Groups Acting with Fixity 4

Dissertation

zur Erlangung des Doktorgrades der Naturwissenschaften
(Dr. rer. nat.)

der

Naturwissenschaftlichen Fakultät II
Chemie, Physik und Mathematik

der Martin-Luther-Universität
Halle-Wittenberg

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Tag der Verteidigung: 26.02.2024

Preface

The origin of group theory is closely related to permutation groups, and hence the action of a group always played a role in the study of groups. On the other hand, in finite group theory, and in this thesis all groups are finite if not otherwise stated, the study of the structure of a group is a key element. Therefore, it is natural to investigate which properties of the action influence which structural properties of the group. This thesis deals with one specific aspect of this interplay. Its objective is the determination and description of groups in which all non-trivial elements fix at most four points and that contain an element with exactly four fixed points. These groups act with the so-called *fixity* 4.

The thesis is part of a project that has its origin in a question asked by Kay Magaard. He and Rebecca Waldecker started the project in 2012. Motivated by the study of Riemann surfaces, one central question of their project is what can be said about groups that act faithfully, transitively, and such that all elements fix at most k points, and what are the finite simple groups with this property, where $k \in \{2, 3, 4\}$. Kay Magaard and Rebecca Waldecker answered the question in 2015 for $k \in \{2, 3\}$. The project was later joined by Barbara Baumeister, and the thesis of Patrick Salfeld also belongs in this context and relates the group-theoretic question back to Riemann surfaces.

Answering the question for $k = 4$ is a joint effort by the aforementioned and myself. The submitted paper [7] classifies all finite simple groups acting with fixity 4, and is therefore closely related to Chapter 3 of this thesis. On the one hand, results of the paper are used as described in Chapter 3. On the other hand, the analysis of Chapter 3 resulted in parts of the paper. This thesis contains more results on the way to answering the original question for $k = 4$. They are positioned after Chapter 3. This work uses the classification theorem of finite simple groups directly and indirectly, and it is an integral component of the proof.

The ultimate aim of this thesis is to gain information about the structure of a group G acting with fixity 4 on some set. One of the subgroups capturing a lot of this information of G is the generalised Fitting subgroup $\mathrm{F}^*(G)$ of G, because G can be described as a subgroup of the semi-direct product of the automorphism group of $\mathrm{F}^*(G)$ and $\mathrm{F}^*(G)$ itself. Thus, the objective of the Main Theorem (Theorem 7.7) of this thesis is to give a detailed description of $\mathrm{F}^*(G)$. To understand $\mathrm{F}^*(G)$, both parts of it, the Fitting subgroup of G and the product of the components of G, have to be studied. Since the components are quasi-simple groups, to do so, quasi-simple groups acting with fixity 4 have to be analysed. Finally, for this, simple groups have to be understood, because every non-abelian simple group is quasi-simple. This thesis is structured in the opposite direction, and its outline is as follows.

In the first two chapters the fundamentals are laid out. They contain, in Section 2.5, an explanation of the usage of GAP, and the role of computer code in the proofs. This part might also be used as a reference resource.

Then in Chapter 3, the first step of the strategy towards the proof of the Main Theorem is done by dealing with the simple groups. In its first section, Lie type groups of small dimension, namely $\mathrm{PSL}(2, q)$, $\mathrm{PSL}(3, q)$, $\mathrm{PSU}(3, q)$, and $\mathrm{Sz}(q)$, are analysed regarding their possible fixity-4 actions. Section 3.2 restricts to the case that the point stabilisers have odd order divisible by 3, and the strategy in this section is structured alongside different subcases. The section afterwards analyses the case that the point stabilisers have order coprime to 6, and its structure is divided among the families of the simple groups as they are described in the classification of finite simple groups. The last section of this chapter bundles together the results of the chapter, the classification of finite simple groups, Theorem 1.2 in [7], and the necessary results published elsewhere in the literature. The result is stated in Theorem 3.56, and classifies all finite simple groups that act transitively and with fixity 4 on some set. It is the first milestone in this thesis.

The next chapter deals with quasi-simple groups and, contrary to most of the other chapters, it contains a section about groups acting with fixity 2 and 3. The final result of the fourth chapter is a list of all quasi-simple non-simple groups that act transitively and with fixity 4.

Then the fifth chapter focuses on components of a group acting with fixity 4. It starts with an excursion about centralisers of involutory automorphisms of simple groups in Section 5.1. Then the first step towards the proof of Theorem 5.6 is to show that under some conditions, each component contains a non-trivial element that fixes some point, or is isomorphic to $\mathcal{A}_5$ or $\mathrm{SL}(2, 5)$. This is done in Section 5.2, before the next section first establishes that every group acting with fixity 4 can have at most one component, and then finishes the proof of Theorem 5.6. The content of this theorem is detailed information about the structure of the unique component of a group acting with fixity 4, if it exists.

It remains to analyse the Fitting subgroup of a group acting with fixity 4. This is done in Chapter 6. First, a general investigation about nilpotent groups acting with fixity at most 4 is done. Afterwards, the analysis is split according to whether the Fitting subgroup contains an element with fixed points or not.

All parts are put together in Chapter 7, where first the interplay between the Fitting subgroup and the components is looked at under some condition. Afterwards the Main Theorem is proven by joining together the above-mentioned results.

This thesis finishes with some closing remarks in Chapter 8.

Acknowledgement

First of all, I would like to thank my supervisor Rebecca Waldecker. She gave me the chance to do this research and she helped me to overcome all the obstacles on my way to writing a PhD thesis. I am thankful for her constant support, her understanding and for all our insightful conversations.

I also thank my colleagues at the Institute of Mathematics in Halle for their kindness, and I especially thank my colleagues in "our part of the corridor" for sharing the ups and downs of my life as a PhD student. In particular, I thank Mara Jakob.

I also owe thanks to Gernot Stroth for sharing parts of his knowledge with me, to Chris Parker for agreeing to review this thesis, and to Barbara Baumeister for our fruitful discussions.

I am thankful for the opportunities to talk about my research in the group theory seminar at Halle and its virtual alternative. I thank the participants of these meetings for their interest and incisive questions, in particular I thank Anika Streck and Christoph Möller.

Furthermore, I had the chance to participate and sometimes present at different conferences during my PhD. They always have been an enrichment and I thankfully acknowledge that I was financial supported to visit these conferences by the grant *Frauenfördermittel* of the Martin-Luther-University Halle-Wittenberg in 2020 and in 2022, by the *SFB-TRR 195 "Symbolic Tools in Mathematics and their Application"*, and by the *Isaac Newton Institute*.

I am grateful for the support of my friends and colleagues who helped me, among other times, in the correction process of this thesis, in particular Atle Höhne, Gabriel Guckenbiehl, Imke Toborg, Mathias Grimm, Patrick Salfeld, and Ruth Hoffmann. Most of all, Wilf Wilson helped me with my problems with the English language. I am thankful for all of his help, for his patience, and his kindness.

Last but not least, I thank all my friends and my family for their constant support in my life and for their friendship.

Contents

1 Introduction

Two important ways of describing a group are by its action and by its inner structure. Therefore, one fundamental research interest in group theory is to determine the connection between these two ways of describing a group. This leads to the question of which information about the inner structure of a group is encoded in its action or even just in partial information about its action. One such partial description of a group action is the behaviour of fixed points.

In this context, the work of Frobenius is remarkable. He analysed transitive permutation groups in which no non-trivial element fixes two or more points and the point stabilisers are non-trivial. These groups are now named *Frobenius groups*. Let G be a Frobenius group, and let K be the subset of G that consists of the identity element together with those elements that do not fix any points. Then Frobenius showed in 1901 in [33] that K is a subgroup of G, known as the *Frobenius kernel*, thus revealing structural information about G. A consequence is that Frobenius groups cannot be simple, because the Frobenius kernel of a Frobenius group is a proper non-trivial normal subgroup. This illustrates that even limited information about the number of points which are fixed by an element of a group gives rise to fundamental structural properties of this group. However, the understanding of Frobenius groups did not stop with Frobenius' result.

Later, in 1959, Thompson showed in [98] that the Frobenius kernel of a group is nilpotent, revealing even more inner structure of the group. Nowadays, even the structure of the point stabilisers of Frobenius groups is well understood (see for example Theorem 10.3.1 in [38]). These examples imply that even a seemingly small restriction of the action can have a huge influence on the structure of the group. Thus, generalising this concept might be a fruitful way of gaining information in group theory, and understanding the connection between groups and their actions even better.

There are different ways of generalising the definition of Frobenius groups and asking the same question, namely which inner structure of the group can be determined from its action. One way uses the notion of *fixity*, first introduced by Ronse in 1980 in [84], which is the central concern of this thesis. The definition is as follows.

Definition 1.1
Let k be a non-negative integer and G a finite group acting on a set Ω. Then G acts with *fixity* k on Ω if and only if all non-trivial elements of G have at most k fixed points in Ω and there exists a non-trivial element with exactly k fixed points in Ω.

In most situations the definition is joined with the assumption of transitivity. Additionionally, a faithful action is often taken as granted. Instead of using the latter hypothesis directly, that a group G acts faithfully and with fixity k on some set Ω, it suffices to assume that G acts with fixity k on Ω and that Ω has at least $k+1$ elements. This is, because in that case, no non-trivial element of G can fix all points in Ω, since it otherwise would fix more than k points, violating the fixity-k assumption. Throughout this thesis, either faithfulness or a restriction on the size of the set is used, depending on what is convenient in each particular situation. On the other hand, transitivity is almost always assumed.

In terms of fixity, Frobenius groups are exactly those groups acting transitively and with fixity 1 on a set of size at least 2. Groups acting transitively and with fixity 0 are regular groups, and an example of a group acting with fixity 4 is the symmetric group $\mathcal{S}_6$ acting in its natural action on a set of size 6. More generally, for every integer $n \geq 3$, the symmetric group $\mathcal{S}_n$ acting in its natural action on a set of size n is an example for a group acting with fixity $n-2$. In particular, groups acting with fixity at most 4 form a larger class of groups than that of Frobenius groups. Thus, the concept of fixity is indeed a way of generalising Frobenius groups. Now the question about the connection between the inner structure and the action of a group can be asked again by restricting to groups acting with fixity at most 4.

The question is not purely of theoretical interest in its own right, but additionally, it has implications in other areas of group theory. One example is [22], where results about groups acting transitively and with fixity 2 or 3 are used to calculate the independence number of the Saxl graph of a finite almost simple primitive base-two group with soluble point stabilisers (see Theorem 1.5 in [22]).

Related to the notion of fixity, there are also other ways of generalising the concept of a Frobenius group, which lead and have led to diverse research interests into the connections between the number of fixed points of elements of a group and structural properties of this group. This started soon after Frobenius' result, and continued both before and after Ronse introduced the term fixity.

One of the first generalisations of Frobenius groups are the so-called *Zassenhaus groups*. These are the groups that act with fixity at most 2 but with the additional condition that the action is 2-transitive (see p. 341 in [58]). The first step towards a classification of these groups was done by Zassenhaus in 1935 in [107], in which he classified all sharply 3-transitive groups. In 1960, Suzuki discovered in [94] a family of groups that are Zassenhaus groups but are not sharply 3-transitive. They are now called *Suzuki groups* and represent a family of finite simple groups. The classification of Zassenhaus groups continued in 1960, when Feit showed in Theorem 1 in [31] that every Zassenhaus group acts on a set of size $q+1$, where q is a power of a prime p, or contains a normal subgroup of order $q+1$. In 1962, Suzuki classified all simple Zassenhaus groups in Theorem 15 in [96] for all even q, and Ito proved in [58] that if q is odd, then the Sylow p-subgroups of the Zassenhaus group are abelian, finishing the classification of simple Zassenhaus groups (see Chapter XI in [55]). Later, Glauberman gave in [37] a more direct proof of Ito's result.

The study of Zassenhaus groups set a new direction in the development of research in group theory (cf. [9]), showing how fruitful the analysis of the connection between the action of a group and its structure can be. The idea of Frobenius groups was even further generalised, when after 1962, both Ito and Suzuki generalised the concept of Zassenhaus groups in different ways in [57] and [97], respectively. In addition, Frobenius groups have been generalised in other directions since Zassenhaus started his investigations.

An example of such a different direction was described by Pretzel and Schleiermacher in 1975 in [79]. They defined, for a positive integer n and non-negative integers $a_1, a_2, \ldots, a_n$ with the condition that $a_1 < a_2 < \ldots < a_n$, an $(a_1, a_2, \ldots, a_n)$-*group* to be a group acting on a set in such a way that $a_1, a_2, \ldots, a_n$ are exactly the numbers of fixed points of the non-trivial group elements. In this notation, Frobenius groups are the transitive $(0, 1)$-groups. This took up an idea by Iwahori, who in 1964 in [59] introduced classes of groups that, in the notation of Pretzel and Schleiermacher, are the (k)-groups, for each positive integer k. Iwahori also classified in Theorem III in [59] the (2)-groups, and in 2001 Cameron continued the study of (k)-groups in [25], while Pretzel and Schleiermacher concentrated on faithful and transitive $(0, k)$-groups. More precisely, in [79], they analysed, for a prime p, the faithful and transitive $(0, p)$-groups, after first giving some general information about arbitrary $(0, k)$-groups. Using these results, they classified in [81] all faithful and transitive $(0, 3)$-groups where the point stabilisers are not so-called TI-subgroups of the group. A group H is called a *TI-subgroup* (trivial intersection subgroup) of a group G if $H \leq G$ and, for all $g \in G$, $H^g \cap H = 1$ unless $H^g = H$. Motivated by the consequence of 1.3. in [79] that the point stabilisers in a $(0, 2)$-group are TI-subgroups of the group, they further studied structural properties of groups in this context in [80]. Likewise, the definition of a Frobenius group implies that the point stabilisers are TI-subgroups of the group. Again generalising this idea, Hale established in 1971 in [49] conditions that guarantee that the point stabilisers of a transitive permutation group are TI-subgroups of this group.

In addition to results that aim to classify groups with certain properties of their action, of which we have seen a few, there are also other research interests on the connection of the action of a group and its structure that are centred on the notion of fixity. One such interest is to give bounds of structural quantities in terms of a size related to the action, like for instance the fixity a group is acting with. An example of such research is [90] by Saxl and Shalev, who identified quantifiable properties of a transitive solvable group that are bounded by the fixity the group is acting with. Among the properties they determined is the order of a point stabiliser if the group is nilpotent (see Theorem 1.1 in [90]). Saxl and Shalev also determined conditions that guarantee that the fixity is bounded, and they applied their results to groups acting as automorphism groups of another group. Their proofs give a deeper insight into the concept of fixity. An good overview of this research direction of restricting structural quantities is given in [20] by Burness, which centres on the fixed point ratio of group elements, and considers, amongst other things, the relation to the notion of fixity. The publication by Burness demonstrates how strong the connection between the study of fixity and other research areas is. The notion of the *minimal degree* of a group especially stands out as being closely related to the fixity a group is acting with, because the minimal degree of a transitive group G acting on a set is the minimal number of points moved by any non-trivial element of G, and hence the sum of the minimal degree and the fixity of G equals the size of the set. As a consequence, a lower bound on the minimal degree determines an upper bound on the fixity, and vice versa. The study of bounds for the minimal degree of a permutation group dates back to at least 1870, when Jordan in [60] analysed the minimal degree of multiply transitive groups in Théorème 83 and the subsequent corollaries, and has been a research interest later on (see § 15 in [103]). The structure of these results is to give a list of exceptions and a general bound for the remaining groups. Without the use of the classification theorem of finite simple groups, Theorem 0.3 in [5], proven by Babai in 1981, was the best known lower bound of the minimal degree of primitive permutation groups that do not act 2-transitively. Using the classification theorem of finite simple groups, the result was improved in 1984 by Liebeck in the corollary in [67], and in 1991 by Liebeck and Saxl in Theorem 6.1 in [68], with different lists of exceptions. In the latter publication, also an application of the result to monodromy groups of covers of compact Riemann surfaces is described, proving a conjecture in [48]. The list of exceptions by Liebeck and Saxl was studied even further by Guralnick and Magaard in 1998 in [46], using the fixed point ratio, and again having the connection to Riemann surfaces in mind. Outside of group theory, these bounds are not only used in the context of Riemann surfaces but also have other applications, as for instance in [78]. These are results concentrating on lower bounds on the minimal degree, and thus on upper bounds on the fixity, whereas [70] concentrates on lower bounds on the fixity, and hence establishing even more insights into the notion of fixity.

Instead of looking at the number of fixed points of all elements, it is also interesting to restrict the condition to some class of group elements, for example the involutions. One result in this context was gained by Bender in 1971 in [11], in which he analysed transitive groups in which all involutions fix exactly one point. The paper is a fundamental contribution to the understanding of the inner structure of groups, and introduces the notion of *strongly embedded subgroups* (see Definition 2.6). This shows once more that the question about the link between the action of a group and its inner structure is a driving force in group theory research. Therefore it is unsurprising that many researchers have looked into the connection between the number of fixed points of involutions and the group structure. Among them is Hering, who studied in 1968 in [50] 2-transitive groups in which the maximal number of fixed points of involutions is 2. His analysis is supplemented by [10], in which Bender concentrated on 2-transitive groups in which no involution fixes a point. Then in 1971, Satz 3 in the aforementioned publication [11] complemented the analysis. Continuing the investigation of 2-transitive groups in which the maximal number μ of fixed points of involutions is small, King published results for $\mu = 3$ in [61], and for $\mu = 4$ the analysis was started by Noda in [75] and continued by Bueckenhout in [16]. Similarly, the same structural questions arise for transitive groups in the same way as for 2-transitive groups, and hence trying to classify all transitive groups in which the maximal number μ of fixed points of involutions is small is a research interest.

Again, the remarkable publication [11] gives insights and an answer for $\mu = 1$ in this context. In [17], Bueckenhout looked into the situation for $\mu = 3$, and in a series of three papers ([85], [18], and [19]), Rowlinson and Bueckenhout settled results for $\mu = 4$. Additionally, in [87] Rowlinson started the analysis for the case that $\mu \leq 5$, and continued it in [88] for $\mu \leq 7$. The investigation for $\mu = 5$ was supplemented by Hirmanine in [51], in which he concentrated on primitive groups in which the maximal number of fixed points of involutions is 5. For $\mu = 6$, again Rowlinson continued the analysis in [86]. Their results were summarised and generalised in 1982 in [83] by Ronse, in which he described the situation for $\mu \leq 15$. More recently, Burness and Covato in [21] and Burness and Thomas in [23] have taken up this topic again. As seen earlier, even though it can be formulated as a purely group-theoretic question, it has application beyond finite group theory. For example, [6] describes an application at the interface of finite geometry and permutation group theory. One more note on the result of Buekenhout and Rowlinson, as written in Table 1 in [19], has to be made, since it is of direct use for this thesis, because in a transitive group that acts with fixity 4, every involution can fix at most four points. The result was reviewed by Salfeld in [89] under the additional hypothesis that the group itself acts with fixity 4. This revealed that Table 1 in [19] is missing the fact that the group $\mathrm{PSL}(2, 9)$ can act transitively and such that the maximal number of fixed points of involutions is 4 if the point stabilisers are dihedral groups of order 6. Therefore his result, instead of that of Buekenhout and Rowlinson, is cited when needed.

Besides [89], there are other results aiming to classify transitive groups acting with small fixity where the notion of fixity as given in Definition 1.1 is used. Magaard and Waldecker list all simple groups acting with fixity 2 and 3 in [71] and [72], respectively. Both publications also give, in different degrees of detail, information about the general structure of groups acting with fixity 2 and 3. They started a project (see [100]), of which this thesis is a part. Thus, these papers also motivate the strategy and aim of this thesis, especially in the amount of detail that is reasonably provable in a general structure result. Later, in [8], Baumeister joint Magaard and Waldecker to additionally give an insight into the Sylow structure of transitive groups acting with fixity 4. The motivation of Magaard and Waldecker for the project comes from Riemann surfaces.

Riemann surfaces are complex analytic manifolds (see Definition 3.0.4 in [27]). Their study is used and has applications in different parts of mathematics and even physics. The origin of the concept of Riemann surfaces dates back to Riemann and his analysis of complex functions in [82], and was set on new foundations by Weyl in his lectures [102]. One of the fields of interest in the research topic of Riemann surfaces, more precisely compact Riemann surfaces of genus at least 2, are Weierstrass points (see III.5.9 on page 87 in [30] for a description) because they "carry a lot of information about the surface" (page 8 in [30]). Additionally, Weierstrass points have influence on the automorphism group of compact Riemann surfaces of genus at least 2. This can be seen for example in section V.1.2 in [30], where the interplay between the Weierstrass points and the automorphism group is explained.

There, two proofs for results about the finiteness of the automorphism group of a compact Riemann surfaces of genus g at least 2 are described. One of them was first given by Schwarz in [92], and states that the automorphism group of such a Riemann surface is finite. It was later improved by the result by Hurwitz on page 424 in [56] that gives an explicit bound of the automorphism group of such a Riemann surface, namely $84(g-1)$, and this bound is sharp. This is the second result proven in section V.1.2 in [30], which shows a connection between the theory of Riemann surfaces (and Weierstrass points) with finite group theory. Another connection was found by Schoeneberg in [91], implying that if a non-trivial automorphism of a Riemann surface has more than four fixed points, then each of these fixed points is a Weierstrass point, and hence contains analytic information.

To fill the gap and also gain information in the case that there do not exist Weierstrass points, Magaard asked what can be achieved by looking at this case group-theoretically, and initiated together with Waldecker the aforementioned project to study the problem. Since the automorphism group of a compact Riemann surface of genus at least 2 is finite, the question translates to finite groups in which each non-trivial element has at most four fixed points in its action on the surface. Restricting to the action on orbits rather than the whole surface does not exclude any automorphism groups, because if every element has in total at most four fixed points on the whole surface, then it also has at most four fixed points on each orbit. Thus, the question translates to all groups acting transitively and with fixity at most 4 on some set, and hence to a purely group-theoretic question, which can be studied with the methods of finite group theory. In this general setup, two special kinds of orbits can be identified. The first one is characterised by the condition that the group acts regularly on it and the second is characterised by the condition that the group does not act faithfully on it. In the first case, the study of the notion of fixity will not give further insights, whereas the fact that the group acts with fixity 4 on the whole set implies that each orbit fulfilling the second condition contains at most four elements (and there exists at most four of them). In particular, leaving these two kinds of orbits aside, on each remaining orbit the group acts transitively, faithfully, and with fixity at least 1 and at most 4. Otherwise, if all orbits of the group action belong to one of the two special kinds, then this fact gives additional information of a different variety. Therefore, it is reasonable to restrict the group-theoretical study to transitive and faithful groups acting with fixity at most 4 on some set. As mentioned earlier, there are results about groups acting with fixity 1 (those about Frobenius groups), 2 and 3. Thus, the remaining question concerns finite, transitive, and faithful groups acting with fixity 4, and these groups are the central concern of this thesis.

One step in the project by Magaard and Waldecker is to identify all finite simple groups that act transitively and with fixity 2, 3, or 4. Note that fixity 1 is missing in this list because non-trivial transitive groups acting with fixity 1 are Frobenius groups, which are not simple. There have been different achievements towards a classification. In the first phase of the project by Magaard and Waldecker, they determined all finite simple groups acting transitively and with fixity 2 and 3. Later, some first results for the classification of finite simple groups acting with fixity 4 regarding the Sylow 2- and 3-subgroup structure were done by Baumeister, Magaard and Waldecker. Afterwards, all finite simple groups acting transitively and with fixity 4 and such that an involution fixes exactly four points were identified by Salfeld. However, the classification for fixity 4 remained incomplete until this thesis, and is now part of the joint publication [7]. The publication also contains an analysis of the 3-structure of a finite group acting with fixity 4 done by Baumeister, Magaard, and Waldecker, which exceeds the results in [8].

In addition to its group-theoretic implications, the answer can be related back to the context of Riemann surfaces. There, the question of whether or not a group, or more concretely a simple group, can actually act as an automorphism group of a Riemann surface and with fixity at most 4 is interesting. Patrick Salfeld addressed this question for simple groups in his thesis [89]. Since, at the time of his writing, a result was not known, he used a conjecture of which finite simple groups can act with fixity 4. This thesis also proves that this conjecture was correct, thus achieving that all simple non-abelian automorphism groups of compact Riemann surfaces of genus at least two that act with fixity at most 4 are classified.

2 Preliminaries

This chapter contains concepts and initial results that will be used in the subsequent chapters. Most of these can be found in [65], for example, along with other fundamental group-theoretic terms. This chapter also establishes the notation that will be used throughout.

If a notion can have different meanings, then in most cases a definition is given for reference, either in this chapter, or when the notion is first used. Alternatively, the meaning becomes clear from the context, and no definition will be given explicitly.

2.1 Notation

The notation usually follows [65] and [105], with a few exceptions. One of the exceptions is the notation of the finite simple classical groups of Lie type, for which the notation in [54] is used. For the purpose of this section let n always be a positive integer, let q always be a prime power, and let p always be a prime. The symmetric group acting on the set $\{1,\dots,n\}$ is denoted by $\mathcal{S}_n$ and the alternating group acting on $\{1,\dots,n\}$ is denoted by $\mathcal{A}_n$. They both have degree n. In general, a permutation group is a group acting faithfully on a set Ω, and has degree $|\Omega|$.

There exist different notations for the structure of a group. Usually an isomorphism type is described by specifying a concrete group of that type. Thus a group of type $\mathrm{PSL}(2,5) \times \mathrm{PSL}(2,5)$ is some group that is isomorphic to the permutation group $\langle (1,2,3),(3,4,5),(6,7,8),(8,9,10)\rangle \leq \mathcal{S}_{10}$. Furthermore, D_n always denotes a dihedral group of order n, E_q denotes an elementary abelian group of order q, and C_n denotes a cyclic group of order n. Moreover, Q_{2^n} denotes a generalised quaternion group of order 2^n and SD_{2^n} denotes a semi-dihedral group of order 2^n. If the structure of a group is not specified further, then the order is written in square brackets. Thus, for example, a group of type [4] can be of type E_4 or C_4.

For the extension of one group by another group, the notation in Section 5.2 in [28] is used, except in the case of a central product of two groups A and B, which is denoted by $A * B$.

While describing groups, their orders are often of importance. Thus, some notation to specify and relate group orders is needed. The order of a single group element x will be denoted by $\mathrm{o}(x)$. Furthermore, the highest positive integer k such that n is divisible by p^k will be denoted by $|n|_p$.

We will see one last notation, which will be useful in the context of fixed points of a group G that acts on a set Ω. For a subset X of G, the set $\mathrm{fix}_\Omega(X)$ contains exactly those points in Ω that are fixed by all elements in X. If X contains only one element x, then $\text{fix}_\Omega(x)$ is an abbreviation for $\text{fix}_\Omega(X)$. Using this notation, a non-trivial group G acting on a set Ω acts with fixity $\max\limits_{x\in G\setminus\{1\}} |\text{fix}_\Omega(x)|$.

2.2 Groups, Subgroups, and Actions

The following two lemmas state basic properties about subgroups and normal subgroups, and will normally be used without citing. Nevertheless proofs are given.

Lemma 2.1
Let G be a finite group and let A, B, and C be subgroups of G such that $A \geq B$. Then $|A \cap C : B \cap C| \leq |A : B|$.

Proof:
The order formula for products of groups yields that

$$|A \cap C : B \cap C| = \frac{|A| \cdot |C| \cdot |BC|}{|AC| \cdot |B| \cdot |C|} \leq \frac{|A| \cdot |BC|}{|BC| \cdot |B|} = |A : B| \,.$$

Lemma 2.2
Let G be a finite group acting transitively and faithfully on a set Ω. Let N be a normal subgroup of G and let $\alpha \in \Omega$.

Then for all $\beta \in \Omega$, there is a one-to-one correspondence between the action of N on α^N and the action of N on β^N. More precisely, for every $\beta \in \Omega$, there exists an element $g \in G$ and an automorphism ψ of N such that for all $\omega \in \alpha^N$ and all $h \in N$, $\omega^g \in \beta^N$ and $(\omega^h)^g = (\omega^g)^{h^\psi}$. In particular, for all $\beta \in \Omega$, the following hold:

(a) $|N_\alpha| = |N_\beta|$ and $|\alpha^N| = |\beta^N|$.

(b) N acts faithfully on α^N if and only if N acts faithfully on β^N.

(c) If k is a non-negative integer, then N acts with fixity k on α^N if and only if N acts with fixity k on β^N.

Proof:
Let $\beta \in \Omega$. Since G acts transitively on Ω, there exists an element $g \in G$ such that $\alpha^g = \beta$. Let ψ denote the function on N defined by conjugation by g. Then ψ is an automorphism on N.

Let $\omega \in \alpha^N$ and $h \in N$. Then there exists an element $a \in N$ such that $\omega = \alpha^a$. Hence, $\omega^g = \alpha^{ag} = \alpha^{gg^{-1}ag} = (\alpha^g)^{a^g} \in \beta^N$ and $(\omega^h)^g = (\omega^g)^{h^g} = (\omega^g)^{h^\psi}$.

All other statements in the lemma are direct consequences.

Another theorem whose statement will normally be used without referencing it is the Theorem in [32], known as the Feit-Thompson Theorem. It is a fundamental result and states that every group of odd order is solvable.

One of the use cases of the Feit-Thompson Theorem is in the context of coprime action. Some results are summarised in the next lemma.

Lemma 2.3
Let G be a finite group. Let A and K be subgroups of G such that A acts by conjugation on K and such that $|A|$ and $|K|$ are coprime. Then the following hold:

(a) If $K \trianglelefteq G$, then $\mathrm{N}_{G/K}(AK/K) = \mathrm{N}_G(A)K/K$.

(b) If A is non-cyclic abelian, then $K = \langle \mathrm{C}_K(a) \mid a \in A \setminus \{1\} \rangle$.

(c) If K is a p-group for some odd prime p and A acts trivially on $\Omega_1(K)$, then A acts trivially on K.

Proof:
For part (a), suppose that $K \trianglelefteq G$. Then $\mathrm{N}_G(A)K/K \leq \mathrm{N}_G(AK)K/K \leq \mathrm{N}_{G/K}(AK/K)$. Let $g \in G$ be such that $Kg \in \mathrm{N}_{G/K}(AK/K)$. Then for all $a \in AK$, it follows that $Ka^g = (Ka)^{Kg} \in AK$. In particular, $a^g \in AK$ and hence $Kg \in \mathrm{N}_G(AK)K/K$. Thus, $\mathrm{N}_G(AK)K/K = \mathrm{N}_{G/K}(AK/K)$. The Schur-Zassenhaus Theorem (see 6.2.1 in [65]) applied to the group AK yields that K has complements in AK (one of them is A) and that AK acts transitively on the set of these complements. Then by a Frattini argument (see 3.1.4 in [65]), $\mathrm{N}_G(AK) = \mathrm{N}_{\mathrm{N}_G(AK)}(A) \cdot AK = \mathrm{N}_{\mathrm{N}_G(AK)}(A)K$. As a consequence,

$$\mathrm{N}_G(AK)K/K = \mathrm{N}_{\mathrm{N}_G(AK)}(A)K/K \leq \mathrm{N}_G(A)K/K \leq \mathrm{N}_{G/K}(AK/K)\,,$$

thus (a) is true.

Part (b) holds by 8.3.4 in [65] and part (c) follows directly from Theorem 5.3.10 in [38].

In the previous lemma, we have seen the use of one version of a Frattini argument. In the next lemma, a different version gives us information about the orders of normalisers in normal subgroups of a group.

Lemma 2.4
Let G be a finite group with a normal subgroup N of index m. Let $p \in \pi(N)$ and $P \in \mathrm{Syl}_p(N)$. Then $|\,\mathrm{N}_G(P)| = m \cdot |\,\mathrm{N}_N(P)|$.
Proof:
By a Frattini argument (see 3.2.7 in [65]), $G = N \cdot \mathrm{N}_G(P)$. Then

$$m \cdot |N| = |G| = |N \cdot \mathrm{N}_G(P)| = \frac{|N| \cdot |\,\mathrm{N}_G(P)|}{|\,\mathrm{N}_G(P) \cap N|} = \frac{|N| \cdot |\,\mathrm{N}_G(P)|}{|\,\mathrm{N}_N(P)|}$$

and the lemma follows.

We have seen an example of how information about a normal subgroup can give rise to information about the group. Likewise, the absence of normal subgroups can have significant implications. One example is the following result about the indices of subgroups in quasi-simple groups.

Lemma 2.5
Let E be a quasi-simple group and U a proper subgroup of E. Then $|E : U| > 4$.

Proof:

Assume, for a contradiction, that $|E:U| \leq 4$. Then E acts transitively on E/U. Let K be the normal subgroup of E that fixes all elements in E/U. Since E is quasi-simple, either $E = K$ or $K \leq \mathrm{Z}(E)$. The first case contradicts the transitive action of E. Thus, $K \leq \mathrm{Z}(E)$ and $E/\mathrm{Z}(E)$ acts faithfully on E/U. Therefore, $E/\mathrm{Z}(E)$ is isomorphic to a subgroup of $\mathcal{S}_4$. Since $E/\mathrm{Z}(E)$ is non-abelian simple, this is a contradiction.

A quasi-simple group E that acts transitively and with fixity 4 contains an element x that has exactly four fixed points. Thus, the previous result can be strengthened for the point stabilisers. The fact that the index of a point stabiliser of E in E is at least 5 implies that E acts on a set with at least five elements but x fixes exactly four points, and hence moves a fifth point to a sixth. In particular, E acts on a set of size at least 6, the index of each point stabiliser of E in E is at least 6, and since E does not contain non-trivial elements with more than four fixed points, E acts faithfully. The latter is true for every transitive quasi-simple group acting with fixity at most 4, in general. In other words, by Lemma 2.5, a quasi-simple group that acts transitively and with fixity at most 4 additionally acts faithfully.

In the context of subgroups that give additional information, there is another important concept that will be used frequently, namely that of a strongly p-embedded subgroup. We will first see the definition and afterwards some equivalent characterisations.

Definition 2.6

Let G be a finite group, H a proper subgroup of G, and p a prime divisor of $|H|$. Then H is *strongly p-embedded in G* if and only if for all $g \in G \setminus H$ the order of $H \cap H^g$ is not divisible by p.

Usually the term strongly 2-embedded is abbreviated as *strongly embedded*.

Using this definition as the point of reference, the following lemma will state the equivalence to some other definitions in the literature.

Lemma 2.7

Let G be a finite group, H a proper subgroup of G, p a prime divisor of $|H|$, and $P \in \mathrm{Syl}_p(H)$. Then the following statements are equivalent.

(1) H is strongly p-embedded in G.

(2) $\mathrm{N}_G(P) \leq H$, and for every element y of P of order p, $\mathrm{C}_G(y) \leq H$.

(3) For every non-trivial subgroup Q of P, $\mathrm{N}_G(Q) \leq H$.

Proof:

This follows from Proposition 17.11 in [42] together with Definition 17.1 in [42].

The next lemma states a sufficient condition for a group to contain a strongly p-embedded subgroup. The proof also gives some understanding of strongly p-embedded subgroups.

Lemma 2.8
Let G be a finite group and let p be a prime dividing $|G|$. Suppose that G has cyclic Sylow p-subgroups. Then either every subgroup of order p of G is normal in G or G has a strongly p-embedded subgroup.

Proof:

Suppose that there exists a subgroup U of order p of G that is not normal in G. Let $P \in \mathrm{Syl}_p(G)$ and $H = \mathrm{N}_G(\Omega_1(P))$. We will see that H is strongly p-embedded in G. Since U is conjugate to $\Omega_1(P)$, the only subgroup of order p of P, it follows that $|H| = |\mathrm{N}_G(\Omega_1(P))| = |\mathrm{N}_G(U)| < |G|$. Furthermore, the order of H is divisible by p.

Let $y \in P$ be of order p. Then by Lemma 2.7, the only two assertions left to be shown are that $\mathrm{C}_G(y) \leq H$ and that $\mathrm{N}_G(P) \leq H$. Since $\langle y \rangle = \Omega_1(P)$, it follows that $\mathrm{C}_G(y) \leq \mathrm{N}_G(\Omega_1(P)) = H$. The remaining statement that $\mathrm{N}_G(P) \leq H$ follows from the fact that $\Omega_1(P)$ is a characteristic subgroup of P.

The previous lemma can be rephrased for odd primes p using the notion of the p-rank of a group. In the literature, there exist different notions for the p-rank of a group. Here we follow the definition in [2] on page 5. Hence, for a prime p the p-rank of a group G is always the largest integer k such that a Sylow p-subgroup of G contains an elementary abelian group of order p^k. Similarly the sectional p-rank of a group G is the largest integer k such that there exists two subgroups K and L of G such that $K \trianglelefteq L$ and that L/K is elementary abelian of order p^k. For every odd prime p, all p-groups of p-rank 1 are cyclic (see 5.3.8 in [65]). As a consequence, the previous lemma shows that a simple non-abelian group G contains a strongly p-embedded subgroup if p is an odd prime and the p-rank of G is 1.

2.3 Counting Arguments for Fixed Points and Number Theory

In different situations, it will be quite useful to determine the number of fixed points of an element. The most common situations are dealt with in the next few lemmas.

Lemma 2.9
Let G be a finite group acting transitively on a set Ω. Let $\alpha \in \Omega$. Then the number of fixed points of an element $x \in G_\alpha$ is

$$\frac{|\{\langle x \rangle^g \mid g \in G \text{ and } \langle x \rangle^g \leq G_\alpha\}| \cdot |\mathrm{N}_G(\langle x \rangle)|}{|G_\alpha|}.$$

Proof:

Let $y \in G$. Then $G_\alpha y$ is a fixed point of x under its action on G/G_α if and only if $x^{y^{-1}} \in G_\alpha$. Therefore, the number of fixed points of x is

$$\frac{|\{y \in G \mid x^{y^{-1}} \in G_\alpha\}|}{|G_\alpha|} = \frac{|\{g \in G \mid x^g \in G_\alpha\}|}{|G_\alpha|} = \frac{|\{\langle x\rangle^g \leq G_\alpha \mid g \in G\}| \cdot |\mathrm{N}_G(\langle x\rangle)|}{|G_\alpha|}.$$

A special case of the previous lemma is when a point stabiliser is cyclic. This is the situation studied in the following corollary.

Corollary 2.10

Let G be a finite group acting transitively on a set Ω. Let $\alpha \in \Omega$. If G_α is cyclic then the number of fixed points of an element $x \in G_\alpha$ is $\frac{|\mathrm{N}_G(\langle x\rangle)|}{|G_\alpha|}$.

Proof:

Suppose G_α is cyclic and let $x \in G_\alpha$. Then $\langle x\rangle$ is the only subgroup of G_α with order $\mathrm{o}(x)$, hence $|\{\langle x\rangle^g \leq G_\alpha \mid g \in G\}| = 1$. Thus, by Lemma 2.9, the number of fixed points of x is $\frac{|\{\langle x\rangle^g \leq G_\alpha | g \in G\}| \cdot |\mathrm{N}_G(\langle x\rangle)|}{|G_\alpha|} = \frac{|\mathrm{N}_G(\langle x\rangle)|}{|G_\alpha|}$.

Another special case is that the point stabilisers are Frobenius groups with cyclic Frobenius complements. In this case the next lemma provides a way to count the number of fixed points of elements in the Frobenius complements.

Lemma 2.11

Let G be a group acting transitively on a set Ω. Let $\alpha \in \Omega$. Suppose that G_α is a Frobenius group with cyclic Frobenius complement and with Frobenius kernel K. If x is an element of a Frobenius complement of G_α, then x has exactly $\frac{|K| \cdot |\mathrm{N}_G(\langle x\rangle)|}{|G_\alpha|}$ fixed points on Ω.

Proof:

Let G_α be a Frobenius group with Frobenius kernel K and cyclic Frobenius complement C. Then $G_\alpha = KC$.

Let $c \in C$ be non-trivial. Since C is cyclic, C has only one subgroup of order $\mathrm{o}(c)$, and hence $|\{\langle c\rangle^g \leq G_\alpha \mid g \in G\}| = |\{C^g \leq G_\alpha \mid g \in G\}|$. Let $g \in G$ be such that $C^g \leq G_\alpha$. Then C^g is also a Frobenius complement of G_α. By 8.3.7 in [65], all Frobenius complements are conjugate in a Frobenius group. Thus, the number $|\{C^g \leq G_\alpha \mid g \in G\}|$ is exactly the number of Frobenius complements in KC. By the calculation on page 79 in [65], this number is $\frac{(|KC|-1)-(|K|-1)}{|C|-1} = \frac{|K| \cdot |C| - |K|}{|C|-1} = |K|$. Therefore by Lemma 2.9 the number of fixed points of c is $\frac{|\{\langle c\rangle^g \leq G_\alpha | g \in G\}| \cdot |\mathrm{N}_G(\langle c\rangle)|}{|G_\alpha|} = \frac{|\{C^g \leq G_\alpha | g \in G\}| \cdot |\mathrm{N}_G(\langle c\rangle)|}{|G_\alpha|} = \frac{|K| \cdot |\mathrm{N}_G(\langle c\rangle)|}{|G_\alpha|}$. Since all Frobenius complements are conjugated, the result follows.

At one point we will need the following number-theoretic result. It can be stated without any group-theoretic context, and the arguments used are of a different nature than in the place the result is used. Therefore it is stated here.

Lemma 2.12
The only pairs (k, l) of non-negative integers with the property that $2^k + 1 = 3^l$ are $(1, 1)$ and $(3, 2)$.

Proof:
Let (k, l) be a pair of non-negative integers such that $2^k + 1 = 3^l$. Then $2^k = 3^l - 1$.

First suppose that l is odd. We can factorise $3^l - 1$ such that $2^k = 3^l - 1 = (3-1)(3^{l-1} + 3^{l-2} + \ldots + 3 + 1)$. Since the last factor has l odd summands, it is odd. Hence, to be a divisor of 2^k, it must be 1 and therefore $l = 1$. Thus, $k = 1$ and $(k, l) = (1, 1)$.

Suppose instead that l is even. Then there exists a non-negative integer m such that $l = 2m$. Thus, $2^k = 3^l - 1 = (3^m - 1)(3^m + 1)$. Hence, both factors are powers of 2. Assume for a contradiction, that m is even. Then $3^m + 1 \equiv (-1)^m + 1 \equiv 1 + 1 \equiv 2 \mod 4$. Hence, $3^m + 1 = 2$ and therefore $m = 0$. However, it is not possible that $2^k = 3^0 - 1 = 0$ for any non-negative integer k. Therefore m is odd, and the same calculation as at the beginning of the proof shows that $m = 1$. Thus, $l = 2$ and hence $k = 3$. Then $(k, l) = (3, 2)$ and the lemma follows.

2.4 Some Properties of Groups Acting with Low Fixity

There are some direct consequences that can be drawn if a group acts with some fixity. In this section we collect and prove some of them. They will be highly useful in the following analysis.

Lemma 2.13
Let k be a positive integer, let G be a finite group acting with fixity k on a set Ω, let $\alpha \in \Omega$, and let X be a non-trivial subgroup of G_α. Then $|\,\mathrm{N}_G(X) : \mathrm{N}_{G_\alpha}(X)| \leq |\,\mathrm{fix}_\Omega(X)| \leq k$.

Proof:
The normaliser of X acts on the set of orbits of X and hence leaves the set of fixed points of X invariant. Since α is one of the fixed points of X, it follows that $\alpha^{\mathrm{N}_G(X)} \subseteq \mathrm{fix}_\Omega(X)$. Then $|\,\mathrm{N}_G(X) : \mathrm{N}_{G_\alpha}(X)| = |\alpha^{\mathrm{N}_G(X)}| \leq |\,\mathrm{fix}_\Omega(X)| \leq k$.

The previous lemma is most useful in combination with Lemma 2.1 because for a positive integer k, a group G acting with fixity k on a set Ω, a non-trivial subgroup X of G_α, and an arbitrary subgroup A of G, the lemmas together imply that $|\,\mathrm{N}_G(X) \cap A : \mathrm{N}_{G_\alpha}(X) \cap A| \leq |\,\mathrm{N}_G(X) : \mathrm{N}_{G_\alpha}(X)| \leq k$. In particular, since $\mathrm{C}_G(X) \leq \mathrm{N}_G(X)$, it holds that $|\,\mathrm{C}_G(X) : \mathrm{C}_{G_\alpha}(X)| \leq k$. Throughout this thesis, in most situations Lemma 2.13 will be used not directly for the normaliser itself but for its intersection with a subgroup, implicitly applying Lemma 2.1 without stating it. Additionally, the previous lemma has an influence on the Sylow subgroup structure, as we will see in the next corollary.

Corollary 2.14
Let k be a positive integer, let G be a finite group acting with fixity k on a set Ω, let $\alpha \in \Omega$, and let p be a prime dividing $|G_\alpha|$ such that $p > k$. Then G_α contains a Sylow p-subgroup of G.

Proof:
Let $Q \in \mathrm{Syl}_p(G_\alpha)$ and let $P \in \mathrm{Syl}_p(G)$ such that $Q \leq P$. Then by Lemma 2.13, $|\,\mathrm{N}_P(Q) : Q| = |\,\mathrm{N}_P(Q) : \mathrm{N}_{P \cap G_\alpha}(Q)| \leq |\,\mathrm{N}_G(Q) : \mathrm{N}_{G_\alpha}(Q)| \leq k < p$. Hence $Q = P$.

After looking at the point stabilisers, the next lemma concentrates on four-point stabilisers of groups acting with fixity 4. They will play an important role, especially for point stabilisers with order coprime to 6, but the next result does not need this additional hypothesis.

Lemma 2.15
Let G be a finite group acting transitively and faithfully on a set Ω and let H be the element-wise stabiliser of a set of size 4. Then the following hold:

(a) If $g \in G$ acts on $\mathrm{fix}_\Omega(H)$, then $g \in \mathrm{N}_G(H)$.

(b) For all $g \in G$, either $H \cap H^g = H$ or $H \cap H^g = 1$.

Proof:
Let $g \in G$ act on $\mathrm{fix}_\Omega(H)$, let $h \in H$, and let $\delta \in \mathrm{fix}_\Omega(H)$. Then $\delta^{h^g} = (\delta^{g^{-1}})^{hg}$. Since $\delta^{g^{-1}} \in \mathrm{fix}_\Omega(H)$, it follows that $\delta^{h^g} = (\delta^{g^{-1}})^g = \delta$ and therefore h^g fixes all points in $\mathrm{fix}_\Omega(H)$. Hence $h^g \in H$ and $g \in \mathrm{N}_G(H)$. This is part (a).

For part (b) let $g \in G$ and $h \in H \cap H^g$. Then h fixes all elements in $\mathrm{fix}_\Omega(H)$ and in $\mathrm{fix}_\Omega(H^g) = \mathrm{fix}_\Omega(H)^g$. If h is non-trivial, then h has at most four fixed points, hence $\mathrm{fix}_\Omega(H) = \mathrm{fix}_\Omega(H)^g$. In particular, g acts on the set of fixed points of H. Thus, by part (a), $g \in \mathrm{N}_G(H)$ and therefore $H = H^g$.

From looking at Lemma 2.13, it is apparent that there are restrictions on the centre of a group acting with small fixity. The next two lemmas formalise this.

Lemma 2.16
Let G be a group acting transitively and faithfully on a set Ω. Then for all $\alpha \in \Omega$, $\mathrm{Z}(G)_\alpha = 1$.

Proof:
Assume otherwise. Then there exists an element $\alpha \in \Omega$ such that $\mathrm{Z}(G)_\alpha$ contains a non-trivial element a. Then $\langle a \rangle \trianglelefteq G$ and $\langle a \rangle$ is a subgroup of G_α. Since G acts transitively, it follows that $\langle a \rangle$ fixes all points in Ω, contradicting the faithfulness of the action of G.

Lemma 2.17
Let k be a positive integer and G be a finite group acting transitively and faithfully

on a set Ω. If there exists a subgroup $U \leq G$ that fixes exactly k points, then $|\,\mathrm{Z}(G)|$ is a divisor of k. In particular, if G acts with fixity k, then $|\,\mathrm{Z}(G)|$ is a divisor of k.

Proof:

Let $U \leq G$ be such that U fixes exactly k points. Let $\alpha_1, \alpha_2, \ldots, \alpha_k \in \Omega$ be the k distinct points that are fixed by U. Since $\mathrm{Z}(G) \leq \mathrm{C}_G(U)$, $\mathrm{Z}(G)$ acts on $\mathrm{fix}_\Omega(U)$. Let $i \in \{1, 2, \ldots, k\}$. Then $|{\alpha_i}^{\mathrm{Z}(G)}| = \frac{|\,\mathrm{Z}(G)|}{|\,\mathrm{Z}(G)_{\alpha_i}|} = |\,\mathrm{Z}(G)|$, because $\mathrm{Z}(G)_{\alpha_i}$ is trivial by Lemma 2.16. Since $\mathrm{fix}_\Omega(U)$ is a disjoint union of all its $\mathrm{Z}(G)$-orbits, each of them having length $|\,\mathrm{Z}(G)|$, it follows that $|\,\mathrm{Z}(G)|$ is a divisor of $|\,\mathrm{fix}_\Omega(U)| = k$.

If G acts with fixity k, then by definition there exists an element x in G with exactly k fixed points, hence $\langle x \rangle \leq G$ is a subgroup that fixes exactly k points. Then the first part implies the rest of the lemma.

2.5 GAP

In some proofs and examples, GAP [36], a system for computational discrete algebra, will be used. In some cases, the motivation is to shorten calculations for groups of small order that in principle could be carried out by hand. In other cases, some of GAP's data libraries such as the Primitive Permutation Groups Library [53] will be used because GAP, or the corresponding package, provides an interface to access this research data. Whenever there is code printed it will be GAP code, even if it is not specified as such.

The output will always be suppressed, usually because it is long and unreasonable to print. For example, sometimes it is a long list of groups in GAP's own notation, depending on the package used. Thus, the notation might vary and be inconsistent. Therefore unique identifiers like the group ID in the Small Groups Library [13] or the number in the Transitive Groups Library [52] are used when applicable. Otherwise an isomorphism test can answer the question of what group GAP returned. Additionally, the command `StructureDescription(G);` can be used carefully for a group `G` to help determine some of its structure. When GAP is used in an example or proof and the result is a group or a list of groups, the isomorphism types will be specified if they are needed, but the code to obtain these types will be suppressed. For checking the correctness of the specified isomorphism type for a group `G`, the command `IsomorphismGroups(G,h);` can be used, where `h` is any group of the stated isomorphism type. If an isomorphism is returned, then the group has the stated isomorphism type, and otherwise `fail` is returned. However, since the notation of an isomorphism type can be inconclusive, it is necessary to check for each option that fits the description whether or not some group `h` of this type is isomorphic to the returned group `G`. An inconclusive notation of an isomorphism type happens especially for products of groups, for instance, if a semi-direct product is denoted, then usually the action on the normal subgroup is not specified, thus the structure of different non-isomorphic groups can be denoted in the same way.

A GAP session could look like the following.

```
gap> li:=AllTransitiveGroups(NrMovedPoints, [4]);
[ C(4) = 4, E(4) = 2[x]2, D(4), A4, S4 ]
gap> h:=Group((1,2),(3,4));;
gap> List(li,x->IsomorphismGroups(x,h));
[ fail, [ (1,4)(2,3), (1,2)(3,4) ] -> [ (1,2), (3,4) ], fail, fail,
↪  fail ]
```

Thereby all transitive groups of degree 4 are determined and only the second is of isomorphism type E_4. The corresponding GAP code as printed for example in a proof would only contain the line `AllTransitiveGroups(NrMovedPoints, [4]);` and everything else would not be printed.

One frequent usage of GAP is to test whether or not a group acts with fixity 4. A first naive way is described in the following remark, and determines for a permutation group G that acts on a set whether or not this action is a faithful fixity-k action for a positive integer k.

Remark 2.18
The GAP function in Program Code 2.1 takes as input a group `G`, a set `set` on which the group acts transitively, and a positive integer `k`. It returns `true` or `false` according to whether the group acts faithfully and with the specified fixity `k` on the given set.

To see the correctness of the Program Code 2.1, first suppose that a group `G` acts faithfully and with fixity `k` on the given set. Then `G` contains a non-trivial element with exactly `k` fixed points. As a consequence, the size of the set is at least `k` + 2. In particular, the condition in line 3 is not fulfilled. Then in line 7 all non-trivial elements of the group are determined. Each of them has at most `k` fixed points, and hence the condition in line 10 is not fulfilled. Since there exists an element with exactly `k` fixed points, the code in lines 13–15 sets `exactly` to `true`, and since this cannot be overwritten by another value, it follows that the code returns `true` in line 17. Therefore it correctly returns `true` if a group acts faithfully and with fixity `k` on the given set.

On the other hand, if the code returns `true`, then the value of `exactly` is `true`. Therefore, line 14 is executed. Thus, the condition in line 13 was fulfilled for some non-trivial element. In particular, the group contains a non-trivial element with exactly `k` fixed points. Another consequence of the fact that `true` is returned is that line 11 had not been executed, and hence the condition in line 10 did not hold for any non-trivial element. Thus, every non-trivial element has at most `k` fixed points. Similarly, line 4 implies that the condition in line 3 did not hold. Therefore the set contains at least `k` + 1 elements. Since none of the non-trivial elements fixes more than `k` points, this implies, that the action of the group is faithful. As a consequence, the group acts with fixity `k` and faithfully on the given set if the code returns `true`.

Thus, after implementing the function, we can use `TestFixity(G,set,k);` to determine whether `G` acts faithfully and with fixity `k` on a set `set`.

The GAP program in the previous remark is especially useful in combination with the Transitive Groups Library [52] because in this library for small degrees (only

```
TestFixity:=function(G,set,k)
  local elements, g, fixnr, exactly;
  if Length(set)<=k
    then return false;
  fi;
  exactly:= false;
  elements:=Difference(Elements(G),[Identity(G)]);
  for g in elements do
    fixnr:=Length(Difference(set,MovedPoints(g)));
    if fixnr>k
      then return false;
    fi;
    if fixnr=k
      then exactly:=true;
    fi;
  od;
  return exactly;
end;
```

Program Code 2.1: `TestFixity`

degrees up to 28 and degree 40 will be needed later), all transitive groups are listed. Thus, with the help of the function in the previous remark, we can, for instance, determine all groups that act transitively, faithfully, and with fixity 4 on a set of size at most 28.

However, to use the function in Program Code 2.1, we need the set, and hence the specific action of a group. Since every transitive action is equivalent to a coset action, we can look at all transitive actions of a group by looking at its subgroups and the actions of the group on the cosets of its subgroups. Some information about these actions is covered by the so-called table of marks. We will first see a definition and afterwards in Lemma 2.20 how information about the action, especially about the fixity, is encoded in the table of marks.

The definition of the table of marks is derived from Definition 1.1 and Proposition 1.2 in [77].

Definition 2.19
Let G be a finite group, let n be a positive integer, and let $\{G_1, \dots, G_n\}$ be a system of representatives of the conjugacy classes of subgroups of G. Then a *table of marks* M of G is an $n \times n$-matrix, where for all $i, j \in \{1, \dots, n\}$ the j-th entry in the i-th row is the number of elements in $\{G_i g \mid g \in G \text{ and for all } h \in G_j, G_i gh = G_i g\}$.

The order of the entries in a table of marks depends on the ordering of the representatives of the conjugacy classes of subgroups of G and we say that $U \leq G$ corresponds to the i-th row (and i-th column) of M if and only if U is in the conjugacy class of G_i.

In the following lemma some properties of tables of marks are presented. They will be used afterwards to describe a GAP program that determines for a table of marks of a group G and for a positive integer k all transitive and faithful fixity-k actions of G.

Lemma 2.20
Let G be a finite group, let M be a table of marks of G, let U be a non-trivial subgroup of G, let $x \in U$, and let k be a positive integer. Furthermore let i be such that U corresponds to the i-th row of M. Then the following hold:

(a) If j is such that $\langle x \rangle$ corresponds to the j-th row of M, then the j-th entry in the i-th row of M is the number of fixed points of x in G/U.

(b) If H is a subgroup of G containing x, and if j is such that H corresponds to the j-th row of M, then the j-th entry in the i-th row of M is at most the number of fixed points of x in G/U.

(c) G acts faithfully on G/U if and only if all entries in the i-th row of M, except for the entry in the column corresponding to the trivial group, are smaller than $|G : U|$.

(d) G acts with fixity k on G/U if and only if k is the highest entry in the i-th row, except for the entry in the column corresponding to the trivial group.

Proof:
For part (a), suppose that j is such that $\langle x \rangle$ corresponds to the j-th row of M. Then by the definition of the table of marks, there exist v and y in G such that the j-th entry in the i-th row of M is the number m of elements in $\{U^v g \mid g \in G \text{ and for all } h \in \langle x \rangle^y, U^v gh = U^v g\}$. Then $m = \frac{|\{g \in G | h^{g^{-1}} \in U^v \text{ for all } h \in \langle x \rangle^y\}|}{|U^v|}$. Since $x^{yg^{-1}} \in U^v$ if and only if for all $h \in \langle x \rangle^y$, $h^{g^{-1}} \in U^v$, it follows that $\{g \in G \mid h^{g^{-1}} \in U^v \text{ for all } h \in \langle x \rangle^y\} = \{g \in G \mid x^{yg^{-1}} \in U^v\}$. We recall that y and v are fixed, and hence $|\{g \in G \mid x^{yg^{-1}} \in U^v\}| = |\{g \in G \mid x^{yg^{-1}v^{-1}} \in U\}| = |\{g \in G \mid x^{g^{-1}} \in U\}|$. Thus, $m = \frac{|\{g \in G | x^{g^{-1}} \in U\}|}{|U|} = \{Ug \mid g \in G \text{ and } Ugx = Ug\}|$. In particular, m is the number of fixed points of x in G/U.

For part (b), suppose that H is a subgroup of G containing x, and that j is such that H corresponds to the j-th row of M. Then there exist v and y in G such that the j-th entry in the i-th row of M is $m := |\{U^v g \mid g \in G \text{ and for all } h \in H^y, U^v gh = U^v g\}| = |\{Uvg \mid g \in G \text{ and for all } h \in H^y, Uvgh = Uvg\}| = |\{Ug \mid g \in G \text{ and for all } h \in H^y, Ugh = Ug\}|$. Since $x^y \in H^y$, this number is at most $|\{Ug \mid g \in G \text{ and } Ugx^y = Ug\}|$. In particular, x^y fixes in its action on G/U at least m points. Since conjugation does not change the number of fixed points of an element, m is at most the number of fixed point of x.

For the first direction of (c), suppose that G acts faithfully on G/U. Assume for a contradiction that there exists a non-trivial group H such that H corresponds to a column j such that the j-th entry in the i-th row of M is not smaller than $|G : U|$. Let h be a non-trivial element of H. By part (b), the number of fixed points of h is at least $|G : U|$ and hence h fixes all elements in G/U, contrary to the assumption that G acts faithfully on G/U. For the other direction of (c), suppose that all entries of M except for the entry in the column corresponding to the trivial group are smaller than $|G : U|$. Let $g \in G$ be such that g fixes all elements in G/U. Then g has exactly $|G : U|$ fixed points. By (a), the entry of M in the i-th row and the column corresponding to $\langle g \rangle$ is $|G : U|$. Then the condition on the i-th row implies that g is the trivial element. As a consequence, G acts faithfully on G/U. This finishes the proof of part (c).

It remains to prove (d). Suppose that G acts with fixity k on G/U. Then there exists a non-trivial element $g \in G$ with exactly k fixed points, and by (a), the entry in the column corresponding to $\langle g \rangle$ of the i-th row of M is k. Assume for a contradiction that there exists a non-trivial subgroup H such that the entry in the column corresponding to H in the i-th row of M is greater than k. Let $h \in H$ be non-trivial. Then by (b), h fixes more than k points, contrary to the fact that G acts with fixity k on G/U. This proves the first direction of (d). For the other direction, suppose that k is the highest entry in the i-th row except for the entry in the column corresponding to the trivial group. Since k is the highest entry, there exists a non-trivial subgroup H corresponding to the column that has the entry k in the i-th row. Let h be a non-trivial element of H. Then by (b), h fixes at least k elements. Thus by part (a), the entry in the i-th row of M in the column corresponding to $\langle h \rangle$ is at least k. Since it is also at most k, part (a) implies that h fixes exactly k points. Assume for a contradiction that there exists a non-trivial element with more than k fixed points. Then by (a), the i-th row contains an entry greater than k. Since this is not possible, part (d) follows.

In GAP, for a group G, a system $\{G_1, \ldots, G_n\}$ of representatives of the conjugacy classes of subgroups of G is normally sorted in a way such that if G_j is contained in a conjugate of G_i, then $j \leq i$. As a consequence, the table of marks M of G is a lower triangle matrix, because if i and j are such that the j-th entry in the i-th row of M is non-zero, then by the definition of the table of marks, there exists an element $g \in G$ such that ${G_j}^{g^{-1}} \leq G_i$, and hence $j \leq i$.

The purpose of the following example it to demonstrate how a table of marks can be understood in GAP. As noted earlier, in the subsequent chapters the output will normally be suppressed. Therefore, the example also explains how a table of marks displayed in GAP can be read and interpreted in general, because later on only the consequences will be stated.

Example 2.21
We look at the table of marks of the group $\mathcal{A}_6$. It is part of the GAP package TomLib [74], which contains numerous precomputed tables of marks, and which will be used frequently in this thesis.

```
gap> Display(TableOfMarks("a6"));
 1:  360
 2:  180 4
 3:  120 . 6
 4:  120 . . 6
 5:   72 . . . 2
 6:   90 2 . . . 2
 7:   90 6 . . . . 6
 8:   90 6 . . . . . 6
 9:   60 4 3 . . . . . 1
10:   60 4 . 3 . . . . . 1
11:   45 5 . . . 1 3 3 . . 1
12:   40 . 4 4 . . . . . . . 4
13:   36 4 . . 1 . . . . . . . 1
14:   30 2 6 . . . . 2 . . . . . 2
15:   30 2 . 6 . . 2 . . . . . . . 2
16:   20 4 2 2 . . . . 2 2 . 2 . . . 2
17:   15 3 3 . . 1 3 1 1 . 1 . . 1 . . 1
18:   15 3 . 3 . 1 1 3 . 1 1 . . . 1 . . 1
19:   10 2 1 1 . 2 . . 1 1 . 1 . . . 1 . . 1
20:    6 2 3 . 1 . . 2 1 . . . 1 2 . . . . . 1
21:    6 2 . 3 1 . 2 . . 1 . . 1 . 2 . . . . . 1
22:    1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
```

The first line of the table of marks corresponds to the trivial group, and hence there is only one entry, the order of the group. We see that there is only one conjugacy class of subgroups of index 180. Thus, $\mathcal{A}_6$ contains a unique conjugacy class of involutions and every subgroup of $\mathcal{A}_6$ generated by an involution corresponds to the second row and second column of the table of marks. Therefore the table of marks implies that every involution t in $\mathcal{A}_6$ fixes four points in $\mathcal{A}_6/\langle t\rangle$. In particular, this action is a fixity-4 action. We can see in row 9 of the table of marks that there exists a subgroup U of index 60 in $\mathcal{A}_6$ such that every involution fixes four points in $\mathcal{A}_6/U$. Since the third entry in row 9 is 3, since the subgroup corresponding to column 3 is the subgroup corresponding to row 3, and since this subgroup has index 120 in $\mathcal{A}_6$, some elements of order 3 have three fixed points in $\mathcal{A}_6/U$. Since row 4 and column 4 also correspond to subgroups of order 3, there additionally exist elements of order 3 that do not fix any points in $\mathcal{A}_6/U$ because the fourth entry in row 9 is 0. Inspecting the whole row, we see that $\mathcal{A}_6$ acts with fixity 4 on $\mathcal{A}_6/U$.

This way of determining fixity-4 actions can be automated, as we will see in the next remark.

Remark 2.22
The GAP function in Program Code 2.2 takes as an input a table of marks `t` of a group G and a positive integer `k`. It returns a list of lists. Each of the lists

```
1  TestTom:=function(t,k)
2    local marks,g,fin;
3    marks:=MarksTom(t);;
4    fin:=[];;
5    for g in [1..Length(marks)] do
6      if ForAll([2..Length(marks[g])],i->marks[g][i]<k+1)
7        and (k in marks[g])
8        and marks[g][1]>k
9      then Add(fin,[StructureDescription(RepresentativeTom(t,g)),
       ↪  marks[g]]);
10     fi;
11   od;
12   return fin;
13 end;
```

Program Code 2.2: `TestTom`

corresponds to one transitive and faithful fixity-`k` action of G and has as a first entry some information about the structure of the point stabiliser, that is the subgroup corresponding to the row of the table of marks, and then a list of all non-zero entries of the row in the table of marks follows. In particular, if an empty list is returned, then G cannot act transitively, faithfully, and with fixity `k` on any set.

Since in GAP the trivial group always corresponds to the first row and column, the first entry in every row is the size of the set on which G acts. By construction, the table of marks contains information exactly about the transitive actions. Therefore it remains to show that the program in Program Code 2.2 determines correctly the faithful fixity-`k` actions among them. The beginning of the loop in line 5 ensures that all rows of the table of marks are looked at. Line 6 checks that all entries in a row, except the first one, which corresponds to the trivial group, are at most `k`. Then line 7 tests whether or not `k` appears in the row, and line 8 tests whether or not this occurrence is in the first entry. Hence, by Lemma 2.20 (d), G acts with fixity `k` if all three conditions are satisfied. By Lemma 2.20 (c), G acts faithfully if and only if all entries in a row are smaller than the first entry. This is checked by line 8 in Program Code 2.2, because line 6 already tested that all entries except for the first one are at most `k`. In particular, if G acts faithfully and with fixity `k`, then the condition in line 8 is satisfied, and by Lemma 2.20 (d), the conditions in line 6 and 7 are additionally fulfilled. Therefore, the program executes line 9 if and only if G acts faithfully and with fixity `k` on the cosets of G modulo the subgroup U that corresponds to the row under consideration. Thus the resulting list only contains entries of transitive and faithful fixity-`k` actions.

Computing all conjugacy classes of all subgroups can be quite computationally expensive. Therefore it is useful that some of the tables of marks are already precomputed in the TomLib package [74]. The groups can be accessed by their names in the package AtlasRep [104]. This package contains information about most of the finite simple groups, their coverings, and outer automorphism groups, which are also part of [28]. The package [104] gives an interface to access this information and to be able to compute with those groups.

Similarly to the function in Remark 2.18, the function in Remark 2.22 is not part of GAP. Hence it has to be implemented before it can be used. Whenever one of these functions is used in one of the examples or proofs, it will be referenced. In the same way, normally, if a package is used, it is referenced but maybe some of the package dependencies are not mentioned, because the referenced package contains the information about its dependencies.

All other specificities in the context of the usage of GAP will be explained when they appear.

3 Simple Groups

This chapter addresses the question which finite simple groups act transitively and with fixity 4. The general strategy was developed by Barbara Baumeister, Kay Magaard and Rebecca Waldecker. They analysed the Sylow 2- and Sylow 3-subgroup structure of finite groups acting with fixity 4 in general before restricting to finite simple groups. Their analysis reveals a case distinction (see Lemma 3.1) and is part of [7]. Therefore, to prove Theorem 3.56, the classification of finite simple groups acting transitively and with fixity 4, we have to consider each of these cases and determine in each case all finite simple groups that fulfil the hypothesis of the case and act with fixity 4.

The paper [7] contains, amongst the mentioned general analysis, the full classification of finite simple groups that act transitively and with fixity 4. Many of the cases were analysed by Patrick Salfeld and myself and some version of our proofs became part of [7]. Therefore, the list of authors is Barbara Baumeister, myself, Kay Magaard, Patrick Salfeld, and Rebecca Waldecker. The content of this chapter here is my main contribution to the paper, in particular the analysis of the case that point stabilisers have odd order divisible by 3, and the analysis of the case that the group is a finite simple group of Lie type and the point stabilisers have order coprime to 6. In the paper [7] the proofs stand in a greater context and were jointly rewritten. In particular, they use far more details of the analysis of the Sylow 3-structure. In this chapter here, the proofs are self-contained in the sense that except for Lemma 3.1 (respectively Theorem 1.2 in [7]) nothing else of [7] is needed.

Some of the proofs in this chapter are similar to the version in [7] and some differ more. This is caused by the fact that the reader of the following sections is not expected to have read and understood the analysis in [7]. The only result needed is the following.

Lemma 3.1
Let G be a finite simple group that acts transitively and with fixity 4 on a set Ω. Let $P \in \mathrm{Syl}_3(G)$, let $S \in \mathrm{Syl}_2(G)$, and let f denote the maximum number of fixed points of involutions in G.

Then one of the following holds.

(1) $f \geq 1$ and G has a strongly embedded subgroup.

(2) $1 \leq f \leq 3$ and S is dihedral or semi-dihedral.

(3) $f = 4$ and G has sectional 2-rank at most 4.

(4) The order of the point stabilisers is odd and divisible by 3. One of the following occurs:

(a) G has a strongly 3-embedded subgroup.

(b) P is elementary abelian of order 9.

(c) P is extra-special of exponent 3 and order 27.

(d) $P \cong C_3 \wr C_3$.

(5) The point stabilisers have order coprime to 6.

Proof:
If G is abelian, then the point stabilisers of G are either trivial, contradicting the assumption that G acts with fixity 4, or the whole group, contradicting the transitivity of G. Consequently, G is non-abelian. Then Lemma 2.5 implies that $|\Omega| \geq 5$, and hence, G acts faithfully on Ω. Therefore, the hypotheses of Theorem 1.2 in [7] are fulfilled. If case (4) (b) in Theorem 1.2 in [7] holds, then $|P| = 3$. Thus, P is cyclic and, as G is simple, Lemma 2.8 proves that G contains a strongly 3-embedded subgroup. Hence, case (4) (a) holds. All other cases of Theorem 1.2 in [7] correspond directly to one of the cases in this lemma.

In each case of the previous lemma we need as a first step a list of all simple groups that fulfil the hypothesis of the case. In some cases, such as (1)-(3), there already exists such a classification for finite simple groups. In other cases we first have to use the classification of finite simple groups to obtain this list. Afterwards we have to determine for every group of these lists whether or not the group can act transitively and with fixity 4.

The case (3) of the previous lemma was analysed by Patrick Salfeld in his PhD thesis [89]. Therefore, the remainder of this chapter deals with the remaining cases before in Section 3.4 all parts are put together to prove Theorem 3.56.

3.1 Some Small Cases

Some groups appear in many cases of this strategy. Instead of dealing with them separately in each case, we look at them once. This makes the later steps of the classification much easier because these cases normally cause problems in general results.

The families of groups we deal with in the next sections are $\mathrm{PSL}(2, q)$, $\mathrm{PSL}(3, q)$, $\mathrm{PSU}(3, q)$, and $\mathrm{Sz}(q)$. All of them are of Lie type and act on a vector space of small dimension. For each family of groups, we will determine all the fixity-4 actions in a single lemma. In each lemma, the strategy will be quite similar to the general strategy for proving the classification of finite simple groups acting with fixity 4. However, some parts of the proofs are shorter because the groups are well understood and therefore the arguments are more concrete.

3.1.1 PSL(2,q)

In some sense $\mathrm{PSL}(2, q)$ is special. Since the subgroup structure is quite restricted, many arguments can be used more directly than for other families of groups, and therefore the proof can be structured differently. Additionally, this family has a quite rich series of fixity-4 actions. However, the used arguments will again appear in later proofs. Therefore it is useful to see them applied in Lemma 3.2, and thus a proof is stated, even though Patrick Salfeld already analysed the situation for $\mathrm{PSL}(2, q)$ in Lemma 2.19 and Lemma 2.30 in his PhD thesis [89]. His strategy differs from the one presented here. Nevertheless, our proofs are highly influenced by each other, due to the many conversations we had about the problem.

Lemma 3.2
Let $q \geq 4$ be a prime power and let $G = \mathrm{PSL}(2, q)$. Suppose that G acts transitively on a set Ω. Then G acts with fixity 4 on Ω if and only if one of the following holds.

(1) $G = \mathrm{PSL}(2, 7)$ and the point stabilisers are of type C_2 or $\mathcal{S}_3$.

(2) $G = \mathrm{PSL}(2, 8)$ and the point stabilisers are of type C_2, $\mathcal{S}_3$, D_{14}, or D_{18}.

(3) $G = \mathrm{PSL}(2, 9)$ and the point stabilisers are of type C_2, $\mathcal{S}_3$, D_{10}, E_9, or $E_9 : C_2$.

(4) $G = \mathrm{PSL}(2, 11)$ and the point stabilisers are of type C_3 or $\mathcal{A}_4$.

(5) $G = \mathrm{PSL}(2, 13)$ and the point stabilisers are of type C_3, $C_{13} : C_3$, or $\mathcal{A}_4$.

(6) $q \geq 17$ is odd. If $q \equiv 1 \mod 4$, then the point stabilisers are either cyclic of order $\frac{q-1}{4}$ or the semi-direct product of an elementary abelian group of order q with a cyclic group of order $\frac{q-1}{4}$. If $q \equiv -1 \mod 4$, then the point stabilisers are cyclic of order $\frac{q+1}{4}$.

Proof:
Using the GAP package TomLib [74] through the algorithm in Remark 2.22, the answer to the following command proves the statement of the lemma for all $q \leq 41$.

```
List([4,5,7,8,9,11,13,16,17,19,23,25,27,29,31,32,37,41],
↪  x->TestTom(TableOfMarks(Concatenation("L2(",String(x),")")),
↪  4));
```

Therefore, throughout the rest of the proof, suppose that $q \geq 43$.

