Theories of Bounded and Unbounded Rationality in Games of Conflict and Coordination

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Summary

This dissertation combines four studies which investigate the strategic behavior in games. The first two studies are based on experiments and develop *theories of bounded rationality* in order to explain the observed behavior. This behavior violates the minimal conditions of game-theoretic rationality, i.e., that equilibrium strategies must be chosen. Instead, real behavior is driven by heuristics like fairness standards and rules of thumb. Game-theoretic rationality is too strong or it is simply inadequate to explain the behavior. The two other studies are concerned with equilibrium selection. For this purpose, game-theoretic rationality is too weak to be able to single out a unique equilibrium as the *rational solution* of a game. A theory of *unbounded rationality* is applied to determine the solutions of specific games.

The first study (Selten, Mitzkewitz & Uhlich, 1997) is concerned with a finite repetition of a Cournot duopoly. Experienced subjects program strategies for this game and computer tournaments among these programs were performed. The typical structure of these programmed strategies reveals that no expectations are formed and nothing is optimized. Instead, fairness criteria are used to determine cooperative goals which are supported by a policy of *fairness and firmness. Measure-for-measure strategies* respond to opponent's movements towards and away from the cooperative goal by similar movements. Strategies tend to be more successful in the tournaments the more typical they are.

In the second study (Mitzkewitz & Nagel, 1993) experiments on two versions of ultimatum games with incomplete information are reported and analyzed. As in the study above, subjects must determine complete strategies for the game. Game theory predicts very similar outcomes for both versions of the game, but the experimental results show significant differences. A theory of boundedly rational theory based on *expectation fairness* is proposed which is well supported by the experimental data.

The third study (Mitzkewitz, 2017) analyzes a class of simple signaling games. The *Harsanyi-Selten theory of equilibrium selection* in games is applied and for each generic game of this class a unique equilibrium is singled out as its rational solution. The results can be used as building stones for the analysis of much more complicated signaling games. The study also contains a brief introduction to the Harsanyi-Selten theory.

The last study (Potters, van Winden & Mitzkewitz, 1991) is an application of game theory to political science. A pressure group tries to influence the behavior of the government by threatening or carrying out punishments. A two-period

model shows a multiplicity of equilibria. However, most of them seem to be implausible. The *Kohlberg-Mertens refinement concept of stable equlibria* is used to reduce this multiplicity. The *Harsanyi-Selten theory of equilibrium selection* for this model is also calculated. It turns out that for a fairly large set of parameter values both solution concepts coincide and show that punishment is executed even if the government makes in advance concessions to the pressure group.

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Duopoly Strategies Programmed by Experienced Players

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DUOPOLY STRATEGIES PROGRAMMED BY EXPERIENCED PLAYERS

BY REINHARD SELTEN, MICHAEL MITZKEWITZ, AND GERALD R. UHLICH

The strategy method asks experienced subjects to program strategies for a game. This paper reports on an application to a 20-period supergame of an asymmetric Cournot duopoly. The final strategies after three programming rounds show a typical structure. Typically, no expectations are formed and nothing is optimized. Instead of this, fairness criteria are used to determine cooperative goals, called "ideal points." The subjects try to achieve cooperation by a "measure-for-measure policy," which reciprocates movements towards and away from the ideal point by similar movements. A strategy tends to be more successful the more typical it is.

KEYWORDS: Duopoly, strategy method, computer tournament.

1. INTRODUCTION

AFTER 150 YEARS SINCE COURNOT (1838) the duopoly problem is still open. An empirically well supported duopoly theory has not yet emerged. Field studies meet the difficulty that cost functions, demand functions, and prices are often insufficiently observable. Game playing experiments permit the control of these basic data. However, only plays are observed and strategies remain hidden. Usually, any given play of a duopoly supergame can be the result of a great multitude of strategy pairs.

More than 20 years ago, one of the authors described a method of experimentation which makes strategies observable (Selten (1967)). This procedure, called the "strategy method," first exposes a group of subjects to the repeated play of a game, and then asks them to design strategies on the basis of their experiences. The strategy method was applied to an oligopoly situation with investment and price variation (Selten (1967)). In view of the special character of the dynamic oligopoly game investigated there, the issue of cooperation which will be important in the paper did not arise in this earlier study. Here we are concerned with a much more basic duopoly situation, namely a finite supergame of an asymmetric Cournot duopoly. Asymmetry is essential for this study, because we are interested in whether and how cooperation can evolve in such situations.

Cournot's quantity variation model is the most popular one in the oligopoly literature. Many theories have been developed in this framework. Supergames of the Cournot model have also been explored in the newer game-theoretical literature (e.g., Friedman (1977), Radner (1980), Abreu (1986), Segerstrom (1988)). Therefore, it seems to be interesting to apply the strategy method to a supergame of an asymmetric Cournot duopoly.

Infinite supergames cannot be played in the laboratory. Attempts to approximate the strategic situation of an infinite game by the device of a supposedly fixed stopping probability are unsatisfactory since a play cannot be continued beyond the maximum time available. The stopping probability cannot remain

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fixed but must become one eventually. Therefore, we decided to base our study on a finite supergame. The experimental literature shows that apart from the end effect there seems to be no significant behavioral difference between infinite and sufficiently long finite supergames (Stoecker (1983), Selten and Stoecker (1986)).

Our subjects were participants of a seminar who first gained experience in playing a 20-period supergame in the Bonn laboratory of experimental economics which is equipped with a network of personal computers. After having gained experience with the game, the participants had to program strategies. These strategies were played against each other in computer tournaments. The participants had the opportunity to improve their strategies in the light of their experience in such tournaments.

Our evaluation will mainly concern the strategies programmed for the final computer tournament. We shall only shortly report on some interesting phenomena observed in the initial game playing rounds and the intermediate tournaments.

The first step in the evaluation of the final tournament strategies was a classification according to structural properties. These properties, called "characteristics," were suggested by a close look at the strategies. We found 13 characteristics, all of which are present in the majority of cases to which they can be applied. A typical structure of a strategy emerges from these characteristics. The programs usually distinguish among an initial phase, a main phase, and an end phase. The initial phase consists of the first one to four periods with outputs depending only on the number of the period. In the main phase, outputs were made dependent on the opponent's previous outputs. By the initial phase the strategies try to prepare cooperation with the opponent to be reached in the main phase. In an end phase of the last one to four periods cooperation is replaced by noncooperative behavior.

Typically, the participants tried to approach the strategic problem in a way which is very different from that suggested by most oligopoly theories. These theories almost always involve the maximization of profits on the basis of expectations on the opponent's behavior. It is typical that the final tournament strategies make no attempt to predict the opponent's reactions and nothing is optimized. Instead of this, a cooperative goal is chosen by fairness considerations and then pursued by an appropriate design of the strategy. Cooperative goals take the form of "ideal points." An ideal point is a pair of outputs at which a player wants to achieve cooperation with his opponent. Such ideal points guide the behavior in the main phase. A move of the opponent towards the player's ideal point usually leads to responses which move the player's output in the direction of his ideal point. Similarly, a move of the opponent away from the ideal point is usually followed by a response which shifts the output away from the ideal point. We refer to this kind of behavior as a "measure-for-measure policy."

The fairness criteria underlying the selection of ideal points are different for different participants, but in most cases not completely arbitrary. Measure-for-

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measure policies for the effectuation of ideal points may be quite different in detail, but they are all based on the same general idea.

On the basis of the 13 characteristics which express structural properties common to most of the strategies we have constructed a measure of typicity which is applied both to characteristics and strategies. The typicity of a strategy is proportional to the sum of the typicities of its characteristics and the typicity of a characteristic is porportional to the sum of the typicities of the final tournament strategies with this characteristic. It was an unexpected result of our investigation that there is a highly significant positive correlation between the typicity of a final tournament strategy and its success in the final tournament. Moreover, it turned out that for each of the 13 characteristics separately those strategies which have it are more successful than those which do not have it.

In order to get a better insight into the implications of the typical structure of final tournament strategies, we constructed a family of "simple typical strategies." In these strategies the details left open by the 13 characteristics are filled in the simplest possible way. The behavior in the main phase is described by a piecewise-linear continuous reaction function.

Two game-theoretical requirements on simple typical strategies are discussed: "conjectural equilibrium conditions" and "stability against short-run exploitation." These requirements impose restrictions on the ideal points. The first requirement is rarely satisfied but the second one is fulfilled by the vast majority of the ideal points used in final strategies. This condition also turns out to be of descriptive value for the profit combinations reached in the last tournament.

We do not claim that our results are transferable to real duopolies. First of all, it is doubtful whether a supergame of the Cournot duopoly is a realistic description of duopolistic markets. Nevertheless, the structure of behavior in such supergames is of great theoretical interest. Our results throw a new light on the duopoly problem posed in this framework. The choice of an ideal point by fairness consideration combined with the pursuit of this cooperative goal by a measure-for-measure policy constitutes a surprisingly simple approach which avoids optimization and the prediction of the opponent's behavior. The connection between typicity and success in the final tournament shows that this approach is not only simple and practicable but also advisable in the pursuit of high profits.

The participants of our seminar did not develop their strategy programs independently of each other. Interaction in the game playing rounds and the preliminary tournaments was unavoidable. It cannot be completely excluded that our results are due to a cultural evolution which might have a different outcome in a different experimental group. One application of the strategy method alone is not sufficient to establish a firm basis for far-reaching behavioral conclusions.

The tit-for-tat strategy which did so well in Axelrod's tournaments (Axelrod (1984)) is the natural consequence of the transfer of the strategic approach emerging from this study to the prisoner's dilemma supergame. There one finds only one reasonable ideal point, namely the cooperative choice taken by both

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players, and only one measure-for-measure policy fitting this cooperative goal, namely tit-for-tat.

More recently, a paper by Fader and Hauser (1988) reports on programs written for two symmetric price triopolies. The players had no opportunity to play the games before writing their strategies and submitted a program only once for each of both models. Fader and Hauser classified strategies according to "features," but it cannot be said that a typical structure emerges from this classification. Perhaps the lack of a typical structure is due to the fact that in comparison to our students the participants of the tournaments were much less experienced with their task. Maybe it is necessary to provide the opportunity to gain extensive game-playing experience and to permit repeated program revisions after preliminary tournaments in order to obtain strategies which show a typical structure.

Nevertheless, this study shows that strategies based on the measure-for-measure principle are very successful against the strategies submitted. The agreement of our findings with those of Axelrod and of Fader and Hauser confirms our impression that the pursuit of ideal points by measure-for-measure policies is more than the accidental result of an isolated study.

The model and the experimental procedure are described in Sections 2 and 3. Then the results of the game playing rounds and the results of the tournaments are discussed in Sections 4 and 5. The evaluation of the strategies programmed for the final tournament begins with Section 6. There the 13 characteristics are introduced and explained in detail. The strategic approach underlying typical strategies is discussed. Section 7 is devoted to the connection between typicity and success. A family of simple typical strategies is introduced in Section 8 as an idealization of the general pattern observed in the programmed strategies. Theoretical properties of these strategies are discussed and game-theoretic stability conditions are compared with the data of the final tournament. Section 9 looks at the implications of our results for duopoly theory. A summary of our findings is given in Section 10.

2. THE MODEL

The experiment is based on a fixed nonsymmetric Cournot duopoly with linear cost and demand functions. Strategies had to be programmed for the 20-period supergame of this Cournot duopoly. The duopolists were fully informed about cost and demand functions, the length of the supergame, and the opponent's decisions in past periods. The decision variable of duopolist i in period t is the quantity $x_i(t)$ for $i=1,2$ and $t=1,\ldots,20$. Quantities must be chosen from nonnegative real numbers. The costs $C_1(t)$ and $C_2(t)$ of duopolists 1 and 2 and the price $p(t)$ in period t are given as follows:

$$C_1(t) = 9820 + 9x_1(t), \qquad x_1(t) \geq 0,$$
$$C_2(t) = 1260 + 51x_2(t), \qquad x_2(t) \geq 0,$$
$$p(t) = \max\,(0; 300 - x_1(t) - x_2(t)).$$

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TABLE I

SOME THEORETICAL POINTS IN THE SOURCE GAME

Concept	Player 1's Output	Player 2's Output	Price	Player 1's Profit	Player 2's Profit
Cournot	111.0	69.0	120.0	2501.0	3501.0
Monopoly of player 1	145.5	0.0	154.5	11350.3	−1260.0
Monopoly of player 2	0.0	124.5	175.5	−9820.0	14240.3
Stackelberg with player 1 as leader	166.5	41.3	92.3	4041.1	441.6
Stackelberg with player 2 as leader	93.8	103.5	102.7	−1030.9	4096.1
Nash product maximum	86.8	49.5	163.7	3615.0	4313.5
Pareto optimum A of Figure 1	79.1	56.1	164.8	2503.8	5124.2
Pareto optimum B of Figure 1	94.8	42.7	162.5	4731.8	3501.0

The supergame payoff of each duopolist is the sum of his profits over all twenty periods.

Table I and Figure 1 show some theoretical features of the Cournot duopoly described above. The row "Nash product maximum" presents the output combination which maximizes the Nash product with the Cournot solution as fixed threat point. Point A in Figure 1 is the Pareto optimum which yields Cournot equilibrium profits for player 1. Analogously, B is the Pareto optimum which yields Cournot profits for player 2. Figure 1 shows that the model is quite asymmetric. Even point A is below the 45-degree line.

3. EXPERIMENTAL PROCEDURE

The experiment was performed in a seminar lasting over the whole summer semester 1987 at Bonn University, Federal Republic Germany. The subjects were 24 students of economics in the third or fourth year with some knowledge of micro- and macroeconomics and some experience with computer programming, but without special training in price theory and game theory. No introduction in these fields was given in the seminar and no references to the relevant literature was supplied. The seminar was organized in five plenary sessions, three rounds of game playing, and three computer tournaments of programmed strategies.

Plenary sessions: In the first plenary session the participants were informed about the organization of the seminar and the model presented in Section 2,

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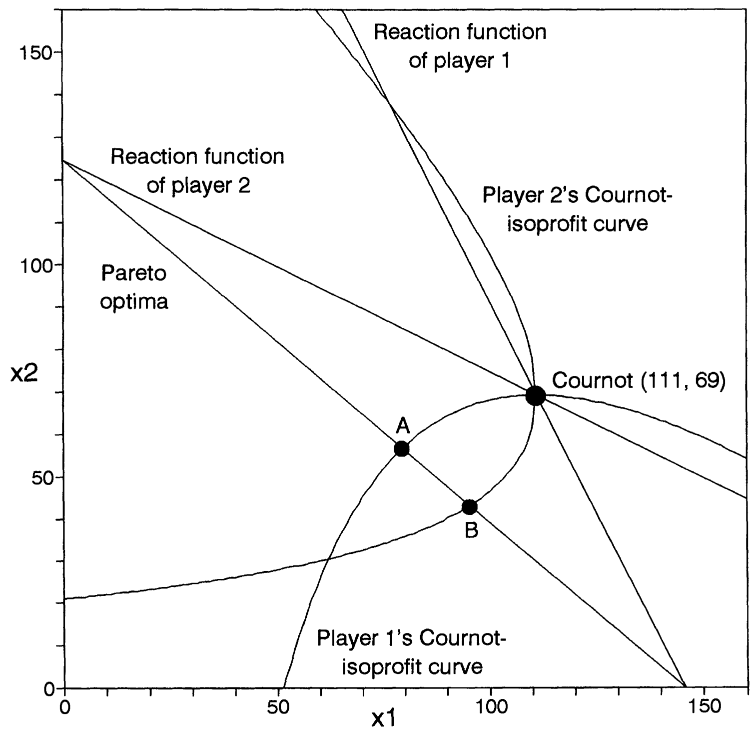

FIGURE 1.—Graphical representation of theoretical features of the one-period Cournot duopoly.

but, of course, without the theoretical features. Moreover, an introduction to the programming of strategies in PASCAL was given. It was not necessary to explain more than an excerpt of PASCAL, since strategies were conceived as subroutines in a game program.[1]

The participants had the task first to gain experience by three rounds of playing the 20-period supergame and then to program strategies for both players in the 20-period supergame. They were told that their objective should be to attain a sum of profits as high as possible in a final tournament in which the strategies of all participants compete against each other. Final strategies had to be documented and reasons had to be given for the decisions embodied in the strategies.

The second plenary session took place after two rounds of game playing. The results of these games were presented, but in a way which left players anonymous. The participants were asked to comment on their experiences.

[1]The Pascal source code of the students' strategies is available on request.

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Each of the three tournaments was followed by one plenary session. Results were presented and students received printouts of the games in which their own strategies were involved. Opponents remained unidentified. The participants were encouraged to discuss strategic problems.

In the last of the five plenary sessions, the most successful participant explained his strategy. Anonymity was not completely preserved in this final plenary session at the end of the seminar.

Game playing rounds: Twenty-two subjects played three 20-period supergames against changing anonymous opponents, two subjects played only two supergames. The subjects were visually isolated from each other in cubicles containing computer terminals. The players interacted only by their decisions via the computer network. The decision time for each period was limited to three minutes. One week passed between one supergame and the next one. In this time the participants had the opportunity to reflect on their experiences. Each subject played with each of both cost functions at least once.

Strategy programming: After the game playing rounds the students had to program strategies in PASCAL for the 20-period supergame. Every student had to write a pair of strategies, one for each player of the supergame. We shall refer to this pair as the student's strategy. PC-owners could program at home, but all participants had the opportunity to develop their strategies at the Bonn laboratory of experimental economics with our technical assistance. A special program called "trainer" could be used by the students to play against their own programmed strategies. The "trainer" was a valuable tool for the development of strategies. No restrictions were imposed on strategies. Decisions could depend on the whole previous history of the play.

Computer tournaments: At three fixed dates the students had to hand in a programmed strategy. In the first two tournaments all workable strategies submitted at this date competed with each other. In the third tournament the last workable strategy of each participant was used. Each of the 24 students succeeded in writing at least one workable strategy.

The tournament program proceeded as follows: Let n be the number of workable strategies. Each of the n strategies played against all others in the role of both players. Payoff sums for player 1 and player 2 were computed on the basis of the $n-1$ games played in the concerning role.

The procedure has the consequence that for each pair of strategies and each assignment of player roles, two supergames are simulated even if the payoff summation for one strategy makes use only of one of these games. Since sometimes random decisions are used in strategy programs, both games may be different. Altogether, $2n(n-1)$ supergames were simulated in a tournament. The success of a strategy can be measured for the roles of both players separately by the corresponding payoff sums. The sum of these two measures is a measure for the overall success of a strategy in a tournament. This measure of overall success was the goal variable in the tournament. Strategies were ranked according to the measure of overall success, but also for the success of both player roles separately. Each participant received period-by-period printouts of

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all $2(n-1)$ games underlying the computation of his success measure. Moreover, all participants received lists of success measures, but without identification of the other writers of strategy programs. On the basis of this information the students could try to improve their strategy programs from one tournament to the next one.

Motivation: In view of the length of the experiment, it was not possible to provide an appropriate financial incentive. Presumably, money payoffs in the framework of a student seminar are not legal anyhow. The students were told that their grades would strongly depend on their success in the last tournament. It was emphasized that the absolute payoff sum rather than the rank was important in this respect. We had the impression that for almost all participants the task itself provided a high intrinsic motivation.

4. RESULTS OF THE GAME PLAYING ROUNDS

In this section we give a brief summary of the results of the game playing rounds. The games served the purpose to provide experiences which could be used in the development of strategy programs. Of course, it is plausible to assume that the subjects were intrinsically motivated by the game payoffs, but it is also possible that some of the behavior in these games was exploratory rather than directly payoff-oriented. Nevertheless, it is interesting to look at the results of the game playing rounds. However, our discussion will not be very detailed because our main interest is in the investigation of the final strategy programs.

First game playing round: Although the participants had been informed one week in advance about the structure of the game, their behavior seemed to be confused. Figure 2 shows the supergame payoffs of the 11 groups (two partici-

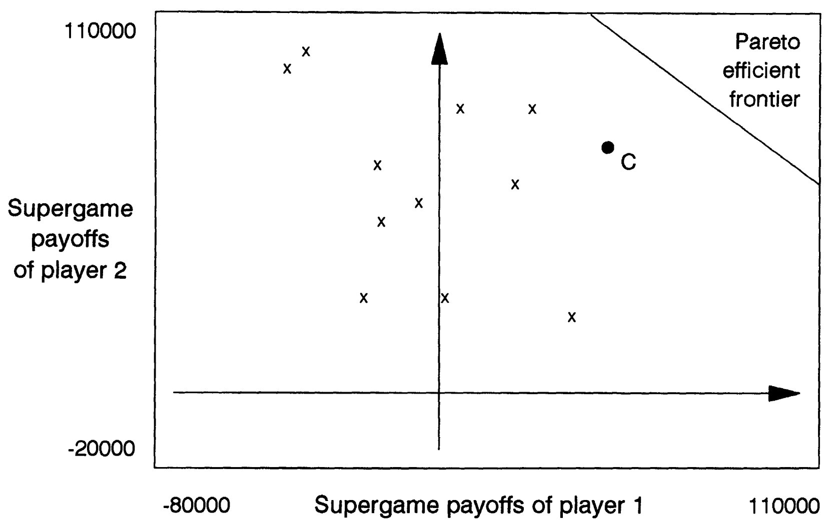

FIGURE 2.—Supergame payoff pairs in the first game playing round.

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pants were absent). The repeated Cournot solution (point C in the diagram) yields 50020 for player 1 and 70020 for player 2. It never happened that *both* players achieved at least their Cournot payoffs. Furthermore, in all 11 cases the sum of both supergame payoffs was below the sum of the Cournot payoffs. In seven cases both players earned less than the Cournot payoffs. The role of player 1 (low variable and high fixed costs) was relatively less successful than the role of player 2. In the mean, subjects in the role of player 1 earned 79% of the Cournot gross profit (gross profit is profit plus fixed costs), whereas the corresponding figure for player 2 is 91%. The correlation coefficient between the payoffs of the two players within the groups is $-.36$. This suggests that some players succeeded to exploit their opponents. Figure 2 also shows part of the Pareto efficient frontier.

Second game playing round: The results of the second game playing round are shown in Figure 3. Here, two groups reached a Pareto improvement over the Cournot payoffs. In one group both players supplied the Cournot outputs in almost all periods. "It's the best thing you can do," they commented afterwards. In the remaining nine groups, both players sustained a loss in comparison with the Cournot solution. The mean gross profits of subjects in the role of player 1 was higher than in the first game playing round (87% of the Cournot gross profit), but the mean gross profit of subjects in the role of player 2 was lower than in the first game playing round (77% of the Cournot gross profit). The mean joint profit of both players was only slightly improved compared with the first game playing round.

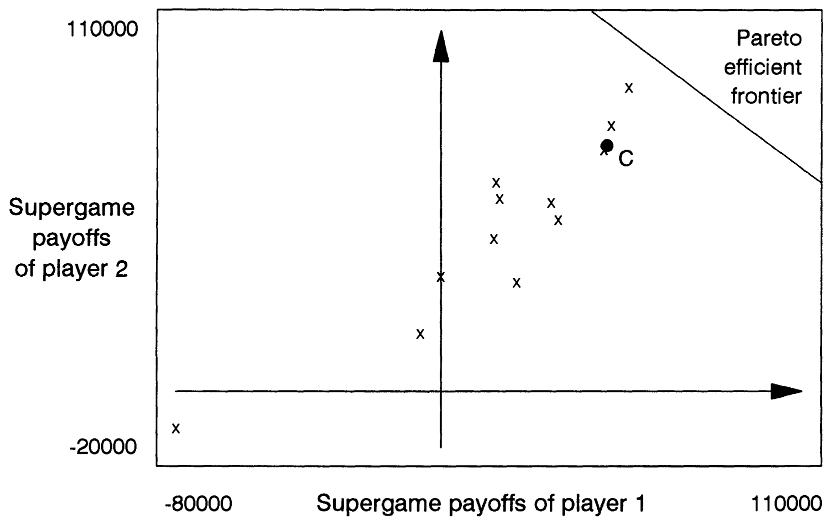

FIGURE 3.—Supergame payoff pairs in the second game playing round.

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There is one striking difference to the first game playing round. In the second game playing round, the correlation coefficient between both players' payoffs is now +.91. This suggests that in the second game playing round the aggressiveness of both players shows a stronger coordination than in the first one. Even if most of the subjects did not yet succeed to play the game well, they seemed to have learned something about the power relationship in the game.

Third game playing round: This round shows an enormous improvement in mean payoffs (Figure 4). Now, subjects in the role of player 1 achieved 101% of the Cournot gross profit and the corresponding figure for those in the role of player 2 is 107%. Eight of the twelve groups succeeded to obtain Pareto improvements over the Cournot payoffs. One group reached a result almost at the Pareto efficient frontier. This group was the only one among those with Pareto improvements over the Cournot payoffs which did not show an end effect. The end effect consists in the breakdown of cooperation in the last periods of the supergame. It is clear that payoffs at the Pareto efficient frontier cannot be achieved if an end effect occurs.

The correlation coefficient between the payoffs of both players is +.72 in the third game playing round. In this respect, the third game playing round is similar to the second one.

It is clear that most of the subjects had learned to cooperate in the supergame in the third game playing round. The results of the three game playing rounds are not dissimilar to those obtained in other experimental studies where finite supergames were repeatedly played against changing anonymous opponents (Stoecker (1983), Selten and Stoecker (1986)). Subjects tend to learn to cooperate but they also learn to exhibit end effect behavior.

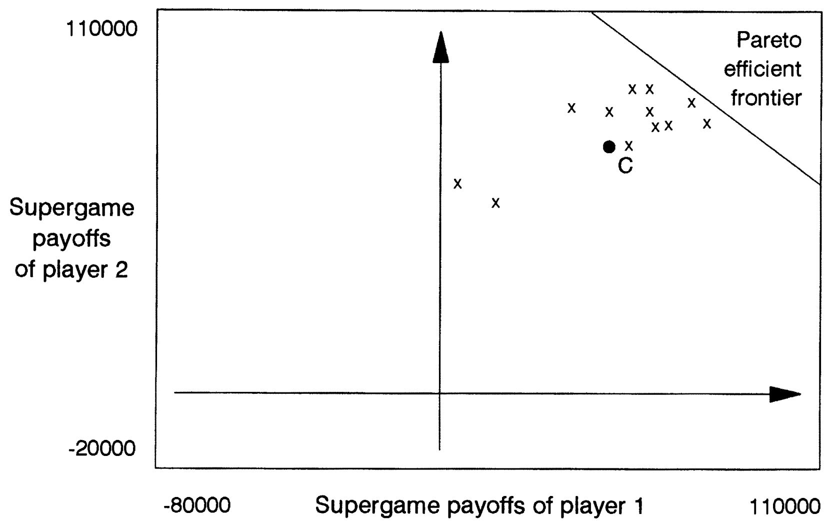

FIGURE 4.—Supergame payoff pairs in the third game playing round.

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5. RESULTS OF THE TOURNAMENTS

In the following section we shall discuss the results of the tournaments without giving a detailed account of the strategies used. The typical structure of the strategies of the final tournament will be examined in the next section.

First tournament: Two weeks after the third game playing round the participants had to hand in a programmed strategy for the supergame. Unfortunately, 4 of the 24 strategies had to be excluded from the first tournament since programming errors like dividing by zero or taking the root of a negative number prevented the execution of these programs. The outcome of the first tournament is presented in Figure 5. The significance of the points in Figure 5 is not the same as in Figures 2, 3, and 4. A point now shows the combination of mean payoffs achieved by one participant's strategy in both player roles. Moreover, a larger scale has been chosen. One of the 20 strategies competing in tournament 1 is not shown in Figure 5 since it achieved a very bad result, namely (−6484, +58178), which is outside the scope of the drawing. We omitted this point in order to be able to present the results of all three tournaments with the same scale without losing the distinguishability of different points.

The participant with the omitted bad result programmed a strategy which supplied the respective Stackelberg leader output each period regardless of the behavior of the other player. Only a few times he succeeded in forcing his opponent to the Stackelberg follower position. In most cases his "aggressive" behavior was punished by high opponent's outputs.

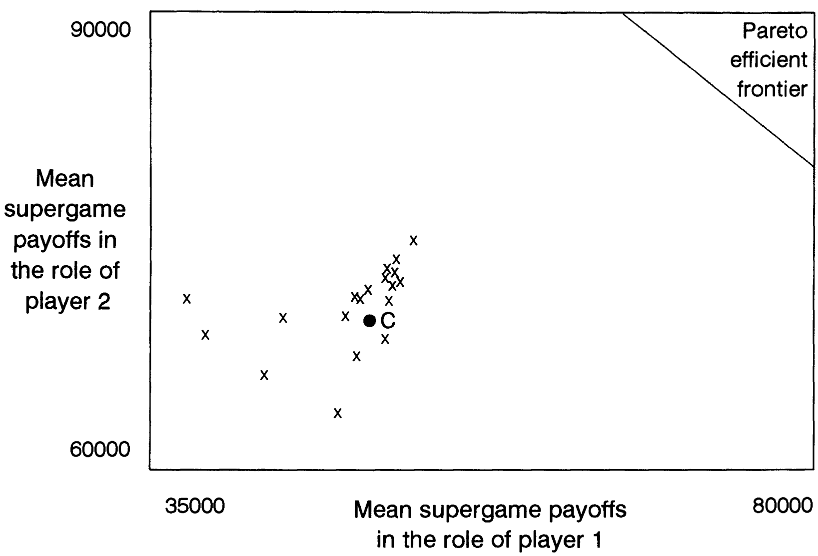

FIGURE 5.—Mean supergame payoffs for both player roles in the first tournament. Each "X" refers to one participant.

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The mean gross profit over the whole simulation was 98% of the Cournot gross profit for the role of player 1 and 99.8% of the Cournot gross profit for the role of player 2. The mean performance is inferior to the third game playing round. Maybe the subjects did not yet succeed sufficiently well to mold their game playing intuition into computer programs.

Second tournament: Within three weeks after the first tournament the participants had the opportunity to improve their strategies. Unfortunately, this time only 16 participants presented workable strategies. In the same way as in Figure 5, the results are shown in Figure 6. One point, namely (23860, 63691) is omitted in Figure 6. Each of the other 15 subjects achieved results higher than Cournot payoffs in both player roles. The mean gross profit was now 104% of Cournot gross profit for Player 1 and 109% of Cournot gross profit for player 2. This is a considerable improvement in comparison with the first tournament. It must be admitted, however, that the comparison with the first tournament is difficult in view of the smaller number of workable strategies. Moreover, the result of the second tournament is also influenced by a "conspiracy" of two subjects represented by the two points nearest the right border of Figure 6. In the first period both participants used special outputs specified up to many decimal places in an unusual way. With the help of this code they recognized each other when they played together in the tournament. They then played in the remaining periods the output combination that maximizes joint profits. In order to prevent this type of behavior in the final tournament, we replaced the 8th digit behind the decimal point of each output decision by a random number.

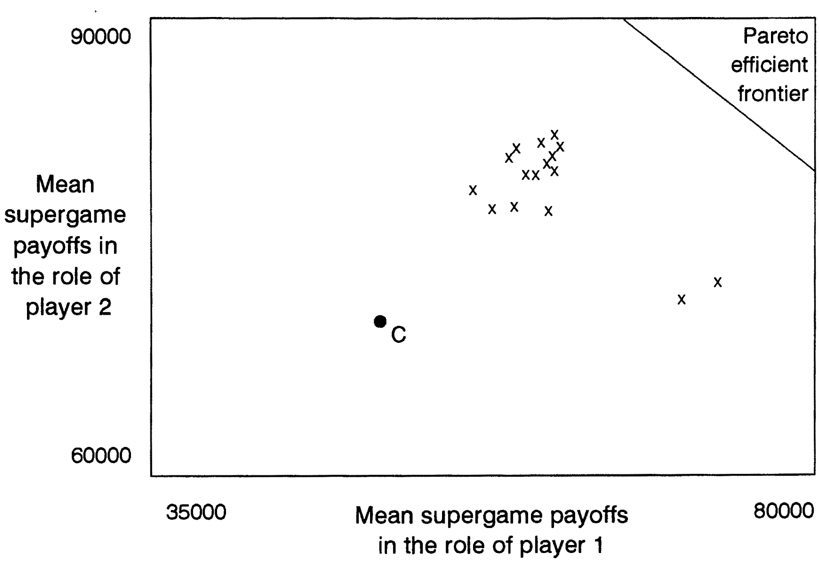

FIGURE 6.—Mean supergame payoffs for both player roles in the second tournament. Each "X" refers to one participant.

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This has a negligible influence on the computation of profits. In the plenary session after the second tournament, we announced that similar conspiracies will be prevented in the future. We did not observe any attempt to conspire in the final tournament.

Third tournament: After two more weeks the final strategies had to be turned in. Again four participants did not succeed to program a workable strategy. Fortunately, each of these participants had completed at least one workable strategy in the two preceding tournaments. The last workable strategy entered the final tournament.

A superficial examination of the programs revealed that one strategy consisted of two sequences of fixed outputs for every period, one sequence for each player. The numbers varied unsystematically from period to period. The seminar paper of this student loosely described a completely different strategy which was much more reasonable. Obviously, this student wanted to avoid investing time and effort into the programming of the strategy described in his paper. The irregularity of the output sequences served the purpose of hiding the discrepancy between the program and its description in the seminar paper. Obviously, the programmed strategy cannot be taken seriously and therefore has been excluded from the third tournament for the purposes of this paper.

The mean gross profit was 105% of the Cournot gross profit for player 1 and 111% of the Cournot gross profit for player 2. These figures are only slightly higher than those of the second tournament. Figure 7 shows the results of the third tournament. Computations of standard deviations of mean payoffs confirm the visual impression that the points in Figure 7 are more strongly concentrated than those in Figure 6.

In 983 of the 1012 supergames simulated in the third tournament, the payoffs of *both* players were greater than their Cournot payoffs. In this sense, we can speak of successful cooperation in 97.1% of all cases. It is also worth mentioning that in none of the remaining 29 supergames did *both* players obtain smaller payoffs than their Cournot payoffs.

In the third game playing round only eight out of twelve supergames resulted in payoffs which were greater than the corresponding Cournot payoffs for both players. The comparison with the results of the third tournament shows that the final programmed strategies tend to be much more cooperative than the behavior in the third game playing round. This suggests that the learning process which began with the three game playing rounds was continued in the three tournaments. The results of the third tournament do not seem to be very different from that which could be expected as the outcome of spontaneous game playing after a comparable amount of experience.

6. THE STRUCTURE OF PROGRAMMED STRATEGIES

In this section we shall concentrate our attention on the structure of the final strategies. We shall not be concerned with the success of the strategies. For the reasons which have been discussed in Section 5 (third tournament), one of

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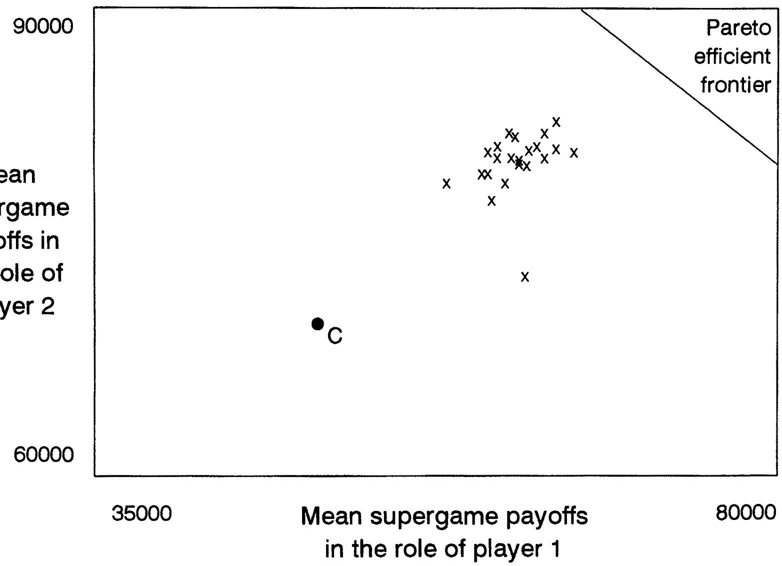

FIGURE 7.—Mean supergame payoffs for both player roles in the third tournament. Each "X" refers to one participant.

the programs will not be considered here. The remaining 23 programs and the underlying ideas expressed in the seminar papers are the basis of the evaluation of structural properties.

A preliminary examination of the strategies and the seminar papers conveyed the impression of a typical structure which is more or less present in almost all programs. Most programs deviate from this typical structure in some respects but the degree of conformity is remarkable.

Usually a program distinguishes three phases of the supergame: an initial phase, a main phase, and an end phase. The initial phase consists of one to four periods and the end phase is formed by the one to four last periods. The main phase covers the periods between the initial phase and the end phase. Different methods of output determination are used in the three phases. The initial phase is characterized by fixed outputs which do not depend on the behavior of the opponent. In the main phase the decisions are responsive to previous developments with the purpose to establish cooperation. In the end phase decisions are guided by the attempt to maximize short-run payoffs.

Different strategies approach the decision problems of the three phases in different ways, but nevertheless a typical structure emerges in this respect, too. In order to describe similarities and differences among the 23 strategies, we introduce 13 *characteristics*. A characteristic is a property of a strategy whose presence or absence can be objectively determined by the examination of a program and its description in the seminar paper. In some cases our characteristics are indicators of strategic ideas underlying the program; in other cases the

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characteristics directly refer to the structure of decision rules. We shall distinguish characteristics concerning general principles and the three phases of the supergame.

All characteristics are typical in the sense that they are present in the majority of all strategies to which they can be meaningfully applied. Characteristic 7 is meaningful only if Characteristic 6 holds, too, and Characteristics 12 and 13 presuppose that the strategy has an end phase. These three characteristics are present only in the majority of all relevant cases. All other characteristics hold for the majority of all final strategies.

6.1. *General Principles*

The first three characteristics are indicators of general principles underlying the typical approach to the problem of writing a strategy program.

CHARACTERISTIC 1: *No prediction.*

Many oligopoly theories proceed from the assumption that a player has a method to predict his opponent's behavior and tries to optimize against his predictions. The predictions may involve reactions to own output changes and the payoff maximization may be long-term rather than short-term. In the final tournament, only 5 of 23 strategies involved any predictions of the opponent's behavior.

In the first two tournaments, predictions were more widespread. Subjects tried to obtain a satisfactory payoff against the predicted output of the opponent in the next period. Several subjects who initially wrote programs involving predictions later expressed the opinion that it is useless to try to predict the opponent's behavior. It seems to be more important to react in a way which indicates willingness to cooperate and resistance to exploitation.

The fact that the absence of any predictions is a typical feature of final strategies seems to be of great significance, precisely because it is in contrast with most oligopoly theories.

CHARACTERISTIC 2: *No random decisions.*

At the beginning of the seminar we observed that several students preferred to build random decisions into their strategies. They motivated this by the belief that a deterministic strategy could possibly be outguessed and exploited by the opponent. In the course of the seminar, most of them learned that in an attempt to achieve cooperation, it is important to signal one's intentions. It may be preferable to be outguessed by the opponent. Cooperation requires reliability and random decisions may be counterproductive in this respect. Twenty-two of the 23 final strategies never make a random decision.

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CHARACTERISTIC 3: *Non-integer outputs*.

It is natural that real persons playing at computer terminals use mostly integer outputs. This was actually the case in the game playing rounds. Usually, a programmed strategy employs functions which make the output dependent on previous quantities. In general, the values obtained are not integers. However, four of the final strategies did not specify such functions but rather made case distinctions; for each case a different integer output or integer output change was prescribed. Since only relatively few cases are distinguished, this way of programming output decisions is less flexible than the specification of a function. In this light, Characteristic 3 is an indicator of smoothness and flexibility of the response pattern.

6.2. *The Initial Phase*

Two characteristics describe the typical behavior in the initial phase.

CHARACTERISTIC 4: *Fixed outputs for at least the first two periods*.

If no randomization takes place the first period is always a fixed output. Therefore, Characteristic 4 is almost equivalent to a nontrivial initial phase where fixed outputs are chosen. Ten strategies make their decision for the second period dependent on their opponent's choice in the first period, but 13 strategies have fixed amounts for more than one period. The length of the initial phase with fixed outputs is two periods for seven strategies, three periods for four strategies, and four periods for two strategies. Twelve of the 13 strategies with nontrivial initial phases play successively reduced outputs. The participants explained this behavior as a signal of their willingness to cooperate. If one's own output is a response to that of the opponent too early, an unsatisfying decision of the opponent in the first period could lead to an aggressive reaction of oneself in the second period that again could annoy the opponent and so forth, so that no cooperation might evolve over the 20 periods. Some subjects observed such unfavorable oscillations in the printouts of the first two tournaments.

CHARACTERISTIC 5: *The last fixed output decision is at least 8% below the Cournot quantity of the concerning player*.

The percentage by which the last fixed output in the initial phase is below the Cournot output can be regarded as a rough measure of a strategy's initial cooperativeness. A Pareto optimum is reached if both players' outputs are about 24.5% below the Cournot output. The criterion of the 8% limit of Characteristic 5 goes roughly a third of the way towards this Pareto optimum. Admittedly, it is arbitrary to measure cooperativeness by percentages of the Cournot output and to fix the limit at exactly 8%. Characteristic 5 is present in 13 of the 23 strategies. If the limit were increased to 10%, only a minority of 10 strategies would meet the criterion.

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6.3. *The Main Phase*

The decision rules for the main phase are the most important part of a strategy program. Characteristics 6 to 11 concern the main phase. The rules given there do not apply to the initial phase and the end phase. This will not be mentioned explicitly in the text of the characteristics.

Typically participants approached the problem of decision making in the main phase by first looking at the question of where cooperation should be achieved. They tried to find an output combination which gives higher profits than Cournot profits to both players and can be considered as a reasonable compromise between the interests of both players. An output combination of this kind which guides the decisions in the main phase will be called an "ideal point." Ideal points are usually not far away from Pareto optimality. They are often based on considerations of equity which will be described below. Some participants used different ideal points for the roles of both players.

In Characteristic 6 we shall speak of "decisions guided by ideal points." With these words we want to express that the strategy program makes explicit use of an ideal point in order to determine the next output as a function of the past history. This can be done in many ways. One possible method connects the ideal point and the Cournot point by a straight line segment in the quantity or profit space. The next output then matches the opponent's last output on the line segment as long as the opponent's last output is in the range where this is possible.

