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Research Paper

On pro-zero homomorphisms and sequences in local (co-)homology

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Let $\underline{x}$ denote a system of elements of a commutative ring R. For an R-module M we investigate when $\underline{x}$ is M-pro-regular resp. M-weakly pro-regular as generalizations of M-regular sequences. This is done in terms of Čech co-homology resp. homology, defined by $H^i(\check{C}_{\underline{x}} \otimes_R \cdot)$ resp. by $H_i(\mathrm{R}\operatorname{Hom}_R(\check{C}_{\underline{x}}, \cdot)) \cong H_i(\operatorname{Hom}_R(\mathcal{L}_{\underline{x}}, \cdot))$, where $\check{C}_{\underline{x}}$ denotes the Čech complex and $\mathcal{L}_{\underline{x}}$ is a bounded free resolution of it as constructed in [17] resp. [16]. The property of $\underline{x}$ being M-pro-regular resp. M-weakly pro-regular follows by the vanishing of certain Čech co-homology resp. homology modules, which is related to completions. This extends previously work by Greenlees and May (see [5]) and Lipman et al. (see [1]). This contributes to a further understanding of Čech (co-)homology in the non-Noetherian case. As a technical tool we use one of Emmanouil's results (see [4]) about the inverse limits and its derived functor. As an application we prove a global variant of the results with an application to prisms in the sense of Bhatt and Scholze (see [3]).

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1. Introduction

Let R denote a commutative ring with $\underline{x} = x_1, \ldots, x_r$ a system of elements. For an R-module M we study generalizations of a M-regular sequence called M-pro-regular sequence and M-weakly pro-regular sequence. To this end we denote by $\check{C}_{\underline{x}}$ the Čech complex with respect to $\underline{x}$ (see e.g. [17, 6.1]). It is a bounded complex of flat R-modules. For an R-module M we write $\check{C}_{\underline{x}}(M) = \check{C}_{\underline{x}} \otimes_R M$. We call $\check{H}^i_{\underline{x}}(M) = H^i(\check{C}_{\underline{x}}(M)), i \in \mathbb{Z}$, the Čech cohomology of M. Dually we look at the complex $\mathrm{R}\operatorname{Hom}_R(\check{C}_{\underline{x}}, M)$ in the derived category. There is a free resolution of $\check{C}_{\underline{x}}$ by a bounded complex $\mathcal{L}_{\underline{x}}$ and $\operatorname{Hom}_R(\mathcal{L}_{\underline{x}}, M)$ is a representative of $\mathrm{R}\operatorname{Hom}_R(\check{C}_{\underline{x}}, M)$ (see [17] and [16]). We define $\check{H}^{\underline{x}}_i(M) = H_i(\operatorname{Hom}_R(\mathcal{L}_{\underline{x}}, M)) \cong H_i(\mathrm{R}\operatorname{Hom}_R(\check{C}_{\underline{x}}, M)), i \in \mathbb{Z}$, as the Čech homology of M. For the case of R a Noetherian ring let $\mathfrak{a} = \underline{x}R$ then it follows that $\check{H}^i_{\underline{x}}(M) \cong H^i_{\mathfrak{a}}(M)$, the i-th local cohomology of M with support in $\mathfrak{a}$. At first this was established by Grothendieck (see [6] and [7]). Dually, for Noetherian rings R we have $\check{H}^{\underline{x}}_i(M) \cong \Lambda^{\mathfrak{a}}_i(M)$, where $\Lambda^{\mathfrak{a}}_i(\cdot)$ denotes the left derived functors of the completion $\Lambda^{\mathfrak{a}}(\cdot)$. Contributions were done by Matlis (see [9]), Simon (see [18]), Greenlees and May (see [5]) and others.

Starting with Greenlees and May (see [5]) and Lipman et al. (see [1]) there were extensions to non-Noetherian rings with sequences $\underline{x}$ that are called pro-regular resp. weakly pro-regular (see below for the definitions). In particular, when $\underline{x}$ is weakly pro-regular the isomorphisms $\check{H}^i_{\underline{x}}(M) \cong H^i_{\mathfrak{a}}(M)$ and $\check{H}^{\underline{x}}_i(M) \cong \Lambda^{\mathfrak{a}}_i(M)$ hold for any $i \in \mathbb{Z}$ and any R-module M and more generally for any complex $X \in D(R)$ (see [11], [12], [14] and [17] for more details).

In the situation of $\underline{x}$ an R-regular sequence there is a corresponding property of $\underline{x}$ being an M-regular sequence (see e.g. [10]). This is a challenge for the study of the relative version that $\underline{x}$ is weakly M-regular for modules instead of $M = R$. Namely, $\underline{x}$ is called an M-weakly pro-regular sequence (see also [17, 7.3.1]) provided the inverse system $\{H_i(\underline{x}^{(n)}; M)\}_{n \geq 1}$ is pro-zero for $i = 1, \ldots, r$, i.e. for each n there is an integer $m \geq n$ such that the natural map $H_i(\underline{x}^{(m)}; M) \to H_i(\underline{x}^{(n)}; M)$ is zero. Here $\underline{x}^{(n)} = x_1^n, \ldots, x_r^n$ and $H_i(\underline{x}^{(n)}; M)$ denotes the Koszul homology. An R-weakly pro-regular sequence is called weakly pro-regular. For a first description of M-weakly pro-regular sequences see [15, Theorem 4.2]. Let $\widehat{M}^{\underline{x}} = \Lambda^{\underline{x}}(M)$ denote the $\underline{x}$-adic completion of M.

Theorem 1.1. *For an R-module M and a sequence $\underline{x} = x_1, \ldots, x_r$ the following is equivalent:*

(i) *$\underline{x}$ is M-weakly pro-regular.*
(ii) *$\check{C}_{\underline{x}}(\operatorname{Hom}_R(M, I))$ is a right resolution of $\operatorname{Hom}_R(\Lambda^{\underline{x}}(M), I)$ for any injective R-module I.*
(iii) *$\operatorname{Hom}_R(\mathcal{L}_{\underline{x}}, M \otimes_R F)$ is a left resolution of $\Lambda^{\underline{x}}(M \otimes_R F)$ for any free R-module F.*
(iv) *$\operatorname{Hom}_R(\mathcal{L}_{\underline{x}}, X)$ is a left resolution of $\Lambda^{\underline{x}}(X)$ for $X = M, M[T]$.*
(v) *$\operatorname{Hom}_R(\mathcal{L}_{\underline{x}}, M[T])$ is a left resolution of $\Lambda^{\underline{x}}(M[T])$.*

Note that the equivalence of (i), (iii) and (iv) in the particular case of $M = R$ was shown by Positselski (see [12, Theorem 3.6]), that is in the case when $\underline{x}$ is R-weakly pro-regular (or weakly pro-regular for short). Then the complexes $\operatorname{Hom}_R(\mathcal{L}_{\underline{x}}, X)$ and $\mathrm{L}\Lambda^{\underline{x}}(X)$ are isomorphic in the derived category for all $X \in D(R)$ (see [11] generalizing the case of bounded complexes shown in [16]). For the proof of 1.1 and the notion of left/right resolution see the comments after 3.6.

The notion of a weakly pro-regular sequence $\underline{x} = x_1, \ldots, x_r$ is defined in terms of the Koszul homology of the whole sequence $\underline{x}$. An M-regular sequence is defined by the vanishing of $\underline{x}_{i-1}M :_M x_i/\underline{x}_{i-1}M$ for $i = 1, \ldots, r$, where $\underline{x}_{i-1} = x_1, \ldots, x_{i-1}$. As a generalization of that Greenlees and May (see [5]) resp. Lipman et al. (see [1]) invented the notion of an M-pro-regular sequence. Note that both of the definitions are equivalent (see [15, Proposition 2.2]). A sequence $\underline{x}$ is called M-pro-regular if the inverse system $\{\underline{x}_{i-1}^{(n)}M :_M x_i^n/\underline{x}_{i-1}^{(n)}M)\}_{n\geq 1}$ with multiplication by x_i^n is pro-zero for $i = 1, \ldots, r$. Note that if $\underline{x}$ is M-regular it is also M-weakly pro-regular since $\underline{x}_{i-1}^{(n)}M :_M x_i^n/\underline{x}_{i-1}^{(n)}M = 0$ (see [10, 16.1]). A characterization of pro-regular sequences in terms of Čech cohomology is known (see [15, Theorem 3.2] and 4.4). Here there is a description in the terms of Čech homology. See 4.5 for the following:

Theorem 1.2. *Let $\underline{x} = x_1, \ldots, x_r$ denote a sequence of elements of R. For an R-module M the following conditions are equivalent:*

(i) *The sequence $\underline{x}$ is M-pro-regular.*
(ii) $\check{H}_0^{x_i}(\Lambda^{\underline{x}_{i-1}}(M\otimes_R F)) \cong \Lambda^{\underline{x}_i}(M\otimes_R F)$ *and* $\check{H}_1^{x_i}(\Lambda^{\underline{x}_{i-1}}(M\otimes_R F)) = 0$ *for* $i = 1, \ldots, r$ *and any free R-module F.*
(iii) $\check{H}_0^{\underline{x}_i}(X) \cong \Lambda^{\underline{x}_i}(X)$ *and* $\check{H}_1^{\underline{x}_i}(X) = 0$ *for* $i = 1, \ldots, r$ *and* $X = M, M[T]$.
(iv) $\Lambda^{\underline{x}_{i-1}}(M[T])$ *is of bounded x_i-torsion for* $i = 1, \ldots, r$.

