ORIGINAL PAPER

The role of variables in relational thinking: an interview study with kindergarten and primary school children

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Abstract
Relational thinking and dealing with variables are two essential aspects of algebraic thinking. Relational thinking means viewing mathematical expressions and equations as a whole rather than as individual computing processes. It is characterized by using relationships between mathematical objects, and refers to the relations of equality and inequality. In this study, to examine the relational thinking of kindergarten and primary school children, this perspective was applied using non-symbolic representations in the form of boxes and marbles. Using multiple variables is a very powerful but also difficult tool of algebra. The study had the aim of examining how kindergarten children and primary school children establish relationships between several variables which are represented with real materials. The interview study was conducted with children aged 5–10 years. Marbles and different colored boxes represented equations with unknowns and quantities depending on each other. Initially, two approaches could be differentiated, namely, number-oriented and structure-oriented approaches. It could be shown that certain conceptualizations of variables were related to children's ability to show relational thinking. Kindergarteners are stimulated to think relationally by unknown quantities which can be determined. This process was observed in primary school children dealing with quantities that depended on each other. In addition, the conceptualization of the variables represented as boxes was examined. The concepts of general number and variable as changing quantity were categorized. Further conceptualizations resulted from the interview data, namely, categories of the undeterminable, the specific number, and the quasi-general.

1 Introduction

Contemporary researchers have shown great interest in early algebra (e.g., Cai & Knuth, 2011; Kaput, 2008; Kieran, 2018). There is increasing consensus that the separation of arithmetic and algebra in primary school and the differences in meaning and concept between these areas make it very difficult to cope with algebraic requirements in later school years (e.g., Carpenter et al., 2005; Carraher et al., 2001). This can be seen in the major problems with algebra experienced by students in secondary education (e.g., Carraher & Schliemann, 2007). Addressing algebraic activities at an early stage should prevent problems later on and enrich mathematics lessons with an algebraic perspective.

Studies show a positive significant effect of early algebra intervention on algebra performance in later school years (cf. Blanton et al., 2019). When describing algebraic thinking, different aspects have been emphasized. Kieran (2011) described the focus of algebraic thinking as "thinking about the general in the particular", "thinking rule-wised about pattern", "thinking relationally about quantity, number, and numerical operations" and "thinking conceptually about the procedural". As an essential way of thinking in early algebra, Kieran (2004a) named the consideration of relational aspects of operations instead of their computation. In contrast to arithmetic thinking, the structural view of mathematical objects is of major importance (Kieran, 2004b; Steinweg, 2013). It is precisely this way of thinking that is specified by relational thinking as part of early algebra. Relational thinking is characterized by a structural rather than an operational view of mathematical elements, by establishing relationships between them and using them to find a solution to a task (Carpenter et al., 2005; Molina & Ambrose, 2008).

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Comparing mathematical expressions and a relational understanding of the equal sign were also part of it. However, there is a lack of research literature that examines the skills of relational thinking in kindergarten children. Nevertheless, the previous experiences of this age group appear to be particularly relevant for research, since they form the basis for later learning of mathematics.

Studies show that this is also possible without the algebraic symbol language through natural language expressions (Akinwunmi, 2012; Radford, 2011). Studies on relational thinking showed that tasks with more than one unknown variable have a positive influence on relational thinking (e.g., Stephens & Wang, 2008). A study by Melzig (2013) showed that the use of materials such as boxes and beans seemed suitable for paving the way for an initial understanding of the variables. For secondary school students, this could build an understanding of the interdependence between multiple variables. There is a lack of research on kindergarten children's thinking in this regard, and their early interpretation and expression of variables.

The subject of this research was therefore the connection of two essential sub-areas of early algebra, namely, relational thinking by using various forms of variables. The aim was an exploratory study to record the early abilities of kindergarten and primary school children regarding the algebraic sub-areas of relational thinking and an understanding of variables.

2 Theoretical framework

In this section, firstly, relational thinking is characterized. Then the essential aspects of variables are discussed, as these are relevant for the evaluation of the study. In the last section, the research interest is derived from the theoretical framework.

2.1 Relational thinking

To describe relational reasoning as an aspect of algebraic reasoning, it is necessary to explain the difference between algebraic and non-algebraic (arithmetic) reasoning. Similarly to Sfard's (1991) distinction between operational and structural perspectives on mathematical concepts, Tall et al. (2001) distinguished between a process view and a concept view regarding the use of mathematical symbols. They described the development of thinking from arithmetic to algebraic. The process-oriented aspects (*process*) of mathematics concern the routine manipulation of objects. This stage is characterized by the algorithmic discovery of a solution and describes arithmetic thinking. Conceptual knowledge (*concept*), on the other hand, is more difficult to grasp because it describes knowledge that exists in a variety of

relationships of objects with one another (Tall et al., 2001). This stage focuses on the relationships between mathematical structures and describes algebraic thinking. Arithmetic thinking and algebraic thinking are brought together in the step of a '*procept*', which describes the transition between the two ways of thinking. While Tall et al. describe the relationship between algebraic and arithmetic thinking and characterize relational thinking as algebraic, this relationship can also be linked to the concept of structure sense. The concept of structure sense described by Hoch and Dreyfus (2004) and Linchevski and Livneh (1999) refers to the recognition and use of structures within equations in general, whereas Sfard's (1991) presentation refers to mathematical concepts. According to Hoch and Dreyfus (2004), structure sense is about recognizing how a mathematical whole consists of parts and the relationships between these parts. According to Greeno (1991), references can be made to the concept of number sense and that of operation sense (Slavit, 1999). What both concepts have in common is that they concern the relationships that exist between numbers and operations. This perspective makes it clear that not only equations as a whole, but also numbers and operations as parts of it can be considered structural. This aspect highlights the importance of recognizing relationships between the parts of an equation.

While this description of algebraic reasoning is general, relational thinking describes this way of thinking in terms of equality relations. Compared to arithmetic thinking, algebraic thinking is characterized by a structural rather than an operational view of mathematical objects (Kieran, 2004b; Steinweg, 2013). Carpenter et al. (2005) defined relational thinking as looking "at expressions and equations in their entirety rather than as a process to be carried out step by step" (p. 54). Molina and Ambrose (2008) introduced the concept of "analyzing expressions" to separate the concept of relational thinking from a relational understanding of the equal sign. This concept is characterized by focusing on arithmetic relations instead of calculating them. Students use their number sense and operation sense to view arithmetic expressions from a structural rather than a procedural perspective.

Blanton's et al. (2019) explanations allow relational thinking to be embedded and specified in an algebraic curriculum. One of the four fundamental algebraic activities, that should be a framework for curriculum development is "reasoning with mathematical structure and relationships", which can be compared to relational thinking. This algebraic activity occurs in the content domain of "equivalence, expressions, equations, and inequalities". Learning goals for these big algebraic ideas are, for example, to recognize and represent variable quantities in problem situations and also, to examine the role of variables of an unknown, fixed, or varying quantity in an expression. Reasoning in a structural way by

solving an equation for a missing value is also an important learning goal.

All in all, relational thinking means viewing mathematical expressions and equations as a whole rather than as individual computing processes. It is characterized by using relationships between mathematical objects and refers to the relation between equality and inequality. It is precisely this structural view that has been described that characterizes relational thinking as an algebraic way of thinking.

2.2 Relational thinking about variables

The study presented in this paper combines this view of relational thinking with an understanding of variables. Therefore, a classification of variables is discussed with regard to their use in studies. Referring to Freudenthal (1973) and Malle (1993), three different variable aspects are presented. First of all, the variable aspect of the *unknowns* describes a specific but undetermined number, whose value can be evaluated (e.g., Freudenthal, 1973). According to Malle (1993), this corresponds to the object aspect of variables. The second aspect, *variable as a changeable* or varying quantity describes a range of values and a relationship between two sets of values, as in functional relationships. Malle (1993) described this as a range aspect, where all numbers are represented in chronological order. When all numbers are represented at the same time in the range aspect of variables, this describes the variable as a *general number*. These numbers describe undetermined numbers which appear in generalizations. While in the case of the unknown, a specific, not yet determined number is sought, the point of the general number is to make a general statement.

