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# Local boundedness of minimizers under unbalanced Orlicz growth conditions

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**Abstract**

Local minimizers of integral functionals of the calculus of variations are analyzed under growth conditions dictated by different lower and upper bounds for the integrand. Growths of non-necessarily power type are allowed. The local boundedness of the relevant minimizers is established under a suitable balance between the lower and the upper bounds. Classical minimizers, as well as quasi-minimizers are included in our discussion. Functionals subject to so-called $p, q$-growth conditions are embraced as special cases and the corresponding sharp results available in the literature are recovered.


*MSC:* 49N60

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## 1. Introduction

We are concerned with the local boundedness of local minimizers, or quasi-minimizers, of integral functionals of the form

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$$\mathcal{F}(u,\Omega)=\int_{\Omega} f(x,u,\nabla u)\,dx, \tag{1.1}$$

where $\Omega$ is an open set in $\mathbb{R}^n$, with $n\geq 2$, and $f:\Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ is a Carathéodory function subject to proper structure and growth conditions. Besides its own interest, local boundedness is needed to ensure certain higher regularity properties of minimizers. Interestingly, some regularity results for minimizers admit variants that require weaker hypotheses under the a priori assumption of their local boundedness.

Local boundedness of local minimizers of the functional $\mathcal{F}$ is classically guaranteed if $f(x,t,\xi)$ is subject to lower and upper bounds in terms of positive multiples of $|\xi|^p$, for some $p\geq 1$. This result can be traced back to the paper [36], which, in turn, hinges upon methods introduced by De Giorgi in his regularity theory for linear elliptic equations with merely measurable coefficients.

The study of functionals built on integrands $f(x,t,\xi)$ bounded from below and above by different powers $|\xi|^p$ and $|\xi|^q$, called with $p,q$-growth in the literature, was initiated some fifty years ago. A regularity theory for minimizers under assumptions of this kind calls for additional structure conditions on $f$, including convexity in the gradient variable. These conditions are needed in the derivation of a Caccioppoli-type inequality. In its proof, the different lower and upper bounds on $f$ force an absorption argument for the gradient terms to be performed via direct use of the functional $\mathcal{F}(u,\Omega)$, and this is only possible if $f$ enjoys extra properties.

As shown in various papers starting from the nineties of the last century, local minimizers of functional with $p,q$-growth are locally bounded under diverse structure conditions, provided that the difference between $q$ and $p$ is not too large, depending on the dimension $n$. This issue was addressed in [49,51] and, more recently, in [25,26]. Related questions are considered in [8,34,53] in connection with anisotropic growth conditions. By contrast, counterexamples show that unbounded minimizers may exist if the exponents $p$ and $q$ are too far apart [35,43,45,46]. The gap between the assumptions on $p$ and $q$ in these examples and in the regularity results available until recently has been filled in the paper [40], where the local boundedness of minimizers is established for the full range of exponents $p$ and $q$ excluded from the relevant counterexamples. A generalization of the techniques from [40] has been applied in [31] to extend the boundedness result to obstacle problems.

In the present paper, the conventional realm of polynomial growths is abandoned and the question of local boundedness of local minimizers, and quasi-minimizers, is addressed under bounds on $f$ of Orlicz type. More specifically, the growth of $f$ is assumed to be governed by Young functions, namely nonnegative convex functions vanishing at 0. The local boundedness of minimizers in the case when lower and upper bounds on $f$ are imposed in terms of the same Young function follows via a result from [20], which also deals with anisotropic Orlicz growths. The same problem for solutions to elliptic equations is treated in [42]. The local Hölder regularity of bounded local solutions to elliptic equations subject to Orlicz growth conditions is treated in [44].

Our focus here is instead on the situation when different Young functions $A(|\xi|)$ and $B(|\xi|)$ bound $f(x,t,\xi)$ from below and above. Functionals with $p,q$-growth are included as a special instance. A sharp balance condition between the Young functions $A$ and $B$ is exhibited for any local minimizer of the functional $\mathcal{F}$ to be locally bounded. Bounds on $f(x,t,\xi)$ depending on a function $E(|t|)$ are also included in our discussion. Let us mention that results in the same spirit can be found in the paper [27], where, however, more restrictive non-sharp assumptions are imposed.

The global boundedness of global minimizers of functionals and of solutions to boundary value problems for elliptic equations subject to Orlicz growth conditions has also been examined in the literature and is the subject e.g. of [1,2,19,54,55]. Note that, unlike those concerning the local boundedness of local minimizers and local solutions to elliptic equations, global boundedness results in the presence of prescribed boundary conditions just require lower bounds in the gradient variable for integrands of functionals or equation coefficients. Therefore, the question of imposing different lower and upper bounds does not arise with this regard.

Beyond boundedness, several further aspects of the regularity theory of solutions to variational problems and associated Euler equations, under unbalanced lower and upper bounds, have been investigated. The early influential papers [46,47] have been followed by various contributions on this topic, a very partial list of which includes [3,4,6,7,9–12,16,17,24,28,30,32,33,38,48]. A survey of investigations around this area can be found in [50]. In particular, results from [3,9,16,29,39] demonstrate the critical role of local boundedness for higher regularity of local minimizers, which we alluded to above.

## 2. Main result

We begin by enucleating a basic case of our result for integrands in (1.1) which do not depend on $u$. Namely, we consider functionals of the form

$$\mathcal{F}(u,\Omega)=\int_{\Omega} f(x,\nabla u)\,dx, \tag{2.1}$$

where

$$f:\Omega\times\mathbb{R}^n\to\mathbb{R}.$$

A standard structure assumption to be fulfilled by $f$ is that

$$\text{the function}\quad \mathbb{R}^n\ni\xi\mapsto f(x,\xi)\quad \text{is convex for a.e. } x\in\Omega. \tag{2.2}$$

Next, an $A, B$-growth condition on $f$ is imposed, in the sense that

$$A(|\xi|)-L\le f(x,\xi)\le B(|\xi|)+L\quad \text{for a.e. } x\in\Omega \text{ and every } \xi\in\mathbb{R}^n, \tag{2.3}$$

where $L$ is a positive constant, $A$ is a Young function and $B$ is a Young function satisfying the $\Delta_2$-condition near infinity. By contrast, the latter condition is not required on the lower bound $A$.

The function $A$ dictates the natural functional framework for the trial functions $u$ in the minimization problem for $\mathcal{F}$. It is provided by the Orlicz-Sobolev class $V^1_{\rm loc}K^A(\Omega)$ of those weakly differentiable functions on $\Omega$ such that

$$\int_{\Omega'} A(|\nabla u|)\,dx<\infty$$

for every open set $\Omega'\Subset\Omega$.

Besides standard local minimizers, we can as well deal with so-called quasi-minimizers, via the very same approach. A function $u \in V^1_{\mathrm{loc}}K^A(\Omega)$ is said to be a local quasi-minimizer of $\mathcal{F}$ if

$$\mathcal{F}(u, \Omega') < \infty$$

for every open set $\Omega' \Subset \Omega$, and there exists a constant $Q \geq 1$ such that

$$\mathcal{F}(u, \operatorname{supp}\varphi) \leq Q\mathcal{F}(u+\varphi, \operatorname{supp}\varphi) \tag{2.4}$$

for every $\varphi \in V^1_{\mathrm{loc}}K^A(\Omega)$ such that $\operatorname{supp}\varphi \Subset \Omega$. Plainly, $u$ is a standard local minimizer of $\mathcal{F}$ provided that the inequality (2.4) holds with $Q = 1$.

Throughout the paper, we shall assume that

$$\int^{\infty} \left(\frac{t}{A(t)}\right)^{\frac{1}{n-1}} dt = \infty. \tag{2.5}$$

Indeed, if $A$ grows so fast near infinity that

$$\int^{\infty} \left(\frac{t}{A(t)}\right)^{\frac{1}{n-1}} dt < \infty, \tag{2.6}$$

then every function $u \in V^1_{\mathrm{loc}}K^A(\Omega)$ is automatically locally bounded, irrespective of whether it minimizes $\mathcal{F}$ or not. This is due to the inclusion

$$V^1_{\mathrm{loc}}K^A(\Omega) \subset L^\infty_{\mathrm{loc}}(\Omega), \tag{2.7}$$

which holds as a consequence of a Sobolev-Poincaré inequality in Orlicz spaces.

Heuristically speaking, our result ensures that any local quasi-minimizer of $\mathcal{F}$ as in (2.1) is locally bounded, provided that the function $B$ does not grow too quickly near infinity compared to $A$. The maximal admissible growth of $B$ is described through the sharp Sobolev conjugate $A_{n-1}$ of $A$ in dimension $n-1$, whose definition is recalled in the next section. More precisely, if

$$n \geq 3 \quad \text{and} \quad \int^{\infty} \left(\frac{t}{A(t)}\right)^{\frac{1}{n-2}} dt = \infty, \tag{2.8}$$

then $B$ has to be dominated by $A_{n-1}$ near infinity, in the sense that

$$B(t) \leq A_{n-1}(Lt) \qquad \text{for } t \geq t_0, \tag{2.9}$$

for some positive constants $L$ and $t_0$.

On the other hand, in the regime complementary to (2.8), namely in either of the following cases

$$\begin{cases} n = 2 \\ n \geq 3 \quad \text{and} \quad \displaystyle\int^{\infty} \left(\frac{t}{A(t)}\right)^{\frac{1}{n-2}} dt < \infty, \end{cases} \tag{2.10}$$

no additional hypothesis besides the $\Delta_2$-condition near infinity is needed on $B$. Notice that, by an Orlicz-Poincaré-Sobolev inequality on $\mathbb{S}^{n-1}$, both options in (2.10) entail that $V^1_{\rm loc}K^A(\mathbb{S}^{n-1}) \subset L^\infty_{\rm loc}(\mathbb{S}^{n-1})$, and $V^1K^A(\mathbb{S}^{n-1}) \subset L^\infty(\mathbb{S}^{n-1})$.

Altogether, our boundedness result for functionals of the form (2.1) reads as follows.

**Theorem 2.1.** *Let $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$ be a Carathéodory function satisfying the structure assumption (2.2). Suppose that the growth condition (2.3) holds for some Young functions $A$ and $B$, such that $B \in \Delta_2$ near infinity. Assume that either the condition (2.10) is in force, or the condition (2.8) is in force and $B$ fulfills the estimate (2.9). Then any local quasi-minimizer of the functional $\mathcal{F}$ in (2.1) is locally bounded in $\Omega$.*

Assume now that $\mathcal{F}$ has the general form (1.1), and hence

$$f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}.$$

Plain convexity in the gradient variable is no longer sufficient, as a structure assumption, for a local boundedness result to hold. One admissible strengthening consists of coupling it with a kind of almost monotonicity condition in the $u$ variable. Precisely, one can suppose that

$$\begin{cases} \text{the function} \quad \mathbb{R}^n \ni \xi \mapsto f(x,t,\xi) \quad \text{is convex for a.e. } x \in \Omega \text{ and every } t \in \mathbb{R}, \\ f(x,t,\xi) \le Lf(x,s,\xi) + E(|s|) + L \quad \text{if } |t| \le |s|, \text{ for a.e. } x \in \Omega \text{ and every } \xi \in \mathbb{R}^n, \end{cases} \tag{2.11}$$

where $L$ is a positive constant and $E : [0,\infty) \to [0,\infty)$ is a non-decreasing function fulfilling the $\Delta_2$-condition near infinity.

An alternate condition which still works is the joint convexity of $f$ in the couple $(t,\xi)$, in the sense that

$$\text{the function} \quad \mathbb{R} \times \mathbb{R}^n \ni (t,\xi) \mapsto f(x,t,\xi) \quad \text{is convex for a.e. } x \in \Omega. \tag{2.12}$$

The growth of $f$ is governed by the following bounds:

$$\begin{aligned} A(|\xi|) - E(|t|) - L \le f(x,t,\xi) \le B(|\xi|) + E(|t|) + L \\ \text{for a.e. } x \in \Omega \text{ and every } t \in \mathbb{R} \text{ and } \xi \in \mathbb{R}^n, \end{aligned} \tag{2.13}$$

where $A$ is a Young function, $B$ is a Young function satisfying the $\Delta_2$-condition near infinity, and $E$ is the same function as in (2.11), if this assumption is in force.

The appropriate function space for trial functions in the definition of quasi-minimizer of the functional $\mathcal{F}$ is still $V^1_{\rm loc}K^A(\Omega)$, and the definition given in the special case (2.1) carries over to the present general framework.