CHARACTERISTIC 6: *Decisions are guided by one or two ideal points.*

The property expressed by Characteristic 6 holds for 18 of the 23 final strategies. Twelve strategies use only one ideal point for both players, whereas 6 strategies specify different ideal points for the two player roles.

Table II gives an overview over the equity concepts underlying the construction of ideal points as far as such concepts could be identified on the basis of the seminar papers. The reasons for the choice of 10 of the 24 ideal points are at least partially unclear. To some extent ideal points were adapted to the learning experience of the first two tournaments and thereby shifted away from equity concepts.

The participants who based their ideal points on equity considerations often did not correctly compute the intended ideal points. They rarely used analytical methods but rather relied on more or less systematic numerical search. The values used instead of the correct ones are given in the footnotes below Table II.

The concept described by the first row of Table II looks at equal profit increases in comparison to Cournot profits as a fair compromise. The Pareto optimum corresponding to this idea is the intended ideal point. The concept of the second row requires profit increases proportional to Cournot profits at a Pareto optimal point.

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TABLE II

CONCEPTS UNDERLYING IDEAL POINTS

Concept	Quantities		Number of Strategies
	Player 1	Player 2	
Maximal equal absolute additional profits compared to Cournot profits	85.61	50.50	3[a]
Maximal profits proportional to Cournot profits	84.37	51.56	2[b]
Profit monotonic quantity reduction along the straight line through the intersections of both Cournot-isoprofit curves	86.53	49.71	2
Profit monotonic quantity reduction proportionally to Cournot quantities	89.73	55.77	1[c]
Maximal equal profits	89.70	47.01	2[d]
Prominent numbers	85.00	50.00	2
	90.00	50.00	2
Unclear	—	—	10

[a]Approximated by (85.50) in all three cases.
[b]Approximated by (86.53) and (84.33, 51.55).
[c]Approximated by (89.76, 55.80).
[d]In one case approximated by (89.0, 46.5).

The third and fourth rows of Table II involve a procedure referred to as *profit monotonic quantity reduction*. Along a prespecified positively inclined straight line through the Cournot point in the quantity diagram, quantities are gradually reduced as long as both profits are increased in this way. The output combination reached by the procedure is the ideal point. In the case of row 3 of Table II, the prespecified straight line connects the intersections of both Cournot isoprofit curves. In the case of row 4 the prespecified straight line connects the Cournot point and the origin.

The concept of row 3 yields a Pareto optimum even if Pareto optimality is not a part of the underlying idea. Contrary to this, profit monotonic quantity reduction proportional to Cournot quantities yields an ideal point which is not even approximately Pareto optimal.

The concept of maximal equal profits determines the Pareto optimum where both profits are equal. Obviously, this ideal point does not only depend on variable costs but also on fixed costs. The same is true for maximal profits proportional to Cournot profits. Two of the ideal points classified as unclear also were based on equal profits but without an attempt towards maximization.

Some participants chose pairs of prominent quantities as ideal points. Roundness in the sense of divisibility by 5 seems to be the prominence criterion. More detailed discussions of prominence in the decimal system can be found in the literature (Schelling (1960), Albers and Albers (1983), Selten (1987)).

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Figure 8 shows the ideal points used by the final strategies. The ideal points are given as quantity combinations. In the quantity diagram the Cournot-isoprofit curves of the two players enclose a lens-shaped area. The ideal points used by final strategies are in a relatively small area in the middle of this lens. The mean of all ideal points is located at (87.02, 49.43). This combination is almost Pareto optimal.

Characteristics 7 to 11 are described as rules to be followed by a programmer of a strategy.

CHARACTERISTIC 7: *If your opponent has chosen an output below his output specified by your ideal point, then choose your ideal point quantity in the next period.*

If a strategy is based on two ideal points then the words "your ideal point" refer to the ideal points for the concerning player role. The interpretation of Characteristic 7 is simple. If your opponent is even more cooperative than required by your ideal point, then there is no reason to deviate from your own ideal point quantity. Ten of the 18 final strategies based on ideal points have this characteristic. However, some other strategies increase the output in the situation of Characteristic 7 in order to test the opponent's willingness to cooperate at a point more favorable for oneself.

The remaining characteristics will be applicable to strategies without ideal points, too. Even if a strategy is not based on an ideal point, it may still involve a

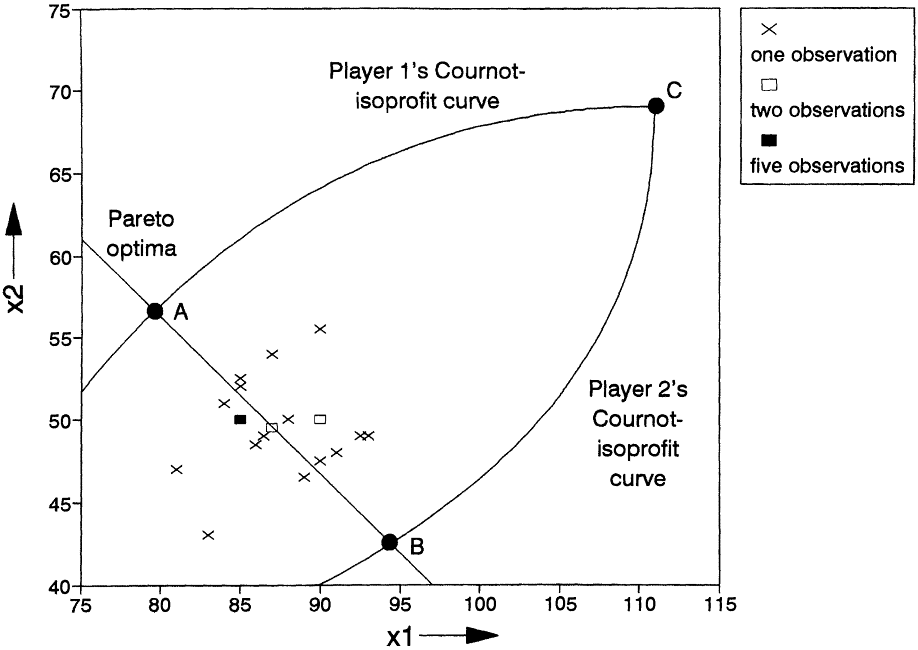

FIGURE 8.—Ideal points used by final strategies.

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measure which permits an interpersonal comparison of cooperativeness. Thus a strategy may look at the profit difference achieved at the beginning of the main phase as a standard reference. Both players are judged to be equally cooperative if this profit difference is attained.

CHARACTERISTIC 8: *If your opponent has chosen an output above his Cournot quantity, then in the next period choose your Cournot quantity.*

Twelve of the 23 strategies obey this rule. The other strategies do not use more rigorous methods of punishment; instead, if they realize that their opponent plays permanently above his Cournot quantity, they abandon the idea of punishment after some periods and reduce their own output below their Cournot quantity to increase their profits. Such strategies run the danger of becoming exploitable by attempts to establish Stackelberg leadership. Characteristic 8 on the one hand avoids excessive aggressiveness and on the other hand provides protection against exploitative opponents.

CHARACTERISTIC 9: *If your opponent has chosen his Cournot quantity, then in the next period choose a quantity not higher than your Cournot quantity and 5% at most below your Cournot quantity.*

It can be seen with the help of Figure 8 that Characteristic 9 limits the response to the opponent's choice of his Cournot quantity to a relatively small interval. Sixteen of the 23 final strategies satisfy the requirement of Characteristic 9. Among these 16 strategies there are 10 which respond to Cournot quantities by Cournot quantities. The remaining 6 strategies want to indicate their willingness to cooperate by a slightly smaller output. Of course, the number of 5% in Characteristic 9 is to some extent arbitrary.

The following two Characteristics 10 and 11 apply to situations in which the following four conditions hold.

(i) The last period was a period of your main phase.

(ii) Up to now you always followed your strategy.

(iii) In the last period your opponent's output was below his Cournot output.

(iv) If you have an ideal point (for the relevant player role), then your opponent's output was above his output in your ideal point.

CHARACTERISTIC 10: *Suppose that conditions (i), (ii), (iii) and (iv) hold. If in the last period your opponent has raised his output, then your decision raises your output to a quantity below your Cournot output.*

CHARACTERISTIC 11: *Suppose that conditions (i), (ii), (iii), and (iv) hold. If in the last period your opponent has lowered his output, then your decision lowers your output. If you have an ideal point, then your new output remains above your ideal point output.*

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To illustrate Characteristics 10 and 11, let us give an example: Consider a strategy which in the main phase matches the opponent's last output on a straight line between the Cournot point and an ideal point in the quantity space, of course, only as long as the opponent's last output was in the relevant range. A strategy of this kind satisfies Characteristics 10 and 11. However, it is necessary to impose condition (i) since in the first period of the main phase matching on the line may require an increase of output even if the opponent has lowered his output.

As long as condition (ii) is satisfied matching on the line in later periods of the main phase will move in the right direction. Conditions (iii) and (iv) make sure that Characteristics 10 and 11 apply only in the relevant range.

Both characteristics can be satisfied for strategies not based on a line between the Cournot point and an ideal point in any space. They may even be satisfied for strategies without ideal points. Thus a strategy's response may be guided by the criterion of a profit difference equal to that at the Cournot point without any regard to Pareto optimality. Two of the final strategies were of this kind.

Fourteen final strategies have Characteristic 10. The number of final strategies with Characteristic 11 is also 14, but only 11 final strategies have both characteristics.

6.4. *The End Phase*

A strategy with an end phase has a special method of output determination for the last one to four periods. Attempts towards cooperation which are typical for the main phase are not continued in the end phase. Instead of this, short-run profit goals are pursued.

Only 2 of the 23 final strategies do not have an end phase. One of these 2 strategies was typical in many other respects but the other was the most atypical. This atypical strategy tries to estimate response functions of the opponent and then computes the output decision by an elaborate approximative method for the solution of the dynamic program of maximizing expected profits for the remainder of the game. Even if something like an end effect is automatically produced by the dynamic programming approach, no end phase is present here since the method of output determination is always the same.

CHARACTERISTIC 12: *The strategy has an end phase of at least two periods.*

Characteristic 12 is shared by 11 of the 21 final strategies with end phases. Ten of these strategies planned an end effect only for the last period.

CHARACTERISTIC 13: *The strategy has an end phase and specifies the Cournot output of the relevant player as the output for all periods of the end phase.*

This characteristic is present in 12 final strategies. Other strategies sometimes optimized short-run profits against the opponent's last output or approached the Cournot output in several fixed steps.

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6.5. *The Strategic Approach Underlying Typical Strategies*

A typical strategy does not try to optimize against expectations on the opponent's behavior (Characteristic 1). The strategic problem is not viewed as an optimization problem but rather as a bargaining problem. The first question to be answered concerns the point where cooperation should be achieved. Of course, cooperation should be favorable for oneself but it also must be acceptable for the opponent. A failure to reach cooperation is expected to lead to Cournot behavior. Therefore, cooperation requires that both players obtain more than their Cournot profits. Ideal points are constructed as reasonable offers of cooperation within these constraints. Various kinds of fairness considerations but also prominence (divisibility by five) and prior experience may influence the selection of ideal points.

After the choice of an ideal point the question arises as to how cooperation at this point or in its neighborhood can be achieved. It is necessary to indicate one's willingness to cooperate there and to show that one is not going to accept less favorable terms.

A decreasing sequence of outputs in the initial phase is a natural signal indicating cooperativeness. In the main phase a typical strategy evaluates the cooperativeness of the opponent's last output and responds by an output of a similar degree of cooperativeness according to some criterion. The response may depend on whether the opponent decreased or increased his output. If there is such a difference, it is natural to respond more aggressively to the same output after an increase.

One may say that main-phase behavior is guided by a principle of "measure for measure." Small changes of the opponent's output lead to small reactions and big changes cause big reactions.

Many oligopoly theories are based on the idea that a player anticipates the reaction of his opponent in order to maximize his profits. Contrary to this, a strategy based on an ideal point and a response rule guided by the principle "measure for measure" does not involve any anticipation of the opponent's reactions. The aim is to exert influence on the opponent rather than to adapt to his behavior. In order to achieve this aim one's own behavior has to provide a clear indication of one's own intentions. If the implied offer of cooperation is reasonable, one can hope that the aim will be reached. A response guided by the principle "measure for measure" protects against attempts to exploit one's own cooperativeness and rewards cooperative moves of the other player.

Of course, cooperation breaks down in the end phase. The strategies have been written for the 20-period supergame. This game permits only one subgame perfect equilibrium path, namely Cournot outputs in every period. The participants were aware of the backward induction argument which came up in the discussions of the plenary sessions. They accepted the idea that cooperation must break down in the last periods but as the strategies show they did not accept the full force of the backward induction argument. An explanation of this phenomenon is given elsewhere (Selten (1978a)).

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7. TYPICITY AND SUCCESS

All characteristics are typical for the final strategies in the sense that they are present in the majority of the cases to which they are applicable. Of course, they are not all equally typical. Some appear in more of the final strategies than others. Moreover, the extent to which a characteristic is typical should not only be judged by the number of strategies with this characteristic, but also by the extent to which these strategies are typical. In the following, we shall construct a measure of typicity applicable to both characteristics and strategies which tries to do justice to these considerations.

The measure of typicity assigns a real number to each characteristic and to each strategy. The sum of the typicities of all 13 characteristics is normed to 1. The measure of typicity can be thought of as the outcome of an iterative procedure. At the beginning, all characteristics have the same typicity 1/13. Then, in each step, first a new typicity is computed for each strategy as the sum of the typicities of its characteristics. Afterwards, a new typicity for each characteristic is computed as proportional to the sum of the typicities of the strategies with this characteristic. The sum of the typicities of all characteristics is again normed to 1.

In order to give a more precise mathematical definition of our measure, it is necessary to introduce some notation. The typicity of characteristic i is denoted by c_i and s_j stands for the typicity of strategy j. The symbol c is used for the column vector with the components $c_1, \dots, c_{13}$ and s denotes the column vector with the components $s_1 \cdots s_{23}$. Let A be the 13×23-matrix with entries a_{ij} as follows: $a_{ij} = 1$ if strategy j has characteristic i, and $a_{ij} = 0$ otherwise. In our case c and s are uniquely determined by the following equations.

$$c = \alpha A s,$$
$$s = A^T c,$$
$$\sum_{i=1}^{13} c_i = 1,$$

where A^T is the transpose of A and $1/\alpha$ is the greatest eigenvalue of AA^T. It is a consequence of elementary facts of linear algebra that the iterative process described above converges to vectors c and s which can be described as the solution of this system of equations.

Table III shows which strategy has which characteristics. The rows correspond to the 13 characteristics and the columns to the 23 final strategies. The strategies have been numbered according to the success in the final tournament. Strategy 1 is the most successful one, strategy 2 the second most successful one, etc. A black mark indicates that the strategy corresponding to the column has the characteristic corresponding to the row.

Obviously, the black marks in Table III describe the matrix A. A black mark corresponds to an entry 1 and the absence of a black mark corresponds to an entry 0. The typicities of the characteristics are given at the right margin and the typicities of the strategies can be found at the bottom of Table III.

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TABLE III

TYPICITY OF CHARACTERISTICS AND STRATEGIES[a]

Characteristics	Strategies																						Typicity
1	—	—	—	—	—	—		—	—	—	—		—	—	—		—	—	—		—		—	.0917
2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—		—	—	.1062
3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—				—	—		.1005
4	—	—		—			—				—	—	—	—	—		—	—			—			.0609
5	—	—	—	—		—	—			—	—	—		—	—	—	—							.0710
6	—	—	—	—	—	—	—	—	—	—				—	—	—	—	—	—		—		—	.0927
7	—	—	—	—	—				—						—	—					—		—	.0545
8	—		—		—	—		—	—			—	—	—		—			—		—			.0653
9	—	—	—		—	—		—	—	—		—	—	—	—	—		—			—	—		.0851
10	—		—		—		—	—	—	—	—	—	—			—					—	—	—	.0729
11	—	—	—			—	—	—	—	—		—	—			—	—					—	—	.0749
12	—	—	—	—		—	—				—		—			—	—	—						.0591
13	—		—					—	—	—	—	—		—		—	—		—		—			.0652
Ranking of success	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	
Ranking of typicity	1	5	2	17	14	8	16	7	4	6	18	12	11	9	15	3	10	19	22	23	13	21	20	
Typicity	1.0000	.7966	.9391	.6366	.6689	.7465	.6382	.7545	.8090	.7602	.6275	.7020	.7166	.7386	.6626	.8474	.7222	.4957	.4211	.1062	.6888	.4396	.4929	

[a] The Spearman rank correlation coefficient between typicity and success of strategies is $r_s = .619$.

The table also shows the ranking of success in the final tournament and the ranking of typicity of the 23 strategies. The Spearman rank correlation coefficient between success and typicity is +.619. This value is significant at the 1% level (two-tailed test).

It is an unexpected phenomenon that *there is a strongly significant positive correlation between the typicity and the success of final strategies*. In principle, the

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opposite relationship would also seem to be possible. It is not inconceivable that typical characteristics reflect nothing else than typical mistakes. However, in our case the characteristics seem to embody advisable strategic principles. Maybe the positive correlation between typicity and success is the result of the learning process which produced the final strategies.

For each characteristic the mean success rank of those strategies which have it is smaller than the mean success rank of those which do not have it. This shows that *each of the characteristics separately is positively connected to the success in the final tournament.* In this sense all 13 characteristics are favorable structural properties of a strategy.

Our judgments of the advisability of the characteristics must be understood relative to the strategies developed by the participants of our experiment. We cannot exclude the possibility that a very atypical strategy can be found which turns out to be very good in a tournament against the 23 final strategies. In fact, the participant who wrote a strategy with success rank 20 firmly believes that this approximative dynamic programming approach based on an estimated response function of the opponent can be improved to a degree which will make it superior to all final strategies in a tournament against them. We doubt that this is the case. The difficulty with the dynamic programming approach is the problem of forming a correct estimate of the opponent's behavior. A best response to a wrong prediction can have disastrous consequences.

Admittedly, our experiment does not really justify strong conclusions since the final strategies have not been developed independently of each other. Perhaps a different picture of a typical strategy would emerge in a repetition of the experiment. Nevertheless, the results reported in this section seem to be of considerable significance for the further development of oligopoly theory.

8. A FAMILY OF SIMPLE TYPICAL STRATEGIES

The 13 characteristics do not completely determine a strategy. Many details are left open. In this section we shall construct a family of strategies which are typical in the sense that they have all 13 characteristics and the missing details are furnished in a particularly simple way. The members of the family differ only by the pair of ideal points used for both player roles. The special case of only one ideal point is not excluded.

For our family of simple typical strategies we shall discuss the question of what happens if two strategies with different ideal points play against each other. This exercise conveys some insight into the strategic properties implied by the 13 characteristics. We shall also look at the question of what is a reasonable choice of ideal points. For this purpose we have determined that member of the family which did best in a tournament against 22 of the final strategies. (The only strategy which involved random decisions was eliminated in order to avoid time consuming Monte-Carlo simulation.)

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8.1. *Description of the Simple Typical Strategies*

The ideal points are described by output pairs u and v, one for each player role:

Ideal point for the role of player 1: $u = (u_1, u_2)$.

Ideal point for the role of player 2: $v = (v_1, v_2)$.

The first components of the vectors u and v denote player 1's output and the second stands for player 2's output. As mentioned above, the special case $u = v$ is not excluded. We also introduce the following notation for the output combination in the Cournot equilibrium of the underlying duopoly.

Cournot equilibrium: $c = (c_1, c_2)$.

We now can describe the decision $x_i(t)$ specified by the *simple typical strategy* with ideal points u and v. The following conditions (i) and (ii) have been imposed on the ideal points:

(i) The ideal points u and v are Pareto superior to the Cournot equilibrium.

(ii) $u_1 \leq .92c_1$ and $v_2 \leq .92c_2$.

Condition (ii) is necessary to make the specification of the initial phase compatible with Characteristic 5.

Initial phase:

$$x_1(t) = \frac{t}{3}u_1 + \frac{3-t}{3}c_1,$$

$$x_2(t) = \frac{t}{3}v_2 + \frac{3-t}{3}c_2 \qquad \text{for} \quad t = 1, 2, 3.$$

Main phase:

$$x_1(t) = \begin{cases} u_1 & \text{for } x_2(t-1) \leq u_2, \\ c_1 & \text{for } x_2(t-1) \geq c_2, \\ u_1 + \dfrac{c_1 - u_1}{c_2 - u_2}(x_2(t-1) - u_2) & \text{otherwise;} \end{cases}$$

$$x_2(t) = \begin{cases} v_2 & \text{for } x_1(t-1) \leq v_1, \\ c_2 & \text{for } x_1(t-1) \geq c_1, \\ v_2 + \dfrac{c_2 - v_2}{c_1 - v_1}(x_1(t-1) - v_1) & \text{otherwise.} \end{cases}$$

End phase:

$$x_i(t) = c_i \quad \text{for } i = 1, 2 \text{ and } t = 19, 20.$$

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The initial phase can be thought of as a sequence of three equal "concessions" moving from the Cournot output c_i to the ideal point output u_1 or v_2 respectively. The first period already makes the first concession. Obviously, the initial phase satisfies Characteristic 4 which requires at least two periods. Characteristic 5 is satisfied since u_1 and v_2 are not greater than $.92c_1$ and $.92c_2$, respectively.

Characteristic 6 requires that the strategy make use of ideal points. Obviously, this is the case for our family of simple typical strategies.

We now turn our attention to the equation for the main phase. The upper line on the right-hand side secures Characteristic 7. The middle line is in agreement with Characteristics 8 and 9. Characteristic 9 concerns the special case $x_j(t-1)=c_j$ and permits a response $x_i(t)$ up to 5% lower than c_i. As has been pointed out before, the majority of these final strategies which conformed to Characteristic 9 specified a response of exactly c_i. Therefore, this response can be considered as typical.

The lower line on the right-hand side of the equation for the main phase is a very simple version of the principle "measure for measure." The last output of the opponent is matched by the corresponding output on the straight line which connects the ideal point and the Cournot point in the quantity space. Obviously, this has the consequence that Characteristics 10 and 11 are present in the strategies of our family.

The end phase has two periods and, therefore, conforms to Characteristic 12. The output in the end phase is always c_i, as required by Characteristic 13.

The strategies of our family also have the Characteristics 1, 2, and 3. In accordance with Characteristic 1, no attempt is made to predict the opponent's behavior and to optimize against this prediction. As required by Characteristic 2, the strategies are completely deterministic. In the main phase the strategies permit a continuum of possible responses and therefore have Characteristic 3.

8.2. *Simple Typical Strategies Playing Against Each Other*

Consider a play of the 20-period supergame where each of both players uses a member of the family described above as his strategy. Let u and v be the ideal points of the strategy of player 1. Similarly, let u^* and v^* be the ideal points of the strategy of player 2. Actually, only u and v^* are of interest here since we have fixed the player roles.

The behavior in the main phase can be described by two "reaction functions," r and r^*:

$$r(x_2)=\begin{cases} u_1 & \text{for } x_2 \le u_2, \\ c_1 & \text{for } x_2 \ge c_2, \\ u_1+\dfrac{c_1-u_1}{c_2-u_2}(x_2-u_2) & \text{otherwise;} \end{cases}$$

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$$r^*(x_1) = \begin{cases} v_2{}^* & \text{for } x_1 \le v_1{}^*, \\ c_2 & \text{for } x_1 \ge c_1, \\ v_2^* + \dfrac{c_2 - v_2{}^*}{c_1 - v_1{}^*}(x_1 - v_1{}^*) & \text{otherwise.} \end{cases}$$

The development of the play in the main phase is given by the following equations:

$$x_1(3) = u_1,$$

$$x_2(3) = v_2{}^*,$$

$$x_1(t) = r(x_2(t-1)) \quad \text{for } t = 4,\ldots,18,$$

$$x_2(t) = r^*(x_1(t-1)) \quad \text{for } t = 4,\ldots,18.$$

Figure 9 shows four examples for the development of this system of difference equations. In Figures 9a and 9b the path of output combinations moves towards the Cournot equilibrium. In Figure 9c the path stays at $(u_1, v_2{}^*)$ for $t = 3,\ldots,18$.

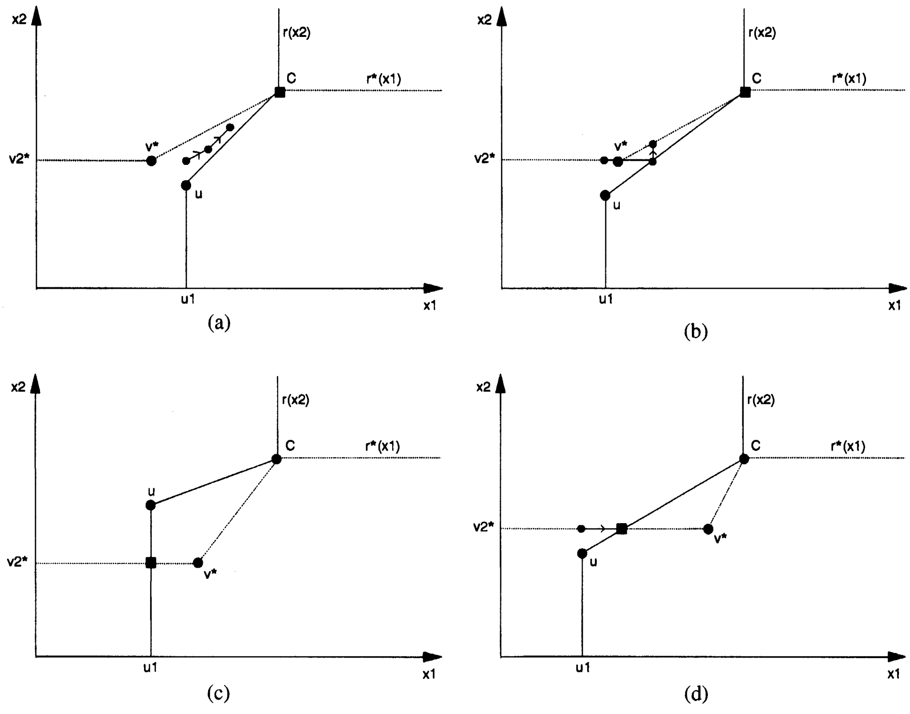

FIGURE 9.—Simple typical strategies playing against each other. Four examples with different ideal point pairs u and v^*.

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In Figure 9d we have

$$x_1(t) = r(v_2^*) \quad \text{for } t = 4, \ldots, 18,$$
$$x_2(t) = v_2^* \quad \text{for } t = 4, \ldots, 18.$$

We shall speak of a *conflict case* if the output combination path moves towards the Cournot equilibrium, and of an *agreement case* if the output path becomes stationary in periods 4 to 18.

It can be seen without difficulty that a conflict case is obtained whenever the Cournot output combination is the only common point of r and r^*. All other cases are agreement cases. An agreement case is also characterized by the condition that player 2's ideal point is not above the straight line through the Cournot point and player 1's ideal point. This is the case if and only if player 1's ideal point is not below the straight line through the Cournot point and player 2's ideal point. From what has been said it follows that an agreement case is obtained if and only if the following inequality holds:

$$\frac{c_2 - v_2^*}{c_1 - v_1^*} \geq \frac{c_1 - u_2}{c_1 - u_1}.$$

Figure 9 will illustrate the consequences of this condition: In the special case in which both ideal points are Pareto optimal, an agreement case is reached if each player does not ask for more than the other player will grant him. The ideal points are like bargaining offers. The less one asks for oneself and the more one grants to the other player, the better are the chances for agreement.

In view of the condition for an agreement case it seems to be quite reasonable to specify two different ideal points for the two player roles in such a way that player 1's ideal point is more favorable for player 2 and vice versa. However, those 6 participants who specified two different ideal points did this in a way which leads to a conflict case if the strategy plays against itself. In each player role these subjects wanted more for themselves than they would grant to the other player if he were in this role.

It can be seen without difficulty that the condition which distinguishes agreement cases from conflict cases does not depend crucially on the special way in which our simple typical strategies specify the initial phase. As long as at the end of the initial phase both outputs are below the respective Cournot outputs, the output combination path moves towards the Cournot point in a conflict case and towards stationary cooperation in an agreement case.

8.3. *The Best Ideal-Point Selection Against the Final Strategies*

It is interesting to ask the question of what is the best selection of ideal points within the family of simple typical strategies defined above in a tournament against the final strategies. Actually, we simulated tournaments only against 22 of the final strategies since we omitted the only strategy which uses random

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choices. The best choice of ideal points turned out to be as follows:

$$u = (89.4, 55.6),$$
$$v = (86.6, 50.4).$$

Both components of u are greater than the corresponding components of v, but if this strategy plays against itself an agreement case is obtained; the quantity combination (89.4, 52.6) is played in periods 4 to 18.

The ideal point (86.6, 50.4) is nearly Pareto optimal whereas $u = (89.4, 55.6)$ is relatively far from the Pareto optimal line. However, $u = (89.4, 55.6)$ has the advantage that it yields agreement cases against all ideal points which have been specified for the role of player 2 by those of the 22 participants who used ideal points. This is due to the fact that $u_2 = 55.6$ is rather large.

The ideal point $v = (86.6, 50.4)$ does yield conflict cases against some of the ideal points specified for the role of player 1 by participants. These ideal points for player 1 are too aggressive to make it worthwhile to reach agreement with them by a more generous ideal-point choice which, of course, would diminish payoffs against other strategies.

The simple typical strategy with $u = (89.4, 55.6)$ and $v = (86.6, 50.4)$ is not only the best among its family but is also the winner of the tournament against the 22 final strategies. This seems to indicate that the way in which the simple typical strategies fill in the details left open by the 13 characteristics is not an unreasonable one. One may say that the structure of these strategies provides an appropriate idealized image of typical behavior of experienced strategy programmers, at least as far as our experiment is concerned.

8.4. *Game-Theoretic Properties of Simple Typical Strategies*

The 20-period supergame has only one subgame perfect equilibrium point. In this equilibrium point both players always choose their Cournot quantities regardless of the previous history. If both players use simple typical strategies of the family described above the resulting strategy pair is always a disequilibrium, simply because it would be advantageous to deviate in the fourth last period.

Game theoretically there is a fundamental difference between finite and infinite supergames. It is known from the experimental literature that this difference seems to have little behavioral relevance. In sufficiently long finite experimental supergames cooperation is possible until shortly before the end, even if the source game has only one equilibrium point (Stoecker (1983), Selten and Stoecker (1986)). If one wants to connect finite supergame behavior with game-theoretical equilibrium notions, one has to take the point of view that the players behave as if they were in an infinite supergame.

It is shown in another paper of one of the authors that it is possible to construct equilibrium points for the infinite supergame of our duopoly model based on the main phase of our simple typical strategies (Mitzkewitz (1988)). In these equilibrium points both players have the same ideal point. This ideal point

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is chosen in the first period of the game; later the strategies respond to the previous period as specified by the reaction functions r and r^*. Under certain conditions which have to be imposed on the ideal points, equilibrium points are obtained in this way. However, these equilibrium points are not subgame perfect. This is a consequence of a result in the literature which shows that equilibria where output continuously depends on the opponent's last-period output only cannot be subgame perfect unless the Cournot output is specified regardless of the previous history (Stanford (1986), Robson (1986)). Mitzkewitz (1988) shows that an appropriate modification of the main phase of the simple typical strategies yields subgame perfect equilibrium points for a wide range of ideal points.

Among the newer game-theoretical literature on the duopoly problem we have only found one paper which shows some similarities with the approach taken here (Friedman and Samuelson (1988)).

8.5. *Reasonable Conditions for Ideal Points*

One may ask the question whether it is possible to impose reasonable restrictions on the choice of ideal points in our simple typical strategies. A strategy programmer who considers an ideal point for one of the player roles will probably explore what happens if his opponent uses the same ideal point for the opposite player role. Therefore, it is natural to focus on the case in which both opponents use the same ideal point $u = (u_1, u_2)$ for both player roles.

Suppose player 1 knows that player 2 plays a simple typical strategy as defined above with the ideal point $u = (u_1, u_2)$. Suppose that for some output x_1 the profit $G_1(x_1, r^*(x_1))$ is greater than $G_1(u_1, u_2)$. Then player 1 has a better alternative than to agree to player 2's ideal point (u_1, u_2). This consideration and an analogous one for player 2 lead to the following conditions:

$$G_1(u_1, u_2) = \max_{x_1} G_1(x_1, r^*(x_1)),$$

$$G_2(u_1, u_2) = \max_{x_2} G_2(r(x_2), x_2).$$

We refer to these two equations as "conjectural equilibrium conditions" since there is an obvious relationship to conjectural oligopoly theories (see Selten (1980)).

Another reasonable condition on ideal points is connected to the possibility of attempts of short-run exploitation. Suppose that a player deviates just once from the ideal point and then returns to cooperation at the ideal point. It should not be possible to improve profits in this way. This leads to the following conditions:

$$2G_1(u_1, u_2) = \max_{x_1} [G_1(x_1, u_2) + G_1(u_1, r^*(x_1))],$$

$$2G_2(u_1, u_2) = \max_{x_2} [G_2(u_1, x_2) + G_2(r(x_2), u_2)].$$

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We refer to these equations as "stability against short-run exploitation." In our numerical case the conjectural equilibrium conditions imply stability against short-run exploitation, but this is not the case for all possible parameter values.

As has been explained in subsection 8.4 it will be shown elsewhere (Mitzkewitz (1988)) that subgame perfect equilibrium points for the infinite supergame can be constructed on the basis of the reaction functions (but with memory also of the own behavior) embodied in the main phase of simple typical strategies if certain conditions on the ideal point are satisfied. These conditions are nothing else than the conjectural equilibrium conditions and the stability against short-run exploitation.

Perhaps it is also of interest that only one Pareto optimal point satisfies the conjectural equilibrium conditions, namely the point described in the third row of Table II: profit monotonic quantity reduction along the straight line through the intersections of both Cournot-isoprofit curves (see Mitzkewitz (1988)). It is tempting to look at this ideal point as distinguished among others by its special theoretical properties. In the final strategies it has been employed twice. However, as can be seen in Table II, other ideal points based on different principles have proved to be at least as attractive to the participants.

Figure 10 shows the ideal points used in final strategies of the participants and the restrictions imposed by the conjectural equilibrium conditions (the smaller lens-shaped area) and by stability against short-run exploitation (the greater lens-shaped area). The equations for these curves will be discussed elsewhere

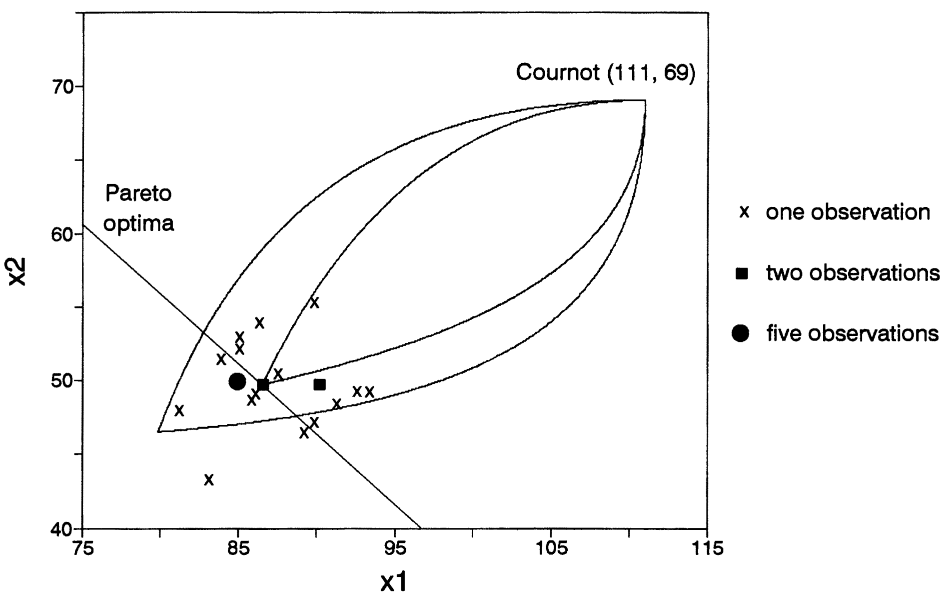

FIGURE 10.—The set of ideal points satisfying the conjectural equilibrium conditions (smaller lens), the set of ideal points stable against short-run exploitations (greater lens), and the ideal points used in final strategies.

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(Mitzkewitz (1988)). Only 4 of the 24 ideal points satisfy the conjectural equilibrium conditions, but 21 of the ideal points are stable against short-run exploitation.

Obviously, the participants were not concerned about the conjectural equilibrium conditions. Maybe a violation of these conditions is not perceived as a serious danger since in the case of an optimization of the other player along one's own reaction function, cooperation will still be reached, even if the resulting output levels are higher than in the ideal point.

Some strategies which were not yet the final ones contained attempts at short-run exploitation. Most participants seemed to be aware of this possibility since the "trainer"-program enables them to play against their own strategy. They were able to check short-run exploitability without analytical computations. Of course, such numerical checks will sometimes fail to reveal the right answer. Maybe it is of interest in this connection that two of the three ideal points without stability against short-run exploitation are very near to the corresponding area in Figure 10.

8.6. *Stability against Short-Run Exploitation and Outcomes of Plays in the Final Tournament*

In the tournament among 23 final strategies (including the strategy with random choices) 1012 plays were simulated (two plays for each strategy pair). Table IV shows the distribution of the pairs of total profits in the 1012 plays. The inner cells of the table correspond to profit intervals of four thousand for both players.

The curve superimposed on this table is connected to stability against short-run exploitations. The curve encloses all profit pairs which can be reached by plays in which the same ideal point with the property of stability against short-run exploitation is played in all 20 periods. We call the region enclosed by this curve the "exploitation stability region."

Consider two simple typical strategies whose ideal points are stable against short-run exploitation. Whenever such strategies are played against each other, the resulting profit combination of the 20-period supergame must be in the exploitation stability region, regardless of whether the ideal points of both players are equal or not. However, the set of all profit combinations which can be reached in this way is a proper subset of the exploitation stability region. This is due to the behavior in the initial phase and the end phase. The exploitation stability region can be obtained by pairs of modified simple typical strategies, strategies in which the initial phase and the end phase are of different length, but the behavior in the main phase remains the same.

In the final tournament 983 (97.1%) of the 1012 plays resulted in total profit combinations in the exploitation stability region. In those few total profit combinations outside the exploitation stability region, one of both profits is below the corresponding Cournot profit.

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TABLE IV

SUPERGAME PROFIT PAIRS IN THE FINAL TOURNAMENT AND THE EXPLOITATION STABILITY REGION.

Total Profits of Player 2 (in thousands)	<46	46-50	50-54	54-58	58-62	62-66	66-70	70-74	74-78	78-82	82-86	86-90	90-94	94-98	>98
>106	–	–	–	–	–	–	–	–	–	–	–	–	–	–	
102-106	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
98-102	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
94-98	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
90-94	2	–	1	2	–	4	–	–	–	–	–	–	–	–	–
86-90	–	–	–	5	9	86	48	–	–	–	–	–	–	–	–
82-86	–	4	4	11	36	146	93	10	2	–	–	–	–	–	–
78-82	2	–	1	25	61	60	66	35	14	–	–	–	–	–	–
74-78	4	–	17	75	37	15	8	9	9	–	–	–	–	–	–
70-74	6	2	33	37	10	8	2	4	–	–	–	–	–	–	–
66-70	–	–	3	–	–	–	–	–	–	–	–	–	–	–	–
<66	–	–	–	–	–	–	2	–	–	–	–	–	–	–	4

Cournot Profits fo Player 2 (between 70-74 and 66-70)

Cournot Profits of Player 1 (between 46-50 and 50-54)

Total Profits of Player 1 (in thousands)

Pareto-efficient frontier

The evidence of Figure 10 and Table IV strongly suggests that stability against short-run exploitation has some relevance for the prediction of outcomes of plays between strategies written by experienced players.

9. IMPLICATIONS FOR DUOPOLY THEORY

The results presented in this paper suggest a new view of the duopoly problem. Traditional duopoly theories and game-theoretical approaches rely heavily on optimization ideas. Usually, a duopolist is assumed to optimize against expectations on his opponent's behavior. Contrary to this, it is typical for the strategies programmed by the experienced players in our experiment that no expectations are formed and nothing is optimized.

The approach to the duopoly problem suggested by our results can be described as the "active pursuit of a cooperative goal." First, one has to answer the question of where one wants to cooperate. The goal of cooperation is made precise by the concept of an ideal point. The ideal point should be a reasonable

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compromise between both players' interests; otherwise, one cannot hope to achieve cooperation. Concepts of fairness such as those listed in Table II are the basis for judgments on the reasonableness of compromises.

It is well known in the experimental literature that considerations of fairness have a strong influence on observed behavior. Many of the empirical and experimental phenomena can be subsumed under an equity principle (Selten (1978b)). Further literature can be found there and in a newer paper which contains many illustrative examples (Kahneman, Knetsch, and Thaler (1986)). Fairness considerations also have been proved to be useful in the explanation of behavior in duopoly experiments (Friedmann (1970), Selten and Berg (1970)).

Once an ideal point has been chosen one has to determine a policy for its effectuation. Formally, an effectuation policy may be described by a reaction function as in the simple typical strategies of Section 8. However, contrary to conjectural oligopoly theory, such reaction functions are not to be interpreted as hypotheses on the opponent's behavior. Effectuation policies are more like reinforcement schedules which serve the purpose to guide the opponent's behavior rather than to optimize against it.

The typical structure of an effectuation policy is based on the principle of measure for measure. This principle requires an interpersonal comparison of the degree of cooperativeness of the players' actions. The degree of cooperativeness measures the nearness to the ideal point. The response matches the opponent's last action according to this measure.

A player who plays the dynamic game may try to learn how to do best against his opponent's behavior. A player who does this takes a "learning approach." It is also possible to take a "teaching approach," which means that one behaves in a way which induces the other player to conform to one's own goals.

