Analysis, approximation, and stability of electromagnetic surface waves

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Contents

1 Introduction **7**
- 1.1 Some notation and preliminaries 10
- 1.2 Maxwell equations in linear and nonlinear media 14
- 1.3 Linear, evanescent surface modes and dispersion relation 17
- 1.4 The framework of evolutionary equations 20

2 Nonlinear evolutionary equations **27**
- 2.1 Lipschitz and locally Lipschitz nonlinearities 27
- 2.2 Small solutions of a cubic Ginzburg–Landau equation 30
- 2.3 Multilinear Volterra operators: local and global well-posedness 33
- 2.4 Initial values for problems with memory 36
- 2.5 Comments and open problems 39

3 Well-posedness and exponential stability for Maxwell systems **41**
- 3.1 Maxwell operator, interface and boundary conditions 41
- 3.2 Well-posedness of nonlinear evolutionary Maxwell systems 46
- 3.3 Exponential stability 49
  - 3.3.1 Exponential stability in the non-magnetic case 50
  - 3.3.2 On the closedness of the range of the curl operator 59
  - 3.3.3 Nonlinear perturbations 61
- 3.4 Comments and open problems 63

4 Approximation of broad surface wavepackets in nonlinear magnetooptics **67**
- 4.1 Ansatz, residual, and amplitude equation 70
  - 4.1.1 Setup and basic ansatz 70
  - 4.1.2 Sum-frequency generation and correction 76
- 4.2 Norm estimates for the residual 84
- 4.3 Small solutions of the error equation 85
- 4.4 Comments 88

5 The Drude–Lorentz model of electric permittivity **91**
- 5.1 Accretivity 92
- 5.2 Dispersion relation 94
- 5.3 Comments 95

6 Discussion **97**

1 Introduction

Electromagnetic fields are ubiquitous in nature, many sciences, and engineering, as they are the forces underlying the principles of electricity, magnetism, and optics. These fields are in turn described by a set of four partial differential equations, named "Maxwell equations" or "Maxwell system" after James Clerk Maxwell; it was the effort of Maxwell in the late nineteenth century (for example, [Max65]) that was significant in unifying the latter theories, namely electricity, magnetism, and optics, into one theory of electromagnetism and that ultimately led to the current formulation ((1.2.1), (1.2.2)) of these equations.

From the point of view of engineering, the ability of these forces to propagate in vacuum and in various media makes the transmission and processing of information possible and has brought forth a large number of technological advancements, creating and shaping the 'digital age' we currently live in.

From a mathematician's perspective, Maxwell's equations continue to offer interesting and challenging opportunities for study, varying both in scope as well as in method. Relevant questions include the local existence and numerical approximation of solutions, long-time stability, and ill-posed problems, see [DHK+23].

Electromagnetic surface waves are a field in physics that is more than a century old, [Zen07]. Today, the most prominent instance of such waves is perhaps the surface plasmon polariton (SPP), which is an evanescent wave propagating along the interface of a metal (the 'plasmonic material') and a dielectric. It has seen numerous applications, typically on the nanoscale, ranging from biochemical sensing [CK21], microscopy [RK88], to laser technology [AKM+20]; see also [ZSM05], [JML13, Chapter 6]. While the SPP is not the only type of surface wave (we refer to Table 1.1 in [JML13] for an overview), it is arguably the simplest to derive explicitly, at least in the case of a planar interface and linear scalar permittivity, see [Rae88]. Nonlinear SPP have been considered in [KZ12].

In this work, we are interested in the study of electromagnetic waves, especially wavepackets, localized to and propagating along the interface between two different optical or magnetic media. Specifically, the primary aim of this thesis is the justification of a modulation equation for nonlinear Maxwell systems with memory. This is done by showing that the approximation of a solution of Maxwell's equations with a suitable model function (or "ansatz"), determined by a solution of the amplitude equation, is valid on a large time scale. Among the distinctive features of the Maxwell equations governing this setup are the following:

(a) discontinuities of the fields due to distinct media,

(b) materials with memory, i.e., a nonlocal behavior in time due to a delayed material response (*temporal dispersion*), as well as a nonlocal behavior in space (*spatial dispersion*),

(c) the presence of nonlinear, especially quadratic, terms.

Introducing any of these features into a partial differential equation usually complicates matters and requires different approaches (for instance, nonlinear equations cannot, in general, be dealt with using linear methods, such as the Fourier transform).

Before we lay out our strategy and the structure of this thesis, let us collect a sample of related works.

A solution theory (or well-posedness, i.e., existence, uniqueness, and continuous dependence of the solution on the given data) is the prerequisite for any analysis of differential equations. For Maxwell equations, such theories fall into various categories, that may also overlap. Among those, an analysis in 'low regularity' spaces (i.e., based on L^2 and first-order Sobolev spaces) is well-suited for linear, anisotropic equations (in particular, with or without an interface) and for numerical problems. In this respect, general functional analytic aspects and a 'classical' treatment of linear Maxwell equations (without memory) can be found in the monographs [DL90a, DL90b, Lei86, Mon03].

Many results deal with linear time-harmonic Maxwell equations, e.g. [PWW01]. In [CHJ17, CHJ22], time-harmonic linear Maxwell equations with memory are considered, with a focus on interface problems for plasmonic waves and metamaterials; see also [CJK17] and the references therein. A general approach to linear Maxwell equations for materials with (non-)continuous memory, based on an abstract theory of integro-differential equations, is shown in Sections 9.6 and 13.3 of [Prü12].

For time-harmonic nonlinear problems, variational techniques can yield the existence of specific solutions (like ground states), see for instance [BM17, BDPR16], sometimes requiring a compactness argument (see Section 3.3.2).

Function spaces with higher regularity are used especially for nonlinear problems. In [SS22], a local well-posedness theory in H^m ($m \geq 3$) for quasilinear Maxwell systems with interface and without memory was developed. [BF03] contains a solution theory for continuous nonlinear materials with memory; see also [BS22] for a recent approach to quasilinear Maxwell equations with memory. The analysis in [SS22] displays some of the difficulties when dealing with interface problems in the context of higher spatial regularity, see also [Web81, DITW23].

Asymptotic methods for electromagnetic nonlinear waves, in particular for proving the validity of modulation equations, have been employed in [Sch00, SU03] for the analysis of modulating pulses in optical fiber, in [DR21b] concerning the existence of solitons in photonics crystals, and in [DST22, DR21a] for surface waves; see also [HH16]. We remark that in many instances, the leading-order nonlinearity is of cubic Kerr-type. Quadratic nonlinearities can lead to failure of the validity, e.g. [Sch05], and the analysis usually involves a set of non-resonance conditions, cf. [Kal88, vH91, DHSZ16].

Similar to the results in [SU03, DST22, DR21b], we want to derive an amplitude equation for interface wave packets in nonlinear media with memory and prove its validity. This specific problem is motivated by two factors: the usual dependence of plasmons on the frequency, i.e., the material response is in general dispersive (as noted in [KZ12]), and quadratic effects created at least by the interface setup (see the discussion in Section 4.4). We will address the difficulties (a), (b), (c) above by

(a′) working in function spaces of low regularity, (b′) use of the framework of evolutionary equations, which explicitly permits nonlocal operators modelling materials with memory and spatial dispersion, for the linearized problem,

(c′) use of perturbative approaches to tackle the nonlinear problem.

These strategies will prove fruitful in answering questions regarding well-posedness, stability, and specific approximations related to Maxwell's equations.

This document is structured as follows.

Section 1.3 contains a first analysis of linear Maxwell systems under the simplifying assumptions of a planar interface and scalar material laws. As a result we obtain the linear dispersion relation and a family of transverse-magnetic or transverse-electric evanescent waves, which serve as building blocks for the wavepacket ansatz in Chapter 4. Although inspired by linear SPP, this derivation is more general, as we do not necessarily assume a metal-dielectric setup, and moreover, a possibly nontrivial magnetization is included.

The remaining part of this introductory chapter is devoted to a short review of the framework of (linear) evolutionary equations (in the sense of Picard, e.g. [Pic00, Pic09, STW22]) and their solution theory (Theorem 1.4.11) in exponentially weighted function spaces that is used throughout this paper.

In Chapter 2 we consider abstract nonlinear evolutionary equations as perturbations of the linear case. Our focus lies on (local) Lipschitz nonlinearities and Volterra-type nonlinear operators, for which we derive local or global (depending on the weight) well-posedness of the associated equation. This includes an equation of Ginzburg–Landau type that appears as an effective amplitude equation in Chapter 4. In addition, we take a look at the connection between evolutionary equations and Cauchy problems with memory, see Section 2.4; Here we observe that, in many cases, the transition from the Cauchy problem to an evolutionary formulation (and back) is seamless.

The first parts of Chapter 3 deal specifically with the well-posedness of nonlinear Maxwell systems with memory, featuring nonlinear operators used in Chapter 2. In Section 3.3 we study exponential stability for linear Maxwell systems without dispersive magnetization. We focus on two cases: materials with explicit electric conductivity, which in turn provides the exponential damping, and those sharing some characteristics with the Lorentz permittivity model. The transition to nonlinear systems, Section 3.3.3, is made possible through a fixed-point argument, obtaining global existence of small solutions. Large parts of this section are based on the paper [DITW23].

The subject of Chapter 4 is an amplitude approximation of wavepacket solutions to nonlinear Maxwell systems on the whole domain $\mathbb{R}^3$. We construct a multiple-scale ansatz U_ε ($0 < \varepsilon \ll 1$) based on the linear modes obtained in Section 1.3, which is effectively described by its amplitude, a solution of the Ginzburg–Landau equation mentioned above. The ansatz is constructed in such a way that, after inserting U_ε into the Maxwell system, the remaining terms (the *residual*) are asymptotically small for $\varepsilon \to 0$.

Following this formal analysis, more rigorous estimates are able to justify this approximation; this means that the ansatz U_ε remains close to an exact solution U of the Maxwell system. This is done by showing that the equation for the error $R = U - U_\varepsilon$ is well-posed and admits a small (and, in fact, exponentially decaying) solution (Theorem 4.3.3).

Finally, Chapter 5 contains some discussion on the (Drude–)Lorentz model of electric permittivity. Specifically, we check the compatibility of the model with the spectral conditions established in Section 3.3, as well as with the assumptions on the dispersion curves needed for the amplitude formalism in Chapter 4.

1.1 Some notation and preliminaries

Most objects used in this work are defined in-place. For convenience, we list some frequent pieces of notation.

Throughout, $\mathbb{R}^n$ and $\mathbb{C}^n$ ($n \in \mathbb{N}$) denote the real, respectively complex, n-dimensional Euclidean space. In $\mathbb{C}^n$ we have the inner product $(x, y) \mapsto x \cdot \overline{y} = \sum_{j=1}^n x_j \overline{y_j}$, and the induced norm $|x| = \sqrt{x \cdot \overline{x}}$. In $\mathbb{R}^n$, as a subset of $\mathbb{C}^n$, the inner product is inherited and simplifies to $(x, y) \mapsto x \cdot y$. More generally, we write $\langle u, v\rangle_X$ for the inner product of $u, v \in X$, and $\|u\|_X$ for the norm of u, if X is an inner product space or a normed space, respectively. The imaginary unit is denoted by $i = \sqrt{-1}$. We define the positive and negative half-lines $\mathbb{R}^+ = \{t \in \mathbb{R} : t > 0\}$, $\mathbb{R}^- = \{t \in \mathbb{R} : t < 0\}$, and the right half-plane $\mathbb{C}_{\mathrm{Re}>\varrho} = \{z \in \mathbb{C} : \operatorname{Re} z > \varrho\}$.

For $x \in \mathbb{C}^n$, $\delta > 0$ we denote by $B(x, \delta) = \{y \in \mathbb{C}^n : |x - y| < \delta\}$ the open ball with radius δ around x, and by $B[x, \delta] = \overline{B(0, \delta)} = \{y \in \mathbb{C}^n : |x - y| \leq \delta\}$ its closure.

A *domain* is a nonempty open and connected subset $\Omega \subseteq \mathbb{R}^n$. Various attributes of a domain, such as 'smooth' or 'Lipschitz', refer to its boundary. In particular, a bounded Lipschitz domain is an open subset $\Omega \subseteq \mathbb{R}^n$, for which its boundary is locally the graph of a Lipschitz-continuous function.

Function spaces

Unless otherwise specified, all functions are complex-valued. If the function f is real-valued (for instance, a physically relevant solution of Maxwell's equations) it is customary to write $f = u + \text{c.c.}$, where $u = \frac{1}{2} \operatorname{Re} f$ and c.c. stands for the complex conjugate, $f = u + \overline{u}$.

For a domain Ω we denote by $L^p(\Omega)^n = L^p(\Omega, \mathbb{C}^n)$ the usual Lebesgue space of measurable functions $u\colon \Omega \to \mathbb{C}^n$, for which the norm

$$\|u\|_{L^p} := \begin{cases} \left(\int_{\mathbb{R}} |u(x)|^p \,\mathrm{d}x\right)^{1/p}, & \text{if } p \in [1, \infty) \\ \operatorname{ess\,sup}_{x\in\Omega} |u(x)|, & \text{if } p = \infty \end{cases}$$

is finite, and where functions equal almost everywhere (a.e.) are identified. $L^p(\Omega)^n$ is a Banach space, and a Hilbert space for $p = 2$ with inner product given by

$$\langle u, v\rangle_{L^2} = \int_\Omega u(x)\overline{v(x)} \,\mathrm{d}x.$$

We mostly deal with $p = 1$, $p = 2$ and $p = \infty$.

For functions $u\colon I \subseteq \mathbb{R} \to \mathbb{C}^n$ on the line defined on an open interval $I \subseteq \mathbb{R}$, we denote by $\partial_t u$ the weak, or more generally, the distributional derivative. In three (spatial) variables, for a domain $\Omega \subseteq \mathbb{R}^3$ and smooth functions $u\colon \Omega \to \mathbb{C}$, $v\colon \Omega \to \mathbb{C}^3$,

$$\operatorname{grad} u = \nabla u, \quad \operatorname{curl} v = \nabla \times v, \quad \operatorname{div} v = \nabla \cdot v$$

denote the gradient, curl, and divergence, respectively. Weak, closed versions of these operators are introduced in Chapter 3.

The notation $H^1(\Omega) = H(\mathrm{grad}, \Omega) = \{u \in L^2(\Omega) : \mathrm{grad}\, u \in L^2(\Omega)^3\}$, and similarly $H(\mathrm{div}, \Omega)$, $H(\mathrm{curl}, \Omega)$, for L^2-based Sobolev spaces is common. In general, the Sobolev space $H^k(\Omega)$ $(= W^{k,2}(\Omega),\ k \in \mathbb{N})$ is defined as the set of functions $u \in L^2(\Omega)$ for which all partial derivatives $\partial^\alpha u$ (with $\alpha \in \mathbb{N}^d$ a multiindex with $|\alpha| \leq k$) belong to $L^2(\Omega)$. For standard properties of Lebesgue and Sobolev spaces (such as density results, embeddings, and trace theorems) we refer to [AF03].

Fourier and Laplace transforms

Throughout this work, we make use of several equivalent, but subtly different versions of the Fourier transform (see also [AF03, Kat04, Eva10]). While it is possible to unify them all into a single transform, we chose to use different notation and restrict each variant to specific use cases. This decision comes at a cost, namely when two transforms need to be compared that are defined on the same domain, notably in Chapter 5. We introduce these transforms in the following.

In optics, a time-delayed linear response $R(u)$ to an applied field u is often modeled using a real-valued susceptibility function χ (e.g. $\chi \in L^1(\mathbb{R})$) as

$$R(u) = \int_{\mathbb{R}} \chi(\tau) u(\,\cdot\, - \tau)\, \mathrm{d}\tau.$$

We introduce

$$\check{\chi}(\omega) = \int_{\mathbb{R}} \chi(\tau) e^{i\omega\tau}\, \mathrm{d}\tau \tag{1.1.1}$$

where $\omega \in \mathbb{R}$, and consider a 1-dimensional, time-harmonic, monochromatic wave $u(t,x) = Ae^{i(kx-\omega t)}$ travelling in positive x-direction. The response to such a field is then given by

$$R(u)(t) = \int_{\mathbb{R}} \chi(\tau)\, Ae^{i(kx-\omega(t-\tau))}\, \mathrm{d}\tau = \int_{\mathbb{R}} \chi(\tau) e^{i\omega\tau}\, \mathrm{d}\tau\, u(t,x) = \check{\chi}(\omega) u(t,x).$$

The transform (1.1.1) is mainly used in Section 1.3 and Chapter 4. In the latter, we also make use of a related Fourier transform in space, denoted by

$$\mathcal{F}_{x\to k} u(k) = \hat{u}(k) = \int_{\mathbb{R}} u(x) e^{-ikx}\, \mathrm{d}x. \tag{1.1.2}$$

In Section 1.4 we consider a weighted version, called the (unitary) Fourier–Laplace transform,

$$\mathcal{L}_\varrho u(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(t) e^{-(\varrho + i\xi)t}\, \mathrm{d}t, \tag{1.1.3}$$

which is used mainly for the transition of operator-valued functions between the time and complex frequency domain.

There should be no confusion between $\check{\chi}$ and $\mathcal{L}_\varrho \chi(\xi)$, as different notation for the functions and variables are used. However, when we extend (1.1.1) to the complex domain, the two are evidently related by

$$\check{\chi}(iz) = \sqrt{2\pi}\mathcal{L}_\varrho \chi(\xi), \quad z = \varrho + i\xi.$$

All three transforms, (1.1.1), (1.1.2), and (1.1.3), extend naturally to (weighted) L^2-spaces.

Functions with values in a Banach space

Bochner spaces are a generalization of the Lebesgue spaces above. The following definitions and results can be found in [Eva10, Yos80]; see also [Boc33]. Let $I \subseteq \mathbb{R}$ be an interval and X a Banach space. A *simple function* $s\colon I \to X$ is one that can be written in the form

$$s(t) = \sum_{j=1}^{n} \mathbf{1}_{E_j}(t)\, v_j, \quad \text{a.e. } t \in I,$$

where $E_j \subseteq I$ is of finite Lebesgue measure $|E_j|$ and $v_j \in X$ for each $j \in \{1, \dots, n\}$. Its integral is given by the sum

$$\int_I \sum_{j=1}^{n} \mathbf{1}_{E_j}(t)\, v_j \,\mathrm{d}t = \sum_{j=1}^{n} |E_j| v_j \in X.$$

A function $u\colon I \to X$ is said to be (Bochner) measurable, if it can be approximated by simple functions, i.e., if there exists a sequence $(s_n)_{n\in\mathbb{N}}$ of simple functions such that

$$\|s_n(t) - u(t)\|_X \to 0 \quad (n \to \infty), \quad \text{a.e. } t \in I.$$

Such function u is called Bochner integrable, write $u \in L^1(I, X)$, if in addition

$$\int_I \|s_n(t) - u(t)\|_X \,\mathrm{d}t \to 0 \quad (n \to \infty).$$

In this case, $\int_I u(t)\,\mathrm{d}t := \lim_{n\to\infty} \int_I s_n(t)\,\mathrm{d}t \in X$ is well-defined. By a theorem of Bochner (see [Yos80, Theorem V.5.1]), $u \in L^1(I, X)$ if and only if the map $t \mapsto \|u(t)\|_X$ belongs to $L^1(I)$. Analogous to the finite-dimensional case above, we define for $p \in [1, \infty)$

$$L^p(I, X) = \{u\colon I \to X \text{ measurable} : \|u\|_{L^p} = \Big(\int_I \|u(t)\|_X^p \,\mathrm{d}t\Big)^{1/p} < \infty\}$$

(and the analogous modification for $p = \infty$), and denote by $L^p_{\mathrm{loc}}(\mathbb{R}, X)$ the set of all measurable $u\colon \mathbb{R} \to X$ such that $u|_I \in L^p(I, X)$ for all relatively compact subsets $I \subset\subset \mathbb{R}$.

Bounded and unbounded linear operators in Hilbert spaces

If X is a Banach space, $\mathcal{B}(X)$ denotes the space of bounded (equivalently: continuous) linear operators from X to itself. $\mathcal{B}(X)$ is again a Banach space, indeed, a Banach algebra. The norm in $\mathcal{B}(X)$ will occasionally be denoted by $\|\cdot\|_{\mathcal{B}(X)}$ or $\|\cdot\|_{X\to X}$, or for example by $\|\cdot\|_{L^2\to L^2}$ if $X = L^2(\Omega)^3$, or simply by $\|\cdot\|$ if clear from the context.

If $\mathcal{H}_1, \mathcal{H}_2$ are Hilbert spaces, we write $T\colon \operatorname{dom}(T) \subseteq \mathcal{H}_1 \to \mathcal{H}_2$ to denote a linear operator, not necessarily bounded, defined on a (dense) subspace $\operatorname{dom}(T)$. We collect some results regarding unbounded operators on a Hilbert space, see [Kat80, §3.V.10], [Yos80, Chapters VII, VIII], [Bre11, §2.6].

A densely defined operator $T\colon \operatorname{dom}(T) \subseteq \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$ is called *accretive*, if $\operatorname{Re}\langle Tv, v\rangle_{\mathcal{H}} \geq 0$ for all $v \in \operatorname{dom}(T)$ (equivalently, the operator $-T$ is *dissipative*). Since $\operatorname{dom}(T)$ is dense in $\mathcal{H}$, the adjoint T^* is well-defined and closed with $\operatorname{dom}(T) \subseteq \operatorname{dom}(T^*)$.

With the help of the symmetric operator

$$\operatorname{Re} T := \frac{1}{2}(T + T^*), \quad \operatorname{dom}(\operatorname{Re} T) = \operatorname{dom}(T),$$

the condition of accretivity can be formulated as

$$\operatorname{Re} T \geq 0$$

in the sense of positive definiteness. T is called *strictly* accretive, if $T - c$ is accretive for some positive number $c > 0$, i.e.,

$$\operatorname{Re} \langle Tv, v \rangle_{\mathcal{H}} \geq c \, \|v\|_{\mathcal{H}}^2 \quad \text{for all } v \in \operatorname{dom}(T);$$

or in short, $\operatorname{Re} T \geq c$. Strictly accretive operators are useful due to their invertibility, as the following result reveals.

Lemma 1.1.1. *Suppose* $T\colon \operatorname{dom}(T) \subseteq \mathcal{H} \to \mathcal{H}$ *is a closed, densely defined operator with dense range, which satisfies*

$$\operatorname{Re} T \geq c > 0.$$

Then T is boundedly invertible with $\|T^{-1}\| \leq 1/c$.

Proof. Using the strict accretivity and the Cauchy–Schwarz inequality,

$$\|Tv\| \|v\| \geq \operatorname{Re} \langle Tv, v \rangle \geq c\|v\|,$$

which shows that T is injective, hence can be inverted on its range, and substituting $w = Av$ we have $\|T^{-1}w\| \leq c^{-1}\|w\|$ for all $w \in \operatorname{ran}(T)$. Moreover, since T is closed and $\operatorname{ran}(T)$ is dense, in fact $\operatorname{ran}(T) = \overline{\operatorname{ran}(T)} = \mathcal{H}$, so $T^{-1} \in \mathcal{B}(\mathcal{H})$. □

There are multiple variants of this result, some using the closed range theorem. For a densely defined and closed operator T, recall that $\ker(T) = \operatorname{ran}(T^*)^\perp$ and $\ker(T^*) = \operatorname{ran}(T)^\perp$ are closed subspaces (where $X^\perp$ denotes the orthogonal complement of the subspace X). The closed range theorem ([Yos80, VII.5]) states that $\operatorname{ran}(T)$ is closed if and only if $\operatorname{ran}(T^*)$ is closed, in which case $\operatorname{ran}(T) = \ker(T^*)^\perp$ and $\operatorname{ran}(T^*) = \ker(T)^\perp$.

Lemma 1.1.2. *Let T be densely defined, closed, and such that*

(i) T and T^* are strictly accretive, or

(ii) T is bounded and strictly accretive, or

(iii) T is bounded, selfadjoint and strictly positive definite.

Then T is boundedly invertible.

Proof. (i) The condition $\operatorname{Re} T, \operatorname{Re} T^* \geq c > 0$ implies that both T and T^* are injective and thus $\operatorname{ran}(T)^\perp = \ker(T^*) = 0$. Hence, T is boundedly invertible on its range, which is dense by $\overline{\operatorname{ran}(T)} = \ker(T^*)^\perp = 0^\perp = \mathcal{H}$. Lemma 1.1.1 gives the conclusion.

(ii) If $T \in \mathcal{B}(\mathcal{H})$, then also $T^* \in \mathcal{B}(\mathcal{H})$ and $\operatorname{Re} T = \frac{1}{2}(T + T^*) = \operatorname{Re} T^*$ is defined everywhere. Thus, both T and T^* are strictly accretive; the claim follows with (i).

(iii) If $T^* = T \in \mathcal{B}(\mathcal{H})$, then $\overline{\operatorname{ran}(T)} = \ker(A^*)^\perp = \ker(A)^\perp = \mathcal{H}$ and the claim follows again with Lemma 1.1.1. (Alternatively, the claim follows directly from (ii).) □

The assumption of selfadjointness and strict positivity will be used multiple times, notably for stationary material functions, out of simplicity.

If T is accretive and $T + \lambda$ is onto for some (and thereby for all) $\lambda > 0$, then T is called m-accretive. Note in particular that skew-selfadjoint operators are m-accretive, since $T^* = -T$ implies $\operatorname{Re} T = \frac{1}{2}(T - T) = 0$ and thus $\operatorname{Re}(T + \lambda) = \lambda = \lambda I$ is onto for all $\lambda > 0$. If T is closed and m-accretive, then $T + \lambda$ is boundedly invertible for $\operatorname{Re}\lambda > 0$ by Lemma 1.1.1, with $\|(T + \lambda)^{-1}\| \le (\operatorname{Re}\lambda)^{-1}$.

Example 1.1.3. Examples of skew-selfadjoint operators are the derivative $\partial_t \colon H^1(\mathbb{R}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ on the line, as well as the Maxwell operator $\mathcal{A} \colon (E, H) \mapsto (-\operatorname{curl} H, \operatorname{curl} E)$ with domain $H_0(\operatorname{curl}, \Omega) \times H(\operatorname{curl}, \Omega)$ in $L^2(\Omega)^6$ (see Lemma 3.1.1).

The Laplace operator

$$-\Delta = -\operatorname{div}\operatorname{grad} \colon H^2(\mathbb{R}^d) \subseteq L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$$

is selfadjoint and m-accretive (as can be seen, for example, by considering the Fourier-representation of $(\Delta + \lambda)^{-1}$ for $\lambda > 0$). More generally, if $a_1, \ldots, a_d \in C^\infty(\mathbb{R}^d)$ are smooth bounded functions with bounded derivatives and $\operatorname{Re} a_j \ge 0$ for $j \in \{1, \ldots, d\}$, then setting $a = \operatorname{diag}(a_1, \ldots, a_d)$, the operator

$$\mathcal{D}_a = -\operatorname{div}(a \operatorname{grad}) = -\sum_{j=1}^{d} \partial_{x_j} a_j \partial_{x_j} \colon \operatorname{dom}(\mathcal{D}_a) \subseteq L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$$

with maximal domain $\operatorname{dom}(\mathcal{D}_a)$ is m-accretive by Lemma 1.1.2; indeed, for $\lambda > 0$ we have

$$\begin{aligned}
\operatorname{Re}\langle(\mathcal{D}_a + \lambda)u, u\rangle &= \operatorname{Re}\langle(\mathcal{D}_a + \lambda)^* u, u\rangle \\
&= \lambda \|u\|^2 + \sum_{j=1}^{d} \int_{\mathbb{R}^d} (\operatorname{Re} a_j(x))\, |\partial_{x_j} u(x)|^2 \,\mathrm{d}x \\
&\ge \lambda \|u\|^2 .
\end{aligned}$$

▲

More examples of accretive operators arise from bilinear forms, see [McI70].

1.2 Maxwell equations in linear and nonlinear media

The macroscopic Maxwell equations (see Equation (6.2.8) on page 218 in [Jac75])

$$\begin{aligned}
\partial_t D - \nabla \times H &= -J & \qquad \nabla \cdot D &= \rho \\
\partial_t B + \nabla \times E &= 0 & \qquad \nabla \cdot B &= 0
\end{aligned} \tag{1.2.1}$$

describe the relationship between the electric field E, magnetic field H, displacement field D, and magnetic induction B in the presence of given functions J and ρ, the latter being the current density and charge density, respectively. The fields D, B constitute the material response to the intrinsic fields E, H. All fields in (1.2.1) are functions of time $t \in \mathbb{R}$ and position $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and take values in $\mathbb{R}^3$.

The differential form (1.2.1) of Maxwell's equations provides a full description of the electromagnetic fields for continuous materials. However, in the presence of defects or sharp interfaces, the correct[1] description is provided by the integral form

$$\begin{aligned} &\partial_t \int_\Omega D - \int_{\partial\Omega} n \times H = -\int_\Omega J \qquad && \int_{\partial\Omega} n \cdot D = \int_\Omega \rho \\ &\partial_t \int_\Omega B + \int_{\partial\Omega} n \times E = 0 && \int_{\partial\Omega} n \cdot B = 0 \end{aligned} \tag{1.2.2}$$

of (1.2.1), where Ω is an arbitrary volume with (smooth) boundary $\partial\Omega$ and outward normal field n. If Γ is an interface, also sufficiently smooth, between two different, but otherwise continuous, media and if J_Γ, ρ_Γ are current and charge densities at the interface, then the integral form (1.2.2) gives rise to the transmission conditions

$$[n \times E]_\Gamma = 0, \quad [n \times H]_\Gamma = -J_\Gamma, \quad [n \cdot D]_\Gamma = -\rho_\Gamma, \quad [n \cdot B]_\Gamma = 0. \tag{1.2.3}$$

Here, with Ω_1, Ω_2 denoting the different parts of Ω on each side of Γ, and n_1, n_2 denoting the unit normal to $\partial\Omega_1, \partial\Omega_2$, the tangential jump

$$[n \times F]_\Gamma = (n_1 \times F)|_{\Omega_1} + (n_2 \times F)|_{\Omega_2} = n_1 \times (F|_{\Omega_1} - F|_{\Omega_2}),$$

and similarly the normal jump

$$[n \cdot F]_\Gamma = n_1 \cdot (F|_{\Omega_1} - F|_{\Omega_2})$$

across Γ are understood in the distributional sense (usually in the sense of traces). Following [DL90a, §4.2], these relations can be derived from the assumption that (1.2.2) must be valid for any domain. Thus we also have, for example,

$$\int_{\Omega_i} \partial_t D - \int_{\partial\Omega_i} n \times H = -\int_{\Omega_i} J \qquad (i \in \{1, 2\}).$$

Now since

$$\int_\Omega \partial_t D - \int_{\partial\Omega} n \times H = \int_{\Omega_1} \partial_t D - \int_{\partial\Omega_1} n \times H + \int_{\Omega_2} \partial_t D - \int_{\partial\Omega_2} n \times H + \int_{\Omega\cap\Gamma} [n \times H]_\Gamma$$

is equal to

$$-\int_\Omega J = -\int_{\Omega_1} J - \int_{\Omega_2} J - \int_{\Omega\cap\Gamma} J_\Gamma$$

and, assuming the integrals above are well-defined, we may conclude that $[n \times H]_\Gamma = -J_\Gamma$. The remaining conditions in (1.2.3) are obtained analogously.

We have thus two versions of the Maxwell equations: the integral formulation (1.2.2), which is 'self-contained', and the differential formulation (1.2.1) supplemented by the transmission conditions (1.2.3). The equivalence "(1.2.2) $\iff$ (1.2.1) & (1.2.3)" depends on the functional

[1] This form is also more natural from a physical standpoint, as the current $\int J$ and charge $\int \rho$ are directly measurable quantities.

setting together with an application of Gauß' theorem, cf. Section 3.1.

The fields (D, B) and (E, H) are related through *constituent relations*, and we assume throughout the direct functional dependence

$$D = D(E) = \epsilon_0 E + P_{\mathrm{el}}(E), \qquad B = B(H) = \mu_0 H + P_{\mathrm{m}}(H), \tag{1.2.4}$$

where $\epsilon_0, \mu_0 > 0$ are positive numbers (the permittivity and permeability in vacuum), and where $P_{\mathrm{el}}, P_{\mathrm{m}}$ are the electric and magnetic polarization (density), respectively. In non-idealized materials, the material response given by $P_{\mathrm{el}}, P_{\mathrm{m}}$ is non-instantaneous: there is a delay between the change of the applied field and the material response. In other terms, the material responds differently at different frequencies; this leads to the notion of dispersion. For $P \in \{P_{\mathrm{el}}, P_{\mathrm{m}}\}$, our focus will lie on operators with continuous memory, either linear, such as

$$P(u)(t) = \int_0^\infty \chi(\tau)\, u(t-\tau)\, \mathrm{d}\tau, \tag{1.2.5}$$

or nonlinear, such as

$$P(u)(t) = \int_0^\infty \chi(\tau)\, Q(u(t-\tau))\, \mathrm{d}\tau \tag{1.2.6}$$

or a linear combination of (1.2.5) and (1.2.6), with a time-independent nonlinearity Q. Here we have omitted the spatial variable x, and the evaluation $u(t)$ is understood as an element in some function space $\mathcal{H}$. In this vein, Q is a map on $\mathcal{H}$ and may also depend nonlocally on its argument. In most cases, the operator $\chi(\tau) = \chi(\tau, x)$ is just a scalar- or matrix-valued map for $\tau > 0$; in general, each $\chi(\tau)$ is a bounded linear operator on $\mathcal{H}$. We like to point out that, by construction, every such operator of the form (1.2.6) is *causal* (or *non-anticipative*), i.e., $P(u)(t)$ depends on past values $u(\tau)$, $\tau \le t$, of its argument, but not on future times $\tau > t$. This property makes sense not only from a physical point of view, but plays an important part mathematically in the solution theory reviewed in Section 1.4.

One further example of such material operators is studied in Section 2.3 and is given by multilinear Volterra-type operators $P^{(n)}$ with

$$P^{(n)}(u)(t) = \int_0^\infty \cdots \int_0^\infty \chi^{(n)}(\tau_1, \dots, \tau_n)\, Q^{(n)}(u(t-\tau_1), \dots, u(t-\tau_n))\, \mathrm{d}\tau_1 \cdots \mathrm{d}\tau_n \tag{1.2.7}$$

for $n \in \mathbb{N}$, where $Q^{(n)}$ is n-linear and $\chi^{(n)}(\tau_1, \dots, \tau_n)$ is again a bounded linear operator. Similar models are used in the field of nonlinear optics, see [Boy08], the idea being that a given nonlinearity P is approximated by a Volterra series $P(u) = \sum_{n=1}^\infty P^{(n)}(u)$. This will also be the heuristic adopted in this work, although we will not deal with questions of convergence (mainly because we will work with finite sums) or for which operators such a series exists. For a slightly more detailed heuristic for functions in one time variable, see Appendix 1.2 in [Rug81] and [BC85]. As it turns out, and as a rule of thumb, the more compact model (1.2.6) is better suited for questions of (global) well-posedness, if Q and χ are subjected to suitable Lipschitz- and integrability conditions, respectively (see Section 2). On the other hand, the form (1.2.7) is practical for concrete computation and approximation of solutions, due to its multilinear nature, and will be used extensively in Chapter 4.

We will treat Maxwell's equations as an evolutionary system in the unknowns E, H. This means, starting with values of E, H at initial time $t = 0$, one is interested in the evolution of the fields for positive time $t > 0$. In fact, due to the nonlocality of the polarization, the whole history of the fields E, H, i.e., their values for $t \leq 0$, is assumed to be prescribed.

1.3 Linear, evanescent surface modes and dispersion relation

We now want to derive explicit, classical solutions to the linear Maxwell equations at a planar interface, in the form of travelling surface waves. This derivation is motivated by surface plasmon polaritons and other surface wave phenomena in the physics literature, and is similar to that in [JML13, §2.2] and [Rae88].

Let $\Gamma = \{x \in \mathbb{R}^3 : n \cdot x = 0\}$ $(n \in \mathbb{R}^3,\ |n| = 1)$ be a plane between the disjoint domains Ω^+, Ω^-. We consider the homogeneous Maxwell system

$$\begin{aligned} \partial_t D - \operatorname{curl} H &= 0 \qquad \operatorname{div} D = 0 \\ \partial_t B + \operatorname{curl} E &= 0 \qquad \operatorname{div} B = 0 \end{aligned} \tag{1.3.1}$$

with the transmission conditions

$$[n \times E]_\Gamma = [n \times H]_\Gamma = 0, \qquad [n \cdot D]_\Gamma = [n \cdot B]_\Gamma = 0, \tag{1.3.2}$$

and with linear constituent relations given by

$$\left.\begin{aligned} D(E)(t,x) &= \epsilon_0 E(t,x) + \int_{\mathbb{R}} \chi_{\mathrm{el}}^{\pm}(\tau) E(t-\tau, x)\,\mathrm{d}\tau \\ B(H)(t,x) &= \mu_0 H(t,x) + \int_{\mathbb{R}} \chi_{\mathrm{m}}^{\pm}(\tau) H(t-\tau, x)\,\mathrm{d}\tau \end{aligned}\right\} \quad x \in \Omega^{\pm},$$

where $\epsilon_0, \mu_0 \in \mathbb{R}^+$ and $\chi_{\mathrm{el}}^+, \chi_{\mathrm{el}}^-, \chi_{\mathrm{m}}^+, \chi_{\mathrm{m}}^- \colon \mathbb{R} \to \mathbb{R}$ satisfy $\chi_{\mathrm{el}}^{\pm}(\tau) = \chi_{\mathrm{m}}^{\pm}(\tau) = 0$ for $\tau < 0$. For (nontrivial) solutions E, H of (1.3.1) in the form of surface waves, propagating in direction parallel to the interface Γ, we make the (complex) ansatz

$$E(t,x) = \xi(n \cdot x) e^{i(k_{\|} \cdot x - \omega t)} \quad \text{and} \quad H(t,x) = \zeta(n \cdot x) e^{i(k_{\|} \cdot x - \omega t)}, \tag{1.3.3}$$

with profile functions $\xi, \zeta \colon \mathbb{R} \to \mathbb{C}^3$, where $n \cdot k_{\|} = 0$, and $\operatorname{Re} \omega > 0$. Due to the exponential terms in our ansatz we then obtain

$$\begin{aligned} D(E)(t) &= \epsilon^{\pm}(\omega) \xi(n \cdot x) e^{i(k_{\|} \cdot x - \omega t)} \\ B(H)(t) &= \mu^{\pm}(\omega) \zeta(n \cdot x) e^{i(k_{\|} \cdot x - \omega t)}, \end{aligned}$$

having defined the frequency-dependent permittivity $\epsilon^{\pm}(\omega)$ and permeability $\mu^{\pm}(\omega)$ by

$$\epsilon^{\pm}(\omega) := \epsilon_0 + \int_0^\infty \chi_{\mathrm{el}}^{\pm}(t) e^{i\omega t}\,\mathrm{d}t, \qquad \mu^{\pm}(\omega) := \mu_0 + \int_0^\infty \chi_{\mathrm{m}}^{\pm}(t) e^{i\omega t}\,\mathrm{d}t$$

on each domain $\Omega^{\pm}$. We assume nontrivial material jumps, thus

$$\epsilon^+(\omega) \neq \epsilon^-(\omega) \quad \text{or} \quad \mu^+(\omega) \neq \mu^-(\omega).$$

Let us fix coordinates and assume w.l.o.g. that $x_1 = n \cdot x$ and $k_{\|} \cdot x = k x_2$, $k > 0$. Now the only spatial dependence is on the variables x_1, x_2. Inserting our ansatz into (1.3.1) we obtain the systems

$$\left.\begin{cases} i\omega\mu^{\pm}(\omega)\zeta_1 = ik\xi_3 \\ i\omega\mu^{\pm}(\omega)\zeta_2 = -\xi_3' \\ i\omega\mu^{\pm}(\omega)\zeta_3 = \xi_2' - ik\xi_1 \end{cases}\right\} \quad \text{and} \quad \left\{\begin{aligned} -i\omega\epsilon^{\pm}(\omega)\xi_1 &= ik\zeta_3 \\ -i\omega\epsilon^{\pm}(\omega)\xi_2 &= -\zeta_3' \\ -i\omega\epsilon^{\pm}(\omega)\xi_3 &= \zeta_2' - ik\zeta_1 \end{aligned}\right\} \tag{1.3.4}$$

understood for $x_1 > 0$ and $x_1 < 0$ separately. (Listed are only the dynamic equations on the left-hand side of (1.3.1); note that the div-equations are obtained by summation of the first two lines in each block and taking derivatives.) Solving the first lines in (1.3.4) for

$$\xi_3 = \frac{\omega\mu^{\pm}(\omega)}{k}\zeta_1 \quad \text{and} \quad \zeta_3 = -\frac{\omega\epsilon^{\pm}(\omega)}{k}\xi_1 \tag{1.3.5}$$

these equations can be decoupled, and with

$$G^{\pm}(\omega,k) := \begin{pmatrix} 0 & -ik \\ ik - \dfrac{i\omega^2\epsilon^{\pm}(\omega)\mu^{\pm}(\omega)}{k} & 0 \end{pmatrix}$$

we find

$$\partial_{x_1}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = G^{\pm}(\omega,k)\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}, \qquad \partial_{x_1}\begin{pmatrix}\zeta_1\\ \zeta_2\end{pmatrix} = G^{\pm}(\omega,k)\begin{pmatrix}\zeta_1\\ \zeta_2\end{pmatrix}. \tag{1.3.6}$$

Diagonalizing $G^{\pm}(\omega,k)$ yields the eigenvectors $(k,\lambda^{\pm})$ and $(k,-\lambda^{\pm})$, corresponding to eigenvalues $i\lambda^{\pm}$ and $-i\lambda^{\pm}$; the latter are the complex roots of $z_{\pm} = k^2 - \omega^2\epsilon^{\pm}(\omega)\mu^{\pm}(\omega)$. If $z_{\pm} \notin \mathbb{R}_0^-$, we can select λ^+, λ^- such that

$$\operatorname{Re} i\lambda^+ < 0 \quad \text{and} \quad \operatorname{Re} i\lambda^- > 0, \tag{1.3.7}$$

and the general bounded solutions of (1.3.6) are evanescent waves given by

$$\begin{pmatrix}\xi_1(x_1)\\ \xi_2(x_1)\end{pmatrix} = a^{\pm}e^{i\lambda^{\pm}x_1}\begin{pmatrix}k\\ \lambda^{\pm}\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}\zeta_1(x_1)\\ \zeta_2(x_1)\end{pmatrix} = b^{\pm}e^{i\lambda^{\pm}x_1}\begin{pmatrix}k\\ \lambda^{\pm}\end{pmatrix}, \tag{1.3.8}$$

where $x_1 \in \mathbb{R}^{\pm}$ and $a^{\pm}$, $b^{\pm} \in \mathbb{C}$. Now taking into account the jump conditions (1.3.2), these translate to $[\xi_2]_\Gamma = [\xi_3]_\Gamma = [\zeta_2]_\Gamma = [\zeta_3]_\Gamma = 0$, so with (1.3.5) and (1.3.8) we obtain

$$\begin{aligned} a^+\lambda^+ - a^-\lambda^- &= 0 \\ a^+\epsilon^+(\omega) - a^-\epsilon^-(\omega) &= 0 \\ b^+\lambda^+ - b^-\lambda^- &= 0 \\ b^+\mu^+(\omega) - b^-\mu^-(\omega) &= 0. \end{aligned} \tag{1.3.9}$$

Since $\lambda^{\pm}, \epsilon^{\pm}(\omega), \mu^{\pm}(\omega)$ are all nonzero, either a^+, a^- are both nonzero or vanish identically; the same is true for b^+, b^-. This leads to the following characterization.

1. *Transverse-magnetic (TM) modes:* If $b^+ = b^- = 0$ and $a^+, a^- \neq 0$, then $E = (E_1, E_2, 0)$ and $H = (0, 0, H_3)$. From (1.3.9) we obtain $\epsilon^+(\omega)\lambda^- = \epsilon^-(\omega)\lambda^+$ (so that $\epsilon^+(\omega) \neq$

$\epsilon^-(\omega)$, since $\lambda^+ \neq \lambda^-$), and taking the square yields the dispersion relation

$$k^2 = \omega^2 \frac{\epsilon^+(\omega)\epsilon^-(\omega)}{\epsilon^+(\omega)+\epsilon^-(\omega)} \cdot \frac{\epsilon^-(\omega)\mu^+(\omega)-\epsilon^+(\omega)\mu^-(\omega)}{\epsilon^-(\omega)-\epsilon^+(\omega)}. \tag{1.3.10}$$

2. *Transverse-electric (TE) modes:* If $a^+ = a^- = 0$ and $b^+, b^- \neq 0$, then $E = (0,0,E_3)$ and $H = (H_1, H_2, 0)$. From (1.3.9) we get $\mu^+(\omega)\lambda^- = \mu^-(\omega)\lambda^+$ (so $\mu^+(\omega) \neq \mu^-(\omega)$) and taking the square yields the dispersion relation

$$k^2 = \omega^2 \frac{\mu^+(\omega)\mu^-(\omega)}{\mu^+(\omega)+\mu^-(\omega)} \cdot \frac{\epsilon^+(\omega)\mu^-(\omega)-\epsilon^-(\omega)\mu^+(\omega)}{\mu^-(\omega)-\mu^+(\omega)}. \tag{1.3.11}$$

There are no other cases; indeed, suppose that a^+, a^-, b^+, b^- are all nonzero, then

$$\begin{aligned}\epsilon^+(\omega)\lambda^- - \epsilon^-(\omega)\lambda^+ &= 0\\ \mu^+(\omega)\lambda^- - \mu^-(\omega)\lambda^+ &= 0.\end{aligned}$$

But since $\lambda^+, \lambda^- \neq 0$, this implies $\epsilon^+(\omega)\mu^-(\omega) - \epsilon^-(\omega)\mu^+(\omega) = 0$. The dispersion relation (either of (1.3.10) or (1.3.11)) is reduced to $k^2 = 0$, contradicting the assumption on k. In conclusion, linear modes of the form (1.3.3) are only possible if k, ω fulfill either (1.3.10) or (1.3.11).

Remark 1.3.1. The non-magnetic setting where $\mu^+ = \mu^- \equiv \mu_0 \in \mathbb{R}_{>0}$ is covered by the TM-setting: In this case $b^+ = b^- = 0$ follows directly from (1.3.9), and (1.3.10) simplifies to

$$k^2 = \omega^2 \frac{\epsilon^+(\omega)\epsilon^-(\omega)}{\epsilon^+(\omega)+\epsilon^-(\omega)},$$

which is the known dispersion relation governing the existence of surface plasmon polaritons, see [Rae88]. Like in the general TM-setting, the dispersion relation is derived from the more basic condition

$$\epsilon^+\lambda^- = \epsilon^-\lambda^+. \tag{1.3.12}$$

This condition is the basis for the often cited requirement that $\operatorname{Re}\epsilon^+$ and $\operatorname{Re}\epsilon^-$ must have different signs (for instance, [JML13, §1.3.1], [BDE03]). Indeed, this follows from (1.3.12), but only if the imaginary parts of $\epsilon^+, \epsilon^-, \lambda^+, \lambda^-$ are neglected. Among other cases, this approximation is assumed for metal-dielectric interfaces. In general, the relationship between $\epsilon^\pm, \lambda^\pm$ is more complex; writing $\epsilon^\pm = \epsilon_r^\pm + i\epsilon_i^\pm$, $\lambda^\pm = \lambda_r^\pm + i\lambda_i^\pm$ with real parameters $\epsilon_r^\pm, \epsilon_i^\pm, \lambda_r^\pm, \lambda_i^\pm$, then comparing real and imaginary parts in (1.3.12) we obtain

$$\begin{aligned}\epsilon_r^+\lambda_r^- - \epsilon_i^+\lambda_i^- &= \epsilon_r^-\lambda_r^+ - \epsilon_i^-\lambda_i^+\\ \epsilon_i^+\lambda_r^- + \epsilon_r^+\lambda_i^- &= \epsilon_i^-\lambda_r^+ + \epsilon_r^-\lambda_i^+.\end{aligned}$$

Recall that the assumption (1.3.7) needed for evanescent waves was that $\lambda_i^+ > 0$ and $\lambda_i^- < 0$, which is compatible even with $\epsilon_r^+, \epsilon_r^- > 0$ by suitable choice of the other parameters. $\Diamond$

Finally, we remark that in the derivation above we have assumed that ϵ^+, ϵ^- are homogeneous and scalar-valued. There are other settings in which the existence of linear surface waves can be derived, for instance at the interface between a homogeneous dielectric and a uniaxial crystal, with optical axis parallel to the interface. In this case, surface waves exist (introduced in [Dya88] and now called *Dyakonov waves*) that can propagate in certain angles to the optical axis. For other examples we refer to [JML13].

