Non-classical Error Bounds in the Central Limit Theorem

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Preface

The classical central limit theorem states the uniform convergence of the distribution functions of the standardized sums of independent and identically distributed square integrable real-valued random variables to the standard normal distribution function. While first versions of the central limit theorem are already due to Moivre (1730) and Laplace (1812), a systematic study of this topic started at the beginning of the last century with the fundamental work of Lyapunov (1900, 1901). Meanwhile, extensions of the central limit theorem are available for a multitude of settings. This includes, e.g., Banach space valued random variables as well as substantial relaxations of the assumptions of independence and identical distributions. Furthermore, explicit error bounds are established and asymptotic expansions are employed to obtain better approximations.

Classical error estimates like the famous bound of Berry and Esseen are stated in terms of absolute moments of the random summands and therefore do not reflect a potential closeness of the distributions of the single random summands to a normal distribution. Non-classical approaches take this issue into account by providing error estimates based on, e.g., pseudomoments. The latter field of investigation was initiated by work of Zolotarev in the 1960's and is still in its infancy compared to the development of the classical theory. For example, non-classical error bounds for asymptotic expansions seem not to be available up to now.

In the present work we first establish a new non-classical bound for the central limit theorem error in the case of multidimensional random summands, which are not necessarily identically distributed. Up to now the most fargoing result in this general setting is due Rotar (1977, 1978) and we improve upon his result w.r.t. the exponent of the pseudomoment.

Second, we study short asymptotic Edgeworth expansions in the case of real valued random summands, which are not necessarily identically distributed. Here we obtain a non-classical error bound, which to our best knowledge is the first result of this type in the literature.

We briefly describe the content of this thesis. In Chapter 1 we provide an introduction to the topic of non-classical error bounds and we sketch the historical development of the essential achievements in this field up to now including the new result obtained in this thesis. Chapter 2 deals with non-classical error bounds in the multidimensional central limit theorem. Our new estimate is stated in Theorem 2.1 in Section 2.1 and a detailed comparison with the bound of Rotar is carried out in Section 2.2. Section 2.3 contains the proof of Theorem 2.1. Chapter 3 is devoted to non-classical error bounds for short asymptotic expansions in the one-dimensional central limit theorem. The new estimate is presented in Section 3.1 and proven in Section 3.2. Auxiliary results, which are used as technical tools for the proofs of Theorem 2.1 and Theorem 3.1 are gathered in the Appendix.

I am very grateful to my supervisor, Professor Ulyanov from the Department of Computational Mathematics and Cybernetics at the Moscow State University, for advising me during working on this topic and for providing me with a lot of scientific support and helpful discussions.

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Notation

For $x, y \in \mathbb{R}^d$ we use $|x|$ and $\langle x, y \rangle$ to denote the Euclidean norm of x and the scalar product of x and y, respectively.

For a square matrix $V = (v_{ij}) \in \mathbb{R}^{d\times d}$ we use $|V|$, $\operatorname{tr} V$ and $\|V\|$ to denote the determinant of V, the trace of V and the maximum eigenvalue of V, respectively. Furthermore, $|V^{ij}|$ is used to abbreviate the algebraic adjunct to the element v_{ij} of V. The identity matrix in $R^{d\times d}$ is denoted by I_d.

For $A \subset \mathbb{R}^d$, $x \in \mathbb{R}^d$ and $\varepsilon > 0$ we put

$$\begin{aligned} \rho(x, A) &= \inf_{y\in A} |x - y|, \\ A^{\varepsilon} &= \{x \in \mathbb{R}^d : \rho(x, A) \le \varepsilon\}, \\ B_{\varepsilon}(x) &= \{x\}^{\varepsilon}, \\ A^{-\varepsilon} &= \{x \in A : B_{\varepsilon}(x) \subset A\}. \end{aligned}$$

For a k-times continuously differentiable $f : \mathbb{R}^d \to \mathbb{R}$, $x \in \mathbb{R}^d$ and $h \in \mathbb{R}^k$ we put

$$f^{(k)}(x)h^k = \left[\left(\sum_{m=1}^{d} h_m \frac{\partial}{\partial x_m}\right)^k f\right](x).$$

For a non-negative definite matrix $V \in \mathbb{R}^{d\times d}$ we use N_V, Φ_V and η_V to denote the centered normal distribution with covariance matrix V, the corresponding distribution function and the corresponding Lebesgue density, respectively. In particular, we write N, Φ and η for N_{I_d}, Φ_{I_d} and η_{I_d}, respectively, if the dimension d is clear from the context.

The convolution of two probability measures μ_1 and μ_2 on $\mathbb{R}^d$ is denoted by $\mu_1 * \mu_2$.

For d-dimensional random vectors X and Y we use f_X to denote the characteristic function of X and we write $X \stackrel{\mathrm{d}}{=} Y$ if X and Y have the same distribution on $\mathbb{R}^d$.

Throughout this work we use c, $c(d)$, $c(d, n)$ etc. to denote unspecified positive constants that only depend on the parameters explicitly stated as arguments.

For sequences of non-negative numbers a_n and b_n we write $a_n \ll b_n$ if $a_n \leq c \cdot b_n$ holds for every $n \in \mathbb{N}$.

Chapter 1

Introduction

Consider a sequence of centered, independent and identically distributed random variables $X_1, \ldots, X_n$ taking values in $\mathbb{R}$ with the common distribution function F such that

$$\sigma^2 = \mathbf{E}X_1^2 \in (0, \infty),$$

and let $F^{[n]}$ denote the distribution function of the corresponding standardized sum, i.e.,

$$F^{[n]}(x) = P\big((\sqrt{n} \cdot \sigma)^{-1} \cdot (X_1 + \ldots + X_n) \leq x\big).$$

The central limit theorem then states that

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} |F^{[n]}(x) - \Phi(x)| = 0. \tag{1.1}$$

Already at the beginning of the last century Lyapunov (1900, 1901) obtained a bound of order $\log n/\sqrt{n}$ for the central limit theorem error. Cramér (1928) proved that the $\log n$-term can be omitted if the characteristic function f of the random summands satisfies the condition

$$\limsup_{|t|\to\infty} |f(t)| < 1, \tag{C}$$

which excludes, e.g., discrete distributions F. Berry (1941) and Esseen (1942) independently of Berry showed that the latter restriction is superfluous and established the famous estimate

$$\sup_{x\in\mathbb{R}} |F^{[n]}(x) - \Phi(x)| \leq \frac{c}{\sqrt{n}} \cdot \frac{\mathbf{E}|X_1|^3}{\sigma^3}, \tag{1.2}$$

where c is an absolute positive constant. The order of convergence is therefore at least $n^{-1/2}$ and this is the best possible general result, see the following Example.

Example 1.1. Let $n \in 2\mathbb{N}$ and consider independent random variables $X_1, \ldots, X_n$ with

$$P(X_k = 1) = P(X_k = -1) = 1/2$$

for $k = 1, \ldots, n$. Then $\mathbf{E}X_k = 0$, $\sigma^2 = \mathbf{E}X_k^2 = 1$, and $F^{[n]}$ is a step function with a jump of height

$$P\left(\sum_{k=1}^{n} X_k = 0\right) = \binom{n}{n/2} \cdot \frac{1}{2^n}$$

at the point $x = 0$. Applying the Stirling formula we obtain

$$|F^{[n]}(0) - \Phi(0)| \geq \binom{n}{n/2} \cdot \frac{1}{2^{n+1}} \sim \frac{1}{\sqrt{2\pi n}}.$$

For appropriate extensions of the Berry-Esseen bound (1.2) to the case of independent, multivariate and non-identically distributed random variables as well as for similar estimates under weaker absolute moment conditions we refer to, e.g., Esseen (1945), Bergström (1945, 1949, 1969), Katz (1963), Sazonov (1968), Rotar (1970) and Bikyalis (1971). See also Wallace (1958), Petrov (1972) and Hall (1982) for an overview.

Classical central limit theorem error bounds of the Berry-Esseen type as in (1.2) are of the form κ/n^{α} with $0 < \alpha \leq 1/2$ and κ depending only on absolute moments of X_1 up to the order $2 + 2\alpha$, and thus only the number n of random summands is used for establishing the closeness of $F^{[n]}$ and Φ. However, it was already noted by Paul Lévy (1937) that

$$\sup_{x\in\mathbb{R}} |F^{[n]}(x) - \Phi(x)| \leq n \cdot \sup_{x\in\mathbb{R}} |F(x) - \Phi_{\sigma^2}(x)|.$$

Hence $F^{[n]}$ can be close to Φ even for moderate n if the common distribution of the single random summands is close to a normal distribution. In the extreme case of normally distributed $X_1, \ldots, X_n$ we have $F^{[n]} = \Phi$ and the left hand side in (1.2) is equal to zero, while the right hand side is bounded below by $c/\sqrt{n}$.

The first non-classical approach is due to Zolotarev (1965), who employed the absolute third pseudomoment

$$\nu_3 = \int_{\mathbb{R}} |x/\sigma|^3 |F - \Phi_{\sigma^2}|(dx) \tag{1.3}$$

as a measure of closeness of F and Φ_{σ^2} and obtained the error bound

$$\sup_{x\in\mathbb{R}} |F^{[n]}(x) - \Phi(x)| \leq \frac{c}{n^{1/8}} \cdot \nu_3^{1/4}. \tag{1.4}$$

Note that $\nu_3 = 0$ in the case of normally distributed random summands. Furthermore, we always have

$$\nu_3 \leq 3\frac{\mathbf{E}|X_1|^3}{\sigma^3}.$$

The order $n^{-1/8}$ of convergence provided in (1.4) is far from being optimal and was improved by Paulauskas (1969a), who established the first non-classical central limit theorem error bound with the best possible order of convergence $n^{-1/2}$, namely

$$\sup_{x\in\mathbb{R}} |F^{[n]}(x) - \Phi(x)| \leq \frac{c}{\sqrt{n}} \cdot \max(\nu_3, \nu_3^{1/4}). \tag{1.5}$$

If $\nu_3 < 1$ then the bound (1.5) is given by $c/\sqrt{n} \cdot \nu_3^{1/4}$. Since this case is of main interest it is natural to ask whether, in general, the exponent $1/4$ can be increased. The following example from Zolotarev (1968) provides a negative answer to this question in the case $n = 1$.

Example 1.2. Let $\varepsilon > 0$ and consider a real-valued random variable X with the symmetric distribution function F given by

$$F(x) = \begin{cases} 1/2, & \text{if} \quad 0 \le x \le a\varepsilon, \\ \Phi(\varepsilon), & \text{if} \quad a\varepsilon < x \le \varepsilon, \\ \Phi(x), & \text{if} \quad x > \varepsilon, \end{cases}$$

and $F(x) = 1 - F(-x)$ for $x < 0$, where $a \in (0,1)$ is the unique solution of the equation

$$\int_0^\varepsilon x^2 d\Phi(x) = (a\varepsilon)^2 \cdot \int_0^\varepsilon d\Phi(x).$$

Note that the latter property implies

$$\sigma^2 = \int_{\mathbb{R}} x^2 F(dx) = \int_{\mathbb{R}} x^2 d\Phi(dx) = 1.$$

For $\varepsilon \le 1$ we therefore have

$$\sup_{x\in\mathbb{R}} |F(x) - \Phi_{\sigma^2}(x)| \ge \frac{1}{2} P(X = a\varepsilon) = \frac{1}{2} \int_0^\varepsilon d\Phi(x) \ge c \cdot \varepsilon$$

while

$$\nu_3 = (a\varepsilon)^3 \cdot 2 \int_0^\varepsilon d\Phi(x) + 2 \int_0^\varepsilon x^3 d\Phi(x) \le c \cdot \varepsilon^4$$

with some absolute constant $c > 0$.

However, the bound (1.5) can be substantially improved for $n > 1$. Due to Ulyanov (1978) we have

$$\sup_{x\in\mathbb{R}} |F^{[n]}(x) - \Phi(x)| \le \frac{c}{\sqrt{n}} \cdot \max\bigl(\nu_3, \nu_3^{\min(n/4,1)}\bigr). \tag{1.6}$$

The following example generalizes Example 1.2 and shows that the exponent $\min(n/4, 1)$ in the bound (1.6) is optimal for $n \le 4$.

Example 1.3. Let $n \in \mathbb{N}$ and consider independent real-valued random variables $X_1, \dots, X_n$ with the common distribution function F from Example 1.2. For $\varepsilon \le 1$ we get

$$\sup_{x\in\mathbb{R}} |F^{[n]}(x) - \Phi(x)| \ge \frac{1}{2}\big(P(X_1 = a\varepsilon)\big)^n \ge (c \cdot \varepsilon)^n \ge c^{3n/4} \cdot \nu_3^{n/4}.$$

Next we omit the assumption that the random summands $X_1, \dots, X_n$ are identically distributed. Let F_k denote the distribution function of X_k and assume that

$$\sigma_k^2 = \mathbf{E}X_k^2 < \infty$$

for $k = 1, \dots, n$.

Furthermore, we put

$$\overline{\sigma} = \big(\mathrm{Var}\big((X_1 + \ldots + X_n)/\sqrt{n}\big)\big)^{1/2} = \left(\frac{1}{n}\sum_{k=1}^{n} \sigma_k^2\right)^{1/2}$$

and we assume that $\overline{\sigma} > 0$. As previously, we use $F^{[n]}$ to denote the distribution function of the standardized sum $(\sqrt{n} \cdot \overline{\sigma})^{-1} \cdot (X_1 + \ldots + X_n)$. Next, let

$$\nu_{3,k} = \int_{\mathbb{R}} |x/\overline{\sigma}|^3 |F_k - \Phi_{\sigma_k^2}|(dx) \tag{1.7}$$

for $k = 1, \dots, n$ and define

$$\nu_3 = \frac{1}{n}\sum_{k=1}^{n} \nu_{3,k}, \tag{1.8}$$

to appropriately extend the notion (1.3) of the absolute third pseudomoment to the present case. Note that $\sigma_1 = \ldots = \sigma_n = \sigma$ implies $\overline{\sigma} = \sigma$. Finally, put

$$L = \frac{\sigma_1^3 + \ldots + \sigma_n^3}{(\sigma_1^2 + \ldots + \sigma_n^2)^{3/2}}. \tag{1.9}$$

Nagaev and Rotar (1973) obtained

$$\sup_{x\in\mathbb{R}} |F^{[n]}(x) - \Phi(x)| \le \frac{c}{\sqrt{n}} \cdot \max\big(\nu_3, \nu_3^{1/4} \cdot (\sqrt{n} \cdot L)^{3/4}\big), \tag{1.10}$$

which generalizes the result (1.5) of Paulauskas, since $L = 1/\sqrt{n}$ if the random summands are identically distributed. Ulyanov (1978) established the estimate

$$\sup_{x\in\mathbb{R}} |F^{[n]}(x) - \Phi(x)| \le \frac{c}{\sqrt{n}} \cdot \max\bigl(\nu_3, \nu_3^{1/2\cdot(1-2^{-q})} \cdot (\sqrt{n}\cdot L)^{1/2\cdot(1+2^{-q})}\bigr), \quad (1.11)$$

where

$$q = \left\lfloor \frac{\sigma_1^2 + \ldots + \sigma_n^2}{\max_{1\le k\le n} \sigma_k^2} \right\rfloor. \quad (1.12)$$

Note that $q \ge 1$ and therefore the bound (1.11) improves upon (1.10) w.r.t. to the power of the pseudomoment ν_3. However, in the case of equal variances we have $q = n$ and the estimate (1.11) reduces to

$$\sup_{x\in\mathbb{R}} |F^{[n]}(x) - \Phi(x)| \le \frac{c}{\sqrt{n}} \cdot \max\bigl(\nu_3, \nu_3^{1/2\cdot(1-2^{-n})}\bigr),$$

which does not provide the optimal exponent $\min(1, n/4)$ for ν_3 if $n \ge 2$, see (1.6) and Example 1.3. We add that the following improvement of (1.11) is announced in Ulyanov (1979), namely,

$$\sup_{x\in\mathbb{R}} |F^{[n]}(x) - \Phi(x)| \le \frac{c}{\sqrt{n}} \cdot \max\bigl(\nu_3, \nu_3^{\min(q/4,1)}\bigr),$$

which clearly generalizes the result (1.6).

We now turn to the multidimensional case. To this end we let $d \in \mathbb{N}$ and we consider a sequence of centered, independent and identically distributed $\mathbb{R}^d$-valued random vectors $X_1, \ldots, X_n$ with the common distribution Q on $\mathbb{R}^d$ and positive definite covariance matrix $\mathrm{Cov}(X_1) > 0$. Put

$$\Sigma = (\mathrm{Cov}(X_1))^{1/2}.$$

We generalize the notion of the absolute third pseudomoment by defining

$$\nu_3 = \int_{\mathbb{R}^d} |\Sigma^{-1}x|^3 \, |Q - \mathrm{N}_{\Sigma^2}|(dx) \quad (1.13)$$

and we use $Q^{[n]}$ to denote the distribution of the standardized sum of the random vectors, i.e.,

$$Q^{[n]}(A) = P\bigl((\sqrt{n}\cdot\Sigma)^{-1} \cdot (X_1 + \ldots + X_n) \in A\bigr)$$

for every Borel set $A \subset \mathbb{R}^d$.

A first result for the d-dimension case is due to Paulauskas (1969b), who obtained a non-classical bound for the maximum deviation of $Q^{[n]}$ from N on the class of d-dimensional intervals. In the present work we analyze the closeness of $Q^{[n]}$ and N on the class $\mathcal{C}$. Paulauskas (1969c) showed that

$$\sup_{A\in\mathcal{C}} |Q^{[n]}(A) - \mathrm{N}(A)| \le \frac{c(d)}{\sqrt{n}} \cdot \max\bigl(\nu_3, \nu_3^{1/4}\bigr), \tag{1.14}$$

where the constant $c(d)$ only depends on the dimension d. Clearly, (1.14) extends the result (1.5) of the same author. A first improvement of the exponent 1/4 of the pseudomoment in the latter bound is due to Sazonov (1972), who obtained

$$\sup_{A\in\mathcal{C}} |Q^{[n]}(A) - \mathrm{N}(A)| \le \frac{c(d)}{\sqrt{n}} \cdot \max\bigl(\nu_3, \nu_3^{d/(d+3)}\bigr). \tag{1.15}$$

Ulyanov (1978) then managed to increase the exponent further. He proved that

$$\sup_{A\in\mathcal{C}} |Q^{[n]}(A) - \mathrm{N}(A)| \le \frac{c(d)}{\sqrt{n}} \cdot \max\bigl(\nu_3, \nu_3^{\kappa(n,d)}\bigr) \tag{1.16}$$

with

$$\kappa(n,d) = \min\Bigl(1, \frac{nd}{d+3}\Bigr), \tag{1.17}$$

which generalizes the result (1.6) he obtained for the one-dimensional case. Note that $\kappa(n,d) = 1$ for $n \ge 4$.

The following multidimensional extension of Example 1.3 shows that the exponent $\kappa(n,d)$ is optimal in the case $nd/(d+3) \le 1$, see also Sazonov (1981).

Example 1.4. Let $\varepsilon > 0$ and put

$$B = \{x \in \mathbb{R}^d : |x| \le \varepsilon\}.$$

Define a probability measure Q on $\mathbb{R}^d$ by

$$Q(A) = \mathrm{N}(A \cap B^c) + \frac{1}{2d}\mathrm{N}(B) \cdot \sum_{j=1}^{d} (1_A(a\varepsilon \cdot e_j) + 1_A(-a\varepsilon \cdot e_j)),$$

where $e_1, \dots, e_d$ are the unit vectors in $\mathbb{R}^d$ and $a \in (0,1)$ is the unique solution of the equation

$$\int\limits_B |x|^2 \mathrm{N}(dx) = (a\varepsilon)^2 \cdot \mathrm{N}(B).$$

Clearly, Q is centered, and employing the latter equality we obtain

$$\Sigma^2 = \int\limits_{\mathbb{R}^d} x \cdot x^{\mathrm{T}} Q(dx) = I_d,$$

where $I_d \in \mathbb{R}^{d \times d}$ denotes the identity matrix.

Let $n \in \mathbb{N}$ and assume $\varepsilon \leq 1$. Then

$$Q^{[n]}(\{\sqrt{n} \cdot a\varepsilon \cdot e_1\}) \geq \big(Q(\{a\varepsilon \cdot e_1\})\big)^n = (\mathrm{N}(B)/(2d))^n \geq c(d,n) \cdot \varepsilon^{dn}$$

and

$$\nu_3 = (a\varepsilon)^3 \cdot \mathrm{N}(B) + \int\limits_B |x|^3 \mathrm{N}(dx) \leq \varepsilon^3 \cdot 2\mathrm{N}(B) \leq c(d,n) \cdot \varepsilon^{d+3},$$

where the constant $c(d,n) > 0$ only depends on d and n. Hence

$$\begin{aligned} \sup_{A \in \mathcal{C}} |Q^{[n]}(A) - \mathrm{N}(A)| &\geq |Q^{[n]}(\{\sqrt{n} \cdot a\varepsilon \cdot e_1\}) - \mathrm{N}(\{\sqrt{n} \cdot a\varepsilon \cdot e_1\})| \\ &\geq c(d,n) \cdot (\nu_3 / c(d,n))^{nd/(d+3)}. \end{aligned}$$

Finally we turn to the setting of interest for the present work. We assume that $X_1, \dots, X_n$ are independent, centered, $\mathbb{R}^d$-valued random vectors with

$$\mathbf{E}|X_k|^2 < \infty$$

for $k = 1, \dots, n$. We use Q_k to denote the distribution of X_k on $\mathbb{R}^d$ and we put

$$\Sigma_k = (\mathrm{Cov}(X_k))^{1/2}.$$

Thus, as in the one-dimensional case we drop the condition of identically distributed random summands. Furthermore, we assume that

$$\overline{\Sigma} = \big(\mathrm{Cov}\big((X_1 + \ldots + X_n)/\sqrt{n}\big)\big)^{1/2} = \left(\frac{1}{n}\sum_{k=1}^n \Sigma_k^2\right)^{1/2} > 0,$$

and, as previously, we use $Q^{[n]}$ to denote the distribution of the standardized sum of the random vectors, i.e.,

$$Q^{[n]}(A) = P\big((\sqrt{n} \cdot \overline{\Sigma})^{-1} \cdot (X_1 + \ldots + X_n) \in A\big)$$

for every Borel set $A \subset \mathbb{R}^d$. In order to extend the definitions (1.8) and (1.13) of absolute third pseudomoments to the present setting we put

$$\nu_{3,k} = \int_{\mathbb{R}^d} |\overline{\Sigma}^{-1} x|^3 |Q_k - \mathrm{N}_{\overline{\Sigma}_k^2}|(dx) \tag{1.18}$$

and we define ν_3 to be the arithmetic mean of $\nu_{3,1}, \ldots, \nu_{3,n}$ as in the one-dimensional case, see (1.8).

