Journal of Applied and Computational Topology (2023) 7:397–425
https://doi.org/10.1007/s41468-022-00107-4

Geodesic complexity via fibered decompositions of cut loci

Stephan Mescher[1] · **Maximilian Stegemeyer**[2,3]

Received: 17 June 2022 / Revised: 1 November 2022 / Accepted: 3 November 2022 /
Published online: 23 November 2022

Abstract

The geodesic complexity of a Riemannian manifold is a numerical isometry invariant that is determined by the structure of its cut loci. In this article we study decompositions of cut loci over whose components the tangent cut loci fiber in a convenient way. We establish a new upper bound for geodesic complexity in terms of such decompositions. As an application, we obtain estimates for the geodesic complexity of certain classes of homogeneous manifolds. In particular, we compute the geodesic complexity of complex and quaternionic projective spaces with their standard symmetric metrics.

Keywords Geodesic complexity · Cut locus · Topological complexity · Motion planning

Mathematics Subject Classification 55M30 · 53C22

Contents

✉ Stephan Mescher
stephan.mescher@mathematik.uni-halle.de

Maximilian Stegemeyer
maximilian.stegemeyer@mis.mpg.de

1 Institut für Mathematik, Martin-Luther-Universität Halle-Wittenberg, Theodor-Lieser-Strasse 5, 06120 Halle (Saale), Germany

2 Max Planck Institute for Mathematics in the Sciences, Inselstrasse 22, 04103 Leipzig, Germany

3 Mathematisches Institut, Universität Leipzig, Augustusplatz 10, 04109 Leipzig, Germany

1 Introduction

The geodesic complexity of a complete Riemannian manifold is an integer-valued isometry invariant. It is given as a geometric analogue of the notion of topological complexity as introduced by M. Farber in Farber (2003). Geodesic complexity was originally defined by D. Recio-Mitter in the more general framework of metric spaces in Recio-Mitter (2021). Given a complete Riemannian manifold (M, g) we denote its space of length-minimizing geodesic segments by GM, seen as a subspace of the path space $C^0([0, 1], M)$ with the compact-open topology. Consider the endpoint evaluation map

$$\mathrm{ev}\colon GM \to M \times M, \qquad \mathrm{ev}(\gamma) = (\gamma(0), \gamma(1)).$$

The geodesic complexity of (M, g), denoted by $\mathsf{GC}(M, g)$, is defined as the smallest integer k for which there exists a decomposition of $M \times M$ into locally compact subsets $A_1, \dots, A_k$ with each A_i admitting a continuous local section of ev.

In the same way that topological complexity is motivated by a topological abstraction of the motion planning problem from robotics, geodesic complexity is motivated by an abstract notion of *efficient* motion planning. Sections of ev can be seen as geodesic motion planners since they assign to a pair of points $(p, q) \in M \times M$ a length-minimizing path connecting these two points.

As noted by Recio-Mitter in (Recio-Mitter 2021, p. 144), the main problem in determining the geodesic complexity of (M, g) lies in understanding the geodesic motion planning problem on its total cut locus. The latter is defined as

$$\mathrm{Cut}(M) = \{(p, q) \in M \times M \mid q \in \mathrm{Cut}_p(M)\} \subseteq M \times M,$$

where $\mathrm{Cut}_p(M) \subset M$ denotes the cut locus of $p \in M$ with respect to the given metric g. In this article we introduce the notion of a *fibered decomposition of the total cut locus*. If such a decomposition exists, it gives rise to a new upper bound for the geodesic complexity of M. The main applications of this upper bound are estimates for the geodesic complexity of certain homogeneous Riemannian manifolds. Similar situations were already studied by the authors in Mescher and Stegemeyer (2021). The upper bounds in the present article are however independent of the ones given in Mescher and Stegemeyer (2021). Various estimates from that article can be improved using the new results.

Let (M, g) be a complete Riemannian manifold and consider the extended exponential map

$$\mathrm{Exp}\colon TM \to M \times M, \qquad \mathrm{Exp}(v) = (\mathrm{pr}(v), \exp_{\mathrm{pr}(v)}(v)),$$

where $\mathrm{pr} : TM \to M$ denotes the bundle projection. We say that the total cut locus *admits a fibered decomposition* if there is a decomposition of $\mathrm{Cut}(M)$ into locally compact subsets $A_1, \dots, A_k$, such that the restriction

$$\pi_i := \mathrm{Exp}|_{\widetilde{A}_i} \colon \widetilde{A}_i \to A_i,$$

where $\widetilde{A}_i = \mathrm{Exp}^{-1}(A_i) \cap \widetilde{\mathrm{Cut}}(M)$, is a fibration for each $i \in \{1, \ldots, k\}$. Here, $\widetilde{\mathrm{Cut}}(M)$ denotes the *total tangent cut locus* which will be defined below. We will establish the following result.

Theorem (Theorem 3.2) *Let* (M, g) *be a complete Riemannian manifold. If the total cut locus* $\mathrm{Cut}(M)$ *admits a fibered decomposition* $A_1, \ldots, A_k$ *with fibrations* $\pi_i : \widetilde{A}_i \to A_i$ *for* $i \in \{1, \ldots, k\}$*, then the geodesic complexity of* M *can be estimated by*

$$\mathsf{GC}(M, g) \leq \sum_{i=1}^{k} \mathsf{secat}(\pi_i : \widetilde{A}_i \to A_i) + 1.$$

Here, secat denotes the sectional category of a fibration, which was introduced by A. Schwarz in Albert (1966) as the genus of a fiber space.

Evidently, this raises the question whether there are any interesting cases of Riemannian manifolds whose total cut loci admit fibered decompositions. For a homogeneous Riemannian manifold we will establish a tangible criterion on the cut locus of a single point implying that its total cut locus admits a fibered decomposition. We will further show that each irreducible compact simply connected symmetric space satisfies this condition, providing a large class of examples whose total cut loci admit fibered decompositions. By applying this result, we are able to compute the geodesic complexity of complex and quaternionic projective spaces with respect to their standard symmetric metrics.

Theorem (Theorem 4.6) *Let* $M = \mathbb{C}P^n$ *or* $\mathbb{H}P^n$ *equipped with its standard or Fubini-Study metric* g_{sym}*, where* $n \in \mathbb{N}$*. Then its geodesic complexity satisfies*

$$\mathsf{GC}(M, g_{\mathrm{sym}}) = 2n + 1.$$

In particular, the geodesic complexity of (M, g_{sym}) *equals the topological complexity of* M.

Moreover, using results by V. Ozols from Ozols (1974), we study the total cut locus of three-dimensional lens spaces with metrics of constant sectional curvature. We show that lens spaces of the form $L(p; 1)$, where $p \geq 3$, are further examples of homogeneous manifolds whose total cut loci admit fibered decompositions. As these spaces are not globally symmetric, this shows that fibered decompositions are not exclusively obtained in the globally symmetric case. A detailed analysis of the fibrations involved in the fibered decompositions of $\mathrm{Cut}(L(p; 1))$ shows that

$$6 \leq \mathsf{GC}(L(p; 1), g) \leq 7,$$

see Theorem 5.8, where g is a metric of constant sectional curvature.

This manuscript is organized as follows. In Sect. 2 we review the definitions of the total cut locus and of geodesic complexity and note some basic properties of these objects.

The central notion of a fibered decomposition of the total cut locus is introduced in Sect. 3. In that section we also prove the above mentioned upper bound on geodesic complexity and study a criterion for the existence of a fibered decomposition.

Symmetric spaces are studied in Sect. 4. After recalling some properties of root systems and related notions we prove that the total cut loci of irreducible compact simply connected symmetric spaces admit fibered decompositions and derive an upper bound on geodesic complexity. This will be applied to the examples of complex and quaternionic projective spaces and a particular complex Grassmannian.

Finally, in Sect. 5 we discuss the total cut loci of three-dimensional lens spaces and study a fibered decomposition to derive an upper bound on the geodesic complexity of these spaces.

2 Geodesic complexity and the total cut locus

In this section we quickly introduce the basic notions of geodesic complexity and of the total cut locus. For more properties of geodesic complexity and of the relation between cut loci and geodesic complexity we refer to Recio-Mitter (2021) and Mescher and Stegemeyer (2021).

Under a *locally compact decomposition* of a topological space X we understand a cover $A_1, \dots, A_k$ of X such that the A_i are pairwise disjoint and each $A_i, i \in \{1, \dots k\}$, is a locally compact subspace of X. As usual we equip the path space $C^0(I, M)$ with the compact-open topology, where $I = [0, 1]$ is the unit interval. For a Riemannian manifold (M, g) we let $GM \subseteq C^0(I, M)$ be the space of length-minimizing paths in M equipped with the subspace topology of $C^0(I, M)$, i.e.

$$GM = \{\gamma \in C^0(I, M) \mid \gamma \text{ is a length-minimizing geodesic in } M\}.$$

Definition 2.1 Let (M, g) be a complete Riemannian manifold and let

$$\mathrm{ev} : GM \to M \times M, \qquad \mathrm{ev}(\gamma) = (\gamma(0), \gamma(1)).$$

(1) A local section of ev is called a *geodesic motion planner*.
(2) Let $B \subseteq M \times M$ be a subset. The *subspace geodesic complexity of* B *in* (M, g) is defined to be the smallest integer k for which there is a locally compact decomposition $A_1, \dots, A_k$ of B with the following property: for each $i \in \{1, \dots, k\}$ there exists a continuous geodesic motion planner $A_i \to GM$. The subspace geodesic complexity of B in (M, g) is denoted by $\mathsf{GC}_{(M,g)}(B)$. If no such k exists, we put $\mathsf{GC}_{(M,g)}(B) := +\infty$.
(3) The *geodesic complexity of* (M, g) is defined to be the subspace geodesic complexity of $M \times M$ itself and is denoted by $\mathsf{GC}(M, g)$, i.e. $\mathsf{GC}(M, g) = \mathsf{GC}_{(M,g)}(M \times M)$.

Remark 2.2 (1) By the definition of topological complexity via locally compact decompositions, see (Farber 2008, Section 4.3), it is clear that the geodesic complexity of a Riemannian manifold (M, g) is bounded from below by the topological complexity $\mathsf{TC}(M)$ of M.

(2) If the metric under consideration is apparent, then we will drop the metric from the notation and simply write $\mathsf{GC}(M) := \mathsf{GC}(M, g)$ or $\mathsf{GC}_M(B) := \mathsf{GC}_{(M,g)}(B)$.

(3) Geodesic complexity was introduced by D. Recio-Mitter in Recio-Mitter (2021) for more general geodesic spaces, i.e. metric spaces in which any two points are connected by a length-minimizing path. Since every complete Riemannian manifold is a geodesic space, our definition is nothing but a particular case of Recio-Mitter's definition. Note however that our definition of geodesic complexity differs from the one in Recio-Mitter (2021) by one. More precisely, while in Recio-Mitter (2021) a geodesic space of geodesic complexity $k \in \mathbb{N}$ is decomposed into at least $k + 1$ locally compact subsets admitting geodesic motion planners, our definition requires the existence of a decomposition into k subsets having this property.

As pointed out in Recio-Mitter (2021) the geodesic complexity of a Riemannian manifold (M, g) crucially depends on the cut loci of M. We next recall the notion of the cut locus of a point as well as the total cut locus and the total tangent cut locus of a Riemannian manifold. The latter two notions were introduced in Recio-Mitter (2021).

Definition 2.3 Let (M, g) be a complete Riemannian manifold and let $p \in M$.

(1) Let $\gamma\colon [0, \infty) \to M$ be a unit-speed geodesic with $\gamma(0) = p$. We say that the *cut time of* γ is

$$t_{\mathrm{cut}}(\gamma) = \sup\{t > 0 \mid \gamma|_{[0,t]} \text{ is minimal}\}.$$

In case that $t_{\mathrm{cut}}(\gamma) < \infty$ we say that $\gamma(t_{\mathrm{cut}}(\gamma))$ is a *cut point of* p *along* γ and that $t_{\mathrm{cut}}(\gamma)\dot{\gamma}(0) \in T_pM$ is a *tangent cut point* of p.