The bound to be imposed on the function $B$ is the same as in the $u$-free case described above. On the other hand, the admissible growth of the function $E$ is dictated by the Sobolev conjugate $A_n$ of $A$ in dimension $n$. Specifically, we require that

$$E(t) \le A_n(Lt) \qquad \text{for } t \ge t_0, \tag{2.14}$$

for some positive constants $L$ and $t_0$.

Our comprehensive result then takes the following form.

**Theorem 2.2.** *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a Carathéodory function satisfying either the structure assumption (2.11) or (2.12). Suppose that the growth condition (2.13) holds for some Young functions $A$ and $B$ and a non-decreasing function $E$, such that $B, E \in \Delta_2$ near infinity. Assume that either the condition (2.10) is in force, or the condition (2.8) is in force and $B$ fulfills estimate (2.9). Moreover, assume that $E$ fulfills the estimate (2.14). Then any local quasi-minimizer of the functional $\mathcal{F}$ in (1.1) is locally bounded in $\Omega$.*

Our approach to Theorems 2.1 and 2.2 follows along the lines of De Giorgi's regularity result for linear equations with merely measurable coefficients, on which, together with Moser's iteration technique, all available proofs of the local boundedness of local solutions to variational problems or elliptic equations are virtually patterned. The main novelties in the present framework amount to the use of sharp Poincaré and Sobolev inequalities in Orlicz spaces and to an optimized form of the Caccioppoli-type inequality. The lack of homogeneity of non-power type Young functions results in Orlicz-Sobolev inequalities whose integral form necessarily involves a gradient term on both sides. This creates new difficulties, that also appear, again because of the non-homogeneity of Young functions, in deriving the optimized Caccioppoli inequality. The latter requires an ad hoc process in the choice of trial functions in the definition of quasi-minimizers. The advantage of the use of the relevant Caccioppoli inequality is that its proof only calls into play Sobolev-type inequalities on $(n-1)$-dimensional spheres, instead of $n$-dimensional balls. This allows for growths of the function $B$ dictated by the $(n-1)$-dimensional Sobolev conjugate of $A$. By contrast, a more standard choice of trial functions would only permit slower growths of $B$, not exceeding the $n$-dimensional Sobolev conjugate of $A$. Orlicz-Sobolev and Poincaré inequalities in dimension $n$ just come into play in the proof of Theorem 2.2, when estimating terms depending on the variable $u$. The trial function optimization strategy refines that used in diverse settings in recent years. The version exploited in [40] – a variant of [5] – to deal with functionals subject to $p, q$-growth conditions is sensitive to the particular growth of the integrand. The conditions imposed in the situation under consideration here are so general to force us to resort to a more robust optimization argument, implemented in Lemma 5.1, Section 5. The latter is inspired to constructions employed in [13] in the context of div-curl lemmas, and in [41] in the proof of absence of Lavrientiev-phenomena in vector-valued convex minimization problems.

We conclude this section by illustrating Theorems 2.1 and 2.2 with applications to a couple of special instances. The former corresponds to functionals with $p, q$-growth. It not only recovers the available results but also augments and extends them in some respects. The latter concerns functionals with "power-times-logarithmic" growths, and provides us with an example associated with genuinely non-homogenous Young functions.

**Example 2.1.** In the standard case when

$$A(t) = t^p,$$

with $1 \le p \le n$, Theorem 2.1 recovers a result of [40]. Indeed, if $n \ge 3$ and $1 \le p < n-1$, we have that $A_{n-1}(t) \approx t^{\frac{(n-1)p}{(n-1)-p}}$, and the assumption (2.9) is equivalent to

$$B(t) \lesssim t^{\frac{(n-1)p}{(n-1)-p}} \quad \text{near infinity.} \tag{2.15}$$

Here, the relations $\lesssim$ and $\approx$ mean domination and equivalence, respectively, in the sense of Young functions.

If $p = n-1$, then $A_{n-1}(t) \approx e^{t^{\frac{n-1}{n-2}}}$ near infinity, whereas if $p > n-1$, then the second alternative condition (2.10) is satisfied. Hence, if either $n = 2$ or $n \geq 3$ and $p \geq n-1$, then any Young function $B \in \Delta_2$ near infinity is admissible.

The condition (2.15) is sharp, since the functionals with $p, q$-growth exhibited in [35,43,45,46] admit unbounded local minimizers if the assumption (2.15) is dropped.

Let us point out that the result deduced from Theorem 2.1 also enhances that of [40], where the function $\xi \mapsto f(x,\xi)$ is assumed to fulfill a variant of the $\Delta_2$-condition, which is not imposed here.

On the other hand, Theorem 2.2 extends the result of [40], where integrands only depending on $x$ and $\nabla u$ are considered. The conclusion of Theorem 2.2 hold under the same bound (2.15) on the function $B$. Moreover, $A_n(t) \approx t^{\frac{np}{n-p}}$ if $1 \leq p < n$ and $A_n(t) \approx e^{t^{\frac{n}{n-1}}}$ near infinity if $p = n$. Hence, if $1 \leq p < n$, then the assumption (2.14) reads:

$$E(t) \lesssim t^{\frac{np}{n-p}} \quad \text{near infinity.}$$

If $p = n$, then any non-decreasing function $E$ satisfying the $\Delta_2$-condition near infinity satisfies the assumption (2.14), and it is therefore admissible.

**Example 2.2.** Assume that

$$A(t) \approx t^p (\log t)^\alpha \quad \text{near infinity,}$$

where $1 < p < n$ and $\alpha \in \mathbb{R}$, or $p = 1$ and $\alpha \geq 0$, or $p = n$ and $\alpha \leq n-1$. Observe that these restrictions on the exponents $p$ and $\alpha$ are required for $A$ to be a Young function fulfilling the condition (2.5). From an application of Theorem 2.2 one can deduce that any local minimizer of $\mathcal{F}$ is locally bounded under the following assumptions.

If $n \geq 3$ and $p < n-1$, then we have to require that

$$B(t) \lesssim t^{\frac{(n-1)p}{(n-1)-p}} (\log t)^{\frac{(n-1)\alpha}{(n-1)-p}} \quad \text{near infinity.}$$

If either $n = 2$, or $n \geq 3$ and $n-1 \leq p < n$, then any Young function $B \in \Delta_2$ near infinity is admissible.

Moreover, if $p < n$, then our assumption on $E$ takes the form:

$$E(t) \lesssim t^{\frac{np}{n-p}} (\log t)^{\frac{n\alpha}{n-p}} \quad \text{near infinity.}$$

If $p = n$, then any non-decreasing function $E \in \Delta_2$ near infinity is admissible.

## 3. Orlicz-Sobolev spaces

This section is devoted to some basic definitions and properties from the theory of Young functions and Orlicz spaces. We refer the reader to the monograph [52] for a comprehensive presentation of this theory. The Sobolev and Poincaré inequalities in Orlicz-Sobolev spaces that play a role in our proofs are also recalled.

Orlicz spaces are defined in terms of Young functions. A function $A : [0, \infty) \to [0, \infty]$ is called a Young function if it is convex (non trivial), left-continuous and $A(0) = 0$.

The convexity of $A$ and its vanishing at 0 imply that

$$\lambda A(t) \le A(\lambda t) \quad \text{for } \lambda \ge 1 \text{ and } t \ge 0, \tag{3.1}$$

and that the function

$$\frac{A(t)}{t} \quad \text{is non-decreasing in } (0, \infty). \tag{3.2}$$

The Young conjugate $\widetilde{A}$ of $A$ is defined by

$$\widetilde{A}(t) = \sup\{\tau t - A(\tau) : \tau \ge 0\} \qquad \text{for} \qquad t \ge 0\,.$$

The following inequalities hold:

$$s \le A^{-1}(s)\widetilde{A}^{-1}(s) \le 2s \qquad \text{for } s \ge 0, \tag{3.3}$$

where $A^{-1}$ and $\widetilde{A}^{-1}$ denote the generalized right-continuous inverses of $A$ and $\widetilde{A}$, respectively.

A Young function $A$ is said to satisfy the $\Delta_2$-condition globally – briefly $A \in \Delta_2$ globally – if there exists a constant $c$ such that

$$A(2t) \le cA(t) \quad \text{for } t \ge 0. \tag{3.4}$$

If the inequality (3.4) just holds for $t \ge t_0$ for some $t_0 > 0$, then we say that $A$ satisfies the $\Delta_2$-condition near infinity, and write $A \in \Delta_2$ near infinity. One has that

$$\begin{aligned} A \in \Delta_2 \quad & \text{globally [near infinity] if and only if there exists } q \ge 1 \\ & \text{such that } \frac{tA'(t)}{A(t)} \le q \text{ for a.e. } t > 0\ [t \ge t_0]. \end{aligned} \tag{3.5}$$

A Young function $A$ is said to dominate another Young function $B$ globally if there exists a positive constant $c$ such that

$$B(t) \le A(ct) \tag{3.6}$$

for $t \ge 0$. The function $A$ is said to dominate $B$ near infinity if there exists $t_0 \ge 0$ such that (3.6) holds for $t \ge t_0$. If $A$ and $B$ dominate each other globally [near infinity], then they are called equivalent globally [near infinity]. We use the notation $B \lesssim A$ to denote that $A$ dominates $B$, and $B \approx A$ to denote that $A$ and $B$ are equivalent. This terminology and notation will also be adopted for merely nonnegative functions, which are not necessarily Young functions.

Let $\Omega$ be a measurable set in $\mathbb{R}^n$. The Orlicz class $K^A(\Omega)$ built upon a Young function $A$ is defined as

$$K^A(\Omega) = \left\{ u : u \text{ is measurable in } \Omega \text{ and } \int_\Omega A(|u|)\, dx < \infty \right\}. \tag{3.7}$$

The set $K^A(\Omega)$ is convex for every Young function $A$.

The Orlicz space $L^A(\Omega)$ is the linear hull of $K^A(\Omega)$. It is a Banach function space, equipped with the Luxemburg norm defined as

$$\|u\|_{L^A(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega A\left(\frac{|u|}{\lambda}\right) dx \le 1\right\} \tag{3.8}$$

for a measurable function $u$. These notions are modified as usual to define the local Orlicz class $K^A_{\mathrm{loc}}(\Omega)$ and the local Orlicz space $L^A_{\mathrm{loc}}(\Omega)$.

If either $A \in \Delta_2$ globally, or $|\Omega| < \infty$ and $A \in \Delta_2$ near infinity, then $K^A(\Omega)$ is, in fact, a linear space, and $K^A(\Omega) = L^A(\Omega)$. Here, $|\Omega|$ denotes the Lebesgue measure of $\Omega$.

Notice that, in particular, $L^A(\Omega) = L^p(\Omega)$ if $A(t) = t^p$ for some $p \in [1, \infty)$, and $L^A(\Omega) = L^\infty(\Omega)$ if $A(t) = 0$ for $t \in [0, 1]$ and $A(t) = \infty$ for $t \in (1, \infty)$.

The identity

$$\|\chi_E\|_{L^A(\Omega)} = \frac{1}{A^{-1}(1/|E|)} \tag{3.9}$$

holds for every Young function $A$ and any measurable set $E \subset \Omega$. Here, $\chi_E$ stands for the characteristic function of $E$.

The Hölder inequality in Orlicz spaces tells us that

$$\int_\Omega |uv|\, dx \le 2\|u\|_{L^A(\Omega)} \|v\|_{L^{\widetilde{A}}(\Omega)} \tag{3.10}$$

for $u \in L^A(\Omega)$ and $v \in L^{\widetilde{A}}(\Omega)$.

