Existence, multiplicity and behaviour of solutions of some elliptic differential equations of higher order

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Zusammenfassung

Wir sind an Problemen interessiert, die mit der Existenz, Multiplizität, Positivität und dem Verhältnis von Lösungen von elliptischen partiellen Differentialgleichungen zweiter und höherer Ordnung zu tun haben.

In allgemeinem erfüllen Probleme in der Form $(-\Delta)^m u = f$ in $\Omega \subset \mathbb{R}^2$, $\partial^j/(\partial\nu)^j u = 0$ auf $\partial\Omega$, mit $m > 1$, $0 \leq j \leq m-1$ weder die Maximumprinzipien noch die Positivitätserhaltungseigenschaft. Wir werden zeigen, dass die Positivitätserhaltungseigenschaft für Gebiete erfüllt wird, die zu einer Scheibe nah sind.

Dann werden wir einige Ergebnisse von Existenz und Multiplizität von Lösungen des Steklov Problems von zweiter und vierter Ordnung darstellen.

Abschließend werden wir die singulären radialen Lösungen von $\Delta^2 u = \lambda e^u$ in der Einheitsscheibe mit den Randbedingungen $u = \partial u/\partial\nu = 0$ charakterisieren. Wir werden zeigen, dass diese Lösungen schwach singulär sind, das heißt, dass $\lim_{r\to 0} ru'(r) \in \mathbb{R}$ existiert.

Abstract

We are interested in questions related with existence, multiplicity, positivity and behaviour of solutions of elliptic boundary value problems of second and higher order.

In general problems $(-\Delta)^m u = f$ in $\Omega \subset \mathbb{R}^2$, $\partial^j/(\partial\nu)^j u = 0$ on $\partial\Omega$, where $m > 1$, $0 \leq j \leq m-1$ do not satisfy a maximum principle or the positivity preserving property. We will show that for domains near to a circle positivity preserving property is satisfied.

Then we will give some results of existence and multiplicity of solutions of the Steklov problem of second and fourth order.

Finally we will characterize singular radial solutions of $\Delta^2 u = \lambda e^u$ in the unit disk, with boundary conditions $u = \partial u/\partial\nu = 0$. We will show that its radial singular solutions are weakly singular, it means $\lim_{r\to 0} ru'(r) \in \mathbb{R}$ exists.

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1 Introduction

We are interested in questions related with existence, multiplicity, positivity and behaviour of solutions of boundary value problems of the kind

$$\begin{cases} (-\Delta)^m u = f(u) & \text{in } \Omega, \\ \left(\frac{\partial}{\partial\nu}\right)^j u = 0 & \text{on } \partial\Omega, \quad \text{for } j = 0, \dots, m-1 \end{cases}$$

and related eigenvalue problems. Here is $\Omega \subset \mathbb{R}^n$ a sufficient smooth domain with external normal unitary vector ν; $n, m \in \mathbb{N}$.

Many techniques familiar from second order equations do not extend even to biharmonic equations, we just mention any form of a strong maximum principle. We think that it is this reason that - up to now - the theory of higher order nonlinear elliptic problems is by far less well developed than the theory of second order elliptic equations.

On the other hand, significant progress has been achieved in the past years, as far as e.g. comparison principles [40], positivity preserving properties, existence for semilinear biharmonic problems [32, 27] are concerned.

Among these questions we shall address the following

- For which domains do polyharmonic problems with homogeneous boundary conditions assume positive solutions?
- When do exist solutions for the Steklov problem?
- Which is the behaviour of critical solutions for the nonlinear biharmonic eigenvalue problem with exponential growth?

In what follows we sketch in which direction the mentioned questions are investigated in the present thesis.

1.1 Positivity in perturbations of the two dimensional disk

Strong maximum principles are known for elliptic equations of second order, it means, given a linear elliptic differential operator of the form

$$Lu = a_{ij}(x)D_{ij}u + b_i(x)D_iu + c(x)u$$

with coefficients a_{ij}, b_i, c, where $i, j = 1 \dots, n$ defined on a bounded domain $\Omega \subset \mathbb{R}^n$, with the matrix $[a_{ij}]$ symmetric positive everywhere in Ω, which smallest eigenvalue is $\lambda(x)$, such that $\frac{|b_i(x)|}{\lambda(x)} \leq const \leq \infty$ is satisfied for every $i = 1, \ldots, n$, $c = 0$ and $Lu \geq 0$, if u achieves its maximum in the interior of Ω, then u is constant (see [35, Theorem 3.5])

For elliptic equations of higher order, the principle is not more available, like for the polyharmonic function $\tilde{u} := -|x|^2 + 1$ shows: on the domain $B = \{x \in \mathbb{R}^n : |x| \leq 1\}$ we have that $\Delta^2 \tilde{u} = 0$ on B, but $\sup_B \tilde{u} = 1$ is achieved in the interior of Ω.

For more than one century mathematicians are asking, if and when maximum and comparison principles can be extended to problems of higher order, for example in order to study the physical problem of the clamped plate: an elastic horizontally clamped plate $\Omega \subset \mathbb{R}^2$ subject to a vertical force f is described by the system

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \frac{\partial}{\partial \nu} u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

We could suppose that, in reasonable regular domains, with a positive load on the plate (it means with $f \geq 0$), then the complete body should move up, like conjectured Boggio [12] in 1901 or Hadamard [44] in 1908. This hypothesis is correct in the case of Ω equal to a ball B, like Boggio [13] proved. Even in the more general case, with $\Omega = B \subset \mathbb{R}^n$ and substituting Δ^2 with $(-\Delta)^m$. In [13] (see also [37]), positivity of the Green function on the ball B was shown. But in 1909 Hadamard [45] displayed that in an annulus with small inner radius, the solution u could be negative, also if $f \geq 0$. Even assuming convexity for the domain Ω is not enough to prove the positivity of the solution. Duffin [25] in 1949 was the first to disprove this conjecture in an unbounded domain, then were found other examples of convex domains in which, for suitable $f \geq 0$, the solution changes sign, like in [19, 20, 47, 54, 57, 63, 67]. In [31] is proved that the Green function for (1) changes sign in oblong ellipses, Coffman and Duffin obtained the same result in the case of a square.

But the circle is not the only domain that guarantees positivity for the Green function for the clamped plate, like was explained by Grunau and Sweers in [42]. Their work proved that if the domain is sufficiently near to a disk in $\mathbb{R}^2$ in a certain sense, then $0 \not\equiv f \geq 0 \Rightarrow u \geq 0$. In Section 2 we will relax the required notion of closeness: it will be enough that the two-dimensional domain has a curvature close to a constant in $C^{0,\alpha}$ and no more in C^2.

Our results are restricted to two dimensions, because we will work with conformal maps: in $\mathbb{R}^2$ the conformal maps are the holomorphic functions with non-zero derivative in $\mathbb{C}$ and we can use a suitable bjiective conformal function that maps the domain Ω onto a unitary disk. In $\mathbb{R}^n$, with $n \neq 2$, the conformal functions map balls onto another ball, it means we can't find any bjiective conformal function from Ω onto a unitary ball.

1.2 Steklov boundary eigenvalue problems

Elliptic problems with parameters in the boundary conditions are called Steklov problems from their first appearance in [64]. The system

$$\begin{cases} \Delta^2 u = g & \text{in } \Omega, \\ u = \Delta u - (1-\sigma)\kappa u_\nu = 0 & \text{on } \partial\Omega \end{cases} \tag{2}$$

is interesting for its physical applications: when Ω is a planar domain with smooth boundary, (2) describes the deformation of a linear elastic supported plate Ω under the action of a vertical load $g = g(x)$ is described by (2), where κ is the curvature of its boundary and $\sigma \in (-1, 1/2)$ is the Poisson ratio, a measure for the transversal expansion or contraction when the material is under the load of an external force. The Poisson ratio is given by the negative transverse strain divided by the axial strain in the direction of the stretching force. We refer to [51, 69] for more details. There are some materials (see [51]) which have a negative Poisson ratio. This problem is connected to the eigenvalue problem

$$\begin{cases} \Delta^2 u = 0 & \text{in } \Omega, \\ u = \Delta u - \delta u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

Moreover, as pointed out by [49], the least positive eigenvalue δ_1 of (3) is the sharp constant for a priori estimates for the Laplace equation

$$\begin{cases} \Delta v = 0 & \text{in } \Omega, \\ v = g & \text{on } \partial\Omega, \end{cases}$$

where $g \in L^2(\Omega)$.

The boundary conditions of (3) are in some sense intermediate between Dirichlet conditions (corresponding to $\delta = -\infty$) and Navier conditions (corresponding to $\delta = 0$). Berchio, Gazzola, Mitidieri in [8] had shown that, for suitable values of δ, (3) enjoys of the positivity preserving property.

In Section 3 we will study some Steklov problems of second and fourth order.

1.3 Semilinear biharmonic eigenvalue problems with exponential growth

For many years nonlinear second order elliptic problems have been studied in bounded and unbounded domains, looking for existence and multiplicity of solutions, using many different techniques, like variational and topological methods.

The Gelfand problem

$$\begin{cases} -\Delta u = \lambda e^u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where Ω is a bounded smooth domain in $\mathbb{R}^n$ and λ a nonnegative parameter, was first considered in 1853 by Liouville in [53] for the case $n = 1$, then by Bratu in [14] for $n = 2$ and by Gelfand in [34] for $n \geq 1$. For this reason is also known as Liouville-Gelfand problem and as Bratu-Gelfand problem.

It has been deeply studied for its applications, like in the Chandrasekhar model for the expansion of the universe (see [18]), or for the connection with combustion problem, for example with the quasilinear parabolic problem of the solid fuel ignition model

$$\begin{cases} u_t = \Delta u + \lambda(1 - \varepsilon u)^m e^{(u/(1+\varepsilon u))}, & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

Equation (5) describes the thermal reaction process in a combustible non-deformable material of constant density during the ignition period, where u is the temperature, $1/\varepsilon$ is the activation energy, λ is the Frank-Kamenetskii parameter, a parameter determined by the reactivity of the reactants. The system answers to the question to model a combustible medium placed in a vessel whose walls are mantained at a fixed temperature, see [29]. Nontrivial solutions of (4) arise as steady-state solutions of (5), with the approximation $\varepsilon \ll 1$.

Problem (4) may have both unbounded (singular) and bounded (regular) solutions ([16, 30]) and from the works [15, 23] we know, there exists a $\lambda^\star > 0$ such that for $\lambda > \lambda^\star$ there is not any solution of (4) and for $0 \leq \lambda < \lambda^\star$ there exists a minimal regular solution U_λ for (4) and the map $\lambda \mapsto U_\lambda$ is smooth and increasing.

The study of fourth order equations has often a physical application, as it is explained in [59]: they can model cellular flows, water waves driven by gravity and capillarity or travelling waves in suspension bridges.

In order to gain a better comprehension of the behaviour of fourth order equations, we study the problem

$$\begin{cases} \Delta^2 u = \lambda e^u & \text{in } B, \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial B; \end{cases} \tag{6}$$

here B denotes the unit ball in $\mathbb{R}^n$ ($n \geq 5$) centered at the origin and $\frac{\partial}{\partial \nu}$ the differentiation with respect to the exterior unit normal, i.e. the radial direction. $\lambda \geq 0$ is a parameter.

In particular, we will characterize the behaviour of critical solution of (6) near the origin, extending the results obtained by Arioli, Gazzola, Grunau, Mitidieri ([6]), using techniques of Ferrero, Grunau ([27]).

Simultaneously and independently Davila, Dupagne, Guerra, Montenegro obtained quite similar results by different techniques, in [21].

1.4 Acknowledgment

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2 Positivity in perturbations of the two dimensional disk

2.1 Introduction

We are looking for positivity preserving property for the polyharmonic operator of arbitrary order under homogeneous Dirichlet boundary conditions on domains $\Omega \subset \mathbb{R}^2$:

$$\begin{cases} (-\Delta)^m u = f & \text{in } \Omega, \\ \frac{\partial^j}{\partial \nu^j} u = 0 & \text{on } \partial\Omega, \qquad 0 \leq j \leq m-1. \end{cases}$$

We ask, which condition do we have to impose on the domain Ω, such that nonnegativity of the right-hand side $0 \not\equiv f \geq 0$ implies a positivity of the solution u.

The analogous problem with the Laplacian operator is solved by the strong maximum principle, if the boundary of Ω is sufficiently smooth.

Looking at the past works, we can find that Boggio [13] in 1905 determined explicitly the Green function $G_{m,n}$ for $(-\Delta)^m$ on the unit ball $B \subset \mathbb{R}^n$ and proved the positivity $G_{m,n}(x, y) > 0$ for $x, y \in B$, $x \neq y$. Some years ago, a work of Grunau and Sweers [42] gave conditions for regularity and closeness of the two-dimensional domain for polyharmonic operators, such that the positivity preserving property holds. In particular, Ω has to be close to a circle. Here, we will improve their results, showing that the property holds also for domains that differ a bit more from B. In the first subsection of this work is demonstrated the existence of a biholomorphic function h from $\overline{B}$ to $\overline{\Omega}$, while closeness of Ω to B implies closeness of the map to the identity. In the second subsection we will pull back the differential operator $(-\Delta)^m$ from Ω to B using h. We obtain a new operator, whose principal part is polyharmonic, such that we can involve results that ensure the positivity of the solution for such an operator on the disk.

2.2 Perturbation of the domain

In order to estimate the regularity of a domain we recall the following definition of [35, section 6.2]:

Definition 2.1 *A bounded domain $\Omega \subset \mathbb{R}^n$ and its boundary are of class $C^{m,\gamma}$, $0 \leq \gamma \leq 1$, if at each point $x_0 \in \partial\Omega$ there is a ball $B_0 = B(x_0)$ and a one-to-one mapping ψ of B_0 onto $D \subset \mathbb{R}^n$ such that:*

i) $\psi(B_0 \cap \Omega) \subset \mathbb{R}^n_+$;

ii) $\psi(B_0 \cap \partial\Omega) \subset \partial\mathbb{R}^n_+$;

iii) $\psi \in C^{m,\gamma}(B_0)$, $\psi^{-1} \in C^{m,\gamma}(D)$.

Here we will explain the meaning of Ω close to a ball:

Definition 2.2 *Let $\varepsilon \geq 0$. We call Ω ε-close in $C^{m,\gamma}$-sense to $\Omega^\star$, if there exists a $C^{m,\gamma}$ mapping $g : \overline{\Omega^\star} \to \overline{\Omega}$ such that $g\left(\overline{\Omega^\star}\right) = \overline{\Omega}$ and*

$$\|g - Id\|_{C^{m,\gamma}\left(\overline{\Omega^\star}\right)} \leq \varepsilon.$$

We are now ready to introduce our first result:

Theorem 2.3 *Let δ be given. Then there is some $\varepsilon_0 = \varepsilon_0(\delta, m) > 0$ such that for $\varepsilon \in [0, \varepsilon_0)$ we have the following:*

If the $C^{m,\gamma}$ domain Ω is ε-close in $C^{m,\gamma}$-sense to B, then there is a biholomorphic mapping $h : \overline{B} \to \overline{\Omega}$, $h \in C^{m,\gamma}(\overline{B}), h^{-1} \in C^{m,\gamma}(\overline{\Omega})$ with $\|h - Id\|_{C^{m,\gamma}(\overline{B})} \leq \delta$.

Comparing this result with the analogous one by Grunau and Sweers [40], we gain an order of derivative in the estimate for $h - Id$. There are some similar results also in [60, 62].

In order to build the function h, we introduce the following lemma:

Lemma 2.4 *Let Ω be a domain ε-close to a disk B in $C^{m,\gamma}$-sense. Let g be a map satisfying $g : \overline{B} \to \overline{\Omega}$, with $\|g - Id\|_{C^{m,\gamma}(\overline{B})} < \varepsilon$, and let $\varphi_1(x) = \log|x|$ on the boundary of Ω.*

Then there exists a function $\hat{\varphi} \in C^{m,\gamma}(\overline{B})$ such that $\hat{\varphi} = \varphi_1 \circ g$ on ∂B and $\|\hat{\varphi}\|_{C^{m,\gamma}(\overline{B})} \leq \mathcal{O}(\varepsilon)$.

Proof: Let

$$\psi_b(\theta) := \varphi_1(g(\cos\theta, \sin\theta)), \qquad \psi_i(x) := \varphi_1\left(\left|g\left(\frac{x}{|x|}\right)\right|\right).$$

So ψ_b takes the values of φ_1 from the boundary of Ω to the boundary of B and ψ_i is the radial extension of these values in the interior of B. Namely if we evaluate the function $\psi_i(x)$ when $x := f(\theta) = (\cos(\theta), \sin(\theta))$, that is when $x \in \partial B$:

$$\begin{aligned} \psi_i(x) &= \frac{1}{2}\log\left(g_1^2\left(\frac{x}{|x|}\right) + g_2^2\left(\frac{x}{|x|}\right)\right) \\ &= \log\left(g_1^2(\cos(\theta), \sin(\theta)) + g_2^2(\cos(\theta), \sin(\theta))\right) = \psi_b(\theta). \end{aligned}$$

Notice that ψ_i is not defined in the origin and has not a compact support. For these reasons we choose a function φ_2 such that

$$\varphi_2 \in C_0^\infty(\mathbb{R}^2), \quad \varphi_2(x) = \begin{cases} 1 & \frac{1}{2} \le |x| \le 2, \\ 0 & |x| < \frac{1}{4},\ |x| > 4, \\ 0 < \varphi_2 < 1 & \text{otherwise.} \end{cases}$$

Define

$$\hat{\varphi}(x) := \varphi_2(x)\psi_i(x).$$

It is the aimed function: on ∂B is $\hat{\varphi} = \varphi_1 \circ g$, in $B_{\frac{1}{4}}$ is $\hat{\varphi} = 0$ and $\hat{\varphi} \in C^{m,\gamma}(\overline{B})$. Then, the norm

$$\begin{aligned} \|\hat{\varphi}\|_{C^{m,\gamma}(\overline{B})} &= \|\hat{\varphi}\|_{C^{m,\gamma}\left(\overline{B}\setminus B_{\frac{1}{4}}\right)} \\ &\le C_1 \left(\sum_{j=0}^{m} \left(\sum_{k=0}^{j} \left| \frac{\partial^j \hat{\varphi}}{\partial r^k \partial \theta^{j-k}} \right|_{0;\Omega} \right) + \left[\frac{\partial^m \hat{\varphi}}{\partial r^j \partial \theta^{m-j}} \right]_{\gamma;\Omega} \right), \end{aligned} \tag{7}$$

with polar coordinates (r,θ) and a suitable constant C_1. Because of $\hat{\varphi} = \varphi_2 \psi_i$, the regularity of φ_2 and all the derivatives of ψ_i with respect to r being zero, we obtain

$$(7) \le C_2 \left(\sum_{j=0}^{m} \left\| \frac{\partial^j \psi_i}{\partial \theta^j} \right\|_{C^0(\Omega)} + \left\| \frac{\partial^m \psi_i}{\partial \theta^m} \right\|_{C^{0,\gamma}(\Omega)} \right),$$

again with a suitable constant C_2. Because of the radial indipendence of φ_1, then

$$\frac{\partial^j \psi_i}{\partial \theta^j}(x) = \frac{\partial^j \psi_i}{\partial \theta^j}\left(\frac{x}{|x|}\right) = \frac{\partial^j \psi_b}{\partial \theta^j}(\theta).$$

Let $\tilde{g}(\theta) := g(f(\theta)) = g(\cos(\theta), \sin(\theta))$; then

$$\begin{aligned} \left(\frac{d}{d\theta}\right)^j \psi_b &= \left(\frac{d}{d\theta}\right)^j (\varphi_1 \circ \tilde{g}) \\ &= \sum_{|\vec{\alpha}|=1}^{j} ((D^{\vec{\alpha}} \varphi_1) \circ \tilde{g}) \left(\sum_{\substack{p_1+\cdots+p_{|\vec{\alpha}|}=j \\ 1 \le p_l}} d_{j,\vec{\alpha},\vec{p}} \prod_{l=1}^{|\vec{\alpha}|} \left(\frac{d}{d\theta}\right)^{p_l} \tilde{g}^{(\beta_l)} \right), \end{aligned}$$

with some suitable coefficients $d_{j,\vec{\alpha},\vec{p}}$, $\beta_l = 1$ for $l = 1, \ldots, \alpha_1$ and $\beta_l = 2$ for $l = \alpha_1 + 1, \ldots, |\vec{\alpha}|$. Let $\tilde{g}_0(\theta) := Id(f(\theta))$. We observe first that

$$\begin{aligned}
\|\tilde{g} - \tilde{g}_0\|_{C^{1,0}([0,2\pi])} &= \sup_{\theta\in[0,2\pi]} |\tilde{g}(\theta) - \tilde{g}_0(\theta)| + \sup_{\theta\in[0,2\pi]} |\tilde{g}'(\theta) - \tilde{g}_0'(\theta)| \\
&\leq \sup_{\theta\in[0,2\pi]} |g(f(\theta)) - Id(f(\theta))| \\
&\quad + \sup_{i=1,2} \sup_{\theta\in[0,2\pi]} |\partial_{x_i}(g(f(\theta)))f_i(\theta) - \partial_{x_i}(Id(f(\theta)))f_i(\theta))| \\
&\leq \sup_{x\in\overline{B}} |g(x) - Id(x)| \\
c7 &\quad + \sup_{i=1,2} \sup_{\substack{x\in\overline{B} \\ \theta\in[0,2\pi]}} |\partial_{x_i}(g(x) - Id(x))f_i(\theta)| \\
&\leq \sup_{x\in\overline{B}} |g(x) - Id(x)| + \sup_{i=1,2} \sup_{x\in\overline{B}} |\partial_{x_i}(g(x) - Id(x))| \\
&= \|g - Id\|_{C^{1,0}(\overline{B})} \leq \mathcal{O}(\varepsilon).
\end{aligned}$$