(2) The set of tangent cut points of p is called the *tangent cut locus of* p and is denoted by $\widetilde{\mathrm{Cut}}_p(M) \subset T_pM$. The set of cut points of p is called the *cut locus of* p and is denoted by $\mathrm{Cut}_p(M)$.

(3) The *total tangent cut locus of* M is given by

$$\widetilde{\mathrm{Cut}}(M) = \bigcup_{p\in M} \widetilde{\mathrm{Cut}}_p(M) \subseteq TM.$$

The *total cut locus of* M is defined as

$$\mathrm{Cut}(M) = \bigcup_{p\in M} (\{p\} \times \mathrm{Cut}_p(M)) \subseteq M \times M.$$

Remark 2.4 Let (M, g) be a complete Riemannian manifold.

(1) By definition of the Riemannian exponential map $\exp_p\colon T_pM \to M$ at $p \in M$ we have

$$\exp_p(tv) = \gamma_v(t) \quad \text{for all } t > 0 \text{ and } v \in T_pM,$$

where γ_v is the unique geodesic starting at p with $\dot{\gamma}(0) = v$. Consequently, $\exp_p$ maps the tangent cut locus $\widetilde{\mathrm{Cut}}_p(M)$ onto the cut locus $\mathrm{Cut}_p(M)$.

(2) We recall the definition of the *global Riemannian exponential map*, see e.g. (Lee 2018, p. 128), which is given by

$$\mathrm{Exp}\colon TM \to M \times M, \quad \mathrm{Exp}(v) = (\mathrm{pr}(v), \exp_{\mathrm{pr}(v)}(v)).$$

Here, $\mathrm{pr}\colon TM \to M$ denotes the bundle projection. It is clear from the definitions that Exp maps the total tangent cut locus $\widetilde{\mathrm{Cut}}(M)$ onto the total cut locus $\mathrm{Cut}(M)$.

Finally, we want to note how the total cut locus of a Riemannian manifold (M, g) can be used to study the geodesic complexity of M. As Recio-Mitter argues in (Recio-Mitter 2021, Theorem 3.3) there is a unique continuous geodesic motion planner on $(M \times M) \setminus \mathrm{Cut}(M)$. By (Błaszczyk and Carrasquel-Vera 2018, Lemma 4.2) the latter is an open subset of $M \times M$, from which one derives the estimate

$$\mathsf{GC}_{(M,g)}(\mathrm{Cut}(M)) \leq \mathsf{GC}(M) \leq \mathsf{GC}_{(M,g)}(\mathrm{Cut}(M)) + 1. \tag{2.1}$$

Hence, in order to find bounds on the geodesic complexity of a complete Riemannian manifold (M, g) one can study the subspace geodesic complexity of its total cut locus $\mathrm{Cut}(M)$.

3 Fibered decompositions of cut loci

In this section we introduce the notion of a fibered decomposition of the total cut locus of a Riemannian manifold M and show that such a fibered decomposition of $\mathrm{Cut}(M)$ can be used to derive upper and lower bounds on the geodesic complexity of M. After that we give a condition on the cut locus of a point $p \in M$ of a homogeneous Riemannian manifold which implies that the total cut locus admits a fibered decomposition.

Definition 3.1 Let (M, g) be a complete Riemannian manifold.

(1) A locally compact decomposition $A_1, \ldots, A_k$ of $\mathrm{Cut}(M)$ is called a *fibered decomposition of* $\mathrm{Cut}(M)$ if the following holds: for each $i \in \{1, \ldots, k\}$ the restricted exponential map

$$\pi_i = \mathrm{Exp}\,|_{\widetilde{A}_i} \colon \widetilde{A}_i \to A_i$$

is a fibration, where $\widetilde{A}_i = \mathrm{Exp}^{-1}(A_i) \cap \widetilde{\mathrm{Cut}}(M)$.

(2) Similarly, if $p \in M$, then a locally compact decomposition $B_1, \ldots, B_k$ of $\mathrm{Cut}_p(M)$ is called a *fibered decomposition of* $\mathrm{Cut}_p(M)$ if

$$\exp_p|_{\widetilde{B}_i} \colon \widetilde{B}_i \to B_i$$

is a fibration, where $\widetilde{B}_i = \exp_p^{-1}(B_i) \cap \widetilde{\mathrm{Cut}}_p(M)$.

Here, under a fibration we always understand a Hurewicz fibration in the sense of homotopy theory. Next we will discuss how fibered decompositions of cut loci yield new lower and upper bounds for geodesic complexity.

Theorem 3.2 *Let (M, g) be a complete Riemannian manifold. If the total cut locus* $\mathrm{Cut}(M)$ *admits a fibered decomposition $A_1, \ldots, A_k$ with fibrations $\pi_i : \widetilde{A}_i \to A_i$ for $i \in \{1, \ldots, k\}$ as in Definition 3.1.(1), then the geodesic complexity of M can be estimated by*

$$\mathsf{GC}(M) \leq \sum_{i=1}^{k} \mathsf{secat}(\pi_i) + 1.$$

Proof We begin by showing that continuous local sections of π_i induce continuous geodesic motion planners. Let $C \subseteq A_i$ be a locally compact subset of A_i and assume that $s : C \to \widetilde{A}_i$ is a continuous section of the fibration π_i. In particular, we have for $(p, q) \in C$ that

$$\mathrm{Exp}(s(p, q)) = (p, q).$$

We define $\sigma : C \to GM$ by

$$\sigma((p, q))(t) = \mathrm{pr}_2(\mathrm{Exp}(ts(p, q))) \quad \text{for } \ t \in [0, 1],$$

where $\mathrm{pr}_2 : M \times M \to M$ denotes the projection onto the second component. This is clearly a geodesic motion planner. In order to see that map σ is also continuous note that the map

$$\widetilde{\sigma} : C \times I \to M, \quad ((p, q), t) \mapsto \mathrm{pr}_2(\mathrm{Exp}(ts(p, q)))$$

is continuous since it is a composition of continuous maps. By a general property of the compact-open topology, the continuity of $\widetilde{\sigma}$ implies the continuity of the induced map $\sigma : C \to GM$, see e.g. [Bredon 2013, Theorem VII.2.4].

For each $i \in \{1, \ldots, k\}$ we put $m_i := \mathsf{secat}(\pi_i)$. Then, see e.g. (Mescher and Stegemeyer 2021, Lemma 4.1), for each i there is a locally compact decomposition $C_{i,1}, \ldots, C_{i,m_i}$ of A_i for which there is a continuous section of π_i on each $C_{i,j}$, $j \in \{1, \ldots, m_i\}$. Since the sets $A_1, \ldots, A_k$ form a decomposition of $\mathrm{Cut}(M)$, we see that the sets

$$\{C_{i,j} \mid i \in \{1, \ldots, k\}, j \in \{1, \ldots, m_i\}\}$$

are a decomposition of $\mathrm{Cut}(M)$ with each $C_{i,j}$ locally compact. By the first part of the proof we see that each $C_{i,j}$ admits a continuous geodesic motion planner. This shows that

$$\mathsf{GC}_{(M,g)}(\mathrm{Cut}(M)) \leq \sum_{i=1}^{k} m_i = \sum_{i=1}^{k} \mathsf{secat}(\pi_i).$$

Combining this inequality with the inequality (2.1) completes the proof. □

In the subsequent sections we will see examples of upper bounds on geodesic complexity by virtue of Theorem 3.2. The next result however shows how a fibered decomposition of the total cut locus $\mathrm{Cut}(M)$ gives rise to a lower bound on $\mathsf{GC}_{(M,g)}(\mathrm{Cut}(M))$. Before we state the result, we recall the definition of the *velocity map*, see [Mescher and Stegemeyer 2021, Definition 3.1], i.e. the map given by

$$v : GM \to TM, \qquad v(\gamma) = \dot{\gamma}(0).$$

The velocity map is continuous by [Mescher and Stegemeyer 2021, Proposition 3.2]. Furthermore, we recall that the sectional category of a fibration $p\colon E \to B$ is defined by considering open covers $U_1, \ldots, U_k$ of B such that each U_i, $i \in \{1, \ldots, k\}$ admits a continuous local section of p. The geodesic complexity of a complete Riemannian manifold M however is defined via locally compact decompositions of $M \times M$. In order to compare these two concepts in the following theorem, we employ the notion of *generalized sectional category* as introduced by J. M. García Calcines in [García-Calcines 2019, Definition 2.1].

Definition 3.3 Let $p\colon E \to B$ be a fibration. The *generalized sectional category* $\mathsf{secat}_g(p)$ is defined as the smallest integer k for which there exists a cover $A_1, \ldots, A_k$ of B such that each A_i, $i \in \{1, \ldots, k\}$, admits a continuous local section of p.

Note that the sets A_i in the above definition can be arbitrary subsets of B. García-Calcines shows in [García-Calcines 2019, Theorem 2.7] that if $p\colon E \to B$ is a fibration and if E and B are absolute neighborhood retracts, one has

$$\mathsf{secat}_g(p) = \mathsf{secat}(p).$$

Theorem 3.4 *Let (M, g) be a complete Riemannian manifold. Assume that the total cut locus $\mathrm{Cut}(M)$ admits a fibered decomposition $A_1, \ldots, A_l$ with fibrations $\pi_i\colon \widetilde{A}_i \to A_i$ for $i \in \{1, \ldots, l\}$. Furthermore, assume that all $\widetilde{A}_i$ and A_i are absolute neighborhood retracts. Then*

$$\mathsf{GC}_{(M,g)}(\mathrm{Cut}(M)) \geq \max\{\mathsf{secat}(\pi_i) \mid i \in \{1, \ldots, l\}\}.$$

Proof Let $m \in \mathbb{N}$ be the maximum of $\{\mathsf{secat}(\pi_i) \mid i \in \{1, \ldots, l\}\}$ and choose $i_0 \in \{1, \ldots, l\}$ such that $\mathsf{secat}(\pi_{i_0}) = m$. Assume that the assertion of the theorem is false. Then there are a locally compact decomposition $B_1, \ldots, B_k$ of $\mathrm{Cut}(M)$ with $k < m$ and continuous geodesic motion planners $s_j\colon B_j \to GM$ for $j \in \{1, \ldots, k\}$. For $i \in \{1, \ldots, k\}$ set $C_i = B_i \cap A_{i_0}$. It is possible that there are $i \in \{1, \ldots, k\}$ with $C_i = \emptyset$. By reordering the B_i we can arrange that $C_1, \ldots, C_r \neq \emptyset$ and $C_{r+1}, \ldots, C_k = \emptyset$ for some $1 \leq r \leq k$. The sets $C_1, \ldots, C_r$ form a cover of A_{i_0}. For $j \in \{1, \ldots, r\}$ we define a map

$$\sigma_j\colon C_j \to \widetilde{A}_{i_0}, \qquad \sigma_j = v \circ s_j|_{C_j},$$

where v denotes the velocity map. It is clear that σ_j is continuous. We claim that it is a section of $\widetilde{A}_{i_0}$. For any $(p,q) \in C_j$ the path $s_j(p,q)$ is a minimal geodesic. Thus, there is $w \in \widetilde{\mathrm{Cut}}_p(M)$ with

$$s_j(p,q)(t) = \exp_p(tw).$$

By definition of the velocity map, we obtain

$$\sigma_j(p,q) = (v \circ s_j)(p,q) = w$$

and by definition of $\widetilde{A}_{i_0}$ it is clear that $w \in \widetilde{A}_{i_0}$. Consequently,

$$(\pi_{i_0} \circ \sigma_j)(p,q) = (\mathrm{Exp}\,|_{\widetilde{A}_{i_0}} \circ \sigma_j)(p,q) = (p, \exp_p(w)) = (p,q),$$

which shows that σ_j is a continuous section of π_{i_0}. Hence, we obtain

$$\mathsf{secat}_g(\pi_{i_0}) \leq r \leq k < m.$$

Since $\pi_{i_0}\colon \widetilde{A}_{i_0} \to A_{i_0}$ is a fibration with $\widetilde{A}_{i_0}$ and A_{i_0} being absolute neighborhood retracts, we derive from [García-Calcines 2019, Theorem 2.7] that

$$\mathsf{secat}(\pi_{i_0}) = \mathsf{secat}_g(\pi_{i_0}) < m,$$

which is a contradiction. This completes the proof. □

Corollary 3.5 *Let (M,g) be a complete Riemannian manifold. Assume that*

$$\pi = \mathrm{Exp}\,|_{\widetilde{\mathrm{Cut}}(M)} \colon \widetilde{\mathrm{Cut}}(M) \to \mathrm{Cut}(M)$$

is a fibration and assume that $\widetilde{\mathrm{Cut}}(M)$ and $\mathrm{Cut}(M)$ are absolute neighborhood retracts. Then

$$\mathsf{GC}_{(M,g)}(\mathrm{Cut}(M)) = \mathsf{secat}(\pi) \quad \text{and} \quad \mathsf{secat}(\pi) \leq \mathsf{GC}(M,g) \leq \mathsf{secat}(\pi) + 1.$$

Proof It is clear by Theorem 3.4 that

$$\mathsf{GC}_{(M,g)}(\mathrm{Cut}(M)) \geq \mathsf{secat}(\pi).$$

The reverse inequality follows from the proof of Theorem 3.2. The second asserted inequality follows from equation (2.1). □

In Sect. 4 we will show that the symmetric metrics on complex and quaternionic projective spaces are examples for which the conditions of Corollary 3.5 are satisfied.