Assume now that $\Omega$ is an open set. The homogeneous Orlicz-Sobolev class $V^1K^A(\Omega)$ is defined as the convex set

$$V^1K^A(\Omega) = \left\{u \in W^{1,1}_{\mathrm{loc}}(\Omega) : |\nabla u| \in K^A(\Omega)\right\} \tag{3.11}$$

and the inhomogeneous Orlicz-Sobolev class $W^1K^A(\Omega)$ is the convex set

$$W^1K^A(\Omega) = K^A(\Omega) \cap V^1K^A(\Omega). \tag{3.12}$$

The homogenous Orlicz-Space $V^1L^A(\Omega)$ and its inhomogenous counterpart $W^1L^A(\Omega)$ are accordingly given by

$$V^1L^A(\Omega) = \left\{u \in W^{1,1}_{\mathrm{loc}}(\Omega) : |\nabla u| \in L^A(\Omega)\right\} \tag{3.13}$$

and

$$W^1L^A(\Omega) = L^A(\Omega) \cap V^1L^A(\Omega). \tag{3.14}$$

The latter is a Banach space endowed with the norm

$$\|u\|_{W^{1,A}(\Omega)} = \|u\|_{L^A(\Omega)} + \|\nabla u\|_{L^A(\Omega)}. \tag{3.15}$$

Here, and in what follows, we use the notation $\|\nabla u\|_{L^A(\Omega)}$ as a shorthand for $\| |\nabla u| \|_{L^A(\Omega)}$.

The local versions $V^1_{\rm loc}K^A(\Omega)$, $W^1_{\rm loc}K^A(\Omega)$, $V^1_{\rm loc}L^A(\Omega)$, and $W^1_{\rm loc}L^A(\Omega)$ of these sets/spaces is obtained by modifying the above definitions as usual. In the case when $L^A(\Omega) = L^p(\Omega)$ for some $p \in [1,\infty]$, the standard Sobolev space $W^{1,p}(\Omega)$ and its homogeneous version $V^{1,p}(\Omega)$ are recovered.

Orlicz and Orlicz-Sobolev classes of weakly differentiable functions $u$ defined on the $(n-1)$-dimensional unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$ also enter our approach. These spaces are defined as in (3.7), (3.8), (3.11), (3.13), and (3.14), with the Lebesgue measure replaced with the $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$, and $\nabla u$ replaced with $\nabla_{\mathbb{S}} u$, the vector field on $\mathbb{S}^{n-1}$ whose components are the covariant derivatives of $u$.

As highlighted in the previous section, sharp embedding theorems and corresponding inequalities in Orlicz-Sobolev spaces play a critical role in the formulation of our result and in its proof. As shown in [19] (see also [18] for an equivalent version), the optimal $n$-dimensional Sobolev conjugate of a Young function $A$ fulfilling

$$\int_0 \left(\frac{t}{A(t)}\right)^{\frac{1}{n-1}} dt < \infty \tag{3.16}$$

is the Young function $A_n$ defined as

$$A_n(t) = A(H_n^{-1}(t)) \qquad \text{for } t \geq 0, \tag{3.17}$$

where the function $H_n : [0,\infty) \to [0,\infty)$ is given by

$$H_n(s) = \left(\int_0^s \left(\frac{t}{A(t)}\right)^{\frac{1}{n-1}} dt\right)^{\frac{n-1}{n}} \qquad \text{for } s \geq 0. \tag{3.18}$$

The function $A_{n-1}$ is defined analogously, by replacing $n$ with $n-1$ in the equations (3.17) and (3.18).

In the statements of Theorems 2.1 and 2.2, the functions $A_n$ and $A_{n-1}$ are defined after modifying $A$ near 0, if necessary, in such a way that the condition (3.16) be satisfied. The assumptions (2.3) and (2.13) are not affected by the choice of the modified function $A$, thanks to the presence of the additive constant $L$. Membership of a function in an Orlicz-Sobolev local class or space associated with $A$ is also not influenced by this choice, inasmuch as the behavior of $A$ near 0 is irrelevant (up to additive and/or multiplicative constants) whenever integrals or norms over sets with finite measure are concerned.

An optimal Sobolev-Poincaré inequality on balls $\mathbb{B}_r \subset \mathbb{R}^n$, centered at 0 and with radius $r$ reads as follows. In its statement, we adopt the notation

$$u_{\mathbb{B}_r} = \fint_{\mathbb{B}_r} u(x)\, dx,$$

where $\fint$ stands for integral average.

**Theorem A.** Let $n \geq 2$, let $r > 0$, and let $A$ be a Young function fulfilling the condition (3.16). Then, there exists a constant $\kappa = \kappa(n)$ such that

$$\int_{\mathbb{B}_r} A_n\left(\frac{|u - u_{\mathbb{B}_r}|}{\kappa\left(\int_{\mathbb{B}_r} A(|\nabla u|)dy\right)^{\frac{1}{n}}}\right) dx \leq \int_{\mathbb{B}_r} A(|\nabla u|)\, dx \tag{3.19}$$

for every $u \in V^1K^A(\mathbb{B}_r)$.

As a consequence of the inequality (3.19) (which continues to hold for balls centered at any point in $\mathbb{R}^n$) and of Lemma 4.1, Section 4, one has that

$$V^1_{\text{loc}}K^A(\Omega) \subset K^A_{\text{loc}}(\Omega) \tag{3.20}$$

for any open set $\Omega \subset \mathbb{R}^n$ and any Young function $A$. Thereby,

$$V^1_{\text{loc}}K^A(\Omega) = W^1_{\text{loc}}K^A(\Omega).$$

Hence, in what follows, the spaces $V^1_{\text{loc}}K^A(\Omega)$ and $W^1_{\text{loc}}K^A(\Omega)$ will be equally employed.

Besides the Sobolev-Poincaré inequality of Theorem A, a Sobolev type inequality is of use in our applications and is the subject of the following theorem. Only Part (i) of the statement will be needed. Part (ii) substantiates the inclusion (2.7).

**Theorem B.** Let $n \geq 2$, let $r > 0$, and let $A$ be a Young function fulfilling the condition (3.16).

(i) Assume that the condition (2.5) holds. Then, there exists a constant $\kappa = \kappa(n, r)$ such that

$$\int_{\mathbb{B}_r} A_n\left(\frac{|u|}{\kappa\left(\int_{\mathbb{B}_r} A(|u|) + A(|\nabla u|)dy\right)^{\frac{1}{n}}}\right) dx \leq \int_{\mathbb{B}_r} A(|u|) + A(|\nabla u|)\, dx \tag{3.21}$$

for every $u \in W^1K^A(\mathbb{B}_r)$.

(ii) Assume that the condition (2.6) holds. Then, there exists a constant $\kappa = \kappa(n, r, A)$ such that

$$\|u\|_{L^\infty(\mathbb{B}_r)} \leq \kappa\left(\int_{\mathbb{B}_r} A(|u|) + A(|\nabla u|)\, dx\right)^{\frac{1}{n}} \tag{3.22}$$

for every $u \in W^1K^A(\mathbb{B}_r)$.

In particular, if $r \in [r_1, r_2]$ for some $r_2 > r_1 > 0$, then the constant $\kappa$ in the inequalities (3.21) and (3.22) depends on $r$ only via $r_1$ and $r_2$.

A counterpart of Theorem B for Orlicz-Sobolev functions on the sphere $\mathbb{S}^{n-1}$ takes the following form.

**Theorem C.** Let $n \geq 2$ and let $A$ be a Young function such that

$$\int_0 \left(\frac{t}{A(t)}\right)^{\frac{1}{n-2}} dt < \infty \qquad \text{if } n \geq 3. \tag{3.23}$$

(i) Assume that $n \geq 3$ and

$$\int^\infty \left(\frac{t}{A(t)}\right)^{\frac{1}{n-2}} dt = \infty. \tag{3.24}$$

Then, there exists a constant $\kappa = \kappa(n)$ such that

$$\int_{\mathbb{S}^{n-1}} A_{n-1}\left(\frac{|u|}{\kappa\left(\int_{\mathbb{S}^{n-1}} A(|u|) + A(|\nabla_{\mathbb{S}} u|) d\mathcal{H}^{n-1}(y)\right)^{\frac{1}{n-1}}}\right) d\mathcal{H}^{n-1}(x) \leq \int_{\mathbb{S}^{n-1}} A(|u|) + A(|\nabla_{\mathbb{S}} u|)\, d\mathcal{H}^{n-1}(x) \tag{3.25}$$

for $u \in W^1 K^A(\mathbb{S}^{n-1})$.

(ii) Assume that one of the following situations occurs:

$$\begin{cases} n = 2 & \text{and} \quad \lim_{t\to 0^+} \frac{A(t)}{t} > 0 \\ n \geq 3 & \text{and} \quad \int^\infty \left(\frac{t}{A(t)}\right)^{\frac{1}{n-2}} dt < \infty. \end{cases} \tag{3.26}$$

Then, there exists a constant $\kappa = \kappa(n, A)$ such that

$$\|u\|_{L^\infty(\mathbb{S}^{n-1})} \leq \kappa \left(\int_{\mathbb{S}^{n-1}} A(|u|) + A(|\nabla_{\mathbb{S}} u|)\, d\mathcal{H}^{n-1}(x)\right)^{\frac{1}{n-1}} \tag{3.27}$$

for $u \in W^1 K^A(\mathbb{S}^{n-1})$.

Theorems A and B are special cases of [22, Theorems 4.4 and 3.1], respectively, which hold in any Lipschitz domain in $\mathbb{R}^n$ (and for Orlicz-Sobolev spaces of arbitrary order). The assertions about the dependence of the constants can be verified via a standard scaling argument. Theorem C can be derived via arguments analogous to those in the proof of [22, Theorem 3.1]. For completeness, we offer the main steps of its proof.

**Proof of Theorem C.** *Part (i).* Let us set

$$u_{\mathbb{S}^{n-1}} = \fint_{\mathbb{S}^{n-1}} u(x)\, d\mathcal{H}^{n-1}(x).$$

A key step is a Sobolev-Poincaré type inequality, a norm version of (3.19) on $\mathbb{S}^{n-1}$, which tells us that

$$\|u - u_{\mathbb{S}^{n-1}}\|_{L^{A_{n-1}}(\mathbb{S}^{n-1})} \leq c\|\nabla_{\mathbb{S}} u\|_{L^A(\mathbb{S}^{n-1})} \tag{3.28}$$

for some constant $c = c(n)$ and for $u \in V^1L^A(\mathbb{S}^{n-1})$. A proof of inequality (3.28) rests upon the following symmetrization argument combined with a one-dimensional Hardy-type inequality in Orlicz spaces.

Set

$$c_n = \mathcal{H}^{n-1}(\mathbb{S}^{n-1}) \tag{3.29}$$

and denote by $u^\circ : [0, c_n] \to [-\infty, \infty]$ the signed decreasing rearrangement of $u$, defined by

$$u^\circ(s) = \inf\{t \in \mathbb{R} : \mathcal{H}^{n-1}(\{u > t\}) \leq s\} \quad \text{for } s \in [0, c_n].$$

Moreover, define the signed symmetral $u^\sharp : \mathbb{S}^{n-1} \to [-\infty, \infty]$ of $u$ as

$$u^\sharp(x) = u^\circ(V(x)) \quad \text{for } x \in \mathbb{S}^{n-1},$$

where $V(x)$ denotes the $\mathcal{H}^{n-1}$-measure of the spherical cap on $\mathbb{S}^{n-1}$, centered at the north pole on $\mathbb{S}^{n-1}$, whose boundary contains $x$. Thus, $u^\sharp$ is a function, which is equimeasurable with $u$, and whose level sets are spherical caps centered at the north pole.

The equimeasurability of the functions $u$, $u^\circ$ and $u^\sharp$ ensures that

$$\|u - u_{\mathbb{S}^{n-1}}\|_{L^{A_{n-1}}(\mathbb{S}^{n-1})} = \|u^\sharp - u_{\mathbb{S}^{n-1}}\|_{L^{A_{n-1}}(\mathbb{S}^{n-1})} = \|u^\circ - u_{\mathbb{S}^{n-1}}\|_{L^{A_{n-1}}(0,c_n)}. \tag{3.30}$$

Moreover, since $u^\circ(c_n/2)$ is a median of $u^\circ$ on $(0, c_n)$ and $u_{\mathbb{S}^{n-1}}$ agrees with the mean value of $u^\circ$ over $(0, c_n)$, one has that

$$\|u^\circ - u^\circ(c_n/2)\|_{L^{A_{n-1}}(0,c_n)} \geq \tfrac{1}{2}\|u^\circ - u_{\mathbb{S}^{n-1}}\|_{L^{A_{n-1}}(0,c_n)} = \tfrac{1}{2}\|u - u_{\mathbb{S}^{n-1}}\|_{L^{A_{n-1}}(\mathbb{S}^{n-1})}, \tag{3.31}$$

see e.g. [23, Lemma 2.2].

