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Imke Toborg, Rebecca Waldecker & Clemens B. Tietze

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L_9-free groups

Imke Toborg, Rebecca Waldecker, and Clemens B. Tietze*

Mathematics, Martin-Luther-Universitat Halle-Wittenberg, Halle, Germany

ABSTRACT
In this article we classify all L_9-free finite groups.

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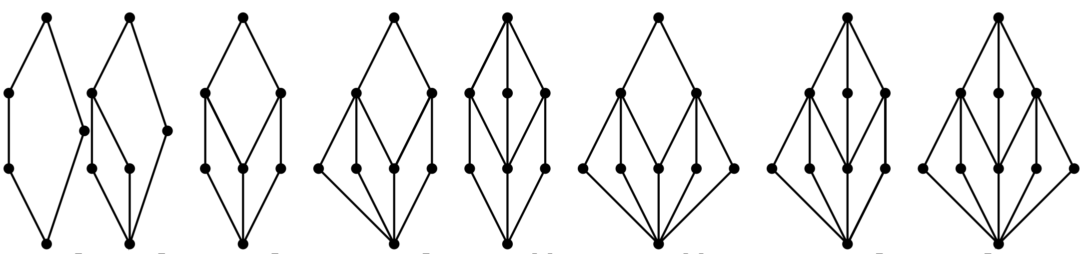
Introduction

There are some algebraic laws that hold in a lattice L if and only if L does not have a sublattice of a specific shape. For example, a lattice is modular if and only if it does not have a sublattice isomorphic to the so-called pentagon L_5.

If L is a lattice, then we call a group L-free if and only if its subgroup lattice does not contain a lattice isomorphic to L. For example, the finite L_5-free groups are exactly the modular groups, and these have been classified by Iwasawa in 1941, see [4]. The subgroup lattice of the dihedral group of order 8, often denoted by L_{10}, and some of its sublattices are of particular interest. One reason is that, if p is a prime number, then a finite p-group is L_5-free if and only if it is L_{10}-free.

There are several sublattices of L_{10} containing L_5:



In 1999 Baginski and Sakowicz [2] studied finite groups that are L_6-free and L_7-free at the same time, and later Schmidt [8] classified the finite groups that are L_6- or L_7-free. Together with Andreeva and the first author he also characterized, in [1], all finite groups that are L_8-free or M_8-free. Finally, the finite M_9-free groups have been classified by Pölzing and the second author in [6]. Furthermore, there is a general discussion of L_{10}-free groups by Schmidt, which can be found in [9] and [10].

In this paper we investigate finite L_9-free groups. Since L_9 is a sublattice of L_{10}, the groups that we consider are L_{10}-free and therefore we can use Corollary C in [9] as a starting point for our analysis:

CONTACT Rebecca Waldecker rebecca.waldecker@mathematik.uni-halle.de Mathematics, Martin-Luther-Universitat Halle-Wittenberg, Halle, Germany.
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Every finite L_{10}-free group G has normal Hall subgroups $N_1 \leq N_2$ such that $N_1 = \langle P \in \mathrm{Syl}(G) \mid P \trianglelefteq G\rangle$, N_2/N_1 is a 2-group and G/N_2 is meta-cyclic.

Our strategy is to choose $N := N_1$ maximal with respect to the above constraints, and then we show that N has a complement K that is a direct and coprime product of groups of the following structure: cyclic groups, groups isomorphic to Q_8 or semi-direct products $Q \rtimes R$, where Q has prime order and R is cyclic of prime-power order such that $1 \neq \Phi(R) = C_Q(R)$. Furthermore, $[N, K] \cap C_N(K)$ is a 2-group and $C_{O_2(N)}(K)$ is cyclic or elementary abelian of order 4 and every Sylow subgroup of $[N, K]$ is elementary abelian or isomorphic to Q_8. If the action of K on N satisfies some more conditions, then we say that NK is in class $\mathfrak{L}$.

The aim of our article is to prove the following theorem:

Main Theorem. *A finite group is in class $\mathfrak{L}$ if and only if it is L_9-free.*

1. Notation and preliminary results

In this article we mostly follow the notation from Schmidt's book [7] and from [5]. All groups considered are finite and G always denotes a finite group, moreover p and q always denote prime numbers. We quickly recall some standard concepts:

$L(G)$ denotes the **subgroup lattice of** G, consisting of the set of subgroups of G with inclusion as the partial ordering. The **infimum** of two elements $A, B \in L(G)$ is $A \cap B$ (their intersection) and the **supremum** is $A \cup B = \langle A, B\rangle$ (the subgroup generated by A and B).

If L is any lattice, then G is said to be L-**free** if and only if $L(G)$ does not have any sub-lattice that is isomorphic to L.

A lattice L is said to be **modular** if and only if for all $X, Y, Z \in L$ such that $X \leq Z$, the following (also called the **modular law**) is true: $(X \cup Y) \cap Z = X \cup (Y \cap Z)$. We say that a group G is **modular** if and only if $L(G)$ is modular.

The modular law is similar to Dedekind's law (see 1.1.11 of [5]). For all $X, Y, Z \leq G$ such that $X \leq Z$ it says that $XY \cap Z = X(Y \cap Z)$. We will use Dedekind's law frequently throughout this article without giving an explicit reference each time.

If $N \leq G$, then we say that an element $g \in G$ induces **power automorphisms** on N if and only if $U^g = U$ for all subgroups U of N. Furthermore $\mathrm{Pot}_G(N) := \{g \in G \mid U^g = U \text{ f.a. } U \leq N\}$ is a subgroup of G because $\mathrm{Pot}_G(N) = \bigcap_{U \leq N} N_G(U)$.

Lemma 1.1. *Let K be a finite group that acts coprimely on the p-group P. Then $P = [P, K]C_P(K)$. If P is abelian, then this product is direct. If $[P, K] \leq \Phi(P)$, then $[P, K] = 1$. Furthermore $[P, K] = [P, K, K]$ and for all K-invariant normal subgroups N of P we have that $C_{P/N}(K) = C_P(K)N/N$.*

Proof. These statements are a collection from 8.2.2, 8.2.7, 8.2.9, and 8.4.2 of [5]. □

Lemma 1.2. *Let $p \in \pi(G)$ and suppose that $G = PK$, where P is a normal Sylow p-subgroup of G, $K \leq G$ is a p'-group and $P_0 := [P, K] \neq 1$. Suppose further that $\Phi(P_0) \leq C_P(K)$ and that K acts irreducibly on $P_0/\Phi(P_0)$. If $g \in P \setminus C_P(K)$, then $P_0 \leq \langle [g, K], K\rangle \leq \langle g, K\rangle$.*

Proof. Let $g \in P \setminus C_P(K)$ and $R := \langle [g, K], K\rangle$. Then we first remark that $R \leq \langle g, K\rangle$. Lemma 1.1 shows that $P = C_P(K)P_0$ and hence we have elements $c \in C_P(K)$ and $h \in P_0$ such that $g = ch$. We note that $P_0 \trianglelefteq G$ and therefore $\Phi(P_0)$ is a normal subgroup of G, moreover P_0 is a p-group and hence $P_0/\Phi(P_0)$ is elementary abelian. Let $\bar{\ } : G \to G/\Phi(P_0)$ denote the natural homomorphism. As $\overline{P_0} = [\bar{P}, \bar{K}]$ is abelian and $\bar{K}$ acts coprimely on it, we see that $C_{[\bar{P},\bar{K}]}(\bar{K}) \cap [\bar{P}, \bar{K}, \bar{K}] = 1$, again by Lemma 1.1. Therefore $C_{[\bar{P},\bar{K}]}(\bar{K}) = C_{[\bar{P},\bar{K},\bar{K}]}(\bar{K}) = 1$. We recall that $ch = g \notin C_P(K)$ and thus $h \notin C_P(K)$, and then by hypothesis $h \notin \Phi(P_0)$ and in particular $1 \neq \bar{h} \in \overline{P_0}$. It follows that $[\bar{h}, \bar{K}] \neq 1$ because $C_{[\bar{P},\bar{K}]}(\bar{K}) = 1$, see above. We conclude that $1 \neq [\bar{h}, \bar{K}] = [\bar{g}, \bar{K}] = \overline{[g, K]} \leq \overline{P_0} \cap \bar{R}$. By hypothesis K acts irreducibly on $\overline{P_0}$, hence $\bar{K}$ does as well and we see that $\overline{P_0} = \overline{P_0 \cap R\Phi(P_0)} = \overline{(P_0 \cap R)\Phi(P_0)}$. The main property of the Frattini subgroup (see for example 5.2.3 in [5]) finally gives that $P_0 = P_0 \cap R$. □

Lemma 1.3. *Let Q be a 2-group that is elementary abelian, cyclic or isomorphic to Q_8. Then Q does not admit power automorphisms of odd order.*

Proof. If Q is abelian, then the assertion follows from 2.2.5 of [5]. If $Q \cong Q_8$, then $Aut(Q) \cong \mathrm{Sym}_4$, and any automorphism of order 3 interchanges the maximal subgroups of Q. □

Lemma 1.4. *Suppose that $G = NK$, where N is a normal Hall subgroup of G and K is a complement. Let $N_1, N_2 \leq N$, $Q, R \leq K$ and $x \in N$. Then the following hold:*

(a) If N_1Q and N_2R^x are subgroups of G, then $N_1Q \cap N_2R^x = N_1(Q \cap R) \cap N_2(Q \cap R)^x$.
(b) If $N_2R^x \leq G$ and $\langle x^{Q\cap R}\rangle \cap N_2 = 1$, then $Q \cap N_2R^x = Q \cap R^x \leq C_{Q\cap R}(x)$.
(c) If K is abelian and acts irreducibly on the abelian group $N/\Phi(N)$ or if N is abelian and K induces power automorphisms on it, then $C_K(N) = C_K(x)$ or $x \in \Phi(N)$.
(d) If $Q \leq R$, then $\langle Q, R^x\rangle = \langle [x,Q]^{R^x}\rangle R^x$.

Proof. Suppose that N_1Q and N_2R^x are subgroups of G.

For (a) we do the following calculation:

$$\begin{aligned}
N_1Q \cap N_2R^x &= (N_1Q \cap NQ) \cap N_2R^x = N_1Q \cap (NQ \cap N_2R^x) = N_1Q \cap N_2(NQ^x \cap R^x) \\
&= N_1Q \cap N_2(NQ^x \cap (K^x \cap R^x)) = N_1Q \cap N_2((NQ^x \cap K^x) \cap R^x) \\
&= N_1Q \cap N_2(Q^x \cap R^x) = N_1Q \cap (N(Q \cap R)^x \cap N_2(Q \cap R)^x) \\
&= (N_1Q \cap N(Q \cap R)) \cap N_2(Q \cap R)^x = N_1(Q \cap N(Q \cap R)) \cap N_2(Q \cap R)^x \\
&= N_1((Q \cap K) \cap N(Q \cap R)) \cap N_2(Q \cap R)^x \\
&= N_1(Q \cap (K \cap N(Q \cap R))) \cap N_2(Q \cap R)^x \\
&= N_1(Q \cap (Q \cap R)) \cap N_2(Q \cap R)^x = N_1(Q \cap R) \cap N_2(Q \cap R)^x.
\end{aligned}$$

Then (a) yields that $Q \cap N_2R^x = (Q \cap R) \cap N_2(Q \cap R)^x$. Therefore, if $\langle x^{Q\cap R}\rangle \cap N_2 = 1$ and if $a, b \in Q \cap R$ and $y \in N_2$ are such that $a = yb^x = yx^{-1}x^{b^{-1}}b$, then the fact that $ab^{-1} = yx^{-1}x^{b^{-1}} \in K \cap N = 1$ implies that $a = b$ and that $y^{-1} = [x, b^{-1}] \in N_2 \cap \langle x^{Q\cap R}\rangle = 1$. We deduce that $a = b^x = a^x \in Q \cap R^x$ and that $[x, a] = 1$. Hence $Q \cap N_2R^x = (Q \cap R) \cap N_2(Q \cap R)^x = Q \cap R^x \leq C_{Q\cap R}(x)$. This is (b).

If K is abelian, then $x \in C_N(C_K(x))$ and $C_N(C_K(x))$ is K-invariant. Thus, if K acts irreducibly on $N/\Phi(N)$ and $x \notin \Phi(N)$, then $C_N(C_K(x))\Phi(N) = N$, and the fact that $[C_K(x), N] = 1$ implies that $C_K(x) = C_K(N)$. If K induces power automorphisms on the abelian group N, then 1.5.4 of [7] implies that these are universal. If $x = 1$, then $x \in \Phi(N)$, and otherwise $x \neq 1$ and every element of K that centralizes x also centralizes N. Altogether (c) holds.

Suppose finally that $Q \leq R$. Then, for all $g \in Q$, we have that $g = g^x \cdot (g^{-1})^x \cdot g \in R^x[x, Q] \leq \langle [x,Q]^{R^x}\rangle R^x$. It follows that $\langle Q, R^x\rangle \leq \langle [x,Q]^{R^x}\rangle R^x$.

On the other hand $(g^{-1})^x \in Q^x \leq R^x$ for all $g \in Q$ and therefore $[x, g] = (g^{-1})^x g \in \langle Q, R^x\rangle$.

This implies that $[x, Q] \leq \langle Q, R^x\rangle$. Since $R^x \leq \langle Q, R^x\rangle$, we deduce that $\langle [x,Q]^{R^x}\rangle R^x \leq \langle Q, R^x\rangle$, which is (d). □

2. Battens and batten groups

Definition 2.1. (a) We say that G is a **batten** if and only if G is a cyclic p-group, or isomorphic to Q_8, or $G = QR$, where Q is a normal subgroup of prime order and R is a cyclic p-group of order coprime to $|Q|$ and such that $C_R(Q) = \Phi(R) \neq 1$.

(b) We say that G is a **batten group** if and only if G is a direct product of battens of pairwise coprime order.

(c) If G is a batten group, then we say that $B \leq G$ is a **batten of** G if and only if B is a batten that is one of the direct factors of G.

Warning: It is possible for a subgroup of a batten group G to be a batten, abstractly, but not to be a batten of G. This can happen when it is a p-subgroup for some prime p of a batten as in the third case of Definition 2.1.

Example 2.2. (a) Suppose that $Q := \langle x \rangle$ is a group of order 19 and that $R = \langle y \rangle$ is a subgroup of $\mathrm{Aut}(X)$ of order 27. Further suppose that $x^y := x^7$. Then $B := QR$ is a non-nilpotent batten. For this we calculate that $x^{y^3} = (x^7)^{y^2} = (x^{49})^y = (x^{11})^y = x^{77} = x$. Then the fact that $x^y = x^7 \neq x$ implies that $C_R(Q) = \langle y^3 \rangle = \Phi(R)$. We note that B is a batten group and that Q is a subgroup of B that is a batten, but not a batten of B because $[Q, R] \neq 1$.
(b) Let $B = QR$ be as in (a), let $T \cong Q_8$ and let S be a cyclic group of order 625. Then $B \times T \times S$ is a batten group.

Remark 2.3. *Let G be a batten group and let B be a batten of G such that $|\pi(B)| = 2$.*
(a) B is not nilpotent, but B has a unique normal Sylow subgroup.
(b) A Sylow subgroup Q of B is cyclic, and therefore Q is batten. But Q is not a direct factor of G and hence Q is not a batten of G.

Definition 2.4. Suppose that G is a non-nilpotent batten. Then there is a unique prime $q \in \pi(G)$ such that G has a normal Sylow q-subgroup Q, and Q is cyclic of order q. In this case we set $\mathcal{B}(G) := Q$.

From the definition we can immediately see that $\mathcal{B}(G)$ is a characteristic subgroup of a non-nilpotent batten G and that it has prime order.

Lemma 2.5. *Suppose that G is a non-nilpotent batten, that $r \in \pi(G)$ and that $R \in \mathrm{Syl}_r(G)$ has order at least r^2. Then $Z(G) = C_R(\mathcal{B}(G)) = \Phi(R) = O_r(G)$.*

Proof. Since $|R| \geq r^2$, we see that $R \neq \mathcal{B}(G)$. Then Definition 2.1 implies that R is cyclic and that there is a prime $q \in \pi(G) \setminus \{r\}$ such that $Q := \mathcal{B}(G) \in \mathrm{Syl}_q(G)$.

Now $G = Q \rtimes R$ and $C_R(\mathcal{B}(G)) = \Phi(R)$. We recall that R is cyclic, and then this implies that $\Phi(R) \leq Z(G)$. Since G is not nilpotent, we see that $\mathcal{B}(G)$ is not contained in $Z(G)$. Thus $Z(G)$ is an r-group, because $\mathcal{B}(G)$ has prime order. In addition $R \nleq Z(G)$, because G is not nilpotent. Since $\Phi(R)$ is a maximal subgroup of the cyclic group R, it follows that $\Phi(R) = Z(G)$. Moreover we have that $[Q, O_r(G)] \leq Q \cap R = 1$, whence $O_r(G) \leq C_R(Q) = \Phi(R) = Z(G) \leq C_R(Q)$. This proves all statements. □

Lemma 2.6. *If G is a batten group and $P \leq G$ is a Sylow p-subgroup of G for some prime p, then G has a subgroup of order p. In addition, $\Omega_1(P) \leq Z(G)$ or there is some non-nilpotent batten B of G such that $\Omega_1(P) = P = \mathcal{B}(B)$.*

Proof. Since $P \leq G$, it follows that P is cyclic or isomorphic to Q_8. Therefore $\Omega_1(P)$ has order p.

If P is a batten, then $\Omega_1(P) \leq Z(P) \leq Z(G)$. In particular $\Omega_1(P)$ is the unique subgroup of G of its order. Otherwise there is a non-nilpotent batten B of G such that $P \leq B$. If P has order p, then $P = \Omega_1(P) = \mathcal{B}(B)$ is a normal subgroup of B and so of G. Again $\Omega_1(P)$ is the unique subgroup of K of order q. Otherwise we have that $\Omega_1(P) \leq \Phi(P) = Z(B) \leq Z(G)$ by Lemma 2.5. In particular $\Omega_1(P)$ is the unique subgroup of K of its order. □

Lemma 2.7. *Suppose that H is a batten and that $U \lneq H$. Then U is a cyclic batten group. Furthermore, all subgroups of batten groups are batten groups.*

Proof. Assume for a contradiction that U is not a cyclic batten group. Then U is not a cyclic batten, and therefore H is neither cyclic of prime power order nor isomorphic to Q_8. Thus H is not nilpotent, in particular $|\pi(H)| = 2$ and all Sylow subgroups of H are cyclic. This implies that U is not a p-group. Let $\pi(H) = \{q, r\}$, let $Q := \mathcal{B}(H)$ and $R \in \mathrm{Syl}_r(H)$ be such that $H = QR$ and $C_R(Q) = \Phi(R) \neq 1$. Now $\pi(U) = \{q, r\}$ as well and therefore $\mathcal{B}(H) \leq U$. Then Dedekind's law gives that $U = \mathcal{B}(H) \cdot (U \cap R)$ is a proper subgroup of $H = \mathcal{B}(H) \cdot R$, and it follows that $U \cap R$ is a proper subgroup of R. In particular, since R is cyclic, we have that $1 \neq U \cap R \leq \Phi(R)$. Then Lemma 2.5 gives that $\Phi(R) = Z(H)$. Altogether $U \leq \mathcal{B}(H)\Phi(R) = \mathcal{B}(H) \times \Phi(R)$. But then U is a direct product of cyclic groups of prime power order, i.e. a cyclic batten group, and this is a contradiction.

Next suppose that G is a batten group and that U is a subgroup of G. Then U is a direct product of subgroups of the battens of G whose orders are pairwise coprime. Consequently U is a batten group as well, by the arguments above. □

We remark that sections of battens, or batten groups, are not necessarily batten groups. For example, $Q_8/Z(Q_8)$ is not a batten group.

Lemma 2.8. *Suppose that K is a batten group and that $Q \leq K$ is a q-group.*

If Q is not normal in K, then there is a non-nilpotent batten B of K such that $B = \mathcal{B}(B)Q$ and $N_K(Q) = C_K(Q)$.

If $Q \trianglelefteq K$, then $|K : C_K(Q)| \in \{1, 4\}$ or this index is a prime number.

Proof. Since K is a batten group, there is a batten B of K such that $Q \leq B$. Moreover there is a subgroup L of K such that $K = L \times B$. Then $L \leq C_K(B) \leq C_K(Q)$ $(*)$.

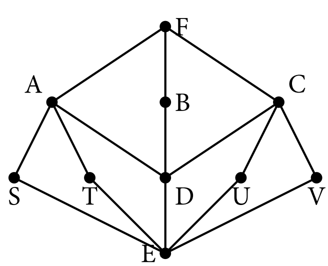
We first suppose that Q is not a normal subgroup of K. Since K is a direct product of battens, it follows that Q is not normal in B. Thus B is neither abelian nor hamiltonian (otherwise all subgroups of B would be normal), and it follows that B is not nilpotent. Now Q is a proper subgroup of B because $Q \not\trianglelefteq B$. We conclude from Lemma 2.5 that neither $Q \leq \mathcal{B}(B)$ nor $Q \leq Z(B)$, whence $B = \mathcal{B}(B)Q$ and therefore $N_B(Q) = Q = C_B(Q)$. Consequently $(*)$ and Dedekind's modular law yield that $N_K(Q) = LN_B(Q) = LC_B(Q) = C_K(Q)$.

Suppose now that $Q \trianglelefteq K$. Using $(*)$ we see that $|K : C_K(Q)| = |B : C_B(Q)|$. Hence we may suppose that B is not abelian. If B is not nilpotent, then $Z(B)$ and a Sylow q-subgroup of B centralize Q. Thus Lemma 2.5 yields that $|K : C_K(Q)| = |B : C_B(Q)|$ equals the prime in $\pi(B)\setminus\{q\}$. Let $B \cong Q_8$ and suppose that $Q \not\leq Z(B)$. Then Q has order 4 or 8. In the first case $|K : C_K(Q)| = |B : C_B(Q)| = |B : Q| = 2$ and in the second case $|K : C_K(Q)| = |B : C_B(Q)| = |B : Z(B)| = 4$, which completes the proof. □

3. L_{10} and its sublattices

Throughout this article we will use the notation from the next definition whenever we refer to L_{10} and its sublattices:

Definition 3.1. The lattice L_{10} is defined to be isomorphic to $L(D_8)$, with notation as indicated in the picture.



Now we define
(a) $L_5 := \{E, S, U, A, F\}$,
(b) $L_6 := L_5 \cup \{T\}$,
(c) $L_7 := L_5 \cup \{D, C\}$,
(d) $L_8 := L_{10} \setminus \{B, V\}$,
(e) $M_8 := L_{10} \setminus \{T, V\}$,
(f) $L_9 := L_{10} \setminus \{V\}$,
(g) $M_9 := L_{10} \setminus \{B\}$,
with the corresponding inclusion relations induced from the lattice L_{10}.

Definition 3.2. Let L be a lattice. An equivalence relation $\equiv$ on L is called a congruence relation if and only if, for all $A, B, C, D \in L$ such that $A \equiv B$ and $C \equiv D$, we have that $\langle A, C\rangle \equiv \langle B, D\rangle$ and $A \cap C \equiv B \cap D$.

Lemma 3.3. *Let $\equiv$ be a congruence relation on $L_9 = \{A, B, C, D, E, F, U, T, S\}$ as in Definition 3.1 and suppose that $\equiv$ is not equality. Then $E \equiv D$.*

Proof. Let $\equiv$ be a congruence relation on L_9 and suppose that $E \not\equiv D$.

If $X \in \{A, B, C, F\}$, then $E \cap D = E \not\equiv D = X \cap D$ and therefore $E \not\equiv X$.

First we assume that $X_0 \in L_9 \setminus \{F\}$ is such that $F \equiv X_0$. Then there is some $X \in \{A, B, C\}$ such that $X_0 \leq X$, and then $X = \langle X_0, X\rangle \equiv \langle F, X\rangle = F$. We choose $Y \in \{T, U\}$ such that $X \cap Y = E$. Then $E = X \cap Y \equiv F \cap Y = Y$ and therefore, if $Z \in \{T, U\} \setminus \{Y\}$, then $Z = \langle Z, E\rangle \equiv \langle Z, Y\rangle = F$. Now it follows that $E = Z \cap D \equiv F \cap D = D$, which gives a contradiction.