1.4 The framework of evolutionary equations

The Maxwell equations with memory will be embedded into a functional analytic framework of operator equations over exponentially weighted function spaces (an explicit connection is established in Section 2.4). For such problems, the term *evolutionary equations* has been established. A well-posedness theory has been developed for linear equations, which, to some extent, can be applied to nonlinear problems. Recall that a (differential) equation is well-posed, in the sense of Hadamard, if it admits a unique solution, which depends continuously on the given data.

This section aims to provide an overview and some background regarding evolutionary equations. Some of the basic results needed for the solution theory are reviewed without proof. For more insight into the subject and further details we refer to [STW22], specifically Chapters 3, 5, 6, and 8.

Throughout, let $\mathcal{H}$ be a Hilbert space. For $\mathcal{H}$-valued functions $u\colon \mathbb{R} \to \mathcal{H}$, we write $u \in C(\mathbb{R}, \mathcal{H})$ if u is continuous, and $u \in L^2_{\mathrm{loc}}(\mathbb{R}, \mathcal{H})$ if u is Bochner measurable and square integrable over compact intervals. For a real parameter $\varrho \in \mathbb{R}$ exponentially weighted variants of these spaces are defined as

$$C_\varrho(\mathbb{R}, \mathcal{H}) := \{u \in C(\mathbb{R}, \mathcal{H}) : \|u\|_{C_\varrho} := \sup_{t\in\mathbb{R}} \|u(t)\|_{\mathcal{H}}\, e^{-\varrho t} < \infty\}$$

$$L^2_\varrho(\mathbb{R}, \mathcal{H}) := \{u \in L^2_{\mathrm{loc}}(\mathbb{R}, \mathcal{H}) : \|u\|_{L^2_\varrho} := \Big(\int_{\mathbb{R}} \|u(t)\|^2_{\mathcal{H}}\, e^{-2\varrho t}\, \mathrm{d}t\Big)^{1/2} < \infty\}.$$

The latter is a Hilbert space with the Hermitian product $\langle u, v\rangle_{L^2_\varrho} = \int_{\mathbb{R}} \langle u(t), v(t)\rangle_{\mathcal{H}} e^{-2\varrho t}\, \mathrm{d}t$. Analogously, $L^p_\varrho(\mathbb{R}, \mathcal{H}) \subseteq L^p_{\mathrm{loc}}(\mathbb{R}, \mathcal{H})$ can be defined for $1 \leq p \leq \infty$, where $\|u\|_{L^p_\varrho} = \|u \cdot \exp(-\varrho\, \cdot)\|_{L^p(\mathbb{R},\mathcal{H})}$.

The space $C_c^\infty(\mathbb{R}, \mathcal{H})$ of smooth $\mathcal{H}$-valued functions with compact support in $\mathbb{R}$ is dense in $L^p_\varrho(\mathbb{R}, \mathcal{H})$ for all $p \geq 1$ and $\varrho \in \mathbb{R}$. For $u \in C_c^\infty(\mathbb{R}, \mathcal{H})$ we denote by $\partial_t u(t) = \dot{u}(t) \in \mathcal{H}$ the pointwise derivative with respect to the time variable t. Furthermore, we define its (weighted) *Fourier–Laplace transform* $\mathcal{L}_\varrho u = \mathcal{L}_\varrho[u]$ by the integral

$$(\mathcal{L}_\varrho u)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(t) e^{-(\varrho+i\xi)t}\, \mathrm{d}t.$$

Notice that if $u(t) = 0$ for $t < 0$, the integral $(\mathcal{L}_\varrho u)(\xi)$ is the (unitary) Laplace transform of u in the complex parameter $z = \varrho + i\xi$. For $\varrho = 0$, $\mathcal{L}_0$ is the standard unitary Fourier transform.

Lemma 1.4.1. *The following statements hold.*

(i) The time-derivative is a closable operator in $L^2_\varrho(\mathbb{R}, \mathcal{H})$; its closure, denoted again by ∂_t, is the weak time-derivative

$$\partial_t\colon\ \mathrm{dom}(\partial_t) \subseteq L^2_\varrho(\mathbb{R}, \mathcal{H}) \to L^2_\varrho(\mathbb{R}, \mathcal{H})$$

with maximal domain $\mathrm{dom}(\partial_t) = H^1_\varrho(\mathbb{R}, \mathcal{H}) := \{u \in L^2_\varrho(\mathbb{R}, \mathcal{H}) : \partial_t u \in L^2_\varrho(\mathbb{R}, \mathcal{H})\}$.

(ii) The transform $\mathcal{L}_\varrho$ extends to a unitary operator $\mathcal{L}_\varrho\colon L^2_\varrho(\mathbb{R},\mathcal{H}) \to L^2_0(\mathbb{R},\mathcal{H})$ via the Plancherel identity

$$\langle \mathcal{L}_\varrho u, \mathcal{L}_\varrho v\rangle_{L^2_0} = \langle u, v\rangle_{L^2_\varrho}.$$

(iii) The derivative rule

$$\mathcal{L}_\varrho[\partial_t u](\xi) = (\varrho + i\xi)\mathcal{L}_\varrho[u](\xi)$$

holds for $u \in H^1_\varrho(\mathbb{R},\mathcal{H})$, thus $\mathcal{L}_\varrho[H^1_\varrho(\mathbb{R},\mathcal{H})] = \{v \in L^2(\mathbb{R},\mathcal{H}) : (\varrho + i\,\cdot)v(\cdot) \in L^2(\mathbb{R},\mathcal{H})\}$.

(iv) The convolution theorem

$$\mathcal{L}_\varrho[u * v] = \sqrt{2\pi}(\mathcal{L}_\varrho u)(\mathcal{L}_\varrho v)$$

holds for $u \in L^1_\varrho(\mathbb{R},\mathcal{H}),\ v \in L^2_\varrho(\mathbb{R},\mathcal{H})$.

(v) For $\varrho > 0$, the operator ∂_t is boundedly invertible; its inverse $\partial_t^{-1}\colon L^2_\varrho(\mathbb{R},\mathcal{H}) \to L^2_\varrho(\mathbb{R},\mathcal{H})$ satisfies $\|\partial_t^{-1}\|_{\mathcal{B}(L^2_\varrho(\mathbb{R},\mathcal{H}))} \le \varrho^{-1}$ and is given by the causal integral

$$(\partial_t^{-1}u)(t) = \int_{-\infty}^t u(\tau)\,\mathrm{d}\tau.$$

Proof. Most assertions are analogous to the unweighted case $\varrho = 0$, where $L^2_\varrho(\mathbb{R},\mathcal{H}) = L^2_0(\mathbb{R},\mathcal{H}) = L^2(\mathbb{R},\mathcal{H})$, and can be reduced to this case by making use of the unitary operator

$$\exp_{-\varrho}\colon L^2_\varrho(\mathbb{R},\mathcal{H}) \to L^2(\mathbb{R},\mathcal{H}), \quad u \mapsto (t \mapsto u(t)e^{-\varrho t}).$$

For instance, since the diagram

$$\begin{tikzcd}
L^2_\varrho(\mathbb{R},\mathcal{H}) \arrow[r, "\exp_{-\varrho}"] \arrow[dr, dashed, "\mathcal{L}_\varrho"'] & L^2(\mathbb{R},\mathcal{H}) \arrow[d, "\mathcal{L}_0"] \\
 & L^2(\mathbb{R},\mathcal{H})
\end{tikzcd}$$

is commutative, $\mathcal{L}_\varrho$ is indeed unitary. In the same vein, the diagram

$$\begin{tikzcd}
L^2_\varrho(\mathbb{R},\mathcal{H}) \arrow[r, "\exp_{-\varrho}"] \arrow[d, dashed, "\partial_t"'] & L^2(\mathbb{R},\mathcal{H}) \arrow[r, "\mathcal{L}_0"] \arrow[d, "(\partial_t + \varrho)"] & L^2(\mathbb{R},\mathcal{H}) \arrow[d, "(\varrho + i\,\cdot)"] \\
L^2_\varrho(\mathbb{R},\mathcal{H}) & L^2(\mathbb{R},\mathcal{H}) \arrow[l, "\exp_\varrho"] & L^2(\mathbb{R},\mathcal{H}) \arrow[l, "\mathcal{L}_0^*"]
\end{tikzcd}$$

is commutative and we have $\partial_t = \exp_\varrho(\partial_t + \varrho)\exp_{-\varrho}$ (in the sense of the diagram, i.e., ∂_t on the right denotes the weak derivative in $L^2(\mathbb{R},\mathcal{H})$; the identity is valid on a dense subspace and defines a closed operator). Realizing that $\partial_t + \varrho = \mathcal{L}_0^*(\varrho + i\,\cdot)\mathcal{L}_0$ is boundedly invertible in $L^2(\mathbb{R},\mathcal{H})$ yields the bounded invertibility of ∂_t in $L^2_\varrho(\mathbb{R},\mathcal{H})$, together with the norm estimate. The formula for ∂_t^{-1} follows from the variation of constants formula. □

Remark 1.4.2. Since ∂_t is skew-selfadjoint in $L^2(\mathbb{R})$, the proof gives the formula

$$\partial_t^* = (\exp_\varrho(\partial_t + \varrho)\exp_{-\varrho})^* = \exp_\varrho(-\partial_t + \varrho)\exp_{-\varrho} = -\partial_t + 2\varrho.$$

Thus, $\partial_t + \partial_t^* = 2\varrho$, or $(\partial_t - \varrho)^* = -(\partial_t - \varrho)$. Put differently, the operator $\partial_t - \varrho$ is skew-selfadjoint in $L^2_\varrho(\mathbb{R}, \mathcal{H})$, but not boundedly invertible. ♢

Remark 1.4.3. In fact, $\|\partial_t^{-1}\|_{L^2_\varrho \to L^2_\varrho} = 1/|\varrho|$ for $\varrho \neq 0$. For $\varrho > 0$, this can be seen with $u_\nu(t) := \sqrt{2(\varrho - \nu)}\theta(t)e^{\nu t}$, $\nu < \varrho$, where we have $\|u_\nu\|_{L^2_\varrho} = 1$ and $\lim_{\nu \nearrow \varrho} \|\partial_t^{-1} u_\nu\| = 1/\varrho$. Analogously for $\varrho < 0$; here the bounded inverse is given by $(\partial_t^{-1}u)(t) = -\int_t^\infty u(\tau)\,\mathrm{d}\tau$ and is thus *anti*-causal, i.e., depending only on future times $\tau > t$. ♢

Remark 1.4.4. Although ∂_t and its inverse, as operators in L^2_ϱ, depend formally on ϱ, we will not make this dependence explicit. This is further justified in a more general case, see Lemma 1.4.10 below. Furthermore, we will always denote by ∂_t^{-1} the causal operator above for some $\varrho > 0$. ♢

Proposition 1.4.5 (Sobolev embedding)**.** *Let $\varrho > 0$ and define*

$$C_{\varrho,0}(\mathbb{R}, \mathcal{H}) := \{u \in C_\varrho(\mathbb{R}, \mathcal{H}) : \lim_{|t| \to \infty} \|u(t)\|_{\mathcal{H}}\, e^{-\varrho t} = 0\}.$$

Then for all $u \in H^1_\varrho(\mathbb{R}, \mathcal{H})$ we have $u \in C_{\varrho,0}(\mathbb{R}, \mathcal{H})$ and

$$\|u\|_{C_{\varrho,0}} = \sup_{t \in \mathbb{R}} \|u(t)\|\, e^{-\varrho t} \leq \frac{1}{\sqrt{2\varrho}}\, \|u\|_{H^1_\varrho}\,.$$

Proof. We refer to [STW22, Theorem 4.1.2] or [Tro18, Proposition 1.1.8] for the proof. □

Linear material laws

Abstract material functions, in particular convolution operators, are introduced by the following notion.

Definition 1.4.6. A *linear material law* on $\mathcal{H}$ is a complex-analytic map $M\colon \operatorname{dom}(M) \subseteq \mathbb{C} \to \mathcal{B}(\mathcal{H})$ which is uniformly bounded in a right half-plane, i.e.,

$$\exists \varrho_0 \in \mathbb{R} : \quad \sup_{\operatorname{Re} z > \varrho_0} \|M(z)\|_{\mathcal{B}(\mathcal{H})} < \infty. \tag{1.4.1}$$

For such M we define $M(\partial_t) := \mathcal{L}_\varrho^* M(\varrho + i\,\cdot)\mathcal{L}_\varrho$ for $\varrho > \varrho_0$.

Remark 1.4.7. Introducing the multiplication operator m by $(\mathsf{m}u)(\xi) = \xi u(\xi)$, Lemma 1.4.1 (iii) can be formulated as $\mathcal{L}_\varrho \partial_t = (\varrho + i\mathsf{m})\mathcal{L}_\varrho$. From this one derives the spectral representation $\partial_t^{-1} = \mathcal{L}_\varrho^*(\varrho + i\mathsf{m})^{-1}\mathcal{L}_\varrho$. For analytic maps $f\colon \operatorname{dom}(f) \subseteq \mathbb{C} \to \mathbb{C}$ this formula readily generalizes to

$$f(\partial_t^{-1}) = f(\mathcal{L}_\varrho^*(\varrho + i\mathsf{m})^{-1}\mathcal{L}_\varrho) = \mathcal{L}_\varrho^* f((\varrho + i\mathsf{m})^{-1})\mathcal{L}_\varrho.$$

In this respect, a linear material law can be understood as a holomorphic functional calculus for the bounded operator ∂_t^{-1} on $L^2_\varrho(\mathbb{R}, \mathcal{H})$ for $\varrho > 0$ by viewing $f(\partial_t^{-1}) = M(\partial_t)$ as a function of ∂_t^{-1}. This is the approach to linear material laws initially established in [Pic09].

On a small note, a prerequisite for defining $f(\partial_t^{-1})$ is that f be well-defined and holomorphic around the spectrum $\sigma(\partial_t^{-1})$ of ∂_t^{-1}. We find that

$$\begin{aligned}\|(z-\partial_t^{-1})^{-1}\|_{\mathcal{B}(L^2_\varrho)} &= \sup_{\xi\in\mathbb{R}} \|(z-(\varrho+i\xi)^{-1})^{-1}\|_{L^2_0} < \infty \\ &\iff \inf_{\xi\in\mathbb{R}} |z-(\varrho+i\xi)^{-1}| > 0.\end{aligned}$$

Since the inversion $w \mapsto w^{-1}$ maps $\varrho+i\mathbb{R}$ onto the circle $\partial B(\frac{1}{2\varrho},\frac{1}{2\varrho})$ around $\frac{1}{2\varrho}$ with radius $\frac{1}{2\varrho}$, we obtain $\sigma(\partial_t^{-1}) = \partial B(\frac{1}{2\varrho},\frac{1}{2\varrho})$. Thus, to define $f(\partial_t^{-1})$ on $L^2_\varrho(\mathbb{R},\mathcal{H})$ for all $\varrho > \varrho_0$, the map f should be at least holomorphic on the disc $B[\frac{1}{2\varrho},\frac{1}{2\varrho}]$ (the closure of $\bigcup_{\varrho\geq\varrho_0} \partial B(\frac{1}{2\varrho},\frac{1}{2\varrho})$). $\Diamond$

We will also need a slightly more general definition of what it means for a—possibly nonlinear—operator to be causal.

Definition 1.4.8. A map $f\colon \operatorname{dom}(f) \subseteq L^2_{\mathrm{loc}}(\mathbb{R},\mathcal{H}_1) \to L^2_{\mathrm{loc}}(\mathbb{R},\mathcal{H}_2)$ is *causal*, if for all $t_0 \in \mathbb{R}$ and $u, v \in \operatorname{dom}(f)$, the condition $u = v$ in $(-\infty, t_0]$ implies $f(u) = f(v)$ in $(-\infty, t_0]$.

Causality and complex analyticity (holomorphy) in L^2-spaces are intimately related by the Paley–Wiener theorem (e.g. [Kat04, VI.7]). Consider the following weighted version.

Theorem 1.4.9 (Paley–Wiener, [STW22, Corollary 8.1.3])**.** *Let $\varrho \in \mathbb{R}$. There is an isometric isomorphism between $L^2_\varrho(\mathbb{R}^+,\mathcal{H}) = \{u \in L^2(\mathbb{R},\mathcal{H}) : u|_{(-\infty,0]} = 0\}$ and the Hardy space*

$$H_2(\mathbb{C}_{\mathrm{Re}>\varrho},\mathcal{H}) = \{\zeta\colon \mathbb{C}_{\mathrm{Re}>\varrho} \to \mathcal{H} \text{ analytic}, \sup_{\varrho'>\varrho} \|\zeta(\varrho'+i\,\cdot)\|_{L^2_0} < \infty\}$$

on the right half-plane $\mathbb{C}_{\mathrm{Re}>\varrho} = \{z \in \mathbb{C} : \operatorname{Re} z > \varrho\}$, given by $u \mapsto (\varrho + i\xi \mapsto (\mathcal{L}_\varrho u)(\xi))$.

Lemma 1.4.10. *Let M be a linear material law satisfying (1.4.1) with $\varrho_0 \in \mathbb{R}$. Then, for $\varrho > \varrho_0$ the operator $M(\partial_t)\colon L^2_\varrho(\mathbb{R},\mathcal{H}) \to L^2_\varrho(\mathbb{R},\mathcal{H})$ is bounded, causal, and independent of ϱ, i.e., for all $\varrho_1, \varrho_2 > \varrho_0$ and $u \in L^2_{\varrho_1}(\mathbb{R},\mathcal{H}) \cap L^2_{\varrho_2}(\mathbb{R},\mathcal{H})$ we have*

$$\mathcal{L}^*_{\varrho_1} M(\varrho_1 + i\,\cdot)\mathcal{L}_{\varrho_1} u = \mathcal{L}^*_{\varrho_2} M(\varrho_2 + i\,\cdot)\mathcal{L}_{\varrho_2} u.$$

Proof. Boundedness: Since $\mathcal{L}_\varrho$ is unitary and (1.4.1) holds, $M(\partial_t) = \mathcal{L}^*_\varrho M(\varrho + i\,\cdot)\mathcal{L}_\varrho$ is bounded on $L^2_\varrho(\mathbb{R},\mathcal{H})$, uniformly in $\varrho > \varrho_0$.

Causality: Let $u_1, u_2 \in L^2_\varrho(\mathbb{R},\mathcal{H})$ with $u := u_1 - u_2 = 0$ on $(-\infty, t_0]$. We can assume w.l.o.g. that $t_0 = 0$ (else consider $u(\,\cdot\, + t_0)$ instead of u in the following). We have thus $u \in L^2_\varrho(\mathbb{R}^+,\mathcal{H})$ and obtain successively

$$\begin{aligned} \mathcal{L}_\varrho u &\in H_2(\mathbb{C}_{\mathrm{Re}>\varrho},\mathcal{H}) && \text{by Theorem 1.4.9,} \\ M(\varrho + i\,\cdot)\mathcal{L}_\varrho u &\in H_2(\mathbb{C}_{\mathrm{Re}>\varrho},\mathcal{H}) && \text{by uniform boundedness of } M, \\ \mathcal{L}^*_\varrho M(\varrho + i\,\cdot)\mathcal{L}_\varrho u &\in L^2(\mathbb{R}^+,\mathcal{H}) && \text{again by Theorem 1.4.9.} \end{aligned}$$

This shows that $M(\partial_t)u_1 = M(\partial_t)u_2$ on $(-\infty, 0]$.

Independence of ϱ: First, let $u \in C_c^\infty(\mathbb{R},\mathcal{H})$ with $\operatorname{supp} u \subseteq [0,\infty)$. Then we have $u \in L^2_\varrho(\mathbb{R},\mathcal{H})$ and $\mathcal{L}_\varrho u \in H_2(\mathbb{C}_{\mathrm{Re}>\varrho},\mathcal{H})$ for all $\varrho > \varrho_0$. Defining (formally) the translations

$$\tau_\nu f(\xi) := f(\xi - i\nu) \qquad (\nu, \xi \in \mathbb{R}),$$

we obtain for $\varrho_1, \varrho_2 > \varrho_0$,

$$\begin{aligned}\mathcal{L}_{\varrho_1}^* M(\varrho_1 + i\,\cdot\,)\mathcal{L}_{\varrho_1} u &= \mathcal{L}_{\varrho_1}^* M(\varrho_2 + (\varrho_1 - \varrho_2) + i\,\cdot\,)\mathcal{L}_{\varrho_1} u \\ &= \mathcal{L}_{\varrho_1}^* \tau_{\varrho_1 - \varrho_2} M(\varrho_2 + i\,\cdot\,)\tau_{\varrho_2 - \varrho_1}\mathcal{L}_{\varrho_1} u \\ &= \mathcal{L}_{\varrho_2}^* M(\varrho_2 + i\,\cdot\,)\mathcal{L}_{\varrho_2} u,\end{aligned}$$

which follows from manipulation of the integral representation of $\mathcal{L}_{\varrho_j} u$ ($j \in \{1,2\}$), i.e., all expressions are well-defined pointwise. By substituting $\tilde{u} = u(\,\cdot\, + t_0)$, the identity extends to $\operatorname{supp} u \subseteq [t_0, \infty)$, and finally to $u \in L^2_{\varrho_j}(\mathbb{R}, \mathcal{H})$, $j \in \{1,2\}$, by density[2] of $C_c^\infty(\mathbb{R}, \mathcal{H})$ in $L^2_\varrho(\mathbb{R}, \mathcal{H})$. This concludes the proof. □

Solution theory for linear evolutionary equations

A linear evolutionary equation is an operator equation of the form

$$(\partial_t M(\partial_t) + \mathcal{A})u = g \in L^2_\varrho(\mathbb{R}, \mathcal{H}) \tag{1.4.2}$$

with a given inhomogeneity g, where M is a linear material law and $\mathcal{A}\colon \operatorname{dom}(\mathcal{A}) \subseteq \mathcal{H} \to \mathcal{H}$ is a densely defined operator, which is extended pointwise to $L^2_\varrho(\mathbb{R}, \operatorname{dom}(\mathcal{A}))$ via $(\mathcal{A}u)(t) = \mathcal{A}(u(t))$. The sum $\partial_t M(\partial_t) + \mathcal{A}$ is defined on $H^1_\varrho(\mathbb{R}, \mathcal{H}) \cap L^2_\varrho(\mathbb{R}, \operatorname{dom}(\mathcal{A}))$ for $\varrho > \varrho_0$. The idea underlying the solution theory for such equations is to establish the inverse of $\partial_t M(\partial_t) + \mathcal{A}$ in a suitable sense. To this end, recall the discussion about accretive operators in Section 1.1.

Theorem 1.4.11 (Picard's Theorem). *Let $\mathcal{A}\colon \operatorname{dom}(\mathcal{A}) \subseteq \mathcal{H} \to \mathcal{H}$ be skew-selfadjoint and M a linear material law. Let $\varrho_0 \in \mathbb{R}$ be such that on the half plane $\mathbb{C}_{\mathrm{Re} > \varrho_0}$, M is uniformly bounded and $z \mapsto zM(z)$ is uniformly strictly accretive, i.e.,*

$$\exists c > 0\ \forall z \in \mathbb{C}_{\mathrm{Re}>\varrho_0}: \quad \operatorname{Re} zM(z) \geq c. \tag{1.4.3}$$

Then for all $\varrho > \varrho_0$ the operator $\partial_t M(\partial_t) + \mathcal{A}$ is closable and

$$S_\varrho := \left(\overline{\partial_t M(\partial_t) + \mathcal{A}}\right)^{-1} \colon L^2_\varrho(\mathbb{R}, \mathcal{H}) \to L^2_\varrho(\mathbb{R}, \mathcal{H})$$

is well-defined and bounded, with $\|S_\varrho\|_{L^2_\varrho \to L^2_\varrho} \leq c^{-1}$. Moreover, S_ϱ is causal and for all $\varrho, \varrho' > \varrho_0$ the following implications hold:

(i) If $g \in L^2_\varrho(\mathbb{R}, \mathcal{H}) \cap L^2_{\varrho'}(\mathbb{R}, \mathcal{H})$, then $S_\varrho g = S_{\varrho'} g \in L^2_\varrho(\mathbb{R}, \mathcal{H}) \cap L^2_{\varrho'}(\mathbb{R}, \mathcal{H})$.

(ii) If $g \in H^1_\varrho(\mathbb{R}, \mathcal{H})$, then $S_\varrho g = (\partial_t M(\partial_t) + \mathcal{A})^{-1} g \in H^1_\varrho(\mathbb{R}, \mathcal{H}) \cap L^2_\varrho(\mathbb{R}, \operatorname{dom}(\mathcal{A}))$.

Remark 1.4.12. We comment briefly on the proof of Picard's theorem (cf. [STW22, Theorem 6.2.1]). The strategy consists in establishing the operator $S(z) := (zM(z) + \mathcal{A})^{-1}$ as a linear material law on $\mathbb{C}_{\mathrm{Re}>\varrho_0}$; it then follows from Lemma 1.4.10 that

$$S(\partial_t) = S_\varrho = \mathcal{L}_\varrho^* \overline{((\varrho + i\mathrm{m})M(\varrho + i\mathrm{m}) + \mathcal{A})}^{-1} \mathcal{L}_\varrho = \left(\overline{\partial_t M(\partial_t) + \mathcal{A}}\right)^{-1}$$

[2] It follows by standard smoothing and cutoff arguments that for each element $u \in L^2_{\varrho_1}(\mathbb{R}, \mathcal{H}) \cap L^2_{\varrho_2}(\mathbb{R}, \mathcal{H})$ there exists a sequence (u_n) in $C_c^\infty(\mathbb{R}, \mathcal{H})$ such that simultaneously $u_n \to u$ in $L^2_{\varrho_1}$ and in $L^2_{\varrho_2}$. Alternatively, one can perform the approximation over the set of simple functions with compact support, cf. [STW22, Lemma 4.2.1].

is bounded and causal on $L^2_\varrho(\mathbb{R},\mathcal{H})$ for $\varrho > \varrho_0$, and independent of ϱ. Here the uniform boundedness of S follows from the skew-selfadjointness of $\mathcal{A}$ and strict accretivity of $zM(z)$ for $\operatorname{Re} z > \varrho_0$. Together with the holomorphy of M, this also shows that S is holomorphic. Moreover, this method makes it clear that the result can be generalized to m-accretive operators $\mathcal{A}$. ♢

Remark 1.4.13. To be more exact, Theorem 1.4.11 provides a sufficient condition for the spectral operator

$$\tilde{S}\colon \operatorname{dom}(M) \cap \mathbb{C}_{\operatorname{Re}>\varrho_0} \to \mathcal{B}(\mathcal{H}), \quad z \mapsto (zM(z) + \mathcal{A})^{-1}$$

to possess an analytic and bounded extension to $\mathbb{C}_{\operatorname{Re}>\varrho_0}$. If such an extension exists for some ϱ_0, we say that problem (1.4.2) is *well-posed in* $\bigcup_{\varrho>\varrho_0} L^2_\varrho(\mathbb{R},\mathcal{H})$, or simply *well-posed*[3]. ♢

In general, the operator sum $\partial_t M(\partial_t) + \mathcal{A}$ is not closed, even if $\partial_t M(\partial_t)$ and $\mathcal{A}$ are closed. For $g \in L^2_\varrho(\mathbb{R},\mathcal{H})$, the solution u of (1.4.2) yielded by Theorem 1.4.11 thus satisfies

$$(\overline{\partial_t M(\partial_t) + \mathcal{A}})u = g.$$

By (ii), however, it is seen that the closure can be omitted if $g \in H^1_\varrho(\mathbb{R},\mathcal{H})$, in which case

$$(\partial_t M(\partial_t) + \mathcal{A})u = \partial_t M(\partial_t)u + \mathcal{A}u = g$$

holds in $L^2_\varrho(\mathbb{R},\mathcal{H})$ with $u \in H^1_\varrho(\mathbb{R},\mathcal{H}) \cap \operatorname{dom}(\mathcal{A})$, and moreover, u is also continuous by the Sobolev embedding, Proposition 1.4.5. Since $S_\varrho = S(\partial_t)$ is analytic, ∂_t commutes with S_ϱ and we have $\partial_t u = S_\varrho \partial_t g$. In general, $S_\varrho\colon H^k_\varrho(\mathbb{R},\mathcal{H}) \to H^k_\varrho(\mathbb{R},\mathcal{H})$ is also bounded for $k \in \mathbb{Z}$, where $H^{-k}(\mathbb{R},\mathcal{H})$ denotes the dual space of $H^k(\mathbb{R},\mathcal{H})$. Thus, the issue of time-regularity is already built into the solution theory provided by Theorem 1.4.11. For later reference, we summarize this fact with the following result, see [PM11, Section 3.1] and [STW22, Section 6.3].

Proposition 1.4.14. *Let $(\partial_t M(\partial_t) + \mathcal{A})\, u = g$ be well-posed in $\bigcup_{\varrho>\varrho_0} L^2_\varrho(\mathbb{R},\mathcal{H})$ with $\varrho_0 \in \mathbb{R}$. If $\varrho > \varrho_0$ and $g \in H^1_\varrho(\mathbb{R},\mathcal{H})$, then $u = \left(\overline{\partial_t M(\partial_t) + \mathcal{A}}\right)^{-1} g \in H^1_\varrho(\mathbb{R},\mathcal{H})$, with continuous dependence on the data:*

$$\|u\|_{L^2_\varrho} \lesssim \|g\|_{L^2_\varrho}, \quad \|\partial_t u\|_{L^2_\varrho} \lesssim \|\partial_t g\|_{L^2_\varrho}.$$

In fact, $\partial_t u = \left(\overline{\partial_t M(\partial_t) + \mathcal{A}}\right)^{-1} \partial_t g$. Moreover, $u \in C_\varrho(\mathbb{R},\mathcal{H})$ by the Sobolev embedding theorem.

The framework reviewed in this section provides a unified solution theory for a large class of equations in mathematical physics. These include the classical equations (namely those with trivial material laws $M(\partial_t) = \mathrm{id}$), such as the heat and wave equation, as well as integro-differential equations and equations with delay, see for example Section 6.2 and Chapter 7 in [STW22].

[3] This is the definition of well-posedness in [Tro18].

2 Nonlinear evolutionary equations

In this chapter we turn our attention to various equations of the form

$$(\partial_t M(\partial_t) + \mathcal{A})u = f(u), \tag{2.0.1}$$

where f is now a nonlinear function, defined on some subspace of $L^2_{\mathrm{loc}}(\mathbb{R}, \mathcal{H})$. We assume that $M(\partial_t)$ and $\mathcal{A}$ satisfy the conditions of Picard's theorem 1.4.11, i.e., $\mathcal{A}$ is skew-selfadjoint in $\mathcal{H}$ and $M(\partial_t)$ is a linear material law satisfying

$$\operatorname{Re} z > \varrho_0 \implies \operatorname{Re} zM(z) \geq c > 0$$

for some $\varrho_0 \in \mathbb{R}$, $c > 0$. Thus, the linear equation induced by $\partial_t M(\partial_t) + \mathcal{A}$ is well-posed, i.e., the operator

$$S_\varrho = \left(\overline{\partial_t M(\partial_t) + \mathcal{A}}\right)^{-1} : L^2_\varrho(\mathbb{R}, \mathcal{H}) \to L^2_\varrho(\mathbb{R}, \mathcal{H})$$

is uniformly bounded with $\|S_\varrho\|_{L^2_\varrho \to L^2_\varrho} \leq c^{-1}$ and causal for all $\varrho > \varrho_0$. Hence we can reformulate (2.0.1) as a fixed-point equation,

$$u = S_\varrho f(u). \tag{2.0.2}$$

Of course, the latter expression is only meaningful if $\operatorname{dom}(f) \cap L^2_\varrho(\mathbb{R}, \mathcal{H}) \neq \varnothing \neq \operatorname{ran}(f) \cap L^2_\varrho(\mathbb{R}, \mathcal{H})$.

2.1 Lipschitz and locally Lipschitz nonlinearities

The case of uniformly Lipschitz continuous f is one of the simplest, and will serve as a basis for dealing with other settings.

Definition 2.1.1. A (continuous) map $f\colon \operatorname{dom}(f) \subseteq L^2_{\mathrm{loc}}(\mathbb{R}, \mathcal{H}) \to L^2_{\mathrm{loc}}(\mathbb{R}, \mathcal{H})$ is called *uniformly Lipschitz continuous in* $\bigcup_{\varrho > \varrho_0} L^2_\varrho(\mathbb{R}, \mathcal{H})$, if f maps each $L^2_\varrho(\mathbb{R}, \mathcal{H})$ into itself and satisfies

$$\forall u, v \in L^2_\varrho(\mathbb{R}, \mathcal{H}) : \quad \|f(u) - f(v)\|_{L^2_\varrho} \leq L\|u - v\|_{L^2_\varrho} \tag{2.1.1}$$

for all $\varrho > \varrho_0$, with L independent of ϱ.

Remark 2.1.2. In the more 'interior' definition of uniform Lipschitz continuity in [STW22], the map f is defined a priori on the (dense) subset of simple functions with compact support in $L^2_{\mathrm{loc}}(\mathbb{R}, \mathcal{H})$, and then uniquely extended to each $L^2_\varrho(\mathbb{R}, \mathcal{H})$. Defined in this way, causality is in fact a consequence of uniform Lipschitz continuity, see [STW22, Theorem 4.2.5].

We will ignore this technicality (also because we want to consider maps that are not uniformly Lipschitz continuous) and instead always assume causality as a prerequisite, for instance by considering Volterra-type operators which are causal by definition. $\Diamond$

Proposition 2.1.3. *Let $f\colon \operatorname{dom}(f) \subseteq L^2_{\mathrm{loc}}(\mathbb{R},\mathcal{H}) \to L^2_{\mathrm{loc}}(\mathbb{R},\mathcal{H})$ be a causal map and let $\varrho_0 \in \mathbb{R}$ be such that*

1. *f is uniformly Lipschitz continuous on $\bigcup_{\varrho>\varrho_0} L^2_\varrho(\mathbb{R},\mathcal{H})$, and*

2. *$\operatorname{Re} z \geq \varrho > \varrho_0 \implies \operatorname{Re} zM(z) \geq c\varrho$ for some $c > 0$.*

Then, there exists $\varrho_1 > 0$ such that the equation (2.0.2) admits a unique solution in $L^2_\varrho(\mathbb{R},\mathcal{H})$ for all $\varrho \geq \varrho_1$, which is independent of ϱ.

Proof. The second condition implies the that $\|S_\varrho\|_{L^2_\varrho\to L^2_\varrho} \leq (c\varrho)^{-1}$ by Theorem 1.4.11. Together with the uniform Lipschitz continuity of f we have

$$\|S_\varrho f(u) - S_\varrho f(v)\|_{L^2_\varrho} = \|S_\varrho(f(u) - f(v))\|_{L^2_\varrho} \leq \frac{1}{c\varrho}\|f(u) - f(v)\|_{L^2_\varrho} \leq \frac{L}{c\varrho}\|u - v\|_{L^2_\varrho}.$$

Hence, $S_\varrho \circ f$ is Lipschitz continuous, with Lipschitz constant $L/(c\varrho) < 1$ for large $\varrho > 0$. This shows that $S_\varrho \circ f$ becomes a contraction on $L^2_\varrho(\mathbb{R},\mathcal{H})$ for large ϱ, and consequently, by the Banach fixed-point theorem, (2.0.2) has a unique solution. Since $S_\varrho u = S_{\varrho'} u$ for $\varrho, \varrho' > \varrho_0$, this solution is independent of ϱ. □

Remark 2.1.4. If the Lipschitz constant

$$L = L(\varrho) = \inf_{u\neq v} \frac{\|f(u) - f(v)\|_{L^2_\varrho}}{\|u - v\|_{L^2_\varrho}}$$

of f is allowed to vary with ϱ, the result can be generalized in two ways. First, notice that the contraction argument remains valid if

$$\limsup_{\varrho\to\infty} \frac{L(\varrho)}{\varrho} < 1.$$

Second, if instead we have

$$\limsup_{\varrho\to\infty} L(\varrho) = o(1),$$

then the norm estimate on S_ϱ can be relaxed by allowing

$$\sup_{\xi\in\mathbb{R}} \|(\varrho + i\xi + \mathcal{A})^{-1}\|_{\mathcal{H}\to\mathcal{H}} = O(1), \quad \text{as } \varrho \to \infty.$$

An example of such f is given in [MP02] for a time-shift operator. ◊

Example 2.1.5. For Maxwell's equations we will be mainly interested in the case $\mathcal{H} = L^2(\Omega)^n$ for some domain $\Omega \subseteq \mathbb{R}^3$. Two relevant examples of uniformly Lipschitz continuous mappings $f\colon L^2_\varrho(\mathbb{R},\mathcal{H}) \to L^2_\varrho(\mathbb{R},\mathcal{H})$ are given in the following.

(a) (Instantaneous saturable nonlinearities) Let $\eta\colon \mathbb{R} \to \mathbb{R}$ be continuously differentiable and bounded, with bounded derivative such that $|\eta'(r)| = O(r^{-1})$ for $r \to \infty$ (for instance $\eta(r) = r^p/(1 + r^s)$, $0 \leq p \leq s$), and consider $Q\colon \mathbb{R}^3 \to \mathbb{R}^n$ given by

$$Q(v) = \eta(|v|)v.$$

Then Q is pointwise Lipschitz continuous, since for $v, w \in \mathbb{R}^3$ (assuming $|v| \geq |w|$ and using the mean-value inequality),

$$\begin{aligned}
|\eta(|v|)v - \eta(|w|)w| &\leq |\eta(|v|)v - \eta(|v|)w| + |\eta(|v|)w - \eta(|w|)w| \\
&\leq \sup_r |\eta(r)|\, |v-w| + |\eta(|v|) - \eta(|w|)|\, |w| \\
&\leq \sup_r |\eta(r)|\, |v-w| + \big(\sup_{r \geq |w|} |\eta'(r)|\, |w|\big)\, ||v| - |w|| \\
&\leq \big(\sup_r |\eta(r)| + \sup_{r \geq |w|} |\eta'(r)|\, r\big)\, |v-w| \\
&\leq C\, |v-w|\,.
\end{aligned}$$

The pointwise extension of Q (first to $\mathcal{H}$, and then) to $L^2_\varrho(\mathbb{R}, \mathcal{H})$ is uniformly Lipschitz continuous for all $\varrho \in \mathbb{R}$.

(b) (Saturable Volterra operator) A nonlocal version of the above is given by the Volterra-type operator

$$f(u)(t) = g(t) + \int_{\mathbb{R}} \kappa(t-\tau) Q(u(\tau))\, \mathrm{d}\tau,$$

where $\kappa \in L^1_{\varrho_\kappa}(\mathbb{R}, \mathcal{B}(\mathcal{H}))$ with $\varrho_\kappa \in \mathbb{R}$ and $\operatorname{supp} \kappa \subseteq [0, \infty)$, $g \in L^2_{\mathrm{loc}}(\mathbb{R}, \mathcal{H})$, and $Q \colon \mathcal{H} \to \mathcal{H}$ is Lipschitz continuous. Taking $\varrho > \varrho_\kappa$ we compute for $u, v \in L^2_\varrho(\mathbb{R}, \mathcal{H})$ (cf. [MP02])

$$\begin{aligned}
&\|f(u) - f(v)\|^2_{L^2_\varrho} \\
&\leq \int_{\mathbb{R}} \|\kappa(t-s)\,(Q(u(s)) - Q(v(s)))\, \mathrm{d}s\|^2_{\mathcal{H}}\, e^{-2\varrho t}\, \mathrm{d}t \\
&\leq |Q|^2_{\mathrm{Lip}} \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} \|\kappa(t-s)\|_{\mathcal{B}(\mathcal{H})}\, \|u(s) - v(s)\|_{\mathcal{H}}\, \mathrm{d}s\Big)^2 e^{-2\varrho t}\, \mathrm{d}t \\
&\overset{(\star)}{\leq} |Q|^2_{\mathrm{Lip}}\, \|\kappa\|_{L^1_{\varrho_\kappa}} \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} \|\kappa(t-s)\|_{\mathcal{B}(\mathcal{H})}\, \|u(s) - v(s)\|^2_{\mathcal{H}}\, e^{\varrho_\kappa (t-s)}\, \mathrm{d}s\Big) e^{-2\varrho t}\, \mathrm{d}t \\
&= |Q|^2_{\mathrm{Lip}}\, \|\kappa\|_{L^1_{\varrho_\kappa}} \int_{\mathbb{R}} \int_{\mathbb{R}} \|\kappa(t-s)\|_{\mathcal{B}(\mathcal{H})}\, e^{-\varrho_\kappa (t-s)}\, \underbrace{e^{-2(\varrho - \varrho_\kappa)(t-s)}}_{\leq 1 \text{ for } t-s \geq 0}\, \mathrm{d}t\, \|u(s) - v(s)\|^2_{\mathcal{H}}\, e^{-2\varrho s}\, \mathrm{d}s \\
&\leq |Q|^2_{\mathrm{Lip}}\, \|\kappa\|_{L^1_{\varrho_\kappa}} \int_{\mathbb{R}} \|\kappa(r)\|_{\mathcal{B}(\mathcal{H})}\, e^{-\varrho_\kappa r}\, \mathrm{d}r \int_{\mathbb{R}} \|u(s) - v(s)\|^2_{\mathcal{H}}\, e^{-2\varrho s}\, \mathrm{d}s \\
&= |Q|^2_{\mathrm{Lip}}\, \|\kappa\|^2_{L^1_{\varrho_\kappa}}\, \|u - v\|^2_{L^2_\varrho}
\end{aligned}$$

using Tonelli's theorem, where $(\star)$ follows from the Cauchy–Schwarz inequality after writing $\|\kappa(t-s)\| = \|\kappa(t-s)\|^{\frac{1}{2}+\frac{1}{2}} e^{-\varrho_\kappa (t-s)(\frac{1}{2}-\frac{1}{2})}$. For fixed $g \in \bigcap_{\varrho > \varrho_0} L^2_\varrho(\mathbb{R}, \mathcal{H})$, this shows the uniform Lipschitz continuity of f in $\bigcup_{\varrho > \max\{\varrho_\kappa, \varrho_0\}} L^2_\varrho(\mathbb{R}, \mathcal{H})$. ▲

A refinement of the fixed-point argument in Proposition 2.1.3 yields well-posedness for nonlinearities that satisfy only a local Lipschitz estimate. We remark that, unlike the linear solution operator S_ϱ, for which uniform boundedness in $\bigcup_{\varrho > \varrho_0} L^2_\varrho(\mathbb{R}, \mathcal{H})$ is a necessary consequence of causality, the nonlinear map need not satisfy the Lipschitz estimate for all $\varrho > \varrho_0$, if one is interested in solving the nonlinear equation only in *some* $L^2_\varrho(\mathbb{R}, \mathcal{H})$.

Proposition 2.1.6. *Let $A \colon \operatorname{dom}(A) \subset \mathcal{H} \to \mathcal{H}$ be a densely defined and closed operator and M a linear material law, such that the linear equation $(\partial_t M(\partial_t) + A)u = g$ is well-posed in $\bigcup_{\varrho > \varrho_0} L^2_\varrho(\mathbb{R}, \mathcal{H})$ and $\|(\overline{\partial_t M(\partial_t) + A})^{-1}\|_{L^2_\varrho \to L^2_\varrho} \leq c^{-1}$ with $c > 0$. Let $\varrho > \varrho_0$ and*

$f\colon L^2_\varrho(\mathbb{R},\mathcal{H}) \to L^2_\varrho(\mathbb{R},\mathcal{H})$ be a nonlinear map satisfying $f(0) = 0$ and, for some $d, \alpha > 0$,

$$\|f(u) - f(v)\|_{L^2_\varrho} \le d(\|u\|_{L^2_\varrho} + \|v\|_{L^2_\varrho})^\alpha \|u - v\|_{L^2_\varrho}$$

for all $u, v \in L^2_\varrho(\mathbb{R},\mathcal{H})$. Then, for all $g \in L^2_\varrho(\mathbb{R},\mathcal{H})$ with $\|g\|_{L^2_\varrho} < \frac{d}{2}\left(\frac{c}{d}\right)^{1+\frac{1}{\alpha}}(1 - 2^{-\alpha})$, the nonlinear equation $(\partial_t M(\partial_t) + A)u = f(u) + g$ admits a unique solution $u \in L^2_\varrho(\mathbb{R},\mathcal{H})$ with $\|u\|_{L^2_\varrho} \le \frac{1}{2}\left(\frac{c}{d}\right)^{1/\alpha}$.

Proof. Denote by $T_\varrho\colon L^2_\varrho(\mathbb{R},\mathcal{H}) \to L^2_\varrho(\mathbb{R},\mathcal{H})$ the operator given by

$$T_\varrho(u) = \left(\overline{\partial_t M(\partial_t) + A}\right)^{-1}(f(u) + g).$$

Then, on an arbitrary closed ball $B_r := \{u \in L^2_\varrho(\mathbb{R},\mathcal{H}) : \|u\|_{L^2_\varrho} \le r\}$ the Lipschitz constant of T_ϱ can be estimated by

$$L_{\varrho,r} := \sup_{u,v \in B_r, u \ne v} \frac{\|T_\varrho(u) - T_\varrho(v)\|_{L^2_\varrho}}{\|u - v\|_{L^2_\varrho}} \le \frac{d}{c}(2r)^\alpha,$$

thus $L_{\varrho,r} < 1$ if $r < \frac{1}{2}\left(\frac{c}{d}\right)^{1/\alpha}$. Now to have $T_\varrho(u) \in B_r$ if $u \in B_r$, we demand that

$$\|T_\varrho(u)\|_{L^2_\varrho} \le \frac{1}{c}(d\|u\|^{\alpha+1}_{L^2_\varrho} + \|g\|_{L^2_\varrho}) \le \frac{1}{c}(dr^{\alpha+1} + \|g\|_{L^2_\varrho}) \overset{!}{\le} r.$$

Replacing r with $\frac{1}{2}\left(\frac{c}{d}\right)^{1/\alpha}$ in the last inequality leads to the condition

$$\frac{\|g\|_{L^2_\varrho}}{d} \overset{!}{<} \frac{1}{2}\frac{c}{d}\left(\frac{c}{d}\right)^{\frac{1}{\alpha}} - \left(\frac{1}{2}\left(\frac{c}{d}\right)^{\frac{1}{\alpha}}\right)^{\alpha+1} = \frac{1}{2}\left(\frac{c}{d}\right)^{1+\frac{1}{\alpha}}(1 - 2^{-\alpha}),$$

which, if fulfilled, establishes T_ϱ as a contraction on B_r for some $r < \frac{1}{2}\left(\frac{c}{d}\right)^{1/\alpha}$. The Banach fixed-point theorem gives the conclusion. □

2.2 Small solutions of a cubic Ginzburg–Landau equation

We will apply Proposition 2.1.6 in the following to treat an evolutionary problem without memory. A particular instance of this problem appears in Chapter 4 as an amplitude equation. Consider the evolutionary problem

$$\partial_t u + \sigma u + \mathcal{D}u = \gamma|u|^2 u + g, \tag{2.2.1}$$

where $\sigma \in \mathbb{R}^+$, $\gamma \in \mathbb{C}$, and $\mathcal{D}\colon \operatorname{dom}(\mathcal{D}) \subset H^k(\mathbb{R}^d) \to H^k(\mathbb{R}^d)$, with $k > d/2$ fixed, is an m-accretive operator. We demonstrate how to obtain solutions $u\colon \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ of (2.2.1) subject to some smoothness and decay. Substituting $v(t) = u(t)e^{\sigma t}$ transforms the equation into

$$(\partial_t + \mathcal{D})v = \gamma e^{-2\sigma t}|v|^2 v + g =: f(v) + g. \tag{2.2.2}$$

We will show that this evolutionary equation is well-posed in the space

$$\mathscr{H}_\varrho := \{u \in H^1_\varrho(\mathbb{R}, H^k(\mathbb{R}^d)) : u = 0 \text{ in } (-\infty, 0]\}, \quad \|\cdot\|_{\mathscr{H}_\varrho} := \|\cdot\|_{H^1_\varrho(\mathbb{R}, H^k(\mathbb{R}^d))} \tag{2.2.3}$$

for $\varrho \in (0, \sigma]$. To this aim, first observe from the general solution theory, Theorem 1.4.11, and time-regularity, Proposition 1.4.14, that for all $\varrho_0 > 0$ the linear equation $(\partial_t + \mathcal{D})u = g$ is well-posed in $\bigcup_{\varrho > \varrho_0} H^1_\varrho(\mathbb{R}, H^k(\mathbb{R}^d))$, with

$$\|(\partial_t + \mathcal{D})^{-1}\|_{H^1_\varrho \to H^1_\varrho} \leq \frac{1}{\varrho}.$$

Moreover, due to causality, $(\partial_t + \mathcal{D})^{-1}$ leaves $\mathscr{H}_\varrho$ invariant for all $\varrho > 0$. For the nonlinear equation we employ again a fixed-point argument. Note that since $k > d/2$, the space $H^k(\mathbb{R}^d)$ is a multiplication algebra, i.e., in particular

$$\|u \cdot v\|_{H^k} \leq a_k \|u\|_{H^k} \|v\|_{H^k} \qquad (u, v \in H^k(\mathbb{R}^d)) \tag{2.2.4}$$

for some $a_k > 0$, see [AF03, Theorem 4.39].