For this general setting, the first non-classical estimate of the deviation of $Q^{[n]}$ from the standard normal distribution N on the class of convex sets $\mathcal{C}$ is due to Paulauskas (1969c), who showed

$$\sup_{A \in \mathcal{C}} |Q^{[n]}(A) - \mathrm{N}(A)| \leq \frac{c(d)}{n^{1/8}} \cdot \nu_3^{1/4}. \tag{1.19}$$

Clearly, this bound is too large if the random vectors are identically distributed. In this case the pseudomoment ν_3 does not depend on n and the bound (1.19) only provides the order of convergence $n^{-1/8}$ while the best possible order is $n^{-1/2}$, see (1.16). Furthermore, the pseudomoment exponent 1/4 is smaller than the optimal exponent (1.17) for $n > 1$.

The first result with the best order of convergence $1/\sqrt{n}$ is due to Rotar (1977, 1978) who essentially obtained an estimate of the form

$$\sup_{A \in \mathcal{C}} |Q^{[n]}(A) - \mathrm{N}(A)| \leq \frac{c(d)}{\sqrt{n}} \cdot \max\big(\nu_3, \nu_3^{\widetilde{\chi}(q)} \cdot (\sqrt{n} \cdot L)^{1-\widetilde{\chi}(q)}\big), \tag{1.20}$$

where L and q are defined by (1.9) and (1.12), respectively, with σ_k^2 replaced by $\mathfrak{s}_k^2 = \operatorname{tr} \operatorname{Cov}(\overline{\Sigma}^{-1} X_k)$, and the pseudomoment exponent $\widetilde{\chi}(q)$ satisfies $1/4 \leq \widetilde{\chi}(q) < 1/3$ if q is sufficiently large. See Section 2.2 for a more precise formulation..

In the present work, we present a substantial improvement of the estimate of Rotar w.r.t. the power of the pseudomoment, see Theorem 2.1 in Section 2.1 and Yaroslavtseva (2006). We show that (1.20) holds with the pseudomoment exponent $\widetilde{\chi}(q)$ replaced by an exponent $\chi(q) \in [1/4, 1/3)$ such that

$$1/3 - \widetilde{\chi}(q) \geq \frac{128(d+4) \cdot 2^{\lfloor q/(2d) \rfloor}}{(q+1)^3} \cdot (1/3 - \chi(q))$$

for sufficiently large q. Hence $1/3 - \widetilde{\chi}(q)$ is exponentially larger than $1/3 - \chi(q)$ as q tends to infinity.

The bounds (1.11) and (1.16) obtained by Ulyanov for one-dimensional summands and for identically distributed multi-dimensional summands, respectively, show that the pseudomoment exponent $\chi(q)$ is not optimal in general. Note that $L = 1/\sqrt{n}$ in the latter case. We conjecture that the bound (1.20) holds with a pseudomoment exponent

$$\chi^*(q) \in [1/4, 1]$$

in place of $\widetilde{\chi}(q)$ such that $\chi^*(q) = 1$ for sufficiently large q.

As already mentioned at the beginning of this chapter, the order of convergence $n^{-1/2}$ is the best possible general result on the asymptotic behavior of the error in the central limit theorem, see Example 1.1. However, for specific distributions a higher order of convergence is possible. This suggests to provide a more detailed analysis of the speed of convergence in the central limit theorem in general and to obtain better approximations to $F^{[n]}$ hereby as well. The following classic approach to achieve these goals uses asymptotic expansions of $F^{[n]}$ in terms of Φ and its derivatives and was already initiated around the beginning of the last century, see, e.g., Chebyshev (1890), Charlier (1905) and Edgeworth (1905).

Consider again a sequence of real-valued, centered, independent and identically distributed random variables $X_1, \ldots, X_n$ with common distribution function F and finite, positive variance $\sigma^2 = \mathbf{E}X_1^2$. Let f denote the characteristic function of X_1 and define the r-th order cumulant γ_r by

$$\gamma_r = \frac{1}{i^r} \cdot \frac{d^r}{dt^r} \log f(0)$$

for $r \in \mathbb{N}$, where log denotes the principal value of the complex logarithm. For example,

$$\gamma_1 = 0, \quad \gamma_2 = \sigma^2, \quad \gamma_3 = \mathbf{E}X_1^3, \quad \gamma_4 = \mathbf{E}X_1^4 - 3\sigma^4.$$

A formal Taylor expansion of the logarithm of the characteristic function of the normalized sum $(\sqrt{n} \cdot \sigma)^{-1}(X_1 + \ldots + X_n)$ leads to a formal expansion of the corresponding distribution function $F^{[n]}$, which is of the form

$$F^{[n]} = \Phi + \sum_{r=1}^{\infty} \frac{P_r(-\Phi)}{n^{r/2}}, \tag{1.21}$$

where P_r a polynomial of degree $3r$ with coefficients depending on the cumulants of order $k = 3, \ldots, r+2$ and powers of Φ are interpreted as derivatives. Hence $P_r(-\Phi(x))$ can be expressed as the product of the standard normal density function η and a polynomial in x. Moreover, $\gamma_3 = \ldots = \gamma_{r+2} = 0$ implies $P_r = 0$. For instance,

$$\begin{aligned} P_1(-\Phi(x)) &= \frac{\gamma_3 \cdot \Phi^{(3)}(x)}{6}, \\ P_2(-\Phi(x)) &= \frac{\gamma_4 \cdot \Phi^{(4)}(x)}{24} + \frac{\gamma_3^2 \cdot \Phi^{(6)}(x)}{72}. \end{aligned}$$

The first result on the validity of the expansion (1.21) is due to Cramér (1928). Let $k \geq 3$ and assume that $\mathbf{E}|X_1|^k$ is finite and that the characteristic function f satisfies the condition (C). Then

$$\sup_{x \in \mathbb{R}} \left| F^{[n]}(x) - \Phi(x) - \sum_{r=1}^{k-3} \frac{P_r(-\Phi(x))}{n^{r/2}} \right| \leq c(F, k) \cdot n^{-(k-2)/2} \tag{1.22}$$

with an unspecified constant $c(F, k)$ that depends on F and k. The most fargoing general result w.r.t. the speed of convergence of the Edgeworth expansion (1.21) is due to Esseen (1945), who improved upon (1.22) by showing that even

$$\lim_{n \to \infty} n^{(k-2)/2} \cdot \sup_{x \in \mathbb{R}} \left| F^{[n]}(x) - \Phi(x) - \sum_{r=1}^{k-2} \frac{P_r(-\Phi(x))}{n^{r/2}} \right| = 0 \tag{1.23}$$

holds under the above conditions on the distribution of the random summands.

Classical results on explicit error bounds are available under the assumption of a finite absolute moment $\mathbf{E}|X_1|^{k+2\alpha}$ with $k \geq 3$ and $0 < \alpha \leq 1/2$, and are typically of the form

$$\sup_{x\in\mathbb{R}}\left|F^{[n]}(x) - \Phi(x) - \sum_{r=1}^{k-2}\frac{P_r(-\Phi(x))}{n^{r/2}}\right| \leq \kappa \cdot n^{-(k/2-1+\alpha)} + \delta_n. \tag{1.24}$$

Here, κ depends only on absolute moments of X_1 up to the order $k+2\alpha$ and δ_n, depending on F and k, converges to zero exponentially fast if the Cramér condition (C) is satisfied. As an example we state a bound of Osipov (1967) with $\alpha = 1/2$. If $\mathbf{E}|X_1|^{k+1} < \infty$ then

$$\sup_{x\in\mathbb{R}}\left|F^{[n]}(x) - \Phi(x) - \sum_{r=1}^{k-2}\frac{P_r(-\Phi(x))}{n^{r/2}}\right| \leq \frac{\mathbf{E}|X_1|^{k+1}}{\sigma^{k+1}} \cdot n^{-(k-1)/2} + \delta_n. \tag{1.25}$$

We add that asymptotic expansions were also studied for non-identically distributed random summands, see, e.g., Cramér (1937), Statulyavichus (1965), Survila (1965), Pipiras and Statulyavichus (1968) and Pipiras (1970). Multivariate Edgeworth expansions and respective truncation error estimates have been obtained by, e.g., Rao (1960, 1961), von Bahr (1967), Bikyalis (1968) and Bhattacharya (1968, 1971, 1972). For an overview and further results on asymptotic expansions we refer to Petrov (1972) and Bhattacharya and Ranga Rao (1976). Asymptotic expansions in the central limit theorem are also investigated for random summands with values in function spaces, see, e.g., Bentkus (1984a, b), Ulyanov (1986), Nagaev and Chebotarev (1993), Bogatyrev, Götze and Ulyanov (2006), and also, Bentkus, Götze, Paulauskas and Rachkauskas (2000) for an overview.

Consider the case of identically normally distributed random summands $X_1, \ldots, X_n$. Then all of the respective cumulants γ_r of order $r \geq 3$ are zero, and, consequently, the maximum error on the left hand side in (1.25) is zero as well. However, the corresponding right hand side bound only provides convergence to zero of order $n^{-(k-1)/2}$. Thus, similar to the Berry-Esseen inequality (1.2), the closeness of F to the normal distribution function $\Phi_{\sigma^2}$is not taken into account by classical error estimates of the type (1.24).

Non-classical bounds for the error of truncated asymptotic expansions in the central limit theorem seem to be unknown up to now. Define the pseudomoments ν_r of order r by

$$\nu_r = \int_{\mathbb{R}} |x/\sigma|^r |F - \Phi_{\sigma^2}|(dx)$$

for $r \in \mathbb{N}$ and assume that $\mathbf{E}|X_1|^4 < \infty$. In Chapter 3 we provide the first non-classical error estimate for so-called short asymptotic expansions, namely

$$\left|F^{[n]}(x) - \Phi(x) - P_1(-\Phi(x))/\sqrt{n}\right| \leq \kappa_1/n + \kappa_2/n^2 + \delta_n, \tag{1.26}$$

where

$$\kappa_1 = \nu_4 + \nu_4^{\frac{1}{2}} + \nu_3^2 + \nu_3^{\frac{4}{3}}$$

depends only on the pseudomoments of order 3 and 4, κ_2 is determined by σ and $E|X_1|^3$, and δ_n, depending on F, tends to zero exponentially fast if the Cramér condition (C) is satisfied. Even more general, in Theorem 3.1 we present the first non-classical Edgeworth expansion error bound for the case of independent, centered and real-valued, but not necessarily identically distributed random variables, which implies (1.26), see Corollary 3.1.

For the proof of both Theorem 2.1 and Theorem 3.1 we employ a new method of Bentkus (2003a) to represent the difference of expected values $\mathbf{E}\varphi(S) - \mathbf{E}\varphi(Z)$ of a smooth function $\varphi : \mathbb{R}^d \to \mathbb{C}$ applied to the standardized sum S of the d-dimensional random summands $X_1, \ldots, X_n$ and to a d-dimensional standard normal vector Z, respectively. In Chapter 2 this representation is used for smoothed indicator functions φ_A of convex sets $A \subset \mathbb{R}^d$, and it is combined with the method of compositions, which was introduced by Bergström (1944) and is further developed in Bergström (1945, 1949), Sazonov (1968, 1972), Paulauskas (1969a, 1969b, 1996c), Ulyanov (1978) and Sazonov and Ulyanov (1982). In Chapter 3 we apply characteristic functions techniques, see, e.g., Esseen (1945), Hsu (1945), Rotar (1977, 1978), Sazonov and Ulyanov (1995) and we use the representation method of Bentkus for the functions $\varphi_t(x) = e^{itx}$ with $t \in \mathbb{R}$. We refer to Sazonov (1981) for a detailed description of the composition method as well as characteristic function techniques and an overview on applications in the literature.

Chapter 2

Non-classical Error Bounds in the CLT in $\mathbb{R}^d$

Let $d, n \in \mathbb{N}$ and consider a sequence $X_1, \ldots, X_n$ of d-dimensional, centered and independent random vectors with

$$\mathbf{E}|X_k|^2 < \infty$$

for $k = 1, \ldots, n$. We use Q_k to denote the distribution of X_k on $\mathbb{R}^d$ and we put

$$\Sigma_k = (\mathrm{Cov}(X_k))^{1/2}.$$

Let $Y_1, \ldots, Y_n$ be d-dimensional, centered and independent Gaussian vectors, such that

$$\mathrm{Cov}(Y_k) = \Sigma_k$$

and define

$$S = X_1 + \ldots + X_n, \qquad Z = Y_1 + \ldots + Y_n.$$

Throughout this chapter we assume that

$$\mathrm{Cov}(S) = \mathrm{Cov}(Z) > 0$$

and we put

$$\overline{\Sigma} = \left(\mathrm{Cov}(S/\sqrt{n})\right)^{1/2}.$$

For $r \geq 0$ we define

$$\nu_{r,k} = \int\limits_{\mathbb{R}^d} \left|\overline{\Sigma}^{-1} z\right|^r \left|Q_k - N_{\Sigma_k^2}\right|(dz),$$

and we put

$$\nu_r = \frac{1}{n} \sum_{k=1}^{n} \nu_{r,k}.$$

Let $Q^{[n]}$ denote the distribution of the standardized sum of the random vectors X_k on $\mathbb{R}^d$, i.e.,

$$Q^{[n]} = P_{\overline{\Sigma}^{-1} \cdot S/\sqrt{n}}.$$

We are interested in the maximum deviation of $Q^{[n]}$ from the d-dimensional standard normal distribution N on the class $\mathcal{C}$ of all convex subsets of $\mathbb{R}^d$, i.e.,

$$\Delta_n(\mathcal{C}) = \sup_{A \in \mathcal{C}} |Q^{[n]}(A) - \mathrm{N}(A)|.$$

2.1 Main Result

Put

$$\mathfrak{s}_k^2 = \operatorname{tr} \operatorname{Cov}(\overline{\Sigma}^{-1} X_k)$$

and define

$$L_n = \frac{\mathfrak{s}_1^3 + \ldots + \mathfrak{s}_n^3}{(\mathfrak{s}_1^2 + \ldots + \mathfrak{s}_n^2)^{3/2}} = \frac{\mathfrak{s}_1^3 + \ldots + \mathfrak{s}_n^3}{(nd)^{3/2}}$$

as well as

$$q_n = \left\lfloor \frac{\mathfrak{s}_1^2 + \ldots + \mathfrak{s}_n^2}{\max_{1 \leq k \leq n} \mathfrak{s}_k^2} \right\rfloor.$$

Furthermore, put

$$\chi(x) = \frac{1}{3} - \frac{1}{12 \cdot 2^{\lfloor \frac{x}{2d} \rfloor}}$$

for $x \geq 0$.

Our main result is the following estimate:

Theorem 2.1. *There exists an absolute constant $M > 1$ such that*

$$\Delta_n(\mathcal{C}) \leq M \cdot \frac{d^3}{\sqrt{n}} \cdot \left(\nu_3 + \nu_3^{\chi(q_n)} \cdot (\sqrt{n} \cdot L_n)^{1-\chi(q_n)}\right). \tag{2.1}$$

2.2 Discussion

We briefly discuss basic properties of the bound (2.1) and then we turn to a detailed comparison of our estimate with an estimate obtained by Rotar.

Clearly, the ratio L_n satisfies

$$1/\sqrt{n} \le L_n \le 1$$

and for the pseudomoment exponent $\chi(q_n)$ we have

$$1/4 \le \chi(q_n) < 1/3.$$

Moreover, $\chi(q_n)$ tends to 1/3 exponentially fast as q_n tends to infinity.

If the random summands $X_1, \ldots, X_n$ are identically distributed then $L_n = 1/\sqrt{n}$ and $q_n = n$ such that the bound (2.1) reduces to

$$\Delta_n(\mathcal{C}) \le M \cdot \frac{d^3}{\sqrt{n}} \cdot \Big(\nu_3 + \nu_3^{\frac{1}{3} - \frac{1}{12 \cdot 2^{\lfloor n/(2d) \rfloor}}}\Big). \tag{2.2}$$

A comparison with the results of Ulyanov (1.16) shows that this estimate is suboptimal w.r.t. the power of pseudomoment.

Let

$$\widetilde{\mathfrak{s}}_k^2(\theta) = \mathbf{E}\big(\theta^T (\mathrm{Cov}(S))^{-1/2} X_k\big)^2$$

for $\theta \in \mathbb{R}^d$ and $k = 1, \ldots, n$, and define

$$\widetilde{L}_n(\theta) = \frac{\widetilde{\mathfrak{s}}_1^3(\theta) + \ldots + \widetilde{\mathfrak{s}}_n^3(\theta)}{(\widetilde{\mathfrak{s}}_1^2(\theta) + \ldots + \widetilde{\mathfrak{s}}_n^2(\theta))^{3/2}} = \widetilde{\mathfrak{s}}_1^3(\theta) + \ldots + \widetilde{\mathfrak{s}}_n^3(\theta)$$

as well as

$$\widetilde{L}_n = \sup_{|\theta|=1} \widetilde{L}_n(\theta), \qquad \widetilde{q}_n = 1/\widetilde{L}_n^2.$$

Furthermore, put

$$\widetilde{\chi}_\delta(x) = \frac{\lfloor x \cdot \delta/64 \rfloor}{3\lfloor x \cdot \delta/64 \rfloor + 2d + 8}$$

for $\delta \in (0,1)$ and $x \ge 0$. Note that

$$1/\sqrt{n} \le \widetilde{L}_n \le 1, \qquad \widetilde{\chi}_\delta(\widetilde{q}_n) < 1/3.$$

The following bound is due to Rotar (1977, 1978).

Theorem 2.2. *For every $\delta \in (0,1)$ there exists a positive constant $c(\delta, d)$ such that*

$$\widetilde{q}_n \geq \frac{d+5}{(1-\sqrt[3]{\delta})^3}$$

implies

$$|\Delta_n(\mathcal{C})| \leq \frac{c(\delta,d)}{\sqrt{n}} \cdot \max\big(\nu_3, \nu_3^{\widetilde{\chi}_\delta(\widetilde{q}_n)} \cdot (\sqrt{n} \cdot \widetilde{L}_n)^{1-\widetilde{\chi}_\delta(\widetilde{q}_n)}\big). \tag{2.3}$$

In Propostion 1 we compare the bounds (2.1) and (2.3) by comparing the respective pseudomoment exponents $\chi(q_n)$ and $\widetilde{\chi}_\delta(\widetilde{q}_n)$ as well as the ratios L_n and $\widetilde{L}_n$ for a fixed dimension d. While the latter ratios are shown to be asymptotically equivalent, it turns out that $1/3 - \widetilde{\chi}_\delta(\widetilde{q}_n)$ is exponentially larger than $1/3 - \chi(q_n)$ for sufficiently large q_n.

Proposition 1. *The ratios L_n and $\widetilde{L}_n$ satisfy*

$$L_n \leq \widetilde{L}_n \leq d^{3/2} \cdot L_n. \tag{2.4}$$

The pseudomoment exponents $\chi(q_n)$ and $\widetilde{\chi}_\delta(\widetilde{q}_n)$ satisfy

$$1/3 - \widetilde{\chi}_\delta(\widetilde{q}_n) \geq \frac{128(d+4)}{(q_n+1)^3} \cdot 2^{\lfloor q_n/(2d) \rfloor} \cdot (1/3 - \chi(q_n)), \tag{2.5}$$

if $\widetilde{q}_n \geq 128(d+4)/\delta$ and

$$\widetilde{\chi}_\delta(\widetilde{q}_n) < \chi(q_n) \tag{2.6}$$

otherwise.

Proof. Put $Y_k = (\mathrm{Cov}(S))^{-1/2} X_k$ and let $\theta \in \mathbb{R}^d$ with $|\theta| = 1$. By the Hölder inequality,

$$|\theta^T Y_k|^2 \leq |\theta|^2 \cdot |Y_k|^2 = |Y_k|^2.$$

Hence

$$\widetilde{\mathfrak{s}}_k^2(\theta) = E|\theta^T Y_k|^2 \leq \operatorname{tr} \mathrm{Cov}(Y_k) = \mathfrak{s}_k^2/n$$

and consequently

$$\widetilde{L}_n \leq \frac{1}{n^{3/2}} \sum_{k=1}^n \mathfrak{s}_k^3 = d^{3/2} \cdot L_n.$$

Put

$$\theta_0 = (1/\sqrt{d}, \ldots, 1/\sqrt{d}).$$

Then $|\theta_0| = 1$ and we have

$$\widetilde{\mathfrak{s}}_k^2(\theta_0) = \frac{1}{d} \cdot \operatorname{tr} \operatorname{Cov}(Y_k) = \frac{\mathfrak{s}_k^2}{d \cdot n}.$$

Hence

$$\widetilde{L}_n \geq \widetilde{L}_n(\theta_0) = \widetilde{\mathfrak{s}}_1^3(\theta_0) + \ldots + \widetilde{\mathfrak{s}}_n^3(\theta_0) = \frac{\mathfrak{s}_1^3 + \ldots + \mathfrak{s}_n^3}{(n \cdot d)^{3/2}} = L_n,$$

which finishes the proof of (2.4).

Note that the latter inequality implies

$$\widetilde{q}_n = \frac{1}{\widetilde{L}_n^2} \leq \frac{1}{L_n^2} \leq \frac{(n \cdot d)^3}{\max_{1 \leq k \leq n} \mathfrak{s}_k^6} \leq (q_n + 1)^3. \tag{2.7}$$

For the proof of (2.5) we put

$$a = (2d + 8)/\lfloor \widetilde{q}_n \cdot \delta/64 \rfloor$$

and we first consider the case that $a > 1$. Then (2.6) follows from

$$\widetilde{\chi}_\delta(\widetilde{q}_n) = \frac{1}{3 + a} < 1/4 \leq \chi(q_n). \tag{2.8}$$

If $a \leq 1$, we obtain

$$\widetilde{\chi}_\delta(\widetilde{q}_n) = \frac{1}{3 + a} \leq \frac{1}{3} - \frac{a}{12} \leq \frac{1}{3} - \frac{128(d + 4)}{12 \widetilde{q}_n}.$$

Thus, by (2.7),

$$\frac{1}{3} - \widetilde{\chi}_\delta(\widetilde{q}_n) \geq \frac{128(d + 4)}{12(q_n + 1)^3} = \frac{128(d + 4)}{(q_n + 1)^3} \cdot 2^{\lfloor q_n/(2d) \rfloor} \cdot (1/3 - \chi(q_n)),$$

which finishes the proof of proposition. □

2.3 Proof of Theorem 2.1

We first state some facts, which are easy to check or well known from the literature and are used throughout in this section.

We will use the following two properties of the class $\mathcal{C}$ of convex subsets of $\mathbb{R}^d$.