The order of G is $q \cdot \frac{q^2-1}{\gcd(2,q-1)}$. Detailed information about the subgroup structure of G is stated in Hauptsatz II 8.27 in [54]. First we look at some more properties of G.

Let p be the prime dividing q and let $P \in \mathrm{Syl}_p(G)$. Then P is elementary abelian of order q and, by Theorem 6.5.1 in [43], $\mathrm{N}_G(P)$ is a Frobenius group of order $q \cdot \frac{q-1}{\gcd(2,q-1)}$. The Frobenius kernel of $\mathrm{N}_G(P)$ is P. If $p = 2$, then every non-trivial element in $\mathrm{N}_G(P)$ of odd order lies in a Frobenius complement of $\mathrm{N}_G(P)$, hence does not centralise any element of P. Therefore, if $p = 2$, then for every element $t \in P$, $|t^{\mathrm{N}_G(P)}| = |\,\mathrm{N}_G(P) : \mathrm{C}_{\mathrm{N}_G(P)}(t)| = \frac{q(q-1)}{q} = q - 1 = |P| - 1$. Thus, all non-trivial elements of P are conjugate. Hence, all involutions in G are conjugate if $p = 2$.

By Satz II 8.5 in [54], a non-trivial element of G either lies in a Sylow p-subgroup of G, in a cyclic group of order $\frac{q-1}{\gcd(2,q-1)}$, or in a cyclic group of order $\frac{q+1}{\gcd(2,q-1)}$. Each of these groups is conjugate to all other groups of the same order and any two different conjugates have trivial intersection. Let $g \in G$ and let $\varepsilon \in \{-1, 1\}$ be such that the order of g divides $q - \varepsilon$. Then by Satz II 8.3 and Satz II 8.4 in [54], $\mathrm{N}_G(\langle g \rangle)$ is a dihedral group of order $2 \cdot \frac{q-\varepsilon}{\gcd(2,q-1)}$.

All of these properties will be used without further reference.

For the first direction of the lemma, suppose that G acts with fixity 4 on Ω. Let $\alpha \in \Omega$. We will make a case distinction depending on whether q and $|G_\alpha|$ are coprime or not.

First suppose that q and $|G_\alpha|$ have a common prime divisor p. Let $x \in G_\alpha$ be of order p. Let $Q \in \mathrm{Syl}_p(G)$ be such that $x \in Q$ and $Q_\alpha \in \mathrm{Syl}_p(G_\alpha)$. Then Q is elementary abelian and therefore $Q \leq \mathrm{N}_G(\langle x \rangle)$. By Lemma 2.13, this implies that $|Q : Q_\alpha| \leq |\,\mathrm{N}_G(\langle x \rangle) : \mathrm{N}_{G_\alpha}(\langle x \rangle)| \leq 4$. Let f be a positive integer such that $p^f = q$. If $p = 2$, then the order of G_α is divisible by $\frac{q}{4} = 2^{f-2}$, and $f \geq 5$ thus $f - 2 > \frac{f}{2}$. If $p = 3$, then the order of G_α is divisible by $\frac{q}{3} = 3^{f-1}$, and $f \geq 3$ thus $f - 1 > \frac{f}{2}$. Finally, if $p \geq 5$, then the order of G_α is divisible by q. In particular, $|Q_\alpha| > p^{f/2}$ and $|Q_\alpha| \geq \frac{q}{4} \geq \frac{43}{4} > 10$.

We now show that G_α cannot be a p-group. The strategy is different depending on whether q is even or odd.

Suppose that q is odd. Assume for a contradiction that G_α is a p-group. Then there exists an element $a \in G_\alpha$ of order p with exactly four fixed points. By Lemma 2.9, the number of fixed points of a is $\frac{|\{\langle a \rangle^g \leq G_\alpha | g \in G\}| \cdot |\,\mathrm{N}_G(\langle a \rangle)|}{|G_\alpha|} = 4$. Since $|\,\mathrm{N}_G(\langle a \rangle)|$ is divisible by q and since 4 is coprime to q, $|G_\alpha|$ is divisible by q and hence $G_\alpha = Q_\alpha = Q$. Then Lemma 2.13 gives the contradiction that $4 \geq |\,\mathrm{N}_G(Q) : \mathrm{N}_{G_\alpha}(Q)| = \frac{q \cdot \frac{q-1}{2}}{q} = \frac{q-1}{2} \geq \frac{42}{2} = 21$. Hence, G_α is not a p-group if q is odd.

Suppose that q is even. Since all involutions in $\mathrm{PSL}(2, q)$ are conjugate, the size of $\{\langle x \rangle^g \leq Q_\alpha \mid g \in G\}$ is the number of involutions in Q_α and equals $|Q_\alpha| - 1$. Then Lemma 2.9 implies that the number of fixed points of x is $\frac{|\{\langle x \rangle^g \leq G_\alpha | g \in G\}| \cdot |\,\mathrm{N}_G(\langle x \rangle)|}{|G_\alpha|} \geq \frac{|\{\langle x \rangle^g \leq Q_\alpha | g \in G\}| \cdot |\,\mathrm{N}_G(\langle x \rangle)|}{|G_\alpha|} = \frac{(|Q_\alpha|-1) \cdot |\,\mathrm{N}_G(\langle x \rangle)|}{|G_\alpha|} \geq \frac{(|Q_\alpha|-1) \cdot q}{|G_\alpha|}$. Since x fixes at most four points, $|G_\alpha| \geq \frac{q \cdot (|Q_\alpha|-1)}{4}$ and since $q \geq 64$, it follows that $|G_\alpha| \geq \frac{q \cdot (|Q_\alpha|-1)}{4} \geq q \cdot \frac{\frac{q}{4}-1}{4} \geq q \cdot \frac{15}{4} > q$. Thus, the point stabilisers cannot be p-groups.

Therefore the information about the order of Q_α together with an inspection of the list of subgroups of $\mathrm{PSL}(2, q)$ reveals that G_α is a subgroup of a Frobenius group of order $q \cdot \frac{q-1}{\gcd(2,q-1)}$, hence, G_α itself is a Frobenius group. The Frobenius kernel is Q_α, the unique Sylow p-subgroup of G_α. Let $y \in G_\alpha$ have prime order and such that $\mathrm{o}(y)$ divides $\frac{q-1}{\gcd(2,q-1)}$. Then by Lemma 2.11, the number of fixed points of y is $\frac{|Q_\alpha|\cdot|\mathrm{N}_G(\langle y\rangle)|}{|G_\alpha|} = \frac{|Q_\alpha|\cdot 2\cdot\frac{q-1}{\gcd(2,q-1)}}{|G_\alpha|}$. By Lemma 2.9, x fixes exactly $\frac{|\{\langle x\rangle^g \leq G_\alpha | g\in G\}|\cdot|\mathrm{N}_G(\langle x\rangle)|}{|G_\alpha|}$ points. Since all p-elements of G_α lie in Q_α, this number coincides with $\frac{|\{\langle x\rangle^g \leq Q_\alpha | g\in G\}|\cdot|\mathrm{N}_G(\langle x\rangle)|}{|G_\alpha|}$. The number $|\{\langle x\rangle^g \leq Q_\alpha \mid g \in G\}|$ is bounded above by the number of distinct subgroups of order p of Q_α and the order of $\mathrm{N}_G(\langle x\rangle)$ is bounded above by $|\,\mathrm{C}_G(x)|\cdot|\,\mathrm{Aut}(\langle x\rangle)| = q\cdot(p-1)$. Thus x fixes at most $\frac{\frac{|Q_\alpha|-1}{p-1}\cdot q\cdot(p-1)}{|G_\alpha|} = \frac{q(|Q_\alpha|-1)}{|G_\alpha|}$ points in Ω. Hence the number of fixed points of x is at most the number of points that are fixed by y. Since the maximal number of fixed points of non-trivial elements in G_α is reached by an element of prime order, an element of prime order dividing $\frac{q-1}{\gcd(2,q-1)}$ fixes four points. By the calculation above, it follows that $|G_\alpha| = |Q_\alpha|\cdot\frac{q-1}{2\cdot\gcd(2,q-1)}$. This implies that $\frac{q-1}{2\cdot\gcd(2,q-1)}$ is an integer. Thus, q is odd, more precisely, $q \equiv 1 \mod 4$. Since G_α is a Frobenius group with Frobenius kernel Q_α, $|Q_\alpha| \equiv 1 \mod \frac{q-1}{4}$, implying that $|Q_\alpha| = q$ because $\frac{q-1}{4} \geq \frac{42}{4} > 10$, $|Q_\alpha| \in \{q, \frac{q}{2}, \frac{q}{3}, \frac{q}{4}\}$ and $q \equiv 1 \mod \frac{q-1}{4}$. Hence, $|G_\alpha| = q\cdot\frac{q-1}{4}$. This case is listed as part of statement (6). This finishes the analysis in the case that q and $|G_\alpha|$ have a common prime factor.

Therefore now instead suppose that q and $|G_\alpha|$ are coprime. Then $|G_\alpha|$ divides q^2-1. Let $x \in G_\alpha$ be of prime order r such that x fixes exactly four points. Then r divides $q-1$ or $q+1$. Let $\varepsilon \in \{-1,1\}$ be such that r divides $\frac{q-\varepsilon}{\gcd(2,q-1)}$. The normaliser of $\langle x\rangle$ is a dihedral group of order $2\cdot\frac{q-\varepsilon}{\gcd(2,q-1)}$. Let C be the cyclic subgroup of order $\frac{q-\varepsilon}{\gcd(2,q-1)}$ of that normaliser. Then Lemma 2.13 yield that $|C : C_\alpha| \leq |\,\mathrm{N}_G(\langle x\rangle) : \mathrm{N}_{G_\alpha}(\langle x\rangle)| \leq 4$. Hence, G_α contains a cyclic subgroup of order at least $\frac{q-\varepsilon}{4\cdot\gcd(2,q-1)} \geq \frac{42}{8} > 5$. Since $|G_\alpha|$ and q are coprime, an inspection of the list of subgroups of G shows that G_α either is cyclic of order dividing $\frac{q-\varepsilon}{\gcd(2,q-1)}$ or a dihedral group.

For a contradiction, assume that G_α is a dihedral group. Then there is an involution $t \in G_\alpha$ outside the cyclic subgroup C_α of G_α. Since $|G_\alpha|$ and q are coprime, this implies that q is odd. Then t lies in a cyclic group D of order $\frac{q-\delta}{2}$, where $\delta \in \{-1,1\}$. Thus, D has trivial intersection with C_α because otherwise $C_\alpha \leq D$ and hence $C = D$ contradicting the fact that $t \notin C_\alpha = D_\alpha$. Therefore, by Lemma 2.13, $|D : D_\alpha| \leq |\,\mathrm{N}_G(t) : \mathrm{N}_{G_\alpha}(t)| \leq 4$. On the other hand, $|D : D_\alpha| = \frac{|D|}{2} = \frac{q-\delta}{4} \geq \frac{42}{4} > 4$. This contradiction shows that G_α is not a dihedral group.

Therefore G_α is cyclic. Then by Corollary 2.10, the number of fixed points of a non-trivial element c of G_α is $\frac{|\,\mathrm{N}_G(\langle c\rangle)|}{|G_\alpha|} = \frac{2\cdot\frac{q-\varepsilon}{\gcd(2,q-1)}}{|G_\alpha|}$. Since G_α contains non-trivial elements that fix exactly four points, now every element in G_α fixes exactly four points and $|G_\alpha| = \frac{q-\varepsilon}{2\cdot\gcd(2,q-1)}$. Since $q-\varepsilon$ is only divisible by 2 if q is odd, $\gcd(2,q-1) = 2$ and hence $|G_\alpha| = \frac{q-\varepsilon}{4}$. Thus, $q-\varepsilon$ is divisible by 4 and hence $q \equiv \varepsilon \mod 4$. This case is listed as part of statement (6).

For the other direction of the lemma, suppose that q is odd. First suppose that $q \equiv 1 \mod 4$ and suppose that $U \leq G$ is the semi-direct product of an elementary abelian group K of order q with a cyclic group of order $\frac{q-1}{4}$. Then U is a Frobenius group with Frobenius kernel K. Let $x \in U$ be non-trivial. Since all powers of x fix the same points as x, in order to identify the fixity with which G is acting on G/U, it suffices to determine the number of fixed points of all elements of U that have prime order. Let $u \in U$ be of prime order r. If r divides q, then $u \in K$. Let $y \in G$ be such that $Uy \in G/U$ is fixed by u. Then $u^{y^{-1}} \in U$ is an r-element, and hence $u^{y^{-1}} \in K$. Thus, $u \in K \cap K^y$. Since different conjugates of K have trivial intersection, $y \in \mathrm{N}_G(K)$. Thus, u fixes at most $\frac{|\mathrm{N}_G(K)|}{|U|} = \frac{q \cdot \frac{q-1}{2}}{q \cdot \frac{q-1}{4}} = 2$ points in G/U. If r does not divide q, then r divides $q-1$ and by Lemma 2.11, the number of fixed points of u is $\frac{|K| \cdot |\mathrm{N}_G(u)|}{|U|} = \frac{q \cdot 2 \cdot \frac{q-1}{2}}{q \cdot \frac{q-1}{4}} = 4$. Thus, G acts with fixity 4 on G/U.

Therefore, now suppose that there exists $\varepsilon \in \{-1, 1\}$ such that U is cyclic of order $\frac{q-\varepsilon}{4}$. Let $x \in U$. By Lemma 2.10, the number of fixed points of x is $\frac{|N_G(\langle x \rangle)|}{|U|} = \frac{2 \cdot \frac{q-1}{2}}{\frac{q-1}{4}} = 4$ and hence G acts with fixity 4 on G/U. This finishes the proof.

3.1.2 PSL(3,q) and PSU(3,q)

The groups $\mathrm{PSL}(3, q)$ and $\mathrm{PSU}(3, q)$ have a lot of structural properties in common. Thus, instead of dealing with them separately, we look at their possible fixity-4 actions simultaneously and only distinguish at certain parts of the proof when their structures differ. We will see that, with the exception of $\mathrm{PSL}(3, 2) \cong \mathrm{PSL}(2, 7)$ and $\mathrm{PSU}(3, 3)$, none of the simple groups $\mathrm{PSL}_\varepsilon(3, q)$ exhibits a fixity-4 action.

For the special case that an involution fixes exactly four points, Patrick Salfeld states in Lemma 2.21 in [89] that $\mathrm{PSU}(3, 2^f)$ cannot act with fixity 4 under this hypothesis if $f \geq 2$. His proof is a specialisation of the strategy of the first part of the analysis in Lemma 3.4. In his Section 2.4, he also analysed for every odd prime power q whether $\mathrm{PSL}_\varepsilon(3, q)$ can act with fixity 4 and such that involutions fix exactly four points. His approach is more general and written in the context of simple groups of Lie type in odd characteristic. For the purpose of Lemma 3.4, we can use a more direct approach, using the structure information about $\mathrm{PSL}_\varepsilon(3, q)$ and analysing all fixity-4 actions without restriction on the number of fixed points of involutions.

Since $\mathrm{PSL}(3, 2) \cong \mathrm{PSL}(2, 7)$ was dealt with in the previous section and since $\mathrm{PSU}(3, 2)$ is not simple, the analysis in Lemma 3.4 is restricted to $q \geq 3$.

However, first we need some properties of $\mathrm{PSL}_\varepsilon(3, q)$ regarding elements of order 3, when 3 divides $q - \varepsilon$.

Lemma 3.3
Let $q \geq 2$ be a prime power, let $\varepsilon \in \{-1, 1\}$ be such that 3 divides $q - \varepsilon$, and let $G = \mathrm{PSL}_\varepsilon(3, q)$. If $g \in G$ has order 3, then $|\mathrm{N}_G(\langle g \rangle)|$ is even.

Proof:

Since 3 divides $q-\varepsilon$, $|\operatorname{Z}(\operatorname{SL}_\varepsilon(3,q))| = \gcd(3, q-\varepsilon) = 3$. We will use the generic character table of $\operatorname{PSL}_\varepsilon(3,q)$, see Table 2 in [93] with the notation given there in Section 7. Then $\delta = \varepsilon$, $d = 3$, $r = q-\varepsilon$, $r' = \frac{q-\varepsilon}{3}$, $s = q+\varepsilon$, $t = q^2+\varepsilon q+1$, and $t' = \frac{q^2+\varepsilon q+1}{3}$. In particular, t' is not divisible by 3. For every conjugacy class of G a representative in $\operatorname{SL}_\varepsilon(3,q)$ is given in Table 2 in [93]. We will now go through the conjugacy classes of G and determine if they contain elements of order 3 and if they do, we will also look at the order of the normaliser in G.

The conjugacy class $\mathcal{C}_1$ contains only the trivial element. Since 3 does not divide q, the third power of the representative of conjugacy class $\mathcal{C}_2$ does not lie in $\operatorname{Z}(\operatorname{SL}_\varepsilon(3,q))$. The same is true for the representatives of the conjugacy classes $\mathcal{C}_3^{(k)}$. Therefore neither $\mathcal{C}_1$, $\mathcal{C}_2$ nor $\mathcal{C}_3^{(k)}$ contain elements of order 3 of G.

The centraliser order of elements of the conjugacy classes $\mathcal{C}_4^{(k)}$ is $qr'rs = r'q(q-\varepsilon)(q+\varepsilon)$, an even number. Thus, for every element g contained in one of these conjugacy classes, $|\operatorname{N}_G(\langle g\rangle)|$ is even.

For the representatives of the conjugacy classes $\mathcal{C}_5^{(k)}$, their third power lies outside of $\operatorname{Z}(\operatorname{SL}_\varepsilon(3,q))$, again implying that $\mathcal{C}_5^{(k)}$ does not contain elements of order 3 of G.

The subgroup generated by the representative $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$ of the conjugacy class $\mathcal{C}_6'$ where ω is a primitive third root of unity is normalised by the involution $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & \omega \\ 0 & \omega^2 & 0 \end{pmatrix} \in \operatorname{SL}_\varepsilon(3,q) \setminus \operatorname{Z}(\operatorname{SL}_\varepsilon(3,q))$. Hence, for every $g \in \mathcal{C}_6'$ the order of $\operatorname{N}_G(\langle g\rangle)$ is even.

The matrix $\begin{pmatrix} \rho^k & 0 & 0 \\ 0 & \rho^l & 0 \\ 0 & 0 & \rho^m \end{pmatrix}$ where ρ is a primitive r-th root of unity is a representative of conjugacy class $\mathcal{C}_6^{(k,l,m)}$. The third power is in $\operatorname{Z}(\operatorname{SL}_\varepsilon(3,q))$ if and only if $\rho^{3k} = \rho^{3l} = \rho^{3m}$, thus if r divides $3k-3l$. This is the case when $r' = \frac{r}{3}$ divides $k-l$, but this is not possible by the condition that $1 \leq k < l \leq r'$. Therefore, the conjugacy classes $\mathcal{C}_6^{(k,l,m)}$ do not contain elements of order 3 of G.

The third power of an element in the conjugacy class $\mathcal{C}_7^{(k)}$ is described by the matrix $\begin{pmatrix} \rho^{3k} & 0 & 0 \\ 0 & \sigma^{-3\varepsilon k} & 0 \\ 0 & 0 & \sigma^{-3qk} \end{pmatrix}$ where σ is a primitive rs-th root of unity and ρ is again a primitive r-th root of unity. If this element is in $\operatorname{Z}(\operatorname{SL}_\varepsilon(3,q))$, then $\sigma^{-9\varepsilon k} = 1$, and hence rs divides $9k$. Since s is not divisible by 3, this implies that s must divide k contradicting the condition that $k \not\equiv 0 \mod s$. Therefore, the conjugacy classes $\mathcal{C}_7^{(k)}$ do not contain elements of order 3.

The remaining classes $\mathcal{C}_8^{(k)}$ contain elements with centraliser order t'. Since t' is not divisible by 3, they do not contain elements of order 3.

As a result, either a conjugacy class does not contain elements of order 3 or the normaliser of every element of the conjugacy class in G has even order.

Lemma 3.4
Let $q \geq 3$ be a prime power, let $\varepsilon \in \{-1, 1\}$, and let $G = \mathrm{PSL}_\varepsilon(3, q)$. Suppose that G acts transitively on a set Ω. Then G acts with fixity 4 on Ω if and only if $G = \mathrm{PSU}(3, 3)$ and the point stabilisers are of type $(E_9 : C_3) : C_8$.

Proof:

Using the GAP package TomLib [74] through the algorithm in Remark 2.22, the answers to the following commands prove the statement of the lemma for all $q \leq 11$.

```
List([3,4,5,7,8,9,11], x->TestTom(
↪  TableOfMarks(Concatenation("L3(",String(x),")")), 4 ));
List([3,4,5,7,8,9,11], x->TestTom(
↪  TableOfMarks(Concatenation("U3(",String(x),")")), 4 ));
```

Therefore, throughout the rest of the proof, suppose that $q \geq 13$.

We first collect some information about G that will be useful throughout the proof. The order of G is $q^3 \cdot (q^2 + \varepsilon q + 1)(q + \varepsilon) \cdot \frac{(q-\varepsilon)^2}{\gcd(3, q-\varepsilon)}$ and the maximal subgroups of $\mathrm{SL}_\varepsilon(3, q)$ are classified in Tables 8.3, 8.4, 8.5, and 8.6 in [15] including the errata. If q is odd, then also $q^2 + \varepsilon q + 1$ is odd. Since $|\mathrm{Z}(\mathrm{SL}_\varepsilon(3, q))| = \gcd(3, q - \varepsilon)$ is odd, the Sylow 2-subgroups of G are isomorphic to the Sylow 2-subgroups of $\mathrm{GL}_\varepsilon(2, q)$.

Let p be the prime dividing q. Then the Sylow p-subgroups of G have order q^3 and contain a normal elementary abelian subgroup of order q with elementary abelian factor group. Together with the information on page 67 in [105] and calculations in $\mathrm{SL}(3, q)$, this implies that the centre of a Sylow p-subgroup of G has order q.

To prove the lemma for $q \geq 13$, assume for a contradiction that G acts with fixity 4 on Ω. Let $\alpha \in \Omega$. We will first see that G_α has odd order before further analysing the prime divisors of $|G|$ that could possibly divide $|G_\alpha|$.

Assume for a first contradiction that G_α has even order. Let $t \in G_\alpha$ be an involution.

The analysis now depends on whether q is even or odd. First suppose that q is even. Since G_α has even order, one of the cases (1)-(3) in Lemma 3.1 holds. In case (1), Satz 1 in [11] implies that $G = \mathrm{PSU}(3, q)$. Theorem 2 in [45] and the Third Main Theorem in [1] yield that case (2) does not hold under the hypothesis that q is even and the Main Theorem in [40] shows that case (3) is not possible for all even $q \geq 13$. As a consequence, $G = \mathrm{PSU}(3, q)$ and G has a strongly embedded subgroup. Let f be a positive integer such that $q = 2^f$. Let $T \in \mathrm{Syl}_2(G)$ be such that $T_\alpha \in \mathrm{Syl}_2(G_\alpha)$ and such that $t \in T_\alpha$. Then by statement (4) on page 535 in [11], $t \in \mathrm{Z}(T)$. Since t has at most 4 fixed points, $4 \geq |\,\mathrm{C}_T(t) : \mathrm{C}_{T_\alpha}(t)| = |T : T_\alpha|$. Hence, $|T_\alpha|$ is either $|T| = q^3$, $\frac{|T|}{2} = \frac{q^3}{2} = 2^{3f-1}$, or $\frac{|T|}{4} = \frac{q^3}{4} = 2^{3f-2}$. Thus, the order of G_α is divisible by $\frac{q^3}{4} \geq 2^{10}$. An inspection of the maximal subgroups of $\mathrm{SU}(3,q)$ reveals that G_α lies in a maximal subgroup M that has a full pre-image in $\mathrm{SU}(3,q)$ of type $[q^3] : C_{q^2-1}$. Let I be the set of all involutions of G. Assume for a contradiction that $\mathrm{N}_G(T)$ contains I. Then $\langle I\rangle \leq \mathrm{N}_G(T)$. Since G is simple and $\langle I\rangle$ is a normal non-trivial subgroup of G, this implies that $\langle I\rangle = G$. Thus, $\mathrm{N}_G(T) = G$ but then $T \trianglelefteq G$ and this contradicts the fact that G is simple. Therefore, the assumption was incorrect and there exists an involution s of G outside of $\mathrm{N}_G(T)$. By statement (6) on page 534 in [11], $|\,\mathrm{C}_G(s)|$ is divisible by $\frac{q+1}{\gcd(3,q+1)}$.

Since by Lemma 4.1 (i) in [11], all involutions in G are conjugate, there exists $h \in G$ such that $s = t^h \in (G_\alpha)^h = G_{\alpha^h}$. Let $\beta = \alpha^h$. Then $s \in G_\beta$. Since G acts with fixity 4, the element s has at most 4 fixed points, and therefore $|\,\mathrm{C}_G(s) : \mathrm{C}_{G_\beta}(s)| \leq 4$. The order of $\mathrm{C}_G(s)$ is divisible by $\frac{q+1}{\gcd(3,q+1)}$ and since $q = 2^f$, this number is odd and only divisible by 3 if $q+1$ is divisible by 9. Thus, $|\,\mathrm{C}_{G_\beta}(s)|$ is divisible by $\frac{q+1}{\gcd(9,q+1)}$, and hence $|G_\beta| = |G_\alpha|$ is divisible by $\frac{q+1}{\gcd(9,q+1)} > 4$.

If $\frac{q+1}{\gcd(9,q+1)}$ were a 3-power, then Lemma 2.12 would yield that $f \in \{1,3\}$, implying $q \in \{2,8\}$. Since we suppose that $q \geq 13$, $\frac{q+1}{\gcd(9,q+1)}$ is not a 3-power. Since $\frac{q+1}{\gcd(9,q+1)}$ is odd, it must be divisible by a prime greater than 3.

Let $r \geq 5$ be a prime that divides $q+1$ and $|G_\alpha|$. Then G_α contains a Sylow r-subgroup of G. Let k be the greatest positive integer such that r^k divides the order of G_α. Since $|G_\alpha|$ divides $|M| = q^3(q+1)\frac{(q-1)}{\gcd(3,q+1)}$ and since q, $q-1$, and 3 are coprime to r, it follows that r^k is a divisor of $q+1$. Then the fact that $|G| = q^3 \cdot (q^2-q+1)(q-1) \cdot \frac{(q+1)^2}{\gcd(3,q+1)}$ implies that r^{2k} is a divisor of $|G|$ and, as a consequence, Sylow r-subgroups of G have order at least r^{2k}. This contradicts the fact that G_α contains a Sylow r-subgroup. Therefore, G cannot be $\mathrm{PSU}(3,2^f)$. This was the last remaining option in the case that q is even.

Therefore suppose instead that q is odd. Then, by our observation above, the 2-rank of G equals the 2-rank of $\mathrm{GL}_\varepsilon(2,q)$ and by [26] pp. 142-143 the latter is 2. Hence, the hypothesis of Proposition 2.1 in [44] is fulfilled. Therefore, all involutions in G are conjugate, thus t lies inside the centre of a group U that has a full pre-image in $\mathrm{SL}_\varepsilon(3,q)$ isomorphic to $\mathrm{GL}_\varepsilon(2,q)$. As a consequence, $|\,\mathrm{C}_G(t)|$ is divisible by $\frac{q\cdot(q-\varepsilon)^2\cdot(q+\varepsilon)}{\gcd(3,q-\varepsilon)}$. Since $|\,\mathrm{C}_G(t) : \mathrm{C}_{G_\alpha}(t)| \leq 4$ and $\frac{q}{4} > 2$, this implies that there exists an element $x \in G_\alpha$ of order p such that p divides q. Let $Q \in \mathrm{Syl}_p(G)$ be such that $x \in Q$. Then $\mathrm{Z}(Q) \leq \mathrm{C}_G(x)$ and hence $|\,\mathrm{Z}(Q) : \mathrm{Z}(Q) \cap G_\alpha| \leq |\,\mathrm{C}_G(x) : \mathrm{C}_{G_\alpha}(x)| \leq 4$. Thus, the fact that $|\,\mathrm{Z}(Q)| = q$ implies that $\mathrm{Z}(Q) \cap G_\alpha$ contains a non-trivial element z. Then $Q \leq \mathrm{C}_G(z)$ and hence $|Q : Q_\alpha| \leq |\,\mathrm{C}_G(z) : \mathrm{C}_{G_\alpha}(z)| \leq 4$. As a consequence, G_α is divisible by $\frac{q^3}{\gcd(3,q)}$, because q is odd. Since $|G_\alpha|$ is also divisible by $\frac{q\cdot(q-\varepsilon)^2\cdot(q+\varepsilon)}{\gcd(3,q-\varepsilon)}$ and $q \geq 13$, an inspection of the maximal subgroups of $\mathrm{SL}_\varepsilon(3,q)$ shows that $G = \mathrm{PSL}(3,q)$ and that G_α lies in a maximal subgroup M that has a full pre-image in $\mathrm{SL}(3,q)$ of type $E_{q^2} : \mathrm{GL}(2,q)$. Let $E \leq M$ be the normal elementary abelian subgroup of order q^2. Since $|Q_\alpha| > q^2$, there exists a complement L of E in M that has non-trivial intersection with Q_α. Thus, there exists a non-trivial element $a \in Q_\alpha \cap L$. Since a acts non-trivially on E, $|\,\mathrm{C}_E(a)| = q$, and therefore the p-part of $|\,\mathrm{C}_M(a)|$ is q^2. Since $\mathrm{Z}(L) \leq \mathrm{C}_M(a)$, $|\,\mathrm{C}_M(a)|$ is divisible by $\frac{q-1}{\gcd(3,q-1)}$. By Table 2 in [93], all non-trivial elements with a centraliser in G of order divisible by $q^2 \cdot \frac{q-1}{\gcd(3,q-1)}$ have centraliser order $q^3 \cdot \frac{q-1}{\gcd(3,q-1)}$. Thus, $|\,\mathrm{C}_G(a)| = q^3 \cdot \frac{q-1}{\gcd(3,q-1)}$. Since $\mathrm{C}_{G_\alpha}(a) \leq \mathrm{C}_M(a)$, it follows that q divides $|\,\mathrm{C}_G(a) : \mathrm{C}_{G_\alpha}(a)|$, contradicting Lemma 2.13. Consequently, the last remaining option under the assumption that G_α has even order is impossible. This implies that $|G_\alpha|$ is odd.

The order of G_α is divisible by a prime dividing at least one of the numbers $q-\varepsilon$, $q+\varepsilon$, q or $q^2+\varepsilon q+1$. We will eliminate each of the possibilities one after another.

First suppose that there exists an element x in G_α of prime order r such that r divides $q-\varepsilon$. As a first step in this case, we will see that G_α contains a Sylow r-subgroup. If $r \geq 5$, this is already proven in Lemma 2.14. Therefore suppose that $r = 3$. Let P be a Sylow 3-subgroup of G containing x. Since $|G| = q^3 \cdot (q^2+\varepsilon q+1)(q+\varepsilon) \cdot \frac{(q-\varepsilon)^2}{\gcd(3,q-\varepsilon)}$ and $q^2+\varepsilon q+1$ is divisible by 3, $|P| \geq 9$. Either $x \in \mathrm{Z}(P)$ or $\langle x, \mathrm{Z}(P)\rangle$ is of order at least 9. In both cases $\mathrm{C}_G(x)$ is divisible by 9. Then Lemma 3.3 implies that $|\,\mathrm{N}_G(\langle x\rangle)|$ is divisible by 18. Since $|\,\mathrm{N}_G(\langle x\rangle) : \mathrm{N}_{G_\alpha}(\langle x\rangle)| \leq 4$ and $|G_\alpha|$ is odd, this means that G_α contains a Sylow 3-subgroup of G because otherwise the index would be divisible by 3 and 2. Therefore, the point stabiliser G_α contains a Sylow r-subgroup even if $r = 3$.

Then an inspection of the list of maximal subgroups of $\mathrm{SL}_\varepsilon(3,q)$ yields that G_α contains an r-element u in the centre of a subgroup U whose pre-image in $\mathrm{SL}_\varepsilon(3,q)$ is of type $\mathrm{GL}_\varepsilon(2,q)$. Hence, $U \leq \mathrm{C}_G(u)$ and $|\,\mathrm{C}_G(u)|$ is divisible by $\frac{q\cdot(q-\varepsilon)^2\cdot(q+\varepsilon)}{\gcd(3,q-\varepsilon)}$. If q is even, then $q \geq 16$ and if q is odd, then $(q-\varepsilon)(q+\varepsilon)$ is divisible by 8. Thus, in both cases $|\,\mathrm{C}_G(u)|$ is divisible by 8, and since $|G_\alpha|$ is odd, this contradicts the fact that $|\,\mathrm{C}_G(u) : \mathrm{C}_{G_\alpha}(u)| \leq 4$. This contradiction shows that $|G_\alpha|$ and $q-\varepsilon$ are coprime.

Next suppose that there exists an element x in G_α of prime order r such that r divides $q+\varepsilon$. If $r \geq 5$, then Lemma 2.14 again yields that G_α contains a Sylow r-subgroup and, consequently, an r-element b in a subgroup U whose pre-image in $\mathrm{SL}_\varepsilon(3,q)$ is of type $\mathrm{GL}_\varepsilon(2,q)$. If $r = 3$, then (10-2) in [41] shows that G has cyclic Sylow 3-subgroups. Since all Sylow 3-subgroups are conjugate and each of them only has one subgroup of order 3, this implies that all subgroups of order 3 are conjugate in G. Thus, G_α contains a 3-element b in a maximal subgroup U whose pre-image in $\mathrm{SL}_\varepsilon(3,q)$ is of type $\mathrm{GL}_\varepsilon(2,q)$. Then in both cases $|\,\mathrm{C}_G(b)|$ is divisible by $\frac{(q-\varepsilon)}{\gcd(3,q-\varepsilon)}$ and hence the facts that $|\,\mathrm{C}_G(b) : \mathrm{C}_{G_\alpha}(b)| \leq 4$ and that $|G_\alpha|$ and $q-\varepsilon$ are coprime yield that $13-\varepsilon \leq q-\varepsilon \leq 4\cdot\gcd(3,q-\varepsilon) \leq 12$. As a consequence, $q = 13$ and $G = \mathrm{PSL}(3,13)$. Then $r = 7$ and the GAP code `Order(Normalizer(PSL(3,13),SylowSubgroup(PSL(3,13),7)));` implies that $|\,\mathrm{N}(\langle b\rangle)|$ is divisible by 16. This is a contradiction to the facts that $|G_\alpha|$ is odd and that $|\,\mathrm{N}_G(\langle b\rangle) : \mathrm{N}_{G_\alpha}(\langle b\rangle)| \leq 4$. Therefore, $|G_\alpha|$ and $(q-\varepsilon)\cdot(q+\varepsilon)$ are coprime.

Now suppose that there exists an element x in G_α of prime order r such that r divides q. If $r \geq 5$, then by Lemma 2.14, G_α contains a Sylow r-subgroup Q of G. Since Q lies in a subgroup whose pre-image in $\mathrm{SL}_\varepsilon(3,q)$ is of order $q^3\cdot(q-1)\cdot(q-\varepsilon)$ and has a normal subgroup of order q^3, $|\,\mathrm{N}_G(Q)|$ is divisible by $\frac{(q-1)\cdot(q-\varepsilon)}{\gcd(3,q-\varepsilon)} > 4$. This is a contradiction to Lemma 2.13 and the fact that $|G_\alpha|$ is coprime to $(q-\varepsilon)\cdot(q+\varepsilon)$. Thus, $r=3$ and $q \geq 27$. Let $Q \in \mathrm{Syl}_r(G)$ be such that $x \in Q$. Then $|\,\mathrm{Z}(Q) : \mathrm{Z}(Q)\cap G_\alpha| \leq |\,\mathrm{C}_G(x) : \mathrm{C}_{G_\alpha}(x)| \leq 4$, and since $|\,\mathrm{Z}(Q)| = q$, $\mathrm{Z}(Q)\cap G_\alpha$ contains a non-trivial element z. Thus $|\,\mathrm{C}(z)|$ is divisible by q^3, and hence Table 2 in [93] implies that $|\,\mathrm{C}(z)|$ is in fact divisible by $q^3\cdot\frac{q-\varepsilon}{\gcd(3,q-\varepsilon)}$. Since $\frac{(q-\varepsilon)}{\gcd(3,q-\varepsilon)} \geq 26$, the fact that $|\,\mathrm{C}_G(z) : \mathrm{C}_{G_\alpha}(z)| \leq 4$ yields that G_α contains an element of order dividing $q-\varepsilon$, which has proven to be impossible. Therefore, $|G_\alpha|$ and $q\cdot(q-\varepsilon)\cdot(q+\varepsilon)$ are coprime. In particular, $|G_\alpha|$ divides $q^2+\varepsilon q+1$ and since one of the numbers q, $q-\varepsilon$, and $q+\varepsilon$ is divisible by 3, it follows that $|G_\alpha|$ is coprime to 6.

Let $x \in G_\alpha$ be of prime order r and such that x fixes exactly four points. Then r divides $q^2+\varepsilon q+1$. Since $\mathrm{SL}_\varepsilon(3,q)$ contains a maximal subgroup of type $C_{q^2+\varepsilon q+1} : C_3$ and the image of this group in G contains a Sylow r-subgroup, x lies in a cyclic subgroup $C \leq G$ of order $\frac{q^2+\varepsilon q+1}{\gcd(3,q-\varepsilon)}$ where $|\,\mathrm{N}_G(C)| = \frac{q^2+\varepsilon q+1}{\gcd(3,q-\varepsilon)}\cdot 3$ because $r \geq 5$. Since $\langle x\rangle$ is a characteristic subgroup of C, it follows that $|\,\mathrm{N}_G(\langle x\rangle)|$ is divisible by $\frac{q^2+\varepsilon q+1}{\gcd(3,q-\varepsilon)}\cdot 3$. Therefore, Lemma 2.9 implies that the number of fixed points of x is divisible by $\frac{|\,\mathrm{N}_G(\langle x\rangle)|}{|G_\alpha|}$. Since $|G_\alpha|$ is not divisible by 3, the number of fixed points of x is divisible by 3, contradicting the assumption that x has exactly 4 fixed points, completing the proof.

3.1.3 Sz(q)

The last family of groups we look at are the Suzuki groups $\mathrm{Sz}(q)$. Here we will again see that it is helpful to separate the analysis according to whether point stabilisers have even order, odd order divisible by 3, or order coprime to 6. Since the orders of the Suzuki groups are coprime to 3, the case that point stabiliser have odd order divisible by 3 will not occur. This makes our analysis easier. Again Patrick Salfeld considered in [89] the situation for $\mathrm{Sz}(q)$ under the additional hypothesis that an involution fixes exactly four points. The proof presented in Lemma 3.5 here differs from the one of Lemma 2.23 in [89] because the arguments are chosen in a way such that they do not depend on the knowledge of the number of points that are fixed by involutions.

Lemma 3.5
Let n be a positive integer, let $q = 2^{2n+1}$, and let $G = \mathrm{Sz}(q)$. Suppose that G acts transitively on a set Ω. Then G acts with fixity 4 on Ω if and only if the point stabilisers are cyclic of order $q+\sqrt{2q}+1$ or of order $q-\sqrt{2q}+1$.

Proof:

Information about the maximal subgroups is stated in Theorem 4.1 in [105] and in Theorem 6.5.4 in [43] and will be used without further reference. The order of G is $(q^2+1)q^2(q-1)$ (see p. 117 in [105]). In particular, Sylow 2-subgroups have order q^2, and since $q = 2^{2n+1}$, $|G|$ is not divisible by 3. Using the GAP package TomLib [74] through the algorithm in Remark 2.22, the answer to the command `TestTom(TableOfMarks("Sz(8)"),4);` proves the statement of the lemma if $q = 8$. Therefore, throughout the rest of the proof, suppose that $q \geq 32$.

For the first direction, additionally suppose that G acts with fixity 4 on Ω. Let $\alpha \in \Omega$.

As a first step, we will see that G_α has odd order. Assume for a contradiction that $|G_\alpha|$ is even. Let $T \in \mathrm{Syl}_2(G)$ be such that $T_\alpha \in \mathrm{Syl}_2(G_\alpha)$. Let $t \in T_\alpha$ be an involution. Then $t \in \Omega_1(T)$. By Theorem 2.4 (c) in [99], $\Omega_1(T) = \mathrm{Z}(T)$ is elementary abelian of order q, and by part (d), $\mathrm{N}_G(T)$ acts transitively on the set of involutions of T. The first part implies, together with Theorem 2.4 (e) in [99], that $T = \mathrm{C}_G(t)$ and the second part implies that all involutions in G are conjugate.

By Lemma 2.13, $|T : T_\alpha| = |\,\mathrm{C}_G(t) : \mathrm{C}_{G_\alpha}(t)| \leq 4$, hence, $|T_\alpha|$ is divisible by $\frac{q^2}{4}$. By Lemma 2.9, the number of fixed points of t is $\frac{|\{\langle t\rangle^g \leq G_\alpha | g \in G\}| \cdot |\,\mathrm{N}_G(\langle t\rangle)|}{|G_\alpha|}$. Since $|\{\langle t\rangle^g \leq G_\alpha \mid g \in G\}|$ is the number of involutions in G_α, it is at least the number of involutions in T_α. Thus, $|\{\langle t\rangle^g \leq G_\alpha \mid g \in G\}| \geq \frac{q}{4} - 1$. Since $\mathrm{N}_G(\langle t\rangle) = \mathrm{C}_G(t) = T$, it follows that $4 \geq |\,\mathrm{fix}_\Omega(t)| \geq \frac{(\frac{q}{4}-1)\cdot q^2}{|G_\alpha|}$. If G_α were a 2-group, then $|G_\alpha| \leq q^2$ and hence $4 \geq \frac{q}{4} - 1 \geq \frac{32}{4} - 1$ would imply a contradiction. Therefore, G_α contains an element $y \in G_\alpha$ of odd order. Since $T_\alpha \leq G_\alpha$ has order divisible by $\frac{q^2}{4} = 2^{2(2n+1)-2}$, G_α does not lie in a maximal subgroup isomorphic to $\mathrm{Sz}(2^{2m+1})$ where m is a non-negative integer such that $\frac{2n+1}{2m+1}$ is a prime r, because otherwise $2m+1 \geq 2(2n+1) - 2 = 2r(2m+1) - 2$ and hence $2 \geq 2(r-1)(2m+1)$, contradicting the observation that r must be odd. Thus, G_α lies in a Frobenius group of order $q^2(q-1)$ with Frobenius kernel K of order q^2 and cyclic Frobenius complements (see Theorem 2.4 (d) in [99]). Then $T_\alpha \leq K$, more precisely $T_\alpha = K_\alpha$ by the choice of T. Hence, there exists an integer c dividing $q-1$ such that $|G_\alpha| = |T_\alpha| \cdot c$. Since y has odd order, $\mathrm{o}(y)$ divides c and y lies in a cyclic Frobenius complement. In particular, G_α is itself a Frobenius group with Frobenius kernel T_α by 4.1.8 in [65]. Then Lemma 2.11 shows that y has exactly $\frac{|T_\alpha| \cdot |\,\mathrm{N}_G(\langle y\rangle)|}{|G_\alpha|}$ fixed points. Since $\mathrm{o}(y)$ does not divide q^2+1, y lies maximal subgroup that is a dihedral group of order $2(q-1)$. It follows that $|\,\mathrm{N}_G(\langle y\rangle)| = 2 \cdot (q-1)$. Then y has exactly $\frac{|T_\alpha| \cdot 2(q-1)}{|T_\alpha| \cdot c} = \frac{2(q-1)}{c}$ fixed points. Since c is odd and y fixes at least one and at most four points, $c = q-1$ and y fixes exactly two points.

Therefore, $|T_\alpha| \equiv 1 \mod q-1$ and hence $|T_\alpha| = q^2$ because $q > 5$. This implies that $|G_\alpha| = q^2(q-1)$ and that $T = T_\alpha \trianglelefteq G_\alpha$. As a consequence, all involutions in G_α lie in T, hence G_α has $q-1$ involutions. Thus, the number of fixed points of t is $\frac{|\{\langle t\rangle^g \le G_\alpha | g\in G\}|\cdot|\,\mathrm{N}_G(\langle t\rangle)|}{|G_\alpha|} = \frac{(q-1)\cdot q^2}{q^2(q-1)} = 1$. Since G acts with fixity 4, there exists an element $x \in G_\alpha$ with four fixed points. If x has even order, then a power of x is an involution, and hence, conjugate to t, but then it can fix only one point and hence x can fix at most one point. If x is odd, then x lies in a Frobenius complement and has exactly $\frac{|T_\alpha|\cdot|\,\mathrm{N}_G(\langle x\rangle)|}{|G_\alpha|} = \frac{q^2\cdot 2(q-1)}{q^2(q-1)} = 2$ fixed points. Since x has either even or odd order, this is a contradiction. As a result, the assumption that $|G_\alpha|$ is even was wrong.

This means that G_α has odd order and there exists an element $x \in G_\alpha$ of odd prime order p that fixes exactly four points.

The three numbers $(q+\sqrt{2q}+1), (q-\sqrt{2q}+1)$ and $q-1$ are pairwise coprime. Let M_0 be a maximal subgroup of G that is dihedral of order $2(q-1)$ with a cyclic subgroup C_0 of order $q-1$ and for every $\varepsilon \in \{-1,1\}$ let M_ε be a maximal subgroup of order $4(q+\varepsilon\sqrt{2q}+1)$ with a cyclic normal subgroup C_ε of order $q+\varepsilon\sqrt{2q}+1$. Each of the maximal subgroups M_0, M_{-1}, M_{+1} either has a trivial Sylow p-subgroup or C_0, C_{-1}, C_1, respectively, contains a Sylow p-subgroup of G and, consequently, an element a conjugate to x. Then there exists $\beta \in \Omega$ such that $a \in G_\beta$ because G acts transitively on Ω.

Assume for a contradiction that p divides $q-1$. Since $\langle a\rangle \operatorname{char} C_0 \trianglelefteq M_0$ and M_0 is a maximal subgroup of the simple group G, it follows that $\mathrm{N}_G(\langle a\rangle) = M_0$. Then $|M_0 : M_0 \cap G_\beta| = |\,\mathrm{N}_G(\langle a\rangle) : \mathrm{N}_{G_\beta}(\langle a\rangle)| \le 4$ and $q-1$ is odd and coprime to 3 because $|G|$ is coprime to 3. Since $|M_0|$ is divisible by $q-1$, the inequality above implies that $|G_\beta|$ is divisible by $q-1$. An inspection of the list of maximal subgroups shows that the only subgroups of G of odd order divisible by $q-1$ are cyclic of order $q-1$. Hence $|G_\alpha| = |G_\beta| = q-1$. Since x has order p and four fixed points, $4 \equiv |\Omega| \equiv |G/G_\alpha| \equiv \frac{q^2(q^2+1)(q-1)}{q-1} \equiv q^2(q^2+1) \equiv 1^2\cdot(1^2+1) \equiv 2 \mod p$. Since $p \ge 5$, this is a contradiction.

Therefore, there exists $\varepsilon \in \{-1,1\}$ such that p divides $q+\varepsilon\sqrt{2q}+1$. Similar to the previous case the facts that $\langle a\rangle \operatorname{char} C_\varepsilon \trianglelefteq M_\varepsilon$ and that M_ε is a maximal subgroup of the simple group G implies that $\mathrm{N}_G(\langle a\rangle) = M_\varepsilon$. Then $4 \ge |\,\mathrm{N}_G(\langle a\rangle) : \mathrm{N}_{G_\beta}(\langle a\rangle)| = |M_\varepsilon : M_\varepsilon \cap G_\beta| = \frac{(q+\varepsilon\sqrt{2q}+1)\cdot 4}{|M_\varepsilon\cap G_\beta|}$. Since $|G_\beta|$ is odd, this yields that $|M_\varepsilon \cap G_\beta| = q+\varepsilon\sqrt{2q}+1$. Thus, G_β is divisible by $q+\varepsilon\sqrt{2q}+1$. Since the only subgroups of G of odd order divisible by $q+\varepsilon\sqrt{2q}+1$ are cyclic of order $q+\varepsilon\sqrt{2q}+1$, this implies the statement of the lemma.

For the other direction let $\varepsilon \in \{-1,1\}$ and $U \le G$ be such that U is a cyclic group of order $q+\varepsilon\sqrt{2q}+1$. Then G acts transitively on the cosets of G/U by right multiplication. Let $x \in U$ be non-trivial. Let $z \in \langle x\rangle$ be of prime order. Then the order of z divides $q+\varepsilon\sqrt{2q}+1$. Therefore, G has a maximal subgroup M of order $4(q+\varepsilon\sqrt{2q}+1)$ with a cyclic normal subgroup C of order $q+\varepsilon\sqrt{2q}+1$ containing a Sylow $\mathrm{o}(z)$-subgroup and, consequently, a conjugate b of z. Then $\langle b\rangle \operatorname{char} C \trianglelefteq M$ implies that $M \le \mathrm{N}_G(\langle b\rangle)$, and since M is a maximal subgroup of the simple group G, this means that $\mathrm{N}_G(\langle b\rangle) = M$. Hence, $|\,\mathrm{N}_G(\langle z\rangle)| = |M| = 4(q+\varepsilon\sqrt{2q}+1)$.

Thus, by Lemma 2.10, the element z has exactly $\frac{|\mathrm{N}_G(\langle z\rangle)|}{|U|} = \frac{(q+\varepsilon\sqrt{2q}+1)\cdot 4}{q+\varepsilon\sqrt{2q}+1} = 4$ fixed points. Since all elements fixed by x are also fixed by z, the number of fixed points of x is at most 4. Therefore, G acts on G/U transitively and with fixity 4.

3.2 The Case that Point Stabilisers have Odd Order Divisible by 3

We now look at case (4) of Lemma 3.1, namely at the situation that the order of a point stabiliser is odd and divisible by 3. Lemma 3.1 states that then one of four sub-cases holds. We will look at each of them separately.

The description of these sub-cases is in terms of some properties of the group G and not of its action. Thus, we will first collect most of the information about the Sylow 3-subgroup structure of finite simple groups that will us enable to determine which finite simple groups fulfil the requirements of the sub-cases (a)-(d). For this we will make use of the classification of finite simple groups. Afterwards in each case, we investigate for each of the remaining groups whether or not these groups can act transitively, with fixity 4, and such that a point stabiliser has odd order divisible by 3 on some set.

However, beforehand we need some information about the 3-structure of $\mathrm{SL}_\varepsilon(3,q)$ that will us enable to determine the Sylow 3-subgroup structure of $\mathrm{PSL}_\varepsilon(3,q)$. Furthermore the next lemma also contains some statements that will be of use in the analysis of the fixity-4 action of the family $\mathrm{G}_2(q)$, which will come up in the case that the Sylow 3-subgroups are extra-special of order 27.

Lemma 3.6
Let q be a prime power and let $\varepsilon \in \{-1,1\}$ be such that $q \equiv \varepsilon \mod 3$. Let $G = \mathrm{SL}_\varepsilon(3,q)$ and let $P \in \mathrm{Syl}_3(G)$. Then $\mathrm{Z}(P) = \mathrm{Z}(G)$. If $|q-\varepsilon|_3 = 1$, then for all non-trivial $h \in P$, the order of h is 3 and $|\,\mathrm{N}_G(\langle h\rangle)|$ is divisible by $2^{|q-\varepsilon|_2+1}$.

Proof:

Since 3 divides $q-\varepsilon$, it follows that $|\,\mathrm{Z}(\mathrm{SL}_\varepsilon(3,q))| = \gcd(3, q-\varepsilon) = 3$. We will use the generic character table of $\mathrm{SL}_\varepsilon(3,q)$, see Table 1a in [93] with the notation given there in Section 7. Then $\delta = \varepsilon$, $d = 3$, $r = q-\varepsilon$, $s = q+\varepsilon$, and $t = q^2+\varepsilon q+1$. In Table 1a in [93], for every conjugacy class of G a representative is given.

The information about the centraliser orders imply that central elements are only contained in the conjugacy classes $\mathcal{C}_1^{(0)}$, $\mathcal{C}_1^{(1)}$, and $\mathcal{C}_1^{(2)}$. Since none of the other centraliser orders is divisible by 3^3 (or r^2t), the central elements of G are the only elements in the centre of P. Thus, $\mathrm{Z}(P) = \mathrm{Z}(G)$ and for all non-trivial $h \in \mathrm{Z}(P)$, the order of h is 3 and $|\,\mathrm{N}_G(\langle h\rangle)|$ is divisible by $2^{|q-\varepsilon|_2+1}$.

From now on suppose that $|q-\varepsilon|_3 = 1$. Let $h \in P \setminus \mathrm{Z}(G)$. Then the order of $\mathrm{C}_G(h) \geq \langle h\rangle\,\mathrm{Z}(G)$ is divisible by 9.

By Table 1a in [93], h is in one of the conjugacy classes $\mathcal{C}_4^{(k)}$ or $\mathcal{C}_6^{(k,l,m)}$. If there exists a positive integer $k < r$ such that $k \not\equiv 0 \mod \frac{r}{3}$ and such that h is in $\mathcal{C}_4^{(k)}$, then h is conjugate to $\begin{pmatrix} \rho^k & 0 & 0 \\ 0 & \rho^k & 0 \\ 0 & 0 & \rho^{-2k} \end{pmatrix}$ where ρ is a primitive r-th root of unity. Thus, h^r is the identity matrix. Since $|r|_3 = 1$, h has order at most 3. Furthermore Table 1a in [93] implies that then $|\,\mathrm{C}_G(h)|$ is divisible by qr^2s, and hence by $2^{|q-\varepsilon|_2+1}$. Thus, the lemma holds in this case.

Therefore suppose that there exist positive integers k, l, and m, such that $k < l < m \leq r$, such that $k+l+m \equiv 0 \mod r$ and such that h is an element of the conjugacy class $\mathcal{C}_6^{(k,l,m)}$. Then h is conjugate to $a := \begin{pmatrix} \rho^k & 0 & 0 \\ 0 & \rho^l & 0 \\ 0 & 0 & \rho^m \end{pmatrix}$ where ρ is a primitive r-th root of unity. Since a^r is the identity matrix and since $|r|_3 = 1$, h has order at most 3. More precisely, the third power of a is the identity matrix if and only if $\rho^{3k} = \rho^{3l} = \rho^{3m} = 1$. Since $1 \leq k < l < m \leq r$ and $|r|_3 = 1$, this implies that $k = \frac{r}{3}$, $l = \frac{2r}{3}$, and $m = r$. Furthermore Table 1a in [93] implies that then $|\,\mathrm{C}_G(h)|$ is divisible by r^2, and hence by $2^{|q-\varepsilon|_2+1}$ if q is odd. Therefore assume that q is even. Then the involution $t := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}_\varepsilon(3,q)$ that inverts a has determinant 1. Therefore $t \in \mathrm{SL}_\varepsilon(3,q) = G$, and hence $\mathrm{N}_G(\langle h \rangle)$ contains an involution. Thus the lemma follows.

For the description of the Sylow 3-subgroup structure of finite simple groups, we recall that for a positive integer n and a prime p, $|n|_p$ denotes the highest positive integer such that n is divisible by $p^{|n|_p}$.

Lemma 3.7
Let G be a finite simple group, let $P \in \mathrm{Syl}_3(G)$, and let k be a non-negative integer such that $|P| = 3^k$. Then k and the r-rank of G are as stated in Table 3.1, where $\varepsilon \in \{-1, 1\}$, p is a prime, n and f are positive integers, and $q = p^f$.

Proof:

We use the classification theorem of finite simple groups (see page 3 in [105]). If G is cyclic, then the statement of the lemma follows directly.

Suppose that G is an alternating group. Then $|G| = \frac{n!}{2}$. If $G = \mathcal{A}_5$, then G has cyclic Sylow 3-subgroups of order 3. If G is $\mathcal{A}_6$, $\mathcal{A}_7$, or $\mathcal{A}_8$, then a Sylow 3-subgroup of G is $\langle (1,2,3), (4,5,6) \rangle$ and we can derive the numbers in Table 3.1 from this. If G is $\mathcal{A}_9$, $\mathcal{A}_{10}$, or $\mathcal{A}_{11}$, then a Sylow 3-subgroup of G is $\langle (1,2,3), (1,4,7)(2,5,8)(3,6,9) \rangle$ and the lemma holds in this case. All other alternating groups contain the 3-group $\langle (1,2,3), (4,5,6), (7,8,9), (10,11,12) \rangle$, and hence have p-rank at least 4.

Next we analyse the Sylow 3-subgroups of simple groups of Lie type over a field of order q. We will frequently use Theorem 3.3.3 in [43] in the case that

G	k	3-rank of G
C_3	1	1
C_p, $p \neq 3$	0	0
$\mathcal{A}_5$	1	1
$\mathcal{A}_6, \mathcal{A}_7, \mathcal{A}_8$	2	2
$\mathcal{A}_9, \mathcal{A}_{10}, \mathcal{A}_{11}$	4	3
$\mathcal{A}_n$, $n \geq 12$	≥ 5	≥ 4
$\mathrm{PSL}(2, 3^f)$	f	f
$\mathrm{PSL}(2, q)$, $p^f = q \geq 4$, $p \neq 3$	$\lvert q-1\rvert_3 + \lvert q+1\rvert_3$	1
$\mathrm{PSL}_\varepsilon(3, 3^f)$	$3f$	$2f$
$\mathrm{PSL}_\varepsilon(3, q)$, $q \equiv \varepsilon \mod 3$	$2 \cdot \lvert q-\varepsilon\rvert_3$	2
$\mathrm{PSL}_\varepsilon(3, q)$, $q \equiv -\varepsilon \mod 3$	$\lvert q+\varepsilon\rvert_3$	1
$\mathrm{PSL}_\varepsilon(n, 3^f)$, $n \geq 4$	$\geq 6f$	$\geq 4f$
$\mathrm{PSL}_\varepsilon(4, q)$, $q \equiv \varepsilon \mod 3$	$3 \cdot \lvert q-\varepsilon\rvert_3 + 1$	3
$\mathrm{PSL}_\varepsilon(n, q)$, $q \equiv -\varepsilon \mod 3$, $n \in \{4, 5\}$	$2 \cdot \lvert q+\varepsilon\rvert_3$	2
$\mathrm{PSL}_\varepsilon(n, q)$, $q \equiv \varepsilon \mod 3$, $n \geq 5$	≥ 4	≥ 4
$\mathrm{PSL}_\varepsilon(n, q)$, $q \equiv -\varepsilon \mod 3$, $n \in \{6, 7\}$	$3 \cdot \lvert q+\varepsilon\rvert_3 + 1$	3
$\mathrm{PSL}_\varepsilon(n, q)$, $q \equiv -\varepsilon \mod 3$, $n \geq 8$	≥ 5	≥ 4
$\mathrm{PSp}(4, 3^f)$	$4f$	$3f$
$\mathrm{PSp}(4, q)$, $q = p^f$, $p \neq 3$	$2 \cdot \lvert q^2-1\rvert_3$	2
$\mathrm{PSp}(6, 3^f)$, $\mathrm{P\Omega}(7, 3^f)$	$9f$	$\geq 5f$
$\mathrm{PSp}(6, q)$, $\mathrm{P\Omega}(7, q)$, $q = p^f$, $p \neq 3$	$3 \cdot \lvert q^2-1\rvert_3 + 1$	3
$\mathrm{PSp}(2n, q)$, $\mathrm{P\Omega}(2n+1, q)$, $n \geq 4$,	≥ 5	≥ 4
$\mathrm{P\Omega}_+(2n, q)$, $n \geq 4$	≥ 4	≥ 4
$\mathrm{P\Omega}_-(8, q)$, $q = p^n$, $p \neq 3$	$3 \cdot \lvert q^2-1\rvert_3 + 1$	3
$\mathrm{P\Omega}_-(8, 3^f)$, $\mathrm{P\Omega}_-(2n, q)$, $n \geq 5$	≥ 4	≥ 4
$\mathrm{Sz}(2^{2n+1})$	0	0
${}^3\mathrm{D}_4(3^f)$, $\mathrm{G}_2(3^f)$	$\geq 6f$	$\geq 4f$
${}^3\mathrm{D}_4(q)$, $q = p^f$, $p \neq 3$	≥ 4	2
$\mathrm{G}_2(q)$, $q = p^f$, $p \neq 3$	$2 \cdot \lvert q^2-1\rvert_3 + 1$	2
${}^2\mathrm{G}_2(3^{2n+1})$	$3(2n+1)$	$2(2n+1)$
${}^2\mathrm{F}_4(2^{2n+1})$	$2 \cdot \lvert 2^{2n+1}+1\rvert_3 + 1$	2
${}^2\mathrm{F}_4(2)'$	3	2
$\mathrm{F}_4(q)$, $\mathrm{E}_6(q)$, ${}^2\mathrm{E}_6(q)$, $\mathrm{E}_7(q)$, $\mathrm{E}_8(q)$	≥ 4	≥ 4
M_{11}, M_{22}, M_{23}, HS	2	2
$\mathrm{M}_{12}, \mathrm{M}_{24}$, J_2, J_4, He, Ru	3	2
J_1	1	1
J_3, Co_1, Co_2, Co_3, McL, HN	≥ 5	≥ 2
O'N	4	4
Suz, Ly, Fi_{22}, Fi_{23}, Fi_{24}', Th, $\mathbb{B}$, $\mathbb{M}$	≥ 5	≥ 5

Table 3.1: The Order of Sylow 3-Subgroups and the 3-rank of Simple Groups

q is a 3-power and (10-2) in [41] in the other cases. If q is not a 3-power, then $q^2 \equiv 1 \mod 3$, and hence the m_o in (10-2) in [41] is 1 or 2. More precisely, m_o is 1 if $q \equiv 1 \mod 3$ and 2 if $q \equiv -1 \mod 3$. Thus, we can read of the 3-rank in Table 10:1 and 10:2 in [41] but we have to take special care if 3 divides the centre of the group X as defined in (10-2) in [41]. We will usually use these theorems without further reference. We start by investigating the Sylow 3-subgroup structure of simple classical groups of Lie type.

Suppose that $n \geq 2$ and $G = \mathrm{PSL}_\varepsilon(n, q)$. Then (by Theorem 3.3.3 in [43]) the 3-rank of $\mathrm{PSL}(2, 3^f)$ is f, the 3-rank of $\mathrm{PSL}_\varepsilon(3, 3^f)$ is $2f$, and for all $n \geq 4$, the 3-rank of $\mathrm{PSL}_\varepsilon(n, 3^f)$ is at least $4f$. By Table 5.1.A in [62], the order of $\mathrm{PSL}_\varepsilon(n, q)$ is

$$\frac{q^{n(n-1)/2}}{\gcd(n, q-\varepsilon)} \prod_{i=2}^{n} (q^i - \varepsilon^i)$$

and we can read of the order of a Sylow 3-subgroup of G if q is a 3-power. This finishes the proof for $\mathrm{PSL}_\varepsilon(n, 3^f)$. Thus, we may suppose that q is not a 3-power.

Additionally suppose that $n = 2$. Then $G = \mathrm{PSL}(2, q)$, and the unique factor of $|G|$ divisible by 3 is $q - 1$ or $q + 1$. Furthermore the 3-rank of G is 1 (by (10-2) in [41]) as stated in Table 3.1.

Therefore suppose that $n \geq 3$. Then $\mathrm{PSL}(n, q)$ and $\mathrm{PSU}(n, q)$ are different groups with similar properties, and we will analyse them simultaneously. If $G = \mathrm{PSL}(3, q)$ and $q \equiv 1 \mod 3$, then 3 divides the order of $\mathrm{Z}(\mathrm{SL}(3, q))$, and hence the 3-rank of G is 1 or 2, and $|G| = q^3 \cdot (q-1)^2 \cdot (q+1) \cdot \frac{q^2+q+1}{3}$. If $G = \mathrm{PSU}(3, q)$ and $q \equiv -1 \mod 3$, then the 3-rank of G is also 1 or 2, and $|G| = q^3 \cdot (q+1)^2 \cdot (q-1) \cdot \frac{q^2-q+1}{3}$. Thus, to summarise, if $G = \mathrm{PSL}_\varepsilon(3, q)$ and $q \equiv \varepsilon \mod 3$, then the 3-rank of G is 1 or 2 and $|G| = q^3 \cdot (q-\varepsilon)^2 \cdot (q+\varepsilon) \cdot \frac{q^2+\varepsilon q+1}{3}$. Since q, $q+\varepsilon$, and $\frac{q^2+\varepsilon q+1}{3}$ are not divisible by 3 if $q \equiv \varepsilon \mod 3$, the order of a Sylow 3-subgroup of $\mathrm{PSL}_\varepsilon(3, q)$ is $2 \cdot |q-\varepsilon|_3$ as stated in the lemma. If the 3-rank of $G = \mathrm{PSL}_\varepsilon(3, q)$ were 1, then P would be cyclic. Let R be a Sylow 3-subgroup of $\mathrm{SL}_\varepsilon(3, q)$. Then by Lemma 3.6, $\mathrm{Z}(\mathrm{SL}_\varepsilon(3, q)) = \mathrm{Z}(R)$. Hence $R/\mathrm{Z}(R)$ is cyclic, and thus R abelian, contrary to the fact that $|\mathrm{Z}(R)| = |\mathrm{Z}(\mathrm{SL}_\varepsilon(3, q))| = 3$. A similar case distinction yields that if $G = \mathrm{PSL}_\varepsilon(3, q)$ and $q \equiv -\varepsilon \mod 3$, then the 3-rank of G is 1 and $|G| = q^3 \cdot (q+\varepsilon) \cdot (q-\varepsilon)^2 \cdot (q^2+\varepsilon q+1)$. In particular, the lemma holds for $\mathrm{PSL}_\varepsilon(3, q)$ if $q \equiv -\varepsilon \mod 3$.

We have seen that the cases $q \equiv \varepsilon \mod 3$ and $q \equiv -\varepsilon \mod 3$ behave quite differently. Therefore we split the analyses for $\mathrm{PSL}_\varepsilon(n, q)$ with $n \geq 4$ according to whether q is congruent to ε or $-\varepsilon$ modulo 3. First additionally suppose that $q \equiv \varepsilon \mod 3$. If $G = \mathrm{PSL}_\varepsilon(4, q)$, then the 3-rank of G is 3 and $|G| = \frac{q^3}{\gcd(4, q-\varepsilon)} \cdot (q-\varepsilon)^3 \cdot (q+\varepsilon)^2 \cdot (q^2+\varepsilon q+1) \cdot (q^2+1)$. Since $q^2+\varepsilon q+1$ is divisible by 3 and not by 9 and since q^3, $\gcd(4, q-\varepsilon)$, $q+\varepsilon$, and q^2+1 are not divisible by 3, $k = 3 \cdot |q-\varepsilon|_3 + 1$ as stated in the lemma. If $n \geq 5$, then $\mathrm{PSL}_\varepsilon(n, q)$ has 3-rank at least 4. In particular, the order of the Sylow 3-subgroups is also at least 4 and the entry in Table 3.1 is correct.

Thus, suppose instead that $q \equiv -\varepsilon \mod 3$. If G is $\mathrm{PSL}_\varepsilon(4,q)$ or $\mathrm{PSL}_\varepsilon(5,q)$, then the 3-rank of G is 2 and the order formula for G yields that $k = 2 \cdot |q+\varepsilon|_3$ as stated in the lemma. If G is $\mathrm{PSL}_\varepsilon(6,q)$ or $\mathrm{PSL}_\varepsilon(7,q)$, then the 3-rank of G is 3 and the order formula for G yields that $k = 3 \cdot |q+\varepsilon|_3 + 1$ because $q^6 - 1 = (q^2-1)\cdot(q^4+q^2+1)$ and q^4+q^2+1 is divisible by 3 but not by 9. If $n \geq 8$, then the 3-rank of $\mathrm{PSL}_\varepsilon(n,q)$ is at least 4 and $|\mathrm{PSL}_\varepsilon(n,q)|$ is divisible by 3^5. As a consequence, if $G = \mathrm{PSL}_\varepsilon(n,q)$ the lemma holds in all cases.

Therefore suppose instead that $G = \mathrm{PSp}(2n,q)$. By Table 5.1.A in [62],

$$\mathrm{PSp}(2n,q) = \frac{q^{n^2}}{\gcd(2,q-1)} \cdot \prod_{i=1}^{n} (q^{2i}-1)\,.$$

Thus, the order of Sylow 3-subgroups of G is q^{n^2} if q is a 3-power. Since the 3-rank of $\mathrm{PSp}(4,3^f)$ is $3f$, the entries for $\mathrm{PSp}(4,3^f)$ are correct. Furthermore, the 3-rank of $\mathrm{PSp}(2n,3^f)$ is at least $6f$ if $n \geq 3$. Thus, we may suppose that q is not a 3-power. Then the 3-rank of G is n. Since $|\mathrm{PSp}(4,q)| = \frac{q^4}{\gcd(2,q-1)} \cdot (q^2-1)^2 \cdot (q^2+1)$ and the only factor of this product that is divisible by 3 is $(q^2-1)^2$, it follows that $k = 2 \cdot |q^2-1|_3$ if $G = \mathrm{PSp}(4,q)$. The order of $|\mathrm{PSp}(6,q)|$ is $|\mathrm{PSp}(4,q)| \cdot q^5 \cdot (q^2-1)\cdot(q^4+q^2+1)$, and hence $k = 3\cdot|q^2-1|_3+1$ if $G = \mathrm{PSp}(6,q)$. Similarly, $|\mathrm{PSp}(2n,q)|$ is divisible by 3^5 if $n \geq 4$, and hence the lemma holds for $\mathrm{PSp}(2n,q)$ in all cases.

Instead suppose that $G = \mathrm{P\Omega}(2n+1,q)$ and $n \geq 3$. Then the 3-rank of $\mathrm{P\Omega}(2n+1,3^f)$ is at least $5f$, and if q is not a 3-power, then the 3-rank of $\mathrm{P\Omega}(2n+1,q)$ is n. Therefore it only remains to analyse the order of Sylow 3-subgroups of G, but by Bemerkung II 10.16 c) in [54], it holds that $|\mathrm{PSp}(2n,q)| = |\mathrm{P\Omega}(2n+1,q)|$, thus we can copy the entries of $\mathrm{PSp}(2n,q)$.

The next group we consider is $\mathrm{P\Omega}_+(2n,q)$ with $n \geq 4$. The 3-rank of $\mathrm{P\Omega}_+(2n,3^f)$ is at least $6f \geq 6$ and if q is not a 3-power, the 3-rank of $\mathrm{P\Omega}_+(2n,q)$ is at least 4. In particular, the order of a Sylow 3-subgroup of $\mathrm{P\Omega}_+(2n,q)$ is at least 3^4.

The last remaining classical group of Lie type is $\mathrm{P\Omega}_-(2n,q)$ with $n \geq 4$. As before, we use Theorem 3.3.3 in [43] to see that the 3-rank of $\mathrm{P\Omega}_-(2n,q)$ is at least $6f$. If q is not a 3-power, then, as in the other cases, (10-2) in [41] implies that the 3-rank of $\mathrm{P\Omega}_+(8,q)$ is 3 and that for $n \geq 5$ the 3-rank of $\mathrm{P\Omega}_-(2n,q)$ is at least 4. In particular, if G is $\mathrm{P\Omega}_-(8,3^f)$ or $\mathrm{P\Omega}_-(2n,q)$ with $n \geq 5$, then the order of Sylow 3-subgroups of G is at least 3^4, and hence the lemma holds in this cases. Thus, we may suppose that $G = \mathrm{P\Omega}_-(8,q)$ and q is not a 3-power. Then by Table 5.1.A in [62], $|G| = \frac{q^6}{\gcd(4,q^4+1)}(q^4+1)\cdot(q^2-1)\cdot(q^4-1)\cdot(q^6-1) = (q^2-1)^3\cdot(q^4+q^2+1)\cdot\frac{q^6}{\gcd(4,q^4+1)}\cdot(q^2+1)\cdot(q^4+1)$. Therefore $k = 3\cdot|q^2-1|_3+1$. Since this was the last remaining simple classical group of Lie type, we can turn our attention to the simple exceptional groups of Lie type over a field of order q.

The order of $\mathrm{Sz}(2^{2n+1})$ is $(2^{4n+2}+1)q^{4n+2}(2^{2n+1}-1)$ by the information on page 117 in [105]. In particular, $|\mathrm{Sz}(2^{2n+1})|$ is not divisible by 3.

Once more we use Theorem 3.3.3 in [43] and see that the 3-rank of ${}^3\mathrm{D}_4(3^f)$, $\mathrm{G}_2(3^f)$, ${}^2\mathrm{G}_2(3^{2n+1})$, $\mathrm{F}_4(3^f)$, $\mathrm{E}_6(3^f)$, ${}^2\mathrm{E}_6(3^f)$, $\mathrm{E}_7(3^f)$, and $\mathrm{E}_8(3^f)$ is $5f$, $4f$, $2(2n+1)$, $9f$, $16f$, $12f$, $27f$, and $36f$, respectively. In particular, their Sylow 3-subgroups have order at least 3^4. By Table 5.1.B in [62], the highest 3-power dividing $|\mathrm{G}_2(3^f)|$ is 3^{6f} and the highest 3-power dividing $|{}^3\mathrm{D}_4(3^f)|$ is 3^{12f}. If $G = {}^2\mathrm{G}_2(3^{2n+1})$, then by Table 5.1.B in [62], the order of a Sylow 3-subgroup is $3^{3(2n+1)}$. In particular, the entries in Table 3.1 that correspond to exceptional groups of Lie type over a field of characteristic 3 are correct. Therefore, we may suppose that q is not a 3-power.