Concerning activities with equations containing a variable that is to be understood as an unknown, the task can always be solved arithmetically by calculation or by trial and error. Children do not have to use relational thinking. A study by Stephens and Wang (2008) done with 6th and 7th graders showed that the presence of two variables that depend on each other challenges relational thinking. In the equation, $18 + \square = 20 + \square$, values should be put in both iconic boxes so that the equation is true. Further tasks suggested comparing the numbers used. They found that tasks with two variables served the purpose of moving students beyond computations to more in-depth thinking (Stephens & Wang, 2008). The variables in this study can be viewed as changeable. Students are then encouraged to state a general relationship between the two variables, which addresses the variable aspect of the general number. Stephens and Wang's study relates to symbolic representations.

The next two relevant pieces of research used physical material in the form of real boxes, which seem accessible to younger children. Studies by Schliemann et al. (2006) showed that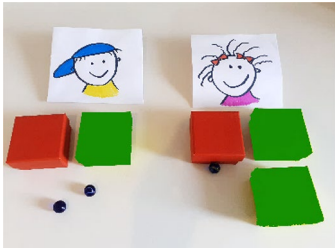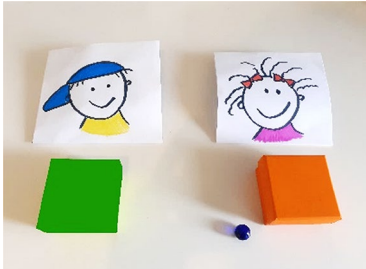 children between the ages of 7 and 11 show an understanding of equivalence between configurations of boxes with different contents of marbles. Children could also solve linear equations containing variables without the use of algebraic notation. In their tasks, they differentiated between the occurrence of known, unknown, and partially known values. The evaluation relates to which different forms of representation are conducive to relational thinking. In their study, it remains unclear how the children understood the variables. A distinction between different variable aspects was not made. A closer look at the children's understanding of the variables was not yet taken. In the study with 7th graders by Melzig (2013), boxes and beans were used to provide a wide range of variable means of access. Two different boxes had to be filled with beans so that both configurations contained the same amount of beans overall. It was shown that beans and boxes are suitable for building up an understanding of variables in particular as changeable entities (Melzig, 2013).

2.3 Research interest

The research interest developed from the previous descriptions as follows.

1. Within algebraic thinking, relational thinking is an essential sub-area, as it describes structural thinking regarding mathematical expressions and situations.
2. Relational thinking is particularly stimulated when dealing with several variables since calculations are no longer possible.
3. Since there is a research gap regarding such thinking in kindergarten children, it seems particularly interesting to investigate their preschool abilities regarding relational thinking and their understanding of variables. This age group does not yet have sufficient experience with symbolic representations. Whereas relational thinking is very closely related to symbolic representations and the use of algebraic notation, it does not have to be limited to this mode of expression. Existing studies showed that real materials appear to be profitable, particularly for young children in kindergarten and primary school, and can challenge relational thinking. It is assumed that dealing with real, manipulable objects helps children understand mathematical concepts (e.g., Bruner, 1964). Particularly in representations without algebraic notation and with the help of materials, the understanding of equality, more-less comparisons, and part-whole relationships can be seen as the basis of mathematical learning. Boxes and beans represent a suitable means of enabling initial access to variables. It is also relevant to investigate what understanding of variables young children have.

Table 1 Task design

Type of task	Tasks with underlying structure and a sample image
Type B: How many? The children had to specify how many marbles were in a box so that both children had the same number of marbles. Children could answer with a specific number. The variable appeared as an unknown that could be clearly determined	B1: boy: $x + y$, girl: $x + y$ B2: boy: 6, girl: $x + 4$ B3: boy: 4, girl: $2x$ B4: boy: $x + y + 2$, girl: $x + 2y + 1$ Task B4: How many marbles have to be in a green box so that both children have the same number of marbles? How did you solve that?
Type C: How many? The content of the boxes is not known. In comparison to task type B, the contents of the boxes could not be determined clearly. Several possible pairs of values can be given, whereby the variable can be understood as a changeable. The numbers of marbles in the boxes depend on each other, which is in the following referred to as 'interdependent'	C1: boy: x, girl: $y + 1$ C2: boy: $2x + 3$, girl: $x + y + 2$ C3: boy: x, girl: $2y$ C4: boy: $x + 2$, girl: $2y$ Task C1: How many marbles have to be in the boy's box so that both children have the same number of marbles? How did you get that?

This foundation gives rise to the following research questions:

1. To what extent do kindergarten and elementary school children manifest relational thinking when dealing with tasks using tangible materials, in which relationships between variables can also be established?

    a. How do children show relational thinking when dealing with tasks in which the variable is an unknown and can therefore be determined clearly?

    b. How do children show relational thinking when dealing with tasks that involve multiple variables that are dependent on each other?

2. Can differences in relational thinking be identified regarding the occurrence of different variable aspects in the tasks? Can these differences enable researchers to identify task types that challenge relational thinking more than others?

3. How do children understand the variables represented by the materials? Which variable concepts can be reconstructed in children's statements?

3 Method

In this study the aim was to examine relational thinking combined with the use of variables within a representation without algebraic notation.

3.1 Task design

Various equations with one or more variables were converted into a form of representation that even very young children could deal with. Marbles and different colored boxes were chosen as materials. Marbles represented quantity values, but the number of marbles in the boxes was not known.

The research reported in this paper dealt with 8 tasks in two different task types (Table 1). The following framing (with simplified wording) was explained to the children in the study:

> "Here you see two children. They are playing with marbles. Some of them are inside different colored boxes and some marbles are separate. Boxes with the same color within one task always contain the same number of marbles. However, boxes of the same color do not have to contain the same number of marbles in subsequent tasks."

An equation may be viewed as a comparison between two quantities, one on each side of an equal sign. Similarly, the model using boxes and marbles involves a comparison between two quantities, one associated with each child. The boxes represent containers. The marbles represent items or units of a discrete quantity. The approach encourages students to draw inferences about the number of items in containers or the total number of items a particular child has. This result can be based on premises about counting items (consisting of shown items as well as items presumably) hidden inside the boxes. The tasks should stimulate an understanding of variables as unknowns and as changeable. In addition, it should be pointed out that the tasks can stimulate only a certain understanding of variables according to the variable aspects presented. The variables are then conceptualized by the children and their task processing.

The tangible materials presented in real life enabled the given quantities to be changed. This way, the children could move or remove individual objects. Access via marbles is very poor in context so barriers can hardly arise due to a failure to understand a factual context.

3.2 Data collection method

To gain a deep insight into the skills of relational thinking, the usage of a qualitative survey method was necessary. Semi-standardized interviews took place in order to understand children's thoughts on solving the tasks. The interview study was performed with 80 children between the ages of 5 and 10 years. Kindergarten children (N = 25) between the ages of 5 and 6 took part. In primary school, 7 to 8-year-old 2nd graders (N = 29) and 9 to 10-year-old 4th graders (N = 26) were involved. Three somewhat equally spaced age groups were chosen to identify similarities, differences, and any development of relational thinking and conceptualization of the variables. Children from two kindergartens and four classes in two schools in different socio-economic locations of a large German city were selected. The educators

or teachers selected the children so that they showed different general and mathematical achievements. The children did not receive any special algebra lessons beforehand and participated voluntarily. The researcher herself conducted the interviews, which were videotaped and then transcribed.

3.3 Data analysis

The data analysis was based on transcribed interviews. A content analysis method was used to record children's skills in relational thinking and the conceptualization of variables. The main component of content analysis work is the application of a category system to the data material.

The evaluation followed the method for analyzing interviews according to Schmidt (2005) and was also based on qualitative content analysis according to Mayring (2010). The analysis was also rule-based and theory-based to allow it to be founded on fixed rules and a question based on the theoretical analysis (Mayring, 2010). In a first step, from the interview transcripts, categories were formed that described the children's abilities regarding relational thinking and dealing with variables. An overview table was created, which initially contained all verbal utterances and gestures concerning the material by the children. This overview gave the first indications of notably simple and demanding tasks of the investigation in order to enable a focus for the later analyses. A category system was developed labeling the procedures described by the children and conceptualizing the variables. All tasks were initially considered separately. After that step, categorizations were developed labeling children's procedures and their conceptualization of the variables across all task types. Categories were created deductively based on preliminary theoretical considerations and inductively obtained from the data. After the categories had been created using prototypical examples, examples of consensual validations of cases previously selected as particularly controversial were carried out within a working group of mathematics didactics specialists. This validation process served to ensure the quality of the categories formed. In the next step, the categories of the evaluation were recorded in a coding guide. This guide formed the basis for coding all the material again, which was done in a third step. In a fourth step, an overview of the categories of the entire data corpus was given in frequency tables. These were used for further analysis by pointing out possible relationships (Schmidt, 2005, p. 455ff).

