Dependence properties of sequential order statistics

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Abstract

In this thesis we study the dependence properties of sequential order statistics based on exchangeable random variables. One of the main application areas of sequential order statistics is reliability theory where they can be used for describing the lifetime of a technical k-out-of-n system. Modeling the system lifetime with the help of sequential order statistics allows to take into account such factors as the interdependence of components and the impact that a failure has on the lifetimes of surviving components. In this respect, the knowledge about the influence of dependence properties of component lifetimes on sequential order statistics would provide an important insight into structural properties of the model.

Our work presents the results on the analysis of such dependence properties as multivariate total positivity of order two (MTP_2), hazard rate increasing upon failure and lifetimes conditionally increasing in sequence.

We provide necessary and sufficient conditions for the above-mentioned dependence properties of sequential order statistics. In particular, we derive sufficient conditions for the MTP_2 of sequential order statistics for several types of distributions distinguished by the special forms of their conditional hazard rates. Among others we provide sufficient conditions for the MTP_2 of sequential order statistics based on Schur-constant random variables and Archimedean copulas.

Obtained results shed light on cases when dependence of component lifetimes induces the corresponding properties of sequential order statistics. Thus, our work gives a better understanding of the relation between the component lifetimes and the lifetime of the system and provides a basis for further analysis of reliability and aging properties of sequential order statistics based on exchangeable random variables.

Zusammenfassung

In dieser Dissertation untersuchen wir die Abhängigkeitseigenschaften von sequentiellen Ordnungsstatistiken basierend auf austauschbaren Zufallsvariablen.
Eines der wichtigsten Anwendungsgebiete von sequentiellen Ordnungsstatistiken ist die Zuverlässigkeitstheorie, in der sie zur Beschreibung der Lebensdauer eines technischen k-von-n-Systems verwendet werden können. Die Modellierung der Systemlebensdauer mit Hilfe von sequentiellen Ordnungsstatistiken erlaubt es, Faktoren wie die Abhängigkeit von Komponenten und die Auswirkungen von Ausfällen auf die Lebensdauern noch intakter Komponenten zu berücksichtigen. Kenntnisse über den Einfluss von Abhängigkeitseigenschaften der Lebensdauern von Komponenten auf sequentielle Ordnungsstatistiken liefern wichtige Einsichten in strukturelle Eigenschaften des Modells.

In unserer Arbeit werden Ergebnisse der Untersuchung von Abhängigkeitseigenschaften wie der mehrdimensionalen totalen Positivität der Ordnung 2 (MTP_2), „hazard rate increasing upon failure“und „conditionally increasing in sequence“vorgestellt.

Wir erhalten notwendige und hinreichende Bedingungen für die genannten Abhängigkeitseigenschaften von sequentiellen Ordnungsstatistiken. Insbesondere leiten wir hinreichende Bedingungen für MTP_2 von sequentiellen Ordnungsstatistiken für mehrere Typen von Verteilungen her, die sich durch die spezielle Form ihrer bedingten Ausfallrate unterscheiden. Unter anderem liefern wir hinreichende Bedingungen für MTP_2 von sequentiellen Ordnungsstatistiken basierend auf Schur-konstanten Zufallsvariablen und Archimedischen Copulas.

Unsere Ergebnisse werfen Licht auf die Fälle, in denen Abhängigkeiten der Lebensdauern von Komponenten entsprechende Eigenschaften für sequentiellen Ordnungsstatistiken bewirken. Hierdurch trägt unsere Arbeit zu einem besseren Verständnis der Zusammenhänge zwischen Lebensdauern von Komponenten und der Systemlebensdauer bei und stellt eine Basis für weitere Untersuchungen von Zuverlässigkeits- und Alterungseigenschaften von sequentiellen Ordnungsstatistiken basierend auf austauschbaren Zufallsvariablen dar.

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Contents

Introduction

Stochastic dependence arises in a variety of areas such as medicine, economics, finance and engineering. For instance, consider a technical system that consists of several identical components working simultaneously. The lifetimes of components are not predefined and, as a result, possess a stochastic nature. Since components share the environment and load, a failure of one likely influences the life-lengths of the remaining. This observation indicates the existence of stochastic dependence between components lifetimes. Thus, the reliability and safety of such system is closely related to the notion of dependence between its parts. In past decades, the topic of stochastic dependence has received a major attention. It has found applications in probability theory and statistics, reliability theory, mathematical physics, etc.

In this thesis we study the dependence properties of sequential order statistics based on exchangeable random variables – a probabilistic model for ordered data that was introduced in Burkschat (2009). One of its main application areas is reliability theory where sequential order statistics provide an expression for the life-time of a technical k-out-of-n system.

In general, a k-out-of-n system consists of n identical components that start working simultaneously. It functions as long as at least k out of n components are working. A machine with 4 engines that works as longs as at least 2 engines are running can be considered as 2-out-of-4 system. k-out-of-n systems were thoroughly investigated in context of the coherent systems theory. Their analysis can be found in Barlow & Proschan (1981) and Meeker & Escobar (1998).

From the definition of a k-out-of-n system follows that the lifetime of the system is represented by the $(n - k + 1)$-th ordered lifetime of its components and can be obtained in terms of an ordered data model. Because of the widely developed statistical theory, order statistics based on iid random variables are often chosen for this purpose. Their detailed description can be found in David & Nagaraja (2003), Arnold et al. (1992). However, order statistics based on iid random variables do not reflect possible effects of failures on the remaining components. To include such interactions sequential order statistics based on conditionally iid random variables were introduced in Kamps (1995). Further, in Burkschat (2009) conditional independence was replaced by exchangeability. It resulted in formulation of a universal model for lifetimes in a k-out-of-n system. The corresponding ordered random quantities were named sequential order statistics based on exchangeable random variables.

As mentioned above, reliability of a system is closely related to dependence relations between its parts. The strength of the relations can be assessed with the help of different dependence notions. In this regard we will focus on positive dependence properties. They measure the tendency of random variables to take on concordant values. For instance, an indicator of positive dependence is positive correlation. An overview of positive dependence concepts can be found in Colangelo et al. (2005). Many dependence properties describe the joint behavior of components from the reliability point of view. For instance, describing lifetimes as "hazard rate increasing upon failure" indicates that with a failure of one component the danger of another breakdown increases. Assumptions about system dependence aspects also lead to a more sophisticated analysis of the effects of different component treatments, like repair or replacement policies, see Lai & Xie (2006). Thus, dependence properties play an important role in understanding the reliability models and in the further development of their statistical applications, see Barlow & Proschan (1981) and Block et al. (1990).

Since the lifetime of a system can be expressed as the $(n-k+1)$-th order statistic, the following question arises: How does the dependence structure of underlying lifetime distributions influence the properties of order statistics? For order statistics based on iid random variables and sequential order statistics based on conditionally iid random variables an extensive answer can be found in the literature, see Avèrousa et al. (2005), Boland et al. (1996) and Cramer (2006). In particular, both models reveal positive dependence and they are based on positively dependent random variables. With this thesis we address a similar problematic for sequential order statistics based on exchangeable random variables. Our goal is to explore their dependence properties in comparison to other models of ordered data and in reference to the real-life qualities of k-out-of-n systems. We will proceed with the narrative as outlined below.

In Chapter 1 we look at several models for ordered data, the assumptions that they impose on a k-out-of-n system, their key properties and connections between different models. Here we start with the consideration of order statistics based on iid random variables and proceed to the models that impose weaker conditions on underlying distributions of random lifetimes. We finish this chapter with the description of sequential order statistics based on exchangeable random variables.

Chapter 2 builds a foundation for the latter analysis of dependence properties of sequential order statistics based on exchangeable random variables. We start this chapter by introducing multivariate conditional hazard rates. These objects represent the risk of a breakdown taking into account previous failures in the system. They also provide an alternative representation for densities of multivariate survival distributions, see Spizzichino (2001). It is important to note that the joint density of sequential order statistics based on exchangeable random variables can be expressed in terms of multivariate densities that describe component lifetimes between two successive failures. Therefore, further in this chapter we look at different types of multivariate survival distributions. They are grouped according to the complexity of interactions between the lifetimes which is assessed with the help of conditional hazard rates.

In Chapter 3 we give an overview of dependence concepts, related stochastic orders and their properties. A special focus is on the dependence properties of ordered random variables as well as random variables described by multivariate distributions from Chapter 2. This chapter provides the motivation for the analysis conducted in Chapter 4.

Chapter 4 represents the results of our research on the topic of dependence properties of sequential order statistics based on exchangeable random variables. As mentioned above the problem can be summarized by the question: Do the sequential order statistics inherit dependence properties from the underlying distributions? First we consider this question in a general set up and provide conditions for several dependence properties of sequential order statistics. Then we proceed with the analysis of multivariate total positivity of order two (MTP_2), which is one of the strongest positive dependence properties. Here we derive the conditions for the MTP_2 of sequential order statistics under the assumption that the underlying lifetimes follow a distribution belonging to one of the groups described in Chapter 2.

Conventions and notation

For $x = (x_1, \ldots, x_n)$ and $x = (y_1, \ldots, y_n)$ under $x \leq y$ we understand $x_i \leq y_i$ for all $i = 1, \ldots, n$.

By increasing (decreasing) functions we mean non-decreasing (non-increasing) functions.

For all integrals or expectations in this thesis we assume that requirements of measurability and integrability are automatically met without further mention. Moreover, considering a product $\mathcal{X}$ of measurable spaces $\mathcal{X}_i,\ i = 1, ..., n$ denote by $\sigma = \sigma_1 \times \ldots \times \sigma_n$ a product measure on $\mathcal{X}$ with σ_i representing σ-finite measures on $\mathcal{X}_i, i = 1, \ldots, n$. Then we will use a shorter notation

$$d\sigma_i(x_i) = dx_i,\ d\sigma(x) = dx$$

and

$$\int_{\mathcal{X}} f(x) d\sigma(x) = \int_{\mathcal{X}} f(x) dx\ .$$

In this thesis we consider random vectors of the form $Y = (Y_1, \ldots, Y_n),\ n \in \mathbb{N}$

defined on some fixed probability space $(\Omega, \mathcal{F}, P)$. In this thesis by absolutely continuous random variables we understand random variables with the joint distribution that is absolutely continuous with respect to the n-dimensional Lebesgue measure.

Notation

$\mathbb{N}$	$\{1,2,3,\ldots\}$
$\mathbb{R}$	$(-\infty,\infty)$
$\mathbb{R}_+$	$[0,\infty)$
$\mathbb{R}^n_{\leq}$	$\{(t_1,\ldots,t_n) \in \mathbb{R}^n \mid t_1 \leq \cdots \leq t_n\}$
C_n^j	$\frac{n!}{j!(n-j)!}$
ln	natural logarithm
a.s.	almost sure
iid	independent and identically distributed
iff	if and only if
S_n	set of all permutations on $\{1,\ldots,n\}$
$X \sim F$	X is a random variable with distribution function F
$\overline{F}$	$1 - F$

1 Order statistics

In this chapter we consider several models for ordered data along with assumptions that they impose in the context of specific applications from reliability theory. The overview is hinged on the objects called order statistics. As they are well studied in the literature, we provide only the information needed for the argumentation in the following chapters. For deeper insights into the theory of ordered data models consult the literature cited below.

Let us start by considering n random variables $Y_1, \ldots, Y_n$. Arranging them in the ascending order of magnitude we obtain order statistics $Y_{1:n} \leq \ldots \leq Y_{n:n}$ (based on $Y_1, \ldots, Y_n$). There are different ways to define order statistics formally. Below the definition from Kamps (1995), using the permutation matrices, is cited. Another one, using pseudo-inverse functions, can be found in David & Nagaraja (2003).

Def 1.1. *For* $(y_1, \ldots, y_n) \in \mathbb{R}^n$ *let a function* $T : \mathbb{R}^n \to \mathbb{R}^n$ *be defined by*

$$T(y_1, \ldots, y_n) = (y_{(1)}, \ldots, y_{(n)})$$

with

$$(y_1, \ldots, y_n)P = (y_{(1)}, \ldots, y_{(n)}) \, ,$$

where $y_{(1)} \leq \ldots \leq y_{(n)}$ *and* P *is* $n \times n$ *permutation matrix (which results from permuting the columns of a* $n \times n$ *identity matrix and is defined according to the positions of* y_i *in* $(y_{(1)}, \ldots, y_{(n)})$, $i = 1, \ldots, n$*). Moreover, for* $i = 1, \ldots, n$ *let the functions* $T_i : \mathbb{R}^n \to \mathbb{R}$ *be defined by*

$$T_i(y_1, \ldots, y_n) = y_{(i)} \, .$$

Then, for real valued random variables $Y_1, \ldots, Y_n$

$$Y_{(i)} = T_i(Y_1, \ldots, Y_n)$$

is called the i*-th order statistic,* $i = 1, \ldots, n$*, and*

$$(Y_{(1)}, \ldots, Y_{(n)}) = T(Y_1, \ldots, Y_n)$$

is the vector of order statistics (based on $Y_1, \ldots, Y_n$).

Remark 1.2. *Note that the occurrence of identical outcomes for order statistics based on absolutely continuous random variables has probability zero. Therefore considering absolutely continuous $Y_1, \ldots, Y_n$ events, such that two or more order statistics attain the same value, can be neglected.*

Order statistics find applications in many areas of mathematics and engineering science such as statistical inference, life testing, extreme value theory, image processing (see David & Nagaraja (2003), Castillo (1988), Pitas & Venetsanopoulos (1992)). In particular, in reliability theory order statistics are used to model lifetimes of components in a technical system.

Consider a system of n identical components that start working simultaneously. In addition, assume that the system works as long as at least k of n components are functioning. In the literature such systems are called k-out-of-n systems, they play a significant role in reliability theory and are well investigated. For a detailed description of different types of k-out-of-n systems we refer to Way & Ming (2002). Below we cite several examples provided in that book. For instance, consider an automobile with eight cylinder engine that can be driven if at least four cylinders are firing, i.e. it functions as 4-out-of-8 system. Considering a communication system with three transmitters, to ensure the delivery of some critical messages at least two of the transmitters should be operational at any time. Then the transmission system behaves as 2-out-of-3 system. Other examples of k-out-of-n systems are multi-engine systems in airplanes, multi-display systems in cockpits, multi-pump subsystems in hydraulic control systems, etc. From the probabilistic point of view we observe that the lifetime of an arbitrary k-out-of-n system is given by the $(n - k + 1)$-th order statistic based on the random variables representing the lifetimes of components. For example, the eight-cylinder car can be driven for the period of time defined by the fourth order statistic.

Many research questions involving ordered data models are motivated by practical applications. Therefore, we will often interpret facts concerning order statistics in terms of lifetimes in a technical system. In the following sections we will specify different models for order statistics by looking at the assumptions that they impose on the corresponding k-out-of-n system.

1.1 Order statistics based on iid random variables

The simplest model for ordered lifetimes arises if we assume components to be identical and have no influence on each other. In other words their lifetimes are represented by continuous iid random variables and the lifetime of the whole system is modeled with the help of order statistics based on them. Ordered failure times in such system are described by the joint density function presented in David & Nagaraja (2003).

Lemma 1.3. *Let $Y_{1:n}, \ldots, Y_{n:n}$ represent order statistics from absolutely continuous iid random variables $Y_1, \ldots, Y_n$. Then the joint density function of $Y_{1:n}, \ldots, Y_{n:n}$ can be calculated as*

$$f^{Y_{1:n},\ldots,Y_{n:n}}(t_1, \ldots, t_n) = n! \prod_{i=1}^{n} f(t_i) \ ,$$

where $t_1 \leq \ldots \leq t_n$ and f represents the density of Y_i, $i = 1, \ldots, n$.

Moreover, $Y_1, \ldots, Y_n$ form a Markov chain with transition densities

$$f^{Y_{r+1:n} \,|\, Y_{r:n}}(t_{r+1}|t_r) = (n-r) \left(\frac{1 - F(t_{r+1})}{1 - F(t_r)} \right)^{n-r-1} \frac{f(t_{r+1})}{1 - F(t_r)} \ ,$$

where F is the distribution function corresponding to f, $r = 1, \ldots, n-1$.

An important advantage of this model is the simple form of the joint density function, which allows to draw conclusions about the system properties at large. However, the model is not able to reflect two important aspects of k-out-of-n systems. First, the components, e.g. engines, share the same working environment and therefore are dependent in many real-life systems. Second, a failure likely influences the work of remaining components. This influence can manifest itself through increased load or damages caused by failures.

Therefore, in the following section we will look at structures that arise from relaxing the independence assumption.

1.2 Order statistics based on exchangeable random variables

Let lifetimes of components in the system be modeled with absolutely continuous random variables $Y_1, \ldots, Y_n$ such that their joint density function is symmetric, e.g.

$$f^{Y_1,\ldots,Y_n}(t_1, \ldots, t_n) = f^{Y_1,\ldots,Y_n}(t_{\pi(1)}, \ldots, t_{\pi(n)}) \tag{1.1}$$

for all permutations π defined on the set $\{1, \ldots, n\}$, $n \in \mathbb{N}$. According to Bernardo (1996) such random variables are called exchangeable. Thus, components are regarded as identical or, in other words, independent of the labels given to them. This assumption corresponds to a system of components with identical technical characteristics.

Representation (1.1) indicates that the density is a symmetric function. As a result, the distribution of order statistics based on exchangeable $Y_1, \dots, Y_n$ can be described by the density function

$$f^{Y_{1:n},\dots,Y_{n:n}}(t_1, \dots, t_n) = n! f(t_1, \dots, t_n) \;, \tag{1.2}$$

for $t_1 \leq \cdots \leq t_n$ (see David & Nagaraja (2003)).

Moreover, exchangeability allows to address a wide class of joint distributions for the lifetimes of underlying components, for instance mixture models and copulas. Note that due to representation (1.2) properties of order statistics are closely related to those of underlying variables.

1.3 Sequential order statistics based on conditionally iid random variables

Consider the assumption that lifetime distributions in the system can change from failure to failure. It reflects the idea of a failure causing damage to surviving components and therefore changing their properties or weakening the system. The corresponding model for ordered data represents an extension of ordinary order statistics based on iid random variables. It was first introduced in Kamps (1995). The underlying concept was explained in terms of the triangular scheme which we introduce next.

$$\begin{array}{ccccccccc}
X_*^{(1)} & \leftarrow & Z_1^1 & Z_2^1 & \cdots Z_{n-1}^1 & Z_n^1 & \sim & F_1(\cdot) \\
X_*^{(2)} & \leftarrow & Z_1^2 & Z_2^2 & \cdots Z_{n-1}^2 & & \sim & \frac{F_2(\cdot)-F_2(z_{1,n}^1)}{1-F_2(z_{1,n}^1)} \\
\vdots & & \vdots & \vdots & & & \vdots & \\
X_*^{(n-1)} & \leftarrow & Z_1^{n-1} & Z_2^{n-1} & & & \sim & \frac{F_{n-1}(\cdot)-F_{n-1}(z_{1,3}^{n-2})}{1-F_{n-1}(z_{1,3}^{n-2})} \\
X_*^{(n)} & \leftarrow & Z_1^{n} & & & & \sim & \frac{F_n(\cdot)-F_n(z_{1,2}^{n-1})}{1-F_n(z_{1,2}^{n-1})}
\end{array}$$

For the sake of simplicity consider the scheme in a context of a k-out-of-n system. Then every line in the scheme corresponds to the state of the system in the time period between two consecutive failures. The order in which the states occur is reflected by the upper index of $X_*^{(\cdot)}$.

Let $F_1, \ldots, F_n$ be continuous distribution functions and $z^{(1)}_{1,n} \leq z^{(2)}_{1,n-1} \leq \ldots \leq z^{(n-1)}_{1,2}$ be real numbers. The first line of the scheme contains random variables $Z^1_1, \ldots, Z^1_n$ that represent the lifetimes of components before the first failure.

Moreover, it is assumed that $Z^1_1, \ldots, Z^1_n$ are iid with distribution function

$$F_1(\cdot) = \frac{F_1(\cdot) - F_1(z^{(0)}_{1,n+1})}{1 - F_1(z^{(0)}_{1,n+1})}, \quad z^{(0)}_{1,n+1} = -\infty \ .$$

Then the first ordered failure time $X^{(1)}_*$ is obtained as a minimum in the sample $Z^1_1, \ldots, Z^1_n$. To pass to the second state, suppose that the first failure has happened at time $z^{(1)}_{1,n}$. Then there remains $n-1$ functioning components with lifetimes $Z^2_1, \ldots, Z^2_{n-1}$. Given the first failure time $z^{(1)}_{1,n}$, the random variables $Z^2_1, \ldots, Z^2_{n-1}$ are iid with distribution functions

$$\frac{F_2(\cdot) - F_2(z^{(1)}_{1,n})}{1 - F_2(z^{(1)}_{1,n})} \ .$$

By analogy, second failure time $X^{(2)}_*$ is modeled as the minimum of $Z^2_1, \ldots, Z^2_{n-1}$.

In general, the triangular scheme consists of random variables

$$\left(Z^{(r)}_j\right)_{1 \leq r \leq n,\ 1 \leq j \leq n-r+1} ,$$

where $\left(Z^{(r)}_j\right)_{1 \leq j \leq n-r+1}$ are iid according to the distribution function

$$\frac{F_r(\cdot) - F_r(z^{(r-1)}_{1,n-r+2})}{1 - F_r(z^{(r-1)}_{1,n-r+2})}, \quad 1 \leq r \leq n, \quad z^{(0)}_{1,n+1} = -\infty ,$$

which is F_r truncated on the left at the occurrence time $z^{(r-1)}_{1,n-r+2}$ of the $(r-1)$-th failure. Given $z^{(r-1)}_{1,n-r+2}$ the next failure time $X^{(r)}_*$ is modeled as the minimum in the sample $Z^{(r)}_1, \ldots, Z^{(r)}_{n-r+1}$, which represents lifetimes of remaining $n-r+1$ components. Failure times $X^{(1)}_*, \ldots, X^{(n)}_*$ obtained according to the triangular scheme are called sequential order statistics. The definition of sequential order statistics contains pseudo inverse functions that we introduce next.

Def 1.4. *For a univariate distribution function* $F : \mathbb{R} \to [0,1]$ *the pseudo-inverse* $F^{-1} : [0,1] \to \mathbb{R}$ *is defined by*

$$F^{-1}(y) = \inf\{x : F(x) \geq y\}$$

for $y \in (0,1)$ *and* $F^{-1}(0) = \lim_{y \to 0+} F^{-1}(y)$, $F^{-1}(1) = \lim_{y \to 1-} F^{-1}(y)$.

Then sequential order statistics can be formally defined as follows.

Def 1.5. *Let* $\left(Y_j^{(r)}\right)_{1 \leq r \leq n, 1 \leq j \leq n-r+1}$ *be independent random variables with*

$$\left(Y_j^{(r)}\right)_{1 \leq j \leq n-r+1} \sim F_r \ ,$$

where $r = 1, \ldots, n$ *and* $F_1, \ldots, F_n$ *are continuous distribution functions with*

$$F_1^{-1}(1) \leq \ldots \leq F_n^{-1}(1) \ .$$

Moreover, for $j = 1, \ldots, n$ *let* $X_j^{(1)} = Y_j^{(1)}$,

$$X_*^{(1)} = \min\left\{X_1^{(1)}, \ldots, X_n^{(1)}\right\}$$

and for $r = 2, \ldots, n$ *define*

$$X_j^{(r)} = F_r^{-1}\left(F_r\left(Y_j^{(r)}\right)\left(1 - F_r\left(X_*^{(r-1)}\right)\right) + F_r\left(X_*^{(r-1)}\right)\right) \ ,$$

$$X_*^{(r)} = \min_{1 \leq j \leq n-r+1} X_j^{(r)} \ .$$

Then the random variables $X_*^{(1)}, \ldots, X_*^{(n)}$ *are called sequential order statistics.*

Remark 1.6. *In Definition 1.5 random variables* $X_1^{(r)}, \ldots, X_{n-r+1}^{(r)}$ *represent the lifetimes of surviving components after the* $(r-1)$*-th failure. Random variables* $Y_1^{(r)}, \ldots, Y_{n-r+1}^{(r)}$

are designed to obtain conditional distributions

$$P\big(X_j^{(r)} \le t \,|\, X_*^{(r-1)} = z\big) = \frac{F_r(t) - F_r(z)}{1 - F_r(z)},$$

that we have seen in the triangular scheme. Indeed

$$\begin{aligned}
&P\big(X_j^{(r)} \le t \,|\, X_*^{(r-1)} = z\big) \\
&\quad = P\bigg(F_r^{-1}\Big(F_r\big(Y_j^{(r)}\big)\Big(1 - F_r\big(X_*^{(r-1)}\big)\Big) + F_r\big(X_*^{(r-1)}\big)\Big) \le t \,|\, X_*^{(r-1)} = z\bigg) .
\end{aligned}$$

Since $F_r\big(Y_j^{(r)}\big)(1 - F_r(z)) + F_r(z)$ is independent of $X_*^{(r-1)}$, the expression above can be reduced to

$$\begin{aligned}
P\Big(F_r\big(Y_j^{(r)}\big)(1 - F_r(z)) + F_r(z) \le F_r(t)\Big) &= P\left(F_r\big(Y_j^{(r)}\big) \le \frac{F_r(t) - F_r(z)}{1 - F_r(z)}\right) \\
&= P\left(Y_j^{(r)} \le F_r^{-1}\left(\frac{F_r(t) - F_r(z)}{1 - F_r(z)}\right)\right) \\
&= \frac{F_r(t) - F_r(z)}{1 - F_r(z)}
\end{aligned}$$

and we obtain the needed representation.

To sum up, the structure of sequential order statistics allows to take into account both the influence of a failure through the changes in distribution and the knowledge about previous failure time incorporated in conditional distributions.

Theorem 1.7. *If $F_1, \dots, F_n$ are continuous distribution functions with densities $f_1, \dots, f_n$ respectively, then the joint density of the first r sequential order statistics $X_*^{(1)}, \dots, X_*^{(r)}$ is given by*

$$\begin{aligned}
&f^{X_*^{(1)},\dots,X_*^{(r)}}(t_1, \dots, t_r) \\
&\qquad = \frac{n!}{(n-r)!}\prod_{i=1}^{r}\left(\frac{1 - F_i(t_i)}{1 - F_i(t_{i-1})}\right)^{n-i}\frac{f_i(t_i)}{1 - F_i(t_{i-1})} ,
\end{aligned}$$

where $r = 1, \ldots, n,\ t_0 = -\infty$ *and* $t_1, \ldots, t_r \in \mathbb{R}$, $t_0 < t_1 \leq \ldots \leq t_r$.

Remark 1.8.

(i) Construction of sequential order statistics ensures not only specific Markov properties of underlying lifetime distributions, but also a Markov property of sequential order statistics themselves. Indeed, as noted in Cramer & Kamps (2001) from Theorem 1.7 follows that sequential order statistics form a Markov chain with transition probabilities

$$P\big(X_*^{(r)} > t \,|\, X_*^{(r-1)} = s\big) = \left(\frac{1 - F_r(t)}{1 - F_r(s)}\right)^{n-r+1} , \tag{1.3}$$

where $r = 2, \ldots, n$ *and* $s < t$.

(ii) In Cramer & Kamps (2003) an alternative representation for sequential order statistics is given. It is built basing on the connection of random variables forming a Markov chain and uniform distributed random variables (see for example Pfeifer (1989)). In more detail, let $F_1, \ldots, F_n$ *be continuous distribution functions with*

$$F_1^{-1}(1) \leq \ldots \leq F_n^{-1}(1) ,$$

where F_i^{-1} *is a pseudo-inverse of* F_i, $i = 1, \ldots, n$. *Let* $V_1, \ldots, V_n$ *be independent random variables with* $V_r \sim Beta(n - r + 1, 1)$, $r = 1, \ldots n$. *Then sequential order statistics (based on* $F_1, \ldots, F_n$*) can be defined as*

$$\begin{aligned} X_*^{(r)} &= F_r^{-1}(X^{(r)}) \\ X^{(r)} &= 1 - V_r \overline{F}_r(X_*^{(r-1)}) , \end{aligned} \tag{1.4}$$

where $r = 1, \ldots, n, \quad X_*^{(0)} = -\infty$.

Example 1.9. *In the definition of sequential order statistics let*

$$F_r(t) = 1 - (1 - F(t))^{\gamma_r/(n-r+1)} , \tag{1.5}$$

where $r = 1, \ldots, n$, F *is a continuous distribution function and* $\gamma_1, \ldots, \gamma_n$ *are positive numbers.*

Then, sequential order statistics $X_*^{(1)}, \dots, X_*^{(n)}$ *based on* $F_1, \dots, F_n$ *of the form (1.5) correspond to a specific class of ordered random variables called generalized order statistics. Generalized order statistics represent a unified approach to a range of models for ordered data and allow to consider their properties in more global context, for deeper insights see Kamps (1995), Cramer & Kamps (2001).*

Applying Theorem 1.7 we can calculate the joint density function of generalized order statistics as

$$\begin{aligned} f^{X_*^{(1)},\dots,X_*^{(n)}}(t_1,\dots,t_n) &= n!\prod_{i=1}^{n}\frac{\gamma_i(1-F(t_i))^{\gamma_i-1}f(t_i)}{(n-i+1)(1-F(t_{i-1}))^{\gamma_i}} \\ &= \gamma_n(1-F(t_n))^{\gamma_n-1}f(t_n)\prod_{i=1}^{n-1}\gamma_i(1-F(t_i))^{\gamma_i-\gamma_{i+1}-1}f(t_i)\ , \end{aligned}$$

where f *is the density corresponding to* F*.*

Thus, the form (1.5) allows to reduce uncertainty in the model to the choice of parameters $\gamma_1, \dots, \gamma_n$ *and the distribution function* F*.*

Sequential order statistics reflect a wide variety of dependence relations that can be observed in systems with identical components. Additionally they possess a graceful density function, which encourages their application in reliability theory. Their construction scheme has formed a background for a more general model that is described in the next section.

1.4 Sequential order statistics based on exchangeable random variables

The structure of sequential order statistics is closely associated with conditional independence of lifetimes given the previous failure time. In many cases, however, this assumption is too strong and it would be desirable to have a similar model basing on a broader class of lifetime distributions. This issue is addressed by the construction introduced in Burkschat (2009), that assumes exchangeable distributions of lifetimes for the periods between consecutive failures.

Let us start the consideration of the new model with a simple example.

Example 1.10. *Consider a system of three identical components that start working simultaneously. In the beginning of their run let the lifetimes of components be modeled by random variables*

$$Y_1^{(1)} = \frac{Z_1}{\Theta_1}, \quad Y_2^{(1)} = \frac{Z_2}{\Theta_1}, \quad Y_3^{(1)} = \frac{Z_3}{\Theta_1}\ ,$$

where iid random variables Z_1, Z_2, Z_3 represent lifetimes of components in perfect, separated environments, and random quantity Θ_1 describes the load that the system goes through, which is independent of Z_1, Z_2, Z_3.

Assume that the first failure appears at a time point $t_1 \geq 0$ and changes the distributions so that the lifetimes are next represented by

$$Y_1^{(2)} = \frac{Z_1}{\Theta_2}, \quad Y_2^{(2)} = \frac{Z_2}{\Theta_2}, \quad Y_3^{(2)} = \frac{Z_3}{\Theta_2},$$

where Θ_2 is independent of Z_1, Z_2, Z_3.

By analogy, after the second failure occurs at a time point $t_2 \geq t_1$, let the lifetimes be distributed as

$$Y_1^{(3)} = \frac{Z_1}{\Theta_3}, \quad Y_2^{(3)} = \frac{Z_2}{\Theta_3}, \quad Y_3^{(3)} = \frac{Z_3}{\Theta_3},$$

where Θ_3 is independent of Z_1, Z_2, Z_3.

Thus, we have modeled the lifetimes of both surviving and failed components $Y_1^{(i)}, Y_2^{(i)}, Y_3^{(i)}$ for the periods between $(i-1)$-th and i-th failure, $i = 1, 2, 3$. The effects induced by each failure are reflected by changes in the distribution of the environmental variable Θ_i.

Next, by analogy to the triangular scheme, for $i = 1, 2, 3$ we construct order statistics $Y_{1:3}^{(i)}, Y_{2:3}^{(i)}, Y_{3:3}^{(i)}$ from the exchangeable random variables representing lifetimes on each level. Finally, to connect the failure times between the levels, for $0 \leq t_1 \leq t_2 \leq t$ we define conditional probabilities

$$\begin{aligned}
&P\big(X_*^{(1)} \leq t\big) = P\big(Y_{1:3}^{(1)} \leq t\big) \\
&P\big(X_*^{(2)} \leq t \,|\, X_*^{(1)} = t_1\big) = P\big(Y_{2:3}^{(2)} \leq t \,|\, Y_{1:3}^{(2)} = t_1\big) \\
&P\big(X_*^{(3)} \leq t \,|\, X_*^{(2)} = t_2, X_*^{(1)} = t_1\big) = P\big(Y_{3:3}^{(3)} \leq t \,|\, Y_{2:3}^{(3)} = t_2, Y_{1:3}^{(3)} = t_1\big),
\end{aligned}$$

where $X_*^{(1)}, X_*^{(2)}, X_*^{(3)}$ denote sequential failure times in the system. Such random variables $X_*^{(1)}, X_*^{(2)}, X_*^{(3)}$ are called sequential order statistics based on exchangeable random variables $Y_1^{(i)}, Y_2^{(i)}, Y_3^{(i)}$, $i = 1, 2, 3$.

In general, according to Burkschat (2009) sequential order statistics based on exchangeable random variables are defined as follows.

Def 1.11. *Let $\big(Y_1^{(r)}, \ldots, Y_n^{(r)}\big)$, $r = 1, \ldots, n$, $n \in \mathbb{N}$ be random vectors with values in $\mathbb{R}^n$ that satisfy the following conditions:*

(i) *Random vectors* $\big(Y_1^{(r)}, \ldots, Y_n^{(r)}\big)$ *have the same support.*

(ii) *For* $r = 1, \ldots, n$ *random variables* $Y_1^{(r)}, \ldots, Y_n^{(r)}$ *fulfill*

$$P\big(Y_i^{(r)} = Y_j^{(r)}\big) = 0 \ ,$$

where $i, j = 1, \ldots, n, \ i \neq j$.

(iii) *Random variables* $Y_1^{(r)}, \ldots, Y_n^{(r)}$ *are exchangeable for* $r = 1, \ldots, n$.

Then, for $r = 1, \ldots, n$ *let* $\big(Y_{1:n}^{(r)}, \ldots, Y_{n:n}^{(r)}\big)$ *be the random vectors of the corresponding order statistics. Then random variables* $X_*^{(1)}, \ldots, X_*^{(n)}$ *are called sequential order statistics based on exchangeable components if their distribution satisfies*

$$P\big(X_*^{(1)} \leq t_1\big) = P\big(Y_{1:n}^{(1)} \leq t_1\big), \quad t_1 \in \mathbb{R},$$

and

$$\begin{aligned} &P\big(X_*^{(r+1)} \leq t_{r+1} \,|\, X_*^{(r)} = t_r, \ldots, X_*^{(1)} = t_1\big) \\ &\quad = P\big(Y_{r+1:n}^{(r+1)} \leq t_{r+1} \,|\, Y_{r:n}^{(r+1)} = t_r, \ldots, Y_{1:n}^{(r+1)} = t_1\big), \quad t_{r+1} \in \mathbb{R} \ , \end{aligned} \tag{1.6}$$

for $P^{X_*^{(r)}, \ldots, X_*^{(1)}}$*-almost all* $(t_r, \ldots, t_1) \in \mathbb{R}^r$, *for every* $r = 1, \ldots, n-1$.

Remark 1.12.

(i) *For the sake of simplicity we will refer to the system state between the* $(r-1)$*-th and* r*-th failure as the "*r*-th level". For example, the system before the first failure is on the first level, between the first and the second failure – on the second level and so on. After the* n*-th failure we do not assign any level to the system. On the* r*-th level the lifetimes of the components are described by random variables* $Y_1^{(r)}, \ldots, Y_n^{(r)}$, $(r-1)$ *of the corresponding components have already failed at that time,* $r = 1, \ldots, n$.

(ii) *For* $r = 1, \ldots, n, \ 0 \leq t_1 \leq \ldots \leq t_r$ *Definition 1.11 ensures*

$$f^{X_*^{(r)} \,|\, X_*^{(r-1)}, \ldots, X_*^{(1)}}(t_r | t_{r-1}, \ldots, t_1) = f^{Y_{r:n}^{(r)} \,|\, Y_{r-1:n}^{(r)}, \ldots, Y_{1:n}^{(r)}}(t_r | t_{r-1}, \ldots, t_1) \ . \tag{1.7}$$

The last conditional density can be calculated as

$$
\begin{aligned}
&f^{Y^{(r)}_{r:n}\,|\,Y^{(r)}_{r-1:n},\ldots,Y^{(r)}_{1:n}}(t_r|t_{r-1},\ldots,t_1)\\
&\quad= \frac{f^{Y^{(r)}_{1:n},\ldots,Y^{(r)}_{r-1:n},Y^{(r)}_{r:n}}(t_1,\ldots,t_{r-1},t_r)}{f^{Y^{(r)}_{1:n},\ldots,Y^{(r)}_{r-1:n}}(t_1,\ldots,t_{r-1})}\\
&\quad= \frac{\int_{t_r}^{\infty}\int_{y_{r+1}}^{\infty}\cdots\int_{y_{n-1}}^{\infty} f_r(t_1,\ldots,t_r,y_{r+1},y_{r+2},\ldots,y_n)dy_n\cdots dy_{r+2}dy_{r+1}}{\int_{t_{r-1}}^{\infty}\int_{y_r}^{\infty}\cdots\int_{y_{n-1}}^{\infty} f_r(t_1,\ldots,t_{r-1},y_r,y_{r+1},\ldots,y_n)dy_n\cdots dy_{r+1}dy_r}\ ,
\end{aligned}
$$

where $f_r(t_1,\ldots,t_n)$ *is a shorter notation for the density* $f^{Y_1^{(r)},\ldots,Y_n^{(r)}}(t_1,\ldots,t_n)$.

The following lemma derives an alternative representation for the integrals in (1.7) .

Lemma 1.13. *For an integrable and symmetric function* $f(t_1,\ldots,t_n)$ *holds*

$$
\begin{aligned}
&\int_t^{\infty}\int_{y_j}^{\infty}\cdots\int_{y_{n-1}}^{\infty} f(t_1,\ldots,t_{j-1},y_j,\ldots,y_n)dy_n\ldots dy_{j+1}dy_j\\
&\qquad= \frac{1}{(n-j+1)!}\int_t^{\infty}\int_t^{\infty}\cdots\int_t^{\infty} f(t_1,\ldots,t_{j-1},y_j,\ldots,y_n)dy_n\ldots dy_{j+1}dy_j\ .
\end{aligned}
$$

The Proof of Lemma 1.13 is primarily technical, for the sake of completeness it can be found in Appendix.

Applying Lemma 1.13 to the integrals in (1.7) we obtain

$$
\begin{aligned}
&f^{X_*^{(r)}\,|\,X_*^{(r-1)},\ldots,X_*^{(1)}}(t_r|t_{r-1}\ldots,t_1)\\
&\quad= (n-r+1)\frac{\int_{t_r}^{\infty}\int_{t_r}^{\infty}\cdots\int_{t_r}^{\infty} f_r(t_1,\ldots,t_r,y_{r+1},y_{r+2},\ldots,y_n)dy_n\cdots dy_{r+2}dy_{r+1}}{\int_{t_{r-1}}^{\infty}\int_{t_{r-1}}^{\infty}\cdots\int_{t_{r-1}}^{\infty} f_r(t_1,\ldots,t_{r-1},y_r,y_{r+1},\ldots,y_n)dy_n\cdots dy_{r+1}dy_r}\ .
\end{aligned}
\tag{1.8}
$$

The following theorem yields a representation for the joint density of sequential order statistics basing on the densities of underlying distributions.

Theorem 1.14. *For* $r=1,\ldots,n$ *let* $\left(Y_1^{(r)},\ldots,Y_n^{(r)}\right)$ *possess a joint density function* f_r *with respect to the* n*-dimensional Lebesgue measure. Then the density of sequential order* *statistics is given by*

$$
\begin{aligned}
& f^{X_*^{(1)},\dots,X_*^{(n)}}(t_1,\dots,t_n) \\
&= n!\frac{f_n(t_1,\dots,t_n)}{\int_{t_{n-1}}^{\infty} f_n(t_1,\dots,t_{n-1},y_n)dy_n} \\
&\quad\times \prod_{i=2}^{n-1}\frac{\int_{t_i}^{\infty}\int_{y_{i+1}}^{\infty}\cdots\int_{y_{n-1}}^{\infty} f_i(t_1,\dots,t_i,y_{i+1},\dots,y_n)dy_n dy_{n-1}\cdots dy_{i+1}}{\int_{t_{i-1}}^{\infty}\int_{y_i}^{\infty}\cdots\int_{y_{n-1}}^{\infty} f_i(t_1,\dots,t_{i-1},y_i,\dots,y_n)dy_n dy_{n-1}\cdots dy_i} \\
&\quad\times \int_{t_1}^{\infty}\int_{y_2}^{\infty}\cdots\int_{y_{n-1}}^{\infty} f_1(t_1,y_2,\dots,y_n)dy_n\cdots dy_2\ ,
\end{aligned}
\tag{1.9}
$$

for $t_1 \le \cdots \le t_n$.

Proof. The proof is deduced from the fact that

$$
f^{X_*^{(1)},\dots,X_*^{(n)}}(t_1,\dots,t_n) = f^{X_*^{(1)}}(t_1)\prod_{i=2}^{n} f^{X_*^{(i)}\,|\,X_*^{(i-1)},\dots,X_*^{(1)}}(t_i|t_{i-1},\dots,t_1)
$$

by applying (1.7). For a detailed reasoning see Burkschat (2009). □

Example 1.15. *Consider the random variables from Example 1.10. Let* Z_1, Z_2, Z_3 *be independent, exponentially distributed with mean 1, and environmental variables* Θ_i *be gamma distributed with parameters* $\alpha_i, \lambda_i,\ i = 1,2,3$. *Then for sequential order statistics* $X_*^{(1)}$, $X_*^{(2)}, X_*^{(3)}$ *it follows by (1.8) that*

$$
\begin{aligned}
f^{X_*^{(1)}}(t_1) &= f^{Y_{1:3}^{(1)}}(t_1) \\
&= \frac{3\,\alpha_1\,\lambda_1^{\alpha_1}}{(3t_1+\lambda_1)^{\alpha_1+1}} \\
f^{X_*^{(2)}\,|\,X_*^{(1)}}(t_2|t_1) &= f^{Y_{2:3}^{(2)}\,|\,Y_{1:3}^{(2)}}(t_2|t_1) \\
&= \frac{2\,(\alpha_2+1)\,(3t_1+\lambda_2)^{\alpha_2+1}}{(t_1+2t_2+\lambda_2)^{\alpha_2+2}} \\
f^{X_*^{(3)}\,|\,X_*^{(2)},X_*^{(1)}}(t_3|t_2,t_1) &= f^{Y_{3:3}^{(3)}\,|\,Y_{2:3}^{(3)},Y_{1:3}^{(3)}}(t_3|t_2,t_1) \\
&= \frac{(\alpha_3+2)(t_1+2t_2+\lambda_3)^{\alpha_3+2}}{(t_1+t_2+t_3+\lambda_3)^{\alpha_3+3}}\ ,
\end{aligned}
$$

where $0 \leq t_1 \leq t_2 \leq t_3$. *Theorem 1.14 yields*

$$
\begin{aligned}
f^{X_*^{(1)},X_*^{(2)},X_*^{(3)}}&(t_1,t_2,t_3) \\
&= f^{X_*^{(3)}\,|\,X_*^{(2)},X_*^{(1)}}(t_3|t_2,t_1) f^{X_*^{(2)}\,|\,X_*^{(1)}}(t_2|t_1) f^{X_*^{(1)}}(t_1) \\
&= 3!\,\lambda_1^{\alpha_1}\,\alpha_1\,(\alpha_2+1)\,(\alpha_3+2) \\
&\quad\times \frac{(t_1+2t_2+\lambda_3)^{\alpha_3+2}(3t_1+\lambda_2)^{\alpha_2+1}}{(t_1+t_2+t_3+\lambda_3)^{\alpha_3+3}(t_1+2t_2+\lambda_2)^{\alpha_2+2}(3t_1+\lambda_1)^{\alpha_1+1}},
\end{aligned}
\tag{1.10}
$$

where $0 \leq t_1 \leq t_2 \leq t_3$. *For the computational details see Appendix.*

Note that the density $f^{X_*^{(1)},X_*^{(2)},X_*^{(3)}}$ *contains parameters* λ_i *and* α_i *from all the model levels, which indicates that* $f^{X_*^{(1)},X_*^{(2)},X_*^{(3)}}$ *inherits some properties from all the vectors* $\big(Y_1^{(i)}, Y_2^{(i)}, Y_3^{(i)}\big)$, $i = 1,2,3$.

According to Burkschat (2009) there exist the following connections between sequential order statistics based on exchangeable random variables and the models for ordered data discussed earlier in this chapter.

Remark 1.16.

(i) Assigning the same joint distributions to the lifetimes $\big(Y_1^{(r)},\dots,Y_n^{(r)}\big)$ *on each level* r, $r = 1,\dots,n$ *we obtain usual order statistics based on exchangeable random variables described in Section 1.2.*

(ii) The connection to sequential order statistics based on conditionally iid random variables appears in the following context:

For $r = 1,\dots,n$ *assume the vectors* $\big(Y_1^{(r)},\dots,Y_n^{(r)}\big)$ *consist of iid components with continuous cumulative distribution functions* F_r. *Note that* $Y_1^{(r)},\dots,Y_n^{(r)}$ *can be seen as exchangeable random variables. Let* $Y_{1:n}^{(r)},\dots,Y_{n:n}^{(r)}$ *represent the order statistics based on* $\big(Y_1^{(r)},\dots,Y_n^{(r)}\big)$. *Then, according to Definition 1.11 for sequential order statistics* $X_*^{(1)},\dots,X_*^{(n)}$ *based on* $\big(Y_1^{(r)},\dots,Y_n^{(r)}\big)$, $r = 1,\dots,n$ *holds*

$$P\big(X_*^{(1)} \leq t_1\big) = P\big(Y_{1:n}^{(1)} \leq t_1\big) = 1 - (1 - F_1(t_1))^n$$

and by Lemma 1.3

$$P\big(X_*^{(r+1)} \leq t_{r+1}\,|\,X_*^{(r)} = t_r,\dots,X_*^{(1)} = t_1\big) = P\big(Y_{r+1:n}^{(r+1)} \leq t_{r+1}\,|\,Y_{r:n}^{(r+1)} = t_r\big)$$

$$= 1 - \left(\frac{1 - F_{r+1}(t_{r+1})}{1 - F_{r+1}(t_r)}\right)^{n-r},$$

where $0 \leq t_1 \leq \ldots \leq t_n$ *and* $r = 1, \ldots, n-1$.