In the following we will derive a tangible criterion in order to find fibered decompositions of the total cut locus. In the setting of homogeneous Riemannian manifolds we want to use a fibered decomposition of the cut locus of a point to obtain a fibered decomposition of the total cut locus, whose fibrations will in fact be fiber bundles. We will see applications of this idea in Sects. 4 and 5.

Note that if a compact group of isometries acts transitively on a Riemannian manifold, then the manifold is necessarily complete. If K is a group of isometries of a Riemannian manifold which fixes a point $p \in M$, then $k \cdot \mathrm{Cut}_p(M) = \mathrm{Cut}_p(M)$ for all $k \in K$.

Definition 3.6 Let (M, g) be a Riemannian manifold and assume that G is a group of isometries acting transitively on M. Let $p \in M$ be a point and let $K \subseteq G$ be its isotropy group. Let $B_1, \dots, B_m$ be a locally compact decomposition of $\mathrm{Cut}_p(M)$. We say that the decomposition is *isotropy-invariant* if $k \cdot B_i = B_i$ for all $i = 1, \dots, m$ and all $k \in K$.

In the following let (M, g) be a Riemannian manifold and let G be a group of isometries of M acting transitively on M. We denote the group action by $\Phi \colon G \times M \to M$. We shall use the shorthand notation $\Phi_g = \Phi(g, \cdot) \colon M \to M$ as well as $\Phi(g, p) = g \cdot p$ for $g \in G$, $p \in M$.

Our aim is to use the homogeneity of M to construct a fibered decomposition of the total cut locus $\mathrm{Cut}(M)$ out of a fibered decomposition of the cut locus of one single point in M.

In the following, we fix a point $p \in M$ and let $B_1, \dots, B_k$ be a decomposition of $\mathrm{Cut}_p(M)$ *which is both isotropy-invariant and a fibered decomposition such that the associated fibrations $\widetilde{B}_i \to B_i$ are fiber bundles for $i \in \{1, \dots, k\}$.*

Let K be the isotropy group of p and let $\mathrm{pr} : G \to M \cong G/K$ denote the canonical projection. For $i \in \{1, \dots, k\}$ set

$$A_i = \{(q, r) \in \mathrm{Cut}(M) \mid r \in \Phi_g(B_i) \text{ for some } g \in G \text{ with } \mathrm{pr}(g) = q\}$$

and

$$\widetilde{A}_i = \{(q, v) \in \widetilde{\mathrm{Cut}}(M) \mid v \in (D\Phi_g)_p(\widetilde{B}_i) \text{ for some } g \in G \text{ with } \mathrm{pr}(g) = q\}.$$

We further consider the maps

$$\pi_i : A_i \to M, \quad \pi_i(q, r) = q, \qquad \widetilde{\pi}_i : \widetilde{A}_i \to M, \quad \widetilde{\pi}_i(q, v) = q, \qquad i \in \{1, \dots, k\}.$$

Lemma 3.7 *In the present setting the following holds for each $i \in \{1, \dots, k\}$:*

1. *$\pi_i : A_i \to M$ is a fiber bundle with typical fiber B_i.*
2. *$\widetilde{\pi}_i : \widetilde{A}_i \to M$ is a fiber bundle with typical fiber $\widetilde{B}_i$.*

Note that by fiber bundle, we mean a fiber bundle in the continuous category. We do not assume that the sets B_i carry any differentiable structure.

Proof We want to show that both A_i and $\widetilde{A}_i$ are locally trivial. Fix an $i \in \{1, \dots, k\}$ and let

$$\pi_i \colon A_i \to M, \quad \pi_i(q, r) = q,$$

be the projection on the first factor. Let $U \subseteq M$ be an open set on which there exists a continuous section $s\colon U \to G$ of pr. Define $\varphi_i\colon A_i|_U \to U \times B_i$ by

$$\varphi_i(q, r) = (q, s(q)^{-1} \cdot r) \quad \text{for} \quad (q, r) \in A_i|_U.$$

This is a well-defined map since if $(q, r) \in A_i$, then there is a $b \in B_i$ such that $r = g \cdot b$ for some $g \in G$ with $g \cdot p = q$. Therefore, by the isotropy invariance of the decomposition $B_1, \ldots, B_k$,

$$s(q)^{-1} \cdot r = (s(q)^{-1} g) \cdot b \in B_i$$

since $s(q)^{-1} g \in K$. Evidently, φ_i is a homeomorphism. For each point $(q, r) \in A_i$ there is such an open neighborhood U of q admitting a continuous section $s\colon U \to G$ of pr. Thus, the above construction shows that $A_i \to M$ is a continuous fiber bundle. The proof for $\widetilde{A}_i$ is analogous. One defines local trivializations of the form $\psi_i\colon A_i|_U \to U \times \widetilde{B}_i$, where U is an open subset of M admitting a continuous section $s\colon U \to G$ of pr, by

$$\psi_i(q, v) = (q, (D\Phi_{s(q)^{-1}})_q v) \quad \text{for} \quad (q, v) \in \widetilde{A}_i.$$

As for φ_i one shows that ψ_i is well-defined and a homeomorphism. □

Theorem 3.8 *Let (M, g) be a Riemannian manifold and G be a group of isometries of M acting transitively on M. Fix a point $p \in M$. Let $B_1, \ldots, B_k$ be a decomposition of $\mathrm{Cut}_p(M)$ which is both isotropy-invariant and a fibered decomposition such that the associated fibrations $\widetilde{B}_i \to B_i$ are fiber bundles. For $i = 1, \ldots, k$ let C_i be the typical fiber of the bundle $\widetilde{B}_i \to B_i$. Define the sets $A_i \subseteq \mathrm{Cut}(M)$ as above. Then the decomposition of $\mathrm{Cut}(M)$ into $A_1, \ldots, A_k$ is a fibered decomposition. More precisely, the restriction $\mathrm{Exp}\,|_{\widetilde{A}_i}\colon \widetilde{A}_i \to A_i$ is a fiber bundle with typical fiber C_i.*

Proof Fix $i \in \{1, \ldots, k\}$ and let $p \in M$. As discussed in the proof of Lemma 3.7, we can find an open neighborhood $U \subseteq M$ of p and local trivializations $\varphi_i\colon A_i|_U \to U \times B_i$ and $\psi_i\colon \widetilde{A}_i|_U \to U \times \widetilde{B}_i$. If φ_i and ψ_i are given as in that proof, then the inverse of φ_i is explicitly given by

$$\varphi_i^{-1}\colon U \times B_i \to A_i|_U, \quad \varphi_i^{-1}(q, b) = (q, s(q) \cdot b),$$

where $s\colon U \to G$ is a local section of $\mathrm{pr}\colon G \to M$. We claim that the diagram

$$\begin{tikzcd}
\widetilde{A}_i|_U \arrow[r, "\mathrm{Exp}\,|_{\widetilde{A}_i|_U}"] \arrow[d, "\psi_i"] & A_i|_U \arrow[d, "\varphi_i"'] \\
U \times \widetilde{B}_i \arrow[r, "(\mathrm{id}_U, \exp_p)"'] & U \times B_i
\end{tikzcd}$$

commutes. To see this, let $(q, v) \in \widetilde{A}_i|_U$. Then

$$\psi_i(q, v) = (q, (D\Phi_{s(q)^{-1}})_q v) = (q, (D\Phi_{s(q)})_p^{-1} v) \in U \times \widetilde{B}_i.$$

By naturality of the exponential map, see [Lee 2018, Proposition 5.20], it thus holds that

$$\begin{aligned}(\varphi_i^{-1} \circ (\mathrm{id}_U, \exp_p) \circ \psi_i)(q, v) &= \varphi_i^{-1}(q, \exp_p((D\Phi_{s(q)})_p^{-1} v)) \\ &= (q, s(q)s(q)^{-1} \cdot \exp_q(v)) \\ &= (q, \exp_q(v)) \\ &= \mathrm{Exp}(q, v).\end{aligned}$$

By assumption the restriction $\exp_p|_{\widetilde{B}_i} : \widetilde{B}_i \to B_i$ is a fiber bundle. Hence, by choosing an open subset $V \subseteq B_i$ such that $\widetilde{B}_i|_V$ is trivial and considering $\varphi_i^{-1}(U \times V)$ we obtain an open set in A_i over which the map $\mathrm{Exp}|_{\widetilde{A}_i} : \widetilde{A}_i \to A_i$ is trivial. Since A_i is covered by such trivializations, this proves the claim. □

4 The total cut loci of symmetric spaces

In this section we turn to the study of cut loci in irreducible compact simply connected symmetric spaces and show that the total cut locus of these spaces always admits a fibered decomposition. Furthermore, we derive a new upper bound for the geodesic complexity of symmetric spaces. Note that this section is related to (Mescher and Stegemeyer 2021, Sect. 3) where the authors proved an upper bound for irreducible compact simply connected symmetric spaces in terms of the sectional category of the isometry bundle $\mathrm{Isom}(M) \to M$ over a symmetric space M and certain subspace geodesic complexities. The upper bound in the current section is derived independently of this previous result.

We briefly recall the most important notions related to root systems of symmetric spaces. Let $M = G/K$ be a symmetric space with (G, K) being a Riemannian symmetric pair. Denote the canonical projection by $\pi : G \to G/K \cong M$. There is a decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ of the Lie algebra $\mathfrak{g}$ of G such that $\mathfrak{m} \cong T_{\pi(e)}M$ is a linear isometry. We set $o = \pi(e)$, where e is the unit element of G. Consider the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$ and choose a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}_{\mathbb{C}}$. A *root* of $\mathfrak{g}_{\mathbb{C}}$ is an element $\alpha \in \mathfrak{h}^*$ of the dual space of $\mathfrak{h}$ for which there exists an $X \in \mathfrak{g}_{\mathbb{C}} \setminus \{0\}$ with

$$[H, X] = \alpha(H)X \quad \text{for all} \quad H \in \mathfrak{h}.$$

If $\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{m}$, then consider the restriction $\alpha|_{\mathfrak{a}}$ of a root of $\mathfrak{g}_{\mathbb{C}}$. If this restriction is non-zero, we call it a *root* of the symmetric pair (G, K). We choose and fix a set of *simple roots* of the symmetric pair (G, K) and denote it by $\pi(G, K)$. We further let δ denote its *highest root*. See [Helgason 1978, Section X.3] or (Bröcker and Dieck 1995, Section V.4) for details on these notions. Due to the compactness of G we can choose an Ad_G-invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ and identify the roots with vectors in $\mathfrak{a}$ via this inner product. Then a *Weyl chamber* of $\pi(G, K)$ can be defined as

$$W := \{X \in \mathfrak{a} \mid \langle \gamma, X \rangle > 0 \ \ \forall \gamma \in \pi(G, K)\}.$$

Note that one can define the other Weyl chambers by choosing other systems of simple roots. The *Weyl group* $W(G, K)$ of the symmetric pair (G, K) is generated by the reflections s_α on the hyperplanes

$$\{H \in \mathfrak{a} \mid \alpha(H) = 0\}.$$

It is a finite group and acts simply transitively on the set of Weyl chambers of (G, K).