And further:

$$\begin{aligned}
\left(\frac{d}{d\theta}\right)^j \psi_b &= \sum_{|\vec{\alpha}|=1}^{j} ((D^{\vec{\alpha}}\varphi_1) \circ \tilde{g} - (D^{\vec{\alpha}}\varphi_1) \circ \tilde{g}_0 + (D^{\vec{\alpha}}\varphi_1) \circ \tilde{g}_0) \\
&\quad \times \sum_{\substack{p_1+\cdots+p_{|\vec{\alpha}|}=j \\ 1\leq p_l}} d_{j,\vec{\alpha},\vec{p}} \prod_{l=1}^{|\vec{\alpha}|} \left(\left(\frac{d}{d\theta}\right)^{p_l} \tilde{g}^{(\beta_l)} - \left(\frac{d}{d\theta}\right)^{p_l} \tilde{g}_0^{(\beta_l)} \right. \\
&\quad \left. + \left(\frac{d}{d\theta}\right)^{p_l} \tilde{g}_0^{(\beta_l)} \right).
\end{aligned}$$

Observing that $\varphi_1(\tilde{g}_0(\theta)) = \log|(\cos(\theta), \sin(\theta))| \equiv 0$, then all derivatives of $\varphi_1(\tilde{g}_0(\theta))$ are zero. It remains to study

$$\left(\frac{d}{d\theta}\right)^{p_l} \tilde{g}^{(\beta_l)} - \left(\frac{d}{d\theta}\right)^{p_l} \tilde{g}_0^{(\beta_l)} \tag{8}$$

and

$$\left(D^{\vec{\alpha}}\varphi_1\right) \circ \tilde{g} - \left(D^{\vec{\alpha}}\varphi_1\right) \circ \tilde{g}_0. \tag{9}$$

Because of the sufficient regularity of φ_1, it is much easier to estimate (9) in the norm $\|\,.\,\|_{C^{1,0}([0,2\pi])}$, than in $\|\,.\,\|_{C^{0,\gamma}([0,2\pi])}$:

$$\begin{aligned}
&\left\|\left(D^{\vec{\alpha}}\varphi_1\right)\circ\tilde{g}-\left(D^{\vec{\alpha}}\varphi_1\right)\circ\tilde{g}_0\right\|_{C^{1,0}([0,2\pi])}\\
&\quad=\sup_{\theta\in[0,2\pi]}\left|\left(D^{\vec{\alpha}}\varphi_1\right)\circ\tilde{g}(\theta)-\left(D^{\vec{\alpha}}\varphi_1\right)\circ\tilde{g}_0(\theta)\right| \qquad (10)\\
&\qquad+\sup_{\theta\in[0,2\pi]}\left|\left[\left(D^{\vec{\alpha}}\varphi_1\right)\circ\tilde{g}(\theta)-\left(D^{\vec{\alpha}}\varphi_1\right)\circ\tilde{g}_0(\theta)\right]'\right|. \qquad (11)
\end{aligned}$$

The equation $\varphi_1(x)=\log(|x|)$ implies

$$\sup_{\vec{\tilde{\alpha}}}\max_{\substack{x\in B_{1+\varepsilon}(0)\\ x\notin B_{1-\varepsilon}(0)}}|D^{\vec{\alpha}+\vec{\tilde{\alpha}}}\varphi_1(x)|<C_3,\qquad \sup_{|\vec{\bar{\alpha}}|=2}\max_{\substack{x\in B_{1+\varepsilon}(0)\\ x\notin B_{1-\varepsilon}(0)}}|D^{\vec{\alpha}+\vec{\bar{\alpha}}}\varphi_1(x)|<C_4,$$

with suitable constants C_3 and C_4. Then

$$\begin{aligned}
(10)\quad&\le\quad\sup_{|\vec{\tilde{\alpha}}|=1}\max_{\substack{x\in B_{1+\varepsilon}(0)\\ x\notin B_{1-\varepsilon}(0)}}|D^{\vec{\alpha}+\vec{\tilde{\alpha}}}\varphi_1(x)|\sup_{\theta\in[0,2\pi]}|\tilde{g}(\theta)-\tilde{g}_0(\theta)|\\
&\le\quad C_3\sup_{\theta\in[0,2\pi]}|\tilde{g}(\theta)-\tilde{g}_0(\theta)|\le\mathcal{O}(\varepsilon); \qquad (12)\\
(11)\quad&=\quad\sup_{\theta\in[0,2\pi]}\left|\sum_{i=1}^{2}\left((\partial_{x_i}D^{\alpha}\varphi_1)\circ\tilde{g}(\theta)\cdot\tilde{g}_i'(\theta)\right.\right.\\
&\qquad\left.\left.-(\partial_{x_i}D^{\alpha}\varphi_1)\circ\tilde{g}_0(\theta)\cdot\tilde{g}_{0i}'(\theta)\right)\right|. \qquad (13)
\end{aligned}$$

We subtract and add $\sum_{i=1}^{2}[((\partial_{x_i}D^{\alpha}\varphi_1)\circ\tilde{g}(\theta))\cdot\tilde{g}_{0i}'(\theta)]$ to (13) and recall that $|\tilde{g}_0'(\theta)|=|(-\sin(\theta),\cos(\theta))|=1$:

$$\begin{aligned}
(13)\quad&=\quad\sup_{\theta\in[0,2\pi]}\left|\sum_{i=1}^{2}\left\{(\partial_{x_i}D^{\alpha}\varphi_1)\circ\tilde{g}(\theta)\cdot[\tilde{g}(\theta)-\tilde{g}_0(\theta)]_i'\right.\right.\\
&\qquad\left.\left.-\left[(\partial_{x_i}D^{\alpha}\varphi_1)\circ\tilde{g}_0(\theta)-(\partial_{x_i}D^{\alpha}\varphi_1)\circ\tilde{g}(\theta)\right]\cdot\tilde{g}_{0i}'(\theta)\right\}\right|\\
&\le\quad\sup_{|\tilde{\alpha}|=1}\max_{\substack{x\in B_{1+\varepsilon}(0)\\ x\notin B_{1-\varepsilon}(0)}}\left|D^{\alpha+\tilde{\alpha}}\varphi_1(x)\right|\sup_{\theta\in[0,2\pi]}|[\tilde{g}(\theta)-\tilde{g}_0(\theta)]'|\\
&\qquad+\sup_{|\overline{\alpha}|=2}\max_{\substack{x\in B_{1+\varepsilon}(0)\\ x\notin B_{1-\varepsilon}(0)}}\left|D^{\alpha+\overline{\alpha}}\varphi_1(x)\right|\sup_{\theta\in[0,2\pi]}|[\tilde{g}(\theta)-\tilde{g}_0(\theta)]|\\
&\le\quad C_3\sup_{\theta\in[0,2\pi]}|[\tilde{g}(\theta)-\tilde{g}_0(\theta)]'|+C_4\sup_{\theta\in[0,2\pi]}|[\tilde{g}(\theta)-\tilde{g}_0(\theta)]|\\
&\le\quad C_3\mathcal{O}(\varepsilon)+C_4\mathcal{O}(\varepsilon)\le\mathcal{O}(\varepsilon). \qquad (14)
\end{aligned}$$

So, combining (12) and (14), we have

$$\|(D^\alpha\varphi)\circ\tilde{g}-(D^\alpha\varphi)\circ\tilde{g}_0\|_{C^{1,0}([0,2\pi])}\leq\mathcal{O}(\varepsilon).$$

It rest to evaluate (8) with respect to the norm $\|\cdot\|_{C^{0,\gamma}([0,2\pi])}$:

$$\begin{aligned}\left\|\left(\frac{d}{d\theta}\right)^{p_l}\left(\tilde{g}^{(\beta_l)}-\tilde{g}_0^{(\beta_l)}\right)\right\|_{C^{0,\gamma}([0,2\pi])} \quad\leq\quad & \max_{\substack{x\in\partial B\\|\alpha|=m}}\left|D^\alpha\left(g(x)-Id(x)\right)\right| \\ &+\max_{\substack{x,y\in\partial B\\|\alpha|=m}}\frac{1}{|x-y|^\gamma}\left|D^\alpha\left(g(x)-Id(x)\right)\right. \\ &\left.-D^\alpha\left(g(y)-Id(y)\right)\right|. \end{aligned}\tag{15}$$

The maximum, estimated only for elements on the boundary of B is smaller than the maximum evaluated on $\overline{B}$, so

$$(15)\leq\|g-Id\|_{C^{m,\gamma}(\overline{B})}\leq\mathcal{O}(\varepsilon).$$

□

Now we can proceed with:
Proof of Theorem 2.3: According to [22, 66], the holomorphic mapping h, which has the desired qualitative properties, may be constructed in the following way. By Lemma 2.4 there is a function $\hat{\varphi}$ such that

$$\|\hat{\varphi}\|_{C^{m,\gamma}(\overline{B})}\leq\mathcal{O}(\varepsilon).$$

We know, there exists a solution r for the problem

$$\begin{cases}\Delta r=0 & x\in B,\\ r(x)=\hat{\varphi}(x) & x\in\partial B\end{cases}$$

and in view of [35, Corollary 6.7, Paragraph 6.4] we obtain the estimation $\|r(x)\|_{C^{m,\gamma}(\overline{B})}\leq\mathcal{O}(\varepsilon)$. Let $G(x,0)$ be the Green function for $-\Delta$ in B under homogeneous Dirichlet condition, it means

$$G(x,0):=-\frac{1}{2\pi}(\log|x|-r(x))$$

and set

$$\omega(x):=2\pi G(x,0).$$

Then we define the harmonic conjugated function of ω

$$\omega^*(x) := \int_{\frac{1}{2}}^{x} \left(-\frac{\partial}{\partial \xi_2}\omega(\xi)\, d\xi_1 + \frac{\partial}{\partial \xi_1}\omega(\xi)\, d\xi_2 \right),$$

where the integral is taken with respect to any curve from $\frac{1}{2}$ to x in $\Omega \setminus \{0\}$. The integral is well defined up to multiples of 2π and we can define

$$h^{-1}(x) := e^{-\omega(x) - i\omega^*(x)},$$

such that its inverse is the function that satisfies Theorem 2.3: h^{-1} is holomorphic, $h^{-1}(\Omega) \subset B$. One finds that $h^{-1}(0) = 0$, $h^{-1}\left(\frac{1}{2}\right) \in \mathbb{R}_+$ and if $x \in \partial\Omega$, then $|h^{-1}(x)| = |e^{-i\omega^*(x)}| = 1$, it means that $h^{-1}(\partial\Omega) = \partial B$, for $x \in \Omega \setminus \{0\}$, $\omega(x) > 0$, $|h^{-1}(x)| < 1$ and then $h^{-1}(\Omega) \subset B$. $\square$

2.3 Pull back of the operator

The purpose of this subsection is to find a property that ensures the positivity preserving property for the problem

$$\begin{cases} (-\Delta)^m u = f & \text{in } \Omega, \\ \frac{\partial^j}{\partial \nu^j} u = 0 & \text{on } \partial\Omega, \qquad 0 \le j \le m-1. \end{cases} \tag{16}$$

In Theorem 2.3 we have seen, if Ω is sufficiently close to the unit ball B, then there is a biholomorphic function h that maps B on Ω. We can use h to pull back the polyharmonic operator from Ω to the ball, where we know that the positivity preserving property applies. We will see in details what happens.

By Theorem 2.3 let $h : B \to \Omega$, $h : (\xi_1, \xi_2) \mapsto (x_1, x_2)$, $h \in C^{m,\gamma}(\overline{B})$. We compose both parts of the first equation of (16) with h:

$$((-\Delta)^m u) \circ h = f \circ h \quad \text{on} \quad B, \tag{17}$$

where Δ is the Laplacian with respect to $x = (x_1, x_2)$. Let us denote by Δ^* the Laplacian with respect to $z = \xi_1 + i\xi_2 = (\xi_1, \xi_2)$, such that we can identify the real and the complex variables $z \in \mathbb{C}$; let $v := (u \circ h)$ and h' the complex derivative of h. The Laplacian in complex coordinates can be rewritten as

$$\Delta^* = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}}.$$

Let g be a twice continuously differentiable function, $g : \mathbb{C} \to \mathbb{R}$;

$$\begin{aligned}
\frac{\partial}{\partial z} g &= \frac{1}{2}\left(\frac{\partial}{\partial \xi_1} - i\frac{\partial}{\partial \xi_2}\right) g;\\
\frac{\partial}{\partial \bar{z}} g &= \frac{1}{2}\left(\frac{\partial}{\partial \xi_1} + i\frac{\partial}{\partial \xi_2}\right) g;\\
4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} g &= \frac{4}{4}\left[\left(\frac{\partial}{\partial \xi_1} - i\frac{\partial}{\partial \xi_2}\right)\left(\frac{\partial}{\partial \xi_1} + i\frac{\partial}{\partial \xi_2}\right) g\right]\\
&= \frac{\partial}{\partial \xi_1}\left(\frac{\partial}{\partial \xi_1} g + i\frac{\partial}{\partial \xi_2} g\right) - i\frac{\partial}{\partial \xi_2}\left(\frac{\partial}{\partial \xi_1} g + i\frac{\partial}{\partial \xi_2} g\right)\\
&= \left(\frac{\partial}{\partial \xi_1}\right)^2 g + \left(\frac{\partial}{\partial \xi_2}\right)^2 g = \Delta g.
\end{aligned}$$

We see that if equation (17) holds in B then

$$\begin{aligned}
\Delta^*(u \circ h) &= \Delta^* v = [(\Delta u) \circ h]\,|h'|^2;\\
-\frac{1}{|h'|^2}\Delta^*(u \circ h) &= (\Delta u) \circ h;\\
f \circ h &= \left(\frac{-1}{|h'|^2}\Delta^*\right)^m (u \circ h) = \left(\frac{-1}{|h'|^2}\Delta^*\right)^m v.
\end{aligned}$$

We turn our attention to the operator $\left(-\frac{1}{|h'|^2}\Delta^*\right)^m$:

Lemma 2.5 *Let h be a holomorphic function and let v be a function in $C^{2m}(B)$, then the operator*

$$\left(\frac{-1}{|h'|^2}\Delta\right)^m v$$

contains no derivatives of h of order larger than m. Here h' denotes the complex derivative of h.

Proof: We identify $\mathbb{C}$ with $\mathbb{R}^2$ and denote $z = \xi_1 + i\xi_2$ for $(\xi_1, \xi_2) \in \mathbb{R}^2$.

We proceed by induction: obviously the claim holds for $m = 1$. In order to show the underlying idea how to exploit h being holomorphic, we first treat the case $m = 2$:

$$\begin{aligned}
\left(\frac{-1}{|h'|^2}\Delta\right)^2 v &= \frac{-4}{|h'|^2}\partial_z\partial_{\overline{z}}\left[\left(-\frac{1}{|h'|^2}\Delta\right)v\right] \\
&= \frac{4}{|h'|^2}\left[(\partial_z\partial_{\overline{z}}\Delta v)\frac{1}{|h'|^2} + (\partial_z\Delta v)\left(\partial_{\overline{z}}\frac{1}{\overline{h'}}\right)\frac{1}{h'}\right. \\
&\quad \left. + (\partial_{\overline{z}}\Delta v)\left(\partial_z\frac{1}{h'}\right)\frac{1}{\overline{h'}} + \Delta v\left(\partial_z\frac{1}{h'}\right)\left(\partial_{\overline{z}}\frac{1}{\overline{h'}}\right)\right] \\
&= \frac{1}{|h'|^2}\left[\frac{1}{|h'|^2}\Delta^2 v - 4\frac{1\overline{h''}}{h'\overline{h'}^2}\partial_z\Delta v - 4\frac{h''}{h'^2\overline{h'}}\partial_{\overline{z}}\Delta v + 4\frac{|h''|^2}{|h'|^4}\Delta v\right].
\end{aligned}$$

Now, we suppose that our hypothesis is true for some $m \geq 2$.

$$\begin{aligned}
\left(\frac{-1}{|h'|^2}\Delta\right)^m v &= (-1)^m \sum_{\beta_1,\beta_2=1}^{m} \overset{(m)}{F}_{\beta_1\beta_2}\left(h^{(m+1-\beta_1)}, \ldots, h', \overline{h^{(m+1-\beta_2)}}, \ldots, \overline{h'}\right) \\
&\quad \times \partial_z^{\beta_1}\partial_{\overline{z}}^{\beta_2} v, \qquad (18)
\end{aligned}$$

where $\overset{(m)}{F}_{\beta_1\beta_2} = \overset{(m)}{F}_{\beta_1\beta_2}(\eta_1, \ldots, \eta_{m+1-k}, \zeta_1, \ldots, \zeta_{m+1-j})$ is a smooth function of both h and $\overline{h}$ derivatives except for $h' = 0$ and with $\beta_1, \beta_2 \leq m$. Then we show that the (18) is also true replacing m by $m+1$:

$$\begin{aligned}
&\left(\frac{-1}{|h'|^2}\Delta\right)^{m+1} v = (-1)^{m+1}\left(\frac{1}{|h'|^2}\Delta\right)\left[\left(\frac{1}{|h'|^2}\Delta\right)^m v\right] \\
&= 4\frac{(-1)^{m+1}}{|h'|^2}\partial_z\partial_{\overline{z}} \sum_{\beta_1,\beta_2=1}^{m}\Bigg[\overset{(m)}{F}_{\beta_1\beta_2}\left(h^{(m-\beta_1+1)},\ldots,h',\overline{h^{(m-\beta_2+1)}},\ldots,\overline{h'}\right) \\
&\quad \times\partial_z^{\beta_1}\partial_{\overline{z}}^{\beta_2} v\Bigg] \\
&= 4\frac{(-1)^{m+1}}{|h'|^2}\partial_z\Bigg\{\sum_{\beta_1,\beta_2=1}^{m}\left[\frac{\partial \overset{(m)}{F}_{\beta_1\beta_2}}{\partial\zeta_1}\overline{h^{(m-\beta_2+2)}}+\cdots+\frac{\partial \overset{(m)}{F}_{\beta_1\beta_2}}{\partial\zeta_{m+1-\beta_2}}\overline{h''}\right] \\
&\quad \times\partial_z^{\beta_1}\partial_{\overline{z}}^{\beta_2} v + \left[\sum_{\beta_1,\beta_2=1}^{m}\overset{(m)}{F}_{\beta_1\beta_2}\,\partial_z^{\beta_1}\partial_{\overline{z}}^{\beta_2}\partial_{\overline{z}} v\right]\Bigg\} \\
&= (-1)^{m+1}\frac{4}{|h'|^2}\partial_z\Bigg\{\sum_{\beta_1,\beta_2=1}^{m}\left[\sum_{k=1}^{m+1-\beta_2}\frac{\partial \overset{(m)}{F}_{\beta_1\beta_2}}{\partial\zeta_k}\overline{h^{(m+3-\beta_2-k)}}\right]\partial_z^{\beta_1}\partial_{\overline{z}}^{\beta_2} v \\
&\quad + \left[\sum_{\beta_1,\beta_2=1}^{m}\overset{(m)}{F}_{\beta_1\beta_2}\left(h^{(m+1-\beta_1)},\ldots,h',\overline{h^{(m+1-\beta_2)}},\ldots,\overline{h'}\right)\partial_z^{\beta_1}\partial_{\overline{z}}^{\beta_2}\partial_{\overline{z}} v\right]\Bigg\} \\
&= (-1)^{m+1}\frac{4}{|h'|^2}\left\{\sum_{\beta_1,\beta_2=1}^{m}[A+B+C+D]\right\},
\end{aligned}$$

where

$$\begin{aligned}
A &= \left(\sum_{j=1}^{m+1-\beta_1}\sum_{k=1}^{m+1-\beta_2}\frac{\partial^2 \overset{(m)}{F}_{\beta_1\beta_2}}{\partial\eta_j\partial\zeta_k}h^{(m+3-\beta_1-j)}\overline{h^{(m+3-\beta_2-k)}}\right)\partial_z^{\beta_1}\partial_{\overline{z}}^{\beta_2} v, \\
B &= \left(\sum_{k=1}^{m+1-\beta_2}\frac{\partial \overset{(m)}{F}_{\beta_1\beta_2}}{\partial\zeta_k}\overline{h^{(m+3-\beta_2-k)}}\right)\partial_z^{\beta_1+1}\partial_{\overline{z}}^{\beta_2} v, \\
C &= \left(\sum_{j=1}^{m+1-\beta_1}\frac{\partial \overset{(m)}{F}_{\beta_1\beta_2}}{\partial\eta_j}h^{(m+3-\beta_1-j)}\right)\partial_z^{\beta_1}\partial_{\overline{z}}^{\beta_2+1} v, \\
D &= \left(\overset{(m)}{F}_{\beta_1\beta_2}\right)\partial_z^{\beta_1+1}\partial_{\overline{z}}^{\beta_2+1} v.
\end{aligned}$$

Note that every $\overset{(m)}{F}_{\beta_1\beta_2}$ and its derivatives contain derivatives of h (respectively $\overline{h}$) of order at most $m-\beta_1+1$ (resp. $m-\beta_2+1$). The derivatives of $\overset{(m)}{F}_{\beta_1\beta_2}$ are multiplied by $\overline{h^{(m+3-\beta_2-k)}}$ and/or $h^{(m+3-\beta_1-j)}$, so the highest derivatives of h and $\overline{h}$ have order $m+1$. $\square$

We report here a result obtained by Grunau and Sweers, [42, Theorem 5.1], which we will involve in proving our main result:

Let $\tilde{n} \geq 1$ and $\mathcal{B}$ the unit ball in $\mathbb{R}^{\tilde{n}}$. Consider the equation

$$\begin{cases} ((-\Delta)^m + \mathcal{A})u = f & \text{in } \mathcal{B}, \\ \mathcal{D}_m u = 0 & \text{on } \partial\mathcal{B}, \end{cases} \tag{19}$$

where

$$\mathcal{A} = \sum_{|\alpha|<2m} a_\alpha(x) D^\alpha, \quad \mathcal{D}_m u = (D^k u)_{k \in \mathbb{N}^n, |k| \leq m-1}$$

and $a_\alpha \in C(\overline{\mathcal{B}})$. The operator $\mathcal{A}$ is a lower order perturbation of $(-\Delta)^m$.