We have seen that E is not congruent to any of the elements A, B, C, D, F. Next we assume that $X \in \{S, T, U\}$ is such that $E \equiv X$ and we choose $Y \in \{A, C\}$ such that $X \not\leq Y$. Then $Y = \langle Y, E\rangle \equiv \langle Y, X\rangle = F$, which gives another contradiction. We conclude that $\{E\}$ and $\{F\}$ are singleton classes with respect to $\equiv$.

If there are $X, Y \in L \setminus \{E, F\}$ such that $X \neq Y$ and $X \equiv Y$, $X \cap Y = E$ or $\langle X, Y\rangle = F$, then this implies that $X = X \cap X \equiv X \cap Y = E$ or $X = \langle X, X\rangle \equiv \langle X, Y\rangle = F$. As this is impossible, we conclude that such elements X, Y do not exist.

In particular A, B, C are pairwise non-congruent and D, S, T, U are also pairwise non-congruent.

We assume that $D \equiv X \in \{A, B, C\}$. Then we choose $Y \in \{T, U\}$ such that $Y \not\leq X$, and this gives the contradiction $F \neq \langle Y, D\rangle \equiv \langle Y, X\rangle = F$. Hence $\{D\}$ is a singleton as well.

Finally, we assume that there are $X \in \{A, B, C\}$ and $Y \in \{T, S, U\}$ such that $X \equiv Y$. We have seen that $X \cap Y \neq E$ and then it follows that $Y \leq X$ by the structure of L_9. Now $D = X \cap D \equiv Y \cap D = E$, which is impossible. In conclusion, for all $X, Y \in L_9$, we have that $X \equiv Y$ if and only if $X = Y$. This means that $\equiv$ is equality. □

In the following lemma we argue similarly to Lemma 2.2 in [8].

Lemma 3.4. *Suppose that $n \in \mathbb{N}$ and that $G_1, \ldots, G_n$ are normal subgroups of G of pair-wise coprime order such that $G = G_1 \times \cdots \times G_n$. Then G is L_9-free if and only if, for every $i \in \{1, \ldots, n\}$, the group G_i is L_9-free.*

Proof. Since subgroups of L_9-free groups are L_9-free we just need to verify the "if" part.

Suppose that $G_1,\ldots,G_n$ are L_9-free. Then Lemma 1.6.4 of [7] implies that $L(G) \cong L(G_1) \times \cdots \times L(G_n)$. By induction we may suppose that $n = 2$. Assume that $L = \{E, T, S, U, D, A, B, C, F\}$ is a sublattice of $L(G) \cong L(G_1) \times L(G_2)$ that is isomorphic to L_9 as in Definition 3.1. Then the projections φ_1 and φ_2 of L into $L(G_1)$ and $L(G_2)$, respectively, are not injective, because $L(G_1)$ and $L(G_2)$ are L_9-free. Let $i \in \{1, 2\}$ and define, for all $X, Y \in L$:

$$X \equiv_i Y :\Leftrightarrow \varphi_i(X) = \varphi_i(Y).$$

Then $\equiv_i$ is a congruence relation on L, because φ_i is a lattice homomorphism, but it is not equality because ϕ_i is not injective. Then Lemma 3.3 implies that $\varphi_1(D) = \varphi_1(E)$ and $\varphi_2(D) = \varphi_2(E)$, and hence $D = E$. This is a contradiction. □

Lemma 3.5. *$L_9 = \{A, B, C, D, E, F, U, T, D\}$ as in Definition 3.1 is completely characterized by the following:*

L9 (i) $D \neq E$.
L9 (ii) $\langle S, T\rangle = \langle S, D\rangle = \langle T, D\rangle = A$ *and* $S \cap T = S \cap D = T \cap D = E$.
L9 (iii) $\langle D, U\rangle = C$ *and* $D \cap U = E$.

L9 (iv) $\langle S, U\rangle = \langle T, U\rangle = F$.
L9 (v) $\langle A, B\rangle = \langle B, C\rangle = F$ *and* $A \cap B = A \cap C = B \cap C = D$.

Proof. We first remark that L_9 satisfies the relations given in L9 (i) – L9 (v).

Suppose conversely that a lattice $L = \{A, B, C, D, E, F, S, T, U\}$ satisfies the relations given in L9 (i) – L9 (v). Then we see that $E \le A, B, C, D, S, T, U \le F$ and $D \le A, B, C$ as well as $S, T \le A$ and $U \le C$. If these are the unique inclusions and $|L| = 9$, then $L \cong L_9$.

For all $X, Y \in L$ such that $X \le Y$ we have that $X \cap Y = X$ and $\langle X, Y\rangle = Y$. Thus L9 (ii) shows that S, T, and D are pair-wise not subgroups of each other.

Using L9 (iii) we obtain that $D \not\le U$ and $U \not\le D$, and L9 (iv) gives that also S, T and U are pair-wise not subgroups of each other. In addition, by L9 (v), we have that A, B and C are pair-wise not subgroups of each other. Together with the fact that $S, T \le A$ and $U \le C$, this implies that $A \not\le U$, $C \not\le S, T$ and $B \not\le S, T, U$. Moreover we have that $D \le A, B, C$ and $S, T \le A$ and $U \le C$ and $A \ne F \ne B$ and $C \ne F$. Together with L9 (ii) and L9 (iii), this information yields that $A \not\ge U$ and $C \not\ge S, T$ as well as $B \not\ge S, T, U$.

We conclude that there is a lattice homomorphism φ from L_9 to L. Hence we obtain a congruence relation $\equiv$ on L_9 by defining that $X \equiv Y$ if and only if $\varphi(X) = \varphi(Y)$, for all $X, Y \in L_9$.

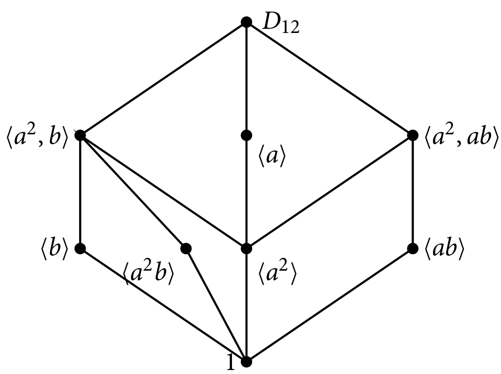
If φ is not injective, then Lemma 3.3 implies that $E = D$. This contradicts L9 (i). Consequently φ is injective and $L \cong L_9$. □

The next lemma gives an example of a group that is not L_9-free.

Lemma 3.6. *D_{12} is not L_9-free.*

Proof. Let G be isomorphic to D_{12} and let $a, b \in G$ be such that $o(a) = 6$, $o(b) = 2$ and $G = \langle a, b\rangle$. Then we find a sublattice in $L(G)$ isomorphic to L_9 by checking the equations from Lemma 3.5.

We let $L := \{1, \langle b\rangle, \langle a^2b\rangle, \langle a^2\rangle, \langle ab\rangle, \langle a^2, b\rangle, \langle a\rangle, \langle a^2, ab\rangle, G\}$ and we define $A := \langle a^2, b\rangle$ and $C := \langle a^2, ab\rangle$.



L9 (i): We see that $a^2 \ne 1$ and hence $\langle a\rangle \ne 1$.
L9 (ii): We notice that $A \le G$ is isomorphic to Sym_3 with cyclic normal subgroup $\langle a^2\rangle$ of order 3 and distinct subgroups $\langle b\rangle$, $\langle a^2b\rangle$ of order 2. Then $\langle\langle b\rangle, \langle a^2b\rangle\rangle = \langle\langle b\rangle, \langle a^2\rangle\rangle = \langle\langle a^2b\rangle, \langle a^2\rangle\rangle = A$ and $\langle b\rangle \cap \langle a^2b\rangle = \langle b\rangle \cap \langle a^2\rangle = \langle a^2b\rangle \cap \langle a^2\rangle = 1$.
L9 (iii): The subgroup C is also isomorphic to Sym_3, the subgroup $\langle ab\rangle \le C$ has order 2, and moreover $\langle\langle a^2\rangle, \langle ab\rangle\rangle = C$ and $\langle a^2\rangle \cap \langle ab\rangle = 1$.
L9 (iv): We first see that $\langle\langle b\rangle, \langle ab\rangle\rangle = \langle a, b\rangle = G$ and then $\langle\langle a^2b\rangle, \langle ab\rangle\rangle = \langle a^2b(ab)^{-1}, ab\rangle = \langle a, b\rangle = G$.
L9 (v): $\langle a\rangle$ is a cyclic normal subgroup of G of order 6, and the subgroups A and C also have order 6. These three subgroups are maximal in G. Hence $\langle A, \langle a\rangle\rangle = \langle C, \langle a\rangle\rangle = G$. Assume for a contradiction that $A = C$. Then $b \in C = \{1, a^2, a^4, ab, a^3b, a^5b\}$, which is false. Since $\langle a^2\rangle$ is the unique subgroup of order 3 of G, we conclude that $A \cap \langle a\rangle = A \cap C = \langle a\rangle \cap C = \langle a^2\rangle$.

Altogether it follows, with Lemma 3.5, that L is isomorphic to L_9, and then G is not L_9-free. □

The next lemma shows how we can construct an entire class of groups that are not L_9-free.

Lemma 3.7. *Suppose that $p \neq q$ and that $G = PQ$, where P is an elementary abelian normal Sylow p-subgroup of G and Q is a cyclic Sylow q-subgroup of G. Suppose that Q acts irreducibly on $[P, Q] \neq 1$ and that $|C_P(Q)| \geq 3$. Then G is not L_9-free.*

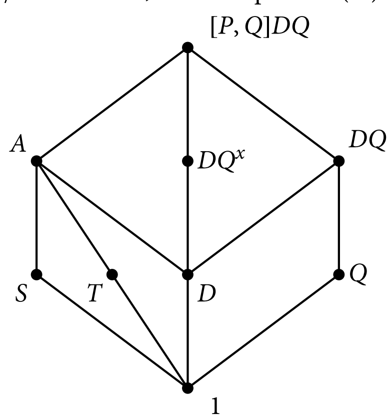
Proof. Since Q is abelian, we see that $E := C_Q(P) \trianglelefteq G$. We claim that $G/C_Q(P)$ is not L_9-free. Therefore we may suppose that $E = 1$.

Our hypotheses imply that $[P, Q]$ is not centralized by Q, and in particular $|[P, Q]| \geq 3$. Moreover $|C_P(Q)| \geq 3$ by hypothesis. Since P is elementary abelian, Lemma 1.1 gives that $P = [P, Q] \times C_P(Q)$.

We let $V \leq [P, Q]$ and $D \leq C_P(Q)$ be subgroups of minimal order such that $|V| \geq 3$ and $|D| \geq 3$ and we set $A := V \times D$. If p is odd, then A has order p^2, and if $p = 2$, then $|A| = 2^4 = 16$. In the first case A has $\frac{p^2-1}{p-1} = p + 1 \geq 4$ subgroups isomorphic to V. In the second case A has $\frac{15 \cdot 14}{3} = 70$ subgroups isomorphic to V, where $3 \cdot \frac{14}{2} - 2 = 19$ of these subgroups intersect V non-trivially and 19 of them intersect D non-trivially. In both cases, we find subgroups T and S of A isomorphic to V such that $|\{D, T, S, V\}| = 4$ and $T \cap V = T \cap D = T \cap S = S \cap D = S \cap V = 1 = E$.

We recall that A is elementary abelian, and then it follows that L9 (i) and L9 (ii) hold and that $A = \langle S, V \rangle = \langle T, V \rangle$ (∗).

We further set $U := Q$. Then $U \cap D \leq Q \cap P = E = 1$ and $\langle U, D \rangle = UD =: C$, which implies L9 (iii). If $X \in \{T, S\}$, then $X \not\leq C_P(Q)$ and then the irreducible action of Q on $[P, Q]$ and Lemma 1.2 yield that $V \leq [P, Q] \leq \langle X, U \rangle$. Using (∗) it follows that $D \leq A \leq \langle V, X, U \rangle = \langle X, U \rangle$. Combining all this information gives that $\langle X, U \rangle = [P, Q]DQ$. Now if we set $F := [P, Q]DQ$, then we have L9 (iv).



To prove our claim, it remains to show that property L9 (v) of Lemma 3.5 is satisfied.

We set $B := DQ^x$ for some $x \in [P, Q]^\#$.

If $y \in \{1, x\}$, then $D \leq A \cap DQ^y = D(A \cap Q^y) = D$ and hence $A \cap B = A \cap C = D$. We further have that $D \leq B \cap C = D(Q \cap DQ^x) = DC_Q(x) = D$, by Lemma 1.4 (b) and (c), because Q acts irreducibly on $[P, Q]$.

In addition, the irreducible action of Q^x on $[P, Q]$ and Lemma 1.2 yield that $[P, Q] \leq \langle A, Q^x \rangle$. It follows that $\langle A, B \rangle = [P, Q]DQ^x = F$. Finally we deduce from Part (d) of Lemma 1.4 that

$$\langle B, C \rangle = D\langle Q, Q^x \rangle = D\langle [x, Q]^{Q^x} \rangle Q^x = D[P, Q]Q^x = F.$$

□

4. Group orders with few prime divisors

Much of our analysis will focus on non-nilpotent groups with a small number of primes dividing their orders. The next lemma sheds some light on why this situation naturally occurs.

Lemma 4.1. *Suppose that G is L_9-free. Then G possesses a normal Sylow subgroup.*

Proof. Assume that this is false. Since G is L_9-free and hence L_{10}-free, [9, Corollary C] is applicable. Then G is metacyclic because it does not have any normal Sylow subgroup, and it follows that G is supersoluble. Then Satz VI.9.1(c) in [3] gives a contradiction. □

Lemma 4.2. *Suppose that G is a p-group. Then the following statements are equivalent:*

- *$L(G)$ is modular.*
- *G is L_5-free.*

- *G is L_9-free.*
- *G is L_{10}-free.*

Proof. This lemma follows from Theorem 2.1.2 in [7] and Lemma 2.1 in [9], since L_9 is a sublattice of L_{10} containing L_5. □

Lemma 4.3. *Suppose that $p \neq q$ and that G is an L_9-free $\{p,q\}$-group. Let P be a normal Sylow p-subgroup of G and let $Q \in \mathrm{Syl}_q(G)$. If G is not nilpotent, then Q is cyclic or $Q \cong Q_8$.*

Proof. First we note that all subgroups and sections of G are L_9-free and that G is L_{10}-free.

Let G be non-nilpotent and assume for a contradiction that Q is neither cyclic nor isomorphic to Q_8. Given that G is L_{10}-free, we may apply Theorem B of [9] and we see that neither (a), (b) nor (c) hold. Therefore $p = 3$ and $q = 2$. Now there are $a, b \in Q$ such that $\langle a, b\rangle$ is not cyclic and b is an involution. If, for all choices of b, we have that $C_P(b) = C_P(Q)$, then $\Omega_1(Q)$ acts element-wise fixed-point-freely on $P/C_P(Q)$, contradicting 8.3.4 (b) of [5]. Therefore we may choose b such that a does not centralize $C_P(b)$, and we also choose a of minimal order under these constraints. Then a^2 centralizes P and a inverts an element $x \in C_P(b)$ by a result of Baer (e.g. 6.7.7 of [5]). If follows that a inverts $\Omega_1(\langle x\rangle)$ and we may suppose that x has order 3. Now $\langle x, a, b\rangle/C_{\langle a\rangle}(x)$ is isomorphic to D_{12}, contrary to Lemma 3.6. □

Lemma 4.4. *Suppose that $p \neq q$ and that G is an L_9-free $\{p,q\}$-group. Furthermore, let P be a normal Sylow p-subgroup of G and let $Q \in \mathrm{Syl}_q(G)$ be cyclic such that $1 \neq [P,Q]$ is elementary abelian.*

Then every subgroup of Q acts irreducibly or by inducing (possibly trivial) power automorphism on $[P,Q]$. Moreover, $C_P(Q)$ is a cyclic 2-group and P is abelian.

Proof. First we note that all subgroups and sections of G are L_9-free and that all subgroups and sections of $[P,Q]$ are elementary abelian, by hypothesis. In addition Lemma 2.2 of [9] yields that $P = C_P(Q) \times [P,Q]$, as G is also L_{10}-free. In particular P is abelian if $C_P(Q)$ is.

Assume that the lemma is false and let G be a minimal counterexample.

Since $[P,Q]$ is elementary abelian, we introduce the following notation with Maschke's theorem: Let $n \in \mathbb{N}$ and let $M_1, \dots, M_n \le [P,Q]$ be Q-invariant and such that $[P,Q] = M_1 \times \cdots \times M_n$ and that Q acts irreducibly on $M_1, \dots, M_n$, respectively. Lemmas 2.3.5 of [7] and 4.2 yield that $\Omega_1(C_P(Q))$ is elementary abelian. Now there are $r \in \mathbb{N}$ and cyclic subgroups $M_{n+1}, \dots, M_{n+r}$ of $\Omega_1(C_P(Q))$ such that $\Omega_1(C_P(Q)) = M_{n+1} \times \cdots \times M_{n+r}$.

We set $H_1 := (M_1 \times \cdots \times M_{n+r-1})Q$ and $H_2 := (M_2 \times \cdots \times M_{n+r})Q$.

Then for every $i \in \{1,2\}$ the group $O_p(H_i)$ is elementary abelian. Moreover H_i is a proper subgroup of G and then the minimal choice of G implies that every subgroup of Q either induces (possibly trivial) power automorphisms on $[O_p(H_i), Q]$ or acts irreducibly on it.

(1) $C_P(Q) = 1$ and $n \le 2$.

Proof. We assume for a contradiction that $n + r \ge 3$.

Then Q does not act irreducibly on both $O_p(H_1)$ and $O_p(H_2)$, and it follows that Q induces (possibly trivial) power automorphisms on $[O_p(H_i), Q]$.

We suppose first that $C_P(Q) = 1$. Then $[O_p(H_i), Q] = O_p(H_i)$ for both $i \in \{1,2\}$. Therefore Lemma 1.5.4 of [7], together with the fact that $1 \neq M_2 \le O_p(H_1) \cap O_p(H_2)$, provides some $k \in \mathbb{N}$ such that $a^y = a^k$ for every $a \in O_p(H_1)O_p(H_2) = [P,Q]\Omega_1(C_P(Q))$. But this means that Q, and hence every subgroup of Q, induces (possibly trivial) power automorphism on $O_p(H_1)O_p(H_2) = [P,Q]$ in this case. Thus G is not a counterexample, which is a contradiction.

We conclude that $C_P(Q) \neq 1$ and now there is some $i \in \{1,2\}$ such that $1 \neq [O_p(H_i), Q]$ and $C_{O_p(H_i)}(Q) \neq 1$. Then H_i satisfies the hypotheses of our lemma and it follows that $C_{O_p(H_i)}(Q)$

is an non-trivial 2-group. In particular $p = 2$. But then Lemma 1.3 provides the contradiction that $[O_p(H_i), Q] = 1$.

For the proof of (1), we assume for a further contradiction that $r = n = 1$. Then Q acts irreducibly on the elementary abelian group $[P, Q]$ and Lemma 3.7, applied to $([P, Q] \times \Omega_1(C_P(Q)))Q$, gives that $|\Omega_1(C_P(Q))| = 2$. In particular we have that $p = 2$. Thus the minimal choice of G and Lemma 1.3 yield, for every proper subgroup U of Q, that U centralizes P or acts irreducibly on $[P, Q] = [P, U]$.

Since G is a counterexample, it follows that $C_P(Q)$ is not cyclic. But $|\Omega_1(C_P(Q))| = 2$ and therefore $C_P(Q)$ is a generalized quaternion group. It follows that $C_P(Q) \cong Q_8$ by Lemma 4.2. In this case $1 \neq Z := Z(C_P(Q)) \trianglelefteq G$ and G/Z satisfies the hypotheses of our lemma, but not the conclusion. Thus G is not a minimal counterexample, contrary to our choice. □

(2) Q acts irreducibly on $P = [P, Q]$.

Proof. Assume for a contradiction that $n = 2$. By hypothesis Q is cyclic, and then we may suppose that $C_Q(M_1) \leq C_Q(M_2) =: Q_0$. If $C_Q(M_1) = C_Q(M_2)$, then Lemma 2.8 of [9] implies that Q induces power automorphisms on P. Thus G is not a counterexample, which is a contradiction.

Therefore $C_Q(M_1) \lneq Q_0$ and $1 \neq [M_1, Q_0] \leq [P, Q_0]$. The minimal choice of G yields that Q_0 acts irreducibly or by inducing power automorphisms on $[P, Q_0]$ and that $C_P(Q_0)$ is a cyclic 2-group. Now we notice that $M_2 \leq C_P(Q_0)$, but $M_2 \not\leq 1 = C_P(Q)$, whence we deduce a contradiction from 2.2.5 of [5]. □

Since $G = PQ$ is a counterexample to the lemma, Q has a proper subgroup U that does not act irreducibly on $[P, Q] = P$ and it also does not induce power automorphisms on $[P, Q]$. In particular it does not act trivially. Since PU is a proper subgroup of our minimal counterexample G, it follows that $1 \neq C_P(U) \neq P$. But $C_P(U)$ is Q-invariant, because Q is abelian. This is a final contradiction with regard to (2). □

Lemma 4.5. *Suppose that q is odd and that G is an L_9-free $\{2, q\}$-group. Suppose further that P is a normal Sylow 2-subgroup of G such that $[P, Q]$ is hamiltonian and let $Q \in \mathrm{Syl}_q(G)$. Then one of the following holds:*

(a) G is nilpotent or

(b) $[P, Q] \cong Q_8$ and there exists a group I of order at most 2 such that $P = [P, Q] \times I$ and Q is a cyclic 3-group. Moreover $[P, Q]Q/Z([P, Q]Q) \cong \mathrm{Alt}_4$.

Proof. We suppose that G is not nilpotent.

Then Q is not normal in G and Lemma 4.3 implies that Q is cyclic. Furthermore, P is L_9-free and hence it is modular by Lemma 4.2. Since $[P, Q]$ is hamiltonian, Theorem 2.3.1 of [7] provides subgroups $P_0, I \leq P$ such that $P_0 \cong Q_8$, I is elementary abelian and $P = P_0 \times I$.

We recall that the automorphism group of Q_8 is isomorphic to Sym_4. Thus, if $Q_8 \cong P_1 \leq P$ is Q-invariant, but not centralized by Q, then $Q_8 \cong P_1 \leq [P, Q]$ and $1 \neq |Q/C_Q(P_1)| = 3$. It follows that Q is a cyclic 3-group and $[P_1, Q]Q/Z([P_1, Q]Q) \cong \mathrm{Alt}_4$.

We conclude that our assertion holds if $I = 1$. Now suppose that $I \neq 1$. We recall that $P \trianglelefteq G$ and therefore $\Phi(P_0) = \Phi(P) \trianglelefteq G$. Since $|\Phi(P_0)| = 2$, it follows that $\Phi(P_0) \leq Z(G)$, and then $\bar{G} := G/\Phi(P_0)$ is not nilpotent because G is not. Furthermore, $\bar{G}$ is L_9-free, $\bar{I} \cong I \neq 1$, $\bar{Q} \cong Q$ and $\bar{P}_0$ is elementary abelian of order 4. In particular $\bar{P}$ is elementary abelian, hence it is a non-hamiltonian 2-group of order at least 8. Lemma 4.4 states that $\bar{Q}$ acts irreducibly on $[\bar{P}, \bar{Q}] \neq 1$ or induces power automorphisms on it. The second case is not possible by Lemma 1.3.