4 Relational thinking with unknown quantities

In the following, the first research question is addressed. It was used to examine how children use relational thinking when dealing with tasks in which the variable can be regarded as an unknown. For this purpose, the categorization of the children's answers to the type B tasks is presented first.

4.1 Number-oriented and structure-oriented approaches

With the help of qualitative content analysis, the children's explanations of how they solved the tasks were categorized. The two main categories[1] were the number-oriented approach and the structure-oriented approach. These approaches are illustrated with the help of transcript excerpts from task B4 (see Table 1).

4.1.1 Number-oriented approach

The number-oriented approach focused on the specific number of the given quantities. This is illustrated, for example, by the approach of the fourth-grader Desiree[2]:

> I: And now I want to know how many marbles are in a green box so that both children have the same number.
> D: They are the same number (*points to both red boxes*). I guess there are three in there. [...] (D. gives different numerical values for the boxes, changes them) No, there one (*taps the girl's red box*), there two (*taps the girl's red box*). Two (*taps the boy's red box*), that's four (*points to the boy's loose marbles*). There is one (*points to the boy's green box*), there is one, that's five (*points to the free space in front of the boy's green box*). One (*lifts the girl's single marble*), two (*taps the girl's red box*), three (*holds her finger on the girl's red box*), four (*taps the girl's rear green box*) five (*taps the girl's front green box*). There should be one in the green [box] and two in the red [box].
> I: […] why do you have to know how many marbles there are in the red box?
> D: Because otherwise, I can't calculate how many there will be with the two of them.

Desiree came up with a numerical value for the red box without being able to know it exactly. This enabled her to calculate the total number of marbles both children had. She

described a computational procedure for determining the answer, which was classified as a number-oriented approach. The total number of marbles for both children in the task design was calculated and compared. Since the number-oriented approach focused on the calculation of sums instead of relating quantities, this could be characterized as an arithmetic approach. This was justified by the fact that the calculation process predominates, and this procedural approach was to be regarded as non-relational and thus also as non-algebraic.

4.1.2 Structure-oriented approach

Anton explained his answer to task B4:

> A: So...same amount (*taps the red boxes*)...same amount (*taps the green boxes in the back row*)...same amount (*takes a marble from each child in hand*)... then this has to be the same amount (*holds the boy's marble in his left hand and taps it on the girl's front green box*).

In contrast to Desiree's approach, Anton didn't have to provide numerical values for the red boxes (Fig. 1). He also did not name the total number of marbles of both children. Instead, he described which quantities are equivalent, but without going into detail about their specific numerical value. Anton's approach was an example of a structure-oriented approach. With this approach, equal numbers of marbles or boxes were related to each other using gestures or words. The focus was on the quantities themselves and not on their value ("they are the same" instead of naming the specific number). In this sense, the viewpoint is focused on the whole, and given amounts are structured. The structure-oriented approach could be characterized as a procedure corresponding to relational thinking, due to the structural
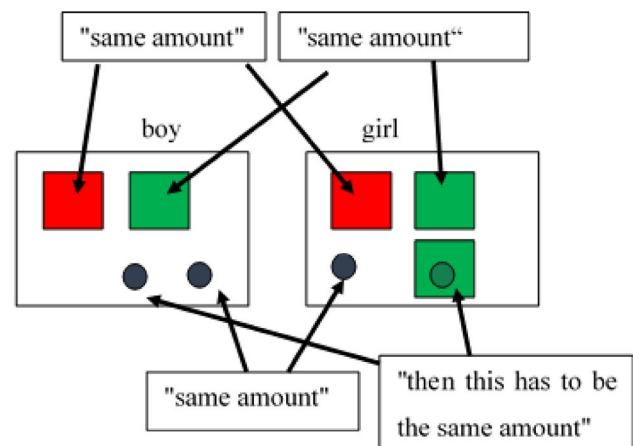

Fig. 1 Anton's approach to task B4. The arrows show Anton's gestural references

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[1] Further categories have been grouped under "Other" such as guessing or a different understanding of the task.

[2] The transcripts were translated from German by the author.

perspective of the task taken by the children. Instead of proceeding procedurally and looking at the specific quantities of marbles in individual boxes, connections were made between the sub-structures of the task. According to Molina and Ambrose (2008), this represented relational thinking.

The presented distinction between the structure-oriented approach as relational thinking and the number-oriented approach as arithmetic thinking is not a strict demarcation, but a transition can be determined. For example, in task B4, some children explain the sum of the marbles using only the green boxes, and thus adopt a number-oriented perspective. However, since they can also describe the additional red boxes as 'equivalent', a structuring of the subsets can be seen. This transition is illustrated by Kevin's approach: "because the two red boxes have the same amount. And if there's a marble in the two green boxes, that's three marbles for both children". This analysis in in agreement with the representations of Tall et al. (2001), who located a transition called '*procept'* on the way from arithmetic thinking to algebraic thinking.

4.2 Findings on relational thinking with unknown quantities

The following analysis examines how the approaches shown above are represented in the other age groups for the four tasks of type B. To answer the second research question, this study served to determine what influences the existence of unknowns which clearly can be determined (as in the boxes in task type B) and quantities, which depend on each other (as in task type C) have on the occurrence of relational thinking (Fig. 2). The charts show the children's categorized responses. To take up the special feature of the additional boxes in tasks B1 and B4, the category 'partially number' was used. The children explained a number-oriented approach, but referred to the red boxes in a structuring manner and thus already showed a transition to the structure-oriented approach.

Task type B showed how children dealt with the unknown quantities, in which the number of the marbles in the requested box could be clearly determined. Thus, the boxes represented unknowns according to the conceptualization of the variables. Only the number of marbles in the red boxes in tasks B1 and B4 are not known and cannot be determined. In task B1 (Fig. 3), children of all ages mainly explained a structure-oriented approach. The red boxes were either structured as "the same" or children let them be without any verbal or gestural reference. At this point, the inclusion of an unknown which cannot be determined did not seem to be an obstacle for the children to answer the question correctly but promoted structure-oriented approaches.

The proportion of number-based approaches increased in tasks B2 (Fig. 4) and B3 (Fig. 5). This could be because in tasks B2 and B3, the quantities could clearly be determined
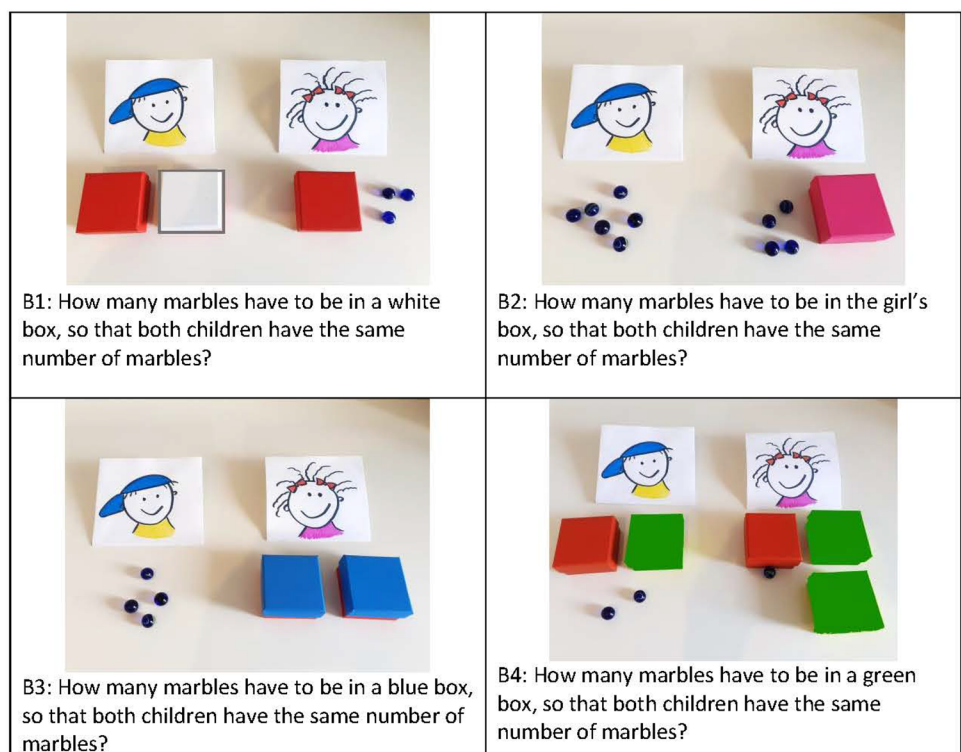
Fig. 2 Overview of the tasks of type B

B1: How many marbles have to be in a white box, so that both children have the same number of marbles?