It seems to be very difficult to design a reasonable strategy which takes the learning approach. One participant tried to do this in a sophisticated way. His strategy involved an approximate intertemporal optimization against statistical estimates of his opponent's strategy. His success rank was 20. As Table III shows, his strategy has only one of the thirteen characteristics, namely the absence of random decisions. Obviously, the optimization attempt, of this participant failed badly. The reason for this lies in the difficulty of forming an accurate estimate of the opponent's behavior on the basis of relatively few observations.

The difficulties connected to the learning approach point in the direction of a teaching approach. Of course, somebody who takes the teaching approach does not necessarily expect that the other player takes a learning approach. The other player may very well take a teaching approach, too. This will not lead to difficulties if both players pursue compatible cooperative goals. However, if the opponent tries to adapt to my strategy, this should not endanger my cooperative goal.

Maybe in a very long supergame of thousands of periods, a good strategy would involve both, teaching and learning, but within 20 periods not much can

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be learned which still can be used within this time. Real duopoly situations rarely are analogous to very long supergames. Maybe a relatively short supergame more adequately captures the decision problem of managers who want to be successful within a foreseeable time.

The new view of the duopoly problem emerging from our results may be described by the slogan "fairness and firmness." One must first choose a fair goal of cooperation and then devise an effectuation policy which shows one's willingness to cooperate and firmly communicates resistance to unfair behavior.

As we have seen, the requirement of stability against short-run exploitation seems to be a restriction obeyed by the participants' choices of ideal points, even if their effectuation policies were not exactly the same as those of the simple typical strategies. It is clear that one should not give rise to the possibility of being exploited. Moreover, in the case in which the other player selects one's own ideal point, he should not be exploitable. This criterion of stability against short-run exploitation is in good agreement with our data.

It is clear that the theory of fairness and firmness can be easily transferred to different contexts, e.g. price-variation duopoly supergames. The tit-for-tat strategy which was the winner of Axelrod's contests (1984) is in harmony with the fairness-and-firmness theory. In the prisoner's dilemma the choice of an ideal point is not an issue. In view of the symmetry of the situation there is only one natural cooperative goal. Since there are only two choices available, measure for measure cannot mean anything else than tit-for-tat.

It must be admitted that no strong conclusions can be drawn from our data since the final strategies cannot be regarded as statistically independent observations. The participants interacted in game playing rounds and tournaments. Moreover, there was some verbal communication, even if the participants seemed to be reluctant to reveal the principles underlying their strategies.

More studies similar to the investigation presented here are necessary to establish the empirical relevance of the fairness-and-firmness theory. It should also be kept in mind that the final strategies of our participants are the result of a long experience with the game situation. It is quite possible that real duopolists have much less experience with their strategic situation and therefore do not achieve the same extent of cooperation. The experimental literature shows that only after a considerable amount of experience, subjects learn to cooperate (Stoecker (1980), Friedman and Hogatt (1980), Alger (1984, 1986), Benson and Faminow (1988)).

It would be wrong to assert that there is no difference between a programmed strategy and spontaneous behavior. The strategy method cannot completely reveal the structure of spontaneous behavior. However, it seems to be plausible that somebody who writes a strategy program is guided by the same motivational forces which would influence his spontaneous behavior. Of course, a strategy program is likely to be more systematic. Obviously this is an advantage from the point of view of theory construction.

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10. SUMMARY OF RESULTS

1. Mean profits increased from one game playing round to the next.

2. The correlation between both player profits was negative in the first game playing round and became positive in the second and the third game playing round. This can be interpreted as a growth of understanding of the strategic situation.

3. Mean profits increased from one computer tournament to the next. In the final tournament 97.1% of all plays had profits above Cournot profits for both players.

4. Typically, a strategy program for the final tournament distinguishes among an initial phase, a main phase, and an end phase. Outputs independent of the opponent's previous behavior are specified for the initial phase of one to four periods. In the main phase the strategies aim at a cooperation with the opponent. Noncooperative behavior characterizes an end phase of one to four periods.

5. Typical structural features of strategies programmed for the final tournament can be described by 13 characteristics. These characteristics imply a strategic approach which begins with the selection of a cooperative goal described by an "ideal point." (A different ideal point may be chosen for each player role.) Cooperation at the ideal point is then pursued by a "measure-for-measure policy." If the opponent moves towards the ideal point or away from it, the response of a measure-for-measure policy is of similar force in the same direction. In the end phase a typical strategy always chooses Cournot outputs.

6. Typically, no predictions about the opponent's behavior are made and nothing is optimized.

7. The extent to which a strategy or a characteristic is typical can be measured by an index of typicity. There is a highly significant positive rank correlation between the index of typicity and the success of a strategy in the final tournament.

8. For each of the 13 characteristics separately those final strategies which have this characteristic have a higher average success rank than those which do not have it.

9. Ideal points are often based on various fairness considerations (see Table II).

10. A family of "simple typical strategies" has been introduced as an idealized description of the structure implied by the 13 characteristics. The simple typical strategy which performed best against the final tournament strategies was determined by a computer simulation. This "best" simple typical strategy is also the winner in the tournament against the final strategies.

11. Two game-theoretical requirements for simple typical strategies impose restrictions on ideal points. One of these restrictions, the "conjectural equilibrium conditions," is rarely satisfied by the ideal points in the final strategies.

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However, most of these ideal points satisfy the weaker restriction of "stability against short-run exploitation."

12. An "exploitation stability region" for profit combinations reached in the supergame can be derived from the requirement of stability against short-run exploitation. The profit combinations of all plays in the final tournament in which both players received more than their Cournot profits are in the exploitation stability region. These are 97.1% of all plays in the final tournament.

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Experimental Results on Ultimatum games

with Incomplete Information

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Experimental Results on Ultimatum Games with Incomplete Information

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Abstract: This paper is about experiments on two versions of ultimatum games with incomplete information, called the offer game and the demand game. We apply the strategy method, that is, each subject had to design a complete strategy in advance instead of reacting spontaneously to a situation which occurs in the game. Game theory predicts very similar outcomes for the offer and the demand games. Our experiments, however, show significant differences in behavior between both games. Using the strategy method, allows us to explore the motivations leading to those differences. Since each subject played the same version of the game eight rounds against changing anonymous opponents we can also study subjects' learning behavior. We propose a theory of boundedly rational behavior, called the "anticipation philosophy", which is well supported by the experimental data.

I Introduction

This paper reports experiments on ultimatum games with incomplete information. An ultimatum game is a two-person game in which player A proposes a division of a "cake" and player B can then either accept the proposal or reject it. If the proposal is rejected both players get nothing, otherwise the proposed division is implemented. In experimental studies the cake is usually a sum of money.

Following Güth, Schmittberger and Schwarze (1982), a number of experimental papers have studied such games, or games in which ultimatum games arise as subgames. For surveys of the literature see Thaler (1988), Güth and Tietz (1990), and Roth (1992). All subgame perfect equilibria of an ultimatum game are characterized by an extremely unequal split of the cake: player B can earn no more than the smallest money unit. Güth and Tietz, among other authors, throw doubt on the empirical relevance of the game-theoretical solution and claim that behavior is mainly driven by considerations of distributive justice (see, e.g., Güth, Ockenfels and Tietz (1990)). On the other hand, some authors insist that strategic issues cannot be neglected in studying human behavior (see, e.g., Binmore, Shaked and Sutton (1985 and 1988)). The state of the art is that something in between the equal split and the equilibrium split occurs. In almost all previous experiments the modal choice of

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player A is the fifty-fifty split, whereas the mean of choices gives him a higher proportion of the cake (between 60% and 70%).

In this study we shall deal with ultimatum games with incomplete information, in which only player A knows the size of the cake precisely. Player B is only informed about the probability distribution of possible cake sizes when responding to player's A proposal. Two versions of such a game are considered:

- In an offer game, player A offers a sum of money to player B. Thus, player B is only informed of her possible gain and does not know the residual to player A.
- In a demand game, player A demands a sum of money for himself. Thus, player B is only informed about player's A payoff, without knowing her own payoff, when making her decision.

Over the last 20 years, games of incomplete information have been of great interest for both normative game theorists and economists applying game theory. However, only little experimental research on bargaining has been done in this direction despite the fact that in real world bargaining private information is the rule and not the exception. Experimental studies which are concerned with incomplete information are Rapoport, Kahan and Stein (1973); Hoggatt, Selten, Crockett, Gill and Moore (1978); Rapoport, Erev and Zwick (1991) and Forsythe, Kennan and Sopher (1991). Subsequent to our experiments Rapoport, Sundali and Potter (1992) also investigate the offer game.

Game theory predicts similar equilibrium outcomes for both offer and demand games (see section II). These predictions are also similar to the equilibrium outcomes in ultimatum games with complete information. Contrary to game theoretic predictions, our experimental data reveal significant differences in actual behavior between offer and demand games.

Selten (1967) proposed an experimental technique called the "strategy method". According to this method, subjects have to specify complete strategies for a given game, after they gained experience with this game by actual play. We follow this experimental technique, but without the phase of actual play since our games have rather simple structures. The advantage of the strategy method is that it makes strategies observable and not only actual choices at those information sets that arise in the course of a play. The structure of an observed strategy allows a deeper insight into the reasoning behind the particular choices.

Each participant in our experiments was matched in eight subsequent rounds against changing anonymous opponents. In all periods a participant remained in the same player role and in the same type of game. In each period, a player could alter his strategy in view of his previous results.

It is the aim of this paper to contribute to a descriptive theory of human behavior in simple bargaining situations with incomplete information. Perhaps such a theory is also valuable for the understanding of behavior in ultimatum games with complete information on the one hand and of behavior in more complex bargaining situations with incomplete information on the other hand. For the explanation of our data we develop a theory of the individual decision process in which issues of fairness, strategic reflections and anticipation of opponent's behavior are involved.

In section II we define the games under consideration and discuss the game-theoretic solution. In section III we present the experimental design. Section IV presents the results aggregated over the eight repetitions. Section V studies the dynamics of the behavior over time. In section VI we look at the individual learning behavior. Section VII concludes.

II The Games

The extensive form of an offer game is described below. The variations for a demand game are indicated in parentheses.

1. The cake may be one of six amounts: 1, 2, 3, 4, 5 or 6 Taler, a fictitious currency. Its size is determined by throwing a die. 1 Taler is equivalent to 1.20 DM. At the time of the experiments this was worth about $ 0.70.
2. Player A is informed of the cake size. Player B knows only how the size of the cake is determined, but is not informed of the actual throw of the die.
3. Player A proposes an offer (demand) to player B. The offers (demands) are restricted to the set {0, 0.5, 1, ..., 5.5, 6} and player A may not offer (demand) more than the whole cake. The smallest money unit of 0.5 is about one cup of coffee in the students' cafeteria.
4. Player B can accept or reject the offer (demand).
5. If player B rejects the proposal, both receive nothing. If she accepts the proposal then player A gets the whole cake minus his offer (he gets just the demand), and B receives the offer (she gets the whole cake minus the demand).

Thus, in the offer game player B is informed about her payoff if she accepts the offer but not about A's payoff (unless she is offered 5.5 or 6.0, in which case she can deduce from the rules of the game that the cake was of size 6). In the demand game player B knows what player A will receive if she accepts his demand, but she does not know her own payoff (unless she is faced with a demand of 5.5 or 6.0, in which case she can deduce that her payoff will be 0.5 or 0.0, respectively).

The sets of Nash equilibria for both versions are very large. The sequential equilibria (Kreps and Wilson, 1982) restrict the equilibrium payoffs of player B to 0.5 at most. However, in the offer game, we neglect those equilibria where player A sometimes offers 0 and player B sometimes accepts 0. These equilibria are weak, since B loses nothing when rejecting 0. We concentrate on those sequential equilibria that are strict on the equilibrium path. This means that the equilibrium strategies induce strict local best replies at all information sets reached in equilibrium. We refer to such an equilibrium as a path-strict sequential equilibrium. It turns out that in both games the concept of path-strict sequential equilibrium yields a unique solution. In both games player B always receives 0.5 at this equilibrium and player A receives the residual of the cake. More specifically, in the offer game, player A offers 0.5, the smallest money unit, regardless of the size of the cake and player B accepts every positive offer. In the demand game, player A demands the entire cake minus 0.5 and player B accepts all .5 demands and rejects all integer demands. Note that in the demand game, the path-strict sequential equilibrium is not perfect in the sense of Selten (1975). Therefore, we also consider the perfect equilibrium at which player A demands the entire amount for cakes 1 to 5 and 5.5 for cake 6 and player B accepts every demand but 6. In equilibrium *no* disagreements occur in either game, unlike in the bargaining game studied by Forsythe, Kennan and Sopher (1991), where the *un*informed player makes the proposal. The path-strict sequential equilibria of both games yield the same payoff division as at the unique subgame perfect, path-strict equilibrium of the ultimatum game with complete information.

III Experimental Design

We ran 10 experiments: 5 on offer games (called O1–O5) and 5 on demand games (D1–D5). Each experiment involved 16 different subjects (making 160 in all) from various faculties of the University of Bonn. Each subject participated in only one experiment.

In each experiment 8 subjects were randomly assigned to be players A and the remaining 8 were players B. Players A and players B were then placed in two large separate classrooms. No communication was possible within a player group. Each subject received a set of written instructions (see appendix), the respective decision sheet (figure 1a, b) and an explanation sheet. The instructions were read aloud to the subjects. Questions concerning the rules of the games were answered. Each subject in a player group played one of the two different ultimatum games eight rounds against a new and anonymous opponent of the other player group each time. A subject played only one type of game and was always in the same position (A or B).

Before the toss of the die in a round and before players B were informed of the decision of players A, both player groups were required to write complete strategies into their decision sheets. Thus, a player A made an offer or demand for each of the 6 possible cakes. The smallest money unit 0.5 restricted the number of information sets of player B to only thirteen. A player B had to decide which of the possible offers or demands from 0.0 to 6.0 she would accept or reject. The decision sheets were collected and matched according to a prepared plan in such a way that a player A played only once against the same player B. The matching plan was not told to the subjects. Afterwards, the die was thrown in the room of players A. The given strategies in a round determined the course of play. Then each decision sheet was returned to the respective subject. On their sheets the subjects were informed about their own last-period payoff, but not about their last-period opponent's complete strategy. After 8 rounds the gains of each player were added, revalued into DM and paid out. It was our intention that all information not specifically withheld by the rules of the game should be common knowledge among the subjects.

The strategies fall into four classes since it is necessary to distinguish offer and demand games, and A strategies from B strategies. We observed 320 strategies for each class (5 experiments × 8 players × 8 rounds). The raw data are available and we will be pleased to supply them on request.

Decision sheet Player A

Player's number: ______
Number of experiment: ______

I demand the following amount dependent on the total amount for myself:

period \ total amount	1	2	3	4	5	6
1						
2						
3						
4						
5						
6						
7						
8						

to be filled in by the experimenter

total amount	your demand	player A's reply	your pay-off

Decision sheet Player B

Player's number: ______
Number of experiment: ______

I accept (a) or reject (r) the following demande made by player A

period \ demand	0	0,5	1	1,5	2	2,5	3	3,5	4	4,5	5	5,5	6
1													
2													
3													
4													
5													
6													
7													
8													

to be filled in by the experimenter

player A's demand	your reply	total amount	your pay-off

Fig. 1. Decision sheets for player A and player B in demand games.

IV Results Pooled Over Eight Periods

A. Choices of Players A

Tables 1a and b present the frequencies of offers and demands, respectively, for the six cakes pooled over all 8 rounds. Figure 2 shows the mean required cake proportions that players A asked for themselves, again aggregated over all 8 rounds, and separated for offer and demand games. To compare this with theoretic predictions, we also plot the path-strict sequential equilibrium split in % to player A and the fifty-fifty split for each cake size.

Table 1a. Frequencies of offers for the six cakes over all offer games

	offers												
	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5	5.5	6
cake													
1	30	273	17										
2	13	101	176	29	1								
3	2	56	108	133	20	1	0						
4	2	42	65	97	107	3	3	1	0				
5	2	40	41	76	101	51	4	4	0	1	0		
6	1	41	38	51	96	40	46	2	1	3	0	1	0

Table 1b. Frequencies of demands for the six cakes over all demand games

	demands												
	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5	5.5	6
cake													
1	1	106	213										
2	1	5	73	54	187								
3	0	0	0	53	74	69	124						
4	0	0	0	4	59	81	87	53	36				
5	0	0	0	4	23	80	84	54	31	32	12		
6	1	0	0	1	10	31	103	53	36	32	21	24	8

In the offer games, the modal choices are (0.5, 1, 1.5, 2, 2, 2) for cakes (1, 2, 3, 4, 5, 6), as indicated by the underline instead of overline numbers in table 1a. Thus, the modal offer is an equal split for cakes 1 to 4 and an offer of 2 for cakes 5 and 6. The mean proportion of the cake required by player A increases with cake size, as predicted by the path-strict sequential equilibrium. This increase is significant at the 5% level for all experiments on offer games (O1–O5), separately, using the Spear-

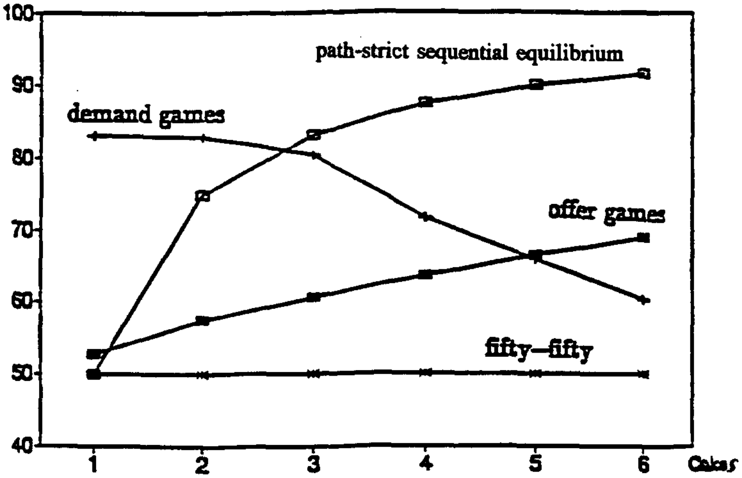

Fig. 2. Mean required cake proportions of player A for the six cakes, over all offer games and demand games.

man rank correlation coefficient[2]. However, the observed cake divisions are far away from the equilibrium prediction, except for cake 1 where 0.5 is the only plausible offer.

In the demand games, the modal choices are (1, 2, 3, 3, 3, 3) for the respective cakes. This means that most players A demand the entire cake for cakes 1 to 3 and continue with a demand of 3 for the remaining cakes. The mean observed proportions of the cake for players A show a reverse trend to the path-strict sequential solution, significant at the 1% level for D1–D4 and insignificant for D5, using the Spearman rank correlation coefficient. In both offer and demand games, players' A average behavior deviates extremely from the fifty-fifty split.

The behavior for any particular cake size differs markedly between offer and demand games. In the demand games for cakes 1 to 4 players' A average request is higher than in the offer games, significant at the 1% level for cakes 1 to 3 and at 2.5% for cake 4, using the Mann-Whitney U-test applied to the comparison of mean required proportions at each cake of for the five offer and five demand games. For cake 6 the reverse holds, that is, players A ask for more in the offer games (significant at 2%). For cake 5 there are no significant differences.

In offer games, 4.7% of the choices leave more than half of the cake to player B. In demand games the respective number is 4.2%. Note that these generous proposals are concentrated on the small cakes in offer games and on large cakes in demand games. Equal split proposals arise in offer games 41% and in demand games only in 24% of all proposals. The entire cake is required in 0.9% of all proposals in offer games, but in 30% in demand games. We will explain these differences later.

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[2] These tests and the following ones are one-tailed tests if not stated otherwise.

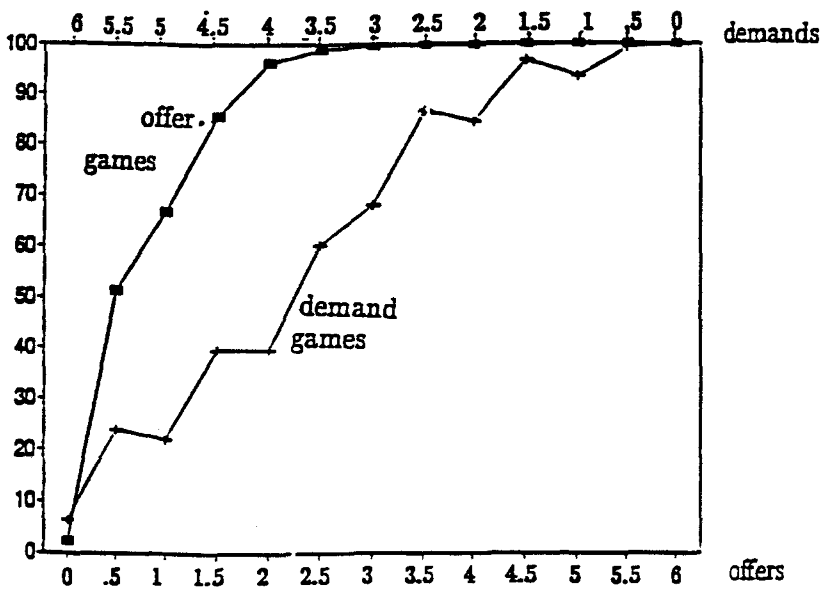

Fig. 3. Relative frequencies of acceptance in % for the 13 possible offers (demands) from players B, over all offer and demand games.

B. Choices of Players B

Figure 3 shows the relative frequencies with which players B accepted possible offers (demands). From the game theoretic point of view, it is interesting to compare the behavior on an offer of 0.5 and on a demand of 5.5, since player B is sure to obtain 0.5 in both cases. In the offer games, 51% accept this lowest positive offer, whereas only 24% accept a demand of 5.5. Also a demand of 4.5 is accepted less frequently (40%) than the lowest positive offer. The reason might be that, in the offer game, 0.5 could be an equal split of cake 1, while a demand of 5.5 or 4.5 certainly aims at an unequal division.

In an offer game, player's B strategy can be sufficiently characterized by her lowest accepted offer if all higher offers are also accepted. Only 10 out of the 320 observed strategies deviated from this monotonicity. In the demand games, 82.5% of the strategies can be characterized just by their highest accepted demand. This means that these strategies accept also all lower demands. The acceptance rate of some integer demands are slightly smaller than that rate of the next higher .5 number, see kinks at integer demands 1, 2, 4, and 5 in figure 3. This indicates that some players reject integers whereas they accept higher .5 demands as the path-strict sequential equilibrium strategy would recommend. This phenomenon is perhaps due to the simple observation that an integer-plus-0.5 demand guarantees a positive payoff while an integer demand may give nothing to player B. We observed that 13.75% of the actual strategies are of this kind.

In the offer games, more players B are in accordance with game theoretic predictions than in the demand games: 51.3% accept all offers from 0 or 0.5 in the offer games, whereas 19.7% accept all demands until 5.5 or 6 or use the strategy of the path-strict sequential equilibrium in the demand games (the difference is significant at the 2% level over the 10 experiments, using Mann-Whitney U-test).

C. Interaction of Actual Strategies

In order to calculate the mean expected payoffs for players A at each cake, we match each strategy of player A of a given period and experiment against the given B-population of the same period and experiment and take the average over all period- and experiment-payoffs of all players A, separately for each cake and offer and demand games, as shown in tables 2a, b. The payoffs of players B are calculated similarly.

Table 2a. Mean expected payoffs of players A and B and mean expected acceptance rate for the respective cakes, over all offer games

cake	1	2	3	4	5	6	mean
expected payoff of A	0.22	0.65	1.30	1.99	2.75	3.51	1.74
expected payoff of B	0.25	0.57	0.91	1.26	1.51	1.73	1.04
expected acceptance rate	0.47	0.61	0.74	0.81	0.85	0.87	0.76

Table 2b. Mean expected payoffs of players A and B and mean expected acceptance rate for the respective cakes, over all demand games

cake	1	2	3	4	5	6	mean
expected payoff of A	0.79	1.45	1.88	1.99	1.91	1.83	1.64
expected payoff of B	0.17	0.33	0.52	0.88	1.27	1.49	0.76
expected acceptance rate	0.95	0.89	0.80	0.72	0.64	0.56	0.76

For cakes 1 to 3, players' A earn less in offer games than in demand games; the reverse is true for cake 5 and 6. Whereas the expected payoffs of players A increase monotonically with cake size in offer games, in demand games the expected payoff takes the maximum at cake 4. The expected payoffs of players B are for each cake larger in the offer games than in the demand games. The average acceptance rate within the demand and offer games, respectively, are about 76%. For a more detailed discussion on the single experiments see Mitzkewitz and Nagel (1992).

D. Classification of the Strategies of Players A

In section IV.A we showed that the behavior of players A is significantly different between the offer and demand games. In this section we will describe the pattern of the observed strategies in order to explain this difference. The number of possible pure strategies of player A is 135 135 ($3 \times 5 \times 7 \times 9 \times 11 \times 13$). This number reduces to

7752 if we consider only monotonic strategies, that is, player A offers (demands) an equal or a higher amount with increasing cake size. Out of the 640 observed strategies (10 experiments × 8 players A × 8 periods) only 29 strategies are nonmonotonic.

We describe below the typical strategies that explain most of the behavior of players' A and we compare them with strategies of theoretic and descriptive interest, which are already mentioned. There are the path-strict sequential equilibrium strategies of both games and the perfect equilibrium in the demand game (as explained in section II) and the fifty-fifty strategy. Although the equilibrium strategies and the fifty-fifty strategy have been observed, other typical patterns of strategies explain more of the data.

First, we consider a version of the fifty-fifty strategy to which we refer as the "fifty-fifty-0.5" strategy. It consists of equal-split offers minus 0.5 (equal-split demands plus 0.5) for each cake, except for cake 1 in offer games, where a zero offer will certainly be rejected. That is, player A uses the fifty-fifty strategy as a focal point, but he additionally keeps 0.5 for himself for each cake. The simple reasoning behind this strategy might be that player A realizes that he is in a stronger position and, hence, should earn a little more than the equal split.

In section IV.A, table 1 presented the observed modal choices for each cake. Although these choices and the resulting requirements are quite different in the two games, both sets of observations may be explained by the same type of thought process of player A. This thought process is based on an own *aspiration criterion* and an *anticipated level of acceptance* of player B. Player's A *aspiration criterion* is a rule which determines his minimal acceptable payoff contingent on the cake. The *anticipated level of acceptance* is player's A expectation on the smallest offer (respectively largest demand) that player B will accept. The thought process consists of three steps:

Step 1: The aspiration criterion and the anticipated level of acceptance are formed.
Step 2: The best reply to the anticipated level of acceptance is intended.
Step 3: For all cakes where the best reply does not satisfy the aspiration criterion, the intended strategy is altered such that player A aims at his minimal acceptable payoffs in the case of acceptance.

Consider two examples: In an offer game player A forms at step 1 the aspiration criterion "At least half of the cake" and the anticipated level of acceptance "2". Thus, he computes at step 2 the best-reply strategy (1, 2, 2, 2, 2, 2). (The choice "1" is arbitrary.) Since his aspiration criterion requires always for at least half of the cake, at step 3 he alters his strategy to (0.5, 1, 1.5, 2, 2, 2). Compare this strategy with the modal choices shown in table 1a.

Let us consider that in a demand game player A has the same aspiration criterion as above and an anticipated level of acceptance "3". At step 2 the strategy (1, 2, 3, 3, 3, 3) is intended. Since for no cake the aspiration criterion is violated, no alterations of the intended strategy take place at step 3. Compare this strategy with the modal choices in demand games presented in table 1b.

We found that some widely used strategies can be explained by the thought process described above and by reasonable specifications of the terms "aspiration criterion" and "anticipated level of acceptance". In the offer game, these are three strategies based on the anticipated levels of acceptance 1, 1.5 and 2 together with the aspiration criterion "At least half of the cake". In the demand games five strategies are of importance, which are best replies to the anticipated levels of acceptance 2, 2.5, 3, 3.5 and 4. We will call these eight strategies *anticipation strategies*, and we will speak of *anticipation strategy x* (in the offer or demand games) if it is based on an anticipated level of acceptance *x*.

Table 3a. Typical strategies of players A in the offer game

Typical strategies	cake					
	1	2	3	4	5	6
a.) strict seq. equil.	0.5	0.5	0.5	0.5	0.5	0.5
b.) fifty-fifty	0.5	1	1.5	2	2.5	3
c.) fifty-fifty-0.5	0.5	0.5	1	1.5	2	2.5
d.) anticipation 1	0.5	1	1	1	1	1
e.) anticipation 1.5	0.5	1	1.5	1.5	1.5	1.5
f.) anticipation 2	0.5	1	1.5	2	2	2

Table 3b: Typical strategies of players A in the demand game

Typical strategies	cake					
	1	2	3	4	5	6
a.) perfect equil.	1	2	3	4	5	5.5
b.) strict seq. equil.	0.5	1.5	2.5	3.5	4.5	5.5
c.) fifty-fifty	0.5	1	1.5	2	2.5	3
d.) fifty-fifty-0.5	1	1.5	2	2.5	3	3.5
e.) anticipation 4	1	2	3	4	4	4
f.) anticipation 3.5	1	2	3	3.5	3.5	3.5
g.) anticipation 3	1	2	3	3	3	3
h.) anticipation 2.5	1	2	2.5	2.5	2.5	2.5
i.) anticipation 2	1	2	2	2	2	2

Tables 3a and b recapitulate the strategies we want to look at. We group them into three classes:

(1) The *equilibrium philosophy* relies on the conviction of common knowledge of rationality (see table 3a, strategy a) for the offer game and table 3b, strategies a) and b) for the demand game).

(2) The *fifty-fifty-philosophy* is based on a superficial analysis of the situation, resulting in simple rules of thumb without explicit strategic considerations (strategies b) and c) in the offer game of table 3a and strategies c) and d) in the demand game of table 3b).

(3) The *anticipation philosophy* is based on player's A expectation of an acceptance level by player B and a strategic reply to it, restricted by his own aspirations.

Table 4a. Relative frequencies for the typical strategies and the classes in offer games with the area of the typical strategies in parentheses

	Offer Games	Perfect hits	Up to one .5 deviations	Up to two .5 deviations	Up to three .5 deviations
Strategies	Path-Strict seq. equ.	11.6 (.01)	12.2 (.04)	12.9 (.08)	12.9 (.13)
	Fifty-fifty	12.5 (.01)	14.9 (.17)	17.5 (.88)	18.8 (2.4)
	Fifty-fifty-0.5	3.3 (.01)	8.9 (.14)	13.5 (.74)	18.8 (1.9)
	Anticipation 1	6.6 (.01)	9.9 (.06)	12.2 (.13)	14.5 (.23)
	Anticipation 1.5	5.3 (.01)	8.6 (.09)	16.8 (.26)	31.4 (.48)
	Anticipation 2	8.9 (.01)	16.2 (.12)	20.8 (.40)	34.3 (.89)
Classes	Equilibrium	11.6 (.01)	12.2 (.04)	12.9 (.08)	12.9 (.13)
	Fifty-fifty	15.8 (.03)	23.8 (.31)	31.0 (1.6)	36.3 (4.1)
	Anticipation	20.8 (.04)	34.7 (.27)	46.2 (.75)	58.1 (1.5)
All three classes		48.2 (.08)	70.6 (.62)	86.5 (2.3)	92.1 (5.4)
Others		51.8 (99.9)	29.4 (99.4)	13.5 (97.7)	7.9 (94.6)

Table 4b. Relative frequencies for the typical strategies and the classes in demand games with the area of the typical strategies in brackets

	Demand Games	Perfect hits	Up to one .5 deviations	Up to two .5 deviations	Up to three .5 deviations
Strategies	Perfect equi.	2.6 (.01)	5.8 (.17)	7.8 (.93)	9.1 (2.9)
	Path-Strict seq. equ.	1.6 (.01)	1.9 (.17)	3.2 (.94)	4.5 (3.0)
	Fifty-fifty	8.1 (.01)	8.8 (.17)	9.4 (.88)	12.0 (2.4)
	Fifty-fifty-0.5	1.0 (.01)	5.8 (.17)	8.1 (.82)	14.2 (2.2)
	Anticipation 4	3.2 (.01)	4.2 (.12)	6.5 (.46)	9.4 (1.1)
	Anticipation 3.5	1.6 (.01)	4.2 (.12)	6.8 (.45)	24.7 (1.0)
	Anticipation 3	13.0 (.01)	15.3 (.09)	17.5 (.28)	19.8 (.58)
	Anticipation 2.5	3.2 (.01)	6.2 (.09)	9.4 (.26)	11.4 (.49)
	Anticipation 2	2.6 (.01)	2.6 (.06)	3.2 (.14)	5.2 (.27)
Classes	Equilibrium	4.2 (.03)	7.8 (.34)	11.0 (1.9)	11.4 (5.4)
	Fifty-fifty	9.1 (.03)	14.6 (.34)	17.5 (1.7)	25.6 (4.5)
	Anticipation	23.7 (.06)	32.5 (.48)	39.9 (1.5)	44.8 (2.9)
All three classes		37.0 (.12)	54.9 (1.1)	68.5 (5.1)	78.9 (13.0)
Others		63.0 (99.9)	45.1 (98.9)	31.5 (94.9)	21.1 (87.0)

We have thus highlighted 6 out of the 7752 possible monotonic strategies (0.08%) in the offer game and 9 (0.12%) in the demand game. Tables 4a and b classify the actual strategies in offer and demand games, respectively, of players A under the types of strategies and under the three philosophies. We exclude the non-monotonic observations here. In the first column of tables 4a and b, upper part, we state the relative frequencies of the actual strategies of the offer games and demand games, respectively, which correspond exactly to the types of strategies mentioned above. In the middle part, they are aggregated to the three classes, and in the lower part the relative frequencies of all three classes together are stated. Since it cannot be expected that actual strategies always coincide with the idealized strategies, small deviations to them are also of interest. In columns 2 to 4 we count additionally those actual strategies that are in the "neighborhood" of the described types of strategies or of strategies belonging to one of the three classes, by allowing minor deviations: up to three choices of the six may differ by ±0.5 from the respective choices of a prescribed strategy provided that monotonicity is not destroyed. However, no choice may differ by more than ±0.5 from the respective choice of a given strategy. Strategies that cannot be classified with our classification scheme are labelled "other strategies".

In each cell of tables 4a and b we state in parentheses the proportion of strategies out of the possible 7752 monotonic strategies considered in the specific neighborhood, in order to show how many possible strategies are captured with our classification scheme and its derivations. This means, for example, that allowing up to one .5 deviation from the fifty-fifty strategy in the offer game 13 out of 7752 (0.17%) strategies are considered. We will call these proportions, following Selten (1991), the "areas". Using Selten's terminology, "areas" are the ratio between the outcomes predicted by a descriptive theory and the set of all possible outcomes. "Hits" are the experimental observations in accordance with a descriptive theory. The "hitrate" of a descriptive theory is its percentage of hits, given a set of observations. Selten (1991) axiomatically proves that the difference between "hitrate" and "area" is the appropriate measure for the descriptive success of a proposed theory. The three philosophies we are considering always induce negligible areas; thus we concentrate only on the hitrates.

In columns 3 and 4 of tables 4a and b the neighborhoods of the types of strategies are not mutually exclusive. Thus, if an actual strategy is minimally deviating from more than one type of strategy we state it under all neighborhoods which it belongs to. Multifold counting is of course avoided when aggregating for a class: therefore, e.g., the relative frequencies of actual strategies for the anticipation philosophy up to three deviations is not simply the sum of the relative frequencies of the respective strategies up to three deviations.

Considering only perfect hits (column 1) we can classify 48% of the behavior in the offer games and 37% in the demand games, whereas the areas are only 0.08% and 0.12%, respectively. In the demand games, the highest relative frequency (13%) of the classified strategies belongs to anticipation strategy 3. In the offer games, the fifty-fifty strategy shows the highest relative frequency. However, this is due to the first period behavior, where about 40% of the actual strategies are fifty-fifty strategies (see section V).

Only 4% of the observed strategies are exactly the sequential strategies in demand games, but in the offer games 11.6% belong to this class. In both offer and demand games, no more than 13% of the strategies are chosen in the neighborhood of the equilibrium strategies. In both types of games the anticipation class is the prevailing one. The frequent use of anticipation strategies also explains the increase of players' A required share in offer games, respectively the decrease of players' A demanded cake proportions according to increasing cakes, as shown in figure 2, since anticipation strategies induce the respective division scheme.

In the ultimatum game with complete information there is an obvious standard of fairness, namely the equal split. Such obvious standards of fairness do not exist in the games with incomplete information because the payoffs of the two players are not comparable for player B. So she must *construct* such fairness standards contingent on the available information. Player A has to simulate this construction when anticipating player's B level of acceptance.

In the offer games, the highest frequency of observations is in the neighborhood of anticipation strategies 1.5 and 2. Note that half of the expected cake is 1.75. Since in the offer games player B is only informed about her own payoff, it seems to be a natural standard of fairness to take half of the expected cake as her acceptance level. We will call this concept *expectation fairness.* The data indicate that many players A anticipate that players B will form their acceptance level according to expectation fairness.

The informational situation for player B is completely different in the demand game than in the offer game. For all demands (except 5.5 and 6) player B is uncertain about her own payoff when she accepts. Therefore, she cannot use the concept of expectation fairness to form her acceptance level. Instead player B may use a concept which we call *resistance to visible unfairness:* She rejects all demands where she is sure to obtain less than half of the cake, i.e. all demands larger than 3. If player A expects that player B will be resistant to visible unfairness, he will choose anticipation strategy 3. As table 4b shows, anticipation strategy 3 is the mostly used strategy up to two deviations.

Now we want to investigate the performance of the typical strategies against players' B actual strategies. For this purpose we calculate the expected payoff of each typical strategy given the 64 strategies of players B within each experiment. We also state the mean expected payoffs of players' A actual strategies by matching these strategies with players' B strategies in the respective period for each experiment. Also the best-reply payoffs are shown, which are the mean of the expected payoffs of the best replies to the population of players' B strategies in each period.

The underlined numbers in tables 5a and b indicate the highest payoffs among the typical strategies, mentioned in table 3a and b. For example, in O1, if player A chooses anticipation strategy 1.5 in each period he will receive an expected payoff of 1.95 per period. Notice that in all experiments the underlined payoffs are obtained by choosing an anticipation strategy and these payoffs are exactly the best-reply payoffs or close to them. Thus, in the offer games, it is a good policy for player A to choose anticipation strategy 1, 1.5 or 2, and in the demand games to choose anticipation strategy 2.5, 3 or 3.5 against the given B-strategies. The equilibrium strategies or the fifty-fifty strategy yield on average the lowest payoffs. On the other hand, the fifty-fifty-0.5 strategy is not a bad policy.

As regards the payoffs of player B, the best she can do is accepting every positive offer in the offer games. Given the actual strategies of players A, she will then receive an expected payoff of 1.25 per period. The actual payoff is 1.04 against the given A-population. In the demand games, she would receive 1.08 if she accepted every demand and 0.53 if she accepted all .5 numbers and no integers. This is contrary to the game theoretical prediction since she would receive 0.083 at the perfect equilibrium mentioned in section 2 and 0.50 at the path-strict equilibrium. The actual payoff is 0.76.

Table 5a. Expected payoffs of the typical strategies, of the actual strategies, and of the best-reply strategies of players A, given the actual distribution of the strategies of players B, separately for each offer game

Typical strategies	expected payoffs in ...					
	O1	O2	O3	O4	O5	mean
sequent. equilibrium	1.27	1.78	1.36	1.27	2.13	1.56
fifty-fifty	1.62	1.59	1.53	1.54	1.70	1.60
fifty-fifty-0.5	1.81	1.83	1.69	1.68	2.02	1.81
anticipation 1	1.60	1.92	1.37	1.29	2.40	1.72
anticipation 1.5	1.95	1.80	1.75	1.68	2.15	1.87
anticipation 2	1.87	1.78	1.74	1.72	1.95	1.81
actual strategies A	1.72	1.72	1.64	1.60	2.02	1.74
best-reply strategy	1.95	1.95	1.79	1.76	2.41	1.97

Table 5b. Expected payoffs of the typical strategies, of the actual strategies, and of the bestreply strategies of players A, given the actual distribution of the strategies of players B, separately for each demand game

Typical strategies	expected payoffs in ...					
	D1	D2	D3	D4	D5	mean
perfect equilibrium	1.13	1.59	0.98	1.40	2.13	1.45
strict equilibrium	1.41	1.55	1.31	1.45	2.02	1.55
fifty-fifty	1.49	1.43	1.19	1.57	1.63	1.46
fifty-fifty-0.5	1.78	1.69	1.37	1.81	2.00	1.73
anticipation 4	1.35	1.58	0.92	1.83	2.18	1.57
anticipation 3.5	1.88	1.83	1.40	1.82	2.27	1.84
anticipation 3	1.69	1.83	1.26	2.03	2.19	1.80
anticipation 2.5	2.06	1.67	1.58	2.01	2.09	1.88
anticipation 2	1.57	1.60	1.23	1.70	1.72	1.56
actual strategies A	1.68	1.57	1.23	1.71	1.98	1.63
best-reply strategy	2.06	1.83	1.58	2.03	2.27	1.95

V Dynamics of Behavior Over Time

So far we have only considered the data aggregated over all rounds. We now consider the distribution of the types of strategies by players A and players B over time, in order to see whether there is some change in behavior. At the end of this section we test whether or not players A tend to require more over time for themselves and players B accept lower offers or higher demands.

Figures 4a–f show the relative frequencies of the three classes and the types of strategies over time. We classify an actual strategy[3] in that class of strategy from which it has the least (and no more than three) 0.5 choice deviations and no greater deviations. Otherwise, a strategy is classified as "other". If an actual strategy has the same number of 0.5 deviations (but no more than three) from more than one type of strategy, we give equal weight to each type of strategy. In Figure 4a, b we plot the distribution of the three classes of strategies of the offer and demand game, respectively. In all rounds, except round 1 in offer games, the anticipation class predominates over time. In the offer games, the frequency of the equilibrium strategy is slowly increasing. In demand games the increase of that class is not regular. In both games the sequential equilibrium strategies are chosen about 20% of the time in the last two rounds and thus are among the most used strategies in these rounds. Notice that such an increase of the use of the equilibrium strategy, although still small, has never been observed in experiments on the ultimatum game with complete information when playing against real subjects (see e.g., Roth, Prasnikar, Okuno-Fujiwara, and Zamir, 1991). Harrison and McCabe (1992) observe that behavior evolves in the direction of the perfect equilibrium only when subjects played against automated robots playing near equilibrium strategies.