$(P1)$ For every $A \in \mathcal{C}$, every $a \in \mathbb{R}^d$ and every symmetric and invertible matrix $D \in \mathbb{R}^{d\times d}$ we have

$$DA + a \in \mathcal{C}.$$

$(P2)$ For every $A \in \mathcal{C}$ and every $\varepsilon > 0$ we have

$$A^{\varepsilon}, A^{-\varepsilon} \in \mathcal{C}.$$

Furthermore, we will employ the following version of the Taylor expansion formula for a sufficiently smooth function $f : \mathbb{R}^d \to \mathbb{R}$. For $t, h \in \mathbb{R}^d$ and $s \in \mathbb{N}_0$,

$$f(t+h) = \sum_{\ell=0}^{s} \frac{1}{\ell!} \cdot f^{(\ell)}(t)h^{\ell} + \frac{1}{s!} \int_0^1 (1-u)^s f^{(s+1)}(t+uh)h^{s+1} du. \tag{2.9}$$

For positive integers $r_2 \geq r_1$ we have

$$\nu_{r_1,k} \leq 2^{(r_2-r_1)/r_2} \cdot \nu_{r_2,k}^{r_1/r_2} \leq 2 \cdot \nu_{r_2,k}^{r_1/r_2}, \tag{2.10}$$

see Christoph and Wolf (1992).

Throughout the proof of Theorem 2.1 we let

$$\mu_k = Q_k - N_{\Sigma_k^2}$$

denote the difference of the distributions of X_k and Y_k. Recall that the latter random vectors are centered and have the same covariance matrix. For every linear function $G : \mathbb{R}^d \to \mathbb{R}$ and every bilinear form $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$,

$$\int_{\mathbb{R}^d} \mu_k(dz) = \int_{\mathbb{R}^d} G(z)\mu_k(dz) = \int_{\mathbb{R}^d} H(z,z)\mu_k(dz) = 0. \tag{2.11}$$

Finally, we put

$$\widetilde{\nu}_{r,k} = \nu_{r,k}/n^{r/2}, \quad \widetilde{\mathfrak{s}}_k = \mathfrak{s}_k/n^{1/2} \tag{2.12}$$

as well as

$$\widetilde{\nu}_r = \sum_{k=1}^{n} \widetilde{\nu}_{r,k} = \nu_r/n^{r/2-1} \tag{2.13}$$

Due to Property $(P1)$ of the class $\mathcal{C}$ we may assume without loss of generality that

$$\mathrm{Cov}(S) = I_d,$$

which implies

$$Q^{[n]} = P_S.$$

Furthermore, we may assume that the sequences $Y_1, \ldots, Y_n$ and $X_1, \ldots, X_n$ are independent.

In order to prove Theorem 2.1 we proceed by induction on the number n of random vectors $X_1, \ldots, X_n$, and we distinguish the cases $q_n < 2d$ and $q_n \geq 2d$. Note that $q_0^1 = 1$. To carry out the induction step in the case $q_n \geq 2d$ we show that if (2.1) holds for any subsequence of $X_1, \ldots, X_n$ of length $n-1$ with some constant $M > 1$ then (2.1) also holds for the whole sequence $X_1, \ldots, X_n$ with M replaced by $c \cdot \sqrt{M}$, where the constant $c > 0$ is independent of d, n, M and the random vectors X_k.

Case 1: $q_n < 2d$.

Let $T > 0$. Choosing $Q = P_S$ in Lemma A.5 we get

$$\Delta_n(\mathcal{C}) \leq 2 \sup_{A \in \mathcal{C}} |(P_S - \mathrm{N}) * \mathrm{N}_{T^{-2} \cdot I_d}(A)| + \frac{24 \cdot d^{\frac{3}{2}} \cdot \Gamma(\frac{d+1}{2})}{\sqrt{\pi} \cdot T \cdot \Gamma(\frac{d}{2})}. \tag{2.14}$$

For $m \geq l \geq 1$ put

$$Q_{l,m} = Q_l * Q_{l+1} * \ldots * Q_m$$

and

$$\mathrm{N}_{l,m} = \mathrm{N}_{\Sigma_l^2} * \mathrm{N}_{\Sigma_{l+1}^2} * \ldots * \mathrm{N}_{\Sigma_m^2}.$$

In order to estimate the first summand on the right hand side of (2.14) we use the representation

$$(P_S - \mathrm{N}) * \mathrm{N}_{T^{-2} \cdot I_d} = \sum_{j=1}^{n} U_j * \mu_j,$$

where

$$U_j = Q_{1,j-1} * \mathrm{N}_{j+1,n} * \mathrm{N}_{T^{-2} \cdot I_d}$$

and both $Q_{1,0}$ and $\mathrm{N}_{n+1,n}$ denote the Dirac measure in the point $0 \in \mathbb{R}^d$.

Fix $A \in \mathcal{C}$, put

$$W_j = \mathrm{Cov}(X_{j+1} + \ldots + X_n) + T^{-2} \cdot I_d$$

and define

$$g_j(x) = U_j(A + x)$$

for $x \in \mathbb{R}^d$ and $j = 1, \ldots, n$. Since

$$\begin{aligned} U_j(A + x) &= \int_{\mathbb{R}^d} Q_{1,j-1}(A + x - y) \eta_{W_j}(y) dy \\ &= \int_{\mathbb{R}^d} Q_{1,j-1}(A + t) \eta_{W_j}(x - t) dt, \end{aligned}$$

the function g_j is smooth. Therefore, applying the Taylor formula (2.9) to g_j and using (2.11) we obtain

$$\begin{aligned} |U_j * \mu_j(A)| &= \left| \int_{\mathbb{R}^d} g(-x) \mu_j(dx) \right| \\ &\leq \frac{1}{2} \int_0^1 \int_{\mathbb{R}^d} (1-u)^2 |g'''(-ux)x^3| |\mu_j|(dx) du \\ &\leq \frac{d^{\frac{3}{2}}}{2} \sup_{1 \leq l,m,p \leq d} \sup_{x \in \mathbb{R}^d} \left| \frac{\partial^3 g_j(x)}{\partial x_l \partial x_m \partial x_p} \right| \cdot \int_{\mathbb{R}^d} |x|^3 |\mu_j|(dx). \end{aligned}$$

Clearly,

$$\frac{\partial^3 g_j(x)}{\partial x_l \partial x_m \partial x_p} = \int\limits_{\mathbb{R}^d} Q_{1,j-1}(A+t) \cdot \frac{\partial^3 \eta_{W_j}(x-t)}{\partial x_l \partial x_m \partial x_p} dt$$

for all $1 \le l, m, p \le d$. For $d = 1$ we have

$$\left| \frac{\partial^3 g_j(x)}{\partial x_1^3} \right| \le \int\limits_{\mathbb{R}} \left(\frac{3|x_1 - t|}{W_j^2} + \frac{|x_1 - t|^3}{W_j^3} \right) \eta_{W_j}(x_1 - t) dt \le \frac{5}{W_j \sqrt{W_j}} \le 5T^3.$$

For $d \ge 2$ we use Lemma A.6 as well as Lemma A.7 with $V_1 = T^{-2} \cdot I_d$ and $V_2 = W_j - T^{-2} \cdot I_d$ to derive that

$$\begin{aligned} \left| \frac{\partial^3 g_j(x)}{\partial x_l \partial x_m \partial x_p} \right| &\le \sqrt{\frac{6|W_j^{ll}||W_j^{mm}||W_j^{pp}|}{|W_j|^3}} \\ &\le \sqrt{\frac{6|V_1^{ll}||V_1^{mm}||V_1^{pp}|}{|V_1|^3}} \\ &= \sqrt{6} \cdot T^3. \end{aligned}$$

Therefore

$$|U_j * Q_j(A)| \le \frac{5d^{3/2}}{2} \cdot T^3 \cdot \widetilde{\nu}_{3j}$$

and consequently,

$$\Delta_n(\mathcal{C}) \le 5d^{3/2} \cdot T^3 \cdot \widetilde{\nu}_3 + \frac{24d^{3/2} \cdot \Gamma(\frac{d+1}{2})}{\sqrt{\pi} \cdot T \cdot \Gamma(\frac{d}{2})},$$

due to (2.14). According to the Stirling formula we have $\Gamma(\frac{d+1}{2})/\Gamma(\frac{d}{2}) \le 2d^{1/2}$. Choose $T = d^{1/8} \cdot \widetilde{\nu}_3^{-1/4}$ to obtain

$$\Delta_n(\mathcal{C}) \le c_1 \cdot d^{15/8} \cdot \widetilde{\nu}_3^{1/4},$$

where $c_1 = 5 + 48/\sqrt{\pi}$. By assumption, $q_n < 2d$, and therefore

$$\max_{1 \le i \le n} \widetilde{\mathfrak{s}}_i^2 > \frac{1}{2},$$

which implies

$$L_n = \frac{\widetilde{\mathfrak{s}}_1^3 + \ldots + \widetilde{\mathfrak{s}}_n^3}{d^{3/2}} > \frac{1}{2\sqrt{2} \cdot d^{3/2}}.$$

Hence

$$\Delta_n(\mathcal{C}) \leq 2\sqrt[8]{2} \cdot c_1 \cdot d^3 \widetilde{\nu}_3^{1/4} \cdot L_n^{3/4} \leq c_2 \cdot d^3 \cdot \big(\widetilde{\nu}_3 + \widetilde{\nu}_3^{1/4} \cdot L_n^{3/4}\big),$$

where $c_2 = 2\sqrt[8]{2} \cdot c_1$. Moreover, $\left\lfloor \frac{q_n}{2d} \right\rfloor = 0$, so that $\chi(q_n) = 1/4$, which yields (2.1) with $M = c_2$.

Case 2: $q_n \geq 2d$.

For $k = 1, \ldots, n$ we define

$$S_k = \sum_{i \neq k} X_i, \qquad Z_k = \sum_{i \neq k} Y_i,$$

and we put

$$C_k^2 = \mathrm{Cov}(S_k) = \mathrm{Cov}(Z_k).$$

Note that in the case under consideration, the covariance matrices C_k^2 are invertible with

$$\|C_k^{-1}\| \leq \sqrt{2}, \qquad k = 1, \ldots, n. \tag{2.15}$$

Indeed, the condition $q_n \geq 2d$ is equivalent to $\max_{1 \leq k \leq n} \mathrm{tr}\,\mathrm{Cov}(X_k) \leq 1/2$, which implies that all eigenvalues of $\mathrm{Cov}(X_k)$ are bounded by 1/2. Therefore all eigenvalues of C_k^2 are not less than 1/2.

Let $0 < \varepsilon < 1$. We apply Lemma A.2 as well as Lemma A.3 to obtain

$$\begin{aligned}\Delta_n(\mathcal{C}) \leq& \sup_{A \in \mathcal{C}} |\mathbf{E}\varphi_{\varepsilon,A}(S) - \mathbf{E}\varphi_{\varepsilon,A}(Z)| \\ &\quad + \max\Big\{\sup_{A \in \mathcal{C}} P(Z \in A^\varepsilon \setminus A), \sup_{A \in \mathcal{C}} P(Z \in A \setminus A^{-\varepsilon})\Big\} \\ \leq& \sup_{A \in \mathcal{C}} |\mathbf{E}\varphi_{\varepsilon,A}(S) - \mathbf{E}\varphi_{\varepsilon,A}(Z)| + \sqrt{2/\pi} \cdot d^{3/2} \cdot \varepsilon, \end{aligned} \tag{2.16}$$

where the functions $\varphi_{\varepsilon,A} : \mathbb{R}^d \to \mathbb{R}$ are chosen according to Lemma A.1.

Fix $A \in \mathcal{C}$. For convenience we write φ instead of $\varphi_{\varepsilon,A}$ in the sequel. To estimate $|\mathbf{E}\varphi(S) - \mathbf{E}\varphi(Z)|$ we use the following representation, introduced by Bentkus (2003a). Put

$$H_k = S_k \cdot \cos\alpha + Z_k \cdot \sin\alpha$$

for $k = 1, \ldots, n$, where α is uniformly distributed on $[0, \pi/2]$ and independent of the vector $(X_1, \ldots, X_n, Y_1, \ldots, Y_n)$. Then

$$\mathbf{E}\varphi(S) - \mathbf{E}\varphi(Z) = -\frac{\pi}{2}\sum_{k=1}^{n} \Theta_k \tag{2.17}$$

with

$$\Theta_k = \mathbf{E}\varphi'(H_k + X_k \cdot \cos\alpha + Y_k \cdot \sin\alpha) \cdot (-X_k \cdot \sin\alpha + Y_k \cdot \cos\alpha).$$

Below we will show that

$$\begin{aligned} |\Theta_k| \ll \varepsilon^{-1} \cdot (\widetilde{\nu}_{3,k} + \widetilde{\mathfrak{s}}_k^2 \cdot \widetilde{\nu}_{1,k}) \\ \times \big(Md^3 \cdot \big(\widetilde{\nu}_3 + \widetilde{\nu}_3^{2\chi(q_n)-1/3} \cdot L_n^{4/3-2\chi(q_n)}\big) + d^{3/2} \cdot \varepsilon\big) \\ + \widetilde{\nu}_{3,k}. \end{aligned} \tag{2.18}$$

Summing (2.18) with respect to $k = 1, \ldots, n$ and using (2.17) we obtain

$$\begin{aligned} &|\mathbf{E}\varphi(S) - \mathbf{E}\varphi(Z)| \\ &\quad \ll \frac{1}{\varepsilon} \cdot \left(\widetilde{\nu}_3 + \sum_{k=1}^{n} \widetilde{\mathfrak{s}}_k^2 \widetilde{\nu}_{1,k}\right) \\ &\qquad \times \big(Md^3 \cdot \big(\widetilde{\nu}_3 + \widetilde{\nu}_3^{2\chi(q_n)-1/3} \cdot L_n^{4/3-2\chi(q_n)}\big) + d^{3/2} \cdot \varepsilon\big) \\ &\qquad + \widetilde{\nu}_3. \end{aligned} \tag{2.19}$$

Note that $\widetilde{\nu}_{1,k}^3 \leq 8\widetilde{\nu}_{3,k}$ due to (2.10). Hence

$$\begin{aligned}\sum_{k=1}^{n} \widetilde{\nu}_{1,k} \cdot \widetilde{\mathfrak{s}}_k^2 &\leq \left(\sum_{k=1}^{n} \widetilde{\nu}_{1,k}^3\right)^{\frac{1}{3}} \cdot \left(\sum_{k=1}^{n} \widetilde{\mathfrak{s}}_k^3\right)^{\frac{2}{3}} \\ &\leq 2\left(\sum_{k=1}^{n} \widetilde{\nu}_{3,k}\right)^{\frac{1}{3}} \cdot \left(\sum_{k=1}^{n} \widetilde{\mathfrak{s}}_k^3\right)^{\frac{2}{3}} \\ &= 2 \cdot \widetilde{\nu}_3^{\frac{1}{3}} \cdot L_n^{\frac{2}{3}} \cdot d. \end{aligned} \tag{2.20}$$

Combining (2.16), (2.19) and (2.20) we derive

$$\begin{aligned}\Delta_n(\mathcal{C}) \ll \frac{Md^4}{\varepsilon} \cdot \left(\widetilde{\nu}_3 + \widetilde{\nu}_3^{\frac{1}{3}} \cdot L_n^{\frac{2}{3}}\right) \cdot \left(\widetilde{\nu}_3 + \widetilde{\nu}_3^{2\chi(q_n)-\frac{1}{3}} \cdot L_n^{\frac{4}{3}-2\chi(q_n)}\right) \\ + d^2 \cdot \varepsilon + d^3 \cdot \left(\widetilde{\nu}_3 + \widetilde{\nu}_3^{\frac{1}{3}} \cdot L_n^{\frac{2}{3}}\right). \end{aligned} \tag{2.21}$$

Let

$$\delta = \sqrt{Md^2 \cdot \left(\widetilde{\nu}_3 + \widetilde{\nu}_3^{\frac{1}{3}} \cdot L_n^{\frac{2}{3}}\right) \cdot \left(\widetilde{\nu}_3 + \widetilde{\nu}_3^{2\chi(q_n)-\frac{1}{3}} \cdot L_n^{\frac{4}{3}-2\chi(q_n)}\right)}.$$

First, assume $\delta \geq 1$. Note that $\Delta_n(\mathcal{C}) \leq 1$. Hence

$$\Delta_n(\mathcal{C}) = \sqrt{M}d \cdot \sqrt{\widetilde{\nu}_3^2 + \widetilde{\nu}_3^{\frac{4}{3}} \cdot L_n^{\frac{2}{3}} + \widetilde{\nu}_3^{2\chi(q_n)+\frac{2}{3}} \cdot L_n^{\frac{4}{3}-2\chi(q_n)} + \widetilde{\nu}_3^{2\chi(q_n)} \cdot L_n^{2-2\chi(q_n)}}.$$

Since $1/4 \leq \chi(q_n) \leq 1/3$ we have

$$\widetilde{\nu}_3^{\frac{4}{3}} L_n^{\frac{2}{3}} \leq \widetilde{\nu}_3^2 + \widetilde{\nu}_3^{2\chi(q_n)} \cdot L_n^{2-2\chi(q_n)}$$

and

$$\widetilde{\nu}_3^{2\chi(q_n)+\frac{2}{3}} \cdot L_n^{\frac{4}{3}-2\chi(q_n)} \leq \widetilde{\nu}_3^2 + \widetilde{\nu}_3^{2\chi(q_n)} \cdot L_n^{2-2\chi(q_n)},$$

which yields

$$\begin{aligned}\Delta_n(\mathcal{C}) &\ll \sqrt{M}d\sqrt{\widetilde{\nu}_3^2 + \widetilde{\nu}_3^{2\chi(q_n)} \cdot L_n^{2-2\chi(q_n)}} \\ &\leq \sqrt{M}d \cdot \left(\widetilde{\nu}_3 + \widetilde{\nu}_3^{\chi(q_n)} \cdot L_n^{1-\chi(q_n)}\right) \\ &\leq \sqrt{M}d^3 \cdot \left(\widetilde{\nu}_3 + \widetilde{\nu}_3^{\chi(q_n)} \cdot L_n^{1-\chi(q_n)}\right). \end{aligned} \tag{2.22}$$

Next, assume $\delta < 1$. Take $\varepsilon = \delta$ in (2.21) and observe

$$\widetilde{\nu}_3^{\frac{1}{3}} \cdot L_n^{\frac{2}{3}} \leq \widetilde{\nu}_3 + \widetilde{\nu}_3^{\chi(q_n)} \cdot L_n^{1-\chi(q_n)}$$

to derive

$$\begin{aligned}
\Delta_n(\mathcal{C}) &\ll d^2 \cdot \sqrt{Md^2 \cdot \left(\widetilde{\nu}_3 + \widetilde{\nu}_3^{\frac{1}{3}} \cdot L_n^{\frac{2}{3}}\right) \cdot \left(\widetilde{\nu}_3 + \widetilde{\nu}_3^{2\chi(q_n)-\frac{1}{3}} \cdot L_n^{\frac{4}{3}-2\chi(q_n)}\right)} \\
&\qquad + d^3 \cdot \left(\widetilde{\nu}_3 + \widetilde{\nu}_3^{\chi(q_n)} \cdot L_n^{1-\chi(q_n)}\right) \\
&\ll \sqrt{M} d^3 \cdot \left(\widetilde{\nu}_3 + \widetilde{\nu}_3^{\chi(q_n)} \cdot L_n^{1-\chi(q_n)}\right) + d^3 \cdot \left(\widetilde{\nu}_3 + \widetilde{\nu}_3^{\chi(q_n)} \cdot L_n^{1-\chi(q_n)}\right) \\
&\ll \sqrt{M} d^3 \cdot \left(\widetilde{\nu}_3 + \widetilde{\nu}_3^{\chi(q_n)} \cdot L_n^{1-\chi(q_n)}\right) \qquad (2.23)
\end{aligned}$$

since $M > 1$. We can rewrite (2.22) and (2.23) as

$$\Delta_n(\mathcal{C}) \leq c_3 \cdot \sqrt{M} d^3 \cdot \left(\widetilde{\nu}_3 + \widetilde{\nu}_3^{\chi(q_n)} \cdot L_n^{1-\chi(q_n)}\right)$$

with some absolute constant c_3. Recall the absolute constant c_2 from Case 1. Choosing $M = \max(c_2, c_3^2)$ ensures

$$c_3 \sqrt{M} \leq M$$

and completes the proof of the theorem.

It remains to obtain the bound (2.18). Fix $k \in \{1, \ldots, n\}$ and define

$$\xi = Y_k' \cos\alpha + Y_k \sin\alpha, \qquad \zeta = (-Y_k' \sin\alpha + Y_k \cos\alpha)$$

with a d-dimensional centered Gaussian vector Y_k' such that

$$\operatorname{Cov}(Y_k') = \operatorname{Cov}(X_k)$$

and $Y_1', \ldots, Y_n', Y_1, \ldots, Y_n, X_1, \ldots, X_n, \alpha$ are independent. Put

$$R = (S_k, Z_k, \alpha).$$

Then H_k is a function of R, and Y_k, Y'_k, S_k, Z_k, α are independent. Hence, conditioned on R, the vector $(H_k+\xi, \zeta)$ is normally distributed and satisfies

$$\begin{aligned}\mathrm{Cov}(H_k+\xi,\zeta|R) &= \mathbf{E}(\xi\zeta^T|\alpha)\\ &= -\mathbf{E}(Y'_k(Y'_k)^T)\cos\alpha\sin\alpha + \mathbf{E}(Y'_kY_k^T)\cos^2\alpha\\ &\qquad - \mathbf{E}(Y_k(Y'_k)^T)\sin^2\alpha + \mathbf{E}(Y_kY_k^T)\cos\alpha\sin\alpha\\ &= \mathbf{E}Y'_k\mathbf{E}Y_k^T\cos^2\alpha - \mathbf{E}Y_k\mathbf{E}(Y'_k)^T\sin^2\alpha\\ &= 0.\end{aligned}$$

We conclude that, conditioned on R, the random variables $\varphi'(H_k+\xi)$ and ζ are independent, which implies

$$\begin{aligned}\mathbf{E}\varphi'(H_k+\xi)\zeta &= \mathbf{E}\mathbf{E}(\varphi'(H_k+\xi)\zeta|R)\\ &= \mathbf{E}(\mathbf{E}(\varphi'(H_k+\xi)|R)\cdot\mathbf{E}(\zeta|R))\\ &= \mathbf{E}(\mathbf{E}(\varphi'(H_k+\xi)|R)\cdot(-\mathbf{E}Y'_k\sin\alpha+\mathbf{E}Y_k\cos\alpha))\\ &= 0.\end{aligned}$$

Consequently,

$$\begin{aligned}\Theta_k &= \mathbf{E}\varphi'(H_k+X_k\cos\alpha+Y_k\sin\alpha)(-X_k\sin\alpha+Y_k\cos\alpha)\\ &\qquad - \mathbf{E}\varphi'(H_k+Y'_k\cos\alpha+Y_k\sin\alpha)(-Y'_k\sin\alpha+Y_k\cos\alpha)\\ &= \mathbf{E}\int\limits_{\mathbb{R}^d}\varphi'(H_k+z\cos\alpha+Y_k\sin\alpha)(-z\sin\alpha+Y_k\cos\alpha)\mu_k(dz). \qquad (2.24)\end{aligned}$$

Define

$$\mathrm{I}_1 = \mathrm{I}_{[0,\gamma)}(\alpha), \qquad \mathrm{I}_2 = \mathrm{I}_{[\gamma,\pi/2]}(\alpha),$$

where $\gamma = \arcsin\varepsilon$. Moreover, put

$$I_1 = \mathbf{E}\mathrm{I}_1\int\limits_{\mathbb{R}^d}\varphi'(H_k+z\cos\alpha+Y_k\sin\alpha)(-z\sin\alpha+Y_k\cos\alpha)\mu_k(dz),$$

$$I_2 = \mathbf{E}\mathrm{I}_2\int\limits_{\mathbb{R}^d}\varphi'(H_k+z\cos\alpha+Y_k\sin\alpha)(-z\sin\alpha+Y_k\cos\alpha)\mu_k(dz),$$

as well as

$$I_3 = \mathbf{E}\mathrm{I}_2 \int\limits_{\mathbb{R}^d} \varphi'(C_k Y + z\cos\alpha + Y_k \sin\alpha)(-z\sin\alpha + Y_k\cos\alpha)\mu_k(dz),$$

where Y is a d-dimensional standard normal vector such that the sequence $Y, X_1 \dots, X_n, Y_1, \dots, Y_n, Y_1', \dots, Y_n', \alpha$ is independent. Clearly,

$$|\Theta_k| = |I_1 + I_2| \le |I_1| + |I_2 - I_3| + |I_3|.$$

We separately estimate the quantities $|I_1|$, $|I_2 - I_3|$ and $|I_3|$.