Hence $Q \cong \bar{Q}$ acts irreducibly on $[\bar{P}, \bar{Q}] = \overline{[P, Q]}$ and, by Lemma 4.4, we see that $C_{\bar{P}}(\bar{Q})$ is a cyclic 2-group. Since $I \neq 1$ and $\Omega_1(P) = \Phi(P_0) \times I$, we have that $1 \neq \bar{I} = \overline{\Omega_1(P)}$ is $\bar{Q}$-invariant and $\bar{P}_0$ is a non-cyclic complement of $\bar{I}$ in $\bar{P}$. This implies that $\bar{I} = C_{\bar{P}}(\bar{Q})$ is cyclic and elementary abelian at the same time. Thus $I \cong \bar{I}$ has order 2 and with Lemma 1.1 we deduce that $C_P(Q)\Phi(P_0) = I\Phi(P_0) = \Omega_1(P)$ is elementary abelian of order 4. Then we deduce that $C_P(Q) = \Omega_1(P)$ and then $[P,Q] \cap C_P(Q) \neq 1$. Moreover, since $[P,Q]C_P(Q) = P = P_1 \times I \leq P_1C_Q(P)$, it follows that $|[P,Q]| \leq |P_1| \leq 8$. In conclusion, $[P,Q]$ is a subgroup of order at most 8 admitting an automorphism of odd order that centralizes $\Omega_1([P,Q])$. It follows that $[P,Q] \cong Q_8$ and then, together with the fact that $I \leq Z(G)$, our assertions follow. □

Definition 4.6. Suppose that Q is a cyclic q-group that acts coprimely on the p-group P. We say that the action of Q on P **avoids** L_9 (and we indicate more technical details by writing **"of type (·)"**) if and only if one of the following is true:

- (std) Every subgroup of Q acts irreducibly or by inducing (possibly trivial) power automorphisms on the elementary abelian group $[P,Q] = P$.
- (cent) Every subgroup of Q acts irreducibly or trivially on the elementary abelian group $[P,Q]$, P is abelian and $C_P(Q)$ is a nontrivial cyclic 2-group.
- (hamil) $[P,Q] \cong Q_8$, $P = [P,Q] \times I$, where I is a group of order at most 2, and Q is a cyclic 3-group such that $[P,Q]Q/Z([P,Q]Q) \cong \mathrm{Alt}_4$.

Lemma 4.7. *Suppose that p is an odd prime and that G is an L_9-free $\{2,p\}$-group. Let further P be a normal Sylow p-subgroup of G and let $Q \in \mathrm{Syl}_2(G)$ be isomorphic to Q_8 and such that $1 \neq [P,Q]$ is elementary abelian.*

Then $p \equiv 3 \mod 4$, $|P| = p^2$ and Q acts faithfully on P.

Proof. We set $Z := \Omega_1(Q)$. If $Z \trianglelefteq G$, then $Z \leq Z(G)$ and we consider $\bar{G} := G/Z$. Then $\bar{G}$ is an L_9-free $\{2,p\}$-group, $\bar{P}$ is a normal Sylow p-subgroup of $\bar{G}$ and $\bar{Q} \in \mathrm{Syl}_2(\bar{G})$. Since $\bar{Q}$ is neither cyclic nor isomorphic to Q_8, Lemma 4.3 is applicable and we see that $\bar{G}$ is nilpotent. But then G is also nilpotent, contrary to our hypothesis that $[P,Q] \neq 1$. Thus $\Omega_1(Q)$ is not normal in G and Q acts faithfully on P. Now, for all $y \in Q$ of order 4, we apply Lemma 4.4 on $[P,Q]\langle y\rangle$ to deduce that $\langle y\rangle$ either induces power automorphisms on $[P,Q]$ or acts irreducibly on it. Theorem 1.5.1 of [7] states that $\mathrm{Pot}_G(P)$ is abelian, but $Q \cong Q_8$ is not, which means that we may choose y such that y does not induce power automorphisms on P. In particular P is not cyclic of prime order. Moreover p is odd and therefore 4 divides $(p+1)(p-1) = p^2 - 1$, and Satz II 3.10 of [3] yields that $|P| \leq p^2$. It follows that $|P| = p^2$. More precisely, as $|P| \neq p$, the result implies that $p \equiv 3 \pmod 4$, and then the proof is complete. □

Definition 4.8. Suppose that $Q \cong Q_8$ acts coprimely on the p-group P. We say that the action of Q on P **avoids** L_9 if and only if $p \equiv 3 \mod 4$, $|P| = p^2$ and Q acts faithfully on P.

Lemma 4.9. *Suppose $Q \cong Q_8$ and that P is a p-group on which Q acts avoiding L_9. Then P is elementary abelian, $\Omega_1(Q)$ inverts P, and every subgroup of Q of order at least 4 acts irreducibly on P.*

Proof. Since a cyclic group of order p^2 has an abelian automorphism group by 2.2.3 of [5], it follows that P is elementary abelian. If $1 \neq R$ is a cyclic subgroup of P, then $|\mathrm{Aut}(R)| = p-1$ and therefore R does not admit an automorphism of order 4. Additionally, $[P,\Omega_1(Q)]$ is Q-invariant and, since Q acts faithfully on P, we see that $[P,\Omega_1(Q)] \neq 1$. Furthermore, Lemma 1.1 gives that $[P,\Omega_1(Q)] \cap C_P(\Omega_1(Q)) = 1$ because P is abelian. Moreover, Q has rank 1, and then it follows that Q acts faithfully on $[P,\Omega_1(Q)]$. This implies that $|[P,\Omega_1(Q)]| \neq p$ and consequently $[P,\Omega_1(Q)] = P$. Hence 8.1.8 of [5] states that $\Omega_1(Q)$ inverts P. In particular $\Omega_1(Q)$ inverts every cyclic subgroup R of P.

In addition, these arguments show that every subgroup U of order 4 of Q does not normalize any nontrivial proper subgroup of the elementary abelian group P. This means that U acts irreducibly on P. □

Corollary 4.10. *Suppose that $p \neq q$ and that G is an L_9-free $\{p,q\}$-group such that $P \in \mathrm{Syl}_p(G)$ is normal in G and $Q \in \mathrm{Syl}_q(G)$.*

Then either G is nilpotent and P and Q are modular or Q is a batten and it acts on P avoiding L_9.

In particular, if G is not nilpotent, then Q is isomorphic to Q_8 or cyclic and $[P,Q]$ is elementary abelian or isomorphic to Q_8, where in the second case $q = 3$.

Proof. By hypothesis G is L_9-free, hence P and Q are, too. Then Lemma 4.2 implies that P and Q are modular.

Suppose that G is not nilpotent. Then Lemma 4.3 applies: Q is cyclic or isomorphic to Q_8 and hence it is a batten. Moreover Lemma 2.2 of [9] states that $[P,Q]$ is a hamiltonian 2-group or elementary abelian. In the first case Lemma 4.5 gives the assertion. In the second case our statement follows from Lemmas 4.4 and 4.7. □

Lemma 4.11. *Let Q be a nilpotent batten that acts on the p-group P avoiding L_9, and suppose that U is a subgroup of Q.*

Then U induces power automorphisms on P or it acts irreducibly on $[P,Q]/\Phi([P,Q])$.

Proof. First suppose that $Q \cong Q_8$. Then Lemma 4.9 implies that every subgroup of order at least 4 of Q, and in particular Q itself, acts irreducibly on $P = [P,Q]/\Phi([P,Q])$. Moreover, the involution of Q inverts P by Lemma 4.9, and then the statement holds.

Next we suppose that Q is cyclic. Then Definition 4.6 gives the assertion unless the action of Q on P avoids L_9 of type (hamil). In this case every proper subgroup of Q centralizes P, while Q acts irreducibly on $[P,Q]/Z([P,Q]) = [P,Q]/\Phi([P,Q])$. □

Next we investigate groups of order divisible by more than two primes. This needs some preparation.

Lemma 4.12. *Suppose that P and R are distinct Sylow subgroups of G, that $Q \in \mathrm{Syl}_q(G)$ is cyclic and that it normalizes P and R, but does not centralize them. Suppose further that R normalizes every Q-invariant subgroup of P. If $C_Q(P) = C_Q(R)$, then G is not L_9-free.*

Proof. We suppose that $C_Q(P) = C_Q(R) =: E$. Then E is a normal subgroup of G because Q is abelian.

We claim that G/E is not L_9-free, and for this we may suppose that $E = 1$. Then $Q \neq 1$ acts faithfully on P and R. Now we need a technical step before we move on:

There are a Q-invariant subgroup D of P and elements $x, y \in D$ such that $C_Q(x) = C_Q(y) = C_Q(xy^{-1}) = 1$ and $D = [x,Q] = [y,Q] = [xy^{-1},Q]$. $(*)$

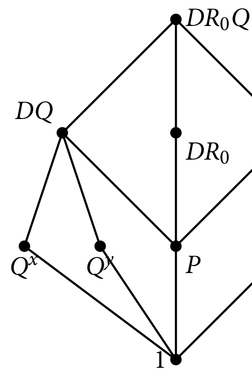
Since Q is cyclic, there is some $u \in Q$ such that $\langle u \rangle = Q$. Let $x_0 \in [P,Q]$ be such that $\Omega_1(Q)$ does not centralize x_0.

Then $[x_0,Q] = [x_0,\langle u\rangle,Q] = [\langle [x_0,u]\rangle,Q] = [[x_0,u],Q]$ by Lemma 1.1, and for all integers n we have the following: $(x_0^{-1}x_0^u)^{u^n} = 1$ iff $x_0^{u^n} = x_0^{u^{n+1}}$ iff $x_0^u = 1$. It follows that $[x_0,u] \in [[x_0,u],Q] = [x,Q]$ and $C_Q([x_0,u]) = C_Q(x_0) = 1$.

Now we set $x := [x_0,u]$, $y := x^u$ and $D := [x,Q]$. Then D is Q-invariant and we have that $x, y \in D$, $C_Q(x) = C_Q(y) = 1$ and $D = [x,Q] = [y,Q]$. If we set $z := xy^{-1}$, then $z = [x^{-1},u]$ and we can use the information from the end of the previous paragraph:
$[z,Q] = [x^{-1},Q] = D$ and $C_Q(z) = C_Q(x^{-1}) = C_Q(x) = 1$. This concludes the proof of $(*)$.

We use $(*)$ and its notation and, similarly, we find a Q-invariant subgroup R_0 of R and an element $h \in R_0$ such that $C_Q(h) = 1$ and $R_0 = [h,Q]$.

We set $S := Q^x$, $T = Q^y$, $A := DQ$, $B := DR_0$, $U := Q^h$, $C := DQ^h$ and $F := DR_0Q$, and we claim that $\{A, B, C, D, E, F, S, T, U\}$ is isomorphic to L_9. The properties L9 (i) and L9 (iii) of Lemma 3.5 follow from the choice of D, since h and Q normalize D.



For L9 (ii) we first note that $D \cap Q^x = D \cap Q^y = 1 = E$ and $\langle D, Q^x\rangle = \langle D, Q^y\rangle = DQ$, since $x \in D$ and hence $y = x^u \in D$. Next, Lemma 1.4 (b) yields that $T \cap S = Q^x \cap Q^y \leq C_Q(xy^{-1})^y = 1$. Part (d) of the same lemma shows that $\langle T, S\rangle = \langle Q^x, Q^y\rangle = \langle [xy^{-1}, Q]^{Q^{xy^{-1}}}\rangle Q^{xy^{-1}} = DQ^{xy^{-1}} = DQ = A$, as $xy^{-1} \in D$.

For all $z \in \{x, y\}$ we calculate that $\langle Q^z, Q^h\rangle = \langle [zh^{-1}, Q]^{Q^{zh^{-1}}}\rangle Q^{zh^{-1}} = \langle([z, Q]^{h^{-1}}[h^{-1}, Q])^{Q^{zh^{-1}}}\rangle Q^{zh^{-1}} = \langle D, R_0\rangle Q^{zh^{-1}} = F$ by Lemma 1.4 (d). Thus L9 (iv) of Lemma 3.5 is true.

We moreover have that $\langle A, B\rangle = \langle D, Q, R_0\rangle = F = \langle D, Q^h, R_0\rangle$ and $A \cap B = DQ \cap DR_0 = D(Q \cap DR_0) = D = D(Q^h \cap DR_0) = C \cap B$. Finally $A \cap C = DQ \cap DQ^h = D(Q \cap DQ^h) \leq DC_Q(h) = D$ by Lemma 1.4 (b).

Using Lemma 3.5 we conclude that G/E is not L_9-free, and hence G is not L_9-free. □

Corollary 4.13. *Suppose that p, q and r are pairwise distinct primes and that G is a directly indecomposable L_9-free $\{p, q, r\}$-group. Suppose further that $P \in \mathrm{Syl}_p(G)$ and $R \in \mathrm{Syl}_r(G)$ are normal in G and let $Q \in \mathrm{Syl}_q(G)$.*

Then Q is cyclic and $C_Q(P) \neq C_Q(R)$.

Proof. Since G is directly indecomposable, we see that Q acts non-trivially on both P and R. Moreover, PQ and RQ are L_9-free by hypothesis, and then we conclude that Q is cyclic or isomorphic to Q_8.

In the first case, our assertion follows from Lemma 4.12, and in the second case, we choose a maximal subgroup Q_1 of Q. Then Q_1 acts irreducibly on P and R by Lemma 4.9, and the same Lemma shows that $\Phi(Q)$ inverts P and R. Thus Q_1 acts on P and on R avoiding L_9, respectively, and it acts faithfully. This contradicts Lemma 4.12. □

We explain another example where a subgroup lattice contains L_9.

Lemma 4.14. *Suppose that p, q and r are pairwise distinct primes and that G is a $\{p, q, r\}$-group. Suppose further that $P \in \mathrm{Syl}_p(G)$ is normal in G and that $Q \in \mathrm{Syl}_q(G)$ and $R \in \mathrm{Syl}_r(G)$ are cyclic groups such that $R \trianglelefteq RQ$. Suppose that $|R| = r$ and $C_Q(R) = 1$.*

If R acts irreducibly on P, but non-trivially, and if $1 \neq [P, Q]$ is elementary abelian, then G is not L_9-free.

Proof. We first remark that G is soluble, because $P \trianglelefteq PR \trianglelefteq PRQ = G$. We will construct the lattice L_9 in $L(G)$ using Lemma 3.5. For this we set $E := 1$ and $D := P$. Then $D \neq E$ and we see that L9 (i) is true.

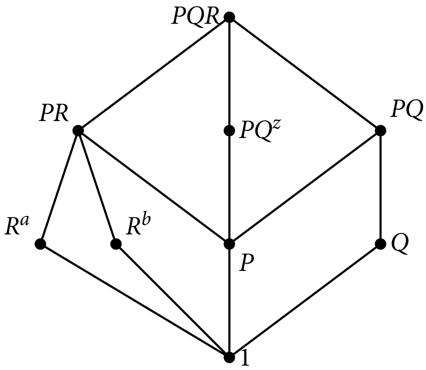
Next, we recall that $1 \neq [P, Q]$ by hypothesis. Assume that $|[P, Q]| = 2$. Then Q, which normalizes $[P, Q]$, must centralize it, and then Lemma 1.1 gives a contradiction.

Therefore $|[P, Q]| \gneq 2$. As a consequence, we find $a, b \in [P, Q]^\#$ such that $a \neq b$, and then we set $S := R^a$ and $T := R^b$. Now $D \cap S = 1 = E = D \cap T$ and $\langle D, T\rangle = PR^b = PR = PR^a = \langle D, S\rangle$. We set $A := PR$. In addition, since R acts irreducibly, but non-trivially on P, it follows that $C_R(ba^{-1}) \lneq R$. Then the fact that $|R| = r$ gives that $C_R(ba^{-1}) = 1 = E$.

Lemma 1.4 (b) shows that $S \cap T = (R \cap R^{ba^{-1}})^a \leq C_R(ba^{-1})^a = E$, and now we recall that $[ba^{-1}, R] \neq 1$. Moreover, R acts irreducibly on P, and then Part (e) of the same lemma yields the following:

$\langle T, S\rangle = \langle R, R^{ba^{-1}}\rangle^a = (\langle [ba^{-1}, R]^{R^{ba^{-1}}}\rangle R^{ba^{-1}})^a = PR = A$. We conclude that L9 (ii) holds.

For L9 (iii) we set $U := Q$ and $C := \langle D, Q\rangle = PQ$. Then we note that $U \cap D = Q \cap P = 1 = E$.



Assume for a contradiction that $X := \langle R^c, Q\rangle$ has odd order for some $c \in \{a, b\}$.

In both cases X is a p'-Hall subgroup of the soluble group $G = PRQ$ and therefore $R^c = O_r(X)$. It follows that Q normalizes R^c and then that $Q^{c^{-1}}$ and Q normalize R. Since $N_P(R)$ is R-invariant and R acts irreducibly, but non-trivially on P, we conclude that $N_G(R) = RQ$. Thus Sylow's theorem provides some $y \in R$ such that $Q^{c^{-1}} = Q^y$. Now $[yc, Q] \le PR \cap Q = 1$. In addition $|R| = r$ and $[R, Q] \neq 1$ by hypothesis. Together this gives that $yc \in C_G(Q) \cap PR \le PQ \cap PR = P(Q \cap PR) = P$, by Dedekind's modular law. Altogether we have that $yc \in C_P(Q)$. We recall that $c \in \{a, b\} \subseteq P$, and then $y = ycc^{-1} \in P$. But we chose $y \in R$ and now $y \in R \cap P = 1$, whence $Q^{c^{-1}} = Q$. In other words, $c \in N_P(Q)$, and this means that $[Q, c] \le Q \cap P = 1$ and $c \in C_P(Q)$. We recall that $c \in [P, Q]^\#$ and that $[P, Q]$ is elementary abelian by hypothesis. Then Lemma 1.1 implies that $[P, Q] = [P, Q, Q] \times C_{[P,Q]}(Q)$, and this contradicts the fact that $c \in C_P(Q) \cap [P, Q]$.

It follows that X has even order and since R^c acts irreducibly on P, we conclude that $P \le X$. This implies that $\langle S, U\rangle = \langle R^a, Q\rangle = PRQ = G = \langle R^b, Q\rangle = \langle T, Q\rangle$ and then L9 (iv) holds for $F := G$.

We finally set $B := PQ^z$ for some $z \in R^\#$. Then $R = \langle z\rangle$ and Lemma 1.4 (b) and (c), together with our hypothesis, show that $P \le PQ \cap PQ^z = P(Q \cap PQ^z) \le PC_Q(z) = PC_Q(R) = P$. Thus we have that $B \cap C = P = D$. We further see that $A \cap C = PR \cap PQ = P(R \cap PQ) = P = D$ and $A \cap B = PR \cap PQ^z = P(R \cap PQ^z) = P = D$. Since $\langle A, B\rangle = \langle PR, PQ^z\rangle = PQR = G$ and $\langle B, C\rangle = \langle PQ, PQ^z\rangle = P\langle [Q, z]^{Q^z}\rangle Q^z = P[Q, R]Q = PRQ = G$ by Lemma 1.4 (d), we finally obtain L9 (v).

Altogether Lemma 3.5 gives the assertion. □

Proposition 4.15. *Suppose that p, q, and r are pairwise distinct primes and that G is a non-nilpotent L_9-free $\{p, q, r\}$-group with normal Sylow p-subgroup P. Suppose further that $R \in \mathrm{Syl}_r(G)$ and $Q \in \mathrm{Syl}_q(G)$ are not normal in G, that $R \trianglelefteq RQ$ and $[R, Q] \neq 1$.*

Then RQ is a batten, P is elementary abelian of order p^r, R and Q act irreducibly on P and $\Phi(Q)$ induces non-trivial power automorphisms on P.

Proof. We proceed in a series of steps.

(1) The groups PQ and PR are not nilpotent, Q is cyclic and $R \cong Q_8$ or $|R| = r$. In addition $R = [R, Q]$ and $[R, C_Q(P)] \le C_R(P)$.

Proof. By hypothesis R is not normal in G, but Q normalizes R. Hence $P \not\le N_G(R)$ and in particular PR is not nilpotent. But PR is L_9-free, because G is. Moreover, RQ is non-nilpotent and L_9-free, again by hypothesis. Then Corollary 4.10 implies that R and Q are battens, that R acts on P avoiding L_9 and that Q acts on R avoiding L_9. More specifically, R and Q are cyclic or isomorphic to Q_8, and $[P, R]$ as well as $[R, Q]$ are elementary abelian or isomorphic to Q_8.

It follows that $R \cong Q_8$ or that R is cyclic of order r. In both cases Lemma 1.1 yields $R = [R, Q]$ and the avoiding L_9 action of Q on R gives that Q is cyclic.

In addition $[P, C_Q(P), R] = 1$, $[P, R, C_Q(P)] \le [P, C_Q(P)] = 1$ and then the Three Subgroups Lemma (see for example 1.5.6 of [5]) implies that $1 = [R, C_Q(P), P]$. Thus $[R, C_Q(P)] \le C_R(P)$.

If it was true that $[P, Q] = 1$, then $R = [R, Q] = [R, C_Q(P)]$ would centralize P. But this is a contradiction. □

(2) $C_R(P) = 1$ and $C_Q(P) \leq Z(G)$.

Proof. Since $[P, R] \neq 1$ by (1), we can apply Corollary 4.10 to PR, and this shows that R acts on P avoiding L_9. Otherwise (1) implies that $R \cong Q_8$ and then Definition 4.8 gives that R acts faithfully on P. If $|R| = r$, then R acts faithfully of P because $[P, R] \neq 1$. In both cases we see that $C_R(P) = 1$, and then the last statement of (1) implies that $[R, C_Q(P)] \leq C_R(P) = 1$. Then $C_Q(P)$ centralizes P and R, and Q is cyclic by (1), and therefore it follows that $C_Q(P) \leq Z(G)$. □

(3) $Z(G) = 1$ or $p \neq 2$.

Proof. We suppose that $p = 2$. Let $- : G \to G/Z(G)$ be the natural homomorphism. We show that $\bar{G}$ satisfies the hypotheses of our lemma.

From (1) we see that none of the groups P, Q or R is contained in $Z(G)$. We even have that $R \cap Z(G) = 1$ by (1). In particular $p, q, r \in \pi(\bar{G})$ and $\bar{G}$ is L_9-free. We see that $\bar{P}$ is a normal Sylow p-subgroup of G and that $\bar{Q} \in \mathrm{Syl}_q(G)$ and $\bar{R} \in \mathrm{Syl}_r(G)$ are such that $\bar{R} \trianglelefteq \bar{R}\bar{Q} \leq G$. Let $X \in \{R, Q\}$. If $\bar{X} \trianglelefteq \bar{G}$, then $XZ(G) \trianglelefteq G$ and X is a characteristic Sylow subgroup of $XZ(G)$ and hence normal in G. This is a contradiction.

We deduce that all hypotheses of the lemma hold for $\bar{G}$ and that $[\bar{R}, \bar{Q}] = \overline{[R, Q]} = \bar{R} \neq 1$. Therefore, if $Z(G) \neq 1$, then the minimal choice of G implies that $\Phi(\bar{Q})$ induces non-trivial power automorphisms on the elementary abelian group $\bar{P}$. Then Lemma 1.3 yields that $p \neq 2$. □

(4) $C_P(R) = 1$, and the groups $[P, Q]$ and $P = [P, R]$ are elementary abelian.

Proof. As PR and PQ are not nilpotent by (1), Corollary 4.10 implies that X acts on P avoiding L_9 and that $[P, X]$ is elementary abelian or isomorphic to Q_8 for both $X \in \{Q, R\}$.

Assume for a contradiction that $[P, X]$ is isomorphic to Q_8 for some $X \in \{Q, R\}$. Then X acts on P of type (hamil). Hence we obtain a group I of order 1 or 2 such that $P \cong Q_8 \times I$. It follows that $\mathrm{Aut}(P)$ is a $\{2, 3\}$-group. But this is impossible because Q and R both act coprimely and non-trivially on P by (1), and $p = 2$, q and r are pairwise distinct.

We conclude that $[P, Q]$ and $[P, R]$ are elementary abelian, and then it follows that P is abelian, by Lemma 4.4, applied to PR.