In the final section we apply the previous results to a global situation. To this end we consider a pair $(\mathcal{I}, x)$ consisting of an effective Cartier divisor $\mathcal{I} \subseteq R$ and an element $x \in R$ (see 5.1 for the definitions). We call it pro-regular whenever the inverse system $\{H_1(x^n; R/\mathcal{I}^n)\}_{n\geq 1}$ is pro-zero. Then our investigations (see 5.5) yield the following:

Corollary 1.3. *With the previous notation the following conditions are equivalent:*

(i) *$R/\mathcal{I}$ is of bounded x-torsion.*
(ii) *$(\mathcal{I}, x)$ is pro-regular.*
(iii) $\check{H}_0^{x}(\Lambda^{\mathcal{I}}(F)) \cong \Lambda^{(x,\mathcal{I})}(F))$ *and* $\check{H}_1^{x}(\Lambda^{\mathcal{I}}(F)) = 0$ *for any free R-module F.*
(iv) $\Lambda^{\mathcal{I}}(R)$ *and* $\Lambda^{\mathcal{I}}(R[T])$ *are of bounded x-torsion.*

As shown in [15] this has applications to prisms in the sense of Bhatt and Scholze (see [3]). The equivalent conditions in 1.3 are improvements of the results shown in [15, Corollary 5.7].

In the paper we start with recollections about inverse limits. In particular we include a different proof of one of Emmanouil's results (see [4]) about inverse systems needed in the paper. In the third section we prove additional statements about weakly pro-regular sequences, extending those known before. In section 4 we study pro-regular sequences, continuing the results shown in [15]. Moreover, we prove a necessary and sufficient condition for the isomorphism $\Lambda^x(\Lambda^{\mathcal{I}}(M)) \cong \Lambda^{(x,\mathcal{I})}(M)$ for an ideal $\mathcal{I} \subset R$ and an element $x \in R$ generalizing a result by Greenlees and May (see [5, Lemma 1.6]). Finally in section 5 we study when a pair $(\mathcal{I}, x)$ consisting of an effective Cartier divisor $\mathcal{I}$ and an element $x \in R$ is pro-regular. Finally we apply these results to prisms in the sense of [3] generalizing partial results of [15].

In the terminology we follow that of [17]. In our approach we prefer to work in the category of modules instead of the derived category. For that reason we use a bounded free resolution of the Čech complex (see 3.1).

2. Recollections about inverse limits

Notation 2.1. (A) Let R denote a commutative ring. Let $\{M_n\}_{n\geq 0}$ be an inverse system of R-modules with $\phi_{n,m} : M_m \to M_n$ for all $m \geq n$. Then there is an exact sequence

$$0 \to \varprojlim M_n \to \prod_{n\geq 0} M_n \xrightarrow{\Phi} \prod_{n\geq 0} M_n \to \varprojlim{}^1 M_n \to 0,$$

where Φ denotes the transition map and $\varprojlim{}^1 M_n$ is the first left derived functor of the inverse limit (see e.g. [20, 3.5] or [17, 1.2.2]).
(B) Let M denote an R-module. Let T be a variable over R. In the following we use $M[|T|]$, the formal power series R-module over M. That is, the R-module $M[|T|]$ consists of all formal series $\sum_{i\geq 0} x_i T^i$ with $x_i \in M$ for all $i \geq 0$. Correspondingly, the R-module $M[T]$ consists of all polynomials over M. Therefore, $\sum_{i\geq 0} x_i T^i \in M[T]$ if only finitely many x_i are non-zero. Whence there is an injection $0 \to M[T] \to M[|T|]$ of R-modules.
(C) The inverse system $\{M_n\}_{n\geq 0}$ is called pro-zero if for each n there is an integer $m \geq n$ such that the homomorphism $\phi_{n,m} : M_m \to M_n$ is zero. If $\{M_n\}_{n\geq 0}$ is pro-zero, then it is well known that $\varprojlim M_n = \varprojlim^1 M_n = 0$ since Φ is an isomorphism (see e.g. [17, 1.2.4]).
(D) Let $\{M_n\}_{n\geq 0}$ be an inverse system. Then clearly $\operatorname{Im} \phi_{n,m'} \subseteq \operatorname{Im} \phi_{n,m} \subseteq M_n$ for all $m' \geq m \geq n$. We say that $\{M_n\}_{n\geq 0}$ satisfies the *Mittag-Leffler condition* if for each n the sequence of submodules $\{\operatorname{Im} \phi_{n,m} | m \geq n\}$ stabilizes. For instance, this holds if the maps $\phi_{n,m}$ are surjective or $\{M_n\}_{n\geq 0}$ is an inverse system of Artinian R-modules. It is well-known that $\varprojlim^1 M_n = 0$ if $\{M_n\}_{n\geq 0}$ satisfies the Mittag-Leffler condition (see e.g. [17, 1.2.3]).

For more details about inverse systems we refer to Jensen's exposition in [8] and to [4]. It is remarkable that the vanishing in 2.1 (C) does not imply that $\{M_n\}_{n\geq 0}$ is pro-zero. To this end see the example [17, 1.2.5] or the following generalization:

Example 2.2. Let $(R, \mathfrak{m})$ denote a complete local Noetherian ring with $x \in R$ a non-unit. We consider the direct system $\{R_n\}_{n\geq 0}$ with $R_n = R$ and $\psi_{n,n+1} : R_n \to R_{n+1}$ the multiplication by x. Then $\varinjlim R_n \cong R_x$ and there is a short exact sequence

$$0 \to \oplus_{n\geq 0} R_n \to \oplus_{n\geq 0} R_n \to R_x \to 0.$$

Now we apply $\operatorname{Hom}_R(\cdot, R)$ and obtain the inverse system $\{M_n\}_{n\geq 0}$ with $M_n = \operatorname{Hom}_R(R_n, R)$ and with the multiplication $M_{n+1} \xrightarrow{x} M_n$. By applying $\operatorname{Hom}_R(\cdot, R)$ to the previous short exact sequence it yields the exact sequence

$$0 \to \operatorname{Hom}_R(R_x, R) \to \prod_{n\geq 0} M_n \to \prod_{n\geq 0} M_n \to \operatorname{Ext}^1_R(R_x, R) \to 0.$$

Since R is also xR-complete $\varprojlim M_n = \operatorname{Hom}_R(R_x, R) = 0$ and $\varprojlim^1 M_n = \operatorname{Ext}^1_R(R_x, R) = 0$ (see [17, 3.1.10]) while the inverse system $\{M_n\}_{n\geq 0}$ is neither pro-zero nor satisfies the Mittag-Leffler condition.

In the following we shall discuss necessary and sufficient conditions for an inverse system to be pro-zero. This extends known results. We need a technical construction.

Remark 2.3. An R-module M induces a short exact sequence

$$0 \to M[T] \xrightarrow{T} M[T] \to M \to 0,$$

where T denote the shift operator defined by $\sum_{n\geq 0}^k x_n T^n \mapsto \sum_{n\geq 0}^k x_n T^{n+1}$. The inverse system $\{M_n\}_{n\geq 0}$ induces a short exact sequence of inverse systems

$$0 \to \{M_n[T]\}_{n\geq 0} \xrightarrow{T} \{M_n[T]\}_{n\geq 0} \to \{M_n\}_{n\geq 0} \to 0,$$

induced by the shift operator. Then we have the six-term long exact sequence associated to the inverse limit

$$0 \to \varprojlim M_n[T] \to \varprojlim M_n[T] \to \varprojlim M_n \to \varprojlim{}^1 M_n[T] \to \varprojlim{}^1 M_n[T] \to \varprojlim{}^1 M_n \to 0$$

(see e.g. [17, 1.2.2]).

