Smoothing singularities of Riemannian metrics while preserving lower curvature bounds

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Abstract

In this work, we describe a smoothing technique for singular Riemannian metrics, which almost preserves nonnegative curvature. Combined with results of M. Simon [27], [28], it gives rise to some geometric applications.

In the first part, we glue two smooth Riemannian manifolds along isometric boundaries. We show that, provided that the sum of the second fundamental forms of the boundaries is nonnegative, lower bounds on certain curvatures are preserved under the gluing operation up to an arbitrary small error term. These curvatures include the Riemannian curvature operator, Ricci curvature, scalar curvature, isotropic curvature, and bi-curvature.

In the second part, we study the evolution of the curvatures from the first part under the Ricci flow on compact manifolds. Under the assumption that the scalar curvature satisfies a bound of the form C/t (where $C > 0$ is small), we show that initial lower bounds on these curvatures do not become too bad on a well controlled time interval. This result holds for all curvatures from the first part, except for the Ricci curvature. Combining the first and second part with results from [27] and [26], we show that manifolds which arise from gluing two manifolds with nonnegative curvature admit a smooth metric of nonnegative curvature, which allows a topological classification of such manifolds.

In the third part, we are concerned with metrics of nonnegative Riemannian curvature on three manifolds, which are possibly singular (discontinuous) at one point, where the singularity has a certain cone-like structure. Using a gluing construction, we smooth out such singularities while keeping the curvature operator almost nonnegative. As an application, by combining this with a result from [28], we show that a manifold with such a singular metric admits a smooth metric of nonnegative Ricci curvature.

Zusammenfassung

In der vorliegenden Arbeit beschreiben wir eine Glättungstechnik für singuläre Riemannsche Mannigfaltigkeiten, bei der nichtnegative Riemannsche Krümmung fast erhalten bleibt. Diese Technik liefert in Kombination mit den Resultaten von M. Simon [27] [28] einige geometrische Anwendungen.

Im ersten Teil dieser Arbeit kleben wir zwei glatte Riemannsche Mannigfaltigkeiten entlang isometrischer Ränder. Unter der Voraussetzung, dass die Summe der zweiten Fundamentalformen der Ränder nichtnegativ ist, werden dabei untere Schranken bestimmter Krümmungen bis auf einen beliebig kleinen Fehlerterm erhalten. Zu diesen Krümmungen zählen der Riemannsche Krümmungsoperator, Ricci-Krümmung, skalare Krümmung, isotropische Krümmung und Bi-Krümmung.

Im zweiten Teil untersuchen wir die Evolution dieser Krümmungen unter dem Ricci-Fluss. Unter der Annahme, dass die Skalarkrümmung eine Schranke der Form C/t erfüllt (wobei $C > 0$ klein), zeigen wir, dass untere Anfangsschranken dieser Krümmungen (ausgenommen die Ricci-Krümmung) auf einem kontrollierten Zeitintervall nicht zu schlecht werden können. Wir kombinieren das mit den Ergebnissen aus dem ersten Teil und den Resultaten aus [27] und [26], und zeigen, dass eine Mannigfaltigkeit, die durch Kleben zweier Mannigfaltigkeiten mit nichtnegativer Krümmung entsteht, eine glatte Metrik mit nichtnegativer Krümmung besitzt, was eine topologische Klassifikation solcher Mannigfaltigkeiten erlaubt.

Im dritten Teil beschäftigen wir uns mit Metriken mit nichtnegativer Riemannscher Krümmung (in Dimension 3), die in einem Punkt singulär (nicht stetig) sein können, wobei die Singularität eine gewisse kegelähnliche Struktur hat. Unter Verwendung der Klebe-Technik aus dem ersten Teil können wir solche Singularitäten glätten, während wir den Krümmunsoperator fast nichtnegativ halten. Als Anwendung, in Kombination mit den Ergebnissen von M. Simon [28], zeigen wir, dass eine solche Mannigfaltigkeit eine glatte Metrik mit nichtnegativer Ricci Krümmung besitzt.

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Introduction

The current work is divided into three main parts.

In Chapter 1, we describe a gluing technique for two smooth Riemannian manifolds of curvature $\geq \kappa \in \mathbb{R}$, which have isometric boundaries. Even though the resulting manifold admits a smooth differentiable structure, one can only expect the glued metric to be C^0 across the common boundary. In particular, it makes no sense to speak of the Riemannian curvature operator of such a metric in the classical sense. One way of dealing with the non-smoothness is to view the glued metric as a C^0 limit of smooth metrics. One of the main results in Chapter 1 is that, under the assumption that the sum of the second fundamental forms of the boundaries is nonnegative, there exists such an approximating sequence of smooth metrics g_i whose curvature operators are $\geq \kappa - \varepsilon_i$, where ε_i tends to zero (see Thm. 1.1.2). Analogous results hold for various other curvatures, including Ricci curvature, scalar curvature, (1- and 2-) isotropic curvature, and bi-curvature. In the scalar curvature case it suffices to assume that the sum of the mean curvatures of the boundaries is nonnegative.

A similar problem has been addressed in a number of works in the framework of Alexandrov spaces, which generalizes the notion of bounded sectional curvature for abstract metric spaces (we refere to [6] for a detailed discussion). In [21], Yu. G. Reshetnyak has shown that upper curvature bounds in the sense of Alexandrov are preserved under gluing, if the glued boundaries are convex. In [19], A. Petrunin has shown that lower curvature bounds in the sense of Alexandrov are preserved under gluing. In [15], N. N. Kosovskiĭ studied the case where the glued spaces are Riemannian manifolds with sectional curvature $\geq \kappa$ in the classical sense. Using an approximating sequence of smooth Riemannian metrics, he has shown that the resulting space is an Alexandrov space of curvature $\geq \kappa$ if and only if the sum of the second fundamental forms of the glued boundaries is nonnegative. The method of the proofs in Chapter 1 is similar to the one in [15].

Chapter 2 is devoted to almost nonnegative curvature conditions which are preserved under the Ricci flow. Ricci flow invariant (weakly) positive curvature conditions have been studied in a number of works, and gave rise to various geometric applications. In [12], R. Hamilton proved that a compact three-manifold with positive Ricci curvature is diffeomorphic to a spherical space form, where a crucial step of the proof was to show that nonnegative Ricci curvature is preserved under the Ricci flow in dimension three. Similar results were obtained in [13] for four-manifolds with positive Riemannian curvature operator, where Hamilton proved that nonnegative curvature operator is preserved under the Ricci flow, and classified all compact four-manifolds with nonnegative curvature operator. In [8], H. Chen generalized Hamilton's results from [13], showing that 2-nonnegative curvature is preserved under the Ricci flow. In [4], S. Brendle and R. Schoen proved the Differentiable Sphere Theorem, where the proof strongly relied on the fact that nonnegative isotropic curvature is preserved under the Ricci flow, which was also shown independently by H. T. Nguyen [18].

In [22], T. Richard studied curvature conditions which are invariant under the Ricci flow, and lie between nonnegative Riemannian curvature operator and nonnegative Ricci curvature (such conditions include nonnegative Riemannian curvature operator itself, 2-nonnegative curvature operator, and nonnegative 1- and 2-isotropic curvature). One of the results of his work was that the corresponding almost nonnegative curvature conditions are preserved under the Ricci flow on a well controlled time interval, provided one has a bound of the form $|S(t)| \leq C/t$ (where $C > 0$ is small) on the scalar curvature. In certain cases, the method of the proof in [22] (which mainly involves Hamilton's maximum principle for systems) still can be applied to curvature conditions which do not necessarily imply nonnegative Ricci curvature, such as nonnegative isotropic curvature. We shall verify this in Chapter 2. As an application, combining results from Chapter 1 and 2 with M. Simon's results from [26] and [27], we show that glued manifolds with curvatures ≥ 0 as in Chapter 1 admit a smooth metric of nonnegative curvature.

In Chapter 3, we are concerned with point singularities of Riemannian metrics. In [28], M. Simon studied a class of complete non-collapsed three manifolds with Ricci curvature uniformly bounded from below and controlled geometry at infinity. He showed that a solution to the Ricci flow of such manifolds exists on a well controlled time interval, which made it possible to introduce a notion of Ricci flow for (possibly singular) metric spaces (X, d_X) arising as Gromov-Hausdorff limits of sequences of such manifolds. An important result of [28] is that in particular X is a manifold (cf. [28] Thm 9.2), which shows that the conjecture of M.Anderson-J.Cheeger-T.Colding-G.Tian is correct in dimension three. Moreover, if the lower bounds on the Ricci curvature of the manifolds in the sequence tend to zero, then X admits a Riemannian metric of nonnegative Ricci curvature, which allows a topological classification of such spaces in view of the works of W. X. Shi [24] and R. Hamilton [12].

In the current work, we study Riemannian three manifolds (M, g) such that g is smooth everywhere except at a point $o \in M$, where g is possibly discontinuous, and such that, where defined, the curvature operator of g is nonnegative. We show that, under some additional assumptions on the structure of the singularity, one can approximate g by a sequence of smooth metrics with almost nonnegative curvature operators, converging to g in the C^0 sense on $M \setminus o$ (see Thm. 3.4.1).

Let us briefly describe the smoothing procedure in Chapter 3. We require that the singularity of the metric g at o has a certain cone-like structure. Essentially, we assume that the distance function $\mathrm{dist}_g(\cdot, o) : M \to \mathbb{R}$ arising from the metric g is continuous at o and smooth on a neighborhood of o (except at o), that its level sets $\Gamma(r) = \{\mathrm{dist}(\cdot, o) = r\}$ are homeomorphic to the standard sphere S^2, and that the second fundamental form of $\Gamma(r)$ approaches $\frac{1}{r}\, g|_{\Gamma(r)}$ as r tends to zero (note that $\frac{1}{r}\, g|_{\Gamma(r)}$ is just the second fundamental form of $\Gamma(r)$ if g is a standard cone metric). This enables us to replace a neighborhood of the singularity by a standard cone with nonnegative curvature operator, using the gluing technique described in Chapter 1. Even though the standard cone has a singularity at the vertex, due to its well controlled geometry it can be smoothed out while keeping the curvature operator nonnegative.

As an application, we consider manifolds (M, g) as above, such that (M, g) is non-collapsed at infinity (that is, balls of radius one lying outside some neighborhood of the singular point satisfy a uniform lower volume bound > 0), and such that the curvature operator of g is bounded at infinity. We show that such manifolds can be viewed as Gromov-Hausdorff limits of sequences of manifolds with almost nonnegative curvature operator as in M. Simon's work [28]. In particular, M admits a smooth metric of nonnegative Ricci curvature, and hence it can be assigned to a certain topological class.

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Chapter 1.

Gluing Riemannian manifolds with curvature operators $\geq \kappa$

1.1. Introduction and preliminaries

In [15], N. N. Kosovskiĭ studied the gluing of two Riemannian manifolds with sectional curvature $\geq \kappa$ along isometric boundaries. He showed that the resulting space has curvature $\geq \kappa$ in the sense of Alexandrov, if and only if the sum of the second fundamental forms of the boundaries is nonnegative. In this chapter, we shall examine a similar setup for smooth Riemannian manifolds with smooth compact boundaries and curvature operators $\geq \kappa$. The method being used in [15] can be applied with some modifications.

Let us introduce some notations before stating the main result of this chapter. Let (M, g) be a smooth Riemannian manifold with a smooth metric g, and $\Lambda^2(TM) \subset TM \otimes TM$ be the bundle of two-vectors over M. Given a point $p \in M$ and a basis $\{e_1, \dots, e_n\}$ of T_pM, the space $\Lambda^2(T_pM)$ is generated by

$$\{e_i \wedge e_j = e_i \otimes e_j - e_j \otimes e_i \,|\, 1 \leq i < j \leq n\}.$$

The metric g induces an inner product $\mathcal{I}^g$ on $\Lambda^2(TM)$, defined by

$$\mathcal{I}^g(e_i \wedge e_j, e_k \wedge e_l) := g_{ik}g_{jl} - g_{jk}g_{il}, \tag{1.1.1}$$

where $g_{ik} = g(e_i, e_k)$. Note that if the vectors e_i are orthonormal with respect to g, then the two-vectors $e_i \wedge e_j$ are orthonormal with respect to $\mathcal{I}^g$. Let R^g be the Riemannian curvature tensor[1] of g and $R^g_{ijkl} = R^g(e_i, e_j, e_k, e_l)$. R^g induces a symmetric bilinear form $\mathcal{R}^g$ on $\Lambda^2(TM)$ via

$$\mathcal{R}^g(e_i \wedge e_j, e_k \wedge e_l) = R^g_{ijkl}.$$

The Riemannian curvature operator on $\Lambda^2(TM)$, which we shall also denote by $\mathcal{R}^g$, is defined by the property

$$\mathcal{I}^g(\cdot, \mathcal{R}^g \cdot) = \mathcal{R}^g(\cdot, \cdot).$$

By $\mathcal{R}^g \geq \kappa \in \mathbb{R}$ (or $\mathcal{R}^g \geq \kappa\mathcal{I}^g$) we mean that all eigenvalues of $\mathcal{R}^g$ are at least κ, or equivalently that

$$\mathcal{R}^g(\alpha, \alpha) \geq \kappa\mathcal{I}^g(\alpha, \alpha)$$

for all $\alpha \in \Lambda^2(TM)$. We refer to Appendix B.1 for a more detailed discussion on the connection between (4,0)-tensors and linear operators.

Let M_0 and M_1 be smooth Riemannian manifolds with smooth boundaries Γ_0 and Γ_1, and smooth metrics g_0 and g_1. Suppose that there exists an isometry

[1] We adopt the sign convention $R^g(X, Y) = \nabla^g_Y \nabla^g_X - \nabla^g_X \nabla^g_Y + \nabla^g_{[X,Y]}$.

$\phi : (\Gamma_0, g_0|_{\Gamma_0}) \to (\Gamma_1, g_1|_{\Gamma_1})$ of the boundaries. By gluing M_0 and M_1 along ϕ we mean identifying points $p \in \Gamma_0$ and $\phi(p) \in \Gamma_1$. The resulting space $M = M_0 \cup_\phi M_1$ can be equipped with a smooth differentiable structure, such that M_0 and M_1 are smooth submanifolds of M (see Section 1.2). Moreover, with respect to this structure, $\Gamma := \Gamma_0 =_\phi \Gamma_1$ is a smooth hypersurface of M. Let L_0 and L_1 be the second fundamental forms of $\Gamma_0 \subset M_0$ and $\Gamma_1 \subset M_1$ with respect to the inward normals. In view of the above construction, L_0 and L_1 can be regarded as $(2,0)$-tensors on $T\Gamma$, which enables us to consider their sum $L_0 + L_1$.

Let us define the metric g on the glued manifold M by $g|_{M_i} = g_i$, $i = 0, 1$. In what follows, we use the notation $g =: g_0 \cup_\phi g_1$. Due to the isometry of the boundaries, g is continuous, but fails to be C^2-smooth in general. In this case we can not speak of the Riemannian curvature operator of g in the classical sense. In [15], Kosovskiĭ made use of the fact that nevertheless M can be equipped with a length structure induced by g and instead of bounded sectional curvature in the classical sense one has the notion of bounded curvature in the sense of Alexandrov (see [6]). However, there is no analogue of this notion for bounds on the Riemannian curvature operator. We introduce the following definition:

Definition 1.1.1. *Let M be a Riemannian manifold, equipped with a continuous metric g. We say that the Riemannian curvature operator of g is at least κ, if there exists a family of C^∞ metrics $(g_{(\delta)})$ on M which converge to g uniformly on every compact subset as δ tends to zero and*

$$\mathcal{R}(g_{(\delta)}) \geq \big(\kappa - \varepsilon(\delta)\big)\mathcal{I}(g_{(\delta)})$$

holds with $\varepsilon(\delta) \to 0$.

In view of the above definitions the main result of this chapter is the following

Theorem 1.1.2. *Let M_0 and M_1 be smooth Riemannian manifolds with (at least C^2-)smooth metrics g_0 and g_1 and smooth compact boundaries Γ_0 and Γ_1, respectively. Suppose that there exists an isometry $\phi : \Gamma_0 \to \Gamma_1$, and let $M = M_0 \cup_\phi M_1$, and $g = g_0 \cup_\phi g_1$. Let L_0 and L_1 be the second fundamental forms of $\Gamma_0 \subset M_0$ and $\Gamma_1 \subset M_1$, respectively, and let $L := L_0 + L_1$ on $\Gamma := \Gamma_0 =_\phi \Gamma_1$. Suppose that $\mathcal{R}(g_0)$ and $\mathcal{R}(g_1)$ are at least κ. If L is positive semidefinite, then $\mathcal{R}(g) \geq \kappa$ in the sense of Definition 1.1.1.*

Analogous results hold for manifolds with lower bounds on Ricci curvature, scalar curvature (in this case it suffices to require only that $\mathrm{tr}_g L \geq 0$ on Γ), bi-curvature (the sum of the two smallest eigenvalues of the curvature operator), and isotropic curvature, respectively.

Plan of the proof of Theorem 1.1.2:

We proceed similarly to [15]:

- In Section 1.2, we sum up auxiliary constructions. We introduce a smooth structure on M relative to which M_0, M_1 and their common boundary Γ are smooth submanifolds. The metric g on M induced by g_0 and g_1 is continuous. By modifying the metric g_0 near Γ, we construct a new metric g_δ on M_0, such that the coefficients of the metric $g_{(\delta)} := g_\delta \cup_\phi g_1$ belong to the Sobolev class $W^{2,\infty}_{loc}$. The constructions in this section were adopted from [15] (cf. §§ 3-6) to the greatest extent.

- In Section 1.3, we compare the Riemannian curvature operators of g_δ and g_0 on M_0. This section corresponds with § 7 in [15].
- In Sections 1.4 and 1.5, we estimate the curvature operator of g_δ, showing that $\mathcal{R}(g_\delta) \geq \kappa - \varepsilon(\delta)$ holds on M_0, which implies that the weakly defined curvature operator of the $W^{2,\infty}_{loc}$ metric $g_{(\delta)}$ satisfies $\mathcal{R}(g_{(\delta)}) \geq \kappa - \varepsilon(\delta)$ a.e. on M.
- In Section 1.6, we mollify $g_{(\delta)}$ and construct a family of smooth metrics as required in Definition 1.1.1.

1.2. Definitions and auxiliary identities

Throughout this section, we use the notation from [15]. Consider two Riemannian manifolds M_0, M_1 with smooth compact boundaries Γ_0, Γ_1, and smooth metrics g_0, g_1, such that there exists an isometry $\phi : \Gamma_0 \to \Gamma_1$. First, let us introduce a smooth structure on $M = M_0 \cup_\phi M_1$, such that $M_0, M_1 \subset M$ are smooth submanifolds, and $\Gamma_0 =_\phi \Gamma_1 =: \Gamma \subset M$ is a smooth hypersurface with respect to this structure (cf. [15], Lemma 3.1). Let us fix a coordinate chart $(x^1, \dots, x^{n-1})$ of Γ. The distance functions $\operatorname{dist}_{g_0}$ and $\operatorname{dist}_{g_1}$ of g_0 and g_1 are smooth near Γ on M_0 and M_1, respectively. For a point $p \in M_0$ near Γ we put $x^n(p) = \operatorname{dist}_{g_0}(p, \Gamma)$, and $x^i(p) = x^i(\hat{p})$ for $i = 1, \dots, n-1$, where $\hat{p}$ is the point of Γ satisfying $\operatorname{dist}_{g_0}(p, \Gamma) = \operatorname{dist}_{g_0}(p, \hat{p})$. Note that $\hat{p}$ is unique, if p is close enough to Γ. We then repeat this construction on M_1, putting $x^n(p) = -\operatorname{dist}_{g_1}(p, \Gamma)$ for points $p \in M_1$ near Γ. The collection of all such coordinate charts $(x^1, \dots, x^n)$, where $(x^1, \dots, x^{n-1})$ is a coordinate chart of Γ, is compatible with the smooth structures of M_0 and M_1, and gives us the smooth structure on M with the desired properties. The coordinates $(x^1, \dots, x^n)$ are also known as Fermi coordinates. We refer to Appendix A for a more detailed discussion. Throughout this chapter, all computations will be carried out in these coordinates, unless noted differently.

Lemma 1.2.1. *The metric $g = g_0 \cup_\phi g_1$ is continuous. In coordinates defined above, it has the form*

$$(g_{ij})_{1 \leq i,j \leq n} = \begin{pmatrix} g_{1,1} & \cdots & g_{1,n-1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ g_{n-1,1} & \cdots & g_{n-1,n-1} & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \tag{1.2.1}$$

on a neighborhood of Γ.

Proof. In our coordinates, the metrics g_0 and g_1 have the form $g_0 = \begin{pmatrix} \hat{g}_0 & 0 \\ 0 & 1 \end{pmatrix}$ and $g_1 = \begin{pmatrix} \hat{g}_1 & 0 \\ 0 & 1 \end{pmatrix}$, where $\hat{g}_0$ and $\hat{g}_1$ are the restrictions of g_0 and g_1 to the equidistant hypersurfaces of Γ (see Appendix A.1). By assumption we have $\hat{g}_0 = \hat{g}_1$ on Γ, which shows the continuity of g. □

Notation 1.2.2. We denote the basis vector fields of TM with respect to the coordinate charts $(x^1, \dots, x^n)$ by $\partial_i = \frac{\partial}{\partial x^i}$ for $1 \leq i \leq n-1$ and $N = \frac{\partial}{\partial x^n}$. Note that near Γ the vector field N is smooth, has unit length, and is orthogonal to the equidistant hypersurfaces of Γ (cf. Appendix A.1).

Lemma 1.2.3 ([15], Lemma 3.1)**.** *The metric g_1 smoothly extends to a metric g_1' on a small neighborhood of Γ in M_0 in such a way that in our coordinates g_1' has the same form as g in Lemma 1.2.1, that is, $(g_1')_{in} = \delta_{in}$ for all $1 \le i \le n$.*

Proof. In coordinates defined above the metric g_1 on M_1 is of the same form as in (1.2.1). Locally in a small enough coordinate neighborhood U of some point of Γ we may smoothly extend $(g_1)_{1\le i,j\le n-1}$ to $U \cap M_0$ in such a way that the extended matrix $(g_1')_{1\le i,j\le n-1}$ is positive definite, and put $(g_1')_{in} = \delta_{in}$ for $1 \le i \le n$. We then cover Γ by finitely many such neighborhoods and define g_1' near Γ using a subordinate partition of unity. One easily checks that the obtained metric has the desired property. □

Throughout this chapter, we will use the following

Notation 1.2.4. Given a $(2,0)$ tensor A on T_pM, $p \in M$, we denote by $\mathbf{A}$ the corresponding linear endomorphism of T_pM satisfying

$$A(v,w) = \langle v, \mathbf{A}w\rangle_g.$$

If $\{e_1,\ldots,e_n\}$ is a basis of T_pM and $\mathbf{A}e_i = \mathbf{A}_i^j e_j$, then $\mathbf{A}_i^j = A_{ki}g^{kj}$, where $A_{ki} = A(e_k, e_i)$, and $(g^{kl})_{1\le k,l\le n}$ is the inverse of the matrix $(g(e_k,e_l))_{1\le k,l,\le n}$. The operator $\mathbf{A}$ is self-adjoint iff the tensor A is symmetric.

Definition, Lemma 1.2.5 (The operator $\mathbf{L}$, cf. [15], 3.4 and 3.5)**.** *Let L be the sum of the second fundamental forms on Γ with respect to the inward normals on M_0 and M_1 (or the difference of the second fundamental forms with respect to the common normal N), and $\mathbf{L}$ be the corresponding self-adjoint operator on $T\Gamma$, i.e. $L(\cdot,\cdot) = \langle \cdot, \mathbf{L}\cdot\rangle_0$. On a small neighborhood of Γ in M_0, the operator $\mathbf{L}$ extends to TM_0 in such a way that $\mathbf{L}N = 0$ and $\nabla_N \mathbf{L} = 0$.*

Proof. At a point $p \in \Gamma$ we may extend $\mathbf{L}$ to T_pM_0 by linearity such that $\mathbf{L}N = 0$. Given $q \in M_0$ near Γ and $X \in T_qM_0$, we use parallel transportation P along the integral curves of the vector field N and put $\mathbf{L}X := P^{-1}\mathbf{L}PX$. Then clearly the extended operator satisfies $\mathbf{L}N = 0$. The fact that $\nabla_N\mathbf{L} = 0$ is shown in Lemma D.1. □

Note that if the initial operator is positive semidefinite, then so is its extension. Indeed,

$$\langle X, \mathbf{L}X\rangle_0 = \langle X, P^{-1}\mathbf{L}PX\rangle_0 = \langle PX, \mathbf{L}PX\rangle_0 \ge 0.$$

The following C^∞ functions will be used to modify the metric g_0 near Γ:

Definition 1.2.6 (Auxiliary functions f_δ, F_δ and $\mathcal{F}_\delta$, cf. [15], 3.3)**.** *For small $\delta > 0$, we find C^∞ functions $f_\delta, F_\delta, \mathcal{F}_\delta : [0,\infty) \to \mathbb{R}$ with the following properties:*

- $\mathcal{F}_\delta' = F_\delta$ *and* $F_\delta' = f_\delta$ *on* $[0,\infty)$
- $f_\delta(0) = 1$, $0 \le f_\delta \le 1$ *on* $[0,\delta^2]$, *and* $|f_\delta| \le \delta$ *on* $[\delta^2,\infty)$
- $f_\delta' \le \delta$ *on* $[0,\infty)$
- $F_\delta(0) = \mathcal{F}_\delta(0) = 0$, $|F_\delta|, |\mathcal{F}_\delta| \le \delta$ *on* $[0,\infty)$, *and* $f_\delta = F_\delta = \mathcal{F}_\delta = 0$ *on* $[\delta,\infty)$.

The existence of such functions f_δ, F_δ, $\mathcal{F}_\delta$ is shown in Appendix D, Lemma D.2. Figure 1.1 below shows the function f_δ.

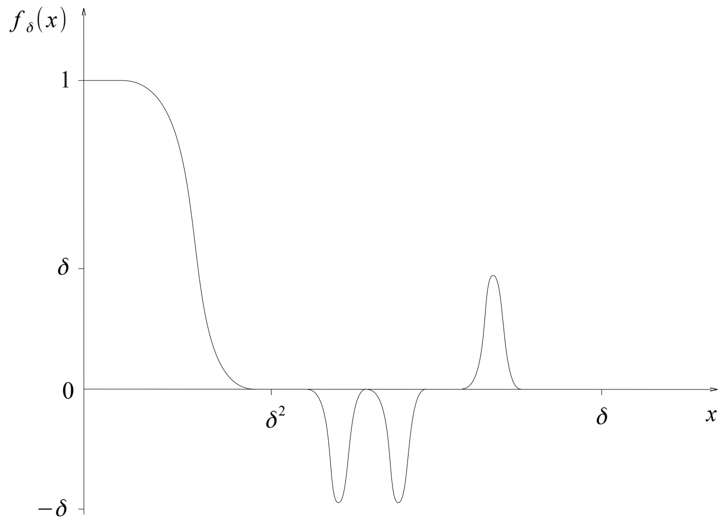

Figure 1.1.: The function f_δ

Remark 1.2.7. The functions f_δ, F_δ, $\mathcal{F}_\delta$ we use here are slightly different from the ones used in [15]. Our functions satisfy similar properties as those in [15], and, in addition, $\mathcal{F}_\delta = 0$ on $[\delta, \infty)$.

Notation 1.2.8 (Projection operators). For small distances $d > 0$, we denote by $\Gamma(d)$ the equidistant hypersurfaces of Γ in (M_0, g_0), that is,

$$\Gamma(d) = \{p \in M_0 \,|\, \operatorname{dist}_{g_0}(p, \Gamma) = d\}.$$

Furthermore, we define the projection operators

$$\mathbf{P}^T : TM_0 = T\Gamma(d) \oplus T\Gamma(d)^\perp \to T\Gamma(d)$$

and

$$\mathbf{P}^N : TM_0 = T\Gamma(d) \oplus T\Gamma(d)^\perp \to T\Gamma(d)^\perp,$$

where $\perp = \perp_{g_0}$. The coefficients of the corresponding $(2,0)$-tensors (with respect to the coordinates chosen above) are

$$(P^T)_{ij} = \begin{pmatrix} (g_{ij})_{1 \le i,j \le n-1} & 0 \\ 0 & 0 \end{pmatrix}$$

and $(P^N)_{ij} = \delta_{in}\delta_{jn}$.

Definition 1.2.9 (The modified metric g_δ, [15], 3.6)**.** *Let* $\mathbf{I}$ *denote the identity operator on* TM_0*. We define the self-adjoint endomorphism* $\mathbf{G}_\delta$ *by*

$$\mathbf{G}_\delta = \mathbf{I} + 2F_\delta(x^n)\mathbf{L} - 2C\mathcal{F}_\delta(x^n)\mathbf{P}^T, \tag{1.2.2}$$

and the modified inner product $\langle \cdot, \cdot \rangle_\delta$ *on* TM_0 *by*

$$\langle \cdot, \cdot \rangle_\delta = \langle \cdot, \mathbf{G}_\delta \cdot \rangle_0,$$

i.e. in coordinates we have

$$g^\delta_{ij} = g^0_{ij} + 2F_\delta(x^n)L_{ij} - 2C\mathcal{F}_\delta(x^n)(P^T)_{ij}. \tag{1.2.3}$$

The constant C in the definition of $\mathbf{G}_\delta$ is to be chosen later. Note that regardless of the fact that x^n may be defined only on a neighborhood $\{\operatorname{dist}_{g_0}(\cdot,\Gamma)\le d_0\}\subset M_0$, $d_0>0$, we may nevertheless consider $\mathbf{G}_\delta$ as an operator on M_0, since F_δ and $\mathcal{F}_\delta$ vanish on $[\delta,\infty)\supset[d_0,\infty)$ for small enough δ. This also shows that $\mathbf{G}_\delta=\mathbf{I}$ off a δ-neighborhood of Γ in M_0.

Lemma 1.2.10. $\mathbf{G}_\delta$ *has the following properties:*

(i) As δ tends to zero, $\mathbf{G}_\delta$ converges to $\mathbf{I}$ uniformly on M_0.

(ii) In our coordinates, $(g_\delta)_{ij}$ has the same form as g (cf. Lemma 1.2.1), that is, $(g_\delta)_{in}=\delta_{in}$ for all $1\le i\le n$.

(iii) The coefficients of the metric $g_{(\delta)}:=\begin{cases} g_\delta \text{ on } M_0\\ g_1 \text{ on } M_1\end{cases}$ *belong to* $W^{2,\infty}_{loc}$.

Proof. (i): This is because $\mathbf{L}$ and $\mathbf{P}^T$ are bounded near Γ, and $F_\delta,\mathcal{F}_\delta\to 0$ uniformly as $\delta\to 0$.

(ii): This follows from (1.2.3) and the fact that $L_{in}=(P^T)_{in}=0$ for all $1\le i\le n$.

(iii): Note that on Γ we have $g_\delta=g_0=g_1$ since $F_\delta(0)=\mathcal{F}_\delta(0)=0$, so $g_{(\delta)}$ is well defined. Clearly, the first derivatives of $g_{(\delta)}$ are locally Lipschitz off Γ, since g_δ and g_1 are at least C^2 smooth by assumption. Furthermore, the first derivatives of g_δ and g coincide on Γ, which implies that $g_{(\delta)}$ is C^1 on M. Indeed, on Γ we have

$$\partial_k g^\delta_{ij}=\partial_k g^0_{ij}=\partial_k g^1_{ij}$$

for $k=1,\dots,n-1$, since $g_\delta=g_0=g_1$ on Γ. At a point of Γ, using $L^0_{ij}=-\langle\nabla^0_{\partial_i}N,\partial_j\rangle_0$ and $L^1_{ij}=\langle\nabla^1_{\partial_i}N,\partial_j\rangle_1$ we compute

$$\partial_n g^0_{ij}=-2L^0_{ij}$$

and

$$\partial_n g^1_{ij}=2L^1_{ij}.$$

Thus, on Γ we have

$$\partial_n g^\delta_{ij}=\partial_n g^0_{ij}+2L_{ij}=2(L_{ij}-L^0_{ij})=2L^1_{ij}=\partial_n g^1_{ij},$$

where we used that $F'_\delta(0)=f_\delta(0)=1$ and $F_\delta(0)=0=\mathcal{F}_\delta(0)$. Let $p\in\Gamma$. Since $\Gamma\subset M$ is a smooth hypersurface, we may cover Γ with coordinate neighborhoods (U,φ), where $\varphi: M\supset U\to V\subset\mathbb{R}^n$, such that

$$\varphi(U\cap\Gamma)=V\cap(\mathbb{R}^{n-1}\times\{0\})$$

$$\varphi(U\cap M_0)=V\cap(\mathbb{R}^{n-1}\times\mathbb{R}_{\ge 0})=:V_0$$

$$\varphi(U\cap M_1)=V\cap(\mathbb{R}^{n-1}\times\mathbb{R}_{\le 0})=:V_1.$$

Moreover, after choosing U even smaller, we may assume that V is convex, and $\partial_k g^\delta_{ij}:V_0\to\mathbb{R}$ and $\partial_k g^1_{ij}:V_1\to\mathbb{R}$ are Lipschitz with constants $C_0,C_1<\infty$. Let $x,y\in V$. If $x,y\in V_0$, then

$$|\partial_i g^{(\delta)}_{kl}(x)-\partial_i g^{(\delta)}_{kl}(y)|=|\partial_i g^\delta_{kl}(x)-\partial_i g^\delta_{kl}(y)|\le C_0|x-y|.$$

Similarly, if $x, y \in V_1$, then

$$|\partial_i g^{(\delta)}_{kl}(x) - \partial_i g^{(\delta)}_{kl}(y)| = |\partial_i g^1_{kl}(x) - \partial_i g^1_{kl}(y)| \leq C_1|x-y|.$$

Suppose that $x \in V_0$ and $y \in V_1$. By construction, the straight line segment connecting x and y is contained in V, and intersects $\varphi(U \cap \Gamma) = V \cap (\mathbb{R}^{n-1} \times \{0\}) = V_0 \cap V_1$ in some point z, so that $x, z \in V_0$ and $y, z \in V_1$. Using the fact that the derivatives of g_δ and g_1 coincide on Γ, we then compute

$$\begin{aligned}
|\partial_i g^{(\delta)}_{kl}(x) - \partial_i g^{(\delta)}_{kl}(y)| &= |\partial_i g^{(\delta)}_{kl}(x) - \partial_i g^{(\delta)}_{kl}(z)| + |\partial_i g^{(\delta)}_{kl}(z) - \partial_i g^{(\delta)}_{kl}(y)| \\
&= |\partial_i g^{\delta}_{kl}(x) - \partial_i g^{\delta}_{kl}(z)| + |\partial_i g^1_{kl}(z) - \partial_i g^1_{kl}(y)| \\
&\leq C_0|x-z| + C_1|z-y| \\
&\leq (C_0 + C_1)(|x-z| + |z-y|) \\
&= (C_0 + C_1)|x-y|.
\end{aligned}$$

Thus, the derivatives of $g_{(\delta)}$ are locally Lipschitz, and $g_{(\delta)} \in W^{2,\infty}_{loc}$.

□

Definition 1.2.11. *Given two endomorphisms $\mathbf{S}_\delta, \mathbf{T}_\delta$ of TM_0 which depend on δ, we say that*

$$\mathbf{S}_\delta \approx \mathbf{T}_\delta$$

if $\mathbf{S}_\delta|_\Gamma = \mathbf{T}_\delta|_\Gamma$ and all eigenvalues of $\mathbf{S}_\delta - \mathbf{T}_\delta$ tend to zero uniformly on compact subsets of M_0 as $\delta \to 0$.

For two vector fields X_δ, Y_δ on M_0, we say that $X_\delta \approx Y_\delta$ if $X_\delta|_\Gamma = Y_\delta|_\Gamma$ and $\|X_\delta - Y_\delta\|_0 \to 0$ uniformly on compact subsets as $\delta \to 0$.

Note that $\mathbf{S}_\delta \approx \mathbf{T}_\delta$ ($X_\delta \approx Y_\delta$) holds iff in local coordinates $(S_\delta)_{ij} = (T_\delta)_{ij}$ on Γ and $|(S_\delta)_{ij} - (T_\delta)_{ij}| \to 0$ ($X^i_\delta = Y^i_\delta$ on Γ and $|X^i_\delta - Y^i_\delta| \to 0$).

Lemma 1.2.12 (Auxiliary identities, cf. [15], Lemma 6.1, 6.2, 6.3)**.** *Let*

$$X, Y \in \{\partial_1, \dots, \partial_{n-1}\} \subset T\Gamma(d) \subset TM_0$$

and

$$N = \partial_n \in (T\Gamma(d))^\perp \subset TM_0.$$

The following (approximate and exact) identities hold:

$$\begin{aligned}
&\mathbf{G}_\delta \approx \mathbf{I}, \quad \nabla_X \mathbf{G}_\delta \approx 0, \quad \nabla_N \mathbf{G}_\delta \approx 2f_\delta(x^n)\mathbf{L} \\
&\nabla_X \nabla_N \mathbf{G}_\delta \approx 2f_\delta(x^n)\nabla_X \mathbf{L} \\
&\nabla_N \nabla_N \mathbf{G}_\delta \approx 2f'_\delta(x^n)\mathbf{L} - 2Cf_\delta(x^n)\mathbf{P}^T
\end{aligned} \tag{1.2.4}$$

$$\begin{aligned}
&\langle \nabla^\delta_X N, Y\rangle_\delta = \langle \nabla^\delta_N X, Y\rangle_\delta \\
= \; &\frac{1}{2}\big(\langle \nabla_N X, \mathbf{G}_\delta Y\rangle + \langle X, \mathbf{G}_\delta \nabla_N Y\rangle + \langle X, (\nabla_N \mathbf{G}_\delta)Y\rangle\big)
\end{aligned} \tag{1.2.5}$$

$$\nabla^\delta_N N = 0 \tag{1.2.6}$$

$$\nabla^\delta_N X = \nabla^\delta_X N \approx \nabla_X N + f_\delta(x^n)\mathbf{L}X \tag{1.2.7}$$

$$\mathbf{P}^T(\nabla^\delta_X Y) \approx \mathbf{P}^T(\nabla_X Y). \tag{1.2.8}$$

Proof. The first identity in (1.2.4) follows from the fact that $\mathbf{G}_\delta = \mathbf{I}$ on $\Gamma = \{x^n = 0\}$, since $F_\delta(0) = \mathcal{F}_\delta(0) = 0$, and Lemma 1.2.10 (i).

Let us verify $\nabla_X \mathbf{G}_\delta \approx 0$. We have

$$\nabla_X \mathbf{G}_\delta = \nabla_X(\mathbf{I} + 2F_\delta \mathbf{L} - 2C\mathcal{F}_\delta \mathbf{P}^T) = \nabla_X \mathbf{I} + 2F_\delta \nabla_X \mathbf{L} - 2C\mathcal{F}_\delta \nabla_X \mathbf{P}^T,$$

since F_δ and $\mathcal{F}_\delta$ depend only on x^n. For any $\xi, \zeta \in TM_0$ we then have

$$(\nabla_\zeta \mathbf{I})\xi = \nabla_\zeta(\mathbf{I}\xi) - \mathbf{I}(\nabla_\zeta \xi) = \nabla_\zeta \xi - \nabla_\zeta \xi = 0.$$

Moreover, $\nabla_X \mathbf{L}$ and $\nabla_X \mathbf{P}^T$ are locally bounded, so the result follows since $F_\delta, \mathcal{F}_\delta \to 0$ as $\delta \to 0$.

One verifies the remaining identities using similar arguments. Detailed computations are given in Appendix D, Lemma D.3.

□

1.3. The Riemannian curvature operator of g_δ

In this section, we compare the Riemannian curvature operators of g_δ and g_0 on M_0 (cf. §§ 7-8 of [15]).

Let us briefly recall the connection between $(4,0)$-tensors on a finite dimensional vector space V and the corresponding linear operators on $\Lambda^2 V$ (we refere to Appendix B.1 for a detailed discussion). Any $(4,0)$-tensor $\{T_{ijkl}\}$ which is antisymmetric in i,j and k,l, respectively, induces a bilinear form $\mathcal{T}$ on $\Lambda^2 V$ via

$$\mathcal{T}(e_i \wedge e_j, e_k \wedge e_l) := T(e_i, e_j, e_k, e_l) = T_{ijkl},$$

where $e_1, \dots, e_n$ is a basis of V, and $e_i \wedge e_j = e_i \otimes e_j - e_j \otimes e_i$, $1 \le i < j \le n$ is the induced basis of $\Lambda^2 V$. The antisymmetries of T ensure that

$$\mathcal{T}(e_i \wedge e_j, e_k \wedge e_l) = -\mathcal{T}(e_j \wedge e_i, e_k \wedge e_l) = -\mathcal{T}(e_i \wedge e_j, e_l \wedge e_k),$$

that is, $\mathcal{T}$ is well defined. If in addition $T_{ijkl} = T_{klij}$, then the induced bilinear form $\mathcal{T}$ is symmetric. For arbitrary $\alpha, \beta \in \Lambda^2 V$, $\alpha = \sum_{i<j} \alpha^{ij} e_i \wedge e_j = \alpha^{ij} e_i \otimes e_j$, $\beta = \sum_{i<j} \beta^{ij} e_i \wedge e_j = \beta^{ij} e_i \otimes e_j$ ($\alpha^{ij} = -\alpha^{ji}$ and $\beta^{ij} = -\beta^{ji}$) one has

$$\mathcal{T}(\alpha, \beta) = \frac{1}{4} T_{ijkl} \alpha^{ij} \beta^{kl} \tag{1.3.1}$$

(see Lemma B.1.1), where here and in what follows we make use of the summation convention.

Conversely, any bilinear form $\mathcal{T}$ on $\Lambda^2 V$ (or the corresponding linear operator) induces a $(4,0)$-tensor on V via

$$T(e_i, e_j, e_k, e_l) := \mathcal{T}(e_i \wedge e_j, e_k \wedge e_l) = \mathcal{I}^g(e_i \wedge e_j, \mathcal{T}(e_k \wedge e_l)),$$

where $\mathcal{I}^g$ is the inner product on $\Lambda^2 V$ induced by g,

$$\mathcal{I}^g(e_i \wedge e_j, e_k \wedge e_l) = g_{ik} g_{jl} - g_{jk} g_{il}.$$

The such defined tensor has the symmetries $T_{ijkl} = -T_{jikl} = -T_{ijlk}$, and if in addition the bilinear form is symmetric, then we also have $T_{ijkl} = T_{klij}$.

Using the inner product $\mathcal{I}^g$, we may identify linear operators and bilinear forms on $\Lambda^2 V$ by putting

$$\mathcal{I}^g(e_i \wedge e_j, \mathcal{T}(e_k \wedge e_l)) = \mathcal{T}(e_i \wedge e_j, e_k \wedge e_l).$$

The bilinear form is symmetric iff the operator is self-adjoint. In view of these identifications, in what follows we will often switch between operators and bilinear forms on $\Lambda^2(TM)$ and $(4,0)$-tensors on TM.

We will also make use of the Kulkarni-Nomizu product on $\mathrm{End}(TM)$ (see Appendix B.2 for a detailed discussion). The Kulkarni-Nomizu product of two linear endomorphisms $\mathbf{A}$, $\mathbf{B}$ of V is the linear endomorphism $\mathbf{A}\wedge\mathbf{B} : \Lambda^2 V \to \Lambda^2 V$, which is defined by

$$(\mathbf{A}\wedge\mathbf{B})(e_i \wedge e_j) := \frac{1}{2}\big(\mathbf{A}(e_i)\wedge\mathbf{B}(e_j) + \mathbf{B}(e_i)\wedge\mathbf{A}(e_j)\big)$$

for basis vectors $e_i \wedge e_j$, and extends to $\Lambda^2 V$ by linearity. The factor $\frac{1}{2}$ ensures that we have $\mathrm{id}_V \wedge \mathrm{id}_V = \mathrm{id}_{\Lambda^2 V}$. The corresponding bilinear form on $\Lambda^2 V$ is given by

$$\begin{aligned}\mathbf{A}\wedge\mathbf{B}(e_i \wedge e_j, e_k \wedge e_l) &:= \mathcal{I}^g\big(e_i \wedge e_j, (\mathbf{A}\wedge\mathbf{B})(e_k \wedge e_l)\big)\\ &= \frac{1}{2}(A_{ik}B_{jl} - A_{jk}B_{il} + B_{ik}A_{jl} - B_{jk}A_{il}),\end{aligned}$$

where A, B are the bilinear forms on V corresponding with $\mathbf{A}, \mathbf{B}$ (cf. Notation 1.2.4). Note that the induced $(4,0)$-tensor $\{(\mathbf{A}\wedge\mathbf{B})_{ijkl}\}$ is antisymmetric in i, j and k, l, respectively. If in addition A and B are symmetric, then we also have the symmetry $(\mathbf{A}\wedge\mathbf{B})_{ijkl} = (\mathbf{A}\wedge\mathbf{B})_{klij}$.

Let us now consider the Riemannian curvature operator of g_δ. For ease of notation, here and in what follows we shall suppress the index 0 for quantities related to M_0. For example, we write $\langle\cdot,\cdot\rangle$ for $\langle\cdot,\cdot\rangle_0$ and $\mathcal{R}$ for $\mathcal{R}_0$. Similarly as in Definition 1.2.11, given operators $\mathcal{S}_\delta, \mathcal{T}_\delta : \Lambda^2(TM_0) \to \Lambda^2(TM_0)$ which depend on δ, we say that $\mathcal{S}_\delta \approx \mathcal{T}_\delta$, if $\mathcal{S}_\delta|_\Gamma = \mathcal{T}_\delta|_\Gamma$, and $\mathcal{S}_\delta - \mathcal{T}_\delta \to 0$ uniformly on compact sets as $\delta \to 0$. Note that this is the case if and only if in local coordinates the coefficients of the corresponding $(4,0)$-tensors satisfy $S^\delta_{ijkl} \approx T^\delta_{ijkl}$.

The main result of this section is

Proposition 1.3.1. *Let $\mathcal{R}_\delta = \mathcal{R}(g_\delta)$. Then*

$$\mathcal{R}_\delta \;\approx\; \mathcal{R} - f_\delta^2\mathcal{A} + f_\delta\mathcal{B} - 2f_\delta'\mathcal{L} + 2f_\delta^2\mathcal{L}^2 + 2Cf_\delta\hat{\mathcal{I}} \tag{1.3.2}$$

holds on M_0, where

$$\begin{aligned}\mathcal{A} &:= \mathbf{L}\wedge\mathbf{L}\\ \mathcal{L} &:= \mathbf{L}\wedge\mathbf{P}^N\\ \mathcal{L}^2 &:= \mathbf{L}^2\wedge\mathbf{P}^N\\ \hat{\mathcal{I}} &:= \mathbf{P}^T\wedge\mathbf{P}^N\end{aligned}$$

(cf. Notation 1.2.8 for the definitions of $\mathbf{P}^T$ and $\mathbf{P}^N$), and $\mathcal{B}$ is a smooth operator on $\Lambda^2(TM_0)$ which we will define later.

In order to prove this statement, we compute the coefficients of the corresponding $(4,0)$-tensor of $\mathcal{R}_\delta$ locally in coordinates $(x^1,\ldots,x^n)$ from the previous section.

Lemma 1.3.2. *For $i,j,k,l \in \{1,\dots,n-1\}$ we have*

$$R^\delta_{ijkl} \approx R_{ijkl} - f_\delta^2(\mathbf{L}\wedge\mathbf{L})_{ijkl} - 2f_\delta(\mathbf{L}\wedge\nabla N)_{ijkl}, \tag{1.3.3}$$

where ∇N is the endomorphism $X \in TM \mapsto \nabla_X N \in TM$ (recall that N is the unit vector field orthogonal to the hypersurfaces of M_0 equidistant to Γ, cf. Notation 1.2.2).

Proof. We proceed as in [15], Lemma 7.1. Let $p \in M_0$ be a point near Γ and $d = \mathrm{dist}\,(x,\Gamma) = x^n(p)$. Let $k,l \le n-1$. Recall that by Definition 1.2.2 we have

$$g^\delta_{kl} = g_{kl} + 2F_\delta(x^n)L_{kl} - 2C\mathcal{F}_\delta(x^n)g_{kl}.$$

Therefore, for $i,j \le n-1$ we have

$$\partial_i g^\delta_{kl} \approx \partial_i g_{kl}$$

and

$$\partial_i\partial_j g^\delta_{kl} \approx \partial_i\partial_j g_{kl},$$

which implies that the curvature tensors of $g_\delta|_{\Gamma(d)}$ and $g|_{\Gamma(d)}$ satisfy

$$R^\delta_{\Gamma(d)} \approx R_{\Gamma(d)}.$$

Using the Gauss theorem and (1.2.7), at p we compute

$$\begin{aligned}
R^\delta_{ijkl} &= \langle R^\delta(\partial_i,\partial_j)\partial_k,\partial_l\rangle_\delta \\
&= \langle R^\delta_{\Gamma(d)}(\partial_i,\partial_j)\partial_k,\partial_l\rangle_\delta - \langle\nabla^\delta_{\partial_i}N,\partial_k\rangle_\delta\langle\nabla^\delta_{\partial_j}N,\partial_l\rangle_\delta + \langle\nabla^\delta_{\partial_j}N,\partial_k\rangle_\delta\langle\nabla^\delta_{\partial_i}N,\partial_l\rangle_\delta \\
&\approx \langle R_{\Gamma(d)}(\partial_i,\partial_j)\partial_k,\partial_l\rangle - (\langle\nabla_{\partial_i}N,\partial_k\rangle + f_\delta\langle\partial_i,\mathbf{L}\partial_k\rangle)(\langle\nabla_{\partial_j}N,\partial_l\rangle + f_\delta\langle\partial_j,\mathbf{L}\partial_l\rangle) \\
&\quad +(\langle\nabla_{\partial_j}N,\partial_k\rangle + f_\delta\langle\partial_j,\mathbf{L}\partial_k\rangle)(\langle\nabla_{\partial_i}N,\partial_l\rangle + f_\delta\langle\partial_i,\mathbf{L}\partial_l\rangle) \\
&= \langle R_{\Gamma(d)}(\partial_i,\partial_j)\partial_k,\partial_l\rangle - \langle\nabla_{\partial_i}N,\partial_k\rangle\langle\nabla_{\partial_j}N,\partial_l\rangle + \langle\nabla_{\partial_j}N,\partial_k\rangle\langle\nabla_{\partial_i}N,\partial_l\rangle \\
&\quad -f_\delta^2\big(\langle\partial_i,\mathbf{L}\partial_k\rangle\langle\partial_j,\mathbf{L}\partial_l\rangle - \langle\partial_j,\mathbf{L}\partial_k\rangle\langle\partial_i,\mathbf{L}\partial_l\rangle\big) \\
&\quad -f_\delta\big(\langle\partial_i,\mathbf{L}\partial_k\rangle\langle\nabla_{\partial_j}N,\partial_l\rangle - \langle\partial_j,\mathbf{L}\partial_k\rangle\langle\nabla_{\partial_i}N,\partial_l\rangle \\
&\quad +\langle\nabla_{\partial_i}N,\partial_k\rangle\langle\partial_j,\mathbf{L}\partial_l\rangle - \langle\nabla_{\partial_j}N,\partial_k\rangle\langle\partial_i,\mathbf{L}\partial_l\rangle\big).
\end{aligned}$$

□

Lemma 1.3.3. *For $i,j,l \in \{1,\dots,n-1\}$ we have*

$$R^\delta_{ijnl} \approx R_{ijnl} + f_\delta\big(\langle\partial_i,(\nabla_{\partial_j}\mathbf{L})\partial_l\rangle - \langle\partial_j,(\nabla_{\partial_i}\mathbf{L})\partial_l\rangle\big). \tag{1.3.4}$$

Proof. We proceed as in [15], Lemma 7.3. Let $i,j,l \in \{1,\dots,n-1\}$. By definition of the Riemannian curvature tensor we have

$$\begin{aligned}
&\langle R^\delta(\partial_i,\partial_j)\partial_n,\partial_l\rangle_\delta \\
=\ & \langle\nabla^\delta_{\partial_j}\nabla^\delta_{\partial_i}N,\partial_l\rangle_\delta - \langle\nabla^\delta_{\partial_i}\nabla^\delta_{\partial_j}N,\partial_l\rangle_\delta \qquad (1.3.5)\\
=\ & \partial_j\langle\nabla^\delta_{\partial_i}N,\partial_l\rangle_\delta - \partial_i\langle\nabla^\delta_{\partial_j}N,\partial_l\rangle_\delta - \langle\nabla^\delta_{\partial_i}N,\nabla^\delta_{\partial_j}\partial_l\rangle_\delta + \langle\nabla^\delta_{\partial_j}N,\nabla^\delta_{\partial_i}\partial_l\rangle_\delta.
\end{aligned}$$

1) For the first two terms on the right hand side we compute using (1.2.5)

$$\begin{aligned}
&\partial_j\langle\nabla^\delta_{\partial_i}N,\partial_l\rangle_\delta - \partial_i\langle\nabla^\delta_{\partial_j}N,\partial_l\rangle_\delta \\
=\ & \frac{1}{2}\partial_j(\langle\nabla_N\partial_i,\mathbf{G}_\delta\partial_l\rangle + \langle\partial_i,\mathbf{G}_\delta\nabla_N\partial_l\rangle + \langle\partial_i,(\nabla_N\mathbf{G}_\delta)\partial_l\rangle) \\
-\ & \frac{1}{2}\partial_i(\langle\nabla_N\partial_j,\mathbf{G}_\delta\partial_l\rangle + \langle\partial_j,\mathbf{G}_\delta\nabla_N\partial_l\rangle + \langle\partial_j,(\nabla_N\mathbf{G}_\delta)\partial_l\rangle).
\end{aligned}$$

After termwise differentiation we get three different types of terms:

a) Terms in which $\mathbf{G}_\delta$ is not differentiated: Since $\mathbf{G}_\delta \approx \mathbf{I}$, their sum is

$$\approx \partial_j \langle \nabla_{\partial_i} N, \partial_l \rangle - \partial_i \langle \nabla_{\partial_j} N, \partial_l \rangle.$$

b) Terms in which $\mathbf{G}_\delta$ is differentiated with respect to ∂_i or ∂_j: Since $1 \le i \le n-1$, these terms are ≈ 0 by (1.2.4).

c) Terms which involve mixed derivatives of $\mathbf{G}_\delta$ with respect to both ∂_i and N: In view of (1.2.4), their sum is

$$\begin{aligned} &\approx f_\delta\big(\langle \nabla_{\partial_j}\partial_i, \mathbf{L}\partial_l\rangle + \langle \partial_i, (\nabla_{\partial_j}\mathbf{L})\partial_l\rangle + \langle \partial_i, \mathbf{L}(\nabla_{\partial_j}\partial_l)\rangle \\ &\quad -\langle \nabla_{\partial_i}\partial_j, \mathbf{L}\partial_l\rangle - \langle \partial_j, (\nabla_{\partial_i}\mathbf{L})\partial_l\rangle - \langle \partial_j, \mathbf{L}(\nabla_{\partial_i}\partial_l)\rangle\big) \\ &= f_\delta\big(\langle \partial_i, (\nabla_{\partial_j}\mathbf{L})\partial_l\rangle + \langle \partial_i, \mathbf{L}(\nabla_{\partial_j}\partial_l)\rangle - \langle \partial_j, (\nabla_{\partial_i}\mathbf{L})\partial_l\rangle - \langle \partial_j, \mathbf{L}(\nabla_{\partial_i}\partial_l)\rangle\big), \end{aligned}$$

where we used that ∂_i and ∂_j commute. Combining a), b) and c) gives us

$$\begin{aligned} &\partial_j \langle \nabla^\delta_{\partial_i} N, \partial_l \rangle_\delta - \partial_i \langle \nabla^\delta_{\partial_j} N, \partial_l \rangle_\delta \\ \approx\ &\partial_j \langle \nabla_{\partial_i} N, \partial_l \rangle - \partial_i \langle \nabla_{\partial_j} N, \partial_l \rangle \qquad (1.3.6) \\ +\ &f_\delta\big(\langle \partial_i, (\nabla_{\partial_j}\mathbf{L})\partial_l\rangle + \langle \partial_i, \mathbf{L}(\nabla_{\partial_j}\partial_l)\rangle - \langle \partial_j, (\nabla_{\partial_i}\mathbf{L})\partial_l\rangle - \langle \partial_j, \mathbf{L}(\nabla_{\partial_i}\partial_l)\rangle\big). \end{aligned}$$

2) Let us now consider the last two terms on the right hand side of (1.3.5). Using the fact that $\langle \nabla^\delta_{\partial_i} N, N\rangle_\delta = 0$, we have

$$\langle \nabla^\delta_{\partial_i} N, \nabla^\delta_{\partial_j} \partial_l \rangle_\delta = \langle \nabla^\delta_{\partial_i} N, \mathbf{P}^T(\nabla^\delta_{\partial_j}\partial_l)\rangle_\delta.$$

Therefore, in view of (1.2.7) and (1.2.8)

$$\begin{aligned} &-\langle \nabla^\delta_{\partial_i} N, \nabla^\delta_{\partial_j}\partial_l\rangle_\delta + \langle \nabla^\delta_{\partial_j} N, \nabla^\delta_{\partial_i}\partial_l\rangle_\delta \\ =\ &-\langle \nabla^\delta_{\partial_i} N, \mathbf{P}^T(\nabla^\delta_{\partial_j}\partial_l)\rangle_\delta + \langle \nabla^\delta_{\partial_j} N, \mathbf{P}^T(\nabla^\delta_{\partial_i}\partial_l)\rangle_\delta \\ \approx\ &-\big(\langle \nabla_{\partial_i} N, \mathbf{P}^T(\nabla_{\partial_j}\partial_l)\rangle + f_\delta\langle \mathbf{L}\partial_i, \mathbf{P}^T(\nabla_{\partial_j}\partial_l)\rangle\big) \qquad (1.3.7) \\ &+\big(\langle \nabla_{\partial_j} N, \mathbf{P}^T(\nabla_{\partial_i}\partial_l)\rangle + f_\delta\langle \mathbf{L}\partial_j, \mathbf{P}^T(\nabla_{\partial_i}\partial_l)\rangle\big) \\ \approx\ &-\langle \nabla_{\partial_i} N, \nabla_{\partial_j}\partial_l\rangle + \langle \nabla_{\partial_j} N, \nabla_{\partial_i}\partial_l\rangle + f_\delta\big(\langle \mathbf{L}\partial_j, \nabla_{\partial_i}\partial_l\rangle - \langle \mathbf{L}\partial_i, \nabla_{\partial_j}\partial_l\rangle\big), \end{aligned}$$

where in the last line we used $\langle \nabla_{\partial_i} N, N\rangle = 0$ and $\langle \mathbf{L}\partial_i, N\rangle = \langle \partial_i, \mathbf{L}N\rangle = 0$. Combining (1.3.6) and (1.3.7) we obtain the desired result. □

Lemma 1.3.4. *For $j, l \in \{1, \dots, n-1\}$ we have*

$$\begin{aligned} R^\delta_{njnl} \approx\ & R_{njnl} - 2f'_\delta(\mathbf{L}\wedge\mathbf{P}^N)_{njnl} + 2f^2_\delta(\mathbf{L}^2\wedge\mathbf{P}^N)_{njnl} + 2Cf_\delta(\mathbf{P}^T\wedge\mathbf{P}^N)_{njnl} \\ & -f_\delta\big(\langle \mathbf{L}\partial_j, \nabla_{\partial_l} N\rangle + \langle \nabla_{\partial_j} N, \mathbf{L}\partial_l\rangle\big). \end{aligned}$$

Proof. We proceed as in [15], Lemma 7.2. Using Lemma 1.2.12 we compute

$$\begin{aligned}
R^\delta_{njnl} &= \langle R^\delta(N,\partial_j)N,\partial_l\rangle_\delta \\
&= \langle \nabla^\delta_{\partial_j} \underbrace{\nabla^\delta_N N}_{\equiv 0}, \partial_l\rangle_\delta - \langle \nabla^\delta_N \nabla^\delta_{\partial_j} N, \partial_l\rangle_\delta \\
&= -N\langle \nabla^\delta_{\partial_j} N, \partial_l\rangle_\delta + \langle \nabla^\delta_{\partial_j} N, \nabla^\delta_N \partial_l\rangle_\delta \\
&\overset{(1.2.5),(1.2.7)}{\approx} -N\Big[\frac{1}{2}\big(\langle \nabla_N\partial_j, \mathbf{G}_\delta \partial_l\rangle + \langle \partial_j, \mathbf{G}_\delta(\nabla_N \partial_l)\rangle + \langle \partial_j, (\nabla_N \mathbf{G}_\delta)\partial_l\rangle\big)\Big] \\
&\quad + \langle \nabla_N \partial_j + f_\delta \mathbf{L}\partial_j, \nabla_N\partial_l + f_\delta \mathbf{L}\partial_l\rangle \\
&\overset{(1.2.4)}{\approx} \langle R(N,\partial_j)N,\partial_l\rangle - f'_\delta\langle \partial_j, \mathbf{L}\partial_l\rangle + f_\delta^2\langle \mathbf{L}\partial_j, \mathbf{L}\partial_l\rangle + Cf_\delta\langle\partial_j,\partial_l\rangle \\
&\quad - f_\delta\big(\langle \mathbf{L}\partial_j, \nabla_{\partial_l}N\rangle + \langle \nabla_{\partial_j}N, \mathbf{L}\partial_l\rangle\big).
\end{aligned}$$

□

We are now ready to prove Proposition 1.3.1.

