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Cost Optimal Maintenance in Systems with Imperfect Maintenance

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Abstract

This thesis is concerned with the statistical modeling of optimal maintenance actions in repairable systems with either continuous or discrete lifetime distribution.
Five imperfect maintenance models are considered, that is, the impact of a preventive maintenance (PM) action is not minimal (as bad as old) and not perfect (as good as new) but lies in between these boundary cases. Two of the imperfect maintenance models examined, uses a periodic imperfect PM policy with failure type specific conditional maintenance (CM) actions. In these models the underlying system has two failure types whereby minor failures can only be removed through minimal repair and major failures can only be removed through replacement. Two further imperfect maintenance models uses a sequential failure limit PM policy with imperfect CM and imperfect PM actions. Moreover, an imperfect maintenance model with sequential PM policy and minimal CM actions is investigated. This model takes into account that with increasing age PM actions have to be done more often. Cost optimal maintenance policies for some cost functions and different continuous and discrete lifetime distributions are considered.

Zusammenfassung

Die vorliegende Arbeit beschäftigt sich mit der Modellierung optimaler Instandhaltungsmaßnahmen für reparierbare Systeme, die entweder eine stetige oder eine diskrete Ausfallverteilung besitzen.
Insgesamt werden fünf unvollständige Reparaturmodelle betrachtet, bei denen die vorbeugenden Instandhaltungen Reparaturgrade zwischen den beiden Extremen vollständige Erneuerung und minimale Reparatur zulassen. In zwei der betrachteten unvollständigen Reparaturmodellen wird eine periodische Instandhaltungsstrategie verwendet und das zugrundeliegende System besitzt zwei unterschiedliche Typen von Systemausfällen. Kleine Ausfälle können durch minimale Reparatur behoben werden, wohingegen große Ausfälle nur durch eine Erneuerung behoben werden können. Zwei weitere unvollständige Reparaturmodelle verwenden eine sequenzielle Ausfalllimit Instandhaltungsstrategie mit unvollständigen ausfallbedingten und vorbeugenden Instandhaltungen. Darüber hinaus wird ein weiteres Reparaturmodell betrachtet, bei dem eine sequenzielle Instandhaltungsstrategie mit unvollständigen vorbeugenden und minimalen ausfallbedingten Instandhaltungen zugrunde liegt. In diesem Modell wird berücksichtigt, dass ein System mit fortschreitendem Alter häufiger instand gesetzt werden muss.
Für die betrachteten unvollständigen Reparaturmodelle werden, unter Verwendung verschiedener Kostenfunktionen und verschiedener stetiger und diskreter Lebensdauerverteilungen, kostenoptimale Instandhaltungsstrategien bestimmt und ausgewertet.

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List of Abbreviations

CDF cumulative distribution function

CM corrective maintenance

DLFRD discrete linear failure rate distribution

DMWD discrete modified Weibull distribution

DRD discrete Rayleigh distribution

DRMWD discrete reduced modified Weibull distribution

DWD discrete Weibull distribution

e.g. for example (abbreviation of Latin ”exempli gratia”)

et al. and others (abbreviation of Latin ”et alli”)

i.e. that is (Latin ”id est”)

LFRD linear failure rate distribution

MSS multi-state system

MWD modified Weibull distribution

PDF probability density function

PM preventive maintenance

PMF probability mass function

RD Rayleigh distribution

RMWD reduced modified Weibull distribution

WD Weibull distribution

Notation

$C(\cdot)$ function of the average maintenance costs per unit time

c_{CM} costs for corrective maintenance (CM) action

c_F constant cost value

c_I constant cost value

c_k realization of the kth maintenance time

c_M constant costs for a minimal repair

c_{PM} costs for PM action

$c_{PM}(k)$ costs of the kth PM action

c_R costs of a replacement

c_S constant cost value

Δ_n rectangular distributed random number

$E(\cdot)$ expected value

$F^T(\cdot)$ cumulative distribution function of the random variable T

$f^T(\cdot)$ probability density function of the random variable T

$f^{T_1^*,\ldots,T_k^*}(t_1,\ldots,t_k)$ joint probability density function of the random variables $T_1^*,\ldots,T_k^*$

$\bar{F}^T(\cdot)$ survival function of the random variable T

$H^T(\cdot)$ cumulative hazard function of the random variable T

$h^T(\cdot)$ hazard function or failure rate of the random variable T

L cycle length

$\Lambda^N(\cdot)$ mean function of the counting process $N=(N_t)_{t\geq 0}$

$\lambda^N(\cdot)$ intensity function of the counting process $N=(N_t)_{t\geq 0}$

$\lambda^T(\cdot)$ alternative failure rate of T

M state space of a Markov chain

$N=(N_t)_{t\geq 0}$ failure counting process for a repairable system without PM

$\mathbb{N}$ set of all natural numbers $\{0, 1, 2, \dots\}$

N after $N-1$ PM actions a preventive replacement is done

n index or number of states of a MSS in which the system can fail

$N' = (N'_t)_{t \geq 0}$ type 1 failure counting process for a repairable system without PM

$N'' = (N''_t)_{t \geq 0}$ type 2 failure counting process for a repairable system without PM

$N_{\max}$ upper limit of N

$\mathbb{N}^+$ set of all natural numbers without 0, i.e. $\{1, 2, \dots\}$

$N^* = (N^*_t)_{t \geq 0}$ failure counting process for a repairable system with PM

$N^\star$ cost optimal value of N

N''_t random number of type 2 failures up to time t for a repairable system without PM

N_t random number of failures up to time t for a repairable system without PM

N^*_t random number of failures up to time t for a repairable system with PM

$N^{*'}_t$ random number of type 1 failures up to time t for a repairable system with PM

$N^{*''}_t$ random number of type 2 failures up to time t for a repairable system with PM

P probability measure

p probability of a type 2 failure

$p_{ij}(m, n)$ transition probabilities of the Markov chain from state e_i at time m to state e_j at time n

$p_i(n)$ probability that at time n the system has state e_i

$p_n^N(s, t)$ probability of n failures in the interval $(s, t]$ of a repairable system without PM

$p_n^{N^*}(s, t)$ probability of n failures in the interval $(s, t]$ of a repairable system with PM

S_t random state at t of a system without PM

s_t realisation of S_t or S^*_t

S^*_t random state at t of a system with PM

T random lifetime of a nonrepairable system without PM

t point in time

τ time between two PM actions

t_k time of the kth repair

$T_k^{*'}$ random time of the kth type 1 failure of a repairable system with PM

$T_k^{*''}$ random time of the kth type 2 failure of a repairable system with PM

T_n nth failure time of a repairable system without PM

$T_n^{'}$ random time of the nth type 1 failure of a repairable system without PM

$T_n^{''}$ random time of the nth type 2 failure of a repairable system without PM

T_n^* random time of the nth failure of a repairable system with PM and without distinction of failure types

t_n^* realization of the random variable T_n^*

T^* random time to failure of MSS with PM and without distinction of failure types

T^v remaining lifetime of a repairable system after a maintenance action that reduces the virtual age to v

v constant virtual age of a repairable system after PM action

v_k virtual age of a repairable system after the kth repair

X random variable

ξ degree of repair

ξ_k degree of the kth repair

x_k interval length after which the kth PM action is carried out

X_n one-step transition matrix of the Markov chain $(S_t)_{t \in \mathbb{N}}$

X_n^* one-step transition matrix of the Markov chain $(S_t^*)_{t \in \mathbb{N}}$

y_k virtual age immediately before the kth PM action

Z random number of type 1 failures until the first type 2 failure occurs

Z_t random number of type 1 failures in an interval with length $\min\{T_1^{*''}, t\}$

1. Introduction

In our days we use a multitude of technical systems not only in industry but also in our daily life. Especially for the systems used in production, transportation services and communication services, reliability plays a key role because most people see reliability as one of the most important quality characteristics. Maintenance actions can help to improve the reliability of a system and therefore, in the past several decades a multitude of maintenance models have been discussed in literature.
This research is concerned with the statistical modeling and optimization of imperfect maintenance models for repairable deteriorating systems with continuous or discrete lifetime distributions. The considered systems are assumed to be a unity in relation to the emergence, the occurrence and the localization of failures as well as related to the planing and the performing of maintenance actions [9]. When a failure occurs a repairable system can be restored to an operating condition by some repair process. Therefore, it is not necessary to replace the whole system and the failure intensity of the system depends on the history of repairs. In general, there are two kinds of maintenance actions. Preventive or planned maintenance actions and unplanned corrective maintenance actions. Preventive maintenance (PM) occurs when the system is operating and corrective maintenance (CM), also called repair, is carried out after a failure of the system. The aim of CM actions is to retain the system in or restore it to an acceptable operating condition [39].
In general, two steps are needed to build maintenance models. The first one has to define the effect of PM and CM actions and the second one has to choose a PM policy, which defines the link between PM and CM actions [17].
The modeling of the maintenance effect can be done, for example, through reduction of failure intensity or virtual age (see for example Doyen and Gaudoin [16]). In this thesis it is assumed that maintenance actions have an influence on the failure intensity of the system in such a way that they adjust the virtual age of the system in a Kijima type manner. Kijima [24] proposed that the state of the system just after a maintenance action can be described by its virtual age, which is smaller or equal than the real age of the system. Therefore, Kijima [24] constructed two virtual age models depending on how the maintenance action affects the virtual age. In Kijima model I it is assumed that a repair cannot remove the damage that incurred before the previous repair action. In Kijima model II it is assumed that a repair action can remove the whole damage accumulated up to the time of the repair action. This means that in Kijima model II the repair actions can reset the virtual age of the system to a state between as good as new and as bad as old. The two most common assumptions on the influence of maintenance actions on the failure intensity of the system is minimal repair or as bad as old and perfect repair or as good as new. After minimal repair the failure intensity of the system is the same it had when it failed. Perfect repair means the failure intensity of the system after repair is that of a new system [5]. Beichelt [10],

for example, contains different basic and sophisticated perfect maintenance models. In reality the state of the system after maintenance will often not be as good as new and not as bad as old, but something in between. In this case the maintenance action or repair is called to be imperfect and these imperfect maintenance actions form the basis of all maintenance models in this thesis. Imperfect maintenance is still a relative young field of science in comparison to perfect maintenance. Doyen and Gaudoin [17], Pham and Wang [39], Wang [47] or Pham and Wang [40] give an overview of different perfect and imperfect maintenance models.
Regarding the selection of the PM policy there are plenty of choices. PM actions can be, for instance, time-based like age-dependent, periodic or sequential, though the age-dependent PM policy is the most popular one. Beside this, the PM actions can also be condition-based and therefore occur at unscheduled times, for example, when the failure intensity or reliability of a system reaches a predetermined level. A review of possible maintenance policies is presented, for example, in Pham and Wang [39], Wang [47] and Sarkar et al. [44]. The maintenance models discussed in this thesis, use periodic, sequential and failure limit PM policies.
Another significant aspect of the models investigated in this thesis is the lifetime distribution of the underlying system. Most models in reliability theory use continuous lifetime distributions, as is the case in three out of five maintenance models in this research. This is appropriate because in reality most of the lifetimes are continuous. But there are also several situations where discrete failure data arise. This is the case if the life length of a system is measured in cycles and the number of cycles successfully completed prior to failure is observed. The same holds if we have a multi-state system (MSS) and the number of states prior to failure is observed. For these cases discrete lifetime distributions are needed to model optimal maintenance actions.
MSS are an important area in modern reliability theory. They provide a flexible tool for modeling engineering systems in real life. Therefore, two out of five models in this thesis are concerned with the modeling of optimal maintenance in MSS with a fixed number of states as described in Kahle [22]. Multi-state systems were first introduced in Barlow and Wu [6] and El-Neveihi [18]. A historical overview of MSS and an overview of ideas for MSS reliability theory can be found for example in Lisnianski and Levitin [31]. A recent contribution on the subject is Lisnianski [30].
Besides the distinction of the used lifetime distributions in continuous and discrete ones, another significant differentiator of the used lifetime distributions in this research is the shape of the failure rate. The general quantitative shape of the failure rate consists of three intervals. The first time interval just after going into operation consists of early failures (infant mortality) with a decreasing failure rate. During the second stage of usable life, systems will often fail at an approximately constant rate. During this period, failures of the system are usually caused by external forces. Then the phase of wear out failures starts and the failure rate will increase again. This type of failure rate with all three aging stages is well known as the bathtub curve [9]. In this research, some of the most commonly used distributions in survival analysis such as Rayleigh, linear failure rate, Weibull and modified Weibull distribution are used to model the time to the first failure. Besides these distributions that can have only constant, increasing or decreasing failure rates, a reduced modified Weibull distribution introduced by Almalki [1] which can also have a bathtub curved failure rate is used to model the

time to the first failure.
The optimization of PM policies requires an optimization criterion. Possible optimality criteria are, for example, maximum availability or limit on failure rate or minimum cost rate. The latter represents the criterion applied in this thesis. To compute this cost rate the mean costs per cycle are set in relation to the mean cycle length. Therefore, the operational life of the system is disassembled in cycles, i.e. in relation to cost and length statistically equivalent time periods.
This research analyzes five different imperfect maintenance models. Chapter 2 contains general definitions that are required for modeling in further chapters. Furthermore, both some continuous and discrete lifetime distributions and some special cost functions for maintenance actions are introduced. They form the basis for later calculations.
In what follows, five imperfect maintenance models that are divided in models with continuous and models with discrete lifetime distributions are analyzed and cost optimal maintenance strategies for several cost functions are computed.
The first model with continuous lifetime distribution is introduced in Chapter 3. This model is based on the perfect maintenance model of Beichelt [7] that was the first maintenance model with two different failure types. Analogously to Beichelt's model it is assumed that type 1 failures can only be removed by minimal repair and type 2 failures can only be removed through replacement. The model of Beichelt [7] is extended by imperfect PM actions that reduces the virtual age of the system. Hence, for modeling the cost optimization problem a periodic imperfect PM policy with failure type specific CM actions and imperfect PM actions is used.
The repairable system under consideration in Chapter 4 has also a continuous lifetime distribution but only one failure type. A sequential failure limit PM policy is used to model the cost optimization problem. The fundamental element of this model are both imperfect CM and imperfect PM actions.
The last model with continuous lifetime distribution is examined in Chapter 5. Here the model from Nakagawa [34] is extended with non-constant costs for PM actions. The repairable system again has only one failure type and a sequential PM policy with minimal repair as CM and imperfect PM is used. Therefore, this model takes into account that with increasing time PM actions have to be done more often.
The second part of this thesis contains two imperfect maintenance models with discrete lifetime distributions. The model in Chapter 6 is the discrete analogue of the model from Chapter 3. Hence, the underlying system is a multi-state system with two failure types. Here a periodic PM policy is used whereby CM actions are minimal or perfect and PM actions are imperfect.
The last model of this thesis in Chapter 7 is the discrete version of the model from Chapter 4. Therefore, the underlying repairable multi-state system has one failure type and undergoes imperfect CM and imperfect PM. The PM policy in this chapter is a sequential failure limit PM policy.
Finally, in Chapter 8 the conclusion summarizes the findings of this thesis and presents the reasons of possible deviations between the optimal maintenance strategies of the models with continuous lifetime distribution and the corresponding models with discrete lifetime distribution. Further, the conclusion gives a short overview of areas for further development and research.

2. Basics

2.1. Basic Terminology

Consider a *nonrepairable system*, which means that there is no repair and the system is discarded after its one and only failure. Let T be the lifetime of that system. Then T is a positive random variable and $F^T(t) = P(T \leq t)$ is the cumulative distribution function (CDF) of T. Therefore, $F^T(t)$ is the probability that the system failure time is before time t and $F^T(t) = 0$ if $t \leq 0$. Note that for all systems in this research it is assumed that simultaneous failures cannot occur and therefore, it is assumed that $F^T(0) = 0$. If the derivative of the CDF exists, $f^T(t) = dF^T(t)/dt$ is the probability density function (PDF). The survival or reliability function of the system is the probability that the system failure time is beyond time $t \geq 0$. Thus, for the reliability function it holds $\bar{F}^T(t) = 1 - F^T(t) = P(T > t)$. An important function in modeling the failure behavior, that was even defined in the 60s of the last century in e.g. Barlow and Proschan [5], is the hazard function, also called failure rate.

Definition 2.1 ([41, p. 8] **Continuous Hazard Function/Failure Rate)**
The hazard function or failure rate of the continuous random lifetime T is

$$h^T(t) = \lim_{\Delta t \to 0} \frac{P(t \leq T \leq t + \Delta t | T \geq t)}{\Delta t}. \tag{2.1}$$

Equation (2.1) can be rewritten as follows

$$h^T(t) = \lim_{\Delta t \to 0} \frac{F^T(t + \Delta t) - F^T(t-)}{\Delta t} \cdot \frac{1}{\bar{F}^T(t-)} = \frac{f^T(t)}{\bar{F}^T(t)}.^1 \tag{2.2}$$

The failure rate then has the following property

$$P(t < T \leq t + \Delta t) = h^T(t)\Delta t + o(\Delta t).^2 \tag{2.3}$$

Thus, for Δt being sufficiently small $h^T(t)\Delta t$ is approximately the probability that the system fails in $[t, t + \Delta t]$ if it has survived to the beginning of the interval [10]. Integration of equation (2.2) yields to the following result

$$\begin{aligned}\int_0^t h^T(x)dx &= \int_0^t \frac{f^T(x)}{\bar{F}^T(x)}dx \overset{y:=F^T(x)}{=} \int_0^{F^T(t)} \frac{1}{1-y}dy = [-\ln(1-y)]_0^{F^T(t)} \\ &= -\ln(1 - F^T(t))\end{aligned}$$

[1] Here $F^T(t-) = \lim_{x \nearrow t} F^T(x)$.

[2] It holds: $g(x)$ is said to be $o(x) \Leftrightarrow \lim_{x \to a} \left|\frac{g(x)}{x}\right| = 0$ and $a \in \mathbb{R} \cup \{-\infty, +\infty\}$.

$$\Leftrightarrow \quad F^T(t) = 1 - \exp\left(-\int_0^t h^T(x)dx\right) \tag{2.4}$$

$$\Rightarrow \quad \bar{F}^T(t) = \exp\left(-\int_0^t h^T(x)dx\right). \tag{2.5}$$

Definition 2.2 ([41, p. 11] Continuous Cumulative Hazard Function)
Let T be a continuous random lifetime. Then the quantity

$$H^T(t) = \int_0^t h^T(x)dx \tag{2.6}$$

is called the cumulative hazard function.

If T is a discrete random variable the failure rate and the cumulative hazard function is defined as follows.

Definition 2.3 ([26, p. 168] Discrete Hazard Function/Failure Rate)
The hazard function or failure rate of the discrete random lifetime T is

$$h^T(t) = P(T = t|T \geq t) = \frac{P(T = t)}{P(T \geq t)}. \tag{2.7}$$

Definition 2.4 ([26, p. 171] Discrete Cumulative Hazard Function)
Let T be a discrete random lifetime. Then the quantity

$$H^T(t) = \sum_{j=1}^{t} h^T(j) \tag{2.8}$$

is called the discrete time cumulative hazard function.

Note that (2.7) is the commonly used discrete time failure rate function, that was first defined in Barlow et al. [4]. For discrete failure time distributions the failure rate is a conditional probability and therefore $h^T(t) \leq 1$, whereas in the continuous case the failure rate is no probability and can be unbounded in some situations. Furthermore, the cumulative hazard function in the discrete case is not equal $-\ln(1 - F^T(t))$ as in the continuous case (see (2.4)), i.e.

$$H^T(t) = \sum_{j=1}^{t} h^T(j) \neq -\ln(1 - F^T(t)), \quad t = 1, 2, \ldots. \tag{2.9}$$

Therefore, several authors including Roy and Gupta [42] and Xie et al. [49] have proposed an alternative definition of the discrete failure rate function, denoted by

$$\lambda^T(t) = \ln \frac{\bar{F}^T(t-1)}{\bar{F}^T(t)}, \quad t = 1, 2, \ldots. \tag{2.10}$$

Using the alternative failure rate (2.10) the corresponding discrete cumulative hazard function now equals $-\ln(1 - F^T(t))$. Since the discrete failure rate will be used in the modeling of transition probabilities of a Markov chain, it is necessary in what follows

to use the discrete failure rate function from Definition 2.3.
Now consider a *repairable system*. This means that there could be more than one failure during the useful life of the system. In the following the random variables $T_1, T_2, \ldots, T_n$ denotes the random failure times of a repairable system. Point processes that count the failures through time are used for formulating stochastic models in mathematical reliability theory.

Definition 2.5 ([36, p. 16] Counting process)
Define the random variable

$$\begin{aligned} N_t &= \sum_{n \geq 1} \mathbf{1}_{\{T_n \leq t\}} \\ &= \max\{n \in \mathbb{N}^+ : T_n \leq t\} \end{aligned} \tag{2.11}$$

which is the number of failures in the interval $[0, t]$*, where* $\mathbf{1}_{\{.\}}$ *is the indicator function. Then the stochastic point process* $N = (N_t)_{t \geq 0}$ *is called a counting process.*

Note that because of the previous definition the number of failures in the interval $(a, b]$ can be calculated as $N_b - N_a$.

Definition 2.6 ([41, p. 23] Mean Function of a Point Process)
The mean function of a point process $N = (N_t)_{t \geq 0}$ *is defined to be the expectation*

$$\Lambda^N(t) = E(N_t), \quad \forall t \geq 0. \tag{2.12}$$

Definition 2.7 ([41, p. 27] Intensity Function)
The intensity function of a point process $N = (N_t)_{t \geq 0}$ *is*

$$\lambda^N(t) = \lim_{\Delta t \to 0} \frac{P(N_{t+\Delta t} - N_t \geq 1)}{\Delta t}. \tag{2.13}$$

The difference between the intensity function (2.13) and the hazard rate (2.1) is that $h^T(t)\Delta t$ is approximately the conditional probability that the one and only failure will occur in a small interval, whereas $\lambda^N(t)\Delta t$ is approximately the unconditional probability of a failure in a small interval. This failure do not necessarily need to be the first one.
For every maintenance model of this research the mean cycle length is an essential element in the calculation of the average maintenance costs per unit time. To calculate this value the following formula is used.

Remark 2.1
Suppose X *is a positive continuous random variable with PDF* $f^X(x)$ *and CDF* $F^X(x)$ *with* $x \geq 0$*. Now suppose* $t \geq 0$ *is a positive real number. For the CDF of the positive random variable* $L = \min\{X, t\}$ *it holds*

$$F^L(x) = \begin{cases} F^X(x) & , \textit{if } 0 \leq x < t \\ 1 & , \textit{if } x \geq t \end{cases}. \tag{2.14}$$

Thus, for the expected value of $L = \min\{X, t\}$ *it holds*

$$E(L) = \int_0^\infty (1 - F^L(x))dx = \int_0^t (1 - F^X(x))dx + \int_t^\infty (1-1)dx = \int_0^t (1 - F^X(x))dx. \tag{2.15}$$

2.2. Poisson Process

Definition 2.8 ([41, p. 35] **Poisson Process)**
A counting process $N = (N_t)_{t\geq 0}$ *is said to be a Poisson process if*

1. $N_0 = 0$.
2. *The process* $N = (N_t)_{t\geq 0}$ *has stochastically independent increments, i.e. for any* $n \in \mathbb{N}^+$ *and any* $0 \leq t_0 < t_1 < \cdots < t_n$ *the random variables* $N_{t_i} - N_{t_{i-1}}$ *(i=1,...,n) are stochastically independent.*
3. *There is a function* $\lambda^N(t)$ *such that*
$$\lambda^N(t) = \lim_{\Delta t \to 0} \frac{P(N_{t+\Delta t} - N_t = 1)}{\Delta t}. \tag{2.16}$$
The function $\lambda^N(t)$ *is called the intensity function of the Poisson process.*
4. $$\lim_{\Delta t \to 0} \frac{P(N_{t+\Delta t} - N_t \geq 2)}{\Delta t} = 0. \tag{2.17}$$

Note that in case of a constant intensity function, the Poisson process is referred to as *homogeneous* and if the intensity function is not constant it is *inhomogeneous*. Assumption 4 precludes the possibility of simultaneous failures and therefore the Poisson process is a simple process.

Theorem 2.2
Properties 1. through 4. of the Poisson process imply that

$$P(N_t = n) = \frac{1}{n!}\left(\int_0^t \lambda^N(x)dx\right)^n \exp\left(-\int_0^t \lambda^N(x)dx\right), \quad n = 1, 2, \ldots. \tag{2.18}$$

Proof. See for example [41, p. 36]. □

Equation (2.18) shows that for a Poisson process the random variable N_t has a Poisson distribution with mean function

$$\Lambda^N(t) = E(N_t) = \int_0^t \lambda^N(x)dx. \tag{2.19}$$

Lemma 2.3
Let $N = (N_t)_{t\geq 0}$ be a Poisson process with intensity function $\lambda^N(t)$. For $0 \leq s < t$ the random variable $N_t - N_s$ has a Poisson distribution with expected value

$$E(N_t - N_s) = \Lambda^N(t) - \Lambda^N(s) = \int_0^t \lambda^N(x)dx - \int_0^s \lambda^N(x)dx = \int_s^t \lambda^N(x)dx. \quad (2.20)$$

Define the function $p_n^N(s,t)$ to be the probability that in the interval $(s,t]$ n failures occur. Then it holds

$$\begin{aligned} p_n^N(s,t) &= P(N_t - N_s = n) \\ &\overset{\text{Lemma 2.3}}{=} \frac{1}{n!}\left(\int_s^t \lambda^N(x)dx\right)^n \exp\left(-\int_s^t \lambda^N(x)dx\right), \end{aligned} \quad (2.21)$$

for $0 \leq s < t$ and $n = 1, 2, \ldots$.

Remark 2.4 ([**29, p. 536**])
Consider a repairable system with failure times $T_1, T_2, \ldots$ which form a simple point process. Let $N = (N_t)_{t\geq 0}$ be the corresponding failure counting process. If all failures are removed through minimal repair, the failure intensity of the counting process is equal to the hazard function of the time to the first failure of a new system, i.e.

$$\lambda^N(t) = h^{T_1}(t) \quad (2.22)$$

for $t \geq 0$. The process $N = (N_t)_{t\geq 0}$ is then an inhomogeneous Poisson process with intensity function $h^{T_1}(t)$.

2.3. Markov Chain

Definition 2.9 ([**38, p. 695**] **Discrete-Time Markov Chain**)
Let S_t be an integer valued random variable, called the random state at time t. Suppose the sample values for each random variable S_t, $t = 0, 1, 2, \ldots$, form a countable set M called state space. The process $(S_t)_{t\in\mathbb{N}}$ is a discrete-time Markov chain if the following condition is fulfilled

$$P(S_t = s_t | S_0 = s_0, S_1 = s_1, \ldots, S_{t-1} = s_{t-1}) = P(S_t = s_t | S_{t-1} = s_{t-1}), \quad (2.23)$$

for all $t \geq 2$ and $s_0, s_1, \ldots, s_t \in M$ with $P(S_0 = s_0, S_1 = s_1, \ldots S_{t-1} = s_{t-1}) > 0$. The initial state S_0 has an arbitrary probability distribution.

The condition (2.23) is also known as the Markov condition. Thus, in a Markov chain the future evolution of the process $(S_t)_{t\in\mathbb{N}}$ depends only on the present state and not on how the system arrived at that state.

Definition 2.10 ([**38, p. 697**] **Transition Probabilities**)
Suppose $(S_t)_{t\in\mathbb{N}}$ is a discrete-time Markov chain with state space M. Then,

$$p_i(n) = P(S_n = i) \quad (2.24)$$

represents the probability that at time n the system occupies state i and

$$p_{ij}(m,n) = P(S_n = j | S_m = i) \tag{2.25}$$

the probability that the system goes into state j at time n given that is was in state i at time m with $i, j \in M$, $m < n$ and $n, m \in \mathbb{N}$. The numbers $p_{ij}(m,n)$ represent the transition probabilities of the Markov chain from state i at time m to state j at time n.

If the transition probabilities are arranged in a matrix form $X(m,n)$ with

$$X(m,n) = \begin{pmatrix} p_{11}(m,n) & p_{12}(m,n) & \dots & p_{1j}(m,n) & \dots \\ p_{21}(m,n) & p_{22}(m,n) & \dots & p_{2j}(m,n) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{i1}(m,n) & \dots & \dots & p_{ij}(m,n) & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \tag{2.26}$$

the matrix $X(m,n)$ represents a stochastic matrix, i.e. each row summing to 1 and every entry is a nonnegative real number representing a probability. Since for $m < n < r$ with $m, n, r \in \mathbb{N}$ and $i, j, k \in M$ it holds

$$P(S_r = i, S_n = j, S_m = k) = p_{ji}(n,r)p_{kj}(m,n)p_k(m), \tag{2.27}$$

the probabilities (2.24) and (2.25) completely determine the Markov chain.
In the further research the one-step transition probabilities $p_{ij}(n-1,n)$ are needed. For $i, j \in M$ and $n \in \mathbb{N}^+$, the one-step transition probabilities are defined as follows

$$p_{ij}(n-1,n) = P(S_n = j | S_{n-1} = i). \tag{2.28}$$

Let $(X_n)_{n \in \mathbb{N}^+}$ be the corresponding one-step transition matrices. Thus

$$X_n = \begin{pmatrix} p_{11}(n-1,n) & p_{12}(n-1,n) & \dots & p_{1j}(n-1,n) & \dots \\ p_{21}(n-1,n) & p_{22}(n-1,n) & \dots & p_{2j}(n-1,n) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{i1}(n-1,n) & \dots & \dots & p_{ij}(n-1,n) & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}. \tag{2.29}$$

2.4. Kijima Type Repairs

Kijima et al. [25] and Kijima [24] proposed that the state of the system just after repair can be described by its virtual age. There two ways are described how repair actions can affect the virtual age of a system.

Definition 2.11 (Kijima's type I imperfect repair model)
Suppose ξ_k is the degree of the kth repair that takes place at time t_k. For the virtual age after the kth repair v_k it holds

$$v_k = v_{k-1} + \xi_k \cdot (t_k - t_{k-1}), \quad t_{k-1} < t_k,\ k \geq 1,\ 0 \leq \xi_k \leq 1. \tag{2.30}$$

Thus, in Kijima's type I imperfect repair model a repair cannot remove the damage that incurred before the previous repair action and the virtual age of the system after the kth repair is always greater or equal the virtual age after the $(k-1)$th repair.

Definition 2.12 (Kijima's type II imperfect repair model)
Suppose ξ_k is the degree of the kth repair that takes place at time t_k. For the virtual age after the kth repair v_k it holds

$$v_k = \xi_k \cdot (v_{k-1} + t_k - t_{k-1}), \quad t_{k-1} < t_k,\, k \geq 1,\, 0 \leq \xi_k \leq 1. \tag{2.31}$$

Therefore, in Kijima's type II imperfect repair model a repair action can remove the whole damage accumulated up to the time of the repair action. Hence, the virtual age of the system after repair can obtain values between zero and the calendar age of the system. In the first case the degree of repair is zero and therefore the repair makes the system as good as new. In the second case, when the degree of repair is one, the system is minimally repaired.
Let $N = (N_t)_{t\geq 0}$ be a failure counting process. Using the virtual age v_k, for the intensity function of N between the kth and the $(k+1)$th repair it holds

$$\lambda^N(t) = \lambda^N(t - t_k + v_k), \quad t_k \leq t < t_{k+1},\, k \geq 0. \tag{2.32}$$

2.5. Continuous Lifetime Distributions

In this section some continuous lifetime distributions are introduced that will be used later to model the time to the first failure. First the modified Weibull distribution (MWD) that was introduced by Sarhan and Zaindin [43] is described. Then the reduced modified Weibull distribution (RMWD) that was introduced by Almalki [1] is described.

2.5.1. The Modified Weibull Distribution

The modified Weibull distribution (MWD) that was introduced by Sarhan and Zaindin [43] generalizes some most commonly used distributions in survival analysis such as exponential, Rayleigh, linear failure rate and Weibull distribution.
In the following, the notation MWD(α, β, γ) is used to denote the modified Weibull distribution with the scale parameter α and the two shape parameters β and γ. Let X be modified Weibull distributed MWD(α, β, γ). Then the cumulative distribution function (CDF) of X is

$$F^X(x; \alpha, \beta, \gamma) = 1 - \exp\left(-\alpha x - \beta x^\gamma\right), \quad \forall x \geq 0 \tag{2.33}$$

and the probability density function (PDF) of X is

$$f^X(x; \alpha, \beta, \gamma) = (\alpha + \beta\gamma x^{\gamma-1}) \exp\left(-\alpha x - \beta x^\gamma\right), \quad \forall x > 0, \tag{2.34}$$

where $\gamma > 0$, α, $\beta \geq 0$ such that $\alpha + \beta > 0$. The hazard function of the MWD(α, β, γ) is then

$$h^X(x; \alpha, \beta, \gamma) = \alpha + \beta\gamma x^{\gamma-1}, \quad \forall x > 0. \tag{2.35}$$

Note that the hazard function is constant if $\beta = 0$ or $\gamma = 1$, increasing in x if $\beta > 0$ and $\gamma > 1$ and decreasing in x if $\beta > 0$ and $0 < \gamma < 1$. If $\gamma = 1$ it holds $f^X(0) = \alpha + \beta$ and $h^X(0) = \alpha$, if $\gamma > 1$ it holds $f^X(0) = h^X(0) = \alpha$ and if $\gamma < 1$ the probability density function and the hazard function tends to infinity if x goes to zero.
As mentioned above, the MWD(α, β, γ) includes other important lifetime distributions. If $\gamma = 2$ the MWD(α, β, γ) becomes the linear failure rate distribution (LFRD) with parameters α and β. By setting $\alpha = 0$ and $\gamma = 2$ we get the Rayleigh distribution (RD) with parameter β. In the case of $\alpha = 0$ we obtain the Weibull distribution (WD) with parameters β and γ. Finally, if $\beta = 0$, we obtain the exponential distribution with parameter α.

2.5.2. The Reduced Modified Weibull Distribution

This subsection describes the reduced modified Weibull distribution (RMWD), that was introduced by Almalki [1]. Almalki reduces the number of parameters of the modified Weibull distribution, that was introduced by Almalki and Yuan [3], from five to three and that is why this distribution have the prefix "reduced". In contrast to the MWD from Section 2.5.1, both the initial modified Weibull distribution from Almalki and Yuan [3] and the RMWD from Almalki [1] allow increasing, decreasing and even bathtub shapes of the hazard function.
In the following, the notation RMWD(α, β, γ) is used to denote the reduced modified Weibull distribution with scale parameters α and β and acceleration parameter γ. Let X be reduced modified Weibull distributed RMWD(α, β, γ). Then the CDF of X is

$$F^X(x; \alpha, \beta, \gamma) = 1 - \exp\left(-\alpha\sqrt{x} - \beta\sqrt{x}\exp(\gamma x)\right), \quad \forall x \geq 0 \tag{2.36}$$

and the PDF of X is

$$f^X(x; \alpha, \beta, \gamma) = \frac{1}{2\sqrt{x}}\left(\alpha + \beta(1 + 2\gamma x)\exp(\gamma x)\right)\exp\left(-\alpha\sqrt{x} - \beta\sqrt{x}\exp(\gamma x)\right), \tag{2.37}$$

where $x > 0$, α, $\beta > 0$ and $\gamma > 0$. The hazard function or failure rate of the RMWD(α, β, γ) is then

$$h^X(x; \alpha, \beta, \gamma) = \frac{1}{2\sqrt{x}}\left(\alpha + \beta(1 + 2\gamma x)\exp(\gamma x)\right), \quad \forall x > 0. \tag{2.38}$$

Figure 2.1 shows the different probability density functions. It can be seen that the PDF of the LFRD and the RD have a similar shape. This is true also for the hazard and the cumulative hazard functions as seen in Figures 2.2 and 2.3. As can be seen in Figure 2.2, the chosen parameters for the RMWD leads to a bathtub shaped failure rate. Note that the parameters for all distributions are chosen so that the expected value is equal to 5. Table 2.1 summarizes formulas for the PDF, CDF and failure rate for all continuous lifetime distributions that are used later in this research.

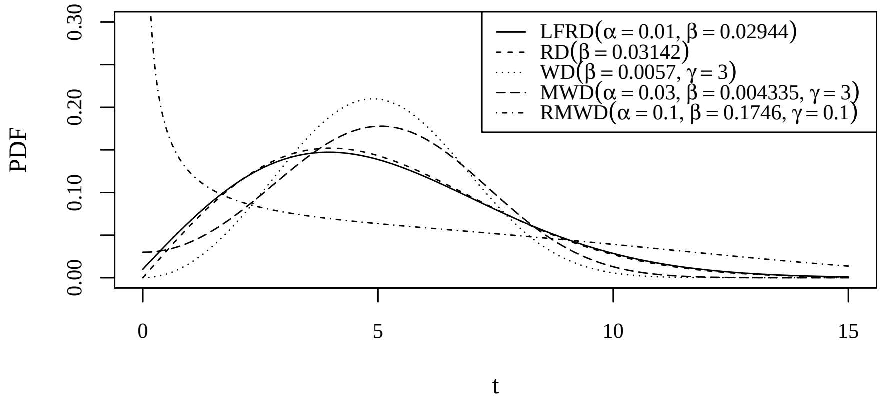


Figure 2.1.: Probability density functions.

