Beyond Perfection:
On Relaxations and Superclasses

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Zusammenfassung

Perfekte Graphen, Anfang der 1960er Jahre von Claude Berge eingeführt, bilden eine Graphenklasse mit reichen strukturellen Eigenschaften. Charakterisierungen perfekter Graphen bezüglich so verschiedener Konzepte wie

- Färbungen von Graphen,
- Additivität der Graph-Entropie,
- Ganzzahligkeit von Polytopen,

verdeutlichen diese besonderen Eigenschaften und bilden gleichzeitig eine Schnittstelle zwischen den Gebieten Graphentheorie, Informationstheorie, Kombinatorischer Optimierung, Ganzzahliger Programmierung und Polyedertheorie, siehe Sektion 1.1.

Leider sind die meisten Graphen nicht perfekt und besitzen keine solche herausragenden Eigenschaften. Es ist daher interessant zu erforschen, welche Graphen zumindest hinsichtlich einiger Eigenschaften 'fast perfekt' sind und wie man diese Nähe zu Perfektheit messen kann. Wir *relaxieren* dafür den Perfektheitsbegriff bezüglich oben genannter Konzepte und untersuchen die so erhaltenen *Oberklassen* perfekter Graphen sowie verschiedene Wege, um den *Grad von Imperfektheit* auszudrücken.

Färbungen von Graphen. Das Färben der Knoten von Graphen ist ein wichtiges Konzept mit vielfältigen Anwendungen, das Bestimmen der Färbungszahl $\chi(G)$ eines Graphen ist jedoch i.a. NP-schwer. Die Cliquenzahl $\omega(G)$ ist eine natürliche untere Schranke für $\chi(G)$; für perfekte Graphen G gilt stets Gleichheit (für alle induzierten Untergraphen), im Allgemeinen können die beiden Parameter jedoch beliebig weit auseinander liegen [70]. Eine natürliche Frage ist also, für welche Graphenklassen die Differenz zwischen Cliquenzahl $\omega(G)$ und Färbungszahl $\chi(G)$ unter Kontrolle ist. Wir beschäftigen uns mit zwei Konzepten, um diese Frage zu beantworten: oberen Schranken für $\chi(G)$ als Funktion von $\omega(G)$ und dem Imperfektheitsgrad eines Graphen.

Eine Graphenklasse $\mathcal{G}$ ist χ-*bound* mit Bindingfunktion b, falls $\chi(G') \leq b(\omega(G'))$ für alle induzierten Untergraphen G' von Graphen $G \in \mathcal{G}$ gilt [51]. Perfekte Graphen sind genau die Graphen mit Bindingfunktion $b(x) = x$ und Klassen mit linearer Bindingfunktion $b(x) = bx + c$ haben ähnlich gute Färbungseigenschaften. Wir untersuchen in Kapitel 2 verschiedene Klassen mit Bindingfunktion $b(x) = x + 1$.

Gerke und McDiarmid führten in [45] als ähnliches Konzept den *Imperfektheitsgrad* eines Graphen G ein als

$$\mathrm{imp}(G) = \max\left\{\frac{\chi_f(G,c)}{\omega(G,c)} \mid c : V(G) \to \mathbb{N} \setminus \{0\}\right\}$$

wobei $\chi_f(G,c)$ die fraktionale gewichtete Färbungszahl und $\omega(G,c)$ die gewichtete Cliquenzahl ist. Jeder perfekte Graph G hat $\mathrm{imp}(G) = 1$ und alle Graphen mit einem kleinen Imperfektheitsgrad können als 'fast perfekt' angesehen werden. Wir geben für einige Klassen obere Schranken für den Imperfektheitsgrad an, siehe Sektion 5.2. Weiter leiten wir für Klassen mit unbeschränktem Imperfektheitsgrad eine hinreichende Bedingung für die Nichtexistenz von Bindingfunktionen her, siehe Sektion 2.3.

Additivität der Graph-Entropie. Körner [56] führte die *Graph-Entropie*

$$H(G,p) = \limsup_{k\to\infty} \min\left\{\frac{1}{k}\log_2 \chi(G^k[U]) : U \subseteq V(G^k), \sum_{x\in U} p^k(x) > 1-\varepsilon\right\}$$

als Gütemaß für ein Kodierungsproblem ein, das sowohl vom Graphen G als auch einer Wahrscheinlichkeitsverteilung p abhängt. Die Graph-Entropie ist subadditiv bezüglich der Vereinigung von Graphen auf der gleichen Knotenmenge, insbesondere gilt für Komplementärgraphen

$$H(p) \leq H(G,p) + H(\overline{G},p)\ \forall p$$

wobei $H(p) = H(K_n,p)$ die sog. Shannon-Entropie (d.h. die Entropie der Wahrscheinlichkeitsverteilung p selbst) ist. Csiszár et al. [31] zeigten, dass letztere Ungleichung genau dann für alle Wahrscheinlichkeitsverteilungen p mit Gleichheit erfüllt ist, wenn G perfekt ist.

Dies legt nahe, mithilfe des Wertes $H(G,p) + H(\overline{G},p) - H(p)$ die Imperfektheit eines Graphen G auszudrücken. Es gilt

$$\min\{H(G,p) + H(\overline{G},p) - H(p) : p\} = 0$$

genau dann, wenn G *normal* ist. Normale Graphen bilden eine bisher wenig untersuchte Oberklasse perfekter Graphen. Körner und de Simone [59] vermuten, dass jeder Graph normal ist, der weder C_5, C_7 noch $\overline{C}_7$ als induzierte Untergraphen enthält (Normale-Graphen-Vermutung). Wir zeigen diese Vermutung für einige erste Graphenklassen (Sektion 3.3) und geben verschiedene Wege zur Konstruktion normaler Graphen an (Sektion 3.2). Daraus resultierende Konsequenzen zeigen leider, dass normale Graphen nicht als ‘fast perfekt’ angesehen werden können, wie bisher vermutet wurde (Sektion 3.4). Insbesondere kann der Imperfektheitsgrad für normale Graphen nicht beschränkt werden. Da

$$\max\{H(G,p) + H(\overline{G},p) - H(p) : p\} = \log_2 \operatorname{imp}(G)$$

gilt, existieren also normale Graphen G, für welche die Differenz zwischen $\max\{H(G,p)+H(\overline{G},p)-H(p) : p\}$ und $\min\{H(G,p)+H(\overline{G},p)-H(p) : p\}$ beliebig groß ist. Damit hängt der Wert $H(G,p) + H(\overline{G},p) - H(p)$ für normale Graphen G stark von der Wahrscheinlichkeitsverteilung p ab, und wir folgern, dass nicht normale Graphen, sondern solche mit kleinem Imperfektheitsgrad in dieser Hinsicht ‘fast perfekt’ sind (siehe Sektion 3.4).

Das Stabile-Mengen-Polytop. Das *Stabile-Mengen-Polytop* STAB(G) ist definiert als die konvexe Hülle der Inzidenzvektoren aller stabilen Mengen von G; die Beschreibung durch facetten-definierende Ungleichungen ist für die meisten Graphen unbekannt. Für alle Graphen bilden Nichtnegativitätsbedingungen und Cliquebedingungen assoziiert mit maximalen Cliquen Facetten von STAB(G), diese Facettentypen reichen jedoch nur genau für perfekte Graphen aus [21, 40, 75]. Eine natürliche LP-Relaxierung von STAB(G) ist daher das *fraktionale Stabile-Mengen-Polytop*

$$\operatorname{QSTAB}(G) = \left\{ x \in \mathbb{R}_+^{|G|} : \sum_{i \in Q} x_i \leq 1 \ \forall Q \subseteq G \text{ Clique} \right\}$$

und es gilt $\operatorname{STAB}(G) \subset \operatorname{QSTAB}(G)$ für alle imperfekten Graphen. Die Differenz der beiden Polytope ist also ein natürliches Maß für die Imperfektheit. Wir untersuchen dafür

- die *Facettenmenge* von STAB(G) (hinsichtlich Anzahl und Art der zusätzlich benötigten Facetten),
- den *disjunktiven Index* von QSTAB(G) (als die kleinste Anzahl von Disjunktionen, um QSTAB(G) in ein ganzzahliges Polytop zu überführen, was dem Imperfektheitsindex $\operatorname{imp}_{\mathrm{I}}(G)$ entspricht [1]),

- den *Dilationsgrad* von STAB(G) und QSTAB(G) (der nach Gerke und McDiarmid [45] eine weitere Darstellung des Imperfektheitsgrades als $\text{imp}(G) = \min\{t : \text{QSTAB}(G) \subseteq t\ \text{STAB}(G)\}$ liefert).

Wir geben verschiedene Resultate zu allen drei Konzepten an: wir beschreiben die Facetten der Stabile-Mengen-Polytope verschiedener Graphen (Kapitel 4) und untersuchen sowohl Schranken für den Imperfektheitsgrad als auch für den Imperfektheitsindex (Kapitel 5). Es zeigt sich, dass viele Graphenklassen mit 'einfach' zu beschreibenden Stabile-Mengen-Polytopen auch einen kleinen Imperfektheitsgrad aufweisen, während der Imperfektheitsindex für fast alle untersuchten Graphenklassen unbeschränkt ist. Wir folgern, dass der Imperfektheitsgrad auch im polyedertheoretischen Sinne ein sinnvolles Maß für die Imperfektheit darstellt.

Schlussfolgerung. Der Imperfektheitsgrad hat eine Verbindung zu allen untersuchten Konzepten, da er

- sowohl in seiner ursprünglichen Definition als auch in Verbindung mit Bindingfunktionen gute Färbungseigenschaften wiedergibt,
- eine obere Schranke für den Wert $H(G,p) + H(\overline{G},p) - H(p))$ liefert, die unabhängig von der Wahrscheinlichkeitsverteilung p ist,
- der Dilationsgrad von STAB(G) und QSTAB(G) ist und Graphen mit 'einfach' zu beschreibenden Stabile-Mengen-Polytopen auch einen kleinen Imperfektheitsgrad aufweisen.

Damit ist der Imperfektheitsgrad mit allen untersuchten Konzepten kompatibel und liefert ein geeignetes Maß für die Imperfektheit eines Graphen, da Graphen mit kleinem Imperfektheitsgrad tatsächlich hinsichtlich mehrerer Eigenschaften als 'fast perfekt' angesehen werden können.

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Contents

Chapter 1

Introduction

Perfect graphs, introduced in the early 1960s by Claude Berge, constitute a well-studied graph class with a rich structure. This is reflected by characterizations of perfect graphs with respect to such different concepts as coloring properties, forbidden subgraphs, the integrality of certain polyhedra, or splitting graph entropies, see Section 1.1. In addition, several otherwise hard combinatorial optimization problems can be solved for perfect graphs in polynomial time.

Thus, perfect graphs play a role in such various mathematical disciplines as graph theory, information theory, combinatorial optimization, integer and semidefinite programming, polyhedral and convexity theory, thereby linking those disciplines in a truly unexpected way, see Section 1.1 as well.

Unfortunately, most graphs are imperfect and do not have such nice properties. It is, therefore, natural to ask which imperfect graphs are close to perfection in some sense and how to measure that. A canonical way is to *relax* several concepts which characterize perfect graphs and investigate the corresponding *superclasses* of perfect graphs, see Section 1.2.

We know that perfect graphs are exceptional with respect to *all* the studied concepts. A canonical question is which graphs in the considered superclasses are 'almost' perfect in *several* respects, and which are close to perfection w.r.t. *one* concept only, but not w.r.t. the others.

In addition, we are also interested in linking the concepts that generalize the notion of perfection in different ways, e.g., in polyhedral terms, by means of splitting graph entropies, or w.r.t. more general coloring concepts. As conclusion, it will turn out that the imperfection ratio is an appropriate measure for imperfection in all these respects.

1.1 Why perfect graphs deserve their name

Berge introduced perfect graphs in 1960, motivated from Shannon's famous information-theoretical problem of finding the zero-error capacity of a discrete memoryless channel [97]. Shannon's problem has a graph-theoretical formulation, regarding the asymptotic growth of the maximum cliques in the co-normal product G^n of $G = (V, E)$, where G^2 has $V \times V$ as node set and

$$\{(a_1, b_1), (a_2, b_2) : (a_1, a_2) \in E \text{ or } (b_1, b_2) \in E\}$$

as edge set. The Shannon capacity of G is

$$C(G) = \lim_{n \to \infty} \frac{1}{n} \log \omega(G^n)$$

where $\omega(G^n)$ denotes the size of a maximum clique in G^n. Shannon observed that $\omega(G^n) = (\omega(G))^n$ holds for graphs G with $\omega(G) = \chi(G)$ which makes the otherwise difficult problem of determining $C(G)$ tractable ($\chi(G)$ is the chromatic number and denotes the least number of stable sets covering V). This led Berge [6] introduce *perfect graphs* as those graphs G, where $\omega(G')$ equals $\chi(G')$ for each induced subgraph $G' \subseteq G$.

Berge observed that all chordless odd cycles C_{2k+1} with $k \geq 2$, called *odd holes*, and their complements, the *odd antiholes* $\overline{C}_{2k+1}$, satisfy $\omega(G) < \chi(G)$, see Figure 1.1. (The complement $\overline{G}$ has the same node set as G, but two nodes are adjacent in $\overline{G}$ if and only if they are non-adjacent in G.)

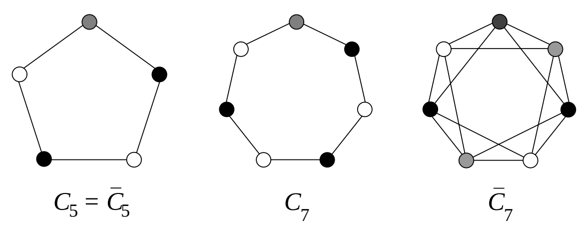


Figure 1.1: Small odd holes and odd antiholes

This motivated Berge's famous Strong Perfect Graph Conjecture:

G is perfect $\Leftrightarrow$ G has no odd hole or odd antihole as induced subgraph.

In particular, Berge conjectured that the class of perfect graphs is closed under taking complements (Perfect Graph Conjecture). Developing the antiblocking theory of polyhedra, Fulkerson launched a massive attack to this conjecture, see [39, 40], before it was turned to the Perfect Graph Theorem by Lovász [65], who gave two different proofs and established, in addition, the following characterization of perfect graphs:

$$G \text{ is perfect} \Leftrightarrow \omega(G')\omega(\overline{G}') \geq |G'| \text{ holds for all induced subgraphs } G' \subseteq G.$$

Many efforts to prove the Strong Perfect Graph Conjecture stimulated the study of perfect graphs, but were not successful for over 40 years. Finally, Chudnovsky, Robertson, Seymour, and Thomas [17] turned this conjecture into the Strong Perfect Graph Theorem, thereby exploring the structure of odd hole- and odd antihole-free graphs.

During the last decades, many fascinating structural properties of perfect graphs and interesting relationships to other fields of scientific enquiry have been discovered, see [88] for a recent survey. In particular, both in general hard to compute parameters $\omega(G)$ and $\chi(G)$ can be determined in polynomial time if G is perfect [50]. The latter result relies on the characterization of the stable set polytope of perfect graphs by means of facet-inducing inequalities.

The *stable set polytope* STAB(G) of a graph G is defined as the convex hull of the incidence vectors of all stable sets of G. It is easy to see that STAB(G) has a different representation, namely,

$$\text{STAB}(G) = \text{conv}\{x \in \{0,1\}^{|G|} : x(Q) = \sum_{i \in Q} x_i \ \leq \ 1, \ Q \subseteq G \text{ clique}\}$$

as a clique and a stable set have clearly at most one node in common and, thus, all clique constraints $x(Q) \leq 1$ are valid for STAB(G). A canonical relaxation of STAB(G) is, therefore, the *clique constraint polytope*

$$\text{QSTAB}(G) = \{x \in \mathbb{R}^{|G|} : \sum_{i \in Q} x_i \ \leq \ 1, \ Q \subseteq G \text{ clique}\}$$

obtained by dropping the integrality requirements. We have STAB(G) $\subseteq$ QSTAB(G) for all graphs, but *equality* for perfect graphs only [21, 40, 75]:

$$G \text{ is perfect} \Leftrightarrow \text{STAB}(G) = \text{QSTAB}(G).$$

Since the stable set problem, that is computing a stable set of maximum size or weight $\alpha(G, c)$, is NP-hard one is tempted to look at the linear relaxation $\max c^T x, x \in$ QSTAB(G) for determining $\alpha(G, c)$. The following chain of inequalities and equations is typical for integer/linear programming approaches to combinatorial problems:

$$\begin{aligned}
\alpha(G,c) = \ & \max\{\textstyle\sum_{i\in S} c_i : S \subseteq G \text{ stable}\} \\
= \ & \max\{c^T x : x \in \mathrm{STAB}(G)\} \\
= \ & \max\{c^T x : x(Q) \leq 1 \ \forall \text{cliques } Q \subseteq G,\ x \geq 0,\ x \in \{0,1\}^{|G|}\} \\
\leq \ & \max\{c^T x : x(Q) \leq 1 \ \forall \text{cliques } Q \subseteq G,\ x \geq 0\} \\
= \ & \min\{\textstyle\sum_Q y_Q : \sum_{Q \ni i} y_Q \geq c_i \ \forall i \in G,\ y_Q \geq 0 \ \forall \text{cliques } Q \subseteq G\} \\
\leq \ & \min\{\textstyle\sum_Q y_Q : \sum_{Q \ni i} y_Q \geq c_i \ \forall i \in G,\ y_Q \geq 0,\ y_Q \in \mathbb{Z}_+ \\
& \forall \text{cliques } Q \subseteq G\} \\
= \ & \overline{\chi}(G,c)
\end{aligned}$$

The inequalities come from dropping or adding integrality constraints, one of the equations is implied by linear programming duality. The last program can be interpreted as an integer programming formulation of to determine the weighted clique cover number $\overline{\chi}(G,c)$.

It follows from the Perfect Graph Theorem that equality holds throughout the whole chain for all 0/1-vectors c if and only if G is perfect. This, in turn, is equivalent to

$$G \text{ is perfect} \Leftrightarrow \text{the value } \max c^T x, x \in \mathrm{QSTAB}(G) \text{ is integral } \forall c \in \{0,1\}^{|G|}$$

and results of Fulkerson [39] and Lovász [65] imply that this is even true for all $c \in \mathbb{Z}^{|G|}$. This proves particularly that the constraint system defining $\mathrm{QSTAB}(G)$ is totally dual integral for perfect graphs G.

However, maximizing a linear objective function $c^T x, x \in \mathrm{QSTAB}(G)$ in polynomial time does not work directly [50]. For the class of perfect graphs, though, the optimization problem for $\mathrm{QSTAB}(G)$ (and, therefore, for $\mathrm{STAB}(G)$) can be solved in polynomial time–albeit via a detour involving a geometric representation of graphs introduced by Lovász [67] in 1979.

Let $G = (V,E)$ be a graph. An *orthonormal representation* of G is a sequence $(u_i : i \in V)$ of $|V|$ vectors $u_i \in \mathbb{R}^N$, where N is some positive integer, such that

- $||u_i|| = 1$ for all $i \in V$ and
- $u_i^T u_j = 0$ for all $ij \notin E$.

Trivially, every graph has an orthonormal representation: just take all the vectors u_i mutually orthogonal in $\mathbb{R}^{|V|}$, but also less trivial orthonormal representations with $N < |V|$ exist.

For any orthonormal representation $(u_i : i \in V)$, $u_i \in \mathbb{R}^N$ of G and any additional vector $c \in \mathbb{R}^N$ of unit length, the *orthonormal representation constraint (ONRC)*

$$\sum_{i \in V} (c^T u_i)^2 x_i \leq 1$$

is valid for STAB(G) due to the following reason. For any stable set S of G, the vectors $u_i, i \in S$ are mutually orthogonal by construction and, therefore, $\sum_{i \in S} (c^T u_i)^2 \leq 1$ follows. We obtain

$$\sum_{i \in V} (c^T u_i)^2 \chi_i^S = \sum_{i \in S} (c^T u_i)^2$$

for the incidence vector of any stable set S of G yielding the validity of the orthonormal representation constraints for STAB(G).

Moreover, taking an orthonormal basis $B = \{e_1, \ldots, e_{|V|}\}$ of $\mathbb{R}^{|V|}$ and a clique Q of G, we obtain an orthonormal representation by setting $u_i = e_1$ for all $i \in Q$ and assigning different vectors of $B - \{e_1\}$ to all the remaining nodes $j \in G - Q$ (where e_i denotes the i-th unit vector). Then the corresponding orthonormal representation constraint for $c = e_1$ is just the clique constraint associated with Q (by $c^T u_i = 1$ for $i \in Q$ and $c^T u_j = 0$ otherwise). Hence, every clique constraint is a special orthonormal representation constraint.

For any graph $G = (V, E)$, the set

$$\text{TH}(G) = \{x \in \mathbb{R}_+^V : x \text{ satisfies all ONRC's}\}$$

is the intersection of infinitely many half-spaces (since G admits infinitely many orthonormal representations), so TH(G) is a convex set but no polytope in general. The above remarks imply

$$\text{STAB}(G) \subseteq \text{TH}(G) \subseteq \text{QSTAB}(G)$$

and all three convex sets coincide if and only if G is perfect. This result is particularly remarkable since it states that a graph

$$G \text{ is perfect} \Leftrightarrow \text{the convex set TH}(G) \text{ is a polytope.}$$

The key property of TH(G) for linear programming was established by Grötschel, Lovász, and Schrijver [49]: If $c \in \mathbb{R}_+^V$ is a vector of node weights, the optimization problem (with infinitely many linear constraints) $\max c^T x$, $x \in$ TH(G) can be solved in polynomial time for any graph G. This deep result rests on the fact that the value $\vartheta(G, c) = \max\{c^T x : x \in \text{TH}(G)\}$ can be characterized in many equivalent ways, e.g., as the maximum

- value of a semidefinite program,
- eigenvalue of a certain set of symmetric matrices,
- value of some function involving orthonormal representations,

see [50] for the details. As we have $\alpha(G, c) = \vartheta(G, c)$ for all *perfect* graphs G, this finally implies that the stable set problem can be solved in polynomial time for perfect graphs.

Therefore, the clique cover number $\overline{\chi}(G) = \alpha(G)$, the chromatic number $\chi(G) = \overline{\chi}(\overline{G})$, and the clique number $\omega(G) = \alpha(\overline{G})$ can be computed in polynomial time for perfect graphs G, even in the weighted versions.

A further important characterization of perfect graphs is obtained in information theory and relies on the generalization of Fulkerson's antiblocking theory [39, 40] to convex corners.

A subset $\mathcal{A} \subset \mathbb{R}^n_+$ is called *convex corner* if $\mathcal{A}$ is convex, compact, and down-monotone in $\mathbb{R}^n_+$, i.e., if $a \in \mathcal{A}, a' \in \mathbb{R}^n_+$ and $a' \leq a$ implies $a' \in \mathcal{A}$. (The sets STAB(G), TH(G), and QSTAB(G) are examples of convex corners.)

Let $p \in \mathbb{R}^n_+$ be a probability distribution, that is a vector where the components sum up to one. The *entropy* of a convex corner $\mathcal{A}$ w.r.t. p is given by

$$H_{\mathcal{A}}(p) = \min \left\{ \sum_{i \leq n} p_i \log_2 \frac{1}{a_i} : a \in \mathcal{A} \right\}.$$

Körner [56] showed that the entropy of the stable set polytope of a graph G is exactly

$$H(G, p) = H_{\text{STAB}(G)}(p)$$

where $H(G, p)$ denotes the entropy of G w.r.t. a probability distribution $p \in \mathbb{R}^{|G|}_+$ on its node set. The graph-entropy was originally defined by Körner [56] as

$$H(G, p) = \limsup_{k \to \infty} \min \left\{ \frac{1}{k} \log_2 \chi(G^k[U]) : U \subseteq V(G^k), \sum_{x \in U} p^k(x) > 1 - \varepsilon \right\}$$

(involving the growth of the chromatic number of co-normal products of certain subgraphs) as a performance measure for a certain coding problem: The graph $G = (V, E)$ reflects the distinguishabilty of symbols of an alphabet V and p their probabilities, $k \cdot H(G, p)$ is the minimal length of 0/1-code words required for encoding the distinguishable words of length k over V having a certain probability $> 1 - \varepsilon$.

The most important property of the graph-entropy is its sub-additivity w.r.t. graphs on the same node set, in particular for complementary graphs:

$$H(p) \leq H(G, p) + H(\overline{G}, p) \ \forall p$$

where

$$H(p) = \sum_{i \leq n} p_i \log_2 \frac{1}{p_i}$$

stands for the entropy of the complete graph, i.e., for the entropy of p itself. Körner raised the question for which graphs G the bound $H(p)$ is attained, that means when *equality* holds rather than just sub-additivity. According to Cziszár, Körner, Lovász, Marton, and Simonyi [31], this is true for all probability distributions p if $\mathrm{STAB}(G)$ is the antiblocker of $\mathrm{STAB}(\overline{G})$, i.e., if

$$\mathrm{abl}(\mathrm{STAB}(\overline{G})) = \{x \in \mathbb{R}_+^V : x^T y \leq 1 \ \forall y \in STAB(\overline{G})\} = \mathrm{STAB}(G).$$

As $\mathrm{abl}(\mathrm{STAB}(\overline{G})) = \mathrm{QSTAB}(G)$ holds for all graphs, we obtain for *perfect* graphs G that, indeed,

$$H(p) = H_{\mathrm{STAB}(G)}(p) + H_{\mathrm{STAB}(\overline{G})}(p)$$

for all p, since $\mathrm{abl}(\mathrm{STAB}(\overline{G})) = \mathrm{STAB}(G)$ holds in this case. This yields the information-theoretic characterization of perfect graphs, namely, a graph

$$G \text{ is perfect} \iff H(p) = H(G, p) + H(\overline{G}, p) \text{ for all } p$$

obtained by Cziszár et al. [31]. Perfect graphs are, therefore, also called *strongly splitting graphs*, as splitting their graph entropies yields the Shannon entropy for all probability distributions.

In summary, perfect graphs do not only have nice graph-theoretical properties and behave nicely from an algorithmic point of view, but the above characterizations of perfect graphs also establish links to

- polyhedral theory (G is perfect iff certain polyhedra are identical);
- integer programming (a graph G is perfect iff certain linear programs have integral objective values);
- semidefinite programming (a graph is perfect iff the feasible region of a certain semidefinite program is a polytope);
- information theory (a graph G is perfect iff the entropies of G and $\overline{G}$ add up to the Shannon-entropy for all probability distributions);

which indeed reflects the importance of perfect graphs in many different fields of scientific enquiry.

1.2 Beyond perfection

The above considerations show that perfect graphs are a class with an extraordinarily rich structure. Unfortunately, most graphs are imperfect and do not admit such nice properties. For instance, a result of Prömel and Steger [86] shows that a random graph is with high probability perfect only if it is very sparse or, due to the invariance of perfection by complementation, very dense.

Thus, it is natural to ask which imperfect graphs are close to perfection in some sense and how to measure that. Canonical ways are to look for imperfect graphs G

- where the maximal perfect induced subgraph is as large as possible (e.g., such that removing only one node from G yields a perfect graph);
- with certain properties almost as nice as for perfect graphs (with respect to, e.g., coloring or entropy splitting);
- where the difference between the polytopes $\mathrm{STAB}(G)$ and $\mathrm{QSTAB}(G)$ is not 'too large' (as, e.g., only a few constraints have to be added to $\mathrm{QSTAB}(G)$ in order to obtain $\mathrm{STAB}(G)$).

For that, we *relax* several concepts which characterize perfect graphs and investigate the corresponding *superclasses* of perfect graphs whether they still share structural properties with perfect graphs or admit as good bounds for certain interesting graph parameters as perfect graphs.

1.2.1 'Almost' perfect graphs

Padberg was the first who asked which graphs are 'almost' perfect [75, 76]. He studied imperfect graphs with the property that all of their proper induced subgraphs are perfect. Such graphs are nowadays called *minimally imperfect*.

Using this term, the Strong Perfect Graph Conjecture reads that odd holes and odd antiholes are the only minimally imperfect graphs. Thus, characterizing minimally imperfect graphs was one possibility to verify or falsify the Strong Perfect Graph Conjecture. Before the conjecture was settled by Chudnovsky et al. [17], many fascinating properties of minimally imperfect graphs have been discovered. First, the Perfect Graph Theorem implies that a graph is minimally imperfect iff its complement is. Further properties reflecting an extraordinary symmetry of their maximum cliques and stable sets were given by Lovász [65] and Padberg [75]: Every minimally imperfect graph G with $\alpha = \alpha(G)$ and $\omega = \omega(G)$ has

- exactly $\alpha\omega + 1$ nodes,
- for every node x of G, the graph $G-x$ can be partitioned into α cliques of size ω and into ω stable sets of size α,
- precisely $|G|$ maximum stable sets and precisely $|G|$ maximum cliques,
- each node of G is contained in precisely $\alpha(G)$ maximum stable sets and in precisely $\omega(G)$ maximum cliques,
- for every maximum clique Q (maximum stable set S) there is a unique maximum stable set S (maximum clique Q) with $Q \cap S = \emptyset$.

Unfortunately, minimally imperfect graphs are not characterized by those properties but share them with other graphs. Bland, Huang, and Trotter suggested in [8] to call a graph *partitionable* if it satisfies the first two conditions for some integers α, ω and verified the remaining properties for all partitionable graphs (see Figure 1.2 for two partitionable graphs which are not minimally imperfect). Thus, all potential counterexamples to the Strong Perfect Graph Conjecture have to be partitionable, which caused the interest in this class until the conjecture was settled (see [85] for more information on minimally imperfect and partitionable graphs).

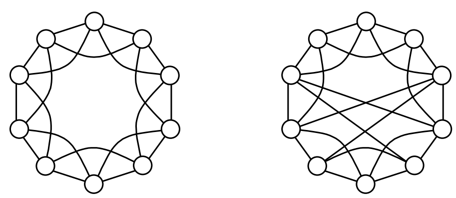


Figure 1.2: Examples of partitionable graphs.

Minimally imperfect graphs can also be seen as imperfect graphs such that removing an arbitrary node yields a perfect graph. This motivated us to generalize minimally imperfect graphs to *almost-perfect graphs* G where one node v exists such that removing v yields a perfect graph $G-v$, see [61].

Clearly, every perfect graph is in particular almost-perfect and every minimally imperfect graph as well (as even removing an *arbitrary* node yields a perfect graph). Thus, almost-perfect graphs built the smallest *superclass* of perfect graphs with respect to this concept (as the class of minimally imperfect graphs does not contain any perfect graph). It is natural to expect that almost-perfect graphs also satisfy properties almost as nice as perfect graphs. We address this question in Section 2.2.1 and give several positive answers.

1.2.2 Graphs with 'nice' coloring properties

Coloring the nodes of a graph is an important concept with a large variety of applications, but calculating $\chi(G)$ is an NP-hard problem in general. In a clique all nodes have to be colored differently, thus the clique number $\omega(G)$ is a trivial lower bound on $\chi(G)$. This bound is, in general, hard to evaluate as well and can be arbitrarily bad [70].

For perfect graphs G, the chromatic number $\chi(G')$ equals this lower bound $\omega(G')$ for all induced subgraphs $G' \subseteq G$. Thus, a natural question is for which other classes of graphs the difference between the clique number $\omega(G)$ and the chromatic number $\chi(G)$ is under control. This motivated Gyárfás [51] to introduce a concept using functions in $\omega(G)$ as *upper bound* on $\chi(G)$: A class $\mathcal{G}$ of graphs is called χ-*bound* with χ-*binding function* b if $\chi(G') \leq b(\omega(G'))$ holds for all induced subgraphs G' of all graphs $G \in \mathcal{G}$.

Thus, perfect graphs built exactly the class with χ-binding function $b(x) = x$ and graph classes with a linear binding function $b(x) = bx + c$ can be considered as close to perfect graphs with respect to coloring-properties. We address such problems in Chapter 2. We show, for instance, that almost-perfect graphs are χ-bound with the smallest non-trivial χ-binding function $b(x) = x + 1$; the same is true for so-called circular-perfect graphs obtained via a more general coloring concept (see Section 2.2 for more details).

As a concept similar to χ-binding functions, Gerke and McDiarmid introduced in [45] the *imperfection ratio* of a graph G as

$$\mathrm{imp}(G) = \max\left\{\frac{\chi_f(G,c)}{\omega(G,c)} \mid c : V(G) \to \mathbb{N} \setminus \{0\}\right\}$$

where $\chi_f(G,c)$ denotes the fractional weighted chromatic number and $\omega(G,c)$ the weighted clique number (thus, the imperfection ratio is some asymptotic slope of a χ-binding function).

By definition, every perfect graph G has $\mathrm{imp}(G) = 1$ and all graphs with an imperfection ratio close to one can be considered as 'not so imperfect'. For instance, Gerke and McDiarmid showed in [45] that every minimally imperfect graph G has $\mathrm{imp}(G) = \frac{|G|}{|G|-1}$ which also reflects the fact that long odd (anti)holes admit a larger perfect subgraph as short ones.

We discuss the relation of the imperfection ratio and χ-binding functions for certain classes of graphs in Section 2.3, thereby exploring some conditions for the (non-)existence of χ-binding functions. Furthermore, we present several results on upper bounds for the imperfection ratio of several graph classes in Section 5.2.

1.2.3 Weakly splitting graphs

According to the information-theoretical characterization obtained by Cziszár et al. [31], perfect graphs are exactly those graphs G with

$$H(p) = H(G, p) + H(\overline{G}, p)\ \forall p$$

and are, therefore, called strongly splitting graphs as they split graph entropies for *all* probability distributions. It is natural to call a graph G *weakly splitting* if equality holds for at least one positive probability distribution $p > 0$. Körner [56] proved that weakly splitting graphs are exactly the so-called *normal graphs* which come up in a natural way in an information-theoretic context [57, 31] and are, in graph-theoretic terms, defined by cross-intersecting families $\mathcal{Q}$ of cliques and $\mathcal{S}$ of stable sets, that is every clique in $\mathcal{Q}$ intersects every stable set in $\mathcal{S}$.

We investigate several problems concerning normal graphs in Chapter 3, involving the problem how to construct normal graphs (Section 3.2), the so-called Normal Graph Conjecture as a natural analogue to the Strong Perfect Graph Conjecture addressing forbidden subgraphs for normal graphs (Section 3.3), and the question how close normal graphs are to perfection (Section 3.1 and 3.4).

In particular, it is natural to consider the value

$$\max\{H(G, p) + H(\overline{G}, p) - H(p) : p\}$$

as a possible measure for imperfection (since this value is zero for perfect graphs). Indeed, Simonyi [100] established the following link

$$\log_2 \operatorname{imp}(G) = \max\{H(G, p) + H(\overline{G}, p) - H(p) : p\}$$

between the imperfection ratio and the graph entropy for any graph G (note that this implies the invariance of the imperfection ratio under complementation). Thus, one might expect that normal graphs have a small imperfection ratio since

$$0 = \min\{H(G, p) + H(\overline{G}, p) - H(p) : p\}$$

holds for every normal graph G, as they are weakly splitting.

However, we showed in [114] that the imperfection ratio of normal graphs cannot be bounded. This result is fairly unexpected as it shows in particular the existence of normal graphs G where the difference between the values $\max\{H(G,p)+H(\overline{G},p)-H(p) : p\}$ and $\min\{H(G,p)+H(\overline{G},p)-H(p) : p\}$ taken over all positive probability distributions p tends, in fact, to *infinity*, see Section 3.4 for more details.

1.2.4 The difference between $\mathrm{STAB}(G)$ and $\mathrm{QSTAB}(G)$

Padberg [75, 76] investigated in the early seventies general set packing problems and studied the case when the polyhedron

$$P(A) = \{x \in \mathbb{R}^n_+ : Ax \leq \mathbb{1}\}$$

associated with an $m \times n$ 0/1-matrix A has *integral* extreme points only (where $\mathbb{1} = (1, \ldots, 1)$). He proved in [75] that $P(A)$ coincides with $P_I(A)$, the convex hull of the integer points of $P(A)$, if and only if A is a perfect 0/1-matrix.

In order to translate this result in graph-theoretical terms [75], consider the graph G associated with A where the nodes of G correspond to the n columns of A and two nodes of G are linked by an edge if the corresponding columns of A have a 1-entry in common. Consequently, A is the clique-node incidence matrix of G and $P(A) = \mathrm{QSTAB}(G)$. Furthermore, we have that $P_I(A) = \mathrm{STAB}(G)$ and the result on perfect 0/1-matrices shows that $\mathrm{STAB}(G) = \mathrm{QSTAB}(G)$ if and only if G is perfect.

For all imperfect graphs G it holds that

$$\mathrm{STAB}(G) \subset \mathrm{QSTAB}(G)$$

and it is, therefore, natural to use the *difference* between the two polytopes in order to decide how far an imperfect graph is away from being perfect; we discuss different concepts, involving the facet set of $\mathrm{STAB}(G)$, the Chvátal-rank and the disjunctive index of $\mathrm{QSTAB}(G)$, and the dilation ratio of the two polytopes.

Facet descriptions of $\mathrm{STAB}(G)$. Padberg was, again, the first who studied imperfection in this context. He introduced in [75, 76] almost perfect matrices as the clique-node incidence matrices of minimally imperfect graphs and obtained the following characterization of minimally imperfect graphs in polyhedral terms (long time before the graph-theoretical characterization via the Strong Perfect Graph Theorem was achieved): A graph G is minimally imperfect if and only if $\mathrm{QSTAB}(G)$ has exactly one fractional extreme point (namely, $\frac{1}{\omega(G)}\mathbb{1}$ which is adjacent to the $|G|$ integer extreme points coming from the maximum stable sets of G) and

$$\mathrm{STAB}(G) = \mathrm{QSTAB}(G) \cap \{x \in \mathbb{R}^{|G|}_+ : x(G) = \sum_{i \in G} x_i \leq \alpha(G)\}$$

holds. This shows that, for minimally imperfect graphs, the two polytopes are as close as possible and, hence, minimally imperfect graphs are indeed 'almost perfect'.

Inspired by Padberg's results, Shepherd [98] introduced the notions of near-perfect matrices and graphs, where only the constraint $x(G) \leq \alpha(G)$ has to be added to QSTAB(G) in order to obtain STAB(G).

To generalize this concept further we consider 0/1-inequalities $x(G') \leq \alpha(G')$ associated with arbitrary induced subgraphs $G' \subseteq G$, called *rank constraints*, and rank-perfect graphs where such inequalities suffice as facets of the stable set polytope. By restricting the facet set to rank constraints associated with certain subgraphs only, several well-known graph classes are defined, e.g., t-perfect graphs [21] and h-perfect graphs [50]. Further well-known classes of rank-perfect graphs are, e.g., line graphs [33] and antiwebs [110], see Section 4.1 for more details and more results.

A further way to generalize clique constraints is the concept of *clique family inequalities*, investigating valid inequalities for the stable set polytope which rely on the intersection of cliques within the family. As clique family inequalities can be seen as a generalization of these constraints describing the matching polytope, there is a strong link to line graphs and their superclasses as, e.g., quasi-line graphs and claw-free graphs, see Section 4.2.

Chvátal-rank of QSTAB(G). A further way to see how 'easy' STAB(G) can be obtained starting from QSTAB(G) is based on the following more general concept. Chvátal [20] (and implicitly Gomory [48]) introduced a general method to obtain approximations of $P_I(A)$ outgoing from a polytope $P(A)$. If $\sum a_i x_i \leq b$ is a valid inequality for $P(A)$ and has integer coefficients only, then $\sum a_i x_i \leq \lfloor b \rfloor$ is a Chvátal-Gomory cut for $P(A)$. Define the *Chvátal-closure* $P'(A)$ of $P(A)$ as the set of points satisfying all Chvátal-Gomory cuts for $P(A)$, and let $P^0(A) = P(A)$ and $P^{t+1}(A) = (P^t)'(A)$ for all non-negative integers t. Obviously,

$$P_I(A) \subseteq P^t(A) \subseteq P(A)$$

holds for every t. An inequality $\sum a_i x_i \leq b$ is said to have Chvátal-rank at most t if it is a valid inequality for the polytope $P^t(A)$. Chvátal showed that for each rational polyhedron $P(A)$ there exists a finite $t \geq 0$ with $P^t(A) = P_I(A)$. The smallest such t is the *Chvátal-rank* of $P(A)$ which can be seen as an indicator for the quality of the linear relaxation $P(A)$. The fractional matching polytope is a famous example of a polytope with Chvátal-rank one (see Section 4.2 for more details).

The Chvátal-rank of the clique constraint polytope $P(A) = \mathrm{QSTAB}(G)$ is a further way to express the difference between $\mathrm{QSTAB}(G)$ and $\mathrm{STAB}(G)$. We say that a graph class $\mathcal{G}$ has Chvátal-rank t if t is the minimum value such that $\mathrm{QSTAB}(G)^t = \mathrm{STAB}(G)$ holds for all $G \in \mathcal{G}$. Hence, perfect graphs form exactly the class of graphs G where $\mathrm{QSTAB}(G)$ has Chvátal-rank zero. Minimally imperfect graphs, t-perfect graphs, h-perfect graphs, and line graphs are known to have Chvátal-rank one. However, the Chvátal-rank cannot be bounded for general rank-perfect graphs [24]. We address the Chvátal-rank for several subclasses of rank-perfect graphs (see Section 4.1) and for subclasses of claw-free graphs (see Section 4.2). Our main result is to provide an upper bound for the Chvátal-rank of clique family inequalities which particularly implies that all *rank* clique family inequalities have Chvátal-rank one [82].

Disjunctive index of QSTAB(G). Besides considering the set of facets which has to be added to $\mathrm{QSTAB}(G)$ in order to obtain $\mathrm{STAB}(G)$, one can also look at the number and structure of the fractional extreme points of $\mathrm{QSTAB}(G)$: the clique constraint polytope of a perfect graph has no fractional extreme points, whereas $\mathrm{QSTAB}(G)$ of a minimally imperfect graph G has exactly one.

More generally, Balas et al. [3] introduced the *disjunctive procedure* for binary linear programs as a way to obtain a complete description of the integer polytope $P_I(A)$ starting from the polytope $P(A)$. Let $V = \{1, \ldots, n\}$ denote the set of binary variables. For a subset $J = \{i_1, \ldots, i_j\}$ of the variables,

$$P_J(P(A)) = \mathrm{conv}\{x \in P(A) : x_j \in \{0,1\}, j \in J\}$$

holds. Balas et al. [3] showed that $P_J(P(A)) = P_{i_1}(P_{i_2}(\ldots P_{i_j}(P(A))))$. Clearly, $P_V(P(A)) = \mathrm{conv}(P(A) \cap \{0,1\}^n)$, but also proper subsets can have this property. This result allows to define the *disjunctive index* of a polytope $P(A)$ as the minimum size of a set $J \subseteq V$ such that $P_J(P(A)) = \mathrm{conv}(P(A) \cap \{0,1\}^n)$.

We can clearly apply this procedure to $\mathrm{QSTAB}(G)$ and Aguilera et al. [1] suggested to define the *imperfection index* $\mathrm{imp_I}(G)$ of a graph G as the disjunctive index of $\mathrm{QSTAB}(G)$. Thus, we have $\mathrm{imp_I}(G) = 0$ iff G is perfect and $\mathrm{imp_I}(G) = 1$ iff G is almost-perfect, including all minimal imperfect graphs. Moreover, it is proved in [1] that $\mathrm{imp_I}(G) = \mathrm{imp_I}(\overline{G})$ holds for all graphs. The imperfection index has a graph-theoretical interpretation as the minimum cardinality of a node subset $J \subset V(G)$ such that $G - J$ is perfect or, equivalently, as the cardinality of a minimum node subset meeting all minimal imperfect subgraphs of G (see Section 5.3.1). This provides a new and simpler proof of the invariance of the imperfection index under taking complements and establishes the link of this concept to the previously mentioned question of maximal perfect induced subgraphs of imperfect graphs.

We further address the problem which graph classes have a small imperfection index. Unfortunately, we obtain in [61] that the imperfection index cannot be bounded for many graph classes which are close to perfect graphs in some sense. In particular, our results indicate that there are many more graph classes with an unbounded imperfection index than with an unbounded imperfection ratio, including near-perfect graphs, t-perfect and h-perfect graphs, line graphs, antiwebs, and general rank-perfect graphs (see Section 5.4 for more details and some suggestions for refining the concept).

Dilation ratio of $\mathrm{STAB}(G)$ **and** $\mathrm{QSTAB}(G)$**.** In order to measure the difference between two polytopes of antiblocking type or, more generally, between two convex corners $\mathcal{A}, \mathcal{B} \subset \mathbb{R}^n_+$, one can use their *dilation ratio*

$$\begin{aligned} \mathrm{dil}(\mathcal{A}, \mathcal{B}) &= \min\{t : \mathcal{B} \subseteq t\ \mathcal{A}\} \\ &= \max\{x^T y : x \in \mathrm{abl}(\mathcal{A}), y \in \mathcal{B}\}. \end{aligned}$$

In particular, there is the following link between the dilation ratio of two convex corners and their entropies

$$\log_2 \mathrm{dil}(\mathcal{A}, \mathcal{B}) = \max\{H_{\mathcal{A}}(p) - H_{\mathcal{B}}(p) : p\}.$$

Recalling the previously mentioned characterization of the imperfection ratio

$$\log_2 \mathrm{imp}(G) = \max\{H(G, p) + H(\overline{G}, p) - H(p) : p\}$$

established by Simonyi [100], we shall combine this with Körner's [56] characterization of the entropy of a graph as the entropy of its stable set polytope $H(G, p) = H_{\mathrm{STAB}(G)}(p)$ and with the fact that $\mathrm{abl}(\mathrm{STAB}(\overline{G})) = \mathrm{QSTAB}(G)$. Indeed, we finally obtain the following alternative characterization of the imperfection ratio

$$\begin{aligned} \mathrm{imp}(G) &= \min\{t : \mathrm{QSTAB}(G) \subseteq t\ \mathrm{STAB}(G)\} \\ &= \max\{x^T y : x \in \mathrm{QSTAB}(G), y \in \mathrm{QSTAB}(\overline{G})\} \end{aligned}$$

as the dilation ratio of $\mathrm{STAB}(G)$ and $\mathrm{QSTAB}(G)$ [45]. This shows in particular, that $\mathrm{imp}(G) = \mathrm{imp}(\overline{G})$ holds for all graphs and that computing $\mathrm{imp}(G)$ is NP-hard.

For most graph classes, it is even unknown whether it can be bounded, but the above characterization suggests that the knowledge on the facet system of STAB(G) should help to determine imp(G). Indeed, there are upper bounds known for the imperfection ratio of some classes of rank-perfect graphs, including minimally imperfect graphs, t-perfect, h-perfect, and line graphs [45], antiwebs and several of their superclasses [27] (and the corresponding complementary classes). We are going to discuss these results in Section 5.2.

The main interest for bounding the imperfection ratio for all graphs in a class $\mathcal{G}$ is due to the following: if imp(G) $\leq q$ for all $G \in \mathcal{G}$, then $\vartheta(G)$ is at least a q-approximation of $\alpha(G)$ due to STAB(G) $\subseteq$ TH(G) $\subseteq$ QSTAB(G). Conversely, if there is no approximation-algorithm for $\alpha(G)$ for all $G \in \mathcal{G}$, then the imperfection ratio of the graphs in $\mathcal{G}$ is clearly unbounded.

1.2.5 Comparing different imperfection measures

Throughout the previous subsections, we surveyed several ways to relax such different concepts as

- the relation of clique and chromatic number,
- splitting graph entropies of complementary graphs,
- the relation of the stable set polytope and its LP-relaxation QSTAB(G),

all characterizing perfect graphs, and how to measure imperfection of a graph accordingly, namely, by means of

- χ-binding functions or the imperfection ratio,
- the value $\max\{H(G,p)+H(\overline{G},p)-H(p) : p\}$,
- the disjunctive index or the Chvátal-rank of QSTAB(G), or the dilation ratio of STAB(G) and QSTAB(G).

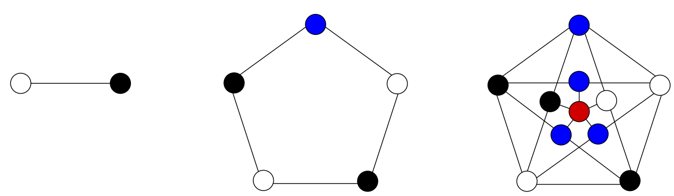
Perfect graphs are exceptional in all these respects, and minimally imperfect graphs are, indeed, close to perfect graphs by means of *all* these concepts. The objective of the following chapters is to explore the previously introduced superclasses of perfect graphs in this sense. The main question is whether graphs in those superclasses are close to perfection w.r.t. the *one* studied concept only or share more (structural or algorithmic) properties with perfect graphs w.r.t. the other concepts as well.

To streamline the presentation of the ideas and results, we refrain from presenting proofs within the next chapters, but provide some selected proofs in the appendix.

Chapter 2

Classes of χ-bound Graphs

Coloring the nodes of a graph is an important concept with a large variety of applications, but calculating $\chi(G)$ is an NP-hard problem in general. In a clique all nodes have to be colored differently, thus the clique number $\omega(G)$ is a trivial lower bound on $\chi(G)$. This bound can be arbitrarily bad: Mycielski [70] constructed a famous series of graphs $G_0, G_1, G_2, \ldots$ with



G_0 G_1 G_2

and showed $\omega(G_i) = 2$ for all i but $\chi(G_i) = 2 + i$, thus

$$\chi(G_i) - \omega(G_i) \to \infty \text{ if } i \to \infty$$

follows. A natural question is, therefore, for which classes of graphs the difference between the clique number $\omega(G)$ and the chromatic number $\chi(G)$ is under control. Perfect graphs are the most famous such class, as we have equality of the two parameters for all induced subgraphs. This motivated Gyárfás [51] to introduce the concept of χ-bound classes of graphs where a function in $\omega(G)$ is used as *upper* bound on $\chi(G)$.

The next section surveys several questions raised by Gyárfás [51] in this context. In Section 2.2 we discuss graph classes with the smallest possible binding-function for a class containing imperfect graphs. In Section 2.3 we relate the imperfection ratio and χ-binding functions and discuss consequences for the existence of binding-functions.

2.1 Problems concerning χ-bound classes of graphs

A class $\mathcal{G}$ of graphs is χ-*bound* if there is a binding-function b with

$$\chi(G') \leq b(\omega(G'))$$

for all induced subgraphs G' of $G \in \mathcal{G}$. Here, $b : \mathbb{N} \to \mathbb{N}$ should satisfy $b(1) = 1$ and $b(x) \geq x$ for all $x \in \mathbb{N}$. Thus, $b(x) = x$ is the smallest possible binding-function and the class of χ-bound graphs with this binding-function is precisely the class of perfect graphs.

Gyárfás addressed in [51] the following natural questions:

Problem 2.1 [51] *For a given class $\mathcal{G}$ of graphs,*

- *is there a binding-function for $\mathcal{G}$ at all?*
- *what is the smallest possible binding-function for $\mathcal{G}$?*
- *is there a linear binding-function for $\mathcal{G}$?*

Gyárfás presented in [51] several examples of χ-bound graph classes; however, in most cases the order of magnitude or linearity of their smallest binding function is not known.

The significance from an algorithmic point of view is, that χ-bound classes of graphs are canonical candidates for polynomial approximation algorithms for the coloring problem. Typically, the proof of the existence of a binding-function b for a graph class $\mathcal{G}$ provides a polynomial algorithm for coloring the graphs $G \in \mathcal{G}$ with at most $b(\omega(G))$ colors. This gives a polynomial approximation algorithm with a performance ratio of at most

$$\frac{b(\omega(G))}{\omega(G)}$$

and, for graph classes with a *linear* binding-function, the performance ratio is even constant. This causes particular interest in graph classes $\mathcal{G}$ with the smallest possible non-trivial binding-function $b(x) = x + 1$. We discuss such classes in Section 2.2.

In Section 2.3 we present a condition for the non-existence of linear binding-functions for certain graph classes.

Gyárfás [51] further addressed the problem of *complementary* binding-functions, i.e., the case when both $\mathcal{G}$ and its complementary graph class $\overline{\mathcal{G}}$ are χ-bound. Using this notion, the Perfect Graph Theorem states that $b(x) = x$ is a self-complementary binding function. The question is for which other binding-functions there is a complementary binding-function at all.

Gyárfás [51] showed that this is possible for "small" binding-functions only, since $\inf \frac{b(x)}{x} = 1$ is required if $b(x)$ has a complementary binding-function. This provides a necessary condition for the existence of a complementary binding-function. As a sufficient condition, Gyárfás conjectures the following:

Conjecture 2.2 [51] *The function $b(x) = x + c$ admits a complementary binding-function for any fixed positive integer c.*

At present, this conjecture is open even in the case $c = 1$. Any self-complementary class $\mathcal{G}$ with binding-function $b(x) = x+c$ clearly has $b(x) = x + c$ also as complementary binding-function. We present two classes with this property for $c = 1$: almost-perfect graphs and strongly circular-perfect graphs (see Section 2.2). But we also exhibit a class of χ-bound graphs with binding-function $b(x) = x + 1$ such that there is no complementary binding-function $b(x) = x + c$ for any fixed positive integer c. In Section 2.3 we present a condition for the non-existence of binding functions for certain self-complementary graph classes.

In addition, a generalization of the Perfect Graph Theorem states that a graph G is perfect iff $\alpha(G')\omega(G') \geq |G'|$ holds for all induced subgraphs $G' \subseteq G$. Gyárfás raised the following problem:

Problem 2.3 [51] *Is it true that a graph class $\mathcal{G}$ is χ-bound if every graph $G \in \mathcal{G}$ satisfies $\alpha(G')\omega(G') + 1 \geq |G'|$ for all induced subgraphs $G' \subseteq G$?*

We cannot answer this question completely, but we give an affirmative answer for almost-perfect graphs and strongly circular-perfect graphs, the two self-complementary classes with binding-function $b(x) = x+1$ discussed in Section 2.2.

Finally, classes of perfect graphs are often characterized by means of forbidden induced subgraphs. Gyárfás formulated the analogue question:

Problem 2.4 [51] *Which forbidden induced subgraphs make a graph class χ-bound?*

The case of an infinite or self-complementary set of forbidden subgraphs is, thereby, of particular interest; the most prominent example was the Strong Perfect Graph Conjecture itself. We address the problem of possible obstructions for χ-bound classes during the next sections.

2.2 Classes with the binding-function $b(x) = x+1$

From an algorithmic point of view, there is a particular interest in graph classes $\mathcal{G}$ with the smallest possible non-trivial binding-function $b(x) = x+1$; we discuss some classes with this property in detail.

2.2.1 Almost-perfect graphs

The class of almost-perfect graphs was introduced in [61] as a smallest possible superclass of perfect graphs: that are graphs G containing a node v such that removing v yields a perfect graph $G - v$.

Clearly, every perfect graph is in particular almost-perfect and every minimally imperfect graph as well (such graphs even satisfy the property that removing an *arbitrary* node yields a perfect graph).

It is natural to expect that almost-perfect graphs also satisfy properties almost as nice as perfect graphs. Indeed, it is a routine to check that the class of almost-perfect graphs is χ-bound with the smallest non-trivial binding-function $b(x) = x + 1$. Since the class of almost-perfect graphs is self-complementary due to the Perfect Graph Theorem, it clearly also admits a complementary binding-function and is also $\overline{\chi}$-bound with binding-function $b(x) = x + 1$.