Lemma 2.6 *There exists $\varepsilon_0 > 0$ such that, if $\|a_\alpha\|_\infty \leq \varepsilon_0$ for all α with $|\alpha| < 2m$, then the following holds.*

- *i) For all $f \in L^p(\mathcal{B})$ there exists a solution $u \in W^{2m,p}(\mathcal{B}) \cap W_0^{m,p}(\mathcal{B})$ of* (19)*.*
- *ii) Moreover, if $f \in L^p(\mathcal{B})$ and $0 \not\equiv f \geq 0$ in $\mathcal{B}$, then the solution of* (19) *satisfies $u > 0$ in $\mathcal{B}$.*

Now we can introduce our main result:

Theorem 2.7 *Let $m \geq 2$, $0 < \gamma < 1$. Then there is some $\varepsilon_0 = \varepsilon_0(m, \gamma) > 0$ such that for $0 < \varepsilon < \varepsilon_0$, the following holds:*

If the domain $\Omega \subset \mathbb{R}^2$ is $C^{m,\gamma}-$ smooth and ε-close to the disk in $C^{m,\gamma}$-sense and $f \in C^{0,\gamma}(\overline{\Omega})$, $0 \not\equiv f \geq 0$, then the uniquely determined solution $u \in C^{m,\gamma}(\overline{\Omega})$ of

$$\begin{cases} (-\Delta)^m u = f & \text{in } \Omega, \\ \frac{\partial^j}{\partial \nu^j} u = 0 & \text{on } \partial\Omega, \qquad 0 \leq j \leq m-1 \end{cases}$$

is strictly positive.

Proof: The existence and unicity of the solution of this problem has been proved by Agmon, Douglas and Nirenberg, [2, Section 8], cfr. also [35]. Theorem 2.3 ensures the existence of a sufficient regular and close to identity map h, that we can use to pull back $(-\Delta)^m$ to the unit disk. Here the differential operator becomes

$$\left(-\frac{1}{|h'|^2}\Delta\right)^m v,$$

and contains derivatives of order no more than m of h and $\overline{h}$, that are next to disappear. So we can apply Theorem 2.6 in the case $\tilde{n}=2$ and obtain the positivity of the solution. $\square$

Theorems 2.3, 2.7 use Definition 2.2 of *ε-closeness*. It needs the knowledge of a sufficient regular map $g\left(\overline{\Omega^\star}\right)=\overline{\Omega}$, that is a quite unconfortable requireness. It is easier and more natural to define the closeness only with the boundaries:

Lemma 2.8 *Let $\partial\Omega$ be given by the curve: $\alpha:\mathbb{R}\to\mathbb{R}^2$, α $2\pi-$periodic, $\alpha\in C^{m,\gamma}$, with $\|\alpha(t)-(\cos(t),\sin(t))\|_{C^{m,\gamma}([0,2\pi])}\le\varepsilon$. Then Ω is $\tilde{\varepsilon}$-close to $B_1(0)$ in $C^{m,\gamma}$-sense, where $\tilde{\varepsilon}=\tilde{\varepsilon}(\varepsilon,m)=\mathcal{O}(|\varepsilon|)$.*

Proof: Let $(x_1,x_2)=(\rho\cos(\varphi),\rho\sin(\varphi))\in B_1(0)$; we set

$$g:(\rho\cos(\varphi),\rho\sin(\varphi))\mapsto(\rho\alpha_1(\varphi),\rho\alpha_2(\varphi)).$$

Its differential is

$$\begin{aligned} Dg &= \begin{pmatrix} \alpha_1(\varphi) & \rho\alpha_1'(\varphi) \\ \alpha_2(\varphi) & \rho\alpha_2'(\varphi) \end{pmatrix} \frac{1}{\rho} \begin{pmatrix} \rho\cos(\varphi) & \rho\sin(\varphi) \\ -\sin(\varphi) & \cos(\varphi) \end{pmatrix} \\ &= \begin{pmatrix} \alpha_1(\varphi)\cos(\varphi)-\alpha_1'(\varphi)\sin(\varphi) & \alpha_1(\varphi)\sin(\varphi)+\alpha_1'(\varphi)\cos(\varphi) \\ \alpha_2(\varphi)\cos(\varphi)-\alpha_2'(\varphi)\sin(\varphi) & \alpha_2(\varphi)\sin(\varphi)+\alpha_2'(\varphi)\cos(\varphi) \end{pmatrix} \end{aligned} \tag{20}$$

So we obtain

$$\begin{aligned} \frac{\partial g_1}{\partial x_1} &= \alpha_1(\varphi)\cos(\varphi)-\alpha_1'(\varphi)\sin(\varphi) \\ &= (\cos(\varphi)+\mathcal{O}(\varepsilon))\cos(\varphi)+(\sin(\varphi)+\mathcal{O}(\varepsilon))\sin(\varphi)=1+\mathcal{O}(\varepsilon). \end{aligned}$$

That is

$$(20)=\begin{pmatrix} 1+\mathcal{O}(\varepsilon) & \mathcal{O}(\varepsilon) \\ \mathcal{O}(\varepsilon) & 1+\mathcal{O}(\varepsilon) \end{pmatrix}.$$

Calculating the second derivatives we obtain

$$
\begin{aligned}
D\frac{\partial g}{\partial x_1} &= \begin{pmatrix} 0 & (\alpha_1'(\varphi)-\alpha_1'(\varphi))\cos(\varphi)-(\alpha_1(\varphi)+\alpha_1''(\varphi))\sin(\varphi) \\ 0 & (\alpha_2'(\varphi)-\alpha_2'(\varphi))\cos(\varphi)-(\alpha_2(\varphi)+\alpha_2''(\varphi))\sin(\varphi) \end{pmatrix} \\
&\quad \times\frac{1}{\rho}\begin{pmatrix} \rho\cos(\varphi) & \rho\sin(\varphi) \\ -\sin(\varphi) & \cos(\varphi) \end{pmatrix} \\
&= \frac{1}{\rho}\begin{pmatrix} \sin^2(\varphi)(\alpha_1(\varphi)+\alpha_1''(\varphi)) & -\sin(\varphi)\cos(\varphi)(\alpha_1(\varphi)+\alpha_1''(\varphi)) \\ \sin^2(\varphi)(\alpha_2(\varphi)+\alpha_2''(\varphi)) & -\sin(\varphi)\cos(\varphi)(\alpha_2(\varphi)+\alpha_2''(\varphi)) \end{pmatrix}.
\end{aligned}
$$

We observe, the second derivatives have a discontinuity in the point $(0,0)$. In order to solve this problem of regularity in a neighborhood of the origin, we define a cut-off function $\psi \in C^\infty$ with

$$
\begin{cases} \psi(x) = 0 & |x| < \frac{1}{4}, \\ 0 \le \psi(x) \le 1 & \frac{1}{4} \le |x| \le \frac{1}{2}, \\ \psi(x) = 1 & |x| > \frac{1}{2}. \end{cases}
$$

We introduce the function

$$
\tilde{g}(x) := \begin{cases} g(x) & |x| > \frac{1}{2}, \\ g(x)\psi(x) + x(1-\psi(x)) & \frac{1}{4} \le |x| \le \frac{1}{2}, \\ x & |x| < \frac{1}{4}. \end{cases}
$$

We shall now prove that $\tilde{g}$ has the desired properties. On $\Omega \setminus B_{\frac{1}{2}}(0)$

$$
\frac{\partial^{|j|}\left(\tilde{g}-Id\right)_i}{\partial x_1^{j_1}\partial x_2^{j_2}} = \frac{\partial^{|j|}\left\{\rho\left[\alpha-\frac{Id}{\rho}\right]\right\}_i}{\partial x_1^{j_1}\partial x_2^{j_2}} = \frac{1}{\rho^{|j|-1}}\mu_{\vec{j}}(\varphi), \tag{21}
$$

where

$$
\begin{aligned}
\mu_{\vec{j}}(\varphi) &= \sum_{h=0}^{|j|}\nu_{h,i,\vec{j}}(\varphi)\sigma_{h,i,\vec{j}}(\varphi), \\
\nu_{h,i,\vec{j}}(\varphi) &= \left(\frac{\partial}{\partial\varphi}\right)^h\left(\alpha_1(\varphi)-\cos(\varphi), \alpha_2(\varphi)-\sin(\varphi)\right)_i, \\
\sigma_{h,i,\vec{j}}(\varphi) &= \sum_{\substack{k_1+k_2=|j| \\ k_1,k_2\ge 0}} c_{h,i,\vec{j},\vec{k}}(\cos(\varphi))^{k_1}(\sin(\varphi))^{k_2},
\end{aligned}
$$

with some suitable coefficients $c_{h,i,\vec{j},\vec{k}}$. We imposed

$$
\|\alpha(\varphi)-(\cos(\varphi),\sin(\varphi))\|_{C^{m,\gamma}([0,2\pi])} \le \mathcal{O}(\varepsilon).
$$

So for every $\vec{j}$, such that $|j| \leq m$ we have

$$\left(\frac{\partial}{\partial\varphi}\right)^h (\alpha_1(\varphi) - \cos(\varphi), \alpha_2(\varphi) - \sin(\varphi)) \leq \mathcal{O}(\varepsilon)$$

and then (21) $\leq \mathcal{O}(\varepsilon)$. We consider only the Hölder seminorm of the highest order derivative, because lower order derivatives are more regular. If we set $w := \rho_1(\cos(\varphi_1), \sin(\varphi_1))$ and $z := \rho_2(\cos(\varphi_2), \sin(\varphi_2))$, then

$$\begin{aligned} [\tilde{g} - Id]_{m,\gamma,\Omega\setminus B_{\frac{1}{2}}(0)} &= \sup_{\substack{|j|=m \\ w,z\in\Omega\setminus B_{\frac{1}{2}}(0)}} \frac{\left|\frac{\partial^{|j|}(\tilde{g}-Id)_i}{\partial x_1^{j_1}\partial x_2^{j_2}}(w) - \frac{\partial^{|j|}(\tilde{g}-Id)_i}{\partial x_1^{j_1}\partial x_2^{j_2}}(z)\right|}{|w-z|^\gamma} \\ &= \sup_{\substack{|j|=m \\ w,z\in\Omega\setminus B_{\frac{1}{2}}(0)}} \frac{\left|\frac{1}{\rho_1^{m-1}}\mu_{\vec{j}}(\varphi_1) - \frac{1}{\rho_2^{m-1}}\mu_{\vec{j}}(\varphi_2)\right|}{|w-z|^\gamma}. \end{aligned} \tag{22}$$

Applying the notation of (21), we add and subtract to the numerator the quantity $\frac{1}{\rho_1^{m-1}}\mu_{\vec{j}}(\varphi_2)$:

$$
\begin{aligned}
(22) \quad \leq \quad & \sup_{\substack{|j|=m \\ w,z\in\Omega\setminus B_{\frac{1}{2}}(0)}} \frac{\left|\frac{1}{\rho_1^{m-1}}\Big(\mu_{\vec{j}}(\varphi_1)-\mu_{\vec{j}}(\varphi_2)\Big)\right|}{|w-z|^\gamma} \\
& + \sup_{\substack{|j|=m \\ w,z\in\Omega\setminus B_{\frac{1}{2}}(0)}} \frac{\left|\left(\frac{1}{\rho_1^{m-1}}-\frac{1}{\rho_2^{m-1}}\right)\mu_{\vec{j}}(\varphi_2)\right|}{|w-z|^\gamma} \\
\leq \quad & \sup_{\substack{|j|=m \\ w,z\in\Omega\setminus B_{\frac{1}{2}}(0)}} \frac{\left|\frac{1}{\rho_1^{m-1}}\sum_{h=0}^{|j|}\left[\nu_{h,i,\vec{j}}(\varphi_1)\big(\sigma_{h,i,\vec{j}}(\varphi_1)-\sigma_{h,i,\vec{j}}(\varphi_2)\big)\right]\right|}{|w-z|^\gamma} \\
& + \sup_{\substack{|j|=m \\ w,z\in\Omega\setminus B_{\frac{1}{2}}(0)}} \frac{\left|\frac{1}{\rho_1^{m-1}}\sum_{h=0}^{|j|}\left[\big(\nu_{h,i,\vec{j}}(\varphi_1)-\nu_{h,i,\vec{j}}(\varphi_2)\big)\sigma_{h,i,\vec{j}}(\varphi_2)\right]\right|}{|w-z|^\gamma} \\
& + \sup_{\substack{|j|=m \\ w,z\in\Omega\setminus B_{\frac{1}{2}}(0)}} \frac{\left|\left(\frac{1}{\rho_1^{m-1}}-\frac{1}{\rho_2^{m-1}}\right)\cdot\mu_{\vec{j}}(\varphi_2)\right|}{|w-z|^\gamma} \\
\leq \quad & \sup_{\substack{|j|=m \\ w,z\in\Omega\setminus B_{\frac{1}{2}}(0)}} \frac{1}{\rho_1^{m-1}}\sum_{h=0}^{|j|}\left[\left|\nu_{h,i,\vec{j}}(\varphi_1)\right|\frac{\left|\big(\sigma_{h,i,\vec{j}}(\varphi_1)-\sigma_{h,i,\vec{j}}(\varphi_2)\big)\right|}{|w-z|^\gamma}\right] \quad (23) \\
& + \sup_{\substack{|j|=m \\ w,z\in\Omega\setminus B_{\frac{1}{2}}(0)}} \frac{1}{\rho_1^{m-1}}\sum_{h=0}^{|j|}\left[\frac{\left|\nu_{h,i,\vec{j}}(\varphi_1)-\nu_{h,i,\vec{j}}(\varphi_2)\right|}{|w-z|^\gamma}\left|\sigma_{h,i,\vec{j}}(\varphi_2)\right|\right] \quad (24) \\
& + \sup_{\substack{|j|=m \\ w,z\in\Omega\setminus B_{\frac{1}{2}}(0)}} \frac{\left|\frac{1}{\rho_1^{m-1}}-\frac{1}{\rho_2^{m-1}}\right|}{|w-z|^\gamma}\left|\mu_{\vec{j}}(\varphi_2)\right|. \quad (25)
\end{aligned}
$$

Studying term by term (23), (24) and (25), we can show that they are all sufficiently small:

$$(23) \leq \underbrace{\sup_{\substack{|j|=m \\ w,z\in\Omega\setminus B_{\frac{1}{2}}(0)}} \frac{1}{\rho_1^{m-1}}}_{\leq 2^{m-1}} \sum_{h=0}^{|j|} \Bigg[\underbrace{\sup_{\substack{|j|=m \\ w,z\in\Omega\setminus B_{\frac{1}{2}}(0)}} \left|\nu_{h,i,\vec{j}}(\varphi_1)\right|}_{\leq \mathcal{O}(\varepsilon)}$$

$$\times \underbrace{\sup_{\substack{|j|=m \\ w,z\in\Omega\setminus B_{\frac{1}{2}}(0)}} \frac{\left|\overbrace{\sigma_{h,i,\vec{j}}(\varphi_1) - \sigma_{h,i,\vec{j}}(\varphi_2)}^{\in C^\infty}\right|}{|w-z|^\gamma}}_{\leq C_5} \Bigg] \leq \mathcal{O}(\varepsilon),$$

$$(24) \leq \underbrace{\sup_{\substack{|j|=m \\ w,z\in\Omega\setminus B_{\frac{1}{2}}(0)}} \frac{1}{\rho_1^{m-1}}}_{\leq 2^{m-1}} \sum_{h=0}^{|j|} \Bigg[\underbrace{\sup_{\substack{|j|=m \\ w,z\in\Omega\setminus B_{\frac{1}{2}}(0)}} \frac{\left|\overbrace{\nu_{h,i,\vec{j}}(\varphi_1) - \nu_{h,i,\vec{j}}(\varphi_2)}^{\in C^{0,\gamma}}\right|}{|w-z|^\gamma}}_{\leq \mathcal{O}(\varepsilon)}$$

$$\times \underbrace{\sup_{\substack{|j|=m \\ w,z\in\Omega\setminus B_{\frac{1}{2}}(0)}} \left|\overbrace{\sigma_{h,i,\vec{j}}(\varphi_2)}^{\in C^\infty}\right|}_{\leq C_6} \Bigg] \leq \mathcal{O}(\varepsilon),$$

$$(25) \leq \underbrace{\sup_{\substack{|j|=m \\ w,z\in\Omega\setminus B_{\frac{1}{2}}(0)}} \frac{\left|\overbrace{\frac{1}{\rho_1^{m-1}} - \frac{1}{\rho_2^{m-1}}}^{\in C^\infty}\right|}{|w-z|^\gamma}}_{C_7} \underbrace{\sup_{\substack{|j|=m \\ w,z\in\Omega\setminus B_{\frac{1}{2}}(0)}} \left|\overbrace{\mu_{\vec{j}}(\varphi_2)}^{\in C^{0,\gamma}}\right|}_{\leq \mathcal{O}(\varepsilon)} \leq \mathcal{O}(\varepsilon).$$

It means, $(22) \leq \mathcal{O}(\varepsilon)$. On $A := \left\{x : \frac{1}{4} \leq |x| \leq \frac{1}{2}\right\}$ we have the same estimate, multiplied with another constant:

$$\begin{aligned} \|\tilde{g} - Id\|_{C^{m,\gamma}(A)} &= \|g\cdot\psi + Id\cdot(1-\psi) - Id\|_{C^{m,\gamma}(A)} \\ &= \|(g - Id)\cdot\psi\|_{C^{m,\gamma}(A)} \leq \mathcal{O}(\varepsilon). \end{aligned}$$

Finally, $\tilde{g}$ on $B_{\frac{1}{4}}(0)$ is the Id, so

$$\|\tilde{g} - Id\|_{C^{m,\gamma}(B_{\frac{1}{4}}(0))} = 0.$$

$\square$

3 Steklov boundary value problems

3.1 Introduction

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded domain with $\partial\Omega \in C^2$, let $d, \delta \in \mathbb{R}$ and consider the linear problems

$$(P2) \quad \begin{cases} -\Delta u = f & \text{in } \Omega, \\ u_\nu - du = 0 & \text{on } \partial\Omega\ , \end{cases} \qquad (P4) \quad \begin{cases} \Delta^2 u = g & \text{in } \Omega, \\ u = \Delta u - \delta u_\nu = 0 & \text{on } \partial\Omega\ , \end{cases}$$

where u_ν denotes the outer normal derivative of u on $\partial\Omega$ and $f, g \in L^2(\Omega)$. If $d = 0$, then $(P2)$ becomes a Neumann problem.

By solution of $(P2)$ and $(P4)$ we mean, respectively, a function $u \in H^1(\Omega)$ and $u \in H^2 \cap H^1_0(\Omega)$ such that

$$\int_\Omega \nabla u \nabla v \, dx = d \int_{\partial\Omega} uv \, dS + \int_\Omega fv \, dx \qquad \text{for all } v \in H^1(\Omega)$$

and

$$\int_\Omega \Delta u \Delta v \, dx = \delta \int_{\partial\Omega} u_\nu v_\nu \, dS + \int_\Omega gv \, dx \qquad \text{for all } v \in H^2 \cap H^1_0(\Omega).$$

We obtain these formulations by multiplying the first equation of $(P2)$ by a function $v \in H^1(\Omega)$ and integrating by parts on Ω:

$$\begin{aligned} \int_\Omega fv \, dx &= \int_\Omega -\Delta uv \, dx = -\int_{\partial\Omega} u_\nu v \, dS + \int_\Omega \nabla u \nabla v \, dx \\ &= -\int_{\partial\Omega} duv \, dS + \int_\Omega \nabla u \nabla v \, dx. \end{aligned}$$

In the same way, multiplying by $v \in H^2 \cap H^1_0(\Omega)$, problem $(P4)$ becomes

$$\int_\Omega gv \, dx = \int_\Omega \Delta^2 uv \, dx = \int_\Omega \nabla \Delta u \nabla v \, dx = \int_{\partial\Omega} \Delta u v_\nu \, dS - \int_\Omega \Delta u \Delta v \, dx.$$

A crucial role in the solvability of $(P2)$ and $(P4)$ is played by the eigenvalue problems

$$(E2) \quad \begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u_\nu - du = 0 & \text{on } \partial\Omega\ , \end{cases} \qquad (E4) \quad \begin{cases} \Delta^2 u = 0 & \text{in } \Omega, \\ u = \Delta u - \delta u_\nu = 0 & \text{on } \partial\Omega\ . \end{cases}$$

We say that d (resp. δ) is an eigenvalue of $(E2)$ (resp. $(E4)$) if the problem admits nontrivial solutions $u \not\equiv 0$, the corresponding eigenfunctions.