If $G = {}^3\mathrm{D}_4(q)$, then as usual we can use (10-2) in [41] to see that the 3-rank of G is 2. Since by Table 5.1.B in [62], $|G| = q^{12}\cdot(q^8+q^4+1)\cdot(q^4+q^2+1)\cdot(q^2-1)^2$, and since (q^8+q^4+1), (q^4+q^2+1), and (q^2-1) are divisible by 3, the order of a Sylow 3-subgroup of G is at least 3^4.

If $G = \mathrm{G}_2(q)$, then Table 5.1.B in [62] shows that $|G| = q^6(q^2-1)^2(q^4+q^2+1)$. Since q^4+q^2+1 is divisible by 3 but not by 9, $k = 2\cdot|q^2-1|_3+1$. The fact that the 3-rank of G is 2 finishes the proof that the entries in Table 3.1 for $\mathrm{G}_2(q)$ are correct.

Since ${}^2\mathrm{F}_4(2)'$ has index 2 in ${}^2\mathrm{F}_4(2)$ by page 167 in [105], the Sylow 3-subgroup of ${}^2\mathrm{F}_4(2)'$ and ${}^2\mathrm{F}_4(2)$ are isomorphic. The 3-rank of ${}^2\mathrm{F}_4(2^{2n+1})$ is 2. Let $q = 2^{2n+1}$ and $G = {}^2\mathrm{F}_4(q)$. Then $q \equiv -1 \mod 3$ and by Table 5.1.B in [62], $|G| = q^{12}\cdot(q^6+1)(q^4-1)(q^3+1)(q-1)$. The only factors of this product that are divisible by 3 are $q^4-1 = (q^2+1)(q+1)(q-1)$ and $q^3+1 = (q+1)(q^2-q+1)$. In particular, $k = 2\cdot|q+1|_3+1$.

For all remaining simple groups of exceptional Lie type over a field of order $q = p^f$ with $p \neq 3$, their 3-rank is at least 4, and hence the order of their Sylow 3-subgroups is at least 3^4. As a consequence, the lemma holds for all simple groups of Lie type.

Therefore finally suppose that G is a sporadic simple group. By Table 5.3a in [43] the Sylow 3-subgroups of M_{11} are elementary abelian of order 9. The information in Table 5.3b, Table 5.3e, Table 5.3g, and Table 5.3i in [43] imply that the Sylow 3-subgroups of M_{12}, M_{24}, J_2 and J_4 have 3-rank 2 and order 3^3. The Sylow 3-subgroups of M_{22} and M_{23} are elementary abelian of order 9 by Table 5.3c and Table 5.3d in [43].

The order of J_1 is divisible by 3 but not by 9 (see Table 5.3f in [43]), and therefore the entries for J_1 in Table 3.1 follow. Table 5.3h in [43] implies that J_3 has Sylow 3-subgroups of order 3^5 and contains a subgroup isomorphic to $C_3 \times \mathcal{A}_6$. Thus the 3-rank of J_3 is at least 3.

Since HS has a subgroup isomorphic to M_{22} and the Sylow 3-subgroups of HS have order 9 (see Table 5.3m in [43]), the Sylow 3-subgroups of HS are elementary abelian of order 9. The information in Table 5.3j and Table 5.3k in [43] indicate that Co_3 and Co_2 both have Sylow 3-subgroups of order divisible by 3^6 and contain a subgroup isomorphic to HS. Therefore their Sylow 3-subgroups have 3-rank at least 2. Since by Table 5.3l in [43] the group Co_1 contains a subgroup isomorphic Co_2, the entries in Table 3.1 are correct for the three Conway groups.

For McL, Table 5.3n in [43] shows that the 3-rank of McL is at least 4 and that the Sylow 3-subgroups of McL have order 3^6.

The groups Suz, Ly, and Fi_{22} all contain an elementary abelian group of order 3^5 (see Table 5.3o, Table 5.3q, and Table 5.3t in [43]). Similarly, by Table 5.3u, Table 5.3v, and Table 5.3x in [43], Fi_{23}, Fi_{24}', and Th contains an elementary abelian subgroup of order 3^6, 3^7, and 3^5, respectively.

The information in Table 5.3p and Table 5.3r in [43] indicate that He and Ru both have Sylow 3-subgroup of order 3^3 and 3-rank 2. Since by Table 5.3s, O'N has an elementary abelian subgroup of order 3^4, the lemma holds in this case.

By Table 5.3w in [43] the group HN contains a subgroup isomorphic $\mathcal{A}_{12}$, and hence has 3-rank at least 4. Since 3^6 divides $|\mathrm{HN}|$, the entries in Table 3.1 for HN are correct.

Since $\mathbb{B}$ has a subgroup isomorphic to Fi_{23} (see Table 5.3y in [43]), it also contains an elementary abelian 3-subgroup of order 3^6. The information in Table 5.3z in [43] indicate that $\mathbb{M}$ has a subgroup isomorphic to Th, and hence the Sylow 3-subgroups of $\mathbb{M}$ also contain an elementary abelian 3-subgroup of order 3^5. Since this was the last remaining group that had to be analysed, the entries in Table 3.1 are correct in all cases.

3.2.1 Strongly 3-embedded Subgroups

We start with the analysis of case (4) (a) of Lemma 3.1. In this case the simple group G contains a strongly 3-embedded subgroup. By Lemma 2.8, this case contains amongst others all non-abelian finite simple groups with cyclic non-trivial Sylow 3-subgroups. Therefore we first determine which finite simple groups have cyclic Sylow 3-subgroups. This is a direct consequence of Lemma 3.7.

Lemma 3.8
Let G be a finite simple group. Then G has cyclic Sylow 3-subgroups if and only if there exists a prime p and a positive integer f such that G is isomorphic to C_p, J_1, $\mathrm{PSL}(2,p^f)$ where $p^f \neq 2$ and $p \neq 3$, $\mathrm{PSL}_\varepsilon(3,p^f)$ where $p^f \equiv -\varepsilon \mod 3$, or $\mathrm{Sz}(2^{2f+1})$.

Proof:
We use Lemma 3.7 and see that for a prime $p \neq 3$ and a positive integer n, C_p and $\mathrm{Sz}(2^{2n+1})$ are the only finite simple groups with trivial Sylow 3-subgroups. Since by 5.3.8 in [65] the 3-rank of G is 1 if and only if the Sylow 3-subgroups of G are non-trivial and cyclic, the same lemma implies that G has non-trivial cyclic Sylow 3-subgroup if and only if there exists a prime p and an integer f such that G is isomorphic to C_3, $\mathcal{A}_5$, $\mathrm{PSL}(2,3)$, $\mathrm{PSL}(2,p^f)$ where $p \neq 3$, $\mathrm{PSL}_\varepsilon(3,p^f)$ where $p^f \equiv -\varepsilon \mod 3$, or J_1. Since $\mathcal{A}_5 \cong \mathrm{PSL}(2,4)$ and since neither $\mathrm{PSL}(2,3)$ nor $\mathrm{PSL}(2,2)$ is simple, the lemma follows.

As a next step, we will see a full list of all finite simple groups that contain a strongly 3-embedded subgroup.

Lemma 3.9
Let G be a finite simple group that contains a strongly 3-embedded subgroup. Then there exists a prime power q and a positive integer n such that G is isomorphic to J_1, $\mathrm{PSL}(3,4)$, M_{11}, $\mathrm{PSL}(2,q)$ where $q \notin \{2,3\}$, $\mathrm{PSL}(3,q)$ where $q \equiv 2 \mod 3$, $\mathrm{PSU}(3,q)$ where $q \not\equiv 2 \mod 3$, or ${}^2\mathrm{G}_2(3^{2n+1})$.

Proof:
Since G contains a strongly 3-embedded subgroup, G is non-abelian and $|G|$ is divisible by 3. In particular, G can be neither cyclic nor a Suzuki group (of Lie type). The result (24-1) in [41] implies that G has cyclic Sylow 3-subgroups or there exists a positive integer f such that G is isomorphic to $\mathrm{PSL}(2,3^f)$ with $f \geq 3$, $\mathrm{PSU}(3,3^f)$, ${}^2\mathrm{G}_2(3^{2f+1})$, $\mathrm{PSL}(3,4)$, $\mathcal{A}_6 \cong \mathrm{PSL}(2,9)$, or M_{11}. The latter groups are all listed in the assertion and Lemma 3.8 shows that all non-abelian simple groups with cyclic Sylow 3-subgroups of order at least 3 are also stated in the lemma.

We have studied $\mathrm{PSL}(2,q)$, $\mathrm{PSL}(3,q)$, and $\mathrm{PSU}(3,q)$ in the previous section and we have analysed all their fixity-4 actions. Therefore the next family of groups we have to deal with is ${}^2\mathrm{G}(3^{2n+1})$.

Lemma 3.10
Let n be a positive integer, let $q = 3^{2n+1}$, and let $G = {}^2\mathrm{G}_2(q)$. Suppose that G acts transitively on a set Ω. Then G acts with fixity 4 and such that the order of a point stabiliser is odd and divisible by 3 if and only if the point stabilisers are Frobenius groups of order $q^3 \cdot \frac{q-1}{2}$.

Proof:
For the first direction suppose that G acts with fixity 4 and such that the order of a point stabiliser is odd and divisible by 3. Let $\alpha \in \Omega$ and let $a \in G_\alpha$ be of order 3. Let $P \in \mathrm{Syl}_3(G)$ be such that $a \in P$ and $P_\alpha \in \mathrm{Syl}_3(G_\alpha)$. By the Theorem in [101] part (2), $\mathrm{Z}(P)$ is elementary abelian of order $q \geq 27$. By Lemma 2.13, $|\mathrm{Z}(P) : \mathrm{Z}(P) \cap G_\alpha| = |\mathrm{Z}(P) \cap \mathrm{N}_G(\langle a \rangle) : \mathrm{Z}(P) \cap \mathrm{N}_{G_\alpha}(\langle a \rangle)| \leq |\mathrm{N}_G(\langle a \rangle) : \mathrm{N}_{G_\alpha}(\langle a \rangle)| \leq 4$. Therefore G_α contains an element $z \in \mathrm{Z}(P)$ of order 3. Then $P \leq \mathrm{N}_G(\langle z \rangle)$, and hence $|P : P_\alpha| \leq |\mathrm{N}_G(\langle z \rangle) : \mathrm{N}_{G_\alpha}(\langle z \rangle)| \leq 4$. As a consequence, $|P : P_\alpha| \in \{1,3\}$.

Therefore P/P_α is abelian, and hence $P' \leq P_\alpha$. Since P' is a characteristic subgroup of P, $\mathrm{N}_G(P) \leq \mathrm{N}_G(P')$. By (3) of the Theorem in [101], $|\mathrm{N}_G(P)|$ has order $q^3(q-1)$. Since G is simple and P' is non-trivial, $\mathrm{N}_G(P')$ lies in a maximal subgroup of G, and hence Theorem 6.5.5 in [43] implies that $\mathrm{N}_G(P') = \mathrm{N}_G(P)$ is a Frobenius group of order $q^3(q-1)$. Since $q-1 \equiv 3^{2n+1}-1 \equiv (-1)^{2n+1}-1 \equiv 2 \mod 4$, the order of $\mathrm{N}_G(P')$ is divisible by 2 but not by 4. Then Lemma 2.13 proves that $|\mathrm{N}_G(P') : \mathrm{N}_{G_\alpha}(P')| \leq 4$, and since $|G_\alpha|$ is odd, it follows that $|\mathrm{N}_G(P') : \mathrm{N}_{G_\alpha}(P')| = 2$, and hence the order of $\mathrm{N}_{G_\alpha}(P')$ is $q^3 \cdot \frac{q-1}{2}$. To summarise, $G_\alpha \geq \mathrm{N}_{G_\alpha}(P')$ has odd order divisible by $q^3 \cdot \frac{q-1}{2}$ and lies in a maximal subgroup of G. Then Theorem 6.5.5 in [43] again implies that G_α is a subgroup of a Frobenius group of order $q^3(q-1)$ or more precisely a Frobenius group of order $q^3 \cdot \frac{q-1}{2}$. This finishes the first direction of the proof.

For the other direction suppose that G acts on Ω such that the point stabilisers are Frobenius groups of order $q^3 \cdot \frac{q-1}{2}$. Let U be one of the point stabilisers. Then the action of G on Ω is equivalent to the action of G on G/U and $|U|$ is odd and divisible by 3.

Let K denote the Frobenius kernel of U. Then $|K| = q^3$ and the Frobenius complements are cyclic of order $\frac{q-1}{2}$. Let $x \in U$ be non-trivial. Since all powers of x fix the same points as x, in order to identify the fixity with which G acts on G/U, it suffices to determine the number of fixed points of all elements of U that have prime order. Let $u \in U$ be of prime order r. If r divides q, then $u \in K$. Let $y \in G$ be such that $Uy \in G/U$ is fixed by u. Then $u^{y^{-1}} \in U$ is an r-element, and thus $u^{y^{-1}} \in K$. Hence, $u \in K \cap K^y$. Since K is a Sylow 3-subgroup of G, (3) of the Theorem in [101] implies that different conjugates of K have trivial intersection. Therefore $y \in \mathrm{N}_G(K)$. Thus, u fixes at most $\frac{|\mathrm{N}_G(K)|}{|U|} = \frac{q^3 \cdot q-1}{q^3 \cdot \frac{q-1}{2}} = 2$ points in G/U. If r does not divide q, then u lies in a Frobenius complement J of order $\frac{q-1}{2}$. Since then r is odd and divides q^2-1, [63] p. 62 implies that u lies in a Sylow r-subgroup that is a subgroup of a group M of type $C_2 \times \mathrm{PSL}(2,q)$. Let $L \leq M$ be isomorphic to $\mathrm{PSL}(2,q)$. By Satz II 8.3 in [54], the order of $\mathrm{N}_L(\langle u \rangle) = 2 \cdot \frac{q-1}{2}$. Therefore $|\,\mathrm{N}_M(\langle u \rangle)| = 2(q-1)$. More precisely, the information on pages 61 and 62 in [63] show that $\mathrm{N}_G(\langle u \rangle)$ is a subgroup of a maximal subgroup of type $C_2 \times \mathrm{PSL}(2,q)$ and hence $|\,\mathrm{N}_G(\langle u \rangle)| = |\,\mathrm{N}_M(\langle u \rangle)| = 2(q-1)$. Then by Lemma 2.11, it follows that u fixes exactly $\frac{|K| \cdot |\,\mathrm{N}_G(\langle u \rangle)|}{|U|} = \frac{q^3 \cdot 2(q-1)}{q^3 \cdot \frac{q-1}{2}} = 4$ points in Ω. Thus, G acts with fixity 4 on G/U.

We finish this subsection by answering the question of which finite simple groups that contain a strongly 3-embedded subgroup can act with fixity 4 but we leave out a result for $\mathrm{PSL}(2,q)$ because Lemma 3.2 summarises the situation for these groups sufficiently.

Lemma 3.11
Let G be a finite simple group that contains a strongly 3-embedded subgroup. Suppose that for all prime power q, G is not isomorphic to $\mathrm{PSL}(2,q)$ and suppose that G acts transitively on a set Ω. Then G acts with fixity 4 and such that the order of a point stabiliser is odd and divisible by 3 if and only if G is isomorphic to J_1 and the point stabilisers are cyclic of order 15 or there exists a positive integer n such that G is isomorphic to ${}^2\mathrm{G}_2(3^{2n+1})$ and the point stabilisers are Frobenius groups of order $3^{3(2n+1)} \cdot \frac{3^{2n+1}-1}{2}$.

Proof:
The group G fulfils the hypothesis of Lemma 3.9, and hence G is one of the groups listed there. For some groups of this list we can use the GAP package TomLib [74] together with the program in Remark 2.22 to determine whether the

group can act transitively and with fixity 4 on some set. Among those groups are J_1 and M_{11}. The answer to the command `TestTom(TableOfMarks("J1"),4);` shows that if $G = J_1$, then G acts with fixity 4 if and only if the point stabilisers are cyclic of order 15. If $G = M_{11}$, then the answer to the GAP command `TestTom(TableOfMarks("M11"),4);` implies that G cannot act transitively, with fixity 4, and such that the order of a point stabiliser is odd and divisible by 3 on any set.

Lemma 3.4 proves that for all prime power $q \geq 3$ neither $\mathrm{PSL}(3,q)$ nor $\mathrm{PSU}(3,q)$ can act transitively, with fixity 4, and such that the order of a point stabiliser is odd and divisible by 3 on any set. Since $\mathrm{PSL}(3,2) \cong \mathrm{PSL}(2,7)$, this covers all groups of Lemma 3.9 except for ${}^2\mathrm{G}_2(3^{2n+1})$ where n is a positive integer. For this last remaining family of groups, Lemma 3.10 yields the correctness of the statement of this lemma.

3.2.2 Elementary Abelian Sylow 3-Subgroups

Similarly to the previous subsection, we will also for case (4) (b) of Lemma 3.1 first determine the finite simple groups that fulfil the conditions of this case before we turn towards possible fixity-4 actions. The next result is Proposition (1.2) in [64] but since there the proof is not elucidated, it is given in the following.

Lemma 3.12
Let G be a finite simple group. Then G has elementary abelian Sylow 3-subgroups of order 9 if and only if there exists a prime power q and $\varepsilon \in \{-1, 1\}$ such that G is isomorphic to

1. $\mathcal{A}_6$, $\mathcal{A}_7$, M_{11}, M_{22}, M_{23}, HS,
2. $\mathrm{PSL}_\varepsilon(3,q)$ where $|q-\varepsilon|_3 = 1$ (and $\mathrm{PSU}(3,2)$ is not simple),
3. $\mathrm{PSL}_\varepsilon(4,q)$ where $|q+\varepsilon|_3 = 1$,
4. $\mathrm{PSL}_\varepsilon(5,q)$ where $|q+\varepsilon|_3 = 1$, or
5. $\mathrm{PSp}(4,q)$ where $|q^2-1|_3 = 1$ (and $\mathrm{PSp}(4,2)$ is not simple).

Proof:
Lemma 3.7 yields that G has 3-rank 2 and Sylow 3-subgroups of order 3^2 if and only if there exist a prime power q and $\varepsilon \in \{-1, 1\}$ such that G is isomorphic to $\mathcal{A}_6$, $\mathcal{A}_7$, $\mathcal{A}_8$, $\mathrm{PSL}(2,9)$, $\mathrm{PSL}_\varepsilon(3,q)$ where $q \equiv \varepsilon \mod 3$ and $2 \cdot |q-\varepsilon|_3 = 2$, $\mathrm{PSL}_\varepsilon(4,q)$ where $q \equiv -\varepsilon \mod 3$ and $2 \cdot |q+\varepsilon|_3 = 2$, $\mathrm{PSL}_\varepsilon(5,q)$ where $q \equiv -\varepsilon \mod 3$ and $2 \cdot |q+\varepsilon|_3 = 2$, $\mathrm{PSp}(4,q)$ where q is not a 3-power and $2 \cdot |q^2-1|_3 = 2$, M_{11}, M_{22}, M_{23}, or HS. The fact that $|q-\varepsilon|_3 = 1$ is equivalent to the facts that $2 \cdot |q-\varepsilon|_3 = 2$ and that $q \equiv \varepsilon \mod 3$. A similar conclusion holds for the fact that $|q+\varepsilon|_3 = 1$. Analogously $|q^2-1|_3 = 1$ if and only if $2 \cdot |q^2-1|_3 = 2$ and q is not a 3-power. Since $\mathcal{A}_6 \cong \mathrm{PSL}(2,9)$ and $\mathcal{A}_8 \cong \mathrm{PSL}(4,2)$, the lemma follows.

In the next lemma, we will see that the existence of a subgroup of G isomorphic to $\mathrm{SL}(2,q)\times\mathrm{SL}(2,q)$ under additional conditions implies that G cannot act transitively, with fixity 4, and such that the order of a point stabiliser is odd and divisible by 3 on any set.

Lemma 3.13
Let q be a prime power such that $|q^2-1|_3 = 1$ and let $\hat{G}$ be a finite group that has a subgroup U isomorphic to $\mathrm{SL}(2,q)\times\mathrm{SL}(2,q)$. Let $Z = \mathrm{Z}(\hat{G})$, $d = |Z|$, and $G = \hat{G}/Z$. Suppose that UZ/Z contains a Sylow 3-subgroup P of G and that d is coprime to 3. Further suppose that if q is even, then d is odd and if q is odd, then d is not divisible by 8. If $q \geq 3$, then there does not exists a set Ω such that G acts transitively, with fixity 4 on Ω, and such that the order of the point stabilisers is odd and divisible by 3.

Proof:

Let U_1 and U_2 both be isomorphic to $\mathrm{SL}(2,q)$ and such that $U = U_1 \times U_2$. Since the order of $\mathrm{SL}(2,q)$ is $q(q^2-1)$ and since $|q^2-1|_3 = 1$, it follows that U has elementary abelian Sylow 3-subgroups of order 9. As a consequence, P is elementary abelian of order 9 because d is coprime to 3. Let $\varepsilon \in \{-1,1\}$ be such that 3 divides $q+\varepsilon$.

Assume for a contradiction that G acts transitively, with fixity 4 on a set Ω, and such that the order of the point stabilisers is odd and divisible by 3.

Then there exists $\alpha \in \Omega$ and a non-trivial element $b \in P$ such that $b \in P_\alpha \in \mathrm{Syl}_3(G_\alpha)$. Since $|Z|$ is coprime to 3, there exists an element $a \in U$ of order 3 such that $Za = b$. Let $a_1 \in U_1$ and $a_2 \in U_2$ be such that $a = a_1a_2$. Since a has order 3, it follows that a_1 and a_2 each have order dividing 3, and hence for $i \in \{1,2\}$, the order of $\mathrm{C}_{U_i}(a_i)$ is divisible by $q+\varepsilon$ by Satz II. 8.3 and Satz II. 8.4 in [54] and Lemma 2.3 (a). Then $\mathrm{C}_{U_i}(a_i) \leq \mathrm{C}_{U_i}(a)$, and thus $\mathrm{C}_U(a) \geq \mathrm{C}_{U_1}(a_1) \times \mathrm{C}_{U_2}(a_2)$. For all $i \in \{1,2\}$, let $t_i \in U_i$ be such that $t_i^2 \in \mathrm{Z}(U_i)$ and that $a_i^{t_i} = a_i^{-1}$. Let $t = t_1t_2$. Then t is an involution and $a^t = a_1^t a_2^t = a_1^{t_1} a_2^{t_2} = a_1^{-1}a_2^{-1} = a^{-1}$. In particular, $t \in \mathrm{N}_U(\langle a\rangle) \setminus \mathrm{C}_U(a)$. Therefore $|\,\mathrm{N}_U(\langle a\rangle)|$ is divisible by $2(q+\varepsilon)^2$. As a consequence, $|\,\mathrm{N}_G(\langle b\rangle)|$ is divisible by $2\frac{(q+\varepsilon)^2}{|Z\cap U|}$. If q is even, then d is odd and coprime to 3, and hence $|\,\mathrm{N}_G(\langle b\rangle)|$ is divisible by $2\cdot 3^2$. If q is odd, then $q+\varepsilon$ is even, d is not divisible by 8, and d is coprime to 3. Therefore $|\,\mathrm{N}_G(\langle b\rangle)|$ is again divisible by $2\cdot 3^2$. On the other hand, Lemma 2.13 shows that $|\,\mathrm{N}_G(\langle b\rangle) : \mathrm{N}_{G_\alpha}(\langle b\rangle)| \leq 4$. Since G_α has odd order, it follows that $|G_\alpha|$ is divisible by 9 because otherwise $|\,\mathrm{N}_G(\langle b\rangle) : \mathrm{N}_{G_\alpha}(\langle b\rangle)|$ would be divisible by 2 and 3. As a consequence, $P_\alpha = P$.

Let $R \in \mathrm{Syl}_3(U)$ be such that $RZ/Z = P$. Since $R \cap U_1 \in \mathrm{Syl}_3(U_1)$, there exists a non-trivial element $u \in R\cap U_1$. Then $U_2 \leq \mathrm{C}_U(u)$ and once more by Satz II. 8.3 and Satz II. 8.4 in [54] and Lemma 2.3 (a), $|\,\mathrm{N}_{U_1}(\langle u\rangle)| = 2(q+\varepsilon)$. Thus, the order of $\mathrm{N}_U(\langle u\rangle) \geq \mathrm{N}_{U_1}(\langle u\rangle) \times U_2$ is divisible by $2(q+\varepsilon)q(q^2-1)$. Therefore, $|\,\mathrm{N}_G(\langle Zu\rangle)|$ is divisible by $\frac{2(q-\varepsilon)q(q^2-1)}{|Z\cap U|}$. If q is odd, this number is divisible by $\frac{2\cdot 2\cdot 8}{4} = 8$ and if q is even and greater than 2, this number is also divisible by $2{\cdot}4 = 8$. This contradicts the facts that $|\,\mathrm{N}_G(\langle Zu\rangle) : \mathrm{N}_{G_\alpha}(\langle Zu\rangle)| \leq 4$ and that G_α has odd order. Therefore $q = 2$, and the lemma follows.

Our next objective is to determine for each group of Lemma 3.12 whether or not this group can act with fixity 4 and such that point stabilisers have odd order divisible by 3. In most cases the previous lemma will be useful.

Lemma 3.14
Let G be a finite simple group with elementary abelian Sylow 3-subgroups of order 9. Suppose that G acts transitively on a set Ω. Then G acts with fixity 4 on Ω and such that the order of the point stabilisers is odd and divisible by 3 if and only if G is isomorphic to $\mathcal{A}_6$ and the point stabilisers are elementary abelian of order 9.

Proof:
We go trough the list of finite simple groups in Lemma 3.12. Since $\mathcal{A}_6 \cong \mathrm{PSL}(2,9)$, Lemma 3.2 proves that the only transitive fixity-4 action of $\mathcal{A}_6$ where the order of the point stabilisers is odd and divisible by 3 is when the point stabilisers are elementary abelian of order 9.

For some of the other groups, we can use the GAP function described in Remark 2.22 together with the GAP package TomLib [74]. Among them are $\mathcal{A}_7$ and M_{11}. The result of `List(["a7","m11"],x->TestTom(TableOfMarks(x),4));` shows that neither $\mathcal{A}_7$ nor M_{11} can act transitively, with fixity 4 and such that the order of the point stabilisers is odd and divisible by 3 on any set. Similarly, the GAP command `TestTom(TableOfMarks("m22"),4);` yields that M_{22} does not act transitively, with fixity 4 and such that point stabilisers have order divisible by 3 on any set. For M_{23} and HS the answer to the GAP command `List(["m23","HS"],x->TestTom(TableOfMarks(x),4));` implies that none of these two groups can act transitively and with fixity 4 on any set. Similarly, we can use `List(["L4(2)","L5(2)"],x->TestTom(TableOfMarks(x),4));` to see that neither $\mathrm{PSL}(4,2)$ nor $\mathrm{PSL}(5,2)$ can act transitively and with fixity 4 on any set.

If G is isomorphic to $\mathrm{PSL}_\varepsilon(3,q)$ where $|q-\varepsilon|_3 = 1$, then q is not a 3-power and G is not isomorphic to $\mathrm{PSL}(3,2)$. Since $\mathrm{PSU}(3,2)$ is not simple, we can suppose that $q \geq 4$. Therefore Lemma 3.4 shows that G does not act transitively and with fixity 4 on any set.

Suppose that G is isomorphic to one of the remaining groups $\mathrm{PSL}_\varepsilon(4,q)$ where $|q+\varepsilon|_3 = 1$, $\mathrm{PSL}_\varepsilon(5,q)$ where $|q+\varepsilon|_3 = 1$, or $\mathrm{PSp}(4,q)$ where $|q^2-1|_3 = 1$. Then in all cases $|q^2-1|_3 = 1$. Let $\hat{G}$ be the corresponding matrix group ($\mathrm{SL}_\varepsilon(4,q)$ for $\mathrm{PSL}_\varepsilon(4,q)$, $\mathrm{SL}_\varepsilon(5,q)$ for $\mathrm{PSL}_\varepsilon(5,q)$, and $\mathrm{Sp}(4,q)$ for $\mathrm{PSp}(4,q)$). In particular, $G = \hat{G}/\,\mathrm{Z}(\hat{G})$. Then $\hat{G}$ has a subgroup U that is isomorphic to $\mathrm{SL}(2,q) \times \mathrm{SL}(2,q)$ because $\mathrm{SL}(2,q) \cong \mathrm{SU}(2,q) \cong \mathrm{Sp}(2,q)$. Let $Z = \mathrm{Z}(\hat{G})$ and $d = |Z|$. Then $d = \gcd(4, q-\varepsilon)$ for $\mathrm{PSL}_\varepsilon(4,q)$, $d = \gcd(5, q-\varepsilon)$ for $\mathrm{PSL}_\varepsilon(5,q)$, and $d = \gcd(2, q-1)$ for $\mathrm{PSp}(4,q)$. In all cases d is coprime to 3 and not divisible by 8. If q is even, then $q+1$ and $q-1$ are odd, and therefore d is odd. Since $|U| = q^2(q^2-1)^2$, since the order of Z is coprime to 3, and since

$|q^2-1|_3 = 1$, it follows that U has Sylow 3-subgroups of order 9, and hence UZ/Z contains a Sylow 3-subgroup of $G = \hat{G}/Z$. Therefore Lemma 3.13 proves that if $q \geq 3$, then none of these groups can act transitively, with fixity 4, and such that the order of the point stabilisers is odd and divisible by 3 on any set. As a consequence, the only case that remains to be analysed is $q = 2$.

Since $\mathrm{PSp}(4,2)$ is not simple and for $\mathrm{PSU}(4,2)$ and $\mathrm{PSU}(5,2)$, $|q-1|_3 = 0$, the only remaining groups are $\mathrm{PSL}(4,2)$ and $\mathrm{PSL}(5,2)$ but these groups have already been excluded. Hence, the lemma follows because all groups in Lemma 3.12 have been considered.

3.2.3 Extra-Special Sylow 3-Subgroups

The objective of this subsection is to show that none of the simple groups that fulfil case (4) (c) of Lemma 3.1 can act transitively, with fixity 4, and such that the order of a point stabiliser is odd and divisible by 3 on any set. We proceed as in the previous subsection.

Lemma 3.15
Let G be a finite simple group. Suppose that G has extra-special Sylow 3-subgroups of order 27. Then there exists a prime power q and a positive integer n such that G is isomorphic to

1. $\mathrm{PSL}(3,3)$, $\mathrm{PSU}(3,3)$, $\mathrm{M}_{12}, \mathrm{M}_{24}$, J_2, J_4, He, Ru,
2. $\mathrm{G}_2(q)$ where $|q^2-1|_3 = 1$,
3. ${}^2\mathrm{F}_4(2^{2n+1})$ where $|2^{2n+1}+1|_3 = 1$, or
4. ${}^2\mathrm{F}_4(2)'$.

Proof:
Let P be a Sylow 3-subgroup of G. Since P is extra-special of order 3^3, it follows that P is non-abelian. In particular, P is neither cyclic nor elementary abelian. Therefore the 3-rank of P (and thus of G) is 2. Then Lemma 3.7 yields that there exist a prime power q and $\varepsilon \in \{-1, 1\}$ such that G is isomorphic to $\mathrm{PSL}_\varepsilon(3,3)$, $\mathrm{G}_2(q)$ where q is not a 3-power and $2 \cdot |q^2-1|_3 + 1 = 3$, ${}^2\mathrm{F}_4(2^{2n+1})$ where $2 \cdot |2^{2n+1}+1|_3 + 1 = 3$, ${}^2\mathrm{F}_4(2)'$, M_{12}, M_{24}, J_2, J_4, He, or Ru. Since $2 \cdot |q^2-1|_3 + 1 = 3$ implies that $|q^2-1|_3 = 1$, the lemma follows.

We have a closer look at the family $\mathrm{G}_2(q)$.

Lemma 3.16
Let q be a prime power, such that $|q^2-1|_3 = 1$ and let $G = \mathrm{G}_2(q)$. Then G cannot act transitively, with fixity 4, and such that the order of the point stabilisers is odd and divisible by 3 on any set.

Proof:

Assume for a contradiction that there exists a set Ω such that G acts transitively and with fixity 4 on Ω and such that the order of the point stabilisers is odd and divisible by 3. Let $\varepsilon \in \{-1, 1\}$ be such that $q \equiv \varepsilon \mod 3$.

By the information on pages 125 and 126 and by Table 4.1 in [105], G has a subgroup M_ε of type $\mathrm{SL}_\varepsilon(3, q) : C_2$. Let N_ε be a subgroup of M_ε of index 2 and isomorphic to $\mathrm{SL}_\varepsilon(3, q)$. Since the order of N_ε is $q^3(q^3 - \varepsilon)(q^2 - 1) = q^3 \cdot (q - \varepsilon)^2 \cdot (q^2 + \varepsilon q + 1) \cdot (q + \varepsilon)$, it follows that N_ε contains a Sylow 3-subgroup P of G. Let $\alpha \in \Omega$ be such that $P_\alpha \in \mathrm{Syl}_3(G_\alpha)$. Then $|P_\alpha| \geq 3$.

By Lemma 3.6, $\mathrm{Z}(N_\varepsilon) = \mathrm{Z}(P)$. In particular $\mathrm{Z}(P)$ is a characteristic subgroup of $N_\varepsilon \trianglelefteq M_\varepsilon$.

Assume for a contradiction that there exists a non-trivial $x \in \mathrm{Z}(P) \cap G_\alpha$. Then $\langle x \rangle = \mathrm{Z}(P)$. Thus, $\mathrm{N}_G(\langle x \rangle)$ contains M_ε and therefore a subgroup of order $q^3 \cdot (q - \varepsilon)^2 \cdot (q^2 + \varepsilon q + 1) \cdot (q + \varepsilon) \cdot 2$. Independent of the parity of q this number is divisible by 2^4. By Lemma 2.13, it follows that $|\,\mathrm{N}_G(\langle x \rangle) : \mathrm{N}_{G_\alpha}(\langle x \rangle)| \leq 4$, contradicting the fact that $|G_\alpha|$ is odd. As a consequence, $\mathrm{Z}(P) \cap G_\alpha = 1$.

In particular, $P_\alpha = P \cap G_\alpha$ is a non-normal subgroup of P. Since $|P| = 27$, it follows that $|P_\alpha| = 3$. Hence, all point stabilisers have order divisible by 3 but not by 9.

Let $y \in P_\alpha \setminus \mathrm{Z}(P)$. Then $\langle y \rangle \times \mathrm{Z}(P) \leq \mathrm{N}_{N_\varepsilon}(\langle y \rangle)$. Thus, Lemma 3.6 implies that the order of $\mathrm{N}_{N_\varepsilon}(\langle y \rangle)$ is divisible by $2 \cdot 9$. Since $|G_\alpha|$ is odd and not divisible by 9, the index $|\,\mathrm{N}_G(\langle y \rangle) : \mathrm{N}_{G_\alpha}(\langle y \rangle)|$ is divisible by 6 contradicting Lemma 2.13. This final contradiction proves the Lemma.

Lemma 3.17
Let G be a finite simple group. Suppose that G has extra-special Sylow 3-subgroups of order 27. Then G does not act transitively, with fixity 4, and such that the order of the point stabilisers is odd and divisible by 3 on any set.

Proof:

By Lemma 3.15, G is one of the groups mentioned there.

Lemma 3.4 proves that neither $\mathrm{PSL}(3, 3)$ nor $\mathrm{PSU}(3, 3)$ can act transitively, with fixity 4, and such that point stabilisers have odd order on any set. For M_{12}, we can use the GAP program in Remark 2.22 and the GAP package TomLib [74]. Then the GAP command `TestTom(TableOfMarks("m12"),4);` shows that M_{12} cannot act transitively, with fixity 4, and such that point stabilisers have odd order on any set. Similarly, the answer to the GAP command `List(["m24","J2","He","2F4(2)'"], x->TestTom(TableOfMarks(x),4));` implies that neither M_{24}, J_2, He, nor ${}^2\mathrm{F}_4(2)'$ can act transitively and with fixity 4 on any set.

Assume for a contradiction that G acts transitively, with fixity 4, and such that the order of a point stabiliser is odd and divisible by 3 on a set Ω.

If G is isomorphic to J_4, then by [28] p. 188, G has just one conjugacy class of elements of order 3 and the centraliser of such an element has order 2661120, which is divisible by 8. Since by Lemma 2.13, $|\,\mathrm{C}_G(x) : \mathrm{C}_{G_\alpha}(x)| \leq 4$ and since the order of G_α is odd, this gives a contradiction. If G is isomorphic to Ru, then [28] p. 127 implies that the centraliser of every 3-element has order 2160, which is divisible by 8. Since $|\,\mathrm{C}_G(x) : \mathrm{C}_{G_\alpha}(x)| \leq 4$ and the order of G_α is odd, this again gives a contradiction.

Assume for a contradiction that G is isomorphic to ${}^2\mathrm{F}_4(2^{2n+1})$, where $n \geq 1$. Then there exists an element $x \in G_\alpha$ of order 3. Let $q = 2^{2n+1}$. By p. 54 in [73] all 3-elements in G are conjugate and the normaliser of $\langle x \rangle$ in G has order $|\,\mathrm{SU}(3,q)| \cdot 2 = q^3(q^2-1)(q^3+1) \cdot 2$, hence is divisible by 8. This is another contradiction to Lemma 2.13.

Therefore, by Lemma 3.15, G is isomorphic to $\mathrm{G}_2(q)$, where $|q^2-1|_3 = 1$, but this contradicts Lemma 3.16. As a consequence, the assumption that G can act transitively, with fixity 4, and such that the order of a point stabiliser is odd and divisible by 3 was false and the lemma holds.

3.2.4 Sylow 3-Subgroups that are a Wreath Product

We will again first classify the simple groups that have a Sylow 3-subgroup of type $C_3 \wr C_3$ before we will see that none of them can act transitively, with fixity 4, and such that point stabilisers have odd order divisible by 3.

Lemma 3.18
Let G be a finite simple group. Suppose that G has Sylow 3-subgroups of type $C_3 \wr C_3$. Then there exists a prime power q and a positive integer n such that G is isomorphic to

1. $\mathcal{A}_9$, $\mathcal{A}_{10}$, $\mathcal{A}_{11}$, $\mathrm{PSp}(4,3)$,
2. $\mathrm{PSL}_\varepsilon(4,q)$ where $|q-\varepsilon|_3 = 1$,
3. $\mathrm{PSL}_\varepsilon(6,q)$ where $|q+\varepsilon|_3 = 1$,
4. $\mathrm{PSL}_\varepsilon(7,q)$ where $|q+\varepsilon|_3 = 1$,
5. $\mathrm{PSp}(6,q)$ where $|q^2-1|_3 = 1$,
6. $\mathrm{P\Omega}(7,q)$ where q is odd and $|q^2-1|_3 = 1$, or
7. $\mathrm{P\Omega}_-(8,q)$ where $|q^2-1|_3 = 1$.

Proof:
Let P be a Sylow 3-subgroup of G. Since P is of type $C_3 \wr C_3$, it follows that $|P| = 3^4$ and that P is non-abelian and has an elementary abelian subgroup of order 3^3. Therefore the 3-rank of P (and thus of G) is 3. Then Lemma 3.7 yields that there exist a prime power q and $\varepsilon \in \{-1,1\}$ such that G is isomorphic to $\mathcal{A}_9$, $\mathcal{A}_{10}$, $\mathcal{A}_{11}$, $\mathrm{PSL}_\varepsilon(4,q)$ where $q \equiv \varepsilon \mod 3$ and $3 \cdot |q-\varepsilon|_3 + 1 = 4$, $\mathrm{PSL}_\varepsilon(6,q)$ where $q \equiv -\varepsilon \mod 3$ and $3 \cdot |q+\varepsilon|_3 + 1 = 4$, $\mathrm{PSL}_\varepsilon(7,q)$ where $q \equiv -\varepsilon \mod 3$ and $3 \cdot |q+\varepsilon|_3 + 1 = 4$, $\mathrm{PSp}(4,3)$, $\mathrm{PSp}(6,q)$ where q is not a 3-power and

$3 \cdot |q^2 - 1|_3 + 1 = 4$, $\mathrm{P\Omega}(7, q)$ where q is not a 3-power and $3 \cdot |q^2 - 1|_3 + 1 = 4$, or $\mathrm{P\Omega}_-(8, q)$ where q is not a 3-power and $3 \cdot |q^2 - 1|_3 + 1 = 4$. Since for every 2-power q, $\mathrm{PSp}(6, q) \cong \mathrm{P\Omega}(7, q)$, all of these groups are listed in the lemma.

For proving that none of the groups in the previous lemma can act transitively, with fixity 4, and such that point stabilisers have odd order divisible by 3, we will determine in a sequence of two lemmas sufficient conditions. Therefore the next lemma formulates such a condition on 3-elements of a simple group G with Sylow 3-subgroups of type. $C_3 \wr C_3$.

Lemma 3.19

Let G be a finite simple group with Sylow 3-subgroups of type $C_3 \wr C_3$, let $P \in \mathrm{Syl}_3(G)$, and let $Q \leq P$ be elementary abelian of order 3^3. Suppose that there exists an element $x \in P \setminus Q$ of order 3 such that $|\,\mathrm{N}_G(\langle x \rangle)|$ is even and further suppose that there exists a non-trivial element $y \in P$, such that $|\,\mathrm{N}_G(\langle y \rangle)|$ is divisible by 8. Then G does not act transitively, with fixity 4, and such that the order of a point stabiliser is odd and divisible by 3 on any set.

Proof:

Assume for a contradiction that there exists a set Ω such that G acts transitively, with fixity 4, and such that the order of a point stabiliser is odd and divisible by 3 on Ω. Let $\alpha \in \Omega$ be such that $P_\alpha \in \mathrm{Syl}_3(G_\alpha)$. The automorphism group of P is a $\{2, 3\}$-group. (This can been seen for example by the GAP command `AutomorphismGroup(SylowSubgroup(AlternatingGroup(9),3));`.) By Corollary 1.2 in [47], there exists an element $d \in \mathrm{N}_G(P) \setminus P\,\mathrm{C}_G(P)$, and hence d acts as a non-trivial automorphism on P. Therefore d acts as a $\{2, 3\}$-element, and since $d \notin P \in \mathrm{Syl}_3(P)$, the order of d is even. As a consequence, $\mathrm{N}_G(P)$ contains an involution t. The order of $\mathrm{Z}(P)$ is 3, and since $\mathrm{Z}(P)$ is a characteristic subgroup of P, $t \in \mathrm{N}_G(\mathrm{Z}(P))$.

If $P_\alpha = P$, then $y \in G_\alpha$. Thus, by Lemma 2.13, $|\,\mathrm{N}_G(\langle y \rangle) : \mathrm{N}_{G_\alpha}(\langle y \rangle)| \leq 4$, contradicting the assumptions that $|G_\alpha|$ is odd and that $|\,\mathrm{N}_G(\langle y \rangle)|$ is divisible by 8. Therefore $|P : P_\alpha| \geq 3$.

If $\mathrm{Z}(P)$ fixes a point $\omega \in \Omega$, then $\mathrm{Z}(P) \leq P_\omega$. Since $\mathrm{N}_G(\mathrm{Z}(P))$ has even order but $|G_\alpha| = |G_\omega|$ is odd, $|\,\mathrm{N}_G(\mathrm{Z}(P)) : \mathrm{N}_{G_\omega}(\mathrm{Z}(P))|$ is divisible by 2. Thus the fact that $|\,\mathrm{N}_G(\mathrm{Z}(P)) : \mathrm{N}_{G_\omega}(\mathrm{Z}(P))| \leq 4$ implies that $P \leq G_\omega$ contrary to the facts that $|P : P_\alpha| \geq 3$ and $P_\alpha \in \mathrm{Syl}_3(G_\alpha)$. Thus, $\mathrm{Z}(P)$ acts semi-regularly on Ω. In particular, $\mathrm{Z}(P) \cap P_\alpha = 1$, and hence P_α is not normal in P. As a consequence, $|P : P_\alpha| \geq 9$.

Since $|P| = 81$, Lemma 10 in [8] implies that only case (c) of this Lemma 10 can hold and thus $|P_\alpha| = 3$.

Assume for a contradiction that Q fixes a point $\omega \in \Omega$. Then $Q_\omega = P_\omega$, and hence $Q \leq \mathrm{N}_G(P_\omega)$ because Q is abelian. Therefore Lemma 2.13 implies that $|Q : Q_\omega| \leq |\,\mathrm{N}_G(P_\omega) : \mathrm{N}_{G_\omega}(P_\omega)| \leq 4$. Since $|Q| = 3^3$ and Q_ω has order 3, this is a contradiction. Therefore Q acts semi-regularly on Ω.

The following GAP code shows, that P contains only one conjugacy class of subgroups of order 3 that has trivial intersection with Q.

```
P:=SylowSubgroup(AlternatingGroup(9),3);;
Q:=Filtered(NormalSubgroups(P),x->IsElementaryAbelian(x) and
↪  Order(x)=27)[1];;
li:=List(ConjugacyClassesSubgroups(P),Representative);;
lf:=Filtered(li,x->Order(x)=3);;
List(lf,x->IsTrivial(Intersection(x,Q)));
```

Therefore the element $x \in P \setminus Q$ of order 3 is conjugate to an element in P_α. Thus, there exists $\beta \in \Omega$ such that $x \in G_\beta$. Since $\langle x \rangle \,\mathrm{Z}(P) \leq \mathrm{N}_G(\langle x \rangle)$ and $|\,\mathrm{N}_G(\langle x \rangle)|$ is even, $|\,\mathrm{N}_G(\langle x \rangle)|$ is divisible by $9 \cdot 2$. By Lemma 2.13, $|\,\mathrm{N}_G(\langle x \rangle) : \mathrm{N}_{G_\beta}(\langle x \rangle)| \leq 4$ but $|G_\beta|$ is odd and not divisible by 9. This final contradiction finishes the proof.

With the previous result, it remains to show that in every group that is mentioned in Lemma 3.18 the normalisers of certain 3-elements have order divisible by a suitable power of 2. In the next lemma, we will determine certain subgroups that guarantee the conditions on the 3-elements.

Lemma 3.20
Let q be a prime power and $\varepsilon \in \{-1, 1\}$ such that $|q - \varepsilon|_3 = 1$. Let C be cyclic of order at most 2, let N be isomorphic to $C.\,\mathrm{SL}_\varepsilon(3, q)$ and let $\hat{G}$ be a finite group that has a subgroup M such that $N \trianglelefteq M$. Let $Z \leq \mathrm{Z}(\hat{G})$, $d = |Z|$, and $G = \hat{G}/Z$. Suppose that MZ/Z contains a Sylow 3-subgroup of G of type $C_3 \wr C_3$ and that d is coprime to 3. Furthermore suppose that d is not divisible by $2^{|q-\varepsilon|_2+1}$. Then there does not exists a set Ω such that G acts transitively and with fixity 4 on Ω, and such that the order of the point stabilisers is odd and divisible by 3.

Proof:
Let $P \in \mathrm{Syl}_3(M)$. Since $d = |Z|$ is coprime to 3, $PZ/Z \cong P \cong C_3 \wr C_3$ and $PZ/Z \in \mathrm{Syl}_3(G)$. Let $Q \leq P$ be elementary abliean of order 3^3. Since $P \cap N \in \mathrm{Syl}_3(N)$, the order of $P \cap N$ is 27. By Lemma 3.6, all non-trivial elements in $(PC/C) \cap (N/C)$ have order 3. As a consequence, all non-trivial elements in $P \cap N$ have order 3. If all of them were contained in Q, then $P \cap N = Q$, and thus N and N/C would have elementary abelian Sylow 3-subgroups contrary to Lemma 3.6. Therefore let $x \in (P \cap N) \setminus Q$ be of order 3. Then $\langle x \rangle$ acts coprimely on C, and hence with the use of Lemma 2.3 (a), Lemma 3.6 implies that $|\,\mathrm{N}_N(\langle x \rangle)/C| = |\,\mathrm{N}_{N/C}(\langle Cx \rangle)|$ is divisible by $2^{|q-\varepsilon|_2+1}$. As a consequence, $\mathrm{N}_G(\langle Zx \rangle) \geq \mathrm{N}_{NZ/Z}(\langle Zx \rangle) \geq \mathrm{N}_N(\langle x \rangle)Z/Z$ has order divisible by 2 because d is not divisible by $2^{|q-\varepsilon|_2+1}$. Since PZ/Z is isomorphic to P, all elements of order 3 in PZ/Z that are not an element of the elementary abelian subgroup of order 27 of PZ/Z have a normaliser in G of even order.

Let $y \in \mathrm{Z}(N)$ be of order 3. Then $N = \mathrm{N}_N(\langle y \rangle)$, and hence $\mathrm{N}_G(\langle Zy \rangle) \geq \mathrm{N}_{NZ/Z}(\langle Zy \rangle) \geq \mathrm{N}_N(\langle y \rangle)Z/Z$ has order divisible by $\frac{|N|}{|N \cap Z|}$. Since $|N|$ is divisible by $8 \cdot 2^{|q-\varepsilon|_2}$, $|\,\mathrm{N}_G(\langle Zy\rangle)|$ is divisible by 8. As a consequence, PZ/Z contains an element of order 3 whose normaliser in G has order divisible by 8. Therefore Lemma 3.19 implies that G does not act transitively, with fixity 4, and such that the order of the point stabilisers is odd and divisible by 3 on any set.

We will see in the next lemma, that for most of the groups of Lemma 3.18, a situation as in the previous lemma can be found.

Lemma 3.21
Let G be a finite simple group. Suppose that G has Sylow 3-subgroups of type $C_3 \wr C_3$. Then G does not act transitively, with fixity 4, and such that the order of the point stabilisers is odd and divisible by 3 on any set.

Proof:

For $\mathcal{A}_9$, $\mathcal{A}_{10}$, and $\mathcal{A}_{11}$, we can use the GAP program in Remark 2.22 together with the GAP package TomLib [74]. Then the answer to the GAP command `List(["a9","a10","a11"],x->TestTom(TableOfMarks(x),4));` shows that none of the three groups can act transitively and with fixity 4 on any set. Similarly, the GAP command `TestTom(TableOfMarks("S4(3)"),4);` implies that $\mathrm{PSp}(4,3)$ cannot act transitively, with fixity 4, and such that point stabiliser have order divisible by 3 on any set.

Following Lemma 3.18, suppose that there exists a prime power q and $\varepsilon \in \{-1,1\}$ such that $|q-\varepsilon|_3 = 1$ and such that G is isomorphic to $\mathrm{PSL}_{\varepsilon(4,q)}$. Set $\hat{G} = \mathrm{SL}_\varepsilon(4,q)$ and $Z = \mathrm{Z}(\hat{G})$. Then $|Z| = \gcd(4, q-\varepsilon)$ and this number is not divisible by $2^{|q-\varepsilon|_2+1}$. By Table 8.8 and Table 8.10 in [15], $\hat{G}$ has a subgroup M isomorphic to $\mathrm{GL}_\varepsilon(3,q)$. Since $|\,\mathrm{GL}_\varepsilon(3,q)| = q^3(q-\varepsilon)^3(q^2+\varepsilon q+1)(q+\varepsilon)$ and since $|Z|$ is coprime to 3, MZ/Z contains a Sylow 3-subgroup of G. Then Lemma 3.20 shows that $G = \hat{G}/Z$ does not act transitively, with fixity 4, and such that the order of a point stabiliser is odd and divisible by 3 on any set.

Therefore instead suppose that there exists a prime power q and $\varepsilon \in \{-1,1\}$ such that $|q+\varepsilon|_3 = 1$ and such that G is isomorphic to $\mathrm{PSL}_\varepsilon(6,q)$. Let $\hat{G} = \mathrm{SL}_\varepsilon(6,q)$ and let $Z = \mathrm{Z}(\hat{G})$. Then $|\,\mathrm{Z}(\hat{G})| = \gcd(6, q-\varepsilon)$ and this number is divisible by neither 3 nor $2^{|q^2-1|_2+1}$. By Table 8.24 and Table 8.26 in [15], $\hat{G}$ has a subgroup M isomorphic to $\mathrm{SL}(3,q^2).C_{q+\varepsilon}$. Since $|M| = q^6(q+\varepsilon)^3(q^2-\varepsilon q+1)(q-\varepsilon)^2(q^2+1)(q^2+\varepsilon q+1)$ and since $|Z|$ is coprime to 3, it follows that MZ/Z contains a Sylow 3-subgroup of G. Then Lemma 3.20 shows that $G = \hat{G}/Z$ does not act transitively, with fixity 4, and such that the order of a point stabiliser is odd and divisible by 3 on any set.

Therefore instead suppose that there exists a prime power q and $\varepsilon \in \{-1,1\}$ such that $|q+\varepsilon|_3 = 1$ and such that G is isomorphic to $\mathrm{PSL}_\varepsilon(7,q)$. Set $\hat{G} = \mathrm{SL}_\varepsilon(7,q)$ and $Z = \mathrm{Z}(\hat{G})$. Then $|Z| = \gcd(7, q-\varepsilon)$ and this number is coprime to 6. Then $\hat{G}$ has a subgroup isomorphic to $\mathrm{SL}_\varepsilon(6,q)$, and hence as above $\hat{G}$ has a subgroup M isomorphic to $\mathrm{SL}(3,q^2).C_{q+\varepsilon}$ and MZ/Z contains a Sylow 3-subgroup of G. Then Lemma 3.20 again shows that $G = \hat{G}/Z$ does not act transitively, with fixity 4, and such that the order of a point stabiliser is odd and divisible by 3 on any set.

Therefore instead suppose that there exists a prime power q such that G is isomorphic to $\mathrm{PSp}(6,q)$ and such that $|q^2-1|_3 = 1$. Let $\hat{G} = \mathrm{Sp}(6,q)$ and let $Z = \mathrm{Z}(\hat{G})$. Then $|Z| = \gcd(2, q-1)$ and this number is divisible by neither $2^{|q^2-1|_2+1}$nor 3. Let $\varepsilon \in \{-1,1\}$ be such that $|q-\varepsilon|_3 = 1$. Additionally suppose that q is odd. Then by Table 8.28 in [15], $\hat{G}$ contains a subgroup isomorphic to $\mathrm{GL}_\varepsilon(3,q)$. Then Lemma 3.20 yields that $G = \hat{G}/\mathrm{Z}(\hat{G})$ does not act transitively, with fixity 4, and such that the order of a point stabiliser is odd and divisible by 3 on any set. Therefore we may suppose that q is even. Then by Table 8.28 in [15], $\hat{G} \cong G$ contains a subgroup isomorphic to $\mathrm{SO}_\varepsilon(6,q)$, and hence a subgroup isomorphic to $\Omega_\varepsilon(6,q)$. Since by Table 8.31 and Table 8.33 in [15], $\Omega_\varepsilon(6,q) \cong \mathrm{PSL}_\varepsilon(4,q) \cong \mathrm{SL}_\varepsilon(4,q)$, G contains a subgroup isomorphic to $\mathrm{GL}_\varepsilon(3,q)$ and Lemma 3.20 implies that G does not act transitively, with fixity 4, and such that the order of a point stabiliser is odd and divisible by 3 on any set.

Therefore instead suppose that there exists an odd prime power q such that $|q^2-1|_3 = 1$ and such that G is isomorphic to $\mathrm{P}\Omega(7,q)$. Then by page 80 in [105], $\mathrm{P}\Omega(7,q) \cong \Omega(7,q)$ and we set $\hat{G} = \Omega(7,q)$. Then $Z = \mathrm{Z}(\hat{G}) = 1$. Let $\varepsilon \in \{-1,1\}$ be such that $|q-\varepsilon|_3 = 1$. By Table 8.39 in [15], $\hat{G}$ contains a subgroup isomorphic to $\Omega_\varepsilon(6,q)$. Therefore, by Table 8.31 and Table 8.33 in [15], $\hat{G}$ contains a subgroup isomorphic to $C_2.\,\mathrm{GL}_\varepsilon(3,q)$. Then Lemma 3.20 implies that $G = \hat{G}/Z$ does not act transitively, with fixity 4, and such that the order of a point stabiliser is odd and divisible by 3 on any set.

Finally suppose that there exists a prime power q such that $|q^2-1|_3 = 1$ and such that G is isomorphic to $\mathrm{P}\Omega_-(8,q)$. Set $\hat{G} = \Omega_-(8,q)$ and $Z = \mathrm{Z}(\hat{G})$. Then $|\mathrm{Z}(\hat{G})| = 1$ by page 80 in [105]. Let $\varepsilon \in \{-1,1\}$ be such that $|q-\varepsilon|_3 = 1$. By Table 8.52 in [15], $\hat{G}$ contains a subgroup isomorphic to $\mathrm{P}\Omega(7,q)$. Since this group has a subgroup isomorphic $\Omega_\varepsilon(6,q)$ by Table 8.39 in [15] and this subgroup has a subgroup isomorphic to $C_{\gcd(2,q-1)}.\,\mathrm{GL}_\varepsilon(3,q)$ by Table 8.31 and Table 8.33 in [15], Lemma 3.20 implies that $G = \hat{G}/Z$ does not act transitively, with fixity 4, and such that the order of a point stabiliser is odd and divisible by 3 on any set.

By Lemma 3.18, the statement of this lemma follows.

3.3 The Case that Point Stabilisers have Order Coprime to 6

In the case of Lemma 3.1 that the point stabilisers have order coprime to 6 no further information about the group structure is given. Nevertheless, we can state some properties of subgroups of an arbitrary finite simple group G that acts transitively, with fixity 4, and such that the point stabilisers have order coprime to 6. In contrast to the previous sections, in which a point stabiliser was our main subgroup of interest, in this section often some non-trivial four-point stabiliser will be most useful for the analysis. Therefore, the next lemma contains statements related to the four-point stabiliser that will be needed in this section but are also of general interest. The following lemma strengthens the results of Lemma 2.15 for groups, not necessarily simple, that have point stabilisers of order coprime to 6. Afterwards we will use it in Lemma 3.23 to derive structural information about G.

Lemma 3.22
Let G be a finite group acting transitively, faithfully, and with fixity 4 on a set Ω, and such that the point stabilisers have order coprime to 6. Let H be the element-wise stabiliser of a set of size 4 and let $\alpha \in \mathrm{fix}_\Omega(H)$. Then the following hold:

(a) If $X \leq H$ is non-trivial, then $\mathrm{N}_{G_\alpha}(X) = \mathrm{N}_H(X)$.

(b) Either $G_\alpha = H$ or G_α is a Frobenius group.

(c) If $X \leq G_\alpha$ is non-trivial, then $|\,\mathrm{N}_G(X)|$ is divisible by neither 8, 9, nor 6.

(d) If $X \leq H$ is non-trivial, then $|\,\mathrm{N}_G(X) : \mathrm{N}_H(X)| \leq 4$. Furthermore, $|\,\mathrm{N}_G(X)|$ is divisible by neither 3 nor 8.

(e) If $p \in \pi(H)$, then $\mathrm{N}_G(H)$ is strongly p-embedded in G.

Proof:
Suppose that $X \leq H$ and that $1 \neq X$. Let $g \in \mathrm{N}_{G_\alpha}(H)$. Then $\mathrm{N}_{G_\alpha}(X)$ acts on $\mathrm{fix}_\Omega(X) \setminus \{\alpha\} = \mathrm{fix}_\Omega(H) \setminus \{\alpha\}$, a set of size 3. Since G_α has order coprime to 6, g fixes every element in $\mathrm{fix}_\Omega(H) \setminus \{\alpha\}$, and hence is an element of H. In particular, $\mathrm{N}_{G_\alpha}(X) \leq \mathrm{N}_H(X)$ and (a) follows.

For part (b) suppose that $G_\alpha \neq H$. Let $g \in G_\alpha$ be such that $H \cap H^g \neq 1$. By Lemma 2.15 (b), $g \in \mathrm{N}_G(H)$, and by part (a), $\mathrm{N}_{G_\alpha}(H) = \mathrm{N}_H(H) = H$. Since $g \in G_\alpha \cap \mathrm{N}_G(H)$, this implies that $g \in H$. Thus, G_α is a Frobenius group with Frobenius complement H.

Let $X \leq G_\alpha$ be non-trivial. By Lemma 2.13, $|\,\mathrm{N}_G(X) : \mathrm{N}_{G_\alpha}(X)| \leq 4$. Since $|G_\alpha|$ is coprime to 6, part (c) follows.

Suppose that $X \leq H$ and that $1 \neq X$. Then by (a), $|\,\mathrm{N}_{G_\alpha}(X) : \mathrm{N}_H(X)| = 1$. Thus, $|\,\mathrm{N}_G(X) : \mathrm{N}_H(X)| = |\,\mathrm{N}_G(X) : \mathrm{N}_{G_\alpha}(X)| \cdot |\,\mathrm{N}_{G_\alpha}(X) : \mathrm{N}_H(X)| \leq 4 \cdot 1$. Assume, for a contradiction, that $|\,\mathrm{N}_G(X)|$ is divisible by 3. Then $\mathrm{N}_G(X)$ contains an element x of order 3. Thus, x acts on $\mathrm{fix}_\Omega(X) = \mathrm{fix}_\Omega(H)$, a set of size 4. Therefore x fixes a point, but the point stabiliser have order coprime to 6. This proves together with part (c) the remaining statements in (d).

For part (e) suppose that p is a prime dividing $|H|$. Since G acts transitively, faithfully, and with fixity 4 on Ω, there exists an element ω that is not fixed by H and an element $g \in G$ such that $\alpha^g = \omega$. Then $g \notin \mathrm{N}_G(H)$, and thus, $\mathrm{N}_G(H) \neq G$. Let $P \in \mathrm{Syl}_p(\mathrm{N}_G(H))$ and let $Q \leq P$ be non-trivial. Then Q acts on the set of fixed points of H, but since H fixes four points and every element in Q has p-power order with $p \geq 5$, every element in Q fixes all four fixed points of H. Let $y \in \mathrm{N}_G(Q)$. Then y acts on $\mathrm{fix}_\Omega(Q) = \mathrm{fix}_\Omega(H)$, and hence by Lemma 2.15 (a), $y \in \mathrm{N}_G(H)$. Therefore, by Lemma 2.7, $\mathrm{N}_G(H)$ is strongly p-embedded in G.

The last part in the previous lemma enables us to show that with some exceptions the Sylow p-subgroups of G are cyclic if p divides the order of a four-point stabiliser. The full strength of this result will be used in the later subsections about groups of Lie type, but since it holds for all finite simple groups, it is already stated here.

Lemma 3.23
Let G be a finite simple group acting transitively and with fixity 4 on a set Ω and such that the point stabilisers have order coprime to 6. Let p be a prime that divides the order of the element-wise stabiliser of a set of size 4. Then G has cyclic Sylow p-subgroups or there exists a positive integer n such that $G \in \{\mathrm{PSL}(2,p^n), \mathrm{PSU}(3,p^n), \mathcal{A}_{2p}, \mathrm{J}_4, \mathrm{Fi}_{22}\}$.

In particular, if G is a finite simple group of Lie type such that there does not exist a prime power q such that G is isomorphic to $\mathrm{PSL}(2,q)$ or $\mathrm{PSU}(3,q)$, then G has cyclic Sylow p-subgroups.

Proof:

By Lemma 3.22 (e), G contains a strongly p-embedded subgroup. Since the hypothesis yields that $p \geq 5$, (24-1) in [41] implies that G has cylcic Sylow p-subgroups or that $G \in \{\mathrm{PSL}(2,p^n), \mathrm{PSU}(3,p^n), {}^2\mathrm{F}_4(2)', \mathcal{A}_{2p}, \mathrm{McL}, \mathrm{Fi}_{22}, \mathrm{J}_4\}$. Then the following GAP command uses the GAP package TomLib [74] through the GAP function in Remark 2.22 and shows that both ${}^2\mathrm{F}_4(2)'$ and McL do not act with fixity 4 on any set.

```
List(["2F4(2)'","McL"],x->TestTom(TableOfMarks(x),4));
```

The remaining groups are listed in the lemma and except for $\mathrm{PSL}(2,q)$ and $\mathrm{PSU}(3,q)$ none of them are of Lie type.

All fixity-4 actions of $\mathrm{PSL}(2,q)$ and $\mathrm{PSU}(3,q)$ have been determined in section 3.1. Therefore the restriction that G is none of these groups is no obstacle in the forthcoming analysis.

Since sporadic simple groups are fairly well analysed, there are other resources than Lemma 3.23 to understand their structure. In particular, for J_4 and Fi_{22}, Lemma 3.25 uses a different approach to show that these groups do not act with fixity 4 and such that point stabilisers have order coprime to 6.

Thus, the only interesting exception in the previous lemma are the alternating groups. They are the topic of the next subsection.

3.3.1 Alternating Groups

In the proof of Lemma 3.24, we will see how the property that the point stabilisers have order coprime to 6 can be exploit. A key concept is to fix the smallest prime that divides the order of a point stabiliser (or a four-point stabiliser) and derive a contradiction by showing that the order of a point stabiliser is divisible by a smaller prime. This idea will also be used for the other finite simple groups. Thus, the following lemma gives a first insight in the arguments.

This is the motivation to prove the result here, even though it is, with a different proof, part of [7].

Lemma 3.24
Let $n \geq 5$ be an integer and let $G = \mathcal{A}_n$. Suppose that G acts transitively on a set Ω. Then G acts with fixity 4 and such that the point stabilisers have order coprime to 6 if and only if $G = \mathcal{A}_7$ and the point stabilisers are cyclic of order 5.

Proof:

Using the GAP package TomLib [74] through the algorithm in Remark 2.22, the answer to the following command proves the statement of the lemma for all $n \leq 12$.

```
List([5..12],x->TestTom(TableOfMarks(Concatenation( "A",String(x)
 ↪  )), 4));
```

Therefore, throughout the rest of the proof, suppose that $n \geq 13$.

Assume, for a contradiction, that G acts with fixity 4 and such that the point stabilisers have order coprime to 6. Let $\alpha \in \Omega$ and let p be the smallest prime that divides $|G_\alpha|$. Thus, $p \geq 5$ and hence by Lemma 2.14, G_α contains a Sylow p-subgroup of G. Therefore, G_α also contains an element x conjugate to the p-cycle $\sigma := (1, 2, \dots, p)$.

Additionally, assume that $p \leq n-4$. Then $\mathrm{C}_G(\sigma)$ contains a subgroup isomorphic to $\mathcal{A}_{n-(n-4)} = \mathcal{A}_4$, and hence has order divisible by 12. Therefore, $\mathrm{N}_G(\langle x\rangle) \geq \mathrm{C}_G(x)$ has order divisible by 12 contrary to Lemma 3.22 (b). This contradiction shows that $p \geq n-3$.

Since $\mathcal{S}_p$ acts transitively on the set $\{1, 2, \dots, p\}$, $\mathrm{N}_{\mathcal{S}_n}(\langle\sigma\rangle)$ contains the full automorphism group of $\langle\sigma\rangle$, and hence has order divisible by $p-1$. Since $\mathcal{A}_n$ has index 2 in $\mathcal{S}_n$, $|\,\mathrm{N}_{\mathcal{A}_n}(\langle\sigma\rangle)| = |\,\mathrm{N}_G(\langle x\rangle)|$ is divisible by $\frac{p-1}{2} \geq \frac{n-4}{2} > 4$. By Lemma 2.13, $|\,\mathrm{N}_G(\langle x\rangle) : \mathrm{N}_{G_\alpha}(\langle x\rangle)| \leq 4$. Thus, there exists a prime r that divides $\frac{p-1}{2}$ and $|G_\alpha|$ contrary to the hypothesis that p is the smallest prime dividing $|G_\alpha|$. This final contradiction proves the lemma.

3.3.2 Sporadic Groups

Another aspect of proving that certain finite simple groups cannot act transitively, with fixity 4, and such the that point stabilisers have order coprime to 6 can be seen in the following lemma about the sporadic simple groups. Mainly Lemma 3.22 (c) is exploit.

The result is part of [7] but the proof here differs from the proof given there and contains more details.

Lemma 3.25
Let G be a sporadic finite simple group. Suppose that G acts transitively on a set Ω. Then G acts with fixity 4 and such that the point stabilisers have order coprime to 6 if and only if one of the following holds.

(1) $G = \mathrm{M}_{11}$ and the point stabilisers are of isomorphism type C_5 or $C_{11} : C_5$.

(2) $G = \mathrm{M}_{22}$ and the point stabilisers are of isomorphism type C_5 or $C_{11} : C_5$.

Proof:

For some sporadic simple groups, we can use the GAP package TomLib [74] together with the GAP function in Remark 2.22 to determine all transitive fixity-4 actions of these groups. Among them are M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, J_1, J_2, and J_3. The following GAP command implies that of these groups only M_{11}, M_{12}, M_{22}, and J_1 can act transitively and with fixity 4.

```
List(["M11","M12","M22","M23","M24","J1","J2","J3"],
↪  x->TestTom(TableOfMarks(x),4));
```

Furthermore, the result of the command states the sizes of the point stabilisers. The only cases where the point stabilisers have order coprime to 6 are exactly those stated in the lemma. Therefore, suppose that G is a sporadic simple group other than M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, J_1, J_2, and J_3.

Assume, for a contradiction, that G acts with fixity 4 and such that the point stabilisers have order coprime to 6. Let $\alpha \in \Omega$ and let p be the smallest prime that divides $|G_\alpha|$. Then $p \geq 5$ and by Lemma 2.14, G_α contains a Sylow p-subgroup of G. Let Y be a subgroup of order p of G. Then there exists a subgroup $X \leq G_\alpha$ conjugate to Y. By Lemma 2.13, $|\,\mathrm{N}_G(X) : \mathrm{N}_{G_\alpha}(X)| \leq 4$. Thus, every prime $r \geq 5$ that divides $|\,\mathrm{N}_G(X)|$ additionally divides $|G_\alpha|$, and hence by the minimality of p, it follows that $p \leq r$. Therefore, whenever a prime r divides $|\,\mathrm{N}_G(Y)|$, either $r \in \{2,3\}$ or $p \leq r$. On the other hand, by Lemma 3.22 (b), $|\,\mathrm{N}_G(Y)| = |\,\mathrm{N}_G(X)|$ is divisible by neither 8, 9, nor 6.

We look at all remaining sporadic simple groups and in each case we derive a contradiction in one way or another. We also keep the notation that p is the smallest prime dividing $|G_\alpha|$ and that Y denotes a subgroup of G of order p.

Suppose that $G = \mathrm{J}_4$. Then Table 5.3i in [43] gives information about the normalisers of subgroups of prime order. Furthermore the table implies that $p \in \{5, 7, 11, 23, 29, 31, 37, 43\}$. If $p = 43$, then $|\,\mathrm{N}_G(Y)| = 43{\cdot}14$. Thus, $|\,\mathrm{N}_G(Y)|$ is divisible by $r := 7$ but $r = 7 \leq 43 = p$ contrary to the observation above that either $r \in \{2,3\}$ or $p \leq r$. Therefore, $p \neq 43$. If $p = 37$, then $|\,\mathrm{N}_G(Y)| = 37 \cdot 12$, and hence $|\,\mathrm{N}_G(Y)|$ is divisible by 6, giving another contradiction. If $p = 31$, then we can choose Y such that $|\,\mathrm{N}_G(Y)| = 31 \cdot 10$. Thus, 5 divides $|\,\mathrm{N}_G(Y)|$ but neither $5 \in \{2,3\}$ nor $p \leq 5$, contrary to our earlier observation. Similarly, if $p \in \{29, 23, 11, 7\}$, then we can choose Y such that $|\,\mathrm{N}_G(Y)|$ is divisible by a prime grater than 3 and smaller than p and derive a contradiction. As a consequence, $p = 5$ but then for a subgroup $X \leq G_\alpha$ of order 5, $|\,\mathrm{N}_G(X)|$ is divisible by 8, giving a last contradiction in this case. Therefore, $G \neq \mathrm{J}_4$.

The group Co_3 is another sporadic simple group where we can use the GAP package TomLib [74] together with the program in Remark 2.22. Then the GAP command `TestTom(TableOfMarks("Co3"),4);` shows that Co_3 does not act transitively and with fixity 4 on any set.

Suppose that $G = \mathrm{Co}_2$. Then we use Table 5.3k in [43]. Thus, $p \in \{5, 7, 11, 23\}$. If $p \in \{23, 11\}$, then $|\,\mathrm{N}_G(Y)|$ is divisible by a prime smaller than p and greater than 3. Thus, $p \in \{5, 7\}$ but in both cases we can choose Y such that $|\,\mathrm{N}_G(Y)|$ is divisible by 8 and this also gives a contradiction. Therefore, $G \neq \mathrm{Co}_2$.

Suppose that $G = \mathrm{Co}_1$. Then in Table 5.3l in [43], information about G are collected. Thus, $p \in \{5, 7, 11, 13, 23\}$. If $p \in \{23, 11\}$, then $|\,\mathrm{N}_G(Y)|$ is divisible by a prime smaller than p and greater than 3. If $p \in \{13, 7\}$, then $|\,\mathrm{N}_G(Y)|$ is divisible by 6 and this gives a contradiction, too. Thus, $p = 5$ but then G_α contains a subgroup of order p whose normaliser has order divisible by 8, giving another contradiction. Therefore $G \neq \mathrm{Co}_1$.