T. Sakai has studied the cut loci of compact simply connected symmetric spaces in Sakai (1978), see also Sakai (1977) and Sakai (1978). We summarize the main results. If there are two or more simple roots of (G, K), put

$$\mathcal{D} := \{\Delta \subset \pi(G, K) \mid \Delta \neq \emptyset, \ \ \delta \notin \Delta\}.$$

In case there is only one simple root γ, this is then also the highest root and we set

$$\mathcal{D} := \{\{\gamma\}\}.$$

If there are two or more simple roots, we set

$$S_\Delta := \left\{X \in \overline{W} \mid \langle \gamma, X \rangle > 0 \ \ \forall \gamma \in \Delta, \ \langle \gamma, X \rangle = 0 \ \ \forall \gamma \in \pi(G, K) \setminus \Delta, \ 2\langle \delta, X \rangle = 1\right\}$$

for each $\Delta \in \mathcal{D}$. In case there is a single simple root γ, we define

$$S_{\{\gamma\}} := \{X \in \mathfrak{a} \mid 2\langle \gamma, X \rangle = 1\}.$$

As usual, we denote by $\exp : \mathfrak{g} \to G$ the exponential map of G and define

$$\overline{\exp} : \mathfrak{m} \to M, \qquad \overline{\exp} := \pi \circ \exp|_{\mathfrak{m}}.$$

This in fact agrees with the Riemannian exponential at the point o under the canonical identification $\mathfrak{m} \cong T_oM$ and is often denoted by Exp. In order not to confuse it with the global Riemannian exponential map used in Section 3, we denote it by $\overline{\exp}$. For $\Delta \in \mathcal{D}$ set

$$\widetilde{\Phi}_\Delta : K \times S_\Delta \to M, \qquad \widetilde{\Phi}_\Delta(k, X) = \overline{\exp}(\mathrm{Ad}_k(X)),$$

and

$$\widetilde{\Psi}_\Delta : K \times S_\Delta \to \mathfrak{m}, \qquad \widetilde{\Psi}_\Delta(k, X) = \mathrm{Ad}_k(X).$$

Furthermore, we define

$$Z_\Delta := \{k \in K \mid \overline{\exp}(\mathrm{Ad}_k(X)) = \overline{\exp}(X)\ \forall X \in S_\Delta\}$$

and

$$K_\Delta = \{k \in K \mid \mathrm{Ad}_k(X) = X\ \forall X \in S_\Delta\}.$$

Evidently, Z_Δ and K_Δ are closed subgroups of K with $K_\Delta \subseteq Z_\Delta$. Sakai shows in (Sakai 1978, Proposition 4.10), that if $\Delta \in \mathcal{D}$, then the map $\widetilde{\Phi}_\Delta$ induces a differentiable embedding

$$\Phi_\Delta : K/Z_\Delta \times S_\Delta \to M.$$

Define $C_\Delta := \operatorname{im} \Phi_\Delta$ for each $\Delta \in \mathcal{D}$. The cut locus of the point $o = \pi(e) \in M$ is then given by

$$\mathrm{Cut}_o(M) = \bigcup_{\Delta \in \mathcal{D}} C_\Delta$$

see (Sakai 1978, Theorem 5.3). Moreover, the set $\{C_\Delta\}_{\Delta \in \mathcal{D}}$ forms a locally compact decomposition of $\mathrm{Cut}_o(M)$.

Lemma 4.1 *The map $\widetilde{\Psi}_\Delta$ induces a continuous embedding $\Psi_\Delta \colon K/K_\Delta \times S_\Delta \to \mathfrak{m}$ and for $\widetilde{C}_\Delta := \mathrm{im}(\Psi_\Delta)$ we have that*

$$\widetilde{C}_\Delta = \overline{\exp}^{-1}(C_\Delta) \cap \widetilde{\mathrm{Cut}_o}(M).$$

Proof By definition of K_Δ it is clear that $\widetilde{\Psi}_\Delta$ induces a continuous map $\Psi_\Delta : K/K_\Delta \times S_\Delta \to \mathfrak{m}$. To prove that Ψ_Δ is an embedding, we closely follow the proof of (Sakai 1978, Proposition 4.10). For the injectivity of Ψ_Δ, let $k, k' \in K$ and $X, X' \in S_\Delta$ such that $\mathrm{Ad}_{k'} X' = \mathrm{Ad}_k X$. We need to show that $[k'] = [k]$ in K/K_Δ and that $X = X'$. Clearly, it holds that

$$\mathrm{Ad}_{k^{-1}k'} X' = X.$$

Therefore, by (Helgason 1978, Proposition VI.2.2) we know that there is an element s of the Weyl group $W(G, K)$ of the Riemannian pair (G, K) such that $sX' = X$. But since X and X' are in the closure of the same Weyl chamber, they have to be equal, see (Sakai 1978, p. 131). This also shows that $k^{-1}k' \in K_\Delta$, so $[k'] = [k]$ in K/K_Δ.

In order to show that Ψ_Δ is an embedding, let $(k_n)_{n\in\mathbb{N}}$ be a sequence in K and $(X_n)_{n\in\mathbb{N}}$ be a sequence in S_Δ such that $\mathrm{Ad}_{k_n}(X_n) \to \mathrm{Ad}_k X$ for $n \to \infty$, where $k \in K$ and $X \in S_\Delta$. We want to show that $[k_n] \to [k]$ in K/K_Δ and $X_n \to X$ for $n \to \infty$. Assume that this does not hold. Then by compactness of K there are $k_0 \in K$ and $Y \in \overline{W}$ and there are subsequences $(k_{n_i})_{i\in\mathbb{N}}$ and $(X_{n_i})_{i\in\mathbb{N}}$ with $k_{n_i} \to k_0$ and $X_{n_i} \to Y$ for $i \to \infty$ with $([k_0], Y) \neq ([k], X)$. By continuity of Ad we have

$\mathrm{Ad}_{k_0}Y = \mathrm{Ad}_k X$ so as argued above for the injectivity, we obtain $X = Y$ and $[k_0] = [k]$ in K/K_Δ which gives a contradiction. This shows the sequential continuity of Φ_Δ^{-1}, thereby yielding that Ψ_Δ is an embedding.

Finally, by (Sakai 1978, p.133) we have that $\widetilde{C}_\Delta = \mathrm{im}(\Psi_\Delta) \subseteq \widetilde{\mathrm{Cut}}_o(M)$. Moreover, it is clear by construction that $\overline{\exp}(\widetilde{C}_\Delta) = C_\Delta$. In order to show that

$$(\overline{\exp}|_{\widetilde{\mathrm{Cut}}_o(M)})^{-1}(C_\Delta) \subseteq \widetilde{C}_\Delta$$

let $X \in \widetilde{\mathrm{Cut}}_o(M)$ such that $\overline{\exp}(X) \in C_\Delta$. Then there is $k \in K$ with

$$k \cdot \overline{\exp}(X) = \overline{\exp}(\mathrm{Ad}_k(X)) = \overline{\exp}(Y)$$

for some $Y \in S_\Delta$. We set $q = \overline{\exp}(Y)$ and $\widetilde{X} = \mathrm{Ad}_k X$. Clearly, $\widetilde{X} \in \widetilde{\mathrm{Cut}}_o(M)$ and since $\widetilde{X} \in \overline{\exp}^{-1}(q)$ we have by (Sakai 1978, Lemma 4.7) that there is an $h \in Z_\Delta$ with $\widetilde{X} = \mathrm{Ad}(h)(Y)$. But this implies that $X = \mathrm{Ad}(k^{-1}h)(Y)$ which shows that $X \in \widetilde{C}_\Delta$. □

It is clear by construction that the decomposition $\{C_\Delta\}_{\Delta\in\mathcal{D}}$ of $\mathrm{Cut}_o(M)$ is isotropy-invariant. The next theorem shows that it is a fibered decomposition of $\mathrm{Cut}_o(M)$.

Theorem 4.2 *Let $M = G/K$ be an irreducible compact simply connected symmetric space with (G, K) being a Riemannian symmetric pair and let $p \in M$. Then the cut locus of p admits a decomposition which is both isotropy-invariant and a fibered decomposition with the associated fibrations being fiber bundles.*

Proof As we have already argued, the decomposition of $\mathrm{Cut}_o(M)$ into the C_Δ, $\Delta \in \mathcal{D}$, is a decomposition into locally compact subsets and is isotropy-invariant. Hence, it remains to show that it is a fibered decomposition. Let $\Delta \in \mathcal{D}$ and consider the map

$$\chi : K/K_\Delta \times S_\Delta \to K/Z_\Delta \times S_\Delta, \qquad \chi(kK_\Delta, X) = (kZ_\Delta, X).$$

We derive from Lemma 4.1 that the diagram

$$\begin{array}{ccc} K/K_\Delta \times S_\Delta & \xrightarrow{\chi} & K/Z_\Delta \times S_\Delta \\ \downarrow{\scriptstyle \Psi_\Delta} & & \downarrow{\scriptstyle \Phi_\Delta} \\ \widetilde{C}_\Delta & \xrightarrow{\overline{\exp}|_{\widetilde{C}_\Delta}} & C_\Delta \end{array}$$

commutes where the vertical arrows are homeomorphisms. It is well-known, see e.g. (Steenrod 1951, Theorem I.7.4), that the canonical map $K/K_\Delta \to K/Z_\Delta$ is a fiber bundle with typical fiber Z_Δ/K_Δ. Consequently, the above commutative diagram shows that $\overline{\exp}|_{\widetilde{C}_\Delta} : \widetilde{C}_\Delta \to C_\Delta$ is a fiber bundle with typical fiber Z_Δ/K_Δ. Since this holds for all $\Delta \in \mathcal{D}$ we have shown that the decomposition $\{C_\Delta\}_{\Delta\in\mathcal{D}}$ is a fibered decomposition with the associated fibrations being fiber bundles. □

Combining Theorems 3.8 and 4.2 we obtain the following.

Corollary 4.3 *Let M be an irreducible compact simply connected symmetric space. Then the total cut locus* $\mathrm{Cut}(M)$ *admits a fibered decomposition and the associated fibrations are fiber bundles.*

For $\Delta \in \mathcal{D}$ let $A_\Delta \subseteq \mathrm{Cut}(M)$ and $\widetilde{A}_\Delta \subseteq \widetilde{\mathrm{Cut}}(M)$ be the subsets of the total cut locus and the total tangent cut locus, resp., induced by the C_Δ as described in Sect. 3. The set $\pi(G, K)$ consists of precisely $r = \mathrm{rank}\, M$ elements. For each $i \in \{1, 2, \ldots, r\}$ we set

$$\mathcal{D}_i := \{\Delta \in \mathcal{D} \mid \#\Delta = i\} \qquad \text{and} \qquad A_i := \bigcup_{\Delta \in \mathcal{D}_i} A_\Delta.$$

Note that by (Sakai 1978, Lemma 5.2), we have for all $i \in \{1, \ldots, r\}$ that

$$\overline{C}_\Delta \cap C_{\Delta'} = \emptyset \qquad \text{for } \Delta, \Delta' \in \mathcal{D}_i,\ \Delta \neq \Delta'.$$

It is easy to see that the same relation then holds for the A_Δ, i.e.

$$\overline{A}_\Delta \cap A_{\Delta'} = \emptyset \qquad \text{for } \Delta, \Delta' \in \mathcal{D}_i,\ \Delta \neq \Delta'. \tag{4.1}$$

Therefore, if we have a locally compact decomposition of all A_Δ, $\Delta \in \mathcal{D}_i$, then we can combine geodesic motion planners in the following way.