In figures 4c and d we show the relative frequencies of the different anticipation strategies over time and in figures 4e, f we disaggregate the fifty-fifty class. In the offer games, anticipation strategy 2 and in the demand games anticipation strategy 3 prevails over time. The fifty-fifty strategy is one of the most frequently used in the first three rounds, but decreases and belongs to the least frequently used strategies in the remaining rounds.

Figures 5a and b show the development over time of the relative frequencies of players' B lowest acceptance levels in the offer games and highest acceptance levels in the demand games, respectively. In the offer games, the sequential equilibrium strategies, i.e. to accept all offers or all positive offers, is the most frequently used right from the beginning, and there is a sharp increase in its use in the last period. Up to the two last rounds strategies with acceptance level 1.5 and 2 together are used by about 30% of players B. It might be that these players are guided by the idea to accept half of the expected cake. Indeed, 10 out of 40 players B wrote on their explanation sheets that they calculated half of the expected cake size. Whereas in the demand games, strategies in which demands up to 1.5 or 2 are accepted are of minor importance. Accepting demands up to 5.5 or 6 increase slightly over time, but remain below 30%. In all but period 2 and 5, acceptance levels 2.5–3 are predominant.

To analyze the average behavior over time, we introduce a requirement index for player A and a rejection index for player B and calculate the average indexes over the eight respective strategies for each period within each experiment. For any given strategy of player A the requirement index in the offer game is defined as the difference of the expected cake size and player's A mean offer over the six cakes:

[3] Again, we consider only those actual strategies that are monotonic.

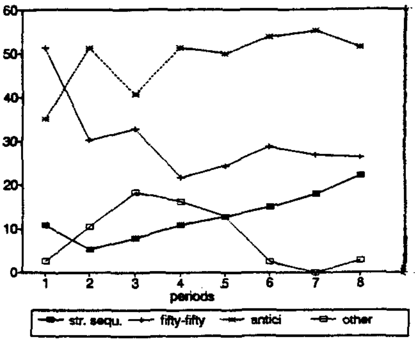

Fig. 4a. The three classes in offer games

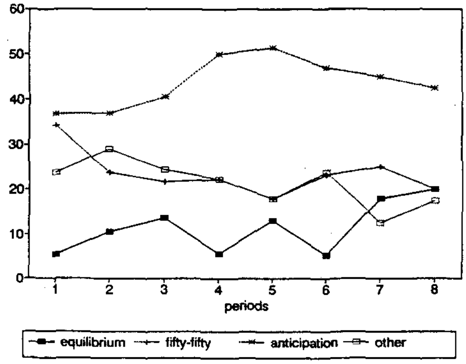

Fig. 4b. The three classes in demand games

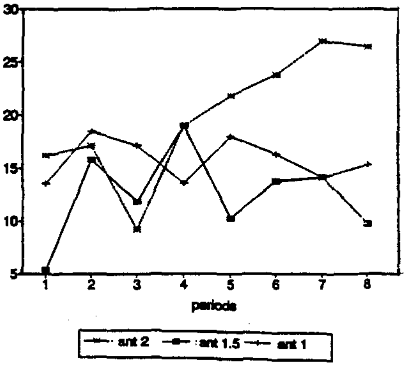

Fig. 4c. Anticipation strategies in offer games

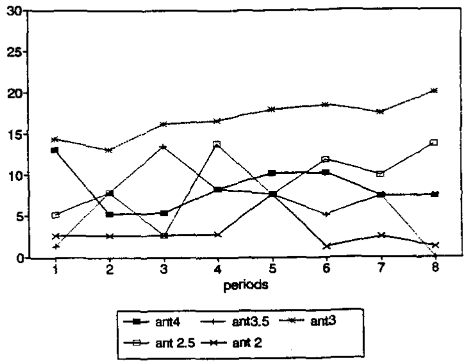

Fig. 4d. Anticipation strategies in demand games

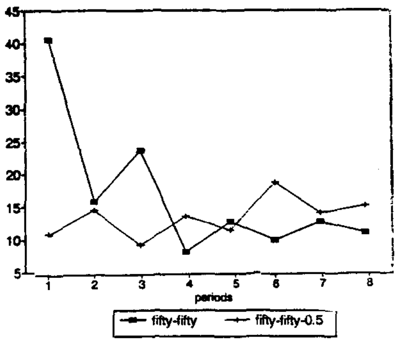

Fig. 4e. Fifty-fifty philosophy in offer games

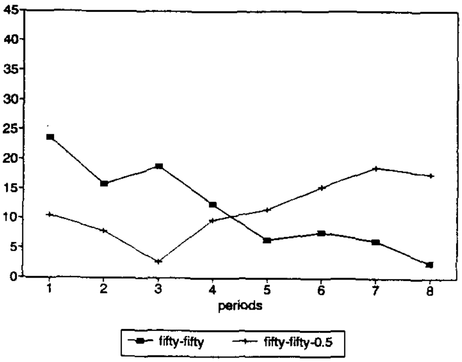

Fig. 4f. Fifty-fifty philosophy in demand games

Fig. 4. Frequencies of the three classes and types of strategies over time

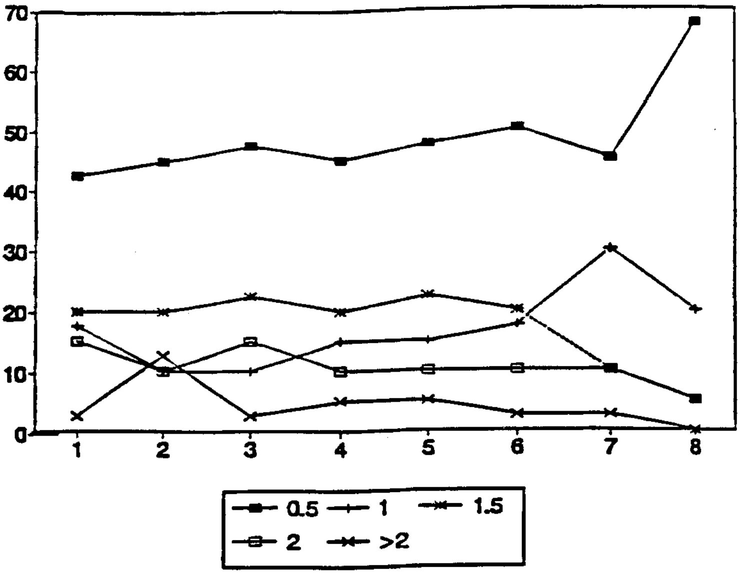

Fig. 5a. Relative frequencies of players B acceptance level over time in offer games

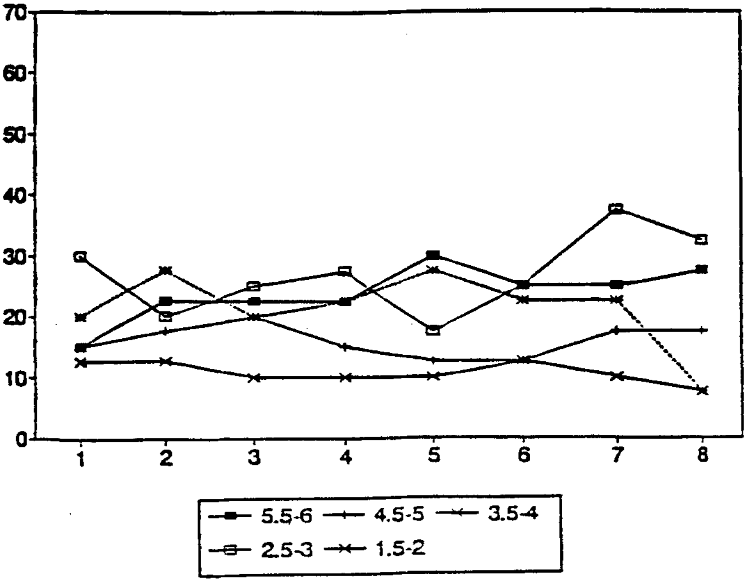

Fig. 5b. Relative frequencies of players B acceptance level over time in demand games

$3.5-\sum$ offers/6, and in the demand game as player's A mean demand over the six cakes: $\sum$ demands/6. The rejection index for a strategy of player B in the offer game is defined as the number of player's B rejections, and in the demand game as the sum of rejected demands (to count only the number of rejections is not sensible because of the presence of strategies that are not solely characterized by the highest accepted demand, see section IV.B). Table 6 presents the Spearman rank-correlation coefficients between the periods and the players' average behavior over time in each experiment. In all experiments, but D3, the coefficients between periods and players' A requirements are positive, that means they increase their requirements over time. Over all ten experiments, this is significant at the 2% level, using the binomial test (two-tailed). However, only four of these ten rank correlation coefficients are significant at the 1% or 5% level. As regards the behavior of players B, no coherent trend can be observed in the demand games. In the offer games there is a weak tendency to accept more offers over time.

Table 6. Spearman rank-correlation coefficients between round and indexes of requirement and rejections

	Players' A increase of requirement	Players' B decrease of rejections
O1	+.60	−.12
O2	+.67*	+.71*
O3	+.10	+.30
O4	+.05	+.59
O5	+.97**	+.23
D1	+.95**	+.87**
D2	+.20	+.91**
D3	−.19	−.31
D4	+.82*	−.19
D5	+.29	+.55

* significant on 5% level,
** significant on 1% level

VI Individual Learning Behavior

In this section, we want to propose a simple theory that tries to explain the *direction* of individual strategy changes from period to period. We have seen in the last section that a strict tendency over the periods towards the game-theoretic predicted behavior cannot be claimed significantly if we look at the mean behavior of independent player groups. Here, instead, we consider the effects of individual experience. Our proposed theory is in the tradition of Bush and Mosteller (1955) and Suppes and Atkinson (1960), which can be labelled "stochastic learning theories" or "Markovian learning theories". In the latter study, only 2×2 games in normal form were analyzed, whereas our games, taken as normal form games, are 135135×8192 games. So, in our case a learning theory cannot be derived on a level of specifying the probability of changing from one particular strategy to another, or on a level of specifying a probability distribution over all pure strategies, based on history. But our notions of requirement index and rejection index will be useful when trying to explain the *direction* of strategy changes.

First, the notions of "success" and "failure" must be defined in our context. A success for a player A in both types of games is interpreted as an agreement being achieved (that is player B has accepted player's A proposal). A failure for him always means that no agreement has been achieved (that is player B has rejected the offer or demand).

For player B, it is not so clear how to define success or failure. If she accepted the last-period proposal and received a positive payoff, we assume that her intention has been satisfied and thus she achieved a success. If she rejected a proposal, one might say that she succeeded in punishing a greedy player A. But this seems to be an artificial interpretation, and we will understand such an event as *failure by rejection*. In the demand game, but not in the offer game it may happen that player B receives nothing although he accepted the proposal and expected to receive a positive payoff. We will call this *failure by accepting*. So we will deal with two kinds of failures separately in demand games.

In order to describe the reaction on success or failure by the two player groups within the two games, we use the definitions of the indexes of requirements and rejections as defined at the end of section V.

We distinguish only between the direction of change of the respective indexes by each subject without being concerned with the magnitude of changes. We obtained 56 observations from each experiment (8 subjects had 7 opportunities to alter their strategies). The number of actual rejections in the previous periods was 67 out of 280 (5 experiments×8 players×7 rounds) over all offer games. Over all demand games, the number of rejections was 64 out of 280 observations.

Tables 7a, b present the relative frequencies of the direction of index changes by players A over all offer (demand) games, conditioned on the last period event, separately for all experiments. Figures 6a, b show the mean results.

The reactions by players A on success or failure are very similar between offer and demand games. After a success, only 9% (15%) of the time, players A decrease their requirements in offer games (demand games). They rather require more (37% in offer games, 34% in demand games) or keep their previous strategy (about 50% in both games). However, after a failure, about 50% of players A require less than in the previous period in both games and only 7% in offer games and 17% in demand games increase their requirements. Thus, in both games, when player A has experienced a disagreement in the previous period and if he wants to change his strategy, he is most likely to moderate his requirements, whereas, after a success, the reverse is more probable. A theory that predicts a steady increase of requirements towards the game theoretic solution would expect small conditional probability of decreases of requirements both after successes and failures in either type of game.

The data for the index changes of players B can be found in tables 8a, b and figures 7a, b. It can be seen that the reactions by players B differ in some aspects between the two types of games. In the offer games, increases in the number of

Table 7a. Relative frequencies of changes of requirement index by players A depending on the last round event and total number of observations of an event for each offer game, separately

	Player B accepted in the last round				Player B rejected in the last round			
exp.	increase requirement index	unchange	decrease	# of observ.	increase requirement index	unchange	decrease	# of obs.
O1	0.36	0.58	0.07	45	0.00	0.36	0.64	11
O2	0.33	0.58	0.10	40	0.19	0.31	0.50	16
O3	0.53	0.33	0.14	36	0.00	0.55	0.45	20
O4	0.32	0.57	0.11	44	0.17	0.33	0.50	12
O5	0.35	0.58	0.06	48	0.00	0.63	0.38	8
mean	0.37	0.54	0.09	42.6	0.07	0.43	0.49	13.4

Table 7b. Relative frequencies of changes of requirement index by players A depending on the last round event and total number of observations of an event for each demand game, separately

	Player B accepted in the last round				Player B rejected in the last round			
exp.	increase requirement index	unchange	decrease	# of observ.	increase requirement index	unchange	decrease	# of obs.
D1	0.40	0.23	0.23	48	0.38	0.25	0.38	8
D2	0.39	0.45	0.16	38	0.17	0.22	0.61	18
D3	0.31	0.55	0.14	42	0.21	0.43	0.36	14
D4	0.45	0.47	0.08	28	0.06	0.33	0.61	17
D5	0.20	0.68	0.12	50	0.17	0.33	0.50	6
mean	0.34	0.51	0.15	43.2	0.17	0.31	0.52	12.8

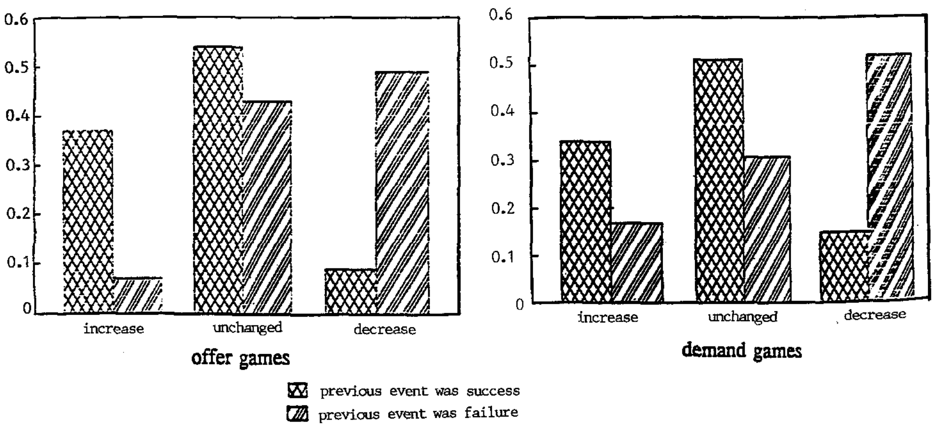

Fig. 6a, b. Relative frequencies of changes of requirements by players'A due to the last period event.

Table 8a. Relative frequencies of changes of rejection index by players B depending on the last round event and total number of observations of an event for each offer game, separately

exp.	Player B accepted in the last round: increase (rejection index)	unchange (rejection index)	decrease (rejection index)	# of observ.	Player B rejected in the last round: increase (rejection index)	unchange (rejection index)	decrease (rejection index)	# of obs.
O1	0.11	0.84	0.04	45	0.00	0.82	0.18	11
O2	0.15	0.75	0.10	40	0.00	0.31	0.69	16
O3	0.11	0.81	0.08	36	0.05	0.60	0.35	20
O4	0.09	0.80	0.11	44	0.08	0.50	0.42	12
O5	0.15	0.79	0.06	48	0.11	0.44	0.44	8
mean	0.12	0.80	0.08	42.6	0.04	0.53	0.43	13.4

Table 8b. Relative frequencies of changes of rejection index by players B depending on the last round event and total number of observations of an event for each demand game, separately

	Player B accepted in the last round and received positive payoffs				Player B rejected in the last round				Player B accepted in the last round and received zero payoffs			
Exp	incr (rejection index)	unch (rejection index)	decr (rejection index)	# of obs	inc (rejection index)	unch (rejection index)	decr (rejection index)	# of obs	incr (rejection index)	unch (rejection index)	decr (rejection index)	# of obs
D1	0.27	0.53	0.20	30	0.33	0.17	0.50	8	0.35	0.45	0.20	18
D2	0.16	0.81	0.03	31	0.11	0.50	0.39	18	0.14	0.71	0.14	7
D3	0.20	0.60	0.20	20	0.31	0.23	0.46	14	0.48	0.30	0.22	22
D4	0.21	0.59	0.21	29	0.07	0.71	0.21	17	0.23	0.54	0.23	10
D5	0.23	0.62	0.15	39	0.17	0.33	0.50	6	0.36	0.27	0.36	11
mean	0.21	0.63	0.15	29.8	0.18	0.44	0.39	12.6	0.35	0.42	0.23	13.6

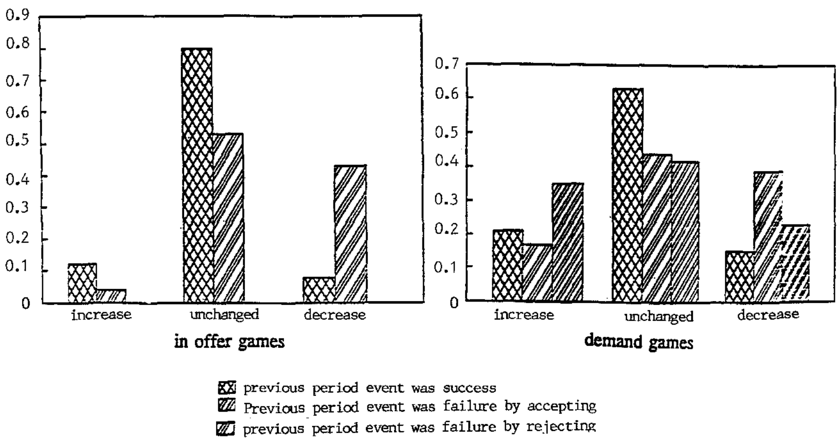

Fig. 7a, b. Relative frequencies of changes of rejections by players' B due to the last period event.

rejections are rare phenomena, independent of the previous event. In the demand games, player B is much more inclined to increase the rejection index, both after success and failure in the previous period. This is especially true after a failure by accepting. In both types of games, decreases in the number of rejections mainly happen when they failed to reach an agreement. Thus, as with players A, players B play less tough than in the previous period when they have experienced a failure by rejection. Here, a theory that predicts a decrease of the rejection index, i.e. a tendency towards equilibrium behavior, would expect small conditional probabilities of increases of the rejection indexes independent of the previous experience. This holds for the offer games, but not for the demand games after any kind of experience.

VII Summary

In this paper we investigated two versions of ultimatum games with one-sided incomplete information. Although game theory predicts similar outcomes for both versions, our experiments reveal significant differences in behavior:

- In offer games players' A mean required cake proportions increase with cake size, but in demand games the reverse holds.
- In offer games we observe many fifty-fifty choices (mainly at the small cakes), but in demand games such choices are less frequently made.
- In offer games we observe almost no requests of the entire cake, but in demand games one third of the choices are of this kind (mainly at small cakes).
- In offer games half of the observed strategies of players B are in accordance with game theoretic predictions, but in demand games this proportion is only one fifth.
- In offer games the expected acceptance rate increases with cake size, but in demand games it decreases.

We proposed a theory of boundedly rational behavior, called the anticipation philosophy, that tries to explain these differences. An anticipation strategy of player A is based on an own *aspiration criterion* and on an *anticipated level of acceptance* of player B. Since the informational situation of player B is quite different in the two games, the acceptance levels have to be constructed in two different ways. In the offer game we suggest the concept of *expectation fairness,* i.e. that player B will accept only offers of at least half of the expected cake. In the demand game the concept of *resistance to visible unfairness* is introduced, where each demand larger than half of the maximal cake is rejected. Player A intends to respond optimally to the anticipated level of acceptance, but he adjusts choices which do not fulfill his aspiration criterion, e.g. to obtain at least as much as player B.

We observe that anticipation strategies, especially those which are based on expectation fairness and resistance to visible unfairness, are widely used in all eight periods, and are nearly optimal replies to the population of actual strategies of players B.

The strategy method made it possible to analyze the planned behavior extensively. The strategy method is sometimes criticized to build up an artificial decision situation and lead to results deviating from spontaneous behavior. (For a discussion of the methodological aspects see Roth (1992)). The results of Rapoport, Sundali and Potter (1992), who ran experiments of the offer game in the usual sequential way, suggest that for the case at hand no significant differences in behavior are caused by the application of the strategy method.

Appendix

Offer Games

(The instructions for the demand game are similar.)

Instructions for Player A

You participate in a simple decision experiment. By chance you are chosen to be player A during the whole experiment. In this room you are together only with players A. Another group with players B is sitting in a different room.

The experiment consists of 8 rounds. In each round you play together with a player B, who will be different from round to round and not identifiable to you. The rules of the game are the same for all 8 rounds. Player B knows what kind of information you get.

The game is as follows:

In each round a certain total amount is to be divided between player A and B. But only you as player A are informed about this amount.

Player B knows only that the amount can either be 1, 2, 3, 4, 5, 6 with equal probability. The actual amount is determined by the throw of a die in each round in this room not visible to players B. The thrown number is the total amount in taler which has to be divided. 1 taler is equivalent to 1.20 DM.

It is your task as player A to offer to player B an amount which he should get. Player B has to decide whether to accept (a) or to reject (r) the offer. If he accepts he receives the offered amount. You as player A get the residual of the total amount. Whereas in case of rejection both of you do not get anything. In both cases player B is never informed about the actual total amount.

An example:

The die has been 4, so that 4 taler must be divided in that round. Player A offers 1.5 to player B. Player B, who does not know that 4 taler is the total sum, accepts the offer of 1.5 taler and thus receives it in that round. Player A gets the resting $4-1.5=2.5$ taler. Yet if player B had rejected the offer both players would have gained zero.

For your offer you have to follow certain rules:

- the offer may not be negative
- the offer may not be greater than the total amount

- the offer must be a whole number or with "point five"; therefore 0; 0.5; 1; 1.5; 2; 2.5; ... until the total amount are possible. I.e. for the total amount of 2 the following offers can be made: 0; 0.5; 1; 1.5; or 2.

Player B can therefore receive one of the following 13 offers: 0; 1.5; 1; 1.5; ...; 5.5; 6.

But in our experiment in each round you have to make your decision before the die has determined the actual amount. That means that you have to decide *in advance* on an offer to player B for each possible total amount of 1 to 6 taler. In each round you fill in your offers for the 6 total amounts on your decision sheet. The sheets are collected after each round. Afterwards one of you will throw a die visible for all players A.

At the same time players B have to make their decisions filling in their decision sheets, that means they have to decide which of the possible offers they will accept or reject, before they receive the actual offer.

At the end of each period every decision sheet of player A is matched with one of player B by chance. The offer of the actual total amount made by player A will be written on the matched decision sheet of player B. Player A is told player B's reaction to this offer and the gains to each player is written in the decision sheet respectively. Afterwards your decision sheet is returned to you and the next round starts.

Between the rounds you should shortly explain your decisions on a separate explanation sheet (aims, motives etc.).

After 8 rounds your taler gains of each round are added, revalued into DM and paid to you. (1 taler = 1.20 DM).

For the first two rounds you will have 10 minutes to fill in your decision sheet, for rounds 3–8 about 5 min. each.

During the experiments you must not talk. If you have any questions you should ask them now.

Have fun!

Offer Games

Instructions for Player B

You participate in a simple decision experiment. By chance you are chosen to be a player B during the whole experiment. In this room you are together only with players B. Another group of players A is sitting in a different room.

The experiment consists of 8 rounds of a game. In each round you play together with a player A, who will change from round to round and is not identifiable to you. The rules of the game are the same for all 8 rounds.

The game is as follows:

In each round a certain total amount is to be divided between players A and B. Only players A are informed about this amount. The actual amount in each round is determined by the throw of a die in the room of players A and thus not visible to you as player B. The thrown number is the total amount in "taler" which has to be divided between both players. The actual amount can either be 1, 2, 3, 4, 5, or 6 with equal probability. Players A know that you are never informed about the actual amount and that you know the random rule.

For both players 1 taler is equivalent to 1.20 DM.

It is the task of player A to offer to you an amount which you will get if you accept. You have to decide whether to accept (a) or to reject (r) the offer. If you reject both of you get nothing. If you accept you receive the offered amount and he gets the residual of the total amount.

Whether you reject or accept you are not informed about the actual total amount thrown.

An example:

The die came up 4, so that 4 taler must be divided in this round. Player A offers 1.5 taler to player B. Player B, who does not know that 4 taler is the total sum, accepts the offer of 1.5 taler and thus receives that amount in this round. Player A gets the remaining $4-1.5=2.5$ taler. If player B had rejected the offer, both players would have gained zero.

Player A has to follow certain rules for his offer:

- the offer may not be negative
- the offer may not be greater than the total amount
- the offer must be a whole number or with "point five"; therefore possible offers are 0; 0.5; 1; 1.5; 2; 2.5; ... up to the amount thrown. I.e. for the total amount thrown 2 the following offers can be made: 0; 0.5; 1; 1.5; or 2.

You as player B can therefore receive one of the following 13 offers: 0; 1.5; 1; 1.5; ...; 5.5; 6.

In our experiment in each round you have to make your decision before you receive the actual offer from player A. This means you have to decide *in advance* which of the 13 possible offers you will accept (a) or reject (r). In each round you fill in your decisions for all 13 possible offers on your decision sheet. The sheets are collected after each round.

At the same time players A have to make their decisions filling in their decision sheets. This means they have to decide in advance on an offer to player B for each possible total amount of 1 to 6 taler *before* the die is thrown.

At the end of each period every decision sheet of a player A is matched randomly with one of a player B. Player A's offer of the actual total amount will be written on the matched decision sheet of player B. Player B's reaction to that offer will be written on the sheet of player A and the gains to each player is written in his decision sheet. Then the decision sheet of each person is returned to him and the next round will start.

Between the rounds we will ask you to briefly explain your decisions on a separate explanation sheet (aims, motives etc.).

After 8 rounds your taler gains of each round are added, revalued into DM and paid to you. (1 taler = 1.20 DM).

For each of the first two rounds will have 10 minute to fill in your decision sheet, for rounds 3–8 about 5 min. each.

During the experiments you must not talk. If you have any questions you should ask them now.

Have fun!

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Equilibrium Selection and Simple Signaling Games

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Equilibrium Selection and Simple Signaling Games

Michael Mitzkewitz

Abstract

This paper calculates the Harsanyi-Selten solutions for a class of simple signaling games. This means that for each generic game belonging to this class one of its equilibrium points is selected according to the principles developed by John C. Harsanyi and Reinhard Selten (Harsanyi & Selten, A General Theory of Equilibrium Selection in Games, 1988). For almost fifty years signaling games have been of great interest for both normative game theorists and scientists interested in the analysis of social, cultural and biological phenomena. The paper provides an introduction into the Harsanyi-Selten theory, solves all generic games and subsumes the results. Thus comparisons to Nash refinement concepts can easily be done and the solution of more complex games is facilitated.

JEL Classification Number: C 72

Keywords: Noncooperative Game Theory, Signaling Games, Equilibrium Selection, Harsanyi-Selten theory, Risk Dominance, Tracing Procedure

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Contents

1. Introduction

A *signaling game* is a game where one of the players (called the "sender") can be of different types. The actual type is chosen by random and is informed about his identity. The type can choose some action (called the "signal") observable for an other player (called the "receiver"). The receiver does not know the actual type but the probability distribution of the possible types (or, as a Bayesian, he forms prior beliefs about the probabilty distribution). The receiver can use the observed signal to update his beliefs about the actual type. Hence the actual type can choose the signal strategically to influence the receiver's updated beliefs about his identity. It is easy to imagine situations where a type has a strong incentive that his true identity becomes public and other situations where the type is interested to feign an honourable character.

Signaling games have been of great attractiveness in the last decades for both economists and game theorists, and the interest seems to increase unbrokenly. Starting from the pioneering works of Akerlof (1970) and Spence (1973, 1974) economists have realized that many situations of substantial economic significance are characterized by incomplete information where privately informed agents can strategically choose actions to affect the beliefs of uninformed agents about the true state of the world. A series of papers of Harsanyi (1967-1968) provided the framework to analyze situations of incomplete information with the appropriate game-theoretical tools. Harsanyi demonstrated that a game of incomplete information can be sensibly transformed into a game of imperfect information. This was a breakthrough because before no satisfying solution concept existed for games of incomplete information. Selten (1965, 1975) refined the concept of Nash equlibrium point (Nash (1950, 1951)) by eliminating incredible threats and he proposed the concept of subgame perfect equilibrium point and (especially for games of imperfect information) the concept of perfect equilibrium point. So almost at the same time the insight into the necessity to analyze models of incomplete information and the possibility to do this in an appropriate way appeared.

In the subsequent years a large number of articles and books has been published that apply signaling games in different economic arenas. Michael Spence, one of the pioneers, dedicated his 2001 Nobel prize lecture to "Signaling in Retrospect and the Informational Structure of Markets" (Spence (2002)). From the vast literature let me list only a small sample of economic or related fields to which signaling games of the described structure or similar structures have been applied and some of the corresponding arcticles:

- Labor market (Spence (1973, 1974, 1976), Nöldeke & van Damme (1990), Austen-Smith & Fryer (2005), Delfgaauw & Dur (2007))
- Market entry (Milgrom & Roberts (1982a, 1982b), De Bijl (1997))
- Competition in product quality (Gal-Or (1989), Bagwell & Riordan (1991))
- Advertising (Milgrom & Roberts (1986), Bagwell (2001), Anand & Shachar (2009))
- Insurances (Wilson (1977), Puelz & Snow (1996), Aarbu (2017))
- Finance ((Ross (1977), Allen & Morris (2001), Levine & Hughes (2005))
- Economics of Law (Reinganum & Wilde (1986), Schweizer (1989), Friedman & Wittman (2007), Dari-Mattiacci & Saraceno (2017))
- Money Laundering (Takáts (2011))
- Bargaining (Rubinstein (1985a, 1985b), Admati & Perry (1987), Feinberg & Skrzypacz (2005))
- Political Science (Banks (1991), Potters, van Winden & Mitzkewitz (1991), Prat (2002), Gavious & Mizrahi (2003)).

Of course the articles mentioned above are usually not solely based on "pure" signaling games as described before but on games with more sophisticated signaling structures or on games where simple signaling games are embedded.

Let me mention that (besides the fact that game theory as a whole has an unexpected predictive power in evolutionary biology) signaling games provide also a useful framework to study animal behavior. Impressive examples are presented e. g. by Grafen (1990), Godfray (1991) and Getty (2006). The philosophical theories of the evolution of conventions (Lewis (1969)) and of the emergence of language (Zollman (2005), Huttegger (2007), Skyrms (2010)) benefit also from the analysis of signaling games.

Signaling games are, however, also under special observation of *pure* game theorists not mainly driven by interests in economic or whatever applications. The point is that simple numerical examples for some signaling games reveal the weakness of certain equilibrium concepts, especially of the *sequential equilibrium* (Kreps & Wilson (1982)). This means that a nontrivial signaling game can have (or usually has) sequential equilibrium points labelled "unreasonable", "nonsensible" or "counterintuitive" by some straightforward criteria. This gave rise to doubts on the claim stated above that the appropriate tool to analyze games of imperfect (and, à la Harsanyi, incomplete) information is not really given by perfect equilibrium or its non-uniovalur twin sequential equilibrium.

After realizing this in the 1980ies a series of papers was published which demonstrated the weakness of existing equilibrium concepts and tried to overcome this weakness by "refining" these concepts. I will call this the "refinement programme". Refinements are usually made in notions of the sequential equilibrium concept and are concerned with restrictions on the beliefs a player can sensibly form at information sets *off* the equilibrium path. The aim of the refinement program is to reduce the multiplicity of sequential equilibria by putting more and more requirements to the players' "rational" choices.

Contributions to the refinement program are for example Banks & Sobel (1987), Cho (1987), Cho & Kreps (1987), Cho & Sobel (1990) and Okunu-Fujiwara, Postlewaite, & Suzumura (1990). The most important contribution to the refinement program was the introduction of *stable equilibria* by Kohlberg & Mertens (1986). Stable equilibria are based on *forward induction.* This means that a player's *past behavior indicates his future behavior* (which is something different from that his *past behavior indicates his identity).* Many of the papers mentioned above are concerned among other things with the question of how the set of stable equilibria can be characterized for signaling games. Surveys on the different refinement concepts and their implications for signaling games are presented by van Damme (1987) and Kreps & Sobel (1994).

Completely different to the refinement program John C. Harsanyi and Reinhard Selten claimed that in any case the *rational solution* for a game must be a *unique* equilibrium point and that this solution cannot be derived by putting more and more restrictions on the equilibrium concept. Instead, given a particuliar equilibrium concept, one and only one equilibrium points out of the set of all equilibrium points of this kind. Hence, the problem of normative game theory is not to create sophisticated refinement procedures but to develop reasonable selection criteria. This should be done from the point of view of an "expert" outside the game who is asked by the players (or by some of the players and, maybe, independent of each other) for a *rational strategic recommendation*. A professional game theorist must be an expert for "*how to play a game*", and, of course, he has to recommend each of his clients *an equilibrium strategy* and, if he tries to live on his new job, he has to recommend *strategies belonging to the same equilibrium point.* Therefore, *a game theorist should have a theory which equilibrium point is the solution of a given game*. Of course, such a theory has to reflect carefully all the strategic relationships and opportunities the game includes. Harsanyi & Selten (1988) present a theory that selects a unique equilibrium point for each finite game as its solution. To quote from Robert J. Aumann's foreword of the Harsanyi-Selten book: "The major implication, like that of the first heavier-than-air flying machine, is that it can be done."

In this study we will calculate the Harsanyi-Selten solution for a class of simple signaling games. This class is characterized as follows: There are just two types of the *sender* possible, each of these two types has just two different choices, and only after one of these two choices, called the "*inside*" choice, the *receiver* comes into play, not knowing, which type has sent the signal. After being alarmed the receiver has two different responses which both terminate the game. If the active type chooses his "*outside*" choice the game ends immediately. The game tree for this class of signaling games is later shown in section 3. Probably this is the simplest class of games which can capture the essence of signaling.

In the following we calculate the Harsanyi-Selten solution for all *generic* games of the class described above. What "generic" means in our context is explained at the end of section 4. The author, however, also find the solution for the *nongeneric games* but to write down all the calculations will exceed the limits of this study. The results are available on request.

As the reader will see, even for the generic games it takes much effort (not only for the author) to go through all the case distinctions which appear to be necessary. The reader may ask whether the aim of this study is not too modest to justify such a fatiguing exertion. I give four answers to this question.

- First, despite its frugal game-theoretical structure the class of signaling games we will consider can be applied to different elementary situations of economic relevance. Having computed the solutions for the whole class, the solutions for games of special interest are easily available in our overview of results.
- Secondly, more complex and interesting economic and other models may have games belonging to our class as subgames. The Harsanyi-Selten theory has the property that the solution of a game prescribes for all agents in a subgame the same local strategies as if the subgame is solved as a game by itself. This *subgame-consistency property* makes it valuable to have complete overviews of the solutions of simple games in order to facilitate to solve more complex games where the simple games arise as subgames.
- Thirdly, it would be interesting to compare the results of the Harsanyi-Selten theory with the results of certain refinement concepts in the latters' domain, the signaling games. Unfortunately, an overwiew how the sets of, e.g., stable equilibria for the whole class of signaling games considered here is not available. It is obvious that for a large part of the parameter space the refinement concepts fail to contract successfully the set of equilibrium points contrary to the ingenious numerical examples

presented in the literature. For a special model such a comparison is made in Potters, van Winden & Mitzkewitz (1991).

- Finally, a lot of the concepts introduced by Harsanyi and Selten are involved in solving our class of signaling games. The interested reader can observe the concepts “at work”. So this study can also been taken as a learning-by-doing introduction to the Harsanyi-Selten theory.

This paper is organized as follows. After this introdution section 2 presents a brief digest of the Harsanyi-Selten theory. Section 3 defines the class of games we will consider and presents the solution for special members of this class, called the “decomposable and reducible games”. In section 4 we normalize the “indecomposable and irreducible games” and in section 5 we compute for generic cases, i. e. “for almost all” games, their solutions. Section 6 presents an overview of the results and section 7 summarizes.

2. Relevant Elements of the General Theory of Equilibrium Selection

The theory of equilibrium selection developed by John C. Harsanyi and Reinhard Selten (Harsanyi & Selten (1988)) singles out a unique equilibrium point for each finite noncooperative game as its solution. In this section we will sketch the Harsanyi-Selten theory only briefly. Some important ingredients of this theory which will not be involved in the course of our analysis, like "*strategic distance*", are not mentioned here. Other components are explained only to such a degree of complexity which is sufficient to understand the procedures in the following sections. We omit detailed discussions and justifications of the concepts and refer the interested reader to the book of Harsanyi and Selten. Given these limitations, this section could be considered as a small user's guide for the Harsanyi-Selten theory.

In the class of games we will analyze each player has just one information set, so there is no distinction between a player and his single agent. Because of this *normal-form structure* we can omit the explanation of the *"standard form"* of a game which distinguishes thoroughly between players and their agents.

2.1. Some Notations and Definitions

NORMAL FORM. A n-player game in *normal form* $G = (\Phi_1, \dots, \Phi_n; H)$ consists of n nonempty finite sets $\Phi_1, \dots, \Phi_n$ and a *payoff function* H. The set of *pure strategies* of player i $(i = 1, \dots, n)$ is represented by Φ_i. A *pure strategy combination* is denoted by φ:

$$\boldsymbol{\varphi} = (\boldsymbol{\varphi}_1, \dots, \boldsymbol{\varphi}_n) \text{ } \boldsymbol{with} \text{ } \boldsymbol{\varphi}_i \in \boldsymbol{\Phi}_i \qquad (1)$$

The payoff function $\boldsymbol{H}$ assigns a payoff vector $\boldsymbol{H}(\boldsymbol{\varphi})$ to each $\boldsymbol{\varphi}$:

$$\boldsymbol{H}(\boldsymbol{\varphi}) = (\boldsymbol{H}_1(\boldsymbol{\varphi}), \dots, \boldsymbol{H}_n(\boldsymbol{\varphi})) \qquad (2)$$

MIXED STRATEGIES. A *mixed strategy* of player i is a probability distribution over Φ_i and is denoted by q_i. The notation $q_i(\varphi_i)$ represents the probability that player i will choose his pure strategy φ_i. Given a *mixed strategy combination* $q = (q_1, \dots, q_n)$, a particular pure strategy combination $\varphi = (\varphi_1, \dots, \varphi_n)$ occurs with the following probability:

$$\boldsymbol{q}(\boldsymbol{\varphi}) = \boldsymbol{q}_1(\boldsymbol{\varphi}_1) \cdot \ldots \cdot \boldsymbol{q}_n(\boldsymbol{\varphi}_n) \tag{3}$$

Thus the payoff function H can be extended to mixed strategy combinations in the following way:

$$\boldsymbol{H}(\boldsymbol{q}) = \sum_{\boldsymbol{\varphi} \in \boldsymbol{\Phi}} \boldsymbol{q}(\boldsymbol{\varphi}) \boldsymbol{H}(\boldsymbol{\varphi}) \tag{4}$$

Here Φ represents the set of all pure strategy combinations.

In the class of games we will consider each player has just two pure strategies. For this reason we can represent a mixed strategy of player i by a single number q_i, which means the probability to choose the player's *first* pure strategy (it will always be clear what is meant by "*first*"). Hence $1 - q_c$ is the probability to choose his second pure strategy. Pure strategy choices can be represented by $q_i = 1$ and $q_i = 0$. Therefore we can describe any strategy combination (pure or mixed) by a n-tuple of the following kind:

$$\boldsymbol{q} = (\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n) \;\; \boldsymbol{with} \;\; \mathbf{0} \leq \boldsymbol{q}_i \leq \mathbf{1} \quad \boldsymbol{for} \;\; \boldsymbol{i} = \mathbf{1}, \ldots, \boldsymbol{n} \tag{5}$$

***i*-INCOMPLETE MIXED STRATEGY COMBINATIONS.** An *i-incomplete mixed strategy combination*q_{-i} is a $(n-1)$-tuple of mixed strategies:

$$\boldsymbol{q}_{-i} = (\boldsymbol{q}_1, \ldots, \boldsymbol{q}_{i-1}, \boldsymbol{q}_{i+1}, \ldots, \boldsymbol{q}_n) \tag{6}$$

Using this notation a mixed strategy combination q can also be written as follows:

$$\boldsymbol{q} = (\boldsymbol{q}_i \boldsymbol{q}_{-i}) \tag{7}$$

This means that q contains player i's mixed strategy q_i and the other players' mixed strategies in q_{-i} as its components.

BEST REPLIES. A mixed strategy r_i is called a *best reply* to the i-incomplete strategy combination q_{-i} if:

$$\boldsymbol{H}_i(\boldsymbol{r}_i \boldsymbol{q}_{-i}) = \max_{\boldsymbol{q}_i} \boldsymbol{H}_i(\boldsymbol{q}_i \boldsymbol{q}_{-i}) \tag{8}$$

We say that r_i is a *strong best reply* to q_{-i} if all other strategies yield a lower payoff than r_i. Of course a strong best reply must be a pure strategy.

EQUILIBRIUM POINTS. A mixed strategy combination $q = (q_1, \dots, q_n)$ is called an *equilibrium point* of game G if for each player i ($i = 1, \dots, n$) his mixed strategy q_i is a best reply to q_{-i}. If all q_i are strong best replies to q_{-i}, than q is called a *strong equilibrium point*. Notice that we use the term "strong equilibrium point" different from Aumann (1959).