In the following subsection we will study $(E2)$, $(E4)$, $(P2)$ and $(P4)$ in one dimension: when do these problems admit solutions and their equations. In the second and third subsections we will describe the spectrum of $(E2)$ and $(E4)$ in a general domain and in the case of Ω equal to the unit ball B. In the fourth subsection we will report a result of existence of solution for problem $(P4)$ when δ is equal to an eigenvalue. In the fifth we introduce a nonlinearity in $(P2)$ and $(P4)$, we will see, there are infinitely many solutions in this case.

3.2 The one-dimensional case

Dimension 1 is a special case with particular properties. Namely,

Proposition 3.1 *System* $(E2)$ *becomes*

$$\begin{cases} u''(x) &= 0 \qquad \text{on } (-1,1), \\ u'(-1) &= -du(-1), \\ u'(1) &= du(1) \end{cases} \tag{26}$$

and it admits nontrivial solutions only for the two eigenvalues $d_1 = 0$ *and* $d_2 = 1$ *with eigenfunctions respectively* $\varphi_1(x) = b$, *for any* $b \in \mathbb{R}$ *and* $\varphi_2(x) = ax$, *for any* $a \in \mathbb{R}$.

Proof: From $u''(x) = 0$ it follows that the solution u is a polynomial of first order $u(x) = ax + b$. Consequently the boundary conditions of (26) become

$$\begin{cases} a = da - db, \\ a = da + db, \end{cases} \quad \text{that is} \quad \begin{pmatrix} 1-d & d \\ 1-d & -d \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The system admits nontrivial solution only for the eigenvalues, that are $d_1 = 0$ and $d_2 = 1$. Substituting these values in the boundary conditions, it is easy to obtain the eigenfunctions. $\square$

Proposition 3.2 *System* $(E4)$ *becomes*

$$\begin{cases} u^{(4)}(x) &= 0 \qquad \text{on } (-1,1), \\ u''(-1) &= -\delta u'(-1), \\ u''(1) &= \delta u'(1), \\ u(-1) &= 0, \\ u(1) &= 0 \end{cases} \tag{27}$$

and it admits nontrivial solution only for the two eigenvalues $\delta_1 = 1$, $\delta_2 = 3$ *with eigenfunction respectively* $\psi_1(x) = bx^2 - b$, *for any* $b \in \mathbb{R}$ *and* $\psi_2(x) = ax^3 - ax$, *for any* $a \in \mathbb{R}$.

Proof: From $u^{(4)}(x) = 0$ it follows that the solution u is a polynomial of the third grade $u(x) = ax^3 + bx^2 + cx + d$.

Consequently the boundary conditions of (27) become

$$\begin{cases} -6a + 2b & = & -3\delta a + 2\delta b - \delta c, \\ 6a + 2b & = & 3\delta a + 2\delta b + \delta c, \\ -a + b - c + d & = & 0, \\ a + b + c + d & = & 0, \end{cases}$$

that is

$$\begin{pmatrix} -6+3\delta & 2-2\delta & \delta & 0 \\ 6-3\delta & 2-2\delta & -\delta & 0 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{28}$$

The system has nontrivial solutions if $\delta_1 = 1$ or $\delta_2 = 3$. Solving (28) with δ_1 we determine the eigenfunctions $\psi_1(x) = bx^2 - b$ for any $b \in \mathbb{R}$. In case of $\delta_2 = 3$, we obtain the eigenfunctions $\psi_2(x) = ax^3 - ax$ for any $a \in \mathbb{R}$. $\square$

In the one dimensional case, $(P2)$ and $(P4)$ become

$$(ODE2) \begin{cases} -u'' = f & \text{on } (-1,1), \\ -u'(-1) - du(-1) = 0, \\ u'(1) - du(1) = 0; \end{cases} \qquad (ODE4) \begin{cases} u^{(4)} = g & \text{on } (-1,1), \\ u(-1) = 0, \\ u(1) = 0, \\ u''(-1) + \delta u'(-1) = 0, \\ u''(1) - \delta u'(1) = 0. \end{cases}$$

The second order system has the following property:

Proposition 3.3 *Let $f \in L^1([-1,1])$.*

a) If $d \notin \{0,1\}$, then for every f problem $(ODE2)$ admits a unique solution given by

$$\begin{aligned} u(x) \;=\; & \frac{(1-d-dx)\int_{-1}^{1}\left[f(t) - d\int_{-1}^{t} f(\tau)\,d\tau\right]dt}{2d^2 - d} \\ & - \int_{-1}^{x}\int_{-1}^{t} f(\tau)\,d\tau\,dt. \end{aligned} \tag{29}$$

b) If $d = 0$, then $(ODE2)$ admits solutions if and only if

$$\int_{-1}^{1} f(t)\,dt = 0;$$

in such case there are infinitely many solutions given by

$$u(x) = -\int_{-1}^{x}\int_{-1}^{t} f(\tau)\,d\tau\,dt + C, \quad (\ C \in \mathbb{R}\ ).$$

c) If $d = 1$, *then* $(ODE2)$ *admits solutions if and only if*

$$\int_{-1}^{1}\int_{-1}^{t} f(\tau)\,d\tau\,dt = \int_{-1}^{1} f(t)\,dt;$$

in such case there are infinitely many solutions given by

$$u(x) = -\int_{-1}^{x}\int_{-1}^{t} f(\tau)\,d\tau\,dt - Cx, \quad (\ C \in \mathbb{R}\ ).$$

Proof: We assume that u solves $(ODE2)$. System $(ODE2)$ implies

$$\begin{aligned} u'(x) &= -\int_{-1}^{x} f(t)\,dt + u'(-1) = -\int_{-1}^{x} f(t)\,dt - u(-1)d, \\ u(x) &= -\int_{-1}^{x}\int_{-1}^{t} f(\tau)\,d\tau\,dt + u(-1)(1 - d - dx). \end{aligned} \tag{30}$$

On the boundary we obtain

$$\begin{aligned} u'(1) &= -\int_{-1}^{1} f(t)\,dt - du(-1), \\ u(1) &= -\int_{-1}^{1}\int_{-1}^{t} f(\tau)\,d\tau\,dt + u(-1)(1 - 2d). \end{aligned} \tag{31}$$

Imposing

$$0 = u'(1) - du(1)$$

then

$$0 = \int_{-1}^{1}\left(f(t) - d\int_{-1}^{t} f(\tau)\,d\tau\right)dt + u(-1)(2d - 2d^2). \tag{32}$$

a) In case of $d \notin \{0, 1\}$, we have

$$u(-1) = \frac{\int_{-1}^{1}\left(f(t) - d\int_{-1}^{t} f(\tau)\,d\tau\right)dt}{2d^2 - 2d}.$$

Substituting this value in (30) we obtain (29) and indeed, this gives a solution of $(ODE2)$.

b) If $d=0$, then $u'(-1)=u'(1)=0$, so (31) becomes

$$\int_{-1}^{1} f(t)\,dt = 0 \tag{33}$$

and there is no condition on $u(-1)$. Hence we have solutions if and only if (33) is satisfied and there for any $C\in\mathbb{R}$

$$u(x) = -\int_{-1}^{x}\int_{-1}^{t} f(\tau)\,d\tau\,dt + C.$$

c) If $d=1$, condition (32) yields

$$0 = \int_{-1}^{1}\left(f(t) - \int_{-1}^{t} f(\tau)\,d\tau\right)dt,$$

it means there is no condition on $u(-1)$. Hence (30) becomes

$$u(x) = -\int_{-1}^{x}\int_{-1}^{t} f(\tau)\,d\tau\,dt - xu(-1),$$

for any $u(-1)\in\mathbb{R}$.

$\square$

Proposition 3.4 *Let $g\in L^1([-1,1])$.*
a) If $\delta\neq\{1,3\}$, then for every g $(ODE4)$ admits a unique solution given by

$$\begin{aligned} u(x) &= \int_{-1}^{x}\int_{-1}^{\gamma}\int_{-1}^{\zeta}\int_{-1}^{\eta} g(\xi)\,d\xi\,d\eta\,d\zeta\,d\gamma + \frac{u'''(-1)}{6}(x+1)^3 \\ &\quad + u'(-1)(1+x)\left[1-\frac{\delta}{2}(x+1)\right], \end{aligned} \tag{34}$$

where

$$\begin{aligned} u'''(-1) &= \frac{\int_{-1}^{1}\int_{-1}^{\gamma}\left[-g(\zeta)+\int_{-1}^{\zeta}\left(\delta g(\eta)+\int_{-1}^{\eta}-\delta g(\xi)\,d\xi\right)d\eta\right]d\zeta\,d\gamma}{-\frac{2}{3}\delta+2}, \\ u'(-1) &= \frac{\int_{-1}^{1}\int_{-1}^{\gamma}[g(\zeta)+\int_{-1}^{\zeta}(-\delta g(\eta)+\int_{-1}^{\eta}\left(\frac{3}{2}\delta-\frac{3}{2}\right)g(\xi)\,d\xi)d\eta]d\zeta\,d\gamma}{(\delta-1)(\delta-3)}. \end{aligned}$$

b) If $\delta = 1$, then $(ODE4)$ admits solutions if and only if

$$\int_{-1}^{1}\int_{-1}^{\gamma}\left(g(\zeta)-\int_{-1}^{\zeta}g(\eta)\,d\eta\right)\,d\zeta\,d\gamma=0$$

and the solutions are as in (34), where

$$u'''(-1)=-\frac{3}{4}\int_{-1}^{1}\int_{-1}^{\gamma}\int_{-1}^{\zeta}\int_{-1}^{\eta}g(\xi)\,d\xi\,d\eta\,d\zeta\,d\gamma$$

and $u'(-1)$ is arbitrary.
c) If $\delta = 3$, then $(ODE4)$ admits solutions if and only if

$$\int_{-1}^{1}\int_{-1}^{\gamma}\left[g(\zeta)+\int_{-1}^{\zeta}\left(-3g(\eta)+\int_{-1}^{\eta}3g(\xi)\,d\xi\right)\,d\eta\right]\,d\zeta\,d\gamma=0 \tag{35}$$

and the solutions are as in (34) where $u'(-1)$ is arbitrary and

$$u'''(-1)=-\frac{3}{4}\int_{-1}^{1}\int_{-1}^{\gamma}\int_{-1}^{\zeta}\int_{-1}^{\eta}g(\xi)\,d\xi\,d\eta\,d\zeta\,d\gamma+3u'(-1).$$

Proof: We assume that u solves $(ODE4)$. System $(ODE4)$ implies

$$\begin{aligned}
u'''(x) &= \int_{-1}^{x}g(\gamma)\,d\gamma+u'''(-1),\\
u''(x) &= \int_{-1}^{x}\int_{-1}^{\gamma}g(\zeta)\,d\zeta\,d\gamma+u'''(-1)(x+1)+u''(-1),\\
u'(x) &= \int_{-1}^{x}\int_{-1}^{\gamma}\int_{-1}^{\zeta}g(\eta)\,d\eta\,d\zeta\,d\gamma+\frac{u'''(-1)}{2}(x+1)^2\\
&\quad +u''(-1)(x+1)+u'(-1),\\
u(x) &= \int_{-1}^{x}\int_{-1}^{\gamma}\int_{-1}^{\zeta}\int_{-1}^{\eta}g(\xi)\,d\xi\,d\eta\,d\zeta\,d\gamma+\frac{u'''(-1)}{6}(x+1)^3\\
&\quad +\frac{u''(-1)}{2}(x+1)^2+u'(-1)(x+1)+u(-1).
\end{aligned}$$

Then on the boundary we obtain that

$$\begin{aligned}
u''(1) &= \int_{-1}^{1}\int_{-1}^{\gamma}g(\zeta)\,d\zeta\,d\gamma+2u'''(-1)-\delta u'(-1),\\
u'(1) &= \int_{-1}^{1}\int_{-1}^{\gamma}\int_{-1}^{\zeta}g(\eta)\,d\eta\,d\zeta\,d\gamma+2u'''(-1)+(1-2\delta)u'(-1),\\
u(1) &= \int_{-1}^{1}\int_{-1}^{\gamma}\int_{-1}^{\zeta}\int_{-1}^{\eta}g(\xi)\,d\xi\,d\eta\,d\zeta\,d\gamma+\frac{4}{3}u'''(-1)+(2-2\delta)u'(-1).
\end{aligned}$$

Since $u(1)=0$ and $u''(1)-\delta u'(1)=0$, we obtain the system

$$\begin{aligned}
&\begin{pmatrix} \frac{4}{3} & 2-2\delta \\ 2-2\delta & 2\delta^2-2\delta \end{pmatrix}\begin{pmatrix} u'''(-1) \\ u'(-1) \end{pmatrix} \\
&= \begin{pmatrix} -\int_{-1}^{1}\int_{-1}^{\gamma}\int_{-1}^{\zeta}\int_{-1}^{\eta} g(\xi)\,d\xi\,d\eta\,d\zeta\,d\gamma \\ \int_{-1}^{1}\int_{-1}^{\gamma}\left[-g(\zeta)+\int_{-1}^{\zeta}\delta g(\eta)\,d\eta\right]\,d\zeta\,d\gamma \end{pmatrix},
\end{aligned} \tag{36}$$

that is not singular if $\delta \neq \{1,3\}$.

a) If $\delta \neq \{1,3\}$, solving system (36) we obtain

$$\begin{aligned}
u'(-1) &= \frac{1}{2(\delta-1)(\delta-3)}\int_{-1}^{1}\int_{-1}^{\gamma}\left[2g(\zeta)+\int_{-1}^{\zeta}\left((-2\delta)g(\eta)\right.\right. \\
&\quad \left.\left.+\int_{-1}^{\eta}3(\delta-1)g(\xi)\,d\xi\right)\,d\eta\right]\,d\zeta\,d\gamma, \\
u'''(-1) &= \frac{\int_{-1}^{1}\int_{-1}^{\gamma}\left[-g(\zeta)+\int_{-1}^{\zeta}\left(\delta g(\eta)+\int_{-1}^{\eta}(-\delta)g(\xi)\,d\xi\right)\,d\eta\right]\,d\zeta\,d\gamma}{-\frac{2}{3}\delta+2}.
\end{aligned}$$

With these choices, u indeed is a solution.

b) If $\delta = 1$, then (36) implies

$$\int_{-1}^{1}\int_{-1}^{\gamma}\left[g(\zeta)+\int_{-1}^{\zeta}(-1)g(\eta)\,d\eta\right]\,d\zeta\,d\gamma = 0, \tag{37}$$

and

$$u'''(-1) = -\frac{3}{4}\int_{-1}^{1}\int_{-1}^{\gamma}\int_{-1}^{\zeta}\int_{-1}^{\eta} g(\xi)\,d\xi\,d\eta\,d\zeta\,d\gamma, \tag{38}$$

then there is no condition on $u'(-1)$. Hence, we have solutions if and only if condition (37) is satisfied and there are

$$\begin{aligned}
u(x) &= \int_{-1}^{x}\int_{-1}^{\gamma}\int_{-1}^{\zeta}\int_{-1}^{\eta} g(\xi)\,d\xi\,d\eta\,d\zeta\,d\gamma + \frac{u'''(-1)}{6}(x+1)^3 \\
&\quad + u'(-1)(1+x)\left[1-\frac{1}{2}(x+1)\right],
\end{aligned}$$

where $u'''(-1)$ as in (38) and $u'(-1)$ free.

c) If $\delta = 3$, (36) yields

$$\begin{aligned}
4u'''(-1)-12u'(-1) &= -3\int_{-1}^{1}\int_{-1}^{\gamma}\int_{-1}^{\zeta}\int_{-1}^{\eta} g(\xi)\,d\xi\,d\eta\,d\zeta\,d\gamma \\
&= -\int_{-1}^{1}\int_{-1}^{\gamma}\left[-g(\zeta)+\int_{-1}^{\zeta}3g(\eta)\,d\eta\right]\,d\zeta\,d\gamma.
\end{aligned}$$

So we obtain condition (35) and $u'''(-1)$. □

3.3 The spectrum in general domains

It is known (see [17, Theorem 3], that the first nontrivial eigenvalue $d_1 = d_1(\Omega)$ of problem $(E2)$ is defined by

$$d_1(\Omega) := \inf_{u\in\mathbb{H}(\Omega)} \frac{\displaystyle\int_\Omega |\nabla u|^2}{\displaystyle\int_{\partial\Omega} u^2} \quad \text{with} \quad \mathbb{H}(\Omega) := \left\{u \in H^1(\Omega) \setminus H^1_0(\Omega), \int_{\partial\Omega} u = 0\right\}.$$

The inverse of its square root $d_1^{-1/2}$ is the norm of the compact linear operator $\mathbb{H}(\Omega) \to L^2(\partial\Omega)$, defined by $u \mapsto u|_{\partial\Omega}$. A lower bound for $d_1(\Omega)$ was obtained by [48, 58] and extended to more dimensional cases by [26].

In order to characterize the other eigenvalues, we seek solutions of $(E2)$ in the Hilbert space $H^1(\Omega)$, endowed with the scalar product

$$(u,v) := \int_{\partial\Omega} uv\, dS + \int_\Omega \nabla u \nabla v\, dx. \tag{39}$$

Consider the subspace

$$Z_2 := \{v \in C^\infty(\overline{\Omega}) : \Delta u = 0\} \tag{40}$$

and denote by V_2 its completion with respect to the scalar product (39). Then we have the following:

Proposition 3.5 *Assume that $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) is an open bounded domain with C^2 boundary. Then problem $(E2)$ admits infinitely many (countable) eigenvalues. The only eigenfunction of one sign is the one corresponding to the first eigenvalue. The set of the eigenfunctions forms a complete orthonormal system in V_2. Moreover, the space $H^1(\Omega)$ admits the following orthogonal decomposition with respect to the scalar product (39):*

$$H^1(\Omega) = V_2 \oplus H^1_0(\Omega).$$

Finally, if $v \in H^1(\Omega)$ and $v = v_1 + v_2$ with $v_1 \in V_2$ and $v_2 \in H^1_0(\Omega)$, then v_1 and v_2 are weak solutions of

$$\begin{cases} \Delta v_1 = 0 & \text{in } \Omega, \\ v_1 = v & \text{on } \partial\Omega; \end{cases} \quad \text{and} \quad \begin{cases} \Delta v_2 = \Delta v & \text{in } \Omega, \\ v_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

Proof:
Let Z_2 be as in (40) and define on Z_2 the scalar product given by

$$(u,v)_{W_2} = \int_{\partial\Omega} uv\, dS, \quad \forall u, v \in Z_2$$

and we denote by W_2 the completion of Z with respect to this scalar product. Then V_2 is compactly embedded into the space W_2:

$$\|u\|_{W_2} = \int_{\partial\Omega} u^2\, dS \leq \int_{\partial\Omega} u^2\, dS + \int_{\Omega} |\nabla u|^2\, dx = \|u\|_{V_2}$$

and hence any Cauchy sequence in Z_2 with respect to the norm of V_2 is a Cauchy sequence with respect to the norm of W_2. Since V_2 is the completion of Z_2 with respect to (39), it follows immediately that $V_2 \subset W_2$.

More, the embedding is compact. Let $u_m \rightharpoonup u$ in V_2, so that $u_m \rightharpoonup u$ in H^1. Then by trace embedding and compact embedding we obtain $u_m \to u$ in W_2.

We denote by $I_1 : V_2 \to W_2$ the embedding $V_2 \subset W_2$ and $I_2 : W_2 \to V_2'$ the continuous linear operator defined by

$$\langle I_2 u, v\rangle = (u,v)_{W_2}, \quad \forall u \in W_2, \forall v \in V_2.$$

Let $L_2 : V_2 \to V_2'$ be the linear operator given by

$$\langle L_2 u, v\rangle = \int_{\partial\Omega} uv\, dS + \int_{\Omega} \nabla u \cdot \nabla v\, dx, \quad \forall u, v \in V_2.$$

Then L_2 is an isomorphism and the linear operator $K := L_2^{-1} I_2 I_1 : V_2 \to V_2$ is compact. Since for $n \geq 2$, V_2 is an infinite dimensional Hilbert space and K is a compact self-adjoint operator with strictly positive eigenvalues, then V_2 admits an orthonormal base of eigenfunctions of K. Moreover, the set of eigenvalues of K can be ordered in a strictly decreasing sequence $\{\mu_i\}$ which converges to zero. It follows, problem $(E2)$ admits an infinite set of eigenvalues given by $d_i = \frac{1}{\mu_i}$ and the eigenfunctions of $(E2)$coincide with the eigenfunctions of K.

