j = n - i$.

If $j = 2$, the result follows from the fact that $W_{n-2}(\lambda K + (1 - \lambda)E; E)$ is a polynomial of degree at most two, which coincides with the polynomial $\lambda W_{n-2}(K; E) + (1 - \lambda)W_{n-2}(E; E)$ at (at least) the points $0, \lambda_0, 1$, and hence they are really the same polynomial.

Now we assume $2 < j + 1 \leqslant n$ and that the result is true for j, i.e., $W_{n-j}\,(K(\lambda); E) = \lambda W_{n-j}(K; E) + (1 - \lambda)W_{n-j}(E; E)$ for all $\lambda \in [0, 1]$. Then, by Lemma 2.3.1, we have that

$$f_{K;E}^{(n-j)}(-1) = \cdots = f_{K;E}^{(n-2)}(-1) = 0,$$

and so, $W_{n-j-1}(K(\lambda); E) = a + b\lambda + c\lambda^{j+1}$. From the identities at $0, \lambda_0$ and 1, it follows

$$\begin{aligned} a + \lambda_0(b + c) &= W_{n-j-1}(E; E) + \lambda_0\big(W_{n-j-1}(K; E) - W_{n-j-1}(E; E)\big) \\ &= W_{n-j-1}(K(\lambda_0); E) = a + b\lambda_0 + c\lambda_0^{j+1}, \end{aligned}$$

which yields that $c = 0$ and thus, $W_{n-j-1}(K(\lambda); E)$ is linear. This concludes the proof. □

Remark 2.3.16 ([65, Remark 4.1]). *In the case in which K and E are both n-dimensional convex bodies, Theorems 2.3.13 and 2.3.14 follow from [9, Theorems 1.4 and 1.5].*

2.4 LINEARITY OF THE DETERMINANT

In this section, we show the characterization of *linearity of the determinant* -in the same sense as for the volume function $\mathrm{vol}_{K;E}$- of positive definite symmetric matrices via 'sausages' of matrices, i.e., the sum of a matrix of rank (at most) 1 and another matrix. Notice that like for $\mathrm{vol}_{K;E}(\lambda)$, where for $\lambda \notin [0, 1]$ we lose the geometry, for positive definite symmetric matrices, we would lose the positivity if we let λ run outside $[0, 1]$.

The Brunn-Minkowski inequality has also its counterpart for matrices (see e.g. [34]).

Theorem 2.4.1 ([34]). *Let A, B be positive definite symmetric $n \times n$ matrices. Then*

$$\det(A + B)^{1/n} \geqslant \det(A)^{1/n} + \det(B)^{1/n}.$$

However, conditions for positive definite symmetric matrices A, B to fulfill a result of the type of Theorem 2.1.2 are not known to the author. Of course, assumptions on common/equal volume projection onto a hyperplane (or maximal volume sections through parallel hyperplanes to a given one) of the parallelepipeds whose volume is given by the determinants of A and B are enough (for the volume of the convex combination of those parallelepipeds). Nevertheless, it cannot be read in terms of the determinant of $\lambda A + (1 - \lambda)B$. For further information on these topics see, e.g., [1] and the references inside.

We first prove the following property for diagonal matrices.

Proposition 2.4.2 ([65, Proposition 5.1]). *(Linearity case for orthogonal boxes) Let $A, B \in \mathbb{R}^{n \times n}$ be diagonal matrices. Then*

$$\det(\lambda A + (1 - \lambda)B) = \lambda \det(A) + (1 - \lambda) \det(B),$$

if and only if $B = L + A$, where L is a diagonal matrix such that $\operatorname{rank} L \leqslant 1$.

Proof. Let $A = \mathrm{diag}(\lambda_1 + \varepsilon_1, \dots, \lambda_n + \varepsilon_n)$, where $B = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ and $(\varepsilon_i)_{i=1}^n \subset \mathbb{R}$. Then, for all $\lambda \in [0, 1]$, we have

$$\begin{aligned}\prod_{i=1}^{n}(\varepsilon_i\lambda + \lambda_i) &= \det(\lambda A + (1-\lambda)B)\\ &= \lambda \det(A) + (1-\lambda)\det(B)\\ &= \prod_{i=1}^{n}\lambda_i + \lambda\left(\prod_{i=1}^{n}(\varepsilon_i + \lambda_i) - \prod_{i=1}^{n}\lambda_i\right).\end{aligned}$$

Comparing the coefficients of both polynomials in λ, we get that the set $\{1 \leqslant i \leqslant n : \varepsilon_i \neq 0\}$ has at most one element, which implies that at least $n-1$ of the ε_i vanish. It concludes the proof. □

From this result, it immediately follows:

Corollary 2.4.3 ([65, Corollary 5.1]). *Let* $K, E \in \mathcal{K}^n$ *be orthogonal boxes with parallel edges. Then*

$$\mathrm{vol}(\lambda K + (1-\lambda)E) = \lambda \mathrm{vol}(K) + (1-\lambda)\mathrm{vol}(E),$$

if and only if $K = L + E$, *with* $\dim L \leqslant 1$, *i.e., the pair* K, E *is a sausage.*

Theorem 2.4.4 ([65, Theorem 5.1]). *Let* $A, B \in \mathbb{R}^{n\times n}$ *be positive definite (symmetric) matrices. Then*

$$\det(\lambda A + (1-\lambda)B) = \lambda \det(A) + (1-\lambda)\det(B),$$

if and only if $B = L + A$, *with* $\mathrm{rank}\, L \leqslant 1$.

Proof. Let $T \in \mathbb{R}^{n\times n}$ be an orthogonal matrix such that the matrix T^tAT is the diagonal matrix $\mathrm{diag}(a_1, \dots, a_n)$, where $a_i > 0$ are the eigenvalues of A. With $\widetilde{T} = T\,\mathrm{diag}\left(1/\sqrt{a_1}, \dots, 1/\sqrt{a_n}\right)$, we get that $\widetilde{T}^tA\widetilde{T} = In$, with In the identity matrix.

Since $\widetilde{T}^tB\widetilde{T}$ is positive definite and symmetric, there exists an orthogonal matrix $S \in \mathbb{R}^{n\times n}$ such that $S^t\widetilde{T}^tB\widetilde{T}S = \mathrm{diag}(y_1, \dots, y_n)$, with $y_i > 0$, and it is

$$\prod_{i=1}^{n} y_i = \det(B)\det(\widetilde{T}^t)\det(\widetilde{T}) = \frac{\det(B)}{\det(A)}.$$

Therefore,

$$S^t\widetilde{T}^t(\lambda A + (1-\lambda)B)\widetilde{T}S = \mathrm{diag}(\lambda + (1-\lambda)y_1, \dots, \lambda + (1-\lambda)y_n)$$

and hence,

$$\begin{aligned}\det\big(\lambda In + (1-\lambda)\mathrm{diag}(y_1, \dots, y_n)\big) &= \det\big(S^t\widetilde{T}^t(\lambda A + (1-\lambda)B)\widetilde{T}S\big)\\ &= \frac{1}{\det(A)}\det(\lambda A + (1-\lambda)B)\\ &= \lambda + (1-\lambda)\frac{\det(B)}{\det(A)}\\ &= \lambda \det(In) + (1-\lambda)\det\left(\mathrm{diag}(y_1, \dots, y_n)\right).\end{aligned}$$

From Proposition 2.4.2, i.e., linearity of the determinant for diagonal matrices, we have that $\mathrm{diag}(y_1, \ldots, y_n) = L_1 + In$, with $\mathrm{rank}\, L_1 \leqslant 1$, or equivalently $S^t \widetilde{T}^t B \widetilde{T} S = L_1 + S^t \widetilde{T}^t A \widetilde{T} S$. From that, it follows that $B = P L_1 Q + A$, where P and Q are invertible matrices, which implies that $L = P L_1 Q$ has rank at most 1. Indeed, since B is symmetric, L will be of the form $\mu\, u\, u^t$ for some $u \in \mathbb{R}^n$ of length one and $\mu \in \mathbb{R}$. $\square$

Part II

INNER PARALLEL BODIES: OLD AND NEW

PROLOGUE TO PART II

In this second part of the work, we focus on decompositions of convex bodies via their inner parallel bodies.

Assertions on functions defined on the one-parameter family of inner parallel bodies play an important role in several classical proofs in Convexity (e.g., differentiability assertions with respect to the parameter of a system of inner parallel bodies [68, p. 415]). Indeed, reproducing words of Schneider [68, Section 7.5]: "some of the deeper investigations of inequalities for mixed volumes and of the equality cases make essential use of the method of inner parallel bodies".

Inner parallel bodies and their properties have been studied in [6, 16, 17, 31, 32, 62] among others. Although it seems restrictive to ask for a decomposition of a convex body by means of its inner parallel bodies, it is *geometrically natural*, since inner parallel bodies of a convex body K share with it several geometrical properties and provide us with a one-parameter family of convex bodies *close* to K in different senses.

In order to understand the possibility of decomposing a convex body using its inner parallel bodies, some special issues about the boundary structure of the convex body (and of its inner parallel bodies) will play an important role. For K ,E convex bodies, using (the closure of) the outer normal vectors at regular points of K, the so-called form body of K, with respect to E, is constructed. This convex body, in general ,different from both, K and E, happens to be, in some sense, geometrically connected to the inner parallel bodies of K with respect to E. Indeed, it will be fundamental for most of the decomposition results throughout Chapter 3.

Along the way in Chapter 3, we study certain differentiability properties of the quermassintegrals of the inner parallel bodies of $K \in \mathcal{K}^n$, relative to a fixed gauge body $E \in \mathcal{K}^n$, with respect to the parameter of definition of the inner parallel bodies. More precisely, some differentiability properties of the functions $\lambda \mapsto W_i(K_\lambda; E)$, $1 \leqslant i \leqslant n-1$, are used to classify convex bodies in classes. This classification, in the case of $\mathbb{R}^3$ and $E = B_3$, goes back to Bol [6], who asked for a (geometrical) description of the convex bodies lying in each of these classes. The complete description of all the classes in the general case, i.e., $\mathbb{R}^n$ and arbitrary E remains open. However, in Chapter 3 we provide a complete answer to the original question in [6] and fully describe one of the non-trivial classes in the general case.

We consider also decompositions of polytopes by their inner parallel bodies when the gauge body E is the Euclidean unit ball. In this case, we can use some *more effective* criteria for summands (see e.g.

[68, page 163]), obtaining results which slightly improve the ones we obtain for general convex bodies K, E. In particular, in the third section of Chapter 3, where we restrict to polytopes and $E = B_n$, we will prove that, strongly based on the defining finiteness of the boundary structure of polytopes, some characterization results about decompositions of convex bodies via inner parallel bodies, which are still open for general convex bodies, can be proven. We characterize the polytopes P satisfying that P_λ is a summand of P_μ with $\lambda < \mu \leqslant 0$, providing an explicit decomposition of those polytopes involving the form bodies of their inner parallel bodies.

The second chapter of this part of the work, Chapter 4, focusses on "applications" of inner parallel bodies. As mentioned in the introduction, an extension of Bonnesen-Blaschke's (inradius) inequality (N.24) to higher dimensions was conjectured by Wills [72] and proven simultaneously by Bokowski [5] and Diskant [20] for $E = B_n$. Later, Sangwine-Yager [63] proved the Wills conjecture inequality

$$\mathrm{vol}(K) - n r(K;E) W_1(K;E) + (n-1) r(K;E)^n \mathrm{vol}(E) \leqslant 0,$$

for a general gauge body E with interior points. In the more general inequality proved by Sangwine-Yager in [63], sufficient conditions for equality were also provided. These conditions are strongly related to the decomposition of the convex body K in terms of its inner parallel bodies and the form body (with respect to E). These results have been strengthened in [11] and [43], by means of considering further aspects of the convex body K and its inner parallel bodies. Indeed, for the proofs of many of those results, inner parallel bodies play a crucial role. In Chapter 4, we prove sharp inequalities for the volume of the inner parallel bodies of a convex body, involving mixed volumes of those, and also, inequalities which relate a convex body to its inner parallel bodies, its kernel, and its form body. The proofs of these inequalities use the *technique of inner parallel bodies*, namely, integration of some Steiner polynomial type inequalities of mixed volumes involving the appropriate convex bodies. We show, in particular, that equality conditions rely on the decomposition of the convex body through its kernel. Moreover, we obtain further refinements of the Wills conjecture inequality:

$$\mathrm{vol}(K) \leqslant n W_1(K;E) r(K;E) - n \sum_{k=0}^{n-2} W_{k+2}(K;E) \frac{r(K;E)^{k+2}}{(k+1)(k+2)},$$

for $E \in \mathcal{K}^n_n$, and $K \in \mathcal{K}^n$.

Finally, we introduce the selector $\kappa : \mathcal{K}^n_n \longrightarrow \mathbb{R}^n$, defined by means of the "subsequent kernels". The map κ associates to every convex body a point in the kernel of it, with respect to the Euclidean ball. In other words, it selects the center of one of the largest balls contained in a convex body. We study some properties of this selector, such as additivity and continuity.

3 DECOMPOSITION VIA INNER PARALLEL BODIES

The most subversive people are those who ask questions.
"Sophie's World", J. Gaarder

3.1 INNER PARALLEL BODIES, FORM BODIES AND DECOMPOSITIONS

As already introduced in the Notation and Basics part, the Minkowski difference of $K \in \mathcal{K}^n$ and an appropriate positive dilation of $E \in \mathcal{K}^n$ constitutes the so-called inner parallel bodies of K with respect to E. More precisely, for two convex bodies K, E and $0 \leqslant \lambda \leqslant r(K;E)$ the inner parallel body of K (relative to E) at distance λ is the set

$$K \sim \lambda E = \{x \in \mathbb{R}^n : \lambda E + x \subset K\}.$$

The convex body $K \sim r(K;E)E$ is the set of relative incenters of K, usually called *kernel* of K with respect to E and denoted by $\ker(K;E)$. The dimension of $\ker(K;E)$ is strictly less than n (see (N.8)). If $\lambda = 0$ the original body K is obtained. As described by (N.7), the family of (relative) inner and outer parallel bodies can be defined by a single parameter λ, as follows:

$$K_\lambda := \begin{cases} K \sim |\lambda|\, E & \text{for } -r(K;E) \leqslant \lambda \leqslant 0, \\ K + \lambda E & \text{for } 0 \leqslant \lambda < \infty. \end{cases} \tag{3.1}$$

It is natural to ask whether, with an appropriate fixed *gauge body* E, there is a convex body K, such that its inner parallel bodies with respect to E are homothetic to K. Tangential bodies provide us with a positive answer to this question. Indeed, inner parallel bodies and tangential bodies (see Appendix A for details) happen to be intrinsically connected by means of this matter. The following result illustrates the close connection between inner parallel bodies and tangential bodies, answering the above question. We notice, that if K is a tangential body of E, then $r(K;E) = 1$.

Theorem 3.1.1 ([68, Lemma 3.1.14]). *Let $K, E \in \mathcal{K}^n_n$ be convex bodies, and let $\lambda \in \big(-r(K;E), 0\big)$. Then K_λ is homothetic to K if and only if K is homothetic to a tangential body of E.*

It is not surprising that both, inner parallel bodies, and tangential bodies, play an important role in the decomposition results presented in this chapter. There is a vast amount of literature dealing with (different aspects of) decomposition of convex bodies. For a description of the situation we refer to [68, Section 3.2] and the references therein.

Let us fix a gauge body $E \in \mathcal{K}^n$. Our aim in this chapter is to investigate decompositions of a convex body K via its parallel bodies

K_λ, $-r(K;E) \leqslant \lambda \leqslant 0$. From now on we will write $r = r(K;E)$ for the sake of brevity, unless the convex bodies K, E are not clear from the context. Let $K, L \in \mathcal{K}^n$. We recall that the convex body L is called a summand of K if there exists $M \in \mathcal{K}^n$, such that $K = L + M$. We notice that, due to the relation $(K \sim L) + L \subseteq K$, the question whether K_λ is a summand of K, arises naturally. A summand $L \in \mathcal{K}^n$ of a convex body $K \in \mathcal{K}^n$ is said to be *trivial* if L is homothetic to K, i.e., it is a (possibly translated) dilatate of K.

Given a convex body $K \in \mathcal{K}^n$, its inner parallel bodies strongly depend on the geometry of the gauge body. In particular, we will often need to assume that the convex body E, with respect to which the inner parallel bodies are taken, is strictly convex and regular, i.e., its boundary, $\mathrm{bd}\, E$, does not contain a line segment, and the supporting hyperplane to E at any boundary point is unique.

Indeed, the boundary structure of the convex bodies involved in the decomposition plays a crucial role in the next. More precisely, we will need the following classification of support planes of a convex body.

A vector $u \in \mathbb{S}^{n-1}$ is an r-extreme normal vector of the convex body K, $0 \leqslant r \leqslant n-1$, if it cannot be written as $u = u_1 + \cdots + u_{r+2}$, with u_i linearly independent normal vectors at one and the same boundary point of K. We write $\mathcal{U}_r(K)$ to denote the set of r-extreme normal vectors of K and notice that, for any $0 \leqslant r < s \leqslant n-1$, $\mathcal{U}_r(K) \subseteq \mathcal{U}_s(K)$. Then a support plane is said to be r-extreme if its outer normal vector is r-extreme.

The following characterization of extreme normal vectors in terms of the support function of K will be needed later on.

Lemma 3.1.2 ([62, Lemma 2.3])**.** *Let $K \in \mathcal{K}^n$, and let $u \in \mathbb{S}^{n-1}$ be an outer normal vector to K. Then $u \in \mathcal{U}_0(K)$ if and only if for any distinct vectors $u_1, u_2 \in \mathbb{S}^{n-1}$ and $\alpha, \beta > 0$ such that $u = \alpha u_1 + \beta u_2$,*

$$h(K, u) < \alpha h(K, u_1) + \beta h(K, u_2).$$

In the next lemma, we collect some properties relating extreme normal vectors of a Minkowski sum and difference of convex bodies to the extreme normal vectors of the involved convex bodies. We refer to [62, Lemma 2.4, Lemma 4.5]) and [43, Lemma 3.1, Lemma 3.2] for the proofs.

Lemma 3.1.3 ([43, 62])**.** *Let $K, L \in \mathcal{K}^n$. The following facts hold:*

(i) $\mathcal{U}_0(K) \cup \mathcal{U}_0(L) \subseteq \mathcal{U}_0(K + L)$.

(ii) $\mathcal{U}_0(K + L) = \mathcal{U}_0(K + \mu L), \quad \mu > 0$.

(iii) If $L \in \mathcal{K}^n_n$, *then* $\mathcal{U}_0(K \sim L) \subseteq \mathcal{U}_0(K)$.

(iv) If $L \in \mathcal{K}^n_n$, *then* $\mathcal{U}_0(K_\lambda) \subseteq \mathcal{U}_0(K)$ *for* $-r(K;L) < \lambda \leqslant 0$.

Next, we introduce a useful tool for the decomposition analysis we aim to carry out in this chapter, the so-called *form body* of a convex body.

The (relative) *form body* of a convex body $K \in \mathcal{K}^n_n$, *with respect to* $E \in \mathcal{K}^n_n$, denoted by K^*, is defined as (see e.g. [17])

$$K^* = \bigcap_{u \in \mathcal{U}_0(K)} \bigl\{x : \langle x, u\rangle \leqslant h(E,u)\bigr\}, \tag{3.2}$$

and it can also be constructed with the closure of the set of outer unit normal vectors at regular boundary points (cf. Remark P and [68, p. 386]), i.e.,

$$K^* = \bigcap_{u \in \Omega} \bigl\{x : \langle x, u\rangle \leqslant h(E,u)\bigr\}, \tag{3.3}$$

where $\Omega = \mathrm{cl}\bigl\{u \in N_K(x) : x \in \mathrm{bd}\, K \text{ regular}\bigr\}$. Notice that K^* depends also on the fixed convex body E. Nevertheless, and for the sake of simplicity, we again omit E in the notation, unless the convex body E is not clear from the context.

Notice that the form body K^* of a convex body K is always a tangential body of E. This allows us to restate Theorem 3.1.1 as follows:

Theorem 3.1.4 ([68, Lemma 3.1.14]). *Let $K, E \in \mathcal{K}^n_n$ and $\lambda \in (-r(K;E), 0)$. Then K_λ is homothetic to K if and only if K is homothetic to K^*.*

Conditions for extreme normal vectors will often appear in our results on decompositions of convex bodies.

The following result connecting the support of the area measures of K and the extreme unit normal vectors will be of great utility in this chapter.

Theorem 3.1.5 ([68, Theorem 4.5.3], [67, pp. 135-136]). *Let $K \in \mathcal{K}^n$, and let $m \in \{0, \dots, n-1\}$. The support of the m-th area measure $S_m(K;\cdot)$ is the closure of the set of all $(n-1-m)$-extreme unit normal vectors of K. If $E \in \mathcal{K}^n$ is a regular and strictly convex body, then for all $i = 0, \dots, n-1$*

$$\mathrm{supp}\, S\bigl(K[n-i-1], E[i]; \cdot\bigr) = \mathrm{cl}\,\mathcal{U}_i(K).$$

Next we show a property similar to Lemma 3.1.3 (ii), now, regarding the $(n-2)$-extreme normal vectors of the Minkowski sum.

Lemma 3.1.6 ([45, Lemma 2.1]). *Let $K, L \in \mathcal{K}^n$. Then, for any $\mu > 0$, the following equality holds*

$$\mathrm{cl}\,\mathcal{U}_{n-2}(K + \mu L) = \mathrm{cl}\,\mathcal{U}_{n-2}(K) \cup \mathrm{cl}\,\mathcal{U}_{n-2}(L).$$

Proof. Let $E \in \mathcal{K}^n$ be a regular and strictly convex body. Then, using Theorem 3.1.5, we can assert that for any convex body $K \in \mathcal{K}^n$ and for $i = 0, \dots, n-1$,

$$\mathrm{supp}\, S\bigl(K[n-i-1], E[i]; \cdot\bigr) = \mathrm{cl}\,\mathcal{U}_i(K). \tag{3.4}$$

Here $\operatorname{supp}\nu$ denotes the support of the measure ν. Therefore, in particular, $\mathrm{cl}\,\mathcal{U}_{n-2}(K+\mu L) = \operatorname{supp} S\big(K+\mu L, E[n-2];\cdot\big)$. The linearity of the area measure (N.16) in each argument, i.e., here,

$$S\big(K+\mu L, E[n-2];\cdot\big) = S\big(K, E[n-2];\cdot\big) + \mu S\big(L, E[n-2];\cdot\big),$$

allows us to conclude that

$$\begin{aligned}\operatorname{supp} S\big(K{+}\mu L, E[n-2];\cdot\big) \\ &= \operatorname{supp} S\big(K, E[n-2];\cdot\big) \cup \operatorname{supp} S\big(L, E[n-2];\cdot\big) \\ &= \mathrm{cl}\,\mathcal{U}_{n-2}(K) \cup \mathrm{cl}\,\mathcal{U}_{n-2}(L),\end{aligned}$$

as required. □

Let $K, E \in \mathcal{K}^n_n$. Since the (relative) form body of K is defined via the 0-extreme normal vectors of K, it is natural to ask whether there exists a connection between the extreme normal vectors of K and those of K^*. The following result can be found in [62, Lemma 4.6]. The equality case was treated in [45, Lemma 2.1].

Lemma 3.1.7 ([62, Lemma 4.6], [42, Lemma 2.1])**.** *Let $K, E \in \mathcal{K}^n_n$. Then*

$$\mathrm{cl}\,\mathcal{U}_0(K) \supseteq \mathcal{U}_0(K^*).$$

If E is regular, then there is equality for any $K \in \mathcal{K}^n_n$.

In [62, Lemmata 4.3 and 4.8] the following statement, relating a convex body, its inner parallel bodies, and the form body was proven.

Proposition 3.1.8 ([62, Lemmata 4.3 and 4.8])**.** *Let $K \in \mathcal{K}^n$, $E \in \mathcal{K}^n_n$, and let $-r < \lambda \leqslant 0$. Then*

(i) $K \supseteq K_\lambda + |\lambda|\, K^*$,

(ii) $h(K_\lambda, u) = h(K, u) - |\lambda|\, h(E, u)$ *for every* $u \in \mathcal{U}_0(K_\lambda)$.

Aiming to decompose convex bodies via inner parallel bodies, it is practical to observe, that most relations of the type contained in Proposition 3.1.8, which hold for $-r < \lambda \leqslant 0$, remain true for $\lambda = -r$. In particular, we will use the following remark.

Remark 3.1.9 ([45, Lemma 2.2])**.** *Let $K \in \mathcal{K}^n$, and let $E \in \mathcal{K}^n_n$. If for every $-r < \lambda \leqslant 0$, $K = K_\lambda + |\lambda|\, K^*$, then equality $K = K_{-r} + rK^*$ also holds true.*

There exist several relations between inner parallel bodies, form bodies and extreme normal vectors. One of them arises through the so-called Riemann-Minkowski integral (see [18] and [62, Lemma 3.2]).

We will write $K^*_\lambda = (K_\lambda)^*$ to denote the form body of the inner parallel body of K at distance $|\lambda|$, $-r < \lambda \leqslant 0$; notice that K^*_{-r} can be unbounded or empty.

For a convex body K with inradius r, the Riemann-Minkowski integral of K^*_λ in $-r \leqslant \lambda \leqslant 0$, denoted by $\int_{-r}^{0} K^*_\lambda d\lambda$, is the convex body whose support function is given by

$$h\left(\int_{-r}^{0} K^*_\lambda d\lambda, u\right) = \int_{-r}^{0} h(K^*_\lambda, u)d\lambda, \quad \text{for all } u \in \mathbb{S}^{n-1}.$$

For further details about the Riemann-Minkowski integral in this particular context, we refer to [19].

Sangwine-Yager [62] proved the following result, improving Proposition 3.1.8.

Theorem 3.1.10 ([62, Corollary to Lemma 4.8 and Lemma 4.9]). *Let $K \in \mathcal{K}^n$ and let $E \in \mathcal{K}^n_n$. Then*

$$\frac{d}{d\lambda} h(K_\lambda, u) \geqslant h(K^*_\lambda, u), \tag{3.5}$$

for all $u \in \mathbb{S}^{n-1}$ and a.e. in $[-r, 0]$, and

$$K \supseteq K_{-r} + \int_{-r}^{0} K^*_\lambda d\lambda. \tag{3.6}$$

If for all $-r < \lambda \leqslant 0$, equality

$$\mathrm{cl}\,\mathcal{U}_0(K_\lambda) = \mathrm{cl}\,\mathcal{U}_0(K_\lambda + K^*_\lambda) \tag{3.7}$$

holds, then:

(i) $K \subseteq K_\lambda + |\lambda|\, K^*_\lambda$,

(ii) $\dfrac{d}{d\lambda} h(K_\lambda, u) = h(K^*_\lambda, u)$ *for every* $u \in \mathbb{S}^{n-1}$,

(iii) $K = K_{-r} + \int_{-r}^{0} K^*_\lambda d\lambda$,

for all $-r < \lambda \leqslant 0$.

However, a complete characterization for the equality cases in (3.5) and (3.6) is, to the best of the author knowledge, not known. For $n = 2$, it is known that condition (3.7) holds for all planar convex bodies.

Remark 3.1.11. *Notice that integration of the expression in Theorem 3.1.10 (ii) states that, if condition (3.7) holds for all $\lambda \in (-r, 0]$, then*

$$K = K_\lambda + \int_{\lambda}^{0} K^*_\mu \, d\mu;$$

for any $\lambda \in [-r, 0]$.

We state next a decomposition result for a convex body $K \in \mathcal{K}^n$, involving inner parallel bodies, and the form body of K. This was proven in [43, Theorem 2.2], and provides a characterization of the convex bodies $K \in \mathcal{K}^n_n$ which satisfy that $K = K_\lambda + |\lambda|\, K^*$ for every $-r \leqslant \lambda \leqslant 0$.

Theorem 3.1.12 ([43, Theorem 2.2]). *Let $K, E \in \mathcal{K}^n_n$, and let E be regular. Then $K = K_\lambda + |\lambda|\, K^*$ for every $-r \leqslant \lambda \leqslant 0$ if and only if K is a tangential body of $K_{-r} + rE$ satisfying that, for all $-r \leqslant \lambda \leqslant 0$,*

$$\mathcal{U}_0(K) = \mathcal{U}_0(K_\lambda + K^*). \tag{3.8}$$

We notice that condition (3.8) does not involve the form body of the inner parallel bodies of K, but just the inner parallel bodies, and the form body of K, unlike (3.7). Indeed, if condition (3.8) is satisfied, using Lemma 3.1.3 (i) and Lemma 3.1.7 we easily get

$$\begin{aligned}\mathcal{U}_0(K) &= \mathcal{U}_0(K_\lambda + K^*) \supseteq \mathcal{U}_0(K_\lambda) \cup \mathcal{U}_0(K^*)\\ &= \mathcal{U}_0(K_\lambda) \cup \mathrm{cl}\,\mathcal{U}_0(K) \supseteq \mathrm{cl}\,\mathcal{U}_0(K),\end{aligned}$$

which implies that $\mathcal{U}_0(K) = \mathrm{cl}\,\mathcal{U}_0(K)$, i.e., $\mathcal{U}_0(K)$ is closed. Thus, the last theorem can be rewritten using

$$\mathrm{cl}\,\mathcal{U}_0(K) = \mathcal{U}_0(K_\lambda + K^*) \tag{3.9}$$

in the following way.

Theorem 3.1.13 ([43, Theorem 2.2]). *Let $K, E \in \mathcal{K}^n_n$ with E regular. Then $K = K_\lambda + |\lambda|\, K^*$ for every $-r \leqslant \lambda \leqslant 0$ if and only if K is a tangential body of $K_{-r} + rE$ satisfying condition (3.9) for all $-r \leqslant \lambda \leqslant 0$.*

Remark 3.1.14 ([43, Proof of Theorem 2.2]). *If $K, E \in \mathcal{K}^n_n$ are such that $K = K_{-r} + rK^*$, then the inner parallel bodies of K relative to E do inherit such a decomposition, namely, $K_\lambda = K_{-r} + (r+\lambda)K^*$ for $-r \leqslant \lambda \leqslant 0$.*

The following lemma shows the "almost" equivalence between condition $K = K_\lambda + |\lambda|\, K^*$ and certain linearity of the family $\{K_\lambda\}_{-r\leqslant\lambda\leqslant 0}$. Notice that condition (3.9) plays a crucial role.

Lemma 3.1.15 ([45, Lemma 2.5]). *Let $E \in \mathcal{K}^n_n$ be a regular convex body, and let $K \in \mathcal{K}^n$. Then $K = K_\lambda + |\lambda|\, K^*$ for any $-r \leqslant \lambda \leqslant 0$ if and only if*

$$K_\lambda = \frac{|\lambda|}{r}K_{-r} + \left(1 - \frac{|\lambda|}{r}\right)K, \tag{3.10}$$

for every $-r \leqslant \lambda \leqslant 0$, and condition (3.9) holds.

Proof. First we assume that $K = K_\lambda + |\lambda|\, K^*$. Then from Theorem 3.1.13 we get condition (3.9). Further, equality $K_\lambda = K_{-r} + \big(r - |\lambda|\big)K^*$ holds for any $-r \leqslant \lambda \leqslant 0$. Thus,

$$\begin{aligned}\frac{|\lambda|}{r}K_{-r} + \left(1 - \frac{|\lambda|}{r}\right)K &= \frac{|\lambda|}{r}K_{-r} + \left(1 - \frac{|\lambda|}{r}\right)\left(K_\lambda + |\lambda|\, K^*\right)\\ &= \frac{|\lambda|}{r}K_{-r} + \left(\frac{r-|\lambda|}{r}\right)K_\lambda + \frac{|\lambda|}{r}\big(r-|\lambda|\big)K^*\\ &= \frac{|\lambda|}{r}\Big[K_{-r} + \big(r-|\lambda|\big)K^*\Big] + \frac{r-|\lambda|}{r}K_\lambda\\ &= \frac{|\lambda|}{r}K_\lambda + \frac{r-|\lambda|}{r}K_\lambda = K_\lambda.\end{aligned}$$

Conversely, now we assume (3.9) and (3.10). Using Lemma 3.1.3 (i), (ii), and (iv), we get, from (3.10), that

$$\mathcal{U}_0(K_\lambda) \supseteq \mathcal{U}_0(K_{-r}) \cup \mathcal{U}_0(K) \supseteq \mathcal{U}_0(K) \supseteq \mathcal{U}_0(K_\lambda),$$

for every $-r < \lambda \leqslant 0$, i.e., $\mathcal{U}_0(K_\lambda) = \mathcal{U}_0(K)$.

On the other hand, using Proposition 3.1.8, we have, for $u \in \mathcal{U}_0(K_\lambda)$, that $h(K_\lambda, u) = h(K, u) - |\lambda|\, h(E, u)$. This fact implies that for every $u \in \mathcal{U}_0(K) = \mathcal{U}_0(K_\lambda)$ we have

$$h(K, u) - |\lambda|\, h(E, u) = h(K_\lambda, u) = \frac{|\lambda|}{r} h(K_{-r}, u) + \left(1 - \frac{|\lambda|}{r}\right) h(K, u),$$

for any $-r < \lambda \leqslant 0$, i.e.,

$$\frac{|\lambda|}{r} h(K, u) = \frac{|\lambda|}{r} h(K_{-r}, u) + |\lambda|\, h(E, u),$$

which leads to

$$h(K, u) = h(K_{-r}, u) + r\, h(E, u) = h(K_{-r} + rE, u)$$

for every $u \in \mathcal{U}_0(K)$. It proves that K is a tangential body of $K_{-r} + rE$ (see Remark A.2) satisfying, by hypothesis, condition (3.9). Theorem 3.1.13 allows us to conclude that $K = K_\lambda + |\lambda|\, K^*$. □

As a by-product, we obtain that if a convex body satisfies, for all $-r \leqslant \lambda \leqslant 0$, that $K = K_\lambda + |\lambda|\, K^*$, then not only has K all its inner parallel bodies as summands, but also (up to a dilatation) K itself is a summand of any of its inner parallel bodies K_λ, for $-r < \lambda \leqslant 0$.

3.2 A PARTICULAR DECOMPOSITION VIA INNER PARALLEL BODIES: THE CLASS $\mathcal{R}_{n-2}$

In [42] the following definition was introduced.

Definition 3.2.1 ([42, Definition 1.1]). *Let $E \in \mathcal{K}^n_n$ and let p be an integer, $0 \leqslant p \leqslant n-1$. A convex body $K \in \mathcal{K}^n$* belongs to the class $\mathcal{R}_p$ *if, for all $0 \leqslant i \leqslant p$ and for $-r \leqslant \lambda < \infty$, the following equality holds:*

$$\frac{d^-}{d\lambda} W_i(\lambda) = \frac{d^+}{d\lambda} W_i(\lambda) = (n-i) W_{i+1}(\lambda). \tag{3.11}$$

As usual, $\frac{d^-}{d\lambda} W_i$ and $\frac{d^+}{d\lambda} W_i$ denote, respectively, the left and right derivatives of the function $W_i(\lambda) := W_i(K_\lambda; E)$, and for $\lambda = -r$, only the second identity for the right derivative in (3.11) is considered. Notice that the class $\mathcal{R}_p$ depends on the fixed convex body E. Nevertheless, and for the sake of simplicity, we omit E in the notation, unless it is not clear from the context. Definition 3.2.1 is natural, since from the concavity of the family (N.7) of parallel bodies (cf. (N.10)) and the general Brunn-Minkowski theorem for relative quermassintegrals, i.e., Theorem L, the validity of inequalities

$$\frac{d^-}{d\lambda}W_i(\lambda) \geqslant \frac{d^+}{d\lambda}W_i(\lambda) \geqslant (n-i)W_{i+1}(\lambda) \tag{3.12}$$

follows for $i = 0, \ldots, n-1$. From now on, whenever we write f' for a function f, we mean that the left and right derivatives do exist and coincide, whenever it makes sense in its domain. The volume function, $\mathrm{vol}(\lambda) = W_0(\lambda) = W_0(K_\lambda; E)$, is always differentiable with respect to λ and $\mathrm{vol}'(\lambda) = nW_1(\lambda)$ (see e.g. [6, 54]), which implies that $\mathcal{R}_0 = \mathcal{K}^n$. Directly from the definition, it follows that $\mathcal{R}_{i+1} \subset \mathcal{R}_i$, for $i = 0, \ldots, n-2$, and also (see [45]), that all these inclusions are strict, since there exist $(n-i-1)$-tangential bodies of E lying in $\mathcal{R}_i$ which are not in $\mathcal{R}_{i+1}$ (see Appendix A for further details about tangential bodies).

The problem of studying the differentiability of the quermassintegrals $W_i(K_\lambda; B_3)$ of a convex body K, with respect to the parameter λ of definition of the full system of parallel bodies of K, in the 3-dimensional case, and with respect to the Euclidean unit ball B_3, goes back to Bol [6]. In [31], Hadwiger addressed a close related question, providing some partial solutions to it. We notice that for $n = 3$ and $E = B_3$, Definition 3.2.1 coincides with the definition of the *classes of convex bodies* considered in [6] and [31].

In [42], the general n-dimensional problem (with respect to any full-dimensional gauge body) is studied. In particular, it is shown that the smallest class, namely, $\mathcal{R}_{n-1}$, is given by

$$\mathcal{R}_{n-1} = \left\{K = L + \lambda E : L \in \mathcal{K}^n,\ \dim L \leqslant n-1,\ \lambda \geqslant 0\right\}, \tag{3.13}$$

for all $E \in \mathcal{K}^n_n$. Also necessary conditions for a convex body to belong to the other classes, for special types of sets E, are stated in terms of the support function of the relative form body of K_λ, its mixed area measures and the set of its r-extreme normal vectors, in the above mentioned work.

Tangential bodies, as intimately related to inner parallel bodies, do also play an important role for the description of the classes $\mathcal{R}_p$. The next result deals with the special class of 1-tangential bodies (see Appendix A for details), also named cap-bodies. It proves, roughly speaking, that the property of being a cap-body is, in some cases, "transferred" to the inner parallel bodies and the form body.