Lemma 2.1. *We have*

$$|I_3| \ll \widetilde{\nu}_{3,k}.$$

Proof. Recall that C_k is invertible and use partial integration to obtain

$$\begin{aligned} I_3 &= \mathbf{E}\mathrm{I}_2 \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} \varphi'(C_k u + z\cos\alpha + Y_k\sin\alpha)(-z\sin\alpha + Y_k\cos\alpha)\mu_k(dz)\eta(u)du \\ &= -\mathbf{E}\mathrm{I}_2 \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} \varphi(C_k u + z\cos\alpha + Y_k\sin\alpha) \\ &\qquad\qquad \times \eta'(u) C_k^{-1}(-z\sin\alpha + Y_k\cos\alpha)\mu_k(dz)du \\ &= -\mathbf{E}\mathrm{I}_2 \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} \varphi(C_k x)\eta'(x - C_k^{-1}(z\cos\alpha + Y_k\sin\alpha)) \\ &\qquad\qquad \times C_k^{-1}(-z\sin\alpha + Y_k\cos\alpha)\mu_k(dz)dx. \end{aligned}$$

Put

$$W_x = x - C_k^{-1} Y_k \sin\alpha$$

for $x \in \mathbb{R}^d$. Apply the Taylor formula (2.9) to η' to obtain

$$
\begin{aligned}
&\eta'(W_x - C_k^{-1} z \cos\alpha) C_k^{-1}(-z \sin\alpha + Y_k \cos\alpha) \\
&\quad = \eta'(W_x) C_k^{-1}(-z \sin\alpha + Y_k \cos\alpha) \\
&\qquad - \eta''(W_x) C_k^{-1}(-z \sin\alpha + Y_k \cos\alpha) C_k^{-1} z \cos\alpha \\
&\qquad + \frac{1}{2}\eta'''(W_x) C_k^{-1} Y_k \cos\alpha (C_k^{-1} z \cos\alpha)^2 \\
&\qquad - \int_0^1 (1-u)\eta'''(W_x - u C_k^{-1} z \cos\alpha) C_k^{-1} z \sin\alpha (C_k^{-1} z \cos\alpha)^2 du \\
&\qquad - \frac{1}{2}\int_0^1 (1-u)^2 \eta''''(W_x - u C_k^{-1} z \cos\alpha) C_k^{-1} Y_k \cos\alpha (C_k^{-1} z \cos\alpha)^3 du.
\end{aligned}
$$

Observing (2.11) we have

$$
\int_{\mathbb{R}^d} \eta'(W_x) C_k^{-1}(-z \sin\alpha + Y_k \cos\alpha) \mu_k(dz) = 0,
$$

$$
\int_{\mathbb{R}^d} \eta''(W_x) C_k^{-1}(-z \sin\alpha + Y_k \cos\alpha) C_k^{-1} z \cos\alpha = 0,
$$

$$
\int_{\mathbb{R}^d} \eta'''(W_x) C_k^{-1} Y_k \cos\alpha (C_k^{-1} z \cos\alpha)^2 = 0,
$$

and therefore,

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \eta'(W_x - C_k^{-1} z \cos\alpha) C_k^{-1}(-z \sin\alpha + Y_k \cos\alpha) \mu_k(dz) \\
&\qquad = -\sin\alpha \cos^2\alpha \int_0^1 (1-u) \int_{\mathbb{R}^d} \eta'''(W_x - u C_k^{-1} z \cos\alpha) \\
&\qquad\qquad\qquad \times (C_k^{-1} z)^3 \mu_k(dz) du \\
&\qquad\quad - \frac{1}{2}\cos^4\alpha \int_0^1 (1-u)^2 \int_{\mathbb{R}^d} \eta''''(W_x - u C_k^{-1} z \cos\alpha) C_k^{-1} Y_k \\
&\qquad\qquad\qquad \times (C_k^{-1} z)^3 \mu_k(dz) du.
\end{aligned}
$$

Since $0 \le \varphi \le 1$ we conclude that

$$\begin{aligned}
|I_3| \le \mathbf{E} \int\limits_{\mathbb{R}^d} \varphi(C_k x) \int\limits_0^1 \int\limits_{\mathbb{R}^d} |\eta'''(W_x - uC_k^{-1} z \cos\alpha)(C_k^{-1} z)^3| |\mu_k|(dz) du dx \\
+ \mathbf{E} \int\limits_{\mathbb{R}^d} \varphi(C_k x) \int\limits_0^1 \int\limits_{\mathbb{R}^d} |\eta''''(W_x - uC_k^{-1} z \cos\alpha) C_k^{-1} Y_k (C_k^{-1} z)^3| \\
\times |\mu_k|(dz) du dx \\
\le \mathbf{E} \int\limits_0^1 \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} |\eta'''(W_x - uC_k^{-1} z \cos\alpha)(C_k^{-1} z)^3| dx |\mu_k|(dz) du \\
+ \mathbf{E} \int\limits_0^1 \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} |\eta''''(W_x - uC_k^{-1} z \cos\alpha) C_k^{-1} Y_k (C_k^{-1} z)^3| dx |\mu_k|(dz) du \\
= \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} |\eta'''(x)(C_k^{-1} z)^3| dx |\mu_k|(dz) \\
+ \mathbf{E} \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} |\eta''''(x) C_k^{-1} Y_k (C_k^{-1} z)^3| dx |\mu_k|(dz).
\end{aligned}$$

It is easy to check that

$$\eta'''(x) y^3 = (3|y|^2 x^T y - (x^T y)^3)\eta(x) \tag{2.25}$$

and

$$\eta''''(x)\widetilde{y} y^3 = (3|y|^2 \widetilde{y}^T y - 3(x^T y)^2 \widetilde{y}^T y - 3|y|^2 x^T \widetilde{y} x^T y + x^T \widetilde{y}(x^T y)^3)\eta(x) \tag{2.26}$$

for all $x, y, \widetilde{y} \in \mathbb{R}^d$. Let $\overline{Y}$ be a d-dimensional standard normal vector independent of Y_k. Then (2.25) and (2.26) imply

$$\int\limits_{\mathbb{R}^d} |\eta'''(x)(C_k^{-1} z)^3| dx \ll |C_k^{-1} z|^2 \mathbf{E}|\overline{Y}^T C_k^{-1} z| + \mathbf{E}|\overline{Y}^T C_k^{-1} z|^3$$

and

$$\mathbf{E}\int\limits_{\mathbb{R}^d} |\eta''''(x)C_k^{-1}Y_k(C_k^{-1}z)^3|dx$$
$$\ll |C_k^{-1}z|^2\mathbf{E}|(C_k^{-1}Y_k)^T C_k^{-1}z| + \mathbf{E}|\overline{Y}^T C_k^{-1}z|^2\mathbf{E}|(C_k^{-1}Y_k)^T C_k^{-1}z|$$
$$+ |C_k^{-1}z|^2\mathbf{E}|\overline{Y}^T C_k^{-1}Y_k \cdot \overline{Y}^T C_k^{-1}z| + \mathbf{E}|\overline{Y}^T C_k^{-1}Y_k(\overline{Y}^T C_k^{-1}z)^3|.$$

Clearly, $\overline{Y}^T C_k^{-1}z$ is a centered normal random variable with variance $|C_k^{-1}z|^2$. In particular,

$$\mathbf{E}|\overline{Y}^T C_k^{-1}z|^2 = |C_k^{-1}z|^2, \quad \mathbf{E}|\overline{Y}^T C_k^{-1}z|^3 \ll |C_k^{-1}z|^3. \tag{2.27}$$

Using the independence of $\overline{Y}$ and Y_k we obtain

$$\begin{aligned}\mathbf{E}|\overline{Y}^T C_k^{-1}Y_k(\overline{Y}^T C_k^{-1}z)^j| &\le (\mathbf{E}(\overline{Y}^T C_k^{-1}Y_k)^2)^{\frac{1}{2}}(\mathbf{E}(\overline{Y}^T C_k^{-1}z)^{2j})^{\frac{1}{2}}\\ &= (\mathbf{E}\mathbf{E}((\overline{Y}^T C_k^{-1}Y_k)^2|Y_k))^{\frac{1}{2}}\sqrt{(2j-1)!!}|C_k^{-1}z|^j\\ &= (\mathbf{E}|C_k^{-1}Y_k|^2)^{\frac{1}{2}}\sqrt{(2j-1)!!}|C_k^{-1}z|^j\\ &\le ||C_k^{-1}||\widetilde{\mathfrak{s}}_k \cdot \sqrt{(2j-1)!!}|C_k^{-1}z|^j\end{aligned}$$

for $j \in \mathbb{N}$. Additionally,

$$\mathbf{E}|(C_k^{-1}Y_k)^T C_k^{-1}z| \le ||C_k^{-1}|| \cdot |C_k^{-1}z|\mathbf{E}|Y_k| \le ||C_k^{-1}|| \cdot |C_k^{-1}z| \cdot \widetilde{\mathfrak{s}}_k.$$

Collecting the bounds, using (2.15) and the fact that $\widetilde{\mathfrak{s}}_k \le 1/\sqrt{2}$ in the case under consideration we derive

$$\int\limits_{\mathbb{R}^d} |\eta'''(x)(C_k^{-1}z)^3|dx \ll |z|^3 \tag{2.28}$$

as well as

$$\mathbf{E}\int\limits_{\mathbb{R}^d} |\eta''''(x)C_k^{-1}Y_k(C_k^{-1}z)^3|dx \ll |z|^3.$$

It remains to observe that $|z|^3 = n^{-3/2} \cdot |\overline{\Sigma}^{-1}z|^3$ to complete the proof of the lemma. $\square$

Lemma 2.2. *We have*

$$|I_1| \ll \frac{1}{\varepsilon} \cdot (\widetilde{\nu}_{3,k} + \widetilde{\mathfrak{s}}_k^2 \widetilde{\nu}_{1,k}) \cdot \big(M d^3 \cdot \big(\widetilde{\nu}_3 + \widetilde{\nu}_3^{2\chi(q_n) - \frac{1}{3}} \cdot L_n^{\frac{4}{3} - 2\chi(q_n)} \big) + d^{\frac{3}{2}} \varepsilon \big).$$

Proof. Clearly, we may assume

$$C_k^{-1} Z_k = \frac{1}{\sqrt{2}} (\overline{Y} + \widetilde{Y}),$$

where $\overline{Y}, \widetilde{Y}$ are d-dimensional standard normal vectors such that the sequence $\overline{Y}, \widetilde{Y}, X_1, \ldots, X_n, Y_k, \alpha$ is independent. Then

$$H_k = S_k \cos\alpha + \frac{1}{\sqrt{2}} C_k \overline{Y} \sin\alpha + \frac{1}{\sqrt{2}} C_k \widetilde{Y} \sin\alpha.$$

Put

$$T = S_k \cos\alpha + \frac{1}{\sqrt{2}} C_k \overline{Y} \sin\alpha.$$

We have

$$\begin{aligned}
I_1 &= \mathbf{E} \mathrm{I}_1 \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} \varphi'(T + \frac{1}{\sqrt{2}} C_k y \sin\alpha + z\cos\alpha + Y_k \sin\alpha)(-z\sin\alpha + Y_k\cos\alpha) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times \eta(y) \mu_k(dz) dy \\
&= \mathbf{E} \mathrm{I}_1 \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} \varphi'(T + \frac{1}{\sqrt{2}} C_k u \sin\alpha)(-z\sin\alpha + Y_k\cos\alpha) \\
&\qquad\qquad\qquad \times \eta(u - \sqrt{2}\,\frac{\cos\alpha}{\sin\alpha} C_k^{-1} z - \sqrt{2} C_k^{-1} Y_k) \mu_k(dz) du \\
&= -\mathbf{E} \mathrm{I}_1 \sin\alpha \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} \varphi'(T + \frac{1}{\sqrt{2}} C_k u \sin\alpha) z \\
&\qquad\qquad\qquad \times \eta(u - \sqrt{2}\,\frac{\cos\alpha}{\sin\alpha} C_k^{-1} z - \sqrt{2} C_k^{-1} Y_k) \mu_k(dz) du \\
&\quad + \mathbf{E} \mathrm{I}_1 \cos\alpha \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} \varphi'(T + \frac{1}{\sqrt{2}} C_k u \sin\alpha) Y_k \\
&\qquad\qquad\qquad \times \eta(u - \sqrt{2}\,\frac{\cos\alpha}{\sin\alpha} C_k^{-1} z - \sqrt{2} C_k^{-1} Y_k) \mu_k(dz) du. \qquad (2.29)
\end{aligned}$$

Apply the Taylor formula (2.9) to η and η' to obtain the representations

$$\begin{aligned}
&\eta(u-\sqrt{2}\,\frac{\cos\alpha}{\sin\alpha}C_k^{-1}z-\sqrt{2}C_k^{-1}Y_k)\\
&\qquad=\eta(u-\sqrt{2}C_k^{-1}Y_k)\\
&\qquad\quad+\eta'(u-\sqrt{2}C_k^{-1}Y_k)(-\sqrt{2}\,\frac{\cos\alpha}{\sin\alpha}C_k^{-1}z)\\
&\qquad\quad+\int_0^1(1-u_1)\eta''(u-\sqrt{2}u_1\frac{\cos\alpha}{\sin\alpha}C_k^{-1}z-\sqrt{2}C_k^{-1}Y_k)\\
&\qquad\qquad\times(-\sqrt{2}\,\frac{\cos\alpha}{\sin\alpha}C_k^{-1}z)^2du_1
\end{aligned}$$

and

$$\begin{aligned}
&\eta(u-\sqrt{2}\,\frac{\cos\alpha}{\sin\alpha}C_k^{-1}z-\sqrt{2}C_k^{-1}Y_k)\\
&\qquad=\eta(u-\sqrt{2}\,\frac{\cos\alpha}{\sin\alpha}C_k^{-1}z)\\
&\qquad\quad+\int_0^1\eta'(u-\sqrt{2}u_1C_k^{-1}Y_k)(-\sqrt{2}C_k^{-1}Y_k)du_1\\
&\qquad\quad+\int_0^1\int_0^1\eta''(u-\sqrt{2}u_2\frac{\cos\alpha}{\sin\alpha}C_k^{-1}z-\sqrt{2}u_1C_k^{-1}Y_k)(-\sqrt{2}C_k^{-1}Y_k)\\
&\qquad\qquad\times(-\sqrt{2}\,\frac{\cos\alpha}{\sin\alpha}C_k^{-1}z)du_1du_2,
\end{aligned}$$

which are used for the first and the second integral in (2.29), respectively. Thus, observing (2.11) and $\mathbf{E}Y_k=0$, we derive

$$\begin{aligned}
I_1=&-2\mathbf{E}\mathrm{I}_1\frac{\cos^2\alpha}{\sin\alpha}\int_0^1(1-u_1)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\varphi'(T+\frac{1}{\sqrt{2}}C_ku\sin\alpha)z\\
&\times\eta''(u-\sqrt{2}u_1\frac{\cos\alpha}{\sin\alpha}C_k^{-1}z-\sqrt{2}C_k^{-1}Y_k)(C_k^{-1}z)^2\mu_k(dz)dudu_1\\
&+2\mathbf{E}\mathrm{I}_1\frac{\cos^2\alpha}{\sin\alpha}\int_0^1\int_0^1\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\varphi'(T+\frac{1}{\sqrt{2}}C_ku\sin\alpha)Y_k\\
&\times\eta''(u-\sqrt{2}u_2\frac{\cos\alpha}{\sin\alpha}C_k^{-1}z-\sqrt{2}u_1C_k^{-1}Y_k)C_k^{-1}Y_kC_k^{-1}z\mu_k(dz)dudu_1du_2.
\end{aligned}$$

Hence

$$|I_1| \ll \mathbf{E}\mathrm{I}_1 \frac{\cos^2\alpha}{\sin\alpha} \Bigg(\int_0^1 \int_{\mathbb{R}^d} \Big| \int_{\mathbb{R}^d} \varphi'(T + \frac{1}{\sqrt{2}} C_k u \sin\alpha) z$$

$$\times \eta''(u - \sqrt{2} u_1 \frac{\cos\alpha}{\sin\alpha} C_k^{-1} z - \sqrt{2} C_k^{-1} Y_k)(C_k^{-1} z)^2 du \Big| |\mu_k|(dz) du_1$$

$$+ \int_0^1 \int_0^1 \int_{\mathbb{R}^d} \Big| \int_{\mathbb{R}^d} \varphi'(T + \frac{1}{\sqrt{2}} C_k u \sin\alpha) Y_k$$

$$\times \eta''(u - \sqrt{2} u_2 \frac{\cos\alpha}{\sin\alpha} C_k^{-1} z - \sqrt{2} u_1 C_k^{-1} Y_k) C_k^{-1} Y_k C_k^{-1} z du \Big|$$

$$\times |\mu_k|(dz) du_1 du_2 \Bigg).$$

By the properties of φ' we have

$$\Big| \varphi'(T + \frac{1}{\sqrt{2}} C_k u \sin\alpha) x - \varphi'(T + \frac{1}{\sqrt{2}} C_k v \sin\alpha) x \Big| \leq \frac{4\sqrt{2}\sin\alpha}{\varepsilon^2} \cdot |u - v| \cdot |x|$$

for all $u, v, x \in \mathbb{R}^d$ and

$$\mathrm{supp}(\varphi') \subset A^\varepsilon \backslash A,$$

see Lemma A.1. Employ Lemma A.4 and use (2.15) to get

$$|I_1| \ll \frac{1}{\varepsilon^2} \mathbf{E}\mathrm{I}_1 \cos^2\alpha \Bigg(\int_0^1 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathrm{I}_{A^\varepsilon \backslash A} \Big(T + \frac{1}{\sqrt{2}} C_k u \sin\alpha \Big) \tag{2.30}$$

$$\times |\eta'(u - \sqrt{2} u_1 \frac{\cos\alpha}{\sin\alpha} C_k^{-1} z - \sqrt{2} C_k^{-1} Y_k) C_k^{-1} z| |z|^2 |\mu_k|(dz) du du_1$$

$$+ \int_0^1 \int_0^1 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathrm{I}_{A^\varepsilon \backslash A} \Big(T + \frac{1}{\sqrt{2}} C_k u \sin\alpha \Big)$$

$$\times |\eta'(u - \sqrt{2} u_2 \frac{\cos\alpha}{\sin\alpha} C_k^{-1} z - \sqrt{2} u_1 C_k^{-1} Y_k) C_k^{-1} Y_k| |z| |Y_k|$$

$$\times |\mu_k|(dz) du du_1 du_2 \Bigg).$$

The random variables S_k, $\overline{Y}$, Y_k and α are independent. Therefore

$$\begin{aligned} P\Big(T + \frac{1}{\sqrt{2}} C_k u \sin\alpha \in A^\varepsilon \setminus A \big| Y_k, \alpha\Big) \\ &= P\Big(T + \frac{1}{\sqrt{2}} C_k u \sin\alpha \in A^\varepsilon \setminus A \big| \alpha\Big) \\ &\le \sup_{x \in \mathbb{R}^d} P(T + x \in A^\varepsilon \setminus A \,|\, \alpha) \\ &= \sup_{x \in \mathbb{R}^d} P(T \in (A - x)^\varepsilon \setminus (A - x) \,|\, \alpha) \\ &\le J_1, \end{aligned}$$

where

$$J_1 = \sup_{\substack{E \in \mathcal{C}, \\ \beta \in [0, \pi/2]}} P\Big(S_k \cos\beta + \frac{1}{\sqrt{2}} C_k \overline{Y} \sin\beta \in E^\varepsilon \setminus E\Big).$$

Thus (2.30) implies

$$\begin{aligned} |I_1| \ll \frac{J_1}{\varepsilon^2} \cdot \mathbf{E} \mathrm{I}_1 \cos^2\alpha \\ \times \Bigg(\int_0^1 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Big| \eta'(u - \sqrt{2} u_1 \frac{\cos\alpha}{\sin\alpha} C_k^{-1} z - \sqrt{2} C_k^{-1} Y_k) C_k^{-1} z \Big| \\ \times |z|^2 |\mu_k|(dz) du du_1 \\ + \int_0^1 \int_0^1 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Big| \eta'(u - \sqrt{2} u_2 \frac{\cos\alpha}{\sin\alpha} C_k^{-1} z - \sqrt{2} u_1 C_k^{-1} Y_k) C_k^{-1} Y_k \Big| \\ \times |z| |Y_k| |\mu_k|(dz) du du_1 du_2 \Bigg) \\ = \frac{J_1}{\varepsilon^2} \cdot \mathbf{E} \mathrm{I}_1 \cos^2\alpha \Bigg(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\eta'(u) C_k^{-1} z| |z|^2 |\mu_k|(dz) du \\ + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\eta'(u) C_k^{-1} Y_k| |z| |Y_k| |\mu_k|(dz) du \Bigg). \end{aligned}$$

Use

$$\int_{\mathbb{R}^d} |\eta'(u)C_k^{-1}z|du = \int_{\mathbb{R}^d} |u^T C_k^{-1}z|\eta(u)du = \mathbf{E}|\overline{Y}^T C_k^{-1}z| \le |C_k^{-1}z| \le \sqrt{2}|z|$$

as well as

$$\int_{\mathbb{R}^d} |\eta'(u)C_k^{-1}Y_k|du \le \sqrt{2}|Y_k|$$

and

$$\mathbf{E}\mathrm{I}_1 \cos^2\alpha \le \mathbf{E}\mathrm{I}_1 \cos\alpha = \frac{2}{\pi}\varepsilon,$$

and observe the independence of Y_k and α again to conclude that

$$\begin{aligned} |I_1| &\ll \frac{J_1}{\varepsilon^2} \cdot \mathbf{E}\mathrm{I}_1 \cos^2\alpha \cdot \left(\int_{\mathbb{R}^d} |z|^3 |\mu_k|(dz) + |Y_k|^2 \int_{\mathbb{R}^d} |z||\mu_k|(dz) \right) \\ &\le \frac{J_1}{\varepsilon} \cdot (\widetilde{\nu}_{3k} + \widetilde{\mathfrak{s}}_k^2 \widetilde{\nu}_{1k}). \end{aligned} \tag{2.31}$$