In addition, an easy argumentation shows that any almost-perfect graph G satisfies $\alpha(G')\omega(G') + 1 \geq |G'|$ for all induced subgraphs $G' \subseteq G$. This gives one affirmative answer to the above mentioned Problem 2.3 of Gyárfás.

Thus, almost-perfect graphs are indeed very close to perfect graphs and it seems worth to explore which graphs are almost-perfect.

Every graph containing exactly one minimally imperfect subgraph is clearly almost-perfect; we call such graphs *uniquely imperfect*. This class includes all odd holes, odd antiholes, odd wheels, and odd antiwheels. The latter graphs $C_{2k+1} * v$ and $\overline{C}_{2k+1} * v$ are obtained by completely joining an odd hole or odd antihole with a single node v. Actually, all graphs obtained as complete join of an odd (anti)hole and a perfect graph are uniquely perfect.

More generally, *every* almost-perfect graph is of the form $G(P, V', v)$ where P is a perfect graph, $V' \subseteq V(P)$ a node subset of P, and v a node totally joined to V'; see Figure 2.1 for two complementary almost-perfect graphs (the node v is black-filled). In particular, if P is bipartite, then $G(P, V', v)$ is a so-called *almost-bipartite graph* since removing v yields a bipartite graph.

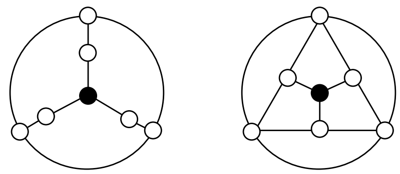


Figure 2.1: Two complementary almost-perfect graphs.

Finally, it is worth to address possible obstructions for almost-perfect graphs. One infinite but easy to describe class of obstructions obviously consists of all combinations of two disjoint minimally imperfect graphs. But there exist also highly connected obstructions, namely, the imperfect webs and antiwebs different from odd holes and odd antiholes.

A *web* is a graph W_n^k with n nodes $1, \ldots, n$ where ij is an edge if i and j differ by at most k (i.e., if $|i - j| \leq k \bmod n$) and $i \neq j$. W_n^1 is a hole and W_{2k+1}^{k-1} an odd antihole for $k \geq 2$. Webs are also called *circulant graphs* C_n^k in [22]. The complements of webs are called *antiwebs* or, alternatively, *circular cliques* $K_{n/k}$, see [115] (note that $K_{n/k} = \overline{W}_n^{k-1}$ holds). Figure 2.2 shows some examples of webs and antiwebs.

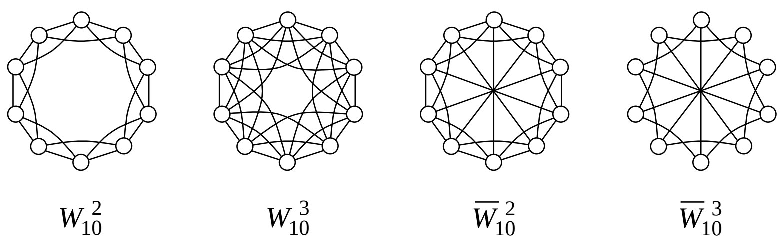


Figure 2.2: Some examples of webs and antiwebs.

Due to the cyclic symmetry of webs and antiwebs, it is easy to see that removing an arbitrary node from an imperfect but not minimally imperfect web or antiweb still yields an imperfect graph. This implies in particular:

Lemma 2.5 *The only imperfect almost-perfect webs and antiwebs are odd holes and odd antiholes.*

Note that odd holes and odd antiholes are the only graphs which are webs and antiwebs at the same time [104]. Thus, the intersection of the classes of all webs and antiwebs belongs to the class of almost-perfect graphs.

2.2.2 Line graphs and their superclasses

A canonical example of graphs with χ-binding function $b(x) = x + 1$ is the class of line graphs, since we can 'translate' a famous result of Vizing [108] on edge-colorings.

The *line graph* $L(G)$ of a root graph G is obtained by taking the edges of G as nodes of $L(G)$ and joining two nodes of $L(G)$ if the corresponding edges of G are incident.

In an edge-coloring of a graph G, incident edges have to be colored differently, thus the maximum degree $\Delta(G)$ is a trivial lower bound for the minimum number of required colors, called chromatic index $\gamma(G)$. A famous result of Vizing [108] shows that $\Delta(G) + 1$ is an upper bound on $\gamma(G)$, that is we have

$$\Delta(G) \leq \gamma(G) \leq \Delta(G) + 1$$

for every graph G. Turning to the line graph of G, we immediately see that an edge-coloring of G corresponds to a node-coloring of its line graph $L(G)$ and, thus, $\gamma(G) = \chi(L(G))$ holds. Furthermore, pairwise incident edges of G correspond to cliques of $L(G)$, thus cliques of $L(G)$ come from edge-stars or triangles in G. This implies $\omega(L(G)) = \max\{\Delta(G), 3\}$ (if G contains a triangle) and it is easy to see that

$$\omega(L(G)) \leq \chi(L(G)) \leq \omega(L(G)) + 1$$

follows. As Vizing's bound holds for all graphs, the latter inequality is true for all induced subgraphs of line graphs which implies that line graphs are χ-bound with binding-function $b(x) = x + 1$.

Choudom [15] raised the intriguing question whether this property also holds for superclasses of line graphs. As line graphs are characterized by a set $\mathcal{F}$ of nine forbidden subgraphs [5], it is natural to define such superclasses by using *sub*sets of the nine obstructions in $\mathcal{F}$ only. Choudom [15] considered two such classes using four forbidden subgraphs from $\mathcal{F}$ and showed that both classes are χ-bound with binding-function $b(x) = x + 1$.

Choudom further asked for smallest possible sets of obstructions with this property. There is only one class of χ-bound graphs with binding-function $b(x) = x + 1$ with exactly *one* forbidden subgraph, namely, the P_4-free graphs–which are perfect and have even binding-function $b(x) = x$. This motivated the question which *pairs* of forbidden subgraphs from $\mathcal{F}$ imply the studied bound.

Two first results with this spirit show that a triangle-free graph G is 3-colorable if G also does not contain a P_5 [103] or a $K_{1,4}$ [11].

Randerath answered in [89] this problem by giving a complete list of forbidden pairs from $\mathcal{F}$ that guarantee the studied bound. He proved that one of the two obstructions has to be a tree, the other obstruction one of the graphs depicted in Figure 2.3.

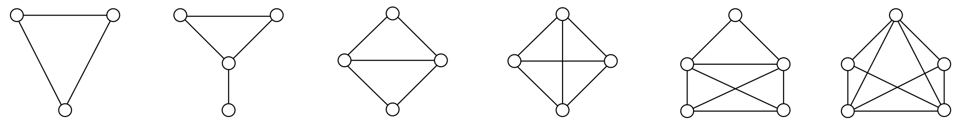


Figure 2.3: List of obstructions

There is another well-known superclass of line graphs, the *quasi-line graphs*, where the neighbors of every node split into two cliques. Alternatively, that are claw-free graphs not containing a node totally joined to an odd antihole (i.e., quasi-line graphs form a superclass of line graphs with an infinite set of forbidden subgraphs).

We exhibit that quasi-line graphs are not χ-bound with binding-function $b(x) = x+1$, as a subclass of quasi-line graphs, the webs, does not have this property.

Example 2.6 Consider the following sequence of webs W_{6l+2}^{2l} for $l \geq 1$. For any web W_n^k, we have that $\omega(W_n^k) = k+1$ and $\chi(W_n^k) = k+1+\lceil \frac{r}{\alpha} \rceil$ holds with $r = n \bmod (k+1)$ and $\alpha = \lfloor \frac{n}{k+1} \rfloor$. A simple computation yields $r = 2l$ and $\alpha = 2$ (since $n = 6l+2 = 2(2l+1)+2l$) and, therefore,

$$\chi(W_{6l+2}^{2l}) = 2l+1+\left\lceil \frac{2l}{2} \right\rceil = \omega(W_{6l+2}^{2l}) + l$$

for all $l \geq 1$.

This implies:

Corollary 2.7 *For the class of webs there is no binding-function $b(x) = x+c$ for any positive integer c.*

The same is obviously true for all superclasses; in particular we have:

Corollary 2.8 *For the class of quasi-line graphs there is no binding-function $b(x) = x+c$ for any positive integer c.*

2.2.3 Circular-perfect graphs

As generalization of perfect graphs, Zhu [115] introduced recently the class of circular-perfect graphs based on the following more general coloring concept.

Define a (k,d)-*circular coloring* of a graph $G = (V,E)$ as a mapping $f : V \to \{0,\dots,k-1\}$ with $|f(u) - f(v)| \geq d \bmod k$ if $uv \in E$. The *circular chromatic number* $\chi_c(G)$ is the minimum $\frac{k}{d}$ taken over all (k,d)-circular colorings of G; we immediately obtain $\chi_c(G) \leq \chi(G)$ since every $(k,1)$-circular coloring is a usual k-coloring of G. (Note that $\chi_c(G)$ is sometimes called the star chromatic number in the literature, see [9, 107].)

In order to obtain a lower bound on $\chi_c(G)$, Zhu generalized cliques to circular cliques $K_{k/d}$. Recall that circular cliques are the antiwebs and include all cliques $K_k = K_{k/1}$, odd antiholes $\overline{C}_{2k+1} = K_{2k+1/2}$, and odd holes $C_{2k+1} = K_{2k+1/k}$, see Figure 2.4. The *circular clique number* is defined as $\omega_c(G) = \max\{\frac{k}{d} : K_{k/d} \subseteq G\}$ and we immediately obtain $\omega(G) \leq \omega_c(G)$.

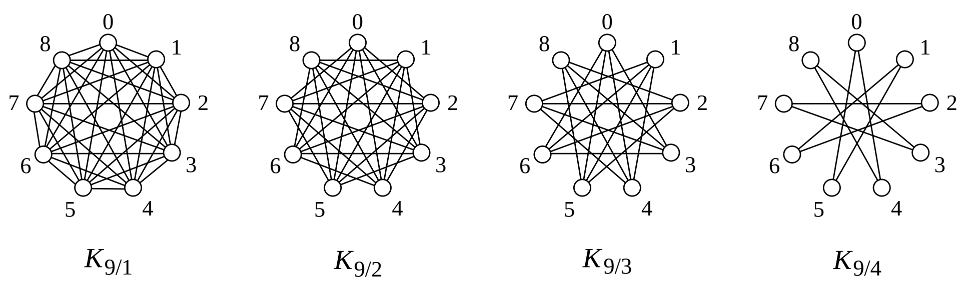


Figure 2.4: The circular cliques on nine nodes.

Every circular clique $K_{k/d}$ clearly admits a (k,d)-circular coloring (simply take the node numbers as colors, as in Figure 2.4) but no (k',d')-circular coloring with $\frac{k'}{d'} < \frac{k}{d}$ by [9]. Thus we obtain, for any graph G, the following chain of inequalities:

$$\omega(G) \leq \omega_c(G) \leq \chi_c(G) \leq \chi(G). \tag{2.1}$$

Zhu [115] called a graph G *circular-perfect* if, for each induced subgraph $G' \subseteq G$, circular clique number $\omega_c(G')$ and circular chromatic number $\chi_c(G')$ coincide.

Obviously, every perfect graph has this property by (2.1) as $\omega(G')$ equals $\chi(G')$. Also, every odd hole and odd antihole C is circular-perfect since all proper induced subgraphs are perfect and $\omega_c(C) = \chi_c(C)$ follows from [9]. Thus, circular-perfect graphs form a proper superclass of perfect graphs.

Since, in addition, for any graph G it holds $\omega(G) = \lfloor \omega_c(G) \rfloor$ by [115] and $\chi(G) = \lceil \chi_c(G) \rceil$ by [107], we obtain that circular-perfect graphs G satisfy

$$\omega(G) \leq \chi(G) \leq \omega(G) + 1 \tag{2.2}$$

and, thus, built a further class of χ-bound graphs with the smallest non-trivial χ-binding function $b(x) = x + 1$. Hence, also circular-perfect graphs admit coloring properties almost as nice as perfect graphs.

Our aim is to look for other parallels between the classes of perfect and circular-perfect graphs.

First note that the class of circular-perfect graphs is not closed under taking complements. For instance, the left graph in Figure 2.1 is circular-perfect whereas its complement is not. So far, it is not clear whether there is also a (linear) complementary binding function for circular-perfect graphs; we show that it cannot be $b(x) = x + c$ for any integer c if it exists at all.

As analogue to the Strong Perfect Graph Theorem, one might be tempted to ask for an appealing conjecture on minimal forbidden subgraphs in circular-perfect graphs. In addition, this would help to figure out which forbidden subgraphs make a graph class χ-bound, as asked by Gyárfás [51]. We call a graph G *minimal circular-imperfect* if G is not circular-perfect but every proper induced subgraph is. The hope is to identify all classes of minimal circular-imperfect graphs in order to characterize circular-perfect graphs by means of forbidden subgraphs.

Alternatively, we try to identify subclasses of circular-perfect graphs.

Circular cliques and their superclasses. Zhu [115] verified that every circular clique is circular-perfect. In [4] this has been extended to *convex-round graphs*: that are graphs G where the node set can be labeled in a cyclic order such that all neighbors of any node are consecutive w.r.t. this order. (Circular cliques obviously admit such an order, thus convex-round graphs constitute a proper superclass.)

Remark 2.9 In [27] we could show that every convex-round graph is the complement of a so-called fuzzy circular interval graph (see Section 4.1.2 for the definition). So fare, it is unknown whether also all fuzzy circular interval graphs are circular-perfect. At least, there are no minimal circular-imperfect graphs known within this class.

Note, in addition, that circular cliques different from odd holes and odd antiholes are obstructions for almost-perfect graphs, but not for the class of χ-bound graphs with binding function $b(x) = x + 1$.

However, the circular cliques show that the class of circular-perfect graphs does not admit a complementary binding-function of the form $b(x) = x + c$.

Recall from Example 2.6 the sequence of webs W^{2l}_{6l+2} for $l \geq 1$ with

$$\chi(W^{2l}_{6l+2}) = 2l + 1 + \left\lceil \frac{2l}{2} \right\rceil = \omega(W^{2l}_{6l+2}) + l$$

for all $l \geq 1$. This implies

$$\chi(W^{2l}_{6l+2}) - \omega(W^{2l}_{6l+2}) \to \infty \text{ if } l \to \infty,$$

and, hence, we obtain:

Corollary 2.10 *For the circular cliques (and their superclasses), there is no complementary binding-function $b(x) = x + c$ for any positive integer c.*

Partitionable graphs. Recall that Lovász [65] and Padberg [75] introduced partitionable graphs as a tool to study properties of minimal imperfect graphs, as every minimal imperfect graph is in particular partitionable. Since, further, all circular cliques $K_{\alpha\omega+1,\alpha}$ are partitionable and every partitionable graph G satisfies $\chi(G) = \omega(G) + 1$, one might expect that at least some subclasses of partitionable graphs are circular-perfect. However, we obtained in [81] the following.

Theorem 2.11

- *All partitionable graphs different from circular cliques are circular-imperfect.*
- *The partitionable graphs $\overline{K}_{3\omega+1,3}$ are minimal circular-imperfect for all $\omega \geq 3$.*

Even worse, the webs $W^{4l}_{3(4l+1)+1}$ for $l \geq 1$ form a sequence of partitionable graphs containing induced subgraphs with arbitrarily large difference between clique and chromatic number. Trotter [104] showed that $W^{k'}_{n'}$ is an induced subgraph of W^k_n if and only if

$$\frac{k+1}{k'+1} n' \leq n \leq \frac{k}{k'} n'$$

holds. With the help of this formula it is easy to check that the above considered web W^{2l}_{6l+2} is an induced subgraph of $W^{4l}_{3(4l+1)+1}$ for every $l \geq 1$. As the class of partitionable graphs is closed under complementation, this implies:

Corollary 2.12 *The class of partitionable graphs is neither χ-bound nor $\overline{\chi}$-bound with binding-function $b(x) = x + c$ for any positive integer c.*

Planarity and circular-perfection. As the only planar circular cliques are odd holes, K_3, and K_4 it is easy to determine the circular clique number of a planar graph [81]:

Lemma 2.13 *The circular clique number of a planar graph G is*

- $\omega_c(G) = 1, 2$ *if G is bipartite,*
- $\omega_c(G) = 4$ *if G has an induced K_4, or else*
- $\omega_c(G) = 2 + \frac{1}{d}$ *if C_{2d+1} is a shortest chordless odd cycle in G.*

Note that, as a consequence, we obtain that a planar graph G can be minimal circular-imperfect only if either $\omega(G) = 2$ and $\chi(G) = 3$ or $\omega(G) = 3$ and $\chi(G) = 4$ holds.

One result from [81] exhibits the circular-perfection of an interesting subclass of planar graphs: the *outerplanar graphs* where all nodes lie on the outer face.

Theorem 2.14 *Outerplanar graphs are circular-perfect.*

The proof relies on the fact that outerplanar graphs are closed under clique-identification. As a by-product of Theorem 2.14, the circular chromatic number of an outerplanar graph is equal to 2 if all cycles have even size, or $2 + \frac{1}{d}$ where $2d + 1$ is the size of the smallest odd cycle. This gives a different proof of a recent result by Kemnitz and Wellmann [54].

Outgoing from the circular-perfection of outerplanar graphs, we introduced in [81] the following class of minimal circular-imperfect planar graphs. For every positive integers k and l such that $(k, l) \neq (1, 1)$, let $T_{k,l}$ denote the planar graph with $2l + 1$ inner faces $F_1, F_2, \ldots F_{2l+1}$ of size $2k + 1$ arranged in a circular fashion around a central node, where all other nodes lie on the outer face, as depicted in Figure 2.5.

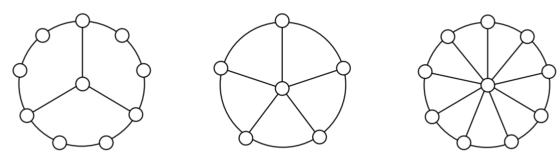
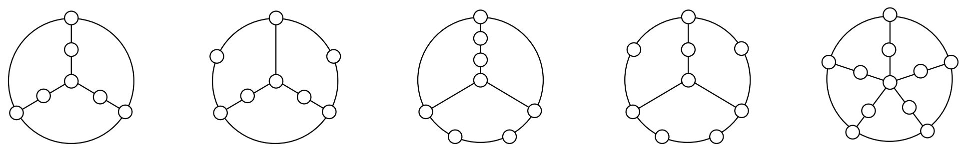


Figure 2.5: Examples of graphs $T_{k,l}$

Lemma 2.15 *For every positive integers k and l such that $(k, l) \neq (1, 1)$, the graph $T_{k,l}$ is minimal circular-imperfect.*

Remark 2.16 Note that we showed circular-imperfection for such graphs by $\omega_c(T_{k,l}) < \chi_c(T_{k,l})$, whereas minimality follows from Theorem 2.14 as removing any node of $T_{k,l}$ yields an outerplanar graph. The latter seems to be a crucial property due to the following. One can interpret the graphs $T_{k,l}$ as certain subdivisions of odd wheels, where only the "rim edges" are subdivided. One could consider more general subdivisions of odd wheels, where all inner faces should become odd cycles of the same length, see Figure 2.6 for examples. However, as soon as spokes of the wheel become paths, we loose the property that removing any node yields an outerplanar graph, see Figure 2.6. In fact, among such subdivisions of odd wheels, there are circular-perfect graphs (a), minimal circular-imperfect graphs (b),(c),(d), as well as circular-imperfect graphs which are not minimal (e).



(a) (b) (c) (d) (e)

Figure 2.6: More general subdivisions of odd wheels.

It is not known yet which subdivisions of odd wheels are minimal circular-imperfect. This shows in particular, that it is even hard to characterize the (planar) minimal circular-imperfect graphs with clique number two. In addition, there are also infinite sequences of planar minimal circular-imperfect graphs with clique number three. From the previously mentioned series, all odd wheels are clearly planar. Furthermore, the partitionable graphs $\overline{K}_{3\omega+1,3}$ admit an embedding in the plane for any odd $\omega \geq 3$. Two additional sequences are shown below.

Example 2.17 We call the graphs in Figure 2.7 *diamonded* as they can be constructed from a chain of diamonds by linking the first and the last node by an edge. It is a routine to check that such graphs are minimal circular-imperfect (note that adding the dashed edges yields minimal circular-imperfect graphs again).

Example 2.18 The graphs $M(C_{2k+1})$ in Figure 2.8 can be obtained via the following Mycielski-like construction. Take a chordless odd cycle C_{2k+1}

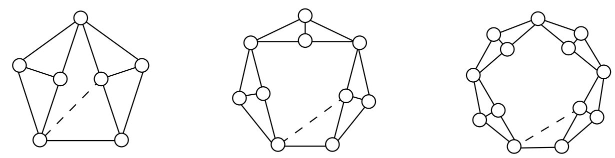


Figure 2.7: Three diamonded graphs.

with nodes $x_1, \ldots, x_{2k+1}$. Add the nodes $y_1, \ldots, y_{2k+1}$ and connect every y_i exactly with x_i and x_{i+1} (indices are considered modulo $2k+1$). Finally, add a node z and link it to each of the nodes $y_1, \ldots, y_{2k+1}$. By case analyses, it is easy to verify that the graphs $M(C_{2k+1})$ are minimal circular-imperfect for all $k \geq 1$.

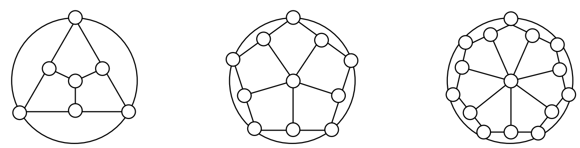


Figure 2.8: Three Mycielski-like graphs.

Complete joins and circular-perfection. Motivated from the observation that all (imperfect) odd wheels are minimal circular-imperfect, we studied in [81] the behavior of circular-perfect graphs under taking complete joins and completely characterized complete joins w.r.t. circular-(im)perfection as follows:

Theorem 2.19 *The complete join $G_1 * G_2$ of two graphs G_1 and G_2 is*

(i) *circular-perfect if and only if both G_1 and G_2 are perfect;*

(ii) *minimal circular-imperfect if and only if G_1 is an odd hole or odd antihole and G_2 is a single node (or vice versa).*

Note that odd antiwheels $\overline{C}_{2k+1} * v$ are examples of minimal circular-imperfect graphs with arbitrarily large clique and chromatic number. Recall further that odd wheels and odd antiwheels are almost-perfect graphs. Thus they are certainly forbidden subgraphs for circular-perfect graphs, but not for χ-bound graphs with binding-function $b(x) = x + 1$ in general.

Conclusions. The previous results show that, at first sight, there is no straightforward common structure in the known classes of minimal circular-imperfect graphs. Thus, formulating an analogue to the Strong Perfect Graph Theorem for circular-perfect graphs seems to be difficult.

However, there is an interesting link between minimal circular-imperfect graphs G with $\omega(G) = \omega_c(G)$ and minimal *k-chromatic graphs*, that are graphs G with $\chi(G) = k$ and $\chi(G - v) = k - 1$ for all nodes v of G. We have the following:

Observation 2.20 *A graph G with $\omega_c(G) = k$ is minimal circular-imperfect only if G is minimal $(k+1)$-chromatic.*

The reason is the following. G is minimal circular-imperfect only if $\omega_c(G) < \chi_c(G) \leq \chi(G) = k + 1$ and $\omega_c(G') = \chi_c(G')$ holds for all proper induced subgraphs $G' \subset G$. In particular, $\omega_c(G') \leq k$ and $\lceil \chi_c(G') \rceil = \chi(G)$ imply that all $G' \subset G$ are k-colorable. Hence, every minimal circular-imperfect graph G with $\omega_c(G) = k$ is in particular minimal $(k+1)$-chromatic.

Furthermore, being minimal k-chromatic is a necessary condition for a graph G to be *k-critical*, that is $\chi(G) = k$ and $\chi(G - e) = k - 1$ for all edges e of G. A famous result of Hajós [52] says that every k-critical graph is *Hajós-k-constructable*, that means, it can be obtained from a clique K_k by repeated applications of the following two operations:

(H1) If G_1 and G_2 are graphs constructed that way, remove an edge $x_i y_i$ of G_i for $i = 1, 2$, identify x_1 and x_2, join y_1 and y_2 by a new edge.

(H2) If G is a previoulsy constructed graph, identify two independent nodes of G.

Several of the above minimal circular-imperfect graphs are 4-critical, e.g.,

- odd wheels,
- diamonded graphs (without the dashed edges, see Figure 2.7),
- the Mycielski-like graphs $M(C_{2k+1})$ for all $k \geq 1$.

(but neither odd antiwheels nor the graphs $\overline{K}_{3\omega+1,3}$ with $\omega \geq 3$).

This implies that at least some minimal circular-imperfect graphs G with $\omega_c(G) = k$ are Hajós-$(k+1)$-constructable. It is worth to check which minimal circular-imperfect graphs admit this property. Furthermore, this link to Hajós' construction raises the interesting question about construction techniques for minimal circular-imperfect graphs, in particular, when the application of one of the two Hajós-operations (H1) and (H2) to a minimal circular-imperfect graph preserves this property.

2.2.4 Strongly circular-perfect graphs

As circular-perfect graphs are, in contrary to perfect graphs, not stable under complementation, we shall study the complementary core of circular-perfect graphs in order to get a better analogue to perfect graphs.

For that, we introduced *strongly circular-perfect graphs* as all those circular-perfect graphs G where $\overline{G}$ is circular-perfect as well. Thus, strongly circular-perfect graphs are a further class of χ-bound graphs admitting a complementary binding function; here $b(x) = x + 1$ is both a χ- and a $\overline{\chi}$-binding function.

The class of strongly circular-perfect graphs clearly entails all perfect graphs, odd holes, and odd antiholes. In [28] we showed that, in fact, odd holes and odd antiholes are the *only* prime circular cliques which can occur in a strongly circular-perfect graph. As the class is closed under complementation, this is equivalent to prove that the only prime circular-perfect webs W_n^k are odd holes and odd antiholes (recall $\overline{W}_n^k = K_{n,k+1}$). We fully characterized the circular-(im)perfection of webs as follows:

Theorem 2.21 *The web W_n^k is*

(i) *circular-perfect if $k = 1$ or $n \leq 2(k+1)+1$,*

(ii) *circular-perfect if $k = 2$ and $n = 0 \pmod 3$,*

(iii) *minimal circular-imperfect if $k = 2$ and $n = 1 \pmod 3$,*

(iv) *circular-imperfect if $k = 2$ and $n = 2 \pmod 3$,*

(v) *circular-imperfect if $k \geq 3$ and $n \geq 2(k+2)$.*

As a consequence, we obtained the following characterization of the strongly circular-perfect circular cliques [28] .

Corollary 2.22 *A circular clique is strongly circular-perfect if and only if it is a clique, an odd antihole, an odd hole, a stable set, or of the form $K_{3k/3}$ with $k \geq 3$.*

In particular, we have for circular clique and chromatic number of every strongly circular-perfect graph:

Corollary 2.23 *Let G be a strongly circular-perfect graph.*

(i) *If $\omega(G) = 2$, then $\omega_c(G) = 2$ follows iff G is perfect and $\omega_c(G) = 2 + \frac{1}{k}$ iff G is imperfect and C_{2k+1} the shortest odd hole in G.*

(ii) *If $\omega(G) \geq 3$, then $\omega_c(G) = \max\{\omega(G), k' + \frac{1}{2}\}$ where $\overline{C}_{2k'+1}$ is the shortest odd antihole in G.*

So fare, we were able to completely characterize the strongly circular-perfect graphs G with $\omega(G) = 2$ as follows [28]:

Theorem 2.24 *A triangle-free graph G is strongly circular-perfect if and only if G is either bipartite or an interlaced odd hole.*

Here, a graph G is an *interlaced odd hole* if and only if the node set of G admits a partition $((A_i)_{1\leq i\leq 2p+1}, (B_i)_{1\leq i\leq 2p+1})$ into $2p+1$ non-empty sets $A_1, \ldots, A_{2p+1}$ and $2p+1$ possibly empty sets $B_1, \ldots, B_{2p+1}$ such that $p \geq 2$ and for all $1 \leq i \leq 2p+1$

1. $|A_i| > 1$ implies $|A_{i-1}| = |A_{i+1}| = 1$, (indices modulo $2p+1$),
2. $B_i \neq \emptyset$ implies $|A_i| = 1$,

and the edge set of G is equal to $\cup_{i=1,\ldots,2p+1}(E_i \cup E'_i)$, where E_i (resp. E'_i) denotes the set of all edges between A_i and A_{i+1} (resp. between A_i and B_i); see Figure 2.9 for an example (the sets of nodes in B_i are grey).

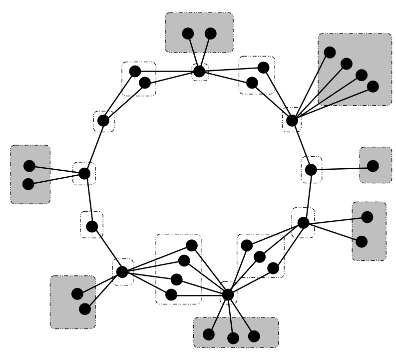


Figure 2.9: An interlaced odd hole

In particular, we could prove in [28]:

Lemma 2.25 *Interlaced odd holes are almost-bipartite.*

As bipartite and almost-bipartite graphs are almost-perfect, this implies:

Corollary 2.26 *All strongly circular-perfect graphs G with $\omega(G) = 2$ or $\alpha(G) = 2$ are almost-perfect.*

Moreover, almost-bipartite graphs are a subclass of the well-known t-perfect graphs for which a maximum weighted stable set can be found in polynomial time [50] (see Section 4.1.2 for more details). Thus, we have:

Corollary 2.27 *The weighted stable set problem can be solved in polynomial time for triangle-free strongly circular-perfect graphs.*

Further, a polynomial recognition algorithm for triangle-free strongly circular-perfect graphs was derived in [28], outgoing from the above characterization of the class. Thus, both the stable set problem and the recognition problem are solvable in polynomial time for such graphs.

There is also one class of strongly circular-perfect graphs known which does not consist of triangle-free graphs: the webs $W^2_{3\alpha}$ for $\alpha \geq 2$ and their complements $K_{3\alpha,3}$ according to Theorem 2.21(ii).

By case analysis, it is also easy to show that every induced subgraph G' of a web $W^2_{3\alpha}$ satisfies $\alpha(G')w(G')+1 \geq |G'|$. Thus, the webs $W^2_{3\alpha}$ and their complements $K_{3\alpha,3}$ are a further graph classes giving an affirmative answer to Gyárfás' above mentioned Problem 2.3.

Remark 2.28 The webs $W^2_{3\alpha}$ are not almost-perfect, as for any node v of $W^2_{3\alpha}$ holds that $C_{2\alpha-1} \subset W^2_{3\alpha} - v$. Thus, not all strongly circular-perfect graphs are almost-perfect. The converse is also true, as the two complementary almost-perfect graphs in Figure 2.1 are not strongly circular-perfect (for the right graph G, we have $3 = \omega(G) = \omega_C(G) < \chi_c(G) \leq \chi(G) = 4$).

Finally, we addressed in [28] the problem of characterizing strongly circular-perfect graphs by means of forbidden subgraphs. Recall that the results from [81] suggest that formulating a corresponding conjecture for circular-perfect graphs is difficult; it is even unknown which triangle-free graphs are minimal circular-imperfect. For (general) minimally not strongly circular-perfect graphs, there is also no conjecture at hand, but we were able to give a complete answer in the triangle-free case again [28]:

Theorem 2.29 *A triangle-free graph G is minimal strongly circular-imperfect if and only if G is either the disjoint union of an odd hole and a singleton or an extended odd hole.*

Here, a graph G is called an *extended odd hole* if it admits a proper partition into an induced odd hole $\mathcal{O} = \{o_1, \ldots, o_{2p+1}\}$ and a pair of nodes $\{x, y\}$ which is connected to $\mathcal{O}$ in one of the following ways:

(a)	$\{o_1x, xy, o_4y\}$	(d)	$\{o_1x, o_3x, xy, o_2y\}$
(b)	$\{o_1x, xy, o_2y\}$	(e)	$\{o_1x, o_3x, xy, o_2y, o_4y\}$
(c)	$\{o_1x, o_3x, xy, o_4y\}$	(f)	$\{o_1x, o_3x, o_2y, o_4y\}$

2.3 On the existence of linear binding-functions

Gyárfás raised in [51] the question to find conditions for the existence of a χ-binding function for a given graph class $\mathcal{G}$ (Problem 2.1).

The existence of *linear* binding-functions $b(x) = ax + c$ is of particular interest since this gives approximation algorithm with a constant performance ratio of at most $\frac{ax+c}{x} = a$ and, in addition, complementary binding functions can exist only in the case $a = 1$ according to [51].

We present a sufficient condition from [114] for the *non*-existence of a linear binding-function for certain graph classes, starting with the imperfection ratio

$$\operatorname{imp}(G) = \max\left\{\frac{\chi_f(G,c)}{\omega(G,c)} \mid c : V(G) \to \mathbb{N} \setminus \{0\}\right\}$$

where $\chi_f(G,c)$ stands for the fractional weighted chromatic number and $\omega(G,c)$ for the weighted clique number. Let

$$\operatorname{imp}(\mathcal{G}) = \sup\{\operatorname{imp}(G) : G \in \mathcal{G}\}$$

stand for the supremum over the imperfection ratios of the graphs G in a class $\mathcal{G}$.

We call a graph class $\mathcal{G}$ *simple* if it suffices to consider the *un*weighted versions of $\chi_f(G,c)$ and $\omega(G,c)$ in order to determine $\operatorname{imp}(\mathcal{G})$, that is if

$$\operatorname{imp}(\mathcal{G}) = \sup\left\{\frac{\chi_f(G)}{\omega(G)} \mid G \in \mathcal{G}\right\}$$

holds. For instance, triangle-free graphs form a simple class by [45]; in addition, all graph classes $\mathcal{G}$ which are closed under replication (that is substituting cliques for nodes) are simple as replicating every node v_i of a graph G in $\mathcal{G}$ by a clique of size c_i yields a graph in $\mathcal{G}$ again.

For simple classes with *un*bounded imperfection ratio, we obtain in [114]:

Observation 2.30 *If $\mathcal{G}$ is a simple class with $\operatorname{imp}(\mathcal{G}) = \infty$, then $\mathcal{G}$ has no linear χ-binding function.*

The reason is that, for every integer $k \geq 0$, there is a graph $G_k \in \mathcal{G}$ with

$$k < \frac{\chi_f(G_k)}{\omega(G_k)} \leq \frac{\chi(G_k)}{\omega(G_k)}$$

and, thus, $k\omega(G_k) < \chi(G_k)$ follows. Combining this result with Gyárfás' necessary condition for the existence of complementary binding-functions, we obtain further:

Corollary 2.31 *If $\mathcal{G}$ is a simple class with* $\operatorname{imp}(\mathcal{G}) = \infty$*, then $\mathcal{G}$ has no complementary binding-function.*

If $\mathcal{G}$ is in addition *self*-complementary, this implies:

Corollary 2.32 *If $\mathcal{G}$ is a simple, self-complementary class with* $\operatorname{imp}(\mathcal{G}) = \infty$*, then $\mathcal{G}$ does not have a binding-function at all.*

Every graph class $\mathcal{G}$ which is closed under substitution and contains at least on imperfect graph satisfies both conditions: $\mathcal{G}$ is simple and $\operatorname{imp}(\mathcal{G}) = \infty$ holds. Thus, we finally deduce:

Corollary 2.33 *If a graph class $\mathcal{G}$ is closed under substitution and contains at least on imperfect graph, then $\mathcal{G}$ has no complementary binding-function. If $\mathcal{G}$ is in addition self-complementary, it has no binding-function at all.*

Chapter 3

Normal Graphs

Normal graphs come up in a natural way in an information-theoretical context [57, 31] and are, in graph-theoretical terms, defined by cross-intersecting families of cliques and stable sets.

The interest in normal graphs is caused by the fact that they form a 'weaker variant' of perfect graphs, e.g., by means of co-normal products [56] or splitting graph entropies [31, 58], see next section for more details. This motivated Körner and de Simone [59] to ask whether the similarity of the two classes is also reflected in terms of forbidden subgraphs. In analogy to the Strong Perfect Graph Conjecture, they conjectured that every $(C_5, C_7, \overline{C}_7)$-free graph is normal (Normal Graph Conjecture).

At present, not many graphs are known to be normal. We are going to address this issue in two ways: by considering several techniques to construct normal graphs (Section 3.2) and by verifying the Normal Graph Conjecture for certain graph classes (Section 3.3).

Finally, we discuss possible other parallels between perfect and normal graphs (Section 3.4). Our results imply, unfortunately, that normal graphs cannot be characterized by means of decomposition techniques or forbidden subgraphs. Moreover, we exhibit negative consequences for the algorithmic behavior of normal graphs and bounds for certain graph parameters, reflected by the fact that neither the imperfection ratio can be bounded for normal graphs nor a χ-binding function exists. We show that the latter is also true for the class of (C_5, C_7, $\overline{C}_7$)-free graphs.

3.1 Weaker Perfect Graphs

A graph G is called *normal* if G admits a clique cover $\mathcal{Q}$ and a stable set cover $\mathcal{S}$ s.t. every clique in $\mathcal{Q}$ intersects every stable set in $\mathcal{S}$.

Two normal graphs are shown in Figure 3.1 (the bold edges are the clique covers, $\{\{1,3,5\},\{1,4,6\},\{2,4,5,7\}\}$ resp. $\{\{0,2,4,6\},\{0,3,5,7\},\{1,3,6\}\}$ the stable set covers).

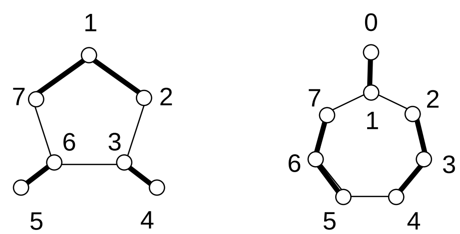


Figure 3.1: Two normal graphs

The interest in normal graphs is caused by the fact that they form, in many ways, a "weaker variant" of perfect graphs.

Recall that Berge introduced perfect graphs in 1960, motivated from Shannon's information-theoretical problem of finding the zero-error capacity of a discrete memoryless channel [97] which can be reformulated as

$$C(G) = \lim_{n\to\infty} \frac{1}{n} \log \omega(G^n)$$

regarding the asymptotic growth of the maximum cliques in the co-normal product G^n. Shannon observed that $\omega(G^n) = (\omega(G))^n$ holds for graphs G with $\omega(G) = \chi(G)$ which makes the otherwise difficult problem of determining $C(G)$ tractable. This led Berge [6] introduce perfect graphs as those graphs G, where $\omega(G')$ equals $\chi(G')$ for each induced subgraph $G' \subseteq G$.

Outgoing from the fact that $\omega(G^n) = (\omega(G))^n$ holds for all graphs G with $\omega(G) = \chi(G)$, one might expect that the class of perfect graphs is closed under taking co-normal products. This is not true as Körner and Longo [57] showed that all co-normal products of a graph G are perfect only if G is the union of disjoint cliques. However, all co-normal products of normal (and, therefore, of perfect) graphs are *normal* by [56].

Thus, normal graphs built the *closure* of perfect graphs by taking co-normal products.

Another information-theoretical link between perfect and normal graphs has been established by means of the graph entropy $H(G, p)$ w.r.t. a probability distribution p on its node set, which is defined by

$$H(G,p) = \limsup_{k\to\infty} \min \left\{ \frac{1}{k} \log_2 \chi(G^k[U]) : U \subseteq V(G^k), \sum_{x\in U} p^k(x) > 1-\epsilon \right\}$$

and is sub-additive w.r.t. complementary graphs, i.e.,

$$H(p) \leq H(G,p) + H(\overline{G},p) \ \forall p$$

holds for all graphs G. An interesting question is when the minimum $H(p)$ is attained, i.e., when equality holds rather than just sub-additivity. By Cziszár et al. [31], perfect graphs are characterized as precisely those graphs G where equality holds for *all* probability distributions, thus

$$H(p) = H(G,p) + H(\overline{G},p) \text{ for all } p \Leftrightarrow G \text{ is perfect}$$

(called strong additivity). The relaxed condition that equality holds for *at least one* nowhere vanishing probability distribution is true exactly for all normal graphs by Körner and Marton [58], that is

$$H(p) = H(G,p) + H(\overline{G},p) \text{ for at least one } p > 0 \Leftrightarrow G \text{ is normal}$$

(called weak additivity). Thus, normal graphs form a superclass of perfect graphs by means of splitting graph entropies.

Since normal graphs are "weaker" perfect graphs in several ways, Körner and de Simone [59] asked for a similarity of the two classes in terms of forbidden subgraphs. Körner [56] showed that an odd hole C_{2k+1} is normal iff $k \geq 4$. In particular, C_5 and C_7 are not normal, and so neither $\overline{C}_7$ is. These three graphs are even *minimally* not normal since all of their proper induced subgraphs are perfect and, hence, normal. This led Körner and de Simone conjecture:

Conjecture 3.1 (Normal Graph Conjecture [59]) *Any graph without C_5, C_7, or $\overline{C}_7$ as induced subgraph is normal.*

The validity of this conjecture would imply that the only minimally not normal graphs are precisely C_5, C_7, and $\overline{C}_7$ – as the only minimally imperfect graphs are precisely all odd holes and odd antiholes due to the Strong Perfect Graph Theorem. However, the validity of the Normal Graph Conjecture would provide us a *sufficient* condition for normality only, but no characterization: The non-existence of C_5, C_7, and $\overline{C}_7$ in a graph is *not* necessary for its normality, see the graphs in Figure 3.1.

Note that the Normal Graph Conjecture claims that the (C_5, C_7, $\overline{C}_7$)-free graphs built exactly the hereditary core of the normal graphs, called *strongly normal graphs*. In particular, as Körner and de Simone remark in [59], the recognition problem for the class of strongly normal graphs would be solvable in polynomial time if the Normal Graph Conjecture is true.

In order to treat the Strong Perfect Graph Conjecture from a probabilistic point of view, Prömel and Steger [86] asked for the relation of perfect and odd hole, odd antihole-free graphs on the same number of nodes. For that, they proved the following result.

Theorem 3.2 [86] Almost all C_5-free graphs are perfect.

This theorem verified the Strong Perfect Graph Conjecture asymptotically since every odd hole, odd antihole-free graph is in particular C_5-free. Since every $(C_5, C_7, \overline{C}_7)$-free graph is C_5-free as well, Theorem 3.2 implies further that almost all $(C_5, C_7, \overline{C}_7)$-free graphs are perfect and, therefore, normal. As a consequence, the Normal Graph Conjecture is *asymptotically true.*

On the other hand, this result suggests that there are not many more normal than perfect graphs. In fact, not many graphs are known yet to be normal apart from perfect graphs, odd holes, and odd antiholes of length ≥ 9. We are going to address this issue in two ways: In Section 3.2 we explore several ways to construct normal graphs outgoing from normal and even not normal ones via substitution, composition, and clique identification and consider decomposition techniques along the corresponding structures homogeneous set, skew partition, and clique cutset. Furthermore, we verify the Normal Graph Conjecture for certain graph classes, namely webs, 1-trees, cacti, and line graphs, thereby showing that those classes do contain many more normal than perfect graphs (Section 3.3).

Our results imply that normal graphs cannot be characterized by means of decomposition techniques or forbidden subgraphs (Section 3.4). Moreover, we address negative consequences for the algorithmic behavior of normal graphs and bounds for certain graph parameters, reflected by the fact that neither the imperfection ratio can be bounded for normal graphs nor a χ-binding function exists. The latter is, unfortunately, also true for the class of (C_5, C_7, $\overline{C}_7$)-free graphs.

3.2 Composing and Decomposing Normal Graphs

As observed above, the class of normal graphs is closed under taking co-normal products (by Körner [56]), yielding a closure of perfect graphs that way. In particular, taking co-normal products of known normal graphs is a way to obtain further normal graphs. In order to extend the knowledge on examples of normal graphs, other graph construction techniques are discussed in [114], namely some which are known to preserve perfection (see next subsection).

On the other hand, it is possible to decompose graphs along certain structural faults, as it was done for odd hole, odd antihole-free graphs for the proof of the Strong Perfect Graph Theorem [17]. Hence we also address the problem how to decompose normal graphs.

In contrast to perfect graphs, we obtained in [114] that it is possible to use *non-normal* components to built normal graphs and, analogously, obtain *non-normal* blocks by decomposing normal graphs in certain ways.

3.2.1 Constructing normal graphs

From the definition of normal graphs, it is clear that they are closed under taking complements, as perfect graphs are. Furthermore, both perfection and normality are closed under *substitution* by Lovász [65] and Körner, Simonyi, and Tuza [60]. Let v be a node of a graph G_1 then substituting v by another graph G_2 means to delete v and to join every neighbor of v in G_1 with every node of G_2. If G_1 and G_2 are two graphs, then substituting G_2 for *all* nodes of G_1 yields their *lexicographic product* $G_1 \times G_2$. Thus, perfect and normal graphs are closed under taking lexicographic products as well.

We are going to check for two further well-known perfection preserving graph transformations, *composition* [7] and *clique identification* [23], whether they also preserve normality.

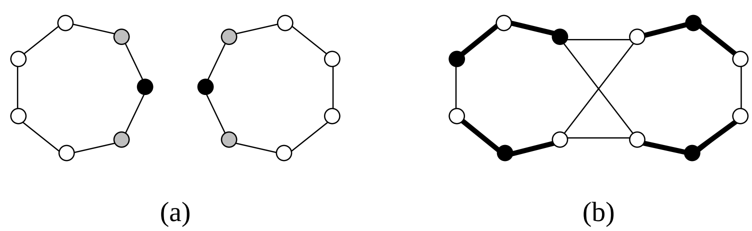


Figure 3.2: Composing two graphs

Composing two disjoint graphs G_1 and G_2 w.r.t. two nodes v_1 and v_2 means to delete v_1 and v_2 and to join every neighbor of v_1 in G_1 with every node of G_2 and every neighbor of v_2 in G_2 with every node of G_1 (see the composition of two 7-holes w.r.t. the black-filled nodes in Figure 3.2).

G arises by identification of two disjoint graphs G_1 and G_2 in a clique if there are cliques $Q_1 \subseteq G_1$ and $Q_2 \subseteq G_2$ with $|Q_1| = |Q_2|$ and a bijection $\phi : Q_1 \to Q_2$ identifying every node $v \in Q_1$ with $\phi(v) \in Q_2$, see Figure 3.3.

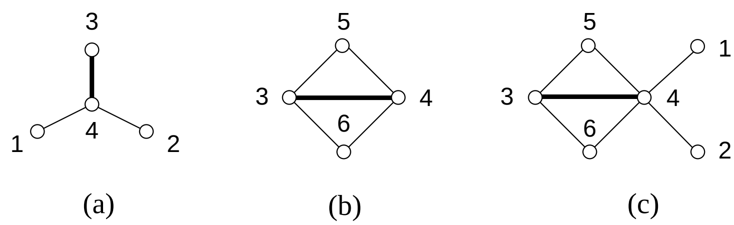


Figure 3.3: Identifying two graphs in an edge

The canonical technique to show normality for a graph G obtained in one of these ways is to construct a clique cover and a cross-intersecting stable set cover of G outgoing from the corresponding covers of its building blocks G_1 and G_2. We present several such ways to construct new normal graphs from *normal* ones. In addition, there exist normal graphs obtained by composition or clique identification where one or even both of the building blocks are *not normal*. We show that this cannot happen by applying substitutions or taking lexicographic products.

Finally, we provide a technique that allows us to construct normal graphs from *arbitrary* ones.

In the sequel, we call a clique cover $\mathcal{Q}$ (resp. stable set cover $\mathcal{S}$) of a graph G *valid* if there exists a cross-intersecting stable set cover $\mathcal{S}$ (resp. clique cover $\mathcal{Q}$) and say that $(\mathcal{Q}, \mathcal{S})$ is a *valid pair* of G.

Substitution and lexicographic products

The result of Körner, Simonyi, and Tuza [60] implies that the normality of both building blocks is sufficient for the normality of a graph constructed by substitution. We established in [114] that it is, in addition, also a necessary condition.

Lemma 3.3 *A graph G obtained by substituting a node v of a graph G_1 by G_2 is normal* only if *G_1 and G_2 are normal.*

This implies the following:

Theorem 3.4 *A graph G obtained by substituting a node of a graph G_1 by G_2 is normal if and only if G_1 and G_2 are normal.*

Corollary 3.5 *Any lexicographic product of two graphs G_1 and G_2 is normal if and only if G_1 and G_2 are normal.*

Composition and normal graphs

This subsection treats the case of composition. A first result from [114] is that the class of normal graphs is also closed under composition.

Lemma 3.6 *A graph G obtained by composing two normal graphs G_1 and G_2 w.r.t. the nodes v_1 and v_2 is normal.*

Thus, the normality of the building blocks is sufficient for the normality of the resulting graph. In contrast to the case of substitution this is, however, *not* a necessary condition. Figure 3.2 shows a normal graph obtained by composing *two non-normal* graphs, namely two 7-holes (to verify normality, take the bold edges in Figure 3.2(b) as clique cover and four stable sets of the form indicated by the black nodes).

However, the building blocks in this example are not too fare away from being normal: they are graphs G such that G has a stable set cover $\mathcal{S}$, $G-v$ has a clique cover $\mathcal{Q}_v$, for some node v, and $\mathcal{S}$ and $\mathcal{Q}_v$ are cross-intersecting. We call such a graph G *almost normal*, $(\mathcal{Q}_v, \mathcal{S})$ an *almost valid pair* of G, and v an *unnormal node* of G. Also this weaker form of normality suffices for constructing a normal graph by composition, as shown in [114]:

Lemma 3.7 *A graph G obtained by composing two almost normal graphs G_1 and G_2 w.r.t. two unnormal nodes v_1 and v_2 is normal.*

A natural question is whether there exist further ways to construct normal graphs by composition. The next lemma gives a negative answer.

Lemma 3.8 *Let G be a normal graph obtained by composing two graphs G_1 and G_2 w.r.t. the nodes v_1 and v_2.*

(1) *If G admits a valid pair $(\mathcal{Q}, \mathcal{S})$ such that $\mathcal{Q}$ contains a clique meeting neighbors of both v_1 and v_2, then G_1 and G_2 are normal;*

(2) *If G admits a valid pair $(\mathcal{Q}, \mathcal{S})$ such that $\mathcal{Q}$ contains no clique meeting neighbors of both v_1 and v_2, then G_1 and G_2 are almost normal with unnormal nodes v_1 and v_2.*

The above results finally imply the following characterization of normal graphs obtained by composition.

Theorem 3.9 *Let G be the graph obtained by composing two graphs G_1 and G_2 w.r.t. the nodes v_1 and v_2. G is normal if and only if at least one of the following conditions is satisfied:*

(1) *G_1 and G_2 are normal;*

(2) *G_1 and G_2 are almost normal with unnormal nodes v_1 and v_2.*

Clique identification and normal graphs

Similar to substitution and composition, also clique identification is a well-known perfection preserving graph transformation. As in the previous subsections one might expect that the class of normal graphs is closed under clique identification, too, and that a valid pair of the resulting graph can be constructed starting from valid pairs of its building blocks. The most natural way to do this would be the following. Consider a graph G obtained by identifying G_1 and G_2 in a clique Q^* and let $(\mathcal{Q}(G_i), \mathcal{S}(G_i))$ be valid pairs of G_i for $i = 1, 2$. Canonically, one would choose

$$\mathcal{Q}(G) = \mathcal{Q}(G_1) \cup \mathcal{Q}(G_2)$$

as clique cover of G (possibly with removing cliques properly contained in another one). Furthermore, one would combine the stable sets $S_i \in \mathcal{S}(G_i)$ according to their intersection with the common clique Q^*, i.e.,

$$\mathcal{S}(G) = \{S_1 \cup S_2 : S_i \in \mathcal{S}(G_i), S_1 \cap Q^* = S_2 \cap Q^*\}$$

to obtain a cross-intersecting stable set cover. However, this construction works *only if* $\mathcal{S}(G_1)$ and $\mathcal{S}(G_2)$ either both contain a stable set avoiding Q^* or both contain only stable sets meeting Q^*. This is not always the case, as the following example from [114] shows.

Example 3.10 The two graphs G_1 and G_2 depicted in Figure 3.3(a) and (b), respectively, admit both a *unique* valid pair, namely,

$$\begin{array}{rcl}
\mathcal{Q}(G_1) & = & \{\{1,4\},\{2,4\},\{3,4\}\} \\
\mathcal{S}(G_1) & = & \{\{1,2,3\},\{4\}\} \\
\mathcal{Q}(G_2) & = & \{\{3,4,5\},\{3,4,6\}\} \\
\mathcal{S}(G_2) & = & \{\{3\},\{4\},\{5,6\}\}.
\end{array}$$

Identifying G_1 and G_2 in the edge $\{3,4\}$ (emphasized as bold line in the picture) yields the graph G drawn in Figure 3.3(c). According to the above construction, we would obtain

$$\begin{aligned}\mathcal{Q}(G) &= \{\{1,4\},\{2,4\},\{3,4,5\},\{3,4,6\}\}\\ \mathcal{S}(G) &= \{\{1,2,3\},\{4\},\{5,6\}\}.\end{aligned}$$

Thus, $\mathcal{Q}(G)$ is a clique cover and $\mathcal{S}(G)$ a stable set cover of G, but they are not cross-intersecting. (However, G is certainly perfect and, hence, normal.)

Furthermore, similar to the case of composition, it is possible to create normal graphs by identifying two not necessarily normal graphs in a clique.

The two graphs in Figure 3.1 are examples for normal graphs, constructed by identifying a *non-normal* C_5 and C_7 with two edges and one edge, respectively. Moreover, Figure 3.4(b) shows a normal graph obtained by identifying *two non-normal* graphs in an edge (the bold edges are chosen as clique cover and $\{\{0,2,4,6,9\},\{0,3,5,7,9\},\{1,3,6,8\}\}$ as stable set cover).

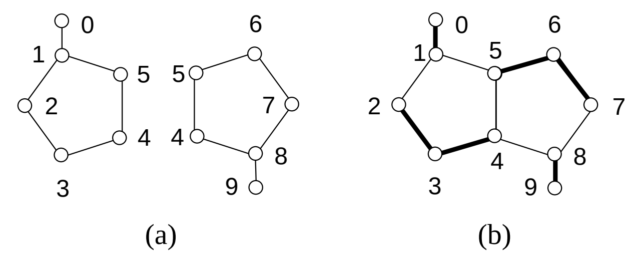


Figure 3.4: Identifying two non-normal graphs in an edge

However, the non-normal building blocks in the latter examples are again not too fare away from being normal, but almost normal. Looking at the above construction and the latter examples, it is obviously not required that both clique families cover all nodes of G_1 and G_2: it suffices if the nodes in the common clique Q^* are covered in *one* of the two building blocks.

This suggests to relax the condition of the normality for the building blocks appropriately. Let G be a graph such that G has a stable set cover $\mathcal{S}$, $G - Q'$ has a clique cover $\mathcal{Q}_{Q'}$, for some clique Q', and $\mathcal{S}$ and $\mathcal{Q}_{Q'}$ are cross-intersecting. We call such a graph G *nearly normal*, $(\mathcal{Q}_{Q'}, \mathcal{S})$ a *nearly valid pair* of G, and Q' an *unnormal clique* of G. (Note that a nearly normal graph is normal (resp. almost normal) if its unnormal clique is empty (resp. consists of one node only).)