Lemma 3.2.2 ([45, Lemma 2.4])**.** *Let* $E \in \mathcal{K}^n_n$ *be a regular convex body, and let* $K \in \mathcal{K}^n$ *be a cap-body of* $K_{-r} + rE$ *satisfying condition* (3.9) *for all* $-r \leqslant \lambda \leqslant 0$. *Then*

(i) K^* *is a cap-body of* E *and*

(ii) K_λ *is a cap-body of* $K_{-r} + (r+\lambda)E$.

Proof.

(i) Since K is a cap-body of $K_{-r} + rE$ satisfying (3.9), Theorem 3.1.13 ensures that $K = K_\lambda + |\lambda|\, K^*$ for every $-r \leqslant \lambda \leqslant 0$. Then, by Lemma 3.1.6, we get, in particular, that

$$\mathrm{cl}\,\mathcal{U}_{n-2}(K^*) \subseteq \mathrm{cl}\,\mathcal{U}_{n-2}\big(K_\lambda + |\lambda|\, K^*\big) = \mathrm{cl}\,\mathcal{U}_{n-2}(K).$$

From the regularity of E we know that $\mathrm{cl}\,\mathcal{U}_0(K) = \mathcal{U}_0(K^*)$ (cf. Lemma 3.1.7) and moreover, $K_{-r} + rE$ is also regular. Hence, since K is a cap-body of $K_{-r} + rE$, it follows that $\mathcal{U}_0(K) = \mathcal{U}_{n-2}(K)$ and thus, we get

$$\mathcal{U}_{n-2}(K^*) \subseteq \mathrm{cl}\,\mathcal{U}_{n-2}(K^*) \subseteq \mathrm{cl}\,\mathcal{U}_{n-2}(K) = \mathrm{cl}\,\mathcal{U}_0(K) = \mathcal{U}_0(K^*).$$

Then K^* is a tangential body of E satisfying $\mathcal{U}_0(K^*) = \mathcal{U}_{n-2}(K^*)$, which shows that K^* is a cap-body of E.

(ii) Using Remark 3.1.14, from the decomposition $K = K_\lambda + |\lambda|\, K^*$, we get, for all $\lambda \in [-r, 0]$, that

$$K_\lambda = K_{-r} + (r + \lambda)K^*, \tag{3.14}$$

for all $\lambda \in [-r, 0]$. In our case, Theorem 3.1.13 ensures the equality $K = K_\lambda + |\lambda|\, K^*$ for every $\lambda \in [-r, 0]$, and thus, (3.14) holds. Moreover, since K is a tangential body of $K_{-r} + rE$, it is known (see Lemma A.1 in Appendix A) that

$$\mathcal{U}_0(K_\lambda) = \mathcal{U}_0(K) \quad \text{for} \quad -r < \lambda \leqslant 0, \tag{3.15}$$

which implies, using the alternative representation of the form body given in (3.3), that $K^* = K^*_\lambda$, for $-r < \lambda \leqslant 0$. Thus, we get $K_\lambda = K_{-r} + (r+\lambda)K^* = K_{-r} + (r+\lambda)K^*_\lambda$, and Theorem 3.1.13 applied to K_λ ensures that any inner parallel body of K is a tangential body of $K_{-r} + (r+\lambda)E$. It remains to prove that, moreover, it is a cap-body of $K_{-r} + (r+\lambda)E$. From (3.14) it follows that, for any $-r \leqslant \lambda \leqslant 0$, and every $u \in \mathcal{U}_0(K)$

$$\begin{aligned} h(K_\lambda, u) &= h(K_{-r}, u) + (r+\lambda)h(K^*, u) \\ &= h(K_{-r}, u) + (r+\lambda)h(E, u). \end{aligned}$$

Then it is enough to prove that

$$\mathcal{U}_0(K_\lambda) = \mathcal{U}_{n-2}(K_\lambda) \quad \text{for every} \quad -r < \lambda \leqslant 0, \tag{3.16}$$

since these last two assertions, together with (3.15), will imply that

$$h(K_\lambda, u) = h(K_{-r}, u) + (r+\lambda)h(E, u) = h\big(K_{-r} + (r+\lambda)E, u\big),$$

for every $u \in \mathcal{U}_0(K) = \mathcal{U}_0(K_\lambda) = \mathcal{U}_{n-2}(K_\lambda)$, and $-r < \lambda \leqslant 0$. This shows that K_λ is a cap-body of $K_{-r} + (r+\lambda)E$.

Thus, we have to prove (3.16). From Lemma 3.1.6, we get that

$$\begin{aligned}\mathrm{cl}\,\mathcal{U}_{n-2}(K) &= \mathrm{cl}\,\mathcal{U}_{n-2}(K_{-r} + rK^*) = \mathrm{cl}\,\mathcal{U}_{n-2}\big(K_{-r} + (r+\lambda)K^*\big)\\ &= \mathrm{cl}\,\mathcal{U}_{n-2}(K_\lambda).\end{aligned}$$

Moreover, since K is a cap-body of a regular convex body, we have that $\mathcal{U}_0(K) = \mathcal{U}_{n-2}(K)$, and with (3.15) we can conclude that

$$\mathrm{cl}\,\mathcal{U}_0(K_\lambda) = \mathrm{cl}\,\mathcal{U}_0(K) = \mathrm{cl}\,\mathcal{U}_{n-2}(K) = \mathrm{cl}\,\mathcal{U}_{n-2}(K_\lambda)$$

for every $-r < \lambda \leqslant 0$. Finally, we show that the closures can be omitted. Indeed, since $K_\lambda = K_{-r} + (r+\lambda)K^*$, then, by Lemma 3.1.3 (i), we get that, in particular, $\mathcal{U}_0(K^*) \subseteq \mathcal{U}_0(K_\lambda)$. Thus, together with (3.15) and Lemma 3.1.7 we obtain that

$$\begin{aligned}\mathrm{cl}\,\mathcal{U}_0(K_\lambda) = \mathrm{cl}\,\mathcal{U}_0(K) = \mathcal{U}_0(K^*) &\subseteq \mathcal{U}_0(K_\lambda)\\ &\subseteq \mathcal{U}_{n-2}(K_\lambda) \subseteq \mathrm{cl}\,\mathcal{U}_{n-2}(K_\lambda).\end{aligned}$$

Since $\mathrm{cl}\,\mathcal{U}_0(K_\lambda) = \mathrm{cl}\,\mathcal{U}_{n-2}(K_\lambda)$, the inclusions in the middle also coincide, i.e., $\mathcal{U}_0(K_\lambda) = \mathcal{U}_{n-2}(K_\lambda)$, as required. □

Remark 3.2.3 ([45, Remark 2.1]). *Note that condition (3.9) cannot be omitted in Lemma 3.2.2 for either of the items: the example provided in [43, Remark 3.2] proves it. We reproduce it here for completeness. Let $\sigma \subset \mathbb{R}^3$ be a line segment of length not smaller than 2 and take a point x lying outside the solid cylinder with circular cross section of radius 1 and axis the line $\mathrm{aff}\,\sigma$. The convex body $K = \mathrm{conv}\{\sigma + B_3, x\}$ (see Figure 2) satisfies that*

- $\ker K = \sigma$ *and* $r(K; B_3) = 1$,
- K^* *is the convex hull of* B_3 *and a suitable segment, and*
- K *is a cap-body of* $\sigma + B_3 = K_{-1} + B_3$,

but condition (3.9) does not hold.

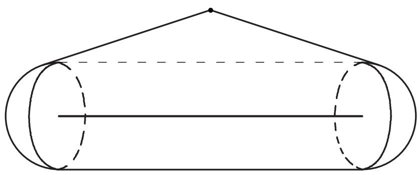

Figure 2: A cap-body of $K_{-r} + rB_3$ not satisfying (3.9).

Next, we prove that condition (3.9) is necessary for a convex body to lie in $\mathcal{R}_{n-2}$, as well as characterize the tangential bodies of the convex bodies $K_{-r} + rE$ lying in $\mathcal{R}_{n-2}$.

Proposition 3.2.4 ([45, Proposition 3.1]). *Let $E \in \mathcal{K}^n_n$ be a regular and strictly convex body and let $K \in \mathcal{R}_{n-2}$. Then $\mathrm{cl}\,\mathcal{U}_0(K) = \mathcal{U}_0(K_\lambda + K^*)$ for every $-r < \lambda \leqslant 0$.*

Proof. First, notice that, since E is regular, then (cf. Lemmata 3.1.7 and 3.1.3 (i))

$$\mathrm{cl}\,\mathcal{U}_0(K) = \mathcal{U}_0(K^*) \subseteq \mathcal{U}_0(K_\lambda) \cup \mathcal{U}_0(K^*) \subseteq \mathcal{U}_0(K_\lambda + K^*).$$

Thus, it is enough to prove the reverse inclusion. If $K \in \mathcal{R}_{n-1}$, then it is an outer parallel body of a lower-dimensional convex body (cf. (3.13)), and hence, condition (3.9) holds trivially since K is regular, i.e., since $\mathcal{U}_0(K) = \mathbb{S}^{n-1}$ (notice that if $x \in \mathrm{bd}\,K$ is a regular point, then the only outer normal vector at x is 0-extreme). Therefore, we may assume that $K \in \mathcal{R}_{n-2}\backslash\mathcal{R}_{n-1}$. Then it follows from [45, Theorem 1.2 (iv)] that

$$\mathrm{cl}\,\mathcal{U}_0(K_\lambda) = \mathrm{cl}\,\mathcal{U}_1(K_\lambda) = \cdots = \mathrm{cl}\,\mathcal{U}_{n-2}(K_\lambda) \tag{3.17}$$

for every $-r < \lambda \leqslant 0$. Now, since E is regular and strictly convex, equality $\mathrm{cl}\,\mathcal{U}_0(K_\lambda + K^*) = \mathrm{supp}\,S\big(K_\lambda + K^*[n-1];\cdot\big)$ (cf. (3.4)) holds. The linearity of the area measure in each argument, i.e., (N.16), yields

$$\begin{aligned}\mathrm{supp}\,S\big(K_\lambda + K^*[n-1];\cdot\big) &= \mathrm{supp}\,S\big(K_\lambda[n-1];\cdot\big) \cup \mathrm{supp}\,S\big(K^*[n-1];\cdot\big)\\ &\quad\cup \left[\bigcup_{i=1}^{n-2} \mathrm{supp}\,S\big(K_\lambda[i], K^*[n-i-1];\cdot\big)\right],\end{aligned}$$

and hence, we get, together with (3.4), that

$$\begin{aligned}\mathcal{U}_0(K_\lambda + K^*) &\subseteq \mathrm{cl}\,\mathcal{U}_0(K_\lambda + K^*) = \mathrm{supp}\,S\big(K_\lambda + K^*[n-1];\cdot\big)\\ &= \mathrm{cl}\,\mathcal{U}_0(K_\lambda) \cup \mathrm{cl}\,\mathcal{U}_0(K^*) \cup \left[\bigcup_{i=1}^{n-2} \mathrm{supp}\,S\big(K_\lambda[i], K^*[n-i-1];\cdot\big)\right].\end{aligned}$$

In [67, Lemma 3.4], the following content is proven: for any $n-1$ convex bodies $K, K_1, \ldots, K_{n-2} \in \mathcal{K}^n$,

$$\mathrm{supp}\,S\big(K, K_1, \ldots, K_{n-2};\cdot\big) \subseteq \mathrm{supp}\,S\big(E, K_1, \ldots, K_{n-2};\cdot\big),$$

provided $E \in \mathcal{K}^n_n$ is regular and strictly convex. Hence, we have

$$\begin{aligned}\mathrm{supp}\,S\big(K_\lambda[i], K^*[n-i-1];\cdot\big) &\subseteq \mathrm{supp}\,S\big(K_\lambda[i], E[n-i-1];\cdot\big)\\ &= \mathrm{cl}\,\mathcal{U}_{n-i-1}(K_\lambda),\end{aligned}$$

for $i = 1, \ldots, n-2$ (cf. (3.4)), and thus, together with (3.17), Lemma 3.1.7 and Lemma 3.1.3 (iv), it follows that

$$\begin{aligned}\mathcal{U}_0(K_\lambda + K^*) &\subseteq \mathrm{cl}\,\mathcal{U}_0(K_\lambda) \cup \mathrm{cl}\,\mathcal{U}_0(K^*) \cup \left[\bigcup_{i=1}^{n-2} \mathrm{cl}\,\mathcal{U}_{n-i-1}(K_\lambda)\right]\\ &= \mathrm{cl}\,\mathcal{U}_0(K_\lambda) \cup \mathrm{cl}\,\mathcal{U}_0(K^*) = \mathrm{cl}\,\mathcal{U}_0(K_\lambda) \cup \mathrm{cl}\,\mathcal{U}_0(K) \subseteq \mathrm{cl}\,\mathcal{U}_0(K),\end{aligned}$$

which shows the result. □

Proposition 3.2.5 ([45, Proposition 3.2]). *Let $E \in \mathcal{K}^n_n$ be a regular and strictly convex body, and let $K \in \mathcal{K}^n$ be a tangential body of $K_{-r} + rE$. Then $K \in \mathcal{R}_{n-2}$ if and only if K is a cap-body of $K_{-r} + rE$ satisfying (3.9) for $-r \leqslant \lambda \leqslant 0$.*

Proof. First we assume that $K \in \mathcal{R}_{n-2}$. Proposition 3.2.4 ensures that K satisfies (3.9) for $-r < \lambda \leqslant 0$. On the other hand, since K is a tangential body of $K_{-r} + rE$, we have the equality $\mathcal{U}_0(K_\lambda) = \mathcal{U}_0(K)$ for all $-r < \lambda \leqslant 0$ (cf. (3.15)) and hence, $K^* = K^*_\lambda$ for $-r < \lambda \leqslant 0$. Thus, condition (3.9) can be rewritten as $\mathrm{cl}\,\mathcal{U}_0(K_\lambda) = \mathcal{U}_0(K_\lambda + K^*_\lambda)$, and, by Theorem 3.1.10 (i), we have $K \subseteq K_\lambda + |\lambda|\, K^*_\lambda$ for $-r < \lambda \leqslant 0$. Since $K \supseteq K_\lambda + |\lambda|\, K^*$ always holds (cf. Proposition 3.1.8 (i)), both inclusions, together with $K^* = K^*_\lambda$, prove that $K = K_\lambda + |\lambda|\, K^*$ for all $-r < \lambda \leqslant 0$.

Notice that we have shown the above equality for the half-open interval $(-r, 0]$ and so we can apply Remark 3.1.9 to get the equality $K = K_\lambda + |\lambda|\, K^*$ for every $\lambda \in [-r, 0]$. Then Theorem 3.1.13 ensures that, in particular, condition (3.9) holds for all $-r \leqslant \lambda \leqslant 0$. It remains to be proven that K is a cap-body of $K_{-r} + rE$. Since $K = K_\lambda + |\lambda|\, K^*$ for every $\lambda \in [-r, 0]$, we can apply Lemma 3.1.15 to get

$$K_\lambda = \frac{|\lambda|}{r} K_{-r} + \left(1 - \frac{|\lambda|}{r}\right) K,$$

and this representation of K_λ can be used to easily compute $W_{n-1}(\lambda)$, $W_{n-2}(\lambda)$, and to differentiate $W_{n-2}(\lambda)$:

$$\begin{aligned} W_{n-1}(\lambda) &= -\frac{\lambda}{r} W_{n-1}(K_{-r}; E) + \left(1 + \frac{\lambda}{r}\right) W_{n-1}(K; E), \\ W_{n-2}(\lambda) &= \left(\frac{\lambda}{r}\right)^2 W_{n-2}(K_{-r}; E) - 2\frac{\lambda}{r}\left(1 + \frac{\lambda}{r}\right) V\big(K_{-r}, K, E[n-2]\big) \\ &\quad + \left(1 + \frac{\lambda}{r}\right)^2 W_{n-2}(K; E), \end{aligned}$$

and hence,

$$\begin{aligned} W'_{n-2}(\lambda) = \frac{2}{r}\bigg[\frac{\lambda}{r} W_{n-2}(K_{-r}; E) &- \left(1 + 2\frac{\lambda}{r}\right) V\big(K_{-r}, K, E[n-2]\big) \\ &+ \left(1 + \frac{\lambda}{r}\right) W_{n-2}(K; E)\bigg]. \end{aligned}$$

Since $K \in \mathcal{R}_{n-2}$, we have $W'_{n-2}(\lambda) = 2W_{n-1}(\lambda)$. Identifying the corresponding coefficients in the above polynomials, we get, in particular,

$$r\, W_{n-1}(K; E) = W_{n-2}(K; E) - V\big(K_{-r}, K, E[n-2]\big),$$

or equivalently, $V\big(K[2], E[n-2]\big) = V\big(K, K_{-r} + rE, E[n-2]\big)$. Thus, using the formula for the mixed volumes given in (N.13), we get

$$\int_{\mathbb{S}^{n-1}} \big[h(K, u) - h(K_{-r} + rE, u)\big]\, dS\big(K, E[n-2]; u\big) = 0, \tag{3.18}$$

and, since $K_{-r} + rE \subseteq K$, then (3.18) is equivalent to the equality $h(K, u) = h(K_{-r} + rE, u)$ for all $u \in \operatorname{supp} S\big(K, E[n-2]; \cdot\big) = \mathrm{cl}\,\mathcal{U}_{n-2}(K)$. It shows that K is a cap-body of $K_{-r} + rE$.

In order to prove the converse, we assume that K is a cap-body of $K_{-r} + rE$ satisfying (3.9) for $-r \leqslant \lambda \leqslant 0$, and we have to prove that K lies in $\mathcal{R}_{n-2}$, i.e., for every $i = 0, \dots, n-2$, the real function $W_i(\lambda)$ is differentiable and $W_i'(\lambda) = (n-i)W_{i+1}(\lambda)$, for any $-r \leqslant \lambda \leqslant 0$.

By Theorem 3.1.13, we know that $K = K_{-r} + rK^*$, which implies (cf. (3.14)) $K_\lambda = K_{-r} + (r+\lambda)K^*$. Hence, for $i = 0, \dots, n-2$, we can write

$$W_i(\lambda) = \sum_{k=0}^{n-i} \binom{n-i}{k} (r+\lambda)^k V\big(K_{-r}[n-i-k], K^*[k], E[i]\big),$$

which is clearly differentiable, and thus,

$$\begin{aligned} W_i'(\lambda) &= \sum_{k=1}^{n-i} \binom{n-i}{k} k(r+\lambda)^{k-1} V\big(K_{-r}[n-i-k], K^*[k], E[i]\big) \\ &= \sum_{k=0}^{n-i-1} \binom{n-i}{k+1} (k+1)(r+\lambda)^k V\big(K_{-r}[n-i-k-1], K^*[k+1], E[i]\big). \end{aligned}$$

Therefore, $K \in \mathcal{R}_{n-2}$ if and only if $W_i'(\lambda) = (n-i)W_{i+1}(\lambda)$, i.e.,

$$\begin{aligned} &\binom{n-i}{k+1} (k+1) V\big(K_{-r}[n-i-k-1], K^*[k+1], E[i]\big) \\ &\qquad = (n-i) \binom{n-i-1}{k} V\big(K_{-r}[n-i-k-1], K^*[k], E[i+1]\big), \end{aligned}$$

for every $i = 0, \dots, n-2$, and any $k = 0, \dots, n-i$. It is an immediate computation to check that $(k+1)\binom{n-i}{k+1} = (n-i)\binom{n-i-1}{k}$. Thus, K lies in the class $\mathcal{R}_{n-2}$ if and only if

$$\begin{aligned} V\big(K_{-r}[n-i-k-1], &K^*[k], E[i+1]\big) \\ &= V\big(K_{-r}[n-i-k-1], K^*[k+1], E[i]\big), \end{aligned} \tag{3.19}$$

for every $i = 0, \dots, n-2$, and any $k = 0, \dots, n-i$. Thus, in order to conclude, we will prove (3.19). Notice that the case $i = n-2$ and $k = 1$ in (3.19), i.e., the identity $V\big(K^*, E[n-1]\big) = V\big(K^*[2], E[n-2]\big)$, is equivalent to the fact that K^* is a cap-body of E, which we already know by Lemma 3.2.2.

Since $K_\lambda = K_{-r} + (r+\lambda)K^*$ is a cap-body of $K_{-r} + (r+\lambda)E$ by Lemma 3.2.2, we can assure that

$$h(K_\lambda, u) = h\big(K_{-r} + (r+\lambda)E, u\big) \qquad \text{for every } \ u \in \mathcal{U}_{n-2}(K_\lambda).$$

Then, for all $u \in \operatorname{supp} S\big(K_\lambda[n-i-1], E[i]; \cdot\big)$, and any $i = 0, \dots, n-2$, using (N.13), we get that

$$V\big(K_\lambda[n-i-1], K_{-r} + (r+\lambda)E, E[i]\big) = V\big(K_\lambda[n-i], E[i]\big),$$

which, together with the linearity of mixed volumes (N.15), leads to

$$\begin{aligned} V\big(K_\lambda[n-i-1], &K_{-r}, E[i]\big) + (r+\lambda)V\big(K_\lambda[n-i-1], E[i+1]\big) \\ &= V\big(K_\lambda[n-i-1], K_{-r} + (r+\lambda)E, E[i]\big) = V\big(K_\lambda[n-i], E[i]\big) \\ &= V\big(K_\lambda[n-i-1], K_{-r} + (r+\lambda)K^*, E[i]\big) \\ &= V\big(K_\lambda[n-i-1], K_{-r}, E[i]\big) + (r+\lambda)V\big(K_\lambda[n-i-1], K^*, E[i]\big). \end{aligned}$$

This is,

$$V\big(K_\lambda[n-i-1],E[i+1]\big)=V\big(K_\lambda[n-i-1],K^*,E[i]\big).$$

Finally, writing $K_\lambda=K_{-r}+(r+\lambda)K^*$ in the above equality, and using (N.12), we get the identity

$$\sum_{k=0}^{n-i-1}(r+\lambda)^k V\big(K_{-r}[n-i-k-1],K^*[k],E[i+1]\big)$$
$$=\sum_{k=0}^{n-i-1}(r+\lambda)^k V\big(K_{-r}[n-i-k-1],K^*[k+1],E[i]\big),$$

for every $-r\leqslant\lambda\leqslant 0$. Thus, it follows that

$$V\big(K_{-r}[n-i-k-1],K^*[k],E[i+1]\big)$$
$$=V\big(K_{-r}[n-i-k-1],K^*[k+1],E[i]\big)$$

for every $i=0,\dots,n-2$ and any $k=0,\dots,n-i$, which shows (3.19) and finishes the proof. □

Remark 3.2.6 ([45, Remark 4.1]). *Notice that if $K\in\mathcal{R}_p$, $0\leqslant p\leqslant n-1$, then $K+\rho E\in\mathcal{R}_p$ for all $\rho\geqslant 0$. Indeed, for $\rho\geqslant 0$ fixed, it is clear that $r(K+\rho E;E)=r+\rho$, and the inner parallel bodies of $K+\rho E$ are given by*

$$(K+\rho E)_\lambda:=\begin{cases}K+(\rho+\lambda)E & \text{for } -\rho\leqslant\lambda\leqslant 0,\\ K_{\rho+\lambda} & \text{for } -(r+\rho)\leqslant\lambda\leqslant-\rho.\end{cases}$$

Then the quermassintegral $W_i\big((K+\rho E)_\lambda\big)$, $0\leqslant i\leqslant p$, $1\leqslant p\leqslant n-1$, is just the i-th quermassintegral of the inner/outer parallel bodies of $K\in\mathcal{R}_p$, with a linear change of parameter. In that case, the convex body K corresponds to $\lambda=-\rho$. It is then straightforward that $K+\rho E\in\mathcal{R}_p$.

Our next result provides a characterization of the class $\mathcal{R}_{n-2}$.

Theorem 3.2.7 ([45, Theorem 1.1]). *Let $E\in\mathcal{K}^n_n$ be regular and strictly convex. The only sets K in $\mathcal{R}_{n-2}$ are cap-bodies of convex bodies lying in $\mathcal{R}_{n-1}$ which satisfy*

$$\mathrm{cl}\,\mathcal{U}_0(K)=\mathcal{U}_0(K_\lambda+K^*)$$

for $-r\leqslant\lambda\leqslant 0$, and their outer parallel bodies.

We notice that the above condition is exactly (3.9).

Proof. By Remark 3.2.6, we can assume that (a dilatation of) E is not a summand of K, i.e., that K cannot be written as $K=L+\rho E$.

From Proposition 3.2.5, it follows that cap-bodies of $K_{-r}+rE$ satisfying condition (3.9) lie in $\mathcal{R}_{n-2}$.

Conversely, let $K\in\mathcal{R}_{n-2}\backslash\mathcal{R}_{n-1}$. Then we know that (cf. (3.17))

$$\mathrm{cl}\,\mathcal{U}_0(K_\lambda)=\mathrm{cl}\,\mathcal{U}_1(K_\lambda)=\cdots=\mathrm{cl}\,\mathcal{U}_{n-2}(K_\lambda),\quad\text{for all } -r<\lambda\leqslant 0.$$

Since $K \in \mathcal{R}_{n-2}$, also all inner parallel bodies $K_\lambda \in \mathcal{R}_{n-2}$, for $-r < \lambda \leqslant 0$, because their quermassintegrals satisfy the same differentiability conditions. Notice that the case $\lambda = -r$ is excluded here, since K_{-r} has no inner parallel bodies. Hence, applying Proposition 3.2.4 to K_λ we get

$$\mathrm{cl}\mathcal{U}_0(K_\lambda) = \mathcal{U}_0(K_\lambda + K_\lambda^*), \quad -r < \lambda \leqslant 0, \tag{3.20}$$

and, with Lemma 3.1.10 (i), we conclude that

$$K \subseteq K_\lambda + |\lambda|\, K_\lambda^* \tag{3.21}$$

for every $-r < \lambda \leqslant 0$. Since the content $K_\lambda + |\lambda|\, K^* \subseteq K$ holds (cf. Proposition 3.1.8), by means of condition (3.21), we obtain that, for all $-r < \lambda \leqslant 0$,

$$K_\lambda + |\lambda|\, K^* \subseteq K \subseteq K_\lambda + |\lambda|\, K_\lambda^*. \tag{3.22}$$

Notice that the left inclusion also holds for $\lambda = -r$.

At this point, we observe that, in order to conclude the proof, it suffices to show that

$$\mathrm{cl}\mathcal{U}_0(K) = \mathrm{cl}\mathcal{U}_0(K_\lambda) \quad \text{for} \quad -r < \lambda \leqslant 0. \tag{3.23}$$

Indeed, from (3.23) and using again the representation of the form body given in (3.3), we get $K^* = K_\lambda^*$. This shows, by using (3.22), that $K = K_\lambda + |\lambda|\, K^*$ for $-r < \lambda \leqslant 0$, and with Remark 3.1.9 we get the validity of the identity $K = K_\lambda + |\lambda|\, K^*$ for all $-r \leqslant \lambda \leqslant 0$. Then Theorem 3.1.13 implies that K is a tangential body of $K_{-r} + rE$ satisfying (3.9). Finally, Proposition 3.2.5 gives the required result.

It remains to prove (3.23) for a convex body K lying in $\mathcal{R}_{n-2}$. The inclusion $\mathrm{cl}\mathcal{U}_0(K_\lambda) \subseteq \mathrm{cl}\mathcal{U}_0(K)$ clearly holds for $-r < \lambda \leqslant 0$ (cf. Lemma 3.1.3 (iv)). Thus, we only have to prove the reverse inclusion. In order to do so, we assume that there exists a vector $u_0 \in \mathrm{cl}\mathcal{U}_0(K) \backslash \mathrm{cl}\mathcal{U}_0(K_{\lambda'})$ for some $\lambda' < 0$. We observe that this implies that $u_0 \notin \mathrm{cl}\mathcal{U}_0(K_\lambda)$ for all $\lambda \in [-r, \lambda']$. Since condition (3.17) is satisfied, such a vector $u_0 \notin \mathrm{cl}\mathcal{U}_{n-2}(K_{\lambda'})$, i.e., u_0 is an $(n-1)$-extreme normal vector of $K_{\lambda'}$ which does not lie in the closure of its $(n-2)$-extreme normal vectors. Geometrically, the latter corresponds to the fact that u_0 is a normal vector at a non-regular point of $K_{\lambda'}$, lying in the *interior* of the n-th dimensional normal cone of that point.

Since $u_0 \in \mathrm{cl}\mathcal{U}_0(K)$, there exists $\varepsilon > 0$ such that $u_0 \in \mathrm{cl}\mathcal{U}_0(K_\lambda)$ for all $\lambda \in (-\varepsilon, 0]$. Then, since $u_0 \notin \mathrm{cl}\mathcal{U}_0(K_{\lambda'})$, there exists

$$\lambda_0 = \max\{-r \leqslant \lambda < 0 : u_0 \notin \mathrm{cl}\mathcal{U}_0(K_\lambda)\}$$

satisfying $\lambda' \leqslant \lambda_0 \leqslant -\varepsilon < 0$, and so $u_0 \in \mathrm{cl}\mathcal{U}_0(K_\lambda)$ for all $\lambda_0 < \lambda < 0$. Notice that if $\lambda_0 = -r$, then (3.23) holds, and we may assume that $\lambda_0 > -r$. Using Lemma 3.1.8 (ii), we can ensure that, for $\lambda_0 < \lambda \leqslant 0$,

$h(K_\lambda, u_0) = h(K, u_0) + \lambda h(E, u_0)$. A continuity argument (cf. Remark 3.1.9) leads to

$$h(K_\lambda, u_0) = h(K, u_0) + \lambda h(E, u_0) \quad \text{for all } \lambda_0 \leqslant \lambda \leqslant 0. \tag{3.24}$$

Observe that since (3.20) holds, we can apply Theorem 3.1.10 (ii) to get

$$\frac{d}{d\lambda} h(K_\lambda, u) = h(K_\lambda^*, u) \quad \text{for every } u \in S^{n-1}. \tag{3.25}$$

Then, taking derivatives (right derivative for $\lambda = \lambda_0$) in (3.24), and using the above expression (3.25), we get that

$$h(K_\lambda^*, u_0) = \frac{d}{d\lambda} h(K_\lambda, u_0) = h(E, u_0),$$

for all $\lambda_0 \leqslant \lambda \leqslant 0$. Now, since $h(K_{\lambda_0}^*, u_0) = h(E, u_0)$, u_0 cannot lie in the interior of an n-dimensional normal cone at a boundary point of K_{λ_0}, but in the boundary of the cone itself, in other words, we have $u_0 \in \operatorname{cl}\mathcal{U}_{n-2}(K_{\lambda_0}) = \operatorname{cl}\mathcal{U}_0(K_{\lambda_0})$, which gives the required contradiction. This concludes the proof. □

Using Theorems 3.2.7 and 3.1.13 , we obtain that every inner parallel body of a convex body K, lying in $\mathcal{R}_{n-2}$, is a summand of K.

Corollary 3.2.8 ([45, Corollary 4.1]). *Let $E \in \mathcal{K}_n^n$ be regular and strictly convex and $K \in \mathcal{R}_{n-2}$. Then, for any $-r \leqslant \lambda \leqslant 0$, K_λ is a summand of K.*

Remark 3.2.9 ([45, Remark 4.2]). *How does a convex body $K \in \mathcal{R}_{n-2}$ look like? From the previous results, it is clear that K is a cap-body of an outer parallel body of a (strictly) lower dimensional convex body. But any of these cap-bodies is not valid: the additional points which determine the set when constructing the convex hull with $K_{-r} + rE$ cannot lie anywhere. For instance, if $\dim K_{-r} = 1$, then those points should lie in the (infinite) cylinder containing $K_{-r} + rE$ with $(n-2)$-dimensional spherical cross section rB_{n-2} (see Figure 3); otherwise the kernel K_{-r} would not be a summand of K and, moreover, 1-extreme normal vectors would appear when considering $K_{-r} + rK^*$, contradicting condition (3.9). Figure 3 shows a cap-body of $K_{-r} + rB_3$ in $\mathbb{R}^3$, lying in $\mathcal{R}_1$; on the contrary, the one shown in Figure 2 does not lie in $\mathcal{R}_1$.*

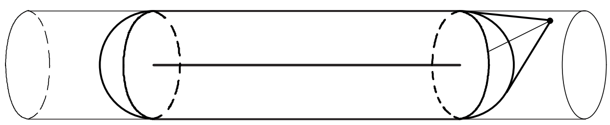

Figure 3: A cap-body of $K_{-r} + rB_3$ lying in $\mathcal{R}_1$.

A similar reasoning gives an idea of the situation for any dimension of the kernel. In general, when K_{-r} is not strictly convex in its affine hull, the "allowed" positions for the points determining the convex hull have many restrictions, because of the segments contained in the (relative) boundary of K_{-r}. Figure 4 shows another example of a convex body lying in $\mathcal{R}_1$.

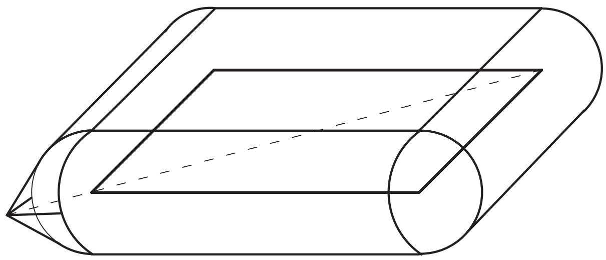

Figure 4: Another convex body lying in $\mathcal{R}_1$.

3.3 DECOMPOSITION OF POLYTOPES VIA INNER PARALLEL BODIES

In this section we deal with the more particular case of decompositions of polytopes, by inner parallel bodies of those, when $E = B_n$. We will prove that, in this case, some of the previous results can be better described and understood.

A fundamental tool to address this problem will be the following criterion (see [70]) that characterizes the polytopes which can be summands of a given one. Let $K \in \mathcal{K}^n$, $u \in S^{n-1}$ and $H(K, u)$ be the supporting hyperplane to K with outer normal vector u. We denote by $F(K, u) = K \cap H(K, u)$ the corresponding face of K cut off by $H(K, u)$.

Theorem 3.3.1 ([70]). *(Shephard's decomposition criterion) Let the convex bodies $P, Q \in \mathcal{K}^n$ be two polytopes. Then Q is summand of P if and only if the following two conditions hold:*

(i) $\dim F(P, u) \geqslant \dim F(Q, u)$ *for every* $u \in S^{n-1}$.

(ii) For every edge $F(P, u)$ of P, it is

$$\mathrm{vol}_1(F(P, u)) \geqslant \mathrm{vol}_1(F(Q, u)).$$

This result has been generalized and proven to be equivalent to conditions having very different flavor, as intersections of translates or monotonicity of mixed volumes (see e.g. [68, Section 3.2]).

In the next, we will use more precise nomenclature and equivalent definitions for some of the already introduced notions, now for the particular case we are dealing with, namely, the convex body P is a polytope and E is the Euclidean unit ball B_n.

Let P be a polytope, and let $u \in S^{n-1}$ be a 0-extreme normal vector. From the definition, it follows that the unique normal vector to the face $F(P, u)$ is precisely the vector u. Hence, the set of 0-extreme normal vectors of the polytope P coincides with the set of outer normal vectors to the facets of P, i.e., $(n-1)$-dimensional faces of P.

As $E = B_n$, using Remark P and Lemma 3.1.3 (iii), we can write the inner parallel body of any polytope P, at distance $|\lambda|$, as

$$P_\lambda = \{x \in \mathbb{R}^n : \langle x, u \rangle \leqslant h(P, u) - |\lambda|, \text{ for all } u \in \mathcal{U}_0(P)\}.$$

Thus, if P is the polytope

$$P = \{x \in \mathbb{R}^n : \langle x, u_i \rangle \leqslant b_i, 1 \leqslant i \leqslant m\},$$

with outer normal vectors $u_i \in \mathbb{S}^{n-1}$, $1 \leqslant i \leqslant m$, to the facets, for $b_i \in \mathbb{R}$, $1 \leqslant i \leqslant m$ (i.e., none of the inequalities $\langle x, u_i \rangle \leqslant b_i$, $1 \leqslant i \leqslant m$ is redundant), then, for $-r \leqslant \lambda \leqslant 0$, the inner parallel body of P at distance $|\lambda|$ is

$$P_\lambda = \{x \in \mathbb{R}^n : \langle x, u_i \rangle \leqslant b_i - |\lambda| \ , 1 \leqslant i \leqslant m\};$$

i.e., P_λ is the polytope which arises *by moving* inwards the facets of P all at distance $|\lambda|$.

We obtain directly from the previous results in these chapter, particularizing to polytopes, the following decomposition result.

Theorem 3.3.2 (Theorem 3.1.13 and Theorem 3.1.10)**.** *Let $P \in \mathcal{K}^n$ be a polytope and $E = B_n$. Then*

(i) $P = P_\lambda + |\lambda| P^*$ *for every* $-r \leqslant \lambda \leqslant 0$ *if and only if*

$$h(P, u) = h(P_{-r}, u) + r, \text{ for all } u \in \mathcal{U}_0(P),$$

and condition (3.9) *holds for all* $-r \leqslant \lambda \leqslant 0$*.*

(ii) If condition (3.9) *holds for all* $-r \leqslant \lambda \leqslant 0$*, then*

$$P = P_\lambda + \int_\lambda^0 P_\mu^* d\mu,$$

for all $-r \leqslant \lambda \leqslant 0$*.*

We observe that we have rewritten Theorem 3.1.10 using Remark A.2 for the form body of K. In the next section, we will prove that the integral expression appearing in Theorem 3.3.2 (ii) is, as it may probably be expected for a polytope, a finite sum. Indeed, it provides us with a precise decomposition of P using the form bodies of its inner parallel bodies. This, in turn, shows that condition (3.7), namely, the equality $\mathrm{cl}\mathcal{U}_0(K_\lambda) = \mathrm{cl}\mathcal{U}_0(K_\lambda + K_\lambda^*)$, is, for polytopes, fully understood via decompositions by inner parallel bodies.

The following two corollaries are immediate consequences of Theorem 3.1.4.

Corollary 3.3.3 ([49, Corollary 3.1])**.** *Let $P \in \mathcal{K}^n$ be a polytope. Then, if $P = rP^*$, all inner parallel bodies of P are (trivial) summands of P.*

A convex body $K \in \mathcal{K}^n$ is indecomposable if all its summands are trivial, i.e., if a representation $K = M + L$ with $M, L \in \mathcal{K}^n$ is only possible with M, L homothetic to K. For example, simplicial polytopes or pyramids (see e.g. [70, Section 15.1]) are indecomposable while simple polytopes (except for the simplex) are not.