B2: How many marbles have to be in the girl's box, so that both children have the same number of marbles?

B3: How many marbles have to be in a blue box, so that both children have the same number of marbles?

B4: How many marbles have to be in a green box, so that both children have the same number of marbles?

Fig. 3 Approaches to task B1 (boy: $x + y$; girl: $x + 3$) (A large proportion of the responses belonged to the categories "other" and "cannot be categorized", especially among the kindergarten children. Due to limited verbal expression, the children's descriptions could not always be understood and eluded precise categorization)
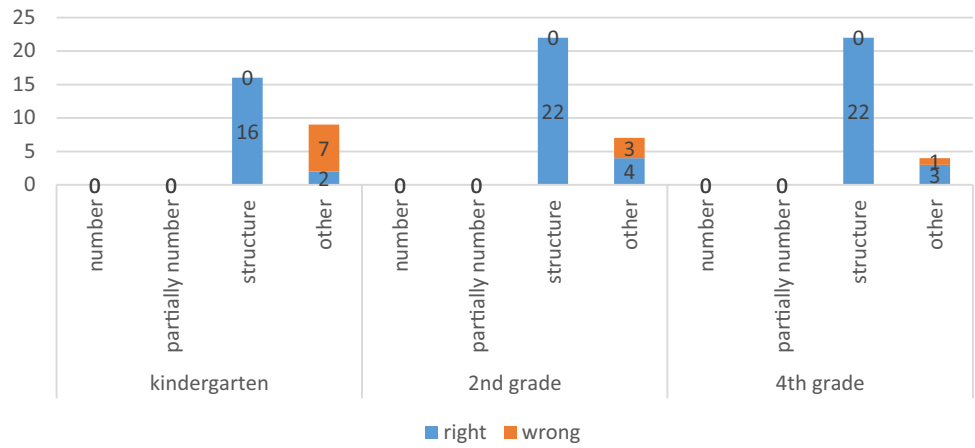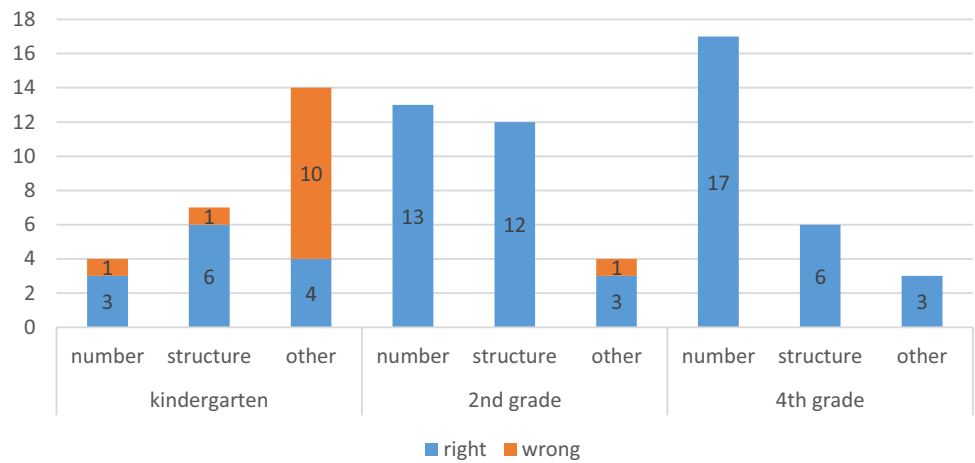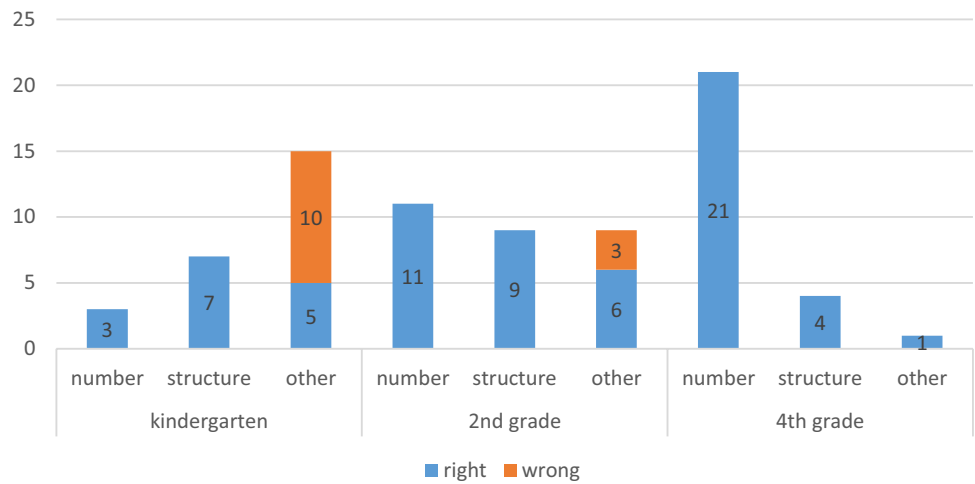

**Fig. 4** Approaches to task B2 (boy:6, girl: $x + 4$)

Fig. 5 Approaches to task B3 (boy:4, girl:2x)

for all boxes. It was thus possible to start from a discrete number of marbles and to justify the answer with that assumption. Although the interviewer asked about their way of thinking, the children explained the correctness of the given answer. This appeared particularly obvious when specifying a calculation (e.g., "two plus four equals six"). This explanation may not yet be available to the kindergarten children so they referred to the represented quantities in a structure-oriented approach and established relationships.

Fig. 6 Approaches to task B4 (boy: $x + y + 2$; girl: $x + 2y + 1$)
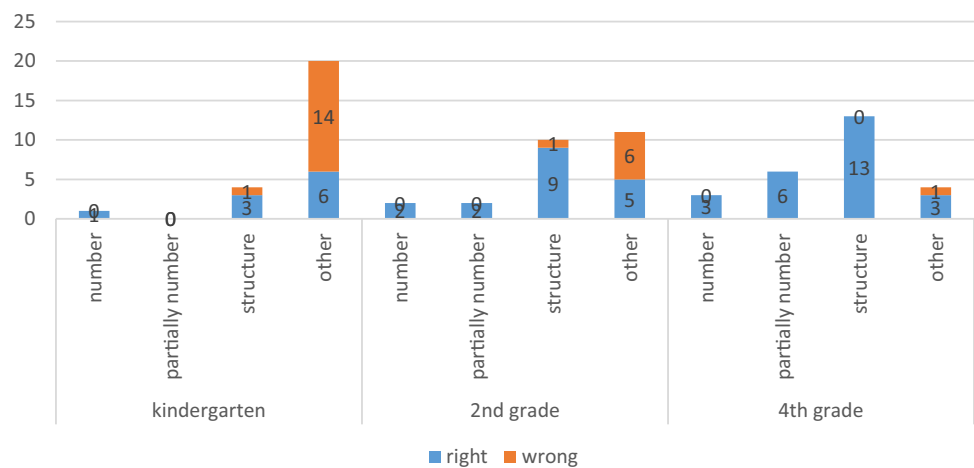
B4 was the most difficult task and, like task B1, contained red boxes. The majority of the children in task B1 described a structure-oriented approach, and no child proceeded in a number-oriented way by determining a total amount. In task B1, the marbles could be assigned directly to the requested box. In comparison, the proportion of the number-oriented approach in task B4 (Fig. 6) increased and was greatest among fourth-graders at almost 35% (9 of 26). 6 children were already able to structure some of the quantities. The complexity of task B4 had to be taken into account because the requested box was present several times and there were also individual marbles lying by both children.

Regarding the high proportion of number-oriented approaches in tasks B2 and B3 and the structure-oriented approaches in tasks B1 and B4, the following finding could be derived:

Discrete quantities prompted students to describe number-oriented approaches. The addition of unknown quantities which did not have to be determined stimulated the formulation of structure-oriented approaches more strongly. This result leads to the conclusion that unknown quantities, which cannot be determined, should be included in order to stimulate relational thinking, especially in a school context. These can encourage the children not only to refer to calculation methods but also to establish connections between the shown values. This result is being discussed again later concerning the second research question about the influence of unknown and interdependent variables.