We show in [114] that this weaker form of normality suffices to construct a normal graph by clique identification, provided the involved stable set covers are suitable.

Lemma 3.11 *Let G be obtained by identifying two nearly normal graphs G_1 and G_2 in a clique Q^* and let $Q_1, Q_2 \subseteq Q^*$ be disjoint unnormal cliques. The resulting graph G is normal if there exist nearly valid pairs $(\mathcal{Q}_{Q_i}(G_i), \mathcal{S}(G_i))$ for $i = 1, 2$ satisfying at least one of the following conditions.*

(1) *$\mathcal{S}(G_i)$ contains a stable set S with $S \cap Q^* = \emptyset$ for $i = 1, 2$;*

(2) *$\mathcal{S}(G_i)$ contains no stable set S with $S \cap Q^* = \emptyset$ for $i = 1, 2$;*

(3) *$\mathcal{S}(G_1)$ contains a stable set S with $S \cap Q^* = \emptyset$ but $\mathcal{S}(G_2)$ does not, and Q_1 is non-empty (or vice versa).*

Remark 3.12 There exist normal graphs G_1, G_2 not satisfying the above conditions (1) or (2), e.g., the two graphs G_1 and G_2 depicted in Figure 3.3(a) and (b), respectively (to check this, look at their *unique* valid pairs given in Example 3.10). However, removing the clique $\{3, 4\}$ from $\mathcal{Q}(G_1)$ yields nearly valid pairs satisfying condition (3).

Lemma 3.11 provides three *sufficient* conditions to obtain a normal graph via clique identification. It is not clear yet whether these conditions are also *necessary.* However, condition (1) of Lemma 3.11 is certainly satisfied if Q^* is a *non-maximal* clique in G_i for $i = 1, 2$: In order to cover a common neighbor v of all nodes in Q^*, $\mathcal{S}(G_i)$ has to contain a stable set S with $v \in S$ and, therefore, $S \cap Q^* = \emptyset$. This implies:

Corollary 3.13 *The class of normal graphs is closed under identifying two graphs in non-maximal cliques.*

So far no proof is known yet that clique identification preserves normality in general, but also no counterexample has been found yet.

As we have seen that we can construct normal graphs by identifying two unnormal ones in a clique (as in Figure 3.4), a natural question is whether there exist further ways to construct normal graphs by clique identification. The next lemma from [114] gives a negative answer showing that the normality or nearly normality of the building blocks is required if the resulting graph is supposed to be normal.

Lemma 3.14 *Let G be a normal graph obtained by identifying two graphs G_1 and G_2 in a clique Q^*.*

(1) *If G admits a valid pair $(\mathcal{Q}, \mathcal{S})$ such that $\mathcal{S}$ contains no stable set avoiding Q^*, then G_1 and G_2 are normal;*

(2) *If G admits a valid pair $(\mathcal{Q}, \mathcal{S})$ such that $\mathcal{S}$ contains a stable set avoiding Q^*, then G_1 and G_2 are nearly normal with unnormal cliques Q_1^* and Q_2^* such that $Q_i^* \subseteq Q^*$ and $Q_1^* \cap Q_2^* = \emptyset$.*

Constructing normal graphs from arbitrary graphs

We were finally able to describe in [114] a way to construct normal graphs from *arbitrary* ones; we include the short proof in order to give insight in the construction.

Theorem 3.15 *For any graph G, there is a normal graph G^* containing G as induced subgraph.*

Proof. Consider a graph G, a clique partition $\mathcal{Q}(G) = \{Q_1, \ldots, Q_k\}$, and a coloring $\mathcal{S}(G) = \{S_1, \ldots, S_l\}$. (The gray-filled ellipses in Figure 3.5 stand for the cliques in $\mathcal{Q}(G)$, where the black nodes represent a stable set in $\mathcal{S}(G)$.) We construct a graph G^* containing G as induced subgraph by adding, for each clique $Q_i \in \mathcal{Q}(G)$, a new node v_i and joining v_i to exactly the nodes in Q_i. Then $\mathcal{Q}(G^*) = \{Q_i \cup v_i : Q_i \in \mathcal{Q}(G)\}$ is obviously a clique partition of G^*, see Figure 3.5. We obtain a cross-intersecting stable set cover $\mathcal{S}(G^*)$ by extending each stable set $S_i \in \mathcal{S}(G)$ by those nodes v_j with $S_i \cap Q_j = \emptyset$, i.e., by constructing $S_i^* := S_i \cup \bigcup_{S_i \cap Q_j = \emptyset} v_j$, see Figure 3.5 again. These sets S_i^* are obviously stable and intersect all cliques in $\mathcal{Q}(G^*)$. Finally, $\mathcal{S}(G^*) = \{S_1^*, \ldots, S_l^*\} \cup \{v_1, \ldots, v_k\}$ covers all nodes of G^*. Hence, G^* admits the valid pair $(\mathcal{Q}(G^*), \mathcal{S}(G^*))$ and is, therefore, normal. □

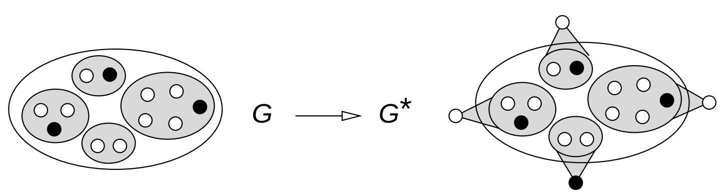


Figure 3.5: Constructing a normal graph G^* from an arbitrary graph G

3.2.2 Decomposing normal graphs

In this section we present straightforward consequences of the previous results for decomposing normal graphs alongside three structures: homogeneous sets, skew partitions, and clique cutsets.

Any graph G obtained by substituting a node v in G_1 by a graph G_2 has a partition of its node set into three subsets N_1 and $\overline{N}_1$ consisting in all neighbors and non-neighbors of v, respectively, and V_2 containing all nodes of G_2. Recall that N_1 and V_2 are totally joined, where $\overline{N}_1$ and V_2 are totally unjoined. If all three subsets are nonempty and V_2 has at least two nodes, we say that V_2 is a *homogeneous set* of G and that G can be decomposed into two blocks: the minor G_1 obtained by contracting V_2 to a single node and the subgraph G_2 induced by all nodes in V_2. Thus, a consequence of Theorem 3.4 is:

Corollary 3.16 *A graph with homogeneous set is normal if and only if its blocks are both normal.*

A graph G obtained by composing two graphs G_1 and G_2 w.r.t. nodes v_1 and v_2 has a *skew partition*: that is a partition of its node set into the four subsets $\overline{N}_1, N_1, N_2, \overline{N}_2$ where N_i and $\overline{N}_i$ stand for the neighbors and non-neighbors of v_i, respectively, for $i = 1, 2$. Recall that N_1 and N_2 are totally joined while $\overline{N}_1$ and $\overline{N}_2$ are totally unjoined. We can decompose G into two blocks: the minor G_1 induced by $N_1, \overline{N}_1$ where N_2 is contracted to a single node and the minor G_2 induced by $N_2, \overline{N}_2$ where N_1 is contracted to a single node. Hence, Theorem 3.9 implies:

Corollary 3.17 *A graph with a skew partition is normal if and only if its blocks are either both normal or both almost normal.*

Finally, a graph G obtained by identifying two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ in a clique Q has Q as *clique cutset*: that is the nodes of G can be partitioned into the three subsets V_1', V_2', Q such that $V_1' = V_1 \setminus Q$ and $V_2' = V_2 \setminus Q$ are totally unjoined. G can be decomposed into two blocks, namely the subgraphs G_i induced by V_i' and Q for $i = 1, 2$.

Thus, a consequence of Lemma 3.14 is the following:

Corollary 3.18 *A graph with a clique cutset Q is normal only if its blocks are either both normal or both nearly normal with disjoint unnormal cliques $Q_1, Q_2 \subseteq Q$.*

3.3 Progress on the Normal Graph Conjecture

Besides perfect graphs, no further graph class is known yet to be completely contained in the class of normal graphs. For odd holes and odd antiholes, we have a characterization which ones are normal (namely those with length ≥ 9). Körner and de Simone [59] found a sufficient and necessary condition for the normality of connected triangle-free graphs: the existence of nice edge covers. (An edge cover $\mathcal{F}$ of a graph G is *nice* if it is minimal w.r.t. set inclusion and if every odd cycle C of G has at least three nodes which are incident to an even number of edges in $E(C) \cap \mathcal{F}$; the bold edges in Figure 3.1 form nice edge covers.) However, this result by [59] does *not* prove the Normal Graph Conjecture for triangle-free graphs.

We are going to characterize all normal graphs within certain graph classes (webs and antiwebs in [113], two classes of sparse graphs in [90]), thereby verifying the Normal Graph Conjecture for those classes. In addition, we verify the conjecture for all line graphs.

3.3.1 Graphs with circular structure

As mentioned above, odd holes and odd antiholes constitute one of the two graph classes for which a characterization of all the normal graphs within the class was previously known. We are going to consider, as common generalization of odd holes and odd antiholes, graphs with a circular structure of their maximum cliques and stable sets, namely webs and antiwebs. Recall that a web W_n^k is the graph with nodes $1, \ldots, n$ where ij is an edge if i and j are distinct and differ by at most k mod n. W_n^1 is the hole C_n and W_{2k+1}^{k-1} the odd antihole $\overline{C}_{2k+1}$. Antiwebs are precisely the complements of webs.

In order to verify the Normal Graph Conjecture for webs and antiwebs, we characterized in [113] all the normal webs. The main idea is to consider special clique covers, construct cross-intersecting stable set covers, and prove that any web not admitting such a valid pair is not normal. Call a clique cover $\mathcal{Q} = \{Q_1, \ldots, Q_l\}$ of W_n^k *cyclic* if each clique Q_i is maximum and has a non-empty intersection with precisely the cliques Q_{i-1} and Q_{i+1} (indices taken modulo l). For any cyclic clique cover $\mathcal{Q}$ of *odd* size $l = 2t - 1$ and any node x belonging to exactly *one* clique $Q \in \mathcal{Q}$, we construct a cross-intersecting set $S(x, \mathcal{Q})$ of size t (as $\mathcal{Q} - Q$ consists of $t - 1$ pairs of intersecting cliques, we choose one node from each intersection and x itself to built $S(x, \mathcal{Q})$). Figure 3.6 shows cyclic clique covers of size 3 (resp. 5, 7) in W_{12}^4 (resp. W_{12}^3, W_{12}^2); the black nodes built such a cross-intersecting set $S(x, \mathcal{Q})$.

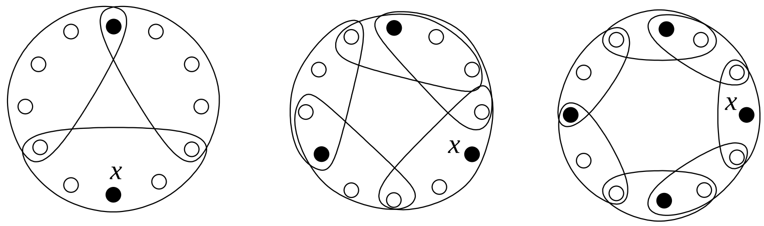


Figure 3.6: Examples of cyclic clique covers

We ensured in [113] that those sets $S(x, \mathcal{Q})$ are stable and that their union covers the whole web, if the parameters n, k, t are suitable:

Theorem 3.19 *For each $t \geq 2$, a web W_n^k with $k \geq 2$ admits*

- *a cyclic clique cover $\mathcal{Q}$ of size $2t-1$ and*
- *a cross-intersecting stable set cover $\mathcal{S}$ of stable sets of size t*

if $t(k+1) \leq n \leq (2t-1)k$ holds.

It is left to check for which webs W_n^k such a suitable t with $t(k+1) \leq n \leq (2t-1)k$ exists. We obtained that this is true for any web W_n^k with $k \geq 2$ except the cases $n = 3k+1, 3k+2$ (gap between the ranges for $t = 2$ and $t = 3$ for any $k \geq 2$) and W_{11}^2 (gap between the ranges for $t = 3$ and $t = 4$ if $k = 2$). Proving the non-normality of these webs W_{3k+1}^k, W_{3k+2}^k for $k \geq 2$ and W_{11}^2 completes the following characterization of the normal webs:

Theorem 3.20 *A web W_n^k is normal if and only if*

- *$k = 1$ and $n \neq 5, 7$,*
- *$k = 2$ and $n \neq 7, 8, 11$,*
- *$k \geq 3$ and $n \neq 3k+1, 3k+2$.*

Since the non-normal webs are not $(C_5, C_7, \overline{C}_7)$-free, Theorem 3.20 verifies the Normal Graph Conjecture for webs. The invariance of normal graphs under taking complements finally implies:

Corollary 3.21 *The Normal Graph Conjecture is true for webs and antiwebs.*

3.3.2 Sparse normal graphs

It is well-known that a random graph is with high probability perfect only if it is very sparse (i.e., it has less edges than nodes) or, due to the invariance of perfection by complementation, very dense. This relies on the result of Prömel and Steger [86] that almost all C_5-free graphs are perfect–and graphs with middle edge densities almost surely contain a C_5 as induced subgraph. This motivated us to study classes of sparse graphs w.r.t. normality in [90].

In particular, sparse graphs are known to consist of many small 1-tree components, that are connected graphs with as many edges as nodes. Thus, we start with 1-trees. Moreover, as adding random edges to such sparse graphs links different 1-trees to larger components, we extend our study to so-called cacti, obtained by linking 1-trees together.

A *1-tree* is a connected graph $G = (V, E)$ with $|V| = |E|$, obtained from a tree by adding one edge since trees are precisely the graphs with $|V| - 1 = |E|$. Hence, G contains exactly one cycle C and $G - C$ is a forest. In other words:

Observation 3.22 *A 1-tree can be obtained from a cycle and certain trees by a sequence of node-identifications.*

Since trees and cycles different from C_5 and C_7 are normal and node-identification preserves normality by Corollary 3.13, this already implies:

Corollary 3.23 *The Normal Graph Conjecture is true for 1-trees and their complements.*

Even more, we were able to fully characterize the normal 1-trees in [90]. Let $G_1 +_v G_2$ denote the graph obtained from G_1 and G_2 by identification in the node v. We obtained the following:

Theorem 3.24 *A 1-tree G is* not *normal iff one of the following holds.*

(i) $G = C_5$.

(ii) $G = C_5 +_v T$ *where T is a tree.*

(iii) $G = (C_5 +_v T) +_{v'} T'$ *where T, T' are trees and v, v' are two nodes of the C_5 at distance two.*

(iv) $G = C_7$.

This implies in particular that there are many more normal than perfect 1-trees, since a 1-tree is perfect iff its only cycle is even or a triangle, whereas almost all 1-trees are normal.

We extend this result further to the larger class of cacti. A *cactus* is a connected graph whose cycles are all edge-disjoint. Thus, a cactus $G = (V, E)$ with k cycles can be considered as a graph obtained from a tree by adding k edges in a certain way (thus cacti admit $|V| - 1 + k$ edges and are still sparse). Alternatively, we can interpret cacti as follows:

Observation 3.25 *Every cactus can be obtained from several 1-trees by a sequence of node-identifications.*

Thus, it is natural to apply our knowledge on the normality of 1-trees and the behavior of normal graphs under node-identification in order to figure out which cacti are normal.

Since all (C_5, C_7)-free 1-trees are in particular normal by Theorem 3.24 and node-identification preserves normality, we can already infer:

Corollary 3.26 *The Normal Graph Conjecture is true for cacti and their complements.*

Obviously, there also exist normal cacti containing a C_5 or a C_7 as induced subgraph, for instance the graph obtained by identifying the two graphs from Figure 3.1 in a node. Our further goal is to decide whether or not a given cactus is normal.

As a first step, we decompose a cactus G accordingly, i.e., into as many 1-trees as G has cycles. For that, we choose a set $\mathcal{C}$ of cut-nodes in G s.t. each building block $B(C)$ contains exactly one cycle C, that is each $B(C)$ is supposed to be a 1-tree. We define the *block-tree* $T(G)$ of G as follows: Take the chosen set $\mathcal{C}$ of cut-nodes and the set $\mathcal{B}$ of all the resulting building blocks $B(C)$ of G as nodes of $T(G)$; join two nodes of $T(G)$ iff one corresponds to a cut-node $q \in \mathcal{C}$, the other one to a block $B(C) \in \mathcal{B}$, and $q \in B(C)$. Figure 3.7 shows a cactus together with its block-tree.

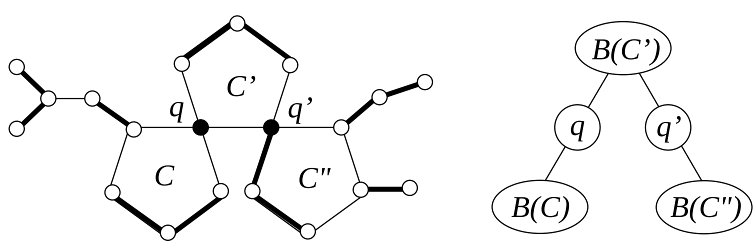


Figure 3.7: A cactus and its block-tree

As a second step, we have to classify each block $B(C)$ of G whether it is normal, almost normal, or none of them. Since each block $B(C)$ is a 1-tree, Theorem 3.24 clearly characterizes all the normal blocks; the failure cases of the theorem are at least almost normal, except the C_5 [90]:

Lemma 3.27 *A 1-tree G is almost normal if and only if $G = C_7$ or its cycle C has length 5 and there are trees attached to C on either exactly one node or on two non-consecutive nodes.*

The main idea for deciding whether or not a given cactus G is normal is to shrink $T(G)$ by a suitable sequence of node-identifications in such a way that we can classify normality of the resulting blocks in each step.

For that, we call a leaf of $T(G)$ an *end-block* of G and say that two blocks of G are adjacent if they share a cut-node in $\mathcal{C}$. In each shrinking step, we

- choose an end-block $B(C)$ and an adjacent block $B(C')$,
- identify the two blocks in their common node q,
- classify how normal the resulting subgraph $B(C) +_q B(C')$ of G is.

In later iterations, the blocks of $T(G)$ are cacti (and no more 1-trees). In order to classify the normality of such a combined block $B(C) +_q B(C')$, we use the following characterizations of normal and almost normal cacti [90]:

Lemma 3.28 *Consider two cacti G_1, G_2 and let $G = G_1 +_q G_2$.*

(1) *G is* normal *if and only if either both G_1 and G_2 are normal or one is normal and the other one almost normal with unnormal node q.*

(2) *G is* almost normal *if and only if G_1 is normal and either G_2 almost normal with all unnormal nodes $\neq q$ or $G_2 = C_5$, or if G_1 and G_2 are almost normal and q is an unnormal node in one of them.*

Starting from this characterization, we can formulate the following sufficient conditions for the *not*-normality of a cactus [90]:

Lemma 3.29 *A cactus G is* not *normal if its block-tree $T(G)$ contains*

(1) *only not normal blocks;*

(2) *a block $B(C)$ s.t. $C = C_5$ has three non-consecutive nodes of degree 2;*

(3) *an almost normal block s.t. all possible unnormal nodes have degree 2.*

There are cacti which satisfy none of the conditions of Lemma 3.29 but are *not* normal, see the graph in Figure 3.7 for an example (the bold edges show an almost nice edge cover). The reason is that "bad" cycles of length five can share a node which has degree > 2 but cannot be covered properly (as the node q in Figure 3.7). Thus, the above conditions characterize only those not normal cacti where all cycles are *node*-disjoint. In the general case, the following algorithm decides in polynomial time, whether or not a given cactus G is normal:

Algorithm (Test for normality of a cactus)
Input: a cactus G;
Output: decision whether or not G is normal.

(0.) *Initialization.* Construct the block-tree $T(G) = (\mathcal{B} \cup \mathcal{C}, E)$ of G and classify each block $B(C) \in \mathcal{B}$ as normal, almost normal, or unnormal.

(1.) *Normality test.* IF all blocks of $T(G)$ are normal, THEN output "G is normal" and STOP. IF one of the conditions in Lemma 3.29 is satisfied, THEN output "G is not normal" and STOP.

(2.) *Shrinking.* Choose an endblock $B(C)$ of $T(G)$ and (one of) its adjacent block(s) $B(C')$ with common node $q \in \mathcal{C}$. Let $B'(C') := B(C) +_q B(C')$ and determine the normality status of $B'(C')$. Shrink $T(G)$ as follows:

- IF q has degree 2 in $T(G)$
  THEN delete $B(C)$ and q and replace $B(C')$ by $B'(C')$;
- IF q has degree > 2 in $T(G)$
  THEN delete $B(C)$ and replace $B(C')$ by $B'(C')$.

GOTO step (1.).

If a cactus G has k cycles, then applying this algorithm to $T(G)$ decides in at most k iterations whether G is normal (in [90], the algorithm is exhibited in more detail and it is shown that it works indeed correctly).

Note that we use the knowledge on the normality of 1-trees only in the initialization step of the algorithm, where all further steps rely on the behavior of normal graphs under node-identification (see Lemma 3.28).

At present, no complete characterization of the normal cacti is known. However, the above results suffice to conclude that there are, indeed, many more normal cacti than perfect ones since a cactus is perfect iff all of its cycles are even or triangles, whereas almost all cacti are normal.

3.3.3 Normal line graphs

Recall that the line graph $L(G)$ of a graph G is obtained by taking the edges of G as nodes of $L(G)$ and joining two nodes of $L(G)$ if the corresponding edges of G are incident. It is well-known that a line graph $L(G)$ is perfect if and only if G has triangles as only odd cycles. The reason is, that any cycle of length k in G turns into a hole C_k in $L(G)$; moreover, no antihole of length ≥ 7 can occur in any line graph. Therefore, verifying the Normal Graph Conjecture for line graphs means to ensure that a line graph $L(G)$ is normal whenever G has no cycles of length 5 or 7. This should be done by exhibiting a clique cover $\mathcal{Q}$ and a cross-intersecting stable set cover $\mathcal{S}$ in every (C_5, C_7)-free line graph $L(G)$. As stable sets in $L(G)$ come from matchings in G and cliques from sets of pairwise adjacent edges, this is, in terms of the root graph, equivalent to the following [95]:

Observation 3.30 *A line graph $L(G)$ is normal if and only if G contains a family $\mathcal{S}$ of matchings and a cross-intersecting family $\mathcal{Q}$ of edge-stars and triangles such that both families cover all edges of G.*

The latter family $\mathcal{Q}$ an be represented as follows. In a graph $G = (V, E)$, let $\mathrm{S}(v) = \{e \in E : e = uv\}$ be the edge-star with central node v and define for a node subset $V' \subseteq V$ the star cover $\mathrm{ES}(V') = \{\mathrm{ES}(v) : v \in V'\}$ where

$$\mathrm{ES}(v) = \begin{cases} T & \text{if there is a triangle } T \text{ with } \mathrm{S}(v) \subset T \\ \mathrm{S}(v) & \text{otherwise.} \end{cases}$$

If V' is a *vertex cover* of G, i.e., a node subset meeting all edges, then $\mathrm{ES}(V')$ clearly covers all edges of G. We introduced in [95] a domination relation of vertex covers and called a vertex cover *good* if it is maximal w.r.t. this relation. The main result from [95] is:

Theorem 3.31 *In any graph G without cycles of length 5 or 7, there exist, for every good vertex cover V' of G, a family of matchings covering all edges of G and cross-intersecting* $\mathrm{ES}(V')$.

This immediately implies:

Corollary 3.32 *The Normal Graph Conjecture is true for line graphs and their complements.*

As the proof is constructive, it can be used to design an algorithm which explicitly constructs the required covers for a given graph without cycles of length 5 or 7 [96].

At present, it is open to fully characterize which line graphs are normal, but we can certainly conclude that there are many more normal than perfect line graphs since a line graph $L(G)$ is perfect iff all cycles of G are even or triangles, whereas the non-existence of cycles of length 5 or 7 in G suffices to make $L(G)$ normal.

3.4 Some Consequences

This section presents some consequences from the above results. On the one hand, the results from the last section indicate that there are many more normal than perfect graphs. On the other hand, we are going to infer from the results in Section 3.2 that normal graphs are not as rich as perfect graphs but indeed only "normal".

The results and examples from Section 3.2 show that there is no chance to characterize normal graphs in a constructive way, by gluing together certain building blocks, as we can also use *not* normal graphs as building blocks. Similarly, we cannot expect to characterize normal graphs by decomposition techniques, as Section 3.2.2 shows the existence of *not* normal blocks if a normal graph has a skew partition or a clique cutset. In addition, Theorem 3.15 implies the following.

Corollary 3.33 *Normal graphs cannot be characterized by means of forbidden subgraphs.*

Another consequence of Theorem 3.15 is that normal graphs can contain arbitrarily "bad" graphs as induced subgraphs. Thus, we cannot expect a better algorithmic behavior of normal graphs than this of general graphs. Moreover, we also cannot expect to find for normal graphs as good bounds for certain interesting graph parameters as for perfect graphs. This is, e.g, reflected by means of χ-binding functions, the imperfection ratio, and the graph-entropy for normal graphs and even for $(C_5, C_7, \overline{C}_7)$-free graphs, as shown in the sequel.

Recall that Mycielski [70] constructed a series of graphs $G_0, G_1, G_2, \ldots$ with $\omega(G_i) = 2$ for all $i \geq 0$ but $\chi(G_i) = 2 + i$. According to Theorem 3.15 we can construct a series of normal graphs $G_0^*, G_1^*, G_2^*, \ldots$ containing all Mycielski graphs $G_0, G_1, G_2, \ldots$, thus

$$\chi(G_i^*) = 2 + i \to \infty \text{ for } i \to \infty$$

follows and implies:

Corollary 3.34 *There exists no χ-binding function for the class of normal graphs.*

Recall that Gerke and McDiarmid [45] introduced the imperfection ratio

$$\operatorname{imp}(G) = \max\left\{\frac{\chi_f(G,c)}{\omega(G,c)} \mid c : V(G) \to \mathbb{N} \setminus \{0\}\right\}$$

as some asymptotic slope of a χ-binding function and that Simonyi [100] established the following link to the graph entropy

$$\log_2 \operatorname{imp}(G) = \max\{H(p) - H(G,p) - H(\overline{G},p) : p\}$$

for any graph G. Every perfect graph G has $\operatorname{imp}(G) = 1$ and one might expect that normal graphs have an imperfection ratio close to 1 (as they are weakly splitting). However, this is not true due to the following reason.

Gerke and McDiarmid [45] studied the behavior of the imperfection ratio under taking lexicographic products $G \times H$ and showed that

$$\operatorname{imp}(G \times H) = \operatorname{imp}(G) \cdot \operatorname{imp}(H)$$

holds for the imperfection ratios. Thus, the imperfection ratio cannot be bounded for any class $\mathcal{G}$ of graphs which is closed under substitution (and, therefore, closed under taking lexicographic products) and contains at least one imperfect graph G as

$$\operatorname{imp}(G^i) \to \infty \text{ for } i \to \infty$$

if $\operatorname{imp}(G) > 1$ (where G^i stands for $G \times \ldots \times G$, i times). As normal graphs are closed under substitution, this implies the following.

Corollary 3.35 *The imperfection ratio cannot be bounded for the class of normal graphs.*

This consequence is fairly unexpected as it shows in particular the existence of normal graphs G where the difference between

$$\max\{H(p) - H(G,p) - H(\overline{G},p) : p\}$$

and

$$0 = \min\{H(p) - H(G,p) - H(\overline{G},p) : p\}$$

taken over all positive probability distributions p tends to *infinity*.

Even worse, the latter observations are also true for the class $\mathcal{G}$ of $(C_5, C_7, \overline{C}_7)$-free graphs due to the following reason. It is easy to check that $\mathcal{G}$ is closed under substitution. Thus, $\mathcal{G}$ is also closed under taking lexicographic products. For instance, the C_9 is clearly $(C_5, C_7, \overline{C}_7)$-free and

$$\text{imp}((C_9)^k) \to \infty \text{ if } k \to \infty$$

follows. This implies:

Corollary 3.36 *The imperfection ratio cannot be bounded for $(C_5, C_7, \overline{C}_7)$-free graphs.*

In addition, the class $\mathcal{G}$ is also closed under replication and $\mathcal{G}$ is, therefore, simple (as it suffices to consider $\chi_f(G,c)$ and $\omega(G,c)$ in their unweighted versions in order to determine $\text{imp}(\mathcal{G})$). Recall from Section 2.3 that, for any simple class with unbounded imperfection ratio, there is no linear χ-binding function (Observation 2.30) and, thus, no complementary binding-function due to Gyárfás [51]. As the class of $(C_5, C_7, \overline{C}_7)$-free graphs is clearly self-complementary, we infer from Corollary 2.32:

Corollary 3.37 *There is no χ-binding function for the class of $(C_5, C_7, \overline{C}_7)$-free graphs.*

Thus, the validity of the Normal Graph Conjecture would certainly provide us a sufficient condition for normality and characterize the hereditary core of the normal graphs, but we even cannot expect nice properties of this special subclass of normal graphs.

As a consequence, we conclude that normal graphs are not as close to perfect graphs as expected–even not in the information-theoretical context of splitting graph entropies, since for a normal graph G the value

$$H(p) - H(G,p) - H(\overline{G},p)$$

strongly depends on the probability distribution p. In contrary, for each minimally imperfect graph G (including the non-normal graphs $C_5, C_7, \overline{C}_7$) we have the small range

$$0 \leq H(p) - H(G,p) - H(\overline{G},p) \leq \log_2\left(\frac{|G|}{|G|-1}\right)$$

for *all* probability distributions p by [45]. This suggests to consider graph classes $\mathcal{G}$ as close to perfection by means of splitting entropies if there is a small upper bound u for $\text{imp}(\mathcal{G})$ since

$$0 \leq H(p) - H(G,p) - H(\overline{G},p) \leq \log_2 u$$

holds for all (normal and non-normal) graphs $G \in \mathcal{G}$ and for all p.

Chapter 4

Rank-Perfect Graphs and Beyond

For all perfect graphs, the stable set polytope STAB(G) coincides with the clique constraint polytope QSTAB(G), whereas STAB(G) $\subset$ QSTAB(G) holds for all imperfect graphs G. Thus, besides clique constraints, additional facets are required to describe the stable set polytope STAB(G) of any imperfect graph.

Following a suggestion of Grötschel, Lovász, and Schrijver [50] one may relax perfection by generalizing clique constraints to other classes of inequalities and investigating all graphs whose stable set polytope is entirely described by nonnegativity constraints and the inequalities in question. We follow up this concept in two different ways: by considering 0/1-constraints associated with arbitrary subgraphs, called *rank constraints*, and constraints associated with families of cliques, called *clique family inequalities*.

Recall that Padberg characterized those imperfect graphs for which the difference between STAB(G) and QSTAB(G) is smallest possible [75, 76]. We develop this idea further and define three polytopes between STAB(G) and QSTAB(G) by allowing certain types of (lifted) rank constraints only. We obtain a chain of three superclasses of perfect graphs and survey in Section 4.1 which graphs are known to belong to one of those superclasses.

Second, we follow the concept of clique family inequalities, investigating valid inequalities for the stable set polytope which rely on the intersection of cliques within the family. As clique family inequalities can be seen as generalization of these constraints describing the matching polytope, there is a strong link to line graphs and their superclasses as, e.g., quasi-line graphs and claw-free graphs, see Section 4.2.

4.1 Rank constraints and rank-perfect graphs

If G is an imperfect graph, $\mathrm{STAB}(G) \subset \mathrm{QSTAB}(G)$ holds and the difference between $\mathrm{STAB}(G)$ and $\mathrm{QSTAB}(G)$ indicates how far a graph is away from being perfect. Recall that, in this sense, Padberg [75, 76] gave the following polyhedral characterization of minimally imperfect graphs: G is minimally imperfect if and only if $\mathrm{QSTAB}(G)$ has *precisely one* fractional vertex which can be cut off by *exactly one* cutting plane, namely, the *full rank constraint*

$$\sum_{i \in G} x_i \;\leq\; \alpha(G) \tag{4.1}$$

associated with G itself. This shows that, for any minimally imperfect graph G, the polytope $\mathrm{QSTAB}(G)$ is as close to $\mathrm{STAB}(G)$ as possible and, hence, minimally imperfect graphs are indeed "almost perfect". We are going to generalize this further.

The next possible case is when $\mathrm{QSTAB}(G)$ may have more than one fractional vertex but, again, the full rank constraint is required as only cutting plane to cut off all of them. This lead Shepherd [98], inspired by Padberg's results, to define near-perfect graphs. Let denote $\mathrm{FSTAB}(G)$ the polytope given by all nonnegativity and clique constraints together with the full rank constraint (4.1). Shepherd [98] called a graph G *near-perfect* if $\mathrm{STAB}(G) = \mathrm{FSTAB}(G)$. Minimally imperfect graphs are obviously near-perfect. Since there is no requirement that $\mathrm{QSTAB}(G)$ has at least one fractional vertex but only that all fractional vertices are cut off by the full rank constraint, perfect graphs are near-perfect, too, see Section 4.1.1 for more examples and considerations on near-perfect graphs.

The next natural step is to generalize clique constraints and the full rank constraint by considering *all* 0/1-inequalities, the *rank constraints*

$$\sum_{i \in G'} x_i \;\leq\; \alpha(G') \tag{4.2}$$

associated with arbitrary induced subgraphs $G' \subseteq G$ (note $\alpha(G') = 1$ holds iff G' is a clique). Every rank constraint is obviously valid for the stable set polytope, hence, the polytope $\mathrm{RSTAB}(G)$ given by all nonnegativity and all rank constraints is a further relaxation of $\mathrm{STAB}(G)$ but contained in $\mathrm{FSTAB}(G)$. We define all graphs G with $\mathrm{STAB}(G) = \mathrm{RSTAB}(G)$ to be *rank-perfect* (i.e., a graph is rank-perfect if we need only 0/1-inequalities to cut off all fractional vertices of $\mathrm{QSTAB}(G)$). Every perfect, every minimally imperfect, and every near-perfect graph is obviously also rank-perfect. Further classes of rank-perfect graphs are discussed in Section 4.1.2.

If a rank constraint is associated with a *proper* subgraph $G' \subset G$, then it does not yield a facet of STAB(G) in general, even if $\sum_{i \in G'} x_i \leq \alpha(G')$ is facet-defining for STAB(G'). In the latter case, we can determine a facet

$$\sum_{i \in G'} x_i + \sum_{i \in G - G'} a_i \, x_i \; \leq \; \alpha(G') \tag{4.3}$$

of the stable set polytope of the whole graph G by computing appropriate coefficients a_i for all nodes i in $G - G'$. We call facets of this form (4.3) *weak rank constraints* if the base rank constraint associated with G' is facet-defining for STAB(G'). Examples are lifted rank constraints where an orthogonal projection is the full rank facet of STAB(G'), and complete joins of different rank facet-producing subgraphs. Clearly, facet-defining rank constraints are weak rank constraints with $a_i = 0$ for $i \in G - G'$. Let WSTAB(G) be the polytope given by all nonnegativity and all weak rank constraints. WSTAB(G) is a further relaxation of STAB(G) but contained in RSTAB(G). We define all graphs G with STAB(G) = WSTAB(G) to be *weakly rank-perfect* (see Section 4.1.3 for classes of weakly rank-perfect graphs).

Moreover, the stable set polytope itself is entirely described by all "trivial" facets $x_i \geq 0$ for all nodes i and all "nontrivial" facets of the general form

$$\sum_{i \in G} a_i \, x_i \; \leq \; \alpha(G, a) \tag{4.4}$$

where we interpret the vector $a = (a_1, \ldots, a_n)$ as a node weighting of G associating the weight a_i to $i \in G$ and denote the weighted graph by (G, a). Furthermore, $\alpha(G, a) = \max\{\sum_{i \in S} a_i : S \subseteq G \text{ stable set}\}$ stands for the weighted stability number. Thus, there is no further relaxation of STAB(G) possible beyond WSTAB(G). By the obtained chain of relaxations of STAB(G)

$$\text{STAB}(G) \subseteq \text{WSTAB}(G) \subseteq \text{RSTAB}(G) \subseteq \text{FSTAB}(G) \subseteq \text{QSTAB}(G)$$

we have finally achieved a hierarchy of polyhedral superclasses of perfect graphs: near-perfect, rank-perfect, and weakly rank-perfect graphs. The difference between QSTAB(G) and the largest polytope coinciding with STAB(G) gives us some information on the stage of imperfection. In the following subsections, we present several aspects of these three classes, and close with some remarks on general, i.e., on not weakly rank-perfect graphs.

4.1.1 Near-perfect graphs

We start with the class of graphs which is, in the polyhedral sense, the smallest superclass of perfect graphs since we only have to add the full rank constraint (4.1) to QSTAB(G) in order to arrive at STAB(G).

As shown by Padberg [75, 76], minimally imperfect graphs are examples of near-perfect graphs, see Section 1.2.4. Before the characterization of minimally imperfect graphs via the Strong Perfect Graph Conjecture was settled, Shepherd [98] found a further polyhedral characterization of minimally imperfect graphs in terms of near-perfection:

Theorem 4.1 (Shepherd [98]) *An imperfect graph G is minimally imperfect if and only if both G and $\overline{G}$ are near-perfect.*

That means, the part of the class of near-perfect graphs which is closed under complementation consists exactly in all perfect and all minimally imperfect graphs. The further goal is to learn more about the remaining part of the class. We call a near-perfect graph G *proper* if G is neither perfect nor minimally imperfect. For such graphs, we proved in [62]:

Lemma 4.2 *If G is a properly near-perfect graph, then both QSTAB(G) and QSTAB($\overline{G}$) have at least two fractional extreme points.*

In order to be (properly) near-perfect, an imperfect graph G has obviously to satisfy the condition that every minimally imperfect subgraph G' of G has the same stability number as G, and every lifting of the rank constraint associated with G' should result in the full rank facet of G. Shepherd [98] conjectured that this property characterizes near-perfection and showed that his conjecture is true if odd holes and odd antiholes are the only minimally imperfect graphs. Hence, by the Strong Perfect Graph Theorem, we have:

Theorem 4.3 (Shepherd [98], Chudnovsky et al [17]) *A graph G is near-perfect if and only if each lifting of a rank constraint associated with a minimally imperfect subgraph of G yields the full rank facet $x(G) \leq \alpha(G)$.*

So fare, no graph-theoretical characterization of near-perfect graphs is known and, besides perfect and minimally imperfect graphs, no other graph class is known to belong (completely) to the class of near-perfect graphs. Hence, it is of interest to study the intersection of near-perfect graphs with such graph classes which are in some sense close to perfect or minimally imperfect graphs. The first result in this direction was obtained by Shepherd for graphs with stability number two.

Theorem 4.4 (Shepherd [98]) *A graph G with $\alpha(G) = 2$ is near-perfect if and only if the neighborhood of every node of G induces a perfect graph.*

This is, e.g., true for all quasi-line graphs G with $\alpha(G) = 2$, as in a quasi-line graph the neighbors of every node induce a co-bipartite graph.

We studied in [111] the intersection of near-perfect graphs with three classes all generalizing the odd holes and odd antiholes: partitionable graphs, webs, and antiwebs.

For every partitionable graph G it is known from Bland, Huang, and Trotter [8] that G and $\overline{G}$ produce the full rank facet, but at most one of G and $\overline{G}$ is near-perfect by Theorem 4.1. Even more, we proved in [111]:

Theorem 4.5 *A partitionable graph G is near-perfect if and only if G is minimally imperfect.*

Hence, being near-perfect is a so-called genuine property that holds *exactly* for all minimally imperfect graphs, but for *none* of the other partitionable graphs. In addition, we have from [62]:

Lemma 4.6 *If G is a partitionable graph but not minimally imperfect, then $STAB(G)$ has at least two non-trivial, non-clique facets and $QSTAB(G)$ has at least two fractional extreme points.*

Next we study two classes which contain all odd holes, all odd antiholes, and many partitionable graphs: the webs and antiwebs. Trotter [104] showed that a web W_n^{k-1} produces the full rank facet iff k is not a divisor of n while the same is true for antiwebs $\overline{W}_n^{k-1}$ iff k and n are relatively prime. We studied in [111] for which webs and antiwebs the full rank facet is the only non-trivial, non-clique facet of the stable set polytope.

Theorem 4.7 *A web is near-perfect if and only if it is perfect, an odd hole, W_{11}^2, or if it has stability number two.*

As all webs are quasi-line, it follows from Theorem 4.4 that all webs with stability number two are near-perfect. In contrary, there are no properly near-perfect antiwebs [111]:

Theorem 4.8 *An antiweb is near-perfect if and only if it is perfect, an odd hole, or an odd antihole.*

4.1.2 Rank-perfect graphs

We now turn to the next superclass of perfect graphs: the class of rank-perfect graphs G where 0/1-inequalities of the form (4.2)

$$x(G') \leq \alpha(G')$$

with $G' \subseteq G$ are needed as only nontrivial facets to describe STAB(G). Since clique constraints are special rank constraints (namely those with $\alpha(G') = 1$), all perfect graphs are rank-perfect in particular. Furthermore, all near-perfect graphs are obviously rank-perfect, too. We survey which other classes of rank-perfect graphs are known.

Chvátal [21] called a graph G *t-perfect* if STAB(G) has rank constraints associated with edges and odd holes as only nontrivial facets. (Note that "t" stands for "trou", the French word for hole, and that every C_{2k+1} with $k \geq 1$ is here considered to be a hole.) Bipartite graphs without isolated nodes are obviously t-perfect. Chvátal conjectured in [21] and Boulala and Uhry proved in [10] that *series-parallel graphs* are t-perfect (that are graphs obtained from disjoint cycle-free subgraphs by repeated application of the following two operations: adding a new edge parallel to an existing edge and subdividing edges, i.e., replacing edges by paths). Further examples of t-perfect graphs are *almost-bipartite graphs* (having a node whose deletion leaves the graph bipartite) due to Fonlupt and Uhry [37] and *strongly t-perfect graphs* (having no subgraph obtained from subdividing edges of a K_4 such that all four cycles corresponding to the triangles of the K_4 are odd) due to Gerards and Schrijver [43]. Further investigations of t-perfect graphs without certain subdivisions of K_4 can be found in Gerards and Shepherd [44]. We identified in [27] *outerplanar graphs* as a new class of strongly t-perfect graphs (recall that such graphs can be embedded into the plane without edge crossing and such that all nodes lay in the outer face).

Theorem 4.9 *Every outerplanar graph is strongly t-perfect.*

By definition, a natural generalization of t-perfect graphs is the class of *h-perfect graphs* (from hole-perfect) where, besides nonnegativity constraints, rank constraints associated with cliques of arbitrary size and odd holes suffice to describe the associated stable set polytopes [50]. One class of nontrivial h-perfect graphs (that are neither perfect, nor t-perfect, nor combinations of these) is the class of (P_5,diamond)-free graphs due to Arbib and Mosca [2]. (For combinations, see Fonlupt and Uhry [37] and Sbihi and Uhry [93].)

One could generalize this concept further to *co-h-perfect* graphs, where rank constraints associated with cliques, odd holes, and odd antiholes are used, or to *p-perfect* graphs, taking rank constraints associated with cliques and arbitrary partitionable graphs. (Note: partitionable graphs are not rank-perfect in general, as observed in [111].)

The key property of t-perfect and h-perfect graphs is that the stable set problem can be solved in polynomial time due to [50]. The reason is that the separation problems are polynomial time solvable for edge and odd hole constraints as well as for orthonormal representation constraints (and, therefore, clique constraints) and odd hole constraints. This result can be extended to co-h-perfect graphs, as a superclass of odd antihole constraints is polynomial time separable [68], but so far not to p-perfect graphs, as at least the separation for partitionable antiweb constraints is NP-hard [14].

Whereas all the above classes are rank-perfect by definition, there are also several "natural" classes of rank-perfect graphs: line graphs [33], semi-line graphs [19], antiwebs [110], convex-round graphs [27], and complements of fuzzy circular interval graphs [112].

Recall that a *line graph* is obtained by taking the edges of an original graph as nodes and connecting two nodes iff the original edges are incident; all facets of the stable set polytopes are known from matching theory [33] implying that line graphs are rank-perfect, (see Section 4.2.2 for details).

Chudnovsky and Seymour [19] recently extended this result to all quasi-line graphs not being fuzzy circular interval graphs. Let $\mathcal{C}$ be a circle and $\mathcal{I} = \{I_1, \ldots, I_m\}$ be a collection of intervals $I_k = [l_k, r_k]$ in $\mathcal{C}$ s.t. there is no proper containment of intervals in $\mathcal{I}$ and no two intervals share an endpoint. Further, take a finite multiset $V = \{v_1, \ldots, v_n\}$ of points in $\mathcal{C}$ (i.e., $v_i \in \mathcal{C}$ may occur in V with a multiplicity > 1). The *fuzzy circular interval graph* $G(V, \mathcal{I}, \mathcal{C}) = (V, E_1 \cup E_2')$ has node set V and edge set $E_1 \cup E_2'$ where

$$\begin{aligned} E_1 &= \{v_i v_j : \exists I_k \in \mathcal{I} \text{ with } v_i, v_j \in I_k \text{ and } \{v_i, v_j\} \neq \{l_k, r_k\}\} \\ E_2' &\subseteq \{v_i v_j : \exists I_k \in \mathcal{I} \text{ with } v_i = l_k, v_j = r_k\} = E_2. \end{aligned}$$

We call a graph *semi-line* if it is a line graph or a quasi-line graph not representable as fuzzy circular interval graph. Chudnovsky and Seymour [19] showed that semi-line graphs are rank-perfect.

Note that there indeed exist quasi-line graphs which are neither line graphs nor fuzzy circular interval graphs: in Figure 4.1, the gray-filled nodes induce an obstruction for line graphs, the squared nodes an obstruction for fuzzy circular interval graphs. Thus, semi-line graphs are a proper superclass of line graphs.

For both classes, the stable set problem can be solved in polynomial time, using Edmonds' matching algorithm for stable sets in line graphs and using the extensions of this algorithm for general claw-free graphs.

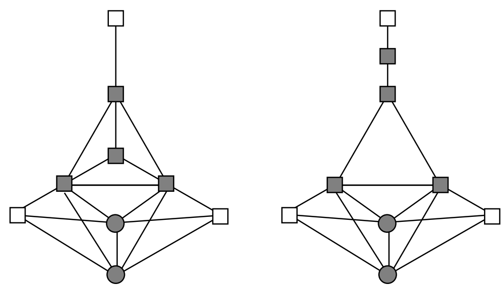


Figure 4.1: Quasi-line graphs not being fuzzy circular interval or line graphs

In [110], we could show that antiwebs are rank-perfect. More precisely:

Theorem 4.10 *The stable set polytope of an antiweb is given by nonnegativity constraints and rank constraints associated with cliques and prime antiwebs only.*

Recall that an antiweb $K_{n/\alpha}$ is prime if $\gcd(n, \alpha) = 1$ and that antiwebs include all cliques $K_k = K_{k/1}$, all odd antiholes $\overline{C}_{2k+1} = K_{2k+1/2}$, and all odd holes $C_{2k+1} = K_{2k+1/k}$. This motivated us to introduce, as common generalization of perfect, t-perfect, h-perfect graphs, and co-h-perfect graphs, the class of *a-perfect graphs* as those graphs whose stable set polytopes are given by nonnegativity constraints and rank constraints associated with cliques and prime antiwebs only (the "a" stands for "antiweb").

We exhibited two further classes of a-perfect graphs: complements of fuzzy circular interval graphs [112] and convex-round graphs [27].

Theorem 4.11 *Co-fuzzy circular interval graphs are a-perfect.*

Recall that convex-round graphs admit a node labeling in a cyclic order such that all neighbors of any node are consecutive w.r.t. this order. As antiwebs obviously admit such an order, convex-round graphs constitute a proper superclass. We showed in [27] that convex-round graphs are special complements of fuzzy circular interval graphs. This implies:

Theorem 4.12 *Every convex-round graph is a-perfect.*

Unfortunately, there is no polynomial time algorithm known to solve the stable set problem for a-perfect graphs, as the separation problem for antiweb constraints is NP-hard by [14].

4.1.3 Weakly rank-perfect graphs

This section deals with the graphs G where, besides nonnegativity constraints, only weak rank constraints (4.3) of the form

$$x(G', \mathbb{1}) + \mathrm{x(G - G', a)} \leq \alpha(\mathrm{G}', \mathbb{1})$$

are required to describe STAB(G). Recall that such inequalities are obtained from the base rank constraint associated with a facet producing subgraph $G' \subseteq G$, i.e., that $x(G', \mathbb{1}) \leq \alpha(\mathrm{G}', \mathbb{1})$ produces the full rank facet of STAB(G'). Since every facet-defining rank constraint $x(G', \mathbb{1}) \leq \alpha(\mathrm{G}', \mathbb{1})$ is a weak rank constraint with $a_i = 0$ for $i \in G - G'$, the class of weakly rank-perfect graphs contains all rank-perfect graphs (and, therefore, all near-perfect and all perfect graphs).

One general way to arrive at classes of weakly rank-perfect graphs is the following. Consider a class of rank-perfect graphs where only nonnegativity constraints and rank constraints associated with subgraphs of a certain type $\mathcal{C}$ are required to describe the stable set polytope. Then define the "corresponding" class of weakly $\mathcal{C}$-perfect graphs by taking weak rank constraints based on those special rank constraints as only nontrivial facets of the stable set polytope. E.g., the class of *weakly h-perfect graphs* can be defined that way to contain all graphs whose stable set polytope is given by nonnegativity constraints, clique constraints, and lifted odd hole constraints. (See Padberg [74] for a general description of the sequential lifting procedure.)

The 5-wheel in Figure 4.2(a) and the graph in Figure 4.2(b) are examples of weakly h-perfect graphs which are not h-perfect. (Note that the classes of weakly t-perfect and weakly h-perfect graphs coincide since clique constraints are liftings of edge constraints.)

Note that the lifted odd hole constraint associated with an odd wheel can also be considered as a complete join facet, obtained by joining two rank facet-producing subgraphs (namely, a clique and an odd hole).

Thus, outgoing from a class of $\mathcal{C}$-perfect graphs, we can also define all *joined $\mathcal{C}$-perfect graphs* as those graphs where all nontrivial facets of the stable set polytope arise from complete joins of graphs $G_1, \ldots, G_k \in \mathcal{C}$ and

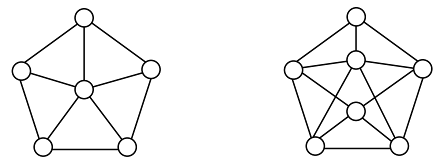


Figure 4.2: Two weakly h-perfect graphs

have the form

$$\sum_{i \le k} \frac{1}{\alpha(G_i)} x(G_i) \le 1 \tag{4.5}$$

(here we normalize the rank constraints associated with all subgraphs G_i to have right hand side equal to 1).

A natural graph class of this type was found by Shepherd [99]. A graph G is *near-bipartite* if removing all neighbors of an arbitrary node leaves the graph bipartite. (That means, for all nodes v of G, the set $G - N(v)$ can be partitioned into two stable sets and near-bipartite graphs are, therefore, the complements of quasi-line graphs.)

Theorem 4.13 (Shepherd [99]) *The only nontrivial facets of stable set polytopes of near-bipartite graphs are constraints*

$$\sum_{i \le k} \frac{1}{\alpha(A_i)} x(A_i) + x(Q) \le 1$$

associated with complete joins of prime antiwebs $A_1, \ldots, A_k$ and a clique Q.

Recall that an antiweb $\overline{W}_n^{k-1}$ is prime if k and n are relatively prime. In particular, all odd holes and odd antiholes are prime. To illustrate Theorem 4.13, consider the complete join of a C_7 and a $\overline{C}_7$ as near-bipartite graph G. Its stable set polytope has as nontrivial facets

$$\begin{array}{rll} x(Q) & \le 1 & \forall \text{ maximal cliques } Q \subseteq G \\ x(\overline{C}_7) + 2x(Q) & \le 2 & \forall \text{ maximal cliques } Q \subseteq C_7 \\ x(C_7) + 3x(Q) & \le 3 & \forall \text{ maximal cliques } Q \subseteq \overline{C}_7 \\ 2x(C_7) + 3x(\overline{C}_7) & \le 6 & \end{array}$$

i.e., constraints (4.5) associated with either a maximal clique of G, the complete join of one antiweb and a maximal clique of the other antiweb, or the complete join of both antiwebs (here, the facets are scaled to have integer coefficients).

The above result shows in particular that near-bipartite graphs form a class of joined a-perfect graphs. For subclasses of near-bipartite graphs, the facet sets can be simpler as in the above theorem only if certain kinds of prime antiwebs or certain complete joins can be excluded.

Shepherd proved in [99] further that odd antiholes are the only prime antiwebs occurring in complements of line graphs. Since all odd antiholes have stability number two, only 0,1,2-valued facets are required for their stable set polytopes:

Theorem 4.14 (Shepherd [99]) *The only nontrivial facets of stable set polytopes of complements of line graphs are constraints*

$$\sum_{i \leq k} x(A_i) + 2x(Q) \leq 2$$

associated with complete joins of odd antiholes $A_1, \dots, A_k$ and a clique Q.

Hence, complements of line graphs are joined p-perfect graphs.

Remark 4.15 Antiwebs are a subclass of near-bipartite graphs, but the occurrence of certain antiwebs in this class can clearly not be excluded. In [110] it was shown that no antiweb contains the complete join of a (smaller) prime antiweb and a single node. Thus, neither complete joins of two prime antiwebs nor of a prime antiweb and a non-empty clique are possible, implying that any facet (4.5) has only one component and is, therefore, a 0,1-valued rank constraint. In [112] this result was extended to the larger class of complements of fuzzy circular interval graphs.

In contrary to antiwebs, the webs (and, thus, also fuzzy circular interval graphs and general quasi-line graphs) are highly not rank-perfect [80]. A recent result of Stauffer [101] implies that webs are weakly rank-perfect; we address this question for fuzzy circular interval graphs in Section 4.2.2.

Further, a description of the facet-system of $\mathrm{STAB}(G)$ for all graphs G with $\alpha(G) = 2$ was found (but not published) by Cook, see [98]. He showed that the stable set polytope of graphs G with $\alpha(G) = 2$ is given by nonnegativity constraints and weak rank constraints of the form

$$x(N'(Q)) + 2x(Q) \leq 2 \tag{4.6}$$

for every clique Q where $N'(Q)$ denotes the set of all nodes v of G with $Q \subseteq N(v)$. We call inequalities of this type *clique neighborhood constraints.*

Hence, graphs G with $\alpha(G) = 2$ are weakly rank-perfect, too. In order to figure out which graphs G with $\alpha(G) = 2$ are rank-perfect, we determine which rank facets may appear. The inequalities (4.6) can be scaled to have no coefficients different from 0 and 1 only if Q is maximal (then $N'(Q) = \emptyset$ follows) or Q is empty (then $N'(Q) = V(G)$ follows). Thus, the only possible rank facets are maximal clique facets and the full rank facet. Hence, we have obtained (see e.g. [111]):

Corollary 4.16 *A graph G with $\alpha(G) = 2$ is rank-perfect if and only if G is near-perfect.*

For further consideration on stable set polytopes of graphs with stability number two, see also Section 4.2.4.

4.1.4 Beyond weakly rank-perfect graphs

We finally discuss for which graph classes considered to be close to perfect graphs in some sense this relation is also reflected in polyhedral terms.