Theorem 4.4 *Let M be an irreducible compact simply connected symmetric space of rank r. Then the geodesic complexity of M can be estimated by*

$$\mathsf{GC}(M) \leq \sum_{i=1}^{r} \max\{\mathsf{secat}(\mathrm{Exp}|_{\widetilde{A}_\Delta} : \widetilde{A}_\Delta \to A_\Delta) \mid \Delta \in \mathcal{D}_i\} + 1.$$

Proof Let $i \in \{1, \ldots, r\}$ and assume that for each $\Delta \in \mathcal{D}_i$ we have a locally compact decomposition $B_{\Delta,1}, \ldots, B_{\Delta,k_\Delta}$ of A_Δ such that for each $j \in \{1, \ldots, k_\Delta\}$ there is a continuous geodesic motion planner $s_{\Delta,j} : B_{\Delta,j} \to GM$. Let $m_i = \max\{k_\Delta \mid \Delta \in \mathcal{D}_i\}$ and set $B_{\Delta,j} = \emptyset$ for $k_\Delta < j \leq m_i$. For $l = 1, \ldots, m_i$ put

$$B_l = \bigcup_{\Delta \in \mathcal{D}_i} B_{\Delta,l}$$

and define a geodesic motion planner $s_l : B_l \to GM$ by

$$s_l(q, r) = s_{\Delta,l}(q, r) \qquad \text{if } (q, r) \in B_{\Delta,l}.$$

It follows from (4.1) that this defines a continuous geodesic motion planner on B_l. Since the sets $B_1, \ldots, B_{m_i}$ form a decomposition of A_i, this shows that $\mathsf{GC}_M(A_i) \leq m_i$. Arguing as in the proof of Theorem 3.2, one further shows that

$$k_\Delta \leq \mathsf{secat}(\mathrm{Exp}|_{\widetilde{A}_\Delta} : \widetilde{A}_\Delta \to A_\Delta) \qquad \forall \Delta \in \mathcal{D}_i,$$

which in turn yields $m_i \leq \max\{\mathsf{secat}(\mathrm{Exp}\,|_{\widetilde{A}_\Delta}) \mid \Delta \in \mathcal{D}_i\}$ for each $i \in \{1, 2, \ldots, r\}$. Eventually, we derive that

$$\mathsf{GC}_M(\mathrm{Cut}(M)) \leq \sum_{i=1}^{r} \mathsf{GC}_M(A_i) \leq \sum_{i=1}^{r} m_i \leq \sum_{i=1}^{r} \max\{\mathsf{secat}(\mathrm{Exp}\,|_{\widetilde{A}_\Delta}) \mid \Delta \in \mathcal{D}_i\}.$$

□

Throughout the following, we shall always write $\cong$ to indicate that two manifolds are diffeomorphic. We further let $\mathbb{S}^n$ denote the n-sphere with its standard differentiable structure for each $n \in \mathbb{N}$.

Example 4.5 Consider the complex Grassmannian $\mathrm{Gr}_2(\mathbb{C}^4)$ which is an irreducible compact symmetric space of rank 2. As shown in (Sakai 1978, p.143) and (Mescher and Stegemeyer 2021, Example 8.5), the cut locus $\mathrm{Cut}_o(M)$ can be decomposed into

$$C_{\Delta_1} \cong \mathbb{S}^2 \times \mathbb{S}^2, \qquad C_{\Delta_2} \cong \{*\}$$

and a six-dimensional manifold C_{Δ_0}. As discussed in (Mescher and Stegemeyer 2021, Example 8.5), these three spaces are simply connected. Note that $\mathcal{D}_1 = \{\Delta_1, \Delta_2\}$. The decomposition of the cut locus of o induces a decomposition of $\mathrm{Cut}(M)$ as in Sect. 3 and we shall call the induced sets A_{Δ_0}, A_{Δ_1} and A_{Δ_2}. In order to apply Theorem 4.4, we need to find upper bounds for

$$\mathsf{secat}(\mathrm{Exp}\,|_{\widetilde{A}_{\Delta_i}} : \widetilde{A}_{\Delta_i} \to A_{\Delta_i}) \quad \text{for } i = 0, 1, 2.$$

Fix $i \in \{0, 1, 2\}$. By (Albert 1966, Theorem 18), we have $\mathsf{secat}(\pi : E \to B) \leq \mathsf{cat}(B)$ for any fibration π where $\mathsf{cat}(B)$ is the Lusternik-Schnirelmann category of B. Consequently, we obtain

$$\mathsf{secat}(\mathrm{Exp}\,|_{\widetilde{A}_{\Delta_i}} : \widetilde{A}_{\Delta_i} \to A_{\Delta_i}) \leq \mathsf{cat}(A_{\Delta_i}).$$

Note that $\mathrm{Gr}_2(\mathbb{C}^4)$ and C_{Δ_i} are simply connected, therefore A_{Δ_i} is simply connected since it is a fiber bundle over $\mathrm{Gr}_2(\mathbb{C}^4)$ with typical fiber C_{Δ_i} by Lemma 3.7. Therefore, we get the estimate

$$\mathsf{cat}(A_{\Delta_i}) \leq \frac{\dim(A_{\Delta_i})}{2} + 1 = \frac{\dim(\mathrm{Gr}_2(\mathbb{C}^4)) + \dim(C_{\Delta_i})}{2} + 1 = \frac{\dim C_{\Delta_i}}{2} + 5$$

by (Cornea et al. 2003, Theorem 1.50). Explicitly, we obtain

$$\mathsf{cat}(A_{\Delta_0}) \leq 8, \quad \mathsf{cat}(A_{\Delta_1}) \leq 7 \quad \text{and} \quad \mathsf{cat}(A_{\Delta_2}) \leq 5.$$

Consequently, by Theorem 4.4, we see that

$$\mathsf{GC}(M) \leq \mathsf{cat}(A_{\Delta_0}) + \max\{\mathsf{cat}(A_{\Delta_1}), \mathsf{cat}(A_{\Delta_2})\} + 1 = 16.$$

Note that this improves the upper bound in (Mescher and Stegemeyer 2021, Example 8.5).

Theorem 4.6 *Let $M = \mathbb{C}P^n$ or $\mathbb{H}P^n$ equipped with the standard or Fubini-Study metric, where $n \in \mathbb{N}$. Then its geodesic complexity satisfies*

$$GC(M) = 2n + 1.$$

In particular, one has

$$GC(M) = TC(M).$$

Proof Since $\mathbb{C}P^n$ and $\mathbb{H}P^n$ are simply connected symmetric spaces of rank one, we know by (Sakai 1978, Theorem 5.3) and Corollary 4.3 that the restriction

$$\mathrm{Exp}\,|_{\widetilde{\mathrm{Cut}}(M)} : \widetilde{\mathrm{Cut}}(M) \to \mathrm{Cut}(M)$$

is a fibration. Moreover for $n \geq 2$, the cut locus of a point satisfies

$$\mathrm{Cut}_p(\mathbb{C}P^n) \cong \mathbb{C}P^{n-1} \quad \text{and} \quad \mathrm{Cut}_q(\mathbb{H}P^n) \cong \mathbb{H}P^{n-1},$$

where $p \in \mathbb{C}P^n$ and $q \in \mathbb{H}P^n$, see (Arthur 1978, Proposition 3.35). By Lemma 3.7 we see that $\mathrm{Cut}(\mathbb{C}P^n)$ is a fiber bundle over $\mathbb{C}P^n$ with typical fiber $\mathbb{C}P^{n-1}$. Since $\mathbb{C}P^n$ is simply connected for each $n \geq 1$, it follows that $\mathrm{Cut}(\mathbb{C}P^n)$ is simply connected as well for all $n \geq 2$. By (Albert 1966, Theorem 18) and (Cornea et al. 2003, Theorem 1.50) we obtain

$$\mathsf{secat}(\widetilde{\mathrm{Cut}}(\mathbb{C}P^n) \to \mathrm{Cut}(\mathbb{C}P^n)) \leq \frac{\dim(\mathrm{Cut}(\mathbb{C}P^n))}{2} + 1 = 2n.$$

Consequently by Theorem 3.2 we obtain

$$\mathsf{GC}(\mathbb{C}P^n) \leq \mathsf{secat}(\widetilde{\mathrm{Cut}}(\mathbb{C}P^n) \to \mathrm{Cut}(\mathbb{C}P^n)) + 1 \leq 2n + 1$$

for $n \geq 2$. Since $\mathsf{TC}(\mathbb{C}P^n) = 2n + 1$ by (Farber 2006, Lemma 28.1), we obtain

$$\mathsf{GC}(\mathbb{C}P^n) = \mathsf{TC}(\mathbb{C}P^n) = 2n + 1$$

for $n \geq 2$. The argument for $\mathbb{H}P^n$ is similar, using that $\mathbb{H}P^n$ is 3-connected for all $n \geq 1$ and that $\mathsf{TC}(\mathbb{H}P^n) = 2n + 1$ by (Basabe et al. 2014, Corollary 3.16). Finally, for $n = 1$ we have that $\mathbb{C}P^1$ is isometric to $\mathbb{S}^2$ and $\mathbb{H}P^1$ is isometric to $\mathbb{S}^4$, where both $\mathbb{S}^2$ and $\mathbb{S}^4$ are equipped with the standard metric. It is well-known that $\mathsf{GC}(\mathbb{S}^2) = \mathsf{GC}(\mathbb{S}^4) = 3$, see (Recio-Mitter 2021, Proposition 4.1), so this proves the assertion for $n = 1$. $\square$

5 Three-dimensional lens spaces

In this section we show that the total cut locus of a lens space of the form $L(p;1)$ with a metric of constant sectional curvature admits a fibered decomposition. It is thus an example of a homogeneous Riemannian manifold which has this property, but which is not a globally symmetric space, see e.g. (Gilkey et al. 2015, p. 105). We will use the explicit fibered decomposition to derive an upper bound for the geodesic complexity of three-dimensional lens spaces of type $L(p;1)$. We start by studying the cut locus of a point in the lens space $L(p;1)$, which was explicitly described by S. Anisov in Anisov (2006). However, we give a self-contained exposition in this section, since we will need a detailed description of the tangent cut locus and of the cut locus in this setting.

We consider the 3-sphere as a subspace of $\mathbb{C}^2$, i.e.

$$\mathbb{S}^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1\overline{z}_1 + z_2\overline{z}_2 = 1\}.$$

In the following we will also consider $\mathbb{S}^3$ as embedded in $\mathbb{R}^4$ under the standard identification $\mathbb{C}^2 \cong \mathbb{R}^4$. The special unitary group $SU(2)$ acts transitively on the 3-sphere. Furthermore, for arbitrary $p \geq 3$, we have an action of $\mathbb{Z}_p$ on $\mathbb{S}^3$ denoted by $\Psi\colon \mathbb{Z}_p \times \mathbb{S}^3 \to \mathbb{S}^3$, where $\mathbb{Z}_p$ is the cyclic group with p elements, given by

$$\Psi(m, (z_1, z_2)) \mapsto (e^{\frac{2\pi i m}{p}} z_1, e^{\frac{2\pi i m}{p}} z_2). \tag{5.1}$$

It is easy to see that this action is properly discontinuous. If we equip $\mathbb{S}^3$ with the standard metric, then Ψ is an action by isometries. Consequently, we can equip the quotient $L(p;1) = \mathbb{S}^3/\mathbb{Z}_p$ with a metric for which $\pi\colon \mathbb{S}^3 \to L(p;1)$ becomes a Riemannian covering. We henceforth always consider $L(p;1)$ as equipped with such a metric. The space $L(p;1)$ is called a *lens space*. Furthermore, note that the metric on $L(p;1)$ constructed in this way is a metric of constant sectional curvature. By the Killing-Hopf theorem all metrics of constant sectional curvature on $L(p;1)$ arise in this way, see e.g. (Lee 2018, Theorem 12.4 and Corollary 12.5).