5 Relational thinking with quantities that are dependent on one another

In the following, it is examined how children show relational thinking when dealing with tasks that involve multiple variables that are dependent on each other to answer the first research question in this way.

5.1 Categorization of relational thinking in task type C

In type C tasks, there were two or more boxes of unknown content, which were dependent on one another, so that a relationship could be established between them. In dealing with these tasks, number-oriented and structure-oriented approaches could be identified. In addition, an intermediate step could also be recognized within the continuum between the number-oriented and the structure-oriented approaches: children recognized the dependency of the quantities, but could not specify this as a relationship.

In the following, the categories that describe how to deal with type C tasks were presented based on task C1 (see Table 1). This was special insofar as no learning effects were to be expected, as in the first type C task.

5.1.1 Category A: children neither describe a relationship nor a dependency

Children neither indicated a generalized relationship nor described a dependency between the quantities. This concerns those categorizations in the course of the interview in which: (i) no answer was given, (ii) the answer was given with specific numerical values for marbles in the box, or (iii) the child could not give the corresponding numerical value for the other box for different numerical values given by the interviewer. The approach of kindergartener-child Christina serves as an example. She said that there are four marbles in both boxes, "because they're so small, only four fit in there". She did not seem to recognize the dependency of the number of marbles but related to the superficial characteristics of the task.

5.1.2 Category B: children display the dependency between interdependent quantities

The children indicated that the contents of one box depend on the contents of another box. This also includes the answers where the children can name different pairs of numbers for both boxes. Children were not assuming a specific, fixed content of the boxes. But it was not possible to generalize the existing relationship between the various possible quantities. This can be confirmed by the explanation of the second-grader Jakob: "… it depends on how many there are inside (*points to the orange box*) because if there are four inside, that would be five in total; there should be five inside (*points to the green box*). But if there is only one inside (*points to the green box*), there must be two inside (*points to the orange box*)". He described that the content of the orange box depends on the green box, and also gave two possible numbers for the boxes. However, he did not explicitly point out the relationship between the boxes so that one more marble must be in the green box.

5.1.3 Category C: children describe a relationship between interdependent quantities

While the dependency of the quantities is named in the previous category, it can be specified in this one. Here, children indicated a general or quasi-general formulated relationship or arithmetic rule between the interdependent quantities. The relationship between the interdependent quantities was recognized and reproduced in a generalized way with the aid of various verbal means. The second-grader Lara described: "No, you don't know how many marbles are in the (*points to the orange box*) … so there should be one more in the green one (*points to the green box*) than in the orange one (*points to the orange box*)". She spontaneously mentioned the relationship between the quantities and could express this in general without having to resort to quasi-generals.

While the dependency between the numbers is recognized in the categorization in B), this relationship can be specified numerically in C). This is not yet possible in category B). In the evaluation of the children's answers, which can indicate a general relationship, a further distinction can be made regarding the linguistic expression. Thus the relationship can be expressed in general or quasi-general terms. The notion of the quasi-general refers to the formulation by Fujii and Stephens (2008). It describes the expression of general relationships without using algebraic formal language. Fujii and Stephens (2008) described this form of expression when elementary school children deal with equations. Before children used the algebraic formula language, they used specific numerical examples. However, these were generally understood and revealed a concept of the variable (Fujii & Stephens, 2008). The kindergartener Aaron's description is an excellent example of this phenomenon: "[…] if there are eight or nine in the orange (*taps on the girl's box*) or the green can, then I just add a pearl, and it's nine or ten".

Regarding the first research question, it can be pointed out that when dealing with tasks similar to C, relational thinking shows itself while describing the relations between the interdependent sets. But even when dealing with these tasks, there is a transition between the poles of relational thinking and arithmetic thinking. This can be seen in category B, where it is already possible to identify the dependency, but no relationship can yet be specified. Figure 7 illustrates the connections between the categorizations and the references to relational thinking.

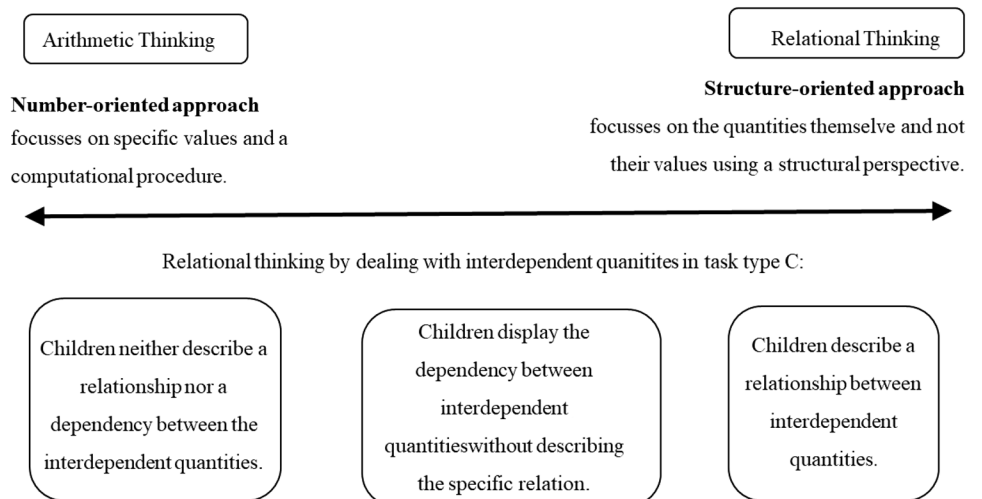**Fig. 7** Overview of categorized approaches

Arithmetic Thinking

Relational Thinking

Number-oriented approach
focusses on specific values and a computational procedure.

Structure-oriented approach
focusses on the quantities themselve and not their values using a structural perspective.

Relational thinking by dealing with interdependent quanitites in task type C:

Children neither describe a relationship nor a dependency between the interdependent quantities.

Children display the dependency between interdependent quantitieswithout describing the specific relation.

Children describe a relationship between interdependent quantities.

Fig. 8 Tasks C1 and C3

C1: How many marbles have to be in the green box, so that both children have the same number of marbles?

C3: How many marbles have to be in the girl's boxes, so that both children have the same number of marbles?
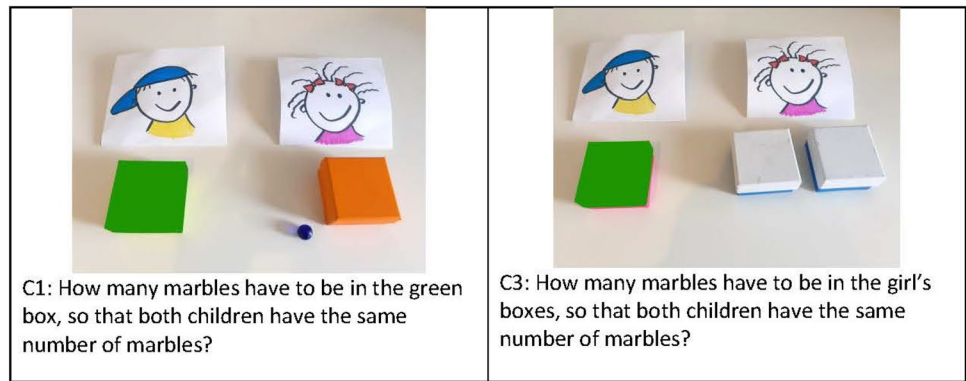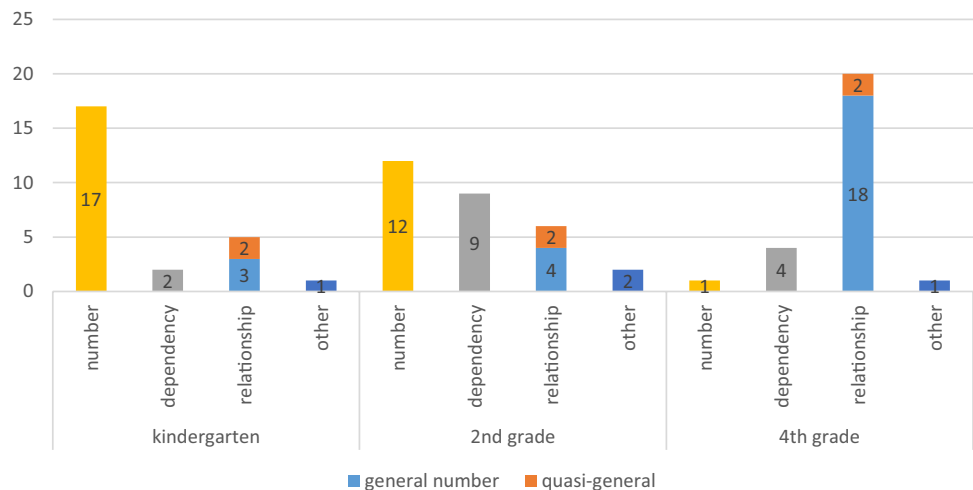
Fig. 9 Categories of C1 (boy:x, girl: x + 1)

5.2 Findings on relational thinking with interdependent quantities

The categorization of the answers to tasks C1 and C3 of all three age groups is presented below.[3] This analysis was done in order to examine the influence of different aspects of variables on relational thinking (second research question). Following the presentation of the categories, a comparison of the most striking results from the evaluation of task type C regarding the three age groups is given (Fig. 8).