Corollary 3.3.4 ([49, Corollary 3.2]). *Let* P *be be an indecomposable polytope, and let* $-r \leqslant \tau \leqslant 0$. *Then* P_τ *is a summand of* P *if and only if* $P = rP^*$.

Let P be a polytope and let F be a face of P. The set

$$N(F) = \mathrm{cl}\left(\mathrm{pos}\{u \in \mathbb{S}^{n-1} : F(P,u) = F\}\right),$$

that consists of all vectors u that are normal to F is called the normal cone of F. The poset of all normal cones of P, ordered by inclusion, is called the normal fan of P, denoted by $\mathcal{N}(P)$. That is, for a non-empty polytope in $\mathbb{R}^n$, $\mathcal{N}(P)$ consists of the normal cones of all faces of P. The union of all such cones is $\mathbb{R}^n$, which means that $\mathcal{N}(P)$ is a complete fan and furthermore, the relative interiors of the cones in the normal fan form a partition of $\mathbb{R}^n$ (see [73, Section 7.1] for a more detailed introduction).

Lemma 3.1.3 (iv) provides a relation between the 1-dimensional cones in $\mathcal{N}(P)$ and those of $\mathcal{N}(P_\tau)$. For the other dimensional cones in $\mathcal{N}(P)$ and $\mathcal{N}(P_\tau)$ no analogous relation holds in general. However, if we ask P_τ to be a summand of P, using Shephard's decomposition criterion, i.e., Theorem 3.3.1, the following known result can be proven (cf. [73, Proposition 7.12]).

Proposition 3.3.5 ([49, Proposition 3.4]). *Let* $P \in \mathcal{K}^n$ *be a polytope. If* P_τ *is a summand of* P, *then the normal fan of* P *is a refinement of the normal fan of* P_τ.

As we shall see in Proposition 3.3.16 (iii), the converse is not true. We recall that the poset σ is a refinement of the poset τ if every $\sigma_i \in \sigma$ is a subset of some $\tau_k \in \tau$.

If $P \in \mathcal{K}^n$ is a polytope, then $\mathcal{U}_0(P)$ is the set of outer normal vectors to the facets of P, which coincides with the 1-dimensional cones in the normal fan. This ensures that there exists $\epsilon > 0$ so that, for $-\epsilon < \lambda \leqslant 0$, $\mathcal{U}_0(P_\lambda) = \mathcal{U}_0(P)$, i.e., there is a range in $(-r, 0]$ in which the polytopes P_λ have exactly the same number of facets as P does. Notice that this is no longer true for a general convex body; see e.g. [62, Figure 2.3].

Using this, we define the following parameters, $\tau_j(P)$, for $j \in \mathbb{N}$, associated to the polytope $P \in \mathcal{K}^n$.

Definition 3.3.6 ([49, Definition 3.5]). *Let* $P \in \mathcal{K}^n$ *be a polytope, let* $\tau_0(P) = 0$. *We define* $\tau_1(P) = \inf\{\mu \in (-r, 0] : \mathcal{U}_0(P_\mu) = \mathcal{U}_0(P)\}$. *We set* $\tau_i(P) = \tau_1(P_{\tau_{i-1}(P)}) = \inf\{\mu \in (-r, 0] : \mathcal{U}_0(P_\mu) = \mathcal{U}_0(P_{\tau_{i-1}(P)})\}$, *inductively.*

Taking into consideration that P_{-r} has dimension strictly less than n (see (N.8)), Lemma 3.1.3 (iv) and the previous comments, it is clear that there exist only finitely many (different) $\tau_i(P)$. If the polytope P has no interior points, i.e., if its inradius is 0, then $\tau_1(P) = 0$. Observe also that $\tau_1(P)$ is not a minimum, i.e., $\mathcal{U}_0(P_{\tau_1(P)}) \neq \mathcal{U}_0(P)$ (unless

$\tau_1(P) = 0$). Indeed, $\tau_1(P)$ can be described geometrically as the largest value on the interval $(-r, 0]$ for which $P_{\tau_1(P)}$ has strictly less facets than P. Hence, $\tau_1(P) = -r$ if and only if

$$h(P_{-r}, u) = h(P, u) - r, \quad \text{for all } u \in \mathcal{U}_0(P).$$

Next, we prove that, for $\lambda \in [\tau_1(P), 0]$, there are necessary and sufficient conditions in order P_λ to be a summand of P, and these rely on the form body of P. Outside this interval, it will be necessary that P_λ, for all λ in at least some open interval of $(-r, 0]$, are summands of P, in order to prove our decomposability conditions.

Proposition 3.3.7 ([49, Proposition 3.6]). *Let $P \in \mathcal{K}^n$ be a polytope, let $\tau_1(P) \leqslant \tau \leqslant 0$, and let P_τ be a summand of P. Then $P = P_\tau + |\tau| P^*$.*

Proof. Let $P = \{x \in \mathbb{R}^n : \langle x, u_i \rangle \leqslant b_i,\ 1 \leqslant i \leqslant m\}$ for $u_i \in \mathbb{S}^{n-1}$, $b_i \in \mathbb{R}$ and $1 \leqslant i \leqslant m$. Then

$$P_\tau = \{x \in \mathbb{R}^n : \langle x, u_i \rangle \leqslant b_i - |\tau|,\ 1 \leqslant i \leqslant m\}.$$

For $\tau_1(P) < \tau \leqslant 0$, we have that $u_i \in \mathcal{U}_0(P_\tau)$ for $1 \leqslant i \leqslant m$. Thus, from Proposition 3.1.8 (ii), it follows that $h(P, u_i) = h(P_\tau, u_i) + |\tau|$, $1 \leqslant i \leqslant m$. The continuity of the support function ensures that this relation holds for $\tau = \tau_1(P)$ too.

Let Q be so that $Q + P_\tau = P$. Then Q is the Minkowski difference of P and P_τ and therefore, we can write

$$\begin{aligned} Q &= \{x \in \mathbb{R}^n : \langle x, u_i \rangle \leqslant h(P, u_i) - h(P_\tau, u_i),\ 1 \leqslant i \leqslant m\} \\ &= \{x \in \mathbb{R}^n : \langle x, u_i \rangle \leqslant |\tau|,\ 1 \leqslant i \leqslant m\} \\ &= |\tau|\, P^*, \end{aligned}$$

for any $\tau_1(P) \leqslant \tau \leqslant 0$, where we have implicitly used Lemma 3.1.3 (iii), i.e., $\mathcal{U}_0(Q) = \mathcal{U}_0(P \sim P_\lambda) \subseteq \mathcal{U}_0(P)$. □

Proposition 3.3.7 provides us with a slight improvement of Theorem 3.3.2 (i) for polytopes, namely, it is not necessary to ask for the precise decomposition of P, but just for P_τ to be a summand of P for all $-r \leqslant \tau \leqslant 0$.

Corollary 3.3.8 ([49, Corollary 3.7]). *Let $P \in \mathcal{K}^n$ be a polytope satisfying $-r = \tau_1(P)$. The following conditions are equivalent:*

(i) *P_τ is a summand of P for all $-r \leqslant \tau \leqslant 0$.*

(ii) *$\mathcal{U}_0(P) = \mathcal{U}_0(P_\tau + P^*)$ for all $-r \leqslant \tau \leqslant 0$.*

Proof. Since $\tau_1(P) = -r$, we obtain the equality $P = P_\tau + |\tau|\, P^*$ for all $-r \leqslant \tau \leqslant 0$, from Proposition 3.3.7 . Thus, it is clear that for all $-r \leqslant \tau \leqslant 0$ we have $\mathcal{U}_0(P) = \mathcal{U}_0(P_\tau + P^*)$.
In order to prove the converse, since $\tau_1(P) = -r$ and $\mathcal{U}_0(P) = \mathcal{U}_0(P_\tau)$ for all $-r < \tau \leqslant 0$, it follows that $h(P, u) = h(P_\tau, u) + |\tau|\, h(P^*, u)$ for every $u \in \mathcal{U}_0(P)$. Condition $\mathcal{U}_0(P) = \mathcal{U}_0(P_\tau + P^*)$ yields that, in fact, $P = P_\tau + |\tau|\, P^*$ (cf. Theorem 3.1.12). □

We notice, that the normal cones of P and P^* are, in general, no refinements one of the other (see [62, Figure 2.2]).

Corollary 3.3.9 ([49, Corollary 3.8]). *Let $P \in \mathcal{K}^n$ be a polytope. If P_τ is a summand of P for some $-r \leqslant \tau_1(P) \leqslant \tau \leqslant 0$, then the normal fan of P is a refinement of the normal fan of P^*.*

The converse of this result is not true, as it will follow from Proposition 3.3.16 (see Figure 8).

If all the inner parallel bodies of P, for $\tau_1(P) \leqslant \tau \leqslant 0$, are summands of P, we get from Proposition 3.3.7 that $P = P_{\tau_1(P)} + |\tau_1(P)|P^*$. This allows us to provide more information on the normal fans of P and P_τ for $\tau_1(P) < \tau \leqslant 0$, improving Proposition 3.3.7. For, we need the following lemma, whose proof is essentially the first part of the proof of Lemma 3.1.15.

Lemma 3.3.10 ([49, Lemma 3.9]). *Let $P \in \mathcal{K}^n$ be a polytope, and let further $P = P_\tau + |\tau|P^*$ for some $-r \leqslant \tau_1(P) \leqslant \tau \leqslant 0$. Then, for all $\tau \leqslant \mu \leqslant 0$,*

$$P_\mu = \left(1 - \frac{|\mu|}{|\tau|}\right) P + \frac{|\mu|}{|\tau|} P_\tau.$$

Proof. Let $P = P_\tau + |\tau|P^*$ for some $\tau_1(P) \leqslant \tau \leqslant 0$. With

$$\begin{aligned} h(P_\mu, u) + |\mu| h(P^*, u) &= h(P_\mu + |\mu|P^*, u) \leqslant h(P, u) \\ &= h(P_\tau + |\tau|P^*, u) = h(P_\tau, u) + |\tau| h(P^*, u), \end{aligned}$$

we follow $P_\mu \subset P_\tau + (|\tau| - |\mu|)P^* \subset P_\mu$, which implies, for $\tau \leqslant \mu \leqslant 0$, that $P_\mu = P_\tau + (|\tau| + \mu)P^*$ and $P = P_\mu + |\mu|\, P^*$ for $\tau \leqslant \mu \leqslant 0$. Thus,

$$\begin{aligned} \frac{|\mu|}{|\tau|} P_\tau + \left(1 - \frac{|\mu|}{|\tau|}\right) P &= \frac{|\mu|}{|\tau|} P_\tau + \left(1 - \frac{|\mu|}{|\tau|}\right) \left(P_\mu + |\mu|\, P^*\right) \\ &= \frac{|\mu|}{|\tau|} P_\tau + \left(\frac{|\tau| - |\mu|}{|\tau|}\right) P_\mu + \frac{|\mu|}{|\tau|} \left(|\tau| - |\mu|\right) P^* \\ &= \frac{|\mu|}{|\tau|} \left[P_\tau + \left(|\tau| - |\mu|\right) P^*\right] + \frac{|\tau| - |\mu|}{|\tau|} P_\mu \\ &= \frac{|\mu|}{|\tau|} P_\mu + \frac{|\tau| - |\mu|}{|\tau|} P_\mu = P_\mu. \end{aligned}$$

□

Remark 3.3.11 ([49, Remark 3.10]). *The proof of Lemma 3.3.10 shows, in particular, that, if P_τ is a summand of P for some $\tau_1(P) \leqslant \tau \leqslant 0$, then P_μ is a summand of P, as well, for all $\tau \leqslant \mu \leqslant 0$.*

Now, together with Proposition 3.3.5, we can say more about the normal fan of a polytope, some of whose inner parallel bodies are summands of it:

Corollary 3.3.12 ([49, Corollary 3.11]). *Let $P \in \mathcal{K}^n$ be a polytope. If P_τ is a summand of P for all $\tau_1(P) \leqslant \tau \leqslant 0$, then the normal fans of P and P_τ coincide for all values $\tau_1(P) < \tau \leqslant 0$.*

Decompositions in dimension 2

In this section we prove that every convex polygon can be written as the sum of its kernel and the Riemann-Minkowski integral of the form bodies of its inner parallel bodies. In order to do it, we establish first the following lemma. Although its proof can be deduced from [68, Lemma 3.2.2], we include it here for completeness.

Lemma 3.3.13 ([49, Lemma 4.1]). *Let* $P \in \mathcal{K}^2$ *be a convex polygon, and let* $-r \leqslant \tau \leqslant 0$. *Then* P_τ *is a summand of* P.

Proof. We use Shephard's criterion. The first condition is obvious. For the second condition, let $u \in \mathbb{S}^{n-1}$ be such that $F(P,u)$ is an edge of P. Then $u \in \mathcal{U}_0(P)$. Since $\mathcal{U}_0(P) \supset \mathcal{U}_0(P_\tau)$, if $u \notin \mathcal{U}_0(P_\tau)$, $F(P_\tau,u)$ is a vertex, and the condition is fulfilled. Let now, otherwise, $u \in \mathcal{U}_0(P_\tau)$. Proposition 3.1.8 (ii) ensures $h(P_\tau,u) = h(P,u)+\tau$. Thus, using that $F(P_\tau,u)+|\tau|B_2 \subset P$, we have

$$\begin{aligned} F(P,u) &\supset \big(F(P_\tau,u)+|\tau|B_2\big)\cap H(P,u)\\ &= \big(F(P_\tau,u)+|\tau|B_2\big)\cap \big(H(P_\tau,u)+|\tau|u\big)\\ &= F(P_\tau,u)+|\tau|u, \end{aligned}$$

which implies the second condition. □

Notice that the above lemma implies, in particular, that if P_τ is a summand of P, then P_τ is also a summand of P_μ for all $\tau \leqslant \mu \leqslant 0$, because, from the definition of inner parallel bodies (see also Lemma E (iv)), it is clear that $P_\tau = (P_\mu)_{\tau-\mu}$. In Proposition 3.3.16 and Corollary 3.3.20, we provide examples of polytopes P all whose inner parallel bodies are summands of them, i.e., P_μ is a summand of P, for the range $-r \leqslant \mu \leqslant P$, but P_μ is not a summand of P_τ for some $\mu < \tau < 0$.

In the next result, we prove an explicit decomposition of any convex polygon P through some of its inner parallel bodies P_τ, $-r \leqslant \tau \leqslant 0$. Although this result is a consequence of Theorem 3.1.10 (see also the comments after Theorem 3.1.10 about the planar case), we provide the proof for the case of polygons, since the same argument works for the general case in the proof of Theorem 3.3.17.

Theorem 3.3.14 ([49, Theorem 4.2]). *Let* P *be a convex polygon, and let* $i \in \mathbb{N}$ *with* $\tau_{i+1}(P) \leqslant \tau \leqslant \tau_i(P)$. *Then*

$$P = P_\tau + |\tau-\tau_i(P)|P^*_{\tau_i(P)} + \sum_{j=1}^{i} |\tau_j(P)-\tau_{j-1}(P)|P^*_{\tau_{j-1}(P)}.$$

Proof. Let $\tau_{i+1}(P) \leqslant \tau \leqslant \tau_i(P)$. Then, by Lemma 3.3.13, we get that P_τ is a summand of $P_{\tau_i(P)}$. By Proposition 3.3.7, we have

$$P_{\tau_i(P)} = P_\tau + |\tau_i(P)-\tau|P^*_{\tau_i(P)}.$$

Again by Lemma 3.3.13, together with Proposition 3.3.7, $P_{\tau_i(P)}$ is a summand of $P_{\tau_{i-1}(P)}$ and

$$\begin{aligned} P_{\tau_{i-1}(P)} &= P_{\tau_i(P)} + |\tau_{i-1}(P) - \tau_i(P)| P^*_{\tau_{i-1}(P)} \\ &= P_\tau + |\tau_i(P) - \tau| P^*_{\tau_i(P)} + |\tau_{i-1}(P) - \tau_i(P)| P^*_{\tau_{i-1}(P)}. \end{aligned}$$

Applying this argument $i - 1$ times yields the theorem. □

Since P^*_τ is constant for $\tau \in (\tau_{i+1}(P), \tau_i(P)]$, this theorem is equivalent to the following result in which the numbers $\tau_i(P)$ are replaced by a Riemann-Minkowski integral.

Corollary 3.3.15 ([49, Corollary 4.3])**.** *Let $P \in \mathcal{K}^2$ be a convex polygon, and let $-r \leqslant \tau \leqslant 0$. Then*

$$P = P_\tau + \int_\tau^0 P^*_\mu d\mu.$$

Decompositions in dimension n

First, we prove that, unlike dimension 2, for $n \geqslant 3$, inner parallel bodies of a convex body may not all be summands of it. This fact will amount to a drastically different behaviour of the summands of a polytope.

Proposition 3.3.16 ([49, Proposition 5.1])**.** *Let $n \geqslant 3$.*

(i) There are n-dimensional polytopes, all of whose inner parallel bodies are summands of them.

(ii) There are n-dimensional polytopes, some of whose inner parallel bodies are summands of them, while others are not.

(iii) There are n-dimensional polytopes, non of whose inner parallel bodies are summands of them.

Proof. For $c \in [\frac{20}{3}, 12]$ let

$$P(c) = \left\{ x \in \mathbb{R}^3 : \begin{array}{rcl} \pm 12x_1 + 35x_3 & \leqslant & 432, \\ \pm 12x_2 + 5x_3 & \leqslant & 60, \\ x_3 & \geqslant & 0, \\ x_3 & \leqslant & c \end{array} \right\}$$

(see Figure 5. To ensure perspicuity, the x_1-axis is dilatated by $\frac{1}{2}$ in all pictures in this proof).

The inradius is $r = \frac{10}{3}$ for all c in the above mentioned range. The inner parallel bodies, for $-r \leqslant \tau \leqslant 0$, are given by

$$P(c)_\tau = \left\{ x \in \mathbb{R}^3 : \begin{array}{rcl} \pm 12x_1 + 35x_3 & \leqslant & 432 + 37\tau, \\ \pm 12x_2 + 5x_3 & \leqslant & 60 + 13\tau, \\ x_3 & \geqslant & 0 - \tau, \\ x_3 & \leqslant & c + \tau \end{array} \right\}$$

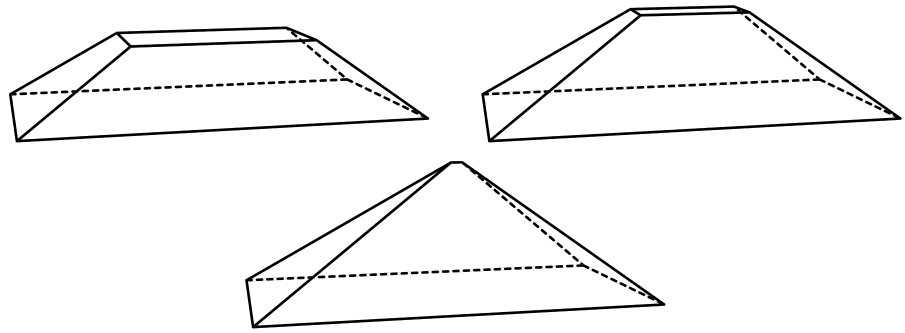

Figure 5: $P(\frac{20}{3})$, $P(9)$ and $P(12)$; (to ensure perspicuity, the x_1-axis is dilatated by $\frac{1}{2}$ in all pictures in this proof).

and

$$P(c)_{-\frac{10}{3}} = conv\left\{\left(\pm 16, 0, \frac{10}{3}\right)^\top\right\}.$$

Furthermore, $\tau_1(P(c)) = -\frac{60-5c}{8}$.

Altogether, we have

$$P(c)_\tau = conv\left\{\begin{pmatrix}\pm(36-6\tau)\\ \pm(5-\frac{3}{2}\tau)\\ \tau\end{pmatrix}, \begin{pmatrix}\pm(36-\frac{35}{12}c-\frac{1}{6}\tau)\\ \pm(5-\frac{5}{12}c-\frac{2}{3}\tau)\\ c-\tau\end{pmatrix}\right\},$$

for $\tau \in (\tau_1(P(c)), 0]$, and

$$P(c)_\tau = conv\left\{\begin{pmatrix}\pm(36-6\tau)\\ \pm(5-\frac{3}{2}\tau)\\ \tau\end{pmatrix}, \begin{pmatrix}\pm(1+\frac{9}{2}\tau)\\ 0\\ 12-\frac{13}{5}\tau\end{pmatrix}\right\}$$

for $\tau \in (-\frac{10}{3}, \tau_1(P(c))]$.

It is clear that all inner parallel bodies satisfy the first condition in Shephard's theorem. To check the second condition, the length of the upper edges with direction $(1,0,0)^\top$ is of importance.

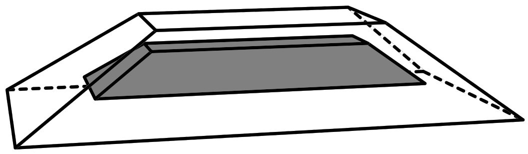

Figure 6: $P(\frac{20}{3})$ and $P(\frac{20}{3})_{\frac{40}{21}}$.

- Let $c \in [\frac{20}{3}, \frac{48}{7}]$. Then it is easy to check, that $P(c)_\tau$ satisfies the second condition in Shephard's theorem for all the values $\tau \in [-\frac{10}{3}, 0]$ (see Figure 6).
- Let $c \in (\frac{48}{7}, 12)$. Then it is easy to check, that for all values $\tau \in [-\frac{70}{9} + \frac{35}{54}c, 0]$, $P(c)_\tau$ satisfies the second condition in Shephard's theorem and, for all $\tau \in [-\frac{10}{3}, -\frac{70}{9} + \frac{35}{54}c)$, $P(c)_\tau$ does not satisfy it (see Figure 7).

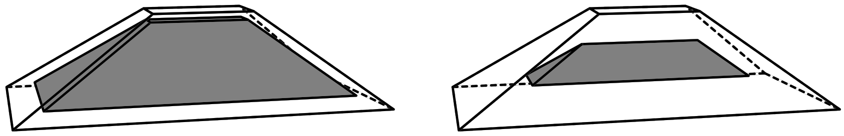

Figure 7: $P(9)$, $P(9)_{\frac{20}{21}}$ and $P(9)_{\frac{50}{21}}$.

- Let $c = 12$. Then for all $\tau \in [-\frac{10}{3}, 0)$, $P(c)_\tau$ does not satisfy the second condition in Shephard's theorem (see Figure 8).

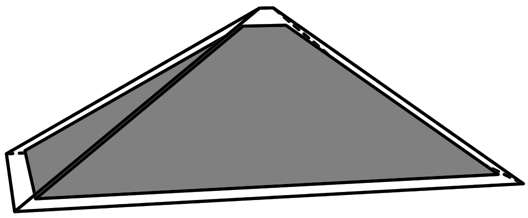

Figure 8: $P(12)$ and $P(12)_{\frac{10}{21}}$.

□

Our main result concerning decomposition of polytopes via their inner parallel bodies is the following one.

Theorem 3.3.17 ([49, Theorem 5.2]). *Let* $P \in \mathcal{K}^n$ *be a polytope, and let* $-r \leqslant \mu_1 < \mu_2 \leqslant 0$. *The following statements are equivalent:*

(i) P_τ *is a summand of* $P_{\tilde{\tau}}$, *for all* $\mu_1 \leqslant \tau \leqslant \tilde{\tau} \leqslant \mu_2$.

(ii) $h(P_{\mu_2}, u) = h(P_\tau, u) + \int_\tau^{\mu_2} h(P^*_\mu, u)\,d\mu$, *for all* $u \in \mathbb{S}^{n-1}$ *and for all* $\mu_1 \leqslant \tau \leqslant \mu_2$.

(iii) $\mathcal{U}_0(P_\tau + P^*_\tau) = \mathcal{U}_0(P_\tau)$, *for all* $\mu_1 \leqslant \tau \leqslant \mu_2$.

(iv) $\frac{d}{d\mu} h(P_\mu, u)|_{\mu=\tau} = h(P^*_\tau, u)$, *for all* $\mu_1 \leqslant \tau \leqslant \mu_2$ *such that the derivative exists for all* $u \in \mathbb{S}^{n-1}$.

(v) $u \mapsto \frac{d}{d\mu} h(P_\mu, u)|_{\mu=\tau}$ *is a support function, for all* $\mu_1 \leqslant \tau \leqslant \mu_2$ *for which the derivative exists for all* $u \in \mathbb{S}^{n-1}$.

Proof. We prove (i) ⇔ (ii), (iv) ⇔ (v), and (i) ⇒ (iii) ⇒ (iv) ⇒ (ii).

- (i) ⇔ (ii): The proof is analogous to the corresponding proof in dimension 2 (see Theorem 3.3.14 and Corollary 3.3.15).

- (i) ⇒ (iii): Let $\mu_1 \leqslant \tau \leqslant \mu_2$. Further, let $i \in \mathbb{N}$, be such that $\tau_{i+1}(P) \leqslant \tau \leqslant \tau_i(P)$. Then by Proposition 3.3.7 and (i) we have $P_{\tau_i(P)} = P_\tau + |\tau_i(P) - \tau| P^*_{\tau_i(P)} = P_\tau + |\tau_i(P) - \tau| P^*_\tau$. Thus $\mathcal{U}_0(P_\tau) = \mathcal{U}_0(P_{\tau_i(P)}) = \mathcal{U}_0(P_\tau + P^*_\tau)$.

- (iii) ⇒ (iv): This follows directly from Theorem 3.1.10.

- (iv) $\Rightarrow$ (ii): This is immediate by integrating the expression in (iv), from τ to μ_2.
- (iv) $\Leftrightarrow$ (v): Assume (v), and let $\mu_1 \leqslant \tau \leqslant \mu_2$ be such that the derivative exists. Then $\frac{d}{d\mu}h(P_\mu, \cdot)|_{\mu=\tau}$ is a support function. Hence, let R_τ be the convex body with support function $\frac{d}{d\mu}h(P_\mu, \cdot)|_{\mu=\tau}$. Since P_μ is a polytope, $h(P_\mu, u)$ is linear in u in all full-dimensional cones in the normal fan $\mathcal{N}(P_\mu)$ and thus, the same is true for $u \mapsto \frac{d}{d\mu}h(P_\mu, u)|_{\mu=\tau}$. Since support functions of polytopes are characterized by being piecewise linear support functions of convex bodies (see [29, Exercise 3.1.19.]), R_τ is a polytope and its normal fan is only coarser than that of P_τ, in other words, $\mathcal{U}_0(R_\tau) \subset \mathcal{U}_0(P_\tau) = \mathcal{U}_0(P_\tau^*)$. To prove (iv) it remains to prove $h(R_\tau, u) = h(P_\tau^*, u)$ for all $u \in \mathcal{U}_0(P_\tau^*)$, i.e., $\frac{d}{d\mu}h(P_\mu, u)|_{\mu=\tau} = 1$ for all $u \in \mathcal{U}_0(P_\tau^*)$. This is true since for any $u \in \mathcal{U}_0(P_\tau^*)$ we have $h(P_\mu, u) = h(P, u) + \mu$ and thus $\frac{d}{d\mu}h(P_\mu, u) = 1$.
The converse direction is straightforward. □

We remark that this result contains the converse of Theorem 3.1.10 for the case of polytopes, namely, it provides necessary conditions for a polytope to have equality in (3.6).

The following two results are consequences of Theorem 3.3.17.

Theorem 3.3.18 ([49, Theorem 1.1]). *Let P be a polytope with inradius r. Then P_τ is a summand of P_μ for all $-r \leqslant \tau \leqslant \mu \leqslant 0$ if and only if*

$$h(P, u) = h(P_\tau, u) + \int_\tau^0 h(P_\mu^*, u)\,d\mu,$$

for all $u \in S^{n-1}$, and all $-r \leqslant \tau \leqslant 0$.

Theorem 3.3.19 ([49, Theorem 1.2]). *Let P be a polytope and $-r < \tau \leqslant 0$. Then P_τ is a summand of P_μ for all $\tau \leqslant \mu \leqslant 0$ if and only if for $\tau \leqslant \mu \leqslant 0$ $\mathcal{U}_0(P_\mu + P_\mu^*) = \mathcal{U}_0(P_\mu)$.*

Notice that Theorem 3.3.18 is exactly the step (i) $\Leftrightarrow$ (ii) and Theorem 3.3.19 is (i) $\Leftrightarrow$ (iii), in both cases for $\mu_2 = 0$, in Theorem 3.3.17.

The assertion in the following corollary is a direct consequence of Theorems 3.3.17 and 3.1.10.

Corollary 3.3.20 ([49, Corollary 5.3]). *Let $P \in \mathcal{K}^n$ be a polytope with $\mathcal{U}_0(P_\tau + P_\tau^*) = \mathcal{U}_0(P_\tau)$ for all $-r < \tau \leqslant 0$. Then P_τ is a summand of P for all $-r \leqslant \tau \leqslant 0$. The converse is not true.*

Proof. By Theorem 3.3.17 (see also Theorem 3.1.10), the condition $\mathcal{U}_0(P_\tau + P_\tau^*) = \mathcal{U}_0(P_\tau)$ for all $-r \leqslant \tau \leqslant 0$, implies that P_τ is a summand of $P_{\tilde{\tau}}$, for all $-r \leqslant \tau \leqslant \tilde{\tau} \leqslant 0$, which proves the assertion. For the converse, let the polytope $P(c)$ be as in the proof of Proposition 3.3.16, and let $c \in (\frac{20}{3}, \frac{48}{7}]$. Then $P(c)_\tau$ is a summand of $P(c)$ for all $\tau \in [-\frac{10}{3}, 0]$. However, $P(c)_\tau$ is not a summand of $P(c)_{\tau(P(c))}$, whenever $\tau \leqslant \tau(P(c))$ and thus, by Theorem 3.3.17, $\mathcal{U}_0(P_\tau + P_\tau^*) = \mathcal{U}_0(P_\tau)$ cannot be fulfilled for all $-r \leqslant \tau \leqslant 0$. □

Moving facets outwards

In this section, we have a brief look at a similar question, if *we move the facets of the polytope outwards*.

For a polytope $P = \{x \in \mathbb{R}^n : \langle x, u_i \rangle \leqslant b_i, 1 \leqslant i \leqslant m\}$ with unit normal vectors to the facets $u_i \in \mathbb{S}^{n-1}$ and $\tau > 0$, we denote -only in this section- by $P_\tau = \{x \in \mathbb{R}^n : \langle x, u_i \rangle \leqslant b_i + \tau, 1 \leqslant i \leqslant m\}$. We remark that, here, P_τ is not an outer parallel body of the polytope P with respect to B_n. Sangwine-Yager [62, p. 55] (referring to a technique used by G. Bol, and using the notation $P(\tau)$) utilizes this construction, and observes that indeed,

$$P + \tau B_n \subseteq P + \tau P^* \subset P_\tau.$$

In the case of positive τ and using, as mentioned, the notation P_τ for this construction, the combinatorial properties of P_τ are easier than for negative τ:

Lemma 3.3.21 ([49, Lemma 6.1]). *Let $P \in \mathcal{K}^n$ be a polytope, and let $\tau > 0$. Then $\mathcal{U}_0(P) = \mathcal{U}_0(P_\tau)$.*

Proof. Let $u_1 \in \mathcal{U}_0(P)$ and assume, without loss of generality, that $F(P_\tau, u_1)$ is not a facet, i.e., $\langle x, u_1 \rangle \leqslant b_1 + \tau$ is redundant.

Hence, there are $\alpha_i > 0$, $2 \leqslant i \leqslant m$, satisfying $\sum_{i=2}^m \alpha_i u_i = u_1$ and $\sum_{i=2}^m \alpha_i (b_i + \tau) \leqslant b_1 + \tau$. Thus, $\sum_{i=2}^m \alpha_i b_i + (m-1)\tau \leqslant b_1 + \tau$. But the latter coincides with $\sum_{i=2}^m \alpha_i b_i \leqslant b_1 - (m-2)\tau \leqslant b_1$. This is a contradiction, since $F(P, u_1)$ is a facet of P. □

We want to answer the question, if and when P is a summand of P_τ, $\tau > 0$. Since, as Lemma 3.3.21 shows, P_τ has the same facet normal vectors as P for all $\tau > 0$, the situation is altogether similar and based on the situation in the interval $[\tau_1(P), 0]$, we have addressed in the previous section.

Theorem 3.3.22 ([49, Theorem 6.2]). *Let $P \in \mathcal{K}^n$ be a polytope. The following statements are equivalent:*

(i) *P is a summand of P_τ for some $\tau > 0$.*

(ii) *P is a summand of P_τ for all $\tau \geqslant 0$.*

(iii) *P is a nested summand of P_τ for all $\tau \geqslant 0$, i.e., P_μ is a summand of P_τ for all $\tau > \mu \geqslant 0$.*

(iv) $\mathcal{U}_0(P + P^*) = \mathcal{U}_0(P)$.

(v) $P_\tau = P_\mu + (\tau - \mu)P^*$ *for all $\tau > \mu \geqslant 0$.*

Proof. We prove (i) ⇒ (iv) ⇒ (ii) ⇒ (iii) ⇒ (v) and the step (v) ⇒ (i) is straightforward.

- (i) $\Rightarrow$ (iv): Let P be a summand of P_τ. By Lemma 3.3.21 we know that $\tau_1(P_\tau) < -\tau$ and thus, by Proposition 3.3.7, we have $P_\tau = (P_\tau)_{-\tau} + \tau P_\tau^* = P + \tau P^*$, which implies (iv) since we have $\mathcal{U}_0(P) = \mathcal{U}_0(P_\tau) = \mathcal{U}_0(P + P^*)$.

- (iv) $\Rightarrow$ (ii): This follows from Theorem 3.1.10.

- (ii) $\Rightarrow$ (iii): Let P be a summand of both, P_τ and P_μ, $\tau > \mu > 0$. Then, by Lemma 3.3.21 and Proposition 3.3.7, we obtain that $h(P_\tau, u) = h(P, u) + \tau h(P^*, u)$ and $h(P_\mu, u) = h(P, u) + \mu h(P^*, u)$ for all $u \in \mathbb{S}^{n-1}$. Subtraction of both equations yields the equality $h(P_\tau, u) = h(P_\mu, u) + (\tau - \mu)h(P^*, u)$ for all $u \in \mathbb{S}^{n-1}$, which implies (iii).

- (iii) $\Rightarrow$ (v): Let $\tau > \mu \geqslant 0$. Then $P_\mu = (P_\tau)_{\mu-\tau}$ is a summand of P_τ. Since $0 > \mu - \tau \geqslant -\tau > \tau_1(P_\tau)$, we get, by Proposition 3.3.7, that $P_\tau = (P_\tau)_{\mu-\tau} + |\mu - \tau| P_\tau^* = P_\mu + (\tau - \mu)P^*$.

□

4 INEQUALITIES AND OTHER APPLICATIONS

4.1 BONNESEN-STYLE INEQUALITIES

Qu'est-ce que signifie "apprivoiser" ? C'est une chose trop oubliée, dit le renard. Ça signifie "créer des liens ..." "Le petit Prince", A. Saint-Éxupery

We start this chapter recalling the Wills conjecture inequality (I.1), namely,

$$\mathrm{vol}(K) - nr(K;E)W_1(K;E) + (n-1)r(K;E)^n\mathrm{vol}(E) \leqslant 0.$$

For $K, E \in \mathcal{K}^n$, and due to the connections between inner parallel bodies and form bodies (of those), it is natural to ask whether the above inequality can be further improved (see [5], [20], and [63]) if these elements, namely, inner parallel bodies, and form bodies, are let play a role in the inequalities. With the aim of (at least partially) answering this question, we will try to use their interplay to obtain new inequalities, of the above type, involving mixed volumes of inner parallel bodies of a given convex body K, their form bodies and, in particular, the kernel of K.

As in the previous chapter, for $K \in \mathcal{K}^n$, $E \in \mathcal{K}^n_n$, we will denote by r the relative inradius of K with respect to E, i.e., $r = r(K;E)$.

Using Proposition 3.1.8 and the definition of the form body, one obtains the following inclusions, which will be important for the proofs along this chapter:

$$K_{-r} + (r+\lambda)K^* \subseteq K_{-r} + (r+\lambda)K^*_\lambda \subseteq K_\lambda \tag{4.1}$$

for $-r \leqslant \lambda \leqslant 0$.

One of the most relevant results used in the proofs of this chapter is that the function $\mathrm{vol} : [-r, 0] \longrightarrow \mathbb{R}_{\geqslant 0}$, defined as $\mathrm{vol}(\lambda) := \mathrm{vol}(K_\lambda)$, is differentiable, and its derivative can be explicitly calculated. Indeed, $\mathrm{vol}'(\lambda) = nW_1(K_\lambda;E)$ (see e.g. [6, 54], and the paragraph after (3.12)). Rewriting the latter in *integral form*, we have

$$\mathrm{vol}(K) = n\int_{-r}^{0} W_1(K_\lambda;E)\,d\lambda. \tag{4.2}$$

This will be thoroughly used in the next.

First, we prove an upper bound for the volume of a convex body in terms of the first quermassintegral and a finite sum of mixed volumes involving K, its kernel K_{-r}, its form body K^*, and the gauge body E. This result strengthens the Wills conjecture inequality (I.1) (see Remark 4.1.4).

Theorem 4.1.1 ([64, Theorem 2.1]). *Let* $K \in \mathcal{K}^n$, $E \in \mathcal{K}^n_n$. *Then*

$$\begin{aligned} \mathrm{vol}(K) \leqslant\ & nW_1(K;E)r \\ & - n\sum_{k=0}^{n-2}\sum_{j=0}^{k} \frac{\binom{k}{j}}{c_{k,j}} V\big(K_{-r}[j], K^*[k-j+1], K[n-k-2], E\big) r^{k-j+2}, \end{aligned} \tag{4.3}$$

where $c_{k,j} = (k-j+1)(k-j+2)$. *If* $K = K_{-r} + rK^*$ *equality holds. If* E *is regular and strictly convex and equality holds then* K *is a tangential body of* $K_{-r} + rE$.