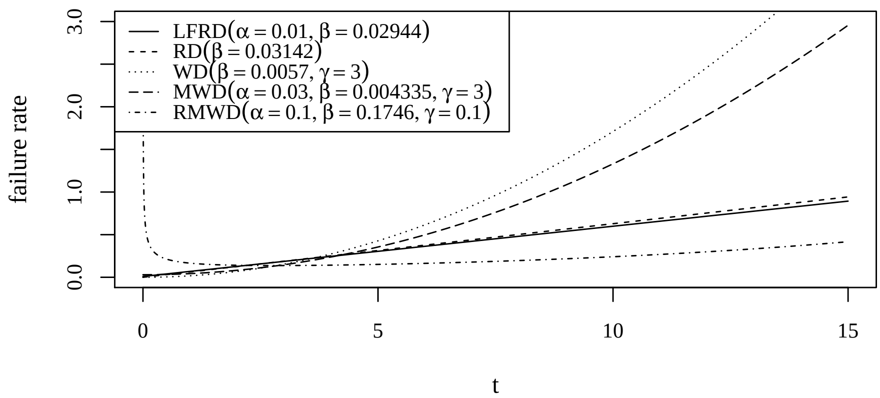


Figure 2.2.: Continuous failure rates.

2.6. Discrete Lifetime Distributions

In this section the discrete versions of the continuous lifetime distributions from Section 2.5 are introduced. These discrete lifetime distributions will be used later to model the lifetime of an operating unit. First the discrete version of the modified Weibull distribution (MWD) from Section 2.5.1 is considered. Then, the discrete version of the RMWD from Section 2.5.2 that was introduced by Almalki and Nadarajah [2] is examined.

2.6.1. The Discrete Modified Weibull Distribution

The discrete version of the MWD that was introduced by Sarhan and Zaindin [43] also generalizes some most commonly used discrete distributions in survival analysis, such

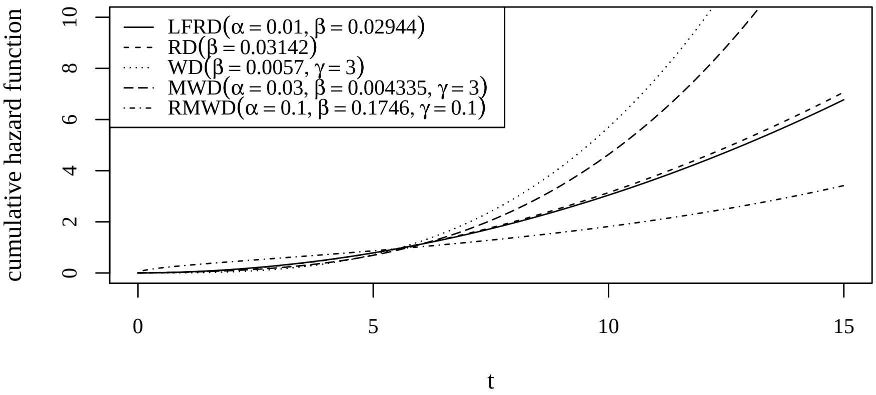


Figure 2.3.: Continuous cumulative hazard functions.

Table 2.1.: Continuous lifetime distributions

Distribution	$f^X(x)$	$F^X(x)$	$h^X(x)$
LFRD	$(\alpha+2\beta x)e^{-\alpha x-\beta x^2}$	$1-e^{-\alpha x-\beta x^2}$	$\alpha+2\beta x$
RD	$(2\beta x)e^{-\beta x^2}$	$1-e^{-\beta x^2}$	$2\beta x$
WD	$(\beta\gamma x^{\gamma-1})e^{-\beta x^\gamma}$	$1-e^{-\beta x^\gamma}$	$\beta\gamma x^{\gamma-1}$
MWD	$(\alpha+\beta\gamma x^{\gamma-1})e^{-\alpha x-\beta x^\gamma}$	$1-e^{-\alpha x-\beta x^\gamma}$	$\alpha+\beta\gamma x^{\gamma-1}$
RMWD	$\frac{\alpha+\beta(1+2\gamma x)e^{\gamma x}}{2\sqrt{x}}e^{-\alpha\sqrt{x}-\beta\sqrt{x}e^{\gamma x}}$	$1-e^{-\alpha\sqrt{x}-\beta\sqrt{x}e^{\gamma x}}$	$\frac{1}{2\sqrt{x}}\left(\alpha+\beta(1+2\gamma x)e^{\gamma x}\right)$

as the geometric, discrete Rayleigh, discrete linear failure rate and the discrete Weibull distribution.
From Section 2.5.1 we have that if X is modified Weibull distributed MWD(α,β,γ) it has the following probability density function

$$f^X(x;\alpha,\beta,\gamma)=\left(\alpha+\beta\gamma x^{\gamma-1}\right)\exp\left(-\alpha x-\beta x^\gamma\right),\quad \forall x>0, \tag{2.39}$$

where $\gamma>0$, α, $\beta\geq 0$ such that $\alpha+\beta>0$.
Let T be the discrete random lifetime of an operating unit. For the discrete version of the MWD we put the probability mass of the interval $(t-1,t]$ into the point t, that is

$$\begin{aligned} P(T=t) &= \int_{t-1}^{t} f^X(s)ds \\ &= \exp\left(-\alpha(t-1)-\beta(t-1)^\gamma\right)-\exp\left(-\alpha t-\beta t^\gamma\right),\quad t=1,2,\ldots. \end{aligned} \tag{2.40}$$

The corresponding CDF is given by

$$F^T(t)=P(T\leq t) = \sum_{j=0}^{t} P(T=j)$$

$$= \quad 1 - \exp\left(-\alpha t - \beta t^{\gamma}\right), \quad t = 1, 2, \ldots \tag{2.41}$$

and $F^T(0) = P(T = 0) = 0$. For the failure rate it holds

$$\begin{aligned} h^T(t) &= P(T = t | T \geq t) \\ &= 1 - \exp\left(\alpha(t-1) + \beta(t-1)^{\gamma} - \alpha t - \beta t^{\gamma}\right), \quad t = 1, 2, \ldots \end{aligned} \tag{2.42}$$

and $h^T(0) = 0$. As mentioned above the DMWD(α, β, γ) includes other important lifetime distributions. If $\gamma = 2$ the DMWD(α, β, γ) becomes the discrete linear failure rate distribution (DLFRD) with parameters α and β. By setting $\alpha = 0$ and $\gamma = 2$ we get the discrete Rayleigh distribution (DRD) with parameter β. In the case of $\alpha = 0$ the discrete Weibull distribution (DWD) with parameters β and γ is obtained. Finally, if $\beta = 0$, we obtain the geometric distribution with success probability $1 - \exp(-\alpha)$.

2.6.2. The Discrete Reduced Modified Weibull Distribution

The discrete version of the RMWD was introduced by Almalki and Nadarajah [2] and based on a three-parameter modified Weibull distribution that was developed by Almalki [1]. In the following, the notation RMWD(α, β, γ) is used to denote the reduced modified Weibull distribution with scale parameters α and β and acceleration parameter γ. Let X be reduced modified Weibull distributed RMWD(α, β, γ). Then, the probability density function of X is

$$f^X(x; \alpha, \beta, \gamma) = \frac{1}{2\sqrt{x}} \left(\alpha + \beta(1 + 2\gamma x)\exp(\gamma x)\right) \exp(-\alpha\sqrt{x} - \beta\sqrt{x}\exp(\gamma x)), \tag{2.43}$$

for $x > 0$, α, $\beta > 0$ and $\gamma > 0$. For the discrete version of the RMWD(α, β, γ) we put the probability mass of the interval $(t-1, t]$ into point t, that is

$$\begin{aligned} P(T = t) &= \int_{t-1}^{t} f^X(s) ds \\ &= \exp\left(-\sqrt{t-1}(\alpha + \beta\exp(\gamma(t-1)))\right) - \exp\left(-\sqrt{t}(\alpha + \beta\exp(\gamma t))\right) \end{aligned} \tag{2.44}$$

for $t = 1, 2, \ldots$. The CDF of the discrete reduced modified Weibull distribution DRMWD(α, β, γ) is given by

$$F^T(t) = P(T \leq t) = 1 - \exp\left(-\sqrt{t}(\alpha + \beta\exp(\gamma t))\right), \quad t = 1, 2, \ldots, \tag{2.45}$$

and $F^T(0) = P(T = 0) = 0$. The failure rate is given by

$$\begin{aligned} h^T(t) &= P(T = t | T \geq t) = \frac{P(T = t)}{P(T \geq t)} \\ &= 1 - \exp\left(\sqrt{t-1}(\alpha + \beta\exp(\gamma(t-1))) - \sqrt{t}(\alpha + \beta\exp(\gamma t))\right), \end{aligned} \tag{2.46}$$

for $t = 1, 2, \ldots$ and $h^T(0) = 0$. As shown in [2] the failure rate of the DRMWD(α, β, γ) can be increasing or has a bathtub shape.

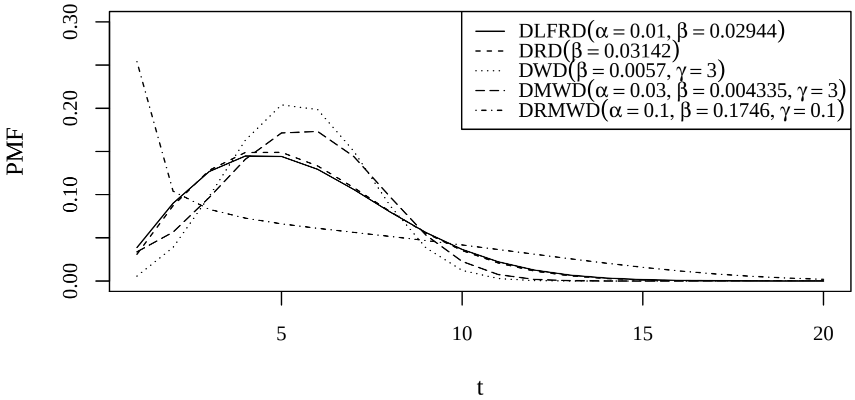


Figure 2.4.: Probability mass functions.

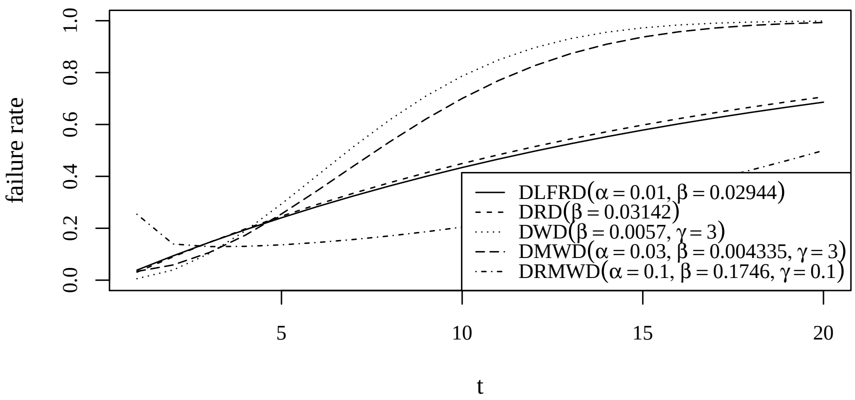


Figure 2.5.: Discrete failure rates.

Figure 2.4 shows the different probability mass functions (PMF). The distribution parameters are the same as in the previous section. Therefore, the plotted discrete distributions are considered to be the discretized distributions from Section 2.5. However, the expected value of these discrete lifetime distributions is no longer identical but slightly larger than five. It can be seen that the PMF of the DLFRD and the DRD have a similar shape. This is true also for the failure rate and the cumulative hazard function as seen in Figure 2.5 and Figure 2.6. As can be seen in Figure 2.5, the chosen parameters for the DRMWD leads to a bathtub shaped failure rate.

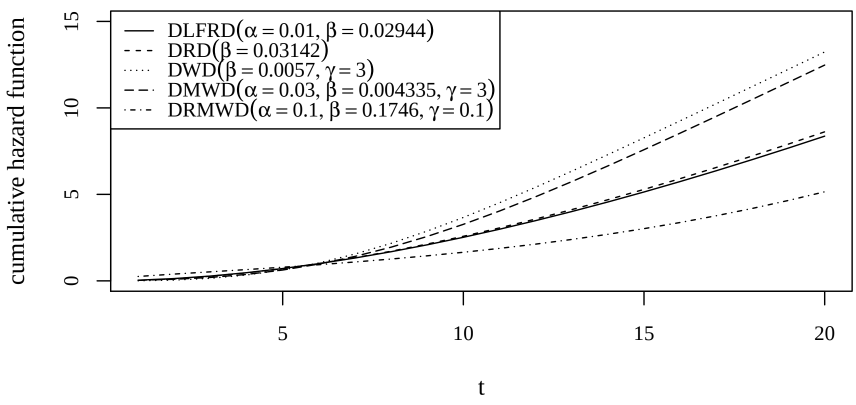


Figure 2.6.: Discrete cumulative hazard functions.

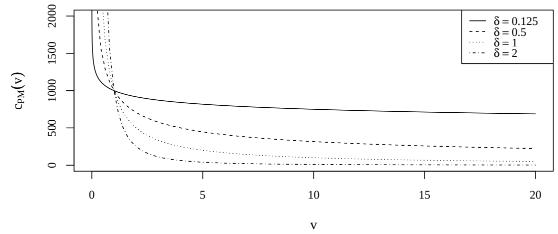
Table 2.2.: Discrete lifetime distributions

Distribution	$P(T=t)$	$F^T(t)$	$h^T(t)$
DLFRD	$e^{-\alpha(t-1)-\beta(t-1)^2} - e^{-\alpha t-\beta t^2}$	$1-e^{-\alpha t-\beta t^2}$	$1-e^{\alpha(t-1)+\beta(t-1)^2-\alpha t-\beta t^2}$
DRD	$e^{-\beta(t-1)^2} - e^{-\beta t^2}$	$1-e^{-\beta t^2}$	$1-e^{\beta(t-1)^2-\beta t^2}$
DWD	$e^{-\beta(t-1)^\gamma} - e^{-\beta t^\gamma}$	$1-e^{-\beta t^\gamma}$	$1-e^{\beta(t-1)^\gamma-\beta t^\gamma}$
DMWD	$e^{-\alpha(t-1)-\beta(t-1)^\gamma} - e^{-\alpha t-\beta t^\gamma}$	$1-e^{-\alpha t-\beta t^\gamma}$	$1-e^{\alpha(t-1)+\beta(t-1)^\gamma-\alpha t-\beta t^\gamma}$
DRMWD	$e^{-\sqrt{t-1}\left(\alpha+\beta e^{\gamma(t-1)}\right)}$ $-e^{-\sqrt{t}\left(\alpha+\beta e^{\gamma t}\right)}$	$1-e^{-\sqrt{t}(\alpha+\beta e^{\gamma t})}$	$1-e^{\sqrt{t-1}\left(\alpha+\beta e^{\gamma(t-1)}\right)-\sqrt{t}\left(\alpha+\beta e^{\gamma t}\right)}$

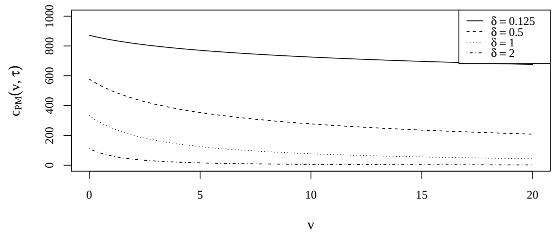
Note that in Figures 2.4, 2.5 and 2.6 the points are connected with lines for better visibility.
Table 2.2 summarizes formulas for the PMF, CDF and failure rate for all discrete lifetime distributions that are used later in this research.
In the further research we use the previously defined discrete lifetime distributions to model the lifetime of a MSS with $n = 20$ states. Usually truncated distributions, i.e. $P(T = t|T \leq n)$, are used to model the lifetime of such systems. Note that for the chosen distribution parameters of all previously defined lifetime distributions, the probability that a failure occurs at a point in time $t > 20$ is close to zero. This means there is no significant difference between the lifetime distribution and the truncated version of it. Hence, it is appropriate here to use the untruncated lifetime distribution instead of the truncated version of it.

2.7. Introduction of Cost Functions

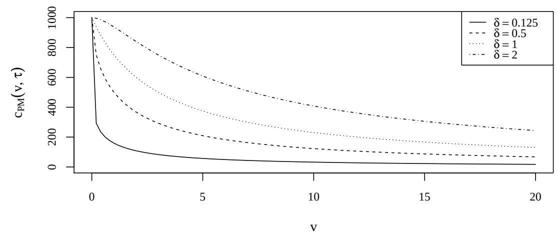
In this thesis different cost functions for preventive and sometimes also for corrective maintenance actions are used. This section summarizes the properties of different cost functions used in further chapters.



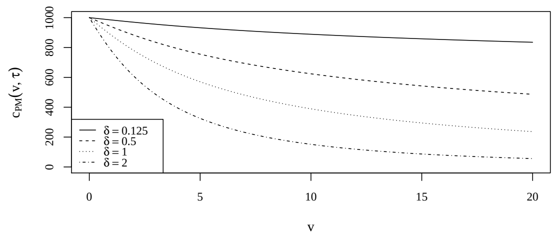
(a) Proportional to the impact of repair



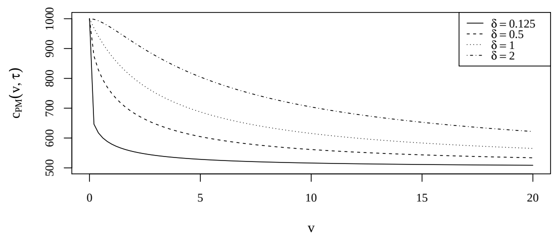
(b) Proportional to the state before repair



(c) Proportional to the degree of repair - 1



(d) Proportional to the degree of repair - 2



(e) Proportional to the degree of repair - 3

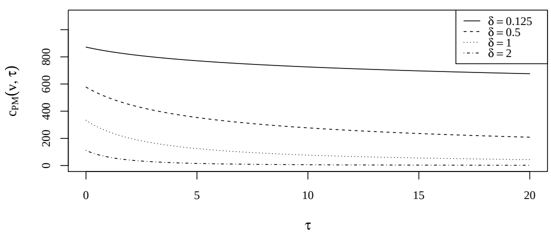
Figure 2.7.: Cost functions for PM depending on v with $c_R = c_I = c_S = 1000$, $c_M = 500$ and $\tau = 3$.

2.7.1. Costs Proportional to the Impact of Repair

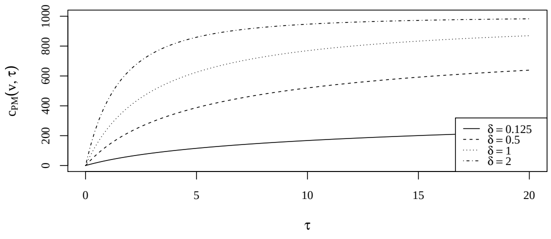
If the preventive maintenance action reduces the virtual age of the system to $v > 0$, the impact of repair can be expressed in terms of v. Therefore, a small value of v corresponds to a high impact of repair and vice versa. If the costs of a PM action depend only on the virtual age after PM, a possible cost function could be

$$c_{PM}(v) = c_I \left(\frac{1}{v}\right)^{\delta}, \tag{2.47}$$

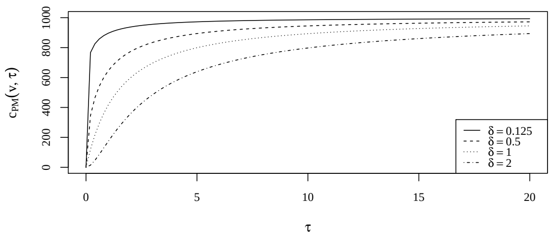
where $v > 0$, $\delta > 0$ and $c_I > 0$ is a constant cost value. This cost function was introduced and used e.g. in Kahle [21], [22] and [23]. Note that in case of costs proportional to the impact of repair, the extreme case of perfect PM actions, i.e. $v = 0$, is excluded. Cost function (2.47) has the following properties:



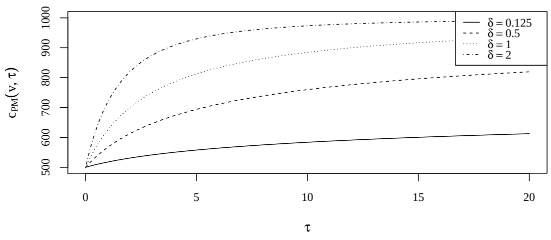
(a) Proportional to the state before repair



(b) Proportional to the degree of repair - 1



(c) Proportional to the degree of repair - 2



(d) Proportional to the degree of repair - 3

Figure 2.8.: Cost functions for PM depending on τ with $c_R = c_I = c_S = 1000$, $c_M = 500$ and $v = 3$.

- The costs of PM are only bounded below by zero, i.e. $c_{PM}(v) > 0$.
- Figure 2.7 (a) shows the better the PM actions, i.e. the smaller v, the higher are the costs for PM. For a perfect repair the costs tend to infinity, i.e. the limit of $c_{PM}(v)$ as v approaches zero is infinity, on condition that δ does not tend towards zero.
- The worse the PM actions, i.e. the higher v, the smaller are the costs of PM. The limit of $c_{PM}(v)$ as v approaches infinity is zero, on condition that δ does not tend towards zero.
- If δ tends to zero the costs of PM converge to c_I for all values of v. Therefore, the costs of relative good PM actions with $v < 1$ decrease and the costs of less good PM actions with $v > 1$ increase.
- The higher δ, the faster the costs of PM converge to infinity as v approaches zero and the faster the costs of PM converge to zero as v approaches infinity.
- If $v = 1$ the costs of PM are always c_I independent of δ.

2.7.2. Costs Proportional to the State before Repair

Assume a PM action that reduces the virtual age of a repairable system to $v \geq 0$. If the PM action is done $\tau \geq 0$ time units after the last maintenance action that reduced the virtual age of the system to v, the state just before repair can be expressed in terms of $v + \tau$. If the costs of PM depend only on the state just before PM, a possible cost function could be

$$c_{PM}(v,\tau) = c_S \left(\frac{1}{v+\tau} \right)^{\delta}, \tag{2.48}$$

where $v, \tau \geq 0$, $v + \tau \neq 0$, $\delta > 0$ and $c_S > 0$ is a constant cost value. This cost function was introduced in Gasmi and Mannai [20] and has the following properties:

- The costs of PM are only bounded below by zero, i.e. $c_{PM}(v, \tau) > 0$.
- Figure 2.7 (b) and Figure 2.8 (a) shows the better the PM actions and the shorter the distance between PM actions, the higher are the costs for PM. In case of perfect repair, i.e. $v = 0$, the costs of PM are $c_{PM} = c_S(1/\tau)^{\delta}$.
- The worse the PM actions and the longer the distance between PM actions, i.e. the higher v and τ, the lower are the costs of PM. The limit of $c_{PM}(v, \tau)$ as v or τ approaches infinity is zero, on condition that δ does not tends towards zero.
- If δ tends to infinity and v or τ tend to infinity the costs of PM converge to zero. The higher δ the faster the costs of PM tend towards zero.
- If δ tends to zero the costs of PM converge to c_S for all possible values of v and τ.
- If δ is increasing the costs of PM tend to infinity if $v + \tau$ is close to zero.

2.7.3. Costs Proportional to the Degree of Repair - 1

It is assumed that a PM action reduces the virtual age of a repairable system to $v \geq 0$. If the PM action is done $\tau \geq 0$ time units after the last maintenance action that reduced the virtual age of the system to v, the degree of repair in a Kijima type II manner is

$$\xi(v,\tau) = \frac{v}{v+\tau}. \tag{2.49}$$

If the costs of PM depend on the degree of repair, a possible cost function could be

$$c_{PM}(v,\tau) = c_R \left(1 - \xi(v,\tau)^{\delta} \right), \tag{2.50}$$

where $v, \tau \geq 0$, $v + \tau \neq 0$, $\delta > 0$ and $c_R > 0$ are the costs of a replacement. This cost function was introduced e.g. in Kahle [21], [22] and [23]. Figure 2.7 (c) and Figure 2.8 (b) show the shape of cost function (2.50) for different parameter constellations. This cost function has the following properties:

- The costs of PM actions are bounded below and above. They are greater than zero and smaller than the costs of a replacement, i.e. $0 \leq c_{PM}(v, \tau) \leq c_R$.
- The better the PM actions, i.e. the smaller v, the higher are the costs for PM. In case of perfect repair, i.e. $\xi(v, \tau) = 0$, implying that $v = 0$, the costs of PM are $c_{PM}(v, \tau) = c_R$. The same holds if v tends to zero and δ does not at the same time tends to zero.

- The worse the PM actions and the lower the distance between PM actions, i.e. the higher v and the smaller τ, the lower are the costs for PM. In case of minimal repair, i.e. $\xi(v, \tau) = 1$, involving that v tends to infinity or $\tau = 0$, there are no costs of PM, i.e. $c_{PM}(v, \tau) = 0$.

- If v and τ tend to infinity and δ does not tend to zero, the costs of PM tend to the cost of a replacement, i.e. $c_{PM}(v, \tau)$ tends to c_R.

- The lower δ the lower are the costs of PM. If $v \neq 0$ and δ tends to zero, the costs of PM tend to zero too.

2.7.4. Costs Proportional to the Degree of Repair - 2

Another way to model the costs of PM actions if they depend on the degree of repair is

$$c_{PM}(v, \tau) = c_R \left(1 - \xi(v, \tau) \exp\left(\xi(v, \tau) - 1\right)\right)^{\delta}, \tag{2.51}$$

where $v, \tau \geq 0$, $v + \tau \neq 0$, $\delta > 0$ and $c_R > 0$ are the costs of a replacement. This cost function was introduced e.g. in Gasmi and Mannai [20]. Figure 2.7 (d) and Figure 2.8 (c) show the shape of cost function (2.51) for different parameter constellations. This cost function has the following properties:

- The costs of PM actions are bounded below and above. They are greater than zero and smaller than the costs of a replacement, i.e. $0 \leq c_{PM}(v, \tau) \leq c_R$.

- The better the PM actions, i.e. the smaller v, the higher are the costs for PM. In case of perfect repair, i.e. $\xi(v, \tau) = 0$, implying that $v = 0$, the costs of PM are $c_{PM}(v, \tau) = c_R$. The same holds if v tends to zero and δ does not at the same time tends to zero.

- The worse the PM actions and the lower the distance between PM actions, i.e. the higher v and the smaller τ, the lower are the costs for PM. In case of minimal repair, i.e. $\xi(v, \tau) = 1$, involving that v tends to infinity or $\tau = 0$, there are no costs of PM, i.e. $c_{PM}(v, \tau) = 0$. The same holds if τ tends to zero and v and δ does not at the same time tend to zero.

- The lower δ, the higher are the costs of PM actions. If $v \neq 0$ and δ tends to zero, the costs of PM tend to the costs of a replacement.

- If v and τ tend to infinity and δ does not tend to zero, the costs of PM tend to the cost of a replacement, i.e. $c_{PM}(v, \tau)$ tends to c_R.

As illustrated in Figure 2.7 (c) and (d), the difference between cost function (2.50) and (2.51) is that if $\delta \leq 1$ the costs of PM actions are now higher than before and if $\delta > 1$ the costs of PM actions are lower than before. In contrast to cost function (2.50), the costs of PM decrease with rising δ.

2.7.5. Costs Proportional to the Degree of Repair - 3

A third possible way to model the costs of PM actions if they depend on the degree of repair is

$$c_{PM}(v,\tau) = c_R - \xi(v,\tau)^{\delta}(c_R - c_M), \tag{2.52}$$

where $v, \tau \geq 0$, $v + \tau \neq 0$, $\delta > 0$, $c_R > 0$ are the costs of a replacement and $c_M > 0$ are the costs of a minimal repair. In Figure 2.7 and Figure 2.8 it can be seen that the shape of both cost functions (2.50) and (2.52) is very similar. The crucial difference to cost function (2.50) is the lower bound for the costs. If $c_M \leq c_R$, the cost function (2.52) has the following properties:

- The costs of PM actions are bounded below and above. They are greater than the costs of a minimal repair and smaller than the costs of a replacement, i.e. $c_M \leq c_{PM}(v,\tau) \leq c_R$.
- The better the PM actions, i.e. the smaller v, the higher are the costs for PM. In case of perfect repair, i.e. $\xi(v,\tau) = 0$, implying that $v = 0$, the costs of PM are $c_{PM}(v,\tau) = c_R$. The same holds if v tends to zero and δ does not at the same time tends to zero.
- The worse the PM actions and the lower the distance between PM actions, i.e. the higher v and the smaller τ, the lower are the costs for PM. In case of minimal repair, i.e. $\xi(v,\tau) = 1$, involving that v tends to infinity or $\tau = 0$, the costs of PM are identical to the costs of a minimal repair, i.e. $c_{PM}(v,\tau) = c_M$.
- If v and τ tend to infinity and δ does not tend to zero, the costs of PM tend to the cost of a replacement, i.e. $c_{PM}(v,\tau)$ tends to c_R.
- The lower δ the lower are the costs of PM and the lower is the cost difference between good and less good PM actions. If $v \neq 0$ and δ tends to zero, the costs of PM tend to the costs of a minimal repair.

Part I.

Models with Continuous Lifetime Distribution

3. System with two Failure Types

In contrast to the most common maintenance models, the model in this chapter assumes that there are different kinds of failures, which cannot all be removed by a minimal repair. In this chapter a repairable system with continuous lifetime distribution and two different failure types is studied. The modeling of the different failure types is analogous to Beichelt [7], who is the pioneer in using different failure types in the modeling of maintenance models. The two failure types are minor failures (type 1) and large failures (type 2). The minor ones can be repaired by minimal repair and the large ones can only be removed through replacement. This idea can be illustrated by the example of a car. If, for instance, the car can not be used because of a defective generator this can easily be removed through a minimal repair. But after a serious traffic accident caused, for instance, by run down tires or brakes, the roadworthiness of the car can surely not be achieved through a minimal repair. In order to avoid such failures and accidents, it is necessary to carry out regular preventive maintenance (PM) actions. Part of such maintenance actions is amongst others to repair or replace all broken and worn items which will soon give up working anyway.

Therefore, it is supposed that the system undergoes both PM and CM actions. A periodic imperfect preventive maintenance policy with finite planning horizon is used to model the occurrence of PM actions. This policy supposes that PM is undertaken at predetermined periodic times. It is assumed that these maintenance actions have a positive influence on the failure intensity. They adjust the virtual age of the system in a Kijima type manner. Furthermore, it is assumed that all CM actions for type 1 failures are minimal repairs which means that the state of the system after repair is the same as just before failure.

In literature there are multiple maintenance models which consider different failure types and there are many ways to differ between these failure types. Colosimo et al. [15], for instance, examine an imperfect maintenance model with two different failure types. Type A failures can be predicted before they happen by a visual inspection. In this way, the repair costs are smaller than the repair costs of type B failures which cannot be predicted. Lin et al. [28] proposed a sequential imperfect preventive maintenance model with two independent failure types. The difference is here that type I failures are maintainable failures and type II failures are non-maintainable. Hence, preventive maintenance actions can reduce the failure rate of maintainable failures but cannot change the failure rate of non-maintainable failures. Based on that model Zequeira and Bérenguer [50] and Castro [13] further assumed for periodic maintenance models that both failure types are dependent. Besides this, there are also models which use repairable and unrepairable failure modes (e.g. in Wang and Zhang [46]). Although there are many publications on systems with multiple types of failures, the imperfect maintenance model described in this section has not yet been discussed in literature.

This chapter is structured as follows. Section 3.1, Section 3.2 and Section 3.3 contain essential assumptions and definitions, which are needed to formulate the cost optimization problem in Section 3.4. Finally, in Section 3.5 different cost functions for PM actions are used to solve the cost optimization problem. Furthermore, the results are computed for several continuous lifetime distributions.

3.1. Modeling the System

According to Beichelt [7] and [10], a system with two failure types is considered. For further research the following assumptions are made.

1. Initially a new repairable system is installed.
2. The system has two failure types, which occur independent of each other. The first failure type are minor failures (type 1) and the second failure type are large failures (type 2).
3. Whenever a failure occurs, it is a minor one (type 1) with probability 1-p and a large one (type 2) with probability p.
4. Type 1 failures can be removed by minimal repair and type 2 failures can only be removed through replacements.
5. The repair times are negligible small.

3.2. Modeling the Failure Counting Process

First consider a repairable system which has only one failure type and which is not preventively maintained. Then let $(T_n)_{n\geq 1}$ be the random failure times of that system. Now introduce the counting process $N = (N_t)_{t\geq 0}$ which counts the CM actions under the assumption that there are no PM actions. The intensity function of this counting process is denoted by $\lambda^N(t)$, $t \geq 0$. Note that if each failure is removed through minimal repair, the process $N = (N_t)_{t\geq 0}$ is an inhomogeneous Poisson process.
Now suppose that the system is preventively maintained. Then the random failure times are denoted by $(T_n^*)_{n\geq 1}$. The corresponding failure counting process is $N^* = (N_t^*)_{t\geq 0}$. Thus, the random variable N_t^* with $t \geq 0$ is the number of failures until t and $\lambda^{N^*}(t)$ with $t \geq 0$ is the intensity function of N^*.
Finally, assume that the system which undergoes PM has different failure types. To construct the failure counting processes of both type 1 and type 2 failures, one have to thin out the process $N^* = (N_t^*)_{t\geq 0}$. This method is described, for example, in Belyaev and Kahle [12]. The random failure times $(T_n^*)_{n\geq 1}$ are a point process. Every realization t_n^* of these random variables is of type 2 with probability p and of type 1 with probability $1-p$. Define a new sequence $(\Delta_n)_{n\geq 1}$ of independent identical rectangular distributed random numbers, i.e. $P(\Delta_n = 1) = p$ and $P(\Delta_n = 0) = 1-p$ for $n \geq 1$. If $\Delta_n = 1$, the failure which occurs at time t_n^* is of type 2, else if $\Delta_n = 0$, the failure at time t_n^* is of type 1. If one divides the random variables $(T_n^*)_{n\geq 1}$ in accordance with the realization of $(\Delta_n)_{n\geq 1}$, this will produce two new point processes $(T_k^{*'})_{k\geq 1}$ and $(T_k^{*''})_{k\geq 1}$

of the type 1 and type 2 failure times, respectively.
Let $(N_t^{*'})_{t\geq 0}$ be the counting process of type 1 failures. Then it holds

$$N_t^{*'} = N_t^* - \sum_{n=1}^{N_t^*} \Delta_n, \quad \forall t \geq 0. \tag{3.1}$$

The same is valid for the counting process of type 2 failures $(N_t^{*''})_{t\geq 0}$, i.e.

$$N_t^{*''} = N_t^* - \sum_{n=1}^{N_t^*} (1 - \Delta_n), \quad \forall t \geq 0. \tag{3.2}$$

The sum of both counting processes leads to the original failure counting process

$$\begin{aligned} N_t^{*'} + N_t^{*''} &= N_t^* - \sum_{n=1}^{N_t^*} \Delta_n + N_t^* - \sum_{n=1}^{N_t^*} (1 - \Delta_n) \\ &= 2N_t^* - \sum_{n=1}^{N_t^*} 1 \\ &= N_t^*, \quad \forall t \geq 0. \end{aligned} \tag{3.3}$$

Note that if $N^* = (N_t^*)_{t\geq 0}$ is an inhomogeneous Poisson process with intensity function $\lambda^{N^*}(t)$, the corresponding stochastic processes $(N_t^{*'})_{t\geq 0}$ and $(N_t^{*''})_{t\geq 0}$ are also inhomogeneous Poisson processes with intensity functions $(1-p)\lambda^{N^*}(t)$ and $p\lambda^{N^*}(t)$, respectively [19, p. 134]. Therefore, using Theorem 2.2 it holds

$$P(N_t^{*''} = n) = \frac{1}{n!} \left(\int_0^t p\lambda^{N^*}(x)dx \right)^n \exp\left(- \int_0^t p\lambda^{N^*}(x)dx \right), \quad n = 1, 2, \ldots. \tag{3.4}$$

3.3. Maintenance Policy

In the following, we consider a periodic imperfect preventive maintenance policy with finite planning horizon. In the periodic PM policy the system is preventively maintained at fixed time intervals and repaired at intervening failures. The maintenance policy used in this chapter is a direct generalization of the one dealt with in Beichelt [8].
Similar maintenance policies were applied, for instance, in Nakagawa [32], Sheu and Lin and Liao [45] and Zequeira and Bérenguer [50].
In particular, the following assumptions are made for the used maintenance policy.