Note that the action of $\mathbb{Z}_p$ on $\mathbb{S}^3$ commutes with the action of $SU(2)$. Thus, $SU(2)$ acts on $L(p;1)$ and in particular this action is transitive, since it is already transitive on $\mathbb{S}^3$. In the following we fix the point $p_0 = \pi(1,0) \in L(p;1)$. Its isotropy group under the $SU(2)$-action on $L(p;1)$ is easily seen to be

$$K = \left\{ \begin{pmatrix} e^{\frac{2\pi i k}{p}} & 0 \\ 0 & e^{-\frac{2\pi i k}{p}} \end{pmatrix} \,\middle|\, k \in \{0, \dots, p-1\} \right\} \cong \mathbb{Z}_p. \tag{5.2}$$

Note that for more general lens spaces of the form $L(p;q)$ where p and q are coprime with $q \neq 1$, see e.g. (Hatcher 2002, Example 2.43), the isometry group does not act transitively in general. See Kalliongis and Miller (2002) for details on the isometry groups of lens spaces.

In order to describe the cut locus of a point $p_0 \in L(p; 1)$, let us first consider the more general situation of a Riemannian covering $\pi : \widetilde{M} \to M$. The following exposition closely follows (Ozols 1974, Section 3).

It is well known that geodesics in $\widetilde{M}$ are mapped to geodesics in M under the Riemanian covering map π. Assume that $M \cong \widetilde{M}/\Gamma$ where Γ is a finite group of isometries of $\widetilde{M}$ acting properly discontinuously. Let $\mathrm{d} : \widetilde{M} \times \widetilde{M} \to \mathbb{R}$ denote the distance function induced by the metric on $\widetilde{M}$. For any two distinct points $q, r \in \widetilde{M}$ we set

$$H_{q,r} = \{u \in \widetilde{M} \mid \mathrm{d}(q,u) < \mathrm{d}(r,u)\}.$$

We recall from (Ozols 1974, Definition 3.1) that

$$\Delta_q = \bigcap_{g \in \Gamma \setminus \{e\}} H_{q, g\cdot q} \subseteq \widetilde{M}$$

is called the *normal fundamental domain of* Γ *centered at* q. The following result by V. Ozols establishes a connection between normal fundamental domains and cut loci.

Theorem 5.1 [(Ozols 1974, Corollary 3.11)] *Let* $\pi : \widetilde{M} \to M$ *be a Riemannian covering, let* $q \in \widetilde{M}$ *and let* $\Delta_q \subset \widetilde{M}$ *be its normal fundamental domain. If its closure satisfies* $\overline{\Delta_q} \cap \mathrm{Cut}_q(\widetilde{M}) = \emptyset$*, then*

$$\mathrm{Cut}_{\pi(q)}(M) = \pi(\partial \Delta_q).$$

Hence, to understand the cut locus of the point $\pi(q) \in M \cong \widetilde{M}/\Gamma$ we can study the boundary of the normal fundamental domain Δ_q. Let $\mathrm{inj}(T_q\widetilde{M}) \subseteq T_q\widetilde{M}$ be the domain of injectivity of the exponential map in $\widetilde{M}$ and put

$$\widehat{\Delta}_q := (\exp_q|_{\mathrm{inj}(T_q\widetilde{M})})^{-1}(\overline{\Delta_q}) \subseteq T_q\widetilde{M}.$$

Assume in the following that $\overline{\Delta_q} \cap \mathrm{Cut}_q(\widetilde{M}) = \emptyset$. Then $\exp_q$ maps $\widehat{\Delta}_q$ homeomorphically onto $\overline{\Delta_q}$, since the restriction of $\exp_q$ to $\mathrm{inj}(T_q\widetilde{M})$ is a homeomorphism onto its image. With $K := \mathrm{inj}(T_{\pi(q)}M) \cup \widetilde{\mathrm{Cut}}_{\pi(q)}(M)$ the diagram

$$\begin{tikzcd} \widehat{\Delta}_q \arrow[r, "D\pi_q", "\approx"'] \arrow[d, "\exp_q"', "\approx"] & K \arrow[d, "\exp_{\pi(q)}"] \\ \overline{\Delta_q} \arrow[r, "\pi"] & M \end{tikzcd}$$

commutes and one checks that the maps $D\pi_q|_{\widehat{\Delta}_q} : \widehat{\Delta}_q \to K$ and $\exp_q|_{\widehat{\Delta}_q} : \widehat{\Delta}_q \to \overline{\Delta_q}$ are homeomorphisms. In particular, we see that $\partial\Delta_q$ is homeomorphic to the tangent cut locus $\widetilde{\mathrm{Cut}}_{\pi(q)}(M)$ and that the exponential map

$$\exp_{\pi(q)}|_{\widetilde{\mathrm{Cut}}_{\pi(q)}(M)} : \widetilde{\mathrm{Cut}}_{\pi(q)}(M) \to \mathrm{Cut}_{\pi(q)}(M)$$

can be understood by considering

$$\pi|_{\partial\Delta_q} : \partial\Delta_q \to \mathrm{Cut}_{\pi(q)}(M).$$

In the following we denote by $\langle\cdot,\cdot\rangle$ the standard inner product on $\mathbb{R}^4$. The next lemma is easily shown by means of elementary geometry. Thus, we omit its proof.

Lemma 5.2 *Let $q, r \in \mathbb{S}^3$ be two distinct points. Let $u = q - r \in \mathbb{R}^4$ and let E_u be the 3-plane of points in $\mathbb{R}^4$ orthogonal to u. Then $H_{q,r} = \{v \in \mathbb{S}^3 \mid \langle v, u\rangle > 0\}$.*

We consider Δ_{q_0}, the normal fundamental domain of $\mathbb{Z}_p$ centered at $q_0 = (1, 0, 0, 0) \in \mathbb{S}^3$. For $k \in \{0, \ldots, p-1\}$, we define

$$u_k = q_0 - k\cdot q_0 = \big(1 - \cos(\tfrac{2\pi k}{p}), -\sin(\tfrac{2\pi k}{p}), 0, 0\big).$$

By Lemma 5.2, it is clear that

$$\Delta_{q_0} = \{r \in \mathbb{S}^3 \mid \langle u_k, r\rangle > 0 \text{ for } k = 1, \ldots, p-1\}.$$

Its boundary is

$$\partial\Delta_{q_0} = \left\{ r \in \mathbb{S}^3 \;\middle|\; \exists k \in \{1, \ldots, p-1\} \text{ with } \langle u_k, r\rangle = 0, \big\langle u_{k'}, r\big\rangle \geq 0\ \forall k' \in \{1, \ldots, p-1\} \setminus \{k\} \right\}.$$

For $l \in \{1, \ldots, p-1\}$ and numbers $1 \leq i_1 < i_2 < \ldots < i_l \leq p-1$, we define

$$\widetilde{D}^{(l)}_{i_1,\ldots,i_l} = \left\{ r \in \mathbb{S}^3 \;\middle|\; \langle u_{i_1}, r\rangle = \cdots = \langle u_{i_l}, r\rangle = 0,\ \langle u_j, r\rangle > 0\ \forall j \in \{1, \ldots, p-1\} \setminus \{i_1, \ldots, i_l\} \right\}.$$

It is clear that

$$\partial\Delta_{q_0} = \bigsqcup_{\substack{l\in\{1,\ldots,p-1\}\\ 1\leq i_1<\ldots<i_l\leq p-1}} \widetilde{D}^{(l)}_{i_1,\ldots,i_l}.$$

Lemma 5.3 *All sets of the form $\widetilde{D}^{(l)}_{i_1,\ldots,i_l}$ are empty except $\widetilde{D}^{(1)}_1$, $\widetilde{D}^{(1)}_{p-1}$ and $\widetilde{D}^{(p-1)}_{1,\ldots,p-1}$. Consequently, $\partial\Delta_{q_0}$ is the disjoint union of $\widetilde{D}^{(1)}_1$, $\widetilde{D}^{(1)}_{p-1}$ and $\widetilde{D}^{(p-1)}_{1,\ldots,p-1}$.*

Proof It is easy to see that

$$\widetilde{D}^{(p-1)}_{1,\ldots,p-1} = \{(0, 0, x, y) \in \mathbb{S}^3 \mid (x, y) \in \mathbb{S}^1\}.$$

Hence, $\widetilde{D}^{(p-1)}_{1,\ldots,p-1}$ is non-empty. For $l \in \{1, \ldots, p-1\}$, $l \neq \frac{p}{2}$, we set

$$\sigma_l = \frac{1 - \cos(\frac{2\pi l}{p})}{\sin(\frac{2\pi l}{p})}.$$

It can be seen directly that

$$\widetilde{D}_1^{(1)} = \{(a, \sigma_1 a, x, y) \in \mathbb{S}^3 \mid a > 0\} \quad \text{and} \quad \widetilde{D}_{p-1}^{(1)} = \{(a, \sigma_{p-1} a, x, y) \in \mathbb{S}^3 \mid a > 0\}.$$

Note that $\sigma_{p-1} = -\sigma_1$. Let $m \in \{2, \dots, p-2\}$. We claim that $\widetilde{D}_m^{(1)} = \emptyset$. Assume that there is a point $r = (a, b, x, y) \in \widetilde{D}_l^{(1)}$. Then, $\langle r, u_m \rangle = 0$ implies that

$$b = \sigma_m a \qquad \text{if } m \neq \frac{p}{2}.$$

For arbitrary $m \in \{2, \dots, p-2\}$, we get from $\langle u_1 + u_{p-1}, r \rangle > 0$ that

$$2\left(1 - \cos(\tfrac{2\pi}{p})\right) a > 0 \tag{5.3}$$

which implies that $a > 0$. In case that p is even and $m = \frac{p}{2}$, it can easily be seen that $a = 0$, yielding a contradiction to inequality (5.3). Thus, $\widetilde{D}_{\frac{p}{2}}^{(1)} = \emptyset$. Therefore, we assume throughout the rest of the proof that $m \neq \frac{p}{2}$. We consider two separate cases, starting with $2 \leq m < \frac{p}{2}$. In this case we have $\sigma_m > 0$, so we see that $b > 0$. We write $r = (\widetilde{a} e^{i\varphi}, x + iy)$ as an element of $\mathbb{C}^2$ with $\widetilde{a} > 0$. It is clear that we have

$$\tan\varphi = \frac{1 - \cos(\frac{2\pi m}{p})}{\sin(\frac{2\pi m}{p})} > 0$$

and we can choose $\varphi \in (0, \frac{\pi}{2})$. Since the third and fourth component of u_1 are trivial, we can use the Euclidean inner product on $\mathbb{R}^2$ to compute that

$$\begin{aligned}
\langle u_1, r \rangle &= \left\langle \begin{pmatrix} 1 - \cos(\frac{2\pi}{p}) \\ -\sin(\frac{2\pi}{p}) \end{pmatrix}, \begin{pmatrix} \widetilde{a}\cos(\varphi) \\ \widetilde{a}\sin(\varphi) \end{pmatrix} \right\rangle_{\mathbb{R}^2} \\
&= \left\langle \begin{pmatrix} 0 \\ -2\sin(\frac{\pi}{p}) \end{pmatrix}, \widetilde{a} \begin{pmatrix} \cos(\varphi - \frac{\pi}{p}) \\ \sin(\varphi - \frac{\pi}{p}) \end{pmatrix} \right\rangle_{\mathbb{R}^2} \\
&= -2\widetilde{a} \sin(\tfrac{\pi}{p}) \sin(\varphi - \tfrac{\pi}{p}),
\end{aligned}$$

where we rotated the vectors by an angle of $-\frac{\pi}{p}$ to get the second equality. Note that by our assumption we have $\langle u_1, r \rangle > 0$ which implies $\sin(\frac{\pi}{p}) \sin(\varphi - \frac{\pi}{p}) < 0$. Since $\varphi < \frac{\pi}{2}$ by assumption, we want to show that $\varphi > \frac{\pi}{p}$. Then $\sin(\frac{\pi}{p}) \sin(\varphi - \frac{\pi}{p}) > 0$, which is thus a contradiction. The inequality $\varphi > \frac{\pi}{p}$ is equivalent to showing that $\tan(\varphi) > \tan(\frac{\pi}{p})$, i.e. that

$$\frac{1 - \cos(\frac{2\pi m}{p})}{\sin(\frac{2\pi m}{p})} \overset{!}{>} \frac{\sin(\frac{\pi}{p})}{\cos(\frac{\pi}{p})}. \tag{5.4}$$