This result is a direct consequence of the following more general one.

Theorem 4.1.2 ([64, Theorem 2.2]). *Let* $K \in \mathcal{K}^n$, $E \in \mathcal{K}^n_n$. *Then*

$$\begin{aligned} \mathrm{vol}(K_\lambda) \leqslant\ & nW_1(K;E)(r+\lambda) \\ & + n\sum_{k=0}^{n-2}\sum_{j=0}^{k} \Bigg[\frac{\binom{k}{j}}{c_{k,j}} V\big(K_{-r}[j], K^*[k-j+1], K[n-k-2], E\big) \\ & \qquad \big[(k-j+1)\lambda - r\big](r+\lambda)^{k-j+1} \Bigg]. \end{aligned} \tag{4.4}$$

for $-r \leqslant \lambda \leqslant 0$.

If $K = K_{-r} + rK^*$, *then equality holds. If* E *is regular and strictly convex and equality holds for some* $-r < \lambda \leqslant 0$, *then* K *is a tangential body of* $K_{-r} + rE$.

Notice that Theorem 4.1.1 is obtained by taking $\lambda = 0$ in Theorem 4.1.2.

For the proof of Theorem 4.1.2, we will need the following inequality contained in [43, Theorem 2.3]. For $i = 0, \dots, n-1$, and $-r \leqslant \lambda \leqslant 0$,

$$W_i(K_\lambda;E) \leqslant W_i(K;E) - |\lambda| \sum_{k=0}^{n-i-1} V\big(K_\lambda[k], K[n-i-k-1], K^*, E[i]\big). \tag{4.5}$$

If $K = K_{-r} + rK^*$, then equality holds in all the inequalities in (4.5). Conversely, if E is regular and strictly convex and equality holds in (4.5) for some $i \in \{0, \dots, n-1\}$, then K is a tangential body of $K_{-r} + rE$.

Proof of Theorem 4.1.2. First we consider inequality (4.5) for the case of W_1, i.e., for every $-r \leqslant \mu \leqslant 0$

$$W_1(K_\mu;E) \leqslant W_1(K;E) - |\mu| \sum_{k=0}^{n-2} V\big(K_\mu[k], K[n-k-2], K^*, E\big).$$

Using (4.2), we can integrate the inequality with respect to μ, and obtain that

$$\begin{aligned}\frac{1}{n}\mathrm{vol}(K_\lambda) &= \int_{-r}^{\lambda} W_1(K_\mu;E)\,d\mu \\ &\leqslant \int_{-r}^{\lambda}\left(W_1(K;E) - |\mu|\sum_{k=0}^{n-2} V\big(K_\mu[k],K[n-k-2],K^*,E\big)\right)d\mu \\ &\leqslant W_1(K;E)(r+\lambda) + \int_{-r}^{\lambda}\left(\mu\sum_{k=0}^{n-2} V\big(K_\mu[k],K[n-k-2],K^*,E\big)\right)d\mu. \end{aligned} \tag{4.6}$$

Notice that inside the integral we have mixed volumes depending on μ where at least three different convex bodies are involved. In order to bound these ones, we observe that (4.1) and the monotonicity of the mixed volumes (N.14) yield

$$V\big(K_\mu[k],K[n-k-2],K^*,E\big) \geqslant V\big(K_{-r}+(r+\mu)K^*[k],K[n-k-2],K^*,E\big)$$

and so, using the linearity of the mixed volumes (N.15), the integral above can be bounded as follows:

$$\begin{aligned}&\int_{-r}^{\lambda}\left(\mu\sum_{k=0}^{n-2} V\big(K_\mu[k],K[n-k-2],K^*,E\big)\right)d\mu \\ &\leqslant \int_{-r}^{\lambda}\left(\mu\sum_{k=0}^{n-2} V\big(K_{-r}+(r+\mu)K^*[k],K[n-k-2],K^*,E\big)\right)d\mu \\ &= \int_{-r}^{\lambda}\sum_{k=0}^{n-2}\sum_{j=0}^{k}\binom{k}{j}\mu(r+\mu)^{k-j}V\big(K_{-r}[j],K^*[k-j],K[n-k-2],K^*,E\big)\,d\mu \\ &= \sum_{k=0}^{n-2}\sum_{j=0}^{k}\binom{k}{j}V\big(K_{-r}[j],K^*[k-j+1],K[n-k-2],E\big)\int_{-r}^{\lambda}\mu(r+\mu)^{k-j}\,d\mu. \end{aligned}$$

Since

$$\int_{-r}^{\lambda}\mu(r+\mu)^{k-j}\,d\mu = \frac{\big[(k-j+1)\lambda - r\big](r+\lambda)^{k-j+1}}{c_{k,j}},$$

plugging this into (4.6), we get the announced bound for the volume.

If $K = K_{-r}+rK^*$, it is clear that equality holds, since the equality $K_\lambda = K_{-r}+(r+\lambda)K^*$ holds true too. If E is regular and strictly convex and equality holds in (4.4) for some $-r < \lambda \leqslant 0$, then equality also holds in (4.5). Thus, from [43, Theorem 2.3] it follows that K is a tangential body of $K_{-r}+rE$. □

As a corollary of inequality (4.3), we obtain the following strengthening of Wills' conjecture inequality (I.1).

Corollary 4.1.3 ([64, Corollary 3.1]). *Let* $E \in \mathcal{K}^n_n$, $K \in \mathcal{K}^n$. *Then*

$$\mathrm{vol}(K) \leqslant nW_1(K;E)r - n\sum_{k=0}^{n-2} W_{k+2}(K;E)\frac{r^{k+2}}{(k+1)(k+2)}. \tag{4.7}$$

Proof. Using Theorem 4.1.1, since $E \subset K^*$ we obtain that

$$\begin{aligned}\mathrm{vol}(K) &\leqslant nW_1(K;E)\,r\\ &\quad - n\sum_{k=0}^{n-2}\sum_{j=0}^{k}\binom{k}{j}V\big(K_{-r}[j], E[k-j+2], K[n-k-2]\big)\frac{r^{k-j+2}}{c_{k,j}}.\end{aligned}$$

Taking just the summands corresponding to $j=0$ for every k, we get the desired bound for the volume:

$$\begin{aligned}\mathrm{vol}(K) &\leqslant nW_1(K;E)r - n\sum_{k=0}^{n-2}V\big(K[n-k-2], E[k+2]\big)\frac{r^{k+2}}{(k+1)(k+2)}\\ &= nW_1(K;E)r - n\sum_{k=0}^{n-2}W_{k+2}(K;E)\frac{r^{k+2}}{(k+1)(k+2)}.\end{aligned}$$

□

Remark 4.1.4 ([64, Remark 3.1]). *We observe that, since $rE \subset K$, the monotonicity of the mixed volumes, i.e., relation (N.14), yields the inequality $r^n\mathrm{vol}(E) \leqslant r^iW_i(K;E) \leqslant \mathrm{vol}(K)$ and thus, the last sum which appears in the proof above can be bounded as*

$$r^n\mathrm{vol}(E)\frac{n-1}{n} \leqslant \sum_{k=0}^{n-2}W_{k+2}(K;E)\frac{r^{k+2}}{(k+1)(k+2)} \leqslant \mathrm{vol}(K)\frac{n-1}{n}.$$

Then it is clear that inequality (4.7) strengthens Wills' conjecture inequality:

$$\begin{aligned}0 &\geqslant \mathrm{vol}(K) - nW_1(K;E)r + n\sum_{k=0}^{n-2}W_{k+2}(K;E)\frac{r^{k+2}}{(k+1)(k+2)}\\ &\geqslant \mathrm{vol}(K) - nW_1(K;E)r + nr^n\mathrm{vol}(E)\sum_{k=0}^{n-2}\frac{1}{(k+1)(k+2)}\\ &= \mathrm{vol}(K) - nW_1(K;E)r + (n-1)r^n\mathrm{vol}(E).\end{aligned}$$

Notice that if $K = rK^*$ (in particular, in this case, K_{-r} is a point), then we have equality. The condition $K = rK^*$ is satisfied if and only if K is a tangential body of E (cf. Remark A.2).

Next, we employ a technique used by Diskant [20] and Brannen [11] in order to prove the following inequality.

Theorem 4.1.5 ([64, Theorem 2.3]). *Let $K \in \mathcal{K}^n$, $E \in \mathcal{K}^n_n$. Then*

$$\mathrm{vol}(K) \leqslant nW_1(K;E)r - n\sum_{j=0}^{n-1}\binom{n-1}{j}\frac{j}{j+1}V\big(K_{-r}[n-j-1], K^*[j], E\big)r^{j+1}. \tag{4.8}$$

If $K = K_{-r} + rK^*$ equality holds.

This result is a consequence, by taking $\lambda = 0$, of the following more general result, which provides us with bounds for the volume of the whole family of inner parallel bodies of K.

Theorem 4.1.6 ([64, Theorem 2.4]). *Let* $K \in \mathcal{K}^n$, $E \in \mathcal{K}^n_n$. *Then, for all* $-r < \lambda < 0$,

$$\begin{aligned} \mathrm{vol}(K_\lambda) \leqslant{} & nW_1(K;E)(r+\lambda) \\ & + n\sum_{j=0}^{n-1}\binom{n-1}{j}V\big(K_{-r}[n-j-1], K^*[j], E\big)\left[\frac{(r+\lambda)^{j+1}}{j+1} - r^j(r+\lambda)\right]. \end{aligned} \tag{4.9}$$

If $K = K_{-r} + rK^*$, *we get equality.*

The proof of Theorem 4.1.6 follows the ideas of the proof of [11, Theorem 4], which will be obtained as a corollary.

Proof of Theorem 4.1.6. First we prove the inequality

$$W_1(K;E) - W_1(K_{-r} + rK^*;E) \geqslant W_1(K_\lambda;E) - W_1(K_{-r} + (r+\lambda)K^*;E). \tag{4.10}$$

Writing $K_{-r} + rK^* = K_{-r} + (r+\lambda)K^* + |\lambda|\, K^*$, we can compute the quermassintegral $W_1(K_{-r} + rK^*;E)$ as follows:

$$\begin{aligned} & W_1(K_{-r} + rK^*;E) \\ & \quad = \sum_{j=0}^{n-1}\binom{n-1}{j}V\big(K_{-r} + (r+\lambda)K^*[j], K^*[n-j-1], E\big)\,|\lambda|^{n-j-1}. \end{aligned}$$

The right hand side can be rewritten as

$$\begin{aligned} & W_1(K_{-r} + (r+\lambda)K^*;E) \\ & \quad + \sum_{j=0}^{n-2}\binom{n-1}{j}V\big(K_{-r} + (r+\lambda)K^*[j], K^*[n-j-1], E\big)\,|\lambda|^{n-j-1}. \end{aligned}$$

Thus, the following identity holds

$$\begin{aligned} & W_1(K;E) + W_1(K_{-r} + (r+\lambda)K^*;E) - W_1(K_{-r} + rK^*;E) \\ & \quad = W_1(K;E) - \sum_{j=0}^{n-2}\binom{n-1}{j}V\big(K_{-r} + (r+\lambda)K^*[j], K^*[n-j-1], E\big)\,|\lambda|^{n-j-1}. \end{aligned}$$

Hence, it is enough to prove that

$$\begin{aligned} & W_1(K;E) - \sum_{j=0}^{n-2}\binom{n-1}{j}V\big(K_{-r} + (r+\lambda)K^*[j], K^*[n-j-1], E\big)\,|\lambda|^{n-j-1} \\ & \quad \geqslant W_1(K_\lambda;E). \end{aligned} \tag{4.11}$$

The monotonicity of the mixed volumes, together with (4.1), yields

$$\begin{aligned}
&W_1(K;E)-\sum_{j=0}^{n-2}\binom{n-1}{j}V\big(K_{-r}+(r+\lambda)K^*[j],K^*[n-j-1],E\big)\,|\lambda|^{n-j-1}\\
&\quad\geqslant W_1(K;E)-\sum_{j=0}^{n-2}\binom{n-1}{j}V\big(K_\lambda[j],K^*[n-j-1],E\big)\,|\lambda|^{n-j-1}\\
&\quad=W_1(K;E)+W_1(K_\lambda;E)\\
&\qquad-\sum_{j=0}^{n-1}\binom{n-1}{j}V\big(K_\lambda[j],K^*[n-j-1],E\big)\,|\lambda|^{n-j-1}\\
&\quad=W_1(K;E)+W_1(K_\lambda;E)-W_1(K_\lambda+|\lambda|\,K^*;E)\geqslant W_1(K_\lambda;E),
\end{aligned}$$

because $K\supseteq K_\lambda+|\lambda|\,K^*$. This proves (4.11) and hence, (4.10). Notice that if $K=K_{-r}+rK^*$ (cf. Theorem 3.1.12), then inequality (4.10) becomes an equality.

Now, integrating (4.10) and using the differentiability of $\mathrm{vol}(\lambda)$, i.e., (4.2), we get, for $-r\leqslant\lambda\leqslant 0$, that

$$\begin{aligned}
&\frac{1}{n}\mathrm{vol}(K_\lambda)=\int_{-r}^{\lambda}W_1(K_\mu;E)d\mu\\
&\quad\leqslant\int_{-r}^{\lambda}\Big[W_1(K;E)+W_1(K_{-r}+(r+\mu)K^*;E)-W_1(K_{-r}+rK^*;E)\Big]d\mu.
\end{aligned}$$

Using again the linearity of the mixed volumes for $W_1(K_{-r}+rK^*;E)$ and $W_1\big(K_{-r}+(r+\lambda)K^*;E\big)$, the previous inequality becomes

$$\begin{aligned}
&\frac{1}{n}\mathrm{vol}(K_\lambda)\leqslant(r+\lambda)W_1(K;E)\\
&\qquad-(r+\lambda)\sum_{j=0}^{n-1}\binom{n-1}{j}V\big(K_{-r}[j],K^*[n-j-1],E\big)r^{n-j-1}\\
&\qquad+\sum_{j=0}^{n-1}\binom{n-1}{j}V\big(K_{-r}[j],K^*[n-j-1],E\big)\int_{-r}^{\lambda}(r+\mu)^{n-j-1}d\mu.
\end{aligned}$$

By computing the integral, it follows that

$$\begin{aligned}
&\frac{1}{n}\mathrm{vol}(K_\lambda)\leqslant W_1(K;E)(r+\lambda)\\
&\quad+\sum_{j=0}^{n-1}\binom{n-1}{j}V\big(K_{-r}[j],K^*[n-j-1],E\big)\left[\frac{(r+\lambda)^{n-j}}{n-j}-(r+\lambda)r^{n-j-1}\right],
\end{aligned}$$

which ends the proof of (4.9).

Sufficient conditions for the equality case in (4.10) provide sufficient conditions for equality in (4.9). Thus, if $K=K_{-r}+rK^*$, we get equality for all $-r\leqslant\lambda\leqslant 0$. □

4.2 THE KERNEL CENTER OF A CONVEX BODY

A selector for a family $\mathcal{X}$ of subsets of a metric space is a function on $\mathcal{X}$ which selects a point from every member of this family. For $n \geqslant 1$, we deal with the subfamily $\mathcal{K}^n_n$ of convex bodies in $\mathbb{R}^n$ (with non-empty interior). Selectors for $\mathcal{K}^n_n$ have been studied by many authors (see, e.g., [56] and the references therein).

As we already did in the next to last section of Chapter 3, we will restrict in this section to the case in which the relative (gauge) body is the Euclidean unit ball, i.e., $E = B_n$. We will define a new selector for $\mathcal{K}^n_n$, the *kernel center map*, whose image belongs to the kernel of the convex body. This selector is constant when restricted to the family of inner parallel bodies of a given convex body.

Since the kernel of K (relative to B_n) needs not be a singleton, there is no analog to the Chebyshev center with balls containing K, replaced by balls contained in K. We recall that the Chebyshev center of $K \in \mathcal{K}^n_n$ (see [56]), is the center of the unique ball in $\mathbb{R}^n$ with minimal radius containing K, that is, the center of the circumball of K.

The kernel center map will select the center of one of the largest balls contained in a convex body. If the convex body has a unique largest ball contained in it, the kernel center may be viewed as the counterpart of the Chebyshev center for the inradius.

Let H be an affine plane of $\mathbb{R}^n$. We will denote the convex bodies contained H, with non-empty relative interior (in H), by $\mathcal{K}_n(H)$, i.e.,

$$\mathcal{K}_n(H) = \{K \in \mathcal{K}^n \ : K \subseteq H\ ,\ \operatorname{relint}_H K \neq \emptyset\}.$$

For a convex body $K \in \mathcal{K}^n$ and $H = \operatorname{aff} K$, we denote by $r_H(K)$ the inradius of K in $\mathcal{K}_n(H)$. That is, $r_H(K)$ is the usual inradius of K in $\operatorname{aff} K$, identified with $\mathbb{R}^k$, $k = \dim H$, with respect to the k-dimensional Euclidean unit ball B_k. Further, $\ker_H(K)$ denotes the kernel of K in $\mathcal{K}_n(H)$, i.e., the set of incenters of K, with respect to B_k, where we have identified the flat H with $\mathbb{R}^k$. If $H = \mathbb{R}^n$, we will write just $r(K)$ and $\ker(K)$ to denote $r(K; B_n)$ and $\ker(K; B_n)$, respectively.

As a map, the kernel, $\ker : \mathcal{K}^n_n \longrightarrow \mathcal{K}^n$ is equivariant under isometries of $\mathbb{R}^n$, but not under affine maps, as a cube and an orthogonal box with edges of different lengths show.

It is natural to ask whether the map ker is Minkowski additive, i.e., whether $\ker(K + L) = \ker(K) + \ker(L)$, for $K, L \in \mathcal{K}^n_n$. The following example shows that the answer is negative.

Example 4.2.1 ([57, Example 3.1]). *We consider* $K_1, K_2 \in \mathcal{K}^n_n$, *given by* $K_1 = [-e_1, e_1] + \sum_{i=2}^n [-2e_i, 2e_i]$ *and* $K_2 = [-2e_1, 2e_1] + \sum_{i=2}^n [-e_i, e_i]$, *two orthogonal boxes. We observe that, on the one hand,* $\ker(K_1 + K_2) = \{0\}$, *while* $\ker(K_1) = \sum_{i=2}^n [-e_i, e_i]$ *and* $\ker(K_2) = [-e_1, e_1]$.

If the convex bodies lie in orthogonal affine flats, i.e., if we are dealing with *direct sums*, we can say more. For the sake of clarity, we will use the symbol $\oplus$ also for the sum of convex bodies lying in orthogonal flats.

Theorem 4.2.2 ([57, Theorem 3.1]). *Let* H_1, H_2 *be orthogonal flats satisfying* $\mathbb{R}^n = H_1 \oplus H_2$. *Let* $K_j \in \mathcal{K}_n(H_j)$ *for* $j = 1, 2$. *Then*

i) $r(K_1 \oplus K_2) = \min_{j=1,2} r_{H_j}(K_j)$,

ii) $\ker_{H_1}(K_1) \oplus \ker_{H_2}(K_2) \subset \ker(K_1 \oplus K_2)$,

iii) $\ker_{H_1}(K_1) \oplus \ker_{H_2}(K_2) = \ker(K_1 \oplus K_2)$ *if and only if* $r_{H_1}(K_1) = r_{H_2}(K_2)$.

Proof. It is easy to see that, up to translation, $B_n \subset B_1 \oplus B_2$, where B_1 and B_2 denote, respectively, the orthogonal projection of B_n onto H_j, for $j = 1, 2$.

Let $r_j := r_{H_j}(K_j)$ for $j = 1, 2$ and $r = r(K_1 \oplus K_2)$. We may assume that $r_1 \leqslant r_2$.

i) Let $x \in \ker(K_1 \oplus K_2)$. Then $x + rB_n \subset K_1 \oplus K_2$. Projecting onto H_j, $j \in \{1, 2\}$, we obtain

$$x|H_j + rB_j \subset (K_1 \oplus K_2)|H_j = K_j,$$

and thus, $r \leqslant r_j$ for $j \in \{1, 2\}$. On the other hand, since $r_1 \leqslant r_2$, it follows that $x_j + r_1 B_j \subset K_j$ for $x_j \in \ker_{H_j}(K_j)$, $j = 1, 2$.

Hence,

$$(x_1 + x_2) + r_1 B_n \subset x_1 + x_2 + r_1(B_1 \oplus B_2) \subset K_1 \oplus K_2, \tag{4.12}$$

which proves that $r_1 \leqslant r$.

Both inequalities show that $r = r_1 = \min\{r_1, r_2\}$.

ii) From (4.12), it follows directly that if $x_j \in \ker_{H_j}(K_j)$, $j = 1, 2$, then $x_1 + x_2 \in \ker(K_1 \oplus K_2)$.

iii) Assume first that $r := r_1 = r_2$. In view of (ii), it suffices to prove that $\ker(K_1 \oplus K_2) \subset \ker_{H_1}(K_1) \oplus \ker_{H_2}(K_2)$.

Since $\ker(K_1 \oplus K_2) + rB_n \subset K_1 \oplus K_2$, by projecting onto H_j for $j \in \{1, 2\}$, we obtain

$$(\ker(K_1 \oplus K_2))|H_j + rB_j \subset K_j.$$

Hence, $(\ker(K_1 \oplus K_2))|H_j \subset \ker_{H_j}(K_j)$ and

$$\begin{aligned}(\ker(K_1 \oplus K_2))|H_1 \oplus & (\ker(K_1 \oplus K_2))|H_2 \\ &= \ker(K_1 \oplus K_2) \subset \ker_{H_1}(K_1) \oplus \ker_{H_2}(K_2).\end{aligned}$$

For the converse, if $\ker_{H_1}(K_1) \oplus \ker_{H_2}(K_2) = \ker(K_1 \oplus K_2)$, from part (i), we know that $r(K_1 \oplus K_2) = r_1$. Projecting onto H_j, we obtain $(\ker(K_1 \oplus K_2))|H_j = \ker_{H_j} K_j$ for $j = 1, 2$. If $r_1 \leqslant r_2$, then

$$\begin{aligned}\ker_{H_1}(K_1) + \ker_{H_2}(K_2) + r_1 B_n + (r_2 - r_1) B_2 \\ &\subset \ker_{H_1}(K_1) + \ker_{H_2}(K_2) + r_1(B_1 + B_2) + (r_2 - r_1) B_2 \\ &\subset K_1 \oplus K_2.\end{aligned}$$

Thus, we get

$$\begin{aligned}\ker(K_1 \oplus K_2) + r_1 B_n + (r_2 - r_1) B_2 \\ &= \ker_{H_1}(K_1) + \ker_{H_2}(K_2) + r_1 B_n + (r_2 - r_1) B_2 \\ &\subset K_1 \oplus K_2,\end{aligned}$$

which implies $r_1 = r_2$. This completes the proof. □

Example 4.2.1 shows that, if the hypotheses of the previous result, Theorem 4.2.2, are satisfied, and if $r_{H_1}(K_1) \neq r_{H_2}(K_2)$, then not only the inclusion $\ker_{H_1}(K_1) \oplus \ker_{H_2}(K_2) \subset \ker(K_1 \oplus K_2)$ is strict, but also the inequality

$$\dim \ker_{H_1}(K_1) + \dim \ker_{H_2}(K_2) \leqslant \dim \ker(K_1 + K_2)$$

may be strict.

The following remark is a direct consequence of the definition of inner parallel bodies of $K \in \mathcal{K}_n^n$ (with respect to the unit ball, i.e., $E = B_n$). We include it for completeness, as it is necessary for the coming definitions.

Remark 4.2.3 ([57, Remark 3.2]). *Let $\{K_\lambda\}_{-r \leqslant \lambda \leqslant 0}$ denote (cf. (3.1)) the system of inner parallel bodies of $K \in \mathcal{K}_n^n$ (relative to $E = B_n$). Then $\ker(K_\lambda) = \ker(K)$ for all $\lambda \in [-r, 0]$.*

Next, we introduce the kernel center selector. To every convex body $K \in \mathcal{K}_n^n$ we assign the following two finite sequences, $(\ker^{(i)}(K))_{i \geqslant 0}$ and $(H_i(K))_{i \geqslant 0}$, defined as follows:

$$\ker^{(0)}(K) := \ker(K) \quad \text{and} \quad H_0(K) := \operatorname{aff} \ker(K); \tag{4.13}$$

if $i \geqslant 1$ and $\dim \ker^{(i-1)}(K) > 0$, then

$$\ker^{(i)}(K) := \ker_{H_{i-1}(K)}(\ker^{(i-1)}(K)) \quad \text{and} \quad H_i(K) := \operatorname{aff} \ker^{(i)}(K). \tag{4.14}$$

Let $m(K) := \min\{i \geqslant 0 \ : \ \dim \ker^{(i)}(K) = 0\}$. It is clear that the sequences $(H_i(K))_{i \geqslant 0}$ and $\left(\ker_{H_i}^{(i)}(K)\right)_{i \geqslant 0}$ are descending.

Moreover, if $\dim \ker^{(i)}(K) > 0$, it follows from (N.8) that

$$\dim \ker^{(i+1)}(K) < \dim \ker^{(i)}(K).$$

Then we define $\kappa(K)$ as the unique point of $\ker^{m(K)}(K)$, or equivalently:

$$\{\kappa(K)\} = \bigcap_{i=0}^{m(K)} \ker^{(i)}(K). \tag{4.15}$$

It is clear that $m(K) \in \{0, \ldots, n\}$, and it provides the number of steps needed to reach $\kappa(K)$ when passing from $\ker(K)$ to the subsequent kernels. Further, $\kappa(K) \in K$, hence, $\kappa : \mathcal{K}^n_n \to \mathbb{R}^n$ is a selector for $\mathcal{K}^n_n$.

Remark 4.2.4 ([57, Remark 4.1]). *Let us notice that for any affine flat* H *in* $\mathbb{R}^n$, *with* $\dim H = k \leqslant n-1$, *by the already mentioned identification of* H *and* $\mathbb{R}^k$, *the functions* $r_H : \mathcal{K}_n(H) \longrightarrow \mathbb{R}$ *and* $\ker_H : \mathcal{K}_n(H) \longrightarrow \mathcal{K}^n$ *are well defined. In the same manner,* $\kappa : \mathcal{K}_n(H) \longrightarrow \mathbb{R}^n$ *is well defined.*

Since the selector κ is defined by means of the "subsequent kernels" of $K \in \mathcal{K}^n_n$, it is natural to ask whether the maps κ and ker behave in a similar way. To answer this, we deal next with some properties of κ. As a direct consequence of the behaviour of the kernel under isometries, we have also that the selector κ is equivariant under isometries of $\mathbb{R}^n$.

As it happens with the kernel, as a map, the selector κ is not equivariant under affine maps. In fact, it is not affine equivariant even when restricted to the family of simplices. For this purpose, we notice that the incenter of a simplex T coincides with $\kappa(T)$. In [21, Theorem 2.1] it is proven that the incenter of a simplex coincides with its centroid if and only if all the facets of the simplex have the same area. The centroid of a convex body is equivariant under affine transformations (see [56, Theorem 12.3.8]). Thus, it is enough to consider any simplex all whose facets do not have the same area, since it is an affine image of a regular simplex.

The next example proves that κ is not continuous with respect to the Hausdorff metric δ_H in $\mathbb{R}^n$ for any $n \geqslant 2$.

Example 4.2.5 ([57, Proposition 4.2]). *The selector* $\kappa : \mathcal{K}^n_n \to \mathbb{R}^n$ *is not continuous with respect to* δ_H.

For every natural k, *let*

$$K_k := \mathrm{conv}\left(B_n \cup \left(2e_1 + \frac{k}{1+k} B_n\right)\right),$$

and $K := \mathrm{conv}(B_n \cup (2e_1 + B_n))$. *It is easy to check, that* $K = \lim K_k$, *while* $e_1 = \kappa(K) \neq \kappa(K_k) = 0$ *for any* k.

Our next example proves that the kernel center map is not Minkowski additive (cf. Example 4.2.1).

Example 4.2.6 ([57, Proposition 4.3]). *The selector* $\kappa : \mathcal{K}^n_n \to \mathbb{R}^n$ *is not Minkowski additive for any* $n \geqslant 2$.

Let K_1, K_2 *be as follows:* $K_1 = \left(\sum_{i=1}^{n-1} [-\rho\, e_i, \rho\, e_i]\right) \oplus [-e_n, e_n]$, *and* $K_2 := \mathrm{conv}(B_n \cup \{2\rho\, e_n\})$, *for* $2 < \rho \in \mathbb{R}$. *It is easy to check that*

$\ker(K_1) = \sum_{i=1}^{n-1}[(-\rho+1)\,e_i,(\rho-1)\,e_i] \subset \mathrm{lin}\{e_1,\dots,e_{n-1}\}$, *whence* $\kappa(K_1) = 0$. *On the other hand,* K_2 *has only one largest ball centered at the origin. Thus,* $\kappa(K_2) = 0$.

It suffices to prove that $0 \notin \ker(K_1 + K_2)$. *Let us consider the Minkowski sum* $K_1 + K_2 = \left(\sum_{i=1}^{n-1}[-\rho\, e_i, \rho\, e_i]\right) + [-e_n, e_n] + \mathrm{conv}\,(B_n \cup \{2\rho\, e_n\})$. *Since* $[-e_n, 2\rho e_n] \subset K_2$, *it follows that*

$$\begin{aligned} M := &\left(\sum_{i=1}^{n-1}[-\rho\, e_i, \rho\, e_i]\right) + [-e_n, e_n] + [-e_n, 2\rho\, e_n] \\ = &\left(\sum_{i=1}^{n-1}[-\rho\, e_i, \rho\, e_i]\right) + [-2e_n, (2\rho+1)e_n] \subset K_1 + K_2. \end{aligned}$$

We observe that $r(M) \geqslant \rho$. *Hence, there exists a ball of radius at least* ρ *in* $K_1 + K_2$, *while the largest ball centered at the origin and contained in* $K_1 + K_2$ *has radius 2. Thus* $0 \notin \ker(K_1 + K_2)$, *whence* $\kappa(K_1 + K_2) \neq 0$.

The selector κ however, exhibits a nice behaviour when dealing with direct sums.

Theorem 4.2.7 ([57, Theorem 4.1])**.** *Let* H_1, H_2 *be orthogonal affine flats with* $\mathbb{R}^n = H_1 \oplus H_2$. *Let* $K_1 \in \mathcal{K}_n(H_1)$ *and* $K_2 \in \mathcal{K}_n(H_2)$. *Then*

$$\kappa(K_1 \oplus K_2) = \kappa(K_1) + \kappa(K_2).$$

Proof. Let $r_1 = r_{H_1}(K_1) \leqslant r_{H_2}(K_2) = r_2$ and $\dim H_1 = k$. By Theorem 4.2.2,

$$r(K_1 \oplus K_2) = \min\{r_1, r_2\} = r_1.$$

We prove that

$$\ker(K_1 \oplus K_2) = \ker_{H_1}(K_1) \oplus (K_2 \sim r_1 B_{n-k}). \tag{4.16}$$

Indeed,

$$\ker_{H_1}(K_1) + (K_2 \sim r_1 B_{n-k}) + r_1 B_n \subset K_1 \oplus K_2.$$

Thus,

$$\ker_{H_1}(K_1) \oplus (K_2 \sim r_1 B_{n-k}) \subset \ker(K_1 \oplus K_2).$$

On the other hand,

$$\ker(K_1 \oplus K_2) + r_1 B_n \subset K_1 \oplus K_2.$$

Projecting onto H_j, $j = 1, 2$, we obtain $\ker(K_1 \oplus K_2)|H_1 \subset \ker_{H_1}(K_1)$ and $\ker(K_2 \oplus K_2)|H_2 \subset (K_2 \sim r_1 B_{n-k})$. Hence,

$$\ker(K_1 \oplus K_2) = \ker_{H_1}(K_1) \oplus (K_2 \sim r_1 B_{n-k}),$$

which proves (4.16).

It is clear that $\dim \ker_{H_1}(K_1) < k$. Since in each step the dimension of one of the two summands decreases, in order to get $\kappa(K_1 \oplus K_2)$, we need to iterate this process a finite number of steps. After i iterations we will have, for $l, m \in \{1,2\}$, $l \neq m$, and $j \in \{1,\dots,i-1\}$, one of the following Minkowski sums: $(K_l)_\mu + \ker_{H_m}^{(i-1)}(K_m)$ for some $-r_{H_l}(K_l) < \mu < 0$ or $\ker_{H_l}^{(j)}(K_l) + \ker_{H_m}^{(i-1-j)}(K_m)$. By Remark 4.2.3, $\ker_{H_l}((K_l)_\mu) = \ker(K_l)$. Thus, after at most $m_{H_1}(K_1) + m_{H_2}(K_2) + 1$ steps we obtain $\kappa(K_1 \oplus K_2) = \kappa(K_1) + \kappa(K_2)$. □

In the next remark we verify that the kernel center map does not coincide with some well known selectors.

Remark 4.2.8 ([57, Proposition 5.1]). *The kernel center map κ is different from the Steiner point map s, the Chebyshev center $\check{c}$, the centroid c_0, the center of the minimal ring c, and the pseudo-center ps.*

(i) Let s be the Steiner point map, that is

$$s(K) := \frac{1}{\mathrm{vol}(B_n)} \int_{S^{n-1}} u h(K,u)\, du,$$

for $K \in \mathcal{K}_n^n$. Since the Steiner point map s is continuous with respect to δ_H and Minkowski additive (see, e.g., [66]), in view of Examples 4.2.5 and 4.2.6, $\kappa \neq s$.

(ii) Let $\check{c}(K)$ be the Chebyshev center of K, i.e., the center of the unique ball with minimal radius containing K. Let K be the cone over the $(n-1)$-dimensional ball $B_n \cap (\mathrm{lin}\, e_n)^\perp$, with vertex e_n. Then $\check{c}(K) = 0 \in \mathrm{bd}\, K$, while $\kappa(K) \in \mathrm{int}\, K$. Thus, $\kappa \neq \check{c}$.

(iii) Let $c_0(K)$ be the centroid of K. By [21, Theorem 3.2] the centroid and the incenter of a simplex coincide if and only if all the facets of the simplex have the same area. Hence, $\kappa \neq c_0$.

(iv) Let $c(K)$ be the center of the minimal ring containing K, that is, the minimizer of $R_K(x) - r_K(x)$, where $R_K(x)$ is the minimal radius of a ball with center x containing K, and $r_K(x)$ is the maximal radius of a ball with center x contained in K (see e.g., [2], [56]). Since the selector c is continuous with respect to the Hausdorff metric ([56, Theorem 12.5.8]), from Example 4.2.5, it follows that $\kappa \neq c$.

(v) Let $ps(K)$ be the pseudo-center of K, i.e., the symmetry center of the centrally symmetric convex body with maximal volume contained in K. The selector ps is equivariant under affine maps (see [56, Theorem 12.6.3]), while the kernel center map is not. Thus $\kappa \neq ps$.

Part III

WITHIN THE L_p-BRUNN-MINKOWSKI THEORY

In 1962, Firey [22] introduced a generalization of the Minkowski addition for $p \geqslant 1$, and $K, E \in \mathcal{K}^n_0$, the p-sum (or L_p-sum). We recall that for K and E convex bodies in $\mathcal{K}^n_0$, the p-sum of K and E is the convex body $K +_p E \in \mathcal{K}^n_0$, whose support function is given by (N.32), i.e.,

$$h(K +_p E, u) = \Big(h(K, u)^p + h(E, u)^p\Big)^{1/p}, \quad u \in \mathbb{R}^n.$$

When $p = 1$, it coincides with the usual Minkowski addition, whereas for $p = \infty$, we have $K +_\infty E = \mathrm{conv}\,(K \cup E)$. In the works [50, 51], a systematic study of means of convex bodies is carried out. This led to the outstanding development of the nowadays known as L_p-Brunn-Minkowski theory. In the last years, many important developments of this theory have come out. For further details and bibliography on the topic, we refer to [68, Chapter 9] and the references therein.

One of our principal aims in this part of the work is to find a feasible p-analog of the Minkowski difference in the context of the L_p-Brunn-Minkowski Theory.

There are several definitions of Minkowski's difference, all of which turn out to be equivalent (see [68, p. 146]). On the one hand, the Minkowski difference of two non-empty convex and compact sets $K, E \subseteq \mathbb{R}^n$ can be defined by (N.4):

$$K \sim E := \{x \in \mathbb{R}^n \, : \, x + E \subseteq K\}.$$

On the other hand, for convex bodies $K, E \in \mathcal{K}^n$, the Minkowski difference $K \sim E$ can be obtained as the Wulff-shape of the function $f(u) = h(K, u) - h(E, u)$, as in (N.41):

$$K \sim E = \bigcap_{u \in \mathbb{S}^{n-1}} \big\{x \in \mathbb{R}^n : \langle x, u\rangle \leqslant h(K, u) - h(E, u)\big\}.$$

The concept of p-sum, together with these two (equivalent) mentioned definitions of the Minkowski difference is the reference to settle down this new notion of p-difference of convex bodies, as we shall see in Chapter 5.

Once equipped with a suitable notion of p-difference, we ask for the p-analog of the concepts and results concerning the Minkowski difference within the classical Brunn-Minkowski Theory. We will first define the family of p-inner parallel bodies, under a single real parameter, following the case $p = 1$, and prove some properties concerning their behaviour with respect to this parameter of definition. In particular, we will prove that the intrinsic connection between inner parallel bodies and tangential bodies contained in Theorem 3.1.1 (and its equivalent formulation in Theorem 3.1.4) does further hold true for p-inner parallel bodies (of suitable convex bodies) and appropriate dilatation factors.

It is of particular interest to observe that the same variational principle which allows obtaining (new) functionals in the classical Brunn-Minkowski theory is a feasible approach in the context of the L_p-Brunn-Minkowski theory (cf. Theorem N). A natural example of this principle within the classical Brunn-Minkowski theory is the surface area, i.e., the 1-st quermassintegral, obtained by *applying* a linear Minkowski combination of convex bodies and a variation of the volume. The latter principle, joined to the notion of p-difference of convex bodies, leads to the natural question, whether analogs of the results about differentiability of quermassintegrals (Chapter 3), and, in particular, of the volume, in terms of the parameter of definition of the p-parallel bodies can be obtained. Unlike the case $p = 1$, where (N.17) proves directly the existence of derivatives for $\lambda > 0$ and any convex bodies K, E, when dealing with p-outer parallel bodies ($p > 1$) the existence of the derivatives is not clear. In Chapter 6, we investigate the existence of these derivatives of the quermassintegrals, as well as the support function of p-parallel bodies of a convex body K (relative to E), providing bounds for them.