Proof of Proposition 1.3.1. We define the $(4,0)$-tensor B by

$$\begin{aligned}
B_{ijkl} &= -2(\mathbf{L}\wedge\nabla N)_{ijkl} \\
&\quad + \langle \partial_i, (\nabla_{\partial_j}\mathbf{L})\partial_l\rangle\langle\partial_k, N\rangle - \langle \partial_j, (\nabla_{\partial_i}\mathbf{L})\partial_l\rangle\langle\partial_k, N\rangle \\
&\quad - \langle \partial_i, (\nabla_{\partial_j}\mathbf{L})\partial_k\rangle\langle\partial_l, N\rangle + \langle \partial_j, (\nabla_{\partial_i}\mathbf{L})\partial_k\rangle\langle\partial_l, N\rangle \\
&\quad + \langle \partial_k, (\nabla_{\partial_l}\mathbf{L})\partial_j\rangle\langle\partial_i, N\rangle - \langle \partial_l, (\nabla_{\partial_k}\mathbf{L})\partial_j\rangle\langle\partial_i, N\rangle \\
&\quad - \langle \partial_k, (\nabla_{\partial_l}\mathbf{L})\partial_i\rangle\langle\partial_j, N\rangle + \langle \partial_l, (\nabla_{\partial_k}\mathbf{L})\partial_i\rangle\langle\partial_j, N\rangle.
\end{aligned}$$

Observe that the tensor B satisfies $B_{ijkl} = -B_{jikl} = -B_{ijlk}$ and $B_{ijkl} = B_{klij}$, thus inducing a symmetric bilinear form $\mathcal{B}$ on $\Lambda^2(TM)$ via $\mathcal{B}(e_i\wedge e_j, e_k\wedge e_l) = B_{ijkl}$ (see the discussion in the beginning of this section). The desired equation

$$\begin{aligned}
\mathcal{R}_\delta &\approx \mathcal{R} - f_\delta^2\mathbf{L}\wedge\mathbf{L} + f_\delta\mathcal{B} \\
&\quad -2f'_\delta\mathbf{L}\wedge\mathbf{P}^N + 2f_\delta^2\mathbf{L}^2\wedge\mathbf{P}^N + 2Cf_\delta\mathbf{P}^T\wedge\mathbf{P}^N
\end{aligned} \tag{1.3.8}$$

follows in view of Lemmas 1.3.2 – 1.3.4. Indeed, note that since the operators on the right hand side (i.e. their corresponding $(4,0)$-tensors) have the same symmetries as the curvature operator, it suffices to evaluate (1.3.8) for $(\partial_i,\partial_j,\partial_k,\partial_l)$, $(\partial_i,\partial_j,\partial_n,\partial_l)$, and $(\partial_n,\partial_j,\partial_n,\partial_l)$, where $1\le i,j,k,l\le n-1$.

Case 1)
Let $i,j,k,l\le n-1$. In this case

$$(\mathbf{L}\wedge\mathbf{P}^N)_{ijkl} = (\mathbf{L}^2\wedge\mathbf{P}^N)_{ijkl} = (\mathbf{P}^T\wedge\mathbf{P}^N)_{ijkl} = 0$$

and $B_{ijkl} = -2(\mathbf{L}\wedge\nabla N)_{ijkl}$. Thus, (1.3.8) follows by Lemma 1.3.2.

Case 2)
Let $i,j,l\le n-1$ and $k=n$. Recall that $L_{in}=0$ for all i and $(P^N)_{in}=0$ for $i\le n-1$. Therefore we have

$$(\mathbf{L}\wedge\mathbf{L})_{ijnl} = (\mathbf{L}\wedge\mathbf{P}^N)_{ijnl} = (\mathbf{L}^2\wedge\mathbf{P}^N)_{ijnl} = (\mathbf{P}^T\wedge\mathbf{P}^N)_{ijnl} = 0.$$

Moreover, $(\nabla N)_{in} = \langle \partial_i, \nabla_N N\rangle = 0 = \langle N, \nabla_{\partial_i} N\rangle = (\nabla N)_{ni}$, and therefore

$$B_{ijnl} = \langle \partial_i, (\nabla_{\partial_j}\mathbf{L})\partial_l\rangle - \langle \partial_j, (\nabla_{\partial_i}\mathbf{L})\partial_l\rangle,$$

and (1.3.8) follows by Lemma 1.3.3.

Case 3)
Let $j, l \leq n-1$ and $i = k = n$. Clearly, $(\mathbf{L}\wedge\mathbf{L})_{njnl} = 0$. As in case 2) we have $(\mathbf{L}\wedge\nabla N)_{njnl} = 0$, and thus

$$\begin{aligned} B_{njnl} &= \langle N, (\nabla_{\partial_j}\mathbf{L})\partial_l\rangle - \langle \partial_j, (\nabla_N\mathbf{L})\partial_l\rangle + \langle N, (\nabla_{\partial_l}\mathbf{L})\partial_j\rangle - \langle \partial_l, (\nabla_N\mathbf{L})\partial_j\rangle \\ &= \langle N, (\nabla_{\partial_j}\mathbf{L})\partial_l\rangle + \langle N, (\nabla_{\partial_l}\mathbf{L})\partial_j\rangle, \end{aligned}$$

where we used that $\nabla_N\mathbf{L} = 0$ (cf. Lemma 1.2.5). Using the fact that $\mathbf{L}$ is self-adjoint and $\mathbf{L}N = 0$ we compute

$$\begin{aligned} \langle N, (\nabla_{\partial_j}\mathbf{L})\partial_l\rangle &= \langle N, \nabla_{\partial_j}(\mathbf{L}\partial_l)\rangle - \langle N, \mathbf{L}(\nabla_{\partial_j}\partial_l)\rangle \\ &= \partial_j\langle N, \mathbf{L}\partial_l\rangle - \langle \nabla_{\partial_j}N, \mathbf{L}\partial_l\rangle \\ &= -\langle \nabla_{\partial_j}N, \mathbf{L}\partial_l\rangle, \end{aligned}$$

which gives us

$$B_{njnl} = -(\langle \nabla_{\partial_j}N, \mathbf{L}\partial_l\rangle) + \langle \nabla_{\partial_l}N, \mathbf{L}\partial_j\rangle).$$

Thus, in this case (1.3.8) follows by Lemma 1.3.4, and we are done. □

1.4. The Riemannian curvature operator of g_1'

In this section, we prove an auxiliary result which we will need in the next section. Recall that g_1' is the extension of g^1 on a small neighborhood of Γ in M_0, as introduced in Lemma 1.2.3. We compare the Riemannian curvature operators on Γ with respect to the metrics g and g_1' (cf. [15], § 9).

We define the self-adjoint operator $\mathbf{G}_1$ on TM_0 by $\langle\cdot, \mathbf{G}_1\cdot\rangle = \langle\cdot,\cdot\rangle_1'$.

Proposition 1.4.1. *Let $\mathcal{R}_1'$ be the Riemannian curvature operator of g_1' on M_0. On Γ we have*

$$\mathcal{R}_1' = \mathcal{R} - \mathcal{A} + \mathcal{B} + 2\mathcal{L}^2 - \nabla_N^2\mathcal{G}_1, \tag{1.4.1}$$

where $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{L}^2$ are as in Theorem 1.3.1 and $\nabla_N^2\mathcal{G}_1 := (\nabla_N^2\mathbf{G}_1)\wedge\mathbf{P}^N$. In particular, since $\mathcal{R}_1' = \mathcal{R}_1$ holds on Γ independently of the extension g_1', and $\mathcal{R}_1 \geq \kappa$ by assumption, we have

$$\mathcal{R} - \mathcal{A} + \mathcal{B} + 2\mathcal{L}^2 - \nabla_N^2\mathcal{G}_1 \geq \kappa\mathcal{I} \tag{1.4.2}$$

on Γ.

Proof. We proceed as in [15], Lemma 9.1. We show that $\mathbf{G}_1$ satisfies similar equalities on Γ as $\mathbf{G}_\delta$ in Lemma 1.2.12, up to the $\nabla_N\nabla_N\mathbf{G}_1$ term (see Lemma 1.4.2 below). We may then repeat the computations from the previous section, where the only difference occurs due to the $\nabla_N\nabla_N\mathbf{G}_1$ term. □

Lemma 1.4.2. *Let $X, Y \in \{\partial_1, \dots, \partial_{n-1}\}$, and $N = \partial_n$. On Γ, the following identities are true:*

$$\mathbf{G}_1 = \mathbf{I} \tag{1.4.3}$$
$$\nabla_X \mathbf{G}_1 = 0 \tag{1.4.4}$$
$$\nabla_N \mathbf{G}_1 = 2\mathbf{L} \tag{1.4.5}$$
$$\nabla_X \nabla_N \mathbf{G}_1 = 2\nabla_X \mathbf{L} \tag{1.4.6}$$
$$\nabla^{1,\prime}{}_N N = 0 \tag{1.4.7}$$
$$\nabla^{1,\prime}{}_X N = \nabla_X N + \mathbf{L}X \tag{1.4.8}$$
$$\mathbf{P}^T(\nabla^{1,\prime}{}_X Y) = \mathbf{P}^T(\nabla_X Y) \tag{1.4.9}$$

Proof. By construction $g_1' = g_1$ on Γ, and by assumption $g_1 = g_0$ on Γ, which shows (1.4.3). The identity (1.4.4) follows from (1.4.3).

Let us show (1.4.5). We have

$$\begin{aligned}
\langle X, (\nabla_N \mathbf{G}_1)Y\rangle &= \langle X, \nabla_N(\mathbf{G}_1 Y)\rangle - \langle X, \mathbf{G}_1(\nabla_N Y)\rangle \\
&\overset{\text{on }\Gamma}{=} N\langle X, \mathbf{G}_1 Y\rangle - \langle \nabla_N X, Y\rangle - \langle X, \nabla_N Y\rangle \\
&= N\langle X, Y\rangle_1' - \langle \nabla_N X, Y\rangle - \langle X, \nabla_N Y\rangle \\
&= \langle \nabla^{1,\prime}{}_N X, Y\rangle_1' + \langle X, \nabla^{1,\prime}{}_N Y\rangle_1' - \langle \nabla_N X, Y\rangle - \langle X, \nabla_N Y\rangle \\
&= 2L_0(X,Y) - 2L_1(X,Y) \\
&= \langle X, 2\mathbf{L}Y\rangle,
\end{aligned} \tag{1.4.10}$$

where we used that in our coordinates the second fundamental forms of Γ in (M_0, g) and (M_0, g_1') with respect to N are $-\langle X, \nabla_N Y\rangle$ and $\langle X, \nabla^{1,\prime}{}_N Y\rangle_1'$ (cf. Lemma 1.2.10). By a similar computation we have

$$\begin{aligned}
\langle N, (\nabla_N \mathbf{G}_1)Y\rangle &\overset{\text{on }\Gamma}{=} N\langle N, \mathbf{G}_1 Y\rangle - \langle \nabla_N N, Y\rangle - \langle N, \nabla_N Y\rangle \\
&= 0 = \langle N, 2\mathbf{L}Y\rangle.
\end{aligned} \tag{1.4.11}$$

Furthermore, $\mathbf{G}_1 N = N$ implies

$$(\nabla_N \mathbf{G}_1)N = \nabla_N(\mathbf{G}_1 N) - \mathbf{G}_1(\nabla_N N) = 0 = 2\mathbf{L}N. \tag{1.4.12}$$

Equation (1.4.5) follows from (1.4.10), (1.4.11), (1.4.12), and the fact that $\nabla_N \mathbf{G}_1$ is self-adjoint.

Equation (1.4.6) is a consequence of (1.4.5).

Equation (1.4.7) follows by a similar computation as $\nabla^\delta_N N = 0$, see Lemma D.3, equation (D.5), since by construction we have $(g_1')_{in} = \delta_{in}$ for all $1 \le i \le n$ (see Lemma 1.2.3).

Let us verify (1.4.8). Using the Koszul formula, similarly as in Lemma D.3, (D.4), one checks that

$$\langle \nabla^{1,\prime}{}_N X, Y\rangle_1' = \frac{1}{2}\big(\langle \nabla_N X, \mathbf{G}_1 Y\rangle + \langle X, \mathbf{G}_1(\nabla_N Y)\rangle + \langle X, (\nabla_N \mathbf{G}_1)Y\rangle\big).$$

Since $\mathbf{G}_1 = \mathbf{I}$ and $\nabla_N \mathbf{G}_1 = 2\mathbf{L}$ on Γ, this implies

$$\langle \nabla^{1,\prime}{}_N X, Y\rangle_1' = \langle \nabla_X N, Y\rangle + \langle X, \mathbf{L}Y\rangle \tag{1.4.13}$$

on Γ. Moreover, using the fact that the vector fields X and N commute, the identity ${\nabla^{1,}}'_N N = 0 = \nabla_N N$, and $\mathbf{L}N = 0$, one checks that

$$\langle {\nabla^{1,}}'_N X, N\rangle_1' = 0 = \langle \nabla_X N, N\rangle + \langle X, \mathbf{L}N\rangle. \tag{1.4.14}$$

We then obtain (1.4.8) by combining (1.4.13) and (1.4.14).

Finally, one verifies (1.4.9) using $g'_{in} = \delta_{in}$ for $1 \le i \le n$ similarly as in Lemma D.3, equation (D.7).

□

1.5. Estimating $\mathcal{R}_\delta$ on M_0

The goal of this section is to show that $\mathcal{R}_\delta \ge \big(\kappa - \varepsilon(\delta)\big)\mathcal{I}_\delta$ holds on M_0.

Lemma 1.5.1 (cf. [15], Lemma 9.2)**.** *We have*

$$\mathcal{R} - f_\delta^2\mathcal{A} + f_\delta\mathcal{B} \quad \ge \quad (\kappa - \varepsilon(\delta))\mathcal{I} + 2f_\delta\big(-\mathcal{L}^2 + \frac{1}{2}\nabla_N^2\mathcal{G}_1\big), \tag{1.5.1}$$

where $\varepsilon(\delta)$ tends to zero as $\delta \to 0$.

Proof. Since Γ is compact, it suffices to show that

$$\mathcal{R}(\alpha,\alpha) - f_\delta^2\mathcal{A}(\alpha,\alpha) + f_\delta\mathcal{B}(\alpha,\alpha) \ge \kappa\mathcal{I}(\alpha,\alpha) + 2f_\delta\big(-\mathcal{L}^2 + \frac{1}{2}\nabla_N^2\mathcal{G}_1\big)(\alpha,\alpha) - \varepsilon(\delta)\mathcal{I}(\alpha,\alpha)$$

holds on a small neighborhood U of a point $p \in \Gamma$ for every two-vector α on U, where $\varepsilon(\delta)$ does not depend on α. Let us fix a coordinate neighborhood (U, φ) of $p \in \Gamma$, where

$$\varphi = (x^1, \dots, x^n) : U \subset M \to V \subset \mathbb{R}^n$$

is as in Section 2. Using this coordinate chart, we identify $U \subset M$ and $\varphi(U) \subset \mathbb{R}^n$, and regard all quantities in the above inequality as functions $V \to \mathbb{R}$. W.l.o.g. we may assume that α has fixed coefficients satisfying $\sum_{i,j=1}^n (\alpha^{ij})^2 = 1$.

We proceed as in Lemma 9.2 of [15]. Off a δ-neighborhood of Γ we have $f_\delta(x^n) = 0$, so the inequality holds without an error term. On $\Gamma = \{x^n = 0\}$ we have $f_\delta(x^n) = 1$, and the inequality follows from (1.4.2).

Let us now fix a point $\hat{x} = (x^1, \dots, x^{n-1}) \in U \cap \Gamma$ and look at the inequality on the line segment $\{(\hat{x}, x^n) : x^n \in [0, \delta]\}$. Let

$$\mathcal{Q} = -\mathcal{L}^2 + \frac{1}{2}\nabla_N^2\mathcal{G}_1.$$

For $x^n \in [0, \delta^2]$ we have $f_\delta(x^n) \in [0, 1]$ (cf. Definition 1.2.6). Suppose for a moment that the quantities $\mathcal{R}(\alpha,\alpha)$, $\mathcal{A}(\alpha,\alpha)$, $\mathcal{B}(\alpha,\alpha)$ and $\mathcal{Q}(\alpha,\alpha)$ do not depend x^n. Then the inequality

$$\mathcal{R}(\alpha,\alpha) - f_\delta^2\mathcal{A}(\alpha,\alpha) + f_\delta\mathcal{B}(\alpha,\alpha) \ge \kappa\mathcal{I}(\alpha,\alpha) + 4f_\delta\mathcal{Q}(\alpha,\alpha) \tag{1.5.2}$$

would hold without an error term. This is because it holds for $f_\delta = 0$ and $f_\delta = 1$, and the function

$$\begin{aligned} [0,1] &\to \mathbb{R} \\ y &\mapsto \mathcal{R}(\alpha,\alpha) - y^2\mathcal{A}(\alpha,\alpha) + y\mathcal{B}(\alpha,\alpha) \end{aligned}$$

is concave (note that $\mathbf{L} \geq 0$ implies $\mathcal{A} = \mathbf{L} \wedge \mathbf{L} \geq 0$, see Lemma B.3.3). Now $\mathcal{R}(\alpha,\alpha)$, $\mathcal{A}(\alpha,\alpha)$, $\mathcal{B}(\alpha,\alpha)$ and $\mathcal{Q}(\alpha,\alpha)$ do depend on x^n, but they are smooth on M_0 and hence almost constant for small x^n. Indeed, one has for instance

$$\begin{aligned} |\mathcal{R}(\alpha,\alpha)(\hat{x},s) - \mathcal{R}(\alpha,\alpha)(\hat{x},t)| &= \frac{1}{4}|R_{ijkl}(\hat{x},s) - R_{ijkl}(\hat{x},t)||\alpha^{ij}\alpha^{kl}| \\ &\leq \delta c(n) \sup_{i,j,k,l} \|R_{ijkl}\|_{C^1(U)} \end{aligned}$$

for all $s,t \in [0,\delta]$, where the right hand side tends to zero since the C^1-norm of the coordinate functions is bounded if we choose U small enough. Therefore (1.5.2) holds up to a small error term $\varepsilon(\delta)$ on the right hand side for $x^n \in [0,\delta^2]$.

For $x^n \in [\delta^2,\delta]$ we have $|f_\delta(x^n)| \leq \delta$. $\mathcal{A},\mathcal{B},\mathcal{I}$ and $\mathcal{Q}$ are uniformly bounded near Γ, therefore (1.5.2) holds for all $x^n \in [0,\delta]$ if we choose δ sufficiently small and subtract another $\varepsilon(\delta)$ on the right hand side.

□

Proposition 1.5.2 (cf. [15], Lemma 10.1)**.** *If the constant C in the definition of g_δ is chosen large enough, then for small $\delta > 0$*

$$\mathcal{R}_\delta \geq \big(\kappa - \varepsilon(\delta)\big)\mathcal{I}_\delta,$$

where $\varepsilon(\delta) \to 0$ as δ tends to zero.

Proof. Since $g_\delta \to g$ uniformly, it suffices to show that $\mathcal{R}_\delta \geq \big(\kappa - \varepsilon(\delta)\big)\mathcal{I}$. From Proposition 1.3.1 and Lemma 1.5.1 we get

$$\begin{aligned} \mathcal{R}_\delta &\approx \mathcal{R} - f_\delta^2\mathcal{A} + f_\delta\mathcal{B} - 2f_\delta'\mathcal{L} + 2f_\delta^2\mathcal{L}^2 + 2Cf_\delta\hat{\mathcal{I}} \\ &\geq \kappa\mathcal{I} + 2f_\delta\big(-\mathcal{L}^2 + \frac{1}{2}\nabla_N^2\mathcal{G}_1 + C\hat{\mathcal{I}}\big) - 2f_\delta'\mathcal{L} + 2f_\delta^2\mathcal{L}^2 - \varepsilon(\delta)\mathcal{I}. \end{aligned}$$

By definition of the operators $\mathcal{L}^2$, $\nabla_N^2\mathcal{G}_1$, $\hat{\mathcal{I}}$ (see Proposition 1.3.1) we have

$$-\mathcal{L}^2 + \frac{1}{2}\nabla_N^2\mathcal{G}_1 + C\hat{\mathcal{I}} = (-\mathbf{L}^2 + \frac{1}{2}\nabla_N^2\mathbf{G}_1 + C\mathbf{P}^T) \wedge \mathbf{P}^N. \tag{1.5.3}$$

Note that the operators $\mathbf{L}^2$ and $\nabla_N^2\mathbf{G}_1$ vanish on $T\Gamma(d)^\perp$. Moreover, these operators are uniformly bounded near Γ. Therefore, the expression in parentheses on the right hand side of (1.5.3) becomes nonnegative, if we choose the constant C large enough. Then $\mathbf{P}^N \geq 0$ implies that the right hand side of (1.5.3) is nonnegative (cf. Lemma B.3.3). Moreover, $-\mathcal{L}^2 + \frac{1}{2}\nabla_N^2\mathcal{G}_1 + C\hat{\mathcal{I}}$ is uniformly bounded near Γ, and $f_\delta \geq -\delta$ by construction. Thus

$$2f_\delta\big(-\mathcal{L}^2 + \frac{1}{2}\nabla_N^2\mathcal{G}_1 + C\hat{\mathcal{I}}\big) \geq -\varepsilon(\delta)\mathcal{I}.$$

The operator $\mathcal{L} = \mathbf{L}\wedge\mathbf{P}^N$ is nonnegative and uniformly bounded near Γ, and $f_\delta' \leq \delta$ by construction. This gives us $-2f_\delta'\mathcal{L} \geq -\varepsilon(\delta)\mathcal{I}$. Obviously, $f_\delta^2\mathcal{L}^2 = f_\delta^2\mathbf{L}^2 \wedge \mathbf{P}^N$ is nonnegative, and we are done.

□

Corollary 1.5.3. *The weakly defined Riemannian curvature operator of the $W^{2,\infty}_{loc}$-metric $g_{(\delta)}$ on M (recall that $g_{(\delta)}|_{M_0} = g_\delta$ and $g_{(\delta)}|_{M_1} = g_1$, cf. Lemma 1.2.10) satisfies*

$$\mathcal{R}(g_{(\delta)}) \geq \kappa - \varepsilon(\delta) \quad a.e.\ on\ M \tag{1.5.4}$$

(everywhere except on Γ).

Proof. In local coordinates the Riemannian curvature tensor of some metric h is given by

$$R(h)_{ijkl} = \partial_j \partial_k h_{il} + \partial_i \partial_l h_{jk} - \partial_j \partial_l h_{ik} - \partial_i \partial_k h_{jl} + (h^{-1} \bullet \partial h \bullet \partial h)_{ijkl}, \quad (1.5.5)$$

where $\bullet$ means contracting tensors using the metric. Since the second derivatives enter (1.5.5) linearly, $\mathcal{R}(g_{(\delta)})$ can be defined on M in the weak sense. $\mathcal{R}(g_{(\delta)}) \geq \kappa - \varepsilon(\delta)$ a.e. follows from Proposition 1.5.2 and the assumption $\mathcal{R}(g_1) \geq \kappa$. □

1.6. Mollifying $g_{(\delta)}$

By mollifying $g_{(\delta)}$ we construct a family of smooth metrics with properties as required in Definition 1.1.1.

Proposition 1.6.1. *There exists a family of smooth metrics $\tilde{g}_{(\delta)}$ such that*

$$\tilde{g}_{(\delta)} \to g$$

as $\delta \to 0$ uniformly on compact subsets of M, and such that

$$\tilde{\mathcal{R}}_{(\delta)} \geq (\kappa - \tilde{\varepsilon}(\delta))\tilde{\mathcal{I}}_{(\delta)},$$

where $\tilde{\varepsilon}(\delta) \to 0$ as $\delta \to 0$.

Proof. Let us fix a small $\delta > 0$. Let U_s, $s \in \mathbb{N}$, be a locally finite open cover of M, such that $U_s \subset\subset U'_s$ for some coordinate neighborhood U'_s. In what follows, we identify the coordinate neighborhoods U'_s with the corresponding neighborhoods in $\mathbb{R}^n$. Since Γ is compact, we may assume w.l.o.g. that $U'_s \cap \Gamma = \emptyset$ for $s > N$ for some $N \in \mathbb{N}$. We denote the coordinate functions of $g_{(\delta)}$ on U'_s by $(g^s_{(\delta)})_{ij}$. After choosing U'_s even smaller if necessary, we may also assume that $\|(g^s_{(\delta)})_{ij}\|_{C^1(U'_s)} \leq K < \infty$ for all $s \leq N$. For $s \leq N$ and $x \in U_s$ let

$$(g^{s,h}_{(\delta)})_{ij}(x) = (\rho_h * (g^s_{(\delta)})_{ij})(x) = \int_{|z| \leq 1} \rho(z)(g^s_{(\delta)})_{ij}(x - hz)dz, \quad (1.6.1)$$

where $\rho \in C_0^\infty(\mathbb{R}^n)$ satisfies $\operatorname{supp} \rho \subset B_1(0)$ and $\int_{\mathbb{R}^n} \rho = 1$, and h is small enough so that for all $s \leq N$ the point $x - hz$ lies in U'_s for all $z \in B_1(0)$. Observe that $g^{s,h}_{(\delta)}$ is a well defined metric on U_s which converges to $g_{(\delta)}|_{U_s}$ in the C^1-sense. Let (η_s) be a partition of unity on M such that $\operatorname{supp} \eta_s \subset U_s$ for all s. For h as above we then define a smooth metric $g^h_{(\delta)}$ on M by

$$g^h_{(\delta)} = \sum_{s \leq N} \eta_s g^{s,h}_{(\delta)} + \sum_{s > N} \eta_s g_{(\delta)}. \quad (1.6.2)$$

Let us now calculate the Riemannian curvature tensor $R(g^h_{(\delta)})$ using the formula (1.5.5). The terms which do not involve any derivatives of the unity functions η_s give us just the mollified Riemannian curvature tensor $(R(g_{(\delta)}))^h$ up to a small error term $\varepsilon(\delta, h) \overset{h \to 0}{\to} 0$, constructed in the same way as $g^h_{(\delta)}$ in (1.6.1) and (1.6.2). Indeed, we have

$$\begin{aligned} & \partial\partial g^h_{(\delta)} + \cdots + (g^h_{(\delta)})^{-1} \bullet \partial g^h_{(\delta)} \bullet \partial g^h_{(\delta)} \\ = \; & \big(\partial\partial g_{(\delta)}\big)^h + \cdots + (g^h_{(\delta)})^{-1} \bullet \partial g^h_{(\delta)} \bullet \partial g^h_{(\delta)} \\ = \; & \big(\partial\partial g_{(\delta)}\big)^h + \cdots + \big((g_{(\delta)})^{-1} \bullet \partial g_{(\delta)} \bullet \partial g_{(\delta)}\big)^h \pm \varepsilon(\delta, h), \end{aligned}$$

where we used that both $(g^h_{(\delta)})^{-1} \bullet \partial g^h_{(\delta)} \bullet \partial g^h_{(\delta)}$ and $\big((g_{(\delta)})^{-1} \bullet \partial g_{(\delta)} \bullet \partial g_{(\delta)}\big)^h$ are C^0 close to $(g_{(\delta)})^{-1} \bullet \partial g_{(\delta)} \bullet \partial g_{(\delta)}$ since $g^h_{(\delta)} \to g_\delta$ in the C^1 sense for any fixed δ as $h \to 0$.

The other terms vanish uniformly on M as h tends to zero. We shall verify this exemplary for the terms involving second derivatives of the unity functions. After fixing a coordinate chart (U, φ) we compute

$$\begin{aligned}
& |\sum_{s\le N} \partial_j\partial_k\eta_s(g^{s,h}_{(\delta)})_{il} + \sum_{s>N} \partial_j\partial_k\eta_s(g_{(\delta)})_{il}| \\
= \ & |\sum_{s\le N} \partial_j\partial_k\eta_s(g_{(\delta)})_{il} + \sum_{s\le N} \partial_j\partial_k\eta_s\big((g^{s,h}_{(\delta)})_{il} - (g_{(\delta)})_{il}\big) + \sum_{s>N} \partial_j\partial_k\eta_s(g_{(\delta)})_{il}| \\
\le \ & |\partial_j\partial_k(\underbrace{\sum_{s=1}^{\infty}\eta_s}_{\equiv 1})(g_{(\delta)})_{il} + \sum_{s\le N} |\partial_j\partial_k\eta_s||(g^{s,h}_{(\delta)})_{il} - (g_{(\delta)})_{il}| \\
\le \ & N\big(\max_{s=1,\dots,N}\|\eta_s\|_{C^2(U_s)}\big)\big(\max_{s=1,\dots,N}\max_{i,l=1,\dots,n}\|(g^{s,h}_{(\delta)})_{il} - (g_{(\delta)})_{il}\|_{C^0(U_s)}\big) \\
\overset{h\to 0}{\to} \ & 0.
\end{aligned}$$

All in all we have

$$|(R(g^h_{(\delta)}))_{ijkl} - (R(g_{(\delta)}))^h)_{ijkl}| \le \varepsilon(\delta, h),$$

where $\varepsilon(\delta,h) \overset{h\to 0}{\to} 0$ for every fixed δ, which implies that

$$\mathcal{R}(g^h_{(\delta)}) \ge (\mathcal{R}(g_{(\delta)}))^h - \tilde{\varepsilon}(\delta,h)\mathcal{I}(g_{(\delta)}), \tag{1.6.3}$$

where $\tilde{\varepsilon}(\delta,h) \overset{h\to 0}{\to} 0$ for every fixed δ. Moreover, Corollary 1.5.3 implies

$$(\mathcal{R}(g_{(\delta)}))^h \ge (\kappa - \varepsilon(\delta))(\mathcal{I}(g_{(\delta)}))^h. \tag{1.6.4}$$

Indeed, for any two-vector $\alpha = \sum_{i<j}\alpha^{ij}\partial_i \wedge \partial_j$ on $U_{s'}$ (w.l.o.g. with fixed coefficients) we compute using (1.3.1)

$$\begin{aligned}
(\mathcal{R}(g_{(\delta)}))^{s,h}(x)(\alpha,\alpha) &= \int_{|z|\le 1} \rho(z)\frac{1}{4}(R(g_{(\delta)}))^s_{ijkl}(x-hz)\alpha^{ij}\alpha^{kl}dz \\
&\overset{(1.5.4)}{\ge} \big(\kappa - \varepsilon(\delta)\big)\int_{|z|\le 1} \rho(z)\frac{1}{4}(\mathcal{I}(g_{(\delta)}))^s_{ijkl}(x-hz)\alpha^{ij}\alpha^{kl}dz \\
&= \big(\kappa - \varepsilon(\delta)\big)(\mathcal{I}(g_{(\delta)}))^{s,h}(x)(\alpha,\alpha).
\end{aligned}$$

Combining (1.6.3) and (1.6.4) we arrive at

$$\begin{aligned}
\mathcal{R}(g^h_{(\delta)}) &\ge (\kappa - \varepsilon(\delta))(\mathcal{I}(g_{(\delta)}))^h - \tilde{\varepsilon}(\delta,h)\mathcal{I}(g_{(\delta)}) \\
&\ge (\kappa - \varepsilon(\delta))(1 \pm \varepsilon(\delta))\mathcal{I}(g^h_{(\delta)}) - \tilde{\varepsilon}(\delta,h)(1+\varepsilon(\delta))\mathcal{I}(g^h_{(\delta)})
\end{aligned}$$

where we used the fact that for every fixed δ both $(\mathcal{I}(g_{(\delta)}))^h$ and $\mathcal{I}(g^h_{(\delta)})$ approach $\mathcal{I}(g_{(\delta)})$ as h tends to zero ($\pm$ referes to the cases $\kappa > 0, \kappa \le 0$, respectively). Since $\tilde{\varepsilon}(\delta,h) \to 0$ as $h \to 0$ for every fixed δ, we may choose h small enough such that $\tilde{\varepsilon}(\delta,h) \le \varepsilon(\delta)$, thereby obtaining

$$\mathcal{R}(g^h_{(\delta)}) \ge \big(\kappa - (|\kappa| + 3)\varepsilon(\delta)\big)\mathcal{I}(g^h_{(\delta)})$$

and the desired result follows with $\tilde{g}_{(\delta)} = g^h_{(\delta)}$ and $\tilde{\varepsilon}(\delta) = (|\kappa|+3)\varepsilon(\delta)$. □

Remark 1.6.2. From the fact that $\tilde{g}_{(\delta)} \to g$ in the C^0 sense on M, and $\tilde{g}_{(\delta)} \equiv g$ off a compact neighborhood $K \supset \Gamma$ it follows that $\tilde{g}_{(\delta)} \to g$ uniformly on M.

The following lemma will be useful for an application in Section 2.4.

Lemma 1.6.3 (Further properties of $\tilde{g}_{(\delta)}$). *Let (M_0, g_0), (M_1, g_1), $M = M_0 \cup_\phi M_1$, $g = g_0 \cup_\phi g_1$, and $\tilde{g}_{(\delta)}$ be as above. The following statements are true:*

(*i*) *If $(M_0, \operatorname{dist}_{g_0})$ and $(M_1, \operatorname{dist}_{g_1})$ are complete (as length-metric spaces), then $(M, \tilde{g}_{(\delta)})$ is a complete manifold for small enough $\delta > 0$.*

(*ii*) *If $\sup_{M_i} |\mathcal{R}(g_i)|_{g_i} < \infty$, $i = 0, 1$, then*

$$\sup_M |\mathcal{R}(\tilde{g}_\delta)|_\delta < \infty,$$

for all $\delta > 0$, where $|\cdot|_\delta$ denotes the norm induced by $\tilde{g}_{(\delta)}$.

(*iii*) *We have $|\Gamma(\tilde{g}_{(\delta_0)}) - \Gamma(\tilde{g}_{(\delta)})|_g \leq c$ for all $\delta_0, \delta > 0$, where $\Gamma(\tilde{g}_{(\delta)})$ refers to the Christoffel symbols of $\tilde{g}_{(\delta)}$, and $c = c(g_0, g_1)$ does not depend on δ, δ_0.*

Proof. (*i*): To show that $(M, \tilde{g}_{(\delta)})$ is complete, by the Hopf-Rinow theorem it suffices to verify that any closed bounded subset $A \subset (M, \operatorname{dist}_{\tilde{g}_{(\delta)}})$ is compact. Since $\tilde{g}_{(\delta)} \to g$ uniformly on M (see the above remark), any such set A is bounded with respect to dist_g. Moreover, since the topologies induced by dist_g and $\operatorname{dist}_{\tilde{g}_{(\delta)}}$ coincide with the initial topology of M, the set A is a closed subset of $(M, \operatorname{dist}_g)$. Since by assumption $(M_0, \operatorname{dist}_{g_0})$ and $(M_1, \operatorname{dist}_{g_1})$ are complete, from construction of $M = M_0 \cup_\phi M_1$ and $g = g_0 \cup_\phi g_1$ it follows that $(M, \operatorname{dist}_g)$ is a complete metric space. A generalized version of the Hopf-Rinow theorem for length-metric spaces (see [10], [2]) says that any closed bounded subset of a complete locally compact length-metric space is compact, so A is compact in $(M, \operatorname{dist}_g)$ (note that M is locally compact since it is a manifold). Using the fact that the topologies of $(M, \operatorname{dist}_g)$ and $(M, \operatorname{dist}_{\tilde{g}_{(\delta)}})$ coincide, we conclude that A is compact in $(M, \operatorname{dist}_{\tilde{g}_{(\delta)}})$.

(*ii*): This follows from the fact that by construction the metrics $\tilde{g}_{(\delta)}$ are smooth and coincide with $g_0 \cup_\phi g_1$ off a compact neighborhood $K \supset \Gamma$.

(*iii*): Since $\tilde{g}_{(\delta)} \to g$ uniformly on M, and off a neighborhood of Γ the metrics $\tilde{g}_{(\delta)}$ coincide with g for small enough $\delta > 0$, it suffices to check that for any $p \in \Gamma$ there exists a coordinate neighborhood of $U \ni p$ such that on U the first derivatives of $(\tilde{g}_{(\delta)})_{ij}$ are bounded independently of δ. Let (U, φ) be a coordinate neighborhood of $p \ni \Gamma$, where $\varphi = (x^1, \ldots, x^n)$ are Fermi coordinates constructed in Section 1.2. Recall that the metric $\tilde{g}_{(\delta)}$ was constructed by mollifying the $W^{2,\infty}_{loc}$ metric $g_{(\delta)}$ near Γ, defined by

$$g_{(\delta)} = \begin{cases} g_\delta & \text{on } M_0 \\ g_1 & \text{on } M_1 \end{cases},$$

where

$$g_\delta = g_0 + 2F_\delta L + 2C\mathcal{F}_\delta \hat{g}.$$

In coordinates φ, the derivatives of g_δ (on $U \cap M_0$) are given by

$$\begin{aligned} \partial_k (g_\delta)_{ij} &= \partial_k (g_{ij} + 2F_\delta L_{ij} + 2C\mathcal{F}_\delta \hat{g}_{ij}) \\ &= \partial_k g_{ij} - \delta_{kn} f_\delta \partial_n g_{ij} - F_\delta \partial_k \partial_n g_{ij} + 2C\delta_{kn} F_\delta \hat{g}_{ij} + 2C\mathcal{F}_\delta \partial_k \hat{g}_{kl}, \end{aligned}$$

where we used $L_{ij} = -\frac{1}{2}\partial_n g_{ij}$. Since the functions f_δ, F_δ and $\mathcal{F}_\delta$ are uniformly bounded independently of δ, this gives us

$$|\partial_k(g_{(\delta)})_{ij}| \le c(\sup_{1\le i,j\le n} |g_{ij}|_{C^2(U\cap M_0)}, \sup_{1\le i,j\le n} |g_{ij}|_{C^1(U\cap M_1)})$$

on U, where the right hand side is finite if we choose U small enough. Since the mollifying procedure in Section 1.6 does not affect the uniform boundedness of the first derivatives, we also have $|\partial_k(\tilde{g}_{(\delta)})_{ij}| \le c$ on U for all $\delta > 0$, where c is as above, and we are done.

□

1.7. Similar results for other curvature operators

As mentioned in the introduction, analogous results hold for manifolds with lower bounds on Ricci curvature, scalar curvature, isotropic curvature, and bi-curvature, respectively.

1.7.1. Manifolds with Ricci curvature $\ge \kappa$

Theorem 1.7.1. *Let $M = M_0 \cup_\phi M_1$, $g = g_0 \cup_\phi g_1$, $\Gamma = \Gamma_0 =_\phi \Gamma_1$, and $L = L_0 + L_1$ be as in Theorem 1.1.2. Suppose that* $\mathrm{Ric}(g_0)$ *and* $\mathrm{Ric}(g_1)$ *are at least κ. If L is positive semidefinite, then* $\mathrm{Ric}(g) \ge \kappa$ *(in a similar sense as in Definition 1.1.1).*

Proof. Given a symmetric bilinear form $\mathcal{T}$ on $\Lambda^2(TM)$ and a metric h, we denote

$$\mathrm{Ric}_h(\mathcal{T}) = h^{jl} T(\cdot, \partial_j, \cdot, \partial_l),$$

where $T(\partial_i, \partial_j, \partial_k, \partial_l) = \mathcal{T}(\partial_i \wedge \partial_j, \partial_k \wedge \partial_l)$. The strategy of the proof is similar as in the proof of Theorem 1.1.2. We show

(a) The curvature operator of the modified metric g_δ on M_0 satisfies $\mathrm{Ric}_{g_\delta}(\mathcal{R}_\delta) \ge \big(\kappa - \varepsilon(\delta)\big) g_\delta$, $\varepsilon(\delta) \to 0$ (this corresponds to Proposition 1.5.2).

(b) By mollifying $g_{(\delta)}$, we construct a family of smooth metrics which approximate g in the C^0 sense and have Ricci curvature at least $\kappa - \varepsilon(\delta)$.

(a): As in the previous sections, when working on M_0, we write g rather than g_0 to simplify the notation. Here we may simplify the argument of the previous sections. Recall that we identify endomorphisms and bilinear forms on TM_0 in the sense of Notation 1.2.4. In view of this identification, we have $g = \mathrm{id}_{TM_0}$. Since $g_\delta \approx g$ on M_0, it suffices to show that

$$\mathrm{Ric}_{g_\delta}(\mathcal{R}_\delta) \ge \big(\kappa - \varepsilon(\delta)\big)\mathrm{id}_{TM_0}.$$

By (1.3.2) we have

$$\begin{aligned} \mathrm{Ric}_{g_\delta}(\mathcal{R}_\delta) \;\ge\; & \mathrm{Ric}_{g_\delta}(\mathcal{R}) - f_\delta^2 \mathrm{Ric}_{g_\delta}(\mathcal{A}) + f_\delta \mathrm{Ric}_{g_\delta}(\mathcal{B}) \\ - \; & 2f_\delta' \mathrm{Ric}_{g_\delta}(\mathcal{L}) + 2f_\delta^2 \mathrm{Ric}_{g_\delta}(\mathcal{L}^2) + 2C f_\delta \mathrm{Ric}_{g_\delta}(\hat{\mathcal{I}}) - \varepsilon(\delta)\mathrm{id}_{TM_0} \end{aligned}$$

(here and in what follows, we suppress constants in the $\varepsilon(\delta)$ term). Since $|f_\delta| \le 1$ and $g_\delta \to g$ uniformly, we may replace Ric_{g_δ} by Ric_g everywhere except in the f_δ' term, i.e. we have

$$\begin{aligned} \mathrm{Ric}_{g_\delta}(\mathcal{R}_\delta) \;\ge\; & \mathrm{Ric}_g(\mathcal{R}) - f_\delta^2 \mathrm{Ric}_g(\mathcal{A}) + f_\delta \mathrm{Ric}_g(\mathcal{B}) \\ - \; & 2f_\delta' \mathrm{Ric}_{g_\delta}(\mathcal{L}) + 2f_\delta^2 \mathrm{Ric}_g(\mathcal{L}^2) + 2C f_\delta \mathrm{Ric}_g(\hat{\mathcal{I}}) - \varepsilon(\delta)\mathrm{id}_{TM_0}. \end{aligned} \tag{1.7.1}$$

Recall that $\hat{\mathcal{I}} = \mathbf{P}^T \wedge \mathbf{P}^N$ (cf. Notation 1.2.8). We compute

$$\begin{aligned}
\big(\mathrm{Ric}_g(\mathbf{P}^T \wedge \mathbf{P}^N)\big)_{ik} &= \frac{1}{2} g^{jl}(P^T_{ik}P^N_{jl} - P^T_{jk}P^N_{il} + P^N_{ik}P^T_{jl} - P^N_{jk}P^T_{il}) \\
&= \frac{1}{2}(\mathrm{tr}_g(P^N)P^T_{jk} + \mathrm{tr}_g(P^T)P^N_{jk}) \\
&= \frac{1}{2}(P^T_{ik} + (n-1)P^N_{ik}), \qquad (1.7.2)
\end{aligned}$$

which implies

$$\mathrm{Ric}_g(\hat{\mathcal{I}}) \geq \frac{1}{2}(\mathbf{P}^T + \mathbf{P}^N) = \frac{1}{2}\mathrm{id}_{TM_0} \qquad (1.7.3)$$

(the case $n = 1$ is trivial, since in this case $\mathcal{R}(g) \equiv 0$). Using (1.7.3) and the assumption $\mathrm{Ric}_g(\mathcal{R}) \geq \kappa$ in (1.7.1), we obtain the estimate

$$\begin{aligned}
\mathrm{Ric}_{g_\delta}(\mathcal{R}_\delta) \;&\geq\; (\kappa - \varepsilon(\delta))\mathrm{id}_{TM_0} - f_\delta^2 \mathrm{Ric}_g(\mathcal{A}) + f_\delta \mathrm{Ric}_g(\mathcal{B}) \\
&-\; 2f'_\delta \mathrm{Ric}_{g_\delta}(\mathcal{L}) + 2f_\delta^2 \mathrm{Ric}_g(\mathcal{L}^2) + C f_\delta \mathrm{id}_{TM_0} \\
&=\; (\kappa - \varepsilon(\delta))\mathrm{id}_{TM_0} - 2f'_\delta \mathrm{Ric}_{g_\delta}(\mathcal{L}) \qquad (1.7.4)\\
&+\; f_\delta\big(-f_\delta \mathrm{Ric}_g(\mathcal{A}) + \mathrm{Ric}_g(\mathcal{B}) + 2f_\delta \mathrm{Ric}_g(\mathcal{L}^2) + C\mathrm{id}_{TM_0}\big).
\end{aligned}$$

The operators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{L}^2$ are smooth and hence uniformly bounded on a neighborhood of Γ in M_0. Therefore, the term in parenthesis in (1.7.4) is nonnegative for large enough fixed C and bounded from above[2]. Since $f_\delta \geq -\delta$, the last line of (1.7.4) is $\geq -\varepsilon(\delta)\mathrm{id}_{TM_0}$, and we arrive at

$$\mathrm{Ric}_{g_\delta}(\mathcal{R}_\delta) \geq (\kappa - \varepsilon(\delta))\mathrm{id}_{TM_0} - 2f'_\delta \mathrm{Ric}_{g_\delta}(\mathcal{L}). \qquad (1.7.5)$$

We now compute the f'_δ term in (1.7.5). Let us fix a point $p \in M_0$ near Γ. Let $\Gamma(p)$ be the equidistant hypersurface of Γ containing p. We choose an orthonormal (w.r.t. g) basis $e_1, \ldots, e_{n-1}$ of $T_p\Gamma(p)$ such that $(L(p))_{1\leq i,j\leq n-1}$ is diagonal. Then $\{e_1, \ldots, e_{n-1}, N\}$ is an orthonormal basis of T_pM and $(L(p))_{1\leq i,j\leq n}$ is diagonal. By construction this implies that $(g_\delta(p))_{1\leq i,j\leq n}$ is diagonal, so $g_\delta(p)_{jl} = \mu_l \delta_{jl}$, where $\mu_l > 0$ since g_δ is positive definite, and $\mu_n = 1$. Moreover, we still have $(P^N)_{ij} = \delta_{in}\delta_{jn}$ in these coordinates. Therefore, given a vector $\xi \in T_pM$, using $L_{kn} = 0$ for $k = 1, \ldots, n$ we compute

$$\begin{aligned}
(\mathrm{Ric}_{g_\delta}(\mathcal{L}))(\xi,\xi) &= g_\delta^{jl}(\mathbf{L} \wedge \mathbf{P}^N)_{ijkl}\xi^i\xi^k = \sum_{l=1}^{n} \frac{1}{\mu_l}(\mathbf{L} \wedge \mathbf{P}^N)_{ilkl}\xi^i\xi^k \\
&= \frac{1}{2}\sum_{l=1}^{n} \frac{1}{\mu_l}(L_{ik}P^N_{ll} - L_{lk}P^N_{il} + P^N_{ik}L_{ll} - P^N_{lk}L_{il})\xi^i\xi^k \\
&= \frac{1}{2}\frac{1}{\mu_n}L(\xi,\xi) + \frac{1}{2}(\xi^n)^2 \sum_{l=1}^{n}\frac{1}{\mu_l}L_{ll} \geq 0
\end{aligned}$$

since $L \geq 0$ by assumption. Using the fact that f'_δ does not exceed δ, and that $\mathrm{Ric}_{g_\delta}(\mathcal{L})$ is uniformly bounded near Γ, we obtain the estimate

$$-2f'_\delta \mathrm{Ric}_{g_\delta}(\mathcal{L}) \geq -\varepsilon(\delta)\mathrm{id}_{TM_0}.$$

[2] Note that at this point we simplified the argument of Section 1.5. $\mathrm{Ric}_g(\mathbf{P}^T \wedge \mathbf{P}^N)$ is estimated from below by the positive definite operator $\frac{1}{2}\mathrm{id}_{TM_0}$, hence the $\mathcal{A}, \mathcal{B}$ and $\mathcal{L}^2$ terms are absorbed by $C\mathrm{id}_{TM_0}$. When considering the full curvature tensor, the corresponding operator $C\mathbf{P}^T \wedge \mathbf{P}^N$ has nontrivial kernel, which is why the concavity argument of Lemma 1.5.1 was necessary.