Moreover, in this setup $X_*^{(1)}, \ldots, X_*^{(n)}$ *possess the Markov property and their transition probabilities coincide with those of sequential order statistics shown in (1.3) . Note that, in general, sequential order statistics based on exchangeable random variables are not Markov, if the underlying distributions of order statistics* $Y_{1:n}^{(r)}, \ldots, Y_{n:n}^{(r)}$ *are not Markov.*

(iii) *In contrast to the other models, sequential order statistics based on exchangeable random variables possess the most complicated form of the density function. For the future investigation of dependence properties it would be desirable to describe sequential order statistics in a simpler way. Therefore, expressions similar to (1.4) are of particular interest. However, the construction of (1.4) requires the Markov property. Therefore a similar representation does not exist for sequential order statistics based on exchangeable random variables in general.*

(iv) *In some* k*-out-of-*n *systems the lifetimes of components are not exchangeable. For instance, consider a system of water pumps with different working capacities, it corresponds to dependent and non-exchangeable lifetimes. An extension of Definition 1.11 for this case can be found in Burkschat (2009).*

Consider a system where lifetimes of components are conditionally independent given some parameter Θ. For instance, a system from Example 1.15 is of this type. As mentioned in Spizzichino (2001), conditional independence summarizes the assumption that there is no "physical interaction" among components and, at the same time, their behavior is influenced by common factors specified by Θ. Distributions of this type are called "mixture models", they will be considered in detail in Chapter 2. The joint density function of sequential order statistics based mixture distributions is shown below.

Theorem 1.17. *Let* $\Theta_1, \ldots, \Theta_n$ *be random variables with distributions* $G_1, \ldots, G_n$, *respectively. For* $r = 1, \ldots, n$ *let* $Y_1^{(r)}, \ldots, Y_n^{(r)}$ *have a joint density function*

$$f_r(t_1, \ldots, t_n) = \int \prod_{i=1}^{n} f_r(y_i|\theta) dG_r(\theta),$$

where $f_r(\cdot|\theta)$ *denotes a density with respect to the Lebesque measure for every* θ *in the* *support of Θ_r. Then the joint density function of sequential order statistics is given by*

$$
\begin{aligned}
&f^{X_*^{(1)},\dots,X_*^{(n)}}(t_1,\dots,t_n)\\
&\quad = n! \int f_1(t_1|\theta)(\overline{F_1}(t_1|\theta))^{n-1} dG_1(\theta)\\
&\qquad \times \prod_{r=1}^{n-1} \frac{\int (\prod_{i=1}^{r+1} f_{r+1}(t_i|\theta))(\overline{F_{r+1}}(t_{r+1}|\theta))^{n-r-1} dG_{r+1}(\theta)}{\int (\prod_{i=1}^{r} f_{r+1}(t_i|\theta))(\overline{F_{r+1}}(t_r|\theta))^{n-r} dG_{r+1}(\theta)}\ ,
\end{aligned}
$$

where $0 \le t_1 \le \cdots \le t_n$ and $F_r(\cdot|\theta)$ denotes the distribution function corresponding to $f_r(\cdot|\theta),\ r = 1,\dots,n$.

Remark 1.18. *Theorem 1.17 leads to the observation that even consideration the lifetime distributions of a simple form does not bring noticeable simplifications to the joint density function of sequential order statistics. The influence of the level distributions on the representation of sequential order statistics density will form the core of Chapter 3.*

In summary, sequential order statistics based on exchangeable random variables reflect a wider range of dependencies between components in a system than other models considered in this thesis. Their properties are influenced by the random variables from all the levels in the model. In order to investigate this connection, in the next section the focus will be placed on special types of distributions for the lifetimes on the levels.

2 Multivariate survival distributions

In the following we will look at several types of lifetime distributions that will be considered throughout this thesis. The distributions are grouped basing on the forms of corresponding conditional hazard rates.

For the sake of completeness we will start with a short introduction to conditional hazard rates and their properties.

2.1 Conditional hazard rates

Conditional hazard rates generalize the concept of usual univariate hazard rates. According to Shaked & Shanthikumar (2007) the latter are defined as follows.

Def 2.1. *Consider a nonnegative random variable Y with an absolutely continuous distribution function F and a density f. Then the hazard rate of Y at a time point $t \geq 0$ is defined as*

$$r(t) = \lim_{\delta \to 0+} \frac{P(t < Y \leq t + \delta \,|\, Y > t)}{\delta} = \frac{f(t)}{\overline{F}(t)} , \tag{2.1}$$

where $\overline{F}(t) = 1 - F(t)$.

Remark 2.2.

- *According to Shaked & Shanthikumar (2007) a hazard rate allows the representation*

$$r(t) = \frac{d}{dt}\big(-\ln \overline{F}(t)\big) .$$

- *From the survival theory point of view a hazard rate can be interpreted as the intensity of failure of a device, with a random lifetime X, at a time point t.*

As mentioned in the previous chapter a breakdown in the system likely affects surviving components and their proneness to failure. For instance, an engine malfunction in a multi-engine system increases the load on the remaining engines and the intensity of their failures. Conditional hazard rates generalize the univariate hazard rates in order to take into account the influence of all preceding incidents.

To introduce the notion of conditional hazard rates let us consider a technical system of n components with not necessary exchangeable random lifetimes $Y_1, \ldots,$ Y_n . We are interested in the failure intensity of the component with the number

k_l and the lifetime Y_{k_l} at a time point $t \geq 0$. A history of failures preceding time point t, ordered by their occurrence, can be described as

$$\mathfrak{h}_t = \{Y_{i_1} = t_1, \ldots, Y_{i_j} = t_j, Y_{k_1} > t, \ldots, Y_{k_{n-j}} > t\} \ , \tag{2.2}$$

where $0 \leq j \leq n, \ 0 < t_1 \leq \cdots \leq t_j \leq t$, and

$$\begin{aligned} I &= \{i_1, \ldots, i_j\} \subset \{1, 2, \ldots, n\} \\ K &= \{k_1, \ldots, k_{n-j}\} = \{1, 2, \ldots, n\} \setminus I \ . \end{aligned} \tag{2.3}$$

In other words, the set I represents components that have failed before the time point t, where $t_1, \ldots, t_j$ are their ordered failure times. $Y_k, k \in K$, on the contrary, are components surviving the time t.

Then, according to Shaked & Shanthikumar (2007), the conditional hazard rate can be defined as follows (see also Spizzichino (2001)).

Def 2.3. *Consider a vector $Y = (Y_1, \ldots, Y_n)$ of absolutely continuous random lifetimes. Given the history of failures $\mathfrak{h}_t$ from (2.2) with I, K defined by (2.3) the conditional hazard rate of the surviving component $Y_{k_l}, k_l \in K$ is defined as*

$$\begin{aligned} &\lambda_{k_l|I}(t|t_1, \ldots, t_j) = \\ &\quad \lim_{\delta \to 0+} \frac{1}{\delta} P(Y_{k_l} \leq t + \delta \,|\, Y_{i_1} = t_1, \ldots, Y_{i_j} = t_j, Y_{k_1} > t, \ldots, Y_{k_{n-j}} > t) \ . \end{aligned} \tag{2.4}$$

For $j = 0$ the conditional hazard rate of the first failure in the system is defined as

$$\lambda_{k_l|\emptyset}(t) = \lim_{\delta \to 0+} \frac{1}{\delta} P(Y_{k_l} \leq t + \delta \,|\, Y_1 > t, \ldots, Y_n > t) \ .$$

Remark 2.4.

(i) *To be able to calculate the limit in (2.4) we can represent the conditional hazard rate* *as*

$$\lambda_{k_l|I}(t|t_1,\ldots,t_j) = \lim_{\delta\to 0+} \frac{1}{\delta} \frac{P(Y_{k_l} \le t+\delta, Y_{k_1} > t, \ldots, Y_{k_{n-j}} > t \,|\, Y_{i_1} = t_1, \ldots, Y_{i_j} = t_j)}{P(Y_{k_1} > t, \ldots, Y_{k_{n-j}} > t \,|\, Y_{i_1} = t_1, \ldots, Y_{i_j} = t_j)}\,, \tag{2.5}$$

where we define the conditional probabilities with the help of the following equality

$$P(Y_{k_1} \in A_{k_1}, \ldots, Y_{k_{n-j}} \in A_{k_{n-j}} \,|\, Y_{i_1} = t_1, \ldots, Y_{i_j} = t_j) = \int_{A_{k_{n-j}}} \cdots \int_{A_{k_1}} f^{Y_{k_1},\ldots,Y_{k_{n-j}} \,|\, Y_{i_1},\ldots,Y_{i_j}}(t_{k_1}, \ldots, t_{k_{n-j}}|t_1,\ldots,t_j)dt_{k_1}\ldots dt_{k_{n-j}}$$

for $A_{k_i} \subset \mathbb{R}_+$, $i = 1, \ldots, n-j$.

The existence of the limits in (2.4) and (2.5) will be addressed in Lemma 2.5 below.

(ii) *Consider the representation (2.5) for exchangeable random variables* $Y_1, \ldots, Y_n$. *It can be shown with the help of Lemma I from Szekli (1995) together with Lemma 1.13 that the limit in (2.5) can be calculated as follows*

$$\begin{aligned}\lambda_{k_l|I}(t \,|\, t_1, \ldots, t_j) &= \lim_{\delta\to 0+} \frac{1}{\delta(n-j)} \frac{P(t < Y_{j+1:n} \le t+\delta \,|\, Y_{1:n} = t_1, \ldots, Y_{j:n} = t_j)}{P(t < Y_{j+1:n} \,|\, Y_{1:n} = t_1, \ldots, Y_{j:n} = t_j)} \\ &= \lim_{\delta\to 0+} \frac{1}{\delta(n-j)} P(Y_{j+1:n} \le t+\delta \,|\, Y_{1:n} = t_1, \ldots, Y_{j:n} = t_j, Y_{j+1:n} > t)\,.\end{aligned} \tag{2.6}$$

Thus, for exchangeable lifetimes conditional hazard rates depend on the number of failures and the set of failure times but not on the particular choice of components. Therefore, from now on a simplified notation will be used for conditional hazard rates based on exchangeable random variables, namely

$$\lambda(t|t_1,\ldots,t_j) = \lambda_{k_l|I}(t|t_1,\ldots,t_j)$$

and

$$\lambda(t) = \lambda_{k_l|\emptyset}(t) \; .$$

(iii) *Consider random variables* $Z_1, \ldots, Z_n$ *such that*

$$Z_1 < \ldots < Z_n \;\; a.s. \tag{2.7}$$

Since the random variables are ordered the identities of failed components are defined by $(i_1, \ldots, i_j) = (1, \ldots, j)$, *where* $i_1, \ldots, i_j$ *are the indexes defined according to (2.2).*

It follows from Lemma I in Szekli (1995) that for $l \neq m$, $l, m \in \{j+1, \ldots, n\}$ *holds*

$$\lim_{\delta \to 0+} \frac{1}{\delta} P(t < Z_l \leq t + \delta, t < Z_m \leq t + \delta, Z_{j+1} > t \,|\, Z_1 = t_1, \ldots, Z_j = t_j) = 0 \; .$$

Therefore, we can summarize the behavior of conditional hazard rates by

$$\lambda_{l,I}(t|t_{i_1}, \ldots, t_{i_k}) \begin{cases} \geq 0 & \text{if } k = l-1 \text{ and } \{i_1, \ldots, i_k\} = \{1, \ldots, l-1\}, \\ = 0 & \text{otherwise.} \end{cases} \tag{2.8}$$

In other words, for ordered random variables a non-trivial case represents conditional hazard rates of the form

$$\lambda_{l,I}(t|t_1, \ldots, t_{l-1}) = \lim_{\delta \to 0+} \frac{1}{\delta} P(Z_l \leq t + \delta \,|\, Z_1 = t_1, \ldots, Z_{l-1} = t_{l-1}, Z_l > t) \, ,$$

where $I = \{1, \ldots, l-1\}$, $l = 1, \ldots, n$.

Note that by construction order statistics fulfill (2.7). Therefore, to emphasize (2.8), we will use the notation

$$\lambda(t|t_1, \ldots, t_{l-1}) = \lim_{\delta \to 0+} \frac{1}{\delta} P(Z_l \leq t + \delta \,|\, Z_1 = t_1, \ldots, Z_{l-1} = t_{l-1}, Z_l > t) \; ,$$

$$\lambda_*(t|t_1,\ldots,t_{l-1}) = \lim_{\delta\to 0+} \frac{1}{\delta} P(Y_{l:n} \le t+\delta \,|\, Y_{1:n}=t_1,\ldots,Y_{l-1:n}=t_{l-1}, Y_{l:n}>t) \ ,$$
$$\lambda_{(j)}(t|t_1,\ldots,t_{l-1}) = \lim_{\delta\to 0+} \frac{1}{\delta} P(Y^{(j)}_{l:n} \le t+\delta \,|\, Y^{(j)}_{1:n}=t_1,\ldots,Y^{(j)}_{l-1:n}=t_{l-1}, Y^{(j)}_{l:n}>t)$$

and

$$\lambda_{(*,l)}(t|t_1,\ldots,t_{l-1}) = \lim_{\delta\to 0+} \frac{1}{\delta} P(X^{(l)}_* \le t+\delta \,|\, X^{(1)}_*=t_1,\ldots,X^{(l-1)}_*=t_{l-1}, X^{(l)}_*>t) \ ,$$

where $j,l=2,\ldots,n$. The empty failure history can be considered by analogy.

The following lemma states the existence of the limits in (2.4) and provides expressions for conditional hazard rates based on exchangeable random variables. For the proof we refer to Szekli (1995) .

Lemma 2.5. *Let $Y=(Y_1,\ldots,Y_n)$ be a vector of exchangeable lifetimes with a joint density function $f(t_1,\ldots,t_n)$. For $1\le j\le n-1$, $0<t_1\le\cdots\le t_j<t$, the limit in (2.4) exists for almost all $(t,t_1,\ldots,t_j)$ and is a measurable function of $(t,t_1,\ldots,t_j)$, with*

$$\lambda(t|t_1,\ldots,t_j) = \frac{\int_t^\infty\cdots\int_t^\infty f(t_1,\ldots,t_j,t,t_{j+2},\ldots,t_n)dt_{j+2}\ldots dt_n}{\int_t^\infty\cdots\int_t^\infty f(t_1,\ldots,t_j,t_{j+1},t_{j+2},\ldots,t_n)dt_{j+1}\ldots dt_n}$$

and

$$\lambda(t) = \frac{\int_t^\infty\cdots\int_t^\infty f(t,t_2,\ldots,t_n)dt_2\ldots dt_n}{\int_t^\infty\cdots\int_t^\infty f(t_1,t_2,\ldots,t_n)dt_1\ldots dt_n}$$

almost sure.

Taking (2.6) into account an alternative expression for a conditional hazard rate can be given.

Lemma 2.6. *For absolutely continuous exchangeable random variables $Y_1,\ldots,Y_n$ holds*

$$\lambda(t|t_1,\ldots,t_j) = -\frac{1}{(n-j)}\frac{\partial}{\partial t}\ln P(Y_{j+1:n}>t \,|\, Y_{1:n}=t_1,\ldots,Y_{j:n}=t_j) \tag{2.9}$$

for $1 \leq j \leq n-1,\ 0 < t_1 \leq \cdots \leq t_j < t$, *and*

$$\lambda(t) = -\frac{1}{n}\frac{\partial}{\partial t}\ln P(Y_{1:n} > t)\ ,$$

almost sure.

Proof. Denote

$$\begin{aligned} p(t|t_1,\ldots,t_j) &= P(Y_{j+1:n} > t|Y_{1:n} = t_1,\ldots,Y_{j:n} = t_j) \\ p(t) &= P(Y_{1:n} > t)\ . \end{aligned}$$

From (2.6) follows

$$\begin{aligned} &\lambda(t|t_1,\ldots,t_j) \\ &\quad = -\frac{1}{(n-j)p(t|t_1,\ldots,t_j)}\lim_{\delta\to 0+}\frac{p(t+\delta|t_1,\ldots,t_j) - p(t|t_1,\ldots,t_j)}{\delta} \\ &\quad = -\frac{1}{(n-j)}\frac{\frac{\partial}{\partial t}p(t|t_1,\ldots,t_j)}{p(t|t_1,\ldots,t_j)} = -\frac{1}{(n-j)}\frac{\partial}{\partial t}\ln p(t|t_1,\ldots,t_j)\ , \end{aligned}$$

for almost all $0 \leq t_1 \leq \ldots \leq t_j$. By analogy we can state

$$\begin{aligned} \lambda(t) &= \frac{1}{n}\lim_{\delta\to 0+}\frac{P(t < Y_{1:n} \leq t+\delta)}{\delta P(t < Y_{1:n})} \\ &= -\frac{1}{n}\frac{\frac{\partial}{\partial t}p(t)}{p(t)} = -\frac{1}{n}\frac{\partial}{\partial t}\ln p(t)\ . \end{aligned} \tag{2.10}$$

□

In the proof of the previous lemma replace $Y_{1:n},\ldots,Y_{n:n}$ by ordered random variables $Z_1,\ldots,Z_n$. Then by analogy to (2.9) we observe the following relation.

Lemma 2.7. *For absolutely continuous random variables* $Z_1,\ldots,Z_n$*, that satisfy (2.7), conditional hazard rates can be expressed as*

$$\lambda(t|t_1,\ldots,t_j) = -\frac{\partial}{\partial t}\ln P(Z_{j+1} > t\,|\,Z_1 = t_1,\ldots,Z_j = t_j) \tag{2.11}$$

for $1 \leq j \leq n-1,\ 0 < t_1 \leq \cdots \leq t_j < t$ *and*

$$\lambda(t) = -\frac{\partial}{\partial t} \ln P(Z_1 > t) \tag{2.12}$$

almost sure.

Comparing (2.9) with (2.11), the only significant difference represents the missing factor in front of the derivative. It arises from the failure history form for ordered random variables that we have discussed in Remark 2.4 (iii).

Lemma 2.7 allows to obtain a simpler representation for the conditional hazard rates of sequential order statistics based on exchangeable random variables.

Lemma 2.8. *Consider sequential order statistics* $X_*^{(1)}, \ldots, X_*^{(n)}$ *based on absolutely continuous exchangeable random lifetimes* $\big(Y_1^{(i)}, \ldots, Y_n^{(i)}\big)$, $i = 1, \ldots, n$. *Then the conditional hazard rates*

$$\lambda_{(*,j+1)}(t|t_1, \ldots, t_j) = \lim_{\delta \to 0+} \frac{1}{\delta} P\big(X_*^{(j+1)} \leq t + \delta \,|\, X_*^{(1)} = t_1, \ldots, X_*^{(j)} = t_j, X_*^{(j+1)} > t\big)$$

can be calculated as

$$\lambda_{(*,j+1)}(t|t_1, \ldots, t_j) = (n-j)\, \lambda_{(j+1)}(t|t_1, \ldots, t_j)\ , \tag{2.13}$$

where by analogy to (2.6) for $j = 1, \ldots, n-1$

$$\lambda_{(j+1)}(t|t_1, \ldots, t_j) = \lim_{\delta \to 0+} \frac{1}{\delta(n-j)} P\big(Y_{j+1:n}^{(j+1)} \leq t + \delta \,|\, Y_{1:n}^{(j+1)} = t_1, \ldots, Y_{j:n}^{(j+1)} = t_j, Y_{j+1:n}^{(j+1)} > t\big)\ .$$

Proof. Since $X_*^{(1)}, \ldots, X_*^{(n)}$ satisfy (2.7), applying Lemma 2.7 we obtain

$$\lambda_{(*,j+1)}(t|t_1, \ldots, t_j) = -\frac{\partial}{\partial t} \ln P(X_*^{(j+1)} \geq t \,|\, X_*^{(1)} = t_1, \ldots, X_*^{(j)} = t_j)\ .$$

Moreover, taking into account (1.6) we can state that

$$\lambda_{(*,j+1)}(t|t_1,\dots,t_j) = -\frac{\partial}{\partial t}\ln P(X^{(j+1)}_{j+1:n} \geq t \mid X^{(j+1)}_{1:n} = t_1,\dots,X^{(j+1)}_{j:n} = t_j)\ .$$

Finally, comparing the last expression with (2.9) leads to

$$\lambda_{(*,j+1)}(t|t_1,\dots,t_j) = (n-j)\,\lambda_{(j+1)}(t|t_1,\dots,t_j)\ ,$$

where $\lambda_{(j+1)}(t|t_1,\dots,t_j)$ corresponds to $\big(Y_1^{(j+1)},\dots,Y_n^{(j+1)}\big)$, $j = 1,\dots,n-1$. □

In the following we define a cumulative hazard of a component as shown in Shaked & Shanthikumar (2007).

Def 2.9. *Consider a system of exchangeable components with the history of failures as in (2.2) and the sets I, K defined by (2.3). The cumulative hazard function of the component $i \in K$ at a time point t is defined as*

$$\Psi_{i|I}(t|t_1,\dots,t_{j-1}) = \int_0^{t_1}\lambda_{i|\emptyset}(u)du + \sum_{k=2}^{j-1}\int_{t_{k-1}}^{t_k}\lambda_{i|i_1,\dots,i_{k-1}}(u|t_1,\dots,t_{k-1})du + \int_{t_{j-1}}^{t}\lambda_{i|I}(u|t_1,\dots,t_{j-1})du\ .$$

Remark 2.10. *Consider ordered random variables $Z_1,\dots,Z_n$ and their conditional hazard rates $\lambda_{i|I}(\cdot|\cdots)$. Taking into account (2.8) for a cumulative hazard of $Z_1,\dots,Z_n$ holds*

$$\Psi_{i|I}(t|t_1,\dots,t_{i-1}) = \begin{cases}\int_{t_{i-1}}^{t}\lambda_{i|I}(u|t_1,\dots,t_{i-1})du & \text{if } I = \{1,\dots,i-1\},\\ 0 & \text{otherwise.}\end{cases}$$

The following lemma states the connection between the joint density of exchangeable random variables and the corresponding conditional hazard rates. It represents a special case of Lemma J in Szekli (1995).

Lemma 2.11. *Let $Y_1,\dots,Y_n$ represent non-negative exchangeable random variables with the joint density function $f(t_1,\dots,t_n)$ and conditional hazard rates $\lambda(t|t_1,\dots,t_{i-1})$, $\lambda(t)$, $i = 2,\dots,n$. Let $Y_{1:n},\dots,Y_{n:n}$ be order statistics based on $Y_1,\dots,Y_n$, for $i = 2,\dots,n$ and almost all $0 \leq t_1 \leq \dots \leq t_n$ holds*

(i)

$$P(Y_{i:n} > t_i \,|\, Y_{i-1:n} = t_{i-1}, \dots, Y_{1:n} = t_1) \\ = \exp\left(-(n-i+1)\int_{t_{i-1}}^{t_i} \lambda(u|t_{i-1},\dots,t_1)du\right),$$

and

$$P(Y_{1:n} > t_1) = \exp\left(-n\int_0^{t_1} \lambda(u)du\right).$$

(ii)

$$f^{Y_{i:n}\,|\,Y_{i-1:n},\dots,Y_{1:n}}(t_i|t_{i-1},\dots,t_1) \\ = (n-i+1)\,\lambda(t_i|t_{i-1},\dots,t_1)\,\exp\left(-(n-i+1)\int_{t_{i-1}}^{t_i} \lambda(u|t_{i-1},\dots,t_1)du\right)$$

and

$$f^{Y_{1:n}}(t_1) = n\,\lambda(t_1)\,\exp\left(-n\int_0^{t_1} \lambda(u)du\right).$$

(iii) As a result,

$$f^{Y_{1:n},\dots,Y_{n:n}}(t_1,\dots,t_n) \\ = n!\,\lambda(t_1)\prod_{h=2}^{n}\lambda(t_h|t_{h-1},\dots,t_1)\,\exp\left(-n\int_0^{t_1}\lambda(u)du\right) \\ \times\exp\left(-\sum_{h=2}^{n}(n-h+1)\int_{t_{h-1}}^{t_h}\lambda(u|t_{h-1},\dots,t_1)du\right).$$

Proof. The first equality is deduced from (2.9) by integrating both parts for $t \in$

$[t_j, t_{j+1}],\ t_j, t_{j+1} \in \mathbb{R}, t_j < t_{j+1}$

$$\int_{t_j}^{t_{j+1}} \lambda(t|t_1,\dots,t_j)dt = -\frac{1}{(n-j)} \int_{t_j}^{t_{j+1}} \frac{\partial}{\partial t} \ln P(Y_{j+1:n} \geq t | Y_{1:n} = t_1, \dots, Y_{j:n} = t_j) dt \ .$$

Calculating the integral on the right we obtain

$$\int_{t_j}^{t_{j+1}} \lambda(t|t_1,\dots,t_j)dt = -\frac{1}{(n-j)} \Big(\ln P(Y_{j+1:n} \geq t_{j+1} \,|\, Y_{1:n} = t_1, \dots, Y_{j:n} = t_j) - \ln P(Y_{j+1:n} \geq t_j \,|\, Y_{1:n} = t_1, \dots, Y_{j:n} = t_j) \Big) \ .$$

Since $P(Y_{j+1:n} \geq t_j \,|\, Y_{1:n} = t_1, \dots, Y_{j:n} = t_j) = 1$, (i) follows.

Next let us look at the conditional density

$$\begin{aligned} f^{Y_{i:n} \,|\, Y_{i-1:n},\dots,Y_{1:n}}(t_i|t_{i-1},\dots,t_1) &= -\frac{\partial}{\partial t_i} P(Y_{i:n} > t_i \,|\, Y_{i-1:n} = t_{i-1}, \dots, Y_{1:n} = t_1) \\ &= -\frac{\partial}{\partial t_i} \exp\Big(-(n-i+1) \int_{t_{i-1}}^{t_i} \lambda(u|t_{i-1},\dots,t_1)du \Big) \end{aligned}$$

and by Leibniz integration rule holds

$$f^{Y_{i:n} \,|\, Y_{i-1:n},\dots,Y_{1:n}}(t_i|t_{i-1},\dots,t_1) = (n-i+1)\, \lambda(t_i|t_{i-1},\dots,t_1) \exp\Big(-(n-i+1) \int_{t_{i-1}}^{t_i} \lambda(u|t_{i-1},\dots,t_1)du \Big) \ .$$

For $i = 1$ the conditional hazard rate coincides with the usual univariate hazard rate of $Y_{1:n}$. Then from (2.10) for $t_1 \geq 0$ follows that

$$P(Y_{1:n} > t_1) = \exp\Big(-n \int_0^{t_1} \lambda(u)du\Big)$$
$$f^{Y_{1:n}}(t_1) = n\,\lambda(t_1)\, \exp\Big(-n \int_0^{t_1} \lambda(u)du\Big)$$

and (ii) is confirmed.

Finally, the expression (iii) is obtained by substituting (i) and (ii) in the joint density representation (1.7). □

Remark 2.12. *Equality (iii) in Lemma 2.11 leads to another representation of the joint density $f^{Y_1,\ldots,Y_n}(t_1,\ldots,t_n)$ in terms of conditional hazard rates.*

Let π be a permutation of $t_1,\ldots,t_n$ such that

$$(t_{(1)},\ldots,t_{(n)}) = \pi(t_1,\ldots,t_n)\ ,$$

where $t_{(1)} \leq \ldots \leq t_{(n)}$.

Since for exchangeable random variables

$$f^{Y_{1:n},\ldots,Y_{n:n}}(t_{(1)},\ldots,t_{(n)}) = n!\, f^{Y_1,\ldots,Y_n}(t_1,\ldots,t_n)\ ,$$

we obtain

$$\begin{aligned} &f^{Y_1,\ldots,Y_n}(t_1,\ldots,t_n) \\ &\quad= \lambda(t_{(1)}) \prod_{h=2}^{n} \lambda(t_{(h)}|t_{(1)},\ldots,t_{(h-1)})\, \exp\Big(-n \int_0^{t_{(1)}} \lambda(u)du\Big) \\ &\qquad\times \exp\Big(-\sum_{h=2}^{n} [n-(h-1)] \int_{t_{(h-1)}}^{t_{(h)}} \lambda(u|t_{(1)},\ldots,t_{(h-1)})du\Big) \end{aligned} \tag{2.14}$$

for $t_1,\ldots,t_n \in \mathbb{R}$.

Lemma 2.13. *For the density of sequential order statistics $X_*^{(1)},\ldots,X_*^{(n)}$ and $0 < t_1 \leq \ldots \leq t_n$ holds*

$$f^{X_*^{(1)},\ldots,X_*^{(n)}}(t_1,\ldots,t_n)$$

$$= n!\, \lambda_{(1)}(t_1) \prod_{h=2}^{n} \lambda_{(h)}(t_h | t_1, \ldots, t_{h-1}) \; \exp\left(-n \int_0^{t_1} \lambda_{(1)}(u) du \right)$$
$$\times \exp\left(-\sum_{h=2}^{n} (n-h+1) \int_{t_{h-1}}^{t_h} \lambda_{(h)}(u | t_1, \ldots, t_{h-1}) du \right) .$$

Proof. The result is obtained by substitution of the representation for $f^{Y_{i:n} \mid Y_{i-1,n}, \ldots, Y_{1:n}}$ from Lemma 2.11 (ii) in (1.7). □

2.2 Distributions with special forms of conditional hazard rates

In the following we are going to consider several types of multivariate distributions by distinguishing the number of failure times on which the conditional hazard rates depend.

2.2.1 Distributions with Markov order statistics

To begin with, let us look at a n-dimensional distribution with hazard rates of the form

$$\lambda(t | t_1, \ldots, t_{k-1}) = g(t)$$

for $k = 2, \ldots, n$. Such conditional hazard rates are independent of the number of previous failures and their failure times. The following lemma describes distributions with conditional hazard rates of this type.

Lemma 2.14. *Consider absolutely continuous exchangeable random variables $Y_1, \ldots, Y_n$ with marginal distribution functions F and corresponding densities f. Denote by*

$$r(t) = \frac{f(t)}{\overline{F}(t)}$$

the univariate hazard rate of Y_j, $j = 1, \ldots, n$. Then $Y_1, \ldots, Y_n$ are iid if and only if for $k = 2, \ldots, n$ holds

$$\lambda(t | t_1, \ldots, t_{k-1}) = r(t) \; . \tag{2.15}$$

Proof. Let $Y_1, \ldots, Y_n$ be iid, then for the history of failures as in (2.2) by Definition

2.3 we obtain

$$\begin{aligned}
&\lambda(t|t_1,\ldots,t_{k-1}) \\
&\quad = \lim_{\delta\to 0+} \frac{1}{\delta} P(Y_{k_l} \le t+\delta \,|\, Y_{i_1}=t_1,\ldots,Y_{i_j}=t_j, Y_{k_1}>t,\ldots,Y_{k_{n-j}}>t) \\
&\quad = \lim_{\delta\to 0+} \frac{1}{\delta} P(Y_{k_l} \le t+\delta \,|\, Y_{k_l}>t)\ .
\end{aligned}$$

Consequently, the conditional hazard rate is equal to the univariate hazard rate defined by (2.1).

It is left to prove that condition (2.15) implies iid. Taking into account (2.14) we can represent the joint density as

$$\begin{aligned}
&f^{Y_1,\ldots,Y_n}(t_1,\ldots,t_n) \\
&\quad = \prod_{i=1}^{n} r(t_{(i)}) \exp\Big(-n\int_0^{t_{(1)}} r(u)du\Big) \exp\Big(-\sum_{h=2}^{n}(n-h+1)\int_{t_{(h-1)}}^{t_{(h)}} r(u)du\Big) \\
&\quad = \prod_{i=1}^{n} r(t_{(i)}) \exp\Big[n\left(\ln\overline{F}(t_{(1)}) - \ln\overline{F}(0)\right)\Big] \\
&\qquad \times \exp\Big[\sum_{h=2}^{n}(n-h+1)\left(\ln\overline{F}(t_{(h)}) - \ln\overline{F}(t_{(h-1)})\right)\Big] \\
&\quad = \prod_{i=1}^{n} r(t_{(i)}) \exp\Big(\sum_{h=1}^{n}\ln\overline{F}(t_{(h)})\Big) = \prod_{i=1}^{n} f(t_{(i)}) = \prod_{i=1}^{n} f(t_i)\ .
\end{aligned}$$

It follows immediately that $Y_1,\ldots,Y_n$ are iid. □

For the next step consider distributions with conditional hazard rates depending on the age of the surviving components and the number of failures, but not on the failure times, i.e.

$$\lambda(t|t_1,\ldots,t_k) = g_k(t)\ , \tag{2.16}$$

where $k=1,\ldots,n-1$.

In the following several examples of distributions that possess such conditional hazard rates are presented.

Example 2.15.

(i) *According to Kotz et al. (2000) the density function of the symmetric Block and Basu multivariate exponential distribution is given by*

$$f(t_1, \dots, t_n) = \frac{1}{n!} \exp\Big(-\lambda \sum_{i=1}^{n} t_i - \gamma t_{(n)}\Big) \prod_{r=1}^{n} (r \cdot \lambda + \gamma) \tag{2.17}$$

for $t_{(n)} = \max\{t_1, \dots, t_n\}$, $t_1, \dots, t_n \in \mathbb{R}_+$, $0 < \lambda \in \mathbb{R}$ *and* $\gamma \in \mathbb{R}_+$.
In Lemma 2.5 let random lifetimes $Y_1, \dots, Y_n$ *follow the Block and Basu distribution. Then substituting (2.17) for* f *we obtain conditional hazard rates*

$$\begin{aligned}
&\lambda(t|t_1, \dots, t_h) \\
&\quad = \frac{1}{(n-h)} \\
&\qquad \times \frac{\exp\Big(-(n-h-1)\lambda t - \gamma t\Big) \exp\Big(-\lambda \sum_{i=1}^{h} t_i - \lambda t\Big)}{\exp\Big(-(n-h)\lambda t - \gamma t\Big) \exp\Big(-\lambda \sum_{i=1}^{h} t_i\Big)} \\
&\qquad \times \frac{\prod_{j=h+1}^{n} \Big(\lambda(n-j+1) + \gamma\Big)}{\prod_{j=h+2}^{n} \Big(\lambda(n-j+1) + \gamma\Big)} \\
&\quad = \frac{\lambda(n-h) + \gamma}{(n-h)} .
\end{aligned}$$

In this case the multivariate conditional hazard rate depends only on the number of failed components and stays constant between two failures.

(ii) *In Spizzichino (2001) Example 2.37, 3.30 the author considers a model by Ross, which can be described by the joint density*

$$f(t_1, \dots, t_n) = \frac{\theta^n}{n!} e^{-\theta t_{(n)}}$$

and conditional hazard rates

$$\lambda(t|t_1, \dots, t_k) = \frac{\theta}{n-k} , \tag{2.18}$$

where $\theta > 0$ is a given quantity and $t_i \geq 0, i = 1, \dots, n$.

From the form of conditional hazard rates we can conclude that for such systems the hazard of a failure increases with every failed component. This can be the case, for example, when failures cause damage to surviving components. Moreover, the conditional hazard rate in (2.18) does not depend on the current time t, i.e. components do not undergo any aging process. This is the case, for instance, when the duration of experiment is small and effects of aging are insignificant in comparison to the failure consequences.

Another group of distributions with conditional hazard rates of the form (2.16) can be found in Navarro & Burkschat (2011). It is of interest due to the close relation to sequential order statistics. More specifically, the distributions arise from the following lemma.

Lemma 2.16. *If $X_*^{(1)}, \dots, X_*^{(n)}$ are sequential order statistics based on $F_1, F_2, \dots, F_n$, then there exists an exchangeable random vector $X^* = (X_1^*, \dots, X_n^*)$ such that the vector of its usual order statistics is equal to the vector $\left(X_*^{(1)}, \dots, X_*^{(n)}\right)$.*

Thus, under consideration are exchangeable random variables such that their order statistics coincide in distribution with sequential order statistics. The joint density of $X_1^*, \dots, X_n^*$ can be obtained as follows.

Lemma 2.17. *If $X_*^{(1)}, \dots, X_*^{(n)}$ are sequential order statistics based on $F_1, F_2, \dots, F_n$, then the exchangeable random variables $X_1^*, \dots, X_n^*$, defined in Lemma 2.16, possess the joint density function*

$$f^{X_1^*,\dots,X_n^*}(t_1, \dots, t_n) = \prod_{i=1}^{n} \left(\frac{1 - F_i(t_{(i)})}{1 - F_i(t_{(i-1)})} \right)^{n-i} \frac{f_i(t_{(i)})}{1 - F_i(t_{(i-1)})} , \tag{2.19}$$

for $t_1, \dots, t_n \in \mathbb{R}$.

Proof. Let $Y_1, \dots, Y_n$ be exchangeable random variables and $Y_{1:n}, \dots, Y_{n:n}$ their order statistics . For $t_1 \leq \dots \leq t_n$ it is known that

$$f^{Y_{1:n},\dots,Y_{n:n}}(t_1, \dots, t_n) = n! \, f^{Y_1,\dots,Y_n}(t_1, \dots, t_n) \ .$$

Then for $t_1, \dots, t_n \in \mathbb{R}$ we can write

$$f^{Y_1,\dots,Y_n}(t_1, \dots, t_n) = \frac{1}{n!} f^{Y_{1:n},\dots,Y_{n:n}}(t_{(1)}, \dots, t_{(n)}) \ , \tag{2.20}$$

where $(t_{(1)}, \ldots, t_{(n)}) = \pi(t_1, \ldots, t_n)$ and $\pi \in S_n$ is such that $t_{(1)} \leq \ldots \leq t_{(n)}$. The result follows by combining (2.20) with the representation of the joint density of sequential order statistics from Theorem 1.7. □

Remark 2.18. *As already mentioned, $X_1^*, \ldots, X_n^*$ provide another example of exchangeable random variables with conditional hazard rates of the form (2.16). Indeed, by (1.3) together with (2.9) almost surely holds*

$$\begin{aligned}\lambda(t|t_1, \ldots, t_h) &= -\frac{1}{(n-h)} \frac{\partial}{\partial t} \ln\left(\frac{\overline{F}_{h+1}(t)}{\overline{F}_h(t_h)}\right)^{n-h} \\ &= \frac{f_{h+1}(t)}{\overline{F}_{h+1}(t)} ,\end{aligned}$$

where $f_{h+1}(t)$ is the density function corresponding to the survival function $\overline{F}_{h+1}(t)$, $h = 1, \ldots, n-1$ and $0 \leq t_1 \leq \ldots \leq t_h < t$. By analogy, combining the result of Theorem 1.7 with (2.12), the conditional hazard rate of the first failure in the system is obtained as

$$\begin{aligned}\lambda(t) &= -\frac{1}{(n-h)} \frac{\partial}{\partial t} \ln\left(\overline{F}_1(t)\right)^n \\ &= \frac{f_1(t)}{\overline{F}_1(t)}\end{aligned}$$

Thus, for $X_1^*, \ldots, X_n^*$ conditional hazard rates depend both on the age of the considered component and the number of failures preceding the current time t.

Example 2.19. *Consider sequential order statistics $X_*^{(1)}, \ldots, X_*^{(n)}$ based on $F_1, \ldots, F_n$. For $\theta_0, \ldots, \theta_{n-1} > 0$ and $i = 1, \ldots, n$ define the distribution functions F_i as*

$$F_i(t) = 1 - (1 - F(t))^{\frac{1}{\theta_{i-1}}} .$$

where $F(t)$ is the distribution function of standard exponential distribution. Then the following representations exist:

$$\begin{aligned}F_i(t) &= 1 - \exp\left(-\frac{t}{\theta_{i-1}}\right) \\ f_i(t) &= \frac{1}{\theta_{i-1}} \exp\left(-\frac{t}{\theta_{i-1}}\right) .\end{aligned}$$

According to Balakrishnan et al. (2008) such $X_*^{(1)}, \ldots, X_*^{(n)}$ *coincide with the order statistics based on Weinman multivariate exponential distribution. They can be described by the following density function taken from Kotz et al. (2000)*

$$f^{X_*^{(1)},\ldots,X_*^{(n)}}(t_1, \ldots, t_n) = n! \prod_{i=0}^{n-1} \frac{1}{\theta_i} \exp\left(\frac{-(n-i)(t_{i+1} - t_i)}{\theta_i} \right) ,$$

with $\theta_i > 0,\ i = 0, \ldots, n-1,\ t_0 = 0$ *and* $t_0 \leq t_1 \leq \ldots \leq t_n$. *Then considering corresponding conditional hazard rates one may observe that*

$$\lambda(t|t_1, \ldots, t_h) = \frac{1}{\theta_h} \qquad a.s.$$

The corresponding exchangeable random variables $X_1^*, \ldots, X_n^*$ *possess the density*

$$f^{X_1^*,\ldots,X_n^*}(t_1, \ldots, t_n) = \prod_{i=0}^{n-1} \frac{1}{\theta_i} \exp\left(\frac{-(n-i)(t_{(i+1)} - t_{(i)})}{\theta_i} \right) ,$$

where $t_1, \ldots, t_n \in \mathbb{R}$. *According to Kotz et al. (2000)* $X_1^*, \ldots, X_n^*$ *follow Freund's multivariate exponential distribution.*

Remark 2.20. *Consider absolutely continuous random variables* $Y_1, \ldots, Y_n$ *with conditional hazard rates of the form (2.16). It can be seen that their order statistics* $Y_{1:n}, \ldots, Y_{n:n}$ *possess the Markov property, i.e.*

$$f^{Y_{i:n}\,|\,Y_{i-1:n},\ldots,Y_{1:n}}(t_i|t_{i-1}, \ldots, t_1) = f^{Y_{i:n}|Y_{i-1:n}}(t_i|t_{i-1}) , \tag{2.21}$$

for $i = 2, \ldots, n$ *and* $0 \leq t_1 \leq \ldots \leq t_n$. *In more detail, from (2.14) the density function of* $Y_1, \ldots, Y_n$ *is of the form*

$$f^{Y_1,\ldots,Y_n}(t_1, \ldots, t_n) = \prod_{i=1}^{n} p_i(t_{(i)}) ,$$

where p_i are appropriately chosen functions. Then the density of order statistics is

$$f^{Y_{1:n},\ldots,Y_{n:n}}(t_1,\ldots,t_n) = n!\prod_{i=1}^{n} p_i(t_i), \quad t_1 < \ldots < t_n$$

and the following equality holds

$$\begin{aligned} & f^{Y_{i:n}\,|\,Y_{i-1:n},\ldots,Y_{1:n}}(t_i|t_{i-1},\ldots,t_1) \\ & = \frac{\int_{t_i}^{\infty}\int_{t_{i+1}}^{\infty}\ldots\int_{t_{n-1}}^{\infty}\prod_{j=i+1}^{n} p_j(t_j)dt_n\ldots dt_{i+2}dt_{i+1}\,p_i(t_i)}{\int_{t_i}^{\infty}\int_{t_i}^{\infty}\int_{t_{i+1}}^{\infty}\ldots\int_{t_{n-1}}^{\infty}\prod_{j=i+1}^{n} p_j(t_j)dt_n\ldots dt_{i+2}dt_{i+1}p_i(t_i)dt_i} \\ & = f^{Y_{i:n}|Y_{i-1:n}}(t_i|t_{i-1}) \end{aligned}$$

almost surely.

Next we are going to look at distributions that satisfy (2.21). For the sake of consistency let us identify these distributions by the form of their conditional hazard rates.

Lemma 2.21. *The order statistics $Y_{1:n},\ldots,Y_{n:n}$, which are based on absolutely continuous exchangeable random variables $Y_1,\ldots,Y_n$, possess Markov property if and only if for $i = 2,\ldots,n$ holds*

$$\lambda(t|t_1,\ldots,t_{i-1}) = g_i(t,t_{i-1})\ ,$$

where $0 \le t_1 \le \ldots \le t_{i-1} < t$.

