Stabilized finite element methods applied to transient convection-diffusion-reaction and population balance equations

Dissertation

zur Erlangung des akademischen Grades

doctor rerum naturalium
(Dr. rer. nat.)

von **M.Phil., Naveed Ahmed**

geb. am 13.04.1981 in Mansehra, Pakistan

genehmigt durch die Fakultät für Mathematik
der Otto-von-Guericke-Universität Magdeburg

Gutachter:
Prof. Dr. rer. nat. habil. Lutz Tobiska
Prof. Dr. rer. nat. habil. Volker John

Eingereicht am: 04.04.2011
Verteidigung am: 30.05.2011

Acknowledgements

I would like to extend my humble and gratitude to Almighty ALLAH who endowed me potential and ability to complete this most difficult task of writing PhD thesis. Specially for me it would be more difficult if I haven't received encouragement and help from supportive people. It is pleasure for me to acknowledge their efforts in my life and specially in this big task.

First and foremost I would like to acknowledge the greatness of my supervisor Prof. Dr. Lutz Tobiska who provided me strong support, encouragement from start to end. He delivered all the technicalities, which I required in a decent and professional way. His attitude was very friendly. Whenever I was feeling any difficulty, he was the person, who took me out from it. A part from his busy schedule, he always gave me proper time whenever I visited him to understand something. He always helped me not only in study but also in my day to day life. His noble advices and his role in my life will be a valuable asset for me and I am thinking that I am extremely lucky to have him as PhD supervisor.

I want to express my sincere gratitude and appreciation to Prof. Dr. Gunar Matthies for all his time given to me for discussions and his valuable suggestions which really helped me out in difficult situations. I am extremely grateful to apl. Prof. Dr. Friedhelm Schieweck as well for his great knowledge, which he shared with me and that will remain a source of inspiration for me in my future life. I would also like to thank all members of the group. It was a wonderful experience for me to have a part of such a good team. I am very thankful to my friends André, Hahni, Zander, Martin, Richard, Hehu and Klim for their support and cooperation. I am also grateful to my friends Umer, Nasir, Sajid, Zaheer, Iftikhar and Tariq for their moral support.

I would like to acknowledge the grant from the Kohat University of Science and Technology and Higher education commission, Pakistan, that played a major role to purse my higher education.

I would like to acknowledge my parents, brothers and sisters who always remains a source of inspiration for me. My parents freed me from all kind of domestics responsibilities to complete this difficult task. Their prayers were always provided me some kind of hidden support. I always feels the powers of their prayers within me. I can confidently say that I completed this task because of them. I don't want to forget saying thanks to all those who supported me the fulfillment of this difficult task in any way.

Finally, I want to pay special thanks to my wife for her endless support and patience in the completion of this degree. She was the person who bears all the direct and indirect tension of my PhD. I also want to acknowledge her for her unconditional love all the time. How can I forget my son Mahen khan for his many inspiration and his innocent amusement.

Abstract

This work focuses on the numerical approximations and analysis of time-dependent convection-diffusion-reaction and population balance equations.

The local projection stabilization method in space and discontinuous Galerkin method in time are proposed for the finite element discretization of the time-dependent convection-diffusion-reaction equations. Error estimates for the semi discrete and fully discrete problems are discussed.

In population balance equations, the distribution of the entities depends not only on space and time but also on its own properties referred to as internal coordinates. An operator splitting and at alternating direction method are developed for the numerical solution of population balance equations. In particular, the operator splitting method transforms the original time-dependent problem into two subproblems: a time-dependent convection-diffusion problem and a transient transport problem with pure advection. The backward Euler time stepping scheme is used to discretize subproblems in time. In addition, the operator splitting method allows to use different type of discretization techniques to solve the subproblems. Since the first subproblem is convection-dominated, stabilization techniques in particular local projection and Streamline-Upwind Petrov-Galerkin methods are used.

First the local projection stabilization method in space with discontinuous Galerkin method in the internal coordinate are used for the finite element discretization of the subproblems. The unconditional stability and convergence analysis for the two-step method are discussed comprehensively.

Second the Streamline-Upwind Petrov-Galerkin method in space together with the discontinuous Galerkin method in the internal coordinate are used. The formulation is strongly consistent in the sense that the time derivative is included in the stabilization term. The stability estimates of the two-step method are proved under the condition that the stabilization parameters depend on the length of the time step. Based on the choice of stabilization parameters, error estimates with the standard order of convergence are derived. Furthermore, the numerical results obtained by Streamline-Upwind Petrov-Galerkin method in space are compared with those computed using local projection stabilization method in space.

The aim in alternating direction scheme is the same as in the operator splitting method, i.e., reducing the high dimensional problem into a set of lower ones. First the problem is discretized in space and internal coordinate using local projection stabilization and discontinuous Galerkin method, respectively. Then the backward Euler time stepping method is used to obtain a fully discrete scheme. The matrices in the fully discrete scheme are tensor

products of the space and internal coordinate direction. Therefore it is possible to derive a two-step alternating direction method. Based on an equivalent one step formulation the stability and convergence of the method are discussed.

Zusammenfassung

Schwerpunkt der vorliegenden Arbeit ist die numerische Approximation und Analysis der zeitabhängigen Konvektions-Diffusions-Reaktions- sowie Populationsbilanzgleichungen.

Für die Finite Elemente Diskretisierung der zeitabhängigen Konvektions-Diffusions-Reaktions-Gleichungen werden die Methode der lokalen Projektion für die räumliche und das unstetige Galerkin-Verfahren für die zeitliche Diskretisierung verwendet. Es werden Fehlerabschätzungen für das semidiskrete und das vollständig diskrete Probleme behandelt.

Bei Populationsbilanzgleichungen hängt die Verteilung der Spezies nicht nur vom Raum und von der Zeit, sondern auch von deren Eigenschaften ab, welche als Eigenschaftskoordinate bezeichnet werden. Zur numerischen Lösung von Populationsbilanzgleichungen wird eine Operator-Splitting-Methode und eine Methode der alternierenden Richtungen entwickelt. Die Operator-Splitting-Methode transformiert das zeitabhängige Problem durch ein zeitabhängiges Konvektions-Diffusions-Problem und durch ein transientes Transportproblem mit reiner Advektion. Zur Diskretisierung der beiden Teilprobleme wird das implizite Euler-Verfahren verwendet. Darüber hinaus erlaubt die Operator-Splitting-Methode die Verwendung verschiedener Diskretisierungstechniken, um die beiden Teilprobleme zu lösen. Da im ersten Teilproblem die Konvektion dominiert, muss eine Stabilisierungsmethode verwendet werden. Dafür wird die Methode der lokalen Projektion und die Streamline-Upwind Petrov-Galerkin Methode verwendet.

Zunächst wird für die Finite Elemente Diskretisierung der Teilprobleme die lokale Projektionsmethode im Raum mit der unstetigen Galerkinmethode für die Eigenschaftskoordinate kombiniert. Die unbedingte Stabilität sowie eine Konvergenzanalyse werden ausführlich für eine Zwei-Schritt Methode behandelt.

Anschließend wird die Streamline-Upwind Petrov-Galerkin Methode im Raum zusammen mit der unstetigen Galerkinmethode für die Eigenschaftskoordinate verwendet. Die Formulierung ist stark konsistent, in dem Sinne, dass die Zeitableitung im Stabilisierungsterm enthalten ist. Die Stabilitätsabschätzungen der Zwei-Schritt Methode sind unter der Voraussetzung bewiesen, dass die Stabilisierungsparameter von der Länge des Zeitschritts abhängen. Basierend auf der Wahl der Stabilisierungsparameter, wird eine Fehlerabschätzung mit der Standardkonvergenzordnung abgeleitet. Darüber hinaus werden numerische Ergebnisse der Streamline-Upwind Petrov-Galerkin Methode mit Resultaten der Methode der lokalen Projektion verglichen.

Das Ziel der Methode der alternierenden Richtungen besteht, wie auch bei der Operator-Splitting-Methode, in der Reduktion eines höher dimensionalen Problems durch Systeme

niedrigerer Dimension. Zunächst wird das Problem räumlich und in der Eigenschaftskoordinate diskretisiert, wobei die LPS-bzw. dG-Methode verwendet werden. Im Anschluss wird das implizite Euler Verfahren angewendet, um eine vollständige Diskretisierung des Systems zu erhalten. Dabei lassen sich die Matrizen des vollständigen diskreten Systems als Tensorprodukte der Systeme aus der räumlichen Diskretisierung und der Diskretisierung in der Eigenschaftskoordinate darstellen. Daher ist es möglich, eine Zwei-Schritt-Methode der alternierenden Richtungen abzuleiten. Basierend auf einer äquivalenten Ein-Schritt-Formulierung wird schließlich die Stabilität und Konvergenz dieser Methode untersucht.

Contents

Chapter 1

Introduction

The modeling of many technical and physical processes leads to descriptions which contain time-dependent convection-diffusion-reaction equations as subproblems. Many engineering problems are governed by these type of equations. A special example is the precipitation process, which involves chemical reaction in a flow field. Such processes are modeled by a population balance system [40], consisting of equations describing the flow field by the Navier-Stokes equations, the chemical reactions by convection-diffusion-reaction equations and the particle size distributions by transport equations. These equations are strongly coupled such that the inaccuracies in the concentration of one species directly effect the concentrations of all other species. These equations are convection dominated in the case that the size of diffusion is smaller by several order of magnitude compare to the flow field and are reaction dominated because of strong chemical reactions. The numerical methods in such situation often produce solutions which contain spurious oscillations.

Therefore, the accurate and efficient solution of time-dependent convection-diffusion-reaction equations is critical for accuracy and efficiency of the whole process. The first objective of this thesis is to address the numerical analysis of algorithms for solving such problems.

In addition to the strong coupling of the equations in population balance systems, the other difficulty in the simulation is that the population balance equation (PBE) depends not only on space and time but also on its own properties referred to as internal coordinates. Consequently, the dimension of the PBE is higher than the other equations in the system. Because of the high dimensionality of the PBE, the numerical simulation of coupled system with standard numerical scheme is a challenge from computational point of view. Thus, a second goal of this thesis is the development and analysis of new efficient numerical methods for the population balance equations which have not been considered in the literature before.

Let Ω be a domain in $\mathbb{R}^d$ $(d = 1, 2, 3)$, with polyhedral boundary $\partial\Omega$ and $T > 0$ is the final time. We consider the scalar convection-diffusion-reaction equation:

Find $u : (0,T) \times \Omega \to \mathbb{R}$:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \varepsilon \Delta u + \mathbf{b} \cdot \nabla u + \sigma u = f \quad \text{in} \quad (0,T] \times \Omega, \\ u(0,\cdot) = u_0 \quad \text{in} \quad \Omega \\ u = 0 \quad \text{on} \quad (0,T] \times \partial\Omega, \end{cases} \tag{1.1}$$

where $\varepsilon > 0$ is the diffusion coefficient, $\mathbf{b}$ is the convection field with $\nabla \cdot \mathbf{b} = 0$, σ is the non-negative reaction coefficient, f is the source function, and u_0 is the initial data.

Population balance equations have many applications in various branches of engineering and science. These equations are widely used in chemical engineering to model processes involving one or more particulate phase. For example, they are used to study crystallization, precipitation, pharmaceutical manufacturing, particle size distribution, aerosol formation, emulsion process and dispersed phase distribution of multiphase flows. A population balance equation describing the particle size distribution z is defined as follows: Find $z : (0,T) \times \Omega_\ell \times \Omega_x \to \mathbb{R}$

$$\begin{cases} \dfrac{\partial z}{\partial t} + \nabla_\ell \cdot \big(Gz\big) - \varepsilon \Delta_x z + \mathbf{b} \cdot \nabla_x z = f \quad \text{in} \quad (0,T] \times \Omega_\ell \times \Omega_x, \\ z(0,\cdot) = z_0 \quad \text{in} \quad \Omega_\ell \times \Omega_x, \quad z = g \ \text{ on } \quad (0,T] \times \partial\Omega_\ell \times \Omega_x, \\ z = 0 \quad \text{on} \quad (0,T] \times \Omega_\ell \times \partial\Omega_x, \end{cases} \tag{1.2}$$

Here, ℓ represents the variable in internal coordinates and x in space (external coordinates) which represents the position of the particle in the physical space. Furthermore, $\varepsilon > 0$ is the diffusion coefficient, Δ_x and ∇_x represents the Laplacian and gradient with respect to the variable x, respectively, and ∇_ℓ the gradient with respect to the variable ℓ. The physical domain $\Omega_x \subset \mathbb{R}^d$, $d = 2, 3$, and internal coordinate domain $\Omega_\ell \subset \mathbb{R}^e$, $e \geq 1$. The vector functions $\mathbf{b}$ is a given d-dimensional velocity field and growth rate $G > 0$, e-dimensional function. The internal coordinates ℓ, often referred to as size, is typically the characteristics length, volume or mass, but it can also represent age, composition and other characteristic of entities in a distribution. The growth rate G can be a function of size and other variables such as temperature and concentration of chemical species in the solution but is independent of space variable x. For more detail, we refer to the book of Ramkrishna [74] where a comprehensive review of the subject of PBE in terms of PBE formulation, application and solution has been discussed.

1.1 Overview

In applications, typically the convection terms are dominant in convection-diffusion or incompressible flow problems and characteristic solutions have sharp layers. In this case, standard finite element methods will lead to solutions which contain global unphysical oscillations. Also standard discretization techniques will not produce an accurate solution on quasi-uniform meshes due to the presence of interior and boundary layers. In order to prevent these difficulties, a-priori choices of meshes and several stabilization techniques were introduced in the literature.

One of the first layer adapted meshes were proposed by Bakhvalov [4]. In 1969, Bakhvalov solved boundary value problems for ordinary differential equations with a small parameter multiplying the second derivative. Solutions to such problems involve boundary layers. The solution was achieved by applying nonuniform grids (Bakhvalov grids) condensing in the boundary layer. A-priori adapted meshes can be used if sufficient information of the structure of the solution is available. The piecewise uniform Shishkin meshes were originally proposed for finite difference methods in [71]. The first analysis of finite element methods on Shishkin meshes is studied in [82]. For more details about the properties and uses of these kinds of meshes, we refer to [76].

As mentioned in [61, 84], the standard discretization methods lack the stability even on the layer-adapted meshes, a stabilization term are added to standard discretization [65].

Several different methods have been devised for the solution of the above mentioned difficulties. One popular method for stabilizing the convection-dominated convection-diffusion-reaction problems is the Streamline-Upwind Petrov-Galerkin method (SUPG) (also known as streamline-diffusion finite element method (SDFEM)). It was introduced by Hughes and Brooks [36] for steady problems. This method provides good stability properties and highly accurate solutions outside the interior and boundary layers. The SUPG method was investigated by many author's, see [23, 44, 45]. The SUPG with higher order finite elements applied to convection-diffusion problems on Shishkin meshes was studied by Stynes and Tobiska [83]. However, the main drawback of the SUPG method is the fact that several additional terms which includes the second order derivatives have to be assembled in order to ensure the strong consistency of the method. In particular, the assembling of the latter ones is time consuming on non-affine meshes. Moreover, the strong consistency requirement leads to a wide (and generally unphysical) coupling of the unknowns.

Alternatives to SUPG are symmetric stabilization methods such as the continuous interior penalty method (CIP) [13], the local projection stabilization (LPS) [6, 8, 67], the subgrid scale modeling (SGS) [31, 59], and others. These methods have been investigated during the last decades.

The idea in the continuous interior penalty method is to add a least squares penalization on the gradient jump between the neighboring elements. Moreover, the Dirichlet boundary conditions are imposed weakly in the discrete problem, unlike all other stabilization methods in which the boundary conditions are incorporated into the finite element space. There exists a huge amount of literature on CIP stabilization method for convection-diffusion problems. Starting from [12] the pure transport problem or convection-dominated problems, which is the extension of interior penalty method proposed by Douglas and Dupont [22], the method was extended to generalized Stokes problem in [14], to the Oseen problem in [11]. The CIP method on Shishkin meshes for convection-diffusion problems with characteristic layers has been analyzed in [28].

The stabilization term of the local projection method is based on a projection $\pi_h : V_h \to \mathcal{D}_h$ of the finite element space V_h which approximates the solution into a discontinuous space $\mathcal{D}_h$. Originally proposed for Stokes problem [5], the local projection method was extended to transport problems in [6]. The application of local projection methods to Oseen problems are studied in [8, 67] and for convection-diffusion-reaction problem with mixed boundary condition in [68]. The local projection method on layer adapted meshes for convection-diffusion problems with exponential boundary layer was studied in [65, 66]. The local projection method provides additional control over the fluctuations of the gradient or parts of it. Although, the method is weakly consistent only, the consistency error can be bounded by choosing the projection spaces rich enough and optimal order of convergence is maintained.

The local projection method was originally proposed as two-level approach where the projection space D_h is defined on a coarser mesh [5, 6, 8]. In this case, standard finite element spaces can be used for both the approximation space and the projection space. The stabilization terms of CIP and the two-level LPS methods introduce additional couplings between degrees of freedoms which do not belong to the same finite element cell. Hence the sparsity of the element matrices decreases and one needs appropriate data structures for an efficient implementation into a given computer code. The general approach presented in [67] allows us to construct a method based on local projection such that the resulting discretization stencil does not increase compared to standard Galerkin or SUPG method. This is done by an enrichment of finite element space compared to the standard finite element space. In this case the enriched and the projection spaces live on the same grid. Although the system looks larger at the first glance. However, the additional degrees of freedom which occur due to the enrichment can be eliminated locally by static condensation. In this way, one can work with the same number of degrees of freedom which are needed to achieve the appropriate approximation order. Furthermore, neither time derivative nor second order derivatives have to be assembled for the stabilization term of LPS method. Recently also exponential enrichment have been considered see [46]. Here we consider the classical one-level LPS method with polynomial enrichment.

The subgrid scale modeling was first applied to a scalar transport problem [31]. This method based on the scale separation of the finite element space $V_h = V_H \oplus V_h^H$, where V_H represents the space of large scales and V_h^H the space of small scales. The standard Galerkin finite element method is stabilized by adding terms which gives a weighted control on the gradient of fluctuations $\mathrm{id} - P_H$ where $P_H : V_h \to V_H \subset V_h$ is a suitable projection operator. Similar to LPS, the scale separation can also be treated as one- and two-level approaches. The difference of SGS to LPS is that in SGS the stabilization term uses gradients of fluctuations while in LPS fluctuations of gradients. For more details and a relation between the stabilizing terms of SGS and LPS, we refer to [76].

The discontinuous Galerkin (dG) method has recently received significant attention and is applied to a wide range of hyperbolic and parabolic problems. The method was first introduced by Reed and Hills [75] for neutron transport problem and analyzed for ordinary differential equation by Dekfour, Hager and Trochu in [19]. The analysis of the dG method for partial differential equation starts with the works of Lesaint & Raviart [58] and Johnson & Pitkäranta [43]. The first work in the context of parabolic problems was done by Eriksson, Johnson & Thomée [24] and Larsson, Thomée & Wahblin [57]. An hp a priori error analysis of the dG time stepping method for initial value problems was studied by Schötzau and Schwab [78] and for hp-version of the dG finite element method for parabolic problems in [79]. The dG method for the spatial discretization of different types of partial differential equation have been investigated by Cockburn and his co-workers [2]. For more details about the dG method we refer to the survey article [16] and the books [17, 85].

In comparison with other numerical methods, e.g. finite element or finite volume methods, the discontinuous Galerkin method has both advantages and disadvantages. This method uses discontinuous piecewise polynomial spaces to approximate the sought solution of partial differential equation on a finite element mesh without any requirement on the continuity between the neighboring element. Like SUPG and other stabilization methods, the dG method is more stable than the standard Galerkin method, when applied to convection-diffusion problems. On the other hand, the construction of the bilinear form in dG finite element formulation is very different from the other finite element methods. The essential boundary conditions are imposed weakly to the weak formulation of the problem without using multipliers and therefore can be applied to domain decomposition. Despite all these advantages, dG methods have not yet made a significant impact for practical applications. Since the bases functions are discontinuous across the element boundaries, the number of unknowns are large. The computational cost associated with discontinuous Galerkin method is larger when compared to continuous finite element or finite volume methods. Recently, the hybridizable discontinuous Galerkin method was introduce to handle this issue, see [72] and their references.

There are several approaches for discretizing time-dependent convection-diffusion-reaction problems by finite element methods. Firstly, space-time elements combined with some stabilization could be used [42, 88]. This results into $(d+1)$-dimensional problems in each space-time slab which are more difficult to handle than the corresponding d-dimensional problems in space. Secondly, semidiscretization as intermediate steps can be used. Here, we distinguish between horizontal and vertical methods of lines. The vertical method of lines discretizes first in space and then in time while the horizontal method of lines (or Rothe's method) applies first a time discretization which is followed by a discretization in space. Stabilization parameters play a critical role in the success of the stabilized methods. The main difficulty in the analysis of SUPG method for time-dependent problems comes from the fact that the time derivative has to be added into the stabilization term and this adds a non-symmetric term that cannot be easily bounded by using standard energy argument.

Stabilized finite element methods for time-dependent convection-diffusion-reaction problems have been investigated by several authors. The stability property of consistent stabilization methods in the small time step limit have been discussed in [7, 35]. The approach in these studies was to discretize the problem in space first with a stabilized method, choose the stabilization parameters for the semi-discrete problem, and then discretize in time. This results in stabilization parameters that depends only on mesh width in space, because the temporal discretization is performed after the choice of stabilization parameters. The stability and convergence properties of the SUPG method in space combined with backward Euler, the Crank-Nicolson or the second order backward differentiation formula in time for transient transport problems are studied in [9]. The error bounds in the L^2 norm and in the norm of material derivative are obtained under a regularity conditions on data and the stabilization parameters depend only on the mesh size in space variable. For non-smooth data, the bounds are valid under the condition that the stabilization parameters depend on the length of the time step. Numerical studies of the different stabilization techniques including SUPG can be found in [18, 62].

On the other hand, if the problem is discretized in space and time first, see [39, 41], and then choosing the stabilization parameters. This gives stabilization parameters which depends on the length of the time step. The stability and error estimates for SUPG method combined with backward Euler time stepping scheme for time-dependent convection-diffusion-reaction problems are discussed in [39]. In particular, two different choices of stabilization parameters are derived. The first choice is to choose the stabilization parameters proportional to the time step length and the second one proportional to some function of time step length. Moreover, the time continuous limit is analyzed under certain conditions on the coefficients of the equation. Numerical studies presented in [39, 41] show that this approach leads to a solution which contains unphysical oscillations for small time steps compared to the approach from [7, 35]. However, the numerical results in time continuous limit case in [39] also suggest that the stabilization parameters can be chosen independent of the length of the time step.

The symmetric stabilization's in space combined with the θ-method and the second order backward differentiation formula in time have been investigated in [10]. In particular, they show that the contribution from stabilization leads to an extended matrix pattern which can be extrapolated from the previous time step. The details of the stability and error estimates are given only for the CIP method due to the same structure of symmetric stabilization methods. The coupling of other stabilization techniques in the one dimensional case with the finite difference time integration in particular the vertical and horizontal method of lines have been discussed in [3]. The standard Galerkin method in space but on a layer adapted Shishkin mesh and different time discretization have been studied in [48]. Besides the finite difference methods, we consider the so-called dG method, based on using a finite element formulation in time with piecewise polynomial of degree q. In the case $q = 0$, i.e., when the approximation functions are piecewise constant in time, the method reduces to the backward Euler scheme with modified right-hand side. One of the advantages of dG finite element method in time is that the method is based on variational formulation, which is useful in the analysis of time discretization error. The dG method has been analyzed in space [20, 27] and in space and time [26]. A numerical study of SUPG applied to time-dependent convection diffusion problems with small diffusion parameter can be found in [41]. Details about the dG time stepping method can be found in book [85].

In population balance equations, the distribution of entities depends not only on space and time but also on their own properties referred to as internal coordinates. Consequently, the PBE given in high dimensional domain is therefore challenging from computational point of view. For example in [40], precipitation processes are modeled by a population balance system consisting of equations describing the flow field by the Navier-Stokes equations, the chemical reaction by convection-diffusion-reaction equations and the particle size distribution by transport equations. In addition to the coupling of these equations the main difficulty in the simulation is the PBE because the dimension of the PBE is higher than the other equations in the system. In recent years several numerical methods have been introduced for the numerical solution of population balance equations. For example, method of moments and its variant, method of characteristics, finite difference etc.

The method of moments, a special class of method of weighted residuals, is applied as a model reduction for the solution of population balance equations, see [63, 81]. This method reduces the population balance equation to a set of ordinary differential equations (ODEs) which can be solved by any ODE solver. However, the resulting model have several drawbacks. The reconstruction of the density function from moments is difficult [37], since the inverse problem is ill-posed. In addition, the use of size dependent functions violate the closure condition. The method of characteristics, in which the spatial discretization is no longer required, was suggested by Kumara and Ramkrishna [77]. This method also transforms the PBE into a system of ODEs that is then solved along the pathline of the particles (characteristic curves). However, in this method the numerical dissipation error caused by the discretization of growth term can be avoided. Most of these and other methods are restricted to only internal coordinates. In addition, these methods are not efficient and cannot be applied to combined processes and multidimensional population balance equations. Hence, it is motivated to find a computationally efficient numerical scheme for solving multidimensional population balance equations.

In order to handle the curse of dimensionality associated with the population balance equations, an operator splitting or alternating direction methods are introduced in [30] where the Streamline-Upwind Petrov-Galerkin method has been combined with the standard Galerkin method. Operator splitting methods can be applied in different context (splitting in coordinates or directions, splitting with respect to physics i.e., diffusion, convection). These methods are widely used for time integration of unsteady problems. The basic theory of operator splitting for one-dimensional problems can be found in [80, 86]. The concept of operator splitting for time-dependent problems is to decompose the spatial operator into a sum of two or more operators. For example in [64], the decomposition of convection-diffusion-reaction problem into pure convection and diffusion-reaction problems was studied. For more details about operator splitting methods for linear and non-linear convection-diffusion problems, see [47, 49, 50, 51, 56]. The main advantage of such splitting is that each of the subproblems can be discretized and stabilized separately by the best suitable method independent of the other subproblem(s). A detailed analysis of an alternating direction implicit (or operator-splitting) scheme is demonstrated in [52] for the Fokker-Planck equation. The basic idea in [52] is to split the high dimensional problem into two low dimensional problems corresponding to the configuration and the physical spaces. The solution of the convection-diffusion type problem in configuration space is obtained by a Galerkin spectral method at each quadrature point corresponding to the physical domain. Furthermore, a type of L^2 projection is used to update the right-hand side vector at the second stage where the solution of advection equation in physical space is approximated by a finite element method.

1.2 Objectives

This thesis is concerned with the study of the stability and convergence of time-dependent problems. We begin with a brief description of the nature of convection and/or reaction dominated problems. In particular, we are interested in the case when the diffusion coefficient in (1.1) and (1.2) is small e.g $0 < \varepsilon \ll 1$ (the process is convection and/or reaction dominant). The smallness of the diffusion coefficient ε reduces the stability for standard numerical methods. We handle this difficulty by using the SUPG and LPS methods.

We start with the time dependent convection-diffusion-reaction problem (1.1) where we combine the local projection stabilization method in space with the discontinuous Galerkin method in time. First we derive the error estimate for the semi-discrete problem after discretizing the problem in space by finite element methods with local projection stabilization. Then, we discretize the problem in time by using a discontinuous Galerkin method. The stability and error estimates for the fully discrete scheme are derived. Theoretical results are confirmed with some numerical tests for smooth solution and we also present the numerical studies for non-smooth data.

A precise review of the literature shows that the error estimates for this kind of local projection stabilization method for time-dependent convection-diffusion-reaction equations are not yet available. Furthermore, the method is unconditionally stable and convergence estimates are half order better than the other finite difference schemes compared to [39, 10].

The population balance equation (1.2) is defined on the domain $\Omega_\ell \times \Omega_x$ which is of Cartesian product structure. The main idea is to decouple a complex equation into two simpler equations and to solve them by best suitable methods. For this we introduce an operator splitting and alternating direction methods. The basic idea in both methods is the same, i.e., reducing the high dimensional problem into a set of lower ones. In operator splitting method, we split the continuous problem first and then discretizes the subproblems in space and internal coordinate. On the other hand, alternating direction method is used after having the fully discrete scheme.

Let $N > 0$ be a given positive integer and consider $0 = t_0 < t_1 < \ldots < t_N = T$ be a uniform partition of $(0, T)$ with the time step size $\tau = T/N$. Then starting with $u(t^0) = z_0$, two subproblems are sequentially solved on the sub-intervals $(t^n; t^{n+1}]$, $n = 0, 1, \ldots, N-1$:

1. **Ω_x-direction:** Given $u(t^n)$ find $\tilde{u} : (t^n; t^{n+1}] \to \mathbb{R}$ such that:

$$\begin{cases} \dfrac{\partial \tilde{u}}{\partial t} + L_x \tilde{u} &= f \qquad \text{in } (t^n, t^{n+1}] \times \Omega_\ell \times \Omega_x \\ \tilde{u} &= 0 \qquad \text{on } (t^n, t^{n+1}] \times \Omega_\ell \times \partial\Omega_x \\ \tilde{u}(t^{n+}) &= u(t^n). \end{cases} \tag{1.3}$$

2. **Ω_ℓ-direction:** Find $u : (t^n, t^{n+1}] \times \Omega_\ell \times \Omega_x \to \mathbb{R}$ such that

$$\begin{cases} \dfrac{\partial u}{\partial t} + L_\ell u &= 0 \qquad \text{in } (t^n, t^{n+1}] \times \Omega_\ell \times \Omega_x \\ u|_{\ell_{\min}} &= z_{\min} \qquad \text{on } (t^n, t^{n+1}] \times \Omega_x \\ u(t^{n+}) &= \tilde{u}(t^{n+1}), \end{cases} \tag{1.4}$$

   where

$$L_\ell z = \nabla_x \cdot (Gz), \qquad L_x z = -\varepsilon \Delta_x z + \mathbf{b} \cdot \nabla_x z.$$

This two-step operator splitting scheme defines $u(t^n)$, $n = 1, \ldots, N$, as an approximation of $z(t^n)$.

The first subproblem (1.3) is a time-dependent convection-diffusion equation and the second subproblem (1.4) is a transport problem with pure advection. Note that on each time interval the solution of (1.3) is obtained by solving a d-dimensional problem parametrized by variable $\ell \in \Omega_\ell$. Similarly, the evaluation of η is given by solving e-dimensional problem (1.4) parametrized by $x \in \Omega_x$. Then the operator splitting scheme is based on spatial and temporal discretization of (1.3) and (1.4).

Since the splitting leads to a sequence of d- and e-dimensional solves at each time step instead of $d+e$-dimensional solve, the curse of dimensionality associated with the numerical solution of population balance equation (1.2) is improved. The same kind of splitting has been considered in [30], where the standard Galerkin method was combined with SUPG method in space and internal coordinate, respectively. The stability and error bounds were derived using an equivalent one-step formulation. Note that, the fully discrete form of operator splitting scheme is equivalent only in the case that stabilization parameters are proportional to the length of the time step, i.e., $\delta = \mathcal{O}(\tau)$ where δ and τ are the stabilization parameter and time step length, respectively. The same choice of stabilization parameter have been considered in [39] for evolutionary equations. On the other hand, if $\delta = \mathcal{O}(\sqrt{\tau}h)$ the second choice of stabilization parameter in [39], the fully discrete form of operator splitting method is not equivalent to one-step formulation. These conditions arise in the stability bounds from the stabilization term with the discretization of time derivative.

In this work we have considered one internal coordinate. Since in our splitting, the first subproblem is convection-dominated, we use Streamline-Upwind Petrov-Galerkin to stabilize the space discretization. The second subproblem is a transport problem with pure advection, one suitable choice is a discontinuous Galerkin method for the discretization with respect to the internal coordinate. Here, we prove that the two-step method is unconditionally stable. Optimal error estimates are obtained for two-step method.

The second goal is to consider SUPG method for space discretization together with dG method in internal coordinate. We have consider the two different choices of stabilization parameters discussed in [39] to derive the stability and convergence results for the two-step method. The numerical results obtained are also compared with those of the LPS method in space.

Finally, we have considered the alternating direction approach for the numerical solution of population balance equation (1.2). We derive the semi-discrete error bounds using LPS method in space and dG method in internal coordinate. Then the alternating direction scheme is established after discretization in time by backward Euler time stepping scheme. Based on an equivalent one-step formulation we derive the unconditional stability and convergence estimates of the method.

1.3 Outline

The contents of the thesis are organized as follows:

In Chapter 2, we present the mathematical notations and functions spaces that will be used throughout the thesis. Further, we explain the one-level variant of local projection stabilization method and the discontinuous Galerkin finite element method for time discretization.

Chapter 3 concerns with the time-dependent convection-diffusion-reaction problem. We combine the local projection stabilization method in space with discontinuous Galerkin method in time. After discretization in space by local projection stabilization method, we give the error estimates for semi-discrete problem. Stability and convergence estimates will then be given by discretizing the semidiscrete problem in time by the discontinuous Galerkin method. Finally, the theoretical results are confirmed by numerical tests.

Chapter 4 deals with the operator splitting method for the population balance equation with one internal coordinate given on high-dimensional domain. The operator splitting method decomposes the original problem into two subproblems. The first subproblem (1.3) is a time-dependent convection-diffusion equation while the second one (1.4) is a transport problem with pure advection. We provide the unconditional stability of the two-step method after discretizing the subproblems in time using the backward Euler time stepping scheme. The fully discrete stabilized scheme is then obtain by applying the local projection stabilization method in space and discontinuous Galerkin finite element method in internal coordinate. Furthermore, based on the unconditional stability of the fully discrete two-step scheme error estimates are proved.

The goal of Chapter 5 is to analyze the SUPG method based on the conditions on the stabilization parameters discussed in [39] for time-dependent convection-diffusion problems. This chapter starts with the SUPG in space and dG method in internal coordinate for the finite element discretization of the two-step method introduced in Chapter 4. After having the fully discrete two-step scheme, the stability estimates based on two different choices of stabilizing parameters are given. Error bounds for the method are obtained under that conditions. Finally, we give a numerical comparison of the results with the results obtained by LPS method in space.

In Chapter 6, we consider the alternating direction Galerkin method for the numerical solution of PBE (1.2). The original work on alternating direction method for the solution of parabolic and elliptic partial differential equation is by Douglas and Dupont [21]. The chapter starts with the discretization in space and internal coordinate using LPS and dG finite element method, respectively. Then we derive the error estimates for the semi-discrete problem. The fully discrete scheme is obtained using the backward Euler method in time. The matrices in the fully discrete scheme are tensor products of the space and internal coordinate direction. Therefore we are able to derive two steps alternating direction method. Based on an equivalent one step formulation we discuss the stability and convergence of the method.

Chapter 2

Preliminaries

The aim of this chapter is to collect several tools that will be needed later. The common link between all results in this chapter is that they are preparatory for the main results in the following chapters.

In Section 2.1 the basic notations and function spaces are summarized. The key idea behind the local projection stabilization method is introduced in Section 2.2. Furthermore, the one-level approach is discussed in detail with some appropriate examples. In Section 2.3 we develop the time discontinuous Galerkin finite element method in a general framework and derive some frequently used properties. Finally, we will state some useful inequalities.

2.1 Function spaces

In this section, we lay down some useful notation for various spaces and their corresponding norms. More details about these spaces can be found in text books about finite element method for partial differential equations and functional analysis [1]. We use the standard notation of function spaces. Let Ω be a bounded domain, we denote by $(\cdot,\cdot)$ the inner product in $L^2(\Omega)$ and by $\|\cdot\|$ the associated norm. Let $H^m(\Omega)$ denote the Sobolev space of functions with derivatives up to order m in $L^2(\Omega)$. The norm in $H^m(\Omega)$ is defined as

$$\|v\|_m = \left(\sum_{|\alpha|\le m} \|D^\alpha v\|^2 \right)^{1/2} .$$

where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d)$ is a multi-index. Since in this work time-dependent problems are considered, we define certain Bochner spaces, for details we refer to [87]. Let X be a

Banach space equipped with norm $\|\cdot\|_X$ and seminorm $|\cdot|_X$. Then we define the following

$$
\begin{aligned}
C\big(0,T;X\big) &= \Big\{v:[0,T]\to X, \quad v \text{ continuous}\Big\},\\
L^2(0,T;X) &= \Big\{v:(0,T)\to X, \quad \int_0^T \|v(t)\|_X^2 dt < \infty\Big\},\\
H^m(0,T;X) &= \Big\{v\in L^2(0,T;X): \frac{\partial^j v}{\partial t^j} \in L^2(0,T;X),\ 1\le j\le m\Big\},
\end{aligned}
$$

where the derivatives $\partial^j v/\partial t^j$ are understood in the sense of distributions on $(0,T)$. Throughout the thesis we will use the short notation

$$Y(X) := Y(0,T;X).$$

The norms and seminorms in the above defined spaces are given by

$$
\|v\|_{C(X)} = \sup_{t\in[0,T]} \|v(t)\|_X, \qquad \|v\|_{L^2(X)} = \left(\int_0^T \|v(t)\|_X^2 dt\right)^{1/2},
$$

$$
|v|_{H^m(X)} = \left(\int_0^T \left\|\frac{\partial^m v}{\partial t^m}\right\|_X^2 dt\right)^{1/2}, \qquad \|v\|_{H^m(X)} = \left(\int_0^T \sum_{j=0}^m \left\|\frac{\partial^j v}{\partial t^j}\right\|_X^2 dt\right)^{1/2}.
$$

2.2 Local projection stabilization (LPS)

It is well known that in convection-dominated convection-diffusion problems the standard finite element methods will lead to solutions which contain global unphysical oscillations. In order to prevent this, stabilization techniques are applied. In this section, we discuss in detail the stabilization method based on local projection.

Let us consider a shape regular triangulation $\{\mathcal{T}_h\}$ of Ω into d-simplices quadrilaterals or hexahedra. The diameter of the cell K will be denoted by h_K and the mesh size parameter h is defined by $h := \max_{K\in\mathcal{T}_h} h_K$. Assume that $V_h \subset H^1(\Omega)$ denotes the approximation space of continuous, piecewise polynomials of degree r, $r \geq 1$, defined over $\mathcal{T}_h$. Let $\mathcal{D}_h$ denotes a finite element space of discontinuous, piecewise polynomials of degree $r-1$ with $r\geq 1$ and let $\mathcal{D}_h(K) = \{q_h|_K :\ q_h \in \mathcal{D}_h\}$ be the local projection space. Let $\pi_K : L^2(K) \to \mathcal{D}_h(K)$ be the local L^2-projection into $\mathcal{D}_h(K)$, which generates the global L^2-projection $\pi_h : L^2(\Omega) \to \mathcal{D}_h$ defined by

$$(\pi_h v)|_K = \pi_K(v|_K) \quad \forall K \in \mathcal{T}_h,\ \forall v \in L^2(\Omega).$$

The fluctuation operator κ_h is given by $\kappa_h := id - \pi_h$ where $id : L^2(\Omega) \to L^2(\Omega)$ is the identity mapping on $L^2(\Omega)$.

Then, the stabilizing term denoted by $S_h(v_h, w_h)$ is defined as

$$S_h(v_h, w_h) = \sum_{K\in\mathcal{T}_h} \mu_K \Big(\kappa_h(\nabla v_h), \kappa_h(\nabla w_h)\Big)_K \tag{2.1}$$

with user defined non-negative constant μ_K, $K \in \mathcal{T}_h$. The optimal mesh dependent choice of the parameter will follow from the error analysis of the methods in the upcoming chapters.

The stabilization S_h gives an additional L^2-control over the fluctuations κ_h of the gradients. Note that one can replace the gradient ∇v_h by the derivative in the streamline direction $\mathbf{b}\cdot\nabla v$ [68]. Then the stabilization term looks like

$$S_h(v_h, w_h) = \sum_{K\in\mathcal{T}_h} \mu_K \Big(\kappa_h(\mathbf{b}\cdot\nabla v_h), \kappa_h(\mathbf{b}\cdot\nabla w_h)\Big)_K \tag{2.2}$$

which represents the fluctuation of the derivative in the streamline direction or even more better ([53, 54]) by $\mathbf{b}_K\cdot\nabla v$ where $\mathbf{b}_K$ is a piecewise constant approximation of b, which leads to similar results.

Stability and convergence properties of LPS methods are based on the following assumptions with respect to the pair $(V_h, \mathcal{D}_h)$ [67, 68].

Assumption A1: There is an interpolation operator $i_h : H^2(\Omega) \to V_h$ such that

$$\|v - i_h v\|_{0,K} + h_K|v - i_h v|_{1,K} \le C h_K^l \|v\|_{l,K}\ \forall K \in \mathcal{T}_h,\ v \in H^l(K),\ 2 \le l \le r+1. \tag{2.3}$$

Assumption A2: The fluctuation operator κ_h satisfies the following approximation property

$$\|\kappa_h q\|_{0,K} \le C h_K^l \|q\|_{l,K} \quad \forall K \in \mathcal{T}_h\, \forall q \in H^l(\Omega),\ 0 \le l \le r. \tag{2.4}$$

The most important factor in the error analysis of the the LPS method is the existence of an interpolant $j_h : H^2(\Omega) \to V_h$ with the error $v - j_h v$ being L^2-orthogonal to the discontinuous projection space $\mathcal{D}_h$ without loosing the standard approximation properties.
Assumption A3: There exists a constant $\beta_1 > 0$ such that for all $h > 0$

$$\inf_{q_h\in\mathcal{D}_h(K)} \sup_{v_h\in V_h(K)} \frac{(v_h, q_h)_K}{\|v_h\|_{0,K}\|q_h\|_{0,K}} \ge \beta_1 > 0 \quad \forall K \in \mathcal{T}_h \tag{2.5}$$

is satisfied where $V_h(K) = \{v_h|_K : v_h \in V_h,\ v_h = 0 \text{ in } \Omega\backslash K\}$.

Note that the Assumptions A1 and A3 guarantee the existence of a interpolant with the usual interpolation properties (2.3) and the orthogonality $v - j_h v \perp D_h$. Assumption

A2 is needed to bound the consistency error [67, 68, 76]. For example, in one-level LPS Assumption A1 and A2 are satisfied if we choose $(V_h, \mathcal{D}_h) = (P_r, P_{r-1}^{\text{disc}})$, i.e., continuous and discontinuous piecewise polynomials of degree r and $r-1$, respectively. In order to guarantee also the Assumption A3, the approximation space V_h is enriched locally by suitable bubble functions. For details see [67].

Theorem 2.2.1. *Let the Assumption A1 and A3 be satisfied. Then there is an interpolation operator $j_h : H^2(\Omega) \to V_h$ satisfying the approximation*

$$\|v - j_h v\|_{0,K} + h_K |v - j_h v|_{1,K} \le C h_K^l \|v\|_{l,K} \ \forall K \in \mathcal{T}_h, \forall v \in H^l(\Omega), \, 2 \le l \le r+1 \quad (2.6)$$

and orthogonality property

$$(v - j_h v, w) = 0 \qquad \forall q_h \in \mathcal{D}_h, \forall v \in H^2(\Omega). \quad (2.7)$$

Proof. The proof of the theorem can be found in [67, 76]. □

In the following we give explicit examples satisfying the assumptions A1-A3. We use mapped finite element spaces [15] where on the reference cell $\widehat{K}$ the enriched spaces are given by

$$\begin{aligned} P_r^{\text{bubble}}(\widehat{K}) &= P_r(\widehat{K}) + \hat{b}_\triangle P_{r-1}(\widehat{K}) \\ Q_r^{\text{bubble}}(\widehat{K}) &= Q_r(\widehat{K}) + \text{span}\big\{\hat{b}_\square \hat{x}_i^{r-1}, \ i = 1, 2\big\}. \end{aligned}$$

Here, $\hat{b}_\triangle$ and $\hat{b}_\square$ are the cubic bubble on the reference triangle and biquadratic bubble on the reference square, respectively, which vanish on the element boundary. The numerical tests are performed in this thesis using for $(V_h, \mathcal{D}_h)$ the pairs $(Q_r^{\text{bubble}}, P_{r-1}^{\text{disc}})$ and $(P_r^{\text{bubble}}, P_{r-1}^{\text{disc}})$ in two dimensional case with $r = 1$, and $r = 2$. An overview of different variants in two dimensional case for $r = 1$ and $r = 2$ are illustrated in figures 2.1 and 2.2.

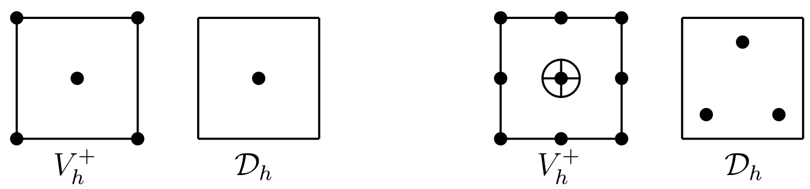


Figure 2.1: Approximation and projection spaces on quadrilaterals (one-level approach).

Figure 2.2: Approximation and projection spaces on triangles (one-level approach).

2.3 Discontinuous Galerkin (dG) method

In the following, we will introduce the key idea behind the discontinuous Galerkin (dG) finite element method for time discretization of the initial value problem

$$u_t + Lu = f, \quad u(0) = u_0 \tag{2.8}$$

where $L : V \to V$ is a bounded operator, independent of time and not necessarily self adjoint. Let us denote by $(\cdot, \cdot)$ the inner product in V and formulate the discontinuous Galerkin method for (2.8). Note that the exact solution of (2.8) satisfies

$$\begin{aligned} \int_0^T \Big\{(u', v) + (Lu, v)\Big\}\, dt &= \int_0^T (f, v) \\ u(0) &= u_0, \end{aligned} \tag{2.9}$$

for sufficiently smooth v. In what follows, we shall denote by f', f'', and $f^{(q)}$ the first, second and q-th order time derivative of f, respectively. Integrating by parts the first term with respect to t, we get

$$\int_0^T \Big\{-(u, v') + (Lu, v)\Big\}\, dt + \big(u(T), v(T)\big) = \big(u(0), v(0)\big) + \int_0^T (f, v)\, dt. \tag{2.10}$$

The idea behind the time discretization by the dG method is to consider a partition of the interval $[0, T]$ into N subintervals $0 = t_0 < t_1 < \cdots < t_N = T$, $J_n = (t_{n-1}, t_n]$ with time step length $k_n = t_n - t_{n-1}$ and $k = \max_n k_n$. For a given nonnegative integer q, we define the following space

$$S_k^q = \Big\{v : [0, T] \to V : v|_{J_n} = \sum_{j=0}^{q} v_j t^j \text{ with } v_j \in V\Big\} \tag{2.11}$$

i.e., on each subinterval J_n, we are looking for an approximation of (2.8) of degree less than or equal to q having values in V. Notice that, by convention, the functions in S_k^q are allowed to be discontinuous at the nodes t_n, $n = 0, \dots, N-1$. For discontinuous in time functions we use the notation

$$\varphi_n^{\pm} = \varphi(t_n^{\pm}) = \lim_{t \to t_n \pm 0} \varphi(t). \tag{2.12}$$

Then the jumps at the end points of J_n are defined as

$$[\varphi]_m := \varphi_m^+ - \varphi_m^-, \qquad m \in \{n-1, n\}. \tag{2.13}$$

Let us introduce the bilinear form B as

$$B(u,v) = \sum_{n=1}^{N} \int_{J_n} \Big\{ (u', v) + (Lu, v) \Big\}\, dt + \sum_{n=1}^{N-1} ([u]_n, v_n^+) + (u_0^+, v_0^+). \tag{2.14}$$

Then the discrete formulation of (2.10) reads:

Given u_0, find $U \in S_k^q$ such that

$$B(U, X) = (u_0, X_0^+) + \int_0^T (f, X)\, dt \qquad \forall X \in S_k^q. \tag{2.15}$$

Lemma 2.3.1. *The bilinear form B can be expressed as*

$$B(u,v) = \sum_{n=1}^{N} \int_{J_n} \Big\{ -(u, v') + (Lu, v) \Big\}\, dt - \sum_{n=1}^{N-1} (u_n^-, [v]_n) + (u_N^-, v_N^-). \tag{2.16}$$

Proof. Integrating by parts the first term in (2.14) we get

$$\begin{aligned}
\sum_{n=1}^{N} \int_{J_n} (u', v)\, dt &= \sum_{n=1}^{N} (u, v)\Big|_{t_{n-1}}^{t_n} - \sum_{n=1}^{N} \int_{J_n} (u, v')\, dt \\
&= \sum_{n=1}^{N} \{ (u_n^-, v_n^-) - (u_{n-1}^+, v_{n-1}^+) \} - \sum_{n=1}^{N} \int_{J_n} (u, v')\, dt.
\end{aligned}$$

Substituting into (2.14)

$$\begin{aligned}
B(u,v) = &\sum_{n=1}^{N} \int_{J_n} \Big\{ -(u, v') + (Lu, v) \Big\}\, dt + \sum_{n=1}^{N} \Big\{ (u_n^-, v_n^-) - (u_{n-1}^+, v_{n-1}^+) \Big\} \\
&+ \sum_{n=1}^{N-1} \Big\{ (u_n^+, v_n^+) - (u_n^-, v_n^+) \Big\} + (u_0^+, v_0^+) \\
= &\sum_{n=1}^{N} \int_{J_n} \Big\{ -(u, v') + (Lu, v) \Big\}\, dt + \sum_{n=1}^{N-1} (u_n^-, v_n^- - v_n^+) + (u_N^-, v_n^- - v_N^-)
\end{aligned}$$

which is equivalent to (2.16). This completes the proof. $\square$

Lemma 2.3.2. *The following holds*

$$B(v,v)=\sum_{n=1}^{N}\int_{J_n}(Lv,v)\,dt+\frac{1}{2}\|v_0^+\|^2+\frac{1}{2}\sum_{n=1}^{N-1}\|[v]_n\|^2+\frac{1}{2}\|v_N^-\|^2. \tag{2.17}$$

Proof. Replacing u by v in (2.14) and (2.16), we get

$$B(v,v)=\sum_{n=1}^{N}\int_{J_n}\Big\{(v',v)+(Lv,v)\Big\}\,dt+\sum_{n=1}^{N-1}([v]_n,v_n^+)+(v_0^+,v_0^+),$$
$$B(v,v)=\sum_{n=1}^{N}\int_{J_n}\Big\{-(v,v')+(Lv,v)\Big\}\,dt-\sum_{n=1}^{N-1}(v_n^-,[v]_n)+(v_N^-,v_N^-).$$

Adding them together and dividing by 2, we get

$$\begin{aligned}B(v,v)&=\sum_{n=1}^{N}\int_{J_n}(Lv,v)\,dt+\frac{1}{2}\sum_{n=1}^{N-1}([v]_n,v_n^+)-\frac{1}{2}\sum_{n=1}^{N-1}(v_n^-,[v]_n)+\frac{1}{2}(v_N^-,v_N^-)+\frac{1}{2}(v_0^+,v_0^+)\\&=\sum_{n=1}^{N}\int_{J_n}(Lv,v)\,dt+\frac{1}{2}\sum_{n=1}^{N-1}([v]_n,[v]_n)+\frac{1}{2}(v_N^-,v_N^-)+\frac{1}{2}(v_0^+,v_0^+).\end{aligned}$$

This completes the proof. □

Gronwall's inequalities are important in the error analysis of time-dependent problems. In this thesis we use continuous as well as discrete versions of the Gronwall's inequality [34].

Lemma 2.3.3 (Continuous Gronwall inequality). *Let f, g, y are piecewise continuous functions defined on (t_0,t). Suppose that g is a non-decreasing function and that there exists a constant α independent of t such that*

$$y(t)+f(t)\le g(t)+\alpha\int_{t_0}^{t}y(s)\,ds\quad\forall t\in(t_0,t).$$

Then,

$$y(t)+f(t)\le\exp\Big(\alpha(t-t_0)\Big)\,g(t)\quad\forall t\in(t_0,t). \tag{2.18}$$

Lemma 2.3.4 (Discrete Gronwall's inequality). *Let k, B, and a_m, b_m, c_m, γ_m, for integer $m\ge 0$, be nonnegative number such that*

$$a_n+k\sum_{m=0}^{n}b_m\le k\sum_{m=0}^{n}\gamma_m a_m+k\sum_{m=0}^{n}c_m+B\qquad\textit{for } n\ge 0.$$

Suppose that $k\gamma_m<1$, for all m, and set $\sigma_m=(1-k\gamma_m)^{-1}$. Then,

$$a_n+k\sum_{m=0}^{n}b_m\le\exp\left(k\sum_{m=0}^{n}\gamma_m\sigma_m\right)\left\{k\sum_{m=0}^{n}c_m+B\right\}\qquad\textit{for } n\ge 0. \tag{2.19}$$

Important tools in the derivation of the error estimates are inverse inequalities.

Lemma 2.3.5 (Inverse inequality). *Let $\{\mathcal{T}_h\}$ be a shape regular family of affine meshes in $\mathbb{R}^d$ with $h \leq 1$. Let $0 \leq m \leq l$ and $1 \leq p, q \leq \infty$, then there exists a constant C independent of h, K, p and q such that (see [25])*

$$|w|_{l,p,K} \leq c_{\mathrm{inv}} h_K^{m-l+d(\frac{1}{p}-\frac{1}{q})} |w|_{m,q,K}, \quad \forall K \in \mathcal{T}_h, \forall w \in V_h. \tag{2.20}$$

Chapter 3

Convection-diffusion-reaction problem

This chapter concerns with the numerical solution of time-dependent convection-diffusion-reaction equations. We apply the vertical method of lines. The aim is to combine the local projection stabilization in space with the discontinuous Galerkin in time, which are discussed in detail in Chapter 2. First we discretize the model problem in space only and investigate the error estimates for the semi-discrete problem. Then the error bounds for the fully discrete scheme are obtained after discretization of the semi-discrete problem in time.

The chapter is organized as follows. Section 3.1 introduces the problem under consideration and derives the weak formulation of the problem. The semi-discretization in space and the local projection stabilization are introduced in Section 3.2. Furthermore, an optimal error estimate for the semi-discretized problems is given. Section 3.3 presents the error analysis for the fully discrete problem after a time discretization by a discontinuous Galerkin method. Finally, numerical results which confirm the theoretical predictions are given in Section 3.4.