Lemma 2.2.1. *For all $\varrho \in (0, \sigma]$ there exists $d_\varrho > 0$ such that the map $f \colon \mathscr{H}_\varrho \to \mathscr{H}_\varrho$, $f(u)(t) = \gamma e^{-2\sigma t}|u(t)|^2 u(t)$, fulfills*

$$\|f(u) - f(v)\|_{\mathscr{H}_\varrho} \leq d_\varrho \big(\|u\|_{\mathscr{H}_\varrho} + \|u\|_{\mathscr{H}_\varrho}\big)^2 \|u - v\|_{\mathscr{H}_\varrho} \tag{2.2.5}$$

for all $u, v \in \mathscr{H}_\varrho$.

Proof. The value of γ plays no role in the following argument; we set $\gamma = 1$. We will use the notation

$$\begin{aligned}
\|\cdot\|_k &= \|\cdot\|_{H^k} \\
\|u\|_{L^2_\varrho} &= \|u\|_{L^2_\varrho(\mathbb{R}, H^k(\mathbb{R}^d))} \\
\|u\|_{C_\varrho} &= \sup_t e^{-\varrho t} \|u(t)\|_{H^k}.
\end{aligned}$$

Now let $\varrho \in (0, \sigma]$. For the map N defined by

$$N(u, v, w)(t) := e^{-2\sigma t} u(t) v(t) w(t)$$

we show that $\|N(u, v, w)\|_{\mathscr{H}_\varrho} \lesssim_{k,\sigma,\varrho} \|u\|_{\mathscr{H}_\varrho} \|v\|_{\mathscr{H}_\varrho} \|w\|_{\mathscr{H}_\varrho}$. Recalling the Sobolev inequality

$$\|u\|_{C_\varrho} \leq \frac{1}{\sqrt{2\varrho}} \|u\|_{H^1_\varrho} \qquad (\varrho > 0),$$

together with (2.2.4) we compute

$$\begin{aligned}
\|N(u, v, w)\|^2_{L^2_\varrho} &= \int_0^\infty \|u(t)v(t)w(t)\|_k^2 e^{-2(2\sigma+\varrho)t} \, \mathrm{d}t \\
&\leq (a_k^2)^2 \int_0^\infty \|u(t)\|_k^2 \, \|v(t)\|_k^2 \, \|w(t)\|_k^2 \, e^{-2(2\sigma+\varrho)t} \, \mathrm{d}t \\
&\leq a_k^4 \, \|u\|^2_{C_\varrho} \, \|v\|^2_{C_\varrho} \int_0^\infty \|w(t)\|_k^2 \, e^{-2(2\sigma-\varrho)t} \, \mathrm{d}t \\
&\leq \frac{a_k^4}{(2\varrho)^2} \, \|u\|^2_{\mathscr{H}_\varrho} \, \|v\|^2_{C_\varrho} \int_0^\infty \|w(t)\|_k^2 \, \underbrace{e^{-2(2\sigma-\varrho)t}}_{\leq e^{-2\varrho t}} \, \mathrm{d}t \\
&\leq \frac{a_k^4}{(2\varrho)^2} \, \|u\|^2_{\mathscr{H}_\varrho} \, \|v\|^2_{\mathscr{H}_\varrho} \, \|w\|^2_{L^2_\varrho}
\end{aligned}$$

$$\leq \frac{a_k^4}{(2\varrho)^2} \|u\|_{\mathscr{H}_\varrho}^2 \|v\|_{\mathscr{H}_\varrho}^2 \|w\|_{\mathscr{H}_\varrho}^2 ,$$

and similarly for the time-derivative, using $(\sum_{j=1}^n b_j)^2 \leq n \sum_{j=1}^n b_j^2$:

$$\begin{aligned}
&\|\partial_t N(u,v,w)\|_{L_\varrho^2}^2 \\
&\quad = \int_0^\infty \|u'(t)v(t)w(t) + u(t)v'(t)w(t) + u(t)v(t)w'(t) - 2\sigma u(t)v(t)w(t)\|_k^2 e^{-2(2\sigma+\varrho)t}\,\mathrm{d}t \\
&\quad \leq a_k^4 \int_0^\infty \Big(\|u'(t)\|_k \|v(t)\|_k \|w(t)\|_k + \|u(t)\|_k \|v'(t)\|_k \|w(t)\|_k \\
&\qquad\qquad + \|u(t)\|_k \|v(t)\|_k \|w'(t)\|_k + 2\sigma \|u(t)v(t)w(t)\|_k^2 \Big)^2 e^{-2(2\sigma+\varrho)t}\,\mathrm{d}t \\
&\quad \leq 3a_k^4 \Big(\|u'\|_{L_\varrho^2}^2 \|v\|_{C_\varrho}^2 \|w\|_{C_\varrho}^2 + \|u\|_{C_\varrho}^2 \|v'\|_{L_\varrho^2}^2 \|w\|_{C_\varrho}^2 \\
&\qquad\qquad + \|u\|_{C_\varrho}^2 \|v\|_{C_\varrho}^2 \|w'\|_{L_\varrho^2}^2 + 4\sigma^2 \|u\|_{C_\varrho}^2 \|v\|_{C_\varrho}^2 \|w\|_{L_\varrho^2}^2 \Big) \\
&\quad \leq \frac{3a_k^4(3+4\sigma^2)}{(2\varrho)^2} \|u\|_{\mathscr{H}_\varrho}^2 \|v\|_{\mathscr{H}_\varrho}^2 \|w\|_{\mathscr{H}_\varrho}^2 .
\end{aligned}$$

Thus,

$$\|N(u,v,w)\|_{\mathscr{H}_\varrho} \leq \frac{a_k^2\sqrt{10+12\sigma^2}}{2\varrho} \|u\|_{\mathscr{H}_\varrho} \|v\|_{\mathscr{H}_\varrho} \|w\|_{\mathscr{H}_\varrho} . \tag{2.2.6}$$

Finally,

$$\begin{aligned}
\|f(u) - f(v)\|_{\mathscr{H}_\varrho} &\leq \big\| (|u|^2 u - |v|^2 v) e^{-2\sigma\cdot} \big\|_{\mathscr{H}_\varrho} \\
&\leq \big\| (|u|^2 (u-v) + (|u|^2 - |v|^2) v) e^{-2\sigma\cdot} \big\|_{\mathscr{H}_\varrho} \\
&\leq \big\| (|u|^2 |u-v| + (|u| + |v|) |v| \, ||u| - |v||) e^{-2\sigma\cdot} \big\|_{\mathscr{H}_\varrho} \\
&\leq \big\| (|u| + |v|)^2 |u-v| \, e^{-2\sigma\cdot} \big\|_{\mathscr{H}_\varrho} \\
&= \| N(|u|+|v|, |u|+|v|, |u-v|) \|_{\mathscr{H}_\varrho} \\
&\leq \frac{a_k^2\sqrt{10+4\sigma^2}}{2\varrho} \left(\|u\|_{\mathscr{H}_\varrho} + \|u\|_{\mathscr{H}_\varrho} \right)^2 \|u-v\|_{\mathscr{H}_\varrho} ,
\end{aligned}$$

where the last estimate follows from (2.2.6). We obtain (2.2.5) with $d_\varrho = \frac{a_k^2\sqrt{10+4\sigma^2}}{2\varrho}$. ☐

Theorem 2.2.2. *For all $\varrho \in (0,\sigma]$ there exists $c_\varrho > 0$ such that if $\|g\|_{\mathscr{H}_\varrho} \leq c_\varrho$, then (2.2.2) admits a unique solution $v \in \mathscr{H}_\varrho = \{v \in H_\varrho^1(\mathbb{R}, H^k(\mathbb{R}^d)) : v = 0 \text{ in } (-\infty, 0]\}$. Thus, $u \in H_{\varrho-\sigma}^1(\mathbb{R}, H^k(\mathbb{R}^d))$, where $u(t) = v(t)e^{-\sigma t}$.*

Proof. By the observations made earlier, $(\partial_t + \mathcal{D})^{-1} \in \mathcal{B}(\mathscr{H}_\varrho)$ with $\|(\partial_t + \mathcal{D})^{-1}\|_{\mathscr{H}_\varrho} \leq 1/\varrho$. Since f satisfies the conditions of Lemma 2.2.1, the fixed-point argument underlying the proof of Proposition 2.1.6 is valid in $\mathscr{H}_\varrho$ with $c = \varrho$, $d = \gamma d_\varrho$, $\alpha = 2$. Hence, we only need to take

$$\|g\|_{\mathscr{H}_\varrho} < \frac{\gamma d_\varrho}{2} \Big(\frac{\varrho}{\gamma d_\varrho} \Big)^{1+\frac{1}{2}} (1 - 2^{-2})$$

small enough to have a unique solution $v \in \mathscr{H}_\varrho$ with $\|v\|_{\mathscr{H}_\varrho} \leq \frac{1}{2} \big(\frac{\varrho}{\gamma d_\varrho} \big)^{1/2}$. ☐

Remark 2.2.3. In view of the application in Chapter 4, the result of Theorem 2.2.2 can be applied to the operator $\mathcal{D} = -\alpha\partial_{x_1}^2$ with maximal domain $\mathrm{dom}(\mathcal{D})$ in $H^k(\mathbb{R}^d)$, where $\alpha \in \mathbb{C}_{\mathrm{Re}>0}$. As such, $\mathcal{D}$ is densely defined and closed. Recall from Example 1.1.3 that $\mathcal{D}$ $(= \mathcal{D}_a$, where $a = \mathrm{diag}\,(\alpha, 0, \dots, 0))$ is m-accretive in $H^k(\mathbb{R}^d)$. $\Diamond$

The results in this section are not intended as an optimal strategy for analysis of Equation (2.2.1). For instance, much more is known about one its most prominent variants, the nonlinear Ginzburg–Landau equation

$$\partial_t u + \sigma u - \alpha\Delta u + \gamma|u|^{q-1}u = 0 \tag{CGL}$$

in $\mathbb{R} \times \mathbb{R}^d$, where $\sigma \in \mathbb{R}$, $\alpha, \gamma \in \mathbb{C}$; see [AK02, LO96] for an overview.

Local and global well-posedness of (CGL) was studied in [SYY16] (and the references therein), and the bound

$$\|u\|_{L^p} \le e^{-\sigma t}\|u_0\|_{L^p} \tag{2.2.7}$$

was proved for $\mathrm{Re}\,\alpha > 0$ and $\gamma \in \mathbb{R}^-$ for a range of parameters p, q, d, for arbitrary initial values $u_0 \in L^p(\mathbb{R}^d)$ and $\sigma \in \mathbb{R}$. Similar estimates were obtained for $\gamma \ge 0$. The existence of special solutions such as traveling pulses is known for $d = 1$ in some cases (notably depending on the sign of γ), see [KS98, vH92]. The existence of such solutions and decay estimates is important for the amplitude formalism reviewed and applied in Chapter 4.

In contrast, the method for the proof of Theorem 2.2.2 cannot produce the decay estimate (2.2.7) due to the singularity of the Lipschitz constant for $\varrho \to 0$ (thus, the bound for $\|g\|_{\mathscr{H}_\varrho}$ vanishes as $\varrho \to 0$). In turn, the proof of the stability result is independent of the sign of γ, and also valid for any m-accretive operator $\mathcal{D}$ in $H^k(\mathbb{R}^d)$. To obtain finer-grained results, the method would need to be adapted to take these parameters into account.

2.3 Multilinear Volterra operators: local and global well-posedness

We now study nonlinearities that cannot be expected to fulfill the Lipschitz estimate in Proposition 2.1.6. This is particularly the case with multilinear operators $\mathcal{V}$ of Volterra-type,

$$\mathcal{V}(u)(t) = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \kappa(t-\tau_1, \dots, t-\tau_n)\, Q(u(\tau_1), \dots, u(\tau_n))\, \mathrm{d}\tau_1 \cdots \mathrm{d}\tau_n, \tag{2.3.1}$$

where $Q\colon (\mathcal{H})^n \to \mathcal{H}$ is a multilinear and bounded map. We use an estimate similar to (2.1.1) to derive the following mapping property.

Lemma 2.3.1. *Let $\varrho_\kappa \in \mathbb{R}$ and let $\kappa\colon \mathbb{R}^n \to \mathcal{B}(\mathcal{H})$ be causal, measurable and such that*

$$L_\kappa := \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \|\kappa(\tau_1, \dots, \tau_n)\|\, e^{-\varrho_\kappa(\tau_1+\ldots+\tau_n)}\, \mathrm{d}\tau_1 \cdots \mathrm{d}\tau_n \tag{2.3.2}$$

$$\ell_\kappa := \sup_{\tau_1,\dots,\tau_n \in \mathbb{R}} \int_{\mathbb{R}} \|\kappa(t-\tau_1, \dots, t-\tau_n)\|\, e^{-\varrho_\kappa(t-\tau_1)} \cdots e^{-\varrho_\kappa(t-\tau_n)}\, \mathrm{d}t \tag{2.3.3}$$

are finite. Let $Q\colon (\mathcal{H})^n \to \mathcal{H}$ be a multilinear bounded map. Then, for all $\varrho \ge \varrho_\kappa$ the nonlinear operator defined by (2.3.1) maps $L^2_\varrho(\mathbb{R}, \mathcal{H})$ continuously into $L^2_{n\varrho}(\mathbb{R}, \mathcal{H})$.

Proof. Let C_q denote the constant in $\|Q(v_1, \dots, v_n)\|_{\mathcal{H}} \le C_q\, \|v_1\|_{\mathcal{H}} \cdots \|v_n\|_{\mathcal{H}}$. We compute for $u \in L^2_\varrho(\mathbb{R}, \mathcal{H})$,

$$\begin{aligned}
&\int_{\mathbb{R}} \left\| \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \kappa(t-\tau_1, \ldots, t-\tau_n)\, Q(u(\tau_1), \ldots, u(\tau_n)) \prod_{j=1}^n \mathrm{d}\tau_j \right\|_{\mathcal{H}}^2 e^{-2n\varrho t}\, \mathrm{d}t \\
&\leq L_\kappa C_q^2 \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \|\kappa(t-\tau_1, \ldots, t-\tau_n)\|_{\mathcal{B}(\mathcal{H})} e^{\varrho_\kappa \sum_j (t-\tau_j)} \prod_{j=1}^n \|u(\tau_j)\|_{\mathcal{H}}^2 \,\mathrm{d}\tau_j \right) e^{-2n\varrho t}\, \mathrm{d}t \\
&\leq L_\kappa C_q^2 \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \|\kappa(t-\tau_1, t-\tau_2)\|_{\mathcal{B}(\mathcal{H})}\, e^{-\varrho_\kappa \sum_j (t-\tau_j)} \underbrace{e^{-2(\varrho-\varrho_\kappa)\sum_j (t-\tau_j)}}_{\leq 1 \text{ for } \tau_j \leq t} \mathrm{d}t \right) \cdot \\
&\qquad\qquad \cdot \prod_{j=1}^n \left(\|u(\tau_j)\|_{\mathcal{H}}^2\, e^{-2\varrho\tau_j}\, \mathrm{d}\tau_j \right) \\
&\leq L_\kappa \ell_\kappa C_q^2 \prod_{j=1}^n \|u\|_{L^2_\varrho}^2 .
\end{aligned}$$

Thus, $\|\mathcal{V}(u)\|_{L^2_{n\varrho}} \leq \sqrt{L_\kappa \ell_\kappa} C_q \, \|u\|_{L^2_\varrho}^n$. □

Lemma 2.3.1 makes it clear that a fixed-point argument in $L^2_\varrho(\mathbb{R}, \mathcal{H})$ for the equation $(\partial_t M(\partial_t) + \mathcal{A})u = \mathcal{V}(u) + g$ cannot be performed in general for $\varrho > 0$, as $L^2_{2\varrho}(\mathbb{R}, \mathcal{H}) \not\subseteq L^2_\varrho(\mathbb{R}, \mathcal{H})$. However, this becomes possible if the linear solution operator leaves $L^2_{-\nu}(\mathbb{R}, \mathcal{H}) \cap L^2_\varrho(\mathbb{R}, \mathcal{H})$ invariant for some $\nu, \varrho > 0$ (more generally, if the linear equation is exponentially stable, see Definition 3.3.1): As in the previous section, we restrict ourselves to functions supported on the positive number line. Letting

$$W_\varrho := \{u \in L^2_\varrho(\mathbb{R}, \mathcal{H}) : u = 0 \text{ in } (-\infty, 0]\},$$

the continuous inclusion $W_{-\nu} \subseteq W_{-\nu'}$ holds for $0 \leq \nu \leq \nu'$ since $\|u\|_{L^2_{-\nu'}} \leq \|u\|_{L^2_{-\nu}}$ for $u \in W_{-\nu}$. This in turn implies, if κ satisfies the conditions in Lemma 2.3.1 with $\varrho_\kappa < 0$, that

$$\forall \nu \in [0, -\varrho_\kappa] : \quad \mathcal{V}(W_{-\nu}) \subseteq W_{-2\nu} \subseteq W_{-\nu}. \tag{2.3.4}$$

We summarize this fact assuming that the problem is well-posed for $\varrho_0 < 0$.

Theorem 2.3.2. *Let $\mathcal{A}\colon \mathrm{dom}(\mathcal{A}) \subseteq \mathcal{H} \to \mathcal{H}$ be m-accretive and M a linear material law with*

$$\exists \nu_0 > 0 : \quad \operatorname{Re} z > -\nu_0 \implies \operatorname{Re} z M(z) \geq c > 0.$$

Let $\mathcal{V}$ be a n-linear Volterra operator with kernel κ satisfying the conditions in Lemma 2.3.1 with $\varrho_\kappa = -\nu_\kappa < 0$. Then there exist $\nu_1 \in (0, \min\{\nu_0, \nu_\kappa\})$, $c_0, r > 0$ such that for each $\nu \in (0, \nu_1)$ and $g \in W_{-\nu}$ with $\|g\|_{L^2_{-\nu}} \leq c_0 r$ the equation $(\partial_t M(\partial_t) + \mathcal{A})u = \mathcal{V}(u) + g$ admits a unique solution $u \in W_{-\nu}$ with $\|u\|_{L^2_{-\nu}} \leq r$.

Proof. The assumptions on M imply that the linear solution operator is boundedly and causally invertible in $L^2_{-\nu}(\mathbb{R}, \mathcal{H})$, uniformly for small ν. As such, it leaves the space $W_{-\nu}$ invariant. By the same argument as in (2.3.4), so does $\mathcal{V}$, and thus the map

$$T(u) := \left(\overline{\partial_t M(\partial_t) + \mathcal{A}}\right)^{-1} (\mathcal{V}(u) + g)$$

is a self-mapping on $W_{-\nu}$. Moreover, due to multilinearity and by Lemma 2.3.1 we obtain the local Lipschitz estimate

$$\|T(u)-T(v)\|_{L^2_{-\nu}} \le \|T(u)-T(v)\|_{L^2_{-n\nu}} \lesssim_{\kappa,Q} \left(\|u\|_{L^2_{-\nu}} + \|v\|_{L^2_{-\nu}}\right)^{n-1} \|u-v\|_{L^2_{-\nu}} .$$

Thus, the restriction of T to a small ball with radius r in $L^2_{-\nu}(\mathbb{R},\mathcal{H})$ becomes a contraction, provided that r and $\|g\| \lesssim r$ are small enough. □

Local well-posedness

The argument can be adapted for $\varrho > 0$ using a cutoff in time. This will produce a local existence and uniqueness result. For $T > 0$, define $\mathcal{V}_T$ by

$$\mathcal{V}_T(u)(t) := \mathbf{1}_{[0,T)}(t)\mathcal{V}(u)(t).$$

Note that $\mathcal{V}_T$ is equivalently obtained from $\mathcal{V}$ by replacing κ in (2.3.1) with

$$\kappa_T(t,\tau_1,\ldots,\tau_n) = \mathbf{1}_{[0,T)}(t)\kappa(\tau_1,\ldots,\tau_n).$$

In addition to finiteness of L_κ, ℓ_κ in (2.3.2) and (2.3.3), we assume that

$$d_\kappa := \operatorname*{ess\,sup}_{\tau_1,\ldots,\tau_n\in\mathbb{R}} \|\kappa(\tau_1,\ldots,\tau_n)\| \, e^{-\varrho_\kappa(\tau_1+\ldots+\tau_n)} < \infty. \tag{2.3.5}$$

We then observe for $t \ge 0$, $\varrho, T > 0$ that

$$\int\cdots\int \|\kappa_T(t,\tau_1,\ldots,\tau_n)\| \, e^{-\varrho_\kappa(\tau_1+\ldots+\tau_n)} \, \mathrm{d}\tau_1\cdots\mathrm{d}\tau_n \le L_\kappa \tag{2.3.6}$$

$$\int \|\kappa_T(t,\tau_1,\ldots,\tau_n)\| \, e^{-\varrho_\kappa(\tau_1+\ldots+\tau_n)} e^{\varrho t} \, \mathrm{d}t \le d_\kappa \int_0^T e^{\varrho t} \, \mathrm{d}t \le d_\kappa T e^{\varrho T}. \tag{2.3.7}$$

Now modifying the estimate in the proof of Lemma 2.3.1 we obtain for $u_1,\ldots,u_n \in L^2_\varrho(\mathbb{R},\mathcal{H})$,

$$\begin{aligned}
&\int_{\mathbb{R}} \left\| \int_{\mathbb{R}}\cdots\int_{\mathbb{R}} \kappa_T(t,t-\tau_1,\ldots,t-\tau_n)\, Q(u_1(\tau_1),\ldots,u_n(\tau_n)) \, \mathrm{d}\tau_1\cdots\mathrm{d}\tau_n \right\|_{\mathcal{H}}^2 e^{-2\varrho t} \, \mathrm{d}t \\
&\overset{(2.3.6)}{\le} L_\kappa C_q^2 \int_0^T \left(\int_{\mathbb{R}}\cdots\int_R \|\kappa(t-\tau_1,\ldots,t-\tau_n)\| \, e^{\varrho_\kappa \sum_j (t-\tau_j)} \prod_{j=1}^n \|u_j(\tau_j)\|_{\mathcal{H}}^2 \, \mathrm{d}\tau_j \right) e^{-2\varrho t} \, \mathrm{d}t \\
&\le L_\kappa C_q^2 \int_{\mathbb{R}}\cdots\int_{\mathbb{R}}\int_0^T \|\kappa(t-\tau_1,\ldots,t-\tau_n)\| \, e^{-\varrho_\kappa \sum_j (t-\tau_j)} e^{2\varrho_\kappa \sum_j (t-\tau_j)} e^{2\varrho \sum_j \tau_j} e^{-2\varrho t} \, \mathrm{d}t \, \cdot \\
&\qquad\qquad \cdot \prod_{j=1}^n \|u_j(\tau_j)\|_{\mathcal{H}}^2 \, e^{-2\varrho\tau_j} \, \mathrm{d}\tau_j \\
&= L_\kappa C_q^2 \int_{\mathbb{R}}\cdots\int_{\mathbb{R}}\int_0^T \|\kappa(t-\tau_1,\ldots,t-\tau_n)\| \, e^{-\varrho_\kappa \sum_j (t-\tau_j)} \underbrace{e^{2(\varrho-\varrho_\kappa)\sum_j (t-\tau_j)}}_{\le 1} e^{2(n-1)\varrho T} \, \mathrm{d}t \, \cdot \\
&\qquad\qquad \cdot \prod_{j=1}^n \|u_j(\tau_j)\|_{\mathcal{H}}^2 \, e^{-2\varrho\tau_j} \, \mathrm{d}\tau_j \\
&\overset{(2.3.7)}{\le} d_\kappa L_\kappa C_q^2 \, T e^{2(n-1)\varrho T} \, \|u_1\|_{L^2_\varrho}^2 \cdots \|u_n\|_{L^2_\varrho}^2 .
\end{aligned}$$

By multilinearity we can again deduce

$$\|\mathcal{V}_T(u) - \mathcal{V}_T(v)\|_{L^2_\varrho} \le L(\kappa, q, T)\left(\|u\|_{L^2_\varrho} + \|v\|_{L^2_\varrho}\right)^{n-1} \|u - v\|_{L^2_\varrho}$$

with $L(\kappa, q, T) \le \sqrt{d_\kappa L_\kappa} C_q \sqrt{T} e^{(n-1)\varrho T}$.

Theorem 2.3.3. *Let $\mathcal{A}\colon \operatorname{dom}(\mathcal{A}) \subseteq \mathcal{H} \to \mathcal{H}$ be m-accretive and M a linear material law with*

$$\operatorname{Re} z > \varrho_0 \implies \operatorname{Re} zM(z) \ge c > 0,$$

where $\varrho_0 \in \mathbb{R}$. Let $\mathcal{V}$ be an n-linear Volterra operator with $\mathcal{V}(0) = 0$, whose kernel satisfies the conditions in Lemma 2.3.1. Then, for given $\varrho > \varrho_0$ there exist $c_0, r, T > 0$ such that for all $g \in L^2_\varrho(\mathbb{R}, \mathcal{H})$ with $\|g\|_{L^2_\varrho} \le c_0 r$ the equation $(\partial_t M(\partial_t) + \mathcal{A})u = \mathcal{V}_T(u) + g$ admits a unique solution $u \in L^2_\varrho(\mathbb{R}, \mathcal{H})$ with $\|u\|_{L^2_\varrho} \le r$.

Proof. Consider the fixed-point equation $u = S_\varrho(\mathcal{V}_T(u) + g) =: F_T(u)$. For $\varrho > \varrho_0$ it follows from $\|S_\varrho\|_{L^2_\varrho \to L^2_\varrho} \le c^{-1}$ and the estimates above that

$$\begin{aligned} \|F_T(u)\|_{L^2_\varrho} &\lesssim_{q,\kappa} \sqrt{T} e^{(n-1)\varrho T} \|u\|^n_{L^2_\varrho} + \|g\|_{L^2_\varrho} \\ \|F_T(u) - F_T(v)\|_{L^2_\varrho} &\lesssim_{q,\kappa} \sqrt{T} e^{(n-1)\varrho T} \left(\|u\|_{L^2_\varrho} + \|v\|_{L^2_\varrho}\right)^{n-1} \|u - v\|_{L^2_\varrho}. \end{aligned}$$

Thus, smallness of $\|g\|_{L^2_\varrho}, T, r$ is sufficient to establish F_T as a contraction on $B_r = \{u \in L^2_\varrho(\mathbb{R}, \mathcal{H}) : \|u\|_{L^2_\varrho} < r\}$. □

2.4 Initial values for problems with memory

We close this chapter with a discussion on how a given initial value problem (with memory) can be formulated as a single evolutionary equation in $L^2_\varrho(\mathbb{R}, \mathcal{H})$. Already in [Pic00], initial value problems were considered in the distributional sense. Our strategy in dealing with initial values and memory is more akin to [Tro18, Tro13], although we follow a more *ad hoc* approach, at the expense of generality and for a selected class of *nonlinear* problems that are relevant to the Maxwell equations (see also [MP02]). The idea is, starting from an initial value problem, to transform the unknown part of the solution, using smooth cutoff functions, to obtain an evolutionary formulation without distributional terms.

Suppose $\mathcal{A}$ is a densely defined and closed operator in the Hilbert space $\mathcal{H}$ and $\mathcal{M}$ a nonlinear operator (specified below). We consider the Cauchy problem

$$\left\{\begin{aligned} \partial_t \mathcal{M}(U)(t) + \mathcal{A}U(t) = g(t), &\quad t > 0 \\ U(t) = \varphi(t), &\quad t \le 0 \end{aligned}\right\} \tag{2.4.1}$$

for $U\colon \mathbb{R} \to \mathcal{H}$, where the inhomogeneity $g\colon \mathbb{R} \to \mathcal{H}$ and the history $\varphi\colon \mathbb{R} \to \mathcal{H}$ are given functions satisfying

$$\operatorname{supp} g \subseteq (0, \infty), \qquad \operatorname{supp} \varphi \subseteq (-\infty, 0].$$

For simplicity, we assume that

$$\mathcal{M}(U) = M_0 U + \mathcal{G}(U), \quad \text{with} \quad \mathcal{G}(U) = \int_{\mathbb{R}} \chi(\tau)\, Q(U(\,\cdot\, - \tau))\, \mathrm{d}\tau,$$

where M_0 is selfadjoint and uniformly positive definite, χ is causal, i.e., $\operatorname{supp}\chi \subseteq [0,\infty)$, rapidly decaying, and smooth on $[0,\infty)$, and $Q\colon \mathcal{H} \to \mathcal{H}$ is Lipschitz continuous with $Q(0) = 0$. We want to convert (2.4.1) into a nonlinear evolutionary equation in $L^2_\varrho(\mathbb{R}, \mathcal{H})$.

To this end, suppose $U \in C(\mathbb{R}, \mathcal{H})$ is a continuous solution of (2.4.1) and also that $f \in C(\mathbb{R}, \mathcal{H})$. Let $\theta = \mathbf{1}_{(0,\infty)}$ denote (multiplication with) the Heaviside step function. We then separate the 'unknown' part

$$U^+ := \theta U$$

of the solution, with $\operatorname{supp} U^+ \subseteq [0,\infty)$, from the given history $\varphi = (1-\theta)\varphi$. With $U = U^+ + \varphi$ we also have $Q(U(t)) = Q(U^+(t)) + Q(\varphi(t))$ for all $t \in \mathbb{R}$, and therefore in fact $\mathcal{M}(U)(t) = \mathcal{M}(U^+)(t) + \mathcal{M}(\varphi)(t)$. Interpreting now ∂_t in the distributional sense, we use the formula

$$\partial_t(\theta h) = \theta \partial_t h + h(0^+)\delta_0 \qquad (h \in C^1(\mathbb{R}, \mathcal{H}))$$

with δ_0 denoting the δ-distribution, to extract from (2.4.1) an equation for U^+ on the whole real line:

$$\begin{aligned}
g = \theta g &= \theta\big[\partial_t \mathcal{M}(U) + \mathcal{A}U\big] \\
&= \partial_t(\theta \mathcal{M}(U)) - \mathcal{M}(U)(0^+)\delta_0 + \mathcal{A}\theta U \\
&= \partial_t(\theta \mathcal{M}(U^+)) + \partial_t(\theta \mathcal{M}(\varphi)) - \mathcal{M}(\varphi)(0^-)\delta_0 + \mathcal{A}U^+ \\
&= \partial_t \mathcal{M}(U^+) + \mathcal{A}U^+ + \partial_t(\theta \mathcal{G}(\varphi)) - M_0\varphi(0^-)\delta_0 - \mathcal{G}(\varphi)(0^-)\delta_0 \\
&= \partial_t \mathcal{M}(U^+) + \mathcal{A}U^+ + \theta\partial_t \mathcal{G}(\varphi) + \mathcal{G}(\varphi)(0^+)\delta_0 - M_0\varphi(0^-)\delta_0 - \mathcal{G}(\varphi)(0^-)\delta_0 \\
&= \partial_t \mathcal{M}(U^+) + \mathcal{A}U^+ + \theta\partial_t \mathcal{G}(\varphi) - M_0\varphi(0^-)\delta_0,
\end{aligned} \tag{2.4.2}$$

where we used $\mathcal{G}(\varphi)(0^-) = \mathcal{G}(\varphi)(0^+)$ due to continuity of the convolution.

The δ_0-term in the last equation can be removed by smoothing the jump of U^+ at $t=0$: Choose $\eta \in C_c^\infty(\mathbb{R})$ with $\eta(0) = 1$, $\eta'(0) = 0$, and set

$$\varphi^+ := \varphi(0^-)\theta\eta, \quad u := U^+ - \varphi^+,$$

see Figure 2.1. Then,

$$\begin{aligned}
\partial_t \mathcal{M}(U^+) = \partial_t \mathcal{M}(u + \varphi^+) &= \partial_t\big(M_0 u + M_0\varphi^+ + \mathcal{G}(u+\varphi^+)\big) \\
&= \partial_t\big(M_0 u + \mathcal{G}(u+\varphi^+)\big) + \theta\partial_t M_0\varphi^+ + M_0\varphi^+(0^+)\delta_0.
\end{aligned}$$

Thus, using that $\varphi^+(0^+) = \varphi(0^-)$, (2.4.2) becomes

$$\begin{aligned}
g &= \partial_t \mathcal{M}(u+\varphi^+) + \mathcal{A}u + \mathcal{A}\varphi^+ + \theta\partial_t\mathcal{G}(\varphi) - M_0\varphi(0^-)\delta_0 \\
&= \partial_t\big(M_0 u + \mathcal{G}(u+\varphi^+)\big) + \mathcal{A}u + \theta\partial_t M_0\varphi^+ + \theta\partial_t\mathcal{G}(\varphi) + \mathcal{A}\varphi^+.
\end{aligned}$$

Finally, the last identity can be written as

$$(\partial_t M_0 + \mathcal{A})u = -\partial_t \mathcal{G}(u+\varphi^+) + g_\varphi, \tag{2.4.3}$$

where

$$g_\varphi := g - \theta\big[\partial_t(M_0\varphi^+ + \mathcal{G}(\varphi)) + \mathcal{A}\varphi^+\big].$$

Now (2.4.3) is a proper reformulation of (2.4.1) as an operator equation in $L^2_\varrho(\mathbb{R},\mathcal{H})$. The well-posedness of (2.4.3) follows by Proposition 2.1.3 from the Lipschitz continuity of $u \mapsto \partial_t \mathcal{G}(u+\varphi^+)$. Since $\varphi^+ = 0$ on $(-\infty, 0]$, the causality of the solution operator and the fixed-point iteration implies $u = 0$ on $(-\infty, 0]$.

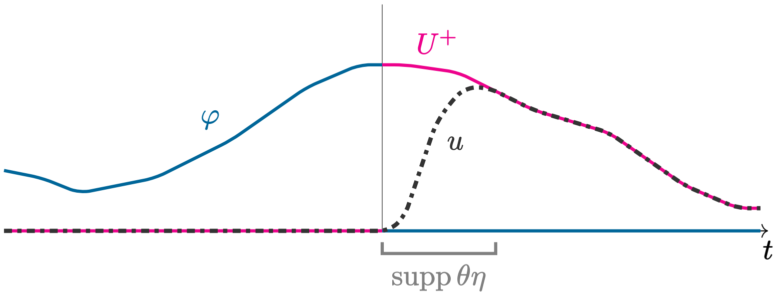

Figure 2.1: Schematic for the conversion of the Cauchy problem to an evolutionary equation.

Remark 2.4.1 (A posteriori justification). If $g_\varphi \in H^1_\varrho(\mathbb{R},\mathcal{H})$, then solutions of (2.4.3) generate continuous solutions of (2.4.1); indeed, in this case Proposition 1.4.14 justifies $u \in H^1_\varrho(\mathbb{R},\mathcal{H})$, and since $\varphi - \varphi^+$ is continuous, $U = u + (\varphi - \varphi^+) \in C(\mathbb{R},\mathcal{H})$. Assuming the history φ is sufficiently regular, i.e., $\varphi \in H^1_\varrho((-\infty,0],\mathcal{H})$ with $\varphi(0^-) \in \operatorname{dom}(\mathcal{A})$, then $g_\varphi \in H^1_\varrho((0,\infty),\mathcal{H})$. In this case, since $g_\varphi = 0$ on $(-\infty, 0]$, a necessary and sufficient condition for $g_\varphi \in H^1_\varrho(\mathbb{R},\mathcal{H})$ is the continuity of g_φ in $t = 0$, i.e.,

$$g_\varphi(0^+) = \lim_{t\searrow 0}\big[g(t) - \partial_t(M_0\varphi^+(t) + \mathcal{G}(\varphi)(t)) + \mathcal{A}\varphi^+(t)\big] = 0. \tag{2.4.4}$$

Under the additional assumption that $(\partial_t\varphi)(0^-) = \lim_{t\nearrow 0}\partial_t\varphi(t)$ exists, we propose the following modification of φ^+. Let $\eta,\gamma \in C_c^\infty(\mathbb{R})$, where $\eta(0) = 1$, $\eta'(0) = 0$, $\gamma(0) = 0$, $\gamma'(0) = 1$, and set

$$\varphi^+ := \varphi(0^-)\theta\eta + (\partial_t\varphi)(0^-)\theta\gamma + M_0^{-1}\chi(0)Q(\varphi(0^-))\theta\gamma.$$

The last term here is connected to the expression

$$\partial_t\mathcal{G}(\varphi)(t) = \begin{cases} \int_{-\infty}^{0} \chi'(t-\tau)\, Q(\varphi(\tau))\,\mathrm{d}\tau, & \tau > 0 \\ \chi(0)\, Q(\varphi(t)) + \int_{-\infty}^{t} \chi'(t-\tau)\, Q(\varphi(\tau))\,\mathrm{d}\tau, & \tau \le 0. \end{cases}$$

Now with g_φ defined as before, (2.4.4) becomes

$$\begin{aligned} g_\varphi(0^+) &= \lim_{t\searrow 0}\big[g(t) - \partial_t M_0(\varphi(0^-)\eta(t) + (\partial_t\varphi)(0^-)\gamma(t)) \\ &\qquad + \chi(0)\, Q(\varphi(t)) + \partial_t\mathcal{G}(\varphi)(t) + \mathcal{A}\varphi(0^-)\eta(t)\big] \\ &= g(0^+) - \big[M_0(\partial_t\varphi)(0^-) + \partial_t\mathcal{G}(\varphi)(0^+) + \mathcal{A}\varphi(0^-)\big] \\ &= g(0^+) - \lim_{t\nearrow 0}\big[\partial_t\mathcal{M}(\varphi)(t) + \mathcal{A}\varphi(t)\big] = 0. \end{aligned} \tag{2.4.5}$$

We can interpret this by saying: *If the history solves the equation at initial time, then there exists an evolutionary formulation equivalent to the Cauchy problem.* ◊

The above derivation is also valid for pure initial value problems without history. Take for example $\mathcal{G} = 0$, $M_0 = \mathrm{id}$ and consider the linear evolution equation

$$(\partial_t + \mathcal{A})u(t) = 0 \quad (t > 0), \qquad u(t=0) = u_0.$$

Analogous to (2.4.3), an evolutionary formulation of this problem reads

$$(\partial_t + \mathcal{A})u = -\theta\big[\partial_t u_0^+ + \mathcal{A}u_0^+\big] =: g_0$$

with $u_0^+(t) = u_0\theta(t)\eta(t)$, $\eta(0) = 1$, where we assume that $g_0 \in H^1_\varrho(\mathbb{R}, \mathcal{H})$ (i.e., no further corrections are necessary for u_0^+). Since $\eta \in C_c^\infty(\mathbb{R})$ is arbitrary as long as $\eta(0) = 1$, in particular this means that $\operatorname{supp}\eta$ can be made arbitrarily small. The conclusion is that perturbative arguments requiring smallness of the data $g_0 \in L^2_\varrho(\mathbb{R}, \mathcal{H})$ can be satisfied independently of u_0. This is not an oversight, however, as instead the H^1_ϱ-norm of u_0^+ will depend sensitively on u_0. Indeed, the previous discussion suggests that $H^1_\varrho(\mathbb{R}, \mathcal{H})$, instead of $L^2_\varrho(\mathbb{R}, \mathcal{H})$, is the proper space to consider initial value problems. Nevertheless, the fixed-point arguments we provide will be formulated mainly with respect to the L^2_ϱ-norm.

2.5 Comments and open problems

In the case of positive weights $\varrho > 0$, we have used a time cutoff in Theorem 2.3.3 to derive local well-posedness for nonlinear problems involving Volterra operators. The mapping property $\mathcal{V}^{(n)}\colon L^2_\varrho \to L^2_{n\varrho} \not\subseteq L^2_\varrho$ renders a contraction mapping for the equation without cutoff impossible in L^2_ϱ.

It may be interesting to see how other fixed-point theorems, such as Schauder's theorem, fare in this respect. If $\mathcal{H} = \mathbb{R}^n$ is finite dimensional and $\mathcal{V}\colon \operatorname{dom}(\mathcal{V}) \subseteq L^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n) \to L^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n)$ is a *compact* Volterra operator, global existence of solutions to the equation

$$u = \mathcal{V}(u)$$

in $L^2_{\mathrm{loc}}(\mathbb{R}, \mathcal{H})$ can be been obtained (by excluding blowup) if $\mathcal{V}$ satisfies a certain growth bound, see Theorem 3.2.2 in [Cor91]. Two immediate questions arise; first, whether this result can be extended to evolutionary equations

$$u = (\overline{\partial_t M(\partial_t) + \mathcal{A}})^{-1}\mathcal{V}(u)$$

on $L^2_\varrho(\mathbb{R}, \mathbb{R}^n)$ (for example, by restricting, once again, to a compact interval $[0, T]$ (but T arbitrary) and noting that $L^2_{\mathrm{loc}}([0, T], \mathcal{H}) \cong L^2_\varrho([0, T], \mathcal{H})$ with equivalent norms; the boundedness of the linear solution operator should then preserve the compactness of $\mathcal{V}$).

Second, how restrictive is the compactness assumption? In finite dimensions, the result provided works for linear Volterra operators on $L^2_{\mathrm{loc}}([0, T), \mathbb{R}^n)$ of the form

$$\mathcal{V}(u)(t) = g(t) + \int_0^t \kappa(t, s)\, u(s)\, \mathrm{d}s$$

for $g \in L^2_{\mathrm{loc}}([0, T), \mathbb{R}^n)$ and generic matrix-valued kernels κ (the extension to multilinear Volterra operators poses no great difficulty) by virtue of the Kolmogorov–Riesz compactness criterion, see [Bre11, Theorem 4.26]:

Lemma 2.5.1 (Kolmogorov–Riesz). *$M \subseteq L^p([0,T],\mathbb{R}^n)$ is compact, if and only if*

$$\sup_{u\in M}\|u\|_{L^p} < \infty \quad \textit{and} \quad \lim_{h\to 0}\sup_{u\in M}\int_0^T |u(t+h)-u(t)|^p \,\mathrm{d}t = 0$$

(i.e., M is bounded and equicontinuous).

In infinite dimensions, e.g. $\mathcal{H} = L^2(\Omega)$, this criterion has to be supplemented by the condition that the set

$$\Big\{\int_I f(t)\,\mathrm{d}t : f \in M\Big\} \subseteq \mathcal{H} \tag{2.5.1}$$

be relatively compact for all bounded intervals I, cf. [Fei84]. Thus to impose compactness on a Volterra operator $\mathcal{V}$, say

$$\mathcal{V}(u) = \int_{\mathbb{R}}\int_{\mathbb{R}} \kappa(\tau_1,\tau_2)\, Q(u(t-\tau_1), u(t-\tau_2))\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2,$$

thus verifying the compactness of $M = \{\mathcal{V}(u) : u \in S\}$ for a bounded set $S \subseteq L^2([0,T],\mathcal{H})$, the condition (2.5.1) amounts to checking that

$$\Big\{\int_I\int_{\mathbb{R}}\int_{\mathbb{R}} \kappa(\tau_1,\tau_2)\, Q(u(t-\tau_1), u(t-\tau_2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2\,\mathrm{d}t : u \in S\Big\}$$

is relatively compact in $\mathcal{H}$.

3 Well-posedness and exponential stability for Maxwell systems

In this chapter we take an evolutionary perspective to Maxwell's equations, utilizing the theory established in Chapter 2 to be able to deal with the equations in nonlinear optics. A special focus is placed in Section 3.3 on exponential stability for systems with simple permeability, for two distinct classes of electric susceptibilities—with and without explicit conduction terms[1].

3.1 Maxwell operator, interface and boundary conditions

In order to formulate a Cauchy problem for the Maxwell equations as an evolutionary system on some domain $\Omega \subseteq \mathbb{R}^3$, we have to specify the underlying Hilbert space $\mathcal{H}$. Here $\mathcal{H} = L^2(\Omega)^3$ is a natural choice. We start by establishing the differential operators grad, div, curl as closed operators in $L^2(\Omega)^3$. There are several variants of these operators. Recall that the spaces

$$C_c^\infty(\Omega) = \{u \in C^\infty(\mathbb{R}^3) : \operatorname{supp} u \subseteq \Omega \text{ compact}\}$$

and

$$C^\infty(\overline{\Omega}) = \{u|_\Omega : u \in C_c^\infty(\mathbb{R}^3)\}$$

(in particular $C^\infty(\overline{\mathbb{R}^3}) = C_c^\infty(\mathbb{R}^3)$) are dense in $L^2(\Omega)$. For $\varphi \in C_c^\infty(\Omega)$, we have the gradient

$$\operatorname{grad} \varphi = \nabla \varphi = \begin{pmatrix} \partial_{x_1}\varphi \\ \partial_{x_2}\varphi \\ \partial_{x_3}\varphi \end{pmatrix}$$

and for $\varphi = (\varphi_1, \varphi_2, \varphi_3) \in C_c^\infty(\Omega)^3$ the divergence and curl, respectively,

$$\begin{aligned} \operatorname{div} \varphi &= \nabla \cdot \varphi = \partial_{x_1}\varphi_1 + \partial_{x_2}\varphi_2 + \partial_{x_3}\varphi_3 \\ \operatorname{curl} \varphi &= \nabla \times \varphi = \begin{pmatrix} \partial_{x_2}\varphi_3 - \partial_{x_3}\varphi_2 \\ \partial_{x_3}\varphi_1 - \partial_{x_1}\varphi_3 \\ \partial_{x_1}\varphi_2 - \partial_{x_2}\varphi_1 \end{pmatrix}. \end{aligned}$$

[1]The presence of a uniformly positive internal conductivity is known to lead to exponential stability in various setups, see [LPS19].