Proof. Let $Y_{1:n},\ldots,Y_{n:n}$ be Markov for $i = 1,\ldots,n$, then by (2.6) holds

$$\begin{aligned} \lambda(t|t_1,\ldots,t_{i-1}) &= \lim_{\Delta t\to 0+}\frac{1}{\Delta t}P(Y_{i:n}\le t+\Delta t\,|\,Y_{1:n}=t_1,\ldots,Y_{i-1:n}=t_{i-1},Y_{i:n}>t) \\ &= \lim_{\Delta t\to 0+}\frac{1}{\Delta t}\frac{P(t<Y_{i:n}\le t+\Delta t\,|\,Y_{1:n}=t_1,\ldots,Y_{i-1:n}=t_{i-1})}{P(t<Y_{i:n}\,|\,Y_{1:n}=t_1,\ldots,Y_{i-1:n}=t_{i-1})} \end{aligned}$$

and due to Markov property of $Y_{1:n}, \ldots, Y_{n:n}$ we obtain

$$\begin{aligned}\lambda(t|t_1,\ldots,t_{i-1}) &= \lim_{\Delta t\to 0+} \frac{1}{\Delta t} \frac{P(t < Y_{i:n} \le t+\Delta t \,|\, Y_{i-1:n} = t_{i-1})}{P(t < Y_{i:n} \,|\, Y_{i-1:n} = t_{i-1})} \\ &= \lim_{\Delta t\to 0+} \frac{1}{\Delta t} P(Y_{i:n} \le t+\Delta t \,|\, Y_{i-1:n} = t_{i-1}, Y_{i:n} > t) \;.\end{aligned}$$

It is left to show the inverse implication. By Lemma 2.11 (i) holds

$$\begin{aligned}&f^{Y_{i:n}\,|\,Y_{1:n},\ldots,Y_{i-1:n}}(t_i|t_1,\ldots,t_{i-1}) \\ &\quad= -\frac{\partial}{\partial t_i} \exp\left(-(n-i+1)\int_{t_{i-1}}^{t_i} g_i(t,t_{i-1})dt\right) \\ &\quad= (n-i+1)\, g_i(t_i,t_{i-1}) \exp\left(-(n-i+1)\int_{t_{i-1}}^{t_i} g_i(t,t_{i-1})dt\right) \;.\end{aligned}$$

For the sake of simplicity denote

$$p_i(t_{i-1},t_i) = g_i(t_i,t_{i-1}) \exp\left(-(n-i+1)\int_{t_{i-1}}^{t_i} g_i(t,t_{i-1})dt\right)$$

so that

$$f^{Y_{i:n}\,|\,Y_{1:n},\ldots,Y_{i-1:n}}(t_i|t_1,\ldots,t_{i-1}) = (n-i+1)\, p_i(t_{i-1},t_i) \tag{2.22}$$

and the proof follows immediately. □

Example 2.22. *Consider non-negative exchangeable random variables $Y_1,\ldots,Y_n$ with joint density function*

$$f(t_1,\ldots,t_n) = l(t_{(1)})\prod_{i=2}^{n} k(t_{(i-1)},t_{(i)}) \;, \tag{2.23}$$

where $\theta \in (0,1]$, $t_1,\ldots,t_n \in \mathbb{R}_+$ and

$$l(t) = 2\,\theta \exp\left[-(2t)^{\theta}\right](2t)^{\theta-1} \;,$$

$$k(t_{(i-1)}, t_{(i)}) = \frac{2\,\theta \exp\Big[-(t_{(i-1)} + t_{(i)})^{\theta}\Big]\Big(t_{(i-1)} + t_{(i)}\Big)^{\theta-2}\Big(\theta(t_{(i-1)} + t_{(i)})^{\theta} - (\theta - 1)\Big)}{2\,\theta \exp\Big[-(2t_{(i-1)})^{\theta}\Big]\Big(2t_{(i-1)}\Big)^{\theta-1}} \; .$$

Note that the numerator of the function $k(\cdot|\cdot)$ *consists of the joint density function of some random variables* Z_1, Z_2 *such that* $Z_1 \leq Z_2$ *a.s. Indeed, since* $\theta \in (0, 1]$ *for* $0 \leq z_1 \leq z_2$ *we can state*

$$2\,\theta \exp\Big[-(z_1 + z_2)^{\theta}\Big](z_1 + z_2)^{\theta-2}\Big(\theta(z_1 + z_2)^{\theta} - (\theta - 1)\Big) \geq 0$$

and

$$\begin{aligned}
&\int_0^{\infty} \int_{z_1}^{\infty} 2\,\theta \exp\Big[-(z_1 + z_2)^{\theta}\Big](z_1 + z_2)^{\theta-2}\Big(\theta(z_1 + z_2)^{\theta} - (\theta - 1)\Big) dz_2 dz_1 \\
&\qquad = \int_0^{\infty} \int_{z_1}^{\infty} 2\,\frac{\partial^2}{\partial z^2} \exp\big(-z^{\theta}\big)\Big|_{z=z_1+z_2} dz_2 dz_1 \\
&\qquad = -\int_0^{\infty} 2\;\frac{\partial}{\partial z} \exp\big(-z^{\theta}\big)\Big|_{z=2z_1} dz_1 \\
&\qquad = -\exp\big(-y^{\theta}\big)\Big|_0^{\infty} = 1 \; .
\end{aligned}$$

Moreover, since the following equality holds

$$\begin{aligned}
&\int_{z_1}^{\infty} 2\,\frac{\partial^2}{\partial z^2} \exp\big(-z^{\theta}\big)\Big|_{z=z_1+z_2} dz_2 \\
&\qquad = 2\,\theta \exp\Big[-(2z_1)^{\theta}\Big]\,(2z_1)^{\theta-1} \; ,
\end{aligned}$$

we can conclude that the denominator of $k(z_1, z_2)$ *corresponds to the marginal density of the variable* Z_1 *. Then* $k(z_1, z_2)$ *represents the conditional density* $f^{Z_2\,|\,Z_1}(z_2|z_1)$*, where* $0 \leq z_1 \leq z_2$.

Let us return to the consideration of exchangeable random variables with the joint density function described by (2.23). According to (1.2) their order statistics possess the joint *density*

$$f^{Y_{1:n},\dots,Y_{n:n}}(t_1,\dots,t_n) = n!\, l(t_1) \prod_{i=2}^{n} k(t_{i-1}, t_i)\,,$$

where $0 \le t_1 \le \ldots \le t_n$. *Then the conditional densities of order statistics can be can be described by*

$$\begin{aligned}
&f^{Y_{i:n}\,|\,Y_{i-1:n},\dots,Y_{1:n}}(t_i|t_{i-1},\dots,t_1) \\
&\quad= \frac{\int_{t_i}^{\infty}\int_{y_{i+1}}^{\infty}\cdots\int_{y_{n-1}}^{\infty} f^{Y_{1:n},\dots,Y_{n:n}}(t_1,\dots,t_i,y_{i+1},\dots,y_n)dy_n\dots dy_{i+1}}{\int_{t_{i-1}}^{\infty}\int_{y_i}^{\infty}\cdots\int_{y_{n-1}}^{\infty} f^{Y_{1:n},\dots,Y_{n:n}}(t_1,\dots,t_{i-1},y_i,\dots,y_n)dy_n\dots dy_i} \\
&\quad= k(t_{i-1},t_i) \\
&\qquad\times \frac{\int_{t_i}^{\infty} k(t_i,y_{i+1}) \int_{y_{i+1}}^{\infty} k(y_{i+1},y_{i+2})\cdots\int_{y_{n-1}}^{\infty} k(y_{n-1},y_n)dy_n\dots dy_{i+1}}{\int_{t_{i-1}}^{\infty} k(t_{i-1},y_i) \int_{y_i}^{\infty} k(y_i,y_{i+1})\cdots\int_{y_{n-1}}^{\infty} k(y_{n-1},y_n)dy_n\dots dy_i}\,.
\end{aligned}$$

Observe that $\int_{y_j}^{\infty} k(y_j, y_{j+1})dy_{j+1} = 1$ *for* $j = 1,\dots,n-1$, *since* $k(y_i, y_{i+1})$ *is a conditional density. Then we can conclude that*

$$f^{Y_{i:n}\,|\,Y_{i-1:n},\dots,Y_{1:n}}(t_i|t_{i-1},\dots,t_1) = k(t_{i-1},t_i)$$

and for $t \ge t_{i-1}$

$$\begin{aligned}
&P(Y_{i:n} > t\,|\,Y_{i-1:n} = t_{i-1},\dots,Y_{1:n} = t_1) \\
&\qquad= \int_t^{\infty} k(t_{i-1},u)du \\
&\qquad= \frac{\int_t^{\infty} 2\,\frac{\partial^2}{\partial z^2}\exp(-z^{\theta})\Big|_{z=t_{i-1}+u} du}{\exp\left[-(2t_{i-1})^{\theta}\right](2t_{i-1})^{\theta-1}} \\
&\qquad= \frac{2\,\theta\exp\left[-(t_{i-1}+t)^{\theta}\right](t_{i-1}+t)^{\theta-1}}{2\,\theta\exp\left[-(2t_{i-1})^{\theta}\right](2t_{i-1})^{\theta-1}}
\end{aligned}$$

$$= \exp\left[(2t_{i-1})^{\theta} - (t_{i-1}+t)^{\theta}\right]\left(\frac{1}{2} + \frac{t}{2t_{i-1}}\right)^{\theta-1} .$$

Finally, according to (2.9) multivariate conditional hazard rates are obtained as

$$\begin{aligned}
&\lambda(t|t_{i-1},\ldots,t_1)\\
&\quad = -\frac{1}{(n-i+1)}\frac{\partial}{\partial t}\ln P(Y_{i:n} > t \,|\, Y_{i-1:n} = t_{i-1})\\
&\quad = -\frac{1}{(n-i+1)}\frac{\partial}{\partial t}\left[(2t_{i-1})^{\theta} - (t_{i-1}+t)^{\theta} + (\theta-1)\ln\left(\frac{1}{2}+\frac{t}{2t_{i-1}}\right)\right]\\
&\quad = \frac{1}{(n-i+1)(t_{i-1}+t)}\left(\theta(t_{i-1}+t)^{\theta} + 1 - \theta\right)
\end{aligned}$$

for $0 \le t_1 \le \ldots \le t_{i-1} \le t$, $i = 1,\ldots,n$. *It depends on the number of previous failures, last failure time and the age of considered component.*

2.2.2 Schur-constant densities

In the following we consider conditional hazard rates that depend on two or more of the preceding failure times.

To begin with we are going to look at Schur-constant random vectors. In Caramellino & Spizzichino (1996) they are defined as follows.

Def 2.23. *A non-negative random vector* $(Y_1,\ldots,Y_n)$ *is called Schur-constant if it possesses a Schur-constant joint survival function, i.e. for a suitable univariate survival function* $\Phi : \mathbb{R}_+ \to [0,1]$ *holds*

$$\overline{F}(t_1,\ldots,t_n) = P(T_1 > t_1,\ldots,T_n > t_n) = \Phi\Big(\sum_{i=1}^{n} t_i\Big) ,$$

for $t_1,\ldots,t_n \in \mathbb{R}_+$.

According to Spizzichino (2001) the Schur-constant property can also be formulated for joint density functions, namely:

Remark 2.24. *A joint density function of non-negative lifetimes is Schur-constant if it* *allows the representation*

$$f(t_1, \dots, t_n) = \phi(\sum_{i=1}^{n} t_i),$$

where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$.

Moreover, an absolutely continuous survival function $\overline{F}(t_1, \dots, t_n)$ *is Schur-constant if and only if its density* $f(t_1, \dots, t_n)$ *is Schur-constant.*

Example 2.25. *Consider the random variables* Y_1, Y_2, Y_3 *distributed similar to* $Y_1^{(i)}, Y_2^{(i)}$, $Y_3^{(i)}$ *from Example 1.15, i.e.*

$$Y_1 = \frac{Z_1}{\Theta}, \quad Y_2 = \frac{Z_2}{\Theta}, \quad Y_3 = \frac{Z_3}{\Theta} \ ,$$

where Z_1, Z_2, Z_3 *are independent, exponentially distributed with mean 1 and* Θ *is a gamma distributed random variable with parameters* α, λ. *Observe that* Y_1, Y_2, Y_3 *are Schur-constant. Indeed, it is shown in Appendix that their survival function is given by*

$$\begin{aligned} P(Y_1 > t_1, Y_2 > t_2, Y_3 > t_3) &= \frac{\lambda^\alpha}{(t_1 + t_2 + t_3 + \lambda)^\alpha} \\ &= \Phi(t_1 + t_2 + t_3) \ , \end{aligned}$$

where

$$\Phi(y) = \frac{\lambda^\alpha}{(y + \lambda)^\alpha} = (1 + \frac{y}{\lambda})^{-\alpha} \ .$$

Since for $\alpha > 0$ *and* $y \geq 0$ *the function* Φ *is a survival function we can conclude that* Y_1, Y_2, Y_3 *are Schur-constant random variables.*

Remark 2.26. *In the context of reliability theory Schur-constant random vectors stand out, among others, by their no-aging property. Namely, Definition 2.23 implies*

$$\begin{aligned} &P(Y_k - t_k > s \,|\, Y_1 > t_1, \dots, Y_k > t_k, \dots, Y_n > t_n) \\ &\qquad = P(Y_l - t_l > s \,|\, Y_1 > t_1, \dots, Y_l > t_l, \dots, Y_n > t_n) \end{aligned} \tag{2.24}$$

for $s > 0$, $(t_1, \dots, t_n) \in \mathbb{R}_+^n$ *and* $k \neq l$.

Let $Y_1, \dots, Y_n$ represent lifetimes of components in a system. Then equation (2.24) states that, under the condition $Y_1 > t_1, \dots, Y_n > t_n$, the distributions of residual lifetimes $(Y_k - t_k)$ and $(Y_l - t_l)$ of two different components are equal, although the ages t_k and t_l can differ. In other words, the age of the component Y_k does not influence the corresponding conditional probability of the component Y_l to survive the next period of length s.

Conditions under which the function $\Phi(\cdot)$ from Definition 2.23 corresponds to a multivariate survival function were studied in Nelsen (2006) and McNeil & Nešlehová (2009). Here we outline some of their results.

Def 2.27. *Consider an interval $I \subset \mathbb{R}$ and denote by $\tilde{I}$ the interior of I. A function $f : \mathbb{R} \to \mathbb{R}_+$ is called d-monotone on I, where $d \geq 2$, if*

(i) it is differentiable on the interior of $\tilde{I}$, up to the order $d-2$,

(ii) for $k = 0, 1, \dots, d-2$ and $x \in \tilde{I}$ the derivatives satisfy

$$(-1)^k f^{(k)}(x) \geq 0 \ ,$$

with $f^{(0)}(x) = f(x)$,

(iii) $(-1)^{d-2} f^{(d-2)}(x)$ is non-increasing and convex on $\tilde{I}$.

For $d = 1$, f is called 1-monotone if it is non-negative and non-increasing on $\tilde{I}$. If f has derivatives of all orders on $\tilde{I}$ and if $(-1)^k f^{(k)}(x) \geq 0$ for all $k \in \mathbb{N}$ and $x \in \tilde{I}$, then f is called completely monotone.

Lemma 2.28. *Let Φ be a real valued function on $[0, \infty)$. The function S, specified for $t_1, \dots, t_n \in \mathbb{R}_+$ by*

$$S(t_1, \dots, t_n) = \Phi(t_1 + \ldots + t_n)$$

is a survival function if and only if Φ is an n-monotone function on $[0, \infty)$ satisfying the boundary conditions $\lim_{t\to\infty} \Phi(t) = 0$ and $\Phi(0) = 1$.

Lemma 2.29. *Let Φ be a continuous univariate survival function. Then $\Phi(t_1 + \ldots + t_n)$ is an n-dimensional survival function for all $n \geq 2$ if and only if Φ is completely monotone on $[0, \infty)$.*

Example 2.30. *Consider a univariate survival function $\Phi(t) = \max(1-t, 0)^{d-1}$, for some $d \in \mathbb{N}, d \geq 2$. Computing partial derivatives we obtain*

$$\Phi^{(h)}(t) = (-1)^h \frac{(d-1)!}{(d-1-h)!} \max(1-t, 0)^{d-1-h}$$

for $h = 0, \dots, d-2$. The left and right derivatives of $\Phi^{(d-2)}(t)$ do not coincide in $t = 1$. Therefore, the derivative of $\Phi^{(d-2)}(t)$ does not exist in $t = 1$. However $(-1)^h \Phi^{(h)}(t) \geq 0$ for $h = 0, \dots, d-2$ and $\Phi^{(d-2)}(t)$ is non-increasing and convex on $(0, \infty)$. Then according to Definition 2.27 Φ is d-monotone but not $(d+1)$-monotone.

The following lemma can be found in Caramellino & Spizzichino (1996). It provides representations for the joint density and conditional hazard rates of Schur-constant random variables.

Lemma 2.31. *Consider absolutely continuous Schur-constant random variables $Y_1, \dots, Y_n$ with the joint survival function*

$$\overline{F}(t_1, \dots, t_n) = P(Y_1 > t_1, \dots, Y_n > t_n) = \Phi\Big(\sum_{i=1}^{n} t_i\Big) ,$$

where $\Phi : \mathbb{R}_+ \to [0,1]$ is n-times differentiable.

Then $Y_1, \dots, Y_n$ possess

(i) a joint density function

$$f^{Y_1,\dots,Y_n}(t_1, \dots, t_n) = (-1)^n \, \Phi^{(n)}\Big(\sum_{i=1}^{n} t_i\Big) ,$$

(ii) conditional hazard rates of the form

$$\lambda(t|t_1, \dots, t_h) = -\frac{\Phi^{(h+1)}(y)}{\Phi^{(h)}(y)} , \tag{2.25}$$

where $y = \sum_{i=1}^{h} t_i + (n-h)t,\ 0 \leq t_1 \leq \dots \leq t_h \leq t$.

Proof. Representation (i) is obtained by direct differentiation of the Schur-constant survival function, i.e.

$$\begin{aligned} f^{Y_1,\dots,Y_n}(t_1, \dots, t_n) &= (-1)^n \frac{\partial^n}{\partial t_1 \dots \partial t_n} \overline{F}(t_1, \dots, t_n) \\ &= (-1)^n \Phi^{(n)}\Big(\sum_{i=1}^{n} t_i\Big) . \end{aligned}$$

To deduce (ii) notice first that the joint density of order statistics based on $Y_1, \dots, Y_n$ can be calculated as

$$f^{Y_{1:n},\dots,Y_{h:n}}(t_1,\dots,t_h) \\ = n! \int_{t_h}^{\infty} \int_{t_{h+1}}^{\infty} \dots \int_{t_{n-1}}^{\infty} (-1)^n \, \Phi^{(n)}\Big(\sum_{i=1}^{n} t_i\Big) dt_n \dots dt_{h+1} \; .$$

Further, applying Lemma 1.13 we can reduce the last integral to the following expression

$$\int_{t_h}^{\infty} \int_{t_{h+1}}^{\infty} \dots \int_{t_{n-1}}^{\infty} (-1)^n \, \Phi^{(n)}\Big(\sum_{i=1}^{n} t_i\Big) dt_n \dots dt_{h+1} \\ = \frac{1}{(n-h)!} \int_{t_h}^{\infty} \int_{t_h}^{\infty} \dots \int_{t_h}^{\infty} (-1)^n \, \Phi^{(n)}\Big(\sum_{i=1}^{n} t_i\Big) dt_n \dots dt_{h+1} \\ = \frac{(-1)^h}{(n-h)!} \, \Phi^{(h)}\left(\sum_{i=1}^{h-1} t_i + (n-h+1)t_h\right) \; .$$

Thus, we have obtained that

$$f^{Y_{1:n},\dots,Y_{h:n}}(t_1,\dots,t_h) \\ = (-1)^h \, \frac{n!}{(n-h)!} \, \Phi^{(h)}\left(\sum_{i=1}^{h-1} t_i + (n-h+1)t_h\right) \; . \tag{2.26}$$

With the help of (2.9) and (2.26) multivariate conditional hazard rates can be described by

$$
\begin{aligned}
&\lambda(t|t_1,\ldots,t_h)\\
&= -\frac{1}{(n-h)}\frac{\partial}{\partial t}\ln\left[\frac{(-1)^{h+1}(n-h)!}{(-1)^h(n-h-1)!}\,\frac{\displaystyle\int_t^\infty \Phi^{(h+1)}\Big(\sum_{i=1}^h t_i+(n-h)y\Big)dy}{\Phi^{(h)}\Big(\sum_{i=1}^{h-1}t_i+(n-h+1)t_h\Big)}\right]\\
&= -\frac{1}{(n-h)}\frac{\partial}{\partial t}\ln\left[\frac{(-1)^h(n-h)!}{(-1)^h(n-h-1)!\,(n-h)}\,\frac{\Phi^{(h)}\Big(\sum_{i=1}^h t_i+(n-h)t\Big)}{\Phi^{(h)}\Big(\sum_{i=1}^{h-1}t_i+(n-h+1)t_h\Big)}\right]\\
&= -\frac{1}{(n-h)}\frac{\partial}{\partial t}\ln\left[(-1)^h\Phi^{(h)}\Big(\sum_{i=1}^h t_i+(n-h)t\Big)\right]\,. \qquad (2.27)
\end{aligned}
$$

Finally the needed result is obtained by differentiation in (2.27). □

Remark 2.32. *Let us look at representation (2.25) from a survival theoretical point of view. For Schur-constant random variables the hazard rate depends on the history of failures only through the number of failures* h *and the total age of the system at the moment* t*, namely*

$$y = \sum_{i=1}^h t_i + (n-h)t\ .$$

Therefore, from now on a shorter notation for conditional hazard rates will be used:

$$
\begin{aligned}
\lambda(h,y) &= \lambda(t|t_1,\ldots,t_h)\\
&= -\frac{\Phi^{(h+1)}(y)}{\Phi^{(h)}(y)}\ .
\end{aligned}
$$

Note that due to (2.27) conditional hazard rates can also be calculated as

$$\lambda(h,y) = -\frac{\partial}{\partial y}\ln|\phi^{(h)}(y)|. \qquad (2.28)$$

A special case represents the densities corresponding to completely monotone survival functions. In fact, they can be expressed as a Laplace transform of some probability distribution. In Nelsen (2006) the Laplace transform is defined as follows.

Def 2.33. *Consider a distribution Π defined on $[0,\infty)$. Then the Laplace transform with a mixing distribution Π is given by*

$$L(t) = \int_0^\infty e^{-\theta t} d\Pi(\theta) \ .$$

Then the connection between Laplace transform and Schur-constant random variables arises from Lemma 4.6.5 in Nelsen (2006).

Lemma 2.34. *A function ϕ on $[0,\infty)$ is the Laplace transform of a probability distribution if and only if it is completely monotone and $\phi(0) = 1$.*

Taking into account Definition 2.23, Lemma 2.34 yields the following result.

Theorem 2.35. *Let $Y_1, \dots, Y_n$ be Schur-constant random variables with survival function*

$$S(t_1, \dots, t_n) = \Phi\left(\sum_{i=1}^n t_i\right) ,$$

where Φ is a completely monotone function on $\mathbb{R}_+$. Then, for a suitable probability distribution $\Pi(\theta)$ on $[0,\infty)$ the joint density of $Y_1, \dots, Y_n$ can be represented as

$$f^{Y_1,\dots,Y_n}(t_1, \dots, t_n) = \int_0^\infty \theta^n \exp\left(-\theta \sum_{i=1}^n t_i\right) d\Pi(\theta) \ , \tag{2.29}$$

where $t_1, \dots, t_n \in \mathbb{R}_+$.

Proof. Note that for the survival function S holds

$$S(0, \dots, 0) = \Phi(0) = 1 \ . \tag{2.30}$$

Then, applying Lemma 2.34 for the function Φ we obtain

$$S(t_1, \dots, t_n) = \int_0^\infty \exp\left(-\theta \sum_{i=1}^n t_i\right) d\Pi(\theta) \ .$$

Finally, with the help of Leibniz integration rule the joint density $Y_1, \ldots, Y_n$ can be described by

$$\begin{aligned} f(t_1, \ldots, t_n) &= (-1)^n \frac{\partial^n}{\partial t_1 \ldots \partial t_n} S(t_1, \ldots, t_n) \\ &= \int_0^\infty \theta^n \exp\left(-\theta \sum_{i=1}^n t_i\right) d\Pi(\theta) \ . \end{aligned}$$

□

Remark 2.36. *The proof presented above is a special case of a similar argument concerning infinite sequences of Schur-constant random variables that can be found in Spizzichino (2001).*

Example 2.37. *Let us look again at the random variables Y_1, Y_2, Y_3 from Example 2.25. They are constructed on the basis of conditionally independent random variables and possess the joint density of the form (2.29). Namely the density is equal to*

$$\begin{aligned} f^{Y_1,Y_2,Y_3}(t_1, t_2, t_3) &= \int_0^\infty e^{-\theta(t_1+t_2+t_3)} \frac{\lambda^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\lambda\theta} d\theta \\ &= \frac{\lambda^\alpha}{(t_1 + t_2 + t_3 + \lambda)^\alpha} \ . \end{aligned}$$

Thereby Y_1, Y_2, Y_3 are distributed according to Pareto distribution of the second kind with parameters $\alpha, \frac{1}{\lambda}$.

According to Kotz et al. (2000) a Pareto survival function of the second kind is defined as

$$\Phi(t) = (1 + \beta t)^{-\alpha}$$

for $\alpha, \beta > 0$ and $t \in \mathbb{R}_+$.

Its h-th derivative has the form

$$\Phi^{(h)}(t) = (-1)^h \frac{(\alpha + h - 1)!}{(\alpha - 1)!} \beta^h (1 + \beta t)^{-\alpha-h} \ . \tag{2.31}$$

By (2.25) conditional hazard rates can be calculate as

$$\lambda(y,h) = -\frac{\Phi^{(h+1)}(y)}{\Phi^{(h)}(y)} = \frac{(\alpha+h)\beta}{1+\beta y}\ , \tag{2.32}$$

where $y \geq 0$. *Since* $\Phi(t)$ *is completely monotone,* $S(t_1,\ldots,t_n) = \Phi(t_1+\ldots+t_n)$ *represents a* n*-dimensional survival function for* $n \geq 2$. *Similar to the three-dimensional case* $S(t_1,\ldots,t_n)$ *corresponds to the density function*

$$f(t_1,\ldots,t_n) = \int_0^\infty \theta^n \exp\Big(-\theta \sum_{i=0}^n t_i\Big)\pi(\theta)d\theta\ , \tag{2.33}$$

where $\pi(\theta)$ *is a density of the gamma distribution with parameters* $(\alpha, \frac{1}{\beta})$.

Remark 2.38. *Let* $Y = (Y_1,\ldots,Y_n)$ *be a Schur-constant random vector with a survival function*

$$\Phi(t_1,\ldots,t_n) = S(t_1+\ldots+t_n)\ ,$$

where S is completely monotone on $[0,\infty)$. *It is shown in Nelsen (2005) that the survival function possesses the representation*

$$S(x_1+\ldots+x_n) = S\big[\phi(S(x_1))+\ldots+\phi(S(x_n))\big]\ , \tag{2.34}$$

where ϕ *is the inverse of* S.

2.2.3 Archimedean copulas

The form (2.34) of the survival function corresponds to a structure known as copula. In the following we introduce copulas as it is done in McNeil & Nešlehová (2009). The definition is formulated in terms of difference operators and quasi monotone sets. For the sake of completeness these objects will be introduced first.

Def 2.39. *Consider a function* $f : \mathbb{R}^n \to \mathbb{R}$, x, $h \in \mathbb{R}^n$ *with* $h > 0$. *A difference operator* $\Delta_h f(x)$ *is defined as*

$$\Delta_h f(x) = \Delta^n_{h_n}\ldots\Delta^1_{h_1} f(x)\ ,$$

where $\Delta^i_{h_i}$ *denotes the first order difference operator given by*

$$\Delta^i_{h_i} f(x) = f(x_1, \ldots, x_{i-1}, x_i + h_i, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) \ .$$

Then, for $A \subset \mathbb{R}^n$ *we will call a function* $f : A \to \mathbb{R}$ *quasi-monotone on* A*, if it satisfies* $\Delta_h f(x) \geq 0$ *for every* x *and* h *such that all vertexes of* $(x, x+h]$ *lie in* A*.*

Then a copula can be defined as follows.

Def 2.40. *A* n*-dimensional copula is a function* $C : [0,1]^n \to [0,1]$ *satisfying*

(i) $C(u_1, \ldots, u_n) = 0$ *whenever* $u_i = 0$ *for at least one* $i = 1, \ldots, n$*,*

(ii) $C(u_1, \ldots, u_n) = u_i$ *if* $u_j = 1$ *for all* $j = 1, \ldots, n$ *with* $j \neq i$*,*

(iii) C *is quasi-monotone on* $[0,1]^n$*.*

Copula functions find applications in a wide range of fields from reliability and survival analysis to actuarial science and finance (see for example Jaworski et al. (2010), Clayton (1978), Frees & Valdez (1997), Cherubini et al. (2004)). Many of these applications originated from Sklar's theorem introduced in Sklar (1959), which we quote next. By the means of copulas it provides a link between multivariate distribution functions and their univariate margins (see also Nelsen (2006)).

Theorem 2.41. *Let* H *be a* n*-dimensional distribution function with margins* F_i*,* $i = 1, \ldots, n$*. Then there exist a copula* C*, referred to as the copula of* H*, such that*

$$H(t_1, \ldots, t_n) = C(F_1(t_1), \ldots, F_n(t_n)) \tag{2.35}$$

for any $t_1, \ldots, t_n \in \mathbb{R}$*. Furthermore, C is uniquely determined on* $D = \{u \in [0,1]^d | u \in ranF_1 \times \cdots \times ranF_n\}$*, where* $ranF_i$ *denotes the range of* F_i*. In addition for any* $u \in D$*,*

$$C(u) = H\big(F_1^{-1}(u_1), \ldots, F_n^{-1}(u_n)\big) \ ,$$

where $F_i^{-1}(u_i) = \inf\{x | F_i(x) \geq u_i\}$*,* $i = 1, \ldots, n$*.*

Conversely, given a copula C and univariate distribution functions $F_i(u_i)$*,* $i = 1, \ldots, n$ *the function* H *defined by (2.35) is an* n*-dimensional distribution function with marginals* $F_1, \ldots, F_n$*.*

An interpretation of Sklar's Theorem for multivariate survival functions can be found in McNeil & Nešlehová (2009).

Theorem 2.42. *Let S be an n-dimensional survival function with marginal survival functions $\overline{F}_i$, $i = 1, \ldots, n$. Then there exist a copula C, referred to as survival copula of S, such that*

$$S(t_1, \ldots, t_n) = C\big(\overline{F}_1(t_1), \ldots, \overline{F}_n(t_n)\big) \tag{2.36}$$

for $t_1, \ldots, t_n \in \mathbb{R}$. Furthermore, C is uniquely determined on $D = \{u \in [0,1]^d | u \in ran\overline{F}_1 \times \cdots \times ran\overline{F}_n\}$, where $ran\overline{F}_i$ denotes the range of $\overline{F}_i$. In addition for any $u \in D$,

$$C(u) = S\Big(\overline{F}_1^{-1}(u_1), \ldots, \overline{F}_n^{-1}(u_n)\Big) \ ,$$

where $\overline{F}_i^{-1}(u_i) = \inf\{x | \overline{F}_i(x) \le u_i\}$, $i = 1, \ldots, n$.

Conversely, given a copula C and univariate survival functions $\overline{F}_i(u_i)$, $i = 1, \ldots, n$, S defined by (2.36) is a n-dimensional survival function with marginals $\overline{F}_1, \ldots, \overline{F}_n$.

Remark 2.43. *With respect to Theorem 2.41 and Theorem 2.42, note that a copula and the corresponding survival copula do not necessarily coincide. This fact is described in more detail in Example 2.49 (ii).*

Next we are going to look at the class of copulas known as Archimedean copulas.

Def 2.44. *Let $\phi : [0,1] \to [0,\infty]$ be a continuous, strictly decreasing function such that $\phi(1) = 0$. Then pseudo-inverse of ϕ is the function $\phi^{[-1]} : [0,\infty] \to [0,1]$ given by*

$$\phi^{[-1]}(t) = \begin{cases} \phi^{-1}(t), & 0 \le t \le \phi(0), \\ 0, & \phi(0) \le t \le \infty. \end{cases} \tag{2.37}$$

Note that $\phi^{[-1]}$ is continuous and non-increasing on $[0,\infty]$ and strictly decreasing on $[0,\phi(0)]$. Furthermore, $\phi^{[-1]}(\phi(u)) = u$ on $[0,1]$, and

$$\begin{aligned} \phi\big(\phi^{[-1]}(t)\big) &= \begin{cases} t, & 0 \le t \le \phi(0), \\ \phi(0), & \phi(0) \le t \le \infty, \end{cases} \\ &= \min\big(t, \phi(0)\big) \ . \end{aligned}$$

Def 2.45. *Let $\phi : [0,1] \to [0,\infty)$ be a continuous, strictly decreasing function such that $\phi(1) = 0$, and let $\phi^{[-1]}$ be the pseudo-inverse of ϕ defined by (2.37). An n-dimensional* *copula C is called Archimedean if it permits a representation*

$$C(u) = \phi^{[-1]}\big(\phi(u_1) + \ldots + \phi(u_n)\big), \quad u \in [0,1]^n . \tag{2.38}$$

Remark 2.46.

(i) The function ϕ from Definition 2.45 is called Archimedean generator. If $\phi(0) = \infty$ then ϕ is called a strict generator. In this case $\phi^{[-1]} = \phi^{-1}$, where ϕ^{-1} is the inverse of ϕ.

According to Widder (1941) a function $g(t)$, completely monotone on $[0, \infty)$ and satisfying $g(c) = 0$ for some finite $c > 0$, must be identically zero on $[0, \infty)$. Then we can conclude that completely monotone $\phi^{[-1]}$ must be positive on $[0, \infty)$, i.e. it corresponds to a strict Archimedean generator and $\phi^{[-1]} = \phi^{-1}$.

(ii) In the frame of Definition 2.45, Remark 2.38 describes the connection between survival functions of Schur-constant random variables and Archimedean survival copulas.

Archimedean copulas have gained popularity due to the simplicity of their construction and nice properties that they possess. The facts presented below are taken from McNeil & Nešlehová (2009) and Nelsen (2006), to which we refer for further detail. In particular, the following statements describe Archimedean generators that produce multivariate copulas.

Lemma 2.47. *Let ϕ be an Archimedean generator. Then $C : [0,1]^d \to [0,1]$ given by*

$$C(u_1, \ldots, u_n) = \phi^{[-1]}\big(\phi(u_1) + \ldots + \phi(u_n)\big) ,$$

is a n-dimensional copula if and only if $\phi^{[-1]}$ is n-monotone on $[0, \infty)$.

Lemma 2.48. *Let ϕ be a continuous strictly decreasing function from $[0,1]$ to $[0,\infty]$ such that $\phi(0) = \infty$ and $\phi(1) = 0$, and let ϕ^{-1} denote the inverse of ϕ. If C is the function from $[0,1]^n$ to $[0,1]$ given by (2.38), then C is an n-copula for all $n \geq 2$ if and only if ϕ^{-1} is completely monotone on $[0, \infty)$.*

Example 2.49.

(i) The survival function from Example 2.30 corresponds to the d-dimensional Archimedean survival copula

$$C(u_1, \ldots, u_d) = \max\left(\sum_{i=1}^{d} u_i^{\frac{1}{d-1}} - (d-1), 0\right)^{d-1} ,$$

where $u_i \in [0,1]$ *for* $i = 1,2,\dots,d$ *and the generator is described by*

$$\phi(t) = 1 - t^{\frac{1}{d-1}} \ ,$$

for $t \in [0,1]$.

(ii) Consider a Gumbel bivariate copula

$$C(u_1,u_2) \\ = u_1 + u_2 - 1 + (1-u_1)(1-u_2)\exp\big(-\theta \ln(1-u_1)\ln(1-u_2)\big) \ ,$$

where $\theta \in (0,1]$. *The corresponding survival copula is shown in Barnett (1980) and has the form*

$$S(u_1,u_2) = u_1 u_2 \exp(-\theta \ln u_1 \ln u_2) \ .$$

It is called Gumbel-Barnett survival copula. It was pointed out in Genest & MacKay (1986) that the Gumbel-Barnett copula is Archimedean with generator

$$\phi(t) = \ln(1 - \theta \ln t)$$

and

$$\phi^{-1}(u) = \exp\left(\frac{1-e^u}{\theta}\right) ,$$

where $t \in [0,1]$, $u \in [0,\infty]$. *However the Gumbel copula is not Archimedean. Furthermore, in Georges et al. (2001) the authors show that the Frank copula is the only Archimedean copula for which the survival copula is also Archimedean (see Frank (1979), Frank (1991), Hutchinson & Lai (1990)).*

Remark 2.50. *As mentioned in Nelsen (2005) there is one-to-one correspondence between Schur-constant survival functions and Archimedean copulas. Thus, a Schur-constant survival function* S *corresponds to a copula with a generator given by the pseudo-inverse of* S.

In particular, a subclass of Archimedean copulas represented by frailty survival functions generalizes completely monotone Schur-constant survival functions. The concept of frailty reflects the idea that survival times of components depend on an unobserved random variable Θ. Moreover, the survival times are assumed to be conditionally independent given Θ. A description of frailty survival distributions can be found in Oakes (1989). In Georges et al. (2001) frailty survival functions are defined as follows.

Def 2.51. *A frailty survival function* $S(t_1, \dots, t_n)$ *with marginal survival functions* $\overline{F_i}$, $i = 1, \dots, n$ *is defined by*

$$S(t_1, \dots, t_n) = \check{C}(\overline{F_1}(t_1), \dots, \overline{F_n}(t_n)) ,$$

where $\check{C}$ *is an Archimedean copula with a generator corresponding to the inverse of a Laplace transform with the mixing distribution of the frailty variable* Θ.

Example 2.52. *For some* $\theta > 0$ *consider an Archimedean survival copula with generator*

$$\phi(t) = \left(t^{-\theta} - 1\right) ,$$
$$\phi^{-1}(u) = (1 + u)^{-\frac{1}{\theta}} ,$$

where $t \in [0, 1]$, $u \in [0, \infty]$. *This copula belongs to the Clayton family. According to Nelsen (2005) it generalizes the case of Schur-constant random variables with Pareto joint survival function of the second kind. In particular, since* ϕ *is completely monotone, it generates an* n*-dimensional Clayton copula*

$$C(u_1, \dots, u_n) = \max\left(\sum_{i=1}^{n} u_i^{-\frac{1}{\theta}} - (n-1), 0\right)^{-\theta} , \tag{2.39}$$

where $u_i \in [0, 1]$, $i = 1, 2, \dots, n$ *(see also Nelsen (2006)).*

For $t_i \in [0, \infty)$, $i = 1, 2, \dots, n$ *let* $u_i = \phi^{-1}(t_i)$. *Then (2.39) turns into a Pareto survival function, i.e.*

$$\begin{aligned} & C\big(\phi^{-1}(t_1), \dots, \phi^{-1}(t_n)\big) \\ & \quad = \max\Big[(1 + t_1)^{-\theta(-\frac{1}{\theta})} + \dots + (1 + t_n)^{-\theta(-\frac{1}{\theta})} - (n-1), 0\Big]^{-\theta} \\ & \quad = (t_1 + \dots + t_n + 1)^{-\theta} . \end{aligned}$$

Moreover, according to Definition 2.51 $C(u_1, \dots, u_n)$ *is a frailty copula. By analogy to*

(2.33) it can be represented as a Laplace transform

$$C(u_1, \ldots, u_n) = \int_0^\infty \exp\Big[-\vartheta \sum_{i=0}^{n} \phi^{-1}(u_i)\Big] \pi(\vartheta) d\vartheta \tag{2.40}$$

with $\pi(\vartheta) = \frac{1}{\Gamma(\alpha)} \vartheta^{\alpha-1} \exp(-\vartheta)$.

Remark 2.53. *In the majority of applications Archimedean copulas with one or two parameters are considered. However, the number of parameters can be increased by so called transformation functions. For a detailed description we refer to Nelsen (2006) .*

3 Stochastic orders and dependence notions

Dependence properties play an important role in understanding of stochastic models. They find applications in such fields as reliability theory, mathematical physics, actuarial and social sciences. In particular, in reliability theory positive dependence defines joint behavior of components. For instance, two components represented by positively dependent random variables tend two behave similarly, i.e. if one tends to longevity then so does the other. In this chapter we will look at several stochastic orders and the dependence properties they generate. Furthermore, we will concentrate our attention on one of the strongest positive dependence properties called multivariate total positivity and illustrate it with the examples of distributions considered in Chapter 2.

3.1 Stochastic orders

Stochastic orders find a wide range of applications in reliability theory, for instance, they play a key role in defining dependence and aging properties. We refer to Shaked & Shanthikumar (1990) for examples of application in reliability theory and to Shaked & Shanthikumar (2007) and Müller & Stoyan (2002) for an exhaustive survey of stochastic orders.

In the following we provide definitions for several stochastic orders (cf. Shaked & Shanthikumar (2007), Spizzichino (2001), Richards (2010)). First a note on notation:

Def 3.1. *For $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ in $\mathbb{R}^n$ define the least upper bound*

$$x \vee y = (\max(x_1, y_1), \dots, \max(x_n, y_n)) \tag{3.1}$$

and the greatest lower bound,

$$x \wedge y = (\min(x_1, y_1), \dots, \min(x_n, y_n)). \tag{3.2}$$

In particular, for $n = 1$

$$x \vee y = \max(x, y) \ ,$$
$$x \wedge y = \min(x, y) \ .$$

Def 3.2. *Let $X = (X_1, \dots, X_n)$ and $Y = (Y_1, \dots, Y_n)$ be two absolutely continuous n-dimensional random vectors of lifetimes and let us denote by*

$\eta_{\cdot|\cdot}(\cdot|\cdot)$ and $\lambda_{\cdot|\cdot}(\cdot|\cdot)$ the corresponding conditional hazard rates,

$\Phi_{\cdot|\cdot}(\cdot|\cdot)$ and $\Psi_{\cdot|\cdot}(\cdot|\cdot)$ the corresponding cumulative hazard functions

described in Definitions 2.3 and 2.9. Then X is said to be smaller than Y in

(i) the usual stochastic order (denoted $X \leq_{st} Y$), if $E(\phi(X)) \leq E(\phi(Y))$ for all increasing functions ϕ for which the expectations exist,

(ii) the multivariate hazard rate order (denoted $X \leq_{hr} Y$) if for all $t > 0$

$$\lambda_{r|I}(t|y_I) \leq \eta_{r|J}(t|x_J) \ , \tag{3.3}$$

whenever $I \subset J \subset \{1, \dots, n\}$, $r \notin J$, $x_i \leq y_i < t$, $x_j < t$ for all $i \in I, j \in J \backslash I$.

(iii) the cumulative hazard rate order ($X \leq_{ch} Y$) if for all $t > 0$

$$\Psi_{r|I}(t|y_I) \leq \Phi_{r|J}(t|x_J) \ , \tag{3.4}$$

whenever $I \subset J \subset \{1, \dots, n\}$, $r \notin J$, $x_i \leq y_i < t$, $x_j < t$ for all $i \in I, j \in J \backslash I$.

(iv) the likelihood ratio order (denoted $X \leq_{lr} Y$) if for any pair of vectors $x, y \in \mathbb{R}^n$

$$f_X(x) f_Y(y) \leq f_X(x \vee y) f_Y(x \wedge y) \ . \tag{3.5}$$

Remark 3.3. *For the sets*

$$A = \{a_1, \dots, a_k\} \subset \{1, \dots, n\} \, ,$$
$$\overline{A} = \{b_1, \dots, b_{n-k}\} = \{1, \dots, n\} \backslash A \, ,$$

where $1 \leq k \leq n$, we will use a shorter notation for the history

$$\{Y_{a_1} = y_{a_1}, \dots, Y_{a_k} = y_{a_k}, Y_{b_1} > t, \dots, Y_{b_{n-k}} > t\} = \{Y_A = y_A, Y_{\overline{A}} > te\} \, .$$

Similarly, for $A = \emptyset$ we denote by

$$\{Y_{\{1,\dots,n\}} > te\} = \{Y_1 > t, \dots, Y_n > t\} \, .$$

Then, considering Definition 3.2 note that:

(i) *in part (ii) the history*

$$\mathfrak{h}_t = \{Y_I = y_I, Y_{\overline{I}} > te\}$$

includes fewer failures with later failure times than

$$\mathfrak{h}'_t = \{X_J = x_J, X_{\overline{J}} > te\} \ ,$$

where $\overline{I} = \{1, \ldots, n\} \backslash I$, $\overline{J} = \{1, \ldots, n\} \backslash J$. *Therefore* $\mathfrak{h}_t$ *is often called less severe than* $\mathfrak{h}'_t$, *see for example Spizzichino (2001), Caramellino & Spizzichino (1996).*

Assume that X *and* Y *represent lifetimes of components in two systems that start working simultaneously. Then (3.6) indicates that the system described by* X *is more prone to failure at the time point* t *whenever it went through at least the same number of failures as the system described by* Y *and the failures occurred earlier than in* Y.

Let X *and* Y *be Schur-constant random variables. Lemma 2.31 states that in this case the multivariate conditional hazard rates depend on the history* $\mathfrak{h}_t$ *only through the number of failed components* h *and the total age of the system* $y = \sum_{i=1}^{h} t_i + (n - h)t$. *Therefore* $X \leq_{hr} Y$ *is equivalent to*

$$\lambda_X(h', y') \geq \lambda_Y(h, y) \ , \tag{3.6}$$

whenever $h' \geq h, y' \leq y$ *for* $h, h' \in \{0, 1, \ldots, n-1\}$ *and* $y, y' \geq 0$.

(ii) *from the definition of likelihood ratio order follows immediately that for* $y \leq x$

$$f_Y(x) f_X(y) \geq f_Y(y) f_X(x).$$

For $n = 1$ *this condition is equivalent to* $X \leq_{lr} Y$. *In general, however, it is weaker than (3.5).*

Example 3.4. *Let us generalize Example 2.1 from Burkschat (2009). Namely, for* $i = 1, 2$ *consider random vectors* $Y_{(i)} = (Y_1^{(i)}, \ldots, Y_n^{(i)})$. *For* $j = 1, \ldots, n$ *let*

$$Y_j^{(i)} \ \sim \ \frac{W_j}{\alpha_i V} \ ,$$

where $W_1, \ldots, W_n$ *are independent exponentially distributed random variables with mean 1, the random variable V follows the gamma distribution with shape parameter* $\alpha > 0$ *and* *scale parameter* $b = \frac{1}{\alpha} > 0$ *and* $\alpha_i > 0$. *Moreover, let* V *be independent of* $W_1, \ldots, W_n$.

Then the joint survival function of $(Y_1^{(i)}, \ldots, Y_n^{(i)})$ *can be expressed as*

$$\begin{aligned}
\overline{F}_i(t_1, \ldots, t_n) &= \int_0^\infty \exp\Big(-\alpha_i v(t_1 + \ldots + t_n)\Big) \frac{1}{\beta^\alpha} \frac{1}{\Gamma(\alpha)} v^{\alpha-1} \exp\Big(-\frac{v}{\beta}\Big) dv \\
&= \frac{\alpha^\alpha}{\Gamma(\alpha)} \int_0^\infty e^{-t} \frac{t^{\alpha-1}}{\big(\alpha_i(t_1 + \ldots + t_n) + \alpha\big)^\alpha} dt \\
&= \Big(\frac{\alpha}{\alpha_i(t_1 + \ldots + t_n) + \alpha}\Big)^\alpha \\
&= \left(1 + \frac{\alpha_i(t_1 + \ldots + t_n)}{\alpha}\right)^{-\alpha} .
\end{aligned} \tag{3.7}$$

We can conclude that $Y_{(i)}$ *is Schur-constant with respective univariate survival function of the form*

$$\Phi_i(y) = \left(1 + \frac{\alpha_i}{\alpha} y\right)^{-\alpha} .$$

The derivatives $\Phi_i^{(l)}$ *can be calculated as*

$$\Phi_i^{(l)}(y) = (-1)^l \frac{(\alpha + l - 1)!}{(\alpha - 1)!} \left(\frac{\alpha_i}{\alpha}\right)^l \left(1 + \frac{\alpha_i}{\alpha} y\right)^{-\alpha-l}$$

and for conditional hazard rates holds

$$\begin{aligned}
\lambda_{(i)}(h, y) &= -\frac{\Phi_i^{(h+1)}(y)}{\Phi_i^{(h)}(y)} \\
&= (\alpha + h) \frac{\alpha_i}{\alpha} \left(1 + \frac{\alpha_i}{\alpha} y\right)^{-1} ,
\end{aligned} \tag{3.8}$$

where $l = 0, \ldots, n,\ h = 0, \ldots, n-1,\ i = 1, 2,\ y \geq 0$.