On the other hand, a version of the Pólya-Szegö principle on $\mathbb{S}^{n-1}$ tells us that $u^\circ$ is locally absolutely continuous, $u^\sharp \in V^1L^A(\mathbb{S}^{n-1})$, and

$$\left\| I_{\mathbb{S}^{n-1}}(s)\left(-\frac{du^\circ}{ds}\right)\right\|_{L^A(0,c_n)} = \|\nabla_{\mathbb{S}} u^\sharp\|_{L^A(\mathbb{S}^{n-1})} \leq \|\nabla_{\mathbb{S}} u\|_{L^A(\mathbb{S}^{n-1})}, \tag{3.32}$$

where $I_{\mathbb{S}^{n-1}} : [0, c_n] \to [0, \infty)$ denotes the isoperimetric function of $\mathbb{S}^{n-1}$ (see [14]). It is well-known that there exists a positive constant $c = c(n)$ such that

$$I_{\mathbb{S}^{n-1}}(s) \geq c \min\{s, c_n - s\}^{\frac{n-2}{n-1}} \quad \text{for } s \in (0, c_n). \tag{3.33}$$

Hence,

$$c\left\| \min\{s, c_n - s\}^{\frac{n-2}{n-1}} \left( -\frac{du^\circ}{ds} \right) \right\|_{L^A(0,c_n)} \leq \|\nabla_{\mathbb{S}} u\|_{L^A(\mathbb{S}^{n-1})}. \tag{3.34}$$

The absolute continuity of $u^\circ$ ensures that

$$u^\circ(s) - u^\circ(c_n) = \int_s^{c_n/2} \left( -\frac{du^\circ}{dr} \right) dr \qquad \text{for } s \in (0, c_n). \tag{3.35}$$

Thanks to the equations (3.30), (3.31), (3.34), (3.35), and to the symmetry of the function $\min\{s, c_n - s\}^{\frac{n-2}{n-1}}$ about $c_n/2$, the inequality (3.28) is reduced to the inequality

$$\left\| \int_s^{c_n/2} r^{-\frac{n-2}{n-1}} \phi(r)\, dr \right\|_{L^{A_{n-1}}(0,c_n/2)} \leq c\|\phi\|_{L^A(0,c_n/2)} \tag{3.36}$$

for a suitable constant $c = c(n)$ and for $\phi \in L^A(0, c_n/2)$. The inequality (3.36) is in turn a consequence of [19, Inequality (2.7)].

Next, by Lemma 4.2, Section 4, applied with $n$ replaced with $n - 1$,

$$\frac{1}{\widetilde{A}^{-1}(t)} \frac{1}{A_{n-1}^{-1}(t))} \leq \frac{1}{t^{\frac{n-2}{n-1}}} \quad \text{for } t > 0.$$

Hence, by the inequality (3.10), with $\Omega$ replaced with $\mathbb{S}^{n-1}$, one has that

$$\begin{aligned} \|u_{\mathbb{S}^{n-1}}\|_{L^{A_{n-1}}(\mathbb{S}^{n-1})} &= |u_{\mathbb{S}^{n-1}}|\|1\|_{L^{A_{n-1}}(\mathbb{S}^{n-1})} \leq \frac{2}{c_n}\|u\|_{L^A(\mathbb{S}^{n-1})}\|1\|_{L^{\widetilde{A}}(\mathbb{S}^{n-1})}\|1\|_{L^{A_{n-1}}(\mathbb{S}^{n-1})} \\ &= \frac{2}{c_n}\frac{1}{\widetilde{A}^{-1}(1/c_n)}\frac{1}{A_{n-1}^{-1}(1/c_n)}\|u\|_{L^A(\mathbb{S}^{n-1})} \leq \frac{2}{c_n^{\frac{1}{n-1}}}\|u\|_{L^A(\mathbb{S}^{n-1})}. \end{aligned} \tag{3.37}$$

Coupling the inequality (3.28) with (3.37) and making use of the triangle inequality entail that

$$\|u\|_{L^{A_{n-1}}(\mathbb{S}^{n-1})} \leq c\big(\|\nabla_{\mathbb{S}} u\|_{L^A(\mathbb{S}^{n-1})} + \|u\|_{L^A(\mathbb{S}^{n-1})}\big) \tag{3.38}$$

for some constant $c = c(n)$ and for $u \in W^1L^A(\mathbb{S}^{n-1})$.

Now set

$$M = \int_{\mathbb{S}^{n-1}} A(|\nabla_{\mathbb{S}} u|) + A(|u|)\, d\mathcal{H}^{n-1}(x),$$

and apply the inequality (3.38) with the function $A$ replaced with the Young function $A_M$ given by

$$A_M(t) = \frac{A(t)}{M} \qquad \text{for } t \geq 0.$$

Hence,

$$\|u\|_{L^{(A_M)_{n-1}}(\mathbb{S}^{n-1})} \le c\big(\|\nabla_{\mathbb{S}} u\|_{L^{A_M}(\mathbb{S}^{n-1})} + \|u\|_{L^{A_M}(\mathbb{S}^{n-1})}\big), \tag{3.39}$$

where $(A_M)_{n-1}$ denotes the function obtained on replacing $A$ with $A_M$ in the definition of $A_{n-1}$. The fact that the constant $c$ in (3.38) is independent of $A$ is of course crucial in deriving the inequality (3.39). Observe that

$$(A_M)_{n-1}(t) = \frac{1}{M} A_{n-1}\Big(\frac{t}{M^{\frac{1}{n-1}}}\Big) \qquad \text{for } t \ge 0. \tag{3.40}$$

On the other hand, by the definition of Luxemburg norm and the choice of $M$,

$$\|u\|_{L^{A_M}(\mathbb{S}^{n-1})} \le 1 \quad \text{and} \quad \|\nabla_{\mathbb{S}} u\|_{L^{A_M}(\mathbb{S}^{n-1})} \le 1. \tag{3.41}$$

Therefore, by the definition of Luxemburg norm again, the inequality (3.39) tells us that

$$\frac{1}{M}\int\limits_{\mathbb{S}^{n-1}} A_{n-1}\Big(\frac{|u(x)|}{2cM^{\frac{1}{n-1}}}\Big)\, d\mathcal{H}^{n-1}(x) \le 1.$$

Hence, the inequality (3.25) follows.

*Part (ii).* First, assume that $n \ge 3$ and the integral condition in (3.26) holds. Let $\overline{A}$ be the Young function defined as

$$\overline{A}(t) = \Bigg(t^{\frac{n-1}{n-2}} \int\limits_t^\infty \frac{\widetilde{A}(r)}{r^{1+\frac{n-1}{n-2}}}\, dr\Bigg)^{\widetilde{}} \qquad \text{for } t \ge 0, \tag{3.42}$$

where $(\cdots)^{\widetilde{}}$ stands for the Young conjugate of the function in parentheses. Notice that the convergence of the integral on the right-hand side of equation (3.42) is equivalent to the convergence of the integral in (3.26), see [21, Lemma 2.3]. Since we are assuming that $A$ fulfills the condition (3.23), the same lemma also ensures that

$$\int\limits_0 \frac{\widetilde{A}(r)}{r^{1+\frac{n-1}{n-2}}}\, dr < \infty. \tag{3.43}$$

From [15, Theorem 4.1] one has that

$$\overline{A}\big(c\|u - u_{\mathbb{S}^{n-1}}\|_{L^\infty(\mathbb{S}^{n-1})}\big) \le \fint\limits_{\mathbb{S}^{n-1}} A(|\nabla_{\mathbb{S}} u|)\, d\mathcal{H}^{n-1} \tag{3.44}$$

for some positive constant $c = c(n)$ and for $u \in V^1K^A(\mathbb{S}^{n-1})$.

Furthermore, by Jensen's inequality,

$$A\big(\|u_{\mathbb{S}^{n-1}}\|_{L^\infty(\mathbb{S}^{n-1})}\big) \le A\bigg(\fint_{\mathbb{S}^{n-1}} |u|\, d\mathcal{H}^{n-1}\bigg) \le \fint_{\mathbb{S}^{n-1}} A(|u|)\, d\mathcal{H}^{n-1}. \tag{3.45}$$

Thanks to [15, Inequality (4.6)],

$$\overline{A}(t) \le A(t) \qquad \text{for } t \ge 0. \tag{3.46}$$

Moreover, the inequality (3.43) ensures that

$$t^{\frac{n-1}{n-2}} \int_t^\infty \frac{\widetilde{A}(r)}{r^{1+\frac{n-1}{n-2}}}\, dr \le c\, t^{\frac{n-1}{n-2}} \qquad \text{for } t \ge 0, \tag{3.47}$$

where we have set

$$c = \int_0^\infty \frac{\widetilde{A}(r)}{r^{1+\frac{n-1}{n-2}}}\, dr.$$

Taking the Young conjugates of both sides of the inequality (3.47) results in

$$\overline{A}(t) \ge ct^{n-1} \qquad \text{for } t \ge 0, \tag{3.48}$$

for some constant $c = c(n, A)$. The inequality (3.27) follows, via the triangle inequality, from the inequalities (3.44), (3.45), (3.46) and (3.48).

Assume next that $n = 2$ and the condition on the limit in (3.26) holds. If we denote by $a$ this limit, then

$$A(t) \ge at \qquad \text{for } t \ge 0. \tag{3.49}$$

A simple one-dimensional argument, coupled with Jensen's inequality and the increasing monotonicity of the function $tA^{-1}(1/t)$ shows that

$$A\big(\tfrac{1}{2\pi}\|u - u_{\mathbb{S}^1}\|_{L^\infty(\mathbb{S}^1)}\big) \le \fint_{\mathbb{S}^1} A(|\nabla_{\mathbb{S}} u|)\, d\mathcal{H}^1 \tag{3.50}$$

for $u \in V^1K^A(\mathbb{S}^1)$ (see [15, Inequality (4.8) and below]). The inequality (3.27) now follows from (3.45) (which holds also when $n = 2$), (3.49) and (3.50). □

## 4. Analytic lemmas

Here, we collect a few technical lemmas about one-variable functions. We begin with two inequalities involving a Young function and its Sobolev conjugate.

**Lemma 4.1.** *Let $n \geq 2$ and let $A$ be a Young function fulfilling the condition (3.16). Then, for every $k > 0$ there exists a positive constant $c = c(k, A, n)$ such that*

$$A(t) \leq A_n(kt) + c \qquad \textit{for } t \geq 0. \tag{4.1}$$

**Proof.** Fix $k > 0$. Since $A_n(t) = A(H_n^{-1}(t))$ and $\lim_{t\to\infty} \frac{H_n^{-1}(t)}{t} = \infty$, there exists $t \geq t_0$ such that $A(t) \leq A_n(kt)$ for $t \geq t_0$. The inequality (4.1) hence follows, with $c = A(t_0)$. □

**Lemma 4.2.** *Let $n \geq 2$ and let $A$ be a Young function fulfilling the condition (3.16). Then,*

$$\frac{1}{\widetilde{A}^{-1}(t)} \frac{1}{A_n^{-1}(t)} \leq \frac{1}{t^{\frac{1}{n'}}} \qquad \textit{for } t > 0. \tag{4.2}$$

**Proof.** Hölder's inequality and the property (3.2) imply that

$$\begin{aligned} t &= \int_0^t \left(\frac{A(r)}{r}\right)^{\frac{1}{n}} \left(\frac{r}{A(r)}\right)^{\frac{1}{n}} dr \leq \left(\int_0^t \frac{A(r)}{r}\, dr\right)^{\frac{1}{n}} \left(\int_0^t \left(\frac{r}{A(r)}\right)^{\frac{1}{n-1}} dr\right)^{\frac{1}{n'}} \\ &\leq \left(\frac{A(t)}{t}\right)^{\frac{1}{n}} t^{\frac{1}{n}} H_n(t) = A(t)^{\frac{1}{n}} H_n(t) \quad \text{for } t > 0. \end{aligned} \tag{4.3}$$

Hence,

$$A^{-1}(t) \leq t^{\frac{1}{n}} H_n(A^{-1}(t)) \quad \text{for } t \geq 0. \tag{4.4}$$

The first inequality in (3.3) and the inequality (4.4) imply that

$$t \leq A^{-1}(t)\widetilde{A}^{-1}(t) \leq t^{\frac{1}{n}} H_n(A^{-1}(t))\widetilde{A}^{-1}(t) \quad \text{for } t \geq 0. \tag{4.5}$$

Hence, the inequality (4.2) follows. □

The next result ensures that the functions $A$, $B$ and $E$ appearing in the assumption (2.13) can be modified near 0 in such a way that such an assumption is still fulfilled, possibly with a different constant $L$, and the conditions imposed on $A$, $B$ and $E$ in Theorem 2.2 are satisfied globally, instead of just near infinity. Of course, the same applies to the simpler conditions of Theorem 2.1, where the function $E$ is missing.