These operators, defined on C_c^∞, are closable[2] in $L^2(\Omega)^3$; their closures will be denoted by

$$\begin{aligned}
\mathrm{grad}_0\colon H_0^1(\Omega) \subseteq L^2(\Omega) &\to L^2(\Omega)^3 \\
\mathrm{div}_0\colon H_0(\mathrm{div},\Omega) \subseteq L^2(\Omega)^3 &\to L^2(\Omega) \\
\mathrm{curl}_0\colon H_0(\mathrm{curl},\Omega) \subseteq L^2(\Omega)^3 &\to L^2(\Omega)^3.
\end{aligned}$$

The adjoints of these operators give rise to the usual weak gradient, divergence, and curl,

$$\mathrm{grad} := -\,\mathrm{div}_0^*, \qquad \mathrm{div} := -\,\mathrm{grad}_0^*, \qquad \mathrm{curl} := \mathrm{curl}_0^*,$$

with maximal domains

$$\begin{aligned}
H^1(\Omega) &= \{u \in L^2(\Omega) : \nabla u \in L^2(\Omega)^3\} \\
H(\mathrm{div},\Omega) &= \{u \in L^2(\Omega)^3 : \nabla \cdot u \in L^2(\Omega)\} \\
H(\mathrm{curl},\Omega) &= \{u \in L^2(\Omega)^3 : \nabla \times u \in L^2(\Omega)^3\}.
\end{aligned}$$

We note that the inclusions $C_c^\infty(\Omega) \subseteq H_0^1(\Omega)$, $C_c^\infty(\Omega)^3 \subseteq H_0(\mathrm{div},\Omega)$, $C_c^\infty(\Omega)^3 \subseteq H_0(\mathrm{curl},\Omega)$ as well as $C^\infty(\overline{\Omega}) \subseteq H^1(\Omega)$, $C^\infty(\overline{\Omega})^3 \subseteq H(\mathrm{div},\Omega)$, and $C^\infty(\overline{\Omega})^3 \subseteq H(\mathrm{curl},\Omega)$, are dense with respect to each graph norm. If $\Omega = \mathbb{R}^3$, the corresponding spaces coincide; we have $\mathrm{grad}_0 = \mathrm{grad}$, $\mathrm{div}_0 = \mathrm{div}$, $\mathrm{curl}_0 = \mathrm{curl}$.

If Ω is a Lipschitz domain, then grad_0, div_0, curl_0 are the weak operators with zero (overall, normal, or tangential) boundary conditions in the sense of traces; in this case, let n denote the outward normal field on $\partial\Omega$, then

$$\begin{aligned}
H_0^1(\Omega) &= \{u \in L^2(\Omega) : \nabla u \in L^2(\Omega)^3,\ u|_{\partial\Omega} = 0\} \\
H_0(\mathrm{div},\Omega) &= \{u \in L^2(\Omega)^3 : \nabla \cdot u \in L^2(\Omega),\ (n \cdot u)|_{\partial\Omega} = 0\} \\
H_0(\mathrm{curl},\Omega) &= \{u \in L^2(\Omega)^3 : \nabla \times u \in L^2(\Omega)^3,\ (n \times u)|_{\partial\Omega} = 0\}.
\end{aligned}$$

For domains with less regular boundary these spaces can be defined nonetheless; in that case the boundary conditions are to be interpreted in a generalized sense. For references of the aforementioned facts, see for instance [STW22, §6.1], [DL90b, Chapter IX].

In $L^2(\Omega)^3$ we consider the Maxwell system

$$\begin{aligned}
\partial_t D(E) - \mathrm{curl}\, H &= -J \qquad & \mathrm{div}\, D(E) &= \rho \\
\partial_t B(H) + \mathrm{curl}_0\, E &= 0 \qquad & \mathrm{div}_0\, B(H) &= 0
\end{aligned} \tag{3.1.1}$$

and introduce the Maxwell operator

$$\mathcal{A} := \begin{pmatrix} 0 & -\,\mathrm{curl} \\ \mathrm{curl}_0 & 0 \end{pmatrix}$$

defined on $H_0(\mathrm{curl},\Omega) \times H(\mathrm{curl},\Omega)$.

[2] Indeed, consider for instance $C = \mathrm{curl}|_{C_c^\infty(\Omega)^3}$. Since C is densely defined, C^* is well-defined and, by the divergence theorem (see the proof of Lemma 3.1.1) $C^\infty(\overline{\Omega})^3 \subseteq \mathrm{dom}(C^*)$, thus also densely defined. Hence $\mathrm{curl}_0 := \overline{C} = C^{**}$ is well-defined and closed. In fact, $H_0^1(\mathrm{curl},\Omega)$ is the closure of $C_c^\infty(\Omega)$ with respect to the graph norm $u \mapsto \|u\|_{H(\mathrm{curl})} = (\|u\|_{L^2}^2 + \|\mathrm{curl}\, u\|_{L^2}^2)^{1/2}$. Similarly for grad and div.

Lemma 3.1.1. *Let $\Omega = \overline{\Omega_1} \sqcup \Omega_2 = \Omega_1 \sqcup \overline{\Omega_2}$ be the disjoint union of nonempty domains $\Omega_1, \Omega_2 \subseteq \mathbb{R}^3$ with connected interface $\Gamma = \overline{\Omega_1} \cap \overline{\Omega_2}$.*

(i) The operator $\mathcal{A}\colon H_0(\operatorname{curl}, \Omega) \times H(\operatorname{curl}, \Omega) \subseteq L^2(\Omega)^6 \to L^2(\Omega)^6$ is skew-selfadjoint.

(ii) If Ω_1, Ω_2 have Lipschitz boundaries and $(u_E, u_H) \in \operatorname{dom}(\mathcal{A})$ are such that $u_E, u_H \in C(\overline{\Omega}_1) \oplus C(\overline{\Omega}_2)$, then

$$[n \times u_E]_\Gamma = [n \times u_H]_\Gamma = 0 \quad \textit{and} \quad (n \times u_E)|_{\partial\Omega} = 0. \tag{3.1.2}$$

Proof. (i) Since curl_0, $\operatorname{curl} = \operatorname{curl}_0^*$ are closed operators,

$$\mathcal{A}^* = \begin{pmatrix} 0 & \operatorname{curl}_0^* \\ -\operatorname{curl}^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & \operatorname{curl} \\ -\operatorname{curl}_0 & 0 \end{pmatrix} = -\mathcal{A}$$

follows by construction.

(ii) Using the divergence theorem on $\overline{\Omega}_1$ and $\overline{\Omega}_2$ separately, we have for all $v_E \in C_c^\infty(\Omega)$, $v_H \in C^\infty(\overline{\Omega})$

$$\begin{aligned}
\int_\Omega (\operatorname{curl}_0 u_E \cdot v_H - u_E \cdot \operatorname{curl} v_H) &= \int_{\Omega_1} \operatorname{div}(u_E \times v_H) + \int_{\Omega_2} \operatorname{div}(u_E \times v_H) \\
&= \int_{\partial\Omega_1} (u_E \times v_H) \cdot n + \int_{\partial\Omega_2} (u_E \times v_H) \cdot n \\
&= \int_\Gamma [(u_E \times v_H) \cdot n]_\Gamma + \int_{\partial\Omega} (u_E \times v_H) \cdot n \\
&= \int_\Gamma [n \times u_E]_\Gamma \cdot v_H + \int_{\partial\Omega} (n \times u_E) \cdot v_H,
\end{aligned}$$

and similarly,

$$\begin{aligned}
\int_\Omega (\operatorname{curl} u_H \cdot v_E - u_H \cdot \operatorname{curl}_0 v_E) &= \int_\Gamma [n \times u_H]_\Gamma \cdot v_E + \int_{\partial\Omega} (n \times u_H) \cdot v_E. \\
&= \int_\Gamma [n \times u_H]_\Gamma \cdot v_E.
\end{aligned}$$

By skew-selfadjointness of $\mathcal{A}$, the left-hand sides must vanish for arbitrary v_E, v_H. Therefore, $[n \times u_E]_\Gamma = [n \times u_H]_\Gamma = 0$ and $(n \times u_E)|_{\partial\Omega} = 0$. □

Using the traces in $H(\operatorname{curl}, \Omega)$ and $H_0(\operatorname{curl}, \Omega)$, equations (3.1.2) can be shown to hold for $u_E \in H_0(\operatorname{curl}, \Omega)$, $u_H \in H(\operatorname{curl}, \Omega)$ in the sense of traces, see also [Lei86, BDPW22]. The domain of $\mathcal{A}$ thus encodes the interface and boundary conditions (3.1.2). If Ω_1, Ω_2 are less regular, these conditions still hold in a generalized sense.

For the divergence equations

$$\operatorname{div} D = \rho, \qquad \operatorname{div}_0 B = 0 \tag{3.1.3}$$

one finds that they are a rather direct consequence of (3.1.1) together with suitable initial values. Indeed (cf. [DL90a, Chapter I Part A §4.1]), applying div to $\partial_t D(t) - \operatorname{curl} H(t) = -J(t)$ and using $\operatorname{div} \operatorname{curl} H = 0$, we can simply define $\rho := \operatorname{div} D$ and realize that ρ and J

are related by the continuity equation

$$\partial_t \rho + \operatorname{div} J = 0, \tag{3.1.4}$$

thus, ρ can be computed from a given J and initial value $\rho(0)$ as

$$\rho(t) = \rho(0) - \int_0^t \operatorname{div} J(s)\,\mathrm{d}s.$$

Similarly, it follows from $\partial_t B(t) + \operatorname{curl}_0 E(t) = 0$ that $\operatorname{div} B(t)$ is constant for all $t > 0$, so $\operatorname{div} B = 0$ reduces to the condition $\operatorname{div} B(0) = 0$ at initial time.

Now suppose J is integrable in time and $J(t) \in H(\operatorname{div}, \Omega)$ for all $t \geq 0$. We then have $\rho(t) \in L^2(\Omega)$ and the equations (3.1.3) imply $D(t) \in H(\operatorname{div}, \Omega)$ and $B(t) \in H_0(\operatorname{div}, \Omega)$ for all $t \geq 0$. Similarly to Lemma 3.1.1 the interface conditions

$$[n \cdot D]_\Gamma = [n \cdot B]_\Gamma = 0$$

follow. Summarizing these observations we have the following result.

Lemma 3.1.2. *Let $I = [0, T]$, let $J\colon I \to H(\operatorname{div}, \Omega)$ be continuous and for given $\rho_0 \in L^2(\Omega)$ define*

$$\rho(t) = \rho_0 - \int_0^t \operatorname{div} J(\tau)\,\mathrm{d}\tau.$$

Suppose $E\colon I \to H_0(\operatorname{curl}, \Omega)$, $H\colon I \to H(\operatorname{curl}, \Omega)$, $D, B\colon I \to L^2(\Omega)^3$ are (continuous) solutions of

$$\begin{aligned} \partial_t D - \operatorname{curl} H &= -J \\ \partial_t B + \operatorname{curl}_0 E &= 0 \end{aligned} \tag{3.1.5}$$

with $\operatorname{div} D(0) = \rho_0$, $\operatorname{div} B(0) = 0$. Then for all $t \in I$ the following holds:

- *$D(t) \in H(\operatorname{div}, \Omega)$, $B(t) \in H_0(\operatorname{div}, \Omega)$ with $\operatorname{div} D(t) = \rho(t)$, $\operatorname{div}_0 B(t) = 0$*
- *$(n \times E(t))|_{\partial\Omega} = [n \times E(t)]_\Gamma = [n \times H(t)]_\Gamma = 0$ and $[n \cdot D(t)]_\Gamma = [n \cdot B(t)]_\Gamma = 0$.*

This result shows that for Cauchy problems at an interface we can focus on the 'dynamic' part (3.1.5) of the Maxwell system, as the remaining equations and (zero) interface conditions can be viewed as a mere statement about regularity, incorporated into the domains of the spatial operators, and initial conditions. In this consideration we thus assume, in view of (1.2.3), that both surface densities ρ_Γ and J_Γ vanish. This assumption will be made throughout this paper, but the following comments provide a heuristic to generalize this.

Distributions and Sobolev chains

Let $C\colon H^1(C) \subseteq \mathcal{H}_1 \to \mathcal{H}_2$ be a linear, densely defined and closed operator with maximal domain $H^1(C) = \{u \in \mathcal{H}_1 : Cu \in \mathcal{H}_2\}$. Then $H^1(C)$ is itself a Hilbert space endowed with the graph inner product $\langle u, v\rangle_{H^1(C)} = \langle u, v\rangle_{\mathcal{H}_1} + \langle Cu, Cv\rangle_{\mathcal{H}_2}$. The adjoint $C^*\colon H^1(C^*) \subseteq \mathcal{H}_2 \to \mathcal{H}_1$ is also closed, and we assume that $H^1(C^*)$ is dense in $\mathcal{H}_2$. Then, both $(H^1(C), \mathcal{H}_1, H^1(C)^*)$ and $(H^1(C^*), \mathcal{H}_2, H^1(C^*)^*)$ are Gelfand triples, i.e., after identifying $\mathcal{H}_1 = \mathcal{H}_1^*$, $\mathcal{H}_2 = \mathcal{H}_2^*$ with their duals, the embeddings

$$H^1(C) \subseteq \mathcal{H}_1 \subseteq H^1(C)^*, \qquad H^1(C^*) \subseteq \mathcal{H}_2 \subseteq H^1(C^*)^*$$

are dense. Using these, C also manifests as an operator $C\colon \mathcal{H}_1 \subseteq H^1(C)^* \to H^1(C^*)^*$, acting weakly by

$$\mathcal{H}_1 \ni v \mapsto (Cv\colon H^1(C^*) \ni u \mapsto \langle v, C^*u\rangle_{\mathcal{H}_1}),$$

thus $Cv \in H^1(C^*)^*$ is well-defined for all $v \in \mathcal{H}_1$. We claim that the operator is again closed. First, it is easy to see that every $v^\dagger \in H^1(C)^*$ can be identified with a pair $(h_1, h_2) \in \mathcal{H}_1 \times \mathcal{H}_2$ such that $v^\dagger(u) = \langle h_1, u\rangle_{\mathcal{H}_1} + \langle h_2, Cu\rangle_{\mathcal{H}_2}$ for all $u \in H^1(C)$. At the same time, every $v \in \mathcal{H}_1 \subseteq H^1(C)^*$ simply acts by $v(u) = \langle v, u\rangle_{\mathcal{H}_1}$. Now suppose

$$\mathcal{H}_1 \ni v_n \to v^\dagger \text{ in } H^1(C)^* \quad \text{and} \quad Cv_n \to w^\dagger \text{ in } H^1(C^*)^*.$$

Then we have with $v^\dagger = (h_1, h_2) \in \mathcal{H}_1 \times \mathcal{H}_2$ and $\|u\|_{H^1(C)} = 1$,

$$\begin{aligned}
\big|(v_n - v^\dagger)(u)\big| &= \big|\langle v_n, u\rangle_{\mathcal{H}_1} - \langle h_1, u\rangle_{\mathcal{H}_1} - \langle h_2, Cu\rangle_{\mathcal{H}_2}\big| \\
&\geq \big||\langle v_n - h_1, u\rangle_{\mathcal{H}_1}| - |\langle h_2, Cu\rangle_{\mathcal{H}_2}|\big| \\
&\geq \big|\langle h_2, Cu\rangle_{\mathcal{H}_2}\big|.
\end{aligned}$$

Since

$$\|v_n - v^\dagger\|_{H^1(C)^*} = \sup_{\|u\|_{H^1(C)}=1} \big|(v_n - v^\dagger)(u)\big| \to 0 \quad \text{as } n \to \infty,$$

it follows that $h_2 = 0$, thus $v^\dagger = h_1 \in \mathcal{H}_1$ and $Cv^\dagger = w^\dagger$. This proves the claim. Replacing C by C^* yields that $C^*\colon \mathcal{H}_2 \subseteq H^1(C^*)^* \to H^1(C)^*$ is densely defined and closed.

Remark 3.1.3. These constructions can be made to arbitrary order: For $k \in \mathbb{N}$ we can define recursively

$$\begin{aligned}
H^k(C) &:= \{u \in H^{k-1}(C) : Cu \in H^{k-1}(C^*)\} \\
H^k(C^*) &:= \{u \in H^{k-1}(C^*) : C^*u \in H^{k-1}(C)\}
\end{aligned}$$

(of course, $H^0(C) = \mathcal{H}_1$, $H^0(C^*) = \mathcal{H}_2$), and $H^{-k}(C) = H^k(C)^*$, $H^{-k}(C^*) = H^k(C^*)^*$. The resulting sequences $(H^k(C))_{k\in\mathbb{Z}}$, $(H^k(C^*))_{k\in\mathbb{Z}}$, which are totally ordered by dense embedding, are called *Sobolev chains*. Moreover, C becomes a closed operator $C\colon H^k(C) \subseteq H^{k-1}(C) \to H^{k-1}(C^*)$ for all $k \in \mathbb{Z}$, indeed, C can be viewed as an operator on the chain $(H^k(C))_{k\in\mathbb{Z}}$ itself.

For more details about Sobolev chains, for the case $\mathcal{H}_1 = \mathcal{H}_2$, we refer to [PM11, Chapter 2.1]. In fact, the above construction is only apparently more general; considering instead one of the operators

$$\begin{pmatrix} 0 & C \\ C^* & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & -C \\ C^* & 0 \end{pmatrix}$$

in $\mathcal{H} := \mathcal{H}_2 \times \mathcal{H}_1$ allows one to reduce to the base case detailed in [PM11]. ◊

Inhomogeneous transmission conditions

Using the notation above for $C \in \{\mathrm{grad}_{(0)}, \mathrm{div}_{(0)}, \mathrm{curl}_{(0)}\}$ we will write

$$H^1(\mathrm{grad}_{(0)}) = H^1_{(0)}(\Omega), \quad H^1(\mathrm{div}_{(0)}) = H_{(0)}(\mathrm{div}, \Omega), \quad H^1(\mathrm{curl}_{(0)}) = H_{(0)}(\mathrm{curl}, \Omega).$$

One way to incorporate nontrivial surface densities ρ_Γ and J_Γ is by introducing them as distributions supported on the surface; this is similar to [SS22], where nontrivial interface charges and currents are considered explicitly. Suppose that the interface is given in local, flat coordinates as $\Gamma = \{x_1 = 0\}$, and

$$J = J_{\text{vol}} + J_{\text{surf}}$$

where $J_{\text{vol}} \in L^2(\Omega)^3$ is a bulk density (which can be neglected in the following) and J_{surf} is given by

$$J_{\text{surf}}(t,x) = J_\Gamma(t,x_2,x_3)\delta_0(x_1)$$

with $J_\Gamma \in L^2(\Gamma)^3$ and δ_0 denoting the Dirac δ-distribution on the line. From Maxwell's equations we demand that

$$[n \times H]_\Gamma = -J_\Gamma,$$

which yields that J_Γ must be tangential to Γ, i.e., it must coincide with its tangential projection: $J_\Gamma = n \times (J_\Gamma \times n)|_\Gamma$. We claim that, if $J_\Gamma = n \times (J_\Gamma \times n)$ is such that $(J_\Gamma \times n) \in H^{1/2}(\Gamma)^3$, then J_{surf} defines an element in $H^1(\text{curl}_0)^*$. Indeed, taking $u \in C_c^\infty(\Omega)^3$ we obtain

$$\begin{aligned}\langle J_{\text{surf}}, u\rangle = \langle J_\Gamma, u\rangle_{L^2(\Gamma)^3} &= \langle J_\Gamma \times n, u \times n\rangle_{L^2(\Gamma)^3} \\ &= \langle J_\Gamma \times n, u \times n\rangle_{H^{1/2}(\Gamma)^3 \times H^{-1/2}(\Gamma)^3}.\end{aligned}$$

By density of $C_c^\infty(\Omega)^3$ in $H^1(\text{curl}_0)$, the last expression is well-defined for $u \in H^1(\text{curl}_0)$, since then $(n \times u)|_\Gamma \in H^{-1/2}(\Gamma)^3$, thus $J_{\text{surf}} \in H^1(\text{curl}_0)^*$.

The surface charge density ρ_Γ for $t \geq 0$ can be derived from the initial value $\rho_\Gamma(0)$ using a relation similar to (3.1.4), resulting in

$$\rho_\Gamma(t) := [n \cdot D]_\Gamma = \rho_\Gamma(0) + \int_0^t \operatorname{div}_\Gamma J_\Gamma(\tau) - [n \cdot J_{\text{vol}}(\tau)]_\Gamma \, \mathrm{d}\tau,$$

see [SS22], where $\operatorname{div}_\Gamma$ denotes the surface divergence.

In conclusion, surface densities can be incorporated into the system by adding a corresponding distributional term to the bulk densities. After changing the underlying function space from $\mathcal{H} = L^2(\Omega)^3 \times L^2(\Omega)^3$ to, e.g., $\mathcal{H} = H^1(\text{curl}_0)^* \times H^1(\text{curl})^*$, the spatial operator is still skew-selfadjoint and the solution theory can be applied analogously.

3.2 Well-posedness of nonlinear evolutionary Maxwell systems

As hinted in the introduction, $D = D(E)$ and $B = B(H)$ are nonlinear material functions with memory, which we write as

$$\begin{aligned}D(E) &= \epsilon_0 E + P_{\text{el}}(E) = \epsilon_0 E + \epsilon_1(\partial_t)E + P_{\text{el,nl}}(E) \\ B(H) &= \mu_0 H + P_{\text{m}}(H) = \mu_0 H + \mu_1(\partial_t)H + P_{\text{m,nl}}(H).\end{aligned}$$

Here we will assume that

- ϵ_0, μ_0 are positive numbers, or in more generality, linear, bounded, selfadjoint, and strictly positive definite operators on $L^2(\Omega)^3$.
- $\epsilon_1(\partial_t)$, $\mu_1(\partial_t)$ are linear material laws on $L^2(\Omega)^3$ according to Definition 1.4.6, hence,

so are

$$\epsilon(\partial_t) := \epsilon_0 + \epsilon_1(\partial_t) \quad \text{and} \quad \mu(\partial_t) := \mu_0 + \mu_1(\partial_t).$$

- $z \mapsto z\epsilon_1(z)$ and $z \mapsto z\mu_1(z)$ are uniformly bounded for $\operatorname{Re} z > \varrho_1 \in \mathbb{R}$.

Explicit conditions for the nonlinear maps $P_{\mathrm{el,nl}}, P_{\mathrm{m,nl}}$ typically involve Lipschitz continuity and will be specified depending on the situation. In view of the previous section our primary focus is a Cauchy problem for the dynamic equations (3.1.5), i.e.,

$$\left.\begin{aligned} \partial_t D(E) - \operatorname{curl} H &= -J \\ \partial_t B(H) + \operatorname{curl}_0 E &= 0 \end{aligned}\right\} \quad t > 0$$
$$\left.\begin{aligned} E(t) &= E_0(t) \\ H(t) &= H_0(t) \end{aligned}\right\} \quad t \leq 0$$

where the history $(E_0(t), H_0(t))$ for $t \leq 0$ is assumed to be known. After making the substitutions as in Section 2.4 for the initial values we have the evolutionary formulation

$$\left(\partial_t \begin{pmatrix} \epsilon(\partial_t) & 0 \\ 0 & \mu(\partial_t) \end{pmatrix} + \begin{pmatrix} 0 & -\operatorname{curl} \\ \operatorname{curl}_0 & 0 \end{pmatrix}\right) \begin{pmatrix} E \\ H \end{pmatrix} + \begin{pmatrix} \partial_t P_{\mathrm{el,nl}}(E) \\ \partial_t P_{\mathrm{m,nl}}(H) \end{pmatrix} = \begin{pmatrix} \phi \\ \psi \end{pmatrix} \tag{3.2.1}$$

for the nonlinear system, or respectively

$$\left(\partial_t \begin{pmatrix} \epsilon(\partial_t) & 0 \\ 0 & \mu(\partial_t) \end{pmatrix} + \begin{pmatrix} 0 & -\operatorname{curl} \\ \operatorname{curl}_0 & 0 \end{pmatrix}\right) \begin{pmatrix} E \\ H \end{pmatrix} = \begin{pmatrix} \phi \\ \psi \end{pmatrix} \tag{3.2.2}$$

for the linear (or linearized) system. Here the history is encoded into ϕ, ψ and we may assume $E(t) = H(t) = 0$ for $t \leq 0$; (3.2.2) and (3.2.1) are understood as systems in $L^2_\varrho(\mathbb{R}, L^2(\Omega)^3)^2$. Setting

$$U = \begin{pmatrix} E \\ H \end{pmatrix}, \quad M(\partial_t) = \begin{pmatrix} \epsilon(\partial_t) & 0 \\ 0 & \mu(\partial_t) \end{pmatrix}, \quad N(U) = \begin{pmatrix} \partial_t P_{\mathrm{el,nl}}(E) \\ \partial_t P_{\mathrm{m,nl}}(H) \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} 0 & -\operatorname{curl} \\ \operatorname{curl}_0 & 0 \end{pmatrix},$$

the system (3.2.1) can be written in the more concise form

$$\partial_t M(\partial_t) U + \mathcal{A} U + N(U) = f,$$

as an equation in $L^2_\varrho(\mathbb{R}, \mathcal{H})$ with $\mathcal{H} = L^2(\Omega)^3 \times L^2(\Omega)^3$. As such, the solution theory for evolutionary equations in Sections 1.4 and Chapter 2 can be applied directly, if uniform conditions on the material functions $M(\partial_t)$ and $N(\cdot)$ are imposed.

Proposition 3.2.1. *Let $\epsilon_1(\partial_t)$, $\mu_1(\partial_t)$ be material laws on $\mathcal{H} = L^2(\Omega)^3$ and $\varrho_1 \in \mathbb{R}$ be such that $z \mapsto z\epsilon_1(z)$ and $z \mapsto z\mu_1(z)$ are bounded for $\operatorname{Re} z > \varrho_1$. Let $\epsilon_0, \mu_0 > 0$ and set $\epsilon(\partial_t) = \epsilon_0 + \epsilon_1(\partial_t)$ and $\mu(\partial_t) = \mu_0 + \mu_1(\partial_t)$. Then the following holds.*

(i) There exists $\varrho_0 > 0$ such that the linear system (3.2.2) is well-posed in $\bigcup_{\varrho > \varrho_0} L^2_\varrho(\mathbb{R}, \mathcal{H})^2$.

(ii) Suppose $\partial_t P_{\mathrm{el,nl}}$, $\partial_t P_{\mathrm{m,nl}} \colon L^2_\varrho(\mathbb{R}, \mathcal{H}) \to L^2_\varrho(\mathbb{R}, \mathcal{H})$ are causal and Lipschitz continuous, uniformly in $\varrho > \varrho_0$, then the nonlinear system (3.2.1) is well-posed, i.e., for some $\varrho_2 \geq \varrho_0$ and for each $\varrho > \varrho_2$ and $\phi, \psi \in L^2_\varrho(\mathbb{R}, \mathcal{H})$ there exists a unique solution $(E, H) \in L^2_\varrho(\mathbb{R}, \mathcal{H})^2$, which depends continuously and causally on (ϕ, ψ).

(iii) Let $\varrho > \varrho_0$ and suppose each $F \in \{\partial_t P_{\mathrm{el,nl}}, \partial_t P_{\mathrm{m,nl}}\}$ is such that

$$\|F(u) - F(v)\|_{L^2_\varrho} \leq d(\|u\|_{L^2_\varrho} + \|v\|_{L^2_\varrho})^\alpha \|u - v\|_{L^2_\varrho}$$

for some $\alpha, d > 0$ and all $u, v \in L^2_\varrho(\mathbb{R}, \mathcal{H})$. Then, for $\phi, \psi \in L^2_\varrho(\mathbb{R}, \mathcal{H})$ sufficiently small, (3.2.1) admits a unique solution $(E, H) \in L^2_\varrho(\mathbb{R}, \mathcal{H})^2$.

Proof. With $M(z) = \begin{pmatrix} \epsilon(z) & 0 \\ 0 & \mu(z) \end{pmatrix}$ we have the equivalence

$$\operatorname{Re} z\epsilon(z), \operatorname{Re} z\mu(z) \geq c \iff \operatorname{Re} zM(z) \geq c.$$

By boundedness of ϵ_1, $\operatorname{Re} z\epsilon(z) \geq \epsilon_0 \operatorname{Re} z - \|z\epsilon_1(z)\|_{\mathcal{B}(\mathcal{H})}$ is uniformly and strictly positive for large $\operatorname{Re} z > 0$, and similarly for $\operatorname{Re} z\mu(z)$. Thus, for $\varrho_0 > 0$ large enough, $\operatorname{Re} zM(z) \geq c > 0$. Since $\mathcal{A}$ is skew-selfadjoint by Lemma 3.1.1, (i) then follows by application of Picard's theorem 1.4.11 to the evolutionary equation $(\partial_t M(\partial_t) + \mathcal{A})u = f = (\phi, \psi)$. (ii) and (iii) are the statements of Proposition 2.1.3 and Proposition 2.1.6, respectively. $\square$

Example 3.2.2. As an admissible nonlinearity $P_{\mathrm{nl}}(U) = (P_{\mathrm{el,nl}}(E), P_{\mathrm{m,nl}}(H))$ satisfying the conditions in (ii) we may take

$$P_{\mathrm{nl}}(U)(t) = \int_{\mathbb{R}} \kappa(\tau)\, Q(U(t-\tau))\, \mathrm{d}\tau$$

and assume that

- $Q\colon L^2(\Omega)^6 \to L^2(\Omega)^6$ is Lipschitz continuous,
- $\kappa\colon \mathbb{R} \to \mathcal{B}(L^2(\Omega)^6)$ with $\kappa(t) = 0$ for $t < 0$,
- κ is differentiable and $\kappa' \in L^1_{\varrho_\kappa}(\mathbb{R}, \mathcal{B}(L^2(\Omega)^6))$,
- $\kappa(0^+) = \lim_{\tau \searrow 0} \kappa(\tau) \in \mathcal{B}(L^2(\Omega)^6)$ exists.

In this case,

$$\partial_t P_{\mathrm{nl}}(U)(t) = \kappa(0^+)\, Q(U(t)) + \int_0^\infty \kappa'(\tau)\, Q(U(t-\tau))\, \mathrm{d}\tau$$

is Lipschitz continuous in U, with

$$\|\partial_t P_{\mathrm{nl}}(U)\|_{\mathrm{Lip}(L^2_\varrho(\mathbb{R}, L^2(\Omega)^6))} \leq \left(\|\kappa(0)\|_{\mathcal{B}(L^2(\Omega)^6)} + \|\kappa'\|_{L^1_{\varrho_\kappa}(\mathbb{R}, L^2(\Omega)^6)}\right) \|Q\|_{\mathrm{Lip}(L^2(\Omega)^6)}\,.$$

This follows analogously as in Example 2.1.5. ▲

Example 3.2.3. Let $P_{\mathrm{nl}} = (P_{\mathrm{el,nl}}, P_{\mathrm{m,nl}})$ be a n-linear Volterra operator

$$P_{\mathrm{nl}}(U)(t) = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \kappa(\tau_1, \ldots, \tau_n)\, Q(U(t-\tau_1), \ldots, U(t-\tau_n)) \prod_{\ell=1}^n \mathrm{d}\tau_\ell.$$

Here we assume that

- $Q\colon [L^2(\Omega)^6]^n \to L^2(\Omega)^6$ is n-linear
- $\kappa\colon \mathbb{R}^n \to \mathcal{B}(L^2(\Omega)^6)$ with $\operatorname{supp} \kappa \subseteq (0, \infty)^n$.

- κ is (Fréchet) differentiable, and there exists $\varrho_\kappa \in \mathbb{R}$ such that the quantities

$$L_\kappa := \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \|(\partial_1 + \ldots + \partial_n)\kappa(\tau_1, \ldots, \tau_n)\| \, e^{-\varrho_\kappa(\tau_1+\ldots+\tau_n)} \, \mathrm{d}\tau_1 \cdots \mathrm{d}\tau_n$$

$$\ell_\kappa := \sup_{\tau_1,\ldots,\tau_n \in \mathbb{R}} \int_{\mathbb{R}} \|(\partial_1 + \ldots + \partial_n)\kappa(t-\tau_1, \ldots, t-\tau_n)\| \, e^{-\varrho_\kappa(t-\tau_1)} \cdots e^{-\varrho_\kappa(t-\tau_n)} \, \mathrm{d}t$$

  are finite.

Then we compute

$$\begin{aligned}\partial_t P_{\mathrm{nl}}(U)(t) &= \sum_{j=1}^{n} \Big[\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \big[\kappa(\tau_1, \ldots, \tau_n) \, Q(U(t-\tau_1), \ldots, U(t-\tau_n))\big]_{\tau_j=0} \prod_{\ell \neq j} \mathrm{d}\tau_\ell \\ &\qquad + \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \partial_j \kappa(\tau_1, \ldots, \tau_n) \, Q(U(t-\tau_1), \ldots, U(t-\tau_n)) \prod_{\ell \neq j} \mathrm{d}\tau_\ell \Big] \\ &= \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \Big(\sum_{j=1}^{n} \partial_j \kappa \Big)(\tau_1, \ldots, \tau_n) \, Q(U(t-\tau_1), \ldots, U(t-\tau_n)) \prod_{\ell \neq j} \mathrm{d}\tau_\ell\end{aligned}$$

and have the following.

1. If $\varrho_\kappa > 0$, this nonlinearity satisfies the conditions of Proposition 2.3.3 of local well-posedness, i.e., defining for $T > 0$ the cutoff

$$P_{\mathrm{nl},T}(U) = \mathbb{1}_{[0,T)} P_{\mathrm{nl}}(U),$$

   the nonlinear system (3.2.1) with P_{nl} replaced by $P_{\mathrm{nl},T}$, admits a unique solution for small data ϕ, ψ and small T.

2. If $\varrho_0 = -\nu_0 < 0$ (meaning the smallest such ϱ_0 in (i)), and $\varrho_\kappa \leq -\nu_0$, then Theorem 2.3.2 applies, and if ϕ, ψ are small in $L^2_{-\nu}(\mathbb{R}, L^2(\Omega)^3)$ for some $\nu \in (0, \nu_0)$, then the nonlinear system (3.2.1) without cutoff admits a solution $(E, H) \in L^2_{-\nu}(\mathbb{R}, L^2(\Omega)^3)^2$. ▲

3.3 Exponential stability

Among one of the strongest forms of stability for a dynamical system is that of exponential stability, which states that each (global) solution with initial values in a neighborhood of an equilibrium approaches it exponentially in time. We assume here that the equilibrium is zero. Similar notions exist for evolutionary equations and systems. A basic definition for linear equations is as follows, cf. [STW22, 11.1].

Definition 3.3.1. The equation $(\partial_t M(\partial_t) + \mathcal{A})u = g$ is called *exponentially stable with decay rate* $\nu_0 > 0$ if, for some $\varrho_0 \in \mathbb{R}$, it is well-posed in $\bigcup_{\varrho > \varrho_0} L^2_\varrho(\mathbb{R}, \mathcal{H})$ and for all $\varrho > \varrho_0$ and $\nu < \nu_0$, the solution operator $(\overline{\partial_t M(\partial_t) + \mathcal{A}})^{-1} \colon L^2_\varrho(\mathbb{R}, \mathcal{H}) \to L^2_\varrho(\mathbb{R}, \mathcal{H})$ maps $L^2_\varrho(\mathbb{R}, \mathcal{H}) \cap L^2_{-\nu}(\mathbb{R}, \mathcal{H})$ continuously into itself.

Exponential stability thus means that the implication

$$g \in L^2_\varrho(\mathbb{R}, \mathcal{H}) \cap L^2_{-\nu}(\mathbb{R}, \mathcal{H}) \implies u = (\overline{\partial_t M(\partial_t) + \mathcal{A}})^{-1} g \in L^2_{-\nu}(\mathbb{R}, \mathcal{H})$$

holds for all $\varrho > \varrho_0$ and $\nu < \nu_0$. Note that requiring $g \in L^2_\varrho(\mathbb{R}, \mathcal{H})$ makes sense, since the equation might still be uniquely solvable in all $L^2_\varrho(\mathbb{R}, \mathcal{H})$, $\varrho \neq 0$, but this solution might not depend causally on the data.

Imposing more time regularity on the data, $g \in H^1_\varrho(\mathbb{R}, \mathcal{H}) \cap H^1_{-\nu}(\mathbb{R}, \mathcal{H})$, $\nu \in (0, \nu_0)$, one can show ([STW22, Proposition 11.1.2]) that $u \in H^1_{-\nu}(\mathbb{R}, \mathcal{H})$, and using the Sobolev embedding (Proposition 1.4.5) $H^1_{-\nu}(\mathbb{R}, \mathcal{H}) \subseteq C_{-\nu,0}(\mathbb{R}, \mathcal{H})$ we have

$$\|u(t)\|_{\mathcal{H}} \, e^{\nu t} \to 0, \quad |t| \to \infty.$$

Hence in this case, the continuous trajectory $t \mapsto u(t)$ decays exponentially in time.

It turns out that Definition 3.3.1 is quite rigid; indeed (see [Tro18, Theorem 2.1.3]), if $\mathbb{C}_{\mathrm{Re}>-\nu_0} \setminus \operatorname{dom}(M)$ is discrete for some $\nu_0 > 0$, then the equation $(\partial_t M(\partial_t) + \mathcal{A})u = g$ is exponentially stable with decay rate ν_0 if and only if it is well-posed in $\bigcup_{\varrho > -\nu_0} L^2_\varrho(\mathbb{R}, \mathcal{H})$ (cf. Remark 1.4.13).

Example 3.3.2. Suppose

$$M(z) = \begin{pmatrix} \epsilon(z) & 0 \\ 0 & \mu(z) \end{pmatrix} = \begin{pmatrix} \epsilon_0 & 0 \\ 0 & \mu_0 \end{pmatrix} + \sum_{j=1}^{N} (z - z_j)^{-1} \begin{pmatrix} \epsilon_j & 0 \\ 0 & \mu_j \end{pmatrix}$$

where ϵ_0, μ_0 and ϵ_j, μ_j are selfadjoint and strictly positive operators in $L^2(\Omega)^3$ and $z_j \in \mathbb{R}_{\leq 0}$, for all $j \in \{1, \ldots, N\}$. If $z_j = 0$ for some $j \in \{1, \ldots, N\}$, then the system

$$\left(\partial_t \begin{pmatrix} \epsilon(\partial_t) & 0 \\ 0 & \mu(\partial_t) \end{pmatrix} + \begin{pmatrix} 0 & -\operatorname{curl} \\ \operatorname{curl}_0 & 0 \end{pmatrix} \right) \begin{pmatrix} E \\ H \end{pmatrix} = \begin{pmatrix} \phi \\ \psi \end{pmatrix}$$

is exponentially stable. Indeed, suppose $z_1 = 0$, then

$$\operatorname{Re} z\epsilon(z) = \epsilon_0 \operatorname{Re} z + \epsilon_1 + \sum_{j=1}^{N} \epsilon_j \frac{|z|^2 - z_j \operatorname{Re} z}{|z - z_j|^2} \geq \epsilon_1 - \delta > 0$$

for $\operatorname{Re} z \geq 0$ (here even with $\delta = 0$), and for $\operatorname{Re} z < 0$ (for some $\delta > 0$) small. Analogously $\operatorname{Re} z\mu(z) \geq \mu_1 - \delta > 0$. Consequently, the system is well-posed in $\bigcup_{\varrho > -\nu_0} L^2_\varrho(\mathbb{R}, L^2(\Omega)^3)^2$ for some $\nu_0 > 0$. Here ϵ_1, μ_1 may be called *damping terms*, since they are apparently the main source of exponential decay of solutions. ▲

The notion of exponential stability becomes meaningful especially when dealing with equations that are not well-posed for negative weights $\varrho_0 < 0$, but for which one can isolate exponentially stable subsystems in order to obtain exponential decay of the solution to the initial problem. We explore this idea for the Maxwell system.

3.3.1 Exponential stability in the non-magnetic case

If the main source for stability in the system is only due to the damping occurring in one component, then it is not obvious why to expect overall exponential stability. The idea here is to find an equivalent formulation of the system which is exponentially stable. We will consider only hyperbolic problems here, as our Maxwell system falls into this category, but similar criteria exist for parabolic systems, for example the heat equation with memory, see [STW22, 11.2].

Linear hyperbolic equations are typically given in the second-order "wave-like" formulation

$$(\partial_t M(\partial_t) + C^*C)u = f, \tag{3.3.1}$$

with a given function f, a densely defined and closed operator $C\colon \operatorname{dom}(C) \subseteq \mathcal{H}_0 \to \mathcal{H}_1$ and a linear material law $M\colon \operatorname{dom}(M) \to \mathcal{B}(\mathcal{H}_0)$. Thus, (3.3.1) is understood as an equation in $L^2_\varrho(\mathbb{R}, \mathcal{H}_0)$. Suppose M is of the form

$$M(z) = M_0(z) + z^{-1}M_1(z)$$

and assume that C is boundedly invertible. Then, introducing a parameter $d > 0$ and the variables

$$q := -Cu, \quad v_d := du + \partial_t u,$$

the equation (3.3.1) can be equivalently written as a system

$$\left(\partial_t \begin{pmatrix} M(\partial_t) & 0 \\ 0 & 1 \end{pmatrix} + d\begin{pmatrix} -M_0(\partial_t) & (M_1(\partial_t) - dM_0(\partial_t))C^{-1} \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix}\right)\begin{pmatrix} v_d \\ q \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix} \tag{3.3.2}$$

in $L^2_\varrho(\mathbb{R}, \mathcal{H}_0 \times \mathcal{H}_1)$ (see [STW22, §2.2]). This motivates to call the second-order equation (3.3.1) exponentially stable, if there exists $d > 0$ such that the first-order system (3.3.2) is exponentially stable. In this case, assuming well-posedness in $\bigcup_{\varrho > \varrho_0} L^2_\varrho(\mathbb{R}, \mathcal{H}_0 \times \mathcal{H}_1)$ and an exponential decay rate $\nu_0 > 0$, we have

$$\begin{aligned} f \in L^2_\varrho(\mathbb{R}, \mathcal{H}_0) \cap L^2_{-\nu}(\mathbb{R}, \mathcal{H}_0) &\implies q \in L^2_{-\nu}(\mathbb{R}, \mathcal{H}_1),\ v_d \in L^2_{-\nu}(\mathbb{R}, \mathcal{H}_0) \\ &\implies Cu \in L^2_{-\nu}(\mathbb{R}, \mathcal{H}_1),\ u, \partial_t u \in L^2_{-\nu}(\mathbb{R}, \mathcal{H}_0) \end{aligned}$$

for all $\varrho > \varrho_0$ and $\nu \in (0, \nu_0)$. Here the latter implication follows since C is boundedly invertible, with $Cu = -q$, $u = -C^{-1}q$, $\partial_t u = v_d + dC^{-1}q$.

Theorem 3.3.3 ([STW22, Theorem 11.5.4]). *Let $C\colon \operatorname{dom}(C) \subseteq \mathcal{H}_0 \to \mathcal{H}_1$ be densely defined, closed, and boundedly invertible. Let M be a material law of the form $M(z) = M_0(z) + z^{-1}M_1(z)$ with $M_0, M_1\colon \operatorname{dom}(M) \subseteq \mathbb{C} \to \mathcal{B}(\mathcal{H}_0)$ analytic and uniformly bounded. Suppose there exists $\nu_0 > 0$ such that $\mathbb{C}_{\mathrm{Re}>-\nu_0} \setminus \operatorname{dom}(M)$ is discrete and*

$$\forall z \in \mathbb{C}_{\mathrm{Re}>-\nu_0} \cap \operatorname{dom}(M): \quad \operatorname{Re} zM(z) \geq c > 0.$$

Then, there exists $d > 0$, $\nu_1 > 0$ such that (3.3.2) is exponentially stable with decay rate ν_1.

Theorem 3.3.4 ([Tro18, Proposition 2.2.5]). *Let $C\colon \operatorname{dom}(C) \subseteq \mathcal{H}_0 \to \mathcal{H}_1$ be densely defined, closed, and boundedly invertible. Let M be given by $M(z) := M_0(z) + z^{-1}M_1(z)$, where $M_0, M_1\colon \operatorname{dom}(M) \subseteq \mathbb{C} \to \mathcal{B}(\mathcal{H}_0)$ are analytic and bounded, $\mathbb{C}_{\mathrm{Re}>-\nu_0} \setminus \operatorname{dom}(M)$ is discrete for some $\nu_0 > 0$, and $\lim_{z\to 0} M_1(z) = 0$. If the condition*

$$\forall \delta > 0\ \exists \nu, c > 0\ \forall z \in \operatorname{dom}(M) \cap \mathbb{C}_{\mathrm{Re}>-\nu} \setminus B[0, \delta]: \quad \operatorname{Re} zM(z) \geq c$$

is met, then there exist $d, \nu_1 > 0$ such that system (3.3.2) is exponentially stable with decay rate ν_1.

In the following, we use these two criteria, Theorem 3.3.3 and Theorem 3.3.4, to study exponential stability for the Maxwell system

$$\left(\partial_t \begin{pmatrix} \epsilon(\partial_t) & 0 \\ 0 & \mu \end{pmatrix} + \begin{pmatrix} 0 & -\operatorname{curl} \\ \operatorname{curl}_0 & 0 \end{pmatrix}\right) \begin{pmatrix} E \\ H \end{pmatrix} = \begin{pmatrix} \phi \\ \psi \end{pmatrix} \tag{3.3.3}$$

where $\mu \in \mathcal{B}(L^2(\Omega)^3)$ is boundedly invertible. We refer to this system as "non-magnetic", meaning that the material law $\mu(\partial_t) = \mu$ does not introduce memory effects or other time-dependence. Using the fact that ∂_t commutes with any of μ, curl, curl_0, we obtain

$$\begin{aligned} \partial_t \phi &= \partial_t \left(\partial_t \epsilon(\partial_t) E - \operatorname{curl} H\right) \\ &= \partial_t^2 \epsilon(\partial_t) E - \operatorname{curl} \partial_t H \\ &= \partial_t^2 \epsilon(\partial_t) E - \operatorname{curl}\left(\mu^{-1}\psi - \mu^{-1} \operatorname{curl}_0 E\right) \end{aligned}$$

and can thus convert (3.3.3) into the second-order system

$$(\partial_t^2 \epsilon(\partial_t) + \operatorname{curl} \mu^{-1} \operatorname{curl}_0) E = \partial_t \phi + \operatorname{curl} \mu^{-1} \psi =: g, \tag{3.3.4}$$

which is the wave equation for the electric field. This derivation is justified and $g \in L^2(\mathbb{R}, L^2(\Omega)^3)$ if ϕ, ψ are regular enough; for instance if $\phi, \psi \in H^1_\varrho(\mathbb{R}, L^2(\Omega)^3)$ and $\mu^{-1}\psi \in L^2_\varrho(\mathbb{R}, H(\operatorname{curl}))$.

As it stands, the criteria above cannot be applied to (3.3.4) directly, since deriving the system (3.3.2) would require that $\operatorname{curl} \mu^{-1} \operatorname{curl}_0 = C^*C$ with C invertible. This cannot be expected, as $\operatorname{curl} \operatorname{grad} \varphi = 0$ for all $\varphi \in C_c^\infty(\Omega)^3$, in particular, curl and curl_0 are not invertible. Our strategy will be to work with invertible versions of these operators.

The subsequent arguments will require that the ranges $\operatorname{ran}(\operatorname{curl})$, $\operatorname{ran}(\operatorname{curl}_0)$ are closed[3] in $L^2(\Omega)^3$, thus, for the moment, we will assume just that. A more detailed discussion follows in Section 3.3.2. Setting

$$\mathcal{H}_0 := \ker(\operatorname{curl}_0)^\perp = \operatorname{ran}(\operatorname{curl}), \qquad \mathcal{H}_1 := \ker(\operatorname{curl})^\perp = \operatorname{ran}(\operatorname{curl}_0)$$

(the orthogonal complement being taken in $L^2(\Omega)^3$) we have the decompositions

$$\mathcal{H} := L^2(\Omega)^3 = \mathcal{H}_0 \oplus \mathcal{H}_0^\perp = \mathcal{H}_1 \oplus \mathcal{H}_1^\perp.$$

Definition 3.3.5. For a closed subspace $U \subseteq \mathcal{H}$ of a Hilbert space $\mathcal{H}$ we denote by

$$\iota_U \colon U \hookrightarrow \mathcal{H}, \qquad \pi_U = (\iota_U)^* \colon \mathcal{H} \to U$$

the canonical embedding of U in $\mathcal{H}$, and the canonical projection of $\mathcal{H}$ onto U, respectively.

Lemma 3.3.6. *Let $\mathscr{H}_0, \mathscr{H}_1$ be Hilbert spaces and $T\colon \operatorname{dom}(T) \subseteq \mathscr{H}_0 \to \mathscr{H}_1$ a linear, densely defined, and closed operator with closed range. Suppose $A \in \mathcal{B}(\mathscr{H}_1)$ is selfadjoint and strictly positive. Then,*

$$S := \pi_{\ker(T)^\perp} T^* A T \iota_{\ker(T)^\perp} \colon \operatorname{dom}(S) \subseteq \ker(T)^\perp \to \ker(T)^\perp$$

[3] In fact, since $\operatorname{curl}, \operatorname{curl}_0$ are densely defined, closed, and adjoint operators, $\operatorname{ran}(\operatorname{curl})$ is closed if and only if $\operatorname{ran}(\operatorname{curl}_0)$ is closed. This is a consequence of the closed range theorem, see [Bre11, Theorem 2.19].

is selfadjoint, boundedly invertible, and strictly positive. Moreover, there exists a boundedly invertible operator $C\colon \operatorname{dom}(C) \subseteq \ker(T)^\perp \to \ker(T)^\perp$ *with* $S = C^*C$.