3.1 Model problem

Let $\Omega \subset \mathbb{R}^d$ be a bounded polygonal for $d = 2$ or polyhedral for $d = 3$ domain with Lipschitz continuous boundary $\Gamma = \partial\Omega$ and $T > 0$. We set $Q_T := \Omega \times (0, T)$ and consider the following time-dependent convection-diffusion-reaction problem:
Find $u : Q_T \to \mathbb{R}$ such that

$$\begin{cases} u_t - \varepsilon\Delta u + \mathbf{b}\cdot\nabla u + \sigma u = f & \text{in } Q_T, \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(\cdot, 0) = u_0 & \text{in } \Omega, \end{cases} \tag{3.1}$$

where $\varepsilon > 0$ is a given positive constant, $\mathbf{b}$ is a given velocity field, f is a source function and u_0 is the initial data. We assume that $\mathbf{b}$, and σ are independent on time t, whereas f may depend on t. Furthermore, let the data $\mathbf{b}$, σ, u_0 and f are sufficiently smooth on Ω and $\Omega \times (0,T)$, respectively. By the transformation $u(x,t) = e^{Kt}v(x,t)$ with a suitable large constant K one obtains always a system for v of the form (3.1) such that

$$\sigma - \frac{1}{2}\text{div } \mathbf{b} \geq \sigma_0 > 0 \quad \text{in } \Omega. \tag{3.2}$$

Let us introduce the space $V = H_0^1(\Omega)$, its dual space $H^{-1}(\Omega)$, and $\langle \cdot, \cdot \rangle$ for the duality product between these two spaces. Then, a function u is a weak solution of problem (3.1), if

$$u \in L^2(H_0^1), \quad u' \in L^2(H^{-1}), \tag{3.3}$$

and for almost all $t \in (0,T)$,

$$\left\{ \begin{array}{rcl} \langle u'(t), v\rangle + a(u(t), v) & = & \langle f(t), v\rangle \quad \forall v \in V, \\ u(0) & = & u_0. \end{array} \right. \tag{3.4}$$

Here the bilinear form a is given by

$$a(u,v) := \varepsilon(\nabla u, \nabla v) + (\mathbf{b} \cdot \nabla u, v) + (\sigma u, v).$$

Note that (3.3) implies the continuity of u as a mapping of $[0,T] \to L^2(\Omega)$ such that the initial condition $u(0) = u_0$ is well-defined.

3.2 Semi-discrete problem

Since we are interested in convection-dominated convection-diffusion-reaction problems, standard finite element methods will lead to solutions which contain global unphysical oscillations. In order to prevent this, we will consider the one-level local projection stabilization method in which the approximation and projection spaces live on the same mesh.

Based on the finite element discretization in Section 2.2, the stabilization term $S_h(u_h, v_h)$ is given by

$$S_h(u_h, v_h) := \sum_{K \in \mathcal{T}_h} \mu_K \Big(\kappa_h(\nabla v_h), \kappa_h(\nabla w_h)\Big)_K \tag{3.5}$$

with user chosen non-negative constants μ_K, $K \in \mathcal{T}_h$. Now, the stabilized semi-discrete problem reads:

For all $t \in (0,T)$, find $u_h(t) \in V_h$ such that

$$\left\{ \begin{array}{rcl} (u_h'(t), v_h) + a_h(u_h(t), v_h) & = & (f(t), v_h) \quad \forall v_h \in V_h, \\ u_h(\cdot, 0) & = & u_{h,0}, \end{array} \right. \tag{3.6}$$

where the stabilized bilinear form a_h is given by

$$a_h(u_h, v_h) := a(u_h, v_h) + S_h(u_h, v_h) \tag{3.7}$$

and $u_{h,0} \in V_h$ is a suitable approximation of $u_0 \in L^2(\Omega)$.

Lemma 3.2.1 (Coercivity)**.** *The stabilized bilinear form $a_h(\cdot,\cdot)$ is coercive*

$$a_h(v_h, v_h) \geq |||v_h|||^2 \tag{3.8}$$

with respect to the mesh dependent norm

$$|||v_h||| = \left\{\varepsilon|v_h|_1^2 + \sigma_0\|v_h\|^2 + S_h(v_h, v_h)\right\}^{1/2}. \tag{3.9}$$

Proof. Setting $u_h = v_h$ in (3.7), we obtain

$$\begin{aligned} a_h(v_h, v_h) &= a(v_h, v_h) + S_h(v_h, v_h) \\ &= \varepsilon(\nabla v_h, \nabla v_h) + (\mathbf{b}\cdot\nabla v_h, v_h) + \sigma(v_h, v_h) + S_h(v_h, v_h) \end{aligned}$$

Integrating by parts the second term and using the inequality (3.2), we get

$$\begin{aligned} a_h(v_h, v_h) &= \varepsilon|v_h|_1^2 + \sigma_0\|v_h\|^2 + S_h(v_h, v_h) \\ &\geq |||v_h|||^2. \end{aligned}$$

This completes the proof. □

In order to analyze the semi-discrete error, we define the Ritz-projection $R_h : V \to V_h$ associated with the stabilized bilinear form a_h as $R_h w \in V_h$ such that

$$a_h(R_h w, v_h) = a(w, v_h) \qquad \forall v_h \in V_h. \tag{3.10}$$

For the stationary problem associated with (3.1) we have

Theorem 3.2.2. *Suppose A1-A3 defined in Chapter 2, $\tau_K \sim h_K$ for all $K \in \mathcal{T}_h$, and let the data of the problem be sufficiently smooth. Then, there exists a positive constant C, independent of ε and h, such that*

$$|||R_h w||| \leq C\|w\|_1 \ \forall w \in H^1(\Omega) \tag{3.11}$$

and

$$|||w - R_h w||| \leq C(\varepsilon^{1/2} + h^{1/2})h^r\|w\|_{r+1} \tag{3.12}$$

for all $w \in H_0^1(\Omega) \cap H^{r+1}(\Omega)$.

Proof. From (3.8) and (3.10) we have

$$
\begin{aligned}
|||R_h w|||^2 &\le a_h(R_h w, R_h w) = a(w, R_h w) \\
&\le C\|w\|_1 \, |||R_h w|||,
\end{aligned}
$$

from which (3.11) follows. For (3.12), see [76, Theorem 3.74]. □

The next theorem states the main result of this section.

Theorem 3.2.3. *Let $u(t)$ and $u_h(t)$ be the solutions of the continuous problem* (3.4) *and the semi-discrete problem* (3.6)*, respectively. Let the assumptions A1-A3 be fulfilled and $u_0 \in H^{r+1}(\Omega)$, $u \in H^1(H^{r+1})$. If $\tau_K \sim h_K$ for all $K \in \mathcal{T}_h$, then there exists a positive constant C independent of t, ε, and h, such that for all $t \in [0,T]$*

$$
\|u_h(t) - u(t)\| \le \|u_{h,0} - u_0\| + C(\varepsilon^{1/2} + h^{1/2})h^r \left\{ \|u_0\|_{r+1} + \int_0^t \|u'\|_{r+1} dt \right\} \tag{3.13}
$$

and

$$
\begin{aligned}
&\int_0^t |||u_h(s) - u(s)|||^2 \, ds \\
&\quad \le C\left[\|u_{h,0} - u_0\|^2 + (\varepsilon + h)h^{2r} \Big\{ \|u_0\|^2_{r+1} + \int_0^t \left(\|u(s)\|^2_{r+1} + \|u'(s)\|^2_{r+1} \right) ds \Big\} \right].
\end{aligned} \tag{3.14}
$$

Proof. We split the error into two parts

$$
u_h(t) - u(t) = u_h(t) - R_h u(t) + R_h u(t) - u(t) = \xi(t) + \eta(t)
$$

where

$$
\xi := u_h - R_h u \qquad \eta := R_h u - u.
$$

The estimate for the projection error $\eta(t)$ follows from (3.13)

$$
\begin{aligned}
\|\eta(t)\| = \|R_h u(t) - u(t)\| &\le C h^r (\varepsilon^{1/2} + h^{1/2}) \|u(t)\|_{r+1} \\
&\le C h^r (\varepsilon^{1/2} + h^{1/2}) \left\{ \|u_0\|_{r+1} + \int_0^t \|u'(s)\|_{r+1} ds \right\}
\end{aligned} \tag{3.15}
$$

where we used in the second step

$$
\|u(t)\|_{r+1} \le \|u_0\|_{r+1} + \int_0^t \|u'(s)\|_{r+1} ds.
$$

In order to bound $\xi(t)$, we use (3.4), (3.6), the definition (3.10) of the Ritz-projection operator R_h, and the fact that R_h commutes with time derivative to get

$$
\begin{aligned}
(\xi'(t), v_h) + a_h(\xi(t), v_h) &= (u_h'(t) - (R_h u)'(t), v_h) + a_h(u_h(t) - R_h u(t), v_h) \\
&= (f(t), v_h) - a_h(R_h u(t), v_h) - ((R_h u)'(t), v_h) \\
&= (u'(t) - (R_h u)'(t), v_h) \\
&= -(\eta'(t), v_h) \qquad \forall v_h \in V_h.
\end{aligned}
$$

Setting $v_h = \xi(t)$ and taking into consideration the non-negativity of the bilinear form a_h, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\xi(t)\|^2 \le (\xi'(t), \xi(t)) + a_h(\xi(t), \xi(t)) = -(\eta'(t), \xi(t)) \le \|\eta'(t)\| \, \|\xi(t)\|.$$

A usual regularization trick to avoid problems with the differentiability of $t \mapsto \|\xi(t)\|$ when $\xi = 0$ and integration over time from 0 to t yields

$$\begin{aligned} \frac{d}{dt}\|\xi(t)\| &\le \|\eta'(t)\| \\ \|\xi(t)\| &\le \|\xi(0)\| + \int_0^t \|\eta'(s)\| \, ds. \end{aligned}$$

The terms on the right hand side can be estimated as follows

$$\begin{aligned} \|\xi(0)\| &\le \|u_h(0) - u(0)\| + \|u(0) - R_h u(0)\| \\ &\le \|u_{h,0} - u_0\| + Ch^r(\varepsilon^{1/2} + h^{1/2})\|u_0\|_{r+1} \\ \|\eta'(s)\| &= \|u'(s) - R_h u'(s)\| \\ &\le Ch^r(\varepsilon^{1/2} + h^{1/2})\|u'(s)\|_{r+1}. \end{aligned}$$

Thus, for the error to the Ritz-projection we have

$$\|\xi(t)\| \le \|u_{h,0} - u_0\| + Ch^r(\varepsilon^{1/2} + h^{1/2}) \left\{ \|u_0\|_{r+1} + \int_0^t \|u'(s)\|_{r+1} ds \right\}.$$

Combining this with (3.15), we get (3.13).

Above we used only the non-negativity of $a_h(\xi(t), \xi(t))$ instead of the stronger coercivity estimate

$$a_h(\xi(t), \xi(t)) \ge |||\xi(t)|||^2.$$

Now starting with

$$\frac{1}{2}\frac{d}{dt}\|\xi(t)\|^2 + |||\xi(t)|||^2 \le \|\eta'(t)\| \, \|\xi(t)\|,$$

applying arithmetic-geometric inequality in the right-hand side

$$\|\eta'(t)\| \, \|\xi(t)\| \le \frac{\sigma_0}{2}\|\xi(t)\|^2 + \frac{1}{2\sigma_0}\|\eta'(t)\|^2$$

where σ_0 from (3.2) and integrating over t, we obtain

$$\|\xi(t)\|^2 + \int_0^t |||\xi(s)|||^2 \, ds \le \|\xi(0)\|^2 + \frac{1}{\sigma_0}\int_0^t \|\eta'(s)\|^2 \, ds.$$

Using again the estimates for $\|\xi(0)\|$ and $\|\eta'(s)\|$ above, we have

$$\|\xi(t)\|^2 + \int_0^t |||\xi(s)|||^2\, ds \le C\|u_{h,0} - u_0\|^2 + C(\varepsilon + h)\, h^{2r} \left\{ \|u_0\|^2_{r+1} + \int_0^t \|u'(s)\|^2_{r+1}\, ds \right\}.$$

Finally, we use (3.12) and the triangle inequality to get (3.14). □

Compared to standard Galerkin finite element method, where $\mu_K = 0$ for all $K \in \mathcal{T}_h$, the local projection method provides additional control over the fluctuation of gradients because of the definition of the triple norm. The additional stabilization term yields improved stability properties. In the special case, when $\varepsilon = 1$, $\mathbf{b} = 0$ and $\sigma = 0$, the corresponding error estimates (3.13) and (3.14) are the same as in [85, Theorem 1.2,1.3].

3.3 Fully discrete problem

For getting a fully discrete version of (3.1) we apply the discontinuous Galerkin method to problem (3.6). Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a partition of the time interval $[0, T]$, $J_n = (t_{n-1}, t_n]$, $k_n = t_n - t_{n-1}$, and $k = \max_n k_n$. For a given non-negative integer q, we define the semi-discrete space as in (2.11)

$$S_k^q := \left\{ v : [0,T] \to V : v\big|_{J_n}(t) = \sum_{j=0}^{q} v_j t^j \quad \text{with } v_j \in V \right\}$$

and the fully discrete space

$$S_{h,k}^{r,q} := \left\{ v : [0,T] \to V_h : v\big|_{J_n}(t) = \sum_{j=0}^{q} v_j t^j \text{ with } v_j \in V_h \right\} \tag{3.16}$$

where V_h consists of elements of order r.

The fully discrete problem reads: Given an approximation $u_{h,0}$ of u_0, find $U_h \in S_{h,k}^{r,q}$ such that

$$B(U_h, X) = (u_{h,0}, X_0^+) + \int_0^T (f, X)\, dt, \quad \forall X \in S_{h,k}^{r,q}, \tag{3.17}$$

where the bilinear form is defined by

$$B(u,v) := \sum_{n=1}^{N} \int_{J_n} \left\{ (u', v) + a_h(u,v) \right\} dt + \sum_{n=1}^{N-1} \left([u]_n, v_n^+ \right) + (u_0^+, v_0^+) \tag{3.18}$$

and integration by parts with respect to t gives

$$B(u,v) = \sum_{n=1}^{N} \int_{J_n} \Big\{ -(u,v') + a_h(u,v) \Big\}\, dt - \sum_{n=1}^{N-1} \big(u_n^-, [v]_n\big) + (u_N^-, v_N^-). \tag{3.19}$$

The jump terms in the bilinear forms are stabilizing operators and have an effect of up-winding information with respect to time.

The exact solution satisfies (3.17) since $[u]_n = 0$, $n = 1, \ldots, N-1$. Since the functions in the fully discrete space $S_{h,k}^{r,q}$ are allowed to be discontinuous at the nodes t_n, $n = 0, \ldots, N-1$, we can choose v to vanish outside the time interval J_n. Therefore, the fully discrete scheme (3.17) becomes

$$\int_{J_n} \big((U_h', v) + a_h(U_h, v)\big)\, dt + (U_{h,n-1}^+, v_n^+) = (U_{h,n-1}^-, v_{n-1}^+) + \int_{J_n} (f, v)\, dt, \quad \forall v \in V_h \tag{3.20}$$

for $1 \leq n \leq N$, where $U_{h,0}^- = u_{h,0}$.

Special case: In the case $q = 0$, the approximation functions are piecewise constants in time on each interval, and in particular we have $U_h' \equiv 0$ and $U_{h,n} = U_{h,n-1}^+$. Therefore, the fully discrete scheme (3.20) reduces to backward Euler method with modified right-hand side, i.e., for all $v \in V_h$ and $1 \leq n \leq N$

$$\begin{aligned} (U_{h,n}, v) + k_n a_h(U_{h,n}, v) &= (U_{h,n-1}, v) + \int_{J_n} (f, v)\, dt, \\ \left(\frac{U_{h,n} - U_{h,n-1}}{k_n}, v\right) + a_h(U_{h,n}, v) &= \frac{1}{k_n} \int_{J_n} (f, v)\, dt. \end{aligned}$$

We consider two mesh-dependent norms given by

$$\|v\|_w = \left(\sum_{n=1}^{N} \int_{J_n} |||v|||^2 dt + \frac{1}{2} \|v_N^-\|^2 \right)^{1/2} \tag{3.21}$$

$$\|v\|_s = \left(\sum_{n=1}^{N} \int_{J_n} |||v|||^2 dt + \frac{1}{2} \|v_0^+\|^2 + \frac{1}{2} \sum_{n=1}^{N-1} \|[v]_n\|^2 + \frac{1}{2} \|v_N^-\|^2 \right)^{1/2}. \tag{3.22}$$

Here, and in the following we assume the regularity of v needed such that B and $\|\cdot\|_s$ are well defined.

Lemma 3.3.1. *The bilinear form B is coercive with respect to the strong norm $\|\cdot\|_s$, i.e.*

$$B(v, v) \geq \|v\|_s^2. \tag{3.23}$$

Proof. Setting $u = v$ in (3.18) and (3.19) and adding them together to get

$$2B(v,v) = 2\sum_{n=1}^{N}\int_{J_n} a_h(v,v)\,dt + \sum_{n=1}^{N-1}([v]_n,[v]_n) + (v_0^+,v_0^+) + (v_N^-,v_N^-).$$

Using the coercivity of a_h with respect to $|||\cdot|||$ and dividing by two, we get

$$B(v,v) \geq \sum_{n=1}^{N}\int_{J_n} |||v|||^2\,dt + \frac{1}{2}\sum_{n=1}^{N-1}\left\|[v]_n\right\|^2 + \frac{1}{2}\|v_0^+\|^2 + \frac{1}{2}\|v_N^-\|^2 = \|v\|_s^2.$$

□

The next lemma gives the stability of the fully discrete scheme (3.17).

Lemma 3.3.2. *The solution U_h of the fully discrete problem* (3.17) *is uniquely determined and satisfies the stability estimate*

$$\|U_h\|_s \leq C\left(\|u_{h,0}\| + \|f\|_{L^2(L^2)}\right). \tag{3.24}$$

Proof. Setting $X = U_h$ in (3.17) and using the coercivity of B, (3.24) follows by means of Cauchy-Schwarz type inequalities

$$\begin{aligned}\left\|U_h\right\|_s^2 &\leq \|u_{0,h}\|\left\|U_h^0\right\| + \int_0^T \|f\|\left\|U_h\right\|dt \\ &\leq \|u_{0,h}\|\left\|U_h^0\right\| + \left(\int_0^T \|f\|^2\,dt\right)\left(\int_0^T \left\|U_h\right\|^2 dt\right) \\ &\leq C\Big(\|u_{0,h}\| + \|f\|_{L^2(L^2)}\Big)\left\|U_h\right\|_s.\end{aligned}$$

□

For analyzing the error of the discontinuous Galerkin method, we define an interpolant $\tilde{u} \in S_k^q$ of the exact solution $u(t)$ of (3.1) defined by

$$\tilde{u}(t_n^-) = u(t_n),\ n = 1,\ldots,N \tag{3.25}$$

$$\int_{J_n}\big(\tilde{u}(t) - u(t)\big)\,t^l\,dt = 0, \qquad l = 0,1,\ldots,q-1,\ n = 1,\ldots,N \tag{3.26}$$

i.e., $\tilde{u}$ interpolates at the nodal points, and the interpolation error is L^2-orthogonal to the space of polynomials of degree $q-1$ on J_n. Note that $\tilde{u}$ is on each J_n a polynomial in t with values in V. For this type of interpolation, we have for $i,j = 0,1$ the error estimates [73].

$$\sup_{0\leq t\leq t_N} |u(t) - \tilde{u}(t)|_j \leq Ck^{q+1}\sup_{0\leq t\leq t_N}|u^{(q+1)}(t)|_j, \tag{3.27}$$

$$\int_{J_n}|u^{(i)}(t) - \tilde{u}^{(i)}(t)|_j^2\,dt \leq Ck^{2(q+1-i)}\int_{J_n}|u^{(q+1)}(t)|_j^2\,dt, \tag{3.28}$$

Lemma 3.3.3. *Suppose A1-A3, and $\tau_K \sim h_K$ for all $K \in \mathcal{T}_h$. Let U_h and u be the solutions of the fully discrete problem* (3.17) *and the continuous problem* (3.1)*. Moreover, let $u_0 \in H^{r+1}(\Omega)$ and $u \in H^1(H^{r+1})$. Then, the following estimates hold true*

$$\|U_h - R_h\tilde{u}\|_s \le C\Big[\|u_{h,0} - R_h u_0^+\| + (\varepsilon^{1/2} + h^{1/2})h^r\|u\|_{H^1(H^{r+1})} + k^{q+1}\|u\|_{H^{q+1}(H^1)}\Big], \tag{3.29}$$

Proof. For $\xi = U_h - R_h\tilde{u} \in S_{h,k}^{r,q}$ we have from the coercivity (3.23) of B

$$\begin{aligned}\|\xi\|_s^2 = \|U_h - R_h\tilde{u}\|_s^2 &\le B\big(U_h - R_h\tilde{u}, U_h - R_h\tilde{u}\big)\\ &\le B\big(U_h - R_h u, \xi\big) + B\big(R_h u - R_h\tilde{u}, \xi\big)\end{aligned} \tag{3.30}$$

Having in mind that U_h and u are the solutions of the fully discrete and continuous problem, respectively, and that the Ritz-projection commutes with the time derivative. From (3.17), (3.4) and (3.10), we have

$$\begin{aligned}B(U_h, \xi) &= (u_{h,0}, \xi_0^+) + \int_0^{t_N} (f, \xi)\,dt\\ &= (u_{h,0}, \xi_0^+) + \int_0^{t_N} \big\{(u', \xi) + a(u, \xi)\big\}\,dt\\ &= (u_{h,0}, \xi_0^+) + \sum_{n=1}^{N}\int_{J_n} \big\{(u', \xi) + a_h(R_h u, \xi)\big\}\,dt.\end{aligned}$$

The regularity assumption $u \in H^1(H^{r+1})$ ensures the continuity of the mapping $t \mapsto R_h u(t)$. Hence, $[R_h u]_n = 0$ for $n = 1, \dots, N-1$. Adding and subtracting the terms $\int_0^{t_N} (R_h u', \xi)$ and $(R_h u_0^+, \xi_0^+)$, we get

$$\begin{aligned}B(U_h, \xi) = &\sum_{n=1}^{N}\int_{J_n} \Big\{(R_h u', \xi) + a_h(R_h u, \xi)\Big\}\,dt + \sum_{n=1}^{N-1}\big([R_h u]_n, \xi_n^+\big) + (R_h u_0^+, \xi_0^+)\\ &+ \sum_{n=1}^{N}\int_{J_n} (u' - R_h u', \xi)\,dt + (u_{h,0} - R_h u_0^+, \xi_0^+).\end{aligned}$$

From (3.18), we get

$$B(U_h, \xi) = B(R_h u, \xi) + \sum_{n=1}^{N}\int_{J_n} (u' - R_h u', \xi)\,dt + (u_{h,0} - R_h u_0^+, \xi_0^+).$$

Hence for the first term in (3.30), we have the relation

$$B(U_h - R_h u, \xi) = \sum_{n=1}^{N}\int_{J_n} (u' - R_h u', \xi)\,dt + (u_{h,0} - R_h u_0^+, \xi_0^+)$$

Now, applying Cauchy-Schwarz's inequality and (3.12), we conclude

$$\begin{aligned} B(U_h - R_h u, \xi) &\le \sqrt{\sum_{n=1}^{N} \int_{J_n} \|u' - R_h u'\|^2 \, dt} \sqrt{\sum_{n=1}^{N} \int_{J_n} \|\xi\|^2 \, dt} + \|u_{h,0} - R_h u_0^+\| \, \|\xi_0^+\| \\ &\le C(\varepsilon^{1/2} + h^{1/2}) h^r \sqrt{\int_0^T \|u'(s)\|_{r+1}^2 \, ds} \sqrt{\sum_{n=1}^{N} \int_{J_n} \|\xi\|^2 \, ds} \\ &\quad + \|u_{h,0} - R_h u_0^+\| \, \|\xi_0^+\|. \end{aligned} \tag{3.31}$$

For the second term in (3.30), we have from the second representation (3.19) of bilinear form B

$$\begin{aligned} B(R_h u - R_h \tilde{u}, \xi) = \sum_{n=1}^{N} \int_{J_n} \Big\{ - \big(R_h (u - \tilde{u}, \xi') + a_h (R_h (u - \tilde{u}), \xi) \Big\} \, dt \\ + \sum_{n=1}^{N-1} \big(R_h (u - \tilde{u})(t_n^-), [\xi]_n \big) + \big(R_h (u - \tilde{u})(t_N^-), \xi_N^- \big). \end{aligned}$$

Taking into consideration that $u(t_n) = \tilde{u}(t_n^-)$, $n = 1, \ldots, N$ and using the second representation of the bilinear form B, we get

$$\begin{aligned} B(R_h u - R_h \tilde{u}, \xi) &= \sum_{n=1}^{N} \int_{J_n} \Big\{ - \big(R_h u - R_h \tilde{u}, \xi' \big) + a_h \big(R_h (u - \tilde{u}), \xi \big) \Big\} \, dt \\ &= \sum_{n=1}^{N} \int_{J_n} a(u - \tilde{u}, \xi) \, dt. \end{aligned}$$

The first term vanishes. Indeed, from

$$a_h \big(R_h (u - \tilde{u}), v_h \big) = a(u - \tilde{u}, v_h) \qquad \forall v_h \in V_h$$

we obtain after multiplying by an arbitrary polynomial p of degree less than or equal to $q-1$ and integration over J_n the relation

$$\int_{J_n} a_h \big(p R_h (u - \tilde{u}), v_h \big) \, dt = \int_{J_n} a \big(p (u - \tilde{u}), v_h \big) \, dt.$$

Applying Fubini's theorem, we get

$$a_h \left(\int_{J_n} p R_h (u - \tilde{u}) dt, v_h \right) = a \left(\int_{J_n} p (u - \tilde{u}), v_h \right) \, dt = 0 \quad \forall v_h \in V_h$$

due to the orthogonality (3.26). Therefore, for all polynomials p of degree $q-1$ we have

$$\int_{J_n} p R_h (u - \tilde{u}) \, dt = 0$$

which implies

$$\int_{J_n} (R_h(u-\tilde{u}), \xi')\, dt = 0.$$

By means of the interpolation error estimate (3.28), we conclude

$$\begin{aligned}\Big|B(R_h u - R_h\tilde{u}, \xi)\Big| &\leq \sqrt{\sum_{n=1}^{N}\int_{J_n} \|u-\tilde{u}\|_1^2\, dt}\sqrt{\sum_{n=1}^{N}\int_{J_n} a_h(\xi,\xi)} \\ &\leq Ck^{q+1}\sqrt{\int_0^T \|u^{q+1}(s)\|_1^2\, ds}\sqrt{\sum_{n=1}^{N}\int_{J_n} |||\xi|||^2\, ds. } \qquad (3.32)\end{aligned}$$

Substituting the estimates (3.31) and (3.32) into (3.30)

$$\begin{aligned}\|\xi\|_s^2 &\leq C(\varepsilon^{1/2}+h^{1/2})h^r\sqrt{\int_0^T \|u'(s)\|_{r+1}^2\, ds}\sqrt{\sum_{n=1}^{N}\int_{J_n} \|\xi\|^2\, ds} \\ &\quad + Ck^{q+1}\sqrt{\int_0^T \|u^{q+1}(s)\|_1^2\, ds}\sqrt{\sum_{n=1}^{N}\int_{J_n} |||\xi|||^2\, ds} + \|u_{h,0} - R_h u_0^+\|\, \|\xi_0^+\| \\ &\leq C\frac{1}{\sigma_0}(\varepsilon^{1/2}+h^{1/2})^2 h^{2r}\int_0^T \|u'(s)\|_{r+1}^2\, ds + \frac{\sigma_0}{4}\sum_{n=1}^{N}\int_{J_n} \|\xi\|^2\, ds \\ &\quad + 2\|u_{h,0} - R_h u_0^+\|^2 + \frac{1}{8}\|\xi_0^+\|^2 + Ck^{2q+2}\int_0^T \|u^{q+1}(s)\|_1\, ds \\ &\quad + \frac{1}{4}\sum_{n=1}^{N}\int_{J_n} |||\xi|||^2\, ds \\ &\leq C\Big[(\varepsilon^{1/2}+h^{1/2})^2 h^{2r}\|u'(s)\|_{L^2(H^{r+1})}^2 + \|u_{h,0} - R_h u_0^+\|^2 + k^{2q+2}\|u\|_{H^{q+1}(H^1)}^2\Big] \\ &\quad + \frac{3}{4}\|\xi\|_s^2 \\ \frac{1}{4}\|\xi\|_s^2 &\leq C\Big[(\varepsilon^{1/2}+h^{1/2})^2 h^{2r}\|u'(s)\|_{L^2(H^{r+1})}^2 + \|u_{h,0} - R_h u_0^+\|^2 + k^{2q+2}\|u\|_{H^{q+1}(H^1)}^2\Big].\end{aligned}$$

From this the required estimate (3.29) follows. □

Lemma 3.3.4. *Suppose A1-A3, and $\tau_K \sim h_K$ for all $K \in \mathcal{T}_h$. Let $u_0 \in H^{r+1}(\Omega)$ and $u \in H^1(H^{r+1})$ be the solution of problem (3.4). Then, we have following estimates*

$$\|R_h\tilde{u} - R_h u\|_s \leq Ck^{q+1/2}|u|_{H^{q+1}(H^1)}, \qquad (3.33)$$

$$\|R_h\tilde{u} - R_h u\|_w \leq Ck^{q+1}|u|_{H^{q+1}(H^1)}, \qquad (3.34)$$

$$\|R_h u - u\|_s \leq C(\varepsilon^{1/2}+h^{1/2})h^r\left(\|u\|_{L^2(H^{r+1})} + \|u\|_{C(H^{r+1})}\right). \qquad (3.35)$$

Proof. From (3.18), we have for the Ritz-projection of the interpolation error $\eta := R_h\tilde{u} - R_h u = R_h(\tilde{u} - u)$

$$\|\eta\|_s^2 \leq B(\eta, \eta) = \sum_{n=1}^{N} \int_{J_n} \Big[-(\eta, \eta') + a_h(\eta, \eta) \Big] \, dt - \sum_{n=1}^{N-1} (\eta_n^-, [\eta]_n) + (\eta_N^-, \eta_N^-).$$

The interpolation $\tilde{u}$ satisfies $\tilde{u}(t_n^-) = u(t_n)$, $n = 1, \dots, N$, thus the last two terms vanishes and for the first two terms applying Cauchy-Schwarz inequality and coercivity of the bilinear form a_h gives

$$\begin{aligned}
\|\eta\|_s^2 &\leq \sum_{n=1}^{N} \int_{J_n} \Big[\|\eta\| \, \|\eta'\| + |||\eta|||^2 \Big] \, dt \\
&\leq \sum_{n=1}^{N} \int_{J_n} \Big[\|\tilde{u} - u\|_1 \, \|\tilde{u}' - u'\|_1 + \|\tilde{u} - u\|^2 \Big] \, dt.
\end{aligned}$$

Here, we used the stability of the Ritz-projection (3.11). We conclude the estimate (3.33) by using the interpolation error estimates (3.28)

$$\|\eta\|_s^2 \leq C k^{2q+1} |u|_{H^{q+1}(H^1)}^2.$$

For the improved error estimate with respect to the weak norm, we have

$$\|\eta\|_w = \Big(\sum_{n=1}^{N} \int_{J_n} |||\eta|||^2 \, dt + \|\eta_N^-\|^2 \Big)^{1/2}$$

Note that $\tilde{u}_N^- = u(t_N^-)$, therefore the second term vanishes. Using again the stability estimate of the Ritz-projection (3.11) and the interpolation estimates (3.27) to get

$$\|\eta\|_w \leq C \left(\sum_{n=1}^{N} \int_{J_n} \|\tilde{u} - u\|_1^2 \right)^{1/2} \leq C k^{q+1} |u|_{H^{q+1}(H^1)},$$

which is (3.34).

Now we estimate the projection error for which the jump terms $[R_h u - u]_n$ vanishes for $n = 1, \dots, N$, we have from (3.22)

$$\|R_h u - u\|_s = \left(\int_0^T |||R_h u - u|||^2 \, ds + \frac{1}{2} \|(R_h u - u)_0^+\|^2 + \frac{1}{2} \|(R_h u - u)_N^-\|^2 \right)^{1/2}.$$

We conclude the final estimate (3.35) by using (3.12)

$$\begin{aligned}
\|R_h u - u\|_s &\leq C(\varepsilon^{1/2} + h^{1/2}) h^r \left(\int_0^T \|u(s)\|_{r+1} \, ds + \|u_0^+\|_{r+1} + \|u_N^-\|_{r+1} \right) \\
&\leq C(\varepsilon^{1/2} + h^{1/2}) h^r \Big(\|u\|_{L^2(H^{r+1})} + \|u\|_{C(H^{r+1})} \Big)
\end{aligned}$$

which completes the proof of the lemma. □

Theorem 3.3.5. *Suppose A1-A3, and $\tau_K \sim h_K$ for all $K \in \mathcal{T}_h$. Let U_h and u be the solutions of fully discrete problem (3.17) and the continuous problem (3.1), respectively. Moreover, let $u_0 \in H^{r+1}(\Omega)$ and $u \in H^1(H^{r+1})$. Then, there exists a positive constant C independent of h, k and ε, such that the following error estimates*

$$\begin{aligned}\|U_h - u\|_s \le \|R_h u_0^+ - u_{h,0}\| &+ Ck^{q+1/2}|u|_{H^{q+1}(H^1)} \\ &+ C(\varepsilon^{1/2} + h^{1/2})h^r\left(\|u\|_{H^1(H^{r+1})} + \|u\|_{C(H^{r+1})}\right)\end{aligned} \tag{3.36}$$

and

$$\begin{aligned}\|U_h - u\|_w \le \|R_h u_0^+ - u_{h,0}\| &+ Ck^{q+1}|u|_{H^{q+1}(H^1)} \\ &+ (\varepsilon^{1/2} + h^{1/2})h^r\left(\|u\|_{H^1(H^{r+1})} + \|u\|_{C(H^{r+1})}\right)\end{aligned} \tag{3.37}$$

hold true.

Proof. The proof follows from the triangle inequality applied to the splitting

$$U_h - u = (U_h - R_h\tilde{u}) + (R_h\tilde{u} - R_h u) + (R_h u - u)$$

and using Lemmas 3.3.3 and 3.3.4. □

Applying discontinuous Galerkin method to ordinary differential equations one gets the error estimates of order $\mathcal{O}(k^{q+1})$. In (3.37) we got the same convergence order in time for weaker norm.

3.4 Numerical results

In this section, we will present some numerical results for the discontinuous Galerkin and LPS methods applied to time dependent convection-diffusion-reaction problems. All numerical calculations were performed with the finite element package MooNMD [38].
Appropriate finite element spaces which fulfill the assumptions A1-A3 are given in [67].
In our numerical computations we use mapped finite element spaces [15] where on the reference cell $\widehat{K}$ the enriched spaces are given by

$$\begin{aligned}P_s^{\text{bubble}}(\widehat{K}) &= P_s(\widehat{K}) + \hat{b}_\triangle P_{s-1}(\widehat{K}) \\ Q_s^{\text{bubble}}(\widehat{K}) &= Q_s(\widehat{K}) + \text{span}\left\{\hat{b}_\square \hat{x}_i^{s-1},\ i = 1, 2\right\}.\end{aligned}$$

Here, $\hat{b}_\triangle$ and $\hat{b}_\square$ are the cubic bubble on the reference triangle and biquadratic bubble on the reference square, respectively.
The numerical tests are performed using for (V_h, D_h) the pairs $(P_1^{\text{bubble}}, P_0^{\text{disc}})$, $(P_2^{\text{bubble}}, P_1^{\text{disc}})$, $(Q_1^{\text{bubble}}, P_0^{\text{disc}})$ and $(Q_2^{\text{bubble}}, P_1^{\text{disc}})$. The stabilization parameters τ_K have been chosen as

$$\tau_K := \tau_0 h_K \quad \forall K \in \mathcal{T}_h$$

where τ_0 denotes a constant which is given further for each of the test calculation. We used $u_{h,0} := j_h u_0$ as discrete initial condition. In order to compare our results with those in the literature, the first example is taken from [10] and the second from [41].
Due to the enrichment of the approximation spaces by bubble functions, quadrature formulas with higher order accuracy have to be applied to calculate the entries of the system matrix, the mass matrix, and the right-hand side vector. Since $Q_r^{\text{bubble}} \subset Q_{r+1}$ on quadrilaterals, the order of Gaussian formula has to be increased just by 1. On simplicial meshes, the inclusion $P_r^{\text{bubble}} \subset P_{r+d}$ holds true. For assembling the mass matrix the use of quadrature formulas which are exact for polynomials of degree r+d on each element can be avoided by computing the mass matrix on the reference cell and transforming it to the current element.

Example 1. In this example, we consider a pure transport problem in two dimension given by $\varepsilon = \sigma = f = 0$, $\mathbf{b} = (-y, x)^T$, $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ with a Gaussian initial condition centered at $(0.3, 0.3)$ given by

$$u_0(x, y) = e^{-10[(x-0.3)^2+(y-0.3)^2]}.$$

The calculations have been performed on triangular meshes which are obtained from an initial triangulation by successive refinement with boundary adaption due to the curved boundary. The initial mesh (level 0) and the mesh on level 3 are shown in Fig. 3.1.

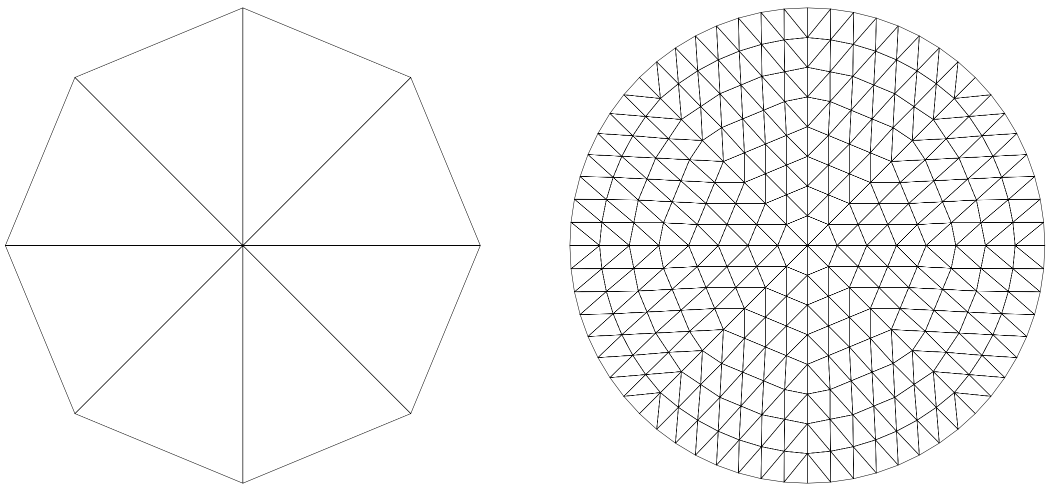


Figure 3.1: Triangular meshes for Example 1: coarsest mesh (left) and mesh after three refinement steps (right).

To find the errors in space and time, we use the standard strategy, i.e., consider the time-step size small enough to find the convergence order in space and vice versa. In Tables 3.1–3.4, we show the convergence results in the strong and weak norms $\|\cdot\|_s$ and $\|\cdot\|_w$, respectively, defined in Section 3.3. For the time discretization, the discontinuous

Galerkin methods of first and second order are used with the final time $T = 2\pi$. Table 3.1 shows the error in space in the strong and weak norms with stabilizing parameter $\tau_0 = 0.1$ and time step length $k = 2\pi \times 10^{-3}$ for $(P_1^{\text{bubble}}, P_0^{\text{disc}})$ and dG(1) in time. In Table 3.2, the convergence results for $(P_2^{\text{bubble}}, P_1^{\text{disc}})$ and dG(2) in time are listed. We see that the expected convergence orders are obtained.

Table 3.1: Errors and convergence orders in space for $(P_1^{\text{bubble}}, P_0^{\text{disc}})$, dG(1), $k = 2\pi \cdot 10^{-3}$ and $\tau_K = 0.1\, h_K$.

level	$\|u-u_h\|_s$		$\|u-u_h\|_w$	
1	6.024193e-01		3.866296e-01	
2	2.060501e-01	1.5478	1.453955e-01	1.4110
3	5.706335e-02	1.8524	4.562678e-02	1.6720
4	1.695608e-02	1.7507	1.495364e-02	1.6094
5	5.535004e-03	1.6152	5.151535e-03	1.5374

Table 3.2: Errors and convergence orders in space for $(P_2^{\text{bubble}}, P_1^{\text{disc}})$, dG(2), $k = 2\pi \cdot 10^{-4}$ and $\tau_K = 0.1\, h_K$.

level	$\|u-u_h\|_s$		$\|u-u_h\|_w$	
1	1.070935e-01		7.581427e-02	
2	1.677886e-02	2.67415	1.263272e-02	2.5853
3	3.046522e-03	2.46141	2.425352e-03	2.3809
4	5.709154e-04	2.41582	4.790734e-04	2.3399
5	1.109878e-04	2.36288	9.813064e-05	2.2875

The numerical errors and convergence orders in time are listed in Table 3.3 with $\tau_0 = 0.1$ for dG(1) and $(P_1^{\text{bubble}}, P_0^{\text{disc}})$ on level 7. The error for dG(2) in time with $(P_2^{\text{bubble}}, P_1^{\text{disc}})$ on level 7 are presented in Table 3.4. We see from the results of weaker norm in Table 3.3 that the expected rates of convergence are achieved for the two largest time step lengths. For smaller time step length the order starts decreasing. This is because the error in space dominates, i.e., the mesh size h is not small enough so that one can see the corresponding convergence rate in time.

Example 2. The second example is the three body rotation used as a test case for advection problem from [41]. We choose $\Omega = (0,1)^2$ and the coefficients $\varepsilon = 10^{-20}$, $\mathbf{b} = (0.5 - y, x - 0.5)^T$, $c = f = 0$. The initial condition consists of three disjoint bodies: a slotted cylinder, a cone and smooth hump, see Fig. 3.2. The position of each body is given by its center (x_0, y_0). Each of the bodies lie within a circle of radius $r_0 = 0.15$ with center (x_0, y_0). The initial condition is zero outside the three bodies. Let

$$r(x,y) = \frac{1}{r_0}\sqrt{(x-x_0)^2 + (y-y_0)^2}.$$

Table 3.3: Errors and convergence orders in time for dG(1) and $(P_1^{\text{bubble}}, P_0^{\text{disc}})$ on level $= 7$ with $\tau_K = 0.1\, h_K$.

k	$\|u - u_h\|_s$		$\|u - u_h\|_w$	
$2\pi/10$	2.006905e-01		4.650374e-02	
$2\pi/20$	9.357820e-02	1.1007	1.198127e-02	1.9565
$2\pi/40$	3.614780e-02	1.3723	2.614460e-03	2.1962
$2\pi/80$	1.337951e-02	1.4338	8.423330e-04	1.6341
$2\pi/160$	4.956802e-03	1.4325	6.429738e-04	0.3897

Table 3.4: Errors and convergence orders in time for dG(2) and $(P_2^{\text{bubble}}, P_1^{\text{disc}})$ on level $= 7$ with $\tau_K = 0.1\, h_K$.

k	$\|u - u_h\|_s$	
$2\pi/10$	4.230334e-02	
$2\pi/20$	8.815594e-03	2.2627
$2\pi/40$	1.669538e-03	2.4006
$2\pi/80$	4.142938e-04	2.0107
$2\pi/160$	1.589712e-04	1.3819

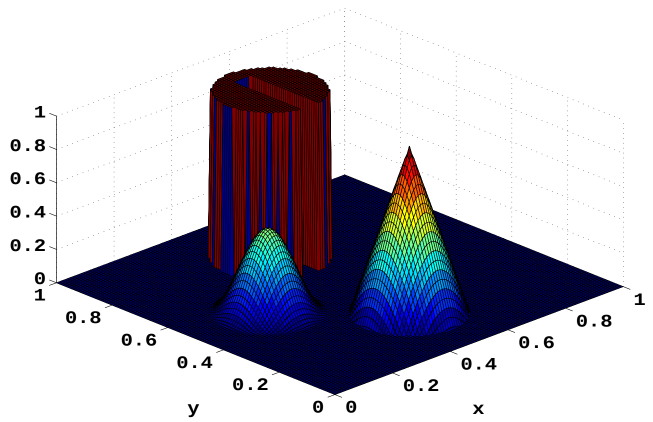


Figure 3.2: Initial condition for rotating body problem.

The center of the slotted cylinder is in $(x_0, y_0) = (0.5, 0.75)$ and its geometry is given by

$$u_0(x,y) = \begin{cases} 1 & \text{if } r(x,y) \le 1,\ |x - x_0| \ge 0.0225 \\ & \text{or} \quad y \ge 0.85, \\ 0 & \text{else.} \end{cases}$$

The conical body at the bottom side is described by its center $(x_0, y_0) = (0.5, 0.25)$ and

$$u_0(x,y) = 1 - r(x,y).$$

Finally, the hump at the left side is given by $(x_0, y_0) = (0.25, 0.5)$ and

$$u_0(x, y) = \frac{1}{4}(1 + \cos(\pi \min\{r(x, y), 1\})).$$

The rotation of the body occurs counter-clockwise and the first full revolution takes place at $T = 2\pi$ which is considered as final time. In the original example [60], the pure transport problem was considered and after each revolution one obtains the initial condition. In our numerical studies, we have used the case of very small diffusion ($\varepsilon = 10^{-20}$). Hence, the results obtained by our method are very closed to the initial condition. The numerical solutions were compared with the initial condition u_0. We present $\|U - u\|_{L^2(L^2)}$ and

$$\mathrm{var}(t) := \max_{(x,y)\in\Omega} U_h(t; x, y) - \min_{(x,y)\in\Omega} U_h(t; x, y),$$

where the maximum and the minimum were computed in the vertices's of the mesh cells. The values $\|U_h - u\|_{L^2(L^2)}$ give some indication of the accuracy of the method and the smearing in the numerical solution whereas $\mathrm{var}(t)$ measures the size of the spurious oscillations. The optimal value is $\mathrm{var}(t) = 1$ for all $t \in [0, T]$.

We have used triangular and quadrilateral meshes which are generated by successive refinement starting from the coarsest meshes (level 0) which are shown in Fig. 3.3.

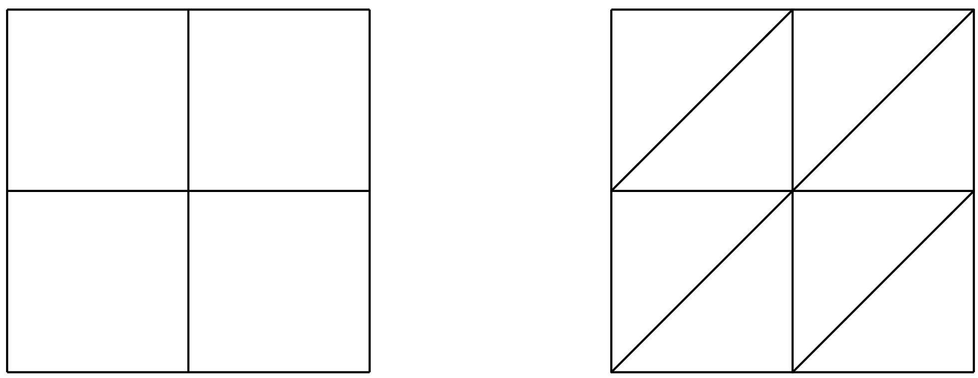


Figure 3.3: Meshes on level 0 for Example 2.

The results computed for the dG(1) in time with time step length $k = 2\pi \times 10^{-3}$ and the pairs $(Q_1^{\mathrm{bubble}}/P_0^{\mathrm{disc}})$ and $(P_1^{\mathrm{bubble}}/P_0^{\mathrm{disc}})$ on level 7 are listed in Tables 3.5 and 3.6 and are plotted in Fig. 3.4.

Table 3.5: Body rotation ($Q_1^{\text{bubble}}/P_0^{\text{disc}}$)

τ_0	$\|U-u\|_{L^2(L^2)}$	var(2π)
0.01	0.131808	1.71336
0.05	0.121924	1.54572
0.1	0.123186	1.42281
0.5	0.136054	1.36074
1.0	0.141408	1.37894
2.0	0.14594	1.37086
5.0	0.152682	1.30647
10.0	0.161308	1.23874

Table 3.6: Body rotation ($P_1^{\text{bubble}}/P_0^{\text{disc}}$).

τ_0	$\|U-u\|_{L^2(L^2)}$	var(2π)
0.01	0.193271	1.88889
0.05	0.161097	1.45071
0.1	0.148976	1.29564
0.5	0.140925	1.33735
1.0	0.144971	1.4265
2.0	0.150848	1.49369
5.0	0.15909	1.59402
10.0	0.164467	1.6663

Note that the same meshes were used in [41]. For the higher order methods ($Q_2^{\text{bubble}}/P_1^{\text{disc}}$) or ($Q_2^{\text{bubble}}/P_1^{\text{disc}}$) with dG(2) using $k = 8\pi \times 10^{-3}$ we list the results in Table 3.7 and 3.8 and plot them in Fig. 3.5.

Table 3.7: Body rotation ($Q_2^{\text{bubble}}/P_1^{\text{disc}}$).

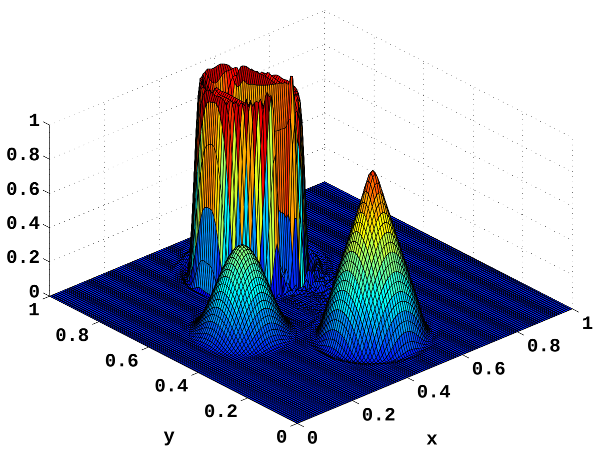
τ	$\|U-u\|_{L^2(L^2)}$	var(2π)
0.01	0.118006	1.41489
0.05	0.119446	1.32256
0.1	0.121823	1.33089
0.5	0.128382	1.38688
1.0	0.132182	1.40098
2.0	0.136541	1.37409
5.0	0.142882	1.35463
10.0	0.147771	1.34645

Table 3.8: Body rotation ($P_2^{\text{bubble}}/P_1^{\text{disc}}$).

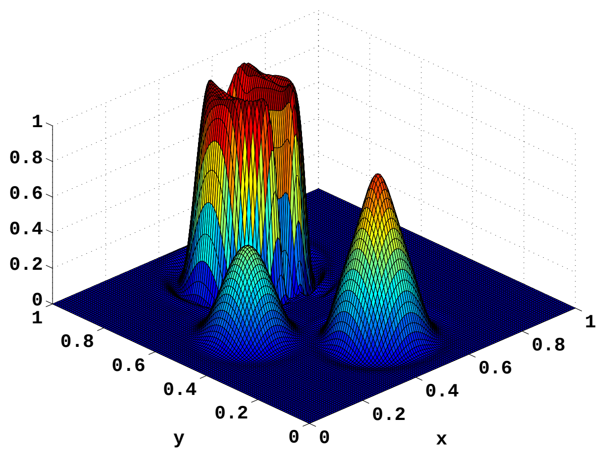
τ	$\|U-u\|_{L^2(L^2)}$	var(2π)
0.01	0.11584	1.30134
0.05	0.116975	1.29865
0.1	0.118591	1.33371
0.5	0.119912	1.40233
1.0	0.119735	1.41362
2.0	0.120005	1.40841
5.0	0.121273	1.41464
10.0	0.12278	1.42961

From Tables 3.5 and 3.6 for the first order discretization, we see that the L^2-error decreased initially since the oscillations becomes smaller. However, increasing τ_0 further, L^2-error increases due to smearing. The results concerning the variations differ on the underlying meshes. On quadrilateral, the variations are decreasing by increasing τ_0, see Table 3.5 and the second picture in Fig. 3.4. On triangles, an increase of τ_0 causes an increase of the variations, see Table 3.6 and last picture in Fig. 3.4.

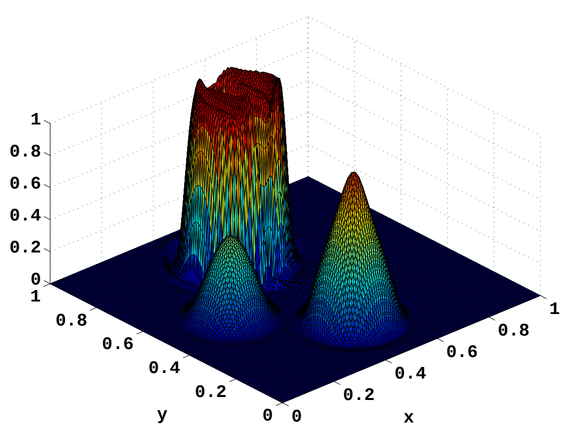
In the second order discretization, the L^2-errors are increasing when τ_0 becomes larger, see Tables 3.7 and 3.8 and Fig. 3.5. The reduction of variation for increasing τ_0 which has been observed on quadrilateral meshes but not on triangular ones is less visible for higher order approximations.