By the Example 2.2 it follows that the vanishing of $\varprojlim^1 M_n$ is necessary but not sufficient for the Mittag-Leffler condition of the inverse system $\{M_n\}_{n\geq 0}$. A characterization of the Mittag-Leffler condition was shown by Emmanouil (see [4]). For our purposes we recall part of Emmanouil's result (see [4, Corollary 6]). In our argument we use a certain exact sequence (see the proof of 2.4) and modify an idea of [19, tag 0CQA] as new ingredients.

Lemma 2.4. *Let $\{M_n\}_{n\geq 0}$ denote an inverse system of R-modules. Then the following conditions are equivalent:*

(i) *$\{M_n\}_{n\geq 0}$ satisfies the Mittag-Leffler condition.*
(ii) *$\{M_n[T]\}_{n\geq 0}$ satisfies the Mittag-Leffler condition.*
(iii) *$\varprojlim^1 M_n = 0$ and $\varprojlim^1 M_n[T] = 0$.*
(iv) *$\varprojlim^1 M_n[T] = 0$.*

Proof. (i) $\Longrightarrow$ (ii): This follows since the inverse system $\{M_n[T]\}_{n\geq 0}$ satisfies the Mittag-Leffler condition too.
(ii) $\Longrightarrow$ (iv): This holds trivially.
(iii) $\Longleftrightarrow$ (iv): This is a consequence of the six-term exact sequence in 2.3.
(iii) $\Longrightarrow$ (i): The injections $0 \to M_n[T] \to M_n[|T|]$ induce a short exact sequence of inverse systems

$$0 \to \{M_n[T]\}_{n\geq 0} \to \{M_n[|T|]\}_{n\geq 0} \to \{M_n[|T|]/M_n[T]\}_{n\geq 0} \to 0.$$

By passing to the inverse limit it provides an exact sequence

$$0 \to \varprojlim M_n[T] \to \varprojlim M_n[|T|] \to \varprojlim M_n[|T|]/M_n[T] \to \varprojlim{}^1 M_n[T].$$

Now suppose that $\{M_n\}_{n\geq 0}$ does not satisfy the Mittag-Leffler condition. Then there is an integer m such that the sequence of submodules $\{\operatorname{Im} \phi_{m,k} | k \geq m\}$ of M_m does not stabilize. Whence there is an infinite sequence $m = m_0 < m_1 < \ldots < m_i < \ldots$ and elements $x_i \in M_{m_i}$ such that $\phi_{m,m_i}(x_i) \in M_m \setminus \phi_{m,m_i+1}(M_{m_i+1})$. Now we define $F = (f_n)_{n\geq 0} \in \prod_{n\geq 0} M_n[|T|]$ with $f_n = \sum_{i\geq n} z_{n,i} T^i$ where we put

$$z_{n,i} = \begin{cases} \phi_{n,m_i}(x_i) & if\ m_i \geq n \\ 0 & else. \end{cases}$$

As easily seen $f_n - \phi_{n,n+1}(f_{n+1}) \in M_n[T]$ and F defines an element $F' \in \varprojlim M_n[|T|]/M_n[T]$. Suppose F' has a preimage $G = (g_n)_{n\geq 0} \in \varprojlim M_n[|T|]$ with $g_n = \sum_{i\geq 0} y_{n,i} T^i$ and $y_{n,i} \in M_n$ for all $i \geq 0$. We have that $y_{n,i} = \phi_{n,n+k}(y_{n+k,i})$ for all $k, i \geq 0$ and therefore $y_{n,i} \in \phi_{n,n+k}(M_{n+k})$. That is, $y_{m,i} \in \phi_{m,m_i+1}(M_{m_i+1})$ and $y_{m,i} \neq \phi_{m,m_i}(x_i)$ since $\phi_{m,m_i}(x_i) \in M_m \setminus \phi_{m,m_i+1}(M_{m_i+1})$. Therefore

$$f_m - g_m = \sum_{i\geq 0} (\phi_{m,m_i}(x_i) - y_{m,i}) T^i \notin M_m[T]$$

and G can not be a preimage of F', a contradiction to the vanishing of $\varprojlim{}^1 M_n[T]$. □

As a consequence of 2.4 a characterization of pro-zero inverse systems follows. The vanishing $\varprojlim M_n = \varprojlim{}^1 M_n = 0$ is not sufficient for $\{M_n\}_{n\geq 1}$ being pro-zero (see 2.2).

As shown next it follows by the vanishing $\varprojlim M_n[T] = \varprojlim{}^1 M_n[T] = 0$ (see 2.5). For the proof we modify Weibel's argument (see the proof [20, 3.5.7]). For an R-module M and a set S we define $M^{(S)} = \oplus_{s \in S} M_s$ with $M_s = M$. Then it is clear that conditions (iii) and (iv) hold also for the inverse system $\{(M_n)^{(S)}\}_{n \geq 0}$ when they hold for $\{M_n\}_{n \geq 0}$.

Corollary 2.5. *Let $\{M_n\}_{n \geq 0}$ denote an inverse system of R-modules. Then the following conditions are equivalent:*

(i) $\{M_n\}_{n \geq 0}$ *is pro-zero.*
(ii) $\{M_n[T]\}_{n \geq 0}$ *is pro-zero.*
(iii) $\varprojlim M_n = \varprojlim^1 M_n = 0$ *and* $\varprojlim M_n[T] = \varprojlim^1 M_n[T] = 0$.
(iv) $\varprojlim M_n[T] = \varprojlim^1 M_n[T] = 0$.

Proof. (i) $\Longrightarrow$ (ii): Because $\{M_n\}_{n \geq 0}$ is pro-zero this holds also for the induced inverse system $\{M_n[T]\}_{n \geq 0}$ as easily seen.
(ii) $\Longrightarrow$ (iv): This is obviously true because $\{M_n[T]\}_{n \geq 0}$ is pro-zero.
(iii) $\Longleftrightarrow$ (iv): This is a consequence of the six-term exact sequence in 2.3.
(iii) $\Longrightarrow$ (i): By view of 2.4 the inverse system $\{M_n\}_{n \geq 1}$ satisfies the Mittag-Leffler condition. We define $N_n = \operatorname{Im} \phi_{n,m}$ where $m = m(n)$ is choosen such that $\{\operatorname{Im} \phi_{n,k}\}_{k \geq n}$ becomes stable. Then $\{N_n\}_{n \geq 1}$ becomes an inverse system with surjective maps. Because the inverse system $\{M_n/N_n\}_{n \geq 1}$ is pro-zero the exact sequence $0 \to N_n \to M_n \to M_n/N_n \to 0$ implies $\varprojlim N_n = \varprojlim M_n = 0$ and therefore $N_n = 0$. $\square$

3. Weakly pro-regular sequences

We start with a few recalls of results and definitions of [17] and [16]. As above R denotes a commutative ring.

Notation 3.1. (A) For a system of elements $\underline{x} = x_1, \ldots, x_r$ of R let $\check{C}_{\underline{x}}$ denote the Čech complex

$$\check{C}_{\underline{x}} := \check{C}_{x_1} \otimes_R \cdots \otimes_R \check{C}_{x_r},$$

where $\check{C}_{x_i} : 0 \to R \to R_{x_i} \to 0$ (see e.g. [7] or [17, 6.1]). In the following we look at the complex $\operatorname{R}\operatorname{Hom}_R(\check{C}_{\underline{x}}, M)$ for an R-module M in the derived category. By virtue of [5] there is a finite free resolution of $\check{C}_{\underline{x}}$. We follow here the one $\mathcal{L}_{\underline{x}}$ as given in [16]. Whence $\operatorname{Hom}_R(\mathcal{L}_{\underline{x}}, M)$ is a representative of $\operatorname{R}\operatorname{Hom}_R(\check{C}_{\underline{x}}, M)$. Define the Čech homology $\check{H}_i^{\underline{x}}(M) = H^{-i}(\operatorname{Hom}_R(\mathcal{L}_{\underline{x}}, M))$ and the Čech cohomology $\check{H}^i_{\underline{x}}(M) = H^i(\mathcal{L}_{\underline{x}} \otimes_R M)$ for all $i \in \mathbb{Z}$ (see [17] and [16] for more details).
(B) Let $\underline{U} = U_1, \ldots, U_r$ denote a sequence of r variables over R. For an R-module M we denote, as above, by $M[|\underline{U}|]$ the module of formal power series in the variables $\underline{U}$. Clearly $M[|\underline{U}|] = \varprojlim M[\underline{U}]/\underline{U}^{(n)} M[\underline{U}]$, where $\underline{U}^{(n)} = U_1^n, \ldots, U_r^n$ and $M[\underline{U}]$ is the

polynomial module over M. For the sequence $\underline{x} = x_1, \ldots, x_r$ we define the sequence $\underline{x} - \underline{U} = x_1 - U_1, \ldots, x_r - U_r$. As one of the main results of the paper [17, Section 8] the following isomorphisms are shown

$$\operatorname{Hom}_R(\mathcal{L}_{\underline{x}}, M) \cong K_\bullet(\underline{x} - \underline{U}; M[|\underline{U}|]) \cong \varprojlim K_\bullet(\underline{x} - \underline{U}; M[\underline{U}]/\underline{U}^{(n)} M[\underline{U}]),$$

where $K_\bullet(\underline{x} - \underline{U}; \cdot)$ denotes the Koszul complex with respect to the sequence $\underline{x} - \underline{U}$. Moreover there are isomorphisms

$$\mathcal{L}_{\underline{x}} \otimes_R M \cong K^\bullet(\underline{x} - \underline{U}; M[\underline{U}^{-1}]) \cong \varinjlim K^\bullet(\underline{x} - \underline{U}; M[\underline{U}]/\underline{U}^{(n)} M[\underline{U}]),$$

where $M[\underline{U}^{-1}]$ denotes the module of inverse polynomials and $K^\bullet(\underline{x} - \underline{U}; \cdot)$ is the Koszul co-complex (see [16, 4.1] for all of the details).