Assume for a further contradiction that $C_P(R) \neq 1$. As R avoids L_9 in its action on P, it follows from Lemma 4.9 that R is not isomorphic to Q_8. Now (1) yields that R is cyclic and we may apply Lemma 4.4 to PR, because $[P, R]$ is elementary abelian. It follows that $C_P(R)$ is a cyclic 2-group and thus q and r are odd.

Furthermore $C_P(R)$ is normalized by Q, because Q normalizes P and R. Since q is odd, it follows that $C_P(R)$ is centralized by Q. We recall that P is abelian, and then (3) yields that $C_P(R) \leq Z(G) = 1$. This is a contradiction.

Altogether $C_P(R) = 1$ and Lemma 1.1 gives that $P = [P, R]$ is elementary abelian. □

(5) $C_P(Q) = 1$ and Q acts on P avoiding L_9 of type (std).

Proof. Since PQ is not nilpotent and Q is cyclic by (1), Corollary 4.10 implies that the action of Q on P avoids L_9. But P is elementary abelian by (4), and therefore the action is not of type (hamil).

We assume for a contradiction that $C_P(Q) \neq 1$. Then it follows that PQ is not of type (std). We consequently have type (cent) and we see that $C_P(Q)$ is a cyclic 2-group. In particular $p = 2$, and thus q and r are odd. Then $|R| = r$ by (1).

Next we claim that G satisfies the hypotheses of Lemma 4.14.

First, q and r are pairwise distinct odd primes and G is a finite $\{2, q, r\}$-group. From above, (1) and our assumption we see that $P \in \mathrm{Syl}_2(G)$ is normal in G and that $Q \in \mathrm{Syl}_q(G)$ and $R \in \mathrm{Syl}_r(G)$ are cyclic groups such that $R \trianglelefteq RQ$. We have shown that R has order r.

We recall that $C_P(Q)$ is a cyclic 2-group (first paragraph). As r is odd and $C_P(R) = 1$ by (4), we see that $C_P(Q)$ is not R-invariant. But $C_P(C_Q(R))$ is R-invariant, and now the irreducible action of R on P and the fact that $1 \neq C_P(Q) \leq C_P(C_Q(R))$ show that $P = C_P(C_Q(R))$. Then (2) and (4) imply that $C_Q(R) \leq C_Q(P) \leq Z(G) = 1$.

By (4) P is an elementary abelian 2-group, and $P = [P, R] \neq 1$ by (1) and (4). In particular R does not induce power automorphism on P by Lemma 1.3. As $C_P(R) = 1$ by (4), we deduce that R acts irreducibly on P (using Lemma 4.4). Finally, (1) and (4) yield that $1 \neq [P, Q]$ is elementary abelian.

All hypotheses of Lemma 4.14 are satisfied now, and we infer that G is not L_9-free. This is a contradiction.

Thus $C_P(Q) = 1$ and we deduce that the action of Q on P is not of type (cen). It remains that Q acts on P avoiding L_9 of type (std). □

(6) R and Q act irreducibly on P. If $X \leq Q$ induces power automorphisms on P, then X centralizes R.

Proof. We recall from (2) that $C_Q(P) \leq Z(G)$ and $C_R(P) = 1$. Consequently $C_{RQ}(P) = C_Q(P) \leq Z(RQ)$ and $RQ/C_Q(P)$ is isomorphic to a subgroup of $\mathrm{Out}(P)$. Since RQ is not nilpotent, we see that $RQ/C_Q(P)$ is not nilpotent.

The group P is an elementary abelian p-group by (4) and hence, if $X \leq RQ$ induces power automorphisms on it, then it follows that $XC_Q(P)/C_Q(P) \leq Z(RQ/C_Q(P))$, see page 177 of [3]. We denote this fact by (∗). Then we deduce that $[R, X] \leq C_Q(P) \leq Z(RQ)$ (see above) and therefore $[R, X] = [X, R] = [X, R, R] = 1$ by Lemma 1.1.

In addition, the fact (∗) shows that neither R nor Q induces power automorphisms on P. It follows from (5) and Definition 4.6 (std) that Q acts irreducibly on P. Furthermore (1), together with Corollary 4.10, yields that R acts on P avoiding L_9. Since $C_P(R) = 1$ by (4), this action has type (std) or (hamil). In the first case, the irreducible action follows from Definition 4.6 (std), and in the second case, it follows from Lemma 4.9. □

(7) $C_Q(R)$ induces nontrivial power automorphisms on P.

Proof. We set $Q_0 := C_Q(R)$ and we assume that Q_0 acts irreducibly on P. Then II 3.11 of [3] implies that $RQ = C_{RQ}(Q_0)$ is isomorphic to a subgroup of the multiplicative group of some field of order $|P|$, and it follows that RQ is cyclic. This is a contradiction.

Thus (6) and Lemma 4.11 show that Q_0 induces power automorphisms on P. Since R normalizes every Q-invariant subgroup of P by (7), we see from Lemma 4.12 and (2) that $C_Q(R) \gneqq C_Q(P)$. □

(8) $|P| = p^q$ and $\Phi(Q) = C_Q(R)$.

Proof. We recall that Q is cyclic, by (1). Moreover $[R, Q] \neq 1$, whence $C_Q(R) < Q$ and therefore we may choose $y \in Q$ such that $C_Q(R) < \langle y \rangle$.

By (5) and Lemma 4.11, it follows that every subgroup of $\langle y \rangle$ either acts irreducibly on P or induces power automorphism on it (in particular normalizing every subgroup of P). Then $P\langle y \rangle$ satisfies (b) of Lemma 3.1 in [8], which implies that it satisfies one of the possibilities 3.1 (i)–3.1 (iii). By (5) and the choice of y, we see that 3.1 (i) is not true. Further (7) provides some $x \in C_Q(R)$ that induces a power automorphism of order q on P. This implies that q divides $p-1$ and therefore $P\langle y \rangle$ satisfies (ii) of Lemma 3.1 (b) in [8]. It follows that $|P| = p^q$ and that, if k is the largest positive integer such that q^k divides $p - 1$, then y induces an automorphism of order q^{k+1} on P. We conclude that $q^{k+1} = |\langle y \rangle : C_{\langle y \rangle}(P)| = |\langle y \rangle : C_R(P)|$, because Q is cyclic. Finally, we deduce that $o(y)$ is uniquely determined, that $Q = \langle y \rangle$ and that$C_Q(P) = \Phi(Q)$. □

By (6), we see that R and Q act irreducibly on P, and (1) gives that Q is cyclic. Then (8) and (7) say that $\Phi(Q)$ induces nontrivial power automorphisms on P. In addition P is elementary abelian by (4), and it has order p^r by (8). If $|R| = r$, then R is a cyclic group of order r and RQ is a batten. In particular G satisfies the assertion of our lemma.

But G is a counterexample, and then it follows that $R \cong Q_8$ and $r = 2$. Then (4) yields that PR fulfills the hypothesis of Lemma 4.7, and consequently $p^r = p^2 = |P| = p^q$ by (8). This is our final contradiction, because $q \neq r$. □

Definition 4.16. Suppose that B is a non-nilpotent batten that acts coprimely on the p-group P. We say that the action of B on P **avoids** L_9 if and only if $[P, Z(B)] \neq 1$ and if one of the following occurs:

(Cy) $[P, \mathcal{B}(B)] = 1$ and Q acts on P avoiding L_9 for every Sylow subgroup Q of B different from $\mathcal{B}(B)$ or

(NN) P is elementary abelian of order $p^{|B:\mathcal{B}(B)Z(B)|}$ and the Sylow subgroups of B act irreducibly on P, while $Z(B)$ induces power automorphisms on P.

As in Definition 4.6, we specify the type of the L_9-avoiding action by writing that "**B acts on P avoiding L_9 of type ($\cdot$)**".

Lemma 4.17. *Let B be a batten that acts non-trivially and avoiding L_9 on the p-group P. Then the following hold:*

(a) If $C_P(B) \neq 1$, then $p = 2$.
(b) Either $P = [P, B] \times C_P(B)$, where $[P, B]$ is elementary abelian and $C_P(B)$ is cyclic, or $P = [P, B] \times I$, where I is a group of order at most 2 and $[P, B] \cong Q_8$.
(c) $C_P(B)$ is centralized by every automorphism of P of order coprime to p that leaves $C_P(B)$ invariant.

Proof. If $B \cong Q_8$, then Definition 4.8 and Lemma 4.9 imply that P is elementary abelian and that B acts irreducibly on it. We conclude that $P = [P, B]$ is elementary abelian and we deduce from Lemma 1.1 that $C_P(B) = 1$. Hence, in this case, all statements of our lemma hold.

Now suppose that B is not nilpotent and that it acts of type (NN). Then Definition 4.16 states that, once more, P is elementary abelian and B acts irreducibly on it. Again we see that $P = [P, B]$ is elementary abelian, and as before all statements hold.

Next we suppose B is not nilpotent and that it acts of type (Cy), or that B is cyclic. In the first case B has a cyclic Sylow subgroup Q that acts on P avoiding L_9 such that $C_P(Q) = C_P(B)$ and $[P, B] = [P, Q]$ by Definition 4.16. In the second case we set $Q := B$.

Then, in both cases, Q is a cyclic group that acts on P avoiding L_9 such that $C_P(Q) = C_P(B)$ and $[P, B] = [P, Q]$. If Q acts of type (std), then $P = [P, Q]$ is elementary abelian by Definition 4.6. Again we deduce the statements of our lemma.

Suppose that Q acts of type (cent). Then Definition 4.6 yields that $[P, Q] = [P, B]$ is elementary abelian, that P is abelian and that $C_P(Q) = C_P(B)$ is a cyclic 2-group. In particular $P = [P, B] \times C_P(B)$ by Lemma 1.1. It also follows that $C_P(B)$ is centralized by every automorphisms of P of odd order that leaves $C_P(B)$ invariant. These are the statements of our lemma.

Finally, suppose that Q acts of type (hamil). Then Definition 4.6 yields that $[P, Q] \cong Q_8$ and $P = [P, Q] \times I$, where I is a group of order at most 2. In particular statement (a) is true. Moreover, we deduce that $C_P(Q) \leq \Omega_1(P) = \Phi([P, Q]) \times I$, where $\Phi([P, Q])$ is cyclic of order 2. In particular $\Phi([P, Q])$ is centralized by B and by every automorphisms of P. We conclude that $C_P(Q)$ is elementary abelian of order at most 4 and that every automorphism of P centralizes a cyclic subgroup of order 2. This implies (b). □

5. Avoiding L_9

We now work toward a classification of arbitrary L_9-free groups, and therefore we need to understand in more detail the group structures that appear when "L_9 is avoided" in the sense of the previous section.

Definition 5.1. Suppose that K is a batten group that acts coprimely on the p-group P. We say that the action of K on P **avoids** L_9 if and only if $[P,K] \neq 1$ and every batten of K either centralizes P or avoids L_9 in its action on P.

Lemma 5.2. *Let K be a batten group that acts coprimely on the p-group P avoiding L_9. Suppose further that $L \leq K$ and $L_0 \trianglelefteq L$ such that $[P,L] \neq 1 = [P,L_0]$. Then L/L_0 acts on P avoiding L_9. In particular L/L_0 is a batten group.*

Proof. By induction we may suppose that K is a batten and that either $L_0 = 1$ and L is a maximal subgroup of K or that L_0 is a minimal normal subgroup of $K = L$. Thus either $|L_0|$ has order q or $|K : L| = q$. Since $L_0 \leq C_K(P)$, we first remark that L/L_0 induces automorphisms on P.

If $K \cong Q_8$, then K acts faithfully on P by Definition 4.8. Thus $L_0 \leq C_K(P) = 1$ and it follows that L is a cyclic group of order 4. Thus Lemma 4.9 yields that $\Omega_1(L)$ inverts P and that L acts irreducibly on the elementary abelian group $P = [P,L]$. Then we see that $L \cong L/L_0$ acts on P avoiding L_9 of type (std).

Next suppose that K is cyclic. Then L/L_0 is cyclic. If K acts of type (std) on P, then L and every subgroup of L act irreducibly or via inducing power automorphisms on the elementary abelian group $P = [P,K]$. Since $[P,L] \neq 1$ and power automorphisms are universal, by Lemma 1.5.4 of [7], it follows that $P = [P,L]$. Moreover, the action of L on P is equivalent to that of L/L_0, and then it follows that L/L_0 acts on P avoiding L_9 of type (std).

If K acts on P of type (cent), then L and all its subgroups act irreducibly or trivially on the elementary abelian group $[P,K]$. Again the fact that $[P,L] \neq 1$ implies that $P = [P,L]$, and then $C_P(L) = C_P(K)$ by Lemma 1.1. Since the action of L on P is equivalent to that of L/L_0, it follows that L/L_0 acts on P avoiding L_9 of type (cent).

We suppose now that K acts of type (hamil). Then K is a cyclic 3-group and $K/C_K(P)$ has order 3. It follows that $L = K$. But again, the action of $L/L_0 = K/L_0$ on P is equivalent to the action of K on P, which means that it has type (hamil).

We finally suppose that K is a non-nilpotent batten. Let R be a Sylow subgroup of K such that $K = \mathcal{B}(K) \cdot R$. Suppose first that L/L_0 is a q-group. Then our choice of L and L_0 implies that $L/L_0 \cong R$. If K acts on P of type (Cy) in this case, then it follows that $L/L_0 \cong R$ is cyclic and that it acts on P avoiding L_9, according to Definition 4.16. Otherwise, if K acts of type (NN), then $\mathcal{B}(K) \not\leq C_K(P)$ and then $L_0 = 1$. It follows that $L = R$ acts irreducibly on the elementary abelian group P, whence $P = [P,L] = [P,L/L_0]$. In addition $\Phi(L)$ and all of its subgroups induce power automorphism on P. Altogether the cyclic group $L/L_0 \cong L$ acts on P of type (std).

Now we suppose that L/L_0 does not have prime power order. Then $L_0 \leq \Phi(R) = Z(K)$. Now if L/L_0 is nilpotent, then $L \neq K$ and therefore $L_0 = 1$. It follows from Lemma 2.5 that $L = Z(K) \times \mathcal{B}(K)$. We have already proven that R acts on P avoiding L_9, and then $Z(K)$ also acts on P avoiding L_9. In addition $\mathcal{B}(K)$ either centralizes P or it acts irreducibly on the elementary abelian group $P = [P,\mathcal{B}(K)]$. Since $\mathcal{B}(K)$ has prime order, it follows that the cyclic group $\mathcal{B}(K)$ acts on P avoiding L_9 of type (std). Altogether $L/L_0 \cong L = Z(K) \times \mathcal{B}(K)$ acts on P avoiding L_9.

Finally, suppose that L/L_0 is not nilpotent. Then L is not nilpotent and hence Lemma 2.7 implies that $L = K$. We conclude that $L_0 \neq 1$. Since $[P,Z(K)] \neq 1$ by Definition 4.16, it follows that L_0 is a proper subgroup of $Z(K)$ and that $L/L_0 = K/L_0 \cong \mathcal{B}(K) \rtimes R/L_0$ is a non-nilpotent batten. If $[P,\mathcal{B}(K)] = 1$, then our investigation above imply that the cyclic group R/L_0 acts on P avoiding L_9, and then K/L_0 acts on P avoiding L_9. Otherwise $1 \neq [P,\mathcal{B}(K)]$ is elementary abelian of order $p^{|K:\mathcal{B}(K)Z(K)|} = p^{|K/L_0:\mathcal{B}(K)Z(K)/L_0|}$, moreover $\mathcal{B}(K) \cong \mathcal{B}(K)L_)/L_0$ and R/L_0 act irreducibly on P. At the same time $Z(K/L_0) = Z(K)/L_0$ induces power automorphisms on P. Altogether K/L_0 acts on P avoiding L_9 of type (NN). □

Lemma 5.3. *Let K be a batten group that acts coprimely on the p-group P avoiding L_9. Then the following assertions are true:*

(a) If $L \trianglelefteq K$, then $[P,K] = [P,L]$ or $[P,L] = 1$.

(b) $[P,K]$ *is elementary abelian or isomorphic to* Q_8*.*

Proof. Let $L \trianglelefteq K$ be such that $[P,L] \neq 1$. Then L is a batten group by Lemma 2.7 and therefore there is a batten B of L such that $[P,B] \neq 1$. Assume for a contradiction that $[P,B] \leq [P,L] \lneq [P,K] \leq P$. Then the fact that $P \neq [P,B]$ implies that $C_P(B) \neq 1$ by Lemma 1.1. In addition B avoids L_9 in its action on P, by Lemma 5.2. Since B is a batten of L, it is characteristic in L, and therefore $B \trianglelefteq K$. In particular $C_P(B)$ is K-invariant and hence it is centralized by K by Lemma 4.17 (c). This implies that $C_P(B) = C_P(K)$. Finally $[P,K] = [[P,B]C_P(B),K] = [[P,B]C_P(K),K] = [[P,B],K] \leq [P,B] \lneq [P,K]$, which is a contradiction. In particular (a) is true.

Together with Lemma 4.17 (b), the statement in (b) follows from (a). □

Lemma 5.4. *Let K be a batten group that acts on the p-group P avoiding L_9, and suppose that H is a subgroup of K. Then H centralizes P or $[P,H] = [P,K]$.*

Moreover, H induces power automorphisms on P or it acts irreducibly on $[P,K]/\Phi([P,K])$.

Proof. Let $H \leq K$. If H centralizes P, then it induces power automorphisms on P. We may suppose that $[P,H] \neq 1$. Then H has a q-subgroup Q such that $[P,Q] \neq 1$. Therefore, if $Q \trianglelefteq K$, then we have that $[P,K] = [P,Q]$ by Lemma 5.3 (a). Then the fact that $[P,Q] \leq [P,H] \leq [P,K]$ yields that $[P,H] = [P,K]$.

Assume for a contradiction that $[P,K] \neq [P,H]$. Then Lemma 2.8 implies that K has a non-nilpotent batten B such that $B = \mathcal{B}(B)Q$. We moreover deduce that $[P,Q] \lneq [P,K]$ and $[P,B] = [P,K]$ by Lemma 5.3 (a), because $B \trianglelefteq K$. Since the action of K on P avoids L_9, the action of B also does. If B acts of type (Cy), then we obtain the contradiction that $[P,Q] = [P,B]$. Thus B acts of type (NN) and in particular Q acts irreducibly on P. But this is impossible as well. It follows that $[P,H] = [P,K]$.

Assume for a further contradiction that $H \leq K$ neither induces power automorphisms on P nor does it act irreducibly on $[P,K]/\Phi([P,K]) = [P,H]/\Phi([P,H])$. Then there is a batten B of H that neither induces power automorphisms on P nor does it act irreducibly on $[P,H]/\Phi([P,H])$. Similarly to the arguments above, we deduce that $[P,K] = [P,H] = [P,B]$, and Lemma 5.2 gives that B avoids L_9 in its action on P. Therefore Lemma 4.11 yields that B is not nilpotent. From Definition 4.16 we further see that B does not act of type (NN), and thus B acts of type (Cy) on P. Consequently $[P,\mathcal{B}(B)] = 1$ and B has a cyclic Sylow subgroup Q such that $B = \mathcal{B}(B)Q$ and Q acts on P avoiding L_9. Again we have $[P,K] = [P,B] = [P,Q]$ and Lemma 4.11 gives that Q induces power automorphisms on P or acts irreducibly on $[P,Q]/\Phi([P,Q]) = [P,K]/\Phi([P,K])$. In the first case $B = C_B(P)Q$ induces power automorphism on P and in the second case B acts irreducibly on $[P,K]/\Phi([P,K])$. This is a contradiction. □

Lemma 5.5. *Let K be a batten group that acts on the p-group P avoiding L_9, and suppose that H is a subgroup of K that acts non-trivially on $R \leq P$.*

Then $C_H(R) = C_H(P)$, $C_P(H) = C_P(K)$ and $[P,H] = [P,K]$.

Proof. From Lemma 5.4 we see that $[P,H] = [P,K]$. In addition $C_P(K) \leq C_P(H) \leq C_P(Q)$ for every q-subgroup Q of H and every prime q. Let Q be a q-subgroup of H for some prime q such that $C_P(Q) \neq P$. Then Q is a batten by Lemma 2.7, and it acts on P avoiding L_9 by Lemma 5.2. If $Q \trianglelefteq K$, then K centralizes $C_P(Q)$ by Lemma 4.17(c). Thus $C_P(K) \leq C_P(H) \leq C_P(Q) \leq C_P(K)$, and this gives that $C_P(K) = C_P(H)$. If Q is not a normal subgroup of K, then Lemma 2.8 provides a non-nilpotent batten B of K such that $B = \mathcal{B}(B)Q$. Since the action of K on P avoids L_9, the action of B also does. If B acts of type (NN), then Q acts irreducibly on P and therefore $C_P(Q) = 1 \leq C_P(K)$. Again we deduce that $C_P(K) = C_P(H)$. If B acts of type (Cy), then $[P,\mathcal{B}(B)] = 1$ and hence $C_P(B) = C_P(Q)$. But now B is a normal subgroup of K, and then Lemma 4.17(c) gives that $C_P(B) = C_P(Q) \leq C_P(K)$. As above we deduce that $C_P(K) = C_P(H)$.

Finally, suppose that $R \leq P$ is H-invariant, but not centralized by H, and set $H_0 := C_H(R) \geq C_H(P)$. Assume for a contradiction that H_0 does not centralize P. Then we deduce, as above, that $C_P(K) = C_P(H_0) \geq R$. This is a contradiction, because H does not centralize R. □

Corollary 5.6. *Let K be a batten group that acts non-trivially and avoiding L_9 on the p-group P. Then the following hold:*

(a) If $C_P(K) \neq 1$, then $p = 2$.
(b) If $C_P(K) = 1$, then $P = [P, K]$ is elementary abelian.
(c) If K induces power automorphisms on P, then $P = [P, K]$ is elementary abelian of odd order. In particular $C_P(K) = 1$ in this case.

Proof. Let B be a batten of K that does not centralize P. Then Lemma 5.5 implies that $C_P(K) = C_P(B)$. Thus Part (a) and (b) of Lemma 4.17 yield the statements (a) and (b) of our lemma. For Part (c) we suppose that K induces power automorphisms on P. Then P is not an elementary abelian 2-group by Lemma 1.3. If $C_P(K) = 1$, then our assertion holds by (b). Otherwise $p = 2$ by (a), and then Lemma 1.3 implies that $[P, K]$ is neither elementary abelian nor isomorphic to Q_8, contradicting Part (b) of Lemma 5.3.

For the final comment we just use that p is odd and then apply (a). □

Lemma 5.7. *Let B be a batten that acts on the p-group P avoiding L_9. Let $R \leq P$ be B-invariant and $R_0 \leq C_P(B)$.*

Then B avoids L_9 in its action on R/R_0.

Proof. Since B centralizes R_0, the action of B on R/R_0 is well-defined.

We first suppose that $B \cong Q_8$. Then Lemma 4.9 yields that B acts irreducibly on P, and then it follows that $R = P$ and $R_0 = 1$. Thus our assertion is true in this case.

Next suppose that B is cyclic. If B acts of type (std) on P, then $R_0 \leq C_P(B) = 1$. If B acts irreducibly on P, then again $P = R$ and there is nothing left to prove. Otherwise B and all of its subgroups induce power automorphisms on P, and hence on $R = [R, B]$ as well. It follows that B also acts of type (std) on $R \cong R/R_0$.

Suppose now that B acts of type (cent). Then, since B does not centralize R and B acts irreducibly on $[P, B]$, it follows that $[P, B] \leq R$. Moreover P is abelian and then we use the fact that $R_0 \leq C_P(B)$. This gives that $R/R_0 = [R/R_0, B] \times C_{R/R_0}(B) \cong [R, B] \times C_R(B)/R_0$, where $C_R(B)/R_0$ is a cyclic 2-group. Since every subgroup of B that does not centralize P acts irreducibly on $[P, B] \cong [R/R_0, B]$ in this case, there are two possibilities for the action of B on $R \cong R/R_0$: If $R_0 \neq C_R(B)$, then B acts of type (cent), and otherwise it acts of type (std).