(b): Let us fix a $\delta > 0$. We construct the metrics $g^h_{(\delta)}$ as in Section 1.6. In view of (1.6.3) and the fact that $g^h_{(\delta)} \to g_{(\delta)}$ uniformly as $h \to 0$ we have

$$\mathrm{Ric}_{g^h_{(\delta)}}(\mathcal{R}(g^h_{(\delta)})) \geq \mathrm{Ric}_{g_{(\delta)}}(\mathcal{R}(g_{(\delta)}))^h - \tilde{\varepsilon}(\delta, h)(g_{(\delta)}),$$

where $\tilde{\varepsilon}(\delta, h) \to 0$ for every fixed δ as $h \to 0$. Given a vector field X on $U_{s'}$ which has constant coefficients not exceeding 1, on U_s we compute using (a) and the mean value theorem

$$\begin{aligned}
& \mathrm{Ric}_{g_{(\delta)}}(\mathcal{R}(g_{(\delta)}))^{h,s}(x)(X,X) \\
= & \int_{|z|\leq 1} \rho(z)(g_{(\delta)})^{jl}(x)(R(g_{(\delta)}))^s_{ijkl}(x-hz)X^iX^k dz \\
= & \int_{|z|\leq 1} \rho(z)(g_{(\delta)})^{jl}(x-hz)(R(g_{(\delta)}))^s_{ijkl}(x-hz)X^iX^k dz \\
+ & \ h\int_{|z|\leq 1} \rho(z)D(g_{(\delta)})^{jl}(\xi_{x,hz})z(R(g_{(\delta)}))^s_{ijkl}(x-hz)X^iX^k dz \\
\geq & \ \big(\kappa - \varepsilon(\delta)\big)g^{s,h}_{(\delta)}(X,X) - hC(\delta) \\
\geq & \ \big(\kappa - 2\varepsilon(\delta)\big)g^{s,h}_{(\delta)}(X,X),
\end{aligned}$$

where $\xi_{x,hz} = (1-t)x + thz$ for some $t \in [0,1]$, and $C(\delta)$ depends on the bound of $\mathcal{R}(g_{(\delta)})$ near Γ, which is finite for every fixed δ. Note that for every fixed δ we may choose h small enough so that $hC(\delta) \leq \varepsilon(\delta)$. Since $U_s \cap \Gamma \neq \emptyset$ only for finitely many s, we deduce

$$\mathrm{Ric}_{g_{(\delta)}}(\mathcal{R}(g_{(\delta)}))^h \geq \big(\kappa - 2\varepsilon(\delta)\big)g^h_{(\delta)}.$$

Thus

$$\mathrm{Ric}_{g^h_{(\delta)}}(\mathcal{R}(g^h_{(\delta)})) \geq \big(\kappa - 2\varepsilon(\delta)\big)g^h_{(\delta)} - \tilde{\varepsilon}(\delta, h)(g_{(\delta)}).$$

Finally, we choose h even smaller such that $\tilde{\varepsilon}(\delta, h) \leq \varepsilon(\delta)$ and $g_{(\delta)} \leq (1+\varepsilon(\delta))g^h_{(\delta)}$, and the result follows with $\tilde{g}_{(\delta)} = g^h_{(\delta)}$ and $\tilde{\varepsilon}(\delta) = 4\varepsilon(\delta)$.

□

1.7.2. Manifolds with scalar curvature $\geq \kappa$

The scalar curvature of a C^2 smooth Riemannian metric g is defined as $S(g) = \mathrm{tr}_g\mathrm{Ric}_g = g^{ik}g^{jl}R^g_{ijkl}$. As mentioned in the introduction, in the scalar curvature case we may weaken the assumption $L \geq 0$ on Γ, requiring only that $\mathrm{tr}_g L \geq 0$ on Γ, i.e. the sum of the mean curvatures of Γ with respect to g_0 and g_1 is nonnegative.

Theorem 1.7.2. *Let $M = M_0 \cup_\phi M_1$, $g = g_0 \cup_\phi g_1$, $\Gamma = \Gamma_0 =_\phi \Gamma_1$, and $L = L_0 + L_1$ be as in Theorem 1.1.2. Suppose that $S(g_0)$ and $S(g_1)$ are at least κ. If $\mathrm{tr}_g L \geq 0$ on Γ, then $S(g) \geq \kappa$ (in a similar sense as in Definition 1.1.1).*

Proof. First, let us assume that $\mathrm{tr}_g L > 0$ on Γ. In analogy to Lemma 1.2.5, we need to verify that the extension of L satisfies $\mathrm{tr}_g L > 0$, if so does the initial operator on Γ. In fact, for $x \in M_0$ near Γ we have $\mathrm{tr}_{g(x)}L(x) = \mathrm{tr}_{g(\hat{x})}L(\hat{x})$, where $\hat{x}$ is the point of Γ nearest to x. Indeed, let $x \in M_0$ be a point near Γ such that the extension $\mathbf{L}$ is well defined at x. Recall that for $X \in T_xM_0$ we defined $\mathbf{L}X = P^{-1}\mathbf{L}PX$, where P is the parallel transportation along the integral curves of the normal field N, which takes $X \in T_x M_0$ to $PX \in T_{\hat{x}} M_0$. Let $e_1, \ldots, e_n$ be an orthonormal basis of $T_x M_0$, and let $g_{ij}(x) = \langle e_i, e_j \rangle_{g(x)} = \delta_{ij}$ and $L_{ij}(x) = \langle \mathbf{L}(x) e_i, e_j \rangle_{g(x)}$. We compute

$$\begin{aligned} \operatorname{tr}_{g(x)} L(x) &= g^{ij}(x) L_{ij}(x) = \sum_{i=1}^{n} \langle \mathbf{L}(x) e_i, e_i \rangle_{g(x)} \\ &= \sum_{i=1}^{n} \langle P^{-1} \mathbf{L}(\hat{x}) P e_i, e_i \rangle_{g(x)} = \sum_{i=1}^{n} \langle \mathbf{L}(\hat{x}) P e_i, P e_i \rangle_{g(\hat{x})} \\ &= \sum_{i=1}^{n} L(\hat{x})(P e_i, P e_i) = \operatorname{tr}_{g(\hat{x})} L(\hat{x}) \end{aligned} \tag{1.7.6}$$

since $Pe_1, \ldots, Pe_n$ is an orthonormal basis of $T_{\hat{x}} M_0$.

Given a metric h and a bilinear form $\mathcal{T} \in \Lambda^2(TM)$ we denote

$$S_h(\mathcal{T}) = h^{ik} h^{kl} T_{ijkl},$$

where $T_{ijkl} = \mathcal{T}(\partial_i \wedge \partial_j, \partial_k \wedge \partial_l)$. As in the Ricci curvature case, the crucial step is to verify that

$$S_{g_\delta}(\mathcal{R}_\delta) \geq \kappa - \varepsilon(\delta) \tag{1.7.7}$$

holds on M_0. By (1.3.2) we have

$$\begin{aligned} S_{g_\delta}(\mathcal{R}_\delta) &\geq S_{g_\delta}(\mathcal{R}) - f_\delta^2 S_{g_\delta}(\mathcal{A}) + f_\delta S_{g_\delta}(\mathcal{B}) \\ &- 2 f'_\delta S_{g_\delta}(\mathcal{L}) + 2 f_\delta^2 S_{g_\delta}(\mathcal{L}^2) + 2C f_\delta S_{g_\delta}(\hat{\mathcal{I}}) - \varepsilon(\delta) \\ &\geq S_g(\mathcal{R}) - f_\delta^2 S_g(\mathcal{A}) + f_\delta S_g(\mathcal{B}) \\ &- 2 f'_\delta S_{g_\delta}(\mathcal{L}) + 2 f_\delta^2 S_g(\mathcal{L}^2) + 2C f_\delta S_g(\hat{\mathcal{I}}) - \varepsilon(\delta), \end{aligned} \tag{1.7.8}$$

where we used that $g_\delta \to g$ in the C^0 sense and the fact that f_δ is bounded independently of δ (note that since this is not the case for f'_δ, we can not replace g_δ by g in the f'_δ term). By (1.7.2) we have

$$S_g(\hat{\mathcal{I}}) = \frac{1}{2} g^{ik} (P^T_{ik} + (n-1) P^N_{ik}) = n - 1 > 0$$

(the case $n = 1$ is trivial). Similarly as in the previous section, in view of the assumption $S_g(\mathcal{R}) \geq \kappa$ and the fact that $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{L}^2$ are bounded near Γ and $f_\delta \geq -\delta$, after choosing C large enough we may estimate (1.7.8) from below by

$$S_{g_\delta}(\mathcal{R}_\delta) \geq \kappa - \varepsilon(\delta) - 2 f'_\delta S_{g_\delta}(\mathcal{L}). \tag{1.7.9}$$

Consider the f'_δ term in the above expression. As in the previous section, at $x \in M_0$ near Γ we may choose local coordinates such that $g_{ij} = \delta_{ij}$, $L_{ij} = \lambda_i \delta_{ij}$, $(g_\delta)_{ij} = \mu_i^\delta \delta_{ij}$ and $P^N_{ij} = \delta_{in} \delta_{jn}$. In these coordinates we have (recall that $\lambda_n = L_{nn} = 0$ and $\mu_n^\delta = 1$)

$$\begin{aligned} (S_{g_\delta}(\mathcal{L})) &= g_\delta^{ik} g_\delta^{jl} (\mathbf{L} \wedge \mathbf{P}^N)_{ijkl} = \sum_{i,j=1}^{n} \frac{1}{\mu_i^\delta} \frac{1}{\mu_j^\delta} (\mathbf{L} \wedge \mathbf{P}^N)_{ijij} \\ &= \frac{1}{2} \sum_{i,j=1}^{n} \frac{1}{\mu_i^\delta} \frac{1}{\mu_j^\delta} (L_{ii} P^N_{jj} - L_{ij} P^N_{ij} + P^N_{ii} L_{jj} - P^N_{ij} L_{ij}) \\ &= \frac{1}{\mu_n^\delta} \sum_{i=1}^{n} \frac{1}{\mu_i^\delta} \lambda_i = \sum_{i=1}^{n-1} \frac{1}{\mu_i^\delta} \lambda_i. \end{aligned} \tag{1.7.10}$$

Note that the eigenvalues $\mu_i^\delta \to 1$ since $g_\delta \to g$ uniformly, and hence $\mathrm{tr}_g(L) = \sum_{i=1}^{n-1} \lambda_i > 0$ implies

$$\sum_{i=1}^{n-1} \frac{1}{\mu_i^\delta} \lambda_i \geq (1 - \varepsilon(\delta)) \sum_{i=1}^{n-1} \lambda_i \geq 0$$

for small enough δ. We then proceed as in the previous section and estimate the f'_δ term in (1.7.9) from below by $-\varepsilon(\delta)$, which gives us (1.7.7) the case $\mathrm{tr}_g L > 0$ on Γ. Using (1.7.7) and the assumption $S(g_1) \geq \kappa$, we then construct the required smooth metric on M by mollifying $g_{(\delta)} = g_\delta \cup_\phi g_1$ as in the previous section.

Let us now study the case where $\mathrm{tr}_g L \geq 0$ on Γ. In this case we may slightly modify either one of the initial metrics g_0 or g_1 near the boundary, so that $\mathrm{tr}_g L$ becomes strictly positive, and then repeat the argument above. More precisely, consider g_0 near Γ. Recall that in local coordinates $(x^1, \ldots, x^n)$ constructed in Section 1.2 the metric g_0 has the form

$$g_0 = \begin{pmatrix} \hat{g}_0 & 0 \\ 0 & 1 \end{pmatrix},$$

where $\hat{g}$ is the restriction of g to the equidistant hypersurfaces $\Gamma(d)$, $d = \mathrm{dist}_g(\Gamma, \cdot) = x^n$. Let $d_0 > 0$ be small enough so that $\Gamma(d)$ is smooth for $d \leq d_0$. We find a smooth function $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying

$$\begin{aligned} &\varphi(0) = 1 \\ &\varphi|_{[d_0, \infty)} \equiv 1 \\ &\varphi'(0) < 0 \\ &|\varphi'|, |\varphi''| \leq \varepsilon \end{aligned}$$

with $\varepsilon > 0$ small, and put

$$\tilde{g}_0 = \begin{pmatrix} \varphi(x^n)\hat{g}_0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Note that in view of $\varphi(0) = 1$ we have $\tilde{g}_0|_\Gamma = g_0|_\Gamma = g_1|_\Gamma$, so that the isometry of the boundaries is preserved. As in Lemma 1.2.10, at a point $p \in \Gamma$ (i.e. $x^n(p) = 0$) we compute

$$\tilde{L}^0_{ij} = -\frac{1}{2}\partial_n \tilde{g}^0_{ij} = -\frac{1}{2}\varphi'(0) g^0_{ij} - \frac{1}{2}\varphi(0)\partial_n g^0_{ij} = -\frac{1}{2}\varphi'(0) g^0_{ij} + L^0_{ij},$$

and thus

$$\mathrm{tr}_{\tilde{g}_0}(\tilde{L}_0) = g_0^{ij}(-\frac{1}{2}\varphi'(0) g^0_{ij} + L^0_{ij}) = -\frac{n}{2}\varphi'(0) + \mathrm{tr}_{g_0} L_0 > \mathrm{tr}_{g_0} L_0,$$

which gives us $\mathrm{tr}_{\tilde{g}_0} \tilde{L}_0 + \mathrm{tr}_{g_1} L_1 > 0$, since by assumption $\mathrm{tr}_{g_0} L_0 + \mathrm{tr}_{g_1} L_1 = \mathrm{tr}_g L \geq 0$ on Γ. Moreover, by construction the new metric $\tilde{g}_0$ is C^2 close to g_0, and thus their scalar curvatures differ only by an error term ε coming from the first two derivatives of φ, which we may choose arbitrary small. We then may replace g_0 by $\tilde{g}_0$ and proceed as in the $\mathrm{tr}_g L > 0$ case.

□

Remark: In [17] P. Miao generalized the positive mass theorem [23] (which says that an asymptotically flat manifold with nonnegative scalar curvature has nonnegative

ADM mass), to metrics which fail to be C^1 across a hypersurface Σ. One of the essential steps of his proof was to smoothen the metric across Σ in such a way that the scalar curvature stays bounded from below by a constant (cf. [17], Proposition 3.1). Theorem 1.7.2 provides a slightly better approximation, since in our case the smooth metrics have scalar curvature $\geq -\varepsilon$.

1.7.3. Manifolds with bi-curvature $\geq \kappa$

The bi-curvature $\mathrm{bi}(g)$ of a C^2 smooth Riemannian metric g is defined as the sum of the two smallest eigenvalues of $\mathcal{R}(g)$. The condition $\mathrm{bi}(g) \geq 0$ is also referred to as '2-nonnegative curvature operator'. Note that $\mathrm{bi}(g) \geq \kappa$ holds on M iff

$$\mathcal{R}(g)(\alpha, \alpha) + \mathcal{R}(g)(\beta, \beta) \geq \kappa$$

for all $\alpha, \beta \in \Lambda^2(TM)$ which are orthonormal with respect to g.

Theorem 1.7.3. *Let $M = M_0 \cup_\phi M_1$, $g = g_0 \cup_\phi g_1$, $\Gamma = \Gamma_0 =_\phi \Gamma_1$, and $L = L_0 + L_1$ be as in Theorem 1.1.2. Suppose that* $\mathrm{bi}(g_0)$ *and* $\mathrm{bi}(g_1)$ *are at least κ. If L is positive semidefinite, then* $\mathrm{bi}(g) \geq \kappa$ *(in a similar sense as in Definition 1.1.1).*

Proof. We proceed as in the previous section and show

(a) The modified metric g_δ on M_0 satisfies $\mathrm{bi}(g_\delta) \geq \kappa - \varepsilon(\delta)$, where $\varepsilon(\delta) \to 0$ as $\delta \to 0$.

(b) By mollifying $g_{(\delta)} = g_\delta \cup_\phi g_1$ we construct smooth metrics which approximate g in the C^0 sense and whose bi-curvature is at least $\kappa - \varepsilon(\delta)$.

As mentioned above, (a) holds iff

$$\mathcal{R}_\delta(\alpha_\delta, \alpha_\delta) + \mathcal{R}_\delta(\beta_\delta, \beta_\delta) \geq \kappa - \varepsilon(\delta) \tag{1.7.11}$$

for all α_δ, β_δ satisfying $\|\alpha_\delta\|_\delta$, $\|\beta_\delta\|_\delta = 1$ and $\langle \alpha_\delta, \beta_\delta \rangle_\delta = 0$ (where $\langle \cdot, \cdot \rangle_\delta = \mathcal{I}(g_\delta)$). In what follows, we will call such two vectors g_δ-orthonormal. Proposition 1.3.1 implies

$$\begin{aligned}
& \mathcal{R}_\delta(\alpha_\delta, \alpha_\delta) + \mathcal{R}_\delta(\beta_\delta, \beta_\delta) \\
= \; & \mathcal{R}(\alpha_\delta, \alpha_\delta) + \mathcal{R}(\beta_\delta, \beta_\delta) - f_\delta^2\big(\mathcal{A}(\alpha_\delta, \alpha_\delta) + \mathcal{A}(\beta_\delta, \beta_\delta)\big) + f_\delta\big(\mathcal{B}(\alpha_\delta, \alpha_\delta) + \mathcal{B}(\beta_\delta, \beta_\delta)\big) \\
- \; & 2f_\delta'\big(\mathcal{L}(\alpha_\delta, \alpha_\delta) + \mathcal{L}(\beta_\delta, \beta_\delta)\big) + 2f_\delta^2\big(\mathcal{L}^2(\alpha_\delta, \alpha_\delta) + \mathcal{L}^2(\beta_\delta, \beta_\delta)\big) \\
+ \; & 2Cf_\delta\big(\hat{\mathcal{I}}(\alpha_\delta, \alpha_\delta) + \hat{\mathcal{I}}(\beta_\delta, \beta_\delta)\big) + \big(\mathcal{E}(\delta)(\alpha_\delta, \alpha_\delta) + \mathcal{E}(\delta)(\beta_\delta, \beta_\delta)\big),
\end{aligned}$$

where $\mathcal{E}(\delta)$ is an operator whose eigenvalues tend to zero uniformly on M_0. Since $g_\delta \to g_0$ uniformly on M_0, for small enough δ any g_δ-orthonormal forms α_δ and β_δ are uniformly bounded with respect to g_0 by some fixed constant. Thus, we can estimate the $\mathcal{E}(\delta)$ terms from below by $-\varepsilon(\delta)$. $\mathcal{L}$ is positive semidefinite and bounded near Γ, and f_δ' does not exceed δ. Therefore, $-2f_\delta'\big(\mathcal{L}(\alpha_\delta, \alpha_\delta) + \mathcal{L}(\beta_\delta, \beta_\delta)\big) \geq -\varepsilon(\delta)$. Finally, the $\mathcal{L}^2$ terms are nonnegative, so we arrive at

$$\begin{aligned}
& \mathcal{R}_\delta(\alpha_\delta, \alpha_\delta) + \mathcal{R}_\delta(\beta_\delta, \beta_\delta) \\
\geq \; & \mathcal{R}(\alpha_\delta, \alpha_\delta) + \mathcal{R}(\beta_\delta, \beta_\delta) - f_\delta^2\big(\mathcal{A}(\alpha_\delta, \alpha_\delta) + \mathcal{A}(\beta_\delta, \beta_\delta)\big) + f_\delta\big(\mathcal{B}(\alpha_\delta, \alpha_\delta) + \mathcal{B}(\beta_\delta, \beta_\delta)\big) \\
+ \; & Cf_\delta\big(\hat{\mathcal{I}}(\alpha_\delta, \alpha_\delta) + \hat{\mathcal{I}}(\beta_\delta, \beta_\delta)\big) - \varepsilon(\delta).
\end{aligned} \tag{1.7.12}$$

By applying the Gram-Schmidt process to α_δ and β_δ and putting

$$\tilde{\alpha}_\delta := \frac{\alpha_\delta}{\|\alpha_\delta\|_0}$$

and

$$\tilde{\beta}_\delta := \frac{\beta_\delta - \langle\tilde{\alpha}_\delta, \beta_\delta\rangle_0 \tilde{\alpha}_\delta}{\|\beta_\delta - \langle\tilde{\alpha}_\delta, \beta_\delta\rangle_0 \tilde{\alpha}_\delta\|_0}$$

we obtain g_0-orthonormal two-vectors $\tilde{\alpha}_\delta$, $\tilde{\beta}_\delta$ satisfying

$$\|\tilde{\alpha}_\delta - \alpha_\delta\|_0,\ \|\tilde{\beta}_\delta - \beta_\delta\|_0 \le \varepsilon(\delta)$$

independently of the initial two-vectors α_δ, β_δ. Since f_δ and all operators on the right hand side of (1.7.12) are uniformly bounded near Γ, we may replace α_δ, β_δ by $\tilde{\alpha}_\delta$, $\tilde{\beta}_\delta$ and the inequality still holds up to $-\varepsilon(\delta)$, that is,

$$\begin{aligned}
& \mathcal{R}_\delta(\alpha_\delta, \alpha_\delta) + \mathcal{R}_\delta(\beta_\delta, \beta_\delta) \\
\ge\ & \mathcal{R}(\tilde{\alpha}_\delta, \tilde{\alpha}_\delta) + \mathcal{R}(\tilde{\beta}_\delta, \tilde{\beta}_\delta) - f_\delta^2\big(\mathcal{A}(\tilde{\alpha}_\delta, \tilde{\alpha}_\delta) + \mathcal{A}(\tilde{\beta}_\delta, \tilde{\beta}_\delta)\big) + f_\delta\big(\mathcal{B}(\tilde{\alpha}_\delta, \tilde{\alpha}_\delta) + \mathcal{B}(\tilde{\beta}_\delta, \tilde{\beta}_\delta)\big) \\
+\ & 2Cf_\delta\big(\hat{\mathcal{I}}(\tilde{\alpha}_\delta, \tilde{\alpha}_\delta) + \hat{\mathcal{I}}(\tilde{\beta}_\delta, \tilde{\beta}_\delta)\big) - \varepsilon(\delta).
\end{aligned}$$

By construction, $\tilde{\alpha}_\delta$ and $\tilde{\beta}_\delta$ are g_0-orthonormal on M_0 and g_1-orthonormal on Γ (recall that $g_0 = g_1$ on Γ). By adopting the argument from Lemma 1.5.1 we obtain

$$\begin{aligned}
& \mathcal{R}_\delta(\alpha_\delta, \alpha_\delta) + \mathcal{R}_\delta(\beta_\delta, \beta_\delta) \\
\ge\ & \kappa + 2f_\delta\big[\big(-\mathcal{L}^2 + \frac{1}{2}\nabla_N^2\mathcal{G}_1 + C\hat{\mathcal{I}}\big)(\tilde{\alpha}_\delta, \tilde{\alpha}_\delta) + \big(-\mathcal{L}^2 + \frac{1}{2}\nabla_N^2\mathcal{G}_1 + C\hat{\mathcal{I}}\big)(\tilde{\beta}_\delta, \tilde{\beta}_\delta)\big] \\
-\ & \varepsilon(\delta).
\end{aligned}$$

Since $-\mathcal{L}^2 + \frac{1}{2}\nabla_N^2\mathcal{G}_1 + C\hat{\mathcal{I}}$ is positive semidefinite for large enough fixed C and uniformly bounded near Γ (cf. proof of Lemma 1.5.2), (a) follows.

(b) Let us fix $\delta > 0$ and define the mollified metric $g^h_{(\delta)}$ in the same way as in Section 1.6. Our goal is to show

$$\mathcal{R}(g^h_{(\delta)})(\alpha, \alpha) + \mathcal{R}(g^h_{(\delta)})(\beta, \beta) \ge \kappa - \varepsilon(\delta) \tag{1.7.13}$$

for all $g^h_{(\delta)}$-orthonormal α, β. The computations in Section 1.6 were carried out for two-vectors with constant coefficients, which we no longer can assume for orthonormal two-vectors.

Using (1.6.3) we obtain

$$\begin{aligned}
& \mathcal{R}(g^h_{(\delta)})(\alpha, \alpha) + \mathcal{R}(g^h_{(\delta)})(\beta, \beta) \\
\ge\ & (\mathcal{R}(g_{(\delta)}))^h(\alpha, \alpha) + (\mathcal{R}(g_{(\delta)}))^h(\beta, \beta) - \tilde{\varepsilon}(\delta, h)\big(\|\alpha\|^2_{(\delta)} + \|\beta\|^2_{(\delta)}\big),
\end{aligned}$$

where $\tilde{\varepsilon}(\delta, h) \to 0$ as $h \to 0$ for every fixed δ. Since α and β have unit length with respect to $g^h_{(\delta)}$ and $g^h_{(\delta)} \overset{h\to 0}{\to} g_{(\delta)}$, we can estimate the last term on the right hand side from below by $-\varepsilon(\delta)$ for small enough h. Thus, (1.7.13) follows if we show

$$(\mathcal{R}(g_{(\delta)}))^h(\alpha, \alpha) + (\mathcal{R}(g_{(\delta)}))^h(\beta, \beta) \ge \kappa - \tilde{\varepsilon}(\delta) \tag{1.7.14}$$

for small enough h and all $g^h_{(\delta)}$-orthonormal α, β. Let us fix a point $x \in M$ and some $g^h_{(\delta)}(x)$-orthonormal α, $\beta \in \Lambda^2(T_xM)$. Recall that in Section 1.6 we mollified $g_{(\delta)}$

and $R(g_{(\delta)})$ only on a small neighborhood of Γ which was covered by finitely many coordinate neighborhoods $U_1,\ldots,U_N$. Off this neighborhood g coincides with $g_{(\delta)}$ and we have

$$(\mathcal{R}(g_{(\delta)}))^h(\alpha,\alpha)+(\mathcal{R}(g_{(\delta)}))^h(\beta,\beta)=\mathcal{R}(g)(\alpha,\alpha)+\mathcal{R}(g)(\beta,\beta)\geq\kappa$$

by assumption. Thus, w.l.o.g. we may assume that $x\notin\bigcup_{s>N}U_s$. For such x we have

$$\begin{aligned}&(\mathcal{R}(g_{(\delta)}))^h(x)(\alpha,\alpha)+(\mathcal{R}(g_{(\delta)}))^h(x)(\beta,\beta) \\ =\;&\sum_{s=1}^N\eta_s(x)\int_{|z|\leq1}\rho(z)(R(g_{(\delta)}))^s_{ijkl}(x-hz)(\alpha_s^{ij}\alpha_s^{kl}+\beta_s^{ij}\beta_s^{kl})dz,\end{aligned}\tag{1.7.15}$$

where the coefficients refer to the charts (U_s',φ_s). We now extend α, β to U_s' in such a way that the extensions are $g^h_{(\delta)}$-orthonormal: We define two-vectors α_s, β_s on U_s', $s=1,\ldots,N$, by putting $\alpha_s^{ij}(y):=\alpha_s^{ij}$ and $\beta_s^{ij}(y):=\beta_s^{ij}$ (here we only have to consider the neighborhoods U_s containing x). Using the Gram-Schmidt process we obtain $g^h_{(\delta)}$-orthonormal two-vectors

$$\tilde{\alpha}_s=\frac{\alpha_s}{\|\alpha_s\|_{g^h_{(\delta)}}}$$

and

$$\tilde{\beta}_s=\frac{\beta_s-\langle\tilde{\alpha}_s,\beta_s\rangle_{g^h_{(\delta)}}\tilde{\alpha}_s}{\|\beta_s-\langle\tilde{\alpha}_s,\beta_s\rangle_{g^h_{(\delta)}}\tilde{\alpha}_s\|_{g^h_{(\delta)}}}.$$

By the mean value theorem the right hand side of (1.7.15) equals to

$$\begin{aligned}&\sum_{s=1}^N\eta_s(x)\int_{|z|\leq1}\rho(z)(R(g_{(\delta)}))^s_{ijkl}(x-hz) \\ &\qquad\cdot\big(\tilde{\alpha}_s^{ij}(x-hz)\tilde{\alpha}_s^{kl}(x-hz)+\tilde{\beta}_s^{ij}(x-hz)\tilde{\beta}_s^{kl}(x-hz)\big)dz \\ +\;&h\sum_{s=1}^N\eta_s(x)\int_{|z|\leq1}\rho(z)(R(g_{(\delta)}))^s_{ijkl}(x-hz)D\big(\tilde{\alpha}_s^{ij}\tilde{\alpha}_s^{kl}+\tilde{\beta}_s^{ij}\tilde{\beta}_s^{kl}\big)(\xi^s_{x,hz})z\,dz,\end{aligned}\tag{1.7.16}$$

where $\xi^s_{x,hz}=(1-t)x+thz$ for some $t\in[0,1]$. Now we apply the Gram-Schmidt process with respect to $g_{(\delta)}$ to the two-vectors $\tilde{\alpha}_s$ and $\tilde{\beta}_s$, and construct $g_{(\delta)}$-orthonormal $\tilde{\tilde{\alpha}}_s,\tilde{\tilde{\beta}}_s$. The first sum in (1.7.16) is estimated from below by

$$\begin{aligned}&\sum_{s=1}^N\eta_s(x)\int_{|z|\leq1}\rho(z)(R(g_{(\delta)}))^s_{jikl}(x-hz) \\ &\qquad\cdot\big(\tilde{\tilde{\alpha}}_s^{ij}(x-hz)\tilde{\tilde{\alpha}}_s^{kl}(x-hz)+\tilde{\tilde{\beta}}_s^{ij}(x-hz)\tilde{\tilde{\beta}}_s^{kl}(x-hz)\big)dz \\ -\;&\varepsilon(\delta,h),\end{aligned}\tag{1.7.17}$$

where

$$\varepsilon(\delta,h)\leq c(n)\|(R(g_{(\delta)}))^s_{ijkl}\|_{L^\infty(U_s')}\|(g_{(\delta)}-g^h_{(\delta)})^s_{ij}\|_{C^0(U_s')}\overset{h\to0}{\to}0$$

for every fixed δ. Moreover, the integrand in (1.7.17) is bounded from below by $\kappa-\varepsilon(\delta)$ in view of (a). Finally, the second integrand in (1.7.16) is bounded by

$$c(n)\|(R(g_{(\delta)}))^s_{ijkl}\|_{L^\infty(U_s')}\|(g_{(\delta)}-g^h_{(\delta)})^s_{ij}\|_{C^1(U_s')},$$

and thus the second expression in (1.7.16) tends to zero uniformly as $h\to0$. For small enough h inequality (1.7.14) follows with $\tilde{\varepsilon}(\delta)=2\varepsilon(\delta)$, and we are done. □

1.7.4. Manifolds with isotropic curvature $\geq \kappa$

Given a smooth Riemannian manifold (M, g), $\dim(M) \geq 4$, we consider the complexification of its tangent bundle $\mathbb{C} \otimes_{\mathbb{R}} TM$ and the complex-linear extensions of the inner product g and the Riemannian curvature tensor R. A complex isotropic two-plane is spanned by two vectors $Z = X + iY$ and $W = U + iV$, where $X, Y, U, V \in TM$ are orthonormal with respect to g. The isotropic curvature of such a two-plane P is given by

$$K(P) = R(Z, W, \bar{Z}, \bar{W}).$$

Using the Bianchi identity

$$R(X, Y, U, V) + R(X, V, Y, U) + R(X, U, V, Y) = 0,$$

one easily verifies

$$K(P) = R(X,U,X,U)+R(X,V,X,V)+R(Y,U,Y,U)+R(Y,V,Y,V)-2R(X,Y,U,V).$$

Given an isotropic two-plane P spanned by $X + iY$ and $U + iV$, one computes using the Bianchi identity

$$K(P) = \mathcal{R}(\alpha, \alpha) + \mathcal{R}(\beta, \beta), \tag{1.7.18}$$

where $\alpha = X \wedge U + V \wedge Y$ and $\beta = X \wedge V + Y \wedge U$. We say that a Riemannian manifold (M, g) has isotropic curvature $\geq \kappa$, if $K(P) \geq \kappa$ holds for all isotropic two-planes of (M, g). Furthermore, we say that (M, g) has 1-isotropic (2-isotropic) curvature $\geq \kappa$ if $(M \times \mathbb{R}, g \oplus dr^2)$ $((M \times \mathbb{R}^2, g \oplus dr^2 \oplus dr^2))$ has isotropic curvature $\geq \kappa$, where dr^2 denotes the standard metric on $\mathbb{R}$.

Theorem 1.7.4. *Let $M = M_0 \cup_\phi M_1$, $g = g_0 \cup_\phi g_1$, $\Gamma = \Gamma_0 =_\phi \Gamma_1$, and $L = L_0 + L_1$ be as in Theorem 1.1.2. Suppose that the isotropic (1-isotropic, 2-isotropic) curvatures of g_0 and g_1 are at least κ. If L is positive semidefinite, then the isotropic (1-isotropic, 2-isotropic) curvatures of g is at least κ (in a similar sense as in Definition 1.1.1).*

Proof. In view of (1.7.18), the proof for the isotropic case is similar as in the previous section.

For the 1-isotropic case, let us examine the manifold resulting from gluing $M_1 \times \mathbb{R}$ and $M_2 \times \mathbb{R}$ along their boundaries. The boundary of M_i, $i = 1, 2$, is given by $\Gamma_i \times \mathbb{R}$. If $\phi : \Gamma_1 \to \Gamma_2$ is some isometry of Γ_1, Γ_2 with respect to g_1, g_2, then

$$\begin{aligned} \tilde{\phi} : \Gamma_1 \times \mathbb{R} &\to \Gamma_2 \times \mathbb{R} \\ (x, s) &\mapsto (\phi(x), s) \end{aligned}$$

is an isometry of $\Gamma_1 \times \mathbb{R}$ and $\Gamma_2 \times \mathbb{R}$ with respect to $g_0 \oplus dr^2$, $g_1 \oplus dr^2$. One easily verifies that

$$(M_1 \times \mathbb{R}) \cup_{\tilde{\phi}} (M_2 \times \mathbb{R}) = (M_1 \cup_\phi M_2) \times \mathbb{R}$$

and

$$(g \oplus dr^2)|_{M_i \times \mathbb{R}} = g|_{M_i} \oplus dr^2.$$

The inward normal on $\Gamma_i \times \mathbb{R}$ with respect to $g_i \oplus dr^2$ is given by $(N_i, 0)$, where N_i is the inward normal on Γ_i with respect to g_i. The second fundamental forms of $\Gamma_i \times \mathbb{R}$ are $L_i \oplus 0$, and therefore their sum is positive semidefinite. We repeat the constructions from Section 1.3 and define the modified metric $(g_0 \oplus dr^2)_\delta = g_\delta \oplus dr^2$ on $M_0 \times \mathbb{R}$. Even though $\Gamma \times \mathbb{R}$ fails to be compact, we may nevertheless proceed as in the isotropic case, since any operator $\mathcal{T}$ on $M_i \times \mathbb{R}$ to occur in the proof satisfies $\sup_{(x,s)\in\Gamma\times\mathbb{R}} |\mathcal{T}(x,s)|_{g\oplus dr^2(x,s)} = \sup_{x\in\Gamma} |\mathcal{T}(x,0)|_{g(x)}$, and therefore is bounded near $\Gamma \times \mathbb{R}$ due to the compactness of Γ. Moreover, any finite covering $(\varphi_i, U_i)_{i=1,\ldots,N}$ of Γ gives us a finite covering $((\varphi_i, \mathrm{id}_\mathbb{R}), U_i \times \mathbb{R})_{i=1,\ldots,N}$ of $\Gamma \times \mathbb{R}$. The desired smooth metric on $M \times \mathbb{R}$, which approximates $g \oplus dr^2$ and has isotropic curvature $\geq \kappa - \varepsilon(\delta)$, is then given by $g_{(\delta)} \oplus dr^2$.

The argument in the 2-isotropic case is similar.

$\square$

Chapter 2.

Preserving lower bounds on curvature operators under the Ricci flow

2.1. Introduction and preliminaries

In this chapter, we study the evolution of some of the operators from Chapter 1 under the Ricci flow. It is well known (see [12], [13], [8], [5]) that these operators remain nonnegative (in the Ricci curvature case in dimension 3) under the Ricci flow, if they are nonnegative at the initial time $t = 0$. We show that, under the additional assumption that the scalar curvature of the evolving metric satisfies a bound of the form $|S(t)| \leq C/t$ for $t > 0$, an arbitrary initial lower bound $-\varepsilon_0$ does not become too negative on a well controlled time interval, which essentially depends on C and ε_0.

A similar result was proved by T. Richard [22], where he considered Ricci flow invariant cones $\mathcal{C}$, such that $\mathcal{C}$ contains the cone of nonnegative operators, and is contained in in the cone of operators with nonnegative Ricci curvature. He showed that if the curvature operator $\mathcal{R}$ satisfies $\mathcal{R} + \varepsilon_0 \mathcal{I} \in \mathcal{C}$ at the initial time $t = 0$, then $\mathcal{R} + \kappa\varepsilon_0 \mathcal{I} \in \mathcal{C}$ on a well controlled time interval. Examples of invariant cones which fit into this framework include the cone of nonnegative curvature operators, nonnegative bi-curvature (2-nonnegative operators), and nonnegative 1- and 2-isotropic curvature. In certain cases, the method of the proof can still be applied if $\mathcal{C}$ is not included in the cone of operators with nonnegative Ricci curvature, which for instance is the case for nonnegative isotropic curvature. Here, we give an explicit proof for the Riemannian curvature case and the isotropic curvature case.

First, let us introduce some notation and background material. For a detailed discussion we refer to [5], [29], or [9]. A one-parameter family of Riemannian metrics $g(t)$, $t \in [0, T)$, on a manifold M is called a solution to the Ricci flow, if

$$\frac{\partial}{\partial t} g(t) = -2\mathrm{Ric}(g(t)).$$

The Riemannian curvature tensor $R = R(g(t))$ of such a solution satisfies the evolution equation

$$\begin{aligned} \partial_t R(X,Y,Z,W) &= \Delta R(X,Y,Z,W) + Q(R)(X,Y,Z,W) \qquad (2.1.1)\\ &- \sum_{k=1}^{n} \mathrm{Ric}(X,e_k)R(e_k,Y,Z,W) - \sum_{k=1}^{n} \mathrm{Ric}(Y,e_k)R(X,e_k,Z,W)\\ &- \sum_{k=1}^{n} \mathrm{Ric}(Z,e_k)R(X,Y,e_k,W) - \sum_{k=1}^{n} \mathrm{Ric}(W,e_k)R(X,Y,Z,e_k) \end{aligned}$$

for all $X, Y, Z, W \in TM$, where $e_1, \dots, e_n$ is a choice of an orthonormal basis (a definition of $Q(R)$ is given below). In what follows, we work with the corresponding operator $\mathcal{R}(g(t))$, rather than the $(4,0)$-tensor $R(g(t))$.

Using moving frames, one can simplify the above evolution equation, omitting the Ric terms (which is known as Uhlenbeck's trick). The following exposition is from from S. Brendle's book [5], Chapter 2. Consider the pullback-bundle E of TM under the projection $\pi : M \times (0,T) \to M$. The fiber of E over a point $(p,t) \in M \times (0,T)$ is given by $E_{(p,t)} = T_pM$. One extends the Levi-Civita connection on TM to a connection on E by defining the covariant time derivative

$$D_{\frac{\partial}{\partial t}} X = \partial_t X - \sum_{i=1}^{n} \mathrm{Ric}(X, e_i) e_i,$$

where $e_1, \dots, e_n$ is an orthonormal frame with respect to $g(t)$. The connection D is compatible with the metric g in the sense that $(D_{\frac{\partial}{\partial t}} g)(X,Y) = 0$ for all sections X, Y. Using $D_{\frac{\partial}{\partial t}}$ instead of $\frac{\partial}{\partial t}$, the evolution equation (2.1.1) simplifies to

$$D_{\frac{\partial}{\partial t}} \mathcal{R} = \Delta \mathcal{R} + \mathcal{Q}(\mathcal{R}). \tag{2.1.2}$$

Here the operator $\mathcal{Q}(\mathcal{R})$ is defined as follows: Let $\eta_1, \dots, \eta_N$, $N = \frac{1}{2}(n-1)n$ be an orthonormal basis of $\Lambda^2(E_{(p,t)})$. Then

$$\mathcal{Q}(\mathcal{R}) = 2(\mathcal{R}^2 + \mathcal{R}^\#),$$

where

$$\mathcal{R}^2(\alpha, \beta) = \sum_{a,b=1}^{N} \mathcal{R}(\alpha, \eta_a) \mathcal{R}(\beta, \eta_b),$$

and

$$\mathcal{R}^\# = \mathcal{R} \# \mathcal{R},$$

where

$$(\mathcal{R} \# \mathcal{S})(\alpha, \beta) = \frac{1}{2} \sum_{a,b=1}^{N} \langle [\mathcal{R}(\eta_a), \mathcal{S}(\eta_b)], \alpha \rangle \cdot \langle [\eta_a, \eta_b], \beta \rangle.$$

(Here we used the notation from [3]: in the expression on the right hand side, we regard 2-vectors as elements of the Lie algebra $\mathfrak{so}(n)$, and $\langle \cdot, \cdot \rangle$ referes to the inner product on $\mathfrak{so}(n)$ given by $\langle \alpha, \beta \rangle = -\frac{1}{2} \mathrm{tr}(\alpha\beta)$.)

The key ingredient when showing that a certain curvature condition is preserved under the Ricci flow is Hamilton's maximum principle for systems (cf. [13] Thm. 4.3, or [29] Thm 9.6.1), which says the following: Let M be a compact manifold, equipped with a time dependent metric g, and let V be a vector bundle over M with a fixed metric h. Furthermore, let A be a time dependent connection on V, which is compatible with h. The Laplacian on V is formed using the the connection A and the Levi-Civita connection on TM. Suppose that a section $f \in \Gamma(V)$ satisfies the PDE

$$\frac{\partial f}{\partial t} = \Delta f + \phi(f),$$

where $\phi(f)$ is a smooth vector field on V, which is tangent to the fibers. Let X be a closed convex subset of V, which is convex in each fiber, and invariant under parallel translation with respect to $A(t)$ for each fixed t. Then a solution of the above PDE, whose initial value lies in X, remains in X, if the solutions to the ODE

$$\frac{df}{dt} = \phi(f)$$

in each fiber remain in X. The latter is the case if and only if $\phi(f)$ lies in the tangent cone of X at f for all $f \in \partial X$ (see Lemma 4.1 of [13]).

2.2. Riemannian curvature bounded from below

Proposition 2.2.1. *Let M be a smooth compact manifold, and $g(t)$, $t \in [0,T)$ a solution to the Ricci flow on M satisfying*

- *$\mathcal{R}(g(0)) \geq -\varepsilon_0$ for some $\varepsilon_0 > 0$*
- *$|S(t)| \leq C/t$ for all $t \in (0,T)$ and some $0 \leq C < \frac{1}{4}$.*

Then there exists $\tilde{T} = \tilde{T}(C, \varepsilon_0, n) > 0$ and $\kappa = \kappa(C) \geq 0$ such that $\mathcal{R}(g(t)) \geq -\kappa\varepsilon_0$ on $[0, \min\{T, \tilde{T}\})$.

Proof. Let

$$\varepsilon(t) = \varepsilon_1 tS + \varepsilon_0(1 + kt),$$

where $\varepsilon_1, k > 0$ are some positive constants we specify later in the proof, and let

$$\mathcal{M} = \mathcal{R} + \varepsilon(t)\mathcal{I}.$$

Let $S^2_B(\Lambda^2 E)$ be the bundle of algebraic curvature operators over $M \times [0,T)$, and let

$$(\mathcal{C}_{\geq 0})_{(p,t)} = \{\mathcal{T} \in S^2_B(\Lambda^2 E_{(p,t)}) \,|\, \mathcal{T} \geq 0\} \subset S^2_B(\Lambda^2 E_{(p,t)})$$

for $(p,t) \in M \times [0,T)$. At $t = 0$ we have

$$\mathcal{M}(p,0) = \mathcal{R}(p,0) + \varepsilon(0)\mathcal{I} = \mathcal{R}(0) + \varepsilon_0\mathcal{I} > 0$$

for all $p \in M$ by assumption, so $\mathcal{M}(p,0) \in (\mathcal{C}_{\geq 0})_{(p,0)}$ for all $p \in M$.

Recall that the evolution equation of the scalar curvature under the Ricci flow is given by

$$(\partial_t - \Delta)S = 2|\mathrm{Ric}|^2.$$

Thus

$$(\partial_t - \Delta)\varepsilon(t) = \varepsilon_1 S + 2\varepsilon_1 t|\mathrm{Ric}|^2 + k\varepsilon_0. \tag{2.2.1}$$

In view of (2.1.2) we then have

$$(D_{\frac{\partial}{\partial t}} - \Delta)\mathcal{M} \;=\; 2(\mathcal{R}^2 + \mathcal{R}^{\#}) + (\varepsilon_1 S + 2\varepsilon_1 t|\mathrm{Ric}|^2 + k\varepsilon_0)\mathcal{I}. \tag{2.2.2}$$

Furthermore, observe that

$$\begin{aligned}
\mathcal{M}^2 + \mathcal{M}^{\#} &= \big(\mathcal{R} + \varepsilon(t)\mathcal{I}\big)^2 + \big(\mathcal{R} + \varepsilon(t)\mathcal{I}\big)^{\#} \\
&= \mathcal{R}^2 + 2\varepsilon(t)\mathcal{R} + \varepsilon(t)^2\mathcal{I} + \mathcal{R}^{\#} + 2\varepsilon(t)\mathcal{R}\#\mathcal{I} + \varepsilon(t)^2\mathcal{I}\#\mathcal{I} \\
&= \mathcal{R}^2 + \mathcal{R}^{\#} + 2\varepsilon(t)\big(\mathcal{R} + \mathcal{R}\#\mathcal{I}\big) + \varepsilon(t)^2\big(\mathcal{I} + \mathcal{I}\#\mathcal{I}\big) \\
&= \mathcal{R}^2 + \mathcal{R}^{\#} + 2\varepsilon(t)\mathrm{Ric} \wedge \mathrm{id} + \varepsilon(t)^2\mathrm{Ric}(\mathcal{I}) \wedge \mathrm{id} \\
&= \mathcal{R}^2 + \mathcal{R}^{\#} + 2\varepsilon(t)\mathrm{Ric} \wedge \mathrm{id} + \varepsilon(t)^2(n-1)\mathcal{I},
\end{aligned} \tag{2.2.3}$$

where we used that $\mathcal{T} + \mathcal{T}\#\mathcal{I} = \mathrm{Ric}(\mathcal{T}) \wedge \mathrm{id}$ for all $\mathcal{T} \in S^2_B(\Lambda^2 E)$ (cf. [3] Lemma 2.1), and $\mathrm{Ric}(\mathcal{I}) \wedge \mathrm{id} = (n-1)\mathrm{id} \wedge \mathrm{id} = (n-1)\mathcal{I}$. Combining (2.2.2) and (2.2.3) gives

$$\begin{aligned}
(D_{\frac{\partial}{\partial t}} - \Delta)\mathcal{M} &= 2\big(\mathcal{M}^2 + \mathcal{M}^{\#}\big) \\
&- 4\varepsilon(t)\mathrm{Ric} \wedge \mathrm{id} + \big(-2(n-1)\varepsilon(t)^2 + \varepsilon_1 S + 2\varepsilon_1 t|\mathrm{Ric}|^2 + k\varepsilon_0\big)\mathcal{I},
\end{aligned}$$

and the corresponding ODE in the fiber is

$$\begin{aligned} \frac{d}{dt}\mathcal{M} &= 2\big(\mathcal{M}^2 + \mathcal{M}^{\#}\big) \\ &- 4\varepsilon(t)\mathrm{Ric} \wedge \mathrm{id} + \big(- 2(n-1)\varepsilon(t)^2 + \varepsilon_1 S + 2\varepsilon_1 t|\mathrm{Ric}|^2 + k\varepsilon_0\big)\mathcal{I}. \end{aligned}$$

Suppose that $\mathcal{M}(p,t) \in \partial(\mathcal{C}_{\geq 0})_{(p,t)}$ at $t \in (0,T)$. Then $(\mathcal{M}^2+\mathcal{M}^{\#})(p,t) \in (\mathcal{C}_{\geq 0})_{(p,t)}$ by [13]. Thus, in order to show that $\frac{d}{dt}\mathcal{M}(p,t) \in (\mathcal{C}_{\geq 0})_{(p,t)}$ it suffices to verify that

$$\mathcal{N} := -4\varepsilon(t)\mathrm{Ric} \wedge \mathrm{id} + \big(- 2(n-1)\varepsilon(t)^2 + \varepsilon_1 S + 2\varepsilon_1 t|\mathrm{Ric}|^2 + k\varepsilon_0\big)\mathcal{I}$$

lies in $(\mathcal{C}_{\geq 0})_{(p,t)}$ at (p,t). First, observe that the assumption $|S| \leq C/t$ for $t > 0$ implies

$$|\varepsilon(t)| = |\varepsilon_1 t S + \varepsilon_0(1+kt)| \leq \varepsilon_1 C + 2\varepsilon_0 =: K \tag{2.2.4}$$

if $t \leq 1/k$. Furthermore, $\mathcal{M}(p,t) \geq 0$ yields

$$0 \leq \mathrm{tr}_{24}\mathcal{M} = \mathrm{Ric} + \varepsilon(t)(n-1)\,\mathrm{id}$$

at (p,t) (see Lemma B.3.2), so that in view of (2.2.4) we have

$$\mathrm{Ric} \geq -K(n-1)\,\mathrm{id}, \tag{2.2.5}$$

and taking the trace gives

$$S \geq -K(n-1)n$$

so that we also have

$$|S| \leq S + 2K(n-1)n \tag{2.2.6}$$

at (p,t). Moreover, (2.2.5) implies

$$\mathrm{Ric} \leq (S + K(n-1)^2)\,\mathrm{id}. \tag{2.2.7}$$

Indeed, since Ric has eigenvalues $\lambda_i \geq -K(n-1)$, we have

$$\lambda_i = \sum_{k=1}^{n} \lambda_i - \sum_{k \neq i} \lambda_i \leq S + K(n-1)^2.$$

Combining (2.2.4) and (2.2.7) yields

$$\begin{aligned} -4\varepsilon(t)\mathrm{Ric} \wedge \mathrm{id} &\geq -4K\big(|S| + K(n-1)^2\big)\,\mathrm{id} \wedge \mathrm{id} \\ &= -4K|S| - 4K^2(n-1)^2, \end{aligned} \tag{2.2.8}$$

where we suppressed $\mathrm{id} \wedge \mathrm{id} = \mathcal{I}$ on the right hand side. Using (2.2.8), (2.2.6) and (2.2.4) we now estimate $\mathcal{N}$ from below by

$$\begin{aligned} \mathcal{N} \geq\ & (-4K + \varepsilon_1)|S| \\ & -4K^2(n-1)^2 - 2(n-1)K^2 - 2\varepsilon_1 K(n-1)n + k\varepsilon_0. \end{aligned} \tag{2.2.9}$$

Recall that $K = \varepsilon_1 C + 2\varepsilon_0$, and $C < \frac{1}{4}$ by assumption, so that

$$\begin{aligned} -4K + \varepsilon_1 &= -4\varepsilon_1 C - 8\varepsilon_0 + \varepsilon_1 \\ &= \varepsilon_1(1-4C) - 8\varepsilon_0 = 0 \end{aligned}$$

for $\varepsilon_1 = \varepsilon_1(C, \varepsilon_0) := \frac{8\varepsilon_0}{1-4C}$, and the second line on the right hand side of (2.2.9) becomes nonnegative if we choose $k = k(C, \varepsilon_0, n)$ large enough.

Recall that in the above computation we only assumed $t \leq \frac{1}{k}$. Thus, we have shown that $\frac{d}{dt}\mathcal{M}(p,t)$ lies in $(\mathcal{C}_{\geq 0})_{(p,t)}$ if $t \leq \min(T, \frac{1}{k})$. By putting $\tilde{T} = \tilde{T}(C, \varepsilon_0, n) = \frac{1}{k}$ we have

$$\mathcal{M} = \mathcal{R} + \varepsilon(t)\mathcal{I} \geq 0$$

on $[0, \tilde{T})$, which in view of (2.2.4) implies

$$\mathcal{R} \geq -\varepsilon_1 C - 2\varepsilon_0 = -\frac{8\varepsilon_0 C}{1-4C} - 2\varepsilon_0 = -\kappa\varepsilon_0$$

on $[0, \tilde{T})$, where $\kappa = \kappa(C) := \frac{8C}{1-4C} + 2$. □

2.3. Isotropic curvature bounded from below

As seen in the previous section, a crucial step when showing that a certain curvature operator $\mathcal{K}(\mathcal{R})$ remains bounded from below under the Ricci flow is controlling the scalar curvature and the operator $\mathrm{Ric} \wedge \mathrm{id}$. In T. Richard's work [22] this was accomplished by making the assumption that the corresponding cone $\mathcal{C}_{\mathcal{K}}$ is contained in the cone of operators with nonnegative Ricci curvature. In certain cases, this assumption can be omitted, when lower bounds of an operator $\mathcal{K}(\mathcal{R})$ provide bounds for the scalar curvature and $\mathrm{Ric} \wedge \mathrm{id}$. This is the case for the isotropic curvature, as we show in Lemmas 2.3.1 and 2.3.3 below.

Lemma 2.3.1. *Let $\mathcal{R} \in S^2_B(\Lambda^2\mathbb{R}^n)$, $n \geq 4$ be an algebraic curvature operator and* $\mathrm{Iso} = \mathrm{Iso}(\mathcal{R})$ *the map which assigns to an orthonormal four frame e_i, e_j, e_k, e_l the real number*

$$\mathrm{Iso}(e_i, e_j, e_k, e_l) = R_{ikik} + R_{ilil} + R_{jkjk} + R_{jljl} - 2R_{ijkl},$$

where $R_{ijkl} = \langle \mathcal{R}(e_i \wedge e_j), e_k \wedge e_l \rangle$. For any $n \geq 4$ there exists a constant $c(n)$ such that for any orthonormal basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$

$$\sum_{i,j,k,l \text{ p.d.}} \mathrm{Iso}(e_i, e_j, e_k, e_l) = c(n)S$$

(where the sum is taken over all pairwise distinct indices $i, j, k, l \in \{1, \ldots, n\}$).

Proof. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $\mathbb{R}^n$ and $R_{ijkl} = R(e_i, e_j, e_k, e_l)$ and $a_{ij} := R_{ijij}$ (note that $a_{ij} = a_{ji}$, $a_{ii} = 0$, and $S = \sum_{i,j} a_{ij} = \sum_{i \neq j} a_{ij}$. Using this notation we have

$$\mathrm{Iso}(e_i, e_j, e_k, e_l) = a_{ik} + a_{il} + a_{jk} + a_{jl} - 2R_{ijkl}.$$

The first Bianchi identity $R_{ijkl} + R_{kijl} + R_{jkil} = 0$ gives us

$$\begin{aligned}
& \mathrm{Iso}(e_i, e_j, e_k, e_l) + \mathrm{Iso}(e_k, e_i, e_j, e_l) + \mathrm{Iso}(e_j, e_k, e_i, e_l) \\
= \;& a_{ik} + a_{il} + a_{jk} + a_{jl} + a_{kj} + a_{kl} + a_{ij} + a_{il} + a_{ji} + a_{jl} + a_{ki} + a_{kl} \\
= \;& 2(a_{ij} + a_{ik} + a_{il} + a_{jk} + a_{jl} + a_{kl}).
\end{aligned}$$

Thus

$$
\begin{aligned}
& 3 \sum_{i,j,k,l \text{ p.d.}} \mathrm{Iso}(e_i, e_j, e_k, e_l) \\
= & \sum_{i,j,k,l \text{ p.d.}} (\mathrm{Iso}(e_i, e_j, e_k, e_l) + \mathrm{Iso}(e_k, e_i, e_j, e_l) + \mathrm{Iso}(e_j, e_k, e_i, e_l)) \\
= & \ 2 \sum_{i,j,k,l \text{ p.d.}} (a_{ij} + a_{ik} + a_{il} + a_{jk} + a_{jl} + a_{kl}) \\
= & \ 12 \sum_{i,j,k,l \text{ p.d.}} a_{ij} \\
= & \ 24 \binom{n-2}{2} \sum_{i \neq j} a_{ij} \\
= & \ 24 \binom{n-2}{2} S, \qquad (2.3.1)
\end{aligned}
$$

where we used that for each pair $i, j \in \{1, \ldots, n\}$, $i \neq j$ there are $\binom{n-2}{2} \cdot 2!$ possible choices of (k, l), $k, l \in \{1, \ldots, n\}$ such that i, j, k, l are pairwise distinct. The result follows by putting $c(n) = 24\binom{n-2}{2}$.