If G is one of the groups HS or McL, then we can use, similarly to the earlier cases, the GAP code `List(["HS","McL"],x->TestTom(TableOfMarks(x),4));` to see that G does not act transitively and with fixity 4 on any set.

Suppose that $G = \mathrm{Suz}$. Then Table 5.3o in [43] gives information about G. Thus, $p \in \{5, 7, 11, 13\}$. If $p \in \{13, 7, 5\}$, then the order of $\mathrm{N}_G(Y)$ is divisible by 6. Therefore, $p = 11$ but then 5 divides $|\,\mathrm{N}_G(Y)|$ giving a contradiction to the minimality of p. As a consequence, $G \neq \mathrm{Suz}$.

Suppose that $G = \mathrm{He}$. We can use Table 5.3p in [43] to gain information about G. Then $p \in \{5, 7, 17\}$. If $p \in \{17, 5\}$, then $|\,\mathrm{N}_G(Y)|$ is divisible by 8. Since this is impossible, $p = 7$. However, then G_α contains a subgroup of order 7 whose normaliser in G has order divisible by 6, giving a contradiction, too. Hence, $G \neq \mathrm{He}$.

Suppose that $G = \mathrm{Ly}$. We use Table 5.3q in [43] to determine the orders of normalisers of subgroups of prime order and see that $p \in \{5, 7, 11, 31, 37, 67\}$. If $p = 67$, then $|\,\mathrm{N}_G(Y)|$ is divisible by 11 contradicting the minimality of p. In the remaining possibilities for p, there exists a subgroup $X \leq G_\alpha$ of order p such that $\mathrm{N}_G(X)$ has order divisible by 6, giving another contradiction. Therefore, $G \neq \mathrm{Ly}$.

Suppose that $G = \mathrm{Ru}$. Then Table 5.3r in [43] contains information regarding G. Thus, $p \in \{5, 7, 13, 29\}$. If $p = 29$, then $|\,\mathrm{N}_G(Y)|$ is divisible by 7, contradicting the minimality of p. If $p \in \{7, 13\}$ then the fact that we can choose Y such that $|\,\mathrm{N}_G(Y)|$ is divisible by 6 gives another contradiction. Thus, $p = 5$ but then there is a subgroup $X \leq G_\alpha$ of order 5 such that $|\,\mathrm{N}_G(X)|$ is divisible by 8. Therefore $G \neq \mathrm{He}$.

Suppose that $G = \mathrm{O'N}$. We use Table 5.3s in [43]. Thus, $p \in \{5, 7, 11, 19, 31\}$. If $p \in \{31, 11\}$, then $|\,\mathrm{N}_G(Y)|$ is divisible by 5. If $p \in \{19, 7\}$, then $\mathrm{N}_G(Y)$ has order divisible by 6. Therefore $p = 5$, but then the order of a normaliser of a subgroup of order 5 is divisible by 8, giving the last contradiction in this case and proving that $G \neq \mathrm{O'N}$.

Suppose that $G = \mathrm{Fi}_{22}$. Then we use the information in Table 5.3t in [43]. Thus, $p \in \{5, 7, 11, 13\}$. If $p \in \{13, 7, 5\}$, then $|\,\mathrm{N}_G(Y)|$ is divisible by 6. Therefore $p = 11$, but then $|\,\mathrm{N}_G(Y)|$ is divisible by 5, contradicting the minimality of p. Hence, $G \neq \mathrm{Fi}_{22}$.

Suppose that $G = \mathrm{Fi}_{23}$. Then by using Table 5.3u in [43], we can derive that $p \in \{5, 7, 11, 13, 17, 23\}$. If $p = 23$, then $|\,\mathrm{N}_G(Y)|$ is divisible by 11. If $p \in \{17, 11, 5\}$, then we can choose Y such that $|\,\mathrm{N}_G(Y)|$ is divisible by 8. Thus $p \in \{7, 13\}$, but then the order of $|\,\mathrm{N}_G(Y)|$ is divisible by 6, giving another contradiction. Therefore, $G \neq \mathrm{Fi}_{23}$.

Suppose that $G = \mathrm{Fi}_{24}'$. Then Table 5.3v in [43] contains information about G. Thus, $p \in \{5, 7, 11, 13, 17, 23, 29\}$. If $p \in \{29, 23, 11\}$, then $|\,\mathrm{N}_G(Y)|$ is divisible by a prime $r \geq 5$ smaller than p. If $p \in \{13, 7, 5\}$, then $|\,\mathrm{N}_G(Y)|$ is divisible by 6. Therefore $p = 17$, but then $|\,\mathrm{N}_G(Y)|$ is divisible by 8, giving another contradiction. Hence, $G \neq \mathrm{Fi}_{24}'$.

Suppose that $G = \mathrm{HN}$. We use Table 5.3w in [43]. Then $p \in \{5, 7, 11, 19\}$. If $p = 5$, then G_α has a subgroup X of order 5 such that $|\,\mathrm{N}_G(X)|$ is divisible by 8. If $p = 7$, then we can choose Y such that $\mathrm{N}_G(Y)$ has order divisible by 6. If $p = 11$, then $|\,\mathrm{N}_G(Y)|$ is divisible by $5 < p$. Thus, $p = 19$ but then $|\,\mathrm{N}_G(Y)|$ is divisible by 9 giving the last contradiction in this case. Therefore $G \neq \mathrm{HN}$.

Suppose that $G = \mathrm{Th}$. Then we use the information in Table 5.3x in [43]. Thus, $p \in \{5, 7, 13, 19, 31\}$. If $p \in \{19, 13, 7, 5\}$, then we can choose Y such that $|\,\mathrm{N}_G(Y)|$ is divisible by 6. Thus $p = 31$, but then $\mathrm{N}_G(Y)$ has order divisible by 5, contradicting the minimality of p. Hence, $G \neq \mathrm{Th}$.

Suppose that $G = \mathbb{B}$. Then Table 5.3y in [43] contains information about G. Thus, $p \in \{5, 7, 11, 13, 17, 19, 23, 31, 47\}$. If $p \in \{47, 31, 23, 11\}$, then $|\,\mathrm{N}_G(Y)|$ is divisible by a prime $r \geq 5$ that is smaller than p. If $p \in \{19, 13, 7\}$, then we can choose Y such that the order of $\mathrm{N}_G(Y)$ is divisible by 6. Thus $p \in \{17, 5\}$, but then G_α contains a subgroup of order p whose normaliser has order divisible by 8. As a consequence, $G \neq \mathbb{B}$.

Therefore, we look at the last remaining sporadic simple group. Suppose that $G = \mathbb{M}$. We use the information in Table 5.3z in [43] and conclude that $p \in \{5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71\}$. If p is one of the primes in $\{71, 59, 47, 41, 31, 29, 23, 11\}$, then $|\,\mathrm{N}_G(Y)|$ is divisible by a prime $r \geq 5$ that is smaller than p. If $p \in \{19, 13, 7\}$, then G_α contains a subgroup of order p whose normaliser has order divisible by 6. Thus, $p \in \{5, 17\}$ but then we can choose Y such that $|\,\mathrm{N}_G(Y)|$ is divisible by 8. This last contradiction shows that $G \neq \mathbb{M}$ and finishes the proof.

3.3.3 Classical Groups of Lie Type

We will take up some of the ideas that we have seen in the previous two sections. The general strategy is the same for all families of classical groups of Lie type. In each case we will assume that a finite simple group G acts transitively, with fixity 4, and such that the point stabilisers have order coprime to 6. Then we will investigate the structure of a four-point stabiliser and derive either a concrete fixity-4 action or a contradiction. In the latter case Lemma 3.22 (d) and the following result are of utmost importance.

Lemma 3.26
Let G be a finite simple classical group of Lie type over a field with q elements such that G acts transitively and with fixity 4 on a set Ω, and such that the point stabilisers have order coprime to 6. Let H be a four-point stabiliser of G. Suppose that $G \notin \{\mathrm{PSL}(2, q), \mathrm{PSL}(3, q), \mathrm{PSU}(3, q)\}$. Then the following hold:

(a) The order of H and $q \cdot (q-1) \cdot (q+1)$ are coprime.

(b) For every prime r that divides $|H|$, $q^{r-1} - 1$ is divisible by r.

Proof:

Assume for a contradiction, that there exists a prime p that divides $|H|$ and $q \cdot (q-1) \cdot (q+1)$. Since $|H|$ is coprime to 6, $p \geq 5$ and p divides exactly one of the numbers q, $q-1$, and $q+1$. By Lemma 3.23, G has cyclic Sylow p-subgroups. If p divides q, then Theorem 3.3.3 in [43] shows that the p-rank of G is at least 3, and hence G cannot have cyclic Sylow p-subgroups. Therefore, p divides $q^2 - 1$. Then the multiplicative order of q modulo p is 1 or 2. Thus, (10-2) in [41] implies together with Table 10:1 in [41] that the p-rank of G is at least 2. This contradiction shows (a).

For part (b), suppose that r is a prime that divides $|H|$. Since part (a) proves that r is coprime to q, Fermat's little theorem yields that q^{r-1} is divisible by r.

Since all fixity-4 actions of $\mathrm{PSL}(2, q)$, $\mathrm{PSL}(3, q)$, and $\mathrm{PSU}(3, q)$ have been analysed in Section 3.1, we do not have to deal with them again. Therefore, it is no problem to exclude them from our forthcoming results.

For every family of classical groups of Lie type, we will first analyse the structure of a certain subgroup to determine the order of the normaliser of a p-group where p is a well-chosen divisor of the order of a four-point stabiliser. Then we will use this information to derive either a concrete fixity-4 action or a contradiction. Even though this strategy works in general for all classical groups of Lie type, the choice of p depends on the specific family of groups. More specifically, we choose a positive integer k minimal with the property that there exists a prime divisor p of the order of a four-point stabiliser that also divides a certain polynomial in q. This polynomial depends on k and on the family of groups and is closely related to Zsigmondi's polynomials in [108]. However, the forthcoming proofs do not need detailed knowledge of Zsigmondi's theory. The strategy is probably most visible in Lemma 3.28, because there it can be used straight forward, since both the polynomial and the subgroup structure spare additional specifics.

However, we first collect information about the normaliser of a subgroup of order p in $\mathrm{PSL}(n, q)$. For this, a cyclic subgroup of $\mathrm{GL}(n, q)$, which is sometimes referred to as singer cycle, is used.

Lemma 3.27

Let $n \geq 2$ be an integer, let q be a prime power, and let $G = \mathrm{PSL}(n, q)$. Let p be a prime divisor of $|G|$ that does not divide $q-1$ and let $k \in \{n-1, n\}$ be such that p divides $q^k - 1$.

(a) If $k = n-1$, then there exists an element $y \in G$ of order p such that $|\mathrm{N}_G(\langle y \rangle)|$ is divisible by $\frac{(q^{n-1}-1)\cdot(n-1)}{\gcd(n,q-1)}$.

(b) If $k = n$, then there exists an element $y \in G$ of order p such that $|\mathrm{N}_G(\langle y \rangle)|$ is divisible by $\frac{q^n-1}{q-1} \cdot \frac{n}{\gcd(q-1,n)}$.

Proof:
Let $Z = \mathrm{Z}(\mathrm{SL}(n,q))$. Then $G = \mathrm{SL}(n,q)/Z$ and $|Z| = \gcd(n, q-1)$. We use Satz II 7.3 a) in [54] and get that $\mathrm{GL}(k,q)$ contains a cyclic subgroup T of order $q^k - 1$ and that $|\,\mathrm{N}_{\mathrm{GL}(k,q)}(T)/T|$ is cyclic of order k.

First suppose that $k = n-1$. Then $\mathrm{SL}(n,q)$ contains a subgroup U isomorphic to $\mathrm{GL}(k,q)$. By the above, there exists a cyclic subgroup C of U of order $q^k - 1$ and $|\,\mathrm{N}_U(C)/C| = k$. Let $a \in C$ be of order p. Then $\langle a\rangle$ is a characteristic subgroup of C, and hence $\mathrm{N}_U(\langle a\rangle) \geq \mathrm{N}_U(C)$. As a consequence, $\mathrm{N}_U(\langle a\rangle)$ is divisible by $(q^k-1)\cdot k$. By assumption, p does not divide $q-1$, and hence $Za \in CZ/Z$ has order p. Therefore, $\mathrm{N}_G(\langle Za\rangle) \geq \mathrm{N}_{\mathrm{SL}(n,q)}(\langle a\rangle)/Z \geq \mathrm{N}_U(\langle a\rangle)/Z$. Thus, $\mathrm{N}_G(\langle Za\rangle)$ has order divisible by $\frac{(q^k-1)\cdot k}{\gcd(n,q-1)}$.

Next, suppose instead, that $k = n$. Let T be a cyclic subgroup of $\mathrm{GL}(n,q)$ of order $q^n - 1$ and let $C = T \cap \mathrm{SL}(n,q)$. By Satz II 7.3 b) in [54], C has order $\frac{q^n-1}{q-1}$ and $\mathrm{N}_{\mathrm{SL}(n,q)}(C) = \mathrm{N}_{\mathrm{GL}(n,q)}(T) \cap \mathrm{SL}(n,q)$. Therefore, $|\,\mathrm{N}_{\mathrm{SL}(n,q)}(C)| = \frac{|\,\mathrm{N}_{\mathrm{GL}(n,q)}(T)|\cdot|\,\mathrm{SL}(n,q)|}{|\,\mathrm{N}_{\mathrm{GL}(n,q)}(T)\,\mathrm{SL}(n,q)|}$, and thus $|\,\mathrm{N}_{\mathrm{SL}(n,q)}(C)|$ is divisible by $\frac{|\,\mathrm{N}_{\mathrm{GL}(n,q)}(T)|\cdot|\,\mathrm{SL}(n,q)|}{|\,\mathrm{GL}(n,q)|} = \frac{(q^n-1)n\cdot|\,\mathrm{SL}(n,q)|}{(q-1)|\,\mathrm{SL}(n,q)|} = \frac{q^n-1}{q-1}\cdot n$. Let $a \in C$ be of order p. Since the order of $\mathrm{N}_G(\langle Za\rangle)$ is again divisible by $|\,\mathrm{N}_{\mathrm{SL}(n,q)}(C)/Z|$. We set $y = Za$ and the lemma follows.

Lemma 3.28
Let $n \geq 4$ be an integer, let q be a prime power, and let $G = \mathrm{PSL}(n,q)$. Then G does not act transitively, with fixity 4, and such that the point stabilisers have order coprime to 6 on any set.

Proof:
We will use that

$$|G| = \frac{q^{\frac{n(n-1)}{2}}}{\gcd(n,q-1)} \prod_{i=2}^{n} (q^i - 1)$$

throughout the proof without further reference. First we deal with $\mathrm{PSL}(4,2)$ and $\mathrm{PSL}(4,3)$ before we turn towards the general proof.

Since $\mathrm{PSL}(4,2) \cong \mathcal{A}_8$, Lemma 3.24 implies that $\mathrm{PSL}(4,2)$ does not act transitively, with fixity 4, and such that the order of the point stabilisers is coprime to 6 on any set. For $\mathrm{PSL}(4,3)$, we can use the algorithm in Remark 2.22 together with the GAP package TomLib [74]. Then the GAP command `TestTom(TableOfMarks("L4(3)"),4);` yields that $\mathrm{PSL}(4,3)$ does not act transitively and with fixity 4 on any set.

Assume, for a contradiction, that G acts transitively, with fixity 4, and such that the point stabilisers have order comprime to 6 on a set Ω. Let $\alpha \in \Omega$ and let H be a non-trivial four-point stabiliser contained in G_α. We have just seen, that if $n = 4$, then $q \geq 4$.

By Lemma 3.26 (a), a prime that divides $|H|$ does not divide q. Since this prime divides the order of G, it divides a factor of $|G|$ coprime to q, and we are in the position to define a positive integer with some minimality condition as described before Lemma 3.27. For that purpose, let k be the smallest positive integer such that there exists a prime $p \in \pi(H)$ with the property that p divides $q^k - 1$. Then, by Lemma 3.26 (a), $k \geq 3$.

First additionally assume, for a contradiction, that $k \leq n - 2$. The group $\mathrm{SL}(n, q)$ has a subgroup U_1 isomorphic to $\mathrm{SL}(n-2, q)$ and a subgroup U_2 isomorphic to $\mathrm{SL}(2, q)$ such that $U_1 \times U_2 \leq \mathrm{SL}(n, q)$. Let $Z = \mathrm{Z}(\mathrm{SL}(n, q))$. Then $U_1Z/Z \times U_2Z/Z \leq G$. Since $3 \leq k \leq n-2$, there exists an element $a \in U_1Z/Z$ of order p. Then $\mathrm{C}_G(a)$ contains U_2Z/Z, and hence $|\,\mathrm{C}_G(a)|$ is divisible by $\frac{q(q^2-1)}{\gcd(n,q-1)}$. Since H contains a Sylow p-subgroup, it contains an element b conjugate to a.

If $q = 2$, then $\frac{q(q^2-1)}{\gcd(n,q-1)} = 6$, and thus $|\,\mathrm{C}_G(b)|$ is divisible by 3, contradicting Lemma 3.22 (d). If $q = 3$, then $\frac{q(q^2-1)}{\gcd(n,q-1)}$ is divisible by 12, giving the same contradiction. If $q \geq 4$, then $\frac{q(q^2-1)}{\gcd(n,q-1)}$ is divisible by $q + 1 \geq 5$. Thus, the fact that $|\,\mathrm{C}_G(b) : \mathrm{C}_H(b)| \leq 4$ implies that $|H|$ is divisible by a prime factor r of $q + 1$, contradicting Lemma 3.26 (a).

As a consequence, $k \geq n - 1$.

Next assume, for a contradiction, that $k = n - 1$. Then Lemma 3.27 shows that there exists an element $y \in G$ of order p such that $|\,\mathrm{N}_G(\langle y\rangle)|$ is divisible by $\frac{(q^{n-1}-1)\cdot(n-1)}{\gcd(n,q-1)} = \frac{q^{n-1}-1}{q-1}\cdot(n-1)\cdot\frac{q-1}{\gcd(n,q-1)}$. Since H contains a Sylow p-subgroup it contains a subgroup Y conjugate to $\langle y\rangle$.

If $n \geq 6$, then $n - 1 \geq 5$, and hence $|\,\mathrm{N}_G(Y)|$ is divisible by $n - 1 \geq 5$. By Lemma 3.22 (d), it follows that $|H|$ is divisible by a prime factor r of $n-1$. Then Lemma 3.26 (b) yields that $q^{r-1}-1$ is divisible by r. Since $r-1 \leq n-1-1 < k$, this contradicts the minimality of k.

If $n = 5$, then $\frac{(q^{n-1}-1)\cdot(n-1)}{\gcd(n,q-1)} = (q^2+1)\cdot(q+1)\cdot 4\cdot\frac{q-1}{\gcd(5,q-1)}$. Hence $|\,\mathrm{N}_G(Y)|$ is divisible by $4(q+1)$. Since $|H|$ is odd and $|\,\mathrm{N}_G(Y) : \mathrm{C}_H(Y)| \leq 4$, it follows that $|H|$ is divisible by a prime factor r of $q+1$, contradicting Lemma 3.26 (a).

Thus, $n = 4$. Then $\frac{(q^{n-1}-1)\cdot(n-1)}{\gcd(n,q-1)} = 3\cdot\frac{(q^3-1)}{\gcd(4,q-1)}$ and hence $|\,\mathrm{N}_G(Y)|$ is divisible by 3. This contradicts Lemma 3.22 (d).

As a consequence, $k = n$.

Then Lemma 3.27 proves that there exists an element $y \in G$ of order p such that $|\,\mathrm{N}_G(\langle y\rangle)|$ is divisible by $\frac{q^n-1}{q-1}\cdot\frac{n}{\gcd(q-1,n)}$. Since H contains a Sylow p-subgroup, it contains a subgroup Y conjugate to $\langle y\rangle$, and hence $|\,\mathrm{N}_G(Y)|$ is divisible by $\frac{q^n-1}{q-1}\cdot\frac{n}{\gcd(q-1,n)}$.

If $n = 4$, then $|\,\mathrm{N}_G(Y)|$ is divisible by $(q^2+1)\cdot(q+1)\cdot\frac{4}{\gcd(q-1,4)}$. We recall that, if $n = 4$, $q \geq 4$. In particular, $|\,\mathrm{N}_G(Y)|$ is divisible by $q + 1 > 4$. Then the fact that $|\,\mathrm{N}_G(Y) : \mathrm{N}_H(Y)| \leq 4$ implies that $|H|$ is divisible by a prime that divides $q + 1$. This again contradicts Lemma 3.26 (a). Therefore, $n \geq 5$. We will spilt the analysis, according to whether n is divisible by a prime r such that $3 \leq r < n$, n is a 2-power, or n is prime.

If n is divisible by a prime r such that $3 \leq r < n$, then $\frac{q^n-1}{q-1} = \prod_{d|n,d\neq 1} \Phi_d(q)$ is divisible by $\Phi_r(q)$. The facts that $\Phi_r(q) \geq q^2+q+1 \geq 2^2+2+1 = 7$ and that $|\,\mathrm{N}_G(Y) : \mathrm{N}_H(Y)| \leq 4$ together imply that $|H|$ is divisible by a prime divisor of $\Phi_r(q)$. Since this prime then also divides $q^r - 1$ and since $r < n = k$, this is a contradiction to the minimality of k.

If n is a 2-power, then $\frac{q^n-1}{q-1} = \prod_{d|n,d\neq 1} \Phi_d(q)$ is divisible by $\Phi_4(q) = q^2 + 1$. Thus, $|\,\mathrm{N}_G(Y)|$ is divisible by $q^2+1 \geq 2^2+1 > 4$. Therefore, by Lemma 3.22 (d), it follows that $|H|$ is divisible by a prime dividing $q^2 + 1$ and hence $q^4 - 1$. However, since $4 < n = k$, this again contradicts the minimality of k.

The last case is that n is prime. Then $\Phi_k(q) = \Phi_n(q) = 1+q+q^2+\ldots+q^{n-1}$. If $q-1$ is divisible by n, then $\Phi_k(q) \equiv n\cdot 1 \equiv 0 \mod n$. Since $|\,\mathrm{N}_G(Y)|$ is divisible by $\frac{q^n-1}{q-1} \cdot \frac{n}{\gcd(q-1,n)}$, which in turn is divisible by $\Phi_k(q)$, this implies that $n \geq 5$ divides the order of $\mathrm{N}_G(Y)$. Therefore by Lemma 3.22 (d), $|H|$ is divisible by n, contradicting Lemma 3.26 (a) because n also divides $q - 1$. Thus, $q - 1$ cannot be divisible by n. Since n is a prime, this implies that $\gcd(q - 1, n) = 1$, and hence $|\,\mathrm{N}_G(Y)|$ is divisible by $\frac{q^n-1}{q-1} \cdot n$. The facts that $n \geq 5$ and that $|\,\mathrm{N}_G(Y) : \mathrm{N}_H(Y)| \leq 4$ together imply that the order of H is divisible by n. Then by Lemma 3.26 (b), which is applicable because n is a prime, n divides $q^{n-1} - 1$. Since $n - 1 < k$, this contradicts the minimality of k for a last time and finishes the proof.

The proof of the previous lemma not just explains the strategy that we will also use for the other classical groups of Lie type but additionally illustrates why we have to look at all families separately. The estimates and number-theoretic arguments were highly dependent on $\mathrm{PSL}(n, q)$ and its order and cannot be conveyed directly for the other families of classical groups of Lie type. We will see the difference more clearly when analysing $\mathrm{PSU}(n, q)$ and its fixity-4 actions. However, beforehand we investigate parts of the subgroup structure of $\mathrm{PSU}(n, q)$.

The subgroups of our main interest in the analysis of $\mathrm{PSU}(n, q)$ and the other classical groups of Lie type arise from the Aschbacher class $\mathcal{C}_3$ in [3]. For a prime power q, a positive integer n, and a prime r that divides n, the field $\mathbb{F}_{q^r}$ can be interpreted as a vector space of dimension r over $\mathbb{F}_q$, and thus a vector space V of dimension n/r over $\mathbb{F}_{q^r}$ can be identified with a vector space W of dimension n over $\mathbb{F}_q$. Therefore $\mathrm{GL}(n/r, q^r)$ is isomorphic to a subgroup of $\mathrm{GL}(n, q)$. This construction harmonises in some way with a form attached to the vector space. Thus, the subgroup of all elements of $\mathrm{GL}(n/r, q^r)$ respecting this, to V attached, form (i. e. the group of all isometries regarding this form) corresponds to a subgroup U of a group K that is the subgroup of all elements in $\mathrm{GL}(n, q)$ that respects the corresponding form attached to W. More details are given in section 7 in [3], especially in (7.2) and (7.4), and in paragraph 4.3 in [62], especially in Table 4.3.A. Then the subgroups we are interested in can be understood as the normaliser of U in K. From this (more geometric) construction, it is clear that $\mathrm{N}_K(U)$ exists, even though it is not necessarily a maximal subgroup of K. However, for the remaining proofs of this subsection we only need the existence and not the maximality. The structure and thus the order of $\mathrm{N}_K(U)$ can be derived from [3] and [62]. This is done in [105], and hence we normally consult an appropriate place there for the structure of $\mathrm{N}_K(U)$.

Lemma 3.29
Let $n \geq 2$ be an integer, let q be a prime power, and let $G = \mathrm{GU}(n, q)$. Let p be a prime divisor of $q^n - (-1)^n$ such that for all positive integers $l < n$ the number $q^l - (-1)^l$ is not divisible by p. Let $P \in \mathrm{Syl}_p(G)$. Then $|\,\mathrm{N}_G(P)|$ is divisible by $(q^n - (-1)^n) \cdot n$.

Proof:

Throughout the proof, we will need the order of $\mathrm{GU}(n, q)$ frequently. By (3.25) on p. 66 in [105], $|G| = q^{\frac{n(n-1)}{2}} \prod_{i=1}^{n} (q^i - (-1)^i)$.

We split the analysis according to whether n is even or odd, because the subgroups that are used in the investigation of each case are different. First suppose that n is even. Then $\mathrm{GU}(n, q)$ has a subgroup M isomorphic to $\mathrm{GL}(n/2, q^2).C_2$ (see Theorem 3.9 (iv) in [105]). Let $L \leq M$ be isomorphic to $\mathrm{GL}(n/2, q^2)$. By Satz II 7.3 in [54], L has a cyclic subgroup T of order $q^{2 \cdot \frac{n}{2}} - 1 = q^n - 1$ such that $\mathrm{N}_L(T)/T$ is cyclic of order $n/2$. Since for all positive integers $l < n$ the number $q^l - (-1)^l$ is not divisible by p, T contains a Sylow p-subgroup Q of G. Thus, by Lemma 2.4, $|\,\mathrm{N}_M(T)| = 2|\,\mathrm{N}_L(T)| = 2 \cdot (q^n - 1) \cdot \frac{n}{2} = (q^n - 1) \cdot n$, and hence $|\,\mathrm{N}_G(T)|$ is divisible by $(q^n - 1) \cdot n$. Since Q is a characteristic subgroup of T, it follows that the order of $\mathrm{N}_G(Q)$ is divisible by $|\,\mathrm{N}_G(T)|$. Thus, $|\,\mathrm{N}_G(Q)|$ is divisible by $(q^n - 1) \cdot n = (q^n - (-1)^n) \cdot n$ in the case that n is even. Since P is conjugate to Q, $\mathrm{N}_G(P)$ has order divisible by $(q^n - (-1)^n) \cdot n$, too.

Now instead suppose that n is odd. Then p divides $q^n + 1$. Let d be a positive integer that divides n. We prove the statement that the normaliser of a Sylow p-subgroup of $\mathrm{GU}(d, q^{n/d})$ is divisible by $(q^n + 1) \cdot d$ by induction. If $d = 1$, then $\mathrm{GU}(d, q^{n/d}) = \mathrm{GU}(1, q^n)$ is a cyclic group of order $q^n + 1$. Thus, the normaliser of a Sylow p-subgroup of $\mathrm{GU}(d, q^{n/d})$ is $\mathrm{GU}(1, q^n)$ itself, and hence has order $q^n + 1 = (q^n + 1) \cdot 1 = (q^n + 1) \cdot d$. For the induction step, additionally suppose that there exists a positive integer d that divides n such that for all $e < d$ that divide d, the normaliser of a Sylow p-subgroup of $\mathrm{GU}(e, q^{n/e})$ has order divisible by $(q^n + 1) \cdot e$. Then by Theorem 3.9 (vii) in [105] (or section 7 in [3]), there exists an odd prime r dividing d such that the group $\mathrm{GU}(d, q^{n/d})$ has a subgroup M isomorphic to $\mathrm{GU}(d/r, (q^{n/d})^r).C_r$. Let $U \leq M$ be isomorphic to $\mathrm{GU}(d/r, (q^{n/d})^r)$. Since n is odd and for all positive integers $l < n$, the number $q^l - (-1)^l$ is not divisible by p, it follows that for all $l_1 < d$ the number $(q^{n/d})^{l_1} - (-1)^{l_1}$ is not divisible by p. Thus, U contains a Sylow p-subgroup Q of $\mathrm{GU}(d, q^{n/d})$. Therefore Lemma 2.4 implies that $|\,\mathrm{N}_M(Q)| = r|\,\mathrm{N}_U(Q)|$. Then by the induction hypothesis (for $U \cong \mathrm{GU}(d/r, q^{\frac{n}{d/r}})$), $|\,\mathrm{N}_U(Q)|$ is divisible by $(q^n + 1) \cdot (d/r)$. As a consequence, $|\,\mathrm{N}_M(Q)|$ is divisible by $r \cdot (q^n + 1) \cdot (d/r) = (q^n + 1) \cdot d$. This proves the statement of the induction. Therefore, $|\,\mathrm{N}_G(P)| = |\,\mathrm{N}_{\mathrm{GU}(n,q)}(P)|$ is divisible by $(q^n + 1) \cdot n = (q^n - (-1)^n) \cdot n$ in the case that n is odd.

Lemma 3.30
Let $n \geq 2$ be an integer, let q be a prime power, and let $G = \mathrm{PSU}(n, q)$. Let p be a prime and k an integer greater than 2 such that p divides $q^k - (-1)^k$ and such that for all positive integers $l < k$, p does not divide $q^l - (-1)^l$. Let $P \in \mathrm{Syl}_p(G)$.

(a) If $k = n-1$, then there exists a subgroup $R \leq P$ such that $|\,\mathrm{N}_G(R)|$ is divisible by $\frac{(q^{n-1}-(-1)^{n-1})\cdot(n-1)}{\gcd(n,q+1)}$.

(b) If $k = n$, then $|\,\mathrm{N}_G(P)|$ is divisible by $\frac{(q^n-(-1)^n)\cdot n}{(q+1)\cdot\gcd(n,q+1)}$.

Proof:
Let $Z = \mathrm{Z}(\mathrm{SU}(n, q))$. Then $G = \mathrm{SU}(n, q)/Z$ and $|Z| = \gcd(n, q + 1)$. Since $k \geq 3$, p does not divide $q + 1$.

First suppose that $k = n$. Let Q be a Sylow p-subgroup of the full preimage of P in $\mathrm{SU}(n, q)$. Then $QZ/Z = P$. Since $\mathrm{SU}(n, q)$ has index $q + 1$ in $\mathrm{GU}(n, q)$ and since p does not divide $q + 1$, Q is a Sylow p-subgoup of $\mathrm{GU}(n, q)$. By Lemma 3.29, $|\,\mathrm{N}_{\mathrm{GU}(n,q)}(Q)|$ is divisible by $(q^n - (-1)^n) \cdot n$. Therefore, $|\,\mathrm{N}_{\mathrm{SU}(n,q)}(Q)| = \frac{|\,\mathrm{N}_{\mathrm{GU}(n,q)}(Q)|\cdot|\,\mathrm{SU}(n,q)|}{|\,\mathrm{N}_{\mathrm{GU}(n,q)}(Q)\,\mathrm{SU}(n,q)|}$. This number is divisible by $\frac{|\,\mathrm{N}_{\mathrm{GU}(n,q)}(Q)|\cdot|\,\mathrm{SU}(n,q)|}{|\,\mathrm{GU}(n,q)|}$ and hence by $\frac{(q^n-(-1)^n)\cdot n}{q+1}$. Since $\mathrm{N}_G(P) \geq \mathrm{N}_{\mathrm{SU}(n,q))}(Q)/Z$, it follows that $|\,\mathrm{N}_G(P)|$ is divisible by $\frac{(q^n-(-1)^n)\cdot n}{(q+1)\cdot\gcd(n,q+1)}$.

Therefore, instead suppose that $k = n - 1$. Then $\mathrm{SU}(n, q)$ contains a subgroup U isomorphic to $\mathrm{GU}(n - 1, q)$. Let Q be a Sylow p-subgroup of U. Then by Lemma 3.29, $|\,\mathrm{N}_U(Q)|$ is divisible by $(q^{n-1} - (-1)^{n-1}) \cdot (n - 1)$. Since $\mathrm{N}_U(Q)Z/Z \leq \mathrm{N}_{\mathrm{SU}(n,q)}(Q)/Z \leq \mathrm{N}_G(QZ/Z)$, this implies that $|\,\mathrm{N}_G(QZ/Z)|$ is divisible by $\frac{(q^{n-1}-(-1)^{n-1})\cdot(n-1)}{\gcd(n,q+1)}$. The lemma follows because all Sylow p-subgroups are conjugate, and therefore P contains a subgroup conjugate to QZ/Z.

With this information collected, we can turn towards the proof of the following lemma about the fixity-4 actions of $\mathrm{PSU}(n, q)$.

Lemma 3.31
Let $n \geq 4$ be an integer, let q be a prime power, and let $G = \mathrm{PSU}(n, q)$. Suppose that G acts transitively on a set Ω. Then G acts with fixity 4 and such that the point stabilisers have order coprime to 6 if and only if $G = \mathrm{PSU}(4, 2)$ or $G = \mathrm{PSU}(4, 3)$ and, in both cases, the point stabilisers are cyclic of order 5.

Proof:
For $\mathrm{PSU}(4, 2)$ the GAP command `TestTom(TableOfMarks("U4(2)"),4);` and for $\mathrm{PSU}(4, 3)$ the command `TestTom(TableOfMarks("U4(3)"),4);` yields the correctness of the statement. The commands use the algorithm in Remark 2.22 together with the GAP package TomLib [74].

To complete the proof, assume, for a contradiction, that G is neither $\mathrm{PSU}(4, 2)$ nor $\mathrm{PSU}(4, 3)$ and acts with fixity 4 and such that the point stabilisers have order coprime to 6 on a set Ω. Let $\alpha \in \Omega$ and let H be a non-trivial four-point stabiliser contained in G_α.

Similarly to the proof of Lemma 3.28, we will define an integer with some minimality property and in most cases derive a contradiction to the minimality. Albeit for this purpose, we will use a family of polynomials different from the one used in Lemma 3.28.

First we note that, by Lemma 3.26 (a), $|H|$ and q are coprime. Then we define k to be the smallest positive integer such that there exists a prime $p \in \pi(H)$ with the property that p divides $q^k - (-1)^k$. By Lemma 3.26 (a), it follows that $k \geq 3$.

Let $Z = \mathrm{Z}(\mathrm{SU}(n,q))$. Then $G = \mathrm{SU}(n,q)/Z$ and p does not divide $|Z|$.

First additionally assume, for a contradiction, that $k = n$. Let $P \in \mathrm{Syl}_p(H)$. Then by Lemma 3.22, $P \in \mathrm{Syl}_p(G)$. Therefore, by Lemma 3.30 (b), $|\,\mathrm{N}_G(P)|$ is divisible by $\frac{(q^n-(-1)^n)\cdot n}{(q+1)\cdot\gcd(n,q+1)}$.

We split the analysis in this case, according to whether n is divisible by 4, n is even but not divisible by 4, or n is odd.

Therefore, additionally assume, for a contradiction, that n is divisible by 4. Then $|\,\mathrm{N}_G(P)|$ is divisible by $\frac{(q^n-1)\cdot n}{(q+1)\cdot\gcd(n,q+1)} = (q^{n/2}+1)\cdot\frac{(q^{n/2}-1)\cdot n}{(q+1)\cdot\gcd(n,q+1)}$. Let $m = \frac{(q^{n/2}-1)\cdot n}{(q+1)\cdot\gcd(n,q+1)}$. If $|H|$ is divisible by a prime r that divides m, then $r \geq 5$ and either r divides $q^{n/2}-1 = q^{n/2}-(-1^{n/2})$, contradicting the minimality of k, or r divides n. By Lemma 3.26 (b), $q^{r-1}-1$ is divisible by r. Thus, in the latter case that r divides n, $r-1 \leq n-1 < k$, and we again get a contradiction to the minimality of k. Therefore $|H|$ and m are coprime. Since $|\,\mathrm{N}_G(P) : \mathrm{N}_H(P)| \leq 4$, this implies that $m \leq 4$.

If $n = 4$, then $m = \frac{(q^2-1)\cdot 4}{(q+1)\cdot\gcd(4,q+1)} = \frac{(q-1)\cdot 4}{\gcd(4,q+1)}$. Therefore, $q-1 \leq \gcd(4,q+1)$. By our assumption that G is neither $\mathrm{PSU}(4,2)$ nor $\mathrm{PSU}(4,3)$, $q \geq 4$ and hence $q-1 > \gcd(4,q+1)$. As a consequence, $n > 4$, and thus $n \geq 8$ because of our assumption that n is divisible by 4. Since $q+1$ divides either $q^{n/4}+1$ or $q^{n/4}-1$, it follows that $\frac{q^{n/2}-1}{q+1} = \frac{(q^{n/4}-1)\cdot(q^{n/4}+1)}{q+1} \geq q^{n/4}-1$. Hence, $m \geq (q^{n/4}-1)\cdot\frac{n}{\gcd(n,q+1)} \geq q^{n/4}-1$. If $q \geq 3$, then $m \geq q^{n/4}-1 \geq 3^2-1 = 8$. If $n > 8$, then $m \geq q^{n/4}-1 \geq q^{12/4}-1 = q^3-1 \geq 2^3-1 = 7$. If $n = 8$ and $q = 2$, then $m = \frac{(2^4-1)\cdot 8}{3\cdot\gcd(8,3)} = 40$. Thus, in all cases $m > 4$. This contradiction shows that n is not divisible by 4, in the case that $k = n$.

Therefore, instead assume, for a contradiction, that n is even but not divisible by 4. Then $|\,\mathrm{N}_G(P)|$ is divisible by $\frac{(q^n-1)\cdot n}{(q+1)\cdot\gcd(n,q+1)} = \frac{q^{n/2}-1}{q-1}\cdot(q-1)\cdot\frac{(q^{n/2}+1)\cdot n}{(q+1)\cdot\gcd(n,q+1)}$. Under this assumptions we set $m = (q-1)\cdot\frac{(q^{n/2}+1)\cdot n}{(q+1)\cdot\gcd(n,q+1)}$. Then $|\,\mathrm{N}_G(P)|$ is divisible by m. Since $n \geq 4$ and $n/2$ is odd, $n \geq 6$, and therefore $\frac{q^{n/2}+1}{q+1} = q^{n/2-1} - q^{n/2-2} + \ldots - q + 1 \geq q^2 - q + 1$. If $q = 2$, then $m = \frac{(2^{n/2}+1)\cdot n}{(2+1)\cdot\gcd(n,2+1)} \geq \frac{(2^{6/2}+1)\cdot 6}{3\cdot 3} = 6 > 4$. If $q \geq 3$, then $m = (q-1)\cdot\frac{(q^{n/2}+1)}{(q+1)}\cdot\frac{n}{\gcd(n,q+1)} \geq (q-1)\cdot(q^2-q+1) \geq 2\cdot 7 = 14 > 4$. Thus, in all cases, $m > 4$.

Since $|\,\mathrm{N}_G(P):\mathrm{N}_H(P)| \leq 4$, $|H|$ and m have a common prime factor $r \geq 5$. By Lemma 3.26 (a), $|H|$ and $q-1$ are coprime, thus r divides $q^{n/2}+1 = q^{n/2}-(-1)^{n/2}$ or n. The first case contradicts the minimality of k and in the second case, by Lemma 3.26 (b), $q^{r-1}-1$ is divisible by r. Since $r-1 \leq n-1 < k$, this is a contradiction to the minimality of k, too. Therefore, n is odd if $k = n$.

Then $|\,\mathrm{N}_G(P)|$ is divisible by $\frac{(q^n+1)\cdot n}{(q+1)\cdot \gcd(n,q+1)}$. If n is a prime then either $\gcd(n,q+1) = 1$ and hence $|\,\mathrm{N}_G(P)|$ divisible by n, or n divides $q+1$. In the latter case, $\frac{q^n+1}{q+1} \equiv q^{n-1}-q^{n-2}+q^{n-3}-\ldots+q^2-q+1 \equiv (-1)^{n-1}-(-1)^{n-2}+(-1)^{n-3}-\ldots+(-1)^2-(-1)+1 \equiv n \equiv 0 \mod n$. As a consequence, if n is prime, then n divides the order of $\mathrm{N}_G(P)$ and $n \geq 5$. Since $|\,\mathrm{N}_G(P):\mathrm{N}_H(P)| \leq 4$, this implies that n divides $|H|$, but since $q^{n-1}-1$ is divisible by n, this contradicts the minimality of k. Therefore, n has an odd prime divisor $s < n$ and $n \geq 9$. Then q^s+1 divides q^n+1. Thus, $\frac{(q^n+1)\cdot n}{(q+1)\cdot \gcd(n,q+1)} = \frac{q^n+1}{q^s+1}\cdot\frac{(q^s+1)}{(q+1)}\cdot\frac{n}{\gcd(n,q+1)}$. Let $m = \frac{(q^s+1)}{(q+1)}\cdot\frac{n}{\gcd(n,q+1)}$. Then $|\,\mathrm{N}_G(P)|$ is divisible by m. If $q \geq 3$, then $m \geq \frac{q^s+1}{q+1} \geq \frac{3^3+1}{3+1} = 7 > 4$. If $q = 2$, then $m = \frac{(2^s+1)}{3}\cdot\frac{n}{\gcd(n,3)} \geq \frac{(2^3+1)\cdot 9}{3\cdot 3} = 9 > 4$ because $n \geq 9$. Since $|\,\mathrm{N}_G(P):\mathrm{N}_H(P)| \leq 4$, it follows that there exists a prime t that divides $(q^s+1)\cdot n$ and $|H|$. The minimality of k forces t to divide n but then $q^{t-1}-1$ is divisible by t and this also contradicts the minimality of k, because $t-1$ is even and $t-1 \leq n-1 < n$. This last contradiction implies that $k < n$.

Therefore, assume instead, for a contradiction, that $k = n-1$. Let P be a Sylow p-subgroup of H. Since $P \in \mathrm{Syl}_p(G)$, Lemma 3.30 yields that there exists a subgroup $R \leq P$ such that $|\,\mathrm{N}_G(R)|$ is divisible by $\frac{(q^{n-1}-(-1)^{n-1})\cdot(n-1)}{\gcd(n,q+1)}$. If a prime s divides $(q^2-1)\cdot(n-1)$ and $|H|$, then $s \geq 5$ and s divides q^2-1 or $n-1$. By Lemma 3.26 (a), s cannot divide q^2-1, and thus s divides $n-1$. However, by Lemma 3.26 (b), $q^{s-1}-1$ is divisible by s and since s is odd and $s-1 \leq n-1-1 = k-1$, this also contradicts the minimality of k. Hence $|H|$ and $(q^2-1)\cdot(n-1)$ are coprime.

Additionally assume, for a contradiction, that $n-1$ is even. Then $|\,\mathrm{N}_G(R)|$ is divisible by $\frac{(q^{n-1}-1)\cdot(n-1)}{\gcd(n,q+1)} = \frac{q^{n-1}-1}{q^2-1}\cdot(n-1)\cdot\frac{(q^2-1)}{\gcd(n,q+1)}$. If $n \geq 6$, then $n-1 > 4$, and hence, by Lemma 3.22 (d), $|H|$ and $n-1$ have a common prime divisor, but this is impossible. Thus, $n-1 = 4$. If $q \geq 3$, then $\frac{(q^2-1)\cdot(n-1)}{\gcd(n,q+1)} = \frac{(q^2-1)\cdot 4}{\gcd(5,q+1)} \geq \frac{(3^2-1)\cdot 4}{5} = \frac{32}{5} > 4$. If $q = 2$, then $\frac{(q^2-1)\cdot(n-1)}{\gcd(n,q+1)} = \frac{(2^2-1)\cdot 4}{\gcd(5,2+1)} = 12 > 4$. Therefore, by Lemma 3.22 (d), $|H|$ and $(q^2-1)(n-1)$ have a common prime divisor. Since this is not possible, $n-1$ is odd.

Then $|\,\mathrm{N}_G(R)|$ is divisible by $\frac{(q^{n-1}+1)\cdot(n-1)}{\gcd(n,q+1)} = \frac{q^{n-1}+1}{q+1}\cdot\frac{(q+1)\cdot(n-1)}{\gcd(n,q+1)}$. If $n-1 \geq 5$, then the fact that $|\,\mathrm{N}_G(R):\mathrm{N}_H(R)| \leq 4$ implies that $n-1$ and $|H|$ again have a prime divisor in common. If $n = 4$, then, by our assumption that G is neither $\mathrm{PSU}(4,2)$ nor $\mathrm{PSU}(4,3)$, $q \geq 4$. If $q = 4$, then $\frac{(q+1)\cdot(n-1)}{\gcd(n,q+1)} = \frac{(4+1)\cdot 3}{\gcd(4,4+1)} = 15 > 4$. If $q \geq 5$, $\frac{(q+1)\cdot(n-1)}{\gcd(n,q+1)} \geq \frac{(5+1)\cdot 3}{\gcd(4,q+1)} \geq \frac{18}{4} > 4$. As a consequence, Lemma 3.22 (d) implies that there exists a prime s dividing $(q+1)\cdot(n-1)$ and $|H|$. However, as seen earlier, this is not possible. Hence, $k \leq n-2$.

As a next step, we will determine a factor of the order of the normaliser of a Sylow p-subgroup of G by starting with $\mathrm{GU}(n,q)$ and a Sylow p-subgroup therein. The group $\mathrm{GU}(n,q)$ has a subgroup M isomorphic to $\mathrm{GU}(n-2,q)\times\mathrm{GU}(2,q)$. Let $U \leq M$ be isomorphic to $\mathrm{GU}(n-2,q)$ and let $L \leq M$ be isomorphic to $\mathrm{GU}(2,q)$ such that $U \times L = M$. Then U has a subgroup V isomorphic to $\mathrm{GU}(k,q)$ and by Lemma 3.29, the normaliser of a Sylow p-subgroup P of V has order divisible by $(q^k-(-1)^k)\cdot k$. Since $L \leq \mathrm{C}_M(U)$, $L \leq \mathrm{N}_M(P)$, and since $L \cap V = 1$, this implies that $|\mathrm{N}_M(P)|$ is divisible by $(q^k-(-1)^k)\cdot k\cdot|L| = (q^k-(-1)^k)\cdot k\cdot q(q^2-1)(q+1)$. By Lemma 3.26 (a), p does not divide $q+1$, and thus $|P \cap \mathrm{SU}(n,q)| = |P|$, and hence $P \leq \mathrm{SU}(n,q)$. Therefore and because $\mathrm{N}_{\mathrm{GU}(n,q)}(P) \geq \mathrm{N}_M(P)$, $\mathrm{N}_{\mathrm{SU}(n,q)}(P)$ has order divisible by $\frac{(q^k-(-1)^k)\cdot k\cdot q(q^2-1)(q+1)}{q+1} = (q^k-(-1)^k)\cdot k\cdot q(q^2-1)$. Since $\mathrm{N}_{\mathrm{SU}(n,q)}(P)Z/Z \leq \mathrm{N}_G(PZ/Z)$, it follows that $|\mathrm{N}_G(PZ/Z)|$ is divisible by $\frac{(q^k-(-1)^k)\cdot k\cdot q(q^2-1)}{\gcd(n,q+1)}$.

Since H contains a Sylow p-subgroup of G, it contains a subgroup Q conjugate to PZ/Z. Hence, $|\mathrm{N}_G(Q)|$ is also divisible by $\frac{(q^k-(-1)^k)\cdot k\cdot q(q^2-1)}{\gcd(n,q+1)}$ which in turn is divisble by $k\cdot q$. Since $k \geq 3$ and $q \geq 2$, $k\cdot q > 4$. Then by Lemma 3.22 (d), $|H|$ is divisible by a prime r that divides $k\cdot q$. Since $|H|$ and q are coprime, r divides k and Lemma 3.26 (b) implies that $q^{r-1}-1$ is divisible by r. This contradicts the minimality of k because $r-1$ is even and $r-1 \leq k-1 < k$. This last contradiction finishes the proof.

The next family of classical groups of Lie type that we look at are the symplectic groups. We will see that $\mathrm{PSp}(4,q)$ can act with fixity 4 and such that the point stabilisers are cyclic of order coprime to 6. However, beforehand, we analyse some parts of the subgroup structure similarly to Lemma 3.29.

Lemma 3.32
Let $n \geq 2$ be an integer, let q be a prime power, and let $G = \mathrm{PSp}(2n,q)$. Let p be a prime such that p divides $q^{2n}-1$ and such that for all positive integers $l < n$, $q^{2l}-1$ is not divisible by p. Let $P \in \mathrm{Syl}_p(G)$ and let $\varepsilon \in \{-1,1\}$ be such that p divides $q^n-\varepsilon$. Then $|\mathrm{N}_G(P)|$ is divisible by $\frac{(q^n-\varepsilon)\cdot 2n}{\gcd(2,q+1)}$.

Proof:
At one point during the proof, we need the order of G. For convenience, we state it here (see pp. 60-61 in [105]).

$$|\mathrm{PSp}(2n,q)| = \frac{q^{n^2}}{\gcd(2,q+1)}\prod_{i=1}^{n}(q^{2i}-1)$$

Let d be a positive integer dividing n. We proof the statement that $\frac{(q^n-\varepsilon)\cdot 2d}{\gcd(2,q+1)}$ divides the order of the normaliser of a Sylow p-subgroup of $\mathrm{PSp}(2d,q^{n/d})$ by induction.

If $d=1$, then $\mathrm{PSp}(2d,q^{n/d}) = \mathrm{PSp}(2,q^n)$. Since, by Hilfssatz II 9.12 in [54], $\mathrm{PSp}(2,q^n)$ is isomorphic to $\mathrm{PSL}(2,q^n)$, the normaliser of a Sylow p-subgroup of $\mathrm{PSp}(2d,q^{n/d})$ is, by Satz II 8.3 and 8.4 in [54], a dihedral group of order $2\cdot\frac{q^n-\varepsilon}{\gcd(2,q+1)} = \frac{(q^n-\varepsilon)\cdot 2d}{\gcd(2,q+1)}$.

For the induction, suppose that there exists a positive integer d such that d divides n and such that for every positive integer $e < d$ that divides d, the normaliser of a Sylow p-subgroup of $\mathrm{PSp}(2e, q^{n/e})$ has order divisible by $\frac{(q^n-\varepsilon)\cdot 2e}{\gcd(2,q+1)}$. Then Theorem 2 in [29] (cf. Theorem 3.7 (iv) and Theorem 3.8 (vi) in [105]) implies that there exists a prime r that divides d and such that $\mathrm{PSp}(2d, q^{n/d})$ has a subgroup M isomorphic to $\mathrm{PSp}(2d/r, (q^{n/d})^r).C_r$. Let $U \leq M$ be isomorphic to $\mathrm{PSp}(2d/r, (q^{n/d})^r) = \mathrm{PSp}(2d/r, q^{\frac{n}{d/r}})$. Since for all positive integers $l < n$, the number $q^{2l} - 1$ is not divisible by p, it follows that for all $l_1 < d$, p does not divide $(q^{n/d})^{2l_1} - 1$. Thus, U contains a Sylow p-subgroup Q of $\mathrm{PSp}(2d, q^{n/d})$. Therefore, $Q \in \mathrm{Syl}_p(M)$. By Lemma 2.4, $|\,\mathrm{N}_M(Q)| = r \cdot |\,\mathrm{N}_U(Q)|$. Then by the induction hypothesis (for $U \cong \mathrm{PSp}(2d/r, q^{\frac{n}{d/r}})$), it follows that $|\,\mathrm{N}_U(Q)|$ is divisible by $\frac{(q^n-\varepsilon)\cdot 2(d/r)}{\gcd(2,q+1)}$. Hence, $|\,\mathrm{N}_M(Q)|$ is divisible by $r \cdot \frac{(q^n-\varepsilon)\cdot 2(d/r)}{\gcd(2,q+1)} = \frac{(q^n-\varepsilon)\cdot 2d}{\gcd(2,q+1)}$. This proves the statement of the induction.

As a consequence, $|\,\mathrm{N}_G(P)| = |\,\mathrm{N}_{\mathrm{PSp}(2n,q)}(P)|$ is divisible by $\frac{(q^n-\varepsilon)\cdot 2n}{\gcd(2,q+1)}$.

For $\mathrm{PSL}(n,q)$ and $\mathrm{PSU}(n,q)$ we did not need any maximality information about the used subgroups. For $\mathrm{PSp}(n,q)$ where we also have to prove that for $n = 4$ a certain action is indeed a fixity-4 action, we need more information than stated in the previous lemma mostly related to the maximal subgroups of $\mathrm{PSp}(4,q)$. The results are collect in the next lemma.

Lemma 3.33
Let $q \geq 7$ be a prime power and let $G = \mathrm{PSp}(4,q)$. Let $U < G$ have order divisible by $\frac{q^2+1}{\gcd(2,q+1)}$. Then either U is a maximal subgroup of G and there exists a positive integer n such that U is isomorphic to $\mathrm{Sz}(2^{2n+1})$ or U lies in a maximal subgroup of type $\mathrm{PSL}(2,q^2).C_2$. In particular, if U has odd order, then it is cyclic of order $\frac{q^2+1}{\gcd(2,q+1)}$.

Proof:

Let $Z = \mathrm{Z}(\mathrm{Sp}(4,q))$. Then $G = \mathrm{Sp}(4,q)/Z$ and $|Z| = \gcd(2, q-1)$. Let $\hat{U}$ be the full pre-image of U in $\mathrm{Sp}(4,q)$. Then $\hat{U} \neq \mathrm{Sp}(4,q)$, and hence $\hat{U}$ lies in a maximal subgroup of $\mathrm{Sp}(4,q)$.

An inspection of the the tables 8.12, 8.13, and 8.14 in [15] shows that the only maximal subgroups of $\mathrm{Sp}(4,q)$ with order divisible by $\frac{q^2+1}{\gcd(2,q+1)}$ have one of the following types (and some are only relevant under certain conditions).

Type	subgroup	conditions
(I)	$\mathrm{Sp}(2,q^2).C_2$	
(II)	$[2^5].\mathcal{S}_5$	q prime and congruent to -1 or 1 modulo 8
(III)	$[2^5].\mathcal{A}_5$	q prime and congruent to -3 or 3 modulo 8
(IV)	$2^{\cdot}\mathcal{A}_6$	$q \neq 7$ prime and congruent to 5 or 7 modulo 12
(V)	$2^{\cdot}\mathcal{S}_6$	q prime and congruent to 1 or 11 modulo 12
(VI)	$2^{\cdot}\mathcal{A}_7$	$q = 7$
(VII)	$\mathrm{SO}_-(4,q)$	$q \geq 4$ even
(VIII)	$\mathrm{Sz}(q)$	there exists an odd integer $e \geq 3$ such that $q = 2^e$

If $\hat{U}$ is a subgroup of a group of type (II) or (III), then the fact that $\frac{q^2+1}{\gcd(2,q+1)}$ is odd implies that $\frac{q^2+1}{\gcd(2,q+1)} \leq 3 \cdot 5$. Hence $q < 6$, but this contradicts the hypothesis that $q \geq 7$. If $\hat{U}$ is a subgroup of a group of type (IV) or (V), then $\frac{q^2+1}{\gcd(2,q+1)} \leq 3 \cdot 5 \cdot 3$. Thus, $q < 10$. This contradicts the conditions in type (V) and for type (IV), it implies that $q = 5$, contradicting the hypothesis. If $\hat{U}$ is a subgroup of a group of type (VI), then the condition states that $q = 7$. Thus, $\frac{q^2+1}{\gcd(2,q+1)} = 5^2$, but $|2^{\cdot}\mathcal{A}_7|$ is not divisible by 5^2, giving a contradiction.

Suppose that $\hat{U}$ is a subgroup of a maximal subgroup ot type (VIII). Then q is even. In particular, $\mathrm{Sp}(2n,q) \cong G$ and $\hat{U} \cong U$. By Table 8.16 in [15], $\mathrm{Sz}(q)$ does not have a proper subgroup divisible by $\frac{q^2+1}{\gcd(2,q+1)}$, and hence $\hat{U} \cong U$ is isomorphic to $\mathrm{Sz}(q)$ itself.

As a next step, we will see that if $\hat{U}$ is a subgroup of a group $\hat{M}$ of type (I) or type (VII), then U is a subgroup of a group isomorphic to $\mathrm{PSL}(2,q^2).C_2$.

For that purpose, first suppose that $\hat{U}$ is a subgroup of a group of type (VII). Then q is even, and thus $G \cong \mathrm{Sp}(4,q)$ and $\hat{U} \cong U$. Let $M \leq G$ be isomorphic to $\mathrm{SO}_-(4,q)$. Let L be the subgroup of index 2 in M isomorphic to $\Omega_-(4,q)$ (see p. 77 in [105]). Since q is even, $L \cong \Omega_-(4,q) \cong \mathrm{P}\Omega_-(4,q) \cong \mathrm{PSL}(2,q^2)$ by (3.57) on page 96 in [105]. Therefore, M is of type $\mathrm{PSL}(2,q^2).C_2$.

Suppose instead that $\hat{U}$ is a subgroup of a maximal subgroup $\hat{M}$ of type (I). Let $\hat{L} \leq \hat{M}$ be isomorphic to $\mathrm{Sp}(2,q^2)$. Then $Z = \mathrm{Z}(\hat{L})$, and hence $L := \hat{L}/Z$ is isomorphic to $\mathrm{PSp}(2,q^2)$ and has order $|\hat{L}/Z|$. Thus, $M := \hat{M}/Z$ has order $2 \cdot |L|$ and L is normal in M. Therefore M is of type $\mathrm{PSp}(2,q^2).C_2$. This finishes the proof of the first part of the statement.

Finally suppose that U has odd order. Then U is not isomorphic to $\mathrm{Sz}(q)$, thus U is a subgroup of a group M that is isomorphic to $\mathrm{PSL}(2,q^2).C_2$. Hence U is a subgroup of the subgroup of index 2 in M that is isomorphic to $\mathrm{PSL}(2,q^2)$. By Hauptsatz II 8.27 in [54], the only subgroups of $\mathrm{PSL}(2,q^2)$ of odd order divisible by $\frac{q^2+1}{\gcd(2,q+1)}$ are cyclic of order $\frac{q^2+1}{\gcd(2,q+1)}$. Hence, U is cyclic of order $\frac{q^2+1}{\gcd(2,q+1)}$.

With the information in the previous lemma collected, we are able to determine all fixity-4 actions of $\mathrm{PSp}(4,q)$. The strategy is to reduce the possibilities for point stabilisers to situations in which the previous lemma is applicable.

Lemma 3.34

Let $n \geq 2$ be an integer, let q be a prime power, and let $G = \mathrm{PSp}(2n,q)$. Suppose that G acts transitively on a set Ω. Then G acts with fixity 4 and such that the point stabilisers have order coprime to 6 if and only if $G = \mathrm{PSp}(4,q)$ and the point stabilisers are cyclic of order $\frac{q^2+1}{\gcd(2,q+1)}$.

Proof:

For $n = 2$ and small values of q we can use the GAP program in Remark 2.22 to establish the correctness of the lemma. The answer to the GAP command `TestTom(TableOfMarks(PSP(4,2)),4);` yields that $\mathrm{PSp}(4,2)$ acts transitively, with fixity 4, and such that the point stabiliser have order coprime to 6 if and only if the point stabilisers are cyclic of order 5. Since $\mathrm{PSp}(4,3) \cong \mathrm{PSU}(4,2)$, Lemma 3.31 implies the statement for $\mathrm{PSp}(4,3)$. For $\mathrm{PSp}(4,4)$ and $\mathrm{PSp}(4,5)$, their tables of marks are provided by the GAP package TomLib [74] and thus the results of the two commands `TestTom(TableOfMarks("S4(4)"),4);` and `TestTom(TableOfMarks("S4(5)"),4);` show that the lemma holds for the groups $\mathrm{PSp}(4,4)$ and $\mathrm{PSp}(4,5)$.

As a consequence, if $n = 2$, then we can suppose that $q > 5$.

For the first direction, suppose that G acts with fixity 4 and such that the point stabilisers have order coprime to 6 on a set Ω. Let $\alpha \in \Omega$ and let H be a non-trivial four-point stabiliser contained in G_α.

By Lemma 3.26 (a), $|H|$ and $q \cdot (q^2 - 1)$ are coprime. Let k be the smallest positive integer such that there exists a prime $p \in \pi(H)$ with the property that p divides $q^{2k} - 1$. Then $k \geq 2$.

Let $Z = \mathrm{Z}(\mathrm{Sp}(2n,q))$. Then $G = \mathrm{Sp}(2n,q)/Z$ and $|Z| = \gcd(2, q-1)$.

Assume, for a contradiction, that $k \leq n-1$. If q is even, then $\mathrm{Sp}(2n,q) \cong \mathrm{PSp}(2n,q) = G$. Then G has a subgroup isomorphic to $\mathrm{Sp}(2,q) \times \mathrm{Sp}(2(n-1),q)$ (see Theorem 3.7 (ii) in [105]). If q is odd, then $\mathrm{PSp}(2n,q) = G$ contains a subgroup isomorphic to $\mathrm{Sp}(2,q) * \mathrm{Sp}(2(n-1),q)$ (see Theorem 3.8 (ii) in [105]). In both cases let M denote the described subgroup and let $L \cong \mathrm{Sp}(2,q)$ and $U \cong \mathrm{Sp}(2(n-1),q)$ be such that $LU = M$. Let $R \in \mathrm{Syl}_p(U)$. Then $L \leq \mathrm{N}_G(R)$, and thus $|\,\mathrm{N}_G(R)|$ is divisible by $|L| = |\,\mathrm{Sp}(2,q)| = q(q^2-1)$.

Since by Corollary 2.14, H contains a Sylow p-subgroup of G, it contains a subgroup Q conjugate to R. Therefore, $|\,\mathrm{N}_G(Q)|$ is divisible by $q(q^2-1) \geq 6$. Then Lemma 3.22 (d) implies that there exists a prime r that divides $|H|$ and $q(q^2-1)$. This contradiction to Lemma 3.26 (a) yields that $k = n$.

Since p divides $(q^k - 1) \cdot (q^k + 1)$, there exists $\varepsilon \in \{-1, 1\}$ such that p divides $q^k - \varepsilon$. Let $P \in \mathrm{Syl}_p(H)$. Then $P \in \mathrm{Syl}_p(G)$, and thus Lemma 3.32 shows that $|\,\mathrm{N}_G((P)|$ is divisible by $\frac{(q^n-\varepsilon)\cdot 2n}{\gcd(2,q+1)}$. In particular, $|\,\mathrm{N}_G(P)|$ is divisible by $2n$. If $n \geq 3$, then $2n \geq 6$. Since $|\,\mathrm{N}_G(P) : \mathrm{N}_H(P)| \leq 4$, this implies that there exists a prime r dividing $|H|$ and $2n$. Then $r \geq 5$ and by Lemma 3.26 (b), $q^{r-1} - 1$ is divisible by r. Since $r - 1$ is even, this contradicts the minimality of k. As a consequence, $n = 2$ and thus $q \geq 7$.

If $\varepsilon = 1$, then p divides $q^2 - 1$ contradicting Lemma 3.26 (a). Hence, $\varepsilon = -1$ and p divides $q^2 + 1$. By Lemma 3.32, the order of $\mathrm{N}_G(P)$ is divisible by $n \cdot 2 \cdot \frac{q^n-\varepsilon}{\gcd(2,q-\varepsilon)} = 4 \cdot \frac{q^2+1}{\gcd(2,q+1)}$. Since $|\,\mathrm{N}_G(P) : \mathrm{N}_H(P)| \leq 4$, this means that $|H|$ is divisible by $\frac{q^2+1}{\gcd(2,q+1)}$. Therefore, the order of $G_\alpha \geq H$ is divisible by $\frac{q^2+1}{\gcd(2,q+1)}$, too. Then Lemma 3.33 implies that G_α is cyclic of order $\frac{q^2+1}{\gcd(2,q+1)}$, because $|G_\alpha|$ is odd. This finishes the proof of the first implication of the lemma.

For the other implication suppose that $G = \mathrm{PSp}(4,q)$ and $U \leq G$ is cyclic of order $\frac{q^2+1}{\gcd(2,q+1)}$. Then G acts transitively on G/U and U is a point stabiliser under this action. Since $q^2 + 1$ is divisible by neither 3 nor 4, $|U| = \frac{q^2+1}{\gcd(2,q+1)}$ is coprime to 6.

Let $y \in U$. By Lemma 2.10, y fixes exactly $\frac{|\,\mathrm{N}_G(\langle y \rangle)|}{|U|}$ points in G/U. Since U is cyclic, $\mathrm{N}_G(\langle y \rangle) \geq U$, and thus $|\,\mathrm{N}_G(\langle y \rangle)|$ is divisible by $|U| = \frac{q^2+1}{\mathrm{gcd}(2,q+1)}$.

Let $Y \leq U$ be non-trivial. Since G is simple, $\mathrm{N}_G(Y) < G$. Then Lemma 3.33 shows that $\mathrm{N}_G(Y)$ lies in a maximal subgroup of type $\mathrm{PSL}(2,q^2).C_2$ or is a maximal subgroup isomorphic to $\mathrm{Sz}(q)$. Since we can suppose that $q \geq 7$, $\mathrm{Sz}(q)$ is simple, and hence not the normaliser of a non-trivial group.

As a consequence, $\mathrm{N}_G(Y)$ lies in a maximal subgroup M of type $\mathrm{PSL}(2,q^2).C_2$. Let $L \leq M$ be isomorphic to $\mathrm{PSL}(2,q^2)$. Since $Y \leq U$ has odd order, $Y \leq L$. By Satz II 8.4 in [54], $\mathrm{N}_L(Y)$ is a dihedral group of order $2 \cdot \frac{q^2+1}{\mathrm{gcd}(2,q+1)}$. Since $|M : L| = 2$, it follows that $|\,\mathrm{N}_M(Y)| \leq 2 \cdot |\,\mathrm{N}_L(Y)| = 2 \cdot 2 \cdot \frac{q^2+1}{\mathrm{gcd}(2,q+1)} = 4 \cdot |U|$. Hence, $|\,\mathrm{N}_G(Y)| \leq 4 \cdot |U|$.

This also implies that for all non-trivial $y \in U$, $|\,\mathrm{N}_G(\langle y \rangle)| \leq 4 \cdot |U|$. Therefore, the number of fixed points of y is $\frac{|\,\mathrm{N}_G(\langle y \rangle)|}{|U|} \leq \frac{4 \cdot |U|}{|U|} = 4$. In particular, all non-trivial elements in U have at most 4 fixed points.

Let p be a prime dividing $|U|$ and let $P \in \mathrm{Syl}_p(U)$. Since $p \geq 5$, $P \in \mathrm{Syl}_p(G)$. Let $x \in P$ be of order p. Since $\langle x \rangle \operatorname{char} P$, $\mathrm{N}_G(\langle x \rangle) \geq \mathrm{N}_G(P)$. We have seen above that $\mathrm{N}_G(P)$ lies in a maximal subgroup M of G that has a subgroup L of index 2 such that $|\,\mathrm{N}_L(P)| = 2 \cdot \frac{q^2+1}{\mathrm{gcd}(2,q+1)}$. By Lemma 2.4, $2 \cdot |\,\mathrm{N}_L(P)| = |\,\mathrm{N}_M(P)|$. Thus, $|\,\mathrm{N}_G(P)| = 2 \cdot |\,\mathrm{N}_L(P)| = 2 \cdot 2 \cdot \frac{q^2+1}{\mathrm{gcd}(2,q+1)} = 4 \cdot |U|$, and hence $|\,\mathrm{N}_G(\langle x \rangle)| \geq |\,\mathrm{N}_G(P)| = 4 \cdot |U|$. Therefore, the number of fixed points of x is $\frac{|\,\mathrm{N}_G(\langle y \rangle)|}{|U|} \geq \frac{4 \cdot |U|}{|U|} = 4$. Since x is an an element of U and hence has at most four fixed points, it has exactly four fixed points in G/U. Therefore, G acts with fixity 4 on G/U. This completes the proof.

Finishing the symplectic groups, the only families of classical groups of Lie type left to be analysed are the orthogonal groups.

Some parts of the analysis of the orthogonal groups work for all three families similarly. They are fused in the next lemma. Afterwards we handle the subgroup structure and the investigation whether the groups act with fixity 4 or not separately for each family.

Lemma 3.35
Let $n \geq 2$ be an integer, let q be a prime power, let $\varepsilon \in \{-1, 1\}$, and let $G = \mathrm{GO}_\varepsilon(2n, q)$. Let p be a prime divisor of $q^n - \varepsilon$ such that for all positive integers $l < n$ the number $q^{2l} - 1$ is not divisible by p. Let $P \in \mathrm{Syl}_p(G)$. Then $|\,\mathrm{N}_G(P)|$ is divisible by $(q^n - \varepsilon) \cdot 2n$.

Proof:

By (3.31) and (3.32) on page 72 in [105], the order of $\mathrm{GO}_\varepsilon(2n, q)$ is

$$2 \cdot q^{n(n-1)} \cdot (q^n - \varepsilon) \cdot \prod_{i=1}^{n-1} (q^{2i} - 1)$$

and we use this fact without further reference.

Let d be a positive integer that divides n. We prove, by induction, the statement that the order of the normaliser of a Sylow p-subgroup of $\mathrm{GO}_\varepsilon(2d, q^{n/d})$ is divisible by $(q^n - \varepsilon) \cdot 2d$.

If $d = 1$, then by p. 71 in [105], $\mathrm{GO}_\varepsilon(2d, q^{n/d}) = \mathrm{GO}_\varepsilon(2, q^n)$ is a dihedral group of order $2(q^n - \varepsilon)$. Thus, the normaliser of a Sylow p-subgroup of $\mathrm{GO}_\varepsilon(2d, q^{n/d})$ is $\mathrm{GO}_\varepsilon(2, q^n)$ itself, and hence has order $2 \cdot (q^n - \varepsilon) = (q^n - \varepsilon) \cdot 2d$.

Therefore, suppose that there exists a positive integer d that divides n and such that for all positive integers $e < d$ that divide d, the order of the normaliser of a Sylow p-subgroup of $\mathrm{GO}_\varepsilon(2e, q^{n/e})$ is divisible by $(q^n - \varepsilon) \cdot 2e$. Then by (7.2) in [3] together with Proposition 4.3.8 (ii) and the proofs of Proposition 4.3.14 and Proposition 4.3.16 in [62] (cf. Theorem 3.11 (ix) and Theorem 3.12 (xi) in [105]), there exists a prime r that divides d and such that the group $\mathrm{GO}_\varepsilon(2d, q^{n/d})$ has a subgroup M isomorphic to $\mathrm{GO}_\varepsilon(2d/r, (q^{n/d})^r).C_r$. Let $U \leq M$ be isomorphic to $\mathrm{GO}_\varepsilon(2d/r, (q^{n/d})^r) = \mathrm{GO}_\varepsilon(2d/r, q^{\frac{n}{d/r}})$. Since for all positive integers $l < n$, the number $q^{2l} - 1$ is not divisible by p, it follows that for all positive integers $l_1 < d$, p does not divide $(q^{n/d})^{2l_1} - 1$. Thus, U contains a Sylow p-subgroup Q of $\mathrm{GO}_\varepsilon(2d, q^{n/d})$. Therefore $Q \in \mathrm{Syl}_p(M)$. Hence, by Lemma 2.4, $|\,\mathrm{N}_M(Q)| = r|\,\mathrm{N}_U(Q)|$. Then the induction hypothesis (for $U \cong \mathrm{GO}_\varepsilon(2d/r, q^{\frac{n}{d/r}})$) implies that $|\,\mathrm{N}_U(Q)|$ is divisible by $(q^n - \varepsilon) \cdot 2(d/r)$. Thus, $|\,\mathrm{N}_{\mathrm{GO}_\varepsilon(2d, q^{n/d})}(Q)|$ is divisible by $r \cdot (q^n - \varepsilon) \cdot 2(d/r) = (q^n - \varepsilon) \cdot 2d$. This proves the statement of the induction.

Therefore, $|\,\mathrm{N}_G(P)| = |\,\mathrm{N}_{\mathrm{GO}_\varepsilon(2n,q)}(P)|$ is divisible by $(q^n - \varepsilon) \cdot 2n$.

From the three families of orthogonal groups we start with the family of orthogonal groups of odd dimension first, because these groups have a close connection to the symplectic groups, which we analysed last.

Lemma 3.36
Let $n \geq 2$ be an integer, let q be an odd prime power, and let $G = \Omega(2n + 1, q)$. Let p be a prime divisor of $q^{2n} - 1$ such that for all positive integers $l < n$, the number $q^{2l} - 1$ is not divisible by p. Let $P \in \mathrm{Syl}_p(G)$ and let $\varepsilon \in \{-1, 1\}$ be such that p divides $q^n - \varepsilon$. Then $|\,\mathrm{N}_G(P)|$ is divisible by $(q^n - \varepsilon) \cdot n$.