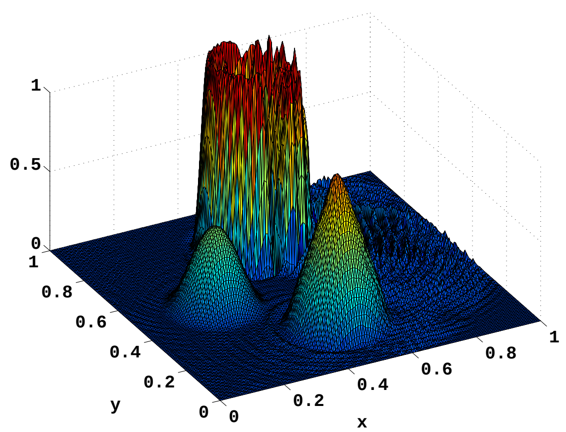
(a) $(Q_1^{\text{bubble}}/P_0^{\text{disc}})$ with $\tau_0 = 0.1$



(b) $(Q_1^{\text{bubble}}/P_0^{\text{disc}})$ with $\tau_0 = 10.0$

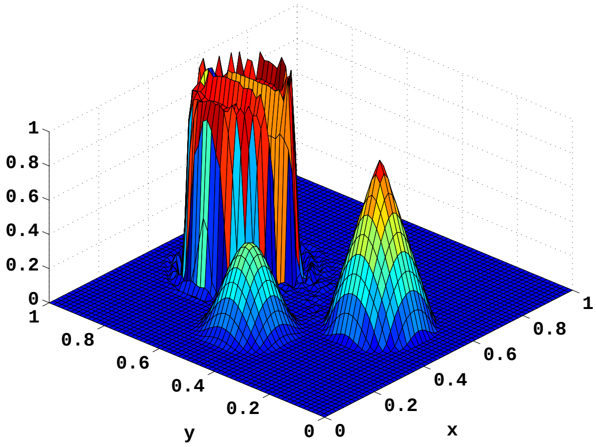


(c) $(P_1^{\text{bubble}}/P_0^{\text{disc}})$ with $\tau_0 = 0.1$

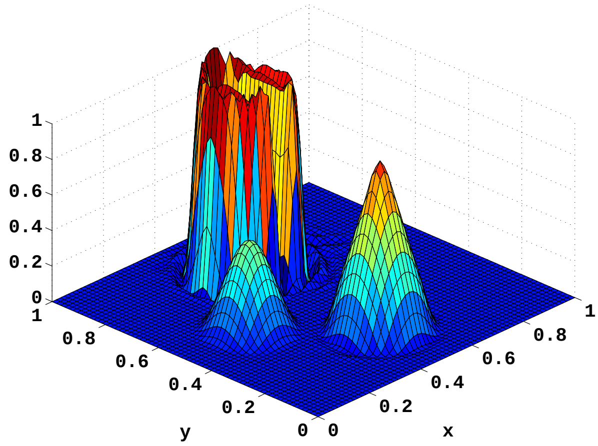


(d) $(P_1^{\text{bubble}}/P_0^{\text{disc}})$ with $\tau_0 = 10.0$

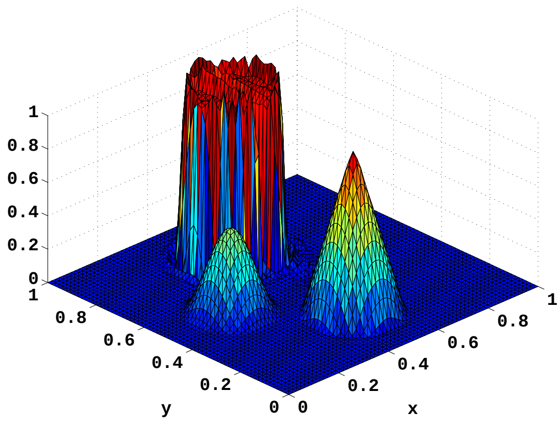
Figure 3.4: Body rotation problem; the computed solution at $t = 2\pi$.



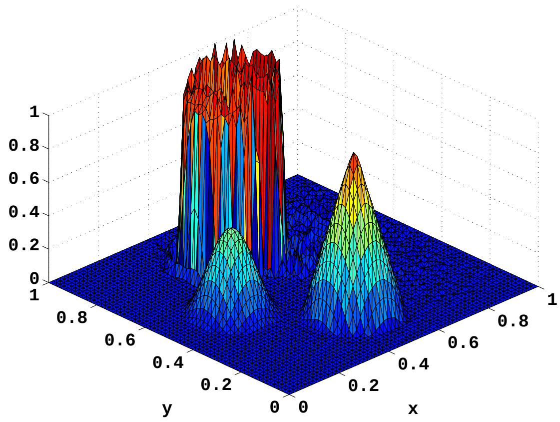
(a) $(Q_2^{\text{bubble}}/P_1^{\text{disc}})$ with $\tau_0 = 0.5$



(b) $(Q_2^{\text{bubble}}/P_1^{\text{disc}})$ with $\tau_0 = 10.0$



(c) $(P_2^{\text{bubble}}/P_1^{\text{disc}})$ with $\tau_0 = 0.05$



(d) $(P_2^{\text{bubble}}/P_1^{\text{disc}})$ with $\tau_0 = 10.0$

Figure 3.5: Body rotation problem; the computed solution at $t = 2\pi$.

Chapter 4

PBE, LPS method in space and dG in internal coordinate

The goal of this chapter is to overcome the curse of dimensionality associated with the numerical solution of population balance equation (1.2). For this, we apply an operator splitting method in the context of finite element methods.

The application of operator splitting method reduces the original high-dimensional problem into a collection of two or more low-dimensional unsteady subproblems of smaller complexity. The first subproblem is a time-dependent convection-diffusion problem while the second one is a transport problem with pure advection. The main advantage of such a splitting is that each of the subproblems can be discretized and stabilized separately by the best suitable method independently of the other subproblem(s). Since in our splitting, the first subproblem is convection-dominated, we use in this chapter the stabilization method based on local projection. In Chapter 5, the SUPG method is considered. The second subproblem in our splitting is a transport problem with pure advection, one suitable choice is the discontinuous Galerkin finite element method for the discretization with respect to internal coordinate. For temporal discretization, we use backward Euler time stepping scheme.

The format of the chapter is as follows: Section 4.1 introduces the model problem under consideration and defines the basic notations. In Section 4.2, the operator splitting technique is applied to decompose the problem into two simpler ones. We shall formulate the backward Euler discretization and derive the weak form of the two subproblems. Further, we derive the unconditional stability of the two-step method. We then discretize the subproblems in space and internal coordinate using local projection stabilization and discontinuous Galerkin methods, respectively, in Section 4.3. We show the unconditional stability of the fully discrete two-step method. Section 4.4 presents the error analysis of the fully discrete two-step scheme. Some implementation issues of the method are discussed in Section 4.5. Finally, we present in Section 4.6 some computational results supporting our theoretical results.

4.1 Model problem

Let Ω_x be a domain in $\mathbb{R}^d$ ($d = 2\,\text{or}\,3$) with boundary $\partial\Omega_x$, $\Omega_\ell = [\ell_{\min}, \ell_{\max}] \subset \mathbb{R}$ and $T > 0$. The state of individual particle in population balance equations consists of the external coordinate x, referring to its position in the physical space, and the internal coordinate ℓ, representing the properties of particles, such as size, temperature, volume etc. A population balance for a solid process such as crystallization with one internal coordinate can be described by the following partial differential equation:
Find $z : (0,T) \times \Omega_\ell \times \Omega_x \to \mathbb{R}$ such that

$$\begin{cases} \dfrac{\partial z}{\partial t} + \dfrac{\partial (Gz)}{\partial \ell} - \varepsilon \Delta_x z + \mathbf{b}(x) \cdot \nabla_x z = f & \text{in } (0,T] \times \Omega_\ell \times \Omega_x, \\ z(0,\cdot) = z_0 & \text{in } \Omega_\ell \times \Omega_x, \\ z|_{\ell_{\min}} = z_{\min} & \text{on } (0,T] \times \Omega_x, \\ z = 0 & \text{on } (0,T] \times \Omega_\ell \times \partial\Omega_x, \end{cases} \tag{4.1}$$

where the diffusion coefficient $\varepsilon > 0$ is a given constant, Δ_x and ∇_x represent the Laplacian and gradient with respect to x, respectively, $\mathbf{b}$ is a given velocity field satisfying $\nabla_x \cdot \mathbf{b} = 0$, and f is a source function. Here $G > 0$ represents the growth rate of the particles that depends on ℓ but is independent of x and t, we also assume that $\partial_\ell G \geq 0$ [69, 70]. Furthermore, let us consider the data of the problem G, $\mathbf{b}$, f, z_0 and $z_{\min}$ to be sufficiently smooth functions.

Here and in the next chapters we denote by $(\cdot,\cdot)$ the L^2-inner product in $L^2(\Omega_\ell \times \Omega_x)$ and by $\|\cdot\|_0$ the corresponding L^2-norm defined by

$$(v,w) = \int_{\Omega_\ell \times \Omega_x} vw \, d\ell \, dx \qquad \text{and} \qquad \|v\|_0^2 = (v,v).$$

Furthermore, to distinguish the inner products and corresponding norms in space and internal coordinate, let us denote by $(\cdot,\cdot)_\ell$ and $\|\cdot\|_{L^2(\Omega_\ell)}$ the L^2-inner product and associated norm in Ω_ℓ, respectively, and by $(\cdot,\cdot)_x$ and $\|\cdot\|_{L^2(\Omega_x)}$ the L^2-inner product and the associated norm in Ω_x. The Bochner spaces defined in Chapter 2 are used for $\Omega_\ell = [\ell_{\min}, \ell_{\max}]$, i.e.,

$$\begin{aligned} C(\Omega_\ell; X) &= \Big\{ v : \Omega_\ell \to X, \quad v \text{ continuous} \Big\}, \\ L^2(\Omega_\ell; X) &= \Big\{ v : \Omega_\ell \to X, \quad \int_{\Omega_\ell} \|v(\ell)\|_X^2 d\ell < \infty \Big\}, \\ H^m(\Omega_\ell; X) &= \Big\{ v \in L^2(\Omega_\ell; X) : \frac{\partial^j v}{\partial \ell^j} \in L^2(\Omega_\ell; X), \quad 1 \leq j \leq m \Big\}, \end{aligned}$$

where the derivatives $\partial^j v / \partial \ell^j$ are understood in the sense of distribution on Ω_ℓ. For spaces X and Y we use the short notation $Y(X) := Y(\Omega_\ell; X)$. The norms in the above defined spaces are given as follows

$$\|v\|_{C(X)} = \sup_{\ell\in\Omega_\ell} \|v(\ell)\|_X, \qquad \|v\|_{L^2(X)} = \left(\int_{\Omega_\ell} \|v(\ell)\|_X^2 d\ell\right)^{1/2},$$
$$\|v\|_{H^m(X)} = \left(\int_{\Omega_\ell} \sum_{j=0}^{m} \left\|\frac{\partial^j v}{\partial \ell^j}\right\|_X^2 d\ell\right)^{1/2}.$$

4.2 Operator splitting method

The numerical method for solving (4.1) in $d+1$ variable is based on an operator splitting with respect to (ℓ, t) and (x, t) in Ω_ℓ and Ω_x direction, respectively. We consider a uniform partition of the time interval $(0, T]$ i.e. $t^n = \tau n$, $n = 1, \ldots, N$, with time step length $\tau = T/N$. Then starting with $u(t^0) = z_0$, two subproblems are sequentially solved on the sub-intervals $(t^n, t^{n+1}]$, $n = 0, 1, \ldots, N-1$:
Given $u(t^n)$ find $\tilde{u} : (t^n, t^{n+1}] \times \Omega_\ell \times \Omega_x \to \mathbb{R}$ such that

$$\begin{cases} \dfrac{\partial \tilde{u}}{\partial t} + L_x \tilde{u} &= f \qquad \text{in } (t^n, t^{n+1}] \times \Omega_\ell \times \Omega_x \\ \tilde{u} &= 0 \qquad \text{on } (t^n, t^{n+1}] \times \Omega_\ell \times \partial\Omega_x \\ \tilde{u}(t^{n+}) &= u(t^n). \end{cases} \tag{4.2}$$

Find $u : (t^n, t^{n+1}] \times \Omega_\ell \times \Omega_x \to \mathbb{R}$ such that

$$\begin{cases} \dfrac{\partial u}{\partial t} + L_\ell u &= 0 \qquad \text{in } (t^n, t^{n+1}] \times \Omega_\ell \times \Omega_x \\ u|_{\ell_{\min}} &= z_{\min} \qquad \text{on } (t^n, t^{n+1}] \times \Omega_x \\ u(t^{n+}) &= \tilde{u}(t^{n+1}), \end{cases} \tag{4.3}$$

where

$$L_\ell z = \frac{\partial (Gz)}{\partial \ell}, \qquad L_x z = -\varepsilon \Delta_x z + \mathbf{b} \cdot \nabla_x z. \tag{4.4}$$

This two-steps operator splitting scheme defines $u(t^n)$, $n = 1, \ldots, N$, as an approximation of $z(t^n)$.

In the framework of PBE, the first subproblem (4.2) is a time-dependent convection-diffusion equation posed on Ω_x parametrized by the variable ℓ in internal coordinate and the second subproblem (4.3) is a one-dimensional transport problem on Ω_ℓ parametrized by the space variable x.

Let us consider the spaces $V = H_0^1(\Omega_x)$ and $W = H^1(\Omega_\ell)$. We introduce the space

$$\mathcal{P} = \left\{ v \in L^2(\Omega_\ell \times \Omega_x) : v \in L^2(\Omega_x; W) \cap L^2(\Omega_\ell; V) \right\}. \tag{4.5}$$

A variational form of (4.2) and (4.3) reads as follows:

First step: Find $\tilde{u} : (t^n, t^{n+1}] \to \mathcal{P}$ with $\tilde{u}(t^{n+}) = u(t^n)$ such that

$$\int_{\Omega_\ell} \big(\tilde{u}_t, v\big)_x + \int_{\Omega_\ell} a(\tilde{u}, v) = \int_{\Omega_\ell} \big(f, v\big)_x \qquad \forall v \in \mathcal{P}, \tag{4.6}$$

where the bilinear form a is defined as

$$a(u, v) = \varepsilon(\nabla_x u, \nabla_x v)_x + (\mathbf{b} \cdot \nabla_x u, v)_x.$$

Second step: Find $u : (t^n, t^{n+1}] \to \mathcal{P}$ with $u(t^{n+}) = \tilde{u}(t^{n+1})$ such that

$$\int_{\Omega_\ell} \big(u_t, v\big)_x + b\big(u, v\big) = \big((Gz)_{\min}, v(\ell_{\min})\big)_x \qquad \forall v \in \mathcal{P}, \tag{4.7}$$

where $w_{\min} = w(\ell_{\min})$ and the bilinear form b is defined as

$$b(u, v) = \int_{\Omega_\ell} \Big(\frac{\partial(Gu)}{\partial\ell}, v\Big)_x + \Big((Gu)(\ell_{\min}), v(\ell_{\min})\Big)_x.$$

Note that we have imposed the boundary condition $(u|_{\ell_{\min}} = z_{\min})$ in ℓ-direction in weak sense.

After discretizing in time by the backward Euler method, the first order accurate implicit scheme is considered as two-step method:

First step: Given $u^n \in \mathcal{P}$, find $\tilde{u}^{n+1} \in \mathcal{P}$ such that

$$\int_{\Omega_\ell} \Big(\frac{\tilde{u}^{n+1} - u^n}{\tau}, v\Big)_x d\ell + \int_{\Omega_\ell} a(\tilde{u}^{n+1}, v) = \int_{\Omega_\ell} (f^{n+1}, v)_x \tag{4.8}$$

for all $v \in \mathcal{P}$.

Second step: Update $\tilde{u}^{n+1}$ from the first step and find the solution $u^{n+1} \in \mathcal{P}$ such that

$$\int_{\Omega_\ell} \Big(\frac{u^{n+1} - \tilde{u}^{n+1}}{\tau}, v\Big)_x + b(u^{n+1}, v) = \Big(G_{\min} z_{\min}^{n+1}, v(\ell_{\min})\Big)_x \tag{4.9}$$

for all $v \in \mathcal{P}$, where $z_{\min}^{n+1} = z_{\min}(t^{n+1}, \cdot)$.
The next paragraph gives the stability of the two-step method (4.8) and (4.9).

Lemma 4.2.1. *Assume that $\tilde{u}^n$, u^n , $n = 1, 2 \ldots, N$, is the solution obtained from the two-step algorithm (4.8) and (4.9). If $\partial_\ell G \geq 0$ and $\tau \leq \frac{1}{4}$, then the following stability estimate holds*

$$\begin{aligned} &\big\|u^N\big\|_0^2 + \tau \sum_{n=0}^{N-1} \int_{\Omega_\ell} \Big\{2\varepsilon \big\|\tilde{u}^{n+1}\big\|^2_{H^1(\Omega_x)} + \partial_\ell G \big\|u^{n+1}\big\|^2_{L^2(\Omega_x)}\Big\} \\ &\quad \leq \exp(3T/2) \left\{ \big\|u^0\big\|_0^2 + \tau \sum_{n=0}^{N-1} \Big(2\big\|f^{n+1}\big\|_0^2 + \big\|G_{\min}^{1/2} z_{\min}^{n+1}\big\|^2_{L^2(\Omega_x)}\Big) \right\}. \end{aligned} \tag{4.10}$$

Proof. Setting $v = \tilde{u}^{n+1}$ in (4.8), yields

$$\int_{\Omega_\ell} (\tilde{u}^{n+1} - u^n, \tilde{u}^{n+1})_x + \tau \int_{\Omega_\ell} a(\tilde{u}^{n+1}, \tilde{u}^{n+1}) = \tau \int_{\Omega_\ell} (f^{n+1}, \tilde{u}^{n+1})_x.$$

Using the relation $2(a-b)a = a^2 - b^2 + (a-b)^2$, one can write

$$\int_{\Omega_\ell} (\tilde{u}^{n+1} - u^n, \tilde{u}^{n+1})_x = \frac{1}{2}\|\tilde{u}^{n+1}\|_0^2 - \frac{1}{2}\|u^n\|_0^2 + \frac{1}{2}\|\tilde{u}^{n+1} - u^n\|_0^2.$$

Integrating by parts with respect to x the second term in the bilinear form $a(\cdot,\cdot)$, one obtains

$$\int_{\Omega_\ell} a(\tilde{u}^{n+1}, \tilde{u}^{n+1}) = \varepsilon \int_{\Omega_\ell} \|\tilde{u}^{n+1}\|_{H^1(\Omega_x)}^2$$

since $\tilde{u}^{n+1}$ vanishes on the boundary $\partial\Omega_x$ and $\nabla_x \cdot \mathbf{b} = 0$. Hence by using Cauchy-Schwarz inequality for the right-hand side, we have for the first step

$$\|\tilde{u}^{n+1}\|_0^2 - \|u^n\|_0^2 + \|\tilde{u}^{n+1} - u^n\|_0^2 + 2\tau\varepsilon \int_{\Omega_\ell} \|\tilde{u}^{n+1}\|_{H^1(\Omega_x)}^2 \\ \leq \tau\|f^{n+1}\|_0^2 + \tau\|\tilde{u}^{n+1}\|_0^2. \tag{4.11}$$

Substituting $v = u^{n+1}$ in the second step (4.9) gives

$$\int_{\Omega_\ell} (u^{n+1} - \tilde{u}^{n+1}, u^{n+1})_x + \tau b(u^{n+1}, u^{n+1}) = \tau\Big(G_{\min} z_{\min}^{n+1}, u^{n+1}(\ell_{\min})\Big)_x. \tag{4.12}$$

Starting from

$$b(u^{n+1}, u^{n+1}) = \int_{\Omega_\ell} \Big(\frac{\partial(G u^{n+1})}{\partial \ell}, u^{n+1}\Big)_x + \Big(G_{\min} u^{n+1}(\ell_{\min}), u^{n+1}(\ell_{\min})\Big)_x$$

an integration by parts twice with respect to ℓ gives

$$b(u^{n+1}, u^{n+1}) = \frac{1}{2}\int_{\Omega_\ell} \partial_\ell G \big\|u^{n+1}\big\|_{L^2(\Omega_x)}^2 + \frac{1}{2}\big\|G_{\max}^{1/2} u^{n+1}(\ell_{\max})\big\|_{L^2(\Omega_x)}^2 \\ + \frac{1}{2}\big\|G_{\min}^{1/2} u^{n+1}(\ell_{\min})\big\|_{L^2(\Omega_x)}^2.$$

where $G_{\max} = G(\ell_{\max})$. Cauchy-Schwarz inequality gives for the right-hand side in (4.12)

$$\Big(G_{\min} z_{\min}^{n+1}, u^{n+1}(\ell_{\min})\Big)_x \leq \frac{1}{2}\big\|G_{\min}^{1/2} z_{\min}^{n+1}\big\|_{L^2(\Omega_x)}^2 + \frac{1}{2}\big\|G_{\min}^{1/2} u^{n+1}(\ell_{\min})\big\|_{L^2(\Omega_x)}^2.$$

Combining these two results in (4.12) and using the same relation $2(a-b)a = a^2 - b^2 + (a-b)^2$ for first term, we get for second step

$$\big\|u^{n+1}\big\|_0^2 - \big\|\tilde{u}^{n+1}\big\|_0^2 + \big\|u^{n+1} - \tilde{u}^{n+1}\big\|_0^2 + \tau \int_{\Omega_\ell} \partial_\ell G \big\|u^{n+1}\big\|_{L^2(\Omega_x)}^2 \\ \leq \tau\big\|G_{\min}^{1/2} z_{\min}^{n+1}\big\|_{L^2(\Omega_x)}^2. \tag{4.13}$$

Adding (4.11) and (4.13), neglecting some contribution of positive terms, and summing over $n = 0, \dots, N-1$, we obtain

$$\begin{aligned} \|u^N\|_0^2 + \tau \sum_{n=0}^{N-1} \int_{\Omega_\ell} \left\{ 2\varepsilon \|\tilde{u}^{n+1}\|_{H^1(\Omega_x)}^2 + \partial_\ell G \|u^{n+1}\|_{L^2(\Omega_x)}^2 \right\} \\ \le \|u^0\|_0^2 + \tau \sum_{n=0}^{N-1} \left\{ \|f^{n+1}\|_0^2 + \|G_{\min}^{1/2} z_{\min}^{n+1}\|_{L^2(\Omega_x)}^2 \right\} + \tau \sum_{n=0}^{N-1} \|\tilde{u}^{n+1}\|_0^2. \end{aligned}$$

From (4.11) we have

$$\|\tilde{u}^{n+1}\|_0^2 \le \frac{\tau}{1-\tau} \|f^{n+1}\|_0^2 + \frac{1}{1-\tau} \|u^n\|_0^2. \tag{4.14}$$

Using (4.14) in the above inequality, we get

$$\begin{aligned} \|u^N\|_0^2 + \tau \sum_{n=0}^{N-1} \int_{\Omega_\ell} \left\{ 2\varepsilon \|\tilde{u}^{n+1}\|_{H^1(\Omega_x)}^2 + \partial_\ell G \|u^{n+1}\|_{L^2(\Omega_x)}^2 \right\} \\ \le \|u^0\|_0^2 + \tau \sum_{n=0}^{N-1} \left\{ \frac{4}{3} \|f^{n+1}\|_0^2 + \|G_{\min}^{1/2} z_{\min}^{n+1}\|_{L^2(\Omega_x)}^2 \right\} + \frac{4\tau}{3} \sum_{n=0}^{N-1} \|u^n\|_0^2, \end{aligned}$$

where we have used $1/(1-\tau) \le 4/3$ for $\tau \le 1/4$. We conclude the statement by applying Gronwall's lemma. This completes the proof. $\square$

The critical issue of the operator splitting method is the overall accuracy of the two-step method. Using Taylor series expansions first order accuracy of the two-step method (4.2) and (4.3) can be shown. A detail error analysis for the first order Lie operator splitting of the sum of two elliptic operators can be found in [32, 33]. Unfortunately, we can't use these results due to the hyperbolic nature of the operator L_ℓ.

4.3 Fully discrete method

In view of different properties of operator L_ℓ and L_x, the operator splitting technique allows us to use different types of discretization methods to solve the problems in Ω_ℓ and Ω_x. Since the first subproblem (4.7) is convection-dominated, we use the local projection method to stabilize the space discretization. While the second subproblem (4.9) is a transport problem with pure advection, one suitable choice is the discontinuous Galerkin method for the discretization with respect to the internal coordinate.

4.3.1 LPS in space

In this subsection, we discretize the subproblem in space. For this, let us denote by $\{\mathcal{T}_h\}$ a family of shape regular decompositions of Ω_x into d-simplices, quadrilateral or hexahedra.

The diameter of a cell $K \in \mathcal{T}_h$ is denoted by h_K and h describes the maximum diameter of cells K. Let $V_h \subset V$ denote the standard finite element space of continuous, piecewise polynomials of degree r. The Galerkin discretization of the problem (4.7) is generally unstable due to dominating advection when the diffusion coefficient is very small $\varepsilon \ll 1$. We handle this difficulty by adding a stabilizing term based on local projection. Details of the local projection stabilization method are given in Chapter 2.

The stabilized bilinear form is then defined as

$$a_h(u,v) = a(u,v) + S_h(u,v). \tag{4.15}$$

where

$$S_h(u,v) = \sum_{K\in\mathcal{T}_h} \mu_K \Big(\kappa_h(\nabla_x u), \kappa_h(\nabla_x v)\Big)_K$$

and μ_K, for all $K \in \mathcal{T}_h$, denote user defined parameters. The bilinear form a_h is coercive on V_h with respect to the mesh dependent norm

$$|||v||| := \Big(\varepsilon|v|^2_{H^1(\Omega_x)} + \sum_{K\in\mathcal{T}_h} \mu_K \|\kappa_h(\nabla_x v)\|^2_{L^2(K)}\Big)^{1/2}, \tag{4.16}$$

that is $a_h(v_h, v_h) \geq |||v_h|||^2$ for all $v_h \in V_h$.

4.3.2 Discontinuous Galerkin method in internal coordinate

To discretize (4.7) and (4.9) in internal coordinate ℓ, we apply a discontinuous Galerkin method. Let $M > 0$ be a given positive integer and $\ell_{\min} = \ell_0 < \ell_1 < \cdots < \ell_M = \ell_{\max}$ is a partition of Ω_ℓ. Here and in the next chapters we denote by $I_i = (\ell_{i-1}, \ell_i]$, $k_i = \ell_i - \ell_{i-1}$, and $k = \max_i k_i$ with respect to the internal coordinate. Also as in (2.11), we denote by S_k^q the function space of discontinuous piecewise polynomials of degree $q \geq 1$ and is defined as

$$S_k^q = \Big\{ v : \Omega_\ell \to \mathbb{R} : v|_{I_i}(\ell) = \sum_{j=0}^{q} v_j \ell^j \quad \text{with} \quad v_j \in \mathbb{R},\ j = 0,\ldots,q \Big\}.$$

Then we give the fully discrete space $S_{h,k}^{r,q}$ as follows

$$\begin{aligned} S_{h,k}^{r,q} &= V_h \otimes S_k^q \\ &= \Big\{ v : \Omega_\ell \times \Omega_x \to \mathbb{R} : v|_{I_i}(\ell) = \sum_{j=0}^{q} v_j \ell^j \quad \text{with} \quad v_j \in V_h,\ j = 0,\ldots,q \Big\}. \end{aligned} \tag{4.17}$$

The functions in these spaces are allowed to be discontinuous at the nodes ℓ_i, $i = 1,\ldots,M-1$. The jumps across the nodes are defined by $[\phi]_i = \phi(\ell_i^+) - \phi(\ell_i^-)$, where

$$\varphi_m^\pm = \varphi(\ell_m^\pm) = \lim_{\ell\to\ell_m\pm 0} \varphi(\ell).$$

In the next paragraph, we define the fully discrete scheme based on the two-step method.

First step: For given $u^n_{h,k} \in S^{r,q}_{h,k}$, find $\tilde{u}^{n+1}_{h,k} \in S^{r,q}_{h,k}$ such that

$$\int_{\Omega_\ell} \Big(\frac{\tilde{u}^{n+1}_{h,k} - u^n_{h,k}}{\tau}, X \Big)_x + \int_{\Omega_\ell} a_h(\tilde{u}^{n+1}_{h,k}, X) = \int_{\Omega_\ell} (f^{n+1}, X)_x \tag{4.18}$$

for all $X \in S^{r,q}_{h,k}$ where $u^0_{h,k}$ is a suitable approximation of z_0 in $S^{r,q}_{h,k}$.

Second step: Update the solution $\tilde{u}^{n+1}_{h,k}$ from (4.18) and find $u^{n+1}_{h,k} \in S^{r,q}_{h,k}$ such that

$$\int_{\Omega_\ell} \Big(\frac{u^{n+1}_{h,k} - \tilde{u}^{n+1}_{h,k}}{\tau}, X \Big)_x + B(u^{n+1}_{h,k}, X) = \Big(G_{\min} z^{n+1}_{\mathrm{min,h}}, X(\ell_0^+) \Big)_x \tag{4.19}$$

for all $X \in S^{r,q}_{h,k}$, where $z^{n+1}_{\mathrm{min,h}} \in S^{r,q}_{h,k}$ is an approximation of $z^{n+1}_{\min}$ and the bilinear form B is defined as

$$B(u,v) = \sum_{i=1}^{M} \int_{I_i} \Big(\frac{\partial (Gu)}{\partial \ell}, v \Big)_x + \sum_{i=1}^{M-1} \Big(\big[(Gu)\big]_i, v(\ell_i^+) \Big)_x + \Big(G_{\min} u(\ell_0^+), v(\ell_0^+) \Big)_x. \tag{4.20}$$

Integrating by parts with respect to ℓ

$$\int_{I_i} \Big(\frac{\partial (Gu)}{\partial \ell}, v \Big)_x = \Big((Gu)(\ell_i^-), v(\ell_i^-) \Big)_x - \Big((Gu)(\ell_{i-1}^+), v(\ell_{i-1}^+) \Big)_x - \int_{I_i} \Big(Gu, \frac{\partial v}{\partial \ell} \Big)_x$$

leads to the representation

$$B(u,v) = -\sum_{i=1}^{M} \int_{I_i} \Big(Gu, \frac{\partial v}{\partial \ell} \Big)_x - \sum_{i=1}^{M-1} \Big(u(\ell_i^-), \big[(Gv)\big]_i \Big)_x + \Big(G_{\max} u(\ell_M^-), v(\ell_M^-) \Big)_x. \tag{4.21}$$

We introduce the mesh dependent norm on $S^{r,q}_{h,k}$

$$\begin{aligned} \|v\|^2_{\mathrm{dG}} = \sum_{i=1}^{M} \int_{I_i} \partial_\ell G \|v\|^2_{L^2(\Omega_x)} + \big\| G^{1/2}_{\min} v(\ell_0^+) \big\|^2_{L^2(\Omega_x)} + \sum_{i=1}^{M-1} \big\| \big[(G^{1/2} v)\big]_i \big\|^2_{L^2(\Omega_x)} \\ + \big\| G^{1/2}_{\max} v(\ell_M^-) \big\|^2_{L^2(\Omega_x)}. \end{aligned} \tag{4.22}$$

Lemma 4.3.1. *The bilinear form B is coercive with respect to the mesh dependent norm $\|\cdot\|_{\mathrm{dG}}$, i.e.,*

$$B(v_h, v_h) \ge \frac{1}{2} \|v_h\|^2_{\mathrm{dG}} \qquad \forall v_h \in S^{r,q}_{h,k}. \tag{4.23}$$

Proof. Setting $u = v$ in (4.20) and in (4.21) to get

$$B(v_h, v_h) = \sum_{i=1}^{M} \int_{I_i} \Big(\frac{\partial (G v_h)}{\partial \ell}, v_h \Big)_x + \sum_{i=1}^{M-1} \big(\big[(G v_h)\big]_i, v_h(\ell_i^+) \big)_x + \big(G_{\min} v_h(\ell_0^+), v_h(\ell_0^+) \big)_x$$

$$B(v_h, v_h) = \sum_{i=1}^{M} \int_{I_i} -\Big(G v_h, \frac{\partial v_h}{\partial \ell} \Big)_x - \sum_{i=1}^{M-1} \big(v_h(\ell_i^-), \big[(G v_h)\big]_i \big)_x + \big(G_{\max} v_h(\ell_M^-), v_h(\ell_M^-) \big)_x.$$

Adding them together and dividing by two, we get the statement of lemma

$$\begin{aligned} B(v_h, v_h) &= \frac{1}{2}\sum_{i=1}^{M}\int_{I_i}\frac{\partial G}{\partial \ell}(v_h, v_h)_x + \frac{1}{2}\sum_{i=1}^{M-1}\left([G^{1/2}v_h]_i, [G^{1/2}v_h]_i\right)_x \\ &\quad + \frac{1}{2}\Big(G_{\min}^{1/2}v_h(\ell_0^+), v_h(\ell_0^+)\Big)_x + \frac{1}{2}\Big(G_{\max}^{1/2}v_h(\ell_M^-), v_h(\ell_M^-)\Big)_x \\ &\geq \frac{1}{2}\|v_h\|_{\mathrm{dG}}^2. \end{aligned}$$

□

The next Lemma gives the stability result of the fully discrete method (4.18) and (4.19).

Lemma 4.3.2. *Let $\partial_\ell G \geq 0$ and $\tau \leq 1/2$, then the solution $\tilde{u}_{h,k}^n$ and $u_{h,k}^n$, $n = 1, 2, \dots, N$, of (4.18) and (4.19),respectively, satisfies*

$$\begin{aligned} &\left\|u_{h,k}^N\right\|_0^2 + 2\tau\sum_{n=0}^{N-1}\int_{\Omega_\ell}\left|\left|\left|\tilde{u}_{h,k}^{n+1}\right|\right|\right|^2 + \tau\sum_{n=0}^{N-1}\left\|u_{h,k}^{n+1}\right\|_{\mathrm{dG}}^2 \\ &\quad\leq \exp(3T/2)\left\{\left\|u_{h,k}^0\right\|_0^2 + \tau\sum_{n=0}^{N-1}\left(\frac{4}{3}\left\|f^{n+1}\right\|_0^2 + 2\left\|(G_{\min}^{1/2}z_{\mathrm{min,h}}^{n+1})\right\|_{L^2(\Omega_x)}^2\right)\right\}. \end{aligned} \tag{4.24}$$

Proof. The proof of the stability estimate for the fully discrete method are obtained following the same steps as in the proof of Lemma 4.2.1. That is, we start by setting $X = \tilde{u}_{h,k}^{n+1}$ in (4.18) and using the equality $2(a-b)a = a^2 - b^2 + (a-b)^2$, to get for the first term

$$\int_{\Omega_\ell}(\tilde{u}_{h,k}^{n+1} - u_{h,k}^n, \tilde{u}_{h,k}^{n+1})_x = \frac{1}{2}\|\tilde{u}_{h,k}^{n+1}\|_0^2 - \frac{1}{2}\|u_{h,k}^n\|_0^2 + \frac{1}{2}\|\tilde{u}_{h,k}^{n+1} - u_{h,k}^n\|_0^2.$$

From the coercivity of the bilinear form a_h we obtain for the second term in (4.18)

$$\int_{\Omega_\ell} a_h(\tilde{u}_{h,k}^{n+1}, \tilde{u}_{h,k}^{n+1}) \geq \int_{\Omega_\ell}|||\tilde{u}_{h,k}^{n+1}|||^2.$$

Combining these estimates and using Cauchy-Schwarz for the right-hand side to get

$$\begin{aligned} \|\tilde{u}_{h,k}^{n+1}\|_0^2 - \|u_{h,k}^n\|_0^2 + \|\tilde{u}_{h,k}^{n+1} - u_{h,k}^n\|_0^2 + 2\int_{\Omega_\ell}|||\tilde{u}_{h,k}^{n+1}|||^2 \\ \leq \|f^{n+1}\|_0^2 + \|\tilde{u}_{h,k}^{n+1}\|_0^2. \end{aligned} \tag{4.25}$$

Similarly setting $X = u_{h,k}^{n+1}$ in (4.19), we get for first term

$$\int_{\Omega_\ell}(u_{h,k}^{n+1} - \tilde{u}_{h,k}^{n+1}, u_{h,k}^{n+1})_x = \frac{1}{2}\|u_{h,k}^{n+1}\|_0^2 - \frac{1}{2}\|\tilde{u}_{h,k}^{n+1}\|_0^2 + \frac{1}{2}\|u_{h,k}^{n+1} - \tilde{u}_{h,k}^{n+1}\|_0^2.$$

For the second term in (4.19), the coercivity property (4.23) of the bilinear form B gives

$$B(u_{h,k}^{n+1}, u_{h,k}^{n+1}) \geq \frac{1}{2}\|u_{h,k}^{n+1}\|_{\mathrm{dG}}^2.$$

Cauchy-Schwarz inequality and Young's inequality gives for the right-hand side

$$\begin{aligned}\Big(G_{\min} z_{\mathrm{min,h}}^{n+1}, u_{h,k}^{n+1}(\ell_0^+)\Big)_x &\leq \big\|G_{\min}^{1/2} z_{\mathrm{min},h}^{n+1}\big\|_{L^2(\Omega_x)}^2 + \frac{1}{4}\big\|G_{\min}^{1/2} u_{h,k}^{n+1}(\ell_0^+)\big\|_{L^2(\Omega_x)}^2 \\ &\leq \big\|G_{\min}^{1/2} z_{\mathrm{min},h}^{n+1}\big\|_{L^2(\Omega_x)}^2 + \frac{1}{4}\big\|u_{h,k}^{n+1}\big\|_{\mathrm{dG}}^2\end{aligned}$$

Combining these results and contributing the dG-norm with the left hand side, we get

$$\big\|u_{h,k}^{n+1}\big\|_0^2 - \big\|\tilde{u}_{h,k}^{n+1}\big\|_0^2 + \big\|u_{h,k}^{n+1} - \tilde{u}_{h,k}^{n+1}\big\|_0^2 + \frac{\tau}{2}\big\|u_{h,k}^{n+1}\big\|_{\mathrm{dG}}^2 \leq 2\tau\big\|G_{\min}^{1/2} u_{h,k}^{n+1}(\ell_0^-)\big\|_{L^2(\Omega_x)}^2. \quad (4.26)$$

Adding (4.25) and (4.26) and summing over $n = 0, \ldots, N-1$, we arrives at

$$\begin{aligned}&\big\|u_{h,k}^N\big\|_0^2 + 2\tau\sum_{n=0}^{N-1}\int_{\Omega_\ell}\big|\big|\big|\tilde{u}_{h,k}^{n+1}\big|\big|\big|^2 + \frac{\tau}{2}\sum_{n=0}^{N-1}\big\|u_{h,k}^{n+1}\big\|_{\mathrm{dG}}^2 \\ &\qquad \leq \big\|u_{h,k}^0\big\|_0^2 + \tau\sum_{n=0}^{N-1}\Big\{2\big\|G_{\min}^{1/2} u_{h,k}^{n+1}(\ell_0^-)\big\|_{L^2(\Omega_x)}^2 + \big\|f^{n+1}\big\|_0^2 + \big\|\tilde{u}_{h,k}^{n+1}\big\|_0^2\Big\}.\end{aligned}$$

We conclude the statement of the lemma using (4.14) and Gronwall's Lemma 2.3.4 in a same fashion as in Lemma 4.2.1. □

The operator splitting based on an equivalent one-step method is studied in [30], wherein SUPG method in internal coordinates is combined with standard Galerkin finite element method in space. This method is conditionally stable because of the time derivative, which needs to add into the stabilization terms in order to ensure the consistency of the method. On the other hand, the dG method includes a natural upwinding that is equivalent to some stabilization and therefore need no extra stabilization. The two-step method presented here is unconditionally stable.

4.4 Error estimates

In this section, we derive the error estimates of the fully discrete two-step scheme (4.18) and (4.19). First we define a special interpolant $\Pi_k w(t, \cdot, x) \in S_k^q$ of a function $w(t, \ell, x)$ by

$$\Pi_k w(\ell_i^-) = w(\ell_i^-), \qquad i = 1, \ldots, M-1, \quad (4.27)$$

$$\int_{I_i}(\Pi_k w - w)\ell^s \, d\ell = 0, \qquad s \leq q-1, \, i \geq 1, \quad (4.28)$$

i.e., $\Pi_k w$ interpolates at the nodal points and the interpolation error is orthogonal to the space of polynomials of degree $q-1$ on I_i. For this type of interpolant we have the following error estimates

$$\sup_{0\le\ell\le\ell_M} |\Pi_k w(\ell) - w(\ell)|_j \le Ck^{q+1} \sup_{0\le\ell\le\ell_M} |w^{(q+1)}(\ell)|_j, \qquad j=0,1, \tag{4.29}$$

$$\int_{I_i} |\Pi_k w^{(s)}(\ell) - w^{(s)}(\ell)|_j^2 \, d\ell \le Ck^{2(q+1-s)} \int_{I_i} |w^{(q+1)}(\ell)|_j \, d\ell, \qquad s,j=0,1, \tag{4.30}$$

see [73, 85]. We denote by $P_{h,k}$ the projection operator which maps onto the tensor product space $S_{h,k}^{r,q}$ and for sufficiently smooth function w is defined by

$$P_{h,k} w = j_h \Pi_k w = \Pi_k j_h w, \tag{4.31}$$

where j_h is the special interpolant in space satisfying assumption A1 and Π_k is the special interpolant with respect to the internal coordinate. In addition, we have the stability property of interpolant Π_k given by

$$\int_{\Omega_\ell} \left\|\Pi_k u\right\|^2_{H^{r+1}(\Omega_x)} \le C \int_{\Omega_\ell} \|u\|^2_{H^{r+1}(\Omega_x)} \tag{4.32}$$

since Π_k acts in ℓ-direction and the norms are with respect to the space direction.

Let us consider $\eta^n := u(t^n) - P_{h,k}u(t^n)$ and $\xi^n := P_{h,k}u(t^n) - u^n_{h,k}$. We also denote $\tilde{\eta}^n := \tilde{u}(t^n) - P_{h,k}\tilde{u}(t^n)$ and $\tilde{\xi}^n := P_{h,k}\tilde{u}(t^n) - \tilde{u}^n_{h,k}$, then the error $u(t^n) - u^n_{h,k}$ can be decomposed as follows

$$e^n = u(t^n) - u^n_{h,k} = \eta^n + \xi^n,$$

where $\tilde{u}^n_{h,k}$ and $u^n_{h,k}$ are the solution for fully discrete scheme (4.18) and (4.19) and $\tilde{u}(t^n)$ and $u(t^n)$ is the solution of (4.2) and (4.3), respectively. Furthermore, to obtain the separate estimates in space and internal coordinate we use the following decomposition of errors

$$P_{h,k}w - w = \big(P_{h,k}w - \Pi_k w\big) + \big(\Pi_k w - w\big) = \vartheta + \varphi. \tag{4.33}$$

Assumption A4: Let u, u_t, u_{tt}, $\tilde{u}$, $\tilde{u}_t$, $\tilde{u}_{tt}$, $z_{\min}$ and z_0 satisfy the following regularity conditions

$$\begin{aligned}
&u,\, \tilde{u} \in H^1\big(L^2(H^{r+1})\big) \cap H^1\big(H^{q+1}(H^1)\big), && u_t,\, \tilde{u}_t \in L^2\big(L^2(H^{r+1})\big) \cap L^2\big(H^{q+1}(L^2)\big),\\
&u_{tt},\, \tilde{u}_{tt} \in L^2\big(L^2(L^2)\big), && z_0 \in L^2\big(\Omega_\ell; H^{r+1}(\Omega_x)\big) \cap H^{q+1}\big(\Omega_\ell; L^2(\Omega_x)\big),\\
&z_{\min} \in H^1\big(0,T; H^{r+1}(\Omega_x)\big).
\end{aligned}$$

Lemma 4.4.1. *Let the assumptions A1-A4 be fulfilled. Then for all $t \in (0,T]$, we have the following estimates for the interpolation error*

$$\|P_{h,k}u(t) - \Pi_k u(t)\|^2_{\mathrm{dG}} \le C\, h^{2r+2} \left\{ \|u(t)\|^2_{L^2(H^{r+1})} + \|u(t)\|^2_{C(H^{r+1})} \right\}, \tag{4.34}$$

$$\|\Pi_k u(t) - u(t)\|^2_{\mathrm{dG}} \le C\, k^{q+1/2} \|u(t)\|^2_{H^{q+1}(L^2)}. \tag{4.35}$$

Proof. For simplicity we skip the dependency t within the proof. From (4.22) we get

$$\frac{1}{2}\|\vartheta\|_{\mathrm{dG}}^2 \le \sum_{i=1}^{M}\int_{I_i}\partial_\ell G\|\vartheta\|^2_{L^2(\Omega_x)} + \left\|G_{\min}^{1/2}\vartheta(\ell_0^+)\right\|^2_{L^2(\Omega_x)} + \left\|G_{\max}^{1/2}\vartheta(\ell_M^-)\right\|^2_{L^2(\Omega_x)}.$$

The jump terms $[j_h u - u]_i$, $i = 1, \dots, M-1$, vanishes due to the continuity of interpolation $j_h u$ in internal coordinate. The interpolation estimates (2.6) and condition (4.32) gives

$$\begin{aligned}\|\vartheta\|_{\mathrm{dG}}^2 &\le Ch^{2r+2}\left(\sum_{i=1}^{M-1}\int_{I_i}\|\Pi_k u\|^2_{H^{r+1}(\Omega_x)} + \|\Pi_k u(\ell_0^+)\|^2_{H^{r+1}(\Omega_x)} + \|\Pi_k u(\ell_M^-)\|^2_{L^2(\Omega_x)}\right)\\ &\le C\,h^{2r+2}\left\{\|u\|^2_{L^2(H^{r+1})} + \|u\|^2_{C(H^{r+1})}\right\}.\end{aligned}$$

Using (4.21), $\Pi_k u(\ell_i^-) = u(\ell_i^-)$, $i = 1\dots, M$, (4.29) and (4.30), we get

$$\begin{aligned}\|\varphi\|_{\mathrm{dG}}^2 \le B(\varphi,\varphi) &= \sum_{i=1}^{M}\int_{I_i} -\Big(G\varphi, \frac{\partial\varphi}{\partial\ell}\Big)_x\\ &\le \sum_{i=1}^{M}\int_{I_i}\|G\varphi\|_{L^2(\Omega_x)}\|\partial_\ell\varphi\|_{L^2(\Omega_x)}\\ &\le C\,k^{2q+1}\|u\|^2_{H^{q+1}(L^2)}\end{aligned}$$

which completes the proof of the lemma. □

Lemma 4.4.2. *Let the assumptions A1-A4 be fulfilled and $\tau_K \sim h_K$. Then for all $t \in (0, T]$, the following estimates hold*

$$\int_{\Omega_\ell} a_h\big((P_{h,k}u - \Pi_k u)(t), \xi(t)\big) \le C\,(\varepsilon^{1/2} + h^{1/2})h^r\|u(t)\|_{L^2(H^{r+1})}\left(\int_{\Omega_\ell}|||\xi(t)|||^2\right)^{1/2} + C\,h^{r+1}\|u(t)\|_{L^2(H^{r+1})}\,\|\xi(t)\|_0, \tag{4.36}$$

$$\int_{\Omega_\ell} a_h\big((\Pi_k u - u)(t), \xi(t)\big) \le C\,(\varepsilon^{1/2} + h^{1/2})\,k^{q+1}\|u(t)\|_{H^{q+1}(H^1)}\left(\int_{\Omega_\ell}|||\xi(t)|||^2\right)^{1/2} + C\,k^{q+1}\|u(t)\|_{H^{q+1}(H^1)}\|\xi(t)\|_0, \tag{4.37}$$

$$B\big((P_{h,k}u - \Pi_k u)(t), \xi(t)\big) \le C\,h^{r+1}\Big\{\|u(t)\|_{H^1(H^{r+1})}\|\xi(t)\|_0 + \|u(t)\|_{C(H^{r+1})}\|\xi(t)\|_{\mathrm{dG}}\Big\}, \tag{4.38}$$

$$B\big((\Pi_k u - u)(t), \xi(t)\big) \le C\,k^{q+1}\|u(t)\|_{H^{q+1}(L^2)}\|\xi(t)\|_0. \tag{4.39}$$

Proof. For simplicity of the presentation we again skip the dependency of the time within the proof. From the definition of the bilinear form a_h, we have for $\vartheta = P_{h,k}u - \Pi_k u$

$$\begin{aligned}\int_{\Omega_\ell} a_h\big(\vartheta, \xi\big) &= \varepsilon\int_{\Omega_\ell}\big(\nabla_x\vartheta, \nabla_x\xi\big)_x + \int_{\Omega_\ell}\big(\mathbf{b}\cdot\nabla_x\vartheta, \xi\big)_x + \int_{\Omega_\ell} S_h\big(\vartheta,\xi\big)\\ &= I_1 + I_2 + I_3.\end{aligned} \tag{4.40}$$

We start by estimating the first term on the right-hand side. Using Cauchy-Schwarz inequality, (2.6) and (4.32), it follows that

$$\begin{aligned}|I_1| &\leq \varepsilon \int_{\Omega_\ell} ||\vartheta||_{H^1(\Omega_x)} ||\xi||_{H^1(\Omega_x)} \leq C\,\varepsilon^{1/2} h^r \int_{\Omega_\ell} \|\Pi_k u\|_{H^{r+1}} \,|||\xi||| \\ &\leq C \varepsilon^{1/2} h^r \left(\int_{\Omega_\ell} \|u\|^2_{H^{r+1}} \right)^{1/2} \left(\int_{\Omega_\ell} |||\xi|||^2 \right)^{1/2} \\ &\leq C\,\varepsilon^{1/2} h^r \|u\|_{L^2(H^{r+1})} \left(\int_{\Omega_\ell} |||\xi|||^2 \right)^{1/2}.\end{aligned}$$

Integrating I_2 in (4.40) by parts with respect to the space variable x, using the orthogonality property (2.7) and Cauchy-Schwarz inequality, we get

$$\begin{aligned}|I_2| &= \Big| \int_{\Omega_\ell} \big(\mathbf{b} \cdot \nabla_x \vartheta, \xi\big)_x \Big| = \Big| \int_{\Omega_\ell} \big(\vartheta, \mathbf{b} \cdot \nabla_x \xi\big)_x \Big| \\ &\leq \Big| \int_{\Omega_\ell} \big(\vartheta, \kappa_h(\mathbf{b} \cdot \nabla_x \xi)\big)_x \Big| \\ &\leq \int_{\Omega_\ell} \sum_{K \in \mathcal{T}_h} \|\vartheta\|_{L^2(K)} \|\kappa_h(\mathbf{b} \cdot \nabla_x \xi)\|_{L^2(K)}.\end{aligned}$$

Let $\overline{\mathbf{b}}$ be the L^2-projection of $\mathbf{b}$ in the space of piecewise constant functions with respect to $\mathcal{T}_h$. Using the L^2-stability of the fluctuation operator κ_h, inverse inequality and $\kappa_h(\overline{\mathbf{b}} \cdot \nabla_x)\xi = \overline{\mathbf{b}} \cdot \kappa_h(\nabla_x \xi)$, we get in a same fashion as in [67] the following estimate

$$\big\|\kappa_h(\mathbf{b} \cdot \nabla_x)\xi\big\|_{L^2(K)} \leq C|\mathbf{b}|_{1,\infty,K} \|\xi\|_{L^2(K)} + \|\mathbf{b}\|_{0,\infty,K} \big\|\kappa_h(\nabla_x \xi)\big\|_{L^2(K)}. \tag{4.41}$$

Thus inserting this in the previous estimate, using (2.6), $\mu_K \sim h_K$, and (4.32) to get

$$\begin{aligned}|I_2| \leq\; & C \int_{\Omega_\ell} \sum_{K \in \mathcal{T}_h} |\mathbf{b}|_{1,\infty,K} \|\vartheta\|_{L^2(K)} \|\xi\|_{L^2(K)} \\ &+ C \int_{\Omega_\ell} \sum_{K \in \mathcal{T}_h} |\mathbf{b}|_{0,\infty,K} \|\vartheta\|_{L^2(K)} \big\|\kappa_h(\nabla_x \xi)\big\|_{L^2(K)} \\ \leq\; & C \int_{\Omega_\ell} \left(\sum_{K \in \mathcal{T}_h} h_K^{2r+2} \big\|\Pi_k u\big\|^2_{H^{r+1}(K)} \right)^{1/2} \left(\sum_{K \in \mathcal{T}_h} \|\xi\|^2_{L^2(K)} \right)^{1/2} \\ &+ C \int_{\Omega_\ell} \left(\sum_{K \in \mathcal{T}_h} \mu_K^{-1} h_K^{2r+2} \big\|\Pi_k u\big\|^2_{H^{r+1}(K)} \right)^{1/2} \left(\sum_{K \in \mathcal{T}_h} \mu_K \|\kappa_h(\nabla_x \xi)\|^2_{L^2(K)} \right)^{1/2} \\ \leq\; & C\, h^{r+1/2} \Big\{ h^{1/2} \|\xi\|_0 + \Big(\int_{\Omega_\ell} |||\xi|||^2 \Big)^{1/2} \Big\} \|u\|_{L^2(H^{r+1})}.\end{aligned}$$

For I_3, the Cauchy-Schwarz inequality and interpolation error estimates give

$$
\begin{aligned}
|I_3| = \Big| \int_{\Omega_\ell} S_h\big(\vartheta, \xi\big) \Big| &\le \int_{\Omega_\ell} S_h\big(\vartheta, \vartheta\big)^{1/2} S_h\big(\xi, \xi\big)^{1/2} \\
&\le C h^{r+1/2} \int_{\Omega_\ell} \|u\|_{H^{r+1}(\Omega_x)} \, |||\xi||| \\
&\le C h^{r+1/2} \left(\int_{\Omega_\ell} \|u\|^2_{H^{r+1}(\Omega_x)} \right)^{1/2} \left(\int_{\Omega_\ell} |||\xi||| \right)^{1/2} \\
&\le C h^{r+1/2} \|u\|_{L^2(H^{r+1})} \left(\int_{\Omega_\ell} |||\xi|||^2 \right)^{1/2}.
\end{aligned}
$$

Combining I_1, I_2 and I_3, we get the desired estimate

$$
\begin{aligned}
\int_{\Omega_\ell} a_h\big(\vartheta, \xi\big) \le C\,(\varepsilon^{1/2} + h^{1/2})\, \|u\|_{L^2(H^{r+1})} \left(\int_{\Omega_\ell} |||\xi|||^2 \right)^{1/2} \\
+ C\, h^{r+1}\, \|u\|_{L^2(H^{r+1})}\, \|\xi\|_0.
\end{aligned}
$$

Next, we find the estimates in internal coordinate. From the definition of the bilinear form a_h, we have for $\varphi = \Pi_k u - u$

$$
\begin{aligned}
\int_{\Omega_\ell} a_h(\varphi, \xi) &= \varepsilon \int_{\Omega_\ell} \big(\nabla_x \varphi, \nabla_x \xi\big)_x + \int_{\Omega_\ell} \big(\mathbf{b} \cdot \nabla_x \varphi, \xi\big)_x + \int_{\Omega_\ell} S_h\big(\varphi, \xi\big) \\
&= I_4 + I_5 + I_6.
\end{aligned}
$$

Then by using the Cauchy-Schwarz inequality, the stability property of the fluctuation operator κ_h, the approximation properties (4.29) of interpolant Π_k and the parameter choice $\mu_K \sim h_K$, we get for I_4, I_5, and I_6 the following estimates

$$
\begin{aligned}
|I_4| &\le \varepsilon \int_{\Omega_\ell} \|\Pi_k u - u\|_{H^1(\Omega_x)} \|\xi\|_{H^1(\Omega_x)} \\
&\le \varepsilon^{1/2} \int_{\Omega_\ell} \|\Pi_k u - u\|_{H^1(\Omega_x)} |||\xi||| \\
&\le \varepsilon^{1/2} \Big(\int_{\Omega_\ell} \|\Pi_k u - u\|^2_{H^1(\Omega_x)} \Big)^{1/2} \Big(\int_{\Omega_\ell} |||\xi|||^2 \Big)^{1/2} \\
&\le C\, \varepsilon^{1/2} k^{q+1} \|u\|_{H^{q+1}(H^1)} \Big(\int_{\Omega_\ell} |||\xi|||^2 \Big)^{1/2}, \\
|I_5| &\le \int_{\Omega_\ell} \|\mathbf{b}\|_{0,\infty} \big\|\nabla_x \vartheta\big\|_{L^2(\Omega_x)} \|\xi\|_{L^2(\Omega_x)} \le C\, k^{q+1} \|u\|_{H^{q+1}(H^1)} \|\xi\|_0
\end{aligned}
$$

and

$$
\begin{aligned}
|I_6| &\le \int_{\Omega_\ell} \sum_{K\in\mathcal{T}_h} \mu_K \big\|\kappa_h(\nabla_x(\Pi_k u - u))\big\|_{L^2(K)} \big\|\kappa_h(\nabla_x \xi)\big\|_{L^2(K)} \\
&\le C\, h^{1/2} \int_{\Omega_\ell} \big\|\nabla_x(\Pi_k u - u)\big\|_{L^2(\Omega_x)} |||\xi||| \\
&\le C\, h^{1/2} k^{q+1} \|u\|_{H^{q+1}(H^1)} \Big(\int_{\Omega_\ell} |||\xi|||^2 \Big)^{1/2}.
\end{aligned}
$$

Hence, combining these estimates we get the second statement of the lemma

$$
\begin{aligned}
\int_{\Omega_\ell} a_h(\varphi,\xi) \le C\,(\varepsilon^{1/2} + h^{1/2})\, k^{q+1} \|u\|_{H^{q+1}(H^1)} \Big(\int_{\Omega_\ell} |||\xi|||^2 \Big)^{1/2} \\
+ C\, k^{q+1} \|u\|_{H^{q+1}(H^1)} \|\xi\|_0.
\end{aligned}
$$

To obtain the last two estimates, we use the two different representations (4.20) and (4.21) of B. Note that the jump terms $[j_h u - u]_i$, $i = 1, \dots, M-1$, vanishes due to the continuity of the interpolant $j_h u$ in ℓ-direction. From (4.20), (2.6), and (4.32), we get for $\vartheta = P_{h,k} u - \Pi_k u$

$$
\begin{aligned}
B(\vartheta,\xi) &= \sum_{i=1}^{M} \int_{I_i} \Big(\frac{\partial (G\vartheta)}{\partial \ell}, \xi \Big)_x + \Big(G_{\min} \vartheta(\ell_0^+), \xi(\ell_0^+) \Big)_x \\
&\le \sum_{i=1}^{M} \int_{I_i} \big\|\partial_\ell (G\vartheta)\big\|_{L^2(\Omega_x)} \|\xi\|_{L^2(\Omega_x)} + \big\|G_{\min}^{1/2} \vartheta(\ell_0^+)\big\|_{L^2(\Omega_x)} \big\|G_{\min}^{1/2} \xi(\ell_0^+)\big\|_{L^2(\Omega_x)} \\
&\le C\, h^{r+1} \Big\{ \|u\|_{H^1(H^{r+1})} \|\xi\|_0 + \|u\|_{C(H^{r+1})} \|\xi\|_{\mathrm{dG}} \Big\}.
\end{aligned}
$$

The interpolation $\Pi_k u$ satisfies $\Pi_k u(\ell_i^-) = u(\ell_i^-)$, $i = 1, \dots, M$. Thus, from (4.21), we get for $\varphi = \Pi_k u - u$

$$
B(\varphi,\xi) = \sum_{i=1}^{M} \int_{I_i} - \Big(G\varphi, \frac{\partial \xi}{\partial \ell} \Big)_x.
$$

Let $\Pi_0 G$ be the L^2-projection of G in a space of piecewise constant functions in ℓ-direction. Using the orthogonality (4.28) of the interpolant Π_k, we get

$$
\begin{aligned}
B(\varphi,\xi) &= \sum_{i=1}^{M} \int_{I_i} \Big(\varphi, (G - \Pi_0 G) \frac{\partial \xi}{\partial \ell} \Big)_x \\
&\le \sum_{i=1}^{M} \int_{I_i} \|\varphi\|_{L^2(\Omega_x)} \big\|(G - \Pi_0 G)\partial_\ell \xi\big\|_{L^2(\Omega_x)}\, dx \\
&\le C\, k^{q+1} \|u\|_{H^{q+1}(L^2)} \|\xi\|_0. \qquad (4.42)
\end{aligned}
$$

Here, we used the Cauchy-Schwarz inequality, the inverse inequality and the interpolation error estimates (4.29). This complete the proof. □

Theorem 4.4.3. *Let $\tilde{u}(t^n)$, $u(t^n)$ and $\tilde{u}^n_{h,k}$, $u^n_{h,k}$, be the solutions of two-step method (4.2), (4.3) and (4.18), (4.19), respectively. Under the assumptions A1-A4 and $\mu_K \sim h_K$ there holds for $\xi^n = P_{h,k}u(t^n) - u^n_{h,k}$ and $\tilde{\xi}^n = P_{h,k}u(t^n) - u^n_{h,k}$*

$$\begin{aligned}\left\|\xi^N\right\|_0^2 + \tau\sum_{n=0}^{N-1}\int_{\Omega_\ell}\left|\left|\left|\tilde{\xi}^{n+1}\right|\right|\right|^2 + \frac{\tau}{2}\sum_{n=0}^{N-1}\left\|\xi^{n+1}\right\|_{\mathrm{dG}}^2 \\ \leq C_u e^{9T/2}\Big[\left\|P_{h,k}z_0 - u^0_{h,k}\right\|_0^2 + \tau^2 + (\varepsilon+h)\,h^{2r} + k^{2q+2}\Big]\end{aligned} \tag{4.43}$$

and for $e^n = u(t^n) - u^n_{h,k}$ and $\tilde{e}^n = \tilde{u}(t^n) - \tilde{u}^n_{h,k}$

$$\begin{aligned}\left\|e^N\right\|_0^2 + \tau\sum_{n=0}^{N-1}\int_{\Omega_\ell}\left|\left|\left|\tilde{e}^{n+1}\right|\right|\right|^2 + \frac{\tau}{2}\sum_{n=0}^{N-1}\left\|e^{n+1}\right\|_{\mathrm{dG}}^2 \\ \leq C_u e^{9T/2}\left[\left\|P_{h,k}z_0 - u^0_{h,k}\right\|_0^2 + \tau^2 + (\varepsilon+h)\,h^{2r} + k^{2q+1}\right]\end{aligned} \tag{4.44}$$

where C_u depends on u, u_t, u_{tt}, $\tilde{u}, \tilde{u}_t$, $\tilde{u}_{tt}$ and $z_{\min}$.