□

Lemma 2.3.2. *Let $n \geq 4$ and* Iso *as in Lemma 2.3.1. Then for any orthonormal basis $e_1, \ldots, e_n$ we have*

$$2 \, \mathrm{Iso}(\mathrm{Ric} \wedge \mathrm{id})(e_1, e_2, e_3, e_4) = \sum_{(i,j,k,l) \in I(n)} (R_{ikik} + R_{ilil} + R_{jkjk} + R_{jljl}),$$

where

$$I(n) \subset \{(i, j, k, l) | 1 \leq i, j \leq n, 1 \leq k, l \leq 4 \text{ and } i, j, k, l \text{ p.d.}\}$$

is an index set which does not depend on the particular basis $e_1, \ldots, e_n$.

Proof. Recall that the Kulkarni-Nomizu product of two bilinear forms A, B on $\mathbb{R}^n$ is the $(4, 0)$-tensor

$$(A \wedge B)_{ijkl} = \frac{1}{2}(A_{ik}B_{jl} - A_{jk}B_{il} + B_{ik}A_{jl} - B_{jk}A_{il}) \qquad (2.3.2)$$

(cf. Section B.2 of the Appendix). In view of (2.3.2)

$$(\mathrm{Ric} \wedge \mathrm{id})_{ijkl} = \frac{1}{2}(\mathrm{Ric}_{ik}\delta_{jl} - \mathrm{Ric}_{jk}\delta_{il} + \delta_{ik}\mathrm{Ric}_{jl} - \delta_{jk}\mathrm{Ric}_{il}),$$

and therefore $(\mathrm{Ric} \wedge \mathrm{id})_{ijij} = \frac{1}{2}(\mathrm{Ric}_{ii} + \mathrm{Ric}_{jj})$ for $i \neq j$ and $(\mathrm{Ric} \wedge \mathrm{id})_{ijkl} = 0$ for pairwise distinct i, j, k, l. Thus

$$
\begin{aligned}
& \mathrm{Iso}(\mathrm{Ric} \wedge \mathrm{id})(e_1, e_2, e_3, e_4) \\
= & \ \mathrm{Ric}_{11} + \mathrm{Ric}_{22} + \mathrm{Ric}_{33} + \mathrm{Ric}_{44} = \sum_{k=1}^{4} \sum_{i=1}^{n} R_{ikik}. \qquad (2.3.3)
\end{aligned}
$$

Let $a_{ik} = R_{ikik}$ as in Lemma 2.3.1. In view of (2.3.3), we have to show that

$$2 \sum_{k=1}^{4} \sum_{i=1}^{n} a_{ik} = \sum_{(i,j,k,l) \in I(n)} (a_{ik} + a_{il} + a_{jk} + a_{jl}),$$

where $I(n) \subset \{(i,j,k,l) | 1 \leq i,j \leq n, 1 \leq k,l \leq 4 \text{ and } i,j,k,l \text{ p.d.}\}$. For $n = 4$ we have

$$\begin{aligned}
2\sum_{k=1}^{4}\sum_{i=1}^{4} a_{ik} &= a_{12} + a_{13} + a_{42} + a_{43} \\
&+ a_{13} + a_{14} + a_{23} + a_{24} \\
&+ a_{12} + a_{14} + a_{32} + a_{34} \\
&+ a_{21} + a_{24} + a_{31} + a_{34} \\
&+ a_{21} + a_{23} + a_{41} + a_{43} \\
&+ a_{31} + a_{32} + a_{41} + a_{42}
\end{aligned}$$

(observe that each a_{ij}, $i \neq j$ appears twice on the right hand side). Similarly, for $n = 5$ one checks that

$$\begin{aligned}
2\sum_{k=1}^{4}\sum_{i=1}^{5} a_{ik} &= a_{12} + a_{14} + a_{32} + a_{34} \\
&+ a_{13} + a_{14} + a_{53} + a_{54} \\
&+ a_{23} + a_{24} + a_{53} + a_{54} \\
&+ a_{21} + a_{24} + a_{31} + a_{34} \\
&+ a_{21} + a_{23} + a_{41} + a_{43} \\
&+ a_{12} + a_{13} + a_{42} + a_{43} \\
&+ a_{31} + a_{32} + a_{51} + a_{52} \\
&+ a_{41} + a_{42} + a_{51} + a_{52}.
\end{aligned}$$

For arbitrary $n \geq 6$ the claim follows by induction: Suppose that for some $n \geq 4$ there exists $I(n) \subset \{(i,j,k,l) | i,j,k,l \text{ p.d. and } 1 \leq k,l \leq 4\}$ such that

$$2\sum_{k=1}^{4}\sum_{i=1}^{n} a_{ik} = \sum_{(i,j,k,l)\in I(n)} (a_{ik} + a_{il} + a_{jk} + a_{jl}).$$

Then

$$\begin{aligned}
2\sum_{k=1}^{4}\sum_{i=1}^{n+2} a_{ik} &= 2\sum_{k=1}^{4}\sum_{i=1}^{n} a_{ik} + 2\sum_{k=1}^{4}\sum_{i=n+1}^{n+2} a_{ik} \\
&= \sum_{(i,j,k,l)\in I(n)} (a_{ik} + a_{il} + a_{jk} + a_{jl}) \\
&+ a_{n+1,1} + a_{n+1,2} + a_{n+2,1} + a_{n+2,2} \\
&+ a_{n+1,3} + a_{n+1,4} + a_{n+2,3} + a_{n+2,4} \\
&+ a_{n+1,1} + a_{n+1,3} + a_{n+2,1} + a_{n+2,3} \\
&+ a_{n+1,2} + a_{n+1,4} + a_{n+2,2} + a_{n+2,4} \\
&= \sum_{(i,j,k,l)\in I(n+2)} (a_{ik} + a_{il} + a_{jk} + a_{jl}),
\end{aligned}$$

where

$$I(n+2) := I(n) \cup \{(n+1,n+2,1,2), (n+1,n+2,3,4), (n+1,n+2,1,3), (n+1,n+2,2,4)\}.$$

$\square$

Lemma 2.3.3. *Suppose that* $\mathrm{Iso}(\mathcal{R}) \geq K \in \mathbb{R}$. *Then*

$$|\mathrm{Iso}(\mathrm{Ric} \wedge \mathrm{id})| \leq c(n)S + c(n,K).$$

Proof. Let $\{e_1, e_2, e_3, e_4\}$ be an orthonormal four frame. We extend this frame to an orthonormal basis $\{e_1, \ldots, e_n\}$. In view of our assumption, for any pairwise distinct $i, j, k, l \in \{1, \ldots, n\}$ we have

$$\begin{aligned} K &\leq \frac{1}{2}\big[\mathrm{Iso}(e_i, e_j, e_k, e_l) + \mathrm{Iso}(e_j, e_i, e_k, e_l)\big] \\ &= \frac{1}{2}\big[R_{ikik} + R_{ilil} + R_{jkjk} + R_{jljl} - 2R_{ijkl} \\ &+ R_{jkjk} + R_{jljl} + R_{ikik} + R_{ilil} - 2R_{jikl}\big] \\ &= R_{ikik} + R_{ilil} + R_{jkjk} + R_{jljl}, \end{aligned}$$

where we used $R_{ijkl} = -R_{jikl}$. Thus, using Lemma 2.3.2 and the fact that for any $\alpha \in \mathbb{R}$, $\alpha \geq K$ we have $|\alpha| \leq \alpha + 2|K|$, we compute

$$|\mathrm{Iso}(\mathrm{Ric} \wedge \mathrm{id})(e_1, e_2, e_3, e_4)| = \frac{1}{2}|\sum_{i,j,k,l \in I(n)} (R_{ikik} + R_{ilil} + R_{jkjk} + R_{jljl})|$$

$$\begin{aligned} &\leq \frac{1}{2}|\sum_{i,j,k,l \in I(n)} (R_{ikik} + R_{ilil} + R_{jkjk} + R_{jljl})| \\ &+ \frac{1}{2}|\sum_{\substack{i,j,k,l \text{ p.d.} \\ (i,j,k,l) \notin I(n)}} (R_{ikik} + R_{ilil} + R_{jkjk} + R_{jljl})| \\ &\leq \frac{1}{2}\sum_{i,j,k,l \text{ p.d.}} (R_{ikik} + R_{ilil} + R_{jkjk} + R_{jljl}) + c(n,K) \\ &= 2\sum_{i,j,k,l \text{ p.d.}} R_{ikik} + c(n,K) \\ &= 4\binom{n-2}{2}\sum_{i \neq j} R_{ikik} + c(n,K) \\ &= c(n)S + c(n,K) \end{aligned}$$

and we are done.

□

Proposition 2.3.4. *Let M be a smooth compact manifold, and $g(t)$, $t \in [0,T)$ a solution to the Ricci flow on M satisfying*

- $\mathrm{Iso}(\mathcal{R}(0)) \geq -\varepsilon_0$ *for some* $\varepsilon_0 > 0$
- $|S(t)| \leq C/t$ *for all* $t \in (0,T)$ *and some* $0 \leq C < \frac{1}{c(n)}$, *where* $c(n) = 4\binom{n-2}{2}$ *is as in Lemma 2.3.3.*

Then there exists $\tilde{T} = \tilde{T}(C, \varepsilon_0, n) > 0$ *and* $\kappa = \kappa(C,n) \geq 0$ *such that*

$$\mathrm{Iso}(\mathcal{R}(t)) \geq -\kappa\varepsilon_0$$

on $[0, \min\{T, \tilde{T}\})$.

Proof. We proceed as in the proof of Proposition 2.2.1. Let

$$\mathcal{M} = \mathcal{R} + \varepsilon(t)\mathcal{I}$$

where $\varepsilon(t) = \varepsilon_1 tS + \varepsilon_0(1 + kt)$.

Let

$$(\mathcal{C}_{\mathrm{Iso}})_{(p,t)} = \{\mathcal{T} \in S^2_B(\Lambda^2 E_{(p,t)}) \,|\, \mathrm{Iso}(\mathcal{T}(p,t)) \geq 0\} \subset S^2(\Lambda^2 E_{(p,t)}).$$

Obviously $\mathrm{Iso}(\mathcal{I}) \equiv 4$, so in view of our assumptions

$$\mathrm{Iso}(\mathcal{M}(0)) = \mathrm{Iso}(\mathcal{R}(0)) + \varepsilon_0 \mathrm{Iso}(\mathcal{I}) \geq -\varepsilon_0 + 4\varepsilon_0 > 0,$$

so that $\mathcal{M}(p,0) \in (\mathcal{C}_{\mathrm{Iso}})_{(p,0)}$ for all $p \in M$. As in the Riemannian curvature case, we need to check that a solution of the ODE

$$\begin{aligned}\frac{d}{dt}\mathcal{M} &= 2\big(\mathcal{M}^2 + \mathcal{M}^{\#}\big) \\ &\quad -4\varepsilon(t)\mathrm{Ric}\wedge\mathrm{id} + \big(-2(n-1)\varepsilon(t)^2 + \varepsilon_1 S + 2\varepsilon_1 t|\mathrm{Ric}|^2 + k\varepsilon_0\big)\mathcal{I} =: \mathcal{V}(\mathcal{M})\end{aligned}$$

satisfies $\mathcal{M}(p,t) \in (\mathcal{C}_{\mathrm{Iso}})_{(p,t)}$ for all $(p,t) \in M \times \tilde{T}$, that is, if $\mathcal{M} \in \partial(\mathcal{C}_{\mathrm{Iso}})_{(p,t)}$, then $\mathcal{V}(\mathcal{M})(p,t)$ lies in the tangent cone of $(\mathcal{C}_{\mathrm{Iso}})_{(p,t)}$ at $M(p,t)$. In order to do so, it suffices to verify that for any orthonormal four frame e_1, e_2, e_3, e_4 of $E_{(p,t)}$ such that $\mathrm{Iso}(\mathcal{M}(p,t))(e_1,e_2,e_3,e_4) = 0$ we have $\mathcal{V}(\mathcal{M})(p,t)(e_1,e_2,e_3,e_4) \geq 0$ (cf. [5], Chapter 7). By [5], Proposition 7.4, we have that $\mathrm{Iso}(\mathcal{M}^2 + \mathcal{M}^{\#})(e_1,e_2,e_3,e_4)$ for any such four frame, so it suffices to show that $\mathrm{Iso}(\mathcal{N})(e_1,e_2,e_3,e_4) \geq 0$, where

$$\mathcal{N} = -4\varepsilon(t)\mathrm{Ric}\wedge\mathrm{id} + \big(-2(n-1)\varepsilon(t)^2 + \varepsilon_1 S + 2\varepsilon_1 t|\mathrm{Ric}|^2 + k\varepsilon_0\big)\mathcal{I}. \tag{2.3.4}$$

As in the proof of Prop. 2.2.1, from our assumptions it follows that

$$|\varepsilon(t)| \leq \varepsilon_1 C + 2\varepsilon_0 =: K \tag{2.3.5}$$

for all $0 \leq t \leq \frac{1}{k}$. Moreover, if $\mathcal{M}(p,t) \in \partial(\mathcal{C}_{\mathrm{Iso}})_{(p,t)}$, we have

$$\begin{aligned}0 \leq \mathrm{Iso}(\mathcal{M}) &= \mathrm{Iso}(\mathcal{R}) + \varepsilon(t)\mathrm{Iso}(\mathcal{I}) \\ &\leq \mathrm{Iso}(\mathcal{R}) + 4K,\end{aligned}$$

so that

$$\mathrm{Iso}(\mathcal{R}) \geq -4K \tag{2.3.6}$$

at (p,t). By Lemma 2.3.3 this implies that

$$|\mathrm{Iso}(\mathrm{Ric}\wedge\mathrm{id})| \leq c(n)S + c(n,K). \tag{2.3.7}$$

Furthermore, by Lemma 2.3.1 and by (2.3.6) we have

$$c(n)S = \sum_{i,j,k,l \text{ p.d.}} \mathrm{Iso}(\mathcal{R})(e_i,e_j,e_k,e_l) \geq -c(n,K),$$

which gives us

$$S \geq |S| - c(n,K) \tag{2.3.8}$$

at (p,t). (Here and in what follows, by $c(n,K)$ we denote all constants depending on n and K.) Using (2.3.5), (2.3.7) and (2.3.8) in (2.3.4), we arrive at

$$\begin{aligned}\mathrm{Iso}(\mathcal{N}) &\geq -4K\big(c(n)|S|+c(n,K)\big)+4\big(-2(n-1)K^2+\varepsilon_1|S|-\varepsilon_1 c(n,K)+k\varepsilon_0\big)\\ &= 4\big(-Kc(n)+\varepsilon_1\big)|S|+\big(k\varepsilon_0-(1+\varepsilon_1)c(n,K)\big). \qquad (2.3.9)\end{aligned}$$

Using $K=\varepsilon_1 C+2\varepsilon_0$ we compute

$$\begin{aligned}-Kc(n)+\varepsilon_1 &= -\varepsilon_1 c(n)C-2c(n)\varepsilon_0+\varepsilon_1\\ &= (1-c(n)C)\varepsilon_1-2c(n)\varepsilon_0,\end{aligned}$$

which vanishes for $\varepsilon_1=\varepsilon_1(n,\varepsilon_0,C)=\frac{2c(n)\varepsilon_0}{1-c(n)C}$ (recall that $C<1/c(n)$ by assumption). The second expression on the right hand side of (2.3.9) becomes nonnegative for large enough $k=k(n,\varepsilon_0,C)$.

In view of the above computations, $\mathrm{Iso}(\mathcal{M}(t))$ stays nonnegative as long as $t\leq \frac{1}{k}=:\tilde{T}(n,\varepsilon_0,C)$. By definition of $\mathcal{M}(t)$ this implies that

$$\mathrm{Iso}(\mathcal{R}(t))\geq -4\varepsilon(t)\geq -4K=-4\Big(\frac{2c(n)\varepsilon_0 C}{1-c(n)C}+2\varepsilon_0\Big)=-\kappa\varepsilon_0,$$

where $\kappa=\kappa(C,n)=4\big(\frac{2c(n)C}{1-c(n)C}+2\big)$, and we are done.

$\square$

2.4. An application for glued manifolds

In this section, we present an application of the results from Chapters 1 and 2 of the current work and M. Simon's results from [26] and [27]. These works are concerned with the evolution of non-smooth Riemannian metrics by the dual Ricci harmonic heat map flow, which is defined as follows. Given a fixed smooth background metric h on a smooth manifold M, and an initial metric g_0 on M, the dual Ricci harmonic heat map flow (h flow) is the solution to the system

$$\begin{aligned}\partial_t g_{ij}(t) &= -2\mathrm{Ric}(g(t))+{}^t\nabla_j V_j+{}^t\nabla_j V_i \ \text{ on } M\times[0,T] \qquad (2.4.1)\\ g(0) &= g_0,\end{aligned}$$

where $V(x,t)_i=g_{ij}(x,t)g^{kl}(x,t)({}^{g(t)}\Gamma^j_{kl}-{}^h\Gamma^j_{kl})(x,t)$. We refer to [26], and [14], Section 6 for a further discussion of the system (2.4.1). A solution to the h flow induces a solution to the Ricci flow via $\tilde{g}(t):=(\phi_t^{-1})^*g(t)$, where $\phi_t:M\to M$ is a one parameter family of diffeomorphisms solving

$$\begin{aligned}\partial_t\phi(p,t) &= \partial_i\phi(p,t)g^{jk}({}^{g(t)}\Gamma^i_{jk}-{}^h\Gamma^i_{jk})(p,t)\\ \phi(\cdot,0) &= \mathrm{id}_M.\end{aligned}$$

In [26], the following definition was introduced.

Definition 2.4.1. *Let M be a complete manifold and g a C^0 metric, and $1\leq\delta<\infty$ a given constant. A metric h is said to be a δ fair background metric for g, if h is C^∞ and there exists a constant k_0 with*

$$\sup_M|\mathcal{R}(h)|_h=k_0<\infty$$

and

$$\frac{1}{\delta}h\leq g\leq\delta h \ \text{ on } M.$$

W.l.o.g. one can always assume that the curvature tensor $R(h)$ of a metric h as in the above definition satisfies $\sup_M |{}^h\nabla^j R(h)|_h = k_j < \infty$, where ${}^h\nabla^j$ is the j-th covariant derivative with respect to h. This is due to the results of Shi [25], see Remark 1 in [26].

The following existence result was proved in [26].

Theorem 2.4.2 (Theorem 5.2 of [26])**.** *There exists a $\varepsilon(n)$ with the following properties. Let g_0 be a complete metric and h a complete metric which is $1 + \frac{\varepsilon(n)}{2}$ fair to g_0. There exists a $T = T(n, k_0)$ and a family of metrics $g(t)$, $t \in (0, T]$ in $C^\infty(M \times (0, T])$ which solves h flow for $t \in (0, T]$, h is $(1 + \varepsilon)$ fair to $g(t)$ for $t \in (0, T]$ and satisfies*

$$\lim_{t\to 0} \sup_{x\in\Omega'} |g(t) - g_0|_h = 0 \quad \textit{and}$$

$$\sup_M |{}^h\nabla^j g|_h^2 \leq \frac{c_j(n, k_0, \dots, k_j)}{t^j} \quad \textit{for all } t \in (0, T],\ j \in \mathbb{N},$$

where Ω' is any open set $\Omega' \subset\subset \Omega$, where Ω is any open subset on which g_0 is continuous.

The following a priori estimates for solutions to the h flow were proved in [27].

Lemma 2.4.3 (Lemma 2.1 of [27])**.** *Let g_0 be a complete smooth metric on M, and let h be a $1 + \frac{\varepsilon(n)}{2}$ fair background metric for g_0 for which $\sup_M |{}^h\nabla g_0|_h \leq c_0$ also holds. Let $g(t)$, $t \in [0, T]$ be a solution to the h flow as in Theorem 2.4.2. Then*

$$\begin{aligned} \sup_M |{}^h\nabla g(\cdot, t)|_h &\leq c(c_0, n, h) \quad \textit{for all } t \in [0, T], \textit{ and} \\ \sup_M |{}^h\nabla^2 g(\cdot, t)|_h &\leq \frac{c(c_0, n, h)}{\sqrt{t}} \quad \textit{for all } t \in [0, T], \end{aligned}$$

which implies

$$\sup_M |R(g(t))|_g \leq \frac{c(c_0, n, h)}{\sqrt{t}}. \tag{2.4.2}$$

The main result of [27] is the following

Theorem 2.4.4 (Theorem 1.3 of [27])**.** *Let M^n be a manifold, and g be a complete locally Lipschitz metric on M satisfying the following properties:*

1) *There exists a family $({}^\alpha g)$, $\alpha \in \mathbb{N}$, of smooth metrics on M such that*
   a) *$\mathcal{R}({}^\alpha g) \geq -\frac{1}{\alpha}$ for all $\alpha \geq 1$*
   b) *$\lim_{\alpha\to\infty} \sup_M |{}^\alpha g - g|_g = 0$*
   c) *$|\Gamma({}^\alpha g) - \Gamma({}^\beta g)|_g \leq c_0$ for all $\alpha, \beta \geq 1$, where $c_0 < \infty$ is some constant which does not depend on α, and $\Gamma({}^\alpha g)$ refers to the Christoffel symbols of ${}^\alpha g$.*
2) *If M is non-compact, we require that $\sup_M |\mathcal{R}({}^\alpha g)|_g < \infty$ for some sufficiently large α.*

Then the solution $g(x, t)$, $t \in (0, T]$ to h flow of g exists (for some smooth metric h, and $T = T(n, c_0, h) > 0$) and satisfies $\mathcal{R}(g(x, t)) \geq 0$ in the usual smooth Riemannian sense. Furthermore, there exists a constant $c = c(n, c_0, h)$ such that $\sup_M |\mathcal{R}(g(\cdot, t))|_g \leq \frac{c}{t}$ for all $t \in (0, T]$.

Let us briefly describe the idea of the proof of Theorem 2.4.4 given in [27]. In view of condition *1b* there exists a large enough $\alpha_0 \in \mathbb{N}$ such that ${}^{\alpha_0}g$ is $1 + \frac{\varepsilon(n)}{2}$ fair to ${}^{\alpha}g$ for all $\alpha \geq \alpha_0$, where $\varepsilon(n)$ is as in Theorem 2.4.2. We then put $h = {}^{\alpha_0}g$, and obtain solutions ${}^{\alpha}g(t)$, $t \in [0,T]$ to h flow of ${}^{\alpha}g$ for all $\alpha \geq \alpha_0$, where $T = T(n, {}^{\alpha_0}g) > 0$ (more precisely, $T = T(n, \sup_M |\mathcal{R}({}^{\alpha_0}g)|_g)$). Using condition *1a* and the a priori estimates from Lemma 2.4.3, one shows that these solutions satisfy $\mathcal{R}({}^{\alpha}g(t)) \geq -\frac{2}{\alpha}$ for all $t \in [0,T)$ (see Lemma 3.1 of [27]) (note that condition *1c* implies $|{}^{h}\nabla^{\alpha}g|_h \leq C(c_0)$ for large enough α_0 and $\alpha \geq \alpha_0$). The desired solution to h flow of g_0 is then obtained as the limit solution by letting $\alpha \to \infty$.

We now give an application of Theorem 2.4.4 for manifolds obtained by a gluing procedure as in the previous chapter.

Theorem 2.4.5. *Let (M_0, g_0), (M_1, g_1) be smooth compact Riemannian manifolds with isometric boundaries $\Gamma_0 =_\phi \Gamma_1$ such that the sum of the second fundamental forms of Γ_0 in M_0 and Γ_1 in M_1 is nonnegative (as in Theorem 1.1.2). Suppose that $\mathcal{T}(g_0), \mathcal{T}(g_1) \geq 0$, where $\mathcal{T}$ is one of the following curvature operators: Riemannian curvature operator, scalar curvature, isotropic curvature, bi-curvature. Then the solution $g(t)$, $t \in (0,T]$ to h flow of $g = g_0 \cup_\phi g_1$ exists on $M = M_0 \cup_\phi M_1$, where $T = T(n, g_0, g_1)$, and satisfies $\mathcal{T}(g(t)) \geq 0$ for $t \in (0,T]$. In particular, M admits a smooth metric of curvature $\mathcal{T} \geq 0$.*

Proof. We prove this statement for the Riemannian curvature operator case, and the proof for the other cases is similar. We proceed as in the proof of 2.4.4 in [27]. Let $(\tilde{g}_{(\delta)})_{\delta > 0}$ be the family of approximating metrics for g constructed in Chapter 1. By putting ${}^{\alpha}g_0 := \tilde{g}_{(1/\alpha)}$, $\alpha \in \mathbb{N}$ we obtain a sequence of metrics satisfying the conditions of Theorem 2.4.4 (see Lemma 1.6.3), where the constant in condition *1c* is $c_0 = c_0(n, g_0, g_1)$. As in the proof of Theorem 2.4.4, we find a large enough α_0 such that ${}^{\alpha}g_0$ is $1 + \frac{\varepsilon(n)}{2}$ fair to g for all $\alpha \geq \alpha_0$, where $\varepsilon(n)$ is as Theorem 2.4.2. Putting $h = {}^{\alpha}g_0$, we find solutions ${}^{\alpha}g : M \times [0,T]$ to h flow of ${}^{\alpha}g_0$ for all $\alpha \geq \alpha_0$, where $T = T(n, {}^{\alpha_0}g_0) > 0$. In view of Lemma 2.4.3, and the fact that ${}^{\alpha}g(t)$ is $1 + 2\varepsilon(n)$ fair to g, we estimate the scalar curvature of ${}^{\alpha}g(t)$ by

$$S({}^{\alpha}g(t)) \leq 2|\mathcal{R}(g_\alpha(t))|_g \leq \frac{c}{\sqrt{t}} = \frac{c\sqrt{t}}{t}$$

for all $t \in (0,T]$, where $c = c(n, c_0, h) = c(n, g_0, g_1, {}^{\alpha_0}g_0)$, and after choosing $T = T(n, g_0, g_1, {}^{\alpha_0}g_0)$ smaller we have $S({}^{\alpha}g(t)) \leq \frac{1/8}{t}$ for $t \in (0,T]$. Thus, the corresponding solutions to Ricci flow of ${}^{\alpha}g_0$, given by ${}^{\alpha}\tilde{g}(t) = (\phi_t^{-1})^{*\alpha}g(t)$ (see the discussion at the beginning of this section) satisfy the conditions of Proposition 2.2.1, which implies that $\mathcal{R}({}^{\alpha}\tilde{g}(t)) \geq -\kappa\frac{1}{\alpha}$ for all $t \in [0,T]$ (where κ is some fixed constant). Then the solutions to h flow satisfy $\mathcal{R}({}^{\alpha}g(t)) \geq -\kappa\frac{1}{\alpha}$ as well, and we obtain a limit solution by letting $\alpha \to \infty$ as in the proof of Theorem 2.4.4.

□

Chapter 3.

Smoothing cone-like singularities

3.1. Introduction

This chapter is devoted to smoothing point singularities of Riemannian metrics while preserving nonnegative curvature up to a small error term.

In Section 3.2, we present an unpublished result by V.S. Matveev [16], which says that in dimension three one can smooth out standard cone metrics near the vertex while preserving nonnegative Riemannian curvature operator.

In Section 3.3, we derive a formula for the curvature operator of metrics modified on equidistant hypersurfaces of a fixed hypersurface Γ (this result is of a rather technical character).

In Section 3.4, we are concerned with three-dimensional Riemannian manifolds (M, g) having a cone-like structure near a singular point o of g, and nonnegative curvature operator (on $M \setminus o$). Using the gluing result from Chapter 1, we replace g by a standard cone metric on a neighborhood of o while keeping the curvature operator almost nonnegative. We then smooth out the resulting metric near the vertex using the results from Section 3.2. In this way, we are able to construct a sequence of smooth Riemannian metrics g_i approximating the initial singular metric in the C^∞ sense on $M \setminus o$, whose curvature operators satisfy $\mathcal{R}(g_i) \geq -\varepsilon_i$, where $\varepsilon_i \to 0$.

In Sections 3.5 and 3.6, we prove distance and volume estimates for the sequence (M, g_i), and present an application of M. Simons results [28]. In particular, we show that M admits a smooth Riemannian metric with nonnegative Ricci curvature, so that in view of the results of W. X. Shi [24] and R. Hamilton [12], M is diffeomorphic to $\mathbb{R}^3$, $S^2 \times \mathbb{R}$ or S^3 modulo a group of fixed point free isometries in the standard metric.

3.2. Smoothing standard cones

A cone $\mathcal{C}_\Gamma$ over a topological space Γ is the quotient of the product space $\Gamma \times [0, \infty)$ obtained by identifying the points of the fiber $\Gamma \times \{0\}$, that is, $\mathcal{C}_\Gamma = \Gamma \times [0, \infty)/_\sim$, where $(x, t) \sim (x, s)$ iff $t = s = 0$. The point $o = [(x, 0)]_\sim$ is called the vertex of the cone.

Consider the case where Γ is a C^k smooth manifold, $k \geq 0$, of dimension $n - 1$. Then $\mathcal{C}_\Gamma \setminus o \cong \Gamma \times (0, \infty)$ is a C^k smooth manifold of dimension n, where the smooth structure is induced by the smooth structure of Γ and the standard smooth structure of $(0, \infty)$. Note that $\mathcal{C}_\Gamma$ is *not* a manifold in general.

Definition 3.2.1 (Euclidean cone)**.** *Given a Riemannian metric $\bar{\gamma}$ on Γ, we denote by γ_e the Riemannian metric on $\mathcal{C}_\Gamma \setminus o$ defined by*

$$\gamma_e = r^2 \bar{\gamma} + dr^2,$$

where dr^2 *is the standard metric on* $(0,\infty)$. *We call* $(\mathcal{C}_\Gamma, \gamma_e)$ *the* Euclidean cone *over* $(\Gamma, \bar\gamma)$ *with vertex* o.

Definition 3.2.2 (Spherical a-cone)**.** *Given* $a \geq 1$, *let* $\mathcal{C}_{\Gamma,a} := (\Gamma \times [0, \frac{\pi}{2}a))/_\sim$, *where* $\sim$ *is as above, and let* γ_a *be the Riemannian metric on* $\mathcal{C}_{\Gamma,a}$ *defined by*

$$\gamma_a = a^2 \sin^2(r/a)\bar\gamma + dr^2.$$

We call $(\mathcal{C}_{\Gamma,a}, \gamma_a)$ *the* spherical a-cone *over* $(\Gamma, \bar\gamma)$ *with vertex* o.

If $\bar\gamma$ is C^l smooth, then the cone metrics γ_e and γ_a are C^l smooth on $\mathcal{C}_\Gamma \setminus o$ and $\mathcal{C}_{\Gamma,a} \setminus o$, respectively. For the case $l \geq 2$ it is a well known fact that an Euclidean (spherical a-) cone over $(\Gamma, \bar\gamma)$ has curvature operator ≥ 0 ($\geq 1/a^2$) in the classical sense off the vertex, if and only if $\bar\gamma$ has curvature operator ≥ 1. We will verify this for spherical cones using Proposition 3.3.1 below (see Example 3.3.2). In what follows, we will consider cones over C^∞ smooth manifolds Γ with C^∞ smooth metrics $\bar\gamma$.

With the above definitions, the main result of this section is as follows: Given a smooth Riemannian two-manifold $(\Gamma, \bar\gamma)$ such that Γ is homeomorphic to the standard sphere S^2 and $\bar\gamma$ has curvature > 1, the Euclidean cone over $(\Gamma, \bar\gamma)$ admits a smooth structure D, and a smooth Riemannian metric which coincides with the cone metric (induced by $\bar\gamma$) off a small neighborhood of the vertex, and has curvature ≥ 0 (see Proposition 3.2.6).

First we consider the case where Γ is a smooth embedded convex hypersurface of the unit sphere $S^{n-1} \subset \mathbb{R}^n$ (convex in the sense that Γ is the boundary of a convex subset of S^{n-1}). In this special case, the statement is true in any dimension (see Corollary 3.2.5). In the general case, our proof involves the Alexandrov embedding theorem, which makes the assumption $\dim(\Gamma) = 2$ necessary.

Definition 3.2.3. *A subset* $K \subseteq S^{n-1}$ *is called convex, if for any* $p, q \in K$ *with* $\operatorname{dist}_{S^{n-1}}(p,q) < \pi$ *the geodesic from* p *to* q *in* S^{n-1} *(with the standard meric) lies entirely in* K.

Lemma 3.2.4. *Let* K *be a closed convex subset of* S^{n-1} *with smooth* $(n-2)$*-dim. boundary* ∂K. *Let*

$$\mathcal{C}_K = \{tq \,|\, q \in K, t \geq 0\} \subset \mathbb{R}^n$$

be the Euclidean cone over K. *Then there exists a hyperplane* $E \subset \mathbb{R}^n$ *such that*

- $\partial\mathcal{C}_K = \operatorname{graph}(u)$, *where* $u : E \to \mathbb{R}$ *is convex and smooth on* $E \setminus 0$.
- *For any* $\varepsilon > 0$ *there exists a smooth convex function* $\tilde u : E \to \mathbb{R}$ *such that* $\tilde u \equiv u$ *on* $E \setminus B_\varepsilon(0)$.

Proof. It is well known that a closed convex subset $K \subset S^{n-1}$ with smooth $(n-2)$-dim. boundary is either a closed half sphere, the boundary of a half sphere, or it is contained in an open half sphere of S^{n-1}, in which case K has nonempty interior (for convenience we give a proof in Lemma D.4). If K is a closed half sphere or the boundary of a half sphere, then clearly the boundary of the Euclidean cone over K is a hypersurface $E \subset \mathbb{R}^n$. In this case the claim follows immediately by putting $u, \tilde u : E \to \mathbb{R}$, $u, \tilde u \equiv 0$.

Let us consider the case where K is contained in an open half sphere of S^{n-1}. W.l.o.g. after rotating K we may assume that

$$K \subset HS^{n-1} := \{x \in \mathbb{R}^n \,|\, |x| = 1, x^n > 0\},$$

and that the point $(0, \ldots, 0, 1) \in \mathbb{R}^n$ lies in the interior of K. Given $\delta > 0$, consider the hyperplane $H_\delta := \{x = (x^1, \ldots, x^n) \in \mathbb{R}^n \mid x^n = \delta\} \subset \mathbb{R}^n$. Let us show that the intersection $I_\delta := \mathcal{C}_K \cap H_\delta$ is a compact convex body in the hyperplane, i.e. $I_\delta \subset H_\delta$ is convex, compact and has nonempty interior. Indeed, since $K \subset S^{n-1}$ is convex by assumption, the cone $\mathcal{C}_K$ is a convex subset of $\mathbb{R}^n$ (see Lemma D.4). Therefore $I_\delta = \mathcal{C}_K \cap H_\delta \subset \mathbb{R}^n$ is convex, since it is an intersection of two convex sets. We show that the map

$$\begin{aligned} \pi_{H_\delta} : HS^{n-1} &\to H_\delta \\ q &\mapsto \frac{\delta}{q^n} q \end{aligned}$$

is a homeomorphism and $I_\delta = \pi_{H_\delta}(K)$, which implies that I_δ is compact and $\mathring{I}_\delta = \pi_{H_\delta}(\mathring{K}) \neq \emptyset$, since $K \subset HS^{n-1}$ is compact and has nonempty interior by assumption.

Clearly, π_{H_δ} is continuous since $q^n \neq 0$ for all $q \in HS^{n-1}$. Suppose that $\pi_{H_\delta}(q) = \pi_{H_\delta}(\tilde{q})$ for $q, \tilde{q} \in HS^{n-1}$. Then

$$\frac{\delta}{q^n} q = \frac{\delta}{\tilde{q}^n} \tilde{q},$$

so that

$$q = \frac{q^n}{\tilde{q}^n} \tilde{q}.$$

Since $|q| = |\tilde{q}| = 1$, this implies $q^n = \tilde{q}^n$, so we have $q = \tilde{q}$. Furthermore, given $\hat{q} \in H_\delta$ we have

$$\hat{q} = \frac{\delta}{\hat{q}^n / |\hat{q}|} \frac{\hat{q}}{|\hat{q}|} = \pi_{H_\delta}(\frac{\hat{q}}{|\hat{q}|})$$

since $\hat{q}^n = \delta$, where $\frac{\hat{q}}{|\hat{q}|} \in HS^{n-1}$. This shows that π_{H_δ} is bijective. One easily checks that the inverse of π_{H_δ} is given by

$$\begin{aligned} \pi_{H_\delta}^{-1} : H_\delta &\to HS^{n-1} \\ \hat{q} &\mapsto \frac{\hat{q}}{|\hat{q}|}, \end{aligned}$$

which is a continuous map since $|\hat{q}| \neq 0$ for all $\hat{q} \in H_\delta$. This shows that π_{H_δ} is a homeomorphism.

Let us now verify that $I_\delta = \pi_{H_\delta}(K)$. Let $\hat{q} \in I_\delta = \mathcal{C}_K \cap H_\delta$. Then $\hat{q} = tq$, where $q \in K \subset HS^{n-1}$ and $t > 0$. Thus

$$\pi_{H_\delta}^{-1}(\hat{q}) = \pi_{H_\delta}^{-1}(tq) = \frac{1}{t|q|} tq = q \in K,$$

so that $\hat{q} \in \pi_{H_\delta}(K)$. Conversely, for any $q \in K$ we have $\pi_{H_\delta}(q) = \frac{\delta}{q^n} q \in \mathcal{C}_K$ by definition of $\mathcal{C}_K$, so that $\pi_{H_\delta}(q) \in \mathcal{C}_K \cap H_\delta = I_\delta$.

Recall that we assumed that the point $(0, 1) = (0, \ldots, 0, 1)$ lies in the interior of K, which implies that the point $(0, \delta)$ lies in the interior of $\mathcal{C}_K$. We identify $x = (x^1, \ldots, x^{n-1}, \delta) = (\hat{x}, \delta) \in H_\delta$ with $\hat{x} \in \mathbb{R}^{n-1}$, which allows us to see I_δ as a subset of $\mathbb{R}^{n-1}$. Consider the Minkowski functional of I_δ

$$\begin{aligned} &F : H_\delta \to [0, \infty) \\ &F(\hat{x}) = \inf\{\lambda > 0 \mid \hat{x} \in \lambda I_\delta\} \end{aligned}$$

(note that in view of the above identification $x = (\hat{x}, \delta) \in H_\delta$, $x \in \lambda I_\delta$ means $x = (\hat{x}, \delta) = (\lambda\hat{y}, \delta)$ for some $(\hat{y}, \delta) \in I_\delta$). Since $I_\delta \subset H_\delta$ is closed, convex and $(0, \delta) = 0_{H_\delta} \in I_\delta$, we have that F is convex and

$$I_\delta = \{x = (\hat{x}, \delta) \in H_\delta \,|\, F(\hat{x}) \leq 1\} \tag{3.2.1}$$

(see Lemma D.5). This implies that the cone $\mathcal{C}_K$ is given by

$$\mathcal{C}_K = \{x = (\hat{x}, x^n) \in \mathbb{R}^n \,|\, \delta F(\hat{x}) \leq x^n\}. \tag{3.2.2}$$

Let us verify this. From the definitions of $\mathcal{C}_K$ and I_δ it follows that

$$x \in \mathcal{C}_K \setminus \{0\} \quad \text{iff} \quad x^n > 0 \;\; \text{and} \;\; \frac{\delta}{x^n} x \in I_\delta \tag{3.2.3}$$

(see Lemma D.6). Let $x \in \mathcal{C}_K$. If $x = 0$ then $\delta F(\hat{x}) = \delta F(0) = 0 = x^n$. If $x \in \mathcal{C}_K \setminus \{0\}$ then by (3.2.3) we have $x^n \neq 0$ and

$$\frac{\delta}{x^n} x = (\frac{\delta}{x^n}\hat{x}, \delta) \in I_\delta$$

so that in view of (3.2.1)

$$\frac{\delta}{x^n} F(\hat{x}) = F(\frac{\delta}{x^n}\hat{x}) \leq 1$$

(where we used that $F(t\hat{x}) = tF(\hat{x})$ for $t \geq 0$, cf. Lemma D.5), which gives us $\delta F(\hat{x}) \leq x^n$. Conversely, suppose that $\delta F(\hat{x}) \leq x^n$ for some $x = (\hat{x}, x^n) \in \mathbb{R}^n$. If $x^n = 0$, then $F(\hat{x}) = 0$ and from definition of F it follows that there exists a sequence of positive numbers $\lambda_n \to 0$ such that

$$(\frac{1}{\lambda_n}\hat{x}, \delta) \in I_\delta$$

for all λ_n. Since I_δ is compact this implies $\hat{x} = 0$, so that $x = (\hat{x}, x^n) = 0 \in \mathcal{C}_K$. Suppose that $x^n \neq 0$. Then

$$F(\frac{\delta}{x^n}\hat{x}) = \frac{\delta}{x^n} F(\hat{x}) \leq 1,$$

and (3.2.1) implies that

$$\frac{\delta}{x^n}(\hat{x}, x^n) = (\frac{\delta}{x^n}\hat{x}, \delta) \in I_\delta.$$

By (3.2.3) we then have $x = (\hat{x}, x^n) \in \mathcal{C}_K$.

The identity (3.2.2) shows that $\mathcal{C}_K$ is the supergraph of the convex function

$$u := \delta F : \mathbb{R}^{n-1} \to [0, \infty).$$

Since u is continuous, this implies $\partial\mathcal{C}_K = \text{graph}(u)$. The assumption that ∂K is a smooth submanifold of S^{n-1} implies that $\partial\mathcal{C}_K \setminus \{0\} = \{tq \,|\, q \in \partial K, t > 0\}$ is a smooth submanifold of $\mathbb{R}^n$. Thus u is smooth on $\mathbb{R}^{n-1} \setminus \{0\}$. This shows the first assertion of the lemma.

Let us prove the second assertion. We find a smooth convex non-decreasing function $\rho : [0, \infty) \to \mathbb{R}$ such that $\rho|_{[0,1/2)} \equiv const$ and $\rho|_{(1,\infty)} \equiv \text{id}$ (see Figure 3.1). Note that such a function satisfies $\rho(y) \geq y$ for all $y \geq 0$. Recall that we

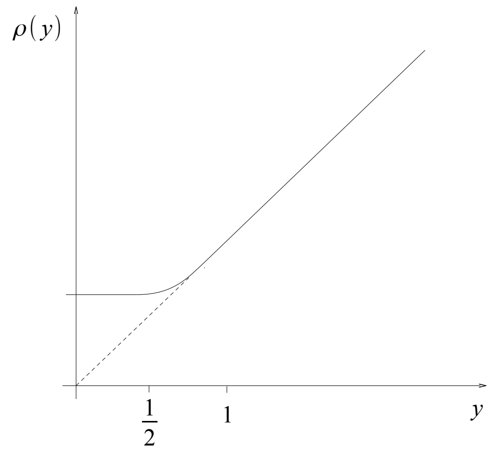

Figure 3.1.: The function ρ

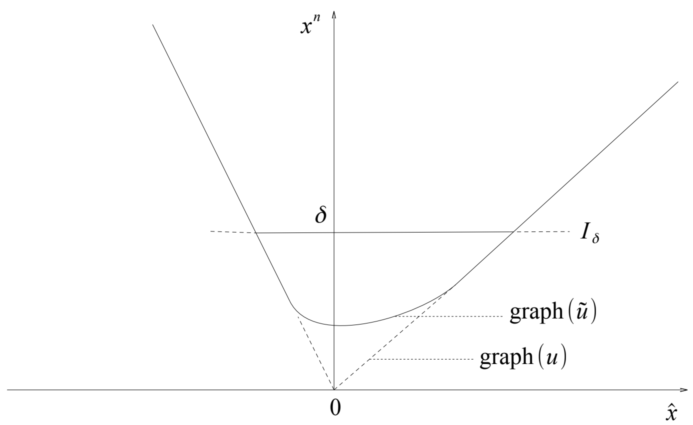

Figure 3.2.: The sets $\mathcal{C}_K = \mathrm{graph}(u)$ and $\mathrm{graph}(\tilde{u})$

identify $(\hat{x}, \delta) \in H_\delta$ and $\hat{x} \in \mathbb{R}^{n-1}$, so that F may be considered as a function $\mathbb{R}^{n-1} \to \mathbb{R}$. Let $\tilde{F} = \rho \circ F$. The function $\tilde{F}$ is convex since F and ρ are convex and ρ is non-decreasing. Moreover, $\tilde{F}$ is smooth since off 0 it is a composition of smooth functions and $\tilde{F} = const$ near 0.

Let $\varepsilon > 0$. We may choose $\delta > 0$ small enough such that $I_\delta \subset B_\varepsilon(0) \subset \mathbb{R}^{n-1}$. This due to the fact that

$$I_\delta = \pi_{H_\delta}(K) = \{\frac{\delta}{x^n}\hat{x} \,|\, x = (\hat{x}, x^n) \in K\}$$

(where we again identify H_δ and $\mathbb{R}^{n-1}$), and $|\hat{x}| \leq 1$ and $x^n > \tilde{\delta} > 0$ for all $x = (\hat{x}, x^n) \in K$ since $\bar{K} = K \subset HS^{n-1}$.

In view of (3.2.1) we then have $F > 1$ on $\mathbb{R}^{n-1} \setminus B_\varepsilon(0) \subset \mathbb{R}^{n-1} \setminus I_\delta$. Let $\tilde{u} = \delta\tilde{F}$. Then $\tilde{u}$ is smooth, convex, and

$$\tilde{u}(\hat{x}) = \delta(\rho(F(\hat{x}))) = \delta F(\hat{x}) = u(\hat{x})$$

on $\mathbb{R}^{n-1} \setminus B_\varepsilon(0)$, since $\rho(y) = y$ for $y > 1$. This proves the second assertion and we are done. $\square$

Corollary 3.2.5. *Let $K \subset S^{n-1}$ and $\mathcal{C}_K$ be as in Lemma 3.2.4, and let*

$$\mathcal{C}_{\partial K} = \partial \mathcal{C}_K = \{tq \,|\, q \in \partial K, t \geq 0\} \subset \mathbb{R}^n$$

be the Euclidean cone over ∂K. Let $g_{\mathcal{C}_{\partial K}}$ be the Riemannian metric on $\mathcal{C}_{\partial K} \setminus 0$ arising from the standard metric of $\mathbb{R}^n$. There exists a smooth structure D on $\mathcal{C}_{\partial K}$ such that

- *D is compatible with the smooth structure of $\mathcal{C}_{\partial K} \setminus 0$ arising from the standard structure of $\mathbb{R}^n$*
- *For any neighborhood $U \ni 0$ there exists a smooth metric $\tilde{g}$ on $(\mathcal{C}_{\partial K}, D)$ such that $\tilde{g} \equiv g_{\mathcal{C}_{\partial K}}$ on $\mathcal{C}_{\partial K} \setminus U$ and $\tilde{g}$ has curvature operator ≥ 0.*

Proof. By Lemma 3.2.4 we have that $\partial \mathcal{C}_K = \mathcal{C}_{\partial K}$ is the graph of a convex function $u : E \to \mathbb{R}$, where E is a hyperplane of $\mathbb{R}^n$ (in particular u is continuous). After rotating K we may assume that $E = \mathbb{R}^{n-1} \times \{0\}$. Thus, the projection

$$\begin{aligned} \pi|_{\mathcal{C}_{\partial K}} : \mathcal{C}_{\partial K} &\to \mathbb{R}^{n-1} \\ x = (\hat{x}, u(\hat{x})) &\mapsto \hat{x} \end{aligned}$$

is a homeomorphism, where π is the projection $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}^{n-1}$, $x = (\hat{x}, x^n) \mapsto \hat{x}$. The map $\pi|_{\mathcal{C}_{\partial K}}$ induces a smooth structure D on $\mathcal{C}_{\partial K}$, which agrees with the smooth structure of $\mathcal{C}_{\partial K} \setminus 0$. This is due to the fact that $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is smooth and $\mathcal{C}_{\partial K} \setminus 0$ is a smooth submanifold of $\mathbb{R}^n$.

Given a point $x \in \mathcal{C}_{\partial K}$, we denote by $\frac{\partial^\varphi}{\partial x^i}(x)$, $i = 1, \ldots, n-1$, the basis of $T_x \mathcal{C}_{\partial K}$ induced by the coordinate chart $\varphi := \pi|_{\mathcal{C}_{\partial K}} : \mathcal{C}_{\partial K} \to \mathbb{R}^{n-1}$. The coefficients of $g_{\mathcal{C}_{\partial K}}$ on $\mathcal{C}_{\partial K} \setminus 0$ with respect to this basis are

$$(g_{\mathcal{C}_{\partial K}})^\varphi_{ij}(x) \;=\; g_{\mathcal{C}_{\partial K}}(x)\big(\frac{\partial^\varphi}{\partial x^i}(x), \frac{\partial^\varphi}{\partial x^j}(x)\big) = \delta_{ij} + \partial_i u(\hat{x}) \partial_j u(\hat{x}).$$

Let U be a neighborhood of 0 in $\mathcal{C}_{\partial K}$. Then $\pi(U) \subset \mathbb{R}^{n-1}$ contains a ball $B_\varepsilon(0)$, $\varepsilon > 0$, and by Lemma 3.2.4 we find a smooth convex function $\tilde{u} : \mathbb{R}^{n-1} \to \mathbb{R}$ such that $\tilde{u} \equiv u$ on $\mathbb{R}^{n-1} \setminus B_\varepsilon(0)$. We define the metric $\tilde{g}$ on $\mathcal{C}_{\partial K}$ by putting

$$\tilde{g}^\varphi_{ij}(x) \;=\; \tilde{g}(x)\big(\frac{\partial^\varphi}{\partial x^i}(x), \frac{\partial^\varphi}{\partial x^j}(x)\big) = \delta_{ij} + \partial_i \tilde{u}(\hat{x}) \partial_j \tilde{u}(\hat{x}). \tag{3.2.4}$$

Clearly, $\tilde{g}$ is a smooth metric on $(\mathcal{C}_{\partial K}, D)$, which coincides with $g_{\mathcal{C}_{\partial K}}$ off U, since

$$\tilde{g}^\varphi_{ij} \circ \varphi^{-1} = \delta_{ij} + \partial_i \tilde{u}\, \partial_j \tilde{u} : \mathbb{R}^{n-1} \to \mathbb{R}$$

is smooth and $\tilde{u} \equiv u$ off $B_\varepsilon(0) \subset \varphi(U)$.

Let us verify that $\tilde{g}$ has nonnegative curvature. Consider the smooth embedded hypersurface $\mathrm{graph}(\tilde{u}) \subset \mathbb{R}^n$ equipped with the smooth metric induced from $\mathbb{R}^n$, which we denote by $g_{\tilde{u}}$. Since $\tilde{u}$ is convex, it follows from the Gauss Theorem that $(\mathrm{graph}(\tilde{u}), g_{\tilde{u}})$ has nonnegative curvature operator. The map

$$\psi = \pi|_{\mathrm{graph}(\tilde{u})} : \mathrm{graph}(\tilde{u}) \to \mathbb{R}^{n-1}$$

is a coordinate chart of $\mathrm{graph}(\tilde{u})$. Let $\frac{\partial^\psi}{\partial x^i}(x)$, $i = 1, \ldots, n-1$ be the basis of $T_x(\mathrm{graph}(\tilde{u}))$. We show that

$$\begin{aligned} J : (\mathcal{C}_{\partial K}, \tilde{g}) &\to (\mathrm{graph}(\tilde{u}), g_{\tilde{u}}) \\ x = (\hat{x}, x^n) &\mapsto (\psi^{-1} \circ \varphi)(x) = (\hat{x}, \tilde{u}(\hat{x})) \end{aligned}$$

is an isometry. Clearly, J is a diffeomorphism, since $\varphi : \mathcal{C}_{\partial K} \to \mathbb{R}^{n-1}$ and ψ : graph$(\tilde{u}) \to \mathbb{R}^{n-1}$ are homeomorphisms, and $\psi \circ J \circ \varphi^{-1} = \mathrm{id}_{\mathbb{R}^{n-1}} = \varphi \circ J^{-1} \circ \psi^{-1}$. Furthermore, at $x = (\hat{x}, \tilde{u}(x)) \in$ graph$(\tilde{u})$ the coefficients of $g_{\tilde{u}}(x)$ with respect to $\frac{\partial^\psi}{\partial x^i}(x)$, $i = 1, \dots, n-1$, are

$$g^\psi_{ij}(x) = g(x)\big(\frac{\partial^\psi}{\partial x^i}(x), \frac{\partial^\psi}{\partial x^j}(x)\big) = \delta_{ij} + \partial_i \tilde{u}(\hat{x}) \partial_j \tilde{u}(\hat{x}). \tag{3.2.5}$$

Given any smooth function f : graph$(\tilde{u}) \to \mathbb{R}$ we compute using $J \circ \varphi^{-1} = \psi^{-1}$ and $\psi \circ J = \varphi$

$$\begin{aligned} DJ(x)\frac{\partial^\varphi}{\partial x^i}(x)(f) &= \frac{\partial(f \circ J \circ \varphi^{-1})}{\partial x^i}|_{\varphi(x)} \\ &= \frac{\partial(f \circ \psi^{-1})}{\partial x^i}|_{\psi(J(x))} = \frac{\partial^\psi}{\partial x^i}(J(x))(f), \end{aligned} \tag{3.2.6}$$

so that

$$DJ(x)\frac{\partial^\varphi}{\partial x^i}(x) = \frac{\partial^\psi}{\partial x^i}(J(x)).$$

Combining (3.2.4), (3.2.5) and (3.2.6) gives

$$\begin{aligned} & g_{\tilde{u}}(J(x))\big(DJ(x)\frac{\partial^\varphi}{\partial x^i}(x), DJ(x)\frac{\partial^\varphi}{\partial x^j}(x)\big) \\ = \; & g_{\tilde{u}}(J(x))\big(\frac{\partial^\psi}{\partial x^i}(J(x)), \frac{\partial^\psi}{\partial x^j}(J(x))\big) \\ = \; & \delta_{ij} + \partial_i \tilde{u}(\psi(J(x)))\partial_j \tilde{u}(\psi(J(x))) \\ = \; & \delta_{ij} + \partial_i \tilde{u}(\hat{x})\partial_j \tilde{u}(\hat{x}) \\ = \; & \tilde{g}(x)\big(\frac{\partial^\varphi}{\partial x^i}(x), \frac{\partial^\varphi}{\partial x^j}(x)\big), \end{aligned}$$

which shows that J is an isometry, and we are done.