1. The system is maintained according to the failure type. Whenever a minor failure (type 1) occurs, a minimal repair will be carried out. If the failure is of type 2, the system will be replaced by a new one.

2. The PM actions are imperfect in the sense that each PM action reduces the virtual age of the system to a constant virtual age of $v \geq 0$.

3. PM is performed at $v+\tau, v+2\tau, \dots, v+(N-1)\tau$ with $\tau > 0$, $v \geq 0$ and $N \in \{1, \dots, N_{\max}\}$.
4. If no type 2 failure occurred in $[0, v+N\tau)$, the system is replaced preventively at $v+N\tau$.

Note that the restriction of N by $N_{\max}$ is appropriate since systems have a finite useful life. Therefore, in our cost optimization problem from Section 3.4 a predefined maximum number of PM actions, i.e. $1 - N_{\max}$, will be taken into account.
The above described maintenance policy contains the age replacement policy and the minimal repair policy as special cases. The first one is obtained if $p = 1$ and $v = 0$. Then the system is replaced at the time of failure or at age τ whichever occurs first. If $p = 0$ and $v = 0$ we have the minimal repair policy which means that the system is always replaced at age τ and failures that occur between the periodic replacements are removed through minimal repair.

3.4. Cost Optimization Problem

Consider a technical system which is maintained with maintenance policy described in Section 3.3. To optimize the maintenance of this system with respect to cost criteria, it is necessary to define the cumulative distribution function of the random time of the first type 2 failure.
Let T_1 be the random time of the first failure of a repairable system without PM. Then T_1^* is the random time of the first failure of a repairable system with PM and only one failure type.
If one takes into account two failure types, one have $T_1^{'}$ and $T_1^{''}$ as the random times of the first occurrence of a type 1 or type 2 failure of a repairable system without PM, respectively. Analogous $T_1^{*'}$ and $T_1^{*''}$ are the random times of the first occurrence of a type 1 or type 2 failure of a repairable system with consideration of PM actions, respectively.
In our modeling the random variable $T_1^{*''}$ is very important because a type 2 failure ends a replacement cycle. In what follows, some properties of the distribution of $T_1^{*''}$ are given.

Lemma 3.1 (**Distribution function of T_1''**)
Suppose T_1 is the random time of the first failure of a repairable system without PM and no distinction in failure types. It is assumed that a failure is of type 1 with probability $1-p$ and of type 2 with probability p. Let $T_1^{''}$ be the random time of the first type 2 failure of a repairable system without PM. Then, $T_1^{''}$ has the following distribution function

$$F^{T_1^{''}}(t) = 1 - \exp\left(- \int_0^t p\lambda^N(x)dx \right), \quad \forall t \geq 0, \tag{3.5}$$

where $\lambda^N(\cdot)$ is the intensity function of the failure counting process $N = (N_t)_{t\geq 0}$.

Proof. Let $N = (N_t)_{t\geq 0}$ be the failure counting process under the assumption that there are no PM actions. As long as each failure is removed through minimal repair,

the process $N = (N_t)_{t\geq 0}$ is an inhomogeneous Poisson process.
In our model we have two failure types and therefore the process N can be written as the sum of the two counting processes $N^{'}$ and $N^{''}$, which counts the type 1 and type 2 failures, respectively. Note that, if $N = (N_t)_{t\geq 0}$ is an inhomogeneous Poisson process with intensity function $\lambda^N(t)$, the corresponding stochastic processes $N^{'} = (N_t^{'})_{t\geq 0}$ and $N^{''} = (N_t^{''})_{t\geq 0}$ are also inhomogeneous Poisson processes with intensity functions $p\lambda^N(t)$ and $(1-p)\lambda^N(t)$, respectively [19, p. 134]. Therefore, as long as no type 2 failure occurred, the processes $N^{'}$ and $N^{''}$ are inhomogeneous Poisson processes and the random variable $N_t^{''}$ is Poisson distributed with mean $\Lambda^{N^{''}}(t) = E(N_t^{''}) = \int_0^t p\lambda^N(x)dx$. Using this, Equation (3.5) can be derived as

$$\begin{aligned} F^{T_1^{''}}(t) = P(T_1^{''} \leq t) &= P(N_t^{''} \geq 1) \\ &= 1 - P(N_t^{''} = 0) \\ &\overset{\text{Theorem 2.2}}{=} 1 - \exp\left(-\int_0^t p\lambda^N(x)dx\right), \quad \forall t \geq 0. \end{aligned}$$

□

Theorem 3.2 (Distribution function of $T_1^{*''}$)
Suppose $T_1^{''}$ is the random time of the first type 2 failure of a repairable system without PM. Let $T_1^{*''}$ be the random time of the first type 2 failure of a repairable system with PM, following PM policy from Section 3.3. Then for the distribution function $F^{T_1^{*''}}(t) = P(T_1^{*''} \leq t)$ it holds

$$F^{T_1^{*''}}(t) = \begin{cases} 0 \text{ , } \textit{if } t < 0 \\ F^{T_1^{''}}(t) \text{ , } \textit{if } t \in [0, v+\tau) \\ F^{T_1^{''}}(v+\tau) + \left(1 - F^{T_1^{''}}(v+\tau)\right)\left(\frac{F^{T_1^{''}}(t-\tau) - F^{T_1^{''}}(v)}{1-F^{T_1^{''}}(v)}\right) \\ \qquad \text{, } \textit{if } t \in [v+\tau, v+2\tau) \\ F^{T_1^{''}}(v+\tau) + \left(F^{T_1^{''}}(v+\tau) - F^{T_1^{''}}(v)\right)\sum_{i=2}^{k}\left(\frac{1-F^{T_1^{''}}(v+\tau)}{1-F^{T_1^{''}}(v)}\right)^{i-1} \\ + \left(\frac{1-F^{T_1^{''}}(v+\tau)}{1-F^{T_1^{''}}(v)}\right)^{k}\left(F^{T_1^{''}}(t-k\tau) - F^{T_1^{''}}(v)\right) \\ \qquad \text{, } \textit{if } t \in [v+k\tau, v+\{k+1\}\tau),\ k \geq 2 \end{cases} . \tag{3.6}$$

Proof. Since $T_1^{*''}$ is the random time of the first type 2 failure of a system with PM actions, it holds that $F^{T_1^{*''}}(t) = 0$, for $t < 0$. Using maintenance policy from Section 3.3, the first PM time is $v + \tau$. Before this point in time, a repairable system with PM actions is identical with an equivalent repairable system without PM. Hence,

$$F^{T_1^{*''}}(t) = P(T_1^{''} \leq t) = F^{T_1^{''}}(t), \quad \forall t \in [0, v+\tau). \tag{3.7}$$

During the time between the first and the second PM action, i.e. $t \in [v+\tau, v+2\tau)$, it holds

$$F^{T_1^{*''}}(t) = P(T_1^{''} \leq v+\tau) + P(T_1^{''} \geq v+\tau)P(T_1^{''} \leq t-\tau | T_1^{''} \geq v)$$

$$= F^{T_1''}(v+\tau) + \left(1 - F^{T_1''}(v+\tau)\right)\left(\frac{F^{T_1''}(t-\tau) - F^{T_1''}(v)}{1 - F^{T_1''}(v)}\right). \quad (3.8)$$

For $t \in [v + k\tau, v + \{k+1\}\tau)$, $k \geq 2$ it holds

$$\begin{aligned} F^{T_1^{*''}}(t) &= P(T_1'' \leq v+\tau) \\ &\quad + \sum_{i=2}^{k} P(T_1'' \geq v+\tau) P(T_1'' \geq v+\tau | T_1'' \geq v)^{i-2} P(T_1'' \leq v+\tau | T_1'' \geq v) \\ &\quad + P(T_1'' \geq v+\tau) P(T_1'' \geq v+\tau | T_1'' \geq v)^{k-1} P(T_1'' \leq t - k\tau | T_1'' \geq v) \\ &= F^{T_1''}(v+\tau) \\ &\quad + \sum_{i=2}^{k} \left(1 - F^{T_1''}(v+\tau)\right)\left(\frac{1 - F^{T_1''}(v+\tau)}{1 - F^{T_1''}(v)}\right)^{i-2}\left(\frac{F^{T_1''}(v+\tau) - F^{T_1''}(v)}{1 - F^{T_1''}(v)}\right) \\ &\quad + \left(1 - F^{T_1''}(v+\tau)\right)\left(\frac{1 - F^{T_1''}(v+\tau)}{1 - F^{T_1''}(v)}\right)^{k-1}\left(\frac{F^{T_1''}(t-k\tau) - F^{T_1''}(v)}{1 - F^{T_1''}(v)}\right) \\ &= F^{T_1''}(v+\tau) + \left(F^{T_1''}(v+\tau) - F^{T_1''}(v)\right)\sum_{i=2}^{k}\left(\frac{1 - F^{T_1''}(v+\tau)}{1 - F^{T_1''}(v)}\right)^{i-1} \\ &\quad + \left(\frac{1 - F^{T_1''}(v+\tau)}{1 - F^{T_1''}(v)}\right)^{k}\left(F^{T_1''}(t-k\tau) - F^{T_1''}(v)\right). \end{aligned} \quad (3.9)$$

□

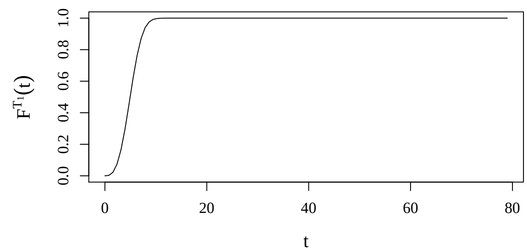
Figure 3.1 shows (a) the distribution function of the time to the first failure of a repairable system without PM and only one failure type, (b) the distribution function of the time to the first type 2 failure of a repairable system without PM and (c) the distribution function of the time to the first type 2 failure of a repairable system with PM. The comparison of (b) and (c) shows how PM actions reduce the probability of the occurrence of failures.

Remark 3.3 (Intensity Function of $N^* = (N_t^*)_{t\geq 0}$)
The intensity function of the counting process $N^* = (N_t^*)_{t\geq 0}$ is

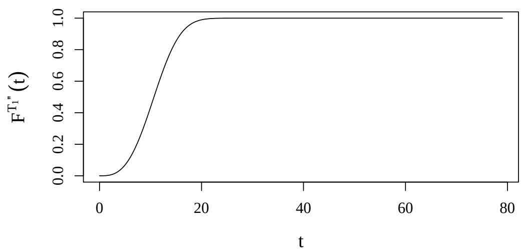
$$\lambda^{N^*}(t) = \begin{cases} 0 & , \text{ if } t < 0 \\ \lambda^N(t) & , \text{ if } t \in [0, v+\tau) \\ \lambda^N(t - k\tau) & , \text{ if } t \in [v + k\tau, v + \{k+1\}\tau),\ k = 1, \ldots, N-1 \end{cases}, \quad (3.10)$$

where $\lambda^N(t)$ is the intensity function of the counting process $N = (N_t)_{t\geq 0}$, which counts the failures of a repairable system without PM.

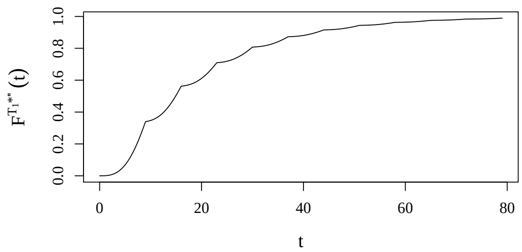
Note that with Remark 2.4 the intensity function of the counting process $N = (N_t)_{t\geq 0}$ is equal to the hazard function of the time to the first failure of a new system, i.e. $\lambda^N(t) = h^{T_1}(t)$ for $t \geq 0$.
Using maintenance policy from Section 3.3, the random cycle length, i.e. the time



(a) Distribution function of T_1



(b) Distribution function of $T_1^{''}$



(c) Distribution function of $T_1^{*''}$

Figure 3.1.: Distribution functions when T_1 is WD($\beta = 0.0057, \gamma = 3$), $v = 2$, $\tau = 7$, $N = 11$ and $p = 0.1$.

between two replacements, is

$$L_{v,\tau,N} \quad = \quad \min\{T_1^{*''}, v + N\tau\} \begin{cases} < v + N\tau & \text{, with } P(T_1^{*''} \leq v + N\tau) \\ = v + N\tau & \text{, with } 1 - P(T_1^{*''} \leq v + N\tau) \end{cases} . \tag{3.11}$$

In this special problem the distribution function of $L_{v,\tau,N}$ is

$$F^{L_{v,\tau,N}}(t) = P(L_{v,\tau,N} \leq t) \quad = \quad \begin{cases} 0 & \text{, if } t < 0 \\ F^{T_1^{*''}}(t) & \text{, if } 0 \leq t < v + N\tau \\ 1 & \text{, if } t \geq v + N\tau \end{cases} . \tag{3.12}$$

Theorem 3.4 (Mean Cycle Length)
In the case of maintenance policy from Section 3.3, the mean cycle length is

$$\begin{aligned} E(L_{v,\tau,N}) \quad = \quad & v + N\tau - \int_0^v F^{T_1^{''}}(t)dt \\ & - \tau F^{T_1^{''}}(v+\tau) \sum_{i=1}^{N-1} (N-i) \left(\frac{1 - F^{T_1^{''}}(v+\tau)}{1 - F^{T_1^{''}}(v)} \right)^{i-1} \\ & + \tau F^{T_1^{''}}(v) \sum_{i=1}^{N-1} (N-i) \left(\frac{1 - F^{T_1^{''}}(v+\tau)}{1 - F^{T_1^{''}}(v)} \right)^{i} \end{aligned}$$

$$- \int_v^{v+\tau} F^{T_1''}(t)dt \sum_{i=1}^{N} \left(\frac{1 - F^{T_1''}(v+\tau)}{1 - F^{T_1''}(v)} \right)^{i-1} . \qquad (3.13)$$

Proof. Using Remark 2.1 we get

$$E(L_{v,\tau,N}) = \int_0^{v+N\tau} \left(1 - F^{T_1^{*''}}(t)\right) dt = v + N\tau - \int_0^{v+N\tau} F^{T_1^{*''}}(t)dt. \qquad (3.14)$$

After inserting (3.6) and using some algebra one get formula (3.13) for the mean cycle length. □

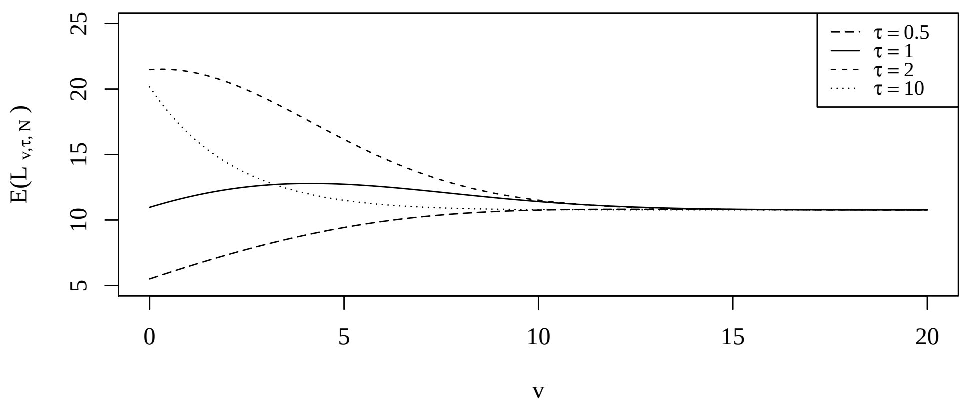


Figure 3.2.: Mean cycle length when T_1 is WD($\beta = 0.0057, \gamma = 3$), $N = 11$ and $p = 0.1$.

In Figure 3.2 the mean cycle length is plotted for the WD with expectation 5. If $v = 0$ and τ is relative small, the probability that a type 2 failure ends the replacement cycle is rather low. Therefore, the mean cycle length approximates $v+N\tau$. If τ increases, the probability that a type 2 failure ends the replacement cycle is increasing and thus the mean cycle length for $v = 0$ is clearly smaller than $v + N\tau$. Note that with increasing $v+N\tau$, the mean cycle length converges to the expected time of the first type 2 failure. To compute a maintenance cost rate, it is necessary to determine the random number of minimal repairs during a replacement cycle.

Theorem 3.5 (Mean Number of Type 1 Failures in a Replacement Cycle) *Suppose $Z_{v+N\tau}$ is the random number of type 1 failures in the replacement cycle with length* $\min\{T_1^{*''}, v + N\tau\}$. *Then it holds*

$$E(Z_{v+N\tau}) = \int_0^{v+N\tau} \left(1 - F^{T_1^{*''}}(x)\right) \lambda^{N^*}(x)dx - F^{T_1^{*''}}(v + N\tau). \qquad (3.15)$$

Proof. The proof of Theorem 3.5 based on the ideas in Beichelt and Franken [11] and Beichelt [10, p. 128 ff.]. It is assumed that after a minimal repair the system starts working immediately. Suppose that the random variables $(T_n^*)_{n\geq 1}$ are the random failure times of a repairable system with PM. These failures are type 1 failures with probability $1-p$ and type 2 failures with probability p. Because of the minimal repair after type 1 failures the state of the system does not change through this type of failure. Therefore, as long as no type 2 failure occurred, the failure counting process $N^* = (N_t^*)_{t\geq 0}$ is an inhomogeneous Poisson process with independent increments and intensity function $\lambda^{N^*}(t)$ for $t \geq 0$. Using Lemma 2.3 and the resulting equation (2.21), it holds

$$\begin{aligned} p_0^{N^*}(s,t) &= P(N_t^* - N_s^* = 0) = \exp\left(-\int_s^t \lambda^{N^*}(x)dx\right) \\ &= \exp\left(-\left(\Lambda^{N^*}(t) - \Lambda^{N^*}(s)\right)\right), \end{aligned} \tag{3.16}$$

where $\Lambda^{N^*}(t) = \int_0^t \lambda^{N^*}(x)dx$ is the mean function of the counting process N^*. Furthermore, by making use of Theorem 2.2, the distribution function of the random time of the first failure of a repairable system with PM becomes

$$\begin{aligned} F^{T_1^*}(t) = P(T_1^* \leq t) &= P(N_t^* \geq 1) = 1 - P(N_t^* = 0) \\ &= 1 - \exp\left(-\int_0^t \lambda^{N^*}(x)dx\right), \quad t \geq 0. \end{aligned} \tag{3.17}$$

The density function of T_1^* can be calculated as follows

$$f^{T_1^*}(t) = \frac{dF^{T_1^*}(t)}{dt} = \lambda^{N^*}(t)\left(1 - F^{T_1^*}(t)\right) = \lambda^{N^*}(t)\exp\left(-\int_0^t \lambda^{N^*}(x)dx\right). \tag{3.18}$$

Suppose that only the last failure at the random time T_k^* is of type 2. Hence, the failure counting process $N^* = (N_t^*)_{t\geq 0}$ up to this time is an inhomogeneous Poisson process. By using induction it can be proved that the following holds for the joint probability density function of $(T_1^*, T_2^*, \ldots, T_k^*)$, $k = 1, 2, \ldots$, (induction hypothesis)

$$f^{T_1^*,T_2^*,\ldots,T_k^*}(t_1, t_2, \ldots, t_k) = \begin{cases} \lambda^{N^*}(t_1)\lambda^{N^*}(t_2)\cdot\dots\cdot\lambda^{N^*}(t_{k-1})f^{T_1^*}(t_k) & \text{, if } 0 \leq t_1 < t_2 < \cdots < t_k \\ 0 & \text{, else} \end{cases}. \tag{3.19}$$

The base case is that the statement holds for $k = 2$. Let $1 - p_0^{N^*}(t, t_2)$ be the probability that between t and t_2 at least one failure occurs. Using (3.16), the conditional distribution function of T_2^* given that $T_1^* = t_1$ becomes

$$F^{T_2^*}(t_2|T_1^* = t_1) = 1 - p_0^{N^*}(t_1, t_2) = 1 - \exp\left(-\left(\Lambda^{N^*}(t_2) - \Lambda^{N^*}(t_1)\right)\right). \tag{3.20}$$

Differentiation with respect to t_2 yields the corresponding conditional density function

$$f^{T_2^*}(t_2|T_1^* = t_1) = \lambda^{N^*}(t_2)\exp\left(-\left(\Lambda^{N^*}(t_2) - \Lambda^{N^*}(t_1)\right)\right), \tag{3.21}$$

where $0 \leq t_1 < t_2$. The joint probability density function of (T_1^*, T_2^*) is then

$$f^{T_1^*,T_2^*}(t_1,t_2) = f^{T_2^*}(t_2|T_1^*=t_1)f^{T_1^*}(t_1) = \begin{cases} \lambda^{N^*}(t_1)f^{T_1^*}(t_2) & \text{, if } 0 \leq t_1 < t_2 \\ 0 & \text{, else} \end{cases}. \quad (3.22)$$

Now assume that the statement (3.19) is true and use this to prove the statement for $k+1$. It holds

$$\begin{aligned} F^{T_{k+1}^*}(t_{k+1}|T_1^*=t_1,\dots,T_k^*=t_k) &= 1 - p_0^{N^*}(t_k,t_{k+1}) \\ &= 1 - \exp\left(-\left(\Lambda^{N^*}(t_{k+1}) - \Lambda^{N^*}(t_k)\right)\right). \quad (3.23) \end{aligned}$$

Differentiation with respect to t_{k+1} yields the corresponding conditional density function

$$f^{T_{k+1}^*}(t_{k+1}|T_1^*=t_1,\dots,T_k^*=t_k) = \lambda^{N^*}(t_{k+1})\exp\left(-\left(\Lambda^{N^*}(t_{k+1}) - \Lambda^{N^*}(t_k)\right)\right), \quad (3.24)$$

where $0 \leq t_1 < \cdots < t_{k+1}$. Thus, for the joint probability density function of $(T_1^*,\dots,T_{k+1}^*)$ it holds

$$\begin{aligned} & f^{T_1^*,T_2^*,\dots,T_{k+1}^*}(t_1,t_2,\dots,t_{k+1}) \\ &= f^{T_{k+1}^*}(t_{k+1}|T_1^*=t_1,\dots,T_k^*=t_k)f^{T_1^*,T_2^*,\dots,T_k^*}(t_1,t_2,\dots,t_k) \\ &= \begin{cases} \lambda^{N^*}(t_1)\lambda^{N^*}(t_2)\cdot\dots\cdot\lambda^{N^*}(t_k)f^{T_1^*}(t_{k+1}) & \text{, if } 0 \leq t_1 < \dots < t_{k+1} \\ 0 & \text{, else} \end{cases}. \quad (3.25) \end{aligned}$$

Hence, hypothesis (3.19) holds for all $k \geq 1$.
Now let Z be the random number of type 1 failures until the first type 2 failure occurs and let p be the probability of type 2 failures. Note that for an integrable function $g(x)$, $x \geq 0$, it holds [9, p. 196]

$$\int_0^t \int_0^{x_k} \dots \int_0^{x_3} \int_0^{x_2} \prod_{i=1}^k g(x_i)dx_1dx_2\dots dx_k = \frac{1}{k!}\left(\int_0^t g(x)dx\right)^k, \quad (3.26)$$

where $k \geq 2$.
From (3.19) and for $k \geq 1$ it follows

$$\begin{aligned} P(Z=k) &= \int_0^\infty \int_0^{x_{k+1}} \dots \int_0^{x_3} \int_0^{x_2} \prod_{i=1}^k (1-p)\lambda^{N^*}(x_i)dx_i p f^{T_1^*}(x_{k+1})dx_{k+1} \\ &\overset{(3.26)}{=} \int_0^\infty \frac{1}{k!}\left(\int_0^{x_{k+1}} (1-p)\lambda^{N^*}(x)dx\right)^k p f^{T_1^*}(x_{k+1})dx_{k+1} \\ &\overset{t:=x_{k+1}}{=} \frac{1}{k!}\int_0^\infty \left(\int_0^t (1-p)\lambda^{N^*}(x)dx\right)^k p f^{T_1^*}(t)dt. \quad (3.27) \end{aligned}$$

This is the probability that the first k failures are of type 1 and the $(k+1)$th failure is of type 2. Let $T_1^{*''}$ be the random time of the first type 2 failure. Suppose Z_t is the random number of type 1 failures in the replacement cycle with length $\min\{T_1^{*''}, t\}$.

Then it holds

$$
\begin{aligned}
P(Z_t = k | T_1^{*''} = t) & \\
= & \lim_{\Delta t \to 0} \frac{P(Z_t = k \cap t \leq T_1^{*''} \leq t + \Delta t)}{P(t \leq T_1^{*''} \leq t + \Delta t)} \\
= & \lim_{\Delta t \to 0} \frac{P(t \leq T_1^{*''} = T_{k+1}^{*} \leq t + \Delta t)}{F^{T_1^{*''}}(t + \Delta t) - F^{T_1^{*''}}(t)} \\
\overset{(3.27)}{=} & \lim_{\Delta t \to 0} \frac{\frac{1}{k!} \int_t^{t+\Delta t} \left(\int_0^y (1-p) \lambda^{N^*}(x) dx \right)^k p f^{T_1^*}(y) dy}{F^{T_1^{*''}}(t + \Delta t) - F^{T_1^{*''}}(t)} \\
= & \lim_{\Delta t \to 0} \left(\frac{\frac{1}{k!} \int_t^{t+\Delta t} \left(\int_0^y (1-p) \lambda^{N^*}(x) dx \right)^k p f^{T_1^*}(y) dy}{\Delta t} \right. \\
& \left. \cdot \frac{\Delta t}{F^{T_1^{*''}}(t + \Delta t) - F^{T_1^{*''}}(t)} \right). \qquad (3.28)
\end{aligned}
$$

If the limits of both factors in equation (3.28) exist, one can rewrite this equation as a product of two limits. For the limit of the first factor it holds

$$
\begin{aligned}
& \lim_{\Delta t \to 0} \left(\frac{\frac{1}{k!} \int_t^{t+\Delta t} \left(\int_0^y (1-p) \lambda^{N^*}(x) dx \right)^k p f^{T_1^*}(y) dy}{\Delta t} \right) \\
\overset{0/0}{=} & \lim_{\Delta t \to 0} \frac{\frac{1}{k!} \left(\int_0^{t+\Delta t} (1-p) \lambda^{N^*}(x) dx \right)^k p f^{T_1^*}(t + \Delta t)}{1} \\
= & \frac{1}{k!} \left(\int_0^t (1-p) \lambda^{N^*}(x) dx \right)^k p f^{T_1^*}(t)
\end{aligned}
$$

and for the limit of the second factor it holds

$$
\begin{aligned}
\lim_{\Delta t \to 0} \frac{\Delta t}{F^{T_1^{*''}}(t + \Delta t) - F^{T_1^{*''}}(t)} & = \frac{1}{f^{T_1^{*''}}(t)} \\
& = \frac{1}{p \lambda^{N^*}(t)(1 - F^{T_1^{*''}}(t))} \\
& = \frac{1 - F^{T_1^*}(t)}{p f^{T_1^*}(t)(1 - F^{T_1^{*''}}(t))}.
\end{aligned}
$$

Therefore, equation (3.28) reduces to

$$
P(Z_t = k | T_1^{*''} = t) = \frac{1}{k!} \left(\int_0^t (1-p) \lambda^{N^*}(x) dx \right)^k \exp \left(- \int_0^t (1-p) \lambda^{N^*}(x) dx \right), \qquad (3.29)
$$

where T_{k+1}^* is the random time of the $(k+1)$th failure absolute.
The expected value of Z_t can be computed with the help of conditional expectation in

the following way

$$E(Z_t) = E(Z_t|T_1^{*''} < t)P(T_1^{*''} < t) + E(Z_t|T_1^{*''} \geq t)P(T_1^{*''} \geq t). \qquad (3.30)$$

The first summand represents the case that the replacement cycle is terminated by a type 2 failure and the second summand represents the case that a preventive replacement ends the replacement cycle.
From (3.29) it follows that, given $T_1^{*''} = t$, the random variable Z_t has a Poisson distribution with mean

$$E(Z_t|T_1^{*''} = t) = \int_0^t (1-p)\lambda^{N^*}(x)dx. \qquad (3.31)$$

This can be used to compute both conditional expected values in (3.30). For the first one it holds

$$E(Z_t|T_1^{*''} < t) = \frac{\int_0^t E(Z_x|T_1^{*''} = x)dF^{T_1^{*''}}(x)}{F^{T_1^{*''}}(t)} \qquad (3.32)$$

and the second one can be rewritten as follows

$$E(Z_t|T_1^{*''} \geq t) = E(Z_t|T_1^{*''} = t). \qquad (3.33)$$

The following term is used in the subsequent calculation

$$\int_0^t p\lambda^{N^*}(y)dy \overset{(3.4)}{=} -\ln P(N_t^{*''} = 0) = -\ln P(T_1^{*''} > t) = -\ln \bar{F}^{T_1^{*''}}(t). \qquad (3.34)$$

Inserting (3.32) and (3.33) in the formula for the (unconditional) mean value of Z_t (3.30), using partial integration and substitution yield

$$\begin{aligned}
E(Z_t) &= \int_0^t \int_0^x (1-p)\lambda^{N^*}(y)dy dF^{T_1^{*''}}(x) + \bar{F}^{T_1^{*''}}(t) \int_0^t (1-p)\lambda^{N^*}(x)dx \\
&\overset{(3.34)}{=} \int_0^t \left(\Lambda^{N^*}(x) + \ln \bar{F}^{T_1^{*''}}(x)\right) dF^{T_1^{*''}}(x) + \bar{F}^{T_1^{*''}}(t)\left(\Lambda^{N^*}(t) + \ln \bar{F}^{T_1^{*''}}(t)\right) \\
&= \int_0^t \Lambda^{N^*}(x)dF^{T_1^{*''}}(x) + \int_0^t \ln \bar{F}^{T_1^{*''}}(x)dF^{T_1^{*''}}(x) \\
&\quad + \bar{F}^{T_1^{*''}}(t)\left(\Lambda^{N^*}(t) + \ln \bar{F}^{T_1^{*''}}(t)\right) \\
&= \int_0^t \Lambda^{N^*}(x)p\lambda^{N^*}(x)\bar{F}^{T_1^{*''}}(x)dx - \int_1^{\bar{F}^{T_1^{*''}}(t)} \ln y \, dy \\
&\quad + \bar{F}^{T_1^{*''}}(t)\left(\Lambda^{N^*}(t) + \ln \bar{F}^{T_1^{*''}}(t)\right) \\
&= \Lambda^{N^*}(t)F^{T_1^{*''}}(t) - \int_0^t F^{T_1^{*''}}(x)d\Lambda^{N^*}(x) - F^{T_1^{*''}}(t) - \bar{F}^{T_1^{*''}}(t)\ln \bar{F}^{T_1^{*''}}(t) \\
&\quad + \Lambda^{N^*}(t)\bar{F}^{T_1^{*''}}(t) + \ln \bar{F}^{T_1^{*''}}(t)\bar{F}^{T_1^{*''}}(t) \\
&= \int_0^t \bar{F}^{T_1^{*''}}(x)d\Lambda^{N^*}(x) - F^{T_1^{*''}}(t). \qquad (3.35)
\end{aligned}$$

□

In the following the optimization criterion of interest will be the average maintenance cost per unit time. In the underlying maintenance policy there are two scenario.
The first one is that there is no type 2 failure up to the first preventive replacement at time $v+N\tau$. In this case the replacement cycle has length $[0, v+N\tau]$. The total costs comprise the costs for minimal repairs during the replacement cycle, the costs for $N-1$ PM actions and the costs for a preventive replacement at $v+N\tau$. The probability of the first scenario is $1 - P(T_1^{*''} \leq v + N\tau)$.
The second scenario is that a type 2 failure occurs before the preventive replacement takes place at $v + N\tau$. In this case the total costs consist of the costs for minimal repairs up to the time of the type 2 failure, the costs of the PM actions and the costs of a replacement because the type 2 failure can only be removed through a replacement. The probability of the second scenario is $P(T_1^{*''} \leq v + N\tau)$.

Definition 3.1 (Cost Optimization Problem)
Let c_M denotes the costs for a minimal repair, c_{PM} the costs of PM and c_R the costs of a replacement. The average maintenance costs per unit time are

$$\begin{aligned} C(v,\tau,N) &= (1 - P(T_1^{*''} \leq v + N\tau)) \\ &\quad \cdot \frac{\left(c_M E(Z_{v+N\tau} | T_1^{*''} \geq v + N\tau) + (N-1)c_{PM} + c_R\right)}{E(L_{v,\tau,N})} \\ &\quad + P(T_1^{*''} \leq v + N\tau) \left(\frac{c_M E(Z_{v+N\tau} | T_1^{*''} < v + N\tau)}{E(L_{v,\tau,N})} \right. \\ &\quad \left. + \frac{\sum_{k=1}^{N-1} c_{PM} \mathbf{1}_{\{v+k\tau < E(T_1^{*''} | T_1^{*''} < v+N\tau)\}} + c_R}{E(L_{v,\tau,N})} \right) \\ &= \frac{c_M E(Z_{v+N\tau}) + c_R}{E(L_{v,\tau,N})} + \bar{F}^{T^{*''}}(v+N\tau) \frac{(N-1)c_{PM}}{E(L_{v,\tau,N})} \\ &\quad + F^{T_1^{*''}}(v+N\tau) \frac{\sum_{k=1}^{N-1} c_{PM} \mathbf{1}_{\{v+k\tau < E(T_1^{*''} | T_1^{*''} < v+N\tau)\}}}{E(L_{v,\tau,N})}, \end{aligned} \tag{3.36}$$

where $E(T_1^{*''} | T_1^{*''} < v + N\tau)$ is the expected time of the first type 2 failure under the condition that a type 2 failure ends the replacement cycle and $\mathbf{1}_{\{.\}}$ is the indicator function. The optimization problem then have the following form

$$\min_{v \in [0,\infty),\, \tau \in (0,\infty),\, N \in \{1,\ldots,N_{\max}\}} C(v,\tau,N). \tag{3.37}$$

Note that the extreme case $\tau = 0$ is excluded from optimization problem (3.37). However, the other extreme case of perfect PM actions, i.e. $v = 0$, is still part of the cost optimization problem.

3.5. Example for Cost Optimal Maintenance

In this section a part of the cost functions introduced in Section 2.7 are used to model the costs of PM. Optimal maintenance strategies are computed for different lifetime

distributions.
The main objective of this section is to compute the cost optimal values for v, τ and N. The number of PM actions is restricted up to 10, i.e. $N_{\max} = 11$, and hence the system will be preventively replaced at the latest after ten PM actions. The probability that a failure is of type 2 is assumed to be ten percent, i.e. $p = 0.1$. The costs of corrective maintenance actions, i.e. c_M and c_R, are assumed to be constant. No general statement whether the costs of PM are always in between c_M and c_R can be made. This individually depends on v, τ, δ and the ratio c_M/c_R. Since the resulting average maintenance costs per unit time are not convex and the cost optimization problem (3.37) could not be solved analytical, complete enumeration is used to find the optimal maintenance strategies. The optimal maintenance strategies in this section are computed with the statistical computing software R and have an accuracy of two decimal places. It is however important to note that all values in the following tables are computed based on a numerical integration routine and are therefore only approximative solutions.
The continuous lifetime distributions used in the following subsections to model the lifetime of the underlying repairable system are described in detail in Section 2.5. The parameters of all used distributions were chosen so that the expectation is 5 for all distributions.

3.5.1. Costs Proportional to the Impact of Repair

Suppose the costs of PM actions depend only on the virtual age after PM, i.e.

$$c_{PM}(v) = c_I \left(\frac{1}{v}\right)^{\delta}, \tag{3.38}$$

where $v > 0$, $\delta > 0$ and $c_I > 0$ is a constant cost value. This cost function is described in detail in Subsection 2.7.1. Further, it is assumed that $c_R = c_I$. Note that in case of costs proportional to the impact of repair, the extreme case of perfect PM, i.e. $v = 0$, is excluded from optimization problem (3.37).
In the following, the optimal values for v and τ and the optimal number of PM actions before a preventive replacement takes place are computed for different cost ratios c_M/c_I and different δ. The numerical results are given in Table 3.1 and lead to the following conclusions:

1. Because of the very high costs of PM actions with a high impact of repair, the cost optimal value of v does not have values less than 1.

2. If $v > 1$ it holds that the smaller δ, the more expensive are PM actions and the lower is the difference between the costs of PM and the cost of a replacement since $c_R = c_I$. Therefore, for small δ it is cost optimal to do no PM, i.e. $N = 1$, and with rising δ it becomes cost optimal to do more PM actions, i.e. the cost optimal N is rising. In comparison to the other distributions the RMWD has sooner $N > 1$, because of the high failure rate at lower ages.