Suppose now that B acts of type (hamil). Then the cyclic 3-group B acts irreducibly on $[P, B]/\Phi([P, B])$, by Lemma 5.4, and we see again that $[P, B] \leq R$. It follows that $R \cong Q_8 \times I$, where I is a group of order at most 2, and then $R_0 \leq C_R(B) \leq \Phi([R, B]) \times I$. We remark that $[R/R_0, B]B/Z([R/R_0, B]B) \cong [R, B]B/Z([R, B]B) \cong [P, B]B/Z([P, B]B) \cong \mathrm{Alt}_4$.

If $R_0 \cap [P, B] = 1$, then $R/R_0 \cong Q_8 \times \tilde{J}$ for some group J of order $\frac{|I|}{|R_0|}$. Thus B acts on R/R_0 of type (hamil) in this case.

Otherwise we have that $R_0 \geq \Phi([P, B])$ and therefore R/R_0 is elementary abelian of order 4 or 8. Moreover B acts irreducibly on $[R/R_0, B]$, which is a group of order 4. In addition every proper subgroup of B centralizes R/R_0. Consequently, if $R/R_0 = [R/R_0, B]$, then B acts on R/R_0 of type (std) or of type (cent).

The final case is that B is not nilpotent, and we suppose that B acts of type (NN) on P. Then Definition 4.16 yields that B acts irreducibly on P. Hence there is nothing left to prove.

Suppose that B acts of type (Cy). Then we choose a Sylow subgroup Q of B such that $B = \mathcal{B}(B)Q$. Then $[R/R_0, \mathcal{B}(B)] = [R, \mathcal{B}(B)] = [P, \mathcal{B}(B)] = 1$ and Q acts on P avoiding L_9 in such a way that $\Phi(Q) = Z(B)$ does not centralize P. Then Lemma 5.5 yields that $\Phi(Q)$ does not centralize R. In particular, we have that $[R/R_0, Z(B)] \neq 1$. In addition Q acts on R/R_0 avoiding L_9, by our arguments above. Altogether B acts on R/R_0 avoiding L_9 of type (Cy) in this final case. □

6. The first implication

We now investigate the general case.

Proposition 6.1. *Let G be a finite L_9-free group. Then $G = NK$, where N is a nilpotent normal Hall-subgroup of G with modular Sylow subgroups and K is a batten group. Moreover, for all $p \in \pi(N)$, every batten of K acts on $O_p(N)$ avoiding L_9 or it centralizes $O_p(N)$.*

Proof. We first remark that G is L_{10}-free, whence Theorem A of [9] implies that G is soluble. Furthermore, Corollary C of [9] provides normal Hall-subgroups N and M of G such that $N \leq M$ and such that N is nilpotent, M/N is a 2-group and G/M is metacyclic. We choose N as large as possible with these constraints. From Lemma 4.1 we see that $N \neq 1$. We also have that every Sylow subgroup of N is L_9-free, and hence it is modular by Lemma 4.2. In addition the Schur-Zassenhaus Theorem (see for example 3.3.1. of [5]) provides a complement K of N in G.

(1) If RQ is a non-nilpotent Hall $\{r, q\}$-subgroup of K, where R is a normal Sylow r-subgroup of RQ and $Q \in \mathrm{Syl}_q(RQ)$, then RQ is a batten. For all $p \in \pi(N)$, the group RQ centralizes $O_p(N)$ or acts on it avoiding L_9.

Proof. If there is some $p \in \pi(N)$ such that $[O_p(N), R] \neq 1$, then we set $P := O_p(N)$. Otherwise the maximal choice of N implies that R is not a normal subgroup of K. Then, using the solubility of G, we find a prime $s \in \pi(K) \setminus \{r\}$ and a normal s-subgroup T of K such that $[T, R] \neq 1$ (see 5.2.2 of [5]). Then $s \neq q$ because $1 \neq [R, Q] \leq R$ and $1 \neq [T, R] \leq T$. In this case we set $P := T$.

In both cases p, r and q are pairwise different primes and PRQ is a non-nilpotent $\{p, r, q\}$-subgroup that satisfies the hypothesis of Proposition 4.15. For this we note that $P \trianglelefteq PRQ$, $P \not\leq N_G(R)$ and $R \not\leq N_G(Q)$. Since $[R, Q] \neq 1$, the assertion in (1) follows. □

(2) Every Sylow subgroup S of K is a batten, and for all $p \in \pi(N)$ it is true that S centralizes $O_p(N)$ or acts on $O_p(N)$ avoiding L_9.

Proof. Let S be a Sylow subgroup of K. If S centralizes N, then the choice of N provides some Sylow subgroup R of K such that RS is not nilpotent. Then (1) implies that RS is a batten, then that S is cyclic and hence that S is a batten.

Let $p \in \pi(N/C_N(S))$. Then $O_p(N)S$ is an L_9-free $\{p, q\}$-group for some prime q. Hence Corollary 4.10 implies the assertion. □

(3) K is a batten group.

Proof. Let $1 \neq B \leq K$ be such that there is some $K_1 \leq K$ such that $K = K_1 \times B$, where $(|K_1|, |B|) = 1$ and B is not a direct product of nontrivial subgroups of coprime order. In particular B is a Hall subgroup of K. If B is nilpotent, then B is a Sylow q-subgroup of K for some prime q. In this case (2) implies that B is a batten.

Assume for a contradiction that B is not a batten of K. Then B is not nilpotent and therefore (1) yields that $|B|$ is divisible by at least three different primes. Since $B \leq G$ is L_9-free, Lemma 4.1 provides a normal Sylow r-subgroup R of B for some prime $r \in \pi(B)$. We remark that R is a normal subgroup of $K = K_1 \times B$. In addition B is is not a direct product of non-trivial subgroups with coprime order and hence there are a prime $q \in \pi(B)$ and some $Q \in \mathrm{Syl}_q(B)$ such that RQ is not nilpotent. Now B is a Hall subgroup of K and thus RQ is a Hall subgroup of K. In particular (1) implies that RQ is a batten and it follows that $|R| = r$ and $1 \neq C_Q(R) = \Phi(Q)$. We further see, from Definition 4.16 and (1), that for every $p \in \pi(N)$ with the property $[O_p(N), R] \neq 1$ we have that $|O_p(N)| = p^q$. Since R is a normal Sylow subgroup of K, the maximal choice of N provides some $p \in \pi(N)$ such that R does not centralize $P := O_p(N)$. In particular we have that $|P| = p^q$.

Let $s \in \pi(B) \setminus \{q, r\}$ and let S be a Sylow s-subgroup of B such that $QS = SQ$. Such a subgroup exists by Satz VI. 2.3 in [3]. If S does not centralize R, then RS is not nilpotent and therefore our arguments above show that $p^s = |P| = p^q$. This is impossible because $r \neq s$. Consequently $[R, S] = 1$.

Since B is directly indecomposable, we conclude that SQ is not nilpotent. But SQ is a Hall subgroup of B and then it is a Hall subgroup of K. In particular (1) yields that SQ is a batten. From the fact that $1 \neq C_Q(R) = \Phi(Q)$ we conclude that $|Q| \neq q$, and then $|S| = s$ and $S \trianglelefteq SQ$. In addition $C_Q(S) = \Phi(Q) = C_Q(R)$. But $(R \times S)Q$ is an L_9-free group, and this situation contradicts Corollary 4.13. □

We conclude that, by construction, $G = NK$, where N is a nilpotent normal subgroup of G with modular Sylow subgroups and K is a batten group, by (3), such that for all $p \in \pi(N)$ it is true that every batten of K centralizes $O_p(N)$ or acts on $O_p(N)$ avoiding L_9, by (1) and (2). □

The converse of Proposition 6.1 is false, as can be seen in the following example and subsequent lemma.

Example 6.2. Let $H = C_{19} \times C_{19}$, $J = C_5 \times C_5$ and let $x, y \in GL(2, 19) \times GL(2, 5)$ be such that

$$x = \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix} \right) \text{ and } y = \left(\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \right).$$

Then $xy = \left(\begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 2 & 3 \end{pmatrix} \right) = yx$. Hence $G := (H \times J) \rtimes (\langle x \rangle \times \langle y \rangle)$ is a group.

Moreover, $N := H \times J$ is a nilpotent normal subgroup of G with modular Sylow subgroups. Since

$$x^8 = (x^2)^4 = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \right)^4 = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right)^2 = 1$$

and

$y^9 = \left(\begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)^3$, it follows that x and y have coprime order.

Thus $K := \langle x \rangle \times \langle y \rangle$ is cyclic, which means that it is a batten group.

We see that x and y induce non-trivial power automorphisms on H. Thus every batten of K acts on $H = O_{19}(N)$ avoiding L_{19} of type (std). In addition x^2 and y^3 induce power automorphisms on J. Since x and y act irreducibly on J, it follows that every batten of K acts on $J = O_5(N)$ avoiding L_{19} of type (std), too.

Altogether G satisfies the conclusion of Proposition 6.1.

On the other hand we observe that $\pi(K) = \{2, 3\} = \pi(\langle x, y \rangle / \langle x^2 \rangle) = \pi(K/C_K(H))$ and that $C_H(C_K(J)) = C_H(\langle y^3 \rangle) = 1$.

If $g \in HJ$ centralizes $\langle x \rangle$ or $\langle y \rangle$, then $g = 1$. Thus the following lemma yields that G is not L_9-free.

Lemma 6.3. *Let H be a non-trivial abelian group where all non-trivial Sylow subgroups are non-cyclic elementary abelian, let L be a cyclic group inducing power automorphism on H such that $\pi(L) = \pi(L/C_L(H))$ and let $1 \neq J$ be an abelian group admitting L as a group of automorphisms such that the action of L on $O_p(J)$ avoids L_9 for every $p \in \pi(J)$ and such that $(|H|, |J|) = 1$.*

Let $\pi := \{q \in \pi(L) \mid C_{O_q(L)}(H) < C_{O_q(L)}(J)\}$. Suppose that $C_H(C_L(J)) = 1$ and, for all $g \in (H \times J)^\#$, suppose that g centralizes neither $O_\pi(L)$ nor $O_{\pi'}(L)$.

Then $(H \times J) \rtimes L$ is not L_9-free.

Proof. We first set $L_1 := O_\pi(L)$ and $L_2 := O_{\pi'}(L)$. Since none of the groups L_1 nor L_2 is centralized by any element of $HJ \setminus \{1\} \neq \varnothing$, it follows that L_1 and L_2 are both non-trivial.

For every odd prime $p \in \pi(H)$ there is an elementary abelian subgroup H_p of H that has order p^2. In particular there are elements a_p and b_p of H such that $H_p = \langle a_p \rangle \times \langle b_p \rangle$.

We set $a := \prod_{p \in \pi(H)} a_p$ and $b := \prod_{p \in \pi(H)} b_p$.

These elements are well-defined (in the sense that the ordering of the primes does not matter) because H is abelian. For every $p \in \pi(J)$, we further see that L acts on $O_p(J)$ avoiding L_9. Since J is abelian, we deduce from Lemma 5.3(b) that $[O_p(J), L]$ is elementary abelian. In addition L acts irreducibly on $[O_p(J), L]$ or it induces power automorphisms on $O_p(L)$, by Lemma 5.4. We choose $x_p \in [O_p(J), L]^{\#}$. Then L acts irreducibly on $P := \langle x_p^L \rangle$. Next we choose $l \in L$ such that $L = \langle l \rangle$. Then $1 \neq x_p^l \in P \leq HJ$ and $1 \neq [l, x_p] \in P \leq HJ$. Thus our hypothesis implies that $1 \neq [[l, x_p], L_1] \leq P$ and $1 \neq [x_p^l, L_2] \leq P$. Altogether we have that $P = \langle x_p^L \rangle = \langle (x_p^l)^L \rangle = \langle [[l, x_p], L_1]^L \rangle = \langle [x_p^l, L_2]^L \rangle$. Moreover, Lemma 5.5 yields that $C_L(x_p) = C_L(P) = C_L(O_p(J))$.

For every $p \in \pi(L)$ we choose $1 \neq x_p \in [O_p(J), L]$, and then we set $x := \prod_{p \in \pi(J)} x_p$.

Since J is abelian, it follows that $C_L(J) = \bigcap_{p\in\pi(J)} C_L(O_p(J)) = \bigcap_{p\in\pi(J)} C_L(x_p) = C_L(x)$. Next we set $y := x^l$. Then our previous arguments show that $J_0 := \langle x^L \rangle = \langle y^L \rangle = \langle [[l, x], L_1]^L \rangle = \langle [y, L_2]^L \rangle$ (∗).

We will construct a subgroup lattice L_9 using Lemma 3.5. For this we set $E := C_L(HJ) \trianglelefteq HJL$ and $D := C_L(J)$.

For every $q \in \pi'$ we have $C_{O_q(L)}(H) \geq C_{O_q(L)}(J)$ and so $C_{O_q(L)}(J) \leq E$. This implies that $C_{L_2}(J) = C_{L_2}(HJ) \leq E$ (∗∗) and that $D = C_L(J) = C_{L_1}(J)$. We conclude that $C_{L_1}(x)E = (L_1 \cap C_L(x))E = (L_1 \cap C_L(J))E = C_{L_1}(J)E = DE = D$.

If $q \in \pi = \pi(L_1)$, then $C_{O_q(L)}(H) < C_{O_q(L)}(J) \leq C_{L_1}(x)$ and therefore $C_{L_1}(H) = C_{L_1}(HJ) \leq E$ (∗ ∗ ∗). Then it follows that $E \cap L_1 = C_{L_1}(HJ) = C_{L_1}(H) < C_{L_1}(x) \leq D \cap L_1$. In particular $E \neq D$ and hence (L9(i)) of Lemma 3.5 holds.

Next we set $A := \langle a \rangle L_1^x E$, $S := L_1^{ax} E$ and $T := L_1^{a^{-1}x} E$. Then we have that A contains the subgroups S, T and $D\ (= C_{L_1}(x)E)$. In addition, if $c \in \{a, a^{-1}, a^2\}$, then we see that $\langle c \rangle = \langle a \rangle$, because $o(a)$ is odd by construction. Since $C_{\langle a \rangle}(D) \leq C_H(D) = C_H(C_L(J)) = 1$, it follows that $\langle a \rangle = \langle c \rangle = [c, D] \times C_{\langle a \rangle}(D) = [c, D]$, by Lemma 1.1. The group L_1 induces power automorphisms on H, which means that it normalizes $[c, D]$. Together with Part (d) of Lemma 1.4 we conclude that

$$\langle D, L_1^c E \rangle = \langle [c, D]^{L_1^c E} \rangle L_1^c E = [c, D] L_1^c E = \langle c \rangle L_1 E = A^{x^{-1}}.$$

In particular, since D centralizes x, it follows that $\langle D, T \rangle = \langle D, S \rangle = A$. Moreover, we have that $A \geq \langle T, S \rangle = \langle L_1, L_1^{a^2} E \rangle^{a^{-1}x} \geq \langle D, L_1^{a^2} E \rangle^{a^{-1}x} = (A^{x^{-1}})^{a^{-1}x} = A$ and therefore we conclude that $\langle S, T \rangle = A$ as well. Next Lemma 5.5 gives that $C_{L_1}(a) = C_{L_1}(H) \leq E$ by (∗ ∗ ∗). Furthermore L_1^c is a π-group for all $c \in \{a, a^{-1}, a^2\}$, whence

$$L_1 E \cap L_1^c \leq O_\pi(L_1 E) \cap L_1^c = L_1 \cap L_1^c = C_{L_1}(c) = C_{L_1}(a) \leq E$$

by Part (b) of Lemma 1.4. Altogether Dedekind's modular law gives that $L_1 E \cap L_1^c E = (L_1 E \cap L_1^c) E \leq E$ for all $c \in \{a, a^{-1}, a^2\}$. We conclude that $T \cap S = L_1^{ax} E \cap L_1^{a^{-1}x} E \leq E^x = E$, that $D \cap T \leq (L_1 E \cap L_1^{a^{-1}} E)^x \leq E^x = E$ and that $D \cap S \leq (L_1 E \cap L_1^a E)^x \leq E$. With all these properties, we see that (L9(ii)) of Lemma 3.5 is true.

Now we set $C := \langle b \rangle D L_2$ and $U := \langle b \rangle L_2 E$. Then $C = \langle D, U \rangle$ and $D \cap U = C_L(J) \cap \langle b \rangle L_2 E = (C_L(J) \cap \langle b \rangle L_2) E = C_{L_2}(J) E$ by Dedekind's modular law and by Part (b) of Lemma 1.4. Hence (∗∗) implies that (L9(iii)) of Lemma 3.5 holds.

Let $c \in \{a, a^{-1}\}$ and let $X := \langle L_1^{cx}, L_2 \rangle$. Then X contains a π-Hall subgroup as well as a $\pi(L_2)$-Hall subgroup of HJL. Since HJL is soluble, there is a $\pi(L)$-Hall subgroup K of X such that $L_2 \leq K$ and some $g \in HJ$ such that $L^g = K$. It follows that $L_2 \leq K \cap L = L^g \cap L = C_L(g)$ by Lemma 1.4(b). The hypothesis of our lemma yields that $g = 1$ and therefore $L = K \leq X$. From there we obtain some $h \in X$ such that $L_1^h = L_1^{cx}$ and hence $L_1 = L_1 \cap L_1^{hx^{-1}c^{-1}} \leq C_L(hx^{-1}c^{-1})$ by Lemma 1.4(b). This forces $cx = h \in X$, and then $c, x \in X$, because H and J have coprime order and centralize each other. Altogether $\langle c \rangle = \langle a \rangle$ and $\langle x^L \rangle = J_0$ are subgroups of X, and we conclude that $X = \langle a \rangle J_0 L$. Thus

$$\langle U, T \rangle = \langle L_1^{a^{-1}x}, \langle b \rangle L_2 E \rangle = \langle b \rangle \langle L_1^{a^{-1}x}, L_2, E \rangle = \langle b \rangle X E = \langle a, b \rangle J_0 L = \langle b \rangle \langle L_1^{ax}, L_2, E \rangle = \langle U, L \rangle.$$

We set $F := \langle a, b\rangle J_0 L$ in order to obtain Part (L9(iv)) of Lemma 3.5. Moreover, Dedekind's law and Part (a) of Lemma 1.4 gives that

$$A \cap C = \langle a\rangle L_1^x E \cap \langle b\rangle DL_2 = (\langle a\rangle L_1^x E \cap \langle b\rangle L_2)D$$

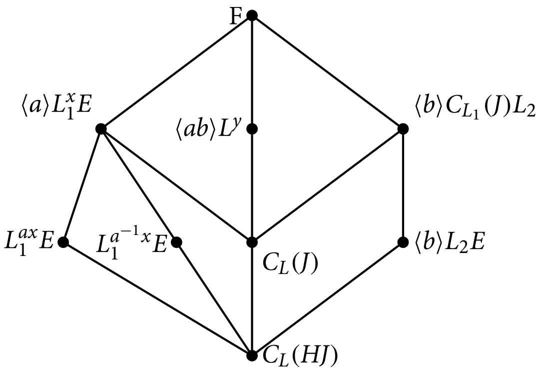
$$= (\langle a\rangle (L_1 E \cap L_2)^x \cap \langle b\rangle (L_1 E \cap L_2))D = (\langle a\rangle C_{L_2}(HJ)^x \cap \langle b\rangle C_{L_2}(HJ))D$$

$$= (\langle a\rangle C_{L_2}(HJ) \cap \langle b\rangle C_{L_2}(HJ))D = (\langle a\rangle \cap \langle b\rangle C_{L_2}(HJ))C_{L_2}(HJ)D = C_{L_2}(HJ)D = D.$$

We set $B := \langle ab\rangle L^y$. Then

$$A \cap B = (\langle a\rangle L_1 E \cap \langle ab\rangle L^{yx^{-1}})^x \le (\langle a\rangle (L_1 E \cap HL^{yx^{-1}}))^x \le (\langle a\rangle C_{L_1E}(yx^{-1}))^x \le (\langle a\rangle C_L(J))^x = \langle a\rangle D$$

by Lemma 1.4(b), because $\langle (yx^{-1})^L\rangle \cap H \le J \cap H = 1$.



In a similar way we obtain that $A \cap B \le \langle ab\rangle D$ and therefore $D \le A \cap B \le \langle a\rangle D \cap \langle ab\rangle D = (\langle a\rangle \cap \langle ab\rangle D)D = D$.

We further calculate that

$$B \cap C = \langle ab\rangle L^y \cap \langle b\rangle DL_2 \le \langle ab\rangle (L^y \cap HDL_2) \le \langle ab\rangle C_{DL_2}(y) \le \langle ab\rangle C_L(J) = \langle ab\rangle D$$

and similarly $B \cap C \le \langle b\rangle D$. Therefore $D \le B \cap C \le \langle ab\rangle D \cap \langle b\rangle D = (\langle ab\rangle \cap \langle b\rangle D)D = D$.

Finally $(*)$ and Part (d) of Lemma 1.4 yield that

$$\langle A, B\rangle = \langle \langle a\rangle L_1^x E, \langle ab\rangle L^y\rangle = \langle a, b\rangle \langle L_1^x E, L^y\rangle = \langle a, b\rangle \langle [yx^{-1}, L_1]^L\rangle L = \langle a, b\rangle \langle [[l, x], L_1]^L\rangle L$$

$$= \langle a, b\rangle J_0 L = F = \langle a, b\rangle \langle [y, L_2]^L\rangle L = \langle a, b\rangle \langle L^y, DL_2\rangle = \langle \langle ab\rangle L^y, \langle b\rangle DL_2\rangle = \langle B, C\rangle.$$

Altogether $\{A, B, C, D, E, F, S, T, U\}$ satisfies every condition of Lemma 3.5, which means that it is isomorphic to L_9. □

The previous lemma and Lemma 4.12 motivate the following definition:

Definition 6.4. Here we define a class $\mathfrak{L}$ of finite groups, and each group in $\mathfrak{L}$ has a type.

We say that $G \in \mathfrak{L}$ has type (N, K) if and only if the following hold:

($\mathfrak{L}$1) $G = N \rtimes K$, where N is a normal nilpotent Hall subgroup of G with modular Sylow subgroups and K is a batten group.

($\mathfrak{L}$2) If $p \in \pi(N)$, then every batten of K centralizes $O_p(N)$ or it acts on it avoiding L_9.

($\mathfrak{L}$3) For all Sylow subgroups Q of K and all distinct Sylow subgroups P and R of N that are not centralized by Q, we have that $C_Q(P) \ne C_Q(R)$.

($\mathfrak{L}$4) Suppose that $H \le N$ is abelian, that its nontrivial Sylow subgroups are not cyclic and that $L \le \mathrm{Pot}_K(H)$ is cyclic and such that $\pi(L) = \pi(L/C_L(H))$. Let $1 \ne J \le N$ be L-invariant and abelian, suppose that $(|H|, |J|) = 1$, $[H, J] = 1$ and $C_H(C_L(J)) = 1$, and set $\pi := \{q \in \pi(L) \mid C_{O_q(L)}(H) < C_{O_q(L)}(J)\}$.

Then there is some $g \in (HJ)^\#$ that centralizes $O_\pi(L)$ or $O_{\pi'}(L)$.

Theorem 6.5. *Let G be a finite L_9-free group. Then $G \in \mathfrak{L}$.*

Proof. From Lemma 6.1 we see that $G = NK$ and that ($\mathfrak{L}1$) and ($\mathfrak{L}2$) are satisfied.

For ($\mathfrak{L}3$) we let Q be a Sylow subgroup of K and we let P and R be distinct Sylow subgroups of N that are not centralized by Q. Since N is a nilpotent normal Hall subgroup of G, it follows that $[P, R] = 1$ and that Q normalizes P and R. Then $(P \times R)Q$ is directly indecomposable and L_9-free, whence Corollary 4.13 gives that $C_Q(P) \neq C_Q(R)$.