□

Proposition 3.2.6. *Let Γ be a smooth 2 dim. Riemannian manifold homeomorphic to the sphere S^2, and let $\bar{\gamma}$ be a smooth Riemannian metric on Γ of curvature > 1. Let $(\mathcal{C}_\Gamma, \gamma_e)$ be the Euclidean cone over $(\Gamma, \bar{\gamma})$ with vertex o as in Definition 3.2.1. Then there exists a smooth structure D_Γ on $\mathcal{C}_\Gamma$ such that*

- *D_Γ is compatible with the smooth structure of the product manifold $\Gamma \times (0, \infty) = \mathcal{C}_\Gamma \setminus o$.*
- *For any open neighborhood $U \ni o$ there exists a smooth metric $\tilde{\gamma}_e$ on $(\mathcal{C}_\Gamma, D_\Gamma)$ such that $\tilde{\gamma}_e = \gamma_e$ on $\mathcal{C}_\Gamma \setminus U$ and $\tilde{\gamma}_e$ has curvature ≥ 0.*

Proof. A result of Pogorelov ([20], §8 Thm. 2), which is a version of Alexandrov's embedding theorem (§2 in Section XII of [1]) for regular surfaces, states that a closed 2-dim. manifold with regular metric of curvature greater than κ is isometric to a regular closed convex surface in the 3-dim. space of constant curvature κ. In our case this implies that there exists a closed convex subset $K \subset S^3$ with smooth boundary ∂K, such that $(\Gamma, \bar{\gamma})$ is isometric to $(\partial K, g_{\partial K})$, where $g_{\partial K}$ denotes the smooth Riemannian metric on ∂K induced by the standard metric of S^3.

Let $\mathcal{C}_{\partial K} = \partial \mathcal{C}_K \subset \mathbb{R}^4$ be the Euclidean cone over ∂K as in Corollary 3.2.5, and let $g_{\mathcal{C}_{\partial K}}$ be the smooth Riemannian metric on $\mathcal{C}_{\partial K} \setminus 0$ induced by the standard metric of $\mathbb{R}^4$. Note that $g_{\mathcal{C}_{\partial K}}$ coincides with the cone metric on $\mathcal{C}_{\partial K} \setminus 0$ induced by $g_{\partial K}$. Let $\bar{H}$ be the isometry $(\Gamma, \bar{\gamma}) \to (\partial K, g_{\partial K})$. Then the map

$$\begin{array}{rcl} H : \mathcal{C}_\Gamma & \to & \mathcal{C}_{\partial K} \\ (p,t) & \mapsto & t\bar{H}(p) \end{array}$$

is a homeomorphism taking $o \in \mathcal{C}_\Gamma$ to $0 \in \mathcal{C}_{\partial K}$, and the restriction

$$H|_{\mathcal{C}_\Gamma \setminus o} : (\mathcal{C}_\Gamma \setminus o, \gamma_e) \to (\mathcal{C}_{\partial K} \setminus 0, g_{\mathcal{C}_{\partial K}})$$

is an isometry. By Corollary 3.2.5 there exists a smooth structure D on $\mathcal{C}_{\partial K}$ which agrees with the smooth structure of $\mathcal{C}_{\partial K} \setminus 0$. By pulling back this structure by H we obtain a smooth structure D_Γ on $\mathcal{C}_\Gamma$ which is compatible with the smooth structure of $\mathcal{C}_\Gamma \setminus o$. Clearly $H : (\mathcal{C}_\gamma, D_\Gamma) \to (\mathcal{C}_{\partial K}, D)$ is a diffeomorphism.

Let $U \ni o$ be a neighborhood of o in $\mathcal{C}_\Gamma$. Then $V = H(U) \subset \mathcal{C}_{\partial K}$ is a neighborhood of 0 in $\mathcal{C}_{\partial K}$, and in view of Corollary 3.2.5 we find a smooth nonnegatively curved metric $\tilde{g}$ on $(\mathcal{C}_{\partial K}, D)$ which coincides with $g_{\mathcal{C}_{\partial K}}$ off V. Then the pullback of $\tilde{g}$ under H is a smooth nonnegatively curved metric on $\mathcal{C}_\Gamma$ which coincides with γ_e off U.

□

Remark 3.2.7 ($\mathcal{C}_\Gamma$ as a metric space). Even though the cone metric γ_e fails to be continuous on $\mathcal{C}_\Gamma$ in general, we can nevertheless define a length metric on $\mathcal{C}_\Gamma$ induced by γ_e in the same way as for smooth Riemannian metrics. We choose a basis $\{v_1, v_2, v_3\}$ of $T_o\mathcal{C}_\Gamma$ and put $\gamma_e(o)(v_i, v_j) := \delta_{ij}$ for $i, j \in \{1, 2, 3\}$. This way γ_e is well defined on $\mathcal{C}_\Gamma$. For $x, y \in \mathcal{C}_\Gamma$ we put

$$\operatorname{dist}_{\gamma_e}(x,y) := \inf\{L_{\gamma_e}(c) \,|\, c : [a,b] \to \mathcal{C}_\Gamma \text{ is a piecewise } C^1 \text{ curve from } x \text{ to } y\},$$

where

$$L_{\gamma_e}(c) = \int_a^b \|\dot{c}(t)\|_{\gamma_e(c(t))} dt.$$

Note that $\operatorname{dist}_{\gamma_e}$ is finite, since for any $x = (p,s) \in \mathcal{C}_\Gamma$ the curve $c : [0,1] \to \mathcal{C}_\Gamma$, $c(t) = (p, ts)$ connects x to the vertex o and has finite length. Indeed, $c(t) = (\hat{c}(t), \tau(t))$, where $\hat{c}$ is the constant curve $\hat{c} \equiv p$, and $\tau(t) = ts$ for $t \in [0,1]$. Then $\dot{\hat{c}}(t) \equiv 0$ and $\dot{\tau}(t)^2 = s^2$. Thus, by definition of γ_e we have

$$L_{\gamma_e}(c) = \int_0^1 \sqrt{\tau(t)^2 \|\dot{\hat{c}}(t)\|_{\bar{\gamma}(\hat{c}(t))} + \dot{\tau}(t)^2}\, dt = \int_0^1 s\, dt = s < \infty.$$

Let us check that in fact $\operatorname{dist}_{\gamma_e}(x, o) = s$ for all $x = (p,s) \in \mathcal{C}_\Gamma$. In view of the definition of $\operatorname{dist}_{\gamma_e}$, the above computation shows that $\operatorname{dist}_{\gamma_e}(x,o) \leq s$. To show the reverse inequality let $x = (p,s) \in \mathcal{C}_\Gamma$, w.l.o.g. $s > 0$, and let $c : [a,b] \to \mathcal{C}_\Gamma$ be a piecewise C^1 curve connecting x and o, where $c(a) = x$, $c(b) = o$. Let

$$\tilde{b} = \inf\{t > a \,|\, c(t) = o\} \in (a, b].$$

Clearly $L_{\gamma_e}(c) \geq L_{\gamma_e}(c|_{[a,\tilde{b}]})$, and $c|_{[a,\tilde{b}]} = (\hat{c}, \tau)$, where $\hat{c} : [a, \tilde{b}] \to \Gamma$ and $\tau : [a, \tilde{b}] \to$

$\mathbb{R}_{\geq 0}$ are piecewise C^1, and τ connects s and 0. Then

$$\begin{aligned} L_{\gamma_e}(c|_{[a,\tilde b]}) &= \lim_{\beta\nearrow\tilde b}\int_a^\beta \|\dot c(t)\|_{\gamma_e(c(t))}dt \\ &= \lim_{\beta\nearrow\tilde b}\int_a^\beta \sqrt{\tau(t)^2\|\dot{\hat c}(t)\|_{\bar\gamma(\hat c(t))}+\dot\tau(t)^2}dt \\ &\geq \lim_{\beta\nearrow\tilde b}\int_a^\beta |\dot\tau(t)|\,dt = \int_a^{\tilde b}|\dot\tau(t)|\,dt = L_{eucl}(\tau)\geq s. \end{aligned}$$

A detailed discussion on length structures on cones in a more general setting can be found in [6].

Lemma 3.2.8. *Let $\mathcal{C}_\Gamma$, γ_e and $\tilde\gamma$ be as in Prop. 3.2.6. Furthermore, let* $\operatorname{dist}_{\gamma_e}$ *be as in Remark 3.2.7, and* $\operatorname{dist}_{\tilde\gamma}$ *be the distance function on $\mathcal{C}_\Gamma$ arising from the smooth metric $\tilde\gamma$. Then*

$$\operatorname{dist}_{\tilde\gamma}(\cdot,o)\geq \operatorname{dist}_{\gamma_e}(\cdot,o)$$

on $\mathcal{C}_\Gamma$.

Proof. Let $x=(p,s)\in\mathcal{C}_\Gamma$, w.l.o.g. $s>0$. Consider the curve $c:[0,1]\to\mathcal{C}_\Gamma$, $c(t)=(p,ts)$ connecting o and x as in Remark 3.2.7. Since $\operatorname{dist}_{\tilde\gamma}(x,o)$ is the infimum of the lengths w.r.t. $\tilde\gamma$ of all piecewise C^1 curves connecting o and x, it suffices to show that $L_{\tilde\gamma}(c)\leq s=\operatorname{dist}_{\gamma_e}(x,o)$.

Let $\mathcal{C}_{\partial K}=\operatorname{graph}(u)$ and $\tilde g$ be as in the proof of Prop. 3.2.6. Recall that by construction $H:(\mathcal{C}_\Gamma,\tilde\gamma)\to(\mathcal{C}_{\partial K},\tilde g)$ is an isometry. Thus, the claim follows if we show that $L_{\tilde g}(H\circ c)\leq s$.

Let $H(x)=(\hat y,u(\hat y))\in\operatorname{graph}(u)$. From construction of H it follows that the curve $\tilde c:=H\circ c$ is given by

$$\begin{aligned} \tilde c:[0,1] &\to \operatorname{graph}(u) \\ \tilde c(t) &= t\cdot(\hat y,u(\hat y))=(t\,\hat y,t\,u(\hat y)). \end{aligned}$$

Consider $\dot{\tilde c}(t)\in T_{\tilde c(t)}\big(\operatorname{graph}(u)\big)$. Let $\varphi=\pi|_{\operatorname{graph}(u)}:\operatorname{graph}(u)\to\mathbb{R}^3$, where π is the projection $\mathbb{R}^4\to\mathbb{R}^3$, be the coordinate chart of graph(u) (as in the proof of Cor. 3.2.5). For any $(\hat z,u(\hat z))\in\operatorname{graph}(u)$ we have $\varphi(\hat z,u(\hat z))=\hat z$ and $\varphi^i(\hat z,u(\hat z))=\hat z^i$, $i=1,2,3$. Thus, the coefficients of $\dot{\tilde c}(t)$ with respect to this chart are

$$\big(\dot{\tilde c}(t)\big)^i=\dot{\tilde c}(t)(\varphi^i)=\frac{d}{dt}\big(\varphi^i(\tilde c(t))\big)=\frac{d}{dt}(t\hat y^i)=\hat y^i.$$

By (3.2.4) we then have

$$\begin{aligned} \|\dot{\tilde c}(t)\|^2_{\tilde g(\tilde c(t))} &= \tilde g(\tilde c(t))\big(\dot{\tilde c}(t),\dot{\tilde c}(t)\big) \\ &= \tilde g_{ij}(\tilde c(t))\big((\dot{\tilde c}(t))^i,(\dot{\tilde c}(t))^j\big) \\ &= \big(\delta_{ij}+\partial_i\tilde u(\pi(\tilde c(t))\,\partial_j\tilde u(\pi(\tilde c(t))\big)\hat y^i\hat y^j \\ &= \big(\delta_{ij}+\partial_i\tilde u(t\hat y)\,\partial_j\tilde u(t\hat y)\big)\hat y^i\hat y^j \\ &= |\hat y|^2+\langle\nabla\tilde u(t\hat y),\hat y\rangle^2 \\ &= |\hat y|^2+\big((\tilde u(t\hat y))'\big)^2. \end{aligned} \tag{3.2.7}$$

Recall that by construction $\tilde u\equiv u$ on $\mathbb{R}^3\setminus B_\varepsilon(0)$, $\varepsilon>0$, so that $\tilde u(t\hat y)=u(t\hat y)$ for $t\geq\tilde\varepsilon$, where $\tilde\varepsilon:=\varepsilon/|\hat y|$. In particular, for $t\geq\tilde\varepsilon$ we have

$$(\tilde u(t\hat y))'=(u(t\hat y))'=(tu(\hat y))'=u(\hat y).$$

Moreover, since $\nabla\tilde{u}$ vanishes near 0 we have $(\tilde{u}(t\hat{y}))' = \langle \nabla\tilde{u}(t\hat{y}), \hat{y}\rangle = 0$ at $t = 0$. From the fact that $\tilde{u}$ is convex it follows that $t \mapsto u(t\hat{y})$ is convex, which implies that $t \mapsto (\tilde{u}(t\hat{y}))'$ is non-decreasing. Thus

$$0 \leq (\tilde{u}(t\hat{y}))' \leq u(\hat{y})$$

for all $t \geq 0$. Using this and (3.2.7), we compute

$$\begin{aligned} L_{\tilde{g}}(\tilde{c}) &= \int_0^1 \|\dot{\tilde{c}}(t)\|_{\tilde{g}(\tilde{c}(t))} dt \\ &= \int_0^1 \sqrt{|\hat{y}|^2 + \left((\tilde{u}(t\hat{y}))'\right)^2} \\ &\leq \int_0^1 \sqrt{|\hat{y}|^2 + u(\hat{y})^2} \\ &= |(\hat{y}, u(\hat{y}))| = |H((p,s))| = s \end{aligned}$$

and we are done.

$\square$

3.3. Modifying metrics on equidistant hypersurfaces

Let M be a smooth n-dim. manifold, equipped with a smooth Riemannian metric g. Let $\Gamma \subset M$ be a smooth hypersurface and $p_0 \in \Gamma$. On a small enough neighborhood $U \ni p_0$ we introduce Fermi coordinates $x = (x^1, \ldots, x^n)$ above Γ (cf. Appendix A.1), such that $|x^n| = \operatorname{dist}_g(\cdot, \Gamma)$. Let $\Gamma(d) = \{p \in U \,|\, x^n(p) = d\}$ denote the equidistant hypersurfaces of Γ. In our coordinates g has the form

$$g = \begin{pmatrix} \hat{g} & 0 \\ 0 & 1 \end{pmatrix},$$

where $\hat{g}$ is the restriction of g to the equidistant hypersurfaces, that is, $\hat{g}(p) = g(p)|_{T_p\Gamma(d)\times T_p\Gamma(d)}$ for $p \in U$ with $x^n(p) = d$. At each point $p \in U$, we have the decomposition

$$T_pU = T_p\Gamma(d) \oplus T_p\Gamma(d)^{\perp_g},$$

where $T_p\Gamma(d)^{\perp_g}$ is the orthogonal complement of $T_p\Gamma(d)$ in T_pU with respect to g. In what follows, we will identify $\hat{g}$ and $\langle \cdot, P^T \cdot \rangle_g$, where P^T denotes the projection $TU \to T\Gamma(d)$, so that $\hat{g}$ may be regarded as a 2-tensor on TU.

Let $\varphi : (-a, a) \to \mathbb{R}_{>0}$ be a smooth function. We may regard φ as a function on U by putting $\varphi(p) = \varphi(x^n(p))$. Since φ is strictly positive, we may define a new metric $\tilde{g}$ on U by

$$\tilde{g} = \begin{pmatrix} \varphi\hat{g} & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.3.1}$$

The goal of the current section is to compute the curvature operator $\mathcal{R}(\tilde{g})$ of the modified metric. First, let us introduce some further notations. Let $L(g)$ denote the second fundamental form of the equidistant hypersurfaces $(\Gamma(d), \hat{g})$ in (U, g) with respect to the normal $N = \frac{\partial}{\partial x^n}$. More precisely, given a point $p \in U$ and the equidistant hypersurface $\Gamma(d)$ containing p (i.e. $d = x^n(p)$) we have

$$\begin{aligned} L(g)(p) &\in (T_p\Gamma(d) \times T_p\Gamma(d))^* \\ (X, Y) &\mapsto -\langle {}^g\nabla_X N, Y\rangle_g \end{aligned}$$

In our coordinates we have $L(g)_{ij}(x(p)) = -\frac{1}{2}\partial_n g_{ij}(x(p))$. Indeed,

$$\begin{aligned} L(g)(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) &= -\langle {}^g\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^n}, \frac{\partial}{\partial x^j}\rangle_g \\ &= -{}^g\Gamma^k_{in} g_{kj} \\ &= -\frac{1}{2} g^{kl}(\partial_i g_{nl} + \partial_n g_{il} - \partial_l g_{in}) g_{kj} \\ &= -\frac{1}{2}(\partial_i g_{nj} + \partial_n g_{ij} - \partial_j g_{in}) \\ &= -\frac{1}{2}\partial_n g_{ij}, \end{aligned} \tag{3.3.2}$$

where we used that $g_{nj} \equiv const$ for all j in our coordinates. Observe that $L(g) = \langle {}^g\nabla_{(\cdot)} N, \cdot\rangle_g$ can also be regarded as a symmetric 2-tensor on TU, where $L(g)(X, N) = 0 = L(g)(N, N)$ for $X \in T\Gamma(d)$.

Similarly, we denote by $L(\tilde{g})$ the second fundamental form of $(\Gamma(d), \varphi\hat{g})$ in $(U, \tilde{g})$ with respect to $\frac{\partial}{\partial x^n}$ (note that $\frac{\partial}{\partial x^n}$ is also the unit normal of $T\Gamma(d)$ with respect to $\tilde{g}$ in view of (3.3.1)). A similar computation as above shows that $L(\tilde{g})_{ij} = -\frac{1}{2}\partial_n \tilde{g}_{ij}$ in our coordinates.

Finally, let P^N denote the projection $TM \to (T\Gamma(d))^\perp$. In what follows, we identify P^N and the two-tensor $\langle \cdot, P^N \cdot\rangle_g$, which in our coordinates is given by $P^N_{ij} = \delta_{ni}\delta_{nj}$ (cf. Notation 1.2.8).

Note that since both $L(g)$ and $\hat{g}$ can be seen as 2-tensors on $T\Gamma(d)$ as well as 2-tensors on $TU \subset TM$, the Kulkarni-Nomizu products (see Section B.2) $L(g) \wedge \hat{g}$ and $\hat{g} \wedge P^N$ may be regarded as sections of $\Lambda^2(T\Gamma(d))$ as well as of $\Lambda^2(TU)$. In what follows, it will be clear from the context which interpretation is being used.

Proposition 3.3.1. *Let g and $\tilde{g}$ be as above, and let $\mathcal{R}(g)$ and $\mathcal{R}(\tilde{g})$ be the Riemannian curvature operators of g and $\tilde{g}$, respectively. We have*

$$\begin{aligned} \mathcal{R}(\tilde{g}) = \varphi\mathcal{R}(g) \quad &+ \quad (1-\varphi)\varphi L(g) \wedge L(g) + \varphi\varphi' L(g) \wedge \hat{g} - \frac{1}{4}(\varphi')^2 \hat{g} \wedge \hat{g} \\ &+ \quad \frac{-2\varphi''\varphi + (\varphi')^2}{2\varphi} \hat{g} \wedge P^N + 2\varphi' L(g) \wedge P^N. \end{aligned} \tag{3.3.3}$$

Proof. We denote by $\partial_i = \frac{\partial}{\partial x^i}$, $i = 1, \ldots, n$, the coordinate vectors with respect to the coordinate chart $(x^1, \ldots, x^n)$. First, let us compute $\mathcal{R}(\tilde{g})$ on $\Lambda^2(T\Gamma(d)) = span\{\partial_i \wedge \partial_j | 1 \leq i < j \leq n-1\}$. By the Gauss theorem we have

$$\mathcal{R}(\tilde{g})|_{\Lambda^2(T\Gamma(d))} = \mathcal{R}(\varphi\hat{g}) - L(\tilde{g}) \wedge L(\tilde{g}). \tag{3.3.4}$$

Since $\varphi|_{\Gamma(d)} = const$, we have $\mathcal{R}(\varphi\hat{g}) = \varphi\mathcal{R}(\hat{g})$. Moreover, in view of the above discussion

$$L(\tilde{g})_{ij} = -\frac{1}{2}\partial_n(\varphi\hat{g}_{ij}) = -\frac{\varphi'}{2}\hat{g}_{ij} - \frac{\varphi}{2}\hat{g}_{ij} = -\frac{\varphi'}{2}\hat{g}_{ij} + \varphi L(g)_{ij}.$$

Using this in (3.3.4) we get

$$\begin{aligned} \mathcal{R}(\tilde{g})|_{\Lambda^2(T\Gamma(d))} &= \varphi\mathcal{R}(\hat{g}) - \big(\varphi L(g) - \frac{\varphi'}{2}\hat{g}\big) \wedge \big(\varphi L(g) - \frac{\varphi'}{2}\hat{g}\big) \qquad (3.3.5) \\ &= \varphi\big(\mathcal{R}(g)|_{\Lambda^2(T\Gamma(d))} + L(g) \wedge L(g)\big) \\ &\quad -\varphi^2 L(g) \wedge L(g) + \varphi\varphi' L(g) \wedge \hat{g} - \frac{1}{4}(\varphi')^2 \hat{g} \wedge \hat{g} \\ &= \varphi\mathcal{R}(g)|_{(\Lambda^2 T\Gamma(d))} \\ &\quad +(1-\varphi)\varphi L(g) \wedge L(g) + \varphi\varphi' L(g) \wedge \hat{g} - \frac{1}{4}(\varphi')^2 \hat{g} \wedge \hat{g}, \end{aligned}$$

which is the desired equation on $\Lambda^2(T\Gamma(d))$, since the operators $\hat{g}\wedge P^N$ and $L(g)\wedge P^N$ vanish on $\Lambda^2(T\Gamma(d))$.

In view of the symmetries of the curvature operator, to verify that (3.3.3) holds on $\left(\Lambda^2(T\Gamma(d))\right)^\perp$ it suffices to compute $\mathcal{R}(\tilde{g})(\partial_i\wedge\partial_j,\partial_k\wedge\partial_n)$ and $\mathcal{R}(\tilde{g})(\partial_i\wedge\partial_n,\partial_k\wedge\partial_n)$, $1\le i,j,k\le n-1$. In the first case all operators on the right hand side of (3.3.3) except for $\mathcal{R}(g)$ vanish, and we have to check

$$\mathcal{R}(\tilde{g})(\partial_i\wedge\partial_j,\partial_k\wedge\partial_n)=\varphi\mathcal{R}(g)(\partial_i\wedge\partial_j,\partial_k\wedge\partial_n). \tag{3.3.6}$$

Let us fix a point p near Γ and choose coordinates $x_1,\dots,x_{n-1}$ on $\Gamma(d(p))$ near p such that $\hat{g}_{ij}(p)=g(p)(\partial_i|_p,\partial_j|_p)=\delta_{ij}$ and $\partial_i g_{jk}(p)=0$ for all $1\le i,j,k\le n-1$ (note that $\partial_n g_{ij}$ does not necessarily vanish in p). We denote by $\tilde{R}_{ijkl}$ the curvature tensor of $\tilde{g}$ and by $\tilde{\Gamma}^k_{ij}$ the Christoffel symbols of $\tilde{g}$. We have

$$\tilde{R}_{ijkl}=(\partial_j\tilde{\Gamma}^s_{ik}-\partial_i\tilde{\Gamma}^s_{jk}+\tilde{\Gamma}^r_{ik}\tilde{\Gamma}^s_{rj}-\tilde{\Gamma}^r_{jk}\tilde{\Gamma}^s_{ri})\tilde{g}_{sl}.$$

Recall that in our coordinates $\tilde{g}_{sn}=\delta_{sn}$ near p for all $s=1,\dots,n$, and consequently $\partial_r\tilde{g}_{sn}(p)=0$ for all $r,s=1,\dots,n$. This gives us

$$\tilde{R}_{ijkn}=\partial_j\tilde{\Gamma}^n_{ik}-\partial_i\tilde{\Gamma}^n_{jk}+\tilde{\Gamma}^r_{ik}\tilde{\Gamma}^n_{rj}-\tilde{\Gamma}^r_{jk}\tilde{\Gamma}^n_{ri} \tag{3.3.7}$$

and

$$\begin{aligned}\tilde{\Gamma}^n_{ik} &= \frac{1}{2}\tilde{g}^{nr}(\partial_i\tilde{g}_{kr}+\partial_k\tilde{g}_{ir}-\partial_r\tilde{g}_{ik})\\ &= \frac{1}{2}(\partial_i\tilde{g}_{kn}+\partial_k\tilde{g}_{in}-\partial_n\tilde{g}_{ik})\\ &= -\frac{1}{2}\partial_n\tilde{g}_{ik}\\ &= -\frac{1}{2}\varphi' g_{ik}-\varphi\frac{1}{2}\partial_n g_{ik}=-\frac{1}{2}\varphi' g_{ik}+\varphi\Gamma^n_{ik}.\end{aligned} \tag{3.3.8}$$

Thus, at p we have

$$\partial_j\tilde{\Gamma}^n_{ik}=-\frac{1}{2}\varphi'\partial_j g_{ik}-\varphi\frac{1}{2}\partial_j\partial_n g_{ik}=-\varphi\frac{1}{2}\partial_j\partial_n g_{ik}=\varphi\partial_j\Gamma^n_{ik},$$

where we used that φ depends only on x^n. By an analogous computation we have

$$\partial_i\tilde{\Gamma}^n_{jk}=\varphi\partial_i\Gamma^n_{jk}$$

at p. Moreover, the last two terms in (3.3.7) vanish at p. Indeed, we have

$$\tilde{\Gamma}^r_{ik}\tilde{\Gamma}^n_{rj}=\sum_{r=1}^{n-1}\tilde{\Gamma}^r_{ik}\tilde{\Gamma}^n_{rj}+\tilde{\Gamma}^n_{ik}\tilde{\Gamma}^n_{nj}=0$$

since $\tilde{\Gamma}^n_{nj}\equiv 0$ for all j, and $\tilde{\Gamma}^r_{ik}(p)=\varphi\Gamma^r_{ik}(p)=0$ by our choice of coordinates. This gives us $\tilde{R}_{ijkn}=\varphi R_{ijkn}$, which shows (3.3.6).

Let us now consider $\tilde{\mathcal{R}}(\partial_i\wedge\partial_n,\partial_k\wedge\partial_n)=\tilde{R}_{inkn}$. In this case (3.3.3) becomes

$$\begin{aligned}\tilde{\mathcal{R}}(\partial_i\wedge\partial_n,\partial_k\wedge\partial_n) &= \mathcal{R}(\partial_i\wedge\partial_n,\partial_k\wedge\partial_n)\\ &\quad+\frac{-2\varphi''\varphi+(\varphi')^2}{2\varphi}\hat{g}\wedge P^N+2\varphi' L(g)\wedge P^N\end{aligned} \tag{3.3.9}$$

since L and $\hat{g}$ vanish on $(T\Gamma(d))^N$. Similar as above, in view of $\tilde{\Gamma}^n_{nk} \equiv 0$ for all k one has

$$\tilde{R}_{inkn} = \partial_n \tilde{\Gamma}^n_{ik} - \tilde{\Gamma}^r_{nk}\tilde{\Gamma}^n_{ir} \tag{3.3.10}$$

at p. Using (3.3.8) and (3.3.2) we compute

$$\begin{aligned} \partial_n \tilde{\Gamma}^n_{ik} &= \partial_n(-\frac{1}{2}\varphi' g_{ik} - \varphi\frac{1}{2}\partial_n g_{ik}) \\ &= -\frac{1}{2}\varphi'' g_{ik} - \varphi'\partial_n g_{ik} - \varphi\frac{1}{2}\partial_n^2 g_{ik} \\ &= -\frac{1}{2}\varphi'' g_{ik} + 2\varphi' L_{ik} + \varphi\partial_n\Gamma^n_{ik}. \end{aligned} \tag{3.3.11}$$

Moreover, for $1 \leq r \leq n-1$ we have

$$\begin{aligned} \tilde{\Gamma}^r_{nk} &= \frac{1}{2}\tilde{g}^{rs}(\partial_n\tilde{g}_{ks} + \partial_k\tilde{g}_{ns} + \partial_s\tilde{g}_{nk}) \\ &= \frac{1}{2}\tilde{g}^{rs}\partial_n\tilde{g}_{ks} = \frac{1}{2\varphi}g^{rs}\partial_n(\varphi g_{ks}) \\ &= \frac{\varphi'}{2\varphi}\delta^r_k + \frac{1}{2}g^{rs}\partial_n g_{ks} \\ &= \frac{\varphi'}{2\varphi}\delta^r_k + \Gamma^r_{nk}, \end{aligned}$$

which gives us

$$\begin{aligned} -\tilde{\Gamma}^r_{nk}\tilde{\Gamma}^n_{ir} &= -(\frac{\varphi'}{2\varphi}\delta^r_k + \Gamma^r_{nk})(-\frac{1}{2}\varphi' g_{ir} + \varphi\Gamma^n_{ir}) \\ &= \frac{(\varphi')^2}{4\varphi}g_{ik} - \frac{\varphi'}{2}\Gamma^n_{ik} + \frac{\varphi'}{2}\Gamma^r_{nk}g_{ir} - \varphi\Gamma^r_{nk}\Gamma^n_{ir} \\ &= \frac{(\varphi')^2}{4\varphi}g_{ik} - \varphi' L_{ik} - \varphi\Gamma^r_{nk}\Gamma^n_{ir}, \end{aligned} \tag{3.3.12}$$

where in the last step we used the fact that in our coordinates

$$L_{ik} = \langle -\nabla_{\partial_i}\partial_n, \partial_k\rangle_g = -\Gamma^r_{in}g_{rk} = -\frac{1}{2}g^{rs}\partial_n g_{is}g_{rk} = -\frac{1}{2}\partial_n g_{ik} = \Gamma^n_{ik}.$$

Combining (3.3.10), (3.3.11) and (3.3.12) gives

$$\begin{aligned} \tilde{R}_{inkn} &= \varphi R_{inkn} + \frac{-2\varphi\varphi'' + (\varphi')^2}{4\varphi}g_{ik} + \varphi' L_{ik} \\ &= \varphi R_{inkn} + \frac{-2\varphi\varphi'' + (\varphi')^2}{2\varphi}(\hat{g} \wedge P^N)_{inkn} + 2\varphi'(L \wedge P^N)_{inkn}, \end{aligned}$$

which shows (3.3.9), and we are done.

□

As an example, we compute the curvature operator of a spherical a-cone.

Example 3.3.2. Consider a spherical a-cone $(\mathcal{C}, \gamma) = (\mathcal{C}_{\Gamma,a}, \gamma_a)$, $a \geq 1$, with vertex o over a smooth Riemannian manifold $(\Gamma, \bar{\gamma})$ (see Definition 3.2.2). We introduce Fermi coordinates $(\hat{x}, x^n)$ above the hypersurface $\Gamma \times \{1\} = \partial B^\gamma_1(o)$, where $\hat{x}$ are local coordinates on Γ, and

$$x^n = 1 - \mathrm{dist}_\gamma(\cdot, o) : \mathcal{C} \setminus o \to (1 - \frac{\pi}{2}a, 1).$$

In this coordinates the metric γ has the form

$$\gamma(\hat{x}, x^n) = \begin{pmatrix} a^2\sin^2(\frac{1-x^n}{a})\bar{\gamma}(\hat{x}) & 0 \\ 0 & 1 \end{pmatrix}.$$

The induced metric on $\Lambda^2(T\mathcal{C})$ is $\gamma\wedge\gamma = \hat{\gamma}\wedge\hat{\gamma} + 2\hat{\gamma}\wedge P^N$, where $\hat{\gamma}$ is the restriction of γ to the equidistant hypersurfaces of $\Gamma\times\{1\}$, in coordinates $\hat{\gamma}(\hat{x}, x^n) = a^2\sin^2(\frac{1-x^n}{a})\bar{\gamma}(\hat{x})$. Suppose that the curvature operator of $\bar{\gamma}$ is ≥ 1, that is, $\mathcal{R}(\bar{\gamma}) \geq \bar{\gamma}\wedge\bar{\gamma}$. Putting $g = \bar{\gamma} + dr^2$ and $\varphi(x^n) = a^2\sin^2(\frac{1-x^n}{a})$ in Proposition 3.3.1 we compute using $L(g)(\hat{x}, x^n) = \frac{1}{2}\partial_n\bar{\gamma}(\hat{x}) \equiv 0$ on $\mathcal{C}\setminus o$

$$\begin{aligned}
\mathcal{R}(\gamma) &= \varphi\mathcal{R}(\bar{\gamma}) - \frac{1}{4}(\varphi')^2\bar{\gamma}\wedge\bar{\gamma} + \frac{-2\varphi''\varphi + (\varphi')^2}{2\varphi}\bar{\gamma}\wedge P^N \\
&= a^2\sin^2\Big(\frac{1-x^n}{a}\Big)\mathcal{R}(\bar{\gamma}) \\
&- a^2\sin^2\Big(\frac{1-x^n}{a}\Big)\cos^2\Big(\frac{1-x^n}{a}\Big)\bar{\gamma}\wedge\bar{\gamma} + 2\sin^2\Big(\frac{1-x^n}{a}\Big)\bar{\gamma}\wedge P^N \\
&\geq a^2\sin^2\Big(\frac{1-x^n}{a}\Big)\bar{\gamma}\wedge\bar{\gamma} \\
&- a^2\sin^2\Big(\frac{1-x^n}{a}\Big)\cos^2\Big(\frac{1-x^n}{a}\Big)\bar{\gamma}\wedge\bar{\gamma} + 2\sin^2\Big(\frac{1-x^n}{a}\Big)\bar{\gamma}\wedge P^N \\
&= a^2\sin^4\Big(\frac{1-x^n}{a}\Big)\bar{\gamma}\wedge\bar{\gamma} + 2\sin^2\Big(\frac{1-x^n}{a}\Big)\bar{\gamma}\wedge P^N \\
&= \frac{1}{a^2}\hat{\gamma}\wedge\hat{\gamma} + \frac{1}{a^2}2\hat{\gamma}\wedge P^N \\
&= \frac{1}{a^2}\gamma\wedge\gamma.
\end{aligned}$$

An analogous computation shows that $\mathcal{R}(\gamma) \geq \frac{1}{a^2}$ implies $\mathcal{R}(\bar{\gamma}) \geq 1$.

3.4. Smoothing cone-like singularities in dimension 3

Let us fix some notation for this section. In what follows, by 'smooth' we mean 'C^∞ smooth', unless noted differently. We are concerned with Riemannian manifolds (M, g), where g is possibly singular (discontinuous) at a fixed point $o \in M$, and smooth on $M\setminus o$. We wish to introduce a distance function dist_g induced by g on M in a similar way as for smooth Riemannian metrics. In order to do so, we put

$$\operatorname{dist}_g(p,q) = \inf\{L_g(c)\,|\,c \text{ is a piecewise } C^1 \text{ curve connecting } p \text{ and } q\},$$

and $\operatorname{dist}_g(o,o) := 0$, where $L_g(c)$ is the length of c with respect to g, which is defined as follows: For a piecewise C^1 curve $c : [a,b] \to M\setminus o$ we put $L_g(c) = \int_a^b \|\dot{c}(t)\|_{g(c(t))}dt$. Furthermore, if $c(t_0) = o$ for some $t_0 \in [a,b]$, we define

$$L_g(c) = \int_a^b \|\dot{c}(t)\|_{g(c(t))}dt := \lim_{\alpha\searrow 0}\int_a^{t_0-\alpha} \|\dot{c}(t)\|_{g(c(t))}dt + \lim_{\alpha\searrow 0}\int_{t_0+\alpha}^b \|\dot{c}(t)\|_{g(c(t))}dt$$

(note that in the definition of dist_g it suffices to consider piecewise C^1 curves which pass through o not more than once).

Observe that, in general, we cannot expect $\operatorname{dist}_g(\cdot,\cdot)$ to be finite since g is possibly singular at o. In what follows, we shall assume that M is connected and $\operatorname{dist}_g(\cdot, o) : M \to \mathbb{R}$ is continuous at o. Using the assumption that g is smooth on $M\setminus o$, one verifies that in this case $\operatorname{dist}_g : M\times M \to \mathbb{R}$ is a continuous (in particular finite) metric, and the topology induced by dist_g coincides with that of M (see Lemma C.2.4).

We denote by B^g_r the metric balls

$$B^g_r := \{p \in M \,|\, \operatorname{dist}_g(p, o) < r\}$$

centered at o. We will also assume that there exists a small neighborhood $U \ni o$ such that $\operatorname{dist}_g(\cdot, o)$ is smooth on $U \setminus o$, and $\nabla(\operatorname{dist}_g(\cdot, o)) \neq 0$ on $U \setminus o$ (where $\nabla = \nabla^g$). This ensures that the level sets of $\operatorname{dist}_g(\cdot, o)$, which we denote by

$$\Gamma_g(r) := \{p \in M \,|\, \operatorname{dist}_g(p, o) = r\} = \partial B^g_r,$$

are smooth hypersurfaces of M for all $0 < r \leq r_0$ for a small enough $r_0 > 0$ (note that this is the case if g is smooth on M and r_0 is less than the injectivity radius of o). This notation is slightly different as compared to the previous section, where we denoted by $\Gamma(d)$ the equidistant hypersurfaces of a fixed hypersurface Γ.

For each $0 < r \leq r_0$, the vector field $-\nabla(\operatorname{dist}_g(\cdot, o))|_{\Gamma_g(r)}$ is the unit normal field on $\Gamma_g(r)$ pointing inside $\bar{B}^g_r$ (see Lemma C.2.6). Similarly as in the previous section, we denote by $\hat{g}$ the restriction of g to the hypersurfaces $\Gamma_g(r)$, that is, $\hat{g}(p) = g(p)|_{T_p\Gamma_g(r) \times T_p\Gamma_g(r)}$ at a point $p \in \Gamma_g(r)$. We denote by $L(g, \Gamma_g(r))$ the second fundamental form of $(\Gamma_g(r), \hat{g})$ in (M, g) with respect to the normal $-\nabla(\operatorname{dist}_g(\cdot, o))|_{\Gamma_g(r)}$. We also regard $\hat{g}$ and $L(g, \Gamma_g(r))$ as tensors on TM after extending them by 0 in normal direction.

In view of these notations, the main result of this chapter reads as follows:

Theorem 3.4.1. *Consider a 3-dim. connected Riemannian manifold (M, D_M, g) (where D_M denotes the smooth structure of M) which satisfies the following conditions:*

1) g is smooth on $M \setminus o$ (and possibly singular at o).

2) a) $\operatorname{dist}_g(\cdot, o)$ is continuous at o

b) There exists a neighborhood U of o such that $\operatorname{dist}_g(\cdot, o)$ is smooth on $U \setminus o$, and $\nabla(\operatorname{dist}_g(\cdot, o)) \neq 0$.

3) There exists a $r_0 > 0$ such that

a) $\Gamma_g(r_0)$ is homeomorphic to the sphere S^2

b) for all $0 < \delta \leq r_0$ the second fundamental form of $\Gamma_g(\delta)$ with respect to the inward normal satisfies

$$(1 - \varepsilon(\delta))\hat{g} \leq \delta\, L(g, \Gamma_g(\delta)) \leq (1 + \varepsilon(\delta))\hat{g},$$

where $\varepsilon(\delta)/\delta^2 \to 0$ as $\delta \to 0$.

4) $\mathcal{R}(g) \geq 0$ on $M \setminus o$.

Then there exists a family of metrics $\{g_i\}_{i\in\mathbb{N}}$ and smooth structures $\{D_i\}_{i\in\mathbb{N}}$ on M such that

- *D_i is compatible with D_M on $M \setminus o$, and (M, D_i) is diffeomorphic to (M, D_M) for all $i \in \mathbb{N}$*
- *g_i is C^2 smooth on (M, D_i) and C^∞ smooth on $M \setminus B^g_{r_0/i}$*

- *$g_i \to g$ off o in the C^∞ sense as $i \to \infty$*
- *$\mathcal{R}(g_i) \geq -\varepsilon_i$, where $\varepsilon_i \to 0$ as $i \to \infty$.*

Remark 3.4.2. *Condition 3* may be interpreted to the effect that the singular metric in some sense becomes cone-like near o. Indeed, consider a standard cone $S^{n-1} \times [0,\infty) = \mathbb{R}^n$ with vertex 0, equipped with the metric $\gamma = r^2\bar{\gamma} \otimes dr^2$, where $\bar{\gamma}$ is a smooth Riemannian metric on S^{n-1} and dr^2 is the standard metric on $[0,\infty)$. Then $\Gamma_\gamma(r) = rS^{n-1}$ for all $r > 0$, and at a point $(x,r) \in S^{n-1} \times (0,\infty)$ the second fundamental form of $\Gamma_\gamma(r)$ with respect to the inward normal is given by $L(\gamma, \Gamma_\gamma(r))(x,r) = r\bar{\gamma}(x) = \frac{1}{r}(r^2\bar{\gamma}(x)) = \frac{1}{r}\gamma|_{\Gamma_\gamma(r)}(x,r)$.

Proof of Theorem 3.4.1. The proof breaks up into the following steps:

Step 1: We consider the rescaled metric $g_\delta = \frac{1}{\delta^2}g$ on M. On $\bar{B}_1^{g_\delta}$, we replace g_δ by the Euclidean cone metric $\gamma_\delta = r^2\bar{g}_\delta + dr^2$, where $\bar{g}_\delta$ is the restriction of g_δ to $\bar{\Gamma}_\delta := \Gamma_{g_\delta}(1)$.

Step 2: By modifying the metrics γ_δ on $\bar{B}_1^{g_\delta}$ and g_δ on $M \setminus B_1^{g_\delta}$ near $\bar{\Gamma}_\delta$ we construct metrics $\tilde{\gamma}_\delta$ and $\tilde{g}_\delta$ such that the new metrics meet the requirements of Theorem 1.1.2.

Step 3: We show that the sum of the second fundamental forms of $\tilde{\gamma}_\delta$ and $\tilde{g}_\delta$ is nonnegative on $\bar{\Gamma}_\delta$.

Step 4: We show that the curvature operators of $\tilde{\gamma}_\delta$ and $\tilde{g}_\delta$ are almost nonnegative.

Step 5: Using constructions from Section 3.2 we introduce a new smooth structure D_δ on M which is compatible with D_M on $M \setminus o$, and find a smooth metric $\gamma_\delta^{(sm)}$ on $\bar{B}_1^{g_\delta}$, which coincides with $\tilde{\gamma}_\delta$ off a small neighborhood of o. The curvature stays almost nonnegative. In this step the assumption $\dim(M) = 3$ is necessary, since here we use Proposition 3.2.6.

Step 6: We apply Theorem 1.1.2 to $(\bar{B}_1^{g_\delta}, \gamma_\delta^{(sm)})$ and $(M \setminus B_1^{g_\delta}, \tilde{g}_\delta)$, and find a C^2 smooth (w.r.t. D_δ) metric $g_\delta^{(sm)}$ which coincides with $\gamma_\delta^{(sm)}$ and $\tilde{g}_\delta$, respectively, off a small neighborhood of $\bar{\Gamma}_\delta$, and has almost nonnegative curvature.

Step 7: By scaling back $g_\delta^{(sm)}$ we construct the sequence g_i with the desired properties.

Step 1: First, observe that for small enough $r_0 > 0$ we can cover $\bar{B}_{r_0}^g \setminus o$ with a set of Fermi coordinates over $\Gamma_g(\delta)$, for any $0 < \delta \leq r_0$. Roughly speaking, this is because on a small neighborhood $U \ni o$ the integral curves of $-\nabla(\mathrm{dist}_g(\cdot, o))$ are unique unit speed geodesics connecting points of $\bar{B}_{r_0}^g \setminus o$ to hypersurfaces $\Gamma_g(\delta)$, so for small enough r_0 the ball $B_{r_0}^g$ enjoys similar properties as a geodesic ball in a smooth Riemannian manifold (we refer to Section C.2 of the appendix for a detailed discussion, see Lemma C.2.6, Remark C.2.7). In particular, in view of *Condition 3a* this implies that the equidistant hypersurfaces $\Gamma_g(\delta)$ are homeomorphic to S^2 for all $0 < \delta \leq r_0$.

In what follows we assume w.l.o.g. $r_0 = 1$ and $\varepsilon(\delta) \leq 1$ (after rescaling the metric g). Let $g_\delta = \frac{1}{\delta^2}g$ be the rescaled metric on M, and $\bar{\Gamma}_\delta := \Gamma_{g_\delta}(1) = \Gamma_g(\delta)$.

Let us fix a small $\delta > 0$. We introduce Fermi coordinates $(\hat{x}, x^n)$ above $\bar{\Gamma}_\delta$ such that $\hat{x} = (x^1, \ldots, x^{n-1})$ is a coordinate chart of $\bar{\Gamma}_\delta$, and

$$x^n(p) = \begin{cases} \operatorname{dist}_{g_\delta}(p, \bar{\Gamma}_\delta) & \text{if } p \in \bar{B}_1^{g_\delta} \setminus o \\ -\operatorname{dist}_{g_\delta}(p, \bar{\Gamma}_\delta) & \text{if } p \in B_{1/\delta}^{g_\delta} \setminus B_1^{g_\delta} \end{cases} .$$

Note that the charts $(x^1, \ldots, x^n)$ do depend on δ, but we shall omit the index δ to simplify the notation. Our coordinates are well defined for $-1/\delta + 1 \leq x^n < 1$. In view of $\operatorname{dist}_{g_\delta}(\cdot, o) = 1 - x^n$ we have that $(\hat{x}, 1-r)$ is a coordinate chart of the equidistant hypersurface $\Gamma_{g_\delta}(r)$ for any fixed $0 < r \leq 1/\delta$.

As before, we use the notation $\frac{\partial}{\partial x^i} = \partial_i$, $i = 1, \ldots, n$ for coordinate vectors. In view of our choice of coordinates ∂_n is the inward normal on the hypersurfaces $\Gamma_{g_\delta}(r)$ in $(\bar{B}_r^{g_\delta}, g_\delta)$, $0 < r \leq 1/\delta$. In particular, ∂_n is the inward normal on $\bar{\Gamma}_\delta$ in $(\bar{B}_1^{g_\delta}, g_\delta)$. Since $g_\delta = \frac{1}{\delta^2} g$, the inward normal on $\Gamma_g(\delta) = \Gamma_{g_\delta}(1)$ in $(\bar{B}_\delta^g, g)$ is given by $\frac{1}{\delta}\partial_n$. Thus the second fundamental form of $\bar{\Gamma}_\delta$ satisfies

$$L(g_\delta, \bar{\Gamma}_\delta) = \frac{1}{\delta} L(g, \bar{\Gamma}_\delta) = \frac{1}{\delta} L(g, \Gamma_g(\delta))$$

and *Condition 3b* reads

$$(1 - \varepsilon(\delta))\bar{g}_\delta \leq L(g_\delta, \bar{\Gamma}_\delta) \leq (1 + \varepsilon(\delta))\bar{g}_\delta, \tag{3.4.1}$$

where $\bar{g}_\delta := g_\delta|_{\bar{\Gamma}_\delta}$ denotes the restriction of g_δ to $\bar{\Gamma}_\delta$.

We now replace g_δ on $\bar{B}_1^{g_\delta}$ by an Euclidean cone metric. Observe that in our coordinates g_δ has the form

$$g_\delta(x) = \begin{pmatrix} \hat{g}_\delta(x) & 0 \\ 0 & 1 \end{pmatrix},$$

where $\hat{g}_\delta(x)$ is the restriction of g_δ to the equidistant hypersurfaces $\Gamma_{g_\delta}(\operatorname{dist}_{g_\delta}(x, o))$. We introduce a new metric γ_δ on $\bar{B}_1^{g_\delta}$ by putting

$$\gamma_\delta(x) = \begin{pmatrix} (1 - x^n)^2 \bar{g}_\delta(\hat{x}) & 0 \\ 0 & 1 \end{pmatrix}$$

for $0 \leq x^n < 1$. Since by definition $\bar{g}_\delta = g_\delta|_{\bar{\Gamma}_\delta}$, the metrics γ_δ and g_δ coincide on $\bar{\Gamma}_\delta$. Moreover, since we modified g_δ only in 'tangential' direction, the distance functions $\operatorname{dist}_{g_\delta}(\cdot, o)$ and $\operatorname{dist}_{\gamma_\delta}(\cdot, o)$ coincide, so that $B_r^{\gamma_\delta} = B_r^{g_\delta}$ and $\Gamma_{\gamma_\delta}(r) = \Gamma_{g_\delta}(r)$ for all $0 < r \leq 1$ (in particular $\bar{\Gamma}_\delta = \Gamma_{g_\delta}(1) = \Gamma_{\gamma_\delta}(1)$). Finally, ∂_n is the inward normal of $\Gamma_{\gamma_\delta}(r)$ in $(B_r^{\gamma_\delta}(o), \gamma_\delta)$ for all $0 < r \leq 1$, and the second fundamental form of $\Gamma_{\gamma_\delta}(r)$ in $(\bar{B}_r^{\gamma_\delta}(o), \gamma_\delta)$ with respect to ∂_n is given by

$$L(\gamma_\delta, \Gamma_{\gamma_\delta}(r))(\hat{x}, 1-r) = -\frac{1}{2}\partial_n|_{x^n = (1-r)}\big((1 - x^n)^2 \bar{g}_\delta(\hat{x})\big) = r\bar{g}_\delta(\hat{x})$$

in our coordinates (cf. lemmas A.2.1 and A.3.1 of the Appendix).

Let us compute the sum of the second fundamental forms of γ_δ and g_δ on $\bar{\Gamma}_\delta$. Let $L^+(\gamma_\delta, \bar{\Gamma}_\delta)$ be the second fundamental form of $\bar{\Gamma}_\delta$ in $(\bar{B}_1^{\gamma_\delta}, \gamma_\delta)$ with respect to the inward normal $N^+ = \partial_n$, and let $L^-(g_\delta, \bar{\Gamma}_\delta)$ be the second fundamental form

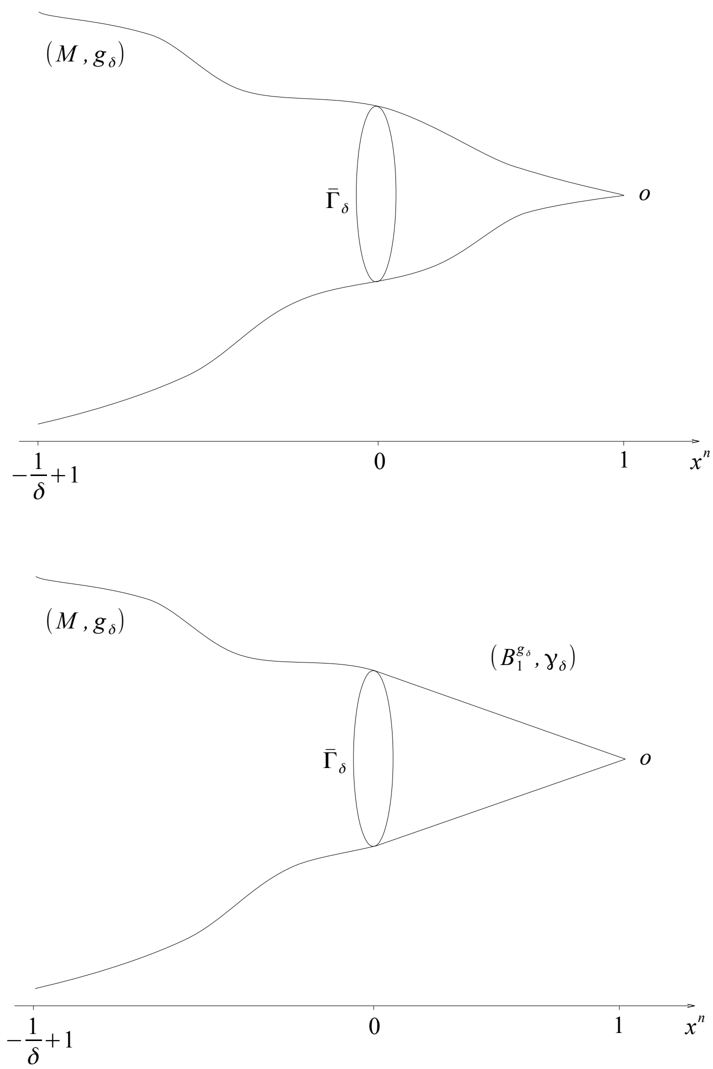

Figure 3.3.: Replacing g_δ by the cone metric γ_δ on $\bar{B}_1^{g_\delta}$

of $\bar{\Gamma}_\delta$ in $M \setminus B_1^{g_\delta}$ with respect to the inward normal $N^- = -\partial_n$. Using (3.4.1) we compute

$$\begin{aligned}
& L^+(\gamma_\delta, \bar{\Gamma}_\delta) + L^-(g_\delta, \bar{\Gamma}_\delta) \\
= \;& -\frac{1}{2}\partial_n|_{x^n=0}\big((1-x^n)^2\bar{g}_\delta\big) + L^-(g_\delta, \bar{\Gamma}_\delta) \\
= \;& \bar{g}_\delta + L^-(g_\delta, \bar{\Gamma}_\delta) \\
\geq \;& \frac{1}{1+\varepsilon(\delta)} L^+(g_\delta, \bar{\Gamma}_\delta) + L^-(g_\delta, \bar{\Gamma}_\delta) \\
= \;& \frac{-\varepsilon(\delta)}{1+\varepsilon(\delta)} L^+(g_\delta, \bar{\Gamma}_\delta) + \underbrace{L^+(g_\delta, \bar{\Gamma}_\delta) + L^-(g_\delta, \bar{\Gamma}_\delta)}_{=0} \\
\geq \;& -\varepsilon(\delta)\bar{g}_\delta.
\end{aligned}$$

In the next step we modify γ_δ and g_δ near $\bar{\Gamma}_\delta$, so that the sum of the second fundamental forms becomes nonnegative, which is a crucial condition of Theorem 1.1.2.

Step 2: Let $\varphi : [0,1) \to \mathbb{R}$ and $\psi : (-\infty, 0] \to \mathbb{R}$ be smooth functions such that

- $\varphi(0) = 1 - 4\varepsilon(\delta) = \psi(0)$
- $\varphi'(0) = -2\varepsilon(\delta)$ and $-2\varepsilon(\delta) \leq \varphi' \leq 0$ and $\varphi' \equiv 0$ on $[\frac{1}{2}, 1)$
- $|\varphi''| \leq 8\varepsilon(\delta)$
- $\psi'(0) = 0$
- $\psi \equiv 1$ on $(-\frac{1}{\delta} + 1, -\frac{2}{3\delta} + 1)$

(see Figure 3.4 below). We will impose further conditions for ψ later in the proof. Note that the first two conditions imply that $1 - 5\varepsilon(\delta) \leq \varphi \leq 1 - 4\varepsilon(\delta)$. A function φ with the required properties can be constructed as follows: We find a nondecreasing function $\tilde{\varphi} : [0, \frac{1}{2}] \to \mathbb{R}$ such that $\tilde{\varphi}(0) = -2\varepsilon(\delta)$, $\tilde{\varphi}(\frac{1}{2}) = 0$, and $\tilde{\varphi}' \leq 8\varepsilon$, $\tilde{\varphi}'(0) = 0 = \tilde{\varphi}'(\frac{1}{2})$. We then extend $\tilde{\varphi}$ to $[0,1)$ by zero, and put $\varphi(x) = 1 - 4\varepsilon(\delta) + \int_0^x \tilde{\varphi}(t)dt$.