This is clearly true for minimally imperfect graphs by Padberg [75, 76], but not for partitionable graphs and almost-perfect graphs: There exist infinitely many not rank-perfect partitionable webs [80] (and it is not known yet whether partitionable graphs are weakly rank-perfect). Even worse, almost-perfect graphs are not weakly rank-perfect, as the two smallest not weakly rank-perfect graphs are the almost-perfect graphs from Figure 2.1, which is certainly unexpected. (Both examples have 1,2-valued facets where the subgraph induced by the nodes with coefficient 1 is perfect.)

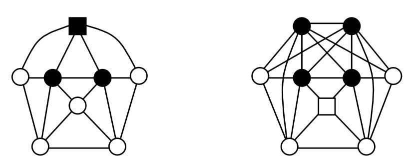
Moreover, there is a whole sequence of almost-perfect but not weakly rank-perfect graphs. Denote by $G(k, v, 1)$ the graph obtained by completely joining a clique K_k with a node v, and inserting one additional node on each edge incident to v (the left graph in Figure 2.1 is $G(3, v, 1)$). Such graphs are obviously almost-perfect (as removing v yields a perfect graph); we showed in [29] that $G = G(k, v, 1)$ is circular-perfect for each $k \geq 1$ and its stable set polytope has a facet of the form

$$x(G - v) + (k - 1)x_v \ \leq \ k$$

which is not a weak rank constraint for any $k \geq 3$ as the subgraph $G - v$ induced by the nodes with coefficient 1 is perfect. Thus, almost-perfect and circular-perfect graphs can certainly be considered to be close to perfect graphs, but this is not reflected in polyhedral terms. However, all the known strongly circular-perfect graphs are rank-perfect [29].

The other way round, there are also near-perfect graphs which are not close to perfection in other respects. For any $k \geq 2$, the lexicographic product $C_{2k+1} \times K_2$ is near-perfect (as near-perfection is closed under replication [98], see Section 5.3.2 for more details). Our results imply that all graphs in this sequence are neither almost-perfect [61] nor circular-perfect [29].

Furthermore, the class of perfect graphs is closed under complementation, but none of its polyhedral superclasses is: Theorem 4.1 shows this for near-perfect graphs, the 5-wheel for rank-perfect graphs as its complement is rank-perfect, and the so-called "wedges" depicted in Figure 4.3 for weakly rank-perfect graphs, as they are not weakly rank-perfect, but their complements are. (These graphs produce 1,2-valued facets where the white nodes in the figure have coefficient 1 and induce a perfect subgraph [47] and are almost-perfect as removing the squared node yields a perfect graph. We are going to discuss wedges in detail in Section 4.2.4.)



(a) (b)

Figure 4.3: Two not weakly rank-perfect graphs.

Line graphs and quasi-line graphs are the only not self-complementary classes where a polyhedral description for the stable set polytope is known for both the graph class and the complementary class. While line graphs are rank-perfect and their complements as joined p-perfect graphs weakly rank-perfect, we only know that near-bipartite graphs are joined a-perfect, whereas we do not know whether the inequalities describing the stable set polytope of quasi-line graphs are weak rank constraints (see next section for more details).

However, it is clear that a further subclass of quasi-line graphs, the claw-free graphs, cannot be weakly rank-perfect as, e.g., all wedges are claw-free but not weakly rank-perfect by Giles and Trotter [47]. In addition, Pulleyblank and Shepherd [87] showed that wedges belong to a certain subclass of claw-free graphs, the distance claw-free graphs, hence such graphs are not weakly rank-perfect, too.

Figure 4.4 illustrates the inclusion relations of the studied graph classes, including the informations which are near-perfect, rank-perfect, or weakly rank-perfect.

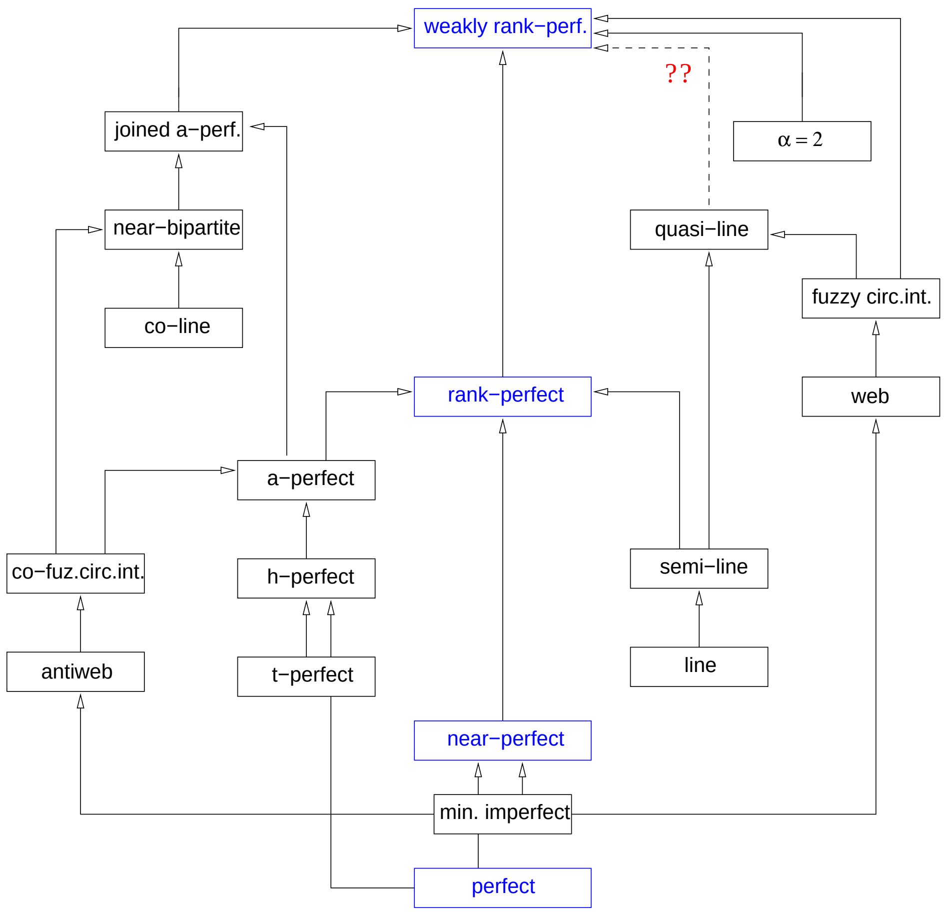


Figure 4.4: Inclusion relations of the studied graph classes.

4.2 Clique Family Inequalities and Classes of Claw-Free Graphs

In this section, we address the stable set problem for claw-free graphs, that is the problem of finding a stable set of maximum size or weight. For members of this class, the stable set problem can be solved in polynomial time [69, 71, 92]; the existing algorithms are extensions of Edmonds matching algorithm [34]. This implies that also the optimization problem over the stable set polytope of a claw-free graph is solvable in polynomial time [50]. Hence, the stable set polytope of claw-free graphs is, in this respect, under control. However, no explicit description by means of a facet-defining system is known for the stable set polytope of claw-free graphs yet; even no conjecture was at hand up to now.

Edmonds' characterization of the matching polytope [33] implies such a description of the facets for the stable set polytope of line graphs and implies that line graphs are rank-perfect and have Chvátal-rank one (see next section for more details). But, in contrary to the algorithmic aspect, this description could not be extended to a facet-defining system for the whole class of claw-free graphs. So far, only the rank facets of the stable set polytopes of claw-free graphs are well-understood due to Galluccio and Sassano [41] who showed that these rank facets essentially come from cliques, line graphs of 2-connected hypomatchable graphs, and partitionable webs. The structure of the *non*-rank facets for stable set polytopes of claw-free graphs is still not well-understood. This apparent asymmetry between the algorithmic and the polyhedral status of the stable set problem in claw-free graphs gives rise to the challenging problem of providing a complete description of the non-rank facets of general claw-free graphs, a long-standing open problem originally posed in [50]; as a first step, we formulate an appealing conjecture.

Edmonds' odd set inequalities for the matching polytope [33] can be extended to so-called clique family inequalities: that are valid inequalities for the stable set polytope which rely on the intersection of cliques within the family and have at most two consecutive non-zero coefficients (see next section for more details). For the intermediate class of quasi-line graphs, Ben Rebea claimed in the early eighties that all non-trivial, non-clique facets of their stable set polytopes belong to this class of clique family inequalities. Oriolo [73] verified this conjecture for line graphs, Chudnovsky and Seymour [19] extended this recently to semi-line graphs. The latter result implies that every quasi-line but not fuzzy circular interval graph is rank-perfect with matching-like facets. For the stable set polytopes of fuzzy circular interval graphs, however, clique family inequalities with arbitrarily high coefficients are required due to Giles and Trotter [47]. Even the subclass of webs is highly not rank-perfect [78, 79, 80] and clique family inequalities with arbitrarily high coefficients are required [63] (see Section 4.2.2).

Ben Rebea's conjecture was recently proved by Eisenbrandt et al. [36]; thus clique family inequalities indeed suffice for this intermediate class. However, even for the stable set polytopes of quasi-line graphs, the following questions remain open.

In analogy to Edmonds' odd set inequalities for the matching polytope, clique family inequalities built a huge class of valid inequalities, but only few of them are essential. Thus, the question remains open which clique family inequalities induce facets. Basing on results in [78, 79, 80] we conjecture which clique family inequalities are the essential ones (see Section 4.2.2). An affirmative answer to this conjecture would show that quasi-line graphs are weakly rank-perfect.

Edmonds conjectured that all facets of the stable set polytopes of a claw-free graph can be obtained through a single application of the Chvátal-Gomory procedure to its clique constraint stable set polytope. This was disproved by Giles and Trotter [47] for general claw-free graphs and by Oriolo [73] for quasi-line graphs. Even worse, the Chvátal-rank of a claw-free graph can be arbitrarily large [24]. We investigate the Chvátal-rank of clique family inequalities for arbitrary graphs. Our main result is that the highest coefficient in a clique family inequality is an upper bound for its Chvátal-rank [82]. This provides an alternative proof for the validity of clique family inequalities and shows that all *rank* clique family inequalities have Chvátal-rank one. Hence, semi-line graphs have Chvátal-rank one, too (see Section 4.2.3 for more details).

However, even for small claw-free but not quasi-line graphs clique family inequalities do not suffice to describe all facets of the stable set polytope. In Section 4.2.4, we collect examples of such claw-free graphs, including the graphs with stability number two, wedges [47], and further small claw-free graphs from [47, 63] producing facets with up to five consecutive non-zero coefficients. (Recall that wedges induce non-weak rank facets, thus claw-free graphs are not weakly rank-perfect.)

Our aim is to formulate an appropriate *conjecture* on the non-rank facets for general claw-free graphs. The facets for graphs G with $\alpha(G) = 2$ are clique neighborhood constraints; claw-free graphs G with $\alpha(G) \geq 4$ are not too far from quasi-line graphs [38] and it is conjectured that rank constraints and certain clique neighborhood constraints suffice to describe their stable set polytope [101]. In fact, all the known difficult facets of claw-free graphs occur if $\alpha(G) = 3$. We analyze these facets in [83] and show that all of them belong to only one class of inequalities, the so-called co-spanning 1-forest constraints. Combining all those results enables us to formulate our conjecture, see Section 4.2.4.

In addition, we extend the concept of clique family inequalities further in order to cover this kinds of facets by more general inequality classes of the same spirit [84]. For that we introduce general clique family inequalities and discuss the validity of several classes of general clique family inequalities, including the original ones.

We prove that all clique neighborhood constraints and some types of co-spanning 1-forest constraints (including the facets induced by wedges) can be expressed as general clique family inequalities. We conjecture that this is true for all co-spanning 1-forest facets of claw-free graphs. An affirmative answer to this conjecture would imply that also the polyhedral aspect of the stable set problem for claw-free graphs is an extension of Edmonds' description of the matching polytope, as every general clique family inequality extends Edmonds' odd set inequalities for the matching polytope.

4.2.1 From odd set inequalities to clique family inequalities

Firstly, we consider Edmonds' odd set inequalities for the matching polytope [33]. This result implies a description for the stable set polytopes of line graphs; matching-like facets can be extended to so-called clique family inequalities. We develop these ideas starting with the matching polytope.

A set of pairwise non-incident edges in a graph G is called *matching*, the *matching polytope* $M(G)$ is defined as the convex hull of the incidence vectors of all matchings in G. Edmonds gave the following characterization:

Theorem 4.17 [33] *The matching polytope $M(G)$ of any graph $G = (V, E)$ is given by*

(0) *nonnegativity constraints* $x_e \geq 0 \; \forall e \in E$,

(i) *star constraints* $x(\delta(v)) \leq 1 \; \forall v \in V$, $\delta(v) = \{e \in E : e \text{ incident to } v\}$,

(ii) *odd set constraints* $x(E(H)) \leq \frac{|H|-1}{2} \; \forall H \subseteq V$ *with* $|H| \geq 3$ *odd.*

The *fractional matching polytope* $M_f(G)$ is defined as the set of all points in $\mathbb{R}^{|G|}$ satisfying the constraints (0) and (i). Chvátal [20] observed that the constraints of type (ii) are obtained through a single application of the Chvátal-Gomory procedure to $M_f(G)$; hence, the odd set constraints (ii) and $M_f(G)$ have Chvátal-rank one.

Note that not all constraints of type (ii) are essential: Edmonds and Pulleyblank [35] proved later that an odd set constraint associated with H defines a facet only if H is a 2-connected, hypomatchable induced subgraph of G (i.e., $H - v$ is connected and admits a matching meeting all nodes, for all v, as the left graph in Figure 4.5).

Edges in a graph G turn to nodes of its line graph $L(G)$, thus matchings in G obviously correspond to stable sets in $L(G)$ and edge-stars in G to cliques in $L(G)$, which implies the following:

Corollary 4.18 *For any graph G, the stable set polytope of $L(G)$ is given by*

(0) *nonnegativity constraints $x_v \geq 0 \ \forall v \in V(L(G))$,*

(i) *clique constraints $x(Q) \leq 1 \ \forall$ cliques $Q \subseteq L(G)$,*

(ii) *rank constraints $x(V(L(H))) \leq \frac{|H|-1}{2}$ associated with the line graphs of 2-connected, hypomatchable induced subgraphs $H \subseteq G$.*

In particular, line graphs are rank-perfect and have Chvátal-rank one. In order to interpret odd set inequalities for STAB($L(G)$), consider an odd set $H \subseteq V(G)$. The edge stars of the nodes in H correspond to a family $\mathcal{Q}$ of $|H|$ cliques in $L(G)$, two such cliques overlap if there is an edge between the respective nodes in G, see the example in Figure 4.5.

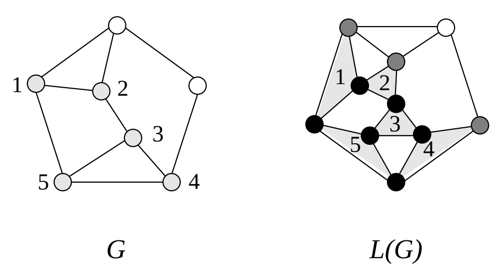


Figure 4.5: From odd sets in G to clique families in $L(G)$.

Denoting the set of nodes in G by $V(\mathcal{Q}, 2)$ which are covered twice by the cliques in $\mathcal{Q}$, we can reformulate the rank constraint (ii) associated with $L(H) \subseteq L(G)$ as $\sum_{i \in V(\mathcal{Q},2)} x_i \leq \lfloor \frac{|\mathcal{Q}|}{2} \rfloor$.

The above corollary shows that such rank constraints are the only non-trivial, non-clique facets required for the stable set polytope of line graphs. Chudnovsky and Seymour [19] recently extended this result to all quasi-line graphs not being fuzzy circular interval graphs, i.e., to the larger class of semi-line graphs.

However, rank constraints do not suffice to describe the stable set polytope for quasi-line graphs; even most webs are not rank-perfect [80]. Ben Rebea generalized the above matching-like constraints to so-called clique family inequalities, involving two consecutive coefficients which are not equal to 1 and 0 in general.

Let $G = (V, E)$ be a graph, $\mathcal{Q}$ be a family of at least three inclusion-wise maximal cliques of G, $p \leq |\mathcal{Q}|$ be an integer, and define two sets

$$\begin{aligned} V(\mathcal{Q}, p) &= \{i \in V : |\{Q \in \mathcal{Q} : i \in Q\}| \geq p\}, \\ V(\mathcal{Q}, p-1) &= \{i \in V : |\{Q \in \mathcal{Q} : i \in Q\}| = p-1\}. \end{aligned}$$

Then the *clique family inequality* $(\mathcal{Q}, p)$ is defined as

$$(p-r) \sum_{i \in V(\mathcal{Q},p)} x_i + (p-r-1) \sum_{i \in V(\mathcal{Q},p-1)} x_i \leq (p-r) \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor \tag{4.7}$$

with $r = |\mathcal{Q}| (\mathrm{mod} p)$ and $r > 0$. For example, choose the five grey triangles of the line graph in Figure 4.5 as clique family $\mathcal{Q}$ and let $p = 2$. Then $V(\mathcal{Q}, p)$ consists in the black nodes, $V(\mathcal{Q}, p-1)$ in the grey nodes (whereas the white node does not belong to any of the two sets). Moreover, $r = 1$ follows and we obtain that the clique family inequality $(\mathcal{Q}, p)$ is the rank constraint

$$1\, x(V(\mathcal{Q}, p)) + 0\, x(V(\mathcal{Q}, p-1)) \leq 2$$

associated with the black nodes.

Oriolo [73] established that clique family inequalities are valid for the stable set polytope of *every* graph. Thus it is natural to ask whether clique and clique family inequalities are sufficient to describe stable set polytopes of other graphs than line graphs.

4.2.2 Clique family inequalities for quasi-line graphs

Ben Rebea (see [73]) claimed that clique family inequalities suffice to describe the stable set polytopes of quasi-line graphs. This was recently proved by Eisenbrandt, Oriolo, Stauffer, and Ventura:

Theorem 4.19 (Ben Rebea's Theorem [36]) *The stable set polytope of any quasi-line graph admits only three types of facets:*

(0) *nonnegativity constraints,*

(i) *clique constraints,*

(ii) *clique family inequalities.*

Ben Rebea's Theorem is the generalization of Edmonds' description of the matching polytope (Theorem 4.17) to quasi-line graphs. However, clique family inequalities built, like Edmonds' odd set inequalities, a huge class of valid inequalities–and it remains open which of them are *essential*, i.e., the inequalities associated with which clique families induce indeed facets. In the matching case, all odd set inequalities associated with 2-connected hypomatchable graphs are the essential ones due to [35]; we address the question which clique families are facet-inducing for quasi-line graphs.

Chudnovsky and Seymour [19] recently showed that semi-line graphs are rank-perfect. Since the rank facets of claw-free graphs are well-understood according to Galluccio and Sassano [41], the interesting part of the class of quasi-line graphs consists in all fuzzy circular interval graphs.

We first summarize results from a series of papers [78, 79, 80] which show that webs are the core of the class of fuzzy circular interval graphs, as already almost all webs are not rank-perfect. Outgoing from these results, we formulate a conjecture which clique family inequalities are essential for the stable set polytopes of fuzzy circular interval graphs. An affirmative answer to this conjecture would imply that quasi-line graphs are weakly rank-perfect and that all facets of fuzzy circular interval graphs rely on webs.

Stable set polytopes of webs

From the literature, the following was known about facets of webs. The webs W_n^1 are, as holes, perfect or rank-perfect [21, 75] and the webs W_n^2 are rank-perfect by Dahl [32]. On the other hand, Kind [55] found (by means of the PORTA software[1]) examples of webs with clique number > 4 which are *not* rank-perfect, e.g., W_{31}^4, W_{25}^5, W_{29}^6, W_{33}^7, W_{28}^8, W_{31}^9. Thus, it was open whether the webs with clique number 3 are rank-perfect and whether there exist only finitely many not rank-perfect webs. Results from [78, 79, 80] answere both questions negatively, as described in the sequel.

Firstly, we presented in [78] an infinite sequence of not rank-perfect webs W_{33}^3, W_{42}^3, W_{51}^3, W_{60}^3, ... with clique number equal to 4.

Next, we introduced in [79] a construction technique for non-rank facets as key tool in order to answer the second question. For that, we need the notion of proper weak non-rank facets. Recall that a facet $\mathbf{a}^T\mathbf{x} \le c\alpha(G')$ of STAB(G) is a weak rank constraint w.r.t. $G' \subseteq G$, if $a_i = c$ holds for every node i of G' and if G' is rank facet-producing (i.e., $\sum_{i \in V(G')} x_i \le \alpha(G')$

[1]By PORTA it is possible to generate all facets of the convex hull of a given set of integer points, see http://www.zib.de

defines a facet of $\mathrm{STAB}(G')$). As any rank facet is a particular weak rank facet (with $a_i = c = 1$ for every $i \in V(G')$ and $a_i = 0$ otherwise), we call a weak rank facet *non-rank* if it cannot be scaled to have 0/1-coefficients only and *proper* if G' is not a clique.

The main consequence of our construction technique introduced in [79] is the following:

Theorem 4.20 *If* $\mathrm{STAB}(W_n^k)$ *admits a proper weak non-rank facet then also* $\mathrm{STAB}(W_{n+k+1}^k)$ *has a proper weak non-rank facet.*

Therefore, if $\mathrm{STAB}(W_n^k)$ has a proper weak non-rank facet then *all* webs $W_{n+l(k+1)}^k$ are not rank-perfect for any $l \geq 0$, too. Hence, an important consequence of Theorem 4.20 is:

Corollary 4.21 *If there is a base set of* $k+1$ *webs* $W_{n_0}^k, \ldots, W_{n_k}^k$ *such that, for* $0 \leq i \leq k$,

- $\mathrm{STAB}(W_{n_i}^k)$ *has a proper weak non-rank facet*
- $n_i = i \pmod{k+1}$

then all webs W_n^k *with* $n \geq \max\{n_0, \ldots, n_k\} - k$ *are not rank-perfect.*

This implies that, for every *fixed* $k \geq 3$, constructing a finite base set of webs $W_{n_0}^k, \ldots, W_{n_k}^k$ with proper weak non-rank facets suffices to show that almost all webs W_n^k are not rank-perfect. In order to present such base sets for all values of $k \geq 3$, we considered special clique family inequalities associated with proper subwebs which all yield proper weak non-rank facets.

A clique family inequality $(\mathcal{Q}, p)$ is *associated* with a proper subweb $W_{n'}^{k'}$ of a web W_n^k if $\mathcal{Q} = \{Q_i : i \in W_{n'}^{k'}\}$ is chosen as clique family and $p = k'+1$, where $Q_i = \{i, \ldots, i+k\}$ denotes the maximum clique of W_n^k starting in node i. (Recall that the clique number of a web W_n^k is $k+1$ and the stability number is $\lfloor \frac{n}{k+1} \rfloor$.) We obtained in [78]:

Lemma 4.22 *Let* $W_{n'}^{k'} \subset W_n^k$ *be a proper induced subweb. The clique family inequality* $(\mathcal{Q}, k'+1)$ *associated with* $W_{n'}^{k'}$ *is*

$$(k'+1-r) \sum_{i \in V(\mathcal{Q}, k'+1)} x_i + (k'-r) \sum_{i \in V(\mathcal{Q}, k')} x_i \leq (k'+1-r)\, \alpha(W_{n'}^{k'})$$

where $r = n' \bmod (k'+1)$ *and* $0 < r < k'+1$ *holds;* $(\mathcal{Q}, k'+1)$ *is a valid inequality for STAB*(W_n^k) *and* $W_{n'}^{k'} \subseteq V(\mathcal{Q}, p)$.

Every clique family inequality $(\mathcal{Q}, k'+1)$ associated with a subweb $W_{n'}^{k'} \subseteq W_n^k$ is particularly a proper weak rank constraint with base rank constraint $x(W_{n'}^{k'}) \leq \alpha(W_{n'}^{k'})$, and is non-rank if $r < k'$.

For illustration, look at the smallest not rank-perfect web W_{25}^5. Its non-rank facets are clique family inequalities associated with induced subwebs $W_{10}^2 \subseteq W_{25}^5$ (the node sets 1, 2, 6, 7, 11, 12, 16, 17, 21, 22 and 1, 3, 6, 8, 11, 13, 16, 18, 21, 23 both induce a $W_{10}^2 \subseteq W_{25}^5$, see the black nodes in Figure 4.6).

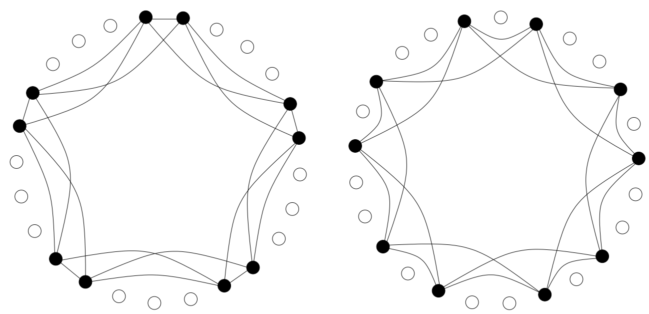


Figure 4.6: The two possible induced subwebs $W_{10}^2 \subseteq W_{25}^5$

Choosing $\mathcal{Q} = \{Q_i : i \in W_{10}^2\}$ yields $p = \omega(W_{10}^2) = 3$ in both cases. All remaining nodes are covered twice, hence $V(\mathcal{Q}, p-1) = W_{25}^5 - W_{10}^2$ follows. The corresponding clique family inequality $(\mathcal{Q}, 3)$ is

$$2 \sum_{i \in W_{10}^2} x_i + 1 \sum_{i \notin W_{10}^2} x_i \leq 2\alpha(W_{10}^2)$$

due to $r = |\mathcal{Q}| \bmod p = 1$ and yields a non-rank facet of $\mathrm{STAB}(W_{25}^5)$.

The main results in [78, 80] prove that several clique family inequalities $(\mathcal{Q}, k'+1)$ associated with different regular subwebs $W_{n'}^{k'}$ induce proper weak non-rank facets

A subweb $W_{n'}^{k'} \subset W_n^k$ is called $(b_1, w_1, \ldots, b_t, w_t)$-*regular*, if the nodes of $W_{n'}^{k'}$ occur in W_n^k in equal blocks where b_i consecutive nodes from $W_{n'}^{k'}$ alternate with w_i consecutive nodes outside $W_{n'}^{k'}$, for $1 \leq i \leq t$. The two subwebs $W_{10}^2 \subseteq W_{25}^5$ presented in Figure 4.6 show a (2,3)-regular and a (1,1,1,2)-regular subweb, respectively.

We proved the existence of a base set of webs W_n^k for all $k \geq 3$ by presenting several clique family inequalities associated with different regular subwebs which all yield proper weak non-rank facets.

Theorem 4.23 [78] *For $k = 3$, consider a $(2,1)$-regular subweb $W^2_{2l} \subset W^3_{3l}$. The clique family inequality $(\mathcal{Q}, 3)$*

$$2 \sum_{i \in W^2_{2l}} x_i + 1 \sum_{i \notin W^1_{1l}} x_i \leq 2\alpha(W^2_{2l})$$

associated with W^2_{2l} is a proper weak non-rank facet of $\mathrm{STAB}(W^3_{3l})$ *if* $l = 2 \pmod 3$ *and $l \geq 11$ holds.*

Thus, the stable set polytopes of the webs W^3_{33}, W^3_{42}, W^3_{51}, W^3_{60} admit proper weak non-rank facets and Corollary 4.21 implies that all webs W^3_n with $n > 56$ are not rank-perfect.

Theorem 4.24 [80] *For any $k \geq 5$, consider a $(k', k - k')$-regular subweb $W^{k'}_{lk'}$ of W^k_{lk} with $2 \leq k' \leq k - 3$ and odd $l \geq 3$. The clique family inequality $(\mathcal{Q}, k' + 1)$*

$$2 \sum_{i \in W^{k'}_{lk'}} x_i + 1 \sum_{i \notin W^{k'}_{lk'}} x_i \leq 2\alpha(W^{k'}_{lk'})$$

associated with $W^{k'}_{lk'}$ is a proper weak non-rank facet of $\mathrm{STAB}(W^k_{lk})$ *if* $l = 2 \pmod{k' + 1}$ *and* $\alpha(W^{k'}_{lk'}) < \alpha(W^k_{lk})$.

As a consequence, we obtain many different infinite sequences of not rank-perfect webs, among them the required base sets for all *even* values of $k \geq 6$ (but not for the odd values $k \geq 5$ since all webs in the latter sequences have an odd number of vertices). For any even $k \geq 6$, choosing $k' = \frac{k}{2}$ if $k = 0 \pmod 4$ and $k' = \frac{k}{2} - 1$ if $k = 2 \pmod 4$ and $l = (k' + 3) + (k' + 1)2j$ for $j \geq 1$ in both cases as odd values of l with $l = 2 \pmod{k' + 1}$ satisfies the precondition of Theorem 4.24. Thus, we obtain the following infinite sequences of not rank-perfect webs:

Theorem 4.25 [80] *Let $k \geq 6$ be even. Then for every integer $j \geq 1$ holds that* $\mathrm{STAB}(W^k_n)$ *has a proper weak non-rank facet if*

- $n = \left(\frac{k+6}{2} + (k+2)j\right) k$ *and* $k = 0 \pmod 4$;
- $n = \left(\frac{k+4}{2} + kj\right) k$ *and* $k = 2 \pmod 4$.

We showed in [80] that these sequences contain the required base sets for the case of webs W^k_n with even $k \geq 6$. The remaining base sets for $k = 4$ and all odd values of $k \geq 5$ are constructed as follows.

Theorem 4.26 [80] *For $k = 4$, consider a (1,1)-regular subweb $W_l^2 \subset W_{2l}^4$. The clique family inequality* $(\mathcal{Q}, 3)$

$$2 \sum_{i \in W_l^2} x_i + 1 \sum_{i \notin W_l^2} x_i \leq 2\,\alpha(W_l^2)$$

associated with W_l^2 is a proper weak non-rank facet of $\mathrm{STAB}(W_{2l}^4)$ *if* $l = 1 \pmod 3$ *and* $l \geq 13$.

Thus, the stable set polytopes of the webs W_{26}^4, W_{32}^4, W_{38}^4, W_{44}^4, and W_{50}^4 have a proper weak non-rank facet and all webs W_n^4 with $n > 45$ are, therefore, not rank-perfect.

For each odd $k \geq 5$, we extended our result for $k = 3$ from [78] by considering the clique family inequality associated with the $(k-1,1)$-regular subweb $W_{l(k-1)}^{k-1} \subset W_{lk}^k$ as follows:

Theorem 4.27 [80] *For any odd $k \geq 5$, consider a $(k-1,1)$-regular subweb $W_{l(k-1)}^{k-1} \subset W_{lk}^k$. The clique family inequality* $(\mathcal{Q}, k)$

$$2 \sum_{i \in W_{l(k-1)}^{k-1}} x_i + 1 \sum_{i \notin W_{l(k-1)}^{k-1}} x_i \;\leq\; 2\,\alpha(W_{l(k-1)}^{k-1})$$

associated with $W_{l(k-1)}^{k-1}$ is a proper weak non-rank facet of STAB(W_{lk}^k) if $l = l'k + 2$ and $l' \geq 3$.

For any odd $k \geq 5$, the sequence of the $k+1$ webs $W_{k(l'k+2)}^k$ with $3 \leq l' \leq 3 + k$ is the required base set. Thus, W_n^k with $n \geq ((k+3)k+1)k$ is not rank-perfect for any odd $k \geq 5$.

In summary, all the above results from [78, 79, 80] show that there is, for each fixed $k \geq 3$, a value $n(k)$ such that all webs W_n^k with $n \geq n(k)$ are not rank-perfect. In particularly, for any $k \geq 3$ there are only finitely many rank-perfect webs W_n^k, which implies:

Corollary 4.28 *Almost all webs with given clique size at least 4 are not rank-perfect.*

Note that the above results use clique family inequalities associated with certain subwebs which yield 1/2-valued facets only. Since the construction technique from [79] does not change the involved coefficients, the stable set polytopes of almost all webs admit 1/2-valued facets.

According to Ben Rebea's Theorem [36], the stable set polytopes of webs have clique and clique family inequalities as only non-trivial facets. We conjectured [80] and Stauffer recently proved [101] that the *essential* ones among them are precisely the clique family inequalities associated with proper subwebs:

Theorem 4.29 [101] *The only facets of STAB(W_n^k) are the following:*

(0) *nonnegativity constraints,*

(i) *clique constraints,*

(ii) *full rank constraint,*

(iii) *clique family inequalities* $(\mathcal{Q}, k'+1)$ *associated with proper subwebs* $W_{n'}^{k'}$ *where* $(k'+1)\not| \, n'$ *and* $\alpha(W_{n'}^{k'}) < \alpha(W_n^k)$ *holds.*

As a web W_n^k can have subwebs $W_{n'}^{k'}$ for *all* values $1 \leq k' < k$, this implies that the stable set polytope of W_n^k admits $(k-2)/(k-1)$-valued facets. In particular, the stable set polytopes of all webs W_n^3 have 1/2-valued facets only, where for all webs W_n^k with $k > 3$ larger coefficients are required. In fact, Liebling et al. [63] proved that, for any odd $k \geq 5$, the stable set polytope of $W_{k^2}^k$ has a $(k-2)/(k-1)$-valued facet. Hence, the webs are indeed the not rank-perfect core of the class of fuzzy circular interval graphs. Moreover, the above theorem combined with Lemma 4.22 yields that all webs are weakly rank-perfect.

A conjecture on facet-defining clique family inequalities

In order to figure out which clique family inequalities define facets for general quasi-line graphs, we shall extend the above result for webs to the whole class of fuzzy circular interval graphs (recall that the stable set polytopes of semi-line graphs are described by [19]).

Firstly, note that we can interpret a clique family inequality $(\mathcal{Q}, k'+1)$ associated with an induced subweb $W_{n'}^{k'} \subset W_n^k$ as follows: we take all the n' maximum cliques $Q'_1, \ldots, Q'_{n'}$ of $W_{n'}^{k'}$ and extend them to a family $\mathcal{Q} = \{Q_1, \ldots, Q_{n'}\}$ of maximal cliques of the whole graph W_n^k, and we choose $p = k' + 1$.

In particular, a clique family $\mathcal{Q}$ constructed this way is k'-*cyclic*, that is, every clique Q_i intersects the k' cliques $Q_{i-k'}, \ldots, Q_{i-1}$ and the k' cliques $Q_{i+1}, \ldots, Q_{i+k'}$ (all indices are taken modulo n') as the cliques $Q'_1, \ldots, Q'_{n'}$ obviously satisfy this property. (Recall that all nodes of $W_{n'}^{k'}$ are covered $(k'+1)$-times by the cliques in $\mathcal{Q}$ and, thus, belong to the set $V(\mathcal{Q}, k'+1)$.)

The task is to figure out which clique families are the crucial ones in general fuzzy circular interval graphs $G(V, \mathcal{I}, \mathcal{C})$. We first show that they are also k'-cyclic for some $k' \geq 1$.

As for the clique families only maximal cliques are required, we first observe that a maximal clique of a fuzzy circular interval graph $G(V, \mathcal{I}, \mathcal{C})$ corresponds to all points of V belonging to an interval $I \in \mathcal{I}$, where the endpoints of I have possibly less multiplicity due to fuzzyness.

Next, the subgraph $G'(\mathcal{Q}, p)$ of $G(V, \mathcal{I}, \mathcal{C})$ induced by the nodes in $V(\mathcal{Q}, p)$ and $V(\mathcal{Q}, p-1)$ is *imperfect*, otherwise $(\mathcal{Q}, p)$ cannot be a (non-clique) facet. Thus $G'(\mathcal{Q}, p)$ contains, according to the Strong Perfect Graph Theorem, an odd hole or odd antihole and, hence, in particular a prime web.

We showed in [112] that every prime subweb $W_{n'}^{k'}$ of a fuzzy circular interval graph $G(V, \mathcal{I}, \mathcal{C})$ admits exactly one representation as fuzzy circular interval graph, namely, the following *canonical* one: Consider a point set $V' = \{1, \ldots, n'\}$ distributed on $\mathcal{C}$ in this order and without multiplicities. Further, let $\mathcal{I}' = \{I_1, \ldots, I_{n'}\}$ be a collection of intervals in $\mathcal{I}$ with $I_i \cap V' = \{i, \ldots, i + k'\}$ for $1 \leq i \leq n'$ (indices are taken modulo n'). Then $G(V', \mathcal{I}', \mathcal{C})$ obviously equals the web $W_{n'}^{k'}$. As an example, Figure 4.7 shows the canonical representation of $\overline{C}_7 = W_7^2$.

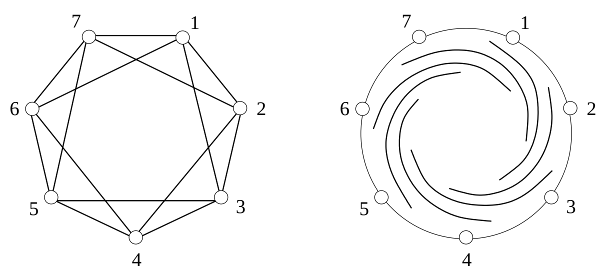


Figure 4.7: The canonical representation of W_7^2.

This implies in particular that the intervals representing the maximum cliques of $W_{n'}^{k'}$ cover $\mathcal{C}$ completely and built a k'-cyclic clique family of $G(V, \mathcal{I}, \mathcal{C})$. Moreover, every prime subweb $W_{n'}^{k'}$ of $G(V, \mathcal{I}, \mathcal{C})$ is dominating [112], i.e., every node outside $W_{n'}^{k'}$ is connected to some node in $W_{n'}^{k'}$.

The subgraph $G'(\mathcal{Q}, p)$ of $G(V, \mathcal{I}, \mathcal{C})$ induced by the nodes in $V(\mathcal{Q}, p)$ and $V(\mathcal{Q}, p-1)$ contains at least one prime subweb $W_{n'}^{k'}$, as it contains at least one odd hole or odd antihole. In particular, the intervals representing the cliques in $\mathcal{Q}$ cover $\mathcal{C}$ completely, as the intervals representing the maximum cliques of $W_{n'}^{k'}$ already do. More precisely, we proved in [84]:

Lemma 4.30 *If $(\mathcal{Q}, p)$ is a facet-defining clique family inequality for a fuzzy circular interval graph $G(V, \mathcal{I}, \mathcal{C})$, then $\mathcal{Q}$ is a k-cyclic clique family for some $k \geq \max\{k' : W_{n'}^{k'} \subseteq G'(\mathcal{Q}, p), prime\}$.*

As only cyclic clique families can be essential, this motivates to consider clique family inequalities $(\mathcal{Q}, k'+1)$ *associated* with prime subwebs $W_{n'}^{k'}$ of fuzzy circular interval graphs, where $\mathcal{Q}$ is obtained by extending the n' maximum cliques of $W_{n'}^{k'}$ to maximal cliques of the whole graph.

All the known non-rank facets for fuzzy circular interval graphs are of this type, e.g., the facets with arbitrarily high coefficients of the following sequence of fuzzy circular interval graphs G^k, $k \geq 1$, introduced in [47]:

Example 4.31 Take the webs W_n^{k+1} and W_n^k with $n = 2k(k+2)+1$ where $V(W_n^{k+1}) = \{1, \ldots, n\}$ and $V(W_n^k) = \{1', \ldots, n'\}$. Construct the graph G^k by taking W_n^{k+1} and W_n^k as induced subgraph and adding the edges $\{i, i'\}$, $\{i, (i+1)'\}$, $\ldots$, $\{i, (i+2k+1)'\}$ for $1 \leq i \leq n$ where all indices are taken modulo n. (The graph G_1 is depicted in Fig. 4.8). Giles and Trotter [47] showed that the clique family inequality $(\mathcal{Q}, k+2)$

$$(k+1)\ x(W_n^{k+1})\ +\ k\ x(W_n^k)\ \leq\ (k+1)\,\alpha(W_n^{k+1})$$

is a facet of $\mathrm{STAB}(G^k)$ for every $k \geq 1$, where $\mathcal{Q}$ is constructed as follows. Let $Q_i = \{i, \ldots, i+k+1\}$ denote the maximum clique of W_n^{k+1} starting in node i and, analogously, $Q'_{j'} = \{j', \ldots, (j+k)'\}$ the maximum clique of W_n^k starting in node j'. Since every node i of W_n^{k+1} is exactly linked to the nodes $i', \ldots, (i+2k+1)'$, it is easy to see that Q_i and $Q'_{(i+k+1)'}$ are totally joined for each $1 \leq i \leq n$. Indeed, the chosen clique family is $\mathcal{Q} = \{Q_i \cup Q'_{(i+k+1)'} : 1 \leq i \leq n\}$, which is associated with W_n^{k+1} (every maximum clique Q_i is extended by $Q'_{(i+k+1)'}$ to a maximal clique of the whole graph G^k) and $(k+1)$-cyclic (see Fig. 4.8 for an example).

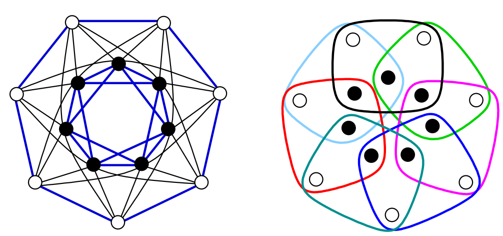


Figure 4.8: The graph G_1 together with its cyclic clique family

To support this feeling further, we extended in [84] our result from [79] that no facet-defining line graphs different from odd holes can occur in webs:

Lemma 4.32 *The only 2-connected, critical hypomatchable graphs H s.t. $L(H)$ occurs in a fuzzy circular interval graph are triangles and odd holes.*

Combining this lemma with the characterization of the rank facets for claw-free graphs due to Galluccio and Sassano [41], all *rank*-facets of fuzzy circular interval graphs rely on webs. We believe that this is also true for the *non-rank* facets and conjecture:

Conjecture 4.33 *Every facet of the stable set polytope of a fuzzy circular interval graph belongs to one of the following classes:*

(0) *nonnegativity constraints,*

(i) *clique constraints,*

(ii) *clique family inequalities associated with prime subwebs.*

An affirmative answer would imply that all facets (not only the rank-facets) of fuzzy circular interval graphs rely on prime webs. The facets of semi-line graphs rely on line graphs of hypomatchable graphs only (they are rank-facets, and webs are excluded by definition). Thus, we would have a real partition of quasi-line graphs: in semi-line graphs with line graph-based rank facets and in fuzzy circular interval graphs with web-based facets:

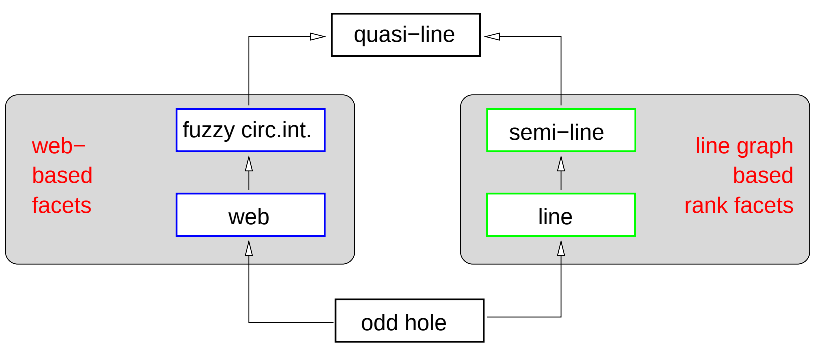


Figure 4.9: Facet types and classes of quasi-line graphs

In addition, we would obtain that all fuzzy circular interval graphs are weakly rank-perfect. This would particularly imply that quasi-line graphs are the largest graph class where weakly rank-perfection is known for the class and the complementary class (the near-bipartite graphs).

4.2.3 The Chvátal-rank of quasi-line graphs

Edmonds conjectured that the Chvátal-rank of a claw-free graph is one. This was disproved by [47, 73]; the Chvátal-rank of a claw-free graph can even be arbitrarily large [24].

We investigate the Chvátal-rank of general clique family inequalities $(\mathcal{Q}, p)$. Our main result is that their Chvátal-rank is at most $\min\{r, p-r\}$ with $r = |\mathcal{Q}| \pmod p$. The argumentation provides an alternative proof for the validity of clique family inequalities and shows that all 0/1-valued clique family inequalities have Chvátal-rank 1. We discuss consequences regarding the Chvátal-rank of quasi-line graphs. In particular, all rank-perfect subclasses of quasi-line graphs have Chvátal-rank 1, which extends Edmonds result on the Chvátal-rank of line graphs to semi-line graphs.

The Chvátal-rank of general clique family inequalities

Our first result from [82] is the following:

Theorem 4.34 *Let $(\mathcal{Q}, p)$ be a clique family inequality and $r = |\mathcal{Q}| \pmod p$. For every $1 \le i \le p-r$, the inequality*

$$i \sum_{v \in V(\mathcal{Q},p)} x_v + (i-1) \sum_{v \in V(\mathcal{Q},p-1)} x_v \le i \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor$$

has Chvátal-rank at most i; thus $(\mathcal{Q}, p)$ has Chvátal-rank at most $p-r$.

The proof of Theorem 4.34 shows also validity of clique family inequalities for the stable set polytope of any graph, involving only standard rounding arguments. Furthermore, as the largest coefficient of a clique family inequality is an upper bound for its Chvátal-rank, we obtain:

Corollary 4.35 *Every* rank *clique family inequality has Chvátal-rank one.*

The latter consequence is particularly nice, as neither general rank constraints nor general clique family inequalities have this property [24, 73], but the combination of both.

However, the upper bound established in Theorem 4.34 gets weaker if r gets smaller; we improved this upper bound for $r < p/2$ in [82] as follows:

Theorem 4.36 *Every clique family inequality $(\mathcal{Q}, p)$ with $r = |\mathcal{Q}| \pmod p$ has Chvátal-rank at most r if $0 \le r < p-r$.*

Thus, Theorem 4.34 and Theorem 4.36 together imply the following:

Corollary 4.37 *Every clique family inequality* $(\mathcal{Q}, p)$ *has Chvátal-rank* $t \leq \min\{r, p-r\} \leq \frac{p}{2}$.

Very recently, Stauffer and Ventura [102] established a lower bound by showing that the Chvátal-rank of $(\mathcal{Q}, p)$ is at least $\log(\frac{1}{2}\min\{r, p-r\})$, provided $(\mathcal{Q}, p)$ defines a facet and $V(\mathcal{Q}, p)$ contains a node subset V' with $|V'| = |\mathcal{Q}|$ and $\omega(G[V']) \leq p$. We call a clique family inequality *good* if it satisfies this property and infer:

Corollary 4.38 *Every good clique family inequality* $(\mathcal{Q}, p)$ *has Chvátal-rank* t *with* $\log(\frac{1}{2}\min\{r, p-r\}) < t \leq \min\{r, p-r\}$.

Consequences for quasi-line graphs

We are clearly interested in consequences of the above results for quasi-line graphs, as all non-trivial, non-clique facets of their stable set polytope are clique family inequalities according to Ben Rebea's Theorem [36].

We conclude from Corollary 4.35 that all *rank*-perfect subclasses of quasi-line graphs have Chvátal-rank one. This reproves Edmonds' result on line graphs and verifies his conjecture for the larger class of semi-line graphs, as they are rank-perfect by [19]:

Corollary 4.39 *Semi-line graphs have Chvátal-rank one.*

Thus, in order to discuss the Chvátal-rank for quasi-line graphs, it suffices to restrict to fuzzy circular interval graphs. The next example from [47, 73] shows that fuzzy circular interval graphs can have Chvátal-rank at least 2.

Example 4.40 Giles and Trotter [47] considered a fuzzy circular interval graph G obtained by joining the webs W_{37}^{7} and W_{37}^{6} in a certain way and showed that there is a clique family $\mathcal{Q}$ of size 37 such that $(\mathcal{Q}, 8)$ is a facet of STAB(G). Oriolo noticed in [73] that this clique family inequality $(\mathcal{Q}, 8)$ has Chvátal-rank *at least* 2.

This example disproves Edmonds' conjecture for fuzzy circular interval graphs. On the other hand, Theorem 4.34 shows that this clique family inequality $(\mathcal{Q}, 8)$ has Chvátal-rank at most 3, since $r = 5$ and so $p - r = 3$.

Furthermore, consider again the sequence of fuzzy circular interval graphs G^k, $k \geq 1$ introduced by Giles and Trotter [47] from Example 4.31.

Example 4.41 Recall that the graph G^k is obtained by joining the webs W_n^{k+1} and W_n^k with $n = 2k(k+2)+1$ in a certain way. For every $k \geq 1$, $\mathrm{STAB}(G^k)$ admits a clique family facet $(\mathcal{Q}, k+2)$ with arbitrarily high coefficients $k+1$, but Theorem 4.36 shows that they have Chvátal-rank 1, since $|\mathcal{Q}| = n = 2k(k+2)+1 = 1 \pmod{k+2}$.

We next discuss the Chvátal-rank of webs. The webs with either clique number three or stability number two are the only further rank-perfect subclasses of quasi-line graphs [32, 111] and have, therefore, Chvátal-rank one. Almost all other webs are not rank-perfect, as shown above. However, the *known* non-rank facets of stable set polytopes of webs are clique family inequalities with Chvátal-rank at most two: Recall that all the non-rank facets from [78, 79, 80] presented above are all 1,2-valued and have, therefore, Chvátal-rank one. Even the following sequence of webs with clique family facets having arbitrarily high coefficients turn out to have Chvátal-rank one:

Example 4.42 Liebling et al. [63] considered the following sequence of webs $W_{(2a+3)^2}^{2(a+2)}$ for every integer $a \geq 1$. They showed that there exists a clique family facet $(\mathcal{Q}, a+2)$ where $\mathcal{Q}$ is of size $(a+2)(2a+3)$. Since $(a+2)(2a+3) = 1 \pmod{a+2}$, Theorem 4.36 shows that even these non-rank inequalities with coefficients a and $a+1$ for all $a \geq 1$ have Chvátal-rank 1.

For any clique family inequality $(\mathcal{Q}, k'+1)$ associated with a proper subweb $W_{n'}^{k'}$ of a web or a general fuzzy circular interval graph G, Corollary 4.37 shows that $(\mathcal{Q}, k'+1)$ has Chvátal-rank at most $\frac{k'}{2}$. Thus, Theorem 4.29 implies that, for any *fixed* k, the Chvátal-rank of all webs W_n^k is at most $\frac{k-1}{2}$. For the whole class of webs, however, there could occur clique family facets $(\mathcal{Q}, p)$ with arbitrarily high p and Chvátal-rank $\frac{p}{2}$.

This was recently shown for fuzzy circular interval graphs by Stauffer and Ventura [102]:

Example 4.43 Consider a web $W_{a(2a+1)+a+1}^{2a}$ in its canonical representation as fuzzy circular interval graph (without multiplicities and fuzzyness). Construct a fuzzy circular interval graph G_a by adding $n = a(2a+1)+a+1$ new nodes, namely, one in between two consecutive nodes of the web w.r.t. the circular ordering. Stauffer and Ventura [102] showed that, for each $a \geq 1$, $\mathrm{STAB}(G_a)$ has a facet-defining clique family inequality $(\mathcal{Q}, p)$ with $|\mathcal{Q}| = n$, $p = 2a+1$, $r = a+1$, and $p - r = a$, with Chvátal-rank $t \geq \log(\frac{a}{2})$.

Consequently, the Chvátal-rank of quasi-line graphs can be arbitrarily large, as for general claw-free graphs.

4.2.4 General claw-free graphs

The *rank* facets of the stable set polytopes of claw-free graphs are well-understood by Galluccio and Sassano [41]. This is, however, not true for the *non*-rank facets; our aim is to formulate an appropriate *conjecture*.

As clique family inequalities are valid for all graphs and can be non-rank, it is natural to ask which role play such inequalities for general claw-free graphs. However, already the smallest not quasi-line graph, the 5-wheel, has a facet not associated with a clique family. We exhibit some small claw-free graphs with non-clique family facets and discuss how the usual clique family inequalities should be generalized to cover such kinds of facets.

The facets for graphs G with $\alpha(G) = 2$ are given by Cook; we show that they are generalized clique family inequalities.

All the known difficult facets of claw-free graphs occur if $\alpha(G) = 3$ and so far their structure was not well-understood. Starting from Giles and Trotter's construction for wedges, we analyze these facets and show that they belong to only one inequality class [83]. In addition, we express several such facets (including those induced by wedges) as generalized clique family inequalities and conjecture that this is possible for all facets of this type.

We finally collect results from [19, 38, 101] about facets of claw-free graphs G with $\alpha(G) \geq 4$ which suggest that such graphs are not too far from quasi-line graphs and have "easy" non-rank facets obtained by lifting 5-wheel constraints. Combining all those results enables us to formulate our conjecture on facets for general claw-free graphs.

General clique family inequalities

We exhibit that more general facets than clique family inequalities $(\mathcal{Q}, p)$ are required to describe the stable set polytope of general claw-free graphs.

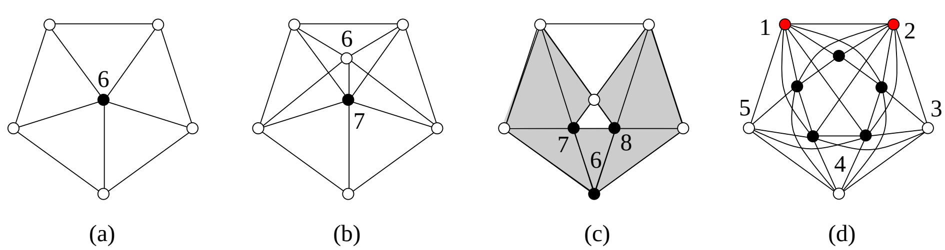


Figure 4.10: Graphs with facets different from clique family inequalities

Example 4.44 The four claw-free graphs depicted in Figure 4.10 produce facets which cannot be expressed as usual clique family inequalities.

(a) The 5-wheel drawn in Figure 4.10(a) induces

$$1(x_1 + \ldots + x_5) + 2x_6 \leq 2$$

as facet. In a clique family inequality, we would need $V(\mathcal{Q}, p) = \{x_6\}$ and $V(\mathcal{Q}, p-1) = \{x_1, \ldots, x_5\}$; the nodes $x_1, \ldots, x_5$ had to be covered at least twice by the cliques in $\mathcal{Q}$. Thus, we had to choose all five maximal cliques for $\mathcal{Q}$ and $p = 3$, yielding $r = 2$ and

$$\begin{aligned} (p-r-1) \sum_{1 \leq i \leq 5} x_i \;+\; (p-r)\; x_6 \leq\; & (p-r) \left\lfloor \tfrac{|\mathcal{Q}|}{p} \right\rfloor \\ 0 \sum_{1 \leq i \leq 5} x_i \;+\; 1\; x_6 \leq\; & 1 \end{aligned}$$

which is only an upper bound for x_6.

We can obtain the required facet by choosing a *smaller* value for r than $|\mathcal{Q}| \bmod p$, namely, $r = 1$.

(b) The graph in Figure 4.10(b) induces

$$1(x_1 + \ldots + x_6) + 2x_7 \leq 2$$

as facet. In a clique family inequality, we would need $V(\mathcal{Q}, p) = \{x_7\}$ and $V(\mathcal{Q}, p-1) = \{x_1, \ldots, x_6\}$; the nodes $x_1, \ldots, x_6$ had to be covered at least twice by the cliques in $\mathcal{Q}$. In order to cover $x_1, \ldots, x_5$ accordingly, we had to take all five maximal cliques for $\mathcal{Q}$ and $p = 3$ again, but then x_6 and x_7 would be covered three times, yielding $r = 2$ and

$$\begin{aligned} (p-r-1) \sum_{1 \leq i \leq 5} x_i \;+\; (p-r)\; (x_6 + x_7) \leq\; & (p-r) \left\lfloor \tfrac{|\mathcal{Q}|}{p} \right\rfloor \\ 0 \sum_{1 \leq i \leq 5} x_i \;+\; 1\; (x_6 + x_7) \leq\; & 1 \end{aligned}$$

but not the required facet.