3. The lower the costs of a repair with a high impact compared to the costs of a minimal repair, i.e. the higher the ratio c_M/c_I, the better are the PM actions,

Table 3.1.: Optimal values in case of costs proportional to the impact of repair

	LFRD	RD	WD	MWD	RMWD
	$\alpha = 0.01$ $\beta = 0.02944$	$\beta = 0.03142$	$\beta = 0.0057$ $\gamma = 3$	$\alpha = 0.03$ $\beta = 0.004335$ $\gamma = 3$	$\alpha = 0.1$ $\beta = 0.1746$ $\gamma = 0.1$
$c_M/c_I = 0.5$					
$\delta = 0.125$	$N = 1$ $v + \tau = 8.87$	$N = 1$ $v + \tau = 8.57$	$N = 1$ $v + \tau = 5.83$	$N = 1$ $v + \tau = 6.4$	$N = 11$ $v = 1.24$ $\tau = 13.16$
$\delta = 0.5$	$N = 2$ $v = 1.71$ $\tau = 7.81$	$N = 2$ $v = 1.82$ $\tau = 7.42$	$N = 1$ $v + \tau = 5.83$	$N = 1$ $v + \tau = 6.4$	$N = 9$ $v = 3.03$ $\tau = 10.09$
$\delta = 1$	$N = 8$ $v = 2.56$ $\tau = 5.85$	$N = 8$ $v = 2.46$ $\tau = 5.86$	$N = 4$ $v = 2.05$ $\tau = 3.82$	$N = 10$ $v = 2.19$ $\tau = 4.41$	$N = 11$ $v = 3.86$ $\tau = 7.12$
$\delta = 2$	$N = 10$ $v = 2.66$ $\tau = 3.98$	$N = 10$ $v = 2.6$ $\tau = 4$	$N = 11$ $v = 2.36$ $\tau = 2.54$	$N = 11$ $v = 2.46$ $\tau = 2.75$	$N = 11$ $v = 3.52$ $\tau = 6.48$
$c_M/c_I = 1$					
$\delta = 0.125$	$N = 1$ $v + \tau = 6.21$	$N = 1$ $v + \tau = 6$	$N = 1$ $v + \tau = 4.61$	$N = 1$ $v + \tau = 5.06$	$N = 8$ $v = 1.1$ $\tau = 10.09$
$\delta = 0.5$	$N = 1$ $v + \tau = 6.21$	$N = 1$ $v + \tau = 6$	$N = 1$ $v + \tau = 4.61$	$N = 1$ $v + \tau = 5.06$	$N = 8$ $v = 2.53$ $\tau = 8.61$
$\delta = 1$	$N = 2$ $v = 2$ $\tau = 4.37$	$N = 1$ $v + \tau = 6$	$N = 1$ $v + \tau = 4.61$	$N = 1$ $v + \tau = 5.06$	$N = 11$ $v = 3.3$ $\tau = 6.7$
$\delta = 2$	$N = 11$ $v = 2.4$ $\tau = 2.7$	$N = 11$ $v = 2.35$ $\tau = 2.65$	$N = 10$ $v = 2.1$ $\tau = 2.15$	$N = 11$ $v = 2.23$ $\tau = 2.3$	$N = 11$ $v = 3.29$ $\tau = 4.85$
$c_M/c_I = 2$					
$\delta = 0.125$	$N = 1$ $v + \tau = 4.37$	$N = 1$ $v + \tau = 4.22$	$N = 1$ $v + \tau = 3.66$	$N = 1$ $v + \tau = 4.01$	$N = 11$ $v = 1.05$ $\tau = 8.59$
$\delta = 0.5$	$N = 1$ $v + \tau = 4.37$	$N = 1$ $v + \tau = 4.22$	$N = 1$ $v + \tau = 3.66$	$N = 1$ $v + \tau = 4.01$	$N = 11$ $v = 2.23$ $\tau = 7.67$
$\delta = 1$	$N = 1$ $v + \tau = 4.37$	$N = 1$ $v + \tau = 4.22$	$N = 1$ $v + \tau = 3.66$	$N = 1$ $v + \tau = 4.01$	$N = 10$ $v = 2.85$ $\tau = 5.46$
$\delta = 2$	$N = 11$ $v = 2.06$ $\tau = 2.11$	$N = 11$ $v = 2.03$ $\tau = 2.09$	$N = 10$ $v = 1.89$ $\tau = 1.74$	$N = 10$ $v = 1.99$ $\tau = 1.86$	$N = 11$ $v = 2.96$ $\tau = 4.15$

i.e. the cost optimal v is decreasing, and the lower is the distance between PM actions, i.e. the cost optimal τ is decreasing too. Therefore, it is favorable to do a good and more frequent PM instead of doing more minimal repairs.

Note that if the cost optimal N is 1, the optimal maintenance strategy is to do no PM actions and to replace the system every $v + \tau$ time units. Hence, the average maintenance costs from (3.36) reduce to $(c_M E(Z_{v+\tau}) + c_R)/E(L_{v,\tau,1})$ and the mean cycle length in this special case is $E(L_{v,\tau,1}) = v + \tau - \int_0^{v+\tau} F^{T_1''}(t)dt$. Thus, the optimal maintenance strategy is not unique because different combinations of v and τ that lead to the same sum $v + \tau$ also lead to the same average maintenance costs. Therefore, if the optimal N is 1, it is proper to show the optimal sum $v + \tau$ instead of the optimal single values for v and τ.

3.5.2. Costs Proportional to the State before Repair

Suppose the cost function c_{PM} depends on the state just before PM. It holds

$$c_{PM}(v, \tau) = c_S \left(\frac{1}{v + \tau} \right)^{\delta}, \tag{3.39}$$

where $\delta > 0$ and $c_S > 0$ is a constant cost value. This function is described in detail in Subsection 2.7.2. It is assumed that the costs of a replacement are equal to c_S, i.e. $c_R = c_S$. In the following, the optimal values for v and τ and the optimal N are computed for different cost ratios c_M/c_S and different δ. The numerical results are given in Table 3.2. These results lead to the following conclusions:

1. The smaller δ, the more expensive are PM actions. Therefore, with rising δ it becomes cost optimal to do more PM actions, i.e. the cost optimal N is rising.

2. The lower the costs of PM compared to the costs of a minimal repair, i.e. the higher the ratio c_M/c_S, the lower is the distance between PM actions, i.e. the cost optimal τ is decreasing. Therefore, it is favorable to do PM more often, instead of doing more minimal repairs.

3. Since the cost difference between good and less good PM actions is comparatively small (see e.g. Figure 2.7 (b)), it is for the LFRD, RD, WD and MWD cost optimal to do perfect PM actions, i.e. $v = 0$. The same is not true for the RMWD because of the higher failure rate at lower ages and the associated larger number of type 1 failures.

4. The LFRD and RD lead nearly to the same optimal solutions. The same holds for the WD and the MWD.

3.5.3. Costs Proportional to the Degree of Repair - 1

In this subsection it is assumed that the costs of a PM action are proportional to the degree of repair $\xi(v, \tau)$. Here a Kijima type II model is considered and therefore the

Table 3.2.: Optimal values in case of costs proportional to the state before repair

	LFRD	RD	WD	MWD	RMWD
	$\alpha = 0.01$	$\beta = 0.03142$	$\beta = 0.0057$	$\alpha = 0.03$	$\alpha = 0.1$
	$\beta = 0.02944$		$\gamma = 3$	$\beta = 0.004335$	$\beta = 0.1746$
				$\gamma = 3$	$\gamma = 0.1$
$c_M/c_S = 0.5$					
$\delta = 0.125$	$N = 8$	$N = 11$	$N = 9$	$N = 9$	$N = 10$
	$v = 0$	$v = 0$	$v = 0$	$v = 0$	$v = 0.22$
	$\tau = 7.54$	$\tau = 7.48$	$\tau = 5.85$	$\tau = 5.91$	$\tau = 13.32$
$\delta = 0.5$	$N = 10$	$N = 10$	$N = 11$	$N = 11$	$N = 11$
	$v = 0$	$v = 0$	$v = 0$	$v = 0$	$v = 0.44$
	$\tau = 6.83$	$\tau = 6.8$	$\tau = 5.1$	$\tau = 5.3$	$\tau = 10.02$
$\delta = 1$	$N = 11$	$N = 11$	$N = 11$	$N = 11$	$N = 11$
	$v = 0$	$v = 0$	$v = 0$	$v = 0$	$v = 0.56$
	$\tau = 5.28$	$\tau = 5.17$	$\tau = 4.42$	$\tau = 5.02$	$\tau = 8.89$
$\delta = 2$	$N = 11$	$N = 11$	$N = 11$	$N = 11$	$N = 11$
	$v = 0$	$v = 0$	$v = 0$	$v = 0$	$v = 0.76$
	$\tau = 4.12$	$\tau = 4.04$	$\tau = 3.71$	$\tau = 3.95$	$\tau = 6.99$
$c_M/c_S = 1$					
$\delta = 0.125$	$N = 9$	$N = 9$	$N = 8$	$N = 11$	$N = 8$
	$v = 0$	$v = 0$	$v = 0$	$v = 0$	$v = 0.36$
	$\tau = 5.64$	$\tau = 5.66$	$\tau = 4.48$	$\tau = 5.02$	$\tau = 10.68$
$\delta = 0.5$	$N = 11$	$N = 11$	$N = 10$	$N = 10$	$N = 11$
	$v = 0$	$v = 0$	$v = 0$	$v = 0$	$v = 0.54$
	$\tau = 4.93$	$\tau = 4.81$	$\tau = 4.15$	$\tau = 4.47$	$\tau = 8.9$
$\delta = 1$	$N = 11$	$N = 11$	$N = 11$	$N = 11$	$N = 10$
	$v = 0$	$v = 0$	$v = 0$	$v = 0$	$v = 0.73$
	$\tau = 4.56$	$\tau = 4.62$	$\tau = 3.76$	$\tau = 4.03$	$\tau = 7.44$
$\delta = 2$	$N = 11$	$N = 11$	$N = 11$	$N = 11$	$N = 11$
	$v = 0$	$v = 0$	$v = 0$	$v = 0$	$v = 0.94$
	$\tau = 3.42$	$\tau = 3.35$	$\tau = 3.21$	$\tau = 3.41$	$\tau = 5.93$
$c_M/c_S = 2$					
$\delta = 0.125$	$N = 11$	$N = 8$	$N = 10$	$N = 10$	$N = 11$
	$v = 0$	$v = 0$	$v = 0$	$v = 0$	$v = 0.52$
	$\tau = 4.56$	$\tau = 4.08$	$\tau = 3.76$	$\tau = 3.92$	$\tau = 8.91$
$\delta = 0.5$	$N = 10$	$N = 10$	$N = 11$	$N = 10$	$N = 10$
	$v = 0$	$v = 0$	$v = 0$	$v = 0$	$v = 0.72$
	$\tau = 3.81$	$\tau = 3.72$	$\tau = 3.41$	$\tau = 3.7$	$\tau = 7.32$
$\delta = 1$	$N = 11$	$N = 11$	$N = 11$	$N = 11$	$N = 11$
	$v = 0$	$v = 0$	$v = 0$	$v = 0$	$v = 0.93$
	$\tau = 3.4$	$\tau = 3.32$	$\tau = 3.17$	$\tau = 3.4$	$\tau = 6.21$
$\delta = 2$	$N = 11$	$N = 11$	$N = 11$	$N = 11$	$N = 11$
	$v = 0$	$v = 0$	$v = 0$	$v = 0$	$v = 1.12$
	$\tau = 2.83$	$\tau = 2.78$	$\tau = 2.77$	$\tau = 2.94$	$\tau = 5.01$

degree of repair corresponds to

$$\xi(v,\tau) = \frac{v}{v+\tau}. \tag{3.40}$$

The following cost function is used

$$c_{PM}(v,\tau) = c_R\left(1 - \xi(v,\tau)^{\delta}\right), \tag{3.41}$$

where $\delta > 0$ and $c_R > 0$ are the costs of a replacement. This function is described in detail in Subsection 2.7.3. In the following, the optimal values for v and τ and the optimal number of PM actions before a preventive replacement takes place are computed for different cost ratios c_M/c_R and different δ. The numerical results are given in Table 3.3. Note that if the cost optimal N is 1, there are many cost optimal values of v and τ, namely the cost optimal values of v and τ form a fixed sum (see the end of Subsection 3.5.1 for a detailed explanation).

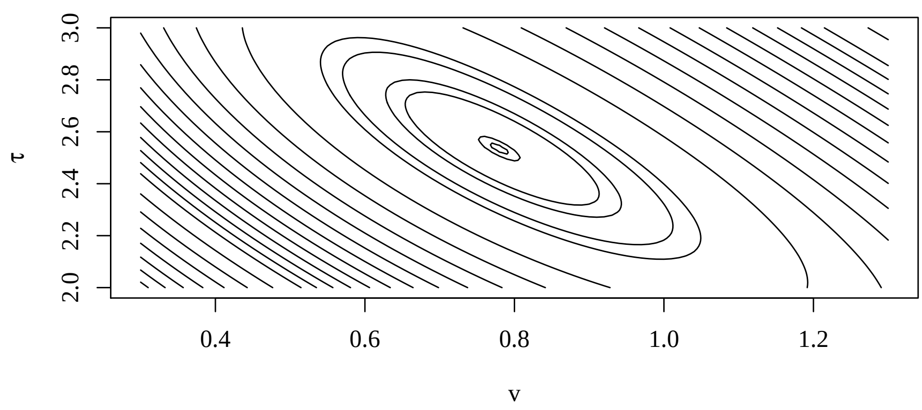


Figure 3.3.: Average maintenance costs for the MWD ($\alpha = 0.03, \beta = 0.004335, \gamma = 3$) if $N = 10$, $c_R = 500$, $c_M = 1000$, $p = 0.1$ and $\delta = 0.5$.

The contour plot in Figure 3.3 shows the dependency of the average maintenance costs on v and τ for the MWD($\alpha = 0.03, \beta = 0.004335, \gamma = 3$), the ratio of costs $c_M/c_R = 2$, $p = 0.1$, $N = 10$, $\delta = 0.5$ and cost function for PM (3.41).
The numerical results in Table 3.3 lead to the following conclusions:

1. If τ does not tend to zero it holds that the higher δ, the more expensive are the PM actions. Therefore, with rising δ it becomes cost optimal to do less PM, i.e. N is decreasing.

2. The lower the costs of a renewal compared to the costs of a minimal repair, i.e. the higher the ratio c_M/c_R, the less expensive are the costs for PM compared to the costs of a minimal repair. Therefore, with rising ratio c_M/c_R the cost optimal distance between PM actions becomes shorter, i.e. τ is decreasing, and the PM actions become better, i.e. v is decreasing too. Hence, it is favorable to do good PM actions more often, instead of doing more minimal repairs.

Table 3.3.: Optimal values in case of costs proportional to the degree of repair - 1

	LFRD	RD	WD	MWD	RMWD
	$\alpha = 0.01$ $\beta = 0.02944$	$\beta = 0.03142$	$\beta = 0.0057$ $\gamma = 3$	$\alpha = 0.03$ $\beta = 0.004335$ $\gamma = 3$	$\alpha = 0.1$ $\beta = 0.1746$ $\gamma = 0.1$
$c_M/c_R = 0.5$					
$\delta = 0.125$	$N = 11$ $v = 0.84$ $\tau = 3.52$	$N = 11$ $v = 0.81$ $\tau = 3.61$	$N = 11$ $v = 0.84$ $\tau = 2.66$	$N = 11$ $v = 0.93$ $\tau = 2.88$	$N = 11$ $v = 2.18$ $\tau = 7.71$
$\delta = 0.5$	$N = 11$ $v = 0.85$ $\tau = 6.63$	$N = 11$ $v = 0.81$ $\tau = 6.6$	$N = 11$ $v = 1.07$ $\tau = 3.88$	$N = 11$ $v = 1.27$ $\tau = 3.77$	$N = 9$ $v = 2.11$ $\tau = 10.94$
$\delta = 1$	$N = 9$ $v = 0$ $\tau = 9.41$	$N = 9$ $v = 0$ $\tau = 9.29$	$N = 9$ $v = 0$ $\tau = 5.85$	$N = 10$ $v = 0$ $\tau = 6.73$	$N = 11$ $v = 0.57$ $\tau = 13.66$
$c_M/c_R = 1$					
$\delta = 0.125$	$N = 11$ $v = 0.61$ $\tau = 2.48$	$N = 11$ $v = 0.59$ $\tau = 2.4$	$N = 11$ $v = 0.68$ $\tau = 2.15$	$N = 11$ $v = 0.74$ $\tau = 2.36$	$N = 10$ $v = 1.91$ $\tau = 5.59$
$\delta = 0.5$	$N = 9$ $v = 0.61$ $\tau = 4.87$	$N = 11$ $v = 0.7$ $\tau = 3.97$	$N = 10$ $v = 0.89$ $\tau = 2.9$	$N = 10$ $v = 0.99$ $\tau = 3.14$	$N = 11$ $v = 2.25$ $\tau = 7.65$
$\delta = 1$	$N = 4$ $v = 0$ $\tau = 6.23$	$N = 4$ $v = 0$ $\tau = 6.37$	$N = 8$ $v = 0$ $\tau = 4.58$	$N = 11$ $v = 0$ $\tau = 5.02$	$N = 9$ $v = 0.58$ $\tau = 12.21$
$c_M/c_R = 2$					
$\delta = 0.125$	$N = 11$ $v = 0.43$ $\tau = 1.8$	$N = 11$ $v = 0.42$ $\tau = 1.74$	$N = 11$ $v = 0.54$ $\tau = 1.72$	$N = 11$ $v = 0.59$ $\tau = 1.89$	$N = 11$ $v = 1.83$ $\tau = 4.4$
$\delta = 0.5$	$N = 10$ $v = 0.51$ $\tau = 3.02$	$N = 10$ $v = 0.49$ $\tau = 2.93$	$N = 11$ $v = 0.74$ $\tau = 2.25$	$N = 10$ $v = 0.78$ $\tau = 2.53$	$N = 11$ $v = 2.09$ $\tau = 5.57$
$\delta = 1$	$N = 11$ $v = 0$ $\tau = 4.56$	$N = 6$ $v = 0$ $\tau = 4.23$	$N = 10$ $v = 0$ $\tau = 3.76$	$N = 8$ $v = 0$ $\tau = 4.01$	$N = 11$ $v = 0.93$ $\tau = 8.67$

3. The cost optimal values of v for the RMWD are higher than for the other distributions because of the higher failure rate at lower ages. The cost optimal values of τ for the RMWD are also higher than for the other distributions. The reason for this is that the failure rate for this distribution remains for a while at a relatively low level before it starts increasing again.

3.5.4. Costs Proportional to the Degree of Repair - 2

Like in the previous subsection it is assumed that the costs for a PM action are proportional to the degree of repair $\xi(v, \tau)$. Again a Kijima type II model is considered and therefore the degree of repair corresponds to (3.40). The following cost function is used

$$c_{PM}(v, \tau) = c_R \left(1 - \xi(v, \tau) \exp\left(\xi(v, \tau) - 1\right)\right)^{\delta}, \tag{3.42}$$

where $\delta > 0$ and $c_R > 0$ are the costs of a replacement. This function is described in detail in Subsection 2.7.4.
In the following, the cost optimal values for v and τ and the optimal number of PM actions before a preventive replacement takes place are computed for different cost ratios c_M/c_R and different δ. The numerical results are given in Table 3.4. As before, if the cost optimal N is 1, there are many cost optimal values of v and τ (see Subsection 3.5.1 for a detailed explanation).
The numerical results in Table 3.4 lead to the following conclusions:

1. The higher δ, the less expensive are PM actions. Therefore, with rising δ it becomes cost optimal to do more PM, i.e. N is increasing. Since also the cost difference between good and less good maintenance actions becomes higher with rising δ, for small values of δ it is cost optimal to do even perfect PM actions and for high values of δ it is cost optimal to do less good PM actions.

2. The lower the costs of a renewal compared to the costs of a minimal repair, i.e. the higher the ratio c_M/c_R, the less expensive are the costs for PM and the shorter is the distance between PM actions, i.e. the smaller is τ, and the better are the PM actions, i.e. v is decreasing.

3. In contrast to the other lifetime distributions, for the RMWD it is not even for small values of δ cost optimal to do perfect PM actions because of the higher failure rate at lower ages. The cost optimal values of τ for the RMWD are higher than for the other distributions. The reason for this is that the failure rate for this distribution remains for a while at a relatively low level before it starts increasing again.

Table 3.4.: Optimal values in case of costs proportional to the degree of repair - 2

	LFRD	RD	WD	MWD	RMWD
	$\alpha = 0.01$ $\beta = 0.02944$	$\beta = 0.03142$	$\beta = 0.0057$ $\gamma = 3$	$\alpha = 0.03$ $\beta = 0.004335$ $\gamma = 3$	$\alpha = 0.1$ $\beta = 0.1746$ $\gamma = 0.1$
$c_M/c_R = 0.5$					
$\delta = 0.5$	$N = 9$ $v = 0$ $\tau = 9.41$	$N = 9$ $v = 0$ $\tau = 9.29$	$N = 9$ $v = 0$ $\tau = 5.85$	$N = 10$ $v = 0$ $\tau = 6.73$	$N = 11$ $v = 0.23$ $\tau = 13.86$
$\delta = 1$	$N = 9$ $v = 0$ $\tau = 9.41$	$N = 9$ $v = 0$ $\tau = 9.29$	$N = 9$ $v = 0$ $\tau = 5.85$	$N = 10$ $v = 0$ $\tau = 6.73$	$N = 11$ $v = 0.31$ $\tau = 13.82$
$\delta = 2$	$N = 9$ $v = 0$ $\tau = 9.41$	$N = 9$ $v = 0$ $\tau = 9.29$	$N = 11$ $v = 3.84$ $\tau = 0.35$	$N = 11$ $v = 4.19$ $\tau = 0.38$	$N = 11$ $v = 0.4$ $\tau = 13.77$
$\delta = 3$	$N = 11$ $v = 3.93$ $\tau = 0.85$	$N = 11$ $v = 3.86$ $\tau = 0.79$	$N = 11$ $v = 2.9$ $\tau = 0.66$	$N = 11$ $v = 3.18$ $\tau = 0.72$	$N = 11$ $v = 7.15$ $\tau = 1.75$
$c_M/c_R = 1$					
$\delta = 0.5$	$N = 4$ $v = 0$ $\tau = 6.23$	$N = 4$ $v = 0$ $\tau = 6.37$	$N = 8$ $v = 0$ $\tau = 4.58$	$N = 11$ $v = 0$ $\tau = 5.02$	$N = 8$ $v = 0.36$ $\tau = 10.68$
$\delta = 1$	$N = 4$ $v = 0$ $\tau = 6.23$	$N = 4$ $v = 0$ $\tau = 6.37$	$N = 8$ $v = 0$ $\tau = 4.58$	$N = 11$ $v = 0$ $\tau = 5.02$	$N = 8$ $v = 0.42$ $\tau = 10.64$
$\delta = 2$	$N = 11$ $v = 4.13$ $\tau = 0.28$	$N = 11$ $v = 4.03$ $\tau = 0.26$	$N = 11$ $v = 3.06$ $\tau = 0.27$	$N = 11$ $v = 3.34$ $\tau = 0.31$	$N = 11$ $v = 7.47$ $\tau = 0.9$
$\delta = 3$	$N = 11$ $v = 2.87$ $\tau = 0.56$	$N = 11$ $v = 2.71$ $\tau = 0.57$	$N = 11$ $v = 2.32$ $\tau = 0.52$	$N = 11$ $v = 2.54$ $\tau = 0.57$	$N = 11$ $v = 5.92$ $\tau = 1.5$
$c_M/c_R = 2$					
$\delta = 0.5$	$N = 11$ $v = 0$ $\tau = 4.56$	$N = 6$ $v = 0$ $\tau = 4.23$	$N = 10$ $v = 0$ $\tau = 3.76$	$N = 8$ $v = 0$ $\tau = 4.01$	$N = 11$ $v = 0.54$ $\tau = 8.9$
$\delta = 1$	$N = 11$ $v = 0$ $\tau = 4.56$	$N = 6$ $v = 0$ $\tau = 4.23$	$N = 10$ $v = 0$ $\tau = 3.76$	$N = 8$ $v = 0$ $\tau = 4.01$	$N = 11$ $v = 0.6$ $\tau = 8.87$
$\delta = 2$	$N = 11$ $v = 2.98$ $\tau = 0.2$	$N = 11$ $v = 2.85$ $\tau = 0.18$	$N = 11$ $v = 2.42$ $\tau = 0.22$	$N = 11$ $v = 2.66$ $\tau = 0.24$	$N = 11$ $v = 0.82$ $\tau = 8.74$
$\delta = 3$	$N = 11$ $v = 1.99$ $\tau = 0.42$	$N = 11$ $v = 1.92$ $\tau = 0.41$	$N = 11$ $v = 1.85$ $\tau = 0.41$	$N = 11$ $v = 2.02$ $\tau = 0.46$	$N = 11$ $v = 4.85$ $\tau = 1.29$

4. System with one Failure Type and Imperfect PM and CM

Most of the imperfect maintenance models that have been investigated in literature use either imperfect preventive maintenance actions or imperfect corrective maintenance actions. Only a few models uses both imperfect preventive and imperfect corrective maintenance actions (such as Wang and Pham [48] and Lie and Chun [27]). Such a maintenance effect modeling forms the basis of the maintenance model in this chapter. In this chapter a sequential failure limit PM policy in the sense of [17, p. 765] with infinite planning horizon and with imperfect preventive and imperfect corrective maintenance actions is used to formulate a cost optimization problem.
This chapter is structured as follows. Section 4.1 and Section 4.2 contain essential assumptions and definitions which are needed to formulate the cost optimization problem in Section 4.3. Finally, in Section 4.4 different cost functions for PM actions are used to solve the cost optimization problem. Furthermore, the results are computed for several continuous lifetime distributions.

4.1. Modeling the System

In this chapter the underlying repairable system has the following properties.

1. Initially a new repairable system is installed.
2. The system has only one failure type which can be removed through imperfect repair actions.
3. The repair times are negligible small.

4.2. Maintenance Policy

The maintenance strategy which is described here is designed for an infinite time horizon. The following assumptions are made.

1. All failures that occurred after installation during the time interval $(0, v]$ are removed through minimal repair.
2. If there is a failure during the time interval $(v, v + \tau)$ a CM action is carried out. Otherwise a PM at time $v + \tau$ will be carried out.
3. If there is no failure during the pre-defined time interval of length $\tau > 0$ after a maintenance action, a PM will be carried out.

4. If a failure occurs during the time interval of the length $\tau > 0$ after a maintenance action, a CM is carried out.

5. The virtual age of the system after both PM and CM actions is always $v \geq 0$. But since PM actions can be planned, they are assumed to be more cost effective than unplanned CM actions.

6. Suppose $c_1, c_2, \ldots$ are the realizations of the general maintenance times, i.e. the times of the CM and PM actions, with $c_1, c_2, \cdots \geq v$ and $c_k < c_{k+1}$ for $k \geq 1$. In terms of Kijima type II model the degree of the kth repair is

$$\xi_k(v, c_k, c_{k-1}) = \frac{v}{v + c_k - c_{k-1}}, \tag{4.1}$$

for $k \geq 1$.

This maintenance policy is a sequential failure limit policy (see [17, p. 765]) because an alternative formulation of Assumption 2 might be: A PM is performed when the failure intensity reaches the predetermined level $\lambda^{N^*}(v + \tau)$.
Note that if $v = 0$, both PM and CM actions are perfect. This is the only case for which PM and CM actions have the same degree of repair, in fact $\xi_k = 0$ for all $k \geq 1$. Since simultaneous failures are excluded and $\tau > 0$, the interval between two maintenance actions is always greater than zero, i.e. $c_k - c_{k-1} > 0$ for all $k \geq 1$. Therefore, it is not possible to have minimal CM or PM actions, i.e. $\xi_k < 1$ for all $k \geq 1$. But if τ tend to zero or a failure occurs directly after a maintenance actions, the degree of repair can have values close to one.
Note that if the kth repair is a PM action, the degree of repair for this PM actions is $v/(v + \tau)$ since a PM action is only carried out if there was no failure during a time interval of length τ, i.e. $c_k - c_{k-1} = \tau$. However, the degree of repair of CM actions is always greater than or equal the degree of repair of PM actions since $c_k - c_{k-1} < \tau$.

4.3. Cost Optimization Problem

Consider a technical system which is maintained with maintenance policy described in Section 4.2. The aim of this section is to formulate a cost optimization problem. The optimization criterion are the average maintenance costs per unit time. For this purpose, the expected maintenance costs per cycle are set in relation to the mean cycle length. Here the cycle length is the time between two maintenance actions. For reasons of simplification, the time between the startup of the system and the age of v is excluded in the following from the modeling of the cost optimization problem.
Suppose $N^* = (N_t^*)_{t \geq 0}$ is the failure counting process, i.e. N_t^* is the random number of failures of a repairable system with PM in the interval $[0, t]$.

Lemma 4.1 (Intensity Function of $N^* = (N_t^*)_{t \geq 0}$)
Suppose $c_1, c_2, \ldots$ are realizations of the general maintenance times. The intensity

function of the counting process $N^* = (N_t^*)_{t\geq 0}$ *is then*

$$\lambda^{N^*}(t) \quad = \quad \begin{cases} 0 & , \textit{ if } t < 0 \\ h^{T_1}(t) & , \textit{ if } t \in [0, c_1) \\ h^{T_1}(v + t - c_k) & , \textit{ if } t \in [c_k, c_{k+1}),\ k \geq 1, \end{cases} \tag{4.2}$$

where $h^{T_1}(\cdot)$ *is the hazard function of the time to the first failure of a new system without maintenance actions.*

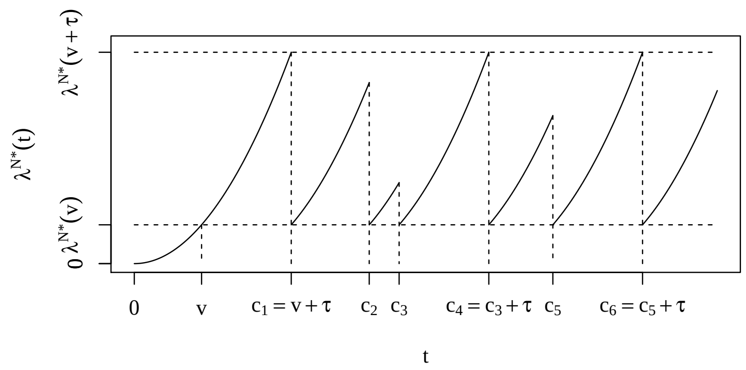


Figure 4.1.: Exemplary shape of the intensity function for the WD($\beta = 0.0057, \gamma = 3$).

In Figure 4.1 an exemplary shape of the intensity function is plotted for the WD with expectation 5. Here CM is carried out at maintenance times c_2, c_3 and c_5 and PM is performed at c_1, c_4 and c_6.
For the computation of the expected maintenance costs per cycle it is necessary to compute the probability that a failure occurs within τ time units after a maintenance action.

Lemma 4.2 (**Distribution function of** T^v)
Let T_1 *be the random time of the first failure of a repairable system without maintenance. Suppose* T^v *is the remaining lifetime of the system after a maintenance action that reduces the virtual age of the system to* v*. Then* T^v *is a truncated random variable with the following cumulative distribution function*

$$F^{T^v}(t) = P(T^v \leq t) = \frac{F^{T_1}(v+t) - F^{T_1}(v)}{1 - F^{T_1}(v)},\ \forall t \geq 0. \tag{4.3}$$

Suppose the random cycle length $L_{v,\tau}$ is the random time between two maintenance actions. Therefore, it is either the time between two PM actions or the time between CM and PM actions. It holds

$$L_{v,\tau} \quad = \quad \min\{T^v, \tau\} \begin{cases} < \tau & , \text{with } P(T^v < \tau) = P(T_1 < v + \tau | T_1 \geq v) \\ = \tau & , \text{with } P(T^v \geq \tau) = 1 - P(T_1 < v + \tau | T_1 \geq v) \end{cases}, \tag{4.4}$$

for all $\tau \geq 0$. Then for the cumulative distribution function of $L_{v,\tau}$ it holds

$$F^{L_{v,\tau}}(t) = P(L_{v,\tau} \leq t) \quad = \quad \begin{cases} 0 & \text{, if } t < 0 \\ F^{T_1}(v+t|T_1 \geq v) & \text{, if } 0 \leq t < \tau \\ 1 & \text{, if } t \geq \tau \end{cases} . \tag{4.5}$$

Theorem 4.3 (**Mean Cycle Length**)
For the expected cycle length it holds

$$E(L_{v,\tau}) \quad = \quad \int_0^\tau \bar{F}^{T_1}(v+t|T_1 \geq v)dt. \tag{4.6}$$

Proof. The mean cycle length (4.6) is derived by using Remark 2.1. □

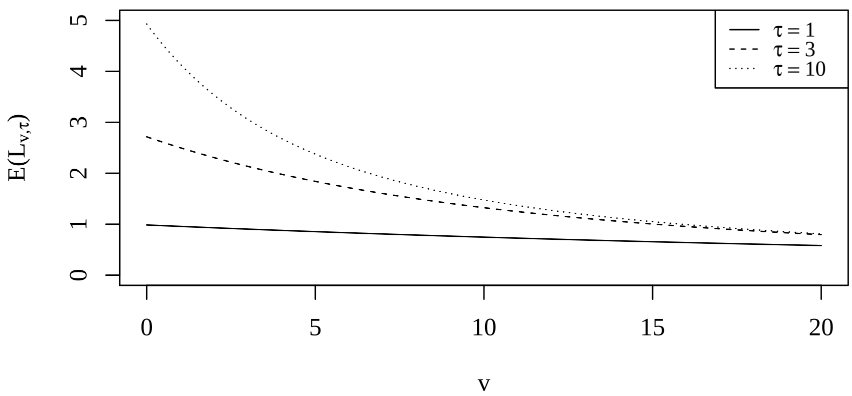


Figure 4.2.: Mean cycle length for the LFRD($\alpha = 0.01, \beta = 0.02944$).

In Figure 4.2 the mean cycle length is plotted for the LFRD with expectation 5. Note that the mean cycle length for $v = 0$ is bounded from above by $\min\{\tau, E(T_1)\}$.

Definition 4.1 (**Cost Optimization Problem**)
Let c_{CM} denotes the costs of a CM action and c_{PM} the costs of a PM action. The average maintenance costs per unit time are

$$\begin{aligned} C(v,\tau) \quad &= \quad \frac{c_{CM}P(L_{v,\tau} < \tau) + c_{PM}P(L_{v,\tau} = \tau)}{E(L_{v,\tau})} \\ &\overset{(4.5)(4.6)}{=} \quad \frac{c_{CM}F^{T_1}(v+\tau|T_1 \geq v) + c_{PM}\bar{F}^{T_1}(v+\tau|T_1 \geq v)}{\int_0^\tau \bar{F}^{T_1}(v+t|T_1 \geq v)dt}. \end{aligned} \tag{4.7}$$

The optimization problem then has the following form

$$\min_{v\in[0,\infty),\, \tau\in(0,\infty)} C(v,\tau). \tag{4.8}$$

Note that the extreme case $\tau = 0$ is excluded from optimization problem. The value $\tau = 0$ would lead to a degree of repair of one, which can be interpreted as minimal PM actions and it is not reasonable to do such maintenance actions. However, the other extreme case of perfect PM actions, i.e. $v = 0$, is still part of the cost optimization problem.

4.4. Example for Cost Optimal Maintenance

The costs of CM and PM actions in optimization problem (4.4) are yet unspecified. In this section a part of the cost functions introduced in Section 2.7 are used to model the costs of PM and CM actions. Then, the cost optimal parameter v and τ are computed with R for different continuous lifetime distributions that have all an expected value of 5. Since the optimal solution of optimization problem could not be computed analytically, complete enumeration is used to determine the cost optimal maintenance strategies.

4.4.1. Costs Proportional to the Impact of Repair

Suppose the costs of maintenance actions depends only on the virtual age of the system after the maintenance action, i.e. the cost functions for PM and CM are

$$c_{PM}(v) = c_I \left(\frac{1}{v}\right)^{\delta}, \tag{4.9}$$

$$c_{CM}(v) = c_F + c_I \left(\frac{1}{v}\right)^{\delta}, \tag{4.10}$$

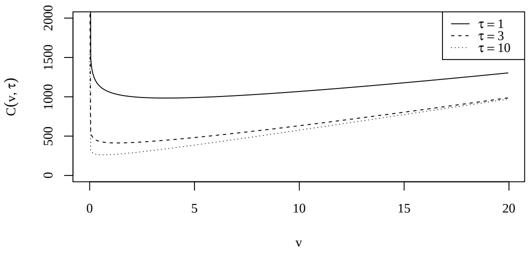
where $v > 0$, $\delta > 0$, $c_I > 0$ is a constant cost value and $c_F > 0$ is the fixed amount by which the costs of CM are higher than for PM. The cost function is described in detail in Subsection 2.7.1. In Figure 4.3 and Figure 4.4 the resulting total costs are plotted for different values of δ. It can be seen that if $v < 1$ and τ is fix, the average maintenance costs increase with increasing δ. If $v = 1$ and τ is fix, the total costs are independent of δ. Further, if $v > 1$ and τ is fix, the average maintenance costs decrease with increasing δ. Note that in case of costs proportional to the impact of repair the extreme case $v = 0$ is excluded from optimization problem (4.4).
The cost optimal values for v and τ are computed for different cost ratios c_F/c_I and different δ. The numerical results in Table 4.1 lead to the following conclusions:

1. The higher the amount by which the costs of CM are higher than for PM, i.e. the higher the ratio c_F/c_I, the better are the maintenance actions, i.e. v is decreasing, and the shorter is the interval before a PM action is carried out, i.e. τ is decreasing.