Note that the error to the interpolant $P_{h,k}u$ is superclose with respect to the internal coordinate (order $k+1$ instead of $k+1/2$).

Proof. From the result of the Lemma 4.3.2, we can write for $\xi^n = P_{h,k}u(t^n) - u^n_{h,k}$

$$\frac{1}{2}\left\|\xi^N\right\|_0^2 - \frac{1}{2}\left\|\xi^0\right\|_0^2 + \tau\sum_{n=0}^{N-1}\int_{\Omega_\ell}\left|\left|\left|\tilde{\xi}^{n+1}\right|\right|\right|^2 + \frac{\tau}{2}\sum_{n=0}^{N-1}\left\|\xi^{n+1}\right\|_{\mathrm{dG}}^2 \leq T_1 + T_2 \tag{4.45}$$

where

$$T_1 = \tau\sum_{n=0}^{N-1}\int_{\Omega_\ell}\left\{\Big(\frac{\tilde{\xi}^{n+1}-\xi^n}{\tau}, \tilde{\xi}^{n+1}\Big)_x + a_h(\tilde{\xi}^{n+1}, \tilde{\xi}^{n+1})\right\}, \tag{4.46}$$

$$T_2 = \tau\sum_{n=0}^{N-1}\left\{\int_{\Omega_\ell}\Big(\frac{\xi^{n+1}-\tilde{\xi}^{n+1}}{\tau}, \xi^{n+1}\Big)_x + B(\xi^{n+1}, \xi^{n+1})\right\}. \tag{4.47}$$

We first consider T_1. Using (4.18) to get

$$\begin{aligned}T_1 = \tau\sum_{n=0}^{N-1}\int_{\Omega_\ell}\bigg\{&\Big(\frac{P_{h,k}\tilde{u}(t^{n+1}) - P_{h,k}u(t^n)}{\tau}, \tilde{\xi}^{n+1}\Big)_x + a_h\big(P_{h,k}\tilde{u}(t^{n+1}), \tilde{\xi}^{n+1}\big) \\ &- \Big(\frac{\tilde{u}^{n+1}_{h,k} - u^n_{h,k}}{\tau}, \tilde{\xi}^{n+1}\Big)_x - \int_{\Omega_\ell} a_h(\tilde{u}^{n+1}_{h,k}, \tilde{\xi}^{n+1})\bigg\} \\ = \tau\sum_{n=0}^{N-1}\int_{\Omega_\ell}\bigg\{&\Big(\frac{P_{h,k}\tilde{u}(t^{n+1}) - P_{h,k}u(t^n)}{\tau}, \tilde{\xi}^{n+1}\Big)_x + a_h\big(P_{h,k}\tilde{u}(t^{n+1}), \tilde{\xi}^{n+1}\big) \\ &- \big(f^{n+1}, \tilde{\xi}^{n+1}\big)_x\bigg\}.\end{aligned}$$

Using (4.6) at $t = t^{n+1}$ for the last term on the right-hand side, we obtain

$$
\begin{aligned}
T_1 &= \tau \sum_{n=0}^{N-1} \int_{\Omega_\ell} \Big(\frac{P_{h,k}\tilde{u}(t^{n+1}) - P_{h,k}u(t^n)}{\tau} - \tilde{u}_t(t^{n+1}), \tilde{\xi}^{n+1} \Big)_x \\
&\quad + \tau \sum_{n=0}^{N-1} \int_{\Omega_\ell} a\big(P_{h,k}\tilde{u}(t^{n+1}) - \tilde{u}(t^{n+1}), \tilde{\xi}^{n+1}\big) + \tau \sum_{n=0}^{N-1} \int_{\Omega_\ell} S_h\big(P_{h,k}\tilde{u}(t^{n+1}), \tilde{\xi}^{n+1}\big) \\
&= \tau \sum_{n=0}^{N-1} \int_{\Omega_\ell} \Big(\frac{P_{h,k}\tilde{u}(t^{n+1}) - P_{h,k}u(t^n)}{\tau} - \tilde{u}_t(t^{n+1}), \tilde{\xi}^{n+1} \Big)_x \\
&\quad + \tau \sum_{n=0}^{N-1} \int_{\Omega_\ell} a_h\big(P_{h,k}\tilde{u}(t^{n+1}) - \tilde{u}(t^{n+1}), \tilde{\xi}^{n+1}\big) + \tau \sum_{n=0}^{N-1} \int_{\Omega_\ell} S_h\big(\tilde{u}(t^{n+1}), \tilde{\xi}^{n+1}\big) \\
&= T_{1,1} + T_{1,2} + T_{1,3}.
\end{aligned} \tag{4.48}
$$

We treat the contribution of the terms on the right-hand side of (4.48) separately. For the first term, using Cauchy-Schwarz inequality, the Young's inequality and the initial condition $\tilde{u}(t^n) = u(t^n)$ for first step

$$
\begin{aligned}
|T_{1,1}| &\leq \tau \sum_{n=0}^{N-1} \int_{\Omega_\ell} \left\| \frac{P_{h,k}\tilde{u}(t^{n+1}) - P_{h,k}u(t^n)}{\tau} - \tilde{u}_t(t^{n+1}) \right\|_{L^2(\Omega_x)} \|\tilde{\xi}^{n+1}\|_{L^2(\Omega_x)} \\
&\leq \frac{\tau}{2} \sum_{n=0}^{N-1} \int_{\Omega_\ell} \left\| \frac{P_{h,k}\tilde{u}(t^{n+1}) - P_{h,k}u(t^n)}{\tau} - \tilde{u}_t(t^{n+1}) \right\|^2_{L^2(\Omega_x)} \\
&\quad + \frac{\tau}{2} \sum_{n=0}^{N-1} \int_{\Omega_\ell} \|\tilde{\xi}^{n+1}\|^2_{L^2(\Omega_x)} \\
&\leq \tau \sum_{n=0}^{N-1} \left\| \frac{P_{h,k}\tilde{u}(t^{n+1}) - P_{h,k}\tilde{u}(t^n)}{\tau} - P_{h,k}\tilde{u}_t(t^{n+1}) \right\|^2_0 \\
&\quad + \tau \sum_{n=0}^{N-1} \big\| P_{h,k}\tilde{u}_t(t^{n+1}) - \tilde{u}_t(t^{n+1}) \big\|^2_0 + \frac{\tau}{2} \sum_{n=0}^{N-1} \|\tilde{\xi}^{n+1}\|^2_0.
\end{aligned} \tag{4.49}
$$

For the first term, applying Taylor's theorem with integral remainder term and for second term the approximation properties of interpolant j_h and Π_k and condition (4.32) yields

$$
\begin{aligned}
\left\| \frac{P_{h,k}\tilde{u}(t^{n+1}) - P_{h,k}\tilde{u}(t^n)}{\tau} - P_{h,k}\tilde{u}_t(t^{n+1}) \right\|^2_0 &= \frac{1}{\tau^2} \left\| \int_{t^n}^{t^{n+1}} (t - t^n) P_{h,k}\tilde{u}_{tt} \, dt \right\|^2_0 \\
&\leq \frac{1}{\tau^2} \left(\left(\int_{t^n}^{t^{n+1}} (t - t^n)^2 \, dt \right)^{1/2} \left(\int_{t^n}^{t^{n+1}} \big\| P_{h,k}\tilde{u}_{tt} \big\|^2_0 \, dt \right)^{1/2} \right) \\
&\leq C\,\tau \int_{t^n}^{t^{n+1}} \big\| \tilde{u}_{tt} \big\|^2_0 \, dt,
\end{aligned}
$$

$$\begin{aligned}\big\|P_{h,k}\tilde{u}_t(t^{n+1})-\tilde{u}_t(t^{n+1})\big\|_0^2 &\le \big\|P_{h,k}\tilde{u}_t(t^{n+1})-\Pi_k\tilde{u}_t(t^{n+1})\big\|_0^2+\big\|\Pi_k\tilde{u}_t(t^{n+1})-\tilde{u}_t(t^{n+1})\big\|_0^2\\ &\le C\,h^{2r+2}\big\|\Pi_k u_t(t^{n+1})\big\|^2_{L^2(H^{r+1})}+Ck^{2q+2}\big\|\tilde{u}_t(t^{n+1})\big\|^2_{H^{q+1}(L^2)}\\ &\le Ch^{2r+2}\big\|\tilde{u}_t(t^{n+1})\big\|^2_{L^2(H^{r+1})}+Ck^{2q+2}\big\|\tilde{u}_t(t^{n+1})\big\|^2_{H^{q+1}(L^2)}.\end{aligned}$$

Combining them together in (4.49), we get

$$\begin{aligned}|T_{1,1}| \le{}& \tau^2\sum_{n=0}^{N-1}\int_{t^n}^{t^{n+1}}\big\|\tilde{u}_{tt}\big\|_0^2\,dt+\frac{\tau}{2}\sum_{n=0}^{N-1}\|\tilde{\xi}^{n+1}\|_0^2\\ &+C\tau\sum_{n=0}^{N-1}\Big[h^{2r+2}\big\|\tilde{u}_t(t^{n+1})\big\|^2_{L^2(H^{r+1})}+k^{2q+2}\big\|\tilde{u}_t(t^{n+1})\big\|^2_{H^{q+1}(L^2)}\Big].\end{aligned}\tag{4.50}$$

To find the estimates for $T_{1,2}$ in (4.48), we use the decomposition (4.33) of errors into space and internal coordinate and get

$$T_{1,2}=\tau\sum_{n=0}^{N-1}\Big\{a_h\big(\tilde{\vartheta}^{n+1},\tilde{\xi}^{n+1}\big)+a_h\big(\tilde{\varphi}^{n+1},\tilde{\xi}^{n+1}\big)\Big\}.$$

Then from the results (4.36) and (4.37) of Lemma 4.4.2, we obtain

$$\begin{aligned}|T_{1,2}| \le{}& C\,(\varepsilon+h)\,\tau\sum_{n=0}^{N-1}\Big[h^{2r}\big\|\tilde{u}(t^{n+1})\big\|^2_{L^2(H^{r+1})}+k^{2q+2}\big\|\tilde{u}(t^{n+1})\big\|^2_{H^{q+1}(H^1)}\Big]\\ &+C\,\tau\sum_{n=0}^{N-1}\Big[h^{2r+2}\big\|\tilde{u}(t^{n+1})\big\|^2_{L^2(H^{r+1})}+k^{2q+2}\big\|\tilde{u}(t^{n+1})\big\|^2_{H^{q+1}(H^1)}\Big]\\ &+\frac{\tau}{4}\sum_{n=0}^{N-1}\int_{\Omega_\ell}|||\tilde{\xi}^{n+1}|||^2+\frac{\tau}{2}\sum_{n=0}^{N-1}\|\tilde{\xi}^{n+1}\|_0^2.\end{aligned}\tag{4.51}$$

The estimate for $T_{1,3}$ in (4.48) follows from the approximation properties of the fluctuation operator κ_h and the choice of the stabilizing parameter $\mu_K\sim h_K$. We obtain

$$\begin{aligned}|T_{1,3}| &\le \tau\sum_{n=0}^{N-1}\int_{\Omega_\ell}S_h\big(\tilde{u}(t^{n+1}),\tilde{u}(t^{n+1})\big)+\frac{\tau}{4}\sum_{n=0}^{N-1}\int_{\Omega_\ell}S_h\big(\tilde{\xi}^{n+1},\tilde{\xi}^{n+1}\big)\\ &\le C\,h^{2r+1}\tau\sum_{n=0}^{N-1}\big\|\tilde{u}(t^{n+1})\big\|^2_{L^2(H^{r+1})}+\frac{\tau}{4}\sum_{n=0}^{N-1}\int_{\Omega_\ell}|||\tilde{\xi}^{n+1}|||^2.\end{aligned}\tag{4.52}$$

Inserting (4.50)-(4.52) into (4.48), we obtain

$$
\begin{aligned}
|T_1| \le{}& \tau^2 \sum_{n=0}^{N-1} \int_{t^n}^{t^{n+1}} \left\|\tilde{u}_{tt}\right\|_0^2 dt + \frac{\tau}{2} \sum_{n=0}^{N-1} \int_{\Omega_\ell} \left|\left|\left|\tilde{\xi}^{n+1}\right|\right|\right|^2 + \tau \sum_{n=0}^{N-1} \left\|\tilde{\xi}^{n+1}\right\|_0^2 \\
&+ C\, h^{2r} \tau \sum_{n=0}^{N-1} \left[(\varepsilon + h) \left\|\tilde{u}(t^{n+1})\right\|^2_{L^2(H^{r+2})} + h^2 \left\|\tilde{u}_t(t^{n+1})\right\|^2_{L^2(H^{r+1})} \right] \\
&+ C\, k^{2q+2}\, \tau \sum_{n=0}^{N-1} \left[(\varepsilon + h + 1) \left\|\tilde{u}(t^{n+1})\right\|^2_{H^{q+1}(H^1)} + \left\|\tilde{u}_t(t^{n+1})\right\|^2_{H^{q+1}(L^2)} \right]. \qquad (4.53)
\end{aligned}
$$

Now we estimate the second term T_2 in (4.47). Using (4.19) we get

$$
\begin{aligned}
T_2 ={}& \tau \sum_{n=0}^{N-1} \int_{\Omega_\ell} \left(\frac{P_{h,k}u(t^{n+1}) - P_{h,k}\tilde{u}(t^{n+1})}{\tau}, \xi^{n+1} \right)_x + \tau \sum_{n=0}^{N-1} B\big(P_{h,k}u(t^{n+1}), \xi^{n+1}\big) \\
&- \tau \sum_{n=0}^{N-1} \Big(G_{\min} z^{n+1}_{\min,\mathrm{h}}, \xi^{n+1}(\ell_0^+) \Big)_x .
\end{aligned}
$$

and (4.7) at $t = t^{n+1}$ gives

$$
\begin{aligned}
T_2 ={}& \tau \sum_{n=0}^{N-1} \int_{\Omega_\ell} \left(\frac{P_{h,k}u(t^{n+1}) - P_{h,k}\tilde{u}(t^{n+1})}{\tau} - u_t(t^{n+1}), \xi^{n+1} \right)_x \\
&+ \tau \sum_{n=0}^{N-1} B\big(P_{h,k}u(t^{n+1}) - u(t^{n+1}), \xi^{n+1}\big) \\
&- \tau \sum_{n=0}^{N-1} \Big(G_{\min} z^{n+1}_{\min} - G_{\min} z^{n+1}_{\min,\mathrm{h}}, \xi^{n+1}(\ell_0^+) \Big)_x \\
={}& T_{2,1} + T_{2,2} + T_{2,3}. \qquad (4.54)
\end{aligned}
$$

Following the same steps as in (4.49), we get for $T_{2,1}$

$$
\begin{aligned}
|T_{2,1}| \le{}& \tau \sum_{n=0}^{N-1} \left\| \frac{P_{h,k}u(t^{n+1}) - P_{h,k}\tilde{u}(t^{n+1})}{\tau} - P_{h,k}u_t(t^{n+1}) \right\|_0^2 \\
&+ \tau \sum_{n=0}^{N-1} \left\| P_{h,k}u_t(t^{n+1}) - u_t(t^{n+1}) \right\|_0^2 + \frac{\tau}{2} \sum_{n=0}^{N-1} \left\|\xi^{n+1}\right\|_0^2 \\
\le{}& \tau^2 \sum_{n=0}^{N-1} \int_{t^n}^{t^{n+1}} \left\|u_{tt}\right\|_0^2 dt + \frac{\tau}{2} \sum_{n=0}^{N-1} \left\|\xi^{n+1}\right\|_0^2 \\
&+ C\tau \sum_{n=0}^{N-1} \left[h^{2r+2} \left\|u_t(t^{n+1})\right\|^2_{L^2(H^{r+1})} + k^{2q+2} \left\|u_t(t^{n+1})\right\|^2_{H^{q+1}(L^2)} \right]. \qquad (4.55)
\end{aligned}
$$

Note that in above estimates we have used the initial condition $u(t^n) = \tilde{u}(t^{n+1})$ from (4.3). The bounds for $T_{2,2}$ in (4.54) are obtained by using the error decomposition (4.33) and the estimates (4.38) and (4.39)

$$
\begin{aligned}
|T_{2,2}| &= \left| \tau \sum_{n=0}^{N-1} \Big\{ B\big(\vartheta^{n+1}, \xi^{n+1}\big) + B\big(\varphi^{n+1}, \xi^{n+1}\big) \Big\} \right| \\
&\leq C\, h^{2r+2}\, \tau \sum_{n=0}^{N-1} \Big[\big\| u(t^{n+1}) \big\|^2_{H^1(H^{r+1})} + \big\| u(t^{n+1}) \big\|^2_{C(H^{r+1})} \Big] \\
&\quad + C\, k^{2q+2}\, \tau \sum_{n=0}^{N-1} \big\| u(t^{n+1}) \big\|^2_{H^{q+1}(L^2)} + \frac{\tau}{2} \sum_{n=0}^{N-1} \big\| \xi^{n+1} \big\|^2_0 + \frac{\tau}{8} \sum_{n=0}^{N-1} \big\| \xi^{n+1} \big\|^2_{\mathrm{dG}}. \qquad (4.56)
\end{aligned}
$$

Applying Cauchy-Schwarz inequality and Young's inequality for $T_{2,3}$ leads to

$$
\begin{aligned}
|T_{2,3}| &\leq \tau \sum_{n=0}^{N-1} \big\| G^{1/2}_{\min} z_{\min}(t^{n+1}) - G^{1/2}_{\min} z^{n+1}_{\min,\mathrm{h}} \big\|_{L^2(\Omega_x)} \big\| G^{1/2}_{\min} \xi^{n+1}(\ell_0^+) \big\|_{L^2(\Omega_x)} \\
&\leq C\, h^{2r+2}\, \tau \sum_{n=0}^{N-1} \big\| z_{\min}(t^{n+1}) \big\|^2_{H^{r+1}(\Omega_x)} + \frac{\tau}{8} \sum_{n=0}^{N-1} \big\| \xi^{n+1} \big\|^2_{\mathrm{dG}}. \qquad (4.57)
\end{aligned}
$$

Substituting (4.55)-(4.57) into (4.54) we get for T_2

$$
\begin{aligned}
|T_2| &\leq \tau^2 \sum_{n=0}^{N-1} \int_{t^n}^{t^{n+1}} \big\| u_{tt} \big\|^2_0 \, dt + \tau \sum_{n=0}^{N-1} \|\xi^{n+1}\|^2_0 + \frac{\tau}{4} \sum_{n=0}^{N-1} \big\| \xi^{n+1} \big\|^2_{\mathrm{dG}} \\
&\quad + C\tau h^{2r+2} \sum_{n=0}^{N-1} \Big[\big\| u(t^{n+1}) \big\|^2_{H^1(H^{r+1})} + \big\| z_{\min}(t^{n+1}) \big\|^2_{H^{r+1}(\Omega_x)} + \big\| u_t(t^{n+1}) \big\|^2_{L^2(H^{r+1})} \\
&\qquad\qquad + \big\| u(t^{n+1}) \big\|^2_{C(H^{r+1})} \Big] \\
&\quad + C\tau k^{2q+2} \sum_{n=0}^{N-1} \Big[\big\| u(t^{n+1}) \big\|^2_{H^{q+1}(L^2)} + \big\| u_t(t^{n+1}) \big\|^2_{H^{q+1}(L^2)} \Big]. \qquad (4.58)
\end{aligned}
$$

Inserting (4.53) and (4.58) in (4.45), adding the triple norm and the dG norm contributions in the left-hand side and using (4.14), we get

$$\begin{aligned}
&\frac{1}{2}\|\xi^N\|_0^2 - \frac{1}{2}\|\xi^0\|_0^2 + \tau\sum_{n=0}^{N-1}\int_{\Omega_\ell}|||\tilde{\xi}^{n+1}|||^2 + \frac{\tau}{2}\sum_{n=0}^{N-1}\|\xi^{n+1}\|_{\mathrm{dG}}^2 \\
&\quad\leq \tau^2\sum_{n=0}^{N-1}\int_{t^n}^{t^{n+1}}\|u_{tt}\|_0^2\,dt + \tau\sum_{n=0}^{N-1}\gamma_n\|\xi^n\|_0^2 + 2\tau\sum_{n=0}^{N-1}\|f^{n+1}\|_0^2 \\
&\quad+ C\,h^{2r}\,\tau\sum_{n=0}^{N-1}\Big[(\varepsilon+h)\,\|u(t^{n+1})\|_{H^1(H^{r+1})}^2 + h^2\|u_t(t^{n+1})\|_{L^2(H^{r+1})}^2 \\
&\qquad\qquad + h^2\|z_{\min}(t^{n+1})\|_{H^{r+1}(\Omega_x)}^2 + \|u(t^{n+1})\|_{C(H^{r+1})}^2\Big] \\
&\quad+ C\,k^{2q+2}\,\tau\sum_{n=0}^{N-1}\Big[(\varepsilon+h+1)\|u(t^{n+1})\|_{H^{q+1}(H^1)}^2 + \|u_t(t^{n+1})\|_{H^{q+1}(L^2)}^2\Big]
\end{aligned}$$

where $\gamma_0 = 2$, $\gamma_N = 1$ and $\gamma_n = 3$, $n = 1, \dots, N-1$. We conclude by applying the Gronwall's Lemma in the same fashion as in Lemma 4.2.1. □

Consider the standard Galerkin finite element method in space, $\mu_K = 0$ for all $K \in \mathcal{T}_h$, and set $\varepsilon = 1$. The error estimates (4.44) is then of $\mathcal{O}(\tau^2 + k^{2q+1} + h^{2r})$. We see that, the same rate of convergence can be derived without using any extra stabilization with respect to internal coordinate as compared to the estimates in [30].

4.5 Implementation of numerical method

This section indicates the implementation of the operator splitting method in the context of finite element methods.
Using the bases

$$V_h = \mathrm{span}\{\phi_i\},\ 1 \leq i \leq N_x, \quad S_k^q = \mathrm{span}\{\psi_k\},\ 1 \leq k \leq N_\ell,$$

the tensor product space $S_{h,k}^{r,q}$ is defined as follows

$$S_{h,k}^{r,q} = \left\{ v = \sum_{i=1}^{N_x}\sum_{k=1}^{N_\ell}\alpha_{ik}\phi_i(x)\psi_k(\ell),\ \alpha_{ik}\in\mathbb{R},\ \ 1 \leq i \leq N_x,\ 1 \leq k \leq N_\ell \right\}.$$

The finite element functions are represented as

$$u_{h,k}^n = \sum_{i=1}^{N_x}\sum_{k=1}^{N_\ell}\xi_{ik}^n\phi_i(x)\psi_k(\ell), \qquad X = \sum_{j=1}^{N_x}\sum_{l=1}^{N_\ell}x_{jl}\phi_j(x)\psi_l(\ell).$$

We define the matrices $M_x, T_x, D_x, S_x \in \mathbb{R}^{N_x \times N_x}$ by

$$(M_x)_{ij} = \big(\phi_i(x), \phi_j(x)\big)_x, \qquad (D_x)_{ij} = \varepsilon\big(\nabla_x\phi_i(x), \nabla_x\phi_j(x)\big)_x$$
$$(T_x)_{ij} = \big(\mathbf{b}\cdot\nabla_x\phi_i(x), \phi_j(x)\big)_x, \qquad (S_x)_{ij} = S_h\big(\phi_i(x), \phi_j(x)\big).$$

Similarly we define the matrices $M_\ell, T_\ell \in \mathbb{R}^{N_\ell \times N_\ell}$ as

$$(M_\ell)_{kl} = \Big(\psi_k(\ell), \psi_l(\ell)\Big)_\ell,$$
$$(T_\ell)_{kl} = \sum_{i=1}^{N_\ell}\Big(\partial_\ell(G\psi_k(\ell)), \psi_l(\ell)\Big)_{I_i} + \sum_{i=1}^{N_\ell-1}[G\psi_k(\ell)]_i\,\psi_l(\ell_i^+) + G\psi_k(\ell_0^+)\psi_l(\ell_0^+).$$

For the ease of presentation let us consider (4.1) with source term $f = 0$. Then the algorithm for the operator splitting method described in (4.18) and (4.19) is as follows: Within each time interval $(t^n, t^{n+1}]$, we begin with the x-direction step where we are looking for the solution of the time-dependent convection-diffusion equation (4.18). We compute $\tilde\eta_j^{n+1} \in \mathbb{R}^{N_x}$, $j = 1, \ldots, N_\ell$, by solving the linear systems

$$(M_x + \tau D_x + \tau T_x + \tau S_x)\tilde\eta_j^{n+1} = M_x\eta_j^n, \qquad j = 1, \ldots, N_\ell.$$

With obtaining the solutions $\tilde\eta_j^{n+1}$, $j = 1, \ldots, N_\ell$, the x-direction step is completed. Then, we continue with the ℓ-direction step where we update the solution from the first step and compute the solution of the one-dimensional transport problem (4.19) by a discontinuous Galerkin method. In this step we solve the linear systems

$$(M_\ell + \tau T_\ell)\eta_j^{n+1} = M_\ell\tilde\eta_j^{n+1}, \qquad j = 1, \ldots, N_x,$$

and the obtained solutions η_j^{n+1}, $j = 1, \ldots, N_\ell$, are used as input for the time interval $(t^{n+1}, t^{n+1}]$.

4.6 Numerical tests

We report in this section the numerical computations illustrating the theoretical results obtained in the previous section. The two-step method (4.18) and (4.19) in the context of finite element method in space and discontinuous Galerkin method in internal coordinate is implemented in the finite element package MooNMD [38].

The tests are made in two plus one dimensions, i.e, we consider $\Omega_x = (0,1) \times (0,1)$ as two dimensional domain in space and $\Omega_\ell = (0,1)$ as one dimensional domain in the internal coordinate. We consider the velocity field $\mathbf{b}$ as $b_1 = b_2 = 0.1$, the growth rate $G(\ell) = 1$ and two different choices of diffusion coefficient ε, $\varepsilon = 1$ and $\varepsilon \ll 1$. The source term f, the boundary and initial conditions are chosen such that the analytical solution of the problem (4.1) is

$$z(t, \ell, x, y) = e^{-0.1t}\sin(\pi\ell)\cos(\pi x)\cos(\pi y).$$

Let $e^n := z(t^n) - u_{h,k}^n$, where z is the exact solution of (4.1)and the numerical solution $u_{h,k}^n$ is obtained by two-step method (4.18) and (4.19). We use the following notations

$$\|e\|_0 = \left(\tau \sum_{n=1}^{N} \|e^n\|_{L^2(L^2)}^2 + \tau \sum_{n=1}^{N} \|e^n\|_{\mathrm{dG}}^2\right)^{1/2},$$

$$\|e\|_1 = \left(\tau \sum_{n=1}^{N} \|e^n\|_{L^2(H^1)}^2 + \tau \sum_{n=1}^{N} \|e^n\|_{\mathrm{dG}}^2\right)^{1/2},$$

$$\|e\|_{\mathrm{DG}} = \left(\tau \sum_{n=1}^{N} \int_{\Omega_\ell} |||e^n|||^2 + \tau \sum_{n=1}^{N} \|e^n\|_{\mathrm{dG}}^2\right)^{1/2}.$$

In order to illustrate the convergence order in time, internal coordinate and space, we use the well known strategy, i.e., the convergence order in time can be obtained by assuming that the mesh sizes k and h are small enough compared to the time-step size τ. In the numerical computations, we have used triangular and quadrilateral meshes which are generated by successive refinement starting from coarsest meshes (level 0) as in Fig. 3.3 for two-dimensional domain Ω_x and a line divided into two cells for one-dimensional domain Ω_ℓ.

Case $\varepsilon = 1$: In this case, the Galerkin finite element method in space is combined with a discontinuous Galerkin method in internal coordinate. For time discretization, the backward Euler time stepping scheme is used with final time $T = 1$. One can expect a convergence for $\|\cdot\|_0$-norm of order $\mathcal{O}(h^{r+1})$ and for $\|\cdot\|_1$-norms of order $\mathcal{O}(h^r)$ using Q_r and P_r finite elements in space with sufficiently small time step length τ and mesh size k. The results are presented in Tables 4.1–4.4.

Tables 4.1 and 4.2 show the second order convergence in the $\|\cdot\|_0$-norm and first order convergence in the $\|\cdot\|_1$-norm for both Q_1 and P_1 finite elements in space with dG(1) in internal coordinate. The length of the time step was set to be $\tau = 10^{-3}$ and mesh size to $k = 1/64$. For Q_2 and P_2 finite elements in space with dG(2) in internal coordinate, the time step length was set to $\tau = 10^{-4}$ and mesh size $k = 1/64$. The results of Tables 4.3 and 4.4 show third order convergence for the $\|\cdot\|_0$-norm and second order for the $\|\cdot\|_1$-norm.

In Tables 4.5 and 4.6, the errors and convergence orders for internal coordinate and time are listed. We expect a convergence of order $\mathcal{O}(k^{q+1/2})$ in the internal coordinate and a convergence of $\mathcal{O}(\tau)$ in time. The errors for dG(1) in internal coordinate with Q_1 on level 7 and time step length $\tau = 2.5 \cdot 10^{-4}$ are presented in Table 4.5. We see that the expected orders of convergence are achieved. The numerical errors and convergence orders in time are listed in Table 4.6 for dG(1) with $k = 1/32$ and Q_1 on level 6. The theoretically predicted convergence order is achieved.

Table 4.1: Errors and rate of convergence in space for Q_1 and dG(1), $k = 1/64$ and $\tau = 10^{-3}$.

Level	$\|e\|_0$ error	$\|e\|_0$ order	$\|e\|_1$ error	$\|e\|_1$ order
0	1.719554e-01	——	1.006185	——
1	4.746460e-02	1.8571	4.892384e-01	1.0403
2	1.206219e-02	1.9764	2.412003e-01	1.0203
3	3.167958e-03	1.9289	1.201483e-01	1.0054

Table 4.2: Errors and rate of convergence in space for P_1 and dG(1), $k = 1/64$ and $\tau = 10^{-3}$.

Level	$\|e\|_0$ error	$\|e\|_0$ order	$\|e\|_1$ error	$\|e\|_1$ order
0	2.353104e-01	——	1.432599	——
1	7.412177e-02	1.6666	7.996426e-01	0.8413
2	1.981996e-02	1.9029	4.113880e-01	0.9589
3	5.144843e-03	1.9458	2.072235e-01	0.9893

Table 4.3: Errors and rate of convergence in space for Q_2 and dG(2), $k = 1/64$ and $\tau = 10^{-4}$.

Level	$\|e\|_0$ error	$\|e\|_0$ order	$\|e\|_1$ error	$\|e\|_1$ order
0	1.916287e-02	——	2.396151e-01	——
1	2.599528e-03	2.8820	6.137457e-02	1.9650
2	3.354662e-04	2.9540	1.561139e-02	1.9750

Table 4.4: Errors and rate of convergence in space for P_2 and dG(2), $k = 1/64$ and $\tau = 10^{-3}$.

Level	$\|e\|_0$ error	$\|e\|_0$ order	$\|e\|_1$ error	$\|e\|_1$ order
0	3.511498e-02	——	5.583590e-01	——
1	4.796648e-03	2.8720	1.526520e-01	1.8710
2	6.138514e-04	2.9661	3.929766e-02	1.9577

Case $\varepsilon = 10^{-9}$: In the case of convection-dominated convection-diffusion, we consider local projection as stabilization in space. Discontinuous Galerkin methods of first and

Table 4.5: Errors and rate of convergence in internal coordinate for dG(1), Q_1 on level 6 and $\tau = 2.5 \cdot 10^{-4}$.

k	$\|e\|_0$	
1/2	6.696513e-02	——
1/4	1.829413e-02	1.7398
1/8	6.521805e-03	1.4880

Table 4.6: Errors and rate of convergence in time for Q_1 and dG(1) on level = 6 and $k = 1/32$.

τ	$\|e\|_0$		$\|e\|_1$	
	error	order	error	order
1/10	1.815303e-01	——	4.027364	——
1/20	9.577853e-02	0.9224	2.170105	0.8921
1/40	4.983170e-02	0.9427	1.141479	0.9269
1/80	2.567753e-02	0.9566	5.869174e-01	0.9597

second order are used for the discretization in internal coordinate. For time discretization, the backward Euler time stepping scheme is used. The numerical tests are performed using for (V_h, D_h) the pairs $(P_1^{\text{bubble}}, P_0^{\text{disc}})$, $(P_2^{\text{bubble}}, P_1^{\text{disc}})$, $(Q_1^{\text{bubble}}, P_0^{\text{disc}})$, and $(Q_2^{\text{bubble}}, P_1^{\text{disc}})$. The stabilization parameters μ_K have been chosen as

$$\mu_K := \mu_0 h_K \qquad \forall K \in \mathcal{T}_h$$

where μ_0 denotes a constant which will be given for each of the test calculations.

In Tables 4.7 and 4.8 we show the convergence results for space in norm $\|\cdot\|_{\text{DG}}$. Table 4.7 shows the error in space with stabilizing parameter $\mu_0 = 5$, time step length $\tau = 10^{-3}$ and mesh size $k = 1/64$ for $(Q_1^{\text{bubble}}, P_0)$ and $(P_1^{\text{bubble}}, P_0)$ with dG(1) in internal coordinate. In Table 4.8, the convergence results for $(Q_2^{\text{bubble}}, P_1^{\text{disc}})$ and $(P_2^{\text{bubble}}, P_1^{\text{disc}})$ with dG(2) in internal coordinate with $\mu_0 = 5$, $k = 1/64$ and $\tau = 10^{-4}$ are listed. We see that the expected orders of convergence $\mathcal{O}(h^{r+1/2})$ are achieved for quadrangles. For smaller mesh size h, the convergence order starts to decrease for triangles. This is because the influence of the error in internal coordinate increases, i.e., the mesh size k is not small enough that one can see the corresponding convergence rate in space for higher order elements.
The numerical errors and convergence orders in internal coordinate are listed in Table 4.9 for dG(1) and $(Q_1^{\text{bubble}}, P_0)$ with $\mu_0 = 5$ on level 7 and $\tau = 2.5 \cdot 10^{-4}$. The convergence order starts to decrease for small mesh size k since the errors in space have increasing influence.
Finally, Table 4.10 shows the errors and convergence orders in time for $(Q_1^{\text{bubble}}, P_0)$ on

Table 4.7: Errors and rate of convergence in space for $(Q_1^{\text{bubble}}, P_0)$ and $(P_1^{\text{bubble}}, P_0)$ and dG(1), $k = 1/64$, $\tau = 10^{-3}$ and $\mu_K = 5h_K$.

	$(Q_1^{\text{bubble}}, P_0)$		$(P_1^{\text{bubble}}, P_0)$	
Level	$\|e^n\|_{\text{DG}}$		$\|e^n\|_{\text{DG}}$	
0	1.756772	——	1.93314	——
1	6.394630e-01	1.4580	7.247844e-01	1.4153
2	2.280495e-01	1.4875	2.661525e-01	1.4453
3	8.245890e-02	1.4678	1.086554e-01	1.2925

Table 4.8: Errors and rate of convergence in space for $(Q_2^{\text{bubble}}, P_1^{\text{disc}})$ and $(P_2^{\text{bubble}}, P_1^{\text{disc}})$ and dG(2), $k = 1/64$, $\tau = 10^{-4}$ and $\mu_K = 5h_K$.

	$(Q_2^{\text{bubble}}, P_1^{\text{disc}})$		$(P_2^{\text{bubble}}, P_1^{\text{disc}})$	
Level	$\|e^n\|_{\text{DG}}$		$\|e^n\|_{\text{DG}}$	
0	1.272972	——	1.234504	——
1	2.558153e-01	2.3151	2.352103e-01	2.3919
2	4.700162e-02	2.4443	5.094834e-02	2.2069
3	8.010563e-03	2.5527	1.222369e-02	2.0593

Table 4.9: Errors and rate of convergence in internal coordinate for dG(1) and $(Q_1^{\text{bubble}}, P_0)$ on level 7 with $\mu_K = 5h_K$ and $\tau = 2.5 \cdot 10^{-4}$.

k	$\|e^n\|_{\text{DG}}$	
1/2	2.493607e-01	——
1/4	9.283060e-02	1.4256
1/8	3.425394e-02	1.4383
1/16	1.446166e-02	1.2441

level 6 with $\mu_0 = 2.5$ and dG(1) with $k = 1/32$. We see that the time stepping scheme is of first order convergent.

Table 4.10: Errors and rate of convergence in time for dG(1) and $(Q_1^{\text{bubble}}, P_0)$ on level with $\mu_K = 2.5h_K$ and $k = 1/32$.

τ	$\|e^n\|_{\text{DG}}$	
1/10	8.017623e-01	——
1/20	4.318566e-01	0.8926
1/40	2.270064e-01	0.9278
1/80	1.166372e-01	0.9607

Chapter 5

PBE, SUPG method in space and dG in internal coordinate

This chapter concentrates on the SUPG method as spatial discretization of the two-step method introduced in Chapter 4 for the population balance equations (4.1). For discretization in internal coordinate and time, dG and backward Euler methods, respectively, are used. The main focus of this chapter is to explore the conditions on the stabilization parameters discussed in [39] for the population balance equations based on an operator splitting method.

The structure of this chapter is as follows. In Section 5.1, we address the full discretization of the subproblems (4.6) and (4.7) by considering the SUPG and dG methods in space and internal coordinate, respectively. Stability bounds are derived assuming $\delta = \mathcal{O}(\tau)$ and $\delta = \mathcal{O}\big(\sqrt{\tau} h\big)$ in Section 5.2, where δ is the stabilization parameter and τ the time step length. In Section 5.3, we use the stability estimates to derive convergence results. Furthermore, we give a comparison of SUPG with LPS method in space. Numerical results illustrating the theory are reported in Section 5.4

5.1 The SUPG and dG method

Let $\mathcal{T}_h$ be a family of an admissible and shape regular triangulations of the polyhedral domain Ω_x. Let $V_h \subset V$ denote the underlying finite element space of piecewise polynomials of order $r \geq 1$. The stabilized bilinear form in the SUPG scheme is defined as follows

$$a_{\mathrm{S}}(u, v) = a(u, v) + \sum_{K \in \mathcal{T}_h} \delta_K(-\varepsilon \Delta_x u + \mathbf{b} \cdot \nabla_x u, \mathbf{b} \cdot \nabla_x v)_K. \tag{5.1}$$

Here, $K \in \mathcal{T}_h$ denotes a mesh cell of the triangulation and δ_K are the local stabilizing parameters which have to be chosen appropriately.
To discretize the subproblems in internal coordinates Ω_ℓ, let $\ell_{\min} = \ell_0 < \ell_1 < \cdots < \ell_M = \ell_{\max}$ be a partition of Ω_ℓ, with $I_i = (\ell_{i-1}, \ell_i]$, $k_i = \ell_i - \ell_{i-1}$ and $k = \max_i k_i$. As in (4.17),

the fully discrete space $S^{r,q}_{h,k}$ is defined as follows

$$S^{r,q}_{h,k} = V_h \times S^q_k = \Big\{ v : \Omega_x \times \Omega_\ell \to \mathbb{R} : v\big|_{I_i}(\ell) = \sum_{j=0}^{q} v_j \ell^j \quad \text{with } v_j \in V_h \Big\}. \tag{5.2}$$

Then, the fully discrete two-step method reads:

First step: For given $u^n_{h,k} \in S^{r,q}_{h,k}$, find $\tilde{u}^{n+1}_{h,k} \in S^{r,q}_{h,k}$ such that

$$\begin{aligned} \int_{\Omega_\ell} \big(\tilde{u}^{n+1}_{h,k} - u^n_{h,k}, \chi\big)_x + \tau \int_{\Omega_\ell} a_{\mathrm{S}}\big(\tilde{u}^{n+1}_{h,k}, \chi\big) + \int_{\Omega_\ell} \sum_{K \in \mathcal{T}_h} \delta_K \Big(\tilde{u}^{n+1}_{h,k} - u^n_{h,k}, \mathbf{b} \cdot \nabla_x \chi\Big)_K \\ = \tau \int_{\Omega_\ell} (f^{n+1}, \chi)_x + \tau \int_{\Omega_\ell} \sum_{K \in \mathcal{T}_h} \delta_K \big(f^{n+1}, \mathbf{b} \cdot \nabla_x \chi\big)_K \end{aligned} \tag{5.3}$$

for all $\chi \in S^{r,q}_{h,k}$ where $u^0_{h,k}$ is a suitable approximation of z_0 in $S^{r,q}_{h,k}$.

Second step: Update the solution $\tilde{u}^{n+1}_{h,k}$ from (5.3) and find $u^{n+1}_{h,k} \in S^{r,q}_{h,k}$ such that

$$\int_{\Omega_\ell} \Big(u^{n+1}_{h,k} - \tilde{u}^{n+1}_{h,k}, \chi\Big)_x + \tau B(u^{n+1}_{h,k}, \chi) = \tau \big(G_{\min} z^{n+1}_{\mathrm{min,h}}, \chi(\ell_0^+)\big)_x \tag{5.4}$$

for all $\chi \in S^{r,q}_{h,k}$, where $z^{n+1}_{\mathrm{min,h}}$ is an approximation of z^{n+1}_{min}. Two different representations of the bilinear form B are given in (4.20) and (4.21).

Assume that for $u \in H^{r+1}(\Omega_x)$ there exists an interpolation operator $\pi_h : V \to V_h$ satisfying the following approximation property [15]

$$\big\|u - \pi_h u\big\|_{H^m(K)} \le C\, h_K^{r+1-m} |u|_{H^{r+1}(K)} \qquad \text{for } m = 0, 1, 2 \tag{5.5}$$

for all $K \in \mathcal{T}_h$.

Lemma 5.1.1 (Coercivity of a_S). *If we choose the SUPG parameter δ_K such that*

$$\delta_K \le \frac{h_K^2}{2\varepsilon c_{\mathrm{inv}}^2} \quad \forall K \in \mathcal{T}_h \tag{5.6}$$

then the bilinear form $a_S(\cdot,\cdot)$ satisfies

$$a_S(v_h, v_h) \ge \frac{1}{2} \|v_h\|^2_{\mathrm{S}} \tag{5.7}$$

with

$$\|v_h\|_{\mathrm{S}} := \left(\varepsilon |v_h|^2_{H^1(\Omega_x)} + \sum_{K \in \mathcal{T}_h} \delta_K \|\mathbf{b} \cdot \nabla_x v_h\|^2_{L^2(K)} \right)^{1/2}. \tag{5.8}$$

Proof. See [76, Lemma 3.25]. □

5.2 Stability of the two-step method

This section studies the stability analysis of the two-step method. We will consider the two different choices of stabilization parameters discussed in [39]. These conditions arises in the stability bounds from the stabilization term with the discretization of the time derivative. In the next lemma, we give the stability of the two-step method based on stabilization parameter proportional to the length of the time step.