In the following there is technical result for the computation of $\check{H}_i^{\underline{x}}(M)$ and $\check{H}^i_{\underline{x}}(M)$ resp.

Lemma 3.2. *We fix the notation of 3.1. Furthermore let $\underline{x}^{(n)} = x_1^n, \ldots, x_r^n$ and let $H_i(\underline{x}^{(n)}; M)$ denote the Koszul homology and $H^i(\underline{x}^{(n)}; M)$ the Koszul cohomology.*

(a) *There are isomorphisms $\check{H}^i_{\underline{x}}(M) \cong \varinjlim H^i(\underline{x}^{(n)}; M)$ and short exact sequences*

$$0 \to \varprojlim{}^1 H_{i+1}(\underline{x}^{(n)}; M) \to \check{H}_i^{\underline{x}}(M) \to \varprojlim H_i(\underline{x}^{(n)}; M) \to 0,$$

for all $i \in \mathbb{Z}$.

(b) *For $i > 0$ we have $\check{H}_i^{\underline{x}}(M) = 0$ if and only if $\varprojlim{}^1 H_{i+1}(\underline{x}^{(n)}; M) = \varprojlim H_i(\underline{x}^{(n)}; M) = 0$ and $\check{H}_0^{\underline{x}}(M) \cong \Lambda^{\underline{x}}(M)$ if and only if $\varprojlim{}^1 H_1(\underline{x}^{(n)}; M) = 0$.*

Proof. For the proof of (a) we refer to [17, 6.1.4, 8.1.7] or [16, 5.6]. Then (b) is a consequence of the exact sequences in (a). □

Next we shall give a further characterization for an element $x \in R$ such that an R-module M is of bounded x-torsion.

Definition 3.3. (A) Let M denote an R-module and $x \in R$ an element. Then M is called of bounded x-torsion if the family of increasing submodules $\{0 :_M x^n\}_{n \geq 0}$ stabilizes, that is

$$0 :_M x^n = 0 :_M x^{n+1} \text{ for all } n \gg 0.$$

Note that this is equivalent to the fact that the inverse system $\{0 :_M x^n\}_{n \geq 0}$ with the multiplication map $0 :_M x^m \xrightarrow{x^{m-n}} 0 :_M x^n, m \geq n$, being pro-zero.
(B) It is obvious that M is of bounded x-torsion if and only if the inverse system of Koszul

homology modules $\{H_1(x^n;M)\}_{n\geq 0}$ with the multiplication map $H_1(x^m;M) \xrightarrow{x^{m-n}} H_1(x^n;M)$ is pro-zero. With this in mind Lipman (see [2]) introduced the generalization of a weakly pro-regular sequence for a ring R. For a generalization to an R-module M see [17, 7.3.1]. That is, a sequence $\underline{x} = x_1, \ldots, x_r$ is called M-weakly pro-regular, if for $i > 0$ the inverse system $\{H_i(\underline{x}^{(n)};M)\}_{n\geq 0}$ is pro-zero, where $H_i(\underline{x}^m;M) \to H_i(\underline{x}^n;M), m \geq n$, denotes the natural map induced by the Koszul complexes. A first systematic study of R-weakly pro-regular sequences has been done in [14].

For a characterization of M-weakly pro-regular sequences see [16]. In fact, this is an extension of R-weakly pro-regular sequences shown in [11] which extended the results of [14] to unbounded complexes. Here we shall prove another characterization of M-weakly pro-regular sequences. It is a slight extension of Potsitselski's result see [12, Section 3]) to the case of an R-module M. As above, for an R-module M and a set S we define $M^{(S)} = \oplus_{s\in S} M_s$ with $M_s = M$. Note that $M[T] \cong M^{(\mathbb{N})}$. Moreover, $\Lambda^{\underline{x}}(M) = \widehat{M}^{\underline{x}} = \varprojlim M/\underline{x}^{(n)}M$ denotes the $\underline{x}R$-adic completion of an R-module M.

Theorem 3.4. *Let $\underline{x} = x_1, \ldots, x_r$ denote a sequence of elements of R. For an R-module M the following conditions are equivalent:*

(i) *$\underline{x}$ is M-weakly pro-regular.*
(ii) *For any set S it holds $\check{H}_i^{\underline{x}}(M^{(S)}) = 0$ for all $i > 0$ and $\check{H}_0^{\underline{x}}(M^{(S)}) = \Lambda^{\underline{x}}(M^{(S)})$.*
(iii) *$\check{H}_i^{\underline{x}}(M[T]) = \check{H}_i^{\underline{x}}(M) = 0$ for all $i > 0$ and $\check{H}_0^{\underline{x}}(M[T]) = \Lambda^{\underline{x}}(M[T])$ and $\check{H}_0^{\underline{x}}(M) = \Lambda^{\underline{x}}(M)$.*
(iv) *$\check{H}_i^{\underline{x}}(M[T]) = 0$ for all $i > 0$ and $\check{H}_0^{\underline{x}}(M[T]) = \Lambda^{\underline{x}}(M[T])$.*

Proof. (i) $\Longrightarrow$ (ii): It is clear that for $i > 0$ the inverse system $\{H_i(\underline{x}^{(n)};M^{(S)})\}_{n\geq 0}$ is pro-zero too. Then $\varprojlim H_i(\underline{x}^{(n)};M^{(S)}) = \varprojlim^1 H_i(\underline{x}^{(n)};M^{(S)}) = 0$ for $i > 0$ and (ii) is a consequence of 3.2.
(ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv): These hold obviously.
(iv) $\Longrightarrow$ (i): By view of 3.2 the assumptions imply that

$$\varprojlim H_i(\underline{x}^{(n)};M[T]) = \varprojlim{}^1 H_i(\underline{x}^{(n)};M[T]) = 0 \text{ for } i > 0.$$

By 2.5 this completes the proof because of $H_i(\underline{x}^{(n)};M[T]) \cong H_i(\underline{x}^{(n)};M)[T]$. □

In the following example we show that it is not sufficient to assume S to be finite in 3.4 for the characterization of weakly pro-regular sequences (see also [15, Example 3.3]).

Example 3.5. Let $R = \Bbbk[|x|]$ denote the formal power series ring in the variable x over the field $\Bbbk$. Then define $A = \prod_{n\geq 1} R/x^nR$. By the component wise operations A becomes a commutative ring. The natural map $R \to A, r \to (r + x^nR)_{n\geq 1}$, is a ring homomorphism with $x \mapsto \mathbf{x} := (x + x^nR)_{n\geq 1}$. As a direct product of xR-complete modules A is an xR-complete R-module (see [17, 2.2.7]). Since R is a Noetherian ring x is R-weakly pro-regular and $\check{H}_i^x(A) \cong H_i(\operatorname{Hom}_R(\mathcal{L}_x, A)) = 0$ for $i > 0$ and $\check{H}_0^x(A) \cong H_0(\operatorname{Hom}_R(\mathcal{L}_x, A)) \cong A$. Moreover, by the change of rings there is an isomorphism $\operatorname{Hom}_R(\mathcal{L}_x, A) \cong \operatorname{Hom}_A(\mathcal{L}_{\mathbf{x}}, A)$. That is, $\check{H}_i^{\mathbf{x}}(A) = 0$ for $i > 0$ and $\check{H}_0^{\mathbf{x}}(A) \cong A$. Now note that A is not of bounded $\mathbf{x}$-torsion as easily seen. It follows that the equivalent conditions in 3.4 do not hold for A and $A[T]$. To be more precise, recall $H_1(x^n; A) = \prod_{i \geq 1}(x^{i-n}R/x^iR)$ with $x^{i-n}R = R$ for $i \leq n$, that is

$$H_1(x^n; A) = (\underbrace{R/xR, \ldots, R/x^nR}_{i \leq n}, \underbrace{xR/x^{n+1}, \ldots, x^{i-n}R/x^iR, \ldots}_{i>n}).$$

Therefore $H_1(x^m; A)$ does not stabilize under the multiplication by x^{m-n} in $H_1(x^n; A)$. Note that the i-component of the image of $H_1(x^m; A)$ under the multiplication by x^{m-n} in $H_1(x^n; A)$ is zero for $i \leq m - n < m$ and non-zero for $i = m - n + 1$. Whence $\{H_1(x^n; A)\}_{n \geq 1}$ does not satisfy the Mittag-Leffler condition. By view of 2.4 we have $\varprojlim{}^1 H_1(x^n; A[T]) \neq 0$ and $\Lambda_0^x(A[T]) \cong \check{H}_0^x(A[T]) \twoheadrightarrow \Lambda^x(A[T])$ is not an isomorphism (see 3.2 (a)).