Let us now estimate J_1. We have

$$J_1 \le J_2 + J_3$$

with

$$\begin{aligned} J_2 = \sup_{\substack{E\in\mathcal{C} \\ \beta\in[0,\pi/2]}} \Big| P\Big(S_k \cos\beta + \frac{1}{\sqrt{2}} C_k \overline{Y} \sin\beta \in E^\varepsilon \setminus E\Big) \\ - P\Big(Z_k \cos\beta + \frac{1}{\sqrt{2}} C_k \overline{Y} \sin\beta \in E^\varepsilon \setminus E\Big) \Big| \end{aligned}$$

and

$$J_3 = \sup_{\substack{E\in\mathcal{C} \\ \beta\in[0,\pi/2]}} P\Big(Z_k \cos\beta + \frac{1}{\sqrt{2}} C_k \overline{Y} \sin\beta \in E^\varepsilon \setminus E\Big).$$

Recall the invariance properties $(P1)$ and $(P2)$ of the class $\mathcal{C}$ and use the independence of (S_k, Z_k) and $\overline{Y}$ to get

$$\begin{aligned} J_2 = &\sup_{\substack{E\in\mathcal{C}\\ \beta\in[0,\pi/2]}} \Big|\mathbf{E}P\Big(S_k\cos\beta + \frac{1}{\sqrt{2}}C_k\overline{Y}\sin\beta \in E^\varepsilon\setminus E \,\Big|\, \overline{Y}\Big) \\ &\qquad - \mathbf{E}P\Big(Z_k\cos\beta + \frac{1}{\sqrt{2}}C_k\overline{Y}\sin\beta \in E^\varepsilon\setminus E \,\Big|\, \overline{Y}\Big)\Big| \\ &\le \mathbf{E}\sup_{\substack{E\in\mathcal{C}\\ \beta\in[0,\pi/2]}} \Big|P\Big(S_k\cos\beta + \frac{1}{\sqrt{2}}C_k\overline{Y}\sin\beta \in E^\varepsilon\setminus E \,\Big|\, \overline{Y}\Big) \\ &\qquad - P\Big(Z_k\cos\beta + \frac{1}{\sqrt{2}}C_k\overline{Y}\sin\beta \in E^\varepsilon\setminus E \,\Big|\, \overline{Y}\Big)\Big| \\ &\le \sup_{\substack{E\in\mathcal{C}\\ \beta\in[0,\pi/2]}} |P(S_k\cos\beta\in E^\varepsilon\setminus E) - P(Z_k\cos\beta\in E^\varepsilon\setminus E)| \\ &\le 2\sup_{\substack{E\in\mathcal{C}\\ \beta\in[0,\pi/2]}} |P(S_k\cos\beta\in E) - P(Z_k\cos\beta\in E)| \\ &= 2\sup_{E\in\mathcal{C}} |P(S_k\in E) - P(Z_k\in E)|. \end{aligned}$$

Recall that $\max_{1\le k\le n}\widetilde{\mathfrak{s}}_k^2 \le 1/2$ in the case under consideration. Hence

$$\widetilde{\mathfrak{s}}_1^2 + \ldots + \widetilde{\mathfrak{s}}_{k-1}^2 + \widetilde{\mathfrak{s}}_{k+1}^2 + \ldots + \widetilde{\mathfrak{s}}_n^2 = d - \widetilde{\mathfrak{s}}_k^2 \ge d - \frac{1}{2} \ge \frac{d}{2}$$

for every $k\in\{1,\ldots,n\}$ and consequently,

$$\begin{aligned} \frac{\widetilde{\mathfrak{s}}_1^3 + \ldots + \widetilde{\mathfrak{s}}_{k-1}^3 + \widetilde{\mathfrak{s}}_{k+1}^3 + \ldots + \widetilde{\mathfrak{s}}_n^3}{(\widetilde{\mathfrak{s}}_1^2 + \ldots + \widetilde{\mathfrak{s}}_{k-1}^2 + \widetilde{\mathfrak{s}}_{k+1}^2 + \ldots + \widetilde{\mathfrak{s}}_n^2)^{3/2}} &\le 2\sqrt{2}\cdot\frac{\widetilde{\mathfrak{s}}_1^3 + \ldots + \widetilde{\mathfrak{s}}_n^3}{d^{3/2}} \\ &= 2\sqrt{2}\cdot L_n. \end{aligned}$$

Put

$$q_k^{(n)} = \Big\lfloor \sum_{i\ne k}\mathfrak{s}_i^2 \Big/ \max_{i\ne k}\mathfrak{s}_i^2 \Big\rfloor$$

for $k = 1,\ldots,n$. Use the last inequality together with the induction assumption to obtain

$$
\begin{aligned}
&\sup_{E\in\mathcal{C}}|P(S_k\in E)-P(Z_k\in E)|\\
&\qquad\le Md^3\sum_{\substack{1\le m\le n\\ m\ne k}}\Bigg(\int\limits_{\mathbb{R}^d}|C_k^{-1}z|^3|\mu_m|(dz)+\Big(\int\limits_{\mathbb{R}^d}|C_k^{-1}z|^3|\mu_m|(dz)\Big)^{\chi(q_k^{(n)})}\\
&\qquad\qquad\times\Big(\frac{\widetilde{\mathfrak{s}}_1^3+\ldots+\widetilde{\mathfrak{s}}_{k-1}^3+\widetilde{\mathfrak{s}}_{k+1}^3+\ldots+\widetilde{\mathfrak{s}}_n^3}{(\widetilde{\mathfrak{s}}_1^2+\ldots+\widetilde{\mathfrak{s}}_{k-1}^2+\widetilde{\mathfrak{s}}_{k+1}^2+\ldots+\widetilde{\mathfrak{s}}_n^2)^{3/2}}\Big)^{1-\chi(q_k^{(n)})}\Bigg)\\
&\qquad\le Md^3\sum_{1\le m\le n}\Bigg(\int\limits_{\mathbb{R}^d}|C_k^{-1}z|^3|\mu_m|(dz)+\Big(\int\limits_{\mathbb{R}^d}|C_k^{-1}z|^3|\mu_m|(dz)\Big)^{\chi(q_k^{(n)})}\\
&\qquad\qquad\times(2\sqrt{2}\cdot L_n)^{1-\chi(q_k^{(n)})}\Bigg)\\
&\qquad\ll Md^3\cdot\big(\widetilde{\nu}_3+\widetilde{\nu}_3^{\chi(q_k^{(n)})}\cdot L_n^{1-\chi(q_k^{(n)})}\big).
\end{aligned}
$$

Let us show that

$$
\widetilde{\nu}_3^{\chi(q_k^{(n)})}L_n^{1-\chi(q_k^{(n)})}\le\widetilde{\nu}_3+\widetilde{\nu}_3^{2\chi(q_n)-\frac{1}{3}}L_n^{\frac{4}{3}-2\chi(q_n)}. \tag{2.32}
$$

Clearly, (2.32) holds if $L_n\le\widetilde{\nu}_3$. Next assume $L_n>\widetilde{\nu}_3$..... Observe that

$$
q_n\le\left\lfloor\frac{\sum\limits_{i\ne k}\widetilde{\mathfrak{s}}_i^2}{\max\limits_{1\le i\le n}\widetilde{\mathfrak{s}}_i^2}+1\right\rfloor\le q_k^{(n)}+1,
$$

which yields

$$
\chi(q_k^{(n)})=\frac{1}{3}-\frac{1}{12\cdot 2^{\lfloor\frac{q_k^{(n)}}{2d}\rfloor}}\ge\frac{1}{3}-\frac{1}{12\cdot 2^{\lfloor\frac{q_n-1}{2d}\rfloor}}\ge\frac{1}{3}-\frac{2}{12\cdot 2^{\lfloor\frac{q_n}{2d}\rfloor}}=2\chi(q_n)-\frac{1}{3}.
$$

Consequently,

$$
\widetilde{\nu}_3^{\chi(q_k^{(n)})}L_n^{1-\chi(q_k^{(n)})}=\Big(\frac{\widetilde{\nu}_3}{L_n}\Big)^{\chi(q_k^{(n)})}L_n\le\Big(\frac{\widetilde{\nu}_3}{L_n}\Big)^{2\chi(q_n)-\frac{1}{3}}L_n=\widetilde{\nu}_3^{2\chi(q_n)-\frac{1}{3}}L_n^{\frac{4}{3}-2\chi(q_n)},
$$

which completes the proof of (2.32). Hence

$$
\sup_{E\in\mathcal{C}}|P(S_k\in E)-P(Z_k\in E)|\ll Md^3\cdot\big(\widetilde{\nu}_3+\widetilde{\nu}_3^{2\chi(q_n)-\frac{1}{3}}\cdot L_n^{\frac{4}{3}-2\chi(q_n)}\big) \tag{2.33}
$$

and

$$J_2 \ll Md^3 \cdot \big(\widetilde{\nu}_3 + \widetilde{\nu}_3^{2\chi(q_n)-\frac{1}{3}} \cdot L_n^{\frac{4}{3}-2\chi(q_n)}\big). \tag{2.34}$$

Next, we estimate J_3. Note that

$$Z_k \cos\beta + \frac{1}{\sqrt{2}} C_k \overline{Y} \sin\beta \overset{d}{=} \frac{1}{\sqrt{2}} C_k \overline{\overline{Y}} \cos\beta + \frac{1}{\sqrt{2}} C_k \overline{Y},$$

where $\overline{\overline{Y}}$ is a d-dimensional standard normal vector such that $\overline{Y}$ and $\overline{\overline{Y}}$ are independent. Hence, by Lemma A.3,

$$\begin{aligned} J_3 = &\sup_{\substack{E\in\mathcal{C} \\ \beta\in[0,\pi/2]}} P\Big(\frac{1}{\sqrt{2}} C_k \overline{\overline{Y}} \cos\beta + \frac{1}{\sqrt{2}} C_k \overline{Y} \in E^{\varepsilon} \setminus E\Big) \\ &\le \sup_{E\in\mathcal{C}} P\Big(\frac{1}{\sqrt{2}} C_k \overline{Y} \in E^{\varepsilon} \setminus E\Big) \\ &\le \sup_{E\in\mathcal{C}} P(\overline{Y} \in E^{2\varepsilon} \setminus E) \\ &\le 2\sqrt{\frac{2}{\pi}} \cdot d^{\frac{3}{2}} \cdot \varepsilon. \end{aligned} \tag{2.35}$$

Combining (2.31), (2.34) and (2.35) we finally get

$$|I_1| \ll \frac{1}{\varepsilon} \cdot \big(\widetilde{\nu}_{3,k} + \widetilde{\mathfrak{s}}_k^2 \cdot \widetilde{\nu}_{1,k}\big) \cdot \big(Md^3 \cdot \widetilde{\nu}_3 + Md^3 \cdot \widetilde{\nu}_3^{2\chi(q_n)-\frac{1}{3}} \cdot L_n^{\frac{4}{3}-2\chi(q_n)} + d^{\frac{3}{2}} \cdot \varepsilon\big),$$

which finishes the proof of the lemma. □

Lemma 2.3. *We have*

$$|I_2 - I_3| \ll \frac{Md^3}{\varepsilon} \cdot \big(\widetilde{\nu}_{3,k} + \widetilde{\mathfrak{s}}_k^2 \cdot \widetilde{\nu}_{1,k}\big) \cdot \big(\widetilde{\nu}_3 + \widetilde{\nu}_3^{2\chi(q_n)-\frac{1}{3}} \cdot L_n^{\frac{4}{3}-2\chi(q_n)}\big).$$

Proof. Using the independence of Y and α we get

$$C_k Y \overset{d}{=} C_k Y \cos\alpha + C_k \widetilde{Y} \sin\alpha,$$

where $\widetilde{Y}$ is a d-dimensional standard normal vector such that $\widetilde{Y}$, Y, Y_k, α

are independent. Therefore

$$
\begin{aligned}
I_2 - I_3 = \mathbf{E}\mathrm{I}_2 \int\limits_{\mathbb{R}^d} \varphi'(S_k \cos\alpha + Z_k \sin\alpha + z\cos\alpha + Y_k \sin\alpha) \\
\times (-z\sin\alpha + Y_k\cos\alpha)\mu_k(dz) \\
- \mathbf{E}\mathrm{I}_2 \int\limits_{\mathbb{R}^d} \varphi'(C_k Y\cos\alpha + C_k\widetilde{Y}\sin\alpha + z\cos\alpha + Y_k\sin\alpha) \\
\times (-z\sin\alpha + Y_k\cos\alpha)\mu_k(dz).
\end{aligned}
$$

Integration by parts and changing variables gives

$$
\begin{aligned}
I_2 - I_3 &= -\mathbf{E}\mathrm{I}_2(\sin\alpha)^{-1} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} \varphi(S_k\cos\alpha + C_k u\sin\alpha + z\cos\alpha + Y_k\sin\alpha) \\
&\qquad \times \eta'(u) C_k^{-1}(-z\sin\alpha + Y_k\cos\alpha)\mu_k(dz)du \\
&\quad + \mathbf{E}\mathrm{I}_2(\sin\alpha)^{-1} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} \varphi(C_k Y\cos\alpha + C_k u\sin\alpha + z\cos\alpha + Y_k\sin\alpha) \\
&\qquad \times \eta'(u) C_k^{-1}(-z\sin\alpha + Y_k\cos\alpha)\mu_k(dz)du \\
&= -\mathbf{E}\mathrm{I}_2(\sin\alpha)^{-1} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} \varphi(S_k\cos\alpha + C_k x\sin\alpha) \\
&\quad \times \eta'(x - \cot\alpha C_k^{-1} z - C_k^{-1} Y_k) C_k^{-1}(-z\sin\alpha + Y_k\cos\alpha)\mu_k(dz)dx \\
&\quad + \mathbf{E}\mathrm{I}_2(\sin\alpha)^{-1} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} \varphi(C_k Y\cos\alpha + C_k x\sin\alpha) \\
&\quad \times \eta'(x - \cot\alpha C_k^{-1} z - C_k^{-1} Y_k) C_k^{-1}(-z\sin\alpha + Y_k\cos\alpha)\mu_k(dz)dx \\
&= \mathbf{E}\mathrm{I}_2 \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} \mathbf{E}(\varphi(S_k\cos\alpha + C_k x\sin\alpha) - \varphi(C_k Y\cos\alpha + C_k x\sin\alpha)|\alpha) \\
&\qquad \times \eta'(x - \cot\alpha C_k^{-1} z - C_k^{-1} Y_k) C_k^{-1} z\mu_k(dz)dx \\
&\quad - \mathbf{E}\mathrm{I}_2\cot\alpha \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} \mathbf{E}(\varphi(S_k\cos\alpha + C_k x\sin\alpha) \\
&\qquad - \varphi(C_k Y\cos\alpha + C_k x\sin\alpha)|\alpha) \\
&\qquad \times \eta'(x - \cot\alpha C_k^{-1} z - C_k^{-1} Y_k) C_k^{-1} Y_k\mu_k(dz)dx. \qquad (2.36)
\end{aligned}
$$

Apply the Taylor formula (2.9) to η' to obtain

$$\begin{aligned}
&\eta'(x - \cot\alpha C_k^{-1}z - C_k^{-1}Y_k)C_k^{-1}z \\
&\quad = \eta'(x - C_k^{-1}Y_k)C_k^{-1}z \\
&\qquad + \eta''(x - C_k^{-1}Y_k)C_k^{-1}z(-\cot\alpha\, C_k^{-1}z) \\
&\qquad + \int_0^1 (1-u)\eta'''(x - \cot\alpha\, uC_k^{-1}z - C_k^{-1}Y_k)C_k^{-1}z(-\cot\alpha\, C_k^{-1}z)^2 du.
\end{aligned}$$

for the first summand and

$$\begin{aligned}
&\eta'(x - \cot\alpha\, C_k^{-1}z - C_k^{-1}Y_k)C_k^{-1}Y_k \\
&\quad = \eta'(x - \cot\alpha\, C_k^{-1}z)C_k^{-1}Y_k \\
&\qquad + \int_0^1 \eta''(x - u_1C_k^{-1}Y_k)(C_k^{-1}Y_k)^2 du_1 \\
&\qquad + \int_0^1\int_0^1 \eta'''(x - \cot\alpha\, u_2C_k^{-1}z - u_1C_k^{-1}Y_k)(C_k^{-1}Y_k)^2 \\
&\qquad\qquad \times (-\cot\alpha\, C_k^{-1}z)du_1du_2
\end{aligned}$$

for the second summand in (2.36). Observing (2.11) and $\mathbf{E}Y_k = 0$ we conclude that

$$\begin{aligned}
|I_2 - I_3| \le{}& \mathbf{E}\mathrm{I}_2 \cot^2\alpha \int_0^1 \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\mathbf{E}(\varphi(S_k\cos\alpha + C_kx\sin\alpha) \\
&\qquad - \varphi(C_kY\cos\alpha + C_kx\sin\alpha)|\alpha)| \\
&\times |\eta'''(x - \cot\alpha\, uC_k^{-1}z - C_k^{-1}Y_k)(C_k^{-1}z)^3||\mu_k|(dz)dxdu \\
&+ \mathbf{E}\mathrm{I}_2 \cot^2\alpha \int_0^1\int_0^1\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\mathbf{E}(\varphi(S_k\cos\alpha + C_kx\sin\alpha) \\
&\qquad - \varphi(C_kY\cos\alpha + C_kx\sin\alpha)|\alpha)| \\
&\times |\eta'''(x - \cot\alpha\, u_2C_k^{-1}z - u_1C_k^{-1}Y_k)(C_k^{-1}Y_k)^2C_k^{-1}z| \\
&\qquad \times |\mu_k|(dz)dxdu_1du_2.
\end{aligned}$$

Using the second part of Lemma A.1 as well as the independence of S_k, Y and α, we have

$$|\mathbf{E}(\varphi(S_k \cos\alpha + C_k x \sin\alpha) - \varphi(C_k Y \cos\alpha + C_k x \sin\alpha)|\alpha)| \\ \leq \sup_{z\in\mathbb{R}^d} |\mathbf{E}(\psi(\rho(S_k \cos\alpha + z, A)/\varepsilon) - \psi(\rho(C_k Y \cos\alpha + z, A)/\varepsilon)|\alpha)|.$$

Fix $z \in \mathbb{R}^d$ and $\beta \in [0, \pi/2]$, and let G_1 and G_2 denote the distribution functions of $\rho(S_k \cos\beta + z, A)/\varepsilon$ and $\rho(C_k Y \cos\beta + z, A)/\varepsilon$, respectively. Then

$$\begin{aligned} &|\mathbf{E}(\psi(\rho(S_k \cos\alpha + z, A)/\varepsilon) - \psi(\rho(C_k Y \cos\alpha + z, A)/\varepsilon)|\alpha = \beta)| \\ &\qquad = \left| \int_{\mathbb{R}} \psi(t) dG_1(t) - \int_{\mathbb{R}} \psi(t) dG_2(t) \right| \\ &\qquad = \left| \int_{\mathbb{R}} \psi'(t)(G_1(t) - G_2(t)) dt \right| \\ &\qquad \leq \int_{\mathbb{R}} |\psi'(t)||G_1(t) - G_2(t)| dt, \end{aligned}$$

by partial integration. Moreover, using the invariance properties of $\mathcal{C}$ we obtain

$$\begin{aligned} |G_1(t) - G_2(t)| &\leq \sup_{E\in C} |P(S_k \cos\beta \in E) - P(C_k Y \cos\beta \in E)| \\ &\leq \sup_{E\in C} |P(S_k \in E) - P(C_k Y \in E)|. \end{aligned}$$

Employing (2.33) we thus conclude that

$$\begin{aligned} &\sup_{z\in R} |\mathbf{E}(\psi(\rho(S_k \cos\alpha + z, A)/\varepsilon) - \psi(\rho(C_k Y \cos\alpha + z, A)/\varepsilon)|\alpha)| \\ &\qquad \leq \sup_{E\in C} |P(S_k \in E) - P(C_k Y \in E)| \cdot \int_{\mathbb{R}} |\psi'(t)| dt \\ &\qquad \ll M d^3 \cdot \left(\widetilde{\nu}_3 + \widetilde{\nu}_3^{2\chi(q_n) - \frac{1}{3}} L_n^{\frac{4}{3} - 2\chi(q_n)} \right). \end{aligned}$$

Hence

$$|I_2 - I_3| \ll Md^3 \cdot \mathbf{E}\mathrm{I}_2 \cot^2 \alpha \bigg(\int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} (|\eta'''(x)(C_k^{-1}z)^3| + |\eta'''(x)(C_k^{-1}Y_k)^2 C_k^{-1}z|)|\mu_k|(dz)dx \bigg) \times \big(\widetilde{\nu}_3 + \widetilde{\nu}_3^{2\chi(q_n)-\frac{1}{3}} \cdot L_n^{\frac{4}{3}-2\chi(q_n)}\big).$$

Since

$$\eta'''(x)y^2\widetilde{y} = (2y^T\widetilde{y}x^Ty + y^Tyx^T\widetilde{y} - x^T\widetilde{y}(x^Ty)^2) \cdot \eta(x)$$

for every $x, \widetilde{y}, y \in \mathbb{R}^d$, we get

$$\mathbf{E} \int\limits_{\mathbb{R}_d} |\eta'''(x)(C_k^{-1}Y_k)^2 C_k^{-1}z| dx \ll \mathbf{E}|(C_k^{-1}Y_k)^T C_k^{-1}zY^TC_k^{-1}Y_k| + \mathbf{E}|C_k^{-1}Y_k|^2\mathbf{E}|Y^TC_k^{-1}z| + \mathbf{E}|Y^TC_k^{-1}z(Y^TC_k^{-1}Y_k)^2|.$$

Use (2.15) and (2.27) to derive

$$\begin{aligned}\mathbf{E}|(C_k^{-1}Y_k)^T C_k^{-1}zY^TC_k^{-1}Y_k| &\le (\mathbf{E}|(C_k^{-1}Y_k)^T C_k^{-1}z|^2)^{\frac{1}{2}} (\mathbf{E}|Y^TC_k^{-1}Y_k|^2)^{\frac{1}{2}} \\ &\le |C_k^{-1}z|\mathbf{E}|C_k^{-1}Y_k|^2 \\ &\le 2\sqrt{2} \cdot \widetilde{\mathfrak{s}}_k^2 \cdot |z|\end{aligned}$$

as well as

$$\begin{aligned}\mathbf{E}|Y^TC_k^{-1}z(Y^TC_k^{-1}Y_k)^2| &= \int\limits_{\mathbb{R}^d} \mathbf{E}|Y^TC_k^{-1}z(Y^TC_k^{-1}y)^2| d\mathrm{N}_{\Sigma_k^2}(y) \\ &\le \int\limits_{\mathbb{R}^d} (\mathbf{E}|Y^TC_k^{-1}z|^2)^{\frac{1}{2}} (\mathbf{E}|Y^TC_k^{-1}y|^4)^{\frac{1}{2}} d\mathrm{N}_{\Sigma_k^2}(y) \\ &= \sqrt{3} \cdot \int\limits_{\mathbb{R}^d} |C_k^{-1}z||C_k^{-1}y|^2 d\mathrm{N}_{\Sigma_k^2}(y) \\ &\le 4\sqrt{2} \cdot \widetilde{\mathfrak{s}}_k^2 \cdot |z|.\end{aligned}$$

Thus

$$\mathbf{E}\int\limits_{\mathbb{R}_d} |\eta'''(x)(C_k^{-1}Y_k)^2 C_k^{-1}z|dx \ll \widetilde{\mathfrak{s}}_k^2 \cdot |z|. \tag{2.37}$$

Furthermore,

$$\mathbf{E}\mathrm{I}_2 \cot^2\alpha \le \int\limits_{\gamma}^{\pi/2} \frac{\cos x}{\sin^2 x}dx \le \frac{1}{\varepsilon}. \tag{2.38}$$

Finally, the bounds (2.28), (2.37) and (2.38) imply

$$|I_2 - I_3| \ll \frac{Md^3}{\varepsilon} \cdot (\widetilde{\nu}_{3k} + \widetilde{\mathfrak{s}}_k^2 \cdot \widetilde{\nu}_{1,k}) \cdot \left(\widetilde{\nu}_3 + \widetilde{\nu}_3^{2\chi(q_n)-\frac{1}{3}} \cdot L_n^{\frac{4}{3}-2\chi(q_n)}\right)$$

as claimed □

Clearly, Lemmas 2.1, 2.2 and 2.3 imply (2.18), which completes the proof to Theorem 2.1.