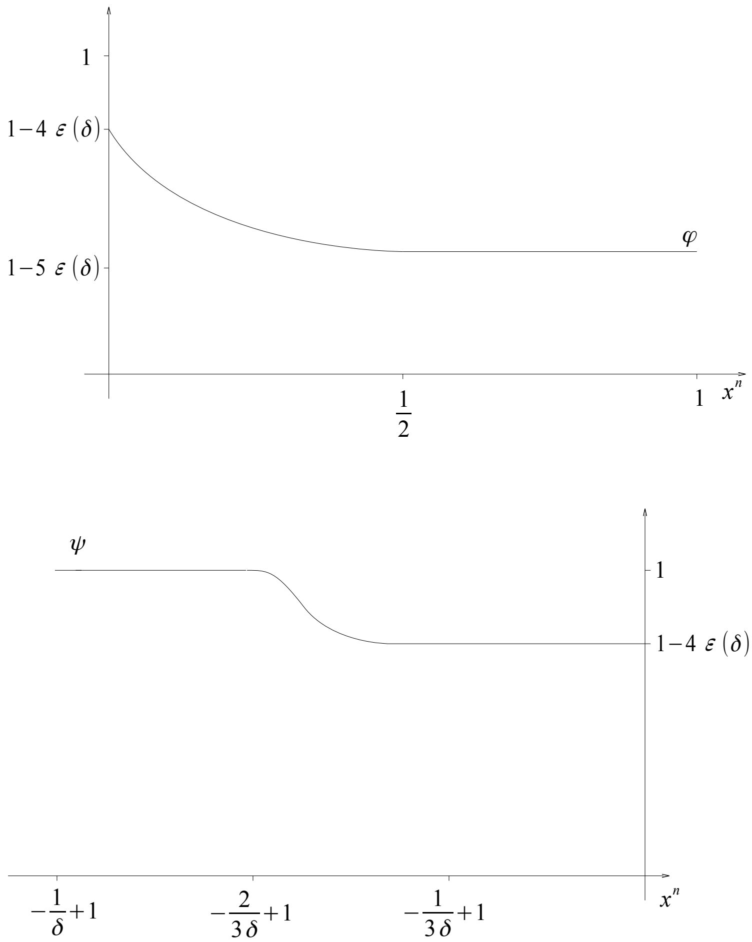

Figure 3.4.: The functions φ und ψ

We replace γ_δ on $\bar{B}_1^{g_\delta}$ and g_δ on $M \setminus B_1^{g_\delta}$ by

$$\tilde{\gamma}_\delta(x) = \begin{pmatrix} \varphi(x^n)(1-x^n)^2 \bar{g}_\delta(\hat{x}) & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\tilde{g}_\delta(x) = \begin{pmatrix} \psi(x^n)\hat{g}_\delta(x) & 0 \\ 0 & 1 \end{pmatrix} \text{ on } B^{g_\delta}_{1/\delta} \setminus B^{g_\delta}_1, \quad \tilde{g}_\delta = g_\delta \text{ on } M \setminus B^{g_\delta}_{1/\delta}$$

(see Figure 3.5 below). The isometry of the boundaries is preserved due to $\varphi(0) = \psi(0)$. As in *Step 1*, the distance functions of the new metrics coincide with $\operatorname{dist}_{g_\delta}(\cdot, o)$ (cf. Lemma A.2.1).

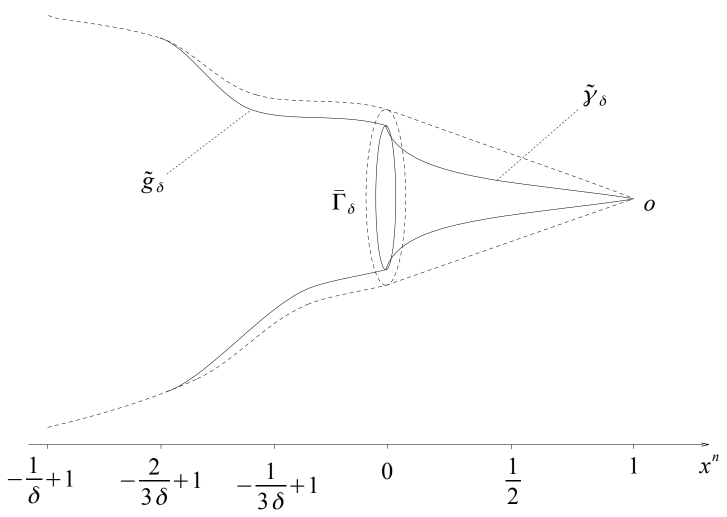

Figure 3.5.: Modified metrics $\tilde{g}_\delta$ and $\tilde{\gamma}_\delta$

Step 3: Let us check that the sum of the second fundamental forms of the new metrics on $\bar{\Gamma}_\delta$ becomes nonnegative. Similar as in *Step 1*, let $L^+(\tilde{\gamma}_\delta, \bar{\Gamma}_\delta)$ be the second fundamental form of $\bar{\Gamma}_{g_\delta}(1)$ in $(\bar{B}^{g_\delta}_1, \tilde{\gamma}_\delta)$ with respect to the inward normal $N^+ = \partial_n$, and let $L^-(\tilde{\gamma}_\delta, \bar{\Gamma}_\delta)$ be the second fundamental form of $\bar{\Gamma}_\delta$ in $M \setminus B^{g_\delta}_1$ with respect to the inward normal $N^- = -\partial_n$. We compute

$$\begin{aligned} L^+(\tilde{\gamma}_\delta, \bar{\Gamma}_\delta) &= -\frac{1}{2}\partial_n|_{x^n=0}(\varphi(x^n)(1-x^n)^2\bar{g}_\delta) \\ &= -\frac{1}{2}\varphi'(0)\bar{g}_\delta + \varphi(0)\bar{g}_\delta \end{aligned} \tag{3.4.2}$$

and

$$\begin{aligned} L^-(\tilde{g}_\delta, \bar{\Gamma}_\delta) &= \frac{1}{2}\partial_n|_{x^n=0}(\psi(x^n)\hat{g}_\delta(x)) \\ &= \psi(0)\frac{1}{2}\partial_n|_{x^n=0}\hat{g}_\delta(x) \\ &= \psi(0)L^-(g_\delta, \bar{\Gamma}_\delta) = -\psi(0)L^+(g_\delta, \bar{\Gamma}_\delta). \end{aligned} \tag{3.4.3}$$

In view of (3.4.2), (3.4.3), (3.4.1) and the properties of φ and ψ, the sum of the second fundamental forms is

$$\begin{aligned}
& L^+(\tilde\gamma_\delta,\bar\Gamma_\delta)+L^-(\tilde g_\delta,\bar\Gamma_\delta)\\
= \;& -\frac{1}{2}\varphi'(0)\bar g_\delta+\varphi(0)\bar g_\delta-\psi(0)L^+(g_\delta,\bar\Gamma_\delta)\\
= \;& \varepsilon(\delta)\bar g_\delta+(1-4\varepsilon(\delta))\big(\bar g_\delta-L^+(g_\delta,\bar\Gamma_\delta)\big)\\
\geq \;& \varepsilon(\delta)\bar g_\delta-(1-4\varepsilon(\delta))\varepsilon(\delta)\bar g_\delta\\
= \;& 4\varepsilon(\delta)^2\bar g_\delta>0.
\end{aligned} \tag{3.4.4}$$

Step 4: In the next step we show that the curvature operators of $\tilde\gamma_\delta$ and $\tilde g_\delta$ are almost nonnegative.

a) Curvature operator of $\tilde\gamma_\delta$ on $\bar B_1^{g_\delta}\setminus o$:
By the Gauss theorem we have

$$\mathcal{R}(\gamma_\delta)(x)=(1-x^n)^2\big(\mathcal{R}(\bar g_\delta)(\hat x)-\bar g_\delta(\hat x)\wedge\bar g_\delta(\hat x)\big).$$

Moreover, $\mathcal{R}(g_\delta)=\frac{1}{\delta^2}\mathcal{R}(g)\geq 0$ by *Condition 4*, so in view of the Gauss theorem and (3.4.1)

$$\begin{aligned}
\mathcal{R}(\bar g_\delta) \;&=\; \mathcal{R}(g_\delta)|_{\Lambda^2(T\bar\Gamma_\delta)}+L(g_\delta,\bar\Gamma_\delta)\wedge L(g_\delta,\bar\Gamma_\delta)\\
&\geq\; (1-2\varepsilon(\delta))\bar g_\delta\wedge\bar g_\delta.
\end{aligned} \tag{3.4.5}$$

Thus

$$\mathcal{R}(\gamma_\delta)(x) \;\geq\; -2\varepsilon(\delta)(1-x^n)^2\bar g_\delta(\hat x)\wedge\bar g_\delta(\hat x). \tag{3.4.6}$$

By putting $g=\gamma_\delta$ und $\hat g=(1-x^n)^2\bar g_\delta$ in Proposition 3.3.1 we obtain

$$\begin{aligned}
\mathcal{R}(\tilde\gamma_\delta) \;=\;& \varphi\mathcal{R}(\gamma_\delta)+\varphi(1-\varphi)(1-x^n)^2\bar g_\delta\wedge\bar g_\delta\\
&+\varphi\varphi'(1-x^n)^3\bar g_\delta\wedge\bar g_\delta-\frac{1}{4}(\varphi')^2(1-x^n)^4\bar g_\delta\wedge\bar g_\delta\\
&+\frac{-2\varphi''\varphi+(\varphi')^2}{2\varphi}(1-x^n)^2\bar g_\delta\wedge P^N+2\varphi'(1-x^n)\bar g_\delta\wedge P^N
\end{aligned} \tag{3.4.7}$$

where $P^N_{ij}=\delta_{in}\delta_{jn}$ in our coordinates.

Consider the first two lines on the right hand side of the above equation (the tangential part or the curvature operator). Using (3.4.6) and the fact that $1-\varphi\geq 4\varepsilon(\delta)$ by construction, we compute

$$\begin{aligned}
& \varphi\mathcal{R}(\gamma_\delta)+\varphi(1-\varphi)(1-x^n)^2\bar g_\delta\wedge\bar g_\delta\\
& +\varphi\varphi'(1-x^n)^3\bar g_\delta\wedge\bar g_\delta-\frac{1}{4}(\varphi')^2(1-x^n)^4\bar g_\delta\wedge\bar g_\delta\\
\geq\;& \varphi(1-x^n)^2\Big(-2\varepsilon(\delta)+(1-\varphi)+\varphi'(1-x^n)-\frac{1}{4}\frac{(\varphi')^2}{\varphi}(1-x^n)^2\Big)\bar g_\delta\wedge\bar g_\delta\\
\geq\;& \varphi(1-x^n)^2\big(-2\varepsilon(\delta)+4\varepsilon(\delta)-2\varepsilon(\delta)-\varepsilon(\delta)\big)\bar g_\delta\wedge\bar g_\delta\geq-\varepsilon(\delta)\bar g_\delta\wedge\bar g_\delta,
\end{aligned} \tag{3.4.8}$$

where we assumed that w.l.o.g. $\varphi\geq 1-5\varepsilon(\delta)\geq\frac{1}{2}$, so that $-\frac{1}{4}\frac{(\varphi')^2}{\varphi}(1-x^n)^2\geq-\varepsilon(\delta)$.

Let us consider the last line on the right hand side of (3.4.7). Observe that since $\varphi(1-x^n)^2\bar g_\delta$ and P^N are the restrictions of $\tilde\gamma_\delta$ to $T\Gamma_{g_\delta}(1-x^n)\times T\Gamma_{g_\delta}(1-x^n)$ and $(T\Gamma_{g_\delta}(1-x^n))^\perp\times(T\Gamma_{g_\delta}(1-x^n))^\perp$, respectively, we have

$$\varphi(1-x^n)^2\bar g_\delta\leq\tilde\gamma_\delta$$

and

$$P^N \leq \tilde{\gamma}_\delta$$

which gives us $\varphi(1-x^n)^2\bar{g}_\delta \wedge P^N \leq \tilde{\gamma}_\delta \wedge \tilde{\gamma}_\delta = \mathcal{I}(\tilde{\gamma}_\delta)$ (see Lemma B.3.3). Furthermore, $\varphi' \geq -2\varepsilon(\delta)$ and $\varphi'|_{[\frac{1}{2},1)} \equiv 0$ implies $\varphi'/(1-x^n) \geq -4\varepsilon(\delta)$. Therefore

$$\begin{aligned}
& \frac{-2\varphi''\varphi + (\varphi')^2}{\varphi}(1-x^n)^2\bar{g}_\delta \wedge P^N + 2\varphi'(1-x^n)\bar{g}_\delta \wedge P^N \\
= \; & \frac{-2\varphi''\varphi + (\varphi')^2}{\varphi^2}\big(\varphi(1-x^n)^2\bar{g}_\delta\big) \wedge P^N + \frac{2\varphi'}{(1-x^n)\varphi}\big(\varphi(1-x^n)^2\bar{g}_\delta\big) \wedge P^N \\
\geq \; & \frac{2}{\varphi}\big(-\varphi'' + \frac{\varphi'}{1-x^n}\big)\mathcal{I}(\tilde{\gamma}_\delta) \\
\geq \; & 4\big(-8\varepsilon(\delta) - 4\varepsilon(\delta)\big)\mathcal{I}(\tilde{\gamma}_\delta) = -48\varepsilon(\delta)\mathcal{I}(\tilde{\gamma}_\delta). \qquad (3.4.9)
\end{aligned}$$

Combining (3.4.8) and (3.4.9) we obtain

$$\mathcal{R}(\tilde{\gamma}_\delta) \geq -50\,\varepsilon(\delta)\mathcal{I}(\tilde{\gamma}_\delta).$$

b) Curvature operator of $\tilde{g}_\delta$ on $M \setminus B_1^{g_\delta}$:
Recall that up to now we only required that ψ is a smooth function on $[-\frac{1}{\delta}+1, 0]$ such that $\psi(0) = 1 - 4\varepsilon(\delta) = \varphi(0)$ and $\psi'(0) = 0$. We now specify some further properties of ψ, namely

- $\psi \equiv 1 - 4\varepsilon(\delta)$ on $[-\frac{1}{3\delta}+1, 0]$
- $\psi \equiv 1$ on $[-\frac{1}{\delta}+1, -\frac{2}{3\delta}+1]$
- $-c_1\,\delta\,\varepsilon(\delta) \leq \psi' \leq 0$
- $|\psi^{(k)}| \leq c_k\,\delta^k\varepsilon(\delta)$ for all $k \geq 1$, in particular $|\psi''| \leq c_2\,\delta^2\varepsilon(\delta)$

where c_k are constants independent of δ (see Figure 3.4 above). A function with these properties can be constructed as follows: We find a smooth nondecreasing cut off function $h : [0,1] \to [0,1]$ such that $h|_{[0,\frac{1}{3}]} \equiv 0$ and $h|_{[\frac{2}{3},1]} \equiv 1$, and put

$$\psi(x) = 1 - 4\varepsilon(\delta) + 4\varepsilon(\delta)\Big(h\big(\delta(1-x^n)\big)\Big).$$

One easily checks that ψ has the required properties, where $c_k = 4\|h\|_{C^k([0,1])}$.
As in a), by applying Proposition 3.3.1 to $\tilde{g}_\delta$ we obtain

$$\begin{aligned}
\mathcal{R}(\tilde{g}_\delta) = \psi\mathcal{R}(g_\delta) \; & + \; (1-\psi)\psi L(g_\delta) \wedge L(g_\delta) + \psi\psi' L(g_\delta) \wedge \hat{g}_\delta - \frac{1}{4}(\psi')^2\hat{g}_\delta \wedge \hat{g}_\delta \\
& + \; \frac{-2\psi''\psi + (\psi')^2}{2\psi}\hat{g}_\delta \wedge P^N + 2\psi' L(g_\delta) \wedge P^N. \qquad (3.4.10)
\end{aligned}$$

Observe that at a point $x = (\hat{x}, x^n) \in B_{1/\delta}^{g_\delta} \setminus B_1^{g_\delta}$ (i.e. $1 \leq 1 - x^n < \frac{1}{\delta}$) we have

$$L(g_\delta) = L(g_\delta, \Gamma_{g_\delta}(1-x^n)) = \frac{1}{\delta}L(g, \Gamma_g(\delta(1-x^n))) \geq 0 \qquad (3.4.11)$$

in view of *Condition 3b* (recall that we assumed $\varepsilon(\delta) \leq 1$), which implies $L(g_\delta) \wedge L(g_\delta) \geq 0$ (see Lemma B.3.3).

Consider the terms $\psi\psi' L(g_\delta) \wedge \hat{g}_\delta$ and $2\psi' L(g_\delta) \wedge P^N$ on the right hand side of (3.4.10). Since ψ' vanishes on $(-\frac{1}{\delta}+1, -\frac{2}{3\delta}+1] \cup (-\frac{1}{3\delta}+1, 0]$, it suffices to estimate these terms on

$$B^{g_\delta}_{2/(3\delta)} \setminus B^{g_\delta}_{1/(3\delta)} = B^g_{2/3} \setminus B^g_{1/3}.$$

In this set we have

$$\begin{aligned} L(g_\delta) = L(g_\delta, \Gamma_{g_\delta}(1-x^n)) &= L(g_\delta, \Gamma_{g_\delta}(\operatorname{dist}_{g_\delta}(\cdot, o))) \\ &= \frac{1}{\delta} L(g, \Gamma_g(\operatorname{dist}_g(\cdot, o))) \\ &\leq \frac{1}{\delta} C\hat{g} = \delta C \hat{g}_\delta, \end{aligned} \tag{3.4.12}$$

where $C < \infty$ is the bound of $L(g, \Gamma_g(\operatorname{dist}_g(\cdot, o)))$ on $\bar{B}^g_{2/3} \setminus B^g_{1/3}$, which is finite since

$$B^g_1 \setminus o \ni p \mapsto L(g, \Gamma_g(\operatorname{dist}_g(\cdot, o))$$

is smooth on $B^g_1 \setminus o$ and $\bar{B}^g_{2/3} \setminus B^g_{1/3}$ is compact. From (3.4.12) we get

$$\psi\psi' L(g_\delta) \wedge \hat{g}_\delta \geq \psi' C\delta\, \hat{g}_\delta \wedge \hat{g}_\delta \geq -c_1 C \delta^2 \varepsilon(\delta)\, \hat{g}_\delta \wedge \hat{g}_\delta \tag{3.4.13}$$

and

$$2\psi' L(g_\delta) \wedge P^N \geq -2c_1 C\delta^2 \varepsilon(\delta) \hat{g}_\delta \wedge P^N. \tag{3.4.14}$$

Combining (3.4.10), (3.4.13), (3.4.14) and $\mathcal{R}(g_\delta) \geq 0$, and using $\psi\hat{g}_\delta \leq \tilde{g}_\delta$ and $P^N \leq \tilde{g}_\delta$, as well as $\psi \geq 1 - 4\varepsilon(\delta) \geq \frac{1}{2}$ for small enough δ and $|\psi''| \leq c\,\delta^2\varepsilon(\delta)$, we arrive at

$$\begin{aligned} \mathcal{R}(\tilde{g}_\delta) \;\geq\;& -c_1 C\delta^2\varepsilon(\delta)\, \hat{g}_\delta \wedge \hat{g}_\delta - \frac{1}{4} c_1^2 \delta^2 \varepsilon(\delta)^2 \hat{g}_\delta \wedge \hat{g}_\delta \\ & -c_2 \delta^2 \varepsilon(\delta) \hat{g}_\delta \wedge P^N - 2c_1 C\delta^2\varepsilon(\delta)\hat{g}_\delta \wedge P^N \\ \geq\;& -c_1 C\delta^2\varepsilon(\delta) \frac{1}{\psi^2} (\psi\hat{g}_\delta) \wedge (\psi\hat{g}_\delta) - \frac{1}{4} c_1^2\delta^2\varepsilon(\delta)^2 \frac{1}{\psi^2} (\psi\hat{g}_\delta) \wedge (\psi\hat{g}_\delta) \\ & -c_2\delta^2\varepsilon(\delta)\frac{1}{\psi}(\psi\hat{g}_\delta) \wedge P^N - 2c_1 C\delta^2\varepsilon(\delta)\frac{1}{\psi}(\psi\hat{g}_\delta) \wedge P^N \\ \geq\;& -11(c_1+c_2+1)^2(C+1)\delta^2\varepsilon(\delta)\tilde{g}_\delta \wedge \tilde{g}_\delta \\ =\;& -\tilde{C}\delta^2\varepsilon(\delta)\mathcal{I}(\tilde{g}_\delta) \end{aligned} \tag{3.4.15}$$

where $\tilde{C} = 11(c_1+c_2+1)^2(C+1)$.

Step 5: Next we smooth out the metric $\tilde{\gamma}_\delta$ on $B^{g_\delta}_1$ near o while preserving the lower curvature bound $-50\varepsilon(\delta)$ (see *Step 4a*). Our method involves Proposition 3.2.6, so that here the assumption $\dim(M) = 3$ is necessary.

Consider the metric

$$\tilde{\tilde{\gamma}}_\delta(\hat{x}, x^n) := \begin{pmatrix} \varphi(\frac{1}{2})(1-x^n)^2 \bar{g}_\delta(\hat{x}) & 0 \\ 0 & 1 \end{pmatrix}$$

on $\bar{B}^{g_\delta}_1$. $(\bar{B}^{g_\delta}_1, \tilde{\tilde{\gamma}}_\delta)$ can be regarded as the unit ball in the Euclidean cone over $(\bar{\Gamma}_\delta, \varphi(\frac{1}{2})\bar{g}_\delta)$ (cf. Definition 3.2.1). Recall that $\varphi'|_{[\frac{1}{2},1)} \equiv 0$, which implies that

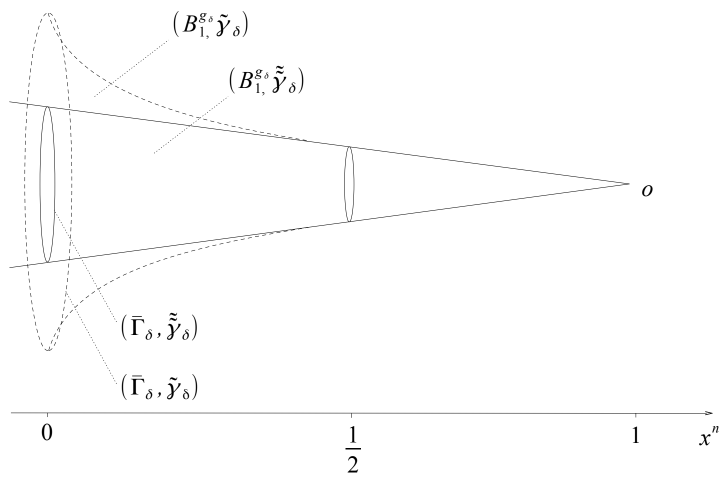

Figure 3.6.: The Euclidean cone $(\bar{B}_1^{g_\delta}, \tilde{\tilde{\gamma}}_\delta)$

$\tilde{\tilde{\gamma}}_\delta = \tilde{\gamma}_\delta$ on $\bar{B}^{g_\delta}_{1/2}$, that is, $\tilde{\tilde{\gamma}}_\delta$ coincides with $\tilde{\gamma}_\delta$ on the 'lower part' of the cone (see Figure 3.6 above).

Observe that the metric $\varphi(\frac{1}{2})\bar{g}_\delta$ has curvature > 1. Indeed, by (3.4.5) we have

$$\mathcal{R}(\bar{g}_\delta) \geq (1 - 2\varepsilon(\delta))\bar{g}_\delta \wedge \bar{g}_\delta$$

which together with $\varphi \leq 1 - 4\varepsilon(\delta)$ gives us

$$\begin{aligned}
\mathcal{R}(\varphi(\frac{1}{2})\bar{g}_\delta) &= \varphi(\frac{1}{2})\mathcal{R}(\bar{g}_\delta) \geq \varphi(\frac{1}{2})(1 - 2\varepsilon(\delta))\bar{g}_\delta \wedge \bar{g}_\delta \\
&= \frac{1}{\varphi(\frac{1}{2})}(1 - 2\varepsilon(\delta))\big(\varphi(\frac{1}{2})\bar{g}_\delta\big) \wedge \big(\varphi(\frac{1}{2})\bar{g}_\delta\big) \\
&\geq \frac{1 - 2\varepsilon(\delta)}{1 - 4\varepsilon(\delta)}\big(\varphi(\frac{1}{2})\bar{g}_\delta\big) \wedge \big(\varphi(\frac{1}{2})\bar{g}_\delta\big) \\
&> \big(\varphi(\frac{1}{2})\bar{g}_\delta\big) \wedge \big(\varphi(\frac{1}{2})\bar{g}_\delta\big) = \mathcal{I}(\varphi(\frac{1}{2})\bar{g}_\delta).
\end{aligned}$$

Given an arbitrary small neighborhood U of o, by applying Proposition 3.2.6 to $(\bar{B}_1^{g_\delta}, \tilde{\tilde{\gamma}}_\delta)$ we find a smooth structure D_δ on $\bar{B}_1^{g_\delta}$, which is compatible with the smooth structure induced by D_M on $\bar{B}_1^{g_\delta} \setminus o$, and a smooth (w.r.t. D_δ) metric on $\bar{B}_1^{g_\delta}$ which coincides with $\tilde{\tilde{\gamma}}_\delta$ on $\bar{B}_1^{g_\delta} \setminus U$ and has curvature ≥ 0. Since $\tilde{\gamma}_\delta$ coincides with $\tilde{\tilde{\gamma}}_\delta$ on $B^{g_\delta}_{1/2}$ and has curvature $\geq -50\varepsilon(\delta)$, this gives us a smooth metric $\gamma_\delta^{(sm)}$ on (M, D_δ) which coincides with $\tilde{\gamma}_\delta$ off U and has curvature $\geq -50\varepsilon(\delta)$ (see Figure 3.7 below).

Since the structure D_δ agrees with D_M on $\bar{B}_1^{g_\delta} \setminus o$, we may regard D_δ as a smooth structure on the entire manifold M and replace D_M by D_δ. Note that this does not affect any of the previous constructions on $M \setminus o$, i.e. all objects considered up to now have the same regularity on $M \setminus o$ w.r.t. D_δ as they did w.r.t. D_M. However, it is important to notice that the distance function of $\gamma_\delta^{(sm)}$ may no longer coincide with the distance function of g_δ, as opposed to the distance functions of all modified metrics we considered up to now, the reason being that the smooth metric constructed in Proposition 3.2.6 does not necessarily coincide with the initial

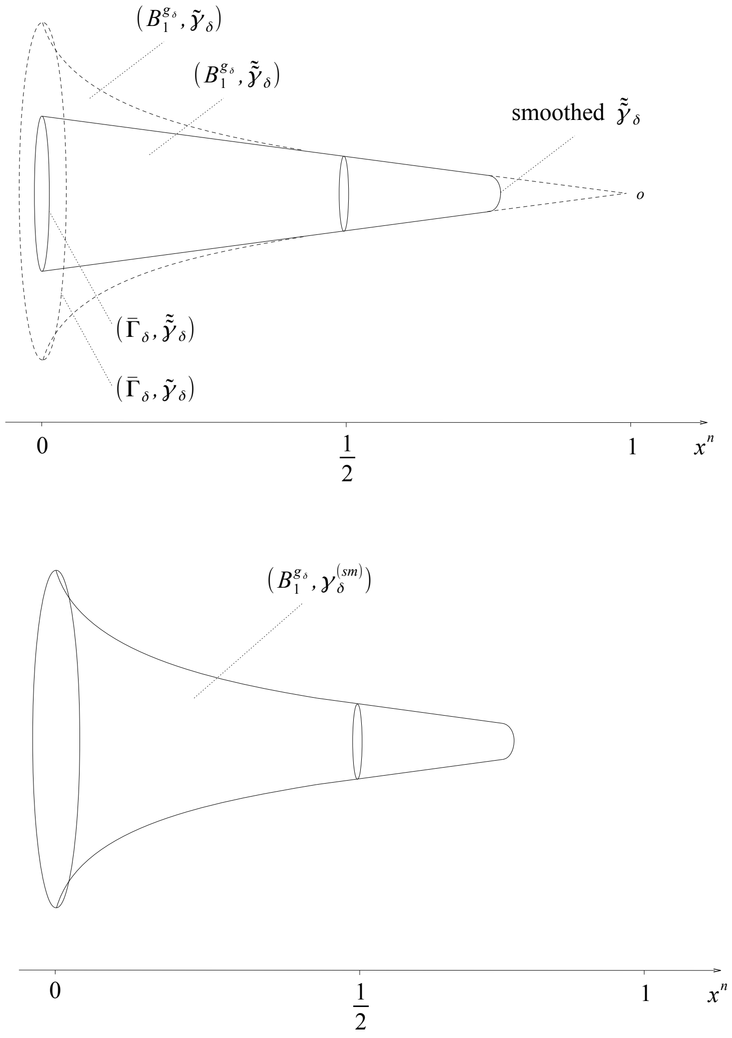

Figure 3.7.: The upper figure shows the metric $\tilde{\tilde{\gamma}}_\delta$ after smoothing near o. The lower figure shows the smoothed metric $\gamma_\delta^{(sm)}$.

metric in normal direction. Note also that (M, D_δ) is diffeomorphic to (M, D_M) for all $\delta > 0$ since $\dim(M) = 3$.

Step 6: Let us sum up the constructions up to this point. We introduced a smooth structure D_δ which agrees with the initial smooth structure D_M on $M \setminus o$, a smooth metric $\tilde{g}_\delta$ on $(M \setminus B_1^{g_\delta}, D_\delta)$, and a smooth metric $\gamma_\delta^{(sm)}$ on $(\bar{B}_1^{g_\delta}, D_\delta)$, such that

- $\tilde{g}_\delta = \gamma_\delta^{(sm)}$ on $\bar{\Gamma}_\delta = \partial B_1^{g_\delta}$.
- The sum of the second fundamental forms of $\tilde{g}_\delta$ and $\gamma_\delta^{(sm)}$ on $\bar{\Gamma}_\delta$ is nonnegative (see *Step 3*).
- The curvature operators satisfy

  $\mathcal{R}(\tilde{g}_\delta) \geq -\tilde{C}\delta^2\varepsilon(\delta)\mathcal{I}(\tilde{g}_\delta)$ (see *Step 4b*)

  $\mathcal{R}(\gamma_\delta^{(sm)}) \geq -50\varepsilon(\delta)\mathcal{I}(\gamma_\delta^{(sm)})$ (see *Step 4a* and *Step 5*).

In what follows, by 'smooth' we mean 'smooth with respect to the structure D_δ', unless noted differently. By choosing δ small enough and applying Theorem 1.1.2 we find a metric $g_\delta^{(sm)}$ on M, such that

- $g_\delta^{(sm)}$ is C^2 smooth,
- $g_\delta^{(sm)}$ coincides with $\tilde{g}_\delta$ and $\gamma_\delta^{(sm)}$, respectively, off a small neighborhood of $\bar{\Gamma}_\delta$, say, $U = B^{g_\delta}_{1+\delta} \setminus B^{g_\delta}_{1-\delta}$. In particular, $g_\delta^{(sm)}$ is C^∞ smooth off U,
- $g_\delta^{(sm)}$ is C^0 close to $\tilde{g}_\delta$ on $M \setminus B_1^{g_\delta}$ and $\gamma_\delta^{(sm)}$ on $\bar{B}_1^{g_\delta} \setminus o$, respectively, say
$$-\delta \tilde{g}_\delta \le g_\delta^{(sm)} - g_\delta \le \delta \tilde{g}_\delta \quad \text{on } M \setminus B_1^{g_\delta}$$
and
$$-\delta \gamma_\delta^{(sm)} \le g_\delta^{(sm)} - \gamma_\delta^{(sm)} \le \delta \gamma_\delta^{(sm)} \quad \text{on } \bar{B}_1^{g_\delta} \setminus o,$$
- the curvature operator of $g_\delta^{(sm)}$ is bounded from below by $-100\,\varepsilon(\delta)$.

(see Figure 3.8 below).

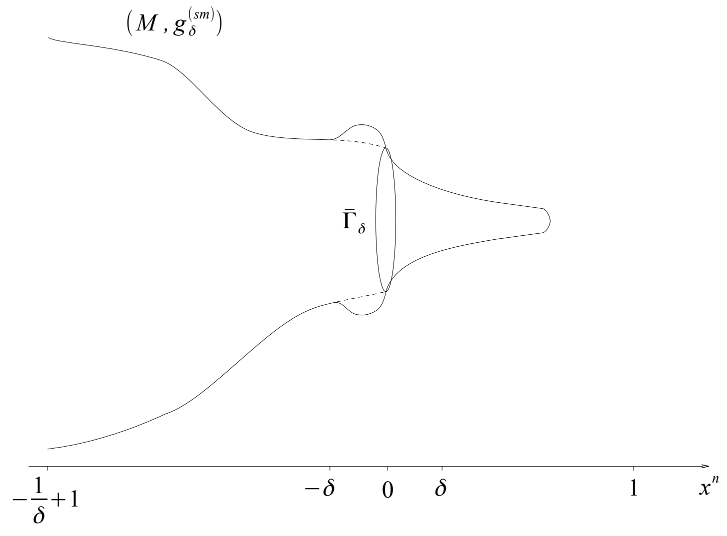

Figure 3.8.: The glued metric after smoothing near $\bar{\Gamma}_\delta$ (cf. Figure 3.5)

Step 7: Finally, let us construct the sequence (M, D_i, g_i). We choose a sequence $\delta_i \to 0$ and define $g_i := \delta_i^2 g_{\delta_i}^{(sm)}$, and $D_i := D_{\delta_i}$, where $g_{\delta_i}^{(sm)}$ and D_{δ_i} are as in the previous step. Let us verify that $g_i \to g$ in the C^∞ sense off o. That is, given a coordinate neighborhood (U, ξ) of (M, D_M) such that $\bar{U}$ is a compact subset of $M \setminus o$, we show that $\|{}^\xi(g_i)_{kl} - {}^\xi g_{kl}\|_{C^k(\xi(U))} \to 0$ as $i \to \infty$ for all $k \ge 0$, where ${}^\xi(g_i)_{kl}$ and ${}^\xi g_{kl}$ are coordinate functions of g_i and g with respect to ξ. Note that this makes sense, since D_i is compatible with D_M on $M \setminus o$, and thus any coordinate chart $(U, \xi) \in D_M$, $U \not\ni o$ lies in D_i for all $i \ge 1$.

Let U be such a coordinate neighborhood. Then $U \subset M \setminus B^g_{2\delta_i}$ for large enough i. From construction g_i coincides with g on $M \setminus B_1^g$, so that w.l.o.g. we may assume that $U \subset B_1^g \setminus B^g_{2\delta_i} = B^{g_{\delta_i}}_{1/\delta_i} \setminus B_2^{g_{\delta_i}}$. In this set we have $\frac{1}{\delta_i^2} g_i = g_{\delta_i}^{(sm)} = \tilde{g}_{\delta_i}$. Let us return to the construction of $\tilde{g}_\delta$ (see *Step 2*). Recall that we introduced Fermi coordinates $x = (\hat{x}, x^n)$ above $\bar{\Gamma}_\delta = \Gamma_{g_\delta}(1) = \partial B_1^{g_\delta}$, where $x^n = 1 - \operatorname{dist}_{g_\delta}(\cdot, o) = 1 - \frac{1}{\delta}\operatorname{dist}_g(\cdot, o)$, and defined

$$\tilde{g}_\delta(x) = \begin{pmatrix} \psi_\delta(x^n)\hat{g}_\delta(x) & 0 \\ 0 & 1 \end{pmatrix}$$

where $\hat{g}_\delta(p)$ is the restriction of $g_\delta(p)$ to $T_p\Gamma_{g_\delta}(r) \times T_p\Gamma_{g_\delta}(r)$ for p with $\operatorname{dist}_{g_\delta}(p, o) = r$. (Note that here we use the subscript δ for the cut-off functions ψ constructed in *Step 4b*.) Thus, in coordinate-free notation

$$\tilde{g}_\delta = \psi_\delta(1 - \frac{1}{\delta}\operatorname{dist}_g(\cdot, o))\hat{g}_\delta + g_\delta^N = \tilde{\psi}_\delta(\operatorname{dist}_g(\cdot, o))\hat{g}_\delta + g_\delta^N, \tag{3.4.16}$$

where $\tilde{\psi}_\delta : [\delta, 1] \to \mathbb{R}$, $\tilde{\psi}_\delta(t) = \psi_\delta(1 - t/\delta)$, and $(g_\delta^N)_{ij} = \delta_{in}\delta_{jn}$ in our coordinates. Putting $\tilde{\psi}_i = \tilde{\psi}_{\delta_i}$, on U we have

$$\begin{aligned} g - g_i &= \delta_i^2 g_{\delta_i} - \delta_i^2 \tilde{g}_{\delta_i} \\ &= \delta_i^2(\hat{g}_{\delta_i} + g_{\delta_i}^N) - \delta_i^2(\tilde{\psi}_i(\operatorname{dist}_g(\cdot, o))\hat{g}_{\delta_i} + g_{\delta_i}^N) \\ &= \big(1 - \tilde{\psi}_i(\operatorname{dist}_g(\cdot, o))\big)\delta_i^2 \hat{g}_{\delta_i} \\ &= \big(1 - \tilde{\psi}_i(\operatorname{dist}_g(\cdot, o))\big)\hat{g}. \end{aligned}$$

Thus, in order to show that $g_i \to g$ in the C^∞-sense on U it suffices to verify that $\tilde{\psi}_i \to 1$, and that all derivatives of $\tilde{\psi}_i$ tend to zero uniformly as $i \to \infty$. In view of the properties of ψ_δ (see *Step 4b*) we have

$$|\tilde{\psi}_i(t) - 1| = |\psi_{\delta_i}(1 - \frac{t}{\delta_i}) - 1| \le 4\varepsilon(\delta_i) \overset{i\to\infty}{\to} 0$$

and

$$|\tilde{\psi}_i^{(k)}(t)| = \frac{1}{(\delta_i)^k}|\psi_{\delta_i}^{(k)}(1 - \frac{t}{\delta_i})| \le c_k\varepsilon(\delta_i) \overset{i\to\infty}{\to} 0$$

and we are done.

Let us show that the metrics g_i have almost nonnegative curvature operator. Since $g_i = g$ on $M \setminus B_1^g$ and $\mathcal{R}(g) \ge 0$ by assumption, it suffices to estimate $\mathcal{R}(g_i)$ on B_1^g. Recall that the curvature operator of $g_\delta^{(sm)}$ is bounded from below by $-100\varepsilon(\delta)$ (see *Step 6*). Thus, on B_1^g we compute

$$\begin{aligned} \mathcal{R}(g_i) &= \mathcal{R}(\delta_i^2 g_{\delta_i}^{(sm)}) = \delta_i^2\mathcal{R}(g_{\delta_i}^{(sm)}) \ge -100\varepsilon(\delta_i)\delta_i^2 g_{\delta_i}^{(sm)} \wedge g_{\delta_i}^{(sm)} \\ &= -100\frac{\varepsilon(\delta_i)}{\delta_i^2}(\delta_i^2 g_{\delta_i}^{(sm)}) \wedge (\delta_i^2 g_{\delta_i}^{(sm)}) = -\varepsilon_i g_i \wedge g_i = -\varepsilon_i \mathcal{I}(g_i), \end{aligned}$$

where $\varepsilon_i := 100\varepsilon(\delta_i)/\delta_i^2 \to 0$ as $i \to \infty$ in view of *Condition 3*.

This concludes the proof of Theorem 3.4.1. □

Remark 3.4.3. Observe that the fact that $g_i \to g$ uniformly on compact subsets and $g_i \equiv g$ on $M \setminus B_1^g$ implies that $g_i \to g$ uniformly on $M \setminus V$ for any neighborhood $V \ni o$. More precisely, given a neighborhood $V \ni o$, for any $\delta > 0$ there exists a $N = N(V, \delta)$ such that on $M \setminus V$

$$-\delta\, g \le g_i - g \le \delta\, g$$

for all $i \ge N(\alpha)$. Note that for small δ this also implies

$$-2\delta\, g_i \le g_i - g \le 2\delta\, g_i.$$

3.5. Distance and volume estimates for g_i

In this section, we discuss some further properties of the sequence (M, g_i) constructed in Theorem 3.4.1. We will use the results from this section for an application of Theorem 3.4.1 involving M. Simon's results from [28]. In particular, we show that $(M_i, \operatorname{dist}_{g_i}, o) \to (M, \operatorname{dist}_g)$ in the pointed Gromov-Hausdorff sense, and that the manifolds (M_i, g_i) are non-collapsed, that is, if unit balls in (M, g) satisfy a uniform lower volume bound > 0, then so do unit balls in (M, g_i), independently of i. Note that this is not necessarily the case for an arbitrary sequence (g_i) converging to g in the C^0 (or even C^k) sense off $o \in M$, since the balls $B_1^{g_i}(o)$ could become very 'small' compared to $B_1^g(o)$. Nevertheless, this does not happen for our sequence, mainly for the reason that in view of our particular construction the distance functions $\operatorname{dist}_{g_i}(\cdot, o)$ are well-controlled near o.

Lemma 3.5.1. *Let M, g, $(g_i)_{i\in\mathbb{N}}$ be as in Theorem 3.4.1. Then for all $p \in B^g_{r_0}$ we have*

$$\operatorname{dist}_{g_i}(p, o) \le \operatorname{dist}_g(p, o) + \tilde{\varepsilon}_i,$$

where $\tilde{\varepsilon}_i \to 0$ as $i \to \infty$. In particular this implies that for all $0 < \alpha \le r_0$

$$B^g_\alpha \subset B^{g_i}_{\alpha+\tilde{\varepsilon}_i}.$$

Proof. As in the proof of Theorem 3.4.1 w.l.o.g. we assume that $r_0 = 1$. We show that for all $\delta > 0$ and $p \in B^{g_\delta}_{1/\delta}$

$$\operatorname{dist}_{g_\delta^{(sm)}}(p, o) \le \operatorname{dist}_{g_\delta}(p, o) + 3. \tag{3.5.1}$$

Since by construction $g_i = \delta_i^2 g^{(sm)}_{\delta_i}$ on $B^g_1 = B^{g_{\delta_i}}_{1/\delta_i}$, this implies

$$\operatorname{dist}_{g_i}(p, o) = \delta_i \operatorname{dist}_{g^{(sm)}_{\delta_i}}(p, o) \le \delta_i(\operatorname{dist}_{g_{\delta_i}}(p, o) + 3) = \operatorname{dist}_g(p, o) + 3\delta_i$$

for all $p \in B^g_1$.

From Lemma 3.2.8 and by construction of $\gamma^{(sm)}_\delta$ (see *Step 5* in the proof of Theorem 3.4.1) we know that for all $q \in B^{g_\delta}_{1/2}$

$$\operatorname{dist}_{g_\delta^{(sm)}}(q, o) \le \operatorname{dist}_{\tilde{\gamma}_\delta}(q, o) = \operatorname{dist}_{g_\delta}(q, o) \le \frac{1}{2}, \tag{3.5.2}$$

so that (3.5.1) holds on $B^{g_\delta}_{1/2}$. Let $p \in B^{g_\delta}_{1/\delta} \setminus B^{g_\delta}_{1/2}$. In view of (3.5.2), given any $q \in \Gamma_{g_\delta}(\frac{1}{2}) = \partial B^{g_\delta}_{1/2}$ we have

$$\operatorname{dist}_{g_\delta^{(sm)}}(p, o) \le \operatorname{dist}_{g_\delta^{(sm)}}(p, q) + \operatorname{dist}_{g_\delta^{(sm)}}(q, o) \le \operatorname{dist}_{g_\delta^{(sm)}}(p, q) + \frac{1}{2}.$$

Since by definition $\operatorname{dist}_{g_\delta^{(sm)}}(p, \Gamma_{g_\delta}(\frac{1}{2})) = \inf\{\operatorname{dist}_{g_\delta^{(sm)}}(p, q) \,|\, q \in \Gamma_{g_\delta}(\frac{1}{2})\}$, this implies

$$\operatorname{dist}_{g_\delta^{(sm)}}(p, o) \le \operatorname{dist}_{g_\delta^{(sm)}}(p, \Gamma_{g_\delta}(\frac{1}{2})) + \frac{1}{2}. \tag{3.5.3}$$

Let $\tilde{g}_\delta \cup \tilde{\gamma}_\delta$ denote the metric on $B^{g_\delta}_{1/\delta}$, given by

$$\tilde{g}_\delta \cup \tilde{\gamma}_\delta = \begin{cases} \tilde{g}_\delta & \text{on } B^{g_\delta}_{1/\delta} \setminus B^{g_\delta}_1 \\ \tilde{\gamma}_\delta & \text{on } \bar{B}^{g_\delta}_1 \end{cases},$$

where $\tilde{g}_\delta$ and $\tilde{\gamma}_\delta$ are the metrics constructed in *Step 2* of the proof of Theorem 3.4.1. Consider the curve

$$\begin{aligned} c : I = [x^n(p), \tfrac{1}{2}] &\to B^{g_\delta}_{1/\delta} \setminus B^{g_\delta}_{1/2} \\ c(t) &= x^{-1}(\hat{x}(p), t). \end{aligned}$$

Then $c(x^n(p)) = p$, $c(\frac{1}{2}) = q \in \Gamma_{g_\delta}(\frac{1}{2})$, and $\dot{c}(t) = \frac{\partial}{\partial x^n}|_{c(t)}$ for all $t \in I$, so that $\|\dot{c}\|_{\tilde{g}_\delta \cup \tilde{\gamma}_\delta} \equiv 1$ and $\|\dot{c}\|_{g_\delta} \equiv 1$. Recall that by construction $g_\delta^{(sm)}$ coincides with $\tilde{g}_\delta \cup \tilde{\gamma}_\delta$ on $B^{g_\delta}_{1/\delta} \setminus B^{g_\delta}_{1/2}$ off a small neighborhood U of $\bar{\Gamma}_\delta = \partial B_1^{g_\delta}$, and $g_\delta^{(sm)}$ and $\tilde{g}_\delta \cup \tilde{\gamma}_\delta$ are C^0 close on U (cf. the construction in *Step 6* in the proof of Theorem 3.4.1). Therefore the length of c with respect to $g_\delta^{(sm)}$ satisfies

$$\begin{aligned} L_{g_\delta^{(sm)}} &= \int_I \|\dot{c}(t)\|_{g_\delta^{(sm)}(c(t))} dt \\ &\leq \int_I \|\dot{c}(t)\|_{\tilde{g}_\delta \cup \tilde{\gamma}_\delta(c(t))} dt + \frac{1}{2} \\ &= \frac{1}{2} - x^n(p) + \frac{1}{2} \\ &= \operatorname{dist}_{g_\delta}(p, \bar{\Gamma}_\delta) + 1 \\ &\leq \operatorname{dist}_{g_\delta}(p, o) + \operatorname{dist}_{g_\delta}(o, \bar{\Gamma}_\delta) + 1 \\ &= \operatorname{dist}_{g_\delta}(p, o) + 2, \end{aligned} \tag{3.5.4}$$

where we used that by construction of the Fermi coordinates $x = (\hat{x}, x^n)$ we have $-x^n(p) = \operatorname{dist}_{g_\delta}(p, \bar{\Gamma}_\delta)$. By definition $\operatorname{dist}_{g_\delta^{(sm)}}(p, \Gamma_{g_\delta}(\frac{1}{2}))$ is the infimum of the lengths with respect to $g_\delta^{(sm)}$ of all piecewise C^1 curves connecting p and $\Gamma_{g_\delta}(\frac{1}{2})$. Thus, (3.5.1) follows from combining (3.5.3) and (3.5.4).

□

Lemma 3.5.2. *Let everything be as above. For any $\alpha > 0$, $\alpha \leq \min\{r_0, 1\}$ there exists a $N = N(\alpha)$ such that for all $i \geq N$ and $p \in M$*

$$|\operatorname{dist}_g(p, o) - \operatorname{dist}_{g_i}(p, o)| \leq \alpha \operatorname{dist}_g(p, o) + \alpha.$$

Proof. Observe that since $g_i \to g$ uniformly on $M \setminus B^g_\alpha$, we find $N = N(\alpha)$ such that $\|\cdot\|_g \leq (1 + \alpha)\|\cdot\|_{g_i}$ and $\|\cdot\|_{g_i} \leq (1 + \alpha)\|\cdot\|_g$ on $M \setminus B^g_\alpha$ for $i \geq N$ (see Remark 3.4.3). Thus, for any piecewise C^1 curve $c : [a, b] \to M \setminus B^g_\alpha$ we have

$$L_g(c) \leq (1 + \alpha) L_{g_i}(c) \quad \text{and} \quad L_{g_i}(c) \leq (1 + \alpha) L_g(c) \tag{3.5.5}$$

for $i \geq N$. Indeed,

$$\begin{aligned} L_g(c) &= \int_a^b \|\dot{c}(t)\|_{g(c(t))} dt \\ &\leq (1 + \alpha) \int_a^b \|\dot{c}(t)\|_{g_i(c(t))} dt \\ &\leq (1 + \alpha) L_{g_i}(c), \end{aligned}$$

and the second estimate follows by a similar computation.

First consider the case where $p \in M \setminus \bar{B}^g_\alpha$. For any fixed $i \geq N$ by definition of $\operatorname{dist}_{g_i}$ we find a piecewise C^1 curve $c : [0, 1] \to M$ connecting p and o such that

$$L_{g_i}(c) \leq \operatorname{dist}_{g_i}(p, o) + \alpha.$$

Let $t_0 = \inf\{t > 0 \,|\, c(t) \in B^g_\alpha\}$ and $c_1 := c|_{[0,t_0]}$. Then $q := c(t_0) \in \partial B^g_\alpha$. Let $c_2 : [t_0, 2] \to M$ be a piecewise C^1 curve connecting q and o such that

$$L_g(c_2) \leq \operatorname{dist}_g(q, o) + \alpha \leq 2\alpha.$$

We denote by $c_1 + c_2$ the concatenation of c_1 and c_2, that is

$$\begin{aligned} c_1 + c_2 : [0,2] &\to M \\ t &\mapsto \begin{cases} c_1(t) & \text{if } t \in [0, t_0] \\ c_2(t) & \text{if } t \in [t_0, 2] \end{cases}. \end{aligned}$$

Since $c_1 + c_2$ is a piecewise C^1 curve connecting p and o we have

$$\begin{aligned} \operatorname{dist}_g(p, o) &\leq L_g(c_1 + c_2) = L_g(c_1) + L_g(c_2) \\ &\leq L_g(c_1) + 2\alpha. \end{aligned} \tag{3.5.6}$$

Furthermore, since by construction $c_1([0, t_0]) \subset M \setminus B^g_\alpha$, by (3.5.5) we have

$$\begin{aligned} L_g(c_1) &\leq (1+\alpha) L_{g_i}(c_1) \\ &= (1+\alpha) L_{g_i}(c|_{[0,t_0]}) \leq (1+\alpha) L_{g_i}(c) \end{aligned}$$

for $i \geq N$. Using this in (3.5.6) we obtain

$$\begin{aligned} \operatorname{dist}_g(p, o) &\leq (1+\alpha) L_{g_i}(c) + 2\alpha \\ &\leq (1+\alpha)\left(\operatorname{dist}_{g_i}(p, o) + \alpha\right) + 2\alpha \\ &\leq (1+\alpha) \operatorname{dist}_{g_i}(p, o) + 4\alpha, \end{aligned}$$

where we used $\alpha \leq 1$. This implies

$$\operatorname{dist}_g(p, o) - \operatorname{dist}_{g_i}(p, o) \leq \alpha \operatorname{dist}_{g_i}(p, o) + 4\alpha \tag{3.5.7}$$

for $p \in M \setminus \bar{B}^g_\alpha$ and $i \geq N$.

Let us show an analogous estimate for $\operatorname{dist}_{g_i}(p, o) - \operatorname{dist}_g(p, o)$. We find a piecewise C^1 curve $\tilde{c} : [0,1] \to M$ such that

$$L_g(\tilde{c}) \leq \operatorname{dist}_g(p, o) + \alpha.$$

Similar as above, we consider the restriction $\tilde{c}_1 := \tilde{c}|_{[0,\tilde{t}_0]}$ where $\tilde{t}_0$ is defined similarly to t_0, such that $\tilde{c}_1([0, \tilde{t}_0]) \subset M \setminus B^g_\alpha$ and $\tilde{q} := \tilde{c}_1(\tilde{t}_0) \in \partial B^g_\alpha$. We then find a piecewise C^1 curve $\tilde{c}_2$ connecting $\tilde{q}$ and o such that

$$L_{g_i}(\tilde{c}_2) \leq \operatorname{dist}_{g_i}(\tilde{q}, o) + \alpha,$$

which gives us

$$\begin{aligned} \operatorname{dist}_{g_i}(p, o) &\leq L_{g_i}(\tilde{c}_1 + \tilde{c}_2) = L_{g_i}(\tilde{c}_1) + L_{g_i}(\tilde{c}_2) \\ &\leq L_{g_i}(\tilde{c}_1) + \operatorname{dist}_{g_i}(\tilde{q}, o) + \alpha. \end{aligned} \tag{3.5.8}$$

Similar as above, since the image of $\tilde{c}_1$ is contained in $M \setminus B^g_\alpha$, in view of (3.5.5) we get the estimate

$$\begin{aligned} L_{g_i}(\tilde{c}_1) &\leq (1+\alpha) L_g(\tilde{c}_1) \\ &= (1+\alpha) L_g(\tilde{c}|_{[0,t_0]}) \\ &\leq (1+\alpha) L_g(\tilde{c}) \\ &\leq (1+\alpha)(\operatorname{dist}_g(p, o) + \alpha) \\ &\leq (1+\alpha)\operatorname{dist}_g(p, o) + 2\alpha \end{aligned} \tag{3.5.9}$$

for $i \geq N$. Moreover, since $\tilde{q} \in \partial B^g_\alpha \subset \bar{B}^g_{r_0}$, by Lemma 3.5.1 we have

$$\operatorname{dist}_{g_i}(\tilde{q}, o) \leq \operatorname{dist}_g(\tilde{q}, o) + \tilde{\varepsilon}_i \leq \operatorname{dist}_g(\tilde{q}, o) + \alpha = 2\alpha \tag{3.5.10}$$

for $i \geq N$, if we choose N even larger. Combining (3.5.8), (3.5.9) and (3.5.10) we arrive at

$$\begin{aligned} \operatorname{dist}_{g_i}(p, o) &\leq (1+\alpha)\operatorname{dist}_g(p, o) + 2\alpha + 2\alpha + \alpha \\ &= (1+\alpha)\operatorname{dist}_g(p, o) + 5\alpha \end{aligned}$$

which gives us

$$\operatorname{dist}_{g_i}(p, o) - \operatorname{dist}_g(p, o) \leq \alpha \operatorname{dist}_g(p, o) + 5\alpha. \tag{3.5.11}$$

Using this in (3.5.7) yields

$$\begin{aligned} \operatorname{dist}_g(p, o) - \operatorname{dist}_{g_i}(p, o) &\leq \alpha \operatorname{dist}_{g_i}(p, o) + 4\alpha \\ &\leq \alpha\big((1+\alpha)\operatorname{dist}_g(p, o) + 5\alpha\big) + 4\alpha \\ &\leq 2\alpha \operatorname{dist}_g(p, o) + 9\alpha. \end{aligned} \tag{3.5.12}$$

Combining (3.5.11) and (3.5.12) gives

$$|\operatorname{dist}_g(p, o) - \operatorname{dist}_{g_i}(p, o)| \leq 10\alpha \operatorname{dist}_g(p, o) + 10\alpha$$

for $p \in M \setminus \bar{B}^g_\alpha$ and $i \geq N(\alpha)$ as above.