Most of the kindergarten children answered with numerical values. But some of them recognized the dependency and were able to formulate a general or quasi-general relationship.

Similarly to the kindergarten children, most second-graders gave numerical values for type C tasks. However, the proportion of formulated relationships and the description of dependencies increased. The fourth-graders were much

more able to indicate relationships between the interdependent quantities. Those who did not succeed in formulating a general relationship could usually state the dependency of them. Only a few children gave specific numerical values (see Figs. 9, 10).

This result shows that fourth-grade children were able to generalize the relationship between interdependent quantities. In summary, in task type C kindergarten children mainly dealt with specific numerical values and so showed a number-oriented approach.

While the kindergarten children in task type B established relationships between discrete quantities and demonstrated relational thinking, in task type C they fall back on numerical values. The fourth-graders, on the other hand, described number-oriented and thus arithmetic procedures in task type B but were able to generalize relationships between the interdependent quantities in task type C, thus manifesting relational thinking. Based on these considerations, the following statement can be made regarding the second research question under the influence of variable aspects in task design.

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[3] Due to the complexity of tasks C2 and C4, these were only given to school children. In order to be able to make a comparison with the kindergarten children, only tasks C1 and C3 will be discussed here.

Fig. 10 Categories of C3 (boy:x, girl:2y)
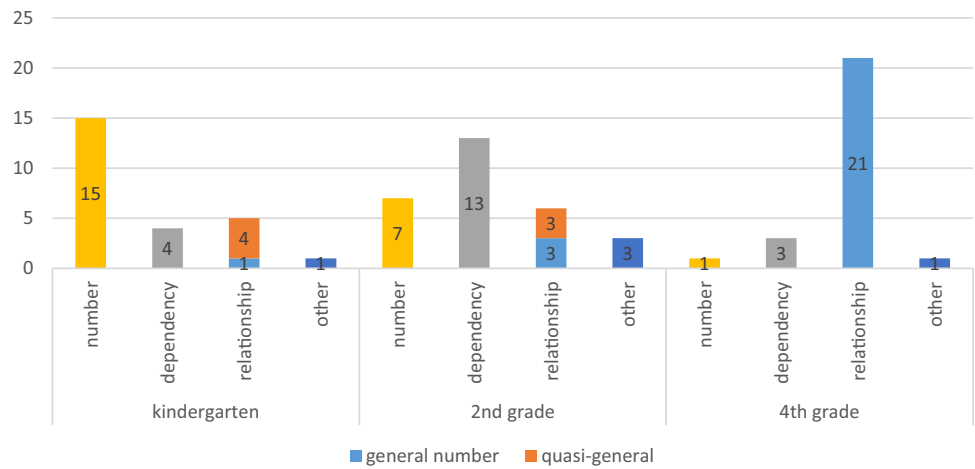
While kindergarten children are stimulated to think relationally by unknown quantities, this propensity should be expected of schoolchildren when dealing with interdependent quantities.

This finding refers to specific differences in the age groups. Task types B and C place significantly different demands on the children, as they require the handling of one or more unknown quantities. To answer the interviewer's questions, kindergarten children named specific numbers of marbles for the different colored boxes. They were not able to deal with interdependent quantities. This problem occurred less often in the responses of the fourth graders. In this age group, many children already manage to deal with the interdependent quantities represented as boxes by being able to generalize the inherent relationship.

Stephens and Wang (2008) emphasized that the inclusion of several variables in formal tasks, in particular, could stimulate relational thinking. Then a computational approach no longer appears to be possible. This claim can be confirmed in the responses of the fourth-graders for the representation with marbles and boxes.

6 Conceptualization of variables that are dependent on one another

In the following, the third research question is addressed, which examines how the variables represented as boxes in task type C are understood by the children. Since the number of marbles in the boxes could not be determined clearly, this offered different possible conceptions of variables. Based on theoretical considerations, the number of marbles in the boxes could be conceptualized as interdependent quantities. If different value pairs were considered, this would correspond to the concept of a changeable variable. However, children's interpretations of the variables represented as boxes differed greatly. The extent to which these answers reflected different conceptualizations of the variables is shown in the following.

Theoretical considerations as well as the data analysis showed that the conceptualizations of the variables depended on the ability to recognize and verbalize relationships between the quantities (Fig. 11). The possible connections between relational thinking and the
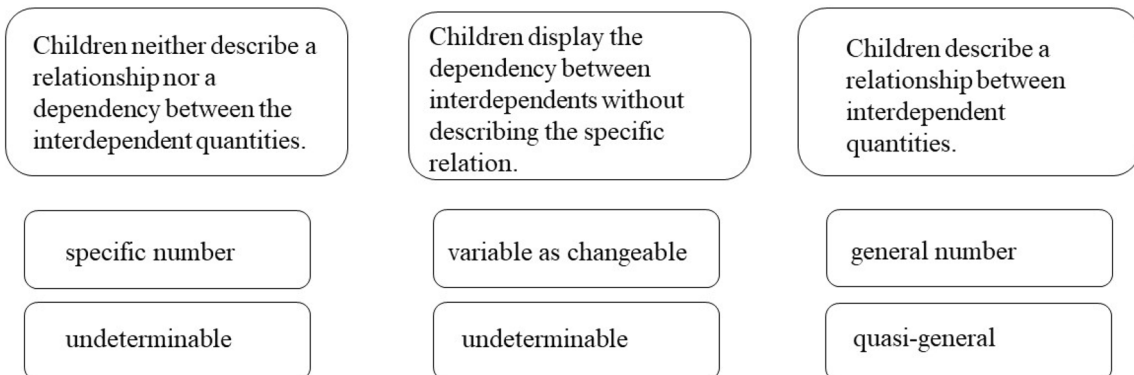

Fig. 11 How do children understand the variables represented by the material? Which variable concepts can be reconstructed in children's statements?

conceptualization of the variables represented as boxes are shown in the following overview and are explained below using transcripts of responses to task C1.

6.1 Conceptualization as a specific number

Children stated specific numbers as values for the boxes and did not understand them as examples. It is assumed that the children thought that the boxes had specific contents. According to the classification of the variable concepts, this category could be compared with the variable aspect of the unknown. Although the contents of the boxes of task type C could not be determined clearly, the children gave specific values corresponding to an unknown. Reference can be made here to the interview transcript of Christina (see Sect. 5.1).

The category of the specific number goes hand in hand with a failure to describe the relationship. If children gave specific numbers as the contents of the boxes, they did not describe the relationship that exists between the interdependent quantities in the boxes. Nor did they describe the interdependence between them. Thus, numerical values were given, which were understood as the real contents of the boxes. The interdependent quantities were conceptualized as a specific number. Another possibility was to state that the contents of the boxes could not be known. The latter describes the conceptualization as undeterminable.

6.2 Conceptualization as undeterminable

This category should be seen in connection with the conceptualization of the general number which follows later. Both conceptualizations have in common that they do not assume a certain number of marbles in the boxes. While in the conceptualization as a general number a relation between the interdependent quantities can be given, this does not happen in the conceptualization as undeterminable. Children indicated that they could not determine the contents of the box or that it was undeterminable. Concerning relational thinking, the category of the undeterminable was possible in two ways.