Chapter 3

Non-classical Error Bounds for Asymptotic Expansions in the CLT in $\mathbb{R}$

Let $n \in \mathbb{N}$. Throughout this chapter we consider a sequence $X_1, \ldots, X_n$ of real-valued, centered and independent random variables with finite absolute third moment

$$\beta_{3,k} = \mathbf{E}|X_k|^3 < \infty$$

and distribution function $F_k : \mathbb{R} \to [0, 1]$ for $k = 1, \ldots, n$. We put

$$\sigma_k^2 = \mathbf{E}X_k^2$$

and we define

$$\overline{\sigma}^2 = \frac{1}{n} \sum_{k=1}^{n} \sigma_k^2.$$

As in Chapter 2, we use $F^{[n]}$ to denote the distribution function of the standardized sum of the random variables X_k, i.e.,

$$F^{[n]}(x) = P\big((\overline{\sigma} \cdot \sqrt{n})^{-1} \cdot (X_1 + \ldots + X_n) \le x\big)$$

for $x \in \mathbb{R}$. Furthermore, we put

$$\tau_n = \mathbf{E}\big((\overline{\sigma} \cdot \sqrt{n})^{-1} \cdot (X_1 + \ldots + X_n)\big)^3 = \frac{\mathbf{E}X_1^3 + \ldots + \mathbf{E}X_n^3}{(\sigma_1^2 + \ldots + \sigma_n^2)^{3/2}},$$

and we consider the first summand $G^{[n]}$ of the asymptotic expansion (1.21) of $F^{[n]}$,

$$G^{[n]}(x) = \Phi(x) + \frac{\tau_n}{6\sqrt{2\pi}} \cdot (1 - x^2) \cdot e^{-x^2/2}.$$

We are interested in the maximum deviation

$$D_n = \sup_{x\in\mathbb{R}} |F^{[n]}(x) - G^{[n]}(x)|$$

of $G^{[n]}$ from $F^{[n]}$.

3.1 Main Result

Let f_k denote the characteristic function of $(\sqrt{n} \cdot \overline{\sigma})^{-1} \cdot X_k$ and put

$$\vartheta_n = \max_{1\le k\le n} \sup_{|t|>\sqrt{n}\cdot\overline{\sigma}^3/\beta_3} |f_k(t)|,$$

where

$$\beta_3 = \frac{1}{n}\sum_{k=1}^{n} \beta_{3,k}.$$

As previously, we define

$$\nu_{r,k} = \int_{\mathbb{R}} \left|\overline{\sigma}^{-1} z\right|^r \left|F_k - \Phi_{\sigma_k^2}\right|(dz)$$

as well as

$$\nu_r = \frac{1}{n}\sum_{k=1}^{n} \nu_{r,k}$$

for $r \ge 0$, and we put

$$L_n = \frac{\sigma_1^3 + \ldots + \sigma_n^3}{(\sigma_1^2 + \ldots + \sigma_n^2)^{3/2}}$$

and

$$K_n = \frac{\sigma_1^4 + \ldots + \sigma_n^4}{(\sigma_1^2 + \ldots + \sigma_n^2)^2}.$$

We have the following non-classical bound for the quantity D_n.

Theorem 3.1. *There exists an absolute constant $M > 0$ such that*

$$D_n \le M \cdot \left(\frac{\nu_4}{n} + \frac{\nu_4^{\frac{1}{2}} \cdot (n \cdot K_n)^{\frac{1}{2}}}{n} + \frac{\nu_3^2}{n} + \frac{\nu_3^{\frac{4}{3}} \cdot (\sqrt{n} \cdot L_n)^{\frac{2}{3}}}{n} \right. \tag{3.1}$$
$$\left. + \frac{1}{n^2} \cdot \left(1 + \frac{\beta_3}{\sqrt{n} \cdot \overline{\sigma}^3}\right) \cdot \left(\frac{\beta_3}{\overline{\sigma}^3}\right)^4 + n^{\frac{1}{2}} \cdot \vartheta_n^n \cdot \frac{\overline{\sigma}^3}{\beta_3} \right).$$

In the case of identically distributed random summands we have

$$\overline{\sigma}^3/\beta_3 = \sigma_1^3/\beta_{3,1}, \quad L_n = 1/\sqrt{n}, \quad K_n = 1/n$$

and Theorem 3.1 implies the following estimate.

Corollary 3.1. *If $X_1, \dots, X_n$ are identically distributed then*

$$\sup_{x \in \mathbb{R}} \left| F^{[n]}(x) - \Phi(x) - \frac{\mathbf{E}X_1^3}{6\sqrt{2\pi n}\sigma_1^3} \cdot (1 - x^2) \cdot e^{-x^2/2} \right|$$
$$\le M \cdot \left(\frac{\nu_4 + \nu_4^{\frac{1}{2}} + \nu_3^2 + \nu_3^{\frac{4}{3}}}{n} + \frac{1}{n^2} \cdot \left(1 + \frac{\beta_{3,1}}{\sqrt{n} \cdot \sigma_1^3}\right) \cdot \left(\frac{\beta_{3,1}}{\sigma_1^3}\right)^4 \right.$$
$$\left. + n^{\frac{1}{2}} \cdot \vartheta_n^n \cdot \frac{\sigma^3}{\beta_{3,1}} \right).$$

Note that ϑ_n^n in Theorem 3.1 and Corollary 3.1 tends to zero exponentially fast if the characteristic functions $f_1, \dots, f_n$ satisfy the Cramér condition (C).

3.2 Proof of Theorem 3.1

For convenience we introduce the notation

$$\widetilde{\beta}_3 = n \cdot \beta_3 = \sum_{k=1}^{n} E|X_k|^3, \quad s_n^2 = n \cdot \overline{\sigma}^2 = \sum_{k=1}^{n} \sigma_k^2,$$

and we put

$$q_n = \left\lfloor \frac{\sigma_1^2 + \ldots + \sigma_n^2}{\max\limits_{1 \le k \le n} \sigma_k^2} \right\rfloor$$

as in Chapter 2. Similar to the proof of Theorem 2.1 we distinguish two cases w.r.t. the size of the quantity q_n.

Case 1: $q_n < 3$.

Clearly,

$$c_1 = \sup_{x\in\mathbb{R}} (1 + x^2) \cdot e^{-x^2/2} < \infty,$$

and

$$\sup_{x\in\mathbb{R}} |F^{[n]}(x) - \Phi(x)| \leq 1.$$

Hence

$$D_n \leq \sup_{x\in\mathbb{R}} |F^{[n]}(x) - \Phi(x)| + \frac{c_1}{6\sqrt{2\pi}} \cdot |\tau_n| \leq 1 + \frac{c_1}{6\sqrt{2\pi}} \cdot \frac{\widetilde{\beta}_3}{s_n^3}.$$

Apply the Hölder inequality and observe that $q_n < 3$ implies

$$\max_{1\leq k\leq n} \sigma_k^2 > s_n^2/3$$

to derive

$$\widetilde{\beta}_3 \geq \sum_{k=1}^{n} (\mathbf{E}|X_k|^2)^{\frac{3}{2}} \geq \max_{1\leq k\leq n} \sigma_k^3 > \frac{s_n^3}{3\sqrt{3}}.$$

Hence $3\sqrt{3} \cdot \widetilde{\beta}_3/s_n^3 > 1$ and therefore,

$$1 + \frac{c_1}{6\sqrt{2\pi}} \cdot \frac{\widetilde{\beta}_3}{s_n^3} < \left(1 + \frac{c_1}{6\sqrt{2\pi}}\right) \frac{3\sqrt{3} \cdot \widetilde{\beta}_3}{s_n^3} < \left(1 + \frac{c_1}{6\sqrt{2\pi}}\right) \left(\frac{3\sqrt{3} \cdot \widetilde{\beta}_3}{s_n^3}\right)^4,$$

which yields (3.1) for every $M \geq 3^6(1 + c_1/(6\sqrt{2\pi}))$.

Case 2: $q_n \geq 3$.

Let f and g denote the Fourier-Stieltjes transforms of the functions $F^{[n]}$ and $G^{[n]}$, respectively. Note that

$$g(t) = \left(1 + \frac{\tau_n}{6} \cdot (it)^3\right) \cdot e^{-t^2/2}, \tag{3.2}$$

see Petrov (1987, page 186).

Clearly,

$$(G^{[n]})'(x) = \Big(1 + \frac{\tau_n}{6} \cdot (x^3 - 3x)\Big) \cdot \eta(x),$$

and therefore,

$$\sup_{x\in\mathbb{R}} |(G^{[n]})'(x)| \ll 1 + \frac{\widetilde{\beta}_3}{s_n^3}.$$

Put

$$\gamma = \frac{s_n^3}{\widetilde{\beta}_3}$$

and let

$$I = \int\limits_{(-\gamma^4/36,\gamma^4/36)} \frac{|f(t) - g(t)|}{|t|} dt.$$

Employing Lemma A.8 with $b = 1/\pi$ and $T = \gamma^4/36$ we derive

$$\sup_{x\in\mathbb{R}} |F^{[n]}(x) - G^{[n]}(x)| \ll I + \left(1 + \frac{1}{\gamma}\right) \frac{1}{\gamma^4}. \tag{3.3}$$

To estimate I we introduce the quantities

$$I_1 = \int\limits_{(-\gamma/36,\gamma/36)} \frac{|f(t) - g(t)|}{|t|} dt,$$

$$I_2 = \int\limits_{(-\gamma^4/36,-\gamma/36)} \frac{|f(t)|}{|t|} dt + \int\limits_{(\gamma/36,\gamma^4/36)} \frac{|f(t)|}{|t|} dt$$

and

$$I_3 = \int\limits_{(-\gamma^4/36,-\gamma/36)} \frac{|g(t)|}{|t|} dt + \int\limits_{(\gamma/36,\gamma^4/36)} \frac{|g(t)|}{|t|} dt.$$

Clearly,

$$I \le I_1 + I_2 + I_3. \tag{3.4}$$

We first provide a bound for the terms I_2 and I_3.

Lemma 3.1. *We have*

$$I_2 \le 6\vartheta_n^n \cdot \gamma$$

and

$$I_3 \ll \frac{1}{\gamma^4}.$$

Proof. Since $\gamma < 1$ implies $I_2 = I_3 = 0$ it suffices to consider the case $\gamma \geq 1$. Note that

$$f(t) = \prod_{k=1}^{n} f_k(t),$$

due to the independence of $X_1, \ldots, X_n$, and consequently,

$$I_2 \leq \vartheta_n^n \cdot \int\limits_{(\gamma/36, \gamma^4/36)} \frac{2}{t}\, dt = 6\vartheta_n^n \cdot \ln\gamma \leq 6\vartheta^n \cdot \gamma.$$

Furthermore,

$$\frac{|g(t)|}{|t|} \leq \left(\frac{1}{|t|} + \frac{t^2}{\gamma}\right) \cdot e^{-\frac{t^2}{2}},$$

due to (3.2), and therefore

$$\begin{aligned} I_3 &\ll \int\limits_{(\gamma/36, \gamma^4/36)} \left(\frac{1}{t} + \frac{t^2}{\gamma}\right) \cdot e^{-\frac{t^2}{2}}\, dt \\ &\ll \int\limits_{(\gamma/36, \gamma^4/36)} \left(\frac{1}{t^5} + \frac{1}{\gamma \cdot t^4}\right) dt \\ &\ll \frac{1}{\gamma^4}, \end{aligned}$$

which completes the proof of the lemma. □

In the sequel we consider a sequence of real-valued centered Gaussian random variables $Y_1, \ldots, Y_n, \overline{Y}_1, \ldots, \overline{Y}_n$ such that

$$\mathbf{E}Y_k^2 = \mathbf{E}\overline{Y}_k^2 = \sigma_k^2$$

for $k = 1, \ldots, n$. Moreover, we consider two random variables α_1 and α_2, which are uniformly distributed on $[0, \pi/2]$, and we assume that all variables $X_1, \ldots, X_n, Y_1, \ldots, Y_n, \overline{Y}_1, \ldots, \overline{Y}_n, \alpha_1, \alpha_2$ are defined on the same probability space and independent. Put

$$S = \sum_{k=1}^{n} X_k, \quad S_\ell = \sum_{k \neq \ell} X_k, \quad S_{\ell,j} = \sum_{k \notin \{\ell, j\}} X_k$$

for $\ell, j \in \{1, \ldots, n\}$ and define Z, Z_ℓ, $Z_{\ell,j}$ and $\overline{Z}$, $\overline{Z}_\ell$, $\overline{Z}_{\ell,j}$ in an analogue way replacing X_k by Y_k and X_k by $\overline{Y}_k$, respectively. Finally, for a sufficiently smooth function $\varphi : \mathbb{R} \to \mathbb{C}$ we put

$$A_{1,\ell}(\varphi) = -\frac{1}{s_n} \cdot \mathbf{E}\varphi'\big(s_n^{-1}(\overline{Z}_\ell \cos\alpha_1 + Z_\ell \sin\alpha_1 + X_\ell \cos\alpha_1 + Y_\ell \sin\alpha_1)\big) \\ \times (-X_\ell \sin\alpha_1 + Y_\ell \cos\alpha_1)$$

and

$$A_{2,\ell,j}(\varphi) = \frac{1}{s_n^2} \cdot \mathbf{E}\varphi''\big(s_n^{-1}(S_{\ell,j} \cos\alpha_2 \cos\alpha_1 + \overline{Z}_{\ell,j} \sin\alpha_2 \cos\alpha_1 + Z_\ell \sin\alpha_1 \\ + X_\ell \cos\alpha_1 + Y_\ell \sin\alpha_1 + X_j \cos\alpha_2 \cos\alpha_1 + \overline{Y}_j \sin\alpha_2 \cos\alpha_1)\big) \\ \times (-X_\ell \sin\alpha_1 + Y_\ell \cos\alpha_1) \cdot (-X_j \sin\alpha_2 + \overline{Y}_j \cos\alpha_2) \cdot \cos\alpha_1$$

for $\ell, j \in \{1, \ldots, n\}$.

We start with the analysis of the integral I_1.

Lemma 3.2. *Let $\varphi_t(x) = e^{itx}$ for $t, x \in \mathbb{R}$. Then*

$$I_1 \ll \sum_{\ell=1}^{n} \int\limits_{(-\gamma/36,\gamma/36)} \left| A_{1,\ell}(\varphi_t) - \frac{\mathbf{E}X_\ell^3}{3\pi \cdot s_n^3} \cdot (it)^3 \cdot e^{-t^2/2} \right| \cdot \frac{1}{|t|} dt \\ + \sum_{\ell=1}^{n} \sum_{j \neq \ell} \int\limits_{(-\gamma/36,\gamma/36)} |A_{2,\ell,j}(\varphi_t)| \cdot \frac{1}{|t|} dt.$$

Proof. By definition,

$$I_1 = \int\limits_{(-\gamma/36,\gamma/36)} \left| f(t) - e^{-t^2/2} - \sum_{\ell=1}^{n} \frac{\mathbf{E}X_\ell^3}{6 s_n^3} \cdot (it)^3 \cdot e^{-t^2/2} \right| \cdot \frac{1}{|t|} dt.$$

To estimate the latter integrand in the interval $(-\gamma/36, \gamma/36)$ we use the fact that

$$\mathbf{E}\varphi({s_n}^{-1} \cdot S) = \mathbf{E}\varphi({s_n}^{-1} \cdot Z) + \frac{\pi}{2} \cdot \sum_{\ell=1}^{n} A_{1,\ell}(\varphi) + \frac{\pi^2}{4} \cdot \sum_{\ell=1}^{n} \sum_{j \neq \ell} A_{2,\ell,j}(\varphi) \quad (3.5)$$

for a sufficiently smooth $\varphi : \mathbb{R} \to \mathbb{C}$, see Bentkus (2003a). Since

$$\mathbf{E}\varphi_t(s_n^{-1} \cdot S) = f(t)$$

and

$$\mathbf{E}\varphi_t(s_n^{-1} \cdot Z) = e^{-t^2/2}$$

we obtain

$$f(t) = e^{-t^2/2} + \frac{\pi}{2} \cdot \sum_{\ell=1}^{n} A_{1,\ell}(\varphi_t) + \frac{\pi^2}{4} \cdot \sum_{\ell=1}^{n} \sum_{j \neq \ell} A_{2,\ell,j}(\varphi_t)$$

from (3.5), and, equivalently,

$$f(t) = e^{-t^2/2} + \frac{\tau_n}{6} \cdot (it)^3 \cdot e^{-t^2/2} + \frac{\pi}{2} \cdot \sum_{\ell=1}^{n} \left(A_{1,\ell}(\varphi_t) - \frac{\mathbf{E}X_\ell^3}{3\pi \cdot s_n^3} \cdot (it)^3 \cdot e^{-t^2/2} \right) + \frac{\pi^2}{4} \cdot \sum_{\ell=1}^{n} \sum_{j \neq \ell} A_{2,\ell,j}(\varphi_t).$$

Therefore

$$\left| f(t) - e^{-t^2/2} - \frac{\tau_n}{6} \cdot (it)^3 \cdot e^{-t^2/2} \right| \ll \sum_{\ell=1}^{n} \left| A_{1,\ell}(\varphi_t) - \frac{\mathbf{E}X_\ell^3}{3\pi \cdot s_n^3} \cdot (it)^3 \cdot e^{-t^2/2} \right| + \sum_{\ell=1}^{n} \sum_{j \neq \ell} |A_{2,\ell,j}(\varphi_t)|, \quad (3.6)$$

which completes the proof of the lemma. □

Next, we provide bounds for the summands in (3.6). Put

$$\overline{\nu}_{r,k} = \overline{\sigma}^r \cdot \nu_{r,k}, \quad \overline{\nu}_r = \sum_{k=1}^{n} \overline{\nu}_{r,k}. \quad (3.7)$$

Lemma 3.3. *We have*

$$\left| A_{1,\ell}(\varphi_t) - \frac{\mathbf{E}X_\ell^3}{3\pi \cdot s_n^3} \cdot (it)^3 \cdot e^{-t^2/2} \right| \ll \frac{\overline{\nu}_{4,\ell} + \overline{\nu}_{3,\ell}\sigma_\ell + \overline{\nu}_{2,l}\sigma_\ell^2}{s_n^4} \cdot t^4 \cdot e^{-\frac{t^2}{3}}.$$

Proof. Since $\overline{Z}_\ell, Z_\ell, \alpha_1$ are independent and $\overline{Z}_\ell$ and Z_ℓ both have a centered normal distribution with variance $s_n^2 - \sigma_\ell^2$, we obtain

$$\overline{Z}_\ell \cos\alpha_1 + Z_\ell \sin\alpha_1 \stackrel{d}{=} Z_\ell.$$

Therefore, observing the independence of $(\overline{Z}_\ell, Z_\ell)$ and (X_ℓ, Y_ℓ) as well, we derive

$$\begin{aligned} A_{1,\ell}(\varphi_t) &= -\frac{it}{s_n}\mathbf{E}e^{its_n^{-1}(\overline{Z}_\ell\cos\alpha_1 + Z_\ell\sin\alpha_1 + X_\ell\cos\alpha_1 + Y_\ell\sin\alpha_1)}(-X_\ell\sin\alpha_1 + Y_\ell\cos\alpha_1) \\ &= -\frac{it}{s_n}\mathbf{E}e^{its_n^{-1}Z_\ell}\int\limits_{\mathbb{R}} e^{its_n^{-1}(z\cos\alpha_1 + Y_\ell\sin\alpha_1)}(-z\sin\alpha_1 + Y_\ell\cos\alpha_1)dF_\ell(z). \end{aligned}$$

Using similar arguments as for obtaining (2.24) we conclude that

$$\begin{aligned} A_{1,\ell}(\varphi_t) = -\frac{it}{s_n}\cdot\mathbf{E}e^{its_n^{-1}Z_\ell}\int\limits_{\mathbb{R}} & e^{its_n^{-1}(z\cos\alpha_1 + Y_\ell\sin\alpha_1)} \\ & \times(-z\sin\alpha_1 + Y_\ell\cos\alpha_1)(F_\ell - \Phi_{\sigma_\ell^2})(dz). \end{aligned}$$

Apply two times the Taylor formula (2.9) to e^{itx} to get

$$\begin{aligned} & e^{its_n^{-1}(z\cos\alpha_1 + Y_\ell\sin\alpha_1)} \\ &\quad = e^{its_n^{-1}Y_\ell\sin\alpha_1} + \frac{it}{s_n}\cdot e^{its_n^{-1}Y_\ell\sin\alpha_1}\cdot z\cos\alpha_1 + \frac{(it)^2}{2s_n^2}\cdot(z\cos\alpha_1)^2 \\ &\qquad + \frac{(it)^3}{2s_n^3}\int\limits_0^1 e^{its_n^{-1}uY_\ell\sin\alpha_1}\cdot(z\cos\alpha_1)^2\cdot Y_\ell\sin\alpha_1 du \\ &\qquad + \frac{(it)^3}{2s_n^3}\int\limits_0^1 (1-u)^2\cdot e^{its_n^{-1}(uz\cos\alpha_1 + Y_\ell\sin\alpha_1)}\cdot(z\cos\alpha_1)^3 du. \end{aligned}$$