As an application we have another characterization that an R-module M is of bounded x-torsion for an element $x \in R$. Note that (iii) in 3.6 is the analogue to 3.4 (iv).

Corollary 3.6. *For an element $x \in R$ and an R-module M the following conditions are equivalent:*

(i) *M is of bounded x-torsion.*
(ii) *$\check{H}_1^x(M[T]) = \check{H}_1^x(M) = 0$ and $\check{H}_0^x(M[T]) \cong \Lambda^x(M[T])$ and $\check{H}_0^x(M) \cong \Lambda^x(M)$.*
(iii) *$\varprojlim 0 :_{M[T]} x^n = \varprojlim{}^1 0 :_{M[T]} x^n = 0$.*

Proof. The equivalence of the first two conditions is a particular case of 3.4. The equivalence of the first and third condition is a particular case of 2.5. □

Moreover, the proof of Theorem 1.1 follows by 3.4 and [16, Proposition 5.3]. To this end note that $\check{H}_i^{\underline{x}}(M) = H_i(\operatorname{Hom}_R(\mathcal{L}_{\underline{x}}, M))$. For an R-module X we call a complex $X_\cdot : \ldots \to X_1 \to X_0 \to 0$ a left resolution whenever $X_\cdot \xrightarrow{\sim} M$. A co-complex $Y^\cdot : 0 \to Y^0 \to Y^1 \to \ldots$ is called a right resolution of X provided $X \xrightarrow{\sim} Y^\cdot$.

With the previous results we have the following slight generalization of Potsitselski's result (see [12, Theorem 3.6]). Note that $\underline{x}$ is R-weakly pro-regular if it is $R[T]$-weakly pro-regular as easily seen.

Corollary 3.7. *For a sequence $\underline{x} = x_1, \ldots, x_r$ of a ring R the following conditions are equivalent:*

(i) *$\underline{x}$ is R-weakly pro-regular.*

(ii) $\operatorname{Hom}_R(\mathcal{L}_{\underline{x}}, M)$ *is a left resolution of* $\Lambda^{\underline{x}}(M)$ *for any free* R*-module* M.
(iii) $\operatorname{Hom}_R(\mathcal{L}_{\underline{x}}, R[T])$ *is a left resolution of* $\Lambda^{\underline{x}}(R[T])$.

Remark 3.8. While the property of R-regular and M-regular sequences are quite "symmetric" this is not the case for the notion of weakly pro-regularity. Let $\underline{x}$ denote a sequence of elements of R. If it is R-weakly pro-regular it follows that $\check{H}^{\underline{x}}_0(M) \cong \Lambda^{\underline{x}}_0(M)$ for any R-module M (see e.g. [17, Chapter 7]). Let $\underline{x}$ be M-weakly pro-regular, then $\check{H}^{\underline{x}}_0(M) \cong \Lambda^{\underline{x}}(M)$ as shown in 3.4. Note that the homomorphism $\Lambda^{\underline{x}}_0(M) \to \Lambda^{\underline{x}}(M)$ is onto (see [17, 2.5.1]) but in general not an isomorphism (see e.g. Example 3.5).

4. Pro-regular sequences

Before we shall investigate pro-regular sequences we need technical results about pro-zero inverse systems. To this end let M denote an R-module with $\{M_n\}_{n\geq 1}$ a decreasing sequence of submodules of M, i.e. $M_{n+1} \subseteq M_n$ for $n \geq 1$. Then $\mathcal{M} = \{M/M_n\}_{n\geq 1}$ forms an inverse system with surjective maps $M/M_{n+1} \to M/M_n$. Moreover, let $\Lambda(\mathcal{M}) = \varprojlim M/M_n$. For a sequence of elements $\underline{x} = x_1, \ldots, x_r \in R$ we consider the induced filtration $\{(\underline{x}^{(n)}M, M_n)\}_{n\geq 1}$, where $\underline{x}^{(n)} = x_1^n, \ldots, x_r^n$. We write $\Lambda(\mathcal{M}/\underline{x}\mathcal{M}) := \varprojlim M/(\underline{x}^{(n)}M, M_n)$ for the inverse limit of the induced filtration. Then there is a natural homomorphism $\Lambda^{\underline{x}}(\Lambda(\mathcal{M})) \to \Lambda(\mathcal{M}/\underline{x}\mathcal{M})$. In the following we will discuss when it is an isomorphism.

Lemma 4.1. *With the previous notation there is a short exact sequence*

$$0 \to \varprojlim{}_n \varprojlim{}^1_m H_1(\underline{x}^{(n)}; M/M_m) \to \Lambda^{\underline{x}}(\Lambda(\mathcal{M})) \to \Lambda(\mathcal{M}/\underline{x}\mathcal{M}) \to 0.$$

Therefore $\Lambda^{\underline{x}}(\Lambda(\mathcal{M})) \cong \Lambda(\mathcal{M}/\underline{x}\mathcal{M})$ *if and only if* $\varprojlim_n \varprojlim{}^1_m H_1(\underline{x}^{(n)}; M/M_m) = 0$.

Proof. Let m, n denote positive integers. We investigate the inverse system of Koszul complexes $\{K_\bullet(\underline{x}^{(n)}; M/M_m)\}_{m\geq 1}$. For its inverse limit there are isomorphisms

$$\varprojlim{}_m K_\bullet(\underline{x}^{(n)}, M/M_m) \cong \operatorname{Hom}_R(K^\bullet(\underline{x}^{(n)}), \Lambda(\mathcal{M})) \cong K_\bullet(\underline{x}^{(n)}; \Lambda(\mathcal{M})).$$

The inverse system $\{K_\bullet(\underline{x}^{(n)}; M/M_m)\}_{m\geq 1}$ is degree-wise surjective. Whence for its 0-th homology there is a short exact sequence

$$0 \to \varprojlim{}^1_m H_1(\underline{x}^{(n)}; M/M_m) \to H_0(\underline{x}^{(n)}; \Lambda(\mathcal{M})) \to \varprojlim{}_m H_0(\underline{x}^{(n)}; M/M_m) \to 0$$

(see [17, 1.2.8]). It forms an exact sequence of inverse systems on n. By passing to the inverse limit it provides the short exact sequence of the statement since $\varprojlim{}^1_n \varprojlim{}^1_m H_1(\underline{x}^{(n)}; M/M_m) = 0$ because of the underlying bi-countable indexed system (see the spectral sequence in [13]). Whence the statement follows. □

The previous result is an extension of [5, Lemma 1.6] to the case of a sequence of elements and a more general filtration. Namely, it was shown by Greenlees and May that the vanishing of $\varprojlim_n \varprojlim^1_m H_1(x^n; M/\mathcal{I}^m M)$ implies the isomorphism $\Lambda^x(\Lambda^{\mathcal{I}}(M)) \cong \Lambda^{(x,\mathcal{I})}(M)$. By 4.1 the vanishing is also necessary for the isomorphism.

For any set S we define also $\Lambda(\mathcal{M}^{(S)}) = \varprojlim M^{(S)}/M_n^{(S)} \cong \varprojlim((M/M_n)^{(S)})$. For an element $x \in R$ we put - as before -

$$\Lambda((\mathcal{M}/x\mathcal{M})^{(S)}) = \varprojlim M^{(S)}/(xM, M_n)^{(S)} \cong \varprojlim((M/(xM, M_n)^{(S)})).$$

Moreover, we study when the inverse system $\{M_n :_M x^n/M_n\}_{n\geq 1}$ with the multiplication by x is pro-zero. That is, when for each $n \geq 1$ there is an $m \geq n$ such that the multiplication map

$$M_m :_M x^m/M_m \xrightarrow{x^{m-n}} M_n :_M x^n/M_n$$

is zero. This is equivalent to the inverse system $\{H_1(x^n; M/M_n)\}_{n\geq 1}$ being pro-zero, where $H_1(x^n; M/M_n)$ denotes the Koszul homology of M/M_n with respect to the element x^n. In other words, for each integer $n \geq 1$ there is an $m \geq n$ such that $M_m :_M x^m \subseteq M_n :_M x^{m-n}$. Note that, if $M_n =: N$ for all $n \geq 1$, then $\{H_1(x^n; M/N)\}_{n\geq 1}$ is pro-zero if and only if M/N is of bounded x-torsion. With this in mind we shall continue with an extension of 3.6.