Some of the obtained results are analogous counterparts of the classical cases, for example, the volume is always differentiable (in this context) for any $1 \leqslant p < \infty$. Other results however, are essentially different: for $\lambda = 0$ every quermassintegral will be differentiable (in this context) for $1 < p < \infty$, which is known not to hold for $p = 1$.

5 p-DIFFERENCE OF CONVEX BODIES

5.1 DEFINITION OF THE p-DIFFERENCE AND FIRST PROPERTIES

Hay que ser muy valiente para pedir ayuda, ¿sabes? Pero hay que ser todavía más valiente para aceptarla.
"Los besos en el pan",
A. Grandes

We start this chapter defining an operation on $\mathcal{K}_0^n$, which is intended to be the p-analog of the Minkowski difference.

Definition 5.1.1 ([53, Definition 2.1]). *Let* $K, E \in \mathcal{K}_0^n$ *be convex bodies containing the origin,* $E \subseteq K$*, and let* $p \geqslant 1$*. The* p-difference *of* K *and* E *is the largest convex body* $K \sim_p E \in \mathcal{K}_0^n$ *such that*

$$(K \sim_p E) +_p E \subseteq K. \tag{5.1}$$

On the one hand, it is clear from the above definition that

$$K \sim_p E = \bigcup_{\substack{M \in \mathcal{K}_0^n \\ M +_p E \subseteq K}} M, \tag{5.2}$$

because the above union is a convex body. Indeed, if we denote by $\mathcal{F} = \{M \in \mathcal{K}_0^n : M +_p E \subseteq K\}$, if $K_1, K_2 \in \mathcal{F}$, then we also have $\mathrm{conv}(K_1 \cup K_2) \in \mathcal{F}$, which implies that the above standard union is a convex set. Now given a sequence of points $(x_n)_n \subseteq \bigcup_{M \in \mathcal{F}} M$ with $\lim_{n\to\infty} x_n = x$, there exists a sequence $(M_n)_n \subseteq \mathcal{F}$ with $x_n \in M_n$ for each $n \in \mathbb{N}$. By Blaschke's Selection theorem, i.e., Theorem C, we can choose $(M_n)_n$ to be convergent to a convex body $\overline{M}$, and it is clear that $\overline{M} \in \mathcal{F}$. Therefore, $x \in \overline{M} \subseteq \bigcup_{M \in \mathcal{F}} M$.

Taking (N.35) into consideration, i.e.,

$$h(K +_\infty L, u) := \max\{h(K, u), h(L, u)\},$$

it is easy to check that $K \sim_\infty E = K$, and, for $p = 1$, we obviously obtain the classical Minkowski difference of K and E.

On the other hand, and looking back at (N.41), namely,

$$K \sim E = \bigcap_{u \in \mathbb{S}^{n-1}} \{x \in \mathbb{R}^n : \langle x, u\rangle \leqslant h(K, u) - h(E, u)\},$$

it would be desirable that such a kind of expression were also possible for the p-difference, in order the properties of the Wulff-shape structure to be used, as well as its connection with the support function. The following theorem shows that this is the case. First, we will assume that $1 \leqslant p < \infty$. The case $p = \infty$ will be treated later.

Theorem 5.1.2 ([53, Theorem 2.1]). *Let* $1 \leqslant p < \infty$ *and let* $K, E \in \mathcal{K}_0^n$ *with* $E \subseteq K$*. Then*

$$K \sim_p E = \bigcap_{u \in \mathbb{S}^{n-1}} \left\{x \in \mathbb{R}^n : \langle x, u\rangle \leqslant \left(h(K, u)^p - h(E, u)^p\right)^{1/p}\right\}. \tag{5.3}$$

Proof. We show (5.3) using the already known expression (5.2) for $K \sim_p E$. Let

$$L = \bigcap_{u \in \mathbb{S}^{n-1}} \left\{ x \in \mathbb{R}^n : \langle x, u \rangle \leqslant \big(h(K,u)^p - h(E,u)^p\big)^{1/p} \right\}.$$

Thus,

$$h(L,u) \leqslant \big(h(K,u)^p - h(E,u)^p\big)^{1/p},$$

and with this, we have $h(L,u)^p + h(E,u)^p \leqslant h(K,u)^p$ for all $u \in \mathbb{S}^{n-1}$, what yields $L \subseteq K \sim_p E$.

Conversely, if $x \in K \sim_p E$, then there exists $M \in \mathcal{K}^n_0$ satisfying $M +_p E \subseteq K$, such that $x \in M$, and from this condition we obtain that

$$h(M,u)^p + h(E,u)^p \leqslant h(K,u)^p \quad \text{for all } \; u \in \mathbb{S}^{n-1}.$$

It implies that $\langle x, u \rangle \leqslant \big(h(K,u)^p - h(E,u)^p\big)^{1/p}$ for all $u \in \mathbb{S}^{n-1}$, that is, $x \in L$, which shows the reverse inclusion and concludes the proof. □

We observe that $K \sim_p E$ is a convex body (cf. (N.40)) whose support function satisfies

$$h(K \sim_p E, u) \leqslant \big(h(K,u)^p - h(E,u)^p\big)^{1/p}. \tag{5.4}$$

For $p = \infty$, the right-hand side in the defining inequality (5.3) shall be seen as the limit when $p \to \infty$. Then the case $p = \infty$ is not achieved in the above result as the following example shows.

Example 5.1.3 ([53, Example 2.1]). *Let C_n be the unit cube. Then*

$$\bigcap_{u \in \mathbb{S}^{n-1}} \left\{ x \in \mathbb{R}^n : \langle x, u \rangle \leqslant \lim_{p\to\infty} \big(h(C_n,u)^p - h(B_n,u)^p\big)^{1/p} \right\} = \{0\},$$

whereas $C_n \sim_\infty B_n = C_n$ using (5.2).

The problem relies on the fact that $h(K,u) = h(E,u)$ for some $u \in \mathbb{S}^{n-1}$ provokes a *devastating geometrical* effect on the intersection expression in (5.3), whereas it is almost unseen by the union used in (5.2). Indeed, if $h(K,u) = h(E,u)$ holds for some $u \in \mathbb{S}^{n-1}$ then

$$\lim_{p\to\infty} \big(h(K,u)^p - h(E,u)^p\big)^{1/p} = 0.$$

However, if $\operatorname{bd} K \cap \operatorname{bd} E = \emptyset$, as $E \subseteq \operatorname{int} K$, we have

$$\lim_{p\to\infty} \big(h(K,u)^p - h(E,u)^p\big)^{1/p} = h(K,u)$$

obtaining that

$$\bigcap_{u \in \mathbb{S}^{n-1}} \left\{ x \in \mathbb{R}^n : \langle x, u \rangle \leqslant \lim_{p\to\infty} \big(h(K,u)^p - h(E,u)^p\big)^{1/p} \right\} = K.$$

Remark 5.1.4 ([53, Remark 2.2]). *(Convention for* $p = \infty$*) From now on we set* $K \sim_\infty E = K$*, i.e.,*

$$\big(h(K,u)^p - h(E,u)^p\big)^{1/p} = h(K,u) \quad \text{for } p = \infty \text{ and all } u \in \mathbb{S}^{n-1},$$

which is the limit when $p \to \infty$ *except if* $h(K,u) = h(E,u) \neq 0$ *for some* $u \in \mathbb{S}^{n-1}$*. With this convention, Theorem 5.1.2 remains true for* $p = \infty$*, too.*

Next we state the p-analog of Lemma E and further properties of the p-difference.

For, we observe first that when combining the p-sum with the scalar multiplication given by (N.33), namely $\lambda \cdot K = \lambda^{1/p}K$, the following basic facts hold: for all $K, E \in \mathcal{K}^n_0$, $\mu, \lambda > 0$ and $p \geqslant 1$,

$$\mu \cdot K +_p \mu \cdot E = \mu \cdot (K +_p E)$$

and

$$\mu \cdot K +_p \lambda \cdot K = \big(\mu^p + \lambda^p\big)^{1/p} K. \tag{5.5}$$

Lemma 5.1.5 ([53, Lemma 2.2 and Proposition 2.1]). *Let* $K, E, M \in \mathcal{K}^n_0$ *and* $p \geqslant 1$*. Then, assuming the suitable inclusions among the sets, the following properties hold:*

- *(i)* $(K \sim_p E) +_p E \subseteq K$*. Equality holds if and only if* E *is a* p-summand *of* K*, i.e., if and only if there exists* $L \in \mathcal{K}^n_0$ *with* $K = L +_p E$*.*
- *(ii)* $(K \sim_p E) +_p M \subseteq (K +_p M) \sim_p E$*.*
- *(iii)* $(K \sim_p E) \sim_p M = K \sim_p (E +_p M)$*.*
- *(iv)* $K +_p E \subseteq M$ *if and only if* $K \subseteq M \sim_p E$*.*
- *(v)* $(K +_p E) \sim_p E = K$*, for* $p \neq \infty$*.*
- *(vi)* $(\lambda K) \sim_p (\mu K) = \big(\lambda^p - \mu^p\big)^{1/p} K$*, for* $0 \leqslant \mu \leqslant \lambda$*.*
- *(vii)* $\lambda(K \sim_p E) = (\lambda K) \sim_p (\lambda E)$*, for all* $\lambda > 0$*.*

Proof. The proofs of (i), (ii) and (iii) are direct applications of (5.3) and (N.32), whereas (iv) follows directly from Definition 5.1.1 (cf. (5.2)).

For (v), we observe that since $h\big((K +_p E) \sim_p E, u\big)^p \leqslant h(K,u)^p$ for all $u \in \mathbb{S}^{n-1}$ (cf. (5.4)), we obtain that $(K +_p E) \sim_p E \subseteq K$. Now, (iv) for $M = K +_p E$ yields $K \subseteq (K +_p E) \sim_p E$, and thus, $K = (K +_p E) \sim_p E$.

In order to prove (vi), we first notice that from the definition of p-sum, (N.32), we get $\big(\lambda^p - \mu^p\big)^{1/p} K +_p \mu K = \lambda K$. Then, by (v), we obtain the result.

Finally we prove (vii). Taking support functions and using (5.4), it is immediate to see that $\big(\lambda(K \sim_p E)\big) +_p \lambda E \subseteq \lambda K$, which yields the inclusion $\lambda\big(K \sim_p E\big) \subseteq (\lambda K) \sim_p (\lambda E)$. Then, applying this relation to λK, λE and $1/\lambda$, we finally get

$$K \sim_p E \subseteq \frac{1}{\lambda}\Big[(\lambda K) \sim_p (\lambda E)\Big] \subseteq K \sim_p E. \qquad \square$$

In [22, Theorem 1] Firey proves that for all $1 \leqslant p \leqslant q \leqslant \infty$,

$$K +_q E \subseteq K +_p E \tag{5.6}$$

and

$$\frac{1}{2^{(p-1)/p}}(K + E) \subseteq K +_p E. \tag{5.7}$$

The following lemma is a straightforward consequence of (5.2) and (5.6).

Lemma 5.1.6 ([53, Lemma 2.3])**.** *Let* $K, E \in \mathcal{K}_0^n$ *be convex bodies with* $E \subseteq K$*, and let* $1 \leqslant p \leqslant q \leqslant \infty$*. Then*

$$K \sim_p E \subseteq K \sim_q E. \tag{5.8}$$

Remark 5.1.7 ([53, Remark 2.3])**.** *We observe here, that the inclusion given in* (5.8) *may be strict, as relation (vi) of Lemma 5.1.5 shows, since the map* $t \mapsto (1 - \varepsilon^t)^{1/t}$, $0 \leqslant \varepsilon \leqslant 1$, $t \geqslant 1$, *is strictly increasing.*

Finally, we deal with the continuity of this new operation in $\mathcal{K}_0^n$. It is known (see [68, Remark 3.1.12]) that Minkowski's subtraction is not continuous with respect to the Hausdorff metric δ_H. Next, we prove that the same holds for the p-difference of convex bodies, for any $1 < p < \infty$. For $p = \infty$, the continuity holds trivially, since $K \sim_\infty E = K$ (cf. Remark 5.1.4).

From now on, taking Example 5.1.3 and Remark 5.1.4 into consideration we will assume $p \neq \infty$.

Proposition 5.1.8 ([53, Proposition 2.1])**.** *Let* $1 < p < \infty$*. The* p*-difference is not continuous with respect to the Hausdorff metric in* $\mathcal{K}_0^n$*.*

We prove the result in the plane, i.e., in $\mathcal{K}_0^2$.

Proof of Proposition 5.1.8. We consider the convex bodies

$$K = \operatorname{conv}\left(B_2 \cup \left\{(2,1)^\top, (2,-1)^\top\right\}\right),$$
$$K_i = \operatorname{conv}\left(B_2 \cup \left\{(2,1)^\top, (2,-1+1/i)^\top\right\}\right), \quad i \in \mathbb{N}.$$

Clearly, K_i converges to K with respect to the Hausdorff metric in $\mathcal{K}_0^2$. Indeed, it can be verified that $\delta_H(K_i, K) \leqslant 1/i$.

On the one hand, we have (see (5.8)) $K \sim_p B_2 \supseteq K \sim B_2 = [0, e_1]$ for all $p > 1$. On the other hand, we claim that $K_i \sim_p B_2 = \{0\}$ for every $i \in \mathbb{N}$ and all $p > 1$, and hence we could conclude that $K_i \sim_p B_2$ does not converge to $K \sim_p B_2$, as required.

In order to prove the claim, let $i \in \mathbb{N}$ and we suppose, by contradiction, that there exists $u = (a, b)^\top \in K_i \sim_p B_2$, $u \neq 0$, which yields $[0, u] +_p B_2 \subseteq K_i$. If $b \neq 0$, then

$$h\big([0, u] +_p B_2, \operatorname{sgn}(b)e_2\big) = \big(1 + |b|^p\big)^{1/p} > 1 = h(K_i, \operatorname{sgn}(b)e_2),$$

where, as usual, sgn denotes the sign function. Clearly, it is not possible, and therefore, $b = 0$, i.e., $u = ae_1$. Now, if $a < 0$, then

$$h\big([0, ae_1] +_p B_2, -e_1\big) = \big(1 + |a|^p\big)^{1/p} > 1 = h\big(K_i, -e_1\big),$$

which is a contradiction. Hence, $a > 0$.

Let $u_i = (\cos\theta_i, \sin\theta_i)^\top \in \mathbb{S}^1$ be the unit outer normal vector to K_i at the "inclined bottom edge", i.e., the unique vector on $\mathbb{S}^1$ with coordinates $\cos\theta_i > 0$, $\sin\theta_i < 0$ (see Figure 9).

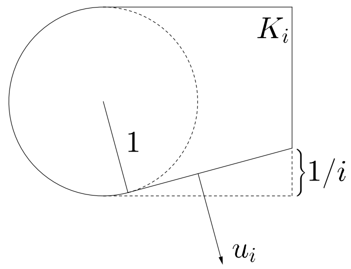

Figure 9: The p-difference is not continuous.

Then we have

$$\begin{aligned} h\big([0, ae_1] +_p B_2, u_i\big) &= (1 + a^p \cos^p\theta_i)^{1/p} > 1 \\ &= h(B_2, u_i) = h(K_i, u_i), \end{aligned}$$

which is impossible. Therefore, $K_i \sim_p B_2 = \{0\}$. □

We conclude the section with an observation on a Brunn-Minkowski type inequality. Using (N.37), the inclusion (5.1) provides, in a straightforward manner, a Brunn-Minkowski type inequality for the p-difference of two convex bodies:

Proposition 5.1.9 ([53, Proposition 2.3])**.** *Let* $K, E \in \mathcal{K}_0^n$ *with* $E \subseteq K$, *and let* $1 \leqslant p < \infty$. *Then*

$$\mathrm{vol}(K \sim_p E)^{p/n} \leqslant \mathrm{vol}(K)^{p/n} - \mathrm{vol}(E)^{p/n}.$$

Equality holds, if K *and* E *are homothetic convex bodies.*

Analogously, a Brunn-Minkowski type inequality for the (relative) quermassintegrals of the p-difference of two convex bodies can be obtained.

5.2 THE FAMILY OF p-INNER PARALLEL BODIES

In this section, we define, using the introduced p-difference, what will be called, from now on, the *full system of* p*-parallel bodies* of K, for $1 \leqslant p < \infty$. We prove, in the spirit of the results for the case $p = 1$ within the classical Brunn-Minkowski theory (cf. Chapters 3 and 4), several properties of such a system.

When dealing with the Minkowski difference, the notions of inradius and kernel play a prominent role. In addition to the classical (relative) inradius, there exists another type of inradius:

Definition 5.2.1 ([2]). *For two convex bodies* $K, E \in \mathcal{K}_0^n$, *the* (relative) inradius at the origin *of* K *with respect to* E *is given by*

$$\rho(K;E) = \max\{\rho \geqslant 0 : \rho E \subseteq K\}.$$

Regarding (any of) the (equivalent) definitions of p-difference, one would be tempted to introduce, for $K, E \in \mathcal{K}_0^n$, $E \subseteq K$, an analog of the relative inradius, i.e., a p-*inradius of* K *relative to* E as

$$\max\bigl\{r \geqslant 0 : M +_p rE \subseteq K \text{ for some } M \in \mathcal{K}_0^n\bigr\}.$$

However, it is immediate to verify that this quantity, for $p \geqslant 1$, coincides with the (relative) inradius at the origin $\rho(K;E)$. Indeed, if there exists $M \in \mathcal{K}_0^n$ such that $M +_p \rho E \subseteq K$, then

$$\rho E = \{0\} +_p \rho E \subseteq M +_p \rho E \subseteq K.$$

We observe that since the "naturally defined" p-inradius does not depend on p, and since, in general, $r(K;E) \neq \rho(K;E)$, in order to develop a structured and systematic study of the p-difference, also valid for $p = 1$, we have the *heuristic necessity* of introducing a subfamily of $\mathcal{K}_0^n$ where also the trivial cases are avoided. Thus, for $E \in \mathcal{K}_0^n$, we define the subfamily, strongly depending on the *geometry* of the body $E \in \mathcal{K}_0^n$, given by

$$\mathcal{K}_{00}^n(E) = \bigl\{K \in \mathcal{K}_0^n : r(K;E) = \rho(K;E)\bigr\} = \bigl\{K \in \mathcal{K}_0^n : 0 \in \ker(K;E)\bigr\}. \tag{5.9}$$

The last equality of sets follows easily observing that, if $0 \in \ker(K;E)$, then $r(K;E)E \subseteq K$, and thus, we have $r(K;E) \leqslant \rho(K;E)$, being the reverse inequality a direct consequence of the definition of inradius. Conversely, if $r(K;E) = \rho(K;E)$ then $r(K;E)E \subseteq K$, which implies that $0 \in \ker(K;E)$.

Definition 5.2.2 ([53]). *For* $E \in \mathcal{K}_0^n$, $K \in \mathcal{K}_{00}^n(E)$, *and* $1 \leqslant p < \infty$, *we define the* p-kernel of K with respect to E *as*

$$\ker_p(K;E) = K \sim_p r(K;E)E.$$

Then, using (5.8), it follows that, for $1 \leqslant p \leqslant q < \infty$,

$$\ker_p(K;E) \subseteq \ker_q(K;E). \tag{5.10}$$

In particular, $\ker(K;E) \subseteq \ker_p(K;E)$, for all $1 \leqslant p < \infty$.

As for the classical kernel, i.e., for $p = 1$, where (N.8) tells us that its dimension is strictly less than n, the following result shows that the p-kernel of $K \in \mathcal{K}_{00}^n(E)$ with respect to $E \in \mathcal{K}_0^n$, for any $1 \leqslant p < \infty$, has always dimension strictly less than n.

Proposition 5.2.3 ([53, Proposition 3.1]). *For* $E \in \mathcal{K}^n_0$, *let* $K \in \mathcal{K}^n_{00}(E)$. *Then, for any* $1 \leqslant p < \infty$,

$$\dim\bigl(\ker_p(K;E)\bigr) \leqslant n-1.$$

Proof. Without loss of generality, we may assume that $r(K;E) = 1$. Consider the set of vectors $U = \bigl\{u \in \mathbb{S}^{n-1} : h(K,u) = h(E,u)\bigr\} \neq \emptyset$. We observe that if we show that

$$\dim \bigcap_{u\in U} \bigl\{x \in \mathbb{R}^n : \langle x,u\rangle \leqslant 0\bigr\} \leqslant n-1, \tag{5.11}$$

then, using (5.3), we would get that

$$K \sim_p E \subseteq \bigcap_{u\in U} \bigl\{x \in \mathbb{R}^n : \langle x,u\rangle \leqslant 0\bigr\},$$

which would finish the proof. Therefore, we have to prove (5.11).

Thus, we assume, by contradiction, that

$$\dim \bigcap_{u\in U} \bigl\{x \in \mathbb{R}^n : \langle x,u\rangle \leqslant 0\bigr\} = n.$$

Let $v_1,\dots,v_n \in \mathbb{R}^n$ be n linearly independent vectors so that

$$A = \operatorname{pos}\{v_1,\dots,v_n\} \subseteq \operatorname{int} \bigcap_{u\in U} \bigl\{x \in \mathbb{R}^n : \langle x,u\rangle \leqslant 0\bigr\},$$

and let $u_1,\dots,u_n \in \mathbb{S}^{n-1}$ be n unit vectors, such that

$$\begin{aligned}&\bigcap_{u\in\{u_1,\dots,u_n\}} \bigl\{x \in \mathbb{R}^n : \langle x,u\rangle \leqslant 0\bigr\}\\ &\qquad = \bigcap_{u\in \operatorname{pos}\{u_1,\dots,u_n\}} \bigl\{x \in \mathbb{R}^n : \langle x,u\rangle \leqslant 0\bigr\} = A.\end{aligned}$$

Let $\widetilde{U} := \operatorname{pos}\{u_1,\dots,u_n\} \cap \mathbb{S}^{n-1}$. Then $U \subseteq \operatorname{relint} \widetilde{U}$ and thus,

$$\varepsilon = \min\Bigl\{h(K,u) - h(E,u) : u \in \operatorname{cl}\bigl(\mathbb{S}^{n-1}\setminus\widetilde{U}\bigr)\Bigr\}$$

is a positive real number. Hence,

$$\begin{aligned}
&A \cap \varepsilon B_n =\\
&= \left(\bigcap_{u\in\widetilde{U}} \bigl\{x \in \mathbb{R}^n : \langle x,u\rangle \leqslant 0\bigr\}\right) \cap \left(\bigcap_{u\in\mathbb{S}^{n-1}} \bigl\{x \in \mathbb{R}^n : \langle x,u\rangle \leqslant \varepsilon\bigr\}\right)\\
&= \left(\bigcap_{u\in\widetilde{U}} \bigl\{x \in \mathbb{R}^n : \langle x,u\rangle \leqslant 0\bigr\}\right) \cap \left(\bigcap_{u\in\mathbb{S}^{n-1}\setminus\widetilde{U}} \bigl\{x \in \mathbb{R}^n : \langle x,u\rangle \leqslant \varepsilon\bigr\}\right)\\
&\subseteq \bigcap_{u\in\mathbb{S}^{n-1}} \Bigl\{x \in \mathbb{R}^n : \langle x,u\rangle \leqslant h(K,u) - h(E,u)\Bigr\} = K \sim E.
\end{aligned}$$

This implies that $K \sim E$ has interior points, which contradicts (N.8). □

For a given $K \in \mathcal{K}^n_{00}(E)$, the dimension of the p-kernel will usually depend on the parameter p. Before stating in a slightly more precise way this property, we prove the following result, which allows us to determine directly the inradius and the p-kernel in a special situation.

Lemma 5.2.4 ([53, Lemma 3.1]). *Let $1 \leqslant p < \infty$. Further, let the convex body $E \in \mathcal{K}^n_0$, and $K \in \mathcal{K}^n_{00}(E)$. If $K = L +_p E$, with $L \in \mathcal{K}^n_0$, and such that $\dim L < \dim(L + E)$, then $r(K; E) = 1$ and $\ker_p(K; E) = L$.*

Proof. Since $E \subseteq L +_p E = K$, then $r(K; E) \geqslant 1$. Moreover, by (5.6) we have that $L +_p E \subseteq L + E$, and since $\dim L < \dim(L + E)$, we get $1 \leqslant r(K; E) \leqslant r(L + E; E) = 1$, i.e., $r(K; E) = 1$. Finally, using Lemma 5.1.5 (v),

$$\ker_p(K; E) = K \sim_p r(K; E)E = K \sim_p E = (L +_p E) \sim_p E = L. \qquad \square$$

For $E \in \mathcal{K}^n_0$, $K \in \mathcal{K}^n_{00}(E)$, and $1 \leqslant p \leqslant q < \infty$, the inequality

$$\dim\bigl(\ker_p(K; E)\bigr) \leqslant \dim\bigl(\ker_q(K; E)\bigr)$$

is an immediate consequence of (5.10).

The following example shows that the above inequality may be strict.

Example 5.2.5 ([53, Proposition 3.2]). *There exist $E \in \mathcal{K}^n_0$, $K \in \mathcal{K}^n_{00}(E)$, and $1 \leqslant p \leqslant q < \infty$, so that*

$$\dim\bigl(\ker_p(K; E)\bigr) < \dim\bigl(\ker_q(K; E)\bigr).$$

Proof. Let $1 \leqslant p < q < \infty$ and let $K = [-e_1, e_1] +_q B_n$. Then, by Lemma 5.2.4, we know that $\ker_q(K; B_n) = [-e_1, e_1]$. Next, we prove that $\ker_p(K; B_n) = \{0\}$, what yields the statement.

Since $K \sim_p B_n \subseteq K \sim_q B_n = [-e_1, e_1]$ (see Lemma 5.1.6), we suppose, by contradiction, that there exists $\lambda e_1 \in K \sim_p B_n$ with $0 < \lambda \leqslant 1$. It implies that $[0, \lambda e_1] \subseteq K \sim_p B_n$, i.e., $[0, \lambda e_1] +_p B_n \subseteq K$, and then

$$h\bigl([0, \lambda e_1] +_p B_n, u\bigr)^p \leqslant h(K, u)^p = h\bigl([-e_1, e_1] +_q B_n, u\bigr)^p,$$

for all $u \in \mathbb{S}^{n-1}$. In particular, taking

$$u = \Bigl(\lambda^{p/(q-p)}, \bigl(1 - \lambda^{2p/(q-p)}\bigr)^{1/2}, 0, \ldots, 0\Bigr)^{\top} \in \mathbb{S}^{n-1},$$

the above inequality is exactly $\lambda^{pq/(q-p)} + 1 \leqslant \bigl(\lambda^{pq/(q-p)} + 1\bigr)^{p/q}$, which is a contradiction because $p < q$ and $\lambda > 0$.

Because of the symmetry, the same argument shows that for all $-1 \leqslant \lambda < 0$, $\lambda e_1 \notin K \sim_p B_n$. Therefore, $K \sim_p B_n = \{0\}$, as claimed. $\square$

Finally, for $E \in \mathcal{K}^n_0$, we define the family of p-parallel bodies of $K \in \mathcal{K}^n_{00}(E)$, which we will also refer to as full-system of p-parallel bodies of $K \in \mathcal{K}^n_{00}(E)$.

Definition 5.2.6 ([53, Definition 4.1]). *Let* $E \in \mathcal{K}^n_0$ *and* $K \in \mathcal{K}^n_{00}(E)$. *Then, for any* $1 \leqslant p < \infty$,

$$K^p_\lambda = \begin{cases} K \sim_p |\lambda| E & \textit{if } -r(K;E) \leqslant \lambda \leqslant 0, \\ K +_p \lambda E & \textit{if } \qquad 0 \leqslant \lambda < \infty. \end{cases}$$

We refer to K^p_λ *as the* p-inner *(respectively,* p-outer*) parallel body of* K *at distance* $|\lambda|$*, relative to* E.

p-difference: continuity, concavity and tangential bodies

Next we show that, similarly as in the case $p = 1$, the full system of p-parallel bodies of a convex body has a certain concavity behaviour with respect to set inclusion. First, we introduce some notation for the p-sum of two real numbers. This sum happens to "fit well" when dealing with p-parallel bodies, and thus, will play an important role in the following. Since negative real numbers are allowed, this definition extends (up to a constant) the classical p-mean of positive real numbers (see [33]).

Definition 5.2.7 ([53]). *Let* $+_p : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ *denote the binary operation defined by*

$$a +_p b = \begin{cases} \operatorname{sgn}_2(a,b)\left(|a|^p + |b|^p\right)^{1/p} & \textit{if } ab \geqslant 0, \\ \operatorname{sgn}_2(a,b)\left(\max\{|a|,|b|\}^p - \min\{|a|,|b|\}^p\right)^{1/p} & \textit{if } ab < 0, \end{cases} \tag{5.12}$$

being $\operatorname{sgn}_2 : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ *the function given by*

$$\operatorname{sgn}_2(a,b) = \begin{cases} \operatorname{sgn}(a) = \operatorname{sgn}(b) & \textit{if } ab > 0, \\ \operatorname{sgn}(a) & \textit{if } ab \leqslant 0 \textit{ and } |a| \geqslant |b|, \\ \operatorname{sgn}(b) & \textit{if } ab \leqslant 0 \textit{ and } |a| < |b|. \end{cases}$$

For $ab \geqslant 0$, this notion corresponds, up to maybe a signed constant, to the classical p-mean ([33, Chapter II]), and does not coincide with any of the more general ϕ-means considered in [33, Chapter III]. Commutativity, associativity and distributivity (with respect to the usual product of real numbers) of $+_p$ can be easily proven distinguishing the sign of the involved real numbers. We collect these properties in the following lemma.

Lemma 5.2.8 ([52, Lemma 2.3.3], [53, p. 13]). *Let* $a, b, c \in \mathbb{R}$. *Then:*

(i) $a +_p b = b +_p a$,

(ii) $(a +_p b) +_p c = a +_p (b +_p c) = (a +_p c) +_p b$,

(iii) $a(b +_p c) = (ab) +_p (ac)$.

In the setting of the L_p-Brunn-Minkowski theory, where we are framing the p-difference, given $K \in \mathcal{K}_0^n$, the corresponding p-multiplication (N.33) is defined by

$$\lambda \cdot K = \lambda^{1/p} K \qquad \text{for } \lambda \geqslant 0.$$

We use the same notation in order to define, for $\lambda \geqslant 0$ and $a \in \mathbb{R}$, the p-product

$$\lambda \cdot a = \lambda^{1/p} a.$$

The following result relating the p-sum and p-product of (arbitrary) real numbers shows that the definition not only makes sense but seems to *fit* in this context.

Lemma 5.2.9 ([53, Lemma 4.1])**.** *Let* $a, b \in \mathbb{R}$, $a \leqslant b$ *and* $\lambda \in [0, 1]$. *Then, for all* $p \geqslant 1$,

$$(1-\lambda) \cdot a +_p \lambda \cdot b \in [a, b].$$

Proof. First, if $ab \geqslant 0$ then

$$\begin{aligned}(1-\lambda) \cdot a +_p \lambda \cdot b &= \left[(1-\lambda)^{1/p} a\right] +_p \left[\lambda^{1/p} b\right] \\ &= \operatorname{sgn}_2\left((1-\lambda)^{1/p} a, \lambda^{1/p} b\right) \left((1-\lambda)|a|^p + \lambda |b|^p\right)^{1/p} \\ &= \operatorname{sgn}(a)\left((1-\lambda)|a|^p + \lambda |b|^p\right)^{1/p},\end{aligned}$$

and thus, in both cases $a \geqslant 0$ and $a \leqslant 0$, we get, from the above identity,

$$a \leqslant (1-\lambda) \cdot a +_p \lambda \cdot b \leqslant b.$$

Let $ab \leqslant 0$, i.e., $a \leqslant 0 \leqslant b$. If $(1-\lambda)^{1/p}|a| \geqslant \lambda^{1/p} b$, then

$$\operatorname{sgn}_2\left((1-\lambda)^{1/p} a, \lambda^{1/p} b\right) = \operatorname{sgn}\left((1-\lambda)^{1/p} a\right) = -1.$$

Therefore (see (5.12)),

$$\begin{aligned}&(1-\lambda) \cdot a +_p \lambda \cdot b \\ &\qquad = \operatorname{sgn}_2\left((1-\lambda)^{1/p} a, \lambda^{1/p} b\right) \left((1-\lambda)|a|^p - \lambda b^p\right)^{1/p} \\ &\qquad = -\left((1-\lambda)|a|^p - \lambda b^p\right)^{1/p} \leqslant 0 \leqslant b,\end{aligned}$$

and

$$(1-\lambda) \cdot a +_p \lambda \cdot b \geqslant -\left((1-\lambda)|a|^p\right)^{1/p} = -(1-\lambda)^{1/p}|a| \geqslant -|a| = a.$$

The proof in the case $(1-\lambda)^{1/p}|a| \leqslant \lambda^{1/p} b$ is analogous to the previous one. □

The defined p-sum of real numbers (5.12) turns out to be the right operation in order to describe the behaviour of the system of p-parallel bodies, as the following proposition shows. The proof follows the one of (N.9) (see also [68, Proof of (3.20)]), just interchanging the Minkowski sum and difference of convex bodies by the p-sum and p-difference, and the usual sum of real numbers by the p-sum defined in (5.12). We include the proof here for completeness.

Proposition 5.2.10 ([53, Proposition 4.1]). *For* $E \in \mathcal{K}_0^n$, *let the convex bodies* $K, L \in \mathcal{K}_{00}^n(E)$, $-r(K;E) \leqslant \mu < \infty$, *and let* $-r(L;E) \leqslant \sigma < \infty$. *Then, for all* $1 \leqslant p < \infty$, *we have*

$$K_\mu^p +_p L_\sigma^p \subseteq (K +_p L)_{\mu +_p \sigma}^p. \tag{5.13}$$

Proof. Let $\mu, \sigma \geqslant 0$. Then $\mu E +_p \sigma E = \left(\mu^p + \sigma^p\right)^{1/p} E = (\mu +_p \sigma)E$, by using (5.5). Thus,

$$\begin{aligned} K_\mu^p +_p L_\sigma^p &= (K +_p \mu E) +_p (L +_p \sigma E) = (K +_p L) +_p (\mu +_p \sigma)E \\ &= (K +_p L)_{\mu +_p \sigma}^p. \end{aligned}$$

Lemma 5.1.5 (i) yields

$$\begin{aligned} K_{-\mu}^p +_p L_{-\sigma}^p +_p (\mu +_p \sigma)E &= (K \sim_p \mu E) +_p \mu E +_p (L \sim_p \sigma E) +_p \sigma E \\ &\subseteq K +_p L, \end{aligned}$$

and hence, by Lemma 5.2.8, we get

$$K_{-\mu}^p +_p L_{-\sigma}^p \subseteq (K +_p L)_{-(\mu +_p \sigma)}^p = (K +_p L)_{(-\mu) +_p (-\sigma)}^p.$$

If $\mu \geqslant \sigma$, using again Lemma 5.1.5 (i) and Lemma 5.2.8 we obtain

$$\begin{aligned} K_\mu^p +_p L_{-\sigma}^p &= (K +_p \mu E) +_p (L \sim_p \sigma E) \\ &= K +_p (L \sim_p \sigma E) +_p \sigma E +_p \big(\mu +_p (-\sigma)\big)E \\ &\subseteq K +_p L +_p \big(\mu +_p (-\sigma)\big)E = (K +_p L)_{\mu +_p (-\sigma)}^p. \end{aligned}$$

Finally, if $\mu \leqslant \sigma$, Lemma 5.1.5 (v) and (i) yield

$$\begin{aligned} K_\mu^p +_p L_{-\sigma}^p &+_p \big(\sigma +_p (-\mu)\big)E \\ &= (K +_p \mu E) +_p (L \sim_p \sigma E) +_p \big(\sigma +_p (-\mu)\big)E \\ &= K +_p (L \sim_p \sigma E) +_p \sigma E \subseteq K +_p L, \end{aligned}$$

which, together with Lemma 5.2.8, implies that

$$\begin{aligned} K_\mu^p +_p L_{-\sigma}^p &\subseteq (K +_p L) \sim_p \big(\sigma +_p (-\mu)\big)E \\ &= (K +_p L)_{-(\sigma +_p (-\mu))}^p = (K +_p L)_{\mu +_p (-\sigma)}^p. \qquad \square \end{aligned}$$

As we already noticed when dealing with the p-difference, its combination with the p-sum is not necessarily commutative if the difference is taken first (cf. Lemma 5.1.5). Next result shows how this fact is translated into the setting of p-parallel bodies. For $p = 1$, i.e., for the classical relative parallel bodies, the result can be found in [31].

Proposition 5.2.11 ([53, Proposition 4.2]). *For* $E \in \mathcal{K}_0^n$, *let* $K \in \mathcal{K}_{00}^n(E)$, *and let* $\lambda, \mu \geqslant 0$. *The following relations hold for any* $1 \leqslant p < \infty$*:*

(i) $\left(K_\lambda^p\right)_\mu^p = K_{\lambda +_p \mu}^p$.

(ii) $\left(K_{-\lambda}^p\right)_\mu^p \subseteq K_{(-\lambda) +_p \mu}^p$, *if* $\lambda \leqslant r(K;E)$.

(iii) $\left(K^p_{-\lambda}\right)^p_{-\mu} = K^p_{(-\lambda)+_p(-\mu)}$, *if* $\lambda +_p \mu \leqslant r(K;E)$.

(iv) $\left(K^p_{\lambda}\right)^p_{-\mu} = K^p_{\lambda+_p(-\mu)}$, *if* $\mu \leqslant r(K;E) +_p \lambda$.

(v) $\lambda K^p_\sigma = \left(\lambda K\right)^p_{\lambda\sigma}$, *for all* $-r(K;E) \leqslant \sigma < \infty$.

Proof. Items (i), (ii) and (iii) follow directly from the definition of p-sum, relation (5.13) with $L = \{0\}$, and Lemma 5.1.5 (iii), respectively, taking (5.5) into consideration.