Proof:

The information on page 75 in [105] implies that $\mathrm{GO}(2n + 1, q)$ has a subgroup isomorphic to $\mathrm{GO}_\varepsilon(2n, q) \times \mathrm{GO}(1, q)$. Let $D \cong \mathrm{GO}_\varepsilon(2n, q)$ and let $D_0 \cong \mathrm{GO}(1, q)$ be such that $D \times D_0 \leq \mathrm{GO}(2n + 1, q)$. Let $Q \in \mathrm{Syl}_p(D)$. Since by [105] p. 70 and p. 80, $G = \Omega(2n + 1, q)$ has index 4 in $\mathrm{GO}(2n + 1, q)$ and since p is odd, it follows that $Q \cap \Omega(2n + 1, q) = Q$. By Lemma 3.35, the order of $\mathrm{N}_D(Q)$ is divisible by $(q^n - \varepsilon) \cdot 2n$, and by [105] p. 71, $|D_0| = |\,\mathrm{GO}(1, q)| = 2$. Thus, $\mathrm{N}_{\mathrm{GO}(2n+1,q)}(Q) \geq \mathrm{N}_D(Q) \times D_0$ has order divisible by $(q^n - \varepsilon) \cdot 2n \cdot 2$. As a consequence, $|\,\mathrm{N}_G(P)|$ is divisible by $\frac{(q^n-\varepsilon)\cdot 4n}{4} = (q^n - \varepsilon) \cdot n$.

Lemma 3.37
Let $n \geq 3$ be an integer and let q be an odd prime power. Then $\mathrm{P}\Omega(2n + 1, q)$ does not act transitively, with fixity 4, and such that the point stabilisers have order coprime to 6 on any set.

Proof:
By [105] p. 80, $\mathrm{P}\Omega(2n+1,q) \cong \Omega(2n+1,q)$, and thus instead of proving the statement of the lemma for $\mathrm{P}\Omega(2n+1,q)$, we can proof it for $G := \Omega(2n+1,q)$.

Assume, for a contradiction, that G acts transitively, with fixity 4, and such that the point stabilisers have order coprime to 6 on a set Ω. Let $\alpha \in \Omega$ and let H be a non-trivial four-point stabiliser contained in G_α. By Lemma 3.26 (a), $|H|$ and q are coprime. Let k be the smallest positive integer such that there exists a prime $p \in \pi(H)$ with the property that p divides $q^{2k}-1$. By Lemma 3.26 (a), $k \geq 2$.

Assume, for a contradiction, that $k \leq n-1$. If $k = n-1$, then p divides $q^{2n-2}-1 = (q^{n-1}-1)(q^{n-1}+1)$, and thus it divides one of the two factors. Let $\varepsilon \in \{-1,1\}$ be such that p divides $q^{n-1}-\varepsilon$. If $k < n-1$, then set $\varepsilon = 1$. The information on page 75 in [105] implies that $\mathrm{GO}(2n+1,q)$ has a subgroup that is isomorphic to $\mathrm{GO}_\varepsilon(2(n-1),q) \times \mathrm{GO}(3,q)$. Let $A \cong \mathrm{GO}_\varepsilon(2(n-1),q)$ and $B \cong \mathrm{GO}(3,q)$ such that $A \times B \leq \mathrm{GO}(2n+1,q)$. Let $P \in \mathrm{Syl}_p(A \cap G)$. Then $B \leq \mathrm{N}_{\mathrm{GO}(2n+1,q)}(P)$, and thus $|\,\mathrm{N}_{\mathrm{GO}(2n+1,q)}(P)|$ is divisible by $|B| = |\,\mathrm{GO}(3,q)|$. The information on pages 70 and 80 in [105] yields that $|\,\mathrm{GO}(2n+1,q) : G| = 4$ and by (3.30) on page 72 in [105], $|\,\mathrm{GO}(3,q)| = 2 \cdot q \cdot (q^2-1)$. Hence, $|\,\mathrm{N}_G(P)|$ is divisible by $\frac{2\cdot q\cdot(q^2-1)}{4}$. Since H contains a Sylow p-subgroup of G, it also contains a subgroup Q conjugate to P. Thus, $|\,\mathrm{N}_G(Q)|$ is divisible by $\frac{q(q^2-1)}{2} \geq \frac{3\cdot 8}{2} > 4$. Then Lemma 3.22 (d) implies that $|H|$ is divisible by a prime dividing $q(q^2-1)$, contradicting Lemma 3.26 (a). As a consequence, $k = n$.

Then p divides $q^{2n}-1 = (q^n-1)(q^n+1)$. Let $\varepsilon \in \{-1,1\}$ be such that p divides $q^n-\varepsilon$. Let $P \in \mathrm{Syl}_p(H)$. Then $P \in \mathrm{Syl}_p(G)$. Thus, by Lemma 3.36, the order of $\mathrm{N}_G(P)$ is divisible by $(q^n-\varepsilon)\cdot n = n \cdot 2 \cdot \frac{q^n-\varepsilon}{2}$. Since q is odd, $q^n-\varepsilon$ is divisible by 2, and therefore $|\,\mathrm{N}_G(P)|$ is divisible by $n \cdot 2$.

Since $n \geq 3$, it follows that $n \cdot 2 \geq 6 > 4$. Since $|\,\mathrm{N}_G(P) : \mathrm{N}_H(P)| \leq 4$, this yields that there exists a prime r that divides $|H|$ and $2n$. Then $r \geq 5$ and r divides n. By Lemma 3.26 (b), $q^{r-1}-1$ is divisible by r and since r is odd this contradicts the minimality of k, because $r-1 \leq n-1 < 2k$. This final contradiction finishes the proof.

Likewise as for the orthogonal groups of odd dimension, we can again use Lemma 3.35 to derive information about some subgroups of $\mathrm{P}\Omega_\varepsilon(2n,q)$ before we analyse, for $\mathrm{P}\Omega_+(2n,q)$ and $\mathrm{P}\Omega_-(2n,q)$ separately, whether some of these groups can act with fixity 4.

Lemma 3.38
Let $n \geq 2$ be an integer, let q be a prime power, let $\varepsilon \in \{-1,1\}$, and let $G = \mathrm{P}\Omega_\varepsilon(2n,q)$. Let $p \in \pi(G)$ and let $k \geq 3$ be an integer such that p divides $q^{2k}-1$ and such that for all positive integer $l < k$, p does not divide $q^{2l}-1$. Let $\eta \in \{-1,1\}$ be such that p divides $q^k-\eta$ and let $P \in \mathrm{Syl}_p(G)$.

(a) If $k = n-1$, then there exists a subgroup $R \leq P$ such that $|\,\mathrm{N}_G(R)|$ is divisible by $\frac{(q^{n-1}-\eta)\cdot 2(n-1)\cdot(q-\varepsilon\cdot\eta)}{\gcd(4,q^n-\varepsilon)}$.

(b) If $k = n$, then $|\,\mathrm{N}_G(P)|$ is divisible by $\frac{n\cdot(q^n-\varepsilon)}{\gcd(4,q^n-\varepsilon)}$.

Proof:
Let $Z = \mathrm{Z}(\Omega_\varepsilon(2n, q))$. Then $G = \Omega_\varepsilon(2n, q)/Z$. By the information on pages 70, 77, and 80 in [105], Z has order 2 if $q^n \equiv \varepsilon \mod 4$ and order 1 otherwise, and $|\,\mathrm{GO}_\varepsilon(2n, q) : \Omega_\varepsilon(2n, q)| = 2 \cdot \gcd(2, q - 1)$.

First suppose that $k = n - 1$. By the information on page 75 in [105] (cf. Theorem 3.11 (ii) and Theorem 3.12 (ii) and (iii) in [105]), $\mathrm{GO}_\varepsilon(2n, q)$ has a subgroup isomorphic to $\mathrm{GO}_\eta(2(n-1), q) \times \mathrm{GO}_{\varepsilon\cdot\eta}(2, q)$. Let $D \cong \mathrm{GO}_\eta(2n-2, q)$ and $D_0 \cong \mathrm{GO}_{\varepsilon\cdot\eta}(2, q)$ be such that $D \times D_0 \leq \mathrm{GO}_\varepsilon(2n, q)$. Let $Q \in \mathrm{Syl}_p(D)$. Then by Lemma 3.35, $|\,\mathrm{N}_D(Q)|$ is divisible by $(q^{n-1}-\eta)\cdot 2(n-1)$. By [105] p. 71, $|D_0| = |\,\mathrm{GO}_{\varepsilon\cdot\eta}(2, q)| = 2(q - \varepsilon \cdot \eta)$. Therefore, the order of $\mathrm{N}_{\mathrm{GO}_\varepsilon(2n,q)}(Q) \geq \mathrm{N}_D(Q) \times D_0$ is divisible by $(q^{n-1} - \eta) \cdot 2(n - 1) \cdot 2(q - \varepsilon \cdot \eta)$. Since p is odd and $|\,\mathrm{GO}_\varepsilon(2n, q) : \Omega_\varepsilon(2n, q)| = 2 \cdot \gcd(2, q - 1)$, it follows that $|\,\mathrm{N}_{\Omega_\varepsilon(2n,q)}(q)|$ is divisible by $\frac{(q^{n-1}-\eta)\cdot 2(n-1)\cdot 2(q-\varepsilon\cdot\eta)}{2\cdot\gcd(2,q-1)} = \frac{(q^{n-1}-\eta)\cdot 2(n-1)\cdot(q-\varepsilon\cdot\eta)}{\gcd(2,q-1)}$. Then the facts that $\mathrm{N}_G(QZ/Z) \geq \mathrm{N}_{\Omega_\varepsilon(2n,q)}(Q)Z/Z$ and that $\gcd(2, q-1)\cdot|Z| = \gcd(4, q^n-\varepsilon)$ imply that $|\,\mathrm{N}_G(Q)|$ is divisible by $\frac{(q^{n-1}-\eta)\cdot 2(n-1)\cdot(q-\varepsilon\cdot\eta)}{\gcd(2,q-1)\cdot|Z|} = \frac{(q^{n-1}-\eta)\cdot 2(n-1)\cdot(q-\varepsilon\cdot\eta)}{\gcd(4,q^n-\varepsilon)}$. Since P is a Sylow p-subgroup of G, it contains a subgroup R conjugate to Q, and the lemma follows in the case $k = n - 1$.

Therefore suppose instead that $k = n$. Then the order formula for G yields that $\eta = \varepsilon$. Let $Q \in \mathrm{Syl}_p(\mathrm{GO}_\varepsilon(2n, q))$. Then $Q \leq \Omega_\varepsilon(2n, q)$ and the order of $\mathrm{N}_{\Omega_\varepsilon(2n,q)}(Q)$ is divisible by $\frac{|\,\mathrm{N}_{\mathrm{GO}_\varepsilon(2n,q)}(Q)|}{2\cdot\gcd(2,q-1)}$. By Lemma 3.35, $|\,\mathrm{N}_{\mathrm{GO}_\varepsilon(2n,q)}(Q)|$ is divisible by $(q^n - \varepsilon) \cdot 2n$. Therefore, the order of $\mathrm{N}_{\Omega_\varepsilon(2n,q)}(Q)$ is divisible by $\frac{2n\cdot(q^n-\varepsilon)}{2\cdot\gcd(2,q-1)}$. Thus, $|\,\mathrm{N}_G(QZ/Z)|$ is divisible by $\frac{2n\cdot(q^n-\varepsilon)}{2\cdot\gcd(2,q-1)\cdot|Z|} = \frac{n\cdot(q^n-\varepsilon)}{\gcd(4,q^n-\varepsilon)}$.

Lemma 3.39
Let $n \geq 4$ be an integer, let q be a prime power, and let $G = \mathrm{P}\Omega_+(2n, q)$. Then G does not act transitively, with fixity 4, and such that the point stabilisers have order coprime to 6 on any set.

Proof:
Let $Z = \mathrm{Z}(\Omega_+(2n, q))$. We recall that $G = \Omega_+(2n, q)/Z$, that Z has order 2 if $q^n \equiv 1 \mod 4$ and order 1 otherwise, and that $|\,\mathrm{GO}_+(2n, q) : \Omega_+(2n, q)| = 2 \cdot \gcd(2, q - 1)$. (See [105] pp. 70, 77, 80.)

Assume, for a contradiction, that G acts transitively, with fixity 4, and such that the point stabilisers have order coprime to 6 on a set Ω. Let $\alpha \in \Omega$ and let H be a non-trivial four-point stabiliser contained in G_α.

By Lemma 3.26 (a), $|H|$ and q are coprime. Let k be the smallest positive integer such that there exists a prime $p \in \pi(H)$ with the property that p divides $q^{2k} - 1$. Then Lemma 3.26 (a) implies that $k \geq 2$.

First assume, for a contradiction, that $k \leq n-2$. If $k = n-2$, then p divides $q^{2n-4}-1 = (q^{n-2}+1)(q^{n-2}-1)$ and hence one of the two factors. Let $\varepsilon \in \{-1, 1\}$ be such that p divides $q^{n-2} - \varepsilon$. If $k < n - 2$, then set $\varepsilon = -1$.

By the information on page 75 in [105] (cf. Theorem 3.12 (ii) and (iii) in [105]), the group $\mathrm{GO}_+(2n,q)$ has a subgroup isomorphic to $\mathrm{GO}_\varepsilon(2(n-2),q)\times\mathrm{GO}_\varepsilon(4,q)$. Let $A\cong\mathrm{GO}_\varepsilon(2(n-2),q)$ and $B\cong\mathrm{GO}_\varepsilon(4,q)$ such that $A\times B\le\mathrm{GO}_+(2n,q)$. Let $P\in\mathrm{Syl}_p(A\cap\Omega_+(2n,q))$. Then $B\le\mathrm{N}_{\mathrm{GO}_+(2n,q)}(P)$, thus $|\,\mathrm{N}_{\mathrm{GO}_+(2n,q)}(P)|$ is divisible by $|B|=|\,\mathrm{GO}_\varepsilon(4,q)|$. By (3.31) and (3.32) in [105], $|\,\mathrm{GO}_\varepsilon(4,q)|=2\cdot q^2\cdot(q^2-1)\cdot(q^2-\varepsilon)$, and hence the order of $\mathrm{N}_{\Omega_+(2n,q)}(P)$ is divisible by $\frac{2\cdot q^2\cdot(q^2-1)\cdot(q^2-\varepsilon)}{2\cdot\gcd(2,q-1)}=\frac{q^2\cdot(q^2-1)\cdot(q^2-\varepsilon)}{\gcd(2,q-1)}$. Since H contains a Sylow p-subgroup, it contains a subgroup Q conjugate to PZ/Z in G. Then the order of $\mathrm{N}_G(Q)$ is divisible by $\frac{q^2\cdot(q^2-1)\cdot(q^2-\varepsilon)}{\gcd(2,q-1)\cdot|Z|}$, because $\mathrm{N}_G(PZ/Z)\ge\mathrm{N}_{\Omega_+(2n,q)}(P)Z/Z$. In particular, $|\,\mathrm{N}_G(Q)|$ is divisible by $\frac{q^2\cdot(q^2-1)}{|Z|}\ge\frac{2^2\cdot(2^2-1)}{2}>4$. Since $|\,\mathrm{N}_G(Q):\mathrm{N}_H(Q)|\le 4$, this implies that $|H|$ is divisible by a prime dividing $q^2\cdot(q^2-1)$ contradicting Lemma 3.26 (a). As a consequence, $k\ge n-1$.

Next assume, for a contradiction, that $k=n-1$. Then p divides $q^{2n-2}-1=(q^{n-1}-1)(q^{n-1}+1)$. Let $\varepsilon\in\{-1,1\}$ be such that p divides $q^{n-1}-\varepsilon$.

Let $P\in\mathrm{Syl}_p(H)$. Then $P\in\mathrm{Syl}_p(G)$, and by Lemma 3.38, there exists a subgroup $Q\le P$ such that the order of $\mathrm{N}_G(Q)$ is divisible by $\frac{(q^{n-1}-\varepsilon)\cdot 2(n-1)\cdot(q-\varepsilon)}{\gcd(4,q^n-1)}=\frac{(q^{n-1}-\varepsilon)\cdot 2(n-1)\cdot(q-\varepsilon)}{\gcd(2,q-1)\cdot|Z|}$. Thus, $|\,\mathrm{N}_G(Q)|$ is divisible by $\frac{2(n-1)\cdot(q-\varepsilon)}{|Z|}$.

If $q>2$, then $\frac{2(n-1)\cdot(q-\varepsilon)}{|Z|}\ge\frac{2(n-1)\cdot(q-\varepsilon)}{2}\ge\frac{2\cdot3\cdot(3-1)}{2}>4$. If $q=2$, then $|Z|=1$, and thus $\frac{2(n-1)\cdot(q-\varepsilon)}{|Z|}=2(n-1)\cdot(2-\varepsilon)\ge 2\cdot 3>4$. Therefore, the fact that $|\,\mathrm{N}_G(Q):\mathrm{N}_H(Q)|\le 4$ implies that there exists a prime r that divides $|H|$ and $2(n-1)\cdot(q-\varepsilon)$. Then $r\ge 5$ and by Lemma 3.26 (a), r does not divide q^2-1. Therefore, r divides $(n-1)$. Then Lemma 3.26 (b) implies that $q^{r-1}-1$ is divisible by r, contradicting the minimality of k, because $r-1$ is even and smaller than $n-1=k$. As a consequence, $k=n$.

Then p divides $q^{2n}-1=(q^n-1)(q^n+1)$ and $|H|$. Since $H\le G$, this implies that p divides q^n-1. If n is even, then this contradicts the minimality of k because then $q^{2\cdot\frac{n}{2}}-1$ is divisible by p and $\frac{n}{2}<n=k$. Therefore n is odd, and hence $n\ge 5$.

Let $Q\in\mathrm{Syl}_p(H)$. Since then $Q\in\mathrm{Syl}_p(G)$, Lemma 3.38 implies that the order of $\mathrm{N}_G(Q)$ is divisible by $\frac{n\cdot(q^n-1)}{\gcd(4,q^n-1)}$. In particular, $|\,\mathrm{N}_G(Q)|$ is divisible by n.

Since $n\ge 5$, the fact that $|\,\mathrm{N}_G(Q):\mathrm{N}_H(Q)|\le 4$ implies that there exists a prime r that divides $|H|$ and n. By Lemma 3.26 (b), $q^{r-1}-1$ is divisible by r, contradicting the minimality of k because $r-1$ is even and $r-1\le n-1<2k$. This is the final contradiction and it proves that G cannot act transitively, with fixity 4, and such that the point stabilisers have order coprime to 6 on any set.

We turn to the last family of classical groups of Lie type, namely $\mathrm{P}\Omega_-(2n,q)$. Even though the reasonings for $\mathrm{P}\Omega_-(2n,q)$ and $\mathrm{P}\Omega_+(2n,q)$ are similar, we will see that for $n=4$, $\mathrm{P}\Omega_-(2n,q)$ can act with fixity 4 and such that the point stabilisers have order coprime to 6. For $\mathrm{P}\Omega_+(8,q)$, the last two section of the proof of Lemma 3.39 revealed that there does not exists a prime that divides the order of a four-point stabiliser and q^n-1 if n is even (because this contradicts the minimality hypothesis in the proof). For $\mathrm{P}\Omega_-(2n,q)$ the number in this step will be q^n+1 and we cannot derive a contradiction to the minimality, but instead we will see that indeed, for $n=4$, $\mathrm{P}\Omega_-(2n,q)$ exhibits a fixity-4 action such that the order of a non-trivial four-point stabiliser is divisible by a prime that divides q^n+1. The details are given in the next lemma.

Lemma 3.40
Let $n \geq 4$ be an integer, let q be a prime power, and let $G = \mathrm{P}\Omega_-(2n,q)$. Suppose that G acts transitively on a set Ω. Then G acts with fixity 4 and such that the point stabilisers have order coprime to 6 if and only if $G = \mathrm{P}\Omega_-(8,q)$ and the point stabilisers are cyclic of order $\frac{q^4+1}{\gcd(2,q+1)}$.

Proof:
Let $Z = \mathrm{Z}(\Omega_-(2n,q))$. Then $G = \Omega_-(2n,q)/Z$. We recall that Z has order 2 if and only if $q^n \equiv -1 \mod 4$, and that $|\,\mathrm{GO}_-(2n,q) : \Omega_-(2n,q)| = 2\cdot\gcd(2,q-1)$. (See [105] pp. 70, 77, 80.)

For the first direction, suppose that G acts with fixity 4 and such that the point stabilisers have order coprime to 6. Let $\alpha \in \Omega$ and let H be a non-trivial four-point stabiliser contained in G_α. By Lemma 3.26 (a), $|H|$ and q are coprime. Let k be the smallest positive integer such that there exists a prime $p \in \pi(H)$ with the property that p divides $q^{2k}-1$. By Lemma 3.26 (a), $k \geq 2$.

Assume, for a contradiction, that $k \leq n-2$. If $k = n-2$, then p divides $q^{2n-4}-1 = (q^{n-2}+1)(q^{n-2}-1)$, and hence one of the factors. Let $\varepsilon \in \{-1,1\}$ be such that p divides $q^{n-2}-\varepsilon$. If $k < n-2$, then set $\varepsilon = -1$.

By the information on page 75 in [105] (cf. Theorem 3.11 (ii) in [105]), the group $\mathrm{GO}_-(2n,q)$ has a subgroup isomorphic to $\mathrm{GO}_\varepsilon(2(n-2),q) \times \mathrm{GO}_{-\varepsilon}(4,q)$. Let $A \cong \mathrm{GO}_\varepsilon(2(n-2),q)$ and $B \cong \mathrm{GO}_{-\varepsilon}(4,q)$ be such that $A\times B \leq \mathrm{GO}_-(2n,q)$. By (3.31) and (3.32) in [105], $|B| = |\,\mathrm{GO}_{-\varepsilon}(4,q)| = 2\cdot q^2\cdot(q^2-1)\cdot(q^2+\varepsilon)$. Let $P \in \mathrm{Syl}_p(A\cap\Omega_-(2n,q))$. Then $B \leq \mathrm{N}_{\mathrm{GO}_-(2n,q)}(P)$, and thus $|\,\mathrm{N}_{\mathrm{GO}_-(2n,q)}(P)|$ is divisible by $|B|$. As a consequence, the order of $\mathrm{N}_{\Omega_-(2n,q)}(P)$ is divisible by $\frac{2\cdot q^2\cdot(q^2-1)\cdot(q^2+\varepsilon)}{2\cdot\gcd(2,q-1)} = \frac{q^2\cdot(q^2-1)\cdot(q^2+\varepsilon)}{\gcd(2,q-1)}$. Since H contains a Sylow p-subgroup, it contains a subgroup Q conjugate to PZ/Z in G. Then the order of $\mathrm{N}_G(Q)$ is divisible by $\frac{q^2\cdot(q^2-1)\cdot(q^2+\varepsilon)}{\gcd(2,q-1)\cdot|Z|}$, because $\mathrm{N}_G(PZ/Z) \geq \mathrm{N}_{\Omega_-(2n,q)}(P)Z/Z$. In particular, $|\,\mathrm{N}_G(Q)|$ is divisible by $\frac{q^2\cdot(q^2-1)}{|Z|} \geq \frac{2^2\cdot(2^2-1)}{2} > 4$. Therefore, Lemma 3.22 implies that $|H|$ is divisible by a prime that divides $q^2\cdot(q^2-1)$, contradicting Lemma 3.26 (a). Hence, $k \geq n-1$.

Thus, assume instead that $k = n-1$. Then $q^{2n-2}-1 = (q^{n-1}-1)(q^{n-1}+1)$ is divisible by p. Let $\varepsilon \in \{-1,1\}$ be such that p divides $q^{n-1}-\varepsilon$ and let $P \in \mathrm{Syl}_p(H)$. Then $P \in \mathrm{Syl}_p(G)$. By Lemma 3.38, there exists a subgroup Q of P such that $|\,\mathrm{N}_G(Q)|$ is divisible by $\frac{(q^{n-1}-\varepsilon)\cdot 2(n-1)\cdot(q+\varepsilon)}{\gcd(4,q^n+1)} = \frac{(q^{n-1}-\varepsilon)\cdot 2(n-1)\cdot(q+\varepsilon)}{\gcd(2,q+1)\cdot|Z|}$.

Thus, $|\operatorname{N}_G(Q)|$ is divisible by $\frac{2(n-1)\cdot(q+\varepsilon)}{|Z|}$. If $q > 2$, then $\frac{(n-1)\cdot 2\cdot(q+\varepsilon)}{|Z|} \geq \frac{(n-1)\cdot 2\cdot(q+\varepsilon)}{2} \geq \frac{3\cdot 2\cdot(3-1)}{2} > 4$. If $q = 2$, then $|Z| = 1$, and hence $\frac{(n-1)\cdot 2\cdot(q+\varepsilon)}{|Z|} \geq (n-1)\cdot 2 \geq 6 > 4$. Therefore, the fact that $|\operatorname{N}_G(Q) : \operatorname{N}_H(Q)| \leq 4$ implies that there exists a prime r that divides $|H|$ and $(n-1)\cdot 2\cdot(q+\varepsilon)$. Then $r \geq 5$ and by Lemma 3.26 (a), r does not divide $q+\varepsilon$. Thus, r divides $n-1$, and hence, by Lemma 3.26 (b), $q^{r-1}-1$ is divisible by r, contradicting the minimality of k because $r-1$ is even and $r-1 \leq n-2 < 2(n-1)$. As a consequence, $k = n$.

Then p divides $q^{2n}-1 = (q^n-1)(q^n+1)$ and $|H|$. Since $H \leq G$, this implies that p divides q^n+1.

Let $Q \in \operatorname{Syl}_p(H)$. Then $Q \in \operatorname{Syl}_p(G)$, and by Lemma 3.38, the order of $\operatorname{N}_G(Q)$ is divisible by $\frac{n\cdot(q^n+1)}{\gcd(4,q^n+1)}$. In particular, $|\operatorname{N}_G(Q)|$ is divisible by n.

If $n \geq 5$, then the fact that $|\operatorname{N}_G(Q) : \operatorname{N}_H(Q)| \leq 4$ implies that there exists a prime r that divides $|H|$ and n. Then $r \geq 5$ and by Lemma 3.26 (b), $q^{r-1}-1$ is divisible by r, contradicting the minimality of k because $r-1$ is even and $r-1 \leq n-1 < 2k$. As a consequence, $n = 4$ and $G = \mathrm{P}\Omega_-(8,q)$.

Since q^4 cannot be congruent to -1 modulo 4, it follows that $|Z| = 1$ and that $\gcd(4, q^4+1) = \gcd(2, q+1)$. Then $|\operatorname{N}_G(Q)|$ is divisible by $\frac{n\cdot(q^n+1)}{\gcd(4,q^n+1)} = \frac{q^4+1}{\gcd(2,q+1)} \cdot 4$. Since $|H|$ is odd and $|\operatorname{N}_G(Q) : \operatorname{N}_H(Q)| \leq 4$, this implies that $|H|$ is divisible by $\frac{q^4+1}{\gcd(2,q+1)}$. Since $|Z| = 1$, G is isomorphic to $\Omega_-(8,q)$. Then Table 8.52 and Table 8.53 in [15] show that the only type of maximal subgroups of $\Omega_-(8,q)$ with order divisible by $\frac{q^4+1}{\gcd(2,q+1)}$ is $\Omega_-(4,q^2).C_2$. By (3.57) on page 96 in [105], $\mathrm{PSL}(2,q^4) \cong \mathrm{P}\Omega_-(4,q^2)$ and since $q^4 \not\equiv -1 \mod 4$, $\mathrm{P}\Omega_-(4,q^2) \cong \Omega_-(4,q^2)$. Therefore, $G_\alpha \geq H$ lies in a subgroup isomorphic to $\mathrm{PSL}(2,q^4).C_2$. Since G_α has odd order, G_α is isomorphic to a subgroup of $\mathrm{PSL}(2,q^4)$. By Hauptsatz II 8.27 in [54] the only subgroups of $\mathrm{PSL}(2,q^4)$ of odd order divisible by $\frac{q^4+1}{\gcd(2,q+1)}$ are cyclic of order $\frac{q^4+1}{\gcd(2,q+1)}$. As a consequence, G_α is a cyclic group of order $\frac{q^4+1}{\gcd(2,q+1)}$.

For the other direction suppose that $G = \mathrm{P}\Omega_-(8,q) \cong \Omega_-(8,q)$ and that $U \leq G$ is a cyclic group of order $\frac{q^4+1}{\gcd(2,q+1)}$. Then G acts by right multiplication transitively on G/U and U is a point stabiliser under this action. Since $q^4+1 \equiv 2 \mod 4$ if q is odd and since q^4+1 is not divisible by 3, $|U| = \frac{q^4+1}{\gcd(2,q+1)}$ is coprime to 6.

Let $y \in U$. Then, by Lemma 2.10, y fixes exactly $\frac{|\operatorname{N}_G(\langle y\rangle)|}{|U|}$ points in G/U. Let $Y \leq U$ be non-trivial. Since G is simple $\operatorname{N}_G(Y) < G$. Then we can again use Table 8.52 and Table 8.53 in [15] to see that the subgroup $\operatorname{N}_G(Y)$ lies in a maximal subgroup M of type $\mathrm{PSL}(2,q^4).C_2$. In particular, $\operatorname{N}_G(Y) = \operatorname{N}_M(Y)$. Let $L \leq M$ be isomorphic to $\mathrm{PSL}(2,q^4)$. Since $Y \leq U$ has odd order, $Y \leq L$. By Satz II 8.4 in [54], it follows that $\operatorname{N}_L(Y)$ is a dihedral group of order $2\cdot\frac{q^4+1}{\gcd(2,q+1)}$. Since $|M : L| = 2$, it follows that $|\operatorname{N}_G(Y)| = |\operatorname{N}_M(Y)| \leq$

$2 \cdot |\,\mathrm{N}_L(Y)| = 2 \cdot 2 \cdot \frac{q^4+1}{\gcd(2,q+1)} = 4 \cdot |U|$. Therefore, the number of fixed points of y is $\frac{|\,\mathrm{N}_G(\langle y \rangle)|}{|U|} \leq \frac{4 \cdot |U|}{|U|} = 4$. In particular, all elements in U have at most four fixed points.

Let p be a prime divisor of $|U|$ and let $P \in \mathrm{Syl}_p(U)$. Then the order formula of G implies that $P \in \mathrm{Syl}_p(G)$. Let $x \in P$ be of order p. Since $\langle x \rangle$ is a characteristic subgroup of P, $\mathrm{N}_G(\langle x \rangle) \geq \mathrm{N}_G(P)$. We have just seen that $\mathrm{N}_G(P)$ lies in a maximal subgroup M of G that has a subgroup L of index 2 such that $|\,\mathrm{N}_L(P)| = 2 \cdot \frac{q^4+1}{\gcd(2,q+1)}$. By Lemma 2.4, $2 \cdot |\,\mathrm{N}_L(P)| = |\,\mathrm{N}_M(P)|$. Thus, $|\,\mathrm{N}_G(P)| = 2 \cdot |\,\mathrm{N}_L(P)| = 2 \cdot 2 \cdot \frac{q^4+1}{\gcd(2,q+1)} = 4 \cdot |U|$. As a consequence, $|\,\mathrm{N}_G(\langle x \rangle)| \geq |\,\mathrm{N}_G(P)| = 4 \cdot |U|$. Hence, the number of fixed points of x is $\frac{|\,\mathrm{N}_G(\langle y \rangle)|}{|U|} \geq \frac{4 \cdot |U|}{|U|} = 4$. Since $x \in U$, x has at most four fixed points, and therefore it has exactly four fixed points in G/U. As a consequence, G acts with fixity 4 on G/U.

We summarise our results.

Lemma 3.41

Let G be a finite simple classical group of Lie type such that for all prime power q, G is neither $\mathrm{PSL}(2,q)$, $\mathrm{PSL}(3,q)$, nor $\mathrm{PSU}(3,q)$. Suppose that G acts transitively on a set Ω. Then G acts with fixity 4 and such the that point stabilisers have order coprime to 6 if and only if there exists a prime power q such that G is $\mathrm{PSU}(4,3)$, $\mathrm{PSp}(4,q)$ with $q \geq 3$, or $\mathrm{P}\Omega_-(8,q)$ and the point stabilisers are cyclic of order 5, $\frac{q^2+1}{\gcd(2,q+1)}$, or $\frac{q^4+1}{\gcd(2,q+1)}$, respectively.

Proof:

Let n be a positive integer and q a prime power. We will use the classification theorem of finite simple groups (see p. 3 in [105]) to restrict the possibilities of G. If $G = \mathrm{PSL}(n,q)$, then the hypothesis implies that $n \geq 4$. Then Lemma 3.28 proves that G does not act transitively, with fixity 4, and such that the point stabilisers have order coprime to 6 on any set. Thus, in this case the lemma holds. If $G = \mathrm{PSU}(n,q)$, then by hypothesis $n \geq 4$. Thus, Lemma 3.31 proves that only $\mathrm{PSU}(4,2)$ and $\mathrm{PSU}(4,3)$ can act transitively, with fixity 4, and such that the point stabilisers have order coprime to 6. Additionally the lemma shows that the point stabilisers in these actions are cyclic of order 5. Since $\mathrm{PSU}(4,2) \cong \mathrm{PSp}(4,3)$ and $5 = \frac{3^2+1}{\gcd(2,3+1)}$, both groups are mentioned in the lemma. If $G = \mathrm{PSp}(n,q)$, then $n \geq 2$ and $G \neq \mathrm{PSp}(4,2)$ because G is simple. Hence, Lemma 3.34 implies the correctness of the lemma in this case. If G is $\mathrm{P}\Omega(2n+1,q)$, then q is odd and $n \geq 3$. Thus, we can use Lemma 3.37 to see that G cannot act transitively, with fixity 4, and such that the point stabilisers have order coprime to 6 on any set. If $G = \mathrm{P}\Omega_+(2n,q)$, then $n \geq 4$ and Lemma 3.39 implies the lemma in this case. Finally, if $G = \mathrm{P}\Omega_-(2n,q)$, then $n \geq 4$. Thus, Lemma 3.40 yields the lemma.

By the classification theorem of finite simple groups, these are the only simple classical groups of Lie type, and hence the lemma holds.

3.3.4 Exceptional Groups of Lie Type

For the exceptional groups of Lie type we will use a similar strategy as for the classical groups of Lie type. Therefore, we need a result equivalent to Lemma 3.26. The next lemma will provide such a result.

Lemma 3.42
Let G be a finite simple exceptional group of Lie type over a field with q elements that acts transitively and with fixity 4 on a set Ω. Let $\alpha \in \Omega$, let H be a four-point stabiliser, and let p be a prime dividing $|H|$. Suppose that $|G_\alpha|$ is coprime to 6. Then p is a divisor of one of the following numbers in each case.

G	p divides
${}^2\mathrm{B}_2(q) = \mathrm{Sz}(q)$	$\Phi_1(q)$, $\Phi_4(q)$
${}^3\mathrm{D}_4(q)$	$\Phi_{12}(q)$
$\mathrm{G}_2(q)$	$\Phi_3(q)$, $\Phi_6(q)$
${}^2\mathrm{G}_2(q)$	$\Phi_1(q)$, $\Phi_2(q)$, $\Phi_6(q)$
$\mathrm{F}_4(q)$	$\Phi_8(q)$, $\Phi_{12}(q)$
${}^2\mathrm{F}_4(q)$	$\Phi_6(q)$, $\Phi_{12}(q)$
$\mathrm{E}_6(q)$	$\Phi_5(q)$, $\Phi_8(q)$, $\Phi_9(q)$, $\Phi_{12}(q)$
${}^2\mathrm{E}_6(q)$	$\Phi_8(q)$, $\Phi_{10}(q)$, $\Phi_{12}(q)$, $\Phi_{18}(q)$
$\mathrm{E}_7(q)$	$\Phi_5(q)$, $\Phi_7(q)$, $\Phi_8(q)$, $\Phi_9(q)$, $\Phi_{10}(q)$, $\Phi_{12}(q)$, $\Phi_{14}(q)$, $\Phi_{18}(q)$
$\mathrm{E}_8(q)$	$\Phi_7(q)$, $\Phi_9(q)$, $\Phi_{14}(q)$, $\Phi_{15}(q)$, $\Phi_{18}(q)$, $\Phi_{20}(q)$, $\Phi_{24}(q)$, $\Phi_{30}(q)$

Proof:
By Lemma 3.23, G has cyclic Sylow p-subgroups. Since $|G_\alpha|$ is coprime to 6, $p \geq 5$. If p divides q, then Theorem 3.3.3 in [43] implies that the p-rank of G is at least 3. In particular, G does not have cyclic Sylow p-subgroups in this case. As a consequence, p does not divide q.

Let k be a positive integer and minimal with the properties that p divides $\Phi_k(q)$ and that $\Phi_k(q)$ is a divisor of $|G|$. By (10-2) in [41], the rank of a Sylow p-subgroup of G is the exponent of $\Phi_k(q)$ in Table 10:2 in [41]. Since G has cyclic Sylow p-subgroups, these exponents have to be 1, and hence the only options for k are those stated in the lemma.

We go through the list of exceptional groups of Lie type, and in each case, either we exclude the option that p divides $\Phi_k(q)$ or we show that under these conditions the group has a fixity-4 action.

Since Lemma 3.5 fully analyses the situation for the family of Suzuki groups ($\mathrm{Sz}(q) = {}^2\mathrm{B}_2(q)$), we start with the groups ${}^3\mathrm{D}_4(q)$ and first emphasis some details about their subgroup structure.

Lemma 3.43
Let q be a prime power and $G = {}^3\mathrm{D}_4(q)$. Let p be a prime divisor of $q^4 - q^2 + 1 = \Phi_{12}(q)$ and let R be a non-trivial p-subgroup of G. Then $\mathrm{N}(R)$ is a maximal subgroup of type $C_{q^4-q^2+1} : [4]$.

Proof:

Since $q^4 - q^2 + 1$ is odd and congruent to 1 modulo 3, $p \geq 5$. By (4.67) on page 142 in [105], the order of G is $q^{12}(q^8 + q^4 + 1)(q^6 - 1)(q^2 - 1) = q^{12}(q^4 - q^2 + 1)(q^4 + q^2 + 1)(q^6 - 1)(q^2 - 1) = q^{12} \cdot \Phi_{12}(q) \cdot (q^6 - 1)^2$. Since $q^2 - 1$ and $q^4 + q^2 + 1$ are coprime to $q^4 - q^2 + 1$, the only factor of $|G|$ that is divisible by p is $q^4 - q^2 + 1$.

By Theorem 4.3 in [105], ${}^3\mathrm{D}_4(q)$ has a maximal subgroup M isomorphic to $C_{q^4-q^2+1} : [4]$. Let $C \leq M$ be of index 4 in M and cyclic. Then C contains a Sylow p-subgroup P of G. In particular, P is cyclic. Since all Sylow p-subgroups are conjugate in G, P contains a conjugate Y of R. Then Y is a characteristic subgroup of C, and hence $\mathrm{N}_M(Y) \geq \mathrm{N}_M(C) = M$. Therefore, the fact that G is simple implies that $\mathrm{N}_G(Y) = M$, and the lemma follows.

With this knowledge of the subgroup structure of ${}^3\mathrm{D}_4(q)$, we can determine all transitive fixity-4 actions of this family of groups. This will be done in the next lemma. Afterwards in Lemma 3.45 the next family of exceptional groups of Lie type, $\mathrm{G}_2(q)$, is studied regarding their fixity-4 action.

Lemma 3.44

Let q be a prime power and let $G = {}^3\mathrm{D}_4(q)$. Suppose that G acts transitively on a set Ω. Then G acts with fixity 4 and such that the point stabilisers have order coprime to 6 if and only if the point stabilisers are cyclic of order $q^4 - q^2 + 1$.

Proof:

For the first direction suppose that G acts with fixity 4 and such that the point stabilisers have order coprime to 6 on a set Ω. Let $\alpha \in \Omega$ and let H be a non-trivial four-point stabiliser contained in G_α. Let $p \in \pi(H)$. Then by Lemma 3.42, p is a divisor of $\Phi_{12}(q) = q^4 - q^2 + 1$. Let $y \in H$ be of order p. By Lemma 3.43, $|\,\mathrm{N}_G(\langle y \rangle)| = 4 \cdot (q^4 - q^2 + 1)$. Since $|\,\mathrm{N}_G(\langle y \rangle) : \mathrm{N}_H(\langle y \rangle)| \leq 4$ and since $|H|$ is odd, $|\,\mathrm{N}_H(\langle y \rangle)| = q^4 - q^2 + 1$. Thus, $G_\alpha \geq H$ has order divisible by $q^4 - q^2 + 1$. An inspection of the list in Theorem 4.3 in [105] shows that $C_{q^4-q^2+1} : [4]$ is the only type of maximal subgroups of ${}^3\mathrm{D}_4(q)$ that has order divisible by $q^4 - q^2 + 1$. Since $|G_\alpha|$ is odd, this implies that G_α itself is cyclic of order $q^4 - q^2 + 1$, proving the first direction.

For the other direction let $U \leq G$ be cyclic of order $q^4 - q^2 + 1$. Then G acts transitively on G/U and $|U|$ is coprime to 6. Let $x \in U$ be non-trivial and let $y \in \langle x \rangle$ be of prime order. Then every point in G/U that is fixed by x is also fixed by y. Lemma 3.43 implies that $|\,\mathrm{N}_G(\langle y \rangle)| = 4 \cdot (q^4 - q^2 + 1)$. Therefore, by Lemma 2.10, y fixes exactly $\frac{|\,\mathrm{N}_G(\langle y \rangle)|}{|U|} = \frac{4 \cdot (q^4 - q^2 + 1)}{|U|} = \frac{4|U|}{|U|} = 4$ points. Thus, x can fix at most four points and G acts with fixity 4 on G/U. This finishes the proof.

Lemma 3.45

Let $q \geq 3$ be a prime power and $G = \mathrm{G}_2(q)$. Then G does not act transitively, with fixity 4, and such that the point stabilisers have order coprime to 6 on any set.

Proof:

By (4.25) in [105], $|G| = q^6(q^6-1)(q^2-1) = q^6 \cdot (q^2-1)^2 \cdot (q^2+q+1) \cdot (q^2-q+1)$ and every prime greater than 3 divides at most one of the factors.

Assume, for a contradiction, that G acts transitively, with fixity 4, and such that the point stabilisers have order coprime to 6 on a set Ω. Let $\alpha \in \Omega$ and let H be a non-trivial four-point stabiliser contained in G_α. Let $p \in \pi(H)$. Then $p \geq 5$, and by Lemma 3.42, p is a divisor of $\Phi_3(q) = q^2+q+1$ or $\Phi_6(q) = q^2-q+1$. Let $\varepsilon \in \{-1, 1\}$ be such that p divides $q^2 + \varepsilon q + 1$. Then H contains a Sylow p-subgroup P of G, and the analysis of the order of G yields that $|P|$ divides $q^2 + \varepsilon q + 1$. Thus, by Table 4.1 in [105], P lies in a maximal subgroup M of type $\mathrm{SL}_\varepsilon(3, q) : C_2$. Let $S \leq M$ be of index 2 in M and isomorphic to $\mathrm{SL}_\varepsilon(3, q)$. Then $P \leq S$, and hence Lemma 2.4 implies that $|\,\mathrm{N}_M(P)| = 2 \cdot |\,\mathrm{N}_S(P)|$.

As a next step, we will see that $|\,\mathrm{N}_S(P)|$ is divisible by 3. If $\mathrm{Z}(S) \neq 1$, then $|\,\mathrm{Z}(S)| = \gcd(3, q-\varepsilon) = 3$, and since $\mathrm{Z}(S) \leq \mathrm{N}_S(P)$, $|\,\mathrm{N}_S(P)|$ is divisible by 3. If $\mathrm{Z}(S) = 1$, then $S \cong \mathrm{PSL}_\varepsilon(3, q)$. By Theorem 6.5.3 in [43], S has a subgroup F such that $P \leq F$ and such that F is a Frobenius group with Frobenius kernel K of order $q^2 + \varepsilon q + 1$ and Frobenius complement of order 3. Since $p \geq 5$, $P \leq K$ or more precisely, since Frobenius kernels are nilpotent, $P = \mathrm{O}_p(K)$. Hence, $\mathrm{N}_F(P) = F$, and therefore $|\,\mathrm{N}_F(P)|$ is divisible by 3. Thus, $\mathrm{N}_S(P) \geq \mathrm{N}_F(P)$ has order divisible by 3.

As a consequence, the order of $\mathrm{N}_M(P)$ is divisible by 2 and 3. Since $|\,\mathrm{N}_G(P) : \mathrm{N}_H(P)| \leq 4$ and $|H|$ is coprime to 6, this is a contradiction. Thus, G cannot act transitively, with fixity 4, and such that the point stabilisers have order coprime to 6 on any set.

In Lemma 3.10, we have already seen that ${}^2\mathrm{G}_2(q)$ can act transitively, with fixity 4, and such that the point stabilisers have odd order. The assumption in Lemma 3.10 was that there exists an element of order 3 that fixes a point, whereas in Lemma 3.47, we suppose that the point stabilisers have order coprime to 6. We will see that under the latter hypothesis ${}^2\mathrm{G}_2(q)$ can act with fixity 4, too. However, beforehand we again need some more information about the subgroup structure of ${}^2\mathrm{G}_2(q)$.

Lemma 3.46

Let n be a positive integer, $q = 3^{2n+1}$, and $G = {}^2\mathrm{G}_2(q)$. Let p be an odd prime divisor of $q-1$ and let $y \in G$ be of order p. Then $|\,\mathrm{N}_G(\langle y \rangle)| = 2(q-1)$.

Proof:

By the information on page 137 in [105], the order of $|G| = (q^3+1)q^3(q-1) = q^3 \cdot (q-1) \cdot (q+1) \cdot (q^2-q+1)$, and every odd prime divides at most one of these factors. In particular, the order of a Sylow p-subgroup of G divides $q-1$.

Theorem 4.2 in [105] implies that ${}^2\mathrm{G}_2(q)$ has a maximal subgroup M of type $C_2 \times \mathrm{PSL}(2, q)$. Let C be cyclic of order 2 and let L be isomorphic to $\mathrm{PSL}(2, q)$ such that $C \times L \leq M$. Then L contains a Sylow p-subgroup P of G, because $|P|$ divides $q-1$. By Satz II 8.3 in [54], $\mathrm{N}_L(P)$ is a dihedral group of order $2 \cdot \frac{q-1}{2}$.

Thus, P is cyclic and for all non-trivial elements $b \in P$, $\mathrm{N}_M(\langle b\rangle) = C \times \mathrm{N}_L(\langle b\rangle)$ has order $2 \cdot 2 \cdot \frac{q-1}{2}$.

Let $a \in P$ be conjugate to y. Since $\langle a\rangle$ is a characteristic subgroup of P, $\mathrm{N}_G(\langle a\rangle) \geq \mathrm{N}_G(P) \geq \mathrm{N}_M(P)$. Thus, $|\,\mathrm{N}_G(\langle a\rangle)|$ is divisible by $2(q-1)$ and lies in a maximal subgroup of G because G is simple. An inspection of the maximal subgroups in Theorem 4.2 in [105] shows that, since the order of $[q^3] : C_{q-1}$ is not divisible by 4, all maximal subgroups of G containing $\mathrm{N}_G(\langle a\rangle)$ are conjugate to M. As a consequence, $|\,\mathrm{N}_G(\langle a\rangle)| = |\,\mathrm{N}_M(\langle a\rangle)| = 2 \cdot (q-1)$, and hence the lemma follows.

Lemma 3.47

Let n be a positive integer, $q = 3^{2n+1}$, and $G = {}^2\mathrm{G}_2(q)$. Suppose that G acts transitively on a set Ω. Then G acts with fixity 4 and such that the point stabilisers have order coprime to 6 if and only if the point stabilisers are cyclic of order $\frac{q-1}{2}$.

Proof:

For the first direction suppose that G acts with fixity 4 and such that the point stabilisers have order coprime to 6. Let $\alpha \in \Omega$, let H be a non-trivial four-point stabiliser contained in G_α, and let $p \in \pi(H)$. Then by Lemma 3.42, p is a divisor of $\Phi_1(q) = q-1$, $\Phi_2(q) = q+1$, or $\Phi_6(q) = q^2 - q + 1$.

First assume that p divides $q^2 - q + 1 = (q + \sqrt{3q} + 1)(q - \sqrt{3q} + 1)$. Then there exists $\varepsilon \in \{-1, 1\}$ such that p divides $q - \varepsilon\sqrt{3q} + 1$. By Theorem 4.2 in [105], G has a subgroup M of type $C_{q-\varepsilon\sqrt{3q}+1} : [6]$. Let $C \leq M$ be cyclic of order $q - \varepsilon\sqrt{3q} + 1$ and let $y \in C$ be of order p. Since $\langle y\rangle$ is a characteristic subgroup of C, $\mathrm{N}_M(\langle y\rangle) \geq \mathrm{N}_M(C) = M$. Hence, $\mathrm{N}_G(\langle y\rangle) \geq \mathrm{N}_M(\langle y\rangle) \geq M$ has order divisible by 6. Since H contains a Sylow p-subgroup of G, it contains an element z conjugate to y. As a consequence, $|\,\mathrm{N}_G(\langle z\rangle)|$ is divisible by 6, contradicting Lemma 3.22 (d).

Therefore instead assume, for a contradiction, that p divides $q+1$. By Theorem 4.2 in [105], ${}^2\mathrm{G}_2(q)$ has a subgroup A of type $E_4 \times D_{(q+1/2)}$. Let E be elementary abelian of order 4 and D a dihedral group of order $\frac{q+1}{2}$ such that $E \times D \leq A$. Let $y \in D$ be of order p. Then $\langle y\rangle$ is normal in D, and hence $\mathrm{N}_A(\langle y\rangle) = E \times D = A$. Therefore, $|\,\mathrm{N}_A(\langle y\rangle)|$ is divisible by 8, because $|D|$ is even. Since H contains a Sylow p-subgroup of G, it contains an element z conjugate to y. Thus, $\mathrm{N}_G(\langle z\rangle)$ is divisible by 8 and this is again a contradiction to Lemma 3.22 (d). As a consequence, p divides $q-1$.

Let $y \in H$ be of order p. By Lemma 3.46, $|\,\mathrm{N}_G(\langle y\rangle)| = 2(q-1)$. Since $|\,\mathrm{N}_G(\langle y\rangle) : \mathrm{N}_H(\langle y\rangle)| \leq 4$ and since H has odd order, $|H|$ is divisible by $\frac{q-1}{2}$. Therefore $G_\alpha \geq H$ is divisible by $\frac{q-1}{2}$, too. Since by Hauptsatz II 8.27 in [54] the only subgroups of $\mathrm{PSL}(2,q)$ with order coprime to 6 and divisible by $\frac{q-1}{2}$ are cyclic of order $\frac{q-1}{2}$, an inspection of the maximal subgroups of G in Theorem 4.2 in [105] reveals that G_α is cyclic of order $\frac{q-1}{2}$.

For the other direction, let $U \leq G$ be cyclic of order $\frac{q-1}{2}$. Then every transitive action of G with cyclic point stabilisers of order $\frac{q-1}{2}$ is equivalent to the action of G on G/U. Since $q \equiv 3^{2n+1} \equiv (-1)^{2n+1} \equiv -1 \mod 4$, $\frac{q-1}{2} = |U|$ is coprime to 6, and hence the point stabilisers in the action of G on G/U have order coprime to 6.

Let $x \in U$ and let $y \in \langle x \rangle$ be of prime order p. Then y fixes every point that is fixed by x. By Lemma 3.46, $|\,\mathrm{N}_G(\langle y \rangle)| = 2(q-1)$.

Since U is cyclic, Lemma 2.10 implies that the number of fixed points of y is $\frac{|\,\mathrm{N}_G(\langle y \rangle)|}{|U|} = \frac{2\cdot(q-1)}{|U|} = \frac{4|U|}{|U|} = 4$. Therefore there exists an element in $U \leq G$ with exactly four fixed points and all non-trivial elements have at most four fixed points. As a consequence, G acts with fixity 4 on G/U.

In a sequence of lemmas, we will see that none of the remaining simple exceptional groups of Lie type can act transitively and with fixity 4 in such a way that the point stabilisers have order coprime to 6.

Lemma 3.48
Let q be a prime power and $G = \mathrm{F}_4(q)$. Then G does not act transitively, with fixity 4, and such that the point stabilisers have order coprime to 6 on any set.

Proof:

Assume, for a contradiction, that G acts transitively, with fixity 4, and such that the point stabilisers have order coprime to 6 on a set Ω. Let $\alpha \in \Omega$ and let H be a non-trivial four-point stabiliser contained in G_α. Let $p \in \pi(H)$. Then by Lemma 3.42, p is a divisor of $\Phi_8(q) = q^4 + 1$ or $\Phi_{12}(q) = q^4 - q^2 + 1$.

First additionally assume, for a contradiction, that p divides $q^4 - q^2 + 1$. Then by Table 5.1 in [69], G has a subgroup M of type ${}^3\mathrm{D}_4(q).C_3$. Let $D \leq M$ be isomorphic to ${}^3\mathrm{D}_4(q)$ and let $P \in \mathrm{Syl}_p(D)$. By Lemma 2.4, $|\,\mathrm{N}_M(P)| = 3 \cdot |\,\mathrm{N}_D(P)|$, and hence $\mathrm{N}_G(P) \geq \mathrm{N}_M(P)$ has order divisible by 3. Since H contains a Sylow p-subgroup, it contains a subgroup Q conjugate to P, and hence $\mathrm{N}_G(Q)$ is divisible by 3, contradicting Lemma 3.22 (d). As a consequence, p divides $q^4 + 1$.

By Table 5.1 in [69], G has a subgroup M of type $C_{\gcd(2,q-1)}.\Omega(9,q)$. Let $P \in \mathrm{Syl}_p(M)$ and let Z be a normal subgroup of M of order $\gcd(2, q-1)$. Since $p \geq 5$, P acts coprimely on Z. Thus, by Lemma 2.3 (a), $\mathrm{N}_{M/Z}(PZ/Z) = \mathrm{N}_M(P)Z/Z$. Since $Z \leq \mathrm{N}_M(P)$, the order of $\mathrm{N}_G(P) \geq \mathrm{N}_M(P)$ is divisible by $|\,\mathrm{N}_{M/Z}(PZ/Z)| \cdot |Z|$.

Lemma 3.23 and (10-2) in [41] together imply that for all positive integers $l < 4$, p does not divide $q^{2l} - 1$. If q is even, then $M/Z \cong \Omega(9,q) \cong \mathrm{PSp}(8,q)$. Therefore, Lemma 3.32 shows that the order of a normaliser of a Sylow p-subgroup of $\mathrm{PSp}(8,q)$ is divisible by $\frac{(q^4+1)\cdot 2\cdot 4}{\gcd(2,q+1)} = (q^4+1)\cdot 8$. As a consequence, $|\,\mathrm{N}_{M/Z}(PZ/Z)|$ is divisible by 8 if q is even. If q is odd, then Lemma 3.36 shows that the order of a normaliser of a Sylow p-subgroup of $\Omega(9,q)$ is divisible by $(q^4+1)\cdot 4$. Therefore $|\,\mathrm{N}_{M/Z}(PZ/Z)|$ is divisible by 8 if q is odd. As a consequence, in both cases, $|\,\mathrm{N}_G(P)|$ is divisible by 8. Since H contains a Sylow p-subgroup, it has a subgroup Q conjugate to P. Thus, $|\,\mathrm{N}_G(Q)|$ is divisible by 8, contradicting Lemma 3.22 (d). This finial contradiction proves the lemma.

Lemma 3.49
Let n be a positive integer, let $q = 2^{2n+1}$, and let $G = {}^2\mathrm{F}_4(q)$ or $G = {}^2\mathrm{F}_4(2)'$. Then G does not act transitively, with fixity 4, and such that the point stabilisers have order coprime to 6 on any set.

Proof:
For the group ${}^2\mathrm{F}_4(2)'$, we can use the GAP package TomLib [74] and the program in Remark 2.22. Then the command `TestTom(TableOfMarks("2F4(2)'"),4);` shows that ${}^2\mathrm{F}_4(2)'$ does not act transitively and with fixity 4 on any set.

Assume, for a contradiction, that $G = {}^2\mathrm{F}_4(q)$ acts transitively, with fixity 4 and such that a point stabiliser has order coprime to 6 on a set Ω. Let $\alpha \in \Omega$ and let H denote a non-trivial four-point stabiliser contained in G_α. Let $p \in \pi(H)$. Then by Lemma 3.42, p is a divisor of $\Phi_6(q) = q^2 - q + 1$ or $\Phi_{12}(q) = q^4 - q^2 + 1$.

First additionally assume, for a contradiction, that p divides $q^2 - q + 1$. By the Main Theorem in [73], the group ${}^2\mathrm{F}_4(q)$ has a maximal subgroup M of type $\mathrm{SU}(3, q) : C_2$. Let $S \leq M$ be isomorphic to $\mathrm{SU}(3, q)$ and let $Y \in \mathrm{Syl}_p(S)$. Since $q + 1 \equiv 2^{2n+1} + 1 \equiv 0 \mod 3$, it follows that $\gcd(3, q + 1) = 3$, and hence $\mathrm{Z}(S)$ is cyclic of order 3. Therefore, $\mathrm{N}_S(Y)$ is divisible by 3. Since H contains a Sylow p-subgroup, it contains a subgroup Z conjugate to Y, and hence $|\,\mathrm{N}_G(Z)|$ is divisible by 3 contradicting Lemma 3.22 (d). Thus, p divides $q^4 - q^2 + 1$.

Since $q^4 - q^2 + 1 = (q^2 + q + 1 + \sqrt{2q}(q + 1))(q^2 + q + 1 - \sqrt{2q}(q + 1))$, there exists $\varepsilon \in \{-1, 1\}$ such that p divides $(q^2 + q + 1 + \varepsilon\sqrt{2q}(q + 1))$. Then the Main Theorem in [73] shows that G has a maximal subgroup M of type $C_{(q^2+q+1+\varepsilon\sqrt{2q}(q+1))} : [12]$. Let $C \trianglelefteq M$ be cyclic of order $(q^2+q+1+\varepsilon\sqrt{2q}(q+1))$ and let $Y \leq C$ be of order p. Then $\mathrm{N}_G(Y) \geq \mathrm{N}_M(Y) \geq \mathrm{N}_M(C) = M$, and hence $|\,\mathrm{N}_G(Y)|$ is divisible by 12. Since H contains a subgroup conjugate to Y, this contradicts Lemma 3.22 (d) and finishes the proof.

The groups $\mathrm{E}_6(q)$ and ${}^2\mathrm{E}_6(q)$ have a similar subgroup structure, as we will see in the next lemma. Therefore, we analyse for both families together whether these groups can act with fixity 4 and such that the point stabilisers have order coprime to 6 or not. However, first we collect some information about subgroups of $\mathrm{E}_6(q)$ and ${}^2\mathrm{E}_6(q)$ that will also be helpful for the analysis of other families of exceptional groups of Lie type.

Lemma 3.50
Let q be a prime power, let $\varepsilon \in \{-1, 1\}$, and let $G = \mathrm{E}_6^\varepsilon(q)$. Let $p \geq 5$ be a prime that divides neither $q^6 - 1$ nor $q^4 - 1$. Suppose that p divides one of the numbers $q^4 - q^2 + 1$ or $q^6 + \varepsilon q^3 + 1$, the number $q^4 + 1$, or the number $q^4 + \varepsilon q^3 + q^2 + \varepsilon q + 1$. Then there exists a p-subgroup Q of G such that $|\,\mathrm{N}_G(Q)|$ is divisible by 3, 8, or 5, respectively.

Proof:
First suppose that p divides $q^4 - q^2 + 1$ or $q^6 + \varepsilon q^3 + 1$. In both cases we will define subgroups N and M of G. By Table 5.1 in [69], G has a subgroup of type $({}^3\mathrm{D}_4(q) \times C_{q^2+\varepsilon q+1}).C_3$. If p divides $q^4 - q^2 + 1$, then we define M to be this subgroup and fix N as a normal subgroup of M of index 3. If p

divides $q^6 + \varepsilon q^3 + 1$, we set $e = \gcd(3, q-1)$ and by Table 5.1 in [69], G has a subgroup M of type $\mathrm{PSL}_\varepsilon(3, q^3).(C_e \times C_3)$. In this case, let $N \le M$ be isomorphic to $\mathrm{PSL}_\varepsilon(3, q^3)$. Then in both cases, p divides $|N|$ and $|M : N|$ is divisible by 3. Let $Q \in \mathrm{Syl}_p(N)$. By Lemma 2.4, it follows that $|\,\mathrm{N}_M(Q)|$ is divisible by 3. Thus, $\mathrm{N}_G(Q) \ge \mathrm{N}_M(Q)$ has order divisible by 3, as stated in the lemma.

Therefore, instead suppose that p divides $q^4 + 1$ or $q^4 + \varepsilon q^3 + q^2 + \varepsilon q + 1$. Let $h = \gcd(4, q-1)$. Then by Table 5.1 in [69], G has a subgroup M of type $C_h.(\mathrm{P}\Omega_\varepsilon(10, q) \times C_{(q-\varepsilon)/h})$. Let Y be an arbitrary p-subgroup of M and Z a normal subgroup of M of order h. Since $p \ge 5$, Y acts coprimely on Z. Thus, by Lemma 2.3 (a), it follows that $\mathrm{N}_{M/Z}(YZ/Z) = \mathrm{N}_M(Y)Z/Z$. As a consequence, the order of $\mathrm{N}_M(Y)$ is divisible by $|\,\mathrm{N}_{M/Z}(YZ/Z)|$. In particular, since M/Z has a subgroup isomorphic to $\mathrm{P}\Omega_\varepsilon(10, q)$, it holds that for every p-subgroup R of $\mathrm{P}\Omega_\varepsilon(10, q)$, there exists a p-subgroup Y of M such that the order of $|\,\mathrm{N}_G(Y)|$ is divisible by $|\,\mathrm{N}_{\mathrm{P}\Omega_\varepsilon(10,q)}(R)|$.

Suppose that the first alternative holds, that is p divides $q^4 + 1$. Then p divides $q^8 - 1$, and for all $l < 4$, the prime p does not divide $q^{2l} - 1$ because of the hypothesis that p divides neither $q^6 - 1$ nor $q^4 - 1$. Therefore, Lemma 3.38 is applicable and yields that there exists a p-subgroup R of $\mathrm{P}\Omega_\varepsilon(10, q)$ such that $|\,\mathrm{N}_{\mathrm{P}\Omega_\varepsilon(10,q)}(R)|$ is divisible by $\frac{(q^4+1)\cdot 2\cdot 4\cdot (q+\varepsilon)}{\gcd(4,q^5-\varepsilon)}$. If q is even, then $\frac{(q^4+1)\cdot 2\cdot 4\cdot (q+\varepsilon)}{\gcd(4,q^5-\varepsilon)} = (q^4+1)\cdot 2\cdot 4\cdot (q+\varepsilon)$ and this number is divisible by 8. If q is odd, then q^4+1 and $q+\varepsilon$ are both divisible by 2, and hence $\frac{(q^4+1)\cdot 2\cdot 4\cdot (q+\varepsilon)}{\gcd(4,q^5+\varepsilon)}$ is divisible by $\frac{2\cdot 2\cdot 4\cdot 2}{4} = 8$. Thus, in both cases, $|\,\mathrm{N}_{\mathrm{P}\Omega_\varepsilon(10,q)}(R)|$ is divisible by 8, and the lemma follows if p divides $q^4 + 1$.

Thus, instead suppose that p divides $q^4 + \varepsilon q^3 + q^2 + \varepsilon q + 1$. In particular, p divides $q^5 - \varepsilon$ and thus $q^{10} - 1$. If p divides $q^4 + 1$, then p also divides $q^8 - 1$ and thus $(q^{10} - 1) - (q^8 - 1) = q^8 \cdot (q^2 - 1)$. Since p divides neither $q^6 - 1$ nor $q^4 - 1$, this implies that for all $l < 5$, the prime p does not divide $q^{2l} - 1$. Therefore, Lemma 3.38 is applicable. Let $R \in \mathrm{Syl}_p(\mathrm{P}\Omega_\varepsilon(10, q))$. Then $|\,\mathrm{N}_{\mathrm{P}\Omega_\varepsilon(10,q)}(R)|$ is divisible by $\frac{5\cdot(q^5-\varepsilon)}{\gcd(4,q^5-\varepsilon)}$, which in turn is divisible by 5. As a consequence, the lemma follows.

Lemma 3.51

Let q be a prime power, let $\varepsilon \in \{-1, 1\}$, and let $G = \mathrm{E}_6^\varepsilon(q)$. Then G does not act transitively, with fixity 4, and such that the point stabilisers have order coprime to 6 on any set.

Proof:

We will prove the lemma by contradiction. Therefore, assume that there exists a set Ω such that G acts transitively, with fixity 4, and such that the point stabilisers have order coprime to 6 on this set. Let $\alpha \in \Omega$ and let H be a non-trivial four-point stabiliser contained in G_α. Let $p \in \pi(H)$. Since $\Phi_{10}(q) = \Phi_5(-q)$ and $\Phi_{18}(q) = \Phi_9(-q)$, by Lemma 3.42, p is a divisor of $\Phi_5(\varepsilon q) = q^4+\varepsilon q^3+q^2+\varepsilon q+1$, $\Phi_8(q) = q^4+1$, $\Phi_9(\varepsilon q) = q^6+\varepsilon q^3+1$, or $\Phi_{12}(q) = q^4-q^2+1$. Additionally, Lemma 3.42 implies that $|H|$ and q are coprime.

Assume, for a contradiction, that a prime divisor r of $|H|$ divides $q^6 - 1$ or $q^4 - 1$. Then the smallest positive integer k such that r divides $\Phi_k(q)$ is in $\{1, 2, 3, 4, 6\}$. Thus, (10-2) in [41] implies that the r-rank of G is at least 2. This contradicts Lemma 3.23. Therefore, $|H|$ and $q^6 - 1$ are corpime, as are $|H|$ and $q^4 - 1$. In particular, the hypotheses of Lemma 3.50 are fulfilled.

If p divides $q^4 - q^2 + 1$ or $q^6 + \varepsilon q^3 + 1$, then by Lemma 3.50, there exists a p-subgroup R such that $|\,\mathrm{N}_G(R)|$ is divisible by 3. Since H contains a Sylow p-subgroup, it contains a subgroup Q conjugate to R, and hence $|\,\mathrm{N}_G(Q)|$ is divisible by 3, contradicting Lemma 3.22 (d).

If p divides $q^4 + 1$, then Lemma 3.50 proves that there exists a p-subgroup R of G such that $|\,\mathrm{N}_G(R)|$ is divisible by 8. Similarly to the previous case, this contradicts Lemma 3.22 (d).

As a consequence, p divides $q^4 + \varepsilon q^3 + q^2 + \varepsilon q + 1$. Then Lemma 3.50 shows that there exists a p-subgroup R such that $|\,\mathrm{N}_G(R)|$ is divisible by 5. Since H contains a Sylow p-subgroup, it again contains a subgroup Q conjugate to R and hence $|\,\mathrm{N}_G(Q)|$ is divisible by 5. Thus, the fact that $|\,\mathrm{N}_G(Q) : \mathrm{N}_H(Q)| \leq 4$ implies that 5 divides $|H|$. Then Fermat's litte theorem yields that $q^4 - 1$ is divisible by 5, but as shown above $|H|$ and $q^4 - 1$ are coprime. This final contradiction finishes the proof.

Using the same strategy as before, we can gain a similar result for $\mathrm{E}_7(q)$. Again, we will first collect some information about the subgroup structure, before we use them to prove that $\mathrm{E}_7(q)$ cannot act transitively, with fixity 4, and such that the point stabilisers have order coprime to 6 on any set.

Lemma 3.52

Let q be a prime power and $G = \mathrm{E}_7(q)$. Let $\varepsilon \in \{-1, 1\}$ and let $p \geq 5$ be a prime that divides neither $q^6 - 1$ nor $q^4 - 1$. Suppose that p divides one of the numbers $q^4 - q^2 + 1$ or $q^6 + \varepsilon q^3 + 1$, the number $q^4 + 1$, the number $q^4 + \varepsilon q^3 + q^2 + \varepsilon q + 1$, or the number $q^6 + \varepsilon q^5 + q^4 + \varepsilon q^3 + q^2 + \varepsilon q + 1$. Then there exists a p-subgroup Q of G such that $|\,\mathrm{N}_G(Q)|$ is divisible by 3, 8, 5, or 7, respectively.

Proof:

First suppose that p divides one of the numbers $q^4 - q^2 + 1$, $q^4 + 1$, $q^4 + \varepsilon q^3 + q^2 + \varepsilon q + 1$, or $q^6 + \varepsilon q^3 + 1$. Let $e = \gcd(3, q - \varepsilon)$. Then by Table 5.1 in [69], G has a subgroup M of type $C_e.(\mathrm{E}_6^\varepsilon(q) \times C_{(q-\varepsilon)/e})$. Let P be a p-subgroup of M and Z a normal subgroup of M of order e. Since $p \geq 5$, it follows that P acts coprimely on Z. Thus, by Lemma 2.3 (a), $\mathrm{N}_{M/Z}(PZ/Z) = \mathrm{N}_M(P)Z/Z$. As a consequence, the order of $\mathrm{N}_M(P)$ is divisible by $|\,\mathrm{N}_{M/Z}(PZ/Z)|$. Since M/Z contains a subgroup isomorphic to $\mathrm{E}_6^\varepsilon(q)$, we can use Lemma 3.50 to derive the lemma in this case.

Therefore suppose instead that p divides $q^6 + \varepsilon q^5 + q^4 + \varepsilon q^3 + q^2 + \varepsilon q + 1$. Thus, p also divides $q^7 - \varepsilon$. Since, by assumption, p does not divide $q^5(q^2 - 1) = (q^7 - \varepsilon) - (q^5 - \varepsilon)$, p does not divide $q^5 - \varepsilon$. Let $f = \frac{\gcd(4, q-\varepsilon)}{\gcd(2, q-1)}$. Then by Table 5.1 in [69], G has a subgroup M of type $C_f.\,\mathrm{PSL}_\varepsilon(8, q)$. Let Z be a normal subgroup of M of order f. Since M/Z is isomorphic to $\mathrm{PSL}_\varepsilon(8,q)$ and since p divides neither q^6-1, $q^5-\varepsilon$, nor q^4-1, we can apply Lemma 3.27 and Lemma 3.30 for M/Z and get that there exists a p-subgroup R of M/Z such that the order of $\mathrm{N}_{M/Z}(R)$ is divisible by $\frac{(q^7-\varepsilon)\cdot 7}{\gcd(8,q-\varepsilon)}$, which in turn is divisible by 7. Let Q be a Sylow p-subgroup of the full pre-image of R in M. Since $|Z|$ and p are coprime, Lemma 2.3 (a) implies that $|\mathrm{N}_M(Q)|$ is divisible by $|\mathrm{N}_{M/Z}(QZ/Z)| = |\mathrm{N}_{M/Z}(R)|$ and hence by 7. This finishes the proof.

Lemma 3.53
Let q be a prime power and $G = \mathrm{E}_7(q)$. Then G does not act transitively, with fixity 4, and such that the point stabilisers have order coprime to 6 on any set.

Proof:
As usual, assume, for a contradiction, that G acts transitively, with fixity 4, and such that the point stabilisers have order coprime to 6 on a set Ω. Let $\alpha \in \Omega$, let H be a non-trivial four-point stabiliser contained in G_α, and let p be the smallest prime diving $|H|$. Then by Lemma 3.42, $|H|$ and q are coprime and there exists $k \in \{5,7,8,9,10,12,14,18\}$ such that $\Phi_k(q)$ is divisible by p.

Assume, for a contradiction, that p divides q^6-1 or q^4-1. Then the smallest positive integer k such that p divides $\Phi_k(q)$ is in $\{1,2,3,4,6\}$. Thus, (10-2) in [41] implies that the p-rank of G is at least 2. This contradicts Lemma 3.23. Therefore, we can apply Lemma 3.52. Furthermore, since p and q are coprime, Fermat's little theorem yields that $q^{p-1}-1$ is divisible by p. Therefore, $p > 7$.

If p divides $\Phi_{12}(q) = q^4-q^2+1$, $\Phi_9(q) = q^6+q^3+1$, or $\Phi_{18}(q) = q^6-q^3+1$, then Lemma 3.52 proves that there exists a p-subgroup R such that $|\mathrm{N}_G(R)|$ is divisible by 3. Since H contains a Sylow p-subgroup, it contains a subgroup Q conjugate to R. Hence, $|\mathrm{N}_G(Q)|$ is divisible by 3, contradicting Lemma 3.22 (d).

If p divides $\Phi_8(q) = q^4+1$, then by Lemma 3.52, there exists a p-subgroup R of G such that $\mathrm{N}_G(R)$ is divisible by 8. Since H contains a subgroup conjugate to R, this contradicts Lemma 3.22 (d).