Theorem 4.2. *With the previous notation the following conditions are equivalent:*

(i) *The inverse system $\{H_1(x^n; M/M_n)\}_{n\geq 1}$ is pro-zero.*
(ii) *$\check{H}_1^x(\Lambda(\mathcal{M}^{(S)})) = 0$ and $\check{H}_0^x(\Lambda(\mathcal{M}^{(S)})) \cong \Lambda((\mathcal{M}/x\mathcal{M})^{(S)})$ for any set S.*
(iii) *Condition (ii) holds for S a set of a single element and $S = \mathbb{N}$.*
(iv) *$\varprojlim H_1(x^n; Y_n) = \varprojlim^1 H_1(x^n; Y_n) = 0$ for both $Y_n = M/M_n$ and $Y_n = M/M_n[T]$.*
(v) *$\varprojlim H_1(x^n; M/M_n[T]) = \varprojlim^1 H_1(x^n; M/M_n[T]) = 0$.*

Proof. (i) $\Longrightarrow$ (ii): We put $X = M^{(S)}$ and $X_n = (M_n)^{(S)}$. Then it follows that $\{H_1(x^n; X/X_n)\}_{n\geq 1}$ is pro-zero too since the Koszul homology commutes with direct sums, therefore

$$\varprojlim H_1(x^n; X/X_n) = \varprojlim{}^1 H_1(x^n; X/X_n) = 0.$$

Furthermore there are isomorphisms

$$\varprojlim_m H_1(x^n; X/X_m) \cong \varprojlim_m \operatorname{Hom}_R(R/x^n R, X/X_m) \cong H_1(x^n; \Lambda(X))$$

for all $n \geq 1$. We have the bi-indexed system $\{H_1(x^n; X/X_m)\}_{n\geq 1, m\geq 1}$ and the diagonal system $\{H_1(x^n; X/X_n)\}_{n\geq 1}$ cofinal in it. There are the isomorphisms and the vanishing

$$\varprojlim_n H_1(x^n;\Lambda(X)) \cong \varprojlim{}_n \varprojlim{}_m H_1(x^n; X/X_m) \cong \varprojlim_{n,m} H_1(x^n; X/X_m) = 0.$$

By virtue of Roos' spectral sequence (see [13] or [20, 5.8.7]) there is a short exact sequence

$$\begin{aligned} 0 \to \varprojlim{}^1_n \varprojlim{}_m H_1(x^n; X/X_m) &\to \varprojlim{}^1_{n,m} H_1(x^n; X/X_m) \\ &\to \varprojlim{}_n \varprojlim{}^1_m H_1(x^n; X/X_m) \to 0 \end{aligned} \qquad (\#)$$

and a similar one with m, n reversed. This implies the vanishing $\varprojlim{}^1 H_1(x^n;\Lambda(X)) = 0$ and also $\varprojlim{}_n \varprojlim{}^1_m H_1(x^n; X/X_m) = 0$. By view of 3.2 and 4.1 this proves the claim.
(ii) $\Longrightarrow$ (iii): This holds trivially.
(iii) $\Longrightarrow$ (iv): By 3.2 the assumption implies that $\varprojlim H_1(x^n;\Lambda(\mathcal{X})) = \varprojlim^1 H_1(x^n;\Lambda(\mathcal{X})) = 0$ for $\mathcal{X} = \mathcal{M}$ and $\mathcal{M}[T]$. Put $X/X_m = \mathcal{M}_m$. Because $\Lambda(\mathcal{X}) \cong \varprojlim_m X/X_m$ and since the inverse limit commutes (as above) with the first Koszul homology it follows that

$$\varprojlim_n \varprojlim_m H_1(x^n; X/X_m) = \varprojlim_n{}^1 \varprojlim_m H_1(x^n; X/X_m) = 0. \qquad (\star)$$

The first vanishing implies that $\varprojlim H_1(x^n; M/M_n) = 0$. In order to continue note that the isomorphism of the assumption $\check{H}^x_0(\Lambda(\mathcal{X})) \cong \Lambda(\mathcal{X}/x\mathcal{X})$ factors through

$$\check{H}^x_0(\Lambda(\mathcal{X})) \xrightarrow{\beta} \Lambda^x(\Lambda(\mathcal{X})) \xrightarrow{\gamma} \Lambda(\mathcal{X}/x\mathcal{X})$$

surjections β (see 3.2) and γ (see 4.1). Whence $\Lambda^x(\Lambda(\mathcal{X})) \to \Lambda(\mathcal{X}/x\mathcal{X})$ is an isomorphism and $\varprojlim_n \varprojlim{}^1_m H_1(\underline{x}^{(n)}; M/M_m) = 0$ (see 4.1). Therefore

$$\varprojlim{}^1_n \varprojlim{}_m H_1(x^n; X/X_m) = \varprojlim{}_n \varprojlim{}^1_m H_1(x^n; X/X_m) = 0.$$

By Roos' exact sequence above (see (#)) $\varprojlim^1 H_1(x^n; M/M_n) = 0$, as required.
(iv) $\Longrightarrow$ (v): This is obvious.
(v) $\Longrightarrow$ (i): The Koszul homology commutes with direct sums. Therefore the implication follows by virtue of 2.5. □

The implication (i) $\Longrightarrow$ (ii) in 4.2 is a generalization of [5, Proposition 1.7]. Furthermore, a certain generalization of bounded torsion to the study of sequences was invented by Greenlees and May (see [5]) and Lipman et al. (see [1]), namely:

Definition 4.3. (A) Let $\underline{x} = x_1, \ldots, x_r$ denote a sequence of elements of R. For an R-module M it is called M-pro-regular if the inverse systems with the multiplication map by x_i^n

$$\{(x_1^n, \ldots, x_{i-1}^n)M :_M x_i^n/(x_1^n, \ldots, x_{i-1}^n)M\}_{n\geq 1}, \quad i = 1, \ldots, r,$$

are pro-zero. This is equivalent to saying that the inverse systems $\{H_1(x_i^{(n)}; M/\underline{x}_{i-1}^{(n)}M)\}_{n\geq 1}$ are pro-zero for $i = 1, \ldots, r$. For a sequence of elements $\underline{x} = x_1, \ldots, x_r$ we specify the subsystems $\underline{x}_i = x_1, \ldots, x_i$ for $i = 0, \ldots, r-1$.
(B) The notion of pro-zero is equivalent to say that for $i = 1, \ldots, r$ and any positive integer n there is an integer $m \geq n$ such that

$$(x_1^m, \ldots, x_{i-1}^m)M :_M x_i^m \subseteq (x_1^n, \ldots, x_{i-1}^n)M :_M x_i^{m-n}.$$

Note that an element $x \in R$ is M-pro-regular if and only if M is of bounded x-torsion.

For a discussion of the notions of pro-regularity of Greenlees and May (see [5]) resp. Lipman (see [1]) we refer to [15]. Moreover, it follows that an M-pro-regular sequence is also M-weakly pro-regular (see e.g. [15, Theorem 2.4]), while the converse does not hold (see [2]). For a homological characterization of M-pro-regular sequences in terms of injective modules we refer to [15, Theorem 2.1]. Here we add a slight extension of [15, Theorem 2.1].