Note that since $m < \frac{p}{2}$, we have $\frac{\pi m}{p} < \frac{\pi}{2}$. Consequently,

$$2\cos(\tfrac{\pi m}{p})^2 < 2\cos(\tfrac{\pi}{p})^2.$$

By standard trigonometry

$$2\cos(\tfrac{\pi m}{p})^2 = 1 + \cos(\tfrac{2\pi m}{p})$$

and therefore

$$1 - \cos(\tfrac{2\pi m}{p})^2 < 2\cos(\tfrac{\pi}{p})^2(1 - \cos(\tfrac{2\pi m}{p})).$$

One checks by direct computation that this is equivalent to

$$(\sin(\tfrac{2\pi m}{p}))^2(\sin(\tfrac{\pi}{p}))^2 < (1 - \cos(\tfrac{2\pi m}{p}))^2(\cos(\tfrac{\pi}{p}))^2.$$

Since all squared terms were positive before squaring, we see that this is equivalent to

$$\sin(\tfrac{2\pi m}{p})\sin(\tfrac{\pi}{p}) < (1 - \cos(\tfrac{2\pi m}{p}))\cos(\tfrac{\pi}{p})$$

which clearly implies the inequality (5.4). We thus get the desired contradiction in the case $2 \leq m < \frac{p}{2}$. The case $\frac{p}{2} < m \leq p-2$ can be treated similarly. One can argue similarly that all sets of the form $\widetilde{D}^{(l)}_{i_1,\dots,i_l}$ with $2 \leq l \leq p-2$ are empty. $\square$

To shorten our notation we write $\widetilde{D}^{(p-1)}$ for $\widetilde{D}^{(p-1)}_{1,\dots,p-1}$. Set $p_0 = \pi(q_0) \in L(p;1)$ and recall that

$$D\pi_{q_0} \circ (\exp_{q_0}|_{\partial\widehat{\Delta}_{q_0}})^{-1} \colon \partial\Delta_{q_0} \to \widetilde{\mathrm{Cut}}_{p_0}(L(p;1))$$

is a homeomorphism. Moreover, the diagram

$$\begin{array}{ccc} \partial\widehat{\Delta}_{q_0} & \xrightarrow{D\pi_{q_0}} & \widetilde{\mathrm{Cut}}_{p_0}(L(p;1)) \\ {\scriptstyle \exp_{q_0}}\downarrow & & \downarrow{\scriptstyle \exp_{p_0}} \\ \partial\Delta_{q_0} & \xrightarrow{\pi} & \mathrm{Cut}_{p_0}(L(p;1)) \end{array}$$

commutes. Here, we obviously consider the restrictions of the maps to the spaces occurring in the diagram, which we drop from the notation for the sake of readability. We denote the images of the sets $\widetilde{D}^{(1)}_i$ by

$$\widetilde{C}^{(1)}_i = (D\pi_{q_0} \circ (\exp_{q_0}|_{\partial\widehat{\Delta}_{q_0}})^{-1})(\widetilde{D}^{(1)}_i), \quad \text{for } i \in \{1, p-1\}$$

and similarly

$$\widetilde{C}^{(p-1)} = (D\pi_{q_0} \circ (\exp_{q_0}|_{\partial\widehat{\Delta}_{q_0}})^{-1})(\widetilde{D}^{(p-1)}).$$

Proposition 5.4 *Let $p \in \mathbb{N}$ with $p \geq 3$ and consider the lens space $L(p;1)$ with a Riemannian metric of constant sectional curvature. Let $\pi : \mathbb{S}^3 \to L(p;1)$ be the corresponding Riemannian covering and put $p_0 := \pi(1,0,0,0)$.*

(1) The sets $\widetilde{C}_1^{(1)}, \widetilde{C}_{p-1}^{(1)}$ and $\widetilde{C}^{(p-1)}$ form a locally compact decomposition of the tangent cut locus $\widetilde{\mathrm{Cut}}_{p_0}(L(p;1))$. Moreover, $\widetilde{C}_1^{(1)}$ and $\widetilde{C}_{p-1}^{(1)}$ are homeomorphic to open 2-disks and $\widetilde{C}^{(p-1)}$ is homeomorphic to $\mathbb{S}^1$.

(2) The cut locus $\mathrm{Cut}_{p_0}(L(p;1))$ admits a locally compact decomposition into

$$C^{(1)} = \pi(\widetilde{D}_i^{(1)}) = \exp_{p_0}(\widetilde{C}_i^{(1)}),\ i \in \{1, p-1\}$$

and

$$C^{(p-1)} = \pi(\widetilde{D}^{(p-1)}) = \exp_{p_0}(\widetilde{C}^{p-1}).$$

The map $\exp_{p_0}|_{\widetilde{C}_i^{(1)}} : \widetilde{C}_i^{(1)} \to C^{(1)}$ is a homeomorphism for $i \in \{1, p-1\}$. Under suitable identifications of $\widetilde{C}^{(p-1)}$ and $C^{(p-1)}$ with $\mathbb{S}^1$, the map

$$\exp_{p_0}|_{\widetilde{C}^{(p-1)}} : \widetilde{C}^{(p-1)} \to C^{(p-1)}$$

can be identified with the standard p-fold covering of $\mathbb{S}^1$ by $\mathbb{S}^1$.

Hence,

$$\mathrm{Cut}_{p_0}(L(p;1)) = C^{(1)} \sqcup C^{(p-1)}$$

is a fibered decomposition of $\mathrm{Cut}_{p_0}(L(p;1))$ and the associated fibrations are fiber bundles.

Proof The first part is apparent given the identification $\partial\Delta_{q_0} \approx \widetilde{\mathrm{Cut}}_{p_0}(L(p;1))$ and the characterization of the sets $\widetilde{D}_1^{(1)}, \widetilde{D}_{p-1}^{(1)}$ and $\widetilde{D}^{(p-1)}$ in the proof of Lemma 5.3. For the second part, we note that

$$\Psi(1, \widetilde{D}_{p-1}^{(1)}) = \widetilde{D}_1^{(1)}.$$

Consequently, $\widetilde{D}_1^{(1)}$ and $\widetilde{D}_{p-1}^{(1)}$ are identified under π. Furthermore, the restriction of π to $\widetilde{D}_i^{(1)}$, $i \in \{1, p-1\}$ is a homeomorphism onto its image since it is continuous, injective and a local homeomorphism. The same properties therefore hold for $\widetilde{C}_1^{(1)}$, $\widetilde{C}_{p-1}^{(1)}$ and the map $\exp_{p_0}$ under the identification $\partial\Delta_{q_0} \cong \widetilde{\mathrm{Cut}}_{p_0}(L(p;1))$. Recall that

$$\widetilde{D}^{(p-1)} = \{(0,z) \in \mathbb{S}^3 \mid z \in \mathbb{S}^1\},$$

thus it is obviously homeomorphic to $\mathbb{S}^1$ and the $\mathbb{Z}_p$-action on $\mathbb{S}^3$ becomes the standard $\mathbb{Z}_p$-action on $\mathbb{S}^1$ under this identification. Since the map $\mathbb{S}^1 \to \mathbb{S}^1/\mathbb{Z}_p$ is a p-fold covering, this proves the last claim. $\square$

In the following, we want to show that the fibered decomposition of $\mathrm{Cut}_{p_0}(L(p;1))$ is isotropy-invariant with respect to the transitive $SU(2)$-action to obtain a fibered decomposition of the total cut locus of $L(p;1)$ from Theorem 3.8.

Lemma 5.5 *Let (M,g) be a Riemannian manifold and let $q \in M$ be a point. Furthermore, let $m \geq 2$ be an integer. Assume that G is a group of isometries of M which fixes q. Let $S_m \subseteq \mathrm{Cut}_q(M)$ be the set of points $r \in \mathrm{Cut}_q(M)$ such that there are precisely m distinct minimal geodesics between q and r. Then S_m is invariant under G.*

Proof Let $\varphi\colon G \times M \to M$ denote the G-action and let $\Phi\colon G \times GM \to GM$ denote the induced pointwise G-action, given by

$$\Phi(g,\gamma)(t) = \Phi_g(\gamma)(t) = \varphi(g,\gamma(t)), \quad \text{for } t \in [0,1], g \in G, \gamma \in GM.$$

Let $r \in S_m$ and $g \in G$. Since r is a cut point, $s = \varphi(g,r) \in \mathrm{Cut}_q(M)$. Let $\gamma_1,\ldots,\gamma_m$ be the m distinct minimal geodesics between q and r. Then $\Phi_g(\gamma_1),\ldots,\Phi_g(\gamma_m)$ are distinct minimal geodesics between q and s. If there was a minimal geodesic σ between q and s which is distinct from all $\Phi_g(\gamma_i)$, $i \in \{1,\ldots,m\}$, then $\Phi_{g^{-1}}(\sigma)$ would be a minimal geodesic joining q and r distinct from $\gamma_1,\ldots,\gamma_m$. This contradicts $r \in S_m$, hence such a σ does not exist and we derive that $s \in S_m$ as well. This proves the claim. $\square$

Corollary 5.6 *The fibered decomposition of $\mathrm{Cut}_{p_0}(L(p;1)) = C^{(1)} \sqcup C^{(p-1)}$ constructed in Proposition 5.4 is isotropy-invariant.*

Proof By Proposition 5.4 we can characterize $C^{(1)}$ and $C^{(p-1)}$ as

$$\begin{aligned} C^{(1)} &= \{q \in \mathrm{Cut}_{p_0}(L(p;1)) \mid \text{there are precisely two minimal geodesics joining } p_0 \text{ and } q\},\\ C^{(p-1)} &= \{q \in \mathrm{Cut}_{p_0}(L(p;1)) \mid \text{there are precisely } p \text{ minimal geodesics joining } p_0 \text{ and } q\}. \end{aligned}$$

Therefore the isotropy invariance is a direct consequence by Lemma 5.5. $\square$

It follows from Theorem 3.8 and Corollary 5.6 that there is a decomposition of $\mathrm{Cut}(L(p;1))$ into sets $A^{(1)}$ and $A^{(p-1)}$ which form a fibered decomposition of $\mathrm{Cut}(L(p;1))$. We now want to study this decomposition in greater detail.

Recall that we denote the isotropy group of the $SU(2)$-action on $L(p;1)$ by K and computed it in equation (5.2). In order to better distinguish the various group actions, let

$$\Phi\colon SU(2) \times \mathbb{S}^3 \to \mathbb{S}^3 \quad \text{and} \quad \varphi\colon SU(2) \times L(p;1) \to L(p;1)$$

be the actions of $SU(2)$ on $\mathbb{S}^3$ and on $L(p;1)$, respectively. Recall that we denoted the $\mathbb{Z}_p$-action on $\mathbb{S}^3$ be Ψ, see equation (5.1). If $A \in SU(2)$ we shall also write Φ_A for the diffeomorphism $\Phi(A,\cdot)\colon \mathbb{S}^3 \to \mathbb{S}^3$ and similarly for the other actions.