Lemma 5.2.1. *Let* (5.6) *be satisfied and* $\partial_\ell G \geq 0$. *With the condition*

$$\delta_K \leq \frac{\tau}{4} \quad \textit{and} \quad \tau \leq \frac{1}{4} \quad \forall K \in \mathcal{T}_h \tag{5.9}$$

the solution $u_{h,k}^{n+1}$ *of the two-step algorithm* (5.3) *and* (5.4) *satisfies*

$$\begin{aligned}
&\left\|u_{h,k}^{n}\right\|_0^2 + \frac{\tau}{2}\sum_{m=0}^{n-1}\int_{\Omega_\ell}\left\|u_{h,k}^{m+1}\right\|_{\mathrm{S}}^2 + \frac{\tau}{2}\sum_{m=0}^{n-1}\left\|u_{h,k}^{m+1}\right\|_{\mathrm{dG}}^2 \\
&\leq e^{3T/2}\left[\left\|u_{h,k}^{0}\right\|_0^2 + \tau\sum_{m=0}^{n-1}\left\{\frac{4}{3}\left(1+\tau\right)\left\|f^{m+1}\right\|_0^2 + 2\left\|G_{\min}^{1/2}z_{\min,\mathrm{h}}^{m+1}\right\|_{L^2(\Omega_x)}^2\right\}\right].
\end{aligned} \tag{5.10}$$

Proof. The proof starts in the usual way by setting $\chi = \tilde{u}_{h,k}^{n+1}$ in (5.3). This gives

$$\begin{aligned}
&\int_{\Omega_\ell}\left(\tilde{u}_{h,k}^{n+1} - u_{h,k}^{n}, \tilde{u}_{h,k}^{n+1}\right)_x + \tau\int_{\Omega_\ell} a_S\left(\tilde{u}_{h,k}^{n+1}, \tilde{u}_{h,k}^{n+1}\right) \\
&\qquad = \tau\int_{\Omega_\ell}\left(f^{n+1}, \tilde{u}_{h,k}^{n+1}\right) + \tau\int_{\Omega_\ell}\sum_{K\in\mathcal{T}_h}\delta_K\left(f^{n+1}, \mathbf{b}\cdot\nabla_x\tilde{u}_{h,k}^{n+1}\right)_K \\
&\qquad\quad - \int_{\Omega_\ell}\sum_{K\in\mathcal{T}_h}\delta_K\left(\tilde{u}_{h,k}^{n+1} - u_{h,k}^{n}, \mathbf{b}\cdot\nabla_x\tilde{u}_{h,k}^{n+1}\right)_K.
\end{aligned} \tag{5.11}$$

The identity $2(a-b)a = a^2 - b^2 + (a-b)^2$ yields

$$\int_{\Omega_\ell}\left(\tilde{u}_{h,k}^{n+1} - u_{h,k}^{n}, \tilde{u}_{h,k}^{n+1}\right)_x = \frac{1}{2}\left\|\tilde{u}_{h,k}^{n+1}\right\|_0^2 - \frac{1}{2}\left\|u_{h,k}^{n}\right\|_0^2 + \frac{1}{2}\left\|\tilde{u}_{h,k}^{n+1} - u_{h,k}^{n}\right\|_0^2.$$

Using this and (5.7) in (5.11), we get

$$\begin{aligned}
&\frac{1}{2}\left\|\tilde{u}_{h,k}^{n+1}\right\|_0^2 - \frac{1}{2}\left\|u_{h,k}^{n}\right\|_0^2 + \frac{1}{2}\left\|\tilde{u}_{h,k}^{n+1} - u_{h,k}^{n}\right\|_0^2 + \frac{\tau}{2}\int_{\Omega_\ell}\left\|\tilde{u}_{h,k}^{n+1}\right\|_{\mathrm{S}}^2 \\
&\qquad \leq \left|\tau\int_{\Omega_\ell}\left(f^{n+1}, \tilde{u}_{h,k}^{n+1}\right)\right| + \left|\tau\int_{\Omega_\ell}\sum_{K\in\mathcal{T}_h}\delta_K\left(f^{n+1}, \mathbf{b}\cdot\nabla_x\tilde{u}_{h,k}^{n+1}\right)_K\right| \\
&\qquad\quad + \left|\int_{\Omega_\ell}\sum_{K\in\mathcal{T}_h}\delta_K\left(\tilde{u}_{h,k}^{n+1} - u_{h,k}^{n}, \mathbf{b}\cdot\nabla_x\tilde{u}_{h,k}^{n+1}\right)_K\right|.
\end{aligned} \tag{5.12}$$

The first two terms on the right-hand side are estimated by using Cauchy-Schwarz inequality followed by Young's inequality

$$\left|\tau \int_{\Omega_\ell} \left(f^{n+1}, \tilde{u}_{h,k}^{n+1}\right)\right| \leq \tau \int_{\Omega_\ell} \left\|f^{n+1}\right\|_{L^2(\Omega_x)} \left\|\tilde{u}_{h,k}^{n+1}\right\|_{L^2(\Omega_x)} \leq \frac{\tau}{2}\left\|f^{n+1}\right\|_0^2 + \frac{\tau}{2}\left\|\tilde{u}_{h,k}^{n+1}\right\|_0^2$$

and

$$\begin{aligned}
&\left|\tau \int_{\Omega_\ell} \sum_{K\in\mathcal{T}_h} \delta_K \left(f^{n+1}, \mathbf{b}\cdot\nabla_x \tilde{u}_{h,k}^{n+1}\right)_K \right| \\
&\qquad \leq \tau \int_{\Omega_\ell} \sum_{K\in\mathcal{T}_h} \delta_K \left\|f^{n+1}\right\|_{L^2(K)} \left\|\mathbf{b}\cdot\nabla_x \tilde{u}_{h,k}^{n+1}\right\|_{L^2(K)} \\
&\qquad \leq 2\tau \int_{\Omega_\ell} \sum_{K\in\mathcal{T}_h} \delta_K \left\|f^{n+1}\right\|_{L^2(K)}^2 + \frac{\tau}{8} \int_{\Omega_\ell} \sum_{K\in\mathcal{T}_h} \delta_K \left\|\mathbf{b}\cdot\nabla_x \tilde{u}_{h,k}^{n+1}\right\|_{L^2(K)}^2 \\
&\qquad \leq 2\tau \int_{\Omega_\ell} \sum_{K\in\mathcal{T}_h} \delta_K \left\|f^{n+1}\right\|_{L^2(K)}^2 + \frac{\tau}{8} \int_{\Omega_\ell} \left\|\tilde{u}_{h,k}^{n+1}\right\|_{\mathrm{S}}^2.
\end{aligned}$$

The estimate for the last term on the right hand side of (5.11) is obtained by using Cauchy-Schwarz inequality, the Young's inequality and condition (5.9) on the stabilization parameters

$$\begin{aligned}
&\left|\int_{\Omega_\ell} \sum_{K\in\mathcal{T}_h} \delta_K \left(\tilde{u}_{h,k}^{n+1} - u_{h,k}^n, \mathbf{b}\cdot\nabla_x \tilde{u}_{h,k}^{n+1}\right)_K \right| \\
&\qquad \leq \int_{\Omega_\ell} \sum_{K\in\mathcal{T}_h} \delta_K \left\|\tilde{u}_{h,k}^{n+1} - u_{h,k}^n\right\|_{L^2(K)} \left\|\mathbf{b}\cdot\nabla_x \tilde{u}_{h,k}^{n+1}\right\|_{L^2(K)} \\
&\qquad \leq \frac{2}{\tau} \int_{\Omega_\ell} \sum_{K\in\mathcal{T}_h} \delta_K \left\|\tilde{u}_{h,k}^{n+1} - u_{h,k}^n\right\|_{L^2(K)}^2 + \frac{\tau}{8} \int_{\Omega_\ell} \sum_{K\in\mathcal{T}_h} \delta_K \left\|\mathbf{b}\cdot\nabla_x \tilde{u}_{h,k}^{n+1}\right\|_{L^2(K)}^2 \\
&\qquad \leq \frac{1}{2}\left\|\tilde{u}_{h,k}^{n+1} - u_{h,k}^n\right\|_0^2 + \frac{\tau}{8} \int_{\Omega_\ell} \left\|\tilde{u}_{h,k}^{n+1}\right\|_{\mathrm{S}}^2.
\end{aligned} \tag{5.13}$$

Substituting these estimates in (5.12) and contributing the $\|\cdot\|_{\mathrm{S}}$-norm in the left-hand side gives

$$\left\|\tilde{u}_{h,k}^{n+1}\right\|_0^2 - \left\|u_{h,k}^n\right\|_0^2 + \frac{\tau}{2} \int_{\Omega_\ell} \left\|\tilde{u}_{h,k}^{n+1}\right\|_{\mathrm{S}}^2 \leq \tau(1+\tau)\left\|f^{n+1}\right\|_0^2 + \tau\left\|\tilde{u}_{h,k}^{n+1}\right\|_0^2. \tag{5.14}$$

Here we have used the condition (5.9) once more. From (5.14) we have

$$\left\|\tilde{u}_{h,k}^{n+1}\right\|_0^2 \leq \frac{\tau(1+\tau)}{1-\tau}\left\|f^{n+1}\right\|_0^2 + \frac{1}{1-\tau}\left\|u_{h,k}^n\right\|_0^2. \tag{5.15}$$

Next, we consider the second step and setting $\chi = u_{h,k}^{n+1}$ in (5.4) to get

$$\int_{\Omega_\ell} \Big(u_{h,k}^{n+1} - \tilde{u}_{h,k}^{n+1}, u_{h,k}^{n+1} \Big)_x + \tau B(u_{h,k}^{n+1}, u_{h,k}^{n+1}) = \tau \Big(G_{\min} z_{\min,\mathrm{h}}^{n+1}, u_{h,k}^{n+1}(\ell_0^+) \Big)_x$$

Using again the identity $2(a-b)a = a^2 - b^2 + (a-b)^2$ and the coercivity (4.23) of the bilinear form B, we obtain

$$\begin{aligned} \frac{1}{2}\big\|u_{h,k}^{n+1}\big\|_0^2 - \frac{1}{2}\big\|\tilde{u}_{h,k}^{n+1}\big\|_0^2 + \frac{1}{2}\big\|u_{h,k}^{n+1} - \tilde{u}_{h,k}^{n+1}\big\|_0^2 + \frac{\tau}{2}\big\|u_{h,k}^{n+1}\big\|_{\mathrm{dG}}^2 & \\ = \Big|\tau\big(G_{\min} z_{\min,\mathrm{h}}^{n+1}, u_{h,k}^{n+1}(\ell_0^+)\big)_x\Big|. \end{aligned}$$

Then applying Cauchy-Schwarz inequality together with Young's inequality to right hand side gives

$$\begin{aligned} \tau\Big|\big(G_{\min} z_{\min,\mathrm{h}}^{n+1}, u_{h,k}^{n+1}(\ell_0^+)\big)_x\Big| &\le \tau\big\|G_{\min}^{1/2} z_{\min,\mathrm{h}}^{n+1}\big\|_{L^2(\Omega_x)}\big\|G_{\min}^{1/2} u_{h,k}^{n+1}(\ell_0^+)\big\|_{L^2(\Omega_x)} \\ &\le \tau\big\|G_{\min}^{1/2} z_{\min,\mathrm{h}}^{n+1}\big\|_{L^2(\Omega_x)}^2 + \frac{\tau}{4}\big\|G_{\min}^{1/2} u_{h,k}^{n+1}(\ell_0^+)\big\|_{L^2(\Omega_x)}^2 \\ &\le \tau\big\|G_{\min}^{1/2} z_{\min,\mathrm{h}}^{n+1}\big\|_{L^2(\Omega_x)}^2 + \frac{\tau}{4}\big\|u_{h,k}^{n+1}\big\|_{\mathrm{dG}}^2. \end{aligned}$$

Hence, we have for the second step

$$\big\|u_{h,k}^{n+1}\big\|_0^2 - \big\|\tilde{u}_{h,k}^{n+1}\big\|_0^2 + \frac{\tau}{2}\big\|u_{h,k}^{n+1}\big\|_{\mathrm{dG}}^2 \le 2\tau\big\|G_{\min}^{1/2} z_{\min,\mathrm{h}}^{n+1}\big\|_{L^2(\Omega_x)}^2. \tag{5.16}$$

Adding (5.14) and (5.16), summing over $m = 0, \dots, n-1$ and using the relation (5.15), we get

$$\begin{aligned} &\big\|u_{h,k}^{n}\big\|_0^2 + \frac{\tau}{2}\sum_{m=0}^{n-1}\int_{\Omega_\ell}\big\|\tilde{u}_{h,k}^{m+1}\big\|_{\mathrm{S}}^2 + \frac{\tau}{2}\sum_{m=0}^{n-1}\big\|u_{h,k}^{m+1}\big\|_{\mathrm{dG}}^2 \\ &\quad\le \big\|u_{h,k}^{0}\big\|_0^2 + \frac{\tau(1+\tau)}{1-\tau}\sum_{m=0}^{n-1}\big\|f^{m+1}\big\|_0^2 + 2\tau\sum_{m=0}^{n-1}\big\|G^{1/2} u_{h,k}^{m+1}(\ell_0^-)\big\|_{L^2(\Omega_x)}^2 \\ &\qquad + \frac{\tau}{1-\tau}\sum_{m=0}^{n-1}\big\|u_{h,k}^{m}\big\|_0^2 \\ &\quad\le \big\|u_{h,k}^{0}\big\|_0^2 + \frac{4\tau}{3}(1+\tau)\sum_{m=0}^{n-1}\big\|f^{m+1}\big\|_0^2 + 2\tau\sum_{m=0}^{n-1}\big\|G_{\min}^{1/2} z_{\min,\mathrm{h}}^{m+1}\big\|_{L^2(\Omega_x)}^2 \\ &\qquad + \frac{4\tau}{3}\sum_{m=0}^{n-1}\big\|u_{h,k}^{m}\big\|_0^2. \end{aligned}$$

Finally, the statement follows by applying Gronwall's lemma. □

The stability estimate of Lemma holds for the stabilization parameter $\delta_K = \tau/4$. The stabilization becomes small for small time step size when the grid in space variable is fixed and vanishes in time continuous limit case. This behavior has been discussed in more detail in [39].

Lemma 5.2.2. *Let* $\partial_\ell G \geq 0$ *and* (5.6) *hold. With the additional condition*

$$\delta_K = \frac{\sigma(\tau) h_K}{\|\mathbf{b}\|_{0,\infty,K} c_{\text{inv}}}, \quad \forall K \in \mathcal{T}_h \quad \textit{with} \quad 0 < \sigma(\tau) \leq \frac{1}{4} \quad \textit{and} \quad \tau \leq \frac{1}{2}, \tag{5.17}$$

where $\sigma(\tau)$ *is a function to be specified later, the solution of two-step algorithm* (5.3) *and* (5.4) *satisfies*

$$\begin{aligned}
&\left\|u_{h,k}^n\right\|_0^2 + \frac{\tau}{2}\sum_{m=0}^{n-1}\int_{\Omega_\ell}\left\|\tilde{u}_{h,k}^{m+1}\right\|_{\mathrm{S}}^2 + \frac{\tau}{2}\sum_{m=0}^{n-1}\left\|u_{h,k}^{m+1}\right\|_{\mathrm{dG}}^2 \\
&\quad\leq \Big(1 + 2\sigma^2(\tau) + 2\tau + 4\tau\sigma^2(\tau)\Big)^{n+1}\Bigg[\left\|u_{h,k}^0\right\|_0^2 + 2\tau\sum_{m=0}^{n-1}\left\|G_{\min}^{1/2} z_{\min,\mathrm{h}}^{m+1}\right\|_{L^2(\Omega_x)}^2 \\
&\qquad + 2\tau\sum_{m=0}^{n-1}\left\|f^{m+1}\right\|_0^2 + 4\tau\sum_{m=0}^{n-1}\int_{\Omega_\ell}\sum_{K\in\mathcal{T}_h}\delta_K\left\|f^{m+1}\right\|_{L^2(K)}^2\Bigg].
\end{aligned} \tag{5.18}$$

Proof. In order to proof the stability estimate (5.18), we follow the same procedure as in [39, Theorem 3.2] for the first step. The proof starts exactly as the proof of Lemma 5.2.1 till (5.12). The bounds for the first two terms on the right hand side of (5.12) are obtained by Cauchy-Schwarz and Young's inequalities

$$\left|\tau\int_{\Omega_\ell}\left(f^{n+1}, \tilde{u}_{h,k}^{n+1}\right)\right| \leq \frac{\tau}{2}\left\|f^{n+1}\right\|_0^2 + \frac{\tau}{2}\left\|\tilde{u}_{h,k}^{n+1}\right\|_0^2$$

$$\left|\tau\int_{\Omega_\ell}\sum_{K\in\mathcal{T}_h}\delta_K\left(f^{n+1}, \mathbf{b}\cdot\nabla_x\tilde{u}_{h,k}^{n+1}\right)_K\right| \leq \tau\int_{\Omega_\ell}\sum_{K\in\mathcal{T}_h}\delta_K\left\|f^{n+1}\right\|_{L^2(K)}^2 + \frac{\tau}{4}\int_{\Omega_\ell}\left\|\tilde{u}_{h,k}^{n+1}\right\|_{\mathrm{S}}^2.$$

The bounds for the last term in (5.12) are obtained by using inverse inequality (2.20)

$$
\begin{aligned}
&\Big| \int_{\Omega_\ell} \sum_{K\in\mathcal{T}_h} \delta_K \big(\tilde{u}_{h,k}^{n+1} - u_{h,k}^n, \mathbf{b}\cdot\nabla_x \tilde{u}_{h,k}^{n+1}\big)_K \Big| \\
&= \Bigg| \int_{\Omega_\ell} \sum_{K\in\mathcal{T}_h} \delta_K \big(\tilde{u}_{h,k}^{n+1} - u_{h,k}^n, \mathbf{b}\cdot\nabla_x (\tilde{u}_{h,k}^{n+1} - u_{h,k}^n)\big)_K \\
&\qquad + \int_{\Omega_\ell} \sum_{K\in\mathcal{T}_h} \delta_K \big(\tilde{u}_{h,k}^{n+1} - u_{h,k}^n, \mathbf{b}\cdot\nabla_x u_{h,k}^n\big)_K \Bigg| \\
&\le \int_{\Omega_\ell} \sum_{K\in\mathcal{T}_h} \delta_K \frac{\|\mathbf{b}\|_{\infty,K} c_{\mathrm{inv}}}{h_K} \big\|\tilde{u}_{h,k}^{n+1} - u_{h,k}^n\big\|_{L^2(K)}^2 + \frac{1}{4}\int_{\Omega_\ell} \sum_{K\in\mathcal{T}_h} \big\|\tilde{u}_{h,k}^{n+1} - u_{h,k}^n\big\|_{L^2(K)}^2 \\
&\qquad + \int_{\Omega_\ell} \sum_{K\in\mathcal{T}_h} \delta_K^2 \|\mathbf{b}\|_{\infty,K}^2 \big\|\nabla_x u_{h,k}^n\big\|_{L^2(K)}^2 \\
&\le \int_{\Omega_\ell} \sum_{K\in\mathcal{T}_h} \left(\delta_K \frac{\|\mathbf{b}\|_{\infty,K} c_{\mathrm{inv}}}{h_K} + \frac{1}{4}\right) \big\|\tilde{u}_{h,k}^{n+1} - u_{h,k}^n\big\|_{L^2(K)}^2 \\
&\qquad + \int_{\Omega_\ell} \sum_{K\in\mathcal{T}_h} \delta_K^2 \|\mathbf{b}\|_{\infty,K}^2 \big\|\nabla_x u_{h,k}^n\big\|_{L^2(K)}^2.
\end{aligned}
$$

The first term can be hidden in the left hand side of (5.12) if the following holds

$$
\delta_K \frac{\|\mathbf{b}\|_{\infty,K} c_{\mathrm{inv}}}{h_K} + \frac{1}{4} \le \frac{1}{2} \implies \delta_K \le \frac{h_K}{4\|\mathbf{b}\|_{\infty,K} c_{\mathrm{inv}}}.
$$

Setting the stabilization parameter (5.17) in above equation, we get

$$
\Big| \int_{\Omega_\ell} \sum_{K\in\mathcal{T}_h} \delta_K \big(\tilde{u}_{h,k}^{n+1} - u_{h,k}^n, \mathbf{b}\cdot\nabla_x \tilde{u}_{h,k}^{n+1}\big)_K \Big| \le \frac{1}{2}\big\|\tilde{u}_{h,k}^{n+1} - u_{h,k}^n\big\|_0^2 + \sigma(\tau)^2 \big\|u_{h,k}^n\big\|_0^2.
$$

Substituting all the estimates in (5.12) leads to following

$$
\begin{aligned}
&\big\|\tilde{u}_{h,k}^{n+1}\big\|_0^2 - \big\|u_{h,k}^n\big\|_0^2 + \frac{\tau}{2}\int_{\Omega_\ell} \big\|\tilde{u}_{h,k}^{n+1}\big\|_{\mathrm{S}}^2 \\
&\le \tau\big\|f^{n+1}\big\|_0^2 + 2\tau \int_{\Omega_\ell} \sum_{K\in\mathcal{T}_h} \delta_K \big\|f^{n+1}\big\|_{L^2(K)}^2 + \tau\big\|\tilde{u}_{h,k}^{n+1}\big\|_0^2 + 2\sigma^2(\tau)\big\|u_{h,k}^n\big\|_0^2.
\end{aligned}
\tag{5.19}
$$

From this equation, we have the following relation

$$
\big\|\tilde{u}_{h,k}^{n+1}\big\|_0^2 = \frac{1}{1-\tau}\left[\Big(1 + 2\sigma^2(\tau)\Big)\big\|u_{h,k}^n\big\|_0^2 + \tau\big\|f^{n+1}\big\|_0^2 + 2\tau \int_{\Omega_\ell} \sum_{K\in\mathcal{T}_h} \delta_K \big\|f^{n+1}\big\|_{L^2(K)}^2\right].
\tag{5.20}
$$

Adding (5.16) and (5.19) and using the relation (5.20), we get

$$
\begin{aligned}
&\big\|u_{h,k}^{n+1}\big\|_0^2 - \big\|u_{h,k}^{n}\big\|_0^2 + \frac{\tau}{2}\int_{\Omega_\ell} \big\|\tilde{u}_{h,k}^{n+1}\big\|_{\mathrm{S}}^2 + \frac{\tau}{2}\big\|u_{h,k}^{n+1}\big\|_{\mathrm{dG}}^2 \\
&\quad \le 2\sigma^2(\tau)\big\|u_{h,k}^{n}\big\|_0^2 + \tau\big\|\tilde{u}_{h,k}^{n+1}\big\|_0^2 + 2\tau\big\|G_{\min}^{1/2} z_{\mathrm{min,h}}^{n+1}\big\|_{L^2(\Omega_x)}^2 + \tau\big\|f^{n+1}\big\|_0^2 \\
&\qquad + 2\tau\int_{\Omega_\ell}\sum_{K\in\mathcal{T}_h}\delta_K\big\|f^{n+1}\big\|_{L^2(K)}^2 \\
&\quad \le 2\sigma^2(\tau)\big\|u_{h,k}^{n}\big\|_0^2 + \frac{\tau}{1-\tau}\Big(1+2\sigma^2(\tau)\Big)\big\|u_{h,k}^{n}\big\|_0^2 + 2\tau\big\|G_{\min}^{1/2} z_{\mathrm{min,h}}^{n+1}\big\|_{L^2(\Omega_x)}^2 \\
&\qquad + \frac{\tau}{1-\tau}\big\|f^{n+1}\big\|_0^2 + \frac{2\tau}{1-\tau}\int_{\Omega_\ell}\sum_{K\in\mathcal{T}_h}\delta_K\big\|f^{n+1}\big\|_{L^2(K)}^2.
\end{aligned}
$$

Hence, $1/(1-\tau)\le 2$ gives

$$
\begin{aligned}
&\big\|u_{h,k}^{n+1}\big\|_0^2 + \frac{\tau}{2}\int_{\Omega_\ell} \big\|\tilde{u}_{h,k}^{n+1}\big\|_{\mathrm{S}}^2 + \frac{\tau}{2}\big\|u_{h,k}^{n+1}\big\|_{\mathrm{dG}}^2 \\
&\quad \le \Big(1+2\sigma^2(\tau)+2\tau+4\tau\sigma^2(\tau)\Big)\big\|u_{h,k}^{n}\big\|_0^2 + 2\tau\big\|G_{\min}^{1/2} z_{\mathrm{min,h}}^{n+1}\big\|_{L^2(\Omega_x)}^2 + 2\tau\big\|f^{n+1}\big\|_0^2 \\
&\qquad + 4\tau\int_{\Omega_\ell}\sum_{K\in\mathcal{T}_h}\delta_K\big\|f^{n+1}\big\|_{L^2(K)}^2. \qquad (5.21)
\end{aligned}
$$

Now, one obtains by induction

$$
\begin{aligned}
&\big\|u_{h,k}^{n+1}\big\|_0^2 + \frac{\tau}{2}\int_{\Omega_\ell} \big\|\tilde{u}_{h,k}^{n+1}\big\|_{\mathrm{S}}^2 + \frac{\tau}{2}\big\|u_{h,k}^{n+1}\big\|_{\mathrm{dG}}^2 \\
&\le \Big(1+2\sigma^2(\tau)+2\tau+4\tau\sigma^2(\tau)\Big)^{n+1}\big\|u_{h,k}^{0}\big\|_0^2 \\
&+\tau\sum_{m=0}^{n}\Bigg[\Big(1+2\sigma^2(\tau)+2\tau+4\tau\sigma^2(\tau)\Big)^{n-m}\Bigg\{2\big\|f^{m+1}\big\|_0^2 + 4\int_{\Omega_\ell}\sum_{K\in\mathcal{T}_h}\delta_K\big\|f^{m+1}\big\|_{0,K}^2 \\
&\qquad + 2\big\|G_{\min}^{1/2} z_{\mathrm{min,h}}^{n+1}\big\|_{L^2(\Omega_x)}^2\Bigg\}\Bigg] \\
&\le \Big(1+2\sigma^2(\tau)+2\tau+4\tau\sigma^2(\tau)\Big)^{n+1}\Bigg[\big\|u_{h,k}^{0}\big\|_0^2 + \tau\sum_{m=0}^{n}\Bigg\{2\big\|f^{m+1}\big\|_0^2 \\
&\qquad + 4\int_{\Omega_\ell}\sum_{K\in\mathcal{T}_h}\delta_K\big\|f^{m+1}\big\|_{0,K}^2 + 2\big\|G_{\min}^{1/2} z_{\mathrm{min,h}}^{n+1}\big\|_{L^2(\Omega_x)}^2\Bigg\}\Bigg]. \qquad (5.22)
\end{aligned}
$$

Summing (5.21) over $m = 0, \ldots, n-1$ gives

$$
\begin{aligned}
&\left\|u_{h,k}^{n}\right\|_0^2 + \frac{\tau}{2}\sum_{m=0}^{n-1}\int_{\Omega_\ell}\left\|\tilde{u}_{h,k}^{m+1}\right\|_{\mathrm{S}}^2 + \frac{\tau}{2}\sum_{m=0}^{n-1}\left\|u_{h,k}^{m+1}\right\|_{\mathrm{dG}}^2 \\
&\leq \Big(2\sigma^2(\tau) + 2\tau + 4\tau\sigma^2(\tau)\Big)\sum_{m=0}^{n-2}\left\|u_{h,k}^{m+1}\right\|_0^2 + \Big(1 + 2\sigma^2(\tau) + 2\tau + 4\tau\sigma^2(\tau)\Big)\left\|u_{h,k}^{0}\right\|_0^2 \\
&+ \tau\sum_{m=0}^{n-1}\left[2\left\|f^{m+1}\right\|_0^2 + 4\int_{\Omega_\ell}\sum_{K\in\mathcal{T}_h}\delta_K\left\|f^{m+1}\right\|_{L^2(K)}^2 + \left\|G_{\min}^{1/2}z_{\mathrm{min,h}}^{n+1}\right\|_{L^2(\Omega_x)}^2\right].
\end{aligned}
$$

Inserting (5.22) in the equation above to get

$$
\begin{aligned}
&\left\|u_{h,k}^{n}\right\|_0^2 + \frac{\tau}{2}\sum_{m=0}^{n-1}\int_{\Omega_\ell}\left\|\tilde{u}_{h,k}^{m+1}\right\|_{\mathrm{S}}^2 + \frac{\tau}{2}\sum_{m=0}^{n-1}\left\|u_{h,k}^{m+1}\right\|_{\mathrm{dG}}^2 \\
&\leq \Big(2\sigma^2(\tau) + 2\tau + 4\tau\sigma^2(\tau)\Big)\sum_{m=0}^{n-2}\Big(1 + 2\sigma^2(\tau) + 2\tau + 4\tau\sigma^2(\tau)\Big)^{m+1}\left\|u_{h,k}^{0}\right\|_0^2 \\
&+ \Big(1 + 2\sigma^2(\tau) + 2\tau + 4\tau\sigma^2(\tau)\Big)\left\|u_{h,k}^{0}\right\|_0^2 \\
&+ \Big(2\sigma^2(\tau) + 2\tau + 4\tau\sigma^2(\tau)\Big)\sum_{m=0}^{n-2}\Big(1 + 2\sigma^2(\tau) + 2\tau + 4\tau\sigma^2(\tau)\Big)^{m+1}\tau\sum_{j=0}^{m-1}\Bigg[2\left\|f^{j+1}\right\|_0^2 \\
&\qquad + 4\int_{\Omega_\ell}\sum_{K\in\mathcal{T}_h}\delta_K\left\|f^{j+1}\right\|_{L^2(K)}^2 + 2\left\|G_{\min}^{1/2}z_{\mathrm{min,h}}^{j+1}\right\|_{L^2(\Omega_x)}^2\Bigg] \\
&+ 2\tau\sum_{j=0}^{n-1}\left\|f^{j+1}\right\|_0^2 + 4\tau\sum_{j=0}^{n-1}\int_{\Omega_\ell}\sum_{K\in\mathcal{T}_h}\delta_K\left\|f^{j+1}\right\|_{L^2(K)}^2 + 2\tau\sum_{j=0}^{n-1}\left\|G_{\min}^{1/2}z_{\mathrm{min,h}}^{j+1}\right\|_{L^2(\Omega_x)}^2.
\end{aligned}
$$

Using $\displaystyle\sum_{i=1}^{n} a^i = \frac{a^{n+1}-a}{a-1}$, one obtains

$$
\begin{aligned}
&\Big(2\sigma^2(\tau) + 2\tau + 4\tau\sigma^2(\tau)\Big)\sum_{m=1}^{n}\Big(1 + 2\sigma^2(\tau) + 2\tau + 4\tau\sigma^2(\tau)\Big)^{m} \\
&\qquad + \Big(1 + 2\sigma^2(\tau) + 2\tau + 4\tau\sigma^2(\tau)\Big) = \Big(1 + 2\sigma^2(\tau) + 2\tau + 4\tau\sigma^2(\tau)\Big)^{n+1}, \\
&\Big(2\sigma^2(\tau) + 2\tau + 4\tau\sigma^2(\tau)\Big)\sum_{m=1}^{n}\Big(1 + 2\sigma^2(\tau) + 2\tau + 4\tau\sigma^2(\tau)\Big)^{m} + 1 \\
&\qquad\qquad = \Big(1 + 2\sigma^2(\tau) + 2\tau + 4\tau\sigma^2(\tau)\Big)^{n+1} - \Big(2\sigma^2(\tau) + 2\tau + 4\tau\sigma^2(\tau)\Big) \\
&\qquad\qquad \leq \Big(1 + 2\sigma^2(\tau) + 2\tau + 4\tau\sigma^2(\tau)\Big)^{n+1}.
\end{aligned}
$$

Hence, we get

$$
\begin{aligned}
&\left\|u_{h,k}^{n}\right\|_0^2 + \frac{\tau}{2}\sum_{m=0}^{n-1}\int_{\Omega_\ell}\left\|\tilde{u}_{h,k}^{m+1}\right\|_{\mathrm{S}}^2 + \frac{\tau}{2}\sum_{m=0}^{n-1}\left\|u_{h,k}^{m+1}\right\|_{\mathrm{dG}}^2 \\
&\qquad \le \Big(1+2\sigma^2(\tau)+2\tau+4\tau\sigma^2(\tau)\Big)^{n+1}\Bigg[\left\|u_{h,k}^{0}\right\|_0^2 + 2\tau\sum_{m=0}^{n-1}\left\|G_{\min}^{1/2} z_{\mathrm{min,h}}^{m+1}\right\|_{L^2(\Omega_x)}^2 \\
&\qquad\qquad + 2\tau\sum_{m=0}^{n-1}\left\|f^{m+1}\right\|_0^2 + 4\tau\sum_{m=0}^{n-1}\int_{\Omega_\ell}\sum_{K\in\mathcal{T}_h}\delta_K\left\|f^{m+1}\right\|_{L^2(K)}^2\Bigg].
\end{aligned}
$$

□

Consider a finite time interval $[0,T]$ and a fixed time step length. The above estimate blows up for $\sigma(\tau)=\text{const}$ in time continuous limit case. This estimate will not blow up in time continuous limit if $\left(1+2\sigma^2(\tau)+2\tau+4\tau\sigma^2(\tau)\right)^{n+1}$ is bounded uniformly. A possible choice is $\sigma(\tau)=\sigma_0\sqrt{\tau}$ to give the stabilizing parameter

$$
\delta_K = \delta_0\frac{\sqrt{\tau}h_K}{\|\mathbf{b}\|_{\infty,K}c_{\mathrm{inv}}}, \tag{5.23}
$$

where δ_0 has to be chosen such that $\delta_0\sqrt{\tau}\le 1/4$. With this choice of $\sigma(\tau)$ we can get

$$
\begin{aligned}
\Big(1+2\tau+2\tau\sigma^2(\tau)+4\tau\sigma^2(\tau)\Big)^{1+n} &= \Big(1+2\tau+2\tau\sigma^2(\tau)+4\tau\sigma^2(\tau)\Big)^{1+T/\tau} \\
&\le \Big(1+2\tau+2\tau\sigma^2(\tau)+4\tau\sigma^2(\tau)\Big)e^{4T}.
\end{aligned}
$$

The following corollary states the stability bounds of the fully discrete two-step algorithm (5.3) and (5.4).

Corollary 5.2.3. *Let $\partial_\ell G\ge 0$,* (5.6) *and* (5.23) *hold, then the solution of two-step algorithm* (5.3) *and* (5.4) *satisfies*

$$
\begin{aligned}
&\left\|u_{h,k}^{n}\right\|_0^2 + \frac{\tau}{2}\sum_{m=0}^{n-1}\int_{\Omega_\ell}\left\|\tilde{u}_{h,k}^{m+1}\right\|_{\mathrm{S}}^2 + \frac{\tau}{2}\sum_{m=0}^{n-1}\left\|u_{h,k}^{m+1}\right\|_{\mathrm{dG}}^2 \\
&\qquad \le e^{4T}\Bigg[\left\|u_{h,k}^{0}\right\|_0^2 + 2\tau\sum_{m=0}^{n-1}\left\|G_{\min}^{1/2} z_{\mathrm{min,h}}^{m+1}\right\|_{L^2(\Omega_x)}^2 + 2\tau\sum_{m=0}^{n-1}\left\|f^{m+1}\right\|_0^2 \\
&\qquad\qquad + 4\tau\sum_{m=0}^{n-1}\int_{\Omega_\ell}\sum_{K\in\mathcal{T}_h}\delta_K\left\|f^{m+1}\right\|_{L^2(K)}^2\Bigg\}\Bigg].
\end{aligned} \tag{5.24}
$$

Compared with the stability results of an equivalent one-step formulation studied in [30], it is apparent that this two-step scheme provides us flexibility to relax the conditions on stabilization parameters. The main difficulty in the analysis of an equivalent one-step method are the following terms

$$\tau \int_{\Omega_\ell} \sum_{K\in\mathcal{T}_h} \delta_K \big(\partial_\ell(Gz), \mathbf{b}\cdot\nabla_x v\big)_K \quad \text{and} \quad \tau \sum_{K\in\mathcal{T}_h} \delta_K \Big(G_{\min} z_{\min}, \mathbf{b}\cdot\nabla_x v(\ell_0^+)\Big)_K.$$

These terms can be combined with the perturbation term of order τ^2 if $\delta_K \sim \tau$. If $\delta_K \sim \sqrt{\tau} h_K$, then one can see that these terms can not be combined with perturbation terms. This means that the one-step formulation is not equivalent to the fully discrete scheme of the original problem.

5.3 Error analysis

Since the stability bounds derived in Lemma 5.2.1 and Corollary 5.2.3 are similar except with the factor in front of right hand sides of (5.18) and (5.24) are different. The detailed analysis for the error estimates is presented here only for the first case which was discussed in Lemma 5.2.1.
For the solution $u(t^n)$ of (4.2) and (4.3) and $u^n_{h,k}$ of (5.3) and (5.4), we define

$$e^n = \big(u(t^n) - P_{h,k}u(t^n)\big) + \big(P_{h,k}u(t^n) - u^n_{h,k}\big) := \eta^n + \xi^n. \tag{5.25}$$

Note that, here we mean by $P_{h,k}$ a projection operator which is defined through the interpolation π_h and Π_k in space and internal coordinate, respectively, and is defined in a similar fashion as in (4.31), i.e.,

$$P_{h,k}w = \pi_h \Pi_k w = \Pi_k \pi_h w, \tag{5.26}$$

where π_h satisfies (5.5).

Since $\xi^n \in S^{r,q}_{h,k}$, we use $\tilde{\xi}^{n+1} = P_{h,k}\tilde{u}(t^{n+1}) - u^{n+1}_{h,k}$ in the first step of splitting (5.3) to get

$$\begin{aligned}
&\int_{\Omega_\ell} \Big(\tilde{\xi}^{n+1} - \xi^n, \tilde{\xi}^{n+1}\Big)_x + \tau \int_{\Omega_\ell} a_h\big(\tilde{\xi}^{n+1}, \tilde{\xi}^{n+1}\big) \\
&= \int_{\Omega_\ell} \Big(P_{h,k}\tilde{u}(t^{n+1}) - P_{h,k}u(t^n), \tilde{\xi}^{n+1}\Big)_x + \tau \int_{\Omega_\ell} a_h\big(P_{h,k}\tilde{u}(t^{n+1}), \tilde{\xi}^{n+1}\big) - \tau \int_{\Omega_\ell} \big(f^{n+1}, \tilde{\xi}^{n+1}\big) \\
&\quad - \tau \int_{\Omega_\ell} \sum_{K\in\mathcal{T}_h} \delta_K \big(f^{n+1}, \mathbf{b}\cdot\nabla_x \tilde{\xi}^{n+1}\big)_K - \int_{\Omega_\ell} \sum_{K\in\mathcal{T}_h} \delta_K \Big(\tilde{u}^{n+1}_{h,k} - u^n_{h,k}, \mathbf{b}\cdot\nabla_x \tilde{\xi}^{n+1}\Big)_K,
\end{aligned}$$

where the terms containing $\tilde{u}^{n+1}_{h,k}$ and $u^n_{h,k}$ are replaced by the right hand side of (5.3).

Then using (4.6) and (4.2) at $t = t^{n+1}$, we get

$$
\begin{aligned}
&\int_{\Omega_\ell} \Big(\tilde{\xi}^{n+1} - \xi^n, \tilde{\xi}^{n+1}\Big)_x + \tau \int_{\Omega_\ell} a_h\big(\tilde{\xi}^{n+1}, \tilde{\xi}^{n+1}\big) \\
&\quad = \int_{\Omega_\ell} \Big(P_{h,k}\tilde{u}(t^{n+1}) - P_{h,k}u(t^n) - \tau \tilde{u}_t(t^{n+1}), \tilde{\xi}^{n+1}\Big)_x + \tau \int_{\Omega_\ell} a\big(P_{h,k}\tilde{u}(t^{n+1}) - \tilde{u}(t^{n+1}), \tilde{\xi}^{n+1}\big) \\
&\quad + \tau \int_{\Omega_\ell} \sum_{K\in\mathcal{T}_h} \delta_K \Big(L_x\big(P_{h,k} - I\big)\tilde{u}(t^{n+1}), \mathbf{b}\cdot\nabla_x\tilde{\xi}^{n+1}\Big)_K - \int_{\Omega_\ell} \sum_{K\in\mathcal{T}_h} \delta_K \Big(\tilde{\xi}^{n+1} - \xi^n, \mathbf{b}\cdot\nabla_x\tilde{\xi}^{n+1}\Big)_K \\
&\quad + \int_{\Omega_\ell} \sum_{K\in\mathcal{T}_h} \delta_K \Big(P_{h,k}\tilde{u}(t^{n+1}) - P_{h,k}u(t^n) - \tau\tilde{u}_t(t^{n+1}), \mathbf{b}\cdot\nabla_x\tilde{\xi}^{n+1}\Big)_K.
\end{aligned}
$$

Hence, we get

$$
\begin{aligned}
&\frac{1}{2}\|\tilde{\xi}^{n+1}\|_0^2 - \frac{1}{2}\|\xi^n\|_0^2 + \frac{1}{2}\|\tilde{\xi}^{n+1} - \xi^n\|_0^2 + \frac{\tau}{2}\int_{\Omega_\ell} \|\tilde{\xi}^{n+1}\|_{\mathrm{S}}^2 \\
&\leq \Bigg|\tau \int_{\Omega_\ell} \Big(\frac{P_{h,k}\tilde{u}(t^{n+1}) - P_{h,k}u(t^n)}{\tau} - \tilde{u}_t(t^{n+1}), \tilde{\xi}^{n+1}\Big)_x + \tau \int_{\Omega_\ell} a\big(P_{h,k}\tilde{u}(t^{n+1}) - \tilde{u}(t^{n+1}), \tilde{\xi}^{n+1}\big) \\
&\quad + \tau \int_{\Omega_\ell} \sum_{K\in\mathcal{T}_h} \delta_K \Big(L_x\big(P_{h,k} - I\big)\tilde{u}(t^{n+1}), \mathbf{b}\cdot\nabla_x\tilde{\xi}^{n+1}\Big)_K + \int_{\Omega_\ell} \sum_{K\in\mathcal{T}_h} \delta_K \Big(\tilde{\xi}^{n+1} - \xi^n, \mathbf{b}\cdot\nabla_x\tilde{\xi}^{n+1}\Big)_K \\
&\quad + \tau \int_{\Omega_\ell} \sum_{K\in\mathcal{T}_h} \delta_K \Big(\frac{P_{h,k}\tilde{u}(t^{n+1}) - P_{h,k}u(t^n)}{\tau} - \tilde{u}_t(t^{n+1}), \mathbf{b}\cdot\nabla_x\tilde{\xi}^{n+1}\Big)_K\Bigg|.
\end{aligned}
$$

The convective term in the bilinear form $a_S(\cdot,\cdot)$ can be split into two terms, the first term corresponding to the error term in space and the second one in internal coordinate. Then the first term is integrated by parts with respect to x to get

$$
\begin{aligned}
&\Big(\mathbf{b}\cdot\nabla_x\big(P_{h,k}\tilde{u} - \tilde{u}\big), \tilde{\xi}^{n+1}\Big)_x \\
&\qquad = \Big(\mathbf{b}\cdot\nabla_x\big(P_{h,k}\tilde{u} - \Pi_k\tilde{u}\big), \tilde{\xi}^{n+1}\Big)_x + \Big(\mathbf{b}\cdot\nabla_x\big(\Pi_k\tilde{u} - \tilde{u}\big), \tilde{\xi}^{n+1}\Big)_x \\
&\qquad = -\Big(P_{h,k}\tilde{u} - \Pi_k\tilde{u}, \mathbf{b}\cdot\nabla_x\tilde{\xi}^{n+1}\Big) + \Big(\mathbf{b}\cdot\nabla_x\big(\Pi_k\tilde{u} - \tilde{u}\big), \tilde{\xi}^{n+1}\Big)_x \\
&\qquad = -\sum_{K\in\mathcal{T}_h} \delta_K \Big(\frac{P_{h,k}\tilde{u} - \tilde{u}}{\delta_K}, \mathbf{b}\cdot\nabla_x\tilde{\xi}^{n+1}\Big)_K + \Big(\mathbf{b}\cdot\nabla_x\big(\Pi_k\tilde{u} - \tilde{u}\big), \tilde{\xi}^{n+1}\Big)_x.
\end{aligned}
$$

Using this and rearranging the terms on the right hand side of above equation, we get the following error inequality for first step

$$
\begin{aligned}
&\frac{1}{2}\|\tilde{\xi}^{n+1}\|_0^2 - \frac{1}{2}\|\xi^n\|_0^2 + \frac{1}{2}\|\tilde{\xi}^{n+1} - \xi^n\|_0^2 + \frac{\tau}{2}\int_{\Omega_\ell} \|\tilde{\xi}^{n+1}\|_{\mathrm{S}}^2 \\
&\leq \Big|\tau\int_{\Omega_\ell} \big(T_1, \tilde{\xi}^{n+1}\big)_x\Big| + \Big|\varepsilon\tau\int_{\Omega_\ell} \big(T_2, \nabla_x\tilde{\xi}^{n+1}\big)_x\Big| + \Big|\tau\int_{\Omega_\ell}\sum_{K\in\mathcal{T}_h}\delta_K\big(T_3, \mathbf{b}\cdot\nabla_x\tilde{\xi}^{n+1}\big)_{L^2(K)}\Big| \\
&\quad + \Big|\int_{\Omega_\ell}\sum_{K\in\mathcal{T}_h}\delta_K\big(\tilde{\xi}^{n+1} - \xi^n, \mathbf{b}\cdot\nabla_x\tilde{\xi}^{n+1}\big)_{L^2(K)}\Big|,
\end{aligned}
\tag{5.27}
$$

where

$$
\begin{aligned}
T_1 &= \left\{\frac{P_{h,k}\tilde{u}(t^{n+1}) - P_{h,k}u(t^n)}{\tau} - \tilde{u}_t(t^{n+1}) + \mathbf{b}\cdot\nabla_x\big(\Pi_k\tilde{u}(t^{n+1}) - \tilde{u}(t^{n+1})\big)\right\}, \\
T_2 &= \nabla_x\Big(P_{h,k}\tilde{u}(t^{n+1}) - \tilde{u}(t^{n+1})\Big), \\
T_3 &= \Big\{L_x\big(P_{h,k} - I\big)\tilde{u}(t^{n+1})\Big\} + \left\{\frac{P_{h,k}\tilde{u}(t^{n+1}) - P_{h,k}u(t^n)}{\tau} - \tilde{u}_t(t^{n+1})\right\} \\
&\quad + \left\{\frac{P_{h,k}\tilde{u}(t^{n+1}) - \Pi_k\tilde{u}(t^{n+1})}{\delta_K}\right\}.
\end{aligned}
$$

We consider the terms on the right hand side of (5.27) separately. For first three terms, Cauchy-Schwarz inequality and Young's inequality gives

$$
\begin{aligned}
\Big|\tau\int_{\Omega_\ell}\big(T_1, \tilde{\xi}^{n+1}\big)_x\Big| &\leq \tau\int_{\Omega_\ell}\|T_1\|_{L^2(\Omega_x)}\|\tilde{\xi}^{n+1}\|_{L^2(\Omega_x)} \\
&\leq \frac{\tau}{2}\|T_1\|_0^2 + \frac{\tau}{2}\|\tilde{\xi}^{n+1}\|_0^2 \\
\Big|\varepsilon\tau\int_{\Omega_\ell}\big(T_2, \nabla_x\tilde{\xi}^{n+1}\big)_x\Big| &\leq \tau\varepsilon\int_{\Omega_\ell}\|T_2\|_{L^2(\Omega_x)}\|\nabla_x\tilde{\xi}^{n+1}\|_{L^2(\Omega_x)} \\
&\leq 2\varepsilon\tau\|T_2\|_0^2 + \frac{\tau}{8}\int_{\Omega_\ell}\|\tilde{\xi}^{n+1}\|_{\mathrm{S}}^2
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big|\tau\int_{\Omega_\ell}\sum_{K\in\mathcal{T}_h}\delta_K\big(T_3, \mathbf{b}\cdot\nabla_x\tilde{\xi}^{n+1}\big)_{L^2(K)}\Big| \\
&\qquad\leq \tau\int_{\Omega_\ell}\sum_{K\in\mathcal{T}_h}\delta_K\|T_3\|_{L^2(K)}\|\mathbf{b}\cdot\nabla_x\tilde{\xi}^{n+1}\|_{L^2(K)} \\
&\qquad\leq \tau\int_{\Omega_\ell}\sum_{K\in\mathcal{T}_h}\delta_K\|T_3\|_{L^2(K)}^2 + \tau\int_{\Omega_\ell}\sum_{K\in\mathcal{T}_h}\delta_K\|\mathbf{b}\cdot\nabla_x\tilde{\xi}^{n+1}\|_{L^2(K)}^2 \\
&\qquad\leq 2\tau\int_{\Omega_\ell}\sum_{K\in\mathcal{T}_h}\delta_K\|T_3\|_{L^2(K)}^2 + \frac{\tau}{8}\int_{\Omega_\ell}\|\tilde{\xi}^{n+1}\|_{\mathrm{S}}^2.
\end{aligned}
$$

For the last term, we use the same procedure as in (5.13)

$$
\begin{aligned}
&\Big| \int_{\Omega_\ell} \sum_{K\in\mathcal{T}_h} \delta_K \big(\tilde{\xi}^{n+1} - \xi^n, \mathbf{b}\cdot\nabla_x \tilde{\xi}^{n+1}\big)_{L^2(K)} \Big| \\
&\qquad \le \int_{\Omega_\ell} \sum_{K\in\mathcal{T}_h} \delta_K \big\|\tilde{\xi}^{n+1} - \xi^n\big\|_{L^2(K)} \big\|\mathbf{b}\cdot\nabla_x \tilde{\xi}^{n+1}\big\|_{L^2(K)} \\
&\qquad \le \frac{2}{\tau} \int_{\Omega_\ell} \sum_{K\in\mathcal{T}_h} \delta_K \big\|\tilde{\xi}^{n+1} - \xi^n\big\|^2_{L^2(K)} + \frac{\tau}{8} \int_{\Omega_\ell} \sum_{K\in\mathcal{T}_h} \delta_K \big\|\mathbf{b}\cdot\nabla_x \tilde{\xi}^{n+1}\big\|^2_{L^2(K)} \\
&\qquad \le \frac{1}{2} \big\|\tilde{\xi}^{n+1} - \xi^n\big\|_0^2 + \frac{\tau}{8} \int_{\Omega_\ell} \big\|\tilde{\xi}^{n+1}\big\|^2_{\mathrm{S}}.
\end{aligned}
$$

Inserting these estimates in (5.27) yields

$$
\begin{aligned}
&\big\|\tilde{\xi}^{n+1}\big\|_0^2 - \big\|\xi^n\big\|_0^2 + \frac{\tau}{4} \int_{\Omega_\ell} \big\|\tilde{\xi}^{n+1}\big\|^2_{\mathrm{S}} \\
&\qquad \le \tau \big\|T_1\big\|_0^2 + 4\varepsilon\tau \big\|T_2\big\|_0^2 + 4\tau \int_{\Omega_\ell} \sum_{K\in\mathcal{T}_h} \delta_K \big\|T_3\big\|^2_{L^2(K)} + \tau \big\|\tilde{\xi}^{n+1}\big\|_0^2.
\end{aligned}
\tag{5.28}
$$

Now consider the second step. Applying ξ^{n+1} to (5.4)

$$
\begin{aligned}
&\int_{\Omega_\ell} \Big(\xi^{n+1} - \tilde{\xi}^{n+1}, \xi^{n+1}\Big)_x + \tau B(\xi^{n+1}, \xi^{n+1}) \\
&\qquad = \int_{\Omega_\ell} \Big(P_{h,k} u(t^{n+1}) - P_{h,k}\tilde{u}(t^{n+1}), \xi^{n+1}\Big)_x + \tau B\big(P_{h,k} u(t^{n+1}), \xi^{n+1}\big) \\
&\qquad\quad - \tau \Big(G_{\min} z^{n+1}_{\mathrm{min,h}}, \xi^{n+1}(\ell_0^+)\Big)_x
\end{aligned}
$$

where the terms $u^{n+1}_{h,k}$ and $\tilde{u}^{n+1}_{h,k}$ are replaced by the right-hand side of (5.4). Then (4.3) at $t = t^{n+1}$ gives

$$
\begin{aligned}
&\int_{\Omega_\ell} \Big(\xi^{n+1} - \tilde{\xi}^{n+1}, \xi^{n+1}\Big)_x + \tau B(\xi^{n+1}, \xi^{n+1}) \\
&= \int_{\Omega_\ell} \Big(P_{h,k} u(t^{n+1}) - P_{h,k}\tilde{u}(t^{n+1}) - \tau u_t(t^{n+1}), \xi^{n+1}\Big)_x + \tau B\big(P_{h,k} u(t^{n+1}) - u(t^{n+1}), \xi^{n+1}\big) \\
&\quad - \tau \Big(G\big(z_{\min}(t^{n+1}) - z^{n+1}_{\mathrm{min,h}}\big), \xi^{n+1}(\ell_0^+)\Big)_x
\end{aligned}
\tag{5.29}
$$

Using the error decomposition into space and internal coordinate, we can write the bilinear form $B(\cdot,\cdot)$ as

$$
\begin{aligned}
&B\big(P_{h,k} u(t^{n+1}) - u(t^{n+1}), \xi^{n+1}\big) \\
&\qquad = B\big(P_{h,k} u(t^{n+1}) - \Pi_k u(t^{n+1}), \xi^{n+1}\big) + B\big(\Pi_k u(t^{n+1}) - u(t^{n+1}), \xi^{n+1}\big).
\end{aligned}
\tag{5.30}
$$

Then for the first term on the right hand side, we use (4.20) to get

$$
\begin{aligned}
&B\big(P_{h,k}u(t^{n+1}) - \Pi_k u(t^{n+1}), \xi^{n+1}\big) \\
&\qquad = \sum_{i=1}^{M} \int_{I_i} \Big(\frac{\partial}{\partial \ell}\big(GP_{h,k}u(t^{n+1}) - G\Pi_k u(t^{n+1})\big), \xi^{n+1}\Big)_x \\
&\qquad\quad + \Big(G_{\min}\big(P_{h,k}u(t^{n+1}, \ell_0^+) - \Pi_k u(t^{n+1}, \ell_0^+)\big), \xi^{n+1}(\ell_0^+)\Big)_x.
\end{aligned}
$$

Here the jump terms vanishes due to the continuity of $\pi_h u$ in ℓ direction.

Note that the interpolant $\Pi_k u$ satisfies (4.27), therefore using the second representation (4.21) of the bilinear form B, we get for the second term on the right hand side of (5.30)

$$
B\big(\Pi_k u(t^{n+1}) - u(t^{n+1}), \xi^{n+1}\big) = -\sum_{i=1}^{M} \int_{I_i} \Big(\Pi_k u(t^{n+1}) - u(t^{n+1}), G\partial_\ell \xi^{n+1}\Big)_x.
$$

Substituting this in (5.30) and combining with (5.29), we get

$$
\begin{aligned}
&\int_{\Omega_\ell} \Big(\xi^{n+1} - \tilde{\xi}^{n+1}, \xi^{n+1}\Big)_x + \tau B(\xi^{n+1}, \xi^{n+1}) \\
&\qquad = \tau \sum_{i=1}^{M} \int_{I_i} \Big\{\big(T_4, \xi^{n+1}\big)_x + \big(T_5, G\partial_\ell \xi^{n+1}\big)_x\Big\} + \Big(T_6, G_{\min}^{1/2} \xi^{n+1}(\ell_0^+)\Big)_x,
\end{aligned}
$$

where

$$
\begin{aligned}
T_4 &= \Big\{\frac{P_{h,k}u(t^{n+1}) - P_{h,k}\tilde{u}(t^{n+1})}{\tau} - u_t(t^{n+1})\Big\} + \Big\{\frac{\partial}{\partial \ell}\big(GP_{h,k}u(t^{n+1}) - G\Pi_k u(t^{n+1})\big)\Big\} \\
T_5 &= \Pi_k u(t^{n+1}) - u(t^{n+1}) \\
T_6 &= \Big\{G_{\min}\big(P_{h,k}u(t^{n+1}, \ell_0^+) - \Pi_k u(t^{n+1}, \ell_0^+)\big)\Big\} + \Big\{G_{\min}^{1/2}\big(z_{\min}(t^{n+1}) - z_{\mathrm{min,h}}^{n+1}\big)\Big\}.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
&\frac{1}{2}\|\xi^{n+1}\|_0^2 - \frac{1}{2}\|\tilde{\xi}^{n+1}\|_0^2 + \frac{\tau}{2}\|\xi^{n+1}\|_{\mathrm{dG}}^2 \\
&\quad \le \Big|\tau \sum_{i=1}^{M} \int_{I_i} \big(T_4, \xi^{n+1}\big)_x\Big| + \Big|\tau \sum_{i=1}^{M} \int_{I_i} \big(T_5, G\partial_\ell \xi^{n+1}\big)_x\Big| + \Big|\Big(T_6, G_{\min}^{1/2} \xi^{n+1}(\ell_0^+)\Big)_x\Big|. \qquad (5.31)
\end{aligned}
$$

For first and last terms on the right hand side, the Cauchy-Schwarz inequality and Young's inequality gives

$$\Big|\tau \sum_{i=1}^{M} \int_{I_i} \big(T_4, \xi^{n+1}\big)_x\Big| \le \tau \sum_{i=1}^{M} \int_{I_i} \big\|T_4\big\|_{L^2(\Omega_x)}\big\|\xi^{n+1}\big\|_{L^2(\Omega_x)}$$
$$\le \tau \sum_{i=1}^{M} \int_{I_i} \big\|T_4\big\|^2_{L^2(\Omega_x)} + \frac{\tau}{4}\big\|\xi^{n+1}\big\|^2_0,$$
$$\Big|\tau\big(T_6, G^{1/2}_{\min}\xi^{n+1}(\ell_0^+)\big)_x\Big| \le \tau\big\|T_6\big\|_{L^2(\Omega_x)}\big\|G^{1/2}_{\min}\xi^{n+1}(\ell_0^+)\big\|_{L^2(\Omega_x)}$$
$$\le \tau\big\|T_6\big\|^2_{L^2(\Omega_x)} + \frac{\tau}{4}\big\|\xi^{n+1}\big\|^2_{\mathrm{dG}}.$$

In order to bound the second term, let $\Pi_0 G$ be the L^2-projection of G in a space of piecewise constant functions in internal coordinate, using the orthogonality property (4.28) of Π_k, the Cauchy-Schwarz inequality and the inverse inequality to get

$$\Big|\tau \sum_{i=1}^{M} \int_{I_i} \big(T_5, G\partial_\ell \xi^{n+1}\big)_x\Big| \le \Big|\tau \sum_{i=1}^{M} \int_{I_i} \Big(T_5, \big(G - \Pi_0 G\big)\partial_\ell \xi^{n+1}\Big)_x\Big|$$
$$\le \tau \sum_{i=1}^{M} \int_{I_i} \big\|T_5\big\|_{L^2(\Omega_x)}|G - \Pi_0 G|\, k_i^{-1}\big\|\xi^{n+1}\big\|_{L^2(\Omega_x)}$$
$$\le C\,\tau \sum_{i=1}^{M} \int_{I_i} \big\|T_5\big\|^2_{L^2(\Omega_x)} + \frac{\tau}{4}\big\|\xi^{n+1}\big\|^2_0.$$

Inserting all estimates in (5.31) leads to

$$\begin{aligned}&\big\|\xi^{n+1}\big\|^2_0 - \big\|\tilde{\xi}^{n+1}\big\|^2_0 + \frac{\tau}{2}\big\|\xi^{n+1}\big\|^2_{\mathrm{dG}}\\ &\le \tau \sum_{i=1}^{M} \int_{I_i} \Big\{2\big\|T_4\big\|^2_{L^2(\Omega_x)} + C\big\|T_5\big\|^2_{L^2(\Omega_x)}\Big\} + 2\tau\big\|T_6\big\|_{L^2(\Omega_x)} + \tau\big\|\xi^{n+1}\big\|^2_0.\end{aligned} \tag{5.32}$$

Adding (5.28) and (5.32), summing over $n = 0, \ldots, N-1$, using (5.15) and applying Gronwall's Lemma 2.3.4 in the same fashion as in Lemma 5.2.1 to get

$$\begin{aligned}&\big\|\xi^N\big\|^2_0 + \frac{\tau}{4}\sum_{n=0}^{N-1}\int_{\Omega_\ell}\big\|\tilde{\xi}^{n+1}\big\|^2_{\mathrm{S}} + \tau\sum_{n=0}^{N-1}\big\|\xi^{n+1}\big\|^2_{\mathrm{dG}}\\ &\quad\le e^{2T}\big\|\xi_0\big\|^2_0 + e^{2T}\tau\sum_{n=0}^{N-1}\Bigg[\big\|T_1\big\|^2_0 + 4\varepsilon\big\|T_2\big\|^2_0 + 4\int_{\Omega_\ell}\sum_{K\in\mathcal{T}_h}\delta_K\big\|T_3\big\|^2_{L^2(K)}\\ &\qquad + \sum_{i=1}^{M}\int_{I_i}\Big\{2\big\|T_4\big\|^2_{L^2(\Omega_x)} + C\big\|T_5\big\|^2_{L^2(\Omega_x)}\Big\} + 2\big\|T_6\big\|^2_{L^2(\Omega_x)}\Bigg].\end{aligned} \tag{5.33}$$

In the following, the arising terms on the right hand side of above equation have to be bounded by the norms of the continuous solution of (4.2) and (4.3).