Theorem 4.4. *Let $\underline{x} = x_1, \ldots, x_r$ denote an ordered sequence of elements of R. Let M denote an R-module. Then the following conditions are equivalent.*

- (i) *The sequence $\underline{x}$ is M-pro-regular.*
- (ii) *The sequence $\underline{x}$ is $(M \otimes_R F)$-pro-regular for any flat R-module F.*
- (iii) $\check{H}^1_{x_i}(\Gamma_{\underline{x}_{i-1}}(\operatorname{Hom}_R(M, I)) = 0$ *for $i = 1, \ldots, k$ and any injective R-module I.*
- (iv) $\check{H}^1_{\underline{x}_i}(\operatorname{Hom}_R(M, I)) = 0$ *for $i = 1, \ldots, k$ and any injective R-module I.*

Proof. For the equivalence of the first three conditions we refer to [15, Theorem 2.1]. For the proof of (iii) $\Longleftrightarrow$ (iv) we put $X = \operatorname{Hom}_R(M, I)$ and recall the following short exact sequence

$$0 \to \check{H}^1_{x_i}(\check{H}^0_{\underline{x}_{i-1}}(X)) \to \check{H}^1_{\underline{x}_i}(X) \to \check{H}^0_{x_i}(\check{H}^1_{\underline{x}_{i-1}}(X)) \to 0$$

for $i = 1, \ldots, r$, (see [17, 6.1.11] or [16, 8.1 (b)]). Then note that $\Gamma_{\underline{x}_{i-1}}(X) \cong \check{H}^0_{\underline{x}_{i-1}}(X)$. If (iv) holds the claim in (iii) follows easily. For the converse we have $\check{H}^1_{\underline{x}_i}(X) \cong \check{H}^1_{x_i}(\check{H}^0_{\underline{x}_{i-1}}(X)) = 0$ for $i = 1, \ldots, r$ and inductively the vanishing of $\check{H}^1_{\underline{x}_i}(X)$ for $i = 1, \ldots, r$, recall that $\check{H}^1_{\underline{x}_{i-1}}(X) = 0$ by the inductive step. This proves (iii). □

Recall that 4.4 provides a characterization of M-pro-regular sequences in terms of Čech cohomology. In the following we shall prove a characterization in terms of Čech homology. This depends upon the results of pro-zero inverse systems as shown above.

Theorem 4.5. *Let $\underline{x} = x_1, \ldots, x_r$ denote a sequence of elements of R. For an R-module M the following conditions are equivalent:*

(i) *The sequence* $\underline{x}$ *is* M*-pro-regular.*
(ii) $\check{H}_0^{x_i}(\Lambda^{\underline{x}_{i-1}}(M^{(S)})) \cong \Lambda^{\underline{x}_i}(M^{(S)})$ *and* $\check{H}_1^{x_i}(\Lambda^{\underline{x}_{i-1}}(M^{(S)})) = 0$ *for* $i = 1, \ldots, r$ *and any set* S*.*
(iii) $\check{H}_0^{x_i}(\Lambda^{\underline{x}_{i-1}}(X)) \cong \Lambda^{\underline{x}_i}(X)$ *and* $\check{H}_1^{x_i}(\Lambda^{\underline{x}_{i-1}}(X)) = 0$ *for* $i = 1, \ldots, r$ *and* $X = M, M[T]$*.*
(iv) $\check{H}_0^{\underline{x}_i}(X) \cong \Lambda^{\underline{x}_i}(X)$ *and* $\check{H}_1^{\underline{x}_i}(X) = 0$ *for* $i = 1, \ldots, r$ *and* $X = M, M[T]$*.*
(v) $\Lambda^{\underline{x}_{i-1}}(X)$ *is of bounded* x_i*-torsion for* $i = 1, \ldots, r$ *and* $X = M, M[T]$*.*

Proof. First note that $\underline{x}$ is $M^{(S)}$-pro-regular for any set S. It turns out since $R/\underline{x}_i^{(n)}R$ is finitely generated and $\operatorname{Hom}_R(R/\underline{x}_i^{(n)}R, \cdot)$ commutes with direct sums. Because of

$$\underline{x}_{i-1}^{(n)}M^{(S)} :_{M^{(S)}} x_i^n/\underline{x}_{i-1}^{(n)}M^{(S)} \cong H_1(x_i^n; H_0(\underline{x}_{i-1}^{(n)}; M^{(S)}))$$

for all $n \geq 0$ and $i = 1, \ldots, r$, it follows that the corresponding inverse systems are isomorphic and pro-zero. Note that $H_0(\underline{x}_{i-1}^{(n)}; M^{(S)}) \cong M^{(S)}/\underline{x}_{i-1}^{(n)}M^{(S)}$. Moreover the condition and Theorem 4.2 proves the equivalence of the first three statements.
(iii) $\Longleftrightarrow$(iv): By view of [16, 8.1] there are short exact sequences

$$0 \to \check{H}_0^{x_i}(\check{H}_j^{\underline{x}_{i-1}}(X)) \to \check{H}_j^{\underline{x}_i}(X) \to \check{H}_1^{x_i}(\check{H}_{j-1}^{\underline{x}_{i-1}}(X)) \to 0 \qquad (\dagger)$$

for $i = 1, \ldots, r$ and $j = 0, 1$. Then the equivalence is easily seen by the exact sequences. More precisely, (iii) $\Longrightarrow$ (iv) follows by increasing induction on i starting at $i = 1$. The converse follows similarly.
(v) $\Longrightarrow$ (iii): The assumption in (v) implies the vanishing

$$\varprojlim H_1(x_i^n; \Lambda^{\underline{x}_{i-1}}(X)) = \varprojlim{}^1 H_1(x_i^n; \Lambda^{\underline{x}_{i-1}}(X)) = 0.$$

By virtue of 3.2 it follows that $\check{H}_1^{\underline{x}_i}(\Lambda^{\underline{x}_{i-1}}(X)) = 0$ and $\check{H}_0^{\underline{x}_i}(\Lambda^{\underline{x}_{i-1}}(X)) \cong \varprojlim H_0(x_i^n; \Lambda^{\underline{x}_{i-1}}(X))$. Now we have $\varprojlim H_0(x_i^n; \Lambda^{\underline{x}_{i-1}}(X)) \cong \varprojlim_n \varprojlim_m X/(x_i^n, \underline{x}_{i-1}^{(m)})X \cong \Lambda^{\underline{x}_i}(X)$, which proves the claim in (iii).
(iii) $\Longrightarrow$(v): The statement yields $\varprojlim H_1(x_i^n; \Lambda^{\underline{x}_{i-1}}(X)) = \varprojlim{}^1 H_1(x_i^n; \Lambda^{\underline{x}_{i-1}}(X)) = 0$. For a fixed n and $j = 0, 1$ we have the short exact sequences

$$0 \to \varprojlim{}^1_m H_{j+1}(x_i^n; X/\underline{x}_{i-1}^{(m)}X) \to H_j(x_i^n; \Lambda^{\underline{x}_{i-1}}(X)) \to \varprojlim{}_m H_j(x_i^n; X/\underline{x}_{i-1}^{(m)}X) \to 0.$$

This follows since the inverse system for $\varprojlim{}_m K_\bullet(x_i^n; X/\underline{x}_{i-1}^{(m)}X) \cong K_\bullet(x_i^n; \Lambda^{\underline{x}_{i-1}}(X))$ has degree wise surjective maps. For $j = 1$ it yields that

$$0 = \varprojlim{}_n H_1(x_i^n; \Lambda^{\underline{x}_{i-1}}(X)) \cong \varprojlim{}_n \varprojlim{}_m H_1(x_i^n; X/\underline{x}_{i-1}^{(m)}X) \cong \varprojlim{}_n H_1(x_i^n; X/\underline{x}_{i-1}^{(n)}X).$$

It remains to show the vanishing of $\varprojlim{}^1_n H_1(x_i^n; X/\underline{x}_{i-1}^{(n)}X)$. First note that the above short exact sequence for $j = 1$ provides that $\varprojlim{}^1_n \varprojlim{}_m H_1(x_i^n; X/\underline{x}_{i-1}^{(m)}X) = 0$. The same sequence for $j = 0$ yields that $\varprojlim_n \varprojlim^1_m H_1(x_i^n; X/\underline{x}_{i-1}^{(m)} X) = 0$. Then the above sequence (#) (see proof of 4.2) with m, n reversed proves the vanishing $\varprojlim^1_n H_1(x_i^n; X/\underline{x}_{i-1}^{(n)} X) = 0$. □

5. A global variation

As before, let R denote a commutative ring. For an element $f \in R$ we write $D(f) = \operatorname{Spec} R \setminus V(f)$. Note that $D(f)$ is an open set in the Zariski topology of $\operatorname{Spec} R$. For $f \in R$ there is a natural map $\operatorname{Spec} R_f \to \operatorname{Spec} R$ that induces a homeomorphism between $\operatorname{Spec} R_f$ and $D(f)$. Since $\operatorname{Spec} R = \cup_{f \in R} D(f)$ and since $\operatorname{Spec} R$ is quasi-compact there are finitely many $f_1, \ldots, f_r \in R$ such that $\operatorname{Spec} R = \cup_{i=1}^r D(f_i)$. Then we recall the following definitions (see [15]).