Let us look at the sufficient conditions for $Y_{(1)} \leq_{hr} Y_{(2)}$. *According to Definition 3.2* *we have to ensure*

$$(\alpha + h') \frac{\alpha_1}{\alpha} \left(1 + \frac{\alpha_1}{\alpha} y'\right)^{-1} \geq (\alpha + h) \frac{\alpha_2}{\alpha} \left(1 + \frac{\alpha_2}{\alpha} y\right)^{-1} ,$$

for $h' \geq h$, $y' \leq y$. *For such* h *and* h' *holds*

$$\alpha + h' \geq \alpha + h ,$$

then it suffices to show

$$\frac{\alpha_1}{\alpha} \left(1 + \frac{\alpha_1}{\alpha} y'\right)^{-1} \geq \frac{\alpha_2}{\alpha} \left(1 + \frac{\alpha_2}{\alpha} y\right)^{-1} .$$

The last can be written equivalently as

$$\frac{\alpha_1}{\alpha + \alpha_1 y'} \geq \frac{\alpha_2}{\alpha + \alpha_2 y} .$$

Note that by construction $\alpha, \alpha_1, \alpha_2 > 0$ *and* $y, y' \geq 0$, *therefore the last inequality holds iff*

$$\begin{aligned} &\alpha_1 \alpha_2 y + \alpha_1 \alpha \geq \alpha_2 \alpha + \alpha_1 \alpha_2 y' \\ &\Leftrightarrow \alpha_1 \alpha_2 (y - y') + \alpha(\alpha_1 - \alpha_2) \geq 0 . \end{aligned} \tag{3.9}$$

Finally, from (3.9) we can conclude that $\alpha_1 \geq \alpha_2$ *ensures* $Y_{(1)} \leq_{hr} Y_{(2)}$.

The theorem below summarizes the relations between the stochastic orders under consideration. It covers results from Shaked & Shanthikumar (2007) and Spizzichino (2001).

Theorem 3.5. *The following implications hold*

$$\leq_{lr} \ \Rightarrow \ \leq_{hr} \ \Rightarrow \ \leq_{ch} \ \Rightarrow \ \leq_{st} \quad .$$

Remark 3.6. *Assume that random variables* $X_1, \ldots, X_n$ *and* $Y_1, \ldots, Y_n$ *from Definition*

3.2 satisfy almost surely

$$X_1 \leq \ldots \leq X_n$$
$$Y_1 \leq \ldots \leq Y_n \ .$$

Consider Definition 3.2 (ii) and (iii):

(i) *First we are going to look at* $X \leq_{hr} Y$. *Observe that the components fail in accordance to their indexes, i.e. in Definition 3.2* $I = \{1, \ldots, m\}$, $J = \{1, \ldots, l\}$ *where* $m \leq l \leq r-1$. *Taking into account Remark 2.4 (iii) we will distinguish the following cases:*

1. $m \leq l < r-1$, *then (3.3) turns into* $0 \leq 0$.
2. $m < l = r-1$, *then we obtain* $0 \leq \eta_{r|J}(t|x_J)$, *which holds per definition of conditional hazard rate.*
3. $m = l = r-1$, *then should hold*

$$\lambda_{r|\{1,\ldots,r-1\}}(t|y_1, \ldots, y_{r-1}) \leq \eta_{r|\{1,\ldots,r-1\}}(t|x_1, \ldots, x_{r-1}) \ , \tag{3.10}$$

whenever $x_i \leq y_i, i = 1, \ldots, r-1$.

(ii) *Consider* $X \leq_{ch} Y$. *Taking into account to Remark 2.10 let us look at the following cases:*

1. $I, J \neq \{1, \ldots, r-1\}$, *then (3.4) turns into* $0 \leq 0$.
2. $I \neq J$ *and* $J = \{1, \ldots, r-1\}$, *then we obtain*

$$\int_{x_{r-1}}^{t} \lambda_{r|J}(u|x_1, \ldots, x_{r-1}) du \geq 0 \ ,$$

which holds since a conditional hazard rate is a non-negative function.

3. $I, J = \{1, \ldots, r-1\}$, *(3.4) yields*

$$\int_{y_{r-1}}^{t} \lambda_{r|I}(u|y_1, \ldots, y_{r-1}) du \leq \int_{x_{r-1}}^{t} \eta_{r|J}(u|x_1, \ldots, x_{r-1}) du \ , \tag{3.11}$$

where $x_i \leq y_i$, $i = 1, \ldots, r-1$.

(iii) *Consider order statistics* $(X_{(1)}, \dots, X_{(n)}), (Y_{(1)}, \dots, Y_{(n)})$ *based on absolutely continuous exchangeable random vectors* $(X_1, \dots, X_n), (Y_1, \dots, Y_n)$. *Then by (3.10) together with (3.11) from*

$$(X_1, \dots, X_n) \leq_{hr} (Y_1, \dots, Y_n)$$

follows

$$(X_{(1)}, \dots, X_{(n)}) \leq_{hr} (Y_{(1)}, \dots, Y_{(n)}).$$

An analogous observation for random vectors with iid components can be found in Belzunce et al. (2003b).

In some cases conditions defining stochastic orders can be made more specific. For instance, in Belzunce et al. (2003a) stochastic orders for random vectors with mixture distributions are described. In particular, the following statement is given:

Theorem 3.7. *Consider random vectors* $(S_1, \dots, S_n, \Theta_1)$ *and* $(T_1, \dots, T_n, \Theta_2)$, *where* $(S_1, \dots, S_n)$ *and* $(T_1, \dots, T_n)$ *are independent given* $\Theta_1 = \theta$ *and* $\Theta_2 = \theta$ *for any value of* θ, *respectively, and* Θ_1 *and* Θ_2 *are m-dimensional random vectors defined on* $\mathbb{R}^m$. *If, for all* $i = 1, \dots, n$

(i) $S_i(\theta) =_{st} T_i(\theta)$ *for all* θ,

(ii) $S_i(\theta) \leq_{lr} S_i(\theta')$ *for all* $\theta \leq \theta'$,

(iii) $\Theta_1 \leq_{lr} \Theta_2$,

then

$$(S_1, \dots, S_n) \leq_{lr} (T_1, \dots, T_n) .$$

Here $S_i(\theta)$, $(T_i(\theta))$ *denotes a distribution of* S_i (T_i) *under the condition* $\Theta_1 = \theta$ $(\Theta_2 = \theta)$.

For the Schur-constant random variables Theorem 3.7 leads to the following observation.

Lemma 3.8. *Let* $S = (S_1, \dots, S_n)$ *and* $T = (T_1, \dots, T_n)$ *be non-negative n-dimensional random vectors with joint densities* f_1, f_2, *respectively. For* $t_1, \dots, t_n \in \mathbb{R}_+$ *let*

$$f_1(t_1, \dots, t_n) = \int_0^\infty \theta^n \exp\Big(-\theta \sum_{i=1}^n t_i\Big) \pi_1(\theta) d\theta$$

$$f_2(t_1,\dots,t_n) = \int_0^\infty \theta^n \exp\Big(-\theta \sum_{i=1}^n t_i\Big)\pi_2(\theta)d\theta \ ,$$

for some absolutely continuous random variables Θ_1, Θ_2 *with density functions* $\pi_1(\theta)$, $\pi_2(\theta)$. *If* $\Theta_2 \le_{lr} \Theta_1$, *then* $S \le_{lr} T$.

Proof. According to Definition 2.23 $(S_1,\dots,S_n),(T_1,\dots,T_n)$ have Schur-constant densities and survival functions. Rewrite f_1, f_2 with respect to $\Upsilon_1 = -\Theta_1, \Upsilon_2 = -\Theta_2$

$$f_1(t_1,\dots,t_n) = \int_{-\infty}^0 (-1)^n\,\eta^n \exp\Big(\eta \sum_{i=1}^n t_i\Big)\pi_1(-\eta)d\eta$$
$$f_2(t_1,\dots,t_n) = \int_{-\infty}^0 (-1)^n\,\eta^n \exp\Big(\eta \sum_{i=1}^n t_i\Big)\pi_2(-\eta)d\eta\,.$$

Then, without loss of generality we can apply Theorem 3.7 for Υ_1, Υ_2 defined on $(-\infty, 0]$. In terms of which $S_i(\eta) =_{st} T_i(\eta)$ and $f^{S_i|\Upsilon_1}(t|\eta) = f^{T_i|\Upsilon_2}(t|\eta) = -\eta e^{\eta t}$, therefore condition (i) is satisfied. According to (ii) for $\eta \le \eta'$ and $y_1 \le y_2$ is required

$$-\,\eta e^{\eta y_2}\big(-\eta' e^{\eta' y_1}\big) \le -\eta e^{\eta y_1}\big(-\eta' e^{\eta' y_2}\big)\,.$$

Indeed the last inequality is equivalent to

$$e^{\eta(y_2-y_1)} \le e^{\eta'(y_2-y_1)}$$
$$\Leftrightarrow 1 \le e^{(\eta'-\eta)(y_2-y_1)}$$

which holds since $(\eta' - \eta) \ge 0$, $(y_2 - y_1) \ge 0$. It remains to check (iii), namely that $\Upsilon_1 \le_{lr} \Upsilon_2$. For $\eta \le \eta'$ per definition of the lr-order should hold

$$-\pi_1(-\eta')(-\pi_2(-\eta)) \le -\pi_1(-\eta)(-\pi_2(-\eta'))\ .$$

Since $\theta' = -\eta' \le \theta = -\eta$, this inequality transforms into

$$\pi_1(\theta')\pi_2(\theta) \le \pi_1(\theta)\pi_2(\theta')\ ,$$

which is equivalent to $\pi_2 \leq_{lr} \pi_1$.

Thus, all conditions of the Theorem 3.7 are satisfied and $S \leq_{lr} T$. □

3.2 Dependence notions

In the literature stochastic dependence properties of random variables are qualified as positive or negative based on the joint behavior that they reflect. Thus, positive (negative) dependence properties describe the tendency of random variables to attain concordant (discordant) values. In the following we give an overview of several dependence notions and their properties.

3.2.1 Positive dependence properties

The following definitions are taken from Karlin & Rinott (1980a), Spizzichino (2001) and Shaked & Shanthikumar (1987).

Def 3.9. *Let $Y = (Y_1, \dots, Y_n)$ be a random vector with joint density $f(t_1, \dots, t_n)$, conditional hazard rates $\lambda_{\cdot|\cdot}(\cdot|\cdot)$ and cumulative hazard rates $\Phi_{(\cdot|\cdot)}(\cdot|\cdot)$. Then Y is called*

(i) associated if

$$Cov(f(Y), g(Y)) \geq 0\ ,$$

for all bounded increasing functions $f, g : \mathbb{R}^n \mapsto \mathbb{R}$.

(ii) conditionally increasing in sequence (CIS) if, for $i = 2, \dots, n$, it satisfies

$$P(Y_i > t \,|\, Y_{i-1} = t_{i-1}, \dots, Y_1 = t_1) \leq P(Y_i > t \,|\, Y_{i-1} = t'_{i-1}, \dots, Y_1 = t'_1)$$

whenever $t_j \leq t'_j,\ j = 1, \dots, i-1$.

(iii) hazard rate increasing upon failure (HIF), if $Y \leq_{hr} Y$ namely if

$$\lambda_{r|I}(t|y_I) \leq \lambda_{r|J}(t|x_J) \tag{3.12}$$

whenever $I \subset J \subset \{1, \dots, n\},\ r \notin J,\ x_i \leq y_i < t,\ x_j < t$ for all $i \in I, j \in J \backslash I$.

(iv) possessing supportive lifetimes (SL), if $X \leq_{ch} X$ i.e.

$$\Phi_{r|I}(t|y_I) \leq \Phi_{r|J}(t|x_J) \tag{3.13}$$

whenever $I \subset J \subset \{1, \ldots, n\}$, $r \notin J$, $x_i \leq y_i < t$, $x_j < t$ *for all* $i \in I, j \in J \backslash I$.

(v) *multivariate totally positive of order two (*MTP_2*), if* $Y \leq_{lr} Y$ *i.e.*

$$f(x)f(y) \leq f(x \vee y)f(x \wedge y) , \tag{3.14}$$

where $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n)$. *For* $n = 2$ *the density is called* TP_2.

Remark 3.10.

(i) *In general, the concept of multivariate total positivity is legitimate not only for probability densities. According to Khaledi & Kochar (2000), a function* $f : \mathbb{R}^n \to [0, \infty)$ *is called* MTP_2 *if it satisfies the inequality*

$$f(x)f(y) \leq f(x \vee y)f(x \wedge y) ,$$

for $x, y \in \mathbb{R}^n$. *Moreover, as mentioned in Richards (2010) the operations (3.1) and (3.2) induce a partial order on* $\mathbb{R}^n$ *and the set becomes a distributive lattice. Thus, the definition can be generalized further. Specifically, in Karlin & Rinott (1980a)* MTP_2 *functions defined on lattices are considered. In this thesis we will mostly concentrate our attention with respect to the* MTP_2 *property on functions of the type* $f : \mathbb{R}^n_+ \to [0, \infty)$. *For the details in a more general set up we refer to Karlin & Rinott (1980a).*

(ii) *In the literature an* MTP_2 *function* $f : \mathbb{R}^n \to \mathbb{R}_+$ *is also called log-supermodular. Initially this terminology comes from game theory and economics, where the concept of* MTP_2 *functions was developed in parallel to stochastics, see, for example, Topkis (1998).*

Def 3.11. *A function* $f : \mathbb{R}^n \to \mathbb{R}$ *is called supermodular, if*

$$f(x) + f(y) \leq f(x \vee y) + f(x \wedge y) ,$$

for all $x, y \in \mathbb{R}^n$.

The log-supermodular representation leads to a new description for positive valued MTP_2 functions. Namely, the following fact can be found in Richards (2010)

Lemma 3.12. *Consider a function $f : \mathbb{R}^n \to \mathbb{R}_+$ strictly positive, twice differentiable on $\mathbb{R}^n$. Then f is MTP_2, iff*

$$\frac{\partial^2}{\partial x_i \partial x_j} \ln f(x_1, \dots, x_n) \geq 0 \,, \tag{3.15}$$

for all $x_i, x_j \in \mathbb{R}$, $i, j = 1, \dots, n$, $i \neq j$.

Remark 3.13. *As shown in Remark 2.4 exchangeable random variables possess a distinct form of conditional hazard rates. Therefore, for exchangeable $Y_1, \dots, Y_n$ the HIF property can be written as*

$$\lambda(t|x_1, \dots, x_i) \leq \lambda(t|y_1, \dots, y_j) \,, \tag{3.16}$$

where $i, j \in \{0, \dots, n\}$, $i \leq j$ and $y_k \leq x_k < t$ for $k = 1, \dots, i$ and $y_s < t$ for $s = i + 1, \dots, j$.

Theorem 3.14. *Between the dependence properties considered above the following relations exist:*

$$\begin{array}{ccccc} & \nearrow & CIS & \searrow & \\ MTP_2 & & & & Association \\ & \searrow & HIF \to SL & \nearrow & \end{array}$$

We refer to Shaked & Shanthikumar (1990) for the proof of the $MTP_2 \to$ HIF relationship, to Shaked & Shanthikumar (1987) for HIF$\to$ SL relationship and to Block & Ting (1981) for the rest.

Due to their role in development of reliability theory and statistics, dependence properties are well studied in the literature. Here we would like to mention the works of Shaked & Shanthikumar (2007), Block & Ting (1981) and Colangelo et al. (2005), where an overview of existing dependence properties is presented. In the following we cite several theorems providing conditions for different types of positive dependence. A special attention is turned to MTP_2 since it is one of the strongest known dependence properties.

Theorem 3.15. *If $f(x)$ and $g(x)$ are two MTP_2 functions, then $f(x)g(x)$ is also MTP_2.*

Proof. Proof follows directly from the definition of MTP_2, for the details see Karlin & Rinott (1980a). □

The following result can be found in Khaledi & Kochar (2001).

Lemma 3.16. *Consider the functions $f(x,y,z)$ and $g(x,z)$ defined on ordered sets X, Y, Z respectively. If*

(i) $f(x,y,z) > 0$ and f is TP_2 in each pairs of variables when the third variable is held fixed,

(ii) $g(x,z)$ is TP_2,

then the function

$$h(x,y) = \int_Z f(x,y,z)g(x,z)d\mu(z)\,,$$

defined on $X \times Y$ is TP_2 in (x,y).

In this thesis we will investigate the MTP_2 property on different sets, i.e. $\mathbb{R}^n$, $\mathbb{R}^n_+, \mathbb{R}^n_<$. Therefore, in the following we present several results concerning the MTP_2 property in a general way. In particular, we are going to consider functions defined on a product $\mathcal{X} = \prod_{i=1}^n \mathcal{X}_i$ of totally ordered spaces $\mathcal{X}_i$, $i = 1, ..., n$ with a partial ordering on $\mathcal{X}$, which for $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathcal{X}$ states $x \leq y$ if $x_i \leq y_i$ in $\mathcal{X}_i$ for $i = 1, ..., n$.

The following theorem can be found in Karlin & Rinott (1980a). It states that the MTP_2 property is preserved under marginalization.

Theorem 3.17. *Let f be an MTP_2 function on $\mathcal{X} = \prod_{i=1}^{n} \mathcal{X}_i$. Then the marginal function ϕ defined on $\prod_{i=1}^{k} \mathcal{X}_i$ by*

$$\phi(x_1, \ldots, x_k) = \int_{\mathcal{X}_n} \cdots \int_{\mathcal{X}_{k+1}} f(x_1, \ldots, x_n) dx_{k+1} \cdots dx_n$$

is MTP_2.

In particular, Theorem 3.17 leads to the following observations.

Theorem 3.18. *Consider a MTP_2 random vector $(X_1, \ldots, X_n)$. Then for $\{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$ the random vector $(X_{i_1}, \ldots, X_{i_k})$ is MTP_2.*

Remark 3.19. *Consider a MTP$_2$ function $f : \mathbb{R}^n \to \mathbb{R}$. In the inequality (3.14) without the loss of generality assume*

$$(x_1, \dots, x_n) = (x_1, x_2, z_3, \dots, z_n) \ ,$$
$$(y_1, \dots, y_n) = (y_1, y_2, z_3, \dots, z_n) \ ,$$

where $z_3, \dots, z_n \in \mathbb{R}$ are held fix. Then inequality (3.14) turns into

$$f(x_1, x_2, z_3, \dots, z_n)\, f(y_1, y_2, z_3, \dots, z_n) \\ \le f(x_1 \vee y_1, x_2 \vee y_2, z_3, \dots, z_n)\, f(x_1 \wedge y_1, x_2 \wedge y_2, z_3, \dots, z_n) \ .$$

In other words, f is TP$_2$ in any (t_i, t_j), $i, j = 1, \dots, n$, $i \neq j$ if all the other variables are held fix. Thus MTP$_2$ implies TP$_2$ in pairs. The following lemma shows that under a certain additional assumption the reversed implication also holds. For the proof see Karlin & Rinott (1980a).

Lemma 3.20. *Let $f(t) = f(t_1, \dots, t_n), (t_1, \dots, t_n) \in \mathcal{X}$ be TP_2 in every pair of arguments, when the remaining arguments are held constant, and suppose that $f(x)f(y) \neq 0$ implies $f(u)f(v) \neq 0$ for any $x, y \in \mathcal{X}$ and $u, v \in \mathcal{X}$ with*

$$x \wedge y \le u,\ v \le x \vee y \ . \tag{3.17}$$

Then for all $x, y \in \mathcal{X}$

$$f(x)f(y) \le f(x \vee y)f(x \wedge y) \ .$$

Applying a similar reasoning the following statement for the case of ordered random variables can be made.

Lemma 3.21. *Consider a density function $f(t) = f(t_1, \dots, t_n)$, $t = (t_1, \dots, t_n) \in \mathbb{R}^n$, such that*

$$\begin{cases} f(t) > 0 & t \in \mathcal{A} \\ f(t) = 0 & t \notin \mathcal{A} \ , \end{cases} \tag{3.18}$$

where $\mathcal{A} = \{t = (t_1, \dots, t_n) \mid t \in \mathbb{R}^n,\ t_1 \le \dots \le t_n\}$.

Let $f(t_1, \dots, t_n)$ be TP_2 in every pair of arguments, when the remaining arguments *are held constant. Then* $f(t_1, \ldots, t_n)$ *is* MTP_2*.*

Proof. For $f(t)$ to be MTP_2 should hold

$$f(x)f(y) \le f(x \wedge y)f(x \vee y) \ , \tag{3.19}$$

for $x, y \in \mathbb{R}^n$.

Note that if $x \notin \mathcal{A}$ or $y \notin \mathcal{A}$ then (3.19) turns into

$$0 \le f(x \wedge y)f(x \vee y) \ ,$$

which holds due to (3.18). Thus it suffices to consider (3.19) for $x, y \in \mathcal{A}$.

Let us check whether in this case $x \vee y,\ x \wedge y$ belong to $\mathcal{A}$. Since $\min(x_i, y_i) \le x_i, y_i$ and $x_i \le x_{i+1},\ y_i \le y_{i+1}$ it follows immediately that

$$\min(x_i, y_i) \le \min(x_{i+1}, y_{i+1})$$

for $i = 1, \ldots, n-1$. Then $x \wedge y \in \mathcal{A}$. By analogy we can state

$$\max(x_i, y_i) \le \max(x_{i+1}, y_{i+1}),$$

since $x_i, y_i \le \max(x_i, y_i)$ and $x_i \le x_{i+1},\ y_i \le y_{i+1}$ for $i = 1, \ldots, n-1$. Therefore $x \vee y \in \mathcal{A}$.

In the following we are going to show that (3.19) holds for all $x, y \in \mathcal{A}$. The proof is obtained by induction.

The case $n = 2$ represents the induction base. Obviously, if $f(t_1, t_2)$ is TP_2, then it also can be called MTP_2.

For the induction step assume that (3.19) holds for $n = k - 1$. Under this assumption we need to prove (3.19) for $n = k$.

To do so, consider two sets $I, J \subset \{1, \ldots, n\},\ J = \{1, \ldots, n\} \backslash I$, such that $x_i < y_i$ for all $i \in I$ and $y_j \le x_j$ for all $j \in J$. Let $s \in S_n$ be a permutation of $(t_1, \ldots, t_n)$ $\in \mathbb{R}^n$ such that $s(t_1, \ldots, t_n) = (t_{(1)}, \ldots, t_{(n)})$, where $t_{(1)} \le \ldots \le t_{(n)}$. For the set of indexes $I = \{i_1, \ldots, i_l\}$ denote $x_I = (x_{i_1}, \ldots, x_{i_l})$. Under this notation it suffices to show

$$\begin{aligned} \frac{f(x \wedge y)f(x \vee y)}{f(x)f(y)} &= \frac{f(s(x_J, y_I))f(s(x_I, y_J))}{f(s(x_J, x_I))f(s(y_J, y_I))} \\ &\ge 1 \ . \end{aligned} \tag{3.20}$$

For the sake of simplicity denote $J\backslash n = J\backslash\{n\}$. Suppose first that $\max(x_n, y_n) = x_n$ and $J\backslash n \neq \emptyset$. Then we can rewrite the left part of (3.20) as

$$\frac{f(s(x_{J\backslash n}, y_I), x_n) f(s(x_I, y_{J\backslash n}), y_n)}{f(s(x_{J\backslash n}, x_I), x_n) f(s(y_{J\backslash n}, y_I), y_n)} = \frac{f(s(x_{J\backslash n}, y_I), x_n) f(s(x_I, y_{J\backslash n}), x_n)}{f(s(x_{J\backslash n}, x_I), x_n) f(s(y_{J\backslash n}, y_I), x_n)} \times \frac{f(s(x_I, y_{J\backslash n}), y_n) f(s(y_{J\backslash n}, y_I), x_n)}{f(s(x_I, y_{J\backslash n}), x_n) f(s(y_{J\backslash n}, y_I), y_n)} . \tag{3.21}$$

Due to (3.18) holds

$$\begin{aligned} f(s(x_I, y_{J\backslash n}), x_n) &= f(\min(x_1, y_1), \ldots, \min(x_{n-1}, y_{n-1}), \max(x_n, y_n)) \neq 0 , \\ f(s(y_{J\backslash n}, y_I), x_n) &= f(y_1, \ldots, y_{n-1}, \max(x_n, y_n)) \neq 0 \end{aligned}$$

for all $x, y \in \mathcal{A}$. The last expression in (3.21) is a product of two terms which either exceed or are equal to one by the induction hypothesis. To confirm (3.21), hold in the first term x_n fixed and apply the induction hypothesis to the remaining $(n-1)$ variables, by analogy in the second term hold $y_{J\backslash n}$ fixed.

Next consider the case $\max(x_n, y_n) = x_n$ and $J\backslash n = \emptyset$, then $\max(x_{n-1}, y_{n-1}) = y_{n-1}$. By analogy to (3.21) we can write

$$\begin{aligned} &\frac{f(y_{I\backslash n-1}, y_{n-1}, x_n)\, f(x_{I\backslash n-1}, x_{n-1}, y_n)}{f(x_{I\backslash n-1}, x_{n-1}, x_n)\, f(y_{I\backslash n-1}, y_{n-1}, y_n)} \\ &\qquad = \frac{f(y_{I\backslash n-1}, y_{n-1}, x_n)\, f(x_{I\backslash n-1}, y_{n-1}, y_n)}{f(x_{I\backslash n-1}, y_{n-1}, x_n)\, f(y_{I\backslash n-1}, y_{n-1}, y_n)} \\ &\qquad\quad \times \frac{f(x_{I\backslash n-1}, y_{n-1}, x_n)\, f(x_{I\backslash n-1}, x_{n-1}, y_n)}{f(s(x_{I\backslash n-1}, y_{n-1}, y_n)\, f(x_{I\backslash n-1}, x_{n-1}, x_n)} \end{aligned}$$

and the result follows by induction.

It remains to prove that the statement holds in the case $\max(x_n, y_n) = y_n$. Indeed, if $I\backslash n \neq \emptyset$ then the representation exists

$$
\begin{aligned}
\frac{f(x \wedge y) f(x \vee y)}{f(x) f(y)} &= \frac{f(s(x_J, y_{I\backslash n}), y_n) f(s(x_{I\backslash n}, y_J), x_n)}{f(s(x_J, x_{I\backslash n}), x_n) f(s(y_J, y_{I\backslash n}), y_n)} \\
&= \frac{f(s(x_J, y_{I\backslash n}), y_n) f(s(x_{I\backslash n}, y_J), y_n)}{f(s(x_J, x_{I\backslash n}), y_n) f(s(y_J, y_{I\backslash n}), y_n)} \\
&\quad \times \frac{f(s(x_{I\backslash n}, x_J), y_n) f(s(y_J, x_{I\backslash n}), x_n)}{f(s(x_{I\backslash n}, x_J), x_n) f(s(y_J, x_{I\backslash n}), y_n)} \,.
\end{aligned} \tag{3.22}
$$

Then (3.22) is shown by induction similarly to the one above. Note that in this case y_n and $x_{I\backslash n}$ are held fix in the corresponding terms .

Finally, the proof for the case $\max(x_n, y_n) = y_n$ and $I\backslash n = \emptyset$ is obtained by complete analogy to $\max(x_n, y_n) = x_n,\ J\backslash n = \emptyset$. □

For random variables possessing the Markov property a more specific conclusion can be found in Karlin & Rinott (1980a):

Theorem 3.22. *Let $Y = (Y_1, \ldots, Y_n)$ describe the evolution of a Markov chain with TP_2 transition probability densities. Then Y has a MTP_2 joint density.*

In the following we provide several examples of MTP_2 functions.

Example 3.23.

(i) It is shown in Karlin & Rinott (1980a) that an indicator function $I_{\mathcal{A}}$ defined by

$$
I_{\mathcal{A}}(x) = \begin{cases} 1, & x \in \mathcal{A} \\ 0, & x \notin \mathcal{A} \end{cases}
$$

is MTP_2 for a set $\mathcal{A} = \{t = (t_1, \ldots, t_n) \mid t \in \mathbb{R}^n,\ t_1 \leq \ldots \leq t_n\}$.

(ii) Consider absolutely continuous exchangeable random variables $Y_1, \ldots, Y_n$. Assume that $Y_1, \ldots, Y_n$ are MTP_2. Then, it can be seen by Theorem 3.15 together with the fact that indicator functions are MTP_2 that order statistics $Y_{1:n}, \ldots, Y_{n:n}$ are also MTP_2. More concretely, according to (1.1) they possess a joint density

$$
f^{Y_{1:n}, \ldots, Y_{n:n}}(t_1, \ldots, t_n) = n!\, f^{Y_1, \ldots, Y_n}(t_1, \ldots, t_n) \mathbb{I}_{\mathbb{R}^n_\leq}(t_1, \ldots, t_n) \ ,
$$

where both $f^{Y_1,\ldots,Y_n}$ and $\mathbb{I}_{\mathbb{R}^n_<}$ are MTP$_2$. Theorem 3.37 and Example 3.38 (i) below illustrate that the reversed implication does not hold. Namely that MTP$_2$ of order statistics does not ensure the MTP$_2$ of the underlying variables.

(iii) *Let $f(x_1,\ldots,x_n) = \prod_{i=1}^{n} f_i(x_i)$, where $f_i : \mathbb{R} \to \mathbb{R}$, $i = 1,\ldots,n$. Then by Theorem 3.15 f is MTP$_2$ for an arbitrary choice of f_i, $i = 1,\ldots,n$.*

In particular, consider the density of sequential order statistics based on conditionally independent random variables

$$\begin{aligned} f^{X_*^{(1)},\ldots,X_*^{(n)}}(t_1,\ldots,t_n) &= n!\, f_1(t_1)\, \overline{F_1}^{\,n-1}(t_1) \prod_{i=2}^{n} \left(\frac{\overline{F_i}(t_i)}{\overline{F_i}(t_{i-1})} \right)^{n-i} \frac{f_i(t_i)}{\overline{F_i}(t_{i-1})} \\ &= n!\, f_n(t_n) \prod_{i=1}^{n-1} \left(\frac{\overline{F_i}(t_i)}{\overline{F_{i+1}}(t_i)} \right)^{n-i} f_i(t_i)\ , \end{aligned}$$

for $0 \le t_1 \le \ldots \le t_n$.

Then $f^{X_*^{(1)},\ldots,X_*^{(n)}}(t_1,\ldots,t_n)$ is represented by the product of functions

$$\left(\frac{\overline{F_i}(t_i)}{\overline{F_{i+1}}(t_i)} \right)^{n-i} f_i(t_i)$$

and $f_n(t_n)$, each of which depends only on one variable. Therefore, sequential order statistics based on conditionally independent random variables, as well as ordinary order statistics based on iid random variables, are MTP$_2$. For the overview concerning dependence properties of order statistics we refer to Belzunce et al. (2003b) and Cramer (2006).

The following fact is taken from Karlin & Rinott (1980a).

Lemma 3.24. *If $f(x), x \in \mathcal{X} = \prod_{i=1}^{n} \mathcal{X}_i$ is MTP$_2$ and $l_1,\ldots,l_n : \mathcal{X}_i \to \mathcal{X}_i$ are all increasing (decreasing) functions , then the function*

$$f(l_1(x_1),\ldots,l_n(x_n))$$

is also MTP$_2$ on $\mathcal{X}$.

Example 3.25. *Consider absolutely continuous random variables $Y_1, \ldots, Y_n$ with values in $\mathbb{R}_+$ and the joint density function equal to*

$$f^{Y_1,\ldots,Y_n}(t_1, \ldots, t_n) = c\Big(\sum_{i=1}^{n} t_i\Big)^{a-1} \prod_{i=1}^{n} t_i^{a_i-1} e^{-bt_i} ,$$

where $c > 0$ is a normalization constant and $a, b, a_i > 0$ for $i = 1, \ldots, n$. According to Gupta & Richards (1987) this distribution belongs to the family of Liouville distributions.

Let us verify the conditions under which $Y_1, \ldots, Y_n$ are MTP$_2$. Note that MTP$_2$ of $f^{Y_1,\ldots,Y_n}$ is equivalent to the MTP$_2$ of

$$f(t_1, \ldots, t_n) = \left(\sum_{i=1}^{n} t_i\right)^{a-1} .$$

According to Lemma 3.12 we need to find the values of a such that

$$\frac{\partial^2}{\partial t_i \partial t_j} \ln f(t_1, \ldots, t_n) \geq 0 .$$

Calculating the derivative we obtain

$$\frac{\partial^2}{\partial t_i \partial t_j} \ln f(t_1, \ldots, t_n) = -\frac{a-1}{\Big(\sum_{i=1}^{n} t_i\Big)^2} ,$$

which is non-negative for $0 < a \leq 1$.

Thus, we can conclude that $Y_1, \ldots, Y_n$ are MTP$_2$ iff $0 < a \leq 1$. In Gupta & Richards (1987) a generalization of this result for arbitrary Liouville distribution is derived.

3.2.2 Negative dependence properties

As mentioned before negative dependence reflects the behavior whereby two subsets of random variables are "repelling" each other. As the following definition shows, most of the negative dependence concepts represent a negative analogy of the positive dependence notions (see Karlin & Rinott (1980b), Müller & Stoyan (2002)).

Def 3.26. *For $n \in \mathbb{N}$ let $X = (X_1, \dots, X_n)$ be a $\mathbb{R}^n$ valued random vector. Then X is said to be*

(i) negatively associated if for every set $I \subset \{1, \dots, n\}$ and $J = \{1, \dots, n\} \setminus I$,

$$cov\big[f(X_i, i \in I), g(X_i, i \in J)\big] \leq 0 \ ,$$

that is,

$$E\big[f(X_i, i \in I) g(X_i, i \in J)\big] \leq E\big[f(X_i, i \in I)\big] E\big[g(X_i, i \in J)\big] \ ,$$

for all non-decreasing functions $f : \mathbb{R}^{|I|} \to \mathbb{R}$ and $g : \mathbb{R}^{n-|I|} \to \mathbb{R}$ such that the covariance exists.

(ii) conditionally decreasing in sequence (denoted CDS) if, for $i = 2, 3, \dots, n$, it satisfies

$$\begin{aligned} &P(X_i > t \mid X_{i-1} = x'_{i-1}, \dots, X_1 = x'_1) \\ &\qquad \leq P(X_i > t \mid X_{i-1} = x_{i-1}, \dots, X_1 = x_1) \ , \end{aligned}$$

whenever $t \in \mathbb{R}$, $x_j \leq x'_j$, $j = 1, 2, \dots, i-1$.

(iii) multivariate reverse rule of order two (denoted MRR_2) if X is absolutely continuous and its density f is a MRR_2 function, i.e.,

$$f(x) f(y) \geq f(x \vee y) f(x \wedge y)$$

for all $x, y \in \mathbb{R}^n$. It is called reversed rule of order two (denoted RR_2) if $n = 2$ and X is MRR_2.

Remark 3.27. *In Pemantle (2000) the authors analyze the development of negative dependence theory. In particular, it is pointed out that negative dependence does not receive sufficient attention in comparison to the theory of positive dependence. One of the reasons for this difference in development lies in the fact that negative dependence does not necessarily imply negative correlation.*

Example 3.28. *Consider continuous random variables $Y_1, \dots, Y_n$ following a Dirichlet distribution. Then, according to Gupta & Richards (1987), their joint density is described* *by*

$$f^{Y_1,\ldots,Y_n}(t_1,\ldots,t_n) = c\Big(1-\sum_{i=1}^n t_i\Big)^{a-1}\prod_{i=1}^n t_i^{a_i-1}\ ,$$

where $c>0$ *is a normalization constant and* $t_i \geq 0$, $\sum_{i=1}^n t_i < 1$ *and* $a, a_i > 0$ *for* $i = 1,\ldots,n$. *It belongs to the family of Liouville distributions of the second kind.*

The MRR_2 *property of* $Y_1,\ldots,Y_n$ *is equivalent to* MRR_2 *of*

$$f(t_1,\ldots,t_n) = \Big(1-\sum_{i=1}^n t_i\Big)^{a-1}\ ,$$

which can be reformulated as MTP_2 *of* $1/f$. *Applying Lemma 3.12 it is left to consider the sign of the partial derivative*

$$\frac{\partial^2}{\partial t_i \partial t_j}\ln\frac{1}{f(t_1,\ldots,t_n)} = (a-1)\frac{1}{\Big(1-\sum_{i=1}^n t_i\Big)^2}\ .$$

It is non-negative for $a \geq 1$.

Thus, we can conclude that $Y_1,\ldots,Y_n$ *are* MRR_2 *iff* $a \geq 1$.

Remark 3.29. *Since many of negative dependence properties are defined with reversed inequalities of positive dependence, part of the observations that hold for positive dependence have their analogies for negative dependence. Thus, the results described by Theorem 3.14 except for association, Lemma 3.15, Remark 3.19, Lemma 3.20, Lemma 3.21 hold also for the corresponding negative dependence properties. However, Theorem 3.17 and Theorem 3.18 are not valid. In this respect, in Karlin & Rinott (1980b) strongly* MRR_2 *(S-*MRR_2*) random variables are defined. The key property of S-*MRR_2 *random variables is that there densities remain* MRR_2 *under marginalization.*

The concepts of MTP_2 and MRR_2 can be generalized further according to theory of total positivity described in Karlin (1968). In particular, the book provides the following connection between TP_2 and RR_2 random variables:

Lemma 3.30. *Consider function* $f, g, h : \mathbb{R}^2 \to \mathbb{R}_+$ *such that*

$$f(a,b) = \int_C g(a,c)h(c,b)dc\ ,$$

where $a \in A, b \in B,\ A, B, C \subset \mathbb{R}$.

(i) *If* $g(a,c)$ *is* TP_2 (RR_2) *and* $h(c,b)$ *is* TP_2 (RR_2), *then* $f(a,b)$ *is* TP_2 *for* $a \in A, b \in B$.

(ii) *If* $g(a,c)$ *is* TP_2 *and* $h(c,b)$ *is* RR_2, *then* $f(a,b)$ *is* RR_2 *for* $a \in A, b \in B$.

The last theorem also illustrates that integration of a MRR_2 function can lead to a MTP_2 result. This observation represents one of the differences between MTP_2 and MRR_2.

3.3 Special cases of dependence

In this section we will survey the conditions securing different dependence properties for the types of random vectors that were already considered in Chapter 2. Statements derived below will provide the basis for the analysis concerning dependence properties of sequential order statistics in Chapter 4.

3.3.1 HIF and SL

Remark 3.31. *Consider absolutely continuous ordered random variables* $Z_1, \dots, Z_n$. *Following the reasoning of Remark 3.6 we can simplify the definitions of such dependence properties as SL and HIF. Namely,* $Z_1, \dots, Z_n$ *are*

(i) *HIF, if holds*

$$\lambda(t|x_1, \dots, x_{i-1}) \leq \lambda(t|y_1, \dots, y_{i-1}) \tag{3.23}$$

(ii) *SL, if holds*

$$\int_{x_{i-1}}^{t} \lambda(u|x_1, \dots, x_{i-1})du \leq \int_{y_{i-1}}^{t} \lambda(u|y_1, \dots, y_{i-1})du\ , \tag{3.24}$$

for $y_k \leq x_k < t,\ k = 1, \dots, i-1$ *and* $i = 1, \dots, n$. *Here* $i = 1$ *corresponds to the hazard rate with no failure history, i.e.* $\lambda(t)$.

Let us look at the connection between the HIF property of exchangeable random variables and their order statistics.

Lemma 3.32. *Consider absolutely continuous exchangeable random variables* $Y_1, \dots, Y_n$ *and their order statistics* $Y_{1:n}, \dots, Y_{n:n}$. *If* $Y_1, \dots, Y_n$ *possess the HIF property, then so do* $Y_{1:n}, \dots, Y_{n:n}$.

Proof. On one hand, according to the Remark 3.13 for $Y_1, \ldots, Y_n$ to be HIF their hazard rates should satisfy the inequalities

$$\lambda(t|x_1, \ldots, x_{i-1}) \leq \lambda(t|y_1, \ldots, y_{j-1}) \;, \tag{3.25}$$

where $i, j \in \{1, \ldots, n\}$, $i \leq j$ and $y_k \leq x_k < t$, $y_s < t$ for $k = 1, \ldots, i-1$, $s = i$, $\ldots, j-1$. As before $i = 1$ corresponds to $\lambda_1(t)$. On the other hand, due to Remark 3.31 for order statistics to be HIF should hold

$$\lambda_*(t|x_1, \ldots, x_{i-1}) \leq \lambda_*(t|y_1, \ldots, y_{i-1}) \tag{3.26}$$

for $i = 1, \ldots, n$, where by analogy to Lemma 2.8

$$\begin{aligned} \lambda_*(t|x_1, \ldots, x_{i-1}) &= \lim_{\delta \to 0+} \frac{1}{\delta} P\big(Y_{i:n} \leq t + \delta \,|\, Y_{1:n} = x_1, \ldots, Y_{i-1:n} = x_{i-1}, Y_{i:n} > t\big) \\ &= (n - i + 1)\lambda_i(t|x_1, \ldots, x_{i-1}) \;. \end{aligned}$$

Then (3.26) turns into

$$(n - i + 1)\lambda(t|x_1, \ldots, x_{i-1}) \leq (n - i + 1)\lambda(t|y_1, \ldots, y_{i-1}) \;.$$

Note that (3.25) implies (3.26). Thus from the HIF property of $Y_1, \ldots, Y_n$ follows the HIF property of order statistics $Y_{1:n}, \ldots, Y_{n:n}$. □

In Example 3.23 we saw that sequential order statistics based on $F_1, \ldots, F_n$ are MTP_2 and consequently HIF. The following lemma provides the condition for HIF of exchangeable random variables which order statistics coincide with sequential order statistics as in Lemma 2.16. It also demonstrates that in general HIF of order statistics is weaker then the HIF property of the underlying exchangeable random variables.

Lemma 3.33. *For the vector X^* as in Lemma 2.16, X^* is HIF iff*

$$\frac{f_i(t)}{\overline{F}_i(t)} \leq \frac{f_{i+1}(t)}{\overline{F}_{i+1}(t)}$$

for $t > 0$, $i = 1, \ldots, n-1$.

Proof. Taking into account Remark 2.18 conditional hazard rates of X^* have the form

$$\lambda(t|t_1,\ldots,t_h) = \frac{f_{h+1}(t)}{\overline{F}_{h+1}(t)} ,$$

where $f_{h+1}(t)$ is the density corresponding to the survival function $\overline{F}_{h+1}(t)$, $h = 1,\ldots,n-1$ and $0 \le t_1 \le \ldots \le t_h < t$. Taking into account observations from Remark 3.13 we obtain the proof. □

Lemma 3.34. *Absolutely continuous, increasingly ordered random variables $Z_1,\ldots,Z_n$ have supportive lifetimes iff their conditional survival functions*

$$P(Z_i > t \,|\, Z_1 = t_1,\ldots,Z_{i-1} = t_{i-1})$$

are increasing in $t_1,\ldots,t_{i-1} \in \mathbb{R}$ i.e.

$$P(Z_i > t \,|\, Z_1 = x_1,\ldots,Z_{i-1} = x_{i-1}) \le P(Z_i > t \,|\, Z_1 = y_1,\ldots,Z_{i-1} = y_{i-1}) \tag{3.27}$$

for all $x_j, y_j \in \mathbb{R}$ such that $x_k \le x_{k+1}$, $y_k \le y_{k+1}$, $y_{i-1} \le t$, $x_j \le y_j$ for $j = 1,\ldots, i-1$, $k = 1,\ldots,i-2$, $i = 2,\ldots,n$.

Moreover, for such $Z_1,\ldots,Z_n$ SL is equivalent to CIS.

Proof. According to Remark 3.31 (ii) the definition of the SL property for ordered random variables takes form

$$\int_{x_{i-1}}^{t} \lambda(u|x_1,\ldots,x_{i-1})du \ge \int_{y_{i-1}}^{t} \lambda(u|y_1,\ldots,y_{i-1})du \; . \tag{3.28}$$

By Lemma 2.7 conditional hazard rates of $Z_1,\ldots,Z_n$ can be expressed as

$$\lambda(t \,|\, t_1,\ldots,t_{i-1}) = -\frac{\partial}{\partial t} \ln P(Z_i > t \,|\, Z_1 = t_1,\ldots,Z_{i-1} = t_{i-1})$$

almost surely. Then (3.28) turns into

$$\begin{aligned} &-\ln P(Z_i > t \,|\, Z_1 = x_1,\ldots,Z_{i-1} = x_{i-1}) \\ &\quad + \ln P(Z_i > x_{i-1} \,|\, Z_1 = x_1,\ldots,Z_{i-1} = x_{i-1}) \end{aligned}$$

$$\geq -\ln P(Z_i > t \,|\, Z_1 = y_1, \ldots, Z_{i-1} = y_{i-1}) \\ + \ln P(Z_i > y_{i-1} \,|\, Z_1 = y_1, \ldots, Z_{i-1} = y_{i-1}) \; .$$

Note that for ordered random variables

$$P(Z_i > a \,|\, Z_1 = t_1, \ldots, Z_{i-2} = t_{i-2}, Z_{i-1} = a) = 1 \; ,$$

where $0 \leq t_1 \leq \ldots \leq t_{i-2} \leq a$. Then we obtain

$$\ln P(Z_i > t \,|\, Z_1 = x_1, \ldots, Z_{i-1} \\ = x_{i-1}) \leq \ln P(Z_i > t \,|\, Z_1 = y_1, \ldots, Z_{i-1} = y_{i-1}) \; .$$

Finally we can state that

$$P(Z_i > t \,|\, Z_1 = x_1, \ldots, Z_{i-1} = x_{i-1}) \leq P(Z_i > t \,|\, Z_1 = y_1, \ldots, Z_{i-1} = y_{i-1}) \; ,$$

where $x_j, y_j \in \mathbb{R}_+, \; x_j \leq y_j, j = 1, \ldots, i-1, \; i = 2, \ldots, n$.