Proof. Let $\iota_k := \iota_{\ker(T)^\perp}$ and $\iota_r := \iota_{\operatorname{ran}(T)} = \iota_{\ker(T^*)^\perp}$. Then we have $\iota_r\iota_r^*T = T$ and, since ι_r, ι_r^* are bounded, also $T^*\iota_r\iota_r^* = (\iota_r\iota_r^*T)^* = T^*$. Hence,

$$S = \iota_k^*T^*AT\iota_k = \iota_k^*T^*\iota_r\iota_r^*A\iota_r\iota_r^*T\iota_k.$$

Now $\iota_r^*T\iota_k\colon \operatorname{dom}(T) \cap \ker(T)^\perp \subseteq \ker(T)^\perp \to \operatorname{ran}(T)$ is injective, surjective, and closed, hence boundedly invertible by the closed graph theorem. The same is true for its adjoint $\iota_k^*T^*\iota_r\colon \operatorname{dom}(T^*) \cap \ker(T^*)^\perp \subseteq \ker(T^*)^\perp \to \operatorname{ran}(T^*) = \ker(T)^\perp$. Consequently, $S = (\iota_k^*T^*\iota_r)(\iota_r^*A\iota_r)(\iota_r^*T\iota_k)$ is the composition of boundedly invertible operators and itself boundedly invertible; that $\iota_r^*A\iota_r$ is boundedly invertible follows from the selfadjointness and strict positivity of A, which shows that S is also selfadjoint and strictly positive. As a consequence of the spectral theorem for selfadjoint operators, there exists a closed and strictly positive operator C such that $S = C^*C$. Indeed, with $\sqrt{A}$ being the unique positive operator such that $\sqrt{A}^2 = A$, we can choose $C = \sqrt{A}\,T\iota_k$. □

Lemma 3.3.7. *Let $\mathcal{H}$ be a Hilbert space, $\mathcal{H}_0 \subseteq \mathcal{H}$ a closed subspace, and $\iota_0\colon \mathcal{H}_0 \hookrightarrow \mathcal{H}$, $\iota_1\colon \mathcal{H}_0^\perp \hookrightarrow \mathcal{H}$ the canonical embeddings. Let $T \in \mathcal{B}(\mathcal{H})$ be a bounded linear operator and define*

$$T_{ij} := \iota_i^*T\iota_j \quad \text{for } i,j \in \{0,1\}.$$

Suppose $\operatorname{Re} T = \frac{1}{2}(T + T^*) \geq d$ *for some* $d > 0$. *Then also*

$$\operatorname{Re} T_{11} \geq d, \qquad \operatorname{Re}(T_{00} - T_{01}T_{11}^{-1}T_{10}) \geq d.$$

Proof. For $\phi \in \mathcal{H}_0^\perp$ we compute

$$\operatorname{Re}\langle T_{11}\phi, \phi\rangle = \operatorname{Re}\langle T\iota_1\phi, \iota_1\phi\rangle \geq d\langle \iota_1\phi, \iota_1\phi\rangle = d\|\phi\|^2,$$

confirming $\operatorname{Re} T_{11} \geq d$. In particular, T_{11} is boundedly invertible. As an operator on $\mathcal{H}_0 \oplus \mathcal{H}_0^\perp$ we can identify

$$T = \begin{pmatrix} T_{00} & T_{01} \\ T_{10} & T_{11} \end{pmatrix},$$

and setting

$$Q = \begin{pmatrix} 1 & 0 \\ -(T_{01}T_{11}^{-1})^* & 1 \end{pmatrix}, \quad Q^* = \begin{pmatrix} 1 & -T_{01}T_{11}^{-1} \\ 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} T_{00} - T_{01}T_{11}^{-1}T_{10} & 0 \\ T_{10} - T_{11}(T_{11}^{-1})^*T_{01}^* & T_{11} \end{pmatrix}$$

we have the factorization

$$R = Q^*TQ.$$

Now we compute for $\phi \in \mathcal{H}_0$,

$$\operatorname{Re}\langle (T_{00} - T_{01}T_{11}^{-1}T_{10})\phi, \phi\rangle = \operatorname{Re}\langle R\begin{pmatrix}\phi\\0\end{pmatrix}, \begin{pmatrix}\phi\\0\end{pmatrix}\rangle$$

$$
\begin{aligned}
&= \operatorname{Re}\langle Q^*TQ\begin{pmatrix}\phi\\0\end{pmatrix}, \begin{pmatrix}\phi\\0\end{pmatrix}\rangle \\
&= \operatorname{Re}\langle TQ\begin{pmatrix}\phi\\0\end{pmatrix}, Q\begin{pmatrix}\phi\\0\end{pmatrix}\rangle \geq d\langle Q\begin{pmatrix}\phi\\0\end{pmatrix}, Q\begin{pmatrix}\phi\\0\end{pmatrix}\rangle \geq d\|\phi\|^2. \qquad \square
\end{aligned}
$$

We now focus on two classes of material laws, for which one can obtain exponentially decaying solutions for non-magnetic Maxwell systems.

Definition 3.3.8. Let $\epsilon\colon \operatorname{dom}(\epsilon) \subset \mathbb{C} \to \mathcal{B}(\mathcal{H})$ be a material law on a Hilbert space $\mathcal{H}$. We call ϵ a permittivity of *conductivity-type* (*C-type*), if

- $\epsilon(z) = M(z) + z^{-1}\sigma$, where $\sigma \in \mathcal{B}(\mathcal{H})$ (the *electric conductivity tensor*) is strictly accretive, and $M\colon \operatorname{dom}(\epsilon) \subseteq \mathbb{C} \to \mathcal{B}(\mathcal{H})$ is analytic and bounded.
- There exist $\nu_1, c_1, c > 0$ such that $\mathbb{C}_{\mathrm{Re}>-\nu_1} \smallsetminus \operatorname{dom}(\epsilon)$ is discrete and

$$
\forall z \in \mathbb{C}_{\mathrm{Re}>-\nu_1} \cap \operatorname{dom}(\epsilon): \ \operatorname{Re} M(z) \geq c_1 \text{ and } \operatorname{Re} z\epsilon(z) \geq c.
$$

Similarly, ϵ is called a permittivity of *Lorentz-type* (*L-type*), if

- $\epsilon(z) = \epsilon_0 + \epsilon_1(z)$, where $\epsilon_0 \in \mathcal{B}(\mathcal{H})$ is strictly accretive and $\epsilon_1\colon \operatorname{dom}(\epsilon) \subseteq \mathbb{C} \to \mathcal{B}(\mathcal{H})$ is analytic and bounded.
- There exists $\nu_1 > 0$ such that $\mathbb{C}_{\mathrm{Re}>-\nu_1} \smallsetminus \operatorname{dom}(\epsilon)$ is discrete and on $\mathbb{C}_{\mathrm{Re}>-\nu_1} \cap \operatorname{dom}(\epsilon)$ the map $z \mapsto z\epsilon_1(z)$ is bounded, $\operatorname{Re}\epsilon(z) \geq c_1 > 0$, and $\lim_{z\to 0} z\epsilon_1(z) = 0$.
- for all $\delta > 0$ there exist $\nu, c > 0$ with

$$
\forall z \in \mathbb{C}_{\mathrm{Re}>-\nu} \cap \operatorname{dom}(\epsilon) \smallsetminus B[0,\delta]: \ \operatorname{Re} z\epsilon(z) \geq c.
$$

Remark 3.3.9. (i) The attribute 'Lorentz-type' is chosen due to similarities to the Lorentz model, see Section 5.1.

(ii) For all further applications, the conductivity may be a more general material law, $\sigma = \sigma(z)$, as long as it is uniformly bounded and strictly accretive on $\mathbb{C}_{\mathrm{Re}>-\nu_1}$. $\Diamond$

Lemma 3.3.10. *Let ϵ be a material law on the Hilbert space $\mathcal{H}$, let $\mathcal{H}_0, \mathcal{H}_1$ be closed subspaces such that $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, and write $\epsilon_{ij}(z) = \iota_{\mathcal{H}_i}^*\epsilon(z)\iota_{\mathcal{H}_j}$. If ϵ is of C- or L-type, then the operators*

$$
\begin{aligned}
\epsilon_{01}(\partial_t)\epsilon_{11}(\partial_t)^{-1}&\colon L^2_{-\nu}(\mathbb{R}, \mathcal{H}_1) \to L^2_{-\nu}(\mathbb{R}, \mathcal{H}_0) \\
\epsilon_{11}(\partial_t)^{-1}\epsilon_{10}(\partial_t)&\colon L^2_{-\nu}(\mathbb{R}, \mathcal{H}_0) \to L^2_{-\nu}(\mathbb{R}, \mathcal{H}_1)
\end{aligned}
$$

are uniformly bounded and causal for $\nu < \nu_1$.

Proof. If ϵ is of L-type, for $z \in \mathbb{C}_{\mathrm{Re}>-\nu_1} \cap \operatorname{dom}(\epsilon)$ we have $\operatorname{Re}\epsilon_{11}(z) \geq c_1 > 0$ by Lemma 3.3.7, thus $\epsilon_{11}(z)$ is boundedly invertible. Since ϵ is also uniformly bounded on $\mathbb{C}_{\mathrm{Re}>-\nu_1} \cap \operatorname{dom}(\epsilon)$, the uniform boundedness of $\epsilon_{01}(z)\epsilon_{11}(z)^{-1}$ and $\epsilon_{11}(z)^{-1}\epsilon_{10}(z)$ on $\mathbb{C}_{\mathrm{Re}>-\nu_1}$ follows by analytic continuation.

If $\epsilon(z) = M(z) + z^{-1}\sigma$ is of C-type, then for $z \in \mathbb{C}_{\mathrm{Re}>-\nu_1} \cap \mathrm{dom}(\epsilon)$ and with $r > 0$ we can write

$$
\begin{aligned}
\epsilon_{01}(z)\epsilon_{11}(z)^{-1} &= (z\epsilon_{01}(z))(z\epsilon_{11}(z))^{-1} \\
&= (zM_{01}(z) + \sigma_{01})(zM_{11}(z) + \sigma_{11})^{-1} \\
&= \sigma_{01}(zM_{11}(z) + \sigma_{11})^{-1} \\
&\quad + zM_{01}(z)(zM_{11}(z) + \sigma_{11})^{-1}\mathbf{1}_{B[0,r]}(z) \\
&\quad + M_{01}(z)M_{11}(z)^{-1}(1 + z^{-1}M_{11}(z)^{-1}\sigma_{11})^{-1}(1 - \mathbf{1}_{B[0,r]}(z)).
\end{aligned}
$$

Here the first two terms are uniformly bounded by Lemma 3.3.7 and boundedness of σ, and since $zM_{01}(z)$ is bounded on the compact set $B[0,r]$. For the third term, choose r large enough so that

$$
\|z^{-1}M_{11}(z)^{-1}\sigma_{11}\|_{\mathcal{B}(\mathcal{H})} \leq r^{-1}\,\|M_{11}(z)^{-1}\|_{\mathcal{B}(\mathcal{H})}\,\|\sigma_{11}\|_{\mathcal{B}(\mathcal{H})} < 1,
$$

then $1 + z^{-1}M_{11}(z)^{-1}\sigma_{11}$ is boundedly invertible through a Neumann series. Again, the uniform boundedness follows by analytic continuation. The argument for $\epsilon_{11}(z)^{-1}\epsilon_{10}(z)$ is analogous.

In both cases, the inverse Fourier–Laplace transform yields the uniform boundedness of $\epsilon_{01}(\partial_t)\epsilon_{11}(\partial_t)^{-1}$ on $L^2_{-\nu}(\mathbb{R}, \mathcal{H}_1)$ and $\epsilon_{11}(\partial_t)^{-1}\epsilon_{10}(\partial_t)$ on $L^2_{-\nu}(\mathbb{R}, \mathcal{H}_0)$ for $\nu < \nu_1$. □

Remark 3.3.11. Suppose $\epsilon(z) = \epsilon_0 + \epsilon_1(z)$ is of L-type, with $\nu_1 > 0$ as above, and let $\mathcal{H}_0, \mathcal{H}_1$ be closed subspaces such that $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$. Denote $\epsilon_{ij}(z) = \iota^*_{\mathcal{H}_i}\epsilon(z)\iota_{\mathcal{H}_j}$. For $z \in \mathbb{C}_{\mathrm{Re}>-\nu_1} \cap \mathrm{dom}(\epsilon)$ we have $\mathrm{Re}\,\epsilon(z) \geq c$, and invoking Lemma 3.3.7, also $\mathrm{Re}\,\epsilon_{11}(z) \geq c$, $\epsilon_{11}(z)^{-1}$ is uniformly bounded, and with

$$
\tilde{\epsilon}(z) := \epsilon_{00}(z) - \epsilon_{01}(z)\epsilon_{11}(z)^{-1}\epsilon_{10}(z)
$$

also $\mathrm{Re}\,\tilde{\epsilon}(z) \geq c$. Moreover, we find that $\tilde{\epsilon}(z) = M_0(z) + z^{-1}M_1(z)$, where

$$
\begin{aligned}
M_0(z) &= \epsilon_{0,00} - \epsilon_{0,01}\epsilon_{11}(z)^{-1}\epsilon_{0,10} \\
M_1(z) &= z(\epsilon_{1,00}(z) - \epsilon_{0,01}\epsilon_{11}(z)^{-1}\epsilon_{1,10}(z) - \epsilon_{1,01}(z)\epsilon_{11}(z)^{-1}\epsilon_{0,10} - \epsilon_{1,01}(z)\epsilon_{11}(z)^{-1}\epsilon_{1,10}(z))
\end{aligned}
$$

are analytic and bounded, and $\lim_{z\to 0} M_1(z) = 0$. Again by Lemma 3.3.7, $\mathrm{Re}\, z\tilde{\epsilon}(z) \geq c$ whenever $\mathrm{Re}\, z\epsilon(z) \geq c$. ◊

We are now able to state a first result concerning exponential decay of the E-field in the non-magnetic setting, based on the second-order formulation (3.3.4).

Theorem 3.3.12. *Let $\Omega \subseteq \mathbb{R}^3$, $\mathcal{H} = L^2(\Omega)^3$ and suppose $\mathcal{H}_0 = \mathrm{ran}(\mathrm{curl}) \subseteq \mathcal{H}$ is closed. Let $\mu \in \mathcal{B}(\mathcal{H})$ be selfadjoint and strictly positive and let $\epsilon\colon \mathrm{dom}(\epsilon) \subseteq \mathbb{C} \to \mathcal{B}(\mathcal{H})$ be a permittivity of C- or L-type. For $(\phi, \psi) \in \bigcup_{\varrho>\varrho_0} L^2_\varrho(\mathbb{R}, \mathcal{H})^2$ let $(E, H) \in \bigcup_{\varrho>\varrho_0} L^2_\varrho(\mathbb{R}, \mathcal{H})^2$ denote the unique solution of the linear first-order system (3.3.3) and define*

$$
g := \partial_t\phi + \mathrm{curl}\,\mu^{-1}\psi, \quad h := \pi_{\ker(\mathrm{curl}_0)}\partial_t^{-1}\phi.
$$

Then, there exists $\nu_0 > 0$ such that if $\nu < \nu_0$ and $\phi, \psi \in L^2_{-\nu}(\mathbb{R}, \mathcal{H})$ the following holds.

(i) If ϵ is of C-type and $g \in L^2_{-\nu}(\mathbb{R}, \mathcal{H})$, then $E \in L^2_{-\nu}(\mathbb{R}, \mathcal{H})$ and $\|E\|_{L^2_{-\nu}} \lesssim \|g\|_{L^2_{-\nu}} + \|\phi\|_{L^2_{-\nu}}$.

(ii) If ϵ is of L-type, $g \in L^2_{-\nu}(\mathbb{R}, \mathcal{H})$, and $h \in L^2_{-\nu}(\mathbb{R}, \ker(\mathrm{curl}_0))$, then $E \in L^2_{-\nu}(\mathbb{R}, \mathcal{H})$ and $\|E\|_{L^2_{-\nu}} \lesssim \|g\|_{L^2_{-\nu}} + \|h\|_{L^2_{-\nu}}$.

Proof. We consider first $\phi, \psi \in C_c^\infty(\mathbb{R}, \mathcal{H})$, so that $g, h \in L^2_\varrho(\mathbb{R}, \mathcal{H}) \cap L^2_{-\nu}(\mathbb{R}, \mathcal{H})$ for $\varrho, \nu > 0$. Due to this time-regularity, (3.3.3) holds in $L^2_\varrho(\mathbb{R}, \mathcal{H})$ and E is a solution of the second-order system (3.3.4). With respect to the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0^\perp$, this system can be written equivalently as

$$\left[\partial_t^2 \begin{pmatrix} \epsilon_{00}(\partial_t) & \epsilon_{01}(\partial_t) \\ \epsilon_{10}(\partial_t) & \epsilon_{11}(\partial_t) \end{pmatrix} + \begin{pmatrix} \pi_{\mathcal{H}_0^\perp} \,\mathrm{curl}\, \mu^{-1} \,\mathrm{curl}_0\, \iota_{\mathcal{H}_0} & 0 \\ 0 & 0 \end{pmatrix}\right] \begin{pmatrix} E_0 \\ E_1 \end{pmatrix} = \begin{pmatrix} g_0 \\ g_1 \end{pmatrix}. \tag{3.3.5}$$

Since $\mu \in \mathcal{B}(\mathcal{H})$ is selfadjoint and strictly positive, it is boundedly invertible. Moreover, the inverse is also strictly positive. By Lemma 3.3.6, there exists a boundedly invertible operator $C_\mu \colon \mathrm{dom}(C_\mu) \subseteq \mathcal{H}_0 \to \mathcal{H}_0$ such that

$$\pi_{\mathcal{H}_0^\perp} \,\mathrm{curl}\, \mu^{-1} \,\mathrm{curl}_0\, \iota_{\mathcal{H}_0} = C_\mu^* C_\mu.$$

Now applying the operator $\epsilon_{01}(\partial_t)\epsilon_{11}(\partial_t)^{-1}$ to the second line of (3.3.5) and subtracting from the first we obtain

$$\left[\partial_t^2 \begin{pmatrix} \tilde{\epsilon}(\partial_t) & 0 \\ \epsilon_{10}(\partial_t) & \epsilon_{11}(\partial_t) \end{pmatrix} + \begin{pmatrix} C_\mu^* C_\mu & 0 \\ 0 & 0 \end{pmatrix}\right] \begin{pmatrix} E_0 \\ E_1 \end{pmatrix} = \begin{pmatrix} \tilde{g}_0 \\ g_1 \end{pmatrix}, \tag{3.3.6}$$

where

$$\begin{aligned} \tilde{\epsilon}(\partial_t) &:= \epsilon_{00}(\partial_t) - \epsilon_{01}(\partial_t)\epsilon_{11}(\partial_t)^{-1}\epsilon_{10}(\partial_t) \\ \tilde{g}_0 &:= g_0 - \epsilon_{01}(\partial_t)\epsilon_{11}(\partial_t)^{-1} g_1. \end{aligned}$$

By Lemma 3.3.7 we have $\mathrm{Re}\, \tilde{\epsilon}(z) \geq c$ whenever $\mathrm{Re}\, \epsilon(z) \geq c$, and $\mathrm{Re}\, z\tilde{\epsilon}(z) \geq c$ whenever $\mathrm{Re}\, z\epsilon(z) \geq c$. If ϵ is of C-type, $\tilde{\epsilon}$ satisfies the conditions of Theorem 3.3.3. Instead, if ϵ is of L-type, $\tilde{\epsilon}$ satisfies the conditions of Theorem 3.3.4, see Remark 3.3.11. Hence, the system

$$(\partial_t^2 \tilde{\epsilon}(\partial_t) + C_\mu^* C_\mu) E = \tilde{g}_0 \tag{3.3.7}$$

is exponentially stable with some decay rate $\nu_0 > 0$. By Lemma 3.3.10, $\epsilon_{01}(\partial_t)\epsilon_{11}(\partial_t)^{-1}$ maps $L^2_{-\nu}(\mathbb{R}, \mathcal{H}_0^\perp)$ into $L^2_{-\nu}(\mathbb{R}, \mathcal{H}_0)$ for $\nu < \nu_1$. If $\nu < \min\{\nu_0, \nu_1\}$ and $g \in L^2_{-\nu}(\mathbb{R}, \mathcal{H})$, then $\tilde{g}_0 \in L^2_{-\nu}(\mathbb{R}, \mathcal{H}_0)$, which by exponential stability of (3.3.7) gives $E_0 \in L^2_{-\nu}(\mathbb{R}, \mathcal{H}_0)$, and E_0 depends continuously on g via

$$\|E_0\|_{L^2_{-\nu}} \lesssim \|\tilde{g}_0\|_{L^2_{-\nu}} \lesssim \|g\|_{L^2_{-\nu}}.$$

Now the second line in (3.3.6) reads

$$\partial_t^2(\epsilon_{10}(\partial_t) E_0 + \epsilon_{11}(\partial_t) E_1) = g_1, \tag{3.3.8}$$

and we want to solve for E_1. Here we consider two cases: First, suppose ϵ is of C-type. Then from (3.3.8) we obtain

$$E_1 = (\partial_t \epsilon_{11}(\partial_t))^{-1}\partial_t^{-1} g_1 - \epsilon_{11}(\partial_t)^{-1}\epsilon_{10}(\partial_t)E_0 \in L^2_{-\nu}(\mathbb{R}, \mathcal{H}_1),$$

which follows from the uniform boundedness of $\epsilon_{11}(z)^{-1}\epsilon_{10}(z)$ and of $(z\epsilon_{11}(z))^{-1}$ on $\mathbb{C}_{\mathrm{Re}>-\nu_1}$, and since $\partial_t^{-1} g_1 = \pi_{\mathcal{H}_1}\phi \in L^2_{-\nu}(\mathbb{R}, \mathcal{H}_1)$ by assumption. We conclude that

$$\|E_1\|_{L^2_{-\nu}} \lesssim \|\partial_t^{-1} g_1\|_{L^2_{-\nu}} + \|E_0\|_{L^2_{-\nu}} \lesssim \|\pi_{\mathcal{H}_1}\phi\|_{L^2_{-\nu}} + \|g\|_{L^2_{-\nu}}.$$

Suppose now ϵ is of L-type and $h = \partial_t^{-2} g_1 \in L^2_{-\nu}(\mathbb{R}, \mathcal{H})$. Then,

$$E_1 = \epsilon_{11}(\partial_t)^{-1}\partial_t^{-2} g_1 - \epsilon_{11}(\partial_t)^{-1}\epsilon_{10}(\partial_t)E_0 \in L^2_{-\nu}(\mathbb{R}, \mathcal{H}_1),$$

and

$$\|E_1\|_{L^2_{-\nu}} \lesssim \|\partial_t^{-2} g_1\|_{L^2_{-\nu}} + \|E_0\|_{L^2_{-\nu}} \lesssim \|h\|_{L^2_{-\nu}} + \|g\|_{L^2_{-\nu}}$$

by boundedness of $\epsilon_{11}(\partial_t)^{-1}$. In both cases we obtain the desired estimate. The general statement follows now by density of $C_c^\infty(\mathbb{R}, \mathcal{H})$ in $L^2_{-\nu}(\mathbb{R}, \mathcal{H})$. □

Remark 3.3.13. Since the proof of Theorem 3.3.12 relies on Theorems 3.3.3 and 3.3.4, exponential decay is not only implied for $E = E_0 + E_1$, but also for $\partial_t E_0$, $\partial_t E_1$, and $C_\mu E_0$, together with the estimates

$$\begin{aligned} \|\partial_t E_0\|_{L^2_{-\nu}}, \|C_\mu E_0\|_{L^2_{-\nu}} &\lesssim \|g\|_{L^2_{-\nu}} \\ \|\partial_t E_1\|_{L^2_{-\nu}} &\lesssim \|\phi\|_{L^2_{-\nu}} + \|g\|_{L^2_{-\nu}}, \end{aligned}$$

the latter following from $\partial_t E_1 = \epsilon_{11}(\partial_t)^{-1}\partial_t^{-1} g_1 + \epsilon_{11}(\partial_t)^{-1}\epsilon_{10}(\partial_t)\partial_t E_0$. ◊

To obtain exponential decay of the H-field, we must consider again the full first-order system (3.3.3). Assuming still that

$$\mathcal{H}_0 = \mathrm{ran}(\mathrm{curl}) = \ker(\mathrm{curl}_0)^\perp, \qquad \mathcal{H}_1 = \mathrm{ran}(\mathrm{curl}_0) = \ker(\mathrm{curl})^\perp$$

are closed in $\mathcal{H} = L^2(\Omega)^3$, we then observe that, by the same argument as in the proof of Lemma 3.3.6, the operator

$$C := \pi_{\mathcal{H}_1} \mathrm{curl}_0\, \iota_{\mathcal{H}_0} \colon \mathcal{H}_0 \to \mathcal{H}_1$$

and its adjoint

$$C^* = \pi_{\mathcal{H}_0} \mathrm{curl}\, \iota_{\mathcal{H}_1} \colon \mathcal{H}_1 \to \mathcal{H}_0$$

are boundedly invertible. The curl operators can then be identified with the matrices

$$\begin{aligned} \mathrm{curl}_0 &= \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix} : \mathcal{H}_0 \oplus \mathcal{H}_0^\perp \to \mathcal{H}_1 \oplus \mathcal{H}_1^\perp \\ \mathrm{curl} &= \begin{pmatrix} C^* & 0 \\ 0 & 0 \end{pmatrix} : \mathcal{H}_1 \oplus \mathcal{H}_1^\perp \to \mathcal{H}_0 \oplus \mathcal{H}_0^\perp, \end{aligned}$$

and we can rewrite (3.3.3) in the form

$$\left(\partial_t\begin{pmatrix}\epsilon_{00}(\partial_t) & \epsilon_{01}(\partial_t) & 0 & 0\\ \epsilon_{10}(\partial_t) & \epsilon_{11}(\partial_t) & 0 & 0\\ 0 & 0 & \mu_{00} & \mu_{01}\\ 0 & 0 & \mu_{10} & \mu_{11}\end{pmatrix}+\begin{pmatrix}0 & 0 & -C^* & 0\\ 0 & 0 & 0 & 0\\ C & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\right)\begin{pmatrix}E_0\\ E_1\\ H_0\\ H_1\end{pmatrix}=\begin{pmatrix}\phi_0\\ \phi_1\\ \psi_0\\ \psi_1\end{pmatrix}, \tag{3.3.9}$$

where $E = E_0 + E_1$, $\phi = \phi_0 + \phi_1 \in \mathcal{H}_0 \oplus \mathcal{H}_0^{\perp}$ and $H = H_0 + H_1$, $\psi = \psi_0 + \psi_1 \in \mathcal{H}_1 \oplus \mathcal{H}_1^{\perp}$.

Theorem 3.3.14. *Let $\Omega \subseteq \mathbb{R}^3$, $\mathcal{H} = L^2(\Omega)^3$ and suppose $\mathcal{H}_0 = \operatorname{ran}(\operatorname{curl}) \subseteq \mathcal{H}$ is closed. Let $\mu \in \mathcal{B}(\mathcal{H})$ be selfadjoint and strictly positive and let $\epsilon\colon \operatorname{dom}(\epsilon) \subseteq \mathbb{C} \to \mathcal{B}(\mathcal{H})$ be a permittivity of C- or L-type. Then, there exists $\nu_0 > 0$ such that if $\nu < \nu_0$ and*

$$\begin{aligned}\phi, \psi, g := \partial_t\phi + \operatorname{curl}\mu^{-1}\psi &\in L^2_{-\nu}(\mathbb{R}, \mathcal{H})\\ \pi_{\ker(\operatorname{curl})}\partial_t^{-1}\psi &\in L^2_{-\nu}(\mathbb{R}, \ker(\operatorname{curl})),\end{aligned}$$

then either of the conditions

(i) ϵ is of C-type, or

(ii) ϵ is of L-type and $\pi_{\ker(\operatorname{curl}_0)}\partial_t^{-1}\phi \in L^2_{-\nu}(\mathbb{R}, \ker(\operatorname{curl}_0))$

imply that

$$E,\ H,\ \partial_t E,\ \partial_t H,\ \operatorname{curl}_0 E,\ \operatorname{curl} H \in L^2_{-\nu}(\mathbb{R}, \mathcal{H}).$$

Proof. Since the conditions of Theorem 3.3.12 are satisfied by the material laws and the data, we obtain $E_0 \in L^2_{-\nu}(\mathbb{R}, \mathcal{H}_0)$, $E_1 \in L^2_{-\nu}(\mathbb{R}, \mathcal{H}_0^{\perp})$. Moreover, in view of Remark 3.3.13, also $\partial_t E_0 \in L^2_{-\nu}(\mathbb{R}, \mathcal{H}_0)$ and $C_\mu E_0 \in L^2_{-\nu}(\mathbb{R}, \mathcal{H}_1)$. The latter implies that $CE_0 \in L^2_{-\nu}(\mathbb{R}, \mathcal{H}_1)$; indeed, since

$$C_\mu^* C_\mu = \iota_{\mathcal{H}_0}^* \operatorname{curl}\mu^{-1}\operatorname{curl}_0 \iota_{\mathcal{H}_0} = \iota_{\mathcal{H}_0}^* \operatorname{curl}\iota_{\mathcal{H}_1}\iota_{\mathcal{H}_1}^*\mu^{-1}\iota_{\mathcal{H}_1}\iota_{\mathcal{H}_1}^*\operatorname{curl}_0\iota_{\mathcal{H}_0} = C^*\mu_{11}^{-1}C$$

(cf. the proof of Lemma 3.3.6) and $\mu_{11}^{-1} \geq d > 0$ is strictly positive by Lemma 3.3.7, we have

$$\begin{aligned}\|C_\mu E_0\|^2_{L^2_{-\nu}} = \langle C_\mu^* C_\mu E_0, E_0\rangle_{L^2_{-\nu}} &= \langle C^*\mu_{11}^{-1}CE_0, E_0\rangle_{L^2_{-\nu}}\\ &= \langle \mu_{11}^{-1}CE_0, CE_0\rangle_{L^2_{-\nu}} \geq d\,\|CE_0\|^2_{L^2_{-\nu}}.\end{aligned}$$

To obtain the statement for H, we solve for the corresponding terms in (3.3.9) to obtain

$$\begin{aligned}H_0 &= (C^*)^{-1}(\partial_t\tilde{\epsilon}(\partial_t)E_0 - \tilde{\phi})\\ \partial_t H_0 &= -\tilde{\mu}^{-1}(CE_0 + \tilde{\psi})\\ H_1 &= \mu_{11}^{-1}(\partial_t^{-1}\psi_1 - \mu_{10}H_0)\\ \partial_t H_1 &= \mu_{11}^{-1}(\psi_1 - \mu_{10}\partial_t H_0)\\ \operatorname{curl} H = CH_0 &= \partial_t\tilde{\epsilon}(\partial_t)E_0 - \tilde{\phi}.\end{aligned}$$

By boundedness of $(C^*)^{-1}, \mu, \mu_{11}^{-1}, \tilde{\mu}, \tilde{\epsilon}$, the right-hand sides can be controlled recursively by $\|E_0\|_{L^2_{-\nu}} + \|\partial_t E_0\|_{L^2_{-\nu}} + \|CE_0\|_{L^2_{-\nu}}$. ☐

3.3.2 On the closedness of the range of the curl operator

The strategy for deriving exponential stability for the non-magnetic Maxwell system (Theorem 3.3.12 and Theorem 3.3.14) relies on a formulation using the boundedly invertible operator $C = \iota_{\mathrm{ran(curl)}}\, \mathrm{curl}\, \iota_{\ker(\mathrm{curl})^\perp}$. This requires $\mathrm{ran(curl)} = \overline{\mathrm{ran(curl)}}$ to be a closed subspace: If ran(curl) is not closed, $\iota_{\mathrm{ran(curl)}}$ and C are not well-defined; defining instead $C = \iota_{\overline{\mathrm{ran(curl)}}}\, \mathrm{curl}\, \iota_{\ker(\mathrm{curl})^\perp}$, this operator is no longer onto, and the argument equally breaks down. The closedness of $\mathrm{ran(curl)} \subseteq \mathcal{H} = L^2(\Omega)^3$ depends largely on the regularity and boundedness of the domain Ω. We outline two methods by means of which the closedness of ran(curl) can be obtained in the case in which Ω has a (local) Lipschitz boundary and falls into one of the following categories:

(a) Ω is a bounded domain, or

(b) Ω is an unbounded, cylindrical domain.

Curl operator on bounded domains

The first method is based on the following compactness result, sometimes called the Picard–Weber–Weck selection theorem, cf. [Pic84, Web80, Wec74].

Theorem 3.3.15. *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with local Lipschitz boundary. Then the embeddings*

$$H_0(\mathrm{curl}, \Omega) \cap H(\mathrm{div}, \Omega) \hookrightarrow L^2(\Omega)^3, \quad H(\mathrm{curl}, \Omega) \cap H_0(\mathrm{div}, \Omega) \hookrightarrow L^2(\Omega)^3$$

are compact.

Remark 3.3.16. Under some smoothness or convexity assumptions on Ω (e.g. $\partial\Omega \in C^2$), the spaces above are equal to

$$\begin{aligned} H_0(\mathrm{curl}, \Omega) \cap H(\mathrm{div}, \Omega) &= \{u \in H^1(\Omega)^3 : (n \times u)|_{\partial\Omega} = 0\} \\ H(\mathrm{curl}, \Omega) \cap H_0(\mathrm{div}, \Omega) &= \{u \in H^1(\Omega)^3 : (n \cdot u)|_{\partial\Omega} = 0\}, \end{aligned}$$

and moreover, we can identify

$$\begin{aligned} H^1(\Omega)^3 &= \{u \in H(\mathrm{curl}, \Omega) \cap H(\mathrm{div}, \Omega) : (n \times u)|_{\partial\Omega} \in H^{1/2}(\Omega)^3\} \\ &= \{u \in H(\mathrm{curl}, \Omega) \cap H(\mathrm{div}, \Omega) : (n \cdot u)|_{\partial\Omega} \in H^{1/2}(\Omega)\}, \end{aligned}$$

which hints at a deeper connection between the Sobolev spaces based on grad, div, and curl. We refer to Theorem 3 and Corollary 1 in [DL90b, Chapter IX, §1] for these facts. ♢

Lemma 3.3.17. *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with local Lipschitz boundary.*

(i) There exists $C > 0$ such that

$$\textit{for all } u \in H(\mathrm{curl}, \Omega) \cap \ker(\mathrm{curl})^\perp : \quad \|u\|_{L^2} \leq C \|\mathrm{curl}\, u\|_{L^2}.$$

An analogous statement holds for curl_0.

(ii) $\mathrm{ran(curl)} = \{\mathrm{curl}\, u : u \in H(\mathrm{curl})\}$ *is closed in* $L^2(\Omega)^3$.

The second statement is a consequence of the first; see [Pic84, Lemmata 6, 7] or [DITW23, Theorems B.1, B.2] for a proof of the Lemma.

Spectrum of the Maxwell operator in cylindrical domains

A more complete picture is provided by the following characterization, see [KNR08].

Theorem 3.3.18. *Let $T\colon \operatorname{dom}(T) \subseteq \mathcal{H}_1 \to \mathcal{H}_2$ be a densely defined and closed operator between Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$. Then the following are equivalent.*

(i) $\operatorname{ran}(T)$ *is closed.*

(ii) $\operatorname{ran}(T^*T)$ *is closed.*

(iii) 0 *is not an accumulation point of the spectrum $\sigma(T^*T)$ of T^*T.*

Together with the next result concerning the spectrum of the Maxwell operator, this criterion becomes applicable in the case of unbounded cylindrical domains.

Lemma 3.3.19. *Let $\Omega = \Sigma \times \mathbb{R}$, where $\Sigma \subseteq \mathbb{R}^2$ is a bounded and simply connected Lipschitz domain and let $\mathcal{A}_1$ denote the selfadjoint Maxwell operator defined on divergence-free fields in $L^2(\Omega)^3$, i.e., $\mathcal{A}_1 = i\mathcal{A}$ with*

$$\operatorname{dom}(\mathcal{A}_1) = (H_0(\operatorname{curl}, \Omega) \cap \ker(\operatorname{div})) \times (H(\operatorname{curl}, \Omega) \cap \ker(\operatorname{div}_0)).$$

Then there exists $r > 0$ such that $\sigma(\mathcal{A}_1) \subset (-\infty, -r] \cup [r, \infty)$.

For the proof of Lemma 3.3.19 we refer to [Fil20, Corollary 1.6]. In fact, in [Fil20] more general selfadjoint Maxwell operators are considered. Similar results about the spectrum in non-selfadjoint settings are available, see [Las98], [ABMW19].

Proposition 3.3.20. *Let $\Omega = \Sigma \times \mathbb{R}$ be given as in Lemma 3.3.19. Then* $\operatorname{ran}(\operatorname{curl})$ *and* $\operatorname{ran}(\operatorname{curl}_0)$ *are closed in $L^2(\Omega)^3$.*

Proof. The inclusions $\operatorname{ran}(\operatorname{curl}) \subset \ker(\operatorname{div})$ and $\operatorname{ran}(\operatorname{curl}_0) \subset \ker(\operatorname{div}_0)$ imply that

$$\begin{aligned} \ker(\operatorname{div})^\perp &\subseteq \operatorname{ran}(\operatorname{curl})^\perp = \ker(\operatorname{curl}_0), \\ \ker(\operatorname{div}_0)^\perp &\subseteq \operatorname{ran}(\operatorname{curl}_0)^\perp = \ker(\operatorname{curl}), \end{aligned}$$

i.e., $\operatorname{dom}(\mathcal{A})$ and $\operatorname{dom}(\mathcal{A}_1)$ differ only by elements in $\ker(\mathcal{A})$. By Lemma 3.3.19 we conclude that the spectrum $\sigma(\mathcal{A})$ contains 0 as an isolated point. Since $\mathcal{A}$ is closed, for the operator $\mathcal{A}^2$ we have $\sigma(\mathcal{A}^2) = \{\lambda^2 : \lambda \in \sigma(\mathcal{A})\}$ (see [KNR08, Theorem 2.15]), which shows that 0 is also an isolated point in the spectrum of

$$\mathcal{A}^2 = \begin{pmatrix} 0 & -\operatorname{curl} \\ \operatorname{curl}_0 & 0 \end{pmatrix}^2 = \begin{pmatrix} -\operatorname{curl}\operatorname{curl}_0 & 0 \\ 0 & \operatorname{curl}_0\operatorname{curl} \end{pmatrix}.$$

In particular, 0 is an isolated point of $\sigma(\operatorname{curl}_0\operatorname{curl})$. By Theorem 3.3.18 this is equivalent to the closedness of $\operatorname{ran}(\operatorname{curl})$ and $\operatorname{ran}(\operatorname{curl}_0)$. $\square$

3.3.3 Nonlinear perturbations

We conclude this section with a discussion of exponential decay of solutions to nonlinear Maxwell systems, with an emphasis on Volterra-type operators. In essence, such operators have already been dealt with in Section 2.3, but for completeness' sake, we formulate a corresponding result for the non-magnetic case, where the linearized system is exponentially stable.

Theorem 3.3.21. *Let $\Omega \subseteq \mathbb{R}^3$, set $\mathcal{H} = L^2(\Omega)^3$ and suppose that $\operatorname{ran}(\operatorname{curl}) \subseteq \mathcal{H}$ is closed. Let $\epsilon\colon \operatorname{dom}(\epsilon) \subseteq \mathbb{C} \to \mathcal{B}(\mathcal{H})$ be a permittivity of C-type or of L-type. With $\nu_0 > 0$ given as in Theorem 3.3.14, fix $\nu < \nu_0$ and let $P_{\mathrm{el,nl}}$ be a nonlinear operator such that each $F \in \{\partial_t^j P_{\mathrm{el,nl}} : j \in \{0,1,2\}\}$ maps $L^2_{-\nu}(\mathbb{R},\mathcal{H})$ into itself, fulfills $F(0) = 0$, and satisfies the estimate*

$$\|F(u) - F(v)\|_{L^2_{-\nu}} \leq c(\|u\|_{L^2_{-\nu}} + \|v\|_{L^2_{-\nu}})^\alpha \|u - v\|_{L^2_{-\nu}} \tag{3.3.10}$$

for all $u, v \in L^2_{-\nu}(\mathbb{R},\mathcal{H})$ with $\|u\|_{L^2_{-\nu}}, \|u\|_{L^2_{-\nu}} \leq \varepsilon_0$, where $c, \varepsilon_0 > 0$ and $\alpha > 0$ are constants. Then, if $\varepsilon \in (0, \varepsilon_0)$ is sufficiently small and if

$$\begin{aligned}
\phi, \psi &\in L^2_{-\nu}(\mathbb{R},\mathcal{H}) \cap L^2_\varrho(\mathbb{R},\mathcal{H}) \qquad (\varrho > \varrho_0)\\
g := \partial_t\phi + \operatorname{curl}\mu^{-1}\psi &\in L^2_{-\nu}(\mathbb{R},\mathcal{H})\\
h := \pi_{\ker(\operatorname{curl}_0)}\partial_t^{-1}\phi &\in L^2_{-\nu}(\mathbb{R},\ker(\operatorname{curl}_0))\\
f := \pi_{\ker(\operatorname{curl})}\partial_t^{-1}\psi &\in L^2_{-\nu}(\mathbb{R},\ker(\operatorname{curl}))
\end{aligned}$$

are such that $\|\phi\|_{L^2_{-\nu}} + \|g\|_{L^2_{-\nu}} + \|h\|_{L^2_{-\nu}} + \|f\|_{L^2_{-\nu}} \leq \varepsilon/2$, then the nonlinear Maxwell system

$$\left(\partial_t \begin{pmatrix} \epsilon(\partial_t) & 0 \\ 0 & \mu \end{pmatrix} + \begin{pmatrix} 0 & -\operatorname{curl} \\ \operatorname{curl}_0 & 0 \end{pmatrix}\right)\begin{pmatrix} E \\ H \end{pmatrix} = \begin{pmatrix} -\partial_t P_{\mathrm{el,nl}}(E) \\ 0 \end{pmatrix} + \begin{pmatrix} \phi \\ \psi \end{pmatrix}$$

admits a unique solution $(E, H) \in L^2_{-\nu}(\mathbb{R},\mathcal{H})^2$ with $\|E\|_{L^2_{-\nu}}, \|H\|_{L^2_{-\nu}} \leq \varepsilon$.