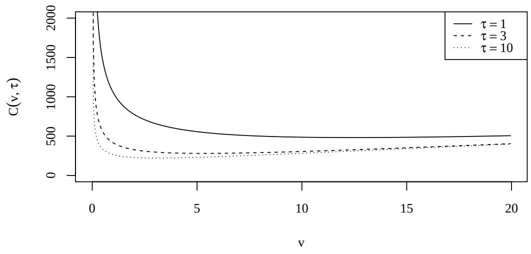
2. Only for $\delta = 0.125$ it is cost optimal to do very good maintenance actions with $v < 1$. For higher δ these maintenance actions are to expensive so that it is cost optimal to have higher values of v. Compared to the other lifetime distributions the cost optimal v for the RMWD is always higher. The reason for this lies in

Table 4.1.: Optimal values in case of costs proportional to the impact of repair

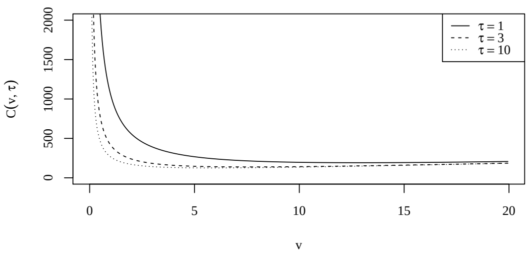
	LFRD	RD	WD	MWD	RMWD
	$\alpha = 0.01$ $\beta = 0.02944$	$\beta = 0.03142$	$\beta = 0.0057$ $\gamma = 3$	$\alpha = 0.03$ $\beta = 0.004335$ $\gamma = 3$	$\alpha = 0.1$ $\beta = 0.1746$ $\gamma = 0.1$
$c_F/c_I = 0.05$					
$\delta = 0.125$	$v = 0.68$ $\tau = 31.36$	$v = 0.64$ $\tau = 30.7$	$v = 0.54$ $\tau = 20.76$	$v = 0.63$ $\tau = 22.28$	$v = 2.39$ $\tau = 45.78$
$\delta = 0.5$	$v = 3.42$ $\tau = 37.18$	$v = 3.24$ $\tau = 35.83$	$v = 1.9$ $\tau = 22.57$	$v = 2.16$ $\tau = 24.14$	$v = 6.04$ $\tau = 23.64$
$\delta = 1$	$v = 7.78$ $\tau = 40.5$	$v = 7.49$ $\tau = 36.64$	$v = 3.39$ $\tau = 10.06$	$v = 3.8$ $\tau = 10.66$	$v = 8.58$ $\tau = 13.86$
$\delta = 2$	$v = 6.42$ $\tau = 5.77$	$v = 6.31$ $\tau = 5.7$	$v = 4.28$ $\tau = 3.98$	$v = 4.55$ $\tau = 4.14$	$v = 7.51$ $\tau = 5.86$
$c_F/c_I = 0.1$					
$\delta = 0.125$	$v = 0.64$ $\tau = 31.22$	$v = 0.61$ $\tau = 30.62$	$v = 0.52$ $\tau = 11.91$	$v = 0.61$ $\tau = 13.44$	$v = 2.32$ $\tau = 26.04$
$\delta = 0.5$	$v = 3.07$ $\tau = 37.01$	$v = 2.92$ $\tau = 35.87$	$v = 1.8$ $\tau = 10.4$	$v = 2.04$ $\tau = 11.42$	$v = 5.61$ $\tau = 18.79$
$\delta = 1$	$v = 5.85$ $\tau = 15.25$	$v = 5.65$ $\tau = 14.99$	$v = 3.04$ $\tau = 7.11$	$v = 3.38$ $\tau = 7.55$	$v = 7.38$ $\tau = 11.13$
$\delta = 2$	$v = 5.07$ $\tau = 4.65$	$v = 4.98$ $\tau = 4.59$	$v = 3.61$ $\tau = 3.31$	$v = 3.85$ $\tau = 3.45$	$v = 6.45$ $\tau = 5.12$
$c_F/c_I = 0.2$					
$\delta = 0.125$	$v = 0.59$ $\tau = 23.25$	$v = 0.56$ $\tau = 22.06$	$v = 0.49$ $\tau = 8.66$	$v = 0.56$ $\tau = 9.74$	$v = 2.19$ $\tau = 20.83$
$\delta = 0.5$	$v = 2.61$ $\tau = 18.67$	$v = 2.49$ $\tau = 18.02$	$v = 1.64$ $\tau = 7.47$	$v = 1.85$ $\tau = 8.18$	$v = 4.99$ $\tau = 14.78$
$\delta = 1$	$v = 4.35$ $\tau = 9.68$	$v = 4.21$ $\tau = 9.49$	$v = 2.62$ $\tau = 5.29$	$v = 2.89$ $\tau = 5.64$	$v = 6.18$ $\tau = 9.11$
$\delta = 2$	$v = 4.07$ $\tau = 3.8$	$v = 3.99$ $\tau = 3.76$	$v = 3.08$ $\tau = 2.79$	$v = 3.28$ $\tau = 2.93$	$v = 5.55$ $\tau = 4.47$



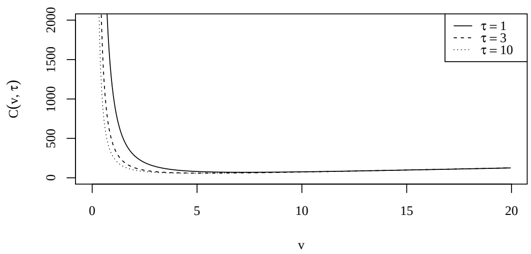
(a) $\delta = 0.125$



(b) $\delta = 0.5$



(c) $\delta = 1$



(d) $\delta = 2$

Figure 4.3.: Average maintenance costs for the LFRD($\alpha = 0.01, \beta = 0.02944$) if $c_I = 1000$ and $c_F/c_I = 0.1$.

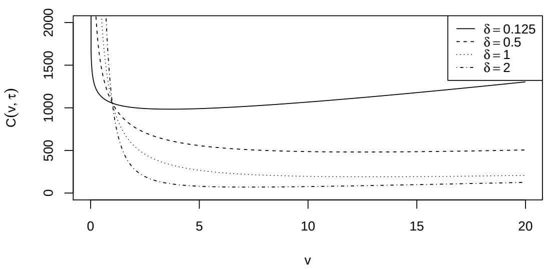


Figure 4.4.: Average maintenance costs for the LFRD($\alpha = 0.01, \beta = 0.02944$) if $c_I = 1000$, $c_F/c_I = 0.1$ and $\tau = 1$.

the bathtub shape of the failure rate for the RMWD and the related high failure rate at lower ages.

3. With rising δ good maintenance actions becomes more expensive compared to less good maintenance actions. Therefore, with rising δ it becomes cost optimal to do less good maintenance actions more often, i.e. v tend to increase and τ is decreasing. This trend can reverse for large δ since the maintenance costs for $v > 1$ are decreasing if δ is increasing.

4.4.2. Costs Proportional to the State before Repair

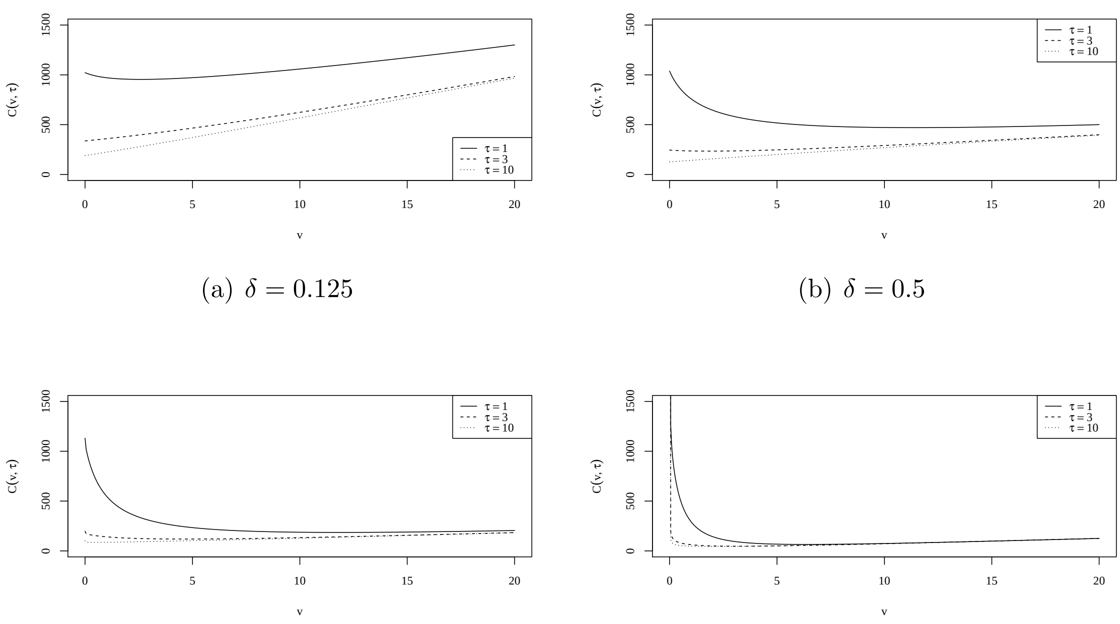
Suppose the cost function c_{PM} depends on the state just before PM. It holds

$$c_{PM}(v,\tau) = c_S \left(\frac{1}{v+\tau}\right)^{\delta}, \tag{4.11}$$

where $\delta > 0$ and $c_S > 0$ is a constant cost value. This cost function is described in detail in Subsection 2.7.2. For CM it is assumed that the costs of a CM action that takes place t time units after the previous maintenance action are $c_F + c_S\,(1/(v+t))^{\delta}$. Here $c_F > 0$ is a fixed amount, by which the costs of CM are higher than for PM. The following cost function is used for CM actions

$$c_{CM}(v,\tau) \;=\; \int_0^{\tau} \left(c_F + c_S \left(\frac{1}{v+t}\right)^{\delta}\right) f^{T^v}(t|T^v \leq \tau)dt. \tag{4.12}$$

In Figure 4.5 the resulting total costs as a function of v are plotted for different values of τ and δ. It can be seen that if $\delta = 2$, the average maintenance costs for very small values of v are very high. This results from the high costs of CM actions if $v + t < 1$ and $\delta = 2$.
In the following the optimal values for v and τ are computed for different cost ratios c_F/c_S and different δ. The numerical results are given in Table 4.2. These results lead to the following conclusions:



(a) $\delta = 0.125$ (b) $\delta = 0.5$

(c) $\delta = 1$ (d) $\delta = 2$

Figure 4.5.: Average maintenance costs for the LFRD($\alpha = 0.01, \beta = 0.02944$) if $c_S = 1000$ and $c_F/c_S = 0.1$.

Table 4.2.: Optimal values in case of costs proportional to the state before repair

	LFRD	RD	WD	MWD	RMWD
	$\alpha = 0.01$ $\beta = 0.02944$	$\beta = 0.03142$	$\beta = 0.0057$ $\gamma = 3$	$\alpha = 0.03$ $\beta = 0.004335$ $\gamma = 3$	$\alpha = 0.1$ $\beta = 0.1746$ $\gamma = 0.1$
$c_F/c_S = 0.05$					
$\delta = 0.125$	$v = 0$ $\tau = 37.91$	$v = 0$ $\tau = 23.31$	$v = 0$ $\tau = 39.65$	$v = 0$ $\tau = 23.47$	$v = 1.34$ $\tau = 46.08$
$\delta = 0.5$	$v = 0.01$ $\tau = 31.73$	$v = 0$ $\tau = 32.04$	$v = 0$ $\tau = 11.74$	$v = 0.07$ $\tau = 13.59$	$v = 3.27$ $\tau = 24.41$
$\delta = 1$	$v = 0.79$ $\tau = 29.32$	$v = 0$ $\tau = 23.75$	$v = 0$ $\tau = 9.08$	$v = 0.5$ $\tau = 10.25$	$v = 5.26$ $\tau = 15.37$
$\delta = 2$	$v = 3.33$ $\tau = 6.94$	$v = 3.18$ $\tau = 6.91$	$v = 0$ $\tau = 6.33$	$v = 1.5$ $\tau = 5.65$	$v = 4.87$ $\tau = 7.13$
$c_F/c_S = 0.1$					
$\delta = 0.125$	$v = 0$ $\tau = 37.78$	$v = 0$ $\tau = 22.77$	$v = 0$ $\tau = 10.66$	$v = 0$ $\tau = 23.34$	$v = 1.31$ $\tau = 26.01$
$\delta = 0.5$	$v = 0.01$ $\tau = 31.73$	$v = 0$ $\tau = 20.46$	$v = 0$ $\tau = 8.86$	$v = 0.06$ $\tau = 10.14$	$v = 3.01$ $\tau = 19.56$
$\delta = 1$	$v = 0.43$ $\tau = 14.67$	$v = 0$ $\tau = 13.99$	$v = 0$ $\tau = 7.15$	$v = 0.43$ $\tau = 7.89$	$v = 4.37$ $\tau = 12.46$
$\delta = 2$	$v = 2.13$ $\tau = 5.76$	$v = 1.98$ $\tau = 5.74$	$v = 0$ $\tau = 5.26$	$v = 1.18$ $\tau = 4.77$	$v = 4.04$ $\tau = 6.24$
$c_F/c_S = 0.2$					
$\delta = 0.125$	$v = 0$ $\tau = 37.56$	$v = 0$ $\tau = 16.92$	$v = 0$ $\tau = 7.97$	$v = 0$ $\tau = 9.02$	$v = 1.26$ $\tau = 20.84$
$\delta = 0.5$	$v = 0.01$ $\tau = 13.09$	$v = 0$ $\tau = 12.31$	$v = 0$ $\tau = 6.88$	$v = 0.05$ $\tau = 7.78$	$v = 2.63$ $\tau = 15.52$
$\delta = 1$	$v = 0.23$ $\tau = 9.53$	$v = 0$ $\tau = 9.11$	$v = 0$ $\tau = 5.77$	$v = 0.35$ $\tau = 6.27$	$v = 3.53$ $\tau = 10.23$
$\delta = 2$	$v = 1.37$ $\tau = 4.81$	$v = 1.22$ $\tau = 4.81$	$v = 0$ $\tau = 4.44$	$v = 0.93$ $\tau = 4.09$	$v = 3.39$ $\tau = 5.45$

1. The lower δ the higher are the costs of PM and CM actions. Therefore, with decreasing δ it becomes costs optimal to do maintenance actions less often, i.e. v is decreasing and τ is increasing.
2. The higher the amount by which the costs of CM are higher than for PM, i.e. the higher the ratio c_F/c_S, the cost optimal interval length until a PM is done becomes shorter, i.e. the cost optimal τ is decreasing, and the better are PM actions, i.e. the cost optimal v is decreasing too. Therefore, it becomes more likely that no failure occurs in the time interval of length τ.
3. Since the cost difference between good and less good PM actions is comparatively small (see e.g. Figure 2.7 (b)), for the LFRD, RD, WD and MWD it is cost optimal to do mostly perfect PM and CM actions, i.e. $v = 0$. The same is not true for the RMWD because of the higher failure rate at lower ages.
4. Compared to the other lifetime distributions the cost optimal v for the WD is zero for all values of δ. Because of the very low failure rate at lower ages, the high costs of CM actions with small values of v and τ if, for example, $\delta = 2$, get a low weight in equation (4.12). Thus, compared to the other lifetime distributions for the WD it needs higher values of δ that the cost optimal v is greater than zero.

4.4.3. Costs Proportional to the Degree of Repair - 1

In this subsection it is assumed that the costs of maintenance actions are proportional to the degree of repair. Here a Kijima type II model is considered and therefore the degree of the kth repair corresponds to (4.1). Since PM actions are always carried out after τ time units without maintenance actions, the degree of repair for every PM action is $\frac{v}{v+\tau}$.

Although both PM and CM actions reduce the virtual age of the system to v, it is assumed that CM actions are slightly more expensive than PM actions. This assumption makes sense since corrective maintenance actions are generally more expensive than planned maintenance actions. The following cost function is used for PM actions

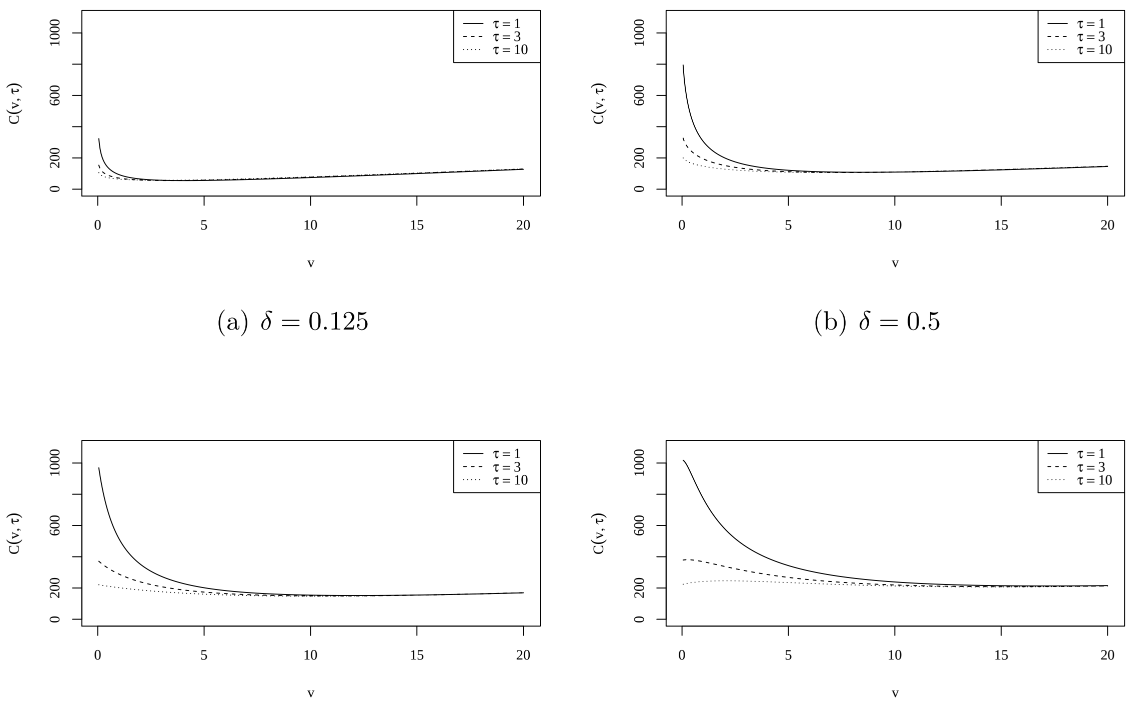
$$c_{PM}(v,\tau) \;=\; c_R\left(1-\left(\frac{v}{v+\tau}\right)^{\delta}\right), \tag{4.13}$$

where $\delta > 0$ and $c_R > 0$ are the costs of a replacement. This function is described in detail in Subsection 2.7.3. For CM it is assumed, that the costs for a CM action, that takes place t time units after the previous maintenance action, are $c_F + c_R\left(1-\left(\frac{v}{v+t}\right)^{\delta}\right)$. Here c_F is the fixed amount, by which the costs of CM are higher than for PM. The following cost function is used for CM actions

$$c_{CM}(v,\tau) \;=\; \int_0^{\tau}\left(c_F + c_R\left(1-\left(\frac{v}{v+t}\right)^{\delta}\right)\right) f^{T^v}(t|T^v \leq \tau)dt. \tag{4.14}$$

Although each costs of PM and costs of CM increase with increasing τ, this does not apply for the total costs $C(v,\tau)$. To get the total costs, the probability weighted costs

of maintenance actions are divided by the mean cycle length, which is also increasing with increasing τ. Therefore, it is non-obvious how the total costs depend on τ.
In Figure 4.6 the resulting total costs are plotted for the LFRD, $c_R = 1000$ and $c_F/c_R = 0.1$. Here it can be seen that for a fixed τ the total costs can be both increasing and decreasing with increasing v. This especially depends on δ. For every δ and τ the average maintenance costs per unit time have a limit as v approaches infinity, since both the costs for maintenance actions and the mean cycle length have a limit as v approaches infinity.



(a) $\delta = 0.125$

(b) $\delta = 0.5$

(c) $\delta = 1$

(d) $\delta = 2$

Figure 4.6.: Average maintenance costs for the LFRD($\alpha = 0.01, \beta = 0.02944$) if $c_R = 1000$ and $c_F/c_R = 0.1$.

In the following the optimal values for v and τ are computed for different cost ratios c_F/c_R and different δ. The numerical results are given in Table 4.3. These results lead to the following conclusions:

1. The higher δ, the more expensive are PM actions. Therefore, with rising δ it becomes cost optimal to do less PM, i.e. τ is increasing.

2. Initially the higher δ, the more expensive are even relatively small repairs, i.e. repairs with large v. That is why with higher δ, the higher is the optimal v. At a certain point this turns around, because the cost difference between small and large maintenance actions becomes smaller. Therefore, it is cost optimal to do even perfect repairs with $v = 0$. This does not apply for the RMWD because of the higher failure rate at lower ages.

Table 4.3.: Optimal values in case of costs proportional to the degree of repair - 1

	LFRD	RD	WD	MWD	RMWD
	$\alpha = 0.01$ $\beta = 0.02944$	$\beta = 0.03142$	$\beta = 0.0057$ $\gamma = 3$	$\alpha = 0.03$ $\beta = 0.004335$ $\gamma = 3$	$\alpha = 0.1$ $\beta = 0.1746$ $\gamma = 0.1$
$c_F/c_R = 0.05$					
$\delta = 0.125$	$v = 5.63$ $\tau = 1.93$	$v = 5.44$ $\tau = 1.88$	$v = 3.71$ $\tau = 0.95$	$v = 4.07$ $\tau = 1.03$	$v = 9$ $\tau = 1.94$
$\delta = 0.5$	$v = 11.19$ $\tau = 7.16$	$v = 10.81$ $\tau = 6.96$	$v = 5.23$ $\tau = 3.44$	$v = 5.78$ $\tau = 3.71$	$v = 12.37$ $\tau = 5.98$
$\delta = 1$	$v = 16.29$ $\tau = 13.47$	$v = 15.75$ $\tau = 13.03$	$v = 6.61$ $\tau = 5.44$	$v = 7.32$ $\tau = 5.86$	$v = 14.69$ $\tau = 8.72$
$\delta = 2$	$v = 23.43$ $\tau = 12.12$	$v = 22.66$ $\tau = 11.94$	$v = 0$ $\tau = 19.91$	$v = 0$ $\tau = 21.5$	$v = 16.87$ $\tau = 12.33$
$c_F/c_R = 0.1$					
$\delta = 0.125$	$v = 3.88$ $\tau = 1.5$	$v = 3.75$ $\tau = 1.46$	$v = 2.92$ $\tau = 0.78$	$v = 3.2$ $\tau = 0.86$	$v = 7.19$ $\tau = 1.69$
$\delta = 0.5$	$v = 7$ $\tau = 6.18$	$v = 6.75$ $\tau = 6.01$	$v = 3.63$ $\tau = 3.39$	$v = 4.06$ $\tau = 3.61$	$v = 9.5$ $\tau = 5.98$
$\delta = 1$	$v = 10.17$ $\tau = 10.88$	$v = 9.8$ $\tau = 10.58$	$v = 3.98$ $\tau = 5.78$	$v = 4.58$ $\tau = 6.06$	$v = 10.96$ $\tau = 9.11$
$\delta = 2$	$v = 14.59$ $\tau = 36.28$	$v = 14.06$ $\tau = 35.56$	$v = 0$ $\tau = 11.34$	$v = 0$ $\tau = 12.92$	$v = 11.7$ $\tau = 13.56$
$c_F/c_R = 0.2$					
$\delta = 0.125$	$v = 2.7$ $\tau = 1.12$	$v = 2.61$ $\tau = 1.09$	$v = 2.3$ $\tau = 0.65$	$v = 2.53$ $\tau = 0.69$	$v = 5.77$ $\tau = 1.38$
$\delta = 0.5$	$v = 4.14$ $\tau = 5.37$	$v = 3.97$ $\tau = 5.25$	$v = 2.49$ $\tau = 3.19$	$v = 2.82$ $\tau = 3.37$	$v = 7.17$ $\tau = 5.71$
$\delta = 1$	$v = 5.49$ $\tau = 9.43$	$v = 5.22$ $\tau = 9.23$	$v = 0.83$ $\tau = 7.32$	$v = 2.1$ $\tau = 6.55$	$v = 7.75$ $\tau = 9.21$
$\delta = 2$	$v = 0$ $\tau = 20.21$	$v = 0$ $\tau = 19.1$	$v = 0$ $\tau = 8.37$	$v = 0$ $\tau = 9.48$	$v = 6.56$ $\tau = 14.77$

3. The higher the amount by which the costs of CM are higher than for PM, i.e. the higher the ratio c_F/c_R, the better are PM actions, i.e. the cost optimal v is decreasing. Therefore, it becomes more likely that no failure occurs in the time interval of length τ.

4.4.4. Costs Proportional to the Degree of Repair - 2

Like in the previous subsection it is assumed that the costs of maintenance actions are proportional to the degree of repair. Again a Kijima type II model is considered and therefore the degree of the kth repair corresponds to (4.1). The following cost function is used for PM actions

$$c_{PM}(v,\tau) \quad = \quad c_R\left(1-\left(\frac{v}{v+\tau}\right)\exp\left(\frac{v}{v+\tau}-1\right)\right)^{\delta}, \tag{4.15}$$

where $\delta > 0$ and $c_R > 0$ are the costs of a replacement. This function is described in detail in Subsection 2.7.4. For CM actions the cost function

$$c_{CM}(v,\tau) = \int_0^{\tau}\left(c_F + c_R\left(1-\left(\frac{v}{v+t}\right)\exp\left(\frac{v}{v+t}-1\right)\right)^{\delta}\right) f^{T^v}(t|T^v \leq \tau)dt \tag{4.16}$$

with $\delta > 0$ is used. Here $c_F > 0$ is the fixed amount, by which the costs of CM are higher than for PM.

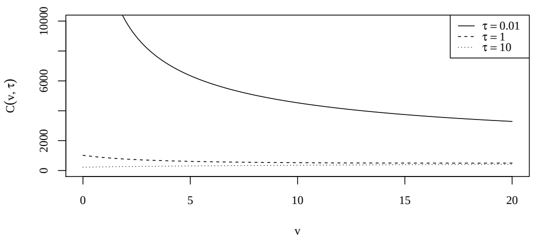
In Figure 4.7 the resulting total costs are plotted for the LFRD, $c_R = 1000$ and $c_F/c_R = 0.1$. Here it can be seen that just like for the costs proportional to the degree of repair -1 for a fixed τ the total costs can be both increasing and decreasing with increasing v. This especially depends on δ. For every δ and τ the average maintenance costs per unit time have a limit as v approaches infinity, since both the costs for maintenance actions and the mean cycle length have a limit as v approaches infinity.

In the following the optimal values for v and τ are computed for different cost ratios c_F/c_R and different δ. The numerical results are given in Table 4.4. These results lead to the following conclusions:

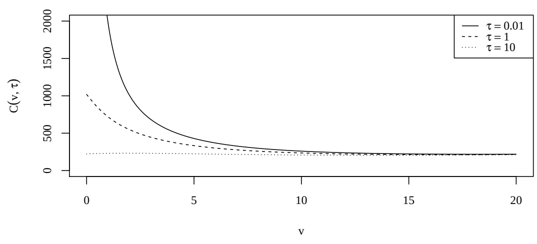
1. The higher δ, the lower are the costs of maintenance actions. Therefore, the higher δ the faster the numerator in (4.7) tends to zero if τ tends to zero and v does not at the same time tends to zero. The denominator in (4.7) also tends to zero if τ tends to zero. For $\delta = 2$ and $\delta = 3$ the costs of maintenance actions are so small that it is cost optimal to do non-stop PM actions, i.e. τ is nearly zero (note that $\tau = 0$ is excluded from optimization problem (4.4)).

2. The higher the amount by which the costs of CM are higher than for PM, i.e. the higher the ratio c_F/c_R, the better are PM actions, i.e. the cost optimal v is decreasing. The cost optimal τ also tend to decrease the higher the ratio c_F/c_R. Therefore, it becomes more likely that no failure occurs in the time interval of length τ.

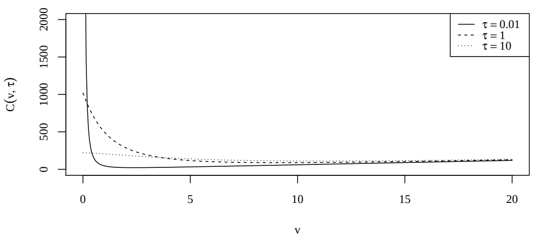
3. For small δ it is cost optimal to do even perfect repair



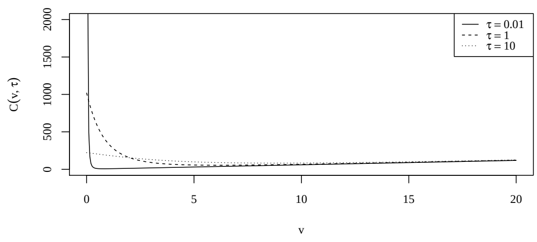
(a) $\delta = 0.5$



(b) $\delta = 1$



(c) $\delta = 2$



(d) $\delta = 3$

Figure 4.7.: Average maintenance costs for the LFRD($\alpha = 0.01, \beta = 0.02944$) if $c_R = 1000$ and $c_F/c_R = 0.1$.

4. The smaller δ, the more expensive are even relatively small repairs, i.e. repairs with large v, and the smaller is the cost difference between small and large maintenance actions. Therefore, for small δ it is cost optimal to do even perfect repairs with $v = 0$. This does not apply for the RMWD because of the higher failure rate at lower ages.

Table 4.4.: Optimal values in case of costs proportional to the degree of repair - 2

	LFRD	RD	WD	MWD	RMWD
	$\alpha = 0.01$ $\beta = 0.02944$	$\beta = 0.03142$	$\beta = 0.0057$ $\gamma = 3$	$\alpha = 0.03$ $\beta = 0.004335$ $\gamma = 3$	$\alpha = 0.1$ $\beta = 0.1746$ $\gamma = 0.1$
$c_F/c_R = 0.05$					
$\delta = 0.5$	$v = 0$ $\tau = 36.02$	$v = 0$ $\tau = 31.13$	$v = 0$ $\tau = 39.83$	$v = 0$ $\tau = 21.5$	$v = 2.38$ $\tau = 56.03$
$\delta = 1$	$v = 22.9$ $\tau = 25.14$	$v = 22.14$ $\tau = 24.74$	$v = 0$ $\tau = 39.83$	$v = 0$ $\tau = 21.5$	$v = 15.88$ $\tau = 13.51$
$\delta = 2$	$v = 2.99$ $\tau = 0.01$	$v = 2.92$ $\tau = 0.01$	$v = 2.6$ $\tau = 0.01$	$v = 2.79$ $\tau = 0.01$	$v = 5.2$ $\tau = 0.01$
$\delta = 3$	$v = 0.93$ $\tau = 0.01$	$v = 0.92$ $\tau = 0.01$	$v = 1.06$ $\tau = 0.01$	$v = 1.12$ $\tau = 0.01$	$v = 2.88$ $\tau = 0.01$
$c_F/c_R = 0.1$					
$\delta = 0.5$	$v = 0$ $\tau = 35.81$	$v = 0$ $\tau = 31.13$	$v = 0$ $\tau = 11.34$	$v = 0$ $\tau = 12.92$	$v = 2.21$ $\tau = 25.64$
$\delta = 1$	$v = 13.74$ $\tau = 38.8$	$v = 13.22$ $\tau = 38.41$	$v = 0$ $\tau = 11.34$	$v = 0$ $\tau = 12.92$	$v = 10.27$ $\tau = 14.92$
$\delta = 2$	$v = 2.37$ $\tau = 0.01$	$v = 2.32$ $\tau = 0.01$	$v = 2.18$ $\tau = 0.01$	$v = 2.34$ $\tau = 0.01$	$v = 4.47$ $\tau = 0.01$
$\delta = 3$	$v = 0.78$ $\tau = 0.01$	$v = 0.77$ $\tau = 0.01$	$v = 0.92$ $\tau = 0.01$	$v = 0.97$ $\tau = 0.01$	$v = 2.84$ $\tau = 0.01$
$c_F/c_R = 0.2$					
$\delta = 0.5$	$v = 0$ $\tau = 20.21$	$v = 0$ $\tau = 19.1$	$v = 0$ $\tau = 8.37$	$v = 0$ $\tau = 9.48$	$v = 1.96$ $\tau = 20.59$
$\delta = 1$	$v = 0$ $\tau = 20.21$	$v = 0$ $\tau = 19.1$	$v = 0$ $\tau = 8.37$	$v = 0$ $\tau = 9.48$	$v = 5.75$ $\tau = 15.18$
$\delta = 2$	$v = 1.1$ $\tau = 0.01$	$v = 1.84$ $\tau = 0.01$	$v = 1.83$ $\tau = 0.01$	$v = 1.97$ $\tau = 0.01$	$v = 3.91$ $\tau = 0.01$
$\delta = 3$	$v = 0.65$ $\tau = 0.01$	$v = 0.64$ $\tau = 0.01$	$v = 0.8$ $\tau = 0.01$	$v = 0.84$ $\tau = 0.01$	$v = 2.82$ $\tau = 0.01$

5. System with one Failure Type and Imperfect PM

In contrast to the other maintenance models of this thesis, the maintenance policy used in this chapter is not periodic but sequential. The main difference is that preventive maintenance is done at fixed predetermined times, which are not necessarily periodic. This concept is very practical because in reality with increasing age of the system preventive maintenance actions have to be done more often. This maintenance policy was first considered from Nguyen and Murthy [37] and Nakagawa [32], [33].
The maintenance model of this chapter is based on the imperfect preventive maintenance model from Nakagawa [34]. The restricted assumption that the costs for preventive maintenance are constant is eliminated and instead it is assumed that the costs of preventive maintenance depend on the degree of repair.
This chapter is structured as follows. Section 5.1 and Section 5.2 contain essential assumptions for the underlying system and maintenance policy. In Section 5.3 the cost optimization problem, which is solved in Section 5.4, is formulated. The results in Section 5.4 are computed for several continuous lifetime distributions.

5.1. Modeling the System

In this chapter the underlying repairable system has the following properties.

1. Initially a new repairable system is installed.
2. The system has only one failure type which can be removed through imperfect repair actions.
3. The repair times are negligible small.

5.2. Maintenance Policy

In the following, we consider a sequential imperfect preventive maintenance policy. This policy was proposed by Nakagawa [35] and Chen [14] and assumes that PM actions are carried out at fixed time intervals. In Nakagawa [34] an incomplete maintenance model was considered with the following assumptions.

1. After pre-defined intervals of length $x_k \geq 0, k = 1, \ldots, N-1$ imperfect PM actions are carried out. Suppose that $x_0 = 0$ and that the Nth preventive maintenance action is a replacement. Therefore, the times of imperfect PM actions are

$$\begin{array}{rl} 1. & x_1 \\ 2. & x_1 + x_2 \\ & \vdots \\ N-1. & x_1 + x_2 + \cdots + x_{N-1} \end{array}$$

and the replacement takes place at

$$N. \qquad x_1 + x_2 + \cdots + x_{N-1} + x_N.$$

2. Failures are removed through minimal repair.

3. The virtual age after the kth imperfect PM action falls to $v_k = \xi_k(v_{k-1}+t_k-t_{k-1})$, whereby $t_k = \sum_{i=1}^{k} x_i$ is the time of the kth imperfect PM action. The PM actions become worse over time, i.e. for the degrees of repair it holds

$$0 = \xi_0 < \xi_1 \le \xi_2 \le \cdots \le \xi_{N-1} < 1 \text{ and } \xi_N = 0. \tag{5.1}$$

5.3. Cost Optimization Problem

Consider a technical system which is maintained with maintenance policy described in Section 5.2. The random failure times of this system are denoted by $T_1^*, T_2^*, \dots$.
The aim of this section is to formulate a cost optimization problem. The optimization criterion are the average maintenance costs per unit time. For this, the expected maintenance costs per cycle are set in relation to the cycle length L. Here the cycle length is the time between two replacements, i.e. $L = \sum_{k=1}^{N} x_k$, and in contrast to the other models discussed in this thesis L is not a random variable.
Suppose $y_k \ge 0$ is the virtual age of the system immediately before the kth PM action. Therefore, it holds

$$\begin{aligned} y_1 &= x_1 \\ y_2 &= x_2 + \xi_1 y_1 = x_2 + \xi_1 x_1 \\ y_3 &= x_3 + \xi_2 y_2 = x_3 + \xi_2 x_2 + \xi_2 \xi_1 x_1 \\ &\vdots \end{aligned}$$

and in general it holds

$$\begin{aligned} y_k &= x_k + \xi_{k-1} y_{k-1} \\ &= x_k + \xi_{k-1} x_{k-1} + \cdots + \xi_{k-1}\xi_{k-2}\dots\xi_2\xi_1 x_1, \quad k = 1, 2, \dots, N. \end{aligned} \tag{5.2}$$

Using (5.2) the cycle length L can be rewritten as follows

$$L = \sum_{k=1}^{N} x_k = \sum_{k=1}^{N} (y_k - \xi_{k-1} y_{k-1}) = \sum_{k=1}^{N-1} (1 - \xi_k) y_k + y_N. \tag{5.3}$$

Suppose $N = (N_t)_{t\geq 0}$ is the failure counting process of the repairable system without PM. This means that all failures are removed through minimal repair and according to Remark 2.4 the process $N = (N_t)_{t\geq 0}$ is an inhomogeneous Poisson process with intensity function $h^{T_1}(t)$.
Suppose $N^* = (N_t^*)_{t\geq 0}$ is the failure counting process of the repairable system with PM. So N_t^* is the random number of failures in the interval $[0, t]$ and for the intensity function the following holds.