We have to drop node x_6 from one of the cliques in $\mathcal{Q}$ in order to obtain $V(\mathcal{Q}, p) = \{x_7\}$ and $V(\mathcal{Q}, p-1) = \{x_1, \ldots, x_6\}$. Thus, choosing a *non*-maximal clique for $\mathcal{Q}$ and again adjusting $r = 1 < |\mathcal{Q}| \bmod p$ is required.

(c) The wedge shown in Figure 4.10(c) induces the facet

$$1(x_1 + \ldots + x_5) + 2(x_5 + x_6 + x_7) \leq 3.$$

In a clique family inequality, we would need $V(\mathcal{Q}, p) = \{x_5, x_6, x_7\}$ and $V(\mathcal{Q}, p-1) = \{x_1, \ldots, x_5\}$; the nodes $x_1, \ldots, x_5$ had to be covered at least twice by the cliques in $\mathcal{Q}$. This could be done by choosing the 7 grey-filled triangles for $\mathcal{Q}$ and $p = 3$, yielding $r = 1$ and

$$\begin{array}{rcl} (p-r-1) \sum\limits_{1 \leq i \leq 5} x_i \;+\; (p-r) \sum\limits_{6 \leq i \leq 8} x_i & \leq & (p-r) \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor \\ 1 \sum\limits_{1 \leq i \leq 5} x_i \;+\; 2 \sum\limits_{6 \leq i \leq 8} x_i & \leq & 4 \end{array}$$

but not the required facet as the right hand side is too weak. Thus, we have to *strengthen* the right hand side appropriately.

(d) The graph in Figure 4.10(d) induces the following facet

$$3(x_1 + x_2) + 1(x_3 + x_4 + x_5) + 2(x_6 + \ldots + x_{10}) \leq 4.$$

As the two different non-zero coefficients of a clique family inequality are clearly not enough to represent this facet, we have to look for a more general setting that allows us to have *more than two* non-zero coefficients.

Thus, there exist claw-free graphs producing facets which cannot be expressed as usual clique family inequalities. We are going to generalize the concept of clique family inequalities by introducing new parameters in such a way that a greater variety of facets can be represented. According to the previous examples, we have to adjust the following:

- allow non-maximal (and possibly multiple) cliques in the family $\mathcal{Q}$;
- define more than the two sets $V(\mathcal{Q}, p)$ and $V(\mathcal{Q}, p-1)$;
- allow values $r < |\mathcal{Q}| \bmod p$;
- choose an appropriate right hand side.

Let $G = (V, E)$ be a graph, $\mathcal{Q}$ be a family of at least three cliques of G (the cliques are not necessarily maximal and distinct). Choose integers

$p \leq |\mathcal{Q}|$, r with $0 \leq r \leq R = |\mathcal{Q}| \bmod p$, and J with $0 \leq J \leq p - r$. Define different types of sets as

$$\begin{aligned} V(\mathcal{Q}, p) &= \{i \in V : |\{Q \in \mathcal{Q} : i \in Q\}| \geq p\}, \\ V(\mathcal{Q}, p-j) &= \{i \in V : |\{Q \in \mathcal{Q} : i \in Q\}| = p - j\ \} \end{aligned}$$

for $1 \leq j \leq J$ (some of the sets $V(\mathcal{Q}, p-j)$ might be empty). We define the *general clique family inequality* $(\mathcal{Q}, p, r, J, b)$ by

$$\sum_{0 \leq j \leq J} (p - r - j)\ x(V(\mathcal{Q}, p - j))\ \leq\ b \tag{4.8}$$

which is valid for the stable set polytope of every graph G by an appropriate choice of the right hand side b, that is if $b \geq \alpha(G, a)$ where $a_i = p - r - j$ for $v_i \in V(\mathcal{Q}, p - j)$.

Thus, two canonical questions arise in this context, namely, firstly which choices of the right hand side b guarantee the validity of the general clique family inequality $(\mathcal{Q}, p, r, J, b)$ and, secondly, for which graphs such inequlities suffice to cover all facets of their stable set polytope. Our goal is to exhibit valid classes of general clique family inequalities which include all the non-rank facets of the stable set polytopes of claw-free graphs.

For that, we shall adjust the parameters in $(\mathcal{Q}, p, r, J, b)$ appropriately. For instance, the original clique family inequalities $(\mathcal{Q}, p)$ are the special cases $(\mathcal{Q}, p, R, 1, b)$ and with $b = (p-r)\lfloor\frac{|\mathcal{Q}|}{p}\rfloor$ as right hand side; Ben Rebea's Theorem shows that this right hand side is best possible in the case of quasi-line graphs.

We exhibit in [84] several valid inequalities of this type, for instance the following (notice that taking $J = 1$ gives back the usual clique family inequalities).

Theorem 4.45 *Every general clique family inequality* $(\mathcal{Q}, p, R, J, b)$ *with* $b = (p - R)\lfloor\frac{|\mathcal{Q}|}{p}\rfloor$ *is valid.*

As Example 4.44(c) shows that the latter choice of the right hand side is too weak for wedges, we consider in [84] also general clique family inequalities $(\mathcal{Q}, p, R, J, b)$ with $b = (p-R)\lfloor\frac{|\mathcal{Q}|}{p}\rfloor - \delta$ for some δ. We can show under which conditions such inequalities are valid if $\delta = \min\{R, p - R\}$. In addition, it turns out that several facets of claw-free graphs G with $\alpha(G) = 3$ are general clique family inequalities of this type with $\delta = J$.

The graphs with stability number two

Recall that Cook described the stable set polytopes of graphs with stability number two as follows (see [99]):

Theorem 4.46 *For every graph G with $\alpha(G) = 2$, all non-trivial facets of* $\mathrm{STAB}(G)$ *are clique-neighborhood constraints $F(Q)$*

$$2x(Q) + 1x(N'(Q)) \leq 2$$

where $Q \subseteq G$ is a clique and $N'(Q) = \{v \in V(G) : Q \subseteq N(v)\}$ s.t. $\overline{G}[N'(Q)]$ has no bipartite component.

The graphs in Example 4.44 (a),(b) show that clique-neighborhood constraints $F(Q)$ are not necessarily clique family inequalities. In order to reformulate $F(Q)$ as general clique family inequality, we are going to analyze the structure of such facets first. Clearly, $F(Q)$ is

- the clique constraint associated with Q if and only if Q is a maximal clique of G (and, therefore, $N'(Q) = \emptyset$),
- the full rank constraint associated with G if and only if $Q = \emptyset$ (and, therefore, $N'(Q) = G$), or
- a proper weak non-rank constraint if both parts Q and $N'(Q)$ are non-empty.

By definition of $N'(Q)$, the subgraph $G[Q \cup N'(Q)]$ is the complete join of Q and $N'(Q)$ and contains an odd antiwheel if $F(Q)$ is a facet. As a claw-free graph G is quasi-line if and only if it does not contain an odd antiwheel, a graph G with $\alpha(G) = 2$ is quasi-line if and only if G is near-perfect, according to the above observations.

Moreover, any non-rank facet $F(Q)$ of a graph G with $\alpha(G) = 2$ is obtained as the complete join of the clique constraint associated with Q and the full rank constraint associated with $N'(Q)$. As a first step towards our goal to express $F(Q)$ as general clique family inequality, we proved the following more general assertion on full rank constraints of graphs with stability number two, which is interesting for its own.

Theorem 4.47 *Let G be a graph with $\alpha(G) = 2$ and $\overline{C}_{2k+1}$ be a shortest odd antihole in G. Then the rank constraint $x(G) \leq 2$ is a general clique family inequality $(\mathcal{Q}, k, r, 1, 2)$ for some $0 \leq r \leq R$.*

With the help of this result, we interprete the clique-neighborhood constraint $F(Q)$ as follows:

- In a first step, we express the rank constraint $x(N'(Q)) \leq 2$ as general clique family inequality $(\mathcal{Q}', k, r, 1, 2)$ associated with a shortest odd antihole $\overline{C}_{2k+1}$ in $N'(Q)$.

- In a second step, we consider the complete join of Q and $N'(Q)$. For that, we adjust the family $\mathcal{Q}'$ in such a way that each node of $N'(Q)$ is covered *exactly* k times (by reducing the cliques in $\mathcal{Q}'$ to appropriate non-maximal cliques); we add Q to each such clique and obtain a new clique family $\mathcal{Q}$.

  By construction, each node in $N'(Q)$ is covered exactly k times by the cliques in $\mathcal{Q}$, and each node in Q exactly $|\mathcal{Q}|$ times. Choosing $p = k+1$ and $J = 1$ yields $V(\mathcal{Q}, p) = Q$ and $V(\mathcal{Q}, p-1) = N'(Q)$.

  By choosing $r = k - 1$ we finally obtain the general clique family inequality $(\mathcal{Q}, k+1, k-1, 1, 2)$

$$\begin{array}{lll} 2\; x(V(\mathcal{Q}, k+1)) & +1\; x(V(\mathcal{Q}, k)) & \leq 2 \left\lfloor \frac{2k+1}{k+1} \right\rfloor \\ 2\; x(Q)) & +1\; x(N'(Q))) & \leq 2 \end{array}$$

  as required.

Thus, starting from the above theorem we have obtained:

Theorem 4.48 *Let G be a graph with $\alpha(G) = 2$, Q be a non-maximal clique of G, and $\overline{C}_{2k+1}$ be a shortest odd antihole in $N'(Q)$. Then the clique-neighborhood constraint $F(Q)$ is a general clique family inequality of the form $(\mathcal{Q}, k+1, k-1, 1, 2)$.*

Note that *all* complete join facets of claw-free graphs are facets of type $F(Q)$ (as already the complete join of a single node and a graph with a stable set of size 3 would contain a claw). Thus, the previous theorem implies the following more general result:

Corollary 4.49 *All complete join facets of claw-free graphs and all facets associated with graphs of stability number two are general clique family inequalities.*

The graphs with stability number three

All the known difficult facets of claw-free graphs occur in the case $\alpha(G) = 3$; our goal is to describe their structure which was not well-understood so far. Starting from Giles and Trotter's construction for wedges, we analyze these facets and show that they belong to only one inequality class [83]. In addition, we express certain cases of such facets (including those induced by wedges) as general clique family inequalities and conjecture that this is possible for all facets of this type.

Giles and Trotter [47] introduced *wedges* as claw-free graphs G s.t. $\overline{G}$ has

- a unique triangle Δ,
- a spanning tree T with either two or three spokes around a central node c with endnodes in Δ where the two spokes have both even length ≥ 4 if $c \in \Delta$ and the three spokes have either all even length (but not all length 2) or all odd length (but at most one length 1) if $c \notin \Delta$,
- additional edges linking inner nodes of T to exactly one node of Δ (to avoid claws in G) which must not create another triangle $\neq \Delta$ in $\overline{G}$;

see Figure 4.11 for the three smallest examples (the dashed lines indicate the additional edges making G claw-free; the complement of the left graph is the wedge from Figure 4.10(c)).

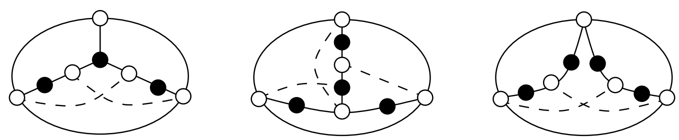


Figure 4.11: Three complements of wedges

Giles and Trotter [47] showed that each wedge G has a non-weak rank facet

$$1x(\circ) + 2x(\bullet) \leq 3$$

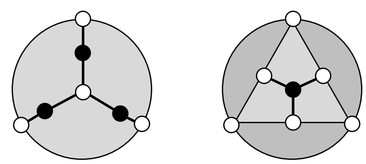
with white and black nodes as in Figure 4.11. The $|G|$ independent tight stable sets, called *roots*, correspond to the following cliques of $\overline{G}$:

- the $|G| - 1$ edges of the spanning tree T and
- the unique triangle Δ

(recall that $\omega(\overline{G}) = 3$ holds and no singleton of $\overline{G}$ can be a root due to the claw-freeness of G).

Our goal is to generalize this idea in order to obtain *all* non-complete join facets of graphs G with $\alpha(G) = 3$. (Note that it is well-known how to built complete join facets [21] and recall that no complete joins occur in claw-free graphs with $\alpha(G) > 2$.)

Consider a graph G with $\alpha(G) = 3$ and a facet $a^T x \leq b$ of STAB(G). We call $a^T x \leq b$ a *co-spanning tree constraint* if all its roots correspond to edges of a spanning tree and a triangle in $\overline{G}$. The facets of wedges are of this type, but there are also graphs G with $\alpha(G) = 3$ containing a claw and admitting such a facet: for instance, the complements of the graphs drawn in Figure 4.11 without the dashed edges are of this type, the smallest graph with this property is depicted in Figure 4.12(a) (recall that it is one of the two smallest graphs which are not weakly rank-perfect).



(a) (b)

Figure 4.12: Complements of graphs with a facet $1x(\circ) + 2x(\bullet) \leq 3$

A simple analysis shows that a co-spanning tree constraint $a^T x \leq b$ is always a non-weak rank constraint with 0,1,2-valued coefficients and right hand side 3. Furthermore, we exhibit in [84] the following:

Theorem 4.50 *Every co-spanning tree facet of the stable set polytope of a claw-free graph G with exactly one stable 3-set is a general clique family inequality* $(\mathcal{Q}, p, R, p-2, b)$ *with* $|\mathcal{Q}| = 7$, $p = 3$, *and* $b = (p-R)\lfloor \frac{|\mathcal{Q}|}{p} \rfloor - J = p$.

Note that the complement of the graph in Figure 4.12(b) has also stability number three but does not induce a co-spanning tree facet. In addition, we have to interprete non-complete join facets $a^T x \leq b$ for graphs G with $\alpha(G) = 3$ involving more than two non-zero coefficients. For that, we extend the idea of Giles and Trotter further. Instead of a spanning tree corresponding to the "edge roots" of $a^T x \leq b$, we consider a spanning forest F in $\overline{G}$ and obtain that also all $|G| - k$ edges of F are independent where k is the number of trees in F. We call a facet $a^T x \leq b$ of STAB(G) a *co-spanning forest constraint* if all its roots correspond to edges of a spanning forest F and as many triangles of $\overline{G}$ as F has components. The complement of the graph in Figure 4.12(b) is the smallest graph with such a facet (the central claw is one tree, the three remaining nodes built three further (edge-less) trees, and all four triangles are roots as well). This graph contains a claw, but there are also claw-free graphs with such facets:

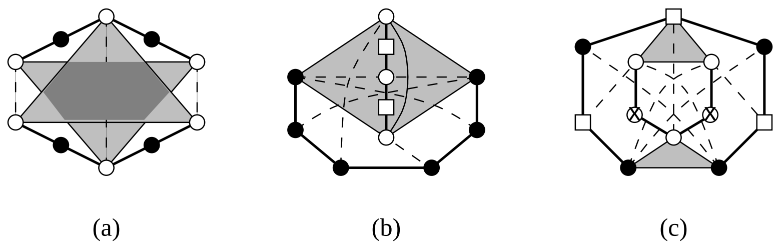


Figure 4.13: Complements of graphs with co-spanning forest facets

Example 4.51 The stable set polytopes of the three claw-free graphs whose complements are depicted in Figure 4.13 admit the following co-spanning forest facets (with node types as indicated in the figure), involving two trees (with the bold edges) and two triangles (grey-filled) each; the dashed edges make G claw-free.

(a) The complement of the graph in Figure 4.13(a) has

$$1x(\circ) + 2x(\bullet) \leq 3$$

as co-spanning forest facet [83].

(b) The complement of the graph in Figure 4.13(b) has

$$1x(\circ) + 2x(\bullet) + 3x(\Box) \leq 4$$

as co-spanning forest facet [83].

(c) The graph G whose complement is depicted in Figure 4.13(b) has

$$1x(\circ) + 2x(\bullet) + 3x(\Box) + 4x(\otimes) \leq 5$$

as co-spanning forest facet [63]; G is known as "fish in a net".

The following facets of (claw-free) graphs G with $\alpha(G) = 3$ can, however, not be interpreted as co-spanning forest constraints:

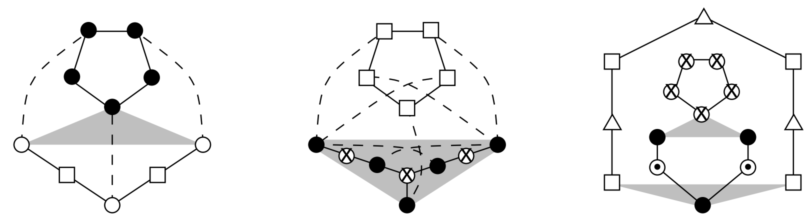


Figure 4.14: Complements of graphs with a co-spanning 1-forest facet

Example 4.52 The stable set polytopes of the graphs whose complements $\overline{G}$ are depicted in Figure 4.14 admit facets different from co-spanning forest constraints: the edge-roots form in $\overline{G}$ trees and odd holes (the bold edges; triangle-roots are again grey-filled, the dashed edges make G claw-free).

(a) The complement of the graph in Figure 4.14(a) has

$$1x(\circ) + 2x(\bullet) + 3x(\Box) \leq 4$$

as facet [47] (note that G is the graph in Figure 4.10(d)).

(b) The complement of the graph in Figure 4.14(b) has

$$2x(\bullet) + 3x(\Box) + 4x(\otimes) \leq 6$$

as facet [83] (it is obtained by combining the facets associated with the 5-hole and the smallest wedge).

(c) The complement of the graph in Figure 4.14(c) has

$$2x(\bullet) + 3x(\Box) + 4x(\otimes) + 5x(\triangle) + 6x(\odot) \leq 8$$

as facet [63]; adding appropriate edges to make G claw-free yields the graph known as "fish in a net with bubble".

These examples motivated us to generalize co-spanning forest constraints further by using not only trees but also 1-trees in $\overline{G}$, built from "root edges" of the facet. Recall that a 1-tree is obtained from a tree by adding one edge, that is we have as many edges as nodes. All edges of a 1-tree are independent if its only cycle is an odd hole; we call such 1-trees *odd*.

A facet $a^T x \leq b$ of STAB(G) is a *co-spanning 1-forest constraint* if all its roots correspond to edges of a spanning 1-forest F consisting of trees and odd 1-trees and as many triangles of $\overline{G}$ as F has tree-components.

The facets of the graphs whose complements are shown in Figure 4.14 are examples of co-spanning 1-forest constraints.

In fact, we are able to show [83] that *all* non-rank, non-complete join facets of graphs G with $\alpha(G) = 3$ are of this type:

Theorem 4.53 *Let G be a graph with $\alpha(G) = 3$ and $a^T x \leq b$ with $b \geq 3$ a non-rank, non-complete join facet of* STAB(G)*. Then $a^T x \leq b$ is a co-spanning forest constraint if b is odd and a co-spanning 1-forest constraint if b is even.*

Clearly, co-spanning tree and co-spanning forest constraints are special co-spanning 1-forest constraints. In addition, there is a class of *rank* co-spanning 1-forest constraints:

Observation 4.54 *Every rank facet associated with a graph G with $\alpha(G) = 2$ is a co-spanning 1-forest constraint.*

The reason is that the condition when a clique neighborhood constraint $F(Q)$ is facet-inducing is equivalent to require that $N'(Q)$ is a spanning 1-forest in the complementary graph: $\overline{G}[N'(Q)]$ has no bipartite component. Thus, we obtain:

Corollary 4.55 *Every clique neighborhood constraint is the complete join of a clique constraint and a co-spanning 1-forest constraint.*

Considering such complete join facets as special kinds of co-spanning 1-forest constraints (where the nodes of the clique correspond to single-node-roots with coefficient b in the complement), we deduce:

Corollary 4.56 *Every non-rank facet $a^T x \leq b$ of the stable set polytope of a claw-free graph G with $\alpha(G) \leq 3$ is a co-spanning 1-forest constraint.*

In addition, we address in [84] the problem of interpreting co-spanning 1-forest constraints as general clique family inequalities. We show:

Theorem 4.57 *Let $a^T x \leq 4$ be a co-spanning 1-forest facet of the stable set polytope of a claw-free graph G with $\alpha(G) = 3$ such that the edge roots built one tree and one odd hole in $\overline{G}$. Then $a^T x \leq 4$ is a general clique family inequality $(\mathcal{Q}, p, R, p-2, p)$ with $|\mathcal{Q}| = 9$ and $p = 4$.*

We conjecture that every co-spanning 1-forest facet of the stable set polytope of a claw-free graph can be interpreted as general clique family inequality of the form $(\mathcal{Q}, p, R, p-2, p)$ with $|\mathcal{Q}| = 2p + 1$ (note that this implies $R = 1$ and $b = (p - R)\lfloor \frac{|\mathcal{Q}|}{p} \rfloor - J$).

A conjecture on the stable set polytope of claw-free graphs

The problem of characterizing the stable set polytope of claw-free graphs has been open for almost three decades and is still open. So far, even no appealing conjecture was at hand; we are now able to formulate such a conjecture.

For that, we combine Cook's result in the case $\alpha(G) = 2$, our results from the previous subsection for the case $\alpha(G) = 3$, and a conjecture of Stauffer [101] for the case $\alpha(G) \geq 4$ which is motivated as follows.

There are two reasons to beliefe that a claw-free graph G is not too far from being quasi-line as soon as G contains a stable set of size four: On the one hand, Chudnovsky and Seymour [19] established a decomposition theorem for claw-free graphs G with $\alpha(G) \geq 4$ asserting that such graphs are either fuzzy circular interval graphs or can be composed from certain linear interval strips. On the other hand, Fouquet [38] showed that a connected claw-free graph with $\alpha(G) \geq 4$ is quasi-line if and only if G does not contain a 5-wheel.

Stauffer exhibited in [101] that the 5-wheels play indeed a central role to describe the facets of claw-free but not fuzzy circular interval graphs with $\alpha(G) \geq 4$, as 5-wheel constraints can be sequentially lifted to more general inequalities of the form $1x(A) + 2x(B) \leq 2$ with suitable node subsets A and B. This led Stauffer to conjecture:

Conjecture 4.58 [101] *The stable set polytope of a claw-free but not fuzzy circular interval graph G with $\alpha(G) \geq 4$ is given by*

- *non-negativity constraints,*
- *rank constraints,*
- *lifted 5-wheel constraints.*

As each lifted 5-wheel constraint is in particular a clique neighborhood constraint, we can combine this conjecture for the case of claw-free graphs with $\alpha(G) \geq 4$ and the results in the cases with $\alpha(G) = 2, 3$ to the following conjecture on non-rank facets for general claw-free graphs:

Conjecture 4.59 *Let G be a claw-free graph, $a^T x \leq b$ a non-rank facet of* STAB(G)*, and G_a the subgraph of G induced by nodes i with $a_i > 0$. Then $a^T x \leq b$ is a*

- *clique neighborhood constraint if $\alpha(G_a) = 2$;*
- *co-spanning 1-forest constraint if $\alpha(G_a) = 3$;*
- *clique family inequality if $\alpha(G_a) \geq 4$.*

Note that an affirmative answer to this conjecture would verify that indeed all complicated facets associated with claw-free graphs occur in the case $\alpha(G_a) = 3$ only, as clique-neighborhood constraints are 0,1,2-valued weak rank constraints and clique family inequalities have also only two different non-zero coefficients. The latter are weak rank constraints, provided Conjecture 4.33 is true. If this is indeed the case, then we have that a claw-free graph is weakly rank-perfect iff it has no non-rank facet associated with a subgraph having stability number three.

Moreover, we can specify Conjecture 4.59 further, taking into account that every clique neighborhood constraint is a special co-spanning 1-forest constraint (Corollary 4.55).

Conjecture 4.60 *A non-rank facet of the stable set polytope of a claw-free graph is either a co-spanning 1-forest constraint or a clique family inequality.*

The structure of rank facets of claw-free graphs is well-understood by Galluccio and Sassano [41]: all rank facets rely on cliques, line graphs, and webs. We combine several results and conjectures together and obtain as counterpart for the non-rank facets:

Conjecture 4.61 *Every non-rank facet of the stable set polytopeof a claw-free graph is based on*

- *an odd antiwheel (clique-neighborhood constraint);*
- *a co-spanning 1-forest;*
- *a web (clique family inequality).*

The latter conjectures provide the answer to the long-time open problem of having an idea of the structure of non-rank facets of claw-free graphs.

In addition, we showed that all clique-neighborhood constraints and several co-spanning 1-forest constraints are general clique family inequalities. We conjecture that similar constructions show that all co-spanning 1-forest constraints can be interpreted as general clique family inequalities. An affirmative answer to this conjecture would imply that all facets of claw-free graphs are certain types of general clique family inequalities. If this is true, also the polyhedral aspect of the stable set problem for claw-free graphs is an extension of Edmonds' description of the matching polytope, as every general clique family inequality extends Edmonds' odd set inequalities for the matching polytope.

Chapter 5

Imperfection ratio and imperfection index

As we have $\mathrm{STAB}(G) \subset \mathrm{QSTAB}(G)$ for all imperfect graphs G, it is natural to use the difference between the two polytopes in order to determine how far a certain imperfect graph is away from being perfect. Two ways to describe the gap between the two polytopes is to look at the constraints that have to be added to $\mathrm{QSTAB}(G)$ in order to arrive at $\mathrm{STAB}(G)$ or on the Chvátal-rank of $\mathrm{QSTAB}(G)$, as done in Chapter 4.

Two other ways are to look at the dilation ratio of the two polytopes or at the disjunctive rank of $\mathrm{QSTAB}(G)$, leading to imperfection ratio and imperfection index of a graph.

Gerke and McDiarmid introduced in [45] the *imperfection ratio* $\mathrm{imp}(G)$ of a graph G as some asymptotic slope of the minimal χ-binding function and showed that $\mathrm{imp}(G)$ can be expressed as the dilation ratio of $\mathrm{STAB}(G)$ and $\mathrm{QSTAB}(G)$ by

$$\mathrm{imp}(G) = \min\{t : \mathrm{QSTAB}(G) \subseteq t\ \mathrm{STAB}(G)\}.$$

Aguilera et al. [1] introduced the *imperfection index* $\mathrm{imp_I}(G)$ as the disjunctive index of $\mathrm{QSTAB}(G)$, that is

$$\mathrm{imp_I}(G) = \min\{|J| : P_J(\mathrm{QSTAB}(G)) = \mathrm{STAB}(G), J \subseteq V\}$$

where $P_J(\mathrm{QSTAB}(G)) = \mathrm{conv}\{x \in \mathrm{QSTAB}(G) : x_j \in \{0,1\}, j \in J\}$.

We firstly discuss possibilities for determining $\mathrm{imp}(G)$ and $\mathrm{imp_I}(G)$ based on the knowledge of facet-defining inequalities for $\mathrm{STAB}(G)$ and the extreme points of $\mathrm{QSTAB}(G)$, respectively, see next section.

If the complete facet-system of STAB(G) is known for a certain graph class, and we are able to determine the strength of all facets, then we certainly can determine the imperfection ratio for such graphs. We use this strategy for several superclasses of perfect graphs to obtain upper bounds on imp(G) for such graphs, see Section 5.2.

Regarding the imperfection index, we first discuss its graph-theoretical interpretation as the minimum cardinality of a node subset $J \subset V(G)$ such that $G - J$ is perfect or, equivalently, as the cardinality of a minimum node subset meeting all minimal imperfect subgraphs of G (see Section 5.3.1).

We further address the problem which graph classes admit a small imperfection index. Unfortunately, we obtain in [61] that the imperfection index cannot be bounded for many graph classes which are close to perfect graphs in some other sense. In particular, our results indicate that there are many graph classes with an unbounded imperfection index, including near-perfect graphs, t-perfect and h-perfect graphs, line graphs, antiwebs, and general rank-perfect graphs.

Comparing the two concepts, we finally conclude that the imperfection index measures imperfection for several graph classes much more rough than the imperfection ratio (see Section 5.4 for more details and some suggestions for refining the concept).

5.1 The extreme points of QSTAB(G)

By the definition of the imperfection index $\mathrm{imp_I}(G)$ as the disjunctive index of QSTAB(G), the knowledge on the extreme points of QSTAB(G) clearly helps to determine $\mathrm{imp_I}(G)$.

We infer the same for the imperfection ratio. Let $\mathcal{F}(G) = \{a \in [0,1]^{|V|} : a^T x \leq 1 \text{ facet of STAB}(G)\}$ denote the set of all normal vectors of nontrivial facets of STAB(G) (scaled to have right hand side equal to one). Hence, t STAB(G) equals $\{x \in \mathbb{R}_+^{|V|} : a^T x \leq t \ \forall a \in \mathcal{F}(G)\}$ and QSTAB(G) fits in it if, for all $y \in \mathrm{QSTAB}(G)$, $a^T y \leq t$ holds. Thus, we have

$$\mathrm{imp}(G) = \max\{a^T y : a \in \mathcal{F}(G), y \in \mathrm{QSTAB}(G)\}$$

as any smaller t violates $a^T y \leq t$ for some $a \in \mathcal{F}(G)$ and $y \in \mathrm{QSTAB}(G)$. It clearly suffices to consider nontrivial facets of STAB(G) and (fractional) extreme points of QSTAB(G) only.

Since QSTAB(G) is the anti-blocker of STAB($\overline{G}$), every facet of STAB($\overline{G}$) is an extreme point of QSTAB(G) [40]. But not all extreme points of QSTAB(G) are conversely of importance for the facet-defining system of its anti-blocker: here it suffices to consider the *dominating* extreme points $x \in \mathrm{STAB}(\overline{G})$ where $y \geq x$ implies $y \notin \mathrm{STAB}(\overline{G})$. That way, we obtain

$$\begin{aligned} \mathrm{imp}(G) &= \max\{a^T y : a \in \mathcal{F}(G), y \in \mathcal{F}(\overline{G})\} \\ &= \max\{a^T y : a \in \mathrm{QSTAB}(\overline{G}), y \in \mathrm{QSTAB}(G)\}. \end{aligned}$$

Since a complete description of STAB(G) (or STAB($\overline{G}$)) is typical not available, we do not know the dominating extreme points of the corresponding clique relaxations either. However, it turns out that also knowledge on *dominated* extreme points is of interest (as only the intersection $\mathrm{supp}(a) \cap \mathrm{supp}(y)$ for $a \in \mathcal{F}(\overline{G})$ and $y \in \mathrm{QSTAB}(G)$ gives a contribution).

This motivates us to fully characterize the extreme points of QSTAB(G) as follows [62].

Theorem 5.1 *A vector $a \neq 0$ is an extreme point of* QSTAB(G) *if and only if there is a subgraph $\overline{G}'$ of $\overline{G}$ such that supp(a) belongs to $\mathcal{F}(\overline{G}')$.*

Note that, for an extreme point a of QSTAB(G), $a^T x \leq 1$ is not required to be a facet of STAB($\overline{G}$) but only supp(a) of STAB($\overline{G}'$).

Example 5.2 Let G be a 5-wheel with center c. Its complement $\overline{G}$ is a 5-hole with an isolated node c. Obviously, $\overline{G}$ has this 5-hole C_5 as only facet-inducing subgraph different from a clique, thus QSTAB(G) has exactly one fractional extreme point, namely $(\chi^{C_5}, \frac{1}{2}) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0)$. Conversely, G has two facet-inducing subgraphs different from a clique, the 5-hole C_5 and G itself, producing the rank constraint $x(C_5) \leq 2$ and the non-rank constraint $x(C_5) + 2x_c \leq 2$. Accordingly, QSTAB($\overline{G}$) has the two fractional extreme points $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0)$ and $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1)$. (Note that $x(C_5) \leq 2$ is not a facet of STAB(G) but only of STAB(C_5).)

Thus, we established a 1-1-correspondence between the extreme points of QSTAB(G) and facet-inducing *sub*graphs of $\overline{G}$. This complete characterization is of interest for our purpose since it helps to identify both

- minimum node subsets $J \subset V$ such that $P_J(\mathrm{QSTAB}(G)) = \mathrm{STAB}(G)$ holds (here it even suffices to consider minimal extreme points which do not dominate any other extreme point of QSTAB(G)),
- facet-defining subgraphs of G and $\overline{G}$ such that the associated extreme point y of QSTAB(G) and normal vector $a \in \mathcal{F}(G)$ satisfy $\mathrm{supp}(y) = \mathrm{supp}(a)$ and $y^T a = \mathrm{imp}(G)$.

With the help of Theorem 5.1 we are, in addition, able to easily reprove several famous results.

5.1.1 Characterizing perfect graphs

The assertion of Theorem 5.1 follows for perfect graphs by the Perfect Graph Theorem and $\mathrm{STAB}(G) = \mathrm{QSTAB}(G)$: If G is perfect, then $\mathrm{QSTAB}(G)$ has integral extreme points only, namely $\chi^{G'}$ where $G' \subseteq G$ is an arbitrary stable set; as $\overline{G}$ is perfect as well, its only facet-inducing subgraphs are all cliques $\overline{G}' \subseteq \overline{G}$.

Conversely, we obtain both the Perfect Graph Theorem and the polyhedral characterization of perfect graphs with the help of Theorem 5.1 as follows. Recall the following chain of inequalities and equations, obtained by dropping or adding integrality constraints and linear programming duality:

$$\begin{aligned}
\alpha(G,c) =\ & \max\{c^T x : x \in \mathrm{STAB}(G)\} \\
=\ & \max\{c^T x : x(Q) \leq 1 \ \forall \text{cliques } Q \subseteq G,\ x \in \{0,1\}^{|G|}\} \\
\leq\ & \max\{c^T x : x(Q) \leq 1 \ \forall \text{cliques } Q \subseteq G,\ x \geq 0\} \\
=\ & \min\{\textstyle\sum_Q y_Q : \sum_{Q \ni i} y_Q \geq c_i \ \forall i \in G,\ y_Q \geq 0 \ \ \forall \text{cliques } Q \subseteq G\} \\
\leq\ & \min\{\textstyle\sum_Q y_Q : \sum_{Q \ni i} y_Q \geq c_i \ \forall i \in G,\ y_Q \in \mathbb{Z}_+ \ \forall \text{cliques } Q \subseteq G\} \\
=\ & \overline{\chi}(G,c)
\end{aligned}$$

The last program is an integer programming formulation of the weighted clique cover problem, the intermediate steps yield the fractional stability and clique cover numbers, $\alpha_f(G,c)$ and $\overline{\chi}_f(G,c)$, which are equal by linear programming duality.

If $\mathrm{STAB}(G) = \mathrm{QSTAB}(G)$ then obviously $\alpha(G,c) = \alpha_f(G,c)$ follows; in particular $\alpha_f(G,c)$ is integer valued for all $c \geq 0$ and comes from an integer solution. Thus, also the optimal solution of the fractional clique cover problem is integral in this case, hence $\overline{\chi}_f(G,c)$ is integer valued for all $c \geq 0$ and coincides with $\overline{\chi}(G,c)$, i.e., we have equality through the whole chain (totally dual integrality). Turning to the complementary graph yields $\omega(\overline{G},c) = \chi(\overline{G},c)$ for all $c \geq 0$. This is particularly true for all 0/1-weightings, implying equality for the unweighted case $\omega(\overline{G}', \mathbb{1}) = \chi(\overline{G}', \mathbb{1})$ for all induced subgraphs $\overline{G}'$ of $\overline{G}$. Hence, $\overline{G}$ cannot contain any minimal imperfect graph as induced subgraph (as clique and chromatic number would differ for such subgraphs) and is, therefore, perfect.

Conversely, if $\overline{G}$ contains a minimal imperfect subgraph $\overline{G}'$, then we have $\omega(\overline{G}', \mathbb{1}) < \chi(\overline{G}', \mathbb{1})$, implying $\alpha(G',c) < \overline{\chi}(G',c)$. We obtain $\mathrm{STAB}(G) \subset \mathrm{QSTAB}(G)$ since otherwise we would have equality through the whole chain, in particular $\alpha(G',c) = \alpha_f(G',c) = \overline{\chi}_f(G',c) = \overline{\chi}(G',c)$, a contradiction.

This implies that $\mathrm{STAB}(G) = \mathrm{QSTAB}(G)$ *if and only if* $\overline{G}$ *is perfect.* With the help of this fact and Theorem 5.1 we easily obtain the following:

Corollary 5.3 *For any graph G, the following assertions are equivalent:*

(1) *G is perfect;*
(2) $\mathrm{STAB}(\overline{G}) = \mathrm{QSTAB}(\overline{G})$*;*
(3) $\mathrm{STAB}(G) = \mathrm{QSTAB}(G)$*;*
(4) *$\overline{G}$ is perfect.*

5.1.2 Near-perfect graphs

For minimally imperfect graphs, Theorem 5.1 corresponds to the well-known characterization of Padberg [75], stating that a graph G is minimally imperfect if and only if STAB(G) has the full rank facet as only nontrivial, nonclique facet and QSTAB(G) has exactly one fractional extreme point, namely $(G, \frac{1}{\omega(G)}\mathbb{1})$.

As $\overline{G}$ is minimally imperfect as well by the Perfect Graph Theorem, its only facet-inducing subgraph different from a clique is $\overline{G}$ itself, producing the full rank constraint $x(G) \leq \alpha(\overline{G}) = \omega(G)$. Therefore, $(G, \frac{1}{\omega(G)}\mathbb{1})$ is the only fractional extreme point of QSTAB(G), and conversely.

For properly near-perfect graphs G, we obtain [62] that both QSTAB(G) and QSTAB($\overline{G}$) have at least two fractional extreme points. Even worse, the following example exhibits a sequence of near-perfect graphs G where the number of fractional extreme points of QSTAB(G) and QSTAB($\overline{G}$) tends to infinity.

Example 5.4 According to Theorem 4.4 and Theorem 4.7, all webs with stability number two are near-perfect, that are the webs W_n^k with $n < 3(k+1)$. With the help of Trotter's formula [104], it is easy to check that the odd antihole $W_{2(l+1)+1}^l$ is an induced subgraph of $W_{2(k+1)+2}^k$ for all $k \geq 2$ if $l \leq \frac{k}{2}$. In particular, we have

$$W_{2(l+1)+1}^l \subset W_{2(2l+1)+2}^{2l} \text{ for all } l \geq 1.$$

Since, by this choice, the number of nodes $2(l+1)+1$ of the odd antihole does not divide the number of nodes $2(2l+1)+2$ of the whole web, we obtain that there are $2(2l+1)+2$ *different* odd antiholes in $W_{2(2l+1)+2}^{2l}$, namely,

$$C(i) = \{i, i+1, (i+1)+2, \ldots, (i+1)+2l, i+2(l+1), \ldots, i+2(2l+1)\}$$

for every node i (thus, we choose i, the next node $i+1$, then l times the next but one node, once more the next node, and finally $l+1$ times the next but one node again).

Thus, the web $W^{2l}_{2(2l+1)+2}$ contains as many different odd antiholes as nodes (resp. the antiweb $\overline{W}^{2l}_{2(2l+1)+2}$ as many different odd holes as nodes). According to Theorem 5.1, each of them yields a fractional extreme point of $\mathrm{QSTAB}(\overline{W}^{2l}_{2(2l+1)+2})$ (resp. $\mathrm{QSTAB}(W^{2l}_{2(2l+1)+2})$). Thus, the number of fractional extreme points is at least $2(2l+1)+2$ for both and tends to infinity if l does. In particular, the odd antiholes in $W^{2l}_{2(2l+1)+2}$ correspond to fractional extreme points of $\mathrm{QSTAB}(\overline{W}^{2l}_{2(2l+1)+2})$ which are *not* dominating (as $W^{2l}_{2(2l+1)+2}$ does not have odd antihole facets).

5.1.3 Half-integral fractional stable set polytopes

We say that an inequality $a^T x \leq b$ is given in its integer form if all entries in a and the right hand side b are integers with greatest common divisor 1 (i.e., if it cannot be scaled down to smaller integer values).

An immediate consequence of Theorem 5.1 is the following.

Corollary 5.5 $\mathrm{QSTAB}(G)$ *is half-integral if and only if any facet-producing subgraph* $\overline{G}'$ *of* $\overline{G}$ *induces a facet having rhs* ≤ 2 *in its integer form.*

This is clearly true for all graphs G such that $\alpha(\overline{G}) \leq 2$ holds. This implies, for any graph G with $\omega(G) \leq 2$, that $\mathrm{QSTAB}(G)$ is half-integral. As for such graphs $\mathrm{QSTAB}(G)$ obviously coincides with the edge constraint stable set polytope $\mathrm{ESTAB}(G)$ (given by nonnegativity and edge constraints only), the above corollary yields the well-known result that $\mathrm{ESTAB}(G)$ has half-integral extreme points only.

Further examples of such graphs are line graphs. Recall that Shepherd [99] gave a complete description of the stable set polytopes of their complements by showing that the only nontrivial facets of stable set polytopes of complements of line graphs are constraints

$$\sum_{i \leq k} x(A_i) + 2x(Q) \leq 2$$

associated with complete joins of odd antiholes $A_1, \ldots, A_k$ and a clique Q. The above corollary implies, that the fractional stable set polytopes of line graphs are half-integral.

As the stable set polytopes of line graphs correspond to the matching polytope introduced and described by Edmonds [33] (see Theorem 4.17), we obtain an alternative proof that the fractional matching polytope has half-integral extreme points only.

5.2 Bounds for the imperfection ratio of several graph classes

There does not exist a general *upper bound* on the imperfection ratio, as observed in [45], due to the following reason. Mycielski constructed a famous series of graphs $G_0, G_1, G_2, \ldots$ with $\omega(G_i) = 2$ for all i but $\chi(G_i) = 2 + i$ (recall that $G_0 = K_2$, $G_1 = C_5$, and G_2 is the well-known Grötzsch graph); Larsen, Propp, and Ullman proved the unexpected recurrence

$$\chi_f(G_{i+1}) = \chi_f(G_i) + \frac{1}{\chi_f(G_i)}.$$

As $\mathrm{imp}(G) = \frac{\chi_f(G)}{2}$ holds for any triangle-free graph G by [45], we obtain

$$\mathrm{imp}(G_i) \to \infty \text{ for } i \to \infty.$$

Thus the imperfection ratio of the class of graphs with clique number two (containing the series $G_0, G_1, G_2, \ldots$) cannot be bounded. Invariance under taking complements yields that there does also not exist an upper bound for the imperfection ratio of the graphs with stability number two; the same is true for all superclasses as, for instance, claw-free graphs and weakly rank-perfect graphs.

The following two *lower bounds* for the imperfection ratio of any graph are known from [45]:

$$\begin{array}{rcl} \mathrm{imp}(G) & \geq & \frac{|G|}{\alpha(G)\omega(G)} \\ \mathrm{imp}(G) & \geq & \mathrm{imp}(G') \ \forall G' \subseteq G \end{array}$$

Combining the two bounds yields

$$\mathrm{imp}(G) \quad \geq \quad \max\{\frac{|G'|}{\alpha(G')\omega(G')} : G' \subseteq G\}. \tag{5.1}$$

For any imperfect graph G, this gives rise to two questions: *which subgraphs are the crucial ones* and *when does equality holds?* Gerke and McDiarmid answered these questions in [45] for line graphs, h-perfect, and co-h-perfect graphs by proving that their imperfection ratio relies on odd (anti)holes only and is, therefore, bounded by $\leq \frac{5}{4}$. We are going to extend these results further, see next subsection. Moreover, we present several graph classes whose imperfection ratio relies on antiwebs only and is bounded by $\leq \frac{3}{2}$, see Subsection 5.2.2.

5.2.1 Graph classes with an imperfection ratio $\leq \frac{5}{4}$

The following results from [45] describe the imperfection ratio by

$$\begin{aligned} \operatorname{imp}(G) &= \max\{\tfrac{2k+1}{2k} : C_{2k+1} \subseteq G\} \\ &= \{\tfrac{2k+1}{2k} : C_{2k+1} \text{ shortest odd hole in } G\} \end{aligned}$$

whenever G is a line graph or h-perfect and by

$$\begin{aligned} \operatorname{imp}(G) &= \max\{\tfrac{2k+1}{2k} : C_{2k+1}, \overline{C}_{2k+1} \subseteq G\} \\ &= \{\tfrac{2k+1}{2k} : 2k+1 \text{ length of shortest odd (anti)hole in } G\} \end{aligned}$$

for all co-h-perfect graphs G where $\operatorname{STAB}(G)$ is given by rank constraints associated with cliques, odd holes, and odd antiholes only. As the C_5 is the shortest odd (anti)hole, this implies that $\operatorname{imp}(G) \leq \frac{5}{4}$ holds for all graphs G belonging to one of these classes. We are going to generalize the above results from [45] for p-perfect graphs (with rank constraints associated with arbitrary partitionable graphs, extending the results on odd (anti)holes, h-perfect, and co-h-perfect graphs) and for semi-line graphs (extending the results on line graphs).

We start with the result on p-perfect graphs from [61]:

Theorem 5.6 *For any p-perfect graph G, we have*

$$\operatorname{imp}(G) = \max\{\tfrac{\alpha'\omega'+1}{\alpha'\omega'} : P \subseteq G \text{ partitionable}\}$$

where $\alpha' = \alpha(P)$ and $\omega' = \omega(P)$ holds.

As a consequence, we have $\operatorname{imp}(G) \leq \frac{5}{4}$ whenever $\operatorname{STAB}(G)$ is given by nonnegativity, clique, and partitionable graph constraints only (since, again, the C_5 is the smallest crucial graph).

Now, we are going to extend the result on the imperfection ratio of line graphs with the help of the following structural result on quasi-line graphs from Chudnovsky and Seymour:

Theorem 5.7 [19] *A connected quasi-line graph G is either a fuzzy circular interval graph or* $\operatorname{STAB}(G)$ *is given only by nonnegativity constraints, clique constraints, and rank clique family inequalities* $(\mathcal{Q}, 2)$

$$\sum_{i \in I(\mathcal{Q},2)} x_i \leq \frac{|\mathcal{Q}|-1}{2}$$

with clique families of odd size $|\mathcal{Q}|$.

This theorem describes the stable set polytopes of a superclass of line graphs, the semi-line graphs. We could extend the result on the imperfection ratio for line graphs from [45] to this larger class in [27]:

Lemma 5.8 *For any semi-line graph G, we have* $\mathrm{imp}(G) \leq \frac{5}{4}$.

In addition, we discussed the imperfection ratio of almost-perfect graphs in [61] and could obtain a rough upper bound of at most two (but we believe that the truth is also $\frac{5}{4}$).

5.2.2 Graph classes with an imperfection ratio $\leq \frac{3}{2}$

In this section we characterize and bound the imperfection ratio for a-perfect graphs, where STAB(G) is given by rank constraints associated with antiwebs only, thereby generalizing the above results from [45] for t-perfect, h-perfect, and co-h-perfect graphs, as cliques, odd holes, and odd antiholes are special antiwebs. We achieved in [27]:

Theorem 5.9 *For any a-perfect graph G, the imperfection ratio is given by*

$$\mathrm{imp}(G) \quad = \quad \max\{\tfrac{n'}{\alpha'\omega'} : K_{n'/\alpha'} \subseteq G\}$$

where $\omega' = \lfloor n'/\alpha' \rfloor$ holds.

Although we do not know yet the characterization of all antiwebs $K_{n/\alpha}$ with $\mathrm{imp}(K_{n/\alpha}) = \frac{n}{\alpha\omega}$, we can provide a polynomial time algorithm to compute $\mathrm{imp}(K_{n/\alpha})$ for any antiweb [27]. In addition, we established an upper bound on the imperfection ratio of antiwebs by

$$\mathrm{imp}(K_{n/\alpha}) < \frac{3}{2}.$$

As a consequence, we can characterize and bound the imperfection ratios of all a-perfect graphs, including all antiwebs [110], convex-round graphs [27], and the complements of fuzzy circular interval graphs [112]:

Corollary 5.10 *For any a-perfect G, the imperfection ratio is* $\mathrm{imp}(G) < \frac{3}{2}$.

Combining this bound with the results from the previous subsection and the invariance of the imperfection ratio under taking complements enables us to bound the imperfection ratio of quasi-line graphs and near-bipartite graphs by $\frac{3}{2}$.

Outgoing from Shepherd's description of the stable set polytopes of near-bipartite graphs (see Theorem 4.13) and our considerations on complete-join-facets, we could further establish the following [27]:

Theorem 5.11 *For any near-bipartite graph G, we have*

$$\mathrm{imp}(G) \quad = \quad \max\{\tfrac{n'}{\alpha'\omega'} : K_{n'/\alpha'} \subseteq G\} \;\; < \;\; \tfrac{3}{2}$$

where $\omega' = \lfloor n'/\alpha' \rfloor$ holds.

Consequently, the imperfection ratio of any near-bipartite (resp. quasi-line) graph is characterized by its induced antiwebs (resp. webs) only and is, therefore, bounded by $\frac{3}{2}$.

Note that this result is best possible as, on the one hand, there exist antiwebs with an imperfection ratio arbitrarily close to $\frac{3}{2}$ [27] and, on the other hand, claw-free graphs certainly constitute a superclass of quasi-line graphs, but there is no upper bound for the imperfection ratio of claw-free graphs (as they contain all graphs with stability number two, including the complements of the Mycielski graphs with unbounded imperfection ratio).

Moreover, there is some hope that the imperfection ratio of two other classes is bounded by $\frac{3}{2}$.

Firstly, the structure of a circular-perfect graph relies, by definition, on its induced circular cliques, i.e., on its induced antiwebs. This indicates that also the imperfection ratio of circular-perfect graphs should rely on its induced antiwebs. According to the above consideration, this would imply that the imperfection ratio of circular-perfect graphs is bounded by $\frac{3}{2}$.

Secondly, Gerke and McDiarmid [45] showed that for any planar graph G we have that $\mathrm{imp}(G) \leq \frac{11}{6}$, but conjectured that the truth is $\mathrm{imp}(G) \leq \frac{3}{2}$.

5.2.3 Imperfection ratio and several graph constructions

Gerke and McDiarmid [45] studied the behavior of the imperfection ratio under well-known perfection-preserving graph composition techniques. The expectation is that the imperfection ratio should be invariant under applying such techniques; the first result in this direction is clearly the invariance under taking complements.

A further well-known perfection-preserving composition technique is the identification of two graphs G_1 and G_2 in a clique Q. To learn about the imperfection ratio of the resulting graph $G_1 +_Q G_2$, it is helpful to make use of Chvátal's characterization [21] of the facet-system describing the stable set polytope of $G_1 +_Q G_2$: it is simply the union of all the facets of $\mathrm{STAB}(G_1)$ and $\mathrm{STAB}(G_2)$. One can easily infer that

$$\mathrm{imp}(G_1 +_Q G_2) = \max\{\mathrm{imp}(G_1), \mathrm{imp}(G_2)\}$$

holds. This result includes in particular the special cases of taking disjoint unions and, by the invariance of the imperfection ratio under complementation, taking complete joins of two graphs.

We extended this result in [27] to complete joins of facet-producing *sub*graphs. Since $\mathrm{imp}(G) = \max\{a^T y : a \in \mathcal{F}(G), y \in \mathrm{QSTAB}(G)\}$, this suggests that

$$\mathrm{imp}(G) = \max\{\mathrm{imp}(G_a) : a \in \mathcal{F}(G)\}$$

where G_a denotes the subgraph of G induced by all nodes i with $a_i \neq 0$. If such a support graph G_a is the complete join of graphs $G_1, \ldots, G_k$, then

$$\mathrm{imp}(G_1 * \ldots * G_k) = \max\{\mathrm{imp}(G_1), \ldots, \mathrm{imp}(G_k)\}$$

holds by [45]. Hence, we have for any graph G where all non-trivial, non-clique facets of $\mathrm{STAB}(G)$ are complete join facets of the form

$$\sum_{i \leq k} \frac{1}{\alpha(G_i)} x(G_i) \leq 1$$

associated with $G_1 * \ldots * G_k$ and $G_i \in \mathcal{C}$, that $\mathrm{imp}(G)$ relies on those subgraphs $G_i \subseteq G$ only which belong to $\mathcal{C}$. This implies:

Theorem 5.12 *For every joined $\mathcal{C}$-perfect graph G, we have that*

$$\mathrm{imp}(G) = \max\{\mathrm{imp}(G_i) : G_i \subseteq G, G_i \in \mathcal{C}\}$$

holds.

This result was, in fact, the key tool to characterize the imperfection ratio of near-bipartite graphs (Theorem 5.11). Moreover, all facets of the stable set polytopes of co-line graphs are complete joins of odd antiholes and a clique by Shepherd [99]. Hence, the above result yields an alternative proof that the imperfection ratio of co-line graphs relies on its induced odd antiholes only and is bounded by $\frac{5}{4}$.

Furthermore, Gerke and McDiarmid [45] studied the behavior of the imperfection ratio under taking lexicographic products. Recall that, for two graphs G and H, substituting H for all nodes of G yields their lexicographic product $G \times H$ and

$$\mathrm{imp}(G \times H) = \mathrm{imp}(G) \cdot \mathrm{imp}(H)$$

holds. Thus, the imperfection ratio cannot be bounded for any class $\mathcal{C}$ of graphs closed under substitution (and, therefore, closed under taking lexicographic products) and containing one imperfect graph G as

$$\operatorname{imp}(G^i) \to \infty \text{ for } i \to \infty$$

since $\operatorname{imp}(G) > 1$. Recall that we achieved this way that the imperfection ratio cannot be bounded for normal graphs and for $(C_5, C_7, \overline{C}_7)$-free graphs (see Section 3.4).

We use this result of Gerke and McDiarmid [45] to address for three further superclasses of perfect graphs, namely for near-perfect, rank-perfect, and circular-perfect graphs, the problem of upper bounds on the imperfection ratio. Such an upper bound on the imperfection ratio for all graphs in a class $\mathcal{G}$ can exist only if $\mathcal{G}$ is at most closed under substituting *perfect graphs* for nodes of any graph $G \in \mathcal{G}$.

We established that the near-perfect, rank-perfect, and circular-perfect graphs admit this property.

Theorem 5.13 *Let G be obtained by substituting a node of a graph G_1 by a graph G_2. G is rank-perfect* if and only if *G_1 is rank-perfect and G_2 is perfect.*

The proof of this result relies on the characterization of the facet-defining system of STAB(G), outgoing from the facts of STAB(G_1) and STAB(G_2) due to [21, 30], see [61].

In addition, we have for the class of circular-perfect graphs:

Theorem 5.14 *Let G be obtained by substituting a node v of a graph G_1 by a graph G_2. G is circular-perfect* only if *G_1 is circular-perfect and G_2 is perfect.*

The proof relies on the easy fact that any imperfect graph G_2 contains an odd hole or odd antihole C and, by the construction of G, each neighbor of v in G_1 is totally joint to G_2 and, thus, to C. Hence, G contains the odd wheel or odd antiwheel $C * v$ and, therefore, a minimal circular-imperfect subgraph due to Theorem 2.19.

Thus, Theorem 5.13 implies that all subclasses of rank-perfect graphs satisfy this necessary condition for the existence of an upper bound on the imperfection ratio, while Theorem 5.14 establishes this for all subclasses of circular-perfect graphs. As circular-perfect graphs strongly rely on antiwebs, we hope particularly that their imperfection ratio is also bounded by $\frac{3}{2}$.