Proof. Consider first the linearized system

$$\left(\partial_t \begin{pmatrix} \epsilon(\partial_t) & 0 \\ 0 & \mu \end{pmatrix} + \begin{pmatrix} 0 & -\operatorname{curl} \\ \operatorname{curl}_0 & 0 \end{pmatrix}\right)\begin{pmatrix} E \\ H \end{pmatrix} = \begin{pmatrix} \phi \\ \psi \end{pmatrix}.$$

Theorem 3.3.12 provides the estimate

$$\|E\|_{L^2_{-\nu}} \lesssim \|\phi\|_{L^2_{-\nu}} + \|g\|_{L^2_{-\nu}} + \|h\|_{L^2_{-\nu}}, \tag{3.3.11}$$

and moreover, adopting the notation of the proof of Theorem 3.3.14, we know that

$$\begin{aligned}
\|H\|_{L^2_{-\nu}} = \|H_0 + H_1\|_{L^2_{-\nu}} &= \|(C^*)^{-1}(\partial_t\tilde{\epsilon}(\partial_t)E_0 - \tilde{\phi}) + \mu^{-1}(\partial_t^{-1}\psi_1 - \mu_{10}H_0)\|_{L^2_{-\nu}}\\
&\lesssim \|E_0\|_{L^2_{-\nu}} + \|\partial_t E_0\|_{L^2_{-\nu}} + \|\tilde{\phi}\|_{L^2_{-\nu}} + \|f\|_{L^2_{-\nu}}\\
&\lesssim \|\phi\|_{L^2_{-\nu}} + \|g\|_{L^2_{-\nu}} + \|h\|_{L^2_{-\nu}} + \|f\|_{L^2_{-\nu}},
\end{aligned} \tag{3.3.12}$$

assuming the norms on the right are finite. Here we have used the estimate from Remark 3.3.13, the boundedness of $\tilde{\epsilon}(\partial_t)$ or $\partial_t\tilde{\epsilon}(\partial_t)$ (depending on L- or C-type), and $\|\tilde{\phi}\|_{L^2_{-\nu}} \lesssim \|\phi\|_{L^2_{-\nu}}$. Now to pass to the nonlinear system, we formally define the nonlinear solution operator by

$$\begin{pmatrix} T_1(E) \\ T_2(E) \end{pmatrix} := \left(\partial_t \begin{pmatrix} \epsilon(\partial_t) & 0 \\ 0 & \mu \end{pmatrix} + \begin{pmatrix} 0 & -\operatorname{curl} \\ \operatorname{curl}_0 & 0 \end{pmatrix}\right)^{-1} \begin{pmatrix} \phi - \partial_t P_{\mathrm{el,nl}}(E) \\ \psi \end{pmatrix}$$

and we will show that T_1 is a contraction on $B_\varepsilon := \{u \in L^2_{-\nu}(\mathbb{R}, \mathcal{H}) : \|u\|_{L^2_{-\nu}} \leq \varepsilon\}$ for small $\varepsilon < \varepsilon_0$. To this end, after performing the substitution $\phi \mapsto \phi - \partial_t P_{\mathrm{el,nl}}(E)$ and using the smallness assumption, we obtain from (3.3.11) and (3.3.12) the following estimates for T_1, T_2, if $\varepsilon \in (0, \varepsilon_0)$:

$$\begin{aligned} \|T_1(E)\|_{L^2_{-\nu}} &\lesssim \|\phi\|_{L^2_{-\nu}} + \|g\|_{L^2_{-\nu}} + \|h\|_{L^2_{-\nu}} + \sum_{j=0}^{2} \|\partial_t^j P_{\mathrm{el,nl}}(E)\|_{L^2_{-\nu}} \\ &\leq \frac{\varepsilon}{2} + 3c\varepsilon^{\alpha+1} = \Big(\frac{1}{2} + 3c\varepsilon^\alpha\Big)\varepsilon \\ \|T_2(E)\|_{L^2_{-\nu}} &\lesssim \|\phi\|_{L^2_{-\nu}} + \|g\|_{L^2_{-\nu}} + \|h\|_{L^2_{-\nu}} + \|f\|_{L^2_{-\nu}} + \sum_{j=0}^{2} \|\partial_t^j P_{\mathrm{el,nl}}(E)\|_{L^2_{-\nu}} \\ &\leq \frac{\varepsilon}{2} + 3c\varepsilon^{\alpha+1} = \Big(\frac{1}{2} + 3c\varepsilon^\alpha\Big)\varepsilon \end{aligned}$$

and moreover,

$$\begin{aligned} \|T_1(u) - T_1(v)\|_{L^2_{-\nu}} &\lesssim \sum_{j=0}^{3} \|\partial_t^j P_{\mathrm{el,nl}}(u) - \partial_t^j P_{\mathrm{el,nl}}(v)\|_{L^2_{-\nu}} \\ &\leq 3c(\|u\|_{L^2_{-\nu}} + \|v\|_{L^2_{-\nu}})^\alpha \|u - v\|_{L^2_{-\nu}} \\ &\leq 3c(2\varepsilon)^\alpha \|u - v\|_{L^2_{-\nu}}. \end{aligned}$$

The constants appearing in these estimates do not depend on ε; we assume without loss that they are equal to unity. Since $\alpha > 0$, we can choose ε so small that simultaneously

$$3c\varepsilon^\alpha < \frac{1}{2} \quad \text{and} \quad 3c(2\varepsilon)^\alpha < 1,$$

in which case T_1 becomes a contraction on B_ε. Thus, $E = T_1(E)$ possesses a fixed point in B_ε, together with $H = T_2(E) \in B_\varepsilon$. □

Remark 3.3.22. Since Theorem 3.3.21 relies on the second-order formulation (3.3.4), nonlinear *magnetic* polarizations $P_{\mathrm{m,nl}}(H)$, cannot, in general, be treated in the same manner. The reason is that $g = \partial_t \phi - \operatorname{curl} \mu^{-1} \psi$ appears on the right-hand side of (3.3.11) and (3.3.12) for the linear system, thus performing the analogous substitution $\psi \mapsto \psi - \partial_t P_{\mathrm{m,nl}}(H)$ creates an extra term $\operatorname{curl} \mu^{-1} \partial_t^{-1} P_{\mathrm{m,nl}}(H)$. If the mapping property

$$H \in H(\operatorname{curl}, \Omega) \implies \mu^{-1} P_{\mathrm{m,nl}}(H) \in \ker(\operatorname{curl}),$$

is imposed, this term vanishes and one can derive an analogous result in this case. In general however, the additional term leads to a loss of spatial regularity for the nonlinear solution operator, prohibiting a direct application of the fixed-point theorem. ◊

Example 3.3.23 (Nonlinear materials with spatial dispersion and fading memory). We take the opportunity to revisit Example 3.2.3; for simplicity we look at a quadratic Volterra operator,

$$P_{\mathrm{el,nl}}(E) := \int_{\mathbb{R}} \int_{\mathbb{R}} \chi^{(2)}(\tau_1, \tau_2)\, Q(E(t-\tau_1), E(t-\tau_2))\, \mathrm{d}\tau_1\, \mathrm{d}\tau_2.$$

with $Q\colon [L^2(\Omega)^3]^2 \to L^2(\Omega)^3$ bilinear and bounded. Relevant instances of such Q are nonlinear operators exhibiting spatial dispersion (see [LL84, §103] for the linear case), for example we can take $Q = (Q_1, Q_2, Q_3)$, where

$$Q_k(u,v)(x) = \sum_{i,j=1}^{3} \iint_{\Omega\times\Omega} \Lambda_{ijk}(x,y,y') u_i(y) v_j(y')\, \mathrm{d}y\, \mathrm{d}y', \quad k \in \{1,2,3\},$$

and with $\Lambda_{ijk} \in L^2(\Omega^3)$. Clearly, Q is bilinear, and the boundedness follows via

$$\iint_{\Omega\times\Omega} \Lambda_{ijk}(x,y,y') u_i(y) v_j(y')\, \mathrm{d}y\, \mathrm{d}y' \leq \|\Lambda_{ijk}(x,\,\cdot\,,\,\cdot\,)\|_{L^2(\Omega^2)} \|u_i\|_{L^2} \|v_j\|_{L^2}$$

from the Cauchy–Schwarz inequality. Thus $\|Q(u,v)\|_{L^2} \leq C_\Lambda \|u\|_{L^2}\|v\|_{L^2}$ with $C_\Lambda > 0$, and Q is indeed bounded. To apply Theorem 3.3.21 we require that $\chi^{(2)}$ is smooth, $\operatorname{supp}\chi^{(2)} \subseteq (0,\infty)^2$, and that

$$L_{ij} := \iint \|\partial_i^j \chi^{(2)}(\tau_1,\tau_2)\|_{\mathcal{B}(L^2(\Omega)^3)}\, e^{\nu(\tau_1+\tau_2)}\, \mathrm{d}\tau_1\, \mathrm{d}\tau_2 < \infty, \quad i \in \{1,2\},\ j \in \{0,1,2\},$$

for some $\nu > 0$. Then, each $\partial_t^j P_{\mathrm{el,nl}}$ (for $j \in \{0,1,2\}$) fulfills the conditions of Lemma 2.3.1, and thus maps $L^2_\varrho(\mathbb{R},\mathcal{H})$ continuously and causally into $L^2_{2\varrho}(\mathbb{R},\mathcal{H})$ for $\varrho \geq -\nu$. In particular, each $F \in \{\partial_t^j P_{\mathrm{el,nl}} : j \in \{0,1,2\}\}$ maps

$$W_{-\nu} = \{u \in L^2_{-\nu}(\mathbb{R},\mathcal{H}) : u = 0 \text{ in } (-\infty, 0]\}$$

into itself, since $W_{-2\nu} \subseteq W_{-\nu}$. On this space, F also satisfies the estimate (3.3.10) with $\alpha = 1$ and some constant c depending on $Q, \chi^{(2)}, \nu$. Consequently, we can perform the same fixed-point argument as in Theorem 3.3.21, if $L^2_{-\nu}$ is replaced by $W_{-\nu}$. ▲

3.4 Comments and open problems

A note on higher regularity

When working with multilinear Volterra operators, such as

$$P_{\mathrm{nl}}(u) = \int_{\mathbb{R}} \int_{\mathbb{R}} \chi^{(2)}(\tau_1,\tau_2)\, Q(u(t-\tau_1), u(t-\tau_2))\, \mathrm{d}\tau_1\, \mathrm{d}\tau_2$$

in Example 3.3.23, we have excluded instantaneous nonlinearities explicitly from the right-hand side of the system

$$(\partial_t M(\partial_t) + \mathcal{A})u = g - \partial_t P_{\mathrm{nl}}(u)$$

by imposing the condition $\operatorname{supp}\chi^{(2)} \subseteq (0,\infty)^2$. This restriction can be removed by working instead in $H^1_{-\nu}$ and using the Sobolev inequality (Proposition 1.4.5) as in the proof of Lemma 2.2.1, to derive the necessary estimates in the perturbation arguments (Example 3.2.3). Note however, that higher derivatives (as they occur in Theorem 3.3.21) pose a problem, since estimating terms such as $\int_{\mathbb{R}} \chi^{(2)}(0,\tau_2)\,\partial_t Q(u(t),u(t-\tau_2)\,\mathrm{d}\tau$ in the H^1_ϱ-norm requires a priori more regularity of u.

The following consideration leads to a similar difficulty: Oftentimes, the effect of spatial dispersion is neglected, thus removing the non-locality from the spatial nonlinearity. If $Q\colon [\mathbb{R}^6]^2 \to \mathbb{R}^6$ is merely taken as a matrix-valued bilinear operator, then it is desired to work in some variant of H^k-Sobolev spaces[4] due to their algebra property (2.2.4). It is generally not possible to infer this required additional *spatial* regularity of the solution from regular data (in contrast to temporal regularity, Proposition 1.4.14). Instead, one can use the structure of the Maxwell system and "trade" temporal for spatial regularity, if the boundary and the interface are smooth enough (cf. [Web81, DITW23, DST22]; see also Remark 3.3.16).

The conclusion we can draw from these remarks is that considering seemingly simpler, instantaneous and local nonlinear material laws (or nonlinear magnetization as in Remark 3.3.22) introduces additional problems that usually require more regularity of the solution. Quasilinear systems (i.e., nonlinearities involving derivatives) are, at present, difficult to impossible to handle in the evolutionary L^2_ϱ-setting[5].

We mention that a loss of temporal regularity has been considered, e.g., in [Pic00], for linear equations, still leading to a well-defined and bounded solution operator, albeit mapping into a different space. It is unclear if such a regularity loss can be incorporated into the theory for nonlinear systems.

Weighted Maxwell systems over exterior domains

We comment briefly on an idea for tackling exponential stability for the non-magnetic Maxwell system on the whole space $\mathbb{R}^3$, based again on the compactness result from case (a) in Section 3.3.2. It turns out that Theorem 3.3.15 generalizes to weighted spaces over unbounded, exterior domains. Let $\Omega = \mathbb{R}^3$ and for some $r > 0$ consider the weight function

$$\gamma\colon \Omega \to \mathbb{R}, \quad \gamma(x) = \begin{cases} 1, & |x| \le r \\ r/|x|, & |x| > r. \end{cases}$$

Let $\mathcal{H} = L^2(\Omega)^3$ and $\mathcal{H}_\gamma = \{u \in L^2_{\mathrm{loc}}(\Omega)^3 : \|\gamma u\|_{L^2} < \infty\}$, then $\mathcal{H}_\gamma$ equipped with the inner product $\langle u, v\rangle_{\mathcal{H}_\gamma} = \langle \gamma u, \gamma v\rangle_{\mathcal{H}}$ is again a Hilbert space. Define the extended curl operator by

$$\operatorname{curl}_\gamma\colon H(\operatorname{curl}_\gamma,\Omega) \subseteq \mathcal{H}_\gamma \to \mathcal{H}, \quad H(\operatorname{curl}_\gamma,\Omega) := \{u \in \mathcal{H}_\gamma : \operatorname{curl} u \in \mathcal{H}\}.$$

Lemma 3.4.1. *The following statements are true.*

(i) There exists $C > 0$ such that

$$\textit{for all } u \in H(\operatorname{curl}_\gamma,\mathbb{R}^3) \cap \ker(\operatorname{curl}_\gamma)^\perp : \quad \|u\|_{\mathcal{H}_\gamma} = \|\gamma u\|_{L^2} \le C\,\|\operatorname{curl} u\|_{L^2}.$$

[4] on each side of the interface, i.e., $\mathcal{H} = H^k(\Omega_1) \oplus H^k(\Omega_2)$

[5] Of course, we can argue that the nonlinearities in Maxwell systems *in nature* are at their core always of continuous memory type; a generalization to quasilinear systems is still an interesting problem from a mathematical perspective.

(ii) The space $\operatorname{ran}(\operatorname{curl}_\gamma) = \{\operatorname{curl} u : u \in H(\operatorname{curl}_\gamma, \mathbb{R}^3)\}$ *is closed in* $L^2(\Omega)^3$.

Proof. We refer to Lemmata 8, 9 in [Pic90]. □

To make use of the operator $\operatorname{curl}_\gamma$, we may consider each of the two weighted versions

$$\mathcal{A}_\gamma := \begin{pmatrix} 0 & -\operatorname{curl}_\gamma \\ \operatorname{curl}_\gamma^* & 0 \end{pmatrix} \quad \text{or} \quad \mathcal{A}^\gamma := \begin{pmatrix} 0 & -\operatorname{curl}_\gamma^* \\ \operatorname{curl}_\gamma & 0 \end{pmatrix}$$

of the Maxwell operator. Note that for a sequence $(u_n)_n \in C_c^\infty(\mathbb{R}^3)^3$ approximating $u \in H(\operatorname{curl}_\gamma, \mathbb{R}^3)$ in $H(\operatorname{curl}_\gamma, \mathbb{R}^3)$ (with norm $u \mapsto \|u\|_{H(\operatorname{curl}_\gamma, \mathbb{R}^3)} = \|\gamma u\|_{L^2}^2 + \|\operatorname{curl}_\gamma u\|_{L^2}^2$) and $\varphi \in C_c^\infty(\mathbb{R}^3)^3$ we have

$$\begin{aligned} \langle \operatorname{curl}_\gamma u, \varphi \rangle_{\mathcal{H}} &= \lim_{n\to\infty} \langle \operatorname{curl} u_n, \varphi \rangle_{\mathcal{H}} \\ &= \lim_{n\to\infty} \langle u_n, \operatorname{curl} \varphi \rangle_{\mathcal{H}} \\ &= \lim_{n\to\infty} \langle \gamma u_n, \tfrac{1}{\gamma} \operatorname{curl} \varphi \rangle_{\mathcal{H}} = \langle \gamma u, \frac{1}{\gamma} \operatorname{curl} \varphi \rangle_{\mathcal{H}} = \langle u, \frac{1}{\gamma^2} \operatorname{curl} \varphi \rangle_{\mathcal{H}_\gamma} \end{aligned}$$

which, by density of $C_c^\infty(\mathbb{R}^3)^3$, shows that

$$\operatorname{curl}_\gamma^* = \frac{1}{\gamma^2} \operatorname{curl}, \quad \operatorname{dom}(\operatorname{curl}_\gamma^*) = \{v \in \mathcal{H} : \tfrac{1}{\gamma} \operatorname{curl} v \in \mathcal{H}\}.$$

Now if $\epsilon(\partial_t), \mu$ are material laws simultaneously defined on $\mathcal{H}$ and $\mathcal{H}_\gamma$ (say, scalar or matrix-valued) then with

$$M(\partial_t) = \begin{pmatrix} \epsilon(\partial_t) & 0 \\ 0 & \mu \end{pmatrix}$$

we may consider three versions of the Maxwell system, namely

$$\begin{aligned} (\partial_t M(\partial_t) + \mathcal{A})U &= g && \text{in } L^2_\varrho(\mathbb{R}, \mathcal{H} \times \mathcal{H}) \\ (\partial_t M(\partial_t) + \mathcal{A}_\gamma)U_\gamma &= g_\gamma && \text{in } L^2_\varrho(\mathbb{R}, \mathcal{H} \times \mathcal{H}_\gamma) \\ (\partial_t M(\partial_t) + \mathcal{A}^\gamma)U^\gamma &= g^\gamma && \text{in } L^2_\varrho(\mathbb{R}, \mathcal{H}_\gamma \times \mathcal{H}). \end{aligned}$$

Here, since $\gamma|_{B(0,r)} \equiv 1$, all three are the same *locally*, i.e., after fixing $\Omega' \subseteq B(0,r) \subseteq \mathbb{R}^3$ and applying the spatial projection $\mathbf{1}_{\Omega'}$ on all sides, the equations coincide. What is still missing is a global mechanism linking all solutions U_γ, U^γ, U to each other, in order to infer decay properties of U from that of U_γ or U^γ.

4 Approximation of broad surface wavepackets in nonlinear magnetooptics

Our aim for this chapter is to derive and justify a wavepacket approximation for a nonlinear Maxwell system on $\Omega = \mathbb{R}^3$ at a planar interface, thereby proving the existence of wavepackets in the nonlinear case. The building blocks for the ansatz are the 2-dimensional linear surface modes in Section 1.3. First we illustrate the general idea.

Consider a nonlinear Cauchy problem

$$\left.\begin{aligned} \partial_t M(\partial_t)U + \mathcal{A}U + N(U) = 0, \quad t > 0 \\ U = V, \quad t \le 0 \end{aligned}\right\} \tag{4.0.1}$$

with a given history V. We want to find a suitable asymptotic ansatz U_ε which is close to an actual solution U of (4.0.1). This can mean different things. One quantity that comes to mind when trying to measure the quality of a given approximation U_ε is the *residual*

$$\operatorname{Res}(U_\varepsilon) := \partial_t M(\partial_t)U_\varepsilon + \mathcal{A}U_\varepsilon + N(U_\varepsilon).$$

After all, $\operatorname{Res}(U) = 0$ (for $t > 0$) if and only if U is an actual solution of the equation. But to infer certain (long time) behaviour of U from the properties of U_ε requires control of the *error*

$$R := U - U_\varepsilon \quad (t > 0)$$

itself. An equation for R can be derived from (4.0.1) for $t > 0$, namely

$$\begin{aligned} 0 &= \partial_t M(\partial_t)U + \mathcal{A}U + N(U) \\ &= \partial_t M(\partial_t)(R + U_\varepsilon) + \mathcal{A}(R + U_\varepsilon) + N(R + U_\varepsilon) - g_V \\ &= \partial_t M(\partial_t)R + \mathcal{A}R + N(R + U_\varepsilon) - N(U_\varepsilon) + \partial_t M(\partial_t)U_\varepsilon + \mathcal{A}U_\varepsilon + N(U_\varepsilon) - g_V \\ &= \partial_t M(\partial_t)R + \mathcal{A}R + N(R + U_\varepsilon) - N(U_\varepsilon) + \operatorname{Res}(U_\varepsilon) - g_V, \end{aligned} \tag{4.0.2}$$

where g_V is related to the history[1]. The task now consists in obtaining a "small" (in a suitable norm) solution R to (4.0.2). Apart from smallness of the residual and the data, the existence of small solutions depends on the form and properties of the nonlinearity $N(\cdot - U_\varepsilon) - N(U_\varepsilon)$, which are largely inherited by those of N. We will subsequently assume

[1] If U_ε and R are supported in $(0, \infty)$, then in fact $g_V = -\theta[\partial_t M(\partial_t)V + N(V)]$. Like in Section 2.4, we assume here the compatibility condition $N(\theta u + (1-\theta)u) = N(\theta u) + N((1-\theta)u)$ for $u \in L^2_{\mathrm{loc}}(\mathbb{R}, \mathcal{H})$, which holds e.g. for continuous Volterra integral operators. However, positive support only makes sense for R; the ansatz should ideally be an approximation of U also for (some) negative times. We will thus assume g_V is small and neglect its role for now.

$N(U)$ to be a sum of symmetric, multilinear terms, say,

$$N(U) = N_2(U,U) + N_3(U,U,U).$$

In this case

$$\begin{aligned} N(R+U_\varepsilon) - N(U_\varepsilon) = N_2(R,R) &+ 2N_2(R,U_\varepsilon) \\ &+ N_3(R,R,R) + 3N_3(R,U_\varepsilon,U_\varepsilon) + 3N_3(R,R,U_\varepsilon), \end{aligned}$$

thus, the genuinely nonlinear terms are given by

$$F_\varepsilon(R) := N_2(R,R) + N_3(R,R,R) + 3N_3(R,R,U_\varepsilon),$$

while the additional linear terms can be collected into

$$M_\varepsilon(\partial_t)R := M(\partial_t)R + \partial_t^{-1}(2N_2(R,U_\varepsilon) + 3N_3(R,U_\varepsilon,U_\varepsilon)).$$

The error equation (4.0.2) now becomes

$$\partial_t M_\varepsilon(\partial_t)R + \mathcal{A}R + F_\varepsilon(R) + \mathrm{Res}(U_\varepsilon) = g_V \qquad (t > 0). \tag{4.0.3}$$

This can be treated as a nonlinear evolutionary equation as in Section 2, provided that

- $M_\varepsilon(\partial_t)$ is again a linear material law—or a small perturbation of such—and the linearized equation $(\partial_t M_\varepsilon(\partial_t) + \mathcal{A})R = g$ is well-posed in $\bigcup_{\varrho>\varrho_0} L^2_\varrho(\mathbb{R},\mathcal{H})$ for some $\varrho_0 \in \mathbb{R}$.
- F_ε is compatible with the perturbation results, i.e., there exists $\varrho > \varrho_0$ such that $F_\varepsilon\colon L^2_\varrho(\mathbb{R},\mathcal{H}) \to L^2_\varrho(\mathbb{R},\mathcal{H})$ is causal and (locally) Lipschitz continuous.
- $\mathrm{Res}(U_\varepsilon)$ is small in $L^2_\varrho(\mathbb{R},\mathcal{H})$.

Ideally, these assumptions should hold with $\varrho_0 < \varrho < 0$, since otherwise R may still grow exponentially with time, even if it is small in $L^2_\varrho(\mathbb{R},\mathcal{H})$. Thus, our aim will be to ensure that

- The linearized error equation is exponentially stable with some decay rate $\nu_0 > 0$.
- There exist $\nu \in (0,\nu_0)$ and a closed subset $W_{-\nu}$ contained in a small ball in $L^2_{-\nu}(\mathbb{R},\mathcal{H})$, such that F_ε is a contraction on $W_{-\nu}$.
- $\mathrm{Res}(U_\varepsilon)$ is small in $W_{-\nu}$.

A fixed-point argument in $W_{-\nu}$ then yields a small solution in $L^2_{-\nu}(\mathbb{R},\mathcal{H})$ for small data.

To some extent, the three points above can be treated independently. We will begin by constructing a first ansatz, which will then be refined using several correction terms to produce a small residual, and will then deal with the error equation.

amplitude equations

Asymptotic and multiple-scale methods are frequently employed in studying nonlinear differential equations. Specifically, the formalism of amplitude equations, or modulation equations, is a prominent tool in describing wavetrain- and wavepacket-like solutions to dispersive equations, by deriving an effective equation for the envelope of the wave, typically a variant of the nonlinear Schrödinger equation

$$i\partial_t A + \Delta A + \gamma |A|^2 A = 0, \tag{NLS}$$

or of the complex Ginzburg–Landau equation

$$\partial_t A + \sigma A - \alpha \Delta A + \gamma |A|^2 A = 0, \tag{CGL}$$

where $\alpha, \gamma \in \mathbb{C}$, $\sigma \in \mathbb{R}$, which can be viewed as a generalized version of (NLS). The underlying mechanism (see, for instance, [vH91, Kal88]) relies on the perturbation of a ground state (in our case $= 0$) near the critical (with respect to stability) value of a (spectral) control parameter in the linear problem. This perturbation results in the creation of a band of solutions, in the form of linear modes (the carrier waves) modulated by a slowly varying amplitude (see Figure 4.1). As a consequence, one expects the dynamics of solutions of the

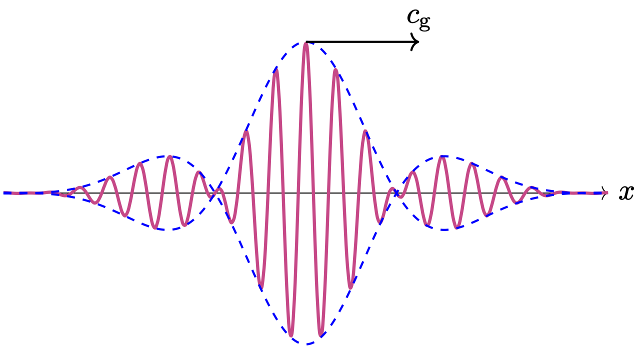

Figure 4.1: One-dimensional wavepacket of the form

$$u(t,x) = \varepsilon A(\varepsilon^2 t, \varepsilon(x - c_g t)) e^{i(kx - \omega t)} + \text{c.c.} \quad (t, x \in \mathbb{R}).$$

The carrier wave $e^{i(kx-\omega t)}$ is modulated by the localized and slowly varying amplitude A (blue dashed). The resulting wavepacket (red solid) moves to the right with group velocity c_g. For $0 < \varepsilon \ll 1$, the scaling determines the height and width of u as being of order $O(\varepsilon)$ and $O(1/\varepsilon)$, respectively.

problems in this spectral regime to be influenced mainly by those of the amplitude, which is a solution of (NLS) or (CGL) or (in the general case) a system of coupled equations of this form.

The structure of these equations and the behavior of solutions are generally well-studied. For example, in the focusing case $\gamma > 0$, (NLS) is known to possess analytical solutions, such as the traveling sech-soliton

$$A(t,x) = \sqrt{2\alpha}\,\mathrm{sech}(\sqrt{\alpha}(x - 2rt - x_0)) e^{i(rx - st + \varphi_0)} \qquad (\alpha = r^2 - s > 0).$$

with constants $\varphi_0 \in \mathbb{R}, x_0, r, s \in \mathbb{R}$, $r, s > 0$ (see [SS99, §1.3.2]). Solutions for (CGL) have been discussed in Section 2.2.

Early applications of this formalism can be found in hydrodynamic problems (e.g. [NW69, IMD89]), followed by numerous uses in other fields, such as chemistry, biology and electrodynamics. For applications close to our use case, we mention here [SU03], where an amplitude approximation is justified for a Maxwell system with memory (see the last section of this chapter for a short summary), and [DST22], where the justification of surface wavepackets is proved using a full quasilinear Maxwell system.

It was shown in [KSM92] that the Ginzburg–Landau approximation is valid in many cases in which the leading nonlinearity is cubic. In contrast, as mentioned in the introductory chapter, resonant quadratic nonlinearities in combination with low regularity of initial values can lead to wrong predictions of the amplitude approximation, see [Sch05, SSZ15].

4.1 Ansatz, residual, and amplitude equation

We will base the analysis in the present section on a model problem, with a multiple-scale ansatz constructed from the linear 2D-modes in Section 1.3.

The asymptotic analysis in this section and the smallness of expressions in terms of $O(\varepsilon^n)$-terms (as $\varepsilon \to 0$) is to be understood, at first, in a purely formal way (or pointwise, if the functions involved in the ansatz are continuous and uniformly bounded). The justification is given a posteriori through rigorous norm estimates.

4.1.1 Setup and basic ansatz

As in Section 1.3, consider the interface

$$\Gamma = \{x \in \mathbb{R}^3 : x_1 = 0\} \quad \text{in} \quad \Omega = \mathbb{R}^3$$

and let $\epsilon(\partial_t), \mu(\partial_t)$ be material laws given by

$$\begin{aligned} \epsilon(\partial_t)E &= \epsilon_0 E + \int_{\mathbb{R}} \chi_{\mathrm{el}}^{\pm}(\tau)\, E(\,\cdot\, - \tau)\,\mathrm{d}\tau \\ \mu(\partial_t)H &= \mu_0 H + \int_{\mathbb{R}} \chi_{\mathrm{m}}^{\pm}(\tau)\, H(\,\cdot\, - \tau)\,\mathrm{d}\tau \end{aligned}$$

and $\epsilon_0, \mu_0 > 0$, where $\chi_{\mathrm{el}}^{\pm}$, $\chi_{\mathrm{m}}^{\pm}$ are scalar, causal kernels, with $\pm$ depending on the side of the interface. Let $N^{(2)}$ be a bilinear Volterra operator of the form

$$N^{(2)}(U,V)(t) = \int_{\mathbb{R}} \int_{\mathbb{R}} \chi^{(2)}(\tau_1, \tau_2)\, Q(U(t-\tau_1), V(t-\tau_2))\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2,$$

where $\chi^{(2)} = \chi_{\pm}^{(2)} \colon \mathbb{R}^2 \to \mathbb{R}^{6\times 6}$ with $\chi^{(2)}(\tau_1,\tau_2) = 0$ whenever $\tau_1 < 0$ or $\tau_2 < 0$, and where we assume that $Q\colon [\mathbb{R}^6]^2 \to \mathbb{R}^6$ is bilinear. For ease of notation, we also take $N^{(2)}$ to be symmetric, specifically, that $\chi^{(2)}$ and Q are symmetric. Now with

$$M(\partial_t) = \begin{pmatrix} \epsilon(\partial_t) & 0 \\ 0 & \mu(\partial_t) \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} 0 & -\operatorname{curl} \\ \operatorname{curl} & 0 \end{pmatrix}, \quad U = \begin{pmatrix} E \\ H \end{pmatrix}$$

we consider the quadratically nonlinear Maxwell system

$$\partial_t(M(\partial_t)U + N^{(2)}(U,U)) + \mathcal{A}U = 0, \quad t > 0. \tag{4.1.1}$$

Remark 4.1.1. Strictly speaking, we do not yet have well-posedness of the latter nonlinear system for this type of nonlinearity, as a multilinear map $Q\colon [\Omega]^2 \to \Omega$ will typically not extend to a bilinear map $Q\colon [L^2(\Omega)^6]^2 \to L^2(\Omega)^6$. Indeed, $N^{(2)}(U,U)$ should be viewed as a local approximation of a nonlocal operator $P^{(2)}(U)$, given as in Example 3.3.23, after the spatial convolution is replaced by suitable effective coefficients. This is done out of convenience, making the explicit computation of the residual easier without having to work in Fourier space (where the convolution is transformed into a product).

In principle, this poses no difficulty, as we will only use equation (4.1.1) to refine the ansatz functions, which are fixed and well-behaved. Similar to Lemma 4.2.1 which is concerned with a convolution in time, this approximation can be justified if the spatial convolution kernel is sufficiently regular. ◊

Recall from Section 1.3 that if $k, \omega \in \mathbb{C} \setminus \{0\}$ satisfy

$$\text{either} \quad k^2 = \omega^2 \frac{\epsilon^+(\omega)\epsilon^-(\omega)}{\epsilon^+(\omega)+\epsilon^-(\omega)} \cdot \frac{\epsilon^-(\omega)\mu^+(\omega)-\epsilon^+(\omega)\mu^-(\omega)}{\epsilon^-(\omega)-\epsilon^+(\omega)} \tag{4.1.2}$$

$$\text{or} \quad k^2 = \omega^2 \frac{\mu^+(\omega)\mu^-(\omega)}{\mu^+(\omega)+\mu^-(\omega)} \cdot \frac{\epsilon^+(\omega)\mu^-(\omega)-\epsilon^-(\omega)\mu^+(\omega)}{\mu^-(\omega)-\mu^+(\omega)}, \tag{4.1.3}$$

with $\epsilon^\pm(\omega) = \epsilon_0 + \check{\chi}^\pm_{\mathrm{el}}(\omega)$, $\mu^\pm(\omega) = \mu_0 + \check{\chi}^\pm_{\mathrm{m}}(\omega)$, where $\check{\chi}(\omega) = \int \chi(t)e^{i\omega t}\,\mathrm{d}t$. Then the linearized system admits a family of solutions

$$U_{\mathrm{2D}}(t,x) = \Phi(x_1)e^{i(kx_2-\omega t)} + \text{c.c.},$$

constant in x_3, with $\Phi\colon \mathbb{R} \to \mathbb{R}^6$ smooth on each half-space and having exponential decay away from the interface. In either of the two cases, the field U_{2D} is transverse magnetic or transverse electric, respectively. On the other hand, if neither of the dispersion relations (4.1.2) or (4.1.3) are fulfilled, then the linear system resulting for Φ admits only the trivial solution $\Phi = 0$.

We reformulate this fact using a more compact notation. With the matrices

$$S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

we have $\operatorname{curl} = S_1\partial_{x_1} + S_2\partial_{x_2} + S_3\partial_{x_3}$, and introducing the family of operators

$$\begin{aligned} &\mathbf{\Lambda}(k,\omega)\colon \left(L^2(\mathbb{R}) \times H^1(\mathbb{R}) \times H^1(\mathbb{R})\right)^2 \subset L^2(\mathbb{R})^6 \to L^2(\mathbb{R})^6, \\ &\mathbf{\Lambda}(k,\omega) := -i\omega \begin{pmatrix} \epsilon_0 + \check{\chi}^\pm_{\mathrm{el}}(\omega) & 0 \\ 0 & \mu_0 + \check{\chi}^\pm_{\mathrm{m}}(\omega) \end{pmatrix} + \begin{pmatrix} 0 & -S_1 \\ S_1 & 0 \end{pmatrix} \partial_{x_1} + \begin{pmatrix} 0 & -S_2 \\ S_2 & 0 \end{pmatrix} ik, \end{aligned} \tag{4.1.4}$$

we observe that

$$\left(\partial_t M(\partial_t) + \mathcal{A}\right)\Phi(x_1)e^{i(kx_2-\omega t)} = \mathbf{\Lambda}(k,\omega)\Phi(x_1)e^{i(kx_2-\omega t)}.$$

The characterization in Section 1.3 gives proof of the following.

Lemma 4.1.2. *Let $k, \omega \neq 0$ be such that $\epsilon^\pm(\omega), \mu^\pm(\omega) \neq 0$ and $k^2 - \omega^2\epsilon^\pm(\omega)\mu^\pm(\omega) \notin \mathbb{R}^-_0$.*

Then,

$$\dim\ker\mathbf{\Lambda}(k,\omega) = \begin{cases} 1, & \textit{if } k,\omega \textit{ satisfy either } (4.1.2) \textit{ or } (4.1.3) \\ 0, & \textit{else.} \end{cases}$$

Although solutions k,ω of the dispersion relation given by either (4.1.2) or (4.1.3) are in general complex, we will restrict our attention to $k \in \mathbb{R}$ to avoid exponential growth as $x_2 \to \pm\infty$. We also assume that $\epsilon^\pm, \mu^\pm$ are rational functions such that solving the dispersion relation for ω with $k \in \mathbb{R}$ yields a finite set of complex-valued curves $k \mapsto \omega(k)$. For the subsequent approximation we will assume that one can isolate one (smooth) dispersion curve and an open interval $I \subset (0,\infty)$ containing a 'critical' wavenumber $k_c > 0$ such that

(D1) $\operatorname{Re}\omega(k) > 0$ and $\operatorname{Im}\omega(k) < 0$ for all $k \in I$.

(D2) $\operatorname{Im}\omega(\cdot)$ attains a local maximum in k_c, thus $c_g := \omega'(k_c) \in \mathbb{R}$ and $\operatorname{Im}\omega''(k_c) < 0$.

The first property means that for all $k \in I$ the wavetrain $e^{i(kx_2-\omega(k)t)}$ has a positive phase velocity $\omega(k)/k$ in x_2-direction and is exponentially damped for $t > 0$. The second ensures that this damping attains a local minimum in k_c, see Figure 4.2.

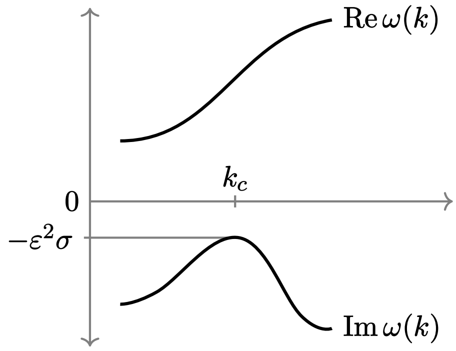

Figure 4.2: Dispersion curve $k \mapsto \omega(k)$ satisfying conditions (D1), (D2) with a critical wavenumber k_c.

For $k \in I$ we write Φ_k to denote an arbitrary but fixed nonzero element in $\ker\mathbf{\Lambda}(k,\omega(k))$ and assume that the mapping $k \mapsto \Phi_k$ is smooth (e.g. as a map $I \to L^2(\mathbb{R})^6$). We write $\mathcal{F}_{x_2\to k}$ to denote the Fourier transform

$$(\mathcal{F}_{x_2\to k}\, u)(k) = \hat{u}(k) = \int_{\mathbb{R}} u(x_2)e^{-ikx_2}\,\mathrm{d}x_2,$$

and $\mathcal{F}^{-1}_{k\to x_2}$ for its inverse. By superposition, integrating over multiple linear modes,

$$\frac{1}{2\pi}\int_{\mathbb{R}} \hat{a}(k)\Phi_k(x_1)e^{i(kx_2-\omega(k)t)}\,\mathrm{d}k = \mathcal{F}^{-1}_{k\to x_2}\left[\hat{a}(k)\Phi_k(x_1)e^{-i\omega(k)t}\right](x_2)$$

yields again a solution of Maxwell's equations, thus $\Phi_k(x_1)e^{-i\omega(k)t}$ can be regarded as a linear mode in Fourier space.

We now introduce a perturbation parameter ε by the properties (D1), (D2) of the dispersion curve (Figure 4.2): we write

$$\omega(k_c) = \omega_c - i\varepsilon^2\sigma, \quad \omega_c, \sigma > 0,\ 0 < \varepsilon \ll 1. \tag{4.1.5}$$

Based on this, the approximation will employ a multiple-scale ansatz of the form

$$U_\varepsilon(t,x) = \varepsilon A(T,X_2,X_3)\,\Phi_{k_c}(x_1)e^{i(k_c x_2-\omega_c t)} + \text{c.c.} + O(\varepsilon^2) \qquad (\varepsilon \to 0),$$

where A is a complex-valued amplitude which depends only on the slow variables

$$X_2 = \varepsilon(x_2 - c_{\mathrm{g}}t), \quad X_3 = \varepsilon^2 x_3, \quad T = \varepsilon^2 t.$$

For a spatially localized amplitude A, the result is a wavepacket travelling with group velocity c_{g} and phase velocity ω_c/k_c, see Figure 4.1. Changes in the overall shape occur on the scale of $O(\varepsilon^{-2})$ in time and in x_3-direction.

However, due to quadratic self-interaction, a modulated linear mode with a single frequency ω_c is not suitable to approximate a solution of the nonlinear system: Take for instance a time-harmonic field $U(t,x) = \Psi(x)e^{-i\omega_0 t}$ oscillating with a base frequency ω_0 and insert it into the nonlinearity $N^{(2)}$. The resulting field is of the form

$$\begin{aligned} N^{(2)}(U,U)(t) &= \int_{\mathbb{R}}\int_{\mathbb{R}} \chi^{(2)}(\tau_1,\tau_2)Q(U(t-\tau_1),U(t-\tau_2))\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2 \\ &= \Big(\int_{\mathbb{R}}\int_{\mathbb{R}} \chi^{(2)}(\tau_1,\tau_2)Q(\Psi,\Psi)e^{i\omega_0(\tau_1+\tau_2)}\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2\Big)e^{-2i\omega_0 t} \end{aligned}$$

and oscillates with doubled frequency $2\omega_0$. To account for this interaction we work with an extended ansatz

$$\begin{aligned} U_\varepsilon(t,x) := {}& \varepsilon A_1(T,X_2,X_3)\phi_1(x_1)e_1 + \varepsilon A_{-1}(T,X_2,X_3)\phi_{-1}(x_1)e_{-1} \\ &+ \varepsilon^2 A_2(T,X_2,X_3)\phi_2(x_1)e_2 + \varepsilon^2 A_{-2}(T,X_2,X_3)\phi_{-2}(x_1)e_{-2} \\ &+ \varepsilon^2 A_0(T,X_2,X_3)\phi_0(x_1)e_0 + O(\varepsilon^3), \end{aligned} \tag{4.1.6}$$

where for all $j \in \{-2,-1,0,1,2\}$,

$$A_{-j} = \overline{A_j}, \quad \phi_{-j} = \overline{\phi_j}, \quad e_j = e_j(x_2,t) := e^{ij(k_c x_2 - \omega_c t)}.$$

Here ϕ_1 should still be (an approximation of) an evanescent profile $\Phi_{k_c} \in \ker \mathbf{\Lambda}(k_c,\omega(k_c))$, while ε^2-terms serve as higher-order corrections; they will depend on A_1 and ϕ_1 in such a way that $\mathrm{Res}(U_\varepsilon)$ is formally of order $O(\varepsilon^4)$. We will subsequently provide a reasoning to refine this ansatz further; a more involved approximation is given in (4.1.27).

Introducing

$$\Psi_j(T,X_2,X_3;t,x_1,x_2) := \varepsilon\, A_j(T,X_2,X_3)\,\phi_j(x_1)e_j, \tag{4.1.7}$$

the expression (4.1.6) can be shortened to

$$U_\varepsilon = \Psi_1 + \Psi_{-1} + \varepsilon(\Psi_2 + \Psi_{-2} + \Psi_0) + O(\varepsilon^3).$$

Furthermore, it will be convenient to work with the Fourier transform $\mathcal{F}_{x_2\to k}$. With

$$K_j := \frac{k - jk_c}{\varepsilon}, \quad f_j = f_j(k,t) := \exp\{-it(j\omega_c + \varepsilon c_{\mathrm{g}} K_j)\}$$

we have the correspondence

$$
\begin{aligned}
\varepsilon \int_{\mathbb{R}} a(X_2) e_j(x_2,t)\, \mathrm{d}x_2 &= \varepsilon \int_{\mathbb{R}} a(X_2) e^{ij(k_c x_2 - \omega_c t) - ikx_2}\, \mathrm{d}x_2 \\
&= \varepsilon \int_{\mathbb{R}} a(X_2) e^{-i((k-jk_c)x_2 + j\omega_c t)}\, \mathrm{d}x_2 \\
&= \int_{\mathbb{R}} a(X_2) e^{-i((k-jk_c)(c_{\mathrm{g}} t + X_2/\varepsilon) + j\omega_c t)}\, \mathrm{d}X_2 \\
&= \int_{\mathbb{R}} a(X_2) e^{-i(X_2(k-jk_c)/\varepsilon)}\, \mathrm{d}X_2\, e^{-it(c_{\mathrm{g}}(k-jk_c)+j\omega_c)} \\
&= \hat{a}\big(\tfrac{k-jk_c}{\varepsilon}\big) e^{-it(j\omega_c + \varepsilon c_{\mathrm{g}}(k-jk_c)/\varepsilon)} \\
&= \hat{a}(K_j) f_j(k,t).
\end{aligned} \tag{4.1.8}
$$

Consequently, we can express the Fourier transformed ansatz $\hat{U}_\varepsilon(t,\hat{x}) = \mathcal{F}_{x_2\to k}[U_\varepsilon(t,x)](k)$, $\hat{x} = (x_1,k,x_3)$, in (4.1.6) as

$$
\begin{aligned}
\hat{U}_\varepsilon(t,\hat{x}) &= \hat{A}_1(T,K_1,X_3)\phi_1(x_1) f_1 + \hat{A}_{-1}(T,K_{-1},X_3)\phi_{-1}(x_1) f_{-1} \\
&\quad + \varepsilon \hat{A}_2(T,K_2,X_3)\phi_2(x_1) f_2 + \varepsilon \hat{A}_{-2}(T,K_{-2},X_3)\phi_{-2}(x_1) f_{-2} \\
&\quad + \varepsilon \hat{A}_0(T,K_0,X_3)\phi_0(x_1) f_0 + O(\varepsilon^2) \\
&= \widehat{\Psi}_1(T,K_1,X_3;t,x_1) + \widehat{\Psi}_{-1}(T,K_{-1},X_3;t,x_1) \\
&\quad + \varepsilon \widehat{\Psi}_2(T,K_2,X_3;t,x_1) + \varepsilon \widehat{\Psi}_{-2}(T,K_{-2},X_3;t,x_1) \\
&\quad + \varepsilon \widehat{\Psi}_0(T,K_0,X_3;t,x_1) + O(\varepsilon^2).
\end{aligned} \tag{4.1.9}
$$

We assume that the amplitudes satisfy the asymptotic $\widehat{A}_j(T,K_j,X_3) \to 0$ as $|K_j| \to \infty$ to ensure that $\widehat{A}_j(T,K_j,X_3)$ is small unless $|k - jk_0| = O(\varepsilon)$. This means that the function $\hat{U}_\varepsilon$ is concentrated around integer multiples of k_c, which enables us to treat each f_j-term separately by localizing $k = jk_c + \varepsilon K_j$ around jk_c (and assuming bounded values for K_j), since then $\hat{U}_\varepsilon$ is very small elsewhere. The same will be true for the Fourier-transformed residual $\widehat{\mathrm{Res}}(U_\varepsilon) := \mathcal{F}_{x_2\to k}\mathrm{Res}(U_\varepsilon)$. To compute $\widehat{\mathrm{Res}}(U_\varepsilon)$ we make some observations.