Lemma 5.1 (Intensity Function of $N^* = (N_t^*)_{t\geq 0}$)
Suppose x_k, $k = 1, \ldots, N$, are the interval lengths between PM actions. The intensity function of the counting process $N^* = (N_t^*)_{t\geq 0}$ is then

$$\lambda^{N^*}(t) = \begin{cases} 0 & , \text{ if } t < 0 \\ h^{T_1}(t) & , \text{ if } t \in [0, x_1) \\ h^{T_1}(\xi_k y_k + t - \sum_{i=1}^{k} x_i) & , \text{ if } t \in \left[\sum_{i=1}^{k} x_i, \sum_{i=1}^{k+1} x_i\right), \\ & \quad k = 1, \ldots, N-1 \end{cases} \tag{5.4}$$

where $h^{T_1}(t)$ is the hazard function of the time to first failure of a new system.

Theorem 5.2 (Mean Number of Failures in a Replacement Cycle)
The mean number of failures in a replacement cycle of length L is

$$E(N_L^*) = \sum_{k=1}^{N} \int_{\xi_{k-1} y_{k-1}}^{y_k} h^{T_1}(t)dt, \tag{5.5}$$

where $h^{T_1}(t)$ is the hazard function of the time to first failure of a new system.

Proof. According to maintenance policy from Section 5.2 all failures are removed through minimal repair and thus the state after repair is the same as immediately before failure. Therefore, the failure counting process $N^* = (N_t^*)_{t\geq 0}$ is an inhomogeneous Poisson process with intensity function $\lambda^{N^*}(t)$ with $t \geq 0$ and independent increments. Suppose $L = \sum_{k=1}^{N} x_k$ is the constant cycle length. Then, from equation (2.19) it follows that

$$\begin{aligned} E(N_L^*) &= \int_0^L \lambda^{N^*}(x)dx \\ &= \int_0^{x_1} h^{T_1}(x)dx + \int_{x_1}^{x_1+x_2} h^{T_1}(\xi_1 y_1 + x - x_1)dx + \ldots \\ &\quad + \int_{\sum_{i=1}^{N-1} x_i}^{\sum_{i=1}^{N} x_i} h^{T_1}\left(\xi_{N-1} y_{N-1} + x - \sum_{i=1}^{N-1} x_i\right) dx \\ &\overset{\text{substitution and (5.2)}}{=} \sum_{k=1}^{N} \int_{\xi_{k-1} y_{k-1}}^{y_k} h^{T_1}(t)dt \end{aligned} \tag{5.6}$$

□

Definition 5.1 (**Cost Optimization Problem**)
Let c_M denotes the costs of a minimal repair, $c_{PM}(k)$ the costs of the kth PM action and c_R the costs of a replacement. The average maintenance costs per unit time are

$$\begin{aligned} C(y_1, y_2, \ldots, y_N) &= \frac{c_M E(N_L^*) + \sum_{k=1}^{N-1} c_{PM}(k) + c_R}{L} \\ &= \frac{c_M \sum_{k=1}^{N} \int_{\xi_{k-1} y_{k-1}}^{y_k} h^{T_1}(t)dt + \sum_{k=1}^{N-1} c_{PM}(k) + c_R}{\sum_{k=1}^{N-1} (1-\xi_k) y_k + y_N}. \end{aligned} \tag{5.7}$$

The optimization problem then has the following form

$$\min_{y_1, y_2, \ldots, y_N} C(y_1, y_2, \ldots, y_N). \tag{5.8}$$

5.4. Example for Cost Optimal Maintenance

The costs for PM actions, the sequence $(\xi_k)_{1 \leq k < N}$ and the hazard function in optimization problem (5.8) are yet unspecified.
In this section it is assumed that the costs of PM actions are proportional to the degree of repair and therefore they are independent of y_k. Under this condition the cost optimal number of PM actions and the cost optimal interval lengths $x_1, \ldots, x_{N-1}$ will be computed using the statistical computing software R.
In what follows, it is assumed that the time to the first failure is modified Weibull distributed MWD(α, β, γ) (see Subsection 2.5.1 for more details). Therefore, the hazard function is

$$h^{T_1}(t) = \alpha + \beta\gamma t^{\gamma-1}, \quad \forall t > 0. \tag{5.9}$$

Theorem 5.3
If the time to the first failure is modified Weibull distributed with hazard function (5.9) *and $\gamma \neq 1$, the optimization problem* (5.8) *reduces to*

$$\min_{N \in \mathbb{N}^+} C(N), \tag{5.10}$$

whereby

$$C(N) = \frac{c_R + \sum_{k=1}^{N-1} c_{PM}(k)}{\sum_{k=0}^{N-1} d_k} \tag{5.11}$$

and $d_k := (1-\xi_k)\left(\frac{1-\xi_k}{1-\xi_k^{\gamma}}\right)^{\frac{1}{\gamma-1}}$.

Proof. Note that if the time to the first failure has a constant hazard function, the optimization problem (5.11) has no unique optimum. In terms of the modified Weibull distribution this means that there exists no unique solution if $\beta = 0$ or $\gamma = 1$. To find the optimal values for $y_1, y_2, \ldots, y_N$, one can set the partial derivatives from (5.7) with respect to y_k, $k = 1, \ldots, N$, equal to zero. Then, the cost optimal values $y_1, y_2, \ldots, y_N$

are a solution of the following system of equations

$$\frac{h^{T_1}(y_k) - \xi_k h^{T_1}(\xi_k y_k)}{1-\xi_k} = h^{T_1}(y_N), \quad k = 1, 2, \ldots, N-1. \tag{5.12}$$

$$c_M h^{T_1}(y_N) = C(y_1, y_2, \ldots, y_N). \tag{5.13}$$

Inserting hazard function (5.9) with $\gamma \neq 1$ into equation (5.12) leads to

$$\frac{\alpha + \beta\gamma y_k^{\gamma-1} - \xi_k\left(\alpha + \beta\gamma(\xi_k y_k)^{\gamma-1}\right)}{1-\xi_k} = \alpha + \beta\gamma y_N^{\gamma-1}$$

$$\Leftrightarrow \quad \frac{y_k^{\gamma-1}\left(1-\xi_k^\gamma\right)}{1-\xi_k} = y_N^{\gamma-1}$$

$$\Leftrightarrow \quad y_k = \left(\frac{1-\xi_k}{1-\xi_k^\gamma}\right)^{\frac{1}{\gamma-1}} y_N, \quad k = 1, 2, \ldots, N-1. \tag{5.14}$$

Substituting here the average maintenance costs $C(y_1, y_2, \ldots, y_N)$ in (5.13) one get

$$c_M h^{T_1}(y_N) = \frac{c_M \sum_{k=1}^{N} \int_{\xi_{k-1}y_{k-1}}^{y_k} h^{T_1}(t)dt + \sum_{k=1}^{N-1} c_{PM}(k) + c_R}{\sum_{k=1}^{N-1}(1-\xi_k)y_k + y_N}$$

$$\Leftrightarrow \quad h^{T_1}(y_N)\left(\sum_{k=1}^{N-1}(1-\xi_k)y_k + y_N\right) - \sum_{k=1}^{N}\int_{\xi_{k-1}y_{k-1}}^{y_k} h^{T_1}(t)dt = \frac{c_R + \sum_{k=1}^{N-1} c_{PM}(k)}{c_M}.$$

Inserting the hazard function (5.9) of the MWD(α, β, γ) leads to

$$\left(\alpha + \beta\gamma y_N^{\gamma-1}\right)\left(\sum_{k=1}^{N-1}(1-\xi_k)y_k + y_N\right) - \sum_{k=1}^{N}\int_{\xi_{k-1}y_{k-1}}^{y_k} \alpha + \beta\gamma t^{\gamma-1}dt = \frac{c_R + \sum_{k=1}^{N-1} c_{PM}(k)}{c_M}$$

$$\Leftrightarrow \left(\alpha + \beta\gamma y_N^{\gamma-1}\right)\left(\sum_{k=1}^{N-1}(1-\xi_k)y_k + y_N\right) - \sum_{k=1}^{N}\left(\alpha(y_k - \xi_{k-1}y_{k-1}) + \beta(y_k^\gamma - \xi_{k-1}^\gamma y_{k-1}^\gamma)\right) = \frac{c_R + \sum_{k=1}^{N-1} c_{PM}(k)}{c_M}$$

$$\Leftrightarrow \left(\alpha + \beta\gamma y_N^{\gamma-1}\right)\left(\sum_{k=1}^{N-1}(1-\xi_k)y_k + y_N\right) - \alpha\left(\sum_{k=1}^{N-1}(1-\xi_k)y_k + y_N\right) - \beta\left(\sum_{k=1}^{N-1}(1-\xi_k^\gamma)y_k^\gamma + y_N^\gamma\right) = \frac{c_R + \sum_{k=1}^{N-1} c_{PM}(k)}{c_M}$$

$$\overset{(5.14)}{\Leftrightarrow} \left(\alpha + \beta\gamma y_N^{\gamma-1}\right)\left(\sum_{k=1}^{N-1}(1-\xi_k)\left(\frac{1-\xi_k}{1-\xi_k^\gamma}\right)^{\frac{1}{\gamma-1}} y_N + y_N\right)$$

$$
\begin{aligned}
&-\alpha\left(\sum_{k=1}^{N-1}(1-\xi_k)\left(\frac{1-\xi_k}{1-\xi_k^{\gamma}}\right)^{\frac{1}{\gamma-1}} y_N + y_N\right) \\
&-\beta\left(\sum_{k=1}^{N-1}(1-\xi_k^{\gamma})\left(\frac{1-\xi_k}{1-\xi_k^{\gamma}}\right)^{\frac{\gamma}{\gamma-1}} y_N^{\gamma} + y_N^{\gamma}\right) &=& \frac{c_R+\sum_{k=1}^{N-1} c_{PM}(k)}{c_M} \\
\Leftrightarrow y_N^{\gamma}\left(\beta\gamma-\beta\right)\left(\sum_{k=1}^{N-1}(1-\xi_k)\left(\frac{1-\xi_k}{1-\xi_k^{\gamma}}\right)^{\frac{1}{\gamma-1}}+1\right) &=& \frac{c_R+\sum_{k=1}^{N-1} c_{PM}(k)}{c_M}
\end{aligned}
$$

$$
\Leftrightarrow y_N^{\gamma} = \frac{c_R+\sum_{k=1}^{N-1} c_{PM}(k)}{c_M\left(\beta\gamma-\beta\right)\left(\sum_{k=1}^{N-1}(1-\xi_k)\left(\frac{1-\xi_k}{1-\xi_k^{\gamma}}\right)^{\frac{1}{\gamma-1}}+1\right)} \tag{5.15}
$$

Using equation (5.13) the following equivalence holds

$$
\min_{y_1,y_2,\cdots,y_N} C(y_1,y_2,\ldots,y_N) \Leftrightarrow \min_N c_M h^{T_1}(y_N). \tag{5.16}
$$

Substituting y_N at the right-hand side through (5.15) results in

$$
c_M h^{T_1}(y_N) \;=\; c_M\left(\left(\frac{c_R+\sum_{k=1}^{N-1} c_{PM}(k)}{c_M\left(\beta\gamma-\beta\right)\left(\sum_{k=0}^{N-1} d_k\right)}\right)^{\frac{\gamma-1}{\gamma}}\beta\gamma+\alpha\right), \tag{5.17}
$$

whereby $d_k := (1-\xi_k)\left(\frac{1-\xi_k}{1-\xi_k^{\gamma}}\right)^{\frac{1}{\gamma-1}}$. Minimizing (5.17) with respect to N provides the same result as minimizing $C(N)$ with respect to N, whereby

$$
C(N) \;:=\; \frac{c_R+\sum_{k=1}^{N-1} c_{PM}(k)}{\sum_{k=0}^{N-1} d_k}. \tag{5.18}
$$

Therefore, the following equivalence holds

$$
\min_{y_1,y_2,\cdots,y_N} C(y_1,y_2,\ldots,y_N) \Leftrightarrow \min_N C(N). \tag{5.19}
$$

□

Theorem 5.4
If $c_{PM}(k) \geq c_{PM}(N)$ for $k < N$ and $\lim_{k\to\infty}\xi_k = 1$, there exists a finite $N^\star$ which satisfies (5.10)*. If the sequence $(\xi_k)_{1\leq k<N}$ is strictly monotonic increasing, $N^\star$ is even unique.*

Proof. In order that formula (5.11) has an unique minimum $C(N^\star)$, the following conditions have to be fulfilled

$$
\forall N \geq N^\star \quad \text{it is:} \quad C(N) \leq C(N+1) \tag{5.20}
$$

$$
\forall N < N^\star \quad \text{it is:} \quad C(N) > C(N+1). \tag{5.21}
$$

The first inequality reduces as follows

$$\begin{array}{rrcl} & C(N) & \leq & C(N+1) \\ \Leftrightarrow & \frac{c_R+\sum_{k=1}^{N-1} c_{PM}(k)}{\sum_{k=0}^{N-1} d_k} & \leq & \frac{c_R+\sum_{k=1}^{N} c_{PM}(k)}{\sum_{k=0}^{N} d_k} \\ \Leftrightarrow & \frac{c_R+\sum_{k=1}^{N-1} c_{PM}(k)}{\sum_{k=0}^{N-1} \frac{d_k}{d_N}} & \leq & \frac{c_R+\sum_{k=1}^{N-1} c_{PM}(k)+c_{PM}(N)}{\sum_{k=0}^{N-1} \frac{d_k}{d_N}+1} \\ \Leftrightarrow & c_R+\sum_{k=1}^{N-1} c_{PM}(k) & \leq & c_{PM}(N)\sum_{k=0}^{N-1} \frac{d_k}{d_N} \\ \Leftrightarrow & \frac{c_R+\sum_{k=1}^{N-1} c_{PM}(k)}{c_{PM}(N)} & \leq & \sum_{k=0}^{N-1} \frac{d_k}{d_N}. \end{array}$$

Suppose $c_{PM}(k) \geq c_{PM}(N)$ for all $k < N$. Then, we have the relation

$$\sum_{k=0}^{N-1} \frac{d_k}{d_N} \geq \frac{c_R+\sum_{k=1}^{N-1} c_{PM}(k)}{c_{PM}(N)} \geq \frac{c_R}{c_{PM}(N)} + (N-1). \tag{5.22}$$

Define $L(N) := \sum_{k=0}^{N-1} \frac{d_k}{d_N} - (N-1)$. Thus, if $L(N)$ is monotonic increasing in N and $\lim_{N\to\infty} L(N) > \frac{c_R}{c_{PM}(N)}$, both inequalities (5.20) and (5.21) are satisfied.
Furthermore, if $\lim_{k\to\infty} \xi_k = 1$, then

$$\lim_{k\to\infty} d_k = \lim_{\xi_k\to 1} (1-\xi_k)\left(\frac{1-\xi_k}{1-\xi_k^\gamma}\right)^{\frac{1}{\gamma-1}} = 0. \tag{5.23}$$

Nakagawa [34] has shown that d_k is decreasing in k, i.e. $d_k \geq d_N$ for $k < N$ and $\frac{d_k}{d_N} \geq 1$. Thus, with $\xi_0 = 0$ it holds that $d_0 = 1$ and

$$\begin{aligned} \lim_{N\to\infty} L(N) &= \lim_{N\to\infty}\left(\sum_{k=0}^{N-1} \frac{d_k}{d_N} - (N-1)\right) = \lim_{N\to\infty}\left(\frac{d_0}{d_N} + \sum_{k=1}^{N-1} \frac{d_k}{d_N} - (N-1)\right) \\ &\geq \lim_{N\to\infty}\left(\frac{d_0}{d_N} + \sum_{k=1}^{N-1} 1 - (N-1)\right) = \lim_{N\to\infty} \frac{d_0}{d_N} = \lim_{N\to\infty} \frac{1}{d_N} = \infty. \end{aligned} \tag{5.24}$$

To summarize, if $\lim_{k\to\infty} \xi_k = 1$ a finite $N^\star$, which minimize equation (5.10), exists. If the sequence $(\xi_k)_{1\leq k<N}$ is strictly monotonic increasing, $N^\star$ is even finite and unique. □

In the following the degree of repair is analogous to Nakagawa [34]

$$\xi_k = \frac{k}{k+1}, \quad \forall k < N. \tag{5.25}$$

Note that with (5.25) the sequence $(\xi_k)_{1\leq k<N}$ is strictly monotonic increasing and that $\lim_{k\to\infty} \xi_k = 1$.

5.4.1. Costs Proportional to the Degree of Repair - 3

In this subsection it is assumed that the costs of PM actions are proportional to the degree of repair. The cost function for PM actions is

$$c_{PM}(k) = c_R - \xi_k^\delta (c_R - c_M), \quad \delta > 0, \tag{5.26}$$

where $c_R > c_M$ and c_M denotes the costs for a minimal repair and c_R are the costs of a replacement. This cost function is described in detail in Subsection 2.7.5. Since the sequence $(\xi_k)_{1\leq k<N}$ is strictly monotonic increasing, $c_{PM}(k) \geq c_{PM}(N)$ for $k < N$. Therefore, all conditions of Remark 5.4 are fulfilled and a finite and unique $N^\star$ exists. The statistical computing software R is used to solve optimization problem (5.19). Inserting the cost optimal N^* in equation (5.15) gives the cost optimal value of y_N that can be used to compute the virtual ages $y_k \geq 0$ of the system immediately before the kth PM actions.

Theorem 5.5
If $\delta \geq 1$ the cost optimal $N^\star$ of optimization problem (5.10) is one.

Proof. Suppose $\gamma \neq 1$. For $N^\star > 1$ the condition $C(1) > C(2)$ must be fulfilled. From (5.11) follows

$$\begin{aligned}
& & C(1) &> C(2) \\
\Leftrightarrow & & c_R &> \tfrac{c_R + c_{PM}(1)}{d_0 + d_1} \\
\Leftrightarrow & & c_R &> \frac{2c_R - \xi_1^\delta (c_R - c_M)}{1 + (1-\xi_1)\left(\frac{1-\xi_1}{1-\xi_1^\gamma}\right)^{\frac{1}{\gamma-1}}} \\
\Leftrightarrow & c_R\left(1 + \left(\tfrac{(1-\xi_1)^\gamma}{1-\xi_1^\gamma}\right)^{\frac{1}{\gamma-1}}\right) - 2c_R & &> -\xi_1^\delta (c_R - c_M) \\
\Leftrightarrow & \tfrac{1}{\xi_1^\delta}\left(1 - \left(\tfrac{(1-\xi_1)^\gamma}{1-\xi_1^\gamma}\right)^{\frac{1}{\gamma-1}}\right) & &< \tfrac{c_R - c_M}{c_R}.
\end{aligned} \tag{5.27}$$

Since $c_R > c_M$ it holds that $0 < \frac{c_R - c_M}{c_R} < 1$. The degree of repair ξ_1 is $0 < \xi_1 < 1$. For the left-hand side of the above inequality it holds

$$\begin{aligned}
& \lim_{\xi_1 \to 0} \frac{1}{\xi_1^\delta}\left(1 - \left(\frac{(1-\xi_1)^\gamma}{1-\xi_1^\gamma}\right)^{\frac{1}{\gamma-1}}\right) \\
= & \lim_{\xi_1 \to 0} \frac{\left(1 - \left(\frac{(1-\xi_1)^\gamma}{1-\xi_1^\gamma}\right)^{\frac{1}{\gamma-1}}\right)}{\xi_1^\delta} \\
\overset{\frac{0}{0}}{=} & \lim_{\xi_1 \to 0} \frac{-\frac{1}{\gamma-1}\left(\frac{(1-\xi_1)^\gamma}{1-\xi_1^\gamma}\right)^{\frac{2-\gamma}{\gamma-1}}\left(-\gamma(1-\xi_1)^{\gamma-1}(1-\xi_1^\gamma) - (1-\xi_1)^\gamma(-\gamma\xi_1^{\gamma-1})\right)}{\delta\xi_1^{\delta-1}(1-\xi_1^\gamma)^2}
\end{aligned}$$

$$= \begin{cases} 0 & \text{, if } \delta < 1 \\ \frac{\gamma}{\gamma-1} & \text{, if } \delta = 1 \text{ and } \gamma > 1 \\ \infty & \text{, if } \delta = 1 \text{ and } \gamma < 1 \\ \infty & \text{, if } \delta > 1 \end{cases} \tag{5.28}$$

and

$$\begin{aligned} \lim_{\xi_1 \to 1} \frac{1}{\xi_1^\delta}\left(1 - \left(\frac{(1-\xi_1)^\gamma}{1-\xi_1^\gamma}\right)^{\frac{1}{\gamma-1}}\right) &= 1 - \left(\lim_{\xi_1 \to 1} \frac{(1-\xi_1)^\gamma}{1-\xi_1^\gamma}\right)^{\frac{1}{\gamma-1}} \\ &\overset{\frac{0}{0}}{=} 1 - \left(\lim_{\xi_1 \to 1} \frac{\gamma(1-\xi_1)^{\gamma-1}}{-\gamma\xi_1^{\gamma-1}}\right)^{\frac{1}{\gamma-1}} \\ &= 1. \end{aligned} \tag{5.29}$$

Define $g(x) \equiv \frac{1}{x^\delta}\left(1 - \left(\frac{(1-x)^\gamma}{1-x^\gamma}\right)^{\frac{1}{\gamma-1}}\right)$. If function $g(x)$ is monotonically decreasing for $\delta \geq 1$, the minimum of $g(x)$ in the interval $[0,1]$ is in $x = 1$ and $g(1) = 1$. In this case, condition (5.27) would not have been complied with and $N^\star = 1$.
In the following the monotony of $g(x)$ is analyzed for $\delta \geq 1$. It holds

$$g(x) \text{ monotonically decreasing } \Leftrightarrow g'(x) = \frac{dg(x)}{dx} < 0. \tag{5.30}$$

The derivative of $g(x)$ with respect to x is

$$g'(x) = \frac{-\delta}{x^{\delta+1}}\left(1 - \left(\frac{(1-x)^\gamma}{1-x^\gamma}\right)^{\frac{1}{\gamma-1}}\right) - \frac{\gamma(1-x)^{\frac{\gamma}{\gamma-1}-1}(1-x^\gamma)^{\frac{1}{\gamma-1}-1}(x^\gamma - x)}{(\gamma-1)x^\delta(1-x^\gamma)^{\frac{2}{\gamma-1}}}. \tag{5.31}$$

Replace $g'(x)$ in (5.30) through (5.31) and get

$$\begin{aligned} g'(x) &< 0 \\ \Leftrightarrow \frac{\gamma(1-x)^{\frac{\gamma}{\gamma-1}-1}(1-x^\gamma)^{\frac{1}{\gamma-1}-1}(x-x^\gamma)}{(\gamma-1)x^\delta(1-x^\gamma)^{\frac{2}{\gamma-1}}} &< \frac{\delta}{x^{\delta+1}}\left(1 - \left(\frac{(1-x)^\gamma}{1-x^\gamma}\right)^{\frac{1}{\gamma-1}}\right) \\ \Leftrightarrow \frac{\gamma(1-x)^{\frac{1}{\gamma-1}}(x-x^\gamma)x}{(\gamma-1)(1-x^\gamma)^{\frac{\gamma}{\gamma-1}}} &< \delta\left(1 - \left(\frac{(1-x)^\gamma}{1-x^\gamma}\right)^{\frac{1}{\gamma-1}}\right) \\ \Leftrightarrow \frac{\gamma(1-x)^{\frac{1}{\gamma-1}}(x-x^\gamma)x}{(\gamma-1)(1-x^\gamma)\left((1-x^\gamma)^{\frac{1}{\gamma-1}} - (1-x)^{\frac{\gamma}{\gamma-1}}\right)} &< \delta \\ \Leftrightarrow \frac{\gamma(1-x)^{\frac{1}{\gamma-1}}(x-x^\gamma)x}{(\gamma-1)(1-x^\gamma)\left((1-x^\gamma)^{\frac{1}{\gamma-1}} - (1-x)^{\frac{\gamma}{\gamma-1}}\right)} &\leq 1 \end{aligned} \tag{5.32}$$

For $\gamma < 1$ this inequality is equivalent to

$$\gamma(1-x)^{\frac{1}{\gamma-1}}(x-x^\gamma)x \geq (\gamma-1)(1-x^\gamma)\left((1-x^\gamma)^{\frac{1}{\gamma-1}} - (1-x)^{\frac{\gamma}{\gamma-1}}\right) \tag{5.33}$$

and since $x \in (0,1)$ it holds that $1-x > 1-x^\gamma$. Therefore, the right-hand side of inequality (5.33) is limited from above through

$$(\gamma-1)(1-x^\gamma)\left((1-x)^{\frac{1}{\gamma-1}} - (1-x)^{\frac{\gamma}{\gamma-1}}\right). \tag{5.34}$$

Then instead of proving inequality (5.33) one can verify the following inequality

$$\begin{aligned}
\gamma(1-x)^{\frac{1}{\gamma-1}}(x-x^\gamma)x &\geq (\gamma-1)(1-x^\gamma)\left((1-x)^{\frac{1}{\gamma-1}} - (1-x)^{\frac{\gamma}{\gamma-1}}\right) \\
\Leftrightarrow \quad \gamma(1-x)^{\frac{1}{\gamma-1}}(x-x^\gamma)x &\geq (\gamma-1)(1-x^\gamma)(1-x)^{\frac{1}{\gamma-1}}x \\
\Leftrightarrow \quad \gamma(x-x^\gamma) &\geq (\gamma-1)(1-x^\gamma) \\
\Leftrightarrow \quad \gamma x - \gamma x^\gamma &\geq \gamma - \gamma x^\gamma - 1 + x^\gamma \\
\Leftrightarrow \quad \gamma &\leq \frac{1-x^\gamma}{1-x} \\
\Leftrightarrow \quad \gamma &\leq x^{\gamma-1} + x^{\gamma-2} + \cdots + x + 1
\end{aligned} \tag{5.35}$$

This inequality is true since $x \in (0,1)$ and each of the γ summands at the right-hand side of (5.35) is greater than one. Therefore, $g'(x) < 0$ and condition (5.27) does not hold. Hence $N^\star$ is always one for $\delta \geq 1$ and $\gamma < 1$.
Analogue if $\gamma > 1$ inequality (5.32) is equivalent to

$$\gamma(1-x)^{\frac{1}{\gamma-1}}(x-x^\gamma)x \leq (\gamma-1)(1-x^\gamma)\left((1-x^\gamma)^{\frac{1}{\gamma-1}} - (1-x)^{\frac{\gamma}{\gamma-1}}\right). \tag{5.36}$$

Using $1-x < 1-x^\gamma$ and $x^{\gamma-1} < 1$, $x^{\gamma-2} < 1$,..., $x < 1$ it can be shown that $g'(x) < 0$ if $\gamma > 1$. Thus also for $\gamma > 1$ and $\delta \geq 1$, $N^\star$ is always one. □

In the following, the cost optimal number of PM actions and the cost optimal interval lengths $x_1, \ldots, x_{N-1}$ are computed for different cost ratios c_M/c_R and different δ. The parameters of all used distributions were chosen so that the expectation is 5 for all distributions. The numerical results are given in Table 5.1, Table 5.2 and Table 5.3. These results lead to the following conclusions:

1. The time between PM actions is decreasing with increasing age of the system.

2. The lower the costs of a replacement compared to the costs of a minimal repair, i.e. the higher the ratio c_M/c_R, the less PM actions are carried out, i.e. the smaller is the cost optimal $N^\star$.

3. With rising δ the factor ξ_k^δ in cost function (5.26) becomes smaller. Therefore, PM actions become more expensive in comparison to minimal repair actions. Hence, less and less PM actions are carried out and beyond a certain point it becomes cost optimal to no longer perform PM actions, i.e. $N^\star = 1$. Instead the time between replacements is increasing and occurring failures are removed through relatively cheap minimal repairs.

4. The LFRD and RD lead nearly to the same optimal solutions. The same holds for the WD and MWD.

Table 5.1.: Optimal values for N and $x_1, \ldots, x_{N-1}$ in case of costs proportional to the degree of repair - 3 and $\delta = 0.2$

	LFRD	RD	WD	MWD
	$\alpha = 0.01$ $\beta = 0.02944$	$\beta = 0.03142$	$\beta = 0.0057$ $\gamma = 3$	$\alpha = 0.03$ $\beta = 0.004335$ $\gamma = 3$
$c_M/c_R = 0.02$				
$N^\star$	11	11	12	12
x_1	25.95	25.12	11.61	12.72
x_2	10.38	10.05	4.77	5.22
x_3	6.67	6.46	3.05	3.34
x_4	4.94	4.78	2.26	2.47
x_5	3.93	3.81	1.79	1.97
x_6	3.27	3.16	1.49	1.63
x_7	2.79	2.71	1.27	1.4
x_8	2.44	2.36	1.11	1.22
x_9	2.17	2.1	0.99	1.08
x_{10}	1.95	1.89	0.89	0.97
x_{11}	20.39	19.74	0.81	0.89
x_{12}			6.88	7.54
$c_M/c_R = 0.05$				
$N^\star$	5	5	5	5
x_1	15.87	15.37	8.41	9.21
x_2	6.35	6.15	3.45	3.78
x_3	4.08	3.95	2.21	2.42
x_4	3.02	2.93	1.64	1.79
x_5	13.23	12.8	5.43	5.95
$c_M/c_R = 0.1$				
$N^\star$	3	3	3	3
x_1	11.68	11.31	6.89	7.55
x_2	4.67	4.52	2.83	3.1
x_3	10.52	10.18	4.93	5.4

Table 5.2.: Optimal values for N and $x_1, \ldots, x_{N-1}$ in case of costs proportional to the degree of repair - 3 and $\delta = 0.5$

	LFRD	RD	WD	MWD
	$\alpha = 0.01$ $\beta = 0.02944$	$\beta = 0.03142$	$\beta = 0.0057$ $\gamma = 3$	$\alpha = 0.03$ $\beta = 0.004335$ $\gamma = 3$
$c_M/c_R = 0.02$				
$N^\star$	2	2	4	4
x_1	29.77	28.82	12.84	14.07
x_2	29.77	28.82	5.27	5.77
x_3			3.38	3.7
x_4			8.61	9.43
$c_M/c_R = 0.05$				
$N^\star$	2	2	2	2
x_1	17.34	16.79	9.01	9.87
x_2	17.34	16.79	7.41	8.12
$c_M/c_R = 0.1$				
$N^\star$	1	1	2	2
x_1	12.29	11.89	7.21	7.9
x_2			5.93	6.5

Table 5.3.: Optimal values for N and $x_1, \ldots, x_{N-1}$ in case of costs proportional to the degree of repair - 3 and $\delta = 0.8$

	LFRD	RD	WD	MWD
	$\alpha = 0.01$ $\beta = 0.02944$	$\beta = 0.03142$	$\beta = 0.0057$ $\gamma = 3$	$\alpha = 0.03$ $\beta = 0.004335$ $\gamma = 3$
$c_M/c_R = 0.02$				
$N^\star$	1	1	1	1
x_1	30.1	29.13	13.15	14.41
$c_M/c_R = 0.05$				
$N^\star$	1	1	1	1
x_1	17.38	16.82	9.12	9.99
$c_M/c_R = 0.1$				
$N^\star$	1	1	1	1
x_1	12.29	11.89	7.24	7.93

Part II.

Models with Discrete Lifetime Distribution

6. System with two Failure Types

The maintenance model of this chapter is the discrete analogue of the model from Chapter 3. The underlying repairable system is assumed to have a discrete lifetime distribution. Under this assumption the system can be interpreted as a repairable multi-state system (MSS) with n states. Therefore, a time scale analogously to Kahle [22] have to be introduced and it is assumed that at time 1 the system is in state one, at time 2 the system is in state two and so on. Using this time scale, one can model imperfect maintenance actions also for multi-state systems.
In this chapter following Beichelt [7] it is assumed that there are two failure types minor and major ones. Minor failures (type 1) can be removed by minimal repair and major failures (type 2) can only be removed by replacement.
Using a periodic PM policy, the system undergoes imperfect PM actions at predetermined periodic times and failures in between are removed according to the failure type. It is assumed that the imperfect PM actions adjust the virtual age of the system in a Kijima type manner.
This chapter is structured as follows. Section 6.1, 6.2 and 6.3 contain essential assumptions and definitions, which are needed to formulate the cost optimization problem in Section 6.4. Finally, in Section 6.5 optimal maintenance strategies are computed for different cost functions for PM and different discrete lifetime distributions.

6.1. Modeling the System

According to Kahle [22], a multi-state system (MSS) is considered. For further research the following assumptions are made.

1. Initially, a new repairable MSS is installed. The MSS has n states in which the system can fail. A time scale is introduced so that the system in time 1 is in state one, in time 2 in state two and so on.

2. At each state the system can fail with some probability. The system has two types of failures, minor failures (type 1) and major failures (type 2).

3. Whenever a failure occurs, it is a minor one (type 1) with probability 1-p and a major one (type 2) with probability p.

4. Type 1 failures can be removed by a minimal repair and type 2 failures can only be removed by replacements.

5. The repair times are negligible small.

6.2. Modeling the Markov Chain

Consider the repairable system from Section 6.1. Let the positive random variables $(T_n^*)_{n\in\mathbb{N}^+}$ be the random failure times of the system with PM and without distinction of failure types.
The corresponding discrete-time counting process $(N_t^*)_{t\in\mathbb{N}}$ is defined by

$$N_t^* = \max\{n \in \mathbb{N}^+ : T_n^* \leq t\} \tag{6.1}$$

for $t \in \mathbb{N}$ and $N_0^* = 0$. This means N_t^* counts the number of failures which occurred up to time t without distinction of type 1 and type 2 failures. Therefore,

$$\Delta N_t^* \quad = \quad N_t^* - N_{t-1}^* = \begin{cases} 1 & \text{, if the MSS fails at } t \\ 0 & \text{, else} \end{cases}. \tag{6.2}$$

It is assumed that after minimal repair the system starts working immediately and that at particular points in time only one failure can occur. Therefore, as long as all failures are removed through minimal repair it holds that $P(\Delta N_t^* = 1) = h^{T^*}(t)$, where T^* is the random time to failure and $h^{T^*}(t)$ is the failure rate of the MSS with PM and without distinction of failure types. It holds

$$h^{T^*}(t) = P(T^* = t|T^* \geq t) = \frac{P(T^* = t)}{P(T^* \geq t)}, \quad t = 0, 1, ..., n. \tag{6.3}$$

Now consider a MSS with two different failure types (see assumptions 2, 3 and 4 from Section 6.1). Then let S_t^* be the random state of the MSS in t. Define the following three states:

$S_t^* = 0$: MSS is operating in t.
$S_t^* = 1$: MSS is under minimal repair in t due to type 1 failure.
$S_t^* = 2$: MSS is not operating because of type 2 failure.