In order to prove (iv) we notice first, that if $\lambda \geqslant \mu$ then, by item (i),

$$K^p_{\lambda+_p(-\mu)} +_p \mu E = K^p_{[\lambda+_p(-\mu)]+_p\mu} = K^p_\lambda,$$

and, using Lemma 5.1.5 (v), we obtain $K^p_{\lambda+_p(-\mu)} = (K^p_\lambda)^p_{-\mu}$.

Now if $\lambda < \mu$, item (ii) yields

$$K^p_{\lambda+_p(-\mu)} +_p \mu E \subseteq K^p_{[\lambda+_p(-\mu)]+_p\mu} = K^p_\lambda.$$

From Lemma 5.1.5 (v), we obtain that $K^p_{\lambda+_p(-\mu)} \subseteq (K^p_\lambda)^p_{-\mu}$. Moreover, using Lemma 5.1.5 (ii) and (v) , we have

$$\begin{aligned}
(K^p_\lambda)^p_{-\mu} +_p |\lambda +_p (-\mu)|E &= (K^p_\lambda \sim_p \mu E) +_p |\lambda +_p (-\mu)|E \\
&\subseteq (K^p_\lambda +_p |\lambda +_p (-\mu)|E) \sim_p \mu E \\
&= K^p_{\lambda+_p|\lambda+_p(-\mu)|} \sim_p \mu E \\
&= K^p_\mu \sim_p \mu E = K,
\end{aligned}$$

which proves the opposite inclusion $(K^p_\lambda)^p_{-\mu} \subseteq K^p_{\lambda+_p(-\mu)}$.

Finally, item (v) is straightforward from the definition of p-sum, if $\sigma \geqslant 0$, and it is a direct consequence of Lemma 5.1.5 (vii), if $\sigma \leqslant 0$. □

Next, we state some basic facts about p-parallel bodies, which will be useful later on:

Lemma 5.2.12 ([52, Lemma 3.2.8]). *Let* $E \in \mathcal{K}^n_0$, $K \in \mathcal{K}^n_{00}(E)$, *and let* $1 \leqslant p < \infty$. *Then, for all* $-r(K;E) \leqslant \lambda < \infty$,

(i) $r(K^p_\lambda;E) = r(K;E) +_p \lambda$,

(ii) $K^p_\lambda \in \mathcal{K}^n_{00}(E)$,

(iii) $\ker_p(K^p_\lambda;E) = \ker_p(K;E)$.

Proof. For the sake of brevity, we write $r = r(K;E)$. We prove (i). Since $+_p$ is associative, if $\lambda \leqslant 0$ we have that $(r +_p \lambda)E +_p |\lambda|E = rE \subseteq K$, whereas, for $\lambda > 0$, we get $(r +_p \lambda)E \subseteq K +_p \lambda E = K^p_\lambda$. Then, in both cases, we conclude that $(r +_p \lambda)E \subseteq K^p_\lambda$ and thus, $r(K^p_\lambda;E) \geqslant r +_p \lambda$. We suppose by contradiction that $r(K^p_\lambda;E) > r +_p \lambda$. Then there exists $\delta > 0$ satisfying $r(K^p_\lambda;E) = \delta +_p (r +_p \lambda)$. Therefore, we obtain that $\big(\delta +_p (r +_p \lambda)\big)E \subseteq K^p_\lambda$. Now, if $\lambda \leqslant 0$, then

$$(\delta +_p r)E = \big(\delta +_p (r +_p \lambda)\big)E +_p |\lambda|E \subseteq K,$$

whereas, for $\lambda > 0$, we have $\big((\delta +_p r) +_p \lambda\big)E \subseteq K +_p \lambda E$, which implies (by the cancellation law) that $(\delta +_p r)E \subseteq K$. Hence, in both cases, $r \geqslant \delta +_p r > r$, a contradiction.

In order to prove (ii) we observe that, by (i) we have $(r +_p \lambda)E \subseteq K$, for all $-r \leqslant \lambda < \infty$. Then $0 \in \ker(K^p_\lambda; E)$ and thus, $K^p_\lambda \in \mathcal{K}^n_{00}(E)$.

Finally, item (iii) is a direct consequence of Proposition 5.2.11 and item (i). □

Using (5.13) and Proposition 5.2.11 (v) we obtain a result about the concavity of the family of p-parallel bodies (cf. (N.10)). In Chapter 6 we will refer to an analog of this as $+_p$-concavity, since the formal property can be obtained from the classical (N.10) replacing the Minkowski sum by the p-sum, and the usual product by positive reals by the p-scalar multiplication. We refer to Definition 6.1.1 and subsequent statements for further facts about this formal replacement of sum and product when dealing with real functions.

Theorem 5.2.13 ([53, Theorem 4.1]). *For $E \in \mathcal{K}^n_0$, let $K \in \mathcal{K}^n_{00}(E)$. The full system of p-parallel sets of K, relative to E, $1 \leqslant p < \infty$, is $+_p$-concave with respect to inclusion, i.e., for $\lambda \in [0,1]$ and $\mu, \sigma \in \big[-r(K;E), \infty\big)$,*

$$(1-\lambda)\cdot K^p_\mu +_p \lambda \cdot K^p_\sigma \subseteq K^p_{(1-\lambda)\cdot\mu +_p \lambda\cdot\sigma}.$$

Proof. We notice that, by Lemma 5.2.9, $(1-\lambda)\cdot\mu +_p \lambda\cdot\sigma \geqslant -r(K;E)$. Then

$$\begin{aligned}(1-\lambda)\cdot K^p_\mu +_p \lambda\cdot K^p_\sigma &= \big((1-\lambda)^{1/p} K^p_\mu\big) +_p \big(\lambda^{1/p} K^p_\sigma\big)\\ &= \big[(1-\lambda)^{1/p} K\big]^p_{(1-\lambda)^{1/p}\mu} +_p \big[\lambda^{1/p} K\big]^p_{\lambda^{1/p}\sigma}\\ &\subseteq K^p_{(1-\lambda)\cdot\mu +_p \lambda\cdot\sigma}.\end{aligned}$$

□

Next, we deal with the continuity of the full system of p-parallel bodies in the parameter λ, with respect to the Hausdorff metric (cf. Proposition 5.1.8).

Proposition 5.2.14 ([53, Proposition 4.3]). *Let $E \in \mathcal{K}^n_0$ and $K \in \mathcal{K}^n_{00}(E)$. The function $\varphi : \big[-r(K;E), \infty\big) \longrightarrow \mathcal{K}^n_{00}(E)$ given by $\varphi(\lambda) = K^p_\lambda$ is continuous with respect to the Hausdorff metric in $\mathcal{K}^n_0$, for $1 \leqslant p < \infty$.*

Proof. From Lemma 5.2.12 (ii), we have that the image of φ is well-defined. We consider a sequence $\{\lambda_i\}_{i=1}^\infty \subseteq \big[-r(K;E), \infty\big)$, such that $\lim_{i\to\infty} \lambda_i = \lambda$. We prove that $\lim_{i\to\infty} \varphi(\lambda_i) = \varphi(\lambda)$. We notice first, that

$$\varphi(\lambda_i) = K^p_{\lambda_i} = [\varphi_{\lambda_i}],$$
$$\varphi(\lambda) = K^p_\lambda = [\varphi_\lambda],$$

where $\varphi_\mu : S^{n-1} \longrightarrow [0,\infty)$ is the (continuous) function given by

$$\varphi_\mu(u) = \Big(h(K,u)^p + \operatorname{sgn}(\mu)|\mu|^p h(E,u)^p\Big)^{1/p}.$$

From the continuity of the functions $\varphi_{\lambda_i}, \varphi_\lambda$ and the compactness of $\mathbb{S}^{n-1}$, we deduce that there exist constants $M_{\lambda_i}, M_\lambda \geqslant 0$, $i \in \mathbb{N}$, such that

$$\varphi_{\lambda_i}\big(\mathbb{S}^{n-1}\big) \subseteq [0, M_{\lambda_i}], \quad \varphi_\lambda\big(\mathbb{S}^{n-1}\big) \subseteq [0, M_\lambda].$$

Since $\lim_{i\to\infty} \lambda_i = \lambda$, the sequence $\{\lambda_i\}_{i=1}^\infty$ is bounded, and then there exists a constant $M > 0$ such that $M \geqslant M_\lambda$ and $M \geqslant M_{\lambda_i}$, $i \in \mathbb{N}$. If $\lambda \neq 0$, then $\operatorname{sgn}(\lambda_i) = \operatorname{sgn}(\lambda)$ for i large enough, whereas, if $\lambda = 0$, then

$$\varphi_{\lambda_i}^p - \varphi_0^p = \operatorname{sgn}(\lambda_i)|\lambda_i|^p h(E, \cdot)^p.$$

Therefore, we have, in both cases, that

$$\begin{aligned}\big\|\varphi_{\lambda_i}^p - \varphi_\lambda^p\big\|_\infty &= \Big\|\operatorname{sgn}(\lambda_i)\big(|\lambda_i|^p - |\lambda|^p\big)h(E, \cdot)^p\Big\|_\infty \\ &= \Big||\lambda_i|^p - |\lambda|^p\Big|\big\|h(E, \cdot)^p\big\|_\infty\end{aligned}$$

for i large enough, and thus, $\lim_{i\to\infty}\big\|\varphi_{\lambda_i}^p - \varphi_\lambda^p\big\|_\infty = 0$.

Since the function $[0, M] \longrightarrow \mathbb{R}$ given by $t \mapsto t^{1/p}$ is uniformly continuous, then

$$\lim_{i\to\infty} \|\varphi_{\lambda_i} - \varphi_\lambda\|_\infty = 0.$$

Now, Theorem O implies that $\lim_{i\to\infty}[\varphi_{\lambda_i}] = [\varphi_\lambda]$, as desired. □

Next, we focus on special families of sets, for which (some) p-parallel bodies can be explicitly determined. The first family we are going to address is the family of tangential bodies. For these convex bodies, their p-inner parallel bodies can be easily obtained (see Figure 10, cf. Lemma 5.1.5 (vi)). We recall that tangential bodies already exhibit an analogous behaviour for $p = 1$, as Theorem 3.1.1 (and its equivalent formulation Theorem 3.1.4) shows.

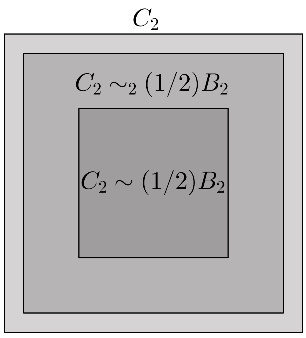

Figure 10: 1 and 2-difference of the square C_2 and the ball $(1/2)B_2$.

Proposition 5.2.15 ([53, Proposition 4.4]). *Let $E \in \mathcal{K}_0^n$ and let $K \in \mathcal{K}_0^n$ be a tangential body of E. Then, for all $1 \leqslant p < \infty$, and any $\lambda \in [0, 1]$,*

$$K_{-\lambda}^p = (1 - \lambda^p)^{1/p} K. \tag{5.14}$$

Proof. We recall that $r(K;E) = 1$. Let $\mathcal{U} \subseteq \mathbb{S}^{n-1}$ be the set of those outer normal vectors of K, for which the support hyperplane to K also supports E, i.e., such that $h(K,u) = h(E,u)$. Since any outer normal vector at a regular point of K is of this type, (N.42) ensures that

$$K = \bigcap_{u \in \mathcal{U}} \{x \in \mathbb{R}^n : \langle x,u \rangle \leqslant h(K,u)\}.$$

Therefore, we get, on the one hand,

$$\begin{aligned} K^p_{-\lambda} &= K \sim_p \lambda E \\ &= \bigcap_{u \in \mathbb{S}^{n-1}} \left\{x \in \mathbb{R}^n : \langle x,u \rangle \leqslant \left(h(K,u)^p - \lambda^p h(E,u)^p\right)^{1/p}\right\} \\ &\subseteq \bigcap_{u \in \mathcal{U}} \left\{x \in \mathbb{R}^n : \langle x,u \rangle \leqslant (1-\lambda^p)^{1/p} h(K,u)\right\} \\ &= (1-\lambda^p)^{1/p} K. \end{aligned}$$

On the other hand, since $E \subseteq K$, then

$$h\big((1-\lambda^p)^{1/p}K,u\big)^p = (1-\lambda^p)h(K,u)^p \leqslant h(K,u)^p - \lambda^p h(E,u)^p$$

for all $u \in \mathbb{S}^{n-1}$. Hence,

$$\begin{aligned} K^p_{-\lambda} &= \bigcap_{u \in \mathbb{S}^{n-1}} \left\{x \in \mathbb{R}^n : \langle x,u \rangle \leqslant \left(h(K,u)^p - \lambda^p h(E,u)^p\right)^{1/p}\right\} \\ &\supseteq \bigcap_{u \in \mathbb{S}^{n-1}} \left\{x \in \mathbb{R}^n : \langle x,u \rangle \leqslant h\big((1-\lambda^p)^{1/p}K,u\big)\right\} \\ &= (1-\lambda^p)^{1/p} K. \end{aligned}$$

□

Indeed, tangential bodies can be characterized by (5.14), i.e., as the only convex bodies such that their p-inner parallel bodies are homothetic copies of them (see Figure 10). The case $p = 1$ is exactly Theorem 3.1.1). In order to prove the result for the case $p > 1$, we need the following auxiliary result, which proves that p-inner parallel bodies of tangential bodies are strongly related to the classical inner parallel ones (with appropriate dilatations). For the sake of brevity we will assume that $r(K;E) = 1$.

Proposition 5.2.16 ([53, Proposition 4.5]). *Let* $K, E \in \mathcal{K}^n_0$ *with* $E \subseteq K$, *and let* $r(K;E) = 1$. *Let further* $1 \leqslant p < \infty$, *and* $\lambda \in [0,1]$. *If* $K^p_{-\lambda} = \theta K$ *for some* $\theta \in [0,1]$, *then* $\theta = (1-\lambda^p)^{1/p}$ *and* $K_{-(1-\theta)} = \theta K$.

Proof. First, we prove that

$$\text{if } K^p_{-\lambda} = \theta K \text{ for } 0 \leqslant \lambda \leqslant 1, \text{ then } \theta = (1-\lambda^p)^{1/p}. \tag{5.15}$$

Indeed, since $(1-\lambda^p)^{1/p}K +_p \lambda E \subseteq (1-\lambda^p)^{1/p}K +_p \lambda K = K$, then we get

$$(1-\lambda^p)^{1/p}K \subseteq K \sim_p \lambda E = \theta K,$$

which yields $\theta \geqslant (1-\lambda^p)^{1/p}$. Moreover, since $r(K;E)=1$, there exists $u \in \mathbb{S}^{n-1}$ such that $h(K,u)=h(E,u)>0$ (see [8, p. 59]). Therefore,

$$\begin{aligned}\theta h(K,u) &= h(\theta K,u) = h(K \sim_p \lambda E, u)\\ &\leqslant \big(h(K,u)^p - \lambda^p h(E,u)^p\big)^{1/p} = (1-\lambda^p)^{1/p} h(K,u),\end{aligned}$$

and since $h(K,u)>0$, we get $\theta \leqslant (1-\lambda^p)^{1/p}$, which proves (5.15).

Now we prove the second statement of the proposition.

First we observe that $\theta K + (1-\theta)E \subseteq \theta K + (1-\theta)K = K$, which yields

$$\theta K \subseteq K \sim (1-\theta)E = K_{-(1-\theta)}.$$

Next, we assume, that there exists $x \in \big(K \sim (1-\theta)E\big)\backslash\theta K$, and argue by contradiction. In particular, we have $x \notin \theta K = K^p_{-\lambda}$, and thus, using (5.3), there is $u_x \in \mathbb{S}^{n-1}$ such that

$$\langle x, u_x\rangle > \big(h(K,u_x)^p - \lambda^p h(E,u_x)^p\big)^{1/p}. \tag{5.16}$$

Moreover, since $x + (1-\theta)E \subseteq K$, taking support functions we get

$$\langle x, u_x\rangle + (1-\theta)h(E,u_x) \leqslant h(K,u_x), \tag{5.17}$$

and joining both inequalities, (5.16) and (5.17), we obtain

$$\big(h(K,u_x)^p - \lambda^p h(E,u_x)^p\big)^{1/p} < h(K,u_x) - (1-\theta)h(E,u_x). \tag{5.18}$$

We notice that $h(K,u_x) > 0$, since $h(K,u_x) \geqslant \langle x,u_x\rangle > 0$ (cf. (5.16)). Thus, writing

$$\alpha = \frac{h(E,u_x)}{h(K,u_x)} \in [0,1],$$

inequality (5.18) becomes

$$(1-\lambda^p\alpha^p)^{1/p} < 1-(1-\theta)\alpha. \tag{5.19}$$

In order to get the contradiction, we define $f(\alpha) = (1-\lambda^p\alpha^p)^{1/p}$ on $[0,1]$. Direct calculations yield

$$f''(\alpha) = -(p-1)\lambda^p\alpha^{p-2}(1-\lambda^p\alpha^p)^{(1-2p)/p} \leqslant 0 \qquad \text{for all } \alpha \in (0,1),$$

i.e., f is a concave function, with $f(0)=1$ and $f(1)=(1-\lambda^p)^{1/p}=\theta$ (cf. (5.15)), which implies that $f(\alpha) \geqslant 1-(1-\theta)\alpha$ for all $\alpha \in [0,1]$. The latter contradicts (5.19), and proves the result. □

Remark 5.2.17 ([53, Remark 4.1]). *Proposition 5.2.16 states that there is a bijection between p-inner parallel bodies and the inner parallel bodies of K, in the case when they all are homothetic to K (cf. Theorem 3.1.1). This bijection is given by*

$$K^p_{-\lambda} \longleftrightarrow K_{-1+(1-\lambda^p)^{1/p}}.$$

Theorem 5.2.18 ([53, Theorem 4.2]). *Let* $K, E \in \mathcal{K}_0^n$, $\operatorname{int} E \neq \emptyset$, *with* $E \subseteq K$ *and* $r(K; E) = 1$. *Let* $1 \leqslant p < \infty$ *and* $\lambda \in (0, 1)$. *Then* K *is a tangential body of* E *if and only if* $K^p_{-\lambda}$ *is homothetic to* K.

Proof. If K is a tangential body of E, then $K^p_{-\lambda} = (1-\lambda^p)^{1/p}K$, by Proposition 5.2.15. Conversely, if $K^p_{-\lambda} = \theta K$ for some $\theta \in (0, 1)$, then Proposition 5.2.16 ensures that $K_{-(1-\theta)} = \theta K$ with $\theta = (1-\lambda^p)^{1/p}$. Finally, Theorem 3.1.1 shows that K is a tangential body of E. □

The second (and last) family of convex bodies for which we will, more explicitly, determine their p-inner parallel bodies is the family of convex bodies which are obtained as p-outer parallel bodies of some lower-dimensional convex body.

Proposition 5.2.19 ([53, Proposition 4.6]). *For* $E \in \mathcal{K}_0^n$, *let* $K \in \mathcal{K}_{00}^n(E)$ *be given by* $K = L +_p \mu E$, *with* $L \in \mathcal{K}_0^n$, $\dim L < \dim(L + E)$, *and* $\mu \geqslant 0$. *Then, for all* $\lambda \in [-\mu, \infty)$,

$$K^p_\lambda = L +_p (\mu +_p \lambda)E.$$

Proof. For $\lambda \geqslant 0$, the result follows directly from the definitions of p-sum of convex bodies and p-sum of real numbers. If $-\mu \leqslant \lambda \leqslant 0$, since $r(L; E) = 0$, we can use Proposition 5.2.11 (iv) in order to obtain

$$K^p_\lambda = K \sim_p |\lambda| E = (L +_p \mu E) \sim_p |\lambda| E = L +_p (\mu +_p \lambda)E. \qquad \square$$

We notice, moreover, that in this case $\ker_p(K; E) = L$ and $r(K; E) = \mu$ (see Lemma 5.2.4). If we remove the assumption $\dim L < \dim(L + E)$, then the result also holds in a suitable range of λ.

We finish this chapter proving the analog of Lemma 3.1.3 (i) for p-sums, $1 < p < \infty$. Lemma 3.1.3 (i) ensures that the 0-extreme vectors of the Minkowski addition of the convex bodies $K, L \in \mathcal{K}^n$ satisfy the following relation:

$$\mathcal{U}_0(K) \cup \mathcal{U}_0(L) \subseteq \mathcal{U}_0(K + L).$$

It is not difficult to see that if the Minkowski sum is replaced by the p-sum, $+_p$, $1 < p < \infty$, the above relation remains true, namely, we have the following proposition.

Proposition 5.2.20 ([52, Proposition 4.1.8]). *Let* $K, L \in \mathcal{K}_0^n$ *and* $1 \leqslant p < \infty$. *Then*

$$\mathcal{U}_0(K) \cup \mathcal{U}_0(L) \subseteq \mathcal{U}_0(K +_p L).$$

Proof. Let $u \in \mathcal{U}_0(K)$ and let $u_1, u_2 \in \mathbb{S}^{n-1}$, $u_1 \neq u_2$, be such that $u = \alpha u_1 + \beta u_2$, with $\alpha, \beta > 0$. Then, by Lemma 3.1.2, we have

$$h(K, u) < \alpha h(K, u_1) + \beta h(K, u_2).$$

For L, the subadditivity of the support function yields

$$h(L, u) \leqslant \alpha h(L, u_1) + \beta h(L, u_2).$$

Using now Minkowski's inequality for sums of non-negative numbers (see, e.g., [33, p. 30], [28, Corollary 1.6]), we obtain

$$\begin{aligned} h(K +_p L, u) &= \big(h(K,u)^p + h(L,u)^p\big)^{1/p} \\ &< \Big(\big[\alpha h(K,u_1) + \beta h(K,u_2)\big]^p + \big[\alpha h(L,u_1) + \beta h(L,u_2)\big]^p\Big)^{1/p} \\ &\leqslant \Big(\big(\alpha h(K,u_1)\big)^p + \big(\alpha h(L,u_1)\big)^p\Big)^{1/p} \\ &\qquad\qquad + \Big(\big(\beta h(K,u_2)\big)^p + \big(\beta h(L,u_2)\big)^p\Big)^{1/p} \\ &= \alpha\Big(h(K,u_1)^p + h(L,u_1)^p\Big)^{1/p} + \beta\Big(h(K,u_2)^p + h(L,u_2)^p\Big)^{1/p} \\ &= \alpha\, h(K +_p L, u_1) + \beta\, h(K +_p L, u_2). \end{aligned}$$

Lemma 3.1.2 implies that $u \in \mathcal{U}_0(K +_p L)$. Thus, $\mathcal{U}_0(K) \subseteq \mathcal{U}_0(K +_p L)$. Analogously, we get $\mathcal{U}_0(L) \subseteq \mathcal{U}_0(K +_p L)$. □

6 DIFFERENTIABILITY AND p-PARALLEL BODIES

6.1 ON $+_p$-CONCAVITY

The human heart is a line, whereas my own is a circle, and I have the endless ability to be in the right place at the right time. The consequence of this is that I'm always finding humans at their best and worst. I see their ugly and their beauty, and I wonder how the same thing can be both.
"The Book Thief", M. Zusak

We start this chapter defining $+_p$-concavity of real functions (cf. Theorem 5.2.13 and the comments before). The motivation for this definition comes from the fact that, in several cases, when dealing with functions which involve p-parallel bodies, inequalities which contain $+_p$ and resemble concavity, appear. We notice that given an interval $I \subseteq \mathbb{R}$, $x, y \in I$ and $\lambda \in [0,1]$, it follows from Lemma 5.2.9 that $(1-\lambda)\cdot x +_p \lambda \cdot y \in I$.

Definition 6.1.1 ([39, Definition 8]). *Let $f : I \longrightarrow \mathbb{R}$ for $I \subseteq \mathbb{R}$ an interval, and let $1 \leqslant p < \infty$. We say that f is* $+_p$-concave *if for all $x, y \in I$ and $\lambda \in [0,1]$,*

$$f\big((1-\lambda)\cdot x +_p \lambda \cdot y\big) \geqslant (1-\lambda)f(x) + \lambda f(y).$$

We say that f is $+_p$-convex *if $-f$ is $+_p$-concave.*

If $p = 1$, it is the usual definition of concavity. Although $+_p$-concave functions may not be as *nice* as concave functions, sometimes they share their good properties. Next, we prove that monotone $+_p$-concave functions, in suitable intervals, are indeed concave. As a consequence, we obtain directly the existence of derivatives almost everywhere (cf. Proposition B.2), as well as absolute continuity (cf. Lemma B.1) for those functions.

Lemma 6.1.2. *[39, Lemma 9] Let $f : I \longrightarrow \mathbb{R}$, with $I \subseteq (-\infty, 0]$ an interval, be an increasing $+_p$-concave function, $1 \leqslant p < \infty$. Then f is a concave function.*

Proof. Let $x, y \in I$ and $\lambda \in [0,1]$. Using the concavity of $t \mapsto t^p$ for $t \geqslant 0$ we get

$$\begin{aligned}(1-\lambda)\cdot x +_p \lambda \cdot y &= -\big((1-\lambda)(-x)^p + \lambda(-y)^p\big)^{1/p} \\ &\leqslant (1-\lambda)x + \lambda y.\end{aligned}$$

Using that f is increasing and $+_p$-concave, we get that f is concave on I. □

Next, we prove that $+_p$-concave functions are quasi-concave, although there is no direct relation between $+_p$-concave functions and concave ones.

Lemma 6.1.3 ([39, Lemma 10]). *Let $1 \leqslant p < \infty$, and let $I \subseteq \mathbb{R}$ be an interval. If $f : I \longrightarrow \mathbb{R}$ is $+_p$-concave, then f is quasi-concave.*

Proof. We observe that, by the intermediate value theorem, there exists $\mu_\lambda \in [0,1]$ such that $(1-\lambda)x+\lambda y = (1-\mu_\lambda)\cdot x +_p \mu_\lambda \cdot y$. Therefore,

$$\begin{aligned} f\big((1-\lambda)x+\lambda y\big) &= f\big((1-\mu_\lambda)\cdot x +_p \mu_\lambda \cdot y\big) \\ &\geqslant (1-\mu_\lambda)f(x)+\mu_\lambda f(y) \geqslant \min\big\{f(x), f(y)\big\}. \square \end{aligned}$$

Remark 6.1.4 ([39, Remark 11]). *In general, there is no relation between concavity and $+_p$-concavity. Indeed, let $f(x) = x^p$, $p > 1$, which is a convex function on $[0,\infty)$. Then:*

(i) f is $+_q$-convex (and not $+_q$-concave) if $1 \leqslant q < p$.

(ii) f is $+_q$-concave (and not $+_q$-convex) if $p < q < \infty$.

(iii) f satisfies $f\big((1-\lambda)\cdot x +_p \lambda \cdot y\big) = (1-\lambda)f(x)+\lambda f(y)$, for $\lambda \in [0,1]$, and $x, y \in [0,\infty)$.

6.2 DIFFERENTIABILITY OF QUERMASSINTEGRALS

In this section we investigate the differentiability of relative quermassintegrals with respect to the one-parameter family of p-parallel bodies of a convex body $K \in \mathcal{K}^n_{00}(E)$, relative to $E \in \mathcal{K}^n_{(0)}$. As it happens in the classical case (Chapter 3), we prove that the volume is always differentiable in the full range $\big(-r(K;E),\infty\big)$. Moreover, although there is no polynomial expression for the quermassintegrals of a p-sum (see e.g. [26] and [55]), we prove that all quermassintegrals are also differentiable for positive values of the parameter, as well as at $\lambda = 0$, when $p > 1$.

For $E \in \mathcal{K}^n_{(0)}$, $K \in \mathcal{K}^n_{00}(E)$ we consider the functions $\lambda \mapsto W_i(K^p_\lambda;E)$, on $\lambda \in \big(-r(K;E),\infty\big)$, for $1 \leqslant p < \infty$ and $0 \leqslant i \leqslant n-1$. For the sake of brevity, we write $W_i(\lambda) = W_i(K^p_\lambda;E)$. Moreover, we also use $W_{p,i}(\lambda, L;E) = W_{p,i}(K^p_\lambda, L;E)$ and $h(\lambda,u) = h(K^p_\lambda,u)$, for $u \in \mathbb{S}^{n-1}$. Finally, we will write, from now on, $r = r(K;E)$ for the relative inradius of K (relative to E).

First, we observe that inequality (N.38) and Theorem 5.2.13 imply that the function $W_i(\lambda)^{p/(n-i)}$ is $+_p$-concave and increasing on $(-r,0)$. Then Lemma 6.1.2 ensures that it is concave on this range. Hence the left and right derivatives of $W_i(\lambda)$ exist in $(-r,0)$.

For the range $[0,\infty)$, we will first prove that the right derivative of $W_i(\lambda)$ always exists. Indeed, in order to do so, we will obtain a stronger result, namely, we prove a lower bound for the right derivative of $W_i(\lambda)$ with respect to λ, for the whole range of definition $[-r,\infty)$. As a by-product, along the proof, we obtain the existence of the mentioned right derivative.

6.2.1 *One-sided derivatives of W_i*

Proposition 6.2.1 ([39, Proposition 14]). *Let* $E \in \mathcal{K}^n_{(0)}$ *and* $K \in \mathcal{K}^n_{00}(E)$. *Let further* $1 \leqslant p < \infty$ *and* $0 \leqslant i \leqslant n-1$. *Then, wherever the right derivative exists,*

$$\frac{d^+}{d\lambda} W_i(\lambda) \geqslant |\lambda|^{p-1}(n-i) W_{p,i}(\lambda, E; E) \quad \text{on } [-r, \infty), \tag{6.1}$$

and equality holds if $\lambda \in [0, \infty)$.

For the proof of this result we need the following property.

Lemma 6.2.2 ([39, Lemma 15]). *Let* $E \in \mathcal{K}^n_{(0)}$, $K \in \mathcal{K}^n_{00}(E)$, $1 \leqslant p < \infty$, $0 \leqslant i \leqslant n-1$, *and let* $\lambda \in [-r, \infty)$ *and* $\varepsilon > 0$. *If there exist suitable positive constants* C, c *not depending on* ε, *such that:*

(i) $K^p_{\lambda+\varepsilon} \supseteq K^p_\lambda +_p (\varepsilon C)^{1/p} E$ *for all* $\varepsilon \leqslant c$, *then*

$$\frac{d^+}{d\lambda} W_i(\lambda) \geqslant C\, \frac{n-i}{p} W_{p,i}(\lambda, E; E);$$

(ii) $K^p_{\lambda+\varepsilon} \subseteq K^p_\lambda +_p (\varepsilon C)^{1/p} E$ *for all* $\varepsilon \leqslant c$, *then*

$$\frac{d^+}{d\lambda} W_i(\lambda) \leqslant C\, \frac{n-i}{p} W_{p,i}(\lambda, E; E).$$

Proof. We prove (i); part (ii) is analogous.

Let $C > 0$ be such that $K^p_{\lambda+\varepsilon} \supseteq K^p_\lambda +_p (\varepsilon C)^{1/p} E$. Then the monotonicity of mixed volumes (N.14) yields

$$\frac{W_i(\lambda+\varepsilon) - W_i(\lambda)}{\varepsilon} \geqslant C \frac{W_i\left(K^p_\lambda +_p (\varepsilon C)^{1/p} E; E\right) - W_i(\lambda)}{\varepsilon C}$$

for $0 < \varepsilon \leqslant c$, and thus, computing the limit as ε approaches 0 to the right and taking (N.36) into consideration, we get

$$\begin{aligned}\frac{d^+}{d\lambda} W_i(\lambda) &\geqslant C \lim_{\eta \to 0^+} \frac{W_i\left(K^p_\lambda +_p \eta^{1/p} E; E\right) - W_i(\lambda)}{\eta} \\ &= C \frac{n-i}{p} W_{p,i}(\lambda, E; E).\end{aligned}$$

□

Proof of Proposition 6.2.1. Let $\varepsilon > 0$, $\alpha \in (0,1)$ and let $\mu(\lambda, \varepsilon)$ satisfy $\lambda + \varepsilon = \lambda +_p \mu(\lambda, \varepsilon)$.

First, we assume $\lambda \in [-r, 0)$ and we observe that, since we aim to take limits as $\varepsilon \to 0$, we may suppose that $-r \leqslant \lambda < \lambda + \varepsilon < 0$. In this case, $\mu(\lambda, \varepsilon) = \left(|\lambda|^p - |\lambda+\varepsilon|^p\right)^{1/p}$. Next, we prove that

$$\mu(\lambda, \varepsilon) \geqslant (\varepsilon\, C_{p,\alpha,\lambda})^{1/p} \quad \text{for all } 0 < \varepsilon \leqslant c(p, \alpha, \lambda), \tag{6.2}$$

with $C_{p,\alpha,\lambda} = p(1-\alpha)|\lambda|^{p-1}$, and

$$c(p,\alpha,\lambda) = \begin{cases} \left[1-(1-\alpha)^{1/(p-1)}\right]|\lambda| & \text{if } p > 1, \\ |\lambda| & \text{if } p = 1. \end{cases}$$

If $p = 1$, then we directly obtain $\mu(\lambda,\varepsilon) = \varepsilon > (1-\alpha)\varepsilon = \varepsilon\, C_{1,\alpha,\lambda}$ for all $\varepsilon \leqslant |\lambda| = c(1,\alpha,\lambda)$, which establishes (6.2) in this case. Thus, let $p > 1$ and $\varepsilon \leqslant c(p,\alpha,\lambda)$. Then

$$(1-\alpha)^{1/(p-1)}|\lambda| \leqslant |\lambda| - \varepsilon = |\lambda+\varepsilon|,$$

i.e., $(1-\alpha)|\lambda|^{p-1} \leqslant |\lambda+\varepsilon|^{p-1}$, and with Lemma B.2 for $a = |\lambda+\varepsilon|$ and $b = |\lambda|$ we get that

$$\mu(\lambda,\varepsilon)^p = |\lambda|^p - |\lambda+\varepsilon|^p \geqslant p\,\varepsilon|\lambda+\varepsilon|^{p-1} \geqslant \varepsilon\, C_{p,\alpha,\lambda}$$

for all $\varepsilon \leqslant c(p,\alpha,\lambda)$, which concludes the proof of (6.2).

Using Proposition 5.2.11 (ii) and (6.2), we immediately get

$$\begin{aligned} K^p_{\lambda+\varepsilon} &= K^p_{\lambda+_p\mu(\lambda,\varepsilon)} \supseteq (K^p_\lambda)^p_{\mu(\lambda,\varepsilon)} = K^p_\lambda +_p \mu(\lambda,\varepsilon)E \\ &\supseteq K^p_\lambda +_p (\varepsilon\, C_{p,\alpha,\lambda})^{1/p}E. \end{aligned}$$

Thus, Lemma 6.2.2 ensures that

$$\begin{aligned} \frac{d^+}{d\lambda}W_i(\lambda) &\geqslant C_{p,\alpha,\lambda}\frac{n-i}{p}W_{p,i}(\lambda,E;E) \\ &= (1-\alpha)|\lambda|^{p-1}(n-i)W_{p,i}(\lambda,E;E) \end{aligned}$$

for all $\alpha \in (0,1)$. It proves (6.1) when $\lambda < 0$.

If $\lambda = 0$, then, writing $\eta = \varepsilon^p$, and using (N.36),

$$\begin{aligned} \left.\frac{d^+}{d\lambda}\right|_{\lambda=0} W_i(\lambda) &= \lim_{\varepsilon\to 0^+}\varepsilon^{p-1}\lim_{\eta\to 0^+}\frac{W_i\left(0+_p\eta^{1/p}\right)-W_i(0)}{\eta} \\ &= \begin{cases} 0 & \text{if } p>1, \\ (n-i)W_{1,i}(0,E;E) & \text{if } p=1. \end{cases} \end{aligned}$$

Therefore (6.1) holds with equality.

Next, we assume $\lambda > 0$. Now $\mu(\lambda,\varepsilon) = \left((\lambda+\varepsilon)^p - \lambda^p\right)^{1/p}$, and therefore, Lemma B.2 yields

$$\left(p\varepsilon\lambda^{p-1}\right)^{1/p} \leqslant \mu(\lambda,\varepsilon) \leqslant \left(p\varepsilon(\lambda+\varepsilon)^{p-1}\right)^{1/p}. \tag{6.3}$$

Using Proposition 5.2.11 (i), the left inequality in (6.3) implies

$$\begin{aligned} K^p_{\lambda+\varepsilon} &= K^p_{\lambda+_p\mu(\lambda,\varepsilon)} = (K^p_\lambda)^p_{\mu(\lambda,\varepsilon)} \supseteq K^p_\lambda +_p \left(\varepsilon p\lambda^{p-1}\right)^{1/p}E \\ &\supseteq K^p_\lambda +_p \left(\varepsilon(1-\alpha)p\lambda^{p-1}\right)^{1/p}E, \end{aligned}$$

for all $\varepsilon > 0$, and Lemma 6.2.2 yields

$$\frac{d^+}{d\lambda} W_i(\lambda) \geqslant (1-\alpha)\lambda^{p-1}(n-i)W_{p,i}(\lambda, E; E)$$

for any $\alpha \in (0,1)$. This proves (6.1) on $(0,\infty)$.

Now we deal with the equality case when $\lambda > 0$. Noticing that $(\lambda+\varepsilon)^{p-1} \leqslant (1+\alpha)\lambda^{p-1}$ if and only if $\varepsilon \leqslant \lambda\left[(1+\alpha)^{1/(p-1)} - 1\right]$, we get from the right inequality in (6.3) that

$$\mu(\lambda,\varepsilon) \leqslant \left(\varepsilon p(1+\alpha)\lambda^{p-1}\right)^{1/p},$$

and hence, by Proposition 5.2.11 (i), that

$$K^p_{\lambda+\varepsilon} = K^p_\lambda +_p \mu(\lambda,\varepsilon)E \subseteq K^p_\lambda +_p \left(\varepsilon p(1+\alpha)\lambda^{p-1}\right)^{1/p} E \tag{6.4}$$

for $\varepsilon \leqslant \lambda\left[(1+\alpha)^{1/(p-1)} - 1\right]$. Now, applying Lemma 6.2.2 we obtain

$$\frac{d^+}{d\lambda} W_i(\lambda) \leqslant (1+\alpha)\lambda^{p-1}(n-i)W_{p,i}(\lambda, E; E)$$

for any $\alpha \in (0,1)$ which, together with (6.1), proves the equality case and concludes the proof. □

We observe, that, as mentioned before, the proof of Proposition 6.2.1 yields that the right derivative always exists on the range $[0,\infty)$. We notice also that if we work on the range $(-r,0]$, the inclusion in (6.4) would be reversed, and we cannot expect to get equality in (6.1). Therefore, in this case we may only have differentiability almost everywhere on $(-r,0)$. More precisely, on the interval $(-r,0)$ we obtain the following result.