5.3 About the imperfection index

We first discuss a graph-theoretical characterization of the imperfection index as the cardinality of a minimum node subset meeting all minimal imperfect subgraphs of G. This motivates to introduce the hypergraph $\mathcal{I}(G)$ with the same node set as G and all node subsets as hyperedges which induce a minimal imperfect subgraph of G. By the invariance of perfection under taking complements, $\mathcal{I}(G)$ clearly equals $\mathcal{I}(\overline{G})$. Our results imply that finding a minimum vertex cover in $\mathcal{I}(G)$ is equivalent to computing $\mathrm{imp_I}(G)$; thus $\mathrm{imp_I}(G) = \mathrm{imp_I}(\overline{G})$ holds for all graphs and evaluating $\mathrm{imp_I}(G)$ is NP-hard (see Section 5.3.1).

Next we consider the behavior of the imperfection index under certain graph composition techniques, namely, under taking disjoint unions, clique identifications, and lexicographic products (Section 5.3.2). For the latter, we characterize how several classes of rank-perfect graphs behave under substitution. Our results have, however, negative consequences for possible upper bounds on the imperfection index for several graph classes which are close to perfection in some (other) sense.

5.3.1 The imperfection index in graph-theoretical terms

Recall that Balas et al. [3] introduced the disjunctive procedure for binary linear programs as a way to obtain a complete description of the integer polytope from the polytope described by the linear relaxation. Let $V = \{1, \ldots, n\}$ denote the set of binary variables. For a subset $J = \{i_1, \ldots, i_j\}$ of the variables,

$$P_J(X) = \mathrm{conv}\{x \in X : x_j \in \{0, 1\}, j \in J\}$$

follows. It is shown in [3] that $P_J(X) = P_{i_1}(P_{i_2}(\ldots P_{i_j}(X)))$. Obviously, $P_V(X) = \mathrm{conv}(X \cap \{0, 1\}^n)$ holds, but also proper subsets can have this property. This result allows to define the disjunctive index of a polytope X as the minimum size of a set $J \subseteq V$ such that $P_J(X) = \mathrm{conv}(X \cap \{0, 1\}^n)$.

As the imperfection index of a graph G is defined as the disjunctive index of $\mathrm{QSTAB}(G)$, the following result directly follows from the definition.

Lemma 5.15 (Ceria [13]) $P_j(\mathrm{QSTAB}(G)) = \mathrm{STAB}(G)$ *holds if and only if* $G[V - j]$ *is perfect.*

Corollary 5.16 *We have* $\mathrm{imp_I}(G) = 1$ *if and only if there exists a node* $j \in V$ *such that* $G[V - j]$ *is perfect.*

This shows in particular that the almost-perfect graphs are exactly those graphs G with and imperfection index at most one (as they are defined to admit one node whose removal results in a perfect graph).

Clearly, Lemma 5.15 can be generalized further as follows (this was independently observed in [61, 64, 72]).

Lemma 5.17 *We have $P_J(\mathrm{QSTAB}(G)) = \mathrm{STAB}(G)$ if and only if $G[V-J]$ is perfect.*

Therefore, J is a subset of nodes meeting all minimal imperfect subgraphs of G. By the Perfect Graph Theorem [65], an induced subgraph G' of G is minimally imperfect if and only if its complement $\overline{G}'$ is minimally imperfect. Hence, the same node-subset J meets all minimal imperfect subgraphs in the complementary graph, which implies:

Corollary 5.18 *Let $G = (V,E)$ be a graph. $P_J(\mathrm{QSTAB}(G)) = \mathrm{STAB}(G)$ holds for a subset of nodes $J \subseteq V$ if and only if $P_J(\mathrm{QSTAB}(\overline{G})) = \mathrm{STAB}(\overline{G})$.*

This reproves the invariance of the imperfection index under taking complements, originally achieved by Aguilera et al. [1].

We formalize the computation of the imperfection index further. For a graph $G = (V,E)$, we introduce the imperfection hypergraph $\mathcal{I}(G) = (V, \mathcal{F})$ on the same node set as G and all node subsets inducing minimally imperfect subgraphs of G as hyperedges. Obviously, we have $\mathcal{I}(G) = \mathcal{I}(\overline{G})$. For our purpose, we look for a minimum vertex cover of $\mathcal{I}(G)$, i.e., for a subset $J \subseteq V$ meeting all hyperedges. Obviously, any vertex cover of $\mathcal{I}(G)$ corresponds to a subset $J \subseteq V$ with $G[V-J]$ perfect resp. with $P_J(\mathrm{QSTAB}(G)) = \mathrm{STAB}(G)$. This implies that the imperfection index of G equals the vertex cover number $\tau(\mathcal{I}(G))$.

Lemma 5.19 *For any graph G, $\mathrm{imp_I}(G) = \mathrm{imp_I}(\overline{G}) = \tau(\mathcal{I}(G)) = \tau(\mathcal{I}(\overline{G}))$.*

As the computation of the vertex cover number of a hypergraph is a well-known NP-hard problem, also determining the imperfection index of a graph is NP-hard.

5.3.2 The imperfection index and graph compositions

Similar to the imperfection ratio, we also consider the behavior of the imperfection index under well-known perfection-preserving graph composition techniques. The expectation is, again, that the imperfection index should be invariant under applying such techniques; the first result in this direction is, again, the invariance under taking complements.

From the above graph-theoretical reformulation of $\mathrm{imp}_{\mathrm{I}}(G)$, we infer:

Lemma 5.20 *The number of disjoint minimally imperfect subgraphs of G is a lower bound on* $\mathrm{imp}_{\mathrm{I}}(G)$.

This immediately implies that for an unconnected graph G, the imperfection index equals the sum of the imperfection indices of its components $G_1, \ldots, G_k$, that is

$$\mathrm{imp}_{\mathrm{I}}(G_1 + \ldots + G_k) = \mathrm{imp}_{\mathrm{I}}(G_1) + \ldots + \mathrm{imp}_{\mathrm{I}}(G_k).$$

Thus, only the imperfection index of perfect graphs is invariant under taking disjoint unions, whereas it cannot be bounded for several other classes of graphs, as we can easily construct graphs within these classes having arbitrarily many disjoint minimally imperfect subgraphs.

Example 5.21 Let kC_9 be the disjoint union of k 9-holes. Then we obviously have $\mathrm{imp}_{\mathrm{I}}(kC_9) = k$ and, in particular,

$$\mathrm{imp}_{\mathrm{I}}(kC_9) \to \infty \text{ if } k \to \infty.$$

As such graphs kC_9, $k \geq 1$ have clique number two and belong to the classes of t-perfect graphs, line graphs, $(C_5, C_7, \overline{C}_7)$-free graphs, normal graphs, as well as planar graphs, the imperfection index cannot be bounded for all these classes and their superclasses.

Similar constructions are possible by linking odd holes through additional edges to a chain; even in highly connected graphs many disjoint odd holes can occur:

Example 5.22 Consider the web W_{5k}^k. For every node $i \in \{1, \ldots, 5k\}$, it contains the 5-hole $C(i) = \{i, i+k, i+2k, i+3k, i+4k\}$. In particular, the k 5-holes $C(i)$ for $1 \leq i \leq k$ are disjoint which implies that $\mathrm{imp}_{\mathrm{I}}(W_{5k}^k) \geq k$ and, thus,

$$\mathrm{imp}_{\mathrm{I}}(W_{5k}^k) \to \infty \text{ if } k \to \infty.$$

Thus, there is also no upper bound of the imperfection index for the classes of webs and antiwebs as well as for any of their superclasses.

The previous examples imply:

Corollary 5.23 *For the following graph classes $\mathcal{G}$, there exists no upper bound on the imperfection index* $\mathrm{imp_I}(G), G \in \mathcal{G}$*:*

- *t-perfect graphs (and, therefore, also h-perfect, a-perfect, p-perfect, rank-perfect graphs);*
- *line graphs (and, therefore, also quasi-line and claw-free graphs);*
- *graphs with clique number two (and, therefore, also graphs with stability number two);*
- *webs and antiwebs (and, therefore, also quasi-line and near-bipartite graphs as well circular-perfect graphs);*
- *$(C_5, C_7, \overline{C}_7)$-free graphs and normal graphs;*
- *planar graphs.*

To learn about the imperfection index of a graph $G_1 +_Q G_2$ obtained by identifying two graphs G_1 and G_2 in a clique Q, we again make use of the graph-theoretical characterization of the imperfection index. Let $\mathcal{J}(G)$ denote the set of all minimum node subsets J of G such that $G - J$ is perfect. Then determining $\mathrm{imp_I}(G)$ means to look for a pair $J_1 \in \mathcal{J}(G_1)$ and $J_2 \in \mathcal{J}(G_2)$ such that $J_1 \cap J_2 \subseteq Q$ is of maximum cardinality. One can easily infer that

$$\mathrm{imp_I}(G_1 +_Q G_2) = \min\{|J_1 \cup J_2| : J_i \in \mathcal{J}(G_i) \text{ for } i = 1, 2\}$$

holds. This result clearly includes the special case of taking disjoint unions.

For almost-perfect graphs G_1 and G_2, the graph $G_1 +_Q G_2$ is again almost-perfect if and only if $(\mathcal{J}(G_1) \cap \mathcal{J}(G_2)) \cap Q \neq \emptyset$.

We finally consider the behavior of the imperfection index under taking lexicographic products $G_1 \times G_2$. In [61] we obtained the following:

Theorem 5.24 *For two graphs G_1, G_2 we have*

$$\mathrm{imp_I}(G_1 \times G_2) = |G_2|\ \mathrm{imp_I}(G_1) + (|G_1| - \mathrm{imp_I}(G_1)) \cdot \mathrm{imp_I}(G_2)$$

as imperfection index of their lexicographic product.

Thus, also the imperfection index cannot be bounded for any class $\mathcal{G}$ of graphs which is closed under substitution (and, therefore, closed under taking lexicographic products) and contains at least one imperfect graph G. In contrary to the imperfection ratio, we have even more:

Corollary 5.25 *Let G_1 be a graph. For any perfect graph G_2, we have*

$$\mathrm{imp_I}(G_1 \times G_2) = |G_2|\ \mathrm{imp_I}(G_1).$$

As this result clearly also applies to the two special cases, namely taking lexicographic products where G_2 is a clique (replicating every node of G_1) or a stable set (multiplying every node of G_1), we immediately obtain the following:

Corollary 5.26 *Let $\mathcal{G}$ be a graph class containing one imperfect graph. If $\mathcal{G}$ is closed under substituting perfect graphs for nodes, replication, or multiplication, then there exists no upper bound for the imperfection index* $\mathrm{imp}_{\mathrm{I}}(G), G \in \mathcal{G}$.

Thus, a sufficient condition for the *non*-existence of an upper bound on the imperfection index is that the graph class $\mathcal{G}$ in question contains an imperfect graph and is closed under substituting certain perfect graphs. This motivates to study the behavior of the remaining graph classes of interest under substitution, in particular for near-perfect and strongly circular-perfect graphs.

We have already seen that precisely substituting perfect graphs for nodes preserves rank-perfection. We are interested whether there are further requirements in order to obtain near-perfect graphs by substitution. Note that Shepherd [98] showed that the class of near-perfect graphs is closed under replication. We ensured in [61] that there is no other way to produce a near-perfect graph by substitution.

Theorem 5.27 *Let G be obtained by substituting a node v of G_1 by G_2. G is near-perfect if and only if either G_1 and G_2 are perfect or G_1 is near-perfect and G_2 is a clique.*

Combining this theorem with Corollary 5.26 yields that also the imperfection index of properly near-perfect graphs cannot be bounded.

Remark 5.28 We provided in [61] for further subclasses of rank-perfect graphs which conditions are required to generate a member of this class by substitution. The results for h-perfect, co-h-perfect, p-perfect, and a-perfect graphs imply that taking lexicographic products with stable sets preserves the membership in all those classes, and yield alternative proofs that the imperfection index cannot be bounded for all those classes.

Finally, we address the imperfection index of strongly circular-perfect graphs. We know from Section 5.2.3 that a graph obtained by substituting a node of a graph G_1 by a graph G_2 is circular-perfect only if G_2 is perfect. However, the class of circular-perfect graphs is not closed under substituting perfect graphs for nodes: Let, e.g., G_1 be one of the circular-perfect graphs in Figure 5.1 and substitute v by a clique of size two. The resulting graph G has $\omega_c(G) = \max\{3, \frac{5}{2}\} = 3$ as triangles and 5-holes are the only prime circular cliques in G. If G would be circular-perfect, we had $\omega_c(G) = \chi_c(G) = 3$ and $\lceil \chi_c(G) \rceil = \chi(G) = 3$, but $\chi(G) = 4$ yields a contradiction.

However, circular-perfectness is certainly closed under multiplication as both parameters $\omega_c(G)$ and $\chi_c(G)$ clearly remain unchanged in this case. This gives an alternative proof that the imperfection index cannot be bounded for circular-perfect graphs.

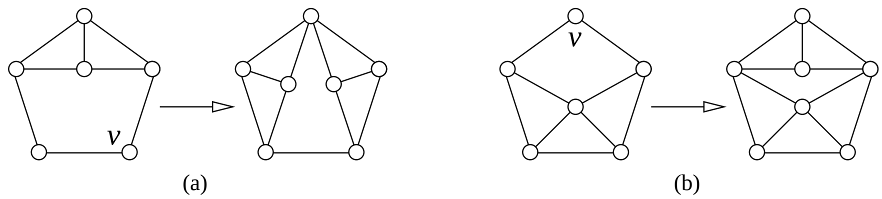


Figure 5.1: Replication does not preserve circular-perfectness

The complement of a graph obtained by multiplying a node v of G_1 k-times equals the graph obtained by replicating v in $\overline{G}_1$ k-times. Thus, as circular-perfect graphs are not closed under replication, the class of strongly circular-perfect graphs is neither closed under multiplication nor under replication and satisfies, therefore, a necessary condition for the existence of an upper bound for the imperfection index. Note that all the strongly circular-perfect graphs presented in Section 2.2.4 have bounded imperfection index.

5.4 Comparing imperfection ratio and imperfection index

In this chapter, we have studied the imperfection ratio and the imperfection index as two different ways to classify how close certain imperfect graphs are to perfection. The imperfection ratio has been shown to be bounded for several classes, whereas the imperfection index turned out to be unbounded for most of them, see Table 5.1 which gives an overview of the results.

An open question is in particular whether there exist a graph class such that the imperfection index of all members is bounded by a constant k with $1 < k < \infty$. The class of strongly circular-perfect graphs could be such a class.

From the achieved results it is fair to conclude that the disjunctive index of QSTAB(G) measures too roughly whether a given graph G is close to the class of perfect graphs. Possible refinements are suggested in [61] as follows:

- The disjunctive procedure can be carried out with any linear combination πx of the variables. The resulting polytope is then defined as

$$P_\pi(X) = \text{conv}(\{x \in X : \pi x \leq \pi_0\} \cup \{x \in X : \pi x \geq \pi_0 + 1\})$$

  For any near-perfect graph G and the full rank constraint $x(G) \leq \alpha(G)$ as $\pi x \leq \pi_0$, it directly follows $P_\pi(\text{QSTAB}(G)) = \text{STAB}(G)$ and the disjunctive index would equal one. Unfortunately, kC_5 still needs k applications of the disjunctive procedure before STAB(G) is reached.

- The unboundedness of the imperfection index for classes of graphs bases in all the above cases on the increase of the number of nodes in the graph without leaving the class (disjoint union, substitution, replication, multiplication). Scaling the imperfection index by the number of nodes $n = |V|$ could resolve this problem.

We, therefore, suggest to consider the *normalized imperfection index*

$$\text{imp}_\text{n}(G) = \frac{\text{imp}_\text{I}(G)}{n}.$$

As there are no imperfect graphs with four or less nodes, $\text{imp}_\text{I}(G)$ can be at most $n - 4$, and thus scaling yields a value $\text{imp}_\text{n}(G) \in [0, 1)$.

All perfect graphs are exactly the graphs with $\text{imp}_\text{n}(G) = 0$; all almost-perfect graphs satisfy $\text{imp}_\text{n}(G) \leq \frac{1}{n}$. Even for $kC_5, k \geq 1$, we obtain as normalized imperfection index $\frac{\text{imp}_\text{I}(kC_5)}{5k} = 0.2$, independent of k. Taking the lexicographic product of k 5-holes yields a sequence with

$$\frac{\text{imp}_\text{I}((C_5)^k)}{|(C_5)^k|} \to 1 \text{ if } k \to \infty$$

(since $\text{imp}_\text{I}((C_5)^k) = 5^k - 4^k$ whereas $|(C_5)^k| = 5^k$), which is consistent with the fact that also the imperfection ratios of these graphs tend to infinity. It is, however, interesting to observe that for the Mycielski graphs $G_0, G_1, G_2, \ldots$ the quotient of imperfection index and number of nodes tends to $\frac{1}{3}$, whereas their imperfection ratios cannot be bounded.

Graph class $\mathcal{G}$	$\sup\{\text{imp}(G) : G \in \mathcal{G}\}$	$\sup\{\text{imp}_{\text{I}}(G) : G \in \mathcal{G}\}$
perfect	$= 1$	$= 0$
minimal imperfect	$\leq \frac{5}{4}$	$= 1$
almost-bipartite	$\leq \frac{5}{4}$	≤ 1
almost-perfect	< 2	≤ 1
t-perfect	$\leq \frac{5}{4}$	∞
h-perfect	$\leq \frac{5}{4}$	∞
co-h-perfect	$\leq \frac{5}{4}$	∞
p-perfect	$\leq \frac{5}{4}$	∞
(semi-)line	$\leq \frac{5}{4}$	∞
antiwebs/webs	$< \frac{3}{2}$	∞
a-perfect	$< \frac{3}{2}$	∞
near-bipartite	$< \frac{3}{2}$	∞
quasi-line	$< \frac{3}{2}$	∞
planar	$\leq \frac{11}{6}$	∞
strongly circular-perfect	??	??
circular-perfect	??	∞
near-perfect	??	∞
rank-perfect	??	∞
$\alpha(G)$=2	∞	∞
weakly rank-perfect	∞	∞
claw-free	∞	∞
$(C_5, C_7, \overline{C}_7)$-free	∞	∞
normal	∞	∞

Table 5.1: Summary of the bounds

Chapter 6

Conclusions

Throughout the previous chapters, we considered superclasses of perfect graphs, obtained by relaxing perfection w.r.t. such different concepts as

- the relation of clique and chromatic number,
- splitting graph entropies of complementary graphs,
- the relation of the stable set polytope and its LP-relaxation QSTAB(G),

which all characterize perfect graphs. We further surveyed several ways to measure imperfection of a graph accordingly, namely, by means of

- χ-binding functions or the imperfection ratio,
- the value $\max\{H(G,p)+H(\overline{G},p)-H(p) : p\}$,
- the facets of STAB(G), the disjunctive index or the Chvátal-rank of QSTAB(G), or the dilation ratio of STAB(G) and QSTAB(G).

It turned out that the imperfection ratio is compatible to all the considered concepts, as it

- reflects nice coloring properties (as some asymptotic slope of the minimal χ-binding function for a family of replications of a graph G),
- is relevant for splitting graph entropies (as a small upper bound u for imp($\mathcal{G}$) yields also a good upper bound for $H(p) - H(G,p) - H(\overline{G},p)$, independent of the probability distribution p),
- measures the difference between STAB(G) and QSTAB(G) (as the dilation ratio of STAB(G) and QSTAB(G)).

Perfect graphs are exceptional in all these respects, and minimally imperfect graphs are, indeed, close to perfect graphs by means of *all* these concepts.

However, the results from the previous chapters imply that the graphs in the studied superclasses tend to be close to perfection w.r.t. *one* concept only (or to a few if they are related), but are typically not close to perfection w.r.t. the other considered concepts.

For instance, almost-perfect graphs are exactly the graphs with imperfection index at most one, and are χ-bound with the smallest non-trivial binding function. Circular-perfect graphs arise as a natural superclass of perfect graphs by means of a more general coloring concept and as a further important class of χ-bound graphs with the smallest non-trivial χ-binding function $\chi(G) \leq \omega(G) + 1$. But the graphs in both classes do not have a nice description of the stable set polytope (see Section 4.1.4) and are not weakly splitting.

Normal graphs are supposed to be 'weaker perfect graphs' by means of splitting graph entropies. Körner and de Simone [59] conjectured a similarity to perfect graphs in terms of forbidden subgraphs: In analogy to the Strong Perfect Graph Conjecture, they conjectured that every $(C_5, C_7, \overline{C}_7)$-free graph is normal. However, neither normal nor $(C_5, C_7, \overline{C}_7)$-free graphs are χ-bound, their imperfection ratio cannot be bounded, and not even the value $\max\{H(G,p)+H(\overline{G},p)-H(p) : p\}$. Thus, the validity of the Normal Graph Conjecture would certainly provide us a sufficient condition for normality and characterize the hereditary core of the normal graphs, but we even cannot expect nice properties of this special subclass of normal graphs.

As a consequence, we conclude that normal graphs are not as close to perfection as expected–even not in the context of splitting graph entropies, since for a normal graph G the value $H(G,p) + H(\overline{G},p) - H(p)$ strongly depends on the probability distribution p.

In contrary, for each minimally imperfect graph G (including the non-normal graphs $C_5, C_7, \overline{C}_7$) we have the small range

$$0 \leq H(p) - H(G,p) - H(\overline{G},p) \leq \log_2 \left(\frac{|G|}{|G| - 1} \right)$$

for *all* probability distributions p by [45]. This suggests to consider graph classes $\mathcal{G}$ as close to perfection by means of splitting entropies if there is a small upper bound u for $\mathrm{imp}(\mathcal{G})$ since

$$0 \leq H(p) - H(G,p) - H(\overline{G},p) \leq \log_2 u$$

holds for all (normal and non-normal) graphs $G \in \mathcal{G}$ and for all p. For instance, all graph classes $\mathcal{G}$ have this property where the imperfection ratio relies on the minimally imperfect subgraphs only as $\mathrm{imp}(\mathcal{G}) \leq \frac{5}{4}$ holds.

Several classes of rank-perfect graphs admit the latter property, namely, line graphs (whose stable set polytopes have as only nontrivial facets rank clique family inequalities) and p-perfect graphs (whose stable set polytopes have as only nontrivial facets rank constraints associated with partitionable graphs). In addition, for a-perfect and near-bipartite graphs, both the imperfection ratio and the nontrivial facets of the stable set polytope rely on induced antiwebs only, thereby bounding the imperfection ratio by $\frac{3}{2}$. Hence, such graphs can be considered as close to perfection.

The latter graph classes are obtained by considering the nontrivial facets of the stable set polytope only. In order to describe the difference between $\mathrm{STAB}(G)$ and $\mathrm{QSTAB}(G)$, also the fractional extreme points of $\mathrm{QSTAB}(G)$ are of interest. The imperfection index $\mathrm{imp_I}(G)$ uses the latter approach, as it equals the minimal number of disjunctions which are required to make $\mathrm{QSTAB}(G)$ integral. In addition, we have for any graph G

$$\chi(G) \leq \omega(G) + \mathrm{imp_I}(G)$$

since $\mathrm{imp_I}(G)$ equals the cardinality of a minimum node subset V' of G such that $G - V'$ is perfect. This suggests that graph classes with a small upper bound u for the imperfection index admit a linear χ-binding function $b(x) = x + u$ (as, for instance, almost-perfect graphs do). Unfortunately, it turned out that the imperfection index cannot be bounded for the most graph classes.

Thus, considering the fractional extreme points of $\mathrm{QSTAB}(G)$ only does not help in general to decide whether a graph is close to perfection. It does even not reflect the difference between $\mathrm{STAB}(G)$ and $\mathrm{QSTAB}(G)$ as, for instance, almost-perfect graphs have imperfection index one, but arbitrarily difficult facets are required to describe their stable set polytope. Conversely, stable set polytopes of near-perfect graphs G have only facets associated with cliques and the graph itself, but $\mathrm{QSTAB}(G)$ can have arbitrarily many fractional extreme points.

We conclude that both knowledge on the facets of $\mathrm{STAB}(G)$ and the fractional extreme points of $\mathrm{QSTAB}(G)$ has to be taken into account in order to describe the difference between $\mathrm{STAB}(G)$ and $\mathrm{QSTAB}(G)$. The imperfection ratio

$$\mathrm{imp}(G) = \max\{a^T y : a \in \mathcal{F}(G), y \in \mathrm{QSTAB}(G)\}$$

takes both into account and is, therefore, also an appropriate measure for imperfection from the polyhedral point of view.

This supports our main assertion further that the imperfection ratio is an appropriate (and probably the best) measure for imperfection and that graphs with a small imperfection ratio are close to perfection by means of several concepts.

We already observed that minimally imperfect graphs admit this property; one further example is the class of line graphs: We can conclude from the previous considerations that line graphs are 'almost perfect' as they

- have an imperfection ratio which relies on induced odd holes only and is bounded by $\frac{5}{4}$,
- are χ-bound with binding function $b(x) = x + 1$,
- satisfy $0 \leq H(G,p) + H(\overline{G},p) - H(p) \leq \log_2(\frac{5}{4})$ where the minimum is attained for all $(C_5, C_7, \overline{C}_7)$-free line graphs,
- have both an easy to describe stable set polytope (with rank clique family inequalities as only nontrivial facets) and an easy clique constraint stable set polytope (with half-integral extreme points only).

Note that all those properties can be translated to the complementary class: Complements of line graphs

- have an imperfection ratio which relies on induced odd antiholes only and is bounded by $\frac{5}{4}$,
- are $\overline{\chi}$-bound with complementary binding function $b(x) = x + 1$,
- satisfy $0 \leq H(G,p) + H(\overline{G},p) - H(p) \leq \log_2(\frac{5}{4})$ where the minimum is attained for all $(C_5, C_7, \overline{C}_7)$-free co-line graphs,
- have both an easy stable set polytope (with (0,1,2)-valued joined odd antihole constraints as only nontrivial facets) and an easy clique constraint stable set polytope (with $(0, \frac{1}{\alpha'})$-valued extreme points only).

Hence, both line graphs and their complements can be seen as 'almost perfect'. Note that the invariance of the imperfection ratio under complementation suggests that, in general, a graph class and its complementary class are either both close to or far from perfection.

Chapter 7

Selected Proofs

7.1 On circular-perfect graphs

Circular-perfect graphs form an important superclass of perfect graphs with the smallest non-trivial χ-binding function $b(x) = x + 1$. As not much is known about members of this class yet, we address in [81] the problem of finding out which graphs are circular-perfect. As partitionable graphs G satisfy the property $\chi(G) = \omega(G) + 1$ one might expect that at least several subclasses of partitionable graphs are circular-perfect. However, we obtained in [81] that all partitionable graphs different from circular cliques are circular-imperfect and the partitionable graphs $\overline{K}_{3q+1,3}$ are minimal circular-imperfect (Theorem 2.11).

More precisely, we prove the following. Given a graph G, an edge e of G is called *indifferent* if e is not contained in any maximum clique of G. The *normalized subgraph* $\mathrm{norm}(G)$ of G is obtained from G by deleting all indifferent edges. We characterize all circular cliques whose normalized subgraph is circular-imperfect, and show which of them are minimal with respect to this property.

Theorem 7.1 *Let $K_{p,q}$ be any prime circular clique. Then* $\mathrm{norm}(K_{p,q})$ *is*

(i) circular-imperfect iff $p \not\equiv -1 \pmod q$ and $\lfloor p/q \rfloor \geq 3$;

(ii) minimal w.r.t. this property iff $p = 3q + 1$ for all $q \geq 3$;

(iii) equal to $\overline{K}_{p,3}$ if $p = 3q + 1$ and $q \geq 3$.

Given an integer p and a subset of integers S of $[0, p-1]$, the *circulant graph* $C(p, S)$ is the graph with node set $\{0, \ldots, p-1\}$ and edge set $\{ij \mid i-j \in S\}$ with arithmetics performed modulo p.

We first state the following observation which relates the normalized sugraph of a partitionable circular clique to its complement.

Lemma 7.2 *If $p = \omega q + 1$, then* $\mathrm{norm}(K_{p,q})$ *is isomorphic to the complement* $\overline{K}_{p,\omega} = C_p^\omega$ *of* $K_{p,\omega}$.

Proof: Both $\mathrm{norm}(K_{p,q})$ and $\overline{K}_{p,\omega}$ are circulant graphs on the node set $V = \{0, 1, \ldots, p-1\}$. The former has generating set

$$S = \{q, q+1, 2q, 2q+1, \ldots, (\omega-1)q, (\omega-1)q+1\}$$

and the latter has generating set

$$S' = \{1, 2, \ldots, \omega-1, p-1, p-2, \ldots, p-\omega+1\}.$$

It is easy to verify that $f : V \to V$ defined as $f(i) = iq \pmod{p}$ satisfies $f(S') = S$. Hence f is an isomorphism from $\overline{K}_{p,\omega}$ to $\mathrm{norm}(K_{p,q})$. □

A *proper variant* of G is a subgraph H' of G obtained by removing a non-empty set of indifferent edges (i.e., any graph H' with $H \subseteq H' \subseteq G$).

We shall now proceed to the proof of Theorem 7.1.

Proof: Consider a circular clique $K_{p,q}$ with $p = \omega q + r$ and $0 \le r \le q-1$ and let $V = V(K_{p,q})$.

Claim 1 $\mathrm{norm}(K_{p,q})$ *is the circulant graph* $C[p, S]$ *whith* $S = \{q, q+1, \cdots, q+r, 2q, 2q+1, \cdots, 2q+r, \cdots, (\omega-1)q, (\omega-1)q+1, \cdots, (\omega-1)q+r\}$.

Consider an edge $0t$. We have $t = kq + r'$, with $1 \le k \le \omega - 1$ and $0 \le r' \le q-1$.

If $0 \le r' \le r$, then the set $\{0, q+r', 2q+r', \cdots, (\omega-1)q+r'\}$ induces a maximum clique containing the edge $0t$, and so the edge $0t$ is not indifferent.

Conversely, if $r+1 \le r' \le q-1$, then let K be a clique containing $0, t$. The other nodes of K belong to the intervals $[q, (k-1)q+r']$ and $[(k+1)q+r', (\omega-1)q+r]$. Therefore, K has at most $\omega - 1$ nodes, namely, at most $k-1$ nodes in the interval $[q, (k-1)q+r']$ and at most $\omega - k - 2$ nodes from the interval $[(k+1)q+r', (\omega-1)q+r]$. Thus K is not a maximum clique and so $0t$ is an indifferent edge. ◇

In particular, $\mathrm{norm}(K_{p,q})$ is isomorphic to $\overline{K}_{p,3}$ if $p = 3q+1$ due to Lemma 7.2 which proves assertion (*iii*).

Claim 2 *Suppose I is a maximal stable set of* $\mathrm{norm}(K_{p,q})$ *and* $i, i+t \in I$ *for some* $t \le r+1$. *Then* $i+j \in I$ *for all* $0 \le j \le t$.

If x is adjacent to $i+j$ in $\mathrm{norm}(K_{p,q})$ for some $0 \le j \le t$, then x is adjacent to either i or $i+t$ in $\mathrm{norm}(K_{p,q})$. $\diamond$

Claim 3 *Suppose I is a stable set of* $\mathrm{norm}(K_{p,q})$. *There is a node i of* $\mathrm{norm}(K_{p,q})$ *such that* $i+j \notin I$ *for any* $1 \le j \le r$.

Otherwise, Claim 2 would imply that all nodes of $\mathrm{norm}(K_{p,q})$ belong to a maximal stable set I' containing I, an obvious contradiction. $\diamond$

Claim 4 *If I is a stable set of* $\mathrm{norm}(K_{p,q})$, *then* $|I| \le q$.

As $\mathrm{norm}(K_{p,q})$ is a circulant graph, by Claim 3, we may assume w.l.o.g. that $S \cap I = \emptyset$, where $S = \{\omega q, \omega q+1, \cdots, \omega q + r - 1\}$.

But $V - S$ can be decomposed into the disjoint union of q cliques of $\mathrm{norm}(K_{p,q})$, namely, $Q_i = \{i, i+q, i+2q, \cdots, i+(\omega-1)q\}$, for $i = 0, 1, \cdots, q-1$. As $|I \cap Q_i| \le 1$ for each $i \in \{0, 1, \cdots, q-1\}$, so $|I| \le q$. $\diamond$

Claim 5 *We have* $\chi_c(\mathrm{norm}(K_{p,q})) = \chi_c(K_{p,q}) = p/q$.

Since $\chi_c(K_{p,q}) = p/q$, we have $\chi_c(\mathrm{norm}(K_{p,q})) \le p/q$. On the other hand, $\chi_c(\mathrm{norm}(K_{p,q})) \ge \chi_f(\mathrm{norm}(K_{p,q})) = p/\alpha(\mathrm{norm}(K_{p,q})) \ge p/q$ due to Claim 4 (where χ_f denotes the fractional chromatic number, a lower bound of the circular chromatic number [116]). So equality holds everywhere. $\diamond$

Therefore the removal of indifferent edges of a circular clique does not alter its circular chromatic number, but clearly its circular clique number. This implies that normalization destroys circular-perfection:

Claim 6 *If* $p \ne -1 \pmod q$ *and* $\lfloor p/q \rfloor \ge 3$ *then* $K_{p,q}$ *is not normalized and every of its proper variants is circular-imperfect.*

We denote by $\Delta(G)$ the maximum degree of a graph G. We have $\Delta(K_{p,q}) = p-(2q-1)$ and $\Delta(\mathrm{norm}(K_{p,q})) = (r+1)(\omega-1)$, where $p = \omega q + r$ and r is the remainder modulo q, by Claim 1. Therefore, if $K_{p,q}$ is normalized (i.e., if $K_{p,q} = \mathrm{norm}(K_{p,q})$) then $p-(2q-1) = (r+1)(\omega-1)$, that is $(\omega-2)q = (r+1)(\omega-2)$. Since $\omega = \lfloor p/q \rfloor \ge 3$, this implies that $r = q-1$, and so $p = -1 \pmod q$, a contradiction.

Hence $K_{p,q}$ is not normalized and the result follows from Claim 5: if H' is any proper variant of $K_{p,q}$ then

$$\omega_c\left(H'\right) < p/q = \chi_c\left(H\right) = \chi_c(H').$$

$\diamond$

This completes the proof of the "if part" of Theorem 7.1 (i). We now treat the "only if part" of assertion (i).

Claim 7 *If* $\lfloor p/q \rfloor < 3$ *or* $p = -1 \pmod q$ *then* $\mathrm{norm}(K_{p,q})$ *is circular-perfect.*

Notice that $\omega = \lfloor p/q \rfloor$ is the clique number of $K_{p,q}$. Therefore, if $\omega < 3$ then $\mathrm{norm}(K_{p,q}) = K_{p,q}$. Thus $\mathrm{norm}(K_{p,q})$ is circular-perfect.

If $p = -1 \pmod q$ then $\mathrm{norm}(K_{p,q}) = K_{p,q}$ follows due to the description of $\mathrm{norm}(K_{p,q})$ for general p and q in Claim 1. Thus $\mathrm{norm}(K_{p,q})$ is circular-perfect. $\diamond$

This completes the proof of Theorem 7.1 (i). We now treat the "only if part" of assertion (ii).

Claim 8 *If* $p \neq 1, -1 \pmod q$ *and* $\omega = \lfloor p/q \rfloor \geq 3$ *then* $K_{p,q}$ *has a circular clique* $K_{(\omega q'+1),q'}$ *as an induced subgraph with at least one indifferent edge of* $K_{p,q}$*, and* $q' \geq 3$.

Let $2 \leq r \leq q-2$ such that $p = q\omega + r$. Notice that $q \neq 2r$ as p and q are relatively prime.

Case 1. If $r < \frac{q}{2}$ then let $q' = \lceil \frac{q}{r} \rceil$. We have $q' \geq 3$. For every $0 \leq i < \omega$, let $X_i = \{iq, iq+r, \ldots, iq+(q'-1)r\}$ and define $X = \left(\bigcup_{0 \leq i < \omega} X_i\right) \cup \{\omega q\}$. We first show that X induces a circular clique $K_{(\omega q'+1),q'} \subseteq K_{p,q}$.

For every $0 \leq x < p$, we denote by S_x the maximum stable set $\{x, x+1, \ldots, x+q-1\}$ of $K_{p,q}$ (arithmetics performed modulo p). Due to Trotter [104], it is enough to check that for every $x \in X$, S_x meets X in exactly q' nodes.

Let $x \in X$: by the definition of X, there exist $0 \leq i \leq \omega$ and $0 \leq \delta < q'$ such that $x = iq + \delta r$.

- If $i < \omega - 1$ then $S_x \subseteq S_{iq} \cup S_{(i+1)q}$. Hence

$$\begin{aligned} S_x \cap X &= (S_{iq} \cap S_x \cap X) \cup (S_{(i+1)q} \cap S_x \cap X) \\ &= \{iq + \lambda r | \delta \leq \lambda < q'\} \cup \{(i+1)q + \lambda r | 0 \leq \lambda < \delta\} \end{aligned}$$

as for every $0 \leq \lambda < q'$, we have $(i+1)q + \lambda r \in S_x$ if and only if

$$0 \leq (i+1)q + \lambda r - x = q + (\lambda - \delta)r < q$$

holds. Therefore S_x meets X in exactly q' nodes.

- If $i = \omega - 1$ and $\delta = 0$ then

$$S_x \cap X = S_{iq} \cap X = \{iq + \lambda r | 0 \leq \lambda < q'\}$$

holds and, again S_x meets X in exactly q' nodes.

- If $i = \omega - 1$ and $\delta > 0$ then $x = (\omega - 1)q + \delta r$. We have $S_x = \{(\omega-1)q+\delta r, (\omega-1)q+\delta r+1, \ldots, (\omega-1)q+\delta r+q-1\}$ (with arithmetics performed modulo p). Hence S_x is the disjoint union $S'_x \cup S"_x$ where $S'_x = \{(\omega - 1)q + \delta r, (\omega - 1)q + \delta r + 1, \ldots, \omega q + r - 1\}$ and $S"_x = \{0, 1, \ldots, (\delta - 1)r - 1\}$ ($S"_x = \emptyset$ if $\delta = 1$). We have

$$\begin{aligned} X \cap S_x &= (X_{\omega-1} \cup X_0 \cup \{\omega q\}) \cap S_x \\ &= \big(X_{\omega-1} \cap S'_x\big) \cup (X_0 \cap S"_x) \cup \{\omega q\} \end{aligned}$$

and thus, $X \cap S$ is of size q' as

$$\begin{aligned} X_{\omega-1} \cap S'_x &= \{(\omega - 1)q + \lambda r | \delta \leq \lambda < q'\} \text{ is of size } q' - \delta; \\ X_0 \cap S"_x &= \{\lambda r | 0 \leq \lambda < \delta - 1\ \} \text{ is of size } \delta - 1. \end{aligned}$$

Therefore S_x meets X in exactly q' nodes.

- If $i = \omega$ and $\delta = 0$ then $x = \omega q$. We have

$$\begin{aligned} S_x \cap X &= (\{\omega q, \omega q + 1, \ldots, \omega q + r - 1\} \cap X) \\ &\quad \cup (\{0, 1, \ldots, q - r - 1\} \cap X) \\ &= \{\omega q\} \cup \{\lambda r | 0 \leq \lambda r \leq q - r - 1 \text{ and } 0 \leq \lambda < q'\} \\ &= \{\omega q\} \cup \{\lambda r | 0 \leq \lambda \leq \lfloor q/r \rfloor - 1 = q' - 2 \text{ and } 0 \leq \lambda < q'\} \\ &= \{\omega q\} \cup \{\lambda r | 0 \leq \lambda \leq q' - 2\} \end{aligned}$$

which also implies that S_x meets X in exactly q' nodes.

As S_x always meets X in exactly q' nodes, X induces a circular clique $G' = K_{(\omega q'+1),q'}$ of $K_{p,q}$ due to [104]. As $\omega \geq 3$ and $0 < r < q/2$, we have $q + r < q + 2r < 2q$ and the edge $\{0, q + 2r\}$ of G' is indifferent by Claim 1.

Case 2. If $r > \frac{q}{2}$, we show that $K_{(3\omega+1),3}$ is an induced subgraph of $K_{p,q}$.

For $j = 0, 1, \ldots, 3\omega$, let $x_j = \lfloor pj/(3\omega+1) \rfloor$. Let $X = \{x_0, x_1, \ldots, x_{3\omega}\}$.

We show that X induces this circular clique $K_{(3\omega+1),3}$: this is equivalent to show that for every $0 \leq i, j \leq 3\omega$, $\{x_i, x_j\}$ is an edge of $K_{p,q}$ if and only if $3 \leq |i-j| \leq 3\omega - 2$.

To prove this, we use the following simple observation several times: if a and b are reals and δ is an integer such that $a - b \geq \delta$ then $\lfloor a \rfloor - \lfloor b \rfloor \geq \delta$.

- Let $0 \leq i, j \leq 3\omega$ such that $\{x_i, x_j\}$ is an edge of $K_{p,q}$ and assume w.l.o.g. that $i < j$. We have $x_i < x_j$ and $q \leq x_j - x_i \leq p - q$.

  If $j - i \leq 2$, then $pj/(3\omega+1) - pi/(3\omega+1) \leq 2(q\omega+r)/(3\omega+1)$ follows. If $2(q\omega+r)/(3\omega+1) > q-1$ then as $\omega \geq 3$ and $q \geq r+2$, a short computation gives $r < 1$, a contradiction. Thus $2(q\omega+r)/(3\omega+1) \leq q-1$ and so $x_j - x_i \leq q-1$, a contradiction. Hence $j - i \geq 3$ follows.

  If $j - i \geq 3\omega - 1$, then $pj/(3\omega+1) - pi/(3\omega+1) \geq (3\omega-1)(q\omega + r)/(3\omega+1) \geq p-q+1$ follows. Thus $x_j - x_i \geq p-q+1$, a contradiction.

  Therefore, we infer $3 \leq j - i \leq 3\omega - 2$.

- Conversely, let $0 \leq i, j \leq 3\omega$ such that $3 \leq j - i \leq 3\omega - 2$ and assume w.l.o.g. that $i < j$. We have $x_i < x_j$ and we need to check that $\{x_i, x_j\}$ is an edge of $K_{p,q}$. On the one hand, $j - i \geq 3$ and $3r \geq q$ imply

  $$pj/(3\omega+1) - pi/(3\omega+1) \geq 3(q\omega+r)/(3\omega+1) \geq q$$

  and, hence, $x_j - x_i \geq q$ follows. On the other hand, $j - i \leq 3\omega - 2$ yields

  $$pj/(3\omega+1) - pi/(3\omega+1) \leq (3\omega-2)(q\omega+r)/(3\omega+1) \leq p-q$$

  and shows $x_j - x_i \leq p - q$. Therefore $\{x_i, x_j\}$ is an edge of $K_{p,q}$ and X induces a circular clique $G' = K_{(3\omega+1),3}$ of $K_{p,q}$, as required.

At last, we need to exhibit an indifferent edge of $K_{p,q}$ in G'.

By Claim 1, the neighbours of 0 in norm$(K_{p,q})$ are the nodes in $S = \{q, q+1, \cdots, q+r, 2q, 2q+1, \cdots, 2q+r, \cdots, (\omega-1)q, (\omega-1)q+1, \cdots, (\omega-1)q+r\}$.

We have $2q - 5p/(3\omega+1) = (\omega q + 2q - 5r)/(3\omega+1) > 0$ as $\omega \geq 3$ and $r \leq q-2$. Hence $x_5 < 2q$.

If $x_5 \geq q+r+1$ then $x_5 \notin S$ and $\{x_0, x_5\}$ is an edge of G' which is indifferent in $K_{p,q}$.

It remains to check the case $x_5 \leq q+r$: identifying an edge of G' which is indifferent in $K_{p,q}$ is more difficult to handle. We are going to exhibit one in an induced circular clique G'' sharing all nodes but one with G'.

For $t = 1, 2, \ldots, \omega-2$, let $\delta_t = x_{3t+2} - (tq + r + 1)$. As $x_5 \leq q + r$, we have $\delta_1 < 0$.

We first check that $\delta_{\omega-2} \geq 0$: we have $\frac{p(3\omega-4)}{3\omega+1} - (\omega-2)q - r - 1 = 2q - 1 - \frac{5p}{3\omega+1}$. If $5p/(3\omega+1) > 2q-1$ then $5r > \omega q - 3\omega + 2q - 1$. From $r \leq q-2$, we get $\omega < 3$, a contradiction. Hence $\frac{p(3\omega-4)}{3\omega+1} - (\omega-2)q - r - 1 \geq 0$ and therefore $\delta_{\omega-2} \geq 0$.

Let t^* be the largest index such that $\delta_{t^*} < 0$: we have $1 \leq t^* < \omega - 2$. Let $x'_{3t^*+2} = t^*q + r + 1$ and let $X' = (X - \{x_{3t^*+2}\}) \cup \{x'_{3t^*+2}\}$. Let G'' be the subgraph of $K_{p,q}$ induced by X'. To prove that G'' is an induced circular clique $K_{(3\omega+1),3}$ of $K_{p,q}$, we have to check that the neighborhood of x'_{3t^*+2} in G'' is the same than the one of x_{3t^*+2} in G', namely $\{x_0, x_1, \ldots, x_{3t^*-1}\} \cup \{x_{3t^*+5}, x_{3t^*+6}, \ldots, x_{3\omega}\}$.

If $\frac{(3t^*+5)p}{3\omega+1} - (t^*q + r + 1) < q$ then we have $\frac{(3(t^*+1)+2)p}{3\omega+1} - ((t^*+1)q + r + 1) < 0$. Thus we infer $\delta_{t^*+1} < 0$, in contradiction with the maximality of t^*. Hence $x_{3t^*+2} \leq x'_{3t^*+2} \leq x_{3t^*+5} - q$, and so x'_{3t^*+2} is adjacent to $\{x_0, x_1, \ldots, x_{3t^*-1}\} \cup \{x_{3t^*+5}, x_{3t^*+6}, \ldots, x_{3\omega}\}$ and x'_{3t^*+2} is not adjacent to x_{3t^*+3} and x_{3t^*+4}.

We have $t^*q + r + 1 - \frac{p3t^*}{3\omega+1} = r + 1 + \frac{t^*(q-3r)}{3\omega+1} < q$ as $r \leq q-2$ and $r > q/3$. Hence x'_{3t^*+2} is not adjacent to x_{3t^*} and x_{3t^*+1}.

Therefore G'' induces a circular clique $K_{(3\omega+1),3}$ of $K_{p,q}$. As $t^*q + r < x'_{3t^*+2} = t^*q + r + 1 < (t^*+1)q$ the edge $\{x_0, x'_{3t^*+2}\}$ of G'' is an indifferent edge of $K_{p,q}$. This finishes the second case.

Thus in both cases $K_{p,q}$ contains an induced circular clique $K_{(\omega q'+1),q'}$ with $q' \geq 3$ and an indifferent edge of $K_{p,q}$. $\diamond$

Claim 9 *If* $\mathrm{norm}(K_{p/q})$ *is minimal circular-imperfect then it is a partitionable web* $W^{\omega-1}_{\omega q+1}$, *and* $q \geq 3$.

Since $\mathrm{norm}(K_{p/q})$ is circular-imperfect we have $p \neq -1 \pmod q$ and $\omega \geq 3$ due to Claim 7.

If $\mathrm{norm}(K_{p/q})$ is not partitionable then $p \neq 1 \pmod q$. By the previous claim, $K_{p,q}$ has an induced subgraph $K_{(\omega q'+1),q'}$ with $q' \geq 3$ and node set W, containing an indifferent edge. As all non-indifferent edges of $K_{(\omega q'+1),q'}$ are non-indifferent in $K_{p,q}$ (since these two graphs have the same maximum clique size), the subgraph $G[W]$ of $K_{p,q}$ induced by W, is a proper variant of $K_{(\omega q'+1),q'}$, and is, therefore, circular-imperfect by Claim 6. Hence $K_{p,q} = K_{(\omega q'+1),q'}$, and $q = q' \geq 3$.

This implies that norm($K_{p,q}$) is partitionable.

It follows that $q \geq 3$ (as $q = 2$ implies that norm($K_{p,q}$) is an odd antihole and, therefore, circular-perfect, a contradiction). Due to Claim 7.2, this shows that norm($K_{p,q}$) is a partitionable web $W_{\omega q+1}^{\omega-1}$ with $q \geq 3$. $\diamond$

Claim 10 *A claw-free graph does not contain any circular cliques different from cliques, odd holes, and odd antiholes.*

Assume $K_{p,q}$ is a circular clique different from a clique, an odd hole, and an odd antihole. Then $q \geq 3$ and $p \geq 2q+2$. Thus $\{1, q+1, q+2, q+3\}$ induces a claw. $\diamond$

Claim 11 *If* norm($K_{p,q}$) *is a minimal circular-imperfect graph, then it has clique number 3.*

We first recall the following result of Trotter [104]: a web $W_{n'}^{k'}$ is an induced subgraph of W_n^k if and only if

$$\frac{k'-1}{k-1}n \quad \leq n' \leq \quad \frac{k'}{k}n \tag{7.1}$$

holds. By Claim 9, norm($K_{p,q}$) is a partitionable web $W_{\omega q+1}^{\omega-1}$, with $q \geq 3$. If $\omega \leq 2$ then norm($K_{p,q}$) is a stable set or an odd hole and is therefore circular-perfect, a contradiction. Hence $\omega \geq 3$.

Assume that $\omega \geq 4$.

Due to Trotter's inequality (7.1), the web W_{3q-1}^2 is an induced subweb of norm($K_{p,q}$) if and only if

$$\frac{2}{\omega-1}(q\omega+1) \leq 3q-1 \leq \frac{3}{\omega}(q\omega+1)$$

holds. Since the right inequality is always satisfied, this may be restated as $\frac{2}{\omega-1}(q\omega+1) \leq 3q-1$ which is equivalent to $1 + 4/(\omega-3) \leq q$.

If $q \geq 5$ (resp. $\omega \geq 5$) then $q \geq 1 + 4/(\omega-3)$ as $4/(\omega-3) \leq 4$ (resp. $q \geq 3$ and $4/(\omega-3) \leq 2$). Hence W_{3q-1}^2 is a proper induced subweb of norm($K_{p,q}$). If W_{2k+1}^{k-1} is any induced odd antihole of W_{3q-1}^2 then $k < 3$ due to Trotter's inequality (7.1). Hence the previous claim implies that $\omega_c(W_{3q-1}^2) = 3$. If W_{3q-1}^2 is 3-colorable, then it admits a partition into three stables sets of size at most $q-1 = \lfloor (3q-1)/3 \rfloor$, a contradiction. Hence $\chi(W_{3q-1}^2) \geq 4$ and so $\chi_c(W_{3q-1}^2) > 3 = \omega_c(W_{3q-1}^2)$. Thus W_{3q-1}^2 is a proper induced circular-imperfect graph of norm($K_{p,q}$), a contradiction.

Therefore, $\omega = 4$ and ($q = 3$ or $q = 4$), that is norm($K_{p,q}$) equals W_{13}^3 or W_{17}^3:

- W_{13}^3 is not minimal circular-imperfect as the subgraph induced by the nodes $\{1, 2, 4, 5, 7, 9, 10, 12\}$ is circular-imperfect, since it has circular-clique number 3 and is not 3-colorable;
- W_{17}^3 is not minimal circular-imperfect as the subgraph induced by the nodes $\{1, 2, 3, 5, 6, 8, 9, 11, 13, 14, 16\}$ is circular-imperfect, since it has circular-clique number 3 and is not 3-colorable.

In both cases, we get a contradiction and infer, therefore, $\omega = 3$. $\diamond$

This completes the proof of the "only if part" of assertion (ii). We now proceed to the proof of the "if part".

Claim 12 *The webs W_{3q+1}^2 with $q \geq 3$ are minimal circular-imperfect.*

Let $q \geq 3$. The web W_{3q+1}^2 is circular-imperfect by Claim 6.

If W_{3q+1}^2 is not minimal circular-imperfect, then there exists a proper induced subgraph W, which is minimal circular-imperfect. Let v be a node of W_{3q+1}^2 not in W.

If $\omega(W) = 3$ then $\omega(W) = 3 \leq \omega_c(W) \leq \chi_c(W) \leq \chi(W_{3q+1}^2 \setminus \{v\}) = 3$, a contradiction with the fact that W is minimal circular-imperfect.

If $\omega(W) = 2$ then let w be any node of W. If w is of degree at least three then w belongs to a triangle of W, as the neighborhood of any node of W_{3q+1}^2 can be covered with only two cliques (since W_{3q+1}^2 is a quasi-line graph), a contradiction. Therefore, the degree of W is at most two and so W is a disjoint union of cycles and paths, and thus is circular-perfect, a contradiction.

Hence W_{3q+1}^2 is minimal circular-imperfect. $\diamond$

This finally proves Theorem 7.1. $\square$

We shall further prove that a partitionable graph G is circular-imperfect unless G is a circular clique.

Proof: If $\omega_c(G) = \omega(G)$, then we have $\chi_c(G) > \omega(G) = \omega_c(G)$ by $\chi(G) = \omega(G) + 1$, therefore G is circular-imperfect.

Assume that $\omega_c(G) = p/q > \omega$ and let $\{0, \ldots, p-1\}$ be the nodes of an induced circular clique $K_{p,q}$ (where the nodes are labeled the usual way). For every $0 \leq i < \omega$, let Q_i be the maximum clique $\{jq | 0 \leq j \leq i\} \cup \{jq + 1 | i < j < \omega\}$. Obviously $Q_0, \ldots, Q_{\omega-1}$ are ω distinct maximum cliques of G containing the node 0.

If $p > \omega q + 1$ then the set $(Q_0 \setminus \{(\omega-1)q+1\}) \cup \{(\omega-1)q+2\}$ is another maximum clique containing 0, a contradiction as 0 belongs to exactly ω maximum cliques of G [8]. Hence $p = \omega q + 1$. This means that G contains the partitionable circular clique $K_{(\omega q+1),q}$ as an induced subgraph. Hence G is the circular clique $K_{(\omega q+1),q}$. □

In addition, it is of interest to know the complements of which circular cliques are also circular-perfect or, equivalently, which circular cliques occur as induced subgraphs of a strongly circular-perfect graph.

For that, we use the following result from [81] which says that a claw-free graph does not contain any prime antiwebs different from cliques, odd antiholes, and odd holes (see Claim 10 above).

This immediately implies the assertion of Corollary 2.23 for circular clique numbers of claw-free graphs G:

1. If $\omega(G) = 2$, then $\omega_c(G) = 2$ follows iff G is perfect and $\omega_c(G) = 2 + \frac{1}{k}$ iff G is imperfect and C_{2k+1} the shortest odd hole in G.
2. If $\omega(G) \geq 3$, then $\omega_c(G) = \max\{\omega(G), k' + \frac{1}{2}\}$ where $\overline{C}_{2k'+1}$ is the shortest odd antihole in G.

This enables us to completely characterize the circular-(im)perfection of webs as follows (Theorem 2.21): The web W_n^k is

(1) circular-perfect if $k = 1$ or $n \leq 2(k+1)+1$,

(2) circular-perfect if $k = 2$ and $n = 0 \pmod 3$,

(3) minimal circular-imperfect if $k = 2$ and $n = 1 \pmod 3$,

(4) circular-imperfect if $k = 2$ and $n = 2 \pmod 3$,

(5) circular-imperfect if $k \geq 3$ and $n \geq 2(k+2)$

(note that the proof of assertion (3) is given in [81], see Claim 12 above).

Proof: For that, we prove the following sequence of claims.