The sequence $(S_t^*)_{t\in\mathbb{N}}$ is a Markov chain with state space $M = \{0, 1, 2\}$ and satisfies the Markov condition (2.23), i.e.

$$P(S_t^* = s_t|S_0^* = s_0, \ldots, S_{t-1}^* = s_{t-1}) = P(S_t^* = s_t|S_{t-1}^* = s_{t-1}) \tag{6.4}$$

for all $t \geq 2$ and $s_0, s_1, \ldots, s_t \in M$ with $P(S_0^* = s_0, S_1^* = s_1, \ldots, S_{t-1}^* = s_{t-1}) > 0$. The MSS starts working with state 0. Therefore, the initial distribution is

$$P(S_0^* = s_0) = \begin{cases} 1 & \text{, if } s_0 = 0 \\ 0 & \text{, if } s_0 \in \{1, 2\} \end{cases}. \tag{6.5}$$

The one-step transition matrices $(X_n^*)_{n\in\mathbb{N}^+}$ are given by

$$X_n^* \quad = \quad \begin{pmatrix} 1 - h^{T^*}(n) & (1-p)h^{T^*}(n) & ph^{T^*}(n) \\ 1 - h^{T^*}(n) & (1-p)h^{T^*}(n) & ph^{T^*}(n) \\ 0 & 0 & 1 \end{pmatrix}. \tag{6.6}$$

Note that state 2 is absorbing because the system has to be replaced after type 2 failures. Using (2.27) one get

$$P(S_k^* = s_k, S_{k+1}^* = s_{k+1}, \ldots, S_n^* = s_n) = P(S_k^* = s_k) \prod_{j=k+1}^{n} X_j^*(s_{j-1}, s_j), \tag{6.7}$$

for $s_k, s_{k+1}, \ldots, s_n \in M$ and with $X_j^*(s_{j-1}, s_j) = P(S_j^* = s_j | S_{j-1}^* = s_{j-1})$ and

$$X_j^*(s_{j-1}, s_j) = \begin{cases} 1 - h^{T^*}(j) & , \text{if } s_{j-1} \in \{0,1\},\ s_j = 0 \\ (1-p)h^{T^*}(j) & , \text{if } s_{j-1} \in \{0,1\},\ s_j = 1 \\ ph^{T^*}(j) & , \text{if } s_{j-1} \in \{0,1\},\ s_j = 2\ , \\ 0 & , \text{if } s_{j-1} = 2,\ s_j \in \{0,1\} \\ 1 & , \text{if } s_{j-1} = 2,\ s_j = 2 \end{cases} \tag{6.8}$$

for $j \geq 1$ and $h^{T^*}(\cdot)$ is the failure rate of T^* and p is the probability of a type 2 failure.

6.3. Maintenance Policy

In the following, we consider a periodic imperfect preventive maintenance policy with finite planning horizon. In the periodic PM policy the system is preventively maintained at fixed time intervals and repaired at intervening failures. Similar maintenance policies were applied for instance in Nakagawa [32], Sheu and Lin and Liao [45] and Zequeira and Bérenguer [50].
In particular, the following assumptions are made for the used maintenance policy.

1. The system is maintained according to the failure type. Whenever a minor failure (type 1) occurs, a minimal repair will be carried out. If the failure is of type 2 the system will be replaced by a new one.

2. The PM actions are imperfect in the sense that each PM action reduces the virtual age of the system to a constant virtual age of v with $v \in \{0, \ldots, n-1\}$.

3. PM is performed at $v + \tau, v + 2\tau, \ldots, v + (N-1)\tau$, with $\tau \in \{1, \ldots, n-v\}$, $v \in \{0, \ldots, n-1\}$ and $N \in \{1, \ldots, N_{\max}\}$. Together with assumption 2 this means that every time when the system reaches the virtual age of $v + \tau$ the PM action resets the system to the virtual age of v. If the MSS has n states the following pairs of v and τ are possible

$$\begin{aligned} \tau = 1; &\qquad v = 0, \ldots, v = n-1, \\ \tau = 2; &\qquad v = 0, \ldots, v = n-2, \\ &\vdots \\ \tau = n-1; &\qquad v = 0 \text{ or } v = 1, \\ \tau = n; &\qquad v = 0. \end{aligned}$$

4. If no type 2 failure occurred in time interval $[0, v + N\tau)$, the system is replaced preventively at $v + N\tau$.

Note that the restriction of N through $N_{\max}$ is appropriate since systems have a finite useful life. Therefore, in our cost optimization problem from Section 6.4 a predefined maximum number of PM actions, i.e. $1 - N_{\max}$, will be taken into account.
The above described maintenance policy contains the age replacement policy and the minimal repair policy as special cases. The first one is obtained if $p = 1$ and $v = 0$. Then the system is replaced at the time of failure or at age τ whichever occurs first. If $p = 0$ and $v = 0$ we have the minimal repair policy which means that the system is always replaced at age τ and failures that occur between the periodic replacements are removed through minimal repair.

6.4. Cost Optimization Problem

Consider a technical system which is maintained with maintenance policy described in Section 6.3. To optimize the maintenance of this system with respect to cost criteria it is necessary to define the cumulative distribution function of the random time of the first type 2 failure.
Let T_1 be the discrete random time of the first failure of a repairable MSS without preventive maintenance and no distinction in failure types. Then T_1^* is the discrete random time of the first failure of a repairable MSS with preventive maintenance and only one failure type.
If one takes into account two failure types, one have $T_1^{'}$ and $T_1^{''}$ as the times of the first occurrence of a type 1 or type 2 failure of a repairable MSS without preventive maintenance, respectively. Analogous $T_1^{*'}$ and $T_1^{*''}$ are the times of the first occurrence of a type 1 or type 2 failure of a repairable MSS with consideration of preventive maintenance actions, respectively.
The random variable $T_1^{*''}$ if of great interest, since a type 2 failure terminates a replacement cycle. In what follows, some properties of the distribution of $T_1^{*''}$ are given.

Remark 6.1 (**Failure rate of T^***)
Suppose $T = T_1$ is the discrete random time of the first failure of a MSS without PM and $h^T(t)$ is the corresponding failure rate. Let $T^* = T_1^*$ be the discrete random time of the first failure of a MSS with PM following the PM policy from Section 6.3. Then the corresponding failure rate is

$$h^{T^*}(t) = \begin{cases} 0 & \text{, if } t < 0 \\ h^T(t) & \text{, if } t \in [0, v + \tau - 1] \\ h^T(t - k\tau) & \text{, if } t \in [v + k\tau, v + (k+1)\tau - 1] \end{cases} \tag{6.9}$$

for $k = 1, 2, \ldots$.

Theorem 6.2 (**Distribution function of $T_1^{*''}$**)
Suppose $T_1^{*''}$ is the discrete random time of the first type 2 failure of a repairable MSS

with PM following PM policy that is described in Section 6.3. Then $T_1^{*''}$ has the following distribution function

$$F^{T_1^{*''}}(t) = P(T_1^{*''} \leq t) = \sum_{j=1}^{t} ph^{T^*}(j)\left(1 - P(S_{j-1}^* = 2)\right), \; t = 1, 2, \ldots, \quad (6.10)$$

where $(S_t^*)_{t\in\mathbb{N}}$ is the Markov chain from Section 6.2.

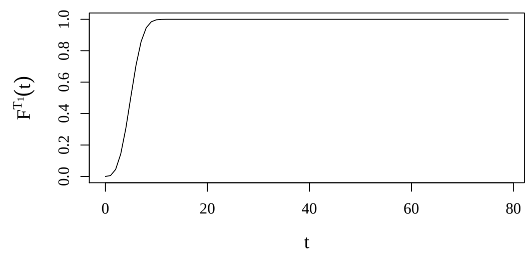
Proof. Suppose a MSS with PM that is done following PM policy from Section 6.3. Let $(S_t^*)_{t\in\mathbb{N}}$ be the Markov Chain from Section 6.2. Thus, the state space is $M = \{e_1, e_2, e_3\} = \{0, 1, 2\}$ and S_t^* is the random state of the MSS in t. For $t = 1, 2, \ldots$ it holds

$$\begin{aligned} P(T_1^{*''} \leq t) &= \sum_{j=1}^{t} P(T_1^{*''} = j) = \sum_{j=1}^{t} P(S_j^* = 2, S_{j-1}^* \neq 2) \\ &= \sum_{j=1}^{t} P(S_j^* = 2 | S_{j-1}^* \neq 2) P(S_{j-1}^* \neq 2) \\ &\overset{(6.8)}{=} \sum_{j=1}^{t} ph^{T^*}(j)\left(1 - P(S_{j-1}^* = 2)\right). \end{aligned} \quad (6.11)$$

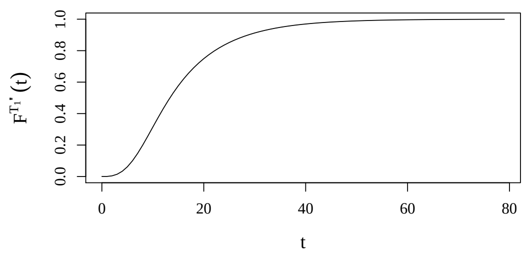
Note that the system starts without failure and therefore $P(T_1^{*''} = 0) = 0$. □

Figure 6.1 shows (a) the distribution function of the time to the first failure of a repairable MSS without PM and only one failure type, (b) the distribution function of the time to the first type 2 failure of a repairable MSS without PM and (c) the distribution function of the time to the first type 2 failure of a repairable MSS with PM. The comparison of (b) and (c) shows how PM actions reduce the probability of failures. Note that in Figure 6.1 the points are connected with lines for better visibility. Comparing Figure 6.1 with Figure 3.1, the values of the distribution functions in (a) are identical at the discrete points in time, since both Weibull distributions have the same parameterization. But in (b) and (c) the distribution functions in the continuous and the discrete case differ. Thus, in the discrete case it takes longer until the first type 2 failure occurs and therefore the mean cycle length in the discrete case is larger than in the continuous case. The reason for these deviations lies in the inequality of the failure rate functions in the discrete and the continuous case.
With the underlying PM policy the random cycle length that means the time between two replacements is

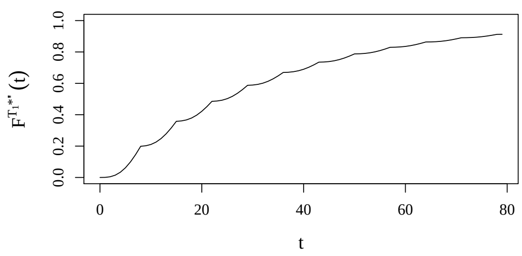
$$L_{v,\tau,N} = \min\{T_1^{*''}, v + N\tau\} \begin{cases} < v + N\tau & \text{, with } P(T_1^{*''} \leq v + N\tau - 1) \\ = v + N\tau & \text{, with } 1 - P(T_1^{*''} \leq v + N\tau - 1) \end{cases}. \quad (6.12)$$



(a) Distribution function of T_1



(b) Distribution function of $T_1^{''}$



(c) Distribution function of $T_1^{*''}$

Figure 6.1.: Distribution functions when T_1 is DWD($\beta = 0.0057, \gamma = 3$), $v = 2$, $\tau = 7$, $N = 11$ and $p = 0.1$.

Theorem 6.3 (Mean Cycle Length)
With the underlying PM policy the mean cycle length is

$$E(L_{v,\tau,N}) \quad = \quad \sum_{j=1}^{v+N\tau-1} jP(T_1^{*''} = j) + (v + N\tau)P(T_1^{*''} \geq v + N\tau). \qquad (6.13)$$

Proof. The random cycle length is a positive discrete random variable and has the following probability mass function

$$P(L_{v,\tau,N} = t) = \begin{cases} 0 & \text{, if } t < 0 \\ P(T_1^{*''} = t) & \text{, if } t \in \{0, \ldots, v + N\tau - 1\} \\ P(T_1^{*''} \geq t) & \text{, if } t = v + N\tau \\ 0 & \text{, if } t > v + N\tau \end{cases}. \qquad (6.14)$$

Using this, the mean cycle length is computed as follows

$$\begin{aligned} E(L_{v,\tau,N}) &= \sum_{j=0}^{\infty} j \cdot P(L_{v,\tau,N} = j) \\ &= \sum_{j=0}^{v+N\tau-1} j \cdot P(T_1^{*''} = j) + (v + N\tau)P(T_1^{*''} \geq v + N\tau) \end{aligned}$$

$$+\sum_{j=v+N\tau+1}^{\infty} j\cdot 0. \tag{6.15}$$

□

To compute a maintenance cost rate, it is necessary to compute the random number of minimal repairs during the replacement cycle and therefore the number of type 1 failures in a replacement cycle.

Theorem 6.4 (**Mean Number of Type 1 failures in a replacement cycle**) *Suppose* $(S_t^*)_{t\in\mathbb{N}}$ *is the Markov Chain from Section 6.2 and* S_t^* *is the random state of the MSS at* t*. Let* Z_t *be the random number of type 1 failures in the replacement cycle with length* $\min\{T_1^{*''}, t\}$ *that have to be removed through minimal repair. Then it holds*

$$E(Z_t) = \sum_{k=1}^{t-1} k \left(\sum_{\substack{j=k+1\\ s_1,\ldots,s_{j-1}\neq 2\\ \sum_{i=1}^{j-1} s_i = k}}^{t-1} P(S_1^* = s_1, \ldots, S_j^* = 2) + \sum_{\substack{s_1,\ldots,s_{t-1}\neq 2\\ \sum_{i=1}^{t-1} s_i = k}} P(S_1^* = s_1, \ldots, S_{t-1}^* = s_{t-1}) \right). \tag{6.16}$$

Proof. Suppose a MSS with PM that is done following PM policy from Section 6.3. Let $(S_t^*)_{t\in\mathbb{N}}$ be the Markov Chain from Section 6.2. Thus, the state space is $M = \{0, 1, 2\}$ and S_t^* is the random state of the MSS at t.
Let Z_t be the random number of type 1 failures of the MSS in a replacement cycle with length $\min\{T_1^{*''}, t\}$. Here we are interested in the number of type 1 failures that are removed through minimal repair. Since $\min\{T_1^{*''}, t\}$ is the length of the replacement cycle, the MSS is replaced preventively at time $t = v+N\tau$ if $T_1^{*''} \geq t$, no matter whether a failure occurs at t or not. Therefore, $t-1$ is the upper limit of the random number of type 1 failures and hence for the number of minimal repairs in the replacement cycle, i.e. $1 \leq Z_t \leq t-1$. For the expected number of minimal repairs in a replacement cycle with length $\min\{T_1^{*''}, t\}$ it holds

$$E(Z_t) = \sum_{k=1}^{t-1} kP(Z_t = k)$$

with

$$P(Z_t = k) = \sum_{\substack{j=k+1\\ s_1,\ldots,s_{j-1}\neq 2\\ \sum_{i=1}^{j-1} s_i = k}}^{t-1} P(S_1^* = s_1, \ldots, S_j^* = 2) + \sum_{\substack{s_1,\ldots,s_{t-1}\neq 2\\ \sum_{i=1}^{t-1} s_i = k}} P(S_1^* = s_1, \ldots, S_{t-1}^* = s_{t-1}).$$

□

The optimization criterion of interest will be the average maintenance costs per unit of time. In the underlying PM policy there are two scenario.
The first one is that there is no type 2 failure up to the preventive replacement at time $v + N\tau$. In this case the replacement cycle has length $v + N\tau$ and there are costs for minimal repairs during this cycle, for $N - 1$ PM actions and for a preventive replacement at $v + N\tau$. The probability of the first scenario is $P(T_1^{*''} \geq v + N\tau)$.
The second scenario is that a type 2 failure occurs before the preventive replacement takes place at $v + N\tau$. In this case one have costs for minimal repairs and costs for PM actions but both only until the expected time of the type 2 failure. Furthermore, one have the costs for replacement because the type 2 failure can only be removed through replacement. The probability of the second scenario is $P(T_1^{*''} < v + N\tau)$.

Definition 6.1 (Cost Optimization Problem)
Let c_M denotes the costs for a minimal repair, c_{PM} the costs of PM and c_R the costs of a replacement. The average maintenance costs per unit of time are

$$\begin{aligned} C(v,\tau,N) &= \frac{c_M E(Z_{v+N\tau}) + c_R}{E(L_{v,\tau,N})} \\ &+ P(T_1^{*''} \geq v + N\tau)\frac{(N-1)c_{PM}}{E(L_{v,\tau,N})} \\ &+ P(T_1^{*''} < v + N\tau)\frac{\sum_{k=1}^{N-1} c_{PM}\mathbf{1}_{\{v+k\tau < E(T_1^{*''}|T_1^{*''}<v+N\tau)\}}}{E(L_{v,\tau,N})}, \end{aligned} \tag{6.17}$$

where $E(T_1^{*''}|T_1^{*''} < v + N\tau)$ is the expected time of the first type 2 failure under the condition that a type 2 failure ends the replacement cycle and $\mathbf{1}_{\{\cdot\}}$ is the indicator function. In case of MSS with n states the optimization problem have the following form

$$\min_{v\in\{0,\dots,n-1\},\, \tau\in\{1,\dots,n-v\},\, N\in\{1,\dots,N_{\max}\}} C(v,\tau,N) \tag{6.18}$$

Note that the extreme case $\tau = 0$ is excluded from optimization problem (3.37). The value $\tau = 0$ would lead to a degree of repair of one, which can be interpreted as minimal PM actions. This is not reasonable in the maintenance model under consideration. However, the other extreme case of perfect PM actions, i.e. $v = 0$, is still part of the cost optimization problem.

6.5. Example for Cost Optimal Maintenance

In this section special cost functions for PM actions are considered and the optimal maintenance strategy for a MSS with $n = 20$ states is computed for different discrete lifetime distributions. In order to provide a comparison with the analogue continuous maintenance model from Chapter 3 the same cost functions for PM used in Chapter 3 are used here.
The main objective of this section is to compute the cost optimal values for v, τ and N. In the following computations the number of PM actions is restricted up to 10. This means $N_{\max} = 11$ and hence the MSS will be preventively replaced at the latest after

ten PM actions. The probability that a failure is of type 2 is assumed to be ten percent, i.e. $p = 0.1$. Since the cost optimization problem (6.18) could not be solved analytical, complete enumeration is used to find the cost optimal maintenance strategies with R. Note that in this section the parameters of all discrete lifetime distributions were identical to the parameters of the continuous lifetime distributions in the previous sections.

6.5.1. Costs Proportional to the Impact of Repair

Suppose the costs of PM actions depend only on the virtual age after PM, i.e.

$$c_{PM}(v) = c_I \left(\frac{1}{v}\right)^{\delta}, \tag{6.19}$$

where $v \geq 1$, $\delta > 0$ and $c_I > 0$ is a constant cost value. This cost function is described in detail in Subsection 2.7.1. Further, it is assumed that $c_R = c_I$. However, it is important to note that in the discrete case v can have only values equal or greater one. Therefore, contrary to the continuous case the costs of PM are not only bounded below but also above by c_I, i.e. $0 < c_{PM}(v) \leq c_I$.
In the following, the optimal values for v and τ and the optimal number of PM actions before a preventive replacement takes place are computed for different cost ratios c_M/c_I and different δ. The numerical results are given in Table 6.1. These results lead to the following conclusions:

1. The smaller δ, the more expensive are PM actions and the lower is the difference between the costs of PM and the costs of a replacement. Therefore, for small δ it is cost optimal to do no PM, i.e. $N = 1$ and with rising δ it becomes cost optimal to do more PM actions, i.e. the cost optimal N is rising. In comparison to the other distributions the DRMWD has sooner $N > 1$, because of the high failure rate at lower states.

2. The lower the costs of a repair with a high impact compared to the costs of a minimal repair, i.e. the higher the ratio c_M/c_I, the better are the PM actions, i.e. the cost optimal v is decreasing, and the lower is the distance between PM actions, i.e. the cost optimal τ is decreasing too. Therefore, it is favorable to do a good and more frequent PM instead of doing more minimal repairs.

3. The DLFRD and DRD lead nearly to the same optimal solutions. The same holds for the DWD and DMWD.

Note that if the cost optimal N is 1, the optimal maintenance strategy is to do no PM actions and to replace the system every $v + \tau$ time units if there has not already occurred a type 2 failure. Hence, the average maintenance costs from (6.17) reduce to $(c_M E(Z_{v+\tau}) + c_R)/E(L_{v,\tau,1})$. Thus, the optimal maintenance strategy is not unique because different combinations of v and τ that lead to the same sum $v + \tau$ also lead to the same average maintenance costs. Therefore, if the cost optimal N is 1, it is proper to show the optimal sum $v + \tau$ instead of the optimal single values for v and τ.

Table 6.1.: Optimal values in case of costs proportional to the impact of repair

	DLFRD	DRD	DWD	DMWD	DRMWD
	$\alpha = 0.01$ $\beta = 0.02944$	$\beta = 0.03142$	$\beta = 0.0057$ $\gamma = 3$	$\alpha = 0.03$ $\beta = 0.004335$ $\gamma = 3$	$\alpha = 0.1$ $\beta = 0.1746$ $\gamma = 0.1$
$c_M/c_I = 0.5$					
$\delta = 0.125$	$N = 1$ $v + \tau = 10$	$N = 1$ $v + \tau = 10$	$N = 1$ $v + \tau = 7$	$N = 1$ $v + \tau = 8$	$N = 2$ $v = 2$ $\tau = 14$
$\delta = 0.5$	$N = 2$ $v = 2$ $\tau = 9$	$N = 2$ $v = 2$ $\tau = 9$	$N = 1$ $v + \tau = 7$	$N = 1$ $v + \tau = 8$	$N = 9$ $v = 3$ $\tau = 12$
$\delta = 1$	$N = 11$ $v = 3$ $\tau = 6$	$N = 11$ $v = 3$ $\tau = 6$	$N = 9$ $v = 2$ $\tau = 5$	$N = 9$ $v = 2$ $\tau = 5$	$N = 11$ $v = 4$ $\tau = 8$
$\delta = 2$	$N = 11$ $v = 3$ $\tau = 4$	$N = 11$ $v = 3$ $\tau = 4$	$N = 11$ $v = 3$ $\tau = 3$	$N = 11$ $v = 3$ $\tau = 3$	$N = 11$ $v = 4$ $\tau = 6$
$c_M/c_I = 1$					
$\delta = 0.125$	$N = 1$ $v + \tau = 7$	$N = 1$ $v + \tau = 7$	$N = 1$ $v + \tau = 5$	$N = 1$ $v + \tau = 6$	$N = 2$ $v = 2$ $\tau = 11$
$\delta = 0.5$	$N = 1$ $v + \tau = 7$	$N = 1$ $v + \tau = 7$	$N = 1$ $v + \tau = 5$	$N = 1$ $v + \tau = 6$	$N = 11$ $v = 3$ $\tau = 9$
$\delta = 1$	$N = 2$ $v = 2$ $\tau = 5$	$N = 2$ $v = 2$ $\tau = 5$	$N = 1$ $v + \tau = 5$	$N = 11$ $v = 2$ $\tau = 4$	$N = 10$ $v = 4$ $\tau = 6$
$\delta = 2$	$N = 11$ $v = 2$ $\tau = 4$	$N = 11$ $v = 2$ $\tau = 3$	$N = 10$ $v = 2$ $\tau = 3$	$N = 11$ $v = 2$ $\tau = 3$	$N = 11$ $v = 4$ $\tau = 5$
$c_M/c_I = 2$					
$\delta = 0.125$	$N = 1$ $v + \tau = 5$	$N = 1$ $v + \tau = 5$	$N = 1$ $v + \tau = 4$	$N = 1$ $v + \tau = 5$	$N = 1$ $v + \tau = 10$
$\delta = 0.5$	$N = 1$ $v + \tau = 5$	$N = 1$ $v + \tau = 5$	$N = 1$ $v + \tau = 4$	$N = 1$ $v + \tau = 5$	$N = 10$ $v = 3$ $\tau = 7$
$\delta = 1$	$N = 1$ $v + \tau = 5$	$N = 1$ $v + \tau = 5$	$N = 1$ $v + \tau = 4$	$N = 1$ $v + \tau = 5$	$N = 10$ $v = 3$ $\tau = 6$
$\delta = 2$	$N = 11$ $v = 2$ $\tau = 2$	$N = 11$ $v = 2$ $\tau = 2$	$N = 11$ $v = 2$ $\tau = 2$	$N = 10$ $v = 2$ $\tau = 2$	$N = 11$ $v = 3$ $\tau = 5$

6.5.2. Costs Proportional to the State before Repair

Suppose the cost function c_{PM} depends on the state just before PM. It holds

$$c_{PM}(v,\tau) = c_S \left(\frac{1}{v+\tau}\right)^{\delta}, \tag{6.20}$$

where $\delta > 0$ and $c_S > 0$ is a constant cost value. This function is described in detail in Subsection 2.7.2. It is assumed that $c_R = c_S$.
The cost optimal maintenance strategies for different cost ratios c_M/c_S and different δ are given in Table 6.2 and lead to the following conclusions:

1. The smaller δ, the more expensive are PM actions. Therefore, with rising δ it becomes cost optimal to do more PM actions, i.e. the cost optimal N is rising.

2. The lower the costs of PM compared to the costs of a minimal repair, i.e. the higher the ratio c_M/c_S, the lower is the distance between PM actions, i.e. the cost optimal τ is decreasing. Therefore, it is favorable to do PM more often, instead of doing more minimal repairs.

3. Since the cost difference between good and less good PM actions is comparatively small, it is cost optimal to do perfect PM actions, i.e. $v = 0$. The same is true also for the DRMWD, since the failure rate is zero at time zero.

4. The DLFRD and DRD lead nearly to the same optimal solutions. The same holds for the DWD and DMWD.

6.5.3. Costs Proportional to the Degree of Repair - 1

Analogue to the continuous case in Subsection 3.5.3 the following cost function is used for PM actions

$$c_{PM}(v,\tau) = c_R \left(1 - \xi(v,\tau)^{\delta}\right), \tag{6.21}$$

where $\xi(v,\tau)$ is the degree of repair (see (3.40)), $\delta > 0$ and $c_R > 0$ are the costs of replacement. This cost function is described in detail in Subsection 2.7.3. The cost optimal maintenance strategies for different cost ratios c_M/c_R and different δ are given in Table 6.3 and lead to the following conclusions:

1. The higher δ, the more expensive are the PM actions. Therefore, with rising δ it becomes cost optimal to do less PM, i.e. N is decreasing.

2. The lower the costs of a replacement compared to the costs of a minimal repair, i.e. the higher the ratio c_M/c_R, the less expensive are the costs for PM compared to the costs of a minimal repair. Therefore, the distance between PM actions becomes smaller, i.e. τ is decreasing.

3. The cost optimal values of v for the DRMWD are higher than for the other distributions because of the higher failure rate at lower states. The cost optimal values of τ for the DRMWD are also higher than for the other distributions. The reason for this is that the failure rate for this distribution remains for a while at a relatively low level before it starts increasing again.

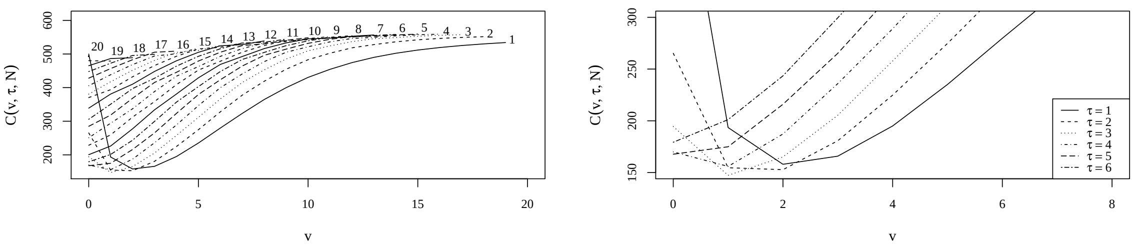
Table 6.2.: Optimal values in case of costs proportional to the state before repair

	DLFRD	DRD	DWD	DMWD	DRMWD
	$\alpha = 0.01$ $\beta = 0.02944$	$\beta = 0.03142$	$\beta = 0.0057$ $\gamma = 3$	$\alpha = 0.03$ $\beta = 0.004335$ $\gamma = 3$	$\alpha = 0.1$ $\beta = 0.1746$ $\gamma = 0.1$
$c_M/c_S = 0.5$					
$\delta = 0.125$	$N = 7$ $v = 0$ $\tau = 9$	$N = 11$ $v = 0$ $\tau = 9$	$N = 6$ $v = 0$ $\tau = 7$	$N = 11$ $v = 0$ $\tau = 7$	$N = 11$ $v = 0$ $\tau = 16$
$\delta = 0.5$	$N = 9$ $v = 0$ $\tau = 7$	$N = 9$ $v = 0$ $\tau = 7$	$N = 10$ $v = 0$ $\tau = 6$	$N = 10$ $v = 0$ $\tau = 6$	$N = 11$ $v = 0$ $\tau = 11$
$\delta = 1$	$N = 11$ $v = 0$ $\tau = 6$	$N = 11$ $v = 0$ $\tau = 6$	$N = 11$ $v = 0$ $\tau = 5$	$N = 11$ $v = 0$ $\tau = 5$	$N = 11$ $v = 0$ $\tau = 11$
$\delta = 2$	$N = 11$ $v = 0$ $\tau = 4$	$N = 11$ $v = 0$ $\tau = 4$	$N = 11$ $v = 0$ $\tau = 4$	$N = 11$ $v = 0$ $\tau = 4$	$N = 11$ $v = 0$ $\tau = 8$
$c_M/c_S = 1$					
$\delta = 0.125$	$N = 11$ $v = 0$ $\tau = 6$	$N = 11$ $v = 0$ $\tau = 6$	$N = 10$ $v = 0$ $\tau = 5$	$N = 8$ $v = 0$ $\tau = 6$	$N = 11$ $v = 0$ $\tau = 11$
$\delta = 0.5$	$N = 11$ $v = 0$ $\tau = 6$	$N = 11$ $v = 0$ $\tau = 6$	$N = 10$ $v = 0$ $\tau = 5$	$N = 10$ $v = 0$ $\tau = 5$	$N = 11$ $v = 0$ $\tau = 11$
$\delta = 1$	$N = 11$ $v = 0$ $\tau = 5$	$N = 11$ $v = 0$ $\tau = 4$	$N = 11$ $v = 0$ $\tau = 4$	$N = 11$ $v = 0$ $\tau = 5$	$N = 10$ $v = 0$ $\tau = 9$
$\delta = 2$	$N = 11$ $v = 0$ $\tau = 4$	$N = 11$ $v = 0$ $\tau = 4$	$N = 11$ $v = 0$ $\tau = 4$	$N = 11$ $v = 0$ $\tau = 4$	$N = 11$ $v = 0$ $\tau = 7$
$c_M/c_S = 2$					
$\delta = 0.125$	$N = 8$ $v = 0$ $\tau = 5$	$N = 10$ $v = 0$ $\tau = 4$	$N = 11$ $v = 0$ $\tau = 4$	$N = 10$ $v = 0$ $\tau = 5$	$N = 10$ $v = 0$ $\tau = 9$
$\delta = 0.5$	$N = 10$ $v = 0$ $\tau = 4$	$N = 10$ $v = 0$ $\tau = 4$	$N = 11$ $v = 0$ $\tau = 4$	$N = 10$ $v = 0$ $\tau = 4$	$N = 10$ $v = 0$ $\tau = 9$
$\delta = 1$	$N = 11$ $v = 0$ $\tau = 4$	$N = 11$ $v = 0$ $\tau = 4$	$N = 11$ $v = 0$ $\tau = 4$	$N = 11$ $v = 0$ $\tau = 4$	$N = 11$ $v = 0$ $\tau = 8$
$\delta = 2$	$N = 11$ $v = 0$ $\tau = 3$	$N = 11$ $v = 0$ $\tau = 3$	$N = 11$ $v = 0$ $\tau = 3$	$N = 11$ $v = 0$ $\tau = 3$	$N = 11$ $v = 0$ $\tau = 6$

Table 6.3.: Optimal values in case of costs proportional to the degree of repair - 1

	DLFRD $\alpha = 0.01$ $\beta = 0.02944$	DRD $\beta = 0.03142$	DWD $\beta = 0.0057$ $\gamma = 3$	DMWD $\alpha = 0.03$ $\beta = 0.004335$ $\gamma = 3$	DRMWD $\alpha = 0.1$ $\beta = 0.1746$ $\gamma = 0.1$
$c_M/c_R = 0.5$					
$\delta = 0.125$	$N = 11$ $v = 1$ $\tau = 4$	$N = 11$ $v = 1$ $\tau = 3$	$N = 11$ $v = 1$ $\tau = 3$	$N = 11$ $v = 1$ $\tau = 3$	$N = 10$ $v = 3$ $\tau = 7$
$\delta = 0.5$	$N = 10$ $v = 1$ $\tau = 7$	$N = 10$ $v = 1$ $\tau = 7$	$N = 11$ $v = 1$ $\tau = 5$	$N = 11$ $v = 2$ $\tau = 4$	$N = 8$ $v = 3$ $\tau = 10$
$\delta = 1$	$N = 10$ $v = 0$ $\tau = 12$	$N = 11$ $v = 0$ $\tau = 12$	$N = 6$ $v = 0$ $\tau = 7$	$N = 4$ $v = 0$ $\tau = 8$	$N = 2$ $v = 0$ $\tau = 16$
$c_M/c_R = 1$					
$\delta = 0.125$	$N = 11$ $v = 1$ $\tau = 2$	$N = 11$ $v = 1$ $\tau = 2$	$N = 11$ $v = 1$ $\tau = 2$	$N = 11$ $v = 1$ $\tau = 3$	$N = 11$ $v = 3$ $\tau = 5$
$\delta = 0.5$	$N = 11$ $v = 1$ $\tau = 5$	$N = 10$ $v = 1$ $\tau = 4$	$N = 10$ $v = 1$ $\tau = 3$	$N = 10$ $v = 1$ $\tau = 4$	$N = 10$ $v = 3$ $\tau = 7$
$\delta = 1$	$N = 4$ $v = 0$ $\tau = 7$	$N = 4$ $v = 0$ $\tau = 7$	$N = 10$ $v = 0$ $\tau = 5$	$N = 6$ $v = 0$ $\tau = 6$	$N = 2$ $v = 0$ $\tau = 13$
$c_M/c_R = 2$					
$\delta = 0.125$	$N = 11$ $v = 1$ $\tau = 2$	$N = 11$ $v = 1$ $\tau = 1$	$N = 11$ $v = 1$ $\tau = 2$	$N = 11$ $v = 1$ $\tau = 2$	$N = 11$ $v = 3$ $\tau = 4$
$\delta = 0.5$	$N = 10$ $v = 1$ $\tau = 3$	$N = 10$ $v = 1$ $\tau = 2$	$N = 10$ $v = 1$ $\tau = 3$	$N = 10$ $v = 1$ $\tau = 3$	$N = 10$ $v = 3$ $\tau = 5$
$\delta = 1$	$N = 6$ $v = 0$ $\tau = 5$	$N = 6$ $v = 0$ $\tau = 5$	$N = 1$ $v + \tau = 4$	$N = 8$ $v = 0$ $\tau = 5$	$N = 2$ $v = 0$ $\tau = 10$

Figure 6.2 shows the dependency of the average maintenance costs on v and τ for the DMWD($\alpha = 0.03, \beta = 0.004335, \gamma = 3$), the cost ratio $c_M/c_R = 2$, $p = 0.1$, $N = 10$ and $\delta = 0.5$ using cost function (6.21). The points are connected with lines for a better visibility. It can be seen that the cost function has a unique minimum for a fixed value of N.



(a) For all τ

(b) For $\tau = 1, \ldots, 6$

Figure 6.2.: Average maintenance costs for the DMWD($\alpha = 0.03, \beta = 0.004335, \gamma = 3$) if $N = 10$, $c_R = 500$, $c_M = 1000$, $p = 0.1$ and $\delta = 0.5$.

6.5.4. Costs Proportional to the Degree of Repair - 2

Like in the previous subsection it is assumed that the costs for a PM action are proportional to the degree of repair $\xi(v, \tau)$, but here the following cost function is used

$$c_{PM}(v, \tau) = c_R \left(1 - \xi(v, \tau) \exp\left(\xi(v, \tau) - 1\right)\right)^{\delta}, \tag{6.22}$$

where $\delta > 0$ and $c_R > 0$ are the costs for a replacement. This cost function is described in detail in Subsection 2.7.4. The cost optimal maintenance strategies are given in table 6.4 and lead to the following conclusions:

1. The higher δ, the less expensive are PM actions. Therefore, with rising δ it becomes cost optimal to do more PM actions, i.e. N is increasing.

2. The lower the costs of a renewal compared to the costs of a minimal repair, i.e. the higher the ratio c_M/c_R, the less expensive are the costs for good PM and the better are the PM actions, i.e. v is decreasing.

3. The cost optimal values of v for the DRMWD are higher than for the other distributions. The reason for this is that the DRMWD has a higher failure rate at lower states. The cost optimal values of τ for the DRMWD are also higher than for the other distributions. This is also caused by the failure rate of the DRMWD which remains for a while at a relatively low level.