**Lemma 4.3.** *Assume that the functions $f$, $A$, $B$ and $E$ are as in Theorem 2.2. Then, there exist two Young functions $\widehat{A}, \widehat{B} : [0, \infty) \to [0, \infty)$, an increasing function $\widehat{E} : [0, \infty) \to [0, \infty)$, and constants $\widehat{L} \geq 1$ and $q > n$ such that:*

$$\widehat{A}(|\xi|) - \widehat{E}(|t|) - \widehat{L} \le f(x,t,\xi) \le \widehat{B}(|\xi|) + \widehat{E}(|t|) + \widehat{L}$$
$$\text{for a.e. } x \in \Omega, \text{ for every } t \in \mathbb{R}, \text{ and every } \xi \in \mathbb{R}^n, \tag{4.6}$$

$$t^{\frac{n}{n-1}} \le \widehat{L}\,\widehat{A}_n(t) \quad \text{for } t \ge 0, \tag{4.7}$$

$$\lim_{t\to 0^+} \frac{\widehat{A}(t)}{t} > 0, \tag{4.8}$$

$$\widehat{E}(2t) \le \widehat{L}\widehat{E}(t) \quad \text{for } t \ge 0, \tag{4.9}$$

$$\widehat{E}(t) \le \widehat{A}_n(\widehat{L}t) \quad \text{for } t \ge 0, \tag{4.10}$$

$$\widehat{B}(\lambda t) \le \lambda^q \widehat{B}(t) \quad \text{for } t \ge 0 \text{ and } \lambda \ge 1. \tag{4.11}$$

*Moreover, if the assumption (2.8) is in force, then the function $B$ satisfies the assumption (2.9) and*

$$\widehat{B}(t) \le \widehat{A}_{n-1}(\widehat{L}t) \qquad \text{for } t \ge 0; \tag{4.12}$$

*if the assumption (2.10) is in force, then*

$$\widehat{B}(t) \le \widehat{L}t^q \quad \text{for } t \ge 0. \tag{4.13}$$

*Here, $\widehat{A}_{n-1}$ and $\widehat{A}_n$ denote the functions defined as $A_{n-1}$ and $A_n$, with $A$ replaced with $\widehat{A}$.*

**Proof.** *Step 1. Construction of $\widehat{A}$.* Denote by $t_1$ the maximum among 1, the constant $t_0$ appearing in the inequalities (2.14) and (2.9), and the lower bound for $t$ in the definition of the $\Delta_2$-condition near infinity for the functions $B$ and $E$. Let us set $a = \frac{A(t_1)}{t_1}$, and define the Young function $\widehat{A}$ as

$$\widehat{A}(t) = \begin{cases} at & \text{if } 0 \le t < t_1 \\ A(t) & \text{if } t \ge t_1. \end{cases} \tag{4.14}$$

Clearly, $\widehat{A}$ satisfies the property (4.8) and

$$A(t) \le \widehat{A}(t) \qquad \text{for } t \ge 0. \tag{4.15}$$

Also, the convexity of $A$ ensures that

$$\widehat{A}(t) \ge at \quad \text{for } t \ge 0. \tag{4.16}$$

Since

$$\widehat{H}_n(s) = \left( \int_0^s \left( \frac{t}{\widehat{A}(t)} \right)^{\frac{1}{n-1}} dt \right)^{\frac{n-1}{n}} \qquad \text{for } s \ge 0,$$

we deduce that

$$\widehat{H}_n(s) \leq a^{-\frac{1}{n}} s^{\frac{n-1}{n}} \quad \text{for } s \geq 0,$$

whence

$$a^{\frac{1}{n-1}} t^{\frac{n}{n-1}} \leq \widehat{H}_n^{-1}(t) \quad \text{for } t \geq 0.$$

Inasmuch as $\widehat{A}_n = \widehat{A} \circ \widehat{H}_n^{-1}$, the latter inequality and the inequality (4.16) yield:

$$\widehat{A}_n(t) \geq \widehat{A}(a^{\frac{1}{n-1}} t^{\frac{n}{n-1}}) \geq (at)^{\frac{n}{n-1}} \quad \text{for } t \geq 0.$$

This shows that the inequality (4.7) holds for sufficiently large $\widehat{L}$.

For later reference, also note that

$$\widehat{A}_n(t) = (at)^{\frac{n}{n-1}} \quad \text{for } t \in [0, t_1]. \tag{4.17}$$

Next, we have that

$$A_n(t) \leq \widehat{A}_n(t) \qquad \text{for } t \geq 0. \tag{4.18}$$

Indeed, the inequality (4.15) implies that

$$\widehat{H}_n(s) \leq H_n(s) \qquad \text{for } s \geq 0.$$

Thus, $H_n^{-1}(t) \leq \widehat{H}_n^{-1}(t)$ for $t \geq 0$, whence the inequality (4.18) follows, on making use of (4.15) again.

Moreover, there exists $t_2 \geq t_1$, depending on $n$ and $A$, such that

$$\widehat{A}_n(t) \leq A_n(2t) \quad \text{for } t \geq t_2. \tag{4.19}$$

Actually, if $s \geq t_1$ and is sufficiently large, then

$$\begin{aligned}
\widehat{H}_n(s) &= \left( \int_0^s \left( \frac{t}{\widehat{A}(t)} \right)^{\frac{1}{n-1}} dt \right)^{\frac{n-1}{n}} \geq \left( \int_{t_1}^s \left( \frac{t}{\widehat{A}(t)} \right)^{\frac{1}{n-1}} dt \right)^{\frac{n-1}{n}} \\
&= \left( \int_{t_1}^s \left( \frac{t}{A(t)} \right)^{\frac{1}{n-1}} dt \right)^{\frac{n-1}{n}} \geq \frac{1}{2} H_n(s).
\end{aligned}$$

Observe that the last inequality holds, for large $s$, thanks to the assumption (2.5). Hence, $\widehat{H}_n^{-1}(t) \leq H_n^{-1}(2t)$ for sufficiently large $t$ and thereby

$$\widehat{A}_n(t) = \widehat{A}(\widehat{H}_n^{-1}(t)) = A(\widehat{H}_n^{-1}(t)) \leq A(H_n^{-1}(2t))) = A_n(2t) \quad \text{for } t \geq t_2,$$

provided that $t_2$ is sufficiently large. The inequality (4.19) is thus established.

*Step 2. Construction of* $\widehat{B}$. First, consider the case when (2.8) and (2.9) hold. Since $B$ is a Young function, there exists $t_3 \geq t_2$, where $t_2$ is the number from Step 2, such that $B(t_3) > A_{n-1}(t_1)$. Define the Young function $\widehat{B}$ as

$$\widehat{B}(t) = \begin{cases} \widehat{A}_{n-1}(t) & \text{if } 0 \leq t < t_2 \\ \frac{t_3 - t}{t_3 - t_2} \widehat{A}_{n-1}(t_2) + \frac{t - t_2}{t_3 - t_2} B(t_3) & \text{if } t_2 \leq t < t_3 \\ B(t) & \text{if } t \geq t_3. \end{cases}$$

We claim that the inequality (4.12) holds with this choice of $\widehat{B}$, provided that $\widehat{L}$ is large enough. If $t \in [0, t_2)$, the inequality in question is trivially satisfied with $\widehat{L} = 1$. If $t \in [t_2, t_3)$, then

$$\widehat{B}(t) \leq \widehat{B}(t_3) = B(t_3) \leq A_{n-1}(Lt_3) \leq \widehat{A}_{n-1}(Lt_3) \leq \widehat{A}_{n-1}((Lt_3/t_2)t),$$

where the third inequality holds thanks to (4.18). Finally, if $t > t_3$, then

$$\widehat{B}(t) = B(t) \leq A_{n-1}(Lt) \leq \widehat{A}_{n-1}(Lt).$$

Altogether, the inequality (4.12) is fulfilled with $\widehat{L} = \max\left\{1, \frac{Lt_3}{t_2}\right\}$.

In order to establish the inequality (4.11), it suffices to show that $\widehat{B}$ satisfies the $\Delta_2$-condition globally. Since $\widehat{B}$ is a Young function, this condition is in turn equivalent to the fact that there exists a constant $c$ such that

$$\frac{t\widehat{B}'(t)}{\widehat{B}(t)} \leq c \quad \text{for a.e. } t > 0. \tag{4.20}$$

Since $B$ is a Young function satisfying the $\Delta_2$-condition near infinity, and $\widehat{B}(t) = B(t)$ for large $t$, the condition (4.20) certainly holds for large $t$. On the other hand, since

$$\lim_{t \to 0+} \frac{t\widehat{B}'(t)}{\widehat{B}(t)} = \lim_{t \to 0+} \frac{t\widehat{A}'_{n-1}(t)}{\widehat{A}_{n-1}(t)} = \frac{n-1}{n-2},$$

the condition (4.20) also holds for $t$ close to 0. Hence, it holds for every $t > 0$.

Next, consider the case when (2.10) holds. The $\Delta_2$-condition near infinity for $B$ implies that there exist constants $q > 1$, $t_4 > 1$ and $c > 0$ such that $B(t) \leq ct^q$ for all $t \geq t_4$. Since $t_4 > 1$, we may suppose, without loss of generality, that $q > n$. Inasmuch as $B(t) \leq \widehat{L}(t^q + 1)$ for $t \geq 0$, provided that $\widehat{L}$ is sufficiently large, the choice $\widehat{B}(t) = \widehat{L}t^q$ makes the inequalities (4.11) and (4.13) true.

*Step 3. Construction of* $\widehat{E}$. We define $\widehat{E}$ analogously to $\widehat{B}$, by replacing $B$ with $E$ and $\widehat{A}_{n-1}$ with $\widehat{A}_n$. The same argument as in Step 2 tells us that the inequalities (4.9) and (4.10) hold for a suitable choice of the constant $\widehat{L}$.

*Step 4. Conclusion*. Since

$$f(x, t, \xi) \leq B(|\xi|) + E(|t|) + L \leq \widehat{B}(|\xi|) + \widehat{E}(|\xi|) + B(t_3) + E(t_3) + L$$

and

$$f(x,t,\xi) \geq A(|\xi|) - E(|t|) - L \geq \widehat{A}(|\xi|) - \widehat{E}(|\xi|) - A(t_1) - E(t_3) - L,$$

for a.e. $x \in \Omega$, and for every $t \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$, equation (4.6) follows, provided that $\widetilde{L}$ is chosen sufficiently large. □

We conclude this section by recalling the following classical lemma – see e.g. [37, Lemma 6.1]:

**Lemma 4.4.** *Let* $Z : [\rho, \sigma] \to [0, \infty)$ *be a bounded function. Assume that there exist constants* $a, b \geq 0$, $\alpha > 0$ *and* $\theta \in [0, 1)$ *such that*

$$Z(r) \leq \theta Z(s) + (s-r)^{-\alpha} a + b \quad \textit{if } \rho \leq r < s \leq \sigma.$$

*Then,*

$$Z(r) \leq c\big((s-r)^{-\alpha} a + b\big) \quad \textit{if } \rho \leq r < s \leq \sigma,$$

*for some constant* $c = c(\alpha, \theta) > 1$.

## 5. Proof of Theorem 2.2

We shall limit ourselves to proving Theorem 2.2, since the content of Theorem 2.1 is just a special case of the former. A key ingredient is provided by Lemma 5.1 below. In the statement, $\Phi_q : [0, \infty) \to [0, \infty)$ denotes the function defined for $q \geq 1$ as

$$\Phi_q(t) = \begin{cases} t & \text{if } 0 \leq t < 1 \\ t^q & \text{if } t \geq 1. \end{cases} \tag{5.1}$$

One can verify that

$$\Phi_q(\lambda t) \leq \lambda^q \Phi_q(t) \qquad \text{for } \lambda \geq 1 \text{ and } t \geq 0. \tag{5.2}$$

Moreover, given a function $u \in W^1K^A(\mathbb{B}_1)$, we set

$$F(u, \rho, \sigma) = \int\limits_{\mathbb{B}_\sigma \setminus \mathbb{B}_\rho} A(|u|) + A(|\nabla u|)\, dx \tag{5.3}$$

for $0 < \rho < \sigma < 1$.