UNIFORM PERTURBATIONS. The Harsanyi-Selten theory is not applied directly to the game G under consideration but to *uniform perturbations* of this game, denoted by G_ε. Each pure strategy of a player must be chosen with a minimal probability ε, where ε is supposed to be close to zero but positive. ε can be interpreted as the probabiltity to choose the "wrong" pure strategy by error due to "trembling hands". The term "uniform" refers to the fact that the perturbation parameter ε is the same for all players and for all pure strategies. This differs from Selten's general definition of perfectness (Selten (1975)).

In the class of games we will consider each player has just two pure strategies. So we can describe each mixed strategy combination which is admissible in the uniformly perturbed game as follows:

$$\boldsymbol{q_\varepsilon = \left(q_{1_\varepsilon}, \dots, q_{n_\varepsilon}\right) \; with \quad \varepsilon \leq q_{i_\varepsilon} \leq 1 - \varepsilon \quad for\ i = 1, \dots, n} \qquad \textbf{(9)}$$

Of course $\varepsilon < 0.5$ is supposed. If player i chooses $q_{i_\varepsilon} = 1 - \varepsilon$ or $q_{i_\varepsilon} = \varepsilon$, we say that he plays an *ε-extreme strategy.* He "tries" to play one of his pure strategies and the other pure strategy can only appear by mistake. We will indicate by φ_ε the *ε-extreme strategy combination* which corresponds to the pure strategy combination φ.

UNIFORMLY PERFECT EQUILIBRIUM POINTS. The *limit equilibrium points* of the uniformly games G_ε for $\varepsilon \to 0$ are called the *uniformly perfect equilibrium points* of the unperturbed game G.

The Harsanyi-Selten theory requires that the solution of a game must be one of its uniformly perfect equilibrium points. But Harsanyi and Selten do not select directly among these equilibrium points (if there are more than one). They first *solve* (i.e., they single out a unique equilibrium of) the perturbations of the game and then, by letting $\varepsilon \to 0$, they obtain the *limit solution* of the game.

Hence it must be kept in mind that in the following descriptions of *how to solve a game* we deal with (uniformly) perturbed games.

2.2. Decomposition and Reduction

The first step in solving a game is to check whether this game is *decomposable.* To understand what this means, we need some further definitions.

CELLS. A proper subset of players *forms a cell* if for each of these players the strategic situation only depends on the other members of the cell and is completely independent of the strategic choices of the players outside the subset. In other words, this subset is closed with respect to the best-reply correspondence. A cell is called *elementary* if it contains no proper subset of players which forms a cell by itself.

DECOMPOSABLE GAMES. A game is called *decomposable* if it has at least one cell. Otherwise it is called *indecomposable.* Obviously an elementary cell is indecomposable.

FIXING A PLAYER. We say that *a player is fixed at a particular strategy* if after this fixing a game is considered which results from the substitution of this player's strategy set by this particular strategy and from modification of the payoff function in the appropriate way. We emphasize that with such a strategy fixing always a *new game* results from a more complex one.

INFERIOR CHOICES. A pure strategy φ_i of player i is called *inferior* if he has a pure strategy ψ_i which is always a best reply whenever φ_i is a best reply, but also in some cases where φ_i is not a best reply. Since in our class of games each player just has two pure strategies, the term "inferior" is here equivalent to "weakly dominated". This is obviously not true for more than two pure strategies. Notice that the original Harsanyi-Selten definition of inferiority is concerned with choices of an agent and not with pure strategies of a player. We do not need such a distinction here because in our games each player has only one information set (and, therefore, no agents). But in order to match Harsanyi's and Selten's terminology, we will speak of *inferior choices* instead of "weakly dominated strategies".

ELIMINATION OF INFERIOR CHOICES. If a player has an inferior choice, this choice is eliminated from his strategy set. But notice that this elimination takes place within the perturbed game. In the class of games we will consider the elimination of an inferior choice means nothing else but fixing the respective player at his ε-extreme strategy concentrated on his superior pure strategy. The inferior choice is still chosen "erroneously" with probability ε.

SEMIDUPLICATE CLASSES. If some pure strategies of a player yield always the same payoff *to him* independent of the strategies chosen by the other players, we say that these pure strategies are *semiduplicates* or that they *form a semiduplicate class.*

CENTROID STRATEGY. The mixed strategy of a player which assigns the same probability to each of his pure strategies is called his *centroid strategy.* Hence, $q_i = {}^1/_2$ is player i's centroid strategy if he has two pure strategies. This is not the exact definition proposed by Harsanyi and Selten, but sufficient for our purposes and more convenient.

ELIMINATION OF SEMIDUPLICATE CLASSES. If the pure strategies of a player form a semiduplicate class, this class is eliminated by fixing this player at his centroid strategy.

IRREDUCIBLE GAMES. A game is called *irreducible* if it is indecomposable and has neither inferior choices nor semiduplicate classes. Otherwise the game is called *reducible*.

DECOMPOSITION AND REDUCTION. The *procedure of decomposition and reduction* tries to facilitate the task of solving games to the simpler task of solving irreducible games. How to solve an irreducible game is explained in the following subsections. The precise procedure of decomposition and reduction is best explained by the flowchart on page 127 in Harsanyi & Selten (1988) or by the flowchart in Güth & Kalkofen (1989) on page 39. For our purposes a much more superficial description is sufficient. It will turn out that games of our class are only decomposable if they contain inferior choices and/or semiduplicate classes. Within our framework we can describe the procedure of decomposition and reduction by the following steps:

- **STEP 1:** If the game is irreducible, carry on with STEP 4. Otherwise carry on with STEP 2.
- **STEP 2:** If the game contains inferior choices, eliminate them and carry on with STEP 1. Otherwise carry on with STEP 3.
- **STEP 3:** Eliminate the semiduplicate classes and carry on with STEP 1.
- **STEP 4:** Compute the solution of the irreducible game (see the following subsections).

Here the term "game" always means the original perturbed game *after* previous elimination steps. So each game will be reduced to an irreducible game after finitely many steps.

2.3. The Linear Tracing Procedure

LINEAR TRACING PROCEDURE. An important component of the Harsanyi-Selten theory is the so-called *linear tracing procedure,* introduced by Harsanyi (1975). The linear tracing procedure is an attempt to extend principles of Bayesian rationality from one-person decision problems to n-person noncooperative games. It is assumed that players form *prior beliefs* about the other players' strategic intentions, maximize their expected payoffs on the base of these beliefs, modify continuously the prior beliefs by "observing" more and more of the other players' maximizing behavior, and change in case of need their own actions on the base of these modified beliefs. Formally, player i's payoff function H_i of a given game G is transformed to:

$$\boldsymbol{H_i^t(q_i q_{-i}) = tH_i(q_i q_{-i}) + (1-t)H_i(q_i p_{-i})} \qquad \textbf{(10)}$$

Here t is the so-called *tracing parameter* with $0 \leq t \leq 1$. The tracing parameter can be loosely interpreted as "time", so $t = 0$ marks the beginning and $t = 1$ marks the end of the process generated by the linear tracing procedure. The prior beliefs (or simply the *priors*) of player i about the other players' strategic intentions are expressed by the i-incomplete mixed strategy combination p_{-i} , whereas q_i and q_{-i} are player i's and the other players' *actual mixed strategies at time t.* Each player is assumed to choose q_i at time t in order to maximize H_i^t. To put it differently, player i plays at time t a best reply to the following i-incomplete mixed strategy (see also Harsanyi & Selten (1988), p. 142n):

$$\boldsymbol{tq_{-i} + (1-t)p_{-i}} \qquad \textbf{(11)}$$

Hence, at time $t = 0$ player i plays a best reply to his priors independent of the other players' actual strategies, which are in fact their best replies to *their* priors. When t increases player i lays less stress on his priors and lays more stress on the "observed" actual strategies of the other players. At time $t = 1$ the influence of the priors completely vanished and, since all players choose best replies to the other players' *actual* strategies, an equilibrium point of the original game G is reached.

PATH AND RESULT OF THE LINEAR TRACING PROCEDURE. We will say that the set of pairs (q, t) for all $0 \leq t \leq 1$ describes the *path of the linear tracing procedure.* In some cases, however, the path of the linear tracing procedure is not well-defined. We will discuss this problem at the end of this subsection. For the moment let us assume that no difficulties of this kind arise. Then it is clear that the path of the linear tracing procedure ends in an equilibrium point of the considered game. This equilibrium point is called the *result of the linear tracing procedure.*

If the strategy combination given by the best replies to the priors (i. e. q at $t = 0$) forms an equilibrium point of the considered game, it is obvious that this strategy combination will be played along the whole path of the linear tracing procedure up to the end (because in this case for each player i his strategy is a best reply to *both* q_{-i} and p_{-i} and, therefore, also a best reply to each convex combination of these two i-incomplete strategy combinations).

DESTABILIZATION POINTS. If the vector of best replies to the priors does not form an equilibrium point of the considered game, it is clear that at least one player must alter his strategy along the path of the linear tracing procedure at some time t. This value for t is called this player's *destabilization point*.

STRATEGY SHIFT. At a player's destabilization point a player *shifts his strategy.* Maybe after the *strategy sh*ift an equilibrium point is reached and then this strategy combination is followed in the further path of the linear tracing procedure. But it is also possible that a series of strategy shifts is necessary to reach an equilibrium point at the end. Notice that with each strategy shift of a player i at time t payoff shifts for all players are usually connected (q_{-j} changes in the modified payoff functions given by **(10)** for all players j with $j \neq i$). So it is important to calculate who is the *first player to shift his strategy.* It is the player with the smallest value for t at his destabilization point. After his strategy shift the new destabilization points are calculated (if there are some) and the *next player to shift his strategy* is determined, etc. Let us mention that even if the path of the linear tracing procedure is well defined, it can have so-called *backward-moving variable segments* (see section 4.19 in Harsanyi & Selten (1988)). In our analysis this phenomenon will not arise and so we omit any discussion of this issue.

The tracing procedure is involved in three ways in Harsanyi's and Selten's solution concept. The three jobs of the linear tracing procedure are:

- *Risk-dominance comparison between two equilibrium points*
- *Forming a substitute of a candidate set*
- *Computation of the solution of a basic game.*

In the next subsection we will explain at which steps of the solution procedure these three jobs come into play. Here we describe the implications for the construction of the priors.

RISK-DOMINANCE COMPARISON. Consider the situation that all players are convinced that the solution of a game is one out of two equilibrium points, say q^1 and q^2, with the property that $q_i^1 \neq q_i^2$ holds for each player *i*. It will turn out that in our analysis only comparisons between *strong* equilibrium points are necessary, so the following explanations are restricted to such a situation. Each player *i* is assumed to be initially doubtful about the "correct" equilibrium point, but he believes that all the other players *know* the correct one and consequently they will *play jointly either* q_{-i}^1 *or* q_{-i}^2. According to Bayesian rationality player *i* must form a *subjective* probability, say z_i, for the event that the other players choose q_{-i}^1 and a *subjective* probability $1 - z_i$ for the event q_{-i}^2. Therefore player *i* is assumed to play initially ($t = 0$) a best reply to the following *i*-incomplete *joint mixture*:

$$\boldsymbol{z_i q_{-i}^1 + (1 - z_i) q_{-i}^2} \qquad \textbf{(12)}$$

Since q^1 and q^2 are strong equilibrium points, there must exist for each player *i* a particular value $\hat{z}_i$ with $0 < \hat{z}_i < 1$, such that q_i^1 is for all $z_i \in (\hat{z}_i, 1]$ a strong best reply to the joint mixture given in (12), but for all $z_i \in [0, \hat{z}_i)$ q_i^2 is a strong best reply.

But how does player *i* form his subjective probability z_i about the "correct" equilibrium point? Or, to put it more precisely, what should the *other* players think about the way player *i* forms z_i? As Bayesians the other players have to construct a distribution function of z_i over the interval [0,1]. Because the initial state must be considered as a situation of complete naivety, there is no reason whatsoever to put more weight on a specific value of z_i than on another one. Hence, Harsanyi and Selten assume that z_i is uniformly distributed over the interval [0,1].

This has the consequence that player *i* is assumed to choose initially (at $t = 0$) q_i^1 with probability $1 - \hat{z}_i$ and q_i^2 with probability $\hat{z}_i$, where as explained above $\hat{z}_i$ is that particular value of z_i of player *i* that makes him indifferent between q_i^1 and q_i^2. So the *prior beliefs about player i* are that he plays the following mixed strategy:

$$\boldsymbol{p_i = (1 - \hat{z}_i) q_i^1 + \hat{z}_i q_i^2} \qquad \textbf{(13)}$$

These priors are also called the *bicentric priors* because just two equilibrium points are compared.

Given these priors for all players, the path of the linear tracing procedure can be computed. If the result of the linear tracing procedure is q^1, than we will say that q^1 *risk-dominates* q^2. If q^2 is the result, q^2 risk-dominates q^1.

SUBSTITUTION OF A CANDIDATE SET. Sometimes in the calculation of the Harsanyi-Selten solution for a given game the linear tracing procedure is used to *substitute* a set of equilibrium points by a single equilibrium point. In such a case the priors about player *i* are formed by the equally weighted average of his mixed strategies used in the equilibrium points of the set that should be substituted. In our case it will turn out that only sets of two pure equilibrium points must be substituted, so the priors are simply given by the players' *centroid strategies* (see subsection 2.2). With the term "*substitution of a candidate set"* used in the next subsection we mean the following: If we replace a set of equilibrium points (the *candidate set*) by that equilibrium point which is the result of the linear tracing procedure using the players' centroid strategies as their priors, then we say that this set of equilibrium points is *substituted*.

SOLUTION OF A BASIC GAME. In the next subsection we introduce the concept of a *basic game*. Here we want to state that the *solution of a basic game* is the result of the linear tracing procedure using the players' centroid strategies in that basic game as their priors.

EXISTENCE OF A WELL-DEFINED PATH OF THE LINEAR TRACING PROCEDURE. Hitherto, we have excluded any discussion about the uniqueness of the path (and, therefore, the result) of the linear tracing procedure. Unfortunately, such a *well-defined path* exists only for "almost all" games. For example, in a game of complete symmetry (or complete asymmetry) between two players, they will have the same destabilization points and the path of the linear tracing procedure does not have a unique continuation after this point (think of a symmetric "battle of sexes" game).

Harsanyi and Selten attempted to single out a unique equilibrium point for *all finite* games and not only for the generic subset. So they could not be satisfied that the linear tracing procedure as one of their most important tools in solving games lead to dubious results in nongeneric cases. Therefore they introduced the *logarithmic tracing procedure.* The logarithmic tracing procedure generates a well-defined path for *all finite* games and the result of the logarithmic tracing procedure is the same as the result of the linear tracing procedure if the latter's path is well-defined. Hence the logarithmic tracing procedure can be considered as a generalization of the linear tracing procedure.

The payoff function along the logarithmic tracing procedure differs from that of the linear tracing procedure (see **(10)**) by an additional logarithmic term which "punishes" to some extent each deviation from the player's centroid strategy. This term ensures that for each $t < 1$ each player has a unique best reply in completely mixed strategies to any strategy combination of the other players.

The logarithmic tracing procedure only comes into play in nongeneric games. In this work we will only determine the solution of the generic elements in our class of signaling games, therefore the linear tracing procedure is sufficient. Since 1988, when Harsanyi's and Selten's book was published, some properties of the tracing procedure and its computability are investigated in more detail (Schanuel, Simon & Zame (1991), van den Elzen & Talman (1995), van den Elzen (1996), Herings & van den Elzen (2002)). However, for our purposes these advances are of no relevance.

2.4. Solution of Irreducible Games

After the preparations given in subsection 2.2 (the process of decomposition and reduction) and in subsection 2.3 (the linear tracing procedure), we want to explain in this subsection how Harsanyi and Selten solve an *irreducible* game (for the definition see subsection 2.2). However, we need some further definitions.

FORMATIONS. Consider a game F which results from a game G by eliminating some pure strategies (and changing the payoff functions in the appropriate way). If for each i-incomplete mixed strategy combination permissible in F player i's best replies in G are all contained in F, and if this holds for each player, we call F a *formation*.

PRIMITIVE FORMATIONS. A formation is called *primitive* if it contains no proper subformations. For example, a strong equilibrium point *generates* a primitive formation. However, strong equilibrium points do not always exist. Harsanyi and Selten introduced the concept of a primitive formation to have a concept with similar stability properties as a strong equilibrium point.

BASIC GAMES. A game is called *basic* if it is irreducible and if it contains no formations. Hence, each irreducible game must be basic or it must contain some primitive formations.

INITIAL CANDIDATES. The *initial candidates* for the solution of an irreducible game are defined as follows: If the game is basic, then the solution of this basic game is the only initial candidate. If the game is not basic, then the solutions of the primitive formations of this game are the initial candidates. The set of initial candidates is also called the *first candidate set*.

It will turn out that in the class of games we consider a game can have two primitive formations at most, and that in this case these two primitive formations must be generated each by a strong pure equilibrium point. So the first candidate set contains either one (pure or mixed) or two (pure) equilibrium points.

If there is only one candidate, this equilibrium point is the solution of the game. If we have two initial candidates, we first look whether one of them strictly payoff-dominates the other one. If this is not the case, a risk-dominance comparison via the linear tracing procedure between these two equilibrium points is necessary (see subsection 2.3). The solution of the game is then the equilibrium point that dominates (strictly payoff-dominates or risk-dominates) the other one, where priority is given to payoff-dominance. However, it is possible that neither (strict) payoff-dominance nor risk-dominance exist between two equilibrium points. No risk-dominance relationship between two equilibrium points is given if the path of the (logarithmic) tracing procedure with the bicentric priors does not end in one of these equilibrium points. This can only happen in degenerate cases. Then a substitution step becomes necessary.

SUBSTITUTION OF A CANDIDATE SET (see also subsection 2.3). If the first candidate set consists of two equilibrium points without dominance relationship, we substitute this set by the equilibrium point which is the result of the tracing procedure using the players' *centroid strategies* as their priors. This equilibrium point is the solution of the game. Notice that this resulting equilibrium is generally not among the two initial candidates. For example, in a symmetric battle-of-sexes game the first candidate set consists of the two pure equilibrium points, but its substitute (and, therefore, the solution of the game) is the mixed equilibrium point.

SUMMARY OF PROCEDURES. In subsection 2.2 we explained how games are transformed to become *irreducible games*. In the present subsection we defined how an irreducible game is *solved*. First, we check whether the game is *basic*. If the game is basic we compute its solution, which is the *result of the linear tracing procedure using the players' centroid strategies as their priors*. If the game is not basic, we compute the *solutions of its primitive formations*. If the game has two primitive formations (generated by two pure equilibrium points), we make a *payoff-dominance comparison* and, if necessary, a *risk-dominance comparison* between the two generating pure equilibrium points. If no dominance relationship exists we compute the *result of the linear tracing procedure using the players' centroid strategies as their priors*. In any case we come out with a unique equilibrium point which is called *the solution of the game.*

Pay special attention to the fact that all procedures mentioned above are done within the *perturbed game*. The *solution of the unperturbed game* is obtained as the limit of the solutions of its perturbations letting ε go to zero.

2.5. Solution of 2x2 Games with Two Strong Equilibrium Points

In many game-theoretical models 2x2-games arise as subgames or cells (see subsection 2.2). Therefore their *solutions* are of special interest. Here we are concerned only with the equilibrium selection problem resulting of a 2x2-game with two strong equilibrium points. Let such a game be given as follows (figure 1):

		Player 2	
		U_2	V_2
Player 1	U_1	a_{11} / b_{11}	a_{12} / b_{12}
	V_1	a_{21} / b_{21}	a_{22} / b_{22}

Figure 1: A 2x2-game with the two strong equilibrium points $(\boldsymbol{U_1}, \boldsymbol{U_2})$ and $(\boldsymbol{V_1}, \boldsymbol{V_2})$ because we assume that $\boldsymbol{a_{11}} > \boldsymbol{a_{21}}$, $\boldsymbol{a_{22}} > \boldsymbol{a_{12}}$, $\boldsymbol{b_{11}} > \boldsymbol{b_{12}}$ and $\boldsymbol{b_{22}} > \boldsymbol{b_{21}}$ hold. For each strategy combination, player 1's payoff is shown in the upper left corner and player 2's payoff is shown in the lower right corner of the respective square.

The game described in figure 1 can be transformed in a *best-reply structure preserving game,* as shown in figure 2:

		Player 2	
		U_2	V_2
Player 1	U_1	u_1, u_2	0, 0
	V_1	0, 0	v_1, v_2

Figure 2: A 2x2-game with the two strong equilibrium points $(\boldsymbol{U_1}, \boldsymbol{U_2})$ and $(\boldsymbol{V_1}, \boldsymbol{V_2})$ which results from the following best-reply structure preserving transformations: $\boldsymbol{u_1} = \boldsymbol{a_{11}} - \boldsymbol{a_{21}} > 0$, $\boldsymbol{v_1} = \boldsymbol{a_{22}} - \boldsymbol{a_{12}} > 0$, $\boldsymbol{u_2} = \boldsymbol{b_{11}} - \boldsymbol{b_{12}} > 0$ and $\boldsymbol{v_2} = \boldsymbol{b_{22}} - \boldsymbol{b_{21}} > 0$.

The term "*best-reply preserving transformations*" simply means that after some payoff manipulations of a game *G* a game *G'* is received with the property that for each player his best replies against all opponents' strategy combinations are the same in both games.

Harsanyi and Selten provide an axiomatic foundation for the risk-dominance comparison between two pure equilibrium points of a 2x2-game like in figure 2. In their book they proof that three plausible requirements on the solution of the selection problem between the two equilibrium points (U_1, U_2) and (V_1, V_2) are only fulfilled by the following criterion:

- (U_1, U_2) is the solution if $u_1 u_2 > v_1 v_2$ holds.
- (V_1, V_2) is the solution if $u_1 u_2 < v_1 v_2$ holds.

The mixed equilibrium of the game is its solution if $u_1 u_2 = v_1 v_2$ holds.

Furthermore, Harsanyi and Selten show that these results are equivalent to those obtained by making a risk-dominance comparison between the two equilibrium points via the linear tracing procedure (see subsection 2.3). This means that the axiomatically founded solution concept for 2x2 games with two strong equilibrium points is embedded into the general solution theory for all games roughly described in this section. The comparison of the *payoff products* $u_1 u_2$ and $v_1 v_2$ is similar to the analysis of the *Nash product* (Nash (1953)). In consequence, this embedding is called the *Nash property* of the Harsanyi-Selten theory.

The solution of a 2x2-game can therefore be obtained without explicitly making use of the tracing procedure. If the game is given as in figure 1, then you have to check whether one of the two equilibrium points strictly payoff-dominates the other one. In this case, the payoff-dominating equilibrium point is the solution of the game. Otherwise, you have to transform the game into a game as in figure 2 preserving the best-reply structure. Then you have to compute which of the equilibrium points yields the higher "Nash product". This one is the solution of the game. If both equilibrium points yield the same Nash product, the mixed equilibrium of the game is its solution.

In the following lemma we show that there exists a simple measure equivalent to the Nash product criterion in 2x2-games, which is in some applications easier to compute (see Potters, van Winden & Mitzkewitz (1991)).

LEMMA. Given a 2x2-game as in figure 2. Then the two pure strategies (one for each player) chosen in the pure equilibrium with the higher product of payoffs (Nash product) are chosen in the mixed equilibrium point of the game with probabilities adding up to *less* than 1.

PROOF. In the mixed equilibrium (q_1, q_2) of the game in figure 2 the strategies U_1 and U_2 are chosen with the following probabilities:

$$\boldsymbol{q_1 = \frac{v_2}{u_2 + v_2}} \qquad \textbf{(14)}$$

$$\boldsymbol{q_2 = \frac{v_1}{u_1 + v_1}} \qquad \textbf{(15)}$$

It follows:

$$\boldsymbol{q_1 + q_2 = 1 + \frac{v_1 v_2 - u_1 u_2}{(u_1 + v_1)(u_2 + v_2)}} \qquad \textbf{(16)}$$

Since all u_i and v_i are greater than zero, it follows that $q_1 + q_2 > 1$ if $v_1 v_2 > u_1 u_2$ and that $q_1 + q_2 < 1$ if $v_1 v_2 < u_1 u_2$. ■

We make use of this result in our analysis.

3. The Class of Games Considered and the Solution of Its Decomposable and Reducible Members

Consider the following class of signaling games. The *sender* is one of two types which occur with known positive probabilities α and $1-\alpha$. Each type has to choose between two alternatives: the move "*inside*" and the move "*outside*". If the activated type chooses "*outside*" the game is finished, but if he chooses "*inside*" a *receiver* observes this *message* without being informed about the sender's type. Afterwards, the receiver has to choose between two responses, called "*left*" and "*right*" to terminate the game. At each of the six possible endpoints of the game the players receive their respective payoffs. Following Harsanyi (1967-1968) we consider the two types as different players, hence the payoff vectors have three components. Figure 3 shows the extensive form of the game without specifying the payoff vectors. The two types of the sender are called player 1 and player 2, and the receiver is called player 3. Nature choosing sender's type by chance is called player 0.

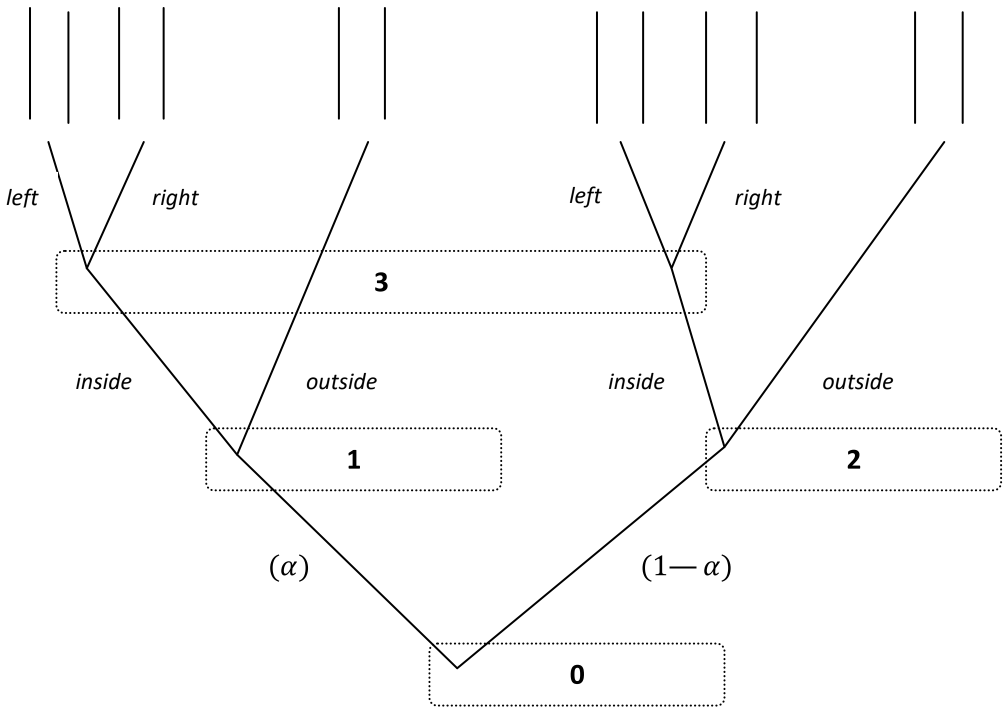

Figure 3: The extensive form of the considered class of games. Information sets are indicated by the dotted lines. Payoff vectors are unspecified.

NORMALIZATION. In this section, we make some steps to normalize this class of signaling games as follows: A type always receives nothing if he is not active. Furthermore, we subtract the payoff vector after an *"outside"* choice of a type from all three payoff vectors which can be achieved if this type has become active. This transformation preserves the best-reply structure for all players. By this procedure the new payoff vectors of the normalized game are obtained. The payoffs are named as in figure 4. We will call this steps *semi-normalization.* In section 4, dealing with the indecomposable and irreducible games, we will proceed with the normalization.

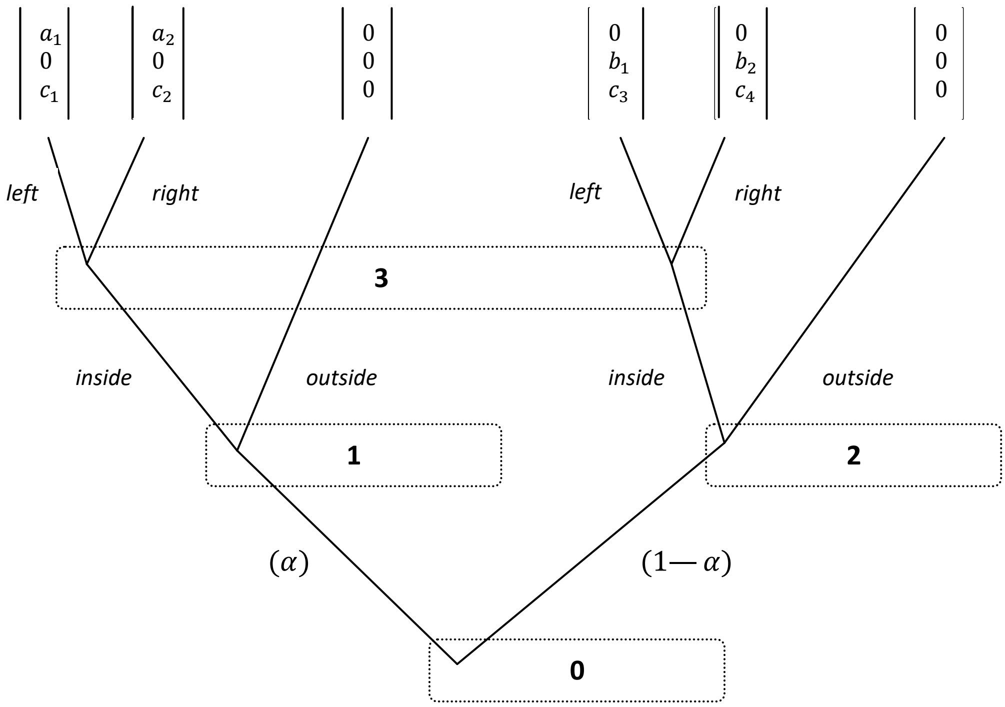

Figure 4: The extensive form of the semi-normalized games.

DECOMPOSITION AND REDUCTION. Now we explain the meaning of *"decomposable"* and *"reducible"* (see subsection 2.2) in the normalized signaling games. Fortunately, for the simple game structure considered here the two concepts are closely connected.

ELEMENTARY CELLS. First we look on possible kinds of *elementary cells* (see subsection 2.2). Obviously, the two types *together* cannot form an elementary cell because their best replies are always independent from each other. Furthermore, the receiver *together with one type* cannot form an elementary cell by the following reason. If they form a cell, the receiver must be independent from the strategy of the other type and, therefore, the receiver should calculate only for the situation after an *"inside"* choice of the cell type. But this means that the receiver's best reply is independent of the probability of this move. Consequently, in this case the receiver forms an elementary cell by himself. It follows, that,*if signaling games of our class are decomposable, an elementary cell is formed by a single player.*

CONDITIONS THAT A TYPE FORMS A CELL. By definition, the best-reply structure of a type forming a cell must be independent of the receiver's strategy. This situation can occur in three ways:

1. The cell type receives in one case more than null and in the other case at least null after an *"inside"* choice in dependence on the receiver's response. This means that the cell type's *"outside"* choice is inferior.
2. The cell type receives in one case less than null and in the other case he receives null at most after an *"inside"* choice in dependence on the receiver's response. This means that the cell type's *"inside"* choice is inferior.
3. The cell type receives always null. This means that his two pure strategies are semiduplicates (see subsection 2.2).

CONDITIONS THAT THE RECEIVER FORMS A CELL. This situation is given in two cases :

1. One of the receiver's choices is (weakly) dominated. Then this pure strategy is, of course, inferior.
2. The payoffs of the receiver only depend on the active type but not on his own choice. Then his two pure strategies are semiduplicate classes.

REDUCTION. In our simple games, the process of solving first the one-person cells is equivalent to the process of reduction. Every player who forms a cell *is fixed* at his superior choice (if he has an inferior choice) or at his centroid strategy (if his pure strategies are semiduplicates). If all three players form cells for themselves the solution of the game is obtained immediately by such strategy fixing. Otherwise, the reduced game has to be analyzed further. Solutions for all decomposable signaling games of our class are developed in the following subsections. The results of the somewhat tedious case-by-case analysis are summarized in an overview presented in section 6 after the results of the indecomposable and irreducible games have also been calculated in section 5.

3.1. At least the receiver forms a cell

The situations are quite similar if only the receiver forms a cell or if the receiver and one type form cells or if all three players form cells. This similarity arises from the fact that *at the latest after the elimination* of the receiver's inferior choice or of his semiduplicate class both types will form cells by themselves. In these cases, the solution of the reduced game is obtained by fixing the types at their superior choices (if they have inferior choices) or at their centroids (if their pure strategies are semiduplicates).

In the remaining subsections those situations are considered in which at least one type forms a cell but the receiver does not.

3.2. Both types form cells

In this case it is necessary to look at the ε-perturbed game. First, both types are fixed at their superior choices or at their centroid strategies. But notice that in the perturbed game inferior choices still occur with probability ε. Table 1 presents the conditional probabilities that the node after player 1's *"inside"* choice (the left node in player 3's information set in figure 3) is reached, given that the receiver has observed an *"inside"* choice.

Probability for player 3's left node after fixing the types		Player 2		
		Inferior choice *"inside"*	Inferior choice *"outside"*	Semiduplicate Class
Player 1	Inferior choice *"inside"*	α	$\frac{\alpha}{\alpha\varepsilon + (1-\alpha)(1-\varepsilon)}$	$\frac{2\alpha\varepsilon}{1-(1-2\varepsilon)\alpha}$
	Inferior choice *"outside"*	$\frac{\alpha(1-\varepsilon)}{\alpha(1-\varepsilon)+(1-\alpha)\varepsilon}$	α	$\frac{2\alpha(1-\varepsilon)}{1+(1-2\varepsilon)\alpha}$
	Semi-duplicate class	$\frac{\alpha}{\alpha+2(1-\alpha)\varepsilon}$	$\frac{\alpha}{\alpha+2(1-\alpha)(1-\varepsilon)}$	α

Table 1: Conditional probabilities that the node after player 1's *"inside"* choice is reached, given that the receiver observed an *"inside"* choice.

Given these conditional probabilities, the receiver is able to compute which of his two responses yields a higher expected payoff. This response is his ε-extreme strategy in the perturbed game. If both responses yield the same expected payoff, the receiver has to choose his centroid strategy. By letting $\varepsilon \to 0$, the limit solution of the game is obtained.

3.3. Only one type forms a cell – he has the inferior choice "outside"

In the remaining parts of section 3 the cell forming type is always called player 1. In this subsection we consider the reduced game after elimination of an inferior *"outside"* choice of player 1. However, this choice occurs with positive probability due to the perturbation. The two responses of the receiver are called r_1 and r_2, and player 2's *"inside"* choice is called m_1 and his *"outside"* choice is called m_2. The payoffs are named as in figure 4.

Since player 2 and player 3 do not initially form cells in the case considered in this subsection, the following conditions for the payoffs must hold:

$$\boldsymbol{b_1 \neq 0, \qquad b_2 \neq 0, \qquad sgn\, b_1 \neq sgn\, b_2} \tag{17}$$

$$\boldsymbol{c_1 \neq c_2, \qquad c_3 \neq c_4, \qquad sgn(c_1 - c_2) \neq sgn(c_3 - c_4)} \tag{18}$$

Without loss of generality we can assume that the receiver's responses are named in such a way that $c_3 - c_4 > 0$ holds. In the reduced perturbed game (after elimination of player 1's inferior *"outside"*) player 2 does obviously not form a cell. But player 3 gets an inferior choice r_1 if the following inequality holds:

$$\boldsymbol{\alpha(c_2 - c_1) \geq (1 - \alpha)(c_3 - c_4)} \tag{19}$$

This inferiority results from the fact that the left node in player 3's information set is reached in the reduced game with probability $\alpha(1 - \varepsilon)$, but the right node is reached with probability $(1 - \alpha)(1 - \varepsilon)$ at most. If **(19)** holds, player 3 is fixed at r_2 and finally player 2 has to choose m_1 if $b_1 < 0$ and m_2 if $b_1 > 0$.

If **(19)** does not hold the reduced game is not further decomposable and reducible. Therefore, the equilibrium points of this game are examined. The probability that player 2 chooses m_1 is called q_2 and the probability that player 3 chooses r_1 is called q_3. Best reply of player 2 is m_1 if either **(20)** or **(21)** holds:

$$q_3 \geq \frac{-b_2}{b_1 - b_2} \equiv b \ \ and \ \ b_1 > 0 \qquad \textbf{(20)}$$

$$q_3 \leq \frac{-b_2}{b_1 - b_2} \equiv b \ \ and \ \ b_1 < 0 \qquad \textbf{(21)}$$

Since **(17)** holds, we have always $0 < b < 1$.

r_1 is a best reply of player 3 if:

$$q_2 \geq (1 - \varepsilon)\frac{\alpha(c_2 - c_1)}{(1 - \alpha)(c_3 - c_4)} \equiv (1 - \varepsilon)c \qquad \textbf{(22)}$$

Since **(18)** holds and **(19)** does not hold in the situation considered, it follows that $0 < c < 1$.

First, consider the case that $b_1 < 0$. The *best-reply correspondences* are shown in figures 5 and 6 for arbitrary values of ε, b and c. For sufficiently small values of ε we must have $\varepsilon < b < 1 - \varepsilon$ and $\varepsilon < (1 - \varepsilon)c < 1 - \varepsilon$.

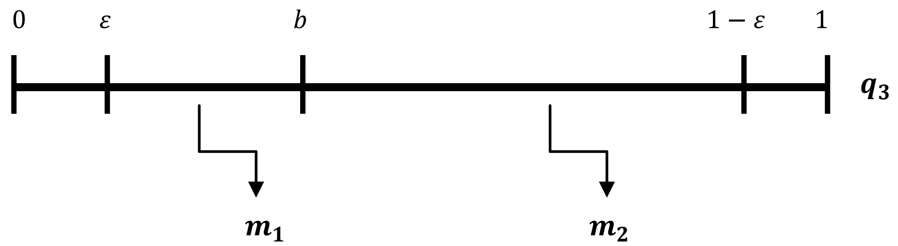

Figure 5: Best-reply correspondence of player 2 in a subclass of subsection 3.2.

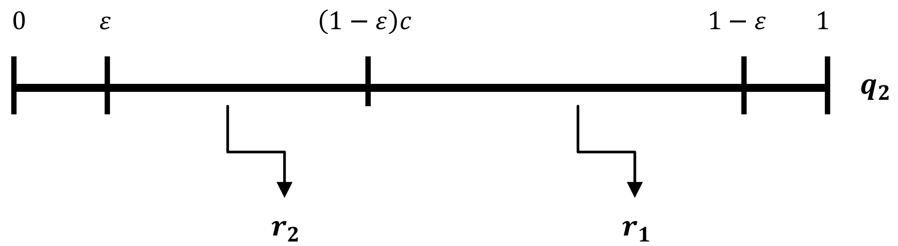

Figure 6: Best-reply correspondence of player 3 in a subclass of subsection 3.2.

Obviously, the (mixed) strategy combination $(q_2, q_3) = ((1-\varepsilon)c, b)$ is the only equilibrium point of the reduced game. Defining q_1 as player 1's probability to choose m_1 (his *"inside"* choice), the limit solution of the whole game is therefore $(q_1, q_2, q_3) = (1, c, b)$.

The situation is quite different for $b_1 < 0$. Player 3's best-reply correspondence is the same as in figure 6, and player 2's best-reply correspondence is obtained by interchanging m_1 and m_2 in figure 5. The reduced game has three strategy combinations (q_2, q_3) as equilibrium points: $(1-\varepsilon, 1-\varepsilon), (\varepsilon, \varepsilon)$ and $((1-\varepsilon)c, b)$. The third one is not in the *first candidate set* for the solution of the reduced game because it is not the solution of a *primitive formation* (see subsection 2.4). Therefore, the first candidate set contains only the two ε-extreme equilibrium points $(1-\varepsilon, 1-\varepsilon)$ and $(\varepsilon, \varepsilon)$. We first analyze under which conditions there is a payoff-dominance relationship between these two equilibrium points.

At the first equilibrium point, player 2's (expected) payoffs are approximately $(1-\alpha)b_1$ for sufficiently small ε, hence they are strictly positive (since $b_1 > 0$). At the second equilibrium point, his expected payoffs are approximately null. Simple computations show that the (expected) payoffs of player 3 are at least as much at the first equilibrium point than at the second one, if the following inequality holds:

$$\boldsymbol{c_3 \geq \frac{\alpha}{1-\alpha}(1-\varepsilon)(c_2 - c_1)} \qquad \textbf{(23)}$$

This inequality is independent of c_4 because the knot at which player 3 receives this payoff is reached in both equilibrium points with the same probability (i.e. $(1-\alpha)(1-\varepsilon)\varepsilon$). In the case we consider, **(19)** does not hold. Therefore we have:

$$\boldsymbol{c_3 - c_4 > \frac{\alpha}{1-\alpha} \cdot (c_2 - c_1)} \qquad \textbf{(24)}$$

Thus, **(23)** always holds if c_4 is nonnegative or if it is negative but its absolute value is small enough. Hence, the equilibrium point $(1-\varepsilon, 1-\varepsilon)$ payoff dominates the equilibrium point $(\varepsilon, \varepsilon)$ for sufficiently small ε if:

$$\boldsymbol{c_3 > \frac{\alpha}{1-\alpha} \cdot (c_2 - c_1)} \qquad \textbf{(25)}$$

In this case, the limit solution of the game is $(q_1, q_2, q_3) = (1,1,1)$.

If **(25)** does not hold (this implies that c_4 is negative and its absolute value is large enough) there is no payoff-dominance relationship between the two equilibrium points in the first candidate set. Therefore a *risk-dominance comparison* between the two candidates becomes necessary.