Then we prove that if d_k is an eigenvalue of (E_2) corresponding to a positive eigenfunction φ_k, then $d_k = d_1$.

Suppose that φ_k is an eigenfunction corresponding to the eigenvalue d_k. Then

$$\begin{aligned} d_k \int_{\partial\Omega} \varphi_k\varphi_1\, dS &= \int_{\partial\Omega} (\varphi_k)_\nu \varphi_1\, dS \\ &= \int_\Omega \nabla\varphi_k \cdot \nabla\varphi_1\, dx + \int_\Omega \underbrace{\Delta\varphi_k}_{=0} \varphi_1\, dx \\ &= \int_{\partial\Omega} \varphi_k (\varphi_1)_\nu\, dS - \int_\Omega \varphi_k \underbrace{\Delta\varphi_1}_{=0}\, dx \\ &= d_1 \int_{\partial\Omega} \varphi_k\varphi_1\, dS; \end{aligned}$$

so $d_k = d_1$.

Let $v_1 \in Z_2$ and $v_2 \in H^1_0(\Omega)$, then $\Delta v_1 \equiv 0$ in Ω and $v_2 \equiv 0$ on $\partial\Omega$; it implies

$$\begin{aligned} (v_1, v_2) &= \int_{\partial\Omega} v_1 v_2\, dS + \int_\Omega \nabla v_1 \cdot \nabla v_2\, dx \\ &= \int_{\partial\Omega} (v_1)_\nu v_2\, dS - \int_\Omega \Delta v_1 v_2\, dx = 0. \end{aligned}$$

Let now $v \in H^1(\Omega)$ and consider the problem

$$\begin{cases} \Delta v_1 = 0 & \text{in } \Omega, \\ v_1 = v & \text{on } \partial\Omega; \end{cases} \quad \Leftrightarrow \quad \begin{cases} -\Delta(v_1 - v) = \Delta v & \text{in } \Omega, \\ v_1 - v = 0 & \text{on } \partial\Omega. \end{cases}$$

By applying the Lax-Milgram theorem to the problem on the right hand side, we find a weak solution $v_1 \in H^1(\Omega)$, i.e. $v_1 \in V_2$. Let $v_2 := v - v_1$, then $v_2 \in H^1(\Omega)$, $(v_2) = 0$ on the boundary of Ω and $v_2 \in H^1_0(\Omega)$. $\square$

For problem $(E4)$ the first eigenvalue δ_1 is

$$\delta_1(\Omega) := \inf_{u \in \mathcal{H}(\Omega)} \frac{\displaystyle\int_\Omega |\Delta u|^2}{\displaystyle\int_{\partial\Omega} u_\nu^2} \qquad \text{with} \qquad \mathcal{H}(\Omega) := [H^2 \cap H^1_0(\Omega)] \setminus H^2_0(\Omega).$$

The norm of the compact linear operator $H^2 \cap H^1_0(\Omega) \to L^2(\partial\Omega)$, defined by $u \mapsto u_\nu|_{\partial\Omega}$ is $\delta_1^{-1/2}$. Moreover, δ_1 has also the following property: given $\varphi \in L^2(\partial\Omega)$ and the Laplace equation

$$\begin{cases} \Delta v = 0 & \text{in } \Omega, \\ v = \varphi & \text{on } \partial\Omega, \end{cases} \tag{41}$$

by using Fichera's principle of duality [28], for the solution v of (41) we obtain

$$\delta_1(\Omega)\|v\|^2_{L^2(\Omega)} \leq \|\varphi\|^2_{L^2(\partial\Omega)},$$

where δ_1 is the largest possible constant for this inequality, as proved by Kuttler in [49].

The eigenvalue δ_1 has also a key role in the positivity preserving property for the biharmonic operator Δ^2 under the boundary conditions $u = \Delta u - \delta u_\nu = 0$ on $\partial\Omega$, as proved in [8, 33]: if $\delta \geq \delta_1$, then the positivity preserving property fails while it holds when δ is in a left neighborhood of δ_1, possibly $\delta \in (-\infty, \delta_1)$. We also refer to [50] for several inequalities between the eigenvalues of $(E4)$ and other eigenvalue problems.

In the case of problem $(E4)$ we endow the Hilbert space $H^2 \cap H^1_0(\Omega)$ with the scalar product

$$(u, v) := \int_\Omega \Delta u \Delta v \, dx. \tag{42}$$

Consider the subspace

$$Z_4 := \left\{ v \in C^\infty(\overline{\Omega}) : \Delta^2 u = 0,\ u = 0 \text{ on } \partial\Omega \right\}$$

and denote by V_4 the completion of Z_4 with respect to the scalar product in (42). Then, we recall from [26] the following two theorems:

Proposition 3.6 *Assume that $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is an open bounded domain with C^2 boundary. Then problem $(E4)$ admits infinitely many (countabel) eigenvalues. The only eigenfunction of one sign is the one corresponding to the first eigenvalue. The set of eigenfunctions forms a complete orthonormal system in V_4.*

Proposition 3.7 *Assume that $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is an open bounded domain with C^2 boundary. Then, the space $H^2 \cap H^1_0(\Omega)$ admits the following orthogonal decomposition with respect to the scalar product (42)*

$$H^2 \cap H^1_0(\Omega) = V_4 \oplus H^2_0(\Omega).$$

Moreover, if $v \in H^2 \cap H^1_0(\Omega)$ and if $v = v_1 + v_2$ is the corresponding orthogonal decomposition, then $v_1 \in V_4$ and $v_2 \in H^2_0(\Omega)$ are the weak solutions of

$$\begin{cases} \Delta^2 v_1 = 0 & \text{in } \Omega \\ v_1 = 0 & \text{on } \partial\Omega \\ (v_1)_\nu = v_\nu & \text{on } \partial\Omega \end{cases} \qquad \text{and} \qquad \begin{cases} \Delta^2 v_2 = \Delta^2 v & \text{in } \Omega \\ v_2 = 0 & \text{on } \partial\Omega \\ (v_2)_\nu = 0 & \text{on } \partial\Omega\ . \end{cases}$$

3.4 The spectrum when Ω is the unit ball

When $\Omega = B$ (the unit ball) we may determine explicitly all the eigenvalues of $(E2)$ and $(E4)$.

To this end, consider the spaces of harmonic homogeneous polynomials, see [5]:

$$\begin{aligned}\mathcal{D}_k \;:=\; &\{P \in C^\infty(\mathbb{R}^n);\ \Delta P = 0 \text{ in } \mathbb{R}^n,\\ &\qquad P \text{ is an homogeneous polynomial of degree } k-1\}.\end{aligned}$$

Also, denote by μ_k the dimension of $\mathcal{D}_k$. In particular, we have

$$\begin{aligned}
&\mathcal{D}_1 = \operatorname{span}\{1\}, \quad \mu_1 = 1,\\
&\mathcal{D}_2 = \operatorname{span}\{x_i;\ (i = 1, \ldots, n)\}, \quad \mu_2 = n,\\
&\mathcal{D}_3 = \operatorname{span}\{x_i x_j;\ x_1^2 - x_h^2;\ (i, j = 1, \ldots, n,\ i \neq j,\ h = 2, \ldots, n)\},\\
&\qquad \mu_3 = \tfrac{n^2+n-2}{2},\\
&\mathcal{D}_k = \operatorname{span}\{p(x) = \textstyle\sum_{|\alpha|=k-1} c_\alpha x^\alpha : \Delta p(x) = 0\},\\
&\qquad \mu_k = \begin{pmatrix} n+k-3 \\ k-1 \end{pmatrix} + \begin{pmatrix} n+k-4 \\ k-2 \end{pmatrix}, \qquad (n \geq 2,\ k \geq 2)
\end{aligned}$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $|\alpha| = \sum_{i=1}^n \alpha_i$, $x = (x_1, x_2, \ldots, x_n)$, $c_\alpha = c_{\alpha_1,\alpha_2,\ldots,\alpha_n}$, $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$ with α_i nonnegative integers.

Harmonic polynomials of different degrees are orthogonal with respect to scalar product (39): let $H_k(x)$ and $H_j(x)$ be homogeneous harmonic polynomials in n variables of degrees k and j respectively with $j \neq k$. Let $r := |x|$ and $\xi := \frac{x}{|x|}$. It was shown in [5, subsection 9.4] that, using the fact that the normal derivative on the sphere is in the radial direction

$$\begin{aligned}\frac{\partial}{\partial \nu} H_k(x)|_{\partial B} &= \frac{\partial}{\partial \nu} H_k(r\xi)|_{\partial B}\, y = \frac{\partial}{\partial r} H_k(r\xi)|_{\partial B} = \frac{\partial}{\partial r} \left(r^k H_k(\xi)\right)\big|_{\partial B}\\ &= k H_k(\xi) = k H_k(x) \end{aligned} \tag{43}$$

and from Green's Theorem follows that

$$\int_{\partial B} H_j H_k \, dS = 0. \tag{44}$$

So combining (43) and (44), we obtain for $j \neq k$

$$\begin{aligned}(H_j, H_k) &= \int_{\partial B} H_j H_k \, dS + \int_B \nabla H_j \cdot \nabla H_k \, dx\\ &= \int_{\partial B} H_j (H_k)_\nu \, dS - \int_B H_j \Delta H_k \, dx\\ &= k \int_{\partial B} H_j H_k \, dS = 0\end{aligned}$$

Let us recall how to determine the eigenvalues of the second order Steklov problem $(E2)$ when $\Omega = B$, the unit ball. In radial and angular coordinates (r, θ), equation $\Delta\varphi = 0$ becomes

$$\frac{\partial^2\varphi}{\partial r^2} + \frac{n-1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\Delta_\theta\varphi = 0 \ ,$$

where $-\Delta_\theta$ denotes the Laplace-Beltrami operator on ∂B. It is known by [9, p.160] the following

Lemma 3.8 *The Laplace-Beltrami operator $-\Delta_\theta$ admits a sequence of eigenvalues $\{\lambda_k\}$ having multiplicity μ_k equal to the number of independent harmonic homogeneous polynomials of degree $k-1$. Moreover, $\lambda_k = (k-1)(n+k-3)$.*

We denote by e_k^ℓ $(\ell = 1, \ldots, \mu_k)$ the independent eigenfunctions corresponding to λ_k such that

$$\int_{\partial B} |e_k^l|^2 \, dS = 1.$$

This system can be chosen to be orthonormal in $L^2(\partial B)$ and is complete in this space. Then, to determine the Steklov eigenvalues and eigenfunctions, one seeks functions $\varphi = \varphi(r, \theta)$ of the kind

$$\varphi(r,\theta) = \sum_{k=1}^{\infty}\sum_{\ell=1}^{\mu_k} \varphi_k^\ell(r) e_k^\ell(\theta) \ .$$

Hence, by differentiating the series, we obtain

$$\Delta\varphi(r,\theta) = \sum_{k=1}^{\infty}\sum_{\ell=1}^{\mu_k}\left(\frac{d^2}{dr^2}\varphi_k^\ell(r) + \frac{n-1}{r}\frac{d}{dr}\varphi_k^\ell(r) - \frac{\lambda_k}{r^2}\varphi_k^\ell(r)\right) e_k^\ell(\theta) = 0 \ .$$

Therefore, we must solve the equations

$$\frac{d^2}{dr^2}\varphi_k^\ell(r) + \frac{n-1}{r}\frac{d}{dr}\varphi_k^\ell(r) - \frac{\lambda_k}{r^2}\varphi_k^\ell(r) = 0, \quad k = 1, 2, \ldots, \quad \ell = 1, \ldots, \mu_k. \tag{45}$$

With the change of variables $r = e^t$ $(t \le 0)$, equation (45) becomes a constant coefficients linear equation. It has two linearly independent solutions, but one is singular. Hence, up to multiples, the only regular solution of (45) is given by $\varphi_k^\ell(r) = a_k r^{k-1}$ because

$$\frac{2 - n + \sqrt{(n-2)^2 + 4\lambda_k}}{2} = k - 1.$$

Since the boundary condition in $(E2)$ reads $\frac{d}{dr}\varphi_k^\ell(1) = d\varphi_k^\ell(1)$ we immediately infer that $d = \bar{k} - 1$ for some integer $\bar{k} \geq 1$.

By means of the Poisson integral formula we see directly that the normalized harmonic polynomials form a complete orthonormal system in the space of the harmonic H^1 functions.

Summarizing, we have

Proposition 3.9 *The number d is an eigenvalue of $(E2)$ with corresponding eigenfunction φ if and only if d is a nonnegative integer and $\varphi \in \mathcal{D}_{d+1}$. In this case, the multiplicity of d is μ_{d+1}.*

By [26] the following results are known:

Proposition 3.10 *The number δ is an eigenvalue of $(E4)$ with corresponding eigenfunction ψ if and only if φ defined by $\psi(x) = (1 - |x|^2)\varphi(x)$ is an eigenfunction of $(E2)$ with $d = \frac{\delta - n}{2}$.*

By combining Propositions 3.9 and 3.10, we infer that the eigenvalues of $(E4)$ are

$$\delta_{\bar{k}} = n + 2(\bar{k} - 1).$$

Theorem 3.11 *If $n \geq 2$ and $\Omega = B$, then for all $k = 1, 2, 3, \ldots$:*

i) the eigenvalues of $(E4)$ are $\delta_k = n + 2(k-1)$;

ii) the multiplicity of δ_k equals μ_k;

iii) for all $\varphi_k \in \mathcal{D}_k$, the function $\psi_k(x) := (1 - |x|^2)\varphi_k(x)$ is an eigenfunction corresponding to δ_k.

Theorem 3.12 *If $n \geq 2$ and $\Omega = B$, then for all $k = 1, 2, 3, \cdots$:*

i) the eigenvalues of $(E2)$ are $d = (k-1)$;

ii) the multiplicity of d_k equals $\mu_k = \begin{pmatrix} n+k-3 \\ k-1 \end{pmatrix} + \begin{pmatrix} n+k-4 \\ k-2 \end{pmatrix}$, that is the dimension of the vector space $\mathcal{D}_k$ of homogeneous polynomials of degree $k-1$;

iii) all $\varphi_k \in \mathcal{D}_k$, are eigenfunctions corresponding to d_k.

Corollary 3.13 *Assume that $n \geq 2$ and that $\Omega = B$. Assume moreover that for all $k \in \mathbb{N}$ the set $\{\varphi_k^\ell : \ell = 1, \ldots, \mu_k\}$ is a basis of $\mathcal{D}_k$ chosen in such a way that the corresponding functions $\psi_k^\ell := r^{k-1}\varphi_k^\ell$ are orthonormal with respect to the scalar product (39). Then, for any $u \in V_4$ there exists a sequence $\{\alpha_k^\ell\} \in \ell^2$ ($k \in \mathbb{N}$; $\ell = 1, \ldots, \mu_k$) such that*

$$u(x) = (1 - |x|^2) \sum_{k=1}^{\infty} \sum_{\ell=1}^{\mu_k} \alpha_k^\ell \varphi_k^\ell(x) \qquad \text{for a.e. } x \in B\ .$$

3.5 Solvability of linear problems at resonance

Here we will report an interesting result on the existence of solution for problem $(P4)$. Before, it is helpful to introduce the following theorem, from [35, p.84, Theorem 5.11] that we will use for the proof:

Theorem 3.14 (Fredholm alternative) *Let $\mathcal{H}$ be a Hilbert space, T a compact mapping of $\mathcal{H}$ into itself and $T^\star$ its adjoint. Then there exists a countable set $\Lambda \subset \mathbb{R}$ having no limit points except possibly $\lambda = 0$, such that if $\lambda \neq 0$, $\lambda \notin \Lambda$ the equations*

$$\lambda x - Tx = y, \qquad \lambda x - T^\star x = y \tag{46}$$

have uniquely determined solutions $x \in \mathcal{H}$ for every $y \in \mathcal{H}$, and the inverse mappings $(\lambda I - T)^{-1}$, $(\lambda I - T^\star)^{-1}$ are bounded. If $\lambda \in \Lambda$, the null spaces of the mappings $\lambda I - T$, $\lambda I - T^\star$ have positive finite dimension and the equations (46) are solvable if and only if y is orthogonal to the null space of $\lambda I - T^\star$ in the first case and $\lambda I - T$ in the other.

Theorem 3.15 *Assume that d is an eigenvalue for $(E2)$ and u its eigenfunction. Then problem $(P2)$ is solvable if and only if f satisfies $\int_\Omega uf\,dx$.*

Proof: We endow the Hilbert space $H^1(\Omega)$ with the scalar product (39) and decompose system $(P2)$ with the help of two weak solution operators:

$$S_2 : H^1(\Omega) \to H^2(\Omega),$$

$$S_2 : w \in H^1(\Omega) \mapsto v_1 \in H^2(\Omega) : \qquad \begin{cases} -\Delta v_1 + v_1 = 0 & \text{in } \Omega, \\ v_{1\nu} = w & \text{on } \partial\Omega, \end{cases}$$

$$G_2 : L^2(\Omega) \to H^2(\Omega),$$

$$G_2 : f \in L^2(\Omega) \mapsto H^2(\Omega) : \qquad \begin{cases} -\Delta v_2 + v_2 = f & \text{in } \Omega, \\ v_{2\nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

Consider U such that

$$\begin{cases} -\Delta U = f & \text{in } \Omega, \\ U_\nu = dU & \text{on } \partial\Omega; \end{cases} \tag{47}$$

it means

$$\begin{cases} -\Delta U + U = f + U & \text{in } \Omega, \\ U_\nu = dU & \text{on } \partial\Omega. \end{cases}$$

Let $U \in H^2(\Omega) \subset H^1(\Omega)$ given; we define

$$\begin{array}{ll} v_1 := dS_2 U, \quad \Rightarrow & \begin{cases} -\Delta v_1 + v_1 = 0 & \text{in } \Omega, \\ v_{1\nu} = dU & \text{on } \partial\Omega; \end{cases} \\ v_2 := G_2 f, \quad \Rightarrow & \begin{cases} -\Delta v_2 + v_2 = f & \text{in } \Omega, \\ v_{2\nu} = 0 & \text{on } \partial\Omega; \end{cases} \\ v_3 := G_2 U, \quad \Rightarrow & \begin{cases} -\Delta v_3 + v_3 = U & \text{in } \Omega, \\ v_{3\nu} = 0 & \text{on } \partial\Omega; \end{cases} \end{array}$$

It follows that

$$\begin{cases} -\Delta(v_1 + v_2 + v_3) + (v_1 + v_2 + v_3) = f + U & \text{in } \Omega, \\ (v_1 + v_2 + v_3)_\nu = dU & \text{on } \partial\Omega. \end{cases}$$

If the solution U of $(P2)$ exists, then

$$\begin{aligned} U &= v_1 + v_2 + v_3 \\ &= dS_2 U + G_2 f + G_2 U. \end{aligned}$$

It is equivalent to

$$(I - dS_2 - G_2)U = G_2 f;$$

the operator $dS + G : H^1 \to H^2 \hookrightarrow H^1$ is compact, so defining $\lambda := \frac{1}{d}$, $T_2 := S_2 + \frac{1}{d} G_2$ and $y := \frac{1}{d} G_2 f$, we obtain a formulation like (46) and we can use Fredholm alternative.

We are looking for the dual $T_2^\star$ of T_2: if we determine it, then there exists a solution for (P_2) if and only if

$$y = \frac{1}{d} G_2 f \in \mathcal{N}\left(\frac{1}{d} I - T_2^\star\right)^\perp.$$

Let $v \in H^1(\Omega)$, we define $\tilde{v} := T_2 v$, then

$$\begin{cases} -\Delta \tilde{v} + \tilde{v} = \frac{1}{d} v & \text{in } \Omega, \\ \tilde{v}_\nu = v & \text{on } \partial\Omega. \end{cases}$$

We will see, T_2 is selfadjoint in H^1:

$$\begin{aligned}
(T_2 v, w) &= \int_\Omega \tilde{v} w \, dx + \int_\Omega \nabla \tilde{v} \cdot \nabla w \, dx \\
&= \int_\Omega \tilde{v} w \, dx + \int_{\partial\Omega} \tilde{v}_\nu w \, dS - \int_\Omega w \Delta \tilde{v} \, dx \\
&= \int_\Omega \tilde{v} w \, dx + \int_{\partial\Omega} v w \, dS + \int_\Omega \left(\frac{1}{d} v - \tilde{v} \right) w \, dx \\
&= \int_{\partial\Omega} v w \, dS + \frac{1}{d} \int_\Omega v w \, dx = (v, T_2 w), \quad \forall v, w \in H^1(\Omega)
\end{aligned}$$

because of the symmetry in v and w.