Comparing the inequality (3.27) with (3.13), the conclusion can be made that for ordered random variables the SL property is equivalent to CIS. □

3.3.2 MTP$_2$

3.3.2.1 Distributions with Markov order statistics

First consider random variables $Y_1, \ldots, Y_n$ with conditional hazard rates of the form

$$\lambda(t|t_1, \ldots, t_h) = r_h(t) \; ,$$

where $h = 0, \ldots, n-1, \; t_1 \leq \ldots \leq t_h < t, \; n \in \mathbb{N}$. According to Remark 2.12 for $t_1, \ldots, t_n \in \mathbb{R}_+$ the joint density of $Y_1, \ldots, Y_n$ can be described by

$$f^{Y_1,\ldots,Y_n}(t_1, \ldots, t_n) = \prod_{i=1}^{n} r_{i-1}(t_{(i)}) \, \exp\Big(-(n-i+1) \int_{t_{(i-1)}}^{t_{(i)}} r_{i-1}(u) du\Big) \; ,$$

where $(t_{(1)}, \ldots, t_{(n)})$ is the permutation of $(t_1, \ldots, t_n)$ such that $t_{(1)} \leq \ldots \leq t_{(n)}$. To emphasize the specific form of the density we will use the notation

$$f^{Y_1,\ldots,Y_n}(t_1, \ldots, t_n) = \prod_{i=1}^{n} g_i(t_{(i)}) \ , \tag{3.29}$$

where

$$\begin{aligned} g_i(t_{(i)}) &= r_{i-1}(t_{(i)}) \exp\big(R_{i-1}(t_{(i)})\big) \\ R_{i-1}(t) &= -(n-i+1)\int_0^t r_{i-1}(u)du + (n-i)\int_0^t r_i(u)du \ . \end{aligned}$$

The following lemma provides a necessary and sufficient condition for the MTP_2 property of random vectors with joint density as in (3.29).

Lemma 3.35. *Consider random variables $Y_1, \ldots, Y_n$ with joint density function*

$$f^{Y_1,\ldots,Y_n}(t_1, \ldots, t_n) = \prod_{i=1}^{n} g_i(t_{(i)})$$

defined on $\mathbb{R}^n$, where for $i = 1, 2, \ldots, n$ $g_i(\cdot)$ are some univariate continuous functions. Moreover $g_i(t) \neq 0$ for $t \in \mathbb{R}$.

Then $Y_1, \ldots, Y_n$ possess the MTP_2 property iff

$$\frac{g_i(t)}{g_{i-1}(t)} \leq \frac{g_i(\tilde{t})}{g_{i-1}(\tilde{t})} \tag{3.30}$$

for $t, \tilde{t} \in \mathbb{R}$, $\tilde{t} < t$, $i = 2, \ldots, n$.

Proof. First we are going to prove the sufficient condition. For the sake of readability we will use the notation

$$\begin{aligned} f(t) &= f(t_1, \ldots, t_n) \\ &= f^{Y_1,\ldots,Y_n}(t_1, \ldots, t_n) \ , \end{aligned}$$

where $t = (t_1, \dots, t_n)$. Note that under the assumptions of the lemma

$$f(x)f(y) = \prod_{i=1}^{n} g_i(x_{(i)}) g_i(y_{(i)}) \neq 0$$

for all $x, y \in \mathbb{R}^n$. Consequently, f satisfies the requirements of Lemma 3.20 and it suffices to prove that $f(t_1, \dots, t_n)$ is TP_2 in every pair of variables. Due to the symmetry of f it is enough to show the TP_2 property with respect to t_1, t_2. To do so, let us consider inequality (3.14) for

$$\begin{aligned} x &= (x_1, x_2, t_1, \dots, t_{n-2}) \\ y &= (y_1, y_2, t_1, \dots, t_{n-2}) \ , \end{aligned} \tag{3.31}$$

where $x_i, y_i, t_j \in \mathbb{R}, x_i \neq y_k \neq t_j$ for $i, k = 1, 2,\ j = 1, \dots, n-2$. TP_2 is then extended by continuity of f to all $x_i, y_i, t_j \in \mathbb{R},\ i = 1, 2,\ j = 1, \dots, n-2$.

In the following we will assume that x_1 and x_2 do not belong simultaneously to $x \wedge y$ or $x \vee y$. Otherwise (3.14) turns into $1 \leq 1$. Due to the symmetry of f, it can also be assumed without the loss of generality that $x_1 < \min(y_1, y_2)$ and $x_2 > y_2$. All the other cases are obtained by renaming the variables. Thus, there are three possible arrangements to consider:

1) $x_1 < y_1 < y_2 < x_2$,

2) $x_1 < y_2 < y_1 < x_2$,

3) $x_1 < y_2 < x_2 < y_1$.

In the succeeding analysis it will be convenient to use a shorter notation. Denote by x_i the i-th element of a vector x (for example from (3.31) $x_3 = t_1$), by $(x_{(1)}, \dots, x_{(n)})$ – the vector of ordered elements of x, $\pi_x(i)$ – the position that x_i takes in $(x_{(1)}, \dots, x_{(n)})$ (for example if $x_{(k)} = x_l$, then $\pi_x(l) = k$).

Then, for f to be TP_2 in the first two variables, it should hold

$$f(x)f(y) \leq f(x \wedge y) f(x \vee y) \ ,$$

for x, y as in (3.31). Taking into account the form of the function f, this inequality can be rewritten as

$$
\begin{aligned}
& g_{\pi_x(1)}(x_1) \times \ldots \times g_{\pi_x(n)}(x_n) \\
& \qquad \times g_{\pi_y(1)}(y_1) \times \ldots \times g_{\pi_y(n)}(y_n) \\
& \leq g_{\pi_{x\wedge y}(1)}(x_1 \wedge y_1) \times \ldots \times g_{\pi_{x\wedge y}(n)}(x_n \wedge y_n) \\
& \qquad \times g_{\pi_{x\vee y}(1)}(x_1 \vee y_1) \times \ldots \times g_{\pi_{x\vee y}(n)}(x_n \vee y_n) \; .
\end{aligned}
\tag{3.32}
$$

To prove (3.32) we are going to look at cases 1) to 3) separately.

1) Consider $x_1 < y_1 < y_2 < x_2$, in other words $\pi_{x\vee y}(1) = \pi_y(1)$, $\pi_{x\wedge y}(1) = \pi_x(1)$ and $\pi_{x\vee y}(2) = \pi_x(2)$, $\pi_{x\wedge y}(2) = \pi_y(2)$. Then inequality (3.32) becomes

$$
\begin{aligned}
& \left[\prod_{j=1}^{\pi_x(1)-1} g_j(t_{(j)}) \cdot g_{\pi_x(1)}(x_1) \prod_{j=\pi_x(1)+1}^{\pi_x(2)-1} g_j(t_{(j-1)}) \cdot g_{\pi_x(2)}(x_2) \prod_{j=\pi_x(2)+1}^{n} g_j(t_{(j-2)}) \right] \\
& \times \left[\prod_{j=1}^{\pi_y(1)-1} g_j(t_{(j)}) \cdot g_{\pi_y(1)}(y_1) \prod_{j=\pi_y(1)+1}^{\pi_y(2)-1} g_j(t_{(j-1)}) \cdot g_{\pi_y(2)}(y_2) \prod_{j=\pi_y(2)+1}^{n} g_j(t_{(j-2)}) \right] \\
& \leq \left[\prod_{j=1}^{\pi_x(1)-1} g_j(t_{(j)}) \cdot g_{\pi_x(1)}(x_1) \prod_{j=\pi_x(1)+1}^{\pi_y(2)-1} g_j(t_{(j-1)}) \cdot g_{\pi_y(2)}(y_2) \prod_{j=\pi_y(2)+1}^{n} g_j(t_{(j-2)}) \right] \\
& \times \left[\prod_{j=1}^{\pi_y(1)-1} g_j(t_{(j)}) \cdot g_{\pi_y(1)}(y_1) \prod_{j=\pi_y(1)+1}^{\pi_x(2)-1} g_j(t_{(j-1)}) \cdot g_{\pi_x(2)}(x_2) \prod_{j=\pi_x(2)+1}^{n} g_j(t_{(j-2)}) \right].
\end{aligned}
$$

Reducing both parts of the inequality by coinciding factors, we obtain

$$
\begin{aligned}
& \prod_{j=\pi_x(1)+1}^{\pi_x(2)-1} g_j(t_{(j-1)}) \cdot \prod_{j=\pi_y(1)+1}^{\pi_y(2)-1} g_j(t_{(j-1)}) \\
& \qquad \leq \prod_{j=\pi_x(1)+1}^{\pi_y(2)-1} g_j(t_{(j-1)}) \cdot \prod_{j=\pi_y(1)+1}^{\pi_x(2)-1} g_j(t_{(j-1)}) \; .
\end{aligned}
\tag{3.33}
$$

Since $\pi_y(2) \leq \pi_x(2)$, the analysis of the last inequality falls into two subcases.

1.1) $\pi_y(2) = \pi_x(2)$ and (3.33) is immediately satisfied.

1.2) $\pi_y(2) < \pi_x(2)$ implies the further reduction of (3.33) to

$$\prod_{j=\pi_y(2)}^{\pi_x(2)-1} g_j(t_{(j-1)}) \leq \prod_{j=\pi_y(2)}^{\pi_x(2)-1} g_j(t_{(j-1)}) \ ,$$

which is equivalent to $1 \leq 1$.

2) Consider $x_1 < y_2 < y_1 < x_2$, then $\pi_{x\vee y}(1) = \pi_y(1) - 1$, $\pi_{x\wedge y}(1) = \pi_x(1)$ and $\pi_{x\vee y}(2) = \pi_x(2)$, $\pi_{x\wedge y}(2) = \pi_y(2) + 1$. In this case inequality (3.32) becomes

$$\begin{aligned}
&\left[\prod_{j=1}^{\pi_x(1)-1} g_j(t_{(j)}) \cdot g_{\pi_x(1)}(x_1) \prod_{j=\pi_x(1)+1}^{\pi_x(2)-1} g_j(t_{(j-1)}) \cdot g_{\pi_x(2)}(x_2) \prod_{j=\pi_x(2)+1}^{n} g_j(t_{(j-2)})\right] \\
&\times \left[\prod_{j=1}^{\pi_y(2)-1} g_j(t_{(j)}) \cdot g_{\pi_y(2)}(y_2) \prod_{j=\pi_y(2)+1}^{\pi_y(1)-1} g_j(t_{(j-1)}) \cdot g_{\pi_y(1)}(y_1) \prod_{j=\pi_y(1)+1}^{n} g_j(t_{(j-2)})\right] \\
&\leq \left[\prod_{j=1}^{\pi_x(1)-1} g_j(t_{(j)}) \cdot g_{\pi_x(1)}(x_1) \prod_{j=\pi_x(1)+1}^{\pi_y(2)} g_j(t_{(j-1)}) \cdot g_{\pi_y(2)+1}(y_2) \prod_{j=\pi_y(2)+2}^{n} g_j(t_{(j-2)})\right] \\
&\times \left[\prod_{j=1}^{\pi_y(1)-2} g_j(t_{(j)}) \cdot g_{\pi_y(1)-1}(y_1) \prod_{j=\pi_y(1)}^{\pi_x(2)-1} g_j(t_{(j-1)}) \cdot g_{\pi_x(2)}(x_2) \prod_{j=\pi_x(2)+1}^{n} g_j(t_{(j-2)})\right].
\end{aligned}$$

Reducing both parts of the inequality by coinciding factors, we obtain

$$\begin{aligned}
&\prod_{j=\pi_x(1)+1}^{\pi_x(2)-1} g_j(t_{(j-1)}) \cdot \prod_{j=1}^{\pi_y(2)-1} g_j(t_{(j)}) \cdot g_{\pi_y(2)}(y_2) \\
&\times \prod_{j=\pi_y(2)+1}^{\pi_y(1)-1} g_j(t_{(j-1)}) \cdot g_{\pi_y(1)}(y_1) \cdot \prod_{j=\pi_y(1)+1}^{n} g_j(t_{(j-2)}) \\
&\leq \prod_{j=\pi_x(1)+1}^{\pi_y(2)} g_j(t_{(j-1)}) \cdot g_{\pi_y(2)+1}(y_2) \cdot \prod_{j=\pi_y(2)+2}^{n} g_j(t_{(j-2)}) \\
&\times \prod_{j=1}^{\pi_y(1)-2} g_j(t_{(j)}) \cdot g_{\pi_y(1)-1}(y_1) \cdot \prod_{j=\pi_y(1)}^{\pi_x(2)-1} g_j(t_{(j-1)}) \ .
\end{aligned} \tag{3.34}$$

Given that $x_1 < y_2 < y_1 < x_2$ we can conclude that $\pi_y(2) < \pi_y(1) \leq \pi_x(2)$, in other words

$$\begin{cases} \pi_y(2) \leq \pi_y(1) - 1 \ , \\ \pi_y(2) \leq \pi_x(2) - 1 \ , \\ \pi_y(1) \leq \pi_x(2) \ . \end{cases}$$

2.1) First let us assume that

$$\begin{cases} \pi_y(2) = \pi_y(1) - 1 \ , \\ \pi_y(2) = \pi_x(2) - 1 \ . \end{cases}$$

Then (3.34) takes form

$$\begin{aligned} & g_{\pi_y(2)}(y_2) \cdot g_{\pi_y(1)}(y_1) \leq g_{\pi_y(2)+1}(y_2) \cdot g_{\pi_y(1)-1}(y_1) \\ \Leftrightarrow \ & \frac{g_{\pi_y(1)}(y_1)}{g_{\pi_y(1)-1}(y_1)} \leq \frac{g_{\pi_y(2)+1}(y_2)}{g_{\pi_y(2)}(y_2)} \ . \end{aligned}$$

Since $\pi_y(2) + 1 = \pi_y(1)$, we can reformulate the last inequality as

$$\frac{g_{\pi_y(1)}(y_1)}{g_{\pi_y(1)-1}(y_1)} \leq \frac{g_{\pi_y(1)}(y_2)}{g_{\pi_y(1)-1}(y_2)} \ ,$$

which is fulfilled for all $\frac{g_{\pi_y(1)}(t)}{g_{\pi_y(1)-1}(t)}$ decreasing in t.

2.2) Let us consider

$$\begin{cases} \pi_y(2) = \pi_y(1) - 1 \ , \\ \pi_y(2) < \pi_x(2) - 1 \ . \end{cases}$$

Then (3.34) turns into

$$\prod_{j=\pi_y(2)+1}^{\pi_x(2)-1} g_j(t_{(j-1)}) \cdot g_{\pi_y(2)}(y_2) \cdot g_{\pi_y(1)}(y_1)$$

$$\leq g_{\pi_y(2)+1}(y_2) \cdot g_{\pi_y(1)-1}(y_1) \cdot \prod_{j=\pi_y(1)}^{\pi_x(2)-1} g_j(t_{(j-1)}) ,$$

which is equivalent to

$$g_{\pi_y(2)}(y_2) \cdot g_{\pi_y(1)}(y_1) \leq g_{\pi_y(2)+1}(y_2) \cdot g_{\pi_y(1)-1}(y_1) .$$

Since $\pi_y(2) = \pi_y(1) - 1$ we obtain

$$\frac{g_{\pi_y(1)}(y_1)}{g_{\pi_y(1)-1}(y_1)} \leq \frac{g_{\pi_y(1)}(y_2)}{g_{\pi_y(1)-1}(y_2)} ,$$

which is true under the conditions of the lemma.

2.3) Finally, if we assume

$$\begin{cases} \pi_y(2) < \pi_y(1) - 1 , \\ \pi_y(2) < \pi_x(2) - 1 , \\ \pi_y(1) \leq \pi_x(2) , \end{cases}$$

then (3.34) is equivalent to

$$g_{\pi_y(2)}(y_2) \prod_{j=\pi_y(2)+1}^{\pi_x(2)-1} g_j(t_{(j-1)}) \prod_{j=\pi_y(2)+1}^{\pi_y(1)-1} g_j(t_{(j-1)}) \cdot g_{\pi_y(1)}(y_1)$$
$$\leq g_{\pi_y(2)+1}(y_2) \prod_{j=\pi_y(2)+2}^{\pi_y(1)} g_j(t_{(j-2)}) \prod_{j=\pi_y(2)}^{\pi_y(1)-2} g_j(t_{(j)}) \cdot g_{\pi_y(1)-1}(y_1) \prod_{j=\pi_y(1)}^{\pi_x(2)-1} g_j(t_{(j-1)}) ,$$

which can be rewritten as

$$g_{\pi_y(2)}(y_2) \prod_{j=\pi_y(2)+1}^{\pi_y(1)-1} g_j(t_{(j-1)}) \prod_{j=\pi_y(2)+1}^{\pi_y(1)-1} g_j(t_{(j-1)}) \cdot g_{\pi_y(1)}(y_1)$$

$$\le g_{\pi_y(2)+1}(y_2) \prod_{j=\pi_y(2)+2}^{\pi_y(1)} g_j(t_{(j-2)}) \prod_{j=\pi_y(2)}^{\pi_y(1)-2} g_j(t_{(j)}) \cdot g_{\pi_y(1)-1}(y_1) \ .$$

Let us rearrange the factors in the following way

$$\prod_{j=\pi_y(2)}^{\pi_y(1)-2} \frac{g_{j+1}(t_{(j)})}{g_j(t_{(j)})} \cdot \frac{g_{\pi_y(1)}(y_1)}{g_{\pi_y(1)-1}(y_1)} \le \frac{g_{\pi_y(2)+1}(y_2)}{g_{\pi_y(2)}(y_2)} \cdot \prod_{j=\pi_y(2)}^{\pi_y(1)-2} \frac{g_{j+2}(t_{(j)})}{g_{j+1}(t_{(j)})} \ .$$

Then we can regroup the factors and write

$$\frac{g_{\pi_y(2)+1}(t_{(\pi_y(2))})}{g_{\pi_y(2)}(t_{(\pi_y(2))})} \prod_{j=\pi_y(2)}^{\pi_y(1)-3} \frac{g_{j+2}(t_{(j+1)})}{g_{j+1}(t_{(j+1)})} \cdot \frac{g_{\pi_y(1)}(y_1)}{g_{\pi_y(1)-1}(y_1)}$$
$$\le \frac{g_{\pi_y(2)+1}(y_2)}{g_{\pi_y(2)}(y_2)} \prod_{j=\pi_y(2)}^{\pi_y(1)-3} \frac{g_{j+2}(t_{(j)})}{g_{j+1}(t_{(j)})} \cdot \frac{g_{\pi_y(1)}(t_{(\pi_y(1)-2)})}{g_{\pi_y(1)-1}(t_{(\pi_y(1)-2)})} \ .$$

Since $t_{(\pi_y(2))} > y_2$, $t_{(j+1)} > t_{(j)}$ for $j = 1, \dots, n-3$ and $y_1 > t_{(\pi_y(1)-2)}$, we can conclude that the inequality is valid under the conditions of the lemma.

3) It is left to consider $x_1 < y_2 < x_2 < y_1$, in other words $\pi_{x\wedge y}(1) = \pi_x(1)$, $\pi_{x\vee y}(1) = \pi_y(1)$ and $\pi_{x\wedge y}(2) = \pi_y(2) + 1, \pi_{x\vee y}(2) = \pi_x(2) - 1$. In this case we

need to verify the following inequality

$$\begin{aligned}
&\left[\prod_{j=1}^{\pi_x(1)-1} g_j(t_{(j)}) \cdot g_{\pi_x(1)}(x_1) \prod_{j=\pi_x(1)+1}^{\pi_x(2)-1} g_j(t_{(j-1)}) \cdot g_{\pi_x(2)}(x_2) \prod_{j=\pi_x(2)+1}^{n} g_j(t_{(j-2)})\right] \\
&\times \left[\prod_{j=1}^{\pi_y(2)-1} g_j(t_{(j)}) \cdot g_{\pi_y(2)}(y_2) \prod_{j=\pi_y(2)+1}^{\pi_y(1)-1} g_j(t_{(j-1)}) \cdot g_{\pi_y(1)}(y_1) \prod_{j=\pi_y(1)+1}^{n} g_j(t_{(j-2)})\right] \\
&\leq \left[\prod_{j=1}^{\pi_x(1)-1} g_j(t_{(j)}) \cdot g_{\pi_x(1)}(x_1) \prod_{j=\pi_x(1)+1}^{\pi_y(2)} g_j(t_{(j-1)}) \cdot g_{\pi_y(2)+1}(y_2) \prod_{j=\pi_y(2)+2}^{n} g_j(t_{(j-2)})\right] \\
&\times \left[\prod_{j=1}^{\pi_x(2)-2} g_j(t_{(j)}) \cdot g_{\pi_x(2)-1}(x_2) \prod_{j=\pi_x(2)}^{\pi_y(1)-1} g_j(t_{(j-1)}) \cdot g_{\pi_y(1)}(y_1) \prod_{j=\pi_y(1)+1}^{n} g_j(t_{(j-2)})\right] .
\end{aligned} \tag{3.35}$$

Since $x_1 < y_2 < x_2 < y_1$ one may observe that

$$\begin{cases} \pi_y(2) \leq \pi_y(1) - 1 \ , \\ \pi_y(2) \leq \pi_x(2) - 1 \ , \\ \pi_x(2) \leq \pi_y(1) \ . \end{cases}$$

3.1) Analog to the case 2.1) let us start by considering

$$\begin{cases} \pi_y(2) = \pi_y(1) - 1 \ , \\ \pi_y(2) = \pi_x(2) - 1 \ . \end{cases}$$

Then (3.35) can be reduced to

$$\begin{aligned}
g_{\pi_x(2)}(x_2) \cdot g_{\pi_y(2)}(y_2) &\leq g_{\pi_y(2)+1}(y_2) \cdot g_{\pi_x(2)-1}(x_2) \\
\Leftrightarrow \frac{g_{\pi_x(2)}(x_2)}{g_{\pi_x(2)-1}(x_2)} &\leq \frac{g_{\pi_y(2)+1}(y_2)}{g_{\pi_y(2)}(y_2)} \ .
\end{aligned} \tag{3.36}$$

Since $\pi_y(2) + 1 = \pi_x(2)$, we obtain

$$\frac{g_{\pi_x(2)}(x_2)}{g_{\pi_x(2)-1}(x_2)} \leq \frac{g_{\pi_x(2)}(y_2)}{g_{\pi_x(2)-1}(y_2)} \ ,$$

which is satisfied if all the $\frac{g_i(t)}{g_{i-1}(t)}$ are decreasing functions.

3.2) Assume now

$$\begin{cases} \pi_y(2) = \pi_x(2) - 1 \ , \\ \pi_y(2) < \pi_y(1) - 1 \ . \end{cases}$$

Then (3.35) reduces to (3.36), i.e.

$$\begin{aligned} & g_{\pi_x(2)}(x_2) \cdot g_{\pi_y(2)}(y_2) \leq g_{\pi_y(2)+1}(y_2) \cdot g_{\pi_x(2)-1}(x_2) \\ \Leftrightarrow \ & \frac{g_{\pi_x(2)}(x_2)}{g_{\pi_x(2)-1}(x_2)} \leq \frac{g_{\pi_y(2)+1}(y_2)}{g_{\pi_y(2)}(y_2)} \ . \end{aligned}$$

3.3) It is left to consider

$$\begin{cases} \pi_y(2) < \pi_x(2) - 1 \ , \\ \pi_y(2) < \pi_y(1) - 1 \ , \\ \pi_x(2) \leq \pi_y(1) \ . \end{cases}$$

Reducing both parts of the inequality (3.35) by coinciding factors, we obtain

$$
\begin{aligned}
&\prod_{j=\pi_x(1)+1}^{\pi_x(2)-1} g_j(t_{(j-1)}) \cdot g_{\pi_x(2)}(x_2) \prod_{j=\pi_x(2)+1}^{n} g_j(t_{(j-2)}) \\
&\times \prod_{j=1}^{\pi_y(2)-1} g_j(t_{(j)}) \cdot g_{\pi_y(2)}(y_2) \prod_{j=\pi_y(2)+1}^{\pi_y(1)-1} g_j(t_{(j-1)}) \\
&\leq \prod_{j=\pi_x(1)+1}^{\pi_y(2)} g_j(t_{(j-1)}) \cdot g_{\pi_y(2)+1}(y_2) \prod_{j=\pi_y(2)+2}^{n} g_j(t_{(j-2)}) \\
&\times \prod_{j=1}^{\pi_x(2)-2} g_j(t_{(j)}) \cdot g_{\pi_x(2)-1}(x_2) \prod_{j=\pi_x(2)}^{\pi_y(1)-1} g_j(t_{(j-1)}) \ .
\end{aligned}
\tag{3.37}
$$

Since $\pi_y(2) < \pi_x(2) - 1$, (3.37) is equivalent to

$$
\begin{aligned}
&\prod_{j=\pi_y(2)+1}^{\pi_x(2)-1} g_j(t_{(j-1)}) \cdot g_{\pi_x(2)}(x_2) \cdot g_{\pi_y(2)}(y_2) \prod_{j=\pi_y(2)+1}^{\pi_x(2)-1} g_j(t_{(j-1)}) \\
&\quad \leq g_{\pi_y(2)+1}(y_2) \prod_{j=\pi_y(2)+2}^{\pi_x(2)} g_j(t_{(j-2)}) \prod_{j=\pi_y(2)}^{\pi_x(2)-2} g_j(t_{(j)}) \cdot g_{\pi_x(2)-1}(x_2)
\end{aligned}
$$

and this is again equivalent to

$$
\begin{aligned}
&\prod_{j=\pi_y(2)}^{\pi_x(2)-2} g_{j+1}(t_{(j)}) \cdot g_{\pi_x(2)}(x_2) \cdot g_{\pi_y(2)}(y_2) \prod_{j=\pi_y(2)+1}^{\pi_x(2)-1} g_j(t_{(j-1)}) \\
&\quad \leq g_{\pi_y(2)+1}(y_2) \prod_{j=\pi_y(2)+1}^{\pi_x(2)-1} g_{j+1}(t_{(j-1)}) \prod_{j=\pi_y(2)}^{\pi_x(2)-2} g_j(t_{(j)}) \cdot g_{\pi_x(2)-1}(x_2) \ .
\end{aligned}
$$

Let us rearrange the factors in the following way

$$\prod_{j=\pi_y(2)}^{\pi_x(2)-2} \frac{g_{j+1}(t_{(j)})}{g_j(t_{(j)})} \cdot \frac{g_{\pi_x(2)}(x_2)}{g_{\pi_x(2)-1}(x_2)} \leq \frac{g_{\pi_y(2)+1}(y_2)}{g_{\pi_y(2)}(y_2)} \cdot \prod_{j=\pi_y(2)+1}^{\pi_x(2)-1} \frac{g_{j+1}(t_{(j-1)})}{g_j(t_{(j-1)})} .$$

Finally, regrouping the factors again we observe

$$\frac{g_{\pi_y(2)+1}(t_{(\pi_y(2))})}{g_{\pi_y(2)}(t_{(\pi_y(2))})} \cdot \prod_{j=\pi_y(2)+1}^{\pi_x(2)-2} \frac{g_{j+1}(t_{(j)})}{g_j(t_{(j)})} \cdot \frac{g_{\pi_x(2)}(x_2)}{g_{\pi_x(2)-1}(x_2)}$$
$$\leq \frac{g_{\pi_y(2)+1}(y_2)}{g_{\pi_y(2)}(y_2)} \cdot \prod_{j=\pi_y(2)+1}^{\pi_x(2)-2} \frac{g_{j+1}(t_{(j-1)})}{g_j(t_{(j-1)})} \cdot \frac{g_{\pi_x(2)}(t_{(\pi_x(2)-2)})}{g_{\pi_x(2)-1}(t_{(\pi_x(2)-2)})} .$$

Since $t_{(\pi_y(2))} > y_2$, $t_{(j)} > t_{(j-1)}$ for $j = 2, \ldots, n-2$ and $x_2 > t_{(\pi_x(2)-2)}$, we can conclude that the inequality holds under the conditions of the lemma.

Thus, the sufficient condition is verified and it is left to show that (3.30) represents a necessary condition.

Assume that the joint density $f(t_1, \ldots, t_n)$ is MTP_2 and consider the inequality (3.34) with x, y as described in 2.1). In this case (3.34) turns into

$$\frac{g_{\pi_y(1)}(y_1)}{g_{\pi_y(1)-1}(y_1)} \leq \frac{g_{\pi_y(1)}(y_2)}{g_{\pi_y(1)-1}(y_2)} .$$

Considering one by one vectors y with $\pi_y(1) = i$, $i = 2, \ldots, n$ we obtain a set of necessary conditions

$$\frac{g_i(y_1)}{g_{i-1}(y_1)} \leq \frac{g_i(y_2)}{g_{i-1}(y_2)} ,$$

for $i = 2, \ldots, n,\ y_1 > y_2$, as was to be proved. □

Theorem 3.36. *Consider a vector X^* as in Lemma 2.16 with continuous densities f_i, $i = 1, \ldots, n$ on $(0, \infty)$. X^* is MTP_2 iff for $0 < x < y$, $i = 2, \ldots, n$ holds*

$$\frac{g_i(y)}{g_{i-1}(y)} \leq \frac{g_i(x)}{g_{i-1}(x)} , \tag{3.38}$$

where $g_i(t) = \left(\frac{1-F_i(t)}{1-F_{i+1}(t)}\right)^{n-i} f_i(t) > 0\,, t > 0, i = 1, \ldots, n.$

Proof. Note that according to (2.19) the joint density of X^* can be represented as

$$f(t_1, \ldots, t_n) = \prod_{i=1}^{n} g_i(t_{(i)})\ ,$$

where $g_i(t) = \left(\frac{1-F_i(t)}{1-F_{i+1}(t)}\right)^{n-i} f_i(t),\ i = 1, \ldots, n.$

Then the result follows from Lemma 3.35. □

Theorem 3.37. *Consider a vector X^* as in Lemma 2.16 with continuous densities $f_i,\ i = 1, \ldots, n$ on $(0, \infty)$. X^* is MRR_2 iff for $0 < x < y,\ i = 2, \ldots, n$ holds*

$$\frac{g_i(y)}{g_{i-1}(y)} \geq \frac{g_i(x)}{g_{i-1}(x)}\,, \tag{3.39}$$

where $g_i(t) = \left(\frac{1-F_i(t)}{1-F_{i+1}(t)}\right)^{n-i} f_i(t) > 0, t > 0,\ i = 1, \ldots, n.$

Proof. The result is obtained analogously to the proof of Theorem 3.36. □

Example 3.38. *Consider random variables $Y_1, \ldots, Y_n$ distributed according to Freund's multivariate exponential distribution (FME) from Example 2.19. Recall that the joint density function of FME distributed random variables can be written as*

$$f^{Y_1,\ldots,Y_n}(t_1, \ldots, t_n) = \prod_{i=0}^{n-1} \frac{1}{\theta_i} \exp\left[\frac{-(n-i)\big(t_{(i+1)} - t_{(i)}\big)}{\theta_i}\right]\,,$$

with $\theta_i \geq 0, i = 0, \ldots, n-1,\ t_{(0)} = 0.$

(i) Let us look at the conditions (3.38). First we will rewrite the joint density as

$$f^{Y_1,\ldots,Y_n}(t_1, \ldots, t_n) = \prod_{i=1}^{n} \frac{1}{\theta_{i-1}} \exp\left[t_{(i)} \left(\frac{n-i}{\theta_i} - \frac{n-i+1}{\theta_{i-1}}\right)\right]\,,$$

where $\theta_n = 1$. Then, in terms of Theorem 3.36 we observe that

$$g_i(t) = \frac{1}{\theta_{i-1}} \exp\left[t\left(\frac{n-i}{\theta_i} - \frac{n-i+1}{\theta_{i-1}}\right)\right]$$

and (3.38) becomes

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{g_i(t)}{g_{i-1}(t)} &= \frac{\theta_{i-2}}{\theta_{i-1}}\frac{\partial}{\partial t}\exp\left[t\left(\frac{n-i}{\theta_i} - 2\frac{n-i+1}{\theta_{i-1}} + \frac{n-i+2}{\theta_{i-2}}\right)\right] \\
&= \frac{\theta_{i-2}}{\theta_{i-1}}\left(\frac{n-i}{\theta_i} - 2\frac{n-i+1}{\theta_{i-1}} + \frac{n-i+2}{\theta_{i-2}}\right) \\
&\quad \times \exp\left[t\left(\frac{n-i}{\theta_i} - 2\,\frac{n-i+1}{\theta_{i-1}} + \frac{n-i+2}{\theta_{i-2}}\right)\right] \le 0\ ,
\end{aligned}$$

which for $i = 2, \ldots, n$ is equivalent to

$$\frac{n-i}{\theta_i} - 2\,\frac{n-i+1}{\theta_{i-1}} + \frac{n-i+2}{\theta_{i-2}} \le 0\ . \qquad (3.40)$$

Thus, we have obtained a system of $n-1$ inequalities of the form (3.40). Next we will show by induction the equivalence of the system (3.40) to the set of conditions

$$\theta_i \le \theta_{i-1}, \quad i = 1, \ldots, n-1\ .$$

First, consider the inequality from (3.40) for $i = n$:

$$\begin{aligned}
&-2\frac{1}{\theta_{n-1}} + \frac{2}{\theta_{n-2}} \le 0 \\
&\Leftrightarrow \frac{1}{\theta_{n-2}} \le \frac{1}{\theta_{n-1}} \\
&\Leftrightarrow \theta_{n-1} \le \theta_{n-2}\ .
\end{aligned}$$

In this case the hypothesis holds.

Assume that for $i = j + 1$ the inequality

$$\frac{n-j-1}{\theta_{j+1}} - 2\,\frac{n-j}{\theta_j} + \frac{n-j+1}{\theta_{j-1}} \leq 0$$

can be replaced by

$$\theta_j \leq \theta_{j-1}\ . \tag{3.41}$$

It remains to prove for $i = j$ the equivalence of

$$\frac{n-j}{\theta_j} - 2\,\frac{n-j+1}{\theta_{j-1}} + \frac{n-j+2}{\theta_{j-2}} \leq 0$$

to

$$\theta_{j-1} \leq \theta_{j-2}\ .$$

Rewrite (3.41) as

$$\frac{1}{\theta_{j-1}} - \frac{1}{\theta_j} \leq 0$$

and consider the system

$$\begin{cases} \frac{1}{\theta_{j-1}} - \frac{1}{\theta_j} \leq 0 \\ \frac{n-j}{\theta_j} - 2\frac{n-j+1}{\theta_{j-1}} + \frac{n-j+2}{\theta_{j-2}} \leq 0\ . \end{cases}$$

Multiplying the first inequality by $(n-j)$ and adding these inequalities together we obtain

$$\begin{cases} \frac{1}{\theta_{j-1}} - \frac{1}{\theta_j} \leq 0 \\ \frac{n-j+2}{\theta_{j-2}} - \frac{n-j+2}{\theta_{j-1}} \leq 0 \end{cases} \Leftrightarrow \begin{cases} \frac{1}{\theta_{j-1}} - \frac{1}{\theta_j} \leq 0 \\ \frac{1}{\theta_{j-2}} - \frac{1}{\theta_{j-1}} \leq 0 \end{cases}$$
$$\Leftrightarrow \begin{cases} \theta_j \leq \theta_{j-1} \\ \theta_{j-1} \leq \theta_{j-2}\ , \end{cases}$$

which is what we wanted to show.

Therefore, the system of inequalities (3.40) for $i = 2, \ldots, n$ *can be replaced by*

$$\theta_i \leq \theta_{i-1}, \quad i = 1, \ldots, n-1. \tag{3.42}$$

Thus, for $\theta_i \leq \theta_{i-1}$ *the resulting* $Y_1, \ldots, Y_n$ *will be* MTP_2*.*

For $\theta_i > \theta_{i-1}$ *their order statistics* $Y_{1:n}, \ldots, Y_{n:n}$ *are still* MTP_2 *according to Theorem 3.15. However the corresponding* $Y_1, \ldots, Y_n$ *are* MRR_2 *since conditions* $\theta_i > \theta_{i-1}$ *for* $i = 1, \ldots, n-1$ *correspond to (3.39).*

(ii) Let us revert the inequality sign in (3.42)

$$\theta_i \geq \theta_{i-1}, \quad i = 1, \ldots, n-1, \tag{3.43}$$

and without the loss of generality consider the covariance of X_1 *and* X_2

$$Cov(X_1, X_2) = \frac{1}{n^2(n-1)} \left(\sum_{i=0}^{n-1} \theta_i^2 \left(n - \frac{n+i}{n-i} \right) - 2 \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} \frac{i\,\theta_i \theta_j}{n-i} \right) \tag{3.44}$$

calculated in the Appendix.

The set of conditions (3.43) yields the following upper bound for the covariance

$$\begin{aligned}
&Cov(X_1, X_2) \\
&\quad = \frac{1}{n^2(n-1)} \left[\sum_{i=0}^{n-1} \theta_i^2 \left(n - \frac{n+i}{n-i} \right) - 2 \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} \frac{i}{n-i} \theta_i \theta_j \right] \\
&\quad \leq \frac{1}{n^2(n-1)} \left[\sum_{i=0}^{n-1} \theta_i^2 \left(n - \frac{n+i}{n-i} \right) - 2 \sum_{i=0}^{n-2} \frac{i}{n-i} \theta_i^2 \left(\sum_{j=i+1}^{n-1} 1 \right) \right] \\
&\quad = \frac{1}{n^2(n-1)} \left[\sum_{i=0}^{n-2} \theta_i^2 \left(n - \frac{n+i}{n-i} - 2\,\frac{i}{n-i}(n-1-i) \right) \right. \\
&\qquad\qquad \left. + (1-n)\theta_{n-1}^2 \right].
\end{aligned}$$

Simplifying the coefficient before θ_i^2 we obtain

$$
\begin{aligned}
&Cov(X_1, X_2) \\
&\quad \leq \frac{1}{n^2(n-1)} \left[\sum_{i=0}^{n-2} \theta_i^2 \, \frac{n^2 - ni - n - i - 2ni + 2i^2 + 2i}{n-i} + (1-n)\,\theta_{n-1}^2 \right] \\
&\quad = \frac{1}{n^2(n-1)} \left[\sum_{i=0}^{n-2} \theta_i^2 \, \frac{(n-i)^2 - i(n-i) - (n-i)}{n-i} + (1-n)\,\theta_{n-1}^2 \right] \\
&\quad = \frac{1}{n^2(n-1)} \left[\sum_{i=0}^{n-2} \theta_i^2 \,(n - 2i - 1) + (1-n)\,\theta_{n-1}^2 \right] .
\end{aligned}
$$

Thus, we come to the conclusion

$$
Cov(X_1, X_2) \leq \frac{1}{n^2(n-1)} \left[\sum_{i=0}^{n-2} \theta_i^2 \,(n - 2i - 1) + (1-n)\,\theta_{n-1}^2 \right] . \tag{3.45}
$$

Obviously, the sequence $(n - 2i - 1)$ is decreasing in i, moreover

$$
\begin{cases}
n - 2i - 1 \geq 0 & \text{for } i \leq [\frac{n-1}{2}] \\
n - 2i - 1 < 0 & \text{for } i > [\frac{n-1}{2}] \, ,
\end{cases}
$$

where $[\cdot]$ denotes the integer part of a fraction. Note also that for $i = n - 1$ holds $n - 2i - 1 = 1 - n$. Taking these facts into account we can rewrite (3.45) as

$$
Cov(X_1, X_2) \leq \frac{1}{n^2(n-1)} \left[\sum_{i=0}^{[\frac{n-1}{2}]} \theta_i^2 \,(n - 2i - 1) - \sum_{i=[\frac{n-1}{2}]+1}^{n-1} \theta_i^2 \,(2i + 1 - n) \right] .
$$

Furthermore, since θ_i is increasing in i,

$$
\begin{aligned}
&Cov(X_1, X_2) \\
&\quad \leq \frac{1}{n^2(n-1)} \left[\sum_{i=0}^{[\frac{n-1}{2}]} \theta_{[\frac{n-1}{2}]}^2 (n - 2i - 1) - \sum_{i=[\frac{n-1}{2}]+1}^{n-1} \theta_{[\frac{n-1}{2}]}^2 (2i + 1 - n) \right]
\end{aligned}
$$

$$= \frac{1}{n^2(n-1)} \left[\theta^2_{[\frac{n-1}{2}]} \sum_{i=0}^{n-1} (n-2i-1) \right]$$

holds.

Observe that

$$\sum_{i=0}^{n-1} (n-2i-1) = n^2 - 2\frac{n-1}{2}n - n = 0 \ .$$

Thus, we have verified that for FME distributed random variables under the conditions of Theorem 3.37 holds

$$Cov(X_i, X_j) \leq 0 \ ,$$

where $i, j = 1, \ldots, n$, $i \neq j$. In other words, although in general MRR_2 does not imply negative covariances, we have proved that for FME random variables this implication holds.

Let us carry on the consideration of special cases for the MTP_2 property. Consider exchangeable random variables $Y_1, \ldots, Y_n$ with conditional hazard rates of the form

$$\lambda(t|t_h, \ldots, t_1) = g(t, h, t_h) \ .$$

In this case we can specify conditions for the MTP_2 property of order statistics based on such $Y_1, \ldots, Y_n$.

By Lemma 2.21 $Y_{1:n}, \ldots, Y_{n:n}$ posses the Markov property. Then applying Theorem 3.22 we can make the following conclusion concerning MTP_2 of order statistics:

Lemma 3.39. *Let $Y_1, \ldots, Y_n$ be exchangeable absolutely continuous random variables possessing the conditional hazard rates of the form*

$$\lambda(t|t_h, \ldots, t_1) = g(t, h, t_h).$$

If all the transition densities $f^{Y_{i:n}|Y_{i-1:n}}$ are TP_2 for $i = 2, \ldots, n$, then $Y_{1:n}, \ldots, Y_{n:n}$ are MTP_2.

Example 3.40. *Consider $Y_{1:n}, \ldots, Y_{n:n}$ based on the random variables from Example 2.22* *with parameter $\theta = 1$. The density $f^{Y_{i:n}|Y_{i-1:n}}$ is calculated as*

$$f^{Y_{i:n}|Y_{i-1:n}}(t_i|t_{i-1}) = exp(t_{i-1} - t_i) \ ,$$

where $0 \le t_{i-1} \le t_i$. Observe that $f^{Y_{i:n}|Y_{i-1:n}}$ is MTP_2 as a product of univariate functions. Then by Lemma 3.39 $Y_{1:n}, \dots, Y_{n:n}$ are also MTP_2.

3.3.2.2 Schur-constant random variables

The dependence properties of Schur-constant random variables are well studied in the literature. For instance, the following facts can be found in Caramellino & Spizzichino (1996):

Lemma 3.41. *Consider absolutely continuous Schur-constant random variables $Y_1, \dots, Y_n$ with joint survival function*

$$\overline{F}(t_1, \dots, t_n) = \Phi(t_1 + \cdots + t_n), \ \ t_1, \dots, t_n \in \mathbb{R}_+$$

and conditional hazard rates $\lambda(h, y)$. Moreover let Φ be n-times differentiable. Then the following conditions are equivalent

- *(i)* $Y_1, \dots, Y_n$ *are* MTP_2,
- *(ii)* $\left|\Phi^{(n)}(t)\right|$ *is log-convex,*
- *(iii)* $\lambda(h, y)$ *are non-decreasing in* h *and non-increasing in* $y \in \mathbb{R}_+$.

Proof. In the following we will prove the equivalence for $h = n$. The result for all the other values of h follows by analogy from the observation that the MTP_2 property of $Y_1, \dots, Y_n$ implies the MTP_2 of $Y_1, \dots, Y_h$.

Lemma 3.12 implies that $Y_1, \dots, Y_n$ are MTP_2 if and only if

$$\frac{\partial^2}{\partial t_i \partial t_j} \ln\Big[(-1)^n \Phi^{(n)}(t_1 + \cdots + t_n)\Big] \ge 0 \ .$$

We can rewrite this fact as

$$\frac{\partial^2}{\partial t^2} \ln\left|\Phi^{(n)}(t)\right|_{t=t_1+\cdots+t_n} \ge 0 \ . \tag{3.46}$$

Thus we have shown the equivalence of (i) and (ii). To state the equivalence of (ii) and (iii), let us consider (3.46) in more detail

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\ln\left|\Phi^{(n)}(t)\right| &= \frac{\partial}{\partial t}\frac{\Phi^{(n+1)}(t)}{\Phi^{(n)}(t)}\\ &= \frac{\Phi^{(n+2)}(t)\Phi^{(n)}(t) - (\Phi^{(n+1)}(t))^2}{(\Phi^{(n)}(t))^2}\\ &\geq 0\end{aligned}$$

for all t such that $\Phi^{(n)}(t) \neq 0$. The last inequality holds iff

$$\Phi^{(n+2)}(t)\Phi^{(n)}(t) - \left(\Phi^{(n+1)}(t)\right)^2 \geq 0\ . \tag{3.47}$$

Consider $\Phi^{(n+1)}(t) = 0$, then (3.47) holds automatically. For $\Phi^{(n+1)}(t) \neq 0$ we can transform (3.47) into

$$\frac{\Phi^{(n+2)}(t)}{\Phi^{(n+1)}(t)} \cdot \frac{\Phi^{(n+1)}(t)}{\Phi^{(n)}(t)} - \left(\frac{\Phi^{(n+1)}(t)}{\Phi^{(n)}(t)}\right)^2 \geq 0\ .$$

In other words (3.47) is equivalent to

$$\lambda(n+1,t)\lambda(n,t) - (\lambda(n,t))^2 \geq 0\ .$$

For $\lambda(n,t) \neq 0$ it can be simplified to

$$\lambda(n+1,t) \geq \lambda(n,t),\ \ t \in \mathbb{R}_+\ .$$

Thus we have shown that $\lambda(h,y)$ is non-decreasing in h. The fact that $\lambda(h,y)$ is non-increasing in y can be verified by combining (3.46) with (2.28), which states

$$\lambda(n,y) = -\frac{\partial}{\partial y}\ln\left|\Phi^{(n)}(y)\right|.$$

Indeed it follows immediately that

$$\frac{\partial}{\partial y}\lambda(n,y) = -\frac{\partial^2}{\partial t^2}\ln\big|\Phi^{(n)}(t)\big| \leq 0$$

as was to be proved. □

Lemma 3.42. *Consider a vector of Schur-constant lifetimes $Y = (Y_1,\ldots,Y_n)$ with multivariate conditional hazard rates $\lambda(h,y),\ h = 0,\ldots,n-1,\ y \in \mathbb{R}_+$. Y is HIF iff*

$$\lambda(h,y) \leq \lambda(h',y'),$$

whenever $h' \geq h, y' \leq y$.