Since the reduced game is a 2x2-game, the lemma of subsection 2.5 can be applied. It implies that the sums of the probabilities chosen in the mixed equilibrium point for those strategies used in the *first* pure equilibrium point determines the result of the risk-dominance comparison. In our case it follows:

- If $b + (1-\varepsilon)c < 1$, the equilibrium point $(q_2, q_3) = (1-\varepsilon, 1-\varepsilon)$ risk-dominates the equilibrium point $(\varepsilon, \varepsilon)$. This condition is satisfied for each $\varepsilon > 0$ if $b + c \leq 1$ holds. Hence, in this case we obtain $(q_1, q_2, q_3) = (1,1,1)$ as the limit solution of the game, too.
- On the other hand, if $b + c > 1$ holds, the inequality $b + (1-\varepsilon)c > 1$ is implied for sufficiently small ε. In this case the equilibrium point $(\varepsilon, \varepsilon)$ risk-dominates the equilibrium point $(1-\varepsilon, 1-\varepsilon)$ and we obtain $(q_1, q_2, q_3) = (1,0,0)$ as the limit solution of the game.

3.4. Only one type forms a cell – he has the inferior choice “inside”

Now we consider the reduced perturbed game after fixing player 1 at his *“outside”* choice. Clearly, **(17)**, **(18)** and $c_3 - c_4 > 0$ still hold. Player 3 obtains (after the fixing) an inferior choice r_2 if the following inequality holds:

$$\boldsymbol{\alpha(c_1 - c_2) \leq (1-\alpha)(c_3 - c_4)} \tag{26}$$

The definition of c given by **(22)** implies that **(26)** is equivalent to $c \leq 1$. If **(26)** holds, player 3 is fixed at his r_1 choice, and, finally, player 2 must choose m_1 if $b_1 > 0$ or m_2 if $b_1 < 0$.

If **(26)** does not hold, we must look at the equilibrium points of the reduced perturbed game. The best-reply structure is still given by **(20)** and **(21)**. Different to **(22)**, r_1 is a best reply of player 3 if the following inequality holds:

$$\boldsymbol{q_2 \geq \varepsilon \frac{\alpha(c_2 - c_1)}{(1-\alpha)(c_3 - c_4)} \equiv \varepsilon c} \tag{27}$$

Since **(26)** does not hold, we have $\varepsilon < \varepsilon c$ for each $\varepsilon > 0$ and we have $\varepsilon c < 1 - \varepsilon$ for sufficiently small ε. For $b_1 < 0$ the situation is illustrated in figures 7 and 8.

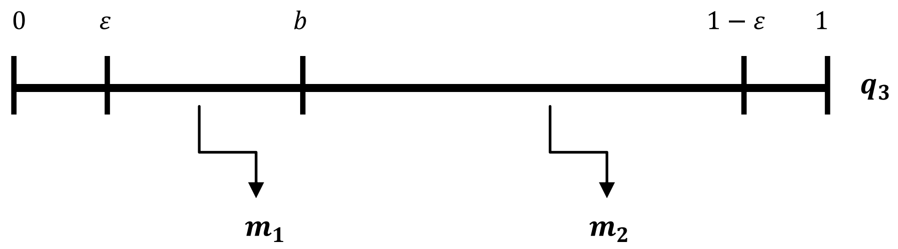

Figure 7: Best-reply correspondence of player 2 in a subclass of subsection 3.3.

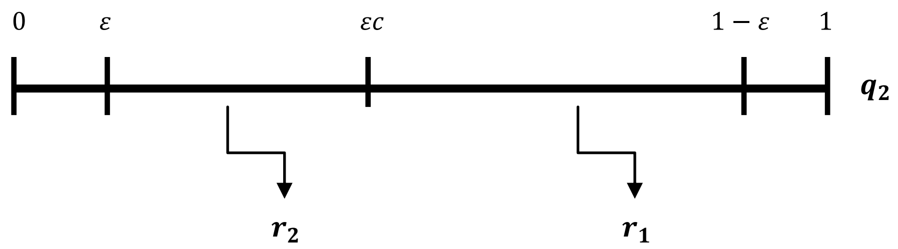

Figure 8: Best-reply correspondence of player 3 in a subclass of subsection 3.3.

The mixed-strategy combination $(q_2, q_3) = (\varepsilon c, b)$ is the only equilibrium point of the reduced perturbed game. Hence, for $\varepsilon \to 0$ we obtain $(q_1, q_2, q_3) = (0,0,b)$ as the limit solution of the whole game.

In the case $b_1 > 0$ the best-reply correspondence of player 2 is obtained by interchanging m_1 and m_2 in figure 7. Now the reduced game has the three equilibrium points $(1-\varepsilon, 1-\varepsilon)$, $(\varepsilon, \varepsilon)$ and $(\varepsilon c, b)$. The first two equilibrium points form the *first candidate set*. Like in subsection 3.3 player 2's (expected) payoffs at the $(1-\varepsilon, 1-\varepsilon)$-equilibrium point are approximately $(1-\alpha)b_1$ (hence, strictly positive) and approximately null at the $(\varepsilon, \varepsilon)$-equilibrium point. The (expected) payoffs of player 3 at the first equilibrium point are not smaller than at the second one if the following holds:

$$\boldsymbol{c_3 \geq \frac{\alpha}{1-\alpha}\varepsilon(c_2 - c_1)} \tag{28}$$

Since $c_2 - c_1 > 0$, inequality **(28)** is fulfilled for sufficiently small ε if $c_3 > 0$ holds. In this case the $(1-\varepsilon, 1-\varepsilon)$-equilibrium point payoff-dominates the $(\varepsilon, \varepsilon)$-equilibrium point. The limit solution of the game is $(q_1, q_2, q_3) = (0,1,1)$.

If $b_1 > 0$ and $c_3 \leq 0$, a risk-dominance comparison between the two ε-extreme equilibrium points is necessary. If ε goes to zero, then in the mixed equilibrium point of the reduced game the sum of the probabilities of the pure strategies used in the equilibrium point $(\varepsilon, \varepsilon)$ approaches $2 - b > 1$, whereas the sum of the probabilities of the pure strategies used in the equilibrium point $(1-\varepsilon, 1-\varepsilon)$ approaches $b < 1$. According to the lemma of subsection 2.5, we obtain the result that the $(1-\varepsilon, 1-\varepsilon)$-equilibrium point risk-dominates the $(\varepsilon, \varepsilon)$-equilibrium point. Again, the limit solution is $(q_1, q_2, q_3) = (0,1,1)$. Hence, for $b_1 > 0$ the limit solution is independent of the sign of c_3.

3.5. Only one type forms a cell – his pure strategies are semiduplicates

In this subsection we consider the reduced game after fixing player 1 at his centroid strategy (because his two pure strategies are semiduplicates). As before, **(13)**, **(14)** and $c_3 - c_4 > 0$ hold. After fixing player 1, player 3 gets an inferior choice r_1 if the following inequality holds:

$$\boldsymbol{\alpha(c_2 - c_1) \geq 2(1-\alpha)(c_3 - c_4)} \qquad \textbf{(29)}$$

This inequality is equivalent to $c \geq 2$ (see the implicit definition of c given in **(22)**). If **(29)** holds player 3 is fixed at his choice r_2. Then, player 2 must choose m_1 if $b_1 > 0$ and m_2 if $b_1 < 0$.

If $c < 2$, player 3 has the best reply r_1 if we have:

$$\boldsymbol{q_2 \geq \frac{\alpha(c_2 - c_1)}{2(1-\alpha)(c_3 - c_4)} \equiv \frac{c}{2}} \qquad \textbf{(30)}$$

The following analysis is quite similar to that of subsection 3.3, replacing $(1-\varepsilon)c$ by $c/2$ (compare **(22)** and **(30)**). Thus, we present the results only briefly. For $c < 2$ and $b_1 < 0$ the mixed strategy combination $(q_2, q_3) = (c/2, b)$ is the only equilibrium point of the reduced game. Therefore, the limit solution of the game is $(q_1, q_2, q_3) = (1/2, c/2, b)$.

For $c < 2$ and $b_1 < 0$ there are three equilibrium points $(1-\varepsilon, 1-\varepsilon)$, $(\varepsilon, \varepsilon)$ and $(c/2, b)$. The $(1-\varepsilon, 1-\varepsilon)$-equilibrium point payoff-dominates the $(\varepsilon, \varepsilon)$-equilibrium point, if (compare with **(23)**):

$$\boldsymbol{c_3 \geq \frac{\alpha}{2(1-\alpha)}(c_2 - c_1)} \qquad \textbf{(31)}$$

$c < 2$ implies:

$$\boldsymbol{c_3 - c_4 > \frac{\alpha}{2(1-\alpha)}(c_2 - c_1)} \qquad \textbf{(32)}$$

Thus **(31**) always holds if $c_4 > 0$ or it holds if c_4 is negative but with a small absolute value. In these cases, $(q_1, q_2, q_3) = (1/2\,, 1{,}1)$ is the limit solution of the game.

If **(31)** does not hold, a risk-dominance comparison between the two ε-extreme equilibrium points becomes necessary. Similar to subsection 3.3 it follows:

- If $b + c/2 < 1$ holds, the equilibrium point $(q_2, q_3) = (1-\varepsilon, 1-\varepsilon)$ risk-dominates the equilibrium point $(\varepsilon, \varepsilon)$. We obtain $(q_1, q_2, q_3) = (1/2\,, 1{,}1)$ as the limit solution of the game.
- If $b + c/2 > 1$ holds, the risk-dominance comparison is the very opposite and we obtain $(q_1, q_2, q_3) = (1/2\,, 0{,}0)$ as the limit solution of the game.
- If $b + c/2 = 1$ holds, the mixed equilibrium point $(q_2, q_3) = (c/2\,, b)$ is obtained as the solution of the reduced game. The limit solution of the whole game is $(q_1, q_2, q_3) = (1/2\,, c/2\,, b)$.

4. Normalization of the Indecomposable and Irreducible Games

In section 3 we characterized the conditions under which one of the types or the receiver form a cell and analyzed these cases. If none of the three players forms a cell by himself the signaling game is *indecomposable and irreducible.* This situation allows the following steps of normalization:

1. Call the "*inside"* choice of the two types m_1 and the *"outside"* choice m_2.
2. Call the receiver *player 3*.
3. A type receives nothing if he is inactive.
4. Subtract the payoff vector after a m_2-choice of each type from those three payoff vectors which can be reached if this type is the active one. By this subtraction the new payoff vectors of the normalized game are obtained.
5. For each type compute the difference of the *receiver's payoffs* achieved after his two responses, given that this type has become active and has chosen m_1. Multiply these differences with the respective probability of occurrence of the two types. Call that type *player 1* who induces the greater absolute value of these "weighted differences". If both types induce the same weighted difference, call by random some type *player 1*. Call the remaining type *player 2.*
6. Call player 1's probability of becoming active α.
7. Call that *response* of player 3 r_1 that yields the *smaller* payoff to him if player 1 becomes active and chooses m_1. Call the other response r_2. Indifference is not possible because the games considered in this section are indecomposable.

The extensive form of the normalized game is shown in figure 9. The following properties of the payoff structure result from the process of normalization described above and from the fact that in this section only indecomposable and irreducible games are considered.

$$\boldsymbol{a_1 \neq 0,\ \ a_2 \neq 0,\ \ sgn\, a_1 \neq sgn\, a_2} \qquad \textbf{(33)}$$

$$\boldsymbol{b_1 \neq 0,\ \ b_2 \neq 0,\ \ sgn\, b_1 \neq sgn\, b_2} \qquad \textbf{(34)}$$

$$\boldsymbol{c_1 < c_2} \qquad \textbf{(35)}$$

$$\boldsymbol{c_3 > c_4} \qquad \textbf{(36)}$$

$$\alpha(c_2 - c_1) \geq (1 - \alpha)(c_3 - c_4) \quad \textbf{(37)}$$

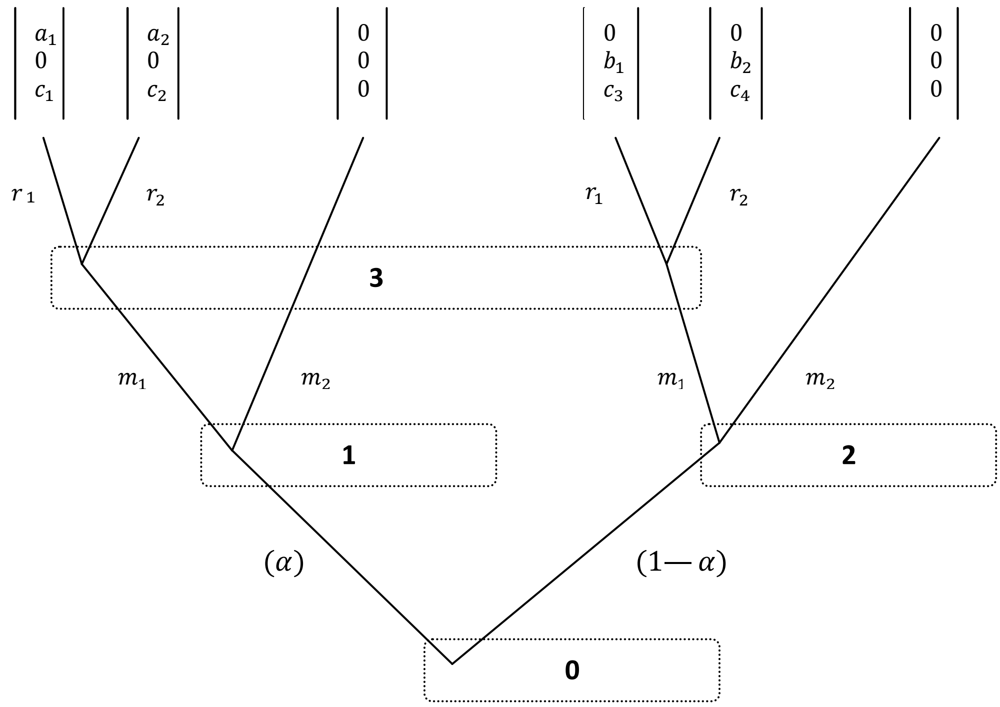

Figure 9: The extensive form of the normalized games.

In the following the probabilities that player 1 and player 2 choose m_1 are called q_1 and q_2, respectively. q_3 is the probability that player 3 chooses r_1. All three players have just two pure strategies, hence, player *i*'s mixed strategy is completely described by q_i. The conditions that a player becomes indifferent between his two choices are calculated now.

INDIFFERENCE POINT OF PLAYER 1. The pure strategies m_1 and m_2 yield the same (expected) payoffs for player 1 if the following holds:

$$q_3 a_1 + (1 - q_3) a_2 = 0 \quad \textbf{(38)}$$

This is equivalent to:

$$q_3 = \frac{-a_2}{a_1 - a_2} \equiv a \quad \textbf{(39)}$$

From **(33)** we can see that $0 < a < 1$ holds. If $a_1 < 0$, then m_1 is a best reply of player 1 if $q_3 \leq a$. If $a_1 > 0$, then m_1 is a best reply of player 1 if $q_3 \geq a$. The next results are obtained in a similar way.

INDIFFERENCE POINT OF PLAYER 2.

$$\boldsymbol{q_3 = \frac{-b_2}{b_1 - b_2} \equiv b} \qquad \textbf{(40)}$$

(34) ensures $0 < b < 1$. A best reply of player 2 is m_1 if $b_1 < 0$ and $q_3 \leq b$ hold simultaneously or if $b_1 > 0$ and $q_3 \geq b$ hold simultaneously.

INDIFFERENCE LINE OF PLAYER 3. The pure strategies r_1 and r_2 yield the same (expected) payoffs for player 3 if the following holds:

$$\boldsymbol{\alpha q_1 c_1 + (1 - \alpha) q_2 c_3 = \alpha q_1 c_2 + (1 - \alpha) q_2 c_4} \qquad \textbf{(41)}$$

This is equivalent to:

$$\boldsymbol{q_2 = \frac{\alpha(c_2 - c_1)}{(1 - \alpha)(c_3 - c_4)} q_1 \equiv c q_1} \qquad \textbf{(42)}$$

Since **(35)**, **(36)** and **(37)** hold, it follows that $1 \leq c < \infty$. If $q_2 \geq c q_1$, then r_1 is a best reply of player 3, if $q_2 \leq c q_1$, then r_2 is a best reply of player 3.

We have to mention that the indifference points for players 1 and 2 and the indifference line for player 3 given **(39)**, **(40)** and **(42)** matter not only for the unperturbed game, but represent also the exact values of the perturbed game. We show this only for player 1. In the perturbed game his two ε-extreme strategies yield the same expected payoffs if:

$$\boldsymbol{(1 - \varepsilon)(q_3 a_1 + (1 - q_3) a_2) = \varepsilon(q_3 a_1 + (1 - q_3) a_2)} \qquad \textbf{(43)}$$

Or, equivalently:

$$\boldsymbol{(1 - 2\varepsilon)(q_3 a_1 + (1 - q_3) a_2) = 0} \qquad \textbf{(44)}$$

Since $\varepsilon < \frac{1}{2}$, equation **(44)** is equivalent to $q_3 = a$ (see **(38)** and **(39)**).

In this work we are only concerned with the generic cases of signaling games. The indecomposable and irreducible games are nongeneric if $a = b$ and/or if $c = 1$. Their solutions have also been calculated by the author (using if necessary the *logarithmic tracing procedure* and numerical methods), but their presentation will go beyond the scope of this work.

5. Solution of the Generic Indecomposable and Irreducible Games

In this section we solve the indecomposable and irreducible games with the additional properties $a \neq b$ and $c > 1$. Let $A_i(r_1)$ for $i = 1, 2$ be player i's best reply to a r_1 choice of player 3. We have to distinguish eight cases which are analyzed in the following subsections:

Subsection 5.1.: Case $a < b,\ A_1(r_1) = A_2(r_1) = m_1$

Subsection 5.2.: Case $a < b,\ A_1(r_1) = m_1,\ A_2(r_1) = m_2$

Subsection 5.3.: Case $a < b,\ A_1(r_1) = m_2,\ A_2(r_1) = m_1$

Subsection 5.4.: Case $a < b,\ A_1(r_1) = A_2(r_1) = m_2$

Subsection 5.5.: Case $a > b,\ A_1(r_1) = A_2(r_1) = m_1$

Subsection 5.6.: Case $a > b,\ A_1(r_1) = m_1,\ A_2(r_1) = m_2$

Subsection 5.7.: Case $a > b,\ A_1(r_1) = m_2,\ A_2(r_1) = m_1$

Subsection 5.8.: Case $a > b,\ A_1(r_1) = A_2(r_1) = m_2$.

Throughout this section we always assume that in the uniformly perturbed game the trembling hand parameter ε is sufficiently small, i.e.:

$$\boldsymbol{\varepsilon < \min\left(a, 1 - a, b, 1 - b, \frac{1}{c + 1}\right)} \qquad \textbf{(45)}$$

5.1. Case $a < b,\ A_1(r_1) = A_2(r_1) = m_1$

The best-reply correspondences of players 1 and 2 in the case considered are given in figure 10. In the following cases we omit the corresponding figures. They can be obtained easily by interchanging m_1 and m_2 for player i if $A_i(r_1) = m_2$ holds instead of $A_i(r_1) = m_1$, and by interchanging the positions of the markings of a and b if $a > b$ holds instead of $a < b$.

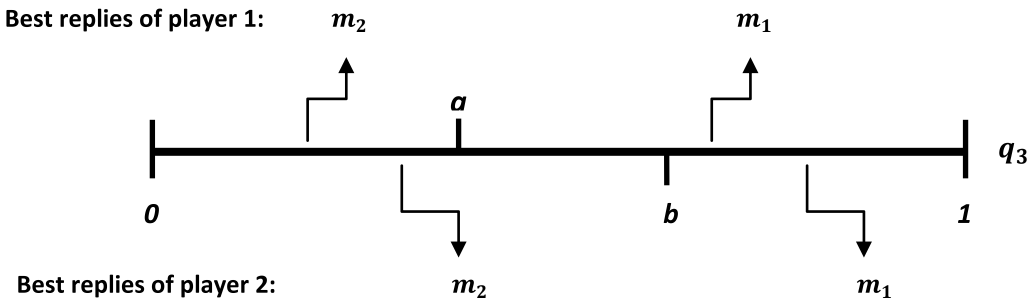

Figure 10: Best-reply correspondences of players 1 and 2 in dependence of player 3's strategy in case 5.1.

The best-reply correspondence of player 3 is shown in figure 11. Such figures are presented for all eight cases considered in this section. The horizontal axis of figure 10 refers to player 1's mixed strategy q_1 and the vertical axis to player 2's mixed strategy q_2. The inner square corresponds to the perturbed game whereas the outer square to the unperturbed game. The straight line $q_2 = cq_1$ shows the set of points at which player 3 is indifferent between his two choices (see **(42)**). Points above this line have the property that r_1 is player 3's unique best reply. The same is true for r_2 if we consider points below the indifference line. This follows from the discussion from **(42)** in section 4 and is true both for the unperturbed and the perturbed game.

In figure 11 and the corresponding figures of the remaining subsections we also mark equilibrium points or connected sets of equilibrium points by the symbol □^{i} if we deal with the unperturbed game and by the symbol ■^{i} if we deal with the uniformly ε-perturbed game. The exact mathematical description of an equilibrium point q^i or of a set of equilibrium points Q^i is given in the text. q^i or Q^i correspond to □^{i} in the following figures. Likewise, q^i_ε or Q^i_ε correspond to ■^{i}. The index i is the number of different equilibrium points or sets of equilibrium points starting with $i = 1$ in case 5.1. It should be clear that equilibrium points in the lower left or the upper right corner of figures correspond to *pooling equilbria* because both types choose the same signal, whereas equilibrium points in the upper left or lower right corner are so-called *separating equilibria.*

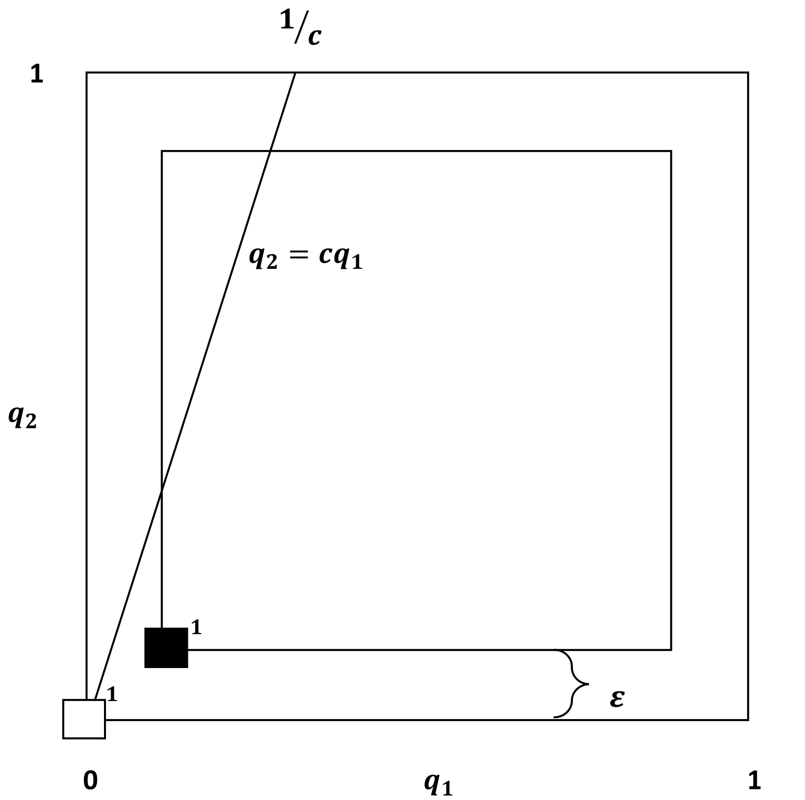

Figure 11: Best-reply correspondences of player 3 in dependence of players' 1 and 2 strategies and equilibrium points in case 5.1.

As one can easily check with the help of figures 10 and 11, in case 5.1 the unperturbed game has the set Q^1, indicated by $\square^1$ in figure 11, as equilibrium points and no others:

$$\boldsymbol{Q^1 = \{(q_1, q_2, q_3) | q_1 = q_2 = 0, q_3 \leq a\}} \qquad \textbf{(46)}$$

But each perturbed game has for sufficiently small ε (see condition **(45)**) a unique equilibrium point, as indicated by $\blacksquare^1$ in figure 11:

$$\boldsymbol{q^1_\varepsilon = (\varepsilon, \varepsilon, \varepsilon)} \qquad \textbf{(47)}$$

Therefore, in case 5.1 the *limit solution* of the game is $q^1 = (0,0,0)$. Of course, $q^1 \in Q^1$.

5.2. Case $a < b$, $A_1(r_1) = m_1$, $A_2(r_1) = m_2$

Figure 12 illustrates the case considered in this subsection. The unperturbed game has a unique equilibrium point $q^2 = (1/c\,, 1, a)$ which is therefore the solution of the game. The unique equilibrium point of the perturbed game is:

$$\boldsymbol{q_\varepsilon^2 = ((1-\varepsilon)/c\,, 1-\varepsilon, a)} \quad \textbf{(48)}$$

Clearly, $\lim_{\varepsilon \to 0} q_\varepsilon^2 = q^2$.

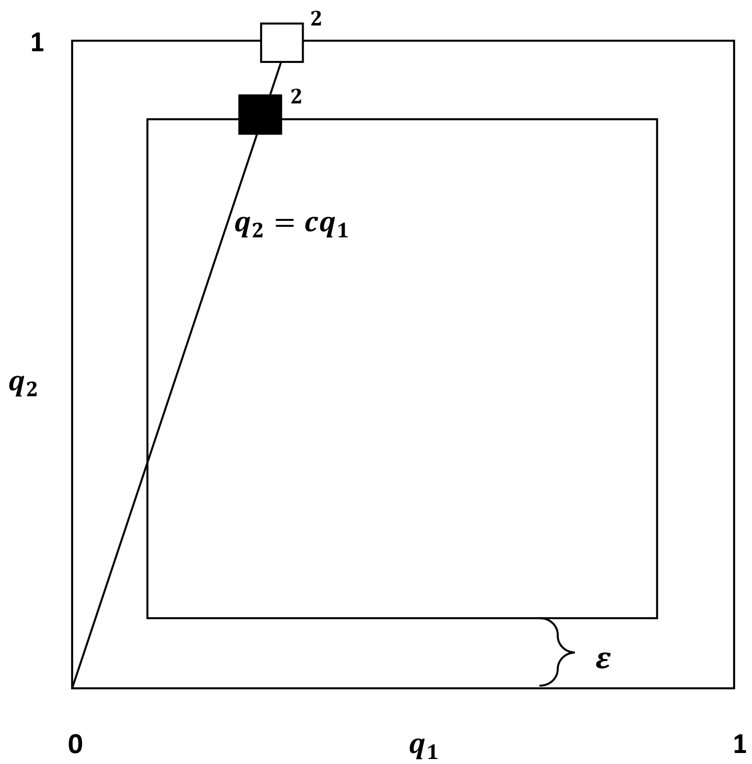

Figure 12: Best-reply correspondences of player 3 in dependence of players' 1 and 2 strategies and equilibrium points in case 5.2.

5.3. Case $a < b$, $A_1(r_1) = m_2$, $A_2(r_1) = m_1$

The equilibrium points in this case are given as follows (see figure 13):

$$\boldsymbol{q^3 = (0, 1, 1)} \quad \textbf{(49)}$$

$$q_\varepsilon^3 = (\varepsilon, 1-\varepsilon, 1-\varepsilon) \quad (50)$$

$$q^4 = (1, 0, 0) \quad (51)$$

$$q_\varepsilon^4 = (1-\varepsilon, \varepsilon, \varepsilon) \quad (52)$$

$$Q^5 = \{(q_1, q_2, q_3) | q_1 = q_2 = 0, a \leq q_3 \leq b\} \quad (53)$$

$$q_\varepsilon^5 = (\varepsilon, c\varepsilon, b) \quad (54)$$

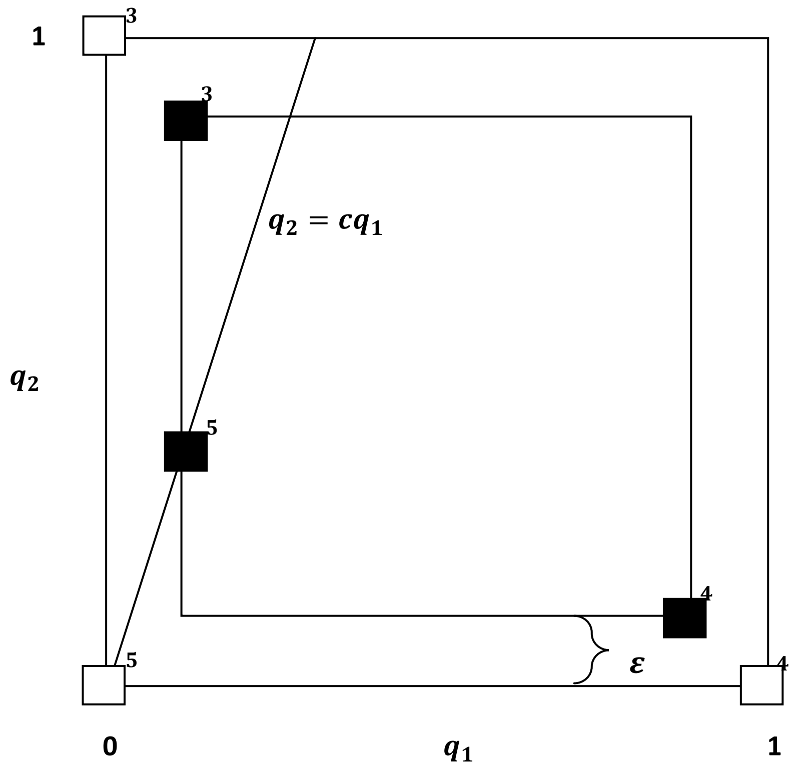

Figure 13: Best-reply correspondences of player 3 in dependence of players' 1 and 2 strategies and equilibrium points in case 5.3.

Since q_ε^5 is not the solution of a *primitive formation*, the *first candidate set* of the perturbed game consists of q_ε^3 and q_ε^4. There is no *payoff-dominance* between these two equilibrium points because player 1 gets positive payoffs at q_ε^4 and zero payoffs at q_ε^3, whereas player 2 gets positive payoffs at q_ε^3 and zero payoffs at q_ε^4. The *linear tracing procedure* (see subsection 2.3) has to decide which equilibrium point *risk-dominates* the other one (see subsection 2.4).

To analyze the *path of the linear tracing procedure* we start with the determination of the *bicentric priors*. In the following pure strategy symbols with an additional lower index "ε" refer to ε-extreme strategies of the perturbed game. The bicentric priors of the first two players can be calculated easily with the help of the appropriate modification of figure 10. We obtain:

Bicentric prior of player 1:

$$\boldsymbol{p_1(m_{1_\varepsilon}) = \frac{a-\varepsilon}{1-2\varepsilon} \equiv \hat{a}} \tag{55}$$

Bicentric prior of player 2:

$$\boldsymbol{p_2(m_{1_\varepsilon}) = 1 - \frac{b-\varepsilon}{1-2\varepsilon} \equiv 1-\hat{b}} \tag{56}$$

To compute the bicentric prior of the third player figure 14 is useful.

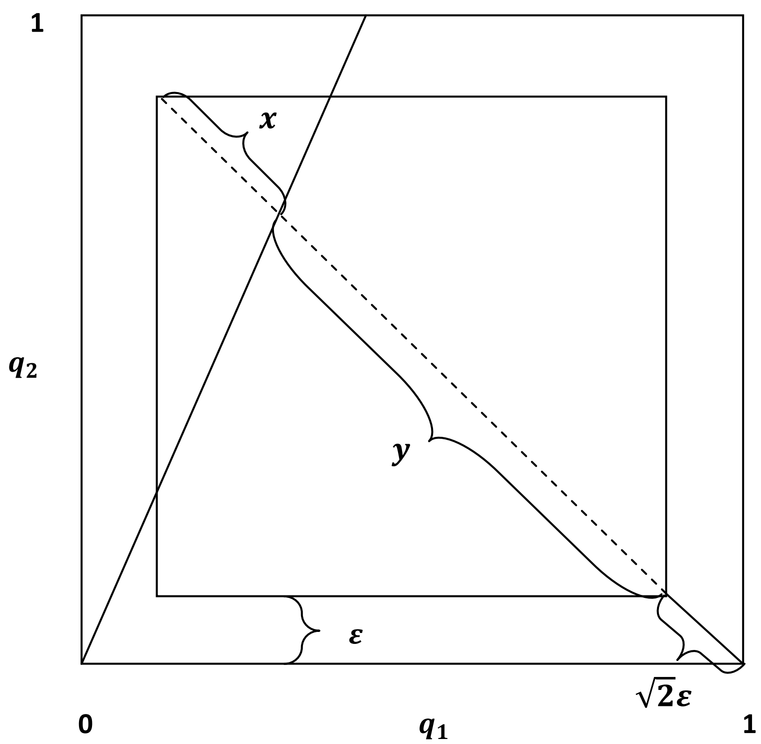

Figure 14: Visualization of player 3's bicentric prior in case 5.3.

Since player 3 assumes that the other player choose either $q^3_{-3_\varepsilon}$ or $q^4_{-3_\varepsilon}$, his expectations are formed along the dashed line in figure 14. x is that part of the whole dashed line at which r_1 is his best reply, y is the rest. Thus his bicentric prior is given by:

$$\boldsymbol{p_3(r_{1_\varepsilon}) = \frac{x}{x+y}} \quad \textbf{(57)}$$

Simple facts of geometry yield:

$$\boldsymbol{\frac{1}{c} = \frac{x+\sqrt{2}\varepsilon}{y+\sqrt{2}\varepsilon}} \quad \textbf{(58)}$$

Furthermore:

$$\boldsymbol{x + y = \sqrt{2}(1-2\varepsilon)} \quad \textbf{(59)}$$

We define $\hat{c}$ as follows:

$$\boldsymbol{\hat{c} \equiv \frac{c-\varepsilon(1+c)}{1-\varepsilon(1+c)}} \quad \textbf{(60)}$$

(57), **(58)**, **(59)** and **(60)** together yield the following result.

Bicentric prior of player 3:

$$\boldsymbol{p_3(r_{1_\varepsilon}) = \frac{1-\varepsilon(1+c)}{1+c-2\varepsilon(1+c)} \equiv \frac{1}{1+\hat{c}}} \quad \textbf{(61)}$$

The "hat" variables $\hat{a}$, $\hat{b}$ and $\hat{c}$, defined in **(55)**, **(56)** and **(60)** converge to a, b and c, respectively, if ε goes to zero. Therefore, $\hat{a} < \hat{b}$ and $\hat{c} > 1$ hold for sufficiently small ε. Let p_{-i} be the i-incomplete bicentric prior resulting from **(55)**, **(56)** and **(61)**. We now analyze what are the players' best replies to the bicentric priors, i.e. the starting point of the linear tracing procedure. Of course, the best replies of the first two players only depend on player 3's bicentric prior.

Best reply to p_{-1} for player 1:

m_{1_ε} *if* $1/(1+\hat{c}) \leq a$

m_{2_ε} *if* $1/(1+\hat{c}) \geq a$

Best reply to p_{-2} for player 2:

m_{1_ε} *if* $1/(1+\hat{c}) \geq b$

m_{2_ε} *if* $1/(1+\hat{c}) \leq b$

Best reply to p_{-3} for player 3:

r_{1_ε} *if* $1-\hat{b} \geq \hat{a}c$

r_{2_ε} if $1-\hat{b} \leq \hat{a}c$

Now we examine which combinations of best replies to the bicentric priors are impossible due to parameter restrictions:

i. If $1/(1+\hat{c}) \leq a$ holds then $1/(1+\hat{c}) \geq b$ is impossible because $a < b$.
ii. If $1/(1+\hat{c}) \leq a$ holds then $1-\hat{b} \geq \hat{a}c$ is impossible. The first inequality implies $1-a \leq a\hat{c}$, but this is a contradiction to $1-\hat{b} \geq \hat{a}c$ because $a\hat{c} \cong \hat{a}c$ and $a < \hat{b}$ for sufficiently small ε.
iii. By a similar argument as above we can conclude that $1/(1+\hat{c}) \geq b$ and $1-\hat{b} \leq \hat{a}c$ cannot hold simultaneously.

Next, we want to show the implications of some relations between a, b and c for their corresponding "hat" variables. From **(60)** it is clear that $\hat{c} > c$ holds for each ε. Thus we can conclude:

$$\mathbf{1/(1+c) \leq a \Rightarrow 1/(1+\hat{c})} < a \quad \textbf{(62)}$$

$$\mathbf{1/(1+c) \leq b \Rightarrow 1/(1+\hat{c})} < b \quad \textbf{(63)}$$

Now assume that $1-b = ac$ holds. With the help of **(55)** and **(56)** one can see that this equation is equivalent to the following one:

$$\mathbf{1-\hat{b} = \hat{a}c + (c-1)\frac{\varepsilon}{1-2\varepsilon}} \quad \textbf{(64)}$$

Since $c > 1$ holds we can conclude:

$$\mathbf{1-b \geq ac \Rightarrow 1-\hat{b} > \hat{a}c} \quad \textbf{(65)}$$

After these preparations we can analyze the best replies to the bicentric priors for the four possible relations among a, b and c. The vector of best replies is denoted by q^0. We obtain:

$$\mathbf{1/(1+c) \leq a} < b \wedge 1-b < ac \Rightarrow \mathbf{q^0 = (1-\varepsilon, \varepsilon, \varepsilon)} \quad \textbf{(66)}$$

$$\mathbf{a < 1/(1+c) \leq b \wedge 1-b} < ac \Rightarrow \mathbf{q^0 = (\varepsilon, \varepsilon, \varepsilon)} \quad \textbf{(67)}$$

$$\mathbf{a < 1/(1+c) \leq b \wedge 1-b \geq ac \Rightarrow q^0 = (\varepsilon, \varepsilon, 1-\varepsilon)} \quad \textbf{(68)}$$

$$\mathbf{a} < b < \mathbf{1/(1+c) \wedge 1-b \geq ac \Rightarrow q^0 = (\varepsilon, 1-\varepsilon, 1-\varepsilon)} \quad \textbf{(69)}$$

No problems arise in the situations given by **(66)** and **(69)** since in both cases the resulting q^0 is one of the two ε-extreme equilibrium points in the first candidate set of the perturbed game and, therefore, no player has an incentive to deviate from this strategy combination along the path of the linear tracing procedure. Thus, in the situation described in **(66)** the limit solution of the game is $q^4 = (1{,}0{,}0)$ and in the situation described in **(69)** the limit solution is $q^3 = (0{,}1{,}1)$.

The best replies to the bicentric priors in the situations described by **(67)** and **(68)** do not yield an equilibrium point. But in **(67)** the analysis is still simple: Player 2 and player 3 have no incentive to deviate from their initial strategies since they are not only best replies to the bicentric priors but also to q^0. This is not true for player 1, in consequence he must change his strategy if *t*, the tracing parameter, becomes sufficiently large. After he has changed his strategy from m_{2_ε} to m_{1_ε} the ε-extreme equilibrium point $q_\varepsilon^4 = (1-\varepsilon, \varepsilon, \varepsilon)$ is reached and no further change of strategies will occur along the remaining path of the tracing procedure. Thus, the limit solution of the game is $q^4 = (1{,}0{,}0)$.

The situation described in **(68)** is more difficult since here two players' (player 2 and player 3) best replies to the bicentric priors are not best replies to q^0. To analyze the path of the linear tracing procedure it must be determined *who is the first to change his strategy*. For this reason we calculate the *destabilization points* (see subsection 2.3) of players 2 and 3. We show that for sufficiently small ε player 2 is the first player to shift to his other strategy.

Player 2's destabilization point t_2 must satisfy the following equation:

$$\boldsymbol{(1-t_2)\frac{1}{1+\hat{c}} + t_2(1-\varepsilon) = b} \qquad \textbf{(70)}$$

Thus t_2 is given as follows:

$$\boldsymbol{t_2 = \frac{b(1+\hat{c})-1}{(1-\varepsilon)(1+\hat{c})-1}} \qquad \textbf{(71)}$$

Since $b > 1/(1+\hat{c})$ holds (see **(63)** and **(68)**), the numerator is positive, and since $b < 1-\varepsilon$ holds, the numerator is smaller than the denominator. Therefore, $0 < t_2 < 1$ holds for sufficiently small ε. We obtain:

$$\boldsymbol{\lim_{\varepsilon\to 0} t_2 = b - \frac{1-b}{c}} \qquad \textbf{(72)}$$

Player 2's destabilization point t_3 can be computed with the help of **(42)**:

$$\boldsymbol{(1-t_3)(1-\hat{b}) + t_3\varepsilon = [(1-t_3)\hat{a} + t_3\varepsilon]c} \qquad \textbf{(73)}$$

Simple computations yield:

$$\boldsymbol{t_3 = \frac{1 - \hat{b} - \hat{a}c}{1 - \hat{b} - \hat{a}c + \varepsilon(c - 1)}} \qquad \textbf{(74)}$$

(68) ensures that the numerator is positive for sufficiently small ε and $c > 1$ ensures that the numerator is smaller than the denominator. Hence, $0 < t_3 < 1$ holds. However,

$$\boldsymbol{\lim_{\varepsilon \to 0} t_3 = 1} \qquad \textbf{(75)}$$

From **(72)** we know that t_2 is positive but smaller than b for sufficiently small ε. Since $b < 1$ we can conclude that $t_2 < t_3$ holds for sufficiently small ε. This means that player 2 is the first player to shift his strategy. But after his shift (from m_{2_ε} to m_{1_ε}) the equilibrium point q_ε^3 is reached and for t with $t_2 < t \leq 1$ no further strategy changes occur. We have shown that in the situation described by **(68**) the limit solution is q^3.

Now our results for case 5.3 can be summarized. Since q^4 is the limit solution in the situations described by **(66)** and **(67)**, and q^3 is the limit solution in the situations described by **(68)** and **(69)**, we can claim: q^3 is the limit solution if $1 - b \geq ac$ holds. Otherwise q^4 is the limit solution.

5.4. Case $\boldsymbol{a < b,\ A_1(r_1) = A_2(r_1) = m_2}$

Figure 15 indicates the equilibrium points in case 5.4.