Then $\mathcal{N}\left(\frac{1}{d} I - T_2^\star\right) = \mathcal{N}\left(\frac{1}{d} I - T_2\right) = \{w \in H^1(\Omega) : -\Delta w = 0 \text{ in } \Omega,\ w_\nu = dw \text{ on } \partial\Omega\}$, that is the eigenspace of d for problem $(E2)$. We need now to determine the orthogonal complement of $\mathcal{N}\left(\frac{1}{d} I - T_2\right)$ in $H^1(\Omega)$. Let $u \in \mathcal{N}\left(\frac{1}{d} I - T_2\right)$ be arbitrary and $y := \frac{1}{d} G_2 f$. Then

$$\begin{aligned}
0 &= (u, y) = \int_\Omega u y \, dx + \int_\Omega \nabla u \cdot \nabla y \, dx \\
&= \int_\Omega u y \, dx + \int_{\partial\Omega} u y_\nu \, dS - \int_\Omega u \Delta y \, dx = \int_\Omega u (y - \Delta y) \, dx = \frac{1}{d} \int_\Omega u f \, dx.
\end{aligned}$$

□

Theorem 3.16 *Assume that δ is an eigenvalue for $(E4)$ and u its eigenfunction. Then, problem $(P4)$ is solvable if and only if g satisfies $\int_\Omega u g \, dx = 0$. When $\Omega = B$ and $\delta = n + 2(m-1)$ for some nonnegative integer m, problem $(P4)$ is solvable if and only if g satisfies $\int_B (1 - |x|^2) \varphi_m g \, dx = 0$.*

Proof: We endow the Hilbert space $H^2(\Omega) \cap H_0^1(\Omega)$ with the scalar product (42) and decompose system $(P4)$ with the help of two weak solution operators:

$$S_4 : H^{1/2}(\partial\Omega) \to H^2(\Omega) \cap H_0^1(\Omega),$$

$$S_4 : w \in H^{1/2}(\partial\Omega) \mapsto v_1 \in H^2(\Omega) \cap H_0^1(\Omega) : \qquad \begin{cases} \Delta^2 v_1 = 0 & \text{in } \Omega, \\ v_1 = 0 & \text{on } \partial\Omega, \\ \Delta v_1 = w & \text{on } \partial\Omega; \end{cases}$$

$$G_4 : L^2(\Omega) \to H^2 \cap H_0^1(\Omega),$$

$$G_4 : g \in L^2(\Omega) \mapsto H^2 \cap H_0^1(\Omega) : \qquad \begin{cases} \Delta^2 v_2 = g & \text{in } \Omega, \\ v_2 = 0 & \text{on } \partial\Omega, \\ \Delta v_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

It means,

$$(v_1, \varphi) \;=\; \int_\Omega \Delta v_1 \Delta \varphi \, dx = 0 \quad \left(\forall \varphi \in H^2 \cap H_0^1(\Omega)\right),$$

If it exists, the solution U of $(P4)$ satisfies

$$\begin{aligned} U &= \delta S_4 U_\nu + G_4 g \\ \Rightarrow U &= \left(I - \delta S_4 \frac{\partial}{\partial \nu}\right)^{-1} G_4 g \\ &= \left(\frac{1}{\delta} I - S_4 \frac{\partial}{\partial \nu}\right)^{-1} \frac{1}{\delta} G_4 g. \end{aligned}$$

Defining $\lambda := \frac{1}{\delta}$, $T_4 := S_4 \frac{\partial}{\partial \nu}$ and $y := \frac{1}{\delta} G_4 g$, we obtain a formulation like (46).

The operator T_4 is compact. We are looking for its dual $T_4^\star$: if we determine it, then there exists a solution for $(P4)$ if and only if

$$y = \frac{1}{\delta} G_4 g \in \mathcal{N}\left(\frac{1}{\delta} I - T_4^\star\right)^\perp.$$

Let $v \in H^2(\Omega) \cap H_0^1(\Omega)$, we define $\tilde{v} := T_4 v$, then

$$\begin{cases} \Delta^2 \tilde{v} = 0 & \text{in } \Omega, \\ \tilde{v} = 0 & \text{on } \partial\Omega, \\ \Delta \tilde{v} = v_\nu & \text{on } \partial\Omega. \end{cases}$$

We will see, T_4 is self-adjoint:

$$\begin{aligned} (T_4 v, w) &= (\tilde{v}, w) = \int_\Omega \Delta \tilde{v} \Delta w \, dx \\ &= \int_\Omega w \Delta^2 \tilde{v} \, dx + \int_{\partial\Omega} w_\nu \Delta \tilde{v} - w \Delta \tilde{v}_\nu \, dS \\ &= \int_{\partial\Omega} v_\nu w_\nu = (v, T_4 w), \qquad \forall v, w \in H^2(\Omega) \cap H_0^1(\Omega) \end{aligned}$$

because of the symmetry in v and w.

Then, $\mathcal{N}\left(\frac{1}{\delta} I - T_4^\star\right) = \mathcal{N}\left(\frac{1}{\delta} I - T_4\right) = \{w \in H^2(\Omega) \cap H_0^1(\Omega) : \Delta^2 w = 0 \text{ in } \Omega,\ w = 0 \text{ on } \partial\Omega,\ \Delta w = \delta w_\nu\}$, that is the eigenspace of δ for problem $(E4)$.

We need now to determine the orthogonal complement of $\mathcal{N}(\delta I - T_4)$ in $H^2(\Omega) \cap H_0^1(\Omega)$. Let $u \in \mathcal{N}\left(\frac{1}{\delta} I - T_4\right)$ be arbitrary and $y := \frac{1}{\delta} G_4 g$:

$$\begin{aligned} 0 &= (u,y) = \int_\Omega \Delta u \Delta y \, dx = \int_{\partial\Omega} u_\nu \Delta y - u(\Delta y)_\nu \, dS + \int_\Omega u \Delta^2 y \, dx \\ &= \int_\Omega u \frac{1}{\delta} g \, dx. \end{aligned} \tag{48}$$

In the special case of the ball $\Omega = B$, then (48) becomes

$$0 = (u,v) = \int_B \Delta u \Delta v \, dS = \int_B (1-|x|^2)\varphi_m g \, dx. \tag{49}$$

□

3.6 Nonlinear problems

Now we consider the nonlinear problems

$$(NL2) \qquad \begin{cases} -\Delta u = |u|^{p-1} u & \text{in } \Omega, \\ u_\nu - du = 0 & \text{on } \partial\Omega; \end{cases}$$

$$(NL4) \qquad \begin{cases} \Delta^2 u = |u|^{q-1} u & \text{in } \Omega, \\ u = \Delta u - \delta u_\nu = 0 & \text{on } \partial\Omega. \end{cases}$$

In order to prove existence results for problems $(NL2)$, $(NL4)$, we will adopt a version of the mountain pass lemma of [4] and introduce some concepts:

Definition 3.17 *Let E a functional on a Banach space V such that $E \in C^1(V)$. A sequence (u_m) in V is a Palais-Smale sequence for E if $|E(u_m)| \le c$ uniformly in m, while $\|DE(u_m))\| \to 0$ as $m \to \infty$.*

The following condition is known as Palais-Smale condition:

$(P.S.)$ Any Palais Smale sequence
has a strongly convergent subsequence.

It plays a fundamental role for the following theorem (see [65, 6.5 Theorem])

Theorem 3.18 *Suppose V is an infinite dimensional Banach space and suppose $E \in C^1(V)$ satisfies $(P.S.)$, $E(u) = E(-u)$ for all u, and $E(0) = 0$. Suppose $V = V^- \oplus V^+$, where V^- is finite dimensional, and assume the following conditions:*

i) $\exists \alpha > 0, \rho > 0 : \|u\| = \rho, u \in V^+ \Rightarrow E(u) \geq \alpha$.

ii) *For any finite dimensional subspace $W \subset V$ there is $R = R(W)$ such that $E(u) \leq 0$ for $u \in W$, $\|u\| \geq R$.*

Then E possesses an unbounded sequence of critical values.

We will apply Theorem 3.18:

Theorem 3.19 *Assume that $1 < p < \frac{n+2}{n-2}$ ($n \geq 3$). Assume moreover that $d < 0$ and $\Omega \in C^2$. Then problem $(NL2)$ admits infinitely many solutions.*

Proof: Consider the energy functional of problem $(NL2)$:

$$J_2(u) := \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{d}{2}\int_{\partial\Omega} u^2\,dS - \frac{1}{p+1}\int_\Omega |u|^{p+1}\,dx$$

in the space $H^1(\Omega)$ endowed with the norm

$$\|u\|_2^2 := \int_\Omega |\nabla u|^2\,dx + \int_{\partial\Omega} |u|^2\,dS.$$

This functional is clearly even, that is $J_2(u) = J_2(-u)$, moreover we have $J_2(0) = 0$. Let now $\{u_m\}_{m\in\mathbb{N}} \subset H^1(\Omega)$ be a Palais-Smale sequence, that is there exists a constant $C > 0$ such that

$$|J_2(u_m)| = \left|\frac{1}{2}\int_\Omega |\nabla u_m|^2\,dx - \frac{d}{2}\int_{\partial\Omega} |u_m|^2\,dS - \frac{1}{p+1}\int_\Omega |u_m|^{p+1}\,dx\right| < C$$

uniformly in m, while

$$J_2'(u_m) \to 0 \quad \text{in } (H^1(\Omega))^* \qquad \text{as } m \to \infty,$$

where $(H^1(\Omega))^*$ is the dual of $H^1(\Omega)$. Then

$$J_2'(u_m)[u_m] = \int_\Omega |\nabla u_m|^2\,dx - d\int_{\partial\Omega} |u_m|^2\,dS - \int_\Omega |u_m|^{p+1}\,dx = o(\|u_m\|_2)$$

We can prove that $\{u_m\}_{m\in\mathbb{N}}$ is bounded:

$$\begin{aligned}(p+1)C + o(1) + o(\|u_m\|_2) &\geq (p+1)J_2(u_m) - J_2'(u_m)[u_m] \\ &= \frac{p-1}{2}\left(\int_\Omega |\nabla u_m|^2\,dx - d\int_{\partial\Omega} |u_m|^2\,dS\right) \\ &\geq \frac{p-1}{2}\min\{1, -d\}\|u_m\|_2^2 = C_1\|u_m\|_2^2,\end{aligned}$$

with $C_1 = \frac{p-1}{2}\min\{1, -d\}$.

It follows, there exists a subsequence $\{u_{m_j}\}_{j\in\mathbb{N}}$ of $\{u_m\}_{m\in\mathbb{N}}$ and a function $u \in H^1(\Omega)$ such that u_{m_j} converges weakly to u in $H^1(\Omega)$. Moreover, by compact embedding, weak convergence in $H^1(\Omega)$ implies strong convergence of u_{m_j} to u in $L^{p+1}(\Omega)$, that is

$$\int_\Omega |u_{m_j}|^{p+1}\,dx \to \int_\Omega |u|^{p+1}\,dx.$$

By [1, Theorem 5.22], we know that the linear trace operator

$$\gamma : H^1(\Omega) \to L^q(\partial\Omega), \quad \gamma : u \mapsto u|_{\partial\Omega}$$

is a compact map if $2 \le q < \frac{2(n-1)}{n-2}$, so it follows that

$$\int_{\partial\Omega} |u_{m_j}|^2\,dx \to \int_{\partial\Omega} |u|^2\,dx.$$

In order to prove that u is a solution of $(NL2)$, we see that

$$\begin{aligned} J_2'(u_{m_j})[\varphi] &= \int_\Omega \nabla u_{m_j}\cdot\nabla\varphi\,dx - d\int_{\partial\Omega} u_{m_j}\varphi\,dS - \int_\Omega |u_m|^{p-1}u_{m_j}\varphi\,dx \\ &\to 0, \qquad \forall\varphi\in H^1(\Omega), \end{aligned}$$

and that

$$\begin{aligned} J_2'(u_{m_j})[\varphi] &\to \int_\Omega \nabla u\cdot\nabla\varphi\,dx - d\int_{\partial\Omega} u\varphi\,dS - \int_\Omega |u|^{p-1}u\varphi\,dx \\ &= J_2'(u)[\varphi], \quad \forall\varphi\in H^1(\Omega). \end{aligned}$$

Finally,

$$\begin{aligned} \mathcal{O}(1) - 0 &= J_2'(u_{m_j})[u_{m_j}] - J_2'(u)[u] \\ &= \int_\Omega \left(|\nabla u_{m_j}|^2 - |\nabla u|^2\right)\,dx \\ &\quad -d\int_{\partial\Omega}\left(|u_{m_j}|^2 - |u|^2\right)\,dS \\ &\quad -\int_\Omega \left(|u_{m_j}|^{p+1} - |u|^{p+1}\right)\,dx \\ &= \int_\Omega \left(\left|\nabla u_{m_j}\right|^2 - |\nabla u|^2\right)\,dx + \mathcal{O}(1) \\ &\to 0 \end{aligned}$$

so that $\int_\Omega |\nabla u_{m_j}|^2\,dx \to \int_\Omega |\nabla u|^2\,dx$ which, combined with weak convergence implies that u_{m_j} converges to u in $H^1(\Omega)$ strongly, that is the Palais-Smale condition is satisfied.

We need now to verify properties i) and ii) of Theorem 3.18 for the functional J_2. First, we define $V^- := \{0\}$ and $V^+ = H^1(\Omega)$.

i) Let $C_2 := \min\left\{\frac{1}{2}, -\frac{d}{2}\right\}$ and $u \in H^1(\Omega)$, such that $\|u\|_2 = \rho$; by Sobolev imbedding (see [35][Corollary 7.11]) there exists a constant $C_3 > 0$ such that $\|u\|_{L^{p+1}(\Omega)} \le C_3\|u\|_2$. So

$$\begin{aligned} J_2(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{d}{2}\int_{\partial\Omega} u^2\,dS - \frac{1}{p+1}\int_\Omega |u|^{p+1}\,dx \\ &\ge C_2\|u\|_2^2 - \frac{C_3^{p+1}}{p+1}\|u\|_2^{p+1}. \end{aligned}$$

Defining $C_4 := \frac{C_3^{p+1}}{p+1}$ and $\psi_1(t) := C_2t^2 - C_4t^{p+1}$, we see that $J_2(u) \ge \psi_1(\|u\|_2)$.

The function $\psi_1(t)$ attains a positive maximum $M = \left(\frac{2C_2}{(p+1)C_4}\right)^{\frac{p+1}{p-1}} \frac{(p-1)}{2}C_4$ at $t_M = \sqrt[p-1]{\frac{2C_2}{(p+1)C_4}}$, so the functional J_2 satisfies the condition i) for $\rho := \sqrt[p-1]{\frac{2C_2}{(p+1)C_4}}$ and $\alpha := M$.

ii) Let W be any finite dimensional subspace of $H^1(\Omega)$ and let $u \in W$, such that $\|u\|_2 = 1$. So

$$J_2(u) \le \frac{1-d}{2} - \frac{1}{p+1}\int_\Omega |u|^{p+1}\,dx$$

and

$$J_2(tu) \le \frac{1-d}{2}t^2 - \frac{t^{p+1}}{p+1}\int_\Omega |u|^{p+1}\,dx.$$

Since W is finite dimensional, there exists

$$C_5 := \inf_{u\in W, \|u\|_2=1}\int_\Omega |u|^{p+1}\,dx > 0.$$

We define $C_6 := \frac{C_5}{p+1}$ and $\psi_2(t) := C_2t^2 - C_6t^{p+1}$, which is negative for $t > \left(\frac{C_2}{C_6}\right)^{\frac{1}{p-1}}$. So we obtain that

$$J_2(tu) \le \psi_2(t) < 0, \quad \forall t > \left(\frac{C_2}{C_6}\right)^{\frac{1}{p-1}},$$

that is $J_2(u) < 0$ if $\|u\|_2 > \left(\frac{C_2}{C_6}\right)^{\frac{1}{p-1}}$ and $ii)$ follows. □

Theorem 3.20 *Assume that* $1 < q < \frac{n+4}{n-4}$ *(*$n \geq 5$*). Assume moreover that* $\delta < 0$ *and* $\Omega \in C^2$*. Then problem* $(NL4)$ *admits infinitely many solutions.*

Proof: We consider the energy functional of problem $(NL4)$:

$$J_4(u) := \frac{1}{2}\int_\Omega |\Delta u|^2\,dx - \frac{\delta}{2}\int_{\partial\Omega} u_\nu^2\,dS - \frac{1}{q+1}\int_\Omega |u|^{q+1}\,dx$$

in the space $H^2(\Omega) \cap H_0^1(\Omega)$ endowed with the norm

$$\|u\|_4^2 := \int_\Omega |\Delta u|^2\,dx$$

This functional is clearly even, that is $J_4(u) = J_4(-u)$, moreover we have $J_4(0) = 0$.

Let now $\{u_m\}_{m\in\mathbb{N}} \subset H^2(\Omega) \cap H_0^1(\Omega)$ be a Palais-Smale sequence, that is there exists a constant $C > 0$ such that

$$|J_4(u_m)| = \left|\frac{1}{2}\int_\Omega |\Delta u_m|^2\,dx - \frac{\delta}{2}\int_{\partial\Omega} |u_{m\nu}|^2\,dS - \frac{1}{q+1}\int_\Omega |u_m|^{q+1}\,dx\right| < C$$

uniformly in m, while

$$J_4'(u_m) \to 0 \quad \text{in } (H^2(\Omega) \cap H_0^1(\Omega))^* \qquad \text{as } m \to \infty,$$

where $(H^2(\Omega) \cap H_0^1(\Omega))^*$ is the dual of $H^2(\Omega) \cap H_0^1(\Omega)$

Then

$$\begin{aligned} J_4'(u_m)[u_m] &= \int_\Omega |\Delta u_m|^2\,dx - \delta\int_{\partial\Omega} |u_{m\nu}|^2\,dS - \int_\Omega |u_m|^{q+1}\,dx \\ &= o(\|u_m\|_4) \end{aligned}$$

We can prove that $\{u_m\}_{m\in\mathbb{N}}$ is bounded:

$$\begin{aligned} (q+1)C + o(1) + o(\|u_m\|_4) &\geq (q+1)J_4(u_m) - J_4'(u_m)[u_m] \\ &= \frac{q-1}{2}\left(\int_\Omega |\Delta u_m|^2\,dx - \delta\int_{\partial\Omega} |u_{m\nu}|^2\,dS\right) \\ &\geq \frac{q-1}{2}\min\{1, -\delta\}\|u_m\|_4^2 = C_1\|u_m\|_4^2, \end{aligned}$$

with $C_1 = \frac{q-1}{2}\min\{1, -\delta\}$.

It follows, there exist a subsequence $\{u_{m_j}\}_{j\in\mathbb{N}}$ of $\{u_m\}_{m\in\mathbb{N}}$ and a function $u \in H^2(\Omega)\cap H^1_0(\Omega)$ such that u_{m_j} converges weakly to u in $H^2(\Omega)\cap H^1_0(\Omega)$:

By compact embedding $H^2(\Omega)$ in $L^{q+1}(\Omega)$ for every $q < \frac{n+4}{n-4}$, then weak convergence in $H^2(\Omega)$ implies strong convergence in $L^{q+1}(\Omega)$ of u_{m_j} to u, for any $q < \frac{n+4}{n-4}$.

Then we can conclude that

$$\int_\Omega |u_{m_j}|^{q+1}\,dx \to \int_\Omega |u|^{q+1}\,dx.$$

In order to prove that u is a solution of $(NL4)$, we see that

$$\begin{aligned} J_4'(u_{m_j})[\varphi] &= \int_\Omega \Delta u_{m_j}\Delta\varphi\,dx - \delta\int_{\partial\Omega}(u_{m_j})_\nu(\varphi)_\nu\,dS \\ &\quad - \int_\Omega |u_m|^{q-1}u_{m_j}\varphi\,dx \to 0, \quad \forall\varphi \in H^2(\Omega)\cap H^1_0(\Omega), \end{aligned}$$

and that

$$\begin{aligned} J_4'(u_{m_j})[\varphi] &\to \int_\Omega \Delta u\Delta\varphi\,dx - \delta\int_{\partial\Omega}u_\nu\varphi_\nu\,dS - \int_\Omega |u|^{q-1}u\varphi\,dx \\ &= J_4'(u)[\varphi], \quad \forall\varphi \in H^2(\Omega)\cap H^1_0(\Omega). \end{aligned}$$

Finally,

$$\begin{aligned} J_4'(u_{m_j})[u_{m_j}] - J_4'(u)[u] &= \int_\Omega \left(|\Delta u_{m_j}|^2 - |\Delta u|^2\right)\,dx \\ &\quad -\delta\int_{\partial\Omega}\left(|(u_{m_j})_\nu|^2 - |u_\nu|^2\right)\,dS \\ &\quad -\int_\Omega \left(|u_{m_j}|^{q+1} - |u|^{q+1}\right)\,dx \to 0, \end{aligned}$$

so that $\int_\Omega |\Delta u_{m_j}|^2\,dx \to \int_\Omega |\Delta u|^2\,dx$ which, combined with weak convergence implies that u_{m_j} converges to u in $H^2(\Omega)\cap H^1_0(\Omega)$ strongly, that is the Palais-Smale condition is satisfied.