Assume, for a contradiction, that there exists $\varepsilon \in \{-1,1\}$ such that p divides $\Phi_5(\varepsilon q) = q^4+\varepsilon q^3+q^2+\varepsilon q+1$ or $\Phi_7(\varepsilon q) = q^6+\varepsilon q^5+q^4+\varepsilon q^3+q^2+\varepsilon q+1$. We note that $\Phi_{10}(q) = \Phi_5(-q)$ and $\Phi_{14}(q) = \Phi_7(-q)$. Then Lemma 3.52 shows that there exists a p-subgroup R such that $|\mathrm{N}_G(R)|$ is divisible by $r \in \{5,7\}$. Since H contains a Sylow p-subgroup, it contains a subgroup Q conjugate to R. Then $|\mathrm{N}_G(Q)|$ is divisible by r, and the fact that $|\mathrm{N}_G(Q) : \mathrm{N}_H(Q)| \leq 4$ yields that $|H|$ is divisible by r, too. This is the final contradiction because $p > 7 \geq r$ is the smallest prime dividing $|H|$. This finial contradiction finishes the proof.

We turn to the last family of exceptional groups of Lie type. Since we do not need the information about the subgroup structure for any other group, it is directly included in the following lemma, in which we analyse the fixity-4 action of $\mathrm{E}_8(q)$.

Lemma 3.54
Let q be a prime power and let $G = \mathrm{E}_8(q)$. Then G does not act transitively, with fixity 4, and such that the point stabilisers have order coprime to 6 on any set.

Proof:

Assume, for a contradiction, that there exists a set Ω such that G acts transitively, with fixity 4, and such that the point stabilisers have order coprime to 6 on it. Let $\alpha \in \Omega$, let H be a non-trivial four-point stabiliser contained in G_α, and let p be the smallest prime dividing $|H|$. Then by Lemma 3.42, there exists $k \in \{7, 9, 14, 15, 18, 20, 24, 30\}$ such that $\Phi_k(q)$ is divisible by p. Additionally, the lemma shows that $|H|$ and q are coprime.

Assume, for a contradiction, that p divides $q^6 - 1$ or $q^4 - 1$. Then by (10-2) in [41], the p-rank of G is at least 4. This contradicts Lemma 3.23. Thus, the hypotheses of Lemma 3.52 are fulfilled. Furthermore, since p and q are coprime, Fermat's little theorem yields that $q^{p-1} - 1$ is divisible by p. Therefore, $p > 7$.

Assume, for a contradiction, that p divides $\Phi_9(q) = q^6 + q^3 + 1$, $\Phi_{18}(q) = q^6 - q^3 + 1$, $\Phi_7(q) = q^6 + q^5 + q^4 + q^3 + q^2 + q + 1$, or $\Phi_{14}(q) = q^6 - q^5 + q^4 - q^3 + q^2 - q + 1$. Let $d = \gcd(2, q-1)$. Then by Table 5.1 in [69], $\mathrm{E}_8(q)$ has a subgroup M of type $C_d.(\mathrm{PSL}(2,q) \times \mathrm{E}_7(q))$. Let Z be a normal subgroup of M of order d. Since M/Z contains a subgroup isomorphic to $\mathrm{E}_7(q)$, Lemma 3.52 yields that M/Z contains a p-subgroup R such that $|\mathrm{N}_{M/Z}(R)|$ is divisible by 3 or 7. Let P be a Sylow p-subgroup of the full pre-image of R in M. Then Lemma 2.3 (a) implies that $|\mathrm{N}_M(P)|$ is divisible by $|\mathrm{N}_{M/Z}(PZ/Z)| = |\mathrm{N}_{M/Z}(R)|$, because P acts corpimely on Z. Since H contains a Sylow p-subgroup of G, it contains a subgroup Q conjugate to P. If p divides $\Phi_9(q)$ or $\Phi_{18}(q)$, then $|\mathrm{N}_M(Q)|$ is divisible by 3, contradicting Lemma 3.22 (d). If p divides $\Phi_7(q)$ or $\Phi_{14}(q)$, then $|\mathrm{N}_M(Q)|$ is divisible by 7. Therefore, the fact that $|\mathrm{N}_G(Q) : \mathrm{N}_H(Q)| \le 4$ implies that $|H|$ is divisible by 7 contradicting the facts that p is the smallest prime divisor of $|H|$ and that $p > 7$.

Next assume, for a contradiction, that p divides $\Phi_{15}(q) = q^8 - q^7 + q^5 - q^4 + q^3 - q + 1$ or $\Phi_{30}(q) = q^8 + q^7 - q^5 - q^4 - q^3 - q + 1$. Let $T \le G$ be as in [69]. Then by Table 5.2 in [69], $|\mathrm{N}_G(T)|$ is divisible by 30. Let $P \in \mathrm{Syl}_p(T)$. Then Lemma 2.4 implies that $|\mathrm{N}_G(P)|$ is divisible by 30. Since H contains a Sylow p-subgroup of G, it has a subgroup Q conjugate to P. Then $|\mathrm{N}_G(Q)|$ is divisible by 30 contradicting Lemma 3.22 (d).

Assume, for a contradiction, that p divides $\Phi_{24}(q) = q^8 - q^4 + 1$. By Table 5.1 in [69], there exists a subgroup M of G such that M is of type $\mathrm{PSU}(3, q^4).[8]$. Let $U \le M$ be isomorphic to $\mathrm{PSU}(3, q^4)$ and $P \in \mathrm{Syl}_p(U)$. Then by Lemma 2.4, $|\mathrm{N}_M(P)| = 8 \cdot |\mathrm{N}_U(P)|$, and hence $\mathrm{N}_G(P) \ge \mathrm{N}_M(P)$ has order divisible by 8. Similarly to the previous case, H contains a subgroup conjugate to Q and this contradicts Lemma 3.22 (d).

Thus, the only remaining case is that p divides $\Phi_{20}(q) = q^8 - q^6 + q^4 - q^2 + 1$. By Table 5.1 in [69], $\mathrm{E}_8(q)$ has a subgroup M of type $\mathrm{SU}(5, q^2).C_4$. Let $U \le M$ be isomorphic to $\mathrm{SU}(5, q^2)$ and let $P \in \mathrm{Syl}_p(U)$. If $\mathrm{Z}(U) \ne 1$, then $|\mathrm{Z}(U)| = \gcd(5, q^2 + 1) = 5$, and hence $\mathrm{N}_U(P) \ge \mathrm{Z}(U)$ has order divisible by 5. If $\mathrm{Z}(U) = 1$, then $U \cong \mathrm{PSU}(5, q^2)$ and if p divides $q^6 + 1$ or $q^8 - 1$, then (10-2) in [41] implies that G has p-rank at least 2, contradicting Lemma 3.23. As a consequence, if $\mathrm{Z}(U) = 1$, then the hypotheses of Lemma 3.30 are fulfilled, and hence $|\,\mathrm{N}_U(P)|$ is divisible by $\frac{(q^{10}+1)\cdot 5}{(q^2+1)\cdot \gcd(5,q^2+1)} = 5 \cdot \frac{q^{10}+1}{(q^2+1)}$. Therefore, in both cases $|\,\mathrm{N}_U(P)|$ is divisible by 5. Since H contains a Sylow p-subgroup, it contains a subgroup Q conjugate to P, and since $|\,\mathrm{N}_G(Q) : \mathrm{N}_H(Q)| \leq 4$, it follows that $|H|$ is divisible by 5. Since $p > 7$ is the smallest prime divisor of $|H|$, this is the final contradiction proving the lemma.

We summarise our results about exceptional groups of Lie type.

Lemma 3.55
Let G be a simple exceptional group of Lie type such that for all 2-power q, G is not isomorphic to $\mathrm{Sz}(q)$. Suppose that G acts transitively on a set Ω. Then G acts with fixity 4 and such that the point stabilisers have order coprime to 6 if and only if there exists a prime power q such that G is isomorphic to ${}^3\mathrm{D}_4(q)$ and the point stabilisers are cyclic of order $q^4 - q^2 + 1$ or there exists a 3-power $q \geq 27$ such that G is isomorphic to ${}^2\mathrm{G}_2(q)$ and the point stabiliser are cyclic of order $\frac{q-1}{2}$.

Proof:
Let q be a prime power. If $G = {}^3\mathrm{D}_4(q)$, then Lemma 3.44 implies the correctness of the lemma in this case. If $G = \mathrm{G}_2(q)$, then $q \geq 3$ (see p. 3 in [105]), and hence Lemma 3.45 shows that G does not act transitively, with fixity 4, and such that the point stabilisers have order coprime to 6 on any set. If n is a positive integer and $G = {}^2\mathrm{G}_2(3^{2n+1})$, then Lemma 3.47 shows that the lemma holds in this case. If $G = \mathrm{F}_4(q)$, then Lemma 3.48 implies that G does not act transitively, with fixity 4, and such that the point stabilisers have order coprime to 6. If $G = {}^2\mathrm{F}_4(q)$ or $G = {}^2\mathrm{F}_4(2)'$, then we use Lemma 3.49. For $\mathrm{E}_6(q)$ and ${}^2\mathrm{E}_6(q)$, Lemma 3.51 implies the statement in these cases. If $G = \mathrm{E}_7(q)$, then by Lemma 3.53, G does not act transitively, with fixity 4, and such that the point stabilisers have order coprime to 6. Finally for $G = \mathrm{E}_8(q)$, we use Lemma 3.54 to derive the statement of the lemma in this case.

Since G is not isomorphic to $\mathrm{Sz}(q) = {}^2\mathrm{B}_2(q)$ the classification theorem of all finite simple groups (see p. 3 in [105]) shows that we dealt with all simple exceptional groups of Lie type, and hence the lemma holds.

3.4 Classification Theorem

Putting together all results of this chapter, we get the following theorem, the classification of all finite simple groups that act transitively and with fixity 4.

Theorem 3.56
Let G be a finite simple group acting transitively on a set Ω. Then G acts with fixity 4 if and only if G is isomorphic to one of the groups in Table 3.2 (under the mentioned conditions) and the point stabilisers are as specified.

group G	condition, remark (q prime power)	structure of point stabilisers
$\mathcal{A}_6$	$\cong \mathrm{PSL}(2,9)$	C_2, $\mathcal{S}_3$, E_9, D_{10}, $E_9 : C_2$
$\mathcal{A}_7$		C_5, $\mathcal{A}_6$
$\mathrm{PSL}(2,7)$	$\cong \mathrm{PSL}(3,2)$	C_2, $\mathcal{S}_3$
$\mathrm{PSL}(2,8)$		C_2, $\mathcal{S}_3$, D_{14}, D_{18}
$\mathrm{PSL}(2,11)$		C_3, $\mathcal{A}_4$
$\mathrm{PSL}(2,13)$		C_3, $C_{13} : C_3$, $\mathcal{A}_4$
$\mathrm{PSL}(2,q)$	$q \geq 17$, $q \equiv 1 \mod 4$	$C_{\frac{q-1}{4}}$, $E_q : C_{\frac{q-1}{4}}$
$\mathrm{PSL}(2,q)$	$q \geq 19$, $q \equiv -1 \mod 4$	$C_{\frac{q+1}{4}}$
$\mathrm{PSU}(3,3)$		$[3^3].C_8$
$\mathrm{PSU}(4,3)$		C_5
$\mathrm{PSp}(4,q)$	$q \geq 3$, $\mathrm{PSp}(4,3) \cong \mathrm{PSU}(4,2)$	$C_{\frac{q^2+1}{\gcd(2,q+1)}}$
$\mathrm{P\Omega}_-(8,q)$		$C_{\frac{q^4+1}{\gcd(2,q+1)}}$
${}^3\mathrm{D}_4(q)$		$C_{q^4-q^2+1}$
$\mathrm{Sz}(q)$	$q = 2^{2n+1}$, n positive integer	$C_{q+\sqrt{2q}+1}$, $C_{q-\sqrt{2q}+1}$
${}^2\mathrm{G}_2(q)$	$q = 3^{2n+1}$, n positive integer	$C_{\frac{q-1}{2}}$, $[q^3].C_{\frac{q-1}{2}}$
M_{11}		C_5, $C_{11} : C_5$, $\mathrm{PSL}(2,11)$
M_{12}		M_{11}
M_{22}		C_5, $C_{11} : C_5$
J_1		C_{15}

Table 3.2: Classification of Finite Simple Groups Acting with Fixity 4

Proof:

If there exists a prime power q such that G is isomorphic to one of the groups $\mathrm{PSL}(2,q)$, $\mathrm{PSL}(3,q)$, $\mathrm{PSU}(3,q)$, or $\mathrm{Sz}(q)$, then Lemma 3.2, Lemma 3.4, and Lemma 3.5 prove the correctness of the theorem. If $G \cong \mathcal{A}_7$, then making use of the GAP package TomLib [74] through the algorithm in Remark 2.22, the answer to the command `TestTom(TableOfMarks("A7"),4);` shows that a transitive action of $\mathcal{A}_7$ is a fixity-4 action if and only if the point stabilisers are isomorphic to C_5 or $\mathcal{A}_6$. If $G = \mathrm{M}_{11}$, then again using the GAP package TomLib [74] through the algorithm in Remark 2.22, `TestTom(TableOfMarks("M11"),4);` gives that the transitive actions of M_{11} are fixity-4 actions if and only if the point stabilisers are isomorphic to C_5, $C_{11} : C_5$, or $\mathrm{PSL}(2,11)$. Therefore from now on suppose that G is none of these groups.

For the first direction of the theorem, additionally suppose that G acts with fixity 4 on a set Ω. Then Lemma 3.1 is applicable. We go through the cases. Let $\alpha \in \Omega$.

First suppose that $|G_\alpha|$ has even order. Then case (1), (2), or (3) holds. If case (1) holds, then Satz 1 in [11] shows that there exists a 2-power $q \geq 4$ such that G is isomorphic to $\mathrm{PSL}(2,q)$, $\mathrm{Sz}(q)$, or $\mathrm{PSU}(3,q)$. If case (2) holds, then Theorem 2 in [45] and the Third Main Theorem in [1] imply that G is $\mathrm{PSL}(2,q)$, $\mathcal{A}_7$, $\mathrm{PSL}(3,q)$, M_{11}, or $\mathrm{PSU}(3,q)$ for a suitable prime power q. We looked at all of these groups earlier. As a consequence, the theorem is true in cases (1) and (2) of Lemma 3.1. If case (3) holds, then an involution of G fixes four points and Satz 2.41 in [89] gives all possibilities for G and G_α, all of them are stated in the theorem.

Therefore, suppose instead that $|G_\alpha|$ is odd and divisible by 3. Then case (4) of Lemma 3.1 holds. Hence, one of the cases (a)-(d) occurs. In case (a), Lemma 3.11 shows that G is one of the groups in Table 3.2 and G_α is as stated. Lemma 3.14 gives the answer in case (b) and Lemma 3.17 in case (c). For the remaining case (d), Lemma 3.21 proves that there are no further examples.

Thus, now suppose that the last remaining case, that $|G_\alpha|$ is coprime to 6, holds. Since G is non-abelian, by the classification of finite simple groups (see p. 3 in [105]), G is an alternating group, a classical group of Lie type, an exceptional group of Lie type, or one of 26 sporadic simple groups. If G is an alternating group, then Lemma 3.24 implies that $G = \mathcal{A}_7$ and G_α is cyclic of order 5. If G is a classical group of Lie type, then Lemma 3.41 gives a list of groups that act transitively, with fixity 4, and such that the point stabilisers have order coprime to 6 together with the structures of their point stabilisers, all of them are stated in Table 3.2. For the exceptional groups of Lie type, Lemma 3.55 gives the answer, and Lemma 3.25 deals with the sporadic simple groups.

For the other direction, we go through the given possibilities for G and show in each case that they indeed exhibit a fixity-4 action on G/G_α. The first group not already dealt with in the beginning of this proof is $\mathrm{PSU}(4,3)$. The GAP command `TestTom(TableOfMarks("U4(3)"),4);` using the GAP package Tom-

Lib [74] through the algorithm in Remark 2.22 gives that $\mathrm{PSU}(4,3)$ acts with fixity 4 if G_α is cyclic of order 5. If $G = \mathrm{PSp}(4,q)$, then Lemma 3.34 proves that $\mathrm{PSp}(4,q)$ acts with fixity 4 on G/U, where U is a cyclic subgroup of G of order $\frac{q^2+1}{\gcd(2,q+1)}$. If $G = \mathrm{P}\Omega_-(8,q)$, then Lemma 3.40 implies that G acts with fixity 4 and cyclic point stabilisers of order $\frac{q^4+1}{\gcd(2,q+1)}$. For $G = {}^3\mathrm{D}_4(q)$, we have seen in Lemma 3.44 that G acts as described. If $G = {}^2\mathrm{G}(q)$, there are two possible point stabiliser structures. If G_α is cyclic, then Lemma 3.47 shows that G indeed acts with fixity 4 on G/G_α. For the other possibility for G_α, Lemma 3.10 implies that ${}^2\mathrm{G}_2(q)$ has a fixity-4 action in this case. For the remaining groups, again making use of the GAP package TomLib [74] through the algorithm stated in Remark 2.22, the answer to the GAP command `List(["M12","M22","J1"],x->TestTom(TableOfMarks(x),4));` implies the statement in the theorem.

4 Quasi-Simple Groups

Before we look into the situation for quasi-simple groups acting with fixity 2, 3, or 4, we analyse the behaviour of groups with non-trivial centre more generally. As we will see in the next lemma, we can describe the action of the factor group of a finite group G modulo a subgroup $Z \leq \mathrm{Z}(G)$ on the set of its Z-orbits in terms of the fixity of G, if the set acted on is big enough. In the context of quasi-simple groups, we can use this lemma to reduce to a situation for simple groups, and then we can use the information we collected in the previous chapter. However, the lemma is useful for reduction in general, because it can be used for all finite groups with non-trivial centre.

Lemma 4.1
Let k be a positive integer and G a finite group acting transitively and with fixity k on a set Ω. Let $Z \leq \mathrm{Z}(G)$. Suppose that $|\Omega| > k \cdot |Z|$. Then G/Z acts transitively, non-regularly, and with fixity at most k on the set $\bar{\Omega} := \{\alpha^Z \mid \alpha \in \Omega\}$, which contains $\frac{|\Omega|}{|Z|}$ elements. Furthermore, the point stabilisers of G in its action on Ω have the same order as the point stabilisers of G/Z in its action on $\bar{\Omega}$.

Proof:
Let $\bar{\Omega} = \{\alpha^Z \mid \alpha \in \Omega\}$ and let $\bar{G} = G/Z$. Since G acts transitively on Ω, $\bar{G}$ acts transitively on $\bar{\Omega}$. By Lemma 2.17, it follows that $|\mathrm{Z}(G)|$ is a divisor of k, hence $|Z|$ is also a divisor of k, and by Lemma 2.16, every non-trivial element of $\mathrm{Z}(G)$ acts fixed-point-freely on Ω. Hence, for all $\alpha \in \Omega$, $|\alpha^Z| = |Z|$. Therefore $|\bar{\Omega}| = \frac{|\Omega|}{|Z|}$.

Let $\beta \in \Omega$. Since k is at least 1, there exists a non-trivial element $b \in G_\beta$. Thus, the fact that $Z_\beta = 1$ implies that $b \in G_\beta \setminus Z$. Then $(\beta^Z)^{Zb} = (\beta^b)^Z = \beta^Z$, and hence Zb fixes a point in $\bar{\Omega}$. Therefore $\bar{G}$ does not act regularly on $\bar{\Omega}$.

Let $g \in G$ be such that Zg has more than k fixed points in $\bar{\Omega}$. Let $\bar{g} = Zg$ and let $\bar{\omega}_1, \bar{\omega}_2, \dots, \bar{\omega}_{k+1}$ be distinct fixed points of $\bar{g}$ in $\bar{\Omega}$. Let $\omega_1, \omega_2, \dots, \omega_{k+1} \in \Omega$ be such that $\omega_i \in \bar{\omega}_i$ for all $i \in \{1, 2, \dots, k+1\}$. Then for every $i \in \{1, 2, \dots, k+1\}$, it follows that ${\omega_i}^g \in \bar{\omega}_i^g = ({\omega_i}^Z)^g = ({\omega_i}^Z)^{Zg} = \bar{\omega}_i^{\bar{g}} = \bar{\omega}_i = {\omega_i}^Z$. Hence for every $i \in \{1, \dots, k+1\}$, there exists an element $z_i \in Z$ such that ${\omega_i}^g = {\omega_i}^{z_i}$. Since $\{\omega_1, \omega_2, \dots, \omega_{k+1}\}$ has size $k+1$, and the order of Z divides $|\mathrm{Z}(G)|$, which in turn divides k, there is an index set $I \subseteq \{1, 2, \dots, k+1\}$ of size at least $\frac{k}{|Z|} + 1$ and an element $z \in Z$ such that $z = z_j$ for every $j \in I$. Then for all $j \in I$, it follows that ${\omega_j}^{gz^{-1}} = ({\omega_j}^g)^{{z_j}^{-1}} = ({\omega_j}^{z_j})^{{z_j}^{-1}} = \omega_j$. For all $c \in Z$, $({\omega_j}^c)^{gz^{-1}} = {\omega_j}^{cgz^{-1}} = {\omega_j}^{gz^{-1}c} = {\omega_j}^c$. Thus, every element in $\bar{\omega}_j = {\omega_j}^Z$ is fixed by gz^{-1}. Therefore, gz^{-1} has $|Z| \cdot |I| \geq |Z| \cdot (\frac{k}{|Z|} + 1) = k + |Z| > k$ fixed points in Ω. Since G acts with fixity k on Ω, $gz^{-1} = 1_G$, and hence $g \in Z$, thus $\bar{g} = Zg = Z$. It follows that every non-trivial element in $\bar{G}$ can have at most k fixed points in $\bar{\Omega}$, hence $\bar{G}$ acts with fixity at most k on $\bar{\Omega}$.

Let U be a point stabiliser of G/Z in its action on $\bar{\Omega}$. Since $|\Omega| = |G : G_\beta|$ and $\frac{|\Omega|}{|Z|} = |\bar{\Omega}| = |G/Z : U|$, we see that $|U| = \frac{|G|}{|\Omega|} = |G_\beta|$, proving the last part of the lemma.

Even though the proof illustrates the situation in some detail, the fixity with which the factor group acts is not determined exactly. This is unsurprising in light of the next example, because different fixities of the factor group can happen for the same group G acting with the same fixity, but permitting different actions.

Example 4.2
Let G be a finite group with a cyclic normal subgroup C of order 2 and a normal subgroup S isomorphic to $\mathrm{PSL}(3,2)$ such that $G = C \times S$. Then $\mathrm{Z}(G) = C$.

In the Small Groups Library [13] of GAP [36], the Group G is identified by the tuple `[336,209]`. Thus, using the algorithm in Remark 2.22, the GAP command `TestTom(TableOfMarks(SmallGroup([336,209])),4);` gives all fixity-4 actions of G. The possible sizes for Ω are 112, 84, 28, 16, and 14. Therefore by Lemma 4.1, $G/\mathrm{Z}(G) \cong \mathrm{PSL}(3,2)$ acts with fixity at most 4 on a set $\bar{\Omega}$ that can have size 56, 42, 14, 8, or 7. Since $\mathrm{PSL}(3,2)$ is simple, it cannot be a Frobenius group, and hence does not act with fixity 1. Again using the algorithm in Remark 2.22, the following three GAP commands determine all fixity-2, fixity-3, and fixity-4 actions of G/Z.

```
TestTom(TableOfMarks(PSL(3,2)),2);
TestTom(TableOfMarks(PSL(3,2)),3);
TestTom(TableOfMarks(PSL(3,2)),4);
```

Since the only fixity-4 actions of $\mathrm{PSL}(3,2)$ are on sets of sizes 84 and 28, G/Z cannot act with fixity 4 on $\bar{\Omega}$. Among the fixity-3 actions of $\mathrm{PSL}(3,2)$ there is one on a set of size 7, but none on sets of sizes 56, 42, 14, or 8. Thus, G/Z acts with fixity 2 on $\bar{\Omega}$ when $\bar{\Omega} \in \{56, 42, 14, 8\}$. On the other hand, $\mathrm{PSL}(3,2)$ does not act with fixity 2 on a set of size 7, and therefore G/Z must act with fixity 3 on $\bar{\Omega}$ when $|\bar{\Omega}| = 7$.

In particular, the size of Ω, and therefore the concrete action of G, influences the fixity with which G/Z is acting on $\bar{\Omega}$.

4.1 Fixity 2 and 3

For fixity 2 and 3, we will see in the next lemma that there do not exists quasi-simple, non-simple groups that act transitively and with one of these fixities. In [71], where groups acting with fixity 2 are analysed in general, the authors have also proven that all quasi-simple groups acting with fixity 2 are simple. In [72], however, there is no such proof. Nevertheless, the authors show many results about groups acting with fixity 3, that will be helpful in the following proof.

Lemma 4.3
Let E be a quasi-simple group acting transitively and with fixity 2 or 3 on a set Ω of size at least 4. Then E is simple.

Proof:
If E acts with fixity 2 on Ω, then, by Theorem 5.1 in [71], the quasi-simple group E is in fact simple. Therefore suppose that E acts with fixity 3 on Ω.

Assume for a contradiction that E is not simple. Then $\mathrm{Z}(E) \neq 1$. Hence by Lemma 2.17 the order of the centre divides 3. Therefore $|\mathrm{Z}(E)| = 3$. Let $\bar{E} = E/\mathrm{Z}(E)$.

We first exclude some special possibilities for $\bar{E}$. If $\bar{E}$ is $\mathcal{A}_6 \cong \mathrm{PSL}(2,9)$, $\mathcal{A}_7$, $\mathrm{PSL}(3,4)$, or M_{22}, then in each case there is, up to isomorphism, only one group that E can be by the information on pages 4, 10, 23, and 39 in [28]. Making use of the GAP package AtlasRep [104] and the GAP function in Remark 2.22, the following commands give an empty list as output, indicating that E cannot act with fixity 3 on any set if $\bar{E}$ is $\mathcal{A}_6$, $\mathrm{PSL}(3,4)$, $\mathcal{A}_7$, or M_{22}.

```
TestTom(TableOfMarks(AtlasGroup("3.a6")),3);
TestTom(TableOfMarks(AtlasGroup("3.a7")),3);
TestTom(TableOfMarks(AtlasGroup("3.L3(4)")),3);
TestTom(TableOfMarks(AtlasGroup("3.m22")),3);
```

As a consequence, whenever we reach a situation in which $\bar{E}$ is one of the groups $\mathcal{A}_6 \cong \mathrm{PSL}(2,9)$, $\mathcal{A}_7$, $\mathrm{PSL}(3,4)$, or M_{22}, then we immediately get a contradiction.

We now look at a different special situation and exclude some possibilities for the size of $|\Omega|$. All transitive groups up to degree 30 are listed in the Transitive Groups Library [52] of GAP [36]. Thus, we can use GAP to generate a list of all groups that are not simple, but quasi-simple, and that act transitively and with fixity 3 on a set of size at most 9. Since E contains a non-trivial element with three fixed points, Ω has at least five elements, and since no non-trivial element in E fixes more than three points, all four-point stabilisers are trivial. Thus, making use of the algorithm in Remark 2.18, we can formulate the following GAP code.

```
AllTransitiveGroups(NrMovedPoints, [5..9], z->(not IsSimple(z))
↪  and IsPerfect(z) and IsSimple(z/Center(z)), true,
↪  x->IsTrivial(Stabilizer(x,[1..4],OnTuples)), true,
↪  y->TestFixity(y, MovedPoints(y), 3), true);
```

The answer is an empty list. This implies that there does not exists a non-simple, quasi-simple group that acts transitively and with fixity 3, when $|\Omega| \leq 9$.

It follows that $|\Omega| > 9$. Then Lemma 4.1 shows that $\bar{E}$ acts with fixity 1, 2, or 3 on a set of size $\frac{|\Omega|}{3}$. Since $\bar{E}$ is simple, it cannot be a Frobenius group, hence it must act with fixity 2 or 3.

Assume for a contradiction that $\bar{E}$ acts with fixity 2. Then by Theorem 1.2 in [71], there exists a prime power q such that $\bar{E}$ is isomorphic to $\mathrm{PSL}(3,4)$, $\mathrm{PSL}(2,q)$, or $\mathrm{Sz}(q)$. Given that the Schur multiplier has to be divisible by 3, a consultation of Table 4.1 in [39] reduces the options for $\bar{E}$ to $\mathrm{PSL}(2,9)$ and $\mathrm{PSL}(3,4)$. As we noticed earlier, this gives a contradiction. Therefore $\bar{E}$ cannot act with fixity 2 and hence must act with fixity 3.

Now Theorem 1.1 in [72] gives a list of all finite simple groups that act transitively, faithfully, and with fixity 3. Hence, there exists a prime power q such that $\bar{E}$ is isomorphic to $\mathcal{A}_5, \mathcal{A}_6, \mathcal{A}_7, \mathcal{A}_8$, $\mathrm{PSL}(2,7), \mathrm{PSL}(2,11)$, $\mathrm{PSL}(3,q)$, $\mathrm{PSU}(3,q)$, $\mathrm{PSL}(4,3)$, $\mathrm{PSU}(4,3)$, $\mathrm{PSL}(4,5)$, M_{11}, or M_{22}. Again, using the information that the Schur multiplier is divisible by 3, we see by Table 4.1 in [39] that the only options for $\bar{E}$ are $\mathcal{A}_6, \mathcal{A}_7$, $\mathrm{PSL}(3,q)$ when $q-1$ is divisible by 3, $\mathrm{PSU}(3,q)$ when $q+1$ is divisible by 3, $\mathrm{PSU}(4,3)$, and M_{22}. As discussed earlier, $\bar{E}$ can be neither $\mathcal{A}_6$, $\mathcal{A}_7$, nor M_{22}.

Assume that $\bar{E}$ is isomorphic to $\mathrm{PSU}(4,3)$. Then the Schur multiplier is 48 by Table 4.1 in [39]. Let $\alpha \in \Omega$. Theorem 1.1 (ii) (d) in [72] specifies that the order of a point stabiliser of $\bar{E}$ is 7. Thus, by Lemma 4.1, $|E_\alpha| = 7$. Then by Lemma 2.14 (c) in [72], E_α is a Sylow 7-subgroup of E. Thus, the GAP command `Order(Normalizer(PSU(4,3),SylowSubgroup(PSU(4,3),7)));` gives that the order of $\mathrm{N}_{\bar{E}}(E_\alpha \mathrm{Z}(E)/\mathrm{Z}(E))$ is 21. Therefore, since $\mathrm{N}_{\bar{E}}(E_\alpha \mathrm{Z}(E)/\mathrm{Z}(E)) = \mathrm{N}_E(E_\alpha)\mathrm{Z}(E)/\mathrm{Z}(E)$, by Lemma 2.3 (a), it follows that $|\mathrm{N}_E(E_\alpha)| = 63$ and hence $|\mathrm{N}_E(E_\alpha) : \mathrm{N}_{E_\alpha}(E_\alpha)| = |\mathrm{N}_E(E_\alpha) : E_\alpha| = 9$, contradicting Lemma 2.13. As a consequence, $\bar{E}$ cannot be isomorphic to $\mathrm{PSU}(4,3)$.

Hence, there exists a prime power q and $\varepsilon \in \{-1,1\}$ such that $\bar{E}$ is isomorphic to $\mathrm{PSL}_\varepsilon(3,q)$ and 3 divides $(q-\varepsilon)$. Then E is $\mathrm{SL}_\varepsilon(3,q)$. Let $\alpha \in \Omega$. By Theorem 1.1 (ii) in [72], the point stabilisers of $\bar{E}$ have order coprime to 6. Then by Lemma 4.1, $|E_\alpha|$ is coprime to 6, too. We look at the cases of Theorem 1.3 in [72]. Since E is quasi-simple with $|\mathrm{Z}(E)| = 3$, it does not contain a normal subgroup of order 27, 9, or index 3. Therefore, only the cases (ii) (b) and (ii) (d) of Theorem 1.3 in [72] remain. If E has a regular normal subgroup N, then $|N| = |\Omega| > 4$, and hence $N = E$, but E does not act regularly. Thus case (ii) (d) must hold, and hence E_α is cyclic. Let p be a prime dividing $|E_\alpha|$. Then by Theorem 1.1 (ii) (a) and (ii) (b) in [72] together with Lemma 4.1, p is a divisor of $\frac{q^2+\varepsilon q+1}{3}$, and by Lemma 2.14 (c) in [72], E_α contains a Sylow p-subgroup R of E. By Theorem 6.5.3 in [43], $\bar{E}$ has a maximal subgroup $\bar{U}$ of order $q^2+\varepsilon q+1$ that is a Frobenius group with Frobenius kernel of order $\frac{q^2+\varepsilon q+1}{3}$ and a Frobenius complement of order 3. Let U be the full preimage of $\bar{U}$. Since neither q, $q-1$, nor $q+1$ can be divisible by p because p is a divisor of $q^2+\varepsilon q+1$ and $p \geq 5$, it follows that U contains a Sylow p-subgroup P of E. Then there exists $g \in E$ such that $P = R^g$, and hence $P = R^g \leq (E_\alpha)^g = E_{\alpha^g}$. Let $\gamma \in \Omega$ be such that $\alpha^g = \gamma$. Then $P \leq E_\gamma$. Since $p \neq 3$, and by Theorem 10.3.1 in [38], every Frobenius kernel is nilpotent, $\bar{P} \coloneqq P\mathrm{Z}(E)/\mathrm{Z}(E) = \mathrm{O}_p(\bar{U})$. Since $\bar{U}$ is a maximal subgroup of the simple group $\bar{E}$, it follows that $\mathrm{N}_{\bar{E}}(\bar{P}) = \mathrm{N}_{\bar{U}}(\bar{P}) = \bar{U}$. Since $|P|$ and $|\mathrm{Z}(E)|$ are coprime, Lemma 2.3 (a) implies that $\mathrm{N}_{\bar{E}}(\bar{P}) = \mathrm{N}_E(P)\mathrm{Z}(E)/\mathrm{Z}(E)$. Together with the fact that $\mathrm{Z}(E) \leq N_E(P)$, this means that $\mathrm{N}_E(P)/\mathrm{Z}(E) = \mathrm{N}_{\bar{E}}(\bar{P}) = \bar{U}$, and thus $|\mathrm{N}_E(P)| = |U| = |\bar{U}| \cdot |\mathrm{Z}(E)| = (q^2 + \varepsilon q + 1) \cdot 3$. Since $q^2 + \varepsilon q + 1$ is divisible by 3, $|\mathrm{N}_E(P)|$ is divisible by 9, thus the fact that $|\mathrm{N}_E(P) : \mathrm{N}_{E_\gamma}(P)| \leq 3$ implies that 3 divides $|E_\gamma| = |E_\alpha|$. This is a contradiction to the fact that $|E_\alpha|$ is coprime to 6. Hence $\bar{E}$ is not isomorphic to $\mathrm{PSL}_\varepsilon(3, q)$, and therefore the last remaining option is excluded. Thus, the assumption that E is not simple was false.

4.2 Fixity 4

For fixity 4, the situation is different, because there exist non-simple, quasi-simple groups. We will first see two special examples before we prove that the whole family $\mathrm{SL}(2, q)$, where q is an odd prime power, gives examples of quasi-simple groups acting with fixity 4.

Example 4.4
The simple group $E \coloneqq \mathrm{PSL}(3, 4)$ acts with fixity 2 on a set of size $2^6 \cdot 3^2 \cdot 7$ with cyclic point stabilisers of order 5. This can be seen for example by using the GAP package TomLib [74] together with the algorithm in Remark 2.22 and the GAP command `TestTom(TableOfMarks("L3(4)"),2);`.

By the information on page 23 in [28], there is, up to isomorphism, only one quasi-simple group G with a centre of order 2 and $G/\mathrm{Z}(G) \cong E = \mathrm{PSL}(3, 4)$. This group is available in the GAP package AtlasRep [104]. Therefore, the GAP command `TestTom(TableOfMarks(AtlasGroup("2.L3(4)")),4);` shows that G has precisely one action with fixity 4, and this is on a set of size $2^7 \cdot 3^2 \cdot 7$ with cyclic point stabilisers of order 5.

Example 4.5
Similarly to the previous example, by using `TestTom(TableOfMarks("Sz(8)"),2);` we see that the simple group $E \coloneqq \mathrm{Sz}(8)$ acts with fixity 2 on a set of size $2^6 \cdot 5 \cdot 13$ with cyclic point stabilisers of order 7. More precisely, we see that E has two different fixity-4 actions, but we are just interested in the described one.

Again using the information in [28], this time on page 28, we see that there is up to isomorphism only one quasi-simple group G with a centre of order 2 and such that $G/\mathrm{Z}(G) \cong E = \mathrm{Sz}(8)$. Using the algorithm in Remark 2.22 and [104] the GAP command `TestTom(TableOfMarks(AtlasGroup("2.Sz(8)")),4);` shows that G has precisely one action with fixity 4 and this is on a set of size $2^7 \cdot 5 \cdot 13$ with cyclic point stabilisers of order 7.

Lemma 4.6
Let q be an odd prime power and let $G = \mathrm{SL}(2, q)$. Suppose that G acts transitively on a set Ω. Then G acts with fixity 4 on Ω if and only if one of the following holds.

(1) $q = 5$ and the point stabilisers are cyclic of order 3 or 5.

(2) $q \equiv 1 \mod 4$ and $q \geq 9$, and the point stabilisers are cyclic of order $\frac{q+1}{2}$.

(3) $q \equiv -1 \mod 4$ and $q \geq 7$, and the point stabilisers are cyclic of order $\frac{q-1}{2}$, or the semi-direct product of an elementary abelian group of order q with a cyclic group of order $\frac{q-1}{2}$.

Proof:

We first collect some general information about G and $G/\mathrm{Z}(G)$ (see for example II §6 in [54]). The order of G is $q(q^2-1)$ and since q is odd, $\mathrm{Z}(G) = 2$. In Hauptsatz II 8.27 in [54], all subgroups of $G/\mathrm{Z}(G) \cong \mathrm{PSL}(2,q)$ are described, and in some cases the version in Theorem 6.5.1 in [43] gives additional structural details of the subgroups. This information will be used without further reference.

For small values of q we can use the GAP code that is described in Remark 2.22 to check whether or not the lemma holds in these cases. For $G = \mathrm{SL}(2,3)$ the command `TestTom(TableOfMarks(SL(2,3)),4);` gives the answer that G does not act transitively and with fixity 4 on any set, and for $G = \mathrm{SL}(2,5)$ the result of `TestTom(TableOfMarks(SL(2,5)),4);` shows that condition (1) of this lemma holds. Therefore now suppose that $q \geq 7$.

For the first direction of the lemma suppose that G acts transitively and with fixity 4 on a set Ω. Let $\alpha \in \Omega$.

Let $T \in \mathrm{Syl}_2(G)$. Then $Z(G) \leq T$. By Satz II. 8.10 a) in [54], T is a generalised quaternion group. Therefore T contains just one involution t. Since $\mathrm{Z}(G) \leq T$ contains an involution, $t \in \mathrm{Z}(G)$. As a consequence, there exists just the one involution t in G.

Assume for a contradiction that G_α has even order. Then G_α contains the only involution t of G, thus $t \in \mathrm{Z}(G) \cap G_\alpha$, but this contradicts Lemma 2.16. Therefore $|G_\alpha|$ is odd.

To complete the proof of this direction of the lemma, we split according to whether $|G_\alpha|$ and q share a prime divisor. First suppose that there exists a prime p that divides $|G_\alpha|$ and q. Let $X \leq G_\alpha$ be of order p and let $P \in \mathrm{Syl}_p(G)$ be such that $X \leq P$. Then P is elementary abelian by Satz II. 8.10 a) in [54], hence $\mathrm{Z}(G)$ and P are subgroups of $\mathrm{N}_G(X)$. Lemma 2.13 yields that $4 \geq |\,\mathrm{N}_G(X) : \mathrm{N}_{G_\alpha}(X)| \geq |P\,\mathrm{Z}(G) : P\,\mathrm{Z}(G) \cap G_\alpha|$. Since G_α has odd order, but $\mathrm{Z}(G)$ and therefore $P\,\mathrm{Z}(G)$ have even order, the last index is divisible by 2. Hence $|P\,\mathrm{Z}(G) : P\,\mathrm{Z}(G) \cap G_\alpha| \in \{2,4\}$, and since p is odd this implies that G_α has order divisible by q, thus it contains a Sylow p-subgroup R of G.

By Lemma 2.3 (a) together with Satz II 8.2 in [54], $|\,\mathrm{N}_G(R)\,\mathrm{Z}(G)/\mathrm{Z}(G)| = |\,\mathrm{N}_{G/\mathrm{Z}(G)}(R\,\mathrm{Z}(G)/\mathrm{Z}(G))| = q \cdot \frac{q-1}{2}$. Since $\mathrm{Z}(G)$ is a subgroup of $\mathrm{N}_G(R)$, this implies that the order of $\mathrm{N}_G(R)$ is $2 \cdot q \cdot \frac{q-1}{2} = q(q-1)$.

Since G acts with fixity 4, Lemma 2.13 implies that $|\,\mathrm{N}_G(R) : \mathrm{N}_{G_\alpha}(R)| \leq 4$. Thus, $|\,\mathrm{N}_{G_\alpha}(R)| \geq \frac{|\mathrm{N}_G(R)|}{4} = \frac{q(q-1)}{4}$. Additionally, $|\,\mathrm{N}_{G_\alpha}(R)|$ divides $q(q-1)$, and hence $|\,\mathrm{N}_{G_\alpha}(R)| \in \{q \cdot (q-1), q \cdot \left(\frac{q-1}{2}\right), q \cdot \left(\frac{q-1}{3}\right), q \cdot \left(\frac{q-1}{4}\right)\}$. Since q is odd, $q-1$ is even and hence $\mathrm{N}_{G_\alpha}(R)$ cannot have order $q(q-1)$ or $q \cdot \frac{q-1}{3}$.

Assume that $|\,\mathrm{N}_{G_\alpha}(R)| = q \cdot \frac{q-1}{4}$. This implies that $q-1$ is divisible by 4, even $q \equiv 5 \mod 8$ because $|G_\alpha|$ is odd. Since $q \geq 7$, this implies that $q \geq 13$ and that

$\mathrm{N}_{G_\alpha}(R)$ contains a subgroup A of odd prime order s such that s divides $\frac{q-1}{4}$. By Lemma 2.3 (a) together with Satz II 8.3 c) in [54], $|\,\mathrm{N}_G(A)\,\mathrm{Z}(G)/\,\mathrm{Z}(G)| = |\,\mathrm{N}_{G/\,\mathrm{Z}(G)}(A\,\mathrm{Z}(G)/\,\mathrm{Z}(G))| = 2\cdot\frac{q-1}{2}$. Since $\mathrm{Z}(G)$ is a subgroup of $\mathrm{N}_G(A)$, this implies that the order of $\mathrm{N}_G(A)$ is $2\cdot 2\cdot\frac{q-1}{2} = 2(q-1)$. Therefore $|\,\mathrm{N}_G(A) : \mathrm{N}_{G_\alpha}(A)|$ is divisible by 8, because $(q-1)$ is divisible by 4 and $|G_\alpha|$ is odd. This contradicts the inequality $|\,\mathrm{N}_G(A) : \mathrm{N}_{G_\alpha}(A)| \leq 4$ of Lemma 2.13. Hence the assumption that $|\,\mathrm{N}_{G_\alpha}(R)| = q\cdot\frac{q-1}{4}$ was false and therefore $|\,\mathrm{N}_{G_\alpha}(R)| = q\cdot\frac{q-1}{2}$.

As a consequence, the order of G_α is divisible by $q\cdot\frac{q-1}{2}$. Since $\mathrm{Z}(G)\cap G_\alpha = 1$, $G_\alpha\,\mathrm{Z}(G)/\,\mathrm{Z}(G)$ is isomorphic to G_α. An inspection of the possible subgroups of $\mathrm{PSL}(2,q)$ shows that $G_\alpha\,\mathrm{Z}(G)/\,\mathrm{Z}(G)$, and hence G_α, is the semi-direct product of an elementary abelian subgroup of order q with a cyclic subgroup of order $\frac{q-1}{2}$, because $q \geq 7$ and therefore $\frac{q-1}{2} \geq 3$, and because $G_\alpha\,\mathrm{Z}(G)/\,\mathrm{Z}(G)$ has odd order. This implies that $\frac{q-1}{2}$ is odd, because $|G_\alpha|$ is odd. Hence $q \equiv -1 \mod 4$ and therefore case (3) of this lemma holds.

Now suppose that there does not exist a prime dividing both $|G_\alpha|$ and q. Let $x \in G_\alpha$ be an element of prime order r that fixes exactly four points in Ω. Then r divides $|G_\alpha|$ and $|G| = q\cdot(q-1)\cdot(q+1)$. By our assumption that r does not divide q, there exists $\varepsilon \in \{-1,1\}$ such that r divides $q-\varepsilon$. Since r is odd, x acts coprimely on $\mathrm{Z}(G)$ and therefore $\mathrm{N}_G(\langle x\rangle)\,\mathrm{Z}(G)/\,\mathrm{Z}(G) = \mathrm{N}_{G/\,\mathrm{Z}(G)}(\langle \mathrm{Z}(G)x\rangle)$ by Lemma 2.3 (a). Then $\mathrm{Z}(G)x$ has order r and therefore by Satz II 8.5 in [54], it lies in a cyclic group of order $\frac{q-\varepsilon}{2}$. By Satz II 8.3 and Satz II 8.4 in [54], $|\,\mathrm{N}_{G/\,\mathrm{Z}(G)}(\langle \mathrm{Z}(G)x\rangle)| = 2\cdot\frac{q-\varepsilon}{2} = q-\varepsilon$. Thus, $\mathrm{N}_G(\langle x\rangle)\,\mathrm{Z}(G)/\,\mathrm{Z}(G)$ has order $q-\varepsilon$ and since $\mathrm{Z}(G) \leq \mathrm{N}_G(\langle x\rangle)$, it follows that $|\,\mathrm{N}_G(x)| = 2(q-\varepsilon)$.

The group G still acts with fixity 4 and hence, by Lemma 2.13, it follows that $|\,\mathrm{N}_G(\langle x\rangle) : \mathrm{N}_{G_\alpha}(\langle x\rangle)| \leq 4$. Thus, $|\,\mathrm{N}_{G_\alpha}(\langle x\rangle)| \geq \frac{|\,\mathrm{N}_G(\langle x\rangle)|}{4} = \frac{2(q-\varepsilon)}{4} = \frac{q-\varepsilon}{2}$. On the other hand, $|\,\mathrm{N}_{G_\alpha}(\langle x\rangle)|$ must be odd and a divisor of $2(q-\varepsilon)$. Since q is odd, this shows that $|\,\mathrm{N}_{G_\alpha}(x)| = \frac{q-\varepsilon}{2}$.

Therefore, the order of G_α is odd and divisible by $\frac{q-\varepsilon}{2}$ but not by any non-trivial factor of q. Since $\mathrm{Z}(G)\cap G_\alpha = 1$, $G_\alpha\,\mathrm{Z}(G)/\,\mathrm{Z}(G)$ is isomorphic to G_α. An inspection of the possible subgroups of $\mathrm{PSL}(2,q)$ shows that $G_\alpha\,\mathrm{Z}(G)/\,\mathrm{Z}(G)$, and hence G_α, is a cyclic group of order $\frac{q-\varepsilon}{2}$. This also implies that $\frac{q-\varepsilon}{2}$ is odd. Hence $q \equiv -\varepsilon \mod 4$ and therefore case (2) or case (3) in the lemma holds.

For the other direction, first suppose that G is isomorphic to $\mathrm{SL}(2,q)$, that $\varepsilon \in \{-1,1\}$, that $q \geq 7$ is congruent to ε modulo 4, that $U \leq G$ is a cyclic subgroup of order $\frac{q+\varepsilon}{2}$, and that $\Omega = G/U$. Let $g \in U$ be non-trivial. Then by Corollary 2.10, g fixes $\frac{|\,\mathrm{N}_G(\langle g\rangle)|}{|U|}$ points in Ω. Since $|\,\mathrm{N}_G(\langle g\rangle)|$ is twice the order of $\mathrm{N}_G(\langle g\rangle)\,\mathrm{Z}(G)/\,\mathrm{Z}(G)$ and since this order equals $|\,\mathrm{N}_{G/\,\mathrm{Z}(G)}(\langle g\rangle\,\mathrm{Z}(G)/\,\mathrm{Z}(G))| = \cdot 2\cdot\frac{q+\varepsilon}{2}$, it follows that $|\,\mathrm{N}_G(\langle g\rangle)| = 2(q+\varepsilon)$. Thus, g has four fixed points on Ω.

Since all point stabilisers of G are conjugate, this means that every non-trivial element in G either does not fix any point or has four fixed points. In particular, G acts with fixity 4 on Ω.

Finally, suppose that G is isomorphic to $\mathrm{SL}(2,q)$, that $q \geq 7$ is congruent to -1 modulo 4, that $U \leq G$ is the semi-direct product of an elementary abelian group of order q with a cyclic group of order $\frac{q-1}{2}$, and that $\Omega = G/U$. Then G acts transitively on Ω and U has odd order. Since $\mathrm{Z}(G)$ has even order, $U \cap \mathrm{Z}(G) = 1$. Thus, U is isomorphic to $U\,\mathrm{Z}(G)/\,\mathrm{Z}(G)$. The subgroup structure of $\mathrm{PSL}(2,q)$ shows that $U\,\mathrm{Z}(G)/\,\mathrm{Z}(G)$, and therefore U, is a Frobenius group. Let K be the Frobenius kernel of U and C a Frobenius complement of U. Then C is cyclic of order $\frac{q-1}{2}$ and K is elementary abelian of order q.

If a point is fixed by an element u of U, then all powers of u fix the same point. Therefore, in order to determine the fixity with which G is acting, it suffices to calculate the maximal number of fixed points of each element of prime order. Let $x \in U$ be of prime order p. Then p divides either q or $\frac{q-1}{2}$.

If p divides $\frac{q-1}{2}$, then x is conjugate to an element $c \in C$. By Lemma 2.11 the number of fixed points of c is $\frac{|K|\cdot|\mathrm{N}_G(\langle c\rangle)|}{|U|} = \frac{q\cdot|\mathrm{N}_G(\langle c\rangle)|}{q\cdot\frac{q-1}{2}} = 2\cdot\frac{|\mathrm{N}_G(\langle c\rangle)|}{q-1}$. Again $|\mathrm{N}_G(\langle c\rangle)| = 2(q-1)$, and hence c fixes exactly four points. As a consequence, x fixes at most four points and there exists an element in U with exactly four fixed points.

If p divides q, then $x \in K$, which is the only Sylow p-subgroup of U. Let $y \in G$ be such that $Uy \in G/U$ is a fixed point under x. Then $x^{y^{-1}} \in U$ is a p-element, thus $x^{y^{-1}} \in K$. Therefore $x \in K \cap K^y$. Then $\mathrm{Z}(G)x$ lies in the intersection $K\,\mathrm{Z}(G)/\,\mathrm{Z}(G) \cap K^y\,\mathrm{Z}(G)/\,\mathrm{Z}(G)$, and together with Satz II 8.2 in [54], this implies that y is an element of $\mathrm{N}_{G/\,\mathrm{Z}(G)}(K\,\mathrm{Z}(G)/\,\mathrm{Z}(G))$. Since $|K| = q$ is coprime to $|\,\mathrm{Z}(G)|$, it follows by Lemma 2.3 (a) that $\mathrm{N}_{G/\,\mathrm{Z}(G)}(K\,\mathrm{Z}(G)/\,\mathrm{Z}(G)) = \mathrm{N}_G\,\mathrm{Z}(G)/\,\mathrm{Z}(G)$, and this group has order $q\cdot\frac{q-1}{2}$. Since $\mathrm{Z}(G) \leq \mathrm{N}_G(K)$, the order of $\mathrm{N}_G(K)$ is $q\cdot(q-1)$. Thus, the number of fixed points of x in its action on G/U is bounded above by $|\,\mathrm{N}_G(K) : U| = \frac{q(q-1)\cdot 2}{q(q-1)} = 2$.

Therefore, x has at most four fixed points and there exist elements in U with exactly four fixed points. As a consequence, G acts with fixity 4 on G/U.

After seeing these examples of non-simple, quasi-simple groups acting with fixity 4, it remains to determine whether or not there exist further such examples. Lemma 4.8 gives the answer, but beforehand we need a more technical lemma, analysing a special situation that will be needed frequently in the proof of Lemma 4.8.

Lemma 4.7
Let G be a finite group acting transitively and with fixity 4 on a set Ω, and let $Z = \mathrm{Z}(G)$. Suppose that $|\Omega| > 4\cdot|Z|$ and that $|Z| \in \{2,4\}$. Further suppose that G/Z acts with fixity 4 on $\Gamma := \{\alpha^Z \mid \alpha \in \Omega\}$. Then G/Z does not have cyclic point stabilisers of order coprime to 6.

Proof:

Let $x \in G$ fix exactly four points in Ω and let $\alpha \in \mathrm{fix}_\Omega(x)$. Assume, for a contradiction, that the point stabilisers of G/Z are cyclic of order coprime to 6. Then the hypotheses of Lemma 4.1 are fulfilled. Therefore, $|G_\alpha|$ is coprime to 6, and hence x has odd order. As a consequence, x acts coprimely on Z and $Zx \neq Z$. Thus, Lemma 2.3 (a) shows that $\mathrm{N}_G(\langle x\rangle)Z/Z = \mathrm{N}_{G/Z}(\langle Zx\rangle)$. Since Z is a subgroup of $\mathrm{N}_G(\langle x\rangle)$, this means that $|\,\mathrm{N}_G(\langle x\rangle)| = |Z|\cdot|\,\mathrm{N}_G(\langle x\rangle)Z/Z| =$

$|Z| \cdot |\,\mathrm{N}_{G/Z}(\langle Zx\rangle)|$. We recall that x has exactly four fixed points. Then Lemma 2.9 implies that $4 = \frac{|\{\langle x\rangle^g \leq G_\alpha | g\in G\}|\cdot|\,\mathrm{N}_G(\langle x\rangle)|}{|G_\alpha|}$, and therefore $|\,\mathrm{N}_G(\langle x\rangle)| = |Z|\cdot|\,\mathrm{N}_{G/Z}(\langle Zx\rangle)|$ is not divisible by 8 because $|G_\alpha|$ is odd. Thus, $|\,\mathrm{N}_{G/Z}(\langle Zx\rangle)|$ is not divisible by $\frac{8}{|Z|}$.

By assumption, the point stabilisers of G/Z are cyclic of order coprime to 6. Therefore Lemma 3.22 (b) implies that every non-trivial element in a point stabiliser of G/Z fixes exactly four points. Since Zx fixes $\alpha^Z \in \Gamma$, it lies in a point stabiliser U of G/Z, thus Zx fixes exactly four points. Then by Lemma 2.10, Zx has exactly $\frac{|\,\mathrm{N}_{G/Z}(\langle Zx\rangle)|}{|U|}$ fixed points in Γ. Hence $|\,\mathrm{N}_{G/Z}(\langle Zx\rangle)| = 4\cdot|U|$, contrary to the fact that $|\,\mathrm{N}_{G/Z}(\langle Zx\rangle)|$ is not divisible by $\frac{8}{|Z|}$.

Lemma 4.8

Let G be a finite quasi-simple group acting transitively and with fixity 4 on a set Ω. Then either G is simple or there exists an odd prime power q such that G is isomorphic to $\mathrm{SL}(2,q)$, $C_2.\,\mathrm{Sz}(8)$, or $C_2.\,\mathrm{PSL}(3,4)$.

Proof:

For the groups $C_2.\mathcal{A}_7$, $C_2.\mathcal{A}_8$, $C_2.\,\mathrm{M}_{12}$, $C_2.\,\mathrm{M}_{22}$, $C_4.\,\mathrm{M}_{22}$, and $E_4.\,\mathrm{Sz}(8)$ the GAP package AtlasRep [104] together with the GAP function in Remark 2.22 and the following commands show that none of the groups can act with fixity 4 on any set.

```
TestTom(TableOfMarks(AtlasGroup("2.a7")),4);
TestTom(TableOfMarks(AtlasGroup("2.a8")),4);
TestTom(TableOfMarks(AtlasGroup("2.m12")),4);
TestTom(TableOfMarks(AtlasGroup("2.m22")),4);
TestTom(TableOfMarks(AtlasGroup("4.m22")),4);
TestTom(TableOfMarks(AtlasGroup("2^2.Sz(8)")),4);
```

Suppose that G is not simple. Let $Z = \mathrm{Z}(G)$ and $\bar{G} = G/Z$. Then $Z \neq 1$ and by Lemma 2.17, $|Z| \in \{2,4\}$.

In the Transitive Groups Library [52] of GAP [36], all transitive groups up to degree 30 are listed. Then we can use GAP to generate a list of all groups that are quasi-simple, but not simple and that act transitively and with fixity 4 on a set of size at most 16. Since G contains a non-trivial element with four fixed points, Ω has at least six elements, and since no non-trivial element in G fixes more than four points, all five-point stabilisers are trivial. Thus, making use of the program in Remark 2.18, we can formulate the following GAP code.

```
AllTransitiveGroups(NrMovedPoints, [6..16], z->(not IsSimple(z))
↪  and IsPerfect(z) and IsSimple(z/Center(z)), true,
↪  x->IsTrivial(Stabilizer(x,[1..5],OnTuples)), true,
↪  y->TestFixity(y, MovedPoints(y), 4), true);
```

This shows that the only such group is $\mathrm{SL}(2,7)$. Hence, we can suppose that $|\Omega| > 16 \geq 4\cdot|Z|$. By Lemma 4.1, the simple group $\bar{G}$ acts non-regularly and with fixity at most 4 on $\Gamma := \{\alpha^Z \mid \alpha \in \Omega\}$, and the order of a point stabiliser of G in its action on Ω is the same as the order of a point stabiliser of $\bar{G}$ in its action on Γ.

Since $\bar{G}$ is simple, it cannot act as a Frobenius group, and therefore $\bar{G}$ acts with fixity 2, 3, or 4 on Γ. We look at the different possibilities one after the other. Let $\alpha \in \Omega$ and let $x \in G_\alpha$ fix exactly four points.

First suppose that $\bar{G}$ acts with fixity 4 on Γ. Then $\bar{G}$ is one of the groups in Theorem 3.56. Since $|Z|$ is divisible by 2, the Schur multiplier of $\bar{G}$ has to be divisible by 2. By Table 4.1 in [39], this leads to the following list of options for $\bar{G}$, where the order of Z is as specified and q is a prime power: $\mathcal{A}_7$ with $|Z| = 2$, $\mathrm{PSL}(2,q)$ with $|Z| = 2$, $\mathrm{PSU}(4,3)$ with $|Z| \in \{2,4\}$, $\mathrm{PSp}(4,q)$ with $|Z| = 2$ and q odd, $\mathrm{P}\Omega_-(8,q)$ with $|Z| = 2$ and q odd, $\mathrm{Sz}(8)$ with $|Z| \in \{2,4\}$, M_{12} with $|Z| = 2$, and M_{22} with $|Z| \in \{2,4\}$. If $\bar{G}$ is isomorphic to $\mathcal{A}_7$, M_{12}, or M_{22}, then there is up to isomorphism just one group for each option for $|Z|$ by the information on pages 10, 31, and 39 in [28]. For each of these groups, we have seen at the beginning of this proof that it cannot act transitively and with fixity 4 on any set. If $\bar{G}$ is isomorphic to $\mathrm{Sz}(8)$, then by the information on page 28 in [28], the only options for G, up to isomorphism, are $C_2.\,\mathrm{Sz}(8)$ and $E_4.\,\mathrm{Sz}(8)$. The first case is listed in the lemma and the second case is already excluded. If $\bar{G}$ is isomorphic to $\mathrm{PSL}(2,q)$ and $|Z| = 2$, then since $\mathrm{PSL}(2,4) \cong \mathrm{PSL}(2,5)$, the prime power q is 4 or odd by Table 4.1 in [39], and G is isomorphic to $\mathrm{SL}(2,q)$ listed in the lemma. Thus, for $\bar{G}$ there remain $\mathrm{PSU}(4,3)$ with $|Z| \in \{2,4\}$, $\mathrm{PSp}(4,q)$ with $|Z| = 2$, and $\mathrm{P}\Omega_-(8,q)$ with $|Z| = 2$. If $\bar{G} = \mathrm{PSU}(4,3)$, then by Theorem 3.56, all point stabilisers of $\bar{G}$ in its action on Γ are cyclic of order 5, and hence have order coprime to 6, contrary to Lemma 4.7. If $\bar{G} = \mathrm{PSp}(4,q)$, then again by Theorem 3.56, a point stabiliser of $\bar{G}$ in its action on Γ is cyclic of order $\frac{q^2+1}{2}$, and hence has order coprime to 6, contrary to Lemma 4.7. If $\bar{G} = \mathrm{P}\Omega_-(8,q)$, then once more by Theorem 3.56, a point stabiliser of $\bar{G}$ in its action on Γ is cyclic of order $\frac{q^4+1}{2}$, and hence has order coprime to 6, and again Lemma 4.7 gives a contradiction. This was the last remaining option in the case that $\bar{G}$ acts with fixity 4 on Γ.

Therefore now instead suppose that $\bar{G}$ acts with fixity 3 on Γ. Then $\bar{G}$ is one of the groups in Theorem 1.1 in [72]. Since $|Z|$ is divisible by 2, the Schur multiplier of $\bar{G}$ has to be even. By Table 4.1 in [39], this leads to the following list of options for $\bar{G}$, where the order of Z is as specified: $\mathcal{A}_5 \cong \mathrm{PSL}(2,5)$ with $|Z| = 2$, $\mathcal{A}_6 \cong \mathrm{PSL}(2,9)$ with $|Z| = 2$, $\mathrm{PSL}(2,7) \cong \mathrm{PSL}(3,2)$ with $|Z| = 2$, $\mathcal{A}_7$ with $|Z| = 2$, $\mathrm{PSL}(2,11)$ with $|Z| = 2$, $\mathrm{PSL}(3,4)$ with $|Z| \in \{2,4\}$, $\mathrm{PSL}(4,3)$ with $|Z| = 2$, $\mathrm{PSU}(4,3)$ with $|Z| \in \{2,4\}$, $\mathrm{PSL}(4,5)$ with $|Z| \in \{2,4\}$, $\mathcal{A}_8$ with $|Z| = 2$, and M_{22} with $|Z| \in \{2,4\}$. If $\bar{G}$ is isomorphic to $\mathrm{PSL}(2,5)$, $\mathrm{PSL}(2,7)$, $\mathrm{PSL}(2,9)$, or $\mathrm{PSL}(2,11)$, then because in all cases $|Z| = 2$, G is a special linear group listed in the lemma. If $\bar{G}$ is isomorphic to $\mathcal{A}_7$, $\mathcal{A}_8$, or M_{22}, there is up to isomorphism just one group for each option for $|Z|$ by the information on pages 10, 22, and 39 in [28], and as seen earlier, these groups cannot act transitively and with fixity 4 on any set. As a consequence, the only options for $\bar{G}$ that remain are $\mathrm{PSL}(3,4)$, $\mathrm{PSL}(4,3)$, $\mathrm{PSU}(4,3)$, and $\mathrm{PSL}(4,5)$. In each case, Theorem 1.1 in [72] also proves that the order of a point stabiliser $\bar{U}$ of $\bar{G}$ in its action on Γ is coprime to 6, and that $\bar{U}$ is cyclic. Let $\bar{u} \in \bar{U}$ be of prime order $p \geq 5$ and such that $\bar{u}$ fixes exactly three points in Γ. Let Y be the full pre-image of $\langle \bar{u} \rangle$ in G and $X \in \mathrm{Syl}_p(Y)$. Then X is cyclic of order p and $XZ/Z = \langle \bar{u} \rangle$. Let $y \in X$ be non-trivial. Then y has order $p \geq 5$, and leaves three orbits of size $|Z|$ invariant, because Zy fixes three points in Γ, the set of Z-orbits in Ω. Since $|Z| \leq 4 < 5 \leq p$, this implies that y fixes every element in an orbit of size $|Z|$, and hence fixes $3 \cdot |Z| \geq 3 \cdot 2 = 6$ points in Ω. This contradicts the fact that G acts with fixity 4 on Ω. This excludes the last remaining options in the case that $\bar{G}$ acts with fixity 3 on Γ.

Finally, suppose that $\bar{G}$ acts with fixity 2 on Γ. Then $\bar{G}$ is one of the groups in Theorem 1.2 in [71]. Thus, there exists a prime power q such that $\bar{G}$ is isomorphic to either $\mathrm{PSL}(2,q)$, $\mathrm{Sz}(8)$, or $\mathrm{PSL}(3,4)$. If $\bar{G} \cong \mathrm{PSL}(2,q)$, then Table 4.1 in [39] together with the facts that $|Z| \in \{2,4\}$ and that $\mathrm{PSL}(2,4) \cong \mathrm{PSL}(2,5)$ shows that q is 4 or an odd prime power and that $G = \mathrm{SL}(2,q)$. If $\bar{G}$ is isomorphic to $\mathrm{Sz}(8)$, then by the information on page 28 in [28] there are, up to isomorphism, only the options $C_2.\mathrm{Sz}(8)$ and $E_4.\mathrm{Sz}(8)$ for G. The first case is listed in the lemma and the second already excluded. Therefore, suppose that $\bar{G} = \mathrm{PSL}(3,4)$. Then with the use of the GAP function in Remark 2.22, the command `TestTom(TableOfMarks(PSL(3,4)),2);` shows that the only action of $\bar{G}$ with fixity 2 is on a set of size $2^6 \cdot 3^2 \cdot 7$ with a point stabiliser of order 5. Then $|G_\alpha| = 5$ and $|\Omega| = \frac{|G|}{5} = \frac{|Z| \cdot |\mathrm{PSL}(3,4)|}{5} = |Z| \cdot 2^6 \cdot 3^2 \cdot 7$. Since $|Z| \in \{2,4\}$, it follows that $|\Omega| \in \{2^7 \cdot 3^2 \cdot 7, 2^8 \cdot 3^2 \cdot 7\}$. Since $2^8 \cdot 3^2 \cdot 7 \equiv 3^2 \cdot 2 \equiv 3 \mod 5$, no element of order 5 can have exactly four fixed points. Therefore $|Z| = 4$ is not possible and hence $|Z| = 2$. Then the information on page 23 in [28] shows that $G = C_2.\mathrm{PSL}(3,4)$, and this group is already listed in the lemma. This completes the analysis of the last remaining case.

Lemmas 4.6 and 4.8 and the two examples above give a full classification of all non-simple, quasi-simple groups acting with fixity 4, and together with Theorem 3.56 they give a full classification of all quasi-simple groups acting transitively and with fixity 4.

The result of this classification suggests that there is more underlying structure than the previous proof revealed. This is made more explicit in the next corollary.

Corollary 4.9
Let G be a finite quasi-simple group acting transitively and with fixity 4 on a set Ω. If G is not simple, then $\mathrm{Z}(G)$ has order 2 and $G/\mathrm{Z}(G)$ acts with fixity 2 on a set of size $\frac{|\Omega|}{2}$.

Proof:
Suppose that G is not simple. Then by Lemma 4.8, there exists an odd prime power q such that G is isomorphic to $\mathrm{SL}(2,q)$, $C_2.\mathrm{Sz}(8)$, or $C_2.\mathrm{PSL}(3,4)$. In all cases $\mathrm{Z}(G)$ has order 2. Let $Z = \mathrm{Z}(G)$. If $G/Z \cong \mathrm{PSL}(3,4)$ or $G/Z \cong \mathrm{Sz}(8)$, then

Example 4.4 and Example 4.5, respectively, show that G/Z acts with fixity 2 on a set half the size of Ω.

Therefore suppose that $G = \mathrm{SL}(2,q)$. Let $\bar{G} = G/\mathrm{Z}(G)$.

If $q = 5$, then $|G| = |\mathrm{SL}(2,5)| = 120$. By Lemma 4.6, $|\Omega| \in \{\frac{120}{3}, \frac{120}{5}\} = \{40, 24\}$. Using the GAP function in Remark 2.22 through the GAP code `TestTom(TableOfMarks(PSL(2,5)),2);` gives that $\bar{G} \cong \mathrm{PSL}(2,5)$ acts (among other actions) transitively and with fixity 2 on sets of order 20 and 12, proving the corollary in this case. Therefore additionally suppose that $q \geq 7$.

By Lemma 4.6, there are two possibilities for the structure of point stabilisers of G. Either they are cyclic, or the semidirect product of an elementary abelian group of order q with a cyclic group of order $\frac{q-1}{2}$. In the first case let $\varepsilon \in \{-1, 1\}$ be such that $q \equiv \varepsilon \mod 4$. Then the point stabilisers of G are cyclic of order $\frac{q+\varepsilon}{2}$. Hence $|\Omega| = \frac{|G|}{\frac{q+\varepsilon}{2}} = 2 \cdot \frac{q(q^2-1)}{q+\varepsilon} = 2q(q-\varepsilon)$. In the latter case $|\Omega| = \frac{|G|}{q \cdot \frac{q-1}{2}} = 2 \cdot \frac{q(q^2-1)}{q(q-1)} = 2(q+1)$.

It remains to prove that $\bar{G}$ has a transitive action with fixity 2 on sets of size $q(q-1)$, $q(q+1)$, and $q+1$ in these cases. The argumentation is quite similar to the argument in the last part of Lemma 4.6. Therefore let $\varepsilon \in \{-1, 1\}$ be such that $q \equiv \varepsilon \mod 4$. By Hauptsatz II 8.27 in [54], the group $\bar{G} \cong \mathrm{PSL}(2,q)$ has a cyclic subgroup $\bar{U}$ of order $\frac{q+\varepsilon}{2}$. Then $\bar{G}$ acts transitively on $\bar{G}/\bar{U}$ and by Lemma 2.10, every non-trivial element $\bar{x} \in \bar{U}$ has $\frac{|N_{\bar{G}}(\langle\bar{x}\rangle)|}{|\bar{U}|}$ fixed points. By Satz II 8.3 c) and Satz II 8.4 c) in [54], the order of $N_{\bar{G}}(\langle\bar{x}\rangle)$ is $2 \cdot \frac{q+\varepsilon}{2} = q + \varepsilon$. Therefore $\bar{x}$ has exactly two fixed points, thus $\bar{G}$ acts with fixity 2 on $\bar{G}/\bar{U}$ and $|\bar{G}/\bar{U}| = \frac{q \cdot \frac{q^2-1}{2}}{\frac{q+\varepsilon}{2}} = q(q-\varepsilon)$. In addition, by Hauptsatz II 8.27 in [54], the group $\bar{G} \cong \mathrm{PSL}(2,q)$ has a subgroup $\bar{U}$ that is a Frobenius group of order $q \cdot \frac{q-1}{2}$. Let $\bar{K}$ be the Frobenius kernel of $\bar{U}$ and let $\bar{C}$ be a Frobenius complement of $\bar{U}$. Then $\bar{C}$ is cyclic of order $\frac{q-1}{2}$ and K is elementary abelian of order q. In order to determine the fixity with which $\bar{G}$ is acting, it suffices to calculate the maximal number of fixed points of each element of prime order. Let $\bar{x} \in U$ be of prime order p. Then p divides either q or $\frac{q-1}{2}$. If p divides $\frac{q-1}{2}$, then $\bar{x}$ is conjugate to an element $\bar{c} \in \bar{C}$. By Lemma 2.11, the number of fixed points of $\bar{c}$ is $\frac{|\bar{K}| \cdot |\mathrm{N}_{\bar{G}}(\langle\bar{c}\rangle)|}{|\bar{U}|} = \frac{q \cdot |\mathrm{N}_{\bar{G}}(\langle\bar{c}\rangle)|}{q \cdot \frac{q-1}{2}} = 2 \cdot \frac{|\mathrm{N}_{\bar{G}}(\langle\bar{c}\rangle)|}{q-1}$. Again the fact that $|\mathrm{N}_{\bar{G}}(\langle\bar{c}\rangle)| = 2 \cdot \frac{q-1}{2}$ implies that $\bar{c}$ fixes exactly two points. Thus, $\bar{x}$ fixes at most two points. If p divides q, then $\bar{x} \in \bar{K}$. Let $\bar{y} \in \bar{G}$ be such that $\bar{U}\bar{y} \in \bar{G}/\bar{U}$ is a fixed point under $\bar{x}$. Then $\bar{x}^{\bar{y}^{-1}} \in \bar{U}$ is a p-element, and hence $\bar{x}^{\bar{y}^{-1}} \in \bar{K}$. Therefore $\bar{x} \in \bar{K} \cap \bar{K}^{\bar{y}}$. By Satz II 8.2 in [54], it follows that $\bar{y}$ is an element of $\mathrm{N}_{\bar{G}}(\bar{K})$, and this group has order $q \cdot \frac{q-1}{2}$. Thus, the number of fixed points of $\bar{x}$ in its action on $\bar{G}/\bar{U}$ is bounded above by $|\mathrm{N}_{\bar{G}}(\bar{K}) : \bar{U}| = \frac{q \cdot \frac{q-1}{2}}{q \cdot \frac{q-1}{2}} = 1$. Therefore $\bar{G}$ acts with fixity 2 on $\bar{G}/\bar{U}$ and $|\bar{G}/\bar{U}| = \frac{q \cdot \frac{q^2-1}{2}}{q \cdot \frac{q-1}{2}} = q+1$. As a consequence, $\bar{G}$ acts transitively and with fixity 2 on sets of size $q(q-1)$, $q(q+1)$, and $q+1$.