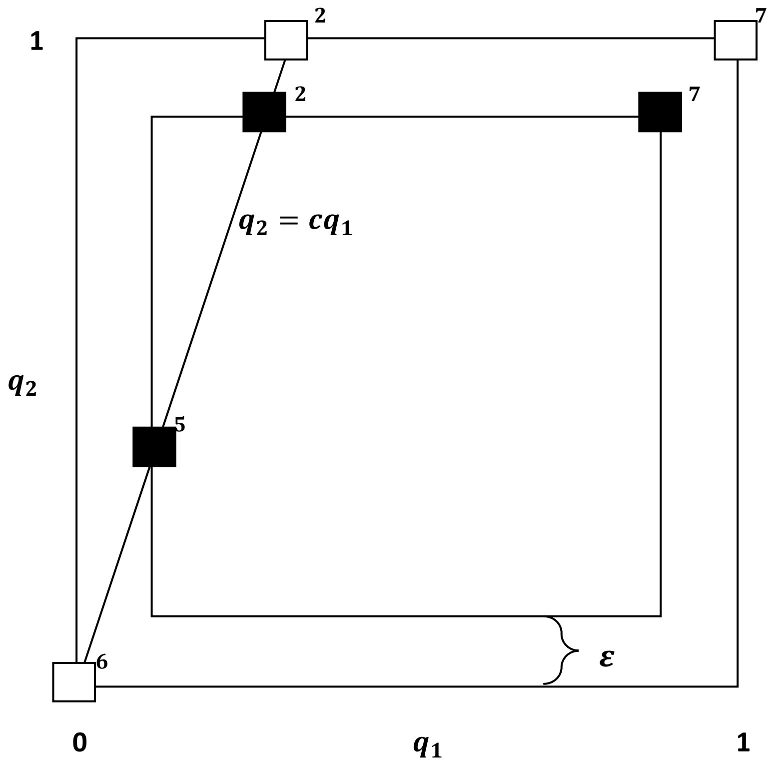

Figure 15: Best-reply correspondences of player 3 in dependence of players' 1 and 2 strategies and equilibrium points in case 5.4.

The formal equivalents to the square symbols in figure 15 for q^2, q_ε^2 and q_ε^5 are still given by **(47)**, **(48)** and **(54)**, respectively. However, Q^6 is not identical with Q^5 given by **(53)**. Instead, we have:

$$\boldsymbol{Q^6} = \{(\boldsymbol{q_1}, \boldsymbol{q_2}, \boldsymbol{q_3}) | \boldsymbol{q_1} = \boldsymbol{q_2} = \boldsymbol{0}, \boldsymbol{q_3} \geq \boldsymbol{b}\} \quad \textbf{(76)}$$

Furthermore:

$$\boldsymbol{q^7} = (\boldsymbol{1}, \boldsymbol{1}, \boldsymbol{0}) \quad \textbf{(77)}$$

$$\boldsymbol{q_\varepsilon^7} = (\boldsymbol{1} - \boldsymbol{\varepsilon}, \boldsymbol{1} - \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}) \quad \textbf{(78)}$$

The equilibrium points q_ε^2 and q_ε^5 of the perturbed game are not solutions of *primitive formations*. Consequently the *first candidate set* contains only q_ε^7. For this reason the limit solution of the game is q^7.

5.5. Case $\boldsymbol{a > b}$, $\boldsymbol{A_1(r_1) = A_2(r_1) = m_1}$

Figure 16 illustrates the case considered now.

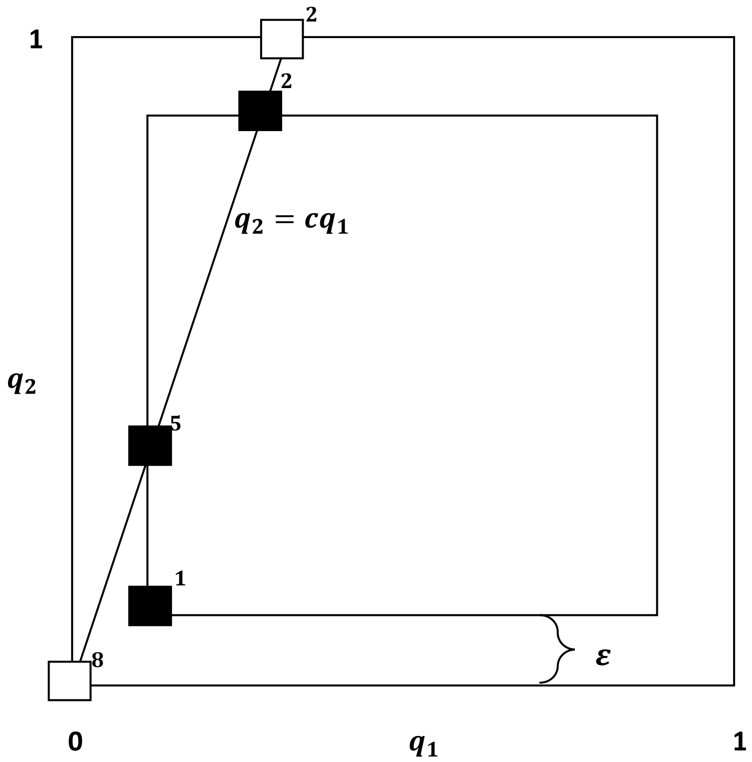

Figure 16: Best-reply correspondences of player 3 in dependence of players' 1 and 2 strategies and equilibrium points in case 5.5.

q_ε^1, q^2, q_ε^2 and q_ε^5 are still given by **(46), (47)**, **(48)** and **(54)**, respectively. Furthermore:

$$\boldsymbol{Q^8 = \{(q_1, q_2, q_3) | q_1 = q_2 = 0, q_3 \leq b\}} \qquad \textbf{(79)}$$

Each perturbed game has three equilibrium points, but only q_ε^1 is the solution of a primitive formation. For this reason, the limit solution of the case 5.5 is $q^1 = (0,0,0)$.

5.6. Case $a > b$, $A_1(r_1) = m_1$, $A_2(r_1) = m_2$

This case is illustrated by figure 17.

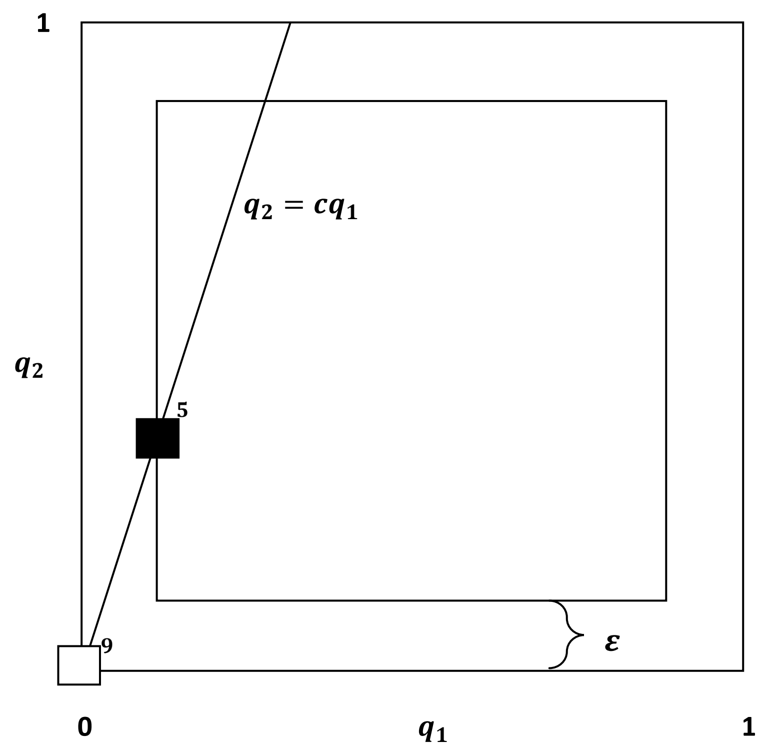

Figure 17: Best-reply correspondences of player 3 in dependence of players' 1 and 2 strategies and equilibrium points in case 5.6.

The set Q^9 is given as follows:

$$\boldsymbol{Q^9 = \{(q_1, q_2, q_3) | q_1 = q_2 = 0, b \leq q_3 \leq a\}} \quad \textbf{(80)}$$

However, the perturbed game has the unique equilibrium point q_ε^5, given by **(54)**. Hence, the limit solution in this case is:

$$\boldsymbol{q^5 = (0, 0, b)} \quad \textbf{(81)}$$

5.7. Case $a > b$, $A_1(r_1) = m_2$, $A_2(r_1) = m_1$

Figure 18 indicates the equilibrium points in case 5.7.

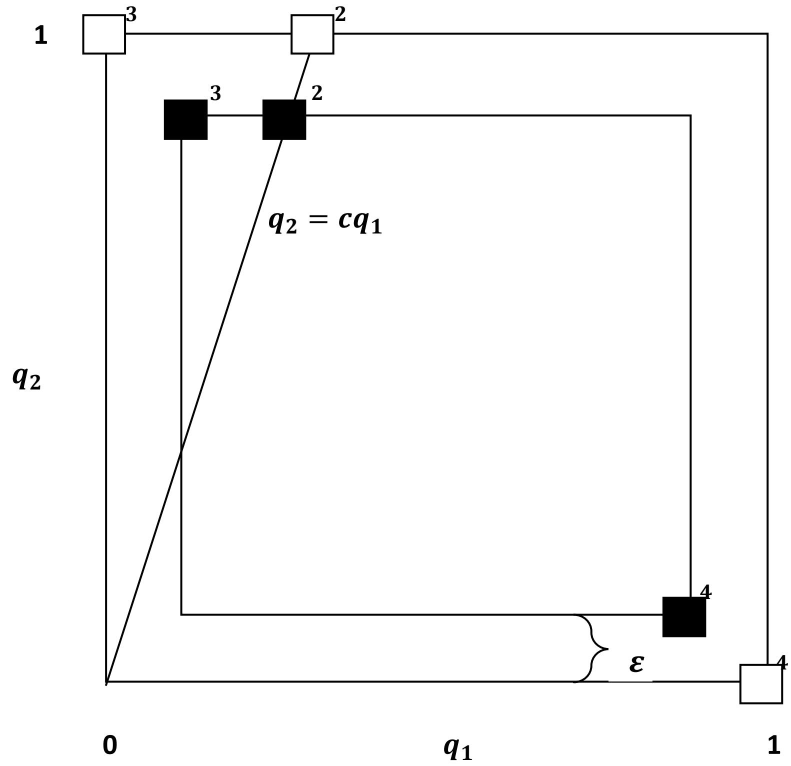

Figure 18: Best-reply correspondences of player 3 in dependence of players' 1 and 2 strategies and equilibrium points in case 5.7.

$q^2 = (1/c\,, 1, a)$, and the other equilibrium points q^2, q_ε^2, q^3, q_ε^3, q^4 and q_ε^4 are given by **(48)** to **(52)**. Since q_ε^2 is not the solution of a primitive formation of the perturbed game, the first candidate set contains only q_ε^3 and q_ε^4. Obviously, there is payoff-dominance relationship between these two equilibrium points. The risk-dominance by means of the linear tracing procedure has to resolve which of the equilibrium points is the solution of the game.

Notice that case 5.7 is similar to case 5.3 except that $a > b$ holds instead of $a < b$. The *bicentric priors* are equivalent to those given by **(55)**, **(56)** and **(61)**. Moreover, the *best replies to the bicentric priors* are exactly the same as in the analysis of case 5.3, and we omit the repetition of the formulas. However, in the case at hand we can exclude *combinations of best replies to the bicentric priors* different from those in case 5.3. The "hat" variables $\hat{a}$, $\hat{b}$ and $\hat{c}$ are defined as in **(55)**, **(56)** and **(60)**.

i. If $1/(1+\hat{c}) \leq b$ holds then $1/(1+\hat{c}) \geq a$ is impossible because $b < a$.

ii. If $1/(1+\hat{c}) \leq b$ holds then $1-\hat{b} \geq \hat{a}c$ is impossible. The first inequality implies $1-b \leq b\hat{c}$, but this is a contradiction to $1-\hat{b} \geq \hat{a}c$ because $b \cong \hat{b}$ and $\hat{a}c > b\hat{c}$ hold for sufficiently small ε.

iii. By a similar argument as above we can conclude that $1/(1+\hat{c}) \geq a$ and $1-\hat{b} \leq \hat{a}c$ cannot hold simultaneously. The first inequality implies $1-a \geq a\hat{c}$, but this is a contradiction to $1-\hat{b} \leq \hat{a}c$ because $a\hat{c} \cong \hat{a}c$ and $a > \hat{b}$ hold for sufficiently small ε.

Note that the implications for the "hat" variables given by **(62)**, **(63)** and **(65)** still matter. Now we can list the possible relations between a, b and c and the resulting vectors of best replies to the bicentric priors, still denoted by q^0.

$$\mathbf{1/(1+c)} \leq \boldsymbol{b} < a \wedge 1-b < ac \Rightarrow \boldsymbol{q^0} = (\mathbf{1-\varepsilon}, \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}) \quad \mathbf{(82)}$$

$$\boldsymbol{b} < \mathbf{1/(1+c)} \leq \boldsymbol{a} \wedge \mathbf{1-b} < ac \Rightarrow \boldsymbol{q^0} = (\mathbf{1-\varepsilon}, \mathbf{1-\varepsilon}, \boldsymbol{\varepsilon}) \quad \mathbf{(83)}$$

$$\boldsymbol{b} < \mathbf{1/(1+c)} \leq \boldsymbol{a} \wedge \mathbf{1-b} \geq \boldsymbol{ac} \Rightarrow \boldsymbol{q^0} = (\mathbf{1-\varepsilon}, \mathbf{1-\varepsilon}, \mathbf{1-\varepsilon}) \quad \mathbf{(84)}$$

$$\boldsymbol{b} < a < \mathbf{1/(1+c)} \wedge \mathbf{1-b} \geq \boldsymbol{ac} \Rightarrow \boldsymbol{q^0} = (\boldsymbol{\varepsilon}, \mathbf{1-\varepsilon}, \mathbf{1-\varepsilon}) \quad \mathbf{(85)}$$

In the situation described by **(82)** the best replies to the bicentric priors establish the equilibrium point q_ε^4, and no player has an incentive to shift his strategy along the path of the tracing procedure. $q^4 = (1,0,0)$ is the limit solution of the game. Similarly, $q^3 = (0,1,1)$ is the limit solution in the situation given by **(85)**.

In the situation described by **(83)** player 2 is the only player whose initial strategy is not a best reply to q^0. Hence, if the tracing parameter t becomes sufficiently large player 2 will shift to his other ε-extreme strategy m_{2_ε}. Then the equilibrium point q_ε^4 is reached and no further strategy changes will occur in the remaining course of the linear tracing procedure. Therefore, the limit solution in this situation is $q^4 = (1,0,0)$.

The situation described by **(84)** is similar to that of **(68)**. Now player 1's and player 3's best replies to the bicentric priors are not best replies to q^0. We have to compute their *destabilization points* t_1 and t_3 to decide *who is the first player to shift to his other strategy*.

The computation of *player 1's destabilization point* t_1 is analogously to that of t_2 in **(71)** for the situation of **(68)**. We obtain:

$$\boldsymbol{t_1 = \frac{a(1+\hat{c})-1}{(1-\varepsilon)(1+\hat{c})-1}} \qquad \textbf{(86)}$$

Since $a > 1/(1+\hat{c})$ (see **(62)** and **(84)**) the numerator is positive and since $a < 1-\varepsilon$ the numerator is smaller than the denominator. Therefore, $0 < t_1 < 1$ holds for sufficiently small ε. It follows:

$$\boldsymbol{\lim_{\varepsilon\to 0} t_1 = a - \frac{1-a}{c}} \qquad \textbf{(87)}$$

Player 3's destabilization point t_3 can be obtained by interchanging the ε's by $(1-\varepsilon)$'s in formulas **(73)** and **(74)** because now player 1 and player 2 both choose m_{1_ε} in q^o instead of m_{2_ε} as in the situation of **(68)**. Simple computations yield:

$$\boldsymbol{t_3 = \frac{1-\hat{b}-\hat{a}c}{1-\hat{b}-\hat{a}c+(1-\varepsilon)(c-1)}} \qquad \textbf{(88)}$$

In view of **(65)**, **(84)** and $c > 1$ it is clear that $0 < t_3 < 1$ holds for sufficiently small ε. However, different to **(75)** we obtain now:

$$\boldsymbol{\lim_{\varepsilon\to 0} t_3 = \frac{1-b-ac}{c-b-ac}} \qquad \textbf{(89)}$$

From **(84)** it follows that $0 \le \lim_{\varepsilon\to 0} t_1 < 1$ and that $0 \le \lim_{\varepsilon\to 0} t_3 < 1$ holds. Unfortunately we cannot identify one player who is *always* the first to shift his strategy. For example, let $a = .4,\ b = .1,\ c = 2.0$ (satisfying the conditions of **(84)**). We obtain $\lim_{\varepsilon\to 0} t_1 = 1/10 > \lim_{\varepsilon\to 0} t_3 = 1/11$. Thus, for these parameter values player 3 is the first to shift his strategy. But for $a = .34,\ b = .22,\ c = 2.0$, satisfying also the conditions of **(84)**, we obtain $\lim_{\varepsilon\to 0} t_1 = 1/100 < \lim_{\varepsilon\to 0} t_3 = 1/11$, and player 1 is the first to shift his strategy. Hence, we have to look closer at the parameters.

The condition that $\lim_{\varepsilon\to 0} t_1 = \lim_{\varepsilon\to 0} t_3$ yields the following relation among the parameters *a, b* and *c*:

$$\boldsymbol{b = \frac{2c - ac(c+1)(2-a)}{(c+1)(1-a)}} \qquad \textbf{(90)}$$

If **(90)** holds with "<" instead of "=", we obtain $\lim_{\varepsilon\to 0} t_1 < \lim_{\varepsilon\to 0} t_3$ and, therefore, $t_1 < t_3$ for sufficiently small ε. Otherwise, if **(90)** holds with ">" instead of "=", we obtain $\lim_{\varepsilon\to 0} t_1 > \lim_{\varepsilon\to 0} t_3$ and, therefore, $t_1 > t_3$ for sufficiently small ε. But now consider the case that **(90)** holds strictly. To answer the question who is the first to change his strategy we look directly at t_1 and t_3 as given by **(86)** and **(88)** and substitute the "hat" variables by their definitions in **(55)**, **(56)** and **(60)**. Some tedious definitions show that $t_1 < t_3$ holds for sufficiently small ε if and only if:

$$\boldsymbol{b < \frac{2c - ac(c+1)(2-a) + a(c+1)(3c - 2 - 2\varepsilon(c-1))}{(c+1)(1-a-\varepsilon)}} \qquad \textbf{(91)}$$

Obviously, the numerator of the right-hand side of **(90)** is greater than that of **(89)** and the denominator of the right-hand side of **(90)** is smaller than that of **(89).** Because all numerators and denominators are positive for sufficiently small ε it follows that **(89)** implies **(90)**. Hence, if **(89)** holds, player 1 is the first player to shift to his other ε-extreme strategy.

Now we must consider the consequences of a strategy shift of player 1 or player 3 along the path of the linear tracing procedure in the situation described by **(84)**. If player 1 is the first to shift his strategy the strategy combination $(\varepsilon, 1-\varepsilon, 1-\varepsilon)$, i.e. the equilibrium point q_ε^3, is reached and no further strategy changes occur afterwards in the remaining course of the linear tracing procedure.

If player 3 is the first to shift his strategy the strategy combination $(1-\varepsilon, 1-\varepsilon, \varepsilon)$ is reached. This is not an equilibrium point of the perturbed game, but now player 2 is the only player whose momentary strategy is not a best reply to the other players' momentary strategies. So player 2 is the next one who changes his strategy. Then the strategy combination $(1-\varepsilon, \varepsilon, \varepsilon)$, i.e. the equilibrium point q_ε^4, is reached und is sustained until the end of the tracing procedure.

The analysis of the situation given by **(84)** can be summarized as follows. The equilibrium point $q^3 = (0,1,1)$ is the limit solution of the game if the following holds:

$$\boldsymbol{b \leq \frac{2c - ac(c+1)(2-a)}{(c+1)(1-a)}} \qquad \textbf{(92)}$$

If **(92)** does not hold, $q^4 = (1,0,0)$ is the limit solution.

Connecting this result for **(84)** with those obtained for **(82)**, **(83)** and **(85)** we can claim:

- If *either* $1 - b \leq ac$ holds *or* if **(84)** holds but **(92)** does not hold, $q^4 = (1,0,0)$ is the limit solution in case 5.7.
- Otherwise, $q^3 = (0,1,1)$ is the limit solution of the game in case 5.7.

5.8. Case $a > b,\ A_1(r_1) = A_2(r_1) = m_2$

The final generic case is illustrated by figure 19.

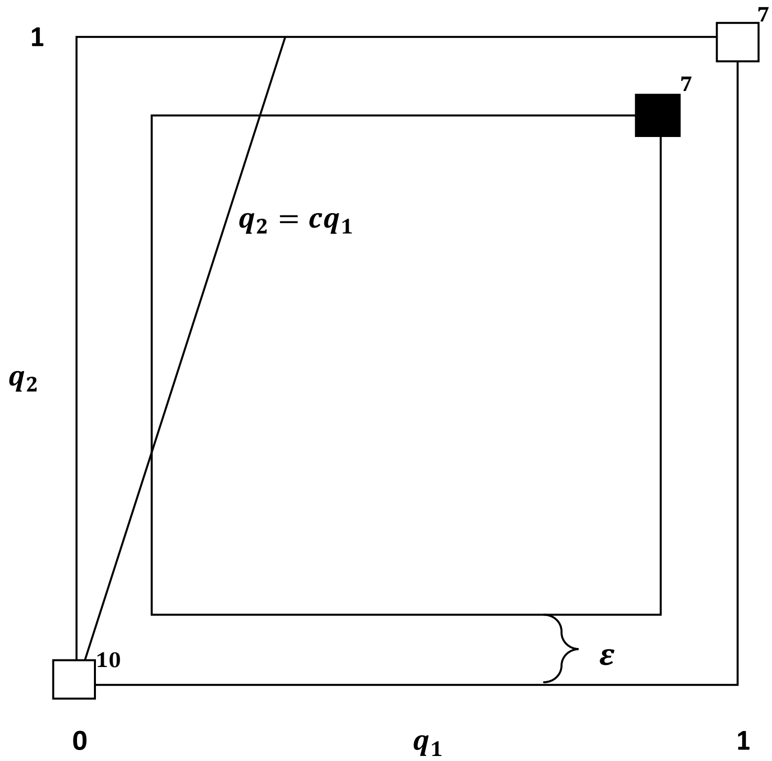

Figure 19: Best-reply correspondences of player 3 in dependence of players' 1 and 2 strategies and equilibrium points in case 5.8.

The equilibrium points q^7 and q_ε^7 are explained by **(77)** and **(78)**. Q^{10} is given as follows:

$$\boldsymbol{Q^{10} = \{(q_1, q_2, q_3) | q_1 = q_2 = 0, q_3 \geq a\}} \qquad \textbf{(93)}$$

However Q^{10} has no corresponding equilibrium points in the perturbed game. Hence, q_ε^7 is the unique equilibrium point of the perturbed game and $q^7 = (1,1,0)$ is the limit solution of the game.

6. Overview of the Results

In this section we present an overview of the limit solutions for all generic games of the class of signaling games investigated. The solutions were derived in sections 3 and 5. If someone is interested in a special game this overview can be used to pick up quickly its solution.

The first step to find the solution for a particular game is to check whether some strategy sets are semiduplicate classes or whether inferior choices exist. If this is the case the particular player forms an elementary cell and the game is decomposable and reducible (**part A** of this overview reports the results of section 3). **Part B** presents the results of the indecomposable and irreducible games calculated in section 5.

Part A: Solutions of Decomposable and Reducible Games

A1: At least the receiver forms an elementary cell.

After fixing the receiver, both types eventually form cells. When they are fixed, the solution is obtained.

The following case distinctions of part A are concerned with situations where the receiver does not initially form a cell but at least one type does.

A2: Both types form cells.

After fixing the two types, the conditional probabilities that the decision node after player 1's "*inside*" choice is reached are given in table 1. With the help of this table the receiver can easily compute his best reply and the solution is obtained. For convenience we repeat table 1 here.

Probability for player 3's left node after fixing the types		Player 2		
		Inferior choice *"inside"*	Inferior choice *"outside"*	Semiduplicate Class
Player 1	Inferior choice *"inside"*	α	$\frac{\alpha}{\alpha\varepsilon + (1-\alpha)(1-\varepsilon)}$	$\frac{2\alpha\varepsilon}{1-(1-2\varepsilon)\alpha}$
	Inferior choice *"outside"*	$\frac{\alpha(1-\varepsilon)}{\alpha(1-\varepsilon)+(1-\alpha)\varepsilon}$	α	$\frac{2\alpha(1-\varepsilon)}{1+(1-2\varepsilon)\alpha}$
	Semi-duplicate class	$\frac{\alpha}{\alpha+2(1-\alpha)\varepsilon}$	$\frac{\alpha}{\alpha+2(1-\alpha)(1-\varepsilon)}$	α

Table 1: Conditional probabilities that the node after player 1's *"inside"* choice is reached, given that the receiver observed an *"inside"* choice.

The remaining case distinctions of part A are concerned with situations where only one type forms a cell. We call him player 1.

A3: Player 1 has the inferior choice "*outside*"

Subcases	**Solution** (q_1, q_2, q_3)
$c \geq 1, b_1 < 0$	(1,1,0)
$c \geq 1, b_1 > 0$	(1,0,0)
$c < 1, b_1 < 0$	(1,c,b)
$c < 1, b_1 > 0, c_3 \geq \alpha(c_2 - c_1)/(1-\alpha)$	(1,1,1)
$c < 1, b_1 > 0, c_3 < \alpha(c_2 - c_1)/(1-\alpha)$, b+c≤ 1	(1,1,1)
$c < 1, b_1 > 0, c_3 < \alpha(c_2 - c_1)/(1-\alpha)$, b+c> 1	(1,0,0)

Table 2: Player 1 has the inferior choice "*outside*".

A4: Player 1 has the inferior choice *"inside"*

Subcases	Solution (q_1, q_2, q_3)
$b_1 > 0$	(0,1,1)
$c > 1, b_1 < 0$	(0,0,b)
$c \leq 1, b_1 < 0$	(0,0,1)

Table 3: Player 1 has the inferior choice "*inside*".

A5: Player 1's choices are semiduplicates

Subcases	Solution (q_1, q_2, q_3)
$c \geq 2, b_1 > 0$	(1/2,1,0)
$c \geq 2, b_1 < 0$	(1,/2,0,0)
$c < 2, b_1 < 0$	(1/2,c/2,b)
$c < 2, b_1 > 0, c_3 > \alpha(c_2 - c_1)/2(1 - \alpha)$	(1/2,1,1)
$c < 2, b_1 > 0, c_3 \leq \alpha(c_2 - c_1)/(1 - \alpha)$, b+$c/2 < 1$	(1/2,1,1)
$c < 2, b_1 > 0, c_3 \leq \alpha(c_2 - c_1)/(1 - \alpha)$, b+$c/2 = 1$	(1/2,c/2,b)
$c < 2, b_1 > 0, c_3 \leq \alpha(c_2 - c_1)/(1 - \alpha)$, b+$c/2 \geq 1$	(1/2,0,0)

Table 4: Player 1's choices are semiduplicates.

Part B: Solutions of Indecomposable and Irreducible Games

Cases	Solution (q_1, q_2, q_3)
B1: $a < b, A_1(r_1) = A_2(r_1) = m_1$	(0,0,0)
B2: $a < b, A_1(r_1) = m_1, A_2(r_1) = m_2$	(1/c,1,a)
B3: : $a < b, A_1(r_1) = m_2, A_2(r_1) = m_1$	
Subcase: $1 - b < ac$	(1,0,0)
Subcase: $1 - b \geq ac$	(0,1,1)
B4: $a < b, A_1(r_1) = A_2(r_1) = m_2$	(1,1,0)
B5: $a > b, A_1(r_1) = A_2(r_1) = m_1$	(0,0,0)
B6: : $a > b, A_1(r_1) = m_1, A_2(r_1) = m_2$	(0,0,b)
B7: : $a > b, A_1(r_1) = m_2, A_2(r_1) = m_1$	
Subcase: $1 - b < ac$	(1,0,0)
Subcase: $1 - b \geq ac, b < \frac{1}{1+c} \leq a,$ $b > (2c - ac(c+)(2 - a)/(c + 1)(1 - a)$	(1,0,0)
Subcase: $1 - b \geq ac, b < \frac{1}{1+c} \leq a,$ $b \leq (2c - ac(c+)(2 - a)/(c + 1)(1 - a)$	(0,1,1)
Subcase: $1 - b \geq ac, a < \frac{1}{1+c}$	(0,1,1)
B8: $a > b, A_1(r_1) = A_2(r_1) = m_2$	(1,1,0)

Table 5: Solutions of indecomposable and irreducible games.

Summary

In this paper we apply the Harsanyi-Selten solution to a class of simple signaling games. Somebody who is not familiar with the theory of Harsanyi and Selten can use this paper as an introduction and can observe different concepts and procedures at work. The overview of the results allows for easy application to economic or other models and for comparisons to the outcomes of alternative equilibrium selection criteria.

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DOES CONCESSION ALWAYS PREVENT PRESSURE?

by

Jan Potters, Frans van Winden and Michael Mitzkewitz*

1. Introduction

In Potters and Van Winden (1989) a repeated game model is used to analyze whether and under which conditions a government would be put under pressure (punished) by a pressure group for not conceding to a certain claim. In the model it is *assumed* that the punitive (aggressive) action is not used by the pressure group if the government concedes to the claim. The same kind of assumption is used in similar models. In the models of Selten (1978), Kreps and Wilson (1982) and Milgrom and Roberts (1982), for example, the monopolist cannot employ an aggressive marketing strategy if the entrant stays out, in Calvert (1987) the political leader cannot punish if the follower obeys, in Alt et al. (1988) the hegemon cannot punish if the ally obeys. In the present paper it will be examined to what extent the outcome of the game changes if this assumption is dropped, that is (in our terminology), if the pressure group is *allowed* to use the punitive strategy even if the government takes the action which is preferred by the group.

It appears that extending the strategy space in this way makes the repeated game much more complex, even in a two-period context. The number of equilibria increases significantly, and some of them are intuitively implausible. In order to reduce the set of equilibria we will employ two completely different concepts which refine the set of Nash-equilibria. First, we will use a concept which restricts "out-of-equilibrium" beliefs of a sequential equilibrium. Following Van Damme (1987, Ch.10) we will call this concept (Kohlberg-Mertens) admissibility. Secondly, we apply the selection theory of Harsanyi and Selten (1988). An additional motivation for the present paper is to see the differences and agreements beteen the two theories. We will start off with sequential equilibria, since for our purposes - the study of pressure, as defined in Potters and

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Van Winden (1989) - it is useful that the beliefs of the players and the payoffs in any single period of the repeated game are explicitly modelled.

It will be seen that, for a fairly large set of parameter values, the use of a punitive action after a concession is part of the equilibrium strategies. However, often it need not come into force because the government anticipates on that strategy and does not concede. As a consequence, the pressure group is often worse off with the extended strategy space. Also the comparative statics of the extended game are different for some admissible equilibria. Hence, the analysis indicates that the *assumption*, mentioned above, of no punishment after a concession may have an important impact on the outcome of the game.

In Section 2 the "standard" game without the possibility to punish a conceding opponent will shortly be discussed. Section 3 presents the "extended" game and the equilibria of the one period game. Section 4 and 5 present the equilibria in case of a repetition of the extended game, using the admissibility concept and Harsanyi-Selten's selection theory, respectively. Section 6 concludes.

2. The standard game

2.1 Structure of the game

In the sequel we will use an interpretation of the model which refers to a context of political pressure (cf. Potters and Van Winden (1989)). Other interpretations, such as those in the papers mentioned in the introduction, are also possible.

Assume that a firm f, at a certain point in time, asks a government g for a subsidy. The firm claims that it can invest profitably only with the subsidy. If the government does not give the subsidy (N) the firm must decide whether to punish (D, not invest) or not (I, invest). If the government subsidizes (S) it is assumed that the firm does not punish (invests). Furthermore, it is assumed that g has incomplete information about f's payoffs. The firm can be of two possible types (in the sense of Harsanyi (1967-8)). With probability 1-P the firm is of the "weak" type (type 1), which prefers to invest (not to punish) even if there is no subsidy. With probability P the firm is of the "strong" type (type 2), which is better off not to invest if there is no subsidy (e.g., because of better opportunities elsewhere). The government prefers not to subsidize if the firm would still invest. Table 1 illustrates this situation.

g's move	f's move	payoffs to g	payoffs to f
g concedes: S	f does not punish: I	0	1
g does not concede: N	f does not punish: I	a	0
	f punishes: D	a−1	b−d

with $0 < a,\ b < 1$

$$\text{and } d = \begin{cases} 1 \text{ with probability } 1-P \text{ (f is type 1, "weak")} \\ 0 \text{ with probability } P \text{ (f is type 2, "strong")} \end{cases}$$

TABLE 1. The standard game

2.2 Equilibrium analysis

It is a matter of simple calculation to show that for all $P \neq a$ there is a unique sequential (and perfect) equilibrium. The government will concede (S) if $P > a$, and will not concede (N) if $P < a$. A firm of type 1 will always invest and a firm of type 2 will invest only if g plays subsidy. The situation becomes more interesting if this game is repeated a number of times because then a weak type of firm can try to mimic the strong type by punishing the government if it does not concede, thereby exerting pressure and building a "reputation" for being strong.

The repeated game has many sequential equilibria, some of which are intuitively "implausible". In a two-fold repetition, for instance, the following is a sequential equilibrium. The government has a high initial belief $P_1 > a$ that the firm is of type 2 but does not give a subsidy in the first period ($x_1 = N$), and plays $x_2 = S$ if and only if the firm invests ($y_1 = I$). Both types of firm respond to this strategy by investing in the first period (if $b < \frac{1}{2}$). This equilibrium rests on an implausible adjustment of the belief P_2 after a deviation from the equilibrium. It is implausible because $y_1 = D$ after $x_1 = N$ should not decrease g's belief P_2 that the firm is of type 2, and therefore g should play $x_2 = S$ after $(x_1,y_1) = (N,D)$. Such equilibria, however, do not pass the test of *admissibility*. This concept requires an equilibrium to be invariant with respect to elimination of never weak best responses. Translated to our signalling game, it requires (cf. Van Damme (1987, Ch.10)) that beliefs after a deviation from the equilibrium are concentrated on the type which is "most easily" induced to make that deviation voluntary. Admissibility is a necessary, but not suffient condition for *stability* in the sense of Kohlberg Mertens' (1986). Furthermore, in our game it leads to the same equilibria as the concept of *universal divinity* by Banks and Sobel (1987).

The following notation will be useful to describe the sequential equilibrium of the repeated game. Let $x_t \in \{S,N\}$ denote g's action in period t and $y_t \in \{I,D\}$ f's action. P_t is g's belief in period t that f is of type 2. Let $\sigma_t(P_t)$ denote the probability (mixed strategy) that g plays S in period t when g's current belief is P_t. Finally, let $\rho_t^k(P_t,x_t)$ indicate the probability that f type k (= 1,2) plays I in period t when g plays x_t (= S,N) and g's belief is P_t. Beliefs are common knowledge and updated according to Bayes' rule. The following proposition is a special case of Proposition 3 in Potters and Van Winden (1989) (see also Van Damme (1987, p.297) and Kreps and Wilson (1982)).

Proposition 1
The unique admissible sequential equilibrium strategies are:

$$\sigma_t(P_t) = \begin{cases} 0 & \text{if } 0 \le P_t < a^{T-t+1} \\ 1-b & \text{if } P_t = a^{T-t+1} \\ 1 & \text{if } a^{T-t+1} < P_t \le 1 \end{cases}$$

$$\rho_t^1(P_t,x_t) = \begin{cases} 1 & \text{if } x_t = S \\ 1 - P_t(1-P_t)^{-1}(a^{-(T-t)}-1) & \text{if } x_t = N \text{ and } P_t \le a^{T-t} \\ 0 & \text{if } x_t = N \text{ and } P_t > a^{T-t} \end{cases}$$

$$\rho_t^2(P_t,x_t) = \begin{cases} 1 & \text{if } x_t = S \\ 0 & \text{if } x_t = N \end{cases} \quad \text{for all } P_t$$

Uniqueness, here and in the sequel, refers to the equilibrium *outcome* (the probability distribution over end-nodes in the extensive form of the game). Off the equilibrium path the equilibrium *strategies* need not be unique.

This - very interesting - equilibrium is extensively analyzed and discussed in Potters and Van Winden (1989). In the present paper it will (for T = 2) mainly be used for comparison with the equilibria of the extended game. The following observations are important in that respect.

2.3 Some observations

Ex ante (i.e., in expected value terms) it is never profitable to *both* players to make binding agreements to a certain course of action. The firm would (only) gain from a binding agreement to play $\rho_t^k(P_t,N) = 0$ if $P_1 < a$. The government, however, would always lose from such a commitment.

According to the definition of pressure in Potters and Van Winden (1989) - which says that pressure involves a costly action aimed at influencing the government's beliefs - the firm can only exert pressure if it is of type 1 in this equilibrium. The firm will then play $y_t = D$ if $x_t = N$ (with a positive probability). This action is costly in the short run for the firm of type 1 ($b-1 < 0$) but not for type 2 ($b > 0$).

The government concedes in a certain period t if $P_t > a^{T-t+1}$. So, the probability of concession is non-decreasing in P_t (the probability that the firm "needs" the subsidy), and non-increasing in a (the cost to g of concession). These result are intuitively very plausible.

3. The extended game

We will now extend the game of Section 2 in the sense that the firm will be allowed to punish (not invest) even if the government concedes (subsidizes). It seems reasonable to assume that the payoff (G(x,y)) to the government in case of subsidy and no investment is less than in case of no subsidy and no investment. Supplying the subsidy but still being punished is the worst alternative to g. So the payoff structure to g is:

$$G(N,I) > G(S,I) > G(N,D) > G(S,D) \quad (3.1)$$

A further reasonable assumption seems to be that the firm gains from a subsidy even if it does not invest: $F(S,D) > F(N,D)$, where $F(x,y)$ denotes the payoff to the firm. Again, it is supposed that the government has incomplete information about the firms' payoffs. With probability 1-P the firm (type 1, with payoff F^1) prefers to invest even if there is no subsidy:

$$F^1(S,I) > F^1(N,I) > F^1(S,D) > F^1(N,D) \quad (3.2)$$

With probability P the firm (type 2, with payoff F^2) prefers to invest if and only if there is a subsidy:

$$F^2(S,I) > F^2(S,D) > F^2(N,D) > F^2(N,I) \quad (3.3)$$

A parametrization of the payoffs is given in Table 2. [This parametrization can be generalized without qualitatively affecting the results, provided that payoffs satisfy the constraints (3.1)-(3.3)].

g	f	y	$G(x,y)$	$F^k(x,y)$
g concedes: S	f does not punish:	I	0	1
	f punishes:	D	-1	c-d
g does not concede: N	f does not punish:	I	a	0
	f punishes:	D	a-1	b-d

where $0 < a < 1$, $0 < b < c < 1$

$$\text{and } d = \begin{cases} 1 \text{ with probability } 1-P \text{ (f is type 1, "weak")} \\ 0 \text{ with probability } P \text{ (f is type 2, "strong")} \end{cases}$$

TABLE 2. The extended game

It is easily seen that the unique sequential equilibrium of the one-period extended game is identical to that of the standard game. In a one-period analysis the *assumption* that $y = I$ if $x = S$ has no effect on the equilibrium strategies or the expected equilibrium payoffs. Hence, the sequential equilibrium strategies in the second period of the repeated game are as follows.

$$\sigma_2(P_2) = \begin{cases} 0 & \text{if } P_2 < a \\ m & \text{if } P_2 = a \\ 1 & \text{if } P_2 > a, \text{ where } m \in [0,1] \end{cases} \tag{3.4}$$

$$\rho_2^1(P_2,x_2) = 1, \text{ for all } P_2 \text{ and } x_2 \in \{S,N\} \tag{3.5}$$

$$\rho_2^2(P_2,x_2) = \begin{cases} 0 \text{ if } x_2 = N \\ 1 \text{ if } x_2 = S \end{cases} \tag{3.6}$$

The next two sections present the equilibrium analysis of the two-period extended game.

4. Admissible equilibria of the two-period extended game

It appears that the analysis of the repeated play of the extended game is far more complex than that of the standard game. A two-fold repetition of the stage game of Table 2, however, will suffice to illustrate our arguments. In this case the normal form of the game contains 2^{25} pure strategy combinations, and if the game is reduced such that players are fixed at dominant strategies we are still left with 2^8 pure strategy combinations. There are many sequential equilibria and even more Nash equilibria, some of which are intuitively very implausible. To give a complete list of all (sequential) equilibria does not seem very useful.

We will concentrate on admissible equilibria. It will be shown that in some cases the concept of admissibility leads to a unique equilibrium, but that in other cases this concept is not strong enough to select a unique equilibrium (outcome).

It is useful for the equilibrium analysis to make a distinction between two sets of the government's initial belief P_1 that the firm is of type 2. We will deal separately with the case $P_1 < a$ and the case $P_1 > a$. [The degenerate case $P_1 = a$ will not be discussed. In the sequel we will disregard all such "knife-edge" cases.]

4.1 A low initial reputation: $P_1 < a$

It is easy to check that the strategies of Proposition 1 for the case T=2 still form a sequential equilibrium of the extended game. However, the following proposition shows that these strategies, at least for certain values of the initial belief P_1, no longer are admissible.