Finally, we look at ($\mathfrak{L}4$) and we assume that it is not true. Then there is an abelian subgroup H of N such that the nontrivial Sylow subgroups are not cyclic, and we find a cyclic group $L \leq \mathrm{Pot}_K(H)$ such that $\pi(L) = \pi(L/C_L(H))$ and a nontrivial L-invariant abelian subgroup J of N such that $(|H|, |J|) = 1$, $[H, J] = 1$ and $C_H(C_L(J)) = 1$. Let $\pi := \{q \in \pi(L) \mid C_{O_q(L)}(H) < C_{O_q(L)}(J)\}$. Then we have, for all $g \in (HJ)^\#$, that g centralizes neither $O_\pi(L)$ nor $O_{\pi'}(L)$ for $\pi := \{q \in \pi(L) \mid C_{O_q(L)}(H) < C_{O_q(L)}(J)\}$.

We note that P does not centralize $O_\pi(L) \leq L$. Then we find a prime $q \in \pi(L)$ such that a Sylow q-subgroup Q of L does not centralize P. Using Lemma 5.2, we see that Q acts on $O_p(N)$ avoiding L_9 and then Lemma 5.7 yields that Q acts non-trivially on P, and avoiding L_9. Now we may apply Corollary 5.6: Since L induces power automorphisms on P, Part (c) shows that P is elementary abelian. Then the hypotheses of Lemma 6.3 are satisfied. It says that $(H \times J) \rtimes L$ is not L_9-free, which is false. We conclude that ($\mathfrak{L}4$) holds. □

7. The class $\mathfrak{L}$

Lemma 7.1. *All groups in the class $\mathfrak{L}$ are soluble.*

Proof. Let $G \in \mathfrak{L}$ be of type (N, K). Then N is nilpotent normal Hall subgroup of G and $G/N \cong K$ is a direct product of p-groups or of groups whose order is divisible by exactly two primes. Thus G/N is soluble as well, and it follows that G is soluble. □

Lemma 7.2. *Let $G \in \mathfrak{L}$ be of type (N, K) and $\pi := \pi([N, K])$. Then every subgroup of $O_\pi(N)$ is normal in N.*

Proof. Let U be a subgroup of $O_\pi(N)$. Then $U \trianglelefteq N$ if and only if $O_p(U) \trianglelefteq O_p(N)$ for all $p \in \pi$, since N is nilpotent. Let $p \in \pi$. We note that this implies that $O_p(N)$ is not centralized by K. In particular there is a batten of K that acts non-trivially on $O_p(N)$ and avoiding L_9. If $O_p(N)$ is abelian, then $O_p(U) \trianglelefteq O_p(N)$. If $O_p(N)$ is not abelian, then we apply Lemma 4.17 (b) to a batten B of K that acts non-trivially on $O_p(N)$. The first possibility described there implies that $O_p(N)$ is abelian, which is not the case here. Thus the second possibility holds, and then $O_p(N) \cong Q_8 \times I$, where I is cyclic of order at most 2. We conclude that $O_p(N)$ is hamiltonian and it follows that $O_p(U) \trianglelefteq O_p(N)$. □

Lemma 7.3. *Let $G \in \mathfrak{L}$ be of type (N, K) and $X \leq G$. Then there is some $x \in [N, K]$ such that $X = (N \cap X)(K^x \cap X)$.*

Proof. We set $M := N \cap X$. Then M is a normal Hall subgroup of X, because N is one of G. Then the Schur-Zassenhaus Theorem provides a complement C of M in X, and we notice that C and M have coprime orders. Therefore $\pi(C) = \pi(X) \setminus \pi(M) \subseteq \pi(G) \setminus \pi(N) = \pi(K)$. It follows that C is contained in a complement for N in G. Since G is soluble by Lemma 7.1, such a complement is conjugate to K, and thus we find $g \in G$ such that $C \leq K^g$. The coprime action of K on N yields, together with Lemma 1.1, that $N = C_N(K)[N, K]$, and therefore $G = KN = KC_N(K)[N, K]$. We notice that $[N, K] \trianglelefteq G$ and we let $x \in [N, K]$ and $y \in KC_N(K) \leq N_G(K)$ be such that $g = yx$. Then $C = K^g \cap X = K^x \cap X$ and hence $X = MC = (N \cap C)(K^x \cap X)$. □

Lemma 7.4. *Let $G \in \mathfrak{L}$ be of type (N, K) and suppose that $U \leq G$. Then U is a group in class $\mathfrak{L}$ of type $(U \cap N, U \cap K^g)$ for some $g \in [N, K]$.*

Proof. Lemma 7.3 provides some $g \in [N, K]$ such that $U = (U \cap N) \cdot (U \cap K^g)$. By conjugation we may suppose that $g = 1$ and we set $K_1 := U \cap K$. Then Lemma 2.7 yields that $K_1 = U \cap K^g \leq K^g \cong K$ is a batten group. Moreover, $M := U \cap N \leq N$ is a normal nilpotent Hall subgroup of U with modular Sylow subgroups, by ($\mathfrak{L}1$). This means that ($\mathfrak{L}1$) holds for U, and now we turn to ($\mathfrak{L}2$) and let $p \in \pi(M)$. Suppose that B is a batten of K_1 that does not centralize $O_p(M)$. Then it does not centralize $O_p(N)$ and therefore Lemma 5.2 implies that $B \cong B/1$ acts on $O_p(N)$ avoiding L_9. Then we may apply Lemma 5.7 to see that B also acts on $O_p(M)/1 \cong O_p(M)$ avoiding L_9.

This gives property ($\mathfrak{L}2$) of Definition 6.4 for U, and ($\mathfrak{L}4$) follows because $M \leq N$ and $K_1 \leq K$.

For ($\mathfrak{L}3$) we let Q_1 be a Sylow subgroup of K_1 and we let $p, r \in \pi(M)$ be different primes such that $[O_p(M), Q_1] \neq 1 \neq [O_r(M), Q_1]$. We need to prove that $C_{Q_1}(O_p(M)) \neq C_{Q_1}(O_r(M))$.

First we let Q be a Sylow subgroup of K that contains Q_1. Then $[O_p(N), Q] \neq 1 \neq [O_r(N), Q]$ and therefore $C_Q(O_p(N)) \neq C_Q(O_r(N))$, using Property ($\mathfrak{L}3$) for G. In particular, these centralizers cannot both be trivial, and we may suppose that $C_Q(O_p(N)) \neq 1$. Then Q does not act faithfully on $O_p(N)$, but the action of Q on $O_p(N)$ avoids L_9. Definition 4.8 immediately gives that $Q \not\cong Q_8$. Then it follows that Q is cyclic, which means that the subgroup lattice $L(Q)$ of Q is a chain, and Q_1 is also cyclic.

We assume for a contradiction that $C_{Q_1}(O_p(M)) = C_{Q_1}(O_r(M))$. Then Lemma 5.5, with Q_1 in the role of H, $O_p(M)$ in the role of R and $O_p(N)$ in the role of P, gives that $C_{Q_1}(O_p(M)) = C_{Q_1}(O_p(N))$. Similarly $C_{Q_1}(O_r(M)) = C_{Q_1}(O_r(N))$, and then it follows that $C_{Q_1}(O_p(N)) = C_{Q_1}(O_r(N))$. We recall that $C_Q(O_p(N)) \neq C_Q(O_r(N))$ and that Q is cyclic, and now we may suppose that $C_Q(O_p(N)) \lneq C_Q(O_r(N))$. This forces $C_{Q_1}(O_p(N)) \lneq C_Q(O_p(N))$, which is impossible. Thus $C_{Q_1}(O_p(M)) \neq C_{Q_1}(O_r(M))$ and ($\mathfrak{L}3$) holds for U as well. □

Lemma 7.5. *Let $G \in \mathfrak{L}$ be of type (N, K) and suppose that S is a normal Sylow q-subgroup of K that centralizes N for some prime q. Let K_1 be a Hall q'-subgroup of K.*

Then G is also of type $(N \times S, K_1)$. In particular, if we choose (N, K) such that $|N|$ is as large as possible, then $\pi(K) = \pi(K/C_K(N))$.

Proof. We show that $G = (N \times S)K_1$ satisfies ($\mathfrak{L}1$)–($\mathfrak{L}4$) of Definition 6.4, and we first note that K_1 is a batten group by Lemma 2.7. The structure of K forces all Sylow subgroups of K to be cyclic or quaternion, more specifically S is cyclic or isomorphic to Q_8. This means that S is modular. Since S is a normal Sylow q-subgroup of K and N is a Hall subgroup of G, by hypothesis, it follows that $N \times S$ is a Hall subgroup of G where all Sylow subgroups are modular.

By hypothesis $[N, S] = 1$ and N is nilpotent, hence $N \times S$ is nilpotent, too. This is ($\mathfrak{L}1$).

For ($\mathfrak{L}2$) we let B be a batten of K_1 and $p \in \pi(N \times S)$. We keep in mind that B is not necessarily a batten of K – if it is, then it centralizes $O_p(N \times S)$ or it acts on it avoiding L_9, because of ($\mathfrak{L}2$) for G.

Now we suppose that B is not a batten of K and that $[O_p(N \times S), B] \neq 1$. Then SB is a non-nilpotent batten of K. If $p \neq q$, then SB acts on $O_p(N)$ avoiding L_9, and $[O_p(N), S] = 1$. Then Definition 4.16 implies that SB acts of type (Cy) and it follows that B acts on $O_p(N)$ avoiding L_9. Finally suppose that $q = p$. Then $\Phi(B)$ centralizes $S = O_p(NS)$, while B induces power automorphisms on the cyclic group S of order p. Thus SB satisfies (std) of Definition 4.6, and we deduce that B acts on S avoiding L_9.

We turn to ($\mathfrak{L}3$). Let Q be a Sylow subgroup of K_1 and let P and R be distinct Sylow subgroups of $N \times S$ that are not centralized by Q. First we note that Q is a Sylow subgroup of K because K_1 is a Hall subgroup of K by hypothesis. Therefore, if $PR \leq N$, then we immediately have that $C_Q(R) \neq C_Q(P)$, by ($\mathfrak{L}3$) in G.

Without loss suppose that $R \not\leq N$, i.e. $R = S$. Then we recall that Q was chosen not to centralize P and $R = S$, which means that Q and S cannot come from distinct battens of K, but their product must be a non-nilpotent batten of K. Moreover, $[P, QS] \neq 1$. We obtain from Lemma 2.5 and Definition 4.16 that $\Phi(Q) = Z(Q) = C_Q(S)$ does not centralize P and therefore $C_Q(R) \neq C_Q(P)$. This is ($\mathfrak{L}3$).

Finally, let $H \leq N \times S$ be such that its nontrivial Sylow subgroups are not cyclic, let $L \leq \mathrm{Pot}_{K_1}(H)$ be cyclic and such that $\pi(L) = \pi(L/C_L(H))$ and let $1 \neq J$ be an L-invariant abelian subgroup of M such that $(|H|, |J|) = 1$, $[H, J] = 1$ and $C_H(C_L(J)) = 1$.

As K is a batten group, the set $\pi(K/C_K(S))$ contains at most one element. We recall that $L \leq K_1 \leq K$, and then it follows that, for every set of primes π, the group S centralizes $O_\pi(L)$ or $O_{\pi'}(L)$. In order to prove ($\mathfrak{L}$4) of Definition 6.4, we may thus suppose that H and J are subgroups of N.

Then $L \leq K_1 \leq K$ shows that $L \leq \mathrm{Pot}_K(H)$. Hence we apply ($\mathfrak{L}$4) of Definition 6.4 to G, i.e. to the type (N, K). If $\pi := \{q \in \pi(L) \mid C_{O_q(L)}(H) < C_{O_q(L)}(J)\}$, then we find some $g \in (HJ)^\#$ that centralizes $O_\pi(L)$ or $O_{\pi'}(L)$.

Altogether, $G = (N \times S)K_1$ satisfies Definition 6.4.

We now suppose that $|N|$ is as large as possible and we assume for a contradiction that $S \leq C_K(N)$ is a Sylow subgroup of K. Then S is not normal in K, hence there is a non-nilpotent batten B of K such that $B = \mathcal{B}(B)S$. For all $p \in \pi(N)$ it follows that $[O_p(N), Z(B)] \leq [O_p(N), S] = 1$, by Lemma 2.5. Then Definition 4.16 yields that B does not act on $O_p(N)$ avoiding L_9, whence B centralizes $O_p(N)$. But now $\mathcal{B}(B) \leq C_K(N)$ and $\mathcal{B}(B)$ is a normal Sylow subgroup of K. This contradicts the maximal choice of N. □

Lemma 7.6. *Let $G \in \mathfrak{L}$ and suppose that $M \trianglelefteq G$. Then $G/M \in \mathfrak{L}$.*

Proof. By induction we may suppose that M is a minimal normal subgroup of G. We recall that G is soluble by Lemma 7.1, and we let r be prime such that M is an elementary abelian r-group. If M has a complement C in G, then $G/M \cong C$ and Lemma 7.4 gives that $C \in \mathfrak{L}$ and hence $G/M \in \mathfrak{L}$.

Consequently we may suppose that M does not have a complement in G. We choose $N, K \leq G$ such that G has type (N, K) and such that $|N|$ is as large as possible. Then $\pi(K) = \pi(K/C_K(N))$ by Lemma 7.5.

First suppose that $r \in \pi(K)$. Then $M \leq K$ and we see that $[N, M] \leq N \cap M \leq N \cap K = 1$, because N is a Hall subgroup of G. Hence $M \leq C_K(N)$. Next we let B be a batten of K that contains M. Then $M \leq C_B(N)$, which means that for all $p \in \pi(N)$, B does not act faithfully on $O_p(N)$.

Since there is some $p \in \pi(N)$ such that B acts on $O_p(N)$ avoiding L_9, it follows from Definition 4.8 that B is not isomorphic to Q_8. In addition $M \lneq Z(B)$, if B is not nilpotent, by Definition 5.1. Therefore, in this case, the section B/M is a non-nilpotent batten as well. We conclude that K/M is a batten group.

Assume for a contradiction that $r \in \pi(N)$, but that $M \nleq C_N(K)$. Then we note that $C_M(K) \trianglelefteq G$ by Lemma 7.2, and we deduce that $C_M(K) = 1$, because M is a minimal normal subgroup of G. This forces $M \leq [O_r(N), K]$. Then $[O_r(N), K]$ is not elementary abelian, because otherwise M would have a complement in this commutator and hence in G. But we are working under the hypothesis that it does not.

Now Lemmas 5.4 and 4.17(b) imply that $M \leq [O_r(N), K] \cong Q_8$. But now $Z([O_r(N), K])$ is the unique subgroup of order 2 of $[O_r(N), K]$, which means that it must be centralized by K and contained in M. But this is a contradiction. Thus $M \leq C_N(K)$ and Corollary 5.6 implies that $p = 2$. We summarise that $M \leq C_K(N)$ and that K/M is a batten group or $M \leq C_N(K)$ and $p = 2$.

Let $- : G \to G/M$ be the natural homomorphism. Then $\bar{G} = \bar{N} \cdot \bar{K}$, where $\bar{N}$ is a normal nilpotent Hall subgroup of G with modular Sylow subgroups, since sections of modular p-subgroups are modular. Moreover $\bar{K}$ is a batten group. Thus $\bar{G}$ satisfies ($\mathfrak{L}$1) of Definition 6.4. We further deduce ($\mathfrak{L}$2) from Lemmas 5.2 and 5.7.

For ($\mathfrak{L}$3) we let Q be a Sylow subgroup of K and we let $p, s \in \pi(\bar{N})$ be distinct primes such that $[O_s(\bar{N}), \bar{Q}] \neq [O_p(\bar{N}), \bar{Q}]$. Then $[O_s(N), Q] \neq [O_p(N), Q]$ and therefore $C_Q(O_s(N)) \neq C_Q(O_p(N))$. Since $M \leq C_N(Q)$ or $M \leq C_K(O_s(N)) \cap C_K(O_p(N))$, it follows that $C_{\bar{Q}}(O_s(\bar{N})) \neq C_{\bar{Q}}(O_p(\bar{N}))$.

We finally let $1 \neq \bar{H} \leq \bar{N}$ be abelian with non-cyclic Sylow subgroups and $\bar{L} \leq \mathrm{Pot}_{\bar{K}}(\bar{H})$ be cyclic with $\pi(\bar{L}) = \pi(\bar{L}/C_{\bar{L}}(\bar{H}))$ and we let $\bar{J}$ be an abelian $\bar{L}$-invariant subgroup of $\bar{N}$ such that $(|\bar{H}|, |\bar{J}|) = 1$ and $C_{\bar{H}}(C_{\bar{L}}(\bar{J})) = 1$. We set $\bar{\pi} := \{q \in \pi(\bar{L}) \mid \forall\, \bar{Q} \in \mathrm{Syl}_q(\bar{L}) : C_{\bar{Q}}(\bar{H}) < C_{\bar{Q}}(\bar{J})\}$.

Then we assume for a contradiction that every nontrivial element $\bar{g} \in \bar{H}\bar{J}$ centralizes neither $O_\pi(\bar{L})$ nor $O_{\pi'}(\bar{L})$. Then Lemma 1.1 yields that $[\bar{H}\bar{J}, \bar{L}] = \bar{H}\bar{J}$. We choose pre-images H, L and J in G of smallest possible order. Then $HJ = [HJ, L]$ and $\pi(\bar{X}) = \pi(X)$ for all $X \in \{H, L, J\}$, because G is soluble by Lemma 7.1. In particular we have that $(|H|, |J|) = 1$.

If $r \in \pi(X)$ for some $X \in \{H, J\}$, then $r = 2$. Then our assumption implies that $C_{O_2(X)}(L) \le M$. It follows that $X \cong \bar{X}$ or that $M \le \Phi(X) = \Phi([X, L])$. In the second case, we apply Lemmas 5.4 and 4.17. Together they show that $[O_2(N), K] = [O_2(N), L] \cong Q_8$ and thus $\pi(L/C_{O_2(N)}(L)) = \{3\}$. This means that $O_2(N)$ centralizes $O_\pi(L)$ or $O_{\pi'}(L)$. But then we also have that $[O_2(\bar{X}), O_\sigma(\bar{L})] = 1$ for some $\sigma \in \{\pi, \pi'\}$, which is a contradiction. We deduce that $\bar{H} \cong H$ and $\bar{J} \cong J$. In particular $H \le N$ is abelian, with non-cyclic Sylow subgroups, and $J \ne 1$ is an abelian L-invariant subgroup of N. Since $\bar{L}$ is cyclic, it follows from our arguments above that L is also cyclic. Moreover $\pi(L) = \pi(\bar{L}) = \pi(\bar{L}/C_{\bar{L}}(\bar{H})) = \pi(L/C_L(H))$, because $M \le C_K(N)$ or $M \cap L = 1$.

We now investigate the action of L on H. Since $\bar{H} \cong H$ and $M \cap L = 1$ or $M \le C_L(H)$, we see that L induces power automorphisms on H. In addition Lemma 1.1 shows that

$$\overline{C_H(C_L(J))} \cong C_{\bar{H}}(\overline{C_L(J)}) \cong C_{\bar{H}}(C_{\bar{L}}(\bar{J})) = 1$$

and then the fact that $H \cap M = 1$ yields that $C_H(C_L(J)) = 1$. Altogether we obtain, by applying ($\mathfrak{L}$4) to G, some $g \in HJ^{\#}$ such that g centralizes $O_\pi(L)$ or $O_{\pi'}(L)$, where $\pi := \{q \in \pi(L) \mid \forall Q \in \mathrm{Syl}_q(L) : C_Q(H) < C_Q(J)\}$. Since $\bar{H}\bar{J} \cong HJ$, it follows that $\bar{g} \ne 1$ and $[O_\pi(\bar{L}), \bar{g}] = 1$ or $[O_{\pi'}(\bar{L}), \bar{g}] = 1$. Again we use that $\bar{L}$ acts on $\bar{H} \cong H$ and $\bar{J} \cong L$ equivalently to L, because $M \cap L \le C_L(N)$. Then we see that

$$\bar{\pi} := \{q \in \pi(\bar{L}) \mid \forall \bar{Q} \in \mathrm{Syl}_q(\bar{L}) : C_{\bar{Q}}(\bar{H}) < C_{\bar{Q}}(\bar{J})\} = \pi.$$

This is a contradiction. □

Lemma 7.7. *Let $G \in \mathfrak{L}$ be of type (N, K) such that $C_K(N) = 1$, let $q \in \pi(K)$ and let $Q \in \mathrm{Syl}_q(K)$. Then $1 \ne [N, \Omega_1(Q)]$ has prime power order.*

Proof. We apply Lemma 7.4 and we see that $N\Omega_1(Q) \in \mathfrak{L}$ has type $(N, \Omega_1(Q))$. Since $\Omega_1(Q)$ does not centralize N, there is a prime $p \in \pi(N)$ such that $[O_p(N), \Omega_1(Q)] \ne 1$. It follows that $C_{\Omega_1(Q)}(O_p(N)) = 1$. Now ($\mathfrak{L}$3) implies that $\Omega_1(Q)$ centralizes $O_r(N)$ for every $r \in \pi(N) \setminus \{p\}$, and this shows that $1 \ne [N, \Omega_1(Q)] \le [O_p(N), \Omega_1(Q)] \le O_p(N)$. □

Lemma 7.8. *Let $G \in \mathfrak{L}$ be of type (N, K) such that $C_K(N) = 1$, let $q \in \pi(K)$ and let $Q \in \mathrm{Syl}_q(K)$. Let $p \in \pi(N)$ be such that $\Omega_1(Q)$ does not centralize $P := O_p(N)$. Then the following hold:*

(a) $N_G(\Omega_1(Q)) = (O_{p'}(N)K) \times C_P(K)$.
(b) $G = [P, \Omega_1(Q)]N_G(\Omega_1(Q))$.
(c) $[P, \Omega_1(Q)] = [P, K]$ *acts transitively on* $\Omega_1(Q)^G = \{Q_0 \le G \mid |Q_0| = q\}$.
(d) If $X \le G$, *then there is some* $x \in [P, \Omega_1(Q)]$ *such that* $X = (X \cap P)N_X(\Omega_1(Q)^x)$.
(e) Suppose that $X \le G$, *that* q *divides* $|X|$ *and that* $x \in P$. *Then* $X = (X \cap P)N_X(\Omega_1(Q)^x)$ *if and only if* $\Omega_1(Q)^x \le X$.

Proof. We set $P_0 := [P, \Omega_1(Q)]$. Then Lemma 7.7 implies that $P_0 = [N, \Omega_1(Q)]$ and so $O_{p'}(N) \le C_G(\Omega_1(Q)) \le N_G(\Omega_1(Q))$. Furthermore K acts on P avoiding L_9 and then we have that $P_0 = [P, Q] = [P, K]$ by Lemma 5.4. Next $K \le N_G(\Omega_1(Q))$ from Lemma 2.6.

Since N is nilpotent, we conclude that $N_G(\Omega_1(Q)) = (O_{p'}(N) \times N_P(\Omega_1(Q)))K$. But we also have that $[N_P(\Omega_1(Q)), \Omega_1(Q)] \le P \cap \Omega_1(Q) = 1$, whence $N_P(\Omega_1(Q)) \le C_P(\Omega_1(Q)) = C_P(K)$ by Lemma 5.5. Consequently $N_P(\Omega_1(Q)) = C_P(\Omega_1(Q))$ and it follows that $N_G(\Omega_1(Q)) = (O_{p'}(N)K) \times C_P(K)$, as stated in (a).

For (b) we recall that, by (a), the subgroups K and $O_{p'}(N)$ normalize $\Omega_1(Q)$. Then $G = PO_{p'}(N)K \le PN_G(Q) \le G$. Moreover $P \trianglelefteq G$ and Lemma 1.1 implies that $P = C_P(\Omega_1(Q))P_0$, where $C_P(\Omega_1(Q)) \le N_G(\Omega_1(Q))$ and therefore $G = P_0N_G(\Omega_1(Q))$ as stated in (b).

From there we deduce that P_0 acts transitively on $\Omega_1(Q)^G$ by conjugation. The second statement of (c) follows because G is soluble (Lemma 7.1), together with the fact that $\Omega_1(Q)$ is the unique subgroup of its order in the Hall subgroup K of G (Lemma 2.6). This means that every subgroup of order q of G is conjugate to $\Omega_1(Q)$, completing (c).