On the one hand, children recognized the dependency of the interdependent quantities. If the children conceptualized the interdependent quantities as changeable variables, it was possible to operate with different values of examples. This action could be supplemented by the children's statement that the contents of the boxes could not be determined, even if it was possible to operate with sample values in some cases. Although the dependency between the interdependent quantities was recognized, the contents of the boxes were indicated as 'undeterminable'. The difference between this case and that of the general number is that the relationship between the number of marbles is not specified. The fourth-grader Amy first describes that she needed to know the contents of the orange box to tell how many were in the green one. This response shows that she considers the contents of the two boxes to be interdependent. After that, she explained: "You can't work that out because you don't know how many are in the box (*points to the orange box*). […] Because the boxes are closed, you can't know". The contents of the boxes not only appeared interdependent to Amy but also undeterminable.

On the other hand, if no relationships or dependencies were formulated between the interdependent quantities, they could also be conceptualized as undeterminable. In this case, children indicated that the content of the boxes could not be known. As an example, consider the response of the second-grader Mara. She guessed that there could be a specific number of marbles in the boxes. However, she herself specified the number of marbles for Anna's box. She could not answer the interviewer's question about more or fewer marbles for the boy's box and stated that she did not know. She could not explain the relationship between the quantities. A description of the dependency of the numbers of marbles was also not made clear.

6.3 Conceptualization of a changeable quantity

According to Malle (1993), variables as changeable quantities describe dynamic changes within functional relationships. This aspect can be seen in tasks of type C when children could operate with different numerical examples. Although it was not possible to describe the relationships between the interdependent quantities of marbles, the contents of the boxes were viewed as changeable so that it was possible to operate with different values of examples. This was shown by the fact that the children gave examples of numerical values for the individual boxes, and could also respond to numerical examples given by the interviewer with the correct values for the other box. Likewise, such responses from children were categorized as changeable when they indicated that the numbers of marbles in the boxes were interdependent. Reference can be made here to the example of the second-grader Jakob (see Sect. 5.1).

When children conceptualized the interdependent number of marbles as changeable, it means that they had recognized the dependency between the boxes. They could operate with various sample values or indicate directly that there was a dependency. In contrast to the next conceptualizations as a general number or quasi-general number, however, the relation could not be specified.

6.4 Conceptualization as a general number

This category included the responses given by those children who made it clear that they could handle the interdependent number of marbles in the boxes. They were aware that there

was no specific solution to the type C tasks, but that different numbers of marbles were possible in the various boxes. Malle (1993) described a variable aspect where all numbers in the range aspect are represented at the same time. They are undetermined and appear especially in generalizations. Here this variable aspect is called the *general number* and describes general relationships. While the term general number is traditionally reserved for variables used in expressing properties and identities (and working with polynomials), I have found it useful to include this term in order to capture the highest level of generalized relational thinking expressed by these young children in their dealing with the functional task of the study—a level of thinking that could not be captured by using the same term of changeable quantity for both the less sophisticated and more advanced levels of thinking.

An understanding of interdependent quantities of marbles in the boxes can only go hand in hand with a recognition of the general relationship that exists between them. Therefore, it could be assumed that those children who managed to formulate a general relationship also conceptualized the quantities of marbles in the boxes as general numbers. These children did not specify the number of marbles for the boxes, but rather directly stated the relationship between the numbers of marbles in the boxes. Using phrases such as "the number that is in here" or "you don't know the content" also partially illustrated the indeterminacy of the number of marbles. The second-grader Julius explained the relationship as follows: "Because one thing (*taps the girl's marble*) and then you still have to calculate that here (*taps the girl's box*), there can be one, two or three (*waving his hand rhythmically in the air*) and there would always have to be one more in it (*taps on the boy's box*) than in this box (*points to the girl's box*) then it would be right".

6.5 Conceptualization as a quasi-general number

This category is to be seen in close relation to the previous one. Similarly to the previous one, the relationship between the interdependent quantities of marbles was recognized and generalized. To clarify the general validity of the existing relationship the formulation of specific numerical values was used. The concept of quasi-generals describes the expression of general relationships without resorting to the algebraic formula language (e.g., Akinwunmi, 2012; Fujii & Stephens, 2008). Children used several numbers combined with a generalizing expression, or the generalization was implicitly made clear by giving several numerical examples (see Aaron's explanation in Sect. 5.1).

Establishing relationships between interdependent quantities presupposes their conceptualization as general numbers. If the children were able to describe a general relationship, they conceptualized the interdependent quantities as general numbers or as quasi-general numbers. The difference between the two conceptualizations lies in the verbal expression. When using quasi-general numbers, the relationship was expressed with the help of the formulation of specific numerical values. The expression of a general number did not need numerical examples.

7 Discussion

The present empirical study shows that relational thinking as an aspect of algebraic thinking is also possible within representations with real material for kindergarten and primary school children of the ages of five to ten years. In the age group of primary school children, former studies concur with this result (cf., Carpenter et al., 2003; Steinweg, 2013). This finding is supplemented in the present study by a differentiated insight into how the children establish relationships between the represented quantities.

Two main approaches for dealing with the tasks can be identified, namely the number-oriented approach and the structure-oriented approach. The structure-oriented approach focuses on the quantities themselves instead of their specific values. Since the structure-oriented approach takes into account the entire task, its subsets, and their interrelationships, this approach can be referred to as relational thinking. In contrast, the number-oriented approach focuses on the specific number of marbles in the boxes. Children refer to specific numbers and thus take a procedural path, which can be assigned to arithmetic thinking. The fact that there is also a transition between these two ways of thinking became particularly clear in task B4 in which some children formed only partial sums.

The analysis of responses to task type B shows that there are differences both concerning the three age groups and the tasks. Tasks in which all values can be determined clearly, stimulate the schoolchildren to adopt number-oriented approaches. This propensity may be due to their access to arithmetical reasoning and therefore may also reflects their school experience. Tasks with variables, whose content does not have to be determined, increasingly encourage the formulation of structure-oriented approaches.

The distinction between the two approaches can be examined in all task types and is also illustrated using task type C. In this type, at least two interdependent variables are shown. It is possible to distinguish whether the children establish relationships between the interdependent quantities, and recognize a dependency, or whether they operate with specific numerical values. This distinction can also be classified in the area of tension between relational thinking and arithmetic thinking.

The present study makes it possible to compare the approaches of kindergarten children and 2nd graders as well

as 4th graders when dealing with the same tasks. The analysis shows that there are large differences between the age groups and allows the identification of trends in children's development. The kindergarten children mainly operated with numerical values. Nevertheless, it must be positively emphasized that a few kindergarten children were already able to indicate relationships between interdependent quantities. The second-graders are already able to formulate these to a greater extent, whereby the indication of numerical values still predominated. Most fourth-graders could formulate general relationships between the interdependent variables.

These statements are to be seen in contrast to results concerning task type B. The kindergarten children showed relational thinking more often in task type B and named numerical values when dealing with several interdependent quantities. However, the results were the opposite for the fourth-graders. In Part B, they more often described arithmetic ways of thinking and were inspired by several interdependent quantities to formulate relationships. Thus, the assumption of Stephens and Wang (2008) about the positive influence of dependent variables on relational thinking can be confirmed, at least for the group of fourth-graders within this study.

In a further analysis, the conceptualizations of the variables adopted by the children were considered. It could be shown that this aspect depends on the ability to recognize and describe relations. In addition to the categorizations of the variable concepts derived from theory (general number, quasi-general number, variable as a changeable quantity), further conceptualizations can be inductively identified from the data (specific number, undeterminable). The conceptualization of the general number and the quasi-general number can only go hand in hand with a recognition of the relationship between the interdependent quantities. If the dependency of the quantities is understood without being able to formulate a relationship, the variables represented as boxes are interpreted either as changeable or undeterminable. If the dependency cannot be recognized either, the children understand the variable as a specific number or undeterminable. Such an investigation of the early conceptualizations of variables is unexplored in the research literature, especially regarding kindergarten children.

8 Conclusion

Algebraic thinking becomes more important in primary school mathematics. Relational thinking in particular describes the basic idea of algebraic thinking when dealing with equations since it looks at the whole rather than at the arithmetic processes to be carried out. In this study I examined relational thinking skills using a non-symbolic type of representation.

The results show that relational thinking as a sub-area of algebraic thinking is already observable in kindergarten and elementary school children through real materials. This research thus confirms the results of previous studies regarding the age group of primary school children (Carpenter et al., 2003; Molina & Ambrose, 2008; Steinweg, 2013).

The examination of kindergarten children is new. They are also capable of establishing relationships, sometimes even between interdependent quantities. The comparison of the three age groups examined is particularly noteworthy. Regarding the kindergarten children, their previous mathematical knowledge was ascertained, while the answers of the school children are to be considered under the influence of their previous school experiences. At the same time, the abilities of kindergarten children are to be seen as a starting point for further mathematical learning regarding early algebra.