**Lemma 5.1.** *Let A and B be Young functions and* $0 < \rho < \sigma < 1$.

(i) *Suppose that the condition* (2.8) *is in force. Assume that there exist constants* $L \geq 1$ *and* $q > 1$ *such that*

$$B(t) \leq A_{n-1}(Lt) \quad \text{and} \quad B(\lambda t) \leq \lambda^q B(t) \quad \text{for } t \geq 0 \text{ and } \lambda \geq 1. \tag{5.4}$$

*Then, for every* $u \in W^1K^A(\mathbb{B}_1)$ *there exists a function* $\eta \in W_0^{1,\infty}(\mathbb{B}_1)$ *satisfying*

$$0 \leq \eta \leq 1 \text{ in } \mathbb{B}_1, \quad \eta = 1 \text{ in } \mathbb{B}_\rho, \quad \eta = 0 \text{ in } \mathbb{B}_1 \setminus \mathbb{B}_\sigma \quad \|\nabla\eta\|_{L^\infty(\mathbb{B}_1)} \leq \frac{2}{\sigma - \rho}, \tag{5.5}$$

*and such that*

$$\int_{\mathbb{B}_1} B(|u\nabla\eta|)\,dx \leq c\,\Phi_q\left(\frac{\kappa F(u,\rho,\sigma)^{\frac{1}{n-1}}}{(\sigma-\rho)^{\frac{n}{n-1}}\rho}\right)F(u,\rho,\sigma) \tag{5.6}$$

*for some constant* $c = c(n, q, L) \geq 1$*. Here,* $\kappa$ *denotes the constant appearing in the inequality* (3.25).

(ii) *Suppose that the condition* (3.26) *is in force. Assume that there exist constants* $L \geq 1$ *and* $q > n$ *such that*

$$B(t) \leq Lt^q \qquad \text{for } t \geq 0. \tag{5.7}$$

*Then, for every* $u \in W^1K^A(\mathbb{B}_1)$ *there exists a function* $\eta \in W_0^{1,\infty}(\mathbb{B}_1)$ *satisfying* (5.5)*, such that*

$$\int_{\mathbb{B}_1} B(|u\nabla\eta|)\,dx \leq \frac{c\kappa^q F(u,\rho,\sigma)^{\frac{q}{n-1}}}{(\sigma-\rho)^{q-1+\frac{q}{n-1}}\rho^{q-(n-1)}} \tag{5.8}$$

*for some constant* $c = c(n, q, L) \geq 1$*. Here,* $\kappa$ *denotes the constant appearing in the inequality* (3.27).

**Proof.** Let $u \in W^1K^A(\mathbb{B}_1)$. Define, for $r \in [0, 1]$, the function $u_r : \mathbb{S}^{n-1} \to \mathbb{R}$ as $u_r(z) = u(rz)$ for $z \in \mathbb{S}^{n-1}$. By classical properties of restrictions of Sobolev functions to $(n-1)$-dimensional concentric spheres, one has that $u_r$ is a weakly differentiable function for a.e. $r \in [0, 1]$. Hence, by Fubini's theorem, there exists a set $N \subset [0, 1]$ such that $|N| = 0$, and $u_r \in W^1K^A(\mathbb{S}^{n-1})$ for every $r \in [0, 1] \setminus N$. Set

$$U_1 = \left\{ r \in [\rho, \sigma] \setminus N : \int_{\mathbb{S}^{n-1}} A(|\nabla_{\mathbb{S}} u_r(z)|)\,d\mathcal{H}^{n-1}(z) \leq \frac{4}{(\sigma-\rho)r^{n-1}} \int_{\mathbb{B}_\sigma\setminus\mathbb{B}_\rho} A(|\nabla u|)\,\mathrm{d}x \right\}. \tag{5.9}$$

From Fubini's Theorem, the inequality $|\nabla_{\mathbb{S}} u_r(z)| \leq |\nabla u(rz)|$ for $\mathcal{H}^{n-1}$-a.e. $z \in \mathbb{S}^{n-1}$, and the very definition of the set $U_1$ we infer that

$$\begin{aligned}\int_{\mathbb{B}_\sigma\setminus\mathbb{B}_\rho} A(|\nabla u|)\,dx &= \int_\rho^\sigma r^{n-1}\int_{\mathbb{S}^{n-1}} A(|\nabla u(rz)|)\,d\mathcal{H}^{n-1}(z)\,dr\\ &\geq \int_{(\rho,\sigma)\setminus U_1} r^{n-1}\int_{\mathbb{S}^{n-1}} A(|\nabla_{\mathbb{S}} u_r(z)|)\,d\mathcal{H}^{n-1}(z)\,dr\\ &\geq \frac{4((\sigma-\rho)-|U_1|)}{(\sigma-\rho)}\int_{\mathbb{B}_\sigma\setminus\mathbb{B}_\rho} A(|\nabla u|)\,dx.\end{aligned}$$

Hence, $|U_1| \geq \frac{3}{4}(\sigma-\rho)$. An analogous computation ensures that the set

$$U_2 = \left\{ r\in[\rho,\sigma]\setminus N : \int_{\mathbb{S}^{n-1}} A(|u_r(z)|)\,d\mathcal{H}^{n-1}(z) \leq \frac{4}{(\sigma-\rho)r^{n-1}}\int_{\mathbb{B}_\sigma\setminus\mathbb{B}_\rho} A(|u|)\,dx \right\} \tag{5.10}$$

has the property that $|U_2| \geq \frac{3}{4}(\sigma-\rho)$. Thereby, if we define the set

$$U = U_1 \cap U_2,$$

then

$$|U| \geq |(\rho,\sigma)| - |(\rho,\sigma)\setminus U_1| - |(\rho,\sigma)\setminus U_2| \geq \frac{1}{2}(\sigma-\rho). \tag{5.11}$$

Next, define the function $\eta : \mathbb{B}_1 \to [0,1]$ as

$$\eta(x) = \begin{cases} 1 & \text{if } 0 \leq |x| < \rho \\ \dfrac{1}{|U|}\displaystyle\int_{|x|}^{\sigma} \chi_U(s)\,ds & \text{if } \rho \leq |x| \leq \sigma \\ 0 & \text{if } \sigma < |x| \leq 1. \end{cases}$$

One has that $0 \leq \eta \leq 1$, $\eta = 1$ in $\mathbb{B}_\rho$, $\eta = 0$ in $\mathbb{B}_1 \setminus \mathbb{B}_\sigma$, $\eta \in W^{1,\infty}_0(\mathbb{B}_1)$ and

$$|\nabla\eta(rz)| = \begin{cases} 0 & \text{for a.e. } r \notin U \\ \frac{1}{|U|} & \text{for a.e. } r \in U, \end{cases} \tag{5.12}$$

and for $z \in \mathbb{S}^{n-1}$. Hence, the function $\eta$ satisfies the properties claimed in (5.5).

Next, set, for $r \in [0,1]\setminus N$,

$$F_r(u) = \int_{\mathbb{S}^{n-1}} A(|u_r(z)|) + A(|\nabla_{\mathbb{S}} u_r(z)|)\,d\mathcal{H}^{n-1}(z). \tag{5.13}$$

By the definition of the set $U$,

$$F_r(u) \le \frac{4}{(\sigma-\rho)r^{n-1}} F(u,\rho,\sigma) \quad \text{for a.e. } r \in U. \tag{5.14}$$

We have now to make use of different inequalities, depending on whether we deal with case (i) or (ii).

Case (i). Owing to the inequality (3.1) and to the second inequality in (5.4),

$$B(\lambda t) \le \Phi_q(\lambda) B(t) \qquad \text{for } \lambda \ge 0 \text{ and } t \ge 0. \tag{5.15}$$

The following chain holds:

$$\begin{aligned}
\int_{\mathbb{B}_1} B(|u\nabla\eta|)\,dx &\le \int_U r^{n-1} \int_{\mathbb{S}^{n-1}} B\left(\left|\frac{2}{(\sigma-\rho)} u_r(z)\right|\right) d\mathcal{H}^{n-1}(z)\,dr \\
&= \int_U r^{n-1} \int_{\mathbb{S}^{n-1}} B\left(\left|\frac{2\kappa u_r(z) F_r(u)^{\frac{1}{n-1}}}{\kappa(\sigma-\rho) F_r(u)^{\frac{1}{n-1}}}\right|\right) d\mathcal{H}^{n-1}(z)\,dr \\
&\le \int_U r^{n-1} \Phi_q\left(\left|\frac{2L\kappa F_r(u)^{\frac{1}{n-1}}}{(\sigma-\rho)}\right|\right) \int_{\mathbb{S}^{n-1}} A_{n-1}\left(\left|\frac{u_r(z)}{\kappa F_r(u)^{\frac{1}{n-1}}}\right|\right) d\mathcal{H}^{n-1}(z)\,dr \\
&\le \int_U r^{n-1} \Phi_q\left(\left|\frac{2L\kappa F_r(u)^{\frac{1}{n-1}}}{(\sigma-\rho)r}\right|\right) F_r(u)\,dr \\
&\le \Phi_q\left(\left|\frac{2L\kappa 4^{\frac{1}{n-1}} F(u,\rho,\sigma)^{\frac{1}{n-1}}}{(\sigma-\rho)^{1+\frac{1}{n-1}}\rho}\right|\right) 4F(u,\rho,\sigma),
\end{aligned} \tag{5.16}$$

where the second inequality is a consequence of the inequality (5.15) and the first inequality in (5.4), the third inequality follows from the Sobolev inequality (3.25), and the last inequality relies upon the inequality (5.14) and the fact that $|U| \le (\sigma-\rho)$. Clearly, the inequality (5.6) follows from (5.16).

Case (ii). The following chain holds:

$$\begin{aligned}
\int_{\mathbb{B}_1} B(|u\nabla\eta|)\,dx &\le L \int_U r^{n-1} \int_{\mathbb{S}^{n-1}} \left|\frac{2}{(\sigma-\rho)} u_r(z)\right|^q d\mathcal{H}^{n-1}(z)\,dr \\
&\le \frac{L 2^q c_n \kappa^q}{(\sigma-\rho)^q} \int_U r^{n-1} F_r(u)^{\frac{q}{n-1}}\,dr \\
&\le \frac{L 2^q c_n \kappa^q}{(\sigma-\rho)^q} \int_U r^{n-1} \left(\frac{4F(u,\rho,\sigma)}{(\sigma-\rho)r^{n-1}}\right)^{\frac{q}{n-1}} dr \\
&\le \frac{L 2^q 4^{\frac{q}{n-1}} c_n \kappa^q}{(\sigma-\rho)^{q-1+\frac{q}{n-1}} \rho^{q-(n-1)}} F(u,\rho,\sigma)^{\frac{q}{n-1}},
\end{aligned} \tag{5.17}$$

where $c_n$ is given by (3.29), the first inequality holds by the inequality (5.7), the second one by the inequality (3.27), the third one by the inequality (5.14), and the last one since $|U| \le (\sigma - \rho)$. The inequality (5.8) follows via (5.17). □

We are now in a position to accomplish the proof of our main result.

**Proof of Theorem 2.2.** Owing to Lemma 4.3, without loss of generality we can assume that the functions $A$, $B$ and $E$ also satisfy the properties stated for the functions $\widehat{A}$, $\widehat{B}$ and $\widehat{E}$ in the lemma. When we refer to properties in the statement of this lemma, we shall mean that they are applied directly to $A$, $B$ and $E$. In particular, $q$ denotes the exponent appearing in the statement of the lemma. Moreover, $Q$ is the constant from the definition of quasi-minimizer.

We also assume that $\mathbb{B}_1 \Subset \Omega$ and prove that $u$ is bounded in $\mathbb{B}_{\frac{1}{2}}$. The general case follows via a standard scaling and translation argument. For ease of presentation, we split the proof into steps.

*Step 1. Basic energy estimate.*

Set, for $r > 0$ and $l > 0$,

$$\mathcal{A}_{l,r} = \mathbb{B}_r \cap \{x \in \Omega : u(x) > l\} \tag{5.18}$$

and

$$J(l,r) = \int_{\mathbb{B}_r} A((u-l)_+) + A(|\nabla(u-l)_+|)\, dx. \tag{5.19}$$

Here, the subscript "+" stands for the positive part.