Proposition 6.2.3 ([39, Proposition 2]). *Let $E \in \mathcal{K}^n_0$, $K \in \mathcal{K}^n_{00}(E)$ and $1 \leqslant p < \infty$. Then $W_i(\lambda)$ is differentiable with the exception of at most countably many points on $(-r,0)$, $0 \leqslant i \leqslant n-1$, and*

$$\frac{d^-}{d\lambda} W_i(\lambda) \geqslant \frac{d^+}{d\lambda} W_i(\lambda) \geqslant |\lambda|^{p-1}(n-i)W_{p,i}(\lambda, E; E).$$

Proof. Inequality (N.38) and Theorem 5.2.13 imply that the function $W_i(\lambda)^{p/(n-i)}$ is $+_p$-concave and increasing on $(-r,0)$. Then Lemma 6.1.2 ensures that it is concave on this range. Hence, there exist left and right derivatives of $W_i(\lambda)$ and they satisfy the required inequality on $(-r,0)$. Finally, (6.1) concludes the proof. □

Next, we note that there exist families of convex bodies for which the functions $W_i(\lambda)$ are differentiable on $(-r,0)$, $0 \leqslant i \leqslant n-1$. This is, for instance, the case of the tangential bodies (see Appendix A).

In Theorem 5.2.18 we have proven that for suitable $K, E \in \mathcal{K}^n$, the convex body K is a tangential body of E if and only if K^p_λ is homothetic to K for all $\lambda \in (-r,0)$, with factor $(1-|\lambda|^p)^{1/p}$. This property, the homogeneity of quermassintegrals (N.15) and the differentiability of $\lambda \mapsto (1-|\lambda|^p)^{1/p}$ on $(-1,0)$ immediately prove the following result. We notice that E is always assumed to lie on $\mathcal{K}^n_0$, and any other assumption complements this one.

Lemma 6.2.4 ([39, Lemma 18]). *Let $E \in \mathcal{K}^n_n$, $K \in \mathcal{K}^n_0$ be a tangential body of E, and let $1 \leqslant p < \infty$. Then $W_i(\lambda)$ is differentiable on $(-1,0)$, $0 \leqslant i \leqslant n-1$, and*

$$W_i'(\lambda) = (n-i)|\lambda|^{p-1}\big(1-|\lambda|^p\big)^{((n-i)/p)-1}W_i(0).$$

In order to get similar properties for the left derivative on $[0,\infty)$ (cf. Proposition 6.2.1), we need a different approach. The following result cannot be obtained as a consequence of the $+_p$-concavity of the full system of p-parallel bodies (Theorem 5.2.13), since there is no analog of Lemma 6.1.2 for $+_p$-concave increasing functions defined on $[0,\infty)$ (see Remark 6.1.4).

Before proceeding, we introduce the following notation, already used in the proof of Proposition 6.2.1. For the sake of brevity, we will often use for given $a, b \in \mathbb{R}$ and $b \geqslant 0$, the notation $\mu(a,b)$ for the real number satisfying

$$\begin{aligned} \text{either}\quad & a+b = a +_p \mu(a,b) \quad \text{– in this case } \mu(a,b) = (a+b) +_p (-a),\\ \text{or}\quad & a-b = a +_p \big(-\mu(a,b)\big) \quad \text{– now } \mu(a,b) = a +_p \big(-(a-b)\big). \end{aligned} \tag{6.5}$$

Of course $\mu(a,b)$ will strongly depend on the "size" of a and b and the sign of a. Which one of the two possibilities is used, will be clearly stated.

Proposition 6.2.5 ([39, Proposition 17]). *Let $E \in \mathcal{K}^n_0$ and $K \in \mathcal{K}^n_{00}(E)$. Let $1 \leqslant p < \infty$, and $0 \leqslant i \leqslant n-1$. Then, wherever the left derivative exists for $\lambda \geqslant 0$,*

$$\frac{d^-}{d\lambda}W_i(\lambda) \geqslant \frac{d^+}{d\lambda}W_i(\lambda).$$

Proof. By Theorem 5.2.13 and Lemma 5.2.8, it is easy to check that

$$K^p_{\lambda+_p(-t)} +_p K^p_{\lambda+_p t} \subseteq 2^{1/p}K^p_\lambda \tag{6.6}$$

for all $t > 0$ such that $\lambda +_p (-t) > -r$. Then (N.38) yields

$$\begin{aligned} &W_i\big(2^{1/p}K^p_\lambda; E\big)^{p/(n-i)}\\ &\qquad \geqslant W_i\big(\lambda +_p (-t)\big)^{p/(n-i)} + W_i(\lambda +_p t)^{p/(n-i)}, \end{aligned}$$

which, by the homogeneity degree of W_i, i.e., (N.15), amounts to

$$\begin{aligned} W_i(\lambda)^{p/(n-i)} &- W_i\big(\lambda +_p (-t)\big)^{p/(n-i)}\\ &\geqslant W_i(\lambda +_p t)^{p/(n-i)} - W_i(\lambda)^{p/(n-i)}. \end{aligned} \tag{6.7}$$

Let $\varepsilon > 0$ with $-r < \lambda - \varepsilon$. Using the notation introduced in (6.5), we write $\lambda - \varepsilon = \lambda +_p \big(-\mu(\lambda, \varepsilon)\big) > -r$. Considering

$$m(a, b) := \frac{W_i(b)^{p/(n-i)} - W_i(a)^{p/(n-i)}}{W_i(b) - W_i(a)},$$

inequality (6.7) implies that

$$\begin{aligned} W_i(\lambda) - W_i(\lambda - \varepsilon) &= \frac{W_i(\lambda)^{p/(n-i)} - W_i(\lambda - \varepsilon)^{p/(n-i)}}{m(\lambda - \varepsilon, \lambda)} \\ &\geqslant \frac{W_i\big(\lambda +_p \mu(\lambda, \varepsilon)\big)^{p/(n-i)} - W_i(\lambda)^{p/(n-i)}}{m(\lambda - \varepsilon, \lambda)} \\ &= \Big(W_i\big(\lambda +_p \mu(\lambda, \varepsilon)\big) - W_i(\lambda)\Big) \frac{m\big(\lambda, \lambda +_p \mu(\lambda, \varepsilon)\big)}{m(\lambda - \varepsilon, \lambda)}. \end{aligned} \tag{6.8}$$

We notice that $m(a, b)$ is the slope in $\mathbb{R}^2$ of the straight line joining the points $\big(W_i(a), W_i(a)^{p/(n-i)}\big)^\top$ and $\big(W_i(b), W_i(b)^{p/(n-i)}\big)^\top$, which yields

$$\lim_{a \to b^-} m(a, b) = \lim_{c \to b^+} m(b, c) = \frac{p}{n-i} W_i(b)^{(p/(n-i))-1}. \tag{6.9}$$

In order to compute the limit in (6.8) we need to *control* the right-hand side in the latter inequality. As $\mu(\lambda, \varepsilon) = \big(\lambda^p - (\lambda - \varepsilon)^p\big)^{1/p}$, given $\alpha \in (0, 1)$, an easy computation proves that, for ε small enough,

$$\lambda +_p \mu(\lambda, \varepsilon) = \big(2\lambda^p - (\lambda - \varepsilon)^p\big)^{1/p} \geqslant \lambda + (1 - \alpha)\, \varepsilon. \tag{6.10}$$

Indeed, if $\lambda = 0$, inequality (6.10) is valid for all $\varepsilon > 0$, whereas if $\lambda > 0$ it suffices to consider

$$\varepsilon \in \left(0, \lambda \frac{1 - (1 - \alpha)^{1/(p-1)}}{1 + (1 - \alpha)^{p/(p-1)}}\right].$$

Thus, for $\varepsilon > 0$ small enough we get

$$\frac{W_i(\lambda) - W_i(\lambda - \varepsilon)}{\varepsilon} \geqslant \frac{W_i\big(\lambda + (1 - \alpha)\, \varepsilon\big) - W_i(\lambda)}{\varepsilon} \frac{m\big(\lambda, \lambda +_p \mu(\lambda, \varepsilon)\big)}{m(\lambda - \varepsilon, \lambda)}.$$

Then, taking limits as $\varepsilon \to 0$ to the right in the above inequality, and using that, by (6.9), we have

$$\lim_{\varepsilon \to 0^+} \frac{m\big(\lambda, \lambda +_p \mu(\lambda, \varepsilon)\big)}{m(\lambda - \varepsilon, \lambda)} = 1,$$

we, finally, obtain that, for all $\alpha \in (0, 1)$,

$$\begin{aligned} \frac{d^-}{d\lambda} W_i(\lambda) &\geqslant (1 - \alpha) \lim_{\varepsilon \to 0^+} \frac{W_i\big(\lambda + (1 - \alpha)\, \varepsilon\big) - W_i(\lambda)}{(1 - \alpha)\varepsilon} \\ &= (1 - \alpha) \lim_{\eta \to 0^+} \frac{W_i(\lambda + \eta) - W_i(\lambda)}{\eta} = (1 - \alpha) \frac{d^+}{d\lambda} W_i(\lambda). \end{aligned}$$

We notice that the above expression is well defined because the right derivative always exists on $[0, \infty)$ by Proposition 6.2.1. □

We observe that, for $\lambda < 0$, inclusion (6.6) does not hold in general.

Remark 6.2.6 ([39, p. 12]). *At this point we remark that, in the classical case* $p = 1$, *the differentiability of the quermassintegral* $W_i(\lambda)$ *on* $(0, \infty)$, $0 \leqslant i \leqslant n-1$, *follows immediately from the fact that* $W_i(K + \lambda E; E)$ *can be written as a polynomial in* $\lambda \geqslant 0$, *namely,* (N.17).

6.2.2 *Differentiability of* W_i *on* $(0, \infty)$

Our main aim is to establish the differentiability of $W_i(\lambda)$ on $(0, \infty)$. In order to do so, and taking Proposition 6.2.5 into consideration, we will prove that the expression for the right derivative given in (6.1) provides also an upper bound for the left derivative. As in Chapter 3, when we write f' for a real function f, we mean that the left and right derivatives do exist and coincide.

Theorem 6.2.7 ([39, Theorem 3]). *Let* $E \in \mathcal{K}^n_{(0)}$ *and* $K \in \mathcal{K}^n_{00}(E)$. *For any* $1 \leqslant p < \infty$, *the function* $W_i(\lambda)$ *is differentiable on* $(0, \infty)$, $0 \leqslant i \leqslant n-1$, *and*

$$W_i'(\lambda) = \lambda^{p-1}(n-i)W_{p,i}(\lambda, E; E).$$

Proof. We are going to prove that

$$\frac{d^-}{d\lambda} W_i(\lambda) \leqslant \lambda^{p-1}(n-i)W_{p,i}(\lambda, E; E) \tag{6.11}$$

which, together with Propositions 6.2.1 (equality case) and 6.2.5, will conclude the proof.

Let $\lambda > 0$ and $\varepsilon > 0$, with $\lambda - \varepsilon > 0$, and let $\lambda - \varepsilon = \lambda +_p \big(-\mu(\lambda, \varepsilon)\big)$, namely, $\mu(\lambda, \varepsilon) = \big(\lambda^p - (\lambda - \varepsilon)^p\big)^{1/p}$ (cf. (6.5)). From Lemma B.2, we get $\mu(\lambda, \varepsilon) \leqslant \big(p\varepsilon\lambda^{p-1}\big)^{1/p}$, and hence,

$$\lambda - \varepsilon \geqslant \lambda +_p \left[-\big(p\varepsilon\lambda^{p-1}\big)^{1/p}\right].$$

It implies, by Proposition 5.2.11 (iv) and the monotonicity of the mixed volumes (N.14), that for all $0 < \varepsilon < \lambda$

$$\frac{W_i(\lambda) - W_i(\lambda - \varepsilon)}{\varepsilon} \leqslant \frac{W_i(\lambda) - W_i\left(\lambda +_p \left[-\big(p\varepsilon\lambda^{p-1}\big)^{1/p}\right]\right)}{\varepsilon}. \tag{6.12}$$

Next, we prove some properties of the quermassintegrals involved in the latter inequality. We argue, where it applies, as in the proof of [50, Theorem (1.1)]. We show the argument for completeness. For the sake of brevity we write $W_{1,i}(\mu, \tau) := W_{1,i}\big(K^p_\mu, K^p_\tau; E\big)$, $\tau, \mu \geqslant 0$. Let $\lambda(\varepsilon) := \lambda +_p \left[-\big(p\varepsilon\lambda^{p-1}\big)^{1/p}\right]$ and

$$g(\varepsilon) := W_i\left(\lambda +_p \left[-(p\varepsilon\lambda^{p-1})^{1/p}\right]\right)^{1/(n-i)} = W_i\big(\lambda(\varepsilon)\big)^{1/(n-i)}.$$

We also define

$$\ell_i := \liminf_{\varepsilon\to 0^+} \frac{W_i(\lambda) - W_{1,i}\big(\lambda, \lambda(\varepsilon)\big)}{\varepsilon},$$
$$\ell_s := \limsup_{\varepsilon\to 0^+} \frac{W_{1,i}\big(\lambda(\varepsilon), \lambda\big) - W_i\big(\lambda(\varepsilon)\big)}{\varepsilon}.$$

Since $K^p_{\lambda(\varepsilon)} \subseteq K^p_\lambda$ for $\varepsilon < \lambda$, the monotonicity of the mixed volumes (N.14) yields that ℓ_i and ℓ_s are the lim inf and lim sup, respectively, of nonnegative functions for $0 < \varepsilon < \lambda$. Using inequality (N.38) we obtain

$$\begin{aligned}\ell_i &\leqslant \liminf_{\varepsilon\to 0^+} \frac{W_i(\lambda) - W_i(\lambda)^{(n-i-1)/(n-i)} W_i\big(\lambda(\varepsilon)\big)^{1/(n-i)}}{\varepsilon}\\ &= W_i(\lambda)^{\frac{n-i-1}{n-i}} \liminf_{\varepsilon\to 0^+} \frac{W_i(\lambda)^{1/(n-i)} - W_i\big(\lambda(\varepsilon)\big)^{1/(n-i)}}{\varepsilon},\end{aligned}$$

and analogously,

$$\ell_s \geqslant \limsup_{\varepsilon\to 0^+} W_i\big(\lambda(\varepsilon)\big)^{\frac{n-i-1}{n-i}} \frac{W_i(\lambda)^{1/(n-i)} - W_i\big(\lambda(\varepsilon)\big)^{1/(n-i)}}{\varepsilon}.$$

The continuity of the full system of p-parallel bodies (Proposition 5.2.14) with respect to the Hausdorff metric, together with the continuity of quermassintegrals W_i on $\mathcal{K}^n$ prove that g is continuous at 0. Hence we may write

$$\begin{aligned}\ell_i &\leqslant W_i(\lambda)^{\frac{n-i-1}{n-i}} \liminf_{\varepsilon\to 0^+} \frac{W_i(\lambda)^{1/(n-i)} - W_i\big(\lambda(\varepsilon)\big)^{1/(n-i)}}{\varepsilon}\\ &\leqslant W_i(\lambda)^{\frac{n-i-1}{n-i}} \limsup_{\varepsilon\to 0^+} \frac{W_i(\lambda)^{1/(n-i)} - W_i\big(\lambda(\varepsilon)\big)^{1/(n-i)}}{\varepsilon} \leqslant \ell_s.\end{aligned} \tag{6.13}$$

Moreover, using the integral expressions of W_i and $W_{1,i}$ given in (N.13) and (N.36), respectively, we can write

$$\ell_i = \liminf_{\varepsilon\to 0^+} \frac{1}{n} \int_{S^{n-1}} \frac{h(\lambda, u) - h\big(\lambda(\varepsilon), u\big)}{\varepsilon}\, dS\left(K^p_\lambda[n-i-1], E[i]; u\right)$$

and

$$\ell_s = \limsup_{\varepsilon\to 0^+} \frac{1}{n} \int_{S^{n-1}} \frac{h(\lambda, u) - h\big(\lambda(\varepsilon), u\big)}{\varepsilon}\, dS\left(K^p_{\lambda(\varepsilon)}[n-i-1], E[i]; u\right).$$

Since

$$\lim_{\varepsilon\to 0^+} \frac{h(\lambda, u) - h\big(\lambda(\varepsilon), u\big)}{\varepsilon} = \lambda^{p-1} h(\lambda, u)^{1-p} h(E, u)^p$$

uniformly on S^{n-1}, the continuity of $\varepsilon \mapsto \big(h(\lambda, u) - h(\lambda(\varepsilon), u)\big)/\varepsilon$ on $(0, \lambda)$, and the weak convergence of $S\big(K^p_{\lambda(\varepsilon)}[n-i-1], E[i]; \cdot\big)$ to

$S\big(K_\lambda^p[n-i-1], E[i]; \cdot\big)$ (see e.g. [68, Theorem 4.2.1] and Proposition 5.2.14) when $\varepsilon \to 0^+$ prove that

$$\ell_i = \ell_s = \frac{\lambda^{p-1}}{n} \int_{S^{n-1}} h(\lambda, u)^{1-p} h(E, u)^p \, dS\big(K_\lambda^p[n-i-1], E[i]; u\big). \tag{6.14}$$

Now, since $\ell_i = \ell_s$, we get from (6.13) that the right derivative of g^{n-i} at 0 does exist and satisfies

$$\lim_{\varepsilon \to 0^+} \frac{g(\varepsilon)^{n-i} - g(0)^{n-i}}{\varepsilon} = (n-i) g(0)^{n-i-1} \frac{d^+}{d\varepsilon}\bigg|_{\varepsilon=0} g(\varepsilon).$$

It implies (cf. (6.13))

$$\lim_{\varepsilon \to 0^+} \frac{W_i(\lambda) - W_i\big(\lambda(\varepsilon)\big)}{\varepsilon} = (n-i)\ell_i = (n-i)\ell_s. \tag{6.15}$$

Thus, (6.12), (6.15), (6.14), and (N.36) yield

$$\begin{aligned}
\frac{d^-}{d\lambda} W_i(\lambda) &= \lim_{\varepsilon \to 0^+} \frac{W_i(\lambda) - W_i(\lambda - \varepsilon)}{\varepsilon} \\
&\leqslant \lim_{\varepsilon \to 0^+} \frac{W_i(\lambda) - W_i\big(\lambda(\varepsilon)\big)}{\varepsilon} \\
&= (n-i)\,\ell_i \\
&= \frac{n-i}{n} \lambda^{p-1} \int_{S^{n-1}} h(E, u)^p h(\lambda, u)^{1-p} \, dS\big(K_\lambda^p[n-i-1], E[i]; u\big) \\
&= (n-i)\lambda^{p-1} W_{p,i}(\lambda, E; E)
\end{aligned}$$

for $\lambda > 0$, which proves (6.11) and concludes the proof. □

Remark 6.2.8 ([52]). *We notice that for $\lambda \in (0, \infty)$, Theorem 6.2.7 ensures that*

$$W_i'(\lambda) = \lambda^{p-1}(n-i) W_{p,i}(\lambda, E; E)$$

which, in general, is not the same function provided by Lemma 6.2.4.

Remark 6.2.9. *We point out that none of the results proven so far provides a proof of the differentiability of W_i at $\lambda = 0$. In order to deal with this we will need a slightly different approach.*

6.2.3 *Differentiability of W_i at $\lambda = 0$*

Next, we will handle the differentiability of $W_i(\lambda)$ at $\lambda = 0$. We will prove that all quermassintegrals are differentiable at 0 for $p > 1$, being the value of the derivative always 0. First we prove a lemma that will be used to provide an upper bound for the left derivative of $W_i(\lambda)$, involving $W_i(\lambda)$ itself. The case $p = 1$ was already obtained in [62, Lemma 4.7].

Lemma 6.2.10 ([39, Lemma 19]). *Let* $E \in \mathcal{K}_n^n$, $K \in \mathcal{K}_{00}^n(E)$ *and* $1 \leqslant p < \infty$. *For all* $-r \leqslant \lambda \leqslant 0$,

$$\frac{r +_p \lambda}{r} K \subseteq K_\lambda^p. \tag{6.16}$$

Equality holds for some $\lambda \in (-r, 0)$ *if and only if* K *is homothetic to a tangential body of* E.

Proof. Let $K \in \mathcal{K}_{00}^n(E)$. As $rE \subseteq K$, we have $rh(E,u) \leqslant h(K,u)$ for all $u \in \mathbb{S}^{n-1}$. Thus, $h(K,u)^p/r^p - h(E,u)^p \geqslant 0$ for all $u \in \mathbb{S}^{n-1}$, and so

$$\frac{r^p - |\lambda|^p}{r^p} h(K,u)^p + |\lambda|^p h(E,u)^p \leqslant h(K,u)^p, \quad \text{for all } u \in \mathbb{S}^{n-1}.$$

It implies, as required, that

$$h\left(\frac{r +_p \lambda}{r} K +_p |\lambda| E, u\right) \leqslant h(K,u), \quad \text{for all } u \in \mathbb{S}^{n-1}.$$

The equality case is provided by Theorem 5.2.18, which ensures that (6.16) holds with equality for some $\lambda \in (-r, 0)$ if and only if K is homothetic to a tangential body of E. $\square$

Now we are ready to prove the mentioned upper bound for the left derivative of $W_i(\lambda)$. The case $p = 1$ of the following result was obtained in [44, Lemma 2.2], where it was proven that

$$\frac{d^-}{d\lambda} W_i(\lambda) \leqslant (n-i) \frac{1}{r - |\lambda|} W_i(\lambda). \tag{6.17}$$

In the following proposition the case $p = 1$ and $\lambda = 0$ should be understood as the just mentioned inequality.

Proposition 6.2.11 ([39, Proposition 20]). *Let* $E \in \mathcal{K}_n^n$ *and* $K \in \mathcal{K}_{00}^n(E)$. *Let* $1 \leqslant p < \infty$, *and* $0 \leqslant i \leqslant n-1$. *Then the left derivative exists on* $(-r, 0]$ *and*

$$\frac{d^-}{d\lambda} W_i(\lambda) \leqslant (n-i) \frac{|\lambda|^{p-1}}{r^p - |\lambda|^p} W_i(\lambda). \tag{6.18}$$

If E *is regular and strictly convex, then for* $0 \leqslant i \leqslant n-2$, *equality holds almost everywhere on* $(-r, 0)$ *if and only if* K *is homothetic to a tangential body of* E.

Proof. As in the proof of Proposition 6.2.3, the existence of the left derivative is ensured by (N.38), Theorem 5.2.13 and Lemma 6.1.2. Let $\lambda \in (-r, 0]$ and $\varepsilon \geqslant 0$ be such that $-r < \lambda - \varepsilon \leqslant \lambda$. By (6.5) and Proposition 5.2.11 (iii) we get

$$K_{\lambda-\varepsilon}^p = K_{\lambda +_p (-\mu(\lambda,\varepsilon))}^p = (K_\lambda^p)_{-\mu(\lambda,\varepsilon)}^p.$$

Then Lemma 6.2.10 and the monotonicity and homogeneity of the mixed volumes, (N.14) and (N.15), yield

$$\left(\frac{r +_p \lambda +_p (-\mu(\lambda,\varepsilon))}{r +_p \lambda}\right)^{n-i} W_i(\lambda) \leqslant W_i(\lambda - \varepsilon).$$

Thus,

$$\begin{aligned}
\frac{d^-}{d\lambda}W_i(\lambda) &= \lim_{\varepsilon\to 0^+}\frac{W_i(\lambda)-W_i(\lambda-\varepsilon)}{\varepsilon}\\
&\leqslant \lim_{\varepsilon\to 0^+}\frac{1-\left(\frac{r^p-|\lambda-\varepsilon|^p}{r^p-|\lambda|^p}\right)^{(n-i)/p}}{\varepsilon}W_i(\lambda)\\
&= (n-i)\frac{|\lambda|^{p-1}}{r^p-|\lambda|^p}W_i(\lambda).
\end{aligned}$$

Next, we deal with the equality case. From Proposition 6.2.3, we know that, with the exception of at most countably many points, the function $W_i(\lambda)$ is differentiable on $(-r,0)$. Hence, assuming equality in (6.18) we have

$$W_i'(\lambda) = (n-i)\frac{|\lambda|^{p-1}}{r^p-|\lambda|^p}W_i(\lambda)$$

almost everywhere on $(-r,0)$. Then, for $\mu\in(-r,0)$,

$$\int_\mu^0 \frac{W_i'(\lambda)}{W_i(\lambda)}\,d\lambda = (n-i)\int_\mu^0 \frac{|\lambda|^{p-1}}{r^p-|\lambda|^p}\,d\lambda,$$

and thus, we obtain

$$W_i(\mu) = \left(\frac{r+_p\mu}{r}\right)^{n-i}W_i(0) = W_i\left(\frac{r+_p\mu}{r}K;E\right). \tag{6.19}$$

Therefore, because of the inclusion provided by Lemma 6.2.10 and the regularity and strict convexity of E, we can conclude that

$$\frac{r+_p\mu}{r}K = K_\mu^p.$$

Now, Theorem 5.2.18 yields that K is homothetic to a tangential body of E.

Conversely, if K is homothetic to a tangential body of E, Lemma 6.2.4 yields

$$\begin{aligned}
W_i'(\lambda) &= (n-i)|\lambda|^{p-1}\frac{(r^p-|\lambda|^p)^{\frac{n-i}{p}-1}}{r^{n-i}}W_i(0)\\
&= (n-i)\frac{|\lambda|^{p-1}}{r^p-|\lambda|^p}W_i(\lambda).
\end{aligned}$$

□

If K is a tangential body of E, for a non-necessarily regular $E\in\mathcal{K}_n^n$, then, doing as in the last part of the proof, we also obtain the equality in (6.18) for $0\leqslant i\leqslant n-2$. We observe that the equality case in (6.18), for $i=n-1$, cannot be deduced from (6.19). The differentiability of W_{n-1} will be treated in a different way in Theorem 6.3.4 of the present chapter.

Finally, we obtain the result concerning the differentiability of the functions $W_i(\lambda)$ at $\lambda=0$ for $1<p<\infty$ and $0\leqslant i\leqslant n-1$.

Corollary 6.2.12 ([39, Corollary 21]). *Let* $E \in \mathcal{K}^n_n$, $K \in \mathcal{K}^n_{00}(E)$, $1 < p < \infty$ *and* $0 \leqslant i \leqslant n-1$. *Then* $W_i(\lambda)$ *is differentiable at* 0 *and* $W_i'(0) = 0$.

Proof. Proposition B.2 and Proposition 6.2.11 yield that the left derivative exists at $\lambda = 0$ and $(d^-/d\lambda)\big|_{\lambda=0} W_i(\lambda) \leqslant 0$. Moreover, using Proposition 6.2.1, we can assure that the right derivative of $W_i(\lambda)$ at $\lambda = 0$ does exist. Finally, the equality case for (6.1) and Proposition 6.2.5 allow us to conclude the result:

$$0 = \frac{d^+}{d\lambda}\bigg|_{\lambda=0} W_i(\lambda) \leqslant \frac{d^-}{d\lambda}\bigg|_{\lambda=0} W_i(\lambda) \leqslant 0. \qquad \square$$

We observe that the above result is not true in the classical case, i.e., $p = 1$. In fact, the argument can be reproduced in order to get, on the one hand, the value

$$\frac{d^+}{d\lambda}\bigg|_{\lambda=0} W_i(\lambda) = (n-i)W_{1,i}(0, E; E) = (n-i)W_{i+1}(K; E)$$

by Proposition 6.2.1. On the other hand, the bound

$$\frac{d^-}{d\lambda}\bigg|_{\lambda=0} W_i(\lambda) \leqslant (n-i)\frac{1}{r}W_i(K; E)$$

(cf. (6.17), and Proposition 6.2.11). However, they are, in general, not equal.

6.2.4 *On the differentiability of the volume*

In order to deal with the differentiability of $\mathrm{vol}(\lambda) = \mathrm{vol}(K^p_\lambda)$, next, we prove Lemma 6.2.13, which will provide us with an *alternative* expression for the left derivative of $W_i(\lambda)$, $0 \leqslant i \leqslant n-1$, involving the p-sum in the computation of the limit.

Lemma 6.2.13 ([39, Lemma 22]). *Let* $E \in \mathcal{K}^n_0$ *and* $K \in \mathcal{K}^n_{00}(E)$. *Let also* $1 \leqslant p < \infty$ *and* $0 \leqslant i \leqslant n-1$. *Then, for all* $\lambda \in (-r, 0)$,

$$\frac{d^-}{d\lambda} W_i(\lambda) = p|\lambda|^{p-1} \lim_{\varepsilon \to 0^+} \frac{W_i(\lambda) - W_i\big(\lambda +_p (-\varepsilon^{1/p})\big)}{\varepsilon}.$$

Proof. Let $\varepsilon > 0$ with $-r < \lambda - \varepsilon$, and let $\lambda - \varepsilon = \lambda +_p \big(-\mu(\lambda, \varepsilon)\big)$, namely, $\mu(\lambda, \varepsilon) = \big(|\lambda - \varepsilon|^p - |\lambda|^p\big)^{1/p}$ (cf. (6.5)). Using Lemma B.2 we obtain the inequalities $p\varepsilon|\lambda|^{p-1} \leqslant \mu(\lambda, \varepsilon)^p \leqslant p\varepsilon|\lambda - \varepsilon|^{p-1}$, and hence

$$K^p_\lambda \sim_p \big(p\varepsilon|\lambda|^{p-1}\big)^{1/p} E \supseteq K^p_{\lambda-\varepsilon} \supseteq K^p_\lambda \sim_p \big(p\varepsilon|\lambda - \varepsilon|^{p-1}\big)^{1/p} E.$$

Thus, using the monotonicity of the mixed volumes (N.14) we can write

$$\begin{aligned} W_i\Big(\lambda +_p \big(-p\varepsilon|\lambda|^{p-1}\big)^{1/p}\Big) &\geqslant W_i(\lambda - \varepsilon) \\ &\geqslant W_i\Big(\lambda +_p \big(-p\varepsilon|\lambda - \varepsilon|^{p-1}\big)^{1/p}\Big). \end{aligned}$$

Therefore, since the left derivative does exist (see the proof of Proposition 6.2.3),

$$p|\lambda|^{p-1} \lim_{\varepsilon\to 0^+} \frac{W_i(\lambda) - W_i\left(\lambda +_p \left(-p|\lambda|^{p-1}\varepsilon\right)^{1/p}\right)}{p|\lambda|^{p-1}\varepsilon} \leqslant \frac{d^-}{d\lambda} W_i(\lambda)$$
$$\leqslant \lim_{\varepsilon\to 0^+} p|\lambda-\varepsilon|^{p-1} \frac{W_i(\lambda) - W_i\left(\lambda +_p \left(-p|\lambda-\varepsilon|^{p-1}\varepsilon\right)^{1/p}\right)}{p|\lambda-\varepsilon|^{p-1}\varepsilon},$$

which proves the result. □

The case $i = 0$ of the previous result can be already found in the literature, directly related to p-sums, though not in the context of p-inner parallel bodies. Lutwak [50] proved the following integral expression for *a p-variation of the volume functional*.

Theorem 6.2.14 ([50, Lemma (3.2)]). *Let $K, E \in \mathcal{K}^n_{(0)}$, and let $1 \leqslant p < \infty$. Then*

$$\begin{aligned}\frac{n}{p} W_{p,0}(K,E;E) &= \lim_{\varepsilon\to 0} \frac{\mathrm{vol}(K +_p \varepsilon\cdot E) - \mathrm{vol}(K)}{\varepsilon}\\ &= \frac{1}{p}\int_{S^{n-1}} h(E,u)^p h(K,u)^{1-p} dS\big(K[n-1];u\big).\end{aligned} \tag{6.20}$$

We recall that $W_{p,0}(K,E;E)$ was introduced in Theorem N by means of (N.36):

$$\begin{aligned}\frac{n-i}{p} W_{p,i}(K,L;E) &:= \lim_{\varepsilon\to 0^+} \frac{W_i(K +_p \varepsilon\cdot L;E) - W_i(K;E)}{\varepsilon}\\ &= \frac{n-i}{p}\frac{1}{n}\int_{S^{n-1}} h(L,u)^p h(K,u)^{1-p}\, dS\big(K[n-i-1],E[i];u\big).\end{aligned}$$

We observe that (6.20) is not a particular case of the just recalled (N.36) when $i = 0$, since here the limit is taking both, from the left and right sides from 0. In the case of the *limit to the left*, the result was established using a variation of the support function. We observe that this argument in the limiting process happens to yield the *same result* as the analogous limiting process using the p-difference we have defined in Chapter 5. Indeed, using Lutwak's proof for an arbitrary $-r \leqslant \lambda \leqslant 0$, we prove in Theorem 6.2.15 that the volume function of the system of parallel bodies, $\mathrm{vol}(\lambda) = \mathrm{vol}(K^p_\lambda)$, is differentiable on its whole range of definition $(-r,\infty)$. We notice that the limit appearing in Theorem 6.2.14 does not coincide with the usual limit defining the derivative, since in the first one the special product $\varepsilon\cdot E = \varepsilon^{1/p}E$ plays a prominent role.

Theorem 6.2.15 ([39, Theorem 24]). *Let $E \in \mathcal{K}^n_{(0)}$ and $K \in \mathcal{K}^n_{00}(E)$. Let also $1 \leqslant p < \infty$. Then, for all $\lambda \in (-r,\infty)$,*

$$\begin{aligned}\mathrm{vol}'(\lambda) &= n|\lambda|^{p-1} W_{p,0}\big(\lambda,E;E\big)\\ &= |\lambda|^{p-1}\int_{S^{n-1}} h(E,u)^p h(\lambda,u)^{1-p}\, dS\big(K^p_\lambda[n-1];u\big).\end{aligned} \tag{6.21}$$

Proof. Theorems 6.2.7 and 6.2.14 ensure that $\mathrm{vol}(\lambda)$ is differentiable on $[0,\infty)$, having the desired derivative. Thus, let $\lambda \in (-r, 0)$. Since the convex body $K_\lambda^p \in \mathcal{K}_{00}^n(E)$ (see Lemma 5.2.12), using Proposition 6.2.3, Lemma 6.2.13 for $i = 0$ and Theorem 6.2.14, we get

$$\begin{aligned} n|\lambda|^{p-1}W_{p,0}(\lambda, E; E) &\leqslant \frac{d^+}{d\lambda}\mathrm{vol}(\lambda) \leqslant \frac{d^-}{d\lambda}\mathrm{vol}(\lambda) \\ &= |\lambda|^{p-1}\int_{\mathbb{S}^{n-1}} h(E,u)^p h(\lambda,u)^{1-p}\, dS\big(K_\lambda^p[n-1];u\big) \\ &= n|\lambda|^{p-1}W_{p,0}(\lambda, E; E), \end{aligned}$$

i.e., the volume function is differentiable and its derivative coincides with (6.21). □

Remark 6.2.16 ([52]). *Since* $\dim K_{-r}^p \leqslant n-1$ *by Proposition 5.2.3, the latter result provides an integral formula for* $\mathrm{vol}(K)$ *in terms of functionals evaluated on the* p*-inner parallel bodies of* K*, which can be considered as a* p*-counterpart of (4.2):*

$$\begin{aligned} \mathrm{vol}(K) &= n\int_{-r}^{0} |\lambda|^{p-1}W_{p,0}(\lambda, E; E)\, d\lambda \\ &= \int_{-r}^{0} |\lambda|^{p-1}\left(\int_{\mathbb{S}^{n-1}} h(E,u)^p h(\lambda,u)^{1-p}\, dS\big(K_\lambda^p[n-1];u\big)\right) d\lambda. \end{aligned}$$

We refer to [68, Lemma 7.5.3] for a different perspective of the differentiability of $W_0(\lambda; p)$ based on the theory of Wulff shapes.

6.3 DIFFERENTIABILITY OF THE SUPPORT FUNCTION

For $K, E \in \mathcal{K}^n$, the concavity of the family of classical parallel bodies of K in $-r \leqslant \lambda < \infty$, i.e., the case $p = 1$, is translated into concavity of the support function, as a function in $\lambda \in (-r, \infty)$, which implies the existence of derivatives almost everywhere. Even more, Chakerian and Sangwine-Yager [14] proved that wherever the derivative exists, it satisfies

$$\frac{d}{d\lambda}h(K_\lambda, u) \geqslant h(E, u), \tag{6.22}$$

and equality holds for all $u \in \mathbb{S}^{n-1}$, all $\lambda \in (0,\infty)$ and almost everywhere on $(-r, 0)$, if and only if $K = K_{-r} + rE$. A slightly better bound for the above derivative (see Theorem 3.1.10) was shown in [62, Lemma 4.9]: wherever the derivative exists, it satisfies

$$\frac{d}{d\lambda}h(K_\lambda, u) \geqslant h\big(K_\lambda^*, u\big). \tag{6.23}$$

We recall that M^* stands for the form body of $M \in \mathcal{K}_n^n$ (see (3.2) and (3.3)). The existence of derivatives of $h(\lambda, u) := h(K_\lambda^p, u)$ almost everywhere for $p \geqslant 1$ is ensured by Lemma 6.1.2. Thus, it makes sense to ask for an analog of (6.22) when $1 \leqslant p < \infty$.

Theorem 6.3.1 ([39, Theorem 4]). *Let* $E \in \mathcal{K}^n_{(0)}$ *and* $K \in \mathcal{K}^n_{00}(E)$. *Let also* $1 \leqslant p < \infty$. *Then, for all* $u \in \mathbb{S}^{n-1}$,

$$\frac{d}{d\lambda} h(\lambda, u) \geqslant \frac{|\lambda|^{p-1} h(E, u)^p}{h(\lambda, u)^{p-1}} \tag{6.24}$$

almost everywhere on $(-r, 0]$. *Equality holds for all* $u \in \mathbb{S}^{n-1}$ *and almost everywhere on* $[-r, 0]$ *if and only if* $K = K^p_{-r} +_p rE$.