Recalling that $\mathrm{curl} = \sum_{j=1}^{3} S_j \partial_{x_j}$, we have for the spatial Maxwell operator

$$
\mathcal{A} = \begin{pmatrix} 0 & -\,\mathrm{curl} \\ \mathrm{curl} & 0 \end{pmatrix} = \sum_{j=1}^{3} \mathbf{S}_j \partial_{x_j}, \quad \text{where} \quad \mathbf{S}_j = \begin{pmatrix} 0 & -S_j \\ S_j & 0 \end{pmatrix}.
$$

Prepending the Fourier transform $\mathcal{F}_{x_2\to k}$ we have

$$
\begin{aligned}
\mathcal{F}_{x_2\to k}\mathcal{A} &= \mathbf{S}_1 \partial_{x_1} + ik\mathbf{S}_2 + \mathbf{S}_3 \partial_{x_3} \\
&= \mathbf{S}_1 \partial_{x_1} + i(jk_c + \varepsilon K_j)\mathbf{S}_2 + \varepsilon^2 \mathbf{S}_3 \partial_{X_3}
\end{aligned} \tag{4.1.10}
$$

for any $j \in \{-2,-1,0,1,2\}$. Formally, this identity determines the action of $\mathcal{A}$ on every Ψ_j in Fourier space, i.e.,

$$
\mathcal{F}_{x_2\to k}\left(\mathcal{A}\Psi_j\right) = \left(\mathbf{S}_1 \partial_{x_1} + i(jk_c + \varepsilon K_j)\mathbf{S}_2 + \varepsilon^2 \mathbf{S}_3 \partial_{X_3}\right)\widehat{\Psi}_j. \tag{4.1.11}
$$

Introducing

$$
\Omega_j := j\omega_c + \varepsilon c_{\mathrm{g}} K_j,
$$

we observe for the time-derivative that

$$\mathcal{F}_{x_2\to k}\left[\partial_t\Psi_j\right] = \partial_t\widehat{\Psi}_j = (-i\Omega_j + \varepsilon^2\partial_T)\widehat{\Psi}_j.$$

The strategy for dealing with convolutions in time will rely on Taylor expansions of the integrands. For sufficiently regular $a, \kappa\colon \mathbb{R}\to\mathbb{R}$ (that is, integrable up to some derivative, cf. Lemma 4.2.1) we have

$$\begin{aligned}
\int_{\mathbb{R}} \kappa(\tau)\, a(\varepsilon^2(t-\tau))\, f_j(t-\tau)\,\mathrm{d}\tau &= \int_{\mathbb{R}} \kappa(\tau)\, a(\varepsilon^2(t-\tau))\, e^{-i(t-\tau)\Omega_j}\,\mathrm{d}\tau \\
&= \int_{\mathbb{R}} \kappa(\tau)\left[a(T) - \varepsilon^2\tau a'(T) + O(\varepsilon^4\tau^2)\right]e^{i\tau\Omega_j}\,\mathrm{d}\tau\, e^{-it\Omega_j} \\
&= \check{\kappa}(\Omega_j)a(T)f_j + i\varepsilon^2\check{\kappa}'(\Omega_j)a'(T)f_j + O(\varepsilon^4)
\end{aligned}$$

where we used $\int_{\mathbb{R}} \tau\kappa(\tau)e^{i\omega\tau}\,\mathrm{d}\tau = -i\check{\kappa}'(\omega)$. Taking derivatives and using

$$\partial_t\left[a(T)f_j(t)\right] = (-i\Omega_j + \varepsilon^2\partial_T)a(T)f_j \tag{4.1.12}$$

we obtain

$$\begin{aligned}
&\partial_t\int_{\mathbb{R}} \kappa(\tau)a(\varepsilon^2(t-\tau))f_j(t-\tau)\,\mathrm{d}\tau \\
&\qquad = (-i\Omega_j + \varepsilon^2\partial_T)\left[\check{\kappa}(\Omega_j)a(T)f_j + i\varepsilon^2\check{\kappa}'(\Omega_j)a'(T)f_j + O(\varepsilon^4)\right] \\
&\qquad = -i\Omega_j\check{\kappa}(\Omega_j)a(T)f_j + \varepsilon^2(\check{\kappa}(\Omega_j) + \Omega_j\check{\kappa}'(\Omega_j))a'(T)f_j + O(\varepsilon^4).
\end{aligned}$$

Thus, setting

$$M_0 := \begin{pmatrix}\epsilon_0 & 0\\ 0 & \mu_0\end{pmatrix}, \quad \chi := \begin{pmatrix}\chi_{\mathrm{el}} & 0\\ 0 & \chi_{\mathrm{m}}\end{pmatrix}$$

we can formally write

$$\begin{aligned}
\mathcal{F}_{x_2\to k}\left[\partial_t M(\partial_t)\Psi_j\right] &= \mathcal{F}_{x_2\to k}\left[\partial_t(M_0\Psi_j + \chi*\Psi_j)\right] \\
&= \partial_t\mathcal{F}_{x_2\to k}\left[(M_0\Psi_j + \chi*\Psi_j)\right] \\
&= -i\Omega_j\left[M_0 + \check{\chi}(\Omega_j)\right]\hat{\Psi}_j(T,K_j) \\
&\quad + \varepsilon^2\left[M_0 + \check{\chi}(\Omega_j) + \Omega_j\check{\chi}'(\Omega_j)\right]\partial_T\hat{\Psi}_j(T,K_j) + O(\varepsilon^4)
\end{aligned} \tag{4.1.13}$$

Now combining (4.1.11) and (4.1.13) we obtain for the linear part

$$\begin{aligned}
\mathcal{F}_{x_2\to k}\left[(\partial_t M(\partial_t) + \mathcal{A})\Psi_j\right] &= \mathcal{F}_{x_2\to k}\left[\partial_t(M_0\Psi_j + \chi*\Psi_j) + \mathcal{A}\Psi_j\right] \\
&= -i\Omega_j\left[M_0 + \check{\chi}(\Omega_j)\right]\Psi_j + \varepsilon^2\left[M_0 + \check{\chi}(\Omega_j) + \Omega_j\check{\chi}'(\Omega_j)\right]\partial_T\Psi_j \\
&\quad + \left[\mathbf{S}_1\partial_{x_1} + i(jk_c + \varepsilon K_j)\mathbf{S}_2 + \varepsilon^2\mathbf{S}_3\partial_{X_3}\right]\Psi_j + O(\varepsilon^4) \\
&= \left[-i\Omega_j(M_0 + \check{\chi}(\Omega_j)) + \mathbf{S}_1\partial_{x_1} + i(jk_c + \varepsilon K_j)\mathbf{S}_2\right]\Psi_j \\
&\quad + \varepsilon^2\left[M_0 + \check{\chi}(\Omega_j) + \Omega_j\check{\chi}'(\Omega_j)\right]\partial_T\Psi_j + \varepsilon^2\mathbf{S}_3\partial_{X_3}\Psi_j + O(\varepsilon^4) \\
&= \mathbf{\Lambda}(jk_c + \varepsilon K_j, \Omega_j)\Psi_j + \varepsilon^2\mathbf{J}(\Omega_j)\partial_T\Psi_j + \varepsilon^2\mathbf{S}_3\partial_{X_3}\Psi_j + O(\varepsilon^4),
\end{aligned} \tag{4.1.14}$$

where

$$\mathbf{J}(\Omega_j) := M_0 + \check{\chi}(\Omega_j) + \Omega_j \check{\chi}'(\Omega_j),$$

and where $\mathbf{\Lambda}(k,\omega)$ was defined in (4.1.4), hence

$$\begin{aligned} -i\Omega_j(M_0 + \check{\chi}(\Omega_j)) + \mathbf{S}_1\partial_{x_1} + i(jk_c + \varepsilon K_j)\mathbf{S}_2 &= \mathbf{\Lambda}(jk_c + \varepsilon K_j, \Omega_j) \\ &= \mathbf{\Lambda}(jk_c + \varepsilon K_j, j\omega_c + \varepsilon c_{\mathrm{g}} K_j). \end{aligned}$$

For future reference, we note that the latter expression can be expanded in two ways. The first consists of a localization around $(jk_c, j\omega_c)$ we will use in the case $j \neq \pm 1$. Here we find for $k = jk_c + \varepsilon K_j$, $\Omega_j = j\omega_c + \varepsilon c_{\mathrm{g}} K_j$,

$$\begin{aligned} \mathbf{\Lambda}(k, \Omega_j) &= \mathbf{\Lambda}(jk_c + \varepsilon K_j, j\omega_c + \varepsilon c_{\mathrm{g}} K_j) \\ &= -i(j\omega_c + \varepsilon c_{\mathrm{g}} K_j)(M_0 + \check{\chi}(j\omega_c + \varepsilon c_{\mathrm{g}} K_j)) + \mathbf{S}_1\partial_{x_1} + i(jk_c + \varepsilon K_j)\mathbf{S}_2 \\ &= -ij\omega_c(M_0 + \check{\chi}(j\omega_c)) + \mathbf{S}_1\partial_{x_1} + ijk_c\mathbf{S}_2 \\ &\quad - i\varepsilon c_{\mathrm{g}}(M_0 + \check{\chi}(j\omega_c) + j\omega_c\check{\chi}'(j\omega_c))K_j + i\varepsilon\mathbf{S}_2 K_j + O(\varepsilon^2) \\ &= \mathbf{\Lambda}(jk_c, j\omega_c) - i\varepsilon c_{\mathrm{g}} K_j \mathbf{J}(j\omega_c) + i\varepsilon K_j \mathbf{S}_2 + O(\varepsilon^2). \end{aligned} \tag{4.1.15}$$

For $j = 1$ (and similarly for $j = -1$ by complex conjugation) we argue that if $k = k_c + \varepsilon K_1$, then $\Omega_1 = \omega_c + \varepsilon c_{\mathrm{g}} K_1$ is $O(\varepsilon)$-close to $\omega(k)$. Recalling from (4.1.5) that $\omega(k_c) = \omega_c - \varepsilon^2 i\sigma$ we obtain the following consecutive expansions:

$$\begin{aligned} \omega(k) = \omega(k_c + \varepsilon K_1) &= \omega(k_c) + \varepsilon\omega'(k_c)K_1 + \tfrac{1}{2}\varepsilon^2\omega''(k_c)K_1^2 + O(\varepsilon^3) \\ &= \omega_c + \varepsilon c_{\mathrm{g}} K_1 + \varepsilon^2(\tfrac{1}{2}\omega''(k_c)K_1^2 - i\sigma) + O(\varepsilon^3) \\ \mathbf{\Lambda}(k, \Omega_1) &= -i\Omega_1(M_0 + \check{\chi}(\Omega_1)) + \mathbf{S}_1\partial_{x_1} + ik\mathbf{S}_2 \\ &= -i\omega(k)(M_0 + \check{\chi}(\omega(k))) + \mathbf{S}_1\partial_{x_1} + ik\mathbf{S}_2 \\ &\quad + \varepsilon^2(\sigma + \tfrac{1}{2}i\omega''(k_c)K_1^2)(M_0 + \check{\chi}(\omega(k)) + \omega(k)\check{\chi}'(\omega(k))) + O(\varepsilon^3) \\ &= \mathbf{\Lambda}(k, \omega(k)) + \varepsilon^2(\sigma + \tfrac{1}{2}i\omega''(k_c)K_1^2)\mathbf{J}(\omega(k)) + O(\varepsilon^3) \\ &= \mathbf{\Lambda}(k, \omega(k)) + \varepsilon^2(\sigma + \tfrac{1}{2}i\omega''(k_c)K_1^2)\mathbf{J}(\omega_c) + O(\varepsilon^3). \end{aligned} \tag{4.1.16}$$

4.1.2 Sum-frequency generation and correction

For the nonlinear terms $N^{(2)}(\Psi_j, \Psi_\ell)$ we approximate the double convolution in time using

$$\begin{aligned} A(\varepsilon^2(t-\tau), \varepsilon(x_2 - c_{\mathrm{g}}(t-\tau))) &= A(T - \varepsilon^2\tau, X_2 + \varepsilon c_{\mathrm{g}}\tau) \\ &= (1 - \varepsilon^2\tau\partial_T + \varepsilon c_{\mathrm{g}}\tau\partial_{X_2})A(T, X_2) + O(\varepsilon^4\tau^2 + \varepsilon^2\tau^2). \end{aligned} \tag{4.1.17}$$

Introducing

$$\check{\chi}^{(2)}(\omega_1, \omega_2) := \int_{\mathbb{R}}\int_{\mathbb{R}} \chi^{(2)}(\tau_1, \tau_2) e^{i\omega_1\tau_1 + i\omega_2\tau_2}\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2,$$

we find for $a_j(t) := \varepsilon A_j(T, X_2)e_j = \varepsilon A_j(\varepsilon^2 t, \varepsilon(x_2 - c_{\mathrm{g}} t))e^{ij(k_c x_2 - \omega_c t)}$, using (4.1.17) and (4.1.8),

$$\mathcal{F}_{x_2\to k}\int_{\mathbb{R}}\int_{\mathbb{R}} \chi^{(2)}(\tau_1, \tau_2) a_j(t - \tau_1) a_\ell(t - \tau_2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2$$

$$
\begin{aligned}
&= \varepsilon(\widehat{A_j A_\ell})(T, K_{j+\ell})\check{\chi}^{(2)}(j\omega_c, \ell\omega_c) f_{j+\ell} \\
&\quad - i\varepsilon^2 c_{\mathrm{g}}\, (\widehat{A_\ell \partial_{X_2} A_j})(T, K_{j+\ell})\, (\partial_1 \check{\chi}^{(2)})(j\omega_c, \ell\omega_c) f_{j+\ell} \qquad (4.1.18)\\
&\quad - i\varepsilon^2 c_{\mathrm{g}}\, (\widehat{A_j \partial_{X_2} A_\ell})(T, K_{j+\ell})\, (\partial_2 \check{\chi}^{(2)})(j\omega_c, \ell\omega_c) f_{j+\ell} + O(\varepsilon^3).
\end{aligned}
$$

Here the higher-order powers of τ_1, τ_2 occurring in the remainder terms of (4.1.17) have been converted into derivatives of $\check{\chi}^{(2)}$, according to

$$
\int_{\mathbb{R}} \int_{\mathbb{R}} \tau_j \chi^{(2)}(\tau_1, \tau_2) e^{i\omega_1\tau_1 + i\omega_2\tau_2}\, \mathrm{d}\tau_1\, \mathrm{d}\tau_2 = -i\partial_j \check{\chi}^{(2)}(\omega_1, \omega_2) \quad (j \in \{1, 2\})
$$

and similar formulas for the higher derivatives. Next, thanks to the bilinearity of Q, we obtain an approximation of

$$
\widehat{N}^{(2)}(\Psi_j, \Psi_\ell) := \mathcal{F}_{x_2 \to k} N^{(2)}(\Psi_j, \Psi_\ell)
$$

by multiplying this last expression by $Q(\phi_j, \phi_\ell)$. For the time-derivative we use again (4.1.12) and obtain

$$
\begin{aligned}
\mathcal{F}_{x_2 \to k} \partial_t N^{(2)}(\Psi_j, \Psi_\ell) &= \partial_t \widehat{N}^{(2)}(\Psi_j, \Psi_\ell) \\
&= -i\Omega_{j+\ell} \widehat{N}^{(2)}(\Psi_j, \Psi_\ell) + O(\varepsilon^3) \\
&= -\hat{A}_{\varepsilon,j,\ell}(T, K_{j+\ell}) Q(\phi_j, \phi_\ell) f_{j+\ell} + O(\varepsilon^3), \qquad (4.1.19)
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{A}_{\varepsilon,j,\ell}(T, K_{j+\ell}) &= i\varepsilon(j+\ell)\omega_c (\widehat{A_j A_\ell})(T, K_{j+\ell})\check{\chi}^{(2)}(j\omega_c, \ell\omega_c) \\
&\quad + i\varepsilon^2 c_{\mathrm{g}} K_{j+\ell} (\widehat{A_j A_\ell})(T, K_{j+\ell})\check{\chi}^{(2)}(j\omega_c, \ell\omega_c) \\
&\quad + \varepsilon^2 (j+\ell)\omega_c c_{\mathrm{g}} (\widehat{A_\ell \partial_{X_2} A_j})(T, K_{j+\ell})(\partial_1 \check{\chi}^{(2)})(j\omega_c, \ell\omega_c) \\
&\quad + \varepsilon^2 (j+\ell)\omega_c c_{\mathrm{g}} (\widehat{A_j \partial_{X_2} A_\ell})(T, K_{j+\ell})(\partial_2 \check{\chi}^{(2)})(j\omega_c, \ell\omega_c).
\end{aligned}
$$

Here (4.1.19) illustrates the mechanism of sum-frequency generation; the interaction of two fields with frequencies $j\omega_c$ and $\ell\omega_c$ creates a field with the frequency $\omega = (j+\ell)\omega_c$, expressed by the $f_{j+\ell}$-term in (4.1.19). We can now sort the residual in Fourier space by these different $f_{j+\ell}$-terms.

Second-harmonic generation

The nonlinear f_2-terms in the residual are given by the interaction of Ψ_1 with itself, and should be compensated by Ψ_2. In detail we consider $k = 2k_c + \varepsilon K_2$, $\Omega_2 = 2\omega_c + \varepsilon c_{\mathrm{g}} K_2$ and have, using (4.1.14) for the linear and (4.1.19) for the nonlinear part,

$$
\begin{aligned}
\widehat{\mathrm{Res}}(U_\varepsilon; f_2) &:= \mathcal{F}_{x_2 \to k}\Big[(\partial_t M(\partial_t) + \mathcal{A})\varepsilon\Psi_2 + \partial_t N^{(2)}(\Psi_1, \Psi_1)\Big] \\
&= \mathbf{\Lambda}(2k_c + \varepsilon K_2, 2\omega_c + \varepsilon c_{\mathrm{g}} K_2)\varepsilon\Psi_2 - i\Omega_2 \widehat{N}^{(2)}(\Psi_1, \Psi_1) + O(\varepsilon^3) \\
&= \varepsilon \widehat{A}_2(K_2)\mathbf{\Lambda}(2k_c, 2\omega_c)\phi_2 f_2 - \varepsilon 2i\omega_c \check{\chi}^{(2)}(\omega_c, \omega_c)\widehat{A_1^2}(K_2)\, Q(\phi_1, \phi_1) f_2 + O(\varepsilon^2).
\end{aligned}
\qquad (4.1.20)
$$

Setting $A_2 := A_1^2$ allows us to extract a common amplitude, and the last expression becomes

$$\widehat{\mathrm{Res}}(U_\varepsilon; f_2) = \varepsilon \widehat{A_1^2}(K_2)\big[\mathbf{\Lambda}(2k_c, 2\omega_c)\phi_2 - 2i\omega_c\check{\chi}^{(2)}(\omega_c, \omega_c)Q(\phi_1, \phi_1)\big] f_2 + O(\varepsilon^2). \tag{4.1.21}$$

A sufficient condition to eliminate the bracket is given below. Recall that ϕ_1 is an evanescent field, i.e., $|\phi_1(x_1)| \lesssim e^{-\lambda x_1}$ $(x_1 \in \mathbb{R})$, for some $\lambda > 0$, which is continuous outside the interface. As such, $|Q(\phi_1, \phi_1)| \lesssim e^{-2\lambda x_1}$; in particular we may assume $Q(\phi_1, \phi_1) \in L^2(\mathbb{R})^6$.

Proposition 4.1.3. *Let $j \in \mathbb{Z}$ and assume the non-resonance condition*

$$\omega(jk_c) + j\omega_c \neq 0, \tag{4.1.22}$$

in the sense that the pair $(jk_c, -j\omega_c)$ does not fulfill the dispersion relation. Then, for every $\delta > 0$ and $\psi \in L^2(\mathbb{R})^6$ there exists $\phi \in \operatorname{dom} \mathbf{\Lambda}$ with $\|\mathbf{\Lambda}(jk_c, j\omega_c)\phi - \psi\|_{L^2} < \delta$.

Proof. Since $\overline{\chi(t)} = \chi(t)$ we have $\overline{\epsilon(\omega)} = \epsilon(-\omega)$ for all $\omega \in \mathbb{R}$, from which it easily follows that $\mathbf{\Lambda}(jk_c, j\omega_c)^* = -\mathbf{\Lambda}(jk_c, -j\omega_c)$. Now (4.1.22) yields

$$\{0\} = \ker \mathbf{\Lambda}(jk_c, -j\omega_c) = \ker \mathbf{\Lambda}(jk_c, j\omega_c)^* = \big(\operatorname{ran} \mathbf{\Lambda}(jk_c, j\omega_c)\big)^\perp$$

by Lemma 4.1.2. Thus, the range of $\mathbf{\Lambda}(jk_c, j\omega_c)$ is dense in $L^2(\mathbb{R})^6$, which proves the claim. □

By an appropriate choice of ϕ_2 we can therefore assume that the f_2-residual $\widehat{\mathrm{Res}}(U_\varepsilon; f_2)$ is formally of order $O(\varepsilon^2)$, provided that the point $(-2k_c, 2\omega_c)$ does not lie on any dispersion curve, i.e., (4.1.22) holds for $j = 2$. To make it even smaller, we expand the nonlinear terms up to order $O(\varepsilon^2)$, using (4.1.19) and $\widehat{A\partial_{X_2} A}(K) = \frac{1}{2} iK\widehat{A^2}(K)$, and find

$$\begin{aligned} -i\Omega_2 \widehat{N}^{(2)}(\Psi_1, \Psi_1) = &-i\varepsilon 2\omega_c \widehat{A_1^2}(K_2)\check{\chi}^{(2)}(\omega_c, \omega_c)Q(\phi_1, \phi_1) f_2 \\ &- i\varepsilon^2 c_{\mathrm{g}} K_2 \widehat{A_1^2}(K_2)\,(1 + \omega_c(\partial_1 + \partial_2))\check{\chi}^{(2)}(\omega_c, \omega_c)\,Q(\phi_1, \phi_1) f_2 + O(\varepsilon^3). \end{aligned}$$

To compensate these higher-order terms, U_ε must be extended by an additional correction. This can be done by replacing Ψ_2 with $\Psi_{2,1} + \varepsilon\Psi_{2,2}$, where

$$\Psi_{2,j}(T, X_2, X_3; t, x_1, x_2) = \varepsilon A_{2,j}(T, X_2, X_3)\phi_{2,j}(x_1)e_2 \quad (j \in \{1, 2\}). \tag{4.1.23}$$

After setting $A_{2,1} = A_1^2$, $A_{2,2} = -i\partial_{X_2} A_1^2$ and using the expansion $\mathbf{\Lambda}(k, \Omega_2) = \mathbf{\Lambda}(2k_c, 2\omega_c) - i\varepsilon c_{\mathrm{g}} K_2 \mathbf{J}(2\omega_c) + i\varepsilon K_2 \mathbf{S}_2 + O(\varepsilon^2)$ from (4.1.15), the f_2-residual (4.1.20) takes the form

$$\begin{aligned} \widehat{\mathrm{Res}}(U_\varepsilon; f_2) &= \mathcal{F}_{x_2 \to k}\big[(\partial_t M(\partial_t) + \mathcal{A})(\varepsilon\Psi_{2,1} + \varepsilon^2\Psi_{2,2}) + \partial_t N^{(2)}(\Psi_1, \Psi_1)\big] \\ &= \mathbf{\Lambda}(2k_c + \varepsilon K_2, \Omega_2)(\varepsilon\widehat{\Psi}_{2,1} + \varepsilon^2\widehat{\Psi}_{2,2}) - i\Omega_2 \widehat{N}^{(2)}(\Psi_1, \Psi_1) + O(\varepsilon^3) \\ &= \varepsilon\widehat{A_1^2}(K_2)\big[\mathbf{\Lambda}(2k_c, 2\omega_c)\phi_{2,1} - 2i\omega_c \check{\chi}^{(2)}(\omega_c, \omega_c)Q(\phi_1, \phi_1)\big] f_2 \\ &\quad + \varepsilon^2 K_2 \widehat{A_1^2}(K_2)\big[\mathbf{\Lambda}(2k_c, 2\omega_c)\phi_{2,2} - (ic_{\mathrm{g}}\mathbf{J}(2\omega_c) - i\mathbf{S}_2)\phi_{2,1} \\ &\qquad - ic_{\mathrm{g}}(1 + \omega_c(\partial_1 + \partial_2))\check{\chi}^{(2)}(\omega_c, \omega_c)Q(\phi_1, \phi_1)\big] f_2 + O(\varepsilon^3). \end{aligned}$$

We can apply the previous argument recursively and find $\phi_{2,1}, \phi_{2,2}$ such that the f_2-residual is of order $O(\varepsilon^3)$; indeed, this process can be repeated to arbitrary order, since it only depends on the same non-resonance condition. The case for the f_{-2}-residual is conjugate.

Optical rectification

The case of the f_0-residual is special since the resonance $\omega(0) = 0$ in the dispersion relation cannot be avoided. Thus, we cannot use Proposition 4.1.3. Nonetheless, this case can be treated similarly.

Since $\Omega_0 = 0 + \varepsilon K_0$, the nonlinear f_0-terms given by the interaction of Ψ_1 with Ψ_{-1},

$$\begin{aligned}\mathcal{F}_{x_2\to k}[\partial_t N^{(2)}(\Psi_1, \Psi_{-1})] &= -i\Omega_0 \widehat{N}^{(2)}(\Psi_1, \Psi_{-1}) + O(\varepsilon^3) \\ &= -i\varepsilon^2 c_{\mathrm{g}} K_0 \widehat{|A_1|^2} \check{\chi}^{(2)}(\omega_c, -\omega_c) Q(\phi_1, \phi_{-1}) f_0 + O(\varepsilon^3),\end{aligned}$$

are of order $O(\varepsilon^2)$, suggesting that a correction of the same order might be sufficient to compensate these terms. However, it becomes apparent below that this is not so. Instead, similar to the f_2-correction, we replace $\varepsilon\Psi_0$ with $\varepsilon\Psi_{0,1} + \varepsilon^2\Psi_{0,2}$, where

$$\Psi_{0,j}(T, X_2, X_3; t, x_1, x_2) = \varepsilon A_{0,j}(T, X_2, X_3)\phi_{0,j}(x_1)e_0 \quad (j \in \{1, 2\}). \tag{4.1.24}$$

Setting $A_{0,1} = |A_1|^2$, $A_{0,2} = \partial_{X_2}|A_1|^2$ and using again the expansion (4.1.15), the f_0-residual amounts to

$$\begin{aligned}\widehat{\mathrm{Res}}(U_\varepsilon; f_0) &= \mathcal{F}_{x_2\to k}[(\partial_t M(\partial_t) + \mathcal{A})(\varepsilon\Psi_{0,1} + \varepsilon^2\Psi_{0,2}) + 2\partial_t(N^{(2)}(\Psi_1, \Psi_{-1})] \\ &= \varepsilon(\mathbf{\Lambda}(0,0) - i\varepsilon K_0(c_{\mathrm{g}}\mathbf{J}(0) - \mathbf{S}_2))\widehat{\Psi}_{0,1} + \varepsilon^2\mathbf{\Lambda}(0,0)\widehat{\Psi}_{0,2} \\ &\quad - 2i\varepsilon c_{\mathrm{g}} K_0 \widehat{N}^{(2)}(\Psi_1, \Psi_{-1}) + O(\varepsilon^3) \\ &= \varepsilon\widehat{|A_1|^2}(K_0)[\mathbf{\Lambda}(0,0)\phi_{0,1}] f_0 \\ &\quad + \varepsilon^2 i K_0 \widehat{|A_1|^2}(K_0)[\mathbf{\Lambda}(0,0)\phi_{0,2} - (c_{\mathrm{g}}\mathbf{J}(0) - \mathbf{S}_2)\phi_{0,1} - c_{\mathrm{g}}\psi_0] f_0 + O(\varepsilon^3),\end{aligned} \tag{4.1.25}$$

where $\psi_0 = 2\check{\chi}^{(2)}(\omega_c, -\omega_c)Q(\phi_1, \phi_{-1}) = \check{\chi}^{(2)}(\omega_c, -\omega_c)Q(\phi_1, \phi_{-1}) + \check{\chi}^{(2)}(-\omega_c, \omega_c)Q(\phi_{-1}, \phi_1)$. Equating the brackets to zero results in the two equations

$$\begin{cases}\mathbf{\Lambda}(0,0)\phi_{0,1} = 0 \\ \mathbf{\Lambda}(0,0)\phi_{0,2} = (c_{\mathrm{g}}\mathbf{J}(0) - \mathbf{S}_2)\phi_{0,1} + c_{\mathrm{g}}\psi_0\end{cases}$$

which we want to solve recursively. First, from $\mathbf{\Lambda}(0,0) = \partial_1\mathbf{S}_1$ we obtain

$$\ker\mathbf{\Lambda}(0,0) = \{v \in L^2(\mathbb{R})^6 : v_2 = v_3 = v_5 = v_6 = 0\},$$

and $\mathbf{\Lambda}(0,0)^* = -\mathbf{\Lambda}(0,0)$ implies further

$$\overline{\mathrm{ran}\,\mathbf{\Lambda}(0,0)} = (\ker\mathbf{\Lambda}(0,0)^*)^\perp = \{w \in L^2(\mathbb{R})^6 : w_1 = w_4 = 0\}.$$

Setting $v = \phi_{0,1} \in \ker\mathbf{\Lambda}(0,0)$ we have

$$(c_{\mathrm{g}}\mathbf{J}(0) - \mathbf{S}_2)v = \left(c_{\mathrm{g}}\epsilon(0)v_1, 0, -v_4, c_{\mathrm{g}}\mu(0)v_4, 0, v_1\right).$$

Thus, assuming $\epsilon(0), \mu(0)$ are invertible, v_1, v_4 can be adjusted in such a way that

$$w := (c_{\mathrm{g}}\mathbf{J}(0) - \mathbf{S}_2)\phi_{0,1} + c_{\mathrm{g}}\psi_0 \in \overline{\mathrm{ran}\,\mathbf{\Lambda}(0,0)}.$$

Consequently, we find $\phi_{0,2} \in \operatorname{dom}(\mathbf{\Lambda})$ with $\|\mathbf{\Lambda}(0,0)\phi_{0,2} - w\|_{L^2} = O(\varepsilon)$, obtaining that the f_0-residual (4.1.25) is $O(\varepsilon^3)$-small.

Remark 4.1.4. As mentioned, the derivation just shown only works if $\epsilon(0), \mu(0)$ are invertible. In some cases however, this cannot be expected. Notably, this is the case if the accretivity condition

$$\operatorname{Re} z > -\nu_0 \implies \operatorname{Re} zM(z) \geq c > 0, \qquad M(z) = \begin{pmatrix} \epsilon(z) & 0 \\ 0 & \mu(z) \end{pmatrix}$$

is imposed, since then $M(z)$ must have a singularity in $z = 0$, so $\epsilon(0), \mu(0)$ are not even well-defined. (Recall that this is a sufficient condition for exponential stability of the linear system, cf. Example 3.3.2.) Still, recalling that

$$\mathbf{\Lambda}(k,\omega) = -i\omega M(-i\omega) + \mathbf{S}_1\partial_{x_1} + ik\mathbf{S}_2,$$

it turns out that $\mathbf{\Lambda}(0,0)$ is in fact boundedly invertible; this follows since $\mathbf{S}_1\partial_{x_1}$ is skew-selfadjoint and

$$\operatorname{Re}(-i\omega M(-i\omega))_{\omega=0} = \operatorname{Re}(zM(z))_{z=0} > 0$$

by the accretivity assumption. Thus $\ker \mathbf{\Lambda}(0,0)^* = \ker \mathbf{\Lambda}(0,0) = \{0\}$ and we have $\phi_{0,1} = 0$. Then, (4.1.25) simplifies to

$$\begin{aligned}\mathcal{F}_{x_2\to k}\big[(\partial_t M(\partial_t) + \mathcal{A})(\varepsilon\Psi_{0,1} + \varepsilon^2\Psi_{0,2}) + \partial_t(N^{(2)}(\Psi_1, \Psi_{-1}) + N^{(2)}(\Psi_{-1}, \Psi_1))\big]& \\ = \varepsilon^2 iK_0\widehat{|A_1|^2}(K_0)\big[\mathbf{\Lambda}(0,0)\phi_{0,2} - c_{\mathrm{g}}\psi_0\big]f_0 + O(\varepsilon^3).&\end{aligned}$$

In this case, the equation $\mathbf{\Lambda}(0,0)\phi_{0,2} = c_{\mathrm{g}}\psi_0$ is solvable in $\phi_{0,2}$ for all right-hand sides $c_{\mathrm{g}}\psi_0 \in L^2(\mathbb{R})^6$. ◊

The amplitude equation

To determine the amplitude $A = A_1$ we now turn to the f_1- and f_{-1}-terms of the residual. We start with the main contribution to the residual (the sum of terms of lowest ε-order), which is the linear part of the Maxwell system applied to Ψ_1. In Fourier space we consider only the f_1-terms (since the setting for f_{-1} is conjugate). By (4.1.14) and the expansion in (4.1.16), these are given by

$$\begin{aligned}&\mathcal{F}_{x_2\to k}\big[(\partial_t M(\partial_t) + \mathcal{A})\Psi_1\big] \\ &\quad = \mathbf{\Lambda}(k, \Omega_1)\widehat{\Psi}_1 + \varepsilon^2\mathbf{J}(\Omega_1)\partial_T\widehat{\Psi}_1 + \varepsilon^2\mathbf{S}_3\partial_{X_3}\widehat{\Psi}_1 \\ &\quad = \mathbf{\Lambda}(k,\omega(k))\widehat{\Psi}_1 + \varepsilon^2\left[(\sigma + \tfrac{1}{2}i\omega''(k_c)K_1^2 + \partial_T)\mathbf{J}(\omega_c) + \mathbf{S}_3\partial_{X_3}\right]\widehat{\Psi}_1 + O(\varepsilon^3).\end{aligned} \tag{4.1.26}$$

Since $\mathbf{\Lambda}(k,\omega(k))\widehat{\Psi}_1 = \hat{A}_1(K_1)\mathbf{\Lambda}(k,\omega(k))\phi_1 f_1$, this term vanishes if $\phi_1 = \Phi_k \in \ker \mathbf{\Lambda}(k,\omega(k))$ (by Lemma 4.1.2, Φ_k is unique up to scaling). Alternatively, we may replace ϕ_1 with an approximation of $\Phi(k,\cdot) := \Phi_k$ given by the first terms in the expansion

$$\Phi(k,\cdot) = \Phi_{k_c} + \varepsilon K_1(\partial_1\Phi)(k_c,\cdot) + \tfrac{1}{2}\varepsilon^2 K_1^2(\partial_1^2\Phi)(k_c,\cdot) + O(\varepsilon^3).$$

This leads to a further refinement of our ansatz:

$$\begin{aligned}U_\varepsilon = \Psi_{1,0} &+ \varepsilon\Psi_{1,1} + \varepsilon^2\Psi_{1,2}\\ &+ \Psi_{-1,0} + \varepsilon\Psi_{-1,1} + \varepsilon^2\Psi_{-1,2}\\ &+ \varepsilon\Psi_{2,1} + \varepsilon^2\Psi_{2,2}\\ &+ \varepsilon\Psi_{-2,1} + \varepsilon^2\Psi_{-2,2}\\ &+ \varepsilon\Psi_{0,1} + \varepsilon^2\Psi_{0,2}\end{aligned} \tag{4.1.27}$$

where $\Psi_{j,\ell}$ are again of the form

$$\Psi_{j,\ell}(t,x) = \Psi_{j,\ell}(T,X_2,X_3;t,x_1,x_2) = \varepsilon A_{j,\ell}(T,X_2,X_3)\phi_{j,\ell}(x_1)e_j.$$

Setting $A_{1,0} = A$, $A_{1,1} = i\partial_{X_2}A$, $A_{1,2} = -\partial_{X_2}^2 A$, and $\Phi_{1,0} = \Phi_{k_c}$, $\Phi_{1,1} = (\partial_1\Phi)(k_c,\cdot)$, $\Phi_{1,2} = \frac{1}{2}(\partial_1^2\Phi)(k_c,\cdot)$, we have thus $\mathcal{F}_{x_2\to k}[\Psi_{1,0} + \varepsilon\Psi_{1,1} + \varepsilon^2\Psi_{1,2}] = \widehat{A}(K_1)\Phi_k f_1 + O(\varepsilon^3)$. Using (4.1.26), the linear part of the f_1-residual is then

$$\begin{aligned}\widehat{\mathrm{Res}}^{(\mathrm{lin})}(U_\varepsilon;f_1) &:= \mathcal{F}_{x_2\to k}\big[(\partial_t M(\partial_t) + \mathcal{A})(\Psi_{1,0} + \varepsilon\Psi_{1,1} + \varepsilon^2\Psi_{1,2})\big]\\ &= \varepsilon^2\left[(\sigma + \tfrac{1}{2}i\omega''(k_c)K_1^2 + \partial_T)\mathbf{J}(\omega_c) + \mathbf{S}_3\partial_{X_3}\right]\widehat{\Psi}_{1,0} + O(\varepsilon^3)\\ &= \varepsilon^2\left[(\sigma + \tfrac{1}{2}i\omega''(k_c)K_1^2 + \partial_T)\widehat{A}(K_1)\mathbf{J}(\omega_c)\Phi_k + \partial_{X_3}\widehat{A}(K_1)\mathbf{S}_3\Phi_k\right]f_1 + O(\varepsilon^3).\end{aligned}$$

Further f_1-terms are generated by nonlinear interaction of e_1 with e_0 and of e_2 with e_{-1}. Recalling that $A_{0,1} = |A|^2$ and $A_{2,1} = A^2$ and using (4.1.19), we see that these terms are given by

$$\begin{aligned}&\mathcal{F}_{x_2\to k}\Big[\partial_t\sum_{j\in\{0,2\}}\big(N^{(2)}(\Psi_{1-j,0},\varepsilon\Psi_{j,1}) + N^{(2)}(\varepsilon\Psi_{j,1},\Psi_{1-j,0})\big)\Big]\\ &\qquad = -2i\Omega_1\big(\widehat{N}^{(2)}(\Psi_{1,0},\varepsilon\Psi_{0,1}) + \widehat{N}^{(2)}(\Psi_{-1,0},\varepsilon\Psi_{2,1})\big) + O(\varepsilon^3)\\ &\qquad = -\varepsilon^2 i\omega_c\widehat{|A|^2A}(K_1)\psi_1 f_1 + O(\varepsilon^3)\end{aligned}$$

where

$$\psi_1 := 2\check{\chi}^{(2)}(\omega_c,0)Q(\phi_{1,0},\phi_{0,1}) + 2\check{\chi}^{(2)}(-\omega_c,2\omega_c)Q(\phi_{-1,0},\phi_{2,1}).$$

Now the smallness of $\widehat{\mathrm{Res}}(U_\varepsilon;f_1) = \widehat{\mathrm{Res}}^{(\mathrm{lin})}(U_\varepsilon;f_1) + \widehat{\mathrm{Res}}^{(\mathrm{nl})}(U_\varepsilon;f_1)$ is obtained in two steps. First, we consider a finite-dimensional reduction. Fix $\Phi_k^* \in \ker \mathbf{\Lambda}(k,\omega(k))^*$ and define[2]

$$\alpha := \frac{1}{2}i\omega''(k_c), \qquad \beta := \frac{\langle \mathbf{S}_3\Phi_k,\Phi_k^*\rangle_{L^2}}{\langle \mathbf{J}(\omega_c)\Phi_k,\Phi_k^*\rangle_{L^2}}, \qquad \gamma := \frac{i\omega_c\langle\psi_1,\Phi_k^*\rangle_{L^2}}{\langle \mathbf{J}(\omega_c)\Phi_k,\Phi_k^*\rangle_{L^2}}.$$

Then, projecting $\mathrm{Res}(U_\varepsilon;f_1)$ onto Φ_k^* and equating the ε^2-terms to zero yields

$$\partial_T\widehat{A} + \sigma\widehat{A} + \alpha K_1^2\widehat{A} + \beta\partial_{X_3}\widehat{A} - \gamma\widehat{|A|^2A} = 0. \tag{4.1.28}$$

[2]provided of course that $\langle \mathbf{J}(\omega_c)\Phi_k,\Phi_k^*\rangle_{L^2} \neq 0$, which we will assume throughout. Recalling that $\mathbf{J}(\omega) = M_0 + \check{\chi}(\omega) + \omega\check{\chi}'(\omega)$, this condition depends on the function $\check{\chi}$ as much as on the frequency range of ω_c. Multiplying instead (4.1.28) with $\langle \mathbf{J}(\omega_c)\Phi_k,\Phi_k^*\rangle_{L^2}$ and then setting it to zero, the amplitude equation, in view of $\beta = 0$, trivially yields $A = 0$.

In fact, since

$$(\mathbf{S}_3 v) \cdot w = v_5 w_1 - v_4 w_2 - v_2 w_4 + v_1 w_5,$$

and since the linear modes are either transverse-magnetic or transverse-electric, for $v := \Phi_k$, $w := \Phi_k^*$ we have

$$\text{either} \quad v_3 = v_4 = v_5 = w_3 = w_4 = w_5 = 0 \quad \text{or} \quad v_1 = v_2 = v_6 = w_1 = w_2 = w_6 = 0.$$

In each case it follows that $(\mathbf{S}_3 v) \cdot w \equiv 0$, which implies

$$\beta = 0.$$

Thus, after taking the inverse Fourier transform, (4.1.28) reduces to the Ginzburg-Landau equation

$$\partial_T A + \sigma A - \alpha \partial_{X_2}^2 A - \gamma |A|^2 A = 0. \tag{4.1.29}$$

Remark 4.1.5. We note that by the assumptions (D1), (D2) on the dispersion relation, $\operatorname{Re} \omega''(k_c)$ is small and $\operatorname{Im} \omega''(k_c) < 0$, thus

$$\operatorname{Re} \alpha = \frac{1}{2} \operatorname{Re} i\omega''(k_c) = -\frac{1}{2} \operatorname{Im} \omega''(k_c) > 0$$

and $\operatorname{Im} \alpha$ is small. By Theorem 2.2.2, equation (4.1.29) admits solutions A which are small in $H^1_{-\nu}(\mathbb{R}, H^s(\mathbb{R}^2))$, where $s > 1$, $\nu \in (0, \sigma)$. $\diamondsuit$

Suppose subsequently that A solves (4.1.29). Then, by construction, $\widehat{\operatorname{Res}}(U_\varepsilon; f_1)$ is $O(\varepsilon^3)$-small in $\ker \mathbf{\Lambda}(k, \omega(k))^*$. In order to make the residual small on the whole space, we extend the ansatz U_ε by a final correction $\varepsilon^2 \Psi_{1,c} + \varepsilon^2 \Psi_{-1,c}$, where $\Psi_{-1,c} = \overline{\Psi_{1,c}}$ and

$$\Psi_{1,c}(T, X_2, X_3; t, x_1, x_2) = -\varepsilon \partial_{X_3} A(T, X_2, X_3) \phi_{1,c}(x_1) e_1 - \varepsilon |A|^2 A(T, X_2, X_3) \psi_{1,c}(x_1) e_1. \tag{4.1.30}$$

This $O(\varepsilon^2)$-correction only changes the linear part $\widehat{\operatorname{Res}}(U_\varepsilon; f_1)$ of the f_1-residual; analogous to the above we have

$$\begin{aligned}\widehat{\operatorname{Res}}(U_\varepsilon; f_1) = \varepsilon^2 \Big[&(\sigma + \tfrac{1}{2} i\omega''(k_c) K_1^2 + \partial_T) \widehat{A}(K_1) \mathbf{J}(\omega_c) \Phi_k + \partial_{X_3} \widehat{A}(K_1) \mathbf{S}_3 \Phi_k \Big] f_1 \\ &+ \mathbf{\Lambda}(k, \omega(k)) \Psi_{1,c} + \widehat{\operatorname{Res}}^{(\mathrm{nl})}(U_\varepsilon; f_1) + O(\varepsilon^3).\end{aligned}$$

Now using (4.1.28) and $\beta = 0$, we may replace $(\sigma + \frac{1}{2} i\omega''(k_c) K_1^2 + \partial_T) \widehat{A}(K_1)$ by $\gamma \widehat{|A|^2 A}(K_1)$. Hence, the above simplifies to

$$\begin{aligned}\widehat{\operatorname{Res}}(U_\varepsilon; f_1) = \varepsilon^2 \widehat{|A|^2 A}(K_1) \big[&\gamma \mathbf{J}(\omega_c) \Phi_k - i\omega_c \psi_1 - \mathbf{\Lambda}(k, \omega(k)) \psi_{1,c} \big] f_1 \\ &+ \varepsilon^2 \partial_{X_3} \widehat{A}(K_1) \big[\mathbf{S}_3 \Phi_k - \mathbf{\Lambda}(k, \omega(k)) \phi_{1,c} \big] f_1 + O(\varepsilon^3),\end{aligned}$$

and the smallness requirement leads to the two equations

$$\begin{aligned}\mathbf{\Lambda}(k, \omega(k)) \psi_{1,c} &= \gamma \mathbf{J}(\omega_c) \Phi_k - i\omega_c \psi_1 \\ \mathbf{\Lambda}(k, \omega(k)) \phi_{1,c} &= \mathbf{S}_3 \Phi_k.\end{aligned}$$

But now the fact that $\langle \gamma \mathbf{J}(\omega_c) \Phi_k - i\omega_c \psi_1, \Phi_k^* \rangle = \langle \mathbf{S}_3 \Phi_k, \Phi_k^* \rangle = 0$ shows that the right-hand sides in this system both lie in $\ker(\mathbf{\Lambda}(k,\omega(k))^*)^\perp = \overline{\operatorname{ran}(\mathbf{\Lambda}(k,\omega(k)))}$. Consequently, we find $\phi_{1,\mathrm{c}}, \psi_{1,\mathrm{c}} \in \operatorname{dom}\mathbf{\Lambda}$ such that

$$\begin{aligned}\mathbf{\Lambda}(k,\omega(k))\psi_{1,\mathrm{c}} &= \gamma\mathbf{J}(\omega_c)\Phi_k - i\omega_c\psi_1 + O(\varepsilon)\\ \mathbf{\Lambda}(k,\omega(k))\phi_{1,\mathrm{c}} &= \mathbf{S}_3\Phi_k + O(\varepsilon),\end{aligned}$$

yielding $\widehat{\operatorname{Res}}(U_\varepsilon; f_1) = O(\varepsilon^3)$.

Summary

Starting with a slowly varying amplitude modulating a linear 2D-mode in Fourier space, we have shown that, imposing non-resonance conditions on the dispersion curve, corrections can be made in such a way that the residual $\widehat{\operatorname{Res}}(U_\varepsilon)$ is overall (formally) of order $O(\varepsilon^3)$, respectively $\operatorname{Res}(U_\varepsilon) = O(\varepsilon^4)$. To summarize, our final and corrected ansatz is

$$\begin{aligned}U_\varepsilon = (\Psi_{1,0} + \varepsilon\Psi_{1,1} + \varepsilon^2\Psi_{1,2}) &+ (\Psi_{-1,0} + \varepsilon\Psi_{-1,1} + \varepsilon^2\Psi_{-1,2}) + (\varepsilon^2\Psi_{1,\mathrm{c}} + \varepsilon^2\Psi_{-1,\mathrm{c}})\\ &+ (\varepsilon\Psi_{2,1} + \varepsilon^2\Psi_{2,2}) + (\varepsilon\Psi_{-2,1} + \varepsilon^2\Psi_{-2,2})\\ &+ (\varepsilon\Psi_{0,1} + \varepsilon^2\Psi_{0,2}).\end{aligned} \tag{4.1.31}$$

Here the symmetry $\Psi_{-i,j} = \overline{\Psi_{i,j}}$ holds, and with

$$T = \varepsilon^2 t, \quad X_2 = \varepsilon(x_2 - c_{\mathrm{g}}t), \quad X_3 = \varepsilon^2 x_3,$$

the individual terms in the parentheses are given by

- The main term (approximation of a modulated 2D-mode):

  $$(\Psi_{1,0} + \varepsilon\Psi_{1,1} + \varepsilon^2\Psi_{1,2})(t,x) = \varepsilon A(T,X_2,X_3)\,\Phi_{k_c}(x_1)\,e^{i(k_c x_2 - \omega_c t)} + O(\varepsilon^3),$$

  where $\Phi_{k_c} \in \ker\mathbf{\Lambda}(k_c,\omega(k_c))$ and A is a solution of (4.1.29).

- The correction (4.1.30) on $(\ker\mathbf{\Lambda}(k,\omega(k)))^\perp$:

  $$\begin{aligned}\varepsilon^2\Psi_{1,\mathrm{c}}(t,x) = \varepsilon(\partial_{X_3}A)&(T,X_2,X_3)\,\phi_{1,\mathrm{c}}(x_1)\,e^{i(k_c x_2 - \omega_c t)}\\ &- \varepsilon(|A|^2A)(T,X_2,X_3)\,\psi_{1,\mathrm{c}}(x_1)\,e^{i(k_c x_2 - \omega_c t)}.\end{aligned}$$

- The correction (4.1.23) due to second-harmonic generation:

  $$\begin{aligned}\varepsilon\Psi_{2,1}(t,x) &= \varepsilon^2A^2(T,X_2,X_3)\,\phi_{2,1}(x_1)\,e^{2i(k_c x_2 - \omega_c t)}\\ \varepsilon^2\Psi_{2,2}(t,x) &= i\varepsilon^2(\partial_{X_2}A^2)(T,X_2,X_3)\,\phi_{2,2}(x_1)\,e^{2i(k_c x_2 - \omega_c t)}.\end{aligned}$$

- The correction (4.1.24) due to optical rectification:

  $$\begin{aligned}\varepsilon\Psi_{0,1}(t,x) &= \varepsilon^2|A|^2(T,X_2,X_3)\,\phi_{0,1}(x_1)\\ \varepsilon^2\Psi_{0,2}(t,x) &= \varepsilon^3(\partial_{X_2}|A|^2)(T,X_2,X_3)\,\phi_{0,2}(x_1).\end{aligned}$$

The profiles $\phi_{i,j}, \phi_{1,\mathrm{c}}, \psi_{1,\mathrm{c}} \in L^2(\mathbb{R})^6$ all have exponential decay towards infinity.

4.2 Norm estimates for the residual

The aim of this section is to justify the previous formal expansions, showing that under suitable conditions the higher order terms are indeed small in $L^2_{-\nu}$ for some $\nu > 0$.

We focus on convolutions in time involving amplitudes depending on long time scales.

Lemma 4.2.1. *Let $\nu > 0$ and let $\chi \in L^2_{-\nu}$ be a causal kernel for which $\check{\chi}'$ is bounded on the strip $\mathcal{S}_\nu := \mathbb{R} + i[-\nu, 0]$. Then for all $a \in H^1_{-\nu}(\mathbb{R})$, $\omega \in \mathbb{R}$, and $\delta \in (0,1)$, the function J_δ with*

$$J_\delta(t) := \int_{\mathbb{R}} \chi(\tau) e^{i\omega\tau} a(\delta(t-\tau))\,\mathrm{d}\tau - \check{\chi}(\omega) a(\delta t)$$

fulfills $J_\delta \in L^2_{-\delta\nu}$ with

$$\|J_\delta\|_{L^2_{-\delta\nu}} \leq \delta \sup_{z \in \mathcal{S}_\nu} |\check{\chi}'(z)|\, \|a'\|_{L^2_{-\nu}}\,.$$

Proof. Recalling that $\check{\chi}(\omega) = \int \chi(\tau) e^{i\omega\tau}\,\mathrm{d}\tau$ and $(\mathcal{L}_\varrho \chi)(\xi) = \frac{1}{\sqrt{2\pi}} \int \chi(\tau) e^{-(\varrho + i\xi)\tau}\,\mathrm{d}\tau$, we have the correspondence $(\mathcal{L}_\varrho \chi)(\xi) = \frac{1}{\sqrt{2\pi}} \check{\chi}(-(\xi - i\varrho))$ for $\varrho, \xi \in \mathbb{R}$. Together with the convolution theorem $\mathcal{L}_\varrho(u * v) = \sqrt{2\pi}(\mathcal{L}_\varrho u)(\mathcal{L}_\varrho v)$, the scaling property $\mathcal{L}_{\delta\varrho}[u(\delta\,\cdot)](\delta\xi) = \delta^{-1}\mathcal{L}_\varrho[u](\xi)$, and the derivative rule $\mathcal{L}_\varrho[u'](\xi) = (\varrho + i\xi)\mathcal{L}_\varrho[u](\xi)$ we compute

$$\begin{aligned}
\delta\mathcal{L}_{-\delta\nu}[J_\delta](\delta\xi) &= \sqrt{2\pi}\delta\mathcal{L}_{-\delta\nu}[\chi e^{i\omega\,\cdot}](\delta\xi)\mathcal{L}_{-\delta\nu}[a(\delta\,\cdot)](\delta\xi) - \delta\check{\chi}(\omega)\mathcal{L}_{-\delta\nu}[a(\delta\,\cdot)](\delta\xi) \\
&= \check{\chi}(\omega - \delta(\xi + i\nu))(\mathcal{L}_{-\nu} a)(\xi) - \check{\chi}(\omega)(\mathcal{L}_{-\nu} a)(\xi) \\
&= (\check{\chi}(\omega - \delta(\xi + i\nu)) - \check{\chi}(\omega))(\mathcal{L}_{-\nu} a)(\xi).
\end{aligned}$$

Due to the boundedness of $\check{\chi}'$ we can estimate this last term by

$$\begin{aligned}
\left|(\check{\chi}(\omega - \delta(\xi + i\nu)) - \check{\chi}(\omega))(\mathcal{L}_{-\nu} a)(\xi)\right| &\leq \sup_{\lambda \in [0,1]} |\check{\chi}'(\omega - \lambda\delta(\xi + i\nu))|\, |\delta(\xi + i\nu)|\, |(\mathcal{L}_{-\nu} a)(\xi)| \\
&\leq \delta \sup_{z \in \mathcal{S}_\nu} |\check{\chi}'(z)|\, |(\xi + i\nu)(\mathcal{L}_{-\nu} a)(\xi)|\,.
\end{aligned}$$

Setting $C_\chi := \sup_{z \in \mathcal{S}_\nu} |\check{\chi}'(z)|$, with $a \in H^1_{-\nu}$ and Plancherel's theorem we conclude

$$\begin{aligned}
\|J_\delta\|_{L^2_{-\delta\nu}} = \|\mathcal{L}_{-\delta\nu} J_\delta\|_{L^2_0} = \left\|\delta\mathcal{L}_{-\delta\nu}\left[J_\delta\right](\delta\,\cdot)\right\|_{L^2_0} &\leq \delta\, C_\chi \left\|(\xi + i\nu)(\mathcal{L}_{-\nu} a)\right\|_{L^2_0} \\
&= \delta\, C_\chi \left\|\mathcal{L}_{-\nu}[a']\right\|_{L^2_0} \\
&= \delta\, C_\chi \|a'\|_{L^2_{-\nu}},
\end{aligned}$$

which yields the assertion. □

Remark 4.2.2. The assumptions can be weakened to $\check{\chi}$ Lipschitz continuous on the strip $\mathcal{S}_\nu$ and $a \in L^2_{-\nu}(\mathbb{R}, \mathcal{H})$, in which case we have the estimate

$$\|J_\delta\|_{L^2_{-\delta\nu}} \leq \delta L_{\chi,\nu}\, \|a\|_{L^2_{-\nu}}\,,$$

with $L_{\chi,\nu}$ denoting the Lipschitz constant of $\check{\chi}$ on $\mathcal{S}_\nu$. ◊

With $\delta = \varepsilon^2$, Lemma 4.2.1 shows that for functions $t \mapsto \varepsilon a(\varepsilon^2 t)$ changing on the long time scale $T = \varepsilon^2 t$, the time convolution with a well-behaved kernel χ can be approximated with instantaneous terms up to an error of $O(\varepsilon^3)$ in $L^2_{-\varepsilon^2\nu}(\mathbb{R})$. To take into account the remaining long scales $X_2 = \varepsilon(x_2 - c_g t)$, $X_3 = \varepsilon^2 x_3$, we replace $\varepsilon a(\varepsilon^2 t)$ by a generic term

$$\Psi_j(t,x) = \varepsilon A_j(T, X_2, X_3)\phi_j(x_1)e_j(t,x_2)$$

occurring in the ansatz U_ε. Here A_j is a polynomial expression in $A, \bar{A}$ and their spatial derivatives, with A being a solution of the amplitude equation (4.1.29). Recall from Theorem 2.2.2 that for such a solution we may assume $A \in H^1_{-\nu}(\mathbb{R}, H^s(\mathbb{R}^2))$ ($s \geq 3$). It is easy to see that due to the scaling $\mathrm{d}x_2\,\mathrm{d}x_3 = \varepsilon^{-3}\,\mathrm{d}X_2\,\mathrm{d}X_3$ the error

$$\begin{aligned} \|\chi * \Psi_j - \check{\chi}(j\omega_c)\Psi_j\|_{L^2_{-\varepsilon^2\nu}} &= \left(\iiint |\chi * \Psi_j(t,x) - \check{\chi}(j\omega_c)\Psi_j(t,x)|^2 \,\mathrm{d}x_2\,\mathrm{d}x_3\, e^{-2\varepsilon^2\nu t}\,\mathrm{d}t\right)^{1/2} \\ &\lesssim \varepsilon^3\varepsilon^{-3/2}\,\|\partial_T A_j\|_{L^2_{-\nu}}\,\|\phi_j\|_{L^2} \end{aligned}$$

becomes of order $O(\varepsilon^{3/2})$. The terms in $\mathrm{Res}(U_\varepsilon)$ which are not related to time convolution can be estimated in a similar way, since those are of the form $\varepsilon^n\Psi_j$ with $n \geq 3$. For such terms, $\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}t = \varepsilon^{-5}\,\mathrm{d}X_2\,\mathrm{d}X_3\,\mathrm{d}T$ gives

$$\|\varepsilon^n\Psi_j\|_{L^2_{-\varepsilon^2\nu}} \lesssim \varepsilon^{n+1}\varepsilon^{-5/2}\,\|A_j\|_{L^2_{-\nu}}\,\|\phi_j\|_{L^2} = O(\varepsilon^{3/2}).$$

Overall we can conclude that

$$\|\mathrm{Res}(U_\varepsilon)\|_{L^2_{-\varepsilon^2\nu}} = O(\varepsilon^{3/2}),$$

which will turn out to be sufficient for a rigorous approximation.