Using triangle inequality, we get

$$\begin{aligned}&\left\|\frac{P_{h,k}\tilde{u}(t^{n+1})-P_{h,k}u(t^n)}{\tau}-\tilde{u}_t(t^{n+1})\right\|_0^2\\&\qquad\le 2\left\|\frac{P_{h,k}\tilde{u}(t^{n+1})-P_{h,k}u(t^n)}{\tau}-P_{h,k}\tilde{u}_t(t^{n+1})\right\|_0^2+2\big\|P_{h,k}\tilde{u}_t(t^{n+1})-\tilde{u}_t(t^{n+1})\big\|_0^2.\end{aligned}$$

The estimates of first term uses the initial condition $\tilde{u}(t^{n+})=u(t^n)$ from first step in (4.2), Taylor's theorem with integral remainder term, commutation and stability property of projection $P_{h,k}$ gives

$$\left\|\frac{P_{h,k}\tilde{u}(t^{n+1})-P_{h,k}\tilde{u}(t^n)}{\tau}-P_{h,k}\tilde{u}_t(t^{n+1})\right\|_0^2\le\tau\int_{t_n}^{t^{n+1}}\big\|P_{h,k}\tilde{u}_{tt}\big\|_0^2\le C\,\tau\int_{t_n}^{t^{n+1}}\big\|\tilde{u}_{tt}\big\|_0^2.$$

Using the splitting (4.33), interpolation error estimates of (5.5), (4.30) and condition (4.32) we get for the second term

$$\begin{aligned}&\big\|P_{h,k}\tilde{u}_t(t^{n+1})-\tilde{u}_t(t^{n+1})\big\|_0^2\\&\quad\le 2\int_{\Omega_\ell}\big\|\pi_h\Pi_k\tilde{u}_t(t^{n+1})-\Pi_k\tilde{u}_t(t^{n+1})\big\|_{L^2(\Omega_x)}^2+2\int_{\Omega_\ell}\big\|\Pi_k\tilde{u}_t(t^{n+1})-\tilde{u}_t(t^{n+1})\big\|_{L^2(\Omega_x)}^2\\&\quad\le Ch^{2r+2}\big\|\tilde{u}_t(t^{n+1})\big\|_{L^2(H^{r+1})}^2+Ck^{2q+2}\big\|\tilde{u}_t(t^{n+1})\big\|_{H^{q+1}(L^2)}^2.\end{aligned}$$

Similarly the estimates for the other terms can be obtained as follows

$$\begin{aligned}&\big\|P_{h,k}\tilde{u}(t^{n+1})-\tilde{u}(t^{n+1})\big\|_0^2\\&\qquad\le 2\int_{\Omega_\ell}\big\|P_{h,k}\tilde{u}(t^{n+1})-\Pi_k\tilde{u}(t^{n+1})\big\|_{L^2(\Omega_x)}^2+2\int_{\Omega_\ell}\big\|\tilde{u}(t^{n+1})-\Pi_k\tilde{u}(t^{n+1})\big\|_{L^2(\Omega_x)}^2\\&\qquad\le Ch^{2r+2}\big|\tilde{u}(t^{n+1})\big\|_{L^2(H^{r+1})}^2+Ck^{2q+2}\big\|\tilde{u}(t^{n+1})\big\|_{H^{q+1}(L^2)}^2,\\&\big\|\mathbf{b}\cdot\nabla_x\big(P_{h,k}\tilde{u}(t^{n+1})-\tilde{u}(t^{n+1})\big\|_0^2\le 2\big\|\mathbf{b}\big\|_{0,\infty}^2\int_{\Omega_\ell}\big\|\nabla_x\big(\Pi_k\pi_h\tilde{u}(t^{n+1})-\Pi_k\tilde{u}(t^{n+1})\big\|_{L^2(\Omega_x)}^2\\&\qquad\qquad+2\big\|\mathbf{b}\big\|_{0,\infty}^2\int_{\Omega_x}\big\|\nabla_x\big(\Pi_k\tilde{u}(t^{n+1})-\tilde{u}(t^{n+1})\big\|_{L^2(\Omega_\ell)}^2\\&\qquad\qquad\le Ch^{2r}\big\|\tilde{u}(t^{n+1})\big\|_{L^2(H^{r+1})}^2+Ck^{2q+2}\big\|\tilde{u}(t^{n+1})\big\|_{H^{q+1}(H^1)}^2.\end{aligned}$$

The estimates for the stabilizing terms can be obtained by using the local approximation properties

$$\begin{aligned}
&\int_{\Omega_\ell} \sum_{K\in\mathcal{T}_h} \delta_K \Big\{ \big\|\varepsilon \Delta_x (P_{h,k} - I)\tilde{u}(t^{n+1})\big\|^2_{L^2(K)} + \big\|\mathbf{b}\cdot\nabla_x (P_{h,k} - I)\tilde{u}(t^{n+1})\big\|^2_{L^2(K)} \Big\} \\
&\quad \le 2\Big(\max_{K\in\mathcal{T}_h}\delta_K\Big) \int_{\Omega_\ell} \Big\{ \varepsilon^2 \big\|\Delta_x (P_{h,k} - \Pi_k)\tilde{u}(t^{n+1})\big\|^2_{L^2(K)} + \varepsilon^2 \big\|\Delta_x (\Pi_k - I)\tilde{u}(t^{n+1})\big\|^2_{L^2(K)} \\
&\qquad\qquad + \big\|\mathbf{b}\cdot\nabla_x (P_{h,k} - \Pi_k)\tilde{u}(t^{n+1})\big\|^2_{L^2(K)} + \big\|\mathbf{b}\cdot\nabla_x (\Pi_k - I)\tilde{u}(t^{n+1})\big\|^2_{L^2(K)} \Big\} \\
&\quad \le C\Big(\max_{K\in\mathcal{T}_h}\delta_K\Big) \Big\{ \varepsilon^2 h^{2r-2} \big\|\tilde{u}(t^{n+1})\big\|^2_{L^2(H^{r+1})} + \varepsilon^2 k^{2q+2} \big\|\tilde{u}(t^{n+1})\big\|^2_{H^{q+1}(H^2)} \\
&\qquad\qquad + h^{2r} \big\|\tilde{u}(t^{n+1})\big\|^2_{L^2(H^{r+1})} + k^{2q+2} \big\|\tilde{u}(t^{n+1})\big\|^2_{H^{q+1}(H^1)} \Big\}.
\end{aligned}$$

The initial condition $\tilde{u}^{n+1} = u(t^n)$ from the second step and Taylor theorem with integral remainder term gives

$$\left\| \frac{P_{h,k}u(t^{n+1}) - P_{h,k}u(t^n)}{\tau} - P_{h,k}u_t(t^{n+1}) \right\|_0^2 \le \tau \int_{t_n}^{t^{n+1}} \big\|P_{h,k}u_{tt}\big\|_0^2 \le C\,\tau \int_{t_n}^{t^{n+1}} \big\|u_{tt}\big\|_0^2$$

and the approximation properties of π_h and Π_k gives

$$\begin{aligned}
\big\|P_{h,k}u_t(t^{n+1}) - u_t(t^{n+1})\big|_0^2 &\le 2\big\|P_{h,k}u_t(t^{n+1}) - \Pi_k u_t(t^{n+1})\big|_0^2 + 2\big\|\Pi_k u_t(t^{n+1}) - u_t(t^{n+1})\big|_0^2 \\
&\le Ch^{2r+2}\big\|u_t(t^{n+1})\big\|^2_{L^2(H^{r+1})} + Ck^{2q+2}\big\|u_t(t^{n+1})\big\|^2_{H^{q+1}(L^2)},
\end{aligned}$$

$$\begin{aligned}
\sum_{i=1}^{M} \int_{I_i} \big\|(\Pi_k - I)u(t^{n+1})\big\|^2_{L^2(\Omega_x)} &\le Ck^{2q+2} \sum_{i=1}^{M} \int_{I_i} \big\|u^{(q+1)}(t^{n+1})\big\|^2_{L^2(\Omega_x)} \\
&\le Ck^{2q+1}\big\|u(t^{n+1})\big\|^2_{H^{q+1}(L^2)}
\end{aligned}$$

and

$$\begin{aligned}
\big\|G_{\min}^{1/2}(P_{h,k} - \Pi_k)u(t^{n+1}, \ell_0^+)\big\|^2_{L^2(\Omega_x)} &\le Ch^{2r+2}\big\|\Pi_k u(t^{n+1}, \ell_0^+)\big\|_{H^{r+1}(\Omega_x)} \\
&\le Ch^{2r+2}\big\|u(t^{n+1})\big\|^2_{C(H^{r+1})}.
\end{aligned}$$

Finally, the bounds for the boundary condition in internal coordinate are obtained as

$$\big\|G_{\min}\big(z_{\min}(t^{n+1}) - z^{n+1}_{\min,\mathrm{h}}\big)\big\|^2_{L^2(\Omega_x)} \le Ch^{2r+2}\big\|z_{\min}(t^{n+1})\big\|^2_{H^{r+1}(\Omega_x)}.$$

Therefore, we have the following optimal error bounds for the terms on the right hand side of (5.33)

$$
\begin{aligned}
\|T_1\|_0^2 &\le C\Big[\tau\int_{t_n}^{t^{n+1}}\|\tilde{u}_{tt}\|_0^2 + h^{2r+2}\|\tilde{u}_t(t^{n+1})\|^2_{L^2(H^{r+1})}\\
&\qquad + k^{2q+2}\left(\|\tilde{u}_t(t^{n+1})\|^2_{H^{q+1}(L^2)} + \|\tilde{u}(t^{n+1})\|^2_{H^{q+1}(H^1)}\right)\Big],\\
\|T_2\|_0^2 &\le C\Big[h^{2r}\|\tilde{u}(t^{n+1})\|^2_{L^2(H^{r+1})} + k^{2q+2}\|\tilde{u}(t^{n+1})\|^2_{H^{q+1}(H^1)}\Big],\\
&\int_{\Omega_\ell}\sum_{K\in\mathcal{T}_h}\delta_K\|T_3\|^2_{L^2(K)}\\
&\le C\Big(\max_{K\in\mathcal{T}_h}\delta_K\Big)\Big[h^{2r}\big(\varepsilon^2h^{-2}+1\big)\|\tilde{u}(t^{n+1})\|^2_{L^2(H^{r+1})} + h^{2r+2}\|\tilde{u}_t(t^{n+1})\|^2_{L^2(H^{r+1})}\\
&\qquad + \tau\int_{t_n}^{t^{n+1}}\|\tilde{u}_{tt}\|_0^2\Big] + \Big(\min_{K\in\mathcal{T}_h}\delta_K\Big)^{-1}h^{2r+2}\|\tilde{u}(t^{n+1})\|^2_{L^2(H^{r+1})}\\
&\quad + C\Big(\max_{K\in\mathcal{T}_h}\delta_K\Big)k^{2q+2}\Big[\big(\varepsilon^2+1\big)\|\tilde{u}(t^{n+1})\|^2_{H^{q+1}(H^2)} + \|\tilde{u}_t(t^{n+1})\|^2_{H^{q+1}(L^2)}\Big],
\end{aligned}
$$

$$
\begin{aligned}
\sum_{i=1}^{M}\int_{I_i}\|T_4\|^2_{L^2(\Omega_x)} &\le C\Big[\tau\int_{t_n}^{t^{n+1}}\|u_{tt}\|_0^2 + h^{2r+2}\Big(\|u_t(t^{n+1})\|^2_{L^2(H^{r+1})} + \|u(t^{n+1})\|^2_{H^1(H^{r+1})}\Big)\\
&\qquad + k^{2q+2}\|u_t(t^{n+1})\|^2_{H^{q+1}(L^2)}\Big],\\
\sum_{i=1}^{M}\int_{I_i}\|T_5\|^2_{L^2(\Omega_x)} &\le Ck^{2q+2}\|u(t^{n+1})\|^2_{H^{q+1}(L^2)},\\
\|T_6\|^2_{L^2(\Omega_x)} &\le Ch^{2r+2}\|z_{\min}(t^{n+1})\|^2_{H^{r+1}(\Omega_x)}.
\end{aligned}
$$

Substituting these bounds in (5.33), applying the triangle inequality and using the interpolation error estimates leads to the following error estimates.

Theorem 5.3.1. *Let $\tilde{u}(t^n)$, $u(t^n)$ and $\tilde{u}^n_{h,k}$, $u^n_{h,k}$ be the solution of (4.2), (4.3) and (5.3), (5.4). Let the stabilization parameter $\delta_K > 0$ satisfies (5.6) and (5.9) for all $K \in \mathcal{T}_h$. Under the regularity Assumption A4 there holds*

$$
\begin{aligned}
&\|u(t^n) - u^n_{h,k}\|_0^2 + \tau\sum_{m=0}^{n-1}\int_{\Omega_\ell}\|\tilde{u}(t^{m+1}) - \tilde{u}^{m+1}_{h,k}\|_{\mathrm{S}}^2 + \tau\sum_{m=0}^{n-1}\|u(t^{m+1}) - u^{m+1}_{h,k}\|^2_{\mathrm{dG}}\\
&\le Ce^{3T/2}\Big[\tau^2 + h^{2r}\big(\varepsilon+\delta+h^2\big) + h^{2r-2}\delta\big(\varepsilon^2+h+h^2\big) + h^{2r+2}\Big(\min_{K\in\mathcal{T}_h}\delta_K\Big)^{-1}\\
&\qquad + k^{2q+1} + \delta k^{2q+2} + \|P_{h,k}z_0 - u^0_{h,k}\|_0^2\Big], \qquad (5.34)
\end{aligned}
$$

where $\delta = \left(\max_{K\in\mathcal{T}_h} \delta_K\right)$ *and* C *is a constant, independent of* ε, h *and* k, *depends only on* u, $\tilde{u}$, u_t, $\tilde{u}_t$, u_{tt} *and* $\tilde{u}_{tt}$.

Using the same procedure as above, we end up with (5.33) with factor e^{4T} on the right hand side. The same analysis as in the proof of Theorem 5.3.1 gives the following error estimates.

Theorem 5.3.2. *Let* $\tilde{u}(t^n)$, $u(t^n)$ *and* $\tilde{u}^n_{h,k}$, $u^n_{h,k}$ *be the solution of* (4.2), (4.3) *and* (5.3), (5.4). *Let the stabilization parameter* $\delta_K > 0$ *satisfies* (5.6) *and* (5.23) *for all* $K \in \mathcal{T}_h$. *Under the regularity Assumption A4 there holds*

$$\begin{aligned}
&\left\|u(t^n) - u^n_{h,k}\right\|_0^2 + \tau\sum_{m=0}^{n-1}\int_{\Omega_\ell}\left\|\tilde{u}(t^{m+1}) - \tilde{u}^{m+1}_{h,k}\right\|_{\mathrm{S}}^2 + \tau\sum_{m=0}^{n-1}\left\|u(t^{m+1}) - u^{m+1}_{h,k}\right\|_{\mathrm{dG}}^2 \\
&\quad\le Ce^{4T}\Bigg[\tau^2 + h^{2r}\left(\varepsilon + \delta + h^2\right) + h^{2r-2}\delta\left(\varepsilon^2 + h + h^2\right) + h^{2r+2}\Big(\min_{K\in\mathcal{T}_h}\delta_K\Big)^{-1} \\
&\qquad + k^{2q+1} + \delta k^{2q+2} + \left\|P_{h,k}z_0 - u^0_{h,k}\right\|_0^2\Bigg],
\end{aligned} \tag{5.35}$$

where $\delta = \left(\max_{K\in\mathcal{T}_h} \delta_K\right)$ *and* C *is a constant, independent of* ε, h *and* k, *depends only on* u, $\tilde{u}$, u_t, $\tilde{u}_t$, u_{tt} *and* $\tilde{u}_{tt}$.

In order to get only one asymptotic order of convergence for the mesh width h and time step length τ in the error estimates (5.34), (5.35) and (4.44), we have to assume that the mesh size k in internal coordinate is small enough. Then, the optimal scaling of mesh width h and time step length τ can be derived from these estimates. Similarly, the optimal scaling for τ and k is obtained by assuming that the mesh width h small.

In SUPG method, the stabilization parameters δ_K depends upon the length of time step, see (5.9) and (5.23). These conditions come from the fact that, in addition to second order derivatives and source term, the time derivative has to be added to the stabilizing term in order to ensures the consistency. This adds a non-symmetric term that can not be easily bounded using standard estimates. Under the assumptions of Lemma 5.2.1 and Corollary 5.2.3, the stabilization parameters are set to $\delta_k = \delta = \tau/4$ and $\delta_K = \sqrt{\tau}h_K/(4\|\mathbf{b}\|_{0,\infty})$, respectively. For fixed mesh width k in internal coordinate, the optimal scaling in convection-dominated regime $\varepsilon \ll h$ is obtained by balancing the terms $\mathcal{O}(\tau)$ and $\mathcal{O}(\tau^{-1/2}h^{r+1})$ in the error estimates (5.34). In the estimate (5.35), the terms $\mathcal{O}(\tau)$ and $\mathcal{O}(\tau^{-1/4}h^{r+1/2})$ have to be balanced to get an optimal scaling.

Similarly, the terms $\mathcal{O}(\tau)$ and $\mathcal{O}(\tau^{1/2}k^{q+1})$ have to be balanced for fixed mesh width h in (5.34) and (5.35). In this case we obtain $\tau = \mathcal{O}(k^{2q+2})$ as optimal choice. For fixed mesh width h, the number of time steps is too large to get the convergence orders for dG(2). Furthermore, the stabilization vanishes in the time-continuous limit.

On the other hand, the LPS method is unconditionally stable, i.e., the stabilization parameters μ_K do not depend on the length of time step. This is because neither time derivatives nor second order derivatives have to be added to the stabilizing terms.

In one-level LPS, the discretization stencil does not increase compared to standard Galerkin or SUPG approach since the approximation and projection spaces live on the same mesh. Although the system looks larger due to the enrichment of the finite element space, the additional degrees of freedom can be eliminated locally by static condensation. In this way, one can work with the same number of degrees of freedom which are needed to achieve the appropriate approximation order.

In convection-dominated regime, the optimal scaling for (4.44) are obtained by balancing the terms $\mathcal{O}(\tau)$ and $\mathcal{O}(h^{r+1/2})$ by assuming k fixed. While keeping h fixed, the terms $\mathcal{O}(\tau)$ and $\mathcal{O}(k^{q+1/2})$ have to be balanced. With this scaling the time step length is larger than in SUPG case and optimal order of convergence in internal coordinate can easily be obtained.

5.4 Numerical studies and comparison

This section presents some numerical results for the Streamline-Upwind Petrov-Galerkin, local projection stabilization and discontinuous Galerkin methods applied to population balance equation. All numerical calculations were performed with the finite element package MooNMD [38].

For the numerical tests, Q_1, Q_2 in SUPG method and for (V_h, D_h) the pairs $(Q_1^{\text{bubble}}, P_0)$ and $(Q_2^{\text{bubble}}, P_1^{\text{disc}})$ in LPS method are used. For discretization in internal coordinate, discontinuous Galerkin method of first and second order are used. The stabilization parameter for the LPS method have been chosen as

$$\mu_K = \mu_0 h_K \qquad \forall K \in \mathcal{T}_h$$

where μ_0 denotes a constant which is specified in the different test calculations. In order to support the theoretical results presented in the previous Sections, the first example is the generalization of an example in [39] and the second one is from [29].

Test example 1: Consider (4.1) with $\Omega_x = (0,1)^2$, $\Omega_\ell = [0,1]$, $T = 1$, $G = 1$, two different choices of ε, $\mathbf{b} = (1, -1)$ and the right-hand side is chosen such that

$$z(t, \ell, x, y) = e^{\sin(2\pi t)} \sin(2\pi\ell) \sin(2\pi x) \sin(2\pi y)$$

is the solution of (4.1). In convection-dominated regime the simulation were performed with $\varepsilon = 10^{-8}$ and in diffusion-dominated regime with $\varepsilon = 1$. In space, uniform quadrilateral grid is used with coarsest grid (level 0) obtained by dividing the unit square into four small quadrilaterals and the initial grid in internal coordinate contains two line segments. The mesh widths h and k are defined by dividing the diameters of the mesh cells by $\sqrt{2}$.

In order to get only one asymptotic order of convergence for the the mesh width h and time step length τ for the error estimates (5.34) and (5.35), one have to assume that the mesh size k in internal coordinate is small enough. Then, the optimal scaling of mesh width h and the time step length τ can be derived from these estimates. Similarly, the optimal scaling for τ and k are obtained by assuming that the mesh width h is small.

The stabilization parameter for the estimates (5.34) under the assumptions of Lemma 5.2.1 is set to $\delta_K = \delta = \tau/4$. In the convection-dominated regime $\varepsilon \ll h$, the terms $\mathcal{O}(\tau)$ and $\mathcal{O}(h^{r+1}\tau^{-1/2})$ have to be balanced to obtain an optimal estimate (5.34). This leads to the scaling $\tau = \mathcal{O}(h^{2(r+1)/3})$, when the mesh size k is very small. In diffusion-dominated regime $h \ll \varepsilon$, the terms $\mathcal{O}(\tau)$, $\mathcal{O}(h^{r-1}\varepsilon\tau^{1/2})$ and $\mathcal{O}(\tau^{-1/2}h^{r+1})$ have to be balanced. This gives the optimal choice of time step length $\tau = \mathcal{O}(h^{2(r+1)/3})$ or $\tau = \mathcal{O}(h^2/\varepsilon)$. On the other hand, the local projection stabilization method is unconditionally stable, i.e., the stabilization parameter μ_K does not depends on the length of the time step. In convection-dominated regime $\varepsilon \ll h$, the terms $\mathcal{O}(\tau)$ and $\mathcal{O}(h^{r+1/2})$ have to be balanced to obtain an optimal estimate (4.44). This leads to the scaling $\tau = \mathcal{O}(h^{r+1/2})$, when the mesh size k assumed to be small enough. The optimal scaling in diffusion-dominated regime is $\tau = \mathcal{O}(h^r)$.

In convection-dominated regime ($\varepsilon \ll h$), the errors and rate of convergence are listed in Tables 5.1 in space for Q_1 and for $(Q_1^{\text{bubble}}, P_0)$ with stabilizing parameter $\mu_0 = 2.5$ and dG(1) in internal coordinate with $k = 1/32$ for the estimates (5.34) and (4.44), respectively. In Table 5.2, the convergence results for Q_2, $(Q_2^{\text{bubble}}, P_1^{\text{disc}})$ and dG(2) in internal coordinate are presented with $k = 1/64$ and stabilizing parameter $\mu_0 = 2.5$. We see that the expected convergence rate $\mathcal{O}(h^{4/3})$, $\mathcal{O}(h^2)$ for the estimates (5.34) and $\mathcal{O}(h^{3/2})$, $\mathcal{O}(h^{5/2})$ for (4.44) can be obtained. In diffusion-dominated regime $h \ll \varepsilon$, the errors and convergence rates are given in Tables 5.3 and 5.4. For $r = 1$, only first order convergence can be expected due to the presence of term $(h^r\varepsilon^{1/2})$. The convergence results in Table 5.3 are calculated in space for Q_1, $(Q_1^{\text{bubble}}, P_0)$ with stabilization parameter $\mu_0 = 2.5$ and dG(1) in internal coordinate on mesh size $k = 1/64$ and in Table 5.4 for Q_2, $(Q_2^{\text{bubble}}, P_1^{\text{disc}})$ and dG(2) in internal coordinate.

Concerning the convergence rate in internal coordinate, the mesh size h is chosen small enough, the stabilizing parameter according to the stability Lemma 5.2.1 are set to $\delta_K = \delta = \tau/4$. In both convection- and diffusion-dominated regime, the terms $\mathcal{O}(\tau)$ and $\mathcal{O}(k^{q+1}\delta^{-1/2})$ in the estimates (5.34), the terms $\mathcal{O}(\tau)$ and $\mathcal{O}(k^{q+1/2})$ in (4.44) have to be balanced. Then the optimal scalings are $\mathcal{O}(\tau) = \mathcal{O}(k^{2(q+1)})$ and $\mathcal{O}(\tau) = \mathcal{O}(k^{q+1/2})$. In both regime the expected convergence orders for dG(1) and dG(2) are of $\mathcal{O}(k^{3/2})$ and $\mathcal{O}(k^{5/2})$ for the error estimates (5.34) and (4.44), respectively. The corresponding results in Tables 5.5 and 5.7 are computed for dG(1), Q_1 and $(Q_1^{\text{bubble}}, P_0)$ and in Tables 5.6 and

Table 5.1: Errors and convergence orders in space for (5.34) and (4.44): $\varepsilon = 10^{-8}$, dG(1) with $k = 1/32$: Q_1 for SUPG, $(Q_1^{\text{bubble}}, P_0)$ for LPS with $\mu_0 = 2.5$.

Level	L^2, Q_1 error	order	SUPG, Q_1 error	order	$L^2, (Q_1^{\text{bubble}}, P_0)$ error	order	LPS, $(Q_1^{\text{bubble}}, P_0)$ error	order
0	1.0364+0		2.7363+0		7.1411-1		3.3569+0	
1	4.9411-1	1.07	9.9410-1	1.46	4.0592-1	0.82	1.1168+0	1.59
2	3.2324-1	0.61	4.7248-1	1.07	1.5716-1	1.37	3.9051-1	1.52
3	1.8328-1	0.82	2.2358-1	1.08	6.0867-2	1.37	1.4058-1	1.48
4	8.9178-2	1.04	1.0071-1	1.15	2.8353-2	1.10	5.3124-2	1.40

Table 5.2: Errors and convergence orders in space for (5.34) and (4.44): $\varepsilon = 10^{-8}$, dG(2) with $k = 1/64$: Q_2 for SUPG, $(Q_2^{\text{bubble}}, P_1^{\text{disc}})$ for LPS with $\mu_0 = 2.5$.

Level	L^2, Q_2 error	order	SUPG, Q_2 error	order	$L^2, (Q_2^{\text{bubble}}, P_1^{\text{disc}})$ error	order	LPS, $(Q_2^{\text{bubble}}, P_1^{\text{disc}})$ error	order
0	5.7266-1		1.3868+0		8.5782-1		4.5615+0	
1	3.6078-1	0.67	5.2445-1	1.40	2.9045-1	2.86	1.6818+0	1.44
2	1.4570-1	1.31	1.6769-1	1.65	6.9940-2	2.06	3.2138-1	2.39
3	4.3851-2	1.73	4.7272-2	1.83	1.4017-2	2.32	5.5968-2	2.52
4	1.1602-2	1.92	1.2314-2	1.94	2.5580-3	2.45	9.3766-3	2.58

5.8 for dG(2), Q_2 and $(Q_2^{\text{bubble}}, P_1^{\text{disc}})$ on level 6 with stabilization parameter $\mu_0 = 2.5$. The results match well with theoretical prediction.

Table 5.3: Errors and convergence orders in space for (5.34) and (4.44): $\varepsilon = 1$, dG(1) with $k = 1/64$: Q_1 for SUPG, $(Q_1^{\text{bubble}}, P_0)$ for LPS with $\mu_0 = 2.5$.

Level	L^2, Q_1 error	order	SUPG, Q_1 error	order	$L^2, (Q_1^{\text{bubble}}, P_0)$ error	order	LPS, $(Q_1^{\text{bubble}}, P_0)$ error	order
2	3.3991-1		2.6562+0		2.5507-1		1.9153+0	——
3	1.5663-1	1.12	1.2543+0	1.08	9.9371-2	1.36	7.2879-1	1.39
4	6.4892-2	1.27	5.6043-1	1.16	3.6532-2	1.44	2.8186-1	1.37
5	2.6657-2	1.28	2.5227-1	1.15	1.4092-2	1.37	1.1382-1	1.31
6	1.1884-2	1.17	1.1603-1	1.12	7.3250-3	0.95	4.9044-2	1.22

According to the stability Lemma 5.2.2, the stabilizing parameters in error estimates (5.35) are set to $\delta_K = \frac{\tau^{1/2} h_K}{4\|\mathbf{b}\|}$. In convection- and diffusion-dominated regime, the terms $\mathcal{O}(\tau)$ and $\mathcal{O}(h^{r+1/2}\tau^{-1/4})$ gives the optimal scaling $\tau = \mathcal{O}(h^{4(r+1/2)/5})$. In Tables 5.9 and 5.10, the errors and convergence orders in space for Q_1 and Q_2 are given. The convergence orders are obtained for Q_1, Q_2 and dG(1), dG(2) with $k = 1/32$, $k = 1/64$,

Table 5.4: Errors and convergence orders in space for (5.34) and (4.44): $\varepsilon = 1$, dG(2) with $k = 1/64$: Q_2 for SUPG, $(Q_2^{\text{bubble}}, P_1^{\text{disc}})$ for LPS with $\mu_0 = 2.5$.

Level	L^2, Q_2		SUPG, Q_2		$L^2, (Q_2^{\text{bubble}}, P_1^{\text{disc}})$		LPS,$(Q_2^{\text{bubble}}, P_1^{\text{disc}})$	
	error	order	error	order	error	order	error	order
0	7.5123-1		6.2969+0		6.9867-1		6.7706+0	
1	3.7996-1	0.98	2.7509+0	1.20	2.1229-1	1.72	1.8473+0	1.87
2	1.1103-1	1.78	7.8478-1	1.81	4.0235-2	2.40	3.2928-1	2.49
3	2.8503-2	1.96	2.0129-1	1.96	7.1660-3	2.49	5.9642-2	2.47
4	7.1778-3	1.99	5.0638-2	1.99	1.2702-3	2.50	1.1430-2	2.38

Table 5.5: Errors and convergence orders in internal coordinate dG(1) for (5.34) and (4.44): $\varepsilon = 10^{-8}$, Q_1 for SUPG, $(Q_1^{\text{bubble}}, P_0)$ for LPS with $\mu_0 = 2.5$ on level 6.

k	L^2, Q_1		SUPG, Q_1		$L^2, (Q_1^{\text{bubble}}, P_0)$		LPS,$(Q_1^{\text{bubble}}, P_0)$	
	error	order	error	order	error	order	error	order
1/2	1.1913+0		1.2111+0		1.0283+0		1.0209+0	
1/4	4.0470-1	1.56	3.9885-1	1.60	4.0804-1	1.33	4.0259-1	1.34
1/8	1.5196-1	1.42	1.5039-1	1.41	1.5248-1	1.42	1.5212-1	1.40

Table 5.6: Errors and convergence orders in internal coordinate dG(2) for (5.34) and (4.44): $\varepsilon = 10^{-8}$, Q_2 for SUPG, $(Q_2^{\text{bubble}}, P_1^{\text{disc}})$ for LPS with $\mu_0 = 2.5$ on level 6.

k	L^2, Q_2		SUPG, Q_2		$L^2, (Q_2^{\text{bubble}}, P_1)$		LPS,$(Q_2^{\text{bubble}}, P_1)$	
	error	order	error	order	error	order	error	order
1/2	1.8438-1		2.0790-1		4.2743-1		4.0093-1	
1/4	8.6508-2	1.09	9.1876-2	1.19	1.4034-1	1.61	1.4273-1	1.49
1/8	2.3084-2	1.91	2.4970-2	1.88	2.8987-2	2.28	3.0517-2	2.23
1/16	6.7152-3	1.78	7.3340-3	1.77	5.2873-3	2.46	5.6203-3	2.44

Table 5.7: Errors and convergence orders in internal coordinate dG(1) for (5.34) and (4.44): $\varepsilon = 1$, Q_1 for SUPG, $(Q_1^{\text{bubble}}, P_0)$ for LPS with $\mu_0 = 2.5$ on level 6.

k	L^2, Q_1		SUPG, Q_1		$L^2, (Q_1^{\text{bubble}}, P_0)$		LPS,$(Q_1^{\text{bubble}}, P_0)$	
	error	order	error	order	error	order	error	order
1/2	1.0294+0		1.1930+0		7.3781-1		2.9389+0	
1/4	1.8345-1	2.49	2.7112-1	2.13	2.3739-1	1.64	6.9701-1	2.08
1/8	6.3522-2	1.53	1.3564-1	1.00	6.9069-2	1.78	1.3693-1	2.35

respectively. From (5.35), the expected rate of convergence for Q_1 and Q_2 are of $\mathcal{O}(h^{6/5})$

Table 5.8: Errors and convergence orders in internal coordinate dG(2) for (5.34) and (4.44): $\varepsilon = 1$, Q_2 for SUPG, $(Q_2^{\text{bubble}}, P_1^{\text{disc}})$ for LPS with $\mu_0 = 2.5$ on level 6.

k	L^2, Q_2		SUPG, Q_2		$L^2, (Q_2^{\text{bubble}}, P_1^{\text{disc}})$		LPS,$(Q_2^{\text{bubble}}, P_1^{\text{disc}})$	
	error	order	error	order	error	order	error	order
1/2	1.9402-1		9.6722-1		4.9673-1		3.1413+0	
1/4	6.1087-2	1.67	2.9147-1	1.73	1.0851-1	2.20	6.5335-1	2.27
1/8	1.5912-2	1.94	8.9782-2	1.70	1.9656-2	2.47	1.1814-1	2.47
1/16	4.4133-3	1.85	2.8066-2	1.68	3.4666-3	2.50	2.1018-2	2.49

and $\mathcal{O}(h^2)$, respectively.

Table 5.9: Errors and rate of convergence in space for the estimates (5.35), $\varepsilon = 10^{-8}$, dG(1) and dG(2) with $k = 1/32$ and $k = 1/64$: Q_1 and Q_2 for SUPG.

Level	L^2, Q_1		SUPG, Q_1	
	error	order	error	order
2	3.8370-1		4.6284-1	
3	2.3993-1	0.68	2.4910-1	0.89
4	1.3123-1	0.87	1.3316-1	0.90
5	6.4660-2	1.02	6.6503-2	1.00
6	3.0378-2	1.09	3.1699-2	1.07

Level	L^2, Q_2		SUPG, Q_2	
	error	order	error	order
0	5.7265-1		1.3868+0	
1	3.6077-1	0.67	5.2443-1	1.40
2	1.4570-1	1.31	1.6769-1	1.65
3	4.3849-2	1.73	4.7270-2	1.83
4	1.1595-2	1.92	1.2308-2	1.94

Table 5.10: Errors and rate of convergence in space for the estimates (5.35), $\varepsilon = 1$, dG(1) and dG(2) with $k = 1/32$ and $k = 1/64$: Q_1 and Q_2 for SUPG.

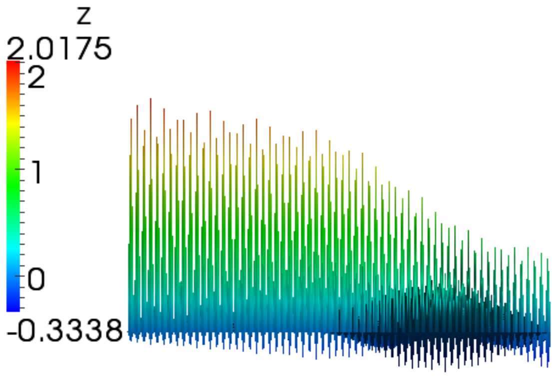
Level	L^2, Q_1		SUPG, Q_1	
	error	order	error	order
3	2.1458-1		1.5424+0	
4	9.9923-2	1.10	7.3057-1	1.08
5	4.4678-2	1.16	3.3605-1	1.12
6	2.0283-2	1.14	1.5437-1	1.12
7	1.0265-2	0.98	7.1537-2	1.11

Level	L^2, Q_2		SUPG, Q_2	
	error	order	error	order
0	7.5120-1		6.2966+0	
1	3.7995-1	0.98	2.7508+0	1.20
2	1.1102-1	1.78	7.8476-1	1.81
3	2.8490-2	2.00	2.0128-1	1.96
4	7.1628-3	2.00	5.0614-2	1.99

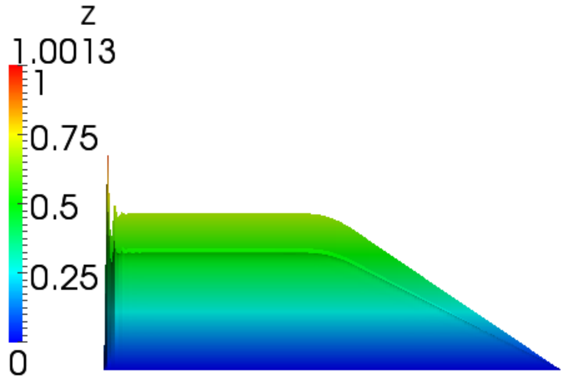
Test example 2: In this example we show numerical experiment for the finite element discretization of population balance equation (4.1). Let us consider (4.1) in the domain $\Omega_x = [0,1] \times [0,1]$ and $\Omega_\ell = [0,1]$ with homogeneous boundary conditions. We take $G = 1$, $\varepsilon = 10^{-8}$, $b_1 = b_2 = 1$, the source term $f = 1$, the initial condition $z_0 = 0$ and the final time $T = 1$.

In the numerical simulation a uniform grid of 16384 quadrilateral cells for the triangulation $\mathcal{T}_h$ and 32 line segments for Ω_ℓ were used. This results in 16 641 degrees of freedom for Q_1 finite elements, 33 025 for Q_1^{bubble} elements including the Dirichlet nodes and 64 degrees of freedom for dG(1). The computational results are obtained with $\delta_K = 0$ for Galerkin finite element method and for SUPG method with $\delta_K = \tau/4$ and $\delta_K = \tau^{1/2}h_K/4$ for all $K \in \mathcal{T}_h$. In LPS method the stabilization parameters are set to $\mu_K = \mu_0 h_K$ for all $K \in \mathcal{T}_h$.

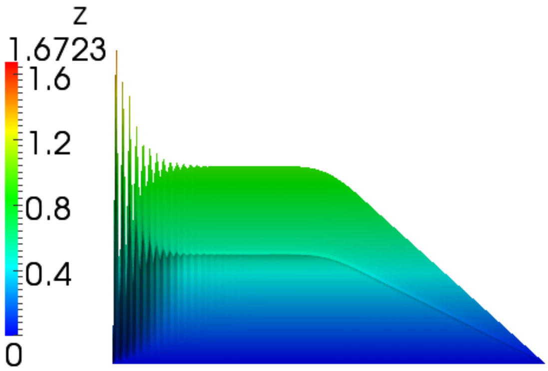
The obtained numerical results for Q_1 and $(Q_1^{\text{bubble}}, P_0)$ with $\mu_0 = 0.1$ at $\ell = 0.0066$ and final time $T = 1$ are plotted in Fig 5.1. To show the effect of the local projection stabilization, the second figure from left to right is generated using only the linear part (Q_1) of the solution. The numerical solution, obtained by two different choice of stabilization parameters δ_K for SUPG method, possess some interior layers. This is because the stabilization effect becomes less for small time steps. In LPS method, the stabilization parameter can be chosen independent of the time step length. Therefore, for suitable choice of stabilization parameter one can remove the unphysical oscillations. Summarizing our numerical studies, one can conclude that the LPS method help to reduce the spurious oscillations in the numerical solution of population balance equation.



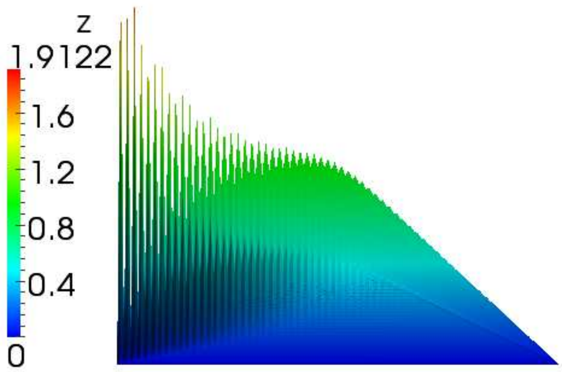
(a) Galerkin finite element method



(b) LPS with $\mu_K = 0.1h_K$



(c) SUPG with $\delta_K = \tau/4$



(d) SUPG with $\delta_K = \sqrt{\tau}h_K/4$

Figure 5.1: Computed solutions for different methods and parameters applied to population balance equation (4.1).

Chapter 6

ADI type methods based on an equivalent one-step formulation

This chapter is concerned with the alternating direction method for the solution of population balance equation (4.1). Alternating direction scheme uses the same principle as the operator splitting method, i.e., it reduces the high dimensional problem into a set of lower ones. At first, LPS and dG methods are used to discretize the whole problem in space and internal coordinate, respectively. Applying backward Euler time stepping method then gives us the fully discrete scheme. The matrices in the fully discrete scheme are tensor product of the space and internal coordinate directions. We discuss the stability and convergence of the method using an equivalent one-step formulation.

This chapter is organized as follows: In Section 6.1 we derive the weak form of the population balance equation. The semi-discretization in space and internal coordinate based on local projection stabilization and discontinuous Galerkin method are introduced. Furthermore, an optimal error estimate for the semi-discretized problem is given. We then introduce Alternating direction Galerkin procedure in Section 6.2 and derive equivalent one-step formulation. In Section 6.3 we derive the stability results and then establish the convergence estimates in Section 6.5.

6.1 Weak and semi-discrete formulation

To derive the weak formulation of problem (4.1), we use the notations and function spaces that were already defined in Chapter 4. Let $z \in L^2(0,T;\mathcal{P})$ and $z_t \in L^2(0,T;\mathcal{P}')$, where $\mathcal{P}'$ is the dual space of $\mathcal{P}$. Then the variational formulation of (4.1) reads:
Find $z \in L^2(0,T;\mathcal{P})$ with $z_t \in L^2(0,T;\mathcal{P}')$ such that for all $v \in \mathcal{P}$

$$\int_{\Omega_\ell} \Big(\frac{\partial z}{\partial t}, v\Big)_x + \mathcal{B}(z,v) = \Big(G_{\min} z_{\min}, v(\ell_{\min})\Big)_x + \int_{\Omega_\ell} (f,v)_x \tag{6.1}$$

with $z(0,\ell,x) = z_0(\ell,x)$. Recalling that

$$\mathcal{P} = \Big\{ v \in L^2(\Omega_\ell \times \Omega_x) : v \in L^2(\Omega_x; W) \cap L^2(\Omega_\ell; V) \Big\}.$$

Here the bilinear form $\mathcal{B}(\cdot,\cdot)$ is given by

$$\mathcal{B}(u,v) = \int_{\Omega_\ell} \Big\{ \Big(\frac{\partial (Gu)}{\partial \ell}, v \Big)_x + a(u,v) \Big\} + \Big(G_{\min} z(\ell_{\min}), v(\ell_{\min}) \Big)_x. \tag{6.2}$$

After discretizing in space and internal coordinate using LPS and dG, the semi-discrete problem is defined as follows: For all $t \in (0,T)$, find $z_{h,k}(t) \in S_{h,k}^{r,q}$ such that for all $X \in S_{h,k}^{r,q}$

$$\int_{\Omega_\ell} \Big(\partial_t z_{h,k}, X \Big)_x + \mathcal{B}_h\big(z_{h,k}, X\big) = \Big(G_{\min} z_{\min,\mathrm{h}}, X(\ell_0^+) \Big)_x + \int_{\Omega_\ell} (f, X)_x, \tag{6.3}$$

where $z_{h,k}(0)$ and $z_{\min,\mathrm{h}}$ are suitable approximations of z_0 and $z_{\min}$, respectively. Here the stabilized bilinear form $\mathcal{B}_h$ is given by

$$\begin{aligned} \mathcal{B}_h(u,v) := & \sum_{i=1}^{N} \int_{I_i} \Big\{ \Big(\frac{\partial (Gu)}{\partial \ell}, v \Big)_x + a_h(u,v) \Big\} + \sum_{i=1}^{N-1} \Big(\big[(Gu)\big]_i, v(\ell_i^+) \Big)_x \\ & + \Big(G_{\min} u(\ell_0^+), v(\ell_0^+) \Big)_x, \end{aligned} \tag{6.4}$$

and the bilinear form a_h is the same as in (4.15). Integration by parts with respect to ℓ gives the second representation of the bilinear form $\mathcal{B}_h$

$$\begin{aligned} \mathcal{B}_h(u,v) := & \sum_{i=1}^{N} \int_{I_i} \Big\{ - \Big(Gu, \frac{\partial v}{\partial \ell} \Big)_x + a_h(u,v) \Big\} - \sum_{i=1}^{N-1} \Big(u(\ell_i^-), \big[(Gv)\big]_i \Big)_x \\ & + \Big(u(\ell_M^-), Gv(\ell_M^-) \Big)_x. \end{aligned} \tag{6.5}$$

Let us introduce the mesh dependent norm

$$\begin{aligned} \|v\|_{\mathrm{DG}} = \Big\{ & \sum_{i=1}^{M} \int_{I_i} \Big(\frac{\partial G}{\partial \ell} \|v\|_{L^2(\Omega_x)}^2 + 2|||v|||^2 \Big) + \sum_{i=1}^{M-1} \big\| \big[(G^{1/2} v)\big]_i \big\|_{L^2(\Omega_x)}^2 \\ & + \big\| G_{\min}^{1/2} v(\ell_0^+) \big\|_{L^2(\Omega_x)}^2 + \big\| G_{\max}^{1/2} v(\ell_M^-) \big\|_{L^2(\Omega_x)}^2 \Big\}^{1/2}, \end{aligned} \tag{6.6}$$

where $|||\cdot|||$ is defined in (4.16).

Lemma 6.1.1. *The bilinear form $\mathcal{B}_h$ is coercive with respect to the mesh dependent norm $\|\cdot\|_{\mathrm{DG}}$, i.e.,*

$$\mathcal{B}_h(v_h, v_h) \geq \frac{1}{2} \|v_h\|_{\mathrm{DG}}^2, \qquad \forall v_h \in S_{h,k}^{r,q}. \tag{6.7}$$

Proof. The statement of the lemma follows by adding the two representations of $\mathcal{B}_h$, setting $u_h = v_h$, and using the coercivity of a_h with respect to the triple norm. □

6.2 Semidiscrete error estimates

Let $z(t)$ be the solution of the continuous problem (6.1) and $z_{h,k}(t)$ be the solution of semidiscrete problem (6.3). As in Section 4.4, we define

$$z(t) - z_{h,k}(t) = z(t) - P_{h,k}z(t) + P_{h,k}z(t) - z_{h,k}(t) = \eta(t) + \xi(t), \tag{6.8}$$

where $\eta := z - P_{h,k}z$, $\xi := P_{h,k}z - z_{h,k}$. As in (4.31), the projection operator $P_{h,k}$ is defined for sufficiently smooth function w by

$$P_{h,k}w = j_h \Pi_k w = \Pi_k j_h w.$$

Furthermore, the separate errors in space and in internal coordinate are decomposed as follows

$$P_{h,k}z(t) - z(t) = P_{h,k}z(t) - \Pi_k z(t) + \Pi_k z(t) - z(t) = \vartheta(t) + \varphi(t) \tag{6.9}$$

where $\vartheta := P_{h,k}z - \Pi_k z$ and $\varphi := \Pi_k z - z$.

Theorem 6.2.1. *Suppose the data of the problem be sufficiently smooth. Let assumptions A1-A3 defined in Chapter 2 be fulfilled. If $\mu_K \sim h_K$ for all $K \in \mathcal{T}_h$, then there exists a constant C, independent of ε and h, such that*

$$|||j_h w - w||| \le C\,(\varepsilon^{1/2} + h^{1/2})\,h^r \|w\|_{H^{r+1}(\Omega_x)} \tag{6.10}$$

for all $w \in H_0^1(\Omega_x) \cap H^{r+1}(\Omega_x)$.

Proof. For proof see [76, Theorem 3.74]. $\square$

The next Lemma states the approximation results based on the interpolation error estimates (2.6) and (4.27)-(4.30).