Consider the case where $p \in \bar{B}^g_\alpha \subset \bar{B}^g_{r_0}$. By Lemma 3.5.1

$$\operatorname{dist}_{g_i}(p, o) \leq \operatorname{dist}_g(p, o) + \varepsilon_i \leq 2\alpha$$

for $i \geq N(\alpha)$ after choosing $N(\alpha)$ even larger. Then clearly

$$|\operatorname{dist}_g(p, o) - \operatorname{dist}_{g_i}(p, o)| \leq 3\alpha \leq 10\alpha \operatorname{dist}_g(p, o) + 10\alpha.$$

Replacing α with $\alpha/10$ gives us the desired estimate.

□

Lemma 3.5.3. *Let everything be as above. For any $\alpha > 0$, $\alpha \leq \min\{r_0, 1\}$ there exists a $N = N(\alpha)$ such that for all $i \geq N$ and $p, q \in M$*

$$|\operatorname{dist}_g(p, q) - \operatorname{dist}_{g_i}(p, q)| \leq \alpha \operatorname{dist}_g(p, q) + \alpha.$$

Proof. The proof is similar as in Lemma 3.5.2. First consider the case where $p, q \in M \setminus \bar{B}^g_\alpha$. We find a piecewise C^1 curve $c : [0, 1] \to M$ connecting p and q such that

$$L_g(c) \leq \operatorname{dist}_g(p, q) + \alpha.$$

We choose $N = N(\alpha)$ such that (3.5.5) holds for all $i \geq N$. If $c([0, 1]) \subset M \setminus B^g_\alpha$, then by (3.5.5)

$$\begin{aligned} \operatorname{dist}_{g_i}(p, q) &\leq L_{g_i}(c) \leq (1+\alpha)L_g(c) \\ &\leq (1+\alpha)(\operatorname{dist}_g(p, q) + \alpha) \\ &\leq (1+\alpha)\operatorname{dist}_g(c) + 2\alpha \end{aligned}$$

for all $i \geq N(\alpha)$, where we used $\alpha \leq 1$, and we have

$$\operatorname{dist}_{g_i}(p,q) - \operatorname{dist}_g(p,q) \leq \alpha \operatorname{dist}_g(p,q) + 2\alpha.$$

Suppose that $c([0,1]) \cap B^g_\alpha \neq \emptyset$. Similar as in the proof of Lemma 3.5.2 we find $0 < t_0 < s_0 < 1$ such that

$$c([0,t_0]), c([s_0,1]) \subset M \setminus B^g_\alpha$$

and

$$\tilde{p} := c(t_0), \tilde{q} := c(s_0) \in \partial B^g_\alpha.$$

We put $c_1 := c|_{[0,t_0]}$ and $c_2 := c|_{[s_0,1]}$. Then by (3.5.5) we have

$$L_{g_i}(c_k) \leq (1+\alpha) L_g(c_k) \quad \text{for} \quad k = 1,2$$

for all $i \geq N$ as above. Let us fix $i \geq N$. We find a piecewise C^1 curve c_α connecting $\tilde{p}$ and $\tilde{q}$ such that

$$\begin{aligned} L_{g_i}(c_\alpha) &\leq \operatorname{dist}_{g_i}(\tilde{p},\tilde{q}) + \alpha \\ &\leq \operatorname{dist}_{g_i}(\tilde{p},o) + \operatorname{dist}_{g_i}(\tilde{q},o) + \alpha \\ &\leq \operatorname{dist}_g(\tilde{p},o) + \operatorname{dist}_g(\tilde{q},o) + 2\tilde{\varepsilon}_i + \alpha \leq 5\alpha, \end{aligned}$$

where we applied Lemma 3.5.1 to $\tilde{p}, \tilde{q} \in \partial B^g_\alpha \subset \bar{B}^g_{r_0}$. Since $c_1 + c_\alpha + c_2$ is a piecewise C^1 curve from p to q we have

$$\begin{aligned} \operatorname{dist}_{g_i}(p,q) &\leq L_{g_i}(c_1 + c_\alpha + c_2) = L_{g_i}(c_1) + L_{g_i}(c_2) + L_{g_i}(c_\alpha) \\ &\leq (1+\alpha)(L_g(c_1) + L_g(c_2)) + 5\alpha \\ &= (1+\alpha) L_g(c|_{[0,t_0]\cup[s_0,1]}) + 5\alpha \\ &\leq (1+\alpha) L_g(c) + 5\alpha \\ &\leq (1+\alpha)(\operatorname{dist}_g(p,q) + \alpha) + 5\alpha \\ &\leq (1+\alpha)\operatorname{dist}_g(p,q) + 7\alpha \end{aligned}$$

which implies

$$\operatorname{dist}_{g_i}(p,q) - \operatorname{dist}_g(p,q) \leq \alpha \operatorname{dist}_g(p,q) + 7\alpha$$

for $i \geq N(\alpha)$. By a similar argument one shows that

$$\operatorname{dist}_g(p,q) - \operatorname{dist}_{g_i}(p,q) \leq \alpha \operatorname{dist}_{g_i}(p,q) + 7\alpha$$

for $i \geq N(\alpha)$. Combining these two estimates we obtain

$$\begin{aligned} \operatorname{dist}_g(p,q) - \operatorname{dist}_{g_i}(p,q) &\leq \alpha \operatorname{dist}_{g_i}(p,q) + 7\alpha \\ &\leq \alpha\big((1+\alpha)\operatorname{dist}_g(p,q) + 7\alpha\big) + 7\alpha \\ &\leq 2\alpha \operatorname{dist}_g(p,q) + 14\alpha. \end{aligned}$$

Thus we have

$$|\operatorname{dist}_g(p,q) - \operatorname{dist}_{g_i}(p,q)| \leq 20\alpha \operatorname{dist}_g(p,q) + 20\alpha \tag{3.5.13}$$

for all $p, q \in M \setminus \bar{B}^g_\alpha$ and $i \geq N(\alpha)$.

Next, consider the case where p is an arbitrary point of M and $q \in \bar{B}^g_\alpha$. Applying Lemma 3.5.1 to q and choosing $N(\alpha)$ even larger, if necessary, yields

$$\operatorname{dist}_{g_i}(q,o) \leq \operatorname{dist}_g(q,o) + \varepsilon_i \leq \operatorname{dist}_g(q,o) + \alpha \leq 2\alpha$$

for $i \geq N(\alpha)$. Then using the triangle inequality and Lemma 3.5.2 we compute

$$\begin{aligned}
|\operatorname{dist}_g(p,q) - \operatorname{dist}_{g_i}(p,q)| &\leq |\operatorname{dist}_g(p,o) - \operatorname{dist}_{g_i}(p,o)| + \operatorname{dist}_g(q,o) + \operatorname{dist}_{g_i}(q,o) \\
&\leq 10\alpha \operatorname{dist}_g(p,o) + 10\alpha + 3\alpha \\
&\leq 10\alpha \operatorname{dist}_g(p,q) + 10\alpha \operatorname{dist}_g(q,o) + 10\alpha + 3\alpha \\
&\leq 10\alpha \operatorname{dist}_g(p,q) + 23\alpha \qquad (3.5.14)
\end{aligned}$$

for all $i \geq N(\alpha)$. Combining (3.5.13) and (3.5.14) gives

$$|\operatorname{dist}_g(p,q) - \operatorname{dist}_{g_i}(p,q)| \leq 30\alpha \operatorname{dist}_g(p,q) + 30\alpha$$

for all $p, q \in M$ and $i \geq N(\alpha)$, and we obtain the desired estimate by replacing α with $\alpha/30$.

□

Proposition 3.5.4. $(M, \operatorname{dist}_{g_i}, o) \to (M, \operatorname{dist}_g, o)$ *in the pointed Gromov-Hausdorff sense.*

Proof. We show that for given $R > 0$ and $\delta > 0$ there exists a $N = N(R,\delta)$ such that for all $i \geq N$ the inclusion $(B^{g_i}_R, \operatorname{dist}_{g_i}) \hookrightarrow (M, \operatorname{dist}_g)$ is a δ-Hausdorff approximation of B^g_R in $(M, \operatorname{dist}_g)$, that is

(1) $|\operatorname{dist}_g(p,q) - \operatorname{dist}_{g_i}(p,q)| \leq \delta$ for all $p, q \in B^{g_i}_R$

(2) $B^g_R \subset T_\delta(B^{g_i}_R) := \{p \in M \mid \operatorname{dist}_g(p, B^{g_i}_R) < \delta\}$.

Let $R > 0, \delta > 0$, and $p, q \in B^{g_i}_R$. By Lemma 3.5.2

$$\operatorname{dist}_g(p,o) \leq (1+\alpha)\operatorname{dist}_{g_i}(p,o) + \alpha \leq (1+\alpha)R + \alpha \leq 2R,$$

and similarly $\operatorname{dist}_g(q,o) \leq 2R$ for small enough $\alpha = \alpha(R)$ and $i \geq N(\alpha)$, where $N(\alpha)$ is as in Lemma 3.5.2, so we have $p, q \in B^g_{2R}$. By Lemma 3.5.3

$$\begin{aligned}
|\operatorname{dist}_g(p,q) - \operatorname{dist}_{g_i}(p,q)| &\leq \alpha \operatorname{dist}_g(p,q) + \alpha \\
&\leq \alpha \operatorname{dist}_g(p,o) + \alpha \operatorname{dist}_g(q,o) + \alpha \\
&\leq 4\alpha R + \alpha \leq \delta
\end{aligned}$$

for small enough $\alpha = \alpha(\delta, R)$ and $i \geq N(\alpha)$ as in Lemma 3.5.3, which shows *(1)*.

Let us show *(2)*. Given $p \in B^g_R$, by Lemma 3.5.2 we have

$$\begin{aligned}
\operatorname{dist}_{g_i}(p,o) &\leq \operatorname{dist}_g(p,o) + \alpha\operatorname{dist}_g(p,o) + \alpha \\
&\leq R + \alpha R + \alpha < R + \frac{\delta}{2}
\end{aligned}$$

for small enough $\alpha = \alpha(R,\delta)$ and $i \geq N(\alpha)$ as in Lemma 3.5.2. This implies that $p \in B^{g_i}_{R+\delta/2}$, so $B^{g_i}_{\delta/2}(p) \cap B^{g_i}_R \neq \emptyset$. Choose $q \in B^{g_i}_{\delta/2}(p) \cap B^{g_i}_R$. Then by Lemma 3.5.3, after choosing α smaller, if necessary, we have

$$\begin{aligned}
\operatorname{dist}_g(p,q) &\leq (1+\alpha)\operatorname{dist}_{g_i}(p,q) + \alpha \\
&\leq (1+\alpha)\frac{\delta}{2} + \alpha < \delta
\end{aligned}$$

for $i \geq N(\alpha)$, which implies $\operatorname{dist}_g(p, B_R^{g_i}) < \delta$, and shows *(2)*.

□

Note that for any ball $B_r^g(p) \subset M \setminus o$ the volume $\mathrm{vol}_g(B_r^g(p))$ with respect to g is well defined since by assumption g is continuous on $M \setminus o$.

Proposition 3.5.5. *Let M, g and g_i be as in Theorem 3.4.1. Suppose that there exist $R > 0$ and $v_0 > 0$ such that*

$$\mathrm{vol}_g(B_1^g(p)) \geq v_0$$

for all balls $B_1^g(p) \subset M \setminus B_R^g(o)$. Then there exists a $v_1 > 0$ and $N \in \mathbb{N}$ such that

$$\mathrm{vol}_{g_i}(B_1^{g_i}(p)) \geq v_1$$

for all $p \in M$, $i \geq N$.

Proof. Recall that by construction $g_i \equiv g$ on $M \setminus B_{r_0}^g(o)$. As in the proof of Theorem 3.4.1, we may assume w.l.o.g. $r_0 = 1$. Consider a ball $B_1^{g_i}(p) \subset M$.

Case 1: $o \in B_{1/2}^{g_i}(p)$
In this case $B_1^{g_i}(p) \supset B_{1/2}^{g_i}(o) \supset B_{1/4}^g(o)$ for large enough i in view of Lemma 3.5.1. Choose $0 < \alpha < \frac{1}{4}$. Using the fact that $g_i \to g$ uniformly on $B_{1/4}^g(o) \setminus B_\alpha^g(o)$, for large enough i we compute

$$\mathrm{vol}_{g_i}(B_1^{g_i}(p)) \geq \mathrm{vol}_{g_i}(B_{1/4}^g(o) \setminus B_\alpha^g(o)) \geq \frac{1}{2}\mathrm{vol}_g(B_{1/4}^g(o) \setminus B_\alpha^g(o)) =: v_{1)} > 0$$

Case 2: $o \notin B_{1/2}^{g_i}(p)$
a) Suppose that

$$B_1^{g_i}(p) \subset M \setminus B_{R+r_0}^g(o),$$

where r_0 is as in Theorem 3.4.1 such that $g_i \equiv g$ for all i on $M \setminus B_{r_0}^g(o)$. Then

$$B_1^{g_i}(p) \subset M \setminus B_{r_0}^g(o),$$

so that $B_1^{g_i}(p) = B_1^g(p)$. Moreover,

$$B_1^{g_i}(p) \subset M \setminus B_R^g(o),$$

so

$$\mathrm{vol}_{g_i}(B_1^{g_i}(p)) = \mathrm{vol}_g(B_1^g(p)) \geq v_0$$

by assumption.

b) Suppose that $B_1^{g_i}(p) \cap B_{R+r_0}^g(o) \neq \emptyset$.
From Lemma 3.5.3 it follows that for small enough $\alpha > 0$ and $i \geq N(\alpha)$

$$\operatorname{dist}_g(p,q) \leq \frac{1}{1-\alpha}\operatorname{dist}_{g_i}(p,q) + \frac{\alpha}{1-\alpha} \leq 2\operatorname{dist}_{g_i}(p,q) + 2\alpha$$

for all $p, q \in M$. Choose $x \in B_1^{g_i}(p) \cap B_{R+r_0}^g(o)$. Then

$$\begin{aligned} \operatorname{dist}_g(p,o) &\leq \operatorname{dist}_g(p,x) + \operatorname{dist}_g(x,o) \\ &\leq 2\operatorname{dist}_{g_i}(p,x) + 2\alpha + R + r_0 \\ &\leq 2 + 2\alpha + R + r_0 \leq 3 + R + r_0 =: \tilde{R} < \infty \end{aligned} \tag{3.5.15}$$

for small enough α and $i \geq N(\alpha)$ as in Lemma 3.5.3. Moreover, $\operatorname{dist}_{g_i}(p,o) \geq \frac{1}{2}$ since $o \notin B^{g_i}_{1/2}(p)$. By Lemma 3.5.3, for small enough α and $i \geq N(\alpha)$ this implies

$$\operatorname{dist}_g(p,o) \geq \frac{1}{1+\alpha}(\operatorname{dist}_{g_i}(p,o) - \alpha) \geq \frac{1/2}{1+\alpha} - \frac{\alpha}{1+\alpha} \geq \frac{1}{4},$$

so

$$B^g_{1/8}(p) \subset M \setminus B^g_{1/8}(o). \tag{3.5.16}$$

For all $y \in B^g_{1/8}(p)$ we have

$$\operatorname{dist}_{g_i}(p,y) \leq (1+\alpha)\operatorname{dist}_g(p,y) + \alpha \leq \frac{1}{8}(1+\alpha) + \alpha < 1$$

which gives us

$$B^{g_i}_1(p) \supset B^g_{1/8}(p). \tag{3.5.17}$$

Combining (3.5.16) and (3.5.17) yields

$$\operatorname{vol}_{g_i}(B^{g_i}_1(p)) \geq \operatorname{vol}_{g_i}(B^g_{1/8}(p)) \geq \frac{1}{2}\operatorname{vol}_g(B^g_{1/8}(p)) \tag{3.5.18}$$

for large enough i since $g_i \to g$ uniformly on $M \setminus B^g_{1/8}(o)$. Moreover, by (3.5.15) and (3.5.16) we have $p \in \bar{B}^g_{\tilde{R}}(o) \setminus B^g_{1/4}(o)$. Since g is continuous on $M \setminus B^g_{1/8}(o)$, the function

$$y \mapsto \operatorname{vol}_g(B^g_{1/8}(y))$$

is continuous on $M \setminus B^g_{1/4}(o)$, and we find $v_{2b)} > 0$ such that $\operatorname{vol}_g(B^g_{1/8}(p)) \geq 2v_{2b)}$ for all $p \in \bar{B}^g_{\tilde{R}}(o) \setminus B_{1/4}(o)$. In view of (3.5.18) this gives us $\operatorname{vol}_{g_i}(B^{g_i}_1(p)) \geq v_{2b)}$.

Putting $v_1 := \min\{v_0, v_{1)}, v_{2b)}\}$ we obtain the desired estimate.

□

3.6. An application of Theorem 3.4.1

In [28], M. Simon studied a class of smooth complete Riemannian three (two) manifolds (M,g) which satisfy

(a) $\operatorname{Ric}(g) \geq k$

(b) $\operatorname{vol}_g\big(B^g_1(x)\big) \geq v_0 > 0$

(c) $\sup_M |\mathcal{R}(g)| < \infty$

and proved uniform estimates for solutions to Ricci flow of such manifolds, showing that there exists a time $T = T(k,v_0) > 0$ such that solutions $(M,g(t))$, $g(0) = g_0$ exist at least on $[0,T)$, and satisfy (a), (b) and (c) with constants $-K$, where $K = K(k,v_0) \geq 0$ and $V = v_0/2$. Moreover, the solutions satisfy $|\mathcal{R}(g_t)| \leq K^2/t$ on $M \times (0,T)$, and $|\operatorname{dist}_{g_t} - \operatorname{dist}_{g_s}| \leq C(K,|t-s|)$ on M for all $t,s \in [0,T)$, where $C(K,|t-s|) \to 0$ as $|t-s| \to 0$ (cf. Thm. 1.9 of [28]). This result was the key step in the proof of the following

Theorem 3.6.1 (Theorem 1.11 of [28]). *Let $k \in \mathbb{R}$ and $v_0 > 0$ be fixed. Let $(M_i, {}^i g_0)$ be a sequence of smooth complete Riemannian three manifolds satisfying* (a), (b) and (c) *(with constants k and v_0 independent of i), and let*

$$(X, d_X, x) = \lim_{i\to\infty} (M, \operatorname{dist}({}^i g_0), x_i)$$

be a pointed Gromov-Hausdorff limit of this sequence. Let $(M_i, {}^i g(t))_{t\in[0,T)}$ be the solutions to Ricci flow as above. Then (after taking a subsequence if necessary) there exists a Hamilton limit solution

$$(N, g(t), x)_{t\in(0,T)} = \lim_{i\to\infty} (M_i, {}^i g(t), x_i)_{t\in(0,T)}$$

satisfying similar estimates as the solutions $(M_i, {}^i g(t))_{t\in[0,T)}$, and

(i) $(N, \operatorname{dist}(g(t)), x) \to (X, d_X, x)$ *in the Gromov-Hausdorff sense as $t \to 0$*

(ii) N is diffeomorphic to X. In particular, X is a manifold.

Moreover, from the proof of [28], Theorem 1.11 it follows that there exists a distance function l on N, such that $\operatorname{dist}(g(t))) \to l$ *as $t \to 0$ in the C^0 sense on N, and (N, l) is isometric to (X, d_X).*

Furthermore, if the metrics ${}^i g_0$ have almost nonnegative Ricci curvature, then the limit solution has nonnegative Ricci curvature for all $t \in (0, T)$. More precisely, we have

Corollary 3.6.2 (Corollary 1.12 of [28]). *Let $(M_i, {}^i g_0)$, $i \in \mathbb{N}$ be a sequence of three manifolds as in Theorem 3.6.1, satisfying*

$$\operatorname{Ric}(M_i, {}^i g_0) \geq -\frac{1}{i}.$$

Let $(X, d_X) = \lim_{i\to\infty}(M_i, \operatorname{dist}({}^i g_0))$ *be the Gromov-Hausdorff limit of this sequence. Then the solution $(N, g(t), x)_{t\in(0,T)}$ obtained in Theorem 3.6.1 satisfies*

$$\operatorname{Ric}(g(t)) \geq 0$$

for all $t \in (0, T)$, and (X, d_X) is diffeomorphic to $(N, g(t))$. In particular, combining this with the results of W. X. Shi [24] and R. Hamilton [12], (X, d_X) is diffeomorphic to $\mathbb{R}^3$, $S^2 \times \mathbb{R}$ or S^3 modulo a group of fixed point free isometries in the standard metric.

Theorem 3.6.3. *Let (M, g) and $(M_i, g_i) = (M, D_i, g_i)$, $i \in \mathbb{N}$ be as in Theorem 3.4.1. Suppose furthermore that* $(M, \operatorname{dist}_g)$ *is complete, and satisfies*

(b') $\operatorname{vol}(B_1^g(p), g) \geq v_0$ *for all balls $B_1^g(p) \subset M \setminus B_R^g(o)$, where $R > 0$ is fixed*

(c') $\sup_{M\setminus B_R^g(o)} |\mathcal{R}(g)| < \infty$

Then the following statements are true:

1) There exists a $N \in \mathbb{N}$ such that for all $i \geq N$ the manifolds (M_i, g_i) are complete and satisfy conditions (b), (c) above, that is

- $\operatorname{vol}_{g_i}(B_1^{g_i}(p)) \geq v_1$ *for all $p \in M_i$, where $v_1 = v_1(R, v_0) > 0$ does not depend on i*

- $\sup_{M_i} |\mathcal{R}(g_i)|_{g_i} < \infty$

Furthermore, $(M_i, \operatorname{dist}_{g_i}, o) \to (M, \operatorname{dist}_g, o)$ *in the pointed Gromov-Hausdorff sense.*

2) There exists a $T = T(R, v_0) > 0$ *such that solutions to Ricci flow* $(M_i, h_i(t))$*,* $h_i(0) = g_i$ *exist at least for* $t \in [0, T)$*, and there exists a Hamilton limit solution* $(N, h(t))_{t\in(0,T)} = \lim_{i\to\infty}(M_i, h_i(t))_{t\in(0,T)}$ *satisfying* $\operatorname{Ric}(h(t)) \geq 0$ *for all* $t \in (0, T)$*.*

3) M *is diffeomorphic to* $(N, h(t))$ *for all* $t \in (0, T)$*. In particular,* M *admits a smooth metric of nonnegative Ricci curvature, and it is diffeomorphic to* $\mathbb{R}^3$*,* $S^2 \times \mathbb{R}$ *or* S^3 *modulo a group of fixed point free isometries in the standard metric. Moreover,* $(M, \operatorname{dist}_g)$ *is isometric to* (N, l)*, where* l *is the* C^0 *limit as* $t \to 0$ *of the distance functions* $\operatorname{dist}_{h(t)}$ *of the limit solution (see Thm. 3.6.1).*

Proof. 1) By assumption $(M, \operatorname{dist}_g)$ is a complete metric space, and M is locally compact since it is a manifold. The generalized Hopf-Rinow Theorem ([10], [2]) implies that any bounded closed subset of M is compact. From Lemma 3.5.3 it follows that for large enough i any bounded subset of $(M_i, \operatorname{dist}_{g_i})$ is bounded in $(M, \operatorname{dist}_g)$, where 'large' does not depend on the particular set. Since for all i the topology of M_i coincides with the topology of M, this implies that for large enough i bounded and closed subsets of $(M_i, \operatorname{dist}_{g_i})$ are compact. From the classical Hopf-Rinow Theorem it then follows that (M_i, g_i) is complete.

The fact that the manifolds (M_i, g_i) are non-collapsed is shown in Proposition 3.5.5. Moreover, the metrics g_i are smooth and coincide with g off a compact neighborhood of o. Therefore condition *(c')* implies that $\sup_{M_i} |\mathcal{R}(g_i)|_{g_i} < \infty$ for all $i \geq 1$. Finally, the Gromov-Hausdorff convergence is shown in Proposition 3.5.4.

In view of 1), assertions 2) and 3) follow by putting $(X, d_X) = (M, \operatorname{dist}_g)$ in Theorem 3.6.1 and Corollary 3.6.2. $\square$

Appendix A.

Fermi coordinates

A.1. Construction of Fermi coordinates about a hypersurface Γ

Fermi coordinates above a submanifold of a Riemannian manifold are a generalization of normal coordinates about a point. A detailed construction for the general case of arbitrary codimension can be found in [11]. Here we sum up constructions from [11] for the case where the submanifold is an embedded hypersurface.

Let M be a smooth n-dim. manifold, equipped with a smooth Riemannian metric g. Let Γ be a smooth embedded hypersurface of M. Let $T\Gamma^{\perp}$ denote the normal bundle of Γ in M, i.e.

$$T\Gamma^{\perp} = \{(\hat{p}, \nu) \,|\, \hat{p} \in \Gamma,\ \nu \in T_{\hat{p}}\Gamma^{\perp}\},$$

where $T_{\hat{p}}\Gamma^{\perp}$ is the orthogonal compliment (with respect to g) of $T_{\hat{p}}\Gamma \subset T_{\hat{p}}M$. Note that Γ can be regarded as a submanifold of $T\Gamma^{\perp}$, after identifying Γ with the zero section of $T\Gamma^{\perp}$.

Let $\exp^{\perp}$ be the restriction to $T\Gamma^{\perp}$ of the exponential map $\exp : TM \to M$. Then $\exp^{\perp}$ maps a neighborhood of $\Gamma \subset T\Gamma^{\perp}$ diffeomorphically onto a neighborhood of $\Gamma \subset M$ (cf. Lemma 2.3 of [11]). This fact allows us to define smooth coordinates (Fermi coordinates) on a small neighborhood $U \subset M$ of a point $p_0 \in \Gamma$ as follows:

Let $\hat{x} = (x^1, \ldots, x^{n-1})$ be a coordinate chart of $\Gamma \cap U$, and let N be a smooth unit section of $T\Gamma^{\perp} \cap TU$. For a point $U \ni p = \exp^{\perp}(tN(\hat{p}))$, where $\hat{p} \in \Gamma \cap U$ and $t \in (-\varepsilon, \varepsilon)$, we put

$$x^i(p) = \begin{cases} x^i(\hat{p}) & i = 1, \ldots, n-1 \\ t & i = n \end{cases}.$$

For any $p \in U$, $s \to \exp^{\perp}(sN(\hat{p}))$, $s \in [0, x^n(p)]$ (or $s \in [x^n(p), 0]$) is the unique shortest geodesic from Γ to p (cf. Lemma 2.7 of [11]), which implies that the distance function from Γ is given by $\mathrm{dist}_g(\cdot, \Gamma) = |x^n|$. Consider the equidistant hypersurfaces $\Gamma(t) = \{p \in M \,|\, x^n(p) = t\}$. Then at each point $p \in U$ we have $T_p\Gamma(t) = \mathrm{span}\{\partial_1(p), \ldots, \partial_{n-1}(p)\}$, where $\partial_i := \frac{\partial}{\partial x^i}$. Moreover, the outward normal from any equidistant hypersurface $\Gamma(t)$, $t \neq 0$ is given by $N = \frac{x^n}{|x^n|}\partial_n$. This is a consequence of the generalized Gauss lemma (Lemma 2.11 of [11]), which states that $N = \nabla^g \mathrm{dist}\,(\cdot, \Gamma)$ off Γ.

In view of the above constructions, in Fermi coordinates the metric tensor g has the form

$$g(p) = \begin{pmatrix} \hat{g}(p) & 0 \\ 0 & 1 \end{pmatrix},$$

where $\hat{g}(p)$ is the restriction of $g(p)$ to $T_p\Gamma(x^n(p))$, given by

$$\hat{g}_{ij}(x(p)) = g(p)(\partial_i(p), \partial_j(p)),$$

$1 \leq i, j \leq n-1$.

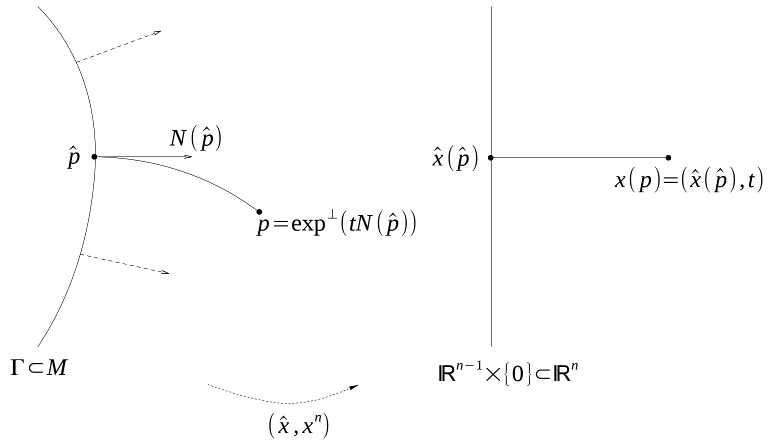

Figure A.1.: Fermi coordinates above Γ

A.2. Modifying the metric on equidistant hypersurfaces

Let $(U, (x^1, \dots, x^n))$ be Fermi coordinates above Γ with respect to g as above, constructed using a coordinate chart $\hat{x} = (x^1, ..., x^{n-1})$ on $\Gamma \cap U$ and a smooth unit section N of $T\Gamma^{\perp_g} \cap TU$. Suppose that h is another smooth metric on TU, which in these coordinates has the form

$$h(p) = \begin{pmatrix} \hat{h}(p) & 0 \\ 0 & 1 \end{pmatrix}.$$

Recall that for any $\hat{p} \in \Gamma \cap U$ we have $N(\hat{p}) = \partial_n(\hat{p})$ from construction, and therefore

$$h(\hat{p})(N(\hat{p}), \partial_i(\hat{p})) = h(\hat{p})(\partial_n(\hat{p}), \partial_i(\hat{p})) = h_{ni}(\hat{p}) = \delta_{ni},$$

which means that N is a smooth unit section of $T\Gamma^{\perp_h}$.

Lemma A.2.1. *Let $(\hat{x}, \tilde{x}^n)$ be Fermi coordinates above $\Gamma \cap U$ induced by $\hat{x}$ and N with respect to h. Then $\tilde{x}^n = x^n$, and in particular* $\operatorname{dist}_h(\cdot, \Gamma) = \operatorname{dist}_g(\cdot, \Gamma)$.

Proof. To verify this it suffices to check that any geodesic γ with respect to g through $\hat{p} \in \Gamma$ with initial velocity $\dot{\gamma}(0) = N(\hat{p})$ is a geodesic with respect to h. In coordinates $(x^1, \dots, x^n)$ we compute using $h_{nl} \equiv const$ for all $l = 1, \dots, n$

$${}^h\Gamma^k_{nn} = \frac{1}{2} h^{kl}(\partial_n h_{nl} + \partial_n h_{nl} - \partial_l h_{nn}) = 0.$$

Moreover, $\dot{\gamma}(t) = \frac{\partial}{\partial x^n}(\gamma(t))$ implies $\dot{\gamma}^i(t) = \delta^i_n$ for $i = 1, \dots, n$, and consequently $\frac{d}{dt}\dot{\gamma}^i \equiv 0$. Thus

$$\begin{aligned} {}^h\nabla_{\dot{\gamma}}\dot{\gamma} &= \big(\frac{d}{dt}\dot{\gamma}^k + \dot{\gamma}^i \dot{\gamma}^j \, {}^h\Gamma^k_{ij} \circ \gamma\big)\partial_k \\ &= \dot{\gamma}^n \dot{\gamma}^n \, {}^h(\Gamma^k_{nn} \circ \gamma)\partial_k = 0 \end{aligned}$$

□

A.3. Second fundamental form of equidistant hypersurfaces

In Fermi coordinates we have a simple expression for the second fundamental forms of equidistant hypersurfaces.

Let $(U,(x^1,\dots,x^n))$ be Fermi coordinates as above and let h be a smooth metric on U as in Lemma A.2.1. By Lemma A.2.1 the equidistant hypersurfaces of g and h coincide, and $\frac{\partial}{\partial x^n}$ is normal to these hypersurfaces. Denote by $\Gamma(p)$ the hypersurface, which is equidistant to Γ and contains the point p. Let ${}^hL(p) \in \big(T_p\Gamma(p)\otimes T_p\Gamma(p)\big)^*$ be the second fundamental form of $(\Gamma(p),\hat{h}) \subset (U,h)$ in p with respect to the normal $\partial_n(p) = \frac{\partial}{\partial x^n}(p)$. Then we have

Lemma A.3.1. ${}^hL_{ij}(x(p)) = -\frac{1}{2}\partial_n\hat{h}_{ij}(x(p))$.

Proof. At p we compute

$$\begin{aligned} {}^hL_{ij} &= -\langle {}^h\nabla_{\partial_i}\partial_n, \partial_j\rangle_h = -{}^h\Gamma^k_{ni}h_{kj} \\ &= -\frac{1}{2}h^{kl}(\partial_n h_{il} + \partial_i h_{nl} - \partial_l h_{ni})h_{kj} \\ &= -\frac{1}{2}\partial_n h_{ij}, \end{aligned}$$

where we used that $h_{ni} \equiv const$ for all $i = 1,\dots,n$. □

Remark A.3.2. Note that $\hat{h}$ and ${}^hL$ can be regarded as a sections of $\big(TM\otimes TM\big)^*$, after identifying $\hat{h} = h(\cdot, P^T\cdot)$, and ${}^hL = {}^hL(P^T\cdot, P^T\cdot)$, respectively, where $P^T(p) : T_pM \to T_p\Gamma(p)$ denotes the projection

$$X = X^k\partial_k(p) \mapsto \sum_{k=1}^{n-1} X^k\partial_k(p).$$

In view of these identifications, in Fermi coordinates we have

$$\hat{h}_{ij} = \begin{pmatrix} h_{ij} & 0 \\ 0 & 0 \end{pmatrix}$$

and

$${}^hL_{ij} = \begin{pmatrix} {}^hL_{ij} & 0 \\ 0 & 0 \end{pmatrix}.$$

Observe that the identification for the second fundamental form is consistent with the fact that ${}^hL_{ij} = -\langle {}^h\nabla_{\partial_i}\partial_n, \partial_j\rangle_h = 0$ if $i = n$ or $j = n$.

Appendix B.

Tensors and linear operators

B.1. Linear operators and $(4,0)$-tensors

Let V be a finite dimensional vector space and T a $(4,0)$-tensor on V. Given a basis $\{e_1, \ldots, e_n\}$ of V, we denote by $T_{ijkl} = T(e_i, e_j, e_k, e_l)$ the coefficients of T with respect to this basis. One has the following connection between $(4,0)$-tensors on V and bilinear forms on $\Lambda^2 V$: Any $(4,0)$-tensor $\{T_{ijkl}\}$ which is antisymmetric in i,j and k,l, respectively, induces a bilinear form $\mathcal{T}$ on $\Lambda^2 V$, which is defined by

$$\mathcal{T}(e_i \wedge e_j, e_k \wedge e_l) := T(e_i, e_j, e_k, e_l) = T_{ijkl}$$

for basis vectors $e_i \wedge e_j = e_i \otimes e_j - e_j \otimes e_i$ and extends to $\Lambda^2 V$ by linearity. Note that the antisymmetries of T ensure that

$$\mathcal{T}(e_i \wedge e_j, e_k \wedge e_l) = -\mathcal{T}(e_j \wedge e_i, e_k \wedge e_l) = -\mathcal{T}(e_i \wedge e_j, e_l \wedge e_k),$$

i.e. $\mathcal{T}$ is well defined. If in addition $T_{ijkl} = T_{klij}$ then the induced bilinear form $\mathcal{T}$ is symmetric. Furthermore, we have

Lemma B.1.1. *Let $\alpha, \beta \in \Lambda^2 V$, $\alpha = \sum_{i<j} \alpha^{ij} e_i \wedge e_j = \alpha^{ij} e_i \otimes e_j$, $\beta = \sum_{i<j} \beta^{ij} e_i \wedge e_j = \beta^{ij} e_i \otimes e_j$ (where $\alpha^{ij} = -\alpha^{ji}$ and $\beta^{ij} = -\beta^{ji}$). Then*

$$\mathcal{T}(\alpha, \beta) = \frac{1}{4} T_{ijkl} \alpha^{ij} \beta^{kl}.$$

Proof. Using the antisymmetries of α^{ij}, β^{kl} and T_{ijkl} we compute

$$\begin{aligned}
\mathcal{T}(\alpha, \beta) &= \mathcal{T}\Big(\sum_{i<j} \alpha^{ij} e_i \wedge e_j, \sum_{k<l} \beta^{kl} e_k \wedge e_l\Big) \\
&= \sum_{i<j} \sum_{k<l} T_{ijkl} \alpha^{ij} \beta^{kl} \\
&= \frac{1}{2} \sum_{i<j} \alpha^{ij} \Big[\sum_{k<l} T_{ijkl} \beta^{kl} + \sum_{k<l} T_{ijkl} \beta^{kl}\Big] \\
&= \frac{1}{2} \sum_{i<j} \alpha^{ij} \Big[\sum_{k<l} T_{ijkl} \beta^{kl} + \sum_{k>l} T_{ijkl} \beta^{kl}\Big] \\
&= \frac{1}{2} \beta^{kl} \sum_{i<j} T_{ijkl} \alpha^{ij} \\
&= \frac{1}{4} T_{ijkl} \alpha^{ij} \beta^{kl}.
\end{aligned}$$

□

Conversely, any bilinear form $\mathcal{T}$ on $\Lambda^2 V$ induces a $(4,0)$-tensor on V via

$$T(e_i, e_j, e_k, e_l) := \mathcal{T}(e_i \wedge e_j, e_k \wedge e_l).$$

The such defined tensor satisfies $T_{ijkl} = -T_{jikl} = -T_{ijlk}$. If in addition the bilinear form $\mathcal{T}$ is symmetric, then we also have $T_{ijkl} = T_{klij}$.

Let us now consider an inner product g on V. The induced inner product $\mathcal{I}^g$ on $\Lambda^2 V$ is defined by

$$\mathcal{I}^g(e_i \wedge e_j, e_k \wedge e_l) = g_{ik}g_{jl} - g_{jk}g_{il}$$

where $g_{ij} = g(e_i, e_j)$. Note that $\mathcal{I}^g$ may also be seen as the bilinear form coming from the $(0,4)$-tensor $\{g_{ik}g_{jl} - g_{jk}g_{il}\}$. Using this inner product we may identify linear operators and bilinear forms on $\Lambda^2 V$ by putting

$$\mathcal{I}^g(e_i \wedge e_j, \mathcal{T}(e_k \wedge e_l)) = \mathcal{T}(e_i \wedge e_j, e_k \wedge e_l). \tag{B.1.1}$$

The bilinear form is symmetric iff the operator is self-adjoint.

B.2. Kulkarni-Nomizu product

Definition B.2.1. *The Kulkarni-Nomizu product of two linear endomorphisms* $\mathbf{A}$, $\mathbf{B}$ *of* V *is the linear endomorphism*

$$\begin{aligned} \mathbf{A} \wedge \mathbf{B} : \Lambda^2 V &\to \Lambda^2 V \\ (\mathbf{A} \wedge \mathbf{B})(e_i \wedge e_j) &:= \frac{1}{2}\big(\mathbf{A}(e_i) \wedge \mathbf{B}(e_j) + \mathbf{B}(e_i) \wedge \mathbf{A}(e_j)\big). \end{aligned}$$

The Kulkarni-Nomizu product of two bilinear forms A, B *on* V *is the* $(4,0)$*-tensor* $A \wedge B$*, defined by*

$$(A \wedge B)_{ijkl} = A \wedge B(e_i, e_j, e_k, e_l) := \frac{1}{2}(A_{ik}B_{jl} - A_{jk}B_{il} + B_{ik}A_{jl} - B_{jk}A_{il}),$$

where $A_{ij} = A(e_i, e_j)$ *and* $B_{ij} = B(e_i, e_j)$*.*

The factor $\frac{1}{2}$ ensures that we have $\mathrm{id}_V \wedge \mathrm{id}_V = \mathrm{id}_{\Lambda^2 V}$. Note that the tensor $\{(A \wedge B)_{ijkl}\}$ is antisymmetric in i, j and k, l, respectively. If in addition A and B are symmetric, then we also have the symmetry $(A \wedge B)_{ijkl} = (A \wedge B)_{klij}$.

Note that Definition B.2.1 is consistent with the identifications in Section B.1. More precisely, we have the following

Lemma B.2.2. *Let* A, B *be the bilinear forms on* V *which correspond to the operators* $\mathbf{A}$, $\mathbf{B}$, *i.e.* $A(\cdot,\cdot) = g(\cdot, \mathbf{A}\cdot)$ *and* $B(\cdot,\cdot) = g(\cdot, \mathbf{B}\cdot)$*. The operator* $\mathbf{A} \wedge \mathbf{B}$ *and the* $(4,0)$*-tensor* $A \wedge B$ *induce the same bilinear form on* $\Lambda^2 V$*:*

$$\mathbf{A} \wedge \mathbf{B}(e_i \wedge e_j, e_k \wedge e_l) = \mathcal{I}^g\big(e_i \wedge e_j, (\mathbf{A} \wedge \mathbf{B})(e_k \wedge e_l)\big) = (A \wedge B)_{ijkl} \tag{B.2.1}$$

Proof. $A(\cdot,\cdot) = g(\cdot, \mathbf{A}\cdot)$ and $B(\cdot,\cdot) = g(\cdot, \mathbf{B}\cdot)$ implies that $\mathbf{A}e_k = A_{rk}g^{rs}e_s$ and $\mathbf{B}e_k = B_{rk}g^{rs}e_s$ for all basis vectors e_k, where g^{rs} is the inverse of the matrix $g_{rs} = g(e_r, e_s)$. We compute

$$\begin{aligned} &\mathcal{I}^g\big(e_i \wedge e_j, (\mathbf{A} \wedge \mathbf{B})(e_k \wedge e_l)\big) = \frac{1}{2}\mathcal{I}^g\big(e_i \wedge e_j, \mathbf{A}(e_k) \wedge \mathbf{B}(e_l) + \mathbf{B}(e_k) \wedge \mathbf{A}(e_l)\big) \\ = \;& \frac{1}{2}(A_{rk}g^{rs}B_{pl}g^{pq} + B_{rk}g^{rs}A_{pl}g^{pq})\mathcal{I}^g(e_i \wedge e_j, e_s \wedge e_q) \\ = \;& \frac{1}{2}(A_{rk}g^{rs}B_{pl}g^{pq} + B_{rk}g^{rs}A_{pl}g^{pq})(g_{is}g_{jq} - g_{js}g_{iq}) \\ = \;& \frac{1}{2}(A_{ik}B_{jl} - A_{jk}B_{il} + B_{ik}A_{jl} - B_{jk}A_{il}). \end{aligned}$$

□

B.3. Inequalities for linear operators

Consider a n-dim. vector space V, equipped with an inner product g, and let $\mathbf{A}$, $\mathbf{B}$ be self-adjoint linear endomorphisms of V with corresponding symmetric bilinear forms A, B, and $\kappa \in \mathbb{R}$. We will use the following notation

Notation B.3.1. We say that $\mathbf{A} \geq \kappa \in \mathbb{R}$, if all eigenvalues of $\mathbf{A}$ are at least κ, or equivalently if

$$A(X,X) \geq \kappa g(X,X)$$

for all $X \in V$. We say that $\mathbf{A} \geq \mathbf{B}$, if $\mathbf{A} - \mathbf{B} \geq 0$, or equivalently if

$$A(X,X) - B(X,X) \geq 0$$

for all $X \in V$.

Lemma B.3.2. *Let $\mathcal{T}$ be a self-adjoint linear endomorphism of $\Lambda^2 V$ and (T_{ijkl}) the corresponding $(0,4)$-tensor on V. If $\mathcal{T} \geq 0$, then the bilinear form*

$$\mathrm{tr}^g_{24} T := g^{jl} T(\cdot, e_j, \cdot, e_l) : V \times V \to \mathbb{R}$$

is positive semidefinite (where $e_1, \dots, e_n$ is a basis of V and $g_{ij} = g(e_i, e_j)$).

Proof. We identify $\mathcal{T}$ with the corresponding symmetric bilinear form on $\Lambda^2 V$ in view of (B.1.1). By assumption $\mathcal{T}(\alpha, \alpha) \geq 0$ for all 2-vectors $\alpha \in \Lambda^2 V$. Let $\{e_1, \dots, e_n\}$ be a basis of V such that $g_{ij} = g(e_i, e_j) = \delta_{ij}$. Let $X = X^k e_k \in V$. For every $1 \leq j \leq n$ we define the 2-vector $\alpha_j = X^k e_k \wedge e_j \in \Lambda^2 V$, and compute

$$\begin{aligned} \mathrm{tr}^g_{24} T(X,X) &= g^{jl} T(X, e_j, X, e_l) = \sum_j X^k X^l T(e_k, e_j, e_l, e_j) \\ &= \sum_j X^k X^l \mathcal{T}(e_k \wedge e_j, e_l \wedge e_j) = \sum_j \mathcal{T}(\alpha_j, \alpha_j) \geq 0. \end{aligned}$$

□

Note that this lemma also holds if we replace $\mathrm{tr}^g_{24} T$ by $\mathrm{tr}^g_{13} T$.

Lemma B.3.3. *Let $\mathbf{A}, \mathbf{B}$ be two self-adjoint endomorphisms of V. If $\mathbf{A}, \mathbf{B} \geq 0$, then $\mathbf{A} \wedge \mathbf{B} \geq 0$. In particular, if $\mathbf{A} \leq \mathbf{C}$ and $\mathbf{B} \geq 0$, then $\mathbf{A} \wedge \mathbf{B} \leq \mathbf{C} \wedge \mathbf{B}$.*

Proof. Since $\mathbf{A}$ is self-adjoint, we find an orthonormal basis $\{e_1, \dots, e_n\}$ of (V, g) such that $A_{ij} = \lambda_i \delta_{ij}$ with respect to this basis (where $\lambda_i \geq 0$ by assumption). Let $\alpha = \sum_{i<j} \alpha^{ij} e_i \wedge e_j = \alpha^{ij} e_i \otimes e_j \in \Lambda^2 V$. Using Lemma B.1.1 and (B.2.1) we compute

$$\begin{aligned} (\mathbf{A} \wedge \mathbf{B})(\alpha, \alpha) &= \frac{1}{8}(A_{ik} B_{jl} \alpha^{ij} \alpha^{kl} - A_{jk} B_{il} \alpha^{ij} \alpha^{kl} + B_{ik} A_{jl} \alpha^{ij} \alpha^{kl} - B_{jk} A_{il} \alpha^{ij} \alpha^{kl}) \\ &= \frac{1}{2} A_{ik} B_{jl} \alpha^{ij} \alpha^{kl} \\ &= \frac{1}{2} \lambda_i B_{jl} \alpha^{ij} \alpha^{il} \geq 0, \end{aligned}$$

where we used $\alpha^{ij} = -\alpha^{ji}$ and the fact that for every fixed i we have $B_{jl} \alpha^{ij} \alpha^{il} \geq 0$ by assumption. In particular, If $\mathbf{A} \leq \mathbf{C}$ and $\mathbf{B} \geq 0$, then $(\mathbf{C} - \mathbf{A}) \wedge \mathbf{B} \geq 0$ and

$$\mathbf{C} \wedge \mathbf{B} = (\mathbf{C} - \mathbf{A}) \wedge \mathbf{B} + \mathbf{A} \wedge \mathbf{B} \geq \mathbf{A} \wedge \mathbf{B}.$$

□

Appendix C.

Length spaces

C.1. Definitions and basic properties

In this section, we sum up some basic properties of length metric spaces. We refer to Chapter 2 of [6] for a detailed discussion.

Definition C.1.1 (Continuous path)**.** *Let X be a topological space. A* path *in X is a continuous map $\gamma : I \to X$, defined on an interval $I \subset \mathbb{R}$, where by 'interval' we mean any connected subset of $\mathbb{R}$.*

Definition C.1.2 (Length structure, see Section 2.1.1 of [6])**.** *Let X be a topological space, let A be a subset of all paths in X, and let L be a map $A \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ (*length *of paths). The pair (A, L) is called a* length structure *on X, if A and L have the following properties:*

A1) A is closed under restrictions: if $\gamma : I \to X$ lies in A, then $\gamma|_J$ lies in A for all $J \subset I$.

A2) A is closed under concatenations: if $\gamma : [a, b] \to X$ and $\sigma : [b, c] \to X$ lie in A, then $\gamma + \sigma : [a, c] \to X$ lies in A, where $(\gamma + \sigma)|_{[a,b]} = \gamma$ and $(\gamma + \sigma)|_{[b,c]} = \sigma$.

A3) A is closed under linear reparameterizations: if $\gamma : [a, b] \to X$ lies in A, and $h : [c, d] \to [a, b]$ is a homeomorphism of the form $c(t) = \lambda t + \mu$, then $\gamma \circ h : [c, d] \to X$ lies in A.

L1) Lengths of paths are additive: $L(\gamma|_{[a,b]}) = L(\gamma|_{[a,c]}) + L(\gamma|_{[c,b]})$ for any $c \in [a, b]$.

L2) The length of a piece of path continuously depends on the piece: for any $\gamma \in A$ such that $L(\gamma) < \infty$, the map $t \mapsto L(\gamma|_{[a,t]})$ is continuous.

L3) The length is invariant under reparameterizations: $L(\gamma) = L(\gamma \circ h)$ for any linear homeomorphism h as in A3.

L4) The length agrees with the topology of X in the following sense: for any neighborhood U of a point $x \in X$, the length of paths connecting x with the compliment of U is separated from zero, that is,

$$\inf\{L(\gamma) \mid \gamma \in A,\, \gamma(a) = x,\, \gamma(b) \in U^c\} > 0.$$

Definition, Lemma C.1.3 (Length space, see Section 2.1.2 of [6])**.** *A length structure (A, L) on a topological space X induces a metric d_L on X via*

$$d_L(x, y) = \inf\{L(\gamma) \mid \gamma : [a, b] \to X,\, \gamma \in A,\, \gamma(a) = x,\, \gamma(b) = y\}$$

(note that d_L is not necessarily finite). If a metric space (X,d) admits a length structure (A,L) such that $d = d_L$, then d is called an intrinsic*, or* length*,* metric*. A metric space whose metric is intrinsic is called a* length space*.*

Remark C.1.4. Observe that the topology induced by d_L can be only finer than that of X, that is, any open set in X is open in (X, d_L) as well. Indeed, given an open set $U \subset X$ and $p \in U$, in view of the property L4 we find a $\varepsilon > 0$ such that $L(\gamma) \geq \varepsilon$ for any path $\gamma \in A$ connecting p and U^c. Therefore, by definition of d_L we have $d_L(p,q) \geq \varepsilon$ for all $q \in U^c$, which implies $B_\varepsilon^{d_L}(p) \subset U$ (where $B_\varepsilon^{d_L}(p) = \{x \in X \mid d_L(x,p) < \varepsilon)\}$).

Definition, Lemma C.1.5 (Induced length, see Section 2.3 of [6])**.** *Let (X,d) be a metric space, and $\gamma : [a,b] \to X$ be a path in X. The* length of γ with respect to the metric d *is defined by*

$$L_d(\gamma) := \sup\{\sum_{i=1}^{N} d(\gamma(t_{i-1}), \gamma(t_i)) \mid N \in \mathbb{N},\, a = t_0 \leq t_1 \leq \cdots \leq t_N = b\}$$

(note that $L_d(\gamma) \geq d(\gamma(a), \gamma(b))$ in view of the triangle inequality). A path γ is called rectifiable *if $L_d(\gamma) < \infty$.*

A metric d induces a length structure (A, L) on X, where A is the set of all paths in X parameterized by closed intervals, and $L = L_d$. Thus, d induces a length metric d_{L_d} on X.

Proposition C.1.6 (Proposition 2.4.1 of [6])**.** *Let (X,d) be a length space (as in Definition C.1.3), and let d_{L_d} be the length metric induced by d (as in Definition C.1.5). Then $d = d_{L_d}$.*

In view of the above proposition, one has an alternative definition of 'intrinsic metric':

Definition C.1.7. *A metric d is intrinsic if and only if for any $x, y \in X$ and any $\varepsilon > 0$ there exists a path γ in X such that $L_d(\gamma) \leq d(x,y) + \varepsilon$.*

Lemma C.1.8 (Induced length is semi-continuous, Proposition 2.3.4 (*iv*) of [6])**.** *If a sequence of rectifiable paths γ_i in (X,d) converges pointwise to a (continuous) path γ, then*

$$\liminf_i L_d(\gamma_i) \geq L_d(\gamma).$$

The following theorem is a version of the Arzela-Ascoli Compactness Theorem.

Theorem C.1.9 (Arzela-Ascoli Theorem, Theorem 2.5.14 of [6])**.** *In a compact metric space, any sequence of curves with uniformly bounded lengths contains a uniformly converging subsequence.*

Lemma C.1.10. *Let (X,d) be locally compact metric space, where d is an intrinsic metric. Then for any $p \in X$ there exists a $r > 0$ such that any $x, y \in \bar{B}_r(p)$ can be connected by a shortest path (that is, a path γ in X satisfying $d(x,y) = L_d(\gamma)$).*

Proof. Let $p \in X$. Since X is locally compact, we find a $r > 0$ such that $\bar{B}_{5r}(p)$ is compact. Since d is intrinsic, we may choose a sequence of paths $\gamma_i : [a,b] \to X$ from x to y such that $L(\gamma_i) \searrow d(x,y)$ (cf. Definition C.1.7). In particular, the lengths of γ_i are uniformly bounded. Moreover, the image of γ_i is contained in $\bar{B}_{5r}(p)$ for large enough i (w.l.o.g. for all i). Indeed, for all $t \in [a,b]$ we have

$$\begin{aligned} d(p,\gamma_i(t)) &\leq d(p,x) + d(x,\gamma_i(t)) \\ &\leq r + L_d(\gamma_i|_{[0,t]}) \\ &\leq r + L_d(\gamma_i) \\ &\leq r + 2d(x,y) \\ &\leq r + 2(d(x,p) + d(p,y)) \leq 5r. \end{aligned}$$

Therefore, the sequence (γ_i) is a sequence of paths in the compact metric space $(\bar{B}_{5r}(p), d|_{\bar{B}_{5r}(p)})$ with uniformly bounded lengths. In view of the Arzela-Ascoli Theorem, a subsequence of (γ_i) (w.l.o.g. the sequence itself) converges to a continuous path γ in $\bar{B}_{5r}(p) \subset X$ connecting x and y. Using semi-continuity of length (see Lemma C.1.8) we conclude

$$d(x,y) \leq L_d(\gamma) \leq \liminf_i L_d(\gamma_i) = d(x,y),$$

which gives us $d(x,y) = L_d(\gamma)$. □

Example C.1.11. Let M be a smooth manifold, and g be a smooth Riemannian metric on M. The class of piecewise C^1 curves in M together with the length function

$$L_g(\gamma) = \int_a^b \|\dot{\gamma}\|_{g(t)} dt$$

for $\gamma : [a,b] \to M$ is a length structure on M.