Definition 5.1. (A) A sequence $\underline{f} = f_1, \ldots, f_r$ of elements of R is called a covering sequence if $\operatorname{Spec} R = \cup_{i=1}^r D(f_i)$. This is equivalent to saying that $R = \underline{f} R$. Moreover, if $\underline{f}$ is a covering sequence then the natural map $M \to \oplus_{i=1}^r M_{f_i}$ is injective for any R-module M as easily seen.
(B) An ideal $\mathcal{I} \subset R$ is called an effective Cartier divisor if there is a covering sequence $\underline{f} = f_1, \ldots, f_r$ such that $\mathcal{I} R_{f_i} = x_i R_{f_i}, i = 1, \ldots, r$, for non-zerodivisors $x_i/1$ of R_{f_i} with $x_i \in R$. It follows that $\mathcal{I} \subseteq (x_1, \ldots, x_r)R$.
(C) Let $\mathcal{I}$ denote an effective Cartier divisor and $x \in R$. The pair $(\mathcal{I}, x)$ is called pro-regular if for any integer n there is an integer $m \geq n$ such that $\mathcal{I}^m : x^m \subseteq \mathcal{I}^n : x^{m-n}$. This is consistent with the definition in [5] (see 4.3) and is equivalent to the fact that for each n there is an integer $m \geq n$ such that the multiplication map $\mathcal{I}^m :_R x^m/\mathcal{I}^m \xrightarrow{x^{m-n}} \mathcal{I}^n :_R x^n/\mathcal{I}^n$ is the zero map. Moreover, the pair $(\mathcal{I}, x)$ is pro-regular if and only if the inverse system $\{H_1(x^n; R/\mathcal{I}^n)\}_{n \geq 1}$ is pro-zero.

For the following we need a technical result about Cartier divisors and their relation to pro-regularity.

Lemma 5.2. *Let $\mathcal{I} \subseteq R$ be an effective Cartier divisor with the covering sequence $\underline{f} = f_1, \ldots, f_r$ such that $\mathcal{I} R_{f_i} = x_i R_{f_i}, i = 1, \ldots, r$, for non-zerodivisors $x_i/1$ of R_{f_i}. For an element $x \in R$ the following conditions are equivalent:*

(i) *$R/\mathcal{I}$ is of bounded x-torsion.*
(ii) *$R_{f_i}/x_i R_{f_i}$ is of bounded $x/1$-torsion for $i = 1, \ldots, r$.*
(iii) *$x_i/1, x/1$ is pro-regular in R_{f_i} for $i = 1, \ldots, r$ in the sense of 4.3.*
(iv) *$(\mathcal{I}, x)$ is pro-regular in the sense of 5.1.*

Proof. (i) $\Longleftrightarrow$ (ii): For each pair of integers $m \geq n \geq 1$ we have the following commutative diagram where the horizontal maps are injections

$$\begin{array}{ccc} \mathcal{I} :_R x^m/\mathcal{I} & \to & \oplus_{j=1}^r (x_i R_{f_i} :_{R_{f_i}} x^m/1)/x_i R_{f_i} \\ \downarrow^{x^{m-n}} & & \downarrow^{\oplus(x^{m-n}/1)} \\ \mathcal{I} :_R x^n/\mathcal{I} & \to & \oplus_{j=1}^r (x_i R_{f_i} :_{R_{f_i}} x^n/1)/x_i R_{f_i} \end{array}$$

which proves the equivalence.
(ii) $\Longleftrightarrow$ (iii): Note that $x_i/1, x/1$ is pro-regular if and only if $R_{f_i}/x_i^k R_{f_i}$ is of bounded $x/1$-torsion for all $k \geq 1$. The equivalence follows easily: First note that $x_i/1$ is R_{f_i}-regular. Then use induction on the short exact sequence

$$0 \to x_i^k R_{f_i}/x_i^{k+1} R_{f_i} \to R_{f_i}/x_i^{k+1} R_{f_i} \to R_{f_i}/x_i^k R_{f_i} \to 0$$

and recall that $x_i^k R_{f_i}/x_i^{k+1} R_{f_i} \cong R_{f_i}/x_i R_{f_i}$.
(iii) $\Longleftrightarrow$ (iv): The equivalence comes out by the following modification of the above commutative diagram

$$\begin{array}{ccc} \mathcal{I}^m :_R x^m/\mathcal{I}^m & \to & \oplus_{j=1}^r (x_i^m R_{f_i} :_{R_{f_i}} x^m/1)/x_i^m R_{f_i} \\ \downarrow^{x^{m-n}} & & \downarrow^{\oplus(x^{m-n}/1)} \\ \mathcal{I}^n :_R x^n/\mathcal{I}^n & \to & \oplus_{j=1}^r (x_i^n R_{f_i} :_{R_{f_i}} x^n/1)/x_i^n R_{f_i}. \end{array}$$

Recall that the horizontal maps are injective (see also [15]). □

Next we apply the previous investigations to the case when the pair $(\mathcal{I}, x)$ is pro-regular in the sense of 5.1.

Lemma 5.3. *Let $\mathcal{I} \subseteq R$ be an effective Cartier divisor with the covering sequence $\underline{f} = f_1, \ldots, f_r$ such that $\mathcal{I} R_{f_i} = x_i R_{f_i}, i = 1, \ldots, r$, for non-zerodivisors $x_i/1$ of R_{f_i}. For an element $x \in R$ the following conditions are equivalent:*

(i) *$R/\mathcal{I}$ is of bounded x-torsion.*
(ii) *$\check{H}_1^x((R/\mathcal{I})[T]) = 0$ and $\check{H}_0^x((R/\mathcal{I})[T]) \cong \Lambda^x((R/\mathcal{I})[T])$.*
(iii) *$\check{H}_1^x(\Lambda^{\mathcal{I}}(X)) = 0$ and $\check{H}_0^x(\Lambda^{\mathcal{I}}(X)) \cong \Lambda^{(x,\mathcal{I})}(X)$ for $X = R, R[T]$.*
(iv) *$\Lambda^{\mathcal{I}}(R)$ and $\Lambda^{\mathcal{I}}(R[T])$ are of bounded x-torsion.*

Proof. First note that by 5.2 $\{H_1(x^n; R/\mathcal{I})\}_{n \geq 1}$ is pro-zero if and only if $\{H_1(x^k; R/\mathcal{I}^k)\}_{k \geq 1}$ is pro-zero. Then the equivalence of (i) and (ii) follows by 3.4. Moreover, by 4.2 the pro-zero property of the second inverse system above implies the equivalence to (iii). Finally the equivalence of (iii) and (iv) is a consequence of 4.5 and 4.1 since $\varprojlim_n \varprojlim^1_m H_1(x^n; R/\mathcal{I}^m) = 0$. □

In the following we shall give a comment of the previous investigations to the recent work of Bhatt and Scholze (see [3]) completing the results of [15]. To this end let $p \in \mathbb{N}$ denote a prime number and let $\mathbb{Z}_p := \mathbb{Z}_{\mathfrak{p}}$ the localization at the prime ideal $(p) = \mathfrak{p} \in \operatorname{Spec} \mathbb{Z}$. In the following let R be a $\mathbb{Z}_p$-algebra.

Definition 5.4. (see [3, Definition 1.1]) A prism is a pair $(R, \mathcal{I})$ consisting of a δ-ring R (see [3, Remark 1.2]) and a Cartier divisor $\mathcal{I}$ on R satisfying the following two conditions.

(a) The ring R is $(p, \mathcal{I})$-adic complete.
(b) $p \in \mathcal{I} + \phi_R(\mathcal{I})R$, where ϕ_R is the lift of the Frobenius on R induced by its δ-structure (see [3, Remark 1.2]).

With the previous definition there is the following application of our results.

Corollary 5.5. *Let $(R, \mathcal{I})$ denote a prism. Then the following conditions are equivalent:*

(i) *$\mathcal{I}$ is of bounded p-torsion.*
(ii) *The pair $(\mathcal{I}, p)$ is pro-regular in the sense of 5.1.*
(iii) *$\check{H}^1_x(\operatorname{Hom}_R(R/\mathcal{I}, I)) = 0$ for any injective R-module I.*
(iv) *$\check{H}^{pR}_0(\Lambda^{\mathcal{I}}(R^{(S)}) \cong \Lambda^{(pR,\mathcal{I})}(R^{(S)}))$ and $\check{H}^{pR}_1(\Lambda^{\mathcal{I}}(R^{(S)}) = 0$ for any set S.*
(v) *$\Lambda^{\mathcal{I}}(R^{(S)})$ and $\Lambda^{\mathcal{I}}(R^{(S)})$ are of bounded p-torsion for any set S.*
(vi) *$\Lambda^{\mathcal{I}}(R)$ and $\Lambda^{\mathcal{I}}(R[T])$ are of bounded p-torsion.*

Proof. This is a consequence of 5.2, 5.3 and 4.4. □

Note that 5.5 is an essential improvement of [15, Corollary 4.5], where it was shown that (i) implies the equivalent conditions (ii) and (iii).

Data availability

No data was used for the research described in the article.

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