We need now to verify properties i) and ii) of Theorem 3.18 for the functional J_4. First, we define $V^- := \{0\}$ and $V^+ = H^2(\Omega)\cap H^1_0(\Omega)$.

i) Let $C_2 := \min\left\{\frac{1}{2}, -\frac{\delta}{2}\right\}$ and $u \in H^2(\Omega)\cap H^1_0(\Omega)$, such that $\|u\|_4 = \rho$; by Sobolev imbedding (see [35][Corollary 7.11]) there exists a constant $C_3 > 0$

such that $\|u\|_{L^{q+1}(\Omega)} \leq C_3\|u\|_4$. So

$$\begin{aligned} J_4(u) &= \frac{1}{2}\int_\Omega |\Delta u|^2\,dx - \frac{\delta}{2}\int_{\partial\Omega} u_\nu^2\,dS - \frac{1}{q+1}\int_\Omega |u|^{q+1}\,dx \\ &\geq C_2\|u\|_4^2 - \frac{C_3^{q+1}}{q+1}\|u\|_4^{q+1}. \end{aligned}$$

Defining $C_4 := \frac{C_3^{q+1}}{q+1}$ and $\psi_1(t) := C_2t^2 - C_4t^{q+1}$, we see that $J_4(u) \geq \psi_1(\|u\|_4)$.

The function $\psi_1(t)$ attains a positive maximum $M = \left(\frac{2C_2}{(q+1)C_4}\right)^{\frac{q+1}{q-1}} \frac{(q-1)}{2}C_4$ at $t_M = \sqrt[q-1]{\frac{2C_2}{(q+1)C_4}}$, so the functional J_4 satisfies the condition i) for $\rho := \sqrt[q-1]{\frac{2C_2}{(q+1)C_4}}$ and $\alpha := M$.

ii) Let W be any finite dimensional subspace of $H^2(\Omega)\cap H_0^1(\Omega)$ and let $u \in W$, such that $\|u\|_4 = 1$. So

$$J_4(u) \leq \frac{1-\delta}{2} - \frac{1}{q+1}\int_\Omega |u|^{q+1}\,dx$$

and

$$J_4(tu) \leq \frac{1-\delta}{2}t^2 - \frac{t^{q+1}}{q+1}\int_\Omega |u|^{q+1}\,dx.$$

Since W is finite dimensional, there exists

$$C_5 := \inf_{u\in W, \|u\|_4=1}\int_\Omega |u|^{q+1}\,dx > 0.$$

We define $C_6 := \frac{C_5}{q+1}$ and $\psi_2(t) := C_2t^2 - C_6t^{q+1}$, which is negative for $t > \left(\frac{C_2}{C_6}\right)^{\frac{1}{q-1}}$. So we obtain that

$$J_4(tu) \leq \psi_2(t) < 0, \quad \forall t > \left(\frac{C_2}{C_6}\right)^{\frac{1}{q-1}},$$

that is $J_4(u) < 0$ if $\|u\|_4 > \left(\frac{C_2}{C_6}\right)^{\frac{1}{q-1}}$ and $ii)$ follows. □

4 Semilinear biharmonic eigenvalue problems with exponential growth

4.1 Introduction

Let $n \geq 5$, B the unit ball centered at the origin and $\frac{\partial}{\partial \nu}$ the differentiation with respect to the exterior unit normal, i.e. the radial direction. We will study in this section the problem

$$\begin{cases} \Delta^2 u = \lambda e^u & \text{in } B, \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial B; \end{cases} \tag{50}$$

where $\lambda \geq 0$ is a parameter.

In particular, we try to give a partial answer to a question of Lions, who in [52, Section4.2(c)] asked if it is possible to describe the solution set of semilinear systems; (50) is a special case of the latter.

Some answers were given by Arioli, Gazzola, Grunau, Mitidieri ([6]). They proved also the existence of a $\lambda^\star$ such that if $\lambda < \lambda^\star$ problem (50) admits a minimal regular solution U_λ and not even a weak solution if $\lambda > \lambda^\star$. Moreover, they gave a characterization of the regular and weakly singular radial solutions, that is the solutions show, in some sense, a limited irregularity in the center of the domain. We will show in the next subsections, that any radial singular solution is also weakly singular i.e. that $\lim_{r\to 0} ru'(r) \in \mathbb{R}$ exists.

The section is organized as follows: in the next subsection a precise formulation of our results is given. In subsection 4.3 system (50) is transformed into an autonomous system of ordinary differential equations. Subsection 4.4 displays properties of regular and weakly singular solutions. In subsection 4.5 we will use energy functions in order to obtain more information on the properties of singular radial solutions.

4.2 Definition and main results

Let p fixed, with $p > \frac{n}{4}$ and $p \geq 2$. According to [6] and using [27] as a guide to solve the problem, we will use the following definitions:

Definition 4.1 *We say that $u \in L^2(B)$ is a solution of (50) if $e^u \in L^1(B)$ and*

$$\int_B u\Delta^2 v = \lambda \int_B e^u v \quad \textit{for all } v \in W^{4,p}(B) \cap H_0^2(B).$$

We say that a solution u of (50) *is regular* (*resp. singular*) *if* $u \in L^\infty(B)$ (*resp.* $u \notin L^\infty(B)$).

Regular and singular solutions are all possible radial solutions of the problem. It follows from elliptic regularity that any regular solution of (50) is in $C^\infty(\overline{B})$.

Definition 4.2 *We say that a radial singular solution $u = u(r)$ of* (50) *is weakly singular if the limit* $\lim_{r\to 0} ru'(r)$ *exists.*

This definition states that weakly singular solutions blow up with a well defined asymptotic behaviour at 0. However, we prove that any radial singular solution has this specified behaviour:

Theorem 4.3 *Any radial singular solution of* (50) *is also weakly singular.*

This result was showed also by Davila, Dupagne, Guerra, Montenegro.

We restrict ourselves study to the ball because of for the proof we need the positivity preserving property of the biharmonic operator, which implies in particular [13], that any solution of (50) is positive. This property holds on balls but fails in general domains.

Finally, we can give an explicit estimate from below for radial singular solutions and the corresponding singular parameter, using the energy considerations of subsection 4.5

Theorem 4.4 *Assume that u_s is a singular radial solution of* (50) *with parameter λ_s. Then $\lambda_s > \lambda_0 = 8(n-2)(n-4)$ and near zero the behaviour of the singular solution is*

$$\begin{aligned} u_s(x) &= -4\log|x| + \log\frac{\lambda_0}{\lambda_s} + o(1), \\ u_s(x) &> -4\log|x| + \log\frac{\lambda_0}{\lambda_s}. \end{aligned}$$

As a direct consequence we can draw a conclusion concerning the regularity of the extremal solution $U^\star \in H_0^2(B)$, $e^{U^\star} \in L^1(B)$.

Theorem 4.5 *If $5 \le n \le 12$, the extremal solution is regular.*

This result was obtained independently and by using different techniques by Davila, Dupagne, Guerra, Montenegro, see [21].

4.3 Autonomous system

We quote here some results of [6], that we will employ for our next proofs. Looking for radial solutions, then (50) can be rewritten in the form (as $0 < r \leq 1$)

$$\begin{cases} u^{(4)}(r) + \frac{2(n-1)}{r}u'''(r) + \frac{(n-1)(n-3)}{r^2}u''(r) - \frac{(n-1)(n-3)}{r^3}u'(r) = \lambda e^{u(r)}, \\ u(1) = 0, \\ u'(1) = 0. \end{cases} \tag{51}$$

Like in the works [6] and [27], we introduce

$$s := \log(r), \qquad v : (-\infty, 0] \to \mathbb{R}, \quad v(s) := u(e^s),$$

such that (51) becomes

$$\begin{aligned} \lambda(e^{v(s)+4s}) &= v^{(4)}(s) + 2(n-4)v'''(s) + (n^2 - 10n + 20)v''(s) \\ &\quad -2(n^2 - 6n + 8)v'(s), \end{aligned} \tag{52}$$

with the conditions $v(0) = 0$ and $v'(0) = 0$. Using the following substitution

$$\begin{cases} v_1(s) = v'(s) + 4, \\ v_2(s) = -v''(s) - (n-2)v'(s), \\ v_3(s) = -v'''(s) + (4-n)v''(s) + 2(n-2)v'(s), \\ v_4(s) = -\lambda e^{v(s)+4s}, \end{cases}$$

we obtain the autonomous system

$$\begin{cases} v_1'(s) = (2-n)v_1(s) - v_2(s) + 4(n-2), \\ v_2'(s) = 2v_2(s) + v_3(s), \\ v_3'(s) = (4-n)v_3(s) + v_4(s), \\ v_4'(s) = v_1(s)v_4(s). \end{cases} \tag{53}$$

together with the initial conditions $v_1(0) = 4$, $v_4(0) = -\lambda$. System (53) has two stationary points

$$P_1 = (4, 0, 0, 0) \text{ and } P_2 = (0, 4n-8, 16-8n, -8(n-2)(n-4)), \tag{54}$$

these correspond to $v_1(0) = 4$ and $v_1(0) = 0$.

Near P_1 we can linearize the problem and substitute the equation

$$v_4'(s) = v_1(s)v_4(s) \quad \text{with} \quad v_4'(s) = 4v_4(s)$$

such that the associated matrix to the problem is

$$M_1 = \begin{pmatrix} 2-n & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 4-n & 1 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

Its eigenvalues are $\mu_1 = 2$, $\mu_2 = 4$ and the negative $\mu_3 = 2-n$ and $\mu_4 = 4-n$. It means, P_1 is a hyperbolic point and that both the stable and unstable manifolds are two-dimensional.

In the same way, near P_2 we adopt the substitution of

$$v_4'(s) = v_1(s)v_4(s) \quad \text{with} \quad v_4'(s) = -8(n-2)(n-4)v_1(s).$$

So we obtain the associated matrix M_2 of the form

$$M_2 = \begin{pmatrix} 2-n & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 4-n & 1 \\ -8(n-2)(n-4) & 0 & 0 & 0 \end{pmatrix}.$$

Its eigenvalues are the solutions of the equation

$$(\nu - 2 + n)(\nu - 2)(\nu - 4 + n)\nu - 8(n-2)(n-4) = 0,$$

that is

$$\nu_{1,2,3,4} = \frac{1}{2}\left(4 - n \pm \sqrt{M_1(n) \pm M_2(n)}\right),$$

where $M_1(n) = n^2 - 4n + 8 > (n-2)^2$ and $M_2(n) = 4\sqrt{68 - 52n + 9n^2}$. Then

$$\nu_1 = \frac{1}{2}\left(4 - n + \sqrt{M_1(n) + M_2(n)}\right), \quad \nu_2 = \frac{1}{2}\left(4 - n - \sqrt{M_1(n) + M_2(n)}\right)$$

are real numbers and $\nu_2 < 0 < \nu_1$ for all $n \geq 4$.

For $5 \leq n \leq 12$ it follows that $M_1(n) - M_2(n) < 0$, while for $n \geq 13$ we have $M_1(n) - M_2(n) > 0$. Then for n between 5 and 12,the eigenvalues

$$\nu_3 = \frac{1}{2}\left(4 - n + \sqrt{M_1(n) - M_2(n)}\right), \quad \nu_4 = \frac{1}{2}\left(4 - n - \sqrt{M_1(n) - M_2(n)}\right)$$

are complex conjugate with real part $\frac{1}{2}(4-n) < 0$. In the case of $n \geq 13$, ν_3 and ν_4 are both real and negative. We can summarize here:

Proposition 4.6 • *For any $n \geq 5$ we have $\nu_1, \nu_2 \in \mathbb{R}$ and $\nu_2 < 0 < \nu_1$.*

- *For any $5 \leq n \leq 12$ we have $\nu_3, \nu_4 \notin \mathbb{R}$ and $\mathrm{Re}(\nu_3) = \mathrm{Re}(\nu_4) < 0$.*
- *For any $n \geq 13$, $\nu_3, \nu_4 \in \mathbb{R}$ and both negative.*

Moreover, the point P_2 is hyperbolic, its stable manifold is 3-dimensional and the unstable 1-dimensional. There is a direction in the stable manifold, along this there is no oscillation.

4.4 Characterisation of regular and weakly singular solutions

Here we recall [6, Theorem 3]:

Theorem 4.7 *Let $u = u(r)$ be a radial solution of* (50) *and let*

$$W(t) = (w_1(t), w_2(t), w_3(t), w_4(t))$$

be the corresponding trajectory relative to (53). *Then*

- *u is regular (i.e. $u \in L^\infty(B)$) if and only if*

$$\lim_{t \to -\infty} W(t) = P_1.$$

- *u is weakly singular if and only if*

$$\lim_{t \to -\infty} W(t) = P_2.$$

In order to prove Theorem 4.3, which states, that every radial singular solution is weakly singular, we introduce the function

$$z : [0, \infty) \to \mathbb{R}, \quad z(s) := v(-s) - 4s + \log \lambda - \log \lambda_0, \tag{55}$$

where $\lambda_0 := 8(n-2)(n-4)$. Consequently equation (52) becomes

$$z^{(4)} - 2(n-4)z''' + (n^2 - 10n + 20)z'' + 2(n^2 - 6n + 8)z' = \lambda_0(e^z - 1). \tag{56}$$

The Theorem 4.8 and Proposition 4.10 will give a qualitative description of the behaviour of z to infinity:

Theorem 4.8 *Let u be a radial solution of the Dirichlet problem* (50) *and define the corresponding function $z = z(s)$ according to* (55). *Then z is bounded from above.*

Proof: *i*) Assume by contradiction that z is not bounded from above. Then it can happen that the limit as $t \to \infty$ does not exist or exists and equals $+\infty$. We assume that the limit does not exist, meaning that

$$\liminf_{t\to\infty} z(t) < \limsup_{t\to\infty} z(t) = +\infty.$$

Then, there is a sequence $t_k \to \infty$ of local maxima for z such that for all k, $z'(t_k) = 0$ and $\lim_{k\to\infty} t_k = \infty$. We define

$$\tilde{z}_k(t) = z(t + t_k), \quad t \in (-t_k, \infty)$$

such that, if $z(t)$ solves (56), then $\tilde{z}_k$ is an admissible solution too. Let now

$$\begin{aligned} \tilde{u}_k(r) &:= \tilde{z}_k(-\log r) - 4\log r - \log\lambda + \log\lambda_0 \\ &= z(-\log r + t_k) - 4\log r - \log\lambda + \log\lambda_0 = u(re^{-t_k}) - 4t_k, \\ \tilde{u}_k'(r) &= -\frac{z'(-\log r + t_k)}{r} - \frac{4}{r}. \end{aligned}$$

When we pull back the function $\tilde{u}_k$ on B we obtain that

$$\begin{cases} \Delta^2 \tilde{u}_k = \lambda e^{\tilde{u}_k} & \text{in } B, \\ \tilde{u}_k(1) = z(t_k) > 0 & \text{on } \partial B, \\ -\frac{\partial \tilde{u}_k}{\partial \nu}(1) = \frac{z'(t_k)}{r} + 4 = 4 > 0 & \text{on } \partial B. \end{cases}$$

We define $U_k(x) := \tilde{u}_k(x) - z(t_k) = u(re^{-t_k}) - t_k - z(t_k)$ and $\lambda_k = \lambda e^{z(t_k)}$, so we obtain

$$\begin{cases} \Delta^2 U_k = \lambda_k e^{U_k} & \text{in } B, \\ U_k(1) = 0 & \text{on } \partial B, \\ -\frac{\partial U_k}{\partial \nu}(1) = \frac{z'(t_k)}{r} + 4 = 4 > 0 & \text{on } \partial B. \end{cases}$$

The boundary problem is solved in weak sense since U_k is a rescaled and translated version of the original weak solution u. We can observe that there is a comparison principle in B with respect to the boundary datum $-\frac{\partial u_k}{\partial \nu}$, see [43]. This shows that U_k is a weak supersolution for the problem

$$\begin{cases} \Delta^2 u = \lambda_k e^u, \ u > 0 & \text{in } B, \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial B. \end{cases} \tag{57}$$

We can infer by standard arguments, like in [7, Lemma 3.3] that for any λ_k problem (57) admits a weak solution. Since $\lambda_k \to \infty$, this contradicts the nonexistence of solutions of (50), like proved in [6] for large λ.

ii) Now, suppose that $\lim_{t\to\infty} z(t) = +\infty$. There exists a $T \in \mathbb{R}$ such that $\forall t \geq T$

$$\begin{aligned} f'(t) &:= z^{(4)}(t) - 2(n-4)z'''(t) + (n^2 - 10n + 20)z''(t) \\ &\quad +(2n^2 - 12n + 16)z'(t) > \frac{\lambda}{2} e^{z(t)}. \end{aligned} \tag{58}$$

By integrating (58) over $[T, t]$, for all $t \geq T$, we get

$$\begin{aligned} f(t) &= z'''(t) - 2(n-4)z''(t) + (n^2 - 10n + 20)z'(t) \\ &\quad +(2n^2 - 12n + 16)z(t) > \frac{\lambda}{2} \int_T^t e^{z(s)}\, ds + C_1, \end{aligned}$$

where

$$\begin{aligned} C_1 &= C_1(T) = f(T) \\ &= z'''(T) - 2(n-4)z''(T) + (n^2 - 10n + 20)z'(T) \\ &\quad +(2n^2 - 12n + 16)z(T). \end{aligned}$$

Because of for $t \to +\infty$ it follows $z(t) \to +\infty$, $e^{z(t)} \to +\infty$ and also $f'(t) \to +\infty$, then there is a $T' \geq T$ such that $f(T') = C(T') > 0$. Since (56) is autonomous, we may assume that $T' = 0$. It is

$$\begin{aligned} \frac{\lambda}{2} \int_T^t e^{z(s)}\, ds &< z'''(t) - 2(n-4)z''(t) + (n^2 - 10n + 20)z'(t) \\ &\quad +(2n^2 - 12n + 16)z(t) \\ C_1(0) &= z'''(0) - 2(n-4)z''(0) + (n^2 - 10n + 20)z'(0) \\ &\quad +(2n^2 - 12n + 16)z(0) > 0. \end{aligned} \tag{59}$$

We now apply the test function method developed by Mitidieri-Pohozahev [55]. More precisely, fix $T_1 > T > 0$ and a nonnegative function $\varphi \in C_c^4[0, \infty)$ such that

$$\varphi(t) = \begin{cases} 1 & \text{for } t \in [0, T], \\ 0 & \text{for } t > T_1. \end{cases}$$

Then $\varphi^{(k)}(T_1) = 0$, for all $k \in \mathbb{N}$. Hence, multiplying (58) by $\varphi(t)$, integrating by parts and recalling (59) yields

$$\begin{aligned} &\int_0^{T_1} \big[\varphi^{(4)}(t) + 2(n-4)\varphi'''(t) + (n^2 - 10n + 20)\varphi''(t) \\ &\quad -(2n^2 - 12n + 16)\varphi'(t)\big]\, z(t)\, dt > \frac{\lambda}{2} \int_0^{T_1} e^{z(t)} \varphi(t)\, dt + C_1(0). \end{aligned} \tag{60}$$

We apply Young's inequality in the following form: For any $\varepsilon > 0$ there exists $C_2 = C_2(\varepsilon) > 0$ such that

$$z\varphi^{(i)} = z\varphi^{\frac{1}{2}}\frac{\varphi^{(i)}}{\varphi^{\frac{1}{2}}} \leq \varepsilon z^2\varphi + C_2(\varepsilon)\frac{|\varphi^{(i)}|^2}{\varphi}, \quad \varphi^{(i)} = \frac{d^i\varphi}{dt^i}, \quad (\forall i = 1,2,3,4).$$

Then, provided ε sufficiently small, (60) becomes

$$C_3\sum_{i=1}^{4}\int_0^{T_1}\frac{|\varphi^{(i)}(t)|^2}{\varphi(t)}\,dt \geq \frac{\lambda}{4}\int_0^T e^{z(t)}\,dt + C(0), \tag{61}$$

where $C_3 = C_3(\varepsilon, n)$. Now we choose $\varphi(t) = \varphi_0\left(\frac{t}{T}\right)$, where $\varphi_0 \in C_c^4([0,\infty))$, $\varphi_0 \geq 0$ and

$$\varphi_0(\tau) = \begin{cases} 1 & 0 \leq \tau \leq 1, \\ 0 & \tau \geq \tau_1 > 1. \end{cases}$$

As noticed in [55], there exists a function φ_0 in such class satisfying moreover

$$\int_0^{\tau_1}\frac{|\varphi_0^{(i)}(\tau)|^2}{\varphi_0(\tau)}\,d\tau = A_i < \infty \quad (i = 1,2,3,4).$$

Then, thanks to a change of variables in the integrals and with the right choose of constants, (61) becomes

$$C_4\sum_{i=1}^{4}A_iT^{1-2i} \geq \frac{\lambda}{4}\int_0^T e^{z(t)}\,dt + C_1(0), \quad \forall T > 0.$$

Letting $T \to \infty$, the previous inequality contradicts $\lim_{t\to\infty} z(t) = +\infty$. $\square$

Proposition 4.9 *Let z be a solution of* (56) *and assume there exists* $L \in (-\infty, +\infty]$ *such that*

$$\lim_{t\to\infty} z(t) = L.$$

Then, $L = 0$.

Proof: For contradiction, assume that $L \neq 0$.

It can not be that $L = +\infty$ because of Theorem 4.8.