Remark 3.43. *Lemma 3.41 and Lemma 3.42 show, in particular, that for Schur-constant random variables MTP_2 is equivalent to HIF.*

The following example illustrates the application of Lemma 3.41.

Example 3.44.

(i) *Let the random variables $Y_1,\ldots,Y_n$ follow the distribution of correlated gamma variables (see Kotz et al. (2000)) of the form*

$$f^{Y_1,\ldots,Y_n}(t_1,\ldots,t_n) = c\,\Big(\sum_{i=1}^{n} t_i\Big)^{n-1}\prod_{i=1}^{n} e^{-bt_i}$$

with a parameter $b > 0$ and a normalization constant $c > 0$ such that $f^{Y_1,\ldots,Y_n}$ is a joint density function. According to the Remark 2.24 $Y_1,\ldots,Y_n$ are Schur-constant and from the form of the joint density we observe that

$$|\Phi^{(n)}(t)| = c\,t^{n-1}\,e^{-bt}\ ,$$

where $\Phi(t)$ is a univariate survival function from Definition 2.23. Consider

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\ln|\Phi^{(n)}(t)| &= c\,\frac{\partial^2}{\partial t^2}\big[(n-1)\ln t - bt\big]\\ &= c\,\frac{\partial}{\partial t}\big[\frac{n-1}{t} - b\big]\end{aligned}$$

$$= -c\,\frac{n-1}{t^2} \le 0 \; .$$

Therefore, by Lemma 3.41 (ii) $Y_1, \dots, Y_n$ are not MTP$_2$. Moreover, according to Gupta & Richards (1987) they are MRR$_2$.

(ii) Consider absolutely continuous random variables $Y_1, \dots, Y_n$ with Pareto survival function of the second kind described in Example 2.37. According to (2.32) conditional hazard rates of $Y_1, \dots, Y_n$ have the form

$$\begin{aligned}\lambda(y,h) &= -\frac{\Phi^{(h+1)}(y)}{\Phi^{(h)}(y)} \\ &= (\alpha + h)\beta(1+\beta y)^{-1} \; ,\end{aligned}$$

where $h = 1, \dots, n-1$, $y \ge 0$. Since it is increasing in h and decreasing in y for all β and α, the condition (iii) of Lemma 3.41 is satisfied and we can state that $Y_1, \dots, Y_n$ are MTP$_2$.

Remark 3.45. *Consider Schur-constant random variables $Y_1, \dots, Y_n$ with completely monotone survival function Φ and the joint density function*

$$f^{Y_1,\dots,Y_n}(t_1, \dots, t_n) = \int_0^\infty \theta^n \exp\left(-\theta \sum_{i=1}^{n} t_i\right) d\pi(\theta)$$

as in (2.29). Note that $\exp(-\theta\, t)$ is RR$_2$ in θ, t. Then Theorem 3.30 (i) yields that $f^{Y_1,\dots,Y_n}(t_1, \dots, t_n)$ is MTP$_2$.

3.3.2.3 Archimedean copulas

An overview of dependence properties for random vectors with joint distribution functions represented by Archimedean copulas can be found in Nelsen (2006) and Müller & Scarsini (2005). In particular, the following results concerning the MTP$_2$ property can be found in Müller & Scarsini (2005).

Theorem 3.46. *Consider absolutely continuous random variables $Y_1, \dots, Y_n$ with a joint distribution function represented by a strict Archimedean copula with generator ϕ and n-times differentiable ϕ^{-1}. Then $Y_1, \dots, Y_n$ are MTP$_2$ if and only if $(-1)^n \frac{\partial^n}{\partial y^n} \phi^{-1}(y)$ is log-convex.*

A similar results holds for Archimedean survival copulas:

Theorem 3.47. *Consider absolutely continuous random variables $Y_1, \dots, Y_n$ with a joint survival function represented by a strict Archimedean survival copula with generator ϕ and n-times differentiable ϕ^{-1}.*

Then $Y_1, \dots, Y_n$ are MTP$_2$ if and only if $(-1)^n \frac{\partial^n}{\partial y^n} \phi^{-1}(y)$ is log-convex.

Example 3.48.

(i) Consider absolutely continuous random variables Y_1, Y_2. Let their distribution function be represented by the Ali-Mikhail-Haq copula with generator

$$\phi(t) = \ln \frac{1-\theta(1-t)}{t} \ ,$$
$$\phi^{-1}(u) = \frac{1-\theta}{e^u - \theta} \ ,$$

where $t \in [0,1]$, $u \in [0,\infty]$, $\theta \in (-1,1]$ (see also Nelsen (2006), Table 4.1, copula 4.2.3). Then by Theorem 3.47 it suffices to check if $\frac{\partial^2}{\partial t^2}\phi^{-1}(t)$ is a log-convex function. The second derivative of ϕ^{-1} is given by

$$\frac{\partial^2}{\partial t^2}\phi^{-1}(t) = \frac{(1-\theta)\, e^t (e^t + \theta)}{(e^t - \theta)^3} \ .$$

Consequently we can state that

$$\frac{\partial^2}{\partial t^2} \ln \frac{\partial^2}{\partial t^2}\phi^{-1}(t) = e^t \cdot \left[\frac{\theta}{(e^t+\theta)^2} + \frac{3\theta}{(e^t-\theta)^2} \right] \geq 0 \ ,$$

for all $\theta \in [0,1]$. Thus, for $\theta \in [0,1]$ random variables Y_1, Y_2 possess the TP$_2$ property.

(ii) Consider random variables Y_1, Y_2 with Gumbel-Barnett survival copula from Example 2.49. By analogy to (i)

$$\frac{\partial^2}{\partial t^2}\phi^{-1}(t) = \exp\left(\frac{1-e^t}{\theta}\right) \frac{e^t}{\theta} \left(\frac{e^t}{\theta} - 1\right)$$
$$\frac{\partial^2}{\partial t^2} \ln \frac{\partial^2}{\partial t^2}\phi^{-1}(t) = -\frac{e^t}{\theta} \left[1 + 1/\Big(\frac{e^t}{\theta} - 1\Big)^2 \right] < 0$$

for all $\theta \in [0,1)$, $t \in [0,\infty]$, *consequently* Y_1, Y_2 *are not* TP_2 *for any* θ.

(iii) *Consider exchangeable random variables* $Y_1, \ldots, Y_n$ *with joint survival function described by an* n*-dimensional Clayton copula from Example 2.52. Let the corresponding marginal survival function* $\overline{F}(\cdot)$ *be absolutely continuous . Due to the integral representation (2.40) the joint density of* $Y_1, \ldots, Y_n$ *is of the form*

$$f(t_1, \ldots, t_n) = \int_0^\infty \theta^n \exp\left[-\theta \sum_{i=0}^n \phi^{-1}\big(\overline{F}(t_i)\big)\right] \times \prod_{i=1}^n \big(\phi^{-1}\big)'\big(\overline{F}(t_i)\big)\, \overline{F}'(t_i)\, \pi(\theta)\, d\theta \,,$$

where $t_1, \ldots, t_n \in \mathbb{R}_+$. *Note that*

$$\begin{aligned} \frac{\partial^2}{\partial\theta\partial t} \ln \exp\left[\theta\phi^{-1}\big(\overline{F}(t)\big)\right] &= \frac{\partial^2}{\partial\theta\partial t}\left[\theta\phi^{-1}\big(\overline{F}(t)\big)\right] \\ &= \big(\phi^{-1}\big)'\big(\overline{F}(t)\big)\overline{F}'(t) \\ &\geq 0 \,. \end{aligned}$$

Therefore, according to Lemma 3.12, the function $\exp\left[\theta\phi^{-1}\big(\overline{F}(t)\big)\right]$ *is* TP_2 *with respect to* θ, t. *Then* $\exp\left[-\theta\phi^{-1}\big(\overline{F}(t)\big)\right]$ *is* RR_2 *and so is*

$$\theta \exp\left[-\theta\, \phi^{-1}\big(\overline{F}(t_i)\big)\right] \big(\phi^{-1}\big)'\big(\overline{F}(t_i)\big)\, \overline{F}'(t_i) \,.$$

Applying Lemma 3.20 together with Lemma 3.30 we can conclude that $f(t_1, \ldots, t_n)$ *is* MTP_2.

4 Dependence properties of sequential order statistics

In previous chapters we have looked at different models for ordered data such as ordinary order statistics based on iid or exchangeable random variables, sequential order statistics based on conditionally iid and sequential order statistics based on exchangeable random variables. As illustrated in Remark 1.16 sequential order statistics based on exchangeable random variables generalize other considered models. Regarding the dependence properties, Example 3.23 and Lemma 3.32 presented cases when ordered random variables inherit the dependence properties from the underlying distributions. The question arises if these results can be extended to the case of sequential order statistics based on exchangeable random variables.

In this respect the upcoming reasoning will be dedicated to the investigation of the relationship between the properties of sequential order statistics $X_*^{(1)}, \dots, X_*^{(n)}$ and the underlying exchangeable random variables $Y_1^{(i)}, \dots, Y_n^{(i)},\ i = 1, \dots, n$. As usual by $Y_{1:n}^{(i)}, \dots, Y_{i:n}^{(i)}$ we will denote order statistics based on $Y_1^{(i)}, \dots, Y_n^{(i)},\ i = 1, \dots, n$.

4.1 CIS and SL

According to Example 3.23 order statistics based on exchangeable random variables are known to be MTP_2. As the CIS property is weaker then MTP_2 we can conclude that such order statistics are also CIS. In the following we provide a similar statement for sequential order statistics based on exchangeable random variables.

Theorem 4.1. *If $Y_{1:n}^{(i)}, \dots, Y_{i:n}^{(i)}$ are CIS for $i = 1, \dots, n$, then $X_*^{(1)}, \dots, X_*^{(n)}$ are CIS.*

Proof. According to Definition 3.9 (ii) we need to verify that for $t_j \le t_j'$ and $j = 1, \dots, i-1$ holds

$$\begin{aligned} &P(X_*^{(i)} > t \mid X_*^{(i-1)} = t_{i-1}, \dots, X_*^{(1)} = t_1) \\ &\qquad \le P(X_*^{(i)} > t \mid X_*^{(i-1)} = t_{i-1}', \dots, X_*^{(1)} = t_1') \ . \end{aligned}$$

Note that according to Definition 1.11 conditional probabilities of $X_*^{(1)}, \dots, X_*^{(n)}$ have the representation

$$P(X_*^{(i)} > t \mid X_*^{(i-1)} = t_{i-1}, \dots, X_*^{(1)} = t_1)$$

$$= P(Y^{(i)}_{i:n} > t \mid Y^{(i)}_{i-1:n} = t_{i-1}, \dots, Y^{(i)}_{1:n} = t_1) \ .$$

Since $Y^{(i)}_{1:n}, \dots, Y^{(i)}_{n:n}$ are CIS, the result follows. □

Remark 4.2. *By analogy we can state that: If $Y^{(i)}_{1:n}, \dots, Y^{(i)}_{i:n}$ are CDS for $i = 1, \dots, n$, then $X^{(1)}_{*}, \dots, X^{(n)}_{*}$ are CDS.*

Theorem 4.3. *If $Y^{(i)}_1, \dots, Y^{(i)}_n$ are MTP_2 for $i = 1, \dots, n$, then $X^{(1)}_{*}, \dots, X^{(n)}_{*}$ are CIS.*

Proof. Recall that MTP_2 of exchangeable $Y^{(i)}_1, \dots, Y^{(i)}_n$ implies the MTP_2 of their order statistics. Moreover, according to Theorem 3.14, MTP_2 is stronger then the CIS property. Then, applying Theorem 4.1, we obtain the result . □

Theorem 4.4. *If for $i = 1, \dots, n$ $Y^{(i)}_1, \dots, Y^{(i)}_n$ are MTP_2, then $X^{(1)}_{*}, \dots, X^{(n)}_{*}$ are SL.*

Proof. Since $X^{(1)}_{*}, \dots, X^{(n)}_{*}$ are absolutely continuous ordered random variables, the proof follows from Theorem 4.3 combined with Lemma 3.34. □

Note that Theorems 4.3 and 4.4 state that $X^{(1)}_{*}, \dots, X^{(n)}_{*}$ inherit at least to some extent the dependence properties of underlying distributions.

4.2 HIF

As we have seen in Lemma 3.32 the HIF property of exchangeable random variables implies the HIF property of their order statistics. It turns out that the relation also holds for sequential order statistics based on exchangeable random variables.

Theorem 4.5. *If $Y^{(i)}_1, \dots, Y^{(i)}_n$ are HIF for $i = 1, \dots, n$, then $X^{(1)}_{*}, \dots, X^{(n)}_{*}$ are HIF.*

Proof. Taking into account (3.23) HIF of $X^{(1)}_{*}, \dots, X^{(n)}_{*}$ can be defined by

$$\begin{aligned} &\lambda_{(*,i)}(t|x_1, \dots, x_{i-1}) \le \lambda_{(*,i)}(t|y_1, \dots, y_{i-1}) \ , \\ &\lambda_{(*,1)}(t) \le \lambda_{(*,1)}(t) \ , \end{aligned} \tag{4.1}$$

where $0 \le y_j \le x_j < t$ for $j = 1, \dots, i-1$, $i = 2, \dots, n$. By (2.13) inequalities in (4.1) are equivalent to

$$\begin{aligned} &\lambda_{(i)}(t|x_1, \dots, x_{i-1}) \le \lambda_{(i)}(t|y_1, \dots, y_{i-1}) \ , \\ &\lambda_{(1)}(t) \le \lambda_{(1)}(t) \ , \end{aligned}$$

respectively. They hold due to the HIF property of $Y_1^{(i)}, \dots, Y_n^{(i)},\ i = 1, \dots, n.$ $\square$

Thus, due to the construction of sequential order statistics their behavior with respect to the HIF property is fully determined by the corresponding properties of distributions on the levels.

4.3 $\mathbf{MTP}_2$

4.3.1 Necessary conditions

Theorem 4.6. *If $X_*^{(1)}, \dots, X_*^{(n)}$ are MTP_2, then $f^{Y_{1:n}^{(i)}, \dots, Y_{i:n}^{(i)}}(t_1, \dots, t_i)$ is TP_2 in t_i, t_k for $t_1, \dots, t_i \in \mathbb{R}_+$ and $k = 1, \dots, i-1,\ i = 2, \dots, n$.*

Proof. By Theorem 3.18 MTP_2 of $X_*^{(1)}, \dots, X_*^{(n)}$ implies the MTP_2 of the marginal densities

$$f^{X_*^{(1)}, \dots, X_*^{(i)}} = \prod_{j=1}^{i} f^{Y_{j:n}^{(j)} \mid Y_{j-1:n}^{(j)}, \dots, Y_{1:n}^{(j)}}(t_j | t_{j-1}, \dots, t_1) ,$$

where $i = 2, \dots, n$.

Note that $f^{X_*^{(1)}, \dots, X_*^{(i)}}$ depends on t_i only through $f^{Y_{1:n}^{(i)}, \dots, Y_{i:n}^{(i)}}(t_1, \dots, t_{i-1}, t_i)$. Then MTP_2 of $f^{X_*^{(1)}, \dots, X_*^{(i)}}$ implies that $f^{Y_{1:n}^{(i)}, \dots, Y_{i-1:n}^{(i)}, Y_{i:n}^{(i)}}(t_1, \dots, t_{i-1}, t_i)$ is TP_2 in t_i, t_k, where $k = 1, \dots, i-1$. $\square$

4.3.2 Sufficient conditions

We will start the analysis of sufficient conditions for the MTP_2 of sequential order statistics by looking at the combination of Lemma 2.13 and 3.12.

Lemma 4.7. *Consider sequential order statistics $X_*^{(1)}, \dots, X_*^{(n)}$ with a density which is twice partially differentiable on*

$$\mathcal{A} = \{(t_1, \dots, t_n) \mid f^{X_*^{(1)}, \dots, X_*^{(n)}}(t_1, \dots, t_n) \neq 0\}$$

in every pair $t_i, t_j,\ i \neq j,\ i, j = 1, \dots, n$. Then $X_*^{(1)}, \dots, X_*^{(n)}$ are MTP_2 on $\mathcal{A}$ iff one of the following conditions is satisfied

(i) $\frac{\partial^2}{\partial t_i \partial t_j} \left[\sum_{k=i}^{n} \ln f^{Y_{k:n}^{(k)} | Y_{k-1:n}^{(k)}, \dots, Y_{1:n}^{(k)}}(t_k | t_{k-1}, \dots, t_1) \right] \geq 0 ,$

(ii) $\frac{\partial^2}{\partial t_i \partial t_j} \sum_{k=i}^{n} \left[\ln \lambda_{(k)}(t_k|t_1,\ldots,t_{k-1}) - (n-k+1) \int_{t_{k-1}}^{t_k} \lambda_{(k)}(u|t_1,\ldots,t_{k-1}) du \right] \geq 0$

for $1 \leq i < j \leq n,\ (t_1,\ldots,t_n) \in \mathcal{A}$.

Proof. Condition (i) is obtained by applying Lemma 3.12 for the joint density of $X_*^{(1)},\ldots,X_*^{(n)}$. Namely, for $1 \leq i < j \leq n$ and $0 \leq t_1 \leq \ldots \leq t_n$ should hold

$$\begin{aligned}
\frac{\partial^2}{\partial t_i \partial t_j} \ln f^{X_*^{(1)},\ldots,X_*^{(n)}}(t_1,\ldots,t_n) &= \frac{\partial^2}{\partial t_i \partial t_j} \ln \prod_{i=1}^{n} f^{X_*^{(i)} \mid X_*^{(i-1)},\ldots,X_*^{(1)}}(t_i|t_{i-1},\ldots,t_1) \\
&= \frac{\partial^2}{\partial t_i \partial t_j} \ln \prod_{i=1}^{n} f^{Y_{i:n}^{(i)} \mid Y_{i-1:n}^{(i)},\ldots,Y_{1:n}^{(i)}}(t_i|t_{i-1},\ldots,t_1) \\
&= \frac{\partial^2}{\partial t_i \partial t_j} \left[\sum_{k=i}^{n} \ln f^{Y_{i:n}^{(i)} \mid Y_{i-1:n}^{(i)},\ldots,Y_{1:n}^{(i)}}(t_i|t_{i-1},\ldots,t_1) \right] \\
&\geq 0\ .
\end{aligned}$$

The second condition is obtained from Lemma 3.12 together with Lemma 2.13. □

The following example illustrates the application of Lemma 4.7.

Example 4.8. *Consider continuous exchangeable random variables* $Y_1^{(i)}, Y_2^{(i)}, Y_3^{(i)},\ i = 1,2,3$ *from Example 1.15. Moreover, let* $\lambda_i = \lambda > 0$ *for* $i = 1,2,3$. *Then, from (1.10) follows that the joint density of sequential order statistics based on* $Y_1^{(i)}, Y_2^{(i)}, Y_3^{(i)}$ *can be obtained as*

$$\begin{aligned}
f^{X_*^{(1)},X_*^{(2)},X_*^{(3)}}(t_1,t_2,t_3) &= 3!\lambda^{\alpha_1}\, \alpha_1(\alpha_2+1)\,(\alpha_3+2) \\
&\quad \times \frac{(t_1+2t_2+\lambda)^{\alpha_3-\alpha_2}(3t_1+\lambda)^{\alpha_2-\alpha_1}}{(t_1+t_2+t_3+\lambda)^{\alpha_3+3}}\ ,
\end{aligned}$$

where $0 \leq t_1 \leq t_2 \leq t_3$.

In the following we will check the MTP$_2$ *property of* $X_*^{(1)}, X_*^{(2)}, X_*^{(3)}$ *by verifying Lemma 4.7 (i). We start by calculating* $\frac{\partial^2}{\partial t_2 \partial t_3} \ln f^{X_*^{(1)},X_*^{(2)},X_*^{(3)}}(t_1,t_2,t_3)$:

$$\frac{\partial^2}{\partial t_2 \partial t_3} \ln f^{X_*^{(1)},X_*^{(2)},X_*^{(3)}}(t_1,t_2,t_3) = -\frac{\partial^2}{\partial t_2 \partial t_3}(\alpha_3+3)\ln(t_1+t_2+t_3+\lambda)$$

$$= -\frac{\partial}{\partial t_2} \frac{\alpha_3 + 3}{t_1 + t_2 + t_3 + \lambda}$$
$$= \frac{\alpha_3 + 3}{(t_1 + t_2 + t_3 + \lambda)^2} \geq 0 \ .$$

It follows by analogy that $\frac{\partial^2}{\partial t_1 \partial t_3} \ln f^{X_*^{(1)}, X_*^{(2)}, X_*^{(3)}}(t_1, t_2, t_3) \geq 0$. *Next, let us look at the sign of* $\frac{\partial^2}{\partial t_1 \partial t_2} \ln f^{X_*^{(1)}, X_*^{(2)}, X_*^{(3)}}(t_1, t_2, t_3)$*:*

$$\begin{aligned}
\frac{\partial^2}{\partial t_1 \partial t_2} \ln f^{X_*^{(1)}, X_*^{(2)}, X_*^{(3)}}(t_1, t_2, t_3) &= \frac{\partial^2}{\partial t_1 \partial t_2} \Big[(\alpha_3 - \alpha_2) \ln(t_1 + 2t_2 + \lambda) \\
&\quad - (\alpha_3 + 3) \ln(t_1 + t_2 + t_3 + \lambda) \Big] \\
&= \frac{\partial}{\partial t_1} \left[\frac{2\,(\alpha_3 - \alpha_2)}{t_1 + 2t_2 + \lambda} - \frac{\alpha_3 + 3}{t_1 + t_2 + t_3 + \lambda} \right] \\
&= \frac{2\,(\alpha_2 - \alpha_3)}{(t_1 + 2t_2 + \lambda)^2} + \frac{\alpha_3 + 3}{(t_1 + t_2 + t_3 + \lambda)^2} \ .
\end{aligned}$$

We can conclude that $\alpha_2 \geq \alpha_3$ *yields* $\frac{\partial^2}{\partial t_1 \partial t_2} \ln f^{X_*^{(1)}, X_*^{(2)}, X_*^{(3)}}(t_1, t_2, t_3) \geq 0$. *Therefore for* $\alpha_2 \geq \alpha_3$ $X_*^{(1)}, X_*^{(2)}, X_*^{(3)}$ *possess an MTP$_2$ density.*

Let us look closer at the case $\alpha_2 < \alpha_3$ *and search for* t_1, t_2, t_3 *that do not satisfy Lemma 4.7 (i). In this respect consider the inequality*

$$\frac{2\,(\alpha_3 - \alpha_2)}{(t_1 + 2t_2 + \lambda)^2} \geq \frac{\alpha_3 + 3}{(t_1 + t_2 + t_3 + \lambda)^2} \ .$$

It can be rewritten in the form

$$(t_1 + t_2 + t_3 + \lambda)^2 \geq \frac{(\alpha_3 + 3)(t_1 + 2t_2 + \lambda)^2}{2\,(\alpha_3 - \alpha_2)} \ .$$

Further, since $t_1, t_2, t_3, \alpha_3, \lambda \geq 0$ *and* $\alpha_3 > \alpha_2$*, we can state equivalently that*

$$t_3 \geq \sqrt{\frac{(\alpha_3 + 3)(t_1 + 2t_2 + \lambda)^2}{2\,(\alpha_3 - \alpha_2)}} - t_1 - t_2 - \lambda \ .$$

Restricting $\alpha_3 > \alpha_2$ we can bound the expression on the right with

$$\sqrt{\frac{(\alpha_3+3)(t_1+2t_2+\lambda)^2}{2\,(\alpha_3-\alpha_2)}} - t_1 - t_2 - \lambda \leq \sqrt{\frac{(\alpha_3+3)(t_1+2t_2+\lambda)^2}{2\,(\alpha_3-\alpha_2)}} < \infty\,.$$

Then for every choice of parameter λ and every $t_1, t_2 \geq 0$ there exists such $t_3 \geq t_2$ that

$$\frac{\partial^2}{\partial t_1 \partial t_2} \ln f^{X_*^{(1)}, X_*^{(2)}, X_*^{(3)}}(t_1, t_2, t_3) \leq 0\ .$$

In other words for $\alpha_3 > \alpha_2$ $X_*^{(1)}, X_*^{(2)}, X_*^{(3)}$ are not MTP$_2$.

Remark 4.9.

(i) Consider the random variables $X_*^{(1)}, X_*^{(2)}, X_*^{(3)}$ from the previous example. Note that according to Remark 3.45 $Y_1^{(i)}, Y_2^{(i)}, Y_3^{(i)}$ are MTP$_2$ for each $i = 1, 2, 3$. Thus, although all the underlying distributions are MTP$_2$, for a specific choice of parameters we obtain sequential order statistics, that are not MTP$_2$. Thereby sequential order statistics based on exchangeable random variables differ from other models of ordered data that we considered in this thesis (recall Example 3.23).

(ii) For $n = 2$ $f^{X_*^{(1)}, X_*^{(2)}}$ is MTP$_2$ iff $f^{Y_{1:2}^{(2)}, Y_{2:2}^{(2)}}$ is MTP$_2$.

Lemma 4.10. *Assume $Y_{1:n}^{(n)}, \ldots, Y_{n:n}^{(n)}$ to be MTP$_2$. Then $X_*^{(1)}, \ldots, X_*^{(n)}$ possess the MTP$_2$ property if $\frac{f^{Y_{1:n}^{(i)}, \ldots, Y_{i:n}^{(i)}}}{f^{Y_{1:n}^{(i+1)}, \ldots, Y_{i:n}^{(i+1)}}}$ is MTP$_2$ for $i = 2, \ldots, n-1$.*

Proof. Recall that the joint density of sequential order statistics is of the form

$$\begin{aligned} f^{X_*^{(1)}, \ldots, X_*^{(n)}}(t_1, \ldots, t_n) &= f^{Y_{1:n}^{(1)}}(t_1) \prod_{i=2}^{n} f^{Y_{i:n}^{(i)} \mid Y_{i-1:n}^{(i)}, \ldots, Y_{1:n}^{(i)}}(t_i | t_{i-1}, \ldots, t_1) \\ &= f^{Y_{1:n}^{(n)}, \ldots, Y_{n:n}^{(n)}}(t_1, \ldots, t_n) \prod_{i=1}^{n-1} \frac{f^{Y_{1:n}^{(i)}, \ldots, Y_{i:n}^{(i)}}(t_1, \ldots, t_i)}{f^{Y_{1:n}^{(i+1)}, \ldots, Y_{i:n}^{(i+1)}}(t_1, \ldots, t_i)}\ . \end{aligned}$$

Then the result follows from Theorem 3.15. □

Remark 4.11. *Applying Lemma 3.12 to twice partially differentiable functions* $f^{Y^{(i)}_{1:n},\dots,Y^{(i)}_{i:n}}$ *and* $f^{Y^{(i+1)}_{1:n},\dots,Y^{(i+1)}_{i:n}}$*, the* MTP_2 *property of* $\frac{f^{Y^{(i)}_{1:n},\dots,Y^{(i)}_{i:n}}}{f^{Y^{(i+1)}_{1:n},\dots,Y^{(i+1)}_{i:n}}}$ *is equivalent to*

$$\frac{\partial^2}{\partial t_k \partial t_l} \ln \frac{f^{Y^{(i)}_{1:n},\dots,Y^{(i)}_{i:n}}(t_1,\dots,t_i)}{f^{Y^{(i+1)}_{1:n},\dots,Y^{(i+1)}_{i:n}}(t_1,\dots,t_i)} \geq 0$$

for $0 \leq t_1 \leq \ldots \leq t_i$ *and* $1 \leq k < l \leq i$.

4.4 Special cases

4.4.1 Distributions with Markov order statistics

Theorem 4.12. *For* $i = 1,\dots,n,\ k = 1,\dots,n-1$ *consider* $Y^{(i)}_1,\dots,Y^{(i)}_n$ *with conditional hazard rates that can be represented as*

$$\lambda_{(i)}(t|t_1,\dots,t_k) = g_i(t,k) \ .$$

Then $X^{(1)}_*,\dots,X^{(n)}_*$ *are* MTP_2.

Proof. Denote

$$l_i(t,h) = \int_0^t g_i(u,h)du \ .$$

Then according to Lemma 2.13 we can write

$$\begin{aligned} f^{X^{(1)}_*,\dots,X^{(n)}_*}&(t_1,\dots,t_n) \\ &= \left(\prod_{i=1}^n g_i(t_i,i-1)\right) \exp\left(-n\int_0^{t_1} g_1(u,0)du\right) \\ &\quad \times \exp\left(-\sum_{h=2}^n (n-h+1)\int_{t_{h-1}}^{t_h} g_h(u,h-1)du\right) \end{aligned}$$

or in terms of $l_i(t,h)$

$$
\begin{aligned}
f^{X_*^{(1)},\dots,X_*^{(n)}}&(t_1,\dots,t_n)\\
&= \left(\prod_{i=1}^{n} g_i(t_i,i-1)\right)\exp\left(-n\Big(l_1(t_1,0)-l_1(0,0)\Big)\right)\\
&\quad\times\exp\left(-\sum_{h=2}^{n}(n-h+1)\Big(l_h(t_h,h-1)-l_h(t_{h-1},h-1)\Big)\right),
\end{aligned}
\tag{4.2}
$$

where $0 < t_1 \le \dots \le t_n$. In other words, $f^{X_*^{(1)},\dots,X_*^{(n)}}(t_1,\dots,t_n)$ can be represented as a product of functions depending only on one of the t_i's each. Then the result follows by Theorem 3.15. □

Recall that by Lemma 2.21 order statistics $Y_{1:n}^{(i)},\dots,Y_{n:n}^{(i)}$ of the random variables $Y_1^{(i)},\dots,Y_n^{(i)}$ from Theorem 4.12 possess the Markov property. Therefore we can state the following generalization of Theorem 4.12.

Theorem 4.13. *Consider $Y_1^{(i)},\dots,Y_n^{(i)}$ such that $Y_{1:n}^{(i)},\dots,Y_{n:n}^{(i)}$ possess the Markov property for every $i=1,\dots,n$. If order statistics $Y_{1:n}^{(i)},\dots,Y_{n:n}^{(i)}$ are MTP_2 for $i=1,\dots,n$, then $X_*^{(1)},\dots,X_*^{(n)}$ are MTP_2 .*

Proof. Since $Y_{1:n}^{(i)},\dots,Y_{n:n}^{(i)}$ are Markov, representation (1.7) turns into

$$
f^{X_*^{(k)}\,|\,X_*^{(k-1)},\dots,X_*^{(1)}}(t_k|t_{k-1},\dots,t_1) = f^{Y_{k:n}^{(k)}\,|\,Y_{k-1:n}^{(k)}}(t_k|t_{k-1})
$$

and the corresponding joint density function can be calculated as

$$
f^{X_*^{(1)},\dots,X_*^{(n)}} = f^{Y_{1:n}^{(1)}}(t_1)\prod_{k=2}^{n} f^{Y_{k:n}^{(k)}\,|\,Y_{k-1:n}^{(k)}}(t_k|t_{k-1}) .
$$

Note that according to Theorem 3.18 MTP_2 property of $Y_{1:n}^{(i)},\dots,Y_{n:n}^{(i)}$ implies the MTP_2 of conditional densities

$$
f^{Y_{k:n}^{(k)}\,|\,Y_{k-1:n}^{(k)}}(t_k|t_{k-1}) = \frac{f^{Y_{k-1:n}^{(k)},Y_{k:n}^{(k)}}(t_{k-1},t_k)}{f^{Y_{k-1:n}^{(k)}}(t_{k-1})} .
$$

By applying Lemma 3.21, we obtain the result. □

Example 4.14. *Let* $Y_1^{(i)}, \dots, Y_n^{(i)}$ *be the exchangeable random variables obtained from sequential order statistics in Example 2.19 with parameters* $\theta_0^i, \dots, \theta_{n-1}^i \geq 0,\ i = 1, \dots, n$. *Then the corresponding conditional hazard rates possess the representation*

$$\lambda_{(i)}(t|t_1, \dots, t_h) = \frac{1}{\theta_h^i} \qquad a.s.$$

By Lemma 4.12 we can conclude that $X_*^{(1)}, \dots, X_*^{(n)}$ *are MTP*$_2$. *Note that although the order statistics* $Y_{1:n}^{(i)}, \dots, Y_{n:n}^{(i)}$ *possess the MTP*$_2$ *property, the random variables* $Y_1^{(i)}, \dots,$ $Y_n^{(i)}$ *are not necessarily MTP*$_2$ *as illustrated in Example 3.38.*

Thus, the MTP$_2$ property of $X_*^{(1)}, \dots, X_*^{(n)}$ is guaranteed by the MTP$_2$ property of underlying distributions of order statistics.

4.4.2 Schur-constant densities

First let us look at the representation for the density of sequential order statistics based on Schur-constant random variables.

Lemma 4.15. *Let* $Y_1^{(i)}, \dots, Y_n^{(i)}$ *be absolutely continuous Schur-constant with* n*-times differentiable survival functions* $\Phi_i,\ i = 1, \dots, n$. *Then the joint density of sequential order statistics can be written as*

$$f^{X_*^{(1)}, \dots, X_*^{(n)}}(t_1, \dots, t_n) = (-1)^n\, n!\, \Phi_1^{(1)}(y_1) \prod_{i=2}^{n} \frac{\Phi_i^{(i)}(y_i)}{\Phi_i^{(i-1)}(y_{i-1})} \ ,$$

where $y_i = \sum_{j=1}^{i-1} t_j + (n - i + 1)t_i$ *and* $0 \leq t_1 \leq \ldots \leq t_n$.

Proof. Combining (2.13) with the representation for conditional hazard rates from Lemma 2.31 (ii) we obtain

$$\begin{aligned}\lambda_{(*,h)}(t|t_1, \dots, t_{h-1}) &= -(n - h + 1) \left.\frac{\Phi_h^{(h)}(y)}{\Phi_h^{(h-1)}(y)}\right|_{y=\sum_{j=1}^{h-1} t_j + (n-h+1)t} \\ &= -(n - h + 1) \left.\frac{\partial}{\partial y} \ln \Phi_h^{(h-1)}(y)\right|_{y=\sum_{j=1}^{h-1} t_j + (n-h+1)t} \ ,\end{aligned}$$

which allows the following representation

$$
\begin{aligned}
&\exp\left(-\int_{t_{h-1}}^{t_h} \lambda_{(*,h)}(t|t_1,\ldots,t_{h-1})dt\right) \\
&\quad = \exp\left((n-h+1)\int_{t_{h-1}}^{t_h} \frac{\partial}{\partial y}\ln\Phi_h^{(h-1)}(y)\Big|_{y=\sum_{j=1}^{h-1}t_j+(n-h+1)t} dt\right) \\
&\quad = \exp\left(\int_{y_{h-1}}^{y_h} \frac{\partial}{\partial y}\ln\Phi_h^{(h-1)}(y)dy\right) \\
&\quad = \frac{\Phi_h^{(h-1)}(y_h)}{\Phi_h^{(h-1)}(y_{h-1})} \; .
\end{aligned}
$$

Then applying Lemma 2.13 we can write

$$
\begin{aligned}
f^{X_*^{(1)},\ldots,X_*^{(n)}}(t_1,\ldots,t_n) &= (-1)^n\, n!\, \frac{\Phi_1^{(0)}(y_1)}{\Phi_1^{(0)}(0)}\,\frac{\Phi_1^{(1)}(y_1)}{\Phi_1^{(0)}(y_1)} \\
&\quad \times \prod_{i=2}^{n}\frac{\Phi_i^{(i-1)}(y_i)}{\Phi_i^{(i-1)}(y_{i-1})}\,\frac{\Phi_i^{(i)}(y_i)}{\Phi_i^{(i-1)}(y_i)} \\
&= (-1)^n\, n!\, \Phi_1^{(1)}(y_1)\prod_{i=2}^{n}\frac{\Phi_i^{(i)}(y_i)}{\Phi_i^{(i-1)}(y_{i-1})} \; .
\end{aligned}
$$

□

With respect to the MTP_2 property of sequential order statistics based on Schur-constant random variables we can state the following:

Lemma 4.16. *Let $Y_{(i)} = (Y_1^{(i)},\ldots,Y_n^{(i)})$ be a random vector of absolutely continuous MTP_2 Schur-constant lifetimes with an n-times differentiable survival function Φ_i and hazard rates*

$$
\lambda_{(i)}(h,y) = -\frac{\Phi_i^{(h+1)}(y)}{\Phi_i^{(h)}(y)} \; ,
$$

where $i = 1,\ldots,n$, $h = 0,\ldots,n-1$ and $y \geq 0$. Moreover let Φ_{n-1},Φ_n be $(n+1)$-times differentiable.

If for $i = 2, \ldots, n-1$ and $y \geq 0$ holds

$$\frac{\partial}{\partial y}\big(\lambda_{(i+1)}(i,y) - \lambda_{(i)}(i,y)\big) \geq 0 \;, \tag{4.3}$$

then $X_*^{(1)}, \ldots, X_*^{(n)}$ are MTP_2.

Proof. Since $Y_{(i)}$ is MTP_2, by Lemma 4.10 it suffices to ensure that

$$\frac{\partial^2}{\partial t_k \partial t_l} \ln \frac{f^{Y_{1:n}^{(i)},\ldots,Y_{i:n}^{(i)}}(t_1,\ldots,t_i)}{f^{Y_{1:n}^{(i+1)},\ldots,Y_{i:n}^{(i+1)}}(t_1,\ldots,t_i)} \geq 0 \;, \tag{4.4}$$

for $0 \leq t_1 \leq \ldots \leq t_i$ and $1 \leq l < k \leq i,\ i = 2, \ldots, n-1$. Due to the representation (2.26) for marginal densities $f^{Y_{1:n}^{(i)},\ldots,Y_{i:n}^{(i)}}$, the expression on the left hand side of (4.4) can be rewritten as

$$\begin{aligned}
&\frac{\partial^2}{\partial t_k \partial t_l} \ln \frac{f^{Y_{1:n}^{(i)},\ldots,Y_{i:n}^{(i)}}(t_1,\ldots,t_i)}{f^{Y_{1:n}^{(i+1)},\ldots,Y_{i:n}^{(i+1)}}(t_1,\ldots,t_i)} \\
&\quad = \frac{\partial^2}{\partial t_k \partial t_l} \left[\ln\Big((-1)^i \Phi_i^{(i)}(y)\Big) - \ln\Big((-1)^i \Phi_{i+1}^{(i)}(y)\Big) \right]\Bigg|_{y=\tau} \;,
\end{aligned}$$

where $\tau = \sum_{j=1}^{i-1} t_j + (n-i+1)t_i$.

Then, for $k \neq i$ $\frac{\partial^2}{\partial t_k \partial t_l} \ln \frac{f^{Y_{1:n}^{(i)},\ldots,Y_{i:n}^{(i)}}}{f^{Y_{1:n}^{(i+1)},\ldots,Y_{i:n}^{(i+1)}}}$ turns into

$$\begin{aligned}
&\frac{\partial^2}{\partial t_k \partial t_l} \ln \frac{f^{Y_{1:n}^{(i)},\ldots,Y_{i:n}^{(i)}}(t_1,\ldots,t_i)}{f^{Y_{1:n}^{(i+1)},\ldots,Y_{i:n}^{(i+1)}}(t_1,\ldots,t_i)} \\
&\quad = \frac{\partial}{\partial t_l} \left[\frac{\Phi_i^{(i+1)}(y)}{\Phi_i^{(i)}(y)} - \frac{\Phi_{i+1}^{(i+1)}(y)}{\Phi_{i+1}^{(i)}(y)} \right]\Bigg|_{y = \sum_{j=1}^{i-1} t_j + (n-i+1)t_i} \\
&\quad = \frac{\partial}{\partial y} \left[\lambda_{(i+1)}(i,y) - \lambda_{(i)}(i,y) \right]\Bigg|_{y = \sum_{j=1}^{i-1} t_j + (n-i+1)t_i} \;.
\end{aligned} \tag{4.5}$$

By analogy, for $k = i$ the derivative can be calculated as

$$\frac{\partial^2}{\partial t_k \partial t_l} \ln \frac{f^{Y_{1:n}^{(i)},\ldots,Y_{i:n}^{(i)}}(t_1,\ldots,t_i)}{f^{Y_{1:n}^{(i+1)},\ldots,Y_{i:n}^{(i+1)}}(t_1,\ldots,t_i)}$$
$$= (n-i+1)\frac{\partial}{\partial y}\left[\lambda_{(i+1)}(i,y) - \lambda_{(i)}(i,y)\right]\Big|_{y = \sum_{j=1}^{i-1} t_j + (n-i+1)t_i} . \tag{4.6}$$

Substituting (4.6) and (4.5) into (4.4) we obtain the required statement. □

Lemma 4.17. *For $i = 1,\ldots,n$ let $Y_{(i)} = (Y_1^{(i)},\ldots,Y_n^{(i)})$ be an absolutely continuous MTP_2 vector of Schur-constant lifetimes with n-times differentiable joint survival function Φ_i. Moreover let Φ_{n-1},Φ_n be $(n+1)$-times differentiable. If for $i = 2,\ldots,n-1$ and $y \geq 0$ holds*

(i) $Y_{(i)} \leq_{hr} Y_{(i+1)}$,

(ii) $\dfrac{\lambda_{(i+1)}(i+1,y)}{\lambda_{(i+1)}(i,y)} \leq \dfrac{\lambda_{(i)}(i+1,y)}{\lambda_{(i)}(i,y)}$,

then the relationship (4.3) is satisfied, i.e.

$$\frac{\partial}{\partial y}\left(\lambda_{(i+1)}(i,y) - \lambda_{(i)}(i,y)\right) \geq 0 .$$

Proof. First, consider the derivative $\frac{\partial}{\partial y}\ln\lambda_{(i)}(h,y)$. Due to (2.25) it can be represented as

$$\frac{\partial}{\partial y}\ln\lambda_{(i)}(h,y) = \frac{\partial}{\partial y}\ln\left[(-1)^{h+1}\Phi_i^{(h+1)}(y)\right] - \frac{\partial}{\partial y}\ln\left[(-1)^{h}\Phi_i^{(h)}(y)\right]$$
$$= \lambda_{(i)}(h,y) - \lambda_{(i)}(h+1,y) .$$

Then for $i = 1,\ldots,n,\ h = 1,\ldots,n-1$ holds

$$\frac{\partial}{\partial y}\lambda_{(i)}(h,y) = \lambda_{(i)}(h,y)\frac{\partial}{\partial y}\ln\lambda_{(i)}(h,y)$$
$$= \lambda_{(i)}(h,y)\big(\lambda_{(i)}(h,y) - \lambda_{(i)}(h+1,y)\big).$$

The last observation allows to represent the expression on the left hand side of (4.3) as

$$\frac{\partial}{\partial y}\big(\lambda_{(i+1)}(i,y)-\lambda_{(i)}(i,y)\big)=\lambda_{(i+1)}(i,y)\big(\lambda_{(i+1)}(i,y)-\lambda_{(i+1)}(i+1,y)\big)\\ -\lambda_{(i)}(i,y)\big(\lambda_{(i)}(i,y)-\lambda_{(i)}(i+1,y)\big)\ .$$

It suffices to show that under the conditions of the lemma holds

$$\lambda_{(i+1)}(i,y)\big(\lambda_{(i+1)}(i+1,y)-\lambda_{(i+1)}(i,y)\big)\\ \leq\lambda_{(i)}(i,y)\big(\lambda_{(i)}(i+1,y)-\lambda_{(i)}(i,y)\big)\ . \tag{4.7}$$

Let us proceed with the verification of (4.7) in several steps.

Consider the case $\lambda_{(i)}(i,y)=0$, due to condition (i) of the lemma it implies $\lambda_{(i+1)}(i,y)=0$ and (4.7) turns into $0\leq 0$.

Next assume that $\lambda_{(i+1)}(i,y)=0$. Note that the MTP_2 property of $Y_{(i)}$ yields

$$\lambda_{(i)}(i+1,y)-\lambda_{(i)}(i,y)\geq 0\ .$$

Thus the expression on the left hand side of (4.7) turns into 0 and the one on the right hand side is non-negative. We can conclude that in this case (4.7) holds.

It is left to consider the case $\lambda_{(i)}(i,y)\lambda_{(i+1)}(i,y)>0$. Dividing both parts of (4.7) by $\lambda^2_{(i)}(i,y)\lambda^2_{(i+1)}(i,y)>0$ we obtain

$$\frac{1}{\lambda^2_{(i)}(i,y)}\left(\frac{\lambda_{(i+1)}(i+1,y)}{\lambda_{(i+1)}(i,y)}-1\right)\\ \leq\frac{1}{\lambda^2_{(i+1)}(i,y)}\left(\frac{\lambda_{(i)}(i+1,y)}{\lambda_{(i)}(i,y)}-1\right)\ . \tag{4.8}$$

Note that from condition (i) follows that

$$\lambda^2_{(i)}(i,y)\geq\lambda^2_{(i+1)}(i,y)\\ \Leftrightarrow\frac{1}{\lambda^2_{(i)}(i,y)}\leq\frac{1}{\lambda^2_{(i+1)}(i,y)}\ . \tag{4.9}$$

In addition, condition (ii) ensures

$$\frac{\lambda_{(i+1)}(i+1,y)}{\lambda_{(i+1)}(i,y)} - 1 \leq \frac{\lambda_{(i)}(i+1,y)}{\lambda_{(i)}(i,y)} - 1 \; . \tag{4.10}$$

Finally, from (4.9) together with (4.10) we can conclude that (4.8) is satisfied. □

Lemma 4.18. *For $i = 1, \ldots, n$ let $Y_{(i)} = (Y_1^{(i)}, \ldots, Y_n^{(i)})$ be an absolutely continuous MTP$_2$ vector of Schur-constant lifetimes with n-times differentiable joint survival functions Φ_i. Let Φ_{n-1}, Φ_n be $(n+1)$-times differentiable functions. If for $i = 2, \ldots, n-1$ and $y \geq 0$ holds*

(i) $Y_{(i)} \leq_{hr} Y_{(i+1)}$,

(ii) $\dfrac{\partial}{\partial y}\dfrac{\lambda_{(i+1)}(i,y)}{\lambda_{(i)}(i,y)} \geq 0$,

then

$$\frac{\partial}{\partial y}\left(\lambda_{(i+1)}(i,y) - \lambda_{(i)}(i,y)\right) \geq 0 \; .$$

Proof. It suffices to show that, under the conditions of the lemma, function

$$\lambda_{(i+1)}(i,y) - \lambda_{(i)}(i,y)$$

is increasing in y, i.e.

$$\begin{aligned} &\lambda_{(i+1)}(i,y_1) - \lambda_{(i)}(i,y_1) \leq \lambda_{(i+1)}(i,y_2) - \lambda_{(i)}(i,y_2) \\ &\Leftrightarrow \lambda_{(i+1)}(i,y_1) + \lambda_{(i)}(i,y_2) \leq \lambda_{(i+1)}(i,y_2) + \lambda_{(i)}(i,y_1) \end{aligned}$$

for $0 \leq y_1 \leq y_2$. In the last inequality denote the summands from left to right as a, b, c, d. Herewith we need to prove that $a + b \leq c + d$.