Proof. The existence of the derivative of $h(\lambda, u)$ almost everywhere on $(-r, 0)$ is ensured by Lemma 6.1.2. Writing $\lambda + \varepsilon = \lambda +_p \mu(\lambda, \varepsilon)$ (cf. (6.5)) and using Proposition 5.2.11(ii), we have

$$\begin{aligned} h(\lambda + \varepsilon, u) - h(\lambda, u) &\geqslant h\big(K^p_\lambda +_p \mu(\lambda, \varepsilon) E, u\big) - h(\lambda, u) \\ &= \big[h(\lambda, u)^p + \mu(\lambda, \varepsilon)^p h(E, u)^p\big]^{1/p} - h(\lambda, u) \\ &\geqslant \frac{\mu(\lambda, \varepsilon)^p h(E, u)^p}{p\big[h(\lambda, u)^p + \mu(\lambda, \varepsilon)^p h(E, u)^p\big]^{(p-1)/p}}, \end{aligned}$$

where the last inequality follows from the right-hand side of (B.1). Since

$$\lim_{\varepsilon \to 0^+} \big[h(\lambda, u)^p + \mu(\lambda, \varepsilon)^p h(E, u)^p\big]^{(p-1)/p} = h(\lambda, u)^{p-1}$$

and $\lim_{\varepsilon \to 0^+} \mu(\lambda, \varepsilon)^p / \varepsilon = p|\lambda|^{p-1}$, we may conclude that

$$\frac{d}{d\lambda} h(\lambda, u) = \lim_{\varepsilon \to 0^+} \frac{h(\lambda + \varepsilon, u) - h(\lambda, u)}{\varepsilon} \geqslant \frac{|\lambda|^{p-1} h(E, u)^p}{h(\lambda, u)^{p-1}}.$$

Now we deal with the equality case in (6.24). If $K = K^p_{-r} +_p rE$, it is not difficult to check that, for all $u \in \mathbb{S}^{n-1}$,

$$h(\lambda, u)^p = h(-r, u)^p + (r +_p \lambda)^p h(E, u)^p,$$

and a direct computation proves that, for all $\lambda \in [-r, 0]$ and $u \in \mathbb{S}^{n-1}$,

$$\frac{d}{d\lambda} h(\lambda, u) = \frac{|\lambda|^{p-1} h(E, u)^p}{h(\lambda, u)^{p-1}}.$$

Conversely, we assume that, for all $u \in \mathbb{S}^{n-1}$ and almost everywhere on $[-r, 0]$, equality holds in (6.24). For $u \in \mathbb{S}^{n-1}$, we consider the function

$$\psi(\lambda) := h(\lambda, u)^p - h(-r, u)^p - (r +_p \lambda)^p h(E, u)^p.$$

Since $h(\lambda, u)^p$ is increasing and $+_p$-concave on $(-r, 0)$, Lemmata B.1 and 6.1.2 yield that ψ is absolutely continuous on $(-r, 0)$. Therefore it is also absolutely continuous on $[-r, 0]$ and, since $\psi(-r) = 0$ and $\psi'(\lambda) = 0$ almost everywhere on $[-r, 0]$, we get that $\psi \equiv 0$ for any $u \in \mathbb{S}^{n-1}$. In particular, $\psi(0) = 0$ for any $u \in \mathbb{S}^{n-1}$, which yields $K = K^p_{-r} +_p rE$. □

We notice that the existence of the derivative, as well as its explicit expression, for the range $\lambda \geqslant 0$, follow directly from the fact that $h(\lambda,u)^p = h(0,u)^p + \lambda^p h(E,u)^p$, i.e., equality holds in (6.24).

We remark that under the assumption of regularity on E, the equality conditions in Theorem 6.3.1 can be slightly relaxed.

Remark 6.3.2 ([52, Proposition 4.3.2]). *Under the (further) assumption that $E \in \mathcal{K}^n_{(0)}$ is regular in Theorem 6.3.1, equality holds in (6.24) almost everywhere on $[-r,0]$ and $S\big(E[n-1];\cdot\big)$-almost everywhere on $\mathbb{S}^{n-1}$ if and only if the convex body K satisfies $K = K^p_{-r} +_p rE$.*

For, it is enough to observe that if $K, L, E \in \mathcal{K}^n$, $K \subseteq L$ with E regular, such that $h(K,u) = h(L,u)$ holds $S\big(E[n-1];\cdot\big)$-almost everywhere on $\mathbb{S}^{n-1}$, then $K = L$.

In the same spirit as (6.23) strengthens (6.22), an improvement of Theorem 6.3.1, using the form body of the p-inner parallel bodies, can be obtained, without a characterization of the equality case, though.

Theorem 6.3.3 ([52, Theorem 4.3.3]). *Let $E \in \mathcal{K}^n_{(0)}$ be regular and strictly convex, $K \in \mathcal{K}^n_{00}(E)$ and $1 \leqslant p < \infty$. Then, for all $u \in \mathbb{S}^{n-1}$,*

$$\frac{d}{d\lambda}h(\lambda,u) \geqslant \frac{|\lambda|^{p-1} h\big((K^p_\lambda)^*,u\big)^p}{h(\lambda,u)^{p-1}} \tag{6.25}$$

almost everywhere on $(-r,0]$.

Proof. Let $-r \leqslant \lambda < 0$, and let $\varepsilon > 0$ be such that $-r \leqslant \lambda < \lambda+\varepsilon < 0$. Then, Proposition 3.1.8 applied to $K^p_{\lambda+\varepsilon} \in \mathcal{K}^n_{00}(E)$, yields

$$K^p_\lambda +_p \mu(\lambda,\varepsilon)(K^p_{\lambda+\varepsilon})^* = (K^p_{\lambda+\varepsilon})^p_{-\mu(\lambda,\varepsilon)} +_p \mu(\lambda,\varepsilon)(K^p_{\lambda+\varepsilon})^* \subseteq K^p_{\lambda+\varepsilon},$$

where $\mu(\lambda,\varepsilon) = \big(|\lambda|^p - (|\lambda|-\varepsilon)^p\big)^{1/p}$ is such, that $\lambda+\varepsilon = \lambda +_p \mu(\lambda,\varepsilon)$ (cf. (6.5)). Now, Proposition 3.1.8 yields

$$K^p_\lambda +_p \mu(\lambda,\varepsilon)(K^p_{\lambda+\varepsilon})^* \subseteq K^p_{\lambda+\varepsilon}. \tag{6.26}$$

Following the proof of Theorem 6.3.1, and due to the continuity of the full system of p-parallel bodies (see Proposition 5.2.14), we obtain that

$$\begin{aligned}\frac{d}{d\lambda}h(\lambda,u) &= \lim_{\varepsilon\to 0^+}\frac{h(\lambda+\varepsilon,u)-h(\lambda,u)}{\varepsilon}\\ &\geqslant \frac{|\lambda|^{p-1}}{h(\lambda,u)^{p-1}}\lim_{\varepsilon\to 0^+} h\left((K^p_{\lambda+\varepsilon})^*,u\right)^p.\end{aligned}$$

Finally, following the argument in [62, Lemma 3.1], we can conclude the result: since $\lambda < \lambda+\varepsilon$, then $(K^p_\lambda)^* \supseteq (K^p_{\lambda+\varepsilon})^* \supseteq E$, which implies that $h\big((K^p_\lambda)^*,u\big)$ is a monotone decreasing function in λ, and continuous almost everywhere on $[-r,0]$. Therefore, the above limit can be computed, $\lim_{\varepsilon\to 0^+} h\left((K^p_{\lambda+\varepsilon})^*,u\right)^p = h\left((K^p_\lambda)^*,u\right)^p$, and we obtain inequality (6.25). □

We finish the chapter by proving the following p-analog of the characterization of the class $\mathcal{R}_{n-1}$ (see Definition 3.2.1) in the setting of p-parallel bodies. The mentioned characterization for the classical case $p = 1$ can be found in [45].

Theorem 6.3.4 ([39, Theorem 25]). *Let the convex body $E \in \mathcal{K}^n_0$ be regular, and let $K \in \mathcal{K}^n_{00}(E)$. Let $1 \leqslant p < \infty$. Then $W_{n-1}(\lambda)$ is differentiable on $(-r, 0)$ with $W'_{n-1}(\lambda) = |\lambda|^{p-1} W_{p,n-1}(\lambda, E; E)$, if and only if K satisfies $K = K^p_{-r} +_p rE$.*

Proof. We first assume that $W'_{n-1}(\lambda) = |\lambda|^{p-1} W_{p,n-1}(\lambda, E; E)$. Integration on the interval $(-r, 0)$, together with (N.36), Fubini and Theorem 6.3.1 yield

$$\begin{aligned} W_{n-1}(K) &- W_{n-1}(K^p_{-r}) \\ &= \frac{1}{n} \int_{-r}^{0} \left(\int_{\mathbb{S}^{n-1}} \frac{|\lambda|^{p-1} h(E, u)^p}{h(\lambda, u)^{p-1}} \, dS\big(E[n-1]; u\big) \right) d\lambda \\ &\leqslant \frac{1}{n} \int_{\mathbb{S}^{n-1}} \left(\int_{-r}^{0} \frac{d}{d\mu}\Big|_{\mu=\lambda} h(\mu, u) \, d\lambda \right) dS(E[n-1]; u) \\ &= W_{n-1}(K) - W_{n-1}(K^p_{-r}). \end{aligned}$$

Hence, we have equality all over the above expression, and thus, the equality

$$\int_{-r}^{0} \frac{|\lambda|^{p-1} h(E, u)^p}{h(\lambda, u)^{p-1}} \, d\lambda = \int_{-r}^{0} \frac{d}{d\mu}\Big|_{\mu=\lambda} h(\mu, u) \, d\lambda$$

holds $S\big(E[n-1]; \cdot\big)$-almost everywhere on $\operatorname{supp} S(E[n-1]; \cdot) = \mathbb{S}^{n-1}$, because E is regular. From Remark 6.3.2, we get $K = K^p_{-r} +_p rE$.

Conversely, if $K = K^p_{-r} +_p rE$ then, for all $\lambda \in (-r, 0)$, by (N.13), Theorem 6.3.1 and (N.36),

$$\begin{aligned} W'_{n-1}(\lambda) &= \frac{1}{n} \int_{\mathbb{S}^{n-1}} \frac{d}{d\mu}\Big|_{\mu=\lambda} h(\mu, u) \, dS\big(E[n-1]; u\big) \\ &= \frac{1}{n} \int_{\mathbb{S}^{n-1}} \frac{|\lambda|^{p-1} h(E, u)^p}{h(\lambda, u)^{p-1}} \, dS\big(E[n-1]; u\big) \\ &= |\lambda|^{p-1} W_{p,n-1}(\lambda, E; E). \end{aligned}$$

□

Part IV

APPENDIX

A

TANGENTIAL BODIES

Uno mismo es quien menos sabe de su existencia. No se existe sino para los demás.
"Niebla",
M. Unamuno

A convex body K is a tangential body of $E \in \mathcal{K}^n$ if and only if through each boundary point of K there exists a support plane to K that also supports E. Tangential bodies are thus, (pairs of) convex bodies that have some, but not all, support planes in common.

Using the notion of extreme vector of a convex body (see Lemma 3.1.2 and the comments before), we can distinguish different types of tangential bodies, according to the type of supporting hyperplanes of K that also support E. More precisely:

Definition A.1. *A convex body $K \in \mathcal{K}^n$ containing $E \in \mathcal{K}^n$ is called a* p-tangential body *of E, $p \in \{0, \dots, n-1\}$, if each $(n-p-1)$-extreme support plane of K supports E.*

From the definition, it follows that a 0-tangential body of E is E itself, and each p-tangential body (of E) is also a q-tangential body (of E) for $p < q \leqslant n-1$. There exist p-tangential bodies of E which are not $(p-1)$-tangential bodies of E (see e.g. [23, p. 163]). An $(n-1)$-tangential body of E is briefly called a *tangential body*. Moreover, the 1-tangential bodies are just the cap-bodies ([68, p. 87]).

There are other equivalent definitions of tangential bodies. For more information we refer the reader to [68, Section 2.2, especially Theorem 2.2.10].

If K is a p-tangential body of E, then the inradius of K relative to E is one, i.e., $r(K; E) = 1$.

The following result characterizes tangential bodies by properties of their mixed volumes.

Theorem A.1 ([68, Theorem 7.6.17]). *Let $K, E \in \mathcal{K}^n_n$ be convex bodies satisfying $E \subseteq K$. Let $1 \leqslant p \leqslant n$. Then K is a p-tangential body of E if and only if*

$$W_0(K; E) = \cdots = W_{n-p}(K; E). \tag{A.1}$$

Indeed, it is enough to ask for $W_{n-p-1}(K; E) = W_{n-p}(K; E)$ in the above result, to obtain a characterization of p-tangential bodies (see [68, Theorem 7.6.17]).

Remark A.1 ([68, Proof of Theorem 7.6.17]). *Let $K, E \in \mathcal{K}^n$ be convex bodies satisfying $E \subseteq K$. Let $1 \leqslant p \leqslant n$. If K is a p-tangential body of E, then (A.1) holds true.*

We observe that, as $E \subseteq K$, the monotonicity of mixed volumes (N.14) yields

$$W_{n-p}(K; E) \leqslant W_j(K; E),$$

for all $0 \leqslant j \leqslant n-p$. The above results provides thus, not just a characterization of tangential bodies, but contributes to the still open problem of characterizing the equality in the Aleksandrov-Fenchel inequality (N.14).

For completeness, we recall Theorem 3.1.1, where a close connection between tangential bodies and inner parallel bodies is brought to light.

Theorem A.2 ([68, Lemma 3.1.14]). *Let* $K, E \in \mathcal{K}^n_n$ *be convex bodies and let* $\lambda \in \big(-r(K;E), 0\big)$. *Then* K_λ *is homothetic to* K *if and only if* K *is homothetic to a tangential body of* E.

If the gauge body E is regular, the equality in Lemma 3.1.3 (iv), namely, the equality in

$$\mathcal{U}_0(K_\lambda) \subseteq \mathcal{U}_0(K),$$

is characterized as follows.

Lemma A.1 ([43, Lemma 3.2]). *Let* $K \in \mathcal{K}^n$ *and* $E \in \mathcal{K}^n_n$ *be regular. If* K *is a tangential body of the outer parallel body* $(K_{-r})_r = K_{-r} + rE$, *then for any* $-r < \lambda \leqslant 0$,

$$\mathcal{U}_0(K) = \mathcal{U}_0(K_\lambda).$$

We notice that a characterization of tangential bodies -in the spirit of Chapter 1- in terms of the roots of the Steiner polynomial $f_{K;E}(z)$ was proven in [37, Proposition 3.1].

The following property has been often used, without further mentioning, through the work.

Remark A.2. *Let* $K, E \in \mathcal{K}^n$ *be convex bodies satisfying* $E \subseteq K$. *Then* K *is a tangential body of* E *if and only if*

$$h(K,u) = h(E,u)$$

for all $u \in \mathcal{U}_0(K)$.

Let $K, E \in \mathcal{K}^n_n$. Since the form body, K^* (see (3.2) for the definition), of K relative to E, is a tangential body of E, $h(K^*,u) = h(E,u)$ for all $u \in \mathcal{U}_0(K^*)$. The form body has played a role in some of the strengthenings of the isoperimetric inequality (e.g. [68, p. 386 and Theorem 7.2.3]), as well as in some of the inequalities contained in Chapter 4.

We conclude this short appendix on tangential bodies with the equality case of the Minkowski inequality (N.22).

Theorem A.3 ([68, Theorem 7.2.1]). *Let* $K \in \mathcal{K}^n$, *and let* $E \in \mathcal{K}^n_n$. *Equality holds in the inequality* (N.22), *i.e.*,

$$V(K[n-1], E)^2 = \mathrm{vol}(K)V(K[n-2], E[2]),$$

if and only if either $\dim K < n-1$, *or* K *is homothetic to an* $(n-2)$-*tangential body of* E.

B CONVEX FUNCTIONS

Camina lento, no te apresures, que a donde tienes que llegar es a ti mismo.
"El Espectador",
J. Ortega y Gasset

This appendix aims to collect properties of convex functions which are (mostly) implicitly used throughout the work. We refer to [10, 61, 68] for proofs of these and accurate studies of convex functions in much more generality.

Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ be the extended system of real numbers. We will adopt the following conventions. For $\lambda \in \mathbb{R}$, $-\infty < \lambda < \infty$, $\infty = \infty + \infty = \lambda + \infty = \infty + \lambda$, further, $\lambda\infty = \mathrm{sgn}(\lambda)\,\infty$, and also $-\infty = -\infty - \infty = \lambda - \infty = -\infty + \lambda = -\infty + (-\infty)$.

For a function $f : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$ and $\alpha \in \overline{\mathbb{R}}$ we write $\{f = \alpha\}$ to denote the set $\{x \in \mathbb{R}^n \,:\, f(x) = \alpha\}$. The sets $\{f < \alpha\}$ and $\{f \leqslant \alpha\}$ are analogously defined. We say that f is proper if $\{f = -\infty\} = \emptyset$ and $\{f = \infty\} \neq \mathbb{R}^n$.

Definition B.1. *A proper function $f : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$ is* convex *if for any $x, y \in \mathbb{R}^n$ and $0 \leqslant \lambda \leqslant 1$,*

$$f\big((1-\lambda)x + \lambda y\big) \leqslant (1-\lambda)f(x) + \lambda f(y).$$

A function f is concave *if $-f$ is convex.*

If the domain of the function f is the convex set $D \subseteq \mathbb{R}^n$, we say that f is convex if the function $\bar{f} : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$ defined as $\bar{f}(x) = f(x)$ for $x \in D$ and $\bar{f}(x) = \infty$ otherwise is convex.

Remark B.1. *Let $f : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$ be convex. Then the sets $\{f < \infty\}$, $\{f < \alpha\}$ and $\{f \leqslant \alpha\}$ for $\alpha \in \mathbb{R}$ are convex.*

Further, the set $\{(x, t) \in \mathbb{R}^n \times \mathbb{R} \,:\, f(x) \leqslant t\}$ is a convex subset of $\mathbb{R}^n \times \mathbb{R}$.

A real function f is affine if and only if it is convex and concave. We recall that affine real functions on $\mathbb{R}^n$ are those which can be written as $f(x) = \langle v, x\rangle + \alpha$, where $v \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$.

Remark B.2. *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a concave function, such that*

$$f\big(\lambda_0 a + (1-\lambda_0)b\big) = \lambda_0 f(a) + (1-\lambda_0)f(b) \;\; \text{for some} \;\; \lambda_0 \in (0, 1).$$

Then f is an affine function on the whole interval $[a, b]$.

Theorem B.1 (Jensen's inequality). *If f is convex, then*

$$f(\lambda_1 x_1 + \cdots + \lambda_k x_k) \leqslant \lambda_1 f(x_1) + \cdots \lambda_k f(x_k)$$

for all $x_1, \ldots, x_k \in \mathbb{R}^n$ and all $\lambda_i \in [0, 1]$, $i \in \{1, \ldots, k\}$, satisfying $\sum_{i=1}^{k} \lambda_i = 1$.

In the next proposition we collect some results about convex functions. Let $\operatorname{dom} f := \{f < \infty\}$.

Proposition B.1. *Let* $f : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$ *be convex. Then*

(i) Every local minimum is a global minimum,

(ii) f *is continuous on* $\operatorname{int} \operatorname{dom} f$*,*

(iii) f *is Lipschitz on any compact subset of* $\operatorname{int} \operatorname{dom} f$*.*

Next we state some differentiability properties of convex functions on the real line.

Proposition B.2. *Let* $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ *be convex, with* I *an open interval. Then*

(i) The left and right derivatives, denoted respectively by $(d^-/dx)f(x)$ *and* $(d^+/dx)f(x)$*, do exist and are finite at each point* $x \in \mathbb{R}$*.*

(ii) $(d^-/dx)f(x)$ *and* $(d^+/dx)f(x)$ *are non-decreasing functions.*

(iii)
$$\frac{d^-}{dx}f(x) \leqslant \frac{d^+}{dx}f(x).$$

(iv) With the exception of at most countably many points,
$$\frac{d^-}{dx}f(x) = \frac{d^+}{dx}f(x).$$

(v) If f *is differentiable on* $\operatorname{int} \operatorname{dom} f$*, then* f *is continuously differentiable.*

Remark B.3. *If* $f : [a, b] \longrightarrow \mathbb{R}$ *is convex, then the left (respectively, right) derivative exists at* $x = b$ *(respectively,* $x = a$*).*

Lemma B.1 ([60, Problem/Remark B, p. 13])**.** *If* $f : I \longrightarrow \mathbb{R}$ *is a convex (or concave) function, where* $I \subseteq \mathbb{R}$ *is an interval, then* f *is absolutely continuous on every* $[c, d] \subseteq I$*.*

The following basic result is useful when dealing with properties of convex funtions.

Remark B.4 (Three-slope-inequality)**.** *Let* $f : \mathbb{R} \longrightarrow \overline{\mathbb{R}}$ *be convex, and let* $x < y < z$*. Then*
$$\frac{f(y) - f(x)}{y - x} \leqslant \frac{f(z) - f(x)}{z - x} \leqslant \frac{f(z) - f(y)}{z - y}.$$

As a consequence of the mean value theorem applied to the function $t \mapsto t^p$, the following numerical inequality, useful in studying properties of convex functions, is easily proven.

Lemma B.2. *Let* $0 \leqslant a \leqslant b$ *and* $1 \leqslant p < \infty$. *Then*

$$p\,(b-a)a^{p-1} \leqslant b^p - a^p \leqslant p\,(b-a)b^{p-1}. \tag{B.1}$$

A function $f : \mathbb{R}^n \longrightarrow [0, \infty)$ is *log-concave* if $f = e^{-\phi}$, for a convex function $\phi : \mathbb{R}^n \longrightarrow (-\infty, \infty]$. There is a strong intertwining between the geometry of convex bodies and (the geometry of) log-concave functions. We refer to [68, Section 9.5] and the references therein for further information about it.

A class of functions which includes log-concave functions, is the class of α-concave functions, for $-\infty \leqslant \alpha \leqslant \infty$. If $\alpha \neq \{0, \pm\infty\}$, the function $f : \mathbb{R}^n \longrightarrow [0, \infty)$ is said to be *α-concave* if f is supported on some convex set Ω, and, for all $x, y \in \Omega$ and $\lambda \in [0, 1]$

$$f\big((1-\lambda)x + \lambda y\big) \geqslant [(1-\lambda)f(x)^\alpha + \lambda f(y)^\alpha]^{1/\alpha}.$$

If $\alpha = 0$ or $\{\pm\infty\}$ the definition is complemented in the limit sense. Using the notion of p-mean of positive numbers, a connection of the defintion of p-concave functions (in the previous sense, i.e., $\alpha = p$) and the definition of p-sum of convex bodies can be seen. For the sake of completeness we state once more an equivalent definition of α-concave functions using the mentioned notion of means.

We recall the definition of the p-th mean of two non-negative numbers, where p is a parameter varying in $\mathbb{R} \cup \{\pm\infty\}$. For this definition we follow [12] (regarding a general reference for p-th means of non-negative numbers, we refer also to the classic text of Hardy, Littlewood, and Pólya [33]).

Consider first the case $p \in \mathbb{R}$ and $p \neq 0$. Given $a, b \geqslant 0$ such that $ab \neq 0$ and $\lambda \in [0, 1]$,

$$M_p(a, b, \lambda) = \big((1-\lambda)a^p + \lambda b^p\big)^{1/p}.$$

For $p = 0$ we set

$$M_0(a, b, \lambda) = a^{1-\lambda} b^\lambda,$$

and the definition is completed for the values $p = \pm\infty$ defining $M_\infty(a, b, \lambda) = \max\{a, b\}$ and $M_{-\infty}(a, b, \lambda) = \min\{a, b\}$. Finally, if $ab = 0$, we will define $M_p(a, b, \lambda) = 0$ for all $p \in \mathbb{R} \cup \{\pm\infty\}$. Note that $M_p(a, b, \lambda) = 0$, if $ab = 0$, is redundant for all $p \leqslant 0$, however it is relevant for $p > 0$. As a consequence of Hölder's inequality one has

$$M_p(a, b, \lambda) \leqslant M_q(a, b, \lambda)$$

for $p \leqslant q$.

Using this definition, a non-negative function $f : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geqslant 0}$ is p-concave, $p \in \mathbb{R} \cup \{\pm\infty\}$, if

$$f\big((1-\lambda)x_1 + \lambda x_2\big) \geqslant M_p\left(f(x_1), f(x_2), \lambda\right)$$

for all $x_1, x_2 \in \mathbb{R}^n$ and all $\lambda \in (0, 1)$. This definition has thus, the following meaning:

(i) for $p = \infty$, f is ∞-concave if and only if f is constant on a convex set and 0 otherwise;

(ii) for $0 < p < \infty$, f is p-concave if and only if f^p is concave on a convex set, and 0 elsewhere;

(iii) for $p = 0$, f is 0-concave if and only if f is log-concave;

(iv) for $-\infty < p < 0$, f is p-concave if and only if f^p is convex;

(v) for $p = -\infty$, the function f is $(-\infty)$-concave if and only if its level sets $\{x \in \mathbb{R}^n : f(x) \geqslant t\}$ are convex (for all $t \in \mathbb{R}$).

The $(-\infty)$-concave functions are called *quasi-concave*. A real function $f : \mathbb{R}^n \longrightarrow [0, \infty)$ is called quasi-concave if

$$f\big((1-\lambda)x + \lambda y\big) \geqslant \min\big\{f(x), f(y)\big\},$$

for all $x, y \in \mathbb{R}^n$ and $0 < \lambda < 1$. For further details and properties on quasi-concave functions, we refer to [68, p. 520] and [10].

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NOTATION INDEX

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Summary

This work is devoted to the study of different aspects of classical Convex Geometry, specially, of the Brunn-Minkowski Theory, as a subfield of the latter. Convex Geometry, as the geometry of convex domains in the Euclidean space, has inherent geometric and analytic connections, as well as further links with other fields within Mathematics, and beyond.

The Habilitationsschrift is divided into three parts, entitled "On the volume of Minkowski combinations", "Inner parallel bodies: old and new", and "Within the L_p theory".

The first part of this work is focussed on different aspects of the Brunn-Minkowski inequality and the Steiner formula for convex bodies, which are some of the most classical results within Convex Geometry and pillars of the so-called Brunn-Minkowski Theory.

In the first chapter, we investigate structural properties of the cone of roots of relative Steiner polynomials of convex bodies, proving that they are closed, monotonic with respect to the dimension, and that they cover the whole upper half-plane, except the positive real axis, when the dimension tends to infinity. Further, there is a clear geometric link between the pairs of convex bodies whose relative Steiner polynomial has a complex root on the (non-real part of the) boundary of the cone of roots and fundamental inequalities within Convex Geometry: these convex bodies have to satisfy some Aleksandrov-Fenchel inequality with equality. A characterization of the polynomials with real coefficients, which can arise as Steiner polynomials of two convex bodies is provided and used as essential tool for the proofs in this chapter.

In the second chapter, we concentrate on the following equality, which can be seen as a particular case of the Brunn-Minkowski inequality, as well as a special case of the Steiner formula, for K, E convex bodies: $\mathrm{vol}(\lambda K + (1-\lambda)E) = \lambda \mathrm{vol}(K) + (1-\lambda)\mathrm{vol}(E)$. We prove that under the sole assumption that K and E have an equal volume projection (or a common maximal volume section) through parallel hyperplanes, if the above equality holds for just one value in $(0, 1)$, then $K = L + E$, or $E = K + L$, with L a convex body, such that $\dim L \leqslant 1$. We prove further, that having equality for all $\lambda \in [0, 1]$, if no extra assumption on K, E is done, such a characterization is not valid. This problem is connected with a conjecture relating the roots of the Steiner polynomial of a pair of convex bodies, the relative inradius of one of them, with respect to the other, and the Minkowski sum, which amounts to the equality case in the Bonnesen inradius inequality, in the two-dimensional case. We provide counterexamples for the general case of this conjecture. In the same line, we provide a counterexample for the equality case of a conjecture of Matheron.

The second part of this work focusses on decompositions of convex bodies, especially via inner parallel bodies. Given a fixed gauge body E, the inner parallel bodies of a convex body K, with respect to this gauge body, are a particular case of the Minkowski difference of two convex bodies, which is a natural subtraction counterpart of the Minkowski addition. In consequence, inner parallel bodies of the convex body K can be considered as potential decomposition elements of K via the Minkowski sum. In the third chapter, and with the aim to understand whether a convex body may be decomposed using its inner parallel bodies, some special issues about the boundary structure of it (and of its inner parallel bodies) are addressed, as they play an important role in the decomposition results. Besides that, the form body of a convex body happens to stand out as a decomposing element in this part and is used as a prominent tool in most of the decomposition results contained in the work. Along the way in this third chapter, we study certain differentiability properties of the quermassintegrals of the inner parallel bodies of a convex body, relative to a fixed gauge body, with respect to the parameter of definition of the inner parallel bodies. As a by-product of this study, we characterize the convex bodies in $\mathbb{R}^n$ whose quermassintegrals satisfy certain differentiability properties. As a consequence, we give an answer to a question posed by Bol in 1943 for the 3-dimensional space. Further, we specialise the decomposition results to the case of polytopes, where the gauge body is now the Euclidean ball, obtaining some improvements of the decomposition results proven for more general convex bodies. In the fourth chapter, we prove sharp bounds for the volume of a convex body, in terms of its surface area and other quermassintegrals. These bounds arise as consequences of inequalities for inner parallel bodies involving mixed volumes and inequalities which relate a convex body to its inner parallel bodies and its form body.

The last part of this work, which consists of two chapters, focusses on the investigation of a p-analog of the Minkowski difference, namely on the analog of the Minkowski difference within the so-called L_p-Brunn-Minkowski theory, an extension of the Brunn-Minkowski theory, in which the Minkowski sum has been replaced by the p-sum of convex bodies (containing the origin). In the fifth chapter, we introduce a subtraction counterpart of the well-known p-sum of convex bodies, the concept of p-difference, for which we prove several properties. In analogy to the classical case, we introduce the notion of p-(inner) parallel bodies, proving an analog of the concavity of the family of classical parallel bodies for the p-parallel ones, as well as the continuity of this new family. Further, we characterize tangential bodies as the only convex bodies such that their p-inner parallel bodies are homothetic copies of them, extending the corresponding (classical) result. Then, in the sixth and last chapter, we investigate the differentiability of the quermassintegrals with respect to the already introduced one-parameter family of p-parallel bodies. We obtain, as in the classical case, that the volume is differentiable. We prove that, although there is no polynomial expression for the p-outer parallel bodies, all other quermassintegrals are differentiable on positive values of the parameter too. We end the chapter proving a sharp lower bound for the derivative of the support function of the p-inner parallel bodies along with equality conditions, complementing the classical analog of this result.

Zusammenfassung der Habilitationsschrift

In der vorliegenden Arbeit beschäftigen wir uns mit verschiedenen modernen Aspekten der klassischen Konvexgeometrie, insbesondere der Brunn-Minkowski-Theorie. Konvexgeometrie ist die Geometrie und Analysis der konvexen Bereiche im Euklidischen Raum und hat somit intrinsische Verknüpfungen und Anwendungen inner- und außerhalb der Mathematik.

Die Habilitationsschrift ist in drei Teile gegliedert: "On the volume of Minkowski combinations", "Inner parallel bodies: old and new" und "Within the L_p theory".

Der erste Teil ist der Brunn-Minkowski-Ungleichung und der Steiner-Formel für konvexe Körper gewidmet; beide gehören zu den zentralen Resultaten innerhalb der klassischen Brunn-Minkowski-Theorie.

Im ersten Kapitel untersuchen wir strukturelle Eigenschaften des (konvexen) Kegels, der von den Nullstellen von Steiner Polynomen konvexer Körper erzeugt wird. Wir beweisen, dass dieser Kegel abgeschlossen ist, sich monoton bezüglich der Dimension verhält, und dass er die komplette obere komplexe Halbebene (bis auf die positive reelle Achse) ausfüllt, wenn die Dimension gegen unendlich geht. Außerdem gibt es eine geometrische Beziehung zwischen den Paaren konvexer Körper, deren Steiner Polynom eine nicht-reelle Nullstelle auf dem Rand des Nullstellenkegels hat, und den fundamentalen Aleksandrov-Fenchel-Ungleichungen: Diese konvexen Körper erfüllen mindestens eine der Ungleichungen mit Gleichheit. Weiterhin geben wir eine Charakterisierung der Steiner Polynome zweier konvexer Körper innerhalb des Raums aller Polynome. Diese Charakterisierung spielt auch eine wichtige Rolle in weiteren Beweisen.

Im zweiten Kapitel konzentrieren wir uns auf die folgende Gleichung: $\mathrm{vol}(\lambda K + (1-\lambda)E) = \lambda \mathrm{vol}(K) + (1-\lambda)\mathrm{vol}(E)$ für konvexe Körper K, E, wobei $\mathrm{vol}(\cdot)$ das Volumen darstellt. Diese Gleichung können wir als Spezialfall der Brunn-Minkowski-Ungleichung als auch der Steiner-Formel betrachten. Wir beweisen unter der alleinigen Annahme, dass K und E eine Projektion auf eine Hyperbene mit gleichem Volumen haben (oder einen gemeinsamen volumenmaximalen Schnitt bezüglich paralleler Hyperebenen haben), dass aus der Gültigkeit der Gleichung für ein $\lambda \in (0,1)$ folgt: $K = E + L$ oder $E = K + L$ für einen konvexen Körper L mit $\dim L \leqslant 1$. Anschließend zeigen wir, dass ohne weitere Annahmen an K, E eine solche Charakterisierung nicht möglich ist, sogar wenn für alle $\lambda \in [0,1]$ Gleichheit gilt. Dieses Problem ist zudem mit einer vermuteten Beziehung zwischen den Nullstellen von Steiner Polynomen zweier konvexer Körper, derem relativen Inradius und ihrer Minkowski-Summe verbunden. Diese vermutete Beziehung entspricht in der Ebene der bekannten Bonnesen-Inradius-Ungleichung. In diesem Zusammenhang geben wir im gleichen Kapitel auch ein Gegenbeispiel für den Gleichheitsfall einer Vermutung von Matheron.

Im zweiten Teil der Arbeit liegt der Fokus auf der Zerlegung konvexer Körper mittels der Minkowski-Summe, insbesondere durch innere Parallelkörper. Innere Parallelkörper bezüglich eines festen Eichkörpers E sind spezielle Beispiele für die Minkowski-Differenz zweier konvexer Körper, die ein natürliches Subtraktionsanalogon der Minkowski-Addition darstellt. Somit sind innere Parallelkörper potenzielle Zerlegungselemente von K mittels der Minkowski-Summe. Um zu entscheiden, ob innere Parallelkörper eines gegebenen konvexen Körpers diesen zerlegen können, werden im dritten Kapitel zunächst verschiedene Fragen bezüglich des Randes des konvexen Körpers (und dessen inneren Parallelkörpern) untersucht. Eine essentielle Rolle bei diesen Zerlegungsarten kommt dem sogenannten Formkörper eines konvexen Körpers zu, der auch sehr häufig ein entscheidendes Element bei den Beweisen in diesem Kapitel ist. Ferner untersuchen wir Differenzierbarkeitseigenschaften von Quermaßintegralen innerer Parallelkörper eines konvexen Körpers bezüglich der reellen Variable, die die inneren Parallelkörper parameterisiert. Hieraus ergibt sich auch eine ausführliche Antwort auf eine von Bol 1943 gestellte Frage im dreidimensionalen Euklidischen Raum.

Weiterhin befassen wir uns auch mit solchen Zerlegungen im Spezialfall von Polytopen und bezüglich der Kugel als Eichkörper. Hier erzielen wir stärke Aussagen als im allgemeinen Fall.

Im vierten Kapitel beweisen wir bestmögliche Schranken für das Volumen eines konvexen Körpers in Abhängigkeit zu seiner Oberfläche und weiteren Quermaßintegralen. Diese Schranken ergeben sich aus Ungleichungen für gemischte Volumina von inneren Parallelkörpern und Ungleichungen, die einen konvexer Körper mit seinem inneren Parallelkörper und seinem Formkörper verbinden.

Im letzten Teil der Arbeit beschäftigen wir uns mit Minkowski-Differenzen innerhalb der allgemeineren L_p Brunn-Minkowski-Theorie. In der L_p-Brunn-Minkowski-Theorie wird die klassische Minkowski-Summe durch sogenannte p-Summen von konvexen Körpern (die den Ursprung im Inneren haben) ersetzt, und sie ist insbesondere eine Erweiterung der klassischen Brunn-Minkowski-Theorie ($p = 1$). Im fünften Kapitel führen wir eine Subtraktion innerhalb der L_p-Brunn-Minkowski Theorie ein. Wir beweisen etliche Eigenschaften für diese Subtraktion und leiten den Begriff der p-Parallelkörpers ab. Analog zu dem klassischen Fall ($p = 1$) beweisen wir für die Familie der p-inneren Parallelkörpers, dass sie stetig bezüglich des Definitionsparameters ist und auch eine zur klassichen Konkavität ähnliche Eigenschaft besitzt. Weiterhin werden Tangentialkörper als die einzigen Körper charakterisiert, deren p-inneren Parallelkörper homothetisch sind. Im sechsten Kapitel untersuchen wir die Differenzierbarkeitseigenschaften der Quermaßintegrale von p-inneren Parallelkörpern bezüglich des Definitionsparameters. Wir weisen nach, dass das Volumen differenzierbar ist und dass alle Quermaßintegrale differenzierbar sind für positive Werte des Definitionsparameters, obwohl es keinen Polynomausdruck dafür gibt. Wir beenden das Kapitel mit einer unteren Schranke für die Ableitung der Stützfunktion von p-inneren Parallelkörpern zusammen mit Gleichheitsbedingungen. Diese erweitert ebenfalls den klassichen Fall $p = 1$.

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Magdeburg, 31. August 2017

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