If the assumption (2.8) holds, then we claim that there exists a constant $c = c(n, q, L, Q) \ge 1$ such that

$$\int_{\mathbb{B}_\rho} A(|\nabla(u-k)_+|)\, dx \le c\left( \frac{\Phi_q(\kappa J(k,\sigma)^{\frac{1}{n-1}})}{(\sigma-\rho)^{\frac{qn}{n-1}}} J(k,\sigma) + \int_{\mathcal{A}_{k,\sigma}} (E(|u|)+1)\, dx \right) \tag{5.20}$$

for $k \ge 0$ and $\frac{1}{2} \le \rho < \sigma < 1$, where $\kappa$ denotes the constant from the inequality (3.25).

If the assumption (2.10) holds, then we claim that there exists a constant $c = c(n, q, L, Q) \ge 1$ such that

$$\int_{\mathbb{B}_\rho} A(|\nabla(u-k)_+|)\, dx \le c\left( \frac{\kappa^q}{(\sigma-\rho)^{\frac{qn}{n-1}}} J(k,\sigma)^{\frac{q}{n-1}} + \int_{\mathcal{A}_{k,\sigma}} (E(|u|)+1)\, dx \right) \tag{5.21}$$

for $k \ge 0$ and $\frac{1}{2} \le \rho < \sigma < 1$, where $\kappa$ denotes the constant from the inequality (3.27).

We shall first establish the inequalities (5.20) and (5.21) under the assumption (2.11).

Given $k \ge 0$ and $\frac{1}{2} \le \rho < \sigma \le 1$, let $\eta \in W^{1,\infty}_0(\mathbb{B}_1)$ be as in the statement of Lemma 5.1, applied with $u$ replaced with $(u-k)_+$. Choose the function $\varphi = -\eta^q (u-k)_+$ in the definition of quasi-minimizer for $u$. From this definition and the first property in (2.11) one infers that

$$\begin{aligned}\int_{\mathcal{A}_{k,\sigma}} f(x,u,\nabla u)\,dx &\le Q \int_{\mathcal{A}_{k,\sigma}} f(x,u+\varphi,\nabla(u+\varphi))\,dx \\ &= Q \int_{\mathcal{A}_{k,\sigma}} f(x,u+\varphi,(1-\eta^q)\nabla u - q\eta^{q-1}\nabla\eta(u-k))\,dx \\ &\le Q \int_{\mathcal{A}_{k,\sigma}} (1-\eta^q) f(x,u+\varphi,\nabla u) + \eta^q f\Big(x,u+\varphi,-\frac{q\nabla\eta}{\eta}(u-k)\Big)\,dx\,.\end{aligned}$$

Hence, since $0 \le u+\varphi \le u$ on $\mathcal{A}_{k,\sigma}$, the second property in (2.11), the upper bound in (2.13), and the monotonicity of the function $E$ ensure that

$$\begin{aligned}\int_{\mathcal{A}_{k,\sigma}} f(x,u,\nabla u)\,dx \le & Q \int_{\mathcal{A}_{k,\sigma}} (1-\eta^q)\big(Lf(x,u,\nabla u) + E(u) + L\big) \\ & + \eta^q \Big(B\Big(\frac{q|\nabla\eta|}{\eta}(u-k)\Big) + E(u) + L\Big)\,dx. \end{aligned} \tag{5.22}$$

Inasmuch as $0 \le \eta \le 1$ and $\eta = 1$ in $\mathbb{B}_\rho$, the use of the inequality (4.11) on the right-hand side of (5.22) yields:

$$\begin{aligned}\int_{\mathcal{A}_{k,\sigma}} f(x,u,\nabla u)\,dx \le & QL \int_{\mathcal{A}_{k,\sigma}\setminus\mathbb{B}_\rho} f(x,u,\nabla u)\,dx + Q\int_{\mathcal{A}_{k,\sigma}} q^q B\big(|\nabla\eta|(u-k)\big) \\ & + E(|u|) + L\,dx. \end{aligned} \tag{5.23}$$

Now, suppose that the assumption (2.8) holds. Combining the inequality (5.23) with the estimate (5.6) (applied to $(u-k)_+$) tells us that

$$\begin{aligned}\int_{\mathcal{A}_{k,\rho}} f(x,u,\nabla u)\,dx \le & QL \int_{\mathcal{A}_{k,\sigma}\setminus\mathbb{B}_\rho} f(x,u,\nabla u)\,dx + cQ\Phi_q\Big(\frac{2\kappa J(k,\sigma)^{\frac{1}{n-1}}}{(\sigma-\rho)^{\frac{n}{n-1}}}\Big) J(k,\sigma) \\ & + Q\int_{\mathcal{A}_{k,\sigma}} (E(u)+L)\,dx \end{aligned} \tag{5.24}$$

for some constant $c = c(n,q,L) \ge 1$. Observe that in deriving the inequality (5.24), we have exploited the inequalities $\frac{1}{2} \le \rho$ and $F((u-k)_+,\rho,\sigma) \le J(k,\sigma)$.

Adding the expression $QL\int_{\mathcal{A}_{k,\rho}} f(x,u,\nabla u)\,dx$ to both sides of the inequality (5.24) and using the inequality (5.2) enable one to deduce that

$$\int_{\mathcal{A}_{k,\rho}} f(x,u,\nabla u)\,dx \le \frac{QL}{QL+1}\int_{\mathcal{A}_{k,\sigma}} f(x,u,\nabla u)\,dx$$

$$+ c\left( \frac{\Phi_q\left(\kappa J(k,\sigma)^{\frac{1}{n-1}}\right)}{(\sigma-\rho)^{\frac{qn}{n-1}}} J(k,\sigma) + \int_{\mathcal{A}_{k,\sigma}} (E(u)+1)\,dx \right),$$

for some constant $c = c(n, q, L, Q) \geq 1$. The estimate (5.20) follows via Lemma 4.4 and the lower bound in (2.13).

Assume next that the assumption (2.10) holds. Hence, the full assumption (3.26) holds, thanks to equation (4.8). One can start again from (5.23), make use of the inequality (5.8), and argue as above to obtain the inequality (5.21). The fact that

$$\frac{1}{(\sigma-\rho)^{q-1+\frac{q}{n-1}}} \leq \frac{1}{(\sigma-\rho)^{\frac{qn}{n-1}}},$$

since $\sigma - \rho \leq 1$, is relevant in this argument.

It remains to prove the inequalities (5.20) and (5.21) under the alternative structure condition (2.12).

Let $\varphi$ be as above, and observe that $u + \varphi = \eta^q k + (1-\eta^q)u$ on $\mathcal{A}_{k,\sigma}$. Hence, by the property (2.12),

$$\begin{aligned} \int_{\mathcal{A}_{k,\sigma}} f(x,u,\nabla u)\,dx \leq& Q \int_{\mathcal{A}_{k,\sigma}} f(x, u+\varphi, \nabla(u+\varphi))\,dx \\ =& Q \int_{\mathcal{A}_{k,\sigma}} f\big(x, (1-\eta^q)u + \eta^q k, (1-\eta^q)\nabla u - q\eta^{q-1}\nabla\eta(u-k)\big)\,dx \\ \leq& Q \int_{\mathcal{A}_{k,\sigma}} (1-\eta^q) f(x,u,\nabla u) + \eta^q f\left(x, k, -\frac{q\nabla\eta}{\eta}(u-k)\right)dx. \end{aligned}$$

Thanks to the assumption (2.13) and the monotonicity of $E$, which guarantees that $E(k) \leq E(u)$ in $\mathcal{A}_{k,\sigma}$, we obtain that

$$\int_{\mathcal{A}_{k,\sigma}} f(x,u,\nabla u)\,dx \leq Q \int_{\mathcal{A}_{k,\sigma}} (1-\eta^q) f(x,u,\nabla u) + \eta^q \left(L + E(u) + B\left(\frac{q|\nabla\eta|}{\eta}(u-k)\right)\right)dx. \tag{5.25}$$

A replacement of the inequality (5.22) with (5.25) and an analogous argument as above yields the same conclusions.

*Step 2. One-step improvement.*

Let us set

$$c_B = \max\{\kappa, 1\},$$

where $\kappa$ denotes a constant, depending only on $n$, such that the inequality (3.21) holds for every $r \in [\frac{1}{2}, 1]$. We claim that, if $h > 0$ is such that

$$c_B L J(h,\sigma)^{\frac{1}{n}} \leq 1, \tag{5.26}$$

then

$$J(k,\rho) \leq c\left(\frac{1}{(\sigma-\rho)^{\frac{qn}{n-1}}} + \frac{1}{(k-h)^{\frac{n}{n-1}}} + L^{\log_2(\frac{k}{k-h})}\right) J(h,\sigma)^{1+\frac{1}{n}} \qquad \text{if } k > h, \tag{5.27}$$

for a suitable constant $c = c(n, q, L, Q, A) \geq 1$.

To this purpose, fix $h > 0$ such that the inequality (5.26) holds. We begin by showing that there exists a constant $c = c(n, L)$ such that

$$|\mathcal{A}_{k,\sigma}| \leq c\frac{J(h,\sigma)^{\frac{n+1}{n}}}{(k-h)^{\frac{n}{n-1}}} \qquad \text{if } k > h. \tag{5.28}$$

The inequality (5.28) is a consequence of the following chain:

$$\begin{aligned} |\mathcal{A}_{k,\sigma}| A_n(k-h) &= \int_{\mathcal{A}_{k,\sigma}} A_n(k-h)\,dx \leq \int_{\mathcal{A}_{h,\sigma}} A_n(u-h)\,dx \\ &\leq \int_{\mathcal{A}_{h,\sigma}} A_n\left(\frac{c_B(u-h)J(h,\sigma)^{\frac{1}{n}}}{c_B J(h,\sigma)^{\frac{1}{n}}}\right)dx \\ &\leq c_B J(h,\sigma)^{\frac{1}{n}} \int_{\mathcal{A}_{h,\sigma}} A_n\left(\frac{u-h}{c_B J(h,\sigma)^{\frac{1}{n}}}\right)dx. \end{aligned} \tag{5.29}$$

Notice that the last inequality holds thanks to the inequality (3.1), applied with $A$ replaced with $A_n$, and to the assumption (5.26). Coupling the inequality (5.29) with (3.21) enables us to deduce that

$$|\mathcal{A}_{k,\sigma}| \leq \frac{c_B J(h,\sigma)^{\frac{n+1}{n}}}{A_n(k-h)}.$$

Hence the inequality (5.28) follows, via (4.7).

Next, by the monotonicity of $E$ and the assumption (4.9),

$$\begin{aligned} \int_{\mathcal{A}_{k,\sigma}} E(u)\,dx &= \int_{\mathcal{A}_{k,\sigma}} E((u-k)+k)\,dx \leq \int_{\mathcal{A}_{k,\sigma}} E(2(u-k)) + E(2k)\,dx \\ &\leq L \int_{\mathcal{A}_{k,\sigma}} E(u-k) + E(k)\,dx \quad \text{for } k > 0. \end{aligned} \tag{5.30}$$

From the inequality (3.1) applied to $A_n$ and the assumption (5.26) one infers that

$$\int_{\mathcal{A}_{k,\sigma}} E(u-k)\,dx \le \int_{\mathcal{A}_{h,\sigma}} E(u-h)\,dx \le \int_{\mathcal{A}_{h,\sigma}} A_n(L(u-h))\,dx \tag{5.31}$$
$$\le c_B L J(h,\sigma)^{\frac{1}{n}} \int_{\mathcal{A}_{h,\sigma}} A_n\left(\frac{u-h}{c_B J(h,\sigma)^{\frac{1}{n}}}\right) dx \le c_B L J(h,\sigma)^{1+\frac{1}{n}} \quad \text{if } k>h.$$

Owing to the assumption (4.9) and the chain (5.31),

$$\int_{\mathcal{A}_{k,\sigma}} E(k) = E\left(\frac{k}{k-h}(k-h)\right)|\mathcal{A}_{k,\sigma}| \le E\left(2^{\lfloor \log_2 \frac{k}{k-h}\rfloor+1}(k-h)\right)|\mathcal{A}_{k,\sigma}| \tag{5.32}$$
$$\le L^{\log_2(\frac{k}{k-h})+1} E(k-h)|\mathcal{A}_{k,\sigma}| \le L^{\log_2(\frac{k}{k-h})+1} \int_{\mathcal{A}_{h,\sigma}} E(u-h)\,dx$$
$$\le L^{\log_2(\frac{k}{k-h})+1} c_B L J(h,\sigma)^{1+\frac{1}{n}} \quad \text{if } k>h,$$

where $\lfloor\cdot\rfloor$ stands for integer part. Combining the inequalities (5.30)–(5.32) yields:

$$\int_{\mathcal{A}_{k,\sigma}} E(u)\,dx \le c L^{\log_2(\frac{k}{k-h})} J(h,\sigma)^{\frac{n+1}{n}} \quad \text{if } k>h, \tag{5.33}$$

for some constant $c=c(n,L)$.