Thus, using (2.11) we obtain

$$\begin{aligned}A_{1,\ell}(\varphi_t) &= \frac{(it)^3}{2s_n^3}\cdot \mathbf{E}e^{its_n^{-1}Z_\ell}\sin\alpha_1\cos^2\alpha_1\int\limits_{\mathbb{R}} z^3(F_\ell-\Phi_{\sigma_\ell^2})(dz)\\ &\quad - \frac{(it)^4}{2s_n^4}\cdot \mathbf{E}e^{its_n^{-1}Z_\ell}\int\limits_0^1\int\limits_{\mathbb{R}} e^{its_n^{-1}uY_\ell\sin\alpha_1}\cdot(-z\sin\alpha_1+Y_\ell\cos\alpha_1)\\ &\qquad\qquad \times (z\cos\alpha_1)^2\cdot Y_\ell\sin\alpha_1(F_\ell-\Phi_{\sigma_\ell^2})(dz)du\\ &\quad - \frac{(it)^4}{2s_n^4}\cdot \mathbf{E}e^{its_n^{-1}Z_\ell}\int\limits_0^1\int\limits_{\mathbb{R}}(1-u)^2\cdot e^{its_n^{-1}\cdot(Y_\ell\sin\alpha_1+uz\cos\alpha_1)}\\ &\qquad\qquad \times(-z\sin\alpha_1+Y_\ell\cos\alpha_1)\cdot(z\cos\alpha_1)^3(F_\ell-\Phi_{\sigma_\ell^2})(dz)du\\ &= A_{1,\ell}^{(1)}(t)+A_{1,\ell}^{(2)}(t)+A_{1,\ell}^{(3)}(t).\end{aligned}$$

In order to complete the proof we show that

$$\left|A_{1,\ell}^{(1)}(t)-\frac{\mathbf{E}X_\ell^3}{3\pi s_n^3}\cdot(it)^3\cdot e^{-t^2/2}\right|\le\frac{\overline{\nu}_{3,\ell}\cdot\sigma_\ell}{s_n^4}\cdot t^4\cdot e^{-\frac{t^2}{3}}, \tag{3.8}$$

$$|A_{1,\ell}^{(2)}(t)|\le\frac{\overline{\nu}_{3,\ell}\cdot\sigma_\ell+\overline{\nu}_{2,\ell}\cdot\sigma_\ell^2}{s_n^4}\cdot t^4\cdot e^{-\frac{t^2}{3}} \tag{3.9}$$

as well as

$$|A_{1,\ell}^{(3)}(t)|\le\frac{\overline{\nu}_{4,\ell}+\overline{\nu}_{3,\ell}\cdot\sigma_\ell}{s_n^4}\cdot t^4\cdot e^{-\frac{t^2}{3}}. \tag{3.10}$$

Let us start with the estimate (3.8). Since

$$\mathbf{E}\sin\alpha_1\cos^2\alpha_1=\frac{2}{3\pi}$$

we have

$$A_{1,\ell}^{(1)}(t)=\frac{(it)^3}{3\pi s_n^3}\cdot\mathbf{E}e^{its_n^{-1}Z_\ell}\int\limits_{\mathbb{R}} z^3(F_\ell-\Phi_{\sigma_\ell^2})(dz).$$

Using $\mathbf{E}Y_\ell^3=0$ and

$$\mathbf{E}e^{its_n^{-1}Z}=e^{-t^2/2}$$

we derive

$$\begin{aligned}
\mathbf{E}X_\ell^3 \cdot e^{-t^2/2} &= \mathbf{E}X_\ell^3 \cdot \mathbf{E}e^{its_n^{-1}Z} \\
&= \mathbf{E}e^{its_n^{-1}\cdot(Z_\ell+Y_\ell)} \int\limits_{\mathbb{R}} z^3 F_\ell(dz) \\
&= \mathbf{E}e^{its_n^{-1}\cdot(Z_\ell+Y_\ell)} \int\limits_{\mathbb{R}} z^3 (F_\ell - \Phi_{\sigma_\ell^2})(dz)
\end{aligned}$$

and, consequently,

$$A_{1,\ell}^{(1)}(t) - \frac{\mathbf{E}X_\ell^3}{3\pi s_n^3} \cdot (it)^3 \cdot e^{-t^2/2} = \frac{(it)^3}{3\pi s_n^3} \cdot \mathbf{E}e^{its_n^{-1}Z_\ell} \cdot (1 - e^{its_n^{-1}Y_\ell}) \int\limits_{\mathbb{R}} z^3 (F_\ell - \Phi_{\sigma_\ell^2})(dz).$$

By the Taylor formula (2.9),

$$e^{its_n^{-1}Y_\ell} = 1 + \frac{it}{s_n} \cdot Y_\ell \int\limits_0^1 e^{itus_n^{-1}Y_\ell} du,$$

which yields

$$\begin{aligned}
\left| A_{1,\ell}^{(1)}(t) - \frac{\mathbf{E}X_\ell^3}{3\pi s_n^3} \cdot (it)^3 \cdot e^{-t^2/2} \right| &\le \frac{\overline{\nu}_{3,\ell}}{s_n^4} \cdot t^4 \cdot \left| \mathbf{E}e^{its_n^{-1}Z_\ell} \cdot Y_\ell \int\limits_0^1 e^{itus_n^{-1}Y_\ell} du \right| \\
&\le \frac{\overline{\nu}_{3,\ell} \cdot \mathbf{E}|Y_\ell|}{s_n^4} \cdot t^4 |\mathbf{E}e^{its_n^{-1}Z_\ell}| \\
&\le \frac{\overline{\nu}_{3,\ell} \cdot \sigma_\ell}{s_n^4} \cdot t^4 \cdot |\mathbf{E}e^{its_n^{-1}Z_\ell}|
\end{aligned}$$

due to the independence of Z_ℓ and Y_ℓ. Note that $s_n^{-1}Z_\ell$ is centered Gaussian with variance $1 - \sigma_\ell^2/s_n^2$. Moreover, we have

$$\max_{1\le\ell\le n} \sigma_\ell^2 \le \frac{s_n^2}{3} \tag{3.11}$$

in the case under consideration. Hence

$$\mathbf{E}e^{its_n^{-1}Z_\ell} = e^{-\frac{s_n^2-\sigma_\ell^2}{2s_n^2}\cdot t^2} \le e^{-\frac{t^2}{3}}, \tag{3.12}$$

which implies (3.8).

Next, we verify (3.9) and (3.10). Use the independence of Z_ℓ and (Y_ℓ, α_1) and observe (3.12) to derive

$$|A_{1,\ell}^{(2)}(t)| = \frac{t^4}{2s_n^4} \cdot \Bigg| \mathbf{E} e^{its_n^{-1}Z_\ell} \cdot \mathbf{E} \int_0^1 \int_{\mathbb{R}} e^{itus_n^{-1}Y_\ell \sin\alpha_1} \cdot (-z\sin\alpha_1 + Y_\ell \cos\alpha_1)$$

$$\times (z\cos\alpha_1)^2 \cdot Y_\ell \sin\alpha_1 (F_\ell - \Phi_{\sigma_\ell^2})(dz)du \Bigg|$$

$$\leq \frac{\overline{\nu}_{3,\ell} \cdot \mathbf{E}|Y_\ell| + \overline{\nu}_{2,\ell} \cdot \mathbf{E}|Y_\ell|^2}{s_n^4} \cdot t^4 \cdot e^{-\frac{t^2}{3}}$$

$$\leq \frac{\overline{\nu}_{3,\ell} \cdot \sigma_\ell + \overline{\nu}_{2,\ell} \cdot \sigma_\ell^2}{s_n^4} \cdot t^4 \cdot e^{\frac{-t^2}{3}}$$

as well as

$$|A_{1,\ell}^{(3)}(t)| = \frac{t^4}{2s_n^4} \cdot \Bigg| \mathbf{E} e^{its_n^{-1}Z_\ell} \cdot \mathbf{E} \int_0^1 \int_{\mathbb{R}} (1-u)^2 \cdot e^{its_n^{-1}(Y_\ell \sin\alpha_1 + uz\cos\alpha_1)}$$

$$\times (-z\sin\alpha_1 + Y_\ell\cos\alpha_1) \cdot (z\cos\alpha_1)^3 (F_\ell - \Phi_{\sigma_\ell^2})(dz)du \Bigg|$$

$$\leq \frac{\overline{\nu}_{4,\ell} + \overline{\nu}_{3,\ell} \cdot \mathbf{E}|Y_\ell|}{s_n^4} \cdot t^4 \cdot e^{-\frac{t^2}{3}}$$

$$\leq \frac{\overline{\nu}_{4,\ell} + \overline{\nu}_{3,\ell} \cdot \sigma_\ell}{s_n^4} \cdot t^4 \cdot e^{\frac{-t^2}{3}},$$

which finishes the proof of the lemma. □

Lemma 3.4. *If $t \in (-\gamma/36, \gamma/36)$ then*

$$|A_{2,\ell,j}(\varphi_t)| \leq \frac{(\overline{\nu}_{3,\ell} + \overline{\nu}_{2,\ell} \cdot \sigma_\ell) \cdot (\overline{\nu}_{3,j} + \overline{\nu}_{2,j} \cdot \sigma_j)}{s_n^6} \cdot t^6 \cdot e^{-\frac{t^2}{12}}.$$

Proof. We have

$$
\begin{aligned}
A_{2,\ell,j}(\varphi_t) &= \frac{(it)^2}{s_n^2} \cdot \mathbf{E} e^{its_n^{-1}(S_{\ell,j}\cos\alpha_2\cos\alpha_1 + \overline{Z}_{\ell,j}\sin\alpha_2\cos\alpha_1 + Z_\ell \sin\alpha_1 + X_\ell\cos\alpha_1 + Y_\ell\sin\alpha_1 + X_j\cos\alpha_2\cos\alpha_1 + \overline{Y}_j\sin\alpha_2\cos\alpha_1)} \\
&\qquad \times (-X_\ell \sin\alpha_1 + Y_\ell\cos\alpha_1)\cdot(-X_j\sin\alpha_2 + \overline{Y}_j\cdot\cos\alpha_2)\cos\alpha_1 \\
&= \frac{(it)^2}{s_n^2} \cdot \mathbf{E} e^{its_n^{-1}(S_{\ell,j}\cos\alpha_2\cos\alpha_1 + \overline{Z}_{\ell,j}\sin\alpha_2\cos\alpha_1 + Z_\ell \sin\alpha_1 + Y_\ell\sin\alpha_1 + \overline{Y}_j\sin\alpha_2\cos\alpha_1)} \\
&\qquad \times \int_{\mathbb{R}}\int_{\mathbb{R}} e^{its_n^{-1}(z_1\cos\alpha_1 + z_2\cos\alpha_2\cos\alpha_1)}\cdot(-z_1\sin\alpha_1 + Y_\ell\cos\alpha_1) \\
&\qquad\qquad \times (-z_2\sin\alpha_2 + \overline{Y}_j\cos\alpha_2)\cdot\cos\alpha_1 dF_\ell(z_1)dF_j(z_2) \\
&= \frac{(it)^2}{s_n^2} \cdot \mathbf{E} e^{its_n^{-1}(S_{\ell,j}\cos\alpha_2\cos\alpha_1 + \overline{Z}_{\ell,j}\sin\alpha_2\cos\alpha_1 + Z_\ell \sin\alpha_1 + Y_\ell\sin\alpha_1 + \overline{Y}_j\sin\alpha_2\cos\alpha_1)} \\
&\quad \times \int_{\mathbb{R}}\int_{\mathbb{R}} e^{its_n^{-1}(z_1\cos\alpha_1 + z_2\cos\alpha_2\cos\alpha_1)}\cdot(-z_1\sin\alpha_1 + Y_\ell\cos\alpha_1) \\
&\quad \times (-z_2\sin\alpha_2 + \overline{Y}_j\cos\alpha_2)\cdot\cos\alpha_1 (F_\ell - \Phi_{\sigma_\ell^2})(dz_1)(F_j - \Phi_{\sigma_j^2})(dz_2),
\end{aligned}
$$

where the last equality is obtained by employing the same arguments as for deriving (2.24). Using two times the Taylor formula (2.9) we get

$$
\begin{aligned}
&e^{its_n^{-1}(z_1\cos\alpha_1+z_2\cos\alpha_2\cos\alpha_1)}\\
&\quad= e^{its_n^{-1}z_1\cos\alpha_1}+\frac{it}{s_n}\cdot e^{its_n^{-1}z_1\cos\alpha_1}\cdot z_2\cos\alpha_2\cos\alpha_1\\
&\qquad+\frac{(it)^2}{s_n^2}\int_0^1 e^{its_n^{-1}uz_2\cos\alpha_2\cos\alpha_1}\cdot(1-u)\cdot(z_2\cos\alpha_2\cos\alpha_1)^2du\\
&\qquad+\frac{(it)^3}{s_n^3}\int_0^1 e^{its_n^{-1}uz_2\cos\alpha_2\cos\alpha_1}\cdot(1-u)\cdot z_1\cos\alpha_1\cdot(z_2\cos\alpha_2\cos\alpha_1)^2du\\
&\qquad+\frac{(it)^4}{s_n^4}\int_0^1\int_0^1 e^{its_n^{-1}(u_1z_1\cos\alpha_1+u_2z_2\cos\alpha_2\cos\alpha_1)}\cdot(1-u_1)\cdot(1-u_2)\\
&\qquad\qquad\times(z_1\cos\alpha_1)^2\cdot(z_2\cos\alpha_2\cos\alpha_1)^2du_1du_2.
\end{aligned}
$$

Therefore, observing (2.11) we conclude that

$$
\begin{aligned}
A_{2,\ell,j}(\varphi_t)&=\frac{(it)^6}{s_n^6}\cdot\mathbf{E}e^{its_n^{-1}(S_{\ell,j}\cos\alpha_2\cos\alpha_1+\overline{Z}_{\ell,j}\sin\alpha_2\cos\alpha_1+Z_\ell\sin\alpha_1+Y_\ell\sin\alpha_1+\overline{Y}_j\sin\alpha_2\cos\alpha_1)}\\
&\quad\times\int_0^1\int_0^1\int_{\mathbb{R}}\int_{\mathbb{R}} e^{its_n^{-1}(u_1z_1\cos\alpha_1+u_2z_2\cos\alpha_2\cos\alpha_1)}\cdot(1-u_1)\cdot(1-u_2)\\
&\quad\times(z_1\cos\alpha_1)^2(z_2\cos\alpha_2\cos\alpha_1)^2\cdot(-z_1\sin\alpha_1+Y_\ell\cos\alpha_1)\\
&\quad\times(-z_2\sin\alpha_2+\overline{Y}_j\cos\alpha_2)\cdot\cos\alpha_1\\
&\quad\times(F_\ell-\Phi_{\sigma_\ell^2})(dz_1)(F_j-\Phi_{\sigma_j^2})(dz_2)du_1du_2.
\end{aligned}
$$

Using that the random variables $X_1,\dots,X_n,Y_1,\dots,Y_n,\overline{Y}_1,\dots,\overline{Y}_n,\alpha_1,\alpha_2$ are independent, we obtain

$$
\begin{aligned}
&|A_{2,\ell,j}(\varphi_t)| \\
&= \frac{t^6}{s_n^6} \cdot \bigg| \mathbf{E}\bigg(\mathbf{E}\big(e^{its_n^{-1}(S_{\ell,j}\cos\alpha_2\cos\alpha_1 + \overline{Z}_{\ell,j}\sin\alpha_2\cos\alpha_1 + Z_{\ell,j}\sin\alpha_1)} \mid \alpha_1,\alpha_2\big) \\
&\qquad \times \mathbf{E}\bigg(\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^1\int_0^1 e^{its_n^{-1}(Y_j\sin\alpha_1 + Y_\ell\sin\alpha_1 + \overline{Y}_j\sin\alpha_2\cos\alpha_1} \\
&\qquad\qquad {}^{+u_1 z_1\cos\alpha_1 + u_2 z_2\cos\alpha_2\cos\alpha_1)} \\
&\qquad \times (1-u_1)\cdot(1-u_2)\cdot(z_1\cos\alpha_1)^2\cdot(z_2\cos\alpha_2\cos\alpha_1)^2 \\
&\qquad \times (-z_1\sin\alpha_1 + Y_\ell\cos\alpha_1)\cdot(-z_2\sin\alpha_2 + \overline{Y}_j\cos\alpha_2) \\
&\qquad \times \cos\alpha_1(F_\ell - \Phi_{\sigma_\ell^2})(dz_1)(F_j - \Phi_{\sigma_j^2})(dz_2)du_1du_2 \,\bigg|\, \alpha_1,\alpha_2\bigg)\bigg)\bigg| \\
&\le \frac{t^6}{s_n^6} \cdot \mathbf{E}\bigg(\big|\mathbf{E}\big(e^{its_n^{-1}(S_{\ell,j}\cos\alpha_2\cos\alpha_1 + \overline{Z}_{\ell,j}\sin\alpha_2\cos\alpha_1 + Z_\ell\sin\alpha_1)} \mid \alpha_1,\alpha_2\big)\big| \\
&\qquad \times \bigg|\mathbf{E}\bigg(\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^1\int_0^1 e^{its_n^{-1}(Y_j\sin\alpha_1 + Y_\ell\sin\alpha_1 + \overline{Y}_j\sin\alpha_2\cos\alpha_1} \\
&\qquad\qquad {}^{+u_1 z_1\cos\alpha_1 + u_2 z_2\cos\alpha_2\cos\alpha_1)} \\
&\qquad \times (z_1\cos\alpha_1)^2\cdot(z_2\cos\alpha_2\cos\alpha_1)^2\cdot(-z_1\sin\alpha_1 + Y_\ell\cos\alpha_1) \\
&\qquad \times (-z_2\sin\alpha_2 + \overline{Y}_j\cos\alpha_2)\cdot\cos\alpha_1 \\
&\qquad \times (F_\ell - \Phi_{\sigma_\ell^2})(dz_1)(F_j - \Phi_{\sigma_j^2})(dz_2)du_1du_2 \,\bigg|\, \alpha_1,\alpha_2\bigg)\bigg|\bigg) \\
&\le \frac{t^6}{s_n^6} \cdot \mathbf{E}\bigg(\big|\mathbf{E}\big(e^{its_n^{-1}(S_{\ell,j}\cos\alpha_2\cos\alpha_1 + \overline{Z}_{\ell,j}\sin\alpha_2\cos\alpha_1 + Z_\ell\sin\alpha_1)} \mid \alpha_1,\alpha_2\big)\big| \\
&\qquad \times \mathbf{E}\bigg(\int_{\mathbb{R}}\int_{\mathbb{R}} z_1^2\cdot z_2^2\cdot|(-z_1\sin\alpha_1 + Y_\ell\cos\alpha_1) \\
&\qquad\qquad\qquad \times(-z_2\sin\alpha_2 + \overline{Y}_j\cos\alpha_2)| \\
&\qquad\qquad \times|F_\ell - \Phi_{\sigma_\ell^2}|(dz_1)|F_j - \Phi_{\sigma_j^2}|(dz_2) \,\bigg|\, \alpha_1,\alpha_2\bigg)\bigg),
\end{aligned}
$$

and therefore,

$$|A_{2,\ell,j}(\varphi_t)| \le \frac{t^6}{s_n^6} \cdot (\overline{\nu}_{3,\ell} + \overline{\nu}_{2,\ell} \cdot \mathbf{E}|Y_\ell|) \cdot (\overline{\nu}_{3,j} + \overline{\nu}_{2,j} \cdot \mathbf{E}|\overline{Y}_j|) \tag{3.13}$$
$$\times \mathbf{E}\Big|\mathbf{E}\Big(e^{its_n^{-1}(S_{\ell,j}\cos\alpha_2\cos\alpha_1 + \overline{Z}_{\ell,j}\sin\alpha_2\cos\alpha_1 + Z_{\ell,j}\sin\alpha_1)} \;\Big|\; \alpha_1,\alpha_2\Big)\Big|$$
$$\le \frac{t^6}{s_n^6} \cdot (\overline{\nu}_{3,\ell} + \overline{\nu}_{2,\ell} \cdot \mathbf{E}|Y_\ell|) \cdot (\overline{\nu}_{3,j} + \overline{\nu}_{2,j} \cdot \mathbf{E}|\overline{Y}_j|)$$
$$\times \mathbf{E}\Big|\prod_{s\ne\ell,j}\mathbf{E}\Big(e^{its_n^{-1}(X_s\cos\alpha_2\cos\alpha_1 + \overline{Y}_s\sin\alpha_2\cos\alpha_1 + Y_s\sin\alpha_1)} \;\Big|\; \alpha_1,\alpha_2\Big)\Big|$$
$$\le \frac{t^6}{s_n^6} \cdot (\overline{\nu}_{3,\ell} + \overline{\nu}_{2,\ell} \cdot \sigma_\ell) \cdot (\overline{\nu}_{3,j} + \overline{\nu}_{2,j} \cdot \sigma_j)$$
$$\times \mathbf{E}\prod_{s\ne l,j}\Big|\mathbf{E}(e^{its_n^{-1}(X_s\cos\alpha_2\cos\alpha_1 + \overline{Y}_s\sin\alpha_2\cos\alpha_1 + Y_s\sin\alpha_1)}\Big|\alpha_1,\alpha_2\Big)\Big|.$$

Let $\omega_1, \omega_2 \in [0, \pi/2]$ and consider independent random variables U_s^0 and $\widetilde{U}_s^0$ such that

$$\widetilde{U}_s^0 \overset{d}{=} U_s^0 \overset{d}{=} X_s\cos\omega_2\cos\omega_1 + \overline{Y}_s\sin\omega_2\cos\omega_1 + Y_s\sin\omega_1.$$

Clearly, the characteristic functions of U_s^0/s_n and $(U_s^0 - \widetilde{U}_s^0)/s_n$ satisfy

$$|f_{U_s^0/s_n}(t)| = \sqrt{f_{(U_s^0-\widetilde{U}_s^0)/s_n}(t)} = \sqrt{\mathbf{E}e^{its_n^{-1}(U_s^0-\widetilde{U}_s^0)}}. \tag{3.14}$$

Using the Taylor expansion (2.9) we get

$$e^{its_n^{-1}(U_s^0-\widetilde{U}_s^0)} = 1 + \frac{it}{s_n}\cdot(U_s^0-\widetilde{U}_s^0) + \frac{(it)^2}{2s_n^2}\cdot(U_s^0-\widetilde{U}_s^0)^2$$
$$+ \frac{(it)^3}{2s_n^3}\cdot(U_s^0-\widetilde{U}_s^0)^3\int_0^1 e^{itus_n^{-1}(U_s^0-\widetilde{U}_s^0)}\cdot(1-u)^2du,$$

which implies

$$|f_{U_s^0/s_n}(t)|^2 = 1 + \frac{it}{s_n} \cdot \mathbf{E}(U_s^0 - \widetilde{U}_s^0) + \frac{(it)^2}{2s_n^2} \cdot \mathbf{E}(U_s^0 - \widetilde{U}_s^0)^2$$
$$+ \frac{(it)^3}{2s_n^3} \cdot \mathbf{E}(U_s^0 - \widetilde{U}_s^0)^3 \int_0^1 e^{itus_n^{-1}(U_s^0 - \widetilde{U}_s^0)} \cdot (1-u)^2 du$$
$$\leq 1 - \frac{t^2}{2s_n^2} \cdot \mathbf{E}(U_s^0 - \widetilde{U}_s^0)^2 + \frac{|t|^3}{6s_n^3} \cdot \mathbf{E}|U_s^0 - \widetilde{U}_s^0|^3. \tag{3.15}$$