Table 6.4.: Optimal values in case of costs proportional to the degree of repair - 2

	DLFRD $\alpha = 0.01$ $\beta = 0.02944$	DRD $\beta = 0.03142$	DWD $\beta = 0.0057$ $\gamma = 3$	DMWD $\alpha = 0.03$ $\beta = 0.004335$ $\gamma = 3$	DRMWD $\alpha = 0.1$ $\beta = 0.1746$ $\gamma = 0.1$
$c_M/c_R = 0.5$					
$\delta = 0.5$	$N = 1$ $v + \tau = 10$	$N = 1$ $v + \tau = 10$	$N = 1$ $v + \tau = 7$	$N = 1$ $v + \tau = 8$	$N = 11$ $v = 0$ $\tau = 17$
$\delta = 1$	$N = 1$ $v + \tau = 10$	$N = 1$ $v + \tau = 10$	$N = 1$ $v + \tau = 7$	$N = 1$ $v + \tau = 8$	$N = 11$ $v = 0$ $\tau = 17$
$\delta = 2$	$N = 1$ $v + \tau = 10$	$N = 1$ $v + \tau = 10$	$N = 1$ $v + \tau = 7$	$N = 1$ $v + \tau = 8$	$N = 11$ $v = 10$ $\tau = 1$
$\delta = 3$	$N = 11$ $v = 5$ $\tau = 1$	$N = 11$ $v = 4$ $\tau = 1$	$N = 11$ $v = 3$ $\tau = 1$	$N = 11$ $v = 4$ $\tau = 1$	$N = 11$ $v = 8$ $\tau = 2$
$c_M/c_R = 1$					
$\delta = 0.5$	$N = 1$ $v + \tau = 7$	$N = 1$ $v + \tau = 7$	$N = 1$ $v + \tau = 5$	$N = 1$ $v + \tau = 6$	$N = 1$ $v + \tau = 13$
$\delta = 1$	$N = 1$ $v + \tau = 7$	$N = 1$ $v + \tau = 7$	$N = 1$ $v + \tau = 5$	$N = 1$ $v + \tau = 6$	$N = 1$ $v + \tau = 13$
$\delta = 2$	$N = 1$ $v + \tau = 7$	$N = 1$ $v + \tau = 7$	$N = 1$ $v + \tau = 5$	$N = 1$ $v + \tau = 6$	$N = 11$ $v = 8$ $\tau = 1$
$\delta = 3$	$N = 11$ $v = 3$ $\tau = 1$	$N = 11$ $v = 3$ $\tau = 1$	$N = 11$ $v = 3$ $\tau = 1$	$N = 11$ $v = 3$ $\tau = 1$	$N = 11$ $v = 7$ $\tau = 1$
$c_M/c_R = 2$					
$\delta = 0.5$	$N = 1$ $v + \tau = 5$	$N = 1$ $v + \tau = 5$	$N = 1$ $v + \tau = 4$	$N = 1$ $v + \tau = 5$	$N = 1$ $v + \tau = 10$
$\delta = 1$	$N = 1$ $v + \tau = 5$	$N = 1$ $v + \tau = 5$	$N = 1$ $v + \tau = 4$	$N = 1$ $v + \tau = 5$	$N = 1$ $v + \tau = 10$
$\delta = 2$	$N = 1$ $v + \tau = 5$	$N = 1$ $v + \tau = 5$	$N = 1$ $v + \tau = 4$	$N = 1$ $v + \tau = 5$	$N = 11$ $v = 7$ $\tau = 1$
$\delta = 3$	$N = 11$ $v = 2$ $\tau = 1$	$N = 11$ $v = 2$ $\tau = 1$	$N = 11$ $v = 2$ $\tau = 1$	$N = 11$ $v = 2$ $\tau = 1$	$N = 11$ $v = 5$ $\tau = 1$

7. System with one Failure Type and Imperfect PM and CM

The maintenance model in this chapter is the discrete analogue of the model in Chapter 4. Like in the previous chapter a repairable multi-state system (MSS) with n states is used and it is assumed that at time 1 the system is in state one, at time 2 the system is in state two and so on (as described in Kahle [22]).
The maintenance model of this chapter includes both imperfect preventive and imperfect corrective maintenance actions. Here a sequential failure limit PM policy with infinite planning horizon is used to formulate a cost optimization problem.
This chapter is structured as follows. Section 7.1 and Section 7.2 contain essential assumptions and definitions that are needed to formulate the cost optimization problem in Section 7.3. Finally, in Section 7.4 different cost functions for PM and CM and different discrete lifetime distributions are used and the optimal maintenance strategy is computed.

7.1. Modeling the System

According to Kahle [22], we consider a repairable multi-state system (MSS). This system has the following properties.

1. Initially a new repairable MSS is installed. The MSS has n states in which the system can fail. A time scale is introduced so that the system at time 1 is in state one, at time 2 in state two and so on.

2. The system has only one failure type which can be removed through imperfect repair actions.

3. The repair times are negligible small.

7.2. Maintenance Policy

The maintenance strategy described here is designed for an infinite time horizon. The following assumptions are made.

1. All failures that occurred after installation during the time interval $(0, v]$ are removed through minimal repair.

2. If there is a failure during the time interval $(v, v+\tau)$ a CM action is carried out. Otherwise a PM action at time $v+\tau$ will be carried out.

3. If there is no failure during the pre-defined time interval of length $\tau > 0$ after a maintenance action, a PM will be carried out. For τ it holds $\tau \in \{1, \ldots, n-v\}$.

4. If a failure occurs during the time interval of the length $\tau > 0$ after a maintenance action, a CM is carried out.

5. The virtual age of the system after both PM actions and CM actions is always $v \geq 0$ and $v \in \{0, \ldots, n-1\}$. Since PM actions can be planned, they are assumed to be more cost effective than unplanned CM actions.

6. Suppose $c_1, c_2, \ldots$ are the realizations of the general maintenance times. In terms of Kijima type II model the degree of the kth repair is

$$\xi_k(v, c_k, c_{k-1}) = \frac{v}{v + c_k - c_{k-1}}, \quad \forall k \geq 1. \tag{7.1}$$

This maintenance policy is a sequential failure limit policy (see [17, p. 765]) because an alternative formulation of Assumption 2 might be: A PM is performed when the failure intensity reaches the predetermined level $\lambda^{N^*}(v + \tau)$.

7.3. Cost Optimization Problem

Consider a technical system that is maintained with maintenance policy described in Section 7.2. The aim of this section is to formulate a cost optimization problem. The optimization criterion are the average maintenance costs per unit time. For this purpose, the expected maintenance costs per cycle are set in relation to the mean cycle length. Here the cycle length is the time between two maintenance actions and for reasons of simplification, the time between the startup of the system and the age of v is excluded from the modeling of the cost optimization problem.
Suppose $N^* = (N_t^*)_{t \geq 0}$ is the failure counting process, i.e. N_t^* is the random number of failures of a repairable system with PM in the interval $[0, t]$.

Lemma 7.1 (**Intensity Function of** $N^* = (N_t^*)_{t \geq 0}$)
Suppose $c_1, c_2, \ldots$ are realizations of the general maintenance times. The intensity function of the counting process $N^* = (N_t^*)_{t \geq 0}$ is then

$$\lambda^{N^*}(t) \;=\; \begin{cases} 0 & , \textit{ if } t < 0 \\ h^{T_1}(t) & , \textit{ if } t \in [0, c_1) \\ h^{T_1}(v + t - c_k) & , \textit{ if } t \in [c_k, c_{k+1}),\ k \geq 1, \end{cases} \tag{7.2}$$

where $h^{T_1}(\cdot)$ is the hazard function of the time to first failure of a new system.

For the computation of the expected maintenance costs per cycle it is necessary to compute the probability that a failure occurs within τ time units after a maintenance action.

Lemma 7.2 (**Distribution function of** T^v)
Let T_1 be the discrete random time of the first failure of a repairable system without

maintenance. Suppose T^v is the remaining discrete lifetime of the system after a maintenance action that reduces the virtual age of the system to v. Then T^v is a truncated discrete random variable with the following cumulative distribution function

$$F^{T^v}(t) = P(T^v \leq t) = \frac{F^{T_1}(v+t) - F^{T_1}(v)}{1 - F^{T_1}(v)}, \forall t = 0, 1, 2, \ldots. \tag{7.3}$$

Suppose the random cycle length $L_{v,\tau}$ is the random time between two maintenance actions. Therefore, it is either the time between two PM actions or the time between CM and PM actions. It holds

$$L_{v,\tau} \quad = \quad \min\{T^v, \tau\} \begin{cases} < \tau & \text{, with } P(T^v < \tau) = P(T_1 < v + \tau | T_1 > v) \\ = \tau & \text{, with } P(T^v \geq \tau) = 1 - P(T_1 < v + \tau | T_1 > v) \end{cases}, \tag{7.4}$$

for $\tau \in \{1, \ldots, n - v\}$. For the cumulative distribution function of $L_{v,\tau}$ it holds

$$F^{L_{v,\tau}}(t) = P(L_{v,\tau} \leq t) \quad = \quad \begin{cases} 0 & \text{, if } t < 0 \\ F^{T_1}(v + t | T_1 > v) & \text{, if } t \in \{0, \ldots, \tau - 1\} \\ 1 & \text{, if } t \geq \tau \end{cases}. \tag{7.5}$$

Theorem 7.3 (Mean Cycle Length)
For the expected cycle length it holds

$$E(L_{v,\tau}) \quad = \quad \sum_{j=0}^{\tau-1} j \cdot P(T_1 = v + j | T_1 > v) + \tau \cdot P(T_1 \geq v + \tau | T_1 > v). \tag{7.6}$$

Proof. The random cycle length is a positive discrete random variable with the following probability mass function

$$P(L_{v,\tau} = t) = \begin{cases} 0 & \text{, if } t < 0 \\ P(T_1 = v + t | T_1 > v) & \text{, if } t \in \{0, \ldots, \tau - 1\} \\ P(T_1 \geq v + t | T_1 > v) & \text{, if } t = \tau \\ 0 & \text{, if } t > \tau \end{cases}. \tag{7.7}$$

Using this, the mean cycle length is computed as follows

$$\begin{aligned} E(L_{v,\tau}) &= \sum_{j=0}^{\infty} j \cdot P(L_{v,\tau} = j) \\ &= \sum_{j=0}^{\tau-1} j \cdot P(L_{v,\tau} = j) + \tau \cdot P(L_{v,\tau} = \tau) \\ &= \sum_{j=0}^{\tau-1} j \cdot P(T_1 = v + j | T_1 > v) + \tau \cdot P(T_1 \geq v + \tau | T_1 > v). \end{aligned} \tag{7.8}$$

□

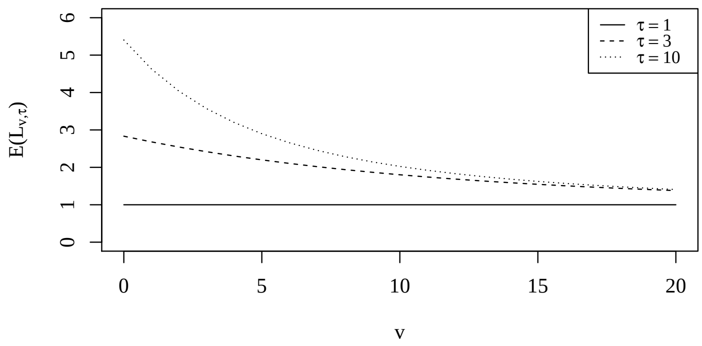


Figure 7.1.: Mean cycle length for the DLFRD($\alpha = 0.01, \beta = 0.02944$).

In Figure 7.1 the mean cycle length is plotted for the DLFRD. Note that for better visibility the points are connected with lines. The mean cycle length in the discrete case is bounded from below by one and for $v = 0$ the mean cycle length is bounded from above by $\min\{\tau, E(T_1)\}$. Note that since $P(T_1 = 0) = 0$, the expected time to first failure of a new system is always greater or equal one, i.e. $E(T_1) \geq 1$. For the same parameterization of the lifetime distribution in the continuous and the discrete case the mean cycle length in the discrete case is always greater or equal the mean cycle length in the continuous case.

Definition 7.1 (**Cost Optimization Problem**)
Let c_{CM} denotes the costs of a CM action and c_{PM} the costs of a PM action. The average maintenance costs per unit time are

$$C(v, \tau) \quad = \quad \frac{c_{CM} P(T^v < \tau) + c_{PM} P(T^v \geq \tau)}{E(L_{v,\tau})}. \tag{7.9}$$

The optimization problem then have the following form

$$\min_{v \in \{0,\dots,n-1\},\, \tau \in \{1,\dots,n-v\}} C(v, \tau). \tag{7.10}$$

Note that if $\tau = 1$, there will be no CM actions, i.e. $P(T^v < 1) = 0$, and $E(L_{v,1}) = 1$. Thus, for the average maintenance costs per unit time it holds $C(v, 1) = c_{PM}$.

7.4. Example for Cost Optimal Maintenance

The costs for CM and PM actions in optimization problem (7.10) are yet unspecified. In this section special cost functions for CM and PM actions are considered and cost optimal parameter v and τ are computed for different discrete lifetime distributions using R and complete enumeration. In order to provide a comparison with the analogue continuous maintenance model from Chapter 4 the same cost functions used in Chapter 4 are used here and the parameterization of the discrete lifetime distributions is identical to the continuous case.

7.4.1. Costs Proportional to the Impact of Repair

In this section it is assumed, that the costs for maintenance actions depends only on the virtual age after repair, i.e. the cost function for PM is

$$c_{PM}(v) = c_I \left(\frac{1}{v}\right)^{\delta}, \tag{7.11}$$

and the cost function for CM is

$$c_{CM}(v) = c_F + c_I \left(\frac{1}{v}\right)^{\delta}, \tag{7.12}$$

where $v > 0$, $\delta > 0$, $c_I > 0$ is a constant cost value and $c_F > 0$ is the fixed amount, by which the costs for CM are higher than for PM. This cost function is described in detail in Subsection 2.7.1. Note that in case of costs proportional to the impact of repair the extreme case of perfect repair, i.e. $v = 0$, have to be excluded from optimization problem (7.10).
The cost optimal maintenance strategies for different cost ratios c_F/c_I and different δ are given in Table 7.1 and lead to the following conclusions:

1. Only for $\delta = 0.125$ it is cost optimal to do very good maintenance actions with $v = 1$. For higher δ these maintenance actions are too expensive compared to worse maintenance actions. Therefore, with rising δ it is cost optimal to have higher values of v.

2. The higher δ the greater is the cost difference between good and less good maintenance actions and the faster the costs of PM tend to zero and the costs of CM tend to c_F. Therefore, for high values of δ it is cost optimal to do nonstop bad PM actions.

3. For lower values of δ the maintenance costs are relatively high and the cost difference between good and less good maintenance actions is relatively small. Therefore, the higher the amount, by which the costs of CM are higher than for PM, i.e. the higher the ratio c_F/c_I, the better are the maintenance actions, i.e. v is decreasing.

4. If the cost optimal values of v for the other distributions are at a low level, the DRMWD has higher cost optimal values of v. The reason for this lies in the high failure rate at lower states for the DRMWD.

7.4.2. Costs Proportional to the State before Repair

Assume the cost function c_{PM} depends on the state just before PM. It holds

$$c_{PM}(v, \tau) = c_S \left(\frac{1}{v+\tau}\right)^{\delta}, \tag{7.13}$$

Table 7.1.: Optimal values in case of costs proportional to the impact of repair

	DLFRD	DRD	DWD	DMWD	DRMWD
	$\alpha = 0.01$ $\beta = 0.02944$	$\beta = 0.03142$	$\beta = 0.0057$ $\gamma = 3$	$\alpha = 0.03$ $\beta = 0.004335$ $\gamma = 3$	$\alpha = 0.1$ $\beta = 0.1746$ $\gamma = 0.1$
$c_F/c_I = 0.05$					
$\delta = 0.125$	$v = 1$ $\tau = 19$	$v = 1$ $\tau = 19$	$v = 1$ $\tau = 10$	$v = 1$ $\tau = 11$	$v = 3$ $\tau = 17$
$\delta = 0.5$	$v = 5$ $\tau = 15$	$v = 4$ $\tau = 16$	$v = 2$ $\tau = 9$	$v = 3$ $\tau = 9$	$v = 7$ $\tau = 13$
$\delta = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 10$ $\tau = 9$
$\delta = 2$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$
$c_F/c_I = 0.1$					
$\delta = 0.125$	$v = 1$ $\tau = 19$	$v = 1$ $\tau = 19$	$v = 1$ $\tau = 8$	$v = 1$ $\tau = 10$	$v = 2$ $\tau = 18$
$\delta = 0.5$	$v = 4$ $\tau = 15$	$v = 4$ $\tau = 14$	$v = 2$ $\tau = 7$	$v = 2$ $\tau = 8$	$v = 6$ $\tau = 14$
$\delta = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$
$\delta = 2$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$
$c_F/c_I = 0.2$					
$\delta = 0.125$	$v = 1$ $\tau = 14$	$v = 1$ $\tau = 13$	$v = 1$ $\tau = 7$	$v = 1$ $\tau = 8$	$v = 2$ $\tau = 18$
$\delta = 0.5$	$v = 3$ $\tau = 10$	$v = 3$ $\tau = 10$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 5$ $\tau = 13$
$\delta = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$
$\delta = 2$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$

where $\delta > 0$ and $c_S > 0$ is a constant cost value. This cost function is described in Subsection 2.7.2. The following cost function is used for CM actions

$$c_{CM}(v,\tau) = \sum_{t=1}^{\tau-1}\left(c_F + c_S\left(\frac{1}{v+t}\right)^{\delta}\right)P(T^v = t|T^v < \tau), \quad (7.14)$$

where $c_F > 0$ is the fixed amount, by which the costs for CM are higher than for PM. The cost optimal maintenance strategies are given in Table 7.2 and lead to the following conclusions:

1. Since the cost difference between good and less good PM actions is comparatively small for lower values of δ (see e.g. Figure 2.7 (b)), for the DLFRD, DRD, DWD and DMWD it is cost optimal to do mostly perfect PM and CM actions, i.e. $v = 0$. The same is not true for the RMWD because of the higher failure rate at lower states.

2. The higher the amount by which the costs of CM are higher than for PM, i.e. the higher the ratio c_F/c_S, the cost optimal interval length until a PM is done becomes shorter, i.e. the cost optimal τ is decreasing, whereas the cost optimal v remains fairly constant. Therefore, it becomes more likely that no failure occurs in the time interval of length τ.

3. The higher δ, the less expensive are maintenance actions. Therefore, for high values of δ it is cost optimal to do nonstop PM actions and no CM actions, i.e. the cost optimal τ is one. Note that if $\tau = 1$ it holds that $C(v,1) = c_{PM}(v,1)$. Since $c_{PM}(v,1)$ is a decreasing function of v, the corresponding cost optimal value of v, if $\tau = 1$, is the highest possible value of v, i.e. $v = 19$.

7.4.3. Costs Proportional to the Degree of Repair - 1

In this subsection it is assumed, that maintenance costs are proportional to the degree of repair. Analogue to the continuous maintenance model from Subsection 4.4.3 the following cost function is used for PM actions

$$c_{PM}(v,\tau) = c_R\left(1-\left(\frac{v}{v+\tau}\right)^{\delta}\right), \quad (7.15)$$

where $\delta > 0$ and $c_R > 0$ are the costs of replacement. This function is described in detail in Subsection 2.7.3. In the discrete case the cost function used for CM actions is

$$c_{CM}(v,\tau) = \sum_{t=1}^{\tau-1}\left(c_F + c_R\left(1-\left(\frac{v}{v+t}\right)^{\delta}\right)\right)P(T^v = t|T^v < \tau), \quad (7.16)$$

where $c_F > 0$ is the fixed amount by which the costs of CM are higher than for PM. In the following the optimal values for v and τ are computed for different cost ratios c_F/c_R and different δ. The numerical results are given in Table 7.3 and lead to the following conclusion:

Table 7.2.: Optimal values in case of costs proportional to the state before repair

	DLFRD	DRD	DWD	DMWD	DRMWD
	$\alpha = 0.01$ $\beta = 0.02944$	$\beta = 0.03142$	$\beta = 0.0057$ $\gamma = 3$	$\alpha = 0.03$ $\beta = 0.004335$ $\gamma = 3$	$\alpha = 0.1$ $\beta = 0.1746$ $\gamma = 0.1$
$c_F/c_S = 0.05$					
$\delta = 0.125$	$v = 0$ $\tau = 20$	$v = 0$ $\tau = 20$	$v = 0$ $\tau = 10$	$v = 0$ $\tau = 11$	$v = 1$ $\tau = 19$
$\delta = 0.5$	$v = 0$ $\tau = 20$	$v = 0$ $\tau = 19$	$v = 0$ $\tau = 9$	$v = 0$ $\tau = 10$	$v = 3$ $\tau = 17$
$\delta = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 0$ $\tau = 8$	$v = 19$ $\tau = 1$	$v = 5$ $\tau = 13$
$\delta = 2$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$
$c_F/c_R = 0.1$					
$\delta = 0.125$	$v = 0$ $\tau = 18$	$v = 0$ $\tau = 17$	$v = 0$ $\tau = 9$	$v = 0$ $\tau = 10$	$v = 1$ $\tau = 19$
$\delta = 0.5$	$v = 0$ $\tau = 14$	$v = 0$ $\tau = 13$	$v = 0$ $\tau = 8$	$v = 0$ $\tau = 9$	$v = 3$ $\tau = 17$
$\delta = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 4$ $\tau = 11$
$\delta = 2$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$
$c_F/c_R = 0.2$					
$\delta = 0.125$	$v = 0$ $\tau = 12$	$v = 0$ $\tau = 12$	$v = 0$ $\tau = 7$	$v = 0$ $\tau = 8$	$v = 1$ $\tau = 18$
$\delta = 0.5$	$v = 0$ $\tau = 10$	$v = 0$ $\tau = 9$	$v = 0$ $\tau = 7$	$v = 0$ $\tau = 7$	$v = 3$ $\tau = 13$
$\delta = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$
$\delta = 2$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$

Table 7.3.: Optimal values in case of costs proportional to the degree of repair - 1

	DLFRD	DRD	DWD	DMWD	DRMWD
	$\alpha = 0.01$ $\beta = 0.02944$	$\beta = 0.03142$	$\beta = 0.0057$ $\gamma = 3$	$\alpha = 0.03$ $\beta = 0.004335$ $\gamma = 3$	$\alpha = 0.1$ $\beta = 0.1746$ $\gamma = 0.1$
$c_F/c_R = 0.05$					
$\delta = 0.5$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$
$\delta = 2$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$
$\delta = 4$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 1$ $\tau = 19$
$\delta = 6$	$v = 0$ $\tau = 20$	$v = 0$ $\tau = 20$	$v = 0$ $\tau = 11$	$v = 0$ $\tau = 12$	$v = 1$ $\tau = 19$
$c_F/c_R = 0.1$					
$\delta = 0.5$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$
$\delta = 2$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$
$\delta = 4$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 1$ $\tau = 19$
$\delta = 6$	$v = 0$ $\tau = 19$	$v = 0$ $\tau = 18$	$v = 0$ $\tau = 9$	$v = 0$ $\tau = 10$	$v = 1$ $\tau = 19$
$c_F/c_R = 0.2$					
$\delta = 0.5$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$
$\delta = 2$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$
$\delta = 4$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$
$\delta = 6$	$v = 0$ $\tau = 13$	$v = 0$ $\tau = 13$	$v = 0$ $\tau = 8$	$v = 0$ $\tau = 8$	$v = 1$ $\tau = 19$

1. The lower δ, the lower are the maintenance costs and the faster the costs of PM tend to zero and the costs of CM tend to c_F. Therefore, it is cost optimal to do nonstop PM actions and no CM actions, i.e. the cost optimal τ is one, and the corresponding cost optimal value of v is the highest possible value of v, i.e. $v = 19$.

2. The higher δ, the higher are the maintenance costs and the smaller is the cost difference between good and less good maintenance actions (see e.g. Figure 2.7 (c)). Therefore, for the DLFRD, DRD, DWD and DMWD it is cost optimal to do perfect PM and CM actions, i.e. $v = 0$. The same is not true for the RMWD because of the higher failure rate at lower states.

3. The higher the amount by which the costs of CM are higher than for PM, i.e. the higher the ratio c_F/c_R, the shorter is the time interval until a PM action will be carried out, i.e. the smaller is τ.

7.4.4. Costs Proportional to the Degree of Repair - 2

In this section it is again assumed that the costs for maintenance actions are proportional to the degree of repair. Analogue to the continuous maintenance model from Subsection 4.4.4 the following cost function is used for PM actions

$$c_{PM}(v,\tau) \quad = \quad c_R\left(1-\left(\frac{v}{v+\tau}\right)\exp\left(\frac{v}{v+\tau}-1\right)\right)^{\delta}, \qquad (7.17)$$

where $\delta > 0$ and $c_R > 0$ are the costs of a replacement. This function is described in detail in Subsection 2.7.4. For CM actions the following cost function is used

$$c_{CM}(v,\tau) = \sum_{t=1}^{\tau-1}\left(c_F + c_R\left(1-\left(\frac{v}{v+t}\right)\exp\left(\frac{v}{v+t}-1\right)\right)^{\delta}\right)P(T^v = t|T^v < \tau). \qquad (7.18)$$

Here $c_F > 0$ is the fixed amount, by which the costs of CM are higher than for PM. In the following the optimal values for v and τ are computed for different cost ratios c_F/c_R and different δ. The numerical results are given in Table 7.4 and lead to the following conclusions:

1. The higher δ, the lower are the maintenance costs and the faster the costs of PM tend to zero and the costs of CM tend to c_F. Therefore, it is cost optimal to do nonstop bad PM actions and no CM actions, i.e. $\tau = 1$ and $v = 19$.

2. The lower δ, the higher are the maintenance costs and the smaller is the cost difference between good and less good maintenance actions (see e.g. Figure 2.7 (d)). Therefore, for the DLFRD, DRD, DWD and DMWD it is cost optimal to do perfect PM and CM actions, i.e. $v = 0$. This does not apply for the RMWD because of the higher failure rate at lower states.

3. The higher the amount by which the costs of CM are higher than for PM, i.e. the higher the ratio c_F/c_R, the shorter is the time interval until a PM action will be carried out, i.e. τ is decreasing, whereas v remains constant.

Table 7.4.: Optimal values in case of costs proportional to the degree of repair - 2

	DLFRD	DRD	DWD	DMWD	DRMWD
	$\alpha = 0.01$ $\beta = 0.02944$	$\beta = 0.03142$	$\beta = 0.0057$ $\gamma = 3$	$\alpha = 0.03$ $\beta = 0.004335$ $\gamma = 3$	$\alpha = 0.1$ $\beta = 0.1746$ $\gamma = 0.1$
$c_F/c_R = 0.05$					
$\delta = 0.125$	$v = 0$ $\tau = 20$	$v = 0$ $\tau = 20$	$v = 0$ $\tau = 11$	$v = 0$ $\tau = 12$	$v = 1$ $\tau = 19$
$\delta = 0.5$	$v = 0$ $\tau = 20$	$v = 0$ $\tau = 20$	$v = 0$ $\tau = 11$	$v = 0$ $\tau = 12$	$v = 1$ $\tau = 19$
$\delta = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$
$\delta = 2$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$
$c_F/c_R = 0.1$					
$\delta = 0.125$	$v = 0$ $\tau = 19$	$v = 0$ $\tau = 18$	$v = 0$ $\tau = 9$	$v = 0$ $\tau = 10$	$v = 1$ $\tau = 19$
$\delta = 0.5$	$v = 0$ $\tau = 19$	$v = 0$ $\tau = 18$	$v = 0$ $\tau = 9$	$v = 0$ $\tau = 10$	$v = 1$ $\tau = 18$
$\delta = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$
$\delta = 2$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$
$c_F/c_R = 0.2$					
$\delta = 0.125$	$v = 0$ $\tau = 13$	$v = 0$ $\tau = 13$	$v = 0$ $\tau = 8$	$v = 0$ $\tau = 8$	$v = 1$ $\tau = 18$
$\delta = 0.5$	$v = 0$ $\tau = 13$	$v = 0$ $\tau = 13$	$v = 0$ $\tau = 8$	$v = 0$ $\tau = 8$	$v = 1$ $\tau = 17$
$\delta = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$
$\delta = 2$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$	$v = 19$ $\tau = 1$

8. Conclusion

This research provides several imperfect maintenance models that can be applied for a variety of maintenance problems in both continuous and discrete time. For two of three maintenance models with continuous lifetime distribution this thesis contains the analogue discrete maintenance model and therefore makes it possible to compare the resulting cost optimal maintenance strategies and to evaluate the effect of discretization on the resulting optimal strategies. As a result it can be concluded that the cost optimal maintenance strategies diverge from each other more or less but the interpretation of results remains unchanged through discretization.
In this thesis it is shown how to use inhomogeneous Poisson processes, discrete time Markov chains and truncated random variables to model different imperfect maintenance models. In the interest of clarity, essential definitions are given in a special chapter called Basics at the beginning of this research. In particular, the impact of preventive maintenance actions is assumed to be not minimal and not perfect but in between these boundary cases. For all imperfect maintenance models investigated, cost optimal maintenance strategies for various cost functions are computed with R and are then interpreted. For this purpose, the costs of preventive maintenance actions are assumed to be proportional to the impact of repair, the state before repair or the degree of repair. Based on such cost functions the costs of preventive maintenance actions increase with increasing maintenance quality. The optimization criterion of interest in this research are always the average maintenance costs per unit time. Therefore, in each model the mean costs per cycle are set in relation to the mean cycle length. Since all imperfect maintenance models use nearly the same cost functions for PM actions, they are introduced in a special section at the beginning of this thesis (see Section 2.7).
In this research both models with continuous and models with discrete lifetime distribution are investigated. All models use the same selection of lifetime distributions and therefore they are described at the beginning of this thesis in Section 2.5 and Section 2.6. In the continuous case the linear failure rate distribution, the Rayleigh distribution, the Weibull Distribution, the modified Weibull distribution and the reduced modified Weibull distribution with an expected value of five are used in calculations that are carried out. In contrast to the other lifetime distributions the last of those distributions allows bathtub curved failure rates and is therefore well-suited to fit the general quantitative shape of the failure rate that consists of the three intervals infant mortality, usable life and phase of wear out failures. To enhance comparability, the discrete counterparts of these distributions are used in the second part of this research to perform calculations for the discrete models. It was deliberately decided against choosing the distribution parameters for the discrete lifetime distributions so that the expected value is five again as in the continuous case. This is useful to exactly evaluate the effect of discretization of continuous lifetime distributions on the cost optimal maintenance strategies. All in all, this proceeding of choosing the same parameteriza-

tion leads to a higher expected time to first failure for the discrete lifetime distributions that is close to 5.5.
In Chapter 3 and Chapter 6 an imperfect maintenance model for systems with two different failure types is modeled and optimal maintenance strategies are determined. These maintenance models provide an extension of a perfect preventive maintenance model from Beichelt [7]. It is assumed that type 1 failures can be removed by minimal repair and that type 2 failures can only be removed through replacement. In addition to corrective maintenance actions that remove failures according to their failure type, the system is preventively maintained at fixed time intervals. These preventive maintenance actions are imperfect in the sense that they reduce the virtual age of the system in a Kijima type manner. In both maintenance models the time between two replacements defines a cycle.
In the continuous case the failure counting processes are modeled as inhomogeneous Poisson processes. The distribution function of the random time to the first type 2 failure and the expected number of minimal repairs in a replacement cycle are determined with the help of the intensity function of these inhomogeneous Poisson processes.
In the discrete case the underlying system is assumed to be a repairable multi-state system with a fixed number of states as described in Kahle [22]. A discrete time Markov chain is used to model the distribution of the random time to the first type 2 failure and the expected number of minimal repairs in the replacement cycle.
Optimal maintenance strategies are computed with R using complete enumeration for both continuous and discrete lifetime distributions. The lifetime distributions used in both cases are parametrized identical to ensure a high degree of comparability between the optimal maintenance strategies in the continuous and the discrete case. As expected, the optimal maintenance strategies that are computed for different lifetime distributions in the continuous and the discrete case are very similar. Nevertheless, there are sometimes more, sometimes less differences in the resulting cost optimal maintenance strategies. One crucial difference between the continuous and the discrete case is the number of minimal repairs that can be expected during the replacement cycle. In the continuous maintenance model from Chapter 3 the expected number of minimal repairs is higher than in the discrete maintenance model from Chapter 6. The reason for this is that in the discrete case the maximum number of type 1 failures and therefore the maximum number of minimal repairs in a replacement cycle is bounded above, since at every point in time only one failure can occur. In the continuous case, however, such a restriction of the number of minimal repairs in a replacement cycle does not exist. Another difference between the continuous and the discrete case is that the mean cycle length in the discrete case is always equal or greater than in the continuous case since the discretization of the continuous lifetime distributions shifts probability mass away from lower values to higher values.
In view of the high complexity, these imperfect maintenance models can be applied for a variety of maintenance problems but it is also a weakness of these models. For this reason it is not possible to determine the optimal maintenance strategies analytically but only through complete enumeration that require long calculation times especially for increasing values of $N_{\max}$. Therefore, a possible starting point for future research could be to find a suitable optimization algorithm that significantly reduces calculation time.

In Chapter 4 and Chapter 7 an imperfect maintenance model with both imperfect preventive and imperfect corrective maintenance actions is investigated. A sequential failure limit PM policy with infinite planning horizon is used to formulate a cost optimization problem.
The derivation of the optimization problem in the continuous (Chapter 4) and the discrete case (Chapter 7) is quite similar. In both cases the remaining lifetime of the system after a maintenance action is modeled as a truncated random variable. With the help of this truncated random variable the costs of corrective maintenance actions, the mean cycle length and thus the average maintenance costs per unit time can be computed. In contrast to all other maintenance models in this thesis, a cycle is defined here as the time between two maintenance actions and not as the time between two replacements.
A comparison of both models shows that the mean cycle length in the discrete case is always higher than in the continuous case since the discretization of the continuous lifetime distributions shifts probability mass away from lower values to higher values. Since a cycle is defined as the time between two maintenance actions, especially for the calculation of the costs of CM actions the time since the last maintenance action have to be taken into special consideration. A failure can occur at every point in time till the PM takes place and therefore one have to compute the expected costs of corrective maintenance per cycle by using the probability distribution of the remaining lifetime of the system after a maintenance action. This will cause the costs of CM to be identical in the discrete and the continuous case if the cost function is independent of the time since the last maintenance action. If the used cost function is an increasing function of the time since the last maintenance action, the costs of CM in the discrete case are higher than in the continuous case and vice versa. Based on the above stated facts, the resulting cost optimal maintenance strategies in the continuous and the discrete case differ considerably in parts.
Optimal maintenance strategies are computed with R using again complete enumeration for both continuous and discrete lifetime distributions since it is not possible to determine the optimal maintenance strategies analytically. This proceeding require long computation time and therefore it is here again meaningful to find a suitable optimization algorithm that reduces calculation time.
The resulting cost optimal maintenance strategies shows that in special parameter constellations it is cost optimal to do quasi-non-stop PM actions if the costs of PM are close to zero. If additionally the cost optimal value of v is very high, it is no longer appropriate to exclude the time between the startup of the system and the age of v from optimization problem, since the costs for minimal repairs in this time interval are likely a significant portion of the total maintenance costs. In further research it can be investigated how this costs can be reasonably embedded in the cost optimization problem.
The maintenance model in Chapter 5 has a maintenance policy with PM that is done at fixed predetermined times which are not necessarily periodic. This model takes into account that in reality with increasing age of the system PM actions have to be done more often. The cost optimization problem is modeled using the intensity function of the failure counting process which is an inhomogeneous Poisson process. In contrast to all other models examined in this thesis, the mean cycle length is not random but

constant. Not least for this reason, the cost optimal maintenance strategies in this model can be determined analytically. Under the assumption that the time to the first failure is modified Weibull distributed, the original cost optimization problem reduces to an optimization problem where only the cost optimal number of PM actions have to be found. It was shown that this optimization problem has an unique optimum under certain conditions. Using this, the cost optimal maintenance strategies are computed with R. A new cost function for PM actions was used so that costs of PM are always greater than the costs of minimal repair and smaller than the costs of a replacement. As a result in this special case, it was proofed that if the cost function for PM is concave it is cost optimal to do no PM actions and only preventive replacements are carried out. Besides, if the cost function of PM is not concave, it becomes possible to have cost optimal maintenance strategies with positive number of PM actions whereby the time between PM actions is decreasing with increasing age of the system.
The corrective maintenance actions in the maintenance model from Chapter 5 are assumed to be minimal. A possible extension of this model could be to include imperfect CM actions. In possible further research similar to the model from Chapter 5, the fact that in reality with increasing age of the system PM actions have to be done more often could be reflected in the models from Chapters 3 and 6 by the use of an increasing sequence of degrees of repairs. This would lead to a sequential instead of periodic maintenance policy. Furthermore, in case of a preventive replacement one can take into consideration a resale value for the exchanged system, since the system is still in working condition and could be used for example as a spare part donator.
All models in this thesis use the minimum cost rate as optimization criterion. Depending on the area of application and the requirements on the repairable system, it could be appropriate to use other optimization criteria such as maximum availability or limit on failure rate. Thus, in future research for the models described in this thesis, the influence on the optimal maintenance strategies could be investigated if another optimization criterion is used.
The maintenance models investigated in this thesis have a multitude of application areas. As a next step these models should be applied to real systems. This will certainly leads to further development of these maintenance models and it is expected that some reports to these subjects will be published in near future.

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