Proposition 2

For $P_1 < a$, admissibility requires that

$$\rho_1^1(P_1,x_1=S) = 1 \text{ and } \rho_1^2(P_1,x_1=S) = 0$$

[The proof is in Appendix A]

This proposition shows the importance of allowing f to punish ($y_1 = D$) even if g concedes ($x_1 = S$). The firm of type 2 will not invest even if the government gives the subsidy. This may seem remarkable in view of the short run cost of this strategy to f: as can be seen in Table 2, also the firm of type 2 prefers to play I if g plays S. This short run cost, however, is outweighed by the longer run benefit of this strategy. Suppose that type 2 would play $\rho_1^2(P_1,x_1) = 1$ if $x_1 = S$. Then Bayesian updating of beliefs after $x_1 = S$, $y_1 = I$ requires:

$$P_2 = \text{Prob}\{\text{f is type } 2 | x_1 = S,\ y_1 = I\} = \frac{P_1\rho_1^2}{(1-P_1)\rho_1^1 + P_1\rho_1^2} = P_1 < a$$

$P_2 < a$ would induce g to play $x_2 = N$ in period 2 and type 2 to play $y_2 = D$. The strategy ρ_1^2 in the proposition, however, leads to $P_2 = 1$ with Bayesian updating after $x_1 = S$, $y_1 = D$, and then g will play $x_2 = S$. The firm of type 2 takes a costly action in order to influence g's beliefs. According to the definition in Potters and Van Winden (1989) the firm *exerts pressure* on the government. Since $F^2(S,D) + F^2(S,I) > F^2(S,I) + F^2(N,D)$ the latter strategy is rational for the firm of type 2 but not for type 1: $F^1(S,D) + F^1(S,I) < F^1(S,I) + F^1(N,D)$. This fact causes that the strategies in proposition 1 are no longer admissible (although they still form a sequential equilibrium).

Proposition 3

For $P_1 < a$ there is a unique admissible equilibrium.

$$\sigma_1(P_1) = \begin{cases} 0 \text{ if } P_1 < \alpha(a) \quad \text{or } a < \alpha(a) \\ 1 \text{ if } \alpha(a) < P_1 < a, \quad \text{where } \alpha(a) := a^2/(1-a^2) \end{cases}$$

$$\rho_1^1(P_1,x_1) = \begin{cases} 1 & \text{if } x_1 = S \\ 1-P_1(1-P_1)^{-1}(a^{-1}-1) & \text{if } x_1 = N \end{cases}$$

$$\rho_1^2(P_1,x_1) = 0 \text{ for } x_1 \in \{S,N\}$$

The strategies in the second period, are as in equations (3.4)-(3.6) with $m = 1-b$.

A first observation that can be made is, that the government supplies the subsidy in the first period only if $\alpha(a) < P_1 < a$. [Note that this condition can only hold if $a < (\surd 5-1)/2$.] It is only in this case that the firm of type 2 will (be able to) exert pressure. In the other cases the threat, of no investment even with a subsidy, deters the government from supplying the subsidy. In these latter cases, however, the firm of type 1 will exert pressure by not investing (with a positive probability). So, whatever the government's action (concession or no concession), pressure cannot be avoided!

A simple comparison of the expected payoffs of the players of the equilibria in Proposition 3 and Proposition 1 (for $P_1 < a$) reveals the following.

Proposition 4

The government is worse off in the extended game if $a^2 < P_1 < a$, and never better off. The firm of type 1 is worse off in the extended game if $a^2 < P_1 < \min\{a,\alpha(a)\}$ and never better off. The firm of type 2 is worse off if $a^2 < P_1 < \min\{a,\alpha(a)\}$ and better off if $\alpha(a) < P_1 < a$.

Hence, for $a^2 < P_1 < \min\{a,\alpha(a)\}$ a binding agreement not to play the punitive action after a concession, would be beneficial to both players. Contrary to the standard game, in the extended game there is scope for mutually benefical binding agreements. Note that, for $a^2 < P_1 < \min\{a,\alpha(a)\}$, the firm would not reveal its type if it expressed a wish to enter such an agreement.

4.2 A high initial reputation: $P_1 > a$

Also in this case the strategy by the firm of type 2 not to invest with a positive probability if there is a subsidy is part of two admissible equilibria (denoted E^2 and E^3). A major difference with the case $P_1 < a$, however, is that now there is also a admissible equilibrium (denoted E^1) in which both types of firms invest if the subsidy is being supplied. Moreover, this equilibium E^1 payoff dominates the other admissible equilibria.

Proposition 5

For $P_1 > a$ there are three admissible equilibria, denoted E^1 and E^2 and E^3.

E^1:

$$\sigma_1(P_1) = 1,\ \rho_1^k(P_1,x_1) = \begin{cases} 0 \text{ if } x_1 = N \\ 1 \text{ if } x_1 = S \end{cases} \quad \text{for } k = 1,\ 2$$

E^2:

$$\sigma_1(P_1) = \begin{cases} 1 \text{ if } a < P_1 < \beta(a) \\ 0 \text{ if } \beta(a) < P_1 < 1, \text{ where } \beta(a) = a^2-a+1 \end{cases}$$

$$\rho_1^1(P_1,x_1) = \begin{cases} 0 \quad \text{if } x_1 = N \\ 1 \quad \text{if } x_1 = S \end{cases} \qquad \rho_1^2(P_1,x_1) = \begin{cases} 0 & \text{if } x_1 = N \\ \frac{1-P_1}{P_1} \cdot \frac{a}{1-a} & \text{if } x_1 = S \end{cases}$$

E^3:

$$\sigma_1(P_1) = \begin{cases} 1 \text{ if } a < P_1 < \gamma(a) \\ 0 \text{ if } \gamma(a) < P_1 < 1 \text{ or } \gamma(a) < a, \text{ where } \gamma(a) = 1/(1+a) \end{cases}$$

$$\rho_1^1(P_1,x_1) = \begin{cases} 0 \text{ if } x_1 = N \\ 1 \text{ if } x_1 = S \end{cases} \qquad \rho_1^2(P_1,x_1) = 0 \quad \text{for } x_1 = N,\ S$$

The second period strategies are given by (3.4)-(3.6), where for equilibrium E^2 g must play $m = (c-b)/(1-b)$ if it is indifferent ($P_2=a$).

Proposition 6

Equilibrium E^1 payoff dominates equilibrium E^2, which in turn (weakly) payoff dominates equilibrium E^3.

[The proof is by simple comparison of expected payoffs.]

The outcome of equilibrium E^1 appears to be identical to the outcome of the equilibrium of the standard game in Proposition 1. Contrary to the case $P_1 < a$ (see Proposition 2), this equilibrium is admissible now. The reason for this is that the higher value of P_1 is sufficient to ensure that the government will again supply a subsidy in period 2, even if P_2

remains unchanged. Therefore, there is no reason for the firm to exert pressure in order to raise P_2.

It is remarkable that for the equilibria E^2 and E^3 the comparative statics results of the standard game need no longer hold. A higher belief P_1 that the firm "needs" the subsidy may decrease the "probability" σ_1 that the government supplies the subsidy in the first period. This, of course, is due to the fact that type 2 does not invest (with a positive probability) if the government supplies the subsidy, which is even worse to g then not supplying the subsidy and being punished.

The players could again benefit from a binding agreement always to play the strategy $\rho_1^k(P_1,S) = 1$. However, as long as the possibility to make binding agreements is not explicitly included in the rules of the game, equilibria E^2 and E^3 cannot be excluded. Although we have not checked all the details yet, we conjecture that not only the outcome of equilibrium E^1 but also that of E^3 is stable in the sense of Kohlberg and Mertens (1986), whereas the equilibrium outcome of E^2 is not. In the next Section we will see that the strategy $\rho_1^2(P_1,S)$ which supports equilibria E^2 and E^3 is eliminated in Harsanyi-Selten's selection theory.

5. The Harsanyi-Selten solution of the two-period extended game

Because the theory of equilibrium selection by Harsanyi and Selten (1988) considers *uniformly perfect equilibria* and not sequential equilibria, we have to change some of the terminology used in the previous sections. In order to facilitate comparison of the results we will use similar notations.

Assume that the firm's agents in the second period are fixed at their dominant choices. Because we are only interested in non-degenerate cases, the avoidance of perturbations in the last stage of the game does not change the results. The truncated game has nine agents. We call government's first-period agent g_1, and $\sigma_1(P)$ is his probability (mixed strategy) to play S. The four agents of the firm in the first period are denoted by f_1^{kx}, where $k\in\{1,2\}$ indicates f's type, and $x\in\{S,N\}$ is g_1's action. The local mixed strategy of f_1^{kx} is $\rho_1^k(P,x)$ and represents his probability to play I. The four second-period agents of the government are called g_2^{xy}, where $x\in\{S,N\}$ is again g_1's action and $y\in\{I,D\}$ is the firm's action in the first period. $\sigma_2(P,x,y)$ is the mixed strategy of g_2^{xy}, the probability to choose S. Harsanyi-Selten's theory does not require posterior beliefs to be specified. Therefore, the argument P, also in the strategies for the second period, refers to the *initial* belief (which is denoted P_1 in the preceding sections).

In Appendix B we will briefly describe those elements of the Harsanyi-Selten theory which are used to calculate the solution of the two-period extended game. Appendix B also includes the proof of the following proposition.

<u>Proposition 7</u>

The Harsanyi-Selten theory prescribes the following local strategies for the agents of the two-period extended game.

$$\sigma_1(P) = \begin{cases} 0 & \text{if } P < \min\{a,\alpha(a)\} \\ 1 & \text{if } P > \min\{a,\alpha(a)\} \end{cases}$$

$$\sigma_2(P,S,I) = \begin{cases} 0 & \text{if } P < a \\ 1 & \text{if } P > a \end{cases}$$

$$\sigma_2(P,S,D) = 1$$

$$\sigma_2(P,N,I) = \begin{cases} 0 & \text{if } P < a \\ b & \text{if } P > a \end{cases}$$

$$\sigma_2(P,N,D) = \begin{cases} 1-b & \text{if } P < a \\ 1 & \text{if } P > a \end{cases}$$

$$\rho_1^1(P,S) = 1$$

$$\rho_1^2(P,S) = \begin{cases} 0 & \text{if } P < a \\ 1 & \text{if } P > a \end{cases}$$

$$\rho_1^1(P,N) = \begin{cases} 1-P(1-P)^{-1}(a^{-1}-1) & \text{if } P < a \\ 0 & \text{if } P > a \end{cases}$$

$$\rho_1^2(P,N) = 0$$

A low initial reputation: $P < a$

A comparison with the results of Propositions 2 and 3 shows that the Harsanyi-Selten theory selects exactly the unique admissible equilibrium.

It should be noted, however, that it is not always the case that a unique admissible (or stable) equilibrium and Harsanyi-Selten's solution coincide (Figure 10.9 in Harsanyi-Selten (1988,p. 351) provides a counterexample.) But, perhaps, the results of these - completely differently constructed - theories more often coincide than one might a priori expect.

A high initial reputation: P > a

The equilibrium selected by Harsanyi-Selten is close to equilibrium E^1 of Proposition 5. Differences only occur *off* the equilibrium paths (consider the strategy of agent g_2^{NI}).

Proposition 6 shows that the E^1-outcome payoff-dominates the outcomes E^2 and E^3. Although payoff-dominance plays a crucial role in Harsanyi-Selten's theory, it is not true that always a payoff-dominant equilibrium of the *whole* game is selected if such an equilibrium exists. The solution of the *whole* game is obtained by comparison of *solely* government's solution payoffs for the "concession cell" and the "non-concession cell" (see Appendix B, Lemma 3). However, the solution of the "concession cell" (see Appendix B, Lemma 1) is determined by payoff-dominance considerations in the case P > a, which indeed eliminates the strategies which sustain equilibria E^2 and E^3.

Comparative statics

If we neglect the degenerate case P = a, the Harsanyi-Selten solution reveals the following properties:

(a) for each agent of the government the probability to supply the subsidy is a nondecreasing function of P, given a, and a nonincreasing function of a, given P (recall that a indicates the cost to g of concession);

(b) for each agent of the firm's weak type (type 1) the probability to invest is a nonincreasing function of P, given a, and a nondecreasing function of a, given P;

(c) for each agent of the firm's strong type (type 2) the probability to invest is a nondecreasing function of P, given a, and a nonincreasing function of a, given P.

These - intuitively appealing - comparative statics results, are of course analoguous to those obtained for the unique admissible equilibrium in the case $P_1 < a$, and the payoff dominant admissible equilibrium E^1 in the case $P_1 > a$.

6. Conclusions

The use of the punitive strategy after a concession by the opponent, may be part of the (unique) equilibrium strategies for the strong type of firm. The short run costs of this strategy may be outweighed by a longer run benefit, because of the effect on the government's beliefs. However, this (remarkable) strategy need not come into force because of anticipation by the government - conceding but still being punished is the worst alternative to the government. Consequently, both the firm and the government can - for a significant range of parameter values - benefit from a commitment by the firm, not to use the punitive action if the government concedes. This fact, which is not visible in the standard model, may provide a rationale for government regulation (e.g., subsidies being coupled with certain conditions).

The probability that pressure is exerted increases in the extended model. Both types of the firm may now exert pressure (i.e., send a costly signal). Moreover, for certain parameter sets pressure cannot be avoided. If the government concedes the strong type of firm will exert pressure, and, if it does not concede the weak type will exert pressure (with a positive probability).

We have also shown that in the extended game the admissibility concept does not always select a unique equilibrium outcome, contrary to the standard game. For some parameter values payoff dominated equilibria survive the admissibility test. Furthermore, the (three) different admissible equilibria mutually disagree on some comparative statics results. Harsanyi-Selten's theory, by definition, selects a unique equilibrium. On the equilibrium path this equilibrium appears to be identical to the admissible equilibrium, when the latter is unique, and identical to the payoff dominant admissible equilibrium, if admissibility does not yield uniqueness (the comparative statics of these equilibria are intuitively plausible). Hence, in our - fairly complex - game there seems to be more agreement between the two theories then one might perhaps a priori expect.

APPENDIX A

First some additional notation will be introduced. $w_t^k(P_t)$, (for $t = 1,2$ and $k = g,1,2$) denotes the total expected payoffs of playing the repeated extended game in period t, for player g, f type 1, and f type 2, respectively. $P_2(P_1,x_1,y_1)$ denotes the updated (posterior) beliefs of g in period 2 as a function of the initial beliefs and the actions in period 1. Using the strategies (3.4)-(3.6) the values of $w_2^k(P_2)$, $k = g,1,2$, can easily be computed. We will use these values in the proofs of Propositions 2 and 4, without presenting them explicitly here.

Proof of Proposition 2

We will prove that $\rho_1^1(P_1,S) = 1$, $\rho_1^2(P_1,S) = 0$ for all $P_1 < a$ by contradiction.

A) Assume that $\rho_1^1(P_1,S) < 1$. This can only be an equilibrium (i.e., optimal) strategy if the total expected payoffs of playing $y_1 = I$ do not exceed those of playing $y_1 = D$: $1 + w_2^1(P_2(P_1,S,I)) \leq c-1 + w_2^1(P_2(P_1,S,D))$ or equivalently: $w_2^1(P_2(P_1,S,D)) - w_2^1(P_2(P_1,S,I)) \geq 2 - c$ (> 1). This, however, gives a contradiction since the left-hand side can never exceed 1. Hence $\rho_1^1(P_1,S) = 1$.

B) Assume $\rho_1^1(P_1,S) = 1$, $\rho_1^2(P_1,S) = 1$. Consistency with Bayes' rule then implies: $P_2(P_1,S,I) = P_1$. $P_2(P_1,S,D)$ cannot be determined by Bayes' rule since $(x_1=S,y_1=D)$ is a "zero-probability event". For the strategies to be optimal it is required that: $1 + w_2^1(P_2(P_1,S,I)) \geq c-1 + w_2^1(P_2(P_1,S,D))$ and $1 + w_2^2(P_2(P_1,S,I)) \geq c + w_2^2(P_2(P_1,S,D))$. Since $P_2(P_1,S,I) = P_1 < a$ these inequalities amount to, respectively: $w_2^1(P_2(P_1,S,D)) \leq 2 - c$ and $w_2^2(P_2(P_1,S,D)) \leq 1 + b - c$. Since $0 < b < c < 1$, the former inequality holds for any $P_2(P_1,S,D) \in [0,1]$ whereas the latter does not. Thus, only type 2 can - for certain values of $P_2(P_1,S,D)$ - be induced to play y_1= D after x_1=S voluntarily. Therefore, admissibility requires that the beliefs after the "zero-probability event" $(x_1=D,y_1=I)$ are concentrated on type 2: $P_2(P_1,S,D) = 1$. But then $w_2^2(P_2(P_1,S,D)) = 1$ which means that $\rho_1^2(P_1,S) = 1$ cannot be optimal.

C) Assume $\rho_1^1(P_1,S) = 1$, $0 < \rho_1^2(P_1,S) < 1$. Consistency implies: $P_2(P_1,S,D) = 1$ and $P_2(P_1,S,I) < P_1$. This means that $1 + w_2^2(P_2(P_1,S,I)) = 1 + b < c + w_2^2(P_2(P_1,S,D)) = c + 1$, so $\rho_1^2(P_1,S) > 0$ cannot be optimal.

Combining A, B, and C establishes the claim.

Proof of Proposition 3

First, by condradiction it will be proved under A, B and C that $\rho_1^1(P_1,N) < 1$ and $\rho_1^2(P_1,N) = 0$.

A) Assume $\rho_1^1(P_1,N) = 1$ and $\rho_1^2(P_1,N) < 1$. Consistency with Bayes' rule implies that: $P_2(P_1,N,D) = 1$ and $P_2(P_1,N,S) < P_1$. Then type 2 prefers to play $y_1 = N$ since $b + w_2^2(P_2(P_1,N,D)) = b + 1 > w_2^2(P_2(P_1,N,I))$. Hence, $\rho_1^2(P_1,N) = 0$. Constistency implies $P_2(P_1,N,I) = 0$. But then $y_1 = I$ cannot be optimal for type 1, since $w_2^1(P_2(P_1,N,I)) = 0 < b-1 + w_2^1(P_2(P_1,N,D))$.

B) Assume $\rho_1^1(P_1,N) = 1$ and $\rho_1^2(P_1,N) = 1$. Consistency implies $P_2(P_1,N,I)) = P_1$ and $P_2(P_1,N,D))$ is undetermined. For these strategies to be optimal, it must hold that,

$$w_2^1(P_2(P_1,N,I)) \geq b-1 + w_2^1(P_2(P_1,N,D)) \text{ and} \tag{A.1}$$

$$w_2^2(P_2(P_1,N,I)) \geq b + w_2^2(P_2(P_1,N,D)) \tag{A.2}$$

The first inequality holds for all $P_2(P_1,N,D))$ whereas the second can only hold if $P_2(P_1,N,D) \geq a$. So, only type 2 can be induced to play y_1 = D voluntary. Therefore, admissibility requires that the posterior beliefs are concentrated on this type: $P_2(P_1,N,D) = 1$. But then inequality (A.2) can never hold.

C) Combining A and B yields $\rho_1^1(P_1,N) < 1$. Type 1 never strictly prefers to play $y_1 = N$ after $x_1 = D$, that is, it must always hold that $w_2^1(P_2(P_1,N,I)) \leq b-1 + w_2^1(P_2(P_1,N,D))$. For this it is necessary that $P_2(P_1,N,D) > P_2(P_1,N,I)$. But then type 2 strictly prefers to play $y_1 = N$ after $x_1 = N$: $\rho_1^2(P_1,N) = 0$. This establishes the claim and also that $P_2(P_1,N,I) = 0$.

D) $\rho_1^1(P_1,N) = 0$ cannot be optimal if $P_1 < a$, since then $P_2(P_1,N,D) = P_1$ and thus the strict inequality in (A.1) would hold. Now, assume $0 < \rho_1^1(P_1,N) < 1$. Then (A.1) must hold with equality, or, equivalently $w_2^1(P_2(P_1,N,D)) = 1-b$. This can only hold if $\sigma_2(P_2) = 1-b$, which requires that g is indifferent in the second period: $P_2(P_1,N,D) = a$. To justify this posterior belief, Bayes' rule requires that

$$P_2(P_1,N,D) = \frac{P_1\rho_1^2(P_1,N)}{(1-P_1)\rho_1^1(P_1,N)+P_1\rho_1^2(P_1,N)} = a \quad \text{or} \quad \rho_1^1(P_1,N) = 1 - \frac{P_1(1-a)}{(1-P_1)a}$$

Since, of course, $\rho_1^1(P_1,N) \leq 1$, this can only hold if $P_1 < a$. If $P_1 > a$, $\rho_1^1(P_1,N) = \rho_1^2(P_1,N) = 0$ are the only admissible strategies.

E) Given the unique admissible equilibrium strategies $\rho_1^k(P_1,x_1{=}N)$, for k = 1, 2, and $P_1 < a$, the optimal response $\sigma_1(P_1)$ to these strategies for player g follow by simple calculation. The optimal strategy for player g is $\sigma_1(P_1) = 1$ if and only if $a^2/(1-a^2) < P_1 < a$. This condition, however, can never hold if $a^2/(1-a^2) > a$, that is, if $a > (\surd 5-1)/2$.

<u>Proof of Proposition 5</u>

The unique admissible equilibrium strategies $\rho_1^k(P_1,x_1{=}N) = 0$, k = 1, 2, for $P_1 > a$ follow from part D of the proof of Proposition 3. Note also that $\rho_1^1(P_1,x_1{=}S) = 1$ is a dominant strategy for type 1 (see part A of the proof of Proposition 2). We will now show that (A) $\rho_1^2(P_1,S) = 1$, (B) $\rho_1^2(P_1,S) = [(1-P_1)a]/[P_1(1-a)]$, and (C) $\rho_1^2(P_1,S) = 0$ are part of the admissible sequential equilibria E^1, E^2, and E^3, respectively.

A) Suppose $\rho_1^2(P_1,S) = 1$ is an equilibrium strategy. Consistency with Bayes' rule requires: $P_2(P_1,S,I) = P_1$. (Since $P_1 > a$ this implies $\sigma_2(P_2(P_1,S,I)) = 1$.) $P_2(P_1,I,D)$ cannot be updated according to Bayes' rule, but, whatever its value neither type of player f can be induced to play $y_1 = D$ after $x_1 = S$: $1 + w_2^1(P_2(P_1,S,I)) = 2 > c-1 + w_2^1(P_2(P_1,S,D))$ and $1 + w_2^2(P_2(P_1,S,I)) = 2 > c + w_2^2(P_2(P_1,S,D))$. Since any posterior belief $P_2(P_1,I,D)$ supports the equilibrium it is admissible.

B) Suppose $\rho_1^2(P_1,S) = [(1-P_1)a]/[P_1(1-a)]$. Consistency with Bayes' rule implies: $P_2(P_1,S,D) = 1$, and $P_2(P_1,S,I) = P_1\rho_1^2(P_1,S)/[P_1\rho_1^2(P_1,S)+(1-P_1)\rho_1^1(P_1,S)] = a$. This makes g indifferent in period 2 after $(x_1,y_1) =$

(S,I). To justify type 2's randomization after x_1=S, $\sigma_2(a)$ = m must be such that type 2's expected payoffs of playing y_1 = D and y_1 = I are equal: c+1 = m2 + (1-m)(b+1). Solving for m yields m = (c-b)/(1-b). There are no zero-probability events in this sequential equilibrium, so it is admissible.

C) Suppose $\rho_1^2(P_1,S) = 0$. Consistency requires $P_2(P_1,S,I) = 0$ and $P_2(P_1,I,D) = 1$. Given these beliefs playing y_1 = I after x_1 = S is optimal for type 1: $1 + w_2^1(P_2(P_1,S,I)) = 1 > c-1 + w_2^1(P_2(P_1,S,D)) = c$, and playing y_1 = D after x_1 = S is optimal for type 2: $1 + w_2^2(P_2(P_1,S,I)) = 1 + b < c + w_2^2(P_2(P_1,S,D)) = 1 + c$. Since there can be no zero-probability events in this sequential equilibrium it also is admissible.

D) The optimal strategies $\sigma_1(P_1)$ in reponse to the strategies $\rho_1^k(P_1,x_1)$ in equilibria E^1, E^2, and E^3 follow by simple calculation. [In equilibrium E^3 the optimal strategy for g is $\sigma_1(P_1) = 1$ if $a < P_1 < 1/(1+a)$. This condition, however, can never hold if $a > 1/(1+a)$, that is, if $a > (\sqrt{5}-1)/2$.]

APPENDIX B

Before proving Proposition 7, some of the concepts and procedures from Harsanyi-Selten's selection theory (HS), used in the proof, will be presented.

Uniformly perfect equilibria. Perturb a game such that each agent has to play each of his choices with probability not smaller than ε. The limit equilibria of this perturbed game as ε goes to zero are the uniformly perfect equilibria of the original game. HS requires the solution to be uniformly perfect.

Cell-consistency. A group of agents forms a cell, if for each of these agents the strategic situation only depends on the other members of the group. In the two-period extended game the four agents f_1^{1S}, f_1^{2S}, g_2^{SI}, and g_2^{SD}, which come into play in case g_1 chooses S, form a cell which we call the "concession cell". The "non-concession cell" is formed by the agents f_1^{1N}, f_1^{2N}, g_2^{NI}, and g_2^{ND}. HS has the property that the solution of the game and the solution of a cell always prescribe the same local strategies to all agents in the cell.

This cell consistency enables us to proceed as follows. First, the solution of the concession cell (Lemma 1) and of the non-concession cell (Lemma 2) are calculated. The eight involved agents are fixed at their solution strategies and the optimal behavior of agent g_1 is determined afterwards (Lemma 3).

Elimination of inferior choices. If an agent has a weakly dominated choice, he is fixed at his other choice. In our game each agent has only two possible choices. This means that the dominating choice is played with probability $1-\varepsilon$. To meet HS's terminology we will use the term "inferior choice" instead of "weakly dominated choice". In our game there is no difference between the two concepts (see HS, p. 118).

Solution of 2x2-games. If a 2x2-game has two pure strategy equilibria and one of them is *payoff dominant*, this is the one selected by HS. If no payoff dominance exists, let (q_1, q_2) be the mixed strategy equilibrium, with q_i player i's probability to play the strategy belonging to the, say, first pure equilibrium. If $q_1+q_2 < 1$ then the first pure equilibrium, if $q_1+q_2 > 1$ then the other pure equilibrium, and if $q_1+q_2 = 1$ then the mixed equilibrium is selected as the solution of the game. This is not an ad hoc rule, but the consequence of an axiomatic approach (see HS, p. 86). Also the *risk dominance* comparison between the two pure equilibria using the *tracing procedure* yields the same result.

Now we are well-prepared to calculate the Harsanyi-Selten solution of the two-period extended game.

Lemma 1

The HS solution for the concession cell is:

$$\rho_1^1(P,S) = 1, \quad \rho_1^2(P,S) = \begin{cases} 0 \text{ if } P < a \\ 1 \text{ if } P > a \end{cases}$$

$$\sigma_2(P,S,D) = 1, \; \sigma_2(P,S,I) = \begin{cases} 0 \text{ if } P < a \\ 1 \text{ if } P > a \end{cases}$$

Proof

Strategies in the perturbed game are indicated by a lower index ε. Agent f_1^{1S} has the inferior choice D since this choice yields him at most c, whereas I yields at least 1. So he is fixed at $\rho_{1\varepsilon}^1(P,S) = 1-\varepsilon$ in the perturbed game. Now consider agent g_2^{SD}. He has the best reply S if:

$$-\varepsilon(1-P)-P(1-\rho_{1\varepsilon}^2(P,S)) \geq \varepsilon(1-P)(a-1)+P(1-\rho_{1\varepsilon}^2(P,S))(a-2)$$

or, equivalently, if:

$$1-\rho_{1\varepsilon}^2(P,S) \geq \frac{(1-P)a}{P(1-a)}\,\varepsilon \tag{B.1}$$

Suppose $P > a$. Then the right-hand side of (B.1) is smaller than ε and the left-hand sight is at least ε. Thus, if $P>a$ (B.1) always holds. Consequently, g_2^{SD} has the inferior choice N and is fixed at $\sigma_{2\varepsilon}(P,S,D) = 1-\varepsilon$. The remaining 2x2-game between f_1^{2S} and g_2^{SI} has two pure strategy equilibria (I,S) and (D,N). The first equilibrium yields the (unperturbed) payoffs 2 for f and 0 for g. The second yields 1+cP for f and a-P-aP for g. Since P>a, the first equilibrium payoff dominates the second one. Thus, $\rho^2_{1\varepsilon}(P,S) = \sigma_{2\varepsilon}(P,S,I) = 1-\varepsilon$ is obtained.

Suppose P < a. Now (B.1) does not always hold. But consider agent g_2^{SI}. He has the best reply N if: $(1-\varepsilon)(1-P)a + P\rho^2_{1\varepsilon}(P,S) \geq 0$, or, equivalently, if:

$$\rho^2_{1\varepsilon}(P,S) \leq \frac{(1-P)a}{P(1-a)}(1-\varepsilon) \qquad (B.2)$$

Since P<a, the right-hand side of (B.2) is always greater than $1-\varepsilon$ and the left-hand side is at most $1-\varepsilon$. This implies (B.2). Hence, agent g_2^{SI} is fixed at $\sigma_{1\varepsilon}(P,S,I) = \varepsilon$. The remaining 2x2-game between f_1^{2S} and g_2^{SD} has two pure equilibria, namely (I,N) and (D,S). (The first of which is only strong in the perturbed game.) Because there is no payoff dominance between them, we consider the mixed equilibrium of the perturbed game, which is given by $\rho^2_{1\varepsilon}(P,S) = 1 - a(1-P)[P(1-a)]^{-1}\varepsilon$ and $\sigma_{2\varepsilon}(P,S,D) = (1-c)(1-b)^{-1} + \varepsilon$. For sufficiently small ε, the sum of the probabilities that f_1^{2S} chooses D and that g_2^{SD} chooses S is strictly less than 1. According to the properties mentioned in the introduction of this appendix (D,S) is selected as the solution of the 2x2-game. We obtain $\rho^2_{1\varepsilon}(P,S) = \varepsilon$ and $\sigma_{2\varepsilon}(P,S,D) = 1-\varepsilon$, if P<a.

By letting ε go to zero the claim of the Lemma follows immediately.

<u>Lemma 2</u>

The HS solution for the non-concession cell is as follows:

$$\rho^1_1(P,N) = \begin{cases} 1 - P(1-P)^{-1}(a^{-1}-1) & \text{if } P < a \\ 0 & \text{if } P > a \end{cases}$$

$$\rho^2_1(P,N) = 0$$

$$\sigma_2(P,N,I) = \begin{cases} 0 & \text{if } P < a \\ b & \text{if } P > a \end{cases}$$

$$\sigma_2(P,N,D) = \begin{cases} 1-b & \text{if } P < a \\ 1 & \text{if } P > a \end{cases}$$

<u>Proof</u>

We start with the determination of the best replies.

Agent f_1^{1N} has the best reply I if:

$$\sigma_{2\varepsilon}(P,N,D) \leq \sigma_{2\varepsilon}(P,N,I) + 1 - b \qquad (B.3)$$

Agent f_1^{2N} has the best reply I if:

$$\sigma_{2\varepsilon}(P,N,D) \leq \sigma_{2\varepsilon}(P,N,I) - b/(1-b) \qquad (B.4)$$

Agent g_2^{NI} has the best reply S if:

$$\rho^2_{1\varepsilon}(P,N) \geq (1-P)a[P(1-a)]^{-1}\rho^1_{1\varepsilon}(P,N) \qquad (B.5)$$

Ag nt g_2^{ND} has the best reply S if:

$\rho^2_{1\varepsilon}(P,N) \leq (1-P)a[P(1-a)]^{-1}\rho^1_{1\varepsilon}(P,N) + (P-a)[P(1-a)]^{-1}$ (B.6)

Suppose $b \geq 1/2$. In that case (B.4) never holds, so agent f_1^{2N} has the inferior choice I and is fixed at $\rho^2_{1\varepsilon}(P,N)=\varepsilon$. Consider the subcase P>a. After fixing f_1^{2N}, (B.6) always holds for sufficiently small ε. Consequently, g_2^{NI} is fixed at $\sigma_{2\varepsilon}(P,N,D)=1-\varepsilon$. The remaining 2x2-game between f_1^{1N} and g_2^{NI} has a unique equilibrium given by:

$\rho^1_{1\varepsilon}(P,N) = P(1-a)[(1-P)a]^{-1}\varepsilon$ and $\sigma_{2\varepsilon}(P,N,I) = b-\varepsilon$.

Now, consider the subcase $b \geq 1/2$ and P<a. After fixing f_1^{2N}, (B.5) could never hold, so agent g_2^{NI} has the inferior choice S and is fixed at $\sigma_{2\varepsilon}(P,N,I)=\varepsilon$. The remaining 2x2-game between agents f_1^{1N} and g_2^{ND} has the following unique equilibrium:

$\rho^1_{1\varepsilon}(P,N) = (a-P)[a(1-P)]^{-1} + P(1-a)[a(1-P)]^{-1}\varepsilon$ and $\sigma_{2\varepsilon}(P,N,D) = 1-b+\varepsilon$.

Letting ε go to zero, the claim follows immediately for the case $b \geq 1/2$.

Now consider the more difficult case $b < 1/2$ in which no inferior choices exist. Figure B1 illustrates (B.3) and (B.4) if $b<1/2$. The letters above and below the indifference curves indicate the best reply regions for the respective choices of the agent associated with this curve. The inner square corresponds to the ε-perturbed game. Similarly, Figures B2 and B3 illustrate (B.5) and (B.6), but here we have to distinguish between P>a and P<a.

Since there are no intersections of the indifference curves, we can conclude that *both* agents of a player cannot be indifferent simultaneously. From the fact that no indifference curve intersects with one of the corners of the inner square (perturbed game) it follows that it is impossible that one agent is indifferent while none of the agents of the other players is indifferent, because then they would have to play a "corner" strategy combination. Consequently, in equilibrium points of the perturbed game either no agent is indifferent (all four agents play ε-extreme strategies) or exactly two agents (one of each player) are indifferent. With the help of this conclusion we can list all strategy combinations for g's agents, the best replies against these for f's agents (with the help of Figure B1), and the best replies against these best replies for g's agents (with the help of Figures B2 and B3). If the "last" best replies are equal to the initial strategy combination, an equilibrium of the perturbed game is found. Tables B1 and B2 present the lists. The symbol "i" is used for a completely mixed strategy of an indifferent agent, chosen appropriately to make one agent of the other player indifferent too.

$(\sigma_{2\varepsilon}(.,I),\sigma_{2\varepsilon}(.,D))$		$(\rho^1_{1\varepsilon}(.),\rho^2_{1\varepsilon}(.))$		$(\sigma_{2\varepsilon}(.,I),\sigma_{2\varepsilon}(.,D))$	
$(1-\varepsilon,1-\varepsilon)$	→	$(1-\varepsilon,\ \varepsilon)$	→	$(\varepsilon,1-\varepsilon)$	no equilibrium
$(1-\varepsilon,\ i)$	→	$(1-\varepsilon,\ i)$	→	$(i,1-\varepsilon)$	no equilibrium
$(1-\varepsilon,\ \varepsilon)$	→	$(1-\varepsilon,1-\varepsilon)$	→	$(1-\varepsilon,1-\varepsilon)$	no equilibrium
$(i,1-\varepsilon)$	→	$(i,\ \varepsilon)$	→	$(i,1-\varepsilon)$	EQUILIBRIUM
$(i,\ \varepsilon)$	→	$(1-\varepsilon,\ i)$	→	$(i,1-\varepsilon)$	no equilibrium
$(\varepsilon,1-\varepsilon)$	→	$(\varepsilon,\ \varepsilon)$	→	$(1-\varepsilon,1-\varepsilon)$	no equilibrium
$(\varepsilon,\ i)$	→	$(i,\ \varepsilon)$	→	$(i,1-\varepsilon)$	no equilibrium
$(\varepsilon,\ \varepsilon)$	→	$(1-\varepsilon,\ \varepsilon)$	→	$(\varepsilon,1-\varepsilon)$	no equilibrium

Table B1. Equilibrium analysis of the perturbed non-concession cell if b < 1/2 and P > a (see Figures B1 and B2).

Table B1 demonstrates that if b<1/2 and P>a the non-concession cell has a unique uniformly perfect equilibrium with $\sigma_{2\varepsilon}(P,N,D) = 1-\varepsilon$ and $\rho^2_{1\varepsilon}(P,N) = \varepsilon$. The specific values for the mixed strategies of agents f_1^{1N} and g_2^{NI} are obtained from (B.3) and (B.5): $\rho^1_{1\varepsilon}(P,N) = P(1-a)[(1-P)a]^{-1}\varepsilon$ and $\sigma_{2\varepsilon}(P,N,I) = b-\varepsilon$. This unique uniformly perfect equilibrium (hence, the solution) is indicated in Figures B1 and B2 by black points.

$(\sigma_{2\varepsilon}(.,I),\sigma_{2\varepsilon}(.,D))$		$(\rho^1_{1\varepsilon}(.),\rho^2_{1\varepsilon}(.))$		$(\sigma_{2\varepsilon}(.,I),\sigma_{2\varepsilon}(.,D))$	
$(1-\varepsilon,1-\varepsilon)$	→	$(1-\varepsilon,\ \varepsilon)$	→	$(\varepsilon,1-\varepsilon)$	no equilibrium
$(1-\varepsilon,\ i)$	→	$(1-\varepsilon,\ i)$	→	$(\varepsilon,\ i)$	no equilibrium
$(1-\varepsilon,\ \varepsilon)$	→	$(1-\varepsilon,1-\varepsilon)$	→	$(\varepsilon,\ \varepsilon)$	no equilibrium
$(i,1-\varepsilon)$	→	$(i,\ \varepsilon)$	→	$(\varepsilon,\ i)$	no equilibrium
$(i,\ \varepsilon)$	→	$(1-\varepsilon,\ i)$	→	$(\varepsilon,\ i)$	no equilibrium
$(\varepsilon,1-\varepsilon)$	→	$(\varepsilon,\ \varepsilon)$	→	$(\varepsilon,\ \varepsilon)$	no equilibrium
$(\varepsilon,\ i)$	→	$(i,\ \varepsilon)$	→	$(\varepsilon,\ i)$	EQUILIBRIUM
$(\varepsilon,\ \varepsilon)$	→	$(1-\varepsilon,\ \varepsilon)$	→	$(\varepsilon,1-\varepsilon)$	no equilibrium

Table B2. Equilibrium analysis of the perturbed non-concession cell if b < 1/2 and P < a (see Figures B1 and B3).

Likewise, there is a unique uniformly perfect equilibrium if P<a, as Table B2 shows. Here $\sigma_{2\varepsilon}(P,N,I) = \varepsilon$ and $\rho^2_{1\varepsilon}(P,N) = \varepsilon$. (B.3) and (B.6) yield: $\rho^1_{1\varepsilon}(P,N) = (a-P)[a(1-P)]^{-1}+P(1-a)[a(1-P)]^{-1}\varepsilon$ and $\sigma_{2\varepsilon}(P,N,D) = 1-b+\varepsilon$. Figures B1 and B3 indicate this equilibrium by black squares.

Hence, there is no difference between the solutions strategies for the cases b ≥ 1/2 and b < 1/2. Let ε go to zero and the claim of the Lemma follows immediately. □

<u>Lemma 3</u>

The HS solution prescribes the following strategy for agent g_1

$$\sigma_1(P) = \begin{cases} 0 \text{ if } P < \min\{a,\alpha(a)\} \\ 1 \text{ if } P > \min\{a,\alpha(a)\} \end{cases}$$

Proof

Easy calculation shows that g's solution payoffs in the concession cell are 0 if P > a and a-P(1+a) if P < a. In the non-concession cell g obtains a-1 if P > a and $2a-P(1+a^{-1})$ if P < a. Consequently, agent g_1 chooses S if P > a. If P < a, agent g_1 prefers S only if P > $\alpha(a)$, as simple computation shows.

Proof of Proposition 7

The proposition summarizes the results of Lemmata 1, 2, and 3.

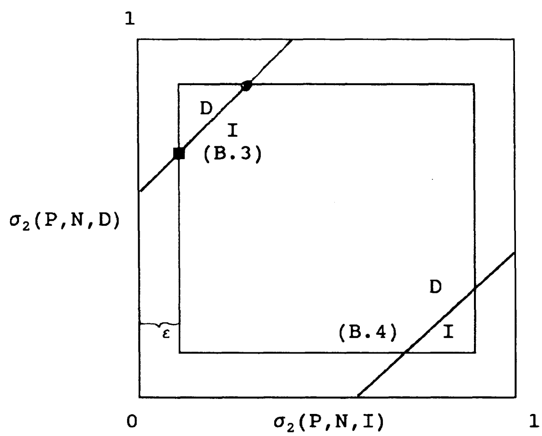

Figure B1. Example (b=1/3) for the best reply structures of agents f_1^{1N} and f_1^{2N} if b < ½.

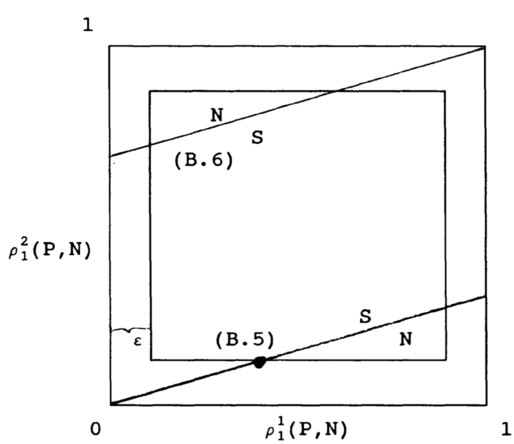

Figure B2. Example (P=2/3,a=1/3) for the best reply structures of agents g_2^{NI} and g_2^{ND} if P > a.

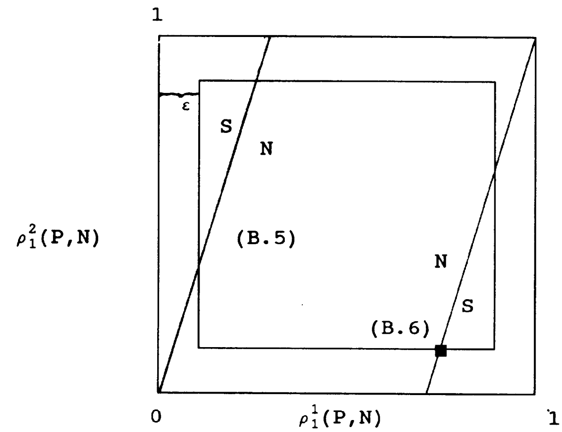

Figure B3. Example (P=1/3,a=2/3) for the best reply structures of agents g_2^{NI} and g_2^{ND} if $P < a$.

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