Claim 13 *Any web W_n^k with $k = 1$ or $n \leq 2(k+1)+1$ is circular-perfect.*

It is a simple observation that the webs W_n^1 are all circular-perfect. Moreover, W_n^k is perfect as well if $n \leq 2(k+1)$ and an odd antihole if $n = 2(k+1)+1$, thus W_n^k is circular-perfect if $n \leq 2(k+1)+1$. ◇

Thus Claim 13 verifies already assertion (1). In the sequel, we have to consider webs W_n^k with $k \geq 2$ and $n \geq 2(k+2)$ only. In [81] it is shown that the webs $W^2_{3\alpha+1}$ are minimal circular-imperfect for $\alpha \geq 3$; this already ensures assertion (3). In order to show circular-perfection for the webs $W^2_{3\alpha}$ with $\alpha \geq 3$ and circular-imperfection for all remaining webs, we need the following.

Claim 14 *W^k_n with $k \geq 2$, $n \geq 2(k+2)$ is circular-perfect only if $\omega(W^k_n) = \chi(W^k_n)$.*

We have $\omega(W^k_n) \geq 3$ and Corollary 2.23(2) implies $\omega_c(W^k_n) = \max\{k+1, k' + \frac{1}{2}\}$ taken over all odd antiholes $W^{k'-1}_{2k'+1}$ in W^k_n. As $W^l_{n'} \subset W^k_n$ holds only if $l < k$ due to Trotter [104], we obtain that $k+1 > k' + \frac{1}{2}$ for any odd antihole $W^{k'-1}_{2k'+1}$ in W^k_n. Thus, $\omega(W^k_n) = k+1 = \omega_c(W^k_n)$ holds, implying the assertion by $\lceil \chi_c(W^k_n) \rceil = \chi(W^k_n)$. $\diamond$

Claim 15 *For a web W^k_n with $n \geq 2(k+2)$, we have $\omega(W^k_n) < \chi(W^k_n)$ if and only if $(k+1) \not| \, n$.*

For any non-complete web W^k_n, it is well-known that $\chi(W^k_n) = \lceil \frac{n}{\alpha} \rceil$ holds where $\alpha = \alpha(W^k_n) = \lfloor \frac{n}{k+1} \rfloor$. Assuming $n = \alpha(k+1) + r$ with $r < k+1$ we obtain

$$\chi(W^k_n) = \left\lceil \frac{n}{\alpha} \right\rceil = \left\lceil \frac{\alpha(k+1) + r}{\alpha} \right\rceil = k + 1 + \left\lceil \frac{r}{\alpha} \right\rceil$$

implying $k+1 = \omega(W^k_n) < \chi(W^k_n)$ whenever $r > 0$, i.e., whenever $(k+1) \not| \, n$. $\diamond$

Combining Claim 14 and Claim 15 proves assertion (4); the only possible circular-perfect webs W^k_n satisfy $(k+1)|n$. This is obviously true for the webs $W^2_{3\alpha}$. In order to show their circular-perfection, we have to ensure that none of them contains a minimal circular-imperfect induced subgraph. By $\omega(W^2_{3\alpha}) = 3 = \chi(W^2_{3\alpha})$, every induced subgraph G' of $W^2_{3\alpha}$ is clearly 3-colorable. Thus, $\omega(G') = 3$ implies $\omega_c(G') = \chi_c(G')$. The next claim also excludes the occurrence of minimal circular-imperfect induced subgraphs with less clique number:

Claim 16 *No web W^2_n contains a (minimal) circular-imperfect graph with clique number 2 as induced subgraph.*

Suppose $G' \subset W^2_n$ is triangle-free. Then G' does not admit any vertex of degree 3 (since every vertex of W^2_n together with three of its neighbors contains a triangle). The assertion follows since all graphs with maximal degree

2 are collections of paths and cycles, thus circular-perfect. $\diamond$

Hence, assertion (2) is true. For the last assertion (5), it is left to show that every web W_n^k with $k \geq 3$ and $(k+1)|n$ contains a circular-imperfect induced subgraph.

Claim 17 *Any web $W_{\alpha(k+1)}^k$ with $k, \alpha \geq 3$ is circular-imperfect.*

We show that all those webs $W_{\alpha(k+1)}^k$ contain a circular-imperfect web as induced subgraph. Claim 15 implies that $W_{\alpha k-1}^{k-1}$ is circular-imperfect as $k \not| (\alpha k - 1)$. We show $W_{3\alpha-1}^2 \subseteq W_{\alpha(k+1)}^k$ if $\alpha < k$ and $W_{\alpha k-1}^{k-1} \subseteq W_{\alpha(k+1)}^k$ if $\alpha \geq k$ with the help of the following result of Trotter [104]:

$$W_{n'}^{k'} \subseteq W_n^k \text{ if and only if } \frac{k'}{k} n \leq n' \leq \frac{k'+1}{k+1} n$$

Hence, we have $W_{3\alpha-1}^2 \subseteq W_{\alpha(k+1)}^k$ for $\alpha < k$ since

$$\frac{2}{k}\alpha(k+1) = 2\alpha + \frac{2\alpha}{k} \leq 3\alpha - 1 \leq \frac{3}{k+1}\alpha(k+1) = 3\alpha$$

holds: the first inequality is satisfied by $2\frac{\alpha}{k} < 2 \leq \alpha - 1$ if $\alpha < k$ and $\alpha \geq 3$; the second one is trivial. Moreover, $W_{\alpha k-1}^{k-1} \subseteq W_{\alpha(k+1)}^k$ follows for $\alpha \geq k$ since

$$\frac{k-1}{k}\alpha(k+1) = \alpha(k-1) + \frac{\alpha(k-1)}{k} \leq \alpha k - 1 \leq \frac{k}{k+1}\alpha(k+1) = \alpha k$$

holds: the first inequality is satisfied since $\frac{\alpha(k-1)}{k} \leq \alpha - 1$ is true due to $\alpha \geq k$; the second inequality obviously holds again. $\diamond$

Thus, a web W_n^k with $k \geq 3$ and $n > 2(k+1)+1$ is circular-imperfect: if $(k+1) \not| n$ by Claim 15 and if $(k+1)|n$ by Claim 17, finally verifying assertion (5). $\square$

As a consequence, we obtain that the only induced prime circular cliques of a strongly circular-perfect graph are cliques, odd antiholes, odd holes, and stable sets. This shows in particular that a circular clique is strongly circular-perfect if and only if it is a clique, an odd antihole, an odd hole, a stable set, or of the form $K_{3k/3}$ with $k \geq 3$ (Corollary 2.22).

7.2 On normal graphs and the Normal Graph Conjecture

At present, not many graphs are known to be normal. We are going to address this issue by verifying the Normal Graph Conjecture for certain graph classes.

Our first goal is to characterize all the normal webs. For that, we explicitly construct the required set families. According to the circular structure of webs, we introduce cyclic clique covers $\mathcal{Q}$ of odd size $2t-1$ and construct the corresponding cross-intersecting covers $\mathcal{S}$ consisting of stable t-sets. We show that such a pair $(\mathcal{Q}, \mathcal{S})$ exists for each web W_n^k satisfying $t(k+1) \leq n \leq (2t-1)k$. Finally, we figure out for which webs W_n^k such an appropriate parameter t exists and for which not. Proving that the latter webs are indeed not normal finishes the characterization.

Let $\mathcal{Q} = \{Q_1, \ldots, Q_l\}$ be a clique cover of W_n^k consisting of maximum cliques only. We call $\mathcal{Q}$ *cyclic* if each clique Q_i has a non-empty intersection with precisely the cliques Q_{i-1} and Q_{i+1} (the indices are taken modulo l), see Figure 7.1 for three examples.

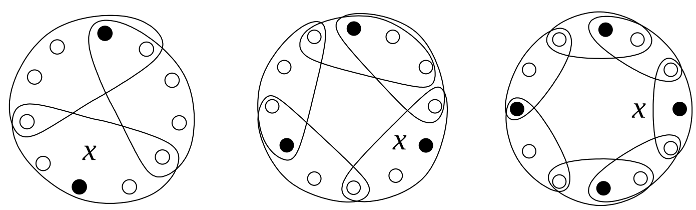


Figure 7.1: Cyclic clique covers of different size in W_{11}^4, W_{12}^3, and W_{12}^2

Lemma 7.3 *W_n^k with $k \geq 2$ admits a cyclic clique cover of size l iff*

$$\tfrac{1}{2}(k+1)l \quad \leq \quad n \quad \leq \quad kl.$$

Proof: Consider a cyclic clique cover $\mathcal{Q} = \{Q_1, \ldots, Q_l\}$ and denote by q_i the first node in Q_i. Then q_i is adjacent to q_{i-1} and q_{i+1} but not to q_j with $i+1 < j < i-1 (\mathrm{mod} l)$ by definition; thus $q_1, \ldots, q_l$ induce an l-hole in W_n^k. On the other hand, consider $W_l^1 \subseteq W_n^k$ with nodes $q_1, \ldots, q_l$ and the maximum cliques $Q(q_i) = \{q_i, \ldots, q_i + k\}$ of W_n^k starting in q_i. A result of Trotter [104] shows that $Q(q_i)$ contains precisely two nodes of W_l^1, namely q_i and q_{i+1}; thus $\mathcal{Q} = \{Q(q_1), \ldots, Q(q_l)\}$ is a cyclic clique cover. Hence cyclic clique covers of size l and holes $W_l^1 \subseteq W_n^k$ correspond to each other. Furthermore, Trotter [104] shows that

$$W_l^1 \subseteq W_n^k \text{ iff } \tfrac{(k+1)}{2}\, l \quad \leq \quad n \quad \leq \quad \tfrac{k}{1}\, l \tag{7.2}$$

holds, as required. □

Note that the assertion of the above lemma remains true if $l = 3$. In this case, inequality (7.2) shows that W_n^k contains a triangle W_3^1 consisting of *non*-consecutive nodes only if $n \leq 3k$ holds. This implies in particular that W_n^k has maximal cliques consisting of non-consecutive nodes only if $n \leq 3k$.

Let $q(x, \mathcal{Q})$ stand for the number of cliques in $\mathcal{Q}$ containing node x. We have $q(x, \mathcal{Q}) \in \{1, 2\}$ for all nodes x (since $\mathcal{Q}$ covers all nodes but no three cliques intersect). We call x a *1-node* (resp. *2-node*) w.r.t. $\mathcal{Q}$ if $q(x, \mathcal{Q}) = 1$ (resp. $q(x, \mathcal{Q}) = 2$) holds.

For our purpose, we are interested in cyclic clique covers $\mathcal{Q}$ of *odd* size $2t-1$ due to the following reason. If a 1-node x belongs to $Q \in \mathcal{Q}$, then $\mathcal{Q} - Q$ consists of $2t-2$ cliques or, in other words, of $t-1$ pairs of intersecting cliques. We denote by $S(x, \mathcal{Q})$ a t-set containing x and one node from the intersection of the $t-1$ pairs of cliques (see the black nodes in Figure 7.1 and Figure 7.2). Thus $S(x, \mathcal{Q})$ intersects all cliques in $\mathcal{Q}$ by construction; we shall show that there exist *stable* sets $S(x, \mathcal{Q})$ whose union *covers* all nodes.

Lemma 7.4 *If $t(k+1) \leq n \leq (2t-1)k$ and $k, t \geq 2$, then W_n^k has a cyclic clique cover $\mathcal{Q}$ of size $2t-1$, and for each 1-node x w.r.t. $\mathcal{Q}$ of W_n^k there is a stable set $S(x, \mathcal{Q})$ of size t in W_n^k.*

Proof: By $\frac{(2t-1)(k+1)}{2} < t(k+1)$, W_n^k has a cyclic clique cover $\mathcal{Q} = \{Q_1, \ldots, Q_{2t-1}\}$ due to Lemma 7.3. Furthermore, $t(k+1) \leq n$ guarantees that W_n^k contains stable sets of size t by $t \leq \alpha(W_n^k) = \lfloor \frac{n}{k+1} \rfloor$.

Consider a 1-node x of W_n^k and assume w.l.o.g. that x belongs to $Q_1 \in \mathcal{Q}$. We construct a stable set $S(x, \mathcal{Q}) = \{x, x_1, \ldots, x_{t-1}\}$ s.t. $x_i \in Q_{2i} \cap Q_{2i+1}$ for $1 \leq i \leq t-1$, see Figure 7.2.

Since $x \in Q_1 - Q_2$, there is a non-neighbor of x in $Q_2 \cap Q_3$ (at least the last node in Q_2 is not adjacent to x but belongs to Q_3). We choose $x_1 = x + (k+1) + d_1 \in Q_2 \cap Q_3$ with $d_1 \in \mathbb{N} \cup \{0\}$ minimal.

In order to construct x_i from x_{i-1} for $2 \leq i \leq t-1$, notice that we have $x_{i-1} \in Q_{2i-2} \cap Q_{2i-1}$, in particular $x_{i-1} \in Q_{2i-1} - Q_{2i}$. As before, there is a non-neighbor of x_{i-1} in $Q_{2i} \cap Q_{2i+1}$ and we choose $x_i = x_{i-1} + (k+1) + d_i \in Q_{2i} \cap Q_{2i+1}$ with $d_i \in \mathbb{N} \cup \{0\}$ minimal. Then $S(x, \mathcal{Q})$ is a stable set if x_{t-1} and x are non-adjacent (all other nodes are non-adjacent by construction).

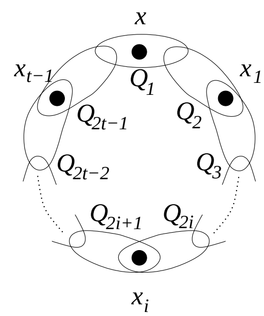


Figure 7.2: Constructing the stable set $S(x, \mathcal{Q})$ for $x \in Q_1$

If $d_i = 0$ for $1 \leq i \leq t-1$, then $x_{t-1} = x + (t-1)(k+1)$. Hence, there are at least $k+1$ nodes between x_{t-1} and x (in increasing order modulo n) due to $n \geq t(k+1)$ and we are done. Otherwise, let j be the smallest index s.t. $d_j > 0$. Then x_j is the first node in Q_{2j+1} since we choose d_j minimal: By $x_{j-1} \notin Q_{2j}$, we have $x_{j-1} + (k+1) \in Q_{2j}$. The only reason for choosing $d_j > 0$ was, therefore, $x_{j-1} + (k+1) + d'_j \notin Q_{2j+1}$ for all $0 \leq d'_j < d_j$ by the minimality of d_j. Hence, x_j is indeed the first node in Q_{2j+1}. This implies that its first non-neighbor is the node $x_j + (k+1)$ belonging to $Q_{2j+2} - Q_{2j+1}$ and $x_{j+1} = x_j + (k+1) + d_{j+1} \in Q_{2j+2} \cap Q_{2j+3}$ is, by the minimality of d_{j+1}, the first node of Q_{2j+3}. The same argumentation shows that *every* further x_i with $i > j+1$ is the first node in Q_{2i+1}; in particular, x_{t-1} is the first node of Q_{2t-1}. Hence, $x \in Q_1 - Q_{2t-1}$ shows that x_{t-1} and x are non-adjacent. Thus $S(x, \mathcal{Q}) = \{x, x_1, \ldots, x_{t-1}\}$ is a stable set of size t and intersects all cliques of $\mathcal{Q}$ by $x \in Q_1$ and $x_i \in Q_{2i} \cap Q_{2i+1}$ for $1 \leq i \leq t-1$. $\square$

This implies that there is, for each 1-node x, *at least* one stable set $S(x, \mathcal{Q})$. It is left to show that the union of *all* such stable sets covers the web.

Lemma 7.5 *Consider a cyclic clique cover $\mathcal{Q}$ of W_n^k of size $2t-1$ where $t(k+1) \leq n \leq (2t-1)k$ and $k, t \geq 2$. Then the union $\mathcal{S}$ of the stable sets $S(x, \mathcal{Q})$, where x is a 1-node of W_n^k w.r.t $\mathcal{Q}$, covers all nodes of W_n^k.*

Proof: Assume to the contrary that there is a node y in W_n^k not covered by $\mathcal{S}$. Then there is no stable set $S(x, \mathcal{Q})$ with $y \in S(x, \mathcal{Q})$. In particular, y is a 2-node w.r.t. $\mathcal{Q}$ by Lemma 7.4. W.l.o.g. let $y \in Q_1 \cap Q_{2t-1}$. We first show $y_l = y + l(k+1) \in Q_{2l+1}$ for $0 \leq l \leq t-2$. Clearly, we have $y = y_0 = y +$

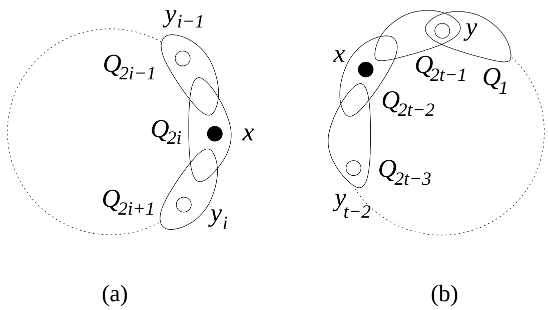


Figure 7.3: Constructing the nodes $y_i \in Q_{2i+1}$

$0(k+1) \in Q_1$ by assumption and prove that $y_{i-1} = y+(i-1)(k+1) \in Q_{2i-1}$ implies $y_i = y + i(k+1) \in Q_{2i+1}$ for $1 \le i \le t-2$.

If there is a 1-node x in $Q_{2i} \backslash (Q_{2i-1} \cup Q_{2i+1})$, then x is adjacent to $y_{i-1} = y + (i-1)(k+1)$, see Figure 7.3(a) (otherwise, there is a stable set $S(x, \mathcal{Q})$ containing x and $y_0, \ldots, y_{i-1}$ in contradiction to our assumption) and $x < y + i(k+1)$ yields $y_i = y + i(k+1) \in Q_{2i+1}$.

If $Q_{2i} \backslash (Q_{2i-1} \cup Q_{2i+1}) = \emptyset$, then $y_i = y + i(k+1)$ clearly belongs to Q_{2i+1} (since we have $y_{i-1} = y+(i-1)(k+1) \in Q_{2i-1}$ and $|Q_{2i-1}| = k+1$).

In particular, we have $y_{t-2} = y + (t-2)(k+1) \in Q_{2t-3}$. Any 1-node x in $Q_{2t-2} \backslash (Q_{2t-3} \cup Q_{2t-1})$ is adjacent to y_{t-2} or to y, see Figure 7.3(b) (otherwise, x together with $y_0, \ldots, y_{t-2}$ would be a set $S(x, \mathcal{Q}) \in \mathcal{S}$ in contradiction to our assumption). We distinguish three cases:

If x is adjacent to y, then $x > y - (k+1) = y_{-1}$ follows and y_{-1} is adjacent to y_{t-2}: either y_{-1} belongs to Q_{2t-3} or is as 1-node adjacent to y_{t-2}; thus, $y_{-1} = y - (k+1) \le y + (t-2)(k+1) + k = y_{t-2} + k$ implies $y \le y + (t-1)(k+1) + k$.

If x is adjacent to y_{t-2}, we obtain $x < y_{t-2} + (k+1) = y_{t-1}$ and y_{t-1} either belongs to Q_{2t-1} or is a 1-node adjacent to y; here, $y_{t-1} \ge y - k$ and, therefore, $y + (t-1)(k+1) \ge y - k$ holds.

The non-existence of a 1-node in Q_{2t-2} implies $y_{t-1} \in Q_{2t-2}$ and, therefore, $y_{t-1} \ge y - k$ follows again.

All three cases imply $n \le (t-1)(k+1) + k$. By $n \ge t(k+1)$, we obtain

$$t(k+1) \le (t-1)(k+1) + k$$

yielding the final contradiction. Hence the union $\mathcal{S}$ of the stable sets $S(x, \mathcal{Q})$, where x is a 1-node of W_n^k w.r.t $\mathcal{Q}$, covers all nodes of W_n^k. □

Since each $S(x, \mathcal{Q})$ meets all cliques in $\mathcal{Q}$ by construction, $\mathcal{S}$ is the required stable set cover. Thus Lemma 7.3, Lemma 7.4, and Lemma 7.5 together imply the assertion of Theorem 3.19 that a web W_n^k with $k \geq 2$ admits, for $t \geq 2$,

- a cyclic clique cover $\mathcal{Q}$ of size $2t - 1$ and
- a cross-intersecting stable set cover $\mathcal{S}$ of stable t-sets

if $t(k+1) \leq n \leq (2t-1)k$ holds.

This shows that a web W_n^k with $k \geq 2$ is normal if there is a $t \geq 2$ with $t(k+1) \leq n \leq (2t-1)k$. It is left to figure out for which W_n^k such a t exists.

Lemma 7.6 *W_n^k with $k \geq 2$ and $n \geq 2(k+1)+2$ is normal if*

- *$k = 2$ and $n \neq 8, 11$,*
- *$k \geq 3$ and $n \neq 3k+1, 3k+2$.*

Proof: We shall ensure, for such a web W_n^k, the existence of a $t \geq 2$ with $t(k+1) \leq n \leq (2t-1)k$. For that we check, for fixed k, whether there are gaps between the ranges $t(k+1) \leq n \leq (2t-1)k$ and $(t+1)(k+1) \leq n \leq (2t+1)k$ for two consecutive values of $t \geq 2$. There is *no* gap between the two ranges if

$$(t+1)(k+1) \quad \leq \quad (2t-1)k+1$$

which is true for $k = 2$ if $t \geq 4$ and for $k \geq 3$ if $t \geq 3$. Thus we have normality for all webs W_n^k with $k \geq 2$ *except* the cases $n = 3k+1, 3k+2$ (gap between the ranges for $t = 2$ and $t = 3$) and W_{11}^2 (gap between the ranges for $t = 3$ and $t = 4$). □

It is a routine to check, by simple case analyzis, that W_{11}^2 and W_{3k+1}^k, W_{3k+2}^k are *not* normal for all $k \geq 2$. As a consequence, we obtain the characterization of all the normal webs (Theorem 3.20), namely, that a web W_n^k is normal if and only if

- $k = 1$ and $n \neq 5, 7$,
- $k = 2$ and $n \neq 7, 8, 11$,
- $k \geq 3$ and $n \neq 3k+1, 3k+2$.

With the help of inequality (7.2) it is a routine to check that all the non-normal webs W_n^k different from C_5, C_7, $\overline{C}_7$ contain either a C_5 (if $n = 3k+1, 3k+2$) or a C_7 (for C_{11}^2) as induced subgraph.

This finally verifies the Normal Graph Conjecturefor webs and, since the class of normal graphs is closed under taking complements, we obtain the same assertion for antiwebs.

Next, we proof the characterization of the normal 1-trees, as this verifies the Normal Graph Conjecturefor such graphs and is the starting point for the forthgoing results on cacti. Recall that a 1-tree can be obtained from one cycle C and certain trees by a sequence of node-identifications. Since trees and cycles of length $\neq 5, 7$ are normal and node-identification preserves normality by Corollary 3.13, we can already conclude that the Normal Graph Conjecture is true for 1-trees.

In order to characterize the normal 1-trees we use, in addition, the result that a connected triangle-free graph is normal iff it has a so-called nice edge cover [59].

Let $G = (V, E)$ be a graph and $\mathcal{F}$ be a minimal edge cover of G, i.e., an inclusion-wise minimal set $\mathcal{F} \subseteq E$ s.t. every node in V is the endnode of some edge in $\mathcal{F}$. Consider a (not necessarily chordless) odd cycle C in G and the distribution of the edges of $\mathcal{F}$ alongside C. We say that a node v of C is *even* w.r.t. $\mathcal{F}$ if v is the endnode of either none or two edges in $\mathcal{F} \cap E(C)$. Sine C is an odd cycle, C has obviously an odd number of even nodes. An edge cover of a graph G is called *nice* if it is minimal and every odd cycle in G has at least three even nodes. Körner and de Simone showed in [59] that the existence of nice edge covers is sufficient for the normality of any graph and also necessary for triangle-free graphs.

Let $G_1 +_v G_2$ denote the graph obtained from G_1 and G_2 by identification in the node v. We show the assertion of Theorem 3.24 that a 1-tree G is *not* normal if and only if one of the following conditions holds:

(i) $G = C_5$,

(ii) $G = C_5 +_v T$ where T is a tree,

(iii) $G = (C_5 +_v T) +_{v'} T'$ where T, T' are trees and v, v' are two nodes of the C_5 at distance two,

(iv) $G = C_7$.

Proof: A 1-tree G can be obtained from a cycle C and certain trees by a sequence of node-identifications. We prove, dependent from the length of C, whether G is normal or not.

Obviously, if $C \neq C_5, C_7$ then G is normal. It remains to consider the cases $C = C_5$ and $C = C_7$. Let G_5 denote the graph depicted in Figure 3.1(a).

Claim 18 *If $C = C_5$ then G is normal if and only if G contains the graph G_5 as subgraph.*

G is triangle-free in both cases, hence G is normal if and only if it admits a nice edge cover. Every minimal edge cover $\mathcal{F}$ of G uses at most three edges of the C_5. All possible types of $\mathcal{F} \cap C_5$ are shown in Figure 7.4 (edges in $\mathcal{F} \cap C_5$ are drawn with bold lines, even nodes w.r.t. $\mathcal{F}$ are black-filled).

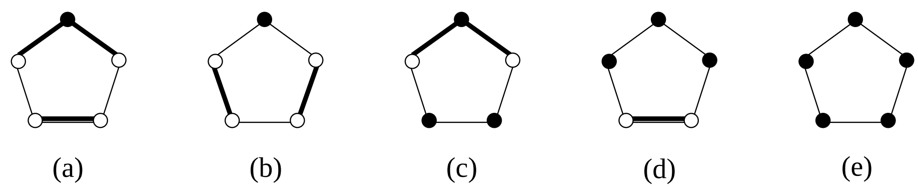


Figure 7.4: Possible types of $\mathcal{F} \cap C_5$

$\mathcal{F}$ is nice if and only if two consecutive nodes v, v' of the C_5 are not covered by the edges in $\mathcal{F} \cap C_5$ (the types (c), (d), and (e)). In order to cover the nodes v, v' by $\mathcal{F}$, there must exist non-empty trees T, T' identified with the C_5 in v resp. v'. In other words, $\mathcal{F}$ is nice if and only if G contains the graph G_5 as subgraph.

(Note that $G = C_5$ is not normal. If $G = C_5 +_v T$ or $G = (C_5 +_v T) +_{v'} T'$ where v, v' are two non-consecutive nodes of the C_5, then $\mathcal{F}$ has to be of type (a) or (b) and is, therefore, not nice.) $\diamond$

Claim 19 *If $C = C_7$ then G is normal if and only if $G \neq C_7$.*

If $G = C_7$ then G is clearly not normal. Otherwise, G can be obtained from the C_7 and certain trees by a sequence of node-identifications. The C_7 is almost normal and all of its nodes are unnormal. Thus identifying the C_7 with a (non-empty) tree in a node yields a normal graph, and adding further trees maintains normality. Hence G is normal. $\square$

This implies in particular that almost all 1-trees are normal. Since every cactus can be obtained from several 1-trees by a sequence of node-identifications, we can also conclude that the Normal Graph Conjecture is true for cacti. Considering the normality of cacti under node-identification leads further to a polynomial time algorithm to decidethe normality of a given cactus.

7.3 Classes of a-perfect graphs

Shepherd [99] achieved the result that the only nontrivial facets of stable set polytopes of near-bipartite graphs are constraints

$$\sum_{i \leq k} \frac{1}{\alpha(\overline{W}_i)} x(\overline{W}_i) + x(Q) \leq 1 \tag{7.3}$$

associated with complete joins of prime antiwebs $\overline{W}_1, \ldots, \overline{W}_k$ and a clique Q.

For subclasses of near-bipartite graphs, the facet sets can admit a simpler structure only if certain kinds of prime antiwebs or certain complete joins can be excluded. The former is true for complements of line graphs, as no prime antiwebs different from odd antiholes occur [99]. We are going to simplify the above constraints for co-fuzzy circular interval graphs by excluding complete joins.

For that, we discuss whether fuzzy circular interval graphs may admit the disjoint union of a prime web and a single node. This is obviously possible in general quasi-line graphs, as they can even *be* the disjoint union of a prime web and a single node. In addition, there exist fuzzy circular interval graphs containing such a disjoint union, involving webs W^k_{2k+2} for $k = 1, 2, 3$. Note that the nodes of any web W^k_{2k+2} can be partitioned into two cliques $\{1, \ldots, 1+k\}$ and $\{k+2, \ldots, 2k+2\}$ allowing a representation by choosing the two cliques as multiple endpoints of one interval. We shall ensure that this construction is possible for the webs W^k_{2k+2} only, but not for any *prime* web.

The key tool is to prove that any web W^k_n with $n > 2k+2$ has precisely *one* representation as fuzzy circular interval graph, namely, the *canonical* one: distribute the point set $V = \{1, \ldots, n\}$ without multiplicities in this order on $\mathcal{C}$ and take a collection $\mathcal{I} = \{I_1, \ldots, I_n\}$ of arcs in $\mathcal{C}$ with $I_i \cap V = \{i, \ldots, i+k\}$ for $1 \leq i \leq n$; then $G(V, \mathcal{I})$ obviously equals the web W^k_n.

Lemma 7.7 *Any web W^k_n with $n > 2k+2$ admits no other representation as fuzzy circular interval graph than the canonical one.*

Proof: Let $G(V, \mathcal{I})$ be a representation of a web W^k_n with $n > 2k+2$ as fuzzy circular interval graph. In order to verify the assertion of the lemma we ensure first that V does not contain multiple points.

Claim 20 *V does not contain points with a multiplicity > 1.*

Assume on the contrary, there are multiple points in V and consider, among them, two points i and $i+t$ at maximum distance in W_n^k. Since multiple points are adjacent, we obtain $0 < t \leq k$.

Consider the node $(i+t)-(k+1)$ in W_n^k: by construction, it is adjacent to i but not to $i+t$ (by $n > 2k+2$). Hence, $\mathcal{I}$ contains an interval I having i and $i+t$ as multiple endpoint and $(i+t)-(k+1)$ as opposite endpoint (then we are free of linking i and $(i+t)-(k+1)$ by an edge but $i+t$ and $(i+t)-(k+1)$ not as both edges belong to E_2); see Figure 7.5.

Further, consider the node $i+(k+1)$ in W_n^k being adjacent to $i+t$ but not to i. As before, this is possible only if $i, i+t$ and $i+(k+1)$ are opposite endpoints of an interval in $\mathcal{I}$. Since i and $i+t$ cannot be endpoints of two intervals in $\mathcal{I}$, this has to be the same interval I implying that $i+(k+1)$ is a multiple point of $(i+t)-(k+1)$, as shown in Figure 7.5. Hence $i+(k+1)$ and $(i+t)-(k+1)$ are adjacent in W_n^k and, by the choice of t, we infer

$$\begin{array}{rcl}(i+t)-(k+1) & \leq & i+(k+1)+t \\ i & \leq & i+(2k+2)\end{array}$$

and $n \leq 2k+2$ follows, yielding a contradiction.

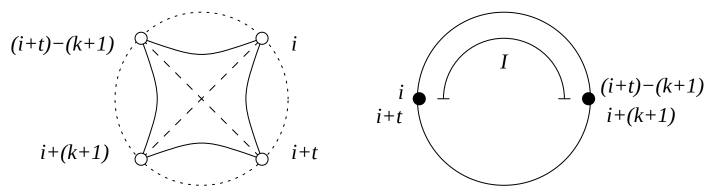


Figure 7.5: The case of multiple points.

Claim 21 *$G(V, \mathcal{I})$ is the canonical representation of W_n^k.*

As Claim 20 shows, adjacencies in W_n^k cannot be realized by multiple points in V but as different points belonging to the same interval in $\mathcal{I}$ only. In particular, consecutive points in V form a clique in W_n^k only if they belong to the same interval in $\mathcal{I}$ (as only other cliques, triangles formed by non-consecutive points are possible). Denote by $Q(i) = \{i, \ldots, i+k\}$ the maximum clique of W_n^k starting in node i. Then, obviously, $Q(i)$ and $Q(i+1)$ intersect precisely in the nodes $\{i+1, \ldots, i+k\}$. Consequently, there exist intervals $I_i, I_{i+1} \in \mathcal{I}$ with

$$\begin{array}{rcl}(I_i \setminus I_{i+1}) \cap V & = & \{i\} \\ (I_i \cap I_{i+1}) \cap V & = & \{i+1, \ldots, i+k\} \\ (I_{i+1} \setminus I_i) \cap V & = & \{i+k+1\}.\end{array}$$

Repeating this argumentation for all maximum cliques $Q(1), \ldots, Q(n)$ finally yields the assertion. □

Thus, every *prime* web W_n^k has only the canonical representation. This implies that the intervals representing a web W_n^k with $n > 2k+2$ as subgraph of a fuzzy circular interval graph G already occupy the whole circle $\mathcal{C}$. Thus every node in $G - W_n^k$ has a neighbor in W_n^k and no disjoint union of W_n^k and a node in $G - W_n^k$ is possible. This implies:

Corollary 7.8 *No fuzzy circular interval graph contains the disjoint union of a prime web and a single node.*

Turning back to the complements we, therefore, obtain the nonexistence of the complete join of a prime antiweb and a single node. Thus, the complete join of two prime antiwebs as well as the complete join of a prime antiweb and a nonempty clique are excluded. Hence, the above lemma implies: the only nontrivial facets of the stable set polytope of a co-fuzzy circular interval graph are constraints (7.3) not consisting of different facet blocks but associated with *either* a prime antiweb *or* a clique. Since both prime antiweb and clique constraints are rank constraints in particular, we obtain the assertion of Theorem 4.11, namely, that the stable set polytope of a co-fuzzy circular interval graph has as only nontrivial facets rank constraints associated either with cliques or with prime antiwebs and is, therefore, a-perfect.

The same is obviously true for all subclasses of co-fuzzy circular interval graphs, in particular, for antiwebs (Theorem 4.10) and convex-round graphs (Theorem 4.12).

7.4 Stable set polytopes of webs

In order to describe the non-rank facets of stable set polytopes of webs, we first provide an appealing construction to obtain, starting from a web with a known non-rank facet, an infinite sequence of not rank-perfect webs with the same clique number. This construction is the main tool to obtain, for all fixed clique numbers ≥ 5, that there are only finitely many rank-perfect webs with this clique number.

Following Galluccio and Sassano [41], a graph G and its rank constraint $x(G) \leq \alpha(G)$ are called *rank-minimal* if and only if G is a clique or G is rank facet-producing and, for each induced subgraph $G' \subset G$, the inequality $x(G') \leq \alpha(G)$ does not define a facet of $\mathrm{STAB}(G')$. All rank-minimal claw-free graphs were described in [41] as cliques, partitionable webs, or line graphs of 2-connected, critical hypomatchable graphs. We show that no rank-minimal line graphs different from odd holes occur as induced subgraphs of webs:

Lemma 7.9 *Let H be a 2-connected, critical hypomatchable graph. If its line graph $L(H)$ is an induced subgraph of a web, then $L(H)$ is a triangle or an odd hole.*

Proof: Consider a 2-connected, critical hypomatchable graph H. Since H is 2-connected, H has at least 3 nodes. Since H is critical hypomatchable, H must not admit parallel edges, i.e., H is simple. If $|H| = 3$, then H as well as $L(H)$ is a triangle. Hence assume $|H| \geq 5$ in the sequel (note: every hypomatchable graph has an odd number of nodes). We show that H as well as $L(H)$ is an odd hole if $L(H)$ is an induced subgraph of a web.

Due to Lovász [66], a graph H is hypomatchable if and only if there is a sequence $H_0, H_1, \ldots, H_k = H$ of graphs such that H_0 is a chordless odd cycle and for $1 \leq i \leq k$, H_i is obtained from H_{i-1} by adding a chordless odd path E_i that joins two (not necessarily distinct) nodes of H_{i-1} and has all internal nodes outside H_{i-1}. The odd paths $E_i = H_i - H_{i-1}$ are called *ears* for $1 \leq i \leq k$ and the sequence $H_0, H_1, \ldots, H_k = H$ an *ear decomposition* of H.

If a hypomatchable graph H is 2-connected and has at least 5 nodes, then H admits an ear decomposition $H_0, H_1, \ldots, H_k = H$ s.t. every H_i is 2-connected for $0 \leq i \leq k$ by Cornuéjols and Pulleyblank [26] and H_0 is an odd hole (i.e. $|H_0| \geq 5$) by [109]. Moreover, in [109] is shown that we can always reorder the ears $E_1, \ldots E_k$ of a given decomposition s.t. the decomposition starts with all ears of length ≥ 3 and ends up with all ears of length one. Thus, every 2-connected hypomatchable graph H with $|V(H)| \geq 5$ has a *proper ear decomposition* $H_0, H_1, \ldots, H_k = H$ where H_0 has length ≥ 5, each H_i is 2-connected, and, if $k > 0$, there is an index j s.t. $E_1, \ldots, E_j$ have length ≥ 3 and $E_{j+1}, \ldots, E_k$ have length one.

Consider a 2-connected hypomatchable graph H with $|V(H)| \geq 5$ and a proper ear decomposition $H_0, H_1, \ldots, H_k = H$ of H. We show in the next two claims: the decomposition of H has neither ears of length 1 nor of length ≥ 3 if H is critical and $L(H)$ is an induced subgraph of a web.

Claim 22 *If $H_0, H_1, \ldots, H_k = H$ contains an ear of length 1, then H is not critical hypomatchable.*

In that case, the last ear E_k of the proper ear decomposition $H_0, H_1, \ldots, H_k = H$ of H is a single edge. Removing the edge E_k from $H_k = H$ yields the hypomatchable graph H_{k-1} with the same node set. Thus, H is not critical hypomatchable. $\diamond$

Claim 23 *If $H_0, H_1, \ldots, H_k = H$ contains an ear of length ≥ 3, then H is not critical hypomatchable or $L(H)$ is not an induced subgraph of a web.*

In that case, the first ear E_1 of the proper ear decomposition $H_0, H_1, \ldots, H_k = H$ of H is a path of length ≥ 3. If the endnodes u_1 and v_1 of E_1 are adjacent in H_0 (see Fig. 7.6(a)), then H admits a proper ear decomposition $H'_0, H'_1, \ldots, H'_k = H$ with $H'_0 = H_0 \cup E_1 - \{u_1v_1\}$ and $E_2, \ldots, E_k, \{u_1v_1\}$ as ear sequence (i.e. $H'_i = H'_{i-1} \cup E_{i+1}$ for $1 \leq i \leq k-1$ and $H'_k = H'_{k-1} \cup \{u_1v_1\}$). Thus, H admits an ear of length 1 and is not critical hypomatchable by Claim 22.

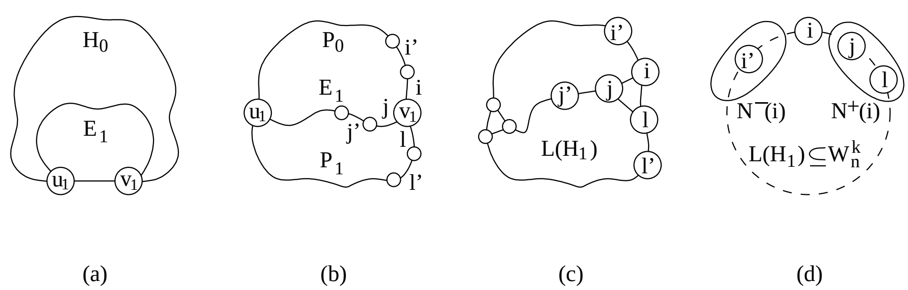


Figure 7.6: Ear decompositions for Claim 23

If the endnodes u_1 and v_1 of E_1 are non-adjacent in H_0 (see Fig. 7.6(b)), then there are 3 internally disjoint paths P_0, P_1, E_1 between u_1 and v_1 in H_1: P_0 with even length ≥ 2 and P_1, E_1 with odd length ≥ 3. Consider in H_1 the edges i, i', j, j', l, l' as shown in Fig. 7.6(b). Then the edges i', j', l' are pairwise disjoint (note: u_1 may be an endnode of i' but neither of j' nor of l' because of the parity of the paths).

Assume $L(H_1)$ is an induced subgraph of a web W_n^k. We have to find a respective order of the nodes i, i', j, j', l, l' in W_n^k (recall that edges of H turn into nodes of $L(H)$, see Fig. 7.6(c)). Moreover, recall that the neighborhood of every node x, denoted by $N(x)$, of a web W_n^k splits into two cliques $N^-(x) = \{x-k, \ldots, x-1\}$ and $N^+(x) = \{x+1, \ldots, x+k\}$ (where all indices are taken modulo n).

Consider $N(i)$ in W_n^k: we have $i', j, l \in N(i)$ where jl is an edge but neither $i'j$ nor $i'l$ (see Fig. 7.6(c)). W.l.o.g. let $i' \in N^-(i)$. Then $j, l \in N^+(i)$ follows since both $N^-(i)$ and $N^+(i)$ are cliques. Furthermore, let $j < l$ (the case $l < j$ goes analogously due to $ij, il \in E$ but $ij', il' \notin E$), i.e., assume $i + 1 \leq j < l \leq i + k$ (see Fig. 7.6(d)).

Now, consider the node j'. We have $j' \in N(j)$ but $j' \notin N(i)$ (see Fig. 7.6(c)). This implies $j' \in N^+(j)$ (since $N^-(j) \subseteq N(i)$ by $j \in N^+(i)$), i.e., we obtain $j' \in \{j+1, \ldots, j+k\}$. But $i + 1 \leq j < l \leq i + k$ implies $N^+(j) \subseteq N(l)$, hence $j' \in N(l)$ in contradiction to j' and l non-adjacent (see Fig. 7.6(c)). Thus, $L(H_1)$ cannot be an induced subgraph of a web W_n^k.

We conclude: if E_1 connects two adjacent nodes of H_0, then H is not critical, if E_1 connects two non-adjacent nodes of H_0, then $L(H)$ is not an induced subgraph of a web. $\diamond$

Hence, we have obtained that for every 2-connected, critical hypomatchable graph H holds the following. If H has 3 nodes, then H and its line graph $L(H)$ are triangles. Otherwise, H admits a proper ear decomposition $H_0, H_1, \ldots, H_k = H$ with and index j s.t. $E_1, \ldots, E_j$ have length ≥ 3 and $E_{j+1}, \ldots, E_k$ have length one. By Claim 22, there is no ear of length 1 (i.e. $j = k$). If the line graph of H is an induced subgraph of a web, then there is no ear of length ≥ 3 by Claim 22 and Claim 23 (i.e. $j = 0$). In conclusion, we obtain $k = 0$, thus H consists in the odd hole H_0 of length ≥ 5 only and $L(H)$ is an odd hole, too. $\square$

Furthermore, we need the following characterization when a valid inequality $\mathbf{a}^T\mathbf{x} \leq b$ is a facet of the stable set polytope of a general graph G. A pair i, j of nodes is $\mathbf{a}$-*critical* in G if there are two roots S_1 and S_2 of $\mathbf{a}^T\mathbf{x} \leq b$ such that $\{i\} = S_1 \backslash S_2$ and $\{j\} = S_2 \backslash S_1$. A subset V' of $V(G)$ is $\mathbf{a}$-*connected* if the graph with node set V' and edge set $\{ij |\ i, j \in V',\ ij\ \mathbf{a}\text{-critical in } G\}$ is connected.

Lemma 7.10 *Let* $\mathbf{a}^T\mathbf{x} \leq b$ *be a valid inequality for* $\mathrm{STAB}(G)$ *with* $b \neq 0$. *Consider a partition* $V_1, \ldots, V_p$ *of* $V(G)$ *such that* V_i *is* $\mathbf{a}$*-connected for every* $1 \leq i \leq p$. *The inequality* $\mathbf{a}^T\mathbf{x} \leq b$ *is facet-defining if and only if there are* p *roots* $S_1, \ldots, S_p$ *with*

$$\begin{vmatrix} |S_1 \cap V_1| & \cdots & |S_1 \cap V_p| \\ \vdots & & \vdots \\ |S_p \cap V_1| & \cdots & |S_p \cap V_p| \end{vmatrix} \neq 0.$$

Proof: In order to prove the If-part, let ${\mathbf{a}'}^T\mathbf{x} \leq b'$ be a facet containing the face induced by the inequality $\mathbf{a}^T\mathbf{x} \leq b$. For every $1 \leq i \leq p$, the set V_i is $\mathbf{a}$-connected and so there exist λ_i such that $a_j = \lambda_i$ for all $j \in V_i$. Since for every stable set S, $\mathbf{a}^T\chi^{\mathbf{S}} = b$ implies that ${\mathbf{a}'}^T\chi^{\mathbf{S}} = b'$, V_i is $\mathbf{a}'$-connected. Therefore there exist λ'_i such that $a'_j = \lambda'_i$ for all $j \in V_i$. Hence we have for every $1 \leq i \leq p$:

$$\begin{array}{rcl} \lambda_1|S_i \cap V_1| + \ldots + \lambda_p|S_i \cap V_p| & = & b \\ \lambda'_1|S_i \cap V_1| + \ldots + \lambda'_p|S_i \cap V_p| & = & b' \end{array}$$

Since

$$\begin{vmatrix} |S_1 \cap V_1| & \cdots & |S_1 \cap V_p| \\ \vdots & & \vdots \\ |S_p \cap V_1| & \cdots & |S_p \cap V_p| \end{vmatrix} \neq 0$$

holds we get $\lambda'_i = \frac{b'}{b}\lambda_i$ for every $1 \leq i \leq b$. Thus $\mathbf{a}^T\mathbf{x} \leq b$ is facet-defining.

Now let us turn to the Only if-part. Since $\emptyset$ is not a root of the facet $\mathbf{a}^T\mathbf{x} \leq b$, there exist n roots $S_1, \ldots S_n$ whose incidence vectors are linearly independent. Let M be the matrix with the incidence vectors of $S_1, \ldots, S_n$ as rows. Let v_i be an element of V_i for $1 \leq i \leq p$. We add to the v_1-th column of M the other columns related to the other elements of V_1; we add to the v_2-th column of M the other columns related to the other elements of V_2 etc. This yields

$$\begin{vmatrix} . & |S_1 \cap V_1| & . & |S_1 \cap V_p| \\ . & \vdots & . & \vdots \\ . & |S_n \cap V_1| & . & |S_n \cap V_p| \end{vmatrix} \neq 0$$

and, thus, the (n, p)-matrix

$$\begin{pmatrix} |S_1 \cap V_1| & \cdots & |S_1 \cap V_p| \\ \vdots & & \vdots \\ |S_n \cap V_1| & \cdots & |S_n \cap V_p| \end{pmatrix}$$

has p linearly independent rows, as required. $\square$

Note that this lemma generalizes the well-known result of Chvátal [21] that a graph produces the full rank facet if it is 1-connected.

We are now able to prove:

Theorem 7.11 *Let* $\mathbf{a}^T\mathbf{x} \leq c\alpha_1$ *be a proper weak rank facet of* $\mathrm{STAB}(W_n^k)$*. Then* $\mathrm{STAB}(W_{n+k+1}^k)$ *has the proper weak rank facet*

$$\sum_{1\leq i\leq n} a_i x_i + \sum_{n<i\leq n+k+1} c\ x_i \ \leq\ c\ (\alpha_1 + 1) \tag{7.4}$$

Proof: By definition, the node set of W_n^k is $\{1,\ldots,n\}$ and the node set of W_{n+k+1}^k is $\{1,\ldots,n+k+1\}$. Hence we may use this convention to identify a node of W_n^k with the corresponding one of W_{n+k+1}^k. Denote by G^1 the web W_n^k and by G^2 the web W_{n+k+1}^k. Let $\omega = k+1$ be the clique number of both G^1 and G^2 and, for every $1 \leq i \leq n$ (resp. $1 \leq i \leq n+\omega$), let $Q_i^1 = [i, i+k]$ (resp. $Q_i^2 = [i, i+k]$) be the maximum clique of G^1 (resp. G^2) with 'first' element i.

Since $\mathbf{a}^T\mathbf{x} \leq c\alpha_1$ is a proper weak rank facet of $\mathrm{STAB}(G^1)$, there exists a subset V_1 of nodes of G^1 such that $\alpha_1 = \alpha(G^1[V_1])$ and $G^1[V_1]$ is rank facet-producing. Moreover, $G^1[V_1]$ has a partitionable web with node set W_1, stability number α_1, and clique number $\omega_1 \geq 2$ as induced subgraph by Lemma 7.9.

Notice that Q_{n-k}^1 is the maximum clique $\{n-k,\ldots,n\}$ of G^1. Let $w_1,\ldots,w_h$ be the elements in increasing order of W_1 in Q_{n-k}^1. We have $h = \omega_1$ or $\omega_1 - 1$, by [104]. For every $1 \leq i \leq h$, let q_i be the element $w_i + \omega$ of Q_{n+1}^2 and define:

$$W_2 = \begin{cases} W_1 \cup \{q_1,\ldots,q_{\omega_1}\} & \text{if } h = \omega_1 \\ W_1 \cup \{n+1\} \cup \{q_1,\ldots,q_{\omega_1-1}\} & \text{if } h = \omega_1 - 1 \end{cases}$$

Let $V_2 = V_1 \cup Q_{n+1}^2 = V_1 \cup \{n+1,\ldots,n+k+1\}$. Let $\mathbf{v}$ be the $(n+\omega)$-column vector $(a_1,\ldots,a_n,c,\ldots,c)$ and $\mathbf{y}$ be the $(n+\omega)$-column vector $(a_1,\ldots,a_n,0,\ldots,0)$.

Claim 24 *Inequality (7.4) is valid for* $\mathrm{STAB}(W_{n+k+1}^k)$*.*

Let S be any stable set of G^2. Let l be the node of S such that $[l+1,n]\cap S = \emptyset$ and let t be the node of S such that $[n+1,t-1]\cap S = \emptyset$. Notice that $S\setminus\{t\}$ is a stable set of G^1. Hence we have $\mathbf{v}^T\chi^S = (\mathbf{y} + c\chi^{Q_{n+1}^2})^T\chi^S \leq c\alpha_1 + x_t \leq c(\alpha_1+1)$ as $x_t \leq c$ if $t \notin Q_{n+1}^2$ and $x_t = c$ if $t \in Q_{n+1}^2$. $\diamond$

Claim 25 *The set of nodes* W_2 *induces a partitionable web with stability number* $\alpha_1 + 1$ *and clique number* ω_1*.*

Let $1 \leq v_1 \leq v_2 \leq \ldots \leq v_{n'} \leq n$ be the nodes of W_1 in increasing order. We discuss the two cases $h = \omega_1$ and $h = \omega_1 - 1$.

If $h = \omega_1$ then let v be any node of W_2. If v is a node q_i of $\{q_1, \ldots, q_{\omega_1}\}$ then the set of nodes Q_v^2 meets W_2 exactly in the ω_1 nodes $\{q_i, \ldots, q_{\omega_1}\} \cup \{v_1, \ldots, v_{i-1}\}$, since W_1 induces a web of G^1 with clique number ω_1 by [104]. If v is a node w_i of $\{w_1, \ldots, w_{\omega_1}\}$ then Q_v^2 meets W_2 precisely in the ω_1 nodes $\{w_i, \ldots, w_{\omega_1}\} \cup \{q_1, \ldots, q_{i-1}\}$. If v is a node of $W_1 \setminus \{w_1, \ldots, w_{\omega_1}\}$, we obviously have $|Q_v \cap W_2| = \omega_1$ since W_1 induces a web of G^1 with clique number ω_1 due to [104].

If $h = \omega_1 - 1$ then notice that $w_1 \neq n - k$ (otherwise we would have $h = \omega_1$). Hence $n + 1 \notin \{q_1, \ldots, q_h\}$ follows. Let v be any node of W_2. If v is a node q_i of $\{q_1, \ldots, q_h\}$ then Q_v^2 meets W_2 exactly in the ω_1 nodes $\{q_i, \ldots, q_h\} \cup \{v_1, v_2, \ldots, v_{i-1}, v_i\}$, since W_1 induces a web of G^1 with clique number ω_1 by [104]. If v is a node w_i of $\{w_1, \ldots, w_h\}$ then Q_v meets W_2 precisely in the ω_1 nodes $\{w_i, \ldots, w_h, n+1\} \cup \{q_1, \ldots, q_{i-1}\}$, as $w_h < n+1 < q_1$. If $v = n+1$ then Q_v meets W_2 exactly in the ω_1 nodes $\{n+1, q_1, \ldots, q_h\}$. If v is a node of $W_1 \setminus \{w_1, \ldots, w_h\}$, we obviously have $|Q_v \cap W_2| = \omega_1$ since W_1 induces a web of G^1 with clique number ω_1 due to [104]. Hence in both cases, W_2 induces a web with clique number ω_1 and $|W|+\omega_1 = (\alpha_1+1)\omega_1+1$ nodes. Thus W_2 is a partitionable web with stability number $\alpha_1 + 1$. $\diamond$

Claim 26 *The node set $V_2 = W_2 \cup Q_{n+1}^2$ is $\mathbf{v}$-connected.*

We first show that W_2 is $\mathbf{v}$-connected. Since $\mathbf{a}^T\mathbf{x} \leq c\alpha_1$ is a weak rank facet of $\mathrm{STAB}(G^1)$, we have by definition $a_i = c$ for every $i \in W_1$. Hence for every $i \in W_2$ follows $v_i = c$. Since W_2 is a partitionable web of stability number $\alpha_1 + 1$ by Claim 25, this implies that W_2 is $\mathbf{v}$-connected.

Let $w_1 < w_2 < \ldots < w_{\omega_1}$ be the elements of W_2 in Q_{n+1}^2 (by definition of W_2 there are exactly ω_1 of them). Let S be a maximum stable set of W_2 disjoint from Q_1^2 (S exists because $W_2 \cap Q_1^2$ is a subset of a maximum clique of W_2, and for every maximum clique Q of a partitionable graph, there exists a unique maximum stable set avoiding Q by [8]). Let s be the element of S with maximal index. Then for every $w_{\omega_1} \leq q \leq n + \omega$, the set $(S \setminus \{s\}) \cup \{q\})$ is obviously a root of inequality (7.4). Hence $W^2 \cup [w_{\omega_1}, n+\omega]$ is $\mathbf{v}$-connected. Likewise, the set $W^2 \cup [n + 1, w_1]$ is $\mathbf{v}$-connected.

For every $1 \leq i < \omega_1$, there exists a maximum stable set of W_2 disjoint from $Q_{w_{i+1}}^2$. Let s be the element of S with maximal index which is less than or equal to w_i. Then for every $w_i \leq q \leq w_{i+1}$, the set $(S \setminus \{s\}) \cup \{q\}$ is a root of inequality (7.4). Hence $W^2 \cup [w_i, w_{i+1}]$ is $\mathbf{v}$-connected and V_2 is $\mathbf{v}$-connected as well. $\diamond$

Let $p = n - |W_1|$ and $\{1, \ldots, n\} \setminus W_1 = \{y_1, \ldots, y_p\}$. Due to Lemma 7.10, there are p roots $S_1, \ldots, S_p$ of $\mathbf{a}^T\mathbf{x} \leq c\alpha_1$ such that the incidence vectors of their restriction to $\{1, \ldots, n\} \setminus W_1 = (\{1, \ldots, n\} \cup Q_n\}) \setminus V_2$ are linearly independent, that is we have:

$$\begin{vmatrix} |S_1 \cap \{y_1\}| & \cdots & |S_1 \cap \{y_p\}| \\ \vdots & & \vdots \\ |S_{k'} \cap \{y_1\}| & \cdots & |S_{k'} \cap \{y_p\}|\} \end{vmatrix} \neq 0$$

Claim 27 *For every $1 \leq i \leq p$, there exists a node q_i of G^2 such that $S'_i = S_i \cup \{q_i\}$ is a root of inequality (7.4).*

For every $1 \leq i \leq p$, let l_i (resp. t_i) be the element of S_i with minimal (resp. maximal) index. Let $q_i = t_i + \omega$. Obviously, q_i is not a neighbor of t_i in G^2. If q_i is a neighbor of l_i in G^2 then $q_i + \omega - 1 - (n + \omega) \geq l_i$. Thus $t_i + \omega - 1 - n \geq l_i$, which implies that t_i is a neighbor of l_i in G^1: a contradiction. Hence $S'_i = S_i \cup \{q_i\}$ is a stable set of G^2. Since q_i is a node of the maximum clique Q_n, it follows that S'_i is a root of inequality (7.4), as required. $\diamond$

Since $G^2[W_2]$ has stability number $\alpha_1 + 1$ (Claim 25), there is a stable set S'_0 of $G^2[V_2]$ which is a root of inequality (7.4).