4.3 Small solutions of the error equation

As mentioned in the introduction to this chapter, our ultimate goal is to be able to perform a fixed-point argument for the error equation on a subset of $L^2_{-\nu}(\mathbb{R}, \mathcal{H})$ for some $\nu > 0$. This includes the exponential stability of the linearized equation and the smallness of the Lipschitz constant of the nonlinearity on small sets in $L^2_{-\nu}(\mathbb{R}, \mathcal{H})$.

As before, we consider a Maxwell system

$$(\partial_t M(\partial_t) + \mathcal{A})U + \partial_t N^{(2)}(U,U) = 0 \qquad (t > 0),$$

with a quadratic and (w.l.o.g.) symmetric nonlinearity $N^{(2)}$ of the form

$$N^{(2)}(u,v)(t) = \iint \chi^{(2)}(t-\tau_1, t-\tau_2)\, Q(u(\tau_1), v(\tau_2))\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2.$$

Recall from (4.0.3) that the error equation takes the form

$$(\partial_t M_\varepsilon(\partial_t) + \mathcal{A})R + F_\varepsilon(R) + \mathrm{Res}(U_\varepsilon) = 0 \qquad (t > 0),$$

with $M_\varepsilon(\partial_t)R = M(\partial_t)R + 2N^{(2)}(R, U_\varepsilon)$, $F_\varepsilon(R) = F(R) = N^{(2)}(R,R)$, and U_ε is the ansatz established in (4.1.31). In order to put the error equation in the framework of evolutionary equations, we make a few observations. We first note that M_ε need not be a linear material law in the strict sense of the definition. For even if the kernel $\chi^{(2)}$ is causal in each variable and Q is a bilinear operator in $L^2(\mathbb{R}^3)^6$, writing $Q(U_\varepsilon(\tau_1), R(\tau_2)) =: Q_\varepsilon(\tau_1)R(\tau_2)$ gives rise to

$$\begin{aligned} N^{(2)}(U_\varepsilon, R)(t) &= \iint \chi^{(2)}(t-\tau_1, t-\tau_2)\, Q(U_\varepsilon(\tau_1), R(\tau_2))\, \mathrm{d}\tau_1\, \mathrm{d}\tau_2 \\ &= \int \Big[\int \chi^{(2)}(t-\tau_1, t-\tau_2)\, Q_\varepsilon(\tau_1)\, \mathrm{d}\tau_1\Big] R(\tau_2)\, \mathrm{d}\tau_2 \\ &= \int \kappa_\varepsilon(t,\tau_2) R(\tau_2)\, \mathrm{d}\tau_2, \end{aligned}$$

with $\kappa_\varepsilon(t,\tau_2) := \int \chi^{(2)}(t-\tau_1, t-\tau_2)\, Q_\varepsilon(\tau_1)\, \mathrm{d}\tau_1$. This non-autonomous convolution operator cannot be described by a linear material law. However, the contribution of $N^{(2)}(U_\varepsilon, R)$ can be viewed as a small Lipschitz perturbation, under some conditions on $\chi^{(2)}$ and U_ε.

Lemma 4.3.1. *Let $\chi^{(2)} \colon \mathbb{R}^2 \to \mathbb{R}$ be measurable and causal such that the map κ with $\kappa(t) = \int |\chi^{(2)}(t,\tau)|\, \mathrm{d}\tau$ fulfills $\kappa \in L^1_{\varrho_0}(\mathbb{R})$ for some $\varrho_0 \in \mathbb{R}$. Let $G \in L^\infty(\mathbb{R}, \mathcal{B}(\mathcal{H}))$. Then, the operator*

$$u \mapsto \int_{\mathbb{R}} \int_{\mathbb{R}} \chi^{(2)}(t-\tau_1, t-\tau_2)\, G(\tau_1) u(\tau_2)\, \mathrm{d}\tau_1\, \mathrm{d}\tau_2$$

is causal and bounded on $L^2_\varrho(\mathbb{R}, \mathcal{H})$ for $\varrho > \varrho_0$.

Proof. The boundedness follows from an estimate completely analogous to those established previously for Lipschitz maps (cf. Example 2.1.5 (b)):

$$\begin{aligned} &\left\| \int_{\mathbb{R}} \int_{\mathbb{R}} \chi^{(2)}(t-\tau_1, t-\tau_2)\, G(\tau_1) u(\tau_2)\, \mathrm{d}\tau_1\, \mathrm{d}\tau_2 \right\|^2_{L^2_\varrho} \\ &\qquad \le \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \int_{\mathbb{R}} |\chi^{(2)}(t-\tau_1, t-\tau_2)|\, \|G(\tau_1)\|_{\mathcal{B}(\mathcal{H})}\, \|u(\tau_2)\|_{\mathcal{H}}\, \mathrm{d}\tau_1\, \mathrm{d}\tau_2 \right)^2 e^{-2\varrho t}\, \mathrm{d}t \\ &\qquad \le \|G\|^2_{L^\infty} \|\kappa\|_{L^1_{\varrho_0}} \int_{\mathbb{R}} \int_{\mathbb{R}} \kappa(t-\tau_2) e^{-\varrho_0(t-\tau_2)} \underbrace{e^{-2(\varrho-\varrho_0)(t-\tau_2)}}_{\le 1}\, \mathrm{d}t\, \|u(\tau_2)\|^2_{\mathcal{H}}\, e^{-2\varrho\tau_2}\, \mathrm{d}\tau_2 \\ &\qquad \le \|G\|^2_{L^\infty} \|\kappa\|^2_{L^1_{\varrho_0}} \|u\|^2_{L^2_\varrho}. \end{aligned}$$

The causality is obvious. □

Consider again a generic term $\Psi_j(t,x) = \varepsilon A_j(T, X_2, X_3)\phi_j(x_1) e_j(t, x_2)$ in the expression for $U_\varepsilon(t,x)$. Assume that $A_j = A_j(T, X_2, X_3)$ is bounded as a function $A \in L^\infty(\mathbb{R}^3)$, and that $\phi_j \in L^\infty(\mathbb{R})^6$. Then the map

$$G(\tau) := Q(\varepsilon^{-1}\Psi_j(\tau), \cdot)$$

fulfills $G \in L^\infty(\mathbb{R}, \mathcal{B}(L^2(\mathbb{R}^3)^6))$, and under suitable decay assumptions on $\chi^{(2)}$, Lemma 4.3.1 yields the boundedness of the operator

$$R \mapsto \varepsilon^{-1} N^{(2)}(\Psi_j, R) = \iint \chi^{(2)}(\cdot - \tau_1, \cdot - \tau_2)\, Q(\varepsilon^{-1}\Psi_j(\tau_1), R(\tau_2))\, \mathrm{d}\tau_1\, \mathrm{d}\tau_2$$

on $\bigcup_{\varrho > \varrho_0} L^2_\varrho(\mathbb{R}, L^2(\mathbb{R}^3)^6)$. By linearity, the operator $R \mapsto \varepsilon^{-1} N^{(2)}(U_\varepsilon, R)$ is also bounded, showing that

$$\partial_t M_\varepsilon(\partial_t) = \partial_t M(\partial_t) + O(\varepsilon), \qquad 0 < \varepsilon \ll 1,$$

in $\mathcal{B}(L^2_\varrho(\mathbb{R}, L^2(\mathbb{R}^3)^6))$ for $\varrho > \varrho_0$. In particular, taking $\varrho_0 = -\nu_0 < 0$ we can conclude the following.

Lemma 4.3.2. *Let $\chi^{(2)}$ be measurable and causal, with $\int_{\mathbb{R}} |\chi^{(2)}(\cdot, \tau)| \, \mathrm{d}\tau \in L^1_{-\nu_0}(\mathbb{R})$, and let U_ε be a uniformly bounded ansatz such that $\mathrm{Res}(U_\varepsilon) = o(\varepsilon)$ in $L^2_{-\nu}(\mathbb{R}, L^2(\mathbb{R}^3)^6)$ for some $\nu > 0$. Then, the linearized error equation is exponentially stable for small $\varepsilon \ll 1$ if the original linearized Maxwell system is exponentially stable.*

What remains is to perform the fixed-point argument for the nonlinear error equation. To this end, the nonlinearity must be established as a mapping on a small and closed space $W_{-\nu} \subseteq L^2_{-\nu}(\mathbb{R}, \mathcal{H})$, $\mathcal{H} = L^2(\mathbb{R}^3)^6$. We can formulate our main approximation result.

Theorem 4.3.3 (Justification of the Ginzburg–Landau approximation for fully nonlocal quadratic nonlinearities)**.** *Let ϵ, μ be (scalar) linear material laws satisfying the following.*

1. *There exists $\nu_0 > 0$ and $c > 0$ such that $\mathbb{C}_{\mathrm{Re}>-\nu_0} \setminus (\mathrm{dom}(\epsilon) \cap \mathrm{dom}(\mu))$ is discrete and*

$$\mathrm{Re}\, z > -\nu_0 \implies \mathrm{Re}\, z\epsilon(z), \mathrm{Re}\, z\mu(z) \geq c$$

   for all $z \in \mathrm{dom}(\epsilon) \cap \mathrm{dom}(\mu)$. In particular, the linearized Maxwell system is well-posed and exponentially stable.

2. *ϵ, μ admit dispersion curves satisfying (D1) and (D2). Hence, the system admits a family of linear (TM) or (TE) surface modes.*

Let $N^{(2)}$ be a quadratic nonlinearity of Volterra type,

$$N^{(2)}(U, V)(t) = \iint \chi^{(2)}(\tau_1, \tau_2)\, Q(U(t - \tau_1), V(t - \tau_2))\, \mathrm{d}\tau_1\, \mathrm{d}\tau_2$$

with $Q\colon \mathcal{H}^2 \to \mathcal{H}$ bilinear, $\mathrm{supp}\,\chi^{(2)} \subseteq (0, \infty)^2$, and $\kappa := (\partial_1 + \partial_2)\chi^{(2)}$ satisfying the integrability conditions of Lemma 4.3.1 and Lemma 2.3.1 with $\varrho_\kappa = -\nu_0$, i.e.,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \|\kappa(\tau_1, \tau_2)\|\, e^{\nu_0(\tau_1+\tau_2)}\, \mathrm{d}\tau_1\, \mathrm{d}\tau_2 < \infty \tag{4.3.1}$$

$$\sup_{\tau_1, \tau_2 \in \mathbb{R}} \int_{\mathbb{R}} \|\kappa(t - \tau_1, t - \tau_2)\|\, e^{\nu_0(t-\tau_1)} e^{\nu_0(t-\tau_2)}\, \mathrm{d}t < \infty. \tag{4.3.2}$$

Finally, let U_ε be an ansatz of the form (4.1.31), i.e.,

$$U_\varepsilon(t, x) = \varepsilon A(\varepsilon^2 t, \varepsilon(x_2 - c_{\mathrm{g}} t), \varepsilon^2 x_3)\, \Phi(x_1)\, e^{i(k_c x_2 - \omega_c t)} + \mathrm{c.c.} + O(\varepsilon^2),$$

such that U_ε is uniformly bounded and $\|\mathrm{Res}(U_\varepsilon)\|_{L^2_{-\varepsilon^2\nu}} = O(\varepsilon^{3/2})$ for small $\varepsilon > 0$. Then, there exist $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0), \nu \in (0, \nu_0)$ the following holds: for small data $g \in L^2_{-\varepsilon^2\nu}(\mathbb{R}, \mathcal{H})^2$, $\mathrm{supp}\, g \subseteq [0, \infty)$, $\|g\|_{L^2_{-\varepsilon^2\nu}} \lesssim \varepsilon$, the error equation (in evolutionary form)

$$(\partial_t M_\varepsilon(\partial_t) + \mathcal{A})R + F(R) + \mathrm{Res}(U_\varepsilon) = g,$$

where $M_\varepsilon(\partial_t)R = M(\partial_t)R + 2N^{(2)}(R, U_\varepsilon)$ and $F(R) = \partial_t N^{(2)}(R, R)$, admits a unique solution $R \in L^2_{-\varepsilon^2\nu}(\mathbb{R}, \mathcal{H})^2$ with $\|R\|_{L^2_{-\varepsilon^2\nu}} \lesssim \varepsilon$.

Proof. Since $\mathrm{Res}(U_\varepsilon) = o(\varepsilon)$ in $L^2_{-\varepsilon^2\nu}(\mathbb{R}, \mathcal{H})^2$, the exponential stability of the linearized Maxwell system implies by Lemma 4.3.2 that $2\partial_t N^{(2)}(U_\varepsilon, R)$ is small in $L^2_{-\varepsilon^2\nu}(\mathbb{R}, \mathcal{H})^2$ and that the linearly perturbed error equation

$$(\partial_t M(\partial_t)R + \mathcal{A})R = g - \mathrm{Res}(U_\varepsilon) - 2\partial_t N^{(2)}(U_\varepsilon, R)$$

is again exponential stable for small $\varepsilon > 0$. The conditions imposed on F establish the local Lipschitz estimate

$$\|F(U) - F(V)\|_{L^2_{-\varepsilon^2\nu}} \leq d(\|U\|_{L^2_{-\varepsilon^2\nu}} + \|V\|_{L^2_{-\varepsilon^2\nu}})\|U - V\|_{L^2_{-\varepsilon^2\nu}}$$

with $d > 0$, see Lemma 2.3.1 and the subsequent comments. The statement follows by a fixed-point argument as in Theorem 2.3.2. $\square$

Remark 4.3.4. Note that to achieve $\mathrm{Res}(U_\varepsilon) = O(\varepsilon^{3/2})$ we have to impose at least Lipschitz continuity of ϵ, μ on a strip $\{z \in \mathrm{dom}(\epsilon) \cap \mathrm{dom}(\mu) : \mathrm{Re}\, z \in (-\nu_0, 0)\}$, cf. Lemma 4.2.1, Remark 4.2.2. $\Diamond$

4.4 Comments

It is reasonable, to some extent, to compare the approximation result in the preceding section to that in [SU03] dealing with electric fields in optical fiber. There, nonlinear Maxwell's equations are considered, in the form of the wave equation

$$\partial_t^2 E = \partial_x^2 E - \partial_t^2 P_{\mathrm{el}}(E)$$

for the E-field in one spatial dimension, featuring materials with continuous memory, modeled by a cubic-quintic electric polarization

$$\begin{aligned} P(E)(t) = &\int \chi^{(1)}(\tau)E(t-\tau)\,\mathrm{d}\tau \\ &+ \varepsilon \int \chi^{(3)}(\tau)|E(t-\tau)|^2 E(t-\tau)\,\mathrm{d}\tau \\ &+ \int \chi^{(5)}(\tau)|E(t-\tau)|^4 E(t-\tau)\,\mathrm{d}\tau, \end{aligned}$$

and where $\chi^{(1)}$, $\chi^{(3)}$, $\chi^{(5)}$ are susceptibilities of a generalized Lorentz-type analogous to the model discussed in Chapter 5. A pair of coupled-mode equations is derived for the amplitudes of an ansatz consisting of right and left travelling wavepackets, which is then simplified (assuming a spatial localization) to a single cubic-quintic Ginzburg–Landau equation. Families of pulse solutions for this equation are used to construct the wavepacket approximation.

The justification result in [SU03] (Theorems 2.8 and 2.9) is similar in spirit to Theorem 4.3.3, albeit with a slightly different scaling: For an ansatz of the order $\varepsilon^{1/2}$ and smallness assumptions on the initial data, one obtains the error bound

$$\|R(t)\|_{H^k(\mathbb{R}^3)} \lesssim \varepsilon^{1/2} e^{-b\varepsilon^2 t}, \quad t \geq 0,$$

for fixed $k \geq 1$ and for some $b > 0$, The method for obtaining this justification is to convert the Maxwell equation into an extended system without memory, made possible by the assumption on the susceptibilities.

Interface effects and centrosymmetric media

We want to highlight another difference between the result in [SU03] and Theorem 4.3.3, that lies in the assumptions on the physical setting. In [SU03], the cubic-quintic electric polarization, in particular, the absence of nonlinearities of even order, is a consequence of *inversion symmetry* imposed on the underlying lattice structure of the material (cf. Figure 4.3): Shifting the origin to the center of symmetry, the material structure looks the same under the inversion $x \mapsto -x$, which means the material response to the field $-E$ (at the point $-x$) is the same as the response to the field E (at the point x), but modulo the inversion itself. Consequently, the polarization must commute with this inversion, i.e., $P(-E) = -P(E)$. Assuming that the polarization is given as a sum $P = \sum_{n\geq 1} P^{(n)}$, with

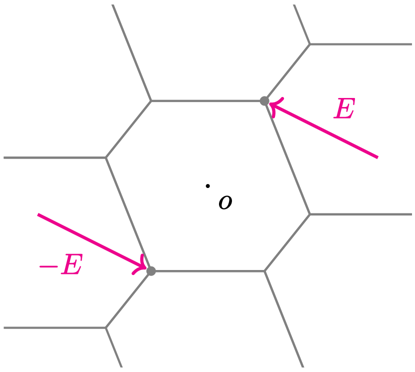

Figure 4.3: Lattice with inversion symmetry around a point o.

$P^{(n)}$ being a tensor of order $n-1$, this means that under inversion symmetry, the even-order polarizations $P^{(2k)}$ ($k \in \mathbb{N}$) must vanish identically. Thus, the lowest-order nonlinearity in the Maxwell system is cubic, and this leads to a main cubic nonlinearity, and otherwise small nonlinear terms, in the error equation. In turn, this makes the general method in [KSM92] applicable, see [SU03, Lemma A.4].

In contrast, the assumption of inversion symmetry is generally not valid in our case, as this symmetry is broken at the interface. The resulting nonlinear effect can be modeled by an effective quadratic polarization and is expected to be concentrated within a thin region of the interface (cf. [Boy08, 2.11], [She89]).

It might be reasonable to assume an overall smallness on this localized interface polarization, while assuming otherwise centrosymmetric, nonlinear media. In this case, the result in Theorem 4.3.3 can be improved as follows. Recall that ε was introduced in (4.1.5) as a perturbation parameter featured in the linear material relations. Now suppose additionally that ε is featured in the nonlinear susceptibility as

$$\partial_t P^{(2)}(U)(t) = \varepsilon^{\alpha} \iint \chi^{(2)}(\tau_1, \tau_2)\, Q(U(t-\tau_1), U(t-\tau_2))\, \mathrm{d}\tau_1\, \mathrm{d}\tau_2 \qquad (4.4.1)$$

with $\alpha > 0$ and a suitably regular kernel $\chi^{(2)}$ (in the sense of the integrability conditions in

Theorem 4.3.3 and the assumption $\operatorname{supp}\kappa \subseteq (0,\infty)^2$). This smallness assumption leads to a smaller Lipschitz constant (from $O(\varepsilon)$ to $o(\varepsilon)$) of the nonlinearity over balls with small radius in $L^2_{-\varepsilon^2\nu}(\mathbb{R},\mathcal{H})$. Under similar assumptions as in Theorem 4.3.3, this provides the improved estimate

$$\|R\|_{L^2_{-\varepsilon^2\nu}} \lesssim \varepsilon^{1+\delta}$$

with $\delta > 0$. In this respect, this estimate is more in line with the results established in [DST22], where the lowest-order nonlinearities are of cubic Kerr-type. However, recalling that the ansatz U_ε is itself exponentially localized around the interface, it is not clear if the smallness assumption in (4.4.1) can be made.

5 The Drude–Lorentz model of electric permittivity

In this part we consider a variant of the Lorentz oscillator model describing the material response of bound electrons to an electric field, and we want to verify the assumptions in Definition 3.3.8.

The basic model for a dipole (cf. [Fox10, §2.2.1]) relies on two assumptions: First, that the polarization[1] P is proportional to the driving force, which in turn is a multiple of the internal field E. Second, regarding the bound electrons as a spring-damper system, that this field obeys the equation of a damped oscillator, i.e., an equation of the form

$$\partial_t^2 P + \gamma \partial_t P + \omega_0^2 P = \alpha E.$$

By Newton's law of motion, $\partial_t^2 P$ is related to the acceleration, $\gamma \partial_t P$ to the damping, $\omega_0^2 P$ is proportional to the restoring force of the "spring", and αE is the forcing term. After taking the Fourier–Laplace transform we obtain that

$$P = \chi_{\mathsf{L}}(\partial_t) E,$$

with the material law χ_{L}, the *Lorentz susceptibility*, which in frequency space is given by

$$\chi_{\mathsf{L}}(z) = \frac{\alpha}{\omega_0^2 + \gamma z + z^2}.$$

The macroscopic model of χ is usually more involved and consists in a finite sum of such terms, or even more generally,

$$\chi(z) = \sum_{j=0}^{n} \frac{\alpha_j + \beta_j z}{\omega_{0,j}^2 + \gamma_j z + z^2}. \tag{5.0.1}$$

The parameters are determined in order to fit the experimental data. A similar model is employed in [SU03] for nonlinear Maxwell equations in an optical fiber. If the parameters $\omega_{0,j}^2$ and γ_j are real and positive, the corresponding operator $\chi(\partial_t)$ acts by convolution of a sum of exponentially damped sine and cosine functions with resonant frequencies $\omega_{0,j}$; this becomes clear after recalling the following identities for the Laplace transform,

$$\begin{aligned} f(t) = \theta(t) e^{-\gamma t} \sin(\omega t) &\implies \int_{\mathbb{R}} f(t) e^{-zt} \,\mathrm{d}t = \frac{\omega}{(z+\gamma)^2 + \omega^2} \\ g(t) = \theta(t) e^{-\gamma t} \cos(\omega t) &\implies \int_{\mathbb{R}} f(t) e^{-zt} \,\mathrm{d}t = \frac{\omega + z}{(z+\gamma)^2 + \omega^2} \end{aligned}$$

[1] or electron displacement, or dipole moment. All these notions differ only by constant factors, like positive or negative charge, volume, etc.

for $\operatorname{Re} z > -\gamma$ and $\omega, \gamma > 0$, where $\theta(t) = \mathbf{1}_{(0,\infty)}(t)$ denotes the Heaviside step function.

5.1 Accretivity

We take for simplicity $n = 1$ in (5.0.1) and consider the material laws χ, ϵ given by

$$\chi(z) = \frac{\alpha + \beta z}{\omega_0^2 + \gamma z + z^2}, \quad \epsilon(z) = \epsilon_0 + \chi(z). \tag{5.1.1}$$

We want to check the positivity conditions of Definition 3.3.8 on half-planes containing the imaginary axis.

Remark 5.1.1. Setting $\omega_0 = \beta = 0$ one obtains the *Drude model*, or free electron model, as a special case of (5.1.1). Here $\sigma(z) = \frac{\alpha}{z+\gamma}$ takes the role of a frequency-dependent conductivity. Since $\lim_{|z|\to\infty} \sigma(z) = 0$, the positivity condition $\operatorname{Re} z\epsilon(z) \geq c > 0$ on a right half-plane is not satisfied. Even when writing instead $\epsilon(z) = \epsilon_0 + \frac{\alpha}{\gamma}(\frac{1}{z} - \frac{1}{z+\gamma}) =: M(z) + z^{-1}\frac{\alpha}{\gamma}$ with a strictly positive "conductivity" $\frac{\alpha}{\gamma}$, the material law $M(z) = \epsilon_0 - \frac{\alpha}{\gamma(z+\gamma)}$ can satisfy $\operatorname{Re} M(z) \geq c > 0$ only for $\operatorname{Re} z \geq \varrho$ with $\varrho > 0$ large enough. As such, none of the conditions in Definition 3.3.8 are fulfilled by this model. ♢

We will assume that $\omega_0 \neq 0$. In this case, the zeros of the denominator are

$$z = \begin{cases} -\frac{\gamma}{2} \pm \frac{i}{2}\sqrt{4\omega_0^2 - \gamma^2}, & \text{if } 2\omega_0 > \gamma \\ -\frac{\gamma}{2} \pm \frac{1}{2}\sqrt{\gamma^2 - 4\omega_0^2}, & \text{if } 2\omega_0 \leq \gamma, \end{cases}$$

and are contained in $\mathbb{C}_{\mathrm{Re}<0}$. Thus, the susceptibility χ and the permittivity $\epsilon(z) = \epsilon_0 + \chi(z)$ are linear material laws with $\mathbb{C}_{\mathrm{Re}\geq -\gamma_0} \subseteq \operatorname{dom}(\chi) = \operatorname{dom}(\epsilon)$, where

$$0 < \gamma_0 := \begin{cases} \frac{\gamma}{2}, & 2\omega_0 > \gamma \\ \frac{\gamma}{2} - \frac{1}{2}\sqrt{\gamma^2 - 4\omega_0^2}, & 2\omega_0 \leq \gamma. \end{cases} \tag{5.1.2}$$

Moreover, it is clear that for $\varrho > 0$ large enough we have $\operatorname{Re} z > \varrho \implies \operatorname{Re} \epsilon(z) \geq c \operatorname{Re} z > 0$, meaning the theorems in Sections 1.4 and 3.2 regarding well-posedness in $\bigcup_{\varrho > \varrho_0} L^2_\varrho(\mathbb{R}, \mathcal{H})$ of the corresponding linear or (Lipschitz-)nonlinear systems are applicable.

If $\beta = 0$, then $\lim_{|z|\to\infty}(z\chi(z)) = 0$, thus $\operatorname{Re} z\epsilon(z) = \operatorname{Re}(\epsilon_0 z + z\chi(z)) \geq c > 0$ cannot hold unless $\operatorname{Re} z \geq \varrho$ for some $\varrho > 0$. We will thus assume subsequently that $\alpha, \beta, \gamma, \omega_0 > 0$.

Lemma 5.1.2. *Let $\alpha, \beta, \gamma, \omega_0 > 0$ and suppose that $\alpha\gamma \geq \beta\omega_0^2$. Then for all $\delta > 0$ there exist $\nu_1 > 0$ and $c > 0$ such that*

$$\forall z \in \mathbb{C}_{\mathrm{Re}>-\nu_1} \setminus B[0,\delta] : \quad \operatorname{Re} z\epsilon(z) \geq c. \tag{5.1.3}$$

Proof. With $\gamma_0 > 0$ defined in (5.1.2), $g(\nu, \xi) := \operatorname{Re}(\nu + i\xi)\chi(\nu + i\xi)$ is well-defined and continuous in $(\nu, \xi) \in \mathbb{R}_{>-\gamma_0} \times \mathbb{R}$. Moreover, we find

$$g(\nu,\xi) = \frac{\beta(\nu^4 + \xi^4) + (2\nu^2 + \alpha\gamma - \beta\omega_0^2)\xi^2 + \nu(\alpha + \beta\gamma)(\nu^2 + \xi^2 + \gamma\nu + \omega_0^2)}{(\omega_0^2 + \nu^2 - \xi^2 + \gamma\nu)^2 + \xi^2(2\nu + \gamma)^2}$$

and obtain that if $\nu > 0$ and (ν, ξ) is bounded away from $(0, 0)$, then $g(\nu, \xi)$ is uniformly positive, since due to $\alpha\gamma - \beta\omega_0^2 \geq 0$ it is a sum of strictly positive terms and $\lim_{|(\nu,\xi)|\to\infty} g(\nu, \xi) = \beta$.

For $\nu = 0$ we have

$$g(0,\xi) = \frac{\beta\xi^4 + (\alpha\gamma - \beta\omega_0^2)\xi^2}{(\omega_0^2 - \xi^2)^2 + \xi^2\gamma^2}, \tag{5.1.4}$$

which is uniformly positive for ξ bounded away from $(0,0)$. Now let $\delta > 0$. By continuity of the map

$$\nu \mapsto \inf_{\nu^2+\xi^2 \geq \delta^2} g(\nu,\xi)$$

there exist $\nu_0 > 0$ and $c_0 > 0$ such that $g(\nu,\xi) \geq c_0 > 0$ for all $\nu \geq -\nu_0$. Choosing $\nu_1 \in (0, \min\{\nu_0, c_0\|\epsilon_0\|^{-1}\})$, we have

$$\operatorname{Re} z\epsilon(z) \geq \epsilon_0 \operatorname{Re} z + c_0 \geq c_0 - \|\epsilon_0\|\nu_1 =: c > 0$$

for $\operatorname{Re} z > -\nu_1$. □

Remark 5.1.3. It follows from (5.1.4) above that the condition $\alpha\gamma \geq \beta\omega_0^2$ is not only sufficient but also necessary for uniform positivity of $\operatorname{Re} z\epsilon(z)$ on some $\mathbb{C}_{\operatorname{Re}>-\nu_1}$; indeed, assuming otherwise, we find $0 \neq \xi_0 \in \mathbb{R}$ such that $g(0,\pm\xi_0) = 0$, and an interval $(-\delta,\delta)$ such that $g(0,\xi) = \operatorname{Re}(i\xi\epsilon(i\xi)) < 0$. Hence, $\operatorname{Re} z\epsilon(z) > 0$ does not hold for $z \in i\mathbb{R} \setminus B[0,\delta/2]$. ◊

Lemma 5.1.4. *Let $\alpha, \beta, \gamma, \omega_0 > 0$ and suppose that $\alpha \leq \beta\gamma$. Then there exist $\nu_1 > 0$ and $c > 0$ such that*

$$\forall z \in \mathbb{C}_{\operatorname{Re}>-\nu_1}: \quad \operatorname{Re}\epsilon(z) \geq c. \tag{5.1.5}$$

Proof. Since $\mathbb{C}_{\operatorname{Re}\geq 0} \subseteq \operatorname{dom}(\chi)$ and $\lim_{|z|\to\infty} \chi(z) = 0$, we know that χ is uniformly bounded on $\mathbb{C}_{\operatorname{Re}\geq 0}$. With $z = \nu + i\xi$, $\nu \geq 0$ we find

$$\operatorname{Re}\chi(\nu + i\xi) = \frac{(\beta\nu + \beta\gamma - \alpha)\xi^2 + \beta\nu^3 + (\alpha + \beta\gamma)\nu^2 + (\alpha\gamma + \beta\omega_0^2)\nu + \alpha\omega_0^2}{(\omega_0^2 + \nu^2 - \xi^2 + \gamma\nu)^2 + \xi^2(2\nu + \gamma)^2}. \tag{5.1.6}$$

Together with the non-negativity of the parameters and with $\beta\gamma - \alpha \geq 0$, this yields $\operatorname{Re}\chi(\nu + i\xi) \geq 0$ for $\nu \geq 0$. Now by continuity of $\nu \mapsto \inf_\xi \operatorname{Re}\chi(\nu + i\xi)$ and since $\epsilon_0 \geq d > 0$, there exists $\nu_1 > 0$ and $c \in (0, d]$ such that

$$\operatorname{Re}(\epsilon_0 + \chi(\nu + i\xi)) \geq c > 0$$

also for small $\nu \in (-\nu_1, 0)$. □

We may immediately combine Lemma 5.1.2 and Lemma 5.1.4 to conclude that $\epsilon(z) = \epsilon_0 + \chi(z)$ with χ as in (5.1.1) is of Lorentz-type according to Definition 3.3.8, provided that

$$\frac{\omega_0^2}{\gamma} \leq \frac{\alpha}{\beta} \leq \gamma \tag{5.1.7}$$

is satisfied. The disadvantage, however, is that this condition is satisfiable only for $\omega_0^2 \leq \gamma^2$. This means that the damping of the oscillator must be stronger than the resonant frequency determining the restoring force. Note that while the leftmost estimate is indeed necessary (see Remark 5.1.3), positivity of $\operatorname{Re}\epsilon(z) = \epsilon_0 + \operatorname{Re}\chi(z)$ can still be achieved for $\alpha/\beta > \gamma$, for example by simultaneously choosing α and β small, such that, by uniform boundedness of $\chi(z)$, the positive term ϵ_0 dominates.

In order to remove the restriction $\omega_0^2 \le \gamma^2$, we can make this last comment (slightly) more precise by providing a rough estimate of

$$|\chi(\nu + i\xi)| = \frac{\left((\alpha + \beta\nu)^2 + \beta^2\xi^2\right)^{1/2}}{\left(\left(\omega_0^2 + \nu^2 - \xi^2 + \gamma\nu\right)^2 + \xi^2(2\nu + \gamma)^2\right)^{1/2}} \tag{5.1.8}$$

in the case of small damping coefficient γ. Specifically, we assume

$$0 < \gamma < 1 < \omega_0, \quad -\frac{\gamma}{2} \ll \nu \ll 1, \quad \omega_0^2\beta \le \alpha\gamma \tag{5.1.9}$$

(the last inequality being the accretivity condition of Lemma 5.1.2). Squaring the denominator in (5.1.8), we find that the minimum of

$$h(\xi^2) = (\omega_0^2 + \nu^2 - \xi^2 + \gamma\nu)^2 + \xi^2(2\nu + \gamma)^2$$

is located at $\xi^2 = \xi_0^2 := \omega_0^2 + \nu^2 - \frac{\gamma}{2}$, and thus

$$\sup_{\xi \in \mathbb{R}} |\chi(\nu + i\xi)| \le |\chi(\nu + i\xi_0)| = \frac{\left((\alpha + \beta\nu)^2 + \beta^2(\omega_0^2 + \nu^2 - \frac{\gamma}{2})\right)^{1/2}}{\left((\frac{\gamma}{2} + \gamma\nu)^2 + (\omega_0^2 + \nu^2 - \frac{\gamma}{2})(2\nu + \gamma)^2\right)^{1/2}}.$$

Now using the assumptions (5.1.9), i.e., the smallness of γ, ν and the bound $\beta \le \alpha\gamma/\omega^2$, the last expression can be estimated to yield

$$\sup_{\xi \in \mathbb{R}} |\chi(\nu + i\xi)| \lesssim \frac{\alpha}{\omega_0\gamma} + O(|\nu|) \quad \text{as } \nu \to 0. \tag{5.1.10}$$

Thus, we may take

$$\frac{\omega_0^2}{\gamma} \le \frac{\alpha}{\beta} \quad \text{and} \quad \frac{\alpha}{\omega_0\gamma} \ll \epsilon_0 \tag{5.1.11}$$

as a plausible replacement for (5.1.7).

5.2 Dispersion relation

Next we turn our attention to the conditions (D1) and (D2) imposed on the dispersion relation. For this purpose, we consider $\mu(\omega) = \mu_0 \in \mathbb{R}$ and ϵ a permittivity of Lorentz-type on each side of the interface. Recall from Section 1.3 that with

$$\epsilon^\pm(\omega) := \epsilon_0 + \int_{\mathbb{R}} \chi^\pm(t)e^{i\omega t}\,\mathrm{d}t$$

the dispersion relation was formulated in terms of the angular frequency $\omega \in \mathbb{R}^+$ as

$$k^2 = \omega^2 \frac{\epsilon^+(\omega)\epsilon^-(\omega)}{\epsilon^+(\omega) + \epsilon^-(\omega)}. \tag{5.2.1}$$

Recall that the material law $\epsilon(z)$ and $\epsilon^\pm(\omega)$ are related by (neglecting a factor of $\sqrt{2\pi}$)

$$\epsilon(z = i\omega) = \epsilon^\pm(\omega).$$

Now if $\epsilon^+(\omega), \epsilon^-(\omega)$ are given by

$$\epsilon^\pm(\omega) = \epsilon_0 + \frac{\alpha^\pm - i\beta^\pm\omega}{\omega_0^\pm - 2i\gamma^\pm\omega - \omega^2}, \quad \omega_0^\pm, \alpha^\pm, \beta^\pm, \gamma^\pm > 0, \tag{5.2.2}$$

then for suitable parameters (5.2.1) gives rise to a dispersion curve satisfying (D1) and (D2), see Figure 5.1.

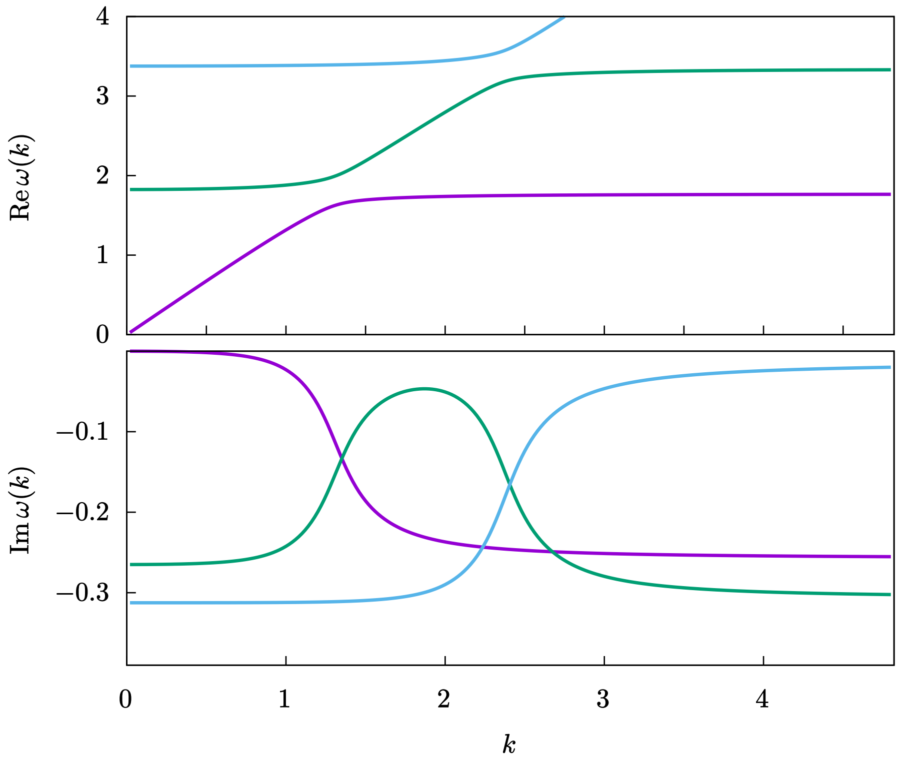

Figure 5.1: Numerically computed dispersion curves $k \mapsto \omega_j(k)$ with positive real part in the transverse-magnetic setting with $\mu(\omega) = \mu_0$ and $\epsilon^+(\omega), \epsilon^-(\omega)$ both given by (5.2.2), where $\mu_0 = 1$ and

$$\begin{aligned} &\epsilon_0^+ = 1, \quad (\omega_0^+)^2 = 11, \quad \alpha^+ = 0.5, \quad \beta^+ = 0.025, \quad \gamma^+ = 0.6 \\ &\epsilon_0^- = 1, \quad (\omega_0^-)^2 = 3, \quad \alpha^- = 0.4, \quad \beta^- = 0.03, \quad \gamma^- = 0.5. \end{aligned}$$

Conditions (D1) and (D2) are satisfied by the green curve. In this example, the parameters satisfy (5.1.11); otherwise, the choice is arbitrary.

5.3 Comments

The preceding discussion shows that the positivity conditions introduced by permittivities of *Lorentz type* (Definition 3.3.8 are, in principle, compatible with the properties (D1), (D2) for the dispersion relation.

Here we have assumed the non-magnetic setting and a 'single-oscillator' model for simplicity. As a general model is usually determined to fit a curve to experimental measurements (from the refractive index of the material), and consists of multiple 'oscillator' terms, in theory there are more degrees of freedom for tuning the desired properties, cf. [SLM17, SLM18]. We make no physical considerations about exact values for the model, nevertheless, we remark that strict positivity of $\operatorname{Re} \epsilon(z)$ and $\operatorname{Re} z\epsilon(z)$ for small $\operatorname{Re} z$ does not seem to be compatible with permittivities of metals in particular, as these usually contain a pure Drude susceptibility term of the form

$$\chi_{\mathsf{D}}(z) = \frac{\sigma}{z(z+\gamma)}$$

which can become negative for small values of $\omega = -iz \in \mathbb{R}^+$ (this is responsible for the reflection in metals at optical frequencies, see [Jac75, §7.5]).

6 Discussion

We take the opportunity to give a short summary of the results in this thesis and present some directions for future work.

Summary

In this paper, a functional analytic perspective to evolutionary equations was adapted to nonlinear Maxwell systems with various types of nonlinear material functions with memory. We explored known criteria for exponential stability and derived new ones for specific linear material laws. The emphasis on Volterra-type nonlinearities allowed to extend known criteria for exponential stability to nonlinear settings for small data.

In the setting of a planar interface and scalar material functions, evanescent surface modes can be derived explicitly and analytically as solutions of the linearized system. A wavepacket ansatz can be constructed from these functions using a slowly-varying amplitude. Accounting for quadratic resonances makes the addition of correction functions necessary; in the end, the wavepacket is determined by a scalar perturbation parameter (in the critical wavenumbers) and by a solution of the amplitude equation.

A rigorous justification of the amplitude approximation was performed for a model problem by showing that the error equation admits a small, exponentially decaying solution. This was done using the stability results established before, imposing sufficient conditions on the data (smallness), the material laws (spectral positivity), and the nonlinearity (compatibility). Under these assumptions, the justification proves the existence and stability of 'broad' surface wavepackets.

Outlook

Apart from the open problems presented in Sections 2.5 and 3.4, the following selected topics can be of interest.

The non-magnetic setting. While frequency-dependent behavior of the linear magnetization (permeability) is well known ([LL35]), and surface waves analogous to surface plasmon polaritons exist in magnetic settings (*surface magnon polaritons*, [MC19]), the magnetization is often assumed trivial in nonlinear optics. It is thus desirable to study non-magnetic settings, or at least weaken the assumptions on μ.

Under the assumption of exponential stability (in the sense of the results in Section 3.3) for the non-magnetic Maxwell system on the whole domain $\Omega = \mathbb{R}^3$, a corresponding result to Theorem 4.3.3 could be formulated, with minimal change (apart from additional regularity conditions on the nonlinear kernel $\chi^{(2)}$ in Example 3.3.23). The (open) problem of deriving exponential stability on exterior domains was discussed in Section 3.4.

Another approach would consist in working on a cylindrical domain, with the unbounded direction being the direction of propagation of the ansatz (the x_2-axis) in (4.1.31). In this case, boundary conditions should be taken into consideration.

Working with(out) exponential stability. If metals are involved the setup, it is known that the permittivity has a negative real part in a wide range of frequencies, see Section 5.3. Even without metals, one would like to consider cases in which the uniform accretivity conditions

$$\operatorname{Re} z\epsilon(z), \operatorname{Re} z\mu(z) \geq c > 0$$

are violated at least in some region of the complex half-plane, hence, where the criteria in Section 3.3 are not able to yield exponential stability. We can address this issue from two directions: either work without exponential stability, or try to derive it in another way.

Following the first idea, the existence of small solutions $R \in L^2_\varrho(\mathbb{R}, \mathcal{H})^2$ to the error equation with $\varrho > 0$ would still be useful, if $\varrho = O(\varepsilon^2)$, say $\varrho = \varepsilon^2 b$. Under the same assumptions of Theorem 4.3.3, this latter assumption would produce, assuming R is continuous, the estimate

$$\|R(t)\|_{L^2} \lesssim \varepsilon e^{\varepsilon^2 bt}.$$

Thus, the error would still remain pointwise of order $O(\varepsilon)$ over an interval of length $O(1/\varepsilon^2)$. One of the main problems with this approach is that Volterra-type operators can only be used with a cutoff over an interval $[0, T_0/\varepsilon^2]$ of the same length scale. Applying Theorem 2.3.3 (local existence), the Lipschitz constant of the nonlinearity gets multiplied with a factor of $O(1/\varepsilon)$, and smallness may not be guaranteed.

We expect that the second approach will require other methods than the ones presented in Section 3.3. For instance, assuming that $\operatorname{Re} z\epsilon(z), \operatorname{Re} z\mu(z) \geq c > 0$ holds only on $\mathcal{B}(L^2(\Omega_1)^6)$, i.e., when restricted to one side of the interface. The task is to infer (exponential) decay on the other side, $L^2(\Omega_2)^6$ from this condition. We suspect that some progress can be made in this direction using boundary control methods, see [AP19, FM96, KM01, PN07].

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Erklärung

Ich erkläre, dass ich meine Arbeit selbständig und ohne fremde Hilfe verfasst und keine anderen als die von mir angegebenen Quellen und Hilfsmittel benutzt habe. Die den benutzten Werken wörtlich oder inhaltlich entnommenen Stellen sind als solche kenntlich gemacht.

Es wurden keine früheren vergeblichen Promotionsversuche unternommen; die Dissertation hat in der gegenwärtigen bzw. in einer anderen Fassung nicht bereits einer anderen Fakultät vorgelegen.

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- Mathias Ionescu-Tira. *Time-Frequency Analysis in the Unit Ball.* (2019) `arXiv:1905.03178`

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