Using the independence and the equality of the first and second moments of the random variables X_s, $\overline{Y}_s$ and Y_s we obtain

$$\mathbf{E}U_s^0 = \mathbf{E}X_s \cos\omega_2 \cos\omega_1 + \mathbf{E}\overline{Y}_s \sin\omega_2 \cos\omega_1 + \mathbf{E}Y_s \sin\omega_1 = 0 \tag{3.16}$$

as well as

$$\mathbf{E}(U_s^0)^2 = \mathbf{E}X_s^2 \cos\omega_2^2 \cos\omega_1^2 + \mathbf{E}\overline{Y}_s^2 \sin\omega_2^2 \cos\omega_1^2 + \mathbf{E}Y_s^2 \sin\omega_1^2 = \sigma_s^2.$$

Hence

$$\mathbf{E}(U_s^0 - \widetilde{U}_s^0)^2 = 2\mathbf{E}(U_s^0)^2 - 2(\mathbf{E}U_s^0)^2 = 2\sigma_s^2. \tag{3.17}$$

Since both $\overline{Y}_s$ and Y_s are $N(0, \sigma_s^2)$-distributed we have

$$\overline{Y}_s \sin\omega_2 \cos\omega_1 + Y_s \sin\omega_1 \stackrel{d}{=} Y_s \sqrt{\sin\omega_2^2 \cos\omega_1^2 + \sin\omega_1^2}$$

and therefore

$$U_s^0 \stackrel{d}{=} X_s \cos\omega_2 \cos\omega_1 + Y_s \sqrt{\sin\omega_2^2 \cos\omega_1^2 + \sin\omega_1^2},$$

which implies

$$\mathbf{E}|U_s^0|^3 \leq \mathbf{E}|X_s|^3 + 3\mathbf{E}|X_s|^2\mathbf{E}|Y_s| + 3\mathbf{E}|X_s|\mathbf{E}|Y_s|^2 + \mathbf{E}|Y_s|^3.$$

Observing

$$\mathbf{E}|Y_s| \leq \sigma_s = (\mathbf{E}|X_s|^2)^{\frac{1}{2}},$$
$$\mathbf{E}|X_s|\mathbf{E}|Y_s|^2 = \mathbf{E}|X_s|\mathbf{E}|X_s|^2 \leq \mathbf{E}|X_s|^3$$

and

$$\mathbf{E}|Y_s|^3 \le 2\sigma_s^3 \le 2\mathbf{E}|X_s|^3,$$

we derive

$$\mathbf{E}|U_s^0|^3 \le 9\mathbf{E}|X_s|^3,$$

and consequently,

$$\begin{aligned}
&\mathbf{E}|U_s^0 - \widetilde{U}_s^0|^3 \\
&\quad = \mathbf{E}(U_s^0 - \widetilde{U}_s^0)^2|U_s^0 - \widetilde{U}_s^0| \\
&\quad \le \mathbf{E}((U_s^0)^2 - 2U_s^0\widetilde{U}_s^0 + (\widetilde{U}_s^0)^2)(|U_s^0| + |\widetilde{U}_s^0|) \\
&\quad = \mathbf{E}(|U_s^0|^3 + |\widetilde{U}_s^0|^3 + (U_s^0)^2|\widetilde{U}_s^0| + (\widetilde{U}_s^0)^2|U_s^0| - 2U_s^0\widetilde{U}_s^0(|U_s^0| + |\widetilde{U}_s^0|)) \\
&\quad = \mathbf{E}(|U_s^0|^3 + |\widetilde{U}_s^0|^3 + (U_s^0)^2|\widetilde{U}_s^0| + (\widetilde{U}_s^0)^2|U_s^0|) \\
&\quad \le 4\mathbf{E}|U_s^0|^3 \\
&\quad \le 36\mathbf{E}|X_s|^3. \qquad (3.18)
\end{aligned}$$

Use (3.17) and (3.18) to obtain

$$|f_{U_s^0/s_n}(t)|^2 \le 1 - \frac{t^2}{s_n^2}\cdot\sigma_s^2 + \frac{6|t|^3}{s_n^3}\cdot\mathbf{E}|X_s|^3 \le e^{-\frac{t^2}{s_n^2}\cdot\sigma_s^2 + \frac{6|t|^3}{s_n^3}\cdot\mathbf{E}|X_s|^3}$$

from (3.15). Note that

$$-\frac{1}{s_n^2}\sum_{s\ne\ell,j}\sigma_s^2 = -1 + \frac{\sigma_\ell^2}{s_n^2} + \frac{\sigma_j^2}{s_n^2} \le -\frac{1}{3}$$

due to (3.11). Hence

$$\begin{aligned}
\left(\prod_{s\ne\ell,j}|\mathbf{E}e^{its_n^{-1}U_s^0}|\right)^2 &= \prod_{s\ne\ell,j}|f_{U_s^0/s_n}(t)|^2 \\
&\le e^{-\frac{t^2}{s_n^2}\sum\limits_{s\ne\ell,j}\sigma_s^2 + \frac{6|t|^3}{s_n^3}\sum\limits_{s\ne\ell,j}\mathbf{E}|X_s|^3} \\
&\le e^{-\frac{t^2}{3} + \frac{6|t|^3}{s_n^3}\sum\limits_{s\ne\ell,j}\mathbf{E}|X_s|^3}
\end{aligned}$$

and therefore

$$\prod_{s\neq\ell,j}\left|\mathbf{E}(e^{its_n^{-1}(X_s\cos\alpha_2\cos\alpha_1+\overline{Y}_s\sin\alpha_2\cos\alpha_1+Y_s\sin\alpha_1)}\;\big|\;\alpha_1=\omega_1,\alpha_2=\omega_2\big)\right|$$

$$=\prod_{s\neq\ell,j}\left|\mathbf{E}e^{its_n^{-1}U_s^0}\right|$$

$$\leq e^{-\frac{t^2}{6}+\frac{3|t|^3}{s_n^3}\sum\limits_{s\neq\ell,j}\mathbf{E}|X_s|^3}. \tag{3.19}$$

Combine (3.19) with (3.13) and observe that $|t|\leq\gamma/36=s_n^3/(36\widetilde{\beta}_3)$ implies

$$\frac{3|t|^3}{s_n^3}\sum_{s\neq\ell,j}\mathbf{E}|X_s|^3\leq\frac{t^2}{12}$$

to complete the proof. □

Let $t\in(-\gamma/36,\gamma/36)$. From (3.6), Lemma 3.3 and Lemma 3.4 it follows that

$$|f(t)-g(t)|\ll\frac{t^4}{s_n^4}\cdot e^{\frac{-t^2}{3}}\sum_{\ell=1}^{n}(\overline{\nu}_{4,\ell}+\overline{\nu}_{3,\ell}\cdot\sigma_\ell+\overline{\nu}_{2,\ell}\cdot\sigma_\ell^2)$$

$$+\frac{t^6}{s_n^6}\cdot e^{-\frac{t^2}{12}}\sum_{\ell=1}^{n}\sum_{j\neq\ell}(\overline{\nu}_{3,\ell}+\overline{\nu}_{2,\ell}\cdot\sigma_\ell)\cdot(\overline{\nu}_{3,j}+\nu_{2,j}\sigma_j). \tag{3.20}$$

By the Hölder inequality and (2.10) we have

$$\sum_{\ell=1}^{n}\overline{\nu}_{3,\ell}\cdot\sigma_\ell\leq\left(\sum_{\ell=1}^{n}\overline{\nu}_{3,\ell}^{\frac{4}{3}}\right)^{\frac{3}{4}}\cdot\left(\sum_{\ell=1}^{n}\sigma_\ell^4\right)^{\frac{1}{4}}\ll\overline{\nu}_4^{\frac{3}{4}}\cdot(s_n^4\cdot K_n)^{\frac{1}{4}},$$

$$\sum_{\ell=1}^{n}\overline{\nu}_{2,\ell}\cdot\sigma_\ell^2\leq\left(\sum_{\ell=1}^{n}\overline{\nu}_{2,\ell}^{2}\right)^{\frac{1}{2}}\cdot\left(\sum_{\ell=1}^{n}\sigma_\ell^4\right)^{\frac{1}{2}}\ll\overline{\nu}_4^{\frac{1}{2}}\cdot(s_n^4\cdot K_n)^{\frac{1}{2}}$$

as well as

$$\begin{aligned}\sum_{\ell=1}^{n}\sum_{j\neq\ell}\overline{\nu}_{2,\ell}\cdot\sigma_\ell\cdot\overline{\nu}_{2,j}\cdot\sigma_j &\leq \sum_{\ell=1}^{n}\sum_{j=1}^{n}\overline{\nu}_{2,\ell}\cdot\sigma_\ell\cdot\overline{\nu}_{2,j}\cdot\sigma_j\\ &= \left(\sum_{\ell=1}^{n}\overline{\nu}_{2,\ell}\cdot\sigma_\ell\right)^2\\ &\leq \left(\sum_{\ell=1}^{n}\overline{\nu}_{2\ell}^{\frac{3}{2}}\right)^{\frac{4}{3}}\cdot\left(\sum_{\ell=1}^{n}\sigma_\ell^3\right)^{\frac{2}{3}}\\ &\ll \overline{\nu}_3^{\frac{4}{3}}\cdot(s_n^3\cdot L_n)^{\frac{2}{3}}\end{aligned}$$

and

$$\begin{aligned}\sum_{\ell=1}^{n}\sum_{j\neq\ell}(\overline{\nu}_{3,\ell}\cdot\overline{\nu}_{2,j}\cdot\sigma_j+\overline{\nu}_{3,j}\cdot\overline{\nu}_{2,\ell}\cdot\sigma_\ell) &= 2\sum_{\ell=1}^{n}\sum_{j\neq\ell}\overline{\nu}_{3,\ell}\cdot\overline{\nu}_{2,j}\cdot\sigma_j\\ &\leq \left(\sum_{\ell=1}^{n}\overline{\nu}_{3,\ell}+\sum_{\ell=1}^{n}\overline{\nu}_{2,\ell}\cdot\sigma_\ell\right)^2\\ &\ll \left(\overline{\nu}_3+\overline{\nu}_3^{\frac{2}{3}}\cdot(s_n^3\cdot L_n)^{\frac{1}{3}}\right)^2.\end{aligned}$$

Moreover,

$$\sum_{\ell=1}^{n}\sum_{j\neq\ell}\overline{\nu}_{3,\ell}\cdot\overline{\nu}_{3,j}\leq\left(\sum_{\ell=1}^{n}\overline{\nu}_{3,\ell}\right)^2=\overline{\nu}_3^2.$$

Furthermore, it is easy to check that

$$\overline{\nu}_4^{\frac{3}{4}}\cdot(s_n^4\cdot K_n)^{\frac{1}{4}}\leq\overline{\nu}_4+\overline{\nu}_4^{\frac{1}{2}}\cdot(s_n^4\cdot K_n)^{\frac{1}{2}}$$

and

$$\overline{\nu}_3^{\frac{5}{3}}\cdot(s_n^3\cdot L_n)^{\frac{1}{3}}\leq\overline{\nu}_3^2+\overline{\nu}_3^{\frac{4}{3}}\cdot(s_n^3\cdot L_n)^{\frac{2}{3}}.$$

Thus

$$
\begin{aligned}
|f(t)-g(t)| &\ll \frac{\overline{\nu}_4+\overline{\nu}_4^{\frac{3}{4}}\cdot(s_n^4\cdot K_n)^{\frac{1}{4}}+\overline{\nu}_4^{\frac{1}{2}}\cdot(s_n^4\cdot K_n)^{\frac{1}{2}}}{s_n^4}\cdot t^4\cdot e^{-\frac{t^2}{3}}\\
&\quad+\frac{\overline{\nu}_3^2+\overline{\nu}_3^{\frac{5}{3}}\cdot(s_n^3\cdot L_n)^{\frac{1}{3}}+\overline{\nu}_3^{\frac{4}{3}}\cdot(s_n^3\cdot L_n)^{\frac{2}{3}}}{s_n^6}\cdot t^6\cdot e^{-\frac{t^2}{12}}\\
&\ll \frac{\overline{\nu}_4+\overline{\nu}_4^{\frac{1}{2}}\cdot(s_n^4\cdot K_n)^{\frac{1}{2}}}{s_n^4}\cdot t^4\cdot e^{-\frac{t^2}{3}}+\frac{\overline{\nu}_3^2+\overline{\nu}_3^{\frac{4}{3}}\cdot(s_n^3\cdot L_n)^{\frac{2}{3}}}{s_n^6}\cdot t^6\cdot e^{-\frac{t^2}{12}}.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
I_1 &\ll \int\limits_{(-\gamma/36,\gamma/36)} \frac{\overline{\nu}_4+\overline{\nu}_4^{\frac{1}{2}}\cdot(s_n^4\cdot K_n)^{\frac{1}{2}}}{s_n^4}\cdot|t|^3\cdot e^{\frac{-t^2}{3}}\,dt\\
&\quad+\int\limits_{(-\gamma/36,\gamma/36)} \frac{\overline{\nu}_3^2+\overline{\nu}_3^{\frac{4}{3}}\cdot(s_n^3\cdot L_n)^{\frac{2}{3}}}{s_n^6}\cdot|t|^5\cdot e^{-\frac{t^2}{12}}\,dt\\
&\ll \frac{\overline{\nu}_4}{s_n^4}+\frac{\overline{\nu}_4^{\frac{1}{2}}\cdot(s_n^4\cdot K_n)^{\frac{1}{2}}}{s_n^4}+\frac{\overline{\nu}_3^2}{s_n^6}+\frac{\overline{\nu}_3^{\frac{4}{3}}\cdot(s_n^3\cdot L_n)^{\frac{2}{3}}}{s_n^6}.
\end{aligned}
$$

Employ Lemma 3.1 and observe (3.4) to conclude that

$$
\begin{aligned}
I &\ll \frac{\overline{\nu}_4}{s_n^4}+\frac{\overline{\nu}_4^{\frac{1}{2}}\cdot(s_n^4\cdot K_n)^{\frac{1}{2}}}{s_n^4}+\frac{\overline{\nu}_3^2}{s_n^6}+\frac{\overline{\nu}_3^{\frac{4}{3}}\cdot(s_n^3\cdot L_n)^{\frac{2}{3}}}{s_n^6}+\vartheta_n^n\cdot\frac{s_n^3}{\widetilde{\beta}_3}+\left(\frac{\widetilde{\beta}_3}{s_n^3}\right)^4\\
&=\frac{\nu_4}{n}+\frac{\nu_4^{\frac{1}{2}}}{n}\cdot(n\cdot K_n)^{\frac{1}{2}}+\frac{\nu_3^2}{n}+\frac{\nu_3^{\frac{4}{3}}}{n}\cdot(\sqrt{n}\cdot L_n)^{\frac{2}{3}}+n^{\frac{1}{2}}\cdot\vartheta_n^n\cdot\frac{\overline{\sigma}_n^3}{\beta_3}+\frac{1}{n^2}\cdot\left(\frac{\beta_3}{\overline{\sigma}_n^3}\right)^4.
\end{aligned}
$$

Finally, use (3.3) to obtain

$$
\begin{aligned}
&\sup_{x\in\mathbb{R}}|F^{[n]}(x)-G^{[n]}(x)|\\
&\qquad\ll \frac{1}{n}\cdot\Big(\nu_4+\nu_4^{\frac{1}{2}}\cdot(n\cdot K_n)^{\frac{1}{2}}+\nu_3^2+\nu_3^{\frac{4}{3}}\cdot(\sqrt{n}\cdot L_n)^{\frac{2}{3}}\\
&\qquad\quad+n^{\frac{3}{2}}\cdot\vartheta_n\cdot\frac{\overline{\sigma}^3}{\beta_3}+\frac{1}{n}\cdot\Big(1+\frac{\beta_3}{\sqrt{n}\cdot\overline{\sigma}^3}\Big)\cdot\Big(\frac{\beta_3}{\overline{\sigma}^3}\Big)^4\Big),
\end{aligned}
$$

which completes the proof of Theorem 3.1.

Appendix A

Auxiliary Results

For the convenience of the reader we compile a number of known facts from the literature, which are used for the proofs of Theorems 2.1 and 3.1. Lemma A.1 is taken from Bentkus (2003b). Lemma A.2 is a straightforward consequence of Lemma 2.1 in the same paper. The proofs of Lemmas A.3 and A.5 are given in Sazonov (1967). For proofs of Lemmas A.4, A.6, A.7, A.8 we refer to Bentkus (2005), Bergström (1949), Paulauskas (1969b), and Petrov (1987, page 154), respectively.

Lemma A.1. *For all $\varepsilon > 0$ and $A \in \mathcal{C}$ there exists a differentiable function $\varphi_{\varepsilon,A} : \mathbb{R}^d \to \mathbb{R}$ such that*

$$0 \le \varphi_{\varepsilon,A} \le 1, \quad \varphi_{\varepsilon,A}(x) = 1 \text{ for } x \in A, \quad \varphi_{\varepsilon,A}(x) = 0 \text{ for } x \notin A^\varepsilon \tag{A.1}$$

and

$$|\varphi'_{\varepsilon,A}(x)| \le \frac{2}{\varepsilon}, \quad |\varphi'_{\varepsilon,A}(x) - \varphi'_{\varepsilon,A}(y)| \le \frac{8|x-y|}{\varepsilon^2} \tag{A.2}$$

for all $x, y \in \mathbb{R}^d$. In particular, one can choose $\varphi_{\varepsilon,A}$ to have the form

$$\varphi_{\varepsilon,A}(x) = \psi(\rho(x,A)/\varepsilon),$$

where $\psi : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable, non-negative and non-increasing function with $\int_{\mathbb{R}} |\psi'(t)| dt = 1$.

Lemma A.2. *Let* $\varepsilon > 0$ *and assume that* $\{\varphi_{\varepsilon,A} : A \in \mathcal{C}\}$ *is a family of functions* $\varphi_{\varepsilon,A} : \mathbb{R}^d \to \mathbb{R}$ *satisfying* (A.1). *For all* $\mathbb{R}^d$*-valued random vectors* X *and* Y *we then have*

$$\sup_{A\in\mathcal{C}} |P(X \in A) - P(Y \in A)| \le \sup_{A\in\mathcal{C}} |\mathbf{E}\varphi_{\varepsilon,A}(X) - \mathbf{E}\varphi_{\varepsilon,A}(Y)| + \max\left\{\sup_{A\in\mathcal{C}} P(Y \in A^\varepsilon \backslash A),\ \sup_{A\in\mathcal{C}} P(Y \in A \backslash A^{-\varepsilon})\right\}.$$

Lemma A.3. *For all* $A \in \mathcal{C}$ *and* $\varepsilon > 0$,

$$\mathrm{N}(A^\varepsilon \setminus A) \le \sqrt{2/\pi} \cdot d^{3/2} \cdot \varepsilon$$

and

$$\mathrm{N}(A \setminus A^{-\varepsilon}) \le \sqrt{2/\pi} \cdot d^{3/2} \cdot \varepsilon.$$

Lemma A.4. *Let* $p : \mathbb{R}^d \to \mathbb{R}$ *be an infinitely differentiable function such that for all* $k, m \in \mathbb{N}_0$

$$\lim_{x\to\infty} |p^{(k)}(x)||x|^m = 0.$$

Furthermore, let $f : \mathbb{R}^d \to \mathbb{R}$ *be Lipschitz continuous with Lipschitz constant* $a > 0$. *Then for every* $h \in \mathbb{R}^d$

$$\left|\int\limits_{\mathbb{R}^d} f(y) \cdot p'(y) h dy\right| \le a \cdot |h| \cdot \int\limits_{\mathbb{R}^d} I_{\{supp(f)\}}(y) \cdot |p(y)| dy, \tag{A.3}$$

where $supp(f) = \{x : f(x) \neq 0\}$.

Lemma A.5. *For every distribution* Q *on* $\mathbb{R}^d$ *and every* $T > 0$,

$$\sup_{A\in\mathcal{C}} |Q(A) - \mathrm{N}(A)| \le 2 \sup_{A\in\mathcal{C}} |(Q - \mathrm{N}) * \mathrm{N}_{T^{-2}\cdot I_d}(A)| + \frac{24 \cdot d^{\frac{3}{2}} \cdot \Gamma(\frac{d+1}{2})}{\sqrt{\pi} \cdot T \cdot \Gamma(\frac{d}{2})}.$$

Lemma A.6. *Let* $d \ge 2$. *For every non-degenerated covariance matrix* $V \in \mathbb{R}^{d\times d}$ *and for all* $l, m, p = 1, \ldots, d$ *we have*

$$\int\limits_{\mathbb{R}^d} \left|\frac{\partial^3 \eta_V(x_1, ..., x_d)}{\partial x_\ell \partial x_m \partial x_p}\right| dx_1 \ldots dx_d \le \sqrt{\frac{6|V^{ll}||V^{mm}||V^{pp}|}{|V|^3}}.$$

Lemma A.7. *Let* $V_1 \in \mathbb{R}^{d\times d}$ *and* $V_2 \in \mathbb{R}^{d\times d}$ *be a non-negative definite and a positive definite matrix, respectively. For every* $i = 1, \dots, d$ *we have*

$$\frac{|V_1 + V_2|}{|V_1^{ii} + V_2^{ii}|} \geq \frac{|V_1|}{|V_1^{ii}|} + \frac{|V_2|}{|V_2^{ii}|}$$

if $|V_1| \neq 0$ *and*

$$\frac{|V_1 + V_2|}{|V_1^{ii} + V_2^{ii}|} \geq \frac{|V_2|}{|V_2^{ii}|}$$

if $|V_1| = 0$.

Lemma A.8. *(Berry-Esseen inequality)*
Let $F_1 : \mathbb{R} \to \mathbb{R}$ *be non-increasing and bounded, and let* $F_2 : \mathbb{R} \to \mathbb{R}$ *be differentiable and of bounded variation, and assume that*

$$\lim_{x\to-\infty} F_1(x) = \lim_{x\to-\infty} F_2(x).$$

Let $T > 0$. *For every* $b > (2\pi)^{-1}$ *we have*

$$\sup_x |F_1(x) - F_2(x)| \leq b \cdot \int\limits_{(-T,T)} \frac{|f_1(t) - f_2(t)|}{|t|} dt + \frac{r(b)}{T} \cdot \sup_x F_2'(x),$$

where f_1 *and* f_2 *denote the Fourier-Stieltjes transforms of* F_1 *and* F_2*, respectively, and the positive constant* $r(b)$ *is only depending on* b.

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