Lemma 6.2.2. *Suppose A1-A4, if $\mu_K \sim h_K$ for all $K \in \mathcal{T}_h$, then for all $t \in (0,T]$ we have the following estimates for the interpolation errors*

$$\left\|P_{h,k}z(t) - \Pi_k z(t)\right\|_{\mathrm{DG}} \le C(\varepsilon^{1/2} + h^{1/2})h^r \left\{ \|z(t)\|_{L^2(H^{r+1})} + \|z(t)\|_{C(H^{r+1})} \right\}, \tag{6.11}$$

$$\left\|\Pi_k z(t) - z(t)\right\|_{\mathrm{DG}} \le C\,k^{q+1/2}\,\|z(t)\|_{H^{q+1}(H^1)}. \tag{6.12}$$

Proof. For the sake of simplicity, we drop the dependency over t within the proof. From (6.6), we have for $\vartheta = P_{h,k}z - \Pi_k z = \Pi_k(j_h z - z)$

$$\begin{aligned}\frac{1}{2}\|\vartheta\|_{\mathrm{DG}}^2 \le \sum_{i=1}^{M} \int_{I_i} \left(\frac{\partial G}{\partial \ell} \|\vartheta\|_{L^2(\Omega_x)}^2 + |||\vartheta|||^2 \right) + \sum_{i=1}^{M-1} \|\left[(G^{1/2}\vartheta)\right]_i\|_{L^2(\Omega_x)}^2 \\ + \left\|G_{\min}^{1/2}\vartheta(\ell_0^+)\right\|_{L^2(\Omega_x)}^2 + \left\|G_{\max}^{1/2}\vartheta(\ell_M^-)\right\|_{L^2(\Omega_x)}^2\end{aligned}$$

Note that the interpolation $j_h z$ is continuous in ℓ-direction, thus the jump terms $[j_h z - z]_i$ vanish for $i = 1, \ldots, M-1$. Hence

$$\begin{aligned}\frac{1}{2}\|\vartheta\|_{\mathrm{DG}}^2 &= \sum_{i=1}^{M} \int_{I_i} \left(\frac{\partial G}{\partial \ell} \|\vartheta\|_{L^2(\Omega_x)}^2 + 2|||\vartheta|||^2 \right) + \left\| G_{\min}^{1/2} \vartheta(\ell_0^+) \right\|_{L^2(\Omega_x)}^2 + \left\| G_{\max}^{1/2} \vartheta(\ell_M^-) \right\|_{L^2(\Omega_x)}^2 \\ &\le \left\| P_{h,k} z - z \right\|_{\mathrm{dG}}^2 + \sum_{i=1}^{M} \int_{I_i} \left|\left|\left| P_{h,k} z - z \right|\right|\right|^2.\end{aligned}$$

We conclude the estimate (6.11) by using the results (4.34), (6.10) and condition (4.32)

$$\begin{aligned}\|P_{h,k} u - \Pi_k u\|_{\mathrm{DG}} &\le C(\varepsilon^{1/2} + h^{1/2}) h^r \Bigg\{ \int_{\Omega_\ell} \left(h^{1/2} \|u\|_{H^{r+1}(\Omega_x)} + \|u\|_{H^{r+1}(\Omega_x)} \right) \\ &\qquad + \|z(\ell_0^+)\|_{H^{r+1}(\Omega_x)} + \|z(\ell_M^-)\|_{H^{r+1}(\Omega_x)} \Bigg\} \\ &\le C(\varepsilon^{1/2} + h^{1/2}) h^r \Big\{ \|z\|_{L^2(H^{r+1})} + \|z\|_{C(H^{r+1})} \Big\}.\end{aligned}$$

From the second representation (6.5) of the bilinear form $\mathcal{B}_h$ and using $\Pi_k z(\ell_i^-) = z(\ell_i^-)$, $i = 1, \ldots, M-1$, we have for $\varphi = \Pi_k z - z$

$$\begin{aligned}\frac{1}{2}\|\varphi\|_{\mathrm{DG}}^2 \le \mathcal{B}_h(\varphi, \varphi) &= \sum_{i=1}^{N} \int_{I_i} \left\{ -\left(G\varphi, \frac{\partial \varphi}{\partial \ell} \right)_x + a_h(\varphi, \varphi) \right\} - \sum_{i=1}^{N-1} \left(\varphi(\ell_i^-), \left[(G\varphi) \right]_i \right)_x \\ &\quad + \left(\varphi(\ell_M^-), G_{\max} \varphi(\ell_M^-) \right)_x = \|\varphi\|_{\mathrm{dG}}^2 + \sum_{i=1}^{N} \int_{I_i} |||\varphi|||^2.\end{aligned}$$

Then, the L^2-stability of the fluctuation operator κ_h and the parameter choice $\mu_K \sim h_K$ gives for the second term on the right-hand side

$$\begin{aligned}|||\varphi|||^2 &= \varepsilon \left\| \nabla_x \varphi \right\|_{L^2(\Omega_x)}^2 + \sum_{K \in \mathcal{T}_h} \mu_K \left\| \kappa_h \nabla_x \varphi \right\|_{L^2(K)}^2 \\ &\le C\,(\varepsilon + h) \left\| \nabla_x \varphi \right\|_{L^2(\Omega_x)}^2.\end{aligned}$$

Incorporating this bound in the above equation, we conclude the second statement of lemma using (4.30) and (4.35)

$$\begin{aligned}\|\varphi\|_{\mathrm{DG}}^2 &= \|\varphi\|_{\mathrm{dG}}^2 + (\varepsilon + h) \sum_{i=1}^{M} \int_{I_i} \|\varphi\|_{H^1(\Omega_x)}^2 \\ &\le C\, k^{2q+1} \|z\|_{H^{q+1}(L^2)}^2 + C\,(\varepsilon + h)\, k^{2q+2} \int_{I_i} \left\| z^{(q+1)} \right\|_{H^1(\Omega_x)}^2 \\ &\le C\,(\varepsilon k + hk + 1)\, k^{2q+1} \, \|z\|_{H^{q+1}(H^1)}^2.\end{aligned}$$

This completes the proof. $\square$

Lemma 6.2.3. *Suppose A1-A4 and $\mu_K \sim h_K$ for all $K \in \mathcal{T}_h$. For the solution $z_{h,k}(t)$ of the semidiscrete problem (6.3) the following estimates hold true for all $t \in [0,T]$*

$$\mathcal{B}_h\Big((P_{h,k}z - \Pi_k z)(t), \xi(t)\Big) \le C\, h^{r+1}\Big[\|z(t)\|_{H^1(H^{r+1})} + \|z(t)\|_{L^2(H^{r+1})}\Big]\, \|\xi(t)\|_0 \\ + C\, h^r\Big[(\varepsilon^{1/2} + h^{1/2})\|z(t)\|_{L^2(H^{r+1})} + h\|z(t)\|_{C(H^{r+1})}\Big]\, \|\xi(t)\|_{\mathrm{DG}}, \tag{6.13}$$

$$\mathcal{B}_h\Big((\Pi_k z - z)(t), \xi(t)\Big) \le C\, k^{q+1}\Big[\|z(t)\|_{H^{q+1}(H^1)} + \|z(t)\|_{H^{q+1}(L^2)}\Big]\|\xi(t)\|_0 \\ + C\,(\varepsilon^{1/2} + h^{1/2})\, k^{q+1}\|z(t)\|_{H^{q+1}(H^1)}\, \|\xi(t)\|_{\mathrm{DG}}. \tag{6.14}$$

Proof. For notation simplicity, we again drop the dependency of t. From (6.4), we have for $\vartheta = P_{h,k}z - \Pi_k z$

$$\begin{aligned}\mathcal{B}_h(\vartheta, \xi) &= \sum_{i=1}^{N}\int_{I_i}\Big\{\Big(\frac{\partial(G\vartheta)}{\partial \ell}, \xi\Big)_x + a_h(\vartheta, \xi)\Big\} + \sum_{i=1}^{N-1}\Big(\big[(G\vartheta)\big]_i, \xi(\ell_i^+)\Big)_x \\ &\quad + \Big(G_{\min}\vartheta(\ell_0^+), \xi(\ell_0^+)\Big)_x \\ &= B\big(\vartheta, \xi\big) + \sum_{i=1}^{M}\int_{I_i} a_h(\vartheta, \xi).\end{aligned}$$

Hence, from (4.36) and (4.38), we get the first statement of the lemma

$$\mathcal{B}_h(\vartheta, \xi) \le C\, h^{r+1}\Big[\|z\|_{H^1(H^{r+1})} + \|z\|_{L^2(H^{r+1})}\Big]\, \|\xi\|_0 \\ + C\, h^r\Big[(\varepsilon^{1/2} + h^{1/2})\|z\|_{L^2(H^{r+1})} + h\|z\|_{C(H^{r+1})}\Big]\, \|\xi\|_{\mathrm{DG}}.$$

Similarly from the second representation (6.5) of bilinear form $\mathcal{B}_h$, we have for $\varphi = \Pi_k z - z$

$$\begin{aligned}\mathcal{B}_h(\varphi, \xi) &= \sum_{i=1}^{N}\int_{I_i}\Big[-\Big(G\varphi, \frac{\partial \xi}{\partial \ell}\Big)_x + a_h(\varphi, \xi)\Big] - \sum_{i=1}^{N-1}\Big(\varphi(\ell_i^-), \big[(G\xi)\big]_i\Big)_x + \Big(\varphi(\ell_M^-), G_{\max}\xi(\ell_M^-)\Big)_x \\ &= B\big(\varphi, \xi\big) + \sum_{i=1}^{M}\int_{I_i} a_h(\varphi, \xi).\end{aligned}$$

Hence, the second statement follows from (4.37) and (4.39)

$$\mathcal{B}_h(\varphi, \xi) \le C\,(\varepsilon^{1/2} + h^{1/2})\, k^{q+1}\|z\|_{H^{q+1}(H^1)}\, \|\xi\|_{\mathrm{DG}} \\ + C\, k^{q+1}\Big\{\|z\|_{H^{q+1}(H^1)} + \|z\|_{H^{q+1}(L^2)}\Big\}\|\xi\|_0.$$

□

The next theorem states the main results of this section.

Theorem 6.2.4. *Let $z(t)$ and $z_{h,k}(t)$ be the solution of the continuous problem* (6.1) *and the semi-discrete problem* (6.3)*, respectively. If $\mu_K \sim h_K$, for all $K \in \mathcal{T}_h$, then there exists a positive constant C independent of t, ε, h, and k such that for all $t \in [0,T]$*

$$
\begin{aligned}
&\big\|z(t) - z_{h,k}(t)\big\|_0^2 + \frac{1}{2}\int_0^t \|z(t) - z_{h,k}(t)\|_{\mathrm{DG}}^2 \le Ce^t\Bigg[\big\|P_{h,k}z_0 - z_{h,k}(0)\big\|_0^2 + \\
&+ (\varepsilon + h)h^{2r}\bigg\{\|z_0\|_{L^2(H^{r+1})}^2 + \int_0^t \Big(\big\|z_t(s)\big\|_{L^2(H^{r+1})}^2 + \big\|z(s)\big\|_{H^1(H^{r+1})}^2 + \big\|z_{\min}(s)\big\|_{r+1}^2\Big)\bigg\}\Bigg] \\
&+ Ce^t k^{2q+1}\Bigg[\big\|z_0\big\|_{H^{q+1}(L^2)}^2 + \int_0^t \Big(\big\|z(s)\big\|_{H^{q+1}(H^1)}^2 + \big\|z_t(s)\big\|_{H^{q+1}(L^2)}^2\Big)\Bigg]
\end{aligned}
$$

Proof. Since $\xi \in S_{h,k}^{r,q}$, we apply (6.3) to $\xi = P_{h,k}z - z_{h,k}$, using (6.1) and the fact that the projection operator $P_{h,k}$ commutes with the time derivative to get

$$
\begin{aligned}
&\int_{\Omega_\ell} \big(\xi_t, v_h\big)_x + \mathcal{B}_h\big(\xi, v_h\big) \\
&\qquad = \int_{\Omega_\ell} \big((P_{h,k}z)_t - \partial_t z_{h,k}, v_h\big)_x + \mathcal{B}_h\big(P_{h,k}z - z_{h,k}, v_h\big) \\
&\qquad = \int_{\Omega_\ell} \Big(P_{h,k}z_t, v_h\Big)_x + \mathcal{B}_h\big(P_{h,k}z, v_h\big) - \Big(G_{\min}z_{\mathrm{min,h}}, v_h(\ell_0^+)\Big)_x - \int_{\Omega_\ell} (f, v_h)_x \\
&\qquad = \int_{\Omega_\ell} \Big(P_{h,k}z_t - z_t, v_h\Big)_x + \mathcal{B}_h\big(P_{h,k}z - z, v_h\big) + \int_{\Omega_\ell} S_h(z, v_h) \\
&\qquad\quad + \Big(G_{\min}(z_{\min} - z_{\mathrm{min,h}}), v_h(\ell_0^+)\Big)_x.
\end{aligned}
$$

Setting $v_h = \xi$, and using (6.7), we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\xi\|_0^2 + \frac{1}{2}\|\xi\|_{\mathrm{DG}}^2 &\le \int_{\Omega_\ell} \Big(P_{h,k}z_t - z_t, \xi\Big)_x + \mathcal{B}_h\Big(P_{h,k}z - z, \xi\Big) + \int_{\Omega_\ell} S_h(z, \xi) \\
&\quad + \Big(G_{\min}(z_{\min} - z_{\mathrm{min,h}}), \xi(\ell_0^+)\Big)_x \\
&= I_1 + I_2 + I_3 + I_4. \qquad (6.15)
\end{aligned}
$$

In the next paragraphs we analyze the terms I_i, $i = 1, \dots, 4$. Using Cauchy-Schwarz inequality, splitting (6.9), interpolation error estimates (2.6), (4.30) and condition (4.32), we get for I_1

$$
\begin{aligned}
|I_1| &= \Big| \int_{\Omega_\ell} \Big(P_{h,k} z_t - z_t, \xi \Big)_x \Big| \le \int_{\Omega_\ell} \| P_{h,k} z_t - z_t \|_{L^2(\Omega_x)} \|\xi\|_{L^2(\Omega_x)} \\
&\le \int_{\Omega_\ell} \Big\{ \| P_{h,k} z_t - \Pi_k z_t \|_{L^2(\Omega_x)} + \| \Pi_k z_t - z_t \|_{L^2(\Omega_x)} \Big\} \|\xi\|_{L^2(\Omega_x)} \\
&\le C\, h^{r+1} \int_{\Omega_\ell} \| z_t \|_{H^{r+1}(\Omega_x)} \|\xi\|_{L^2(\Omega_x)} + C\, k^{q+1} \int_{\Omega_x} \| z_t \|_{H^{q+1}(\Omega_\ell)} \|\xi\|_{L^2(\Omega_\ell)} \\
&\le C \Big\{ h^{r+1} \| z_t \|_{L^2(H^{r+1})} + k^{q+1} \, \| z_t \|_{H^{q+1}(L^2)} \Big\} \, \|\xi\|_0 .
\end{aligned}
$$

For I_2, the error decomposition (6.9) and the results of Lemma 6.2.3 yield

$$
\begin{aligned}
|I_2| &= \Big| \mathcal{B}_h\big(P_{h,k} z - z, \xi \big) \Big| = \Big| \mathcal{B}_h\big(P_{h,k} z - \Pi_k z, \xi \big) + \mathcal{B}_h\big(\Pi_k z - z, \xi \big) \Big| \\
&\le C \Big\{ h^{r+1} \Big(\|z\|_{H^1(H^{r+1})} + \|z\|_{L^2(H^{r+1})} \Big) + k^{q+1} \Big(\|z\|_{H^{q+1}(H^1)} + \|z\|_{H^{q+1}(L^2)} \Big) \Big\} \, \|\xi\|_0 \\
&\quad + C \Big\{ h^r \, \Big((\varepsilon^{1/2} + h^{1/2}) \|z\|_{L^2(H^{r+1})} + h \, \|z\|_{C(H^{r+1})} \Big) \\
&\qquad\qquad + (\varepsilon^{1/2} + h^{1/2}) \, k^{q+1} \|z\|_{H^{q+1}(H^1)} \Big\} \, \|\xi\|_{\mathrm{DG}} .
\end{aligned}
$$

The approximation properties of the fluctuation operator κ_h give for I_3

$$
\begin{aligned}
|I_3| &= \Big| \int_{\Omega_\ell} S_h(z, \xi) \Big| \le \int_{\Omega_\ell} S_h\big(z, z\big)^{1/2} \, S_h\big(\xi, \xi\big)^{1/2} \le C h^{r+1/2} \int_{\Omega_\ell} \|z\|_{H^{r+1}(\Omega_x)} |||\xi||| \\
&\le C\, h^{r+1/2} \|u\|_{L^2(H^{r+1})} \, \|\xi\|_{\mathrm{DG}} .
\end{aligned}
$$

Applying the Cauchy-Schwarz inequality and using the approximation results, we get for I_4

$$
\begin{aligned}
|I_4| &= \Big| \Big(G_{\min} \big(z_{\min} - z_{\mathrm{min,h}} \big), \xi(\ell_0^+) \Big)_x \Big| \le \big\| G_{\min}^{1/2} \big(z_{\min} - z_{\mathrm{min,h}} \big) \big\|_{L^2(\Omega_x)} \big\| G_{\min}^{1/2} \xi(\ell_0^+) \big\|_{L^2(\Omega_x)} \\
&\le C \; h^{r+1} \, \big\| z_{\min} \big\|_{H^{r+1}(\Omega_x)} \|\xi\|_{\mathrm{DG}} .
\end{aligned}
$$

Combining I_1, I_2, I_3 and I_4 in (6.15), we get

$$
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\xi\|_0^2 + \frac{1}{2} \|\xi\|_{\mathrm{DG}}^2 \le C \Big\{ & h^{r+1} \Big(\|z_t\|_{L^2(H^{r+1})} + \|z\|_{H^1(H^{r+1})} + \|z\|_{L^2(H^{r+1})} + \big\| z_{\min} \big\|_{H^{r+1}(\Omega_x)} \Big) \\
& + k^{q+1} \Big(\|z_t\|_{H^{q+1}(L^2)} + \|z\|_{H^{q+1}(H^1)} + \|z\|_{H^{q+1}(L^2)} \Big) \Big\} \, \|\xi\|_0 \\
& + C \Big\{ h^r \, \Big((\varepsilon^{1/2} + h^{1/2}) \|z\|_{L^2(H^{r+1})} + h \, \big\|z\big\|_{C(H^{r+1})} \Big) \\
& \qquad + (\varepsilon^{1/2} + h^{1/2}) \, k^{q+1} \|z\|_{H^{q+1}(H^1)} \Big\} \, \|\xi\|_{\mathrm{DG}} .
\end{aligned}
$$

Young's inequality and integration over 0 to T gives

$$
\begin{aligned}
\|\xi\|_0^2 + \frac{1}{2}\int_0^T \|\xi\|_{\mathrm{DG}}^2 \\
\leq C\int_0^T \Big\{ & h^{2r+2}\Big(\|z_t\|^2_{L^2(H^{r+1})} + \|z\|^2_{H^1(H^{r+1})} + \|z\|^2_{L^2(H^{r+1})} + \big\|z_{\min}\big\|_{H^{r+1}(\Omega_x)}\Big) \\
& + k^{2q+2}\Big(\|z_t\|^2_{H^{q+1}(L^2)} + \|z\|^2_{H^{q+1}(H^1)} + \|z\|^2_{H^{q+1}(L^2)}\Big)\Big\} \\
+ C\int_0^T \Big\{ & h^{2r}\left((\varepsilon+h)\|z\|^2_{L^2(H^{r+1})} + h^2\,\|z\|^2_{C(H^{r+1})}\right) \\
& + (\varepsilon+h)\,k^{2q+2}\|z\|^2_{H^{q+1}(H^1)}\Big\} + \big\|\xi(0)\big\|_0^2 + \int_0^T \|\xi\|_0^2.
\end{aligned}
$$

Applying continuous version of Gronwall's Lemma 2.3.3, we get

$$
\begin{aligned}
\big\|\xi(t)\big\|_0^2 + \frac{1}{2}\int_0^t \|\xi\|_{\mathrm{DG}}^2 \leq e^t\Bigg[& \big\|\xi(0)\big\|_0^2 + C\big(\varepsilon+h\big)h^{2r}\int_0^t \|z(s)\|^2_{H^1(H^{r+1})} \\
& + Ch^{2r+2}\int_0^t \left(\|z_t(s)\|^2_{L^2(H^{r+1})} + \|z\|^2_{C(H^{r+1})} + \big\|z_{\min}(s)\big\|^2_{H^{r+1}}\right) \\
& + Ck^{2q+2}\int_0^t \left((\varepsilon+h+1)\,\|z(s)\|^2_{H^{q+1}(H^1)} + \|z_t(s)\|^2_{H^{q+1}(L^2)}\right)\Bigg].
\end{aligned}
$$

The statement follows by using triangle inequality and interpolation error estimates. □

6.3 Fully discrete problem

In this section, we give a fully discrete scheme. We start with discussing a time discretization of (6.3) using backward Euler scheme. Equivalent one-step method is then shown for the two-step of alternating direction method. Then we give some useful properties which are used in the stability and error estimates in the upcoming sections.

Let $N > 0$ be a given positive integer. We consider a uniform partition of time interval $[0, T]$ with time step size $\tau = T/N$. Further, let $z^n_{h,k} \in S^{r,q}_{h,k}$ be the approximation of $z(t^n)$. Then the backward-Euler time discretization of (6.3) reads as follows:
For given $z^n_{h,k}$, find $z^{n+1}_{h,k} \in S^{r,q}_{h,k}$, for $n = 0, \cdots, N-1$ such that

$$
\int_{\Omega_\ell}\big(\partial_\tau z^{n+1}_{h,k}, X\big)_x + \mathcal{B}_h\big(z^{n+1}_{h,k}, X\big) = \Big(G_{\min} z^{n+1}_{\mathrm{min,h}}, X(\ell_0^+)\Big)_x + \int_{\Omega_\ell}(f^{n+1}, X)_x \tag{6.16}
$$

for all $X \in S_{h,k}^{r,q}$, where $\partial_\tau z_{h,k}^{n+1} = \tau^{-1}(z_{h,k}^{n+1} - z_{h,k}^{n})$.

For deriving the alternating direction scheme, we use the bases functions and matrices defined in Section 4.5. The fully discrete scheme (6.16) in algebraic form can be expressed as follows: Let

$$z_{h,k}^n = \sum_{i=1}^{N_x}\sum_{s=1}^{N_\ell} \xi_{is}^n \phi_i(x)\psi_s(\ell) \in S_{h,k}^{r,q},$$

where

$$\boldsymbol{\xi}^n := \{\xi_{11}^n, \dots, \xi_{1N_\ell}^n, \dots, \xi_{N_x N_\ell}^n\}^T \in \mathbb{R}^{N_x \times N_\ell}.$$

Find $\boldsymbol{\xi}^{n+1} \in \mathbb{R}^{N_x \times N_\ell}$, such that

$$\begin{aligned}(M_x \otimes M_\ell)\Big(\frac{\boldsymbol{\xi}^{n+1} - \boldsymbol{\xi}^n}{\tau}\Big) &+ (M_x \otimes T_\ell)\,\boldsymbol{\xi}^{n+1} \\ &+ \big((T_x + D_x + S_x) \otimes M_\ell\big)\boldsymbol{\xi}^{n+1} = M_x \boldsymbol{\xi}_{\min}^{n+1} \otimes I_x + F^{n+1}, \qquad (6.17)\end{aligned}$$

where the tensor product of matrices is defined as follows

$$M_x \otimes M_\ell = \begin{pmatrix} m_{11}M_\ell & m_{12}M_\ell & \cdots & m_{1n}M_\ell \\ m_{21}M_\ell & m_{22}M_\ell & \cdots & m_{2n}M_\ell \\ \vdots & \vdots & \ddots & \vdots \\ m_{m1}M_\ell & m_{m2}M_\ell & \cdots & m_{mn}M_\ell \end{pmatrix}.$$

Since the matrices in (6.17) are tensor product of the x- and ℓ-direction discretization matrices, we can approximate (6.17) using the following two-step method

$$\Big\{M_x \otimes M_\ell + \tau\Big(D_x + T_x + S_x\Big) \otimes M_\ell\Big\}\tilde{\boldsymbol{\xi}}^{n+1} = (M_x \otimes M_\ell)\,\boldsymbol{\xi}^n + \tau F^{n+1} \qquad (6.18)$$

$$M_x \otimes \Big(M_\ell + T_\ell\Big)\boldsymbol{\xi}^{n+1} = (M_x \otimes M_\ell)\,\tilde{\boldsymbol{\xi}}^{n+1} + M_x \boldsymbol{\xi}_{\min}^{n+1} \otimes I_x. \qquad (6.19)$$

These two equations define the fully discrete alternating direction Galerkin formulation of problem (4.1). Note that, if we write (4.18) and (4.19) in algebraic form we get the (6.18) and (6.19), respectively. The next step is to derive the one-step formulation.

Lemma 6.3.1. *The two-step method* (6.18) *and* (6.19) *is equivalent to the following fully discrete one-step formulation:*

Given $z_{h,k}^n \in S_{h,k}^{r,q}$, for each $n = 0, \cdots, N-1$, find $z_{h,k}^{n+1} \in S_{h,k}^{r,q}$ satisfying

$$\begin{aligned}\int_{\Omega_\ell}\Big(\partial_\tau z_{h,k}^{n+1}, X\Big)_x &+ \mathcal{B}_h\Big(z_{h,k}^{n+1}, X\Big) + \tau K\Big(z_{h,k}^{n+1}, X\Big) \\ = \Big(G_{\min} z_{\min,\mathrm{h}}^{n+1}, X(\ell_0^+)\Big)_x &+ \int_{\Omega_\ell}\Big(f^{n+1}, X\Big)_x + \tau a_h\Big(G_{\min} z_{\min,\mathrm{h}}^{n+1}, X(\ell_0^+)\Big) \qquad (6.20)\end{aligned}$$

for all $X \in S_{h,k}^{r,q}$, *where* $K = K_1 + K_2 + K_3$ *with*

$$K_1(u,v) = \varepsilon \sum_{i=1}^{M} \int_{I_i} \Big(\partial_\ell(G\nabla_x u), \nabla_x v\Big)_x + \varepsilon \sum_{i=1}^{M-1} \Big(\nabla_x\big[(Gu)\big]_i, \nabla_x v(\ell_i^+)\Big)_x + \varepsilon\Big(G_{\min}\nabla_x u(\ell_0^+), \nabla_x v(\ell_0^+)\Big)_x, \tag{6.21}$$

$$K_2(u,v) = \sum_{i=1}^{M} \int_{I_i} \Big(\partial_\ell(G\mathbf{b}\cdot\nabla_x u), v\Big)_x + \sum_{i=1}^{M-1} \Big(\mathbf{b}\cdot\nabla_x\big[(Gu)\big]_i, v(\ell_0^+)\Big)_x + \Big(G_{\min}\mathbf{b}\cdot\nabla_x u(\ell_0^+), v(\ell_0^+)\Big)_x, \tag{6.22}$$

$$K_3(u,v) = \sum_{i=1}^{M} \int_{I_i} S_h\Big(\partial_\ell(Gu), v\Big) + \sum_{i=1}^{M-1} S_h\Big(\big[(Gu)\big]_i, v(\ell_i^+)\Big) + S_h\Big(G_{\min}u(\ell_0^+), v(\ell_0^+)\Big), \tag{6.23}$$

Proof. To obtain the one-step fully discrete formulation, we multiply (6.19) by $\mathbb{I}_\ell \otimes (M_x + \tau D_x + \tau T_x + \tau S_x)M_x^{-1}$ and get

$$\begin{aligned}
&\Big\{(M_x\otimes M_\ell) + \tau\Big((M_x\otimes T_\ell) + (D_x\otimes M_\ell) + (T_x\otimes M_\ell) + (S_x\otimes M_\ell)\Big) \\
&\qquad + \tau^2\Big((D_x\otimes T_\ell) + (T_x\otimes T_\ell) + (S_x\otimes T_\ell)\Big)\Big\}\boldsymbol{\xi}^{n+1} \\
&\qquad = \Big\{(M_x\otimes M_\ell) + \tau\Big((D_x\otimes M_\ell) + (T_x\otimes M_\ell) + (S_x\otimes M_\ell)\Big)\Big\}\tilde{\boldsymbol{\xi}}^{n+1} \\
&\qquad\quad + \tau M_x\boldsymbol{\xi}_{\min}^{n+1}\otimes\mathbb{I}_x + \tau^2\Big(D_x + T_x + S_x\Big)\boldsymbol{\xi}_{\min}^{n+1}\otimes\mathbb{I}_x.
\end{aligned} \tag{6.24}$$

Equating the left-hand side of (6.18) with the right-hand side of (6.24) we obtain

$$\begin{aligned}
&\Big\{(M_x\otimes M_\ell) + \tau\Big((M_x\otimes T_\ell) + (D_x\otimes M_\ell) + (T_x\otimes M_\ell) + (S_x\otimes M_\ell)\Big) \\
&\qquad + \tau^2\Big((D_x\otimes T_\ell) + (T_x\otimes T_\ell) + (S_x\otimes T_\ell)\Big)\Big\}\boldsymbol{\xi}^{n+1} \\
&\qquad = (M_x\otimes M_\ell)\boldsymbol{\xi}^n + \tau M_x\boldsymbol{\xi}_{\min}^{n+1}\otimes\mathbb{I}_x + \tau^2\Big(D_x + T_x + S_x\Big)\boldsymbol{\xi}_{\min}^{n+1}\otimes\mathbb{I}_x + \tau F^{n+1}.
\end{aligned} \tag{6.25}$$

Let us describe separately the cross terms in (6.25)

$$
\begin{aligned}
&\big(D_x \otimes T_\ell\big)\boldsymbol{\xi}^n \\
&\quad = \xi^n_{i,s}\varepsilon\Big(\nabla_x\phi_i(x), \nabla_x\phi_j(x)\Big)_x \Bigg\{ \sum_{\nu=1}^{M} \int_{I_\nu} \frac{\partial}{\partial \ell}\big(G\psi_s(\ell)\big)\psi_p(\ell) + \sum_{\nu=1}^{M-1} [G\psi_s]_\nu \psi_p(\ell_i^+) \\
&\qquad + G_{\min}\psi_s(\ell_0^+)\psi_p(\ell_0^+)\Bigg\} \\
&\quad = \sum_{\nu=1}^{M} \varepsilon \int_{I_\nu} \xi^n_{i,s}\left(\nabla_x \frac{\partial}{\partial \ell}\big(G\phi_i(x)\psi_s(\ell)\big), \nabla_x\big(\phi_j(x)\psi_p(\ell)\big)\right)_x \\
&\qquad + \varepsilon\xi^n_{i,k}\sum_{\nu=1}^{M-1}\Big(\nabla_x[G\phi_i(x)\psi_s(\ell)]_\nu, \nabla_x(\phi_j(x)\psi_p(\ell_\nu^+))\Big)_x \\
&\qquad + \varepsilon\xi^n_{i,s}\Big(\nabla_x\Big(G_{\min}\phi_i(x)\psi_s(\ell_0^+)\Big), \nabla_x(\phi_j(x)\psi_p(\ell_0^+))\Big)_x \\
&\quad = \sum_{\nu=1}^{M} \varepsilon \int_{I_\nu}\left(\nabla_x\Big(\frac{\partial}{\partial \ell}(Gz^n_{h,k})\Big), \nabla_x X\right)_x + \varepsilon\sum_{\nu=1}^{M-1}\Big(\nabla_x[Gz^n_h]_\nu, \nabla_x X(\ell_\nu^+)\Big)_x \\
&\qquad + \varepsilon\Big(G_{\min}\nabla_x z^n_h(\ell_0^+), \nabla_x X(\ell_0^+)\Big)_x \\
&\quad = K_1(z^n_{h,k}, X).
\end{aligned}
$$

Similarly

$$
\big(T_x \otimes T_\ell\big)\boldsymbol{\xi}^n = K_2(z^n_{h,k}, X), \quad \Big(S_x \otimes T_\ell\Big)\boldsymbol{\xi}^n = K_3(z^n_{h,k}, X),
$$

and

$$
(D_x + T_x + S_x)\boldsymbol{\xi}^n_{\min} \otimes \mathbb{I}_x = a_h\big(G_{\min} z^n_{\mathrm{min,h}}, X(\ell_0^+)\big).
$$

Using these expressions, one can write (6.25) in inner product form (6.20). Note that, the one-step formulation (6.20) is equivalent to the fully discrete form (6.16) except the perturbation term of $\mathcal{O}(\tau^2)$ which is presented in (6.20). □

Properties of K

Here we illustrate some basic properties of the bilinear forms K_1, K_2 and K_3 defined in (6.21), (6.22) and (6.23) respectively.

Lemma 6.3.2. *The bilinear forms K_1, K_2 and K_3 can be expressed as*

$$
\begin{aligned}
K_1(u,v) &= \varepsilon \sum_{i=1}^{M} \int_{I_i} -\Big(G\nabla_x u, \partial_\ell \nabla_x v\Big)_x - \varepsilon \sum_{i=1}^{M-1} \Big(\nabla_x u(\ell_i^-), \nabla_x \big[(Gv)\big]_i\Big)_x \\
&\quad + \varepsilon \Big(G_{\max} \nabla_x u(\ell_M^-), \nabla_x v(\ell_M^-)\Big)_x && (6.26)\\
K_2(u,v) &= \sum_{i=1}^{M} \int_{I_i} -\Big(G\mathbf{b}\cdot\nabla_x u, \partial_\ell v\Big)_x - \sum_{i=1}^{M-1} \Big(\mathbf{b}\cdot\nabla_x u(\ell_i^-), \big[(Gv)\big]_i\Big)_x \\
&\quad + \Big(G_{\max}\mathbf{b}\cdot\nabla_x u(\ell_M^-), v(\ell_M^-)\Big)_x && (6.27)\\
K_3(u,v) &= \sum_{i=1}^{M} \int_{I_i} -S_h\Big(Gu, \partial_\ell v\Big) + \sum_{i=1}^{M-1} S_h\Big(u(\ell_i^-), \big[(Gv)\big]_i\Big) + S_h\Big(Gu(\ell_M^-), v(\ell_M^-)\Big). \\
& && (6.28)
\end{aligned}
$$

Proof. Integrating by parts the first term in (6.21), (6.22) and (6.23) with respect to ℓ we get the required results. ☐

Consider the mesh dependent norm

$$\|v\|_K^2 = \|v\|_{K_1}^2 + \|v\|_{K_3}^2$$

with $K_1(v,v) \geq \|v\|_{K_1}^2$, $K_3(v,v) \geq \|v\|_{K_3}^2$ and

$$
\begin{aligned}
\|v\|_{K_1}^2 &= 2\sum_{i=1}^{M} \int_{I_i} \partial_\ell G |||v|||^2 + \varepsilon \sum_{i=1}^{M-1} \big\|\big[(G^{1/2}v)\big]_i\big\|_{H^1(\Omega_x)}^2 + \varepsilon \big\|G_{\min}^{1/2} v(\ell_0^+)\big\|_{H^1(\Omega_x)}^2 \\
&\quad + \varepsilon \big\|G_{\max}^{1/2} v(\ell_M^-)\big\|_{H^1(\Omega_x)}^2, && (6.29)\\
\|v\|_{K_3}^2 &= 2\sum_{i=1}^{M} \int_{I_i} \partial_\ell G |||v|||^2 + \sum_{i=1}^{M-1} S_h\big(\big[(Gv)\big]_i, [v]_i\big) + S_h\big(G_{\min} v(\ell_0^+), v(\ell_0^+)\big) \\
&\quad + S_h\big(G_{\max} v(\ell_M^-), v(\ell_M^-)\big). && (6.30)
\end{aligned}
$$

The next lemma gives the positivity of the form K.

Lemma 6.3.3. *Assume that $G > 0$ and $\partial_\ell G \geq 0$, then the bilinear form K is coercive corresponding to the norm $\|\cdot\|_K$, i.e.*

$$K(v,v) \geq \frac{1}{2}\|v\|_K^2. \tag{6.31}$$

Proof. First we show that $K_2(v,v) = 0$. Setting $u = v$ in (6.22) and (6.27) and then adding them together, we get

$$\begin{aligned}
2K_2(v,v) &= \sum_{i=1}^{M} \int_{I_i} \left\{ \Big(\partial_\ell\big(G\mathbf{b}\cdot\nabla_x v\big), v\Big)_x - \Big(G\mathbf{b}\cdot\nabla_x v, \partial_\ell v\Big)_x \right\} \\
&\quad + \sum_{i=0}^{M-1} \Big(\mathbf{b}\cdot\nabla_x\big[(G^{1/2}v)\big]_i, \big[(G^{1/2}v)\big]_i\Big)_x + \Big(G_{\min}\mathbf{b}\cdot\nabla_x v(\ell_0^+), v(\ell_0^+)\Big)_x \\
&\quad + \Big(G_{\max}\mathbf{b}\cdot\nabla_x v(\ell_M^-), v(\ell_M^-)\Big)_x \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

The velocity field $\mathbf{b}$ does not depend on ℓ, also the growth rate G is independent of space variable x, thus using the fact that ∇_x and ∂_ℓ commute, we can write I_1 as

$$\begin{aligned}
I_1 &= \sum_{i=1}^{M} \int_{I_i} \left\{ \Big(\mathbf{b}\cdot\nabla_x\partial_\ell(Gv), v\Big)_x - \Big(G\mathbf{b}\cdot\nabla_x v, \partial_\ell v\Big)_x \right\} \\
&= \sum_{i=1}^{M} \int_{I_i} \left\{ \partial_\ell G\Big(\mathbf{b}\cdot\nabla_x v, v\Big)_x + G\Big(\mathbf{b}\cdot\nabla_x\partial_\ell v, v\Big)_x - G\Big(\mathbf{b}\cdot\nabla_x v, \partial_\ell v\Big)_x \right\}.
\end{aligned}$$

Since $\nabla_x\cdot\mathbf{b} = 0$, it follows that

$$2\left(\mathbf{b}\cdot\nabla_x v, v\right)_x = -\left(\nabla_x\cdot\mathbf{b}, v^2\right)_x = 0,$$

hence we have

$$\begin{aligned}
I_1 &= \int_{\Omega_\ell} \left\{ G\Big(\mathbf{b}\cdot\nabla_x(\partial_\ell v), v\Big)_x - G\Big(\mathbf{b}\cdot\nabla_x v, \partial_\ell v\Big)_x \right\} \\
&= I_{1,1} - I_{1,2}.
\end{aligned}$$

Next, we show that the two terms on the right-hand side of this equation are the same.

$$\begin{aligned}
I_{1,1} &= \int_{\Omega_\ell} G\Big(\mathbf{b}\cdot\nabla_x(\partial_\ell v), v\Big)_x = \int_{\Omega_\ell\times\Omega_x} \mathbf{b}\cdot\nabla_x\Big(G\partial_\ell v\Big)v = \int_{\Omega_\ell\times\Omega_x} \sum_{i=1}^{d} \mathbf{b}_i \frac{\partial}{\partial x_i}\Big(G\frac{\partial v}{\partial \ell}\Big)v \\
&= \int_{\Omega_\ell\times\Omega_x} \sum_{i=1}^{d} \mathbf{b}_i \frac{\partial}{\partial x_i}\Big(G\frac{\partial(\phi\psi)}{\partial \ell}\Big)\phi\psi = \int_{\Omega_\ell\times\Omega_x} \sum_{i=1}^{d} \mathbf{b}_i \frac{\partial \phi}{\partial x_i}\phi\, G\frac{\partial \psi}{\partial \ell}\psi \\
&= \frac{1}{4}\int_{\Omega_\ell\times\Omega_x} \sum_{i=1}^{d} \mathbf{b}_i \frac{\partial \phi^2}{\partial x_i}\, G\frac{\partial \psi^2}{\partial \ell}\, dx,
\end{aligned}$$

where $\phi = \phi_j(x)$, $1 \le j \le N_x$ and $\psi = \psi_l(\ell)$, $1 \le l \le N_\ell$ are the bases functions defined in Section 4.5. Similarly

$$\begin{aligned}
I_{1,2} &= \int_{\Omega_\ell} G\Big(\mathbf{b}\cdot\nabla_x v, \partial_\ell v\Big)_x = \int_{\Omega_\ell\times\Omega_x} G\Big(\mathbf{b}\cdot\nabla_x v, \partial_\ell v\Big)_x\, d\ell = \int_{\Omega_\ell\times\Omega_x} \sum_{i=1}^{d} \mathbf{b}_i\, \frac{\partial v}{\partial x_i}\, G\frac{\partial v}{\partial \ell} \\
&= \int_{\Omega_\ell\times\Omega_x} \sum_{i=1}^{d} \mathbf{b}_i\, \frac{\partial \phi}{\partial x_i}\phi\, G\frac{\partial \psi}{\partial \ell}\psi = \frac{1}{4}\int_{\Omega_\ell\times\Omega_x} \sum_{i=1}^{d} \mathbf{b}_i\, \frac{\partial \phi^2}{\partial x_i}\, G\frac{\partial \psi^2}{\partial \ell}.
\end{aligned}$$

We deduce from the last two results that

$$I_1 = \int_{\Omega_\ell} \left\{ G\left(\mathbf{b}\cdot\nabla_x\big(\partial_\ell v\big), v\right)_x - G\left(\mathbf{b}\cdot\nabla_x v, \partial_\ell v\right)_x \right\} = 0.$$

For the terms I_2, I_3 and I_4, integrating by parts with respect to x and using $\nabla_x \cdot \mathbf{b} = 0$ we get

$$\begin{aligned}
I_2 &= \sum_{i=0}^{M-1} \Big(\mathbf{b}\cdot\nabla_x\big[(G^{1/2}v)\big]_i, \big[(G^{1/2}v\big]_i\Big)_x = -\frac{1}{2}\sum_{i=0}^{M-1}\Big(\nabla_x\cdot\mathbf{b}, \big(\big[(G^{1/2}v)\big]_i\big)^2\Big)_x = 0,\\
I_3 &= \Big(G\mathbf{b}\cdot\nabla_x v(\ell_0^+), v(\ell_0^+)\Big)_x = 0,\\
I_4 &= \Big(G\mathbf{b}\cdot\nabla_x v(\ell_M^-), v(\ell_M^-)\Big)_x = 0.
\end{aligned}$$

Hence $K_2(v,v) = 0$. The required result is obtained by adding the two different representations of the bilinear forms K_1 and K_3 and then dividing by two. $\square$

6.4 Stability

In this section, we address the stability of the method based on an equivalent one-step scheme (6.20).

Theorem 6.4.1. *Let $z_{h,k}^n$, $n = 1,\ldots,N$, be the solution of (6.20). Assume that $G > 0$ and $\partial_\ell G \geq 0$ then we have the following estimate*

$$\begin{aligned}
\left\|z_{h,k}^n\right\|_0^2 + \sum_{m=0}^{n-1}\left\|z_{h,k}^{m+1} - z_{h,k}^m\right\|_0^2 + \frac{\tau}{2}\sum_{m=0}^{n-1}\left\|z_{h,k}^{m+1}\right\|_{\mathrm{DG}}^2 + \frac{\tau^2}{2}\sum_{m=0}^{n-1}\left\|z_{h,k}^{m+1}\right\|_K^2&\\
\leq e^{2T}\left\|z_{h,k}^0\right\|_0^2 + e^{2T}\sum_{m=0}^{n-1}\tau\Big\{\mathcal{I}\left(z_{\mathrm{min,h}}^{m+1}\right) + \left\|f^{m+1}\right\|_0^2\Big\},& \qquad (6.32)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{I}\left(z_{\mathrm{min,h}}^{m+1}\right) = 4\left\|G_{\min}^{1/2} z_{\mathrm{min,h}}^{m+1}\right\|_{L^2(\Omega_x)}^2 + 4\tau\Big(\varepsilon + \|\mathbf{b}\|_{0,\infty}^2\Big)\left\|G_{\min}^{1/2} z_{\mathrm{min,h}}^{m+1}\right\|_{H^1(\Omega_x)}^2&\\
+ 4\tau S_h\Big(G_{\min} z_{\mathrm{min,h}}^{m+1}, z_{\mathrm{min,h}}^{m+1}\Big).&
\end{aligned}$$

Proof. Set $X = z_{h,k}^{m+1}$ in the one-step formulation (6.20) to get

$$\begin{aligned}
&\int_{\Omega_\ell}\left(z_{h,k}^{m+1} - z_{h,k}^m, z_{h,k}^{m+1}\right)_x + \tau\mathcal{B}_h(z_{h,k}^{m+1}, z_{h,k}^{m+1}) + \tau^2 K(z_{h,k}^{m+1}, z_{h,k}^{m+1})\\
&= \tau\left(G_{\min} z_{\mathrm{min,h}}^{m+1}, z_h^{m+1}(\ell_0^+)\right)_x + \tau\int_{\Omega_\ell}(f^{m+1}, z_{h,k}^{m+1})_x + \tau^2 a_h\left(G_{\min} z_{\mathrm{min,h}}^{m+1}, z_h^{m+1}(\ell_0^+)\right). \qquad (6.33)
\end{aligned}$$

Applying the identity $2(a-b)a = (a^2-b^2)+(a-b)^2$, we obtain for the first term

$$\int_{\Omega_\ell} \left(z_{h,k}^{m+1} - z_{h,k}^{m}, z_{h,k}^{m+1}\right)_x = \frac{1}{2}\left\|z_{h,k}^{m+1}\right\|_0^2 - \frac{1}{2}\left\|z_{h,k}^{m}\right\|_0^2 + \frac{1}{2}\left\|z_{h,k}^{m+1} - z_{h,k}^{m}\right\|_0^2. \tag{6.34}$$

Using Cauchy-Schwarz inequality and Young's inequality with δ_i (for any $\delta_i > 0$), $i = 1, 2, \ldots,$ we get for the first two terms on the right-hand side of (6.33)

$$\begin{aligned}
&\left(Gz_{\mathrm{min,h}}^{m+1}, z_h^{m+1}(\ell_0^+)\right)_x + \int_{\Omega_x}\left(f^{m+1}, z_{h,k}^{m+1}\right)_x \\
&\le \left\|G_{\min}^{1/2} z_{\mathrm{min,h}}^{m+1}\right\|_{L^2(\Omega_x)}\left\|G_{\min}^{1/2} z_h^{m+1}(\ell_0^+)\right\|_{L^2(\Omega_x)} + \left\|f^{m+1}\right\|_0\left\|z_{h,k}^{m+1}\right\|_0 \\
&\le \frac{1}{2\delta_1}\left\|G_{\min}^{1/2} z_{\mathrm{min,h}}^{m+1}\right\|_{L^2(\Omega_x)}^2 + \frac{\delta_1}{2}\left\|G_{\min}^{1/2} z_h^{m+1}(\ell_0^+)\right\|_{L^2(\Omega_x)}^2 + \frac{1}{2\delta_2}\left\|f^{m+1}\right\|_0^2 + \frac{\delta_2}{2}\left\|z_{h,k}^{m+1}\right\|_0^2 \\
&\le \frac{1}{2\delta_1}\left\|G_{\min}^{1/2} z_{\mathrm{min,h}}^{m+1}\right\|_{L^2(\Omega_x)}^2 + \frac{\delta_1}{2}\left\|z_{h,k}^{m+1}\right\|_{\mathrm{DG}}^2 + \frac{1}{2\delta_2}\left\|f^{m+1}\right\|_0^2 + \frac{\delta_2}{2}\left\|z_{h,k}^{m+1}\right\|_0^2.
\end{aligned}\tag{6.35}$$

From the definition of the bilinear form a_h, we write the third term on the right-hand side of (6.33)

$$\begin{aligned}
&\tau a_h\left(G_{\min} z_{\mathrm{min,h}}^{m+1}, z_h^{m+1}(\ell_0^+)\right) \\
&\qquad = \varepsilon\,\tau\left(G_{\min}\,\nabla_x z_{\mathrm{min,h}}^{m+1}, \nabla_x z_h^{m+1}(\ell_0^+)\right)_x + \tau\left(G_{\min}\mathbf{b}\cdot\nabla_x z_{\mathrm{min,h}}^{m+1}, z_h^{m+1}(\ell_0^+)\right)_x \\
&\qquad\quad + \tau S_h\left(G_{\min} z_{\mathrm{min,h}}^{m+1}, z_h^{m+1}(\ell_0^+)\right).
\end{aligned}$$

Then, Cauchy-Schwarz inequality and Young's inequality give

$$\begin{aligned}
&\tau a_h\left(G_{\min} z_{\mathrm{min,h}}^{m+1}, z_h^{m+1}(\ell_0^+)\right) \\
&\qquad \le \varepsilon\,\tau\left\|G_{\min}^{1/2}\nabla_x z_{\mathrm{min,h}}^{m+1}\right\|_{L^2(\Omega_x)}\left\|G_{\min}^{1/2}\nabla_x z_h^{m+1}(\ell_0^+)\right\|_{L^2(\Omega_x)} \\
&\qquad\quad + \tau\left\|G_{\min}^{1/2}\mathbf{b}\cdot\nabla_x z_{\mathrm{min,h}}^{m+1}\right\|_{L^2(\Omega_x)}\left\|G_{\min}^{1/2} z_h^{m+1}(\ell_0^+)\right\|_{L^2(\Omega_x)} \\
&\qquad\quad + \tau S_h\left(G_{\min} z_{\mathrm{min,h}}^{m+1}, z_{\mathrm{min,h}}^{m+1}\right)^{1/2} S_h\left(G_{\min} z_h^{m+1}(\ell_0^+), z_h^{m+1}(\ell_0^+)\right)^{1/2} \\
&\qquad \le \frac{\varepsilon\tau}{2\delta_3}\left\|G_{\min}^{1/2}\nabla_x z_{\mathrm{min,h}}^{m+1}\right\|_{L^2(\Omega_x)}^2 + \frac{\varepsilon\tau\delta_3}{2}\left\|G_{\min}^{1/2}\nabla_x z_h^{m+1}(\ell_0^+)\right\|_{L^2(\Omega_x)}^2 \\
&\qquad\quad + \frac{1}{2\delta_4}\left\|\mathbf{b}\right\|_{0,\infty}^2\left\|G_{\min}^{1/2}\nabla_x z_{\mathrm{min,h}}^{m+1}\right\|_{L^2(\Omega_x)}^2 + \frac{\delta_4}{2}\left\|G_{\min}^{1/2} z_h^{m+1}(\ell_0^+)\right\|_{L^2(\Omega_x)} \\
&\qquad\quad + \frac{\tau}{2\delta_5} S_h\left(G_{\min} z_{\mathrm{min,h}}^{m+1}, z_{\mathrm{min,h}}^{m+1}\right) + \frac{\tau\delta_5}{2} S_h\left(G_{\min} z_h^{m+1}(\ell_0^+), z_h^{m+1}(\ell_0^+)\right) \\
&\qquad \le \tau\left(\frac{\varepsilon}{2\delta_3} + \frac{\left\|\mathbf{b}\right\|_{0,\infty}^2}{2\delta_4}\right)\left\|G_{\min}^{1/2} z_{\mathrm{min,h}}^{m+1}\right\|_{H^1(\Omega_x)}^2 + \frac{\delta_4}{2}\left\|z_{h,k}^{m+1}\right\|_{\mathrm{DG}}^2 + \frac{\tau\delta_3}{2}\left\|z_{h,k}^{m+1}\right\|_{K_1}^2 \\
&\qquad\quad + \frac{\tau\delta_5}{2} S_h\left(G_{\min} z_{\mathrm{min,h}}^{m+1}, z_{\mathrm{min,h}}^{m+1}\right) + \frac{\tau\delta_5}{2}\left\|z_{h,k}^{m+1}\right\|_{K_3}^2.
\end{aligned}\tag{6.36}$$

Hence from (6.7), (6.31) and (6.33)-(6.36), we have

$$\begin{aligned}
&\left\|z_{h,k}^{m+1}\right\|_0^2 - \left\|z_{h,k}^{m}\right\|_0^2 + \left\|z_{h,k}^{m+1} - z_{h,k}^{m}\right\|_0^2 + \tau\left\|z_{h,k}^{m+1}\right\|_{\mathrm{DG}}^2 + \tau^2\left\|z_{h,k}^{m+1}\right\|_K^2 \\
&\quad\le \frac{\tau}{\delta_1}\left\|G_{\min}^{1/2} z_{\mathrm{min,h}}^{m+1}\right\|_{L^2(\Omega_x)}^2 + \tau\Big(\delta_1 + \delta_4\Big)\left\|z_{h,k}^{m+1}\right\|_{\mathrm{DG}}^2 \\
&\qquad + \frac{\tau}{\delta_2}\left\|f^{m+1}\right\|_0^2 + \tau\delta_2\left\|z_{h,k}^{m+1}\right\|_0^2 + \tau^2\Big(\frac{\varepsilon}{\delta_3} + \frac{\|\mathbf{b}\|_{0,\infty}^2}{\delta_4}\Big)\left\|G^{\frac{1}{2}} u_h^{m+1}(\ell_0^-)\right\|_{H^1(\Omega_x)}^2 \\
&\qquad + \tau^2\delta_5 S_h\Big(G_{\min} z_{\mathrm{min,h}}^{m+1}, z_{\mathrm{min,h}}^{m+1}\Big) + \tau^2(\delta_3 + \delta_5)\left\|z_{h,k}^{m+1}\right\|_K^2.
\end{aligned}$$

Setting $\delta_1 = \delta_4 = 1/4$, $\delta_2 = 1$ and $\delta_3 = \delta_5 = 1/4$, multiplying by τ and summing over $m = 0, 1, \ldots, n-1$, we get

$$\begin{aligned}
&\left\|z_{h,k}^{n}\right\|_0^2 + \sum_{m=0}^{n-1}\left\|z_{h,k}^{m+1} - z_{h,k}^{m}\right\|_0^2 + \tau\sum_{m=0}^{n-1}\left\|z_{h,k}^{m+1}\right\|_{\mathrm{DG}}^2 + \tau^2\sum_{m=0}^{n-1}\left\|z_{h,k}^{m+1}\right\|_K^2 \\
&\quad\le \left\|z_{h,k}^{0}\right\|_0^2 + 4\tau\sum_{m=0}^{n-1}\left\|G_{\min}^{1/2} z_{\mathrm{min,h}}^{m+1}\right\|_{L^2(\Omega_x)}^2 + \tau\sum_{m=0}^{n-1}\left\|f^{m+1}\right\|_0^2 \\
&\qquad + 4\tau^2\Big(\varepsilon + \|\mathbf{b}\|_{0,\infty}^2\Big)\sum_{m=0}^{n-1}\left\|G_{\min}^{1/2} z_{\mathrm{min,h}}^{m+1}\right\|_{H^1(\Omega_x)}^2 + 4\tau^2\sum_{m=0}^{n-1} S_h\Big(G_{\min} z_{\mathrm{min,h}}^{m+1}, z_{\mathrm{min,h}}^{m+1}\Big) \\
&\qquad + \frac{\tau}{2}\sum_{m=0}^{n-1}\left\|z_{h,k}^{m+1}\right\|_{\mathrm{DG}}^2 + \frac{\tau^2}{2}\sum_{m=0}^{n-1}\left\|z_{h,k}^{m+1}\right\|_K^2 + \tau\sum_{m=0}^{n-1}\left\|z_{h,k}^{m+1}\right\|_0^2.
\end{aligned}$$

Absorbing $\|\cdot\|_{\mathrm{DG}}$ and $\|\cdot\|_K$ norm contributions in the left-hand side and applying the Gronwall's Lemma 2.3.4, we get

$$\begin{aligned}
&\left\|z_{h,k}^{n}\right\|_0^2 + \frac{\tau}{2}\sum_{m=0}^{n-1}\left\|z_{h,k}^{m+1}\right\|_{\mathrm{DG}}^2 + \frac{\tau^2}{2}\sum_{m=0}^{n-1}\left\|z_{h,k}^{m+1}\right\|_K^2 \\
&\quad\le e^{2T}\left\|z_{h,k}^{0}\right\|_0^2 + e^{2T}\,\tau\sum_{m=0}^{n-1}\Big\{4\left\|G_{\min}^{1/2} z_{\mathrm{min,h}}^{m+1}\right\|_{L^2(\Omega_x)}^2 + \left\|f^{m+1}\right\|_0^2\Big\} \\
&\qquad + 4e^{2T}\tau^2\sum_{m=0}^{n-1}\Big[\Big(\varepsilon + \|\mathbf{b}\|_{0,\infty}^2\Big)\left\|G_{\min}^{1/2} z_{\mathrm{min,h}}^{m+1}\right\|_{H^1(\Omega_x)}^2 + S_h\Big(G_{\min} z_{\mathrm{min,h}}^{m+1}, z_{\mathrm{min,h}}^{m+1}\Big)\Big].
\end{aligned}$$

This completes the proof. □

6.5 Convergence analysis

In this section, we use the equivalent one-step formulation (6.20) and the stability Lemma 6.4.1 to derive the convergence estimates. Define

$$z(t^n) - z_{h,k}^n = \left(z(t^n) - P_{h,k} z(t^n)\right) + \left(P_{h,k} z(t^n) - z_{h,k}^n\right) =: \xi^n + \eta^n, \tag{6.37}$$

where $z(t^n)$ is the solution of continuous problem (4.1) and $z^n_{h,k}$ is the solution of equivalent one-step problem (6.20).

Theorem 6.5.1. *Suppose A1-A4 and $\mu_K \sim h_K$ for all $K \in \mathcal{T}_h$. Let $z(t^n)$ and $z^n_{h,k}$ be the solution of the continuous problem* (4.1) *and the equivalent one-step problem* (6.20)*. Then for $e^n = z(t^n) - z^n_{h,k}$ there holds*

$$\|e^n\|_0^2 + \frac{\tau}{2}\sum_{m=0}^{n-1}\Big\{\|e^{m+1}\|_{\mathrm{DG}}^2 + \tau\|e^{m+1}\|_K^2\Big\} \le Ce^T\Big[\|P_{h,k}z_0 - z^0_{h,k}\|_0^2 + \tau^2 + (\varepsilon + h)h^{2r} + k^{2q+1}\Big],$$

where C only depends on z, z_t, z_0 and $z_{\min}$.