From this point, the argument slightly differs depending on whether the condition (2.8) or (3.26) holds.

Assume first that (2.8) is in force. The assumption (5.26) implies that there exists a constant $c=c(n,q,L)$ such that

$$\Phi_q(\kappa J(k,\sigma)^{\frac{1}{n-1}}) \le c J(k,\sigma)^{\frac{1}{n-1}} \quad \text{if } k>h, \tag{5.34}$$

where $\kappa$ is the constant from the inequality (3.25). Making use of the inequalities (5.28), (5.33) and (5.34) to estimate the right-hand side of (5.20) results in the following bound for its left-hand side:

$$\int_{\mathbb{B}_\rho} A(|\nabla(u-k)_+|)\,dx \le c\left(\frac{J(h,\sigma)^{\frac{n}{n-1}}}{(\sigma-\rho)^{\frac{qn}{n-1}}} + \frac{J(h,\sigma)^{\frac{n+1}{n}}}{(k-h)^{\frac{n}{n-1}}} + L^{\log_2(\frac{k}{k-h})} J(h,\sigma)^{\frac{n+1}{n}}\right) \tag{5.35}$$
$$\le c'\left(\frac{1}{(\sigma-\rho)^{\frac{qn}{n-1}}} + \frac{1}{(k-h)^{\frac{n}{n-1}}} + L^{\log_2(\frac{k}{k-h})}\right) J(h,\sigma)^{\frac{n+1}{n}} \quad \text{if } k>h,$$

for suitable constants $c=c(n,q,L,Q)\ge 1$ and $c'=c'(n,q,L,Q)\ge 1$. From the inequality (4.1) we infer that

$$\int_{\mathbb{B}_\rho} A((u-k)_+)\,dx \le \int_{\mathbb{B}_\rho} A_n((u-k)_+)\,dx + c|\mathcal{A}_{k,\rho}|$$
$$\le \int_{\mathbb{B}_\sigma} A_n((u-h)_+)\,dx + c|\mathcal{A}_{k,\sigma}| \quad \text{if } k > h, \tag{5.36}$$

for some constant $c = c(n, A)$. A combination of the latter inequality with (5.28) and (5.29) tells us that

$$\int_{\mathbb{B}_\rho} A((u-k)_+)\,dx \le cJ(h,\sigma)^{1+\frac{1}{n}} + c\frac{J(h,\sigma)^{1+\frac{1}{n}}}{(k-h)^{\frac{n}{n-1}}} \quad \text{if } k > h, \tag{5.37}$$

for some constant $c = c(n, L, A)$. Coupling the inequality (5.35) with (5.37) yields (5.27).

Assume now that the condition (3.26) holds. The assumption (5.26) and the inequality $q > n$ guarantee that there exists a constant $c = c(n, q, L)$ such that

$$J(k,\sigma)^{\frac{q}{n-1}} \le cJ(k,\sigma)^{\frac{n+1}{n}} \quad \text{if } k > h. \tag{5.38}$$

From the inequalities (5.28), (5.33) and (5.38) one obtains (5.35) also in this case. The inequality (5.27) again follows via (5.35) and (5.37).

*Step 3. Iteration.*

Given $K \ge 1$ and $\ell \in \mathbb{N} \cup \{0\}$, set

$$k_\ell = K(1 - 2^{-(\ell+1)}), \quad \sigma_\ell = \frac{1}{2} + \frac{1}{2^{\ell+2}}, \quad \text{and} \quad J_\ell = J(k_\ell, \sigma_\ell). \tag{5.39}$$

Thanks to the inequality (5.27), if $\ell \in \mathbb{N}$ is such that

$$c_B L J_\ell^{\frac{1}{n}} \le 1, \tag{5.40}$$

then

$$J_{\ell+1} \le c\left(2^{\ell\frac{qn}{n-1}} + K^{-\frac{n}{n-1}} 2^{\ell\frac{n}{n-1}} + L^\ell\right) J_\ell^{1+\frac{1}{n}} \tag{5.41}$$

for a suitable constant $c = c(n, q, L, Q, A) \ge 1$. Clearly, the inequality (5.41) implies that

$$J_{\ell+1} \le c_2 2^{\gamma\ell} J_\ell^{1+\frac{1}{n}}, \tag{5.42}$$

where $\gamma = \max\{q\frac{n}{n-1}, \log_2 L\}$ and $c_2 = c_2(n, q, L, Q, A) \ge 1$ is a suitable constant. Let $\tau = \tau(n, q, L, Q, A) \in (0, 1)$ be such that

$$c_2 2^\gamma \tau^{\frac{1}{n}} = 1. \tag{5.43}$$

Set

$$\varepsilon_0 = \min\{(c_B L)^{-n}, \tau^n\}.$$

We claim that, if

$$J_0 \le \varepsilon_0, \tag{5.44}$$

then

$$J_\ell \le \tau^\ell J_0 \qquad \text{for every } \ell \in \mathbb{N} \cup \{0\}. \tag{5.45}$$

We prove this claim by induction. The case $\ell = 0$ is trivial. Suppose that the inequality (5.45) holds for some $\ell \in \mathbb{N}$. The assumption (5.44) entails that

$$c_B L J_\ell^{\frac{1}{n}} \le c_B L(\tau^\ell J_0)^{\frac{1}{n}} \le c_B L \varepsilon_0^{\frac{1}{n}} \le 1.$$

Therefore, thanks to the equations (5.42), (5.45), and (5.43),

$$J_{\ell+1} \le c_2 2^{\gamma\ell} J_\ell^{1+\frac{1}{n}} \le c_2 (2^\gamma \tau^{\frac{1}{n}})^\ell J_0^{\frac{1}{n}} (\tau^\ell J_0) \le c_2^{1-\ell} \varepsilon_0^{\frac{1}{n}} \tau^\ell J_0 \le \tau^{\ell+1} J_0. \tag{5.46}$$

Notice that the last inequality holds thanks to the inequalities $c_2 \ge 1$, $\ell \ge 1$, and $\varepsilon_0 \le \tau^n$. The inequality (5.45), with $\ell$ replaced with $\ell + 1$, follows from (5.46).

*Step 4. The assumption* (5.44) *holds for large* $K$.

Since

$$J_0 = J(K/2, \mathbb{B}_{\frac{3}{4}}),$$

the inequality (5.44) will follow, for sufficiently large $K$, if we show that

$$\lim_{k\to\infty} J(k, \mathbb{B}_{\frac{3}{4}}) = 0. \tag{5.47}$$

Inasmuch as $u \in V^1_{\mathrm{loc}} K^A(\Omega)$, from the inclusion (3.20) we infer that $\lim_{k\to\infty} |\mathcal{A}_{k,\frac{3}{4}}| = 0$. Hence, the dominated convergence theorem guarantees that

$$\lim_{k\to\infty} \int_{\mathbb{B}_{\frac{3}{4}}} A(|\nabla(u-k)_+|)\, dx = \lim_{k\to\infty} \int_{\mathcal{A}_{k,\frac{3}{4}}} A(|\nabla(u-k)_+|)\, dx = 0. \tag{5.48}$$

It thus suffices to show that

$$\lim_{k\to\infty} \int_{\mathbb{B}_{\frac{3}{4}}} A(|(u-k)_+|)\, dx = 0. \tag{5.49}$$

To this purpose, note that, by the inequality (4.1) and the monotonicity of $A_n$,

$$\int_{\mathbb{B}_{\frac{3}{4}}} A(|(u-k)_+|)\,dx \le c|\mathcal{A}_{k,\frac{3}{4}}| + \int_{\mathbb{B}_{\frac{3}{4}}} A_n(|(u-k)_+|)\,dx \tag{5.50}$$

$$\le c|\mathcal{A}_{k,\frac{3}{4}}| + \int_{\mathbb{B}_{\frac{3}{4}}} A_n\bigg(2\bigg|(u-k)_+ - \fint_{\mathbb{B}_{\frac{3}{4}}}(u-k)_+dy\bigg|\bigg)\,dx$$

$$+ \int_{\mathbb{B}_{\frac{3}{4}}} A_n\bigg(2\bigg|\fint_{\mathbb{B}_{\frac{3}{4}}}(u-k)_+dy\bigg|\bigg)\,dx$$

for some constant $c = c(n, A)$. Moreover,

$$\lim_{k\to\infty} \mathcal{A}_{k,\frac{3}{4}} = 0,$$

and

$$\lim_{k\to\infty} \int_{\mathbb{B}_{\frac{3}{4}}} A_n\bigg(2\bigg|\fint_{\mathbb{B}_{\frac{3}{4}}}(u-k)_+\bigg|\bigg)\,dx \le \lim_{k\to\infty} |\mathbb{B}_{\frac{3}{4}}| A_n\left(\frac{2\|(u-k)_+\|_{L^1(\mathbb{B}_{\frac{3}{4}})}}{|\mathbb{B}_{\frac{3}{4}}|}\right) = 0.$$

It remains to prove that the second addend on the rightmost side of the chain (5.50) vanishes when $k \to \infty$. Thanks to the limit in (5.48), for every $\delta > 0$ there exists $k_\delta \in \mathbb{N}$ such that

$$\int_{\mathbb{B}_{\frac{3}{4}}} A(|\nabla(u-k)_+|)\,dx \le \delta \qquad \text{if } k \ge k_\delta. \tag{5.51}$$

Choose $\delta$ in (5.51) such that $2c_B\delta^{\frac{1}{n}} \le 1$. The property (3.1) applied to $A_n$, and the Sobolev-Poincaré inequality in Orlicz spaces (3.19) applied to the function $(u-k)_+$ ensure that, if $k > k_\delta$, then

$$\int_{\mathbb{B}_{\frac{3}{4}}} A_n\bigg(2\bigg|(u-k)_+ - \fint_{\mathbb{B}_{\frac{3}{4}}}(u-k)_+\bigg|\bigg)\,dx \le 2c_B\delta^{\frac{1}{n}} \int_{\mathbb{B}_{\frac{3}{4}}} A_n\left(\frac{\big|(u-k)_+ - \fint_{\mathbb{B}_{\frac{3}{4}}}(u-k)_+dy\big|}{c_B\big(\int_{\mathbb{B}_{\frac{3}{4}}} A(|\nabla(u-k)_+|)dy\big)^{\frac{1}{n}}}\right)dx$$

$$\le 2c_B\delta^{\frac{1}{n}} \int_{\mathbb{B}_{\frac{3}{4}}} A(|\nabla(u-k)_+|)\,dx.$$

Since the last integral tends to 0 as $k \to \infty$, equation (5.49) is established.

*Step 5. Conclusion.*

The inequality (5.45) tells us that $\inf_{\ell\in\mathbb{N}} J_\ell = 0$. Hence, from the definitions of $J_\ell$ and $J(h,\sigma)$ we deduce that

$$\int_{\mathbb{B}_{\frac{1}{2}}} A((u-K)_+)\,dx \leq J(K, \mathbb{B}_{\frac{1}{2}}) \leq \inf_{\ell \in \mathbb{N}} J_\ell = 0.$$

Therefore, $u \leq K$ a.e. in $\mathbb{B}_{\frac{1}{2}}$.

In order to prove a parallel lower bound for $u$, observe that the function $-u$ is a quasiminimizer of the functional defined as in (1.1), with the integrand $f$ replaced with the integral $\widetilde{f}$ given by

$$\widetilde{f}(x,t,\xi) = f(x,-t,-\xi) \quad \text{for } (x,t,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n.$$

The structure conditions (2.11) and (2.12) and the growth condition (2.13) on the function $f$ are inherited by the function $\widetilde{f}$. An application of the above argument to the function $-u$ then tells us that there exists a constant $K' > 0$ such that $-u \leq K'$ a.e. in $\mathbb{B}_{\frac{1}{2}}$. The proof is complete. □

## Compliance with ethical standards

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## Data availability

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