For every $0 \leq i \leq p$ and $1 \leq j \leq p$, let $\delta_{i,j} = 1$ if $y_j \in S'_i$, 0 otherwise. By Claim 24 and 27, inequality (7.4) is a valid inequality with $p+1$ **v**-critical components V_2, $\{y_1\}, \ldots, \{y_p\}$, and $p+1$ roots $S'_0, S'_1, \ldots, S'_p$ such that

$$\begin{vmatrix} |S'_0 \cap V_2| & \delta_{0,1} & \cdots & \delta_{0,p} \\ |S'_1 \cap V_2| & \delta_{1,1} & \cdots & \delta_{1,p} \\ \vdots & \vdots & & \vdots \\ |S'_p \cap V_2| & \delta_{p,1} & \cdots & \delta_{p,p} \end{vmatrix} = \begin{vmatrix} \alpha_1 + 1 & 0 & \cdots & 0 \\ |S'_1 \cap V_2| & |S_1 \cap \{y_1\}| & \cdots & |S_1 \cap \{y_p\}| \\ \vdots & \vdots & & \vdots \\ |S'_p \cap V_2| & |S_p \cap \{y_1\}| & \cdots & |S_p \cap \{y_p\}| \end{vmatrix}$$

is non-zero and Lemma 7.10 implies that inequality (7.4) defines a facet of $\text{STAB}(G^2)$. To finish the proof, it remains to show that it is a proper weak rank facet.

Claim 28 *The set V_2 is rank facet-producing and $\alpha(G^2[V_2]) = \alpha_1 + 1$.*

We have $\alpha(G^2[V_2]) \leq \alpha(G^2[V_1]) + \alpha(Q_n) \leq \alpha_1 + 1$ and $\alpha(G^2[V_2]) = \alpha(G^2[W_2])$. Let v be any node of $V_2 \setminus W_2$. By the definition of V_2, v is an element of Q^2_{n+1}. Therefore $|N(v) \cap W_2| \geq \omega_1$ as $|W_2 \cap Q^2_{n+1}| = \omega_1$, by the definition of W_2. Let δ be the element of W_2 with maximal index.

If $v < \delta$ then $(\delta-\omega) \in N(v)$. As $\delta-\omega$ is an element of W_2 by the definition of W_2, we get $|N(v) \cap W_2| \geq \omega_1 + 1$. If $v \geq \delta$ then v has at least one neighbor in $Q_v^2 \cap W^2$, as $|Q_v^2 \cap W^2| \geq \omega_1 - 1 \geq 1$. Hence $|N(v) \cap W_2| \geq \omega_1 + 1$.

Thus, in both cases, $|N(v) \cap W_2| \geq \omega_1 + 1$. Hence $\alpha(N(x) \cap W_2) = 2$ and therefore, $G^2[V_2]$ is rank facet-producing by Galluccio & Sassano [41] (recall that W^2 is a partitionable web by Claim 25 and, therefore, rank-minimal). $\diamond$ $\square$

An immediate consequence of Theorem 7.11 is the following: if $\mathrm{STAB}(W_n^k)$ has a proper weak non-rank facet then $\mathrm{STAB}(W_{n+k+1}^k)$ has a proper weak non-rank facet (Theorem 4.20).

Therefore, if $\mathrm{STAB}(W_n^k)$ has a proper weak non-rank facet then *all* webs $\mathrm{STAB}(W_{n+l(k+1)}^k)$ are not rank-perfect for any $l \geq 0$, too. This implies that, for every fixed $k \geq 3$, constructing a finite base set of webs $W_{n_0}^k, \ldots, W_{n_k}^k$ with proper weak non-rank facets and $n_i = i \pmod{k+1}$ suffices to show that almost all webs W_n^k are not rank-perfect (Corollary 4.21).

We prove that several clique family inequalities $(\mathcal{Q}, k'+1)$ associated with different regular subwebs $W_{n'}^{k'}$ induce proper weak non-rank facets (note that $(\mathcal{Q}, k'+1)$ is a proper weak non-rank constraint if $r < k'$). The main result from [80] is Theorem 4.24 stating that for any $k \geq 5$ and a $(k', k-k')$-regular subweb $W_{lk'}^{k'} \subset W_{lk}^k$ with $2 \leq k' \leq k-3$ and odd $l \geq 3$, the clique family inequality $(\mathcal{Q}, k'+1)$

$$2 \sum_{i \in W_{lk'}^{k'}} x_i + 1 \sum_{i \notin W_{lk'}^{k'}} x_i \leq 2\alpha(W_{lk'}^{k'}) \tag{7.5}$$

associated with $W_{lk'}^{k'}$ is a proper weak non-rank facet of $\mathrm{STAB}(W_{lk}^k)$ if $l = 2 \pmod{k'+1}$ and $\alpha(W_{lk'}^{k'}) < \alpha(W_{lk}^k)$.

Proof: By assumption, we have $l = 2 \pmod{k'+1}$ and $\alpha(W_{lk'}^{k'}) < \alpha(W_{lk}^k)$. In order to prove the assertion of the theorem, we have to establish that the inequality (7.5)

$$2 \sum_{i \in W_{lk'}^{k'}} x_i + 1 \sum_{i \notin W_{lk'}^{k'}} x_i \leq 2\alpha(W_{lk'}^{k'})$$

is valid and facet-inducing for $\mathrm{STAB}(W_{lk}^k)$. Validity follows from Lemma 4.22: since $l = 2 \pmod{k'+1}$, we have $lk' = -2 \pmod{k'+1}$ and, therefore, the remainder r of the division of lk' by $k'+1$ is equal to $k'-1$. Thus the valid inequality associated with the subweb $W_{lk'}^{k'}$ is

$$2 \sum_{i \in I(\mathcal{Q}, k'+1)} x_i + \sum_{i \in O(\mathcal{Q}, k'+1)} x_i \leq 2\,\alpha(W_{lk'}^{k'}) \tag{7.6}$$

where $W_{lk'}^{k'} \subseteq I(\mathcal{F}, p)$ holds. Hence, inequality (7.5) is valid. To prove that inequality (7.5) is facet-inducing, we may define the set of nodes V' of the $(k', k-k')$-regular subweb $W_{lk'}^{k'}$ w.l.o.g. as

$$V' = \bigcup_{0 \leq j < l} \{k \cdot j + 1, k \cdot j + 2, \ldots, k \cdot j + k'\}$$

(where $l \geq 5$ and $l = 2 \pmod{k'+1}$).

For convenience, we call the nodes in V' *black nodes* and all remaining nodes *white nodes*. A *black set* is a set of black nodes and likewise a *white set* is a set of white nodes. Further, $[i, i+l]$ denotes the set of $l+1$ consecutive nodes starting in node i.

Claim 29 *The black set V' is* **a***-connected w.r.t. inequality (7.5).*

If $lk' = 0 \pmod{k'+1}$ then $-l = 0 \pmod{k'+1}$ and thus $l = 0 \pmod{k'+1}$, in contradiction with $l = 2 \pmod{k'+1}$, as $k' \geq 2$. Hence $k'+1$ is not a divisor of lk' and we have $lk' = \alpha(G[V'])(k'+1) + r$ with $1 \leq r \leq k'$. Let $S_1 = \{1, 2+(k'+1), 2+2(k'+1), \ldots, 2+(\alpha(G[V'])-1)(k'+1)\}$ and $S_2 = \{2, 2+(k'+1), 2+2(k'+1), \ldots, 2+(\alpha(G[V'])-1)(k'+1)\}$. Since $2+(\alpha(G[V'])-1)(k'+1) = 2+(lk'-r)-(k'+1) \leq lk'-k'$, S_1 and S_2 are both maximum stable sets of $G[V']$. Hence, the edge $\{1,2\}$ of $G[V']$ is α-critical. By the circular symmetry of $G[V']$, this implies that $G[V']$ is α-connected. Since $\alpha(G[V']) = \alpha'$, we obtain that V' is **a**-connected. $\diamond$

Claim 30 *We have $lk' > (\alpha'-2)(k'+1) + 3k'$.*

Since $l = 2 \pmod{k'+1}$, we have $lk' = k'-1 \pmod{k'+1}$. Hence $lk' - \alpha'(k'+1) = k'-1$. It follows that $lk' - \alpha'(k'+1) > 3k' - 2(k'+1)$. Thus $lk' > (\alpha'-2)(k'+1) + 3k'$. $\diamond$

Claim 31 *We obtain $\alpha(W_{lk'}^{k'} \setminus [1, 3k']) \geq \alpha(W_{lk'}^{k'}) - 1$.*

By the previous claim, the set $S' := \{3k'+1, 3k'+(k'+1)+1, \ldots, 3k'+(\alpha'-2)(k'+1)+1\}$ is a stable set of size $\alpha'-1$ of $W_{lk'}^{k'} \setminus [1, 3k']$ and the result follows. $\diamond$

Claim 32 *For every $0 \leq i < l$, the white set $V_i := ik + \{k'+1, \ldots, k\}$ is* **a***-connected.*

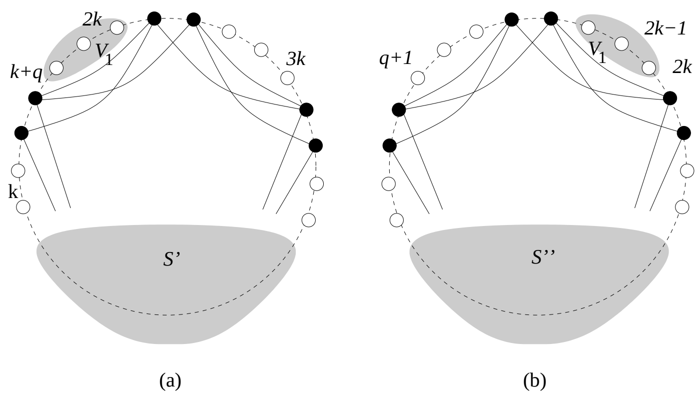


Figure 7.7: : The roots for the proof of Claim 32 with $k = 5$ and $k' = 2$: (a) the roots $S_j = S' \cup \{3k, j\}$ (b) the stable sets $S_1 = S'' \cup \{k' + 1, 2k - 1\}$ and $S_2 = S'' \cup \{k' + 1, 2k\}$

We are going to prove that V_1 is **a**-connected. By the previous claim, there is a black stable set S' of size $\alpha' - 1$ in $G \setminus [k+1, 4k]$. For every $k + k' + 1 \leq j \leq 2k - 1$, the set $S_j := S' \cup \{3k, j\}$ is obviously a root of (7.5), hence the edges $\{k' + k + 1, k' + k + 2\}, \ldots, \{2k - 2, 2k - 1\}$, are **a**-critical (see Fig. 7.7(a)).

It remains to show that the edge $\{2k-1, 2k\}$ is **a**-critical. By the previous claim again, there exists a black stable set S'' of size $\alpha'-1$ in $G\setminus[k'-k+1, 3k]$. The set $S_1 := S'' \cup \{k'+1\} \cup \{2k-1\}$ is a root as $k'+1+k < 2k-1$ (since $k' \leq k-3$). The set $S_2 := S'' \cup \{k'+1\} \cup \{2k\}$ is also a root (see Fig. 7.7(b)). Hence $\{2k - 1, 2k\}$ is **a**-critical and, therefore, V_1 is **a**-connected.

Likewise, the sets $V_0, V_2 \ldots, V_{l-1}$ are **a**-connected. $\diamond$

Claim 33 *For every $0 \leq i < l$ there exists a stable set S_i such that S_i meets V' in exactly $\alpha' - 1$ nodes, V_i in exactly one node, and V_{i+1} in also exactly one node.*

For every $0 \leq i < l$, there exists a black stable set S'_i of size $\alpha' - 1$ in $G\setminus[ik+1, (i+3)k]$. Let S_i be the stable set $S'_i \cup \{ik+k'+1\} \cup \{(i+1)k+k'+2\}$. Then $S_0, \ldots, S_{l-1}$ give the result. $\diamond$

Let S' be a maximum stable set of $G[V']$. Then we have

$$\begin{vmatrix} |S' \cap V'| & |S' \cap V_0| & \cdots & |S' \cap V_{l-1}| \\ |S_0 \cap V'| & |S_0 \cap V_0| & \cdots & |S_0 \cap V_{l-1}| \\ \vdots & \vdots & & \vdots \\ |S_{l-1} \cap V'| & |S_{l-1} \cap V_0| & \cdots & |S_{l-1} \cap V_{l-1}| \end{vmatrix} = \begin{vmatrix} \alpha' & 0 & \cdots & 0 \\ \alpha' - 1 & & & \\ \vdots & & C & \\ \alpha' - 1 & & & \end{vmatrix}$$

where C is the $(2, l)$-circulant matrix with top row $(1, 1, 0, \ldots, 0)$ of size l. The matrix C is invertible as l is odd. Hence the above determinant is non-zero. This finishes the proof. $\square$

As a consequence, we obtain many different infinite sequences of not rank-perfect webs, among them the required base sets for all even values of $k \geq 6$ (but not for the odd values $k \geq 5$ since all webs in the latter sequences have an odd number of vertices). The cases of even values $k \geq 4$ is treated separately in [80].

7.5 The Chvátal-rank of clique family inequalities

Clique family inequalities form an intriguing class of valid inequalities for the stable set polytope of all graphs. We investigate the Chvátal-rank of these inequalities for arbitrary graphs [82]. Our main result is that the highest coefficient involved in the inequality is an upper bound for its Chvátal-rank.

More precisely, we proof that for a clique family inequality $(\mathcal{Q}, p)$ with $r = |\mathcal{Q}| \pmod p$ and every $1 \leq i \leq p - r$, the inequality

$$i \sum_{v \in V_p} x_v + (i-1) \sum_{v \in V_{p-1}} x_v \leq i \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor$$

has Chvátal-rank at most i. In particular, $(\mathcal{Q}, p)$ has Chvátal-rank at most $p - r$ (Theorem 4.34).

The following observation will be used several times in the proof: summing up the clique inequalities corresponding to the cliques in $\mathcal{Q}$ and possibly adding negative constraints $-x(v)$ for those nodes $v \in V_p$ which are contained in more than p cliques, we obtain that

$$p \sum_{v \in V_p} x_v + (p-1) \sum_{v \in V_{p-1}} x_v \quad \leq \quad |\mathcal{Q}| \tag{7.7}$$

is valid for QSTAB(G).

Proof: For every $1 \le i \le p-r$, let $H(i)$ be the assertion:

$$\text{"}i \sum_{v\in V_p} x_v + (i-1)\sum_{v\in V_{p-1}} x_v \le i\left\lfloor \frac{|\mathcal{Q}|}{p}\right\rfloor \text{ has Chvátal-rank at most } i.\text{"}$$

The proof is performed by induction on i.

$H(1)$ is true as Inequality (7.7) implies that $\sum_{V_p} x_v \le \frac{|\mathcal{Q}|}{p}$ is valid for QSTAB(G), hence $\sum_{V_p} x_v \le \left\lfloor \frac{|\mathcal{Q}|}{p}\right\rfloor$ has Chvátal-rank one, as required.

For the induction step, assume that $H(i)$ is true and $i < p-r$. We have to prove that $H(i+1)$ holds. We know that the following inequality has Chvátal-rank at most i:

$$i \sum_{v\in V_p} x_v + (i-1)\sum_{v\in V_{p-1}} x_v \quad\le\quad i\left\lfloor \frac{|\mathcal{Q}|}{p}\right\rfloor \tag{7.8}$$

Scaling Inequality (7.8) $(p-i-1)$-times and adding Inequality (7.7), we derive the following chain of inequalities with Chvátal-rank at most $i+1$:

$$\begin{aligned}
&(p+i(p-i-1))\sum_{v\in V_p} x_v \\
&+(p-1+(i-1)(p-i-1))\sum_{v\in V_{p-1}} x_v &\le\;& |\mathcal{Q}| + (p-i-1)i\left\lfloor \frac{|\mathcal{Q}|}{p}\right\rfloor \\
&(p-i)(i+1)\sum_{v\in V_p} x_v + (p-i)i\sum_{v\in V_{p-1}} x_v &\le\;& |\mathcal{Q}| + (p-i-1)i\left\lfloor \frac{|\mathcal{Q}|}{p}\right\rfloor \\
&(i+1)\sum_{v\in V_p} x_v + i\sum_{v\in V_{p-1}} x_v &\le\;& \left\lfloor \frac{|\mathcal{Q}| + (p-i-1)i\left\lfloor \frac{|\mathcal{Q}|}{p}\right\rfloor}{p-i}\right\rfloor
\end{aligned}$$

It remains to check that the right hand side is at most $(i+1)\left\lfloor \frac{|\mathcal{Q}|}{p}\right\rfloor$. A short computation gives

$$\begin{aligned}
\frac{|\mathcal{Q}| + (p-i-1)i\left\lfloor \frac{|\mathcal{Q}|}{p}\right\rfloor}{p-i} - (i+1)\left\lfloor \frac{|\mathcal{Q}|}{p}\right\rfloor - 1 \;&=\; \frac{|\mathcal{Q}| - p\left\lfloor \frac{|\mathcal{Q}|}{p}\right\rfloor - (p-i)}{p-i} \\
&=\; \frac{i-(p-r)}{p-i}
\end{aligned}$$

which is negative as $i < (p-r)$. Hence

$$\left\lfloor \frac{|\mathcal{Q}| + (p-i-1)i\left\lfloor \frac{|\mathcal{Q}|}{p}\right\rfloor}{p-i}\right\rfloor \le (i+1)\left\lfloor \frac{|\mathcal{Q}|}{p}\right\rfloor$$

follows, as required. □

This provides an alternative proof for the validity of clique family inequalities, involving only standard rounding arguments. In addition, it shows that all rank clique family inequalities have Chvátal-rank one. The latter consequence is particularly nice, as neither general rank constraints nor general clique family inequalities have this property [24, 73], but the combination of both.

However, the upper bound established in Theorem 4.34 gets weaker when r gets smaller; we improve this upper bound for $r < p/2$ as follows: Every clique family inequality $(\mathcal{Q}, p)$ with $r = |\mathcal{Q}| \pmod p$ has Chvátal-rank at most r if $0 \le r < p - r$ (Theorem 4.36).

Proof: Let $(\mathcal{Q}, p)$ be a clique family inequality

$$(p-r)\sum_{v\in V_p} x_v + (p-1-r)\sum_{v\in V_{p-1}} x_v \le (p-r)\left\lfloor \frac{|\mathcal{Q}|}{p}\right\rfloor$$

such that $0 \le r \le p - r$. Let $b_0 = |\mathcal{Q}|$ and for every $1 \le i < r$, define

$$b_{i+1} = \left\lfloor \frac{(p-2i-1)b_i + i\left\lfloor \frac{|\mathcal{Q}|}{p}\right\rfloor}{p-2i} \right\rfloor .$$

Claim 34 *For every* $0 \le i \le r$, *we have* $b_i = (p-i)\left\lfloor \frac{|\mathcal{Q}|}{p}\right\rfloor + (r-i)$.

We use induction on i: let $H(i)$ be the assertion $b_i = (p-i)\left\lfloor \frac{|\mathcal{Q}|}{p}\right\rfloor + (r-i)$.

$H(0)$ is true as $(p-0)\left\lfloor \frac{|\mathcal{Q}|}{p}\right\rfloor + (r-0) = |\mathcal{Q}|$. For the induction step, assume that $0 \le i < r$ and that $H(i)$ is true. We have to check that $H(i+1)$ holds. From $H(i)$, we infer the following:

$$\begin{aligned}
b_{i+1} &= \left\lfloor \frac{(p-2i-1)b_i + i\left\lfloor \frac{|\mathcal{Q}|}{p}\right\rfloor}{p-2i} \right\rfloor \\
&= \left\lfloor \frac{(p-2i-1)(p-i)\left(\left\lfloor \frac{|\mathcal{Q}|}{p}\right\rfloor + (r-i)\right) + i\left\lfloor \frac{|\mathcal{Q}|}{p}\right\rfloor}{p-2i} \right\rfloor \\
&= \left\lfloor \frac{((p-2i-1)(p-i)+i)\left\lfloor \frac{|\mathcal{Q}|}{p}\right\rfloor + (p-2i-1)(r-i)}{p-2i} \right\rfloor \\
&= (p-i-1)\left\lfloor \frac{|\mathcal{Q}|}{p}\right\rfloor + \left\lfloor (r-i) - \frac{r-i}{p-2i} \right\rfloor
\end{aligned}$$

Since $i < r < p - r$, we have $0 < \frac{r-i}{p-2i} \leq 1$. This implies

$$b_{i+1} = (p-i-1)\left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor + r - i - 1$$

and, hence, $H(i+1)$ holds. $\diamond$

In particular, we obtain

$$b_r = (p-r)\left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor. \tag{7.9}$$

Claim 35 *For every $0 \leq i \leq r$, the inequality*

$$(p-i)\sum_{V_p} x_v + (p-i-1)\sum_{V_{p-1}} x_v \leq b_i$$

has Chvátal-rank at most i.

The proof is by induction on i: for every $0 \leq i \leq r$, let $H(i)$ be the assertion: "The inequality

$$(p-i)\sum_{V_p} x_v + (p-i-1)\sum_{V_{p-1}} x_v \leq b_i \tag{7.10}$$

has Chvátal-rank at most i".

$H(0)$ is true due to Inequality (7.7). The induction step goes as follows. let $0 \leq i < r$ and assume that $H(i)$ is true. We have to check that $H(i+1)$ holds. Due to Theorem 4.34, Inequality (7.8) has Chvátal-rank at most i. As $H(i)$ holds, Inequality (7.10) also has Chvátal-rank at most i.

Scaling Inequality (7.10) $(p-2i-1)$-times and adding Inequality (7.8), we get the following chain of inequalities:

$$\begin{aligned}
&(i + (p-2i-1)(p-i))\sum_{V_p} x_v \\
&+((p-2i-1)(p-i-1)+i-1)\sum_{V_{p-1}} x_v \leq (p-2i-1)b_i + i\left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor \\
&(p-2i)(p-i-1)\sum_{V_p} x_v \\
&+(p-2i)(p-i-2)\sum_{V_{p-1}} x_v \leq (p-2i-1)b_i + i\left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor
\end{aligned}$$

Hence the inequality

$$
\begin{aligned}
(p-i-1)\sum_{V_p} x_v + (p-i-2)\sum_{V_{p-1}} x_v \quad &\leq \quad \left\lfloor \frac{(p-2i-1)b_i + i\left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor}{p-2i} \right\rfloor \\
&\leq \quad b_{i+1}
\end{aligned}
$$

has Chvátal-rank at most $i+1$, that is $H(i+1)$ holds. $\diamond$

Combining equation (7.9) and Claim 35 finishes the proof. $\square$

Thus, Theorem 4.34 and Theorem 4.36 together imply that every clique family inequality $(\mathcal{Q}, p)$ has Chvátal-rank at most $\min\{r, p-r\}$ where $r = |\mathcal{Q}| \pmod p$ (Corollary 4.37).

7.6 General clique family inequalities and claw-free graphs

We observed in [84, 83] that clique family inequalities do not suffice to describe the stable set polytope of general claw-free graphs. This motivated us to introduce in [84] the concept of general clique family inequalities in order to describe facets of claw-free but not quasi-line graphs.

One class of inequalities for such claw-free graphs are Cook's clique neighborhood constraints describing the stable set polytopes of graphs G with $\alpha(G) = 2$.

In order to express such constraints as general clique family inequalities, we proved in [84] as a first step that the full rank facet $x(G) \leq 2$ associated with a graph with $\alpha(G) = 2$ is a general clique family inequality $(\mathcal{Q}, k, r, 1, 2)$ where $\overline{C}_{2k+1}$ is the shortest odd antihole in G (Theorem 4.47).

Proof: As G has stability number 2, all of its minimal imperfect subgraphs are odd antiholes. Every odd antihole $\overline{C}_{2l+1}$ has $2l+1$ cliques of maximum size l, namely $Q(i) = \{i, \ldots, i+l-1\}$, for $1 \leq i \leq 2l+1$ (indices are taken modulo $2l+1$); in particular, each node i of $\overline{C}_{2l+1}$ belongs to the l maximum cliques $Q(i-l+1), \ldots, Q(i)$.

In order to present the studied clique family $\mathcal{Q}$, we show that the maximum cliques of a *shortest* odd antihole $\overline{C}_{2k+1} \subseteq G$ can be extended in such a way that every node $v \in G \setminus \overline{C}_{2k+1}$ is covered at least k times (possibly using more than $2k+1$ cliques).

Claim 36 *For any $C_{2l+1} \subseteq G$ and $v \in G \setminus \overline{C}_{2l+1}$, the set of non-neighbors of v on $\overline{C}_{2l+1}$ induces a clique.*

Otherwise, $\alpha(G[\overline{C}_{2l+1} \cup \{v\})] = 3$. $\diamond$

Claim 37 *For any $\overline{C}_{2l+1} \subseteq G$, each node $v \in G \setminus \overline{C}_{2l+1}$ is adjacent to at least $l+t$ consecutive nodes of $\overline{C}_{2l+1}$ where $t \geq 1$; in particular, v is completely joined to $t+1$ maximum cliques of $\overline{C}_{2l+1}$.*

Denote the maximum interval of consecutive neighbors of v on $\overline{C}_{2l+1}$ by $1, \ldots, l+t$. Then v's non-neighbors are among the nodes $l+t+1, \ldots, 2l+1$. As those non-neighbors induce a clique by Claim 36, we have $l+t+1 \geq l+2$, i.e., $t \geq 1$ follows. In particular, v is completely joined to the $t+1$ consecutive maximum cliques $Q(1) = \{1, \ldots, l\}, \ldots, Q(1+t) = \{1+t, \ldots, l+t\}$ of $\overline{C}_{2l+1}$. $\diamond$

Claim 38 *Let $v \in G \setminus \overline{C}_{2l+1}$ and $1, \ldots, l+t$ be the maximum interval of consecutive neighbors of v on $\overline{C}_{2l+1}$. If $t+1 < l$ then $G[\overline{C}_{2l+1} \cup \{v\}]$ contains a shorter odd antihole $\overline{C}_{2(t+1)+1}$.*

In this case, v is certainly not adjacent to $l+t+1$ and $2l+1$ (but v might be adjacent to nodes in between). We show that the node subset $V' = \{2l+1, 1, \ldots, t, v, l+1, \ldots, l+1+t\}$ induces a $\overline{C}_{2(t+1)+1}$ in G. Note that we can rewrite V' as $2l+1, \ldots, (2l+1)+t, v, k+1, \ldots, (l+1)t$. By construction, a node $x \in V'$ has exactly the following non-neighbors in V':

$x \in V'$	$\overline{N}_{V'}(x)$		
v	$l+t+1$	and	$2l+1$
$l+t', 1 \leq t' \leq t$	$2l+t'$	and	$2l+t'+1$
$(l+1)+t$	$2l+t+1$	and	v
$2l+1$	v	and	$l+t$
$(2l+1)+t', 1 \leq t' \leq t$	$l+t'$	and	$l+t'+1$

as required. $\diamond$

Claim 39 *If $\overline{C}_{2k+1} \subseteq G$ is a* shortest *odd antihole in G, then each code $v \in G \setminus \overline{C}_{2k+1}$ is completely joined to at least k maximum cliques of $\overline{C}_{2k+1}$.*

This follows directly from Claim 37 and Claim 38. $\diamond$

Thus, extending the maximum cliques of $\overline{C}_{2k+1}$ appropriatly, we can construct a clique family $\mathcal{Q}$ with $|\mathcal{Q}| \geq 2k+1$ s.t. each node in G is covered at least k times by $\mathcal{Q}$. Choosing $p = k$ yields $V(\mathcal{Q}, k) = V(G)$ and, hence, the cfi $(\mathcal{Q}, k)$ reads as

$$(p-r)x(G) \leq (p-r)\alpha(G) \leq (p-r)\left\lfloor \frac{|\mathcal{Q}|}{k} \right\rfloor$$

wich finally yields $x(G) \leq 2$ by $\lfloor \frac{|\mathcal{Q}|}{k} \rfloor \geq 2$ due to $|\mathcal{Q}| \geq 2k+1$, for any choice of r with $0 \leq r \leq R$. □

As described in Section 4.2.4, this general clique family inequality $(\mathcal{Q}, k, r, 1, 2)$ producing the full rank facet of a graph G with $\alpha(G) = 2$ can easily be extended to a general clique family inequality $(\mathcal{Q}, k+1, k-1, 1, 2)$ representing the complete join of $x(G) \leq 2$ and a clique Q, i.e., the cliqie neighborhood constraint $F(Q)$ (see Theorem 4.48).

In addition, we express in [84] all co-spanning tree constraints

$$1x(\circ) + 2x(\bullet) \leq 3$$

of claw-free graphs G containing exactly one stable set of size three as general clique family inequality $(\mathcal{Q}, p, R, p-2, b)$ with $|\mathcal{Q}| = 7$, $p = 3$, and $b = (p-R)\left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor - J = p$ (Theorem 4.50).

Proof: The roots of this facet correspond in the complementary graph $\overline{G}$ to one triangle $\Delta = \{1, 2, 3\}$ and the edges of a spanning tree T, where the triangle Δ consists of $\circ$-nodes only and in the tree T alternate $\circ$-nodes and $\bullet$-nodes. Note that we have the following in $\overline{G}$:

(i) all inner $\circ$-nodes of T form a stable set;

(ii) all $\bullet$-nodes form a stable set;

(iii) each inner $\circ$-node of T has exactly one neighbor in Δ;

(iv) each $\bullet$-node has exactly one neighbor in Δ.

The latter two conditions hold true since all nodes outside Δ have at least one neighbor in Δ (otherwise G is not claw-free) and at most one neighbor in Δ (otherwise $\overline{G}$ contains a second triangle $\neq \Delta$).

Let $N_\circ(i)$ (resp. $N_\bullet(i)$) denote the set of all $\circ$-neighbors in T (resp. $\bullet$-neighbors) of node i in G. We construct the following clique family $\mathcal{Q}$ in G as follows: we choose $Q_{i,\circ} = \{i\} \cup N_\circ(i)$ and $Q_{i,\bullet} = \{i\} \cup N_\bullet(i)$ for each $i \in \Delta$ and $Q_\bullet$ consisting of all $\bullet$-nodes.

Each set $Q_{i,\circ}$ is a clique by (i) and the definition of $N_\circ(i)$; all three cliques $Q_{1,\circ}$, $Q_{2,\circ}$, and $Q_{3,\circ}$ cover the nodes in Δ once and all inner $\circ$-nodes of T

twice by (iii). Similarly, each set $Q_{i,\bullet}$ is a clique by (ii) and the definition of $N_\bullet(i)$; all three cliques $Q_{1,\bullet}$, $Q_{2,\bullet}$, and $Q_{3,\bullet}$ cover the nodes in Δ once and all $\bullet$-nodes twice by (iv). $Q_\bullet$ is a clique by (ii) and covers all $\bullet$-nodes once.

In total, each $\circ$-node is covered twice and each $\bullet$-node three times. We have $|\mathcal{Q}| = 7$ and choose $p = 3$, $R = 1$, $J = 1$, and $b = (p - R)\left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor - J = p$. Thus, we obtain the studied facet

$$\sum_{0 \le j \le 1} (2 - j)x(V(\mathcal{Q}, 3 - j)) = 1x(\circ) + 2x(\bullet) \le 2\left\lfloor \frac{7}{3} \right\rfloor - 1 \le 3$$

as a general clique family inequality $(\mathcal{Q}, p, R, p-2, p)$. □

With the help of similar techniques we show in [84] that basic co-spanning 1-forest constraints

$$1x(\circ) + 2x(\bullet) + 3x(\square) \le 4$$

of claw-free graphs are general clique family inequalities $(\mathcal{Q}, p, R, p-2, b)$ with $|\mathcal{Q}| = 9$, $p = 4$, and $b = (p - R)\left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor - J = p$ (see Theorem 4.57). We conjecture that every co-spanning 1-forest facet of a claw-free graph can be expressed that way.

The main result from [83] is that every non-rank, non-complete join facet $a^T x \le b$ with $b \ge 3$ of the stable set polytope of a (not necessarily claw-free) graph G with $\alpha(G) = 3$ is a

- co-spanning 1-forest constraint if b is even,
- co-spanning forest constraint if b is odd

(see Theorem 4.53).

Proof: By the assumption $\alpha(G) = 3$, all roots of $a^T x \le b$ have size at most 3.

If there would be a root consisting of one single node s, then $a_s = b$ would follow and s must be adjacent to any other node i of G with $a_i > 0$; a contradiction that $a^T x \le b$ is not a complete join facet.

Thus, all roots of $a^T x \le b$ have size 2 or 3; denote by R_e (resp. $R_\triangle$) the set of edges (resp. triangles) in $\overline{G}$ which correspond to the roots of $a^T x \le b$ in G.

Consider further the graph $F = (V, R_e) \subseteq \overline{G}$. Any component T of F must have at least $|T| - 1$ edges (since it is connected), but at most $|T|$ edges (since all its edges are, as roots, linearly independent by construction). Thus, any component T of F is either a tree (if it has $|T| - |$ edges) or an odd 1-tree (if it has $|T|$ edges). Note that every 1-tree T has as many edges as nodes, but all edges are independent only if the only cycle of T has odd length. Moreover, this odd cycle must have length > 3, otherwise its edges would form a triangle and could not be roots).

Thus, F contains $|G| - k$ roots of $a^T x \leq b$, where k is the number of tree-components of F, and $R_\triangle$ contains the remaining k roots of $a^T x = b$. Hence, $a^T x \leq b$ is a co-spanning 1-forest constraint.

Now suppose that b is odd, but that F has an odd 1-tree component T and consider the odd hole $H \subseteq T$ of length $2l + 1$.

For every edge $v_i v_{i+1}$ of H, we have $a_i + a_{i+1} = b$. In particular, $a_i + a_{i+1} = a_{i+1} + a_{i+2} = b$ implies that all nodes in H with odd (resp. even) index should have the same weight. As also $a_1 + a_{2l+1} = b$ holds, we infer that all nodes v_i of H must have the same weight $a_i = \frac{b}{2}$. This is possible only if $2|b$.

Hence, whenever $a^T x \leq b$ with $b \geq 3$ odd is a non-rank, non-complete join facet of a graph G with $\alpha(G) \geq 3$, then $a^T x \leq b$ must be a co-spanning forest constraint. □

Every co-spanning forest constraint is obviously a co-spanning 1-forest constraint; the only complete join facets in claw-free graphs are clique neighborhood constraints which can also be considered as special co-spanning 1-forest constraints. This immediatly implies that every non-rank facet of the stable set polytope of a claw-free graph G with $\alpha(G) \leq 3$ is a co-spanning 1-forest constraint (Corollary 4.56).

7.7 The extreme points of QSTAB(G)

We observe in [62] that the complete knowledge on the extreme points of QSTAB(G) helps to determine both the imperfection ratio imp(G) and the imperfection index $\text{imp}_\text{I}(G)$ (see Section 5.1).

For that we establish a 1-1 correspondence between the extreme points of QSTAB(G) and facet-inducing subgraphs of $\overline{G}$ in [62]. More precisely, we proof that a vector $a \neq 0$ is an extreme point of QSTAB(G) if and only if there is a subgraph $\overline{G}'$ of $\overline{G}$ such that supp(a) belongs to $\mathcal{F}(\overline{G}')$ (Theorem 5.1).

Proof: *If.* Suppose that $\overline{G}$ contains a subgraph $\overline{G}'$ such that $a^T x \leq 1$ is a facet of STAB($\overline{G}'$) with $0 < a_i \leq 1$ for $i \in \overline{G}'$ (i.e. the facet is scaled to have right hand side 1). Then there exist $n' = |\overline{G}'|$ linearly independent roots $S'_1, \ldots, S'_{n'}$ of $\overline{G}'$

These stable sets clearly correspond to n' cliques $Q'_1, \dots, Q'_{n'}$ of G'. For any such clique Q'_i, choose a maximal clique $Q_i \subseteq G$ with $Q_i \supseteq Q'_i$. Then the vector $x' = (\chi^{G'}, a)$ with $x'_j = a_j$ for $j \in G'$ and $x'_j = 0$ otherwise satisfies the n' clique constraints associated with the maximal cliques $Q_1, \dots, Q_{n'}$ at equality, as

$$x'(Q_i) = \sum_{j \in Q'_i \subseteq Q_i} a_j = a^T \chi^{Q'_i} = a^T \chi^{S'_i} = 1$$

holds by the choice of G'. Furthermore, x' satisfies the $n - n' = |G \setminus G'|$ nonnegativity constraints $-x'_j = 0 \ \forall j \notin G'$ with equality. Hence, x' belongs to $n = |G|$ facets of QSTAB(G). In order to show that x' is an extreme point it remains to ensure that these facets are linearly independent. For that, construct an $(n \times n)$-matrix A as follows: Let the first n' columns of A correspond to nodes in G' and the last $n - n'$ columns to nodes in $G \setminus G'$. Choose further the incidence vectors of the cliques $Q_1, \dots, Q_{n'}$ as first n' rows and the incidence vectors of the nonnegativity constraints $-x'_j = 0 \ \forall j \notin G'$ as last $n - n'$ rows, see Figure 7.8.

$$A = \left(\begin{array}{c|c} A_1 & A_2 \\ \hline 0 & \mathrm{Id} \end{array} \right)$$

Figure 7.8: The $(n \times n)$-matrix A

As the submatrix A_1 corresponds to the independent cliques $Q'_1, \dots, Q'_{n'}$ of G', the whole matrix A is invertible due to its block structure. Thus, x' is indeed an extreme point of QSTAB(G). $\diamond$

Only if. Suppose conversely that $a \neq 0$ is an extreme point of QSTAB(G) with $0 < a_i \leq 1$ for $i \in \overline{G}'$ and $a_i = 0$ otherwise. Then a satisfies n linearly independent facets of QSTAB(G) with equality. Among them are clearly the $n - n'$ nonnegativity constraints $-x_j = 0 \ \forall j \notin G'$. As QSTAB$(G)$ has only two types of facets, a satisfies also n' maximal clique facets with equality, say the clique constraints associated with the maximal cliques $Q_1, \dots, Q_{n'}$ of G. Let $Q'_i = Q_i \cap G'$, then

$$x(Q_i) = \sum_{j \in Q_i} a_j = \sum_{j \in Q'_i \subseteq Q_i} a_j = a^T \chi^{Q'_i} = 1$$

follows. Clearly, the cliques $Q'_1, \dots, Q'_{n'}$ of G' correspond to stable sets $S'_1, \dots, S'_{n'}$ of $\overline{G}'$ and $a^T \chi^{S'_i} = 1$ holds for $1 \leq i \leq n'$. In order to show that $a^T x \leq 1$ is a facet of $\mathrm{STAB}(\overline{G}')$, it remains to verify that $\chi^{S'_1}, \ldots, \chi^{S'_{n'}}$ are linearly independent. For that, construct an $(n \times n)$-matrix A as above, choosing the nodes in G' and in $G \setminus G'$ as first n' and last $n - n'$ columns, respectively, the incidence vectors of the cliques $Q_1, \ldots, Q_{n'}$ as first n' and the unit vectors corresponding to $-x_j = 0\ \forall j \notin G'$ as last $n - n'$ rows, see again Figure 7.8. As a is an extreme point, the matrix A is invertible. In order to show invertibilty for the submatrix A_1, we substract, for each 1-entry in A_2, the corresponding unit vector in $(0, \mathrm{Id})$. That way, we turn A_2 into a matrix with 0-entries only but maintain all entries in A_1. This shows that the rows of A_1 are linearly independent and, therefore, the incidence vectors of the cliques $Q'_1, \ldots, Q'_{n'}$ of G' respectively of the corresponding stable sets $S'_1, \ldots, S'_{n'}$ in $\overline{G}'$. Hence, $a^T x \leq 1$ is indeed a facet of $\mathrm{STAB}(\overline{G}')$. □

With the help of this characterization we can, in addition, easily reprove famous characterizations of perfect and minimally imperfect graphs, as well as the results that the edge constraint stable set polytope and the fractional matching polytope have half-integral extreme points only, see Section 5.1 or [62].

7.8 The imperfection ratio of a-perfect graphs

The imperfection ratio of a graph is NP-hard to compute and, for most graph classes, it is even unknown whether it is bounded. For a-perfect graphs, that are graphs G such that all non-trivial, non-clique facets of $\mathrm{STAB}(G)$ are rank constraints associated with prime antiwebs, we characterize the imperfection ratio as

$$\mathrm{imp}(G) \quad = \quad \max\{\tfrac{n'}{\alpha'\omega'} : K_{n',\alpha'} \subseteq G \text{ prime}\}$$

where $\omega' = \lfloor n'/\alpha' \rfloor$ holds (Theorem 5.9).

Proof: For any a-perfect graph G, we have

$$\mathrm{STAB}(G) = \mathrm{QSTAB}(G) \cap \{x \in \mathbb{R}^n : x(K_{n',\alpha'}) \leq \alpha'\ \forall K_{n',\alpha'} \subseteq G\}$$

where n stands for the number of nodes in G. In particular,

$$\mathcal{F}(G) \quad = \quad \{\tfrac{1}{\alpha'}\chi^{K_{n',\alpha'}} : K_{n',\alpha'} \subseteq G,\ \gcd(n', \alpha') = 1\}$$

follows, where $\chi^{K_{n',\alpha'}}$ stands for the incidence vector of $K_{n',\alpha'}$. This yields

$$\mathrm{imp}(G) \quad = \quad \max\{\tfrac{1}{\alpha'} y(K_{n',\alpha'}) : K_{n',\alpha'} \subseteq G \text{ prime}, y \in \mathrm{QSTAB}(G)\}$$

as we clearly have

$$(\chi^{K_{n',\alpha'}})^T y = \sum_{i \in K_{n',\alpha'}} y_i = y(K_{n',\alpha'}).$$

Furthermore, $y(K_{n',\alpha'}) \leq \frac{n'}{\omega'}$ follows as each node of $K_{n',\alpha'}$ can be covered ω' times by the n' maximum cliques $Q_i = \{i, i+\alpha', \ldots, i+(\omega'-1)\alpha'\}$ for $1 \leq i \leq n'$ of $K_{n',\alpha'}$ (all these cliques are distinct as $\gcd(\alpha', n') = 1$). Thus,

$$\mathrm{imp}(G) \quad \leq \quad \max\{\tfrac{1}{\alpha'}\tfrac{n'}{\omega'} : K_{n',\alpha'} \subseteq G \text{ prime}\}$$

and combining this with inequality (5.1)

$$\mathrm{imp}(G) \quad \geq \quad \max\{\tfrac{n'}{\alpha'\omega'} : K_{n',\alpha'} \subseteq G\}$$

finally yields equality, as required. □

We also give two important upper bounds on $\mathrm{imp}(K_{n,\alpha})$.

Lemma 7.12

(a) $\mathrm{imp}(K_{n,\alpha}) \leq \frac{n}{n-\alpha+1} < 1 + \frac{1}{\omega-1}$

(b) $\mathrm{imp}(K_{n,\alpha}) < \frac{3}{2}$

Proof:

(a) Every odd hole and odd antihole in an antiweb $K_{n,\alpha}$ meets a given set of $\alpha - 1$ consecutives nodes in $K_{n,\alpha}$ by Trotter [104]. Hence, due to the Strong Perfect Graph Theorem [17], any node of the antiweb can be covered $n - \alpha + 1$ times with n induced perfect graphs, thus $\mathrm{imp}(K_{n,\alpha}) \leq n/(n - \alpha + 1)$ by [45].

(b) If $\omega \geq 3$, the result follows from (a). If $\omega = 2$ then $K_{n,\alpha}$ is an odd hole, and therefore its imperfection ratio is bounded by $\frac{5}{4}$.

□

As a consequence, for all subclasses of a-perfect graphs, including all odd holes and odd antiholes, all h-perfect graphs, all antiwebs [110], and the complements of co-fuzzy circular interval graphs [112], the imperfection ratio relies on induced antiwebs only and is bounded by $\frac{3}{2}$.

Furthermore, we also characterize and bound the imperfection ratio for near-bipartite graphs G as

$$\mathrm{imp}(G) \quad = \quad \max\{\tfrac{n'}{\alpha'\omega'} : K_{n'/\alpha'} \subseteq G\} \quad < \quad \tfrac{3}{2}$$

where $\omega' = \lfloor n'/\alpha' \rfloor$ holds (Theorem 5.11).

Proof: Recall that $\mathrm{imp}(G) = \max\{a^T y : a \in \mathcal{F}(G), y \in \mathrm{QSTAB}(G)\}$ taken over all the nontrivial facets $a^T x \leq 1$ of STAB(G) and $y \in \mathrm{QSTAB}(G)$. This suggests that

$$\mathrm{imp}(G) = \max\{\mathrm{imp}(G_a)\} \tag{7.11}$$

taken over all support graphs G_a of nontrivial facets $a^T x \leq 1$ of STAB(G) (i.e., G_a is the subgraph of G induced by all nodes i with $a_i \neq 0$).

Due to Theorem 4.13, all these support graphs G_a are complete joins of prime antiwebs $K_{n_1,\alpha_1}, \ldots, K_{n_k,\alpha_k}$. Further, the imperfection ratio of a complete join $G_1 * G_2$ is simply

$$\mathrm{imp}(G_1 * G_2) = \max\{\mathrm{imp}(G_1), \mathrm{imp}(G_2)\}$$

by [45] (they proved this relation for the disjoint union of two graphs, thanks to the invariance of the imperfection ratio under taking complements, the same applies to complete joins). Hence, we have for any support graph $G_a = K_{n_1,\alpha_1} * \ldots * K_{n_k,\alpha_k}$

$$\mathrm{imp}(G_a) = \max\{\mathrm{imp}(K_{n_1,\alpha_1}), \ldots, \mathrm{imp}(K_{n_k,\alpha_k})\}$$

and equation (7.11) together with Theorem 5.9 imply the assertion. □

Thus, we characterize the imperfection ratio of near-bipartite graphs in terms of induced antiwebs and, by invariance under complementation, the imperfection ratio of any quasi-line graph in terms of its induced webs.

7.9 The imperfection index and graph composition techniques

In order to discuss bounds on the imperfection index, we investigate the behaviour of the imperfection index by means of taking lexicographic products and substituting nodes by other graphs.

Our first result (Theorem 5.24) is that for two graphs G_1, G_2 we have

$$\mathrm{imp}_{\mathrm{I}}(G_1 \times G_2) = |G_2|\ \mathrm{imp}_{\mathrm{I}}(G_1) + (|G_1| - \mathrm{imp}_{\mathrm{I}}(G_1)) \cdot \mathrm{imp}_{\mathrm{I}}(G_2).$$

Proof: Let $V_1' \subseteq V_1$ be a minimum node subset of $G_1 = (V_1, E_1)$ such that $G_1[V_1 - V_1']$ is perfect; in particular we have $\mathrm{imp}_{\mathrm{I}}(G_1) = |V'|$ by Lemma 5.17. Similarly, let $V_2' \subseteq V_2$ be a minimum node subset of $G_2 = (V_2, E_2)$ such that $G_2[V_2 - V_2']$ is perfect.

For each of the nodes $v \in V_1'$ there exists a minimally imperfect subgraph G_v' of G_1 which contains v but none of the other nodes in V_1' (by the minimality of V_1'). Substituting the node v by a graph G_2 creates $|G_2|$ disjoint copies of G_v'; removing *all* $|G_2|$ copies of v is required in order to meet all copies of G_v'.

Moreover, for each of the nodes $v \in V_1 - V_1'$ substitution with G_2 results in a disjoint subgraph isomorphic to G_2. Hence, in order to obtain a perfect subgraph of $G_1 \times G_2$, at least $\mathrm{imp_I}(G_2)$ nodes have to be removed from each of those subgraphs. Let us remove the copies of V_2'. Together, this implies that

$$\begin{aligned} \mathrm{imp_I}(G_1 \times G_2) &\geq |G_2|\ |V_1'| + (|G_1| - |V_1'|) \cdot \mathrm{imp_I}(G_2) \\ &= |G_2|\ \mathrm{imp_I}(G_1) + (|G_1| - \mathrm{imp_I}(G_1)) \cdot \mathrm{imp_I}(G_2). \end{aligned}$$

Now, suppose that $G_1 \times G_2$ is still not perfect after removal of the nodes specified above. Then, there exists a minimally imperfect subgraph G'. If G' is isomorphic to a subgraph of G_2, then $G_2[V_2 - V_2']$ cannot be perfect. Otherwise, G' has to contain nodes from different copies of G_2. If it contains at most one node from every copy, G' is isomorphic to a subgraph of G_1 and $G_1[V_1 - V_1']$ cannot be perfect.

Thus, G' has to contain at least two nodes from one of the copies and nodes from at least two copies. By the Strong Perfect Graph Theorem, G' is either an odd hole or an odd antihole. First, assume G' is an odd hole. Consider a copy of G_2 from which at least two nodes $v_1, \ldots, v_k$ $(k \geq 2)$ belong to G' and let u be a neighbor of one of the nodes, not part of the copy. Node u is adjacent to all nodes $v_1, \ldots, v_k$ which implies that $k = 2$ (otherwise G' is not an odd hole). Moreover, since G' has at least 5 nodes, there has to be another neighbor w of v_1, v_2, not part of the copy. Since w is also adjacent to both v_1 and v_2, we obtain a C_4 as subgraph of G' which violates the assumption G' being an odd hole.

For G' being an odd antihole, a similar argumentation on the complement of $G_1 \times G_2$ can be carried out to prove that G' cannot be an odd antihole as well. Hence, $G_1 \times G_2$ is perfect after removal of the nodes specified above. $\square$

Thus, the imperfection index cannot be bounded for any class $\mathcal{G}$ of graphs which is closed under substitution (and, therefore, under taking lexicographic products) and contains at least one imperfect graph G. Even more for any perfect graph G_2, we have $\mathrm{imp_I}(G_1 \times G_2) = |G_2|\ \mathrm{imp_I}(G_1)$ (see Corollary 5.25).

Thus, a sufficient condition for the *non-existence* of an upper bound on the imperfection index is that the graph class $\mathcal{G}$ in question contains an imperfect graph and is closed under substituting certain perfect graphs. A necessary condition for the *existence* of an upper bound on the imperfection ratio for $\mathcal{G}$ is that $\mathcal{G}$ is closed under substituting *perfect* graphs for nodes only. For the latter, we characterize how several classes of rank-perfect graphs behave under substitution.

For this purpose, we shall make use of the following result:

Theorem 7.13 [21, 30] *Let G be obtained by substituting a node v of a graph $G_1 = (V_1, E_1)$ by a graph $G_2 = (V_2, E_2)$. Then a non-trivial inequality is facet-defining for* $\mathrm{STAB}(G)$ *if and only if it can be scaled to be a facet product of the form*

$$\sum_{i \in V_1 - v} a_i^1 x_i + a_v^1 \sum_{j \in V_2} a_j^2 x_j \le 1 \tag{7.12}$$

where $x(G_i, a^i) \le 1$ is a non-trivial facet of $\mathrm{STAB}(G_i)$ *for $i = 1, 2$.*

Note that Chvátal [21] gave a linear description of $\mathrm{STAB}(G)$ outgoing from the stable set polytopes of the original graphs, whereas Cunningham [30] proved later that each of the inequalities found by Chvátal is indeed facet-defining. We study the consequences of this theorem for several subclasses of rank-perfect graphs. Throughout this section, all non-trivial inequalities are scaled to have right hand side equal to 1 (that means: only clique constraints keep unchanged, rank constraints $x(G', \mathbb{1}) \le \alpha(G')$ turn to $x(G', a) \le 1$ with $a = (\frac{1}{\alpha(G')}, \ldots, \frac{1}{\alpha(G')})$, and non-rank constraints have different non-zero coefficients).

Proposition 7.14 *Consider a graph G obtained by substituting a node v of a graph G_1 by G_2. If there is a non-trivial, non-clique facet of* $\mathrm{STAB}(G_2)$ *then* $\mathrm{STAB}(G)$ *has a non-trivial, non-rank facet.*

Proof: Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ and take the facet product

$$\sum_{i \in Q - v} x_i + \sum_{j \in V_2} a_j^2 x_j \le 1$$

of a clique facet associated with $Q \subseteq V_1$, $v \in Q$ and a non-trivial, non-clique facet $x(G_2, a^2) \le 1$ of $\mathrm{STAB}(G_2)$. Then there is a node $k \in V_2$ with $0 < a_k^2 < 1$ and the above facet product has different non-zero coeffients:

every $i \in Q - v$ has coefficient 1 but $0 < a_k^2 < 1$ (recall: we exclude the case that v does not have any neighbor, hence there is a clique $Q \subseteq V_1$ with $Q - v \neq \emptyset$). Thus, the above facet product is a non-trivial, non-rank facet of STAB(G). □

That means, whenever G_2 is imperfect, the graph obtained by substituting G_2 for a node cannot be rank-perfect. Hence, none of the classes of rank-perfect graphs (different from the class of perfect graphs) is closed under substitution. In addition, we are interested which graphs G_1 and G_2 are allowed in order to produce a rank-perfect graph G by substitution.

Our first result is the following (Theorem 5.13): Let G be obtained by substituting a node v of G_1 by G_2. G is rank-perfect if and only if G_1 is rank-perfect and G_2 is perfect.

Proof: Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. Assume first that G_1 is rank-perfect and G_2 is perfect. Then STAB(G_1) admits only non-trivial facets $x(G_1, a^1) \leq 1$ with $a_i^1 \in \{0, c\}$. Each facet product

$$\sum_{i \in V_1 - v} a_i^1 x_i + a_v^1 \sum_{j \in Q} x_j \leq 1$$

of $x(G_1, a^1) \leq 1$ with an arbitrary clique facet associated with $Q \subseteq V_2$ has again $a_i^1 \in \{0, c\}$ as only coefficients. Thus, the only non-trivial facets of STAB(G) are rank constraints.

Conversely, if G is supposed to be rank-perfect then G_2 has to be perfect (otherwise STAB(G_2) has a non-trivial facet different from a clique constraint and STAB(G) has a non-rank facet by Proposition 7.14). G_1 has to be rank-perfect (otherwise STAB(G_1) has a non-trivial, non-clique facet and its facet product with an arbitrary clique facet of STAB(G_2) yields a non-trivial, non-clique facet of STAB(G)). □

Thus, precisely substituting perfect graphs for nodes preserves rank-perfection and substituting imperfect graphs for nodes in near-perfect, h-perfect, a-perfect, or p-perfect graphs cannot preserve the membership in those classes, too. Thus, the class of perfect graphs is the only class of rank-perfect graphs which is closed under substitution. As a further consequence we obtain that all classes of rank-perfect graphs satisfy a necessary condition for having a bounded imperfection ratio.

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