Proof. We start by applying the equivalent one-step formulation (6.20) to $\xi^m = z(t^m) - z^m_{h,k} - \eta^m$ and setting $X = \xi^{m+1}$, to get

$$\begin{aligned}
&\int_{\Omega_\ell}\Big(\frac{\xi^{m+1}-\xi^m}{\tau}, \xi^{m+1}\Big)_x + \mathcal{B}_h\big(\xi^{m+1},\xi^{m+1}\big) + \tau K\big(\xi^{m+1},\xi^{m+1}\big) \\
&= \int_{\Omega_\ell}\Big(\frac{z(t^{m+1})-z(t^m)}{\tau}, \xi^{m+1}\Big)_x + \mathcal{B}_h\big(z(t^{m+1}),\xi^{m+1}\big) + \tau K\big(z(t^{m+1}),\xi^{m+1}\big) \\
&\quad - \big(G_{\min} z^{m+1}_{\mathrm{min,h}}, \xi^{m+1}(\ell_0^+)\big)_x - \int_{\Omega_\ell}\big(f^{m+1},\xi^{m+1}\big)_x - \tau a_h\big(G_{\min} z^{m+1}_{\mathrm{min,h}}, \xi^{m+1}(\ell_0^+)\big) \\
&\quad - \int_{\Omega_\ell}\Big(\frac{\eta^{m+1}-\eta^m}{\tau}, \xi^{m+1}\Big)_x - \mathcal{B}_h\big(\eta^{m+1},\xi^{m+1}\big) - \tau K\big(\eta^{m+1},\xi^{m+1}\big),
\end{aligned}$$

where the terms containing $z^{m+1}_{h,k}$ and $z^m_{h,k}$ are replaced by the right-hand side of (6.20). Then using the weak form (6.1) at $t = t^{m+1}$, we get

$$\begin{aligned}
&\int_{\Omega_\ell}\Big(\frac{\xi^{m+1}-\xi^m}{\tau}, \xi^{m+1}\Big)_x + \mathcal{B}_h\big(\xi^{m+1},\xi^{m+1}\big) + \sum_{m=0}^{n-1}\tau K\big(\xi^{m+1},\xi^{m+1}\big) \\
&= \int_{\Omega_\ell}\Big(\frac{z(t^{m+1})-z(t^m)}{\tau} - z_t(t^{m+1}), \xi^{m+1}\Big)_x - \int_{\Omega_\ell}\Big(\frac{\eta^{m+1}-\eta^m}{\tau}, \xi^{m+1}\Big)_x \\
&\quad + \tau K\big(z(t^{m+1}),\xi^{m+1}\big) - \mathcal{B}_h\big(\eta^{m+1},\xi^{m+1}\big) - \int_{\Omega_\ell} S_h\big(z(t^{m+1}),\xi^{m+1}\big)_x \\
&\quad - \tau a_h\Big(G_{\min} z^{m+1}_{\mathrm{min,h}}, \xi^{m+1}(\ell_0^+)\Big) - \tau K\big(\eta^{m+1},\xi^{m+1}\big) \\
&\quad - \Big(G_{\min}\big(z^{m+1}_{\mathrm{min,h}} - z_{\min}(t^{m+1})\big), \xi^{m+1}(\ell_0^+)\Big)_x.
\end{aligned}$$

In above we have used the continuity of z. It follows from the coercivity of bilinear forms $\mathcal{B}_h$ and K (Lemma 6.1.1 and 6.3.3) that

$$\frac{1}{2}\|\xi^n\|_0^2 - \frac{1}{2}\|\xi^0\|_0^2 + \frac{\tau}{2}\sum_{m=0}^{n-1}\Big\{\|\xi^{m+1}\|_{\mathrm{DG}}^2 + \tau\|\xi^{m+1}\|_K^2\Big\} \le \sum_{j=1}^{7} T_j, \tag{6.38}$$

where T_j, $j = 1, \dots, 7$, are given as follows

$$T_1 = \sum_{m=0}^{n-1} \tau \int_{\Omega_\ell} \left(\frac{z(t^{m+1}) - z(t^m)}{\tau} - z_t(t^{m+1}), \xi^{m+1} \right)_x,$$

$$T_2 = -\sum_{m=0}^{n-1} \tau \int_{\Omega_\ell} \left(\frac{\eta^{m+1} - \eta^m}{\tau}, \xi^{m+1} \right)_x d\ell,$$

$$T_3 = \sum_{m=0}^{n-1} \tau^2 \Big\{ K\big(z(t^{m+1}), \xi^{m+1}\big) - a_h\big(G_{\mathrm{min,h}} z^{m+1}_{\mathrm{min,h}}, \xi^{m+1}(\ell_0^+)\big) \Big\},$$

$$T_4 = -\sum_{m=0}^{n-1} \tau \mathcal{B}_h\big(\eta^{m+1}, \xi^{m+1}\big), \qquad T_5 = -\sum_{m=0}^{n-1} \tau \int_{\Omega_\ell} S_h\big(z(t^{m+1}), \xi^{m+1}\big)_x,$$

$$T_6 = -\sum_{m=0}^{n-1} \tau^2 K\big(\eta^{m+1}, \xi^{m+1}\big)$$

$$T_7 = \sum_{m=0}^{n-1} \tau \Big(G\big(z^{m+1}_{\mathrm{min,h}} - z_{\mathrm{min}}(t^{m+1})\big), \xi^{m+1}(\ell_0^+) \Big)_x.$$

In the following δ_i, $(i = 1 \dots)$ are arbitrary positive constants to be fixed later. The estimates for the first term are standard, using Cauchy-Schwarz inequality and Taylor's theorem with remainder term we get

$$|T_1| \le C\tau^2 \sum_{m=0}^{n-1} \int_{t^m}^{m+1} \big\| z_{tt}(s) \big\|_0^2 \, ds + \frac{\delta_1}{2} \sum_{m=0}^{n-1} \tau \big\| \xi^{m+1} \big\|_0^2. \tag{6.39}$$

For T_2, the Cauchy-Schwarz inequality, the Young's inequality and the error decomposition (4.33) yield

$$\begin{aligned}
|T_2| &\le \frac{1}{2\delta_2} \sum_{m=0}^{n-1} \tau \Big\| \frac{\eta^{m+1} - \eta^m}{\tau} \Big\|_0^2 + \frac{\delta_2}{2} \sum_{m=0}^{n-1} \tau \big\| \xi^{m+1} \big\|_0^2 \\
&\le \frac{1}{2\delta_2} \sum_{m=0}^{n-1} \int_{t^m}^{t^{m+1}} \big\| \partial_t \eta(t) \big\|_0^2 \, dt + \frac{\delta_2}{2} \sum_{m=0}^{n-1} \tau \big\| \xi^{m+1} \big\|_0^2 \\
&\le \frac{1}{2\delta_2} \sum_{m=0}^{n-1} \int_{t^m}^{t^{m+1}} \Big\{ \big\| \partial_t \vartheta(t) \big\|_0^2 + \big\| \partial_t \varphi(t) \big\|_0^2 \Big\} \, dt + \frac{\delta_2}{2} \sum_{m=0}^{n-1} \tau \big\| \xi^{m+1} \big\|_0^2
\end{aligned}$$

Then the approximation properties (2.6) and (4.30) and the condition (4.32) give

$$\begin{aligned}
|T_2| \le C \sum_{m=0}^{n-1} \int_{t^m}^{t^{m+1}} & \Big\{ h^{2r+2} \big\| z_t(s) \big\|_{L^2(H^{r+1})}^2 + k^{2q+2} \big\| z_t(s) \big\|_{H^{q+1}(L^2)}^2 \Big\} ds \\
&+ \frac{\delta_2}{2} \sum_{m=0}^{n-1} \tau \big\| \xi^{m+1} \big\|_0^2.
\end{aligned} \tag{6.40}$$

For the third term T_3, we use the definitions (6.21)-(6.23) of K_1, K_2, and K_3, respectively. The jump terms are zero due to the continuity of $z(t^{m+1})$. We obtain

$$\begin{aligned}
T_3 &= \sum_{m=0}^{n-1} \tau^2 \Big\{ K_1\Big(z(t^{m+1}), \eta^{m+1}\Big) + K_2\Big(z(t^{m+1}), \xi^{m+1}\Big) + K_3\Big(z(t^{m+1}), \xi^{m+1}\Big) \\
&\qquad\qquad - a_h\big(G_{\min} z^{m+1}_{\mathrm{min,h}}, \xi^{m+1}(\ell_0^+)\big)\Big\} \\
&= \sum_{m=0}^{n-1} \tau^2 \int_{I_i} \Big\{ \varepsilon\Big(\partial_\ell\big(G\nabla_x z(t^{m+1})\big), \nabla_x \xi^{m+1}\Big)_x + \Big(\partial_\ell\big(G\mathbf{b}\cdot\nabla_x z(t^{m+1})\big), \xi^{m+1}\Big)_x \\
&\quad + S_h\Big(\partial_\ell\big(Gz(t^{m+1})\big), \xi^{m+1}\Big)\Big\} + \varepsilon \sum_{m=0}^{n-1} \tau^2 \Big(G_{\min}\nabla_x z_{\min}(t^{m+1}), \nabla_x \xi^{m+1}(\ell_0^+)\Big)_x \\
&\quad + \sum_{m=0}^{n-1} \tau^2 \Big(G_{\min}\mathbf{b}\cdot\nabla_x z_{\min}(t^{m+1}), \xi^{m+1}(\ell_0^+)\Big)_x + \sum_{m=0}^{n-1} \tau^2 S_h\Big(G_{\min} z_{\min}(t^{m+1}), \xi^{m+1}(\ell_0^+)\Big) \\
&\quad - a_h\Big(G_{\min} z^{m+1}_{\mathrm{min,h}}, \xi^{m+1}(\ell_0^+)\Big).
\end{aligned}$$

Then from the weak form (6.1), we get

$$\begin{aligned}
T_3 = \tau^2 &\sum_{m=0}^{n-1} \int_{\Omega_\ell} \Big(\partial_\ell\{Gf^{m+1} - Gz_t(t^{m+1}) - G\partial_\ell(Gz(t^{m+1}))\}, \xi^{m+1}\Big)_x \\
&+ \sum_{m=0}^{n-1} \tau^2 \int_{\Omega_\ell} S_h\Big(\partial_\ell\big(Gz(t^{m+1})\big), \xi^{m+1}\Big) \\
&+ \sum_{m=0}^{n-1} \tau^2 a_h\Big(G_{\min}\big(z_{\min}(t^{m+1}) - z^{m+1}_{\mathrm{min,h}}\big), \xi^{m+1}(\ell_0^+)\Big).
\end{aligned} \tag{6.41}$$

The bounds for the first term are obtained by applying the Cauchy-Schwarz inequality and the Young's inequality

$$\begin{aligned}
&\tau^2 \sum_{m=0}^{n-1} \int_{\Omega_\ell} \Big(\partial_\ell\{Gf^{m+1} - Gz_t(t^{m+1}) - G\partial_\ell\big(Gz(t^{m+1})\big)\}, \xi^{m+1}\Big)_x \\
&\le \tau^2 \sum_{m=0}^{n-1} \int_{\Omega_x} \big\|Gf^{m+1} - Gz_t(t^{m+1}) - G\partial_\ell\big(Gz(t^{m+1})\big)\big\|_{H^1(\Omega_\ell)} \|\xi^{m+1}\|_{L^2(\Omega_\ell)} \\
&\le \frac{\tau^2}{2\delta_3} \sum_{m=0}^{n-1} \int_{\Omega_x} \big\|Gf^{m+1} - Gz_t(t^{m+1}) - G\partial_\ell\big(Gz(t^{m+1})\big)\big\|^2_{H^1(\Omega_\ell)} + \frac{\delta_3 \tau^2}{2} \sum_{m=0}^{n-1} \|\xi^{m+1}\|_0^2 \\
&\le C\,\tau^2 \sum_{m=0}^{n-1} \Big[\big\|f^{m+1}\big\|^2_{H^1(L^2)} + \big\|z_t(t^{m+1})\big\|^2_{H^1(L^2)} + \big\|z(t^{m+1})\big\|^2_{H^2(L^2)} + \frac{\delta_3}{2}\|\xi^{m+1}\|_0^2\Big]
\end{aligned}$$

For the second term we use the fact that ∇_x and ∂_ℓ commute. Then by Cauchy-Schwarz inequality, the Young's inequality, and the approximation properties of fluctuation operator κ_h, we obtain

$$\begin{aligned}
&\tau^2 \sum_{m=0}^{n-1} \int_{\Omega_\ell} S_h\Big(\partial_\ell\big(Gz(t^{m+1})\big), \xi^{m+1}\Big) \\
&\quad \le \tau^2 \sum_{m=0}^{n-1} \left(\int_{\Omega_\ell} S_h\Big(\partial_\ell(Gz(t^{m+1})), \partial_\ell(Gz(t^{m+1}))\Big)\right)^{1/2} \left(\int_{\Omega_\ell} S_h(\xi^{m+1}, \xi^{m+1})\right)^{1/2} \\
&\quad \le \frac{2\tau^3}{\delta_4} \sum_{m=0}^{n-1} \int_{\Omega_\ell} \big\|\partial_\ell\big(G\kappa_h(\nabla_x z(t^{m+1}))\big)\big\|^2_{L^2(K)} + \frac{\delta_4}{8} \sum_{m=0}^{n-1} \tau \int_{\Omega_\ell} |||\xi^{m+1}|||^2 \\
&\quad \le C\, h^{2r+1}\, \tau^3 \sum_{m=0}^{n-1} \big\|z(t^{m+1})\big\|^2_{H^1(H^{r+1})} + \frac{\delta_4}{8} \sum_{m=0}^{n-1} \tau \big\|\xi^{m+1}\big\|^2_{\mathrm{DG}}.
\end{aligned}$$

For the third term in (6.41) using the definition of bilinear form a_h and following the same steps as in (4.40), we get

$$\begin{aligned}
&\tau^2 \sum_{m=0}^{n-1} a_h\Big(G_{\min}\big(z_{\min}(t^{m+1}) - z^{m+1}_{\mathrm{min,h}}, \xi^{m+1}(\ell_0^+)\Big) \\
&\quad = \sum_{m=0}^{n-1} \tau^2 \varepsilon\Big(G_{\min}\nabla_x\big(z_{\min}(t^{m+1}) - z^{m+1}_{\mathrm{min,h}}, \nabla_x\xi^{m+1}(\ell_0^+)\Big)_x \\
&\qquad + \sum_{m=0}^{n-1} \tau^2\Big(G_{\min}\mathbf{b}\cdot\nabla_x\big(z_{\min}(t^{m+1}) - z^{m+1}_{\mathrm{min,h}}, \xi^{m+1}(\ell_0^+)\Big)_x \\
&\qquad + \sum_{m=0}^{n-1} \tau^2 S_h\Big(G_{\min}\big(z_{\min}(t^{m+1}) - z^{m+1}_{\mathrm{min,h}}, \xi^{m+1}(\ell_0^+)\Big) \\
&\quad \le C\big(\tau\varepsilon + \tau h + h\big)\, h^{2r}\tau \sum_{m=0}^{n-1} \big\|z_{\min}(t^{m+1})\big\|^2_{H^{r+1}(\Omega_x)} + \frac{\tau\delta_4}{8} \sum_{m=0}^{n-1} \big\|\xi^{m+1}\big\|^2_{\mathrm{DG}} \\
&\qquad + \frac{\tau^2\delta_5}{2} \sum_{m=0}^{n-1} \big\|\xi^{m+1}\big\|^2_{K}.
\end{aligned}$$

Inserting these bounds into (6.41), we get for T_3

$$\begin{aligned}
|T_3| &\le C\,\tau^3 \sum_{m=0}^{n-1} \Big\{ \big\|f^{m+1}\big\|^2_{H^1(L^2)} + \big\|z_t(t^{m+1})\big\|^2_{H^1(L^2)} + \big\|z(t^{m+1})\big\|^2_{H^2(L^2)} \Big\} \\
&\quad + C\,\Big(\tau\varepsilon + \tau h + h\big)\, h^{2r}\tau \sum_{m=0}^{n-1} \big\|z_{\min}(t^{m+1})\big\|^2_{H^{r+1}(\Omega_x)} + \frac{\tau\delta_3}{2} \sum_{m=0}^{n-1} \big\|\xi^{m+1}\big\|^2_0 \\
&\quad + \frac{\tau\delta_4}{4} \sum_{m=0}^{n-1} \big\|\xi^{m+1}\big\|^2_{\mathrm{DG}} + \frac{\tau^2\delta_5}{2} \sum_{m=0}^{n-1} \big\|\xi^{m+1}\big\|^2_{K}. \qquad (6.42)
\end{aligned}$$

For T_4, the error decomposition (4.33), results of Lemma 6.2.3 and Young's inequality give

$$
\begin{aligned}
T_4 &= \tau \sum_{m=0}^{n-1} \Big\{ \mathcal{B}_h\Big(\vartheta^{m+1}, \xi^{m+1}\Big) + \mathcal{B}_h\Big(\varphi^{m+1}, \xi^{m+1}\Big) \Big\} \\
&\le C\left(\varepsilon + h + h^2\right) h^{2r} \tau \sum_{m=0}^{n-1} \left\| z(t^{m+1}) \right\|^2_{H^1(H^{r+1})} + C\,\tau h^{2r+2} \sum_{m=0}^{n-1} \left\| z(t^{m+1}) \right\|^2_{C(H^{r+1})} \\
&\quad + C\left(\frac{1}{2\delta_6} + \frac{\varepsilon + h}{\delta_7}\right) \tau \sum_{m=0}^{n-1} \left\| z(t^{m+1}) \right\|^2_{H^{q+1}(H^1)} + \frac{\tau \delta_6}{2} \sum_{m=0}^{n-1} \left\| \xi^{m+1} \right\|^2_0 \\
&\quad + \frac{\tau \delta_7}{4} \sum_{m=0}^{n-1} \left\| \xi^{m+1} \right\|^2_{\mathrm{DG}}.
\end{aligned} \tag{6.43}
$$

The approximation properties of fluctuation operator κ_h and the choice of stabilizing parameter $\mu_K \sim h_K$ give for T_5

$$
|T_5| \le C\, h^{2r+1} \tau \sum_{m=0}^{n-1} \left\| z(t^{m+1}) \right\|^2_{L^2(H^{r+1})} + \frac{\tau \delta_8}{4} \sum_{m=0}^{n-1} \left\| \xi^{m+1} \right\|^2_{\mathrm{DG}}. \tag{6.44}
$$

In order to find the bounds for T_6, we write it as

$$
T_6 = \tau^2 \sum_{m=0}^{n-1} \Big\{ K_1(\eta^{m+1}, \xi^{m+1}) + K_2(\eta^{m+1}, \xi^{m+1}) + K_3(\eta^{m+1}, \xi^{m+1}) \Big\}. \tag{6.45}
$$

Then Cauchy-Schwarz inequality and error decomposition (4.33) give for the first term on the right-hand side

$$
\begin{aligned}
\tau^2 \sum_{m=0}^{n-1} K_1(\eta^{m+1}, \xi^{m+1}) &\le \frac{3\tau^2 \delta_9}{2} \sum_{m=0}^{n-1} \left\| \eta^{m+1} \right\|^2_{K_1} + \frac{\tau^2 \delta_9}{6} \sum_{m=0}^{n-1} \left\| \xi^{m+1} \right\|^2_{K_1} \\
&\le \frac{3\tau^2 \delta_9}{2} \sum_{m=0}^{n-1} \Big\{ \left\| \vartheta^{m+1} \right\|^2_{K_1} + \left\| \varphi^{m+1} \right\|^2_{K_1} \Big\} + \frac{\tau^2 \delta_9}{6} \sum_{m=0}^{n-1} \left\| \xi^{m+1} \right\|^2_{K_1}.
\end{aligned}
$$

From Lemma 6.3.3, the coercivity of the bilinear form K_1, the continuity of $j_h z$ in ℓ-direction and the interpolation error estimates, we get

$$
\begin{aligned}
\|\vartheta^{m+1}\|^2_{K_1} &\le K_1(\vartheta^{m+1}, \vartheta^{m+1}) \\
&= \varepsilon \int_{\Omega_\ell} \Big(\partial_\ell (G \nabla_x \vartheta^{m+1}), \nabla_x \vartheta^{m+1} \Big)_x + \varepsilon \Big(G_{\min} \nabla_x \vartheta^{m+1}(\ell_0^+), \nabla_x \vartheta^{m+1}(\ell_0^+) \Big)_x \\
&\le C\, \varepsilon h^{2r} \Big\{ \left\| z(t^{m+1}) \right\|^2_{H^1(H^{r+1})} + \left\| z(t^{m+1}) \right\|^2_{C(H^{r+1})} \Big\}
\end{aligned}
$$

Using $\Pi_k z(\ell_i^-) = z(\ell_i^-)$, $i = 1, 2, \ldots, M$, (6.26) and (4.30), we obtain

$$\begin{aligned}\|\varphi^{m+1}\|_{K_2}^2 \le K_1(\varphi^{m+1}, \varphi^{m+1}) &= \varepsilon \sum_{i=1}^{M} \int_{I_i} \Big(G\nabla_x \varphi^{m+1}, \partial_\ell \big(\nabla_x \varphi^{m+1}\big)\Big)_x \\ &\le C\,\varepsilon\, k^{2q+1} \big\| z(t^{m+1})\big\|^2_{H^{q+1}(H^1)}.\end{aligned}$$

Inserting these two yields

$$\begin{aligned}\sum_{m=0}^{n-1} \tau^2 K_1(\eta^{m+1}, \xi^{m+1}) \le C\,\varepsilon\, h^{2r}\tau^2 \sum_{m=0}^{n-1} &\Big\{ \big\|z(t^{m+1})\big\|^2_{H^1(H^{r+1})} + \big\|z(t^{m+1})\big\|^2_{C(H^{r+1})}\Big\} \\ + C\,\varepsilon\, k^{2q+1}\tau^2 \sum_{m=0}^{n-1} &\big\|z(t^{m+1})\big\|^2_{H^{q+1}(H^1)} + \frac{\tau^2\delta_9}{6}\sum_{m=0}^{n-1} \big\|\xi^{m+1}\big\|^2_K.\end{aligned}$$

Similarly for $K_3(\eta^{m+1}, \xi^{m+1})$

$$\begin{aligned}\sum_{m=0}^{n-1} \tau^2 K_3\Big(\eta^{m+1}, \xi^{m+1}\Big) &\le \frac{3\delta_9}{2}\sum_{m=0}^{n-1}\tau^2 \big\|\eta^{m+1}\big\|^2_{K_3} + \frac{\delta_9}{6}\sum_{m=0}^{n-1}\tau^2\big\|\xi^{m+1}\big\|^2_K \\ &\le C\, h^{2r+1}\tau^2 \sum_{m=0}^{n-1}\Big\{ \big\|z(t^{m+1})\big\|^2_{H^1(H^{r+1})} + \big\|z(t^{m+1})\big\|^2_{C(H^{r+1})}\Big\} \\ &\quad + C\, k^{2q+1}\, h\tau^2 \sum_{m=0}^{n-1} \big\|z(t^{m+1})\big\|^2_{H^{q+1}(H^1)} + \frac{\tau^2\delta_9}{6}\sum_{m=0}^{n-1}\big\|\xi^{m+1}\big\|^2_K.\end{aligned}$$

Next, we bound the second term on the right-hand side of (6.45) as follows. Using the error decomposition (4.33), we get

$$\tau^2 \sum_{m=0}^{n-1} K_2\Big(\eta^{m+1}, \xi^{m+1}\Big) = \tau^2 \sum_{m=0}^{n-1} \Big\{ K_2\Big(\vartheta^{m+1}, \xi^{m+1}\Big) + K_2\Big(\varphi^{m+1}, \xi^{m+1}\Big)\Big\}. \tag{6.46}$$

Then, from (6.22) and continuity of $j_h z$ in internal coordinate, we get for the first term

$$\begin{aligned}\sum_{m=0}^{n-1} \tau^2 K_2\Big(\vartheta^{m+1}, \xi^{m+1}\Big) = \sum_{m=0}^{n-1}\tau^2 &\int_{\Omega_\ell} \Big(\partial_\ell (G\mathbf{b}\cdot\nabla_x \vartheta^{m+1}), \xi^{m+1}\Big)_x \\ &+ \tau^2 \sum_{m=0}^{n-1} \Big(G\mathbf{b}\cdot\nabla_x\vartheta^{m+1}(\ell_0^+), \xi^{m+1}(\ell_0^+)\Big)_x.\end{aligned}$$

Using the same approach as for I_2 in (4.40), we get

$$\tau^2 \sum_{m=0}^{n-1} K_2\Big(\vartheta^{m+1}, \xi^{m+1}\Big)$$
$$\le C\left(\tau h + 1\right) h^{2r+1}\tau^2 \sum_{m=0}^{n-1} \Big\{ \big\|z(t^{m+1})\big\|^2_{H^1(H^{r+1})} + \big\|z(t^{m+1})\big\|^2_{C(H^{r+1})} \Big\}$$
$$+ \frac{\tau^2\delta_9}{6} \sum_{m=0}^{n-1} \big\|\xi^{m+1}\big\|^2_K + \frac{\tau\delta_{10}}{4} \sum_{m=0}^{n-1} \big\|\xi^{m+1}\big\|^2_0 + \frac{\tau\delta_{11}}{4} \sum_{m=0}^{n-1} \big\|\xi^{m+1}\big\|^2_{\mathrm{DG}}.$$

The interpolation $\Pi_k z$ satisfies $\Pi_k z(\ell_i^-) = z(\ell_i^-)$, $i = 1, 2, \dots, M$, thus the last two terms in (6.27) vanish. Following the same steps as in (4.42) we get

$$\Big|\sum_{m=0}^{n-1} \tau^2 K_2\Big(\varphi^{m+1}, \xi^{m+1}\Big)\Big| = \Big| -\tau^2 \sum_{m=0}^{n-1}\sum_{i=1}^{M} \int_{I_i} \Big(\mathbf{b}\cdot\nabla_x \varphi^{m+1}, G\partial_\ell \xi^{m+1}\Big)_x \Big|$$
$$\le C\, k^{2q+2}\tau^3 \sum_{m=0}^{n-1} \big\|z(t^{m+1})\big\|^2_{H^{q+1}(H^1)} + \frac{\tau\delta_{10}}{4} \sum_{m=0}^{n-1} \big\|\xi^{m+1}\big\|^2_0.$$

Combining these two estimates, we get for the second term in (6.45)

$$\tau^2 \sum_{m=0}^{n-1} K_2\Big(\eta^{m+1}, \xi^{m+1}\Big) \le C\left(\tau h + 1\right) h^{2r+1}\tau^2 \sum_{m=0}^{n-1} \Big\{ \big\|z(t^{m+1})\big\|^2_{H^1(H^{r+1})} + \big\|z(t^{m+1})\big\|^2_{C(H^{r+1})} \Big\}$$
$$+ C\, k^{2q+2}\tau^3 \sum_{m=0}^{n-1} \big\|z(t^{m+1})\big\|^2_{H^{q+1}(H^1)} + \frac{\tau^2\delta_9}{4} \sum_{m=0}^{n-1} \big\|\xi^{m+1}\big\|^2_K$$
$$+ \frac{\tau\delta_9}{6} \sum_{m=0}^{n-1} \big\|\xi^{m+1}\big\|^2_0 + \frac{\tau\delta_{11}}{4} \sum_{m=0}^{n-1} \big\|\xi^{m+1}\big\|^2_{\mathrm{DG}}.$$

Inserting the estimates K_1, K_2 and K_3 into (6.45), we get for T_6

$$|T_6| \le C\left(\varepsilon + h + \tau h^2\right) h^{2r}\tau^2 \sum_{m=0}^{n-1} \Big\{ \big\|z(t^{m+1})\big\|^2_{H^1(H^{r+1})} + \big\|z(t^{m+1})\big\|^2_{C(H^{r+1})} \Big\}$$
$$+ C\left(\varepsilon + h\right) k^{2q+2}\tau^2 \sum_{m=0}^{n-1} \big\|z(t^{m+1})\big\|^2_{H^{q+1}(H^1)} + \frac{\tau^2\delta_9}{2} \sum_{m=0}^{n-1} \big\|\xi^{m+1}\big\|^2_K$$
$$+ \frac{\tau\delta_{10}}{2} \sum_{m=0}^{n-1} \big\|\xi^{m+1}\big\|^2_0 + \frac{\tau\delta_{11}}{4} \sum_{m=0}^{n-1} \big\|\xi^{m+1}\big\|^2_{\mathrm{DG}}.$$

Finally for the last term T_7, we obtain

$$|T_7| \le C\, h^{2r+2}\tau \sum_{m=0}^{n-1} \big\|z_{\min}(t^{m+1})\big\|^2_{H^{r+1}(\Omega_x)} + \frac{\tau\delta_{12}}{4} \sum_{m=0}^{n-1} \big\|\xi^{m+1}\big\|^2_{\mathrm{DG}}.$$

Hence, inserting the estimates $T_1, \dots, T_7$ into (6.38), and chose δ_i, $i = 1, \dots, 12$, such that

$$\begin{aligned}
&\delta_4 = \delta_7 = \delta_8 = \delta_{11} = \delta_{12} = \frac{1}{5},\\
&\delta_5 = \delta_9 = \frac{1}{2}, \text{ and } \delta_1 = \delta_2 = \delta_3 = \delta_6 = \delta_{10} = \frac{1}{5}
\end{aligned}$$

we get

$$\begin{aligned}
&\frac{1}{2}\|\xi^n\|_0^2 - \frac{1}{2}\|\xi^0\|_0^2 + \frac{\tau}{2}\sum_{m=0}^{n-1}\left\{\|\xi^{m+1}\|_{\mathrm{DG}}^2 + \tau\|\xi^{m+1}\|_K^2\right\}\\
&\le C\tau^2\sum_{m=0}^{n-1}\left\{\int_{t^m}^{t^{m+1}}\big\|z_t(s)\big\|_0^2 + \tau\big\|f^{m+1}\big\|_{H^1(L^2)}^2 + \tau\big\|z(t^{m+1})\big\|_{H^1(L^2)}^2 + \tau\big\|z_t(t^{m+1})\big\|_{H^2(L^2)}^2\right\}\\
&\quad + \Big(\varepsilon + h + \big(1+\tau\big)h^2\Big)\, h^{2r}\tau\sum_{m=0}^{n-1}\left\{\big\|z(t^{m+1})\big\|_{H^1(H^{r+1})}^2 + \big\|z(t^{m+1})\big\|_{C(H^{r+1})}^2\right\}\\
&\quad + C\Big(\tau\big(\varepsilon+h\big) + h + h^2\Big)h^{2r}\,\tau\sum_{m=0}^{n-1}\big\|z_{\min}(t^{m+1})\big\|_{H^{r+1}(\Omega_x)}^2\\
&\quad + C\,k^{2q+2}\,\tau\sum_{m=0}^{n-1}\left\{\int_{t^m}^{t^{m+1}}\big\|z_t(s)\big\|_{H^{q+1}(L^2)}^2 + \Big(\varepsilon+h+1\Big)\big\|z(t^{m+1})\big\|_{H^{q+1}(H^1)}^2\right\}\\
&\quad + \frac{\tau}{4}\sum_{m=0}^{n-1}\left\{\|\xi^{m+1}\|_{\mathrm{DG}}^2 + \tau\|\xi^{m+1}\|_K^2\right\} + \frac{\tau}{2}\sum_{m=0}^{n-1}\|\xi^{m+1}\|_0^2.
\end{aligned}$$

We conclude by absorbing the $\|\cdot\|_{\mathrm{DG}}$- and $\|\cdot\|_K$-norms contribution in the left-hand side, applying the Gronwall's Lemma 2.3.4 in the same fashion as in the Theorem 6.4.1, triangle inequality and interpolation error estimates. $\square$

Chapter 7

Conclusion

In this thesis, we have considered stability and convergence results for the numerical solution of time-dependent convection-diffusion-reaction problems (1.1) and population balance equations (1.2). We have mainly focused on the finite element method in space with local projection or Streamline-Upwind Petrov-Galerkin stabilization discretization, discontinuous Galerkin in internal variables and backward Euler time stepping methods. We concentrated on the one-level enrichment approach of the local projection stabilization methods.

First of all, we have analyzed a stabilized finite element method for the numerical solution of time-dependent convection-diffusion-reaction equations. We have derived the optimal estimates in the strong and weak norms for the error of the approximate solution by local projection stabilization method in space and discontinuous Galerkin method in time. Using polynomial of degree r in space and q in time, the errors of order $\mathcal{O}\big((\varepsilon^{1/2} + h^{1/2})h^r + k^{q+1/2}\big)$ in strong norm and of order $\mathcal{O}\big((\varepsilon^{1/2} + h^{1/2})h^r + k^{q+1}\big)$ in weak norm have been obtained. Computational results indicate that the error estimates are optimal in strong and weak norms. Furthermore, we observed from our numerical studies that the parameters of LPS lead to different influences on first and second order schemes. First order schemes are more sensitive with respect to changes of LPS parameters than second order schemes.

Then, we have been concerned with the numerical solution of the population balance equation with one internal coordinate posed on domain $\Omega_\ell \times \Omega_x$, where Ω_x was d-dimensional and Ω_ℓ one-dimensional domain. We have considered an operator splitting method which decomposes the original problem into two subproblems. The first subproblem is a time-dependent convection-diffusion problem in physical space parametrized by the variable in internal coordinate and the second one is a transport problem with pure advection in internal coordinate parametrized by the variable in physical space. The method combines the continuous finite element method (and local projection stabilization) in space with discontinuous Galerkin method in internal coordinate. We have considered first order backward Euler time stepping scheme. Under a certain regularity of exact solution, we have derived error estimates for the two-step method, i.e., using polynomials of degree r in space and of degree q in internal coordinate the errors of order $\mathcal{O}(\tau + h^{r+1/2} + k^{q+1/2})$ when $\varepsilon \ll 1$ and $\mathcal{O}(\tau + h^{r} + k^{q+1/2})$ when $\varepsilon = 1$ have been obtained.

Since the operator splitting method allows us to use different techniques to discretize the subproblems in space and internal coordinate. We have used the Streamline-Upwind Petrov-Galerkin method to discretize the two subproblems in space and discontinuous Galerkin method in internal coordinate. The stability and error estimates have been derived for two-step method under the conditions that the stabilization parameters depends on the length of the time step. The mathematical and numerical results have been compared with those obtained by local projection stabilization method in space. Furthermore, the numerical results have been presented for a test problem with known smooth solution. The optimal order of convergence has been obtained for first and second order finite elements in space and first order in internal coordinate. It should be pointed out that for fixed mesh width in space, the optimal scaling in SUPG method gave large numbers of time step, therefore we have only computed the convergence order for dG(1) in internal coordinate. Moreover, it has been shown that the LPS method in space helps to reduce the spurious oscillations which still remains in SUPG method.

Finally, we have considered the alternating direction Galerkin method to derive the stability and convergence estimate for the population balance equation. Local projection stabilization method in space and dG methods in internal coordinate have been used to obtain the semi-discrete error estimates. For the fully discrete problem backward Euler temporal discretization has been considered. Similar error estimates have been proved (as in two-step operator splitting method) for fully discrete scheme based on equivalent one-step formulation.

Note that the resulting fully discrete two-step method obtained using the alternating direction method was similar to that one obtained from the operator splitting method. The difference from the operator splitting method was that the stability and error estimates were derived by using the equivalent one-step formulation obtained from the two-steps alternating direction Galerkin formulation. Whereas in operator splitting scheme we followed the two-step method.

From the analysis and methods presented here in this thesis one can see that further modifications and generalizations are possible, which are as follows

- The operator splitting and alternating direction methods can be used for more than one internal coordinate.
- The method can be extended to higher-order time discretization schemes, for example Crank-Nicolson and discontinuous Galerkin method.
- The presented algorithm can be used to solve coupled multidimensional population

balance systems [40] with suitable numerical methods.

- The operator splitting method facilitate different kind of discretization techniques in space and internal coordinate.
- It is possible to use the present algorithm for the source terms like aggregation and breakage. These terms have to be treated explicitly, where the efficient evaluation of the non-local integral operators are needed [55].

Bibliography

[1] R. A. Adams and J. J. F. Fournier. *Sobolev spaces*, volume 140 of *Pure and Applied Mathematics (Amsterdam)*. Elsevier/Academic Press, Amsterdam, second edition, 2003.

[2] D. N. Arnold, F. Brezzi, B. Cockburn, and L. D. Marini. Unified analysis of discontinuous Galerkin methods for elliptic problems. *SIAM J. Numer. Anal.*, 39(5): 1749–1779, 2001/02.

[3] M. I. Asensio, B. Ayuso, and G. Sangalli. Coupling stabilized finite element methods with finite difference time integration for advection-diffusion-reaction problems. *Comput. Methods Appl. Mech. Engrg.*, 196(35-36):3475–3491, 2007.

[4] N. S. Bakhvalov. The optimization of methods of solving boundary value problems with a boundary layer. *U. S. S. R. Comput. Math. Math. Phys.*, 9(4):139–166, 1969.

[5] R. Becker and M. Braack. A finite element pressure gradient stabilization for the Stokes equations based on local projections. *Calcolo*, 38(4):173–199, 2001.

[6] R. Becker and M. Braack. A two-level stabilization scheme for the Navier-Stokes equations. In *Numerical mathematics and advanced applications*, pages 123–130. Springer, Berlin, 2004.

[7] P. B. Bochev, M. D. Gunzburger, and J. N. Shadid. Stability of the SUPG finite element method for transient advection-diffusion problems. *Comput. Methods Appl. Mech. Engrg.*, 193(23-26):2301–2323, 2004.

[8] M. Braack and E. Burman. Local projection stabilization for the Oseen problem and its interpretation as a variational multiscale method. *SIAM J. Numer. Anal.*, 43(6): 2544–2566 (electronic), 2006.

[9] E. Burman. Consistent SUPG-method for transient transport problems: stability and convergence. *Comput. Methods Appl. Mech. Engrg.*, 199(17-20):1114–1123, 2010. CODEN CMMECC. ISSN 0045-7825. URL `http://dx.doi.org/10.1016/j.cma.2009.11.023`.

[10] E. Burman and M. A. Fernández. Finite element methods with symmetric stabilization for the transient convection-diffusion-reaction equation. *Comput. Methods Appl. Mech. Engrg.*, 198(33-36):2508–2519, 2009.

[11] E. Burman, M. A. Fernández, and P. Hansbo. Continuous interior penalty finite element method for Oseen's equations. *SIAM J. Numer. Anal.*, 44(3):1248–1274, 2006.

[12] E. Burman and P. Hansbo. Edge stabilization for Galerkin approximations of convection-diffusion-reaction problems. *Comput. Methods Appl. Mech. Engrg.*, 193 (15-16):1437–1453, 2004.

[13] E. Burman and P. Hansbo. The edge stabilization method for finite elements in CFD. In *Numerical mathematics and advanced applications*, pages 196–203. Springer, Berlin, 2004.

[14] E. Burman and P. Hansbo. Edge stabilization for the generalized Stokes problem: a continuous interior penalty method. *Comput. Methods Appl. Mech. Engrg.*, 195 (19-22):2393–2410, 2006.

[15] P. G. Ciarlet. *The finite element method for elliptic problems*. North-Holland Publishing Co., Amsterdam, 1978. Studies in Mathematics and its Applications, Vol. 4.

[16] B. Cockburn. Discontinuous Galerkin methods for convection-dominated problems. In *High-order methods for computational physics*, volume 9 of *Lect. Notes Comput. Sci. Eng.*, pages 69–224. Springer, Berlin, 1999.

[17] B. Cockburn, G. E. Karniadakis, and C.-W. Shu. The development of discontinuous Galerkin methods. In *Discontinuous Galerkin methods (Newport, RI, 1999)*, volume 11 of *Lect. Notes Comput. Sci. Eng.*, pages 3–50. Springer, Berlin, 2000.

[18] R. Codina. Comparison of some finite element methods for solving the diffusion-convection-reaction equation. *Comput. Methods Appl. Mech. Engrg.*, 156(1-4):185–210, 1998.

[19] M. Delfour, W. Hager, and F. Trochu. Discontinuous Galerkin methods for ordinary differential equations. *Math. Comp.*, 36(154):455–473, 1981.

[20] V. Dolejší and M. Feistauer. Error estimates of the discontinuous Galerkin method for nonlinear nonstationary convection-diffusion problems. *Numer. Funct. Anal. Optim.*, 26(3):349–383, 2005.

[21] J. J. Douglas and T. Dupont. Alternating-direction Galerkin methods on rectangles. In *Numerical Solution of Partial Differential Equations, II (SYNSPADE 1970) (Proc. Sympos., Univ. of Maryland, College Park, Md., 1970)*, pages 133–214. Academic Press, New York, 1971.

[22] J. J. Douglas and T. Dupont. Interior penalty procedures for elliptic and parabolic Galerkin methods. In *Computing methods in applied sciences (Second Internat. Sympos., Versailles, 1975)*, pages 207–216. Lecture Notes in Phys., Vol. 58. Springer, Berlin, 1976.

[23] K. Eriksson and C. Johnson. Adaptive streamline diffusion finite element methods for stationary convection-diffusion problems. *Math. Comp.*, 60(201):167–188, S1–S2, 1993.

[24] K. Eriksson, C. Johnson, and V. Thomée. Time discretization of parabolic problems by the discontinuous Galerkin method. *RAIRO Modél. Math. Anal. Numér.*, 19(4): 611–643, 1985.

[25] A. Ern and J.-L. Guermond. *Theory and practice of finite elements*, volume 159 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 2004.

[26] M. Feistauer, J. Hájek, and K. Svadlenka. Space-time discontinuous Galerkin method for solving nonstationary convection-diffusion-reaction problems. *Appl. Math.*, 52(3): 197–233, 2007.

[27] M. Feistauer and K. Švadlenka. Discontinuous Galerkin method of lines for solving nonstationary singularly perturbed linear problems. *J. Numer. Math.*, 12(2):97–117, 2004.

[28] S. Franz. Continuous interior penalty method on a Shishkin mesh for convection-diffusion problems with characteristic boundary layers. *Comput. Methods Appl. Mech. Engrg.*, 197(45-48):3679–3686, 2008.

[29] J. d. Frutos, B. García-Archilla, and J. Novo. Stabilization of Galerkin finite element approximations to transient convection-diffusion problems. *SIAM J. Numer. Anal.*, 48(3):953–979, 2010.

[30] S. Ganesan. Population balance equations, Streamline-Upwind Petrov-Galerkin finite element methods, operator-splitting method, backward Euler scheme, error analysis. Preprint 1531, WIAS, Berlin, 2010.

[31] J.-L. Guermond. Stabilization of Galerkin approximations of transport equations by subgrid modeling. *M2AN Math. Model. Numer. Anal.*, 33(6):1293–1316, 1999.

[32] E. Hansen and A. Ostermann. Dimension splitting for evolution equations. *Numer. Math.*, 108(4):557–570, 2008.

[33] E. Hansen and A. Ostermann. Dimension splitting for time dependent operators. *Discrete Contin. Dyn. Syst.*, (Dynamical Systems, Differential Equations and Applications. 7th AIMS Conference, suppl.):322–332, 2009.

[34] J. G. Heywood and R. Rannacher. Finite-element approximation of the nonstationary Navier-Stokes problem. IV. Error analysis for second-order time discretization. *SIAM J. Numer. Anal.*, 27(2):353–384, 1990.

[35] M.-C. Hsu, Y. Bazilevs, V. M. Calo, T. E. Tezduyar, and T. J. R. Hughes. Improving stability of stabilized and multiscale formulations in flow simulations at small time steps. *Comput. Methods Appl. Mech. Engrg.*, 199(13-16):828–840, 2010.

[36] T. J. R. Hughes and A. Brooks. A multidimensional upwind scheme with no crosswind diffusion. In *Finite element methods for convection dominated flows (Papers, Winter Ann. Meeting Amer. Soc. Mech. Engrs., New York, 1979)*, volume 34 of *AMD*, pages 19–35. Amer. Soc. Mech. Engrs. (ASME), New York, 1979.

[37] V. John, I. Angelov, A. Öncül, A., and D. Thévenin. Techniques for the reconstruction of a distribution from a finite number of its moments. *Chemical Engineering Science*, 62(11):2890–2904, 2007.

[38] V. John and G. Matthies. MooNMD—a program package based on mapped finite element methods. *Comput. Vis. Sci.*, 6(2-3):163–169, 2004.

[39] V. John and J. Novo. Error analysis of the SUPG finite element discretization of evolutionary convection-diffusion-reaction equations. (1494), 2010.

[40] V. John, M. Roland, T. Mitkova, K. Sundmacher, L. Tobiska, and A. Voigt. Simulations of population balance systems with one internal coordinate using finite element methods. *Chemical Engineering Science*, 64:733–741, 2009.

[41] V. John and E. Schmeyer. Finite element methods for time-dependent convection-diffusion-reaction equations with small diffusion. *Comput. Methods Appl. Mech. Engrg.*, 198(3-4):475–494, 2008.

[42] C. Johnson, U. Nävert, and J. Pitkäranta. Finite element methods for linear hyperbolic problems. *Comput. Methods Appl. Mech. Engrg.*, 45(1-3):285–312, 1984.

[43] C. Johnson and J. Pitkäranta. An analysis of the discontinuous Galerkin method for a scalar hyperbolic equation. *Math. Comp.*, 46(173):1–26, 1986.

[44] C. Johnson and J. Saranen. Streamline diffusion methods for the incompressible Euler and Navier-Stokes equations. *Math. Comp.*, 47(175):1–18, 1986.

[45] C. Johnson, A. H. Schatz, and L. B. Wahlbin. Crosswind smear and pointwise errors in streamline diffusion finite element methods. *Math. Comp.*, 49(179):25–38, 1987.

[46] D. Juhnke and L. Tobiska. A local projection type stabilization with exponential enrichments applied to one-dimensional advection-diffusion equations. Preprint, Otto-von-Guericke University of Magdeburg, IAN, January 2011.

[47] J. Kačur, B. Malengier, and M. Remešíková. Convergence of an operator splitting method on a bounded domain for a convection-diffusion-reaction system. *J. Math. Anal. Appl.*, 348(2):894–914, 2008.

[48] L. Kaland and H.-G. Roos. Parabolic singularly perturbed problems with exponential layers: robust discretizations using finite elements in space on Shishkin meshes. *Int. J. Numer. Anal. Model.*, 7(3):593–606, 2010.

[49] K. H. Karlsen, K. Brusdal, H. K. Dahle, S. Evje, and K.-A. Lie. The corrected operator splitting approach applied to a nonlinear advection-diffusion problem. *Comput. Methods Appl. Mech. Engrg.*, 167(3-4):239–260, 1998.

[50] K. H. Karlsen and K.-A. Lie. An unconditionally stable splitting scheme for a class of nonlinear parabolic equations. *IMA J. Numer. Anal.*, 19(4):609–635, 1999.

[51] K. H. Karlsen and N. H. Risebro. An operator splitting method for nonlinear convection-diffusion equations. *Numer. Math.*, 77(3):365–382, 1997.

[52] D. J. Knezevic and E. Süli. A heterogeneous alternating-direction method for a micro-macro dilute polymeric fluid model. *M2AN Math. Model. Numer. Anal.*, 43 (6):1117–1156, 2009.

[53] P. Knobloch. On the application of local projection methods to convection-diffusion-reaction problems. In *BAIL 2008—boundary and interior layers*, volume 69 of *Lect. Notes Comput. Sci. Eng.*, pages 183–194. Springer, Berlin, 2009.

[54] P. Knobloch. A generalization of the local projection stabilization for convection-diffusion-reaction equations. *SIAM J. Numer. Anal.*, 48(2):659–680, 2010.

[55] J. Koch. *Effiziente Behandlung von Integraloperatoren bei populationsdynamischen Modellen*. PhD thesis, Otto-von-Guericke-Universität Magdeburg, Fakultät für Mathematik, 2005.

[56] D. Lanser and J. G. Verwer. Analysis of operator splitting for advection-diffusion-reaction problems from air pollution modelling. *J. Comput. Appl. Math.*, 111(1-2): 201–216, 1999. Numerical methods for differential equations (Coimbra, 1998).

[57] S. Larsson, V. Thomée, and L. B. Wahlbin. Numerical solution of parabolic integro-differential equations by the discontinuous Galerkin method. *Math. Comp.*, 67(221): 45–71, 1998.

[58] P. Lasaint and P.-A. Raviart. On a finite element method for solving the neutron transport equation. In *Mathematical aspects of finite elements in partial differential equations (Proc. Sympos., Math. Res. Center, Univ. Wisconsin, Madison, Wis., 1974)*, pages 89–123. Publication No. 33. Math. Res. Center, Univ. of Wisconsin-Madison, Academic Press, New York, 1974.

[59] W. Layton. A connection between subgrid scale eddy viscosity and mixed methods. *Appl. Math. Comput.*, 133(1):147–157, 2002.

[60] R. J. Leveque. High-resolution conservative algorithms for advection in incompressible flow. *SIAM J. Numer. Anal.*, 33(2):627–665, 1996.

[61] T. Linss and M. Stynes. Numerical methods on shishkin meshes for linear convection-diffusion problems. *Computer Methods in Applied Mechanics and Engineering*, 190: 3527–3542, 2001.

[62] G. Lube and D. Weiss. Stabilized finite element methods for singularly perturbed parabolic problems. *Appl. Numer. Math.*, 17(4):431–459, 1995.

[63] D. Marchisio and R. Fox. Solution of population balance equations using the direct quadrature method of moments. *Journal of Aerosol Sciences*, 36:43–73, 2005.

[64] G. I. Marchuk. Splitting and alternating direction methods. In *Handbook of numerical analysis, Vol. I*, Handb. Numer. Anal., I, pages 197–462. North-Holland, Amsterdam, 1990.

[65] G. Matthies. Local projection methods on layer-adapted meshes for higher order discretisations of convection-diffusion problems. *Appl. Numer. Math.*, 59(10):2515–2533, 2009.

[66] G. Matthies. Local projection stabilisation for higher order discretisations of convection-diffusion problems on Shishkin meshes. *Adv. Comput. Math.*, 30(4):315–337, 2009.

[67] G. Matthies, P. Skrzypacz, and L. Tobiska. A unified convergence analysis for local projection stabilisations applied to the Oseen problem. *M2AN Math. Model. Numer. Anal.*, 41(4):713–742, 2007.

[68] G. Matthies, P. Skrzypacz, and L. Tobiska. Stabilization of local projection type applied to convection-diffusion problems with mixed boundary conditions. *Electron. Trans. Numer. Anal.*, 32:90–105, 2008.

[69] A. Mersmann. Batch precipitation of barium carbonate. *Chemical Engineering and Processing*, 38:6177–6184, 1993.

[70] A. Mersmann. Crystallization and precipitation. *Chemical Engineering and Processing*, 38:345–353, 1999.

[71] J. J. H. Miller, E. O'Riordan, and G. I. Shishkin. *Fitted numerical methods for singular perturbation problems.* World Scientific Publishing Co. Inc., River Edge, NJ, 1996. Error estimates in the maximum norm for linear problems in one and two dimensions.

[72] N. C. Nguyen, J. Peraire, and B. Cockburn. Hybridizable Discontinuous Galerkin Methods. In *Spectral and High Order Methods for Partial Differential Equations*, volume 76 of *Lect. Notes Comput. Sci. Eng.*, pages 63–84. Springer, Berlin Heidelberg, 2011.

[73] G. M. Phillips. *Interpolation and approximation by polynomials.* CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 14. Springer-Verlag, New York, 2003.

[74] D. Ramkrishna. *Population Balances: Theory and Applications to Particulate Systems in Engineering:.* Acadamic press: San Diego, CA, 2000.

[75] W. H. Reed and T. R. Hill. Triangular mesh methods for the neutron transport equation. Technical Report LA-UR-73-479, Los Alamos Scientific Laboratory, 1973.

[76] H.-G. Roos, M. Stynes, and L. Tobiska. *Robust numerical methods for singularly perturbed differential equations*, volume 24 of *Springer Series in Computational Mathematics.* Springer-Verlag, Berlin, second edition, 2008. Convection-diffusion-reaction and flow problems.

[77] K. Sanjeev and D. Ramkrishna. On the solution of population balance equations by discretization–iii. nucleation, growth and aggregation of particles. *Chemical Engineering Science*, 52:4659–4679, 1997.

[78] D. Schötzau and C. Schwab. An *hp* a priori error analysis of the DG time-stepping method for initial value problems. *Calcolo*, 37(4):207–232, 2000.

[79] D. Schötzau and C. Schwab. Time discretization of parabolic problems by the *hp*-version of the discontinuous Galerkin finite element method. *SIAM J. Numer. Anal.*, 38(3):837–875, 2000.

[80] G. Strang. On the construction and comparison of difference schemes. *SIAM J. Numer. Anal.*, 5:506–517, 1968.

[81] M. Strumendo and H. Arastoopour. Solution of population balance equations by the finite size domain complete set of trial functions method of moments (FCMOM) for inhomogeneous systems. *Ind. Eng. Chem. Res*, 49(11):5222–5230, 2010.

[82] M. Stynes and E. O'Riordan. A uniformly convergent Galerkin method on a Shishkin mesh for a convection-diffusion problem. *J. Math. Anal. Appl.*, 214(1):36–54, 1997.

[83] M. Stynes and L. Tobiska. Using rectangular Q_p elements in the SDFEM for a convection-diffusion problem with a boundary layer. *Appl. Numer. Math.*, 58(12): 1789–1802, 2008.

[84] P. Sun, L. Chen, and J. Xu. Numerical studies of adaptive finite element methods for two dimensional convection-dominated problems. *J. Sci. Comput.*, 43(1):24–43, 2010.

[85] V. Thomée. *Galerkin finite element methods for parabolic problems*, volume 25 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, second edition, 2006.

[86] N. N. Yaneko. *The method of fractional steps: solution of problems of mathematical physics in several variables*. Springer, Berlin, 1971.

[87] E. Zeidler. *Nonlinear functional analysis and its applications. II/A*. Springer-Verlag, New York, 1990. Linear monotone operators, Translated from the German by the author and Leo F. Boron.

[88] G. H. Zhou. A local L^2-error analysis of the streamline diffusion method for nonstationary convection-diffusion systems. *RAIRO Modél. Math. Anal. Numér.*, 29(5): 577–603, 1995.

Personal Information

Name	**Naveed Ahmed**
Date of birth	13.04.1981
Place of birth	Mansehra, Pakistan

Education

2008 - 2011	PhD Graduation, Faculty of Mathematics, Otto-von-Guericke University, Magdeburg, Germany
2005 - 2007	M. Phil. Applied Mathematics, Quaid-e-Azam University Islamabad, Pakistan
2002 - 2004	M. Sc. Applied Mathematics, , Quaid-e-Azam University Islamabad, Pakistan
1999 - 2001	B.Sc. Mathematics and Statistics, Peshwar University, Pakistan
1997 - 1999	HSSC. WDC Tarbela Dam, Haripur, Pakistan
1986 - 1996	SSC. COMP: High School Tarbela Dam, Pakistan

Academic Honors

- Scholarship by the Higher Education Commision of Pakistan (2008 - 2011)
- Merit Scholarship, Quaid-e-Azam University Islamabad, Pakistan (2006 - 2008)

Publications

- Hayat, T.; Ahmed, Naveed; Sajid, M. Analytic solution for MHD flow of a third order fluid in a porous channel. J. Comput. Appl. Math. 214 (2008), no. 2, 572–582
- Hayat, T.; Ahmed, Naveed; Sajid, M.; Asghar, S. On the MHD flow of a second grade fluid in a porous channel. Comput. Math. Appl. 54 (2007), no. 3, 407–414