C.2. The length metric dist_g

Consider a Riemannian manifold (M,g) as in Section 3.4, that is, M is a smooth connected manifold, and g is a smooth Riemannian metric on $M \setminus o$, which is possibly singular (discontinuous) at o. We define lengths of piecewise C^1 curves in M as follows. For a curve $\gamma : [a,b] \to M$ whose image is contained in $M \setminus o$ we put $L_g(\gamma) = \int_a^b \|\dot{\gamma}(t)\|_{g(\gamma(t))} dt$. If $\gamma(t_0) = o$ for some $t_0 \in [a,b]$, we define

$$L_g(\gamma) = \int_a^b \|\dot{\gamma}(t)\|_{g(\gamma(t))} dt := \lim_{\alpha \searrow 0} \int_a^{t_0-\alpha} \|\dot{\gamma}(t)\|_{g(\gamma(t))} dt + \lim_{\alpha \searrow 0} \int_{t_0+\alpha}^b \|\dot{\gamma}(t)\|_{g(\gamma(t))} dt.$$

We then introduce a distance function induced by g on M in a similar way as for smooth Riemannian metrics, that is, we put

$$\operatorname{dist}_g(p,q) = \inf\{L_g(\gamma) \mid \gamma \text{ is a piecewise } C^1 \text{ curve in } M \text{ from } p \text{ to } q\},$$

and $\operatorname{dist}_g(o,o) := 0$. (Note that in the definition of dist_g it suffices to consider piecewise C^1 curves which pass through o not more than once.) For $p \in M$ and $r > 0$ we denote by $B_r^g(p)$ the set

$$B_r^g(p) = \{x \in M \mid \operatorname{dist}_g(x,p) < r\}.$$

Since g is possibly singular at o, in general one cannot expect $\operatorname{dist}_g(\cdot,\cdot)$ to be finite. However, under some additional assumptions dist_g enjoys similar properties as distance functions coming from smooth Riemannian metrics.

Remark C.2.1. Let dist^o_g be the distance function induced by g on $M \setminus o$ (where g is smooth), that is,

$$\operatorname{dist}^o_g(p,q) = \inf\{L_g(\gamma) \mid \gamma \text{ is a piecewise } C^1 \text{ curve in } M \setminus o \text{ from } p \text{ to } q\},$$

where L_g is as above. Clearly, the class of piecewise C^1 curves in $M \setminus o$ together with the length function L_g is a length structure on $M \setminus o$ (in the sense of Definition C.1.2).

We denote by $B^{g,o}_r(p)$ the open ball of radius r in $(M \setminus o, \operatorname{dist}^o_g)$ centered at $p \in M \setminus o$.

Lemma C.2.2. *Let (M,g), L_g, and dist_g be as above. The class of piecewise C^1 curves in M together with the length function L_g is a length structure on M (cf. Definition C.1.2). Thus, $(M, \operatorname{dist}_g)$ is a length space (cf. Definition C.1.3).*

Proof. We need to verify properties A1 – A3, L1 – L4 from Definition C.1.2. Properties A1 – A3, L1, and L3 are obvious.

Property L2: Clearly, $t \mapsto L_g(\gamma|_{[a,t]})$ is continuous at any t_0 such that $\gamma(t_0) \neq o$. If $\gamma(t_0) = o$, then $L(\gamma|_{[a,t_0]}) = \lim_{t \to t_0} L(\gamma|_{[a,t]})$ holds by definition of L_g.

Property L4: Let $p \in M$ and $U \ni p$ a neighborhood of p in M. Consider the case where $p \neq o$. We find a geodesic ball (with respect to dist^o_g) $B^{g,o}_\varepsilon(p) \subset M \setminus o$ such that $B^{g,o}_\varepsilon(p) \subset U$ and $o \notin \bar{B}^{g,o}_\varepsilon(p)$ (where the closure is with respect to the topology of M). Let $\gamma : [a,b] \to M$ be a piecewise C^1 curve from p to U^c. We then find a $t \in (a,b)$ such that $\gamma([a,t]) \subset \bar{B}^{g,o}_\varepsilon(p)$ and $\gamma(t) \in \partial B^{g,o}_\varepsilon(p)$. In particular, $\gamma|_{[a,t]}$ is a piecewise C^1 curve in $M \setminus o$ connecting p and $\gamma(t) \in \partial B^{g,o}_\varepsilon(p)$. Thus,

$$L_g(\gamma) \geq L_g(\gamma|_{[a,t]}) \geq \operatorname{dist}^o_g(p, \partial B^{g,o}_\varepsilon(p)) = \varepsilon > 0.$$

Consider the case $p = o$. Let U be an open neighborhood of o in M, w.l.o.g. $\bar{U}$ is compact. We find an open neighborhood V of o such that $\bar{V} \subset U$. Let $\gamma : [a,b] \to M$ be a piecewise C^1 curve connecting o and U^c. We find $a < t_0 < t_1 < b$ such that $\gamma([t_0,t_1]) \subset \bar{U} \setminus V$, and $\gamma(t_0) \in \partial V$ and $\gamma(t_1) \in \partial U$. In particular, $\gamma|_{[t_0,t_1]}$ is a piecewise C^1 curve in $M \setminus o$ connecting compact subsets $\partial V, \partial U \subset M \setminus o$. Since $\partial V \cap \partial U = \emptyset$, we have that $\operatorname{dist}^o_g(\partial U, \partial V) := \varepsilon > 0$, so

$$L_g(\gamma) \geq L_g(\gamma|_{[t_0,t_1]}) \geq \operatorname{dist}^o_g(\partial U, \partial V) = \varepsilon > 0.$$

□

Lemma C.2.3. *For all $p \in M \setminus o$ there exists a neighborhood U in $M \setminus o$ such that*

$$\operatorname{dist}^o_g|_U = \operatorname{dist}_g|_U.$$

Proof. Let $p \in M \setminus o$. Observe that $\operatorname{dist}_g \leq \operatorname{dist}^o_g$ holds on any neighborhood of p not containing o, since we take the infimum over a larger set.

Let us check that the inverse inequality holds on a neighborhood of p. Let $U \ni o$ be an open neighborhood of o such that $p \notin \bar{U}$. By Lemma C.2.2 (property L.4) there exists $r > 0$ such that $L_g(\gamma) \geq 4r$ for all piecewise C^1 curves γ connecting o and U^c. After choosing r even smaller, we may assume that $B^{g,o}_{2r}(p) \subset U^c$. We show that $\operatorname{dist}_g \geq \operatorname{dist}^o_g$ on $B^{g,o}_r(p)$. Let $x, y \in B^{g,o}_r(p)$. Choose a sequence of piecewise C^1 curves $\gamma_i : [0,1] \to M$ such that $L_g(\gamma_i) \searrow \operatorname{dist}_g(x,y)$. Suppose that $o \in \gamma_i([0,1])$ for some $i \in \mathbb{N}$. Since $x \in U^c$, we have that

$$L_g(\gamma_i) \geq 4r > 2r + r \geq \operatorname{dist}^o_g(x,y) + r \geq \operatorname{dist}_g(x,y) + r.$$

Thus, $o \notin \gamma([0,1])$ for all $i \geq N$, $N \in \mathbb{N}$ large enough. By definition of dist^o_g we have that $L_g(\gamma_i) \geq \operatorname{dist}^o_g(x,y)$ for all $i \geq N$. Since $L_g(\gamma_i) \searrow \operatorname{dist}_g(x,y)$, this implies $\operatorname{dist}_g(x,y) \geq \operatorname{dist}^o_g(x,y)$. □

Lemma C.2.4. *Let M, g, and dist_g be as above. Additionally, suppose that $\operatorname{dist}_g(\cdot,o) : M \to \mathbb{R}$ is continuous at o. Then $\operatorname{dist}_g : M \times M \to \mathbb{R}$ is finite, continuous, and the topology induced by dist_g agrees with that of M.*

Proof. 1) dist_g is finite: Since $\operatorname{dist}_g(\cdot,o) : M \to \mathbb{R}$ is continuous by assumption, we find a neighborhood $U \ni o$ in M such that $\operatorname{dist}_g(p,o) \leq 1$ for all $p \in U$. By definition of dist_g this implies that for any $p \in U$ there exists a piecewise C^1 curve γ_{po} from p to o satisfying $L_g(\gamma_{po}) \leq 2 < \infty$. Since M is connected, given $x, y \in M \setminus o$ we find points $p, q \in U \setminus o$ and piecewise C^1 curves γ_{xp}, γ_{yq} in $M \setminus o$ connecting x, p and y, q. Since g is smooth on $M \setminus o$, these curves have finite lengths. Then the concatenation of γ_{xp}, γ_{po}, γ_{qo}, and γ_{yq} is a piecewise C^1 curve of finite length from x to y, which implies that $\operatorname{dist}_g(x,y) < \infty$.

2) $\operatorname{dist}_g : M \times M \to \mathbb{R}$ is continuous: First, observe that $\operatorname{dist}_g(\cdot,p) : M \to \mathbb{R}$ is continuous at p for all $p \in M$. Indeed, by assumption $\operatorname{dist}_g(\cdot,o)$ is continuous at o. Moreover, $\operatorname{dist}_g(\cdot,p)$ is continuous at p for all $p \in M \setminus o$ in view of the fact that $\operatorname{dist}^o_g : (M \setminus o) \times (M \setminus o) \to \mathbb{R}$ is continuous, and Lemma C.2.3. Thus, given $p, q \in M$, and $\varepsilon > 0$, we find open (in M) neighborhoods $U_p \ni p$ and $U_q \ni q$ such that $\operatorname{dist}_g(x,p) < \varepsilon$ and $\operatorname{dist}_g(y,q) < \varepsilon$ for all $x \in U_p$, $y \in U_q$. Then $|\operatorname{dist}_g(x,y) - \operatorname{dist}_g(p,q)| \leq 2\varepsilon$ by the triangle inequality.

3) Let $\mathcal{O}_M$ be the topology of M, and $\mathcal{O}_{\operatorname{dist}_g}$ be the topology induced by dist_g on M. The fact that dist_g is continuous on $M \times M$ implies that $\mathcal{O}_{\operatorname{dist}_g} \subset \mathcal{O}_M$. The inverse inclusion is due to the fact that $(M, \operatorname{dist}_g)$ is a length space (see Lemma C.2.2), and Remark C.1.4.

□

Lemma C.2.5. *Let M, g, dist_g be as in Lemma C.2.4 above. There exists a $r > 0$ such that for any point $x \in \bar{B}^g_r(o) \setminus o$ there exists a curve $\gamma : [a,b] \to \bar{B}^g_r(o)$ from x to o, such that $\operatorname{dist}_g(x,o) = L_g(\gamma)$, and $\gamma|_{[a,b)}$ is a geodesic in $(M \setminus o, g|_{M\setminus o})$.*

Proof. Let d be the length metric on M arising from dist_g, that is,

$$d(x,y) = \inf\{L_{\operatorname{dist}_g}(\gamma) \,|\, \gamma : [a,b] \to M \text{ is a } C^0 \text{ path from } x \text{ to } y\},$$

where

$$L_{\operatorname{dist}_g}(\gamma) = \sup\{\sum_{i=1}^{N} \operatorname{dist}_g(\gamma(t_{i-1}), \gamma(t_i)) \,|\, N \in \mathbb{N},\, a = t_0 \leq t_1 \leq \cdots \leq t_N = b\}$$

(cf. Lemma C.1.5). Furthermore, let d^o be the length metric on $M \setminus o$ arising from dist^o_g, defined in a similar way (where we take the infimum over lengths $L_{\operatorname{dist}^o_g}(\gamma)$ of C^0 paths in $M \setminus o$). Since dist_g and dist^o_g are length metrics, we have $d = \operatorname{dist}_g$ on M and $d^o = \operatorname{dist}^o_g$ on $M \setminus o$ in view of Lemma C.1.5. By Lemma C.1.10 there exists a $r > 0$ such that any two points $x, y \in \bar{B}^g_r(o) = \bar{B}^d_r(o)$ can be connected by a shortest path γ w.r.t. d, that is, there exists a C^0 path in M from x to y satisfying $L_{\mathrm{dist}_g}(\gamma) = d(x, y)$.

Let us consider such a shortest path $\gamma : [a, b] \to M$ connecting $x \in \bar{B}^g_r(o) \setminus o$ and o. We may assume that $\gamma(b) = o$ and $o \notin \gamma([a, b))$ (otherwise we put $\tilde{b} = \inf\{t > a \,|\, \gamma(t) = 0\}$ and consider $\gamma|_{[a,\tilde{b}]}$). Then $\gamma|_{[a,b)}$ is a geodesic (in the classical sense) in $(M \setminus o, g|_{M\setminus o})$. Indeed, for any $t \in (a, b)$ we find a neighborhood $U \ni \gamma(t)$ in $M \setminus o$, such that $d = d^o$ on U, and $\delta > 0$ such that $\gamma|_{[t-\delta,t+\delta]} \subset U$. Since $\mathrm{dist}_g = \mathrm{dist}^o_g$ on U, and the restriction $\gamma|_{[t-\delta,t+\delta]}$ is a shortest path in (M, d), we have that

$$d^o(\gamma(t-\delta), \gamma(t+\delta)) = d(\gamma(t-\delta), \gamma(t+\delta)) = L_{\mathrm{dist}_g}(\gamma|_{[t-\delta,t+\delta]}) = L_{\mathrm{dist}^o_g}(\gamma|_{[t-\delta,t+\delta]}),$$

which implies that $\gamma|_{[t-\delta,t+\delta]}$ is a shortest path in $(M \setminus o, d^o)$. It then follows from the fact that d^o arises from the smooth Riemannian metric $g|_{M\setminus o}$, that $\gamma|_{[t-\delta,t+\delta]}$ is a geodesic in $(M \setminus o, g|_{M\setminus o})$.

□

Lemma C.2.6. *Let M, g, dist_g be as in Lemma C.2.4. Additionally, suppose that there exists a neighborhood U of o such that $\mathrm{dist}_g(\cdot, o)$ is smooth on $U \setminus o$ and $\nabla(\mathrm{dist}_g(\cdot, o)) \neq 0$ on $U \setminus o$. Then there exists a $r > 0$ such that $\bar{B}^g_r(o) \setminus o$ can be covered by a set of Fermi coordinates (see Section A) above $\partial B^g_r(o)$.*

Proof. Since the topology of M coincides with that induced by dist_g, we find a $r_0 > 0$ such that $\bar{B}^g_{r_0}(o) \subset \mathring{U}$. In view of Lemma C.2.5, after choosing r_0 smaller we may assume that for any $x \in \bar{B}^g_{r_0}(o) \setminus o$ there exists a shortest curve $\gamma : [0, r] \to \bar{B}^g_r(o)$ from o to x such that $\gamma|_{(0,r]}$ is a unit speed geodesic in $\bar{B}^g_r(o) \setminus o$.

First, let us verify that $\nabla(\mathrm{dist}_g(\cdot, o))|_x = \dot{\gamma}(r)$. Since $\mathrm{dist}_g(\cdot, o)$ is smooth on $U \supset \bar{B}^g_r(o)$ and $\nabla(\mathrm{dist}_g(\cdot, o)) \neq 0$, the level set $\partial B^g_r(o)$ is a smooth hypersurface in $M\setminus o$. Moreover, for any $t \in [0, r]$ the restriction $\gamma|_{[t,r]}$ is a shortest curve connecting $\gamma(t)$ and $\partial B^g_r(o)$ (otherwise there would exists a curve σ connecting $\gamma(t)$ and a point $y \in \partial B^g_r(o)$ such that $L_g(\sigma) < L_g(\gamma|_{[t,r]}) = r - t$, so that $\mathrm{dist}_g(o, \partial B^g_r(o)) \leq L_g(\gamma|_{[0,t]} + \sigma) < t + (r - t) = r$, which is a contradiction). Thus, we have $\dot{\gamma}(r) \perp_g \partial B^g_r(o)$. Furthermore, since the gradient of a function is orthogonal to its level sets, we have $\nabla(\mathrm{dist}_g(\cdot, o))|_x = \lambda\, \dot{\gamma}(r)$ for some $\lambda \in \mathbb{R}$. We compute

$$\begin{aligned} 1 &= \frac{d}{dt}|_{t=r} t = \frac{d}{dt}|_{t=r} \mathrm{dist}_g(\gamma(t), o) \\ &= \langle \nabla(\mathrm{dist}\,(\cdot, o))|_{\gamma(r)}, \dot{\gamma}(r)\rangle_g \\ &= \lambda\, |\dot{\gamma}(r)|^2_g = \lambda, \end{aligned}$$

which implies $\nabla(\mathrm{dist}_g(\cdot, o))|_x = \dot{\gamma}(r)$.

A similar argument as above shows that $\dot{\gamma}(t) = \nabla(\mathrm{dist}_g(\cdot, o))|_{\gamma(t)}$ for all $t \in [0, r)$, that is, $\gamma|_{(0,r]}$ is the integral curve of $\nabla(\mathrm{dist}_g(\cdot, o))$ with $\gamma(r) = x$. In particular, this implies that such a shortest curve connecting o and $x \in \bar{B}^g_{r_0}(o) \setminus o$ must be unique. Moreover, since for any point $x \in B^g_{r_0}(o) \setminus o$ there exists a curve γ from o to x as above, we have $|\nabla(\mathrm{dist}_g(\cdot, o))|_g \equiv 1$ on $B^g_{r_0}(o) \setminus o$.

Since $x \in \partial B^g_r(o) \subset \mathring{U}$, and $\nabla(\mathrm{dist}\,(\cdot, o))$ is smooth on U, we may extend γ to $\tilde{\gamma} : [0, r + \tau) \to M$, $\tau > 0$, such that $\tilde{\gamma}|_{(0,r+\tau)}$ is an integral curve of $\nabla(\mathrm{dist}_g(\cdot, o))$ in $M \setminus o$. Since $\nabla(\mathrm{dist}_g(\cdot, o))$ is a unit vector field, the curve $\tilde{\gamma}|_{(0,r+\tau)}$ is a geodesic (see Lemma D.7). Suppose that the interval $(0, r + \tau)$ is maximal.

Case 1: $r + \tau = \infty$. In this case, $r_0 \in [0, r + \tau)$. Then $\operatorname{dist}_g(\tilde{\gamma}(r_0), o) \leq L_g(\tilde{\gamma}|_{[0,r_0]}) = r_0$, that is, $\gamma(r_0) \in \bar{B}^g_{r_0}(o)$. Since the integral curves of $\nabla(\operatorname{dist}_g(\cdot, o))$ are the unique shortest curves connecting o to points of $\bar{B}^g_{r_0} \setminus o$ (see the argument above), it follows that $\operatorname{dist}_g(\tilde{\gamma}(r_0), o) = L_g(\tilde{\gamma}|_{[0,r_0]}) = r_0$, so $\tilde{\gamma}(r_0) \in \partial B^g_{r_0}(o)$.

Case 2: $r + \tau < \infty$. In this case, $\tilde{\gamma}$ leaves any compact subset of $M \setminus o$. In particular, there exists a $t_0 \in (r, r + \tau)$ such that $\tilde{\gamma}(t_0) \notin \bar{B}^g_{r_0}(o) \setminus B^g_r(o)$. Then $\tilde{\gamma}(t_0) \notin \bar{B}^g_{r_0}(o)$, since $\tilde{\gamma}([t, r + \tau)) \subset M \setminus B^g_t(o)$ for all $t \in [0, r + \tau)$. (Indeed,

$$\frac{d}{dt}(\operatorname{dist}_g(\tilde{\gamma}(t), o)) = \langle \nabla(\operatorname{dist}_g(\cdot, o))|_{\gamma(t)}, \dot{\gamma}(t)\rangle_g = |\nabla(\operatorname{dist}_g(\cdot, o))|_{\gamma(t)}|^2_g = 1 > 0,$$

so $t \mapsto \operatorname{dist}_g(\gamma(t), o)$ is non-decreasing.) This implies that $\tilde{\gamma}(s) \in \partial B^g_{r_0}(o)$ for some $s \in (r, r + \tau)$.

Thus, we have shown that any $x \in B^g_{r_0}(o) \setminus o$ can be connected to a point $\hat{x} \in \partial B^g_{r_0}(o)$ by a distance minimizing geodesic in $B^g_{r_0}(o) \setminus o$ emanating from $\hat{x}$ with initial velocity $-\nabla(\operatorname{dist}_g(\cdot, o))|_{\hat{x}}$ (namely the geodesic $t \mapsto \tilde{\gamma}(-t)$), where $-\nabla(\operatorname{dist}_g(\cdot, o))|_{\partial B^g_{r_0}(o)}$ is a smooth unit vector field on $\partial B^g_{r_0}(o)$. By construction of Fermi coordinates this proves the claim.

$\square$

Remark C.2.7. From the above argument it follows that $\bar{B}^g_{r_0}(o) \setminus o$ can be covered by a set of Fermi coordinates over $\partial B^g_s(o)$, for arbitrary $0 < s \leq r$.

Appendix D.

Technical lemmas

Lemma D.1. *Let* $\mathbf{L}$ *be the extended operator from Lemma 1.2.5. Then* $\nabla_N \mathbf{L} = 0$.

Proof. Let $q \in M_0$ be a point near Γ and γ be the integral curve of N emanating from $p \in \Gamma$ passing through q. Let X be a smooth vector field on a neighborhood of q. Then at q we have

$$\begin{aligned}(\nabla_{N_q}\mathbf{L})(X) &= \nabla_{N_q}(\mathbf{L}X) - \mathbf{L}(\nabla_{N_q}X)\\ &= \nabla_{N_q}(P^{-1}\mathbf{L}PX) - P^{-1}\mathbf{L}P(\nabla_{N_q}X) = 0. \end{aligned}\tag{D.1}$$

Indeed, let $P_s^t : T_{\gamma(s)}M \to T_{\gamma(t)}M$ denote parallel transportation along γ. Let $q = \gamma(t)$. Recall that given smooth vector fields Y and Z the covariant derivative of Y at a point $q \in M$ in the direction of $Z_q \in T_qM$ is given by

$$\nabla_{Z_q} Y = \lim_{h\to 0} \frac{P_{t+h}^t Y_{\gamma(t+h)} - Y_{\gamma(t)}}{h},$$

where $\gamma : (t-\varepsilon, t+\varepsilon) \to M$, $\gamma(t) = q$ is the integral curve of Z passing through q. Then using the fact that $P^{-1}\mathbf{L}P : TM \to TM$ is continuous we compute

$$\begin{aligned}
&\nabla_{N_q}(P^{-1}\mathbf{L}PX) - P^{-1}\mathbf{L}P(\nabla_{N_q}X)\\
=\ & \lim_{h\to 0}\frac{P_{t+h}^t(P^{-1}\mathbf{L}PX)_{\gamma(t+h)} - (P^{-1}\mathbf{L}PX)_{\gamma(t)}}{h}\\
&-P^{-1}\mathbf{L}P\lim_{h\to 0}\frac{P_{t+h}^t X_{\gamma(t+h)} - X_{\gamma(0)}}{h}\\
=\ & \lim_{h\to 0}\frac{P_{t+h}^t(P^{-1}\mathbf{L}PX)_{\gamma(t+h)} - (P^{-1}\mathbf{L}PX)_{\gamma(t)}}{h}\\
&-\lim_{h\to 0}\frac{P^{-1}\mathbf{L}PP_{t+h}^t X_{\gamma(t+h)} - P^{-1}\mathbf{L}PX_{\gamma(t)}}{h}\\
=\ & \lim_{h\to 0}\frac{P_{t+h}^t P^{-1}\mathbf{L}PX_{\gamma(t+h)} - P^{-1}\mathbf{L}PX_{\gamma(t)}}{h}\\
&-\lim_{h\to 0}\frac{P^{-1}\mathbf{L}PP_{t+h}^t X_{\gamma(t+h)} - P^{-1}\mathbf{L}PX_{\gamma(t)}}{h}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
P_{t+h}^t P^{-1}\mathbf{L}PX_{\gamma(t+h)} &= P_{t+h}^t P_0^{t+h}\mathbf{L}P_{t+h}^0 X_{\gamma(t+h)}\\
&= P_0^t\mathbf{L}P_{t+h}^0 X_{\gamma(t+h)}\\
&= P_0^t\mathbf{L}P_t^0 P_{t+h}^t X_{\gamma(t+h)}\\
&= P^{-1}\mathbf{L}PP_{t+h}^t X_{\gamma(t+h)},
\end{aligned}$$

and (D.1) follows. $\square$

Lemma D.2 (Auxiliary functions f_δ, F_δ, $\mathcal{F}_\delta$)**.** *There exist families of C^∞ functions $f_\delta, F_\delta, \mathcal{F}_\delta : [0,\infty) \to \mathbb{R}$, $\delta > 0$, with the following properties:*

- $\mathcal{F}_\delta' = F_\delta$ *and* $F_\delta' = f_\delta$ *on* $[0,\infty)$
- $f_\delta(0) = 1$, $0 \le f_\delta \le 1$ *on* $[0,\delta^2]$, *and* $|f_\delta| \le \delta$ *on* $[\delta^2,\infty)$
- $f_\delta' \le \delta$ *on* $[0,\infty)$
- $F_\delta(0) = \mathcal{F}_\delta(0) = 0$, $|F_\delta|, |\mathcal{F}_\delta| \le \delta$ *on* $[0,\infty)$, *and* $f_\delta = F_\delta = \mathcal{F}_\delta = 0$ *on* $[\delta,\infty)$.

Proof. We choose a smooth cutoff function $\varphi : [-1,1] \to \mathbb{R}$ such that

1. $\varphi = \begin{cases} -1 & \text{on } [-1,-1+\frac{1}{4}] \\ 0 & \text{on } [-\frac{1}{4},\frac{1}{4}] \\ 1 & \text{on } [1,1-\frac{1}{4}] \end{cases}$
2. $\varphi(-x) = -\varphi(x)$ for all $x \in [0,1]$ (that is, φ is an odd function)

(see Figure D.1 below).

Let $C := \|\varphi\|_{C^2([-1,1])}$. Given $0 < \alpha < \frac{1}{16C}$, we put

$$\begin{array}{rcl} \varphi_\alpha : [-\alpha,\alpha] & \to & \mathbb{R} \\ x & \mapsto & \alpha^3\varphi(\dfrac{x}{\alpha}). \end{array}$$

Then

$$\varphi_\alpha = \begin{cases} -\alpha^3 & \text{on } [-\alpha,-\alpha+\frac{\alpha}{4}] \\ 0 & \text{on } [-\frac{\alpha}{4},\frac{\alpha}{4}] \\ \alpha^3 & \text{on } [\alpha-\frac{\alpha}{4},\alpha] \end{cases}$$

and

$$|\varphi_\alpha'(x)| = \alpha^2|\varphi'(\frac{x}{\alpha})| \le \alpha^2 C$$

and

$$|\varphi_\alpha''(x)| = \alpha|\varphi''(\frac{x}{\alpha})| \le \alpha C$$

for all $x \in [-\alpha,\alpha]$.

Furthermore, we find a smooth cutoff function $\psi : [0,2] \to \mathbb{R}$ such that

1. $\psi(0) = 0$ and $\psi|_{[2-\frac{1}{4},2]} \equiv 1$
2. $\psi' \equiv 1$ on $[0,\frac{1}{4}]$, and $0 \le \psi' \le 1$ on $[0,2]$, and $\psi'' \le 0$ on $[0,2]$

(see Figure D.1 below), and put

$$\begin{array}{rcl} \psi_\alpha : [0,2\alpha^3] & \to & \mathbb{R} \\ x & \mapsto & \alpha^3\psi(\dfrac{x}{\alpha^3}). \end{array}$$

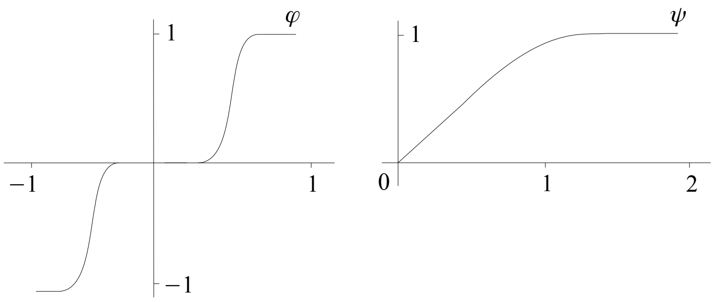

Figure D.1.: The functions φ and ψ

We construct the function $H_\alpha : [0,\infty) \to \mathbb{R}$ by 'pasting together' the functions ψ_α, $\alpha^3 \mathrm{id}|_{[0,\tau]}$, $-\varphi_\alpha$ and $\varphi_\alpha|_{[-\alpha,0]}$, (where we specify $\tau \in [\frac{\alpha}{4} - 2\alpha^3, \alpha]$ later in the proof). More precisely, we put

$$H_\alpha(x) = \begin{cases} \psi_\alpha(x) & \text{if } x \in [0, 2\alpha^3] \\ \alpha^3 & \text{if } x \in [2\alpha^3, 2\alpha^3 + \tau] \\ -\varphi_\alpha(x - 2\alpha^3 - \tau - \alpha) & \text{if } x \in [2\alpha^3 + \tau, 2\alpha^3 + \tau + 2\alpha] \\ \varphi_\alpha(x - 2\alpha^3 - \tau - 3\alpha) & \text{if } x \in [2\alpha^3 + \tau + 2\alpha, 2\alpha^3 + \tau + 3\alpha] \\ 0 & \text{if } x \in [2\alpha^3 + \tau + 3\alpha, \infty) \end{cases}$$

(see Figure D.2 below). Note that H_α vanishes on $[5\alpha, \infty)$, since $5\alpha \geq 2\alpha^3 + \tau + 3\alpha$. Moreover, by construction H_α is smooth and $|H_\alpha| \leq \alpha^3$ on $[0, \infty)$.

We now put $h_\alpha = H'_\alpha$. Clearly, h_α vanishes on $[5\alpha, \infty)$. Moreover, from $\psi'_\alpha(0) = 1$, $0 \leq \psi'_\alpha \leq 1$ and $|\varphi'_\alpha| \leq C\alpha^2$ it follows that

- $h_\alpha(0) = 1$,
- $0 \leq h_\alpha \leq 1$ on $[0, 2\alpha^3]$,
- $h_\alpha = 0$ on $[2\alpha^3, \frac{\alpha}{4}]$,
- $|h_\alpha| \leq C\alpha^2$ on $[\frac{\alpha}{4}, \infty)$,

where we used that $h_\alpha = 0$ on $[2\alpha^3, 2\alpha^3 + \tau]$ and $\tau \geq \frac{\alpha}{4} - 2\alpha^3$. Furthermore, from $\psi''_\alpha \leq 0$ and $|\varphi''_\alpha| \leq \alpha C$ it follows that

$$h'_\alpha \leq C\alpha$$

on $[0, \infty)$.

Next, define the function $\mathcal{H}_\alpha : [0, \infty) \to \mathbb{R}$ by

$$\mathcal{H}(x) = \int_0^x H_\alpha(t) dt.$$

Since $|H_\alpha| \leq \alpha^3$ and $H_\alpha|_{[5\alpha,\infty)} \equiv 0$, we have that $|\mathcal{H}_\alpha| \leq 5\alpha^4$ on $[0, \infty)$. Moreover, observe that

$$\begin{aligned}
&\int_0^{5\alpha} H_\alpha(t)dt \\
= &\int_0^{2\alpha^3} \psi_\alpha(t)dt + \tau\alpha^3 + \int_{-\alpha}^{\alpha} -\varphi_\alpha(t)dt + \int_{-\alpha}^{0} \varphi_\alpha(t)dt \\
= &\int_0^{2\alpha^3} \psi_\alpha(t)dt + \tau\alpha^3 + \int_{-\alpha}^{0} \varphi_\alpha(t)dt \\
= &\tau\alpha^3 - \underbrace{\Big[\int_0^{\alpha} \varphi_\alpha(t)dt - \int_0^{2\alpha^3} \psi_\alpha(t)dt\Big]}_{=:B}, \qquad \text{(D.2)}
\end{aligned}$$

where

$$0 < \frac{1}{4}\alpha^4 - 2\alpha^6 \leq B \leq \alpha^4$$

(here we used that $\varphi_\alpha \equiv \alpha^3$ on $[\frac{3}{4}\alpha, \alpha]$ and $\psi_\alpha \leq \alpha^3$). Thus, (D.2) vanishes if we put $\tau = \frac{B}{\alpha^3}$ (note that $\frac{\alpha}{4} - 2\alpha^3 \leq \tau \leq \alpha$, as required above), which in view of the definition of $\mathcal{H}_\alpha$ implies that $\mathcal{H}_\alpha(x) = 0$ if $x \geq 5\alpha$. Putting $f_\delta = h_{\frac{\delta}{5C}}$, $F_\delta = H_{\frac{\delta}{5C}}$, and $\mathcal{F}_\delta = \mathcal{H}_{\frac{\delta}{5C}}$, one easily checks that these functions have the desired properties.

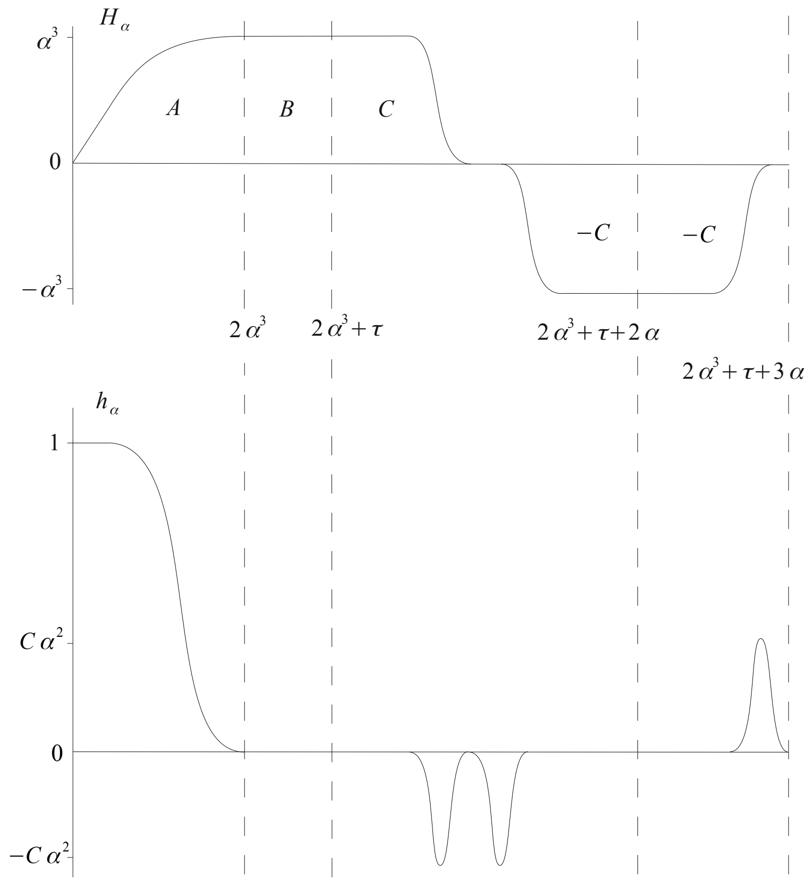

Figure D.2.: The functions H_α and h_α. The areas under the function H_α in the upper figure satisfy A+B=C, which illustrates that the integral of H_α vanishes.

□

Lemma D.3 (Auxiliary identities, cf. [15], Lemma 6.1, 6.2, 6.3)**.** *Let*

$$X, Y \in \{\partial_1, \ldots, \partial_{n-1}\} \subset T\Gamma(d) \subset TM_0$$

and

$$N = \partial_n \in (T\Gamma(d))^\perp \subset TM_0.$$

We have

$$\begin{aligned} \mathbf{G}_\delta &\approx \mathbf{I}, \quad \nabla_X \mathbf{G}_\delta \approx 0, \quad \nabla_N \mathbf{G}_\delta \approx 2f_\delta(x^n)\mathbf{L} \\ \nabla_X \nabla_N \mathbf{G}_\delta &\approx 2f_\delta(x^n)\nabla_X \mathbf{L} \\ \nabla_N \nabla_N \mathbf{G}_\delta &\approx 2f'_\delta(x^n)\mathbf{L} - 2Cf_\delta(x^n)\mathbf{P}^T \end{aligned} \tag{D.3}$$

$$\begin{aligned} \langle \nabla^\delta_X N, Y\rangle_\delta &= \langle \nabla^\delta_N X, Y\rangle_\delta \\ &= \frac{1}{2}\big(\langle \nabla_N X, \mathbf{G}_\delta Y\rangle + \langle X, \mathbf{G}_\delta \nabla_N Y\rangle + \langle X, (\nabla_N \mathbf{G}_\delta)Y\rangle\big) \end{aligned} \tag{D.4}$$

$$\nabla^\delta_N N = 0 \tag{D.5}$$

$$\nabla^\delta_N X = \nabla^\delta_X N \approx \nabla_X N + f_\delta(x^n)\mathbf{L}X \tag{D.6}$$

$$\mathbf{P}^T(\nabla^\delta_X Y) \approx \mathbf{P}^T(\nabla_X Y). \tag{D.7}$$

Proof. Recall that

$$\mathbf{G}_\delta = \mathbf{I} + 2F_\delta \mathbf{L} - 2C\mathcal{F}_\delta \mathbf{P}^T.$$

Proof of (D.3): We have $\Gamma = \{x^n = 0\}$, and $F_\delta(0) = \mathcal{F}_\delta(0) = 0$, and thus, $\mathbf{G}_\delta = \mathbf{I}$ on Γ. Moreover, $F_\delta, \mathcal{F}_\delta \to 0$ uniformly as $\delta \to 0$, which shows $\mathbf{G}_\delta \approx \mathbf{I}$.

Let us verify $\nabla_X \mathbf{G}_\delta \approx 0$: We have

$$\nabla_X \mathbf{G}_\delta = \nabla_X(\mathbf{I} + 2F_\delta \mathbf{L} - 2C\mathcal{F}_\delta \mathbf{P}^T) = \nabla_X \mathbf{I} + 2F_\delta \nabla_X \mathbf{L} - 2C\mathcal{F}_\delta \nabla_X \mathbf{P}^T$$

since F_δ and $\mathcal{F}_\delta$ depend only on x^n. For any $\xi, \zeta \in TM_0$ we then have

$$(\nabla_\zeta \mathbf{I})\xi = \nabla_\zeta(\mathbf{I}\xi) - \mathbf{I}(\nabla_\zeta \xi) = \nabla_\zeta \xi - \nabla_\zeta \xi = 0.$$

Moreover, $\nabla_X \mathbf{L}$ and $\nabla_X \mathbf{P}^T$ are locally bounded, so the result follows since $F_\delta, \mathcal{F}_\delta \to 0$ as $\delta \to 0$.

Let us show $\nabla_N \mathbf{G}_\delta \approx 2f_\delta \mathbf{L}$: Using the product rule we compute

$$\nabla_N \mathbf{G}_\delta = \nabla_N \mathbf{I} + 2f_\delta \mathbf{L} + 2F_\delta \nabla_N \mathbf{L} - 2CF_\delta \mathbf{P}^T - 2C\mathcal{F}_\delta \nabla_N \mathbf{P}^T$$

and the equation follows as above.

$\nabla_X \nabla_N \mathbf{G}_\delta \approx 2f_\delta \nabla_X \mathbf{L}$ follows by a similar argument.

Let us show $\nabla_N\nabla_N\mathbf{G}_\delta \approx 2f_\delta'(x^n)\mathbf{L} - 2Cf_\delta(x^n)\mathbf{P}^T$. Using $\nabla_N\mathbf{L} = 0$, similarly as above we compute

$$\nabla_N\nabla_N\mathbf{G}_\delta = 2f_\delta'\mathbf{L} - 2Cf_\delta\mathbf{P}^T - 2CF_\delta\nabla_N\mathbf{P}^T - 2CF_\delta\nabla_N\mathbf{P}^T - 2C\mathcal{F}_\delta\nabla_N\nabla_N\mathbf{P}^T$$

and the equation follows since $F_\delta(0) = \mathcal{F}_\delta(0) = 0$, and $F_\delta, \mathcal{F}_\delta \to 0$ uniformly as $\delta \to 0$.

Proof of (D.4): We recall the Koszul formula: Let (M,h) be a Riemannian manifold, $\langle\cdot,\cdot\rangle = h$, $\nabla = \nabla^h$, and $\xi,\zeta,\tau \in TM$. Then

$$\langle\nabla_\xi\zeta,\tau\rangle = \frac{1}{2}(\xi\langle\zeta,\tau\rangle - \tau\langle\xi,\zeta\rangle + \zeta\langle\tau,\xi\rangle - \langle\xi,[\zeta,\tau]\rangle + \langle\tau,[\xi,\zeta]\rangle + \langle\zeta,[\tau,\xi]\rangle).$$

Since $X,Y,N \in \{\partial_1,\dots,\partial_n\}$ commute pairwise, it follows from the Koszul formula, that

$$\begin{aligned}
\langle\nabla^\delta_X N, Y\rangle_\delta &= \langle\nabla^\delta_N X, Y\rangle_\delta \\
&= \frac{1}{2}(N\langle X,Y\rangle_\delta - Y\underbrace{\langle N,X\rangle_\delta}_{=0} + X\underbrace{\langle Y,N\rangle_\delta}_{=0}) \\
&= \frac{1}{2}N\langle X,\mathbf{G}_\delta Y\rangle \\
&= \frac{1}{2}(\langle\nabla_N X,\mathbf{G}_\delta Y\rangle + \langle X,(\nabla_N\mathbf{G}_\delta)Y\rangle + \langle X,\mathbf{G}_\delta(\nabla_N Y)\rangle).
\end{aligned}$$

To prove (D.5), and (D.7), recall that in our coordinates we have

$$g^\delta_{in} = g_{in} + 2f_\delta L_{in} - 2C\mathcal{F}_\delta(P^T)_{in} = \delta_{in}$$

for all $i = 1,\dots,n$. Using this, we compute (D.5):

$$\nabla^\delta_N N = \nabla^\delta_{\partial_n}\partial_n = {}^\delta\Gamma^k_{nn}\partial_k = \frac{1}{2}(g^\delta)^{kr}(\partial_n g^\delta_{nr} + \partial_n g^\delta_{nr} - \partial_r g^\delta_{nn})\partial_k = 0.$$

Let us show (D.7): In our coordinates, it suffices to verify that ${}^\delta\Gamma^k_{ij} \approx \Gamma^k_{ij}$ for all $1 \le i,j,k \le n-1$. We use $(g^\delta)^{kn} = 0 = g^{kn}$ for $1 \le k \le n-1$, and the fact that for $1 \le i,j,r \le n-1$ we have

$$\partial_i g^\delta_{jr} = \partial_i g_{jr} + 2F_\delta\partial_i L_{jr} - 2C\mathcal{F}_\delta\partial_i P^T_{jr} \approx \partial_i g_{jr}$$

and compute

$$\begin{aligned}
{}^\delta\Gamma^k_{ij} &= \frac{1}{2}(g^\delta)^{kr}(\partial_i g^\delta_{jr} + \partial_j g^\delta_{ir} - \partial_r g^\delta_{ij}) \\
&= \sum_{r\le n-1}\frac{1}{2}(g^\delta)^{kr}(\partial_i g^\delta_{jr} + \partial_j g^\delta_{ir} - \partial_r g^\delta_{ij}) \\
&\approx \sum_{r\le n-1}\frac{1}{2}g^{kr}(\partial_i g_{jr} + \partial_j g_{ir} - \partial_r g_{ij}) = \Gamma^k_{ij}.
\end{aligned}$$

Finally, (D.6) follows from (D.4) and $\mathbf{G}_\delta \approx \mathbf{I}$ and $\nabla_N\mathbf{G}_\delta \approx 2f_\delta\mathbf{L}$ (see (D.3)). □

Lemma D.4. *Let K be a closed convex (in the sense of Definition 3.2.3) subset of S^n with smooth $n-1$ dimensional boundary ∂K. Then*

(i) K is either a closed half sphere, the boundary of a half sphere, or it is contained in some open half sphere of S^n.

(*ii*) *The Euclidean cone over* K*, given by*

$$\mathcal{C}_K = \{tq \,|\, q \in K, t \geq 0\},$$

is a convex subset of $\mathbb{R}^{n+1}$.

Proof. (*i*): First, let us show that K is contained in a closed half sphere of S^n. Here we denote by $B_r(p)$ a ball (w.r.t. the standard metric of S^n) of radius $r > 0$ in S^n centered at $p \in S^n$. The assumption that K is closed and has nonempty boundary in S^n implies that $S^n \setminus K$ is an nonempty open subset of S^n. Then we find a maximal ball $B_r(p) \subset S^n \setminus K$ of radius $r > 0$ (maximal in the sense that $B_{r+\varepsilon}(q) \cap \partial K \neq \emptyset$ for all $\varepsilon > 0$ and $q \in S^n \setminus K$). Then $B_r(p) \cap \partial K$ contains at least two distinct points $q_1 \neq q_2$. Suppose that $r < \pi$. In this case we have that $\mathrm{dist}_{S^n}(q_1, q_2) < \pi$, and the shortest geodesic in S^n from q_1 to q_2 lies in $B_r(p)$ up to the end points q_1 and q_2. In particular, this geodesic contains points of $S^n \setminus K$, which a contradicts the assumption that K is convex. So the maximal radius r must be at least π. This implies that K is contained in $S^n \setminus B_\pi(p)$, which is a closed half sphere centered at $-p$.

Next, we verify that if K contains opposite points (that is, $p_1, p_2 \in K$, $p_1 = -p_2$), then K must be either a closed half sphere or the boundary of a half sphere. Let $\bar{HS^n}$ be a closed half sphere containing K. Suppose there exist opposite points $p_1, p_2 \in K$. Since $K \subset \bar{HS^n}$, it follows that $p_1, p_2 \in \partial(\bar{HS^n})$, and consequently $p_1, p_2 \in \partial K$. Let $\gamma : [0, \pi] \to \partial(\bar{HS^n})$ be a unit speed geodesic from p_1 to p_2. Since the boundary $\partial K \subset \bar{HS^n}$ is smooth at p_1, such a geodesic γ can be viewed as a limit of shortest geodesics γ_i from $p_i \in \partial K$ to p_2, where $p_i \neq p_1$ for all i. Due to $\mathrm{dist}_{S^n}(p_i, p_2) < \pi$ and the fact that K is convex we have that the geodesics γ_i lie entirely in K, which implies that the limit γ is also contained in K, since K is closed by assumption.

Since $\partial(\bar{HS^n})$ is the union of all such geodesic γ, it follows that $\partial(\bar{HS^n}) \subset K$. On the other hand, the only closed convex subsets of S^n for which this is possible are either closed half spheres or boundaries of half spheres.

Finally, if K does not contain opposite points, then $K \subset \bar{HS^n}$ and $\partial K \cap \partial(\bar{HS^n})$ is either empty, in which case K is contained in the open half sphere HS^n, or it containes at most one single point q, in which case we obtain an open half sphere containing K by slightly rotating HS^n in the direction of the outward normal of $\partial(\bar{HS^n})$ at q.

(*ii*): In the case when K a closed half sphere or the boundary of a half sphere, $\mathcal{C}_K$ is a half space or a hyperplane of $\mathbb{R}^{n+1}$, respectively, and the statement is trivial. Consider the case where K is contained in some open half sphere of S^n, w.l.o.g.

$$K \subset HS^n = \{x = (x^1, \ldots, x^{n+1}) \in S^n \,|\, x^{n+1} > 0\}.$$

The stereographic projection $\pi : HS^n \to \mathbb{R}^n \times \{1\}$, $\pi(x) = \frac{1}{x^{n+1}}x$ is a homeomorphism taking shortest curves in HS^n to straight line segments in $\mathbb{R}^n \times \{1\}$. Thus, π identifies convex subsets of HS^n with convex subsets of $\mathbb{R}^n \times \{1\}$. It then follows that $\mathcal{C}_K = \mathcal{C}_{\pi(K)}$ coincides with the Euclidean cone over a convex subset of $\mathbb{R}^n \times \{1\}$, which is clearly convex. $\square$

The following lemma is a well known result which also holds in a more general setting. We give the prove here for the convenience of the reader:

Lemma D.5 (Minkowski convex functional)**.** *Let A be a closed convex subset of $\mathbb{R}^n$ with non-empty such that $0 \in \mathring{A}$. Then the Minkowski functional*

$$\begin{aligned} F : \mathbb{R}^n &\to [0, \infty) \\ F(x) &= \inf\{\lambda > 0 \,|\, x \in \lambda A\} \end{aligned}$$

satisfies

(i) $F(x+y) \leq F(x) + F(y)$ *for all* $x, y \in \mathbb{R}^n$

(ii) $F(tx) = tF(x)$ *for all* $x \in \mathbb{R}^n$, $t \geq 0$

(iii) F *is convex*

(iv) $A = F^{-1}([0,1])$.

Proof. Note that the condition $0 \in \mathring{A}$ ensures that F is finite. Indeed, we may choose a $\delta > 0$ such that $B_\delta(0) \subset A$. Then for any $x \in \mathbb{R}^n$ we find an $\varepsilon > 0$ such that $\varepsilon x \in B_\delta(0) \subset A$, which implies that $x \in \frac{1}{\varepsilon} A$, so that $F(x) \leq \frac{1}{\varepsilon} < \infty$.

(i): Let $x, y \in \mathbb{R}^n$ and $\varepsilon > 0$. In view of the definition of F we find $\lambda, \mu > 0$ such that $\lambda \leq F(x) + \varepsilon$ and $\mu \leq F(y) + \varepsilon$, and $x \in \lambda A$ and $y \in \mu A$. Thus

$$x + y \in \lambda A + \mu A = (\lambda + \mu) A,$$

so that

$$F(x+y) \leq \lambda + \mu \leq F(x) + F(y) + 2\varepsilon,$$

and the desired inequality follows by letting $\varepsilon \to 0$.

(ii): Given $x \in \mathbb{R}^n$ and $\varepsilon > 0$, we find $\lambda > 0$ such that $\lambda \leq F(x) + \varepsilon$ and $x \in \lambda A$. Then for any fixed $t \geq 0$ we have $tx \in t\lambda A$, so that

$$F(tx) \leq t\lambda \leq tF(x) + t\varepsilon.$$

Letting $\varepsilon \to 0$ we obtain the desired result.

(iii): Let $x, y \in \mathbb{R}^n$ and $t \in [0,1]$. Then by (i) and (ii) we have

$$F((1-t)x + ty) \leq F((1-t)x) + F(ty) = (1-t)F(x) + tF(y).$$

(iv): Suppose that $x \in A$. Then $x \in \lambda A$, $\lambda = 1$, so that $F(x) \leq 1$, which shows $A \subset F^{-1}([0,1])$.

Conversely, suppose that $F(x) = \lambda \in [0,1]$. If $F(x) < 1$, we find a $0 < \lambda < 1$ such that $x \in \lambda A$, that is, $\frac{1}{\lambda} x \in A$. Since $0 \in A$ and A is convex, this implies $x = (1-\lambda) \cdot 0 + \lambda \frac{1}{\lambda} x \in A$. If $F(x) = 1$, then $x \in (1+\varepsilon_n)A$ for all $n \in \mathbb{N}$, where $\varepsilon_n \searrow 1$. Then $\frac{1}{1+\varepsilon_n} x$ is a sequence in A, and $x = \lim_n \frac{1}{1+\varepsilon_n} x \in A$ since A is closed by assumption.

□

Lemma D.6. *Let $\mathcal{C}_K$, $K \subset HS^{n-1}$, $\pi_{H_\delta} : HS^{n-1} \to H_\delta$, I_δ and F be as in the proof of Lemma 3.2.4. Let $x \in \mathbb{R}^n \setminus \{0\}$. Then*

$$x \in \mathcal{C}_K \Leftrightarrow x^n > 0 \ \textit{ and } \ \frac{\delta}{x^n} x \in I_\delta.$$

Proof. Let $x = tq \in \mathcal{C}_K \setminus 0$, $q \in K$, $t > 0$ since $x \neq 0$. Then $x^n = tq^n > 0$ since $q \in HS^{n-1}$. Moreover, $|x| = t|q| = t$, so $K \ni q = \frac{x}{t} = \frac{x}{|x|}$. Since $\pi_{H_\delta}(K) = I_\delta$, we then have

$$I_\delta \ni \pi_{H_\delta}(\frac{x}{|x|}) = \frac{\delta}{x^n/|x|}\frac{x}{|x|} = \frac{\delta}{x^n}x.$$

Conversely, let $x \in \mathbb{R}^n$ such that $x^n \neq 0$ and $\frac{\delta}{x^n}x \in I_\delta$. Then

$$K \ni \pi_{H_\delta}^{-1}(\frac{\delta}{x^n}x) = \frac{1}{|\frac{\delta}{x^n}x|}\frac{\delta}{x^n}x = \frac{x}{|x|}.$$

Thus

$$x = |x|\frac{x}{|x|} \in \mathcal{C}_K$$

by definition of $\mathcal{C}_K$. □

Lemma D.7. *Let (M, g) be a smooth Riemannian manifold, and $f : \Omega \subset M \to \mathbb{R}$ a smooth function satisfying $|\nabla f|_g \equiv const$ (where $\nabla = \nabla^g$). Then integral curves of ∇f are geodesics.*

Proof. Let $\gamma : (-\varepsilon, \varepsilon) \to \Omega$ be an integral curve of ∇f, that is, $\dot\gamma(t) = \nabla f(\gamma(t))$. Given $t \in (-\varepsilon, \varepsilon)$ and $X \in T_{\gamma(t)}M$ we compute at $\gamma(t)$

$$\begin{aligned}
0 &= \frac{1}{2}X(g(\nabla f, \nabla f)) = g(\nabla_X \nabla f, \nabla f) = (\nabla_X \nabla f)(f) \\
&= (\nabla_{\nabla f} X)(f) + [X, \nabla f](f) \\
&= (\nabla_{\nabla f} X)(f) + \underbrace{X(\nabla f(f))}_{\equiv 0} - \nabla f(X(f)) \\
&= g(\nabla_{\nabla f} X, \nabla f) - \nabla f(g(\nabla f, X)) \\
&= -g(\nabla_{\nabla f}\nabla f, X) = -g(\nabla_{\dot\gamma}\dot\gamma, X),
\end{aligned}$$

where we used that $\nabla f(f) = g(\nabla f, \nabla f) \equiv const$.

□

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