As a second focus, the study examined the conceptualization of the variables. The variable aspects according to Freudenthal (1973) and Malle (1993) could be recognized in the children's statements. Likewise, the linguistic generalizations according to Akinwunmi (2012) were recognized and included in a category of quasi-general numbers. The conceptualization of variables could be supplemented by a category of undeterminable quantities. It should be noted that the children's ability to conceptualize variables is always to be seen in their ability to recognize relationships. Both aspects are dependent on each other.

The interviews show that the materials used are particularly suited for making mental structuring visible to others. Regarding this aspect, the form of representation can be helpful, in taking account of the whole and in making this clear to others using gestures. Nevertheless, the representation with the help of real objects proves to be difficult: for example, the children increasingly assumed that the boxes must be filled with a specific number of marbles. This assumption can make it more difficult to switch from number-oriented thinking to relational thinking. At this point, an iconic representation with the help of photos of the tasks needs further investigation.

8.1 Implications for practice

Nevertheless, the task's design has the power to include various aspects of variables. Fujii and Stephens (2001) suggested that teaching should not focus solelyon the variable as the unknown. Rather, they demanded an early inclusion of other variable concepts and viewed quasi-variables within equations as a possible access. Steinweg (2013) also pointed out that by restricting variables to unknowns, a gap can be expected in later grades if the variables are to be regarded as general numbers or changeable.

The task design used in this empirical study demonstrated the possibility of using different variable concepts. According to the diverse answers given by the children, different approaches can be expected in school. But precisely these differences can prove to be fruitful for a school-based consideration of the variables if the different perspectives are addressed in the class discussion and carefully related to one another.

References

Akinwunmi, K. (2012). *Zur Entwicklung von Variablenkonzepten beim Verallgemeinern mathematischer Muster* [Development of variable concepts by generalization of patterns]. Vieweg+Teubner. https://doi.org/10.1007/978-3-8348-2545-2

Blanton, M., Isler-Baykal, I., Stroud, R., Stephens, A., Knuth, E., & Murphy Gardiner, A. (2019). Growth in children's understanding of generalizing and representing mathematical structure and relationships. *Educational Studies in Mathematics, 102*, 193–219. https://doi.org/10.1007/s10649-019-09894-7

Bruner, J. (1964). The course of cognitive growth. *American Psychologist, 19*(1), 1–15. https://doi.org/10.1037/h0044160

Cai, J., & Knuth, E. (Eds.). (2011). *Early algebraization*. Springer. https://doi.org/10.1007/978-3-642-17735-4

Carpenter, T. P., Franke, M. L. & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic & algebra in elementary school*. Heinemann.

Carpenter, T. P., Levi, L., Franke, M. L., & Zeringue Koehler, J. (2005). Algebra in elementary school: Developing relational thinking. *ZDM Mathematics Education, 37*(1), 53–59. https://doi.org/10.1007/BF02655897

Carraher, D., Schliemann, A. D., & Brizuela, B. M. (2001). Can young students operate on unknowns? In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th conference of the international group for the psychology of mathematics education* (Vol. 1, pp. 130–140). Freudenthal Institute.

Carraher, D. W., & Schliemann, A. D. (2007). Early algebra and algebraic reasoning. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 669–705). Information Age Publishing.

Freudenthal, H. (1973). Mathematics as an educational task. *Reidel*. https://doi.org/10.1007/978-94-010-2903-2

Fujii, T., & Stephens, M. (2001). Fostering an understanding of algebraic generalisation through numerical expressions: the role of quasi-variables. In H. Chick, K. Stacey, J. Vincent & J. Vincent (Eds.), *Proceedings of the 12th ICMI study conference: The future of the teaching and learning of algebra* (pp. 258–264). University of Melbourne.

Fujii, T., & Stephens, M. (2008). Using number sentences to introduce the idea of variable. In C. E. Greenes & R. Rubenstein (Eds.), *Algebra and algebraic thinking in school mathematics.* (Seventieth Yearbook, pp. 127–140). The National Council of Teachers of Mathematics.

Greeno, J. G. (1991). Number sense as situated knowing in a conceptual domain. *Journal for Research in Mathematics Education, 22*(3), 170–218.

Hoch, M., & Dreyfus, T. (2004). Structure sense in high school algebra: The effect of brackets. In *Proceedings of the 28th conference of the international group for the psychology of mathematics education* (Vol. 3, pp. 49–56).

Kaput, J. J. (2008). What is algebra? What is algebraic reasoning? In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades: Studies in mathematical thinking and learning* (pp. 5–17). Lawrence Erlbaum Associates.

Kieran, C. (2004a). Algebraic thinking in the early grades: What is it? *The Mathematics Educator, 8*(1), 139–151.

Kieran, C. (2004b). The core of algebra: Reflections on its main activities. In K. Stacey, H. Chick & M. Kendal (Eds.), *The future of the teaching and learning of algebra: The 12th ICMI study* (pp. 21–33). Kluwer. https://doi.org/10.1007/1-4020-8131-6

Kieran, C. (2011). Overall commentary on early algebraization: Perspectives for research and teaching. In J. Cai & E. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives* (pp. 579–593). Springer. https://doi.org/10.1007/978-3-642-17735-4_29

Kieran, C. (Ed.) (2018). *Teaching and learning algebraic thinking with 5- to 12-year olds: The global evolution of an emerging field of research and practice*. Springer. https://doi.org/10.1007/978-3-319-68351-5

Linchevski, L., & Livneh, D. (1999). Structure sense: The relationship between algebraic and numerical contexts. *Educational Studies in Mathematics, 40*, 173–196. https://doi.org/10.1023/A:1003606308064

Malle, G. (1993). *Didaktische Probleme der elementaren Algebra [Didactical problems of elementary algebra]*. Vieweg. https://doi.org/10.1007/978-3-322-89561-5

Mayring, P. (2010). *Qualitative Inhaltsanalyse. Grundlagen und Techniken [Qualitative content analysis. Basics and techniques]*. Beltz.

Melzig, D. (2013). Die Box als Stellvertreter: Ursprüngliche Erfahrungen zum Variablenbegriff [The box as a representive: Original experiences with the concept of variables]. Resource document. Duisburg-Essen Publications online. https://duepublico2.uni-due.de/servlets/MCRFileNodeServlet/duepublico_derivate_00034493/Diss_Melzig_Dagmar.pdf. Accessed 23 Aug 2021.

Molina, M., & Ambrose, R. (2008). From an operational to a relational conception of the equal sign. Third graders' developing algebraic thinking. *Focus on Learning Problems in Mathematics, 30*(1), 61–80.

Radford, L. (2011). Grade 2 students' non-symbolic algebraic thinking. In J. Cai & E. Knuth (Eds.), *Early algebraization* (pp. 303–322). Springer. https://doi.org/10.1007/978-3-642-17735-4_17

Schliemann, A. D., Carraher, D. W., & Brizuela, B. M. (2006). Bringing out the algebraic character of arithmetic: From children's ideas to classroom practice. *Lawrence Erlbaum Associates*. https://doi.org/10.4324/9780203827192

Schmidt, C. (2005). Analyse von Leitfadeninterviews [Analysis of guided interviews]. In U. Flick, E. von Kardoff & I. Steinke (Eds.), *Qualitative Forschung. Ein Handbuch* [Qualitative research. A manual] (pp. 447–456). Rowohlt Taschenbuch Verlag.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*(1), 1–36. https://doi.org/10.1007/BF00302715

Slavit, D. (1999). The role of operation sense in transitions from arithmetic to algebraic thought. *Educational Studies in Mathematics, 37*(3), 251–274.

Steinweg, A. S. (2013). Algebra in der Grundschule [Algebra in primary school]. *Springer.* https://doi.org/10.1007/978-3-8274-2738-0

Stephens, M., & Wang, X. (2008). Investigating some junctures in relational thinking: A study of Year 6 and Year 7 students from Australia and China. *Journal of Mathematics Education, 1*(1), 28–39.

Tall, D., Gray, E., Bin Ali, M., Crowley, L., DeMarois, P., McGowen, M., Pitta, D., Pinot, M., Thomas, M., & Yusof, Y. (2001). Symbols and the bifurcation between procedural and conceptual thinking. *Canadian Journal of Science, Mathematics & Technology Education, 1*(1), 81–104. https://doi.org/10.1080/14926150109556452