Since $Y_{(i)} \leq_{hr} Y_{(i+1)}$ and $Y_{(i)}$ is MTP$_2$ we can state

$$\begin{aligned} \lambda_{(i)}(i,y_1) &\geq \lambda_{(i+1)}(i,y_1) \\ \lambda_{(i)}(i,y_1) &\geq \lambda_{(i)}(i,y_2) \; , \end{aligned}$$

in other words $d \geq a, b$. Note that for $0 \leq y_1 \leq y_2$ condition (ii) is equivalent to

$$\frac{\lambda_{(i+1)}(i, y_1)}{\lambda_{(i)}(i, y_1)} \leq \frac{\lambda_{(i+1)}(i, y_2)}{\lambda_{(i)}(i, y_2)}$$
$$\Leftrightarrow \lambda_{(i+1)}(i, y_1)\lambda_{(i)}(i, y_2) \leq \lambda_{(i)}(i, y_1)\lambda_{(i+1)}(i, y_2) ,$$

in terms of the introduced notation $ab \leq cd$.

Then, we can conclude that

$$(c + d) - (a + b) = \frac{1}{d}[(d - a)(d - b) + (cd - ab)] \geq 0 .$$

Consequently, $a + b \leq c + d$, which is the required result.

The development of the inequality $a + b \leq c + d$ as above can be found in the proof of Theorem 2.1 in Karlin & Rinott (1980a). □

Theorem 4.19. *For $i = 1, \ldots, n$ let $Y_{(i)} = (Y_1^{(i)}, \ldots, Y_n^{(i)})$ be an absolutely continuous MTP_2 vectors of Schur-constant lifetimes with n-times differentiable joint survival functions Φ_i and hazard rates*

$$\lambda_{(i)}(h, y) = -\frac{\Phi_i^{(h+1)}(y)}{\Phi_i^{(h)}(y)} .$$

Moreover let Φ_{n-1}, Φ_n be $(n + 1)$-time differentiable functions.

If for $i = 2, \ldots, n - 1,\ y \geq 0$ one of the following conditions is satisfied

(i)

$$\frac{\partial}{\partial y}(\lambda_{(i+1)}(i, y) - \lambda_{(i)}(i, y)) \geq 0 ,$$

(ii) $Y_{(i)} \leq_{hr} Y_{(i+1)}$ *and*

$$\frac{\lambda_{(i+1)}(i + 1, y)}{\lambda_{(i+1)}(i, y)} \leq \frac{\lambda_{(i)}(i + 1, y)}{\lambda_{(i)}(i, y)} ,$$

(iii) $Y_{(i)} \leq_{hr} Y_{(i+1)}$ *and*

$$\frac{\partial}{\partial y}\frac{\lambda_{(i+1)}(i,y)}{\lambda_{(i)}(i,y)} \geq 0 \ ,$$

then $X_*^{(1)}, \ldots, X_*^{(n)}$ *are* MTP_2*.*

Moreover (ii) ⇒ (i) and (iii) ⇒ (i).

Proof. Direct consequence of Lemma 4.16, Lemma 4.17 and Lemma 4.18. □

Example 4.20. *Consider random vectors* $Y_{(i)} = (Y_1^{(i)}, \ldots, Y_n^{(i)})$ *described in Example 3.4,* $i = 1, \ldots, n$*. Recall that* $Y_{(i)}$ *possess the joint survival function described by (3.7), i.e.*

$$\overline{F_i}(t_1, \ldots, t_n) = \left(1 + \frac{\alpha_i(t_1 + \ldots + t_n)}{\alpha}\right)^{-\alpha} ,$$

where $\alpha_i, \alpha > 0$*,* $i = 1, \ldots, n$*. Next we are going to look at restrictions on* α_i *and* α *that ensure the condition (i) of Theorem 4.19.*

Taking into account (3.8), (i) can be written as

$$\begin{aligned}&\frac{\partial}{\partial y}\left(\lambda_{(i+1)}(i,y) - \lambda_{(i)}(i,y)\right)\\ &\quad = (\alpha + i)\left[-\left(\frac{\alpha_{i+1}}{\alpha}\right)^2\left(1 + \frac{\alpha_{i+1}}{\alpha}y\right)^{-2} + \left(\frac{\alpha_i}{\alpha}\right)^2\left(1 + \frac{\alpha_i}{\alpha}y\right)^{-2}\right] \geq 0 \ .\end{aligned}$$

The last inequality is equivalent to

$$\begin{aligned}&\frac{\alpha_{i+1}^2}{(\alpha + \alpha_{i+1}y)^2} \leq \frac{\alpha_i^2}{(\alpha + \alpha_i y)^2}\\ &\Leftrightarrow \alpha_{i+1}^2\alpha^2 + 2\alpha_{i+1}^2\alpha\,\alpha_i y + \alpha_{i+1}^2\alpha_i^2 y^2 \leq \alpha_i^2\alpha^2 + 2\alpha_i^2\alpha\,\alpha_{i+1}y + \alpha_{i+1}^2\alpha_i^2 y^2\\ &\Leftrightarrow \alpha^2(\alpha_{i+1} - \alpha_i)(\alpha_{i+1} + \alpha_i) \leq 2\alpha\alpha_i\alpha_{i+1}y(\alpha_i - \alpha_{i+1}) \ ,\end{aligned}$$

which holds for all $y \geq 0$ *iff* $\alpha_i \geq \alpha_{i+1}$*. Therefore, sequential order statistics based on such* $Y_{(i)}$ *are* MTP_2 *if* $\alpha_i \geq \alpha_{i+1}$*,* $i = 1, \ldots, n-1$*. Moreover, from Example 3.4 follows that* $Y_{(i)} \leq_{hr} Y_{(i+1)}$*.*

Remark 4.21. *Considering vectors with completely monotone survival functions Lemma 3.8 allows us to give conditions in Theorem 4.19 which may be easier to check. Assume that the joint density of $Y_1^{(i)}, \ldots, Y_n^{(i)}$ has a form*

$$f^{Y_1^{(i)},\ldots,Y_n^{(i)}}(t_1, \ldots, t_n) = \int_0^\infty \theta^n \exp\Big(-\theta \sum_{j=1}^n t_j\Big) \pi_i(\theta) d\theta \ , \tag{4.11}$$

where π_i is a univariate density function, $i = 1, \ldots, n$. First, note that due to representation (4.11) by Remark 3.45 the vectors $Y_{(i)}$ are MTP_2. Second, conditions $Y_{(i)} \leq_{hr} Y_{(i+1)}$ can be replaced by $\pi_{i+1} \leq_{lr} \pi_i$. Taking into account that hr-order is weaker than lr-order, the observation follows from Theorem 4.19 and Lemma 3.8.

Assume that $\Phi_i(y) = \Phi(y, \alpha_1^i, \ldots, \alpha_k^i)$. In other words Φ_i belong to the same distribution family and are distinguished only by values of the parameters $\alpha_1, \ldots, \alpha_k$. In particular, $\alpha_1^i, \ldots, \alpha_k^i$ is a set of parameters that correspond to Φ_i.

Lemma 4.22. *Let $Y_1^{(i)}, \ldots, Y_n^{(i)}$ be absolutely continuous MTP_2 Schur-constant lifetimes with n-times differentiable survival functions $\Phi_i(y) = \Phi(y, \alpha_1^i, \ldots, \alpha_k^i)$, $i = 1, \ldots, n$, $k \in \mathbb{N}$, and hazard rates*

$$\begin{aligned} \lambda_{(i)}(h, y) &= -\frac{\Phi^{(h+1)}(y, \alpha_1^i, \ldots, \alpha_k^i)}{\Phi^{(h)}(y, \alpha_1^i, \ldots, \alpha_k^i)} \\ &= \lambda(h, y, \alpha_1^i, \ldots, \alpha_k^i) \ . \end{aligned}$$

Moreover, let Φ_{n-1}, Φ_n be $(n+1)$-times differentiable in y. If for $y \geq 0$ and $i = 2, \ldots, n-1$ holds

- *(i) $\frac{\partial}{\partial y}\lambda(h, y, \alpha_1, \ldots, \alpha_k)$ is increasing (decreasing) in α_j if y is held fix, $j = 1, \ldots, k$,*
- *(ii) $\alpha_1^i, \alpha_2^i, \ldots, \alpha_n^i$ are increasing (decreasing) in i,*

then $X_*^{(1)}, \ldots, X_*^{(n)}$ are MTP_2.

Proof. In the following we consider an increasing sequence $\alpha_1^i, \alpha_2^i, \ldots, \alpha_n^i$. The proof for a decreasing sequence follows by analogy.

To begin with, rewrite the condition (i) of Theorem 4.19 as

$$\frac{\partial}{\partial y}\lambda(h, y, \alpha_1^i, \alpha_2^i \ldots, \alpha_k^i) \leq \frac{\partial}{\partial y}\lambda(h, y, \alpha_1^{i+1}, \alpha_2^{i+1} \ldots, \alpha_k^{i+1})$$

and denote by

$$g(h, y, \alpha_1, \alpha_2 \ldots, \alpha_k) = \frac{\partial}{\partial y} \lambda(h, y, \alpha_1, \alpha_2 \ldots, \alpha_k) \ .$$

Then, since g is increasing in $\alpha_1, \ldots, \alpha_k$ and $\alpha_1^i, \alpha_2^i, \ldots, \alpha_n^i$ is increasing in i we can state

$$\begin{aligned} g(h, y, \alpha_1^i, \alpha_2^i \ldots, \alpha_k^i) &\leq g(h, y, \alpha_1^{i+1}, \alpha_2^i \ldots, \alpha_k^i) \\ &\leq g(h, y, \alpha_1^{i+1}, \alpha_2^{i+1}, \alpha_3^i \ldots, \alpha_k^i) \\ &\ldots \\ &\leq g(h, y, \alpha_1^{i+1}, \alpha_2^{i+1}, \alpha_3^{i+1} \ldots, \alpha_k^{i+1}) \end{aligned}$$

for $i = 1, \ldots, n-1$. Thus we have shown that Theorem 4.19 (i) is satisfied and consequently $X_*^{(1)}, \ldots, X_*^{(n)}$ are MTP_2. □

Lemma 4.23. *For $i = 1, \ldots, n$ let $Y_{(i)} = (Y_1^{(i)}, \ldots, Y_n^{(i)})$ be an absolutely continuous vector of MTP_2 Schur-constant lifetimes with n-times differentiable survival function $\Phi_i(y) = \Phi(y, \alpha_1^i, \ldots, \alpha_k^i)$ and hazard rates*

$$\begin{aligned} \lambda_{(i)}(h, y) &= -\frac{\Phi^{(h+1)}(y, \alpha_1^i, \ldots, \alpha_k^i)}{\Phi^{(h)}(y, \alpha_1^i, \ldots, \alpha_k^i)} \\ &= \lambda(h, y, \alpha_1^i, \ldots, \alpha_k^i) \ . \end{aligned}$$

Moreover, let Φ_{n-1}, Φ_n be $(n+1)$-times differentiable in y.

If for $i = 2, \ldots, n-1$ and $y \geq 0$ holds

- *(i)* $Y_{(i)} \leq_{hr} Y_{(i+1)}$,
- *(ii)* $\frac{\lambda(i+1, y, \alpha_1, \ldots, \alpha_k)}{\lambda(i, y, \alpha_1, \ldots, \alpha_k)}$ *is decreasing (increasing) in α_j when y is held fix, $j = 1, \ldots, k$,*
- *(iii)* $\alpha_1^i, \alpha_2^i, \ldots, \alpha_n^i$ *are increasing (decreasing) in i,*

then $X_*^{(1)}, \ldots, X_*^{(n)}$ are MTP_2.

Proof. Similar to the proof of Lemma 4.22, Lemma 4.23 implies condition (ii) of Theorem 4.19 . □

Example 4.24. *With the help of Lemma 4.23 we can generalize Example 4.20. Namely, consider a vector of Schur-constant random variables* $Y_{(i)} = (Y_1^{(i)}, \dots, Y_n^{(i)})$ *with Pareto survival function of the second kind*

$$\Phi_i(t) = (1 + \beta_i t)^{-\alpha_i} \ ,$$

where $\alpha_i, \beta_i > 0,\ t \geq 0$. *In particular replacing* (α_i, β_i) *with* $(\alpha, \frac{\alpha_i}{\alpha})$ *we obtain the case considered in Example 4.20. The* h*-th derivative of the survival function is represented by*

$$\Phi_i^{(h)}(t) = (-1)^h \frac{(\alpha_i + h - 1)!}{(\alpha_i - 1)!} \beta_i^h (1 + \beta_i t)^{-\alpha_i - h}$$

and with the help of (2.25) a hazard rate can be calculated as

$$\begin{aligned} \lambda_{(i)}(h, y) &= -\frac{\Phi_i^{(h+1)}(y)}{\Phi_i^{(h)}(y)} \\ &= \frac{(\alpha_i + h + 1 - 1)!\,(\alpha_i - 1)!}{(\alpha_i - 1)!\,(\alpha_i + h - 1)!} \frac{\beta_i^{h+1}}{\beta_i^h} (1 + \beta_i y)^{-\alpha_i - h - 1 + \alpha_i + h} \\ &= (\alpha_i + h)\,\beta_i\,(1 + \beta_i y)^{-1} \ . \end{aligned}$$

Let us look at the requirements of Lemma 4.23. First, consider condition (i). In the following we are going to claim a stronger relation namely that $Y_{(i)} \leq_{lr} Y_{(i+1)}$. *Recall that Pareto survival function of the second kind correspond to the joint density*

$$f^{Y_1^{(i)}, \dots, Y_n^{(i)}}(t_1, \dots, t_n) = \int_0^\infty \theta^n \exp[-\theta \sum_{i=0}^n t_i] \pi_i(\theta) d\theta$$

with

$$\pi_i(\theta) = \frac{1}{\beta_i^{\alpha_i}} \frac{1}{\Gamma(\alpha_i)} \theta^{\alpha_i - 1} \exp[-\frac{\theta}{\beta_i}] \ .$$

According to Lemma 3.8 to verify $Y_{(i)} \leq_{lr} Y_{(i+1)}$ *it suffices to show* $\pi_{i+1} \leq_{lr} \pi_i$. *The last* *is equivalent to* $\frac{\partial}{\partial\theta}\frac{\pi_i(\theta)}{\pi_{i+1}(\theta)} \geq 0$ *for* $\theta \in \mathbb{R}_+$. *Let us look at* $\frac{\partial}{\partial\theta}\frac{\pi_i(\theta)}{\pi_{i+1}(\theta)}$:

$$\begin{aligned}\frac{\partial}{\partial\theta}\frac{\pi_i(\theta)}{\pi_{i+1}(\theta)} &= \frac{\partial}{\partial\theta}\left[\frac{\beta_{i+1}^{\alpha_{i+1}}}{\beta_i^{\alpha_i}}\frac{\Gamma(\alpha_{i+1})}{\Gamma(\alpha_i)}\frac{\theta^{\alpha_i-1}}{\theta^{\alpha_{i+1}-1}}\exp\Big(\frac{\theta}{\beta_{i+1}}-\frac{\theta}{\beta_i}\Big)\right] \\ &= \frac{\beta_{i+1}^{\alpha_{i+1}}}{\beta_i^{\alpha_i}}\frac{\Gamma(\alpha_{i+1})}{\Gamma(\alpha_i)}\theta^{\alpha_i-\alpha_{i+1}-1}\exp\Big(\frac{\theta}{\beta_{i+1}}-\frac{\theta}{\beta_i}\Big) \\ &\quad\times\left[(\alpha_i-\alpha_{i+1})+\Big(\frac{1}{\beta_{i+1}}-\frac{1}{\beta_i}\Big)\theta\right]\end{aligned}$$

Hence $Y_{(i)} \leq_{lr} Y_{(i+1)}$ *if* $\beta_{i+1} \leq \beta_i$ *and* $\alpha_{i+1} \leq \alpha_i$.

Next, consider the ratio of $\lambda_{(i)}(i+1,y)$ *and* $\lambda_{(i)}(i,y)$

$$\begin{aligned}\frac{\lambda_{(i)}(i+1,y)}{\lambda_{(i)}(i,y)} &= \frac{\alpha_i+h+1}{\alpha_i+h} \\ &= 1+\frac{1}{\alpha_i+h}\ .\end{aligned}$$

Then we can state

$$\frac{\partial}{\partial\beta_i}\frac{\lambda_{(i)}(i+1,y)}{\lambda_{(i)}(i,y)} = 0 \tag{4.12}$$

and

$$\begin{aligned}\frac{\partial}{\partial\alpha_i}\frac{\lambda_{(i)}(i+1,y)}{\lambda_{(i)}(i,y)} &= -\frac{1}{(\alpha_i+h)^2} \\ &\leq 0\end{aligned} \tag{4.13}$$

Due to (4.12) and (4.13) conditions (ii) and (iii) are satisfied. Thus, it follows by Lemma 4.23 that sequential order statistics based on random variables with Pareto survival function of the second kind are MTP_2 *if the corresponding joint densities are* lr*-ordered, namely* $f_i \leq_{lr} f_{i+1}$ *or in terms of parameters* $\beta_{i+1} \leq \beta_i$ *and* $\alpha_{i+1} \leq \alpha_i$, $i = 2, \ldots, n-1$. *Note that Lemma 4.22 also leads to this result.*

4.4.3 Archimedean copulas

Lemma 4.25. *For $i = 1, \dots, n$ consider absolutely continuous $Y_1^{(i)}, \dots, Y_n^{(i)}$. Let their survival function be an Archimedean survival copula with strict generator function ϕ_i, i.e.*

$$S_i(t_1, \dots, t_n) = \phi_i^{-1}\Big(\phi_i\big(\overline{F}_i(t_1)\big) + \cdots + \phi_i\big(\overline{F}_i(t_n)\big)\Big) .$$

Then the joint density function of $X_*^{(1)}, \dots, X_*^{(n)}$ can be calculated as

$$\begin{aligned} & f^{X_*^{(1)},\dots,X_*^{(n)}}(t_1, \dots, t_n) \\ &\qquad = n!\, \frac{\partial}{\partial y}\phi_1^{-1}(y)\Big|_{y=n\phi_1\left(\overline{F_1}(t_1)\right)} \phi_1'\big(\overline{F_1}(t_1)\big) f_1(t_1) \\ &\qquad\quad \times \prod_{i=2}^{n} \frac{\frac{\partial^i}{\partial y^i}\phi_i^{-1}(y)\big|_{y=\sum_{k=1}^{i-1}\phi_i\left(\overline{F_i}(t_k)\right)+(n-i+1)\phi_i\left(\overline{F_i}(t_i)\right)} \phi_i'\Big(\overline{F_i}(t_i)\Big) f_i(t_i)}{\frac{\partial^{i-1}}{\partial y^{i-1}}\phi_i^{-1}(y)\big|_{y=\sum_{k=1}^{i-1}\phi_i\left(\overline{F_i}(t_k)\right)+(n-i+1)\phi_i\left(\overline{F_i}(t_{i-1})\right)}} , \end{aligned}$$

for $0 \le t_1 \le \ldots \le t_n$.

Proof. Recall that the joint density function of $Y_1^{(i)}, \dots, Y_n^{(i)}$ can be calculated as

$$\begin{aligned} f_i(t_1, \dots, t_n) &= (-1)^n \frac{\partial^n}{\partial t_1 \cdots \partial t_n} S_i(t_1, \dots, t_n) \\ &= \frac{\partial^n}{\partial y^n}\phi_i^{-1}(y)\Big|_{y=\sum_{k=1}^{n}\phi_i\left(\overline{F_i}(t_k)\right)} \prod_{k=1}^{n} \phi_i'\big(\overline{F_i}(t_k)\big) f_i(t_k) . \end{aligned}$$

In the following, we are going to prove by induction that

$$\begin{aligned} & \int_{t_{h-1}}^{\infty} \cdots \int_{t_{n-1}}^{\infty} f_i(t_1, \dots, t_n) dt_n \dots dt_h \\ &\quad = \frac{1}{(n-h+1)!}\, \frac{\partial^{h-1}}{\partial y^{h-1}}\phi_i^{-1}(y)\Big|_{y=\sum_{k=1}^{h-2}\phi_i\left(\overline{F_i}(t_k)\right)+(n-h+2)\phi_i\left(\overline{F_i}(t_{h-1})\right)} \\ &\quad\ \times \prod_{k=1}^{h-1} \phi_i'\big(\overline{F_i}(t_k)\big) f_i(t_k) . \end{aligned} \tag{4.14}$$

Since $\overline{F_i}(\infty) = 0$ and ϕ_i is a strict generator, for $h = n$ holds

$$\int_{t_{n-1}}^{\infty} f_i(t_1, \ldots, t_n) dt_n$$
$$= \int_{t_{n-1}}^{\infty} \frac{\partial^n}{\partial z_1 \ldots \partial z_n} \phi_i^{-1}(z_1 + \ldots + z_n)\Big|_{\substack{z_k=\phi_i\left(\overline{F_i}(t_k)\right)\\ k=1,\ldots,n}} \prod_{k=1}^{n} \phi_i'\big(\overline{F_i}(t_k)\big) f_i(t_k) dt_n$$

Applying the Leibniz integral rule we obtain

$$\int_{t_{n-1}}^{\infty} f_i(t_1, \ldots, t_n) dt_n$$
$$= \Bigg[-\frac{\partial^{n-1}}{\partial z_1 \ldots \partial z_{n-1}} \int_{z_{n-1}}^{\infty} \frac{\partial}{\partial z_n} \phi_i^{-1}(z_1 + \ldots + z_n) dz_n$$
$$- \frac{\partial^{n-1}}{\partial z_1 \ldots \partial z_{n-2} \partial z_n} \phi_i^{-1}(z_1 + \ldots + z_{n-1} + z_n)\Big|_{z_n = z_{n-1}} \Bigg]\Bigg|_{\substack{z_k=\phi_i\left(\overline{F_i}(t_k)\right)\\ k=1,\ldots,n}}$$
$$\times \prod_{k=1}^{n-1} \phi_i'\big(\overline{F_i}(t_k)\big) f_i(t_k)$$
$$= \Bigg[\frac{\partial^{n-1}}{\partial z_1 \ldots \partial z_{n-1}} \left(\phi_i^{-1}(\sum_{k=1}^{n-2} z_k + 2z_{n-1}) - \phi_i^{-1}(\infty) \right)$$
$$- \frac{\partial^{n-1}}{\partial z_1 \ldots \partial z_{n-2} \partial z_n} \phi_i^{-1}(z_1 + \ldots + z_{n-1} + z_n)\Big|_{z_n = z_{n-1}} \Bigg]\Bigg|_{\substack{z_k=\phi_i\left(\overline{F_i}(t_k)\right)\\ k=1,\ldots,n}}$$
$$\times \prod_{k=1}^{n-1} \phi_i'\big(\overline{F_i}(t_k)\big) f_i(t_k)$$
$$= \left[2\frac{\partial^{n-1}}{\partial y^{n-1}} \phi_i^{-1}(y) - \frac{\partial^{n-1}}{\partial y^{n-1}} \phi_i^{-1}(y) \right]\Bigg|_{y=\sum_{k=1}^{n-2} \phi_i\left(\overline{F_i}(t_k)\right) + 2\phi_i\left(\overline{F_i}(t_{n-1})\right)}$$
$$\times \prod_{k=1}^{n-1} \phi_i'\big(\overline{F_i}(t_k)\big) f_i(t_k)$$

$$= \frac{\partial^{n-1}}{\partial y^{n-1}} \phi_i^{-1}(y) \bigg|_{y=\sum_{k=1}^{n-2} \phi_i\left(\overline{F_i}(t_k)\right)+2\phi_i\left(\overline{F_i}(t_{n-1})\right)} \prod_{k=1}^{n-1} \phi_i'(\overline{F_i}(t_k)) f_i(t_k) \ .$$

According to the induction hypothesis should be satisfied

$$\begin{aligned} &\int_{t_h}^{\infty} \cdots \int_{t_{n-1}}^{\infty} f_i(t_1, \ldots, t_n) dt_n \ldots dt_{h+1} \\ &\qquad = \frac{1}{(n-h)!} \frac{\partial^h}{\partial y^h} \phi_i^{-1}(y) \bigg|_{y=\sum_{k=1}^{h-1} \phi_i\left(\overline{F_i}(t_k)\right)+(n-h+1)\phi_i\left(\overline{F_i}(t_h)\right)} \\ &\qquad\quad \times \prod_{k=1}^{h} \phi_i'(\overline{F_i}(t_k)) f_i(t_k) \ . \end{aligned} \tag{4.15}$$

Then we can represent the integral on the left hand side of (4.14) as

$$\begin{aligned} &\int_{t_{h-1}}^{\infty} \cdots \int_{t_{n-1}}^{\infty} f_i(t_1, \ldots, t_n) dt_n \ldots dt_h \\ &\quad = \int_{t_{h-1}}^{\infty} \frac{1}{(n-h)!} \frac{\partial^h}{\partial y^h} \phi_i^{-1}(y) \bigg|_{y=\sum_{k=1}^{h-1} \phi_i\left(\overline{F_i}(t_k)\right)+(n-h+1)\phi_i\left(\overline{F_i}(t_h)\right)} \\ &\qquad\quad \times \prod_{k=1}^{h} \phi_i'(\overline{F_i}(t_k)) f_i(t_k) dt_h \\ &\quad = \frac{1}{(n-h+1)!} \frac{\partial^{h-1}}{\partial y^{h-1}} \phi_i^{-1}(y) \bigg|_{y=\sum_{k=1}^{h-2} \phi_i\left(\overline{F_i}(t_k)\right)+(n-h+2)\phi_i\left(\overline{F_i}(t_{h-1})\right)} \\ &\qquad\quad \times \prod_{k=1}^{h-1} \phi_i'(\overline{F_i}(t_k)) f_i(t_k) \end{aligned}$$

and the induction is completed. Equality (4.14) yields

$$
\begin{aligned}
&f^{Y^{(i)}_{i:n} \,|\, Y^{(i)}_{i-1:n},\dots,Y^{(i)}_{1:n}}(t_i|t_{i-1},\dots,t_1) \\
&\quad = \frac{\displaystyle\int_{t_i}^{\infty}\int_{y_{i+1}}^{\infty}\cdots\int_{y_{n-1}}^{\infty} f_i(t_1,\dots,t_i,y_{i+1},\dots,y_n)dy_n\cdots dy_{i+2}dy_{i+1}}{\displaystyle\int_{t_{i-1}}^{\infty}\int_{y_i}^{\infty}\cdots\int_{y_{n-1}}^{\infty} f_i(t_1,\dots,t_{i-1},y_i,\dots,y_n)dy_n\cdots dy_{i+1}dy_i} \\
&\quad = \frac{(n-i+1)!}{(n-i)!}\,\frac{\displaystyle\prod_{k=1}^{i}\phi_i'\big(\overline{F}_i(t_k)\big)f_i(t_k)}{\displaystyle\prod_{k=1}^{i-1}\phi_i'\big(\overline{F}_i(t_k)\big)f_i(t_k)} \\
&\qquad \times \frac{\frac{\partial^i}{\partial y^i}\phi_i^{-1}(y)\Big|_{y=\sum_{k=1}^{i-1}\phi_i\left(\overline{F}_i(t_k)\right)+(n-i+1)\phi_i\left(\overline{F}_i(t_i)\right)}}{\frac{\partial^{i-1}}{\partial y^{i-1}}\phi_i^{-1}(y)\Big|_{y=\sum_{k=1}^{i-2}\phi_i\left(\overline{F}_i(t_k)\right)+(n-i+2)\phi_i\left(\overline{F}_i(t_{i-1})\right)}} \\
&\quad = (n-i+1)\phi_i'\big(\overline{F}_i(t_i)\big)f_i(t_i) \\
&\qquad \times \frac{\frac{\partial^i}{\partial y^i}\phi_i^{-1}(y)\Big|_{y=\sum_{k=1}^{i-1}\phi_i\left(\overline{F}_i(t_k)\right)+(n-i+1)\phi_i\left(\overline{F}_i(t_i)\right)}}{\frac{\partial^{i-1}}{\partial y^{i-1}}\phi_i^{-1}(y)\Big|_{y=\sum_{k=1}^{i-2}\phi_i\left(\overline{F}_i(t_k)\right)+(n-i+2)\phi_i\left(\overline{F}_i(t_{i-1})\right)}}\;.
\end{aligned} \tag{4.16}
$$

Then the result follows from the observation that

$$
f^{X^{(1)}_*,\dots,X^{(n)}_*}(t_1,\dots,t_n) = f^{Y^{(1)}_{1:n}}(t_1)\prod_{i=2}^{n} f^{Y^{(i)}_{i:n} \,|\, Y^{(i)}_{i-1:n},\dots,Y^{(i)}_{1:n}}(t_i|t_{i-1},\dots,t_1)\;.
$$

□

Lemma 4.26. *Let $Y^{(i)}_1,\dots,Y^{(i)}_n$ possess a survival function represented by an Archimedean survival copula with a strict generator function $\phi_i,\ i=1,\dots,n$, i.e.*

$$
S_i(t_1,\dots,t_n) = \phi_i^{-1}\Big(\phi_i\big(\overline{F}_i(t_1)\big) + \cdots + \phi_i\big(\overline{F}_i(t_n)\big)\Big)\;.
$$

Then, for $h=2,\dots,n$ and $0 \le t_1 \le \ldots \le t_{h-1} \le t$ multivariate conditional hazard *rates of* $X_*^{(1)}, \dots, X_*^{(n)}$ *are of the form*

$$\begin{aligned}
&\lambda_{(*,h)}(t|t_1, \dots, t_{h-1}) \\
&\quad = (n-h+1)\,\phi_h'\big(\overline{F_h}(t)\big) f_h(t) \\
&\quad \times \frac{\frac{\partial^h}{\partial y^h}\phi_h^{-1}(y)\Big|_{y=\sum_{k=1}^{h-1}\phi_h\left(\overline{F_h}(t_k)\right)+(n-h+1)\phi_h\left(\overline{F_h}(t)\right)}}{\frac{\partial^{h-1}}{\partial y^{h-1}}\phi_h^{-1}(y)\Big|_{y=\sum_{k=1}^{h-1}\phi_h\left(\overline{F_h}(t_k)\right)+(n-h+1)\phi_h\left(\overline{F_h}(t)\right)}}\,.
\end{aligned}$$

Proof. According to Lemma 1.13 and Lemma 2.5 together with equality (2.13) conditional hazard rates of sequential order statistics can be calculated as

$$\begin{aligned}
&\lambda_{(*,h)}(t|t_1, \dots, t_{h-1}) \\
&\quad = \frac{\int_t^\infty \int_{y_{h+1}}^\infty \cdots \int_{y_{n-1}}^\infty f_h(t_1, \dots, t_{h-1}, t, y_{h+1}, \dots, y_n) dy_n \cdots dy_{h+2} dy_{h+1}}{\int_t^\infty \int_{y_h}^\infty \cdots \int_{y_{n-1}}^\infty f_h(t_1, \dots, t_{h-1}, y_h, \dots, y_n) dy_n \cdots dy_{h+1} dy_h}\,.
\end{aligned}$$

Then, applying (4.14) we obtain

$$\begin{aligned}
&\lambda_{(*,h)}(t|t_1, \dots, t_{h-1}) \\
&\quad = \frac{(n-h+1)!}{(n-h)!}\,\frac{\frac{\partial^h}{\partial y^h}\phi_h^{-1}(y)\Big|_{y=\sum_{k=1}^{h-1}\phi_h\left(\overline{F_h}(t_k)\right)+(n-h+1)\phi_h\left(\overline{F_h}(t)\right)}}{\frac{\partial^{h-1}}{\partial y^{h-1}}\phi_h^{-1}(y)\Big|_{y=\sum_{k=1}^{h-1}\phi_h\left(\overline{F_h}(t_k)\right)+(n-h+1)\phi_h\left(\overline{F_h}(t)\right)}} \\
&\quad \times \frac{\prod_{k=1}^{h-1}\phi_h'\big(\overline{F_h}(t_k)\big) f_h(t_k)\phi_h'\big(\overline{F_h}(t)\big) f_i(t)}{\prod_{k=1}^{h-1}\phi_h'\big(\overline{F_h}(t_k)\big) f_h(t_k)} \\
&\quad = (n-h+1)\frac{\frac{\partial^h}{\partial y^h}\phi_h^{-1}(y)\Big|_{y=\sum_{k=1}^{h-1}\phi_h\left(\overline{F_h}(t_k)\right)+(n-h+1)\phi_h\left(\overline{F_h}(t)\right)}\ \phi_h'\big(\overline{F_h}(t)\big) f_h(t)}{\frac{\partial^{h-1}}{\partial y^{h-1}}\phi_h^{-1}(y)\Big|_{y=\sum_{k=1}^{h-1}\phi_h\left(\overline{F_h}(t_k)\right)+(n-h+1)\phi_h\left(\overline{F_h}(t)\right)}}\,,
\end{aligned}$$

which was to be proved. □

Theorem 4.27. *For $i = 1, \ldots, n$ consider absolutely continuous lifetimes $Y_1^{(i)}, \ldots, Y_n^{(i)}$ with univariate marginal distribution functions F_i. Let $Y_1^{(i)}, \ldots, Y_n^{(i)}$ possess an Archimedean survival copula based on strict generator ϕ_i, i.e. the joint survival function of $Y_1^{(i)}$, $\ldots, Y_n^{(i)}$ can be represented as*

$$S_i(t_1, \ldots, t_n) = \phi_i^{-1}\big(\phi_i(\overline{F_i}(t_1)) + \cdots + \phi_i(\overline{F_i}(t_n))\big) \ ,$$

where $\phi_{n-1}^{-1}(t)$ and $\phi_n^{-1}(t)$ are $(n+1)$-times differentiable. Moreover let $Y_{1:n}^{(i)}, \ldots, Y_{n:n}^{(i)}$ possess the MTP_2 property for $i = 2, \ldots, n$. Denote $g_i(t) = \phi_i'\big(\overline{F_i}(t)\big) f_i(t)$.

If for $i, k, l = 2, \ldots, n-1$ such that $2 \le l < k \le i$ and $t_1, \ldots, t_n \in \mathbb{R}$ holds

$$\frac{\partial}{\partial t_l}\left[\lambda_{(i+1)}(t_i|t_1, \ldots, t_{i-1})\frac{g_{i+1}(t_k)}{g_{i+1}(t_i)} - \lambda_{(i)}(t_i|t_1, \ldots, t_{i-1})\frac{g_i(t_k)}{g_i(t_i)}\right] \ge 0 \ ,$$

then $X_*^{(1)}, \ldots, X_*^{(n)}$ are MTP_2.

Proof. According to Lemma 4.10 it suffices to ensure

$$\frac{\partial^2}{\partial t_k \partial t_l} \ln \frac{f^{Y_{1:n}^{(i)}, \ldots, Y_{i:n}^{(i)}}(t_1, \ldots, t_i)}{f^{Y_{1:n}^{(i+1)}, \ldots, Y_{i:n}^{(i+1)}}(t_1, \ldots, t_i)} \ge 0 \tag{4.17}$$

for $0 \le t_1 \le \ldots \le t_i$ and $1 \le l < k \le i$, $i = 2, \ldots, n-1$. For $i, j = 1, \ldots, n$ denote

$$y_{(j,i)} = \sum_{k=1}^{j-1} \phi_i\big(\overline{F_i}(t_k)\big) + (n - j + 1)\phi_i\big(\overline{F_i}(t_j)\big) \ .$$

Observe that the following equality holds

$$\begin{aligned} &\frac{\partial^2}{\partial t_k \partial t_l} \ln \frac{f^{Y_{1:n}^{(i)}, \ldots, Y_{i:n}^{(i)}}(t_1, \ldots, t_i)}{f^{Y_{1:n}^{(i+1)}, \ldots, Y_{i:n}^{(i+1)}}(t_1, \ldots, t_i)} \\ &\quad = \frac{\partial^2}{\partial t_k \partial t_l} \ln\left(\frac{\frac{\partial^i}{\partial y^i}\phi_i^{-1}(y)|_{y=y_{(i,i)}} \prod_{j=1}^i \phi_i'\big(\overline{F_i}(t_j)\big) f_i(t_j)}{\frac{\partial^i}{\partial y^i}\phi_{i+1}^{-1}(y)|_{y=y_{(i,i+1)}} \prod_{j=1}^i \phi_{i+1}'\big(\overline{F_{i+1}}(t_j)\big) f_{i+1}(t_j)}\right) \ . \end{aligned}$$

Noticing that

$$\frac{\partial^2}{\partial t_k \partial t_l} \ln\left(\frac{\prod_{j=1}^{i} \phi_i'(\overline{F_i}(t_j)) f_i(t_j)}{\prod_{j=1}^{i} \phi_{i+1}'(\overline{F_{i+1}}(t_j)) f_{i+1}(t_j)} \right) = 0 \ ,$$

it is left to consider the sign of

$$\frac{\partial^2}{\partial t_k \partial t_l} \ln\left(\frac{\frac{\partial^i}{\partial y^i} \phi_i^{-1}(y)|_{y=y_{(i,i)}}}{\frac{\partial^i}{\partial y^i} \phi_{i+1}^{-1}(y)|_{y=y_{(i,i+1)}}} \right) .$$

For $k < i$ the derivative turns into

$$\begin{aligned} &\frac{\partial^2}{\partial t_k \partial t_l} \ln\left(\frac{\frac{\partial^i}{\partial y^i} \phi_i^{-1}(y)|_{y=y_{(i,i)}}}{\frac{\partial^i}{\partial y^i} \phi_{i+1}^{-1}(y)|_{y=y_{(i,i+1)}}} \right) \\ &\quad = \frac{\partial}{\partial t_l} \left(- \frac{\frac{\partial^{i+1}}{\partial y^{i+1}} \phi_i^{-1}(y)|_{y=y_{(i,i)}} g_i(t_k)}{\frac{\partial^i}{\partial y^i} \phi_i^{-1}(y)|_{y=y_{(i,i)}}} + \frac{\frac{\partial^{i+1}}{\partial y^{i+1}} \phi_{i+1}^{-1}(y)|_{y=y_{(i,i+1)}} g_{i+1}(t_k)}{\frac{\partial^i}{\partial y^i} \phi_{i+1}^{-1}(y)|_{y=y_{(i,i+1)}}} \right) . \end{aligned} \tag{4.18}$$

By analogy, for $k = i$ we obtain

$$\begin{aligned} &\frac{\partial^2}{\partial t_i \partial t_l} \ln\left(\frac{\frac{\partial^i}{\partial y^i} \phi_i^{-1}(y)|_{y=y_{(i,i)}}}{\frac{\partial^i}{\partial y^i} \phi_{i+1}^{-1}(y)|_{y=y_{(i,i+1)}}} \right) \\ &\quad = (n - l + 1) \frac{\partial}{\partial t_l} \left(- \frac{\frac{\partial^{i+1}}{\partial y^{i+1}} \phi_i^{-1}(y)|_{y=y_{(i,i)}} g_i(t_i)}{\frac{\partial^i}{\partial y^i} \phi_i^{-1}(y)|_{y=y_{(i,i)}}} + \frac{\frac{\partial^{i+1}}{\partial y^{i+1}} \phi_{i+1}^{-1}(y)|_{y=y_{(i,i+1)}} g_{i+1}(t_i)}{\frac{\partial^i}{\partial y^i} \phi_{i+1}^{-1}(y)|_{y=y_{(i,i+1)}}} \right) . \end{aligned} \tag{4.19}$$

Taking into account Lemma 4.26, the required statement is a combination of (4.17) with (4.18) and (4.19). □

Remark 4.28. *As follows from Remark 2.38 Archimedean survival copulas generalize the concept of Schur-constant survival functions. In this context it can be seen that Theorem 4.27 yields a generalization of Lemma 4.16.*

Theorem 4.29. *For $i = 1, \dots, n$ let absolutely continuous lifetimes $Y_1^{(i)}, \dots, Y_n^{(i)}$ possess an Archimedean survival copula with strict generator ϕ_i and marginal survival functions $\overline{F}_i$, i.e. the joint survival function of $Y_1^{(i)}, \dots, Y_n^{(i)}$ can be represented as*

$$S_i(t_1, \dots, t_n) = \phi_i^{-1}\Big(\phi_i\big(\overline{F}_i(t_1)\big) + \ldots + \phi_i\big(\overline{F}_i(t_n)\big)\Big) \ .$$

Moreover let the marginal distribution functions be recursively defined as

$$\begin{aligned}\overline{F}_1(t) &= \overline{F}(t)\\ \overline{F}_{i+1}(t) &= \phi_{i+1}^{-1}\Big(\phi_i\big(\overline{F}_i(t)\big)\Big) \ ,\end{aligned} \tag{4.20}$$

for a marginal survival function $\overline{F}$ and $i = 1, \dots, n-1$.

Then, sequential order statistics $X_*^{(1)}, \dots, X_*^{(n)}$ are MTP$_2$, if sequential order statistics $Z_*^{(1)}, \dots, Z_*^{(n)}$ based on absolutely continuous Schur-constant lifetimes $T_1^{(i)}, \dots, T_n^{(i)}$ with survival functions

$$G_i(t_1, \dots, t_n) = \phi_i^{-1}(t_1 + \ldots + t_n) \ ,$$

are MTP$_2$ for $i = 1, \dots, n$.

Proof. For $i = 1, \dots, n$ and the choice of marginal survival functions (4.20) we obtain

$$\phi_{i+1}\big(\overline{F}_{i+1}(t)\big) = \phi_i\big(\overline{F}_i(t)\big) = \phi_1\big(\overline{F}(t)\big) \ .$$

Denote $R(t) = \phi_1\big(\overline{F}(t)\big)$, then by Lemma 4.25 the joint density of $X_*^{(1)}, \dots, X_*^{(n)}$ can be expressed as

$$\begin{aligned} f^{X_*^{(1)},\dots,X_*^{(n)}}(t_1, \dots, t_n) = n! \, &\frac{\partial}{\partial y}\phi_1^{-1}(y)|_{y=nR(t_1)}R'(t_1)\\ &\times \prod_{i=2}^{n} \frac{\frac{\partial^i}{\partial y^i}\phi_i^{-1}(y)|_{y=\sum_{k=1}^{i-1}R(t_k)+(n-i+1)R(t_i)}R'(t_i)}{\frac{\partial^{i-1}}{\partial y^{i-1}}\phi_i^{-1}(y)|_{y=\sum_{k=1}^{i-1}R(t_k)+(n-i+1)R(t_{i-1})}} \ . \end{aligned}$$

Then, the MTP_2 property of the function

$$g(t_1,\ldots,t_n) = n!\frac{\partial}{\partial y}\phi_1^{-1}(y)|_{y=nR(t_1)} \cdot \prod_{i=2}^{n} \frac{\frac{\partial^i}{\partial y^i}\phi_i^{-1}(y)|_{y=\sum_{k=1}^{i-1} R(t_k)+(n-i+1)R(t_i)}}{\frac{\partial^{i-1}}{\partial y^{i-1}}\phi_i^{-1}(y)|_{y=\sum_{k=1}^{i-1} R(t_k)+(n-i+1)R(t_{i-1})}}$$

ensures the same property of $X_*^{(1)},\ldots,X_*^{(n)}$.

In the following we will prove that under the conditions of the theorem the function $g(t_1,\ldots,t_n)$ is MTP_2. Note that applying Lemma 4.25 with the choice of marginal survival functions from (4.20) we obtain the following joint density of sequential order statistics $Z_*^{(1)},\ldots,Z_*^{(n)}$:

$$f^{Z_*^{(1)},\ldots,Z_*^{(n)}}(t_1,\ldots,t_n) = n!\frac{\partial}{\partial y}\phi_1^{-1}(y)|_{y=nt_1}\prod_{i=2}^{n} \frac{\frac{\partial^i}{\partial y^i}\phi_i^{-1}(y)|_{y=\sum_{k=1}^{i-1} t_k+(n-i+1)t_i}}{\frac{\partial^{i-1}}{\partial y^{i-1}}\phi_i^{-1}(y)|_{y=\sum_{k=1}^{i-1} t_k+(n-i+1)t_{i-1}}}$$

This density is assumed to be MTP_2. Since per definition ϕ_i is decreasing on $[0,\infty)$, we can state

$$R'(t) = (\phi_1)'(\overline{F}(t))\overline{F}'(t) \geq 0 \ .$$

Then, applying Lemma 3.24 with $l_i(t) = R(t),\ i = 1,\ldots,n$ we can conclude that $g(t_1,\ldots,t_n)$ is MTP_2, which was to be proved. □

Example 4.30. *For $i = 1,\ldots,n$ let $Y_1^{(i)},\ldots,Y_n^{(i)}$ possess a survival copula from the Clayton copula family with generator $\phi_i(t) = t^{-1\backslash\alpha_i} - 1$ and univariate survival functions $\overline{F}_i(t) = \exp\left(-\frac{\lambda\alpha_i t}{\alpha_1}\right),\ \alpha_i > 0$. Note that in this case holds*

$$\phi_i(\overline{F}_i(t)) = \phi_{i-1}(\overline{F}_{i-1}(t)) = \exp\left(-\frac{\lambda t}{\alpha_1}\right) - 1 \ .$$

Then, the conditions of Theorem 4.29 are satisfied and $X_*^{(1)},\ldots,X_*^{(n)}$ are MTP_2 if the same property holds for $Z_*^{(1)},\ldots,Z_*^{(n)}$ based on exchangeable random variables with Pareto *survival functions*

$$S_i(t_1,\ldots,t_n) = \Big(1 + \sum_{j=1}^{n} t_j\Big)^{-\alpha_i} \ ,$$

which can be represented as a Laplace transform of gamma distribution with the density

$$\pi_i(\theta) = \frac{1}{\Gamma(\alpha_i)}\theta^{\alpha_i - 1}e^{-\theta} \ .$$

From Example 4.24 follows that $Z_*^{(1)},\ldots,Z_*^{(n)}$ *are* MTP_2 *if* $\alpha_i \geq \alpha_{i+1}$.