The fibered decomposition of $\mathrm{Cut}(L(p;1))$ is given as follows. For $l \in \{1, p-1\}$, we have

$$A^{(l)} = \{(q,r) \in \mathrm{Cut}(L(p;1)) \mid r \in \varphi_A(C^{(l)}),\ A \in SU(2) \text{ such that } \mathrm{pr}(A) = q\},$$

where $\mathrm{pr}\colon SU(2) \to L(p;1)$ is the canonical projection. We denote the preimages of $A^{(1)}$ and $A^{(p-1)}$ in the total tangent cut locus by $\widetilde{A}^{(1)}$ and $\widetilde{A}^{(p-1)}$. Explicitly, we have

$$\widetilde{A}^{(1)} = \{(q,v) \in \widetilde{\mathrm{Cut}}(L(p;1)) \mid v \in (D\varphi_A)_{p_0}(\widetilde{C}_1^{(1)} \cup \widetilde{C}_{p-1}^{(1)}),\ \ A \in SU(2) \text{ such that } \mathrm{pr}(A) = q\}$$

By Proposition 5.4, Corollary 5.6 and Theorem 3.8 we obtain that $\widetilde{A}^{(1)} \to A^{(1)}$ is a 2-fold covering and that $\widetilde{A}^{(p-1)} \to A^{(p-1)}$ is a p-fold covering, where we allow coverings to be trivial, i.e. the total space of the covering might not be connected. We want to show that $\widetilde{A}^{(1)}$ consists of two connected components which implies that $\widetilde{A}^{(1)} \to A^{(1)}$ is a trivial covering.

Lemma 5.7 *The set $\widetilde{C}_1^{(1)} \subseteq T_{p_0}L(p;1)$ is isotropy-invariant with respect to the induced $SU(2)$-action in the tangent bundle $TL(p;1)$. More precisely if $A \in K$, then $(D\varphi_A)_{p_0}(\widetilde{C}_1^{(1)}) = \widetilde{C}_1^{(1)}$. The same holds for $\widetilde{C}_{p-1}^{(1)}$.*

Proof Let $x \in \widetilde{C}_1^{(1)} \subseteq T_{p_0}L(p;1)$ and $A \in K$, i.e. there is a $k \in \{0, \ldots, p-1\}$ such that

$$A = \begin{pmatrix} e^{\frac{2\pi i k}{p}} & 0 \\ 0 & e^{-\frac{2\pi i k}{p}} \end{pmatrix}.$$

It holds that $(\Psi_{-k} \circ \Phi_A)(q_0) = q_0$. We want to show that $(D\varphi_A)_{p_0}(x) \in \widetilde{C}_1^{(1)}$. Consider the following diagram.

$$\begin{tikzcd}
\mathbb{S}^3 \arrow[r, "\Phi_A"] & \mathbb{S}^3 \arrow[r, "\Psi_{-k}"] & \mathbb{S}^3 \\
T_{q_0}\mathbb{S}^3 \arrow[u, "\exp_{q_0}"] \arrow[r, "(D\Phi_A)_{q_0}"] \arrow[d, "D\pi_{q_0}"'] & T_{\Phi_A(q_0)}\mathbb{S}^3 \arrow[u, "\exp_{\Phi_A(q_0)}"'] \arrow[r, "(D\Psi_{-k})_{\Phi_A(q_0)}"] \arrow[d, "D\pi_{\Phi_A(q_0)}"'] & T_{q_0}\mathbb{S}^3 \arrow[u, "\exp_{q_0}"'] \arrow[d, "D\pi_{q_0}"] \\
T_{p_0}L(p;1) \arrow[r, "(D\varphi_A)_{p_0}"'] & T_{p_0}L(p;1) \arrow[r, "\mathrm{id}"'] & T_{p_0}L(p;1)
\end{tikzcd}$$

The lower two squares commute by definition of φ and the fact that the induced action of Ψ on $L(p;1)$ is trivial. The upper two squares commute by the naturality of the exponential map. Note that all arrows in the lower two squares are isomorphisms.

If we restrict to $\widetilde{\mathrm{Cut}}_{p_0}(L(p;1))$ and to $\partial\Delta_{q_0}$, respectively, we obtain a commutative diagram

$$\begin{tikzcd}
\partial\Delta_{q_0} \arrow[r, "\Psi_{-k}\circ\Phi_A"] \arrow[d, "D\pi_{q_0}\circ\exp_{q_0}^{-1}"'] & \partial\Delta_{q_0} \arrow[d, "D\pi_{q_0}\circ\exp_{q_0}^{-1}"] \\
\widetilde{\mathrm{Cut}}_{p_0}(L(p;1)) \arrow[r, "(D\varphi_A)_{p_0}"] & \widetilde{\mathrm{Cut}}_{p_0}(L(p;1)).
\end{tikzcd}$$

By the proof of Lemma 5.3 we can write $y = (\exp_{q_0}\circ(D\pi)^{-1}_{q_0})(x) \in \widetilde{D}^{(1)}_1$ as

$$y = ((1+i\sigma_1)a, z) \quad \text{where } \; a > 0,\; z \in \mathbb{C},$$

and where σ_1 was defined in the proof of Lemma 5.3. Then it follows that

$$(\Psi_{-k}\circ\Phi_A)(y) = \left((1+i\sigma_1)a, e^{-\frac{4\pi ik}{p}}z\right),$$

which is again an element of $\widetilde{D}^{(1)}_1$. Consequently, $(D\varphi_A)_{p_0}(x) \in \widetilde{C}^{(1)}_1$. The argument for $\widetilde{C}^{(1)}_{p-1}$ is analogous. □

By the previous lemma, the sets

$$\widetilde{A}^{(1)}_1 = \{(q,v) \in \widetilde{\mathrm{Cut}}(L(p;1)) \mid v \in (D\varphi_A)_q(\widetilde{C}^{(1)}_1),\;\; A \in SU(2) \text{ such that } \mathrm{pr}(A) = q\}$$

and

$$\widetilde{A}^{(1)}_{p-1} = \{(q,v) \in \widetilde{\mathrm{Cut}}(L(p;1)) \mid v \in (D\varphi_A)_q(\widetilde{C}^{(1)}_{p-1}),\;\; A \in SU(2) \text{ such that } \mathrm{pr}(A) = q\}$$

are well-defined. Moreover, we clearly have $\widetilde{A}^{(1)} = \widetilde{A}^{(1)}_1 \sqcup \widetilde{A}^{(1)}_{p-1}$. Since $\widetilde{A}^{(1)} \to A^{(1)}$ is a fiber bundle by Theorem 3.8, we now see that $\widetilde{A}^{(1)} \to A^{(1)}$ is a trivial 2-fold covering. This implies that

$$\mathsf{secat}(\widetilde{A}^{(1)} \to A^{(1)}) = 1.$$

Theorem 5.8 *Let $p \in \mathbb{N}$ with $p \geq 3$ and consider the lens space $L(p;1)$ with a metric of constant sectional curvature. Then*

$$6 \leq GC(L(p;1)) \leq 7.$$

Proof M. Farber and M. Grant have shown in (Farber and Grant 2008, Corollary 15) that the topological complexity of $L(p;1)$ is $\mathsf{TC}(L(p;1)) = 6$, which yields $\mathsf{GC}(L(p;1)) \geq \mathsf{TC}(L(p;1)) = 6$, see (Recio-Mitter 2021, Remark 1.9). By Theorem 3.2 we have

$$\mathsf{GC}(L(p;1)) \leq \mathsf{secat}(\widetilde{A}^{(1)} \to A^{(1)}) + \mathsf{secat}(\widetilde{A}^{(p-1)} \to A^{(p-1)}) + 1. \quad (5.5)$$

As argued above, we have $\mathsf{secat}(\widetilde{A}^{(1)} \to A^{(1)}) = 1$. Recall that $A^{(p-1)}$ is a circle bundle over $L(p; 1)$, therefore it is 4-dimensional and we get

$$\mathsf{secat}(\widetilde{A}^{(p-1)} \to A^{(p-1)}) \leq \mathsf{cat}(A^{(p-1)}) \leq 5$$

by (Albert 1966, Theorem 18) and (Cornea et al. 2003, Theorem 1.50). Thus, the inequality (5.5) gives the upper bound $\mathsf{GC}(L(p; 1)) \leq 7$. □

Funding Open Access funding enabled and organized by Projekt DEAL.

References

Anisov, S.: Cut loci in lens manifolds. C. R. Math. Acad. Sci. Paris **342**(8), 595–600 (2006)

Basabe, I., González, J., Rudyak, Y.B., Tamaki, D.: Higher topological complexity and its symmetrization. Algebr. Geom. Topol. **14**(4), 2103–2124 (2014)

Besse, A.L.: Manifolds all of whose geodesics are closed, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 93. Springer-Verlag, Berlin-New York (1978)

Błaszczyk, Z., Carrasquel-Vera, J.G.: Topological complexity and efficiency of motion planning algorithms. Rev. Mat. Iberoam. **34**(4), 1679–1684 (2018)

Bredon, G.E.: Topology and geometry, Graduate Texts in Mathematics. Springer Science & Business Media, Cham (2013)

Bröcker, T., Tom Dieck, T.: Representations of compact Lie groups, Graduate Texts in Mathematics, vol. 98. Springer-Verlag, New York (1995)

Cornea, O, Lupton, G, Oprea, J, Tanré, D: Lusternik-Schnirelmann category, Mathematical Surveys and Monographs, vol. 103. American Mathematical Society, Providence, RI (2003)

Farber, M.: Invitation to topological robotics, Zurich lectures in advanced mathematics. Eur Math Soc (EMS) **2**, 51 (2008)

Farber, M.: Topology of robot motion planning, Morse theoretic methods in nonlinear analysis and in symplectic topology. NATO Sci Ser II Math Phys Chem **217**, 185–230 (2006)

Farber, M.: Topological complexity of motion planning. Discr. Comput. Geom. **29**(2), 211–221 (2003)

Farber, M., Grant, M.: Robot motion planning, weights of cohomology classes, and cohomology operations. Proc. Am. Math. Soc. **136**(9), 3339–3349 (2008)

Garcia Calcines, J.M: A note on covers defining relative and sectional categories. Topol. Appl. **265**, 106810 (2019)

Gilkey, P., Park, J., Vázquez-Lorenzo, R.: Aspects of differential geometry. II, Synthesis Lectures on Mathematics and Statistics. Morgan & Claypool Publishers, Williston (2015)

Hatcher, A.: Algebraic topology. Cambridge University Press, Cambridge (2002)

Helgason, S.: Differential geometry, Lie groups, and symmetric spaces, Pure and Applied Mathematics, vol. 80. Academic Press Inc, New York (1978)

Kalliongis, J., Miller, A.: Geometric group actions on lens spaces. Kyungpook Math. J. **42**(2), 313–344 (2002)

Lee, J.M.: Introduction to Riemannian manifolds, Graduate Texts in Mathematics, vol. 176. Springer, Cham (2018)

Mescher, S., Stegemeyer, M.: *Geodesic complexity of homogeneous Riemannian manifolds*, http://arxiv.org/abs/2105.09215, to appear in: Algebr. Geom. Topol., (2021)
Ozols, V.: Cut loci in Riemannian manifolds. Tohoku Math. J. **2**(26), 219–227 (1974)
Recio-Mitter, D.: Geodesic complexity of motion planning. J. Appl. Comput. Topol. **5**, 141–178 (2021)
Sakai, T.: Cut loci of compact symmetric spaces, Minimal submanifolds and geodesics (Proc. Japan-United States Sem., Tokyo: North-Holland. Amsterdam-New York **1978**, 193–207 (1978)
Sakai, T.: On cut loci of compact symmetric spaces. Hokkaido Math. J. **6**(1), 136–161 (1977)
Sakai, T.: On the structure of cut loci in compact Riemannian symmetric spaces. Math. Ann. **235**(2), 129–148 (1978)
Schwarz, A.S.: The genus of a fiber space. Amer. Math. Soc. Transl. **55**, 49–140 (1966)
Steenrod, N.: The topology of fibre bundles, Princeton Mathematical Series, vol. 14. Princeton University Press (1951)