For (d) and (e) we let $X \le G$. Lemma 7.3 provides some $g \in G$ such that $X = (N \cap X)(K^g \cap X)$. Moreover, (a) implies that K^g and $O_{p'}(N)(= O_{p'}(N)^g)$ normalize $\Omega_1(Q)^g$, and then we summarise:

$$X = (N \cap X)(K^g \cap X) \le (P \cap X)(O_{p'}(N) \cap X)(K^g \cap X) \le (P \cap X)N_X(\Omega_1(Q)^g) \le X.$$

Using (b) we see that $G = N_G(\Omega_1(Q))P_0$, and then we take $y \in N_G(\Omega_1(Q))$ and $x \in P_0$ such that $g = yx$. Now we deduce that $X = (P \cap X)N_X(\Omega_1(Q)^{yx}) = (P \cap X)N_X(\Omega_1(Q)^x)$, as stated in (d).

Finally, suppose that q divides $|X|$ and that $x \in P$. Suppose first that $X = (X \cap P)N_X(Q^x)$. Then q divides $|N_X(\Omega_1(Q)^x)|$, which provides a subgroup Q_0 of order q in $N_X(\Omega_1(Q)^x) \le N_G(\Omega_1(Q)^x)$ and, by (d), there is some $y \in P_0$ such that $\Omega_1(Q)^y = Q_0 \le N_G(\Omega_1(Q)^x) = N_G(\Omega_1(Q))^x$. Then $\Omega_1(Q)$ and $\Omega_1(Q)^{yx^{-1}}$ are subgroups of $N_G(\Omega_1(Q))$ and therefore

$$\begin{aligned}[yx^{-1}, \Omega_1(Q)] &= [[yx^{-1}, \Omega_1(Q)], \Omega_1(Q)] \le [[P, \Omega_1(Q)] \cap N_G(\Omega_1(Q)), \Omega_1(Q)]\\ &\le [C_P(\Omega_1(Q)), \Omega_1(Q)] = 1\end{aligned}$$

by Lemma 1.1. We conclude that $\Omega_1(Q)^x = \Omega_1(Q)^y = Q_0 \le N_X(\Omega_1(Q)^x) \le X$.

Now, conversely, suppose that $\Omega_1(Q)^x \le X$. Then (d) provides some $z \in P_0$ such that $X = (P \cap X)(N_X(\Omega_1(Q)^z))$. In the paragraph above we have shown that $\Omega_1(Q)^z \le X$. We apply (c) to $X = (X \cap N)(X \cap K^g)$, which is a group in $\mathfrak{L}$ by Lemma 7.4, and we obtain some $y \in P \cap X$ such that $\Omega_1(Q)^{zy} = \Omega_1(Q)^x$. We conclude that

$$X = X^y = (P \cap X)^y(N_{X^y}(\Omega_1(Q)^{zy})) = (P \cap X)(N_X(\Omega_1(Q)^x)),$$

because $P \cap X \trianglelefteq X$. □

Lemma 7.9. *Let $G \in \mathfrak{L}$ be of type (N, K) and let $p \in \pi(N)$ such that K induces non-trivial power automorphisms on $P := O_p(N)$.*

Then for all $X, Y \le G$, there is some $i \in \{0, 1\}$ such that $|\langle X, Y\rangle \cap P| = |(P \cap X)(P \cap Y)| \cdot p^i$.

In addition $i = 0$ if and only if there is some $g \in P$ such that for both $Z \in \{X, Y\}$ we have $Z \le (Z \cap P)O_{p'}(N)K^g$.

Proof. We first remark that Lemma 7.2 gives that every subgroup of P is normal in N, and hence in G, because K normalizes every subgroup of P as well. In addition $P = [P, K]$ is elementary abelian by Corollary 5.6 (c).

Let $X, Y \le G$. Then Lemma 7.3 provides $x, y \in P$ such that $X \le (X \cap P)O_{p'}(N)K^x$ and $Y \le (Y \cap P)O_{p'}(N)K^y$.

This implies that $\langle X, Y\rangle \le (X \cap P)(Y \cap P)\langle x^{-1}y\rangle O_{p'}(N)K^x$, bearing in mind that $X \cap P$ and $Y \cap P$ are normal subgroups of G, and therefore $\langle X, Y\rangle \cap P = (P \cap X)(P \cap Y)\langle x^{-1}y\rangle$.

Since P is elementary abelian, we see that $o(xy^{-1}) \in \{1, p\}$ and we deduce the first assertion.

If it is possible to choose $x = y$, then $\langle X, Y\rangle \cap P = (P \cap X)(P \cap Y)\langle x^{-1}y\rangle = (P \cap X)(P \cap Y)$ and in particular $i = 0$ in the statement of the lemma.

For the converse we suppose that $i = 0$, i.e. $|\langle X, Y\rangle \cap P| = |(P \cap X)(P \cap Y)|$. Then $x^{-1}y \in \langle X, Y\rangle \cap P = (P \cap X)(P \cap Y)$ and thus we find $x_0 \in X \cap P$ and $y_0 \in Y \cap P$ such that $x^{-1}y = x_0y_0$. Then $g := yy_0^{-1} = xx_0 \in P \cap X \cap Y$. We note that x_0 normalizes X, centralizes $P \cap X$ and normalizes $O_{p'}(N)$, which implies that $X = X^{x_0} \le ((X \cap P)O_{p'}(N)K^x)^{x_0} = (X \cap P)O_{p'}(N)K^{xx_0}$. Similarly $Y \le (Y \cap P)O_{p'}(N)K^{yy_0^{-1}}$. □

Lemma 7.10. *Let $G \in \mathfrak{L}$ be of type (N, K). Suppose that X and Y are subgroups of G such that $\langle X, Y\rangle = G$ and let B is a batten of K. Suppose that K has a normal q-complement H. Then one of the following hold:*

(a) $q \nmid |G : X|$,

(b) $q \nmid |G:Y|$, *or*
(c) $q = 2$, *K has a section isomorphic to* Q_8 *and* 4 *divides* $(|X|, |Y|)$.

Proof. Let $Q \in \mathrm{Syl}_q(K)$ and suppose that q divides neither $|G : X|$ nor $q \mid |G : Y|$. Then Lemma 7.3 gives maximal subgroups M_X and M_Y of Q such that $X \leq NHM_X$ and $Y \leq NHM_Y$.

If Q is cyclic, then $M_Y = M_X = \Phi(Q)$ and it follows that $G = \langle X, Y\rangle \leq NH\Phi(Q) \neq NHQ = G$. This is impossible. We conclude that Q is not cyclic and then, since K is a batten group, it follows that $Q \cong Q_8$. We assume for a contradiction that $4 \nmid |X|$. Then $X \leq NH\Phi(Q)$ and hence $G = \langle X, Y\rangle \leq NHM_Y \neq NHQ = G$, which is again a contradiction. □

Lemma 7.11. *Let* $G \in \mathfrak{L}$ *be of type* (N, K) *such that* K *acts irreducibly on* $[O_p(N), K]/\Phi([O_p(N), K])$ *for some prime* $p \in \pi(N)$. *If* $X, Y \leq G$ *are such that* $\langle X, Y\rangle = G$, *then* X *or* Y *acts irreducibly on* $[O_p(N), K]/\Phi([O_p(N), K])$.

Proof. Let $X, Y \leq G$ be such that $\langle X, Y\rangle = G$ and let $P := O_p(N)$. We assume for a contradiction that neither X nor Y act irreducibly on $[P, K]/\Phi([P, K])$. Lemma 7.2 implies that N normalizes every subgroup of P (∗). Moreover, by Lemma 7.3, there are $x, y \in N$ such that $X = (X \cap N)(X \cap K^x)$ and $Y = (X \cap N)(Y \cap K^y)$ and, by assumption, neither $X \cap K^x$ nor $Y \cap K^y$ act irreducibly on $[P, K]/\Phi([P, K])$. It follows from Lemma 5.4 that $X \cap K^x$ and $Y \cap K^y$ both induce power automorphisms on P and that $|P| \neq p$. Thus (∗) yields that X and Y normalize every subgroup of P. Then also $G = \langle X, Y\rangle$ normalizes every subgroup of P, which contradicts the irreducible action of K. □

8. The main result

Theorem 8.1. *If* $G \in \mathfrak{L}$, *then* G *is* L_9*-free.*

Proof. Assume for a contradiction that the statement is false and let G be a minimal counterexample. Then there is a sublattice $\mathcal{L} = \{E, S, T, D, U, A, B, C, F\}$ of $L(G)$ isomorphic to L_9, and in particular $\mathcal{L}$ satisfies the relations in Definition 3.1.

We let G be of type (N, K) where, among the minimal counterexamples, we choose G such that $|N|$ is as large as possible and we set

$$\pi(K)^* := \pi(K) \setminus \{|\mathcal{B}(H)| \mid H \text{ is a non-nilpotent batten of } K\}.$$

Then K has a normal q-complement for every $q \in \pi(K)^*$ and Lemma 7.10 is applicable.

We will first analyze how $\mathcal{L}$ fits into the subgroup lattice of G.

(1) $F = G$, $C_K(N) = 1$ and every subgroup of N is normal in N.

Proof. The group F is a subgroup of G that is not L_9-free, and Lemma 7.4 yields that $F \in \mathfrak{L}$. Hence the minimal choice of G implies that $F = G$.

Similarly, it follows from Lemma 3.4 that G is not a direct product of two non-trivial groups of coprime order. Let $p \in \pi(N)$. Then $N = O_{p'}(N) \times O_p(N)$ because N is nilpotent. If K centralizes $O_p(N)$, then $G = O_{p'}(N)K \times O_p(N)$, where the direct factors have coprime order by ($\mathcal{L}$1). But we just saw above that such a direct decomposition of G is not possible. Therefore $[O_p(N), K] \neq 1$ and p divides $|[O_p(N), K]|$, which divides $|[N, K]|$. We conclude from Lemma 7.2 that every subgroup of $O_p(N)$ is normal in N. This implies that every subgroup of N is a normal subgroup of N, because N is nilpotent.

Since we have chosen N as large as possible, Lemma 7.5 implies that $\pi(K) = \pi(K/C_K(N))$. Let $q \in \pi(C_K(N))$. Then the previous equation forces $q \in \pi(K/C_K(N))$, and therefore a Sylow q-subgroup of K has order at least q^2. In particular, for all non-nilpotent battens V of K, we have that $\mathcal{B}(V) \nleq C_K(N)$. It follows that $q \in \pi(K)^*$. Now Lemma 7.10 provides $X, Y \in \{T, U, B\} \subseteq \mathcal{L}$ such that $X \neq Y$ and $O_q(C_K(N)) \leq X \cap Y = E$. Since $O_q(C_K(N))$ is characteristic in $C_K(N) \trianglelefteq NK = G$, it follows that $G/O_q(C_K(N))$ is not L_{10}-free. Moreover, $G/O_q(C_K(N)) \in \mathfrak{L}$ by Lemma 7.6. Since G is a minimal counterexample, we conclude that $O_q(C_K(N)) = 1$. We recall that G is soluble, by Lemma 7.1, hence $C_K(N)$ is soluble, and then there must exist a prime $q \in \pi(C_K(N))$ such that $O_q(C_K(N)) \neq 1$. This gives a contradiction, and therefore $C_K(N) = 1$. □

We remark that, by (1), we may apply Lemmas 7.7 and 7.8.

(2) For all $p \in \pi(N)$ we have that $O_p(N) \cap D \trianglelefteq G$. In particular $N \cap D \trianglelefteq G$ and $N \cap E = 1$.

Proof. Let $H \in \{E, D\}$, let $p \in \pi(N)$ and set $P := O_p(N)$. If $H = E$, then we set $\mathcal{M} := \{U, T, B\}$ and otherwise we set $\mathcal{M} := \{A, B, C\}$. Then for all distinct $X, Y \in \mathcal{M}$, we have that $X \cap Y = H$ and $\langle X, Y\rangle = F$.

Assume for a first contradiction that $H \cap P$ is not a normal subgroup of G. Since every subgroup of N is normal in N by (1), it follows that K does not induce power automorphism on P. Then Lemma 5.4 implies that K acts irreducibly on $\tilde{P} := [P, K]/\Phi([P, K])$. We apply Lemma 7.11 twice to find some $X \in \mathcal{M}$ and some $Y \in \mathcal{M} \setminus \{X\}$ such that X and Y act irreducibly on $\tilde{P}$. In particular $[P, K] = [H \cap P, Y] = [H \cap P, X] \leq X \cap Y = H$. It follows that $[P, K] \leq H \cap P \leq [P, K]C_P(K)$, which yields that $H \cap P$ is normalized by K and hence $H \cap P \trianglelefteq NK = G$. This is a contradiction. We deduce that $P \cap D \trianglelefteq G$ as stated, in particular $N \cap D \trianglelefteq G$ and also $N \cap E \trianglelefteq G$.

For the final statement in (2) we use that $G/(N \cap E)$ is not L_9-free. Then the minimality of G and Lemma 7.6 give that $N \cap E = 1$. □

(3) For every $q \in \pi(K)$ such that $1 \neq Q \in \mathrm{Syl}_q(D)$, one of the following holds:
$[N, \Omega_1(Q)] \leq D \leq A$ or K induces power automorphisms on $[N, \Omega_1(Q)]$ and $[N, \Omega_1(Q)] \cap A \neq 1$.
Moreover $E = 1$.

Proof. We adopt the same notation as in the previous step, which means that $H \in \{E, D\}$, $p \in \pi(N)$ and $P := O_p(N)$. If $H = E$, then $\mathcal{M} := \{U, T, B\}$, and otherwise $\mathcal{M} := \{A, B, C\}$. Whenever $X, Y \in \mathcal{M}$ are distinct, then $X \cap Y = H$ and $\langle X, Y\rangle = F$.

Let $q \in \pi(K) \cap \pi(H)$ and $Q \in \mathrm{Syl}_q(D)$. By conjugation we may suppose that $Q \leq K$. Then $\Omega_1(Q) = \Omega_1(Q_0)$ for some Sylow q-subgroup Q_0 of K by Lemmas 2.6 and 7.7 provides some $p \in \pi(N)$ such that $1 \neq [N, \Omega_1(Q)]$ is a p-group. Now Lemma 7.8 (e) states that $X = (X \cap P)N_X(\Omega_1(Q))$ for all subgroups $X \in \mathcal{M}$ (∗).

Let X and Y be distinct elements of $\mathcal{M}$ and assume for a contradiction that $X \cap P$ and $Y \cap P$ are subgroups of $C_P(K)$. Then

$$G \overset{(1)}{=} F = \langle X, Y\rangle \overset{(*)}{\leq} \langle C_P(K), N_X(\Omega_1(Q)), N_Y(\Omega_1(Q))\rangle \leq C_P(K)N_G(\Omega_1(Q)) = N_G(\Omega_1(Q)),$$

which contradicts (1).

For the remainder of this proof we let X and Y in $\mathcal{M}$ be such that their intersection with P is not contained in $C_P(K)$. We note that $C_P(K) = C_P(\Omega_1(Q))$ by Lemma 5.5. Then it follows that $1 \neq [P \cap X, \Omega_1(Q)] = [[P \cap X, \Omega_1(Q)], \Omega_1(Q)]$ and hence $[P, K] \cap X \nleq C_P(K)$. In a similar way we observe that $[P, K] \cap Y \nleq C_P(K)$.

If K acts irreducibly on $[P, K]/\Phi([P, K])$, then Lemma 7.11 yields that X or Y, say X, acts irreducibly on $[P, K]/\Phi([P, K])$. It follows that $[P, K] \leq X$ and $[P, K] \cap Y \leq X \cap Y = H$. Let $Z \in \mathcal{M} \setminus \{X, Y\}$. Then Lemma 7.11 yields that Z or Y, say Y, acts irreducibly on $[P, K]/\Phi([P, K])$ and contains $[P, K] \cap Y$. Since $[P, K] \cap Y \nleq C_P(K)$, it follows that $[P, K] \leq V \cap X = H$. By (1), this is only possible if $H = D$, in other words $\pi(K) \cap \pi(E) = \varnothing$. Then the fact that $E \cap N = 1$ (see (1) once more) forces $E = 1$.

The previous paragraph also gives that $[N, \Omega_1(Q)] = [P, \Omega_1(Q)] = [P, K] \leq D \leq A$, by Lemma 5.4, in the case where K acts irreducibly on $[P, K]/\Phi([P, K])$.

Otherwise Lemma 5.4 gives that K and hence $\Omega_1(Q)$ induce power automorphisms on P. Together with (1) this means that every subgroup of P is normal in G. Now, if $V, W \in \mathcal{M}$ are distinct, then

$$PN_G(P) = G = \langle V, W\rangle \overset{(*)}{\leq} (V \cap P)(W \cap P)N_G(\Omega_1(Q))$$

and it follows that $P = (V \cap P)(W \cap P)$. We deduce that $|P| = \frac{|V\cap P|\cdot|W\cap P|}{|H\cap P|}$ for all $W, V \in \mathcal{M}$. In particular $|W \cap P| = |X \cap P| \neq 1$ for all $W \in \mathcal{M}$. This implies all our claims about D, because $P = [P,K] = [P,\Omega_1(Q)] = [N,\Omega_1(Q)]$ by Lemma 5.4 and $A \in \mathcal{M}$.

If, still in the power automorphism case, we have that $H = E$, then we recall that $E \cap P = 1$ by (1). Therefore

$$|(U\cap P)|\cdot|(T\cap P)| = \frac{|(U\cap P)|\cdot|(T\cap P)|}{|E\cap P|} = |(U\cap P)(T\cap P)| = |P| = |(U\cap P)(A\cap P)|$$

$$= \frac{|U\cap P|\cdot|A\cap P|}{|P\cap E|} = |U\cap P|\cdot|A\cap P|.$$

This implies that $A\cap P = T\cap P \neq 1$. But in this case we may interchange T by S in $\mathcal{M}$. Since $1 \neq T\cap P = A\cap P = S\cap P$, we arrive at the contradiction that $1 \neq S\cap T\cap P = E\cap P \leq E\cap N = 1$ (by (1)). Thus, we have that $\pi(K)\cap\pi(E) = \varnothing$ in this case as well. Again it follows that $E = 1$. □

The remainder of the proof is dedicated to constructing a subgroup of G that violates Property ($\mathfrak{L}4$).
(4) If $p \in \pi(N)$, then $A\cap K$ does not centralize $O_p(N)$. In particular $A\cap K \neq 1$.

Proof. We assume for a contradiction that $A\cap K$ centralizes $P := O_p(N)$ for some $p \in \pi(N)$.

It follows from Lemma 7.3 that, for every subgroup X of $G = PO_{p'}(N)K$, there is some $x \in P$ such that $X \leq (X\cap P)O_{p'}(N)K^x$. Since $[P, A\cap K] = 1$, we further have THAT $X \leq (X\cap P)O_{p'}(N)K^y$ for all $X \leq A$ and $y \in P$ (∗).

Assume that $A\cap P = 1$. Then for both $X \in \{U,B\}$, we see that $P \leq G \overset{(1)}{=} F = \langle U,A\rangle \overset{(*)}{\leq} (P\cap X)O_{p'}(N)K^x$ and therefore $P = P\cap U = P\cap B \leq U\cap B = E$. But this contradicts (3).

Thus $A\cap P \neq 1$ and then (1), together with our assumption at the beginning of the proof, imply that every subgroup of $A\cap P$ is normal in A. Suppose that $X, Y \in \{S,T,D\}$ are distinct. We recall that $A\cap P \leq A = \langle X,Y\rangle \leq (X\cap P)(Y\cap P)O_{p'}(N)K$, and then it follows that $A\cap P = (X\cap P)(Y\cap P)$. Since $X\cap Y = E \overset{(2)}{=} 1$, we know more: $T\cap P \cong S\cap P \cong D\cap P$ and $|T\cap P|^2 = |A\cap P|$.

If K induces power automorphisms on P and if $X \in \{A,T,S\}$, then $(X\cap P)$ and $(U\cap P)$ are normal subgroups of G by (1). In addition there is some $u \in P$ such that $U \leq (U\cap P)O_{p'}(N)K^u$ and then $X \leq (X\cap P)O_{p'}(N)K^u$ by (∗). We apply Lemma 7.9 to see that $|P\cap A| = |P : P\cap U| = |P\cap S| = \sqrt{|P\cap A|}$. Now it follows that $P\cap A = 1$, which is a contradiction.

We conclude that K does not induce power automorphisms on P. Then Lemma 5.4 gives that K acts irreducibly on $[P,K]/\Phi([P,K])$. Since $P\cap D \trianglelefteq G$ by (2), we see that either $[P,K] \leq D$ or that $1 \neq P\cap D \leq C_P(K)$.

In the first case $T\cap P \cong D\cap P \geq [P,K]$ and $T\cap D = E \overset{(2)}{=} 1$. This implies, together with Lemma 1.1, that $C_P(K)$ has a subgroup isomorphic to $[P,K]$. We apply Lemma 5.4, in combination with Part (b) of Lemma 4.17, and we deduce that $[P,K]$ is cyclic of order 2 and that, therefore, K centralizes it. This is impossible.

It follows that the second case holds, i.e. $1 \neq P\cap D \leq C_P(K)$. Then $p = 2$ by Corollary 5.6 (a). If $[P,K]$ is not elementary abelian, then Part (b) of Lemmas 4.17 and 5.4 give that $[P,K] \cong Q_8$ and $P = [P,Q]\times I$, where I is a subgroup of P of order at most 2. We note that $T\cap P \cong D\cap P$ and $T\cap D = E \overset{(3)}{=} 1$, and then we conclude that $T\cap P$ and $D\cap P$ are cyclic of order 2. Consequently $A\cap P = (T\cap P)\cdot(D\cap P) = \Omega_1(P) \trianglelefteq G$. Now $1 \overset{(3)}{=} E = U\cap A \geq U\cap\Omega_1(P)$ and therefore $U \leq O_{\pi'}(N)K^u$. We arrive at a contradiction: $P \leq G = \langle U,A\rangle \leq (P\cap A)O_{\pi'}(N)K^u$, because $[P,K] \cong Q_8$ is not elementary abelian.

So we finally have that $[P,K]$ is elementary abelian. Then Lemma 4.17(b) gives that $C_P(K)$ is cyclic and Lemma 1.1 shows that $T\cap P \cong D\cap P \leq C_P(K)$.

Together with the fact that $T\cap D = E \overset{(3)}{=} 1$, we obtain that $D\cap P$ is cyclic of order 2. It follows that $P\cap D$, $T\cap P$ and $P\cap S$ have order 2 and hence $P\cap A$ is elementary abelian of order 4. Moreover

$A \cap [P, K]$ is cyclic of order 2, and therefore it equals one of the subgroups $T \cap P$, $S \cap P$ or $D \cap P$. The last case is not possible because $D \cap P \leq C_P(K)$. By symmetry between S and T in the lattice we may suppose that $T \cap P \leq [P, K]$. Recall that K acts irreducibly on $[P, K]$ while $A \cap K$ centralizes P. In particular (1) implies that A does not act irreducibly on $[P, K]$. Thus Lemma 7.11, together with the fact that $G \overset{(1)}{=} F = \langle A, U\rangle$, gives that U acts irreducibly on $[P, K]$. We also know that $T \cap U = E \overset{(3)}{=} 1$, and this implies that $[P, K] \nleq U$. Then Lemma 1.2 yields that $U \cap P \leq C_P(K)$. But we recall that $C_P(K)$ is cyclic, and its unique involution is contained in D. Then the fact that $U \cap D = E \overset{(3)}{=} 1$ implies that $U \cap P = 1$. Finally, we see that $G \overset{(1)}{=} F = \langle U, T\rangle \overset{(*)}{\leq} [P, K]O_{p'}(N)K^u$, which gives a contradiction. □

By conjugation and by Lemma 7.3 we may suppose that $A = (A \cap N)(A \cap K)$.
We set $\pi := \pi(K)^* \cap \pi(A)$ and we let L_1 be a Hall π-subgroup of $A \cap K$.
Then we let

$$\sigma := \{p \in \pi(N) \mid [O_p(N), Q] \neq 1 \text{ for some } Q \leq L_1, \text{ where } |Q| \in \pi\}.$$

(5) If $p \in \pi(N)$ and $[O_p(A), L_1] = 1$, then $O_p(N) \leq D$. Moreover $\pi \neq \varnothing \neq \sigma$.