5 The Components

In this chapter we will see that a group G that acts with fixity 4 can have at most one component. Since components are quasi-simple, we can then use the information that we gained in the previous chapter to describe the unique component of G. In some cases, we can say even more about G or at least about the point stabilisers of G. However, knowing $\mathrm{E}(G)$ takes us one step forward in understanding $\mathrm{F}^*(G)$, and thus in understanding the structure of G.

5.1 Centralisers of Involutory Automorphisms of Simple Groups

In the previous two chapters we have seen that Lemma 2.13 used for the normaliser or the centraliser of a subgroup of a point stabiliser is most useful. Thus, it is unsurprising that the size of a centraliser will again play a key role in the analysis of the components structure. Therefore, we first look at the centralisers of involutory automorphisms of simple groups before we use this result for quasi-simple groups, or more precisely for components of groups that act with fixity 4. The result of Lemma 5.1 is interesting in its own right and does not need the context of fixity in which we use it.

The idea for the proof is due to Gernot Stroth who suggested to use a version of the Brauer-Fowler Theorem (see [14]) together with a library of primitive groups.

Lemma 5.1
Let E be a non-abelian simple group. Let x be an involution and an automorphism of E. If E is not isomorphic to $\mathcal{A}_5$, then $|\,\mathrm{C}_E(x)| > 4$.

Proof:

Assume, for a contradiction, that E is not isomorphic to $\mathcal{A}_5$ and that the order of $\mathrm{C}_E(x)$ is at most 4.

We first will exclude some finite simple groups that otherwise would appear later on in this proof and have to be dealt with at some point.

Therefore, assume that $E = \mathrm{PSL}(2,7) \cong \mathrm{PSL}(3,2)$. Then the table on page 3 in [28] shows that for every involution $t \in E$, $|\,\mathrm{C}_E(t)| = 8$. The table also shows that the outer automorphism group has order 2 and that for every involution s in $E.C_2$ that is not in E, $|\,\mathrm{C}_E(s)| = 6$. Since x has order 2, it lies in one of the conjugacy classes of involutions of $E.C_2$ or centralises E, and therefore $|\,\mathrm{C}_E(x)| \geq 6$. This contradicts the assumption.

Similarly, we can check the tables in [28] to see that neither M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, $\mathrm{PSL}(2,q)$ where $q \in \{8, 11, 13, 16, 17, 19, 23, 25, 27, 29, 31\}$, $\mathrm{PSL}(3,q)$ where $q \in \{3, 4, 5\}$, $\mathrm{PSL}(5,2)$, $\mathrm{PSU}(3,3)$, $\mathrm{PSU}(4,2)$, nor $\mathrm{PSp}(6,2)$ fulfils our assumptions.

Therefore instead assume that E is an alternating group. Let n be a positive integer such that $E = \mathcal{A}_n$. Then by our assumptions, $n \geq 6$. If $n = 6$, then the table on page 4 in [28] implies the contradiction that $|\,\mathrm{C}_E(x)| \geq 8$. Thus, $n \geq 7$. By Theorem 2.3 in [105], $\mathrm{Aut}(\mathcal{A}_n) \cong \mathcal{S}_n$. In $\mathcal{S}_n$ an involution t can be written as the product of disjoint transpositions. If t is the product of at most two transpositions, it moves at most four points, and hence centralises a subgroup of $\mathcal{S}_n$ isomorphic to $\mathcal{S}_{n-4}$ acting on the remaining $n - 4 \geq 3$ points. Thus, $|\,\mathrm{C}_{\mathcal{S}_n}(t)|$ is divisible by $|\mathcal{S}_3| \cdot |\langle t \rangle| = 12$, and hence the order of $\mathrm{C}_{\mathcal{A}_n}(t)$ is divisible by 6. If t is the product of at least three disjoint transpositions then there are six distinct elements $a_1, a_2, a_3, a_4, a_5, a_6 \in \{1, \ldots, n\}$ and a permutation τ such that τ fixes the six points $a_1, \ldots, a_6$ and such that t is the product of $(a_1, a_2)(a_3, a_4)(a_5, a_6)$ and τ. Then $(a_1, a_2), (a_3, a_4), (a_5, a_6), (a_1, a_3)(a_2, a_4) \in \mathrm{C}_{\mathcal{S}_n}(t)$, and hence $2^4 = 16$ divides the order of $\mathrm{C}_{\mathcal{S}_n}(t)$. Therefore $|\,\mathrm{C}_{\mathcal{A}_n}(t)|$ is divisible by 8. As a consequence, $|\,\mathrm{C}_E(x)| = |\,\mathrm{C}_{\mathcal{A}_n}(t)| \geq 6$ contradicting our assumptions. Thus, E is not an alternating group.

As a consequence, we can suppose that E is none of the described groups.

Let $H = E\langle x \rangle$. Then $|\,\mathrm{C}_H(x)| \leq 8$. By Theorem 1.5 in Chapter 5 of [95] (using the method of Brauer-Fowler), there exists a non-trivial element $w \in H$ such that $|H : \mathrm{C}_H(w)| = \frac{|H|}{|\,\mathrm{C}_H(w)|} \leq |\,\mathrm{C}_H(x)|^2 \leq 8^2 = 64$.

As an intermediate step, we will see that $\mathrm{Z}(H) = 1$. For a contradiction, assume that there exists a non-trivial element $z \in \mathrm{Z}(H)$. Then $\mathrm{Z}(H) \cap E = 1$, and this implies that $|\,\mathrm{Z}(H)| = 2$. In particular, z is an involution. Let $u \in E$ be such that $z = ux$. Then u is an involution. For all elements $g \in E$, $g^u = g^{zx^{-1}} = g^{zx} = g^x$, and hence $|\,\mathrm{C}_E(u)| = |\,\mathrm{C}_E(x)| \leq 4$. Thus, Theorem 1.5 in Chapter 5 of [95] now used for E implies that there exists a non-trivial element $b \in E$ such that $|E : \mathrm{C}_E(b)| = \frac{|E|}{|\,\mathrm{C}_E(b)|} \leq |\,\mathrm{C}_E(u)|^2 \leq 4^2 = 16$. Then E is non-abelian simple and acts transitively and faithfully on a set of size at most 16. The following GAP command uses the Transitive Groups Library [52] and shows that E is either $\mathcal{A}_n$ where n is a positive integer such that $5 \leq n \leq 16$, M_{11}, M_{12}, $\mathrm{PSL}(2,q)$ where $q \in \{7, 8, 11, 13\}$, or $\mathrm{PSL}(3,3)$.

```
AllTransitiveGroups(NrMovedPoints, [1..16], IsSimple, true,
↪  IsAbelian, false);
```

All of these groups have been excluded earlier, either by the assumption that $E \ncong \mathcal{A}_5$ or by the arguments at the beginning of this proof. Therefore $\mathrm{Z}(H) = 1$.

As a next step, we will see that H or E acts transitively, faithfully, and primitively on a set of size at most 64 and such that a point stabiliser contains the centraliser of w, but beforehand we need some information about $\mathrm{C}_H(w)$.

Assume, for a contradiction, that $E \leq \mathrm{C}_H(w)$. If $w \in E$, then $w \in \mathrm{Z}(E) = 1$, contradicting the fact that w is non-trivial. Hence, $w \in H \setminus E$ and $H =$

$\langle w, E\rangle \le \mathrm{C}_H(w)$. Thus, $w \in \mathrm{Z}(H)$, contradicting the fact that $\mathrm{Z}(H) = 1$. As a consequence, $E \nleq \mathrm{C}_H(w)$.

Then there exists a maximal subgroup M of H that contains $\mathrm{C}_H(w)$. We will establish that there exists a non-abelian group G containing E with index at most 2 and a maximal subgroup U of G containing w and $\mathrm{C}_G(w)$ such that G acts transitively, faithfully, and primitively on G/U, a set of size at most 64. If H acts faithfully on H/M, then we set $G = H$ and $U = M$. Therefore, suppose that H does not act faithfully on H/M. Since M contains all elements that act trivially on H/M, it hence follows that M contains a non-trivial normal subgroup N of H. Thus, $N \cap E \trianglelefteq E$. If $N \cap E = 1$, then $|N| = 2$. Since $N \trianglelefteq H$, H contains a central element. As seen earlier this is impossible. Therefore and because E is simple, $N \cap E = E$, and hence $M \ge E$. In particular, $M = E$, $w \in E$ and $|E : \mathrm{C}_E(w)| \le |H : \mathrm{C}_H(w)| \le 64$. We set $G = E$. Let U be a maximal subgroup of E that contains $\mathrm{C}_E(w)$. Then G acts transitively, faithfully, and primitively on $G/U = E/U$, a set of size $|E : U| \le |E : \mathrm{C}_E(w)| \le 64$.

For a contradiction, assume that $U = \mathrm{C}_G(w)$. Then all point stabilisers are isomorphic to $\mathrm{C}_G(w)$. Since $w \in \mathrm{Z}(\mathrm{C}_G(w))$, all point stabilisers have non-trivial centre. Using the Primitive Permutation Groups Library [53], the first line of the following GAP code creates a list of all non-abelian primitive groups G that act on a set of size at most 64. The next line reduces this list to all groups that additionally have point stabilisers with non-trivial centre. Then the next line makes sure that G has a subgroup of index at most 2 that is non-abelian simple, and hence could possibly be E. The last line returns a list of lists of groups that E possibly could be.

```
li:=AllPrimitiveGroups(NrMovedPoints, [1..64], IsAbelian, false);;
li:=Filtered(li, x-> not IsTrivial(Center(Stabilizer(x,1))));;
li:=Filtered(li, x->true in List(LowIndexSubgroups(x,2),
↪  y->(IsSimple(y) and not(IsAbelian(y)))));;
List(li, x->Filtered(LowIndexSubgroups(x,2),IsSimple));
```

The result is that E is either $\mathcal{A}_n$ where n is a positive integer such that $5 \le n \le 11$, $\mathrm{PSL}(2,11)$, $\mathrm{PSL}(3,2)$, $\mathrm{PSU}(3,3)$, $\mathrm{PSU}(4,2)$, or $\mathrm{PSp}(6,2)$. All of these groups have been dealt with earlier, implying in every case a contradiction.

Thus, $U \gneqq \mathrm{C}_G(w)$ and $|\mathrm{C}_G(w)| \le \frac{|U|}{2}$. Hence $|G/U| \le \frac{|G|}{2\cdot|\mathrm{C}_G(w)|} = \frac{|G/\mathrm{C}_G(w)|}{2} \le \frac{64}{2} = 32$. Then we can again use GAP together with the library [53]. The following GAP code shows that E is isomorphic to either $\mathcal{A}_n$ where n is a positive integer such that $5 \le n \le 32$, M_{11}, M_{12}, M_{23}, M_{24}, $\mathrm{PSL}(2,q)$ where $q \in \{7,8,11,13,16,17,19,23,25,27,29,31\}$, $\mathrm{PSL}(3,q)$ where $q \in \{3,4,5\}$, $\mathrm{PSL}(5,2)$, $\mathrm{PSU}(3,3)$, $\mathrm{PSU}(4,2)$, or $\mathrm{PSp}(6,2)$.

```
li:=AllPrimitiveGroups(NrMovedPoints, [1..32], IsAbelian, false);;
li:=Filtered(li, x->true in List(LowIndexSubgroups(x,2),
↪  y->(IsSimple(y) and not(IsAbelian(y)))));;
List(li, x->Filtered(LowIndexSubgroups(x,2),IsSimple));
```

We again see that we excluded all of these groups earlier. This final contradiction finishes the proof.

5.2 Semi-Regular Actions of Components

We will now use Lemma 5.1 to see that the components of a group that acts with fixity 4 can only act semi-regularly if they are isomorphic to $\mathcal{A}_5$ or $\mathrm{SL}(2,5)$. Since most of the implications of a fixity-4 action are not needed to prove this, the formulation of the following lemma is slightly more general and such that it can easily be used in Lemma 5.3.

Lemma 5.2
Let E be a finite quasi-simple group acting on a set Ω. Let x be an involution that normalises E, that acts on Ω and that fixes at least one and at most four points in its action on Ω. If E acts semi-regularly on Ω, then E is isomorphic to $\mathcal{A}_5$ or $\mathrm{SL}(2,5)$.

Proof:

Assume that the lemma is false and let (E,Ω,x) be a minimal counterexample (regarding $|E|$). Then for every triple (L,Γ,v) where L is a quasi-simple group acting on the set Γ, where v is an involution that normalises L, that acts on Γ, and that fixes at most four points in Γ, and such that $|L|<|E|$, L is isomorphic to $\mathcal{A}_5$ or $\mathrm{SL}(2,5)$ if L acts semi-regularly on Ω. Additionally, E acts semi-regularly and is isomorphic to neither $\mathcal{A}_5$ nor $\mathrm{SL}(2,5)$. Then $\mathrm{C}_E(x)$ acts semi-regularly on $\mathrm{fix}_\Omega(x)$, a set with at most four elements, and hence $|\mathrm{C}_E(x)|\leq 4$.

Let $\alpha\in\Omega$ be fixed by x. Let $y\in E\langle x\rangle$ be such that y fixes a point in $\alpha^{E\langle x\rangle}=\alpha^{\langle x\rangle E}=\alpha^E$. Since $E\langle x\rangle$ acts transitively on α^E, y is conjugate to an element in $(E\langle x\rangle)_\alpha=\langle x\rangle E_\alpha=\langle x\rangle$, and hence y fixes at most four points. Thus, there exists $k\in\{1,2,3,4\}$ such that $E\langle x\rangle$ acts transitively, faithfully, and with fixity k on α^E.

Assume, for a contradiction, that $|\mathrm{Z}(E)|$ is even. Since $\mathrm{Z}(E)$ is a characteristic subgroup of E and x normalises E, x acts on $\mathrm{Z}(E)$ by conjugation. Thus the 2-element x acts on the group $\mathrm{Z}(E)$ which has even order, and hence by 8.1.4 in [65], $\mathrm{C}_{\mathrm{Z}(E)}(x)\neq 1$. Let $Z=\mathrm{C}_{\mathrm{Z}(E)}(x)$. Then $Z\leq\mathrm{Z}(E)$ and $Z\trianglelefteq E\langle x\rangle$. By Lemma 4.1 and the fact that $|\alpha^E|=|E|=|E/\mathrm{Z}(E)|\cdot|\mathrm{Z}(E)|\geq|E/\mathrm{Z}(E)|\cdot|Z|\geq 60|Z|>4|Z|$, the group $(E\langle x\rangle)/Z$ acts transitively, non-regularly, and with fixity at most k on $\Gamma\coloneqq\{\alpha^Z\mid\alpha\in\alpha^E\}$. Then Zx fixes α^Z, acts on Γ, has at most $k\leq 4$ fixed points in Γ, normalises E/Z, and is an involution. Additionally $\mathrm{Z}(E/Z)=\mathrm{Z}(E)/Z$ and $(E/Z)/\mathrm{Z}(E/Z)=(E/Z)/(\mathrm{Z}(E)/Z)\cong E/\mathrm{Z}(E)$. In particular, E/Z is quasi-simple and isomorphic to neither $\mathcal{A}_5$ nor $\mathrm{SL}(2,5)$ because E is isomorphic to neither $\mathcal{A}_5$ nor $\mathrm{SL}(2,5)$. Thus, the fact that (E,Ω,x) is a minimal counterexample implies for $(E/Z,\Gamma,Zx)$ that there exists a non-trivial element in $Zg\in E/Z$ that fixes a point $\omega^Z\in\Gamma$. Then $\omega^Z=(\omega^Z)^{Zg}=(\omega^g)^Z$, and hence $\omega^g\in\omega^Z$. This implies that there exists an element $z\in Z$ such that $\omega^g=\omega^z$. Then $gz^{-1}\in E_\omega=1$. Thus $g\in Z$, and hence $Zg=Z$ contradicting the fact that Zg is non-trivial. This contradiction yields that $|\mathrm{Z}(E)|$ is odd.

Therefore, x acts coprimely on $\mathrm{Z}(E)$ and it follows that $\mathrm{C}_{E/\mathrm{Z}(E)}(\mathrm{Z}(E)x)=\mathrm{C}_E(x)\,\mathrm{Z}(E)/\mathrm{Z}(E)$. Thus, $|\mathrm{C}_{E/\mathrm{Z}(E)}(\mathrm{Z}(E)x)|=\frac{|\mathrm{C}_E(x)\,\mathrm{Z}(E)|}{|\mathrm{Z}(E)|}\leq\frac{|\mathrm{C}_E(x)|\cdot|\mathrm{Z}(E)|}{|\mathrm{Z}(E)|}=|\mathrm{C}_E(x)|\leq 4$. Then $\mathrm{Z}(E)x$ has order 2 and is an automorphism of the non-abelian simple group $E/\mathrm{Z}(E)$ and $|\mathrm{C}_{E/\mathrm{Z}(E)}(\mathrm{Z}(E)x)| \leq 4$. By Lemma 5.1, it follows that $E/\mathrm{Z}(E)$ is isomorphic to $\mathcal{A}_5$. As a consequence, E is either $\mathcal{A}_5$ or $\mathrm{SL}(2,5)$ contrary to our assumption that (E, Ω, x) is a counter example.

The assumption that all components act semi-regularly is a strong restriction and leads to a number of conclusions. They are collected in Lemma 5.3. The lemma is slightly more technical but summarises a situation that will appear a number of times in the proofs afterwards.

The first section of the following proof is based on unpublished work by Barbara Baumeister that also inspired parts of Lemma 5.4. She additionally influenced the development process of Lemma 5.2 by pointing out that $\mathrm{SL}(2,5)$ can act semi-regularly and be a component of a group acting transitively and with fixity 4, after the same was already established for $\mathcal{A}_5$.

Lemma 5.3
Let G be a finite group acting transitively, faithfully, and with fixity 4 on a set Ω. Let n be a positive integer and let $L_1, \dots, L_n$ be components of G such that $\mathrm{E}(G) = L_1 * \dots * L_n$. Suppose that for all $i \in \{1, \dots, n\}$, L_i acts semi-regularly on Ω. Let $j \in \{1, \dots, n\}$ and $x \in G$ of prime order such that x fixes a point in Ω. Then the following hold:

(a) The point stabilisers are $\{2,3\}$-groups.

(b) $L_j/\mathrm{Z}(L_j) \cong \mathcal{A}_5$.

(c) $x \in \mathrm{N}_G(L_j)$ and $|\mathrm{C}_{L_j}(x)| \leq 4$.

(d) If x fixes exactly four points in Ω, then x is an involution.

Proof:
Let $j \in \{1, \dots, n\}$ and let x be an element of G of prime order and such that x fixes a point $\alpha \in \Omega$. Assume, for a contradiction, that $x \notin \mathrm{N}_G(L_j)$. Let $L = \{l l^x l^{x^2} \dots l^{x^{p-1}} \mid l \in L_j\}$. Since p is a prime, all elements in the set $\{L_j, L_j^x, \dots, L_j^{x^{p-1}}\}$ are pairwise distinct components of G. Thus they centralise each other, and hence most properties of L_j are conveyed to L, in particular L is a subgroup of G, quasi-simple, and centralised by x. Thus, L acts on $\mathrm{fix}_\Omega(x)$, a set of size at most 4. If $L \not\leq G_\alpha$, then $1 < |\alpha^L| \leq |\mathrm{fix}_\Omega(x)| \leq 4$, and hence L has a proper subgroup of index at most 4 contradicting Lemma 2.5. As a consequence, $L \leq G_\alpha$. Since L is not solvable, there exists a prime $r \geq 5$ such that r divides the order of L. Hence r divides $|\mathrm{E}(G)|$ and $|G_\alpha|$. By Corollary 2.14, G_α contains a Sylow r-subgroup R of G. Then $R \cap \mathrm{E}(G) \in \mathrm{Syl}_r(\mathrm{E}(G))$, and hence $R \cap \mathrm{E}(G)$ is non-trivial. Additionally, for all $i \in \{1, \dots, n\}$, it follows that $R \cap L_i \in \mathrm{Syl}_r(L_i)$ because $L_i \trianglelefteq \mathrm{E}(G)$. Since $|\mathrm{E}(G)| = |L_1 * \dots * L_n|$ and this number divides $|L_1| \cdot \ldots \cdot |L_n|$, there exists $i \in \{1, \dots, n\}$ such that r is a divisor of $|L_i|$. Hence $1 \neq R \cap L_i \leq G_\alpha$ contradicting the fact that $L_i \cap G_\alpha = 1$. So the assumption that $x \notin \mathrm{N}_G(L_j)$ was incorrect.

By Lemma 2.13, for all $j \in \{1, \ldots, n\}$, $|\,\mathrm{C}_{L_j}(x)| = |\,\mathrm{C}_{L_j}(x) : \mathrm{C}_{L_j}(x) \cap G_\alpha| \leq |\,\mathrm{C}_G(x) : \mathrm{C}_{G_\alpha}(x)| \leq 4$.

Assume, for a contradiction, that G_α is not a $\{2,3\}$-group. Let $y \in G_\alpha$ be of prime order $p \geq 5$. Then $|\,\mathrm{C}_{L_1}(y)| \leq 4$ and y acts as an automorphism of prime order $p \geq 5$ on L_1. Thus, the theorem in [34] together with the fact that L_1 is quasi-simple, and hence not solvable, implies that $\mathrm{C}_{L_1}(y)$ is not a 3-group. Thus $|\,\mathrm{C}_{L_1}(y)| \in \{2,4\}$, and therefore $\mathrm{C}_{L_1}(y)$ is an abelian 2-group. However, this is a contradiction to the theorem in [35]. As a consequence, G_α is a $\{2,3\}$-group.

Let $j \in \{1, \ldots, n\}$ and let $z \in G_\alpha$ be of prime order and such that z fixes exactly four points in Ω. Then the earlier proven implies that z has order 2 or 3 and that $z \in \mathrm{N}_G(L_j)$. First assume z has order 3. Then by the theorem in [35] and the fact that L_j is not solvable, $|\,\mathrm{C}_{L_j}(z)| = 3$. Thus, L_j has an order divisible by 3. Let $h \in \mathrm{C}_{L_j}(z)$ be of order 3. Then h acts on $\mathrm{fix}_\Omega(z)$, a set of size 4. On the other hand, h does not fix any points in Ω because L_j acts semi-regular on Ω. This is a contradiction. Therefore z is an involution. Then Lemma 5.2 shows that $L_j \cong \mathcal{A}_5$ or that $L_j \cong \mathrm{SL}(2,5)$.

5.3 The Structure of the Product of the Components

Now we have all results together such that we can prove that a group that acts with fixity 4 has at most one component. Afterwards we can determine the structure of this component even further.

Lemma 5.4
Let G be a finite group that acts transitively, faithfully, and with fixity 4 on a set Ω. Suppose that $\mathrm{E}(G)$ is non-trivial. Then $\mathrm{E}(G)$ is quasi-simple.

Proof:

Assume, for a contradiction, that $\mathrm{E}(G)$ is not quasi-simple. Then there exists a positive integer $n \geq 2$ and components $L_1, \ldots, L_n$ of G such that $\mathrm{E}(G) = L_1 * L_2 * \ldots * L_n$.

First we will prove that all components act semi-regularly on Ω. For a contradiction, assume that there exists $i \in \{1, \ldots, n\}$ and $\omega \in \Omega$ such that $L_i \cap G_\omega \neq 1$. Let $y_i \in L_i \cap G_\omega$ be non-trivial. Then for all $j \in \{1, \ldots, n\} \setminus \{i\}$, it follows that $[L_i, L_j] = 1$, and hence $L_j \leq \mathrm{C}_G(y_i)$. Thus, L_j acts on $\mathrm{fix}_\Omega(y_i)$. If $L_j \nleq G_\omega$, then $1 < |\omega^{L_j}| \leq |\,\mathrm{fix}_\Omega(y_i)| \leq 4$ and hence $L_j \cap G_\omega$ is a proper subgroup of the quasi-simple group L_j of index at most 4, contradicting Lemma 2.5. As a consequence, $L_j \leq G_\omega$. On the other hand, $L_i \leq \mathrm{C}_G(L_j)$. Then Lemma 2.13 implies that $|L_i : L_i \cap G_\omega| \leq |\,\mathrm{C}_G(L_j) : \mathrm{C}_{G_\omega}(L_j)| \leq 4$, and hence by Lemma 2.5, $L_i \leq G_\omega$. Therefore $\mathrm{E}(G) = L_1 * \ldots * L_n \leq G_\omega$, contradicting the assumptions that $\mathrm{E}(G) \neq 1$ and that G acts faithfully and transitively. As a consequence, all components act semi-regularly.

Let $\alpha \in \Omega$ and let $x \in G_\alpha$ be such that x has prime order and fixes exactly four points in Ω. Then Lemma 5.3 proves that G_α is a $\{2,3\}$-group and that for all $j \in \{1, \ldots, n\}$, $L_j / \mathrm{Z}(L_j) \cong \mathcal{A}_5$, $x \in \mathrm{N}_G(L_j)$, $|\,\mathrm{C}_{L_j}(x)| \leq 4$, and that x is an involution.

Let $j \in \{1, \ldots, n\}$. Since x acts on $L_j / \mathrm{Z}(L_j) \cong \mathcal{A}_5$, either x fixes all elements or it acts as an automorphism of order 2 on the simple group $L_j / \mathrm{Z}(L_j)$. Then the table on page 2 in [28] implies that $|\,\mathrm{C}_{L_j/\mathrm{Z}(L_j)}(x)| \geq 4$ and that $|\,\mathrm{C}_{L_j}(x)| \geq 4$. Since $|\,\mathrm{C}_{L_j}(x)| \leq 4$, it follows that $|\,\mathrm{C}_{L_j}(x)| = 4$.

Since $L_1 \cap L_2 \leq \mathrm{Z}(L_1)$ has order at most 2, $|\,\mathrm{C}_{L_1}(x) * \mathrm{C}_{L_2}(x)| = \frac{|\,\mathrm{C}_{L_1}(x)| \cdot |\,\mathrm{C}_{L_2}(x)|}{|\,\mathrm{C}_{L_1}(x) \cap \mathrm{C}_{L_2}(x)|}$ is divisible by $\frac{|\,\mathrm{C}_{L_1}(x)| \cdot |\,\mathrm{C}_{L_2}(x)|}{2} = 8$. On the other hand, since $\mathrm{C}_{L_1}(x) * \mathrm{C}_{L_2}(x) \leq \mathrm{C}_{L_1 * L_2}(x)$, it Lemma 2.13 yields that $|\,\mathrm{C}_{L_1}(x) * \mathrm{C}_{L_2}(x) : (\mathrm{C}_{L_1}(x) * \mathrm{C}_{L_2}(x)) \cap G_\alpha| \leq |\,\mathrm{C}_{L_1 * L_2}(x) : \mathrm{C}_{L_1 * L_2}(x) \cap G_\alpha| \leq 4$. Therefore $(\mathrm{C}_{L_1}(x) * \mathrm{C}_{L_2}(x)) \cap G_\alpha$ has an order divisible by 2, and hence there exists an involution $t \in (\mathrm{C}_{L_1}(x) * \mathrm{C}_{L_2}(x)) \cap G_\alpha$. Then Lemma 5.3 (c) proves that $\mathrm{C}_{L_1}(t)$ and $\mathrm{C}_{L_2}(t)$ both have order at most 4. Since t acts as an automorphism on L_1 and L_2, the table on page 2 in [28] again shows that $|\,\mathrm{C}_{L_1}(t)| = 4 = |\,\mathrm{C}_{L_2}(t)|$. Let $t_1 \in L_1$ and $t_2 \in L_2$ be such that $t = t_1 t_2$. Then $\mathrm{C}_{L_1}(t) = \mathrm{C}_{L_1}(t_1)$ and $\mathrm{C}_{L_2}(t) = \mathrm{C}_{L_2}(t_2)$. Since $L_1 \cap G_\alpha = 1 = L_2 \cap G_\alpha$, both t_1 and t_2 are non-trivial. Then $1 = t^2 = (t_1 t_2)^2 = t_1^2 t_2^2$, and hence $t_1^2 = t_2^{-2} \in L_1 \cap L_2 \leq \mathrm{Z}(L_1) \cap \mathrm{Z}(L_2)$. Thus, either t_1 and t_2 are involutions or t_1^2 and t_2^2 are involutions. In the first case $L_1 \cong \mathcal{A}_5 \cong L_2$ because otherwise t_1 and t_2 would each be the central involution of $\mathrm{SL}(2,5)$ contradicting $|\,\mathrm{C}_{L_1}(t_1)| = 4 = |\,\mathrm{C}_{L_2}(t_2)|$. In the second case t_1 and t_2 both have order 4 and $L_1 \cong \mathrm{SL}(2,5) \cong L_2$ and $L_1 \cap L_2 = \mathrm{Z}(L_1) = \mathrm{Z}(L_2)$.

Calculations in $\mathcal{A}_5 \times \mathcal{A}_5$ and $\mathrm{SL}(2,5) * \mathrm{SL}(2,5)$ done by hand or with the use of lines 10–12 and 23–25 of the GAP code 5.1 imply that $|\,\mathrm{C}_{L_1 * L_2}(t)| = 16$. Thus by Lemma 2.13, $\frac{16}{|\,\mathrm{C}_{L_1 * L_2}(t) \cap G_\alpha|} = |\,\mathrm{C}_{L_1 * L_2}(t) : (\mathrm{C}_{L_1 * L_2}(t)) \cap G_\alpha| \leq 4$, and hence $|(L_1 * L_2) \cap G_\alpha|$ is divisible by 4. Then the results of lines 14–16 and 27–29 of the GAP code 5.1, after defining the groups in lines 2–8 and 20–21, respectively, show that $(L_1 * L_2) \cap G_\alpha$ is isomorphic to E_4, $\mathcal{A}_4$, or $\mathcal{A}_5$. Therefore, $L_1 * L_2$ contains an elementary abelian group V of order 4. Together with lines 17 and 30 of Program Code 5.1, it follows that $|\,\mathrm{N}_{(L_1 * L_2)}(V)|$ is divisible by 24.

If 3 does not divide $|(L_1 * L_2) \cap G_\alpha|$, then $(L_1 * L_2) \cap G_\alpha = V$, and hence $|\,\mathrm{N}_{L_1 * L_2}(V) : \mathrm{N}_{(L_1 * L_2) \cap G_\alpha}(V)| = |\,\mathrm{N}_{L_1 * L_2}(V) : V| \geq \frac{24}{4} = 6$. On the other hand by Lemma 2.13, $4 \geq |\,\mathrm{N}_{L_1 * L_2}(V) : \mathrm{N}_{(L_1 * L_2) \cap G_\alpha}(V)|$. This contradiction yields that $(L_1 * L_2) \cap G_\alpha$ contains an element a of order 3. Then $(L_1 \times L_2) \cap G_\alpha$ is isomorphic to either $\mathcal{A}_4$ or $\mathcal{A}_5$. Since G_α is a $\{2,3\}$-group, $(L_1 \times L_2) \cap G_\alpha$ is isomorphic to $\mathcal{A}_4$, in particular it does not have a subgroup isomorphic to $\mathcal{S}_3$.

Since a fixes α and L_1 and L_2 act semi-regularly on Ω and since $\mathrm{Z}(L_1) = \mathrm{Z}(L_2)$ has order at most 2, there exists $a_1 \in L_1$ and $a_2 \in L_2$ of order 3 such that $a = a_1 a_2$. Let $i \in \{1,2\}$. Then $\langle a_i \rangle$ is a Sylow 3-subgroup of L_i and there exists an element $b_i \in L_i$ that inverts a_i and such that $b_i^2 \in \mathrm{Z}(L_i)$. Let $b = b_1 b_2$. Then b is an involution in $L_1 * L_2$ and $a^b = (a_1 a_2)^{b_1 b_2} = {a_1}^{b_1} {a_2}^{b_2} = {a_1}^{-1} {a_2}^{-1} = (a_1 a_2)^{-1} = a^{-1}$. Thus, $\langle a, b \rangle$ is isomorphic to $\mathcal{S}_3$. Since $|L_1 * L_2|$ is divisible by 9 but $(L_1 * L_2) \cap G_\alpha \cong \mathcal{A}_4$, it follows that $\alpha^{L_1 * L_2}$ has a size divisible by 3.

```
#SL(2,5)*SL(2,5)
D:=DirectProduct(SL(IsPermGroup,2,5),SL(IsPermGroup,2,5));;
li:=Filtered(AllSubgroups(Center(D)), x->Order(x)=2);
z:=li[3];; #depending on the output of the last line
L1L2:=D/z;;
li:=Filtered(List(ConjugacyClassesSubgroups(L1L2),Representative), x->Order(x)=120);;
li:=Filtered(li, x->IdGroup(x)=IdGroup(SL(2,5)));
L1:=li[1]; L2:=li[2];

li:=Filtered(List(ConjugacyClasses(L1L2),Representative), x->Order(x)=2);;
li:=Filtered(li,x->(not x in L1) and (not x in L2));;
List(li,x->Order(Centralizer(L1L2,x)));

li:=List(ConjugacyClassesSubgroups(L1L2),Representative);;
li:=Filtered(li, x->IsTrivial(Intersection(x,L1)) and IsTrivial(Intersection(x,L2)));;
li:=Filtered(li, x->Order(x) mod 4 =0);
List(Filtered(li, x->Order(x)=4), x->Order(Normalizer(L1L2,x)));

#A5*A5
L1:=AlternatingGroup(5);; L2:=Group((6,7,8,9,10),(6,7,8));
L1L2:=DirectProduct(L1,L2);

li:=Filtered(List(ConjugacyClasses(L1L2),Representative), x->Order(x)=2);;
li:=Filtered(li,x->(not x in L1) and (not x in L2));;
List(li,x->Order(Centralizer(L1L2,x)));

li:=List(ConjugacyClassesSubgroups(L1L2),Representative);;
li:=Filtered(li, x->IsTrivial(Intersection(x,L1)) and IsTrivial(Intersection(x,L2)));;
li:=Filtered(li, x->Order(x) mod 4 =0);
List(Filtered(li, x->Order(x)=4), x->Order(Normalizer(L1L2,x)));
```

Program Code 5.1: Structure Details of $L_1 * L_2$

Then the fact, that G acts with fixity 4 implies that every non-trivial 3-element in $L_1 * L_2$ has either none or three fixed points in $\alpha^{L_1 * L_2}$. In particular, a fixes exactly three points in $\alpha^{L_1 * L_2}$. Since $b \in L_1 * L_2$ is an involution that acts on $\langle a \rangle$, it leaves the set of fixed points of a invariant. Thus b fixes a point $\beta \in \alpha^{L_1 * L_2}$ that is also fixed by a, and hence $\langle a, b \rangle \leq (L_1 * L_2) \cap G_\beta$. By the choice of β, the groups $(L_1 * L_2) \cap G_\alpha$ and $(L_1 * L_2) \cap G_\beta$ are conjugate in $L_1 * L_2$. Therefore $\mathcal{A}_4 \cong (L_1 * L_2) \cap G_\alpha$ also contains a subgroup isomorphic to $\langle a, b \rangle \cong \mathcal{S}_3$. This final contradiction proves that $\mathrm{E}(G)$ cannot have more than one component, and hence is quasi-simple.

Lemma 5.5
Let G be a finite group that acts transitively, faithfully, and with fixity 4 on a set Ω and let $\alpha \in \Omega$. Suppose that $\mathrm{E}(G)$ is non-trivial. Then $\mathrm{E}(G) \cap G_\alpha \neq 1$ or $\mathrm{E}(G)/\,\mathrm{Z}(\mathrm{E}(G)) \cong \mathcal{A}_5$.

Proof:
Suppose that $\mathrm{E}(G) \cap G_\alpha = 1$. Since $\mathrm{E}(G) \trianglelefteq G$, it follows that $\mathrm{E}(G)$ acts semi-regularly on Ω. By Lemma 5.4, $\mathrm{E}(G)$ is quasi simple. Thus, G has a unique component. Then by Lemma 5.3 (b), $\mathrm{E}(G)/\,\mathrm{Z}(\mathrm{E}(G)) \cong \mathcal{A}_5$.

If G is a group that acts transitively, faithfully, and with fixity 4, then every element in $\mathrm{E}(G)$ can fix at most four points. Depending on the number of fixed points that non-trivial elements in $\mathrm{E}(G)$ have, we can give further details of the structure of $\mathrm{E}(G)$. This is done in Theorem 5.6.

Theorem 5.6
Let G be a finite group acting faithfully, transitively, and with fixity 4 on a set Ω. Let $E = \mathrm{E}(G)$ and let $\alpha \in \Omega$. Suppose that E is non-trivial. Then E is quasi-simple and one of the following holds.

(1) E is isomorphic to $\mathcal{A}_5$ or to $\mathrm{SL}(2, 5)$ and acts semi-regularly on Ω.

(2) E is simple and acts with fixity 2 on α^E. In particular, there exists a prime power q such that E is isomorphic to $\mathrm{PSL}(3, 4)$, $\mathrm{PSL}(2, q)$, or $\mathrm{Sz}(q)$.

(3) E is isomorphic to $\mathcal{A}_6$ and acts with fixity 3 on α^E. In particular, $|\Omega| \in \{6, 12\}$ and one of the following is true.

   (a) If $|\Omega| = 6$, then G is isomorphic to $\mathcal{S}_6$.

   (b) If $|\Omega| = 12$, then G is isomorphic to one of the following groups:

      (I) M_{10}

      (II) $\mathrm{PGL}(2, 9)$

      (III) $\mathrm{Aut}(\mathcal{A}_6)$

(4) E acts transitively, faithfully, and with fixity 4 on α^E.

Proof:

By Lemma 5.4, E is quasi-simple. Suppose that E acts semi-regularly on Ω. Then Lemma 5.5 proves that $E/\mathrm{Z}(E) \cong \mathcal{A}_5$, and hence E is isomorphic to $\mathcal{A}_5$ or $\mathrm{SL}(2,5)$. This is part (1). Therefore, from now on, suppose that E does not act semi-regularly. Since $E = \mathrm{E}(G)$ is a characteristic subgroup of G, $E_\alpha \neq 1$. Thus, E acts transitively, non-regularly, and with fixity at most 4 on α^E. If α^E had less then five elements, then by Lemma 2.5, $E \leq G_\alpha$ contradicting the faithful action of G. Thus $|\alpha^E| \geq 5$, and hence E acts faithfully on α^E.

Assume, for a contradiction, that $\mathrm{E}(G)$ acts with fixity 1 on α^E. Then E is a Frobenius group, and hence has a non-trivial Frobenius kernel $K \trianglelefteq E$. Then $K = E$ or $K \leq \mathrm{Z}(E)$. In the latter case, $E \leq \mathrm{C}_E(K)$. Thus by 8.1.12 in [65], $E \leq K$. As a consequence, in both cases $E = K$ and E does not have a Frobenius complement. This contradiction implies that E acts transitively, faithfully, and with fixity 2, 3 or 4 on α^E. Then the last case yields statement (4) in this lemma.

Suppose that E acts with fixity 2 on α^E. Then Lemma 4.3 states that E is a simple group. Thus by Theorem 1.2 in [71] there exists a prime power q such that $E/\mathrm{Z}(E)$ is isomorphic to $\mathrm{PSL}(3,4)$, $\mathrm{PSL}(2,q)$, or $\mathrm{Sz}(q)$. This shows statement (2).

Therefore, the only case remaining to be analysed is that E acts with fixity 3 on α^E. Then by Lemma 4.3, the quasi-simple group E is in fact simple. By Lemma 2.2, all E-orbits in Ω have the same size.

Let $a \in E_\alpha$ be of prime order p and such that it fixes exactly three points in α^E. Then $|\alpha^E| \equiv 3 \mod p$. We will make a case distinction on whether $\alpha^E = \Omega$ or not.

First suppose that $\alpha^E \neq \Omega$. Then a can fix up to four points in Ω, because G acts with fixity 4. However, we will first see that a fixes exactly three points in Ω. For a contradiction, assume that a fixes $\delta \in \Omega \backslash \alpha^E$. Then $\mathrm{fix}_{\Omega\backslash\alpha^E}(a) = \{\delta\}$, and hence $1 \equiv |\delta^E| = |\alpha^E| \equiv 3 \mod p$. Therefore $p = 2$, a is an involution, $|E_\alpha|$ is even, and $|\alpha^E|$ is odd. By Theorem 1.1 in [72], one of the following cases holds: $E \cong \mathcal{A}_5$ and $|\alpha^E| = 15$, $E \cong \mathcal{A}_6$ and $|\alpha^E| = 15$, $E \cong \mathrm{PSL}(2,7)$ and $|\alpha^E| = 7$, $E \cong \mathcal{A}_7$ and $|\alpha^E| = 15$, $E \cong \mathrm{PSL}(2,11)$ and $|\alpha^E| = 11$, or $E \cong \mathrm{M}_{11}$ and $|\alpha^E| = 11$. Then the following GAP commands return the table of marks stored in [74] and we can read off that all of these groups have just one conjugacy class of involutions and that all involutions have always exactly three fixed points when E acts on a set of the specified size. Hence no involution can have exactly one fixed point on δ^E. This is a contradiction.

```
Display(TableOfMarks("A5"));
Display(TableOfMarks("A6"));
Display(TableOfMarks("L2(7)"));
Display(TableOfMarks("A7"));
Display(TableOfMarks("L2(11)"));
Display(TableOfMarks("M11"));
```

As a consequence, a fixes exactly three points in Ω. We still suppose $\alpha^E \neq \Omega$. Thus, there exists a point $\delta \in \Omega \setminus \alpha^E$. Hence, $0 \equiv |\delta^E| = |\alpha^E| \equiv 3 \mod p$. Therefore, $p = 3$ and $|E_\alpha|$ and $|\alpha^E|$ are both divisible by 3. By Theorem 1.1 in [72] the group E is isomorphic to $\mathcal{A}_6$ and $|\alpha^E| \in \{6, 15\}$ or $\mathcal{A}_7$ and $|\alpha^E| = 15$. If $|\alpha^E| = 15$, then the GAP commands `Display(TableOfMarks("A6"));` and `Display(TableOfMarks("A7"));` again reveal that in both cases an involution $t \in E$ has three fixed points in α^E and three fixed points in δ^E. Since these E-orbits do not have a point in common, $t \in E \leq G$ has six fixed points contradicting the fact that G acts with fixity 4. As a consequence, $|\alpha^E| = 6$ and E is isomorphic to $\mathcal{A}_6$. Then the command `Display(TableOfMarks("A6"));` shows that every involution $t \in E$ has exactly two fixed points on α^E and therefore on every E-orbit in Ω. Since G acts with fixity 4 on Ω, every involution $t \in E \leq G$ can have at most four fixed points, and hence Ω has at most two E-orbits. Therefore, $|\Omega| = 2 \cdot |\alpha^E| = 2 \cdot 6 = 12$. Using the Transitive Groups Library [52] together with Remark 2.18, the following GAP code implies that G is isomorphic to M_{10}, $\mathrm{PGL}(2,9)$, or $\mathrm{Aut}(\mathcal{A}_6)$, implying (3) (b). It tests for all transitive groups whether or not they act with fixity 4. Since this implies that all five-point stabilisers are trivial, the additional filter does not change the result. The same holds for the condition that the size of G is at least 360 because $E \cong \mathcal{A}_6$ is a normal subgroup of G. This also justifies the second command.

```
li:=AllTransitiveGroups(NrMovedPoints, 12,
↪   x->IsTrivial(Stabilizer(x,[1..5],OnTuples)), true,
↪   z->Order(z)>=360, true, y->TestFixity(y,MovedPoints(y),4),
↪   true);;
Filtered(li, x->360 in List(NormalSubgroups(x),Order));
```

Therefore instead suppose that $\alpha^E = \Omega$. Thus, E acts transitively on Ω. Using Theorem 1.1 in [72], we see that there is a distinction on whether E_α is cyclic or not. Therefore we show, as an intermediate step, that E_α is not cyclic and then a short GAP program will help us to derive (3) (a).

For a contradiction, assume that E_α is cyclic. Thus, by Theorem 1.1 in [72], the order of E_α is coprime to 6. Since every element in $b \in E_\alpha$ centralises a, it stabilises the set of fixed points of a. Thus, the fact that $|\operatorname{fix}_\Omega(a)| = 3$ implies that b fixes all elements in $\operatorname{fix}_\Omega(a)$. As a consequence, E_α is a three-point stabiliser. Then by Corollary 2.10, for every non-trivial element $b \in E_\alpha$ the number of fixed points in Ω is $3 = \frac{|\mathrm{N}_E(\langle b \rangle)|}{|E_\alpha|}$. Thus, for all non-trivial $b \in E_\alpha$, $|\mathrm{N}_E(\langle b \rangle)| = 3|E_\alpha|$. In particular, all normalisers of non-trivial subgroups of E_α in E have the same size $3|E_\alpha|$. By Corollary 2.14, E_α contains a Sylow p-subgroup P of E, and hence $|\mathrm{N}_E(P)| = 3|E_\alpha|$. Since E_α is cyclic, $E_\alpha \leq \mathrm{C}_E(P) \leq \mathrm{N}_E(P)$. If $\mathrm{C}_E(P) = \mathrm{N}_E(P)$, then Burnside's p-complement theorem (see for example 7.2.1 in [65]) implies that the simple group E contains a normal p-complement. This contradiction shows that $\mathrm{C}_E(P) \lneqq \mathrm{N}_E(P)$. As a consequence, $\mathrm{C}_E(P) = E_\alpha$. Assume for a contradiction that $\mathrm{C}_E(a) \neq E_\alpha$. Since $E_\alpha \leq \mathrm{C}_E(a)$ and $|\mathrm{N}_E(\langle a \rangle)| = 3 \cdot |E_\alpha|$, this implies that $\mathrm{N}_E(\langle a \rangle) = \mathrm{C}_E(a)$

has an element d of order 3. Since $\langle a\rangle = \Omega_1(P)$, this element d acts trivially on $\Omega_1(P)$. Thus, Lemma 2.3 (c) yields that d acts trivially on P. Hence $d \in \mathrm{C}_E(P)$, contradicting the fact that $\mathrm{C}_E(P) = E_\alpha$ has order coprime to 6. Therefore $\mathrm{C}_E(a) = E_\alpha$.

Let $x \in G_\alpha$ be of prime order and such that x fixes exactly four points in Ω. Then $x \in \mathrm{N}_G(E_\alpha)$. Let $c \in \mathrm{N}_E(\langle a\rangle)$ be of order 3. Then both c and x map a to some power of a. Therefore $a^{[c,x]} = a$. Since $E \trianglelefteq G$, $c^x \in E$, and thus $[c,x] = c^{-1}c^x \in \mathrm{C}_E(a) = E_\alpha$.

The element c has order 3 and therefore does not fix a point in Ω because otherwise it would lie in a point stabiliser, but their orders are coprime to 6. Let $\beta = \alpha^c$ and let $\gamma = \beta^c$. Since $c \in \mathrm{N}_E(\langle a\rangle)$ acts on the set of orbits of $\langle a\rangle$ in Ω, it follows that $\mathrm{fix}_\Omega(a) = \{\alpha,\beta,\gamma\}$ and $\gamma^c = \alpha$. Then $\gamma^{x^{-1}cx} = \alpha^{[c,x]} = \alpha$ because $[c,x] \in \mathrm{C}_E(a) = E_\alpha$. Since $x \in G_\alpha$, it follows that $\gamma^{x^{-1}} = \alpha^{x^{-1}c^{-1}} = \alpha^{c^{-1}} = \gamma$. Thus, x fixes γ. Then x acts on $\mathrm{fix}_\Omega(a) = \{\alpha,\beta,\gamma\}$, because x normalises $\langle a\rangle$, and x fixes α and γ. Therefore x also fixes β.

Since x has four fixed points there exists $\delta \in \Omega \setminus \mathrm{fix}_\Omega(a)$ such that $\delta^x = \delta$. Now $(\delta^c)^x = \delta^{x^{-1}cx} = \delta^{cc^{-1}x^{-1}cx} = (\delta^c)^{[c,x]} \in (\delta^c)^{E_\alpha}$. Therefore x leaves the E_α-orbit $(\delta^c)^{E_\alpha}$ invariant. Since x has order coprime to $|E_\alpha|$, this implies that x fixes a point ϑ in $(\delta^c)^{E_\alpha}$. Since $(\delta^c)^{E_\alpha}$ and $\mathrm{fix}_\Omega(a)$ do not have a point in common, the fact that x fixes four points yields that $\vartheta = \delta$. Thus, $(\delta^c)^{E_\alpha} = \delta^{E_\alpha}$ and c leaves this orbit invariant. This is impossible because c acts fixed-point-freely on Ω and has order 3 coprime to $|E_\alpha|$. As a consequence, the assumption that E_α is cyclic was false.

Therefore, by Theorem 1.1 in [72], the size of Ω is 15, 6, 7, or 11. Then the following GAP code uses the Transitive Groups Library [52] together with Remark 2.18 and shows that the only non-solvable groups that act with fixity 4 on a set of size 15, 6, 7, or 11 are $\mathcal{S}_6$ and $\mathcal{A}_7$.

```
li:=AllTransitiveGroups(NrMovedPoints, [15,6,7,11],
↪  x->IsTrivial(Stabilizer(x,[1..5],OnTuples)), true, IsSolvable,
↪  false, y->TestFixity(y,MovedPoints(y),4), true);
NrMovedPoints(li[1]);
```

Since $\mathcal{A}_7$ is itself simple, this case implies $E = G$, contradicting the facts that G acts with fixity 4 on Ω but E acts with fixity 3 on Ω. Hence, $G = \mathcal{S}_6$ and $|\Omega| = 6$, implying case (3) (a).

Each of the cases in the previous theorem is necessary and groups that demonstrate this are mostly stated in Chapter 7. There the cases will reappear as cases of the Main Theorem (Theorem 7.7) and they will be accompanied by instructive examples.

However, in the case that the unique component acts semi-regularly, a few more information can be proven regarding the structure of the point stabilisers. This is done in the following lemma and will be illustrated by some examples afterwards.

Lemma 5.7
Let G be a finite group acting transitively, faithfully, and with fixity 4 on a set Ω.

Let $\mathrm{E}(G)$ act semi-regularly on Ω. If $\mathrm{E}(G) \cong \mathcal{A}_5$, then the point stabilisers of G are of isomorphism type $C_2, E_4, C_4, \mathcal{S}_3$, or $\mathcal{A}_4$. If $\mathrm{E}(G) \cong \mathrm{SL}(2,5)$, then the point stabilisers of G are of isomorphim type C_2, E_4, or C_4.

Proof:

Suppose that $\mathrm{E}(G)$ is isomorphic to either $\mathcal{A}_5$ or $\mathrm{SL}(2,5)$. Let $\alpha \in \Omega$. Since $\mathrm{E}(G) \trianglelefteq G$, the point stabiliser G_α acts on $\mathrm{E}(G)$. Let $a \in \mathrm{C}_{G_\alpha}(\mathrm{E}(G))$. Then an element $h \in \mathrm{E}(G)$ of order 5 centralises a, hence acts on the fixed points of a. If a is non-trivial, then a fixes at least one and at most four points. Then h must fix all elements in $\mathrm{fix}_\Omega(a)$ contradicting the semi-regular action of $\mathrm{E}(G)$. Thus, $\mathrm{C}_{G_\alpha}(\mathrm{E}(G)) = 1$ and G_α can be identified with a subgroup of the automorphism group of $\mathrm{E}(G)$. Moreover, by Lemma 5.3, G_α is a $\{2,3\}$-group. Since by page 2 in [28], $\mathrm{Aut}(\mathrm{E}(G)) \cong \mathcal{S}_5$, the point stabilisers are isomorphic to C_2, C_3, E_4, C_4, $\mathcal{S}_3$, C_6, D_8, $\mathcal{A}_4, D_{12}$, or $\mathcal{S}_4$.

Let $x \in G_\alpha$ be of prime order and such that x fixes exactly four points in Ω. Assume, for a contradiction, that x has order 3. Then $|\Omega| \equiv 4 \equiv 1 \mod 3$. Thus, G_α contains a Sylow 3-subgroup P of G, but then the semi-regular action of $\mathrm{E}(G)$ implies that $1 = G_\alpha \cap \mathrm{E}(G) \geq P \cap \mathrm{E}(G) \in \mathrm{Syl}_3(\mathrm{E}(G))$ and gives a contradiction. Therefore, x has order 2 and G_α has even order excluding the option C_3.

If an element $g \in G_\alpha$ of order 2 acts as an outer automorphism on $\mathrm{E}(G)$, then by the table on page 2 (or page xxix) in [28], $|\,\mathrm{C}_{\mathrm{E}(G)}(g)|$ is divisible by 6 contradicting Lemma 5.3.

Assume, for a contradiction, that G_α contains an element c of order 6. Then c^3 can be identified as a transposition in $\mathcal{S}_5 \setminus \mathcal{A}_5$. In particular, c^3 acts as an outer automorphism on $\mathrm{E}(G)$ and has order 2. As shown above, this implies a contradiction. Therefore, G_α is isomorphic to neither C_6 nor D_{12}.

If G_α is isomorphic to D_8 or $\mathcal{S}_4$, then G_α contains a Sylow 2-subgroup of $\mathcal{S}_5$. However, then G_α also contains an outer automorphism of order 2 of $\mathrm{E}(G)$, giving a contradiction, too.

As a consequence, G_α is isomorphic to $C_2, E_4, C_4, \mathcal{S}_3$, or $\mathcal{A}_4$. This implies the statement of this lemma for $\mathrm{E}(G) \cong \mathcal{A}_5$.

Therefore, suppose that $\mathrm{E}(G) \cong \mathrm{SL}(2,5)$. Assume for a contradiction that G_α is isomorphic to $\mathcal{S}_3$. Let $y \in G_\alpha$ be of order 3. Then there exists an involution $t \in G_\alpha$ that inverts y. Since $\mathrm{Z}(G) \cap G_\alpha = 1$, $\langle t \rangle \mathrm{Z}(G)$ has order 4, and hence $\mathrm{N}_G(\langle y \rangle)$ is divisible by 4. On the other hand, the Sylow 3-subgroups of G have order at least 9, and hence $|\,\mathrm{C}_G(y)|$ is also divisible by 9. Thus, the index $|\,\mathrm{N}_G(\langle y \rangle) : \mathrm{N}_{G_\alpha}(\langle y \rangle)|$ is divisible by $\frac{4 \cdot 9}{6} = 6$, contradicting Lemma 2.13. Therefore G_α is not isomorphic to $\mathcal{S}_3$ if $\mathrm{E}(G) \cong \mathrm{SL}(2,5)$.

Assume for a contradiction that G_α is isomorphic to $\mathcal{A}_4$. Let $y \in G_\alpha$ be of order 3. Since the Sylow 3-subgroups of G have order at least 9, the order of $\mathrm{C}_G(y)$ is divisible by 9. Therefore, the fact that $\mathrm{Z}(G) \leq \mathrm{C}_G(y)$ implies that $|\,\mathrm{C}_G(y)|$ is divisible by 18. The structure of G_α yields that $\mathrm{N}_{G_\alpha}(\langle y \rangle)$ has order 3, and hence $|\langle y \rangle^{G_\alpha}| = \frac{12}{3} = 4$ and equals the number of 3-subgroups of G_α. As a consequence, Lemma 2.9 implies that y fixes exactly $\frac{|\{\langle y\rangle^g \leq G_\alpha | g \in G\}| \cdot |\mathrm{N}_G(\langle y\rangle)|}{|G_\alpha|} = \frac{4 \cdot |\mathrm{N}_G(\langle y\rangle)|}{12}$ points. This is not possible because y can fix at most four points but this number is divisible by 6. As a consequence, G_α cannot be isomorphic to $\mathcal{A}_4$ if $\mathrm{E}(G) \cong \mathrm{SL}(2,5)$, and thus the lemma follows.

The following examples will show that all of the cases in the previous lemma do indeed occur.

Example 5.8
Let $E = \mathcal{A}_5$, $F = E_4$, and $G = E \times F$. As described in Lemma 2.20 and Example 2.21, the table of marks of G contains information about all transitive actions of G. The following GAP code uses the table of marks.

```
G:=DirectProduct(AlternatingGroup(5),Group((1,2),(3,4)));;
t:=TableOfMarks(G);; Display(t);
e:=RepresentativeTom(t,67);
U:=[RepresentativeTom(t,12), RepresentativeTom(t,13)];
List(U, x-> IsTrivial(Intersection(e,x)) and
↪  IsTrivial(Intersection(FittingSubgroup(G),x)));
```

We find the component E of G in line 67 of the table of marks because E is the unique group of index 4 in G. Additionally, the entries in the table of marks, in particularly in lines 12 and 13, imply by Lemma 2.20 that the only fixity-4 actions of G are decoded in lines 12 and 13. For both actions the point stabilisers are elementary abelian of order 4 and E and F have trivial intersection with a point stabiliser in each case. As a consequence, there exists a set Ω such that G acts with fixity 4 on Ω, $|\Omega| = 60$, E and F act semi-regularly on Ω and for all $\alpha \in \Omega$, G_α is elementary abliean of order 4.

Example 5.9
Similarly to Example 5.8, we can look at a group isomorphic to $C_2 \times \mathcal{A}_5$. A group G identified by `[120,35]` in the Small Groups Library [13] is such a group. Then $\mathrm{E}(G) \cong \mathcal{A}_5$ and we can use the table of marks as in the previous example.

```
G:=SmallGroup([120,35]); t:=TableOfMarks(G);;
Display(t);
e:=RepresentativeTom(t,21); U:=RepresentativeTom(t,4);
Intersection(e,U);
```

The code first defines G and then creates the table of marks of G. Since $\mathrm{E}(G)$ has index 2 in G, we find $\mathrm{E}(G)$ in line 21 of the table of marks. Additionally, we see a fixity-4 action of G in line 4 and define U to be a corresponding point stabiliser. Then U is cyclic of order 2. Since $\mathrm{E}(G) \trianglelefteq G$, the fact that $U \cap \mathrm{E}(G)$ is trivial implies that $\mathrm{E}(G)$ acts semi-regularly on G/U. Therefore, $C_2 \times \mathcal{A}_5$ is an example of a group that acts transitively and with fixity 4, such that the component is isomorphic to $\mathcal{A}_5$ and acts semi-regularly, and such that the point stabilisers are isomorphic to C_2.

We can adjust the previous GAP code for the group with ID [240,91] which is isomorphic to $\mathcal{A}_5 : C_4$. Then it reveals that line 33 of the table of marks corresponds to the component of the group and that line 10 gives a fixity-4 action. In this actions the point stabilisers are cyclic of order 4 and the component is isomorphic to $\mathcal{A}_5$ and has trivial intersection with a point stabiliser.

For the next example, let G be the group with ID [360,121]. Then $G \cong \mathcal{S}_3 \times \mathcal{A}_5$ and we can use GAP in a similar way, this time $\mathrm{E}(G)$ corresponds to line 42 and we look at a point stabiliser U corresponding to line 18. The action of G on G/U is transitive, with fixity 4, such that $\mathrm{E}(G) \cong \mathcal{A}_5$ acts semi-regularly, and such that U is isomorphic to $\mathcal{S}_3$.

For the last example with a component isomorphic to $\mathcal{A}_5$, we can use the following GAP code.

```
G:=SmallGroup([720,768]); t:=TableOfMarks(G);;
Display(t);
e:=RepresentativeTom(t,55); U:=RepresentativeTom(t,36);
Intersection(e,U);
```

It implies that there exists a group G such that G acts transitively and with fixity 4 on some set, such that $\mathrm{E}(G) \cong \mathcal{A}_5$ acts semi-regularly on this set, and such that the point stabilisers are isomorphic to $\mathcal{A}_4$.

For examples in which the components are isomorphic to $\mathrm{SL}(2,5)$, we can use GAP in a similar way. Let G be the group with ID [240,93]. Then $G \cong \mathrm{SL}(2,5) : C_2$ and this time in the table of marks $\mathrm{E}(G)$ corresponds to line 29 and a point stabiliser U to line 3. The action of G on G/U is transitively, with fixity 4, such that $\mathrm{E}(G) \cong \mathrm{SL}(2,5)$ acts semi-regularly, and such that U is isomorphic to C_2.

The group $\mathrm{GL}(2,5)$ can be identified with ID [480,218] in the Small Groups Library [13]. Then line 46 of the table of marks corresponds to the component $\mathrm{SL}(2,5)$ and in line 8 we see a fixity-4 action with cyclic point stabiliser of order 4. In this action the component acts semi-regularly.

Finally the group identified with [480,959] reveals a fixity-4 action in line 14 of its table of marks. The point stabilisers in this action are elementary abelian of order 4 and the component, which corresponds to line 81 of the table of marks, acts semi-regularly. The group is isomorphic to $\mathrm{SL}(2,5) : E_4$.

6 The Fitting Subgroup

After analysing the situation for the components, in order to totally understand the situation for the generalised Fitting subgroup of a group that acts with fixity 4, we have to study the Fitting subgroup of this group. However, first we will look at nilpotent groups in general. On the one hand because studying them gives a first impression of the analysis of the Fitting subgroup, on the other hand we will need the result of Lemma 6.1 in the proofs concerning the Fitting subgroup.

Afterwards, the analysis depends on whether there exists a non-trivial element in the Fitting subgroup that fixes a point or not. In the first case, we will see that with some exceptions the Fitting subgroup itself acts with fixity 4 on one of its orbits. In the latter case, Lemma 6.9 gives some information.

6.1 Nilpotent Groups Acting with Low Fixity

We collect some information about nilpotent groups that act with fixity at most 4 and we will see that the centre plays an important role.

Lemma 6.1
Let k be a positive integer and let G be a nilpotent group such that G acts transitively, faithfully, and with fixity k on a set Ω such that $1 \leq k \leq 4$. Then one of the following holds.

(1) $k = 2$ and G is a dihedral or semi-dihedral 2-group.

(2) $k = 3$ and G is a 3-group of maximal class.

(3) $k = 4$ and G is a 2-group of sectional 2-rank at most 4.

Proof:
Since G is nilpotent, $\mathrm{Z}(G)$ is non-trivial and $|\mathrm{Z}(G)|$ is divisible by every prime divisor of $|G|$. By Lemma 2.17, $|\mathrm{Z}(G)|$ divides k, in particular $k > 1$. Let $\alpha \in \Omega$.

First suppose that $k = 2$. Then G is a 2-group. Since there exists a non-trivial element in G with exactly two fixed points, $|G : G_\alpha| = |\Omega| \geq 4$. Then by Lemma 2.11 (b) in [71], G is dihedral or semi-dihedral.

Next suppose that $k = 3$. Then G is a 3-group. Therefore both G_α and $|\Omega| = |G : G_\alpha|$ are divisible by 3. Even more, since G contains a non-trivial element with three fixed points $|\Omega| > 3$, and hence $|\Omega|$ is divisible by 9. Following the idea of the proof of Lemma 2.20 in [72], we look at G_α and its normaliser in G. Since Ω is the disjunct union of all G_α-orbits and Ω and G_α are non-trivial

3-powers, $\mathrm{fix}_\Omega(G_\alpha)$ contains exactly three elements. Therefore G_α acts semi-regularly on $\Omega \backslash \mathrm{fix}_\Omega(G_\alpha)$. If $|G_\alpha|$ is divisible by 9, then $0 \equiv |\Omega| \equiv |\,\mathrm{fix}_\Omega(G_\alpha)| = 3 \mod 9$. As a consequence, $|G_\alpha| = 3$. By Lemma 2.13, $|\,\mathrm{N}_G(G_\alpha) : \mathrm{N}_{G_\alpha}(G_\alpha)| \leq 3$, and hence $|\,\mathrm{N}_G(G_\alpha)| = 9$. Then Satz III 14.23 in [54] implies that G is of maximal class.

Finally, suppose that $k = 4$. Then G is a 2-group and there exists an involution $t \in G$ with exactly 4 fixed points. Since no subgroup of G is strongly embedded in any other subgroup of G the Theorem in [83] proves that G has sectional 2-rank at most 4.

6.2 The Fitting Subgroup and Non-Regular Orbits

The main concern of this section are groups acting with fixity 4 and such that the Fitting subgroup contains a non-trivial element that fixes a point. Assuming such a situation, the following two results show that then the group does not contain any components and that the Fitting subgroup either is a 2-group or a 3-group.

Lemma 6.2
Let G act transitively, faithfully, and with fixity 4 on a set Ω. Let $\alpha \in \Omega$ and suppose that $\mathrm{F}(G) \cap G_\alpha \neq 1$. Then $\mathrm{E}(G) = 1$.

Proof:
Let $b \in \mathrm{F}(G) \cap G_\alpha$ be non-trivial. Since $\mathrm{E}(G) \leq \mathrm{C}_G(b)$, Lemma 2.13 implies that $|\,\mathrm{E}(G) : \mathrm{E}(G)_\alpha| \leq |\,\mathrm{C}_G(b) : \mathrm{C}_{G_\alpha}(b)| \leq 4$. Thus by Lemma 2.5, $\mathrm{E}(G) \leq G_\alpha$. Then the faithful and transitive action of G implies that $\mathrm{E}(G) = 1$.

Lemma 6.3
Let G act transitively, faithfully, and with fixity 4 on a set Ω. Let $\alpha \in \Omega$ and suppose that $\mathrm{F}(G) \cap G_\alpha \neq 1$. Then $\mathrm{F}(G)$ is either a 2-group or a 3-group.

Proof:
Since $\mathrm{F}(G) \cap G_\alpha \neq 1$, there exists a non-trivial element $d \in \mathrm{F}(G)$ such that α is fixed by d.

In a first step, we will see that $\mathrm{F}(G)$ is a $\{2,3\}$-group. For a contradiction, assume that $|\,\mathrm{F}(G)|$ is divisible by a prime $r > 3$. Let $b \in \mathrm{Z}(\mathrm{F}(G))$ be of order r. Since $b \in \mathrm{C}_G(d)$, by Lemma 2.13, $|\langle b \rangle : \langle b \rangle \cap G_\alpha| \leq |\,\mathrm{C}_G(d) : \mathrm{C}_{G_\alpha}(d)| \leq 4$. Hence, $b \in G_\alpha$ because b has prime order $r \geq 5$. Therefore, $\Omega_1(\mathrm{O}_r(\mathrm{Z}(\mathrm{F}(G))))$ is a non-trivial subgroup of G_α. Since $\Omega_1(\mathrm{O}_r(\mathrm{Z}(\mathrm{F}(G))))$ is additionally normal in G, this contradicts the transitive and faithful action of G.

Thus, $\mathrm{F}(G)$ is a $\{2,3\}$-group and the only part left to prove is that $|\,\mathrm{F}(G)|$ is not divisible by both primes. Therefore, for a contradiction, assume that $|\,\mathrm{F}(G)|$ is divisible by 6. Then $|\,\mathrm{Z}(\mathrm{F}(G))|$ is divisible by 6, too. Since Lemma 2.13 implies that $|\,\mathrm{Z}(\mathrm{F}(G)) : \mathrm{Z}(\mathrm{F}(G)) \cap G_\alpha| \leq |\,\mathrm{C}_G(d) : \mathrm{C}_{G_\alpha}(d)| \leq 4$, $\mathrm{Z}(\mathrm{F}(G)) \cap G_\alpha$ contains a non-trivial element z. Thus, $|\,\mathrm{F}(G) : \mathrm{F}(G) \cap G_\alpha| \leq |\,\mathrm{C}_G(z) : \mathrm{C}_{G_\alpha}(z)| \leq 4$ and $|\,\mathrm{F}(G) : \mathrm{F}(G) \cap G_\alpha|$ cannot be divisible by both 2 and 3. If $|\,\mathrm{F}(G) : \mathrm{F}(G) \cap G_\alpha|$ is not divisible by 3, then G_α contains $\mathrm{O}_3(G)$, and hence the transitive and faithful action of G implies that $|\,\mathrm{F}(G)|$ is not divisible by 3, giving a contradiction. If $|\,\mathrm{F}(G) : \mathrm{F}(G) \cap G_\alpha|$ is not divisible by 2, then G_α contains $\mathrm{O}_2(G)$, and hence $|\,\mathrm{F}(G)|$ is not divisible by 2, giving a contradiction as well. Thus, $\mathrm{F}(G)$ is either a 2-group or a 3-group.

We will further analyse the structure of the Fitting subgroup of a group G that acts with fixity 4. As a next step, we will study the centre of $\mathrm{F}(G)$. In Lemma 2.16, we have seen that the centre of G acts semi-regularly. Even though the centre of $\mathrm{F}(G)$ can contain more elements than the centre of G, its action has under some conditions the same property. More precisely, we will see in the next lemma that if the set on which G acts is large enough, then the centre of the Fitting subgroup acts semi-regularly on it.

Lemma 6.4
Let G act transitively and with fixity 4 on a set Ω and let $\alpha \in \Omega$. Suppose that $|\Omega| > 28$. Then $\mathrm{Z}(\mathrm{F}(G)) \cap G_\alpha = 1$.

Proof:
Since $|\Omega| > 4$, the hypothesis that G acts with fixity 4 also implies that G acts faithfully on Ω.

For a contradiction, assume that $\mathrm{Z}(\mathrm{F}(G)) \cap G_\alpha$ is non-trivial. Then there exists an element $a \in \mathrm{Z}(\mathrm{F}(G)) \cap G_\alpha$ of prime order. By Lemma 6.3, $\mathrm{F}(G)$ is a 2- or a 3-group. In particular, $V \coloneqq \Omega_1(\mathrm{Z}(\mathrm{F}(G)))$ is elementary abelian and $a \in V_\alpha$. Then the fact that $V \leq \mathrm{C}_G(V_\alpha)$ implies together with Lemma 2.13 that $|\alpha^V| = |V : V_\alpha| \leq |\,\mathrm{C}_G(V_\alpha) : \mathrm{C}_G(V_\alpha)| \leq 4$. If α is a fixed point of V, then $V \leq G_\alpha$. Since V is normal in G, this contradicts the transitive and faithful action of G. Hence $|\alpha^V| \in \{2, 3, 4\}$.

We first look at the elements of V_α and their fixed points. Let $v \in V_\alpha$ and $\beta \in \alpha^V$. Then there exists $u \in V$ such that $\beta = \alpha^u$. Since V is abelian, $\beta^v = (\alpha^u)^v = \alpha^{vu} = \alpha^u = \beta$. Thus, all elements in α^V are fixed by all elements in V_α. Since V is a normal subgroup of G, by Lemma 2.2, this behaviour is not restricted to the orbit α^V but holds for all orbits of V in Ω. As a consequence, every non-trivial element in V that fixes a point, fixes the same set of points as $|V_\alpha| - 2$ other elements of V. Each of these fixed point sets has size at most 4. Therefore elements of V can only fix points in a subset of Ω of size at most $m \coloneqq \frac{|V|-1}{|V_\alpha|-1} \cdot 4$. Since $|V| = |V : V_\alpha| \cdot |V_\alpha| = |\alpha^V| \cdot |V_\alpha|$, it follows that $m = 4 \cdot (|\alpha^V| + \frac{|\alpha^V|-1}{|V_\alpha|-1})$. If $|\alpha^V| \leq |V_\alpha|$, then $m \leq 4 \cdot \left(|\alpha^V| + \frac{|\alpha^V|-1}{|\alpha^V|-1}\right) \leq 4 \cdot (4+1) = 20$ because $|\alpha^V| \in \{2, 3, 4\}$. Otherwise $|\alpha^V| > |V_\alpha|$. Since V_α is non-trivial and $|\alpha^V|$ and $|V_\alpha|$ are powers of the same prime, it follows that $|\alpha^V| = 4$ and $|V_\alpha| = 2$. Thus, $m = 4 \cdot (4 + \frac{4-1}{2-1}) = 28$. Therefore, in all cases, $m \leq 28$. Since $|\Omega| > 28$, there exists an element ϑ not fixed by any non-trivial element of V. Hence, $|\vartheta^V| = |V|$. This is a contradiction because, by Lemma 2.2 (a), $|V| = |\vartheta^V| = |\alpha^V| = |V : V_\alpha|$ and $|V_\alpha|$ is non-trivial.

Remark 6.5
The bound for $|\Omega|$ in Lemma 6.4 is sharp because there exist groups that act transitively and with fixity 4 on a set Ω of size 28 and such that a non-trivial element of $\mathrm{Z}(\mathrm{F}(G))$ fixes a point.

Let G be the semi-direct product of an elementary abelian group F of order 8 with a cyclic group of order 7. Let $U \leq G$ be cyclic of order 2. All involutions in G are conjugate by any of the cyclic groups acting on F. Then G/U has size 28. Since U is cyclic the number of fixed points of the non-trivial element $x \in U$ is, by Lemma 2.10, $\frac{|\mathrm{N}_G(\langle x \rangle)|}{|U|} = \frac{|F|}{2} = 4$. Hence, G acts with fixity 4 on G/U. Then the fact that F is the Fitting subgroup of G implies that $U \leq \mathrm{F}(G) = \mathrm{Z}(\mathrm{F}(G))$. As a consequence, $|\mathrm{Z}(\mathrm{F}(G)) \cap U| = 2$.

The following GAP commands use the algorithm in Remark 2.22 and show that the group G with ID `[168,43]` in the Small Groups Library [13] acts transitively and with fixity 4 on a set of size 28 and such that a non-trivial element of $\mathrm{Z}(\mathrm{F}(G))$ fixes a point.

```
G:=SmallGroup([168,43]);
TestTom(TableOfMarks(G),4);
Order(FittingSubgroup(G)); IsElementaryAbelian(FittingSubgroup(G));
```

The group G is of isomorphism type $E_8 : (C_7 : C_3)$, the Fitting subgroup is elementary abelian of order 8 and the point stabilisers are cyclic of order 6. In particular, the intersection of the Fitting subgroup and any point stabiliser is of order 2 and since $\mathrm{F}(G)$ is abelian, $\mathrm{Z}(\mathrm{F}(G)) = \mathrm{F}(G)$.