Then, applying Lemma 4.23 we can conclude that $X_*^{(1)},\ldots,X_*^{(n)}$ *are* MTP_2 *if* $\alpha_i \geq \alpha_{i+1}$ *for* $i = 2,\ldots,n-1$.

Lemma 4.31. *Consider absolutely continuous exchangeable random variables* $Y_1,\ldots,Y_n$. *For* $l = 1,\ldots,k-2,\ k = 2,\ldots,n$ *the following conditions are equivalent*

(i) $\frac{\partial}{\partial t_l}P(Y_{k:n} > t_k \,|\, Y_{k-1:n} = t_{k-1},\ldots,Y_{1:n} = t_1) = 0$

(ii) $\frac{\partial}{\partial t_l}\lambda(t_k|t_{k-1},\ldots,t_1) = 0$,

under the assumption that $\lambda(t_k|t_{k-1},\ldots,t_1)$ *and* $\frac{\partial}{\partial t_l}\lambda(t_k|t_{k-1},\ldots,t_1)$ *are continuous in* $t_1,\ldots,t_k$ *where* $0 < t_1 \leq \ldots \leq t_k$.

Proof. According to Lemma 2.6 conditional hazard rates allow the following representation

$$\lambda(t_k|t_{k-1},\ldots,t_1) = -\frac{1}{(n-k+1)}\,\frac{\frac{\partial}{\partial t_k}P(Y_{k:n} > t_k \,|\, Y_{k-1:n} = t_{k-1},\ldots,Y_{1:n} = t_1)}{P(Y_{k:n} > t_k \,|\, Y_{k-1:n} = t_{k-1},\ldots,Y_{1:n} = t_1)} \ .$$

Then, for $l = 1,\ldots,k-2$ we can write

$$\begin{aligned}
&\frac{\partial}{\partial t_l}\lambda(t_k|t_{k-1},\ldots,t_1)\\
&\quad = -\frac{1}{n-k+1}\Bigg(\frac{\frac{\partial^2}{\partial t_l \partial t_k}P(Y_{k:n} > t_k \,|\, Y_{k-1:n} = t_{k-1},\ldots,Y_{1:n} = t_1)}{P(Y_{k:n} > t_k \,|\, Y_{k-1:n} = t_{k-1},\ldots,Y_{1:n} = t_1)}\\
&\qquad - \frac{\frac{\partial}{\partial t_l}P(Y_{k:n} > t_k \,|\, Y_{k-1:n} = t_{k-1},\ldots,Y_{1:n} = t_1)}{P(Y_{k:n} > t_k \,|\, Y_{k-1:n} = t_{k-1},\ldots,Y_{1:n} = t_1)^2}
\end{aligned}$$

$$\times \frac{\partial}{\partial t_k} P(Y_{k:n} > t_k \,|\, Y_{k-1:n} = t_{k-1}, \ldots, Y_{1:n} = t_1) \Bigg) \ .$$

where $l = 1, \ldots, k-2$ and $0 < t_1 \leq \ldots \leq t_k$. Thus from (i) follows (ii).

To prove the reversed statement recall that according to Lemma 2.11 (i) holds

$$\begin{aligned} &P(Y_{k:n} > t_k \,|\, Y_{k-1:n} = t_{k-1}, \ldots, Y_{1:n} = t_1) \\ &\qquad = \exp\Big(-(n-k+1) \int_{t_{k-1}}^{t_k} \lambda(u|t_{k-1}, \ldots, t_1) du\Big) \ . \end{aligned}$$

Since $\lambda(t_k|t_{k-1}, \ldots, t_1)$ and $\frac{\partial}{\partial t_l}\lambda(t_k|t_{k-1}, \ldots, t_1)$ are continuous in $t_1, \ldots, t_k$ we can state

$$\begin{aligned} &\frac{\partial}{\partial t_l} P(Y_{k:n} > t_k \,|\, Y_{k-1:n} = t_{k-1}, \ldots, Y_{1:n} = t_1) \\ &\quad = -(n-k+1) \exp\left(-(n-k+1) \int_{t_{k-1}}^{t_k} \lambda(u|t_{k-1}, \ldots, t_1) du\right) \\ &\quad\times \int_{t_{k-1}}^{t_k} \frac{\partial}{\partial t_l} \lambda(u|t_{k-1}, \ldots, t_1) du \ , \end{aligned}$$

and from (ii) follows (i). □

Theorem 4.32. *For $n > 3$ consider exchangeable random lifetimes $Y_1, \ldots, Y_n$ with a joint density $f^{Y_1,\ldots,Y_n} \in C^2$ and a joint survival function $S(t_1, \ldots, t_n)$ represented by an Archimedean survival copula with a strict generator ϕ, i.e.*

$$S(t_1, \ldots, t_n) = \phi^{-1}\Big(\phi\big(\overline{F}(t_1)\big) + \cdots + \phi\big(\overline{F}(t_n)\big)\Big) \ ,$$

where $\overline{F}(t)$ is a marginal survival function of Y_i, $i = 1, \ldots, n$.

Then, for $i, j = 1, \ldots, k-1$, $k > 3$, a conditional density

$$f^{Y_{k:n} \,|\, Y_{k-1:n}, \ldots, Y_{1:n}}(t_k|t_{k-1}, \ldots, t_1)$$

is TP_2 *in every pair of variables* $t_i, t_j,\ i \neq j$ *iff for* $l = 1, \ldots, k-1$ *holds*

$$\frac{\partial}{\partial t_l} \lambda(t_k | t_{k-1}, \ldots, t_1) = 0 \ ,$$

where $0 \leq t_1 < \ldots < t_k$ *such that* $f^{Y_1, \ldots, Y_k}(t_1, \ldots, t_k) \neq 0$.

Proof. Consider the conditional density

$$f^{Y_{k:n} \,|\, Y_{k-1:n}, \ldots, Y_{1:n}}(t_k | t_{k-1}, \ldots, t_1)$$

and assume that it is TP_2 in $t_i, t_j, 1 \leq i \neq j \leq k-1$.

By analogy to (4.16) we can represent the conditional density $f^{Y_{k:n} \,|\, Y_{k-1:n}, \ldots, Y_{1:n}}$ as

$$f^{Y_{k:n} \,|\, Y_{k-1:n}, \ldots, Y_{1:n}}(t_k | t_{k-1}, \ldots, t_1) = \frac{L^{(k)}\Big(\sum_{l=1}^{k-1} R(t_l) + (n-k+1) R(t_k)\Big) R'(t_k)}{L^{(k-1)}\Big(\sum_{l=1}^{k-2} R(t_l) + (n-k+2) R(t_{k-1})\Big)} \ ,$$

where $L^{(k)}(y) = (-1)^k \frac{\partial^k}{\partial y^k} \phi^{-1}(y)$ and $R(t) = \phi(\overline{F}(t))$.

For shortness sake we will use the notation

$$g(s, t_{k-1}, t_k) = \frac{L^{(k)}\Big(s + R(t_{k-1}) + (n-k+1) R(t_k)\Big) R'(t_k)}{L^{(k-1)}\Big(s + (n-k+2) R(t_{k-1})\Big)} \ .$$

First we are going to show that for $1 \leq i < k-1$ holds

$$\frac{\partial}{\partial t_i} f^{Y_{k:n} \,|\, Y_{k-1:n}, \ldots, Y_{1:n}}(t_k | t_{k-1}, \ldots, t_1) = 0 \ .$$

Since $f^{Y_{k:n} \,|\, Y_{k-1:n}, \ldots, Y_{1:n}}(t_k | t_{k-1}, \ldots, t_1)$ is TP_2 in t_i, t_j for $1 \leq i < j < k-1$, we observe that

$$\frac{\partial}{\partial t_i \partial t_j} \ln f^{Y_{k:n} \,|\, Y_{k-1:n}, \ldots, Y_{1:n}}(t_k | t_{k-1}, \ldots, t_1)$$

$$= \frac{\partial}{\partial t_i \partial t_j} \ln \frac{L^{(k)}\Big(\sum_{l=1}^{k-1} R(t_l) + (n-k+1)R(t_k)\Big) R'(t_k)}{L^{(k-1)}\Big(\sum_{l=1}^{k-2} R(t_l) + (n-k+2)R(t_{k-1})\Big)}$$

$$= \frac{\partial^2}{\partial s^2} \ln \frac{L^{(k)}\Big(s + R(t_{k-1}) + (n-k+1)R(t_k)\Big)}{L^{(k-1)}\Big(s + (n-k+2)R(t_{k-1})\Big)}\Bigg|_{s=\sum_{l=1}^{k-2} R(t_l)}$$

$$\times R'(t_k)R'(t_i)R'(t_j) \geq 0 \ .$$

Since ϕ is a decreasing function we can state that

$$R'(t) = -\phi'(\overline{F}(t))f(t) \geq 0 \ .$$

Therefore, for $0 < s < (k-2)R(t_{k-1})$ the following inequality holds

$$\frac{\partial^2}{\partial s^2} \ln \frac{L^{(k)}\Big(s + R(t_{k-1}) + (n-k+1)R(t_k)\Big)}{L^{(k-1)}\Big(s + (n-k+2)R(t_{k-1})\Big)} \geq 0 \tag{4.21}$$

From (4.21) we can conclude that $g(s, t_{k-1}, t_k)$ is log-convex and hence also convex in s on $(0, (k-2)R(t_{k-1}))$ for fixed t_{k-1}, t_k. Then, for $\theta \in [0,1]$ and $0 < x < y < (k-2)R(t_{k-1})$ holds

$$g(\theta x + (1-\theta)y, t_{k-1}, t_k) \leq \theta g(x, t_{k-1}, t_k) + (1-\theta)g(y, t_{k-1}, t_k) \ . \tag{4.22}$$

Let us assume that there exist t_k, t_{k-1}, x, y such that in (4.22) the inequality is strict. Then, due to continuity of $g(s, t_{k-1}, t_k)$, the strict inequality holds in a neighbourhood of $t_k : U(t_k) \subset [t_{k-1}, \infty)$. Integrating both parts of inequality (4.22) in t_k on (t_{k-1}, ∞) we obtain

$$\begin{aligned} &\int_{t_{k-1}}^{\infty} g(\theta x + (1-\theta)y, t_{k-1}, t_k) dt_k \\ &\leq \theta \int_{t_{k-1}}^{\infty} g(x, t_{k-1}, t_k) dt_k + (1-\theta) \int_{t_{k-1}}^{\infty} g(y, t_{k-1}, t_k) dt_k \ . \end{aligned} \tag{4.23}$$

Moreover, we can represent (4.23) as

$$\begin{aligned}
&\int_{U(t_k)} g(\theta x + (1-\theta)y, t_{k-1}, t_k)dt_k + \int_{(t_{k-1},\infty)\setminus U(t_k)} g(\theta x + (1-\theta)y, t_{k-1}, t_k)dt_k \\
&\leq \theta \int_{U(t_k)} g(x, t_{k-1}, t_k)dt_k + (1-\theta)\int_{U(t_k)} g(y, t_{k-1}, t_k)dt_k \\
&\quad + \theta \int_{(t_{k-1},\infty)\setminus U(t_k)} g(x, t_{k-1}, t_k)dt_k + (1-\theta)\int_{(t_{k-1},\infty)\setminus U(t_k)} g(y, t_{k-1}, t_k)dt_k \ ,
\end{aligned}$$

where for $U(t_k) = (a,b), t_{k-1} < a < b < \infty$ we denote by

$$\int_{(t_{k-1},\infty)\setminus U(t_k)} g(x, t_{k-1}, t_k)dt_k = \int_{t_{k-1}}^{a} g(x, t_{k-1}, t_k)dt_k + \int_{b}^{\infty} g(x, t_{k-1}, t_k)dt_k \ .$$

Then, due to the assumption above the inequality in (4.23) is strict, i.e.

$$\begin{aligned}
&\int_{t_{k-1}}^{\infty} g(\theta x + (1-\theta)y, t_{k-1}, t_k)dt_k \\
&\quad < \theta \int_{t_{k-1}}^{\infty} g(x, t_{k-1}, t_k)dt_k + (1-\theta)\int_{t_{k-1}}^{\infty} g(y, t_{k-1}, t_k)dt_k \ .
\end{aligned} \tag{4.24}$$

Due to continuity of ϕ and $\overline{F}$ for every $s \in (0, (k-2)R(t_{k-1}))$ there exist $t'_1, \ldots,$ t'_{k-2}, $0 < t'_1 < \ldots < t'_{k-2} < t_{k-1}$ such that $s = \sum_{l=1}^{k-2} R(t'_l)$. Recall that $g(s, t_{k-1}, t_k)$ represents a conditional density $f^{Y_{k:n} \mid Y_{k-1:n}, \ldots, Y_{1:n}}(t_k | t_{k-1}, t'_{k-2}, \ldots, t'_1)$, therefore

$$\int_{t_{k-1}}^{\infty} g(s, t_{k-1}, t_k)dt_k = 1 \ .$$

Then (4.24) turns into $1 < 1$ and we have reached a contradiction. We can conclude that our assumption was incorrect and there is no such t_k, t_{k-1}, x, y that in (4.22) the inequality is strict, meaning

$$\frac{\partial^2}{\partial s^2} g(s, t_{k-1}, t_k) = 0 \ . \tag{4.25}$$

Recall that we started with the assumption that g is log-convex, i.e.

$$\frac{\partial^2}{\partial s^2} \ln g(s, t_{k-1}, t_k) \geq 0.$$

Note that the above derivative can be calculated as

$$\frac{\partial^2}{\partial s^2} \ln g(s, t_{k-1}, t_k) = \frac{g(s, t_{k-1}, t_k) \frac{\partial^2}{\partial s^2} g(s, t_{k-1}, t_k) - (\frac{\partial}{\partial s} g(s, t_{k-1}, t_k))^2}{g(s, t_{k-1}, t_k)^2} \ .$$

Taking into account (4.25) we obtain the equality

$$\frac{\partial^2}{\partial s^2} \ln g(s, t_{k-1}, t_k) = -\left(\frac{\frac{\partial}{\partial s} g(s, t_{k-1}, t_k)}{g(s, t_{k-1}, t_k)} \right)^2 \ .$$

Then the inequality $\frac{\partial^2}{\partial s^2} \ln g(s, t_{k-1}, t_k) \geq 0$ holds iff $\frac{\partial}{\partial s} g(s, t_{k-1}, t_k) = 0$.

Thus we have shown that if $f^{Y_{k:n}|Y_{k-1:n},\ldots,Y_{1:n}}(t_k|t_{k-1},\ldots,t_1)$ is TP_2 in any pair (t_i, t_j), $1 \leq i < j \leq k-1$, then for $l = 1, \ldots, k-2$

$$\frac{\partial}{\partial t_l} f^{Y_{k:n}|Y_{k-1:n},\ldots,Y_{1:n}}(t_k|t_{k-1},\ldots,t_1) = 0$$

and consequently

$$\frac{\partial}{\partial t_l} P(Y_{k:n} > t | Y_{k-1:n} = t_{k-1}, \ldots, Y_{1:n} = t_1) = 0 \ , \tag{4.26}$$

for $0 < t_1 \leq \ldots \leq t_{k-1} \leq t$.

Moreover, by Lemma 4.31 condition (4.26) is equivalent to

$$\frac{\partial}{\partial t_l} \lambda(t_k | t_{k-1}, \ldots, t_1) = 0 \ ,$$

where $l = 1, \ldots, k-2$.

It is left to prove that

$$\frac{\partial}{\partial t_{k-1}} \lambda(t_k | t_{k-1}, \dots, t_1) = 0$$

Note that $Y_{1:n}, \dots, Y_{n:n}$ possess Markov property due to (4.26). Therefore by analogy to the proof in Section 5 of Lagerås (2010) for $t > t_{k-1}$ should hold

$$\begin{aligned} & P(Y_{k:n} > t | Y_{k-1:n} = t_{k-1}, \dots, Y_{1:n} = t_1) \\ &= \frac{L^{(k-1)}\Big(\sum_{l=1}^{k-1} R(t_l) + (n-k+1)R(t)\Big)}{L^{(k-1)}\Big(\sum_{l=1}^{k-2} R(t_l) + (n-k+2)R(t_{k-1})\Big)} \\ &= \frac{L^{(k-1)}\Big(R(t_{k-1}) + (n-k+1)R(t)\Big)}{L^{(k-1)}\Big((n-k+2)R(t_{k-1})\Big)} \; . \end{aligned} \tag{4.27}$$

Let us denote $x_1 = \sum_{l=1}^{k-2} R(t_l), x_2 = (n-k+2)R(t_{k-1}), x_3 = (n-k+1)R(t) - (n-k+1)R(t_{k-1})$. Then (4.27) can be rewritten as

$$\frac{L^{(k-1)}(x_1 + x_2 + x_3)}{L^{(k-1)}(x_1 + x_2)} = \frac{L^{(k-1)}(x_2 + x_3)}{L^{(k-1)}(x_2)} \; .$$

Let $f(x) = \frac{L^{(k-1)}(x_2+x)}{L^{(k-1)}(x_2)}$, then the above equation turns into

$$\frac{f(x_1 + x_3)}{f(x_1)} = f(x_3) \; ,$$

in other words $f(x_1 + x_3) = f(x_1)f(x_3)$. Consequently for some constant c holds $f(x) = e^{-cx}$.

Therefore from (4.27) we can conclude that

$$P(Y_{k:n} > t | Y_{k-1:n} = t_{k-1}, \dots, Y_{1:n} = t_1) = e^{-c(n-k+1)\Big(R(t) - R(t_{k-1})\Big)} \; .$$

Applying Lemma 2.6 a conditional hazard rate in this case can be calculated as

$$\begin{aligned}\lambda(t|t_{k-1},\dots,t_1) &= -\frac{R'(t)}{n-k+1}\frac{-c(n-k+1)e^{-c(n-k+1)\left(R(t)-R(t_{k-1})\right)}}{e^{-c(n-k+1)\left(R(t)-R(t_{k-1})\right)}} \\ &= cR'(t)\ ,\end{aligned} \tag{4.28}$$

where $k = 1,\dots,n-1$. Then, from representation (4.28) follows that

$$\frac{\partial}{\partial t_{k-1}}\lambda(t_k|t_{k-1},\dots,t_1) = 0\ .$$

It is left to show the reversed statement. Namely, assume that for $0 < t_1 \leq \dots \leq t_k$ and $l = 1,\dots,k-2$ holds

$$\frac{\partial}{\partial t_l}\lambda(t_k|t_{k-1},\dots,t_1) = 0\ .$$

Then, one may observe that for $1 \leq i < j \leq k-1$ holds

$$\begin{aligned}&\frac{\partial^2}{\partial t_i \partial t_j}\ln f^{Y_{k:n}|Y_{k-1:n},\dots,Y_{1;n}}(t_k|t_{k-1},\dots,t_1) \\ &\quad= \frac{\partial^2}{\partial t_i \partial t_j}\ln\left(\lambda(t_k|t_{k-1},\dots,t_1)\exp\left[-(n-k)\int_{t_{k-1}}^{t_k}\lambda(u|t_{k-1},\dots,t_1)du\right]\right) \\ &\quad= \frac{\partial}{\partial t_j}\left(\frac{\frac{\partial}{\partial t_i}\lambda(t_k|t_{k-1},\dots,t_1)}{\lambda(t_k|t_{k-1},\dots,t_1)} - (n-k)\int_{t_{k-1}}^{t_k}\frac{\partial}{\partial t_i}\lambda(u|t_{k-1},\dots,t_1)du\right) \\ &\quad= 0\ ,\end{aligned}$$

which proves the theorem. □

Remark 4.33. *Theorem 4.32 allows to conclude that considering $Y_1^{(i)},\dots,Y_n^{(i)}$ with survival functions represented by Archimedean copulas with a strict generator a necessary* *condition for* MTP_2 *of* $f^{X_*^{(i)} \mid X_*^{(i-1)},\ldots,X_*^{(1)}}$ *is*

$$\frac{\partial}{\partial t_l}\lambda_i(t_i|t_{i-1},\ldots,t_1) = 0$$

for $l = 1,\ldots,i-1$ *and* $i = 4,\ldots,n$.

Taking into account Lemma 2.14 and representation (4.28) we can conclude that such $Y_1^{(i)},\ldots,Y_n^{(i)}$ *are iid. In other words, for* $n > 3$ *the* MTP_2 *property of sequential order statistics* $X_*^{(1)},\ldots,X_*^{(n)}$ *can be achieved without stating any interdependence between the lifetimes* $Y_1^{(i)},\ldots,Y_n^{(i)}$ *on the different levels only if* $Y_1^{(i)},\ldots,Y_n^{(i)}$ *are iid for* $i = 1,\ldots,n$. *Since Schur-constant random variables represent a particular case of Archimedean copulas a similar observation holds also for them.*

Appendix

Proof of Lemma 1.13

Proof. We need to prove that for a symmetric function $f(y_1, \ldots, y_n)$ holds

$$\int_t^\infty \int_t^\infty \cdots \int_t^\infty f(t_1, \ldots, t_{j-1}, y_j, \ldots, y_n) dy_n \ldots dy_{j+1} dy_j$$
$$= (n - j + 1)! \int_t^\infty \int_{y_j}^\infty \cdots \int_{y_{n-1}}^\infty f(t_1, \ldots, t_{j-1}, y_j, \ldots, y_n) dy_n \ldots dy_{j+1} dy_j \; . \tag{4.29}$$

The proof will be conducted by induction.

We will start by proving that the equality holds for $j = n - 1$. Since f is a symmetric function, the integral on the right hand side of the equality (4.29) can be represented as

$$\int_t^\infty \int_t^\infty f(t_1, \ldots, t_{n-2}, y_{n-1}, y_n) dy_n dy_{n-1}$$
$$= \int_t^\infty \int_t^\infty f(t_1, \ldots, t_{n-2}, y_{n-1}, y_n) (\mathbb{I}_{\{y_n \le y_{n-1}\}} + \mathbb{I}_{\{y_n > y_{n-1}\}}) dy_n dy_{n-1}$$
$$= 2 \int_t^\infty \int_{y_{n-1}}^\infty f(t_1, \ldots, t_{n-2}, y_{n-1}, y_n) dy_n dy_{n-1} \; ,$$

which we set out to show.

Now let us assume, that for $j = k + 1$ holds

$$\int_t^\infty \int_t^\infty \cdots \int_t^\infty f(t_1, \ldots, t_k, y_{k+1}, \ldots, y_n) dy_n \ldots dy_{k+2} dy_{k+1}$$
$$= (n - k)! \int_t^\infty \int_{y_{k+1}}^\infty \cdots \int_{y_{n-1}}^\infty f(t_1, \ldots, t_k, y_{k+1}, \ldots, y_n) dy_n \ldots dy_{k+2} dy_{k+1} \; .$$

It is left to prove that for $j = k$ holds

$$\int_t^\infty \int_t^\infty \cdots \int_t^\infty f(t_1, \ldots, t_{k-1}, y_k, \ldots, y_n) dy_n \ldots dy_{k+1} dy_k$$
$$= (n - k + 1)! \int_t^\infty \int_{y_k}^\infty \cdots \int_{y_{n-1}}^\infty f(t_1, \ldots, t_{k-1}, y_k, \ldots, y_n) dy_n \ldots dy_{k+1} dy_k \; . \tag{4.30}$$

Note that for $y_j, \ldots, y_n \in \mathbb{R}$ the following representation exists

$$\begin{aligned} &I_{\mathbb{R}_<^{(n-j)}}(y_{j+1}, \ldots, y_n) \\ &\quad = I_{\mathbb{R}_<^{(n-j+1)}}(y_j, \ldots, y_n) + \sum_{s=j+1}^{n-1} I_{\mathbb{R}_<^{(n-j+1)}}(y_{j+1}, \ldots, y_s, y_j, y_{s+1}, \ldots, y_n) \\ &\qquad + I_{\mathbb{R}_<^{(n-j+1)}}(y_{j+1}, \ldots, y_n, y_j) \ , \end{aligned}$$

where $I_{\mathbb{R}_<^{(n-j)}}(y_{j+1}, \ldots, y_n)$ is an indicator function defined by

$$I_{\mathbb{R}_<^{(n-j+1)}}(y_j, \ldots, y_n) = \begin{cases} 1, & y_j < \ldots < y_n \\ 0, & \text{otherwise} \ . \end{cases}$$

Then we can the integral on the right hand side of (4.30) as

$$\begin{aligned} &\int_t^\infty \int_t^\infty \cdots \int_t^\infty f(t_1, \ldots, t_{j-1}, y_j, \ldots, y_n) dy_n \ldots dy_{j+1} dy_j \\ &\quad = \int_t^\infty \Bigg[(n-j)! \\ &\qquad \int_t^\infty \cdots \int_t^\infty f(t_1, \ldots, t_{j-1}, y_j, \ldots, y_n) \mathbb{I}_{\mathbb{R}_<^{(n-j)}}(y_{j+1}, \ldots, y_n) dy_n \ldots dy_{j+1} \\ &\quad \Bigg] dy_j \\ &\quad = (n-j)! \int_t^\infty \int_t^\infty \cdots \int_t^\infty \Bigg[f(t_1, \ldots, t_{j-1}, y_j, \ldots, y_n) \mathbb{I}_{\mathbb{R}_<^{(n-j+1)}}(y_j, \ldots, y_n) \\ &\qquad + \sum_{s=j+1}^{n-1} f(t_1, \ldots, y_j, \ldots, y_n) \mathbb{I}_{\mathbb{R}_<^{(n-j+1)}}(y_{j+1}, \ldots, y_s, y_j, y_{s+1}, \ldots, y_n) \\ &\qquad + f(t_1, \ldots, t_{j-1}, y_j, \ldots, y_n) \mathbb{I}_{\mathbb{R}_<^{(n-j+1)}}(y_{j+1}, \ldots, y_n, y_j) \\ &\quad \Bigg] dy_n \ldots dy_{j+1} dy_j \ . \end{aligned}$$

Applying the substitution $y'_j = y_{j+1}, y'_{s-1} = y_s, y'_s = y_j$ and switching the integration order according to Fubini's theorem we obtain

$$\begin{aligned}
&\int_t^\infty \int_t^\infty \cdots \int_t^\infty f(t_1,\dots,t_{j-1},y_j,\dots,y_n)dy_n \dots dy_{j+1}dy_j \\
&\quad = (n-j)!(n-j+1) \\
&\qquad \times \int_t^\infty \cdots \int_t^\infty f(t_1,\dots,t_{j-1},y'_j,\dots,y'_n)\mathbb{I}_{\mathbb{R}_<^{(n-j+1)}}(y'_j,\dots,y'_n)dy'_n \dots dy'_j \\
&\quad = (n-j+1)! \int_t^\infty \int_{y_j}^\infty \cdots \int_{y_{n-1}}^\infty f(t_1,\dots,t_{j-1},y_j,\dots,y_n)dy_n \dots dy_{j+1}dy_j
\end{aligned}$$

as was to be proved. □

Complement to Example 1.15

In the following we present the intermediate steps in the calculation of the joint density of sequential order statistics.

The joint survival functions of $Y_1^{(i)}, Y_2^{(i)}, Y_3^{(i)}$ can be computed as

$$\begin{aligned}
P(Y_1 > t_1, Y_2 > t_2, Y_3 > t_3) &= \int_0^\infty e^{-\theta(t_1+t_2+t_3)} \frac{\lambda^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\lambda\theta} d\theta \\
&= \frac{\lambda^\alpha}{(t_1+t_2+t_3+\lambda)^\alpha} \; .
\end{aligned}$$

Then the density f^{Y_1,Y_2,Y_3} is calculated as

$$\begin{aligned}
f^{Y_1,Y_2,Y_3}(t_1,t_2,t_3) &= (-1)^3 \frac{\partial^3}{\partial t_1 \partial t_2 \partial t_3} P(Y_1 > t_1, Y_2 > t_2, Y_3 > t_3) \\
&= \frac{\alpha(\alpha+1)(\alpha+2)\lambda^\alpha}{(t_1+t_2+t_3+\lambda)^{\alpha+3}} \; .
\end{aligned}$$

On basis of this result we can calculate the following densities

$$\begin{aligned}
f^{Y_{1:3},Y_{2:3},Y_{3:3}}(t_1,t_2,t_3) &= 3! f^{Y_1,Y_2,Y_3}(t_1,t_2,t_3) \\
&= 3! \frac{\alpha(\alpha+1)(\alpha+2)\lambda^\alpha}{(t_1+t_2+t_3+\lambda)^{\alpha+3}} \; , \\
f^{Y_{1:3},Y_{2:3}}(t_1,t_2) &= 3! \int_{t_2}^\infty f^{Y_{1:3},Y_{2:3},Y_{3:3}}(t_1,t_2,t_3)dt_3
\end{aligned}$$

$$
\begin{aligned}
&= \frac{3!\,\alpha(\alpha+1)\lambda^{\alpha}}{(t_1+2t_2+\lambda)^{\alpha+2}}\,, \\
f^{Y_{1:3}}(t_1) &= \int_{t_1}^{\infty} f^{Y_{1:3},Y_{2:3}}(t_1,t_2)dt_2 \\
&= \frac{3\,\alpha\,\lambda^{\alpha}}{(3t_1+\lambda)^{\alpha+1}}\,,
\end{aligned}
$$

where $0 \le t_1 \le t_2 \le t_3$.

Derivation of the covariance between X_i and X_j in Example 3.38

Without the loss of generality we restrict the consideration to $Cov(X_1, X_2)$. Since the distribution is symmetric all the other pairs X_i, X_j, $i \neq j$, have the same covariance. Recall that the covariance has the representation

$$
\begin{aligned}
Cov(X_1, X_2) &= E(X_1X_2) - E(X_1)E(X_2) \\
&= E(X_1X_2) - E(X_1)^2\,.
\end{aligned}
\tag{4.31}
$$

In the following we are going to calculate the expectations in (4.31) with the help of moment-generating functions. Specifically, according to Kotz et al. (2000) the moment- generating function for the Freund multivariate distribution of random variables $X_1, \ldots, X_k$ is of the form

$$
E\Big[e^{t_1X_1+\ldots+t_nX_k}\Big] = \frac{1}{k!}\sum_P{}^{*}\prod_{i=0}^{k-1}\left(1-\frac{\Theta_i}{k-i}\sum_{j=i+1}^{k} t_{P(j)}\right)^{-1}, \tag{4.32}
$$

where $\Theta_0, \ldots, \Theta_{k-1} > 0$ and $\{t_{P(1)}, \ldots, t_{P(k)}\}$ is one of the $k!$ possible permutations of $t_1, \ldots, t_k$ and $\sum_P^*$ is a sum over all such permutations.

Taking into account (4.32) we can calculate $E(X_1)$ as

$$
\begin{aligned}
E(X_1) &= \left[\frac{\partial}{\partial t_1}E\Big(\exp(t_1X_1+\ldots+t_kX_k)\Big)\right]_{t_1=\ldots=t_k=0} \\
&= \left[\frac{\partial}{\partial t_1}\frac{1}{k!}\sum_P{}^{*}\prod_{i=0}^{k-1}\Big(1-\frac{\Theta_i}{k-i}\sum_{j=i+1}^{k} t_{P(j)}\Big)^{-1}\right]_{t_1=\ldots=t_k=0}
\end{aligned}
$$

$$= \left[\frac{1}{k!} \sum_P^* \frac{\partial}{\partial t_1} \prod_{i=0}^{k-1} \Big(1 - \frac{\Theta_i}{k-i} \sum_{j=i+1}^{k} t_{P(j)} \Big)^{-1} \right]_{t_1 = \ldots = t_k = 0} .$$

Next we are going to calculate the partial derivative with the help of the formula

$$\frac{\partial}{\partial t} f(t) = f(t) \frac{\partial}{\partial t} \ln |f(t)|$$

for such t that $f(t) \neq 0$. In more detail

$$\begin{aligned} &\frac{1}{k!} \sum_P^* \frac{\partial}{\partial t_1} \prod_{i=0}^{k-1} \left(1 - \frac{\Theta_i}{k-i} \sum_{j=i+1}^{k} t_{P(j)} \right)^{-1} \\ &\quad = \frac{1}{k!} \sum_P^* \prod_{i=0}^{k-1} \left(1 - \frac{\Theta_i}{k-i} \sum_{j=i+1}^{k} t_{P(j)} \right)^{-1} \times \left(- \sum_{i=0}^{k-1} \frac{\partial}{\partial t_1} \ln \left| 1 - \frac{\Theta_i}{k-i} \sum_{j=i+1}^{k} t_{P(j)} \right| \right) . \end{aligned}$$

Without the loss of generality assume $t_1 = t_{P(l)}$, where l is specified by the permutation P. Then the last expression turns into

$$\frac{1}{k!} \sum_P^* \prod_{i=0}^{k-1} \left(1 - \frac{\Theta_i}{k-i} \sum_{j=i+1}^{k} t_{P(j)} \right)^{-1} \times \left(- \sum_{i=0}^{l-1} \frac{-\frac{\Theta_i}{k-i}}{1 - \frac{\Theta_i}{k-i} \sum_{j=i+1}^{k} t_{P(j)}} \right) .$$

For $t = (0, \ldots, 0)$ we obtain

$$E(X_1) = \frac{1}{k!} \sum_P^* \sum_{i=0}^{l-1} \frac{\Theta_i}{k-i} .$$

Note that for each l there exist $(k-1)!$ permutations such that $t_1 = t_{P(l)}$. Consequently we obtain

$$\begin{aligned} E(X_1) &= \frac{(k-1)!}{k!} \sum_{l=1}^{k} \sum_{i=0}^{l-1} \frac{\Theta_i}{k-i} \\ &= \frac{1}{k} \sum_{i=0}^{k-1} \sum_{l=i+1}^{k} \frac{\Theta_i}{k-i} \end{aligned}$$

$$= \frac{1}{k} \sum_{i=0}^{k-1} \Theta_i \ .$$

Then we have established the representation

$$\begin{aligned} E(X_1)E(X_2) &= (E(X_1))^2 \\ &= \frac{1}{k^2}\left(\sum_{i=0}^{k-1}\Theta_i\right)^2 = \frac{1}{k^2}\left(\sum_{i=0}^{k-1}\Theta_i^2 + 2\cdot\sum_{i=0}^{k-2}\sum_{j=i+1}^{k-1}\Theta_i\Theta_j\right) . \end{aligned}$$

It is left to calculate $E(X_1X_2)$. By analogy with the calculation of $E(X_1)$ we can write

$$E(X_1X_2) = \left[\frac{\partial^2}{\partial t_1 \partial t_2} E\big(\exp(t_1X_1 + \ldots + t_kX_k)\big)\right]_{t_1=\ldots=t_k=0} .$$

First let us obtain an explicit representation for the derivative:

$$\begin{aligned} &\frac{\partial^2}{\partial t_1 \partial t_2} E\big(\exp(t_1X_1 + \ldots + t_kX_k)\big) \\ &\quad = \frac{\partial}{\partial t_2}\frac{1}{k!}\sum_P{}^{*}\prod_{i=0}^{k-1}\left(1 - \frac{\Theta_i}{k-i}\sum_{j=i+1}^{k} t_{P(j)}\right)^{-1} \times \left(-\sum_{i=0}^{l-1}\frac{-\frac{\Theta_i}{k-i}}{1-\frac{\Theta_i}{k-i}\sum_{j=i+1}^{k} t_{P(j)}}\right) \\ &\quad = \frac{1}{k!}\sum_P{}^{*}\left[\left(\frac{\partial}{\partial t_2}\prod_{i=0}^{k-1}\left(1 - \frac{\Theta_i}{k-i}\sum_{j=i+1}^{k} t_{P(j)}\right)^{-1}\right) \times \left(\sum_{i=0}^{l-1}\frac{\frac{\Theta_i}{k-i}}{1-\frac{\Theta_i}{k-i}\sum_{j=i+1}^{k} t_{P(j)}}\right)\right. \\ &\qquad\qquad \left. + \prod_{i=0}^{k-1}\left(1 - \frac{\Theta_i}{k-i}\sum_{j=i+1}^{k} t_{P(j)}\right)^{-1} \times \frac{\partial}{\partial t_2}\left(\sum_{i=0}^{l-1}\frac{\frac{\Theta_i}{k-i}}{1-\frac{\Theta_i}{k-i}\sum_{j=i+1}^{k} t_{P(j)}}\right)\right] . \end{aligned}$$

Without the loss of generality assume $t_2 = t_{P(n)}$, where n is specified by P and $n \neq l$, then we can represent the derivative as

$$\frac{\partial^2}{\partial t_1 \partial t_2} E(\exp(t_1X_1 + \ldots + t_kX_k))$$

$$
\begin{aligned}
&= \frac{1}{k!}\sum_P{}^{*}\Bigg[\left(\prod_{i=0}^{k-1}\left(1-\frac{\Theta_i}{k-i}\sum_{j=i+1}^{k}t_{P(j)}\right)^{-1}\right)\\
&\quad\times\left(\sum_{i=0}^{n-1}\frac{\frac{\Theta_i}{k-i}}{1-\frac{\Theta_i}{k-i}\sum_{j=i+1}^{k}t_{P(j)}}\right)\times\left(\sum_{i=0}^{l-1}\frac{\frac{\Theta_i}{k-i}}{1-\frac{\Theta_i}{k-i}\sum_{j=i+1}^{k}t_{P(j)}}\right)\\
&\quad+\prod_{i=0}^{k-1}\left(1-\frac{\Theta_i}{k-i}\sum_{j=i+1}^{k}t_{P(j)}\right)^{-1}\times\left(\sum_{i=0}^{\min(l,n)-1}\frac{(\frac{\Theta_i}{k-i})^2}{(1-\frac{\Theta_i}{k-i}\sum_{j=i+1}^{k}t_{P(j)})^2}\right)\Bigg].
\end{aligned}
$$

For $t=(0,\ldots,0)$ we obtain

$$
E(X_1X_2)=\frac{1}{k!}\sum_P{}^{*}\left[\left(\sum_{i=0}^{n-1}\frac{\Theta_i}{k-i}\right)\times\left(\sum_{i=0}^{l-1}\frac{\Theta_i}{k-i}\right)+\left(\sum_{i=0}^{\min(l,n)-1}\Big(\frac{\Theta_i}{k-i}\Big)^2\right)\right].
$$

Note that for fixed l and n there exist $(k-2)!$ substitutions such that $t_1=t_{P(l)}$, $t_2=t_{P(n)}$. Moreover, the values of $E(X_1X_2)$ for the substitutions with $t_1=t_{P(l)}$, $t_2=t_{P(n)}$ and $t_1=t_{P(n)}, t_2=t_{P(l)}$ are equal. Therefore without the loss of generality we can assume $l<n$ and calculate $E(X_1X_2)$ as

$$
E(X_1X_2)=\frac{2}{k(k-1)}\sum_{l=1}^{k-1}\sum_{n=l+1}^{k}\left[\left(\sum_{i=0}^{n-1}\frac{\Theta_i}{k-i}\right)\times\left(\sum_{i=0}^{l-1}\frac{\Theta_i}{k-i}\right)+\sum_{i=0}^{l-1}\Big(\frac{\Theta_i}{k-i}\Big)^2\right].
$$

The first part of the double sum can be calculated as follows:

$$
\begin{aligned}
&\sum_{l=1}^{k-1}\sum_{n=l+1}^{k}\left(\sum_{i=0}^{n-1}\frac{\Theta_i}{k-i}\right)\times\left(\sum_{i=0}^{l-1}\frac{\Theta_i}{k-i}\right)\\
&=\sum_{l=1}^{k-1}\left[\left(\sum_{i=0}^{l-1}\frac{\Theta_i}{k-i}\right)\times\left(\sum_{n=l+1}^{k}\sum_{i=0}^{n-1}\frac{\Theta_i}{k-i}\right)\right]\\
&=\sum_{l=1}^{k-1}\left[\left(\sum_{i=0}^{l-1}\frac{\Theta_i}{k-i}\right)\times\left((k-l)\sum_{i=0}^{l-1}\frac{\Theta_i}{k-i}+\sum_{i=l}^{k-1}\Theta_i\right)\right]\\
&=\sum_{l=1}^{k-1}(k-l)\left(\sum_{i=0}^{l-1}\frac{\Theta_i}{k-i}\right)^2+\sum_{l=1}^{k-1}\left(\sum_{i=0}^{l-1}\frac{\Theta_i}{k-i}\right)\left(\sum_{i=l}^{k-1}\Theta_i\right)
\end{aligned}
$$

$$= \sum_{l=1}^{k-1}(k-l)\left[\sum_{i=0}^{l-1}\Big(\frac{\Theta_i}{k-i}\Big)^2 + 2\sum_{i=0}^{l-2}\sum_{j=i+1}^{l-1}\frac{\Theta_i\Theta_j}{(k-i)(k-j)}\right] + \sum_{l=1}^{k-1}\sum_{i=0}^{l-1}\sum_{j=l}^{k-1}\frac{\Theta_i\Theta_j}{k-i}$$

$$= \sum_{i=0}^{k-2}\Big(\frac{\Theta_i}{k-i}\Big)^2\left(\sum_{l=i+1}^{k-1}(k-l)\right) + 2\sum_{i=0}^{k-3}\sum_{l=i+2}^{k-1}\sum_{j=i+1}^{l-1}\frac{(k-l)\Theta_i\Theta_j}{(k-i)(k-j)}$$

$$+ \sum_{i=0}^{k-2}\sum_{l=i+1}^{k-1}\sum_{j=l}^{k-1}\frac{\Theta_i\Theta_j}{k-i}$$

$$= \sum_{i=0}^{k-2}\Big(\frac{\Theta_i}{k-i}\Big)^2\frac{(k-i-1)(k-i)}{2} + \sum_{i=0}^{k-3}\sum_{j=i+1}^{k-2}\frac{\Theta_i\Theta_j(k-j-1)}{(k-i)}$$

$$+ \sum_{i=0}^{k-2}\sum_{j=i+1}^{k-1}\frac{\Theta_i\Theta_j(j-i)}{k-i}\ .$$

Then we come to the following formula for $E(X_1X_2)$:

$$E(X_1X_2)$$

$$= \frac{2}{k(k-1)}\left[\sum_{i=0}^{k-2}\Big(\frac{\Theta_i}{k-i}\Big)^2\frac{(k-i-1)(k-i)}{2} + \sum_{i=0}^{k-3}\sum_{j=i+1}^{k-2}\frac{\Theta_i\Theta_j(k-j-1)}{(k-i)}\right.$$

$$\left. + \sum_{i=0}^{k-2}\sum_{j=i+1}^{k-1}\frac{\Theta_i\Theta_j(j-i)}{k-i} + \sum_{i=0}^{k-2}\Big(\frac{\Theta_i}{k-i}\Big)^2\frac{(k-i-1)(k-i)}{2}\right]$$

$$= \frac{2}{k(k-1)}\left[\sum_{i=0}^{k-2}\Big(\frac{\Theta_i}{k-i}\Big)^2(k-i-1)(k-i) + \sum_{i=0}^{k-3}\sum_{j=i+1}^{k-2}\frac{\Theta_i\Theta_j(k-j-1)}{(k-i)}\right.$$

$$\left. + \sum_{i=0}^{k-2}\sum_{j=i+1}^{k-1}\frac{\Theta_i\Theta_j(j-i)}{k-i}\right]\ .$$

Finally we can calculate the covariance as

$$Cov(X_1,X_2)$$

$$= \frac{2}{k(k-1)}\left[\sum_{i=0}^{k-2}\Big(\frac{\Theta_i}{k-i}\Big)^2(k-i-1)(k-i) + \sum_{i=0}^{k-3}\sum_{j=i+1}^{k-2}\frac{\Theta_i\Theta_j(k-j-1)}{(k-i)}\right.$$

$$+ \sum_{i=0}^{k-2} \sum_{j=i+1}^{k-1} \frac{\Theta_i \Theta_j (j-i)}{k-i} \Bigg]$$

$$- \frac{1}{k^2} \left[\sum_{i=0}^{k-1} \Theta_i^2 + 2 \cdot \sum_{i=0}^{k-2} \sum_{j=i+1}^{k-1} \Theta_i \Theta_j \right]$$

$$= \frac{1}{k^2(k-1)} \Bigg[\sum_{i=0}^{k-1} \Theta_i^2 \left(\frac{2k(k-i-1)}{k-i} - k + 1 \right)$$

$$+ 2 \sum_{i=0}^{k-2} \sum_{j=i+1}^{k-1} \Theta_i \Theta_j \left(\frac{k(k-j-1)}{k-i} + \frac{k(j-i)}{k-i} - (k-1) \right) \Bigg]$$

$$= \frac{1}{k^2(k-1)} \left[\sum_{i=0}^{k-1} \Theta_i^2 \Big(k - \frac{k+i}{k-i} \Big) - 2 \sum_{i=0}^{k-2} \sum_{j=i+1}^{k-1} \Theta_i \Theta_j \Big(\frac{i}{k-i} \Big) \right].$$

Thus, we have obtained the formula for the covariance of two random variables belonging to the n-dimensional random vector distributed according to Freund's multivariate exponential distribution:

$$Cov(X_1, X_2) = \frac{1}{k^2(k-1)} \cdot \left[\sum_{i=0}^{k-1} \Theta_i^2 \Big(k - \frac{k+i}{k-i} \Big) - 2 \sum_{i=0}^{k-2} \sum_{j=i+1}^{k-1} \Theta_i \Theta_j \Big(\frac{i}{k-i} \Big) \right].$$

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