Suppose L is finite positive. Then $\lambda_0(e^{z(t)} - 1) \to \alpha = \lambda_0(e^L - 1) > 0$ and for all $\varepsilon > 0$ there exists $T > 0$ such that

$$\begin{aligned} \alpha - \varepsilon \ \leq\ & z^{(4)}(t) - 2(n-4)z'''(t) + (n^2 - 10n + 20)z''(t) \\ & +2(n^2 - 6n + 8)z'(t) \leq \alpha + \varepsilon, \quad \forall t \geq T. \end{aligned} \tag{62}$$

Take $\varepsilon < \alpha$ so that $\alpha - \varepsilon$ is positive and let

$$\delta = \sup_{t \geq T} |z(t) - z(T)| < \infty.$$

Integrating (62) over $[T, t]$ for any $t \geq T$, we obtain

$$\begin{aligned} (\alpha-\varepsilon)(t-T) + C_5 - |K|\delta \;\leq\; & z'''(t) \\ & -2(n-4)z''(t) + (n^2 - 10n + 20)z'(t) \\ \leq\; & (\alpha+\varepsilon)(t-T) + C_5 + |K|\delta, \quad \forall t \geq T, \end{aligned}$$

where $C_5 = C_5(T)$ is a constant containing all the terms $z(T)$, $z'(T)$, $z''(T)$ and $z'''(T)$ and $K = 2(n^2 - 6n + 8)$. Repeating twice more this procedure gives

$$\frac{\alpha-\varepsilon}{6}(t-T)^3 + O(t^2) \leq z'(t) \leq \frac{\alpha+\varepsilon}{6}(t-T)^3 + O(t^2) \quad \text{as } t \to \infty.$$

This contradicts the assumption that z admits a finite positive limit as $t \to \infty$.

Let us now suppose that L is finite negative, where we may proceed similarly. Then $\lambda(e^{z(t)} - 1) \to \alpha = \lambda(e^L - 1) < 0$ and for all $\varepsilon > 0$ there exists $T > 0$ such that

$$\begin{aligned} \alpha - \varepsilon \;\leq\; & z^{(4)}(s) - 2(n-4)z'''(s) + (n^2 - 10n + 20)z''(s) \\ & +2(n^2 - 6n + 8)z'(s) \leq \alpha + \varepsilon, \quad \forall t \geq T. \end{aligned} \tag{63}$$

Take $\varepsilon < |\alpha|$ so that $\alpha + \varepsilon$ is negative and let

$$\delta = \sup_{t \geq T} |z(t) - z(T)| < \infty.$$

Integrating (63) over $[T, t]$ for any $t \geq T$, we obtain

$$\begin{aligned} (\alpha-\varepsilon)(t-T) + C_5 - |K|\delta \;\leq\; & z'''(t) - 2(n-4)z''(t) \\ & +(n^2 - 10n + 20)z'(t) \\ \leq\; & (\alpha+\varepsilon)(t-T) + C_5 + |K|\delta, \quad \forall t \geq T, \end{aligned}$$

Repeating twice more this procedure gives

$$\frac{\alpha-\varepsilon}{6}(t-T)^3 + O(t^2) \leq z'(t) \leq \frac{\alpha+\varepsilon}{6}(t-T)^3 + O(t^2) \quad \text{as } t \to \infty.$$

This contradicts the assumption that z admits a finite negative limit as $t \to \infty$. □

The boundedness of z has interesting influence on its derivatives:

Proposition 4.10 *Assume that* $z : [T_0, \infty) \to \mathbb{R}$ *exists for some* T_0*, solves* (56) *and satisfies* $\lim_{s\to\infty} z(s) = 0$*. Then for all* $k \in \mathbb{N}$*, we have*

$$\lim_{s\to\infty} z^{(k)}(t) = 0.$$

This proposition comes directly from [27, Proposition 1], that we report here:

Proposition 4.11 *Assume that* $z : [T_0, \infty) \to \mathbb{R}$ *exists for some* T_0 *and solves a constant coefficient fourth order equation*

$$z^{(4)}(t) - K_3 z'''(t) + K_2 z''(t) - K_1 z'(t) = f(z(t)), \quad (t > T_0), \tag{64}$$

where $f \in C^1(\mathbb{R})$ *and where the coefficients may be considered as arbitrary real numbers* $K_j \in \mathbb{R}$*. Moreover, let* z_0 *be such that* $f(z_0) = 0$ *and assume that* z *satisfies* $\lim_{t\to\infty} z(t) = z_0$*. Then for* $k = 1, \dots, 4$*, one also has:*

$$\lim_{t\to\infty} z^{(k)}(t) = 0. \tag{65}$$

If $f \in C^{k_0+1}$ *in a neighbourhood of* z_0*, then* (65) *holds true for all* $k < k_0 + 4$*.*

Proof of Proposition 4.10: Equation (56) is equivalent to (64) if we fix $K_3 = 2(n-4)$, $K_2 = (n^2 - 10n + 20)$, $K_1 = -2(n^2 - 6n + 8)$ and $f(z(t)) = \lambda_0(e^{z(t)} - 1)$.

We took $\lim_{t\to\infty} z(t) = 0$; if we fix $z_0 = 0$, we see that $f(z_0) = \lambda_0(e^0 - 1) = 0$, that is all conditions of Proposition 4.11 are satisfied. □

Now, we consider the case z is singular.

Lemma 4.12 *Let* z *be a solution of* (56)*, such that* $\lim_{t\to+\infty} z(t) = -\infty$*, which corresponds to the solution* $u(r) = z(-\log r) - 4\log r + \log\lambda_0 - \log\lambda$ *of* (50)*. Then* u *is regular (at* $r = 0$*).*

Proof:

$$0 \le \Delta^2 u(r) = \lambda_0 e^{z(-\log r)} \cdot r^{-4}, \quad r \in (0, 1].$$

The function z is bounded from above in $[0, +\infty)$, it means $z(t) \le M$ for $t \in [0, \infty)$ and $\lim_{t\to+\infty} z(t) = -\infty$. So we can say that $\forall \varepsilon \ge 0$, $\forall K > 0$ $\exists R_1 \in (0, 1]$:

$$0 \le \Delta^2 u(r) \le \varepsilon r^{-4}, \quad 0 \le u(r) \le -4\log r - K, \quad r \in (0, R_1].$$

Making use of the polar form of the Laplacian for radial functions $\Delta v(r) = r^{1-n}(r^{n-1}v'(r))'$, we obtain

$$\forall r \in (0, R_1]: \quad 0 \le \left(r^{n-1}(\Delta u)'(r)\right)' \le \varepsilon r^{n-5}.$$

Integrating on $[0, r]$ yields with a suitable number $C_1 \in \mathbb{R}$ that

$$\begin{aligned} \forall r \in (0, R_1]: \quad & 0 \le r^{n-1}(\Delta u)'(r) + C_1 \le \frac{\varepsilon}{n-4} r^{n-4}, \\ & 0 \le (\Delta u)'(r) + C_1 r^{1-n} \le \frac{\varepsilon}{n-4} r^{-3}. \end{aligned}$$

We integrate this on $[r, R_1]$ and obtain with suitable numbers $C_2, C_3 \in \mathbb{R}$:

$$\begin{aligned} \forall r \in (0, R_1]: \quad & 0 \le -\Delta u(r) + C_2 + C_3 r^{2-n} \le \frac{\varepsilon}{2(n-4)} r^{-2}, \\ & 0 \le -\left(r^{n-1}u'\right)' + C_2 r^{n-1} + C_3 r \le \frac{\varepsilon}{2(n-4)} r^{n-3}. \end{aligned}$$

Integrating on $[0, r)$, yields with suitable real numbers C_4, C_5, $C_6 \in \mathbb{R}$.

$$\begin{aligned} \forall r \in (0, R_1], \quad & 0 \le -r^{n-1}u'(r) + C_4 + C_5 r^2 + C_6 r^n \le \frac{\varepsilon}{2(n-4)(n-2)} r^{n-2}, \\ & 0 \le -u'(r) + C_4 r^{1-n} + C_5 r^{3-n} + C_6 r \le \frac{\varepsilon}{2(n-4)(n-2)} r^{-1}. \end{aligned}$$

We integrate a last time on $[r, R_1]$ and with suitable numbers $C_7, \ldots, C_{10} \in \mathbb{R}$

$$0 \le u(r) + C_7 + C_8 r^{2-n} + C_9 r^{4-n} + C_{10} r^2 \le -\frac{\varepsilon}{2(n-4)(n-2)} \log(r).$$

Since for r close to 0 we know from the assumption that $0 \le u(r) \le -4\log(r)$ we conclude that $C_8 = C_9 = 0$, so that

$$\forall r \in (0, R_1]: \quad 0 \le u(R) \le -\frac{\varepsilon \log r}{2(n-4)(n-2)} - C_7 - C_{10} r^2.$$

This proves that $u(r) = o(|\log(r)|)$ for $r \searrow 0$.

Hence, for any $\varepsilon > 0$, there exist $R_2 > 0$ and a constant K_2 such that

$$\forall r \in (0, R_2]: \quad 0 \le \Delta^2 u \le K_2 r^{-\varepsilon}.$$

For our purposes it is enough to consider $\varepsilon = 1$:

$$\forall r \in (0, R_2): \quad 0 \le \Delta^2 u \le K_2 r^{-1}.$$

The same procedure as before yields

$$\forall r \in (0, R_2] : 0 \leq u(r) + C_8 r^{2-n} + C_9 r^{4-n} \leq C_{10},$$

where C_8, $C_9 \in \mathbb{R}$, $C_{10} > 0$. As before we conclude that $C_8 = C_9 = 0$, so that

$$0 \leq u(r) \leq C_{10}.$$

This means that u is regular. □

4.5 Energy considerations

Let u solution of (50) and z the function such that it solves (56) for $t > 0$.

We introduce the energy functional

$$E(s) := \frac{1}{2} z''(s)^2 - \frac{1}{2}(n^2 - 10n + 20) z'(s)^2 + \lambda \left(e^{z(s)} - z(s) \right), \tag{66}$$

that is useful in order to prove Theorem 4.3.

Lemma 4.13 *Let u be a radial singular solution of* (50) *and $z : [0, \infty) \to \mathbb{R}$ the corresponding solution of* (56)*. Then z is bounded from above and for $k = 1, \ldots, 4$, the functions $z^{(k)}$ are bounded in* $[0, \infty)$.

Proof: By Proposition 4.8 follows also that $e^{-4s+z(s)}$ is bounded from above in $[0, \infty)$. Hence, by local L^q-estimates for fourth order elliptic equations, we infer that for any $q > 1$ there exists a constant C_q such that, for any $s > 1$, we have

$$\|z^{(i)}(\,.\,)\|_{W^{4,q}(s-1,s+2)} \leq C_q(\lambda e^{\sup z} + 1)$$

Sobolev embedding together with local Schauder estimates give us, there exists a positive constant C_q indipendent of s, such that

$$\|z^{(i)}(\,.\,)\|_{C^{4,\alpha}(s,s+1)} \leq C_q(\lambda e^{\sup z} + 1).$$

□

In the next lemmas we will prove some summability property for the functions z and its derivatives. In what follows we always assume that z corresponds via (55) to a radial singular solution.

Lemma 4.14 *We have*

$$\int_0^\infty |z'(s)|^2\,ds + \int_0^\infty |z''(s)|^2\,ds < \infty$$

Proof: Let $E(t)$ be the function defined in (66). For every $t > 0$, integrating by parts and exploiting, we have

$$\begin{aligned}
E(t) - E(0) &= \int_0^t E'(s)\,ds \\
&= \int_0^t \left(z''z''' - (n^2-10n+20)z'z'' + \lambda(e^z-1)z'\right)\,ds \\
&= z'(t)z'''(t) - z'(0)z'''(0) \\
&\quad + \int_0^t z'(-z^{(4)} - (n^2-10n+20)z'' + \lambda(e^z-1))\,ds \\
&= z'(t)z'''(t) - z'(0)z'''(0) \\
&\quad + \int_0^t z'(-2(n-4)z''' + 2(n-2)(n-4)z')\,ds \\
&= z'(t)z'''(t) - z'(0)z'''(0) - 2(n-4)z'(t)z''(t) \\
&\quad +2(n-4)z'(0)z''(0) \\
&\quad + \int_0^t (2(n-4)z''(s)^2 + 2(n-2)(n-4)z'(s)^2)\,ds.
\end{aligned}$$

By Lemma 4.13 $E(t)$ and the functions $z'(t), z''(t), z'''(t)$ are bounded in $(0,\infty)$ while around 0 they are smooth. In view of Lemmas 4.12 and 4.13, there exists at least a sequence $t_k \nearrow \infty$, where $z(t_k)$ remains bounded. We recall that we always assume $z(\,.\,)$ to correspond to a singular solution.

So, for $t_k \nearrow \infty$, also the left-hand side of the previous calculation remains bounded. This proves the claim. □

Lemma 4.15 *We have*

$$\int_0^\infty |z'''(s)|^2\,ds < \infty.$$

Proof: We multiply equation (56) by z'' and integrate over $(0,t)$:

$$\begin{aligned}
\int_0^t \lambda(e^z-1)z''\,ds &= \int_0^t \left(z^{(4)} + 2(n-4)z''' + (n^2-10n+20)z''\right. \\
&\quad \left.-2(n-2)(n-4)z'\right) z''\,ds
\end{aligned}$$

By Lemma 4.14 and Hölder's inequality we have

$$\begin{aligned}\left|\int_0^t z'(s)z''(s)\,ds\right| &\leq \left(\int_0^t |z'(s)|^2\,ds\right)^2 \left(\int_0^t |z''(s)|^2\,ds\right)^2 \\ &= \mathcal{O}(1) \quad \text{as } t \to \infty\end{aligned}$$

By Lemma 4.13, integrating by parts

$$\begin{aligned}\left|\int_0^t (e^{z(s)}-1)z''(s)\,ds\right| &\leq |(e^{z(t)}-1)z'(t)| + |(e^{z(0)}-1)z'(0)| \\ &\quad + \int_0^t e^z(s)|z'(s)|^2\,ds = \mathcal{O}(1)\end{aligned}$$

as $t \to \infty$ because of $e^z = \mathcal{O}(1)$. By Lemma 4.13 we have

$$\begin{aligned}\left|\int_0^t z'''(s)z''(s)\,ds\right| &\leq \frac{1}{2}|z''(t)|^2 + \frac{1}{2}|z''(0)|^2 = \mathcal{O}(1) \quad \text{as } t \to \infty; \\ \int_0^t |z'''(s)|^2\,ds &= z'''(t)z''(t) - z'''(0)z''(0) - \int_0^t z^{(4)}(s)z''(s)\,ds = \mathcal{O}(1)\end{aligned}$$

as $t \to \infty$. All together these inequalities complete the proof of the lemma. □

Lemma 4.16 *We have*

$$\int_0^\infty |z^{(4)}(s)|^2\,ds < \infty.$$

Proof: We multiply equation (56) by $z^{(4)}$ and integrate over $(0, \infty)$ to obtain

$$\begin{aligned}\int_0^t |z^{(4)}(s)|^2\,ds &= \int_0^t (2(n-4)z'''(s) - (n^2-10n+20)z''(s) \\ &\quad -(2n^2-12n+16)z'(s) - (2n^2-12n+16)z'(s) \\ &\quad +\lambda(e^{z(s)}-1))z^{(4)}(s)\,ds. \qquad (67)\end{aligned}$$

It the same way as in Lemma 4.15, we can prove that (67) remains bounded as $t \to \infty$. □

Lemma 4.17 *We have*

$$\int_0^\infty |e^{z(s)}-1|^2\,ds < \infty.$$

Proof: Using (56) we obtain

$$\begin{aligned}\lambda^2(e^{z(s)}-1)^2 &= \big[z^{(4)}(s)-2(n-4)z'''(s)\\ &\quad +(n^2-10n+20)z''(s)+(2n^2-12n+16)z'(s)\big]^2\end{aligned}$$

Together with Lemmas 4.14, 4.15, 4.16, the proof is done.

□

Proof of Theorem 4.3: Let $W=\{w_1,w_2,w_3,w_4\}$ be the solution of the dynamical system (53), corresponding to a radial singular solution u of (50), and let P_1 and P_2 the stationary points introduced in (54). Because of Lemma 4.12, there exists a sequence $\{\sigma_k\}$, $\sigma_k\to+\infty$ such that $z(\sigma_k)$ is bounded. It follows that $E(\sigma_k)$ is bounded too.

Using Lemmas 4.14-4.17 at least one of the following must be true:

$$\begin{aligned}\exists\{\sigma_k\} \;:\;& \sigma_{k+1}<\sigma_k, \quad \lim_{k\to\infty}\sigma_k=-\infty,\\ & \lim_{k\to\infty}|\sigma_{k+1}-\sigma_k|=0, \quad \lim_{k\to\infty}W(\sigma_k)=P_1\end{aligned}\tag{68}$$

$$\begin{aligned}\exists\{\sigma_k\} \;:\;& \sigma_{k+1}<\sigma_k, \quad \lim_{k\to\infty}\sigma_k=-\infty\\ & \lim_{k\to\infty}|\sigma_{k+1}-\sigma_k|=0, \quad \lim_{k\to\infty}W(\sigma_k)=P_2\end{aligned}\tag{69}$$

Arguing as in [32], we can conclude that

$$\lim_{t\to-\infty}W(t)=P_1 \text{ or } \lim_{t\to-\infty}W(t)=P_2,$$

respectively in the case (68) or (69). From Theorem 4.7, we may exclude the case (69), because we supposed u to be singular. So u must be weakly singular. □

The energy functional defined in (66) is also useful to understand the behaviour of singular solutions of (50) when $r\to 0$. Suppose that

$$\lim_{t\to\infty}z(t)=L.$$

Lemma 4.18 *Let u_s be a weakly singular solution of* (50) *with parameter λ_s and $z:(0,\infty)\to\mathbb{R}$ the corresponding solution to* (56). *Then it cannot happen that $z'(s_0)=0$ for some s_0.*

Proof: Assume for contradiction that $z'(s_0)=0$. Since $z'(0)=-4$, the function z is not constant, z'. So, we suppose now s_0 is a point different from 0, such that $z'(s_0)=0$. We use the the energy functional defined in (66) and, like in Lemma 4.14

$$\begin{aligned} E(t)-E(s_0) &= z'(t)z'''(t)-z'(s_0)z'''(s_0)-2(n-4)\left[z'(t)z''(t)\right.\\ &\quad \left.+z'(s_0)z''(s_0)\right]+\int_{s_0}^{t}\left(2(n-4)z''(s)^2\right.\\ &\quad \left.+2(n-2)(n-4)z'(s)^2\right)\,ds. \end{aligned} \tag{70}$$

If we evaluate (70) for $t\to\infty$, then $z'(t), z''(t), z'''(t), z'(s_0)=0$ for hypotesis, then

$$E(\infty)-E(s_0)=2(n-4)\int_{s_0}^{\infty} z''(s)^2+(n-2)z'(s)^2\,ds>0, \tag{71}$$

It means $E(\infty)>E(s_0)$, that is

$$\lambda>\frac{1}{2}z''(s_0)^2+\lambda(e^{z(s_0)}-z(s_0))\geq\lambda(e^{z(s_0)}-z(s_0)).$$

Then $1>e^{z(s_0)}-z(s_0)\geq\inf_{x\in\mathbb{R}}e^x-x=1$,which is absurd. □

Proof of Theorem 4.4: Since $\lim_{s\to\infty}z(s)=0$, $z'(0)=4$, we have by Lemma 4.18 that $z(s)>0$, i.e.

$$v(-s)-4s+\log\lambda_s-\log\lambda_0>0.$$

When $s=0$, we obtain $\log\lambda_s>\log\lambda_0$, so it follows $\lambda_s>\lambda_0$.

Moreover, $u(e^{-s})-4s>\log\frac{\lambda_0}{\lambda_s}$ and then with the substitution $s=-\log r$ we obtain

$$u(r)>-4\log r+\log\frac{\lambda_0}{\lambda_s}$$

When letting $z\to 0$ we obtain

$$|v(-s)-4s+\log\lambda_s-\log\lambda_0|=\left|u(r)-\left(-4\log r+\log\frac{\lambda_0}{\lambda_s}\right)\right|\to 0.$$

□ *Proof of Theorem* 4.5 From [6] we know that for $\lambda\in[0,\lambda^\star)$ the minimal regular solution is stable, i.e.

$$\forall\varphi\in C_0^\infty(B),\quad \int_B(\Delta\varphi)^2\,dx-\lambda\int_B e^{U_\lambda}\varphi^2\,dx\geq 0.$$

By taking the monotone limit for $\lambda \nearrow \lambda^\star$ we find that

$$\forall \varphi \in C_0^\infty(B), \quad \int_B (\Delta\varphi)^2\,dx - \lambda^\star \int_B e^{U^\star}\varphi^2\,dx \geq 0.$$

Let us assume that $U^\star$ is singular. Then

$$\begin{aligned} \lambda^\star &> 8(n-2)(n-4), \\ U^\star &> -4\log|x| + \log\frac{\lambda_0}{\lambda^\star}, \end{aligned}$$

which gives a Hardy-inequality

$$\forall \varphi \in C_0^\infty \quad \int_B (\Delta\varphi)^2\,dx \geq 8(n-2)(n-4)\int_B r^{-4}\varphi^2\,dx. \tag{72}$$

It is well known that $\frac{1}{16}n^2(n-4)^2$ is the optimal constant for the Hardy inequality (72). So, necessarily it holds that

$$8(n-2)(n-4) \leq \frac{1}{16}n^2(n-4)^2.$$

For integer n this is equivalent to $n \geq 13$. This shows that for $n \leq 12$, the extremal solution is bounded. $\square$

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