Both groups are examples which prove that the condition $|\Omega| > 28$ in Lemma 6.4 is necessary and that this bound cannot be improved.

We will look further into the situation that some non-trivial element of the Fitting subgroup fixes a point and the set has size at least 29. Under this hypothesis the next lemma shows that the Fitting subgroup itself acts with fixity 4 on some set. This further restricts the structure of the Fitting subgroup.

Lemma 6.6
Let G be a finite group acting transitively and with fixity 4 on a set Ω and let $\alpha \in \Omega$. Suppose that $|\Omega| > 28$ and that $\mathrm{F}(G) \cap G_\alpha \neq 1$. Then $\mathrm{E}(G) = 1$ and $\mathrm{F}(G)$ acts faithfully and with fixity 4 on $\alpha^{\mathrm{F}(G)}$. In particular, $\mathrm{F}(G)$ is a 2-group of sectional 2-rank at most 4.

Proof:

Let $F = \mathrm{F}(G)$. Since $|\Omega| > 4$ and G acts with fixity 4, the action of G is faithful. By Lemma 6.2, $\mathrm{E}(G) = 1$. Thus by Theorem 6.5.8 in [65], $\mathrm{C}_G(F) \leq F$, and hence G/F is isomorphic to a subgroup of the automorphism group of F.

Lemma 6.3 yields that F is either a 2- or a 3-group. Let $p \in \{2, 3\}$ be such that F is a p-group. Together with Lemma 6.4 our hypothesis implies that $\mathrm{Z}(F) \cap F_\alpha = \mathrm{Z}(F) \cap G_\alpha = 1$. Therefore, F_α is not a normal subgroup of F and F is not abelian. In particular, $|F : F_\alpha| \geq 4$ and $|F| \geq p^3$.

Assume, for a contradiction, that $4 = |F : F_\alpha| = |\alpha^F|$. Then $p = 2$ and F is a 2-group. Since $|\Omega| > 16 = 4 \cdot |\alpha^F|$, by Lemma 2.2, there exists an element $\omega \in \Omega$ such that F_α acts semi-regularly on ω^F. Thus $|F_\alpha| \leq |\omega^F| = 4$. In particular, F_α is abelian, and hence $F_\alpha \operatorname{Z}(F)$ is also abelian. Therefore, $F_\alpha \operatorname{Z}(F) \neq F$. Since $|F : F_\alpha| = 4$ and $\operatorname{Z}(F) \cap F_\alpha = 1$, it follows that $F_\alpha \operatorname{Z}(F)$ is a subgroup of index 2 in F and that $|\operatorname{Z}(F)| = 2$. Let $s \in F \setminus F_\alpha \operatorname{Z}(F)$ and let $a \in \operatorname{C}_{F_\alpha}(s)$. Then a centralises F_α, s, and $\operatorname{Z}(F)$, and hence a is central in $F_\alpha \operatorname{Z}(F)\langle s \rangle = F$. Thus, $a = 1$, and therefore $\operatorname{C}_{F_\alpha}(s) = 1$. Since $|\operatorname{C}_F(s) : \operatorname{C}_{F_\alpha}(s)| \leq |F : F_\alpha| = 4$, by Satz II 14.23 in [54], F is of maximal class, and hence by Satz III 11.9 in [54] the group F is dihedral, semi-dihedral, or a generalised quaternion group. The only group of one of these types that has a subgroup of index 4 that has trivial intersection with the centre of the group is D_8. Since the automorphism group of D_8 is of order 8 (see for example Theorem 34.8 (a) in [12]), it follows that G is a 2-group. However, then $G = F$ and $|\Omega| = |G : G_\alpha| = |F : F_\alpha| = 4$. This contradiction implies that $|\alpha^F| > 4$, and hence F acts faithfully on α^F.

In summary, F acts transitively, faithfully, non-regularly, and with fixity at most 4 on α^F. Since F is p-group, it cannot act as a Frobenius group, and thus F acts with fixity at least 2 on α^F.

Assume for a contradiction that F acts with fixity 2 on α^F. Then Lemma 6.1 yields that F is a dihedral or semi-dihedral 2-group. In particular, the automorphism group of F is also a 2-group (see Theorem 34.8 in [12]). However, then G is a 2-group and hence $G = F$, contradicting the fact that G acts with fixity 4 but F acts with fixity 2 on $\alpha^F = \alpha^G = \Omega$.

Next assume for a contradiction that F acts with fixity 3 on α^F. Then by Lemma 6.1, F is a 3-group of maximal class. If $|F| \geq 3^5$, then Proposition 3.3 in [76] shows that G is a $\{2,3\}$-group and that 8 does not divide $|G|$ because G/F is isomorphic to a subgroup of the automorphism group of F. If $|F| \in \{3^3, 3^4\}$, then the result of the following GAP commands, using the Small Groups Library [13], also implies that G is a $\{2,3\}$-group and that G is extra-special of order 27 and exponent 3 if 8 divides $|G|$.

```
li:=Concatenation(AllSmallGroups(Size,[3^3],
↪  NilpotencyClassOfGroup,[2]), AllSmallGroups(Size,[3^4],
↪  NilpotencyClassOfGroup,[3]));;
List(li,x->PrimeDivisors(Order(AutomorphismGroup(x))));
li:=Filtered(li,x->Order(AutomorphismGroup(x)) mod 8 =0);
IdGroup(li[1]);
```

Assume, for a contradiction, that $|G|$ is divisible by 8. Then the last line of the above GAP code returns an up to isomorphism unique identifier of F. More precisely, the ID in the Small Groups Library [13] is returned. It is `[27,3]`. By Theorem 1 in [106], the order of $\operatorname{Out}(F)$ is $2 \cdot 3 \cdot (3^2 - 1) = 2^4 \cdot 3$. Thus, $|G|$ divides $|\operatorname{Out}(F)| \cdot |F| = 2^4 \cdot 3^4$. Since we assume that G has order divisible by 8 and that $F \leq G$, $|G| \in \{2^3 \cdot 3^4, 2^4 \cdot 3^4, 2^3 \cdot 3^3, 2^4 \cdot 3^3\}$. The first line of the following GAP code determines all groups of these orders that have a Fitting subgroup with the same ID as F. The second line uses the code in Remark 2.22 and shows that none of these groups can act transitively, faithfully, and with fixity 4 on any set.

```
li:=AllSmallGroups(Size, [2^3*3^4,2^4*3^4,2^3*3^3,2^4*3^3],
↪  x->IdGroup(FittingSubgroup(x)), [[27,3]]);;
List(li,x->TestTom(TableOfMarks(x),4));
```

Since G must be one of these groups and by hypothesis acts with fixity 4, it follows that $|G|$ cannot be divisible by 8.

Let $P \in \mathrm{Syl}_3(G)$ be such that $P_\alpha \in \mathrm{Syl}_3(G_\alpha)$. Since $F \leq P$, $1 \neq F_\alpha$, and $|F| \geq 3^3$, it follows that $|G_\alpha|$ is divisible by 3 and $|P| > 9$. In particular, neither case (a) nor case (b) of Lemma 10 in [8] holds. If case (d) holds, then there exists a unique P-orbit of length 3 and all other orbits are regular. Since $|\alpha^P| \geq |\alpha^F| > 4$, α^P is not the unique orbit of length 3. Thus α^P is a regular orbit, but this contradicts the fact that $P_\alpha \geq F_\alpha > 1$. Therefore case (d) in Lemma 10 in [8] cannot hold. If case (e) holds, then P lies in a point stabiliser of G. However, then $F \leq P$ is also a non-trivial subgroup of a point stabiliser, contradicting the transitive and faithful action of G. Thus, the only remaining case (c) holds, and hence P is of maximal class and since α^P is a non-regular P-orbit, $|P_\alpha| = 3$. Then $F_\alpha = P_\alpha$ and $|\,\mathrm{Z}(P)| = 3$. Since $F \trianglelefteq P$, it follows that $\mathrm{Z}(P) \cap F \neq 1$. As a consequence, $\mathrm{Z}(P) \leq \mathrm{Z}(F)$. Let $z \in \mathrm{Z}(P)$ be of order 3. If z fixes a point $\omega \in \Omega$, then $1 < |\,\mathrm{Z}(F)_\omega| = |\,\mathrm{Z}(F)_\alpha|$ and this contradicts Lemma 6.4. Therefore z acts fix-point-freely on Ω.

Let $x \in G_\alpha$ be of prime order and such that x fixes exactly four points in Ω. Then x acts on $\mathrm{Z}(F)$. If $z^x = z$, z acts on $\mathrm{fix}_\Omega(x)$ a set of size 4 but z acts fix-point-freely and has order 3. This contradiction implies that x inverts z, and hence x has order 2. In particular, $|\Omega| \equiv 4 \mod 2$. Thus, $|\Omega| = |G : G_\alpha|$ is even and G_α does not contain a Sylow 2-subgroup of G, but it does contain x. Since 8 does not divide $|G|$, it follows that $|G_\alpha| = 2$ and that the Sylow 2-subgroups of G have order 4. We recall that $P_\alpha \in \mathrm{Syl}_3(P_\alpha)$ has order 3 and that G is a $\{2,3\}$-group. As a consequence, $|G_\alpha| = 6$. Therefore G has either exactly three or exactly one subgroup of order 2, one of them is $\langle x \rangle$. Hence by Lemma 2.9,

$$4 = |\,\mathrm{fix}_\Omega(x)| = \frac{|\{\langle x \rangle^g \leq G_\alpha \mid g \in G\}| \cdot |\,\mathrm{N}_G(\langle x \rangle)|}{|G_\alpha|}$$

and thus $24 = 4 \cdot |G_\alpha| = |\{\langle x \rangle^g \leq G_\alpha \mid g \in G\}| \cdot |\,\mathrm{N}_G(\langle x \rangle)|$. Since the number $|\{\langle x \rangle^g \leq G_\alpha \mid g \in G\}|$ is odd, this implies that $|\,\mathrm{N}_G(\langle x \rangle)|$ is divisible by 8. However, then $|G|$ is divisible by 8. This contradiction shows that F cannot act with fixity 3 on α^F. Therefore, F acts with fixity 4 on α^F and by Lemma 6.1, F is a 2-group of sectional 2-rank at most 4.

The situation of the previous lemma happens especially when G itself is already a 2-group. However, it can also happen in other instances as the following example shows.

Example 6.7
The group G that is identified by `[192,185]` in the Small Groups Library [13] has a Fitting subgroup of structure $(C_4 \times C_4) : C_2$ and ID `[32,34]`. Making use of the GAP package TomLib [74] through the code in Remark 2.22, the GAP command `TestTom(TableOfMarks(SmallGroup([192,185])),4);` implies that G acts with fixity 4 on a set of size 48. More precisely the command returns all three faithful fixity-4 action of G but we are only interested in the one on a set of size 48 because under this action $\mathrm{F}(G)$ has non-trivial intersection with a point stabiliser.

Under the assumption that there exists a non-trivial element in the Fitting subgroup that fixes a point, only the situation that the group acts on a set of size at most 28 is left to be analysed. Since in the Transitive Groups Library [52] all transitive groups that act on a set of size at most 28 are listed, we can use GAP and the algorithm in Remark 2.18 to determine not only the structure of the Fitting subgroup but the structure of the group itself.

Lemma 6.8
Let G be a finite group acting transitively, faithfully, and with fixity 4 on a set Ω. Let $\alpha \in \Omega$. Suppose that $\mathrm{F}(G) \cap G_\alpha \neq 1$ and that $\mathrm{F}(G)$ does not act faithfully and with fixity 4 on $\alpha^{\mathrm{F}(G)}$. Then either $\mathrm{F}(G)$ is non-cyclic, elementary abelian, and of order at most 16 or $\mathrm{F}(G)$ is of isomorphism type $C_4 \times C_4$ or $(C_4 \times C_4) : C_2$. Moreover, Table 6.1 lists all groups G that fulfil the hypothesis together with all possible point stabiliser structures in each case.

Proof:

By Lemma 6.6, $|\Omega| \leq 28$. Since G acts with fixity 4, no element in G fixes five points, and thus all five-point stabiliser are trivial. Hence, the filter in the GAP program below does not change the result. By Lemma 2.2, the action of $\mathrm{F}(G)$ on $\alpha^{\mathrm{F}(G)}$ is in one-to-one correspondence to the action of $\mathrm{F}(G)$ on $\omega^{\mathrm{F}(G)}$ for every $\omega \in \Omega$. Therefore the calculation is independent of α and we can use the point 1 for our calculations without loss of generality. The following GAP code uses the GAP program in Remark 2.18 together with the Transitive Groups Library [52] and returns a list. This list contains information about ever group G that acts transitively and with fixity 4 on a set of size at most 28, for which $\mathrm{F}(G) \cap G_1$ is non-trivial, and $\mathrm{F}(G)$ acts faithfully and with fixity 4 on $1^{\mathrm{F}(G)}$. The information provided for each group is a list of three IDs in the Small Groups Library [13]. The first entry represents the group itself, the second entry represents its Fitting subgroup, and the third entry represents a point stabiliser.

```
li:=AllTransitiveGroups(NrMovedPoints, [1..28],
↪  x->IsTrivial(Stabilizer(x,[1..5],OnTuples)), true,
↪  y->TestFixity(y,MovedPoints(y),4), true,
↪  z->IsTrivial(Intersection(FittingSubgroup(z),Stabilizer(z,1))),
↪  false, w->TestFixity( w, Orbits(FittingSubgroup(w),
↪  MovedPoints(w) )[1], 4 ), false);;
Set(List(li, x->[IdGroup(x), IdGroup(FittingSubgroup(x)),
↪  IdGroup(Stabilizer(x,1))]));
```

ID in [13]	group G	$\mathrm{F}(G)$	G_α
`[  24,    12]`	$\mathcal{S}_4$	E_4	C_2
`[  24,    13]`	$C_2 \times \mathcal{A}_4$	E_8	C_2, E_4
`[  48,     3]`	$(C_4 \times C_4) : C_3$	$C_4 \times C_4$	C_4
`[  48,    30]`	$C_2 {}^{\cdot} \mathcal{S}_4$	E_8	C_4
`[  48,    48]`	$C_2 \times \mathcal{S}_4$	E_8	C_4, E_4, D_8
`[  48,    49]`	$E_4 \times \mathcal{A}_4$	E_{16}	E_4
`[  48,    50]`	$(E_4 \times E_4) : C_3$	E_{16}	E_4
`[  56,    11]`	$E_8 : C_7$	E_8	C_2
`[  72,    39]`	$E_9 : C_8$	E_9	$\mathcal{S}_3$
`[  72,    40]`	$\mathcal{S}_3 \wr C_2$	E_9	D_{12}
`[  72,    41]`	$E_9 : Q_8$	E_9	$\mathcal{S}_3$
`[  80,    49]`	$E_{16} : C_5$	E_{16}	E_4
`[  96,    64]`	$(C_4 \times C_4) : \mathcal{S}_3$	$C_4 \times C_4$	C_8, $C_4 \times C_2$, D_8, Q_8
`[  96,    72]`	$E_8 {}^{\cdot} \mathcal{A}_4$	$(C_4 \times C_4) : C_2$	D_8
`[  96,   195]`	$C_2 {}^{\cdot} (C_2 \times \mathcal{S}_4)$	E_{16}	D_8
`[  96,   227]`	$(E_4 \times E_4) : \mathcal{S}_3$	E_{16}	$C_4 \times C_2$, D_8, E_8, $\mathcal{A}_4$
`[ 144,   182]`	$E_9 : SD_{16}$	E_9	D_{12}
`[ 144,   184]`	$\mathcal{A}_4 \times \mathcal{A}_4$	E_{16}	$\mathcal{A}_4$
`[ 160,   234]`	$E_{16} : D_{10}$	E_{16}	$C_4 \times C_2$, E_8
`[ 168,    43]`	$E_8 : (C_7 : C_3)$	E_8	C_6
`[ 192,   956]`	$E_8 {}^{\cdot} \mathcal{S}_4$	$(C_4 \times C_4) : C_2$	D_{16}, SD_{16}
`[ 216,   153]`	$E_9 : \mathrm{SL}(2,3)$	E_9	$C_3 \times \mathcal{S}_3$
`[ 288,  1025]`	$\mathcal{A}_4 \wr C_2$	E_{16}	$C_2 \times \mathcal{A}_4$
`[ 288,  1026]`	$(\mathcal{A}_4 \times \mathcal{A}_4) : C_2$	E_{16}	$\mathcal{S}_4$
`[ 320,  1635]`	$E_{16} : (C_5 : C_4)$	E_{16}	$E_4 : C_4$, $C_8 : C_2$
`[ 432,   734]`	$E_9 : \mathrm{GL}(2,3)$	E_9	$\mathcal{S}_3 \times \mathcal{S}_3$
`[ 960, 11357]`	$(E_4 \times E_4) : \mathrm{SL}(2,4)$	E_{16}	$(C_4 \times C_4) : C_3$, $(E_4 \times E_4) : C_3$
`[1344,   814]`	$E_8 {}^{\cdot} \mathrm{PSL}(3,2)$	E_8	$\mathrm{GL}(2,3)$

Table 6.1: Groups Acting Transitively, Faithfully, with Fixity 4, and such that $F(G)$ does not act Faithfully, Semi-Regularly, and with Fixity 4 on $\alpha^{\mathrm{F}(G)}$

Since the IDs are up to isomorphism unique, they can be used as a basis to determine the isomorphism types of the groups they represent. For Table 6.1, the results are sorted in a way that isomorphic groups just appear once but with all their actions that fulfil the hypothesis of this lemma, and the structure of the Fitting subgroup with ID `[32,34]` in the Small Groups Library [13] is denoted by $(C_4 \times C_4) : C_2$.

6.3 The Fitting Subgroup and Regular Orbits

After analysing the Fitting subgroup of a group G that acts with fixity 4 under the condition that some non-trivial element of $\mathrm{F}(G)$ fixes a point, we will now see some information in the case that the Fitting subgroup acts semi-regularly.

Lemma 6.9
Let G be a finite group that acts transitively, faithfully, and with fixity 4 on a set Ω. Let $\alpha \in \Omega$. Suppose that $\mathrm{F}(G)$ acts semi-regularly. Then for every non-trivial element a that fixes a point, $|\,\mathrm{C}_{\mathrm{F}(G)}(a)| \leq 4$. Additionally one of the following holds.

(1) For all $p \in \pi(G_\alpha)$, the Sylow p-subgroups of G_α have p-rank 1.

(2) $\mathrm{F}(G) = \mathrm{O}_2(G) \times \mathrm{O}_3(G)$.

Proof:
Let $a \in G$ be non-trivial and such that a fixes a point $\omega \in \Omega$. Then $\mathrm{F}(G)_\omega = 1$, and hence by Lemma 2.13, $|\,\mathrm{C}_{\mathrm{F}(G)}(a)| = |\,\mathrm{C}_{\mathrm{F}(G)}(a) : \mathrm{C}_{\mathrm{F}(G)_\omega}(a)| \leq 4$.

Suppose that (1) does not hold. Then there exists $r \in \pi(G_\alpha)$ such that G_α contains an elementary abelian subgroup X of order r^2. Assume that there exists a prime $p \in \pi(\mathrm{F}(G))$ such that $p \geq 5$. If $p = r$, then by Corollary 2.14, G_α contains a Sylow p-subgroup, and hence the point stabiliser also contains $\mathrm{O}_p(G) \neq 1$, contradicting the transitive and faithful action of G. Therefore, X acts coprime on $\mathrm{O}_p(G)$. Then Lemma 2.3 (b) implies that $\mathrm{O}_p(G) = \langle \mathrm{C}_{\mathrm{O}_p(G)}(a) \mid a \in X \setminus \{1\}\rangle$. Since for all non-trivial $a \in X$, $4 \geq |\,\mathrm{C}_{\mathrm{F}(G)}(a)| \geq |\,\mathrm{C}_{\mathrm{O}_p(G)}(a)|$ and $p \geq 5$, it follows that $\mathrm{C}_{\mathrm{O}_p(G)}(a) = 1$. Thus, $\mathrm{O}_p(G) = 1$ contrary to the fact that $p \in \pi(\mathrm{F}(G))$.

A finite group G that acts transitively, faithfully, and with fixity 4 fulfils either the conditions of Lemma 6.3 or of Lemma 6.9. In both cases, it holds that $\mathrm{F}(G) = \mathrm{O}_2(G) \times \mathrm{O}_3(G)$ or that for all $p \in \pi(G_\alpha)$ the Sylow p-subgroups of a point stabiliser have p-rank 1.

7 The Generalised Fitting Subgroup

In the previous two chapters, we have analysed the components and the Fitting subgroup of a group that acts with fixity 4, separately. Therefore, we can turn our attention in this chapter to their interplay. As a consequence, a description of the generalised Fitting subgroup, the product of the components and the Fitting subgroup, is within reach. Since the generalised Fitting subgroups of a group G controls the structure of G, the Main Theorem (Theorem 7.7) not only gives details about the structure of the generalised Fitting subgroup of a group that acts with fixity 4 but also about the group itself.

When we look at Example 5.8 from the perspective of the generalised Fitting subgroup, we see that it gives an example of a group G that acts transitively and with fixity 4 on a set Ω and such that both $\mathrm{E}(G)$ and $\mathrm{F}(G)$ act semi-regularly on Ω but $\mathrm{F}^*(G) = \mathrm{E}(G) * \mathrm{F}(G)$ does not. This will be part of one of the cases in Theorem 7.7. However, there are more cases that we have to study. Following Theorem 5.6 we also look at the situation where some non-trivial element of a component fixes a point, and thus we have to consider the cases where the component acts with fixity 2, 3, or 4 on one of its orbits. In each case, we want to derive information about the Fitting subgroup before we collect all results in Theorem 7.7.

7.1 Constraints on the Action of the Fitting Subgroup and Components

The next three lemmas analyse the structure of the Fitting subgroup of a group under the condition that a component of this group acts with fixity 2 on one of its orbits. By Lemma 4.3, in this case, it is sufficient to restrict our study to simple components.

As a first step, in a slightly more general setting, we will see that if the Fitting subgroup stabilises an orbit of the component, then with some exceptions the Fitting subgroup is trivial. Afterwards we will look at these exceptions.

Lemma 7.1
Let E and F be two groups acting faithfully on a set Γ such that $[E, F] = 1$, that $|F| \leq 4$, and that E is non-abelian simple. Suppose that EF acts with fixity at most 4 on Γ, that E acts with fixity 2 on α^E, and that $\alpha^E = \alpha^{EF}$. If $F \neq 1$, then E is isomorphic to $\mathcal{A}_5$, $\mathcal{A}_6$, $\mathrm{PSL}(2,7)$, $\mathrm{PSL}(2,8)$, $\mathrm{PSL}(2,11)$, or $\mathrm{PSL}(2,13)$.

Proof:
Let $G = EF$ and $\Omega = \alpha^{EF}$. Then, since F is abelian and E is non-abelian simple, $\mathrm{Z}(G) = F = \mathrm{F}(G)$. We want to use Lemma 4.1 and therefore we have

to ensure that $|\Omega| > 2 \cdot |F|$. By Lemma 2.5, $|\Omega| = |\alpha^E| \geq 5$. As a consequence, $|\Omega| > 2 \cdot |F|$ if $|F| \leq 2$. If $|F| > 2$, then we use the Transitive Groups Library [52] together with the following GAP command.

```
AllTransitiveGroups(NrMovedPoints, [5..8],
↪  x->Order(x)>=60, [true], y->Order(FittingSubgroup(y)), [3,4]);
```

Since E is non-abelian simple, the order of E is at least 60. Thus, the GAP command returns all groups G that could contain a simple group E, that can act faithfully and transitively on some set of size at least 5 and at most 8, and that have a Fitting subgroup of order 3 or 4. Since no such group is returned, $|\Omega| > 8$ if $|F| > 2$.

Then Lemma 4.1 yields that $EF/F \cong E$ acts non-regularly and with fixity at most 4 on a set $\bar{\Omega}$ of size $\frac{|\Omega|}{|F|}$. Since E is simple, it cannot act as a Frobenius group, and hence it acts with fixity 2, 3, or 4 on $\bar{\Omega}$. In particular, E acts both with fixity 2 on a set of size $|\Omega|$ and with fixity 2, 3, or 4 on a set of size $\frac{|\Omega|}{|F|}$.

By Theorem 1.2 in [71], $E \cong \mathrm{PSL}(3,4)$ or there exists a prime power q such that $E \cong \mathrm{PSL}(2,q)$ or such that $E \cong \mathrm{Sz}(q)$.

If $E \cong \mathrm{PSL}(3,4)$, then the command `TestTom(TableOfMarks("L3(4)"),2);` uses the GAP package TomLib [74] and the program in Remark 2.22 and shows that E has a unique fixity-2 action, namely on a set of size 4032. Thus, if $|F| \neq 1$, then $EF/F \cong E$ cannot act with fixity 2. Since by Theorem 3.56, $\mathrm{PSL}(3,4)$ does not act with fixity 4 on any set, EF/F acts with fixity 3 on a set of size $\frac{4032}{|F|}$. This is a contradiction to Theorem 1.1 in [72], and hence, if $F \neq 1$, then E is not isomorphic to $\mathrm{PSL}(3,4)$.

If $E \cong \mathrm{Sz}(q)$, then $q \geq 8$ and Lemma 3.12 in [71] proves that E can act with fixity 2 only on a set of size q^2+1 and on a set of size $q^2(q^2+1)$. As a consequence, if $F \neq 1$ and EF/F acts with fixity 2, then $|F| = q^2 \geq 64 > 4$. This contradicts the hypothesis and since by Theorem 1.1 in [72], $\mathrm{Sz}(q)$ cannot act with fixity 3, EF/F acts with fixity 4 on a set of size $\frac{q^2+1}{|F|}$ or $\frac{q^2(q^2+1)}{|F|}$. Then this is a contradiction to Theorem 3.56, and therefore, if $F \neq 1$, then E is not isomorphic to $\mathrm{Sz}(q)$.

Therefore suppose that E is isomorphic to $\mathrm{PSL}(2,q)$ but not to any of the groups $\mathcal{A}_5 \cong \mathrm{PSL}(2,4) \cong \mathrm{PSL}(2,5)$, $\mathrm{PSL}(2,7)$, $\mathrm{PSL}(2,8)$, $\mathcal{A}_6 \cong \mathrm{PSL}(2,9)$, $\mathrm{PSL}(2,11)$, and $\mathrm{PSL}(2,13)$. Then $q \geq 16$ and by Lemma 3.11 in [71], E acts with fixity 2 on a set of size $q+1$, $q(q-1)$, or $q(q+1)$. If EF/F acts with fixity 4 on $\bar{\Omega}$, then by Theorem 3.56, q is odd and $|\bar{\Omega}| \in \{2q(q+1), 2(q+1), 2q(q-1)\}$. Thus $|\Omega| \in \{2q(q+1)|F|, 2(q+1)|F|, 2q(q-1)|F|\} \cap \{q+1, q(q-1), q(q+1)\}$. Since $|F| \leq 4$ and $q \geq 16$, this is impossible. If EF/F acts with fixity 3 on $\bar{\Omega}$, then Theorem 1.1 in [72] yields a contradiction. Thus EF/F acts with fixity 2 on $\bar{\Omega}$. Then the size of Ω and of $\bar{\Omega}$ are both in $\{q+1, q(q-1), q(q+1)\}$. Since $q \geq 16$ and since $4 \geq |F| = |\Omega|/|\bar{\Omega}|$, it follows that $|F| = 1$. This proves the lemma.

As a next step, we look at the groups that are explicitly mentioned in the previous lemma. They will appear when we try to apply the result. Since the nature of the arguments in the next proof do not depend on the fixity with which a component acts, we include all three possible fixities in one lemma. This has the advantage that once the statement is established, we have dealt with all upcoming exceptions.

Lemma 7.2
Let G be a finite group acting transitively and with fixity 4 on a set Ω. Let $\alpha \in \Omega$. Suppose that $\mathrm{E}(G)$ acts with fixity at least 2 on $\alpha^{\mathrm{E}(G)}$, that $|\,\mathrm{F}(G)| = 4$, and that $\mathrm{F}(G)$ acts semi-regularly on Ω. Then $\mathrm{E}(G)$ is isomorphic to neither $\mathcal{A}_5$, $\mathcal{A}_6$, $\mathrm{PSL}(2,7)$, $\mathrm{PSL}(2,8)$, $\mathrm{PSL}(2,11)$, nor $\mathrm{PSL}(2,13)$.

Proof:

The hypotheses imply that $\mathrm{F}(G) \cong C_4$ or $\mathrm{F}(G) \cong E_4$ and that $\mathrm{E}(G)\,\mathrm{F}(G)$ acts transitively and with fixity at least 2 and at most 4 on $\alpha^{\mathrm{E}(G)\,\mathrm{F}(G)}$.

Assume, for a contradiction, that $E \cong \mathcal{A}_5$. If $\mathrm{F}(G)$ is elementary abelian of order 4, then Example 5.8 shows that $\mathrm{E}(G)\,\mathrm{F}(G)$ cannot act transitively, with fixity 4, and such that $\mathrm{E}(G)$ has a non-trivial element that fixes a point. Therefore, $\mathrm{F}(G)$ is isomorphic to C_4. We will use the GAP function in Remark 2.22. The following GAP commands show that then $\mathrm{E}(G)\,\mathrm{F}(G)$ does not act transitively and with fixity 2, 3, or 4 on any set.

```
G:=DirectProduct(AlternatingGroup(5), Group((1,2,3,4)));;
List([2..4],x->TestTom(TableOfMarks(G),x));
```

This contradiction implies that $\mathrm{E}(G)$ is not isomorphic to $\mathcal{A}_5$.

Similarly, the following GAP code implies that neither for $\mathrm{F}(G) \cong C_4$ nor for $\mathrm{F}(G) \cong E_4$, $\mathrm{E}(G)$ can be isomorphic to any of the groups $\mathcal{A}_6 \cong \mathrm{PSL}(2,9)$, $\mathrm{PSL}(2,7)$, $\mathrm{PSL}(2,8)$, $\mathrm{PSL}(2,11)$, and $\mathrm{PSL}(2,13)$.

```
li:=[AlternatingGroup(6), PSL(2,7), PSL(2,8), PSL(2,11),
 ↪  PSL(2,13)];;
liC4:=List(li,x->DirectProduct(x,Group((1,2,3,4))));;
liE4:=List(li,x->DirectProduct(x,Group((1,2),(3,4))));;
List(liC4, x->[TestTom(TableOfMarks(x),2),
 ↪  TestTom(TableOfMarks(x),3), TestTom(TableOfMarks(x),4)]);
List(liE4, x->[TestTom(TableOfMarks(x),2),
 ↪  TestTom(TableOfMarks(x),3), TestTom(TableOfMarks(x),4)]);
```

The previous two lemmas together imply the following result which restricts the order of a Fitting subgroup of a group that has a component acting with fixity 2 on one of its orbits.

Lemma 7.3
Let G be a finite group acting transitively and with fixity 4 on a set Ω. Let $\alpha \in \Omega$. Suppose that $\mathrm{E}(G)$ acts with fixity 2 on $\alpha^{\mathrm{E}(G)}$, that $\mathrm{E}(G)$ is simple, and that $\mathrm{F}(G)$ acts semi-regularly. Then $|\,\mathrm{F}(G)| \leq 2$.

Proof:

Let $x \in \mathrm{E}(G)_\alpha$ be such that x fixes exactly two points in $\alpha^{\mathrm{E}(G)}$. Since $\mathrm{E}(G)$ and $\mathrm{F}(G)$ centralise each other, $\mathrm{F}(G)$ acts on the set of fixed points of x. In particular, x fixes every element in $\alpha^{\mathrm{F}(G)}$. Then the fixity-4 action of G implies that x can fix at most four points in Ω, and hence $|\alpha^{\mathrm{F}(G)}| \le 4$. Since Lemma 2.2 yields that x fixes exactly two points in every $\mathrm{E}(G)$-orbit in $\alpha^{\mathrm{E}(G)\,\mathrm{F}(G)}$, the fixity-4 action of G additionally implies that there are at most two $\mathrm{E}(G)$-orbits in $\alpha^{\mathrm{E}(G)\,\mathrm{F}(G)}$. As a consequence, $|\alpha^{\mathrm{F}(G)}| \in \{1,2,4\}$. Since $\mathrm{F}(G)$ acts semi-regularly, it follows that $|\,\mathrm{F}(G)| \in \{1,2,4\}$.

Assume for a contradiction that $|\,\mathrm{F}(G)| = 4$. Then $\alpha^{\mathrm{E}(G)\,\mathrm{F}(G)}$ contains exactly two $\mathrm{E}(G)$-orbits. Let $F = \{f \in \mathrm{F}(G) \mid \alpha^{\mathrm{E}(G)f} = \alpha^{\mathrm{E}(G)}\}$. Then $|F| = 2$. Since F acts semi-regularly and by Lemma 2.5, $\mathrm{E}(G)$ acts faithfully, $\mathrm{E}(G)$ and F fulfil the hypotheses of Lemma 7.1. Thus, $\mathrm{E}(G)$ is isomorphic to $\mathcal{A}_5$, $\mathcal{A}_6$, $\mathrm{PSL}(2,7)$, $\mathrm{PSL}(2,8)$, $\mathrm{PSL}(2,11)$, or $\mathrm{PSL}(2,13)$, but then Lemma 7.2 yields a contradiction. As a consequence, $|\,\mathrm{F}(G)| \le 2$.

We turn our attention towards the case that a component acts with fixity 4 on one of its orbits. Similarly to Lemma 7.1, we will look at the orbits of a component and the orbit of the generalised Fitting subgroup. In the setups of Lemma 7.5 and Lemma 7.6, both orbits will have the same length. The next lemma proves this in a more general situation such that it can be applied by both lemmas and possibly even outside the context of this chapter.

Lemma 7.4

Let E and F be two groups acting on a set Γ such that $[E,F]=1$ and such that $E \cap F = \mathrm{Z}(E)$. Let $\alpha \in \Gamma$. Suppose that there exists an element $x \in E$ such that $|\operatorname{fix}_{\alpha^E}(x)| = |\operatorname{fix}_{\alpha^{EF}}(x)| \neq 0$. Then $\alpha^E = \alpha^{EF}$.

Proof:

Let $G = EF$ and $\beta \in \operatorname{fix}_{\alpha^E}(x)$. Then by Lemma 2.9, $\frac{|\{\langle x\rangle^g \le E_\beta \mid g \in E\}|\cdot|\,\mathrm{N}_E(\langle x\rangle)|}{|E_\beta|} = |\operatorname{fix}_{\alpha^E}(x)| = |\operatorname{fix}_{\alpha^G}(x)| = \frac{|\{\langle x\rangle^g \le G_\beta \mid g \in G\}|\cdot|\,\mathrm{N}_G(\langle x\rangle)|}{|G_\beta|}$. Let M_E denote the set $\{\langle x\rangle^g \le E_\beta \mid g \in E\}$ and let M_G denote $\{\langle x\rangle^g \le G_\beta \mid g \in G\}$. Then $|G_\beta|\cdot|M_E|\cdot|\,\mathrm{N}_E(\langle x\rangle)| = |E_\beta|\cdot|M_G|\cdot|\,\mathrm{N}_G(\langle x\rangle)|$.

We have a closer look at the sets M_E and M_G. Let $Y \in M_G$. Then there exists $g \in G$ such that $Y = \langle x\rangle^g \le G_\beta$. Let $e \in E$ and $f \in F$ be such that $ef = g$. Then $Y = \langle x\rangle^e$. Since $\langle x\rangle^e \le E$, it follows that $Y = \langle x\rangle^e \le E \cap G_\beta = E_\beta$. In particular, $Y \in M_E$. Since $M_E \subseteq M_G$, it follows that $M_E = M_G$, and hence $|G_\beta|\cdot|\,\mathrm{N}_E(\langle x\rangle)| = |E_\beta|\cdot|\,\mathrm{N}_G(\langle x\rangle)|$.

Since F centralises x, $\mathrm{N}_G(\langle x\rangle) = \mathrm{N}_G(\langle x\rangle) \cap EF = F(E \cap \mathrm{N}_G(\langle x\rangle) = F\,\mathrm{N}_E(\langle x\rangle)$. In particular, $|\,\mathrm{N}_G(\langle x\rangle)| = \frac{|F|\cdot|\,\mathrm{N}_E(\langle x\rangle)|}{|F \cap \mathrm{N}_E(\langle x\rangle)|} = \frac{|F|\cdot|\,\mathrm{N}_E(\langle x\rangle)|}{|\,\mathrm{Z}(E)|}$.

As a consequence, $|G_\beta|\cdot|\,\mathrm{N}_E(\langle x\rangle)| = |E_\beta|\cdot\frac{|F|\cdot|\,\mathrm{N}_E(\langle x\rangle)|}{|\,\mathrm{Z}(E)|}$. Thus, $|G_\beta| = \frac{|E_\beta|\cdot|F|}{|\,\mathrm{Z}(E)|}$, hence $|\alpha^{EF}| = |\beta^{EF}| = |G : G_\beta| = \frac{|EF|}{|G_\beta|} = \frac{|E|\cdot|F|\cdot|\,\mathrm{Z}(E)|}{|E\cap F|\cdot|E_\beta|\cdot|F|} = \frac{|E|}{|E_\beta|} = |\beta^E| = |\alpha^E|$. Since $\alpha^E \subseteq \alpha^{EF}$, the lemma follows.

The situation of the previous lemma especially happens when a group G acts with fixity 4 and has a component E that contains an element with exactly four fixed points. This will be used in the next lemma for a simple component and in Lemma 7.6 for a non-simple component.

Lemma 7.5
Let G be a finite group acting transitively and with fixity 4 on a set Ω. Let $\alpha \in \Omega$. Suppose that $\mathrm{E}(G)$ acts with fixity 4 on $\alpha^{\mathrm{E}(G)}$, that $\mathrm{E}(G)$ is simple, and that $\mathrm{F}(G)$ acts semi-regularly on Ω. Then $|\,\mathrm{F}(G)| \leq 2$.

Proof:

Let $E = \mathrm{E}(G)$ and $F = \mathrm{F}(G)$. Let $x \in E$ be such that x fixes exactly four points in α^E. Since F centralises x, it stabilises the set of fixed points, and hence the semi-regular action of F yields that $|F| \in \{1, 2, 4\}$. Assume, for a contradiction, that $|F| = 4$.

Since G acts with fixity 4, x can fix at most four points in Ω and thus no points outside of α^E. Therefore Lemma 7.4 implies that $\alpha^E = \alpha^{EF}$. Let $\Gamma = \alpha^{EF}$.

Since F is abelian and E is non-abelian simple, $\mathrm{Z}(EF) = F = \mathrm{F}(G)$. We want to use Lemma 4.1 and therefore we have to ensure that $|\Gamma| > 4 \cdot |F| = 16$. By Lemma 2.5, $|\Gamma| = |\alpha^E| \geq 5$. We use the Transitive Groups Library [52] together with the following GAP command.

```
AllTransitiveGroups(NrMovedPoints, [5..16],
↪  x->Order(x)>=60, [true], y->Order(FittingSubgroup(y)), [4]);
```

Since E is non-abelian simple, the order of E is at least 60. Thus, the GAP command returns all groups EF that could contain a simple group E, that can act faithfully and transitively on some set of size at least 5 and at most 16, and that have a Fitting subgroup of order 4. Since no such group is returned, $|\Gamma| > 16$.

Then Lemma 4.1 yields that $EF/F \cong E$ acts non-regularly and with fixity at most 4 on a set $\bar{\Gamma}$ of size $\frac{|\Gamma|}{|F|}$. Since E is simple, EF/F cannot act as Frobenius group, and hence acts with fixity 2, 3, or 4 on $\bar{\Gamma}$. In particular, E acts both on a set of size $|\Gamma|$ with fixity 4 and on a set of size $|\bar{\Gamma}| = \frac{|\Gamma|}{|F|} = |\Gamma|/4$ with fixity 2, 3, or 4. Thus, in the latter action, the point stabilisers have order divisible by 4.

First additionally assume, for a contradiction, that E acts with fixity 4 on a set of size $|\Gamma|/4$. In particular, E acts with fixity 4 both on a set of size $|\Gamma|$ and $|\Gamma|/4$. Then Theorem 3.56 implies that E is isomorphic to $\mathrm{PSL}(2, 11)$ or $\mathrm{PSL}(2, 13)$. This is a contradiction to Lemma 7.2.

Therefore instead assume, for a contradiction, that E acts with fixity 3 on a set of size $|\Gamma|/4$. Then E is a group that occurs both in the list of Theorem 1.1 in [72] and in the list of Theorem 3.56. Additionally, in the fixity-3 action, the order of a point stabiliser of E is divisible by 4. As a consequence, E is isomorphic to $\mathcal{A}_6$, $\mathrm{PSL}(2, 7)$, $\mathcal{A}_7$, $\mathrm{PSL}(2, 11)$, or M_{11}. For $E \cong \mathcal{A}_6$ and for $E \cong \mathrm{PSL}(2, 7)$, Lemma 7.2 yields a contradiction. If $E \cong \mathcal{A}_7$, then $|\Gamma| \in \{2^3 \cdot 3^2 \cdot 7, 7\}$. Thus, it follows that $|\bar{\Gamma}| = 2 \cdot 3^2 \cdot 7 = 126$, but according to Theorem 1.1 in [72],

$\mathcal{A}_7$ does not act with fixity 3 on a set with 126 elements. If $E \cong \mathrm{PSL}(2,11)$, then Lemma 7.2 again implies a contradiction. Thus, $E \cong \mathrm{M}_{11}$. Then by Theorem 1.1 in [72], $|\bar{\Gamma}| = 11$, and hence $|\Gamma| = 44$. However, according to Theorem 3.56, M_{11} does not act with fixity 4 on a set of size 44.

As a consequence, E acts with fixity 2 on $\bar{\Gamma}$. Then Theorem 1.2 in [71] implies that E is isomorphic to $\mathrm{PSL}(3,4)$ or there exists a prime power q such that E is isomorphic to $\mathrm{Sz}(q)$ or $\mathrm{PSL}(2,q)$. Since, by Theorem 3.56, $\mathrm{PSL}(3,4)$ cannot act transitively and with fixity 4, there exists a prime power q such that E is isomorphic to $\mathrm{Sz}(q)$ or $\mathrm{PSL}(2,q)$. If $E \cong \mathrm{Sz}(q)$, then by Lemma 3.12 in [71], $|\bar{\Gamma}| \in \{q^2+1, q^2(q^2+1)\}$. Therefore, $|\Gamma| \in \{4(q^2+1), 4q^2(q^2+1)\}$ but this contradicts Theorem 3.56. Thus, $E \cong \mathrm{PSL}(2,q)$. By Lemma 7.2, $q \geq 16$, and hence by Theorem 3.56, q is odd and $|\Gamma| \in \{2q(q+1), 2(q+1), 2q(q-1)\}$. Then $|\bar{\Gamma}| \in \{q(q+1)/2, (q+1)/2, q(q-1)/2\}$, contradicting Lemma 3.11 in [71]. Therefore the last remaining possibility was excluded, and hence the assumption that $|F| = 4$ was false. Thus, $|F| \in \{1,2\}$ and the lemma holds.

Lemma 7.6
Let G be a finite group acting transitively and with fixity 4 on a set Ω. Let $\alpha \in \Omega$. Suppose that $\mathrm{E}(G)$ acts with fixity 4 on $\alpha^{\mathrm{E}(G)}$, that $\mathrm{E}(G)$ is quasi-simple but not simple, and that $\mathrm{F}(G)$ acts semi-regularly. Then $\mathrm{F}(G)$ is cyclic of order 2 or 4.

Proof:
As in Lemma 7.5, let $E = \mathrm{E}(G)$, let $F = \mathrm{F}(G)$, and let $x \in E$ be such that x fixes exactly four points in α^E. Since F centralises x, it stabilises the set of fixed points, and hence the semi-regular action of F and the fact that $\mathrm{Z}(E) \leq F$ yield that $|F| \in \{2,4\}$. Assume, for a contradiction, that F is not cyclic. Then $F \cong E_4$.

By Lemma 4.8, there exists an odd prime power q such that $E \cong \mathrm{SL}(2,q)$, $E \cong C_2.\mathrm{Sz}(8)$, or $E \cong C_2.\mathrm{PSL}(3,4)$. In particular, $|\,\mathrm{Z}(E)| = 2$, and hence there exists a subgroup C of F such that $\mathrm{Z}(E) \times C = F$. As a consequence, $EF = E \times C$.

As in the previous lemma, x can fix at most four points in Ω and thus no points outside of α^E, and hence Lemma 7.4 implies that $\alpha^E = \alpha^{EF}$. Let $\Gamma = \alpha^{EF}$. We use [52] together with the following command to ensure that $|\Gamma| > 4 \cdot |C| = 8$.

```
AllTransitiveGroups(NrMovedPoints, [5..8],
↪  x->Order(x)>=60, [true], y->Order(FittingSubgroup(y)), [4]);
```

Since E is quasi-simple, the order of E is at least 60. Thus, the GAP command returns all groups EF that could contain a quasi-simple group E, that can act faithfully and transitively on some set of size at least 5 and at most 8, and that have a Fitting subgroup of order 4. Since no such group is returned and since by Lemma 2.5, $|\Gamma| = |\alpha^E| \geq 5$, it follows that $|\Gamma| > 8$.

Then Lemma 4.1 yields that $EC/C \cong E$ acts non-regularly and with fixity at most 4 on a set $\bar{\Gamma}$ of size $\frac{|\Gamma|}{|C|}$. Since E is quasi-simple but not simple, Lemma 4.3 implies that EC/C cannot act with fixity 2 or 3 on $\bar{\Gamma}$. Then the fact that quasi-simple groups cannot act as Frobenius groups implies that EC/C acts with fixity 4 on $\bar{\Gamma}$. In particular, E acts with fixity 4 both on a set of size $|\Gamma|$ and $|\bar{\Gamma}| = \frac{|\Gamma|}{|C|} = |\Gamma|/2$.

Since by Example 4.5, $C_2.\,\mathrm{Sz}(8)$ has exactly one transitive fixity-4 action, it cannot act with fixity 4 on two sets of different sizes. Similarly by Example 4.4, $C_2.\,\mathrm{PSL}(3,4)$ has only one fixity-4 action. As a consequence, there exists an odd prime power q such that $E \cong \mathrm{SL}(2,q)$. However, then Lemma 4.6 proves that E cannot act on two sets such that one has twice the size of the other. This contradiction shows that our assumption that $|F|$ is not cyclic was false.

The only case that we have not analysed in this section is when a component acts with fixity 3 on one of its orbits, but unlike for fixity 2 and 4, Theorem 5.6 gives detailed information in this case. In particular, the structure of the Fitting subgroup is a direct consequence of the theorem and will be drawn directly in the proof of Theorem 7.7.

7.2 The Main Theorem

The Main Theorem summarises all the results of the previous section and chapters. It gives detailed information about the generalised Fitting subgroup of a group that acts transitively, faithfully, and with fixity 4. In particular, with this result, important structural information about the group itself are established.

7.2.1 Statement

Theorem 7.7 (Main Theorem)
Let G be a finite group acting transitively, faithfully, and with fixity 4 on a set Ω. Let $\alpha \in \Omega$. Then one of the following cases holds.

(1) $\mathrm{F}(G) \cap G_\alpha \neq 1$ and $\mathrm{E}(G) = 1$. Furthermore, one of the following holds.

  (a) $\mathrm{F}(G)$ is a 2-group with sectional 2-rank at most 4 that acts faithfully and with fixity 4 on $\alpha^{\mathrm{F}(G)}$.

  (b) G is one of the groups under column *group* in Table 6.1, $|\Omega| \leq 28$ and $\mathrm{F}(G)$ is either non-cyclic elementary abelian of order at most 16 or of isomorphism type $C_4 \times C_4$ or $(C_4 \times C_4) : C_2$.

(2) $\mathrm{F}(G)$ acts semi-regularly on Ω. Furthermore, for all $p \in \pi(G_\alpha)$, the Sylow p-subgroups of G have p-rank 1 or $\mathrm{F}(G) = \mathrm{O}_2(G) \times \mathrm{O}_3(G)$. Additionally, one of the following holds.

  (a) $\mathrm{E}(G) = 1$

  (b) $\mathrm{E}(G)/\mathrm{Z}(\mathrm{E}(G)) \cong \mathcal{A}_5$, $\mathrm{E}(G)$ acts semi-regularly on Ω and if $\mathrm{E}(G) \cong \mathcal{A}_5$, then G_α is of isomorphism type C_2, E_4, C_4, $\mathcal{S}_3$, or $\mathcal{A}_4$ and if $\mathrm{E}(G) \cong \mathrm{SL}(2,5)$, then G_α is of isomorphism type C_2, E_4, or C_4.

(c) $\mathrm{E}(G)$ is isomorphic to $\mathrm{PSL}(3,4)$, $\mathrm{PSL}(2,q)$, or $\mathrm{Sz}(q)$, where q is a suitable prime power, $\mathrm{E}(G)$ acts with fixity 2 on $\alpha^{\mathrm{E}(G)}$, and $|\,\mathrm{F}(G)| \leq 2$.

(d) $\mathrm{E}(G) \cong \mathcal{A}_6$ acts with fixity 3 on $\alpha^{\mathrm{E}(G)}$, $\mathrm{F}(G) = 1$, and one of the following holds.

   (i) $|\Omega| = 6$ and $G \cong \mathcal{S}_6$.

   (ii) $|\Omega| = 12$ and G is isomorphic to M_{10}, $\mathrm{PGL}(2,9)$, or $\mathrm{Aut}(\mathcal{A}_6)$.

(e) $\mathrm{E}(G)$ acts with fixity 4 on $\alpha^{\mathrm{E}(G)}$ and additionally one of the following holds.

   (i) $\mathrm{E}(G)$ is isomorphic to one of the groups under column *group* in Table 3.2 and $|\,\mathrm{F}(G)| \leq 2$.

   (ii) $\mathrm{E}(G)$ is isomorphic to $\mathrm{SL}(2,q)$, where q is an odd prime power, to $C_2.\,\mathrm{Sz}(8)$, or to $C_2.\,\mathrm{PSL}(3,4)$, $|\,\mathrm{Z}(\mathrm{E}(G))| = 2$, and $\mathrm{F}(G)$ is cyclic of order 2 or 4.

Proof:
First suppose that $\mathrm{F}(G) \cap G_\alpha \neq 1$. Then by Lemma 6.2, $\mathrm{E}(G) = 1$. If Ω has at least 29 elements, then Lemma 6.6 implies that $\mathrm{F}(G)$ acts with fixity 4 on $\alpha^{\mathrm{F}(G)}$. Thus, Lemma 6.1 proves that $\mathrm{F}(G)$ is a 2-group with sectional 2-rank at most 4 showing case (1) (a). If $|\Omega| \leq 28$, then Lemma 6.8 lists all of these groups and shows also the information about $\mathrm{F}(G)$ in (1) (b).

Therefore, from now on suppose that $\mathrm{F}(G)$ acts semi-regularly. If $\mathrm{F}(G) \neq \mathrm{O}_2(G) \times \mathrm{O}_3(G)$, then by Lemma 6.9, for all $p \in \pi(G_\alpha)$, the Sylow p-subgroups of G have rank 1. Thus, it remains to proof that one of the cases (a) – (e) of (2) holds.

Suppose additionally that (2) (a) does not hold. Then Theorem 5.6 proves that $\mathrm{E}(G)$ is quasi-simple and states that one of four cases happens. If $\mathrm{E}(G)$ acts semi-regularly on Ω, then case (1) of Theorem 5.6 shows together with Lemma 5.7 statement (2) (b) of this theorem.

Thus, suppose that $\mathrm{E}(G)$ fixes a point in Ω. Since $\mathrm{E}(G)$ is a normal subgroup of G and G acts transitively on Ω, we can suppose that $\mathrm{E}(G) \cap G_\alpha \neq 1$.

Case (2) in Theorem 5.6 implies that $\mathrm{E}(G)$ acts with fixity 2 on α^E, that $\mathrm{E}(G)$ is simple and one of the specified groups. Then Lemma 7.3 proves that $\mathrm{F}(G)$ contains at most two elements.

Case (3) in Theorem 5.6 gives all information in case (2) (d) except for the size of the Fitting subgroup of G, but knowing G in all cases yields $\mathrm{F}(G) = 1$.

Hence, the last remaining option for $\mathrm{E}(G)$ is case (4) of Theorem 5.6, and thus $\mathrm{E}(G)$ acts transitively, faithfully, and with fixity 4 on α^E. If $\mathrm{E}(G)$ is simple, then Theorem 3.56 proves that $\mathrm{E}(G)$ is one of the groups described in (2) (e) (i) and Lemma 7.5 shows that $\mathrm{F}(G)$ contains at most two elements. Otherwise $\mathrm{E}(G)$ is quasi-simple and non-simple. Then Lemma 4.8 proves that $\mathrm{E}(G)$ is one of the groups described in (2) (e) (ii) and Lemma 7.6 shows the remaining statements.

7.2.2 Examples

It remains the question whether all of these cases can indeed occur. For some cases we have already seen examples in the previous chapters. Therefore we look at the remaining cases.

In particular, in case 2 (a) of Theorem 7.7, the restriction on the structure allows a wide range of groups. The next example illustrates the situation by stating one group that falls in this case.

Example 7.8
Let $G = \langle (1,2,3,4), (5,6,7), (5,6) \rangle \leq \mathcal{S}_7$. Then $G \cong C_4 \times \mathcal{S}_3$. In particular G is solvable, and hence $\mathrm{E}(G) = 1$. The Fitting subgroup of G is $\langle (1,2,3,4), (5,6,7) \rangle$. Let $U = \langle (5,6) \rangle$. Then G acts transitively on G/U. Since U is a point stabiliser in this action and since $\mathrm{F}(G) \cap U = 1$, the Fitting subgroup of G acts semi-regularly on G/U.

Since $|\,\mathrm{N}_G(\langle (5,6) \rangle)| = 8$, Lemma 2.10 implies that $(5,6)$ fixes exactly four points in G/U. As a consequence, G acts transitively, faithfully, and with fixity 4 on G/U. In particular, G is an example for case (2) (a) in Theorem 7.7.

We now turn our attention towards case 2 (c) of Theorem 7.7. If the Fitting subgroup is trivial, then the following example gives an instance of this case. If the Fitting subgroup has order 2, then we can use Lemma 7.10 to create examples. More generally, the lemma shows a way to construct, for two positive integers r and k, a group acting with fixity $r \cdot k$ out of a group acting with fixity k.

Example 7.9
Let G be the 62nd transitive group of degree 40 in the Transitive Groups Library [52]. Then $G \cong \mathcal{S}_5$ and G acts transitively on a set Ω of size 40. Thus, $\mathrm{E}(G) \cong \mathcal{A}_5$ and $\mathrm{E}(G)$ has index 2 in G. The following GAP code uses the program in Remark 2.18 and shows that G acts with fixity 4 on Ω and that there exists $\alpha \in \Omega$ such that $\mathrm{E}(G)$ acts with fixity 2 on $\alpha^{\mathrm{E}(G)}$.

```
G:=TransitiveGroup(40,62);;
e:=Filtered(NormalSubgroups(G),x->Index(G,x)=2)[1];;
TestFixity(G,MovedPoints(G),4);
TestFixity(e,Orbit(e,1),2);
```

Thus, G is an example of case (2) (c), where $\mathrm{F}(G) = 1$.

Lemma 7.10
Let k be a positive integer and E be a finite group acting faithfully, transitively, and with fixity k on a set Δ. Let r be a positive integer and c be an element of order r such that $G \coloneqq E \times \langle c \rangle$ is a group. Then G acts faithfully, transitively, and with fixity $r \cdot k$ on a set of size $r \cdot |\Delta|$.

Proof:
Let U be a point stabiliser of E under its action on Δ. Then $U \leq E \leq G$ and G acts transitively on $\Omega := G/U$. Let $x \in U$. By Lemma 2.9, $|\operatorname{fix}_\Omega(x)| = \frac{|\{\langle x\rangle^g \leq U | g\in G\}|\cdot|\operatorname{N}_G(\langle x\rangle)|}{|U|}$. Since $\langle c\rangle$ centralises $E \geq U$, it follows that $\operatorname{N}_G(\langle x\rangle) = \operatorname{N}_{E\times\langle c\rangle}(\langle x\rangle) = \operatorname{N}_E(\langle x\rangle)\times\langle c\rangle$ and that $\{\langle x\rangle^g \leq U \mid g \in G\} = \{\langle x\rangle^g \leq U \mid g \in E\}$. Therefore $|\operatorname{fix}_\Omega(x)| = \frac{|\{\langle x\rangle^g \leq U | g\in E\}|\cdot|\operatorname{N}_E(\langle x\rangle)\times\langle c\rangle|}{|U|} = \frac{|\{\langle x\rangle^g \leq U | g\in E\}|\cdot|\operatorname{N}_E(\langle x\rangle)|\cdot|\langle c\rangle|}{|U|} = r\cdot\frac{|\{\langle x\rangle^g \leq U | g\in E\}|\cdot|\operatorname{N}_E(\langle x\rangle)|}{|U|} = r\cdot|\operatorname{fix}_\Delta(x)|$. Since $|\operatorname{fix}_\Delta(x)| \leq k$, every non-trivial element in G can fix at most $r\cdot k$ elements in Ω, and since E contains an element y with exactly k fixed points in Δ, $|\operatorname{fix}_\Omega(y)| = r\cdot k$. Thus, G acts with fixity $r\cdot k$ on Ω.

The size of Δ is the number of fixed points of the trivial element of E, and hence $|\Omega| = |\operatorname{fix}_\Omega(1_G)| = r\cdot|\operatorname{fix}_\Delta(1_G)| = r\cdot|\Delta|$.

Since E acts faithfully on Δ, it follows that $|\Delta| \geq k+2$ and hence Ω has at least $r\cdot(k+2) > r\cdot k$ elements. Therefore, G acts faithfully on Ω.

When we use the previous lemma for $r = 2$ and a simple group E that acts with fixity 2 on some set, then we can construct a group G such that $\operatorname{E}(G) = E$, such that $|\operatorname{F}(G)| = r = 2$, and such that G acts with fixity 4. This is an example of case (2) (c) of Theorem 7.7.

For case (2) (e) on the other hand, we cannot use this construction because in this case the component itself acts with fixity 4. Nevertheless, there are instances of cases (2) (e) (i) and (ii) of Theorem 7.7 with non-trivial Fitting subgroup. We will see some of them in the next two examples.

Example 7.11
Let G be a group that is identified by `[336,209]` in the Small Groups Library [13]. Then $G \cong C_2\times\operatorname{PSL}(2,7)$, and hence $\operatorname{E}(G) \cong \operatorname{PSL}(2,7)$ and $\operatorname{F}(G) \cong C_2$. As described in Lemma 2.20, the table of marks of G contains information about all transitive actions of G. The following GAP code uses the table of marks.

```
G:=SmallGroup([336,209]);
t:=TableOfMarks(G);;
e:=RepresentativeTom(t,39);
U:=RepresentativeTom(t,12);
hom:=FactorCosetAction(G,U);; Gh:=Image(hom,G); eh:=Image(hom,e);
TestFixity(Gh,MovedPoints(Gh),4); TestFixity(eh,Orbit(eh,1),4);
```

The code first defines G and then creates the table of marks of G. Since $\operatorname{E}(G)$ has index 2 in G, we find $\operatorname{E}(G)$ in line 39 of the table of marks. Additionally we see a fixity-4 actions of G in line 12 and define U to be a corresponding point stabiliser. Afterwards the code uses the program in Remark 2.18 to test whether G acts with fixity 4 on G/U or not and also shows that there exists an element $\alpha \in G/U$ such that $\operatorname{E}(G)$ acts with fixity 4 on α^{E}. As a consequence, G is an example of case (2) (e) (i) in Theorem 7.7 where $\operatorname{F}(G) \cong C_2$.

Example 7.12
Let G be the 576th transitive group of degree 24 in the Transitive Groups Library [52]. Then $G \cong \mathrm{SL}(2,5) : C_2$ and G acts transitively on a set Ω of size 24. Thus, $\mathrm{E}(G) \cong \mathrm{SL}(2,5)$ and $\mathrm{E}(G)$ has index 2 in G. The following GAP code uses the program in Remark 2.18 and shows that G acts with fixity 4 on Ω and that there exists $\alpha \in \Omega$ such that $\mathrm{E}(G)$ acts with fixity 4 on $\alpha^{\mathrm{E}(G)}$.

```
G:=TransitiveGroup(24,576);;
e:=Filtered(NormalSubgroups(G),x->Index(G,x)=2)[1];;
TestFixity(G,MovedPoints(G),4);
TestFixity(e,Orbit(e,1),4);
```

Thus, G is an example of case (2) (e) (ii), where $\mathrm{F}(G) \cong C_4$.

Similarly, the 1353rd transitive group of degree 24 provides such an example. This group is isomorphic to $\mathrm{GL}(2,5)$. When we replace the first line of the GAP code above by `G:=TransitiveGroup(24,1353);;`, we can use the GAP commands to see that $\mathrm{GL}(2,5)$ has a fixity-4 action such that $\mathrm{SL}(2,5)$ acts with fixity 4 on one of its orbits.

7.2.3 Remark

Examples have been given throughout this thesis to demonstrate that no case of Theorem 7.7 is superfluous. These examples are collected in the following remark.

Remark 7.13
Example 6.7 together with Lemma 6.6 gives an instance of case (1) (a) of Theorem 7.7. Additionally, every 2-group that acts with fixity 4 falls in this case. One of them is $\langle (2,6)(5,8), (1,2,3,5,4,6,7,8) \rangle \leq \mathcal{S}_8$. In part (b) of case (1) all groups that belong to this case are stated and Lemma 6.8 implies that each of these groups indeed exhibits a fixity-4 action on some set that fulfils the requirements of the case.

Thus, it remains to further investigate the situation of case (2). One instance of case (2) (a) is given in Example 7.8. In part (b) a number of constellations for the component and the point stabiliser are described and Example 5.8 and Example 5.9 show that each of these constellations can indeed occur. Example 7.9 names an instance of case (2) (c) with trivial Fitting subgroup and Lemma 7.10 implies that for instance $\mathrm{PSL}(3,4) \times C_2$ has a fixity-4 action with Fitting subgroup of order 2. Thus, both orders of the Fitting subgroup in case (2) (c) are possible. Part (d) comprises exactly four groups and the following GAP code uses the program in Remark 2.18 and shows that all of them indeed fulfil the requirements of the case.

```
li:=Concatenation([TransitiveGroup(6,16)],List([181,182,220],
↪  x->TransitiveGroup(12,x)));;
comp:=List( li, x->Filtered(LowIndexSubgroups(x,2),y->Index(x,y)=2
↪  )[1]);;
comp[4]:=Filtered( LowIndexSubgroups(comp[4],2),
↪  y->Index(comp[4],y)=2 )[1];;
```

```
List(li,x->TestFixity(x,MovedPoints(x),4));
List(comp,x->TestFixity(x,Orbit(x,1),3));
```

In case (2) (e) (i) of Theorem 7.7 all stated types of components are possible because by Theorem 3.56 all of these simple groups act with fixity 4 on some set (and with trivial Fitting subgroup). Additionally, Example 7.11 gives an instance of case (2) (e) (i) where the Fitting subgroup has order 2. Similarly, Lemma 4.6, Example 4.5, and Example 4.4 show that all quasi-simple groups named in case (2) (e) (ii) can act with fixity 4 and all of these groups have a cyclic Fitting subgroup of order 2. That the Fitting subgroup could also be cyclic of order 4 is illustrated by Example 7.12.

As a consequence, we have seen for each of the cases of Theorem 7.7 an example of a group that fulfils the requirements of the case.

8 Closing Remarks

This thesis gives detailed information about the generalised Fitting subgroup of a faithful and transitive group G that acts with fixity 4, and hence about the structure of G itself. By doing so, it contributes to the understanding of the connection between the action of a group and its structure.

Additionally, it closes an open question in the project started by Magaard and Waldecker that is motivated by the study of automorphism groups of Riemann surfaces. More precisely, since one of the consequences of Theorem 3.56 is that Vermutung 3.16 in [89] holds, the conclusions in [89] drawn with the use of that assumption are true, and hence the classification of all simple non-abelian automorphism groups of compact Riemann surfaces of genus at least two that act with fixity at most 4 is achieved. However, the results of this thesis go beyond the study of simple groups in this setting, and therefore a future point of study could be to use the results to gain information about more than the simple non-abelian automorphism groups of compact Riemann surfaces. As described in the introduction, one aspect of the original project in the study of Riemann surfaces is the case that Schoeneberg's result, which states that all fixed points of a non-trivial automorphism of a Riemann surface that fixes at least five points are Weierstrass points, is not applicable. Therefore, a next step could be to use the information in [89] for finding all Weierstrass points for the examples in which Schoenebergs's result does not give further information. Additionally, in the scope of the project, there are other open problems (see for example pp. 186–196 in [89] for an overview) that possibly could be answered with the results of this thesis. The results can maybe even help to push forward the investigations in other topics connected to Riemann surfaces and groups such as the conjectures in [48] about monodromy groups. On the other hand, in the group-theoretic perspective of the project, this thesis finishes off the first step of understanding groups acting with fixity at most 4.

Even though the Main Theorem of this thesis gives detailed insights into the generalised Fitting subgroup of a finite, faithful, and transitive group acting with fixity 4, the degree of detail in the different cases differs. The information is especially limited in case (2) (a), although the fact that the group does not contain components is a strong restriction. Further studies of this case seem promising. Additionally, the consequences derived in a series of lemmas in Chapter 7 ending in Lemma 7.3, Lemma 7.5, and Lemma 7.6 and their proofs suggest that there is a more fundamental connection between the structure of an (abelian) group F centralising a group E and the different fixities with which the groups EF and E can act. Lemma 7.4 gives a first impression of how this relation could be investigated. The behaviour observed for fixity 4, in comparison to the behaviours for fixity 2 and 3, suggests that the connection could be related to the fact that 4 is not a prime, and that hence a fixity-2 action can be extended to a fixity-4 action in a product with another group. However, the exact relation which also takes higher fixities into account remains to be revealed, and generalises beyond the need of Theorem 7.7. Similarly, in Chapter 4, Corollary 4.9 might be provable without the use of the classification theorem of finite simple groups, and therefore without the use of Lemma 4.8. In both cases this would give an even deeper understanding of the influence that the fixity a group is acting with has on the structure of the group. The behaviour seen is different from what was found for fixity 2 and 3 in [71] and [72], already indicating that the number 4, and the fact that it is not a prime, broadens the structural possibilities of groups acting with fixity 4 compared to smaller fixities.

A natural next step is to ask the same question for fixity 5. However, there arise some difficulties. On the one hand, for the original context of the project and its application to Riemann surfaces the question is not relevant, because if a non-trivial group element (and hence a non-trivial automorphism) fixes five points, these points are by Schoeneberg's result (see [91]) Weierstrass points, and hence contain analytic information. On the other hand, as a group-theoretic research question it is still interesting to ask what can be said about finite, faithful, and transitive groups acting with fixity 5. As indicated earlier, the amount of work needed to understand groups acting with small fixity increased with the fixity, thus analysing the situation for fixity 5 will be even harder than the analysis of groups acting with fixity 4. There are different reasons for this. One of them is that in the analysis, in some cases the setup is reduced to a situation where a group acts with smaller fixity on some set, and thus for each smaller fixity the knowledge of the structure of groups acting with that fixity is needed. Another reason is that, with 5, a third prime has to be taken into account, and hence the analysis of the 2- and 3-structure has to be accompanied by an analysis of the 5-structure, adding another degree of difficulty to it. In light of the study of the 3-structure in [7], the importance of understanding 3-groups of maximal class in the analysis, and the absence of a comparable easily accessible result for 5-groups of maximal class (see [66] for an impression), it currently seems unreasonable to hope for a similar result for fixity 5. However, without such a result, a new strategy for a proof of a classification of all finite simple groups acting transitively and with fixity 5 is needed. A naive way would be to go through the classification of finite simple groups and decide for each group in it whether or not the group acts with fixity 5, but on the one hand, the list has infinite length, and on the other hand, Chapters 3 and 4 of this thesis show how complicated it is to prove that a group can act with a certain fixity. Therefore, a more advanced strategy is indeed needed to make progress in that direction. Even if such a classification could be reached, there are further difficulties for a general structure result. One of these difficulties is that Lemma 5.4 is false if the hypothesis that the group acts with fixity 4 is replaced by the assumption that it acts with fixity 5. The prototypical counterexample is the group $\mathcal{A}_5 \times \mathcal{A}_5$ which can act with fixity 5 on a set of size 60. To see this, we can use the GAP program in Remark 2.22 one last time, together with the commands

```
G:=AlternatingGroup(5);;
TestTom(TableOfMarks(DirectProduct(G,G)),5);
```

and get as result that $\mathcal{A}_5 \times \mathcal{A}_5$ can act transitively and with fixity 5 in two ways and that both of the actions are faithful and on sets of size 60. As a consequence, an equivalent of Theorem 5.6 for fixity 5 would be more complicated, because the analysis cannot be derived out of the current proof by just adding the case that the component can act with fixity 4 but must contain a strategy for dealing with multiple components. Thus, in comparison to the analysis for fixity 3, both the possibility that a component can act semi-regularly and that the group has more than one component have to be taken into account. Alternatively, a totally new approach is needed. Independently of how an alternative for Theorem 5.6 for fixity 5 can be reached, it adds even more complications to a proof of a general structural result. To sum up, trying to answer the question about the structure of groups acting with fixity 5 needs new ideas and might not be as fruitful as desired.

Changing the point of view, instead of asking which groups can act with a certain fixity, another approach is to determine the fixities with which a certain family of groups can act, or at least a bound for them. For example, does there exist a positive integer k such that, for all prime powers q, the group $\mathrm{PSL}(4, q)$ can act transitively and with fixity at most k on some set. The analysis in Section 3.3 seems to be a good starting point to tackle the problem. An answer would give an even deeper understanding of the notion of fixity. Similarly, a generalisation of fixity would also give additional insights. For instance, there are groups that almost act with a certain fixity but have one conjugacy class of non-trivial elements that have more fixed points. A prominent example is $\mathcal{A}_8$ in its natural action, where only the elements conjugate to $(1, 2, 3)$ have five fixed points, and all non-trivial elements in all other conjugacy classes have four or fewer fixed points, making it a group that almost acts with fixity 4. This generalisation raises questions about the structure of those groups, which results about fixity can be adapted (and how), and whether there is a restriction on the number of fixed points of the elements in the exceptional conjugacy class. The study of these and other problems can be supported by GAP, and even more so if the results in [71], [72], [89] and this thesis were directly accessible from inside GAP, like in a package or library.

Another further research direction could be to focus on primitive groups acting with fixity 4. This is a slightly different approach than that of the Main Theorem (Theorem 7.7) of this thesis which concentrates on the generalised fitting subgroup. For the analysis of primitive groups, using the O'Nan-Scott Theorem (see [4] or Theorem 4.1 in [24] for a version of the statement) is a key strategy. The theorem gives different reduction steps in which some information about the action of the group is described, and ends with the situation of an almost simple group, in which case Theorem 7.7 is especially useful. The reason for this is that in the case of an almost simple group G, the product of the components $\mathrm{E}(G)$ is non-trivial, and hence only cases (2) (b)-(e) of Theorem 7.7 can occur, and in all of these cases $\mathrm{E}(G)$ is known. However, to be of even more use, it has to be understood how the properties of a group acting with some fixity can be beneficial throughout the different reduction steps of the O'Nan-Scott Theorem. Since the notation of fixity is already a way of describing some part of the action of a group, it is reasonable to assume that it can be combined with other information about the action, and hence with the notion of a primitive action (as a reduction step) and the further reductions of the O'Nan-Scott Theorem. This is also a reason why primitive groups play an important role in the study of permutation groups, and therefore applying the results of this thesis to primitive groups seems to be the most promising next step.

Additionally, since the results for fixity 2 and 3 are already used as a basis for insights into other studies, the results for fixity 4 can probably support research in a similar way. For example, in light of [22], or more precisely the proof of Theorem 1.5 therein, the results of this thesis might be useful for pushing the boundary of the independence number of the Saxl graph of a permutation group G (denoted by $\alpha(G)$) under certain conditions and with a list of exceptions to $\alpha(G) \geq 5$. Further connections of the notion of fixity to other research areas in group theory and outside were stated in the introduction. Having these connections in mind, it seems conceivable that the results of this thesis can have more applications in future research. Together with the aforementioned possible next steps, there are many interesting further research questions.

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Publikationen

Einreichung	**Barbara Baumeister, Paula Hähndel, Kay Magaard, Patrick Salfeld und Rebecca Waldecker**, *Finite simple permutation groups acting with fixity 4*, In: Journal of Group Theory (Mar. 28, 2023). url: `https://arxiv.org/abs/2207.08788v4`. Eingereicht.
Konferenzbeitrag	**Paula Hähndel und Rebecca Waldecker**, *Questions on Orbital Graphs*, In: James H. Davenport, Manuel Kauers, George Labahn und Josef Urban (Hrsg.) Mathematical Software – ICMS 2018. 6th international conference, South Bend, IN, USA, July 24–27, 2018. Proceedings.
Software	**Paula Hähndel, Markus Pfeifer und Wilf Wilson**, *OrbitalGraphs – GAP package*, `https://github.com/gap-packages/OrbitalGraphs`

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