Proof. First suppose that L_1 centralizes $P := O_p(N)$ for some $p \in \pi(N)$. Then (4) implies that $A \cap K \neq L_1$ and then there is a non-nilpotent batten V of K such that $Q := \mathcal{B}(V) \leq A$ and $[P, Q] \neq 1$. Now $|Q| = q$ is a prime and $[N, Q] = [P, Q]$ by Lemma 7.7. In addition we see, from Definition 4.16, that Q does not induce power automorphisms on P and that $P = [P, Q]$. If q divides $|D|$, then (3) implies that $P = [N, Q] \leq D$, as stated.

Now we suppose that q does not divide $|D|$. Let $R \leq K$ be such that $V = QR$. Up to conjugation we may suppose that $R \cap L_1$ is a Sylow subgroup of L_1. Then R does not centralize P by Definition 5.1 and hence $R \nleq L_1$. It follows that $R \nleq A$, from the definition of L_1. Since $Q \cdot \Phi(R) = Q \cdot Z(V)$ is nilpotent by Lemma 2.5, we deduce that A has a normal q-complement. Moreover $A \in \mathfrak{L}$, by Lemma 7.4, whence we may apply Lemma 7.10 to A. Then we see that S and T have orders divisible by q, because $q \notin \pi(D)$. Let $t, s \in [P, Q]$ be such that $Q^s \leq S$ and $Q^t \leq T$. The irreducible action of Q on P, together with the fact that $P \cap T \cap S \leq E \overset{(3)}{=} 1$, implies that $P \cap T = 1$ or $P \cap S = 1$. Without loss $P \cap T = 1$. Then $T \leq N_G(Q^t)$ by Lemma 7.8 (e). Assume that A normalizes Q^t. Then $Q^s \leq N_G(Q^t)$, which implies that $[ts^{-1}, Q]Q^s = \langle Q^s, Q^t\rangle \leq N_G(Q^t)$ by Lemma 4.1.1 (b) of [7]. Then the irreducible action of Q^t on P forces $t = s$, contradicting the fact that $T \cap S = E \overset{(3)}{=} 1$.

Thus A does not normalize Q^t and $\langle T, D\rangle = A \nleq N_G(Q^t)$, which yields that $D \nleq N_G(Q^t) = (O_{p'}(N)K) \times C_P(K) = O_{p'}(N)K$ by Part (a) of Lemma 7.8. It follows that $D \cap P \neq 1$ and therefore $P \leq D$, because $P \cap D \trianglelefteq G$ by (2) and K acts irreducibly on P.

We turn to the second statement and assume for a contradiction that $\pi = \pi(A) \cap \pi(K) = \varnothing$. Then $A \cap K = 1$, contrary to (4). Hence if $\pi = \pi(A) \cap \pi(K)^* = \varnothing$, then we can draw two conclusions: First $L_1 = 1$ and the statement we just proved gives that $N \leq D$. Second, there must be a prime in $\pi(K) \setminus \pi(K)^*$ dividing $|A|$. By definition of $\pi(K)^*$, such a prime is $|\mathcal{B}(V)|$ for some non-nilpotent batten V of K. We choose such a non-nilpotent batten V and set $Q := \mathcal{B}(V)$. Then there are some prime r and an r-subgroup $R \leq K$ such that $QR = V$, and $r \in \pi(K)^*$ because of the structure of non-nilpotent battens (Definition 2.1). In particular $r \nmid |A|$ by assumption and Lemma 7.10 yields that B and U contain a conjugate of R. We recall that $N \leq D \leq B$ by the first consequence of our assumption and because of the structure of the lattice L_9. Then Lemma 7.8(c) gives that $\Omega_1(R)^G \subseteq B$. Thus $B \cap U = E \overset{(3)}{=} 1$, which is a contradiction. This proves that $\pi \neq \varnothing$.

If $\sigma = \varnothing$, then for all $p \in \pi(N)$, all $q \in \pi$ and all q-subgroups Q of L_1, we have that $[O_p(N), Q] = 1$. Then $[N, L_1] = 1$ by definition of L_1, and $L_1 \neq 1$ because $\pi \neq \varnothing$. But then $1 \neq L_1 \leq C_K(N)$, contrary to (1). □

Next we set $H := O_\sigma(N)$ and we prove that H is a candidate for the desired properties in ($\mathfrak{L}$4).

(6) The non-trivial Sylow subgroups of H are elementary abelian (in particular N is abelian), but not cyclic, and $K \leq \mathrm{Pot}_K(H)$.

Proof. By definition $H \leq N$ is nilpotent. If, for all $p \in \sigma$, the group K does not act irreducibly on $[O_p(N), K]/\Phi([O_p(N), K])$, then Lemma 5.4 gives that $K \leq \mathrm{Pot}_K(H)$. In particular $O_p(N)$ is not cyclic. Moreover, Corollary 5.6 yields that $O_p(N) = [O_p(N), K]$ is elementary abelian. Therefore, our claim is satisfied in this case.

Let us assume for a contradiction that there is some $p \in \sigma$ such that K acts irreducibly on the group $[O_p(N), K]/\Phi([O_p(N), K])$. We set $P := O_p(N)$ and we choose $Q \leq L_1$ of order $q \in \pi$ such that $[P, Q] \neq 1$, by the definition of σ.

Case 1: $[P, Q] \leq A$.

Then Lemma 7.8(c) implies that $Q^g \leq A$ for every $g \in G$. We recall that $U \cap A = E \overset{(3)}{=} 1$, and then it follows first that $q \notin \pi(U)$ and then that $q \mid |B|$, by Lemma 7.10. Here we use that $G = F = \langle B, U\rangle$ by (1). Thus we find some $g \in G$ such that $Q^g \leq D$, because $D = A \cap B$. We apply (3) to observe that $[P, Q] = [N, Q] \leq D$ and then D contains every conjugate of Q by Lemma 7.8(c). Recall that $q \notin \pi(U)$, which implies that $q \in \pi(T)$ by Lemma 7.10. Again we use that $G = F = \langle T, U\rangle$. But this contradicts the fact that $T \cap D = E = 1$ by (3).

Case 2: $[P, Q] \not\leq A$.

Then $[P, Q] \not\leq D$, in particular $P \not\leq D$, and $P \cap D \trianglelefteq G$ by (2). This means that K stabilizes the subgroup $[P \cap D, K]/$ of $[P, K]/\Phi([P, K])$, while acting irreducibly. This forces $[P \cap D, K] \leq \Phi([P, K])$, and together with coprime action (Lemma 1.1) we see that K centralizes $P \cap D$.

By Lemma 7.3 we know that $A \in \mathfrak{L}$, with type $(A \cap N, A \cap K)$. Additionally, since $q \in \pi(K)^*$, the group K has a normal q-complement. Then $A \cap K$ also has a normal q-complement. We may apply Lemma 7.10 to $A = \langle T, S\rangle = \langle T, D\rangle = \langle S, D\rangle$. It yields that at least two of the groups D, T, S have a subgroup of order $|Q|$. As $|Q| \nmid |E|$ by (2), there is some $g \in G$ such that $Q^g \neq Q$ and $Q^g \leq A$. Then Part (e) of Lemma 7.8 implies that $P \cap A \not\leq C_P(Q) = C_P(K)$ by Lemma 5.5. In the present case we have that $[P, A] \not\leq A$, and then Lemma 1.2 gives that A does not act irreducibly on $[P, K]/\Phi([P, K])$. Thus $A \cap K$ induces power automorphism on P by Lemma 5.4, and these automorphisms are not trivial because $Q \leq A \cap K$. Corollary 5.6 (c) implies that $P = [P, A \cap K] \overset{5.4}{=} [P, K]$ is elementary abelian and in particular that $D \cap P \leq C_P(K) \overset{5.5}{=} C_P(A \cap K) = 1$. Hence there is some $d \in [P, K]$ such that $D = N_D(Q^d)$, by Lemma 7.8(b). In addition we see from Lemma 7.11 that B and C act irreducibly on P. If it was true that $P \leq B$, then it would follow that $1 \neq P \cap A \leq P \cap B \cap A \leq P \cap D \leq C_P(K) = 1$, which is a contradiction. We conclude that $P \not\leq B$ and hence $P \cap B = 1$ because of the irreducible action of B on P, and similarly $P \cap C = 1$.

Then Lemma 7.8(b) provides $b, c \in P$ such that $B = N_B(Q^b)$ and $C = N_C(Q^c)$. If $Q^b = Q^c$, then $G = F = \langle B, C\rangle \leq N_G(Q^b)$, which is false.

Consequently $Q^b \neq Q^c$, and we can use Lemma 7.8(a), Dedekind's modular law and Lemma 1.4(c). Together this shows that

$$D \leq N_B(Q^b) \cap N_C(Q^c) = O_{p'}(N)K^b \cap O_{p'}(N)K^c = O_{p'}(N)(K^b \cap O_{p'}(N)K^c)$$

$$\leq O_{p'}(N)C_K(bc^{-1}) \leq C_G(bc^{-1}),$$

because N is nilpotent and because $\langle (bc^{-1})^K\rangle \cap O_{p'}(N) \leq P \cap O_{p'}(N) = 1$.

We recall that $P = [P, K]$ is abelian, hence it is contained in $Z(N)$, and then it follows that D centralizes $1 \neq bc^{-1} \in [P, K]$. Here we also use Lemma 5.4, i.e. that D centralizes P. We conclude that $D = N_D(Q^h)$ for all $h \in P$. Again Lemma 7.8(b) provides $t, s \in P$ such that $T = (T \cap P)N_T(Q^t)$ and $S = (T \cap S)N_C(Q^s)$. Let $X \in \{T, S\}$. Then $P \cap A \leq A = \langle D, X\rangle \leq (X \cap P)N_G(Q^x)$, which implies that $P \cap X = P \cap A$. But now $P \cap A \leq P \cap S \cap T = P \cap E = 1$, by (3), which gives a final contradiction. □

(7) For all $p \in \sigma$ we have that $O_p(N) \cap A \neq 1$.

Proof. We set $P := O_p(N)$ for some $p \in \sigma$. Then there is some $Q \leq L_1$ of order $q \in \pi$ such that $[P, Q] \neq 1$, by the definition of σ. Then $q \in \pi(K)^*$ and we see that K as well as $K \cap A$ have a normal q-complement. Since $A \in \mathfrak{L}$ by Lemma 7.4 ,we may apply Lemma 7.10. We notice that $A = \langle D, T\rangle = \langle D, S\rangle$, and then it follows that $q \in \pi(D)$ or that $q \in \pi(T) \cap \pi(S)$. In the first case (3) implies our assertion. In the second case we use Lemma 7.8(c). It gives that $Q^s \leq S$ and $Q^t \leq T$ for some $t, s \in [P, K]$. We deduce that $[t^{-1}s, Q] \leq \langle Q^s, Q^t\rangle \leq A$ by Lemma 4.1.1 of [7]. In conclusion, our assertion is true or $t^{-1}s$ centralizes Q. But in the second case, we see that $Q^t = Q^s \leq T \cap S = E \overset{(3)}{=} 1$, and this gives a contradiction. □

(8) For all $q \in \pi$ we have that q divides $|S|$, $|T|$, and $|B|$.
Furthermore, if $Q \leq L_1$ and $|Q| = q$, then $[N, Q] \cap T = [N, Q] \cap S = 1$, but $[N, Q] \cap U \neq 1$.

Proof. Suppose that $Q \leq L_1$ has order q. Then $[N, Q]$ is a p-group by Lemma 7.7, for some prime $p \in \sigma$. We set $P = O_p(N)$. Then for every $X \in \{T, S\}$ there is some $x \in P$ such that $X = (X \cap P)N_X(Q^x)$, by Lemma 7.8(b). Using (6) we see that K induces power automorphisms on P. Thus $P = [P, K]$ is elementary abelian by Corollary 5.6(c), and then Lemma 7.9 is applicable.

Assume for a contradiction that $P \gneq (P \cap A)(P \cap U)$. Then, for all $X \in \{T, S, A\}$, we have that

$$P \cap \langle X, U\rangle = P \cap F \overset{(1)}{=} P > (P \cap A)(P \cap U) \geq (P \cap X)(P \cap U).$$

Hence Lemma 7.9 yields that $|P| = |(P \cap X)(P \cap U)|p \overset{(2)}{=} |(P \cap X)||(P \cap U)|p$ and, for all $X \in \{T, S, A\}$, we see that $p \cdot |P \cap X| = |P : P \cap U|$. In particular, we have that $|P \cap A| = |P \cap S| = |P \cap T|$. Therefore, the fact that $T, S \leq A$ implies that $P \cap A = P \cap T = P \cap S \leq P \cap (T \cap S) = P \cap E \overset{(3)}{=} 1$. This contradicts (7).

We conclude that $P = (P \cap A)(P \cap U)$ and hence Lemma 7.9 provides some element $g \in P$ such that, for both $X \in \{A, U\}$, it is true that $X \leq (X \cap P)O_{p'}(N)K^g = (X \cap P)N_G(Q)^g$.

Assume for a contradiction that $q \in \pi(U)$. Then Part (e) of Lemma 7.8 yields that $Q^g \in A \cap U = E \overset{(3)}{=} 1$. This is impossible. Thus $q \notin \pi(U)$ and we deduce that $q \in \pi(X)$ for all $X \in \{T, S, B\}$. Here we use Lemma 7.10 and the fact that $G \overset{(1)}{=} F = \langle U, X\rangle$. Next we apply Lemma 7.9, which gives that $(P \cap A) \neq (P \cap S)(P \cap T)$. Then the same lemma yields, for both combinations of $\{X, Y\} \in \{T, S\}$, that p divides

$$|P \cap Y||P \cap X||P \cap U| = |P \cap A||P \cap U| = |P| \leq |P \cap X||P \cap U| \cdot p.$$

It follows that $p \cdot |P \cap Y| \leq p$ and then that $S \cap P = T \cap P = 1$. Finally (6) and Lemma 7.9 yield that $p^2 \leq |P| \leq |P \cap T||P \cap U| \cdot p = |P \cap U| \cdot p$. We conclude that $P \cap U \neq 1$. □

(9) For every $p \in \sigma$ there is some $r \in \pi(K)^*$ such that a Sylow r-subgroup of U does not centralize $O_p(N)$.

Proof. We set $P := O_p(N)$. Since $p \in \sigma$, there is some $Q \leq L_1$ such that $|Q| = q \in \pi$. In addition (6) implies that P is elementary abelian, and then $P \leq Z(N)$ because N is nilpotent. By Lemma 7.8(b) and (8) there is some $s \in P$ such that $S = N_S(Q^s)$.

Assume for a contradiction that U centralizes P. Then $U = U^s = (U \cap P)N_U(Q^s)$ by Lemma 7.8(d) and hence $G \overset{(1)}{=} F = \langle U, S\rangle \leq (U \cap P)N_G(Q^s)$ implies that $U \cap P = P$. With (7) we obtain the contradiction that $1 \neq P \cap A \leq U \cap A = E \overset{(3)}{=} 1$.

It follows that U does not centralize P. We recall that $P \leq Z(N)$ and then we obtain a prime $r \in \pi(K)$ such that a Sylow r-subgroup of U does not centralize P. Since K induces power automorphisms on P

by (6), Definition 5.1 gives, for every non-nilpotent batten V of K, that $\mathcal{B}(V)$ centralizes P. This implies that $r \in \pi(K)^*$. □

We are now able to define L and J. Let $u \in N$ be such that $U = (U \cap N)(U \cap K^u)$.

We set $\tilde{\pi} := \{r \in \pi(K)^* \mid [H, R] \neq 1 \text{ for some } R \in \mathrm{Syl}_r(U)\}$ and we let L_2 be a Hall $\tilde{\pi}$-subgroup of $(U^{u^{-1}} \cap K)$. Then $L_2 \leq K$ and $L = \langle L_1, L_2 \rangle$ is a subgroup of K. In addition (5) and (9) show that $\tilde{\pi} \neq \varnothing$.

Next we set $\rho := \{p \in \pi(N) \mid [O_p(N), R] \neq 1 \text{ for some } R \leq L_2 \text{ with } |R| \in \tilde{\pi}\}$ and $J := [O_\rho(N), K]$. Then $\rho \neq \varnothing$ by Lemma 7.7, because $\tilde{\pi} \neq \varnothing$, and hence $J \neq 1$. Finally, we note that J is L-invariant by construction.

(10) $\sigma \cap \rho = \varnothing$, i.e. $|H|$ and $|J|$ are coprime.

Proof. We assume for a contradiction that the prime p divides $|H|$ and $|J|$. Then by definition there are $q \in \pi$ and $r \in \tilde{\pi}$ and subgroups $Q \leq L_1$ and $R \leq L_2$ such that the following hold:

$|Q| = q$, $|R| = r$, $1 \neq [N, Q] \leq O_p(N) =: P$ and $1 \neq [N, R] \leq P$.

In particular (6) shows that K induces power automorphisms on P. It follows from Corollary 5.6 (c) and Lemma 5.4 that $P = [P, K] = [P, Q] = [N, Q] = [N, R]$, and then (8) shows that $P \cap S = 1$. Moreover $P \cap A \neq 1$ by (7). We apply Lemma 7.9 to obtain some $i, j \in \{1, 0\}$ such that

$$p^i |P \cap U| = |(P \cap U)(P \cap S)| p^i = |P| = |(P \cap U)(P \cap A)| p^j \overset{(2)}{=} |(P \cap U)||(P \cap A)| p^j.$$

This implies that $|P \cap A| p^j = p^i$, and we obtain that $i = 1$ and $j = 0$.

In particular we see that $P = (U \cap P)(A \cap P)$. Then Lemma 7.9 and the fact that $A \cap U = E \overset{(3)}{=} 1$ give some element $g \in P$ such that $X \leq (P \cap X) K^g = (P \cap X) N_G(R)^g$, where $X \in \{A, U\}$. Assume for a contradiction that $r \in \pi(A)$. Then 7.8(d) forces $R^g \leq U \cap A = E \overset{(3)}{=} 1$. This is impossible, hence $r \notin \pi(A)$ and we apply Lemma 7.10 to $G \overset{(1)}{=} F = \langle A, B \rangle$. Then it follows that $r \in \pi(B)$.

Moreover $q \in \pi(B) \cap \pi(T)$ by (8). Since $B \cap T = E \overset{(3)}{=} 1$, this is not possible, hence Lemma 7.9 and (8) give that $|P| = |P \cap B| \cdot |P \cap T| \cdot p = |P \cap B| \cdot p$. On the other hand we have that $r \in \pi(B) \cap \pi(U)$, which is also impossible because $B \cap U = E \overset{(3)}{=} 1$. Now Lemma 7.9 implies that $|P \cap U| \cdot |P \cap B| \cdot p = |P| = |P \cap B| \cdot p$. In particular we see that $P \cap U = 1$, and this contradicts (8). □

We summarize:

The definitions of σ and H imply that $H \leq N$, and (6) shows that the non-trivial Sylow subgroups of H are not cyclic. In addition $L \leq K$ induces power automorphisms on H.

From (10) we deduce that $\pi \cap \tilde{\pi} = \varnothing$. A Hall $\pi(K)^*$-subgroup of K is nilpotent by Lemma 2.7, which means that L is nilpotent. In particular we see that $L = L_1 \times L_2$. Let $r \in \pi(L) = \pi \cup \tilde{\pi}$ and $R = O_r(L)$. If $r \in \pi$, then $[H, R] \geq [[N, \Omega_1(R)], R] \neq 1$, and if $r \in \tilde{p}$, then $[H, R] \neq 1$ by (9). These arguments show that $\pi(L) = \pi(L/C_L(H))$ $(*)$.

We assume for a contradiction that L is not cyclic. Then, since L is nilpotent and a batten group by Lemma 2.7, we deduce that $O_2(L) \cong Q_8$. Definition 4.8 implies that, for all $p \in \pi(N)$, the group $O_2(L)$ does not induce non-trivial power automorphisms on $O_p(N)$. In particular $O_2(L)$ centralizes H, which contradicts $(*)$. Thus L is cyclic.

Furthermore, we already saw that J is L-invariant and that $1 \neq J$. Then (10) gives that $(|H|, |J|) = 1$, and since N is nilpotent, this forces $[H, J] = 1$.

Assume for a contradiction that J is not abelian. Then Lemma 4.17(b) and Lemma 5.4 show that $O_2(J) = [O_2(N), K] \cong Q_8$ and therefore $2 \in \rho$. We let $r \in \tilde{\pi}$ be such that a Sylow r-subgroup R of K acts faithfully on $O_2(J)$. Then we must have that $|R| = 3$ because $O_2(J) \cong Q_8$. Moreover Lemma 7.8(a) yields that R centralizes $O_{2'}(N) \geq H$. This contradicts $(*)$.

Hence J is abelian. For every $q \in \pi$ and every subgroup $Q \leq L_1$ of order q, the definition of σ and Lemma 7.7 provide some $p \in \sigma$ such that $[N, Q] \leq O_p(N) = P$. Then we see, using (6) and

Corollary 5.6, that $C_P(Q) = 1$. Furthermore (10) yields that Q centralizes $J = O_\rho(N)$, and then it follows that $1 = C_P(Q) \geq C_P(C_L(J))$. Since $H = O_\sigma(N)$ is abelian, we conclude that $C_H(C_L(J)) = 1$.

The previous argument also yields that $\{q \in \pi(L) \mid C_{O_q(L)}(H) < C_{O_q(L)}(J)\} \supseteq \pi$. Let $r \in \tilde{\pi}$ and suppose that $R \leq L$ has order r. We recall the definition of ρ and apply Lemma 7.7: Then we see that $1 \neq [N, R] \leq O_\rho(N) = J$. Thus $r \notin \{q \in \pi(L) \mid C_{O_q(L)}(H) < C_{O_q(L)}(J)\}$ and it follows that $\{q \in \pi(L) \mid C_{O_q(L)}(H) < C_{O_q(L)}(J)\} = \pi$.

Since $G \in \mathfrak{L}$, we can use Property ($\mathfrak{L}$4), which gives some $g \in (HJ)^\#$ that centralizes $O_\pi(L) =: L_1$ or $O_{\pi'}(L) =: L_2$. Let $i \in \{1, 2\}$ be such that $[L_i, g] = 1$. We may suppose that g has prime order p. Then $p \in \sigma \cup \rho$ and hence there is a subgroup $Q \leq L$ of prime order such that $[N, Q] \leq O_p(N) =: P$. Lemma 5.5 gives that $g \in C_P(L_i) = C_P(K)$ or that L_i centralizes P.

In the first case Corollary 5.6 yields that $p = 2$, and then K does not induce power automorphisms on P. Using (6) we deduce that $p \in \rho$ and $g \in O_p(J) \cap C_P(K) = [P, K] \cap C_P(K) \cap J$. Since J is abelian, we obtain a contradiction in this case.

It follows that the second case above holds, i.e. $[P, L_i] = 1$. This means that $i = 1$ if $p \in \rho$ and $i = 2$ if $p \in \sigma$. In addition we see, from (9), that $i \neq 2$. We conclude that $L_i = L_1$, $p \in \rho$ and $q \notin \pi$. In particular $q \notin \pi(A)$. Since $G \overset{(1)}{=} F = \langle A, U\rangle = \langle A, B\rangle$, Lemma 7.10 implies that B and U contain a conjugate of Q. In addition (5) shows that $P \leq D \leq B$ and therefore Lemma 7.8(c) gives that $Q^G \subseteq B$. Finally, we obtain a contradiction, because $B \cap U = E = 1$ by (3). This concludes the proof. □

Main Theorem. *A finite group is in $\mathfrak{L}$ if and only if it is L_9-free.*

Proof. Let G be a finite group. If G is L_9-free, then Theorem 6.5 shows that $G \in \mathfrak{L}$.

Conversely, if $G \in \mathfrak{L}$, then G is L_9